CLIFFORD PROLATE SPHEROIDAL WAVE FUNCTIONS

HAMED BAGHAL GHAFFARI, JEFFREY A. HOGAN, JOSEPH D. LAKEY

Abstract. In the present paper, we introduce the multidimensional Clifford prolate spheroidal wave functions (CPSWFs) defined on the unit ball as eigenfunctions of a Clifford differential operator and provide a Galerkin method for their computation as linear combinations of Clifford-Legendre polynomials. We show that these functions are eigenfunctions of the truncated Fourier transformation. Then we investigate the role of the CPSWFs in the spectral concentration problem associated with balls in the space and frequency domains, the behaviour of the eigenvalues of the time-frequency limiting operator and their spectral accumulation property.

1. Introduction

Prolate spheroidal wave functions (PSWFs) are special functions that have long been used in mathematical physics. They are real-valued functions on the line which arise when solving the Helmholtz equation by separation of variables in prolate spheroidal coordinates (playing the role of Legendre polynomials in spherical coordinates). In a beautiful series of papers published in the Bell Labs Technical Journal in the 1960s, Slepian, Pollak and Landau observed that these same functions were the solutions of the spectral concentration problem which is of enormous importance in communications technologies. This observation allowed for the efficient computation of PSWFs and the eventual incorporation of their digital counterparts (the discrete PSWFs) in computer hardware. Analyticity and asymptotic properties of PSWFs, including numerical evaluations and applications to quadrature and interpolation are investigated in [4,5,13,18,23–25,31]. Other applications of PSWFs can be found in [7,12,14,16,19,20,22,26,30].

In higher dimensions, the computation of PSWFs is more problematic, and the differential equation from which they arise is singular, causing instabilities [27] [21]. Furthermore, the higher dimensional PSWFs, like the one-dimensional PWSFs, are real-valued. Here we seek natural multi-channel versions of the PSWFs of [27] with a view to applications in the treatment of multi-channel signals such as colour images and electromagnetic fields.

This paper is organized as follows. We will review the preliminaries about Clifford Analysis in second section. The third section will be about the definition of the multidimensional Clifford prolate spheroidal wave functions (CPSWFs). In section four, we will have a look at the Sturm-Liouville operator related to the radial part of the CPSWFs. Section five, will be about the proofs stating the CPSWFs are the eigenfunctions of the finite Fourier transformations. In section 6, we will investigate the essential features of the new multidimensional prolate spheroidal wave functions such as the decrease of eigenvalues. In the current section, we will talk about spectral concentration problem. Lastly, we see the spectrum accumulation properties CPSWFs in $m$ dimensions.

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2. Clifford analysis

Let \( \mathbb{R}^m \) be \( m \)-dimensional euclidean space and let \( \{e_1, e_2, \ldots, e_m\} \) be an orthonormal basis for \( \mathbb{R}^m \). We endow these vectors with the multiplicative properties

\[
e_j^2 = -1, \quad j = 1, \ldots, m,
\]

\[
e_j e_i = -e_i e_j, \quad i \neq j, \quad i, j = 1, \ldots, m.
\]

For any subset \( A = \{j_1, j_2, \ldots, j_h\} \subseteq \{1, \ldots, m\} = Q_m \), with \( j_1 < j_2 < \cdots < j_h \) we consider the formal product \( e_A = e_{j_1} e_{j_2} \cdots e_{j_h} \). Moreover for the empty set \( \emptyset \) one puts \( e_\emptyset = 1 \) (the identity element). The Clifford algebra \( \mathbb{R}_m \) is then the \( 2^m \)-dimensional real associative algebra

\[
\mathbb{R}_m = \left\{ \sum_{A \subseteq Q_m} \lambda_A e_A : \lambda_A \in \mathbb{R} \right\}.
\]

Similarly, the Clifford algebra \( \mathbb{C}_m \) is the \( 2^m \)-dimensional complex associative algebra

\[
\mathbb{C}_m = \left\{ \sum_{A \subseteq Q_m} \lambda_A e_A : \lambda_A \in \mathbb{C} \right\}.
\]

Every element \( \lambda = \sum_{A \subseteq Q_m} \lambda_A e_A \in \mathbb{C}_m \) may be decomposed as \( \lambda = \sum_{k=0}^{m} [\lambda]_k \), where

\[
[\lambda]_k = \sum_{|A|=k} \lambda_A e_A
\]

is the so-called \( k \)-vector part of \( \lambda \) \( (k = 0, 1, \ldots, m) \).

Denoting by \( \mathbb{R}_k^m \), the subspace of all \( k \)-vectors in \( \mathbb{R}_m \), i.e., the image of \( \mathbb{R}_m \) under the projection operator \( [\cdot]_k \), one has the multi-vector decomposition \( \mathbb{R}_m = \mathbb{R}_0^m \oplus \mathbb{R}_1^m \oplus \cdots \oplus \mathbb{R}_m^m \) leading to the identification of \( \mathbb{R} \) with the subspace of real scalars \( \mathbb{R}_0^m \) and of \( \mathbb{R}_m^m \) with the subspace of real Clifford vectors \( \mathbb{R}_m \). The latter identification is achieved by identifying the point \( (x_1, \ldots, x_m) \in \mathbb{R}^m \) with the Clifford number \( x = \sum_{j=1}^{m} e_j x_j \in \mathbb{R}_1^m \).

The Clifford number \( e_M = e_1 e_2 \cdots e_m \) is called the pseudoscalar; depending on the dimension \( m \), the pseudoscalar commutes or anti-commutes with the \( k \)-vectors and squares to \( \pm 1 \). The Clifford conjugation on \( \mathbb{C}^m \) is the conjugate linear mapping \( \lambda \mapsto \bar{\lambda} \) of \( \mathbb{C}_m \) to itself satisfying

\[
\bar{\sum \lambda \mu} = \bar{\mu} \bar{\lambda}, \quad \text{for all } \lambda, \mu \in \mathbb{C}_m,
\]

\[
\bar{\lambda_A e_A} = \bar{\lambda}_A \bar{e}_A, \quad \lambda_A \in \mathbb{C},
\]

\[
\bar{e}_j = -e_j, \quad j, \ j = 1, \ldots, m.
\]

The Clifford conjugation leads to a Clifford inner product \( \langle \cdot, \cdot \rangle \) and an associated norm \( |\cdot| \) on \( \mathbb{C}_m \) given respectively by

\[
\langle \lambda, \mu \rangle = [\bar{\lambda} \mu]_0 \quad \text{and} \quad |\lambda|^2 = [\bar{\lambda} \lambda]_0 = \sum_A |\lambda_A|^2,
\]

for \( \lambda = \sum_{A \subseteq Q_m} \lambda_A e_A \in \mathbb{C}_m \).

The product of two vectors \( x, y \in \mathbb{R}_1^m \) can be decomposed as the sum of a scalar and a 2-vector, also called a bivector:

\[
x y = \langle x, y \rangle + x \wedge y,
\]

where \( \langle x, y \rangle = -\sum_{j=1}^{m} x_j y_j \in \mathbb{R}_0^m \), and, \( x \wedge y = \sum_{i=1}^{m} \sum_{j=i+1}^{m} e_i e_j (x_j y_i - x_i y_j) \in \mathbb{R}_2^m \). Note that the square of a vector variable \( x \) is scalar-valued and equals the norm squared.
up to minus sign:

\[ x^2 = -(x, x) = -|x|^2. \]

Clifford analysis offers a function theory which is a higher-dimensional analogue of the theory of holomorphic functions of one complex variable. The functions considered are defined in the Euclidean space \( \mathbb{R}^m \) and take their values in the Clifford algebra \( \mathbb{R}_m \).

The central notion in Clifford analysis is monogenicity, which is a multidimensional counterpart to that of holomorphy in the complex plane.

**Definition 2.1.** A function \( f(x) = f(x_1, \ldots, x_m) \) defined and continuously differentiable in an open region of \( \mathbb{R}^m \) and taking values in \( \mathbb{C}^m \) is said to be left monogenic in that region if

\[ \partial_x f = 0, \]

where \( \partial_x = \sum_{j=1}^m e_j \partial_{x_j} \) is the Dirac operator and \( \partial_{x_j} \) is the partial differential operator \( \frac{\partial}{\partial x_j} \). We also define the Euler differential operator by

\[ E = \sum_{j=1}^m x_j \partial_{x_j}. \]

The Laplace operator is factorized by the Dirac operator as follows:

\[ \Delta_m = -\partial_x^2. \]  \hfill (2.1)

The notion of right monogenicity is defined in a similar way by letting the Dirac operator act from the right. A \( \mathbb{C}^m \)-valued function \( f(x) = \sum_{A \subset Q_m} f_A(x) e_A \) (where each \( f_A \) takes complex values) is right monogenic if and only if its Clifford conjugate \( \overline{f}(x) = \sum_{A \subset Q_m} \overline{f_A(x)} e_A \) is left monogenic. In fact, \( \overline{\partial f} = -f \overline{\partial} \).

**Definition 2.2.** A left, respectively right, monogenic homogeneous polynomial \( P_k \) of degree \( k \) \( (k \geq 0) \) in \( \mathbb{R}^m \) is called a left, respectively right, solid inner spherical monogenic of order \( k \). The set of all left, respectively right, solid inner spherical monogenic of order \( k \) will be denoted by \( M^+_l \), respectively \( M^+_r \).

It can be shown \([11]\) that the dimension of \( M^+_l \) is given by

\[ \dim M^+_l(k) = \frac{(m + k - 2)!}{(m - 2)!k!}. \]

A left, respectively right, monogenic homogeneous function \( Q_k \) of degree \(-(k+m-11)\) in \( \mathbb{R}^m \setminus \{0\} \) is called a left, respectively right, solid outer spherical monogenic of order \( k \).

**Lemma 2.3.** We can see that

\[ M^+_l(k) \cap M^-_l(k) = \{0\}. \]

**Proposition 2.4.** For any two monogenic, homogeneous functions \( Y_k(x) \) and \( Y_l(x) \), we have that

\[ \langle Y_k(x), Y_l(x) \rangle_{L^2(s^{m-1})} = \delta_{k,l}, \]

where \( Y_k(x), Y_l(x) \in M^+_l(k) \cup M^-_l(k) \).

Therefore, we can define the monogenic functions

\[ M_l(k) := M^+_l(k) \oplus M^-_l(k). \]
It’s possible to see that if $P_k(x) \in M_i^+(k)$ then $Q_k = \frac{x}{|x|^m} P_k \frac{x}{|x|^2} \in M_i^-(k)$.

Below, we make some remarks which will be required in proving main theorems. We will need also the following lemma which is easy to obtain by direct calculation.

**Lemma 2.5.** For $P_k \in M_i^+(k)$ and $s \in \mathbb{N}$ the following fundamental formula holds:

\[
\partial_s [x^s P_k] = \begin{cases} -sx^{s-1}P_k & \text{for } s \text{ even,} \\ -(s+2k+m-1)x^{s-1}P_k & \text{for } s \text{ odd.} \end{cases}
\]

**Proof.** For the proof see [11]. □

**Definition 2.6.** A real-valued polynomial $S_k$ of degree $k$ on $\mathbb{R}^m$ satisfying

\[\Delta_m S_k(x) = 0, \quad \text{and,} \quad S_k(tx) = t^k S_k(x) \quad (t > 0),\]

is called a solid spherical harmonic of degree $k$. The collection of solid spherical harmonics of degree $k$ on $\mathbb{R}^m$ is denoted $\mathcal{H}(k)$ (or $\mathcal{H}(m,k)$).

Since $\partial_x^2 = -\Delta_m$, we have that

\[M_i^+(k) \subset \mathcal{H}(k), \quad \text{and,} \quad M_i^-(k) \subset \mathcal{H}(k).\]

Let $H_{(r)}$ be a unitary right Clifford-module, i.e. $(H_{(r)}, +)$ is an abelian group and a law of scalar multiplication $(f, \lambda) \rightarrow f\lambda$ from $H_{(r)} \times \mathbb{C}_m$ into $H_{(r)}$ is defined such that for all $\lambda, \mu \in \mathbb{C}_m$ and $f, g \in H_{(r)}$:

(i) $f(\lambda + \mu) = f\lambda + f\mu$,
(ii) $f(\lambda\mu) = (f\lambda)\mu$,
(iii) $(f + g)\lambda = f\lambda + g\lambda$,
(iv) $f e_0 = f$.

Note that $H_{(r)}$ becomes a complex vector space if $\mathbb{C}$ is identified with $\mathbb{C} e_0 \subset \mathbb{C}_m$. Then a function $\langle \cdot, \cdot \rangle : H_{(r)} \times H_{(r)} \rightarrow \mathbb{C}_m$ is said to be an inner product on $H_{(r)}$ if for all $f, g, h \in H_{(r)}$ and $\lambda \in \mathbb{C}_m$:

(i) $\langle f, g\lambda + h \rangle = \langle f, g \rangle \lambda + \langle f, h \rangle$,
(ii) $\langle f, g \rangle = \overline{\langle g, f \rangle}$,
(iii) $[\langle f, f \rangle]_0 \geq 0$ and $[\langle f, f \rangle]_0 = 0$ if and only if $f = 0$.

From this $\mathbb{C}_m$-valued inner product $\langle \cdot, \cdot \rangle$, one can recover the complex inner product

\[\langle f, g \rangle = [\langle f, g \rangle]_0,\]

on $H_{(r)}$. Putting for each $f \in H_{(r)}$

\[\|f\|^2 = \langle f, f \rangle, \quad (2.2)\]

$\| \cdot \|$ becomes a norm on $H_{(r)}$ turning it into a normed right Clifford-module.

Now, let $H_{(r)}$ be a unitary right Clifford-module provided with an inner product $\langle \cdot, \cdot \rangle$. Then it is called a right Hilbert Clifford-module if $H_{(r)}$ considered as a complex vector space provided with the complex inner product $\langle \cdot, \cdot \rangle$ is a Hilbert space.

We consider the $\mathbb{C}_m$-valued inner product of the functions $f, g : \mathbb{R}^m \rightarrow \mathbb{C}_m$ by

\[\langle f, g \rangle = \int_{\mathbb{R}^m} \overline{f(x)} g(x) \, dx,\]
where $dx$ is Lebesgue measure on $\mathbb{R}^m$ and moreover the associated norm given by (2.2).
The right Clifford-module of $\mathbb{C}_m$-valued measurable functions on $\mathbb{R}^m$ for which $\|f\|_2 < \infty$ is a right Hilbert Clifford-module which we denote by $L^2(\mathbb{R}^m, \mathbb{C}_m)$. Therefore, we obtain the right Hilbert Clifford-module of square integrable functions: $L^2(\mathbb{R}^m, \mathbb{C}_m)$ of functions $f : \mathbb{R}^m \rightarrow \mathbb{C}_m$ for which each component of $f$ is measurable and

$$\|f\|_2 = \left( \int_{\mathbb{R}^m} |f(x)|^2 dx \right)^{\frac{1}{2}} < \infty.$$

The standard tensorial multi-dimensional Fourier transform given by:

$$\mathcal{F} f(\xi) = \int_{\mathbb{R}^m} e^{-2\pi i \langle x, \xi \rangle} f(x) dx.$$

Now, we have two definitions related to the operators.

**Definition 2.7.** Let $L^2(\mathbb{R}^m, \mathbb{C}_m)$ be right Hilbert Clifford-module. For any $f, g \in L^2(\mathbb{R}^m, \mathbb{C}_m)$, the operator $T$ is self-adjoint if

$$\langle Tf, g \rangle = \langle f, Tg \rangle.$$

**Definition 2.8.** Let $L^2(\mathbb{R}^m, \mathbb{C}_m)$ be right Hilbert Clifford-module. The operator $T$ is compact, if $\{T(f_n)\}$ has a convergent subsequence for every bounded sequence of $\{f_n\} \in L^2(\mathbb{R}^m, \mathbb{C}_m)$.

Now we state some fundamental results without proof which will be useful later.

**Theorem 2.9.** The Fourier transform $\mathcal{F}$ is an isometry on the space of the square integrable functions, in other words, for all $f, g \in L^2(\mathbb{R}^m, \mathbb{C}_m)$ the Parseval formula holds:

$$\langle f, g \rangle = \langle \mathcal{F} f, \mathcal{F} g \rangle.$$

In particular, for each $f \in L^2(\mathbb{R}^m, \mathbb{C}_m)$ one has $\|f\|_2 = \|\mathcal{F} f\|_2$.

