Genus-minimal crystallizations of PL 4-manifolds

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Abstract

For \( d \geq 2 \), the regular genus of a closed connected PL \( d \)-manifold \( M \) is the least genus (resp., half of genus) of an orientable (resp., a non-orientable) surface into which a crystallization of \( M \) imbeds regularly. The regular genus of every orientable surface equals its genus, and the regular genus of every 3-manifold equals its Heegaard genus. For every closed connected PL 4-manifold \( M \), it is known that its regular genus \( G(M) \) is at least \( 2\chi(M) + 5m - 4 \), where \( m \) is the rank of the fundamental group of \( M \). In this article, we introduce the concept of “weak semi-simple crystallization” for every closed connected PL 4-manifold \( M \), and prove that \( G(M) = 2\chi(M) + 5m - 4 \) if and only if \( M \) admits a weak semi-simple crystallization. We then show that the PL invariant regular genus is additive under the connected sum within the class of all PL 4-manifolds admitting a weak semi-simple crystallization. Also, we note that this property is related to the 4-dimensional Smooth Poincaré Conjecture.

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1 Introduction and Main Results

First, let us recall that a crystallization of a PL-manifold is a contracted colored graph which represents the manifold (for details and related notations see Subsection 2.1). The existence of crystallizations for every closed connected PL-manifold is ensured by a classical theorem due to Pezzana (see [17], or [14] for subsequent generalizations).

Let \((\Gamma,\gamma)\) be a \((d+1)\)-regular colored graph (i.e., through each vertex of the graph, there are \( d+1 \) edges with different colors from the color set \( \Delta_d := \{0, \ldots, d\} \)). An embedding \(i: \Gamma \hookrightarrow F\) of \(\Gamma\) into a closed surface \(F\) is called regular if there exists a cyclic permutation \(\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_d)\) of the color set \(\Delta_d\), such that the boundary of each face of \(i(\Gamma)\) is a bicolored cycle with edges alternately colored by \(\varepsilon_j, \varepsilon_{j+1}\) for some \(j\) (the addition is modulo \(d+1\)). Then, the regular genus \(\rho(\Gamma)\) of \((\Gamma,\gamma)\) is the least genus (resp., half of genus) of an orientable (resp., a non-orientable) surface into which \(\Gamma\) embeds regularly, and the regular genus \(G(M)\) of a closed connected PL \(d\)-manifold \(M\) is defined as the minimum regular genus of its crystallizations. Note that the notion of regular genus extends classical notions to arbitrary dimension. In fact, the regular genus of a closed connected orientable (resp., non-orientable) surface coincides with its genus (resp., half of its genus), while the regular genus of a closed connected 3-manifold coincides with its Heegaard genus (see [15]).
The invariant regular genus has been intensively studied, yielding some important general results. For example, regular genus zero characterizes the $d$-sphere among all closed connected PL $d$-manifolds (cf. [12]). In particular, in dimension $d \in \{4, 5\}$, a lot of classifying results in PL-category have been obtained, both for closed and bounded PL $d$-manifolds (see, for example, [6, 7, 8]). In fact, many authors give bounds for regular genus of some PL $d$-manifolds (cf. [9, 10, 11, 13, 18]).

Let $(\Gamma, \gamma)$ be a crystallization (with color set $\Delta_4$) of a closed connected PL 4-manifold $M$ and let $m$ be the rank (the minimum size of a generating set) of the fundamental group of $M$. Let $g_{\{i,j,k\}}$ be the number of connected components of the subgraph of $(\Gamma, \gamma)$ restricted in the color set $\{i, j, k\}$. Then, by a result of Gagliardi (cf. Proposition 4 (b)), $g_{\{i,j,k\}} \geq m + 1$ for any distinct $i, j, k \in \Delta_4$. A crystallization $(\Gamma, \gamma)$ of $M$ is called a semi-simple crystallization if $g_{\{i,j,k\}} = m + 1$ for all $i, j, k \in \Delta_4$. Now, generalizing the notion of semi-simple crystallization, we have defined the following.

**Definition 1.** Let $M$ be a closed connected PL 4-manifold and $m$ be the rank of the fundamental group of $M$. A crystallization $(\Gamma, \gamma)$ of $M$ with color set $\Delta_4$ is said to be a weak semi-simple crystallization if $g_{\{i,i+1,i+2\}} = m + 1$ for $i \in \Delta_4$ (addition in subscript of $g$ is modulo 5).

From [3] we know that, for a closed connected PL 4-manifold $M$, its regular genus $G(M) \geq 2\chi(M) + 5m - 4$, and equality holds if $M$ admits a semi-simple crystallization, where $m$ is the rank of the fundamental group of $M$. Here, we have proved the following.

**Theorem 2.** Let $M$ be a closed connected PL 4-manifold and $m$ be the rank of its fundamental group. Then $G(M) = 2\chi(M) + 5m - 4$ if and only if $M$ admits a weak semi-simple crystallization.

