Minimal silting modules and ring extensions

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Abstract  Ring epimorphisms often induce silting modules and cosilting modules, termed minimal silting or minimal cosilting. The aim of this paper is twofold. Firstly, we determine the minimal tilting and minimal cotilting modules over a tame hereditary algebra. In particular, we show that a large cotilting module is minimal if and only if it has an adic module as a direct summand. Secondly, we discuss the behavior of minimality under ring extensions. We show that minimal cosilting modules over a commutative noetherian ring extend to minimal cosilting modules along any flat ring epimorphism. Similar results are obtained for commutative rings of small homological dimensions.

Keywords  minimal silting modules, ring epimorphisms, ring extensions, minimal cosilting modules, tame hereditary algebras

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1 Introduction

Tilting theory and its recent development into silting theory are known to be closely related to localization of rings. For example, every Ore localization \( R \hookrightarrow \Sigma^{-1}R \) of a ring \( R \) with the property that \( \Sigma^{-1}R \) has the projective dimension at most one over \( R \) gives rise to a tilting module \( \Sigma^{-1}R \oplus \Sigma^{-1}R/R \). Of course, such a tilting module will often be large, i.e., it will not be finitely presented, not even up to equivalence.

More generally, ring epimorphisms with nice homological properties give rise to silting modules. Such modules were introduced in [5] as large analogues of the support \( \tau \)-tilting modules studied in representation theory and cluster theory. They can be characterized as zero cohomology of (not necessarily compact) two-term silting complexes.

Building on these connections, it was shown in [6] that the universal localizations of a hereditary ring are parametrized by certain silting modules which are determined by a minimality condition and are called minimal silting. A dual version of this result was recently established in [3], leading to the notion of a minimal cosilting module. The interest in minimal silting or cosilting modules goes well beyond the hereditary case. For example, minimal cosilting modules also parametrize the flat ring epimorphisms starting in a commutative noetherian ring. We refer to Section 2 for details.

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In the present paper, we continue these investigations by analyzing two aspects. The first one concerns an important and widely studied class of hereditary rings: finite-dimensional tame hereditary algebras. The large cotilting modules over such algebras were classified in [15]. They are determined by their indecomposable summands, which can be either finite-dimensional regular modules or infinite-dimensional pure-injective, and thus Prüfer modules, adic modules or the generic module. A classification of the large tilting modules was established in [9]. Both tilting and cotilting modules are parametrized by the pairs \((Y,P)\), where \(Y\) is a branch module, i.e., a finite-dimensional regular module with certain combinatorial features, and \(P\) is a subset of the projective line, when the ground field is algebraically closed, or more generally, a subset of the index set \(X\) of the tubular family \(t = \bigcup_{X \in X} t_X\) in the Auslander-Reiten quiver.

In Section 3, we determine the minimal tilting and minimal cotilting modules over a tame hereditary algebra. Since the finite-dimensional (co)tilting modules are all minimal, we focus on the large ones. We prove that under the parametrization described above, minimal tilting or minimal cotilting modules correspond to the pairs \((Y,P)\), where \(P\) is not empty. This result (see Theorem 3.14) is achieved by an explicit construction of the universal localization corresponding to the tilting module \(T_{(Y,P)}\) when \(P \neq \emptyset\). More precisely, we construct the wide subcategory \(\mathcal{M}\) of the category of finite-dimensional modules that realizes \(T_{(Y,P)}\) as the tilting module \(R_{\mathcal{M}} \oplus R_{\mathcal{M}}/R\) arising from the universal localization \(R \to R_{\mathcal{M}}\) of \(R\) at \(\mathcal{M}\). We also obtain that a large cotilting module is minimal if and only if it has an adic direct summand (see Theorem 3.16).

The second aspect we want to address is the behavior of silting and cosilting modules under ring extensions. A criterion recently established in [13] ensures that every silting module \(T\) over a commutative ring \(R\) extends to a silting module \(T \otimes_R S\) along any ring epimorphism \(R \to S\). In Section 4, we give conditions under which the minimality is preserved. In particular, we show that all the minimal cosilting modules over a commutative noetherian ring extend to minimal cosilting modules along any flat ring epimorphism (see Corollary 4.11). Over a commutative hereditary ring, we see that every cosilting module extends to a minimal cosilting module along any ring epimorphism (see Corollary 4.12).

The rest of this paper is organized as follows. Section 2 contains some preliminaries on ring epimorphisms and a survey on their relation with silting and cosilting modules. In Section 3, we determine the minimal tilting and minimal cotilting modules over a tame hereditary algebra. Section 4 is devoted to extensions of minimal silting (or cosilting) modules along ring epimorphisms. We first study the example of the Kronecker algebra (see Example 4.2). Then we turn to commutative rings and provide some useful criteria for preserving minimality. We close the paper with applications to commutative noetherian rings and commutative rings of small homological dimensions.

### 2 Preliminaries

#### 2.1 Notation

Throughout the paper, denote by \(R\) a ring, \(\text{Mod}R\) (\(R\text{Mod}\)) the category of all the right (left) \(R\)-modules, and \(mod R\) (\(R\text{mod}\)) the category of finitely presented right (left) \(R\)-modules.

We fix a commutative ring \(k\) such that \(R\) is a \(k\)-algebra, together with an injective cogenerator \(W\) in \(\text{Mod}k\), and we denote by \((-)^{\perp} = \text{Hom}_k(-, W)\) the duality functors between \(\text{Mod}R\) and \(R\text{Mod}\). For example, one can choose \(k = \mathbb{Z}\) and \(W = \mathbb{Q}/\mathbb{Z}\). In the case where \(R\) is a finite-dimensional algebra over a field \(k\), we will take the usual vector space duality \((-)^{\perp} = D = \text{Hom}_k(-, k)\).

Let \(C \subset \text{Mod}R\) be a class of modules. Denote by \(\text{Add}C\) (resp. \(\text{add}C\)) the class consisting of all the modules isomorphic to direct summands of (finite) direct sums of elements of \(M\). The class consisting of all the modules isomorphic to direct summands of products of modules of \(C\) is denoted by \(\text{Prod}C\). The class consisting of the right \(R\)-modules which are epimorphic images of arbitrary direct sums of elements in \(C\) is denoted by \(\text{Gen}C\). Dually, we define \(\text{Cogen}C\) as the class of all the submodules of arbitrary direct products of elements in \(C\). Moreover, we write

\[ C^\perp = \{ N_R \mid \text{Ext}^1_R(M, N) = 0 = \text{Hom}_R(M, N) \text{ for each } M \in C \}, \]
A ring homomorphism \( \lambda : R \rightarrow S \) is a ring epimorphism if it is an epimorphism in the category of rings with unit, or equivalently, if the functor given by restriction of scalars \( \lambda_* : \text{Mod}S \rightarrow \text{Mod}R \) is a full embedding.

A ring epimorphism \( \lambda : R \rightarrow S \) is said to be

- homological if \( \text{Tor}_i^R(S,S) = 0 \) for \( i > 0 \), or equivalently, the functor given by restriction of scalars \( \lambda_* : D(\text{Mod}S) \rightarrow D(\text{Mod}R) \) induces a full embedding of the corresponding derived categories,
- (right) flat if \( S \) is a flat right \( R \)-module, and
- pseudoflat if \( \text{Tor}_1^R(S,S) = 0 \).

Two ring epimorphisms \( \lambda : R \rightarrow S \) and \( \lambda' : R \rightarrow S' \) are equivalent if there is a ring isomorphism \( h : S \rightarrow S' \) such that \( \lambda' = h \cdot \lambda \). We say that \( \lambda \) and \( \lambda' \) lie in the same epiclass of \( R \).

Ring epimorphisms are closely related to certain subcategories of \( \text{Mod}R \).

**Definition 2.2.** A full subcategory \( \mathcal{X} \) of \( \text{Mod}R \) is called bireflective if the inclusion functor \( \mathcal{X} \rightarrow \text{Mod}R \) admits both a left and a right adjoint, or equivalently, \( \mathcal{X} \) is closed under products, coproducts, kernels and cokernels.

**Theorem 2.3** (See [11,17]).

1. The map assigning to a ring epimorphism \( \lambda : R \rightarrow S \) the essential image \( \mathcal{X} \) of the functor \( \lambda_* \) defines a bijection between
   (i) epiclasses of ring epimorphisms \( R \rightarrow S \) and
   (ii) bireflective subcategories \( \mathcal{X} \) of \( \text{Mod}R \).

2. The following statements are equivalent for a ring epimorphism \( \lambda : R \rightarrow S \):
   (i) \( \lambda \) is a pseudoflat ring epimorphism;
   (ii) \( \mathcal{X} \) is closed under extensions in \( \text{Mod}R \);
   (iii) the functors \( \text{Ext}_R^1 \) and \( \text{Ext}_S^1 \) agree on \( S \)-modules;
   (iv) the functors \( \text{Tor}_1^R \) and \( \text{Tor}_1^S \) agree on \( S \)-modules.

Classical localization of commutative rings at multiplicative sets provides an important class of examples for flat ring epimorphisms. More generally, the notion of universal localization which we recall below yields a large supply of pseudoflat ring epimorphisms. If \( R \) is a hereditary ring, then \( \lambda : R \rightarrow S \) is a homological ring epimorphism if and only if it is pseudoflat, which is equivalent to being a universal localization of \( R \) by [20, Theorem 6.1].

