A generalization of the Kobayashi-Oshima uniformly bounded multiplicity theorem

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Abstract
Let $P$ be a minimal parabolic subgroup of a real reductive Lie group $G$ and $H$ a closed subgroup of $G$. Then it is proved by T. Kobayashi and T. Oshima that the regular representation $C^\infty(G/H)$ contains each irreducible representation of $G$ at most finitely many times if the number of $H$-orbits on $G/P$ is finite. Moreover, they also proved that the multiplicities are uniformly bounded if the number of $H^C$-orbits on $G^C/B$ is finite, where $G^C, H^C$ are complexifications of $G, H$, respectively, and $B$ is a Borel subgroup of $G^C$. In this article, we prove that the multiplicities of the representations of $G$ induced from a parabolic subgroup $Q$ in the regular representation on $G/H$ are uniformly bounded if the number of $H^C$-orbits on $G^C/Q^C$ is finite. For the proof of this claim, we also show the uniform boundedness of the dimensions of the spaces of group invariant hyperfunctions using the theory of holonomic $D_X$-modules.

Keywords: orbit decomposition, spherical variety, intertwining operator, multiplicity, holonomic system, hyperfunction.

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1 Introduction
Let $G$ be a real reductive Lie group, $H$ a closed subgroup, and $Q$ a parabolic subgroup of $G$. In this article, we prove that the multiplicities of the representations of $G$ induced from $Q$ in the regular representation on $G/H$ are uniformly bounded if the number of $H^C$-orbits on $G^C/Q^C$ is finite, where $G^C, H^C, Q^C$ are complexifications of $G, H$, and $Q$, respectively. For the proof of this claim, we also show the uniform boundedness of the dimensions of the spaces of group invariant hyperfunctions using the theory of holonomic $D_X$-modules. We explain the motivation of this in the following subsections. The main results are stated in Section 1.3.

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1.1 The Finite Multiplicity Theorem

Let $G$ be a real reductive algebraic group and $H$ a real algebraic subgroup of $G$. T. Kobayashi and T. Oshima established a finiteness criterion for multiplicities of the regular representation on the homogeneous space $G/H$.

**Fact 1.1** ([19, Thm. A]). Suppose that $G$ and $H$ are defined algebraically over $\mathbb{R}$. Then the following two conditions on the pair $(G, H)$ are equivalent:

(i) $\dim \text{Hom}_G(\pi, C^\infty(G/H, \tau)) < \infty$ for any $(\pi, \tau) \in \hat{G}_{\text{smooth}} \times \hat{H}$,

(ii) $G/H$ is real spherical.

**Remark 1.2.** In [19], an explicit upper bound of the dimensions in (i) of Fact 1.1 was also given.

Here, $\hat{G}_{\text{smooth}}$ denotes the set of equivalence classes of irreducible smooth admissible Fréchet representations of $G$ with moderate growth, and $\hat{H}$ that of irreducible finite-dimensional representations of $H$. Given $\tau \in \hat{H}$, we write $C^\infty(G/H, \tau)$ for the Fréchet space of smooth sections of the $G$-homogeneous vector bundle over $G/H$ associated to $\tau$. The terminology real sphericity was introduced by Kobayashi [17] in his study of a broader framework for global analysis on homogeneous spaces than the usual (e.g., reductive symmetric spaces).

**Definition 1.3.** A homogeneous space $G/H$ is real spherical if a minimal parabolic subgroup $P$ of $G$ has an open orbit on $G/H$.

The following is one of the characterizations of real spherical homogeneous spaces. This is a consequence of the rank one reduction of T. Matsuki [24] and the classification of real spherical homogeneous spaces of real rank one by B. Kimelfeld [16].

**Fact 1.4** ([3]). For the pair $(G, H)$, the following two conditions are equivalent:

(ii) $G/H$ is real spherical, (i.e., there exists an open $P$-orbit on $G/H$),

(iii) $\#(H \backslash G/P) < \infty$.

Therefore, for a minimal parabolic subgroup $P$, the conditions (i), (ii), and (iii) are equivalent by Facts 1.1 and 1.4 (see Figure 1 below). Thus we ask a question what will happen to the relationship among the three conditions, if we replace $P$ by a general parabolic subgroup $Q$ of $G$. There is an obvious extension of the conditions (ii) and (iii) to a general parabolic subgroup $Q$ (see Definition 1.6 below). In order to formulate a variant of (i) for a parabolic subgroup $Q$ of $G$, we review the notion of $Q$-series.

**Definition 1.5** ([18, Def. 6.6]). Let $\pi \in \hat{G}_{\text{smooth}}$. We say that $\pi$ belongs to $Q$-series if $\pi$ occurs as a subquotient of the degenerate principal series representation $C^\infty(G/Q, \tau)$ for some $\tau \in \hat{Q}$. 

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For a parabolic subgroup $Q$ of $G$, we set
\[
\hat{G}^Q_{\text{smooth}} := \{ \pi \in \hat{G}_{\text{smooth}} \mid \pi \text{ belongs to } Q\text{-series} \}.
\]
Obviously, $\hat{G}^Q_{\text{smooth}} \supset \hat{G}^{Q'}_{\text{smooth}}$ if $Q \subset Q'$. Moreover, $\hat{G}^Q_{\text{smooth}}$ is equal to $\hat{G}_{\text{smooth}}$ if $Q = P$ (minimal parabolic) by Harish-Chandra’s subquotient theorem [9] and to $\hat{G}_f$ if $Q = G$.

**Definition 1.6.** For a parabolic subgroup $Q$ of $G$, we define three conditions (i$_Q$), (ii$_Q$), and (iii$_Q$) as follows:

(i$_Q$) $\dim \text{Hom}_G(\pi, C^\infty(G/H, \tau)) < \infty$ for all $(\pi, \tau) \in \hat{G}^Q_{\text{smooth}} \times \hat{H}_{\text{alg}}$.

(ii$_Q$) $Q$ has an open orbit on $G/H$,

(iii$_Q$) $\#(H\backslash G/Q) < \infty$.

The conditions (i$_Q$), (ii$_Q$), and (iii$_Q$) reduce to (i), (ii), and (iii), respectively, if $Q = P$ (minimal parabolic), and we have seen in Facts 1.1 and 1.4 that the following equivalences hold for $Q = P$,

\[(i_P) \iff (ii_P) \iff (iii_P)\]

Furthermore, if $Q = G$, the condition (i$_Q$) automatically holds by the Frobenius reciprocity, while (ii$_Q$) and (iii$_Q$) are obvious. Hence

\[(i_G) \iff (ii_G) \iff (iii_G)\]

For a general parabolic subgroup $Q$, clearly, (iii$_Q$) implies (ii$_Q$). However, there is an easy counterexample for the converse statement.

**Example 1.7.** The real projective space $\mathbb{RP}^2 = SL(3, \mathbb{R})/Q = G/Q$ splits into an open orbit and continuously many fixed points of the unipotent radical $H$ of the opposite parabolic subgroup $Q$ of $Q$.

On the other hand, the implication (i$_Q$) $\Rightarrow$ (ii$_Q$) holds. To see this, we define a subset $\hat{H}_f(G)$ of $\hat{H}_f$ by

\[\hat{H}_f(G) := \{ \tau \in \hat{H}_f \mid \tau \text{ appears as a quotient of some element of } \hat{G}_f \}.
\]

The implication (i$_Q$) $\Rightarrow$ (ii$_Q$) is derived from the following stronger assertion:
Fact 1.8 ([18 Cor. 6.8]). If $Q$ does not have an open orbit on $G/H$, then, for any $\tau \in \hat{H}(G)$, there exists $\pi \in \hat{G}_\text{smooth}^Q$ such that $\dim \text{Hom}_G(\pi, C^\infty(G/H, \tau)) = \infty$.

This fact implies that $(i_Q) \Rightarrow (ii_Q)$ is true. However, $(ii_Q) \Rightarrow (i_Q)$ is not always true for a general parabolic subgroup $Q$.

Example 1.9. $\dim \text{Hom}_G(C^\infty(G/Q), C^\infty(G/H)) = \infty$ for the pair $(G, H, Q)$ in Example 1.7.

We summarize the known relationship among the three conditions in Figure 2, which shows that the relation between the conditions $(i_Q)$ and $(iii_Q)$ is unsettled.

Question 1. Determine the relationship between the following two conditions:

$(i_Q)$ $\dim \text{Hom}_G(\pi, C^\infty(G/H, \tau)) < \infty$ for all $(\pi, \tau) \in \hat{G}_{\text{smooth}}^Q \times \hat{H}_{\text{alg}}$, 

$(iii_Q)$ $\#(H \setminus G/Q) < \infty$.

For this question, we proved that there exists the pair $(G, H, Q)$ satisfying the condition $(iii_Q)$, although it does NOT satisfy the condition $(i_Q)$ in [22 Thm. 1.8], and proved that $(i_Q) \Rightarrow (ii_Q)$ holds under a certain condition of orientation in [23 Thm. 2.4]. Figure 3 given below summarises the relationship among the conditions $(i_Q)$, $(ii_Q)$ and $(iii_Q)$. In this figure, $\Delta$ on the arrow of $(i_Q) \Rightarrow (iii_Q)$ means that this is proved only under the additional assumption.

Figure 3. $Q$: general parabolic

1.2 Uniformly Bounded Multiplicity

By Fact 1.1, the finiteness of the number of $P$-orbits on $G/H$ guarantees that of multiplicities in the regular representation on $G/H$. Kobayashi and Oshima also proved the criterion for uniformly boundedness of multiplicities in the regular representation. In this article, we say that a complex Lie group $L_{\mathbb{C}}$ is a complexification of a Lie group $L$ if $L_{\mathbb{C}}$ contains $L$ as a closed subgroup and $l_{\mathbb{C}} = l \oplus \sqrt{-1}l$, where $l_{\mathbb{C}}$ is the Lie algebra of $L_{\mathbb{C}}$.

Fact 1.10 ([19 Theorem B]). The following two conditions on the pair $(G, H)$ are equivalent:

(I) $\sup_{\tau \in \hat{H}} \sup_{\pi \in \hat{G}_{\text{smooth}}} \frac{1}{\dim \tau} \dim \text{Hom}_G(\pi, C^\infty(G/H, \tau)) < \infty$,

(II) $G_{\mathbb{C}}/H_{\mathbb{C}}$ is spherical.
Remark 1.11. In [19], an explicit upper bound of (I) of Fact 1.10 was also given, which is optimal in the case that $H$ is the maximal unipotent subgroup $N$ of $G$.

Here we say that a homogeneous space $G_C/H_C$ is spherical if a Borel subgroup of $G_C$ has an open orbit on $G_C/H_C$. It is well known that the condition (II) in Fact 1.10 is characterized by the finiteness of the number of $B$-orbits on $G_C/H_C$.