We have also shown that the class of PL 4-manifolds admitting weak semi-simple crystallizations is closed under connected sum (cf. Lemma 10). Hence, PL 4-manifolds admitting weak semi-simple crystallizations actually constitutes a huge class which comprehends $S^4$, $\mathbb{C}P^2$, $S^2 \times S^2$, $\mathbb{R}P^4$, $\mathbb{R}P^2 \times S^2$, the orientable and non-orientable $S^3$-bundles over $S^1$, an orientable and a non-orientable ($S^2 \times S^1$)-bundles over $S^1$, and the $K3$-surface, together with their connected sums, possibly by taking copies with reversed orientation (cf. Remark 12).

The additivity of regular genus under connected sum has been conjectured for closed connected PL $d$-manifolds. Moreover, the associated (open) problem is significant especially in dimension four. In fact, in dimension four, additivity of regular genus, at least in the simply-connected case, would imply the 4-dimensional Smooth Poincaré Conjecture, in virtue of a well-known Wall’s Theorem (cf. [19]). Here, we have proved the additivity of regular genus for the class of PL 4-manifolds admitting weak semi-simple crystallizations.

**Theorem 3.** Let $M_1$ and $M_2$ be two PL 4-manifolds admitting weak semi-simple crystallizations. Then $G(M_1 \# M_2) = G(M_1) + G(M_2)$.

We have also shown some interesting consequences of the above theorems (cf. Corollaries 8, 9 and 11 and Remarks 12 and 13).
2 Preliminaries

2.1 Crystallizations

A multigraph \( \Gamma = (V(\Gamma), E(\Gamma)) \) is a finite connected graph, with vertex-set \( V(\Gamma) \) and edge-set \( E(\Gamma) \), which can have multiple edges but no loops. A multigraph \( \Gamma \) is called \((d+1)\)-regular if the number of edges adjacent to each vertex is \((d+1)\). For standard terminology on graphs, we refer to [5]. A \((d+1)\)-colored graph is a pair \((\Gamma, \gamma)\), where \( \Gamma = (V(\Gamma), E(\Gamma)) \) is a multigraph of degree at most \(d+1\) and the surjective map \( \gamma : E(\Gamma) \to \Delta_d := \{0, 1, \ldots, d\} \) is a proper edge-coloring (i.e., \( \gamma(e) \neq \gamma(f) \) for any pair \( e, f \) of adjacent edges). The elements of the set \( \Delta_d \) are called the colors of \( \Gamma \).

For each \( B \subseteq \Delta_d \) with \( h \) elements, the graph \( \Gamma_B = (V(\Gamma), \gamma^{-1}(B)) \) is an \( h \)-colored graph with edge-coloring \( \gamma|_{\gamma^{-1}(B)} \). If \( \Gamma_{\Delta_d \setminus \{c\}} \) is connected for all \( c \in \Delta_d \), then \((\Gamma, \gamma)\) is called contracted. For a color set \( \{i_1, i_2, \ldots, i_k\} \subset \Delta_d \), \( g_{\{i_1,i_2,\ldots,i_k\}} \) denotes the number of connected components of the graph \( \Gamma_{\{i_1,i_2,\ldots,i_k\}} \).

Each \((d+1)\)-colored graph uniquely determines a \( d \)-dimensional simplicial cell-complex \( K(\Gamma) \), which is said to be associated to \( \Gamma \):

- for every vertex \( v \in V(\Gamma) \), take a \( d \)-simplex \( \sigma(v) \) and label injectively its \( d+1 \) vertices by the colors of \( \Delta_d \);
- for every edge of color \( i \) between \( v, w \in V(\Gamma) \), identify the \((d-1)\)-faces of \( \sigma(v) \) and \( \sigma(w) \) opposite to \( i \)-labelled vertices, so that equally labelled vertices coincide.

The vector \( f(K(\Gamma)) :=(f_0(K(\Gamma)), f_1(K(\Gamma)), \ldots, f_d(K(\Gamma))) \) is called the \( f \)-vector of \( K(\Gamma) \) where \( f_i(K(\Gamma)) \) is the number of \( i \)-simplices in \( K(\Gamma) \). If the geometrical carrier of \( K(\Gamma) \) is PL-homeomorphic to a PL \( d \)-manifold \( M \), then the \((d+1)\)-colored graph \((\Gamma, \gamma)\) is said to represent \( M \); if, in addition, \((\Gamma, \gamma)\) is contracted, then it is called a crystallization of \( M \). If \((\Gamma, \gamma)\) is a crystallization of a connected PL \( d \)-manifold \( M \), then the number of vertices in \( K(\Gamma) \) is \( d+1 \). If \((\Gamma, \gamma)\) is a crystallization of a closed connected PL \( d \)-manifold then \((\Gamma, \gamma)\) is a \((d+1)\)-regular colored graph, i.e., a \((d+1)\)-colored graph which is also \((d+1)\)-regular.

