ON TWO MODULI SPACES OF SHEAVES SUPPORTED ON QUADRIC SURFACES

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Abstract. We show that the moduli space of semi-stable sheaves on a smooth quadric surface, having dimension 1, multiplicity 4, Euler characteristic 2, and first Chern class $(2, 2)$, is the blow-up at two points of a certain hypersurface in a weighted projective space.

Let $M$ be the moduli space of Gieseker semi-stable sheaves $F$ on $\mathbb{P}^1 \times \mathbb{P}^1$ having Hilbert polynomial $P_F(m) = 4m + 2$, relative to the polarization $\mathcal{O}(1, 1)$, and first Chern class $c_1(F) = (2, 2)$. Let $M_{\mathbb{P}^3}(m^2 + 3m + 2)$ be the moduli space of Gieseker semi-stable sheaves $F$ on $\mathbb{P}^3$ having Hilbert polynomial $P_F(m) = m^2 + 3m + 2$. Such sheaves are supported on quadric surfaces. The purpose of this note is to show that $M_{\mathbb{P}^3}(m^2 + 3m + 2)$ is isomorphic to a certain hypersurface in a weighted projective space (see Proposition 6) and to give an elementary proof of a result of Chung and Moon [3] stating that $M$ is the blow-up of $M_{\mathbb{P}^3}(m^2 + 3m + 2)$ at two regular points.

Let $l, m, n$ be positive integers. Let $V$ be a vector space over $\mathbb{C}$ of dimension $l$. The reductive group $G = (\text{GL}(n, \mathbb{C}) \times \text{GL}(m, \mathbb{C}))/\mathbb{C}^*$ acts by conjugation on the vector space $\text{Hom}(\mathbb{C}^n, \mathbb{C}^m \otimes V)$ of $m \times n$-matrices with entries in $V$. The resulting good quotient $N(V; m, n) = N(l; m, n) = \text{Hom}(\mathbb{C}^n, \mathbb{C}^m \otimes V)^{ss}/G$ is called a Kronecker moduli space. Kronecker moduli spaces arise from the study of moduli spaces of torsion-free sheaves, as in [4]. According to [10, Corollary 3.7] and [3, Lemma 5.2], the map

$$\text{Hom}(2\mathcal{O}_{\mathbb{P}^3}(-1), 2\mathcal{O}_{\mathbb{P}^3})^{ss} \to M_{\mathbb{P}^3}(m^2 + 3m + 2), \quad \langle \varphi \rangle \mapsto \langle \text{Coker}(\varphi) \rangle,$$

is a good quotient modulo $(\text{GL}(2, \mathbb{C}) \times \text{GL}(2, \mathbb{C}))/\mathbb{C}^*$. Thus, the above moduli space is isomorphic to $N(4; 2, 2)$. According to [10, Remark 3.9], $M_{\mathbb{P}^3}(m^2 + 3m + 2)$ is rational; this result was reproved in [3] using the wall-crossing method.

Lemma 1. Assume that $N(l; m, n)$ contains stable points. Then the same is true of $N(k; m, n)$ for all integers $k > l$, and, moreover, $N(k; m, n)$ is birational to $A^{(k-l)m} \times N(l; m, n)$.

Proof. Let $U, V$ be vector spaces over $\mathbb{C}$ of dimension $k - l$, respectively, $l$, and put $W = U \oplus V$. The projection of $W$ onto the second factor induces a $G$-equivariant projection

$$\pi: \text{Hom}(\mathbb{C}^n, \mathbb{C}^m \otimes W) \to \text{Hom}(\mathbb{C}^n, \mathbb{C}^m \otimes V).$$

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From King’s criterion of semi-stability [8] we see that

\[ \pi^{-1}\left( \text{Hom}(\mathbb{C}^n, \mathbb{C}^m \otimes V)^s \right) \subset \text{Hom}(\mathbb{C}^n, \mathbb{C}^m \otimes W)^s. \]

The left-hand-side, denoted by \( E \), is a trivial \( G \)-linearized vector bundle over \( \text{Hom}(\mathbb{C}^n, \mathbb{C}^m \otimes V)^s \) with fiber \( \text{Hom}(\mathbb{C}^n, \mathbb{C}^m \otimes U) \). The geometric quotient map

\[ \text{Hom}(\mathbb{C}^n, \mathbb{C}^m \otimes V)^s \rightarrow N(V; m, n)^s \]

is a principal \( G \)-bundle, so we can apply [7] Theorem 4.2.14 to deduce that \( E \) descends to a vector bundle \( F \) over \( N(V; m, n)^s \). Clearly, \( F \) is the geometric quotient of \( E \) by \( G \), hence \( F \) is isomorphic to an open subset of \( N(W; m, n)^s \). We conclude that \( N(W; m, n) \) is birational to \( \mathbb{A}^{(k-l)mn} \times N(V; m, n) \). \( \square \)

**Proposition 2.**

(i) For \( l \geq 3 \), \( N(l; 2, 2) \) is rational.

(ii) For \( l \geq 3 \) and \( n \geq 1 \), \( N(l; n, n+1) \) is rational.

**Proof.** According to [4, Lemma 25], \( N(3; 2, 2) \) is isomorphic to \( \mathbb{P}^5 \). Identifying \( \mathbb{P}^5 \) with the space of conic curves in \( \mathbb{P}^2 \), the stable points correspond to irreducible conics. Applying Lemma [1] yields (i).