**Theorem 2.4** (See [21, Theorem 4.1]). Let \( R \) be a ring and \( \Sigma \) be a class of morphisms between finitely generated projective right \( R \)-modules. Then there is a pseudoflat ring epimorphism \( \lambda : R \rightarrow R_\Sigma \) called a universal localization of \( R \) at \( \Sigma \) such that

1. \( \lambda \) is \( \Sigma \)-inverting: if \( \sigma \) belongs to \( \Sigma \), then \( \sigma \otimes_R R_\Sigma \) is an isomorphism of right \( R_\Sigma \)-modules, and
2. \( \lambda \) is universal \( \Sigma \)-inverting: for any \( \Sigma \)-inverting morphism \( \lambda' : R \rightarrow S \), there exists a unique ring homomorphism \( g : R_\Sigma \rightarrow S \) such that \( g \circ \lambda = \lambda' \).

Given a bireflective subcategory \( \mathcal{X} \) of \( \text{Mod}R \) and an \( R \)-module \( M \), we will denote by \( \psi_M : M \rightarrow X_M \) the unit of the adjunction given by the left adjoint of the inclusion functor. The map \( \psi_M \) is an \( \mathcal{X} \)-reflection, i.e., \( \text{Hom}_R(\psi_M, X) \) is an isomorphism for all \( X \) in \( \mathcal{X} \). In particular, \( \psi_M \) is a left \( \mathcal{X} \)-approximation which is left minimal, i.e., any endomorphism \( \theta \) of \( X_M \) with \( \theta \circ \psi_M = \psi_M \) is an isomorphism. This entails that every ring epimorphism \( \lambda : R \rightarrow S \) is a left minimal \( R \)-module homomorphism.
2.3 Silting theory

Given a morphism $\sigma: P \to Q$ between projective modules, we define the subcategory
\[ D_{\sigma} = \{ X \in \text{Mod}R \mid \text{Hom}_R(\sigma, X) \text{ is surjective} \}. \]

**Definition 2.5** (See [5, 7, 14]). We say that an $R$-module $T$
- admits a \textit{presilting presentation} if there is a projective presentation $P \xrightarrow{\phi} Q \xrightarrow{\lambda} T \to 0$ such that $\text{Hom}_{D(\text{Mod}R)}(\sigma, \phi^0(1)) = 0$ for all the sets $I$, or equivalently, $\text{Gen}T \subseteq D_{\sigma}$,
- is a \textit{silting module} if it admits a projective presentation $P \xrightarrow{\phi} Q \xrightarrow{\lambda} T \to 0$ with $\text{Gen}T = D_{\sigma}$, in which case we say that $T$ is silting \textit{with respect to} $\sigma$, and
- is a \textit{tilting module} if it is silting with respect to an injective map $\sigma$, or equivalently, $\text{Gen}T = T^\perp$.

These amount to the following conditions:
\begin{enumerate}[(T1)]
\item $\text{proj.dim}(T) \leq 1$;
\item $\text{Ext}_R^1(T, T^{(\kappa)}) = 0$ for any cardinal $\kappa$;
\item there is an exact sequence $0 \to R \to T_0 \to T_1 \to 0$ with $T_0, T_1 \in \text{Add}T$.
\end{enumerate}

Note that every silting module $T$ satisfies $\text{Add}T = \text{Gen}T \cap T^\perp(\text{Gen}T)$. Moreover, $T$ gives rise to a torsion pair with torsion class $\text{Gen}T$ and torsion-free class $T^\perp$. The class $\text{Gen}T$ is called a silting \textit{class}, or a \textit{tilting class} in the case where $T$ is a tilting module. Silting modules having the same silting class are said to be \textit{equivalent}. We say that a silting module is \textit{large} if it is not equivalent to a finitely presented silting module. Cosilting or cotilting modules and classes are defined dually, and equivalence of cosilting or cotilting modules is defined correspondingly.

If $T$ is a silting module with respect to $\sigma$, then by [24] there is a triangle
\begin{equation}
R \xrightarrow{\phi} \sigma_0 \to \sigma_1 \to R[1] \tag{2.1}
\end{equation}
in the derived category $D(\text{Mod}R)$, where $\sigma_0$ and $\sigma_1$ lie in $\text{Add}\sigma$ and $\phi$ is a left $\text{Add}\sigma$-approximation of $R$.

Applying the cohomology functor $H^0(\cdot)$ to this triangle, we obtain an exact sequence
\begin{equation}
R \xrightarrow{\phi} T_0 \xrightarrow{f} T_1 \to 0 \tag{2.2}
\end{equation}
in $\text{Mod}R$, where $T_0, T_1 \in \text{Add}T$ and $f$ is a left $\text{Add}T$-approximation of $R$.

**Definition 2.6** (See [7]). Let $T$ be a silting module with respect to $\sigma$. If the map $\phi$ in the triangle (2.1) can be chosen left minimal, then $T$ is said to be a \textit{minimal silting module}.

For example, all the finite-dimensional silting (i.e., support $\tau$-tilting) modules over a finite-dimensional algebra are minimal (see [7, Remark 1.6]). Minimal silting modules are closely related to pseudoflat ring epimorphisms. Indeed, there is a map assigning a pseudoflat ring epimorphism to every minimal silting module (see [7, Corollary 2.4]). Conversely, every ring epimorphism $\lambda: R \to S$ for which the right $R$-module $S_R$ admits a presilting presentation induces a silting $R$-module of the form $T = S \oplus \text{Coker}\lambda$ (see [7, Proposition 1.3]).

**Definition 2.7.** We say that a silting module $T$ \textit{arises from a ring epimorphism} if there is a ring epimorphism $\lambda: R \to S$ such that $S \oplus \text{Coker}\lambda$ is a silting $R$-module equivalent to $T$.

**Proposition 2.8.** The map assigning to a ring epimorphism $\lambda: R \to S$ the right $R$-module $S \oplus \text{Coker}\lambda$ yields an injection from (i) to (ii), where
\begin{enumerate}[(i)]
\item is epiclasses of ring epimorphisms $\lambda: R \to S$ such that $S_R$ admits a presilting presentation, and
\item is equivalence classes of silting right $R$-modules arising from ring epimorphisms.
\end{enumerate}
If $R$ is right perfect, then this map is a bijection, and all the modules in (ii) are minimal silting modules.

**Proof.** If a silting module $T = S \oplus \text{Coker}\lambda$ arises from a ring epimorphism $\lambda: R \to S$, then the map $\lambda$, viewed as an $R$-module homomorphism, is a minimal left $\text{Add}T$-approximation of $R$ and is thus uniquely determined up to isomorphism. This shows that the equivalence class of $T$ determines the bireflective subcategory $\mathcal{X} = \{ X \in \text{Mod}R \mid \text{Hom}_R(\lambda, X) \text{ is bijective} \}$ and therefore the epiclass of $\lambda$, proving the injectivity of the assignment.
Moreover, every silting module \( T = S \oplus \text{Coker} \lambda \) as in (ii) yields an exact sequence of the form (2.2), where \( f \) is left minimal. Now, if \( R \) is right perfect, we infer from [7, Remark 1.6] that \( T \) is left minimal, and its presilting presentation entails the existence of a presilting presentation for the direct summand \( S_R \) (compare the minimal projective presentations of \( T_R \) and \( S_R \)). Thus the assignment is also surjective.

The bijection for right perfect rings in Proposition 2.8 has a dual version, which holds over arbitrary rings thanks to the existence of minimal injective copresentations. Let us first introduce the necessary terminology. If \( C \) is a cosilting left \( R \)-module, then by [25] there is an exact sequence

\[
0 \to C_1 \to C_0 \xrightarrow{g} R^+,
\]

where \( C_0 \) and \( C_1 \) are in \( \text{Prod}C \), and \( g \) is a right \( \text{Prod}C \)-approximation of \( _RR^+ \).

**Definition 2.9** (See [3]).

(1) A cosilting left \( R \)-module \( C \) is a minimal cosilting module if the exact sequence (2.3) can be chosen such that the subcategory \( \text{Cogen}C \cap \perp C_1 \) is bireflective, and \( \text{Hom}_R(C_0,C_1) = 0 \).

(2) We say that a module \( C \) admits a precosilting copresentation if there is an injective copresentation \( 0 \to C \to Q_0 \xrightarrow{\omega} Q_1 \) such that \( \text{Hom}_{D(A)}(\omega^I,\omega[I]) = 0 \) for all the sets \( I \).

**Theorem 2.10** (See [3, Theorem 4.17]). The map assigning to a ring epimorphism \( \lambda : R \to S \) the left \( R \)-module \( S^+ \oplus \text{Ker} \lambda^+ \) yields a bijection between

(i) epiclasses of ring epimorphisms \( \lambda : R \to S \) such that \( _RS^+ \) has a precosilting copresentation, and

(ii) equivalence classes of minimal cosilting left \( R \)-modules.

Also minimal cosilting modules are intimately related to pseudoflat ring epimorphisms.

**Remark 2.11** (See [3, Example 4.15] and [4, Proposition 4.5]). The ring epimorphisms satisfying the condition (i) above are all pseudoflat, and the converse holds true if \( S_R \) has the weak dimension at most one. If \( R \) is commutative noetherian, then (i) consists precisely of the epiclasses of flat ring epimorphisms.

In the hereditary case, we obtain the following result.

**Theorem 2.12** (See [6, Theorem 5.8 and Corollary 5.17] and [3, Corollary 4.22]). If \( R \) is hereditary, there are bijections between

(i) epiclasses of homological ring epimorphisms \( R \to S \),

(ii) equivalence classes of minimal silting right \( R \)-modules,

(iii) equivalence classes of minimal cosilting left \( R \)-modules, and

(iv) subcategories of \( \text{mod}R \) which are wide, i.e., closed under kernels, cokernels and extensions.

