LOCAL LANGLANDS CORRESPONDENCE FOR REGULAR SUPERCUSPIDAL REPRESENTATIONS OF $GL(n)$

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Abstract. In this paper, we prove the coincidence of Kaletha’s recent construction of the local Langlands correspondence for regular supercuspidal representations with Harris–Taylor’s one in the case of general linear groups. The keys are Bushnell–Henniart’s theory of essentially tame local Langlands correspondence and Tam’s result on Bushnell–Henniart’s rectifier. By combining them, our problem can be reduced to an elementary root-theoretic computation on the difference between two $\chi$-data appearing Kaletha’s theory and Tam’s one.

1. Introduction

One fundamental objective in representation theory of $p$-adic groups is to establish the conjectural local Langlands correspondence, which predicts the existence of a natural connection between $L$-packets (finite sets consisting of irreducible smooth representations) and $L$-parameters for connected reductive groups over a $p$-adic field. From the early days of representation theory of $p$-adic groups, a number of results on the local Langlands correspondence have been obtained. Among them, Harris–Taylor’s construction of the correspondence for the general linear group, which is the most typical example of a connected reductive group, has a particularly significant meaning. Since it was established, it has been playing an important role as an indispensable foundation in a lot of studies.

Recently, several attempts are being made to construct the correspondence for more general connected reductive groups beyond Harris–Taylor’s work on the general linear group. One possible approach to constructing the correspondence for general groups is to restrict the class of representations. For example, in their striking paper [DR09], DeBacker and Reeder constructed $L$-packets consisting of so-called regular depth zero supercuspidal representations and their corresponding $L$-parameters for unramified connected reductive groups. At present, the most general result in this direction is Kaletha’s construction for regular supercuspidal representations [Kal19a]. Regular supercuspidal representations are supercuspidal representations which are obtained by Yu’s construction [Yu01] and satisfy a certain regularity condition. As we can see in the fact that every supercuspidal representation of $GL_n$ is regular when $p$ does not divide $n$, the regularity condition is not so restrictive compared to, for example, the depth zero condition mentioned.

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above. Although not all supercuspidal representations are regular in general, regular supercuspidal representations occupy most of whole supercuspidal representations. By focusing on such representations, Kaletha constructed \( L \)-packets consisting of them and their corresponding \( L \)-parameters, under an assumption that a connected reductive group is tamely ramified. (Recently, a construction for further general supercuspidal representations was announced by him in [Kal19b].) Furthermore, not only constructing the correspondence, he also succeeded in proving various important properties such as the stability, standard endoscopic character relation, and so on (under the assumption of “torality”).

However, it is not so obvious from his construction whether Kaletha’s correspondence recovers Harris–Taylor’s local Langlands correspondence in the case of general linear groups. The goal of this paper is to check it, that is, to prove the following theorem:

**Theorem 1.1 (Theorem 6.1).** Let \( F \) be a \( p \)-adic field with odd residual characteristic. Then the local Langlands correspondence of Kaletha coincides with that of Harris–Taylor for any regular supercuspidal representation of \( GL_n(F) \).

Here we remark that the oddness assumption on the residual characteristic is needed for making Kaletha’s construction of the correspondence work.

We explain the outline of the proof of this result. In the following, we let \( F \) be a \( p \)-adic field with odd residual characteristic and consider the general linear group \( GL_n \) over \( F \). The key ingredients are Bushnell–Henniart’s consecutive works on the essentially tame local Langlands correspondence ([BH05a, BH05b, BH10]) and Tam’s reinterpretation of Bushnell–Henniart’s works ([Tam16]).

In the case of \( GL_n \), regular supercuspidal representations are nothing but so-called essentially tame supercuspidal representations, which have been intensively studied by Bushnell–Henniart. Such representations can be parametrized by pairs \((E, \xi)\) (called \( F \)-admissible pairs) consisting of a degree \( n \) tamely ramified extension \( E \) of \( F \) and a character \( \xi \) of \( E^\times \) satisfying a condition called the \( F \)-admissibility. Then Bushnell–Henniart showed that the essentially tame supercuspidal representation \( \pi_{(E, \xi)} \) corresponding to a pair \((E, \xi)\) maps to \( \text{Ind}^E_F(\xi \mu_{\text{rec}}^{-1}) \) (as an \( n \)-dimensional representation of the Weil group \( W_F \) of \( F \)) under Harris–Taylor’s local Langlands correspondence, where \( \mu_{\text{rec}} \) is a certain tamely ramified character of \( E^\times \) called the rectifier of \((E, \xi)\), and gave an explicit formula for the rectifier.

On the other hand, Kaletha’s construction of the \( L \)-parameter is given as follows. First, we regard \( E^\times \) as the set of \( F \)-valued points of an elliptic maximal torus \( S \) of \( GL_n \) defined over \( F \). Then, by using the local Langlands correspondence for \( S \), we get an \( L \)-parameter \( \phi_\xi \) of \( S \). As \( \phi_\xi \) is a homomorphism from the Weil group \( W_F \) to the \( L \)-group \( ^L S \) of \( S \), we can regard it as an \( L \)-parameter of \( GL_n \) if we can find an \( L \)-embedding of \( ^L S \) into \( ^L GL_n \). In fact, a general procedure for creating such an embedding from certain auxiliary data called “\( \chi \)-data” is known by Langlands–Shelstad. A point here is that a choice of a set of \( \chi \)-data may not be unique, hence Langlands–Shelstad’s construction does not give us an \( L \)-embedding in a canonical way. Then Kaletha constructed a set of \( \chi \)-data (let us write \( \chi_{\text{Kal}} \) for it) carefully by seeing the information of the pair \((E, \xi)\) and defined an \( L \)-parameter of \( GL_n \) by
using Langlands–Shelstad’s $L$-embedding $L_{j_{\chi_{Kal}}}$ determined by $\chi_{Kal}$:

![Diagram](image)

Therefore, in order to investigate the relationship between Harris–Taylor’s construction and Kaletha’s one, it is needed to understand the $L$-parameter obtained in Kaletha’s way as an $n$-dimensional representation of $W_F$. In fact, what Tam pursued in his paper [Tam16] is exactly this point. In his paper [Tam16], he first described the $L$-parameter constructed as above explicitly as an $n$-dimensional representation of $W_F$. More precisely, for a pair $(E, \xi)$ and a set $\chi$ of $\chi$-data, he proved that the $L$-parameter of $GL_n$ obtained as above is given by $\text{Ind}_{W_F}^{W_E} \xi \mu_\chi$ with a character $\mu_\chi$ of $E^\times$ determined by $\chi$ (see [Tam16, Proposition 6.5] or Proposition 3.4 in this paper for details). Second, he indeed constructed a set of $\chi$-data (let us write $\chi_{Tam}$ for it) realizing $\mu_{\text{rect}}$ as $\mu_{\chi_{Tam}}$ from a pair $(E, \xi)$.

Then, where is a nontrivial point left in our problem of comparing the correspondences of Kaletha and Harris–Taylor? In fact, Kaletha’s construction of the correspondence requires some additional twist on $\pi_{(E, \xi)}$ by a certain character. The key object at this point is the character “$\epsilon$” defined by DeBacker–Spice in [DS18] (and also by Kaletha in [Kal19a]). For each $F$-admissible pair $(E, \xi)$, DeBacker and Spice attached to it a tamely ramified character $\epsilon$ of $E^\times$. This character is defined according to a root-theoretic property of the pair $(E, \xi)$ and appears naturally in the context of an explicit character formula of the supercuspidal representation $\pi_{(E, \xi)}$ (which was established by Adler–Spice in [AS09] first and deepened later by DeBacker–Spice and Kaletha). Then Kaletha defined the $L$-parameter $L_{j_{\chi_{Kal}}} \circ \phi_\epsilon$ to be the $L$-parameter corresponding to the representation $\pi_{(E, \xi)}$, that is, the regular supercuspidal representation arising from the twisted pair $(E, \xi_\epsilon)$ (note that this is again $F$-admissible).

Therefore, in fact, Kaletha’s $\chi$-data is not exactly the same as Tam’s one. Our task is to prove that the difference between them (i.e., the ratio of $\mu_{\chi_{Kal}}$ to $\mu_{\chi_{Tam}}^{-1}$) is given by DeBacker–Spice’s character $\epsilon$. To do this, we have to compare the language used in Adler–Spice–DeBacker and Kaletha’s works with the one in Bushnell–Henniart and Tam’s works. In the former one, every notion is defined in a sophisticated way according to the general structure theory of connected reductive groups over $p$-adic fields. On the other hand, in the latter one, since everything can be described explicitly by a speciality of $GL_n$, the corresponding notions are defined concretely in a Galois-theoretic way. Thus, in order to compare $\chi_{Kal}$ to $\chi_{Tam}$, we first have to grasp the relationship between these two languages precisely. If we can do this appropriately, our problem is reduced to a simple case-by-case computation based on a classification of root systems with Galois action arising from elliptic maximal tori of $GL_n$. Then the result follows from elementary properties of several fundamental arithmetic invariants such as the Gauss sum, Jacobi symbol, Langlands constant, and so on.

Thus we would like conclude this introduction by emphasizing that the theoretically difficult part of our problem can be completed almost just by referring to Tam’s work.
Organization of this paper. In Section 2 we explain our notation on fundamental objects and several invariants appearing in this paper. In Section 3 we recall Bushnell–Henniart’s work on Harris–Taylor’s local Langlands correspondence for GL$_n$ and its reinterpretation due to Tam. In Section 4 we review Kaletha’s construction of the local Langlands correspondence and describe it in the case of GL$_n$. In Section 5 we introduce some preliminary results needed for our comparison of two χ-data χ$_{\text{Kal}}$ and χ$_{\text{Tam}}$. What we will do in this section is basically to compare Kaletha’s language, which is abstract and available for general tamely ramified groups, with Bushnell–Henniart–Tam’s one, which is explicit but specialized for GL$_n$. Although we believe most of the contents in this section are well-known to experts, we justify them here for the sake of completeness. In Section 6 we determine the difference between χ$_{\text{Kal}}$ and χ$_{\text{Tam}}$ by a case-by-case computation and complete the proof of our main result.

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2. Notation

First we explain our notation for several notions used in this paper and list their basic properties.

**Prime number:** In this paper, we always assume that $p$ is an odd prime number.

**$p$-adic field:** We fix a $p$-adic field $F$. Let $\mathcal{O}_F$, $\mathfrak{p}_F$, and $k_F$ denote the ring of integers, maximal ideal, and residue field, respectively. Let $q$ be the order of $k_F$. For $x \in \mathcal{O}_F$, $\bar{x}$ denotes the image of $x$ in $k_F$. We write $\mu_F$ for the set of roots of unity in $F$ of order prime to $p$. We often regard an element of $k_F^\times$ as an element of $\mu_F$ by the Teichmüller lift. We write $\text{val}_F$ for the valuation of $F$ and normalize it so that $\text{val}_F(F^\times) = \mathbb{Z}$.

We fix an algebraic closure $\overline{F}$ of $F$ and write $\overline{k_F}$ for the residue field of $F$. We consider every finite extension of $F$ or $k_F$ within these closures. Let $\Gamma_F$ and $W_F$ denote the absolute Galois group of $F$ and the Weil group of $F$, respectively.

For a finite extension $E$ of $F$, we use similar notations to above such as $\mathcal{O}_E$, $\mathfrak{p}_E$, and so on. We write $e(E/F)$ and $f(E/F)$ for the ramification index and the residue degree of the extension $E/F$, respectively.

**Norm 1 subgroup for a finite field:** For a finite field $k$ which contains a subfield $k'$ such that $k$ is quadratic over $k'$ (note that then such a $k'$ is unique), we put $k_1$ to be the kernel of the norm map with respect to $k/k'$:

$$k_1 := \ker(\text{Nr}_{k/k'} : k^\times \to k'^\times).$$

**Filtration on additive and multiplicative groups:** For a finite extension $E$ of $F$, we put

$$E_r := \{x \in E \mid \text{val}_F(x) \geq r\} \quad \text{and} \quad E_{r+} := \{x \in E \mid \text{val}_F(x) > r\}$$

for each $r \in \mathbb{R}$. Similarly, for each $r \in \mathbb{R}_{>0}$, we put

$$E^\times_r := \{x \in 1 + p_E \mid \text{val}_F(x - 1) \geq r\} \quad \text{and} \quad E^\times_{r+} := \{x \in 1 + p_E \mid \text{val}_F(x - 1) > r\}.$$ 

On the other hand, for a positive integer $s \in \mathbb{Z}_{>0}$, we put

$$U^s_E := 1 + p_E.$$ 

Here we note that, if we put $e := e(E/F)$ to be the ramification index of the extension $E/F$, then we have

$$E^\times_s = U^s_E$$

for each $s \in \mathbb{Z}_{>0}$.

**Remark 2.1.** It is well-known that, for finite tamely ramified extensions $E \supset K \supset F$, the norm map $\text{Nr}_{E/K}$ gives a surjection

$$\text{Nr}_{E/K} : U^e(E/K)^s \twoheadrightarrow U^s_K$$

for any integer $s \in \mathbb{Z}_{>0}$ (see, for example, [Ser79, Chapter V, Section 6, Corollary 3]). If we use the above normalizations of filtrations of multiplicative groups, this can be expressed as

$$\text{Nr}_{E/K} : E^\times_{(E/F)^s} \twoheadrightarrow K^\times_{(K/F)^s}.$$


Level and depth of a multiplicative character: For a finite tamely ramified extension $E$ of $F$ and a multiplicative character $\xi$ of $E^\times$, we say

- $\xi$ is of $E$-level $s$ if $\xi|_{U_{E^s}}$ is trivial and $\xi|_{U_{E^t}}$ is nontrivial, and
- $\xi$ is of depth $r$ if $\xi|_{E^r}$ is trivial and $\xi|_{E^s}$ is nontrivial.

Note that if we put the depth of $\xi$ to be $r$, then its $E$-level is given by $e(E/F)^r$.

Additive character: Throughout this paper, we fix an additive character $\psi_F$ on $F$ of level one, i.e., $\psi_F$ is trivial on $p_F$, but not trivial on $O_F$.

For a finite tamely ramified extension $E$ of $F$, let $\psi_E$ denote the additive character $\psi_F \circ \text{Tr}_{E/F}$ on $E$. Note that, by the tameness assumption, this is again of level one, hence induces a nontrivial additive character of the residue field $k_E$ of $E$. By abuse of notation, we again write $\psi_E$ for this induced additive character of $k_E$.

Quadratic character: For a finite cyclic group $C$ of even order, we write $(\cdot)_C$ for the unique nontrivial quadratic character of $C$. Note that if we identify $C$ as a subgroup of $\mathbb{C}$, then we have an equality of sign

$$(c)_C = c|C|^2$$

for any $c \in C$.

Jacobi symbol: Let

$$(\cdot)$$

denote the Jacobi symbol. We note that, for any finite field $k$ with odd order $q_k$, we have $\left( \frac{m}{q_k} \right) = \left( \frac{m}{q_k} \right)$ for any integer $m$ which is prime to $q_k$ (here the left-hand side is the quadratic character defined above and the right-hand side is the Jacobi symbol).

Gauss sum: For a nontrivial additive character $\psi$ of a finite field $k$ with odd order $q_k$, we define its Gauss sum $g(\psi)$ by

$$g(\psi) := \sum_{x \in k^\times} \left( \frac{x}{q_k} \right) \psi(x).$$

We define the normalized Gauss sum $n(\psi)$ by

$$n(\psi) := q_k^{-\frac{1}{2}} g(\psi).$$

Recall that we have the following well-known identity:

$$n(\psi)^2 = \left( \frac{-1}{q_k} \right) = \left( \frac{-1}{q_k} \right)$$

(see, e.g., [BH06 (23.6.3)]).

Langlands constant: For a finite extension $E/K$ of $p$-adic fields and a nontrivial additive character $\psi$ of $K$, we write $\lambda_{E/K}(\psi)$ for the Langlands constant, that is, the ratio of local root numbers:

$$\lambda_{E/K}(\psi) := \frac{\varepsilon(\text{Ind}_{E/K} \chi_E, \frac{1}{2}, \psi)}{\varepsilon(\chi_K, \frac{1}{2}, \psi \circ \text{Tr}_{E/K})}.$$

When $K$ is a finite extension of the fixed $p$-adic field $F$ and $\psi$ is given by $\psi_K = \psi_F \circ \text{Tr}_{K/F}$, we simply write $\lambda_{E/K}$ for $\lambda_{E/K}(\psi_K)$.
General linear group: In this paper, we always write $G$ for the general linear group $\text{GL}_n$ over $F$. Note that the Langlands dual group $\hat{G}$ is then given by $\text{GL}_n$ over the complex number field $C$. We fix an $F$-splitting $\text{spl}_G := (B, T, \{X\})$ of $G$ and a splitting $\text{spl}_G^\prime := (B', T', \{X'\})$ of $\hat{G}$. See, for example, [KS99 Section 1.1] for the precise definition of a splitting. We also fix an isomorphism between the based root data of $G$ determined by $\text{spl}_G$ and the dual of that of $\hat{G}$ determined by $\text{spl}_{\hat{G}}$.

3. Bushnell–Henniart–Tam’s description of Harris–Taylor’s LLC

In this section, we recall Bushnell–Henniart’s work on Harris–Taylor’s local Langlands correspondence for $\text{GL}_n$ and its reinterpretation due to Tam.

3.1. $\chi$-data and Langlands–Shelstad’s construction of $L$-embeddings. Let $S$ be a maximal torus of $G$ defined over $F$. Then we have the set $\Phi(S, G)$ of (absolute) roots of $S$ in $G$, which has an action of $\Gamma_F$. We recall the following terminology of Adler–DeBacker–Spice for roots in $\Phi(S, G)$:

**Definition 3.1.** For each $\alpha \in \Phi(S, G)$, we put $\Gamma_\alpha$ (resp. $\Gamma_{\pm \alpha}$) to be the stabilizer of $\alpha$ (resp. $\{\pm \alpha\}$) in $\Gamma_F$. Let $F_\alpha$ (resp. $F_{\pm \alpha}$) be the subfield of $\hat{F}$ fixed by $\Gamma_\alpha$ (resp. $\Gamma_{\pm \alpha}$):

$$F \subset F_{\pm \alpha} \subset F_\alpha \iff \Gamma_F \supset \Gamma_{\pm \alpha} \supset \Gamma_\alpha.$$  

- When $F_\alpha = F_{\pm \alpha}$, we say $\alpha$ is an symmetric root.
- When $F_\alpha \supseteq F_{\pm \alpha}$, we say $\alpha$ is a symmetric root. Note that, in this case, the extension $F_\alpha/F_{\pm \alpha}$ is necessarily quadratic. Furthermore,
  - when $F_\alpha/F_{\pm \alpha}$ is unramified, we say $\alpha$ is symmetric unramified, and
  - when $F_\alpha/F_{\pm \alpha}$ is ramified, we say $\alpha$ is symmetric ramified.

We write $\Phi(S, G)^\text{sym}$, $\Phi(S, G)^\text{sym}$, $\Phi(S, G)^\text{sym,ur}$, and $\Phi(S, G)^\text{sym,ram}$ for the set of asymmetric roots, symmetric roots, symmetric unramified roots, and symmetric ramified roots, respectively.

**Remark 3.2.** In this paper, we follow the definition of symmetric unramified and ramified roots given by Adler–DeBacker–Spice (see [DS18 Definition 2.6] or also [Kal19a Section 2.2]). Note that this definition is different from Tam’s one used in his paper (see [Tam16 1710 page]). In Section 3.1, we investigate the relationship between Tam’s notion of symmetric unramified and ramified roots and Adler–DeBacker–Spice’s one in the case of $\text{GL}_n$.