According to [5] Propositions 4.5 and 4.6], the subset of \( N(3; n, n+1) \) of matrices whose maximal minors have no common factor is isomorphic to the subset of \( \text{Hilb}_{n+2}(n+1)/2 \) of schemes that are not contained in any curve of degree \( n-1 \). Thus, \( N(3; n, n+1) \) is birational to \( \text{Hilb}_{n+2}(n+1)/2 \), so it is rational. Moreover, \( N(3; n, n+1) \) consists only of stable points. Applying Lemma [1] yields (ii). \( \square \)

**Proposition 3.** For \( l \geq 3 \) and \( n \geq 1 \), \( N(l; n, n) \) is a rational variety.

**Proof.** The argument is inspired by [10, Remark 3.9]. In view of [4, Section 3], \( N(3; n, n) \) contains stable points. This is due to the fact that we have the inequality \( x < n/n < 1/x \), where \( x \) is the smaller solution to the equation \( x^2-3x+1 = 0. \) Thus, we are in the context of Lemma [1] which asserts that \( N(l; n, n) \) is rational for \( l \geq 3 \) if \( N(3; n, n) \) is rational. We may, therefore, restrict to the case when \( l = 3 \). Let \( V \) be a vector space over \( \mathbb{C} \) with basis \( \{ x, y, z \} \). An element \( \varphi \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^n \otimes V) \) can be written uniquely in the form \( \varphi = \varphi_1 x + \varphi_2 y + \varphi_3 z \), where \( \varphi_1, \varphi_2, \varphi_3 \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^n) \). Let

\[ \text{Hom}(\mathbb{C}^n, \mathbb{C}^n \otimes V)_0 \subset \text{Hom}(\mathbb{C}^n, \mathbb{C}^n \otimes V)^s \]

be the open invariant subset given by the condition that \( \varphi_1 \) be invertible. Let \( X \subset \text{Hom}(\mathbb{C}^n, \mathbb{C}^n \otimes V)_0 \) be the closed subset given by the condition \( \varphi_1 = I \). The group \( \text{PGL}(n, \mathbb{C}) \) acts on \( X \) by conjugation. The composite map

\[ X \rightarrow \text{Hom}(\mathbb{C}^n, \mathbb{C}^n \otimes V)_0 \rightarrow \text{Hom}(\mathbb{C}^n, \mathbb{C}^n \otimes V)_0/G \]

is surjective and its fibers are precisely the \( \text{PGL}(n, \mathbb{C}) \)-orbits. Thus, it factors through a bijective morphism

\[ X/ \text{PGL}(n, \mathbb{C}) \rightarrow \text{Hom}(\mathbb{C}^n, \mathbb{C}^n \otimes V)_0/G. \]

In characteristic zero, bijective morphisms of irreducible varieties are birational. We have reduced to the following problem. Let \( U \) be a complex vector space of dimension 2 and let \( \text{PGL}(n, \mathbb{C}) \) act on \( \text{Hom}(\mathbb{C}^n, \mathbb{C}^n \otimes U) \) by conjugation. Then the resulting good quotient is rational.
Choose a basis \( \{ y, z \} \) of \( U \). An element \( \psi \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^n \otimes U) \) can be uniquely written in the form \( \psi = y\psi_1 + z\psi_2 \), where \( \psi_1, \psi_2 \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^n) \). Let

\[
\text{Hom}(\mathbb{C}^n, \mathbb{C}^n \otimes U)_0 \subset \text{Hom}(\mathbb{C}^n, \mathbb{C}^n \otimes U)
\]

be the open invariant subset given by the conditions that \( \psi \) have trivial stabilizer and that \( \psi_1 \) be invertible and have distinct eigenvalues. Let \( Y \subset \text{Hom}(\mathbb{C}^n, \mathbb{C}^n \otimes U)_0 \) be the closed subset given by the condition that \( \psi_1 \) be a diagonal matrix. Let \( S, T \subset \text{PGL}(n, \mathbb{C}) \) be the image of the canonical embedding of the group of permutations of \( n \) elements, respectively, the subgroup of diagonal matrices. Then \( H = ST \) is a closed subgroup of \( \text{PGL}(n, \mathbb{C}) \) leaving \( Y \) invariant. The composite map

\[
Y \ni \text{Hom}(\mathbb{C}^n, \mathbb{C}^n \otimes U)_0 \rightarrow \text{Hom}(\mathbb{C}^n, \mathbb{C}^n \otimes U)_0 / \text{PGL}(n, \mathbb{C})
\]

is surjective and its fibers are precisely the \( H \)-orbits. Thus, it factors through a birational morphism

\[
Y/H \rightarrow \text{Hom}(\mathbb{C}^n, \mathbb{C}^n \otimes U)_0 / \text{PGL}(n, \mathbb{C})
\]

that must be birational. We have reduced the problem to showing that \( Y/H \) is rational.