The bijections (i) \( \to \) (ii), (iii) map a homological ring epimorphism \( \lambda : R \to S \) to the silting right \( R \)-module \( S \oplus \text{Coker} \lambda \) and to the cosilting left \( R \)-module \( S^+ \oplus \text{Ker} \lambda^+ \), and are restricted to bijections between injective homological ring epimorphisms, silting right modules and cotilting left modules. The assignment (ii) \( \to \) (iii) is given by \( T \mapsto T^+ \). The bijection (iv) \( \to \) (i) maps a wide subcategory \( \mathcal{M} \) to the universal localization \( R \mapsto R_\mathcal{M} \) at (projective resolutions of) the modules in \( \mathcal{M} \).

3 Minimal tilting modules over tame hereditary algebras

In this section, let \( R \) be a finite-dimensional tame hereditary algebra over a field \( k \), which we assume to be indecomposable. We want to determine the minimal tilting modules over \( R \). Since every finite-dimensional tilting module is obviously minimal, we will focus on the large tilting modules.

3.1 Preliminaries on tame hereditary algebras

It is well known that the finite-dimensional indecomposable (right) \( R \)-modules are depicted by the Auslander-Reiten quiver of \( \text{mod}R \), which consists of a preprojective and a preinjective component, denoted by \( p \) and \( q \), respectively, and a family of orthogonal tubes \( t = \bigcup_{A \in X} t_A \) containing the regular modules. For details, please refer to [10].
Given a quasi-simple (or simple regular) module $S$, i.e., a module at the mouth of a tube $t_\lambda$, we denote by $S[m]$ the module of regular length $m$ on the ray

$$S = S[1] \subset S[2] \subset \cdots \subset S[m] \subset S[m+1] \subset \cdots$$

and let $S[\infty] = \operatorname{lim}_{m \to \infty} S[m]$ be the corresponding Prüfer module. The adic module $S[-\infty]$ corresponding to $S$ is defined as the inverse limit along the coray ending at $S$. We denote by $G$ the generic module. It is the unique indecomposable infinite-dimensional module which has finite length over its endomorphism ring.

### 3.2 Over the Kronecker algebra

We start by reviewing the case where $R$ is the Kronecker algebra, i.e., the path algebra of the quiver $\bullet \to \cdot$. Denote by $P_i$ (resp. $Q_i$) with $i \in \mathbb{N}$ the (finite-dimensional) indecomposable preprojective (resp. preinjective) right $R$-modules, indexed such that $\dim \operatorname{Hom}_R(P_i, P_{i+1}) = 2$ (resp. $\dim \operatorname{Hom}_R(Q_{i+1}, Q_i) = 2$). Recall that $P_i$ is simple projective and embeds in all the Kronecker modules but the modules in $\operatorname{Add}Q_i$, and $Q_i$ is simple injective with a surjection from all the Kronecker modules but the modules in $\operatorname{Add}P_i$.

By [20, Theorem 6.1], every homological ring epimorphism $R \to S$ is equivalent to a universal localization at a set of (projective resolutions of) finitely presented modules. Here is a complete list of the epiclasses of $R$, together with the corresponding bireflective subcategories $\mathcal{X}$ of $\operatorname{Mod}R$:

- $R \to 0$ and $\operatorname{id}_R : R \to R$;
- the universal localization at $P_i$ with $\mathcal{X} = \operatorname{Add}Q_i$;
- the universal localization at $P_{i+1}$ (i ≥ 1) with $\mathcal{X} = \operatorname{Add}P_i$;
- the universal localization at $Q_i$ (i ≥ 1) with $\mathcal{X} = \operatorname{Add}Q_{i+1}$;
- the universal localization at a non-empty set $\mathcal{U}$ of simple regular modules with $\mathcal{X} = \mathcal{U}^\perp$.

Notice that the epimorphisms in this list are either surjective with an idempotent kernel or injective, and the only non-injective ones are $R \to 0$ and the universal localizations at the projective modules $P_1$ and $P_2$.

The following is a complete list of silting right $R$-modules, up to equivalence:

- $0$, $P_1$ and $Q_1$, the only silting modules that are not tilting;
- $P_i \oplus P_{i+1}$, i ≥ 1;
- $Q_{i+1} \oplus Q_i$, i ≥ 1;
- $R_\mathcal{U} \oplus R_\mathcal{U} / R$, where $\mathcal{U}$ is a non-empty set of simple regular modules;
- the Lukas tilting module $\mathcal{L}$ with $\operatorname{GenL} = ^{\perp \mathcal{p}}$, the unique non-minimal silting module.

For details, please refer to [9] and [6, Example 5.19].

### 3.3 Classification of tilting modules

Let us now return to an arbitrary tame hereditary algebra $R$. The large tilting $R$-modules have been classified in [9]. In contrast to the Kronecker case, they can have finite-dimensional summands. This is due to the existence of (at most three) non-homogeneous tubes. Notice, however, that the finite-dimensional part of a large tilting module can be described explicitly. In order to explain this, we need to recall some terminology.

**Definition 3.1.**

1. An $R$-module $Y$ is said to be **exceptional** if $\operatorname{Ext}_R^1(Y, Y) = 0$.
2. Given a tube $t_\lambda$ of rank $r > 1$ and a module $X = U[m] \in t_\lambda$ of regular length $m < r$, we consider the full subquiver $W_X$ of $t_\lambda$, which is isomorphic to the Auslander-Reiten quiver $\Theta(m)$ of the linearly oriented quiver of type $A_m$ with $X$ corresponding to the projective-injective vertex of $\Theta(m)$. The set $W_X$ is called a **wing** of $t_\lambda$ of size $m$, which is rooted in the vertex $X = U[m]$.
3. A finite-dimensional regular multiplicity-free exceptional $R$-module $Y$ is a **branch module** if it satisfies the following condition: for each quasi-simple module $S$ and $m \in \mathbb{N}$ such that $S[m]$ is a direct summand of $Y$, there exist precisely $m$ direct summands of $Y$ that belong to $W_S[m]$. 


In other words, a branch module is a regular multiplicity-free exceptional module whose indecomposable summands are arranged in disjoint wings, and the number of summands from each wing equals the size of that wing.

It was shown in [9] that the large tilting modules over $R$ are parametrized by the pairs $(Y, P)$, where $Y$ is a branch module, and $P$ is subset of $X$. More precisely, every such pair $(Y, P)$ determines two sets of quasi-simple modules:

- the set $V = V_{(Y, P)}$ given by all the quasi-simple modules in $\bigcup_{\lambda \in P} t_{\lambda}$ and all the regular composition factors of $Y$;
- the set $U = U_{(Y, P)}$ given by all the quasi-simple modules in $\bigcup_{\lambda \in P} t_{\lambda}$ that are not regular composition factors of $\tau^{-1}Y$.

With these sets, one can construct a tilting module

$$T_{(Y, P)} = Y \oplus (L \otimes R V) \oplus \bigoplus_{S \in U} S[\infty],$$

and it turns out that the modules $T_{(Y, P)}$ form a complete irredundant list of all the large tilting modules, up to equivalence. For details, please refer to [1, 9].

Our aim is to show that a large tilting module $T_{(Y, P)}$ is minimal if and only if the set $P \subset X$ is non-empty. The only-if part of this statement has already been contained in a result from [9], which we briefly recall for the reader’s convenience.

**Proposition 3.2 (See [9, Corollary 5.10]).** If $T_{(Y, P)}$ is a minimal tilting module, then the set $P$ is not empty.

**Proof.** We just outline the argument and refer to [9] for details. Assume that $T = T_{(Y, P)}$ is equivalent to a tilting module of the form $S \oplus S/R$ for some ring epimorphism $R \to S$. If $P = \emptyset$, then $T = Y \oplus (L \otimes R V)$. Now one considers the torsion and torsion-free parts of $T$ with respect to the torsion pair $(\text{Gen}_Y, t^{+\infty})$ generated by $t$. It turns out that $Y$ is the torsion and $L \otimes R V$ is torsion-free. Moreover, $\text{Add}Y$ contains all the torsion modules in $\text{Add}T$, and $\text{Add}(L \otimes R V)$ contains all the torsion-free modules in $\text{Add}T$. Next, one shows that $S/R$ is the torsion and therefore lies in $\text{Add}Y$. One then deduces that $(S/R)^{+1} = Y^{+1}$. Notice that $(S/R)^{+1} = \text{Gen}T$, and hence our tilting class $\text{Gen}T$ coincides with $Y^{+1}$. On the other hand, by a well-known result due to Bongartz [12], the finite-dimensional exceptional module $Y$ can be completed to a finite-dimensional tilting module with the tilting class $Y^{+1}$. But this contradicts the assumption that $T$ is large.

\[\square\]

### 3.4 Minimal tilting modules

Let us fix a large tilting module $T = T_{(Y, P)}$ with $P \neq \emptyset$. We want to prove that $T$ is minimal. To this end, we will use Theorem 2.12 and show that $T_{(Y, P)}$ arises from a universal localization at a wide subcategory $M$ of $\text{mod}R$.

We start out with an easy, but useful observation.

**Remark 3.3 (See [9, Example 4.4]).** If $S$ is a quasi-simple module, then $S[\infty]^{+1} = \bigcap_{n \geq 1} S[n]^{+1}$.

More generally, let $E$ be a class of modules, and suppose that a module $X$ lies in $E^{+1}$. Then $X$ lies in $E^{+1}$ for every module $E$ which is filtered by modules from $E$, or which is a submodule of some module in $E$.

This follows immediately from the fact that any class of the form $^{+1}X$ is closed under filtrations by [19, Lemma 3.1.2], and when $X$ has the injective dimension at most one (as in our case), it is also closed under submodules.