**Proof.** See the proof in [11].

**Proposition 2.10.** For any two monogenic, homogeneous polynomials $Y_k$ of homogeneous degree $k$ and $Y_\ell$ of homogeneous degree $\ell$ we have

$$\langle Y_k, Y_\ell \rangle_{L^2(S^{m-1})} := \int_{S^{m-1}} Y_k(\omega) Y_\ell(\omega) d\omega = 0$$

if $k \neq \ell$.

**Theorem 2.11.** *(Clifford-Stokes theorem)* Let $f, g \in C^1(\Omega, \mathbb{C}_m)$. Then for each compact orientable $m$-dimensional manifold $C \subset \Omega$ with boundary $\partial C$,

$$\int_{\partial C} f(x)n(x) g(x) d\sigma(x) = \int_C [f(x) (\partial_\nu g(x)) + f(x) (\partial_\omega g(x))] dx.$$

where $n(x)$ is the outward-pointing unit normal vector on $\partial C$.

**Proof.** For the proof, see [11].

The following result may be obtained as a simple application of the Clifford-Stokes theorem.
Lemma 2.12. Let $f, g$ be defined on a neighbourhood $\Omega$ of the unit ball in $\mathbb{R}^m$ and suppose $f$ is right monogenic on $\Omega$ while $g$ is left monogenic on $\Omega$. Then
\[
\int_{S^{m-1}} f(\omega) g(\omega) \, d\omega = 0. \tag{2.3}
\]

Now here we introduce the representation for functions in $L^2(\mathbb{R}^m, \mathbb{R}_m)$. A proof can be found in [11]

Theorem 2.13. For every function $f \in L^2(\mathbb{R}^m, \mathbb{R}_m)$, we have the following representation
\[
f(x) = \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} \left[ f^{(l)}_k(|x|) Y^{(l)}_k(x) + g^{(l)}_k(|x|) \frac{x}{|x|^m} Y^{(l)}_k \left( \frac{x}{|x|^2} \right) \right], \tag{2.4}
\]
where $Y_k \in M^+_k(k)$ and the radial functions $f^{(l)}_k$ and $g^{(l)}_k$ satisfy
\[
\int_0^{\infty} r^{m+2k-1} |f^{(l)}_k(r)|^2 \, dr < \infty, \quad \int_{\Omega} |g^{(l)}_k(r)|^2 \, dr < \infty.
\]

In what follows, $B(c)$ is the closed ball of radius $c > 0$ in $\mathbb{R}^m$.

Lemma 2.14. Let $f \in L^2(\mathbb{R}^m, \mathbb{R}_m)$, supported on $B(1)$. Then the radial functions $f^{(l)}_k(|x|)$ and $g^{(l)}_k(|x|)$ defined in Theorem 2.13 are supported at $[0, 1]$.

Proof. Since $f$ is supported on $B(1)$, we have
\[
0 = \int_{B(1)^c} \left[ \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} [f^{(l)}_k(|x|) Y^{(l)}_k(x) + g^{(l)}_k(|x|) \frac{x}{|x|^m} Y^{(l)}_k \left( \frac{x}{|x|^2} \right) ]^2 \, dx
\]
\[
= \int_{B(1)^c} \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} f^{(l)}_k(|x|) Y^{(l)}_k(x) \sum_{k'=0}^{d_{k'}} f^{(l')}_{k'}(|x|) Y^{(l')}_{k'}(x) \, dx
\]
\[
+ \int_{B(1)^c} \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} f^{(l)}_k(|x|) Y^{(l)}_k(x) \sum_{k'=0}^{d_{k'}} g^{(l')}_{k'}(|x|) \frac{x}{|x|^m} Y^{(l')}_{k'} \left( \frac{x}{|x|^2} \right) \, dx
\]
\[
+ \int_{B(1)^c} \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} g^{(l)}_k(|x|) \frac{x}{|x|^m} Y^{(l)}_k \left( \frac{x}{|x|^2} \right) \sum_{k'=0}^{d_{k'}} f^{(l')}_{k'}(|x|) Y^{(l')}_{k'}(x) \, dx
\]
\[
+ \int_{B(1)^c} \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} g^{(l)}_k(|x|) \frac{x}{|x|^m} Y^{(l)}_k \left( \frac{x}{|x|^2} \right) \sum_{k'=0}^{d_{k'}} g^{(l')}_{k'}(|x|) \frac{x}{|x|^m} Y^{(l')}_{k'} \left( \frac{x}{|x|^2} \right) \, dx.
\]

The second and third integrals are zero due to the Lemma 2.12. By the orthonormality of $\{Y^l_k : k \geq 0, 1 \leq \ell \leq d_k \}$ we have
\[
\sum_{k=0}^{\infty} \sum_{l=1}^{d_k} \int_0^{\infty} r^{m+2k-1} |f^{(l)}_k(r)|^2 \, dr + \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} \int_0^{\infty} r^{1-m-2k} |g^{(l)}_k(r)|^2 \, dr = 0.
\]
Hence, $f^{(l)}_k(r) = g^{(l)}_k(r) = 0$ for $r > 1$. This completes the proof. \qed
We now give relevant background on the Clifford-Legendre polynomials, which we use to build the CPSWFs. Given $\alpha > -1$ and $Y_k \in M^+_l(k)$, the Clifford-Legendre polynomial $C^\alpha_{n,m}(Y_k)$ is defined by

$$C^\alpha_{n,m}(Y_k) = (1 + x^2)^{-\alpha} \partial_x^n [(1 + x^2)^{\alpha + n} Y_k(x)].$$

Let $L_\alpha$ be the Clifford differential operator

$$L_\alpha f = (1 + x^2) \partial_x^2 f - 2(\alpha + 1)x \partial_x f$$

which acts on $C^2(\mathbb{R}^m, \mathbb{R}_m)$.

**Theorem 2.15.** *(Differential equation for the Gegenbauer polynomials)* For all $n, k \in \mathbb{N}$ and $\alpha > -1$, the Clifford-Gegenbauer polynomial $C^\alpha_{n,m}(Y_k)(x)$ is an eigenfunction of the differential operator $L_\alpha$ with real eigenvalue $C(\alpha, n, m, k)$, i.e.,

$$L_\alpha (C^\alpha_{n,m}(Y_k)) = C(\alpha, n, m, k) C^\alpha_{n,m}(Y_k),$$

where

$$C(\alpha, n, m, k) = \begin{cases} n(2\alpha + n + m + 2k) & \text{if } n \text{ is even}, \\ (2\alpha + n + 1)(n + m + 2k - 1) & \text{if } n \text{ is odd}, \end{cases}$$

and $\{Y_k\}_{k=1}^d$ is any orthonormal basis for $M^+_l(k)$.

**Proof.** See [11].

When $\alpha = 0$, the Clifford-Gegenbauer polynomials are known as the Clifford-Legendre polynomials and are eigenfunctions of the Clifford differential operator $L_0$:

$$L_0 (C^0_{n,m}(Y_k)) = C(0, n, m, k) C^0_{n,m}(Y_k),$$

where $L_0 f = (1 + x^2) \partial_x^2 f - 2x \partial_x f$.

Let $Y_k \in M^+_l(k)$ be normalized in the sense that $\int_{S^{n-1}} |Y_k(\omega)|^2 d\omega = 1$. Using a result from [3], we now define the normalized Clifford-Legendre polynomials $\overline{C}^0_{n,m}(Y_k)$ by

$$\overline{C}^0_{n,m}(Y_k) = \frac{\sqrt{2k + 2n + m}}{2^n n!} C^0_{n,m}(Y_k).$$

We now state some useful results from [3] which we will use in this paper.

**Theorem 2.16.** The Bonnet formula for the normalized Clifford-Legendre polynomials is as follows:

(a) When $n = 2N$ is even,

$$x \overline{C}^0_{2N,m}(Y_k)(x) = A_{N,k,m} \overline{C}^0_{2N+1,m}(Y_k)(x) + B_{N,k,m} \overline{C}^0_{2N-1,m}(Y_k)(x)$$

where

$$A_{N,k,m} = -\frac{(\frac{m}{2} + N + k) \sqrt{m + 4N + 2k}}{(\frac{m}{2} + 2N + k) \sqrt{m + 4N + 2k + 2}},$$

$$B_{N,k,m} = \frac{N \sqrt{m + 4N + 2k}}{(\frac{m}{2} + 2N + k) \sqrt{m + 4N + 2k - 2}}.$$
(b) When \( n = 2N + 1 \) is odd
\[
x C_{2N+1,n}^0(Y_k)(x) = A'_{N,k,m} C_{2N+1,m}^0(Y_k)(x) + B'_{N,k,m} C_{2N,m}^0(Y_k)(x),
\] (2.8)
where
\[
A'_{N,k,m} = \frac{-(N + 1)^2m + 4N + 2k + 2}{(m^2 + 2N + k + 1)^2m + 4N + 2k + 4},
\]
\[
B'_{N,k,m} = \frac{(m^2 + N + k)^2m + 4N + 2k + 2}{(m^2 + 2N + k + 1)^2m + 4N + 2k}.
\]

**Theorem 2.17.** The Fourier transform of the restriction of the Clifford-Legendre polynomial \( C_{n,m}^0(Y_k)(x) \) to the unit ball \( B(1) \) is given by
\[
\mathcal{F}(C_{n,m}^0(Y_k))(\xi) = (-1)^k \ell^{n+k} 2^m n! \xi^m \frac{J_{k+m+n}(2\pi|\xi|)}{|\xi|^{n+k}} Y_k(\xi).
\] (2.9)

**Theorem 2.18.** In dimension \( m = 2 \), the normalised Clifford-Legendre polynomials satisfy
\[
C_{2N+2}^0(Y_k)(x) = -e_1 C_{2N,2}^0(Y_{k+1})(x).
\]

**Lemma 2.19.** There are real constant \( a_{n,m,j}^k \) such that
\[
\partial_x C_{n,m}^0(Y_k)(x) = 4n(n - 1 + k + \frac{m}{2}) C_{n-1,m}^0(Y_k)(x) + \sum_{j=0}^{\lceil \frac{n}{2} \rceil - 1} a_{n,m,j}^k C_{n-2-2j,m}^0(Y_k)(x).
\] (2.10)

**Proof.** The proof is by induction on \( n \). By direct computation, we find that
\[
\partial_x C_{1,m}^0(Y_k)(x) = (m + 2k) Y_k(x) = (m + 2k) C_{1,m}^0(Y_k)(x),
\]
so that (2.10) is verified when \( n = 0 \). Suppose now that (2.10) holds when \( n = N \). By Corollary 3.10 from [3] we have
\[
\partial_x C_{n+1,m}^0(Y_k)(x) = 4(n + 1)(n + k + \frac{m}{2}) C_{n,m}^0(Y_k)(x) - n \partial_x C_{n-1,m}^0(Y_k)(x).
\] (2.11)

With \( n = 2N + 2 \), we have that
\[
\partial_x C_{2N+3,m}^0(Y_k)(x) = 4(2N + 3)(2N + 2 + k + \frac{m}{2}) C_{2N+2,m}^0(Y_k)(x)
\]
\[
- (2N + 2) \partial_x C_{2N+1,m}^0(Y_k)(x)
\]
\[
= 4(2N + 3)(2N + 2 + k + \frac{m}{2}) C_{2N+2,m}^0(Y_k)(x)
\]
\[
- (2N + 2) \sum_{j=0}^{N-1} a_{N,j}^k C_{2N-2-2j,m}^0(Y_k)(x)
\]
\[
= \sum_{j=0}^{N+1} a_{N+1,j}^k C_{2N+2-2j,m}^0(Y_k)(x)
\] (2.12)
where \( a_{N+1,0}^k = 4(2N + 3)(2N + 2 + k + \frac{m}{2}) \) and \( a_{N+1,j}^k = -4(2N + 3)(2N + 2) a_{N,j}^k \) for \( 1 \leq j \leq N - 1 \). Now let’s assume that \( n = 2N + 1 \). Then with the same process
Proposition 3.2. Given and to compute them using Boyd’s algorithm \[6\].

The multidimensional Clifford Prolate Spheroidal Wave Functions (CPSWFs) as the eigenfunctions of second equation. Then, there exists an solution \(u\) for some parameter \(\lambda\) corresponding to different eigenvalues \(\lambda_n, \lambda_m\). Let \(p(x) > 0\) and \(r(x) > 0\) for all \(x \in (a, b)\), and also let \(p(a) = p(b) = 0\). Then \(u_n, u_m\) are orthogonal in \(L_r(a, b)\), i.e.,

\[
\int_a^b r(x)u_n(x)u_m(x) \, dx = 0.
\]

Theorem 2.21. Let \(u_n \) and \(u_m\) be real-valued eigenfunctions for a Sturm-Liouville form defined in \(2.14\) corresponding to different eigenvalues \(\lambda_n, \lambda_m\). Let \(p(x) > 0\) and \(r(x) > 0\) for all \(x \in (a, b)\), and also let \(p(a) = p(b) = 0\). Then \(u_n\) and, \(u_m\) are orthogonal in \(L_r(a, b)\), i.e.,

\[
\int_a^b r(x)u_n(x)u_m(x) \, dx = 0.
\]

Theorem 2.22. Let \(p_i, q_i, i = 1, 2\), be real-valued continuous functions on the interval \([a, b]\) and let

\[
(p_1(x)y')' + q_1(x)y = 0,
\]

\[
(p_2(x)y')' + q_2(x)y = 0,
\]

be two homogeneous linear second order differential equations in self-adjoint form with \(0 < p_2(x) \leq p_1(x)\), and, \(q_1(x) \leq q_2(x)\). Let \(u\) be a non-trivial solution of the first of these equations with successive roots at \(z_1\) and \(z_2\) and let \(v\) be a non-trivial solution of second equation. Then, there exists an \(x \in (z_1, z_2)\) such that \(v(x) = 0\).

3. The Multidimensional Clifford Prolate Spheroidal Wave Functions

The goal of this section is to introduce the Clifford prolate spheroidal wave functions and to compute them using Boyd’s algorithm \[6\].

Definition 3.1. Given \(c \geq 0\), the Clifford differential operator \(L_c\) acts of \(C^2(B(1), \mathbb{R}_m)\) as follows:

\[
L_c f(x) = \partial_x((1 - |x|^2)\partial_x f(x)) + 4\pi^2 c^2 |x|^2 f(x)
\]

where \(B(1)\) is the unit ball in \(\mathbb{R}_m\) and \(\partial_x\) is the Dirac Operator. We define the Clifford Prolate Spheroidal Wave Functions (CPSWFs) as the eigenfunctions of \(L_c\).

Proposition 3.2. The operator \(L_c\) defined in \(3.1\) is self-adjoint.
Proof. First note that $L_c = L_0 + M_c$ where $M_c f(x) = 4 \pi^2 c^2 |x|^2 f(x)$. Clearly $M_c$ is self-adjoint, so we need only show that $L_0$ is self-adjoint. An application of 2.11 yields

$$
\langle f, L_0 g \rangle = \int_{\partial B(1)} \overline{f(x)} \overline{|x|}(1 - |x|^2) \overline{\partial_x g(x)} \, dx
$$

$$
= \int_{\partial B(1)} \frac{\overline{f(x)} \overline{|x|}(1 - |x|^2)(\partial_x g)(x)}{\overline{|x|}} \, d\sigma(x) - \int_{\partial B(1)} \overline{(\partial_x f)(x)(1 - |x|^2)(\partial_x g)(x)} \, dx
$$

$$
= - \int_{\partial B(1)} \frac{[(1 - |x|^2)(\overline{\partial_x f})][\partial_x g](x)}{\overline{|x|}} \, d\sigma(x) + \int_{\partial B(1)} [(1 - |x|^2)(\overline{\partial_x f})] \partial_x g(x) \, dx
$$

$$
= \int_{\partial B(1)} \partial_x[(1 - |x|^2)\partial_x f](x)g(x) \, dx = \langle L_0 f, g \rangle,
$$

which completes the proof. \qed

We now outline an algorithm for the computation of the CPSWFs. Recall that the first Bonnet formula (2.7) of Theorem 2.16 takes the form

$$
x \overline{C^0_{2i,m}(Y^j_k)}(x) = A_{i,k,m} \overline{C^0_{2i+1,m}(Y^j_k)}(x) + B_{i,k,m} \overline{C^0_{2i-1,m}(Y^j_k)}(x) + C_{i,k,m} \overline{C^0_{2i-2,m}(Y^j_k)}(x),
$$

Multiplying both sides of this equation by $-x$ on both sides and applying the second Bonnet formula (2.8) yields

$$
|x|^2 \overline{C^0_{2i,m}(Y^j_k)}(x) = a_i \overline{C^0_{2i+2,m}(Y^j_k)}(x) + b_i \overline{C^0_{2i,m}(Y^j_k)}(x) + c_i \overline{C^0_{2i-2,m}(Y^j_k)}(x),
$$

where $a_i = -A_{i,k,m} A'_{i,k,m}$, $b_i = -(A_{i,k,m} B'_{i,k,m} + A'_{i-1,k,m} B_{i,k,m})$, $c_i = -B_{i,k,m} B'_{i-1,k,m}$.