Let \( Y_0 \subset Y \) be the open \( H \)-invariant subset given by the condition that all entries of \( \psi_2 \) be non-zero. Concretely, \( Y_0 = D \times E \), where \( D, E \subset \text{Hom}(\mathbb{C}^n, \mathbb{C}^n) \) are the subset of invertible diagonal matrices with distinct entries on the diagonal, respectively, the subset of matrices without zero entries. The normal subgroup \( T \leq H \) acts trivially on \( D \), hence \( (D \times E)/T \) is a trivial bundle over \( D \) with fiber \( E/T \). The induced action of \( S = H/T \) is compatible with the bundle structure. The stabilizer in \( S \) of any \( \psi_1 \in D \) acts trivially on the fiber over \( \psi_1 \), because it is trivial. It follows that \( (D \times E)/T \) descends to a fiber bundle \( F \) over \( D/S \). Clearly, \( F \) is isomorphic to \( (D \times E)/H \), hence \( (D \times E)/H \) is birational to \( D/S \times E/T \). Both \( D/S \) and \( E/T \) are rational, namely \( D/S \) is isomorphic to an open subset of \( S^n(\mathbb{A}^1) \cong \mathbb{A}^n \), while \( E/T \cong (\mathbb{A}^1 \setminus \{0\})^{n^2 - n + 1} \). In conclusion, \( Y/H \) is rational. \( \square \)

Let \( r > 0 \) and \( \chi \) be integers. Let \( \text{M}_{22}(r, \chi) \) denote the moduli space of Gieseker semi-stable sheaves on \( \mathbb{P}^2 \) having Hilbert polynomial \( P(m) = rm + \chi \). It is well known that \( \text{M}_{22}(r, 0) \) is birational to \( \text{N}(3, r, r) \) and, if \( r \) is even, \( \text{M}_{22}(r, r/2) \) is birational to \( \text{N}(6; r/2, r/2) \). We obtain the following.

**Corollary 4.** The moduli spaces \( \text{M}_{22}(r, 0) \) and, if \( r \) is even, \( \text{M}_{22}(r, r/2) \), are rational.

The rationality of \( \text{M}_{22}(3, 0) \) and \( \text{M}_{22}(4, 2) \) is already known from [9].

The maps

\[
det: \text{Hom}(\mathbb{C}^2, \mathbb{C}^2 \otimes V) \rightarrow S^2 V, \quad \text{det}(\varphi) = \varphi_{11}\varphi_{22} - \varphi_{12}\varphi_{21},
\]

and

\[
e: \text{Hom}(\mathbb{C}^2, \mathbb{C}^2 \otimes V) \rightarrow \Lambda^4 V, \quad e(\varphi) = \varphi_{11} \wedge \varphi_{22} \wedge \varphi_{12} \wedge \varphi_{21},
\]

are semi-invariant in the sense that for any \( (g, h) \in G \) and \( \varphi \in \text{Hom}(\mathbb{C}^2, \mathbb{C}^2 \otimes V) \),

\[
det((g, h)\varphi) = \det(g)^{-1} \det(h) \det(\varphi), \quad e((g, h)\varphi) = \det(g)^{-2} \det(h)^2 e(\varphi).
\]

Using King’s criterion of semi-stability [8], it is easy to see that \( \varphi \) is semi-stable if and only if \( \text{det}(\varphi) \neq 0 \) and is stable if and only if \( \text{det}(\varphi) \) is irreducible in \( S^* V \). In the case when \( \text{dim}(V) = 3 \), the isomorphism \( \text{N}(V; 2, 2) \rightarrow \mathbb{P}(S^2 V) \) of [11] is given by \( \langle \varphi \rangle \mapsto (\text{det}(\varphi)) \).
In the sequel we will assume that \( \dim(V) = 4 \) and that \( m = 2, n = 2 \). Choose bases \( \{x, y, z, w\} \) of \( V \) and \( \{v_1, v_2, v_3, v_4\} \) of \( V^* \). Consider the semi-invariant functions
\[
\epsilon, \rho : \text{Hom}(\mathbb{C}^2, \mathbb{C}^2 \otimes V) \to \mathbb{C}, \quad \epsilon(\varphi) = i_{v_1} \wedge v_2 \wedge v_3 \wedge v_4 \epsilon(\varphi), \\
\rho(\varphi) = i_{v_1} \wedge v_2 \wedge v_3 \wedge v_4 (i_{v_1} \det(\varphi) \wedge i_{v_2} \det(\varphi) \wedge i_{v_3} \det(\varphi) \wedge i_{v_4} \det(\varphi)).
\]
Here \( i_v \) denotes the internal product with a vector \( v \in V^* \).