Next, we compute the tilting class of $T$. This amounts to computing the subcategory

$$S = {^{+1}\text{Gen}T} \cap \text{mod}R,$$

i.e., the largest subcategory $S$ of $\text{mod}R$ with the property that $\text{Gen}T = S^{+1}$. We are going to see that the indecomposable non-preprojective modules in $S$ either lie on a ray starting in $U = U_{(Y, P)}$, or lie “below” an indecomposable summand of $Y$. Here is the precise statement.
Lemma 3.4. (1) A finitely generated indecomposable $R$-module $M$ belongs to $S$ if and only if one of the following statements holds true:

- $M$ is preprojective;
- $M \cong S[n]$, where $n \geq 1$ and $S$ is in $U$;
- there is a module $S[h] \in \text{Add}Y$ such that $M \cong S[i]$ for some $1 \leq i \leq h$.

(2) $\text{Gen}T = \bigcap_{S \in U} S[\infty]^{-1} \cap Y^{-1}$.

Proof. (1) We know from [9, Theorem 2.7] that $S = \text{add}(p \cup t')$, where $t'$ is a set of regular modules. Take a quasi-simple $S$ whose ray $\{S[n] \mid n \geq 1\}$ contains some modules from $t'$. If the whole ray is contained in $t'$, then by [9, Theorem 4.5], the Prüfer module $S[\infty] \in \text{Add}T$ and $S \in U$. If $t'$ contains some, but not all the modules from the ray, then $t' \cap \{S[n] \mid n \geq 1\} = \{S[i] \mid i \leq h\}$ with $S[h] \in \text{Add}Y$ by [9, Lemma 3.3]. Hence $t'$ and $S$ have the stated shape.

(2) $\text{Gen}T = T^{-1} \subseteq \bigcap_{S \in U} S[\infty]^{-1} \cap Y^{-1}$ since $S[\infty]$, $S \in U$, and $Y$ are submodules of $T$.

Now take $X \in \bigcap_{S \in U} S[\infty]^{-1} \cap Y^{-1}$. By Remark 3.3, it follows that $X$ lies in $\bigcap_{n \geq 1} S[n]^{-1}$ for all $S \in U$. Moreover, if $S[h] \in \text{Add}Y$, then $X$ lies in $S[h]^{-1}$ and also in $S[i]^{-1}$ for each $i \in [1, h]$, due to the inclusion $S[i] \hookrightarrow S[h]$. This shows that $X \in t^{-1} = S^{-1} = \text{Gen}T$. ☐

We want to find a wide subcategory $\mathcal{M} \subseteq \text{mod}R$, which corresponds to $T$ under the bijection in Theorem 2.12. From [8, Corollary 4.13] and [22, Theorem 2.6], we know that the tilting module $R_M \oplus R_M/R$ given by a wide subcategory $\mathcal{M}$ has the tilting class $\text{Gen}(R_M) = \mathcal{M}^{-1}$. Hence $\mathcal{M}$ must satisfy $\text{Gen}T = \mathcal{M}^{-1}$, and in particular it must be contained in $S$. Furthermore, the class $\mathcal{M}^{-1}$ must be contained in $Y^{-1}$ and in each $S[\infty]^{-1}$ with $S \in U$.

In order to construct such an $\mathcal{M}$, we will therefore pick modules from $S$ which filter the Prüfer modules $S[\infty]$ with $S \in U$. This will ensure the inclusion $\mathcal{M}^{-1} \subseteq \bigcap_{S \in U} S[\infty]^{-1}$ by Remark 3.3.

Moreover, for each ray $\{S[n] \mid n \geq 1\}$ containing a module in $\text{Add}Y$, we will include in $\mathcal{M}$ the module $S[h] \in \text{Add}Y$ of maximal regular length on that ray. This will entail the inclusion $\mathcal{M}^{-1} \subseteq Y^{-1}$, again by Remark 3.3. In fact, a closer look shows that it suffices to take care of the rays starting in quasi-simple modules that are not in $U$, because for $S \in U$ we already have $\mathcal{M}^{-1} \subseteq \bigcap_{n \geq 1} S[n]^{-1}$.

So, given a wing $W = W_U[m]$ inside a tube $t_\lambda$, where the vertex $U[m]$ belongs to $\text{Add}Y$, we consider the following two sets:

$$\mathcal{X}_W = \{S[h] \mid S[h] \in \text{Add}Y \text{ of maximal regular length on a ray starting in } S \in W\}$$

and

$$\widetilde{\mathcal{X}}_W = \{S[h] \mid S[h] \in \text{Add}Y \text{ of maximal regular length on a ray starting in } S \in W \setminus \{U\}\}.$$

We will now follow this strategy and construct the category $\mathcal{M}$.

Construction 3.5. Let $Q := \{\lambda \in \mathbb{X} \mid t_\lambda \cap \text{Add}Y \neq \emptyset\}$ and fix $\lambda \in Q$. Let $\{S_1, \ldots, S_r\}$ be a complete irredundant set of the quasi-simple modules in $t_\lambda$. By [9, Proposition 3.7], the summands of $Y$ belonging to $t_\lambda$ are arranged in disjoint wings $W_1, \ldots, W_l$ in $t_\lambda$ whose vertices $S_{n_1}[m_1], \ldots, S_{n_l}[m_l]$ belong to $\text{Add}Y$.

In other words,

$$t_\lambda \cap \text{Add}Y \subseteq \bigcup_{j=1}^l W_j.$$

Observe that $S_{n_1}, \ldots, S_{n_l} \in U$ whenever $\lambda \in Q \cap P$. We now define the set

$$\mathcal{X}(\lambda) = \bigcup_{j=1}^l \mathcal{X}_j,$$

where for each $j \in [1, l]$, the set $\mathcal{X}_j$ is given as follows:

$$\begin{cases} \mathcal{X}_j = \mathcal{X}_{W_j}, & \text{if } \lambda \in Q \setminus P, \\ \mathcal{X}_j = \widetilde{\mathcal{X}}_{W_j}, & \text{if } \lambda \in Q \cap P. \end{cases}$$
Example 3.6. In Figure 1, we illustrate the definition of $\mathcal{R}_\lambda$ by considering the case of $l = 2$ wings $\mathcal{W}_1$ and $\mathcal{W}_2$ rooted in the vertices $S_1[3]$ and $S_7[4]$, respectively, inside a tube $t_\lambda$ of rank $r = 12$. Then $\mathcal{R}_1 = \{S_1[4], S_1[5], S_1[6], S_5, S_6\}$ and $\mathcal{R}_2 = \{S_7[5], S_7[6], S_{12}\}$.

Now suppose that $S_1[2], S_1[3]$ and $S_2$ are the indecomposable summands of $Y$ lying in the wing $\mathcal{W}_1$. Then $\mathcal{X}_1 = \{S_2\}$ if $\lambda \in Q \cap P$, while $\mathcal{X}_1 = \{S_1[3], S_2\}$ if $\lambda \in Q \setminus P$.

Example 3.7. Let $t_\lambda$ be a tube of rank $r > 1$ with quasi-simple modules $S = S_1, \ldots, S_r$.

(1) (See [9, Example 4.4]) Take $T = T_{(Y,P)}$, where $Y = S \oplus \cdots \oplus S[r - 1]$ and $P = \{\lambda\}$. Then $V = \{S_1, \ldots, S_r\}$ and $U = \{S\}$. Moreover, $P = Q$ and $Q = \mathcal{R}(\lambda) = \{S[r]\}$. So, the tilting module

$$T = S \oplus \cdots \oplus S[r - 1] \oplus R_V \oplus S[\infty]$$

is equivalent to $T_M = R_M \oplus R_M/R$ for $\mathcal{M} = \{S[r]\} \cap \text{mod}R$.

(2) Let now $r = 3$. Take $T = T_{(Y,P)}$, where $Y = S_1$ and $P = \{\lambda\}$. Then $V = \{S_1, S_2, S_3\}$ and $U = \{S_1, S_3\}$. Moreover, $P = Q$ and $Q = \mathcal{R}(\lambda) = \{S_1[2], S_1[3], S_3\}$. So, the tilting module

$$T = S_1 \oplus R_V \oplus S_1[\infty] \oplus S_3[\infty]$$

is equivalent to $T_M = R_M \oplus R_M/R$ for $\mathcal{M} = \{S_1[2], S_1[3], S_3\} \cap \text{mod}R$.

(3) Let $r = 4$, and let $t_{\mu}$ be a tube of rank $r_{\mu} = 3$ with quasi-simple modules $\{U_1, U_2, U_3\}$. Take $T_{(Y,P)}$, where $Y = S_1[3] \oplus S_2[2] \oplus S_3[2] \oplus U_1[2] \oplus U_2$ and $P = \{\lambda\}$.

Then

$$Q = \{\lambda, \mu\}, \quad \mathcal{X}(\lambda) = \{S_2[2]\}, \quad \mathcal{X}(\mu) = \{U_1[2], U_2\} \quad \text{and} \quad \mathcal{R}(\lambda) = \{S_1[4]\}.$$

Moreover, $Q = \mathcal{X}(\lambda) \cup \mathcal{X}(\mu) \cup \mathcal{R}(\lambda)$. So, the tilting module $T_{(Y,P)}$ is equivalent to $T_M = R_M \oplus R_M/R$ for $\mathcal{M} = \{S_1[4]\} \cap \text{mod}R$.

We collect some easy observations.
Remark 3.8. (1) Since $Q \subseteq \bigcup_{\lambda \in P \cup Q} t_{\lambda}$, its wide closure $\mathcal{M}$ is contained in $\text{add}(\bigcup_{\lambda \in P \cup Q} t_{\lambda})$.