Let \((\Gamma_1, \gamma_1)\) and \((\Gamma_2, \gamma_2)\) be two disjoint \((d+1)\)-colored graphs with the same color set \( \Delta_d \), and let \( v_i \in V_i \) (\( 1 \leq i \leq 2 \)). The connected sum of \( \Gamma_1, \Gamma_2 \) with respect to vertices \( v_1, v_2 \) (denoted by \((\Gamma_1\#v_1v_2\Gamma_2, \gamma_1\#\gamma_2)\), or simply \((\Gamma_1\#\Gamma_2, \gamma_1\#\gamma_2)\)) is the graph obtained from \((\Gamma_1 \setminus \{v_1\}) \cup (\Gamma_2 \setminus \{v_2\})\) by adding \( d+1 \) new edges \( e_0, \ldots, e_d \) with colors \( 0, \ldots, d \), respectively, such that the end points of \( e_j \) are \( u_{j,1} \) and \( u_{j,2} \), where \( v_i \) and \( u_{j,i} \) are joined in \((\Gamma_i, \gamma_i)\) with an edge of color \( j \) for \( 0 \leq j \leq d \), \( 1 \leq i \leq 2 \). From [14], we know the following.

**Proposition 4.** For \( d \geq 3 \), let \((\Gamma, \gamma)\) be a crystallization of a PL \( d \)-manifold \( M \).

(a) Let \( \nu(\Gamma) \) denote the number of vertices of \( \Gamma \). Then, \[ 2g_{\{i,j,k\}} = g_{\{i,j\}} + g_{\{i,k\}} + g_{\{j,k\}} - \frac{\nu(\Gamma)}{2} \]
for any distinct \( i, j, k \in \Delta_d \).

(b) For any distinct \( i, j \in \Delta_d \), the set of all connected components of \( \Gamma_{\Delta_d \setminus \{i,j\}} \) is in bijection with a set of generators of the fundamental group \( \pi_1(M) \).

(c) If \((\Gamma', \gamma')\) is a crystallization of a PL \( d \)-manifold \( M' \), then the graph connected sum \((\Gamma'\#\Gamma, \gamma'\#\gamma')\) is a crystallization of \( M'\#M' \).
2.2 Regular Genus of PL-manifolds

The notion of regular genus is strictly related to the existence of regular embeddings of crystallizations into closed surfaces, i.e., embeddings whose regions are bounded by the images of bi-colored cycles, with colors consecutive in a fixed permutation of the color set. More precisely, according to [15], if \((\Gamma, \gamma)\) is a crystallization of an orientable (resp., non-orientable) PL \(d\)-manifold \(M (d \geq 3)\), then for each cyclic permutation \(\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d)\) of \(\Delta_d\), a regular embedding \(i_\varepsilon : \Gamma \hookrightarrow F_\varepsilon\) exists, where \(F_\varepsilon\) is the closed orientable (resp., non-orientable) surface with Euler characteristic

\[
\chi_\varepsilon(\Gamma) = \sum_{i \in \mathbb{Z}_{d+1}} g_{\{\varepsilon_i, \varepsilon_{i+1}\}} + (1 - d) \frac{\nu(\Gamma)}{2}.
\]

In the orientable (resp., non-orientable) case, the integer

\[
\rho_\varepsilon(\Gamma) = 1 - \chi_\varepsilon(\Gamma)/2
\]

is equal to the genus (resp., half of the genus) of the surface \(F_\varepsilon\). Then, the regular genus \(\rho(\Gamma)\) of \((\Gamma, \gamma)\) and the regular genus \(\mathcal{G}(M)\) of \(M\) are:

\[
\rho(\Gamma) = \min \{\rho_\varepsilon(\Gamma) \mid \varepsilon\text{ is a cyclic permutation of } \Delta_d\};
\]

\[
\mathcal{G}(M) = \min \{\rho(\Gamma) \mid (\Gamma, \gamma)\text{ is a crystallization of } M\}.
\]

From [3], we know the following.

**Proposition 5.** Let \(M\) be a (closed connected) PL 4-manifold and \(m\) be the rank of its fundamental group. Then, \(\mathcal{G}(M) \geq 2\chi(M) + 5m - 4\).

3 Proof of Main Results and some Consequences

**Lemma 6.** Let \((\Gamma, \gamma)\) be a crystallization of a closed connected PL 4-manifold \(M\). Then \(\nu(\Gamma) = 6\chi(M) + 2\sum_{0 \leq i < j < k \leq 4} g_{\{i,j,k\}} - 30\).

**Proof.** Let \(2p\) be the number of vertices of \(\Gamma\) and \(X = K(\Gamma)\) be the corresponding simplicial cell-complex. Then the Dehn-Sommerville equations in dimension four yield:

\[
\begin{align*}
f_0(X) - f_1(X) + f_2(X) - f_3(X) + f_4(X) &= \chi(M), \\
2f_1(X) - 3f_2(X) + 4f_3(X) - 5f_4(X) &= 0, \\
2f_3(X) - 5f_4(X) &= 0.
\end{align*}
\]

Since \(f_0(X) = 5\) (by construction) and \(f_4(X) = \nu(\Gamma) = 2p\), the following equality holds:

\[
2p = 6\chi(M) + 2f_1(K(\Gamma)) - 30.
\]

Since \(f_1(K(\Gamma)) = \sum_{0 \leq i < j < k \leq 4} g_{\{i,j,k\}}\), we have \(2p = 6\chi(M) + 2\sum_{0 \leq i < j < k \leq 4} g_{\{i,j,k\}} - 30\). \(\square\)