**Proposition 5.** We have the relation \( \epsilon^2 = \rho \).

**Proof.** Let \( \{v'_1, v'_2, v'_3, v'_4\} \) be another basis of \( V^* \) and let \( \nu \in \text{GL}(4, \mathbb{C}) \) be the change-of-basis matrix. With respect to this basis we define the functions \( \nu' \) and \( \epsilon' \) as above. Then \( \epsilon'(\varphi) = \det(\nu) \epsilon(\varphi) \) and \( \rho'(\varphi) = \det(\nu)^2 \rho(\varphi) \), hence \( \epsilon'(\varphi)^2 = \rho'(\varphi) \) if and only if \( \epsilon'(\varphi)^2 = \rho'(\varphi) \). Put \( U = \text{span}\{x, y, z\} \) and let
\[
\pi : \text{Hom}(\mathbb{C}^2, \mathbb{C}^2 \otimes V) \to \text{Hom}(\mathbb{C}^2, \mathbb{C}^2 \otimes U)
\]
be the morphism induced by the projection of \( V = U \oplus Cw \) onto the first factor. It is enough to verify the relation on the Zariski open subset given by the condition that \( \det(\pi(\varphi)) \) be irreducible. Changing, possibly, the basis of \( U \), we may assume that \( \det(\pi(\varphi)) = x^2 - yz \). Since \( \pi(\varphi) \) is stable, and since \( N(U; 2, 2) \) is isomorphic to \( \mathbb{P}(S^2 U) \), we have
\[
\pi(\varphi) \sim \begin{bmatrix} x & y \\ z & x \end{bmatrix}, \quad \text{so we may write} \quad \varphi = \begin{bmatrix} x + aw & y + bw \\ z + cw & x + dw \end{bmatrix}.
\]
We have
\[
\det(\varphi) = x^2 - yz + (a + d)xw - cyw - bz + (ad - bc)w^2,
\]
\[
\epsilon(\varphi) = (d - a)x \wedge y \wedge z \wedge w.
\]
Since we are free to choose the basis of \( V^* \), we choose \( \{v_1, v_2, v_3, v_4\} \) to be the dual of \( \{x, y, z, w\} \). We have
\[
i_{v_1} \det(\varphi) = \frac{\partial}{\partial x} \det(\varphi) = 2x + (a + d)w,
\]
\[
i_{v_2} \det(\varphi) = \frac{\partial}{\partial y} \det(\varphi) = -z - cw,
\]
\[
i_{v_3} \det(\varphi) = \frac{\partial}{\partial z} \det(\varphi) = -y - bw,
\]
\[
i_{v_4} \det(\varphi) = \frac{\partial}{\partial w} \det(\varphi) = (a + d)x - cy - bz + 2(ad - bc)w,
\]
\[
\epsilon(\varphi) = d - a, \quad \rho(\varphi) = \begin{vmatrix} 2 & 0 & 0 & a + d \\ 0 & 0 & -1 & -c \\ 0 & -1 & 0 & -b \\ a + d & -c & -b & 2(ad - bc) \end{vmatrix} = (a - d)^2.
\]
In conclusion, \( \epsilon(\varphi)^2 = (d - a)^2 = \rho(\varphi) \). \hfill \Box

Consider the action of \( \mathbb{C}^* \) on \( S^2 V \oplus \Lambda^4 V \) given by \( t(q, p) = (tq, t^2p) \) and let \( \mathbb{P} \) denote the weighted projective space \( (S^2 V \oplus \Lambda^4 V) \setminus \{0\})/\mathbb{C}^* \). Consider the map
\[
\eta : N(V; 2, 2) \to \mathbb{P}, \quad \eta(\varphi) = \langle \det(\varphi), \epsilon(\varphi) \rangle.
\]
Choose coordinates on \( \mathbb{P} \) given by the choice of basis \( \{x, y, z\} \) of \( V \). In view of Proposition 5, the image of \( \eta \) is contained in the hypersurface \( H \subset \mathbb{P} \) given by the equation \( \text{res}(q) = p^2 \), where \( \text{res}(q) \) denotes the resultant of the quadratic form \( q \).
Proposition 6. Assume that $\dim(V) = 4$. Then the map $\eta: N(V; 2, 2) \to H$ is an isomorphism.

Proof. The singular points of the cone $\hat{H} \subset S^2 V \oplus \Lambda^4 V$ over $H$ are of the form $(q, 0)$, where $q \in S^2 V$ is a singular point of the vanishing locus of the resultant. It follows that $\hat{H}$ is regular in codimension 1. From Serre’s criterion of normality we deduce that $H$ is normal (condition S2 is satisfied because $\hat{H}$ is a hypersurface in a smooth variety). Normality is inherited by a good quotient, hence $H = (\hat{H} \setminus \{0\})/\mathbb{C}^*$ is normal, too. In view of the Main Theorem of Zariski, it is enough to show that $\eta$ is bijective. Since $N(V; 2, 2)$ is complete, and since $N(V; 2, 2)$ and $H$ are irreducible of the same dimension, it is enough to show that $\eta$ is injective.

Assume that $\eta(\langle \varphi_1 \rangle) = \eta(\langle \varphi_2 \rangle)$. Varying $\varphi_1$ and $\varphi_2$ in their respective orbits, we may assume that $\det(\varphi_1) = \det(\varphi_2)$ and $e(\varphi_1) = e(\varphi_2)$. If $\det(\varphi_1)$ is reducible, say $\det(\varphi_1) = uu'$ for some $u, u' \in V$, then it is easy to see that

$$\varphi_1 \sim \begin{bmatrix} u & u' \\ 0 & u' \end{bmatrix}, \quad \varphi_2 \sim \begin{bmatrix} u & u_2 \\ 0 & u' \end{bmatrix}$$

for some $u_1, u_2 \in V$. But then $\langle \varphi_1 \rangle = \langle \varphi_2 \rangle = \langle \text{diag}(u, u') \rangle$. Assume now that $\det(\varphi_1)$ is irreducible. There exists a vector $w \in V$ and a subspace $U \subset V$ such that $V = U \oplus Cw$ and $\det(\pi(\varphi_1))$ is irreducible (notations as at Proposition 5). As mentioned at Proposition 5 we may choose a basis $\{x, y, z\}$ of $U$ such that $\det(\pi(\varphi_1)) = x^2 - yz$, forcing

$$\pi(\varphi_1) \sim \pi(\varphi_2) \sim \begin{bmatrix} x & y \\ z & x \end{bmatrix}.$$