By direct computation, it may be shown that if $X^e_k$ is the collection of functions of the form $F(|x|^2) Y^j_k(x)$ with $F \in C^\infty([0,1], \mathbb{R})$ and $Y^j_k$ a spherical monogenic of degree $k$ on $\mathbb{R}^{2m}$, then $X^e_k$ is invariant under $L_c$. In fact, a direct calculation, using the monogenicity and homogeneity of $Y^j_k$, gives

$$
L_0[F(|x|^2) Y^j_k(x)] = [(-2m + (4 + 4k)|x|^2) F'(|x|^2) - 4k |x|^2 F''(|x|^2)] Y^j_k(x).
$$

Similarly, if $X^o_k$ is the collection of functions of the form $xG(|x|^2) Y^j_k$ with $G \in C^\infty([0,1], \mathbb{R})$ and $Y^j_k$ a spherical monogenic of degree $k$ on $\mathbb{R}^{2m}$, then $X^o_k$ is also invariant under $L_c$. Hence, when searching for eigenfunctions of $L_c$, we may search within the spaces $X^e_k$ and $X^o_k$. Furthermore, the spaces $X^e_k$ and $X^o_k$ are orthogonal for all values of $k$ and $\ell$. The collection $\{C^0_{2i,m}(Y^j_k)\}_{i=0}^\infty$ lies in $X^e_k$ while $\{C^0_{2i+1,m}(Y^j_k)\}_{i=0}^\infty$ lies in $X^o_k$. We may therefore assume that eigenfunctions of $L_c$ will take one of two forms:

$$
f_{N,k}(x) = \sum_{j=0}^\infty C^0_{2i,m}(Y^j_k)(x) \alpha^k_{i,N;}, \quad g_{N,k}(x) = \sum_{i=0}^\infty C^0_{2i+1,m}(Y^j_k)(x) \beta^k_{i,N;},
$$

(3.3)
An application of (3.2) yields

\[ L_c f_{N,k}(x) = L_c \left( \sum_{i=0}^{\infty} C_{2i,m}^0 (Y_k^j)(x) \alpha_{i,N,m}^k \right) \]

\[ = \sum_{i=0}^{\infty} \left( L_0 + 4\pi^2 c^2|x|^2 \right) C_{2i,m}^0 (Y_k^j)(x) \alpha_{i,N,m}^k \]

\[ = \sum_{i=0}^{\infty} \left[ C(0, 2i, m, k) C_{2i,m}^0 (Y_k^j)(x) + 4\pi^2 c^2|x|^2 C_{2i,m}^0 (Y_k^j)(x) \right] \alpha_{i,N,m}^k \]

\[ = \sum_{i=0}^{\infty} \left[ 4\pi^2 c^2 a_i C_{2i+2,m}^0 (Y_k^j)(x) + \left( C(0, 2i, m, k) + 4\pi^2 c^2 b_i \right) C_{2i,m}^0 (Y_k^j)(x) \right. \]

\[ + 4\pi^2 c^2 c_i C_{2i-2,m}^0 (Y_k^j)(x) \right] \alpha_{i,N,m}^k \]

\[ = \sum_{i=0}^{\infty} C_{2i,m}^0 (Y_k^j)(x) \left[ (4\pi^2 c^2 a_{i-1}) \alpha_{i-1,N,m}^k + \left( C(0, 2i, m, k) + 4\pi^2 c^2 b_i \right) \alpha_{i,N,m}^k \right. \]

\[ + \left. (4\pi^2 c^2 c_{i+1}) \alpha_{i+1,N,m}^k \right] \]

\[ = \chi_{2N,m}^{k,c} f_{N,k}(x) = \chi_{2N,m}^{k,c} \sum_{i=0}^{\infty} C_{2i,m}^0 (Y_k^j)(x) \alpha_{i,N,m}^k, \quad (3.4) \]

with the understanding that \( \alpha_{i,N,m}^k = 0 \) if \( i < 0 \). The orthogonality of the Clifford-Legendre polynomials enables us to equate the coefficients on both sides of this equation to obtain the following recurrence formula for \( \alpha_{i,N,m}^k \): \( i=0: \)

\[ (4\pi^2 c^2 a_{i-1}) \alpha_{i-1,N,m} + \left( C(0, 2i, 2, k) + 4\pi^2 c^2 b_i - \chi_{2N,2}^{k,c} \right) \alpha_{i,N,m} + \left( 4\pi^2 c^2 c_{i+1} \right) \alpha_{i+1,N,m} = 0. \]

This recurrence formula is true for all \( i, N \geq 0 \) so the problem reduces to finding the eigenvectors \( \alpha_{i,N,m}^k \) and the eigenvalues \( \chi_{2N,m}^{k,c} \) of the doubly-infinite matrix \( M_{k,m}^c \) with the following entries:

\[ M_{k,m}^c(i,j) = \begin{cases} 
-4\pi^2 c^2 \frac{i(k+i+\frac{m}{2}+1)}{(k+i+\frac{m}{2}+1)(k+i+\frac{m}{2}+2)}, & \text{if } i \geq 1, \quad j = i - 1, \\
4i(k+i+\frac{m}{2}) + 4\pi^2 c^2 \frac{\left( (k+i+\frac{m}{2})^2 + (i+1)^2 \right)}{(k+i+\frac{m}{2}+1)(k+i+\frac{m}{2}+2)}, & \text{if } i = j \geq 0, \\
-4\pi^2 c^2 \frac{(k+i+\frac{m}{2}+1)}{(k+i+\frac{m}{2}+1)(k+i+\frac{m}{2}+2)}, & \text{if } i \geq 0, \quad j = i + 1, \\
0 & \text{else.}
\end{cases} \]

We see directly that \( M_{k,m}^c \) is symmetric and banded in the sense that \( M_{k,m}^c(i,j) = 0 \) if \( |i-j| > 1 \). The symmetry of \( M_{k,m}^c \) is due to the self-adjointness of \( L_c \). With \( g_{N,k} \) as in (3.3), we find that if \( g_{N,k} \) is an eigenfunction of \( L_c \) with eigenvalue \( \chi_{2N+1,m}^{k,c} \) then the vector \( \beta_{i,N}^c \) with \( i \)-th entry \( \beta_{i,N}^c \) is an eigenvector of the doubly-infinite matrix \( M_{k,m}^c \) which has \( (i,j) \)-th entry

\[ M_{k,m}^c(i,j) = \begin{cases} 
-4\pi^2 c^2 \frac{i(k+i+\frac{m}{2})}{(k+i+\frac{m}{2}+1)(k+i+\frac{m}{2}+2)}, & \text{if } i \geq 1, \quad j = i - 1, \\
4(i+1)(k+i+\frac{m}{2}) + 4\pi^2 c^2 \frac{\left( (k+i+\frac{m}{2})^2 + (i+1)^2 \right)}{(k+i+\frac{m}{2}+1)(k+i+\frac{m}{2}+2)}, & \text{if } i = j \geq 0, \\
-4\pi^2 c^2 \frac{(k+i+\frac{m}{2}+1)}{(k+i+\frac{m}{2}+1)(k+i+\frac{m}{2}+2)}, & \text{if } i \geq 0, \quad j = i + 1, \\
0 & \text{else.}
\end{cases} \]
Remark 3.3. We can check directly that the off-diagonal elements of $M^e_{k,m}$ and $M^o_{k,m}$ satisfy
\[
M^e_{k+1,m}(i-1,i) = M^e_{k+1,m}(i,i+1) = -4\pi^2c^2 \frac{(i+1)(k+i + \frac{m}{2} + 1)}{(k+2i + \frac{m}{2} + 1)\sqrt{k+2i + \frac{m}{2} + 3\sqrt{k+2i + \frac{m}{2} + 1}}}
\]  
(3.5)
However, for the diagonal elements we have
\[
M^e_{k,m}(i,i) - M^e_{k+1,m}(i,i) = 4k + 2m. 
\]  
(3.6)
In light of (3.5) and (3.6) we have
\[
M^o_k = M^e_{k+1} + bI,
\]
where $b = 4k+2m$. Therefore, if $v$ is the eigenvector of matrix of $M^e_{k,m}$ with eigenvalue $\lambda$ then $v$ is also an eigenvector of $M^o_{k+1,m}$ with eigenvalue $\lambda + b$.

The matrices $M^e_{k,m}$ and $M^o_{k,m}$ have eigenvectors $\alpha_N$ and $\beta_N$ ($N \geq 0$) respectively, leading to eigenfunctions $\psi_{2N,m}^{k,c}$ and $\psi_{2N+1,m}^{k,c}$ of $L_c$ arising from (3.3):
\[
\psi_{2N,m}^{k,c}(x) = \sum_{j=0}^{\infty} C_{2j+1,m}^0(Y_k(x))\alpha_{i,N,m}^{k,c}; \quad \psi_{2N+1,m}^{k,c}(x) = \sum_{i=0}^{\infty} C_{2i+1,m}^0(Y_k(x))\beta_{i,N,m}^{k,c}, \quad (3.7)
\]
where $(\alpha_N)_i = \alpha_{i,N,m}^{k,c}$ and $(\beta_N)_i = \beta_{i,N,m}^{k,c}$ are the $i$-th entries of the vectors $\alpha_N$ and $\beta_N$ respectively. The relationship between $M^e_{k,m}$ and $M^o_{k,m}$ outlined above leads to the following relationship between the even and off CPSWFs.

Theorem 3.4. Let $\psi_{2N,2}^{k,c}(x)$, and $\psi_{2N+1,2}^{k,c}(x)$, be $L^2$-normalised two-dimensional CP-SWFs. Then we have
\[
\psi_{2N+1,2}^{k,c}(x) = -e_1\psi_{2N,2}^{k+1,c}(x). \quad (3.8)
\]
Proof. We recall equation (3.7) and the fact that $\beta_{k,m}^{k+1,c} = \alpha_{k,N,m}^{k+1,c}$. An application of Theorem 2.18 yields
\[
\psi_{2N+1,2}^{k,c}(x) = \sum_{i=0}^{\infty} C_{2i+1,2}^0(Y_{k+1}(x))\beta_{i,N}^{k,c} = \sum_{i=0}^{\infty} -e_1\alpha_{i+1,N}^{k+1,c}(x) = -e_1\psi_{2N,2}^{k+1,c}(x).
\]
This completes the proof. \qed

As we have seen, each even CPSWF has the form $\psi_{2N,2}^{k,c}(x) = R_N^k(|x|^2)Y_k(x)$ with $R_N^k$ a function defined on $[0, \infty)$ and known as the radial part of $\psi_{2N,2}^{k,c}$. Similarly, the odd CPSWFs have the form $\psi_{2N+1,2}^{k,c}(x) = xS(|x|^2)Y_k(x)$ and we define its radial part to be $|x|S(|x|^2)$. Here we plot the radial parts of the some CPSWFs. In Figures 3 we see the proficiency of Theorem 3.4.

We end this section with an observation about the eigenvalues of the operator $L_c$.

Proposition 3.5. The eigenvalues of $L_c$ are positive real numbers.

Proof. We have already seen that $L_c$ is self-adjoint. Furthermore, the eigenvalues of $L_c$ are the eigenvalues of the symmetric matrices $M^e_k$ and $M^o_k$, and are therefore real numbers. On the other hand, $L_c$ is a positive operator in the sense that
\[
\langle L_c f, f \rangle = [\langle L_c f, f \rangle]_0 \geq 0.
\]
CLIFFORD PROLATE SPHEROIDAL WAVE FUNCTIONS

To see this, note that for each continuously differentiable $\mathbb{C}_m$-valued function defined on $\Omega \subset \mathbb{R}^m$ we have $\overline{\partial f} = -\tilde{f} \partial$. Therefore, by the Clifford-Stokes theorem,

$$
\langle L_0 f, f \rangle = \int_{B(1)} \overline{\partial f}[(1 - \lvert x \rvert^2)\partial f(x)]f(x) \, dx
$$

$$
= -\int_{B(1)} \left[(1 - \lvert x \rvert^2)\overline{\partial f(x)}\right] \partial f(x) \, dx
$$

$$
= -\int_{\partial B(1)} (1 - \lvert x \rvert^2)\overline{\partial f(x)} \frac{x}{|x|} f(x) + \int_{B(1)} (1 - \lvert x \rvert^2)\overline{\partial f(x)} \partial f(x) \, dx
$$

$$
= \int_{B(1)} (1 - \lvert x \rvert^2)\overline{\partial f(x)} \partial f(x) \, dx.
$$

Consequently,

$$
\langle L_c f, f \rangle = \int_{B(1)} (1 - \lvert x \rvert^2)\overline{\partial f(x)} \partial f(x) \, dx + 4\pi^2 c^2 \int_{B(1)} |x|^2 |\overline{f(x)} f(x)|_0 \, dx
$$

$$
= \int_{B(1)} (1 - \lvert x \rvert^2)\lvert \partial f(x) \rvert^2 \, dx + 4\pi^2 \int_{B(1)} \lvert x \rvert^2 \lvert f(x) \rvert^2 \, dx \geq 0.
$$
If \( \lambda \) is a (real) eigenvalue of \( L_0 \) associated with an eigenfunction \( f \), then
\[
\|f\|^2_2 = \lambda \langle f, f \rangle = \langle \lambda f, f \rangle = \langle L_c f, f \rangle \geq 0,
\]
from which we conclude that \( \lambda \geq 0 \).

\[\square\]

4. The Sturm-Liouville Properties of the CPSWFs

In [3] it was shown that \( C_{N,m}^0(Y_k)(x) = P_{N,k,m}(|x|^2)Y_k(x) \), where \( P_{N,k,m}(|x|^2) \) (the radial part of the Clifford-Legendre polynomial) is real-valued. From (2.5) we have
\[
L_0 C_{N,m}^0(Y_k)(x) = C(0, n, m, k) C_{N,m}^0(Y_k)(x),
\]
Putting \( R \)
\[
S_0 R_{N,k,m}(s) = -\frac{C(0, 2N, m, k)}{4} R_{N,k,m},
\]
where \( S_0 \) is the differential operator
\[
S_0 = (1 - s^2) \frac{d^2}{ds^2} + \left[ \left( k + \frac{m}{2} - 1 \right) - \left( \frac{m}{2} + k + 1 \right) s \right] \frac{d}{ds}.
\]

We have a similar result for eigenfunctions \( \psi_{2,m}^{k,c}(x) = P_{N,m}^{k,c}(|x|^2)Y_k(x) \) of the operator \( L_c = L_0 + M_c \), namely,
\[
t(1-t) \frac{d^2}{dt^2} P_{N,m}^{k,c}(t) + \frac{1}{2} (m+2k-t(2m+2k)) \frac{d}{dt} P_{N,m}^{k,c}(t) - \pi^2 c^2 t P_{N,m}^{k,c}(t) = -\frac{\lambda_{2,m}^{k,c}}{4} P_{N,m}^{k,c}(t),
\]
where \(|x|^2 = t \). Putting \( \hat{P}_{N,m}^{k,c}(s) = P_{N,m}^{k,c}(2t - 1) \) \((s \in (-1, 1))\), gives
\[
S_c \hat{P}_{N,m}^{k,c}(s) := S_0 \hat{P}_{N,m}^{k,c}(s) - \frac{\pi^2 c^2}{2} (s + 1) \hat{P}_{N,m}^{k,c}(s) = -\frac{\lambda_{2,m}^{k,c}}{4} \hat{P}_{N,m}^{k,c}(s).
\]

The operator determined by equation (4.2) is not self-adjoint. In order to make it so, we multiply both sides by \( y = y(s) = (s + 1)^{k + \frac{2}{n} - 1} \) to obtain
\[
\frac{d}{ds} \left( p(s) \frac{d}{ds} \hat{P}_{N,m}^{k,c}(s) \right) - \frac{\pi^2 c^2}{2} (s + 1)^{k + \frac{2}{n} - 1} \hat{P}_{N,m}^{k,c}(s) = -\frac{\lambda_{2,m}^{k,c}}{4} \hat{P}_{N,m}^{k,c}(s),
\]
which is a Sturm-Liouville differential equation. From Definition 2.20 and Theorem 2.21 we can conclude the following theorem.

