Relativistic trajectory variables in $1+1$ dimensional Ruijsenaars-Schneider type models

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Abstract
A general algorithm to construct particle trajectories in $1+1$ dimensional canonical relativistic models is presented. The method is a generalization of the construction used in Ruijsenaars-Schneider models and provides a simple proof of the fact that the latter satisfies the world-line conditions granting proper physical Poincaré invariance. The 2-particle case for the rational Ruijsenaars-Schneider model is worked out explicitly. It is shown that the particle coordinates do not Poisson commute, as required by the no-interaction theorem of Currie, Jordan and Sudarshan.
1 Introduction

Relativistic particle mechanics, with instantaneous action-at-a-distance interaction naively contradicts relativistic causality and is thus counter-intuitive. Despite of this apparent difficulty, a consistent theory exists, mainly in the classical domain, but also quantum-mechanically. Nevertheless, relativistic particle physics remains almost completely synonymous with Relativistic Quantum Field Theory. Abandoning the particle alternative is partially due to the famous no-go theorem of Currie, Jordan and Sudarshan [1]. This theorem states that requiring particle positions to satisfy canonical commutation relations excludes the presence of any non-trivial interactions. If we give up this requirement, we can formulate relativistic point mechanics. There are three, essentially equivalent approaches. The first one is called Predictive Relativistic Mechanics (PRM) [2] and is formulated by writing the equations of motion in Newtonian form

$$\ddot{x}_a^i = \mu_a^i(\{x\}, \{\dot{x}\}), \quad (1.1)$$

where $i = 1, 2, 3$ are space indices, $a = 1, 2, \ldots, N$ are particle indices (there are $N$ particles) and the accelerations $\mu_a^i$ occurring in the Newton-equations (1.1) depend on the instantaneous positions $x_a^i$ and velocities $\dot{x}_a^i$ of the particles. Relativistic invariance implies that the accelerations have to satisfy a set of quadratic, partial differential equations, the Currie-Hill (CH) equations [3]. The Currie-Hill equations form a non-linear set of partial differential equations for the instantaneous accelerations $\mu_a^i$. They come from the requirement of relativistic invariance and ensure that if we transform the Newton equations to a Lorentz-boosted new coordinate system, the particle trajectories satisfy Newton equations which are instantaneous action-at-a-distance equations in the boosted coordinate system and moreover the equations in the new system are of the same form as (1.1).

One of the difficulties of the relativistic particle dynamics is that unfortunately not many explicit solutions of the Currie-Hill equations are known, either in $3 + 1$ space-time dimensions or in $1 + 1$ dimensions. Although the most general 2-particle solution has been found in $1 + 1$ dimensions [4], but it is given in a very implicit form. There exist approximate solutions in the $1/c^2$ expansion but for these the study of further questions like the global structure of the phase space, symplectic structure, etc. are difficult. In [5] a completely explicit solution of the Currie-Hill equations in $1 + 1$ dimensional Minkowski space-time was presented. This solution can be written in terms of elementary functions and provides an example in which further questions of the relativistic action-at-a-distance approach (conserved quantities, canonical structure, etc.) can be studied transparently.

An alternative approach to relativistic mechanics [6] can be called canonical. Here a phase space equipped with a symplectic structure is assumed from the beginning, together with the set of 10 generators of the Poincaré group. In this approach
consistent relativistic dynamics can be constructed if we can find the particle positions \(x^i_a\), as functions on the phase space and satisfying the Poisson-bracket relations

\[
\{P_i, x^j_a\} = -\delta^i_j, \quad \{J^i, x^j_a\} = \epsilon^{ijk} x^k_a, \quad \{K_i, x^j_a\} = -\frac{1}{c^2} x^i_a \dot{x}^j_a \tag{1.2}
\]

called the world-line conditions (WLC). Here \(P_i, J^i, K_i\), respectively are the momentum, angular momentum, and Lorentz boost generators, respectively, of the Poincaré group. If we are able to find such particle coordinates, we can calculate the Poisson brackets

\[
\{x^i_a, x^j_b\} \tag{1.3}
\]

which must not vanish, otherwise, due to the no-go theorem, there is no interaction. The advantage of the canonical approach is that only the trajectory variables \(x^i_a\) have to be constructed, the 10 integrals of the Poincaré group are there by construction from the beginning. Provided the set \(\{x^i_a\}, \{\dot{x}^i_a\}\) are good coordinates on the phase space (at least locally), the Newton-equations \eqref{1.1} can be calculated and must satisfy the Currie-Hill equations. There is also a third, essentially equivalent approach \cite{7} which is explicitly covariant. This is not discussed here.