Thus, we may write

$$\varphi_1 = \begin{bmatrix} x + a_1 w & y + b_1 w \\ z + c_1 w & x + d_1 w \end{bmatrix}, \quad \varphi_2 = \begin{bmatrix} x + a_2 w & y + b_2 w \\ z + c_2 w & x + d_2 w \end{bmatrix}.$$

The relation $\det(\varphi_1) = \det(\varphi_2)$ yields the relations $b_1 = b_2, c_1 = c_2, a_1 + d_1 = a_2 + d_2$. The relation $e(\varphi_1) = e(\varphi_2)$ yields the relation $a_1 - d_1 = a_2 - d_2$. We conclude that $\varphi_1 = \varphi_2$, hence $\langle \varphi_1 \rangle = \langle \varphi_2 \rangle$. \hfill \Box

Remark 7. It was already known to Le Potier [10, Remark 3.8] that the map

$$\det: N(V; 2, 2) \to \mathbb{P}(S^2 V)$$

is a double cover branched over the locus of singular quadratic surfaces.

In the sequel, we will use the abbreviations $\mathcal{O}(r, s) = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(r, s), \omega = \omega_{\mathbb{P}^1 \times \mathbb{P}^1}$, and $\mathcal{F}^\omega = \mathcal{E}xt^1_{\mathcal{O}}(\mathcal{F}, \omega)$ for a sheaf $\mathcal{F}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ of dimension 1. We quote below [3 Proposition 3.8].

Proposition 8. The sheaves $\mathcal{F}$ giving points in $\mathbf{M}$ are precisely the sheaves having one of the following three types of resolution:

1. $0 \to 2\mathcal{O}(-1, -1) \xrightarrow{\epsilon} 2\mathcal{O} \to \mathcal{F} \to 0$,
2. $0 \to \mathcal{O}(-2, -1) \to \mathcal{O}(0, 1) \to \mathcal{F} \to 0$,
3. $0 \to \mathcal{O}(-1, -2) \to \mathcal{O}(1, 0) \to \mathcal{F} \to 0$. 

Proof. We have

$$\hat{H} \subset S^2 V \oplus \Lambda^4 V,$$
This proposition was proved in [3] by the wall-crossing method, however, it was also nearly obtained in [1]. At [1] Lemma 20 it is mistakenly claimed that all sheaves in \( M \) have resolution (1). At a closer inspection, the argument of [1] Lemma 20 shows that the sheaves in \( M \) satisfying the conditions \( H^0(\mathcal{F}^\delta(1,0)) = 0 \) and \( H^0(\mathcal{F}^\sigma(0,1)) = 0 \) are precisely the sheaves given by resolution (1). Indeed, the exact sequence (50) in [1] reads

\[
(4) \quad 0 \rightarrow \mathcal{H} \rightarrow 2\mathcal{O} \rightarrow \mathcal{F} \rightarrow 0,
\]

where \( \mathcal{H} \) is a locally free sheaf of rank 2 and determinant \( \omega \). Dualizing this sequence, we get the exact sequence

\[
(5) \quad 0 \rightarrow 2\mathcal{O}(-2,-2) \rightarrow \mathcal{H}^\sigma \simeq \mathcal{H}^* \otimes \omega \simeq \mathcal{H} \otimes \det(\mathcal{H})^* \otimes \omega \simeq \mathcal{H} \rightarrow \mathcal{F}^\sigma \rightarrow 0.
\]

From this we get the relations

\[
h^1(\mathcal{H}(1,0)) = h^0(\mathcal{F}^\sigma(0,1)) \quad \text{and} \quad h^1(\mathcal{H}(0,1)) = h^0(\mathcal{F}^\sigma(0,1)).
\]

The vanishing of \( H^1(\mathcal{H}(1,0)) \) and \( H^1(\mathcal{H}(0,1)) \) implies that \( \mathcal{H} \simeq 2\mathcal{O}(-1,-1) \), in which case (1) yields resolution (1).

