Origin of Structure in the Universe: Quantum Cosmology Reconsidered

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Abstract

Based on a more careful canonical analysis, we motivate a reduced quantization of slightly inhomogeneous cosmology in place of the Dirac quantization in the existing literature, and provide it in the vacuum case. This is attained via consideration of configuration space geometries at various levels of reduction. Some of these have the good fortune of being flat. Geometrically natural coordinates thereupon are interpreted in terms of the original redundant formulation’s well-known mode expansion coefficients.

1 Classical outline

A number of unusual features arose in studying a perturbatively inhomogeneous quantum cosmology model [1] similar to the well-known one of Halliwell and Hawking [2]. Some of these features apply also to [2]’s model itself. These models are perturbations about the spatially closed $S^3$ isotropic model.

1) Something unusual happens upon applying the SVT (scalar–vector–tensor) split to the GR Hamiltonian constraint $\mathcal{H}$ and the GR momentum constraint $\mathcal{M}_i$.

I resolve this issue by a canonical analysis. It is possible for a perturbation, symmetry or mode decomposition’s pieces to transcend from second-class constraints to first-class ones by the obstruction to the closure of the classical brackets not having a contribution from the piece in question. The reverse, however, does not look to be possible. The way out is for some of the pieces to be acting as specifier equations. In retrospective, this is already clear in how those equations are for some of the pieces to be acting as specifier equations. In the models in question, $\mathcal{H}$ and $\mathcal{M}_i$’s SVT pieces no longer all act as first-class constraints. In retroactive, this is already clear in how $\mathcal{H}$ becomes three conditions $S\mathcal{H}$, $N\mathcal{H}$ and $T\mathcal{H}$; the manner in which $\mathcal{H}$ and $\mathcal{M}_i$ function as dynamical entities is changed by the SVT split. This part of the work is unaffected by the differences between [1] and [2]’s modelling. I give a degrees of freedom count in each case in Appendix A.

Thus, despite coming from objects, well known to be first class constraints, not all of $\mathcal{H}$ and $\mathcal{M}_i$’s SVT pieces are themselves first-class constraints. Consequently, proceeding to quantization by just hatting these expressions as if they were first-class constraints becomes a questionable step to take. Thus observation 1) invites a re-analysis of the quantization step of [2] (as well as of a first analysis of [1]’s quantization), and subsequent semiclassical work. Moreover, such models are important as a candidate for the origin of structure in the universe: quantum cosmological fluctuations as amplified by inflation. [2]-type semiclassical regimes represent a more general setting for this than in purely QFT approaches, for a more detailed calculations and to provide foundational justification of the latter.

2) A particular view was taken in [1] as regards what to solve that $\mathcal{M}_i$ for. Namely the thin sandwich prescription[7]: explicitly solve the Lagrangian variables form of $\mathcal{M}_i$ for the shift vector $\beta_i$. The plus side of this procedure is that the elimination just involves very simple algebra. The minus side is that, even allowing for the canonical count’s difference from the non-SVT-split geometrodynamics one, the resultant ‘reduced configuration space’ geometry ensuing for the full Einstein–scalar field system case of [1] turns out to be of the wrong dimension (8-$d$ rather than 7-$d$ per mode). The procedure has ceased to fully eliminate the spatial diffeomorphisms, $\text{Diff}(\Sigma)$. This obstructs progress with careful assessment of how to quantize the full Einstein–scalar field case.

My next new observation is that the vacuum case is free from issue 2) whilst still having issue 1) present. Thus it is an ideal arena in which to start to investigate the new quantization of perturbatively inhomogeneous cosmology.

This model works as follows. Unlike for the case including the scalar field, in the vacuum case the thin sandwich elimination additionally causes the scalar mode $a_{n}$ to drop out from the reduced kinetic term. (This parallels how relational particle models [4, 8] deal with their analogue of the thin sandwich – elimination of translation, rotation and optionally dilatation corrections [4, 8].) To be specific, the outcome of substituting the solutions of the modewise thin sandwich equations (19) into the modewise configuration space line element (17) is the modewise reduced line element

1See [3] for definitions of first-class, second-class, and specifier equations, and for the algorithm for handling all of these in the Hamiltonian formalism.