**Lemma 7.** Let \((\Gamma, \gamma)\) be a crystallization of a closed connected PL 4-manifold \(M\) and \(m\) be the rank of the fundamental group of \(M\). Then, for any cyclic permutation \(\varepsilon = (\varepsilon_0, \ldots, \varepsilon_4 = 4)\) of the color set \(\Delta_4\) we have \(\rho_\varepsilon(\Gamma) = 2\chi(M) + 5m - 4 + \sum_{i \in \mathbb{Z}_5} g_{\{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}\}} - m - 1\).
Proof. From Proposition 4(b), we know that \( g_{(i,j,k)} \geq m + 1 \) for any distinct \( i, j, k \in \Delta_4 \). Therefore, let us assume that \( g_{(i,j,k)} = (m + 1) + t_{(i,j,k)} \), for \( t_{(i,j,k)} \in \mathbb{Z} \) and \( t_{(i,j,k)} \geq 0 \). Thus, from Lemma 6 we have \( \nu(\Gamma) = 6\chi(M) + 2\sum_{0 \leq i < j < k \leq 4} \rho g_{(i,j,k)} - 30 = 6\chi(M) + 20(m + 1) - 30 + 2\sum_{0 \leq i < j < k \leq 4} t_{(i,j,k)} \). We set \( \nu(\Gamma) = 2\bar{p} + 2q \), where \( 2\bar{p} = 6\chi(M) + 20(m + 1) - 30 \) and \( q = \sum_{0 \leq i < j < k \leq 4} t_{(i,j,k)} \). Therefore, from the definition of the regular genus, we can deduce that

\[
\rho_{\varepsilon}(\Gamma) = 1 + \frac{3}{4} \nu(\Gamma) - \frac{1}{2} \sum_{i \in \mathbb{Z}_5} g_{\{\varepsilon_i,\varepsilon_{i+1}\}} = 1 + \frac{3(\bar{p} + q)}{2} - \frac{1}{2} \sum_{i \in \mathbb{Z}_5} g_{\{\varepsilon_i,\varepsilon_{i+1}\}}.
\]  

(1)

Now, by Proposition 4(a), we know that \( 2g_{ijk} = g_{ij} + g_{ik} + g_{jk} - \frac{\nu(\Gamma)}{2} \) for any distinct \( i, j, k \in \Delta_4 \). Thus, we have \( g_{ij} + g_{ik} + g_{jk} = 2g_{ijk} + (\bar{p} + q) \) for \( 0 \leq i < j < k \leq 4 \). This gives ten linear equations which can be written in the following form.

\[
AX = B,
\]

where

\[
A = \begin{bmatrix}
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix},
\]

\[
X = \begin{bmatrix}
g_{(0,1)} \\
g_{(0,2)} \\
g_{(0,3)} \\
g_{(0,4)} \\
g_{(1,2)} \\
g_{(1,3)} \\
g_{(1,4)} \\
g_{(2,3)} \\
g_{(2,4)} \\
g_{(3,4)} \\
\end{bmatrix}
\]

and

\[
B = \begin{bmatrix}
2g_{(0,1,2)} + \bar{p} + q \\
2g_{(0,1,3)} + \bar{p} + q \\
2g_{(0,1,4)} + \bar{p} + q \\
2g_{(0,2,3)} + \bar{p} + q \\
2g_{(0,2,4)} + \bar{p} + q \\
2g_{(0,3,4)} + \bar{p} + q \\
2g_{(1,2,3)} + \bar{p} + q \\
2g_{(1,2,4)} + \bar{p} + q \\
2g_{(1,3,4)} + \bar{p} + q \\
2g_{(2,3,4)} + \bar{p} + q \\
\end{bmatrix}
\]

Therefore,

\[
X = A^{-1}B,
\]

where

\[
A^{-1} = \begin{bmatrix}
1/3 & 1/3 & 1/3 & -1/6 & -1/6 & -1/6 & -1/6 & -1/6 & -1/6 & 1/3 \\
1/3 & -1/6 & -1/6 & 1/3 & 1/3 & -1/6 & -1/6 & -1/6 & 1/3 & -1/6 \\
-1/6 & 1/3 & -1/6 & 1/3 & -1/6 & 1/3 & -1/6 & 1/3 & -1/6 & -1/6 \\
-1/6 & -1/6 & 1/3 & -1/6 & 1/3 & 1/3 & -1/6 & -1/6 & -1/6 & -1/6 \\
1/3 & -1/6 & -1/6 & -1/6 & 1/3 & 1/3 & 1/3 & -1/6 & -1/6 & -1/6 \\
-1/6 & 1/3 & -1/6 & -1/6 & 1/3 & -1/6 & 1/3 & -1/6 & 1/3 & -1/6 \\
-1/6 & -1/6 & 1/3 & -1/6 & 1/3 & -1/6 & -1/6 & -1/6 & 1/3 & 1/3 \\
-1/6 & -1/6 & 1/3 & -1/6 & -1/6 & 1/3 & -1/6 & 1/3 & -1/6 & 1/3 \\
1/3 & -1/6 & -1/6 & -1/6 & -1/6 & 1/3 & -1/6 & -1/6 & 1/3 & 1/3 \\
\end{bmatrix}
\]