According to [1] Theorem 13, if \( \mathcal{F} \) gives a point in \( M \), then \( \mathcal{F}^\sigma(0,1) \) and \( \mathcal{F}^\sigma(1,0) \) give points in the moduli space \( M' \) of semi-stable sheaves on \( \mathbb{P}^1 \times \mathbb{P}^1 \) having Hilbert polynomial \( P(m) = 4m \) and first Chern class \( c_1 = (2,2) \). We claim that the sheaves \( \mathcal{E} \) giving points in \( M' \) and satisfying the condition \( H^0(\mathcal{E}) \neq 0 \) are precisely the structure sheaves of curves \( E \subset \mathbb{P}^1 \times \mathbb{P}^1 \) of type \( (2,2) \). By the argument of [1] Lemma 9, \( \mathcal{O}_E \) gives a stable point in \( M' \). Conversely, if \( \mathcal{E} \) gives a point in \( M' \) and \( H^0(\mathcal{E}) \neq 0 \), then, by the argument of [3] Proposition 2.1.3, there is an injective morphism \( \mathcal{O}_C \rightarrow \mathcal{E} \) for a curve \( C \subset \mathbb{P}^1 \times \mathbb{P}^1 \). If \( C \) did not have type \( (2,2) \), then the semi-stability of \( \mathcal{E} \) would get contradicted. Thus, \( C \) has type \( (2,2) \) and, comparing Hilbert polynomials, we see that \( \mathcal{O}_C \simeq \mathcal{E} \). In conclusion, if \( H^0(\mathcal{F}^\sigma(0,1)) \neq 0 \), then \( \mathcal{F} \simeq \mathcal{O}_E(0,-1)^\sigma \simeq \mathcal{O}_E(0,1) \), hence \( \mathcal{F} \) has resolution (2). If \( H^0(\mathcal{F}^\sigma(1,0)) \neq 0 \), then \( \mathcal{F} \simeq \mathcal{O}_E(-1,0)^\sigma \simeq \mathcal{O}_E(1,0) \), hence \( \mathcal{F} \) has resolution (3).

We denote by \( M_0, M_1, M_2 \subset M \) the subsets of sheaves given by resolution (1), (2), respectively, (3). Clearly, \( M_0 \) is open and \( M_1, M_2 \) are divisors isomorphic to \( \mathbb{P}^2 \). Let \( \text{Hom}(2\mathcal{O}(-1,-1), 2\mathcal{O})_0 \) denote the subset of injective morphisms.

**Corollary 9.** The canonical map from below is a good quotient modulo \( G \):

\[
\gamma: \text{Hom}(2\mathcal{O}(-1,-1), 2\mathcal{O})_0 \rightarrow M_0, \quad \gamma(\varphi) = (\text{Coker}(\varphi)).
\]

**Proof.** According to [1] Lemma 1, for any coherent sheaf \( \mathcal{F} \) on \( \mathbb{P}^1 \times \mathbb{P}^1 \) there is a spectral sequence converging to \( \mathcal{F} \) in degree zero and to 0 in degrees different from zero, similar to the Beilinson spectral sequence. Its first level \( E^1_{ij} \) is given by

\[
E^{ij}_1 = 0 \quad \text{if } i > 0 \text{ or } i < -2,
\]

\[
E^{0j}_1 = H^j(\mathcal{F}) \otimes \mathcal{O}, \quad E^{-2j}_1 = H^j(\mathcal{F}(-1,-1)) \otimes \mathcal{O}(-1,-1),
\]

and by the exact sequences

\[
H^j(\mathcal{F}(0,-1)) \otimes \mathcal{O}(0,-1) \rightarrow E^{-1,j}_1 \rightarrow H^j(\mathcal{F}(-1,0)) \otimes \mathcal{O}(-1,0).
\]

If \( \mathcal{F} \) gives a point in \( M_0 \), then

\[
H^0(\mathcal{F}) \simeq \mathbb{C}^2, \quad H^1(\mathcal{F}) = 0, \quad H^0(\mathcal{F}(-1,-1)) = 0, \quad H^1(\mathcal{F}(-1,-1)) \simeq \mathbb{C}^2,
\]

\[
H^0(\mathcal{F}(0,-1)) = 0, \quad H^1(\mathcal{F}(0,-1)) = 0, \quad H^0(\mathcal{F}(-1,0)) = 0, \quad H^1(\mathcal{F}(-1,0)) = 0.
\]
Thus, $E_1$ has only two non-zero terms: $E_{1,-2,1} = 2\mathcal{O}(-1,-1)$ and $E_{1,0,0} = 2\mathcal{O}$. The relevant part of $E_2$ is represented in the following table:

| 2\mathcal{O}(-1,-1) | 0 | 0 |
|----------------------|---|---|
| 0                    | 0 | 2\mathcal{O} |

The sequence degenerates at $E_3$, hence $\varphi$ is injective and $\text{Coker}(\varphi) \simeq \mathcal{F}$. This shows that resolution $\mathcal{F}$ can be obtained from the Beilinson spectral sequence of $\mathcal{F}$. Arguing as at [6, Theorem 3.1.6], we can see that resolution $\mathcal{F}$ can be obtained for local flat families of sheaves in $M_0$, hence $\gamma$ is a categorical quotient. By the uniqueness of the categorical quotient, we deduce that $\gamma$ is a good quotient map. □

We fix vector spaces $V_1$ and $V_2$ over $\mathbb{C}$ of dimension 2 and we make the identifications

$$
\mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{P}(V_1) \times \mathbb{P}(V_2), \quad H^0(\mathcal{O}(r,s)) = S^r V_1^* \otimes S^s V_2^*, \quad V = V_1^* \otimes V_2^*.
$$