2These differences are due to [1] being relational [4], though for all intents and purposes in the present manuscript, one can take this to be equivalent to the Baierlein–Sharp–Wheeler (BSW) [5] formulation of canonical GR. On the other hand, [2] is performed within the Arnowitt–Deser–Misner (ADM) [6] formulation of GR. In the ADM formalism, the GR Hamiltonian constraint $\mathcal{H}$ arises from variation with respect to the lapse Lagrange multiplier. In the BSW formulation, the lapse is absent, with the Hamiltonian constraint arising instead as a primary constraint. See [3] again for the definitions of primary and secondary constraints, alongside an argument for relationalism’s reparametrization invariance necessitating the presence of at least one primary constraint.

3See Appendices C and D for the notation involved.
This in turn gives rise to a ‘spacetime position dependent mass’ analogue of the Klein–Gordon equation,

\[
\frac{2}{\exp(3\Omega)} ds^2 = \{ -1 + A_n \} d\Omega^2 + \frac{2}{3} d\Omega A_n + ||d\nu_n||^2 .
\]  
(1)

This is 5-d: \( A_n \) has now also dropped out of the line element, so this is now only short by 1 in removing the \( \text{Diff}(S^3) \) degrees of freedom. See Appendix C for the unreduced formulation’s SVT split, and subsequently Appendix D for the definitions of the \( ||\nu_n|| \) and \( A_n \) quantities that are useful in performing the reduction procedure.

Moreover, computing the curvature tensors for (1) [9] confirms that geometrically this is just 5-d Minkowski spacetime. The coordinate transformation to

\[
t_n := \frac{2}{3} \sqrt{A_n - 1} \cosh(\frac{1}{2} \ln(A_n - 1)) ,
\]
(2)

\[
w_n := \frac{2}{3} \sqrt{A_n - 1} \sinh(\frac{1}{2} \ln(A_n - 1))
\]
(3)

then makes this manifest:

\[
\frac{2}{\exp(3\Omega)} ds^2 = -dt_n^2 + dw_n^2 + ||d\nu_n||^2 .
\]
(4)

The cosh and sinh combination here is just the standard manoeuvre from Rindler coordinates. The other individual transformations involved are just basic 1-coordinate transformations, and a diagonalizing transformation.

The corresponding modewise potential – originally (18) in the unreduced formulation – also simplifies in these variables, to

\[
V_n = -\frac{\exp(\Omega)}{2} \{ 1 + s_n^2 - \{ v_n^2 - 1 \} d_n^2 + \frac{A_n}{3} \} + \exp(3\Omega) \Lambda \{ 1 + A_n \} .
\]
(5)

Furthermore, in this vacuum case \( \hat{\mathcal{H}} \) becomes (including pulling out an overall factor of \( c_n^2 \) by which this equation indeed also frees itself of vector mode content),

\[
5\Phi_n^2 - 16\Phi_n\dot{\Omega} + 8\dot{\Omega}^2 + \exp(-2\Omega) = 0
\]
(6)

for

\[
\Phi_n := \exp(3\Omega) \{ 1 + A_n \}/3
\]
(7)

One can then take this as an equation for the thus only temporariljy convenent mixed-SVT variable \( A_n = A_n(\Omega) \). This leads to a fully \( \text{Diff}(S^3) \)-reduced line element of the form

\[
\frac{2}{\exp(3\Omega)} ds^2 = \{ -1 + f_n(\Omega) \} d\Omega^2 + ||d\nu_n||^2 ,
\]
(8)

for

\[
f_n(\Omega) := A_n(\Omega) + \frac{2}{3} \frac{dA_n(\Omega)}{d\Omega}.
\]
(9)

(8) is conformally flat. One can then also define a new scale variable which absorbs the first term’s prefactor:

\[
\zeta_n := \int \sqrt{f_n(\Omega) - 1} d\Omega,
\]
(10)

leaving, up to a conformal factor, the line element

\[
ds^2 = -d\zeta_n^2 + ||d\nu_n||^2 .
\]
(11)

\( \partial/\partial\zeta_n \)'s components \( \partial/\partial s_n, \partial/\partial \zeta_n, \partial/\partial \phi_n, \partial/\partial \phi_n^* \) are then among the 10 conformal Killing vectors.

Finally the corresponding configuration space of the modewise inhomogeneities themselves is also clearly flat \( \mathbb{R}^3 \), with the \( \nu_n \) playing the role of Cartesian coordinates.