**Theorem 4.1.** The differential operator \( \hat{T}_c \) defined by
\[
\hat{T}_c f(s) = \left( 4(s + 1)^{k + \frac{2}{n}} (1-s) f'(s) \right)' - 2\pi^2 c^2 (s + 1)^{k + \frac{2}{n}} f(s)
\]
is self-adjoint and the differential equation (4.2) can be written as
\[
\hat{T}_c \hat{P}_{N,m}^{k,c}(s) + \lambda_{2,m}^{k,c} g(s) \hat{P}_{N,m}^{k,c}(s) = 0,
\]
where \( g(s) = (s + 1)^{k + \frac{2}{n} - 1} \). This is a singular Sturm-Liouville equation, i.e., with \( p(s) = 4(s + 1)^{k + \frac{2}{n}} (1-s) = 0 \) when \( s = \pm 1 \). Therefore, the eigenvalues of \( \hat{T}_c \) are distinct and may be labelled so that \( \chi_{0,m}^{k,c} < \chi_{2,m}^{k,c} < \ldots \), and the eigenfunctions \( \{ \hat{P}_{N,m}^{k,c} \}_{N=0}^{\infty} \) may be normalised so that
\[
\langle \hat{P}_{N,m}^{k,c}, \hat{P}_{N',m}^{k,c} \rangle = \int_{-1}^{1} \hat{P}_{N,m}^{k,c}(s) \hat{P}_{N',m}^{k,c}(s) g(s) \, ds = \delta_{N,N'},
\]
Lemma 4.2. For each integer $k \geq 0$, let $\{Y^k_{i,1}\}_{i=1}^{d_k}$ be an orthonormal basis for $M^+_N(k)$. The even eigenfunctions of the operator $L_e$, i.e., $\{\psi^{k,c,i}_{2N,m} : k \geq 0, 1 \leq i \leq d_k\}$ are orthogonal.

Proof. Note that

$$\int_{B(1)} \frac{\psi^{k,c,i}_{2N,m}(x)}{\psi^{k,c,i'}_{2N,m}(x)} \, dx = \int_{B(1)} \frac{P^{k,c}_{N,m}(|x|^2)}{P^{k,c}_{N,m}(|x|^2)} Y^{k,c}_{i}(x) Y^{k,c'}_{i'}(x) \, dx$$

$$= \int_{B(1)} P^{k,c}_{N,m}(r^2) P^{k,c}_{N,m}(r^2) P^{m+k+k'-1}_{s} \int_{S^{m-1}} Y^{k,c}_{i}(\omega) Y^{k,c'}_{i'}(\omega) \, d\omega \, dr$$

$$= \int_{0}^{1} P^{k,c}_{N,m}(t) P^{k,c}_{N,m}(t) t^{k+\frac{m}{2}-1} \, dt \delta_{kk'} \delta_{ii'}$$

$$= \frac{1}{4} \int_{-1}^{1} Q^{k,c}_{N,m}(s) Q^{k,c}_{N,m}(s) \left( \frac{s+1}{2} \right)^{k+\frac{m}{2}-1} \, ds \delta_{kk'} \delta_{ii'}$$

$$= \frac{1}{2^{k+\frac{m}{2}+1}} \delta_{N,N'} \delta_{kk'} \delta_{ii'}.$$  \(\square\)

Here, we mention that we can obtain a Sturm-Liouville problem for the functions $P^{k,c}_{N,m}$ ($N \geq 0$) without the change of variable.

Lemma 4.3. The radial part of $P^{k,c}_{N,m}(|x|^2)$ of the CPSWF $\psi^{k,c,i}_{2N,2}(x)$, satisfies

$$4t(1-t) \frac{d^2}{dt^2} P^{k,c}_{N,m}(t) + 2(m+2k-2(2m+2k)) \frac{d}{dt} P^{k,c}_{N,m}(t) - 4\pi^2 c^2 t P^{k,c}_{N,m}(t) + \chi_{2N,m} P^{k,c}_{N,m}(t) = 0,$$

which becomes a Sturm-Liouville differential equation after multiplying by $g(t) = t^{k+\frac{m}{2}-1}$. Therefore, $\{P^{k,c}_{N,m}\}_{N=0}^{\infty}$ may be normalised so that

$$\int_{0}^{1} P^{k,c}_{N,m}(t) P^{k,c}_{M,m}(t) t^{k+\frac{m}{2}-1} \, dt = (P^{k,c}_{N,m}, P^{k,c}_{M,m})_{g(t)} = \delta_{MN}.$$  \(4.4\)

Furthermore, for each fixed integer $k \geq 0$, the collection $\{P^{k,c}_{N,m} : N \geq 0\}$ is complete in the weighted $L^2$ space $L^2([0, 1], t^{k+\frac{m}{2}-1})$ of measurable functions $f : [0, 1] \rightarrow \mathbb{R}$ for which $\int_{0}^{1} |f(t)|^2 t^{k+\frac{m}{2}-1} \, dt < \infty$.

Lemma 4.4. If $f$ is as in \(\square\) and is supported on $B(1)$ and $(\psi^{k,c,i}_{2N,m}, f) = 0$ for all $N \geq 0, k \geq 0, 1 \leq j \leq d_k$, where $d_k$ is the dimension of $M^+_N(k)$. Then $f^{(i)} = 0$, for all $k \geq 0, 1 \leq j \leq d_k$. 


Furthermore, the eigenfunctions of the eigenvalues of the differential operator \( \mathcal{L} \) with \( p \) is self-adjoint and can be written, following Sturm-Liouville differential equation

\[
0 = (\psi_{2N,m}^{k,c,i}; f) = \int_{B(1)} \frac{\psi_{2N,m}^{k,c,i}(x)}{\psi_{2N,m}^{k,c,i}(x)} f(x) \, dx
\]

By Lemmas 2.12 and 2.14 we have

\[
0 = \int_{B(1)} P_{N,m}^{k,c}(|x|^2)Y_k^m(x) f(x) \, dx
\]

\[
= \int_{B(1)} P_{N,m}^{k,c}(|x|^2)Y_k^m(x) \sum_{k'=0}^{\infty} \sum_{l=0}^{d_{k'}} f_{k'}^{(l)}(|x|)Y_k^{(l)}(x) \, dx
\]

\[
= \sum_{k'=0}^{\infty} \sum_{l=1}^{d_{k'}} \int_{B(1)} P_{N,m}^{k,c}(|x|^2)f_{k'}^{(l)}(r)r^{k+k'+m-1} \int_{\mathbb{S}^{m-1}} Y_k^{(l)}(\omega)Y_{k'}^{(l)}(\omega) \, d\omega \, dr
\]

\[
= \int_{0}^{1} P_{N,m}^{k,c}(r^2)f_{k}^{(l)}(r)r^{2k+m-1} \, dr
\]

\[
= 2 \int_{0}^{1} P_{N,m}^{k,c}(r)f_{k}^{(l)}(\sqrt{r})t^{k+\frac{m}{2}} \, dt.
\]

Hence by Lemma 4.3 \( f_{k}^{(l)} = 0 \), for all \( k \) and \( i \).

**Proof.** Suppose \( f \in L^2(\mathbb{R}^m, \mathbb{R}_m) \) admits the expansion (2.4) and is supported on \( B(1) \). By Lemmas 2.12 and 2.14 we have

Theorem 4.5. The radial parts \( Q_{N,m}^{k,c} \) of the odd CPSWFs \( Q_{2N+1,m}^{k,c}(Y_k) \) satisfy the following Sturm-Liouville differential equation

\[
(1-s)^2 \frac{d^2}{ds^2} Q_{N,m}^{k,c}(s) + \left[ -s \left( k + \frac{m}{2} + 2 \right) + \left( k + \frac{m}{2} \right) \right] \frac{d}{ds} Q_{N,m}^{k,c}(s)
\]

\[
+ \left[ -\left( \frac{m}{2} + k \right) - \frac{\pi^2}{2}(1+s) + \frac{\lambda_{2N+1,m}^k}{4} \right] Q_{N,m}^{k,c}(s) = 0.
\]

In fact, the differential operator \( \hat{U}_c \) defined by

\[
\hat{U}_c f(s) = \left( 4(s+1)^{k+\frac{m}{2}}(1-s)f'(s) \right)' + 4(s+1)^{k+\frac{m}{2}} \left( -\frac{\pi^2}{2} - \frac{\pi^2}{2} \right) f(s).
\]

is self-adjoint and can be written,

\[
\hat{U}_c Q_{N,m}^{k,c}(s) + \lambda_{2N+1,m}^k g(s)Q_{N,m}^{k,c}(s) = 0,
\]

where \( g(s) = (s+1)^{k+\frac{m}{2}} \). The differential equation in (4.5) is singular Sturm-Liouville with \( p(s) = 4(s+1)^{k+\frac{m}{2}}(1-s) \) which takes the value 0 when \( s = \pm 1 \). Therefore, the eigenvalues of the \( \hat{U}_c \) are distinct and may be ordered so that \( \lambda_{1,m}^k < \lambda_{3,m}^k < \ldots \)

Furthermore, the eigenfunctions \( Q_{N,m}^{k,c} \) may be normalised so that

\[
(Q_{N,m}^{k,c}, Q_{N',m}^{k,c})g(s) = \int \frac{Q_{N,m}^{k,c}(s)Q_{N',m}^{k,c}(s)g(s) \, ds}{\delta_{N,N'}}.
\]
and are complete in the weighted Lebesgue space \( L^2([-1, 1], g) \) of measurable functions \( f : [-1, 1] \to \mathbb{R} \) for which \( \int_0^1 |f(s)|^2 g(s) \, ds < \infty \).

**Lemma 4.6.** The radial part \( Q_{N,m}^{k,c}(|x|^2) \) of the odd CPSWFs \( \psi_{2N+1,2}^{k,c,i}(x) \), satisfy

\[
4t(1-t) \frac{d^2}{dt^2} Q_{N,m}^{k,c}(t) + 2(m+2k+2-t(4+m+2k)) \frac{d}{dt} Q_{N,m}^{k,c}(t) - (4\pi^2 c^2 t + 2(m+2k))Q_{N,m}^{k,c}(t) + \chi_{2N+1,m}^{k,c}(t) = 0, \quad (4.6)
\]

which is a Sturm-Liouville differential equation after multiplication of both sides by \( g(t) = t^{k+\frac{m}{2}} \). Therefore,

\[
\int_0^1 Q_{N,m}^{k,c}(t) Q_{M,m}^{k,c}(t) t^{k+\frac{m}{2}} \, dt = (Q_{N,m}^{k,c}, Q_{M,m}^{k,c})_g = \delta_{MN},
\]

and for each integer \( k \geq 0 \), the collection \( \{Q_{N,m}^{k,c} : N \geq 0\} \) is complete in the weighted Lebesgue space \( L^2([0, 1], t^{k+\frac{m}{2}}) \) of measurable functions \( f : [0, 1] \to \mathbb{R} \) for which \( \int_0^1 |f(t)|^2 t^{k+\frac{m}{2}} \, dt < \infty \).

**Theorem 4.7.** The odd eigenfunctions of \( L_c \), i.e, \( \{\psi_{2N+1,1,m}^{k,c,i} : N \geq 0, \ k \geq 0, \ 1 \leq i \leq d_k\} \) are orthogonal in \( L^2(B(1), \mathbb{R}_m) \).

**Lemma 4.8.** If \( f \) is as in \( (2.4) \) and is supported on \( B(1) \) and \( (\psi_{2N+1,1,m}^{k,c,i}, f) = 0 \) for all \( N \geq 0, \ k \geq 0, \ 1 \leq j \leq d_k \), then \( g_k^{(i)} = 0 \), for all \( k \geq 0, \ 1 \leq i \leq d_k \).

Using the fact that \( \psi_{2N+1,m}^{k,c,i} = P_{N,m}^{k,c}(|x|^2) Y_k^i(x) \) and \( \psi_{2N+1,m}^{k,c,i} = Q_{N,m}^{k,c}(|x|^2) x Y_k^i(x) \), we conclude from Lemma \( 2.12 \) we can conclude that the the even and odd CPSWFs span orthogonal spaces in \( L^2(B(1), \mathbb{R}_m) \).

**Proposition 4.9.** The even and odd eigenfunctions of the \( L_c \), are orthogonal, i.e.,

\[
\int_{B(1)} \overline{\psi_{2N,m}^{k,c,i}(x)} \psi_{2N+1,m}^{k,c,i}(x) \, dx = 0,
\]

for all \( N, N' \geq 0, \ k, k' \geq 0, \ 1 \leq i \leq d_k, \ 1 \leq i' \leq d_{k'} \).

Now we prove that the CPSWFs are not only orthonormal but also a basis for \( L^2(B(1), \mathbb{R}_m) \).

**Theorem 4.10.** The set \( \{\psi_{2N+1,m}^{k,c,i} : N \geq 0, \ k \geq 0, \ 1 \leq j \leq d_k\} \cup \{\psi_{2N+1,1,m}^{k,c,i} : N \geq 0, \ k \geq 0, \ 1 \leq j \leq d_k\} \), is an orthonormal basis for \( L^2(B(1), \mathbb{R}_m) \).

**Proof.** From Proposition \( 4.9 \) and Lemmas \( 4.2 \) and \( 4.7 \) we see that \( \{\psi_{n,m}^{k,c,i} : n \geq 0, \ k \geq 0, \ 1 \leq i \leq d_k\} \) is orthogonal in \( L^2(B, \mathbb{R}_m) \). Completeness is a consequence of Lemmas \( 4.3 \) and \( 4.8 \). \( \square \)

**Theorem 4.11.** The radial part \( P_{N,m}^{k,c}(|x|^2) \) of \( \psi_{2N,m}^{k,c,i}(x) \) and the radial part \( Q_{N,m}^{k,c}(|x|^2) \) of \( \psi_{2N+1,m}^{k,c,i}(x) \) satisfy

\[
P_{N,m}^{k,c}(1) \neq 0 \neq Q_{N,m}^{k,c}(1).
\]
Proof. With $Q = Q_{N,m}^{k,c}$, we assume that $Q(1) = 0$. Putting $s = 1$ in (4.2) gives $-2Q'(1) = 0$ from which we conclude that $Q'(1) = 0$. Taking the $n$-th derivative of (4.2) and evaluating both sides at $s = 1$ gives $-(2n + 2)Q^{(n+1)}(1) = 0$, from which we conclude that $Q^{(n+1)}(1) = 0$. Hence, by induction we have that $Q^{(\ell)}(1) = 0$ for all $\ell \geq 0$. But $Q$ is a solution of (4.2) which has the analytic solution,

$$Q(s) = \sum_{n=0}^{\infty} \frac{Q^{(n)}(1)}{n!} (s - 1)^n,$$

on $(-1, 1)$. Since, $Q^{(n)}(1) = 0$ for all $n \geq 0$, we have $Q(s) \equiv 0$. We conclude that $Q(1) \neq 0$. The proof that $P(1) = P_{N,m}^{k,c}(1) \neq 0$ is similar. \hfill $\square$

Proposition 4.12. The even eigenvalues of the operator $L_c$, i.e., $\chi_{2N,m}^{k,c}$ are also the eigenvalues of $\tilde{T}_c$, which is defined on (4.3). This is true for the odd case too.

Proof. This is essentially the content of Theorem 4.1. \hfill $\square$

According to the 2.20, 2.21, and 2.22, we may conclude some of the features of the SL for the radial parts of the even and odd CPSWFs.

Corollary 4.13. For any fixed $k$, the eigenvalues $\{\chi_{2N,m}^{k,c}\}_{N=0}^{\infty}$ are distinct, as are the eigenvalues $\{\chi_{2N+1,m}^{k,c}\}_{N=0}^{\infty}$.

As a consequence of Sturm-Liouville theory

Corollary 4.14. For any fixed $k$, the zeros of the $\{\psi_{2N,m}^{k,c}\}_{N=0}^{\infty}$ are interlacing as are the zeroes of $\{\psi_{2N+1,m}^{k,c}\}_{N=0}^{\infty}$.

5. CPSWFs as the eigenfunctions of the Finite Fourier Transformation

In this section, we will present the finite Fourier transformation $G_c$ and then prove that the CPSWFs are the eigenfunctions of this operator.