Fact 1.12 ([6, 24, 29]). The condition (II) in Fact 1.10 is equivalent to the following condition:

(III) $\#(B\backslash G_C/H_C) < \infty$.

Therefore, for a Borel subgroup $B$, the three conditions (I), (II), and (II) are equivalent by Facts 1.10 and 1.12 just like the case of Figure 1 (see Figure given below). This equivalence can be interpreted that the $B$-orbit decomposition of $G_C/H_C$ has some information about uniformly boundedness of the multiplicities in the regular representation on $G/H$.

Figure 4. $B$: Borel subgroup

On the other hand, in Section 1.1 we consider the relationship between the $Q$-orbit decomposition of $G/H$ and the multiplicities of $Q$-series representations in the regular representation on $G/H$ motivated by the equivalence in Figure 1. Thus, Figure 4 makes us think about the following question:

Question 2. Determine the relationship between the $Q_C$-orbit decomposition of $G_C/H_C$ and the uniform boundedness of $Q$-series representations in regular representation on $G/H$.

1.3 Main Theorems

In this article, we prove a variant of Fact 1.10 for a (not necessary minimal) parabolic subgroup $Q$ as an answer of Question 2.

Theorem 1.13. Let $Q$ be a parabolic subgroup of a real reductive Lie group $G$, and $H$ a closed subgroup of $G$. We write $G_C$, $H_C$ and $Q_C$ for complexifications of $G$, $H$ and $Q$, respectively. If the number of connected components of $H_C$ is finite and $\#(H_C\backslash G_C/Q_C) < \infty$, then we have

$$\sup_{(\eta, \tau) \in \hat{Q}_t \times \hat{H}_t} \frac{1}{\dim \eta \cdot \dim \tau} \dim \text{Hom}_G(C^\infty(G/Q, \eta), C^\infty(G/H, \tau)) < \infty. \quad (1.1)$$
For the proof of Theorem 1.13, we give an upper bound of the dimensions of the relative invariant Sato hyperfunction spaces with respect to a group action by using the theory of holonomic \( D_X \)-modules. Let \( \mathcal{B}_M \) be the sheaf of Sato’s hyperfunctions on a real analytic manifold \( M \). We say that a complex manifold \( X \) is a complexification of \( M \) if \( X \) contains \( M \) as a real analytic submanifold and \( T_x X = T_x M \oplus \sqrt{-1} T_x M \) for any \( x \in M \).

**Theorem 1.14.** Let \( X \) be a complexification of a real analytic manifold \( M \). Suppose that a complex Lie group \( H_C \) with finitely many connected components acts on \( X \) with \( \#(H_C \backslash X) < \infty \). Then, for any relatively compact semianalytic open subset \( U \subset M \), there exists \( C > 0 \) such that for any finite-dimensional representation \( \tau \) of \( h_C \), we have

\[
\dim(\Gamma(U; \mathcal{B}_M) \otimes \tau)^{hc} \leq C \cdot \dim \tau.
\]

Moreover, we can give an alternative approach of (II) \( \Rightarrow \) (I) in Fact 1.10 as a corollary of Theorem 1.14.

**Remark 1.15.** As stated in Remarks 1.2 and 1.11, explicit upper bounds of multiplicities were already given in [19]. Using the method of this article, one can show that the left-hand side of (1.1) is bounded by

\[
\sum_{p=0}^{\dim G_C/Q_C} \frac{(\dim G_C/Q_C)!}{(\dim G_C/Q_C - p)!} \cdot (2k_0)^p,
\]

which is not optimal because this is the sum of upper bounds of the dimensions of invariant hyperfunctions supported by each \( H \)-orbit on \( G/Q \) (Intertwining operators can be regarded as invariant distributions, see Fact 5.1). Here, \( k_0 \) is the maximum value of heights of the roots of \( g_C \) relative to some Cartan subalgebra and \( c_p \) is the sum of the numbers of connected components of \( \mathcal{O} \cap G/Q \) for all \( p \)-codimensional \( H_C \)-orbits \( \mathcal{O} \) on \( G_C/Q_C \).

**Remark 1.16.** In addition to the original proof of the implication (II) \( \Rightarrow \) (I) in Fact 1.10 given in [19], Kobayashi suggested an alternative approach by using the theory of holonomic \( D \)-modules in Bonn, September 2011. In this direction, there is a recent work by A. Aizenbud, D. Gourevitch and A. Minchenko [1] by an approach of holonomic \( D \)-modules, however, there are also some differences between Theorem 1.14 and their results. Their proof utilizes an argument of the Weil representation and a different filtration from ours, and their estimate of the multiplicity is not strong enough to recover the implication (II) \( \Rightarrow \) (I) in Fact 1.10. We note that [1, Thm. D] for bundle-valued tempered distributions in the algebraic setting can be deduced from Theorem 1.14 for hyperfunctions in the analytic setting.

This article is organized as follows. In Section 2 we recall the theory of sheaves and \( D_X \)-modules. In Section 3 we give an upper bound of the dimensions of the Sato hyperfunction solution spaces for holonomic \( D_X \)-modules. We prove Theorem 1.14 in Section 4. Theorem 1.13 is proved in Section 5. An alternative approach of (II) \( \Rightarrow \) (I) in Fact 1.10 is given in Section 6.
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2 Preliminaries of Sheaves and \( D_X \)-modules

In this section, we recall the basic notions of the theory of sheaves and \( D_X \)-modules. Although almost all the materials in this section are well known, we prove some results for the convenience.

2.1 Sheaves

In this subsection, we briefly recall the basic notions of the theory of sheaves. For further references on this subject, see [13], for example.

Let \( X \) be a good topological space (i.e., a Hausdorff, locally compact space which is countable at infinity and has finite flabby dimension), and \( C_X \) the constant sheaf on \( X \). We write \( \text{Mod}(C_X) \) for the abelian category of sheaves of \( C \)-vector spaces on \( X \) and \( D^b(C_X) \) for the bounded derived category of \( \text{Mod}(C_X) \).

Identifying a sheaf \( S \in \text{Mod}(C_X) \) with a complex of sheaves

\[
\cdots \rightarrow 0 \rightarrow S \rightarrow 0 \rightarrow \cdots \in D^b(C_X)
\]

(with \( S \) in degree 0), we regard \( \text{Mod}(C_X) \) as a full subcategory of \( D^b(C_X) \).

For a closed subset \( \iota_K : K \hookrightarrow X \) and \( S \in \text{Mod}(C_X) \), we define \( S_K \in \text{Mod}(C_X) \) by

\[
S_K := \iota_K \iota_K^{-1}(S),
\]

where \( \iota_K \) and \( \iota_K^{-1} \) are the direct image functor and the inverse image functor, respectively. Then, because \( (\iota_K^{-1}, \iota_K) \) is an adjoint pair of functors, we have

\[
\text{Hom}_{\text{Mod}(C_X)}(\iota_K^{-1}(S), \iota_K^{-1}(S)) \simeq \text{Hom}_{\text{Mod}(C_X)}(S, \iota_K \iota_K^{-1}(S)). \tag{2.1}
\]

Thus, there exists a natural map \( S \rightarrow S_K \), which corresponds to the identity of the left-hand side of (2.1). Then, for an open subset \( U \subset X \), we define \( S_U := \text{Ker}(S \rightarrow S_K) \), where \( K := X \setminus U \). For a locally closed subset \( Z \subset X \), we take an open subset \( U \subset X \) and a closed subset \( K \subset X \) satisfying \( Z = U \cap K \) and define \( S_Z := (S_U)_K \). This definition does not depend on the choice of \( U \) and \( K \). We sometimes abbreviate \( (C_X)_Z \) to \( C_Z \). We write \( \text{Mod}_K(C_X) \) for the full subcategory of \( \text{Mod}(C_X) \) consisting objects whose supports are contained in a closed subset \( K \subset X \).

Lemma 2.1. For \( S \in \text{Mod}_K(C_X) \), we have \( S_K \simeq S \).
Proof. By the definition, we have an exact sequence

\[ 0 \to S_{X\setminus K} \to S \to S_K \to 0. \]  \hspace{1cm} (2.2)

Taking stalks at each point, we have \( S_{X\setminus K} = 0 \). Thus, (2.2) implies the lemma.

\[ \square \]

**Lemma 2.2.** For a closed subset \( \iota_K : K \hookrightarrow X \) and \( S \in \text{Mod}(\mathbb{C}_X) \), we have \( \Gamma(K; \iota_K^{-1}S) = \Gamma(X; S) \), where \( \Gamma(X; S) \) is the space of sections of \( S \) on \( X \).

**Proof.** \( S \simeq S_K \simeq \iota_K \ast \iota_K^{-1}(S) \) implies \( \Gamma(X; S) = \Gamma(X; \iota_K \ast \iota_K^{-1}(S)) = \Gamma(K; \iota_K^{-1}S) \).

\[ \square \]

For a closed subset \( K \) of an open subset \( U \subset X \), we define a subspace of \( \Gamma(U; S) \) by

\[ \Gamma_K(U; S) := \text{Ker}(\Gamma(U; S) \to \Gamma(U \setminus K; S)). \]

For a locally closed subset \( Z \subset X \), we take an open subset \( U \subset X \) and a closed subset \( K \subset X \) satisfying \( Z = U \cap K \) and set \( \Gamma_Z(U; S) := \Gamma_{U \cap Z}(U; S) \). Then, for a locally closed subset \( Z \subset X \), the sheaf \( \Gamma_Z(S) \) is defined by

\[ \Gamma(U; \Gamma_Z(S)) := \Gamma_{Z \cap U}(U; S) \]

for an open subset \( U \subset X \). Then, \( \Gamma_Z \) is an endofunctor on \( \text{Mod}(\mathbb{C}_X) \). We write \( R\Gamma_K : D^b(\mathbb{C}_X) \to D^b(\mathbb{C}_X) \) for the right derived functor of \( \Gamma_K \) and \( R^k\Gamma_Z \) for the \( k \)-th cohomology. We note that \( \Gamma_K(U; S) = j_* \circ j^{-1}(\Gamma(U; S)) \) for an open subset \( j : U \hookrightarrow X \) and that \( \text{supp}(\Gamma_K(S)) \subset K \) for a closed subset \( K \subset X \).

We quote the following two properties of the functor \( \Gamma_Z \). See, [15 Prop. 2.3.9] or [5 Prop 4.14 in Appx. II], for example.