We next recall the notion of $\chi$-data. A set of characters $\{\chi_\alpha: F_\alpha^\times \to \mathbb{C}^\times\}_{\alpha \in \Phi(S, G)}$ indexed by the elements of $\Phi(S, G)$ is called a set of $\chi$-data if it satisfies the following conditions:

- For every $\alpha \in \Phi(S, G)$ and $\sigma \in \Gamma_F$, we have $\chi_\alpha^{-1} = \chi_{-\alpha}$ and $\chi_\sigma(\alpha) = \chi_\alpha \circ \sigma^{-1}$.
- For every $\chi \in \Phi(S, G)^\text{sym}$, the restriction $\chi_\alpha|_{F_\alpha^\times}$ of $\chi_\alpha$ to $F_\alpha^\times$ is equal to the quadratic character corresponding to the quadratic extension $F_\alpha/F_{\pm \alpha}$.

By using a set of $\chi$-data with respect to $(S, G)$, we can construct an embedding of the $L$-group $L^S$ into that $L^G$ of $G$ in the following manner. First, we take an element $g \in G(F)$ which induces an isomorphism $\text{Int}(g): S \cong T$ between $S$ and the maximal torus $T$ in the fixed splitting $\text{spl}_G$ by conjugation (note that this isomorphism is over $\hat{F}$). Then, by considering its dual, we get an isomorphism
between the dual torus $\hat{S}$ of $S$ and $T$ belonging to the splitting $\text{spl}_G$. We write $\hat{j}$ for the embedding

$$\hat{j}: \hat{S} \cong T \hookrightarrow \hat{G}$$

obtained in this way (note that this embedding is defined canonically up to $\hat{G}$-conjugation). Then, if we have a set $\chi := \{\chi_\alpha\}$ of $\chi$-data, we can extend $\hat{j}$ to an $L$-embedding according to Langlands–Shelstad’s construction. More precisely, we can get a $\hat{G}$-conjugacy class of $L$-embeddings of $LS$ into $LG$. Here we do not recall the construction of this extension. See [LS87, Section 2.6] for the details (see also [Tam16, Section 6.2]). We write $Lj_\chi$ for this extended $L$-embedding

$$Lj_\chi: LS \hookrightarrow LG$$

and call it Langlands–Shelstad’s $L$-embedding determined by the $\chi$-data $\chi = \{\chi_\alpha\}$.

Remark 3.3. Strictly speaking, this $L$-embedding $Lj_\chi$ is well-defined only up to $\hat{G}$-conjugation. However, since our purpose is to construct an $L$-parameter of $G$ by using this $L$-embedding, we do not have to take care of this ambiguity. More concretely, taking $\hat{G}$-conjugation does not change the isomorphism classes of $n$-dimensional Weil–Deligne representations.

3.2. Tam’s description of Langlands–Shelstad’s $L$-embeddings. Let us assume that $S$ is elliptic in $G$. Then it is well-known that $S$ corresponds to a finite extension $E$ of $F$ which is of degree $n$, that is, $S$ is isomorphic to $\text{Res}_E/F Gm$. Recall that, by using this field $E$, we can describe the set $\Gamma_F \setminus \Phi(S, G)$ of $\Gamma_F$-orbits of absolute roots of $S$ in $G$ in the following manner (see [Tam16, Section 3.1] for the details). First we fix a set $\{g_1, \ldots, g_n\}$ of representatives of the quotient $\Gamma_F/\Gamma_E$ such that $g_1 = 1$. Then we get an isomorphism

$$S(F) \cong \prod_{i=1}^n F^*$$

which maps $x \in E^* \cong S(F)$ to $(g_1(x), \ldots, g_n(x))$. Then the projections

$$\delta_i: S(F) \cong \prod_{i=1}^n F^* \rightarrow F^*; \quad (x_1, \ldots, x_n) \mapsto x_i$$

form a $\mathbb{Z}$-basis of the group $X^*(S)$ of (absolute) characters of $S$. Under this usage of notation, the set of $\Phi(S, G)$ of absolute roots of $S$ in $G$ is given by

$$\left\{ \begin{bmatrix} g_i \\ g_j \end{bmatrix} : = \delta_i - \delta_j \right| 1 \leq i \neq j \leq n \right\}$$

and the set $\Gamma_F \setminus \Phi(S, G)$ is described as follows:

$$(*) \quad (\Gamma_E \setminus \Gamma_F/\Gamma_E)^{1,1} \rightarrow \Gamma_F \setminus \Phi(S, G); \quad \Gamma_E g_i \Gamma_E \mapsto \Gamma_F \cdot \begin{bmatrix} 1 \\ g_i \end{bmatrix},$$

where $(\Gamma_E \setminus \Gamma_F/\Gamma_E)^{1,1}$ is the set of nontrivial double-$\Gamma_E$-cosets in $\Gamma_F$.

We fix a subset $D$ of $\{g_1, \ldots, g_n\}$ such that the double-$\Gamma_E$-cosets represented by elements of $D$ map to $\Gamma_F \setminus \Phi(S, G)$ bijectively under the map $(*)$, respectively. In other words, we have

$$\Gamma_F \setminus \Phi(S, G) = \left\{ \Gamma_F \cdot \begin{bmatrix} 1 \\ g \end{bmatrix} : g \in D \right\}.$$
Now let us recall Tam’s description of Langlands–Shelstad’s $L$-embeddings determined by $\chi$-data. We suppose that we have a set $\chi$ of $\chi$-data with respect to $S$. Then, as we explained in Section 3.1, we obtain an $L$-embedding $L_{j_\chi} : L_S \hookrightarrow L_G$ from $\chi$. On the other hand, if we have a character $\xi$ of $E^\times (\cong S(F))$, then we can attach to it an $L$-parameter $\phi_\xi : W_F \rightarrow L_S$ by the local Langlands correspondence for tori (see, for example, [Yu09]). By composing it with the $L$-embedding $L_{j_\chi}$, we may regard it as an $L$-parameter of $G$:

$$L_{j_\chi} \circ \phi_\xi : W_F \xrightarrow{\phi_\xi} L_S \xrightarrow{L_{j_\chi}} L_G.$$

As the $L$-group $L_G$ of $G$ is given by $GL_n(\mathbb{C}) \times W_F$, by considering the projection from $L_G$ to $GL_n(\mathbb{C})$, we may regard $L_{j_\chi} \circ \phi_\xi$ as an $n$-dimensional representation of $W_F$. Then it can be described as follows:

**Proposition 3.4 ([Tam16 Proposition 6.5]).** We put $\mu_\chi$ to be the character of $E^\times$ defined by

$$\mu_\chi := \prod_{[\alpha] \in \Gamma_F \setminus \Phi(S, G)} \chi_\alpha|_{E^\times}.$$

Here, for each $[\alpha] \in \Gamma_F \setminus \Phi(S, G)$, we take its representative $\alpha \in \Phi(S, G)$ to be of the form $\begin{bmatrix} 1 \\ g \end{bmatrix}$ for some (unique) $g \in D$. Then the $L$-parameter $L_{j_\chi} \circ \phi_\xi$ is isomorphic to $\text{Ind}^{W_F}_{W_E}(\xi \mu_\chi)$ as an $n$-dimensional representation of $W_F$.

Here we note that, if we suppose that a root $\alpha \in \Phi(S, G)$ is of the form $\begin{bmatrix} g_i \\ g_j \end{bmatrix} = \delta_i - \delta_j$, then the field $F_{\alpha}$ is given by the composite field $g_i(E) \cdot g_j(E)$ of $g_i(E)$ and $g_j(E)$. In particular, under the choice of a representative $\alpha$ of $[\alpha]$ as in Proposition 3.4, the source $F_{\alpha}$ of the character $\chi_\alpha$ contains $E^\times$. Thus restricting $\chi_\alpha$ to $E^\times$ makes sense.

### 3.3. Bushnell–Henniart’s essentially tame LLC

In [HT01], Harris and Taylor constructed a bijective map, which is called the local Langlands correspondence for $GL_n$, from the set $\Pi(G)$ of equivalence classes of irreducible smooth representations of $G(F)$ to the set $\Phi(G)$ of $\hat{G}$-conjugacy classes of $L$-parameters of $G$ (or equivalence classes of $n$-dimensional semisimple Weil–Deligne representations). We write $\text{LLC}^{\text{HT}}_G$ for this map:

$$\text{LLC}^{\text{HT}}_G : \Pi(G) \xrightarrow{1:1} \Phi(G).$$

Then Bushnell–Henniart’s theory of essentially tame local Langlands correspondence gives an explicit description of this map $\text{LLC}^{\text{HT}}_G$ for so-called “essentially tame supercuspidal representations”. Let us recall what these are briefly.

First, from an $F$-admissible pair $(E, \xi)$, which is a pair of

- a finite tamely ramified extension $E$ of $F$ of degree $n$ and
- an $F$-admissible character $\xi$ (see [BH05a Definition in 686 page] for the definition of the $F$-admissibility of a character),

we can obtain an irreducible supercuspidal representation $\pi^{\text{BH}}_{(E, \xi)}$ of $G(F)$ according to a construction of Bushnell–Henniart (see [BH05a Section 2] or [Tam16 Section 5]). We call a representation obtained in this way an essentially tame supercuspidal representation.
Remark 3.5. Precisely speaking, Bushnell and Henniart first defined essentially tame supercuspidal representations by using of the notion of the torsion number of supercuspidal representations (see the beginning of [BH05a, Section 2]). Then they gave a characterization of essentially tame supercuspidal representations in terms of F-admissible pairs, that is, they proved that essentially tame supercuspidal representations are parametrized by F-admissible pairs ([BH05a, Theorem 2.3]). In this paper we do not go back to the original definition since only the latter characterization is important from our viewpoint.

For every essentially tame supercuspidal representation, Bushnell–Henniart proved that its image under the map LLC_{G}^{HT} can be described by using a character $\mu_{rec}$ called the rectifier of $(E, \xi)$ as follows:

Theorem 3.6 ([BH05a BH05b BH10]). There exists a tame character $\mu_{rec}$ of $E^\times$, which can be described explicitly in terms of $E$ and $\xi$, satisfying

$$LLC_{G}^{HT} (\pi_{(E, \xi, \mu_{rec})}) = \text{Ind}_{W_{E}}^{W_{F}} \xi;$$

or equivalently,

$$LLC_{G}^{HT} (\pi_{(E, \xi)}) = \text{Ind}_{W_{E}}^{W_{F}} (\xi^{\mu_{rec}})\xi^{-1}.$$

Note that although we simply write $\mu_{rec}$ for the rectifier, it is defined for each $F$-admissible pair $(E, \xi)$ and depends on $(E, \xi)$.

What Tam noticed is that this mysterious character $\mu_{rec}$ of $E^\times$ has a factorization in terms of $\chi$-data. Let us take a tamely ramified elliptic maximal torus $S$ of $G$ which is isomorphic to $\text{Res}_{E/F} G_{m}$. Then Tam produced a set $\chi_{Tam} = \{\chi_{Tam, \alpha}\}_{\alpha}$ of $\chi$-data with respect to $(S, G)$ from each $F$-admissible pair $(E, \xi)$ and proved the following:

Theorem 3.7 ([Tam16, Theorem 7.1]). Let $\mu_{rec}$ be the rectifier with respect to an $F$-admissible pair $(E, \xi)$. Then we have

$$\mu_{rec} = \mu_{\chi_{Tam}}.$$

Here the right-hand side is a character of $E^\times$ determined by $\chi_{Tam}$ in the manner of Proposition 3.4.

The definition of Tam’s $\chi$-data $\chi_{Tam}$ is given in [Tam16, Theorem 7.1]. Later we will recall it precisely (Sections 6.2, 6.3, and 6.4). As a corollary of Theorems 3.6 and 3.7 we get the following consequence:

Corollary 3.8. Let $S$ be an elliptic maximal torus of $G$ which is isomorphic to $\text{Res}_{E/F} G_{m}$ for a finite tamely ramified extension $E$ of $F$ of degree $n$. Let $\xi$ be an $F$-admissible character of $S(F) \cong E^\times$ and $\pi_{(E, \xi)}^{BH}$ the irreducible supercuspidal representation of $GL_{n}(F)$ arising from the pair $(E, \xi)$. Then we have

$$LLC_{G}^{HT} (\pi_{(E, \xi)}^{BH}) = \text{Ind}_{W_{E}}^{W_{F}} (\xi^{\mu_{\chi_{Tam}}})\xi^{-1}.$$

4. Kaletha’s LLC for regular supercuspidal representations

In this section, we quickly review Kaletha’s construction of the local Langlands correspondence in the case of $GL_{n}$. In [Kal19a], he constructed the local Langlands correspondence for supercuspidal representations satisfying some regularity condition, which are called “regular supercuspidal representations”, of tamely ramified connected reductive groups, under a small restriction on the residual characteristic
In particular, his construction works for $G = \text{GL}_n$ whenever $p$ is an odd prime number.

4.1. **Regular supercuspidal representations.** In this section, let us temporarily consider a general situation where $G$ is a tamely ramified connected reductive group over $F$. In general, *regular supercuspidal representations* are defined to be supercuspidal representations obtained from “regular Yu-data” according to Yu’s construction. Here recall that a Yu-datum consists of

- a sequence $G^0 \subset G^1 \subset \cdots \subset G^d = G$ of tame twisted Levi subgroups of $G$ over $F$;
- an irreducible depth zero supercuspidal representation $\pi_{-1}$ of $G^0 (F)$, and
- characters $\phi_i : G^i (F) \rightarrow \mathbb{C}^\times$ for $0 \leq i \leq d$

satisfying several conditions. We do not recall the precise definition of (regular) Yu-data nor Yu’s construction of supercuspidal representations. See [Kal19a, Section 3.5 and Definition 3.7.3] for details. For a given Yu-datum $\Psi$, let $\pi^\text{Yu}_{\Psi}$ denote the irreducible supercuspidal representation obtained from $\Psi$ by Yu’s construction.

One of important discoveries of Kaletha is the following fact saying that regular Yu-data bijectively correspond to “tame elliptic regular pairs”, which are much simpler objects consisting only of tamely ramified elliptic tori and characters on them (see [Kal19a, Definition 3.7.5] for the precise definition of a tame elliptic regular pair):

**Proposition 4.1** ([Kal19a, Proposition 3.7.8]). Suppose that we have a regular Yu-datum $(G^0 \subset G^1 \subset \cdots \subset G^d, \pi_{-1}, (\phi_0, \ldots, \phi_d))$. Then we can construct a tamely ramified elliptic maximal torus $S$ of $G$ contained in $G^0$ and a character $\phi_{-1}$ on it such that the pair $(S, \prod_{i=-1}^d \phi_i|_{S(F)})$ is tame elliptic regular. Moreover, this procedure

$$(G^0 \subset G^1 \subset \cdots \subset G^d, \pi_{-1}, (\phi_0, \ldots, \phi_d)) \mapsto (S, \prod_{i=-1}^d \phi_i|_{S(F)})$$

gives a bijection from the set of $G$-equivalence classes of regular Yu-data to the set of $G(F)$-conjugacy classes of tame elliptic regular pairs.

Here, “$G$-equivalence” is the equivalence relation for Yu-data introduced by Hakim–Murnaghan in [HM08]. Its important nature is that it describes the “fibers” of Yu’s construction, that is, for any two Yu-data $\Psi_1$ and $\Psi_2$, we have $\pi^\text{Yu}_{\Psi_1} \cong \pi^\text{Yu}_{\Psi_2}$ if and only if $\Psi_1$ is $G$-equivalent to $\Psi_2$. See [HM08, Definition 6.1] or [Kal19a, Section 3.5] for the definition of $G$-equivalence. In particular, by Proposition 4.1 and the definition of regular supercuspidal representations, we can parametrize the set of equivalence classes of regular supercuspidal representations by the set of $G(F)$-conjugacy classes of tame elliptic regular pairs of $G$. When a regular supercuspidal representation $\pi$ arises from a tame elliptic regular pair $(S, \xi)$, we write $\pi = \pi^{(S, \xi)}_{\text{Kal}}$. Thus, if $(S, \xi)$ corresponds to a $G$-equivalence class represented by a Yu-datum $\Psi$, we have $\pi^{(S, \xi)} = \pi^\text{Yu}_{\Psi}$. Then, for any tame elliptic regular pair $(S, \xi)$, we have $\pi^{(S, \xi)} = \pi^\text{Yu}_{\Psi}$.
then we have $\pi_{\text{Kal}}^{(S,\xi)} = \pi_{\Psi}^{\text{Yu}}$ by definition.

\[
\begin{array}{c}
\{\text{irred. s.c. rep'ns of } G(F)\} / \sim \\
\{\text{Yu-data}\} / G\text{-eq.} \xrightarrow{1:1} \{\text{Yu’s s.c. rep’ns of } G(F)\} / \sim \\
\{\text{regular Yu-data}\} / G\text{-eq.} \xrightarrow{1:1} \{\text{regular s.c. rep’ns of } G(F)\} / \sim \\
\{\text{tame elliptic regular pair}\} / G(F)\text{-conj.}
\end{array}
\]

The key point which we have to keep in our mind here is the following connection with Bushnell–Henniart’s construction in the case where $G = \text{GL}_n$. In this case, since every tamely ramified elliptic maximal torus of $G$ is isomorphic to $\text{Res}_{E/F} G_m$ for some finite tamely ramified extension $E$ of $F$ of degree $n$, we can obtain a pair $(E, \xi)$ of such a field $E$ and a character $\xi$ on $E^\times$ from each tame elliptic regular pair $(S, \xi)$ of $G$. In fact, the pair $(E, \xi)$ obtained in this way is $F$-admissible and, conversely, every $F$-admissible pair is obtained from a tame elliptic regular pair (see [Kal19a, Lemma 3.7.7]). Here note that the proof of [Kal19a, Lemma 3.7.7] works for any odd prime number $p$ although the condition $p \nmid n$ is imposed in [Kal19a, Lemma 3.7.7]. Then the supercuspidal representation $\pi_{\text{BH}}^{(E,\xi)}$ constructed from this pair $(E, \xi)$ in Bushnell–Henniart’s way coincides with $\pi_{\text{Kal}}^{(S,\xi)}$. Indeed, Yu’s construction of supercuspidal representations is nothing but Howe’s classical construction when we consider the general linear group (see, for example, [HM08, Section 3.5]). Moreover, as explained in [BH05a, 695 page, Remark 2], Bushnell–Henniart’s construction gives the same supercuspidal representations as those constructed by Howe. As another way to confirm the coincidence of $\pi_{\text{BH}}^{(E,\xi)}$ with $\pi_{\text{Kal}}^{(S,\xi)}$, we can also compare Bushnell–Henniart’s construction with Yu’s one directly. Only nontrivial point in this comparison is how to extend a Heisenberg representation of a certain open compact subgroup written as $J^1$ in [BH05a] (resp. $K^d$ in [Yu01]) to a slightly bigger non-compact subgroup written as $J$ in [BH05a] (resp. $K^d$ in [Yu01]). In Bushnell–Henniart’s construction, they took an extension satisfying several conditions which characterizes the extension uniquely (see [BH05a, Section 2.3]). On the other hand, in Yu’s construction, he defined an extension by using the Heisenberg–Weil representation. Thus, in order to show the coincidence, it suffices to check that Yu’s extension indeed satisfies the conditions characterizing Bushnell–Henniart’s extension, and it can be done easily.

In the following, when a tame elliptic regular pair $(S, \xi)$ of $\text{GL}_n$ corresponds to an $F$-admissible pair $(E, \xi)$, we simply write

\[
\pi^{(S,\xi)} := \pi_{\text{Kal}}^{(S,\xi)} \cong \pi_{\text{BH}}^{(E,\xi)}.
\]

**Remark 4.2.** When $p$ does not divide $n$, every irreducible supercuspidal representation of $\text{GL}_n(F)$ is essentially tame (see [BH05a, Section 2]).
4.2. DeBacker–Spice’s sign and Kaletha’s toral invariant. In this section, we recall two invariants obtained from Yu-data and used in Kaletha’s construction of the local Langlands correspondence for regular supercuspidal representations.