Definition 5.1. We define $G_c$ from $L^2(B(1), \mathbb{C}_m)$ to $L^2(\mathbb{R}_m, \mathbb{R}_m)$ by

$$G_c f(x) = \chi_{B(1)}(x) \int_{B(1)} e^{2\pi i c(x,y)} f(y) \, dy,$$

where $\chi_{B(1)}$ is the characteristic function of $B(1)$. The adjoint $G_c^*$ of $G_c$ is given by

$$G_c^* f(x) = \chi_{B(1)}(x) \int_{\mathbb{R}_m} e^{-2\pi i c(x,y)} f(y) \, dy.$$

Definition 5.2. The “space-limiting” operator $Q : L^2(\mathbb{R}_m, \mathbb{R}_m) \rightarrow L^2(B(1), \mathbb{R}_m)$ is given by

$$Q f(x) = \chi_{B(1)}(x) f(x),$$

and the “bandlimiting” operator $P_c : L^2(\mathbb{R}_m, \mathbb{R}_m) \rightarrow PW_c$ is given by

$$P_c f(x) = \int_{B(1)} \mathcal{F} f(\xi) e^{2\pi i \xi(x)} d\xi.$$

Here, $PW_c$ is the Paley-Wiener space of functions with bandlimit $c$, i.e.,

$$PW_c = \{ f \in L^2(\mathbb{R}_m, \mathbb{R}_m) : \hat{f}(\xi) = 0 \text{ if } |\xi| > c \}.$$
Theorem 5.3. The operators of $L_c$ and $G_c$ commute, i.e., $L_c G_c = G_c L_c$.

Proof. Since $L_c$ is self-adjoint,
\[
G_c L_c f(x) = \int_{B(1)} L_c f(y) e^{2\pi i c(x,y)} dy
\]
\[
= \int_{B(1)} e^{-2\pi i c(x,y)} L_c f(y) dy = \int_{B(1)} L_c(e^{-2\pi i c(x,\cdot)}(y)) f(y) dy. \tag{5.2}
\]
On the other hand,
\[
L_c G_c f(x) = L_c \int_{B(1)} e^{2\pi i c(x,y)} f(y) dy = \int_{B(1)} L_c(e^{2\pi i c(\cdot,y)})(x) f(y) dy. \tag{5.3}
\]
By comparing (5.2) and (5.3) we see that it is sufficient to show that
\[
L_c(e^{2\pi i c(\cdot,y)})(x) = \overline{L_c(e^{-2\pi i c(x,\cdot)})(y)}.
\]
By direct calculation, we find that
\[
L_c(e^{2\pi i c(\cdot,y)})(x) = [-4\pi i cxy + 4\pi^2 c^2(|x|^2 + |y|^2 - |x|^2|y|^2)]e^{2\pi i c(x,y)}
\]
\[
= \overline{L_c(e^{-2\pi i c(x,\cdot)})(y)},
\]
and the proof is complete. \qed

Corollary 5.4. $L_c$, $G_c$ and $G_c^*$ satisfy the following commutation relations:

(a) $G_c^* L_c = L_c G_c^*$

(b) $L_c(G_c^* G_c) = (G_c^* G_c)L_c$ is self-adjoint.

The operators $Q$ and $P_c$ are self-adjoint on $L^2(\mathbb{R}^m, \mathbb{C}_m)$ An equivalent representation of $P_c$ may be achieved as follows:
\[
P_c f(x) = \int_{B(1)} \mathcal{F} f(\xi) e^{2\pi i c(\xi,x)} d\xi
\]
\[
= \int_{B(1)} \left( \int_{\mathbb{R}^m} f(s) e^{-2\pi i (s,\xi)} ds \right) e^{2\pi i c(\xi,x)} d\xi
\]
\[
= c^m \int_{\mathbb{R}^m} f(s) \left( \int_{B(1)} e^{2\pi i c(\xi,s-x)} d\xi \right) ds = c^m \int_{\mathbb{R}^m} f(s) K_c(x - s) ds,
\]
where
\[
K_c(x) = \int_{B(1)} e^{2\pi i c(\xi,x)} d\xi = \left( \frac{2\pi}{c|x|} \right)^{\frac{m}{2}} J_{\frac{m}{2}}(c|x|), \tag{5.4}
\]
where $J_m(x)$ is the Bessel function. Also, we see that
\[
QP_c f(x) = c^m \chi_{B(1)}(x) \int_{\mathbb{R}^m} f(t) K_c(x - t) dt.
\]
$K_c$ is real-valued and even. Now we show that the operator $QP_c : L^2(B(1), \mathbb{C}_m) \rightarrow L^2(B(1), \mathbb{C}_m)$, is self-adjoint. In fact, if $f, g \in L^2(B(1), \mathbb{C}_m)$ then
\[
(f, (QP_c)^* g) = (QP_c f, g) = (P_c f, Q g) = (P_c f, g) = (f, P_c g) = (Q f, P_c g) = (f, Q P_c g).
\]
Furthermore,
\[
\mathcal{G}_c^* \mathcal{G}_c f(x) = \chi_{B(1)}(x) \int_{B(1)} e^{-2\pi i c(y,x)} \mathcal{G}_c f(y) dy
\]
\[
= \chi_{B(1)}(x) \int_{B(1)} e^{-2\pi i c(y,x)} \int_{B(1)} e^{2\pi i c(y,t)} f(t) dt dy
\]
\[
= \chi_{B(1)}(x) \int_{B(1)} f(t) \int_{B(1)} e^{2\pi i c(y,t-x)} dy dt
\]
\[
= \chi_{B(1)}(x) \int_{B(1)} f(t) K_c(x - t) dt = \frac{1}{c^m} QP_c f(x). \quad (5.5)
\]

The operator \( P_cQ \) has the representation
\[
P_cQ f(x) = c^m \int_{\mathbb{R}^m} (Qf)(s) K_c(x - s) ds = c^m \int_{B(1)} f(s) K_c(x - s) ds.
\]

We have also that
\[
Q P_c f(x) = \chi_{B(1)}(x) \int_{B(1)} \chi_{B(1)}(x - y) f(y) dy = \int_{\mathbb{R}^m} M_c(x, y) f(y) dy,
\]
where \( M_c(x, y) = c^m \chi_{B(1)}(x) \chi_{B(1)}(y) \mathcal{F}^{-1} \chi_{B(1)}(x - y) \). We therefore have
\[
\int_{\mathbb{R}^m} \int_{\mathbb{R}^m} |L_c(x, y)|^2 dx dy \leq c^{2m} \int_{B(1)} \int_{\mathbb{R}^m} |\mathcal{F}^{-1} \chi_{B(1)}(x - y)|^2 dy dx
\]
\[
= c^{2m} \int_{B(1)} \int_{\mathbb{R}^m} |\mathcal{F}^{-1} \chi_{B(1)}(x)|^2 dx dy
\]
\[
= c^{2m} |\chi_{B(1)}| \int_{\mathbb{R}^m} |\chi_{B(1)}(x)|^2 dx = c^{2m} |B(1)|^2 < \infty.
\]

Therefore, \( QP_c \) is a Hilbert-Schmidt operator, and therefore compact.

**Proposition 5.5.** Let \( n = 2N \) be an even integer, and \( \psi_{n,m}^{k,c,i}(x) \) be a CPSWF. Then,

(i) \( \psi_{2N,m}^{k,c,i}(x) = P_{2N,m}^{k,c}(|x|^2) Y_k^i(x) \), where \( P_{2N,m}^{k,c}(|x|^2) \) is a radial function.

(ii) \( \mathcal{G}_c \psi_{2N,m}^{k,c,i}(x) = R_{2N,m}^{k,c}(|x|) Y_k^i(x) \), where \( R_{2N,m}^{k,c}(|x|) \) is a radial function.

(iii) \( Q P_c \psi_{2N,m}^{k,c,i}(x) = q_{2N,m}^{k,c,i}(|x|) Y_k^i(x) \), where \( q_{2N,m}^{k,c,i}(|x|) \) is a radial function.

**Proof.**

(i) We’ve seen that \( \psi_{2N,m}^{k,c,i}(x) = \sum_{i=0}^{\infty} b_{i}^{k,c} C_{2N,m}^{i}(Y_k^i)(x) \), but we know that \( C_{2N,m}^{i}(Y_k^i)(x) = P_{2N,m}^{k}(|x|^2) Y_k^i(x) \) with \( P_{2N,m}^{k}(|x|^2) \) a polynomial. Consequently, \( \psi_{2N,m}^{k,c,i}(x) = P_{2N,m}^{k,c}(|x|^2) Y_k^i(x) \).
(ii) Since $\psi_{2N,m}^{k,c,i}(x) = F_{N,m}^{k,c}(|x|^2)Y_k^i(x)$, we have

$$
G_c\psi_{n,m}^{k,c,i}(x) = \chi_B(1) \int_{B(1)} e^{2\pi ic(x,y)} F_{N,m}^{k,c}(|y|^2)Y_k(y) \, dy
$$

$$
= \chi_B(1) \int_0^1 \int_{S^{m-1}} e^{2\pi ic(r^2,\theta)} P_{N,m}^{k,c}(r^2)Y_k(\theta) \, d\theta \, dr
$$

$$
= \chi_B(1) \int_0^1 \int_{S^{m-1}} e^{2\pi ic(r^2,\theta)} P_{N,m}^{k,c}(r^2)Y_k(\theta) \, d\theta \, dr
$$

$$
= \chi_B(1) \int_0^1 \int_{S^{m-1}} e^{2\pi ic(r^2,\theta)} P_{N,m}^{k,c}(r^2)Y_k(\theta) \, d\theta \, dr
$$

$$
= \chi_B(1) \frac{(2\pi)^{m/2}(-i)^k}{(2\pi r^{m-2})} \int_0^1 \int_{S^{m-1}} e^{2\pi ic(r^2,\theta)} P_{N,m}^{k,c}(r^2)Y_k(\theta) \, d\theta \, dr
$$

$$
= \chi_B(1) \frac{(2\pi)^{m/2}(-i)^k}{(2\pi r^{m-2})} \int_0^1 \int_{S^{m-1}} e^{2\pi ic(r^2,\theta)} P_{N,m}^{k,c}(r^2)Y_k(\theta) \, d\theta \, dr
$$

where we have used Lemma 9.10.2 in [2].

(iii) Since $G_c^* f(x) = G_c f(-x)$, we have

$$
G_c^* G_c(\psi_{2N,m}^{k,c,i}(x)) = G_c^* G_c(F_{N,m}^{k,c}(|x|^2)Y_k^i(x))
$$

$$
= G_c^* (P_{2N,m}^{k,c}(|x|)Y_k^i(x))
$$

$$
= G_c^* (P_{2N,m}^{k,c}(|x|)(-1)^kY_k^i(x)) = \psi_{2N,m}^{k,c,i}(x).
$$

In the similar way, we may conclude analogous results for the odd CPSWFs.

\[\square\]

**Proposition 5.6.** Let $n = 2N + 1$ be odd. Then

(i) $\psi_{2N+1,m}^{k,c,i}(x) = Q_{N,m}^{k,c}(|x|^2)xY_k^i(x)$, where $Q_{N,m}^{k,c}(|x|^2)$ is a radial function.

(ii) $G_c\psi_{2N+1,m}^{k,c,i}(x) = S_{2N,m}^{k,c}(|x|)xY_k^i(x)$, where $S_{2N,m}^{k,c}(|x|)$ is a radial function.

(iii) $Q_P\psi_{2N+1,m}^{k,c,i}(x) = T_{2N,m}^{k,c}(|x|)xY_k^i(x)$, where $T_{2N,m}^{k,c}(|x|)$ is a radial function.

**Proof.** The proof of part (i) and (iii) are similar to the proof of Proposition 5.5 so we prove only part (ii). From part (i), we know that $\psi_{2N+1,m}^{k,c,i}(x) = Q_{N,m}^{k,c}(|x|^2)xY_k^i(x)$. Therefore,

$$
G_c\psi_{2N+1,m}^{k,c,i}(x) = \int_{B(1)} Q_{N,m}^{k,c}(|y|^2)yY_k(y)e^{2\pi ic(x,y)} \, dy
$$

$$
= \int_0^1 \int_{S^{m-1}} Q_{N,m}^{k,c}(r^2)\omega Y_k(\omega)e^{2\pi ic(r,\xi)} r^{m-1} \, d\omega \, dr.
$$
Further, Proof. Note that values are positive.

Theorem 5.7. From which we conclude

\[ F(x) := \int_{S^{m-1}} Y_k(\omega)e^{2\pi i c r x} d\omega. \]

Applying the Dirac operator from the left on both sides of this equation yields

\[ \partial_x F(x) := 2\pi i c r \int_{S^{m-1}} \omega Y_k(\omega)e^{2\pi i c r x} d\omega. \]

From Lemma 9.10.2 in [2] we have

\[ \int_{S^{m-1}} Y_k(\omega)e^{2\pi i c r x} d\omega = \frac{(2\pi)(-i)^k}{m-1} \left[ x \right]^{-\left( \frac{m+2k+1}{2} \right)} J_{k+\frac{m}{2}}(2\pi r |x|) Y_k(x), \quad (5.6) \]

and therefore,

\[ \int_{S^{m-1}} \omega Y_k(\omega)e^{2\pi i c r x} d\omega = \frac{(-1)^k (-i)^k}{c r^{\frac{m}{2}} - 1} \partial_x \left[ x \right]^{-\left( \frac{m+2k+1}{2} \right)} J_{k+\frac{m}{2}}(2\pi r |x|) Y_k(x) \]

\[ = \frac{(2\pi)(-1)^k i^{k-1}}{2 \pi - 1} x J_{k+\frac{m}{2}}(2\pi r |x|) \left[ x \right]^{-\left( \frac{m+2k+2}{2} \right)} Y_k(x) \]

from which we conclude

\[ G_e \psi^{k,c,i}_{2N+1,m}(x) = 2\pi (-1)^k i^{k-1} \left( \int_{0}^{1} Q^{k,c,i}_{N,m}(r^2) r^{\frac{m}{2}+1} J_{k+\frac{m}{2}}(2\pi r |x|) \, dr \right) \frac{x}{|x|^{\frac{m+2k+2}{2}}} Y^i_k(x). \]

Theorem 5.7. The even CPSWFs \( \psi^{k,c,i}_{2N+1}(x) \) are eigenfunctions of \( QP_c \). The eigenvalues are positive.