Because of the lack of explicit exact solutions in 3+1 dimensions, it is important to study 1+1 dimensional examples, the most famous of which are the exactly solvable Ruijsenaars-Schneider (RS) models \cite{8}, the relativistic generalizations of the Calogero-Moser systems. The RS approach is canonical, and the RS systems are not only relativistic, but also integrable for any \(N\). The original motivation of constructing the RS models was their relativistic invariance but later the RS literature was almost entirely concerned with their integrable aspects (there are many applications of the RS models in various areas of physics). Here trajectory variables satisfying \eqref{1.2} have been constructed but it is not clear if they are good coordinates on the entire phase space and their explicit form in terms of the canonical variables and their commutation relations \eqref{1.3} are not known. There are also further open questions even in the case of RS models (the question of physical non-relativistic limit, for instance) and for this reason it is important to study these and related models further.

In this paper we present a general algorithm to construct trajectory variables satisfying the 1+1 dimensional world-line conditions in canonical relativistic models. The algorithm is a generalization of that used in RS models but the simple proof of the fact that the WLC are satisfied is also applicable to the RS models themselves. The case of two particles is discussed in detail and in the example of the rational RS model the trajectory variables are given explicitly. This allows us to calculate their Poisson brackets and show that this is nonvanishing as required by the no-interaction theorem. The algebraic complexity of the calculation even in this very simple special case illustrates the difficulties of relativistic particle mechanics mentioned above.
2 Canonical relativistic mechanics in 1 + 1 dimensional Minkowski space-time

The starting point of canonical relativistic mechanics in 1 + 1 dimensions is a phase space equipped with a symplectic (Poisson) structure and the set of 3 generators \( \{ \mathcal{H}, \mathcal{P}, \mathcal{K} \} \) of the 1 + 1 dimensional Poincaré group satisfying

\[
\{ \mathcal{H}, \mathcal{P} \} = 0, \quad \{ \mathcal{H}, \mathcal{K} \} = \mathcal{P}, \quad \{ \mathcal{P}, \mathcal{K} \} = \frac{1}{c^2} \mathcal{H}.
\]  

We will associate differential operators \( \hat{A} \) to functions \( A \) on the phase space in the usual way. Acting on any function \( \mathcal{F} \) we have

\[
\hat{A}\mathcal{F} = \{ A, \mathcal{F} \}
\]

and in particular we will use the notation

\[
\hat{\mathcal{H}}\mathcal{F} = \dot{\mathcal{F}}, \quad \hat{\mathcal{P}}\mathcal{F} = \mathcal{F}'.
\]

The commutator of two such operators satisfies

\[
[\hat{A}, \hat{B}] = \{ A, B \}.
\]

For later use we note that a consequence of the Poincaré commutation relations is the operator identity

\[
\left( \hat{\mathcal{K}} + \frac{x}{c^2} \hat{\mathcal{H}} \right) e^{x\hat{\mathcal{P}}} = e^{x\hat{\mathcal{P}}} \hat{\mathcal{K}}.
\]

We will assume that the canonical coordinates \( q_a, \theta_a \) satisfying

\[
\{ q_a, \theta_b \} = \delta_{ab}, \quad a, b = 1, 2, \ldots, N
\]

are good coordinates on our phase space and phase space functions will be given as \( \mathcal{F}(q, \theta) \).

The dynamics on the phase space is given by the Hamiltonian \( \mathcal{H} \) and we introduce the solution of the equations of motion \( Q_a(t; q, \theta), T_b(t; q, \theta) \) satisfying

\[
\frac{\partial}{\partial t} Q_a(t; q, \theta) = \dot{q}_a(Q, T), \quad \frac{\partial}{\partial t} T_b(t; q, \theta) = \dot{\theta}_b(Q, T)
\]

and the initial conditions

\[
Q_a(0; q, \theta) = q_a, \quad T_b(0; q, \theta) = \theta_b.
\]

The time evolution of any function \( \mathcal{F} \) is now solved by

\[
\left( e^{i\mathcal{H}t} \right) (q, \theta) = \mathcal{F}(Q, T).
\]
Quite analogously we introduce the space “evolution” generated by the momentum $P$

$$
(e^{x \hat{P}} \mathcal{F})(q, \theta) = \mathcal{F}(\bar{Q}, \bar{T}),
$$

(2.10)

where the space evolution is the solution of

$$
\frac{\partial}{\partial x} \bar{Q}_a(x; q, \theta) = q'_a(\bar{Q}, \bar{T}), \quad \frac{\partial}{\partial x} T_b(x; q, \theta) = \theta'_b(\bar{Q}, \bar{T})
$$

(2.11)

and the initial conditions

$$
\bar{Q}_a(0; q, \theta) = q_a, \quad \bar{T}_b(0; q, \theta) = \theta_b.
$$

(2.12)