**Lemma 2.3.** Let \( Z \) be a locally closed subset of \( X \) and \( Z' \) a closed subset of \( Z \). Then, for \( S \in \text{Mod}(\mathbb{C}_X) \), we have an exact sequence

\[ 0 \to \Gamma_Z(S) \to \Gamma_Z(S) \to \Gamma_{Z \setminus Z'}(S). \]

**Lemma 2.4.** Let \( K \) and \( K' \) be closed subsets of \( X \). Then we have

\[ R\Gamma_K \circ R\Gamma_{K'}(\cdot) \simeq R\Gamma_{K \cap K'}(\cdot). \]

For a morphism \( f : X \to Y \) of good topological spaces, we write \( f_! : \text{Mod}(\mathbb{C}_X) \to \text{Mod}(\mathbb{C}_Y) \) for the proper direct image functor. Namely, for \( S \in \text{Mod}(\mathbb{C}_X) \), we have

\[ \Gamma(U; f_!S) = \{ s \in \Gamma(f^{-1}(U); S) \mid f : \text{supp}(s) \to U \text{ is proper} \}, \]

where \( U \) is an open subset of \( Y \). We also write \( f^! \) for the right adjoint functor of the right derived functor \( Rf_! \) of \( f_! \).

Let \( \{ pt \} \) be a topological space with one element. Then, there exists a natural map \( a_X : X \to \{ pt \} \). The dualizing complex \( \omega_X \) on \( X \) is defined by
where \([-\dim X]\) is the shift functor. Note that \(o_X\) is locally isomorphic to \(\mathbb{C}_X\). Namely, for any point \(x\) locally the following isomorphisms as an object of \(D^b(\mathbb{C}_X)\) is defined by

\[
\iota \colon o_X := \omega_X [-\dim X] \in \text{Mod}(\mathbb{C}_X),
\]

(2.3)

where \([-\dim X]\) is the shift functor. Note that \(o_X\) is locally isomorphic to \(\mathbb{C}_X\). For this, we need Lemma 2.5 whose proof can be found in [15, Prop. 3.1.12] for example.

**Lemma 2.5.** Let \(f \colon X \rightarrow Y\) be a morphism of good topological spaces. Assume \(X\) is diffeomorphic to some locally closed subset of \(Y\) by \(f\). Then we have

\[
f^!(\cdot) \simeq f^{-1} \circ \Gamma_{f(X)}(\cdot).
\]

In particular, if \(f \colon X \rightarrow Y\) is a closed embedding, Lemma 2.7 implies

\[
f_\ast f^!(\cdot) \simeq \Gamma_{f(X)}(\cdot)
\]

by exactness of \(f_\ast \circ f^{-1}(\cdot) = (\cdot)_{f(X)}\).

**Lemma 2.6.** Let \(L\) be a real analytic manifold, \(M\) a closed \(m\)-dimensional submanifold of \(L\), and \(N\) a closed \(n\)-dimensional submanifold of \(M\). Then, \(\iota \Gamma_N(C_M)\) is locally isomorphic to \(\mathbb{C}_N[-(m-n)]\) as an object of \(D^b(\mathbb{C}_L)\).

**Proof.** By (2.3), we have locally \(C_M[m] \simeq a^1_M(\mathbb{C}_{pt})\) and \(C_N[n] \simeq a^1_N(\mathbb{C}_{pt})\). Setting \(\iota_N \colon N \rightarrow M\), we have \(a_M \circ \iota_N = a_N\). Therefore, by Lemma 2.5 we have locally the following isomorphisms as an object of \(D^b(\mathbb{C}_M)\):

\[
\iota_\ast \iota_N^! \simeq \iota_\ast a^1_M(\mathbb{C}_{pt}) \simeq \iota_N \ast (a_M \circ \iota_N)^!(\mathbb{C}_{pt}) \simeq \iota_N \ast a^1_N(\mathbb{C}_{pt}) \simeq \iota_N \ast \mathbb{C}_N[n].
\]

By applying the exact functor \(\iota_\ast\), where \(\iota_M \colon M \hookrightarrow L\) is a closed embedding, we have the desired result. \(\square\)

Recall that \(\mathcal{S} \in \text{Mod}(\mathbb{C}_X)\) is called a locally constant sheaf of finite rank on \(X\) if for any \(x \in X\), there exists \(l \in \mathbb{Z}_{\geq 0}\) such that we have \(\mathcal{S} \simeq \mathbb{C}_X^l\) on a sufficiently small open neighborhood of \(x\) in \(X\). For the convenience, we shall often use the following terminology:

**Definition 2.7.** Let \(\mathcal{S} \in \text{Mod}(\mathbb{C}_X)\) and \(\iota_K : K \hookrightarrow X\) a closed subset. Then, we call \(\mathcal{S}\) a locally constant sheaf of finite rank supported in \(K\) if \(\text{supp}(\mathcal{S}) \subset K\) and \(\iota_K^{-1}\mathcal{S}\) is a locally constant sheaf of finite rank on \(K\).
We need the following two lemmas, which show that the dimension of the space of the global sections of a locally constant sheaf supported in \( K \) is bounded by the dimension of its stalk.

**Lemma 2.8.** Let \( S \) be a locally constant sheaf on a connected topological space \( X \). Then, for any \( x \in X \), we have
\[
\dim \Gamma(X; S) \leq \dim S_x,
\]
where \( S_x \) is the germ of \( S \) at \( x \).

**Proof.** It is clear that the natural map \( \Gamma(X; S) \to S_x \) is injective. \( \square \)

**Lemma 2.9.** Let \( K \subset X \) be a connected closed subset, \( S \in \text{Mod}(\mathbb{C}^X) \) a locally constant sheaf supported in \( K \). Then, for any \( x \in K \), we have
\[
\dim \Gamma(X; S) \leq \dim S_x.
\]

**Proof.** By Lemma 2.2, we have \( \Gamma(X; S) = \Gamma(K; \iota_K^{-1}S) \). Thus, the lemma follows from Lemma 2.8. \( \square \)

### 2.2 \( \mathcal{D}_X \)-module

In this subsection, we briefly review some properties of \( \mathcal{D} \)-modules, which is used in later sections. We mainly refer to [11], [15], and [25].

Let \( X \) be a \( d_X \)-dimensional complex manifold. We write \( \mathcal{O}_X \) and \( \mathcal{D}_X \) for the sheaves of holomorphic functions and holomorphic differential operators with finite order, respectively, on \( X \). For a closed \( d_Y \)-dimensional submanifold \( Y \subset X \), recall that the sheaf \( B_Y|X \) is defined by
\[
B_Y|X := R\Gamma_Y(\mathcal{O}_X)[d_X - d_Y]. \tag{2.4}
\]

In particular, we have \( B_X|X = \mathcal{O}_X \) in the case \( Y = X \). Note that the complex \( R\Gamma_Y(\mathcal{O}_X)[d_X - d_Y] \) is concentrated at degree zero [26, Prop 6.2.2]. Namely, we have \( R^k\Gamma_Y(\mathcal{O}_X) = 0 \) for \( k \neq d_X - d_Y \).

If \( X \) is a complexification of some real analytic manifold \( M \), the sheaf \( B_M \) of Sato’s hyperfunctions is defined by
\[
B_M := \mathcal{R}\Gamma_M(\mathcal{O}_X)[d_X] \otimes_{\mathbb{Z}_M} \text{or}_M, \tag{2.5}
\]
where \( \text{or}_M \) is the orientation sheaf. Note that the complex \( \mathcal{R}\Gamma_M(\mathcal{O}_X)[d_X] \) is concentrated at degree zero [26, Prop 7.2.2] and that \( B_M|U = B_U \) for an open subset \( U \subset M \).

Let \( \mathcal{F} \) be the order filtration of \( \mathcal{D}_X \). The associated graded ring \( \text{gr}_\mathcal{F}(\mathcal{D}_X) \) of \( \mathcal{D}_X \) with respect to \( \mathcal{F} \) is defined by
\[
\text{gr}_\mathcal{F}(\mathcal{D}_X) := \bigoplus_{j \in \mathbb{Z}_{\geq 0}} \mathcal{F}_j(\mathcal{D}_X)/\mathcal{F}_{j-1}(\mathcal{D}_X).
\]

Note that there exists an injection \( \pi^{-1}(\text{gr}_\mathcal{F}(\mathcal{D}_X)) \hookrightarrow \mathcal{O}_{T^*X} \), where \( \pi: T^*X \to X \) is the natural projection. For a \( \text{gr}_\mathcal{F}(\mathcal{D}_X) \)-module \( M \), we set
\[
F(M) := \mathcal{O}_{T^*X} \otimes_{\pi^{-1}(\text{gr}_\mathcal{F}(\mathcal{D}_X))} \pi^{-1}(M). \tag{2.6}
\]
Then $F$ defines a functor from the category of $gr_{\mathcal{F}}(\mathcal{D}_X)$-modules to that of $\mathcal{O}_{T_\ast X}$-modules. By the exactness of the inverse image functor and the right exactness of the tensor product, $F$ is right exact.

Let $\mathfrak{M}$ be a coherent $\mathcal{D}_X$-module. The characteristic variety $Ch(\mathfrak{M}) \subset T^\ast X$ of $\mathfrak{M}$ is defined by the support of a $\mathcal{O}_{T_\ast X}$-module $F(gr_{\mathcal{F}_\mathfrak{M}}(\mathfrak{M}))$, where $\mathcal{F}_{\mathfrak{M}}$ is a good filtration of $\mathfrak{M}$. This definition does not depend on the choice of a good filtration. For an irreducible closed analytic subset $V$ of $T^\ast X$, we write $\text{mult}_{V}(\mathfrak{M})$ for the multiplicity of $\mathfrak{M}$ along $V$ \cite[Prop. 1.8.2]{25}, which is given by

$$\text{mult}_{V}(\mathfrak{M}) := \text{mult}_{V}(F(gr_{\mathcal{F}_\mathfrak{M}}(\mathfrak{M}))).$$

Here, $\text{mult}_{V}(F(gr_{\mathcal{F}_\mathfrak{M}}(\mathfrak{M})))$ is the multiplicity of an $\mathcal{O}_{T_\ast X}$-module $F(gr_{\mathcal{F}_\mathfrak{M}}(\mathfrak{M}))$ along $V$ (See \cite[Sec. 1.4 in Appx. V]{12}, for example). Note that if $V$ is not contained in $Ch(\mathfrak{M})$, we have $\text{mult}_{V}(\mathfrak{M}) = 0$.

**Remark 2.10.** It is known that if two holonomic $\mathcal{D}_X$-modules $\mathfrak{M}$ and $\mathfrak{M}'$ satisfy $\mathcal{D}_X^\ast \otimes_{\mathcal{D}_X} \mathfrak{M} \simeq \mathcal{D}_X^\ast \otimes_{\mathcal{D}_X} \mathfrak{M}'$, we have $\text{mult}_{V}(\mathfrak{M}) = \text{mult}_{V}(\mathfrak{M}')$, where $\mathcal{D}_X^\ast$ is the sheaf of holomorphic differential operators of infinite order. This is because the multiplicity of $\mathfrak{M}$ is determined by the perverse sheaf $\mathbb{R}Hom_{\mathcal{D}_X}(\mathfrak{M}, \mathcal{O}_X) \simeq \mathbb{R}Hom_{\mathcal{D}_X}(\mathcal{D}_X^\ast \otimes_{\mathcal{D}_X} \mathfrak{M}, \mathcal{O}_X)$ (see \cite[Sect. 8.2]{12} or \cite[(7.23)]{27}). We use this fact in Appendix to prove Corollary 2.12 given below.