Proof. Note that \( \psi^{k,c,i}_{2N+1}(x) = P^k_{N,m}(|x|^2) Y^i_k(x) \) with \( P^k_{N,m} \) real-valued. Also,

\[ \left( \psi^{k,c,i}_{2N+1,m}, \psi^{k,c,i}_{N,m} \right) = \int_{B(1)} \psi^{k,c,i}_{2N+1,m}(x) \psi^{k,c,i}_{N,m}(x) \, dx = \int_{0}^{1} r^{m+2k-1} P^k_{N,m}(r^2) \, dr > 0. \quad (5.7) \]

Further,

\[ F \psi^{k,c,i}_{2N+1,m}(y) = \int_{B(1)} e^{-2\pi i (x,y)} \psi^{k,c,i}_{2N+1,m}(x) \, dx \]

\[ = \int_{0}^{1} P^k_{N,m}(r^2) r^{m+k-1} \int_{S^{m-1}} e^{-2\pi i r |y| |(\omega,y)|} Y^i_k(\omega) \, d\omega \]

\[ = 2\pi (-i)^k \left( \int_{0}^{1} P^k_{N,m}(r^2) r^{\frac{m}{2}+k} J_{k+\frac{m}{2}-1}(2\pi r |y|) \, dy \right) Y^i_k(y) \]

\[ = (-i)^k S^k_{N,m}(|y|) Y^i_k(y), \]
where $S_{N,m}^k$ is real-valued. Therefore,

$$P_c \psi_{2N,m}^{k,c,i} = \int_{B(c)} F \psi_{2N,m}^{k,c,i}(y) e^{2\pi i (x,y)} dy$$

$$= (-i)^k \int_{B(c)} S_{N,m}^k(|y|) Y_i^i(y) e^{2\pi i (x,y)} dy$$

$$= (-i)^k \int_0^c r^{m+k-1} S_{N,m}^k(r) \int_{S^{m-1}} Y_i^i(\omega) e^{2\pi i r|x|/|\pi^{1/2} \omega|} d\omega dr$$

$$= \frac{2\pi}{|x|^{m+k-1}} \left( \int_0^c r^{m+k} S_{N,m}^k(r) J_{k+m-1}(2\pi r |x|) dr \right) Y_i^i(x)$$

$$= T_{N,m}^k(|x|) Y_i^i(x),$$

with $T_{N,m}^k$ real-valued. Hence,

$$(P_c \psi_{2N,m}^{k,c,i}, P_c \psi_{2N,m}^{k,c,i}) = \int_{\mathbb{R}^m} T_{N,m}^k(|x|)^2 Y_i^i(x) Y_i^i(x) dx$$

$$= \int_0^1 r^{2k+m-1} T_{N,m}^k(r)^2 dr \geq 0.$$

On the other hand,

$$(P_c \psi_{2N,m}^{k,c,i}, P_c \psi_{2N,m}^{k,c,i}) = 0 \iff T_{N,m}^k \equiv 0$$

$$\iff P_c \psi_{2N,m}^{k,c,i} \equiv 0$$

$$\iff F \psi_{2N,m}^{k,c,i}(y) = 0 \text{ for } |y| \leq c.$$

But $\psi_{2N,m}^{k,c,i}$ is non-trivial and compactly supported, so its Fourier transform cannot vanish on $B(c)$. We conclude that

$$(P_c \psi_{2N,m}^{k,c,i}, P_c \psi_{2N,m}^{k,c,i}) > 0. \quad (5.8)$$

Note that

$$L_c(QP_c) \psi_{2N,m}^{k,c,i} = (QP_c) L_c \psi_{2N,m}^{k,c,i} = \chi_{2N}^{k,c}(QP_c) \psi_{2N,m}^{k,c,i}$$

and therefore $QP_c \psi_{2N,m}^{k,c,i} \in E_{\chi_{2N}^{k,c}}(L_c)$ (the $\chi_{2N}^{k,c}$ eigenspace of $L_c$). Note also that $\psi_{2N,m}^{k,c,i} \in H^1_k$ and

$$QP_c \psi_{2N,m}^{k,c,i}(x) = \chi_{B(1)} T_{N,m}^k(|x|) Y_i^i(x) \in H^1_k,$$

so that $QP_c \psi_{2N,m}^{k,c,i} \in E_{\chi_{2N}^{k,c}}(L_c \big| H^1_k)$ which is one-dimensional and contains $\psi_{2N,m}^{k,c,i}$. We conclude that there exists $\lambda \in \mathbb{C}_m$ for which

$$QP_c \psi_{2N,m}^{k,c,i} = \psi_{2N,m}^{k,c,i} \lambda,$$

i.e., each CPSWF $\psi_{2N,m}^{k,c,i}$ is an eigenfunction of $QP_c$. Also,

$$(P_c \psi_{2N,m}^{k,c,i}, P_c \psi_{2N,m}^{k,c,i}) = (\psi_{2N,m}^{k,c,i}, QP_c \psi_{2N,m}^{k,c,i}) = (\psi_{2N,m}^{k,c,i}, \psi_{2N,m}^{k,c,i}) \lambda \quad (5.9)$$

Applying (5.7) and (5.8) to (5.9) then gives $\lambda > 0$.

□
Theorem 5.8. The even CPSWFs $\psi^{k,c,i}_{2N,m}$ are eigenfunctions of $G_c$.

Proof. The proof is similar to that of Theorem 5.7 and uses the commutation relation $G_cL_c = L_cG_c$. \qed

Similarly, we have the following result.

Theorem 5.9. The odd CPSWFs $\psi^{k,c,i}_{2N+1,m}(x)$ are the eigenfunction of both $QP_c$ and $G_c$.

Combining Theorems 4.10, 5.7 and 5.9 gives the following result.

Corollary 5.10. The CPSWFs $\{\psi^{k,c,i}_{n,m} : n \geq 0, k \geq 0, 1 \leq i \leq d_k\}$ is an orthonormal basis for $L^2(B(1), \mathbb{R}_m)$ consisting of eigenfunctions of $QP_c$.

6. Some Features of the CPSWFs

Let’s assume that $G_c\psi_j(x) = \mu_j \psi_j(x)$ where $\psi_j(x)$ is a time-limited Clifford Prolate Spheroidal Wave Functions. We also know that $QP_c\psi_j(x) = \lambda_j \psi_j(x)$. We also see that

$$P_cQ(P_c\psi_j)(x) = \lambda_j(P_c\psi_j)(x),$$

showing that the $QP_c$ and $P_cQ$ have the same eigenvalue $\lambda_j$ with eigenfunctions $\psi_j$ and $P_c\psi_j$ respectively. Now, we are going to show a relationship between the eigenvalues of $G_c$ and $QP_c$. Now we have that

$$|\mu_j|^2 = |\mu|^2(\psi_j, \psi_j) = (\mu_j \psi_j, \mu_j \psi_j) = (G_c \psi_j, G_c \psi_j) = (G_c^* G_c \psi_j, \psi_j) = (\frac{1}{cm} QP_c \psi_j, \psi_j) = \frac{1}{cm} \langle \lambda_j \psi_j, \psi_j \rangle = \frac{1}{cm} \lambda_j.$$

So $|\mu_j|^2 = \frac{1}{cm} \lambda_j$. We know that $\mu_j$ is a complex-valued number. So

$$\mu_j = \alpha_j \frac{\sqrt{\lambda_j}}{c^{\frac{1}{2}}}, \tag{6.1}$$

where $\alpha_j$ is a complex-valued and $|\alpha_j| = 1$. Now, we have already seen that

$$G_c\psi^{k,c,i}_{2N,m}(x) = i^k G_{2N}(|x|)Y_k^i(x) = \mu^{k,c}_{2N,m} \psi^{k,c,i}_{2N,m}(x),$$

$$G_c\psi^{k,c,i'}_{2N+1,m}(x) = i^{k+1} F_{2N+1}(|x|)X_k^{i'}(x) = \mu^{k,c}_{2N+1,m} \psi^{k,c,i'}_{2N+1,m}(x),$$

as a result, I can write

$$\mu_j = \epsilon_j \frac{\sqrt{\lambda_j}}{c^{\frac{1}{2}}} \tag{6.2}.$$ 

Therefore, we have found the relationship between the eigenvalues of the $G_c$, $QP_c$, and, $P_cQ$. At this part of the section, we are going to investigate some features of the CPSWFs related to spectral concentration problem. For this, we firstly are going to prove the non-degeneracy of the eigenvalues of the operator $P_cQ$ for a fixed $k$.

Theorem 6.1. For a fixed $k$ the eigenvalues $\{\lambda^{k,c}_{n,m}\}$ are distinct.
Proof. First suppose that $\lambda_{2N,m}^{k,c} = \lambda_{2N,m}^{k,c}$ with $M \neq N$. Then

$$\lambda_{2N,m}^{k,c} \psi_{2N,m}^{k,c,j}(x) = \int_{B(1)} K_c(x-y) \psi_{2M,m}^{k,c,j}(y) \, dy,$$

so that

$$\lambda_{2N,m}^{k,c} \Delta \psi_{2M,m}^{k,c,j}(x) = \int_{B(1)} \Delta K_c(x-y) \psi_{2M,m}^{k,c,j}(y) \, dy,$$

and as a consequence we have

$$\lambda_{2N,m}^{k,c} \int_{B(1)} \Delta \psi_{2M,m}^{k,c,j}(x) \psi_{2N,m}^{k,c,i}(x) \, dx = \int_{B(1)} \int_{B(1)} \Delta K_c(x-y) \psi_{2M,m}^{k,c,j}(y) \psi_{2N,m}^{k,c,i}(x) \, dy \, dx. \quad (6.3)$$

Similarly, we have

$$\lambda_{2N,m}^{k,c} \int_{B(1)} \psi_{2M,m}^{k,c,j}(x) \Delta \psi_{2N,m}^{k,c,i}(x) \, dx = \int_{B(1)} \int_{B(1)} \psi_{2M,m}^{k,c,j}(x) \Delta K_c(x-y) \psi_{2N,m}^{k,c,i}(y) \, dy \, dx. \quad (6.4)$$

Subtracting (6.4) from (6.3) and applying Stokes’ theorem gives

$$0 = \lambda_{2N,m}^{k,c} \int_{B(1)} \Delta \psi_{2M,m}^{k,c,j}(x) \psi_{2N,m}^{k,c,i}(x) - \psi_{2M,m}^{k,c,j}(x) \Delta \psi_{2N,m}^{k,c,i}(x) \, dx$$

$$= \lambda_{2N,m}^{k,c} \int_{B(1)} \psi_{2M,m}^{k,c,j}(x) \partial_x^2 \psi_{2N,m}^{k,c,i}(x) - \partial_x^2 \psi_{2M,m}^{k,c,j}(x) \psi_{2N,m}^{k,c,i}(x) \, dx$$

$$= \int_{S^{m-1}} \psi_{2M,m}^{k,c,j}(x) x \partial_x \psi_{2N,m}^{k,c,i}(x) + \partial_x \psi_{2M,m}^{k,c,j}(x) x \psi_{2N,m}^{k,c,i}(x) \, d\sigma(x). \quad (6.5)$$

However, since $\psi_{2M,m}^{k,c,j}(x) = P_{M,m}^{k,c}(|x|^2) Y_k^j(x)$, we have $\partial_x \psi_{2M,m}^{k,c,j}(x) = 2x (P_{M,m}^{k,c})'(|x|^2) Y_k^j(x)$ and from (6.5) and the orthonormality of $\{Y_k^j\}_{i=1}^\infty$ in $L^2(S^{m-1}, \mathbb{R}^m)$, we have

$$0 = (P_{M,m}^{k,c})'(1) P_{N,m}^{k,c}(1) - P_{M,m}^{k,c}(1) (P_{N,m}^{k,c})'(1),$$

or equivalently,

$$\frac{(P_{N,m}^{k,c})'(1)}{P_{N,m}^{k,c}(1)} = \frac{(P_{M,m}^{k,c})'(1)}{P_{M,m}^{k,c}(1)}. \quad (6.6)$$

However, from (4.4), $\left(\frac{\chi_{N}^{k,c}}{4} - \pi^2 c^2 \right) P_{N,m}^{k,c}(1)$ and we conclude from (6.6) that $\chi_{2N,m}^{k,c} = \chi_{2M,m}^{k,c}$, which contradicts the eigenvalues $\{\chi_{2N,m}^{k,c}\}_{N=0}^\infty$ of $L_c$ are distinct. We conclude that the eigenvalues $\{\chi_{2N,m}^{k,c}\}_{N=0}^\infty$ of $QP_c$ are distinct.

Suppose now that $\lambda_{2N+1,m}^{k,c} = \lambda_{2M+1,m}^{k,c}$. Then we have

$$0 = \int_{S^{m-1}} \psi_{2M+1,m}^{k,c,j}(x) x \partial_x \psi_{2N+1,m}^{k,c,i}(x) + \partial_x \psi_{2M+1,m}^{k,c,j}(x) x \psi_{2N+1,m}^{k,c,i}(x) \, d\sigma(x). \quad (6.7)$$
However, since \( \psi_{2N+1,m}^{k,c,i}(x) = xQ_{2N,m}^{k,c}(|x|^2)Y_i^k(x) \), by the monogenicity and homogeneity of \( Y_k \) we have

\[
\partial_x \psi_{2N+1,m}^{k,c,i}(x) = [(m + 2k)Q_{N,m}^{k,c}(|x|^2) - 2|x|^2(Q_{N,m}^{k,c})'(|x|^2)]Y_i^k(x). \tag{6.8}
\]

Substituting (6.8) into (6.7), the orthonormality of \( \{Y_k\}_{k=1}^{2N} \) in \( L^2(S^{m-1}) \) yields

\[ 0 = Q_{M,m}^{k,c}(1) \left[ -(m+2k)Q_{N,m}^{k,c}(1) + 2(Q_{N,m}^{k,c})'(1) \right] - Q_{N,m}^{k,c}(1) \left[ -(m+2k)Q_{M,m}^{k,c}(1) + 2(Q_{N,m}^{k,c})'(1) \right] \]

which we rearrange to find

\[ \frac{(m + 2k)Q_{M,m}^{k,c}(1) + 2(Q_{M,m}^{k,c})'(1)}{Q_{M,m}^{k,c}(1)} = \frac{(m + 2k)Q_{N,m}^{k,c}(1) + 2(Q_{N,m}^{k,c})'(1)}{Q_{N,m}^{k,c}(1)}. \tag{6.9} \]

The functions \( \{Q_{N,m}^{k,c}\}_{N=0}^{\infty} \) satisfy (4.5) and therefore,

\[ 2(Q_{N,m}^{k,c})'(1) + (m + 2k)Q_{N,m}^{k,c}(1) = \left( \frac{m}{2} + k - \frac{n^2c^2}{2} - \frac{\lambda_{2N+1}^k}{4} \right), \tag{6.10} \]

and applying (6.10) to (6.9) yields \( \lambda_{2N+1}^k = \lambda_{2M+1,m}^{k,c} \), which contradicts the fact that the eigenvalues \( \{\lambda_{2N+1,m}^k\}_{N=0}^{\infty} \) are distinct. We conclude that \( \lambda_{2N+1}^k \neq \lambda_{2M+1,m}^{k,c} \) unless \( N = M \).

Finally, we suppose \( \lambda_{2N+1}^k = \lambda_{2M}^{k,c} \). Then with an application of Stokes’ theorem, we have

\[
\lambda_{2N,m}^{k,c} \partial_x \psi_{2N,m}^{k,c,i}(x) = \int_{B(1)} \partial_x(K_c(x-y))\psi_{2N,m}^{k,c,i}(y) \, dy \\
= -\int_{B(1)} \partial_y(K_c(x-y))\psi_{2N,m}^{k,c,i}(y) \, dy \\
= -\left\{ \int_{S^{m-1}} K_c(x-y)y\psi_{2N,m}^{k,c,i}(y) \, d\sigma(y) - \int_{B(1)} K_c(x-y)\partial_y\psi_{2N,m}^{k,c,i}(y) \, dy \right\}. \tag{6.11} \]

Multiplying both sides of (6.11) on the left by \( \psi_{2M+1,m}^{k,c,j}(x) \) and integrating over \( B(1) \) gives

\[
\lambda_{2N,m}^{k,c} \int_{B(1)} \psi_{2M+1,m}^{k,c,j}(x)(\partial_x \psi_{2N,m}^{k,c,i})(x) \, dx = \int_{B(1)} \int_{B(1)} \psi_{2M+1,m}^{k,c,j}(x)K_c(x-y)(\partial_x \psi_{2N,m}^{k,c,i})(y) \, dy \, dx \\
- \lambda_{2M+1,m}^{k,c} \int_{S^{m-1}} \psi_{2M+1,m}^{k,c,j}(y)\psi_{2N,m}^{k,c,i}(y) \, d\sigma(y), \tag{6.12} \]

where we have used the fact that

\[
\int_{B(1)} K_c(x-y)\psi_{2M+1,m}^{k,c,j}(x) \, dx = \lambda_{2M+1,m}^{k,c} \psi_{2M+1,m}^{k,c,j}(y). \tag{6.13} \]
On the other hand, conjugating both sides of (6.13), multiplying on the right by \( \partial_x \psi^{k,c,i}_{2N,m}(x) \) and integrating over \( B(1) \) gives

\[
\lambda_{2M+1}^{k} \int_{B(1)} \psi^{k,c,i}_{2M+1,m}(x) (\partial_x \psi^{k,c,i}_{2N,m})(x) = \int_{B(1)} \int_{B(1)} \psi^{k,c,j}_{2M+1,m}(y) K_c(x-y) (\partial_x \psi^{k,c,i}_{2N,m})(x) \, dy \, dx.
\]

If \( \lambda^{k}_{2n,m} = \lambda^{k}_{2M+1} \), then subtracting (6.14) from (6.12) and using the orthonormality of \( \{ Y_{k,i}^{j} \}_{j=1}^{d_c} \) gives

\[
0 = \int_{S^{m-1}} \psi^{k,c,j}_{2M+1,m}(y) y \psi^{k,c,i}_{2N,m}(y) \, d\sigma(y) = Q^k_M(1) \delta_{ij},
\]

which contradicts the fact that \( P^{k,c}_{N,m}(1) Q^{k,c}_{N,m}(1) \neq 0 \). We conclude that \( \lambda^{k,c}_{2n,m} \neq \lambda^{k,c}_{2M+1} \) for all \( N, M \geq 0 \).