Let $W \subset \text{Hom}(\mathcal{O}(-1,-1) \oplus \mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1), \mathcal{O}(-1,0) \oplus \mathcal{O}(0,1) \oplus 2\mathcal{O})$ be the open subset given by the condition that $\psi_{12}$ is injective and $\text{Coker}(\psi)$ is Gieseker semi-stable. We represent $\psi$ by a matrix

$$
\psi = \begin{bmatrix}
\psi_{11} & \psi_{12} \\
\psi_{21} & \psi_{22}
\end{bmatrix} = \begin{bmatrix}
1 \otimes u_{11} & 1 \otimes v_{12} & a_1 & 0 \\
u_{11} \otimes 1 & v_{12} \otimes 1 & 0 & a_2 \\
f_{11} & f_{12} & u_{21} \otimes 1 & v_{22} \otimes 1 \\
f_{21} & f_{22} & v_{22} \otimes 1 & v_{22} \otimes 1
\end{bmatrix},
$$

where $a_1, a_2 \in \mathbb{C}$, $u_{ij}, v_{ij} \in V_j^*$, $f_{ij} \in V$. The algebraic group

$$
G = (\text{Aut}(\mathcal{O}(-1,-1) \oplus \mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1)) \times \text{Aut}(\mathcal{O}(-1,0) \oplus \mathcal{O}(0,1) \oplus 2\mathcal{O})) / \mathbb{C}^*
$$

acts on $W$ by conjugation. We represent elements of $G$ by pairs $(g, h)$, where

$$
g = \begin{bmatrix}
g_{11} & 0 \\
g_{21} & g_{22}
\end{bmatrix}, \quad h = \begin{bmatrix}
h_{11} & 0 \\
h_{21} & h_{22}
\end{bmatrix},
$$

$g_{11} \in \text{Aut}(\mathcal{O}(-1,-1))$, $h_{22} \in \text{Aut}(2\mathcal{O})$, etc.

**Proposition 10.** The canonical map $\theta: W \rightarrow M$, $\theta(\psi) = \langle \text{Coker}(\psi) \rangle$ is a good quotient modulo $G$.

**Proof.** Let $W_0 \subset W$ be the open subset given by the condition that $\psi_{12}$ is invertible. Concretely, $W_0$ is the set of morphisms $\psi$ such that $\psi_{12}$ is invertible and $\alpha(\psi) = \psi_{21} - \psi_{22}^{-1} \psi_{11}$ is injective. In view of Proposition [8], $M_0 = \theta(W_0)$. The restricted map $\theta_0: W_0 \rightarrow M_0$ is the composition

$$
W_0 \twoheadrightarrow \text{Hom}(\mathcal{O}(-1,-1), 2\mathcal{O})_0 \xrightarrow{\gamma} M_0,
$$

where $\gamma$ is the good quotient map from Corollary [8]. Let $G_0 \subset G$ be the closed normal subgroup given by the conditions $g_{11} = cI$, $h_{22} = cI$, $c \in \mathbb{C}^*$. We have the relation $\alpha(h_2^c g_1^{-1}) = h_{22} h_1 \alpha(\psi) g_{11}^{-1}$, hence $\alpha$ is constant on the orbits of $G_0$. Since any $\psi \in W_0$ is equivalent to

$$
\begin{bmatrix}
0 & I \\
\alpha(\psi) & 0
\end{bmatrix},
$$

it follows that the fibers of $\alpha$ are precisely the $G_0$-orbits, and that $\alpha$ has a section. We deduce that $\alpha$ is a geometric quotient modulo $G_0$. Since $\gamma$ is a good quotient
modulo $G/G_0$, we conclude that $\theta_0$ is a good quotient modulo $G$. Let $M_0^2 \subset M_0$ be the subset of stable points. Since $\gamma^{-1}(M_0^2) \to M_0^2$ is a geometric quotient modulo $G/G_0$, we deduce that $\theta^{-1}(M_0^2) \to M_0^2$ is a geometric quotient modulo $G$.

Assume now that $\psi \in W \setminus W_0$. Denote $F = \text{Coker}(\psi)$. Then $\psi_{12} \neq 0$, otherwise $\text{Coker}(\psi_{22})$ would be a destabilizing subsheaf of $F$. Thus, $W \setminus W_0$ is the disjoint union of two subsets $W_1$ and $W_2$. The former is given by the relations $a_1 \neq 0$, $a_2 = 0$; the latter is given by the relations $a_1 = 0$, $a_2 \neq 0$. Assume that $\psi \in W_1$. Then $u_{11}, v_{11}$ are linearly independent, otherwise $F$ would have a destabilizing quotient sheaf of slope zero. Likewise, $u_{22}, v_{22}$ are linearly independent, otherwise $F$ would have a destabilizing subsheaf of slope 1. Consider the morphism

$$\xi \in \text{Hom}(2O(-1, -1) \oplus O(0, -1), O(0, -1) \oplus 2O),$$

$$\xi = \begin{bmatrix} u_{11} \otimes 1 & v_{11} \otimes 1 & 0 \\ f_{11} - a_1^{-1}u_{21} \otimes u_{12} & f_{12} - a_1^{-1}u_{21} \otimes v_{12} & 1 \otimes u_{22} \\ f_{21} - a_1^{-1}v_{21} \otimes u_{12} & f_{22} - a_1^{-1}v_{21} \otimes v_{12} & 1 \otimes v_{22} \end{bmatrix}.$$ 

Clearly, $F \simeq \text{Coker}(\xi)$. Applying the snake lemma to the exact diagram

we obtain resolution $[2]$. This shows that $\theta(W_1) \subset M_1$. It is now easy to see that the restricted map $W_1 \to M_1$ is surjective and that its fibers are precisely the $G$-orbits. By symmetry, the same is true of the restricted map $W_2 \to M_2$.