## 2 Quantum outline

The above flat metrics are useful in a number of applications. As an example of this, starting afresh with the reduced action leads to a reduced Hamiltonian of form

\[
\hat{\mathcal{H}} = \frac{1}{2} \{-\frac{1}{2} p_n^2 + ||p_n^\nu||^2 \} + \hat{V}(\zeta_n, \nu_n) .
\]
(12)

This in turn gives rise to a ‘spacetime position dependent mass’ analogue of the Klein–Gordon equation,

\[
0 = \hat{\mathcal{H}} \Psi_n = \frac{\hbar^2}{2} \{ \partial^2_{\zeta_n} - D^2 \} \Psi_n + \hat{V}(\zeta_n, \nu_n) \Psi_n .
\]
(13)

Note that this still splits into scalar (i.e. now the scalar sum variable \( s_n \)) and tensor parts; each of these is coupled to the scale variable but not directly to each other. Moreover, a Klein–Gordon type equation having a non-constant mass term is in general a significant complication, via this affecting the interpretation of the corresponding QM [10]. Also note that, equations of this form are familiar from e.g. Einstein–minimally coupled scalar isotropic, Bianchi I and diagonal Bianchi IX Quantum Cosmology [11]. Our point is that the regimes covered by such equations additionally extend to modewise perturbatively inhomogeneous Quantum Cosmology. Halliwell and Hawking’s scheme was of this nature too, albeit alongside linear quantum constraints as befits a Dirac quantization. On the other hand, our scheme has no accompanying linear quantum constraints. Instead our reduced configuration space has coordinates whose physical meanings are somewhat more involved, but whose geometrical form is none the less itself simple. Moreover, our scheme does not rely on hatting entities which have ceased to be playing the role of first-class constraints (for which how to Dirac-quantize is in fact less clear).

The scheme presented here straightforwardly admits a semiclassical regime as used to make quantum cosmological calculations and predictions. This shall be the subject of another Article.
A Degrees of freedom counts

| a) | b) | c) SVT split case | d) SVT split case |
|----|----|------------------|------------------|
| $h_{ij}$ | $h_{ij}$ | $\Omega$ | $\phi$ |
| $\beta_i$ | $\beta_i$ | $10$ configuration space | $12$ configuration space |
| $6 + 3$ | $1 + 6 + 3$ | degrees of freedom | degrees of freedom |
| $9$ configuration space | $9$ configuration space | $= 18$ phase space | $= 20$ phase space |
| degrees of freedom | degrees of freedom | degrees of freedom | degrees of freedom |
| $p_{\beta} - 6$ | $p_{\beta} - 6$ | $2 + 18$ S V T | $2 + 20$ S V T |
| $M_{\tilde{a}} - 6$ | $M_{\tilde{a}} - 6$ | mixed | mixed |
| $3$ : superspace | $2$ : superspace + scalar | $6$ | $8$ |
| $H_{\text{rem}}$ | $H_{\text{rem}}$ | $2$ | $4$ |

Figure 1: a) is the usual BSW analysis for vacuum GR. b) is the version additionally including a minimally-coupled scalar field. In each case the ADM analysis just adds 1 configuration space degree of freedom (the lapse $\alpha$) and thus 2 phase space degrees of freedom, but is then subjected to $p_{\alpha} = 0$, removing these 2. In this manner, BSW and ADM are equivalent. c) and d) then consider what happens to the vacuum and scalar field case respectively, upon specializing to perturbatively inhomogeneous cosmology and applying the SVT split. Note the appearance in both c) and d) of the non SVT split variable $A_n$, alongside how the momentum constraints fail to take out a Diff($S^3$)'s worth of degrees of freedom by case-dependent amounts. Some of that freedom has now been transferred to the multiple Hamiltonian constraints. I concentrate upon the descent to modewise Superspace($S^3$); attaining this is marked with grey rectangles for clarity; this objective is presently attained for c) but not for d). This is due to the vacuum case c)'s $V_{\text{H}}$ turning out to be usable to eliminate the non SVT split variable $A_n$, which in this case therefore attains only a temporary significance in performing successive reductions. That the two cases c) and d) behave differently in this regard is an interesting result in its own right. For the thin sandwich is a longstanding problem [5, 7] with further theoretical significance as regards the problem of time in quantum gravity [5, 4, 14].