**Theorem 6.2.** The eigenvalues \( \lambda^{k,c}_{n,m} \) of \( Q P_c \) are continuous functions of \( c \).

**Proof.** Since \( \psi^{k,c,i}_{2N,m}(x) = P^{k,c}_{N,m}(|x|^2) Y_i^k(x) \), we have

\[
G_c \psi^{k,c,i}_{2N,m}(x) = \int_{B(1)} e^{2\pi i c(x,y)} P^{k,c}_{N,m}(|y|^2) Y_i^k(y) \, dy
\]

\[
= \int_{0}^{1} r^{m+k-1} P^{k,c}_{N,m}(r^2) \int_{S^{m-1}} e^{2\pi i c r(x,\omega)} Y_i^k(\omega) \, d\omega \, dr
\]

\[
= \frac{2\pi i^k}{|c||x|^{\frac{m}{2}-1}} \int_{0}^{1} r^{\frac{m}{2}+k} P^{k,c}_{N,m}(r^2) J_{k+\frac{m}{2}-1}(2\pi cr|x|) \, dr \, Y_i^k \left( \frac{x}{|x|} \right),
\]

where we have used (5.6). Similarly,

\[
G_c^* G_c \psi^{k,c,i}_{2N,m}(y) = \int_{B(1)} e^{-2\pi i c(x,y)} G_c \psi^{k,c,i}_{2N,m}(x) \, dx
\]

\[
= \frac{2\pi i^k}{|c|^\frac{m}{2}} \int_{0}^{1} s^{m} \int_{0}^{1} r^{\frac{m}{2}+k} J_{k+\frac{m}{2}-1}(2\pi cs) \int_{S^{m-1}} e^{-2\pi i c s(y,\omega)} Y_i^k(\omega) \, d\omega \, dr \, ds
\]

\[
= \frac{4\pi^2}{c^{m-2} |y|^{\frac{m}{2}}+1} \int_{0}^{1} s^{m} \int_{0}^{1} r^{\frac{m}{2}+k} J_{k+\frac{m}{2}-1}(2\pi cs) J_{k+\frac{m}{2}-1}(2\pi cs |y|) \, dr \, ds \, Y_i^k \left( \frac{y}{|y|} \right)
\]

\[
= \frac{2\pi}{c^{m-1} |y|^{\frac{m}{2}+k-1}} \int_{0}^{1} r^{\frac{m}{2}+k} P^{k}_{n}(r^2) M_c(|y|, r) \, dr \, Y_i^k(y),
\]

(6.15)
Proof. We assume an asymptotic expansion of the form

\[ M_c(r, s) = 2\pi c \int_0^1 sJ_{k+\frac{m}{2}-1}(2\pi cs)J_{k+\frac{m}{2}-1}(2\pi cs|y|) \, ds \]

\[ = \frac{sJ_{k+\frac{m}{2}-2}(2\pi cs)J_{k+\frac{m}{2}-1}(2\pi cs) - rJ_{k+\frac{m}{2}-2}(2\pi cr)J_{k+\frac{m}{2}-1}(2\pi cs)}{r^2 - s^2}. \]  

(6.16)

In calculating (6.16), we have used (6.521) from [15]. The (finite) diagonal values \( M_c(s, s) \) of the kernel \( M_c \) may be obtained by taking the limit \( \lim_{s \to \infty} M_c(s, s) \). Since \( G^r G_c \psi_{2N,m}^{k,c,i}(x) = e^{-m}Qc\psi_{2N}^{k,c,i}(x) = \lambda_{2N,m}^{k,c} \psi_{2N,m}^{k,c,i}(x) \), equation (6.15) may be written as

\[ \frac{\lambda_{2N,m}^{k,c}(c)}{4\pi^2} s^{\frac{m}{2}+k-1} P_{N,m}^{k,c}(s^2) = c \int_0^1 r^{\frac{m}{2}+k} P_{N,m}^{k,c}(r^2) M_c(s, r) \, dr, \]

or equivalently

\[ \frac{\lambda_{2N,m}^{k,c}(c)}{2\pi^2} s^{\frac{m}{2}+k-\frac{1}{2}} P_{N,m}^{k,c}(s) = \int_0^1 r^{\frac{m}{2}+k-\frac{1}{2}} P_{N,m}^{k,c}(r) M_c(s, r) \, dr, \]

i.e., \( r^{\frac{m}{2}+k-\frac{1}{2}} P_{N,m}^{k,c}(r) \) is an eigenfunction of the integral operator on \([0, 1]\) which acts by integration against the kernel \( N_c(s, r) = cM_c(\sqrt{s}, \sqrt{r}) \). Since \( N \) is symmetric and continuous in \( r, s \) and \( c \), we conclude from Chapter III, Section 8.4 of [9] that the eigenvalues \( \lambda_{2N,m}^{k,c} \) vary continuously with \( c \). \( \square \)

**Theorem 6.3.** Let the real constants \( \chi_{n,m}^{k,0}, \chi_{n,m}^{k,c}, \alpha_{n,k}, \beta_{n,k}, \gamma_{n,k} \) be given by

\[ L_0C_{n,m}^0(Y_k) = \chi_{n,m}^{k,0} e^{i\lambda_{n,m}^{k,0}}, \]

\[ L_c\psi_{n,m}^{k,c} = \lambda_{n,m}^{k,c} \psi_{n,m}^{k,c}, \]

\[ x^2C_{n,m}^0(Y_k) = \alpha_{n,k,m} C_{n+2,m}^0(Y_k) + \beta_{n,k,m} C_{n,m}^0(Y_k) + \gamma_{n,k,m} C_{n-2,m}^0(Y_k). \]

Then the asymptotic behaviours of \( \chi_{n,m}^{k,c} \) and \( \psi_{n,m}^{k,c} \) are as follows:

\[ \chi_{n,m}^{k,c} = \chi_{n,m}^{k,0} - 4\pi^2 x^2 \beta_{n,k,m} + O(c^4). \]  

(6.17)

and

\[ \psi_{n,m}^{k,c}(Y_k) = C_{n,m}^0(Y_k) - 4\pi^2 x^2 \left( \frac{\alpha_{n,k,m}}{\gamma_{n,k,m}} C_{n+2,m}^0(Y_k) + \frac{\gamma_{n,k,m}}{\alpha_{n,k,m}} C_{n-2,m}^0(Y_k) \right) + O(c^4). \]  

(6.18)

**Proof.** We assume an asymptotic expansion of the form

\[ \psi_{n,m}^{k,c,i}(x) = C_{n,m}^0(Y_k^i)(x) + x^2 f(x) + O(c^4). \]  

(6.19)

Since \( \|\psi_{n,m}^{k,c,i}\|_2 = \|C_{n,m}^0(Y_k^i)\|_2 = 1 \), we have \( \langle C_{n,m}^0(Y_k^i), f \rangle = 0 \). We then have

\[ \chi_{n,m}^{k,c} \psi_{n,m}^{k,c,i} = L_c C_{n,m}^0(Y_k^i) + c^2 L_c f + O(c^4) \]  

(6.20)

\[ = L_0 C_{n,m}^0(Y_k^i) + 4\pi^2 x^2 \left| C_{n,m}^0(Y_k^i) \right| + c^2 L_0 f + O(c^4) \]

\[ = \chi_{n,m}^{k,0} - 4\pi^2 x^2 \beta_{n,k,m} C_{n,m}^0(Y_k^i) + 4\pi^2 x^2 \left[ \alpha_{n,k,m} C_{n+2,m}^0(Y_k^i) + \gamma_{n,k,m} C_{n-2,m}^0(Y_k^i) \right] \]  

\[ + c^2 L_0 f + O(c^4). \]  

(6.21)
On the other hand,
\[
\chi_{n,m}^{k,c} \psi_{n,m}^{k,c,i} = \chi_{n,m}^{k,c} \overline{C}_{n,m}^{0}(Y_k^{i}) + c^2f + O(c^4).
\] (6.22)

Subtracting (6.22) and (6.21) gives
\[
(\chi_{n,m}^{k,0} - 4\pi^2c^2\beta_{n,k} - \chi_{n,m}^{k,c})\overline{C}_{n,m}^{0}(Y_k^{i}) + c^2(L_0 f - \chi_{n,m}^{k,c} f) - 4\pi^2c^2[\alpha_{nk}\overline{C}_{n+2,m}^{0}(Y_k^{i}) + \gamma_{nk}\overline{C}_{n-2,m}^{0}(Y_k^{i})] = O(c^4).
\] (6.23)

Since \(\langle L_0 f, \overline{C}_{n,m}^{0}(Y_k^{i}) \rangle = \langle f, L_0 \overline{C}_{n,m}^{0}(Y_k^{i}) \rangle = \chi_{n,m}^{k,0} \langle f, \overline{C}_{n,m}^{0}(Y_k^{i}) \rangle = 0\), taking the inner product of both sides on (6.23) against \(\overline{C}_{n,m}^{0}(Y_k^{i})\) gives (6.17).

We now aim to determine the function \(f\) in (6.19). Combining (6.17), (6.19) and (6.21) gives
\[
L_0 f = \chi_{n,m}^{k,0} f - 4\pi^2[\alpha_{nk,m} \overline{C}_{n+2,m}^{0}(Y_k^{i}) + \gamma_{nk,m} \overline{C}_{n-2,m}^{0}(Y_k^{i})],
\] (6.24)

which has solutions of the form
\[
f = \overline{C}_{n,m}^{0}(Y_k^{i}) A + 4\pi^2\left[\frac{\alpha_{nk}}{\chi_{n+2,m} - \chi_{n,m}} \overline{C}_{n+2,m}^{0}(Y_k^{i}) + \frac{\gamma_{nk}}{\chi_{n-2,m} - \chi_{n,m}} \overline{C}_{n-2,m}^{0}(Y_k^{i})\right],
\] (6.25)

where \(A\) is an arbitrary Clifford constant. Substituting (6.25) into (6.19) gives
\[
\psi_{n,m}^{k,c,i} = \overline{C}_{n,m}^{0}(Y_k^{i})(1 + Ac^2) - 4\pi^2c^2\left(\frac{\alpha_{nk}}{\chi_{n+2,m} - \chi_{n,m}} \overline{C}_{n+2,m}^{0}(Y_k^{i}) + \frac{\gamma_{nk}}{\chi_{n-2,m} - \chi_{n,m}} \overline{C}_{n-2,m}^{0}(Y_k^{i})\right) + O(c^4),
\] (6.26)

However, applying \(L_c\) to both sides of (6.26) and applying (6.17) gives
\[
L_c \psi_{n,m}^{k,c,i} - \chi_{n,m}^{k,c} \psi_{n,m}^{k,c,i} = \overline{C}_{n,m}^{0}(Y_k^{i}) Ac^2 + O(c^4),
\]

from which we conclude that \(A = 0\). Putting \(A = 0\) in (6.26) gives (6.18).

For all \(n \geq 0\) we say \(P_{n,m}^{k,c}\) is the radial part of \(\psi_{n,m}^{k,c,i}\), i.e., \(P_{n,m}^{k,c} = P_{n,m}^{k,c}\) and \(R_{2N+1,m}^{k,c} = Q_{N,m}^{k,c}\).

**Theorem 6.4.** For fixed \(k \geq 0\), the eigenvalues \(\lambda_n^{k,c}\) of \(P_c Q\) satisfy \(\lambda_{n+1,m}^{k,c} < \lambda_{n,m}^{k,c}\).

**Proof.** Let’s assume
\[
\chi_{n,m}^{k,c} \psi_{n,m}^{k,c,i}(x) = \int_{B(1)} K_c(x - y) \psi_{n,m}^{k,c,i}(y) dy,
\]

and,
\[
\chi_{n+1,m}^{k,c} \psi_{n+1,m}^{k,c,i}(y) = \int_{B(1)} K_c(y - x) \psi_{n+1,m}^{k,c,i}(x) dx.
\]

Therefore,
\[
\chi_{n,m}^{k,c} \overline{\partial_x \psi_{n,m}^{k,c,i}(x)} = \int_{B(1)} \partial_x K_c(x - y) \overline{\psi_{n,m}^{k,c,i}(y)} dy,
\] (6.27)
Now we use (6.30) and (6.29), where we’ve used the fact that
and, 

$$\lambda_{n+1,m} \partial_y \psi_{n+1,m}^k(y) = \int_{B(1)} \partial_y (K_c(y - x)) \psi_{n+1,m}^k(x) dx.$$ (6.28)

multiply (6.27) by \(\lambda_{n+1,m}^k \psi_{n+1,m}^k(x)\) from the right so we have

$$\lambda_{n+1,m}^k \lambda_{n,m}^k \int_{B(1)} \frac{\partial_x \psi_{n,m}^k(x)}{\psi_{n+1,m}^k(x)} \psi_{n+1,m}^k(x) dx$$

$$= \lambda_{n+1,m}^k \int_{B(1)} \left( \psi_{n,m}^k(y) \partial_x (K_c(x - y)) \right) \psi_{n+1,m}^k(x) dy$$

$$= \lambda_{n+1,m}^k \int_{B(1)} \left( \psi_{n,m}^k(y) \partial_x (K_c(x - y)) \right) \psi_{n+1,m}^k(x) dy$$

$$= \lambda_{n+1,m}^k \int_{B(1)} \left( \psi_{n,m}^k(y) \partial_y (K_c(y - x)) \right) \psi_{n+1,m}^k(x) dy$$

$$= -\lambda_{n+1,m}^k \int_{B(1)} \psi_{n,m}^k(y) \partial_y (K_c(y - x)) \psi_{n+1,m}^k(x) dy$$ (6.29)

where we’ve used the fact that \(\partial_x S(-x) = -\partial_x S(x)\). Now, we multiply (6.28) by \(\lambda_{n,m}^k \psi_{n+1,m}^k(y)\) from the left side. So

$$\lambda_{n+1,m}^k \lambda_{n,m}^k \int_{B(1)} \psi_{n,m}^k(y) \partial_y \psi_{n+1,m}^k(y) dy = \lambda_{n,m}^k \int_{B(1)} \left( \psi_{n,m}^k(y) \partial_y (K_c(y - x)) \right) \psi_{n+1,m}^k(x) dx dy$$ (6.30)

Now we use (6.30) and (6.29),

$$\lambda_{n+1,m}^k \lambda_{n,m}^k \int_{B(1)} \psi_{n,m}^k(y) \partial_y \psi_{n+1,m}^k(y) dy$$

$$= [\lambda_{n,m}^k - \lambda_{n+1,m}^k] \int_{B(1)} \left( \psi_{n,m}^k(y) \partial_y (K_c(y - x)) \right) \psi_{n+1,m}^k(x) dx dy$$

$$= [\lambda_{n,m}^k - \lambda_{n+1,m}^k] \lambda_{n+1,m}^k \int_{B(1)} \psi_{n,m}^k(y) \partial_y \psi_{n+1,m}^k(y) dy,$$

By (6.18), lemma 2.19 and the orthogonality of Clifford Legendre polynomials, we can see that

$$\int_{B(1)} \psi_{n,m}^k(y) \partial_y \psi_{n+1,m}^k(y) dy = 4(n+1)(n+k + \frac{m}{2}) + O(c^2),$$ (6.31)
The integral on the left hand side of (6.31) is real, and is nonzero for sufficiently small \( c \). Therefore we may write

\[
\lambda_{n,m}^{k,c} - \lambda_{n+1,m}^{k,c} = \lambda_{n,m}^{k,c} \left[ 1 + \frac{\int_{B(1)} (\partial_y \bar{\psi}_{n,m}^{k,c,i}(y)) \psi_{n+1,m}^{k,c,i}(y) \, dy}{\int_{B(1)} (\psi_{n,m}^{k,c,i}(y))(\partial_y \bar{\psi}_{n+1,m}^{k,c,i}(y)) \, dy} \right].
\]

(6.32)

Applying Lemma 2.19 and the orthogonality of the Clifford-Legendre polynomials on \( B(1) \), we see that the numerator in the fraction on the right hand side of (6.32) goes to zero as \( c \to 0 \). Hence, for \( c \) sufficiently small, from (6.32) we have

\[
\lambda_{n,m}^{k,c} - \lambda_{n+1,m}^{k,c} > \frac{\lambda_{m,n}^{k,c}}{2} > 0,
\]

and we conclude that for \( c \) sufficiently small,

\[
\lambda_{n,m}^{k,c} > \lambda_{n+1,m}^{k,c} > 0.
\]

(6.33)

Suppose now that for some \( c_1 > 0 \) and some non-negative integer \( n \), \( \lambda_{n,m}^{k,c_1} > \lambda_{n+1,m}^{k,c_1} \). We know that there is a (small) values \( c_0 > 0 \) for which \( \lambda_{n,m}^{k,c_0} > \lambda_{n+1,m}^{k,c_0} \). Since the eigenvalues \( \lambda_{n,m}^{k} \) are continuous functions of \( c \), by the Intermediate Value Theorem, there exists \( c_2 \in (c_0,c_1) \) for which \( \lambda_{n,m}^{k,c_2} = \lambda_{n+1,m}^{k,c_2} \), which contradicts Theorem 6.1. This completes the proof.