To specify particle dynamics we have to find the coordinates (trajectory variables) $x_a(q, \theta)$ for each particle. Their physical meaning is the position of the $a$th particle at $t = 0$ and the full trajectory is given by the evolution

$$
x_a(t; q, \theta) = x_a(Q, T).
$$

(2.13)

We can calculate the velocity and acceleration of the particles:

$$
v_a(q, \theta) = \dot{x}_a(q, \theta), \quad \mu_a(q, \theta) = \dot{v}_a(q, \theta) = \ddot{x}_a(q, \theta).
$$

(2.14)

The proper transformation property of the trajectory variables is obtained from the requirement that by applying a Poincaré transformation generated on the phase space by the generators in (2.1) the full space-time trajectories $x_a(t)$ have to transform by the standard linear Lorentz transformation formulas. These are called the world-line conditions and in the 1 + 1 dimensional case are

$$
x'_a = -1, \quad \hat{K} x_a = -\frac{1}{c^2} x_a v_a, \quad a = 1, 2, \ldots, N.
$$

(2.15)

(No summation over the particle index $a$ is implied.)

To construct a solution of (2.15) we have to associate to each particle a Lorentz-invariant (boost-invariant) quantity $\rho_a$:

$$
\rho_a : \quad \hat{K} \rho_a = 0
$$

(2.16)

and find its space evolution

$$
R_a(x) = e^{x \hat{P}} \rho_a, \quad R_a(x; q, \theta) = \rho_a(\bar{Q}, \bar{T}).
$$

(2.17)

Now the trajectory is defined as the solution of

$$
R_a(x_a) = 0.
$$

(2.18)
This construction works if the solution of (2.18) exists and is unique. If this is the case, we can take the derivative of it with respect to any differential operator \( \hat{L} \):

\[
\left( \hat{L} R_a \right) (x_a) + R'_a(x_a) \hat{L} x_a = 0. \tag{2.19}
\]

Choosing \( \mathcal{L} = \mathcal{P} \) we immediately get from here

\[
\mathcal{P} x_a = -1, \tag{2.20}
\]

i.e. the first world-line condition. Further we get

\[
\mathcal{L} = \mathcal{H} \quad \hat{R}_a + R'_a \dot{x}_a = 0, \\
\mathcal{L} = \mathcal{K} \quad \hat{K} R_a + R'_a \hat{K} x_a = 0, \tag{2.21}
\]

where the argument of \( R_a \) in the above formulas is \( x = x_a \). If we now apply (2.5) to \( \rho_a \) and take it also at \( x = x_a \) we see that the right hand side vanishes and we find

\[
\hat{K} R_a + \frac{x_a}{c^2} \hat{R}_a = 0. \tag{2.22}
\]

Combining the last three equalities we get

\[
R'_a \hat{K} x_a = \frac{x_a}{c^2} \hat{R}_a = -\frac{x_a}{c^2} R'_a \dot{x}_a \tag{2.23}
\]

and simplifying with the factor \( R'_a \) finally gives the second world-line condition

\[
\hat{K} x_a = -\frac{x_a}{c^2} \dot{x}_a. \tag{2.24}
\]

3 The Ruijsenaars-Schneider Ansatz

Ruijsenaars and Schneider found a clever Ansatz [8] for satisfying (2.1):

\[
\mathcal{H} = mc^2 \sum_a \cosh \theta_a V_a, \quad \mathcal{P} = mc \sum_a \sinh \theta_a V_a, \quad \mathcal{K} = -\frac{1}{c} \sum_a q_a, \tag{3.1}
\]

where \( m \) is the mass of the particles and

\[
V_a = \prod_{b \neq a} f(q_a - q_b) \tag{3.2}
\]

is parametrized in terms of a positive, even function of one variable, \( f(q) \). From the relations in (2.1) the only nontrivial one is

\[
\{ \mathcal{H}, \mathcal{P} \} = \frac{m^2 c^3}{2} \sum_a \frac{\partial}{\partial q_a} \prod_{b \neq a} f^2(q_a - q_b) = 0. \tag{3.3}
\]
For the two-particle case ($N = 2$) this gives no further restrictions but for $N > 2$ the functional relations (3.3) are nontrivial. They are satisfied if

$$f^2(q) = a + b \, p(q), \quad (3.4)$$

where $a$ and $b$ are constants and $p(q)$ is the doubly periodic Weierstrass function. Here we will study the degenerate cases where one of the periods (type II, III) or both of them (type I) are sent to infinity and $f$ is characterized by the positive, even “pair potential” $W$ as

$$f(x) = \sqrt{1 + W(x)}. \quad (3.5)$$

In the three degenerate cases we have

$$W(x) = \begin{cases} 
\frac{g^2}{x^2} & \text{type I (rational)}, \\
\frac{\gamma}{\sinh^2 \omega x} & \text{type II (hyperbolic)}, \\
\frac{\gamma}{\sin^2 \omega x} & \text{type III (trigonometric)}. 
\end{cases} \quad (3.6)$$