Since \( g_{(i,j,k)} = (m + 1) + t_{(i,j,k)} \) for any distinct \( i, j, k \in \Delta_4 \), we consider

\[
B = M + T,
\]

5
where

\[
M = \begin{bmatrix}
2(m + 1) + \bar{p} + q \\
2(m + 1) + \bar{p} + q \\
2(m + 1) + \bar{p} + q \\
2(m + 1) + \bar{p} + q \\
2(m + 1) + \bar{p} + q \\
2(m + 1) + \bar{p} + q \\
2(m + 1) + \bar{p} + q \\
2(m + 1) + \bar{p} + q \\
2(m + 1) + \bar{p} + q \\
2(m + 1) + \bar{p} + q \\
2(m + 1) + \bar{p} + q
\end{bmatrix}
\]

and

\[
T = \begin{bmatrix}
2t_{\{0,1,2\}} \\
2t_{\{0,1,3\}} \\
2t_{\{0,1,4\}} \\
2t_{\{0,2,3\}} \\
2t_{\{0,2,4\}} \\
2t_{\{0,3,4\}} \\
2t_{\{1,2,3\}} \\
2t_{\{1,2,4\}} \\
2t_{\{1,3,4\}} \\
2t_{\{2,3,4\}}
\end{bmatrix}.
\]

Thus,

\[
X = A^{-1}M + A^{-1}T.
\]

Therefore, \( g_{\{i,j\}} = \frac{2(m+1) + \bar{p} + q}{3} + 2\sum_{0 \leq k < l < r \leq 4} c_{klr}^{ij} t_{\{k,l,r\}} \) where \( c_{klr}^{ij} \) is the element of \( A^{-1} \) corresponding to \((i,j)\)-row and \((k,l,r)\)-column of \( A^{-1} \). Thus for a given permutation \( \epsilon = (\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 = 4) \) of \( \Delta_4 \), we have

\[
\sum_{i \in \mathbb{Z}_5} g_{\{\epsilon_i, \epsilon_{i+1}\}} = \frac{5}{3}(2(m + 1) + \bar{p} + q) + 2 \sum_{0 \leq k < l < r \leq 4} (\sum_{i \in \mathbb{Z}_5} c_{klr}^{\epsilon_i \epsilon_{i+1}}) t_{\{k,l,r\}}.
\]

Therefore, from Equation (1) we get

\[
\rho_\epsilon(\Gamma) = 1 + \frac{3(\bar{p} + q)}{2} - \frac{5}{6}(2(m + 1) + \bar{p} + q) - \sum_{0 \leq k < l < r \leq 4} (\sum_{i \in \mathbb{Z}_5} c_{klr}^{\epsilon_i \epsilon_{i+1}}) t_{\{k,l,r\}}
\]

\[
= 1 + \frac{2\bar{p} - 5(m + 1)}{3} + \frac{2q}{3} - \sum_{0 \leq k < l < r \leq 4} (\sum_{i \in \mathbb{Z}_5} c_{klr}^{\epsilon_i \epsilon_{i+1}}) t_{\{k,l,r\}}
\]

\[
= 2\chi(M) + 5m - 4 + \frac{2}{3} \sum_{0 \leq k < l < r \leq 4} t_{\{k,l,r\}} - \sum_{0 \leq k < l < r \leq 4} (\sum_{i \in \mathbb{Z}_5} c_{klr}^{\epsilon_i \epsilon_{i+1}}) t_{\{k,l,r\}}.
\]

Now, for a given permutation \( \epsilon = (\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 = 4) \) of \( \Delta_4 \), we have the following values of \( \sum_{i \in \mathbb{Z}_5} c_{klr}^{\epsilon_i \epsilon_{i+1}} \) which is a coefficient of \( t_{\{k,l,r\}} \) in \( \sum_{0 \leq k < l < r \leq 4} (\sum_{i \in \mathbb{Z}_5} c_{klr}^{\epsilon_i \epsilon_{i+1}}) t_{\{k,l,r\}} \).