Let us take a Yu-datum \((G^0 \subseteq G^1 \subseteq \cdots \subseteq G^d, \pi_{-1}, (\phi_0, \ldots, \phi_d))\) of \(G\). Let \(S\) be an elliptic maximal torus of \(G^0\) introduced by Proposition \[11\]. The first invariant which we need is DeBacker–Spice’s sign character \(\epsilon_{\text{sym}} \cdot \epsilon_{\text{sym, ur}}\) defined in \[DS18, Section 4.3\]. This is the product of two characters \(\epsilon_{\text{sym}}\) and \(\epsilon_{\text{sym, ur}}\) of \(S(F)\). In order to define these two characters, we first define a character \(\epsilon_\alpha\) of \(S(F)\) for each asymmetric or symmetric unramified root of \(S\) in \(G\). For each \(\alpha \in \Phi(S, G)\), we can define a set \(\text{ord}_\alpha(\alpha)\) of real numbers as in \[DS18, Definition 3.6\] (we will introduce its definition later, see Section \[5.3\]). On the other hand, for each \(\alpha \in \Phi(S, G)\), there exists a unique index \(i\) such that \(\alpha \in \Phi(S, G^{i+1})\setminus \Phi(S, G^i)\).

We put \(r_i\) to be the depth of the character \(\phi_i\). Then the character \(\epsilon_\alpha : S(F) \to \mathbb{C}^\times\) is defined as

\[
\epsilon_\alpha(\gamma) := \begin{cases} 
\frac{\overline{\epsilon(\gamma)}}{\epsilon_{F_\alpha}} & \text{if } \frac{\gamma}{2} \in \text{ord}_\alpha(\alpha), \\
1 & \text{otherwise},
\end{cases}
\]

when \(\alpha\) is asymmetric, and

\[
\epsilon_\alpha(\gamma) := \begin{cases} 
\frac{\epsilon(\gamma)}{\epsilon_{F_\alpha}} & \text{if } \frac{\gamma}{2} \in \text{ord}_\alpha(\alpha), \\
1 & \text{otherwise},
\end{cases}
\]

when \(\alpha\) is symmetric unramified. Here, in both cases, we regard \(\alpha(\gamma)\), which lies in \(\mathcal{O}_{F_\alpha}^\times\) by the ellipticity of \(S\), as an element of \(k_{F_\alpha}^\times\) by modulo \(p_{F_\alpha}\) reduction. Note that, in the latter case, the definition makes sense since \(\alpha(\gamma)\) always belongs to the kernel of the norm map \(\text{Nr}_{F_\alpha/F_{\pm \alpha}} : F_\alpha^\times \to F_{\pm \alpha}^\times\). Indeed, if we take the nontrivial element \(\tau_\alpha\) of the Galois group of \(F_\alpha/F_{\pm \alpha}\), then we have \(\tau_\alpha(\alpha) = -\alpha\). In other words, we have \(\tau_\alpha(\alpha(\gamma)) = \alpha(\tau_\alpha(\gamma))^{-1}\) for every \(\gamma \in S(F_\alpha)\). Thus, if \(\gamma \in S(F)\), then we get

\[
\text{Nr}_{F_\alpha/F_{\pm \alpha}}(\alpha(\gamma)) = \alpha(\gamma) \cdot \tau_\alpha(\alpha(\gamma)) = \alpha(\gamma) \cdot \alpha(\tau_\alpha(\gamma))^{-1} = \alpha(\gamma) \cdot \alpha(\gamma)^{-1} = 1.
\]

We define \(\epsilon_{\text{sym}}\) and \(\epsilon_{\text{sym, ur}}\) as products of these \(\epsilon_\alpha\)'s:

\[
\epsilon_{\text{sym}} := \prod_{\alpha \in \Gamma_F \times \{\pm 1\} \setminus \Phi(S, G)^{\text{sym}}} \epsilon_\alpha \quad \text{and} \quad \epsilon_{\text{sym, ur}} := \prod_{\alpha \in \Gamma_F \setminus \Phi(S, G)^{\text{sym, ur}}} \epsilon_\alpha,
\]

where the index set of the first product denotes the set of orbits of asymmetric roots via the action of \(\Gamma_F \times \{\pm 1\}\) (the action of \(-1 \in \{\pm 1\}\) is given by \(\alpha \mapsto -\alpha\)).

The second invariant which we have to consider is the character \(\epsilon_{f, \text{ram}}\) defined in \[Kal19a, Definition 4.7.3\]. As explained in \[Kal19a, Lemma 4.7.4\], this is a character of \(S(F)\) expressed as the product of “toral invariants” \(f_{(G, S)}\) for symmetric ramified roots, which are introduced in \[Kal15, Section 4.1\]:

\[
\epsilon_{f, \text{ram}}(\gamma) = \prod_{\alpha \in \Gamma_F \setminus \Phi(S, G)^{\text{sym, ur}}} f_{(G, S)}(\alpha) \quad \text{for } \gamma \in S(F).
\]

In fact, as a special feature of the general linear group, we can show that this invariant is always trivial:
**Proposition 4.3.** For every symmetric root $\alpha \in \Phi(S, G)_\text{sym}$, its toral invariant is trivial, i.e., $f_{(G, S)}(\alpha) = 1$.

**Proof.** We first recall the definition of the toral invariant. Let $\alpha$ be a symmetric root of $S$ in $G$. Then we have the corresponding quadratic extension $F_\alpha/F_{\mathbb{Z}_\alpha}$. We take an element $\tau_\alpha \in \Gamma_{-\alpha}\Gamma_{\alpha}$. If we take a nonzero root vector $X_\alpha \in g_\alpha(F_\alpha)$ for the root $\alpha$, then $\tau_\alpha(X_\alpha)$ is a root vector for the root $\tau_\alpha(\alpha) = -\alpha$. Thus their Lie bracket product $[X_\alpha, \tau_\alpha(X_\alpha)]$ gives a nonzero element of the maximal torus $s(F_\alpha) \subset g(F_\alpha)$.

On the other hand, the differential $d\alpha^\vee$ of the coroot $\alpha^\vee$ corresponding to $\alpha$ defines a nonzero element $H_\alpha := d\alpha^\vee(1)$ of $s(F_\alpha)$. By using these elements, we define the toral invariant $f_{(G, S)}(\alpha)$ of $\alpha$ by

$$f_{(G, S)}(\alpha) := \kappa_{\alpha}\left(\frac{[X_\alpha, \tau_\alpha(X_\alpha)]}{H_\alpha}\right),$$

where $\kappa_{\alpha}$ is the unique quadratic character of $F_{\mathbb{Z}_\alpha}^\times$ corresponding to the quadratic extension $F_\alpha/F_{\mathbb{Z}_\alpha}$. Thus, for our purpose, it suffices to find a root vector $X_\alpha$ satisfying $[X_\alpha, \tau_\alpha(X_\alpha)] = H_\alpha$.

Let $E$ be a tamely ramified extension of $F$ of degree $n$ corresponding to the tamely ramified elliptic maximal torus $S$ of $G$. By taking an $F$-basis of $E$, we obtain an isomorphism

$$g(F) = g_{1n}(F) \cong \text{End}_F(E).$$

Then we may assume that $s(F) \subset g(F)$ corresponds to $E$, which is canonically embedded into $\text{End}_F(E)$ as multiplication maps, in the right-hand side. With this identification in our mind, in order to describe the (absolute) root space decomposition of $g$ via $S$, we consider the action of $(E \otimes_F \overline{F})^\times$ on $\text{End}_F(E) \otimes_F \overline{F}$.

Recall that, in Section 3.2, we took a set $\{g_1, \ldots, g_n\}$ of representatives of $\Gamma_F/\Gamma_E$ and that we have an isomorphism

$$E \otimes_F \overline{F} \cong \bigoplus_{i=1}^n \overline{F}: \quad x \otimes y \mapsto (g_i(x)y)_i.$$

Thus the set of characters and cocharacters of $S$ is described as

$$X^\times(S) = \bigoplus_{i=1}^n \mathbb{Z}\delta_i \quad \text{and} \quad X^\times(S) = \bigoplus_{i=1}^n \mathbb{Z}\delta_i^\vee,$$

where $\delta_i$ and $\delta_i^\vee$ are given by (on $\overline{F}$-valued points)

$$\delta_i: \prod_{i=1}^n \overline{F}^\times \to \overline{F}^\times: \quad (x_i)_i \mapsto x_i,$$

$$\delta_i^\vee: \overline{F}^\times \to \prod_{i=1}^n \overline{F}^\times: \quad x \mapsto (1, \ldots, 1, x_i, 1, \ldots, 1),$$

respectively, and the set of roots of $S$ in $g$ is given by $\{\delta_i - \delta_j \mid 1 \leq i \neq j \leq n\}$. We put $d_i$ and $d_i^\vee$ to be their differentials, i.e.,

$$d_i: \bigoplus_{i=1}^n \overline{F} \to \overline{F}: \quad (x_i)_i \mapsto x_i,$$

$$d_i^\vee: \overline{F} \to \bigoplus_{i=1}^n \overline{F}: \quad x \mapsto (0, \ldots, 0, x_i, 0, \ldots, 0).$$

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Then we can check that $d'_i \circ d_j$ is a root vector for the root $\delta_i - \delta_j$.

Let us assume that our symmetric root $\alpha$ is given by $\delta_i - \delta_j$ and show that $X_\alpha := d'_i \circ d_j$ is a desired root vector. First, this element $X_\alpha \in g_\alpha(F)$ belongs to $g_\alpha(F)$. Indeed, in order to check this, it suffices to show that $\tau(X_\alpha) = X_\alpha$ for every $\tau \in \Gamma_\alpha$. By considering how the $\Gamma_F$-action on $E \otimes_F \overline{F}$ is converted on $\bigoplus_{i=1}^n \overline{F}$, we can easily check that the $\Gamma_F$-action on characters and cocharacters is given by $\tau(\delta_i) = \delta_{\tau(i)}$ (hence $\tau(d_i) = d_{\tau(i)}$) and $\tau(\delta'_i) = \delta'_{\tau(i)}$ (hence $\tau(d'_i) = d'_{\tau(i)}$). Here $\tau(i)$ denotes the unique index satisfying $g_{\tau(i)}g_E = (\tau \circ g_i)g_E$. Thus, for $\tau \in \Gamma_F$, $\tau$ belongs to $\Gamma_\alpha = \operatorname{Stab}_{\Gamma_F}(\delta_i - \delta_j)$ if and only if we have $\tau(i) = i$ and $\tau(j) = j$. On the other hand, we can also check that $\tau(d_i \circ d'_j) = d_{\tau(i)} \circ d'_{\tau(j)}$ easily. Thus $\tau \in \Gamma_\alpha$ implies that $\tau(X_\alpha) = X_\alpha$.

Now let us show that $[X_\alpha, \tau_\alpha(X_\alpha)] = H_\alpha$. By noting that $\mathfrak{g}(\overline{F})$ is embedded into $\operatorname{End}_F(E) \otimes_F \overline{F}$ as

\[
\mathfrak{g}(\overline{F}) \cong \bigoplus_{i=1}^n \overline{F} \hookrightarrow \operatorname{End}_F(E) \otimes_F \overline{F} \cong \operatorname{End}_F(E) \otimes_F \overline{F}
\]

we get $H_\alpha = d'_i \circ d_j - d_j' \circ d_i$. On the other hand, as we have

$\tau_\alpha(\alpha) = \delta_{\tau_\alpha(i)} - \delta_{\tau_\alpha(j)} = -\alpha = \delta_j - \delta_i$, we get $\tau_\alpha(i) = j$ and $\tau_\alpha(j) = i$. Hence we have

$\tau_\alpha(X_\alpha) = d'_{\tau_\alpha(i)} \circ d_{\tau_\alpha(j)} = d'_i \circ d_j$. By noting that $d_i \circ d'_j = \text{id}$ for any $i$, we get

$[X_\alpha, \tau_\alpha(X_\alpha)] = X_\alpha \circ \tau_\alpha(X_\alpha) - \tau_\alpha(X_\alpha) \circ X_\alpha = d'_i \circ d_j - d_i \circ d'_j \circ d_i \circ d'_j \circ d_j = d'_i \circ d_j - d_j' \circ d_i$.

This completes the proof.

4.3. Kaletha’s Construction of $L$-packets and $L$-parameters. In [Kal19a], Kaletha attached an $L$-packet and an $L$-parameter to each “regular supercuspidal $L$-packet datum” of $G$. Let us recall his construction briefly. Here we suppose that $G$ is a quasi-split tamely ramified connected reductive group temporarily (later we will focus on the case where $G = \operatorname{GL}_n$).

First recall that a regular supercuspidal $L$-packet datum of $G$ is a tuple $(S, \tilde{j}, \chi, \xi)$ consisting of

- a tamely ramified torus $S$ over $F$ whose absolute rank is the same as that of $G$,
- an embedding $\tilde{j} : S \hookrightarrow \hat{G}$ whose $\hat{G}$-conjugacy class is $\Gamma_F$-stable,
- a set $\chi$ of $\chi$-data for $(S, G)$, and
- a character $\xi : S(F) \to \mathbb{C}^\times$ satisfying several conditions (see [Kal19a] Definition 5.2.4]). Here the meaning of “a set of $\chi$-data for $(S, G)$” is as follows (see [Kal19a] Section 5.1 for the details): First, by considering the converse of the procedure in Section 4.1, we can get a $\Gamma_F$-stable $G(\overline{F})$-conjugacy class $J$ of embeddings of $S$ into $G$. Then, as explained
in [Kal19a, Section 5.1], the class $J$ has a $\Gamma_F$-fixed element, i.e., an embedding of $S$ into $G$ defined over $F$ because of the quasi-splitness of $G$ ([KotS2 Corollary 2.2]). By taking such an embedding $j: S \hookrightarrow G$ defined over $F$, $j(S)$ gives a maximal torus of $G$ defined over $F$. In particular, by pulling back the set $\Phi(j(S), G)$ via $j$, we get a subset $\Phi(S, G)$ of $X^*(S)$ which is $\Gamma_F$-stable. Since this set with $\Gamma_F$-action does not depend on the choice of $j$, we can define the notion of a set of $\chi$-data in the usual way (see Section 3.1). We note that, in the definition of a regular supercuspidal $L$-packet datum, an ellipticity condition on $S$ is also imposed (i.e., if we regard $S$ as a subtorus of $G$ by using $j$ as above, then it should be elliptic). We also note that we can extend $\tilde{j}$ to an $L$-embedding $L_{\tilde{j}\chi}: L_S \hookrightarrow L_G$ by using a set of $\chi$-data $\chi$ in the same manner as in Section 3.1.

Next let us recall the construction of $L$-packets and $L$-parameters. We take a regular supercuspidal $L$-packet datum $(S, \tilde{j}, \chi, \xi)$ of $G$. First, we can obtain an $L$-parameter from this regular supercuspidal $L$-packet datum $(S, \tilde{j}, \chi, \xi)$ just by using the local Langlands correspondence for tori:

**Construction of $L$-parameters** ([Kal19a, Proof of Proposition 5.2.7]): By applying the local Langlands correspondence for $S$ to $\xi$, we get an $L$-parameter $\phi_S$ of $S$, which is a homomorphism from $W_F$ to $L_S$. On the other hand, by the Langlands–Shelstad construction, we can extend $\tilde{j}$ to an $L$-embedding $L_{\tilde{j}\chi}$ from $L_S$ to $L_G$ by using the set $\chi$ of $\chi$-data. Thus, by composing these two homomorphisms, we get an $L$-parameter $\phi$ of $G$:

$$\phi: W_F \xrightarrow{\phi_S} L_S \xrightarrow{L_{\tilde{j}\chi}} L_G.$$ 

Next we recall the construction of the $L$-packet attached to $(S, \tilde{j}, \chi, \xi)$:

**Parametrization of members in each $L$-packet** ([Kal19a, 1154 page]): As explained in the beginning of this section, from $\tilde{j}$, we can get a $\Gamma_F$-stable $G(F)$-conjugacy class $J$ of embeddings of $S$ into $G$. We put $J$ to be the set of $G(F)$-conjugacy classes of embeddings which belong to $J$ and defined over $F$. Then members of the $L$-packet $\Pi_\phi$ for the $L$-parameter $\phi$ are parametrized by the elements of $J$.

**Construction of members** ([Kal19a, 1153-1154 page]): For each $j \in J$, we call a tuple $(S, \tilde{j}, \chi, \xi, j)$ a regular supercuspidal datum of $G$ (mapping to the regular supercuspidal $L$-packet datum $(S, \tilde{j}, \chi, \xi)$). For each regular supercuspidal datum $(S, \tilde{j}, \chi, \xi, j)$, the pair

$$(j(S), \xi_j) := (j(S), \epsilon \cdot (\xi \cdot \zeta_S^{-1}) \circ j^{-1})$$

is a $(G(F)$-conjugacy class of) tame elliptic regular pair of $G$. Here

$$\epsilon := \epsilon_{\text{sym}} \cdot \epsilon_{\text{sym, ur}} \cdot \epsilon_{\text{f, ram}}$$

and $\zeta_S$ is the character determined by the “$\zeta$-data” measuring the difference between $\chi$ and “Kaletha’s $\chi$-data” $\chi_{\text{Kal}}$. Kaletha’s $\chi$-data $\chi_{\text{Kal}}$ is defined in [Kal19a, the paragraph before Definition 4.7.3] (note that in [Kal19a $\chi_{\text{Kal}}$ is written as $\chi'$). Later we will recall its definition (Sections 6.2, 6.3, and 6.4). On the other hand, we do not recall the definition of the character $\zeta_S$, but point out that it is trivial when $\chi$ is equal to $\chi_{\text{Kal}}$ (see Step 2 and Definition 4.6.5 in [Kal19a for details). We put

$$\Pi_\phi := \{ (j(S), \xi_j) \mid j \in J \}.$$
Remark 4.4. In Kal19a, $L$-packets are constructed according to his formalism of the “rigid inner form”, which was established in Kal16, and compounded in the sense that they consist not only of representations of $G(F)$ but also those of all rigid inner forms of $G$. Thus, strictly speaking, we also have to add a datum of a rigid inner twist to the above explanation of the definition of regular supercuspidal datum (see Kal19a Definition 5.3.2). However, since we treat the local Langlands correspondence only in the case of $G$ itself in this paper, we ignore this point. More precisely, we always take a rigid inner twist $(G',\psi,z)$ in each regular supercuspidal datum $(S,\tilde{j},\chi,\xi,(G',\psi,z),j)$ to be the trivial twist $(G,\text{id},1)$, and write shortly $(S,\tilde{j},\chi,\xi,j)$ for $(S,\tilde{j},\chi,\xi,(G,\text{id},1),j)$.

Here we note that the above map
$$(S,\tilde{j},\chi,\xi,j) \mapsto (j(S),\xi_j) := (j(S),\epsilon \cdot (\xi \cdot \zeta_S^{-1}) \circ j^{-1})$$
gives a bijection from the set of isomorphism classes of regular supercuspidal data of $G$ to the set of $G(F)$-conjugacy classes of tame elliptic regular pairs of $G$. Indeed, the well-definedness of this map is explained (implicitly) in Steps 2 and 3 in Kal19a Section 5.3. Moreover we can check that the inverse map is given by
$$(S,\xi) \mapsto (S,\tilde{j},\chi_{\text{Kal}},\epsilon^{-1} \cdot \xi, j),$$
where $j$ is the inclusion of $S$ into $G$ and $\tilde{j}$ is the embedding constructed in the manner as explained in Section 3.1. As a consequence, we can conclude that Kaletha’s $L$-packets are disjoint from each other and exhausts all of regular supercuspidal representations of $G(F)$.