Because of (6.1), we have a similar conclusion for the absolute values of eigenvalues of \( \mathcal{G}_c \).

**Corollary 6.5.** For a fixed \( k \) the eigenvalues of the \( \mathcal{G}_c \), \( \mu_{n,m}^{k,c} \) are non-degenerate, and

\[
|\mu_{0,m}^{k,c}| > |\mu_{1,m}^{k,c}| > \cdots > |\mu_{n,m}^{k,c}| > |\mu_{n+1,m}^{k,c}| > \cdots.
\]

**Theorem 6.6.** The eigenvalues of \( \mathcal{G}_c \) are given by

\[
\mu_{2N,m}^{k,c} = \frac{\alpha_{0,N}^k(-1)^{m-1} \sqrt{2k + m} \, k \pi^{k+\frac{m}{2}}}{\Gamma\left(k + \frac{m}{2} + 1\right)} \mathcal{P}_{N,m}^{k,c}(0),
\]

(6.34)

and

\[
\mu_{2N+1,m}^{k,c} = \frac{\beta_{0,N}^k(-1)^{m} \sqrt{2k + 2 + m} \, k^{2k+1} \pi^{k+\frac{m}{2}+1}}{\Gamma\left(k + \frac{m}{2} + 2\right)} \mathcal{Q}_{N,m}^{k,c}(0).
\]

(6.35)

where \( \alpha_{0,N}^k \) and \( \beta_{0,N}^k \) have been introduced in (3.3) and \( \mathcal{P}_{N,m}^{k,c}(x) \), and \( \mathcal{Q}_{N,m}^{k,c}(x) \) are the radial parts of the even and odd CPSWFs respectively, at proposition 5.3.

**Proof.** Using the same steps in Theorem 4.6 in [3], we can see that

\[
\mathcal{G}_c(C_{2N,m}^{(j)}(\bar{Y}^{(j)}_k))(\xi) = (-1)^{m-1}i^{k}\sqrt{2k + 4N + m} \frac{J_{k+\frac{m}{2}+2N}(2\pi c|\xi|)}{|\xi|^{\frac{k}{2}+k+c+\frac{m}{2}}} Y^{(j)}_k(\xi).
\]

(6.36)

Since \( \mathcal{G}_c(\psi_{2N,m}^{k,c,j})(\xi) = \mu_{2N,m}^{k,c} \psi_{2N,m}^{k,c,j}(\xi) \) for all \( \xi \in B(1) \) so we compute \( \mu_{2N,m}^{k,c} = \lim_{\xi \to 0} \frac{\mathcal{G}_c(\psi_{2N,m}^{k,c,j})(\xi)}{\psi_{2N,m}^{k,c,j}(\xi)} \). From (6.36) and the fact that

\[
J_{\nu}(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j+\nu+1)} \left( \frac{x}{2} \right)^{2j+\nu},
\]


we see that $G_c(\psi_{2N,m}^{k,c,j})(\xi) = \left[\alpha_{0,N}^k(-1)^{m-1}\sqrt{2k + m}k^{k+\frac{m}{2}}c^k + O(|\xi|^2)\right]Y_k^i(\xi)$ and $\psi_{2N,m}^{k,c,j}(\xi) = P_{N,m}^{k,c}(|\xi|^2)Y_k^i(\xi)$. So

$$G_c(\psi_{2N,m}^{k,c,j})(\xi) = \left[\alpha_{0,N}^k(-1)^{m-1}\sqrt{2k + m}k^{k+\frac{m}{2}}c^k + O(|\xi|^2)\right]Y_k^i(\xi)$$

as $\xi \to 0$. Hence we can get (6.34). The calculation of the $\mu_{2N+1,m}^{k,c}$ is similar.

\[\square\]

**Theorem 6.7.** The eigenvalues of the $G_c$, i.e., $\mu_{n,m}^{k,c}$ enjoy the following relationship

$$\mu_{2N,m}^{k,c} = \mu_{2N+1,m}^{k-1,c}.$$  (6.37)

**Proof.** For the proof, we need to say that (6.34), and (6.35) for $k - 1$, are equal. From the Remark 3.3, we can see that $\alpha_{0,N}^k = \beta_{0,N}^k$. Also, by the normalized version of Theorem 3.12 in [3], we can see that $P_{N,m}^{k,c}(0) = Q_{N,m}^{k-1,c}(0)$. Therefore, we can conclude the (6.37).

\[\square\]

**Theorem 6.8.** For a fixed $n$, the absolute value of the eigenvalues of the $G_c$, i.e., $\mu_{n,m}^{k,c}$ are decreasing in terms of $k = 0, 1, 2, 3, \ldots$, i.e.,

$$|\mu_{n,m}^{0,c}| > |\mu_{n,m}^{1,c}| > |\mu_{n,m}^{2,c}| > \cdots.$$  (6.38)

**Proof.** Let’s assume $n = 2N$. From (6.37) we know that for any $k = 1, 2, 3, \ldots$

$$\mu_{2N,m}^{k,c} = \mu_{2N+1,m}^{k-1,c}.$$  (6.39)

On the other side, from corollary 6.5 we know that $|\mu_{2N,m}^{k-1,c}| > |\mu_{2N+1,m}^{k-1,c}|$, Therefore we can conclude that for $k = 1, 2, 3, \ldots$

$$|\mu_{2N,m}^{k-1,c}| > |\mu_{2N,m}^{k,c}|.$$  (6.40)

Now, let’s assume that $n = 2N + 1$. From corollary 6.5 we know that $|\mu_{2N,m}^{k,c}| > |\mu_{2N+1,m}^{k,c}|$, on the other side from (6.37) we know that for any $k = 1, 2, 3, \ldots$

$$\mu_{2N+1,m}^{k,c} = \mu_{2N,m}^{k-1,c}.$$  (6.41)

Therefore we can conclude that for $k = 1, 2, 3, \ldots$

$$|\mu_{2N,m}^{k-1,c}| > |\mu_{2N+1,m}^{k,c}|.$$  (6.42)

So from (6.39), and, (6.40), we can conclude (6.38).

\[\square\]

Because of corollary 6.5 and theorem 6.8, we can see that for $n \leq n'$, and, $k \leq k'$,

$$|\mu_{n,m}^{k,c}| \geq |\mu_{n',m}^{k',c}|.$$  (6.43)

This means

$$|\mu_{0,m}^{0,c}| \geq |\mu_{n',m}^{k',c}|,$$

for all $k' \geq 0$, and $n' \geq 0$. From (6.1) we can conclude that

$$\lambda_{0,m}^{0,c} \geq \lambda_{n,m}^{k,c},$$

if $(n, k) \neq (0, 0)$.

This will be helpful in proving the Spectral Concentration problem completely.
Remark 6.9. As in one dimension PSWFs, CPSWFs enjoys the dual orthogonality feature and also they are the solution of Spectral Concentration problem. In fact, the way that we constructed $\psi_{n,m}^{k,c,i}(x)$, they are orthonormal spatial-limited functions in $B(1)$. By defining, $\varphi_{n,m}^{k,c,i}(x) = P_c \psi_{n,m}^{k,c,i}(x)$, we obtain the band-limited version of CPSWFs which are orthogonal in $\mathbb{R}^m$. If we assume that $\varphi_{n,m}^{k,c,i}(x)$ are band-limited CPSWFs which are orthogonal on $\mathbb{R}^m$ then we can see that $\tilde{\varphi}_{n,mw}^{k,c,i}(x) = \frac{1}{\sqrt{\lambda_{n,m}}} \varphi_{n,m}^{k,c,i}(x)$ are orthonormal in $\mathbb{R}^m$ and orthogonal in $B(1)$. In fact we have that

$$\langle \tilde{\varphi}_{n,m}^{k,c,i}, \tilde{\varphi}_{n',m'}^{k,c,i} \rangle_{L^2(\mathbb{R}^m)} = \langle \lambda_{n,m}^{k,c} - 1/2 P_c \psi_{n,m}^{k,c,i}, (\psi_{n,m}^{k,c,i})^2 \rangle_{L^2(\mathbb{R}^m)} = (\lambda_{n,m}^{k,c})^{-1/2} (P_c \psi_{n,m}^{k,c,i}, (\psi_{n,m}^{k,c,i})^2)_{L^2(\mathbb{R}^m)}$$

$$= (\lambda_{n,m}^{k,c})^{-1/2} (\psi_{n,m}^{k,c,i}, (\psi_{n,m}^{k,c,i})^2)_{L^2(\mathbb{R}^m)}$$

$$= (\lambda_{n,m}^{k,c})^{-1/2} P_c \varphi_{n,m}^{k,c,i} \varphi_{n',m'}^{k,c,i} \rangle_{L^2(\mathbb{R}^m)}$$

$$= (\lambda_{n,m}^{k,c})^{-1/2} \lambda_{n,m}^{k,c} \delta_{nn'}.$$

Also,

$$\langle \tilde{\varphi}_{n,m}^{k,c,i}, \tilde{\varphi}_{n',m'}^{k,c,i} \rangle_{L^2(\mathbb{R}^m)} = (\lambda_{n,m}^{k,c})^{-1/2} (Q P_c \psi_{n,m}^{k,c,i}, (\psi_{n,m}^{k,c,i})^2)_{L^2(\mathbb{R}^m)}$$

$$= (\lambda_{n,m}^{k,c})^{-1/2} \lambda_{n,m}^{k,c} \delta_{nn'} \rangle_{L^2(\mathbb{R}^m)}.$$
Figure 3. The Graph of Changes of the $\mu_{n,2}^{k,c}$ eigenvalues of the 2-dimension CPSWFs for $k = 0, 1, 2$, $c = 1$

Figure 4. The Graph of Changes of the eigenvalues of the 2-dimension CPSWFs as $c$ changes for different values of the $k = 2, 1$ and $n = 4, 5$

is the maximised. Since $f \in PW_c$, it admits an expansion of the form $f(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=1}^{c_k} d_k \hat{\phi}_{k,c,i}^{k,c}(x)$. Then

$$\int_{B(1)} |f(x)|^2 dx = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=1}^{c_k} |d_k| |\hat{\phi}_{k,c,i}^{k,c}|^2 \lambda_{n,m} = \lambda_{0,m} \|f\|_{L^2(\mathbb{R}^m)}.$$ 

Therefore,

$$\frac{\int_{B(1)} |f(x)|^2 dx}{\int_{\mathbb{R}^m} |f(x)|^2 dx} \leq \lambda_{0,m}^{0,c} \frac{\int_{B(1)} |\hat{\phi}_{0,m}^0(x)|^2 dx}{\int_{\mathbb{R}^m} |\hat{\phi}_{0,m}^0(x)|^2 dx}.$$ 

7. Spectrum Accumulation

In this short section, we present the spectrum accumulation properties of $m$-dimensional CPSWFs. For one-dimensional PSWFs, this property was investigated in [18] and
has important applications in the construction of multi-taper channel estimation algorithms [29, 17].

Recall that \( \lambda_{n,m}^{k,c} \psi_{n,m}^{k,c,i}(x) = \int_{B(1)} K_c(x-y) \psi_{n,m}^{k,c,i}(y) dy \) where \( K_c(x) = e^m \int_{B(1)} e^{2\pi i c(x \omega)} d\omega. \)

Then
\[
\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sum_{i=1}^{d_k} \lambda_{n,m}^{k,c} |\psi_{n,m}^{k,c,i}(x)|^2 = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sum_{i=1}^{d_k} \lambda_{n,m}^{k,c} \left[ \psi_{n,m}^{k,c,i}(x) \psi_{n,m}^{k,c,i}(x) \right]_0
\]
\[
= \left[ \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sum_{i=1}^{d_k} \left( \int_{B(1)} \psi_{n,m}^{k,c,i}(y) K_c(x-y) dy \right) \psi_{n,m}^{k,c,i}(x) \right]_0
\]
\[
= \left[ \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sum_{i=1}^{d_k} \langle \psi_{n,m}^{k,c,i}, K_c(x-\cdot) \rangle \psi_{n,m}^{k,c,i}(x) \right]_0 = K_c(0) = e^m |B(1)|,
\]

On the other hand, if instead of summing over all \( n, k \geq 0 \) in the above calculation, we instead perform a truncated sum by restricting the values of \( n \) and \( k \) so that \( 0 \leq n \leq 2N + 1 \) and \( 0 \leq k \leq K \), then we have
\[
\sum_{k=0}^{K} \sum_{n=0}^{2N+1} \sum_{i=1}^{d_k} \lambda_{n,m}^{k,c} |\psi_{n,m}^{k,c,i}(x)|^2
\]
\[
= \sum_{k=0}^{K} \sum_{n=0}^{N} \sum_{i=1}^{d_k} \lambda_{2n,m}^{k,c} |\psi_{2n,m}^{k,c,i}(x)|^2 + \sum_{k=0}^{K} \sum_{n=0}^{N} \sum_{i=1}^{d_k} \lambda_{2n+1,m}^{k,c} |\psi_{2n+1,m}^{k,c,i}(x)|^2
\]
\[
= \sum_{k=0}^{K} \sum_{n=0}^{N} \sum_{i=1}^{d_k} \lambda_{2n,m}^{k,c} |P_{n,m}^{k,c}(x)|^2 |x|^{2k} \sum_{i=1}^{d_k} \left| Y_i^k \frac{x}{|x|} \right|^2
\]
\[
+ \sum_{k=0}^{K} \sum_{n=0}^{N} \sum_{i=1}^{d_k} \lambda_{2n+1,m}^{k,c} |Q_{n,m}^{k,c}(x)|^2 |x|^{2k+2} \sum_{i=1}^{d_k} \left| Y_i^k \frac{x}{|x|} \right|^2
\]
\[
= \sum_{k=0}^{K} \sum_{n=0}^{N} \sum_{i=1}^{d_k} \lambda_{2n,m}^{k,c} |P_{n,m}^{k,c}(x)|^2 |x|^{2k} \left( \frac{k + m - 2}{|S^{m-1}|(m-2)} \right)
\]
\[
+ \sum_{k=0}^{K} \sum_{n=0}^{N} \sum_{i=1}^{d_k} \lambda_{2n+1,m}^{k,c} |Q_{n,m}^{k,c}(x)|^2 |x|^{2k+2} \left( \frac{k + m - 2}{|S^{m-1}|(m-2)} \right) = G(|x|^2),
\]
where we have used theorem 3.3 from [10]:
\[
\sum_{i=1}^{d_k} Y_i^k(x) Y_i^k(y) = \frac{(k + m - 2)}{|S^{m-1}|(m-2)} |x|^k |y|^k C_k^\mu(t) + (x \wedge y) |x|^{k-1} |y|^{k-1} C_{k-1}^{\mu+1}(t),
\]
in which \( t = \frac{(x \wedge y)}{|x| |y|} \), \( C_k^\mu(t) \) is the Gegenbauer polynomials defined on the line, and \( |S^{m-1}| \) is the Lebesgue measure of \( S^{m-1} \). In the figures [3] and [6] we see numerical computations of teh partial sums in 2 and 3 dimensions, demonstrating the convergence of the partial sums to the constant \( e^m |B(1)| \).

**Remark 7.1.** All codes related to the calculations of CPSWFs, their plots, and their eigenvalues are available in Github. The codes are written in different platforms, namely Matlab, Maple, Mathematica, Python, Julia, Sagemath.
Figure 5. The Radial Graph of numerical computations of spectrum accumulation properties of 2—dimension CPSWFs for $c = 1$. 

Figure 6. The Radial Graph of numerical computations of spectrum accumulation properties of 2—dimension CPSWFs for $c = 2$. 

Figure 7. The Radial Graph of numerical computations of spectrum accumulation properties of 3—dimension CPSWFs for $c = 1$. 
Figure 8. The Radial Graph of numerical computations of spectrum accumulation properties of 3–dimension CPSWFs for $c = 2$.

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