A coherent $\mathcal{D}_X$-module $\mathfrak{M}$ is called holonomic if the characteristic variety $Ch(\mathfrak{M}) \subset T^\ast X$ of $\mathfrak{M}$ is $d_X$-dimensional or $\mathfrak{M} = 0$. We note that if a coherent $\mathcal{D}_X$-module $\mathfrak{M}$ is nonzero, we have $\dim Ch(\mathfrak{M}) \geq d_X$ by the involutivity of the characteristic variety \cite[Thm. 5.3.2]{25}. A stratification $X = \bigsqcup_{\alpha \in A} X_{\alpha}$ of $X$ is called regular with respect to a holonomic $\mathcal{D}_X$-module $\mathfrak{M}$ if the stratification $X = \bigsqcup_{\alpha \in A} X_{\alpha}$ satisfies the regularity conditions of H. Whitney \cite{(a),(b) in Sect. 19]} and $Ch(\mathfrak{M})$ is contained in the union $\bigsqcup_{\alpha \in A} T^\ast_{X_{\alpha}} X$ of the conormal bundles of each stratum \cite[Def. (3.4)]{10}. For any holonomic $\mathcal{D}_X$-module $\mathfrak{M}$, there exists a regular stratification of $X$ with respect to $\mathfrak{M}$ \cite[Lem. (3.2)]{10}.

We use Kashiwara’s constructibility theorem of holonomic $\mathcal{D}_X$-modules:

**Fact 2.11** (\cite[Thms. (3.5) and (3.7)]{10}). Let $\mathfrak{M}$ be a holonomic $\mathcal{D}_X$-module and $X = \bigsqcup_{\alpha \in A} X_{\alpha}$ a regular stratification of $X$ with respect to $\mathfrak{M}$. Then, $\mathbb{R}^k\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \mathcal{O}_X)|_{X_{\alpha}}$ is a locally constant sheaf of finite rank on $X_{\alpha}$ for any $\alpha \in A$ and any $k \in \mathbb{Z}$. Moreover, if $Y := \bigsqcup_{\beta \in B} X_{\beta}$ is a closed subset of $X$ for some subset $B \subset A$, then $\mathbb{R}^k\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \mathcal{B}_{Y_{|X}})|_{X_{\alpha}}$ is also a locally constant sheaf of finite rank on $X_{\alpha}$ for any $\alpha \in A$ and any $k \in \mathbb{Z}$.

**Corollary 2.12.** Let $X = \bigsqcup_{\alpha \in A} X_{\alpha}$ be a regular stratification of $X$ with respect to a holonomic $\mathcal{D}_X$-module. Suppose that $X_{\alpha}$ is closed in $X$. Then, $\mathbb{R}^k\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \mathcal{B}_{X_{\alpha}_{|X}})$ is a locally constant sheaf of finite rank supported in $X_{\alpha}$ for any $k \in \mathbb{Z}$. Moreover, the rank of $\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \mathcal{B}_{X_{\alpha}_{|X}})$ is not greater than $\text{mult}_{T^\ast_{X_{\alpha}} X}(\mathfrak{M})$.

**Proof.** The assertion for $\mathbb{R}^k\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \mathcal{B}_{X_{\alpha}_{|X}})$ is obvious by Fact 2.11. The assertion for the rank of $\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \mathcal{B}_{X_{\alpha}_{|X}})$ is proved in Appendix \cite{A} because we use a different method from this section. \qed
Remark 2.13. Although we use the theory of regular holonomic $\mathcal{D}_X$-modules in this article (see Appendix A), Corollary 2.12 can be proved by using the theory of $\mathcal{E}^X$-modules, see [11] Thm. 3.2.42. Moreover, if the support of $\mathcal{M}$ is a locally constant sheaf supported in $X_\alpha$ (and with the assumption of Corollary 2.12), then the rank of $\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, B_{X_\alpha \mid X})$ is equal to $\text{mult}_{T_{X_\alpha \mid X}}(\mathcal{M})$ by [10] Prop. 3.9.

3 Upper bound of the dimensions of the Sato hyperfunction solution spaces

In this section, we give an upper bound of the dimensions of the Sato hyperfunction solution spaces. First, we estimate the dimension of the space of solutions whose supports are contained in one closed stratum $X_\alpha$. 

Proposition 3.1. Let $M$ be a real analytic manifold, $X$ a complexification of $M$, and $X = \bigsqcup_{\alpha \in A} X_\alpha$ a regular stratification of $X$ with respect to a holonomic $\mathcal{D}_X$-module $\mathcal{M}$. Suppose that $X_\alpha$ is a closed connected subset $X$ and that a subset $M'_{\alpha}$ of $M \cap X_\alpha$ is a closed submanifold of $X_\alpha$. Then, $\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{M'_{\alpha}}(B_M))$ is a locally constant sheaf supported in $M'_{\alpha}$ whose rank is not greater than $\text{mult}_{T_{X_\alpha \mid X}}(\mathcal{M})$.

Before proving, we recall the Grothendieck spectral sequence.

Fact 3.2 ([8] Thm. 2.4.1). Let $\mathcal{C}, \mathcal{C}', \mathcal{C}''$ be abelian categories, $F: \mathcal{C} \rightarrow \mathcal{C}'$ and $G: \mathcal{C}' \rightarrow \mathcal{C}''$ left exact functors. Suppose that $F$ takes injective objects of $\mathcal{C}$ to injective objects of $\mathcal{C}'$. Then for any $A \in \mathcal{C}$, we have a spectral sequence

$$E_2^{p,q} = \mathbb{R}^pG \circ \mathbb{R}^qF(A) \Rightarrow \mathbb{R}^{p+q}(G \circ F)(A).$$

Proof of Proposition 3.1. It is clear that the support of $\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{M'_{\alpha}}(B_M))$ is contained in $M'_{\alpha}$. Therefore, it is sufficient to prove that for any $x_\alpha \in M'_{\alpha}$, there exists an isomorphism

$$\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{M'_{\alpha}}(B_M)) \simeq \mathbb{C}^l_{X_\alpha}$$

on a sufficiently small open neighborhood $U_\alpha \subset X$ of $x_\alpha \in M'_{\alpha}$ satisfying $l \leq \text{mult}_{T_{X_\alpha \mid X}}(\mathcal{M})$. Corollary 2.12 implies that there exists an isomorphism

$$\mathbb{R}^k\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, B_{X_\alpha \mid X}) \simeq \mathbb{C}^{l_0}_{X_\alpha}$$

(3.1)

for some $l_0 \in \mathbb{Z}_{\geq 0}$ on a sufficiently small open neighborhood $U_\alpha$ of $x_\alpha \in X_\alpha$, satisfying

$$l_0 \leq \text{mult}_{T_{X_\alpha \mid X}}(\mathcal{M}).$$

(3.2)

Moreover, we have a chain of isomorphisms on $U_\alpha$

$$\mathbb{R}\Gamma_{M'_{\alpha}}\mathbb{R}\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, B_{X_\alpha \mid X})[d_{X_\alpha}] \simeq \mathbb{R}\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathbb{R}\Gamma_{M'_{\alpha}}(B_{X_\alpha \mid X}))[d_{X_\alpha}] \simeq \mathbb{R}\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathbb{R}\Gamma_{M'_{\alpha}}(\mathcal{O}_X))[d_{X_\alpha}] \simeq \mathbb{R}\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathbb{R}\Gamma_{M'_{\alpha}}(\mathcal{O}_X))[d_{X}] \simeq \mathbb{R}\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathbb{R}\Gamma_{M'_{\alpha}}(B_M)).$$

(3.3)
Here, we omit the orientation sheaf because the isomorphisms are local. For the first isomorphism, see [5, Thm. 7.9 in Appx. II]. The second and fourth isomorphisms follow from (2.4) and (2.5), respectively. In the third isomorphism, we have used $M'_\alpha \cap X_\alpha = M'_\alpha = M'_\alpha \cap M$ and Lemma 2.4. By the definition, we have

$$H^0(\mathbb{R}\text{Hom}_{D_X}(\mathfrak{M}, \mathbb{R}\Gamma_{M'_\alpha}(B_M))) \simeq \mathbb{H}\text{Hom}_{D_X}(\mathfrak{M}, \Gamma_{M'_\alpha}(B_M)).$$

(3.4)

Here, we write $H^0$ for the 0-th cohomology functor. By Fact 2.2 there exists a Grothendieck spectral sequence

$$E^2_{p,q} \simeq \mathbb{R}^p\Gamma_{M'_\alpha}(\mathbb{R}\mathbb{H}\text{om}_{D_X}(\mathfrak{M}, B_{X_\alpha}|X)) \Rightarrow H^{p+q}(\mathbb{R}\Gamma_{M'_\alpha}\mathbb{H}\text{om}_{D_X}(\mathfrak{M}, B_{X_\alpha}|X)).$$

(3.5)

Note that $\mathbb{H}\text{om}_{D_X}(\mathfrak{M}, \ast)$ takes injective objects of the category of $D_X$-modules to injective objects of $\text{Mod}(\mathcal{C}_X)$ ([5, Prop. 6.21 in Appx. II] or [15, Prop. 2.4.6 (vii)]). Lemma 2.6 and (3.1) imply

$$E^2_{p,q} \simeq \mathbb{R}^p\Gamma_{M'_\alpha}(\mathcal{C}^q_{X_\alpha}) \simeq \begin{cases} \mathcal{C}^q_{M'_\alpha} & (p = 2d_{X_\alpha} - \dim M'_\alpha), \\ 0 & (\text{otherwise}). \end{cases}$$

(3.6)

Because $X$ is a complexification of $M$, we have $\dim M'_\alpha \leq d_{X_\alpha}$. Thus, (3.3), (3.4), (3.5) and (3.6) imply

$$\mathbb{H}\text{om}_{D_X}(\mathfrak{M}, \Gamma_{M'_\alpha}(B_M)) \simeq \begin{cases} \mathcal{C}^{\dim M'_\alpha}_{M'_\alpha} & (\dim M'_\alpha = d_{X_\alpha}), \\ 0 & (\dim M'_\alpha \neq d_{X_\alpha}). \end{cases}$$

(3.7)

on a sufficiently small open neighborhood of $x_\alpha \in M'_\alpha$. Therefore, this completes the proof because $l_0 \leq \text{mult}_{D_X/T_{X_\alpha}}(\mathfrak{M})$ by (3.2).