(2) By construction, we have
\[ \bigcup_{j=1}^{l} \mathcal{W}_j \subseteq \mathcal{R}(\lambda)^\perp \cap t_{\lambda}. \]

Moreover, $\text{Hom}_R(X,Y) \neq 0$ for any $X \in \mathcal{R}(\lambda)$ and $Y \in t_{\lambda} \setminus \bigcup_{j=1}^{l} \mathcal{W}_j$. It follows that
\[ \mathcal{R}(\lambda)^\perp \cap t_{\lambda} = \bigcup_{j=1}^{l} \mathcal{W}_j. \]

The indecomposable modules in $\mathcal{M}$ are therefore of the form $M = S[i] \in t_{\lambda}$ with $\lambda \in P \cup Q$. If $\lambda \in P \setminus Q$, all the modules in $t_{\lambda}$ do occur. Let us turn to the remaining cases.

Proposition 3.9. Either any indecomposable module in $\mathcal{M}$ lies on a ray starting in $\mathcal{U}$ or there is some $S[h] \in \bigcup_{\lambda \in Q} \mathcal{X}(\lambda)$ such that $M \cong S[i]$ with $1 \leq i \leq h$.

In particular, $\mathcal{M}$ is contained in $\mathcal{S}$.

Proof. Let $M \in \mathcal{M}$ be indecomposable. Then without loss of generality, $M = S[i] \in t_{\lambda}$ for some $\lambda \in P \cup Q$. If $M$ does not lie on a ray starting in $\mathcal{U}$, we have one of the following cases: (1) $\lambda \in Q \cap P$ and $S \notin \mathcal{U}$, or (2) $\lambda \in Q \setminus P$. We will show that in both cases there is $h \geq i$ such that $S[h] \in \mathcal{X}(\lambda)$.

(1) Assume that $\lambda \in Q \cap P$ and $S \notin \mathcal{U}$. By the assumption,
\[ S[i] \in \perp(Q^\perp) \subseteq \perp(Q^\perp \cap t_{\lambda}) \quad \text{and} \quad Q^\perp = \bigcap_{\mu \in Q} Q(\mu)^\perp \cap \left( \bigcup_{\mu \in P \setminus Q} t_{\mu} \right)^\perp. \]

Now, we have $t_{\lambda} \subseteq Q_{\mu}^\perp$, when $\mu \neq \lambda$ and $t_{\lambda} \subseteq \bigcup_{\mu \in P \setminus Q} t_{\mu}^\perp$ since $\lambda \in P \cap Q$. Thus
\[ Q^\perp \cap t_{\lambda} = Q(\lambda)^\perp \cap t_{\lambda} = \mathcal{R}(\lambda)^\perp \cap t_{\lambda} \cap \mathcal{X}(\lambda)^\perp. \]

From Remark 3.8(2), we infer that
\[ S[i] \in \perp\left( \bigcup_{j=1}^{l} \mathcal{W}_j \cap \mathcal{X}(\lambda)^\perp \right) = \bigcap_{j=1}^{l} (\mathcal{W}_j \cap \mathcal{X}(\lambda)^\perp) = \bigcap_{j=1}^{l} (\mathcal{W}_j \cap \mathcal{X}_j^\perp), \]

keeping in mind that $\mathcal{W}_j \subseteq \mathcal{W}_{k}^\perp \subseteq \mathcal{X}_{k}^\perp$ whenever $j \neq k$.

By the assumption, we further know that $S \notin \mathcal{U}$ is a composition factor of $\tau^{-1}Y$. So there is a $j \in [1,l]$ such that $S \in \{S_{n_j+1}, \ldots, S_{n_j+m_j}\}$. Notice that not all the modules in $\text{Add}Y \cap \mathcal{W}_j$ can lie on the ray starting in $S_{n_j}$, because otherwise $\mathcal{X}_j = X_{\mathcal{W}_j} = \emptyset$ and the assumption
\[ S[i] \in \perp(\mathcal{W}_j \cap \mathcal{X}_j^\perp) = \perp \mathcal{W}_j \]

would yield a contradiction. In particular, this entails $m_j > 1$. So we can apply Proposition 3.11(1) below and obtain $i \leq h < m_j$ such that $S[h] \in \mathcal{X}(\lambda) \subseteq \text{Add}Y$.

(2) Assume that $\lambda \in Q \setminus P$. By construction,
\[ \mathcal{X}(\lambda) = \bigcup_{j=1}^{l} \mathcal{X}_j \subseteq \bigcup_{j=1}^{l} \mathcal{W}_j, \]

so its wide closure $\perp(\mathcal{X}(\lambda)^\perp) \cap \text{mod}R$ is contained in $\text{add}(\bigcup_{j=1}^{l} \mathcal{W}_j)$. As different tubes are Hom- and Ext-orthogonal,
\[ \mathcal{M} \cap t_{\lambda} = \perp(\mathcal{X}(\lambda)^\perp) \cap t_{\lambda} \subseteq \text{add}(\bigcup_{j=1}^{l} \mathcal{W}_j). \]
Since \( S[i] \in \mathcal{M} \cap t_{\lambda} \), there is a \( j \in [1, l] \) such that \( S[i] \in W_j \), and thus \( S \in \{ S_{n_1}, \ldots, S_{n_j+m_j-1} \} \). Let \( W_0 = W_{rS_{n_1[m_1+1]}} \). Note that \( W_0 \subseteq W_k^+ \subseteq X_k^+ \) whenever \( j \neq k \). Hence,
\[
S[i] \in \frac{1}{2}(X(\lambda)^+) \subseteq \frac{1}{2}(W_0 \cap X_0^+).
\]

Now we can apply Proposition 3.11(2) below and obtain \( i \leq h \leq m_j \) such that \( S[h] \in X(\lambda) \subseteq \text{Add}Y \).

**Lemma 3.10.** Let \( W = W_{U_1[m]} \) be a wing rooted in the vertex \( U_1[m] \in \text{Add}Y \). Set
\[
U_2 = \tau^{-1}U_1, \ldots, U_{m+1} = \tau^{-1}U_m.
\]
Define further \( A = \frac{1}{2}(W \cap X_W^+) \). Then the following statements hold true:
1. \( U_1[m] \in W \cap X_W^+ \).
2. Assume that \( m > 1 \), and that there is a subset \( X \subseteq t_{\lambda} \) and integers \( j \in [1, m] \) and \( n \in (0, m-j] \) such that \( W' := W_{U_1[m]} \cap X \cap X_0^+ = X_W \cup X \). Then \( A \subseteq \frac{1}{2}(W' \cap X_W^+) \).

**Proof.** (1) is obvious. For (2), note that
\[
W' \cap X_W^+ \subseteq W \cap X_W^+ \cap X_0^+ = W \cap X_W^+.
\]
Then \( A \subseteq \frac{1}{2}(W' \cap X_W^+) \).

**Proposition 3.11.** Let the assumptions and notation be as in Lemma 3.10.
1. Assume that \( m > 1 \) and that there are \( i \in \mathbb{N} \) and \( S \in \{ U_2, \ldots, U_m, U_{m+1} \} \) such that
\[
S[i] \in A = \frac{1}{2}(W \cap X_W^+).
\]
Then \( S \neq U_{m+1} \) and there is an \( h \in [i, m] \) such that \( S[h] \in \text{Add}Y \).
2. Define \( W_0 = W_{rU_1[m+1]} \), and assume that there are \( i \in \mathbb{N} \) and \( S \in \{ U_1, U_2, \ldots, U_m \} \) such that
\[
S[i] \in B = \frac{1}{2}(W_0 \cap X_W^+).
\]
Then there is an \( h \in [i, m] \) such that \( S[h] \in \text{Add}Y \).

**Proof.** (1) Since \( A \subseteq \frac{1}{2}U_1[m] \) by Lemma 3.10, we have \( S[i] \in W \) and \( S \in \{ U_2, \ldots, U_m \} \). Observe that not all the modules in \( \text{Add}Y \cap W \) can lie on the ray starting in \( U_1 \), as we have already seen in the proof of Proposition 3.9.

We proceed by induction on \( m \). When \( m = 2 \), we know by the observation above that \( W \cap \text{Add}Y = \{ U_2, U_1[2] \} \). Then \( A = \frac{1}{2}(W \cap U_2^+) \) does not intersect \( \{ U_2[t] \mid t \geq 2 \} \). Thus \( S = U_2 \in \text{Add}Y \).

Let \( m > 2 \). We distinguish two cases.

**Case (i)** Suppose that none of the modules \( U_2[m-1], U_3[m-2], \ldots, U_m \) on the right border of the wing \( W \) belongs to \( \text{Add}Y \). By the definition of a branch module, we know that \( \text{Add}Y \) contains precisely \( m \) modules in \( W \); these are \( U_1[m] \) and \( m-1 \) modules in \( W' = W_{U_1[m-1]} \). We obtain \( X_W = \bar{X}_W \). By Lemma 3.10 (here \( X = \emptyset \) ), we have \( A \subseteq \frac{1}{2}(W' \cap X_W^+) \). By the induction assumption, the claim holds true for all \( S \in \{ U_2, \ldots, U_m \} \).

**Case (ii)** Suppose that one of the modules \( U_2[m-1], U_3[m-2], \ldots, U_m \) belongs to \( \text{Add}Y \). Choose \( U_k[h] \in \text{Add}Y \) of maximal regular length. By the definition of a branch module, the module \( U_1[t_1] \) with \( t_1 = m - h - 1 = k - 2 \) must lie in \( \text{Add}Y \). Furthermore, \( \text{Add}Y \) contains precisely \( m \) modules in \( W \); these are \( t_1 \) modules from \( W_1 := W_{U_1[t_1]} \), together with \( h \) modules from \( W_2 := W_{U_1[h]} \) and \( U_1[m] \).

By the induction assumption and Lemma 3.10 (here \( W' = W_1 \) and \( X = X_W \)), the claim holds true for \( S \in \{ U_2, \ldots, U_t, U_{k-1} \} \). Moreover, the claim is clear for \( S = U_k \).