| \sum_{i \in \mathbb{Z}_5} c_{klr}^{\epsilon_i \epsilon_{i+1}} | \begin{array}{cccccccccc}
\text{klr} & 012 & 013 & 014 & 023 & 024 & 034 & 123 & 124 & 134 & 234 \\
\epsilon = (0,1,2,3,4) & 2/3 & -1/3 & 2/3 & -1/3 & 2/3 & 2/3 & -1/3 & -1/3 & 2/3 & 2/3 \\
\epsilon = (0,1,3,2,4) & -1/3 & 2/3 & -1/3 & 2/3 & -1/3 & 2/3 & -1/3 & -1/3 & 2/3 & 2/3 \\
\epsilon = (0,2,3,1,4) & -1/3 & -1/3 & 2/3 & -1/3 & 2/3 & -1/3 & 2/3 & -1/3 & 2/3 & -1/3 \\
\epsilon = (0,2,1,3,4) & 2/3 & -1/3 & -1/3 & 2/3 & -1/3 & 2/3 & -1/3 & -1/3 & 2/3 & -1/3 \\
\epsilon = (0,3,1,2,4) & -1/3 & 2/3 & -1/3 & 2/3 & -1/3 & 2/3 & -1/3 & -1/3 & 2/3 & -1/3 \\
\epsilon = (0,3,2,1,4) & -1/3 & -1/3 & 2/3 & -1/3 & 2/3 & -1/3 & 2/3 & -1/3 & 2/3 & -1/3 \\
\epsilon = (1,0,3,2,4) & -1/3 & 2/3 & -1/3 & 2/3 & -1/3 & 2/3 & -1/3 & -1/3 & 2/3 & -1/3 \\
\epsilon = (1,0,3,2,4) & -1/3 & 2/3 & -1/3 & 2/3 & -1/3 & 2/3 & -1/3 & -1/3 & 2/3 & -1/3 \\
\epsilon = (1,0,2,3,4) & 2/3 & -1/3 & 2/3 & -1/3 & 2/3 & -1/3 & 2/3 & -1/3 & 2/3 & -1/3 \\
\epsilon = (1,2,0,3,4) & 2/3 & -1/3 & 2/3 & -1/3 & 2/3 & -1/3 & 2/3 & -1/3 & 2/3 & -1/3 \\
\epsilon = (1,2,0,3,4) & 2/3 & -1/3 & 2/3 & -1/3 & 2/3 & -1/3 & 2/3 & -1/3 & 2/3 & -1/3 \\
\epsilon = (2,0,1,3,4) & 2/3 & -1/3 & 2/3 & -1/3 & 2/3 & -1/3 & 2/3 & -1/3 & 2/3 & -1/3 \\
\epsilon = (2,0,1,3,4) & 2/3 & -1/3 & 2/3 & -1/3 & 2/3 & -1/3 & 2/3 & -1/3 & 2/3 & -1/3 |
Thus,

$$
\sum_{0 \leq k < l \leq 4} (\sum_{r \in \mathbb{Z}_5} c_{k l r}^{e_i e_{i+1}}) t_{k, l, r} = \frac{2}{3} \sum_{i \in \mathbb{Z}_5} t_{\varepsilon_i, \varepsilon_{i+1}, \varepsilon_{i+2}} - \frac{1}{3} \sum_{i \in \mathbb{Z}_5} t_{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}}.
$$

Since \( \sum_{0 \leq k < l \leq 4} t_{k, l, r} = \sum_{i \in \mathbb{Z}_5} t_{\varepsilon_i, \varepsilon_{i+1}, \varepsilon_{i+2}} + \sum_{i \in \mathbb{Z}_5} t_{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}} \), we have

$$
\rho_\varepsilon(\Gamma) = 2\chi(M) + 5m - 4 + \sum_{i \in \mathbb{Z}_5} t_{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}}.
$$

Now, the result follows from this. \( \square \)

**Proof of Theorem 2** Let \( M \) admit a weak semi-simple crystallization. Thus, we have \( g_{i, i+1, i+2} = m + 1 \) for \( i \in \Delta_4 \) (addition in subscript of \( g \) is modulo 5). Therefore, from Lemma 6, we have \( \rho_\varepsilon(\Gamma) = 2\chi(M) + 5m - 4 + \sum_{i \in \mathbb{Z}_5} (g_{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}} - m - 1) \) for any cyclic permutation \( \varepsilon = (\varepsilon_0, \ldots, \varepsilon_4) \) of the color set \( \Delta_4 \). Let \( \varepsilon = (2, 0, 3, 1, 4) \). Then \( g_{\varepsilon_0, \varepsilon_2, \varepsilon_4} = g_{\varepsilon_1, \varepsilon_3, \varepsilon_0} = m + 1 \) (addition in subscript of \( g \) is modulo 5). Therefore, \( \rho_\varepsilon(\Gamma) = 2\chi(M) + 5m - 4 \) and hence, \( \mathcal{G}(M) \leq 2\chi(M) + 5m - 4 \). Thus, by Proposition 5, \( \mathcal{G}(M) = 2\chi(M) + 5m - 4 \).

On the other hand, let \( \mathcal{G}(M) = 2\chi(M) + 5m - 4 \). Then, from Lemma 7 we have, there exist a crystallization \( \Gamma \) and a cyclic permutation \( \varepsilon = (\varepsilon_0, \ldots, \varepsilon_4) \) of the color set \( \Delta_4 \) such that \( \sum_{i \in \mathbb{Z}_5} (g_{\varepsilon_i, \varepsilon_{i+1}, \varepsilon_{i+2}} - m - 1) = 0 \). Therefore, \( g_{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}} = m + 1 \) (addition in subscript of \( g \) is modulo 5). Let \( \bar{\Gamma} \) be a colored graph, obtained from \( \Gamma \), by replacing the colors \( (\varepsilon_0, \varepsilon_2, \varepsilon_4, \varepsilon_1, \varepsilon_3) \) by \( (0, 1, 2, 3, 4) \). Then \( \bar{\Gamma} \) is a crystallization of \( M \) with \( \bar{g}_{i, i+1, i+2} = m + 1 \) for \( i \in \mathbb{Z}_5 \). Therefore, \( M \) admits a weak semi-simple crystallization. \( \square \)

**Corollary 8.** If a closed connected PL 4-manifold \( M \) admits a crystallization \( (\Gamma, \gamma) \) such that \( \nu(\Gamma) \leq 6\chi(M) + 20m - 6 \), where \( m \) is the rank of the fundamental group of \( M \) then \( \mathcal{G}(M) = \rho(\Gamma) = 2\chi(M) + 5m - 4 \).