We want to apply Proposition 3.1 to the case $M'_\alpha = M \cap X_\alpha$. However, it is impossible because $M_\alpha := M \cap X_\alpha$ may have a singular point in general. In order to overcome this, we consider a stratification $M_\alpha = \bigsqcup_{j=1}^J M_{\alpha}^{(j)}$ such that $M_{\alpha}^{(j)}$ is a submanifold of $X_\alpha$. In fact, such a stratification locally exists by the theory of semianalytic sets.

**Lemma 3.3.** Let $M$ be a real analytic manifold, $X$ a complexification of $M$, and $X = \bigsqcup_{\alpha \in A} X_\alpha$ a regular stratification of $X$. Set $M_\alpha := X_\alpha \cap M$ for any $\alpha \in A$. Then, for any $\alpha \in A$ and any relatively compact semianalytic subset $U_\alpha$ of $X$ (as a real analytic manifold), there exists a stratification $M_\alpha \cap U_\alpha = \bigsqcup_{j=1}^J M_{\alpha}^{(j)}$ such that $J_\alpha < \infty$ and $M_{\alpha}^{(j)}$ is a closed connected submanifold of $(X_\alpha \cap U_\alpha) \setminus \bigsqcup_{j=1}^J M_{\alpha}^{(j)}$.

**Proof.** Note that $M_\alpha$ is a semianalytic subset because $X_\alpha$ is a semianalytic subset of $X$. Thus, this lemma is a direct consequence of [3, Prop. 2.10].

**Corollary 3.4.** Let $M$ be a real analytic manifold, $X$ a complexification of $M$, and $X = \bigsqcup_{\alpha \in A} X_\alpha$ a regular stratification of $X$ with respect to a holonomic
\( \mathcal{D}_X \)-module \( \mathfrak{M} \). Suppose that \( X_\alpha \) is a closed connected subset of \( X \) and set \( M_\alpha := M \cap X_\alpha \). Let \( U_C \) be a relatively compact semianalytic open subset of \( X \) and \( M_\alpha \cap U_C = \bigcup_{j=1}^{j_\alpha} M_\alpha^{(j)} \) a stratification given in Lemma 3.3. Then, we have

\[
\dim \Gamma(U; \mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \Gamma_{M_\alpha}(\mathcal{B}_M))) \leq J_\alpha \cdot \mult_{\mathcal{D}_X X}^{\mathcal{D}_X \mathfrak{M}}
\]

where \( U := U_C \cap M \).

**Proof.** Because we only consider the space of sections over \( U \), we may assume that \( X = U_C \) and \( M = U \). Thus, \( M_\alpha^{(j+1)} \) is a closed connected submanifold of \( X_\alpha \setminus \bigcup_{i=1}^{j} M_\alpha^{(i)} \). For simplicity, we set

\[
X^j := X \setminus \bigcup_{i=1}^{j} M_\alpha^{(i)}, \quad X_\alpha^j := X_\alpha \setminus \bigcup_{i=1}^{j} M_\alpha^{(i)},
\]

\[
M^j := M \setminus \bigcup_{i=1}^{j} M_\alpha^{(i)}, \quad M_\alpha^j := M_\alpha \setminus \bigcup_{i=1}^{j} M_\alpha^{(i)}.
\]

We use the convention \( X^0 := X \), \( X_\alpha^0 := X_\alpha \), \( M^0 := M \), and \( M_\alpha^0 := M_\alpha \). In this notation, we have

\[
\begin{array}{cccc}
X & \supset & X^j & \supset \quad X_\alpha^j \\
\cup & \quad \cup & \quad \cup & \quad \cup \\
M & \supset & M^j = M \cap X^j & \supset \quad M_\alpha^j = M^j \cap X_\alpha^j & \supset \quad M_\alpha^{(j+1)}.
\end{array}
\]

Note that \( M_\alpha^{(j+1)} \subset M^j \cap X_\alpha^j \) is a closed connected submanifold of \( X_\alpha^j \).

We want to prove the corollary by using the filtration by support (cf. [21 (6.10)]) and Proposition 3.3. First, we treat the case of \( \Gamma_{M_\alpha^{(1)}}(\mathcal{B}_M) \). Because \( M_\alpha^{(1)} \) is closed in \( M_\alpha = M_\alpha^0 \), we have an exact sequence

\[
0 \to \Gamma_{M_\alpha^{(1)}}(\mathcal{B}_M) \to \Gamma_{M_\alpha}(\mathcal{B}_M) \to \Gamma_{M_\alpha^1}(\mathcal{B}_M)
\]

(3.8)

by Lemma 2.3. Applying the left exact functor \( \Gamma(M; \mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \cdot)) \), we have

\[
0 \to \Gamma(M; \mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \Gamma_{M_\alpha^{(1)}}(\mathcal{B}_M))) \\
\to \Gamma(M; \mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \Gamma_{M_\alpha}(\mathcal{B}_M))) \to \Gamma(M; \mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \Gamma_{M_\alpha^1}(\mathcal{B}_M))),
\]

which is exact. Therefore, we have

\[
\dim \Gamma(M; \mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \Gamma_{M_\alpha}(\mathcal{B}_M))) \\
\leq \dim \Gamma(M; \mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \Gamma_{M_\alpha^{(1)}}(\mathcal{B}_M))) + \dim \Gamma(M; \mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \Gamma_{M_\alpha^1}(\mathcal{B}_M))).
\]

Because \( M_\alpha^{(1)} \subset M \cap X_\alpha \) is a closed connected submanifold of \( X_\alpha \), we have

\[
\dim \Gamma(M; \mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \Gamma_{M_\alpha^{(1)}}(\mathcal{B}_M))) \leq \mult_{\mathcal{D}_X X}^{\mathcal{D}_X \mathfrak{M}}(\mathfrak{M})
\]

(3.10)
by Lemma 2.9 and Proposition 3.1. By (3.9) and (3.10), we have

\[ \dim \Gamma(M; \mathcal{H}om_{\mathcal{D}X}(\mathcal{M}, \Gamma_{M_1}(\mathcal{B}_M))) \leq \text{mult}_{\mathcal{D}X_{\alpha}}(\mathcal{M}) + \dim \Gamma(M; \mathcal{H}om_{\mathcal{D}X}(\mathcal{M}, \Gamma_{M_2}(\mathcal{B}_M))). \]  

(3.11)

We want to apply the same argument to the last term of (3.11). For this end, we rewrite it. We have a chain of isomorphisms

\[ \Gamma(M; \mathcal{H}om_{\mathcal{D}X}(\mathcal{M}, \Gamma_{M_1}(\mathcal{B}_M))) \cong \Gamma(M; \mathcal{H}om_{\mathcal{D}X}(\mathcal{M}, \Gamma_{M_1}(\mathcal{B}_M))) \cong \Gamma(M; \mathcal{H}om_{\mathcal{D}X}(\mathcal{M}, \Gamma_{M_1}(\mathcal{B}_M))) \cong \Gamma(M; \mathcal{H}om_{\mathcal{D}X}(\mathcal{M}, \Gamma_{M_2}(\mathcal{B}_M))) \cong \Gamma(M; \mathcal{H}om_{\mathcal{D}X}(\mathcal{M}, \Gamma_{M_2}(\mathcal{B}_M))). \]  

(3.12)

where \( \mathcal{M}|_{X^1} \) is the restriction of \( \mathcal{M} \) to \( X^1 \). In the first isomorphism, we have used \( M_1^1 = M \cap M_\alpha \) and [15, Prop. 2.3.9 (ii)]. For the second isomorphism, see [15, (2.3.18)]. The third and last isomorphisms follow from the definitions.

Recall that \( M_\alpha^{(2)} \) is closed in \( M_\alpha \). Thus, we have an exact sequence

\[ 0 \to \Gamma_{M_\alpha^{(2)}}(\mathcal{B}_{M^1}) \to \Gamma_{M_\alpha}(\mathcal{B}_{M^1}) \to \Gamma_{M_\alpha}(\mathcal{B}_{M^1}) \]  

by Lemma 3.3. Applying the left exact functor \( \Gamma(M; \mathcal{H}om_{\mathcal{D}X}(\mathcal{M}|_{X^1}, \cdot)) \), we have

\[ \dim \Gamma(M; \mathcal{H}om_{\mathcal{D}X}(\mathcal{M}|_{X^1}, \Gamma_{M_\alpha}(\mathcal{B}_{M^1}))) \leq \dim \Gamma(M; \mathcal{H}om_{\mathcal{D}X}(\mathcal{M}|_{X^1}, \Gamma_{M_\alpha}(\mathcal{B}_{M^1}))) + \dim \Gamma(M; \mathcal{H}om_{\mathcal{D}X}(\mathcal{M}|_{X^1}, \Gamma_{M_\alpha}(\mathcal{B}_{M^1}))). \]  

(3.13)

We want to apply Proposition 3.1 to the last term of (3.13). For this end, we write \( X_\alpha^1 = \bigsqcup_{k \in K_\alpha} X_{\alpha,k}^1 \) for the connected component decomposition of \( X_\alpha^1 \) and take \( k_0 \in K_\alpha \) such that \( X_{\alpha,k_0}^1 \) is the connected component of \( X_\alpha^1 \) containing \( M_\alpha^{(2)} \). We shall check the assumption of Proposition 3.1. It is clear that \( M_1 \) is a real analytic manifold, \( X^1 \) is its complexification and

\[ X^1 = \left( \bigsqcup_{\beta \neq \alpha} X_\beta \right) \cup \left( \bigsqcup_{k \in K_\alpha} X_{\alpha,k}^1 \right) \]  

is a regular stratification of \( X^1 \) with respect to \( \mathcal{M}|_{X^1} \). Moreover, \( X^1_{\alpha,k_0} \) is a closed connected subset of \( X^1 \) because \( X^1_{\alpha,k_0} \) is the connected component of a closed subset \( X_{\alpha,k_0}^1 \subset X^1 \). Moreover, \( M_\alpha^{(2)} \subset M_1 \cap X_{\alpha,k_0}^1 \) is a closed connected submanifold of \( X_{\alpha,k_0}^1 \) by definition. Thus, we have

\[ \dim \Gamma(M_1; \mathcal{H}om_{\mathcal{D}X}(\mathcal{M}|_{X^1}, \Gamma_{M_\alpha^{(2)}}(\mathcal{B}_{M^1}))) \leq \text{mult}_{\mathcal{D}X_{X_{\alpha,k_0}}}(\mathcal{M}|_{X^1}) \]  

(3.14)
by Lemma 2.9 and Proposition 3.1. Moreover,
\[ \text{mult}_{T_{X_{\alpha_0}, X_0}}^{D_X} X_{\alpha_0} (\mathcal{M}|_{X_1}) = \text{mult}_{T_{X_{\alpha_0}, X_0}}^{D_X} X_{\alpha_0} (\mathcal{M}) \] (3.15)
follows easily from the definition (because the multiplicity of an \(O\)-module is
defined by the length of it at a generic point, see [11, Sect. 2.6] for example). Thus, (3.13), (3.14) and (3.15) imply
\[ \dim \Gamma(U; \mathcal{H}_{\text{Hom}_{D_X}} (\mathbb{M}, \Gamma_{M_1}(\mathcal{B}_{M_1}))) \leq \dim \Gamma(U; \mathcal{H}_{\text{Hom}_{D_X}} (\mathbb{M}, \Gamma_{M_2}(\mathcal{B}_{M_1}))) + \text{mult}_{T_{X_{\alpha_0}, X_0}}^{D_X} X_{\alpha_0} (\mathcal{M}). \] (3.16)
Thus, (3.11), (3.12), and (3.16) imply
\[ \dim \Gamma(U; \mathcal{H}_{\text{Hom}_{D_X}} (\mathbb{M}, \Gamma_{M_1}(\mathcal{B}_{M_1}))) \leq 2 \cdot \text{mult}_{T_{X_{\alpha_0}, X_0}}^{D_X} X_{\alpha_0} (\mathcal{M}) + \dim \Gamma(U; \mathcal{H}_{\text{Hom}_{D_X}} (\mathbb{M}, \Gamma_{M_2}(\mathcal{B}_{M_1}))). \]
Repeating the same argument, we have the corollary.