B Dealing with other than first-class constraints

If one has conditions that are not first class constraints, one deals with them classically prior to quantization. Three alternatives here are as follows.

A) Replace the incipient Poisson brackets with Dirac brackets [3].

B) Extend the phase space with further auxiliary variables so as to ‘gauge-unfix’ second-class constraints into first-class ones. [12].

C) Reduce out the constraints in question. This is not systematically available, but [1]'s case of it is trivially solvable and trivially substitutable, so this is the method chosen for [1] and the current Paper.

C SVT expansions and unreduced formulation

The spatial 3-metric is expanded as [2]

$$h_{ij} = \exp(2\Omega(t))\{S_{ij}(t) + \epsilon_{ij}(\vec{x}, t)\}.$$  \hspace{1cm} (14)

Here, $S_{ij}$ is the standard hyperspherical $S^3$ metric, and $\epsilon_{ij}$ are inhomogeneous perturbations of the form [13]

$$\epsilon_{ij} = \sum_{n, l, m} \left\{ \sqrt{2}a_{nlm}S_{ij}Q_n^l m + \sqrt{6}b_{nlm}(P_j)_l m + \sqrt{2}(c_{nlm}^{\text{o}}(S^o_{ij})_n^l m + c_{nlm}^{\text{e}}(S^e_{ij})_n^l m) + 2(d_{nlm}^{\text{o}}(G_{ij})_n^l m + d_{nlm}^{\text{e}}(G_{ij}^e)_n^l m) \right\}. \hspace{1cm} (15)$$
The gravitational sector configuration space's classical-level configuration space geometric meaning of $A_n$ is the corresponding 3$d$ Laplacian. I also define

$$A_n := -\frac{3}{2} \left\{ a_n^2 - 4 \left\{ \frac{n^2 - 4}{n^2 - 1} b_n^2 + \frac{n^2 - 4}{n^2} c_n^2 + d_n^2 \right\} \right\} .$$

This is the gravitational sector configuration space's volume correction term, in the sense of being the first perturbative correction to the expansion of the unreduced configuration space metric's determinant [1]. It is additionally the sole coupling to the minisuperspace degrees of freedom [c.f. the $d^2$ term in (17)]. This bears some resemblance to the 3-body problem's ellipticity variable $\text{ellip} := p_2^2 - p_1^2$, for $p_i$ the mass-weighted Jacobi relative inter-particle cluster separation vectors [8]. Both are quadratic and comparisons by taking differences. On the other hand, $\text{ellip}$ compares the base and median partial moments of inertia, whereas $A_n$ compares the amount of one of the scalar gravitational modes $a_n$ against that of the other gravitational modes. All three of $\text{ellip}$, $s_n$ and $A_n$ drop out from the reduction as 'ubiquitous groupings', i.e. functional dependencies that feature widespreadly throughout each theory's quantities. Bigger such differences of quadratic expressions also drop out of the theory of larger planar $N$-body problems [8]. Moreover, out of all of these quantities $A_n$ alone subsequently vanishes from the working upon performing further reduction. This goes hand in hand with $A_n$ alone not being a Kuchař observable. This can be spotted since whereas the median and base being compared in $\text{ellip}$ are themselves rotationally invariant entities, 'the other gravitational modes' in $A_n$'s comparison include the unphysical vector modes $c_n$. Nor does the $A_n$ quantity's structure respect the status of $s_n$, rather than the individual $a_n$ and $b_n$, as an invariant quantity.

On the other hand, the final form of the inhomogeneous part of the vacuum potential (5) is another quadratic difference

$$Q_n := s_n^2 - \{n^2 - 1\}d_n^2 .$$

This does specifically compare the amount of the invariant scalar inhomogeneity $s_n$ with the amount of invariant tensor inhomogeneity $d_n$. Moreover, in this case the individual pieces are already Kuchař observables before the difference is taken.

Finally, I note that $A_n$ already occurred within a suite [15] of quadratic difference variables that were found to be useful variables for the quantum wavefunction to depend upon. On the other hand, my study provides the underlying classical-level configuration space geometric meaning of $A_n$. This suite amount to comparing $a_n$ with various subsets of the other modes.
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