**Proof.** From Lemma 6 we have \( 6\chi(M) + 2(\sum_{0 \leq i < j < k \leq 4} g_{i, j, k}) - 30 = \nu(\Gamma) \leq 6\chi(M) + 20m - 6 \). Therefore, \( \sum_{0 \leq i < j < k \leq 4} g_{i, j, k} \leq 10m + 12 \). On the other hand, by Proposition 4(d), we have \( g_{i, j, k} \geq m + 1 \). Thus, at most two \( g_{i, j, k} > m + 1 \). Therefore, there exists a cyclic permutation \( (\delta_0, \ldots, \delta_4) = (4) \) of the color set \( \Delta_4 \) such that \( g_{\delta_i, \delta_{i+1}, \delta_{i+2}} = m + 1 \) for \( i \in \Delta_4 \) (addition in subscript of \( g \) is modulo 5). Let \( \bar{\Gamma} \) be a colored graph, obtained from \( \Gamma \), by replacing the colors \( \delta_i \) by \( i \) for \( i \in \mathbb{Z}_5 \). Then \( \bar{\Gamma} \) is a crystallization of \( M \) with \( \bar{g}_{i, i+1, i+2} = m + 1 \) for \( i \in \mathbb{Z}_5 \). Therefore, \( M \) admits a weak semi-simple crystallization. Now, the result follows from Theorem 2. \( \square \)

**Corollary 9.** Let \( \mathbb{F} \) be a field, let \( (\Gamma, \gamma) \) be a crystallization of a closed \( \mathbb{F} \)-orientable PL 4-manifold \( M \) and, let \( \mathcal{T} \) be the corresponding simplicial cell-complex. If \( \mathcal{T} \) satisfies equality in the Novik-Swartz bound with respect to \( \mathbb{F} \) then \( \mathcal{G}(M) = \rho(\Gamma) = 2\chi(M) + 5m - 4 \).

**Proof.** Set \( f_4(\mathcal{T}) = 2p \), and let \( h(\mathcal{T}) = (h_0(\mathcal{T}), \ldots, h_5(\mathcal{T})) \) be the \( h \)-vector of \( \mathcal{T} \), i.e.,

$$
\sum_{i=0}^{5} h_i(\mathcal{T}) x^{5-i} = \sum_{i=0}^{5} f_{i-1}(\mathcal{T})(x - 1)^{5-i}. \tag{2}
$$
Then, due to Novik and Swartz (cf. [16, Theorem 6.4]), we have

\[ h_j(X) \geq \binom{5}{j} \sum_{i=0}^{j} (-1)^{j-i} \beta_{i-1}(M, \mathbb{Z}). \]  

(3)

By substituting \( x \) by \( x + 1 \) and comparing the coefficients of \( x^i \) on both sides of Equation (2), we obtain \( 2p = \sum_{i=0}^{5} h_i(X) \). Thus, Equation (3) and the Euler characteristic imply

\[
\begin{align*}
\beta_0(M, \mathbb{F}) + 4\beta_1(M, \mathbb{F}) + 6\beta_2(M, \mathbb{F}) + 4\beta_3(M, \mathbb{F}) + \beta_4(M, \mathbb{F}) & \leq 2p, \\
\beta_0(M, \mathbb{F}) - \beta_1(M, \mathbb{F}) + \beta_2(M, \mathbb{F}) - \beta_3(M, \mathbb{F}) + \beta_4(M, \mathbb{F}) & = \chi(M). 
\end{align*}
\]

(4)

Since we have equality in the both equations in (4), it follows that

\[ 5\beta_0(M, \mathbb{F}) - 10\beta_1(M, \mathbb{F}) - 10\beta_3(M, \mathbb{F}) + 5\beta_4(M, \mathbb{F}) = 6\chi(M) - 2p. \]

Let \( m \) be the rank of the fundamental group of \( M \). Then \( m \geq \beta_1(M, \mathbb{F}) = \beta_3(M, \mathbb{F}) \). Thus, \( 2p \leq 6\chi(M) + 10(2m - 1) \). Now, the result follows from Corollary [8].

Lemma 10. Let \( M_1 \) and \( M_2 \) be two PL 4-manifolds admitting weak semi-simple crystallizations. Then \( M_1 \# M_2 \) admits a weak semi-simple crystallization.