**Proposition 3.5.** Let \( M \) be a real analytic manifold, \( X \) a complexification of \( M \), and \( X = \bigsqcup_{\alpha \in A} X_{\alpha} \) a regular stratification of \( X \) with respect to a holonomic \( D_X \)-module \( \mathbb{M} \) such that each stratum \( X_{\alpha} \) is connected. Let \( U \) be a relatively compact semianalytic open subset of \( M \) and \( J_{\alpha} \) the integer given in Lemma 3.3 for any \( \alpha \in A \). Then, we have
\[ \dim \Gamma(U; \mathcal{H}_{\text{Hom}_{D_X}} (\mathbb{M}, \Gamma_{M_1}(\mathcal{B}_{M}))) \leq \sum_{\alpha \in A} J_{\alpha} \cdot \text{mult}_{T_{X_{\alpha_0}, X_0}}^{D_X} X_{\alpha_0} (\mathcal{M}). \]

**Proof.** Take a relatively compact semianalytic open subset \( U_C \) of \( X \) satisfying \( U_C \cap M = U \). Because we only consider the space of sections over \( U \), we may assume that \( X = U_C \) and \( M = U \).

Because \( X = \bigsqcup_{\alpha \in A} X_{\alpha} \) is a regular stratification, there exists \( \alpha_0 \in A \) such that \( X_{\alpha_0} \) is closed in \( X \). Applying Corollary 3.4 we have
\[ \dim \Gamma(U; \mathcal{H}_{\text{Hom}_{D_X}} (\mathbb{M}, \Gamma_{M_{\alpha_0}}(\mathcal{B}_{M_1}))) \leq J_{\alpha_0} \cdot \text{mult}_{T_{X_{\alpha_0}, X_0}}^{D_X} X_{\alpha_0} (\mathcal{M}). \] (3.17)
Because \( M_{\alpha_0} \) is closed in \( M \) and \( \Gamma_M(\mathcal{B}_M) = \mathcal{B}_M \), we have an exact sequence
\[ 0 \to \Gamma_{M_{\alpha_0}}(\mathcal{B}_M) \to \mathcal{B}_M \to \Gamma_{M\setminus M_{\alpha_0}}(\mathcal{B}_M) \]
by Lemma 2.3. Applying the left exact functor \( \Gamma(U; \mathcal{H}_{\text{Hom}_{D_X}} (\mathbb{M}, \cdot)) \), we have
\[ 0 \to \Gamma(U; \mathcal{H}_{\text{Hom}_{D_X}} (\mathbb{M}, \Gamma_{M_{\alpha_0}}(\mathcal{B}_{M}))) \to \Gamma(U; \mathcal{H}_{\text{Hom}_{D_X}} (\mathbb{M}, \mathcal{B}_{M})) \to \Gamma(U; \mathcal{H}_{\text{Hom}_{D_X}} (\mathbb{M}, \Gamma_{M\setminus M_{\alpha_0}}(\mathcal{B}_{M}))), \]
which is exact. Therefore, we have

\[
\dim \Gamma(U; \mathcal{H}om_{D^X}(\mathcal{M}, B_M)) \\
\leq J_{\alpha_0} \cdot \text{mult}_{T_{X_{\alpha_0}}} X(\mathcal{M}) + \dim \Gamma(U; \mathcal{H}om_{D^X}(\mathcal{M}, \Gamma_{M\setminus M_{\alpha_0}}(B_M)))
\]

by (3.17). In the same way, we have

\[
\dim \Gamma(U; \mathcal{H}om_{D^X}(\mathcal{M}, \Gamma_{M\setminus M_{\alpha_0}}(B_M))) \\
\leq J_{\alpha_1} \cdot \text{mult}_{T_{X_{\alpha_1}}} X(\mathcal{M}) + \dim \Gamma(U; \mathcal{H}om_{D^X}(\mathcal{M}, \Gamma_{M\setminus M_{\alpha_0}\cup M_{\alpha_1}}(B_M)))
\]

for \( \alpha_1 \in A \) such that \( X_{\alpha_1} \) is closed in \( X\setminus X_{\alpha_0} \). Repeating this argument, we have the proposition.

We note that \( J_{\alpha} \) only depends on \( U \) and the stratification \( X = \bigsqcup_{\alpha \in A} X_\alpha \).

4 Upper bound of the dimensions of the spaces of group invariant hyperfunctions

In this section, we give an upper bound of the dimensions of the spaces of group invariant hyperfunctions. We want to use Proposition 3.5 for the proof of Theorem 1.14. Therefore, we should show that there exists a holonomic \( D_X \)-module \( \mathcal{M}_\tau \) and a regular stratification \( X = \bigsqcup_{\alpha \in A} X_\alpha \) with respect to \( \mathcal{M}_\tau \) such that

1. \( (\Gamma(U; B_M) \otimes \tau)^{hc} \simeq \Gamma(U; \mathcal{H}om_{D^X}(\mathcal{M}_\tau, B_M)) \) for any open subset \( U \subset X \),
2. the stratification \( X = \bigsqcup_{\alpha \in A} X_\alpha \) does not depend on \( \tau \),
3. each stratum \( X_\alpha \) is connected.

First, we construct a holonomic \( D_X \)-module \( \mathcal{M}_\tau \) satisfying these conditions in Lemma 4.1.

**Lemma 4.1.** In the setting of Theorem 1.14 there exists a holonomic \( D_X \)-module \( \mathcal{M}_\tau \) satisfying the conditions (1), (2) and (3) above.

**Proof.** Let \( U(\mathfrak{h}) \) be the universal enveloping algebra of \( \mathfrak{h}_C \). The action of \( H_C \) on \( X \) induces a Lie algebra homomorphism \( a: U(\mathfrak{h}) \to D_X \). By this homomorphism, we regard \( D_X \) as a right \( U(\mathfrak{h}) \)-module. We define a coherent \( D_X \)-module \( \mathcal{M}_\tau \) by (cf. Beilinson-Bernstein localization [2])

\[
\mathcal{M}_\tau := D_X \otimes_{U(\mathfrak{h})} \tau^\vee.
\]

In other words,

\[
\mathcal{M}_\tau = D_X \otimes_{C_X} \tau^\vee/I_{\tau^\vee},
\]
where $I_\tau$ is a $\mathcal{D}_X$-submodule of $\mathcal{D}_X \otimes \tau^\vee$ defined by

$$I_\tau \ := \ \sum_{H \in h_C, v \in \tau^\vee} \mathcal{D}_X \cdot (a(H) \otimes v - 1 \otimes \tau^\vee(H)v). \quad (4.3)$$

Let $X = \bigsqcup_{\alpha \in A} X_\alpha$ be the $H_C$-orbit decomposition of $X$ and $X_\alpha = \bigsqcup_{k \in K_\alpha} X_{\alpha,k}$ the connected component decomposition of $X_\alpha$. Then, the finiteness of the number of connected components of $H_C$ and $\#(H_C \setminus X) < \infty$ imply that $\mathcal{M}_\tau$ is a holonomic $\mathcal{D}_X$-module and $X = \bigsqcup_{\alpha \in A, k \in K_\alpha} X_{\alpha,k}$ is a regular stratification of $X$ with respect to $\mathcal{M}_\tau$ (see [11, Thm. 5.1.12]) such that each $X_{\alpha,k}$ is connected. Thus $\mathcal{M}_\tau$ satisfies the conditions (2) and (3). By an isomorphism $B_M \simeq \text{Hom}_{\mathcal{D}_X}(\mathcal{D}_X, B_M)$ and the tensor-hom adjunction, we have

$$(\Gamma(M; B_M) \otimes \tau)^b \simeq (\Gamma(M; \text{Hom}_{\mathcal{D}_X}(\mathcal{D}_X, B_M)) \otimes \tau)^b$$

$$\simeq (\text{Hom}_{\mathcal{D}_X}(\mathcal{D}_X, B_M) \otimes \tau)^b$$

$$\simeq \text{Hom}_{\mathcal{U}(b)}(\tau^\vee, \text{Hom}_{\mathcal{D}_X}(\mathcal{D}_X, B_M))$$

$$\simeq \text{Hom}_{\mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{U}(b)} \tau^\vee, B_M)$$

$$\simeq \Gamma(M; \text{Hom}_{\mathcal{D}_X}(\mathcal{M}_\tau, B_M)).$$

The similar argument shows that $\mathcal{M}_\tau$ satisfies the condition (1). \hfill \square

Reindexing, we assume that $X = \bigsqcup_{\alpha \in A} X_\alpha$ is a regular stratification of $X$ with respect to $\mathcal{M}_\tau$ such that each $X_\alpha$ is connected. Then, Proposition 3.5 implies

$$\dim(\Gamma(U; B_M) \otimes \tau)^{bc} \leq \sum_{\alpha \in A} J_\alpha \cdot \text{mult}_{\mathcal{D}_X}^{\mathcal{M}_\tau}(\mathcal{M}_\tau) \quad (4.4)$$

by Lemma 4.1. Note that $J_\alpha$ does not depend on $\tau$. Thus, we want to show that $\text{mult}_{\mathcal{T}_{X_\alpha}}^{\mathcal{D}_X}(\mathcal{M}_\tau)$ is uniformly bounded for the proof of Theorem 1.14.