In this paper we mainly focus on the cases I and II. Physically these cases describe scattering with repulsive interaction and since the order of particles cannot be changed the phase space is restricted to

$$q_1 > q_2 > \cdots > q_N. \quad (3.7)$$

Although the solution (3.4) arose from the requirement of Poincaré invariance, it turned out [8] that the models (3.1) are also Liouville integrable. This means that (beyond $\mathcal{H}$ and $\mathcal{P}$) there are further globally defined, commuting, conserved quantities. Moreover, the corresponding action-angle variables can be found algebraically and the solution of the equations of motion can be given explicitly.

Next we discuss the nonrelativistic (nr) limit of the problem. For this purpose we rescale the variables as

$$\theta_a = \frac{p_a}{mc}, \quad q_a = mc y_a. \quad (3.8)$$

The nr variables are also canonical satisfying $\{y_a, p_b\} = \delta_{ab}$. We also rescale the constant parameters as

$$\omega = \frac{\mu}{mc}, \quad \gamma = \sin \left( \frac{\mu g}{mc} \right) \quad \text{(type II)},$$
$$\omega = \frac{\mu}{mc}, \quad \gamma = \sinh \left( \frac{\mu g}{mc} \right) \quad \text{(type III)}. \quad (3.9)$$
Now we take the nr limit \( c \to \infty \) and find
\[
\lim_{c \to \infty} \mathcal{P} = \mathcal{P}_{\text{nr}} = \sum_a p_a, \quad \mathcal{K} = -m \sum_a y_a
\] (3.10)
and
\[
\lim_{c \to \infty} (\mathcal{H} - Nmc^2) = \mathcal{H}_{\text{nr}} = \frac{1}{2m} \sum_a p_a^2 + \sum_{a<b} V(y_a - y_b),
\] (3.11)
where the nr potential is
\[
V(x) = \begin{cases} 
\frac{\mu^2 g^2}{m \sinh^2 \mu x} & \text{type II,} \\
\frac{\mu^2 g^2}{m \sin^2 \mu x} & \text{type III.}
\end{cases}
\] (3.12)

We see that the nr limit depends on the choice of parametrization (of both the canonical variables and the constant parameters) and it is not obvious if this formal \( c \to \infty \) limit is what one could call the physical nr limit (case of slowly moving particles). It was also questioned [9] if the RS models (which are also called relativistic Calogero-Moser type models) are truly describing relativistic motion of interacting particles. Although the original motivation for studying these models was their relativistic invariance, later mainly their integrability aspects were in the focus of research and not the questions related to Poincaré invariant mechanics.

The above mentioned doubts about true Poincaré invariance can be dispelled by constructing the particle trajectories and showing that the world-line conditions are satisfied. For RS models the choice of the relativistic trajectory variables was motivated by the fact that in a special case, for type II models with \( \gamma = 1 \), the model can be identified with the Sine-Gordon model and the particles with Sine-Gordon solitons. This special case motivated the choice
\[
\rho_a = q_a, \quad R_a = \bar{Q}_a.
\] (3.13)
It is obvious that this \( \rho_a \) is boost invariant. Moreover,
\[
\frac{\partial}{\partial x} \bar{Q}_a = q'_a(\bar{Q}, \bar{T}) = -mc \cosh \bar{T} V_a(\bar{Q}) < 0,
\] (3.14)
hence it is a monotonic function of \( x \) and the solution of (2.18) is unique. Also
\[
v_a = \dot{x}_a = \frac{\bar{Q}_a}{Q'_a} = -c \tanh \bar{T}_a,
\] (3.15)
hence
\[
|v_a| < c
\] (3.16)
as it should.
4 Two-particle RS-type interaction

In this section we will study the construction of trajectories more explicitly in the two-particle case. For $N = 2$ we have

\[ \mathcal{H} = mc^2(\cosh \theta_1 + \cosh \theta_2)f(q_1 - q_2), \]
\[ \mathcal{P} = mc(\sinh \theta_1 + \sinh \theta_2)f(q_1 - q_2), \]
\[ \mathcal{K} = -\frac{1}{c}(q_1 + q_2). \]  

(4.1)

In the two-particle case it is useful to introduce the “external” variables

\[ \zeta = q_1 + q_2, \quad \tau = \frac{\theta_1 + \theta_2}{2} \]  

(4.2)

and the “internal” ones,

\[ q = q_1 - q_2, \quad u = \frac{\theta_1 - \theta_2}{2}. \]  

(4.3)