Let $M^s \subset M$ be the open subset of stable points and $W^s = \theta^{-1}(M^s)$. We have proved above that the fibers of the restricted map $\theta^s: W^s \to M^s$ are precisely the $G$-orbits. Since $M^s$ is normal (being smooth), we can apply [12, Theorem 4.2] to deduce that $\theta^s$ is a geometric quotient modulo $G$. Finally, since $M = M_0 \cup M^s$, we deduce that $\theta$ is a good quotient map.

Choose bases $\{u_1, v_1\}$ of $V_1^*$ and $\{u_2, v_2\}$ of $V_2^*$. Then $x = u_1 \otimes u_2$, $y = v_1 \otimes u_2$, $z = u_1 \otimes v_2$, $w = v_1 \otimes v_2$ form a basis of $V$. An easy calculation shows that the set of injective morphisms

$$\text{Hom}(2O(-1, -1), 2O) \subset \text{Hom}(\mathbb{C}^2, \mathbb{C}^2 \otimes V)$$

is the subset of matrices whose determinant is not a multiple of $xw - yz$. Thus,

$$\text{Hom}(2O(-1, -1), 2O)\!/\!G \simeq N(V; 2, 2) \setminus \det^{-1}\{(xw - yz)\}.$$
According to Remark 1, \( \det^{-1}\{ (xw - yz) \} \) consists of two points \( \nu_1 \) and \( \nu_2 \), where \( \epsilon(\nu_1) = 1, \epsilon(\nu_2) = -1 \). We saw at Corollary 9 that \( \gamma \) induces an isomorphism
\[
\text{Hom}(2\mathcal{O}(-1, -1), 2\mathcal{O}_0) / G \rightarrow M_0.
\]
The inverse of this isomorphism is denoted by
\[
\beta_0: M_0 \rightarrow N(V; 2, 2) \setminus \{ \nu_1, \nu_2 \}.
\]
It is natural to ask whether \( M \) is the blow-up of \( N(V; 2, 2) \) at \( \nu_1 \) and \( \nu_2 \). This is, indeed, one of the main results in [3], where a blowing-down map \( \beta: M \rightarrow N(V; 2, 2) \) is constructed via Fourier-Mukai transforms of sheaves, in view of the identification of \( N(V; 2, 2) \) with \( M_\mathfrak{p}(m^2 + 3m + 2) \). We give below an alternate construction.

**Proposition 11.** The map \( \beta_0 \) extends to a blowing-down map \( \beta: M \rightarrow N(V; 2, 2) \) with exceptional divisor \( M_1 \cup M_2 \) and blowing-up locus \( \{ \nu_1, \nu_2 \} \).

**Proof.** Recall that on \( M_0 = W_0 / G \), \( \beta_0 \) is induced by the map sending \( \psi \) to
\[
\psi_{21} - a_1^{-1} \left[ \begin{array}{c} u_{21} \otimes 1 \\ v_{21} \otimes 1 \end{array} \right] \begin{array}{c} 1 \otimes u_{12} \\ 1 \otimes v_{12} \end{array} - a_2^{-1} \left[ \begin{array}{c} 1 \otimes u_{22} \\ 1 \otimes v_{22} \end{array} \right] \left[ \begin{array}{c} u_{11} \otimes 1 \\ v_{11} \otimes 1 \end{array} \right].
\]
Equivalently, \( \beta_0 \) is induced by the map sending \( \psi \) to
\[
a_2 \psi_{21} - a_1^{-1} a_2 \left[ \begin{array}{c} u_{21} \otimes 1 \\ v_{21} \otimes 1 \end{array} \right] \begin{array}{c} 1 \otimes u_{12} \\ 1 \otimes v_{12} \end{array} - \left[ \begin{array}{c} 1 \otimes u_{22} \\ 1 \otimes v_{22} \end{array} \right] \left[ \begin{array}{c} u_{11} \otimes 1 \\ v_{11} \otimes 1 \end{array} \right]
\]
which is defined on \( W_0 \cup W_1 \). Clearly, this map factors through a morphism \( M_0 \cup M_1 \rightarrow N(V; 2, 2) \), which maps \( M_1 \) to the class of the matrix
\[
\begin{array}{c} 1 \otimes u_2 \\ 1 \otimes v_2 \end{array} \left[ \begin{array}{c} u_1 \otimes 1 \\ v_1 \otimes 1 \end{array} \right] = \begin{array}{c} x \\ z \\ w \end{array},
\]
that is, to \( \nu_1 \). Analogously, \( \beta_0 \) extends to a morphism defined on \( M_0 \cup M_2 \), which maps \( M_2 \) to the class of the matrix
\[
\begin{array}{c} u_1 \otimes 1 \\ v_1 \otimes 1 \end{array} \left[ \begin{array}{c} 1 \otimes u_2 \\ 1 \otimes v_2 \end{array} \right] = \begin{array}{c} x \\ z \\ w \end{array},
\]
that is, to \( \nu_2 \). The two morphisms constructed thus far glue to a morphism \( \beta: M \rightarrow N(V; 2, 2) \). Since \( \nu_1 \) and \( \nu_2 \) are smooth points, \( \beta \) is a blow-down. \( \square \)

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