**Lemma 4.2.** Let $\mathcal{M}_\tau$ be a holonomic $\mathcal{D}_X$-module defined in (4.1). Then, for any $\alpha \in A$, there exists $C_\alpha > 0$, which is independent of $\tau$, satisfying

$$\text{mult}_{\mathcal{T}_{X_\alpha}}^{\mathcal{D}_X}(\mathcal{M}_\tau) \leq \dim \tau \cdot C_\alpha.$$

For the proof of Lemma 4.2, we need some preparation. The order filtration $\mathcal{F}$ of $\mathcal{D}_X$ induces a filtration of a $\mathcal{D}_X$-module $\mathcal{D}_X \otimes \tau^\vee$, which also induces good filtrations of $\mathcal{M}_\tau$ and $I_\tau$. We write $\mathcal{F}$ for these filtration. Namely, we put

$$\mathcal{F}_j(\mathcal{D}_X \otimes \tau^\vee) \ := \ \mathcal{F}_j(\mathcal{D}_X) \otimes \tau^\vee,$$

$$\mathcal{F}_j(\mathcal{M}_\tau) \ := \ (\mathcal{F}_j(\mathcal{D}_X \otimes \tau^\vee) + I_{\tau^\vee})/I_{\tau^\vee},$$

$$\mathcal{F}_j(I_{\tau^\vee}) \ := \ \mathcal{F}_j(\mathcal{D}_X \otimes \tau^\vee) \cap I_{\tau^\vee}.$$

We note that there exists an isomorphism

$$\text{gr}_\mathcal{F}(\mathcal{M}_\tau) \simeq \text{gr}_\mathcal{F}(\mathcal{D}_X \otimes \tau^\vee)/\text{gr}_\mathcal{F}(I_{\tau^\vee}).$$
Lemma 4.3. There exists an $\mathcal{O}_{T,X}$-module $\mathfrak{N}$, which is independent of $\tau$, satisfying the following two conditions:

1. $\dim \text{supp}(\mathfrak{N}) \leq d_X$,

2. there exists a surjective homomorphism $F(gr_{\mathfrak{N}}(\mathfrak{N}_\tau)) \twoheadrightarrow \mathfrak{N} \otimes \tau^\vee$.

Postponing the proof of this lemma, we prove Lemma 4.2

Proof of Lemma 4.4. By the definition of $\text{mult}_{T_{X,a}}^X(\cdot)$, we have

$$\text{mult}_{T_{X,a}}^X(\mathfrak{N}_\tau) = \text{mult}_{T_{X,a}}^X(F(\mathfrak{N}_\tau)).$$

Note that $\text{mult}_{T_{X,a}}^X(\cdot)$ is additive on short exact sequences of $\mathcal{O}_{T,X}$-modules with at most $d_X$-dimensional supports [5, Sect. 1.5 in Appx. V] and that $\mathfrak{N} \otimes \tau^\vee \cong \mathfrak{N}_{\text{dim} \tau}$ as an $\mathcal{O}_{T,X}$-module. Then Lemma 4.3 implies

$$\text{mult}_{T_{X,a}}^X(\mathfrak{N}_\tau) \leq \text{mult}_{T_{X,a}}^X(\mathfrak{N} \otimes \tau^\vee) = \dim \tau \cdot \text{mult}_{T_{X,a}}^X(\mathfrak{N}).$$

This completes the proof.

We shall prove Lemma 4.3 from now on. For any $j \in \mathbb{Z}_{\geq 0}$, we write $\sigma_j : F_j(D_X) \rightarrow gr_{\mathfrak{N}}(D_X)$ for the composition of the natural surjection $F_j(D_X) \rightarrow F_j(D_X)/F_{j-1}(D_X)$ and the injection $F_j(D_X)/F_{j-1}(D_X) \hookrightarrow gr_{\mathfrak{N}}(D_X)$. We use the same symbol $\sigma_j$ for the homomorphism $F_j(D_X \otimes \tau^\vee) \rightarrow gr_{\mathfrak{N}}(D_X \otimes \tau^\vee)$.

Lemma 4.4. For any $H \in \mathfrak{h}_C$ and any $v \in \tau^\vee$, we have

$$\sigma_1(a(H) \otimes v - 1 \otimes \tau^\vee(H)v) = \sigma_1(a(H)) \otimes v.$$

Proof. This is clear by $1 \otimes \tau^\vee(H)v \in F_0(D_X \otimes \tau^\vee)$.

Proof of Lemma 4.4. Recall that $I_{\tau^\vee}$ is a $D_X$-submodule of $D_X \otimes V_{\tau^\vee}$ generated by \{ $a(H) \otimes v - 1 \otimes \tau^\vee(H)v \mid H \in \mathfrak{h}_C, v \in \tau^\vee$ \}. Then we have

$$gr_{\mathfrak{N}}(I_{\tau^\vee}) \supset \sum_{H \in \mathfrak{h}_C, v \in \tau^\vee} gr_{\mathfrak{N}}(D_X) \cdot \sigma_1(a(H) \otimes v - 1 \otimes \tau^\vee(H)v)$$

(cf. [13 Chap. 2]). Thus, Lemma 4.4 implies

$$gr_{\mathfrak{N}}(I_{\tau^\vee}) \supset gr_{\mathfrak{N}}(D_X) \cdot (\sigma_1(a(\mathfrak{h}_C)) \otimes \tau^\vee).$$

This inclusion induces the following surjection:

$$\frac{gr_{\mathfrak{N}}(D_X \otimes \tau^\vee)}{gr_{\mathfrak{N}}(I_{\tau^\vee})} \twoheadrightarrow \frac{gr_{\mathfrak{N}}(D_X \otimes \tau^\vee)}{gr_{\mathfrak{N}}(D_X)(\sigma_1(a(\mathfrak{h}_C)) \otimes \tau^\vee)} = \left( \frac{gr_{\mathfrak{N}}(D_X)}{gr_{\mathfrak{N}}(D_X)(\sigma_1(a(\mathfrak{h}_C)))} \right) \otimes \tau^\vee.

(4.5)
Note that the left-hand side is equal to $\text{gr}_F(\mathfrak{M})$. We define an $\mathcal{O}_{T^*X}$-module $\mathfrak{N}$ by $\mathfrak{N} := \mathcal{O}_{T^*X} / \mathcal{O}_{T^*X} \cdot \sigma_1(a(\mathfrak{h}))$. Applying the right exact functor $F$ (see (2.6)) to the surjection (4.4), we obtain the desired surjection

$$F(\text{gr}_F(\mathfrak{M})) = \mathcal{O}_{T^*X} \otimes_{\mathcal{O}_{T^*X}} \pi^{-1} \left( \frac{\text{gr}_F(D_X)}{\text{gr}_F(D_X) \sigma_1(a(\mathfrak{h}))} \right) \otimes \tau^\vee \simeq \mathfrak{N} \otimes \tau^\vee.$$

Next, we prove $\dim \text{supp}(\mathfrak{N}) \leq d_X$. Let $X = \bigsqcup_{\alpha \in A} X_\alpha$ be the $H_C$-orbit decomposition of $X$. Then we have $\#(A) = \#(H_C \backslash X) < \infty$ by the assumption. Because $\text{supp}(\mathfrak{N})$ is contained in $\bigcup_{\alpha \in A} T_{X_\alpha}^* X$ (cf. [11, Thm. 5.1.12]) and $\dim T_{X_\alpha}^* X = d_X$, we have $\dim \text{supp}(\mathfrak{N}) \leq d_X$.

Proof of Theorem 1.14. By Lemma 4.1, we have (4.4). Note $\#(A) = \#(H_C \backslash X) < \infty$ by the assumption. Then, we have $J := \sup_{\alpha \in A} J_\alpha < \infty$ by Lemma 3.3. Then, we have

$$\dim(\Gamma(U; B_M) \otimes \tau)^{hc} \leq J \sum_{\alpha \in A} \text{mult}_{T^*_{X_\alpha} X}(\mathfrak{M})$$

$$\leq J \sum_{\alpha \in A} \dim \tau \cdot C_\alpha$$

by Lemma 5.2 Setting $C := J \cdot \#(A) \cdot \sup_{\alpha \in A} C_\alpha$, we have Theorem 1.14. □

5 Proof of Theorem 1.13

In this section, we prove Theorem 1.13 by using Theorem 1.14. For this end, we quote the characterization of intertwining operators by invariant distributions.

Fact 5.1 ([21, Prop. 3.2]). Let $G$ be a real Lie group. Suppose that $G'$ and $H$ are closed subgroups of $G$ and that $H'$ is a closed subgroup of $G'$. Let $\tau$ and $\tau'$ be finite-dimensional representations of $H$ and $H'$, respectively.

(1) There is a natural injective map:

$$\text{Hom}_{G'}(C^\infty(G/H, \tau), C^\infty(G'/H', \tau')) \hookrightarrow (D'(G/H, \tau^\vee \otimes \mathbb{C}_{2p}) \otimes \tau')^{H'}.$$

(5.1)

Here $\tau^\vee$ is the contragredient representation of $\tau$, $\mathbb{C}_{2p}$ is the one-dimensional representation of $H$ defined by $h \mapsto |\det(\text{Ad}(h): \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h})|^{-1}$.

(2) If $H$ is cocompact in $G$ (e.g., a parabolic subgroup of $G$ or a uniform lattice), then (5.1) is a bijection.

Proof of Theorem 1.13 By Fact 5.1, we have

$$\text{Hom}_G(C^\infty(G/Q, \eta), C^\infty(G/H, \tau)) \simeq (D'(G/Q, \eta^\vee \otimes \mathbb{C}_{2p}) \otimes \tau)^H.$$

(5.2)
Let \( Q \) act on \( \mathcal{D}'(G) \) from the right. Regarding \( \mathcal{D}'(G) \otimes (\eta^\vee \otimes \mathbb{C}_{2^p}) \) as a tensor representation of \( Q \), we have
\[
\mathcal{D}'(G/Q, \eta^\vee \otimes \mathbb{C}_{2^p}) \simeq (\mathcal{D}'(G) \otimes (\eta^\vee \otimes \mathbb{C}_{2^p}))^Q.
\]
Moreover, letting \( H \) (resp. \( Q \)) act on \( \eta^\vee \otimes \mathbb{C}_{2^p} \) (resp. \( \tau \)) trivially, we have
\[
\begin{align*}
\bigoplus_{\text{dim} (\tau) = 0} \simeq & \quad \left( (\mathcal{D}'(G) \otimes (\eta^\vee \otimes \mathbb{C}_{2^p}))^Q \otimes \tau \right)^H \\
\simeq & \quad (\mathcal{D}'(G) \otimes (\eta^\vee \otimes \mathbb{C}_{2^p}) \otimes \tau)^{H \times Q} \\
\subset & \quad (\mathcal{B}_G(G) \otimes (\eta^\vee \otimes \mathbb{C}_{2^p}) \otimes \tau)^{H \times Q}.
\end{align*}
\]
The last inclusion follows from the fact that the space of Schwartz distributions can be imbedded in the space of Sato’s hyperfunctions. In order to apply Theorem 1.14, we shall construct a relatively compact semianalytic open subset \( U \) of \( G \). Let \( G = KAN \) be the Iwasawa decomposition. This implies \( G \) is diffeomorphic to \( K \times A \times N \simeq K \times \mathbb{R}^k_+ \times \mathbb{R}^j \) for some \( k,j \in \mathbb{N} \), where \( K \) is compact. Define a relatively compact semianalytic open subset \( U \) of \( G \) by \( U := K \times (1,2)^k \times (1,2)^j \subset K \times \mathbb{R}^k_+ \times \mathbb{R}^j \simeq G \). Because \( QU = G \), we have (cf. [21, Thm. 3.16])
\[
(\mathcal{B}_G(G) \otimes (\eta^\vee \otimes \mathbb{C}_{2^p}) \otimes \tau)^{H \times Q} \subset (\mathcal{B}_G(U) \otimes (\eta^\vee \otimes \mathbb{C}_{2^p}) \otimes \tau)^{\mathbb{B} \otimes \mathbb{Q}}. \quad (5.3)
\]
Therefore, Theorem 1.13 follows from Theorem 1.14.