In terms of these,

\[ \mathcal{H} = 2mc^2\varepsilon \cosh \tau, \quad \mathcal{P} = 2mc \varepsilon \sinh \tau, \quad \mathcal{K} = -\frac{1}{c}\zeta. \]  

(4.4)

Here

\[ \varepsilon = \cosh uf(q) > 1 \]  

(4.5)

is the effective mass and $\tau$ is (up to a sign) the center of mass (COM) rapidity. It is easy to see that $\varepsilon$ is Poincaré invariant:

\[ \dot{\varepsilon} = \varepsilon' = \dot{\zeta}\varepsilon = 0. \]  

(4.6)

We also introduce the variable $\psi > 0$ by

\[ \varepsilon = \cosh \psi. \]  

(4.7)

We now want to study the dynamics of the system. First of all we see that

\[ \dot{\tau} = 0, \quad \dot{\zeta} = -2mc^2\varepsilon \sinh \tau, \]  

(4.8)

i.e. $\tau$ is constant and $\zeta$ has linear time dependence. For the internal variables $q$, $u$ we introduce the solution of the equations of motion, $Q(t)$ and $U(t)$. To give them explicitly, we need the following definitions. First of all, we assume that $W(q)$ is positive and monotonically decreasing from $W(0) = \infty$ to $W(\infty) = 0$. This is satisfied in the type I and type II RS model, but since for $N = 2$ there is no
restriction on the functional form of \( W \), our considerations here are valid for any such function. Let us now define the function \( g_\varepsilon(q) \) for \( q > q_\varepsilon \) by

\[
g_\varepsilon(q) = \int_{q_\varepsilon}^{q} \frac{dy}{\sqrt{\varepsilon^2 - f^2(y)}}, \quad f(q_\varepsilon) = \varepsilon. \tag{4.9}
\]

This is a monotonically increasing function and so is its functional inverse \( G_\varepsilon \):

\[
G_\varepsilon(g_\varepsilon(q)) = q. \tag{4.10}
\]

\( G_\varepsilon(\xi) \) is monotonically increasing from \( q_\varepsilon \) to \( \infty \) as \( \xi \) goes from 0 to \( \infty \). We extend the domain of definition of \( G_\varepsilon(\xi) \), which is of the form

\[
G_\varepsilon(\xi) \approx \sqrt{\varepsilon^2 - 1} \xi + D(\varepsilon). \tag{4.11}
\]

The constant term \( D(\varepsilon) \) will be used to characterize the time delay in the scattering process. For the type I (rational) RS model

\[
G_\varepsilon(\xi) = \sqrt{(\varepsilon^2 - 1)\xi^2 + \frac{\gamma^2}{\varepsilon^2 - 1}}, \tag{4.12}
\]

and in this case \( D(\varepsilon) = 0 \). For the type II (hyperbolic) RS model

\[
G_\varepsilon(\xi) = \frac{1}{\omega} \text{arccosh} \left[ \cosh \omega q_\varepsilon \cosh(\omega \sqrt{\varepsilon^2 - 1} \xi) \right], \tag{4.13}
\]

and

\[
D(\varepsilon) = \frac{1}{\omega} \ln \cosh \omega q_\varepsilon = \frac{1}{2\omega} \ln \left( 1 + \frac{\gamma^2}{\varepsilon^2 - 1} \right). \tag{4.14}
\]

The solution of the equations of motion is given by

\[
Q(t) = G_\varepsilon(2mc^2 \cosh \tau t - w), \quad U(t) = -\text{arcsinh} \left( \frac{\dot{Q}}{2mc^2 f(Q) \cosh \tau} \right), \tag{4.15}
\]

where

\[
w = \text{sign}(u)g_\varepsilon(q). \tag{4.16}
\]

The solution of the \( \mathcal{P} \) equations of motion is quite similar.

\[
\tau' = 0, \quad \zeta' = -2mc \varepsilon \cosh \tau, \tag{4.17}
\]

i.e. \( \tau \) is constant and \( \zeta \) has linear \( x \) dependence. Furthermore, defining the \( x \)-dependent \( q, u \) as \( \bar{Q}(x) \) and \( \bar{U}(x) \), we find