Now let us focus on the case where $G = GL_n$. An important observation coming from the speciality of the group $GL_n$ is that every $L$-packet constructed in this way is a singleton. Indeed, as already mentioned before, for every regular supercuspidal $L$-packet datum $(S,\tilde{j},\chi,\xi)$, the set $J$ is nonempty because of the (quasi-)splitness of $G$. Moreover, in general, if we take an element $j$ of $J$, then we can check easily that $J$ is parametrized by the set
$$\text{Ker}(H^1(F,S) \xrightarrow{j} H^1(F,G)),$$
where the map between two cohomology groups are the one induced from $j$. As $S$ is an induced torus when $G$ is $GL_n$, this set is always trivial by Shapiro’s lemma and Hilbert’s theorem 90. Thus $J$ is a singleton, hence so is the corresponding $L$-packet.

In conclusion, Kaletha’s construction of the local Langlands correspondence for regular supercuspidal representations of $G(F) = GL_n(F)$ is summarized as follows:

**Proposition 4.5.** We take a tame elliptic regular pair $(S,\xi)$ of $G$ and consider the regular supercuspidal representation $\pi(S,\xi)$ of $G(F)$ arising from this pair $(S,\xi)$. Let $E$ be a tamely ramified extension of $F$ of degree $n$ corresponding to $S$. We write $\text{LLC}_{G}^{\text{Kal}}(\pi(S,\xi))$ for the $L$-parameter corresponding to $\pi(S,\xi)$ by Kaletha’s construction. Then we have
$$\text{LLC}_{G}^{\text{Kal}}(\pi(S,\xi)) = \text{Ind}^{W_E}_{W_F}(\epsilon^{-1}\xi_{\mu_{\text{Kal}}}).$$

**Proof.** Let $(S,\xi)$ be a pair as in the statement. Then, as explained in this section, the $L$-packet containing $\pi(S,\xi)$ is a singleton and arises from the regular supercuspidal $L$-packet datum $(S,\tilde{j},\chi_{\text{Kal}},\epsilon^{-1}\xi)$. Thus the corresponding $L$-parameter is given
by $L_{j}_{\chi_{K}}\circ\phi_{e-1}\xi$. If we apply Proposition 3.4 to this $L$-parameter, we obtain the induced representation as desired. □

5. Prerequisites for a comparison of $\chi$-data

5.1. Classification of roots of elliptic tori of $\text{GL}_n$. Let $E/F$ be a tamely ramified extension of degree $n$. We simply write $e$ (resp. $f$) for the ramification index $e(E/F)$ (resp. residue degree $f(E/F)$). We take an elliptic maximal torus $S$ in $G$ which is isomorphic to $\text{Res}_{E/F} \mathbb{G}_m$. In this section we classify the symmetric unramified roots and ramified roots of $\Phi(S, G)$. Especially, we will determine the relationship between symmetric unramified/ramified roots in the sense of Adler–DeBacker–Spice, which we adopt in this paper, and those in the sense of Tam. We note that Tam’s definition of the symmetricity for roots coincides with our definition. However, his definition of unramifiedness and ramifiedness for symmetric roots is different from ours. See [Tam16, 1710 page] for the definition of symmetric unramifiedness and ramifiedness of Tam.

We first recall an explicit choice of a set of representatives of $\Gamma_{F}/\Gamma_{E}$, following [Tam16, Section 3.2]. We take uniformizers $\varpi_E$ and $\varpi_F$ of $E$ and $F$ respectively, so that $\varpi_E^{e} = \zeta_{E/F} \varpi_F$ for some $\zeta_{E/F} \in \mu_E$. We fix a primitive $e$-th root $\zeta_e$ of unity and an $e$-th root $\zeta_{E/F,e}$ of $\zeta_{E/F}$, and put $\zeta_{\phi} := \zeta_{E/F,e}^{q^{-1}}$. Then $L := E[\zeta_e, \zeta_{E/F,e}]$ is a tamely ramified extension of $F$ which contains the Galois closure of $E/F$ and is unramified over $E$. The Galois group $\Gamma_{L/F}$ of the extension $E/F$ is given by the semi-direct product $\langle \sigma \rangle \rtimes \langle \phi \rangle$, where

$\sigma: \zeta \mapsto \zeta$ (for $\zeta \in \mu_L$), $\varpi_E \mapsto \zeta_{e}\varpi_E$
$\phi: \zeta \mapsto \zeta^{q}$ (for $\zeta \in \mu_L$), $\varpi_E \mapsto \zeta_{\phi}\varpi_E$

and $\phi \sigma \phi^{-1} = \sigma^{q}$. Moreover, as explained in [Tam16, Proposition 3.3 (i)], we can take a set of representatives of $\Gamma_{F}/\Gamma_{E}$ to be

$$\{\Gamma_{F}/\Gamma_{E} \} := \{\sigma^{k} \phi^{i} | 0 \leq k \leq e - 1, 0 \leq i \leq f - 1\}.$$ 

Here we implicitly regard each $\sigma^{k} \phi^{i} \in \Gamma_{L/F}$ as an element of $\Gamma_{F}$ by taking its extension to $\overline{F}$ from $L$. In the rest of this paper, we always adopt this set of representatives and use the notations defined in Section 3.2. We note that, as $L/E$ is unramified, there exists an integer $c$ such that $\Gamma_{L/E} = \langle \sigma^{c} \phi^{j} \rangle$.

The following criterion of the symmetricity for roots is a slight enhancement of [Tam16, Proposition 3.2]:

$$L = F[\varpi_{E}, \mu_{L}]$$
unramified with Galois group $\langle \sigma^{c} \phi^{j} \rangle$
totally ramified with Galois group $\langle \sigma \rangle$
tamely ramified
$E$
$F[\mu_{L}]$
unramified with Galois group $\langle \phi \rangle$
Proposition 5.1. Let \( \alpha \in \Phi(S, G) \) be a root of the form \( \begin{bmatrix} 1 \\ g \end{bmatrix} \) for some \( g \in \{ \Gamma_F / \Gamma_E \} \).

1. We have \( \Gamma_\alpha = \Gamma_E \cap g\Gamma_E g^{-1} \).
2. The root \( \alpha \) is symmetric if and only if there exists \( x_g \in \Gamma_E \) such that \( gx_g g \in \Gamma_E \). Moreover, in this case we have \( (gx_g)^2 \in \Gamma_\alpha \) and \( \Gamma_\alpha = \langle gx_g, \Gamma_\alpha \rangle \).

Proof. As in the proof of [Tam16, Proposition 3.1], the set \( \Phi(S, G) \) is identified with the set of off-diagonal elements in \( \Gamma_F / \Gamma_E \times \Gamma_F / \Gamma_E \) by the map \( \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \mapsto (g_1 \Gamma_E, g_2 \Gamma_E) \) and the induced \( \Gamma_F \)-action on the latter set is described as \( g(1 \Gamma_E, g_2 \Gamma_E) = (gg_1 \Gamma_E, gg_2 \Gamma_E) \).

The assertion (1) follows immediately from this.

Let us prove the assertion (2). The same description shows that \( \Gamma_F \)-orbit of \(-\alpha\) contains \( \alpha' = \begin{bmatrix} 1 \\ g^{-1} \end{bmatrix} \). Therefore, \( \alpha \) is symmetric if and only if \( \Gamma_E g \Gamma_E = \Gamma_E g^{-1} \Gamma_E \) holds. This latter condition amounts to the existence of \( x_g, y_g \in \Gamma_E \) such that \( y_ggx_g = g^{-1} \), or equivalently, the existence of \( x_g \in \Gamma_E \) such that \( gx_gg \in \Gamma_E \). Moreover, for such a choice of \( x_g \in \Gamma_E \), we have

\[
(gx_g)^2 = (gx_g g) x_g \in \Gamma_E \quad \text{and} \quad (gx_g)^2 = gx_g (gx_g g) g^{-1} \in g \Gamma_E g^{-1},
\]

which shows \( (gx_g)^2 \in \Gamma_\alpha \) by (1). Now again by the description of \( \Gamma_F \)-action above we see that \( gx_g \) sends \( \alpha \) to \(-\alpha\). Hence \( \Gamma_\alpha = \langle gx_g, \Gamma_\alpha \rangle \) as required. \( \square \)

We next give a sufficient condition for a given symmetric root to be unramified.

Lemma 5.2. Let \( \alpha \in \Phi(S, G) \) be a root of the form \( \begin{bmatrix} 1 \\ g \end{bmatrix} \) for some \( g \in \{ \Gamma_F / \Gamma_E \} \).

Moreover we assume that \( \alpha \) is symmetric.

1. Suppose that \( g \not\in \Gamma_K \) for a subfield \( F \subset K \subset E \). Then we have \( F_\alpha = F_{\pm \alpha} \cdot K \).
2. Let \( K = F[\mu_E] \) be the maximal unramified extension of \( F \) in \( E \). If \( g \not\in \Gamma_K \) then \( g \) is symmetric unramified.

Proof. Let us prove (1). Since \( F_\alpha \supset E \supset K \), we clearly have \( F_\alpha \supset F_{\pm \alpha} \cdot K \). Let \( x_g \in \Gamma_E \) be as in Proposition 5.1. As \( g \) does not fix \( K \) while \( x_g \) does, the product \( gx_g \) does not fix \( K \) either. That is, \( K \) is not contained in the fixed field \( F_{\pm \alpha} = F_{\pm \alpha}^{\text{unr}} \). Since \( F_\alpha / F_{\pm \alpha} \) is quadratic, this implies \( F_\alpha = F_{\pm \alpha} \cdot K \).

Let us prove (2). By (1) we have \( F_\alpha = F_{\pm \alpha} \cdot K \). Now \( K / F \) is unramified and so is \( F_\alpha / F_{\pm \alpha} \).

We next investigate symmetric ramified roots. In fact, we can show that they are of a very limited form as follows:

Proposition 5.3. The set \( \Phi(S, G) \) has a symmetric ramified root if and only if \( e \) is even. In this case a \( \Gamma_F \)-orbit of symmetric ramified roots is unique and represented by \( \begin{bmatrix} 1 \\ \sigma^e \end{bmatrix} \).

Proof. We first prove that if \( e \) is even, then the root \( \alpha = \begin{bmatrix} 1 \\ \sigma^e \end{bmatrix} \) is a symmetric ramified root. Let us apply Proposition 5.1 to \( g := \sigma^e \). Note that the behaviour
of $g$ on $E$ is particularly simple:
\[ g = \sigma^g : \zeta \mapsto \zeta \quad (\zeta \in \mu_E) \quad \text{and} \quad \varpi_E \mapsto -\varpi_E. \]
Thus $g$ maps $E$ onto itself and we can take $x_g$ as in Proposition \ref{prop:3.1} (2) to be the identity. In particular, $\alpha$ is symmetric. Moreover, by noting that $F_\alpha = E \cdot g(E) = E$, we have $F_{\pm \alpha} = E^{g_\alpha} = E^g = F[\mu_E, \varpi_E^2]$. Hence $F_\alpha$ is a ramified quadratic extension of $F_{\pm \alpha}$. In other words, $\alpha$ is ramified.

We next prove the converse. To be more precise, let $\alpha \in \Phi(S, G)$ be a symmetric ramified root of the form \[ \begin{bmatrix} 1 \\ g \end{bmatrix} \] for some $g = \sigma^k \phi^i \in \{ \Gamma_E / \Gamma_E \}$. Then it is enough to show that $e$ is even and $g$ is necessarily equal to $\sigma^g$.

First, since our $\alpha$ is symmetric ramified, $g$ fixes $F[\mu_E]$ by Lemma \ref{lem:5.2} (2). In other words, we may assume $g = \sigma^k$ with $0 \leq k \leq e - 1$. Let us show that $e$ is even and that this $k$ is in fact given by $\frac{e}{2}$.

We take $x_g \in \Gamma_E$ as in Proposition \ref{prop:5.1} (2), that is, $g x_g g$ belongs to $\Gamma_E$. Here, since we have $\Gamma_{L/E} = \langle \sigma^e \phi^i \rangle$, we may assume that $x_g$ is given by $\langle \sigma^e \phi^i \rangle^r$ for some integer $r \in \mathbb{Z}$. Then we can easily compute that
\[ x_g = \sigma^{\frac{q^{fr} - 1}{2} \phi^{fr}} \quad \text{and} \quad g x_g g = \sigma^{k(1 + q^{fr}) + \frac{q^{fr} - 1}{2} \phi^{fr}}. \]
Thus the condition that $g x_g g \in \Gamma_E$ can be rephrased as
\[ k(1 + q^{fr}) \equiv 0 \pmod{e}. \tag{1} \]
On the other hand, recall that the unique nontrivial element of the Galois group $\Gamma_{E_\alpha / F_{\pm \alpha}}$ is represented by $g x_g$. Thus our symmetric root $\alpha$ is ramified if and only if
\[ g x_g (\zeta) = \zeta \quad \text{(for any $\zeta \in \mu_{F_\alpha}$).} \tag{2} \]
As we have $F_\alpha = E \cdot g(E) = F[\varpi_E, \zeta_{q^{fr} - 1}, \zeta_E^k]$ (here $\zeta_{q^{fr} - 1} \in \mu_E$ is a primitive $(q^{fr} - 1)$st root of unity), the condition (2) is equivalent to the condition that
\[ g x_g (\zeta_{q^{fr} - 1}) = \zeta_{q^{fr} - 1} \quad \text{and} \quad g x_g (\zeta_E^k) = \zeta_E^k. \]
Since $x_g$ lies in $\Gamma_{L/E}$ and $\sigma$ fixes $\mu_L$, their product $g x_g$ fixes $\mu_E$. In particular, the first equation is automatic. Therefore, by noting that we have $g x_g (\zeta_E^k) = \zeta_E^{k q^{fr}}$, we see that the condition (2) holds if and only if
\[ k(q^{fr} - 1) \equiv 0 \pmod{e}, \]
which in turn is equivalent to $-2k \equiv 0 \pmod{e}$ in view of (1). Recalling that $0 < k \leq e - 1$ we conclude that if $g$ is symmetric ramified then $e$ is even and $k = \frac{e}{2}$.

We finally describe a condition for that a given symmetric root is unramified.

**Proposition 5.4.** Let $\alpha \in \Phi(S, G)$ be a root of the form \[ \begin{bmatrix} 1 \\ g \end{bmatrix} \] for some $g = \sigma^k \phi^i \in \{ \Gamma_E / \Gamma_E \}$. If $\alpha$ is symmetric unramified, then $i$ is given by either 0 or $\frac{e}{2}$ (in the latter case, $f$ must be even). Moreover, in the case where $i = 0$, $k$ is not equal to $\frac{e}{2}$.

**Proof.** The first assertion is nothing but [Tam16, Proposition 3.3 (iii)]. The latter assertion follows from Proposition \ref{prop:5.3}. \qed
Now let us summarize the relationship between Adler–DeBacker–Spice’s notion of unramifiedness/ramifiedness for symmetric roots and Tam’s one. Let \( \alpha = \left[ \frac{1}{\sigma^k \phi^i} \right] \in \Phi(S, G) \) be a symmetric root with \( 0 \leq k \leq e - 1 \) and \( 0 \leq i \leq f - 1 \).

Then \( \alpha \) is ramified in the sense of Adler–DeBacker–Spice only when \( (k, i) = (\frac{r}{2}, 0) \).

In this case, \( \alpha \) is ramified also in the sense of Tam (see [Tam16, 1710 page]). Next let us assume \( \alpha \) is unramified in the sense of Adler–DeBacker–Spice. Then, by Proposition 5.3, \( i \) equals either 0 or \( \frac{r}{2} \). Moreover, the root \( \alpha \) is

\[
\begin{cases} 
\text{ramified in the sense of Tam} & \text{when } i = 0, \\
\text{unramified in the sense of Tam} & \text{when } i = \frac{r}{2}.
\end{cases}
\]

### Table: \( \alpha \) and its Ramifiedness

| \( g = \sigma^k \) | the sense of ADS | the sense of Tam |
|------------------|----------------|----------------|
| \( g = \sigma^k(k \neq \frac{r}{2}) \) | unramified | ramified |
| \( g = \sigma^k \phi^i \) | unramified | unramified |

#### 5.2. Howe factorization.

In this section, we recall the notion of a *Howe factorization* according to Kaletha’s sophisticated definition. Temporarily let \( G \) be a general tamely ramified connected reductive group over \( F \).

Let \( S \) be a tamely ramified elliptic maximal torus of \( G \) defined over \( F \) and \( \xi \) a character of \( S(F) \). For each positive real number \( r \in \mathbb{R}_{>0} \), we define a subset \( \Phi_r \) of \( \Phi(S, G) \) as in [Kal19a, 1107 page, (3.6.1)] by

\[
\Phi_r := \{ \alpha \in \Phi(S, G) \mid \xi \equiv 1 \text{ on } \text{Nr}_{S(L)/S(F)} \circ \alpha_\gamma(L^\times) \},
\]

where \( L \) is the splitting field of the torus \( S \). Note that, this set \( \Phi_r \) does not change even if we replace \( L \) with a finite tamely ramified extension of \( L \). We let \( r_{d-1} > \cdots > r_1 > r_0 \) be the real numbers satisfying

\[
\Phi_r \subset \Phi_{r+} := \bigcap_{r' > r} \Phi_{r'},
\]

and call them the *jumps* of \( \xi \). We put \( r_d := \text{depth}(\xi) \) and \( r_{-1} := 0 \) (note that \( r_d \geq r_{d-1} \)). Since each \( \Phi_{r_i} \) is a Levi subsystem of \( \Phi(S, G) \) by [Kal19a, Lemma 3.6.1], we can take a tame twisted Levi subgroup \( G^i \) of \( G \) which has \( \Phi_{r_i} \) as its roots. We put \( G^{-1} := S \) and \( G^d := G \) and take \( G^i \)'s so that we have \( G^{-1} \subset G^0 \subset \cdots \subset G^d \).

**Definition 5.5 (Howe factorization, [Kal19a, Definition 3.6.2]).** A sequence of characters \( \phi_i : G^i(F) \to \mathbb{C}^\times \) for each \( i = -1, \ldots, d \) is called a *Howe factorization* of \( \xi \) if it satisfies the following conditions:

(i) Each \( \phi_i \) is trivial on \( G^i_{\text{der}}(F) \) (the image of the set of \( F \)-valued points of the simply-connected cover of the derived group of \( G^i \) in \( G^i(F) \)).

(ii) For \( i = -1 \), we have

\[
\phi_{-1} = 1 \quad \text{if } S = G^0, \\
\phi_{-1}|_{S(F)_{0+}} = 1 \quad \text{if } S \subsetneq G^0.
\]

For all \( 0 \leq i \leq d - 1 \), the character \( \phi_i \) is \( G^{i+1} \)-generic of depth \( r_i \),

- For \( i = d \), we have \( \phi_d = 1 \) if \( r_{d-1} = r_d \),

\[
\text{depth}(\phi_d) = r_d \quad \text{if } r_{d-1} < r_d.
\]
(iii) We have
\[ \xi = \prod_{i=-1}^{d} \phi_i|_{S(F)}. \]
Here see [Yu01, Section 9] for the definition of being \( G^{i+1} \)-generic of depth \( r_i \) for each character \( \phi_i \).