We again have two cases.

**Case (i)** Suppose that none of \( U_k[h-1], U_{k+2}[h-2], \ldots, U_m \) belongs to \( \text{Add}Y \). Then \( U_k[h-1] \in \text{Add}Y \). The claim holds true for all \( S \in \{ U_k[1], \ldots, U_m \} \) again by Lemma 3.10 (here \( W' = W_{U_k[h-1]} \) and \( X = X_W \cup \{ U_k[h] \} \)) and the induction assumption.
Case (ii) Suppose that one of $U_{k+1}[h-1], U_{k+2}[h-2], \ldots, U_m$ belongs to $\text{Add} Y$. Choose $U_{k_2}[h_2] \in \text{Add} Y$ of maximal regular length. Then $U_{k_2}[t_2]$ with $t_2 = h-h_2-1$ must lie in $\text{Add} Y$, and the modules in $W_2 \cap \text{Add} Y$ different from the vertex are distributed in the wings $W_3 := W_{U_{k_2}[t_2]}$ and $W_4 := W_{U_{k_2}[h_2]}$. By Lemma 3.10 (here $W' = W_2$ and $X = \tilde{X}_W \cup \tilde{X}_{W_1} \cup \{U_{k_2}[h]\}$) and the induction assumption, we obtain the claim for $S \in \{U_{k+1}, \ldots, U_{k_2-1}\}$. Moreover, the claim is clear if $S = U_{k_2}$. We proceed in this way, obtaining

$$2 \leq k = k_1 < k_2 < k_3 \cdots < m,$$

and we keep distinguishing the cases (i) and (ii). After a finite number (say $s$) of steps, either we reach $k := k_s$ where the case (i) occurs, and then we are done, or we reach $k := k_s \leq m$ such that for $h := h_s$ with $h + h = m + 1$ we have $U_{k}[h] \in \text{Add} Y$ and all the modules on the coray of $U_m$ of regular length less than $h$ are in $\text{Add} T$. Notice that in the step before, it only remains to prove the claim for $S \in \{U_k, \ldots, U_m\}$. But here the claim is trivial, as the intersection of the ray of $S$ with the coray of $U_m$ is in $\text{Add} T$.

(2) Observe that $\tau U_1[m+1] \in W_0 \cap X_{W}^\perp$, and hence $B \subseteq \frac{1}{\tau} U_1[m+1]$. So, $S[\tau] \in W$. Notice that the claim is clear for $S = U_1$. So, we can assume $m > 1$ and $S \in \{U_2, \ldots, U_m\}$. We distinguish two cases.

(i) Suppose that none of the modules $U_m, U_{m-1}[2], \ldots, U_2[m-1]$ belongs to $\text{Add} Y$. Then $U_1[m-1] \in \text{Add} Y$. Let $W' = W_{U_1[m-1]}$. Then $X' = \tilde{X}_{W'} \cup \{U_1[m]\}$. It follows that

$$B \subseteq \frac{1}{\tau} (W' \cap X_{W}^\perp).$$

By the statement (1), the claim holds for $S \in \{U_2, U_3, \ldots, U_m\}$.

(ii) Suppose that one of the modules $U_m, U_{m-1}[2], \ldots, U_2[m-1]$ belongs to $\text{Add} Y$. Choose $U_k[h] \in \text{Add} Y$ of maximal regular length. Then $U_1[t_1]$ with $t_1 = m - h - 1 = k - 2$ must lie in $\text{Add} Y$. So, $\text{Add} Y$ contains precisely $m$ modules in $W$: these are the $t_1$ modules in $W_1 = W_{U_1[t_1]}$, together with $h$ modules from $W_2 = W_{U_k[h]}$ and $U_1[m]$.

Notice that

$$W_1 \cap \tilde{X}_{W_1}^\perp \subseteq W_0 \cap \tilde{X}_{W_1}^\perp \cap X_{W_2}^\perp \cap U_1[m]^\perp = W_0 \cap X_{W_1}^\perp.$$

Therefore, $B \subseteq \frac{1}{\tau} (W_1 \cap \tilde{X}_{W_1}^\perp)$ and the claim follows from the statement (1) for $S \in \{U_2, \ldots, U_{k-1}\}$. Moreover, the claim is clear if $S = U_k$. So, it remains to verify the claim when $S \in \{U_{k+1}, \ldots, U_m\}$.

We again have two cases.

(i) Suppose that none of $U_{k+1}[h-1], U_{k+2}[h-2], \ldots, U_m$ belongs to $\text{Add} Y$. Then $U_k[h-1] \in \text{Add} Y$. It follows that

$$B \subseteq \frac{1}{\tau} (W'' \cap \tilde{X}_{W''}^\perp),$$

where $W'' = W_{U_k[h-1]}$. Then the claim follows again from the statement (1).

(ii) Suppose that one of $U_{k+1}[h-1], U_{k+2}[h-2], \ldots, U_m$ belongs to $\text{Add} Y$. Choose $U_{k_2}[h_2] \in \text{Add} Y$ of maximal regular length. Then $U_{k_2}[t_2]$ with $t_2 = h-h_2-1$ must lie in $\text{Add} Y$. It follows that

$$B \subseteq \frac{1}{\tau} (\tilde{W} \cap \tilde{X}_{\tilde{W}}^\perp),$$

where $\tilde{W} = W_{U_{k_2}[t_2]}$. By the statement (1), we obtain the claim for $S \in \{U_{k+1}, \ldots, U_{k_2-1}\}$. The claim is clear if $S = U_{k_2}$.

Continuing in this fashion and arguing as above, one obtains the claim for the remaining quasi-simple modules $S \in \{U_{k_2+1}, \ldots, U_m\}$.

Next, we have to show that the Prüfer modules corresponding to quasi-simple modules in $\mathcal{U}$ are filtered by modules from $\mathcal{M}$.

**Proposition 3.12.** If $S \in \mathcal{U}$ belongs to a tube $t_\lambda$ of rank $r$, then $S[r] \in \mathcal{M}$.

**Proof.** The claim is clear for $\lambda \in P \setminus Q$ by construction of $\mathcal{M}$. So we assume that $\lambda \in P \cap Q$. Recall that the summands of $Y$ are arranged in the wings $W_1, \ldots, W_l$ rooted in $S_{n_1}[m_1], \ldots, S_{n_l}[m_l]$. Since $S$ is not a composition factor of $\tau - Y$, it either lies in $\{S_{n_j}, S_{n_j+m_j+1}, \ldots, S_{n_j+1}, \ldots, S_{n_j+1-1}\}$ for some $j \in [1, l]$ or in $\{S_n, S_{n+m+1}, \ldots, S_r, S_1, \ldots, S_{n+1}\}$. We can assume without loss of generality that $S$ lies in
\{S_{n_1}, S_{m_1+n_1+1}, \ldots, S_{n_2-1}\}. We will proceed case by case and express \(S[r]\) as an iterated extension of modules from \(R(\lambda)\).

**Case (i)** \(S = S_{n_1}\). We consider the following short exact sequences:

\[
0 \to S_{n_1}[n_2 - n_1] \to S_{n_1}[r] \to S_{n_2}[r + n_1 - n_2] \to 0,
\]

\[
0 \to S_{n_2}[n_3 - n_2] \to S_{n_2}[r + n_1 - n_2] \to S_{n_3}[r + n_1 - n_3] \to 0,
\]

\[
\vdots
\]

\[
0 \to S_{n_{l-1}}[n_l - n_{l-1}] \to S_{n_{l-1}}[r + n_1 - n_{l-1}] \to S_{n_l}[r + n_1 - n_l] \to 0.
\]

Since \(S_{n_i}[r + n_i - n_l]\) and all the first terms of these exact sequences are in \(R(\lambda)\), we obtain \(S_{n_i}[r] \in \mathcal{M}\).

**Case (ii)** \(S = S_{n_1+m_1+1}\). Notice that \(\mathcal{M}\), being extension-closed, contains the wing \(\mathcal{W}'\) rooted in \(S_{n_1+m_1+1}[n_2 - n_1 - m_1 - 1]\). Consider the following short exact sequence

\[
0 \to S_{n_1+m_1+1}[n_2 - n_1 - m_1 - 1] \to S_{n_1+m_1+1}[n_3 - n_1 - m_1 - 1] \to S_{n_2}[n_3 - n_2] \to 0.
\]

It follows that the middle term \(S_{n_1+m_1+1}[n_3 - n_1 - m_1 - 1]\) is in \(\mathcal{M}\). There is an exact sequence

\[
0 \to S_{n_1+m_1+1}[n_3 - n_1 - m_1 - 1] \to S_{n_1+m_1+1}[r] \to S_{n_3}[r + n_1 + m_1 + 1 - n_3] \to 0.
\]

Now we need to prove that

\[
S_{n_3}[r + n_1 + m_1 + 1 - n_3] \in \mathcal{M}.
\]

We have the following exact sequences:

\[
0 \to S_{n_3}[n_4 - n_3] \to S_{n_3}[r + n_1 + m_1 + 1 - n_3] \to S_{n_4}[r + n_1 + m_1 + 1 - n_4] \to 0,
\]

\[
0 \to S_{n_4}[n_5 - n_4] \to S_{n_4}[r + n_1 + m_1 + 1 - n_4] \to S_{n_5}[r + n_1 + m_1 + 1 - n_5] \to 0,
\]

\[
\vdots
\]

\[
0 \to S_{n_r}[r + n_1 - n_l] \to S_{n_l}[r + n_1 + m_1 + 1 - n_l] \to S_{n_1}[m_1 + 1] \to 0.
\]

Since \(S_{n_1}[m_1 + 1]\) and \(S_{n_i}[r + n_1 - n_l]\) are in \(R(\lambda)\), we have \(S_{n_1}[r + n_1 + m_1 + 1 - n_l] \in \mathcal{M}\). It follows that

\[
S_{n_3}[r + n_1 + m_1 + 1 - n_3] \in \mathcal{M}.
\]

So, \(S_{n_1+m_1+1}[r]\) \(\in \mathcal{M}\).