\[
\bar{Q}(x) = G_\varepsilon(2mc \sinh \tau x - w), \quad \bar{U}(x) = -\text{arcsinh} \left( \frac{\bar{Q}'}{2mc f(\bar{Q}) \sinh \tau} \right). \tag{4.18}
\]
The \( x \) evolution of the variables \( q_1 \) and \( q_2 \) is thus
\[
2 \bar{Q}_1(x) = \zeta - 2mc\varepsilon \cosh \tau x + G_\varepsilon(2mc \sinh \tau x - w),
\]
\[
2 \bar{Q}_2(x) = \zeta - 2mc\varepsilon \cosh \tau x - G_\varepsilon(2mc \sinh \tau x - w).
\]
Both solutions are strictly monotonically decreasing (from \(+\infty\) to \(-\infty\)) as \( x \) goes from \(-\infty\) to \(+\infty\) and thus the equations
\[
\zeta - 2mc\varepsilon \cosh \tau x_1 + G_\varepsilon(2mc \sinh \tau x_1 - w) = 0,
\]
\[
\zeta - 2mc\varepsilon \cosh \tau x_2 - G_\varepsilon(2mc \sinh \tau x_2 - w) = 0
\]
have unique solution for the trajectory variables \( x_1, x_2 \). It is easy to see that
\[
x_1 > x_2.
\]
Since
\[
\dot{\zeta} = -2mc^2\varepsilon \sinh \tau \quad \text{and} \quad \dot{w} = -2mc^2 \cosh \tau,
\]
the time evolution of the trajectory variables satisfy
\[
\zeta - 2mc^2 \varepsilon \sinh \tau t - 2mc\varepsilon \cosh \tau x_1(t) + G_\varepsilon(2mc \sinh \tau x_1(t) + 2mc^2 \cosh \tau t - w) = 0,
\]
\[
\zeta - 2mc^2 \varepsilon \sinh \tau t - 2mc\varepsilon \cosh \tau x_2(t) - G_\varepsilon(2mc \sinh \tau x_2(t) + 2mc^2 \cosh \tau t - w) = 0.
\]
This can be used to calculate the large \(|t|\) asymptotics of the trajectories:
\[
x_1(t) \approx x_1^{(-)}(t) = \frac{x_2}{\cosh \beta_1} + ct \tanh \beta_1 + \frac{\delta}{\cosh \beta_1}, \quad (t \to -\infty)
\]
\[
x_2(t) \approx x_2^{(-)}(t) = \frac{x_1}{\cosh \beta_2} + ct \tanh \beta_2 - \frac{\delta}{\cosh \beta_2},
\]
\[
x_1(t) \approx x_1^{(+)}(t) = \frac{x_1}{\cosh \beta_2} + ct \tanh \beta_2 + \frac{\delta}{\cosh \beta_2}, \quad (t \to +\infty)
\]
\[
x_2(t) \approx x_2^{(+)}(t) = \frac{x_2}{\cosh \beta_1} + ct \tanh \beta_1 - \frac{\delta}{\cosh \beta_1},
\]
where
\[
\tilde{x}_1 = \frac{\zeta - \tilde{w}}{2mc}, \quad \tilde{x}_2 = \frac{\zeta + \tilde{w}}{2mc}, \quad \tilde{w} = \sqrt{\varepsilon^2 - 1}w,
\]
and
\[
\beta_1 = -(\psi + \tau), \quad \beta_2 = \psi - \tau, \quad \delta = \frac{D(\varepsilon)}{2mc}.
\]
Classical scattering is characterized by the time delay defined by
\[
x_2^{(+)}(t + \Delta t_1) = x_1^{(-)}(t), \quad x_1^{(+)}(t + \Delta t_2) = x_2^{(-)}(t)
\]
and is given by
\[ c \Delta t_1 = \frac{2\delta}{\sinh \beta_1}, \quad c \Delta t_2 = -\frac{2\delta}{\sinh \beta_2}. \] (4.29)

If we go to the COM frame where
\[ c \tanh \beta_1 = -\bar{v}, \quad c \tanh \beta_2 = \bar{v}, \] (4.30)
we find
\[ \bar{v} \Delta t_1 = \bar{v} \Delta t_2 = -\sqrt{1 - \frac{\bar{v}^2}{c^2}} \frac{D(\varepsilon)}{mc}. \] (4.31)

Using the asymptotic rapidities \( \beta_1 \) and \( \beta_2 \) we can express the energy and momentum of the two-particle system as
\[ E = mc^2(\cosh \beta_1 + \cosh \beta_2) = \mathcal{H}, \quad P = mc(\sinh \beta_1 + \sinh \beta_2) = -\mathcal{P} \] (4.32)
i.e. in our conventions the physical momentum is \( P = -\mathcal{P} \).