Proof. For \( 1 \leq i \leq 2 \), let \( m_i \) be the rank of the fundamental group of \( M_i \). Since \( M_i \) admits a weak semi-simple crystallization, there exists a crystallization \((\Gamma^i, \gamma^i)\) with color set \( \Delta^i \) such that \( g^i_{(j,j+1,j+2)} = m_i + 1 \) for \( j \in \Delta^i \) (addition in subscripts of \( g^i \) is modulo 5) and \( 1 \leq i \leq 2 \). Let \( \Gamma = \Gamma^1 \# \Gamma^2 \). Then \( \tilde{g}_{(j,j+1,j+2)} = m_1 + m_2 + 1 \) for \( j \in \Delta^i \) (addition in subscripts of \( \tilde{g} \) is modulo 5). Now, by Proposition [11(c)], \( \tilde{\Gamma} = \Gamma^1 \# \Gamma^2 \) is a crystallization of \( M_1 \# M_2 \) for some \( u \in V(\Gamma^1) \) and \( v \in V(\Gamma^2) \). Therefore, \( M_1 \# M_2 \) admits a weak semi-simple crystallization.

Now, by using Theorem [2], we have the following.

Corollary 11. For \( i \leq i \leq 2 \), if \( G(M_i) = 2\chi(M_i) + 5m_i - 4 \), where \( m_i \) is the rank of the fundamental group of \( M_i \), then \( G(M_1 \# M_2) = 2\chi(M_1 \# M_2) + 5(m_1 + m_2) - 4 \).

Proof of Theorem [3] For \( 1 \leq i \leq 2 \), let \( m_i \) be the rank of the fundamental group of \( M_i \). Thus, \( G(M_1 \# M_2) = 2\chi(M_1 \# M_2) + 5(m_1 + m_2) - 4 \). Since \( \chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - 2 \), we have \( G(M_1 \# M_2) = (2\chi(M_1) + 5m_1 - 4) + (2\chi(M_2) + 5m_2 - 4) = G(M_1) + G(M_2) \).

Remark 12. From Definition [1], it is clear that, if a PL 4-manifold \( M \) admits a semi simple crystallization (in particular, simple crystallization) then \( M \) admits a weak semi simple crystallization. Therefore, from Theorem [2], we have \( G(M) = 2\chi(M) + 5 \text{rk}(\pi_1(M)) - 4 \).

From [3], we know the class of PL 4-manifolds admitting a weak semi simple crystallization, contains \( S^4, \mathbb{C}P^2, S^2 \times S^2, \mathbb{R}P^4 \), the orientable and non-orientable \( S^3 \)-bundles over \( S^1 \) and the K3-surface, together with their connected sums possibly with reverse orientation also. Moreover,

- from [3] Figure 3, we know \( G(\mathbb{R}P^2 \times S^2) = 5 = 2 \cdot 2 + 5 \cdot 1 - 4 = 2\chi(\mathbb{R}P^2 \times S^2) + 5 \text{rk}(\pi_1(\mathbb{R}P^2 \times S^2)) - 4 \), and hence \( \mathbb{R}P^2 \times S^2 \) admits a weak semi-simple crystallization.

- from [2] Theorem 1, we know that there exists an orientable (resp., a non-orientable) mapping torus \((S^2 \times S^1)_f \) (with fundamental group \( \mathbb{Z} \times \mathbb{Z} \)) of a map \( f : S^2 \times S^1 \rightarrow S^2 \times S^1 \) such that \( G((S^2 \times S^1)_f) = 6 = 2 \cdot 2 + 5 \cdot 1 - 4 = 2\chi((S^2 \times S^1)_f) + 5 \text{rk}(\pi_1(S^2 \times S^1)_f) - 4 \). Therefore, \((S^2 \times S^1)_f \) admits a weak semi-simple crystallization.
Remark 13. Let $M$ be a closed connected PL 4-manifold admitting a semi-simple crystallization $(\Gamma, \gamma)$. Then, regular genus of $M$ depends on the number of vertices of $\Gamma$ and we have only finitely many for such PL 4-manifolds having same regular genus. Thus, we have introduced the notion weak semi-simple crystallization. The regular genus of closed connected PL 4-manifold admitting a weak semi-simple crystallization does not depend on the number of vertices of its crystallizations. Thus, it may be possible to find infinite number of such PL 4-manifolds having same regular genus.

Finally, we note that all PL 4-manifolds whose regular genus is known, admit a weak semi-simple crystallization. Thus, we believe that all closed connected (simply-connected) PL 4-manifolds admit a weak semi-simple crystallization. Therefore, in virtue of Theorem 3, we believe that the conjecture “additivity of regular genus under connected sum for closed connected PL $d$-manifold” is true for $d = 4$.

Remark 14. Lemma 7 has been used only to prove Theorem 2. There is a minor (printing) error in the statement (and in the second last line of the proof) of Lemma 7 in the published version [1] of this article: the value of $\rho_\varepsilon(\Gamma)$ was $2\chi(M) + 5m - 4 + \sum_{i \in \mathbb{Z}_5} (g_{\{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}\}} - m - 1)$ in the published version [1]. The corrected value of $\rho_\varepsilon(\Gamma)$ is $2\chi(M) + 5m - 4 + \sum_{i \in \mathbb{Z}_5} (g_{\{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}\}} - m - 1)$. However, this minor (printing) error does not effect the proof (and hence the statement) of Theorem 2. The author would like to thank Maria Rita Casali for pointing out this error.

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