**Case (iii)** \(S = S_{n_2-1}\). Consider the exact sequence

\[
0 \to S_{n_2-1} \to S_{n_2-1}[n_3 - n_2 + 1] \to S_{n_2}[n_3 - n_2] \to 0.
\]

Since \(S_{n_2}[n_3 - n_2]\) and \(S_{n_2-1}\) are in \(\mathcal{M}\), we obtain

\[
S_{n_2-1}[n_3 - n_2 + 1] \in \mathcal{M}.
\]

There is an exact sequence

\[
0 \to S_{n_2-1}[n_3 - n_2 + 1] \to S_{n_2-1}[r] \to S_{n_3}[r - 1 + n_2 - n_3] \to 0.
\]

Now, we need to prove \(S_{n_3}[r - 1 + n_2 - n_3] \in \mathcal{M}\). We have a series of exact sequences

\[
0 \to S_{n_3}[n_4 - n_3] \to S_{n_3}[r - 1 + n_2 - n_3] \to S_{n_4}[r - 1 + n_2 - n_4] \to 0,
\]

\[
\vdots
\]

\[
0 \to S_{n_{l-1}}[n_l - n_{l-1}] \to S_{n_{l-1}}[r - 1 + n_2 - n_{l-1}] \to S_{n_l}[r - 1 + n_2 - n_l] \to 0,
\]

\[
0 \to S_{n_l}[r + n_1 - n_l] \to S_{n_l}[r - 1 + n_2 - n_l] \to S_{n_1}[n_2 - n_1 - 1] \to 0.
\]
Since $S_n[r + n_1 - n_1]$ and $S_n[n_2 - n_1 - 1]$ are in $\mathcal{R}(\lambda)$, we have $S_n[r - 1 + n_2 - n_1] \in \mathcal{M}$. It follows that

$$S_n[r - 1 + n_2 - n_3] \in \mathcal{M}. $$

So, $S_{n_2-1}[r] \in \mathcal{M}$.

We obtain the remaining cases $S \in \{S_{n_1+m_1+2}, \ldots, S_{n_2-2}\}$ similarly, keeping in mind that $\{S_n[i] \mid m_1 < i \leq n_2 - n_1\} \subseteq \mathcal{M}$. 

We are now ready to prove that $T$ arises from the universal localization $R \to R_{\mathcal{M}}$.

**Theorem 3.13.** $T_{(Y,P)}$ is equivalent to $T_{\mathcal{M}} = R_{\mathcal{M}} \oplus R_{\mathcal{M}}/R$.

**Proof.** By Lemma 3.4, we need to prove

$$\bigcap_{S \in U} S[\infty]^{\perp_1} \cap Y^{\perp_1} = \mathcal{M}^{\perp_1}. $$

Firstly, $\mathcal{M} \subseteq S$ by Proposition 3.9, and hence

$$\bigcap_{S \in U} S[\infty]^{\perp_1} \cap Y^{\perp_1} = S^{\perp_1} \subseteq \mathcal{M}^{\perp_1}. $$

Secondly, if $S \in U$ lies in a tube of rank $r$, then $\mathcal{M}$ contains $S[r]$ by Proposition 3.12, and we deduce inductively from the short exact sequence

$$0 \to S[r(n - 1)] \to S[nr] \to S[r] \to 0$$

that $S[nr]$ belongs to $\mathcal{M}$ for all $n \in \mathbb{N}$. Since $S[\infty]$ is filtered by $S[nr]$ ($n \geq 1$), we conclude from Remark 3.3 that

$$\mathcal{M}^{\perp_1} \subseteq \bigcap_{S \in U} S[\infty]^{\perp_1}. $$

Thirdly, we recall from [9, Proposition 4.2] that there is a decomposition $Y = \bigoplus_{\lambda \in Q} t_\lambda(Y)$ with $t_\lambda(Y) \in \text{add}t_\lambda$. If $\lambda \in Q \setminus P$, then the indecomposable summands of $t_\lambda(Y)$ have the form $S[i]$ such that there is an $h \geq i$ with $S[h] \in \mathcal{X}(\lambda)$. In other words, they are submodules of a module from $\mathcal{X}(\lambda)$. Similarly, when $\lambda \in Q \cap P$, the indecomposable summands of $t_\lambda(Y)$ are either submodules of a module from $\mathcal{X}(\lambda)$ or lie on a ray starting in some quasi-simple $S \in U$.

Thus

$$\mathcal{M}^{\perp_1} \subseteq \bigcap_{\lambda \in Q} t_\lambda(Y)^{\perp_1} = Y^{\perp_1}, $$

again by Remark 3.3. This concludes the proof.

Combining Theorem 3.13 and Proposition 3.2, we obtain the main result of this subsection.

**Theorem 3.14.** For any pair $(Y,P)$, the tilting module $T_{(Y,P)}$ is minimal if and only if the set $P \subseteq \mathcal{X}$ is non-empty.

We correct a result from [9] which contained an error.

**Corollary 3.15 (See [9, Corollary 5.10]).** Let $T$ be a large tilting $R$-module. The following statements are equivalent:

(i) there exists an injective pseudoflat ring epimorphism $\lambda : R \to S$ such that $S \oplus S/R$ is a tilting module equivalent to $T$;

(ii) $T$ is equivalent to a tilting module $T_U = R_U \oplus R_U/R$, where $U \subseteq \mathfrak{t}$ is a set of finite-dimensional indecomposable regular (not necessarily quasi-simple) modules.
3.5 Minimal cotilting modules

The large cotilting modules have been classified in [15]. They are also parametrized by the pairs \((Y,P)\), where \(Y\) is a branch module and \(P\) is a subset of \(X\). More precisely, the following modules form a complete irredundant list of all the large cotilting left \(R\)-modules, up to equivalence (see [1, Subsection 8.5]):

\[
C_{(Y,P)} = Y \oplus G \oplus \bigoplus \{\text{all } S[\infty] \text{ in } Y^\perp \text{ from tubes } t_x, x \notin P\} \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \oplus \prod_{S \in \mathcal{U}} \{\text{all } S[-\infty] \text{ in } Y^\perp \text{ from tubes } t_{\lambda}, \lambda \in P\}.
\]

In fact, the duality \(D = \text{Hom}_k(-, k)\) induces a bijection between (equivalence classes of) tilting right \(R\)-modules and cotilting left \(R\)-modules, which is restricted to a bijection between minimal tilting and minimal cotilting modules (see, e.g., [1, Subsections 6.1 and 6.2] and Theorem 2.12). As shown in [9, Appendix], this bijection maps \(T_{(Y,P)}\) to \(C_{(Y,P)}\).

Notice that a cotilting module is large if and only if it has an infinite-dimensional indecomposable direct summand (see [9, Lemma 2.6]).

**Theorem 3.16.** A large cotilting module is a minimal cotilting module if and only if it has an adic module as a direct summand.

**Proof.** By Theorems 2.12 and 3.14, a large cotilting module \(C\) is minimal if and only if \(C = C_{(Y,P)}\) is equivalent to \(D(T_{(Y,P)})\) for a tilting module \(T = T_{(Y,P)}\) given by a pair \((Y,P)\) with \(P \neq \emptyset\). Notice that \(C_{(Y,P)}\) can be rewritten as follows:

\[
C_{(Y,P)} = Y \oplus G \oplus \bigoplus \{\text{all } S[\infty] \text{ in } Y^\perp \text{ from tubes } t_{\lambda}, \lambda \notin P\} \oplus \prod_{S \in \mathcal{U}} S[-\infty]
\]

and \(P \neq \emptyset\) if and only if \(\mathcal{U} \neq \emptyset\), which amounts to \(C\) having an adic direct summand. \(\square\)

4 Ascent of minimality

The behavior of (co)silting modules under ring extensions has been studied in [13]. In particular, it was shown that over commutative rings every silting module extends to a silting module.

**Theorem 4.1** (See [13, Theorems 2.2 and 2.7]). Let \(\lambda : R \to S\) be a ring homomorphism, and let \(T\) be a silting module with respect to a projective presentation \(\sigma\). Then \(T \otimes_R S\) is a silting \(S\)-module with respect to \(\sigma \otimes_R S\) if and only if \(T \otimes_R S\) viewed as an \(R\)-module, lies in the silting class \(\text{Gen}T\).

The latter condition is verified whenever \(R\) and \(S\) are commutative rings.

Of course, if \(\lambda : R \to S\) is surjective with kernel \(I\), then every silting right \(R\)-module \(T\) verifies the condition \(T \otimes_R S \simeq T/TI \in \text{Gen}T\) and therefore extends to a silting \(S\)-module \(T \otimes_R S\) (see [13, Corollary 2.4]). In general, however, the condition in the theorem above can fail, even when \(T\) is minimal.

**Example 4.2.** Let \(R\) be the Kronecker algebra, and let \(T\) be a silting \(R\)-module. Then the \(R\)-module \(T \otimes_R S\) lies in \(\text{Gen}T\) for every homological ring epimorphism \(\lambda : R \to S\) if and only if \(T\) is not equivalent to the simple projective module \(P_1\).

The statement follows from the classification results reviewed in Subsection 3.2. We proceed in several steps.

**Step 1.** Let \(T = P_1\), and let \(\text{id}_R \neq \lambda : R \to S\) be an injective universal localization. Then the associated bireflective subcategory \(X\) coincides with \(\text{Add}P_i\) for some \(i \geq 2\), or with \(\text{Add}Q_{i+1}\) for some \(i \geq 1\), or with \(\mathcal{U}^{-}\) for a non-empty set \(\mathcal{U}\) of simple regular modules.