## 5 Rational RS model

It is not easy to find the solution of (4.20) in general. In this section we consider the simplest nontrivial case, the type I (rational) RS model. In this case \( x_1 \) and \( x_2 \) are the two solutions of the quadratic equation
\[ (\zeta - 2mc \varepsilon \cosh \tau x)^2 = (\varepsilon^2 - 1)(2mc \sinh \tau x - \bar{w})^2 + \frac{g^2}{\varepsilon^2 - 1} \] (5.1)
given by
\[ x_1 = \frac{p + \sqrt{p^2 + Ah}}{A}, \quad x_2 = \frac{p - \sqrt{p^2 + Ah}}{A}, \] (5.2)
where
\[ A = 4m^2 c^2 (\cosh^2 \psi + \sinh^2 \tau), \quad p = 2mc(\zeta \cosh \psi \cosh \tau - \bar{w} \sinh \psi \sinh \tau), \quad h = \bar{w}^2 - \zeta^2 + \frac{g^2}{\sinh^2 \psi}. \] (5.3)

We see that the physical quantities are expressed in terms of the external canonical variables \( \zeta \) and \( \tau \) and the new internal variables \( \bar{w} \) and \( \psi \). The latter also form a canonically conjugate pair and the non-vanishing Poisson brackets are
\[ \{\zeta, \tau\} = 1, \quad \{\bar{w}, \psi\} = 1. \] (5.4)
The Poincaré generators are given in terms of these variables as
\[
H = 2mc^2 \cosh \psi \cosh \tau, \quad P = 2mc \cosh \psi \sinh \tau, \quad K = -\frac{1}{c} \zeta. \quad (5.5)
\]
Written in terms of these variables, the generators take a general, dynamics-independent form and all dynamics is encoded in the trajectories \([5.2]\). These are equivalent to the relations
\[
x_1 + x_2 = \frac{2p}{A}, \quad x_1 x_2 = -\frac{h}{A}. \quad (5.6)
\]
We can now write the Poisson bracket relations
\[
\{x_1 x_2, x_1 + x_2\} = \left\{ \frac{p}{A} - \frac{h}{A} \right\} = -\left( \frac{x_1 - x_2}{2} \right) \{x_1, x_2\} \quad (5.7)
\]
and by evaluating the \(\{p/A, h/A\}\) Poisson bracket we find
\[
\{x_1, x_2\} = -\frac{g^2}{m^3 c^3 (x_1 - x_2)} \frac{\sinh \tau \cosh \psi}{(\cosh^2 \psi + \sinh^2 \tau)^3}. \quad (5.8)
\]
We see that this Poisson bracket does not vanish, otherwise it would be in contradiction with the no-interaction theorem.

The right hand side of the \(\{x_1, x_2\}\) Poisson bracket is an expression that contains also the canonical variables. It would be nicer to write it in terms of the physical variables \(x_1, x_2\) and their time derivatives \(v_1, v_2\). This raises the question if \(x_1, x_2, v_1, v_2\) form “good” coordinates on the phase space or at least in some part of the phase space. To answer this question we supplement the relations \([5.6]\) with the time derivatives
\[
v_1 + v_2 = \frac{2\dot{p}}{A} = -\frac{c \sinh 2\tau}{\cosh^2 \psi + \sinh^2 \tau} \quad (5.9)
\]
and
\[
x_1 v_2 + x_2 v_1 = \frac{\dot{h}}{A} = \frac{1}{m} \bar{w} \sinh \psi \cosh \tau - \zeta \cosh \psi \sinh \tau \quad (5.10)
\]
We see from \([5.9]\) that
\[
\mu_1 + \mu_2 = 0, \quad (5.11)
\]
i.e. the COM moves with constant velocity. We introduce the notation
\[
u_1 = \frac{v_1}{c}, \quad u_2 = \frac{v_2}{c}, \quad v = \frac{u_1 + u_2}{2}. \quad (5.12)
\]
Using the first equation in \([5.6]\) and \([5.10]\) we can express \(\zeta\) and \(\bar{w}\) in terms of the physical variables and the asymptotic rapidities \((\psi, \tau)\):
\[
\zeta = \left( \cosh^2 \psi + \sinh^2 \tau \right) \frac{mc}{\cosh \psi} \left[ \cosh \tau (x_1 + x_2) + \sinh \tau (x_1 u_2 + x_2 u_1) \right],
\]
\[
\bar{w} = \left( \cosh^2 \psi + \sinh^2 \tau \right) \frac{mc}{\sinh \psi} \left[ \sinh \tau (x_1 + x_2) + \cosh \tau (x_1 u_2 + x_2 u_1) \right]. \quad (5.13)
\]
The variable $\psi$ is determined from (5.9)

$$\cosh^2 \psi = -\frac{\sinh 2\tau}{2v} - \sinh^2 \tau$$

and finally from the second relation in (5.6), substituting the expressions for $\zeta$, $\bar{w}$ and $\cosh^2 \psi$, after some algebra, we get a quintic equation satisfied by the variable $\xi = \tanh \tau$:

$$\xi^2(1 + u_1 u_2) + v\xi(1 + \xi^2) = \frac{\lambda^2 v^2}{(x_1 - x_2)^2}(1 + v\xi)(1 - \xi^2)^2,$$  

where $\lambda = \frac{g}{m c}$. The solution of the quintic is further restricted by the requirements