As proved in [Kal19a, Proposition 3.6.7], we can always take a Howe factorization for any given character \( \xi \) of \( S(F) \). This fact has been known by Howe and Moy classically ([How77, Moy86]) in the \( \text{GL}_n \)-case, and the connection between their definition of Howe factorization and Kaletha’s one in the \( \text{GL}_n \)-case can be roughly explained as follows: Let \( E \) be a tamely ramified extension of \( F \) of degree \( n \) corresponding to \( S \subset \text{GL}_n \). In the definition of Howe and Moy, a Howe factorization of \( \xi \) consists of a sequence of subfields
\[ F = E_d \subsetneq E_{d-1} \subsetneq \cdots \subsetneq E_0 \subset E_{-1} = E \]
and characters \( \xi_i \) of \( E_i^\times \) for each \(-1 \leq i \leq d\) satisfying several conditions such as
\[ \xi = \xi_{-1} \cdot (\xi_0 \circ \text{Nr}_{E/E_0}) \cdots (\xi_d \circ \text{Nr}_{E/E_d}) \]
(see, for example, [HM08, Definition 3.33]). Suppose that we have such a sequence of fields and characters. Then, by considering the centralizers of
\[ F = E_d^\times \subsetneq E_{d-1}^\times \subsetneq \cdots \subsetneq E_0^\times \subset E_{-1}^\times = S(F) \]
in \( \text{GL}_n \), we get a sequence of tame twisted Levi subgroups of \( G \):
\[ \text{GL}_n = G^d \supsetneq G^{d-1} \supsetneq \cdots \supsetneq G^0 \supsetneq G^{-1} = S. \]
Here note that if we put \( n_i := n/[E_i:F] \), then \( G^i \) is isomorphic to \( \text{Res}_{E_i/F} \text{GL}_{n_i,E_i} \). In fact, this sequence of tame twisted Levi subgroups \( G^0 \subsetneq \cdots \subsetneq G^{d-1} \) has \( \Phi_{n_0} \subsetneq \cdots \subsetneq \Phi_{n_{d-1}} \) as its roots. Moreover, if we put \( \phi_i := \xi_i \circ \det \), we get a Howe factorization in the sense of Kaletha. Note that, since \( G_{\text{sc}}^i(F) \) is given by \( \text{SL}_{n_i}(E_i) \), the triviality condition for \( \phi_i \) on \( G_{\text{sc}}^i(F) \) is equivalent to the condition that it factors through the determinant map. Furthermore, as the determinant map is nothing but the norm map \( \text{Nr}_{E/E_i} \) on \( S(F) = E_0^\times \), we have \( \phi_i|_{S(F)} = \xi_i \circ \text{Nr}_{E/E_i} \).

See [HM08, Section 3.5] for more detailed explanation on the relationship between these two apparently different definitions of Howe factorizations.

Remark 5.6. In the language of Bushnell–Henniart, the jumps are defined to be the \( E \)-levels of \( \xi_i \circ \text{Nr}_{E/F} \) (see, for example, [Tam16, Section 5.1]). If we put the \( E \)-levels of \( \xi_i \circ \text{Nr}_{E/F} \) to be \( t_i \), then the relationship between \( t_i \) and the depth \( r_i \) of \( \phi_i \) is expressed as
\[ r_i = t_i / c. \]
Indeed, as Moy–Prasad filtrations behave well under the descent with respect to tame twisted Levi subgroups (see, for example, [AS08, Lemma 3.8]), the depth of
Moreover, we can check each condition so that \( \phi_i \mid_{S(F)} \) is equal to that of \( \phi_i \). Then the tameness of \( E/F \) gives the above equality (see Section [I]).

The following “descent” property of a Howe factorization will be needed later (in Section [6.3]).

**Proposition 5.7.** Let \( G \) be a tamely ramified connected reductive group over \( F \) and \( S \) a tamely ramified elliptic maximal torus of \( G \). Let \( \xi \) be a character of \( S(F) \). We consider a tame twisted Levi subgroup \( H \) of \( G \) containing \( S \). Then the set of jumps \( r_{d-1}' \) of \( \xi \) with respect to \( (S, H) \) is contained in that \( r_{d-1}' \) with respect to \( (S, G) \) (note that we always have \( r_{d-1}' = r_d \) and \( r_{d-1}' = r_{d-1} \)). Moreover we can take

- a Howe factorization \( (\phi_1, \ldots, \phi_d) \) of \( \xi \) with respect to \( S \subset G \) and
- a Howe factorization \( (\phi_1', \ldots, \phi_d') \) of \( \xi \) with respect to \( S \subset H \)

such that we have

\[
\phi_i \mid_{S(F)_r} = \phi_i' \mid_{S(F)_r'}
\]

for any \( 0 \leq i \leq d' \). Here \([i]\) denotes the unique index satisfying \( r_{[i]} = r_i' \).

**Proof.** Let \( \Phi_r \) and \( \Phi_r' \) be the \( r \)-th filtration on the sets \( \Phi(S, G) \) and \( \Phi(S, H) \) of roots, respectively. Then, as \( \Phi(S, H) \) is a subset of \( \Phi(S, G) \), the first assertion on the jumps immediately follows from the definitions of \( \{ \Phi_r \}_{r \in \mathbb{R}_{>0}} \) and \( \{ \Phi_r' \}_{r \in \mathbb{R}_{>0}} \). Moreover, for any index \( 0 \leq i \leq d' \), we have \( \Phi_r_{[i]} \subset \Phi_r'[0]\cap \Phi_r'_{[i]} \subset \Phi_r'[d'-1] \) by the definition of jumps. Thus we may assume that each \( H^{i-1} \) is contained in \( G[i-1] \) and that \( H' \) is contained in \( G[i-1]^{[i]} \).

Note that, for \( i = -1 \), we always have \( G^{-1} = H^{-1} = S \) by definition.

Now we take a Howe factorization \( (\phi_1, \ldots, \phi_d) \) of \( \xi \) with respect to \( S \subset G \). By noting that \( r_d = r_d' \) and \( r_{d-1} = r_{d-1}' \) (i.e., \( [d'] = d \) and \( [-1] = -1 \)), we define \( \phi_i' \) as follows:

\[
\phi_i' := \begin{cases} \phi_{i-1}|_{S(F)} \cdot \phi_{i-2}|_{S(F)} \cdots \phi_0|_{S(F)} & \text{if } S = H^0, \\ \phi_{i-1}|_{H^0(F)} \cdot \phi_{i-2}|_{H^0(F)} \cdots \phi_0|_{H^0(F)} & \text{if } S \subset H^0, \\ \phi_{i-1} & \text{if } S = H^0. \end{cases}
\]

Then \( (\phi_1', \ldots, \phi_d') \) gives a desired Howe factorization. Indeed, as already explained in Remark [5.6], a tamely ramified descent property of Moy–Prasad filtrations assures that the depth of \( \phi_j \mid_{S(F)} \) is equal to that of \( \phi_j \) for any \( j \). Hence the restrictions of \( \phi_{[i-1]} \mid_{S(F)} \) to \( S(F) \) are trivial and we have

\[
\phi_i \mid_{S(F)_r} = \phi_i' \mid_{S(F)_r'}.
\]

Moreover, we can check each condition so that \( (\phi_1', \ldots, \phi_d') \) is a Howe factorization of \( \xi \) with respect to \( (S, H) \) as follows:
(i) In general, for any sequence \( H_1 \subset H_2 \) of connected reductive groups, the map \( H_{1,sc} \to H_1 \subset H_2 \) factors through \( H_{2,sc} \). In particular, the image of \( H_{1,sc}(F) \) in \( H_2(F) \) is contained in the image of that of \( H_{2,sc}(F) \) in \( H_2(F) \). Thus the condition (i) for \( (\phi_{-1}, \ldots, \phi_d) \) implies that for \( (\phi'_{-1}, \ldots, \phi'_d) \).

(ii) The condition for \( i = -1 \) immediately follows from the definition of \( \phi'_{-1} \).

We assume that \( 0 \leq i \leq d' - 1 \). Recall that being \( H^{i+1} \)-generic of depth \( r'_i \) is a condition on the restriction \( \phi'_i|_{H^{i+1}(F)_{x,r_i'}} \) (see [Yu01, Section 9]). Here \( x \) is the point of the reduced Bruhat–Tits building of \( H^{i+1} \) determined by \( S \) (see Section 5.3) and \( H^{i+1}(F)_{x,r_i'} \) denotes the \( r'_i \)-th Moy–Prasad filtration of \( H^{i+1}(F) \) at \( x \). By the same reason as above, we have

\[
\phi'_i|_{H^{i+1}(F)_{x,r_i'}} = \phi_i|_{H^{i+1}(F)_{x,r_i'}}.
\]

Then, as \( H^{i+1} \) is a tame twisted Levi subgroup of \( G^{[i]+1} \) and we have \( r'_i = r_i \), being \( G^{[i]+1} \)-generic of depth \( r_i \) for \( \phi_i \) implies being \( H^{i+1} \)-generic of depth \( r'_i \) for \( \phi'_i \).

Finally, we consider the case where \( i = d' \). If we have \( r'_{d'-1} = r'_{d} \), then we have \( r_{d'-1} = r_{d'}(= r_d) \). Therefore, by noting that the sequence \( r_{i-1} < r_0 < \cdots < r_{d-1} \) is strictly increasing, we get \( r_{d'-1} = r_{d-1} = r_d \).

Thus the condition (ii) for \( \phi_d \) and the definition of \( \phi'_d \) implies that \( \phi'_d = \phi_d|_{H(F)} = 1 \). On the other hand, if we have \( r_{d'-1} < r'_{d} \), then we have \( r_{d-1} < r_d \). Hence the sequence \( r_{d'-1} + 1 < \cdots < r_d \) of depths of \( \phi|_{H^{[d'-1]+1}}(F), \ldots, \phi|_{H^{[d]+1}}(F) \) are strictly increasing and we get

\[
\text{depth}(\phi'_d) = \text{depth}(\phi|_{H^{[d'-1]+1}}(F) \cdots H^{[d]+1}(F)) = \text{depth}(\phi_d) = r_d = r'_{d}.
\]

(iii) This follows immediately by the definition of \( (\phi'_{-1}, \ldots, \phi'_d) \) and the condition (iii) for \( (\phi_{-1}, \ldots, \phi_d) \).

\[ \Box \]

5.3. Notation around Heisenberg groups. In this section, we recall the notion of a t-factor which will be needed in the definition of \( \chi_{\text{Tam}} \) at symmetric roots briefly. The contents of this section are summary of [Tam16, Sections 4 and 5.2].

Let \( E \) be a tamely ramified extension of \( F \) of degree \( n \). Then we get a hereditary order of \( \text{End}_F(E) \) defined by

\[
\mathfrak{A} := \{ X \in \text{End}_F(E) \mid X \cdot p_E^k \subseteq p_E^k \text{ for any } k \in \mathbb{Z} \}.
\]

We put \( \mathfrak{P} \) to be the Jacobson radical of \( \mathfrak{A} \), which is given by

\[
\mathfrak{P} = \{ X \in \text{End}_F(E) \mid X \cdot p_E^k \subseteq p_E^{k+1} \text{ for any } k \in \mathbb{Z} \}.
\]

We consider the quotient

\[
\mathfrak{A} := \mathfrak{A}/\mathfrak{P}
\]

of \( \mathfrak{A} \) by \( \mathfrak{P} \), which admits a natural \( k_E \)-vector space structure. Then the following natural action of \( E^\times \) on \( \text{End}_F(E) \) induces an \( E^\times \)-action on \( \mathfrak{A} \):

\[
X \in \text{End}_F(E), \quad t \in E^\times, \quad t \cdot X := t \circ X \circ t^{-1},
\]

where \( t \) and \( t^{-1} \) in the right-hand side are the multiplication maps via \( t \) and \( t^{-1} \), respectively. As this action factors through the finite group \( E^\times / F^\times U_E^1 \), the space...
$\mathcal{U}$ decomposes into a direct sum of eigenspaces with respect to the $E^\times$-action. This decomposition is described as follows:

**Proposition 5.8 ([Tam16 Proposition 4.4]).** We have a decomposition of $\mathcal{U}$ into a direct sum of $E^\times$-isotypic subspaces

$$\mathcal{U} = \bigoplus_{[g] \in \Gamma_E \backslash \Gamma_F / \Gamma_E} \mathcal{U}_{[g]}.$$  

Here, for each $[g] \in \Gamma_E \backslash \Gamma_F / \Gamma_E$, the subspace $\mathcal{U}_{[g]}$ is canonically identified with the residue field $k_{E,g(E)}$ of the composite field of $E$ and $g(E)$. Moreover, the action of $t \in E^\times$ on $\mathcal{U}_{[g]}$ is given by the $t g(t)^{-1}$-multiplication.

**Remark 5.9.** Recall that we can identify $(\Gamma_E \backslash \Gamma_F / \Gamma_E)'$ with the set $\Gamma_F \backslash \Phi(S, G)$ of $\Gamma_F$-orbits of roots. If a double-$\Gamma_E$-coset $[g]$ corresponds to a root $\alpha$ under this identification, then we can understand the action of $t \in E^\times$ on $\mathcal{U}_{[g]}$ as the $\alpha(t)$-multiplication. In other words, $\mathcal{U}_{[g]}$ is the $\alpha|_{\mathfrak{S}(F)}$-isotypic part of $\mathcal{U}$. Moreover, if we take $\alpha$ to be $[1]_g$ as in Section 5.2, then $E \cdot g(E)$ is nothing but $F_\alpha$.

We next consider an $F$-admissible character $\xi$ of $E^\times$. Recall that then we can attach a sequence

$$F = E_d \subseteq E_{d-1} \subseteq \cdots \subseteq E_0 \subseteq E_{-1} = E$$

of subfields and a sequence

$$r_d \geq r_{d-1} > \cdots > r_0 > r_{-1} = 0$$

of jumps to $\xi$. Note that, in this paper, we put $E_d$ (not $E_{d+1}$ as in [Tam16]) to be $F$. We write $t_i = e \cdot r_i$ for the $E$-level of the $i$-th character $\xi_i$ (see Remark 5.6) and define integers $h_i$ and $j_i$ by

$$h_i := \left\lfloor \frac{t_i}{2} \right\rfloor + 1 = \begin{cases} \frac{t_i}{2} + 1 & \text{if } t_i \text{ is even}, \\ \frac{t_i + 1}{2} & \text{if } t_i \text{ is odd}, \end{cases} \quad j_i := \left\lfloor \frac{t_i + 1}{2} \right\rfloor = \begin{cases} \frac{t_i}{2} & \text{if } t_i \text{ is even}, \\ \frac{t_i + 1}{2} & \text{if } t_i \text{ is odd}. \end{cases}$$

For each $i$, we set

$$\mathfrak{A}_i := \{ X \in \text{End}_{E_i}(E) \mid X \cdot p_i^k \subseteq p_i^k E_i \text{ for any } k \in \mathbb{Z} \}.$$  

We put $\mathfrak{P}_i$ to be the Jacobson radical of $\mathfrak{A}_i$, which is given by

$$\mathfrak{P}_i := \{ X \in \text{End}_{E_i}(E) \mid X \cdot p_i^k \subseteq p_i^{k+1} E_i \text{ for any } k \in \mathbb{Z} \}.$$  

For each positive integer $k \in \mathbb{Z}_{>0}$, we set

$$U^k_{\mathfrak{A}_i} := 1 + \mathfrak{P}_i^k.$$  

Now we define subgroups $H^1$ and $J^1$ as follows:

$$H^1 := U_{\mathfrak{A}_0}^1 U_{\mathfrak{A}_1}^0 \cdots U_{\mathfrak{A}_{d-2}}^{h_{d-2}} U_{\mathfrak{A}_{d-1}}^{h_{d-1}}, \quad J^1 := U_{\mathfrak{A}_0}^1 U_{\mathfrak{A}_1}^0 \cdots U_{\mathfrak{A}_{d-2}}^{j_{d-2}} U_{\mathfrak{A}_{d-1}}^{j_{d-1}}.$$  

Let $\mathfrak{V}$ denote the quotient of $J^1$ by $H^1$. As both of $H^1$ and $J^1$ are $E^\times$-stable, we can consider a decomposition of this $k_F$-vector space $\mathfrak{V}$ into a direct sum of $E^\times$-isotypic parts:

$$\mathfrak{V} = \bigoplus_{[g] \in (\Gamma_E \backslash \Gamma_F / \Gamma_E)' \mathfrak{V}_{[g]}.'}$$
On the other hand, by seeing the definitions of $H^1$ and $J^1$, we have a decomposition

$$\mathfrak{V} := J^1/H^1 = \bigoplus_{i=0}^{d-1} \mathfrak{V}_i$$

of $\mathfrak{V}$ into a direct sum of $k_F$-vector spaces $\mathfrak{V}_i$, which are isomorphic to $U^j_{i, a_i+1}/U^j_{i, a_i+1}$. Note that each component $\mathfrak{V}_i$ is nonzero only when $t_i$ is even. Furthermore, in this case, we can identify $\mathfrak{V}_i$ with $\mathfrak{A}_{i+1}/(\mathfrak{A}_i + \mathfrak{P}_{i+1})$ by using the fixed uniformizer $\varpi_E$ of $E$. As we have

$$\mathfrak{A}_{i+1}/(\mathfrak{A}_i + \mathfrak{P}_{i+1}) \cong \left( (\mathfrak{A}_{i+1} + \mathfrak{P})/(\mathfrak{P}) \right) / \left( (\mathfrak{A}_i + \mathfrak{P})/(\mathfrak{P}) \right),$$

we may furthermore regard $\mathfrak{V}_i$ as a subspace of $\mathfrak{A}$ which is stable under the action of $E^\times$. In this sense, the $[g]$-isotypic part $\mathfrak{V}_{[g]}$ of $\mathfrak{V}$ can be identified with $\mathfrak{U}_{[g]}$ if it is not zero. When $\mathfrak{V}_{[g]} = 0$, we put $\mathfrak{W}_{[g]} = \mathfrak{U}_{[g]}$. Thus, by definition, we have

$$\mathfrak{U}_{[g]} = \mathfrak{W}_{[g]} \oplus \mathfrak{M}_{[g]},$$

and exactly one of $\mathfrak{U}_{[g]}$ or $\mathfrak{M}_{[g]}$ is zero.

Recall that the sequence $F = E_d \subset E_{d-1} \subset \cdots \subset E_0 \subset E_1 = E$ gives rise to a sequence of tame twisted Levi subgroups of $G$:

$$G = G^d \supset G^{d-1} \supset \cdots \supset G^0 \supset G^{-1} = S.$$

Let us suppose that a double-$\Gamma_E$-coset $[g] \in \Gamma_E \backslash \Gamma_F/\Gamma_E$ corresponds to the $\Gamma_F$-orbit of a root $\alpha \in \Phi(S, G)$. Then, for $-1 \leq i < d - 1$, $\mathfrak{U}_{[g]}$ is contained in $\mathfrak{V}_i$ if and only if $\alpha$ belongs to

$$\Phi_{i+1}^i := \Phi(S, G^{i+1}) \setminus \Phi(S, G^i)$$

and $t_i$ is even. Thus $\mathfrak{V}_i = 0$ when $t_i$ is odd and $\mathfrak{V}_i$ can be identified with

$$\bigoplus_{[g] \in (\Gamma_E \backslash \Gamma_F/\Gamma_E)/[g]^{i+1} \in \Phi_{i+1}^i} \mathfrak{U}_{[g]}$$

when $t_i$ is even. Note that, for a double-$\Gamma_E$-coset $[g] \in \Gamma_E \backslash \Gamma_F/\Gamma_E$ corresponding to the $\Gamma_F$-orbit of a root $\alpha \in \Phi_{i+1}^i$, we have

$$\mathfrak{W}_{[g]} = \begin{cases} \mathfrak{U}_{[g]} & \text{if } t_i \text{ is even,} \\ 0 & \text{if } t_i \text{ is odd,} \end{cases} \quad \mathfrak{M}_{[g]} = \begin{cases} 0 & \text{if } t_i \text{ is even,} \\ \mathfrak{U}_{[g]} & \text{if } t_i \text{ is odd.} \end{cases}$$

Now suppose that we have a finite group $\Gamma$ and a symplectic $\mathbb{F}_p[\Gamma]$-module $V$. Then we can attach $t$-factors $t_{\Gamma}^1(V)$ and $t_{\Gamma}^1(V)$:

- a sign $t_{\Gamma}^1(V) \in \{\pm 1\}$
- a sign character $t_{\Gamma}^1(V) : \Gamma \to \{\pm 1\}$

and $V$ in the manner of [BH10 Section 3.4] (or [Tam16 Section 4.2]). We put $t_{\Gamma}(V)$ to be a map from $\Gamma$ to $\{\pm 1\}$ defined by taking their product:

$$t_{\Gamma}(V) := t_{\Gamma}^0(V) \cdot t_{\Gamma}^1(V) : \Gamma \to \{\pm 1\}.$$

If we put

$$\mathfrak{V}_{[g]} := \begin{cases} \mathfrak{W}_{[g]} \oplus \mathfrak{M}_{[g]^{-1}} & \text{if } [g] \text{ corresponds to an asymmetric root,} \\ \mathfrak{W}_{[g]} & \text{if } [g] \text{ corresponds to a symmetric root,} \end{cases}$$

then it admits a symplectic $\mathbb{F}_p[E^\times / E^\times U_E^1]$-module structure (see [Tam16 Sections 4.2.1, 4.2.2, and 4.2.3]). In particular, we can consider $t$-factors for $\mathfrak{V}_{[g]}$ with
\* \( \Gamma = \mu := \mu_E / \mu_F \) or
\* the subgroup \( \varpi \) of \( E^\times / F^\times U_E^1 \) generated by the image of \( \varpi_E \).