6 An alternative approach of (II) \( \Rightarrow \) (I) in Fact 1.10

Proof of (II) \( \Rightarrow \) (I) in Fact 1.10. Let \( \pi^\vee \) be the contragredient representation of \( \pi \in \hat{G} \) smooth in the category of admissible smooth representations with moderate growth. By Casselman’s subrepresentation theorem [7], there exists an injection \( \pi^\vee \hookrightarrow C^\infty(G/P, \eta) \) for some \( \eta \in \hat{P}. \) Then \( \pi \) is isomorphic to an irreducible quotient of \( C^\infty(G/P, \eta^\vee \otimes \mathbb{C}_{2^p}) \) because there exists a natural \( G \)-invariant pairing \( C^\infty(G/P, \eta) \times C^\infty(G/P, \eta^\vee \otimes \mathbb{C}_{2^p}) \to \mathbb{C} \). Moreover by Fact 5.3, we have
\[
\text{Hom}_G(C^\infty(G/P, \eta^\vee \otimes \mathbb{C}_{2^p}), C^\infty(G/H, \tau)) \simeq (\mathcal{D}'(G/P, \eta) \otimes \tau)^H \\
\subset (\mathcal{D}'(G/P_0, \eta|_{P_0}) \otimes \tau)^H \\
\subset (\mathcal{D}'(G/P_0, \eta|_{P_0}) \otimes \tau)^{\mathbb{B} \otimes \mathbb{Q}},
\]
where \( P_0 \) is the identity component of \( P \). We note that any irreducible finite-dimensional representation of \( P \) is the direct sum of at most \( \#(P/P_0) \) irreducible representations of \( P_0 \). Then, it is sufficient to prove
\[
\sup_{\tau \in \hat{H}} \sup_{\eta \in \hat{P}_0} \frac{1}{\dim \tau} \dim (\mathcal{D}'(G/P_0, \eta) \otimes \tau)^{\mathbb{B} \otimes \mathbb{Q}} < \infty. \quad (6.1)
\]
As in the original proof [19], we use the Borel–Weil theorem to deal with the finite-dimensional representation \( \eta \). Let \( P = MAN \) be the Langlands decomposition of \( P \). Take a maximal abelian subspace \( t \) of \( m \) and write \( tc, mc \) for the complexifications of \( t, m \), respectively. We define some positivity on the root system of \((mc, tc)\) and write \( n_m \) for the direct sum of positive root spaces of \( mc \) relative to \( tc \). Then, by the Borel–Weil theorem, there exists \( \lambda \in a^* + \sqrt{-1}t^* \) such that \( \eta \in \mathcal{P}_{0,1} \) is isomorphic to \( C^\infty(P_0/TAN, C_\lambda)^{n_m} \) as a \( P_0 \)-representation. Here \( C_\lambda := (\chi_\lambda, C) \) is a one-dimensional representation of \( TAN \) defined by

\[
\chi_\lambda(e^{T+H}n) := e^{\lambda(T+H)} \quad \text{for } T \in t, H \in a, n \in N
\]

and \( n_m \) acts on \( C^\infty(P_0/TAN, C_\lambda) \) from the right by an isomorphism \( P_0/TAN \simeq M_0/T \simeq M_{0,C}/T_CN_m \). Therefore, we have

\[
(D'(G/P_0, \eta) \otimes \tau)^h \simeq (D'(G/P_0, C^\infty(P_0/TAN, C_\lambda)^{n_m}) \otimes \tau)^h 
\subset (D'(G/TAN, C_\lambda)^{n_m} \otimes \tau)^h. \tag{6.2}
\]

The last inclusion is the composition of an isomorphism (see [20], for example)

\[
(D'(G/P_0, C^\infty(P_0/TAN, C_\lambda)^{n_m})) \simeq (D'(G) \otimes C^\infty(P_0/TAN, C_\lambda)^{n_m})^{P_0} \tag{6.3}
\]

and an injection

\[
(D'(G) \otimes C^\infty(P_0/TAN, C_\lambda)^{n_m})^{P_0} \hookrightarrow D'(G/TAN, C_\lambda)^{n_m},
\]

\[
\sum F(\cdot) \otimes f(\cdot) \mapsto \sum F(\cdot) \otimes f(e). \tag{6.4}
\]

Here, \( P_0 \) acts on \( D'(G) \) and \( C^\infty(P_0/TAN, C_\lambda)^{n_m} \) from the right and the left, respectively, and \( e \in G \) denotes the identity element. Let \( n_m \) act on \( C_\lambda \) trivially. Then, similarly to (6.3), we have

\[
D'(G/TAN, C_\lambda)^{n_m} \simeq (D'(G/N) \otimes C_\lambda)^{n_m+a+t}. \tag{6.5}
\]

Let \( n_m + a + t \) (resp. \( h \)) act on \( \tau \) (resp. \( C_\lambda \)) trivially. Then we have

\[
(6.2) \simeq \left((D'(G/N) \otimes C_\lambda)^{n_m+a+t} \otimes \tau \right)^h
\subset (D'(G/N) \otimes C_\lambda \otimes \tau)^{b \oplus (n_m+a+t)}
\subset (B_{G/N}(G/N) \otimes C_\lambda \otimes \tau)^{b \oplus (n_m+a+t)}. \tag{6.6}
\]

In order to apply Theorem [12,14] we shall construct a relatively compact semianalytic open subset \( U \) of \( G/N \). Let \( K \) be a maximal compact subgroup \( G \), then \( G/N \) is diffeomorphic to \( K \times A \simeq K \times \mathbb{R}^k > 0 \) for some \( k \in \mathbb{N} \) by the Iwasawa decomposition. We define a relatively compact open semianalytic set \( U \) of \( G/N \) by \( U := K \times (1,2)^h \subset K \times \mathbb{R}^k > 0 \simeq G/N \). Then we have

\[
(B_{G/N}(G/N) \otimes C_\lambda \otimes \tau)^{b \oplus (n_m+a+t)} \simeq (B_{G/N}(U) \otimes C_\lambda \otimes \tau)^{b \oplus (n_m+a+t)}. \tag{6.6}
\]
in the same way as \((5.3)\). Moreover, the assumption \#\((H_C \cap G_C/B) < \infty\) implies \#\(((H_C \times A_C T_C N_m) \cap G_C/N_C) < \infty\). Therefore, Theorem \(1.14\) implies that there exists \(C > 0\), which is independent on \(\tau \in \hat{H}_f\) and \(\lambda \in a^* + \sqrt{-1}t^*\), such that

\[
\dim \left( (B_{G/N}(U) \otimes \mathbb{C}_\lambda \otimes \tau)^{h\otimes(n_m+a+1)} \right) \leq C \cdot \dim \tau.
\]

This completes the proof. \(\blacksquare\)

A Appendix

In this section, we prove the remaining assertion of Corollary \(2.12\). For this purpose, we use the theory of regular holonomic \(D_X\)-modules. We recall some facts. See \([5, \text{Thms. 5.5.21 and 22}]\) or \([14, \text{Thm. 6.1.3}]\).

Fact A.1. For any holonomic \(D_X\)-module \(\mathcal{M}\), there exists a regular holonomic \(D_X\)-module \(\mathcal{M}_{\text{reg}}\) such that

1. \(D_X^\infty \otimes_{D_X} \mathcal{M} \simeq D_X^\infty \otimes_{D_X} \mathcal{M}_{\text{reg}}\),
2. \(\text{Hom}_{D_X}(\mathcal{M}, D_X^\infty \otimes_{D_X} \mathcal{M}) \simeq \text{Hom}_{D_X}(\mathcal{M}, \mathcal{M}_{\text{reg}})\) for any regular holonomic \(D_X\)-module \(\mathcal{M}\).

Note that we have \(\text{mult}_{T_X^* X}^D(\mathcal{M}_{\text{reg}}) = 1\). Let \(x \in X_{\alpha}\) and \(f \in \text{Hom}_{D_X}(\mathcal{M}_{\text{reg}}, B_{X_{\alpha}|X}^{l_f})\) with \(f \neq 0\). Take an open neighborhood \(U\) of \(x\) such that \(f\) is defined over \(U\). Consider an exact sequence on \(U\)

\[
0 \to \text{Ker} f \to \mathcal{M}_{\text{reg}} \to B_{X_{\alpha}|X}^{l_f} \to 0, \quad (A.1)
\]

where the exactness at \(B_{X_{\alpha}|X}^{l_f}\) follows from \(\text{mult}_{T_X^* X}^D(B_{X_{\alpha}|X}^{l_f}) = 1\). Then, \(\text{Ker} f\) is regular holonomic \([14, \text{Prop. 1.1.17}]\). Moreover, additivity of \(\text{mult}_{T_X^* X}^D\) with respect to exact sequences of holonomic \(D_X\)-modules \([11, \text{Prop. 2.6.15}]\) implies

\[
\text{mult}_{T_X^* X}^D(\text{Ker} f) = \text{mult}_{T_X^* X}^D(\mathcal{M}_{\text{reg}}) - 1.
\]

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Applying the left exact functor $\mathcal{H}om_{D^X}(*, B^f_{X,a} | X)_x$ to (A.1), we have

$$\dim \mathcal{H}om_{D^X}(\mathcal{M}_{reg}, B^f_{X,a} | X)_x \leq 1 + \dim \mathcal{H}om_{D^X}(\text{Ker} f, B^f_{X,a} | X)_x$$

by $\dim \mathcal{H}om_{D^X}(B^f_{X,a} | X, B^f_{X,a} | X)_x = 1$. Repeating this argument by taking $\mathcal{M} = \text{Ker} f$, we have the corollary because we have $\dim \mathcal{H}om_{D^X}(\mathcal{M}_{reg}, B^f_{X,a} | X)_x = 0$ if $\text{mult}^{D^X}_{T^*_a X}(\mathcal{M}_{reg}) = 0$. □

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