$$\text{sign}(\tau) = -\text{sign}(v), \quad |\xi| > |v|,$$  

coming from (5.14). An interesting problem is to find the subspace spanned by the variables $x_1 - x_2$, $u_1$, $u_2$ such that the quintic has unique solution also satisfying (5.16) there. In this subspace also the accelerations $\mu_1 = -\mu_2$ can be expressed in terms of the instantaneous positions $x_1$, $x_2$ and velocities $v_1$, $v_2$. The accelerations obtained this way must satisfy the Currie-Hill equations. Unfortunately the accelerations are very complicated even though the type I RS model for two particles appears to be the simplest of all cases. The problem is drastically simplified if we go to the COM frame where

$$\tau = 0, \quad \zeta = \text{const.}$$

The following considerations are again valid for any pair potential $W(x)$, not just the one corresponding to the type I RS model. In this frame we have

$$x_1 = \frac{\zeta + q}{2mc\varepsilon}, \quad x_2 = \frac{\zeta - q}{2mc\varepsilon},$$

and consequently

$$x_1 + x_2 = \text{const.}, \quad 2\varepsilon = x_1 - x_2 = \frac{q}{mc\varepsilon}.$$  

The physical equation of motion for this $x$ is obtained in two steps. In the first step we have to solve

$$1 - \frac{\dot{x}^2}{\varepsilon^2} = 1 + W(2mc\varepsilon x)$$

for $\varepsilon$ and then the COM equations of motion are reduced to the Newtonian form

$$\ddot{x} = -\frac{mc^3}{\varepsilon} W'(2mc\varepsilon x).$$
Again, it is not easy to find the solution of (5.20) in general, but for the type I RS model it can be done and we find for the acceleration
\[ \ddot{x} = \frac{c^2 x}{X^2} (R - 1)^2, \] (5.22)
where
\[ R^2 = 1 + \frac{\lambda^2}{x^2} \left( 1 - \frac{\dot{x}^2}{c^2} \right). \] (5.23)
The solution of this equation of motion is given by
\[ x(t) = \frac{1}{\sinh 2\psi} \sqrt{\lambda^2 + (\sigma t)^2}, \quad \sigma = 2c \sinh 2\psi. \] (5.24)
The time variable \( t \) is chosen such that \( x(t) \) is minimal at \( t = 0 \). We see that the solution describes a scattering process with repulsive forces. The two particles, starting from infinity, gradually approach each other and after the turning point, where the particles stop and reach the minimal relative distance, are receding from each other.

6 Summary and Conclusion

In the canonical approach to relativistic mechanics the construction of models describing the motion of interacting particles consists of two main steps. Assuming that the phase space is already equipped with a symplectic structure in the first step we have to find a suitable set of Poincaré generators whose Poisson brackets form the Lie algebra of the Poincaré group. Using these generators an action of the Poincaré group on the phase space can be constructed. The Hamiltonian of the model is identified with the generator of time translations from the Poincaré Lie algebra. In the second step the particle positions (trajectory variables) have to be found as functions on the phase space. Using the given time evolution and starting from the given positions, the complete space-time trajectories of the particles can be determined. The action of the Poincaré group on the phase space induces an action on the particle trajectories and we require that this induced action is identical to the usual linear Poincaré transformation formulas in terms of the space-time coordinates corresponding to the trajectories. This requirement, for infinitesimal transformations, leads to consistency relations called the world-line conditions. These are nontrivial, nonlinear relations which must be satisfied by the trajectory variables. It is not known how to satisfy the world-line conditions in general 3 + 1 dimensional problems.

In this paper we presented a general method for solving the world-line conditions in 1 + 1 dimensional problems. All one has to do is to associate a suitable Lorentz-invariant function on the phase space to each particle and the method provides an
equation the solution of which (provided its solution exists and is unique) gives trajectory variables satisfying the world-line conditions.

Restricting attention to the two-particle problem, we constructed these equations and demonstrated the existence and uniqueness of their solution for a family of models, including the type I (rational) and type II (hyperbolic) Ruijsenaars-Schneider models and generalizations. Further restricting attention to the simplest case, the rational RS model, the trajectory variables were calculated explicitly by solving a quadratic algebraic equation. This provides us with explicit formulas for the two trajectory variables in terms of the original canonical coordinates of the phase space. It would be desirable to use the physical variables (particle position variables and their time derivatives) as coordinates on the phase space (like in Newtonian mechanics) but it is not clear (even in this simple example) what is the domain of these physical variables in which they can be used as coordinates on the phase space and it is even more difficult to find an explicit description of this inverse transformation.

The above mentioned difficulties, which are present already in the very simple model studied in this paper, illustrate the complexity of the construction of relativistic particle models with interaction. We hope to be able to return to these problems in a future publication.

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