On the other hand, for an orthogonal \( k_F[E^\times / F^\times U_E^1] \)-module \( W \), we can attach a fourth root of unity \( t(W) \in \mathbb{C}^\times \) in the manner of [Tam16 Sections 4.3]. Since each

\[
\mathfrak{W}[g] := \begin{cases} \mathfrak{W}[g] \otimes \mathfrak{W}[g^{-1}] & \text{if } [g] \text{ corresponds to an asymmetric root,} \\ \mathfrak{W}[g] & \text{if } [g] \text{ corresponds to a symmetric root,} \end{cases}
\]

has an orthogonal \( k_F[E^\times / F^\times U_E^1] \)-module structure as explained in [Tam16 Section 4.3], we can consider \( t(\mathfrak{W}[g]) \). Here we remark that, only in the case where \([g]\) corresponds to a \( \Gamma_F \)-orbit of a symmetric ramified root, we have to make a special choice of a quadratic form on \( \mathfrak{W}[g] \). In Tam’s paper, this choice is implicitly specified in [Tam16 Remark 7.7] (see also a comment in [Tam16 1755 page (ii)]). Since it is needed to compute the \( t \)-factor in this case explicitly for our purpose, we will recall the precise definition of the quadratic form and the corresponding \( t \)-factor later (see the proof of Proposition 5.2 in the symmetric ramified case in Section 6.4).

Finally, we also have to explain the relationship between these notions and the ingredients used in Kaletha’s construction of the local Langlands correspondence. Thus let us take a tamely ramified elliptic maximal torus \( S \) of \( G = \text{GL}_n \) which corresponds to the tamely ramified extension \( E \) of \( F \) of degree \( n \). Then, by a tamely ramified descent property of Bruhat–Tits building ([Pra01]), we can regard the Bruhat–Tits building \( \mathcal{B}(S, F) \) of \( S \) as a subset of that \( \mathcal{B}(G, F) \) for \( G \). Moreover, as \( S \) is elliptic, the image of \( \mathcal{B}(S, F) \) in the reduced building \( \mathcal{B}_{\text{red}}(G, F) \) consists only of one point. We let \( x \) denote this unique point.

For each \( \alpha \in \Phi(S, G) \) and \( r \in \mathbb{R} \), we put

\[
g_\alpha(F_\alpha)_r := g_\alpha(F_\alpha) \cap g(F_\alpha)_{x, r},
\]

where

\* \( g_\alpha \) is the root subspace of \( g \) attached to \( \alpha \) and
\* \( g(F_\alpha)_{x, r} \) is the \( r \)-th Moy–Prasad filtration of \( g(F_\alpha) \) at \( x \).

By using this filtration on \( g_\alpha(F_\alpha) \), we define a subset \( \text{ord}_x(\alpha) \) of \( \mathbb{R} \) by

\[
\text{ord}_x(\alpha) := \{ r \in \mathbb{R} \mid g_\alpha(F_\alpha)_{r+} \subseteq g_\alpha(F_\alpha)_r \}.
\]

The following proposition is implicitly encoded in the works of Adler–Spice ([AS09]) and DeBacker–Spice ([DS18]), especially the proofs of [AS09 Proposition 3.8] and [DS18 Proposition 4.21]. For the sake of completeness, we give an explanation (in the following proof, we temporarily use the notation of [AS09] [DS18]):

**Proposition 5.10.** Let \( x \in \mathcal{B}_{\text{red}}(G, F) \) be the point determined by \( S \). We take a double-\( \Gamma_F \)-coset \([g]\) and a root \( \alpha \in \Phi_{i+1}^1 \) whose \( \Gamma_F \)-orbit corresponds to \([g]\). Then we have a double-\( \Gamma_F \)-coset \([r] \) and \( \Gamma_F \)-orbit corresponds to \([g]\). Then we have \( r \in \mathbb{Q} \) and \( t \in \mathbb{Q} \) such that \( r \) and \( t \) are relatively prime.

**Proof.** By the definition of \( \text{ord}_x(\alpha) \), we have \( r, t \in \mathbb{Q} \) if and only if the quotient \( g_\alpha(F_\alpha)_{x, \frac{r}{t}} / g_\alpha(F_\alpha)_{x, \frac{t}{t}} \) is not zero. Let \( L \) be the splitting field of \( S \). Note that the \( V_\circ \) in [AS09 Proof of Proposition 3.8], which is the \( \Gamma_\alpha \)-fixed part of the quotient \( g_\alpha(L)_{x, \frac{r}{t}} / g_\alpha(L)_{x, \frac{t}{t}} \) by definition, is nothing but the quotient \( g_\alpha(F_\alpha)_{x, \frac{r}{t}} / g_\alpha(F_\alpha)_{x, \frac{t}{t}} \) by the compatibility of taking...
Thus it suffices to show the equality is defined as follows: 

$$E$$ of characters on $$S$$

Proof. Let $$\pi$$ be a regular supercuspidal representation of $$\text{GL}_n(F)$$, we have

$$\text{Lie}(G^i, G^{i+1})(L)_{x, (r, \tilde{r}_i); (r, \tilde{r}_i+)}$$

by the map

$$X_\alpha \mapsto \sum_{\tau \in \Gamma_F/V_\alpha} \tau(X_\alpha)$$

(note that this isomorphism is $$S(F)$$-equivariant). Again by the tamely ramified descent property of taking the quotient ([Yu01 Corollary 2.3]), the latter space is identified with the $$\alpha|_{S(F)}$$-isotypic part of $$\text{Lie}(G^i, G^{i+1})(F)_{x, (r, \tilde{r}_i); (r, \tilde{r}_i+)}$$, which is furthermore isomorphic to $$\text{Lie}(G^i, G^{i+1})/\text{Lie}(G^i, G^{i+1})$$.

On the other hand, by definition, $$W_{[g]}$$ is the $$\alpha|_{S(F)}$$-isotypic part of the group $$U_{\lambda_i+1}/U_{\lambda_i} \cdot \Upsilon^{h_i}$$, which is isomorphic to $$\Upsilon^{\beta_i}/\Upsilon^{\beta_i} \cdot \Upsilon^{h_i}$$. Therefore the comparison result of Broussous–Lemaire on the lattice filtrations and Moy–Prasad filtrations ([BL02], which justifies the equality 

$$\text{Lie}(G^i, G^{i+1}) = \Upsilon^{e_i}$$

(recall that $$e$$ is the ramification index of $$E/F$$) for any $$r \in \frac{1}{e} \mathbb{Z}$$, gives us an identification

$$\text{Lie}(G^i, G^{i+1})/\text{Lie}(G^i, G^{i+1}) \cong \Upsilon^{\beta_i}/\Upsilon^{\beta_i} \cdot \Upsilon^{h_i}$$

which is $$S(F)$$-equivariant. Thus, in particular, $$V_\alpha \neq 0$$ and only if $$\mathcal{W}_{[g]} \neq 0$$. \(\square\)

6. Comparison of $$\chi$$-data

6.1. Main theorem.

**Theorem 6.1.** We assume that the residual characteristic $$p$$ is odd. Then Kaletha’s local Langlands correspondence coincides with Harris–Taylor’s one for regular supercuspidal representations of $$\text{GL}_n(F)$$. That is, for every regular supercuspidal representation $$\pi$$ of $$\text{GL}_n(F)$$, we have

$$\text{LLC}^{\text{Kal}}_G(\pi) = \text{LLC}^{\text{HT}}_G(\pi).$$

**Proof.** Let $$\pi$$ be a regular supercuspidal representation of $$\text{GL}_n(F)$$ arising from a tame elliptic regular pair $$(S, \xi)$$, i.e., $$\pi \cong \pi^{\text{Kal}}_{(S, \xi)}$$. Here if we let $$E$$ be a finite tamely ramified extension of $$F$$ of degree $$n$$ such that $$S$$ is isomorphic to $$\text{Res}_{E/F} G$$, then we have $$\pi^{\text{Kal}}_{(S, \xi)} \cong \pi^{\text{BH}}_{(E, \xi)}$$ (see Section 4.1). Then, by Corollary 3.8 and Proposition 4.5, we have

$$\text{LLC}^{\text{HT}}_G(\pi) = \text{Ind}^{W_E}_{E}(\xi^{\mu^{-1}}_{\chi_{\text{tam}}}) \quad \text{and} \quad \text{LLC}^{\text{Kal}}_G(\pi) = \text{Ind}^{W_E}_{E}(\xi^{-1}\mu_{\chi_{\text{Kal}}}).$$

Thus it suffices to show the equality

$$\mu^{-1}_{\chi_{\text{tam}}} = \xi^{-1}\mu_{\chi_{\text{Kal}}}$$

of characters on $$E^\times$$. Here let us recall that each character appearing in this equality is defined as follows:
• For a set $\chi$ of $\chi$-data, $\mu_{\chi}$ is defined to be a character of $E^\times$ given by
\[ \mu_{\chi} := \prod_{[\alpha] \in \Gamma_F \setminus \Phi(S, G)} \chi_{\alpha} |_{E^\times}, \]
where, for each $[\alpha] \in \Gamma_F \Phi(S, G)$, we take its representative $\alpha \in \Phi(S, G)$
to be of the form $\begin{bmatrix} 1 \\ g \end{bmatrix}$ for some (unique) $g \in \mathcal{D}$ (see Section 5.2).

• The character $\epsilon$ is defined as the product $\epsilon_{\text{sym}} \cdot \epsilon_{\text{sym, ur}} \cdot \epsilon_{f, \text{ram}}$, where
  - $\epsilon_{\text{sym}}$ is the product of $\epsilon_{\alpha}$ over the set of $\Gamma_F \times \{ \pm 1 \}$-orbits of symmetric roots,
  - $\epsilon_{\text{sym, ur}}$ is the product of $\epsilon_{\alpha}$ over the set of $\Gamma_F$-orbits of symmetric unramified roots, and
  - $\epsilon_{f, \text{ram}}$ is the contribution of symmetric ramified roots, which is in fact trivial by a speciality of GL$_n$ (Proposition 4.3).

Therefore the above equality follows from the following proposition. \[ \square \]

**Proposition 6.2.** Let $(S, \xi)$ be a tame elliptic regular pair of GL$_n$ and $E$ a finite tamely ramified extension of $F$ of degree $n$ corresponding to $S$. Then, for each root $\alpha \in \Phi(S, G)$ of the form $\begin{bmatrix} 1 \\ g \end{bmatrix}$ for some $g \in \mathcal{D}$, we have
\[
\begin{align*}
(\chi_{\text{Tam}, \alpha} \cdot \chi_{\text{Kal}, \alpha}) |_{E^\times}^{-1} &= \epsilon_{\alpha}^{-1} \cdot (\chi_{\text{Kal}, \alpha} \cdot \chi_{\text{Kal}, \alpha}) |_{E^\times} & \text{if } \alpha \in \Phi(S, G)^{\text{sym}}, \\
\chi_{\text{Tam}, \alpha} |_{E^\times} &= \epsilon_{\alpha}^{-1} \cdot \chi_{\text{Kal}, \alpha} |_{E^\times} & \text{if } \alpha \in \Phi(S, G)^{\text{sym, ur}}, \\
\chi_{\text{Tam}, \alpha} |_{E^\times} &= \chi_{\text{Kal}, \alpha} |_{E^\times} & \text{if } \alpha \in \Phi(S, G)^{\text{sym, ram}}.
\end{align*}
\]
Here, when $\alpha$ is asymmetric, we put $\alpha'$ to be the root $\begin{bmatrix} 1 \\ g^{-1} \end{bmatrix}$ (note that then the $\Gamma_F$-orbit of $\alpha'$ contains $-\alpha$).

We will prove this proposition by a case-by-case computation according to whether a root $\alpha$ is symmetric or asymmetric (and furthermore whether unramified or ramified when it is symmetric). In the rest of this section, we fix a tame elliptic regular pair $(S, \xi)$ of GL$_n$. We let $E$ be the finite tamely ramified extension of $F$ of degree $n$ which corresponds to $S$. Moreover, according to Tam (see [Tam16, 1735 page, (5.11)]), we add the following condition to our fixed uniformizer $\varpi_E$:
\[ \varpi_E \in E_0, \]
where $E_0$ the subfield of $E$ appearing in a Howe factorization of $(S, \xi)$ (in the sense of Howe and Moy, see Section 5.2). Note that we can always assume this condition since $E$ is unramified over $E_0$ (see, e.g., [Tam16, Section 5.1]).

6.2. **Asymmetric roots.** Let $\alpha = \begin{bmatrix} 1 \\ g \end{bmatrix} \in \Phi(S, G)^{\text{sym}}$ be an asymmetric root. We take $\alpha'$ as in Proposition 6.2. We recall the definitions of the terms in both sides of the equality in Proposition 6.2.

**Kaletha’s $\chi$-data at $\alpha$:** We take $\chi_{\text{Kal}, \alpha}$ to be the trivial character of $F_\alpha^\times$.

**Tam’s $\chi$-data at $\alpha$:** Since what we need is only the restriction of $\chi_{\text{Tam}, \alpha} \cdot \chi_{\text{Tam}, \alpha'}$ on $E^\times$, here we recall the value of $\chi_{\text{Tam}, \alpha} \cdot \chi_{\text{Tam}, \alpha'}$ at each element of $E^\times$:
\begin{itemize}
  \item $\chi_{\text{Tam}, \alpha} \cdot \chi_{\text{Tam}, \alpha'}$ is trivial on $1 + p_E$,
  \item $(\chi_{\text{Tam}, \alpha} \cdot \chi_{\text{Tam}, \alpha'})(\gamma) = \text{sgn}_\gamma(\mathfrak{M}_{[g]}^\chi) \cdot \text{sgn}_\gamma(\mathfrak{M}_{[g^{-1}]}^\chi)$ for any $\gamma \in \mu_E$, and
\end{itemize}
• \((\chi_{\text{Tam}, \alpha} \cdot \chi_{\text{Tam}, \alpha'})(w_E) = t_w(\mathcal{W}[g](w_E))\).

Here \(\text{sgn}_\epsilon(\mathcal{W}[g])\) (resp. \(\text{sgn}_\mu(\mathcal{W}^{g-1})\)) is the signature of the permutation of \(\mathcal{W}[g]\) (resp. \(\mathcal{W}^{g-1}\)) given by the multiplication by \(\gamma\) (recall that we have identifications \(\mathcal{W}[g] \cong k_{F_\alpha}\) and \(\mathcal{W}^{g-1} \cong k_{F_\alpha'}\), see Proposition 5.8 and Remark 5.9).

\(\varepsilon\) at \(\alpha\): Recall that we put

\[
\epsilon_\alpha(\gamma) := \begin{cases} 
\frac{\sigma(\gamma)}{k_{F_\alpha}} & \text{if } \frac{\gamma}{t} \in \text{ord}_E(\alpha), \\
1 & \text{otherwise},
\end{cases}
\]

for any \(\gamma \in E^\times\).

**Proof of Proposition 6.2: asymmetric case.** If \(\mathcal{W}[g]\) is trivial, then \(\chi_{\text{Tam}, \alpha} \cdot \chi_{\text{Tam}, \alpha'}|_{E^\times}\) is trivial. On the other hand, by Proposition 5.10 \(\epsilon_\alpha\) is also trivial. Thus we are done in this case.

In the following, we assume that \(\mathcal{W}[g] \neq 0\), or equivalently by Proposition 5.10 \(\frac{\gamma}{t} \in \text{ord}_E(\alpha)\). Since \(\epsilon_\alpha\) and \(\chi_{\text{Tam}, \alpha} \cdot \chi_{\text{Tam}, \alpha'}|_{E^\times}\) are trivial on \(1 + p_E\), it is enough to check the equality

\[\epsilon_\alpha(\gamma) = (\chi_{\text{Tam}, \alpha} \cdot \chi_{\text{Tam}, \alpha'})(\gamma)\]

for \(\gamma = w_E\) and each \(\gamma \in \mu_E\).

First, as noted in [Tam16] Theorem 7.1 (i) (b), we have

\[t_w(\mathcal{W}[g](w_E)) = \text{sgn}_\alpha(w_E)(\mathcal{W}[g])\]

by [Tam16] Proposition 4.9 (ii) (a)] (note that \(\mathcal{W}[g]\) in the right-hand side of the last equality in [Tam16] Theorem 7.1 (i) (b) should be correctly \(\mathcal{W}[g]\)). Second, by [Tam16] 1758 page, (7.2)] and the equality preceding [Tam16] 1758 page, (7.2)], we have

\[(\chi_{\text{Tam}, \alpha} \cdot \chi_{\text{Tam}, \alpha'})(\gamma) = \text{sgn}_\alpha(\gamma)(\mathcal{W}[g])\]

for any \(\gamma \in \mu_E\).

Thus, in any case, it suffices to show the equality

\[\text{sgn}_\zeta(\mathcal{W}[g]) = \left(\frac{\zeta}{k_{F_\alpha}}\right)\]

for any \(\zeta \in k_{F_\alpha}^\times\). Furthermore, by noting that \(k_{F_\alpha}^\times\) is a cyclic group, it enough to show the above equality only for the generator \(\zeta\) of \(k_{F_\alpha}^\times\). Then the right-hand side of the above equality is given by \(-1\). On the other hand, since \(k_{F_\alpha}^\times\) acts on \(\mathcal{W}[g]\) though the identification \(\mathcal{W}[g] \cong k_{F_\alpha}\), the action of \(\zeta\) is given by a cyclic permutation of \((|k_{F_\alpha}| - 1)\)-letters (note that only \(0 \in k_{F_\alpha}\) is fixed). Thus it can be written as a product of \((|k_{F_\alpha}| - 2)\) transpositions. As \(|k_{F_\alpha}|\) is odd by the oddness assumption on \(p\), we finally get

\[\text{sgn}_\zeta(\mathcal{W}[g]) = (-1)^{|k_{F_\alpha}| - 2} = -1.\]

\[\square\]

6.3. **Symmetric unramified roots.** Let \(\alpha = \left[\begin{smallmatrix} 1 \\ g \end{smallmatrix}\right] \in \Phi(S, G)_{\text{sym, ur}}\) be a symmetric unramified root with \(g = \sigma k\phi\). Note that, as explained in Section 5.4 (Proposition 5.4), \(i\) equals either 0 or \(\frac{1}{2}\). Moreover, when \(i = 0\), \(k\) is not equal to \(\frac{1}{2}\). We recall the definitions of the terms in both sides of the equality in Proposition 6.2.

**Kaletha’s \(\chi\)-data at \(\alpha\):** We take \(\chi_{\text{Kal}, \alpha}\) to be the unique nontrivial unramified quadratic character of \(F_\alpha^\times\).
Tam’s χ-data at α: As in the asymmetric case, we recall only the values of χ_{\text{Tam},α} on E^\times. By [Tam16] Theorem 7.1 (i), χ_{\text{Tam},α} is a character of F_{α}^\times satisfying the following conditions:

- χ_{\text{Tam},α} is trivial on 1 + p_E,
- χ_{\text{Tam},α}|_{μ_E} = t_μ^1(Ψ|_{[g]}), and
- χ_{\text{Tam},α}(ξE) = t_μ^0(Ψ^0|_{[g]}):t_ξ(Ψ|_{[g]}):t(Ψ|_{[g]}).

Here Ψ^0|_{[g]} denotes the sieve-fixed part of Ψ|_{[g]}.

ε at α: Recall that we put

ε_α(γ) := \begin{cases} \frac{2χ_{\text{Tam}}(γ)}{F_{α}} & \text{if } \frac{γ}{F_{α}} \in \text{ord}_E(α), \\
1 & \text{otherwise}. \end{cases}

Proposition 6.3. We have the equality in Proposition 6.2 which is

χ_{\text{Tam},α}^{-1}(ξ) = ϵ_α^{-1}(ξ) \cdot χ_{\text{Kal},α}(ξ),

for every ζ ∈ E^\times.

Proof. As χ_{\text{Kal},α}|_{μ_E} is trivial, we are to prove

(3) ϵ_α(ξ) = χ_{\text{Tam},α}(ξ)

for ζ ∈ μ_E.

If Ψ|_{[g]} is trivial, then 1/ξ does not belong to ord_ξ(α) by Proposition 5.10. Thus both of ϵ_α and χ_{\text{Tam},α} are trivial on μ_E and the equation (3) follows.

Suppose that Ψ|_{[g]} is nontrivial. Then by [Tam16] Proposition 4.9 (i) (b) and (c) we have

χ_{\text{Tam},α}(ξ) = t_μ^1(Ψ|_{[g]})(ξ) = \begin{cases} \frac{1}{(F_{α})} & \text{if } g = σ^k (k ≠ \frac{ξ}{2}), \\
\frac{ξ}{(F_{α})} & \text{if } g = σ^kφ^{f/2} (f \text{ is even}). \end{cases}

If g = σ^k with k ≠ \frac{ξ}{2}, then we have α(ξ) = ζ \cdot σ^k(ξ)^{-1} = 1 and the equality (3) holds.

If g = σ^kφ^{f/2}, then we have α(ξ) = ζ \cdot σ^kφ^{f/2}(ξ)^{-1} = ζ^{1-q^{f/2}}. Thus we have

ε_α(ξ) = \frac{α(ξ)}{F_{α}} = \frac{ζ^{1-q^{f/2}}}{(F_{α})} = ζ^{\frac{1-q^{f/2}}{F_{α}}},

χ_{\text{Tam},α}(ξ) = \frac{ξ^{r}}{F_{α}} = ζ^{\frac{r}{F_{α}}},

where r = [F_{α} : E]. Hence, in order to check the equality (3), it suffices to show that

\left(1 - q^{\frac{f}{2}}\right)\left(q^{\frac{r}{2}} + 1\right) - \frac{q^f - 1}{2} \equiv 0 \mod q^f - 1.

Here we note that r is odd by [Tam16] Proposition 3.4 (as explained in Section 5.4) our root α is symmetric unramified also in the sense of Tam). Thus we get

q^{\frac{f}{2}} + 1 = \left(1 - q^{\frac{f}{2}} + q^f - \cdots + q^{\frac{f(r-1)}{2}}\right)(1 + q^{\frac{f}{2}}),

hence

\left(1 - q^{\frac{f}{2}}\right)\left(q^{\frac{r}{2}} + 1\right) - \frac{q^f - 1}{2} = \frac{1-q^f}{2} (1 - q^{\frac{f}{2}} + q^f - \cdots + q^{\frac{f(r-1)}{2}} + 1).

As 1 - q^{\frac{f}{2}} + q^f - \cdots + q^{\frac{f(r-1)}{2}} + 1 is even, this is equal to 0 modulo q^f - 1. □
Proposition 6.4. We have the equality in Proposition [6.3] for \( \varpi_E \), that is,
\[
\chi_{\text{Tam}, \alpha}^{-1}(\varpi_E) = \epsilon_{\alpha}^{-1}(\varpi_E) \cdot \chi_{\text{Kal}, \alpha}(\varpi_E).
\]

Proof. Since we have \( \chi_{\text{Kal}, \alpha}(\varpi_E) = -1 \), it is enough to show that
\[
(\epsilon_{\alpha} \cdot \chi_{\text{Tam}, \alpha})^{-1}(\varpi_E) = -1.
\]

Recall that we put
\[
\epsilon_{\alpha}(\varpi_E) = \begin{cases} 
1 & \text{if } V_{[\gamma]} \text{ is trivial}, \\
\left( \frac{\alpha(\varpi_E)}{k_{F, \alpha}} \right) & \text{if } V_{[\gamma]} \text{ is nontrivial}.
\end{cases}
\]

On the other hand, by noting that \( \mathfrak{M}_{[\gamma]} \) is trivial if and only if \( V_{[\gamma]} \) is nontrivial and that we defined \( \epsilon_{\varpi}(\mathfrak{M}_{[\gamma]}) \) to be the product \( t^0_{\varpi}(\mathfrak{M}_{[\gamma]}), t^1_{\varpi}(\mathfrak{M}_{[\gamma]}) \), we have
\[
\chi_{\text{Tam}, \alpha}(\varpi_E) = \begin{cases} 
\epsilon(\mathfrak{M}_{[\gamma]}) & \text{if } V_{[\gamma]} \text{ is trivial,} \\
\left( t^0_{\varpi}(\mathfrak{M}_{[\gamma]}) \cdot t^1_{\varpi}(\mathfrak{M}_{[\gamma]}) \right) & \text{if } V_{[\gamma]} \text{ is nontrivial.}
\end{cases}
\]

If \( V_{[\gamma]} \) is trivial, then we have \( t(\mathfrak{M}_{[\gamma]}) = -1 \) by [Tam16] p.1728 (4.17) and get the desired equality.

In the rest of the proof, we assume that \( V_{[\gamma]} \) is nontrivial and put
\[
\beta := \alpha(\varpi_E) = \begin{cases} 
\zeta_k & \text{if } g = \sigma^k (k \neq \frac{2}{k}), \\
\zeta_k \zeta_{\phi^{-1}} & \text{if } g = \sigma^k \phi^f (f \text{ is even}).
\end{cases}
\]

Note that \( \beta \neq \pm 1 \) in the first case. As explained in [Tam16] Remark 7.3, we have
\[
t^0_{\varpi}(\mathfrak{M}_{[\gamma]}) = \begin{cases} 
-1 & \text{if } g = \sigma^k \phi^f, \beta = 1, \\
1 & \text{otherwise}.
\end{cases}
\]

On the other hand, by [Tam16] Proposition 4.9 (ii) (b) and (c)], we have
\[
t^0_{\varpi}(\mathfrak{M}_{[\gamma]}) = \begin{cases} 
1 & \text{if } g = \sigma^k \phi^f, \beta = \pm 1, \\
-1 & \text{otherwise,}
\end{cases}
\]

\[
t^1_{\varpi}(\mathfrak{M}_{[\gamma]}) = \begin{cases} 
(-1) \frac{\beta}{f_{\gamma}[\beta]} & \text{if } g = \sigma^k \phi^f, \beta = 1, \\
(-1) \frac{\beta}{\phi^{-1}} & \text{if } g = \sigma^k \phi^f, \beta = -1, \\
\text{otherwise.}
\end{cases}
\]

First, if \( g = \sigma^k \phi^f \) and \( \beta = 1 \), then we can easily check the desired equality.
Next suppose that \( g = \sigma^k \phi^f \) and \( \beta = -1 \). Then putting \( r = [F_{\alpha} : E] \), we have
\[
(\epsilon_{\alpha} \cdot \chi_{\text{Tam}, \alpha})^{-1}(\varpi_E) = \frac{1}{k_{F, \alpha}} \cdot (-1)^{\frac{r-2}{2}}
\]
\[
= (-1)^{\frac{r-2}{2}} \cdot (-1)^{\frac{1}{2}}
\]
\[
= (-1)^{\frac{r-2}{2}} \cdot (-1)^{\frac{1}{2}} + 1.
\]

Here, again by noting that \( r \) is odd as explained in the proof of Proposition [6.3] we have
\[
q^\phi - q^\phi = q^\phi(q^\phi - 1)(q^{\frac{r-2}{2}} + \cdots + q^\phi + 1) \equiv 0 \mod 4.
\]
This implies that \( (\epsilon_{\alpha} \cdot \chi_{\text{Tam}, \alpha})^{-1}(\varpi_E) = -1 \), as desired.
Finally suppose that \( \beta \neq \pm 1 \). Then we have \( t^0_\mu : \mathcal{O}_{(3)} \cdot t^0_\mu : \mathcal{O}_{(3)} = -1 \) and thus we are to show that
\[
\left( \frac{\beta}{k_{p_{\alpha}}} \right) = \left( \frac{\beta}{F_{p_{\beta}}} \right).
\]

Before we start our computation, we remark that the fact \( \beta \) belongs to \( F_{p_{\beta}} \) can be checked as follows. First, as already explained in Section 6.2, \( \beta = \alpha(\mathcal{O}_E) \) belongs to \( k_{F_{\alpha}} \). By combining this fact with our assumption that \( \beta \) is not equal to \( \pm 1 \), we know that it does not lie in \( k_{F_{\pm \alpha}} \). Thus \( F_{p_{\beta}} \) gives the unique subfield of \( F_{p_{\beta}} \) such that \( [F_{p_{\beta}} : F_{p_{\beta}}'] = 2 \). Moreover, as the Galois group for \( k_{F_{\alpha}}/k_{F_{\pm \alpha}} \) can be naturally identified with that for \( F_{p_{\beta}}/F_{p_{\beta}}' \) by restriction, we have \( k_{F_{\alpha}} \cap F_{p_{\beta}} = F_{p_{\beta}}' \). Hence \( \beta \) belongs to \( F_{p_{\beta}} \).

Now let us show the above equality (4). If we put \( t := |k_{F_{\alpha}} : F_{p_{\beta}}| \) and \( q_{\beta} := |F_{p_{\beta}}| \), then we have \( |k_{F_{\alpha}}| = q_{\beta}^2 \).

\[
\text{\begin{diagram}
  & k_{F_{\alpha}} & \\
  & k_{F_{\pm \alpha}} & \downarrow \text{degree } t \\
  & F_{p_{\beta}} & \downarrow \text{quadratic} \\
  & F_{p_{\beta}}' (= F_{q_{\beta}}) & \\
\end{diagram}}
\]

Thus we get
\[
\left( \frac{\beta}{k_{F_{\alpha}}} \right) = \beta^{q_{\beta}^t + 1} \quad \text{and} \quad \left( \frac{\beta}{F_{p_{\beta}}'} \right) = \beta^{q_{\beta} + 1}.
\]

Since \( \beta \in F_{p_{\beta}} \) satisfies \( \beta^{q_{\beta} + 1} = 1 \), it suffices to prove
\[
\frac{q_{\beta}^t + 1}{2} - \frac{q_{\beta} + 1}{2} \equiv 0 \mod q_{\beta} + 1.
\]

We have
\[
\frac{q_{\beta}^t + 1}{2} - \frac{q_{\beta} + 1}{2} = \frac{q_{\beta}(q_{\beta} - 1)}{2} \cdot (q_{\beta}^t - 2 + \cdots + q_{\beta} + 1).
\]

Here we note that \( t \) is an odd integer. Indeed, if we suppose that \( t = 2s \) were even, then we would have
\[
\beta^{q_{\beta}^t + 1} = \beta^{q_{\beta}^s + 1} = \beta^{q_{\beta}^s} \cdot \beta = \beta^2.
\]

However, as \( \beta \) belongs to \( k_{F_{\alpha}} \), we have \( \beta^{q_{\beta}^t + 1} = 1 \) and this contradicts the assumption that \( \beta \neq \pm 1 \). Therefore, by the oddness of \( t \), we can conclude that \( q_{\beta} + 1 \) divides \( q_{\beta}^t - 2 + \cdots + q_{\beta} + 1 \) and this completes the proof.

**Proof of Proposition 6.2:** symmetric unramified case. Since the characters on both sides of the equality in Proposition 6.2 are tamely ramified, they are equal by Propositions 6.3 and 6.4.

6.4. **Symmetric ramified roots.** In this section, we consider the case of symmetric ramified roots. First we recall that the elliptic torus \( S \) in \( G \) has a symmetric ramified root only when the ramification index \( e \) of the extension \( E/F \) is even (see Proposition 5.3). Thus, in this section, we assume that the ramification index \( e \) is even. Under this assumption, a \( \Gamma_F \)-orbit of symmetric ramified roots is unique.
and represented by $\alpha := \begin{bmatrix} 1 & 0 \\ \sigma z \end{bmatrix}$. As explained in the proof of Proposition 5.3, the element $g = \sigma \tilde{x}$ of $\Gamma_E$ preserves $E$. Note that thus the field $F_0$ is nothing but $E$.

We first take a Howe factorization $(\phi_{i-1}, \ldots, \phi_d)$ of $(S, \xi)$ (see Section 5.2). Let $i$ be the unique index such that $-1 < i < d - 1$ and the symmetric ramified root $\alpha$ lies in $\Phi_{i+1} = \Phi(S, G^{i+1}) \setminus \Phi(S, G^i)$. We simply write $r$ for the depth $r_i$ of the character $\phi_i$.

Now let us recall the definitions of $\chi_{\text{Kal}, \alpha}$, $\chi_{\text{Tam}, \alpha}$, and $\epsilon_{\alpha}$ for our symmetric ramified root $\alpha$:

**Kaletha’s $\chi$-data at $\alpha$:** We take $\chi_{\text{Kal}, \alpha}$ to be a character of $E^\times$ satisfying the following conditions:
- $\chi_{\text{Kal}, \alpha}$ is trivial on $1 + \mathfrak{p}_E$,
- $\chi_{\text{Kal}, \alpha}|_{\mu_E}$ is the unique nontrivial quadratic character of $\mu_E$, and
- $\chi_{\text{Kal}, \alpha}(2\alpha_a) = \lambda_{E/F_{\pm a}}$,

where $\alpha_a$ is an (any) element of $E_{-r}$ which lifts the unique element $\tilde{a}_\alpha$ of $E_{-r}/E_{r+}$ satisfying the equality

$$\phi_i(N_{S(E)/S(F)}(\alpha^\vee (X + 1))) = \psi_E(\tilde{a}_\alpha \cdot X)$$

for every $X \in E_r/E_{r+}$ (see [Kal19a, 1133 page, (4.7.3)]). Here $\psi_E$ is an additive character of $E$ taken as in Section 2 $\lambda_{E/F_{\pm a}}$ is the Langlands constant (see Section 2, and $N_{S(E)/S(F)}$ is the norm map from $S(E)$ to $S(F)$). Note that the third constraint on the value at $2\alpha_a$ is enough to characterize $\chi_{\text{Kal}, \alpha}$ since $r$ is given by $2s+1$ with some odd integer $2s+1$ (see [Kal19a, Section 4.7]) and $\chi_{\text{Kal}, \alpha}|_{E^\times_{\pm a}}$ should be the quadratic character corresponding to the extension $E/F_{\pm a}$ by the definition of $\chi$-data.

**Tam’s $\chi$-data at $\alpha$:** We take $\chi_{\text{Tam}, \alpha}$ to be the unique character of $E^\times$ satisfying the following conditions:
- $\chi_{\text{Tam}, \alpha}$ is on $1 + \mathfrak{p}_E$,
- $\chi_{\text{Tam}, \alpha}|_{\mu_E}$ is the unique nontrivial quadratic character of $\mu_E$, and
- $\chi_{\text{Tam}, \alpha}(\varpi_E) = t_0(\varpi|^{|_g}) \cdot t_\sigma(\varpi|^{|_g}) \cdot t(\varpi|^{|_g})$.

**Remark 6.5.** A priori the definition of $\chi_{\text{Kal}, \alpha}$ may depend on the choice of a Howe factorization of $\xi$. In fact, also for $\chi_{\text{Tam}, \alpha}$, a priori there may be such a dependence although we cannot see it at this point since we have not recalled the precise definition of $\chi_{\text{Tam}, \alpha}$ (i.e., the definition of $t(\varpi|^{|_g})$) up to now. However, in fact, they are independent of the choice of a Howe factorization. We can easily check this by noting that any two different Howe factorizations are “refactorizations” of each other (see [Kal19a, Lemma 3.6.6], or also [BH05b, Proposition 1.1 (2)]).

In order to compare $\chi_{\text{Kal}, \alpha}$ with $\chi_{\text{Tam}, \alpha}$, we next recall the following factorization of $E/F$, which is used naturally in Bushnell–Henniart’s computation of the rectifier:

$$F = K_{-1} \subset K_0 \subset K_1 \subset \cdots \subset K_{l-1} \subset K_l \subset K_{l+1} = E,$$

where
- $K_0/K_{-1}$ is an unramified extension,
- $K_{l+1}/K_l$ is a quadratic ramified extension for each $0 \leq i \leq l - 1$, and
- $K_{l+1}/K_l$ is a totally ramified extension of an odd degree (we put $m$ to be this degree).
Note that such a factorization is unique and given explicitly by
\[ K_0 = F[\mu_E], \ldots, K_{i-1} = F[\mu_E, \varpi_E^{2m}], K_i = F[\mu_E, \varpi_E^m], K_{i+1} = F[\mu_E, \varpi_E]. \]
Here, by our assumption on the ramification index, we have \( l \geq 1 \), i.e., we have at least one quadratic ramified extension in the above sequence. We also note that, since the action of \( \sigma^2 \) on \( E \) is given by
\[ \zeta \mapsto \zeta \quad (\zeta \in \mu_E) \quad \text{and} \quad \varpi_E \mapsto -\varpi_E, \]
\( \sigma^2 \) belongs to \( \Gamma_{K_{i-1}} \setminus \Gamma_{K_i} \).

Now we consider a tame twisted Levi subgroup \( H := \text{Res}_{E/K} \text{GL}_{2m} \) of \( G \) and a Howe factorization of \( \xi \) with respect to \( S \subset H \). Then, by Proposition 6.2, we can construct a Howe factorization \((\phi_{-1}, \ldots, \phi_d)\) of \( \xi \) with respect to \( S \subset H \) by using the Howe factorization \((\phi_{-1}, \ldots, \phi_d)\) of \( \xi \) with respect to \( S \subset G \) satisfying the following equality for each \( 0 \leq i \leq d' \):
\[ \phi_{[i]}|_{S(F)}|_{r_{[i]}} = \phi_{[i]}|_{S(F)}|_{r_{[i]}}. \]
Here recall that \( r'_0 < \cdots < r'_{d'} \) (resp. \( r_0 < \cdots < r_d \)) is a sequence of jumps of \( \xi \) with respect to \( S \subset H \) (resp. \( S \subset G \)) and \([i]\) denotes the unique index satisfying \( r_{[i]} = r'_i \). If we put \(-1 \leq i' \leq d' - 1\) to be the unique index satisfying \( \alpha \in \Phi(S, H^{i'+1}) \setminus \Phi(S, H^{i'}) \), then we have \([i'] = i\). Note that the existence of such an index (or, in other words, the fact that \( \alpha \) lies in \( \Phi(S, H) \)) is guaranteed by that \( \sigma^2 \) belongs to \( \Gamma_{K_{i-1}} \).

Here we remark that, in general, a Howe factorization of \( \xi \) with respect to \( \text{Res}_{E/F} \mathbb{G}_m \subset \text{Res}_{K_{i-1}/F} \text{GL}_{2m,K_{i-1}} \) can be regarded as that with respect to \( \text{Res}_{E/K_i} \mathbb{G}_m \subset \text{GL}_{2m,K_{i-1}} \) (and vice versa). This can be easily checked by noting that the root system for \( \text{Res}_{E/F} \mathbb{G}_m \subset \text{Res}_{K_{i-1}/F} \text{GL}_{2m,K_{i-1}} \) is given by the several copies of that for \( \text{Res}_{E/K_i} \mathbb{G}_m \subset \text{GL}_{2m,K_{i-1}} \) on which the Galois group \( \Gamma_E \) acts transitively. Then, by seeing our descended Howe factorization \((\phi_{-1}', \ldots, \phi_{d'}')\) as the latter one, it can be also understood as a classical Howe factorization in the sense of Howe and Moy (see Section 7.2). In the language of Howe and Moy, they are described as follows. We have a sequence of subfields
\[ K_{i-1} = E_{d'} \subset \cdots \subset E'_0 \subset E'_{i-1} = E, \]
and characters \( \xi'_i \) of \( E_{i+1}^{\times} \) satisfying
\[ \xi = \xi'_1 \cdot (\xi'_0 \circ \text{Nr}_{E'/E'_0}) \cdots (\xi'_{d'} \circ \text{Nr}_{E/E'_{d'}}). \]
Then index \( i' \) is characterized as the unique index satisfying \( \sigma^2 \in \Gamma_{E'_{i+1}} \setminus \Gamma_{E'_i} \).

The depth of the character \( \phi_{i'} = \xi'_i \circ \text{Nr}_{E/E'_{i'}} \) is given by \( r'_{i'} = r_i (=: r = \frac{2d + 1}{2}) \).

**Proof of Proposition 6.2** symmetric ramified case. Our task is to show that
\[ \chi_{\text{Tam}, \alpha}^{-1} = \chi_{\text{Kal}, \alpha}. \]
For this, by the definitions of \( \chi_{\text{Kal}, \alpha} \) and \( \chi_{\text{Tam}, \alpha} \), it suffices to check that
\[ \chi_{\text{Tam}, \alpha}^{-1}(2a_{\alpha}) = \chi_{\text{Kal}, \alpha}(2a_{\alpha}). \]

We start from rewriting \( \chi_{\text{Kal}, \alpha}(2a_{\alpha}) \). By definition, \( \chi_{\text{Kal}, \alpha}(2a_{\alpha}) \) is equal to the Langlands constant \( \lambda_{E/F_{\pm \alpha}} \). As \( E/F_{\pm \alpha} \) is quadratic ramified, we have
\[ \lambda_{E/F_{\pm \alpha}} = n(\psi_{F_{\pm \alpha}}) \]
(see, for example, [BH05b, Lemma 1.5 (3)]). By noting that \( K_{l-1} \) is contained in \( F_{\pm \alpha} = F[\mu_E, \varpi_E] \) and that the extension \( F_{\pm \alpha}/K_{l-1} \) is totally ramified of degree \( m \), the additive character of \( k_{F_{\pm \alpha}} = k_E \) induced by \( \psi_{F_{\pm \alpha}} \) is given by the composition of the additive character induced by \( \psi_{K_{l-1}} \) and the multiplication map via \( m \). Thus we get
\[
n(\psi_{F_{\pm \alpha}}) = q^{-\frac{1}{2}} \sum_{x \in k_E^\times} \left( \frac{x}{k_E} \right) \psi_{F_{\pm \alpha}}(x)
\]
\[
= q^{-\frac{1}{2}} \sum_{x \in k_E^\times} \left( \frac{x}{k_E} \right) \psi_{K_{l-1}}(mx)
\]
\[
= q^{-\frac{1}{2}} \left( \frac{m-1}{k_E} \right) \sum_{x \in k_E^\times} \left( \frac{x}{k_E} \right) \psi_{K_{l-1}}(x) = \left( \frac{m}{q} \right)^{-1} \cdot n(\psi_{K_{l-1}}).
\]
In summary, we get
\[
(6) \quad \chi_{\text{Tam}, \alpha}(2a_{\alpha}) = n(\psi_{K_{l-1}}) \left( \frac{m}{q} \right).
\]
We next rewrite \( \chi_{\text{Tam}, \alpha}(\varpi_E) \). First, by the oddness of the numerator of \( r \), the module \( \mathcal{M}_{[g]} \) is zero. In particular, two \( t \)-factors \( t_{\mu}(\varpi_E) \) and \( t_\varpi(\varpi_E) \) appearing in the definition of \( \chi_{\text{Tam}, \alpha}(\varpi_E) \) are trivial. Next, we recall from [Tam16, Section 5.3.4, Remark 7.7] the expression of the last \( t \)-factor \( t(\mathcal{M}_{[g]}) \) in terms of a quadratic form and the Gauss sum. For this, observe that for any subextension \( L_1/L_2 \) of \( E/K_{l-1} \) the modules defined by
\[
\mathcal{M}_{L_1/L_2} := \bigoplus_{[h] \in \Gamma_E \setminus \Gamma_{l_2} \Gamma_{l_1}} \mathcal{M}_{[h]} \subset \mathcal{M}_{E/K_{l-1}} := \bigoplus_{[h] \in \Gamma_E \setminus \Gamma_{K_{l-1}} \Gamma_{E}} \mathcal{M}_{[h]},
\]
as well as \( \mathcal{M}_{[g]} \subset \mathcal{M}_{E/K_{l-1}} \), are naturally vector spaces over \( k_{K_{l-1}} = k_E \). Recall that these spaces can be regarded as subspaces of \( \mathfrak{U} = \mathfrak{A} / \mathfrak{P} \). Let \( q_{K_{l-1}}^{(I)} \) be the quadratic form defined by
\[
q_{K_{l-1}}^{(I)} : \mathcal{M}_{E/K_{l-1}}^* \rightarrow k_E;
\]
x \( \mapsto \text{tr}_{A_{K_{l-1}}/k_{K_{l-1}}} ( (x - \varpi_{E}^{-1} x \varpi_{E}) \cdot \varpi_{E}^{-1} x \varpi_{E}^{-1} x )
\]
where we write \( A_{K_{l-1}} := \text{End}_E(K_{l-1}) \) and \( \text{tr}_{A_{K_{l-1}}/k_{K_{l-1}}} \) denotes the reduced trace of \( A_{K_{l-1}}/k_{K_{l-1}} \). Here note that this definition makes sense since the space \( \mathcal{M}_{E/K_{l-1}} \) can be identified with \( (A_{K_{l-1}} \cap \mathfrak{U})/(A_{K_{l-1}} \cap \mathfrak{P}) \subset \mathfrak{U}/\mathfrak{P} \). We also note that we have \( \text{End}_E(K_{l-1}) \cong \text{Mat}_{2m}(K_{l-1}) \) and that \( \text{tr}_{A_{K_{l-1}}/k_{K_{l-1}}} = 0 \) but the trace as a matrix in \( \text{Mat}_{2m}(K_{l-1}) \). This is the quadratic form \( q_{K_{l-1}}^{(I)} \) in [Tam16, (5.19)] for \( F = K_{l-1} \) and \( j = I \) (as remarked before we start this proof, the depth of \( \phi_{j}^I = \xi_{j}^I \circ \text{Nr}_{E/E}^{[j]} \) is equal to \( r = 2s+1 \) and thus \( h_1 \) in [Tam16, p.1738] is equal to \( s+1 \)). Now as explained in [Tam16, Remark 7.7], the \( t \)-factor \( t(\mathcal{M}_{[g]}) \) is given by
\[
(7) \quad \left( \left( \frac{\zeta^{(I)}(\varpi_E)}{k_E} \right) n(\psi_{K_{l-1}}) \right)^{\dim_{k_E} \mathcal{M}_{[g]}} \left( \frac{\det(q_{K_{l-1}}^{(I)} | \mathcal{M}_{[g]})}{k_E} \right).
\]
Here the meanings of the symbols \( \zeta^{(I)}(\varpi_E) \) and \( \det(q_{K_{l-1}}^{(I)} | \mathcal{M}_{[g]}) \) are as follows:
- We consider an element \( \alpha^{(I)}(\varpi_E) \in E_{\varpi_{E}} \times \) satisfying
\[
\xi_{j}^I(1 + X) = \psi_{E_{\varpi_{E}}} \left( \alpha^{(I)}(\varpi_E) \cdot X \right)
\]

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for any \( X \in E_{r'} \). Since such an element is unique up to \( U_{E'}^{1} \), we can find a unique root of unity \( \zeta^{(i)}(\varpi_{E}) \in \mu_{E} \) satisfying
\[
\alpha^{(i)}(\varpi_{E}) = \varpi_{E}^{-2(m+1)\cdot (\zeta^{(i)}(\varpi_{E}))} \mod U_{E}^{1}.
\]

- We put \( \mathfrak{M}^{(i)} = \mathfrak{M}_{K_{1}/K_{1-1}} \cap \mathfrak{M}_{E'/E_{i+1}}^{(i)}. \) Then we define a quadratic form \( q^{(i)}_{K_{1}/K_{1-1}} \) to be the restriction \( q^{(i)}_{K_{1-1}\mid \mathfrak{M}^{(i)}} \) of \( q^{(i)}_{K_{1}/K_{1-1}} \) on \( \mathfrak{M}^{(i)}. \) The symbol \( \det(q^{(i)}_{K_{1}/K_{1-1}\mid \mathfrak{M}^{(i)}}) \) denotes the discriminant of the restriction of \( q^{(i)}_{K_{1}/K_{1-1}} \) to \( \mathfrak{M}^{(i)} \).

We remark that, strictly speaking, the quadratic form \( q^{(i)}_{K_{1}/K_{1-1}} \) in the last quadratic character in [Tam10] Remark 7.7] should be written as \( q^{(i)}_{K_{1}/K_{1-1}\mid \mathfrak{M}^{(i)}} \). Here note that, by definition, \( q^{(i)}_{K_{1}/K_{1-1}\mid \mathfrak{M}^{(i)}} \) is nothing but \( q^{(i)}_{K_{1-1}\mid \mathfrak{M}^{(i)}} \). Now let us compute the discriminant of \( q^{(i)}_{K_{1-1}\mid \mathfrak{M}^{(i)}} \). Since we have \( \dim_{k_{E}} \mathfrak{M}^{(i)} = 1 \) (recall Proposition 3.8, and the definition of \( \mathfrak{M}^{(i)} \)), this is easily done by an explicit computation as follows (cf. the proof of [BH05], Proposition 8.3). We start by extending \( q^{(i)}_{K_{1}/K_{1-1}} \) from \( \mathfrak{M}^{(i)}_{E'/E_{i+1}} \) to \( \mathfrak{M}^{(i)}_{E'/E_{i+1}} \) by the same formula:
\[
\mathfrak{M}^{(i)}_{E'/E_{i+1}} \rightarrow k_{E};
\]
\[
x \mapsto \text{tr}_{A_{K_{1-1}/K_{1-1}}}( (x - \varpi_{E}^{-1} x \varpi_{E}) \cdot \varpi_{E}^{n} x \varpi_{E}^{-x} ),
\]
for which we still write \( q^{(i)}_{K_{1-1}} \). If we take a basis \( \{ \varpi_{E}^{2m-1}, \varpi_{E}^{2m-2}, \ldots, \varpi_{E}, 1 \} \) of \( E \) over \( K_{1-1} = F[\mu_{E}, \varpi_{E}] \) to identify \( A_{K_{1-1}} = \text{End}_{K_{1-1}}(E) \) with \( \text{Mat}_{2m}(K_{1-1}) \), then \( \mathfrak{A}_{K_{1-1}} := \mathfrak{A} \cap A_{K_{1-1}} \) (resp. \( \mathfrak{P}_{K_{1-1}} := \mathfrak{P} \cap A_{K_{1-1}} \)) is identified with the subset of matrices with entries in \( O_{K_{1-1}} \) that are upper triangular (resp. strictly upper triangular) when reduced modulo \( p_{K_{1-1}} \). We have an isomorphism of \( k_{E} \)-algebras:
\[
\mathfrak{M}^{(i)}_{E'/E_{i+1}} \cong \mathfrak{A}_{K_{1-1}} / \mathfrak{P}_{K_{1-1}} \cong k_{E}^{2m};
\]
\[
(x_{ij})_{1 \leq i, j \leq 2m} \mapsto (\varpi_{21}, \varpi_{22}, \ldots, \varpi_{2m, 2m}).
\]
The action of \( \varpi_{E} \in \varpi_{E} \) on the right-hand side induced by this isomorphism is given by the translation
\[
(a_{1}, a_{2}, \ldots, a_{2m}) \mapsto (a_{2}, \ldots, a_{2m}, a_{1}).
\]

Thus the submodule \( \mathfrak{M}^{(i)}_{[g]} \subset \mathfrak{M}^{(i)}_{E'/E_{i+1}} \) is mapped to
\[
(k_{E}^{2m})_{[g]} := \{ v(a) := (a, -a, \ldots, a, -a) \in k_{E}^{2m} \mid a \in k_{E} \}
\]
as it can be characterized as the subspace where \( \varpi_{E} \) acts as multiplication via \( -1 \). Moreover, the trace map \( \text{tr}_{A_{K_{1-1}/K_{1-1}}} \) on \( A_{K_{1-1}} \) induces the summation map
\[
\nu : k_{E}^{2m} \rightarrow k_{E}; \quad (a_{1}, a_{2}, \ldots, a_{2m}) \mapsto \sum_{i=1}^{2m} a_{i}.
\]

Therefore the quadratic form \( q^{(i)}_{K_{1-1}} \) on \( \mathfrak{M}^{(i)}_{[g]} \) induces the following quadratic form on \( (k_{E}^{2m})_{[g]} \):
\[
(k_{E}^{2m})_{[g]} \rightarrow k_{E}; \quad v(a) \mapsto \nu(2v(a) \cdot (-1)^{s}v(a)) = (-1)^{s} 4ma^{2},
\]
whose discriminant is clearly $(-1)^s4m \equiv (-1)^s m \mod (k_E^*)^2$. Now, by substituting $\dim_{k_E} \mathfrak{M}_[\alpha] = 1$ and the discriminant just computed into (7), we get

$$
\chi_{\text{Tam}, \alpha}(\varpi_E) = \left( \frac{\chi^{(i')}(\varpi_E)}{k_E} \right) n(\psi_{K_i}) \left( \frac{(-1)^s m}{q^s} \right).
$$

Before we start our comparison of $\chi_{\text{Kal}, \alpha}$ and $\chi_{\text{Tam}, \alpha}$, we investigate the relationship between $a_\alpha$ and $\xi^{(i')}(\varpi_E)$. Recall that $a_\alpha \in E_r$ is a lift of the unique element $\bar{a}_\alpha \in E_{r'/E_{r'}}$ satisfying the equality

$$
\phi_1(\text{Nr}_{S(E)/S(F)} \circ a^{\gamma}(X + 1)) = \psi_E(\bar{a}_\alpha \cdot X)
$$

for any $X \in E_r/E_{r'}$. Since we have $\phi_1|_{S(F)} = \phi'_1|_{S(F)}$, as explained in (9) and $\phi'_1|_{S(F)} = \xi'_1 \circ \text{Nr}_{E/E'}$, this is equivalent to

$$
\xi'_1 \circ \text{Nr}_{E/E'}(\text{Nr}_{S(E)/S(F)} \circ a^{\gamma}(X + 1)) = \psi_E(\bar{a}_\alpha \cdot X).
$$

Here we note that the image of $X + 1$ under the map $\text{Nr}_{S(E)/S(F)} \circ a^{\gamma}$ from $E^\times$ to $S(F) \cong E^\times$ is given by $(X + 1)(\sigma E(X) + 1)^{-1}$. Indeed, if we consider an isomorphism from $S(F)$ to $\prod_{i=1}^n F_i^{\times}$ as in Section 3.2, the coroot $\alpha^{\vee}$ is described as

$$
E^\times \rightarrow S(E); \quad x \mapsto (x, 1, \ldots, 1, x^{-1}, 1, \ldots, 1).
$$

Here the component having $x^{-1}$ corresponds to $\sigma E \in \{ \Gamma_E/\Gamma_{F,E} \}$. Note that $\alpha^{\vee}$ maps $E^\times$ into $S(E)$ as $E$ is the splitting field $F_\alpha$ of $\alpha$. On the other hand, for each $z \in S(E)$, its norm $\text{Nr}_{S(E)/S(F)}(z)$ is defined as the product of $g(z)$ over $g \in \Gamma_E/\Gamma_{E'}$. Thus, by considering a description of the Galois action on $\prod_{i=1}^n F_i^{\times}$, we can easily check that $\text{Nr}_{S(E)/S(F)} \circ a^{\gamma}(x)$ is given by $x \cdot \sigma E(x)$ for each $x \in E^\times$.

As we have

$$(X + 1)(\sigma E(X) + 1)^{-1} \equiv X - \sigma E(X) + 1$$
in $E_r^\times/E_{r'}^\times$, we have

$$
\text{Nr}_{E/E'}((X + 1)(\sigma E(X) + 1)^{-1}) \equiv \text{Tr}_{E/E'}(X - \sigma E(X)) + 1
$$
in $E_r^\times/E_{r'}^\times$'. By recalling that $r$ is given by $2e + 1$ and $\sigma E$ acts on $E$ via

$$
\sigma E \equiv 1 \quad \text{and} \quad \sigma E(\varpi_E) = -\varpi_E,
$$
we get

$$
\text{Tr}_{E/E'}(X - \sigma E(X)) + 1 \equiv 2 \text{Tr}_{E/E'}(X) + 1
$$
in $E_r^\times/E_{r'}^\times$. Therefore, for any $X \in E_r/E_{r'}$, we have

$$
\xi'_1(2 \text{Tr}_{E/E'}(X) + 1) = \psi_E(\bar{a}_\alpha \cdot X).
$$

On the other hand, by the definitions of $\alpha^{(i')}(\varpi_E)$ and $\zeta^{(i')}(\varpi_E)$, the left-hand side equals

$$
\psi_{E'}(\alpha^{(i')}(\varpi_E) \cdot 2 \text{Tr}_{E/E'}(X)) = \psi_E(\alpha^{(i')}(\varpi_E) \cdot 2X)
$$

$$
= \psi_E((\varpi_E)^{-2} \zeta^{(i')}(\varpi_E) \cdot 2X).
$$

In summary, we have the equality

$$
\psi_E(\bar{a}_\alpha \cdot X) = \psi_E((\varpi_E)^{-2} \zeta^{(i')}(\varpi_E) \cdot 2X)
$$
for any $X \in E_r/E_{r'}$. Therefore we have

$$
\bar{a}_\alpha \equiv 2(\varpi_E)^{-2} \zeta^{(i')}(\varpi_E) \cdot 2X \quad \text{in} \quad E_{r_r}/E_{r_{r+}}.
$$
Now let us complete the proof. By recalling that $\chi_{\text{Tam}, \alpha}$ is given by the nontrivial quadratic character on $\mu_E$, the equality (9) tells us

$$\chi_{\text{Tam}, \alpha}^{-1}(2a_\alpha) = \chi_{\text{Tam}, \alpha}^{-1}(4\varpi_E^{-(2s+1)}c^{(v)}(\varpi_E))$$

$$= \chi_{\text{Tam}, \alpha}(\varpi_E)^{2s+1} \left( \frac{c^{(v)}(\varpi_E)}{k_E} \right)^{-1}.$$  

Then, by the equality (8) and noting that $\chi_{\text{Tam}, \alpha}(\varpi_E)^2 = \frac{-1}{q}$ (this also can be deduced from (8)), we get

$$\chi_{\text{Tam}, \alpha}^{-1}(2a_\alpha) = \left( \frac{-1}{q} \right) s \left( \frac{c^{(v)}(\varpi_E)}{k_E} \right) n(\psi_{K_{i-1}}) \left( \frac{(-1)^s m}{q} \right) \left( \frac{c^{(v)}(\varpi_E)}{k_E} \right)^{-1}$$

$$= n(\psi_{K_{i-1}}) \left( \frac{2a_\alpha}{q} \right).$$

Thus we get $\chi_{\text{Tam}, \alpha}^{-1}(2a_\alpha) = \chi_{\text{Kal}, \alpha}(2a_\alpha)$ by the equality (9) and this completes the proof. \[\square\]

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