Notice that in all the cases, there is a non-trivial map from \(T\) to a module in \(X\). This is clear in the first two cases, and in the third one we can for example choose the embedding of \(P_1\) in the generic module \(G \in X\). It follows that the \(X\)-reflection \(T \to T \otimes_R S\) is non-trivial.

We claim that \(T \otimes_R S \neq 0\) does not lie in \(\text{Gen}T\). Indeed, no non-trivial module in \(X\) can belong to \(\text{Gen}T = \text{Add}P_1\). Again, this is clear in the first two cases, and in the third one we observe that by the
Auslander-Reiten formula
\[ \text{Ext}_R^1(S, P_1) \cong D\text{Hom}_R(P_1, S) \neq 0 \]
for every simple regular module \( S \).

**Step 2.** Let now \( T = Q_1 \), and let \( \lambda : R \to S \) and \( X \) be as above. Then there are no non-trivial maps from \( T \) to \( X \), which is again clear in the first two cases, and in the third one it follows from the fact that \( \text{Hom}_R(S, Q_1) \neq 0 \) for every simple regular module \( S \). We conclude that the \( X \)-reflection \( T \to T \otimes_R S = 0 \) is trivial, and \( T \otimes_R S \) lies in \( \text{Gen}T \).

**Step 3.** By the discussion above, it remains to prove the if-part of the statement, and without loss of generality we can assume that \( \lambda \) is injective, and \( T \) is a tilting module. We have one of the following cases.

**Case (i)** \( \lambda \) is the universal localization at \( P_{i+1} \) (\( i \geq 2 \)) and \( X = \text{Add}P_i \).

If \( \text{Hom}_R(T, P_i) = 0 \), then the \( X \)-reflection \( T \to T \otimes_R S = 0 \) is trivial, and the claim holds true. If \( \text{Hom}_R(T, P_i) \neq 0 \), then \( T \) must be preprojective, because the non-preprojective silting modules all belong to the class \( \text{Add}P \). More precisely, \( T \) must be equivalent to \( P_j \oplus P_{j+1} \) for some \( j \leq i \), and hence it generates \( P_i \), and therefore also \( T \otimes_R S \) which belongs to \( X = \text{Add}P_i \).

**Case (ii)** \( \lambda \) is the universal localization at \( Q_1 \) (\( i \geq 1 \)) and \( X = \text{Add}Q_{i+1} \).

As above we can assume without loss of generality that \( \text{Hom}_R(T, Q_{i+1}) \neq 0 \). If \( T \) is preinjective, then it must be equivalent to \( Q_j \oplus Q_{j+1} \) for some \( j \geq i \), and hence it generates \( Q_{i+1} \), and therefore also \( T \otimes_R S \) which belongs to \( X = \text{Add}Q_{i+1} \).

Now assume that \( T \) is not preinjective. We know from the classification that the non-preinjective silting modules all belong to the class \( \mathfrak{q}^{=0} \) of modules without non-trivial maps from the preinjective component \( \mathfrak{q} = \{Q_1, Q_2, \ldots\} \), which by the Auslander-Reiten formula coincides with the class \( \mathfrak{q}^{=1} \). In particular, \( \text{Ext}_R^1(T, Q_{i+1}) = 0 \) and \( Q_{i+1} \) belongs to the tilting class \( \text{Gen}T \). Hence \( T \) generates \( T \otimes_R S \).

**Case (iii)** \( \lambda \) is the universal localization at a set \( U \neq \emptyset \) of simple regular modules, and \( X = U^{=1} \).

Again we assume without loss of generality that \( \text{Hom}_R(T, X) \neq 0 \). Notice that this implies that \( T \) is not preinjective since \( U^{=1} \subseteq \mathfrak{q}^{=0} \). Moreover, \( U^{=1} \subseteq \mathfrak{q}^{=0} \), which by the Auslander-Reiten formula coincides with the class \( \mathfrak{p}^{=1} \). Hence \( \text{Ext}_R^1(T, T \otimes_R S) = 0 \) and \( T \otimes_R S \) is \( T \)-generated whenever \( T \) is preprojective. Moreover, \( T \otimes_R S \in \mathfrak{p}^{=1} = \text{Gen}L \) is also \( T \)-generated when \( T \) is equivalent to the Lukas tilting module.

It remains to check the case where \( T = R_U \otimes_R V \) for a non-empty set \( V \) of simple regular modules. Then \( \text{Gen}T = V^{=1} \) consists of the modules \( X \) with \( \text{Ext}_R^1(V, X) = 0 \) for all \( V \in V \). Pick a module \( V \in V \). If \( V \) also belongs to \( U \), then \( \text{Ext}_R^1(V, T \otimes_R S) = 0 \). If \( V \) does not belong to \( U \), then it belongs to \( \mathfrak{p}^{=1} = X \), as different tubes are Hom- and Ext-orthogonal. Since \( R_U \to R_U \otimes_R S \) is an \( X \)-reflection, we infer that

\[ \text{Hom}_R(R_U \otimes_R S, V) = \text{Hom}_R(R_U, V), \]

and by the Auslander-Reiten formula \( \text{Ext}_R^1(V, R_U \otimes_R S) \cong \text{Ext}_R^1(V, R_U) = 0 \). We conclude that \( R_U \otimes_R S \) is \( T \)-generated, so are \( R_U \otimes_R S \) and \( T \otimes_R S = (R_U \otimes_R S) \oplus (R_U \otimes_R S) \).

Now we return to the commutative case. If \( \lambda : R \to S \) is a ring epimorphism and \( R \) is commutative, then so is \( S \) by [23]. It follows from Theorem 4.1 that every silting module extends to a silting module along \( \lambda \). In fact, when \( R \) is commutative and hereditary, and \( \lambda \) is pseudoflat, minimality is also preserved. Indeed, \( S \) is then hereditary as well by [11, p. 324], and all the silting modules over commutative hereditary rings are minimal, as we observe next.

**Remark 4.3.** Let \( R \) be a commutative hereditary ring, and let \( T \) be a silting module. Denote by \( C = T^+ \) the corresponding cosilting module. As discussed in [3, the paragraph after Proposition 6.5], there is a flat epimorphism \( \lambda : R \to S \) such that \( S^+ \oplus \text{Ker}\lambda^+ \) is a cosilting module equivalent to \( C \). Moreover, by Theorem 2.12, we also have a minimal silting module \( T' = S \oplus \text{Coker}\lambda \). Now \( T \) and \( T' \) are both mapped to \( C \) under the silting-cosilting-bijection \( T \to T^+ \) established in [2, Corollary 3.6], and hence they are equivalent. This shows that \( T \) is a minimal silting module.

We now want to determine further conditions ensuring that minimality is preserved by ring extensions. Let us first explore how to relax the assumption.
Lemma 4.4. Let $\lambda : R \to S$ be a pseudoflat ring epimorphism such that $S_R$ has the projective dimension at most one. Then $T = S \oplus \text{Coker} \lambda$ is a minimal silting module.

Proof. We know from [3, Example 4.15] that $S_R$ has a presilting presentation. By the proof of [7, Proposition 1.3], the $R$-module homomorphism $\lambda$ can be lifted to a triangle

$$R \xrightarrow{\delta} \sigma_0 \to \sigma_1 \to R[1]$$

in the derived category $D(\text{Mod}R)$ such that $T$ is a silting module with respect to the projective presentation $\sigma = \sigma_0 \oplus \sigma_1$. It follows that $\phi$ is a left $\text{Add}$-approximation. It remains to prove that $\phi$ is left minimal. Consider a morphism $g \in \text{End}_{D(\text{Mod}R)} \sigma_0$ such that $g\phi = \phi$. It holds that

$$R \xrightarrow{\phi} \sigma_0.$$ \hspace{1cm} (4.1)

Applying the cohomology functor $H^0(-)$ to the diagram, we obtain the following commutative diagram:

$$\begin{array}{c}
R \xrightarrow{\lambda} S, \\
\downarrow \quad \downarrow \phi \\
S & \xrightarrow{\text{H}^0(g)} & \text{H}^0(g)
\end{array}$$

where $\text{H}^0(g)$ is an isomorphism since $\lambda$ is left minimal. Since $\text{H}^0(\sigma_0) = 0$ for any $i \neq 0$, we infer that $g$ is an isomorphism. It follows that $T$ is a minimal silting $R$-module.

Lemma 4.5. Let $R$ be an arbitrary ring, and let $\lambda : R \to S$ and $\lambda' : R \to S'$ be ring epimorphisms with the associated bireflective subcategories $\mathcal{X}$ and $\mathcal{X}'$, respectively. Consider the push-out $S \sqcup_R S'$ of $\lambda$ and $\lambda'$ in the category of rings

$$\begin{array}{ccc}
R & \xrightarrow{\lambda} & S \\
\downarrow & \downarrow & \downarrow \\
S' & \xrightarrow{\lambda'} & S \sqcup_R S'.
\end{array} \hspace{1cm} (4.1)

Then $\mu$ and $\mu'$ are ring epimorphisms and the bireflective subcategory associated with the composition $\mu\lambda' = \mu'\lambda$ is given by $\mathcal{X} \cap \mathcal{X}'$. If $\lambda$ is pseudoflat, so is $\mu$. If $\lambda$ is a universal localization, so is $\mu$.

Lemma 4.6. Let $R$ be a commutative ring, and let $\lambda : R \to S$ and $\lambda' : R \to S'$ be ring epimorphisms. Then $\text{Coker} \lambda \otimes_R S' \cong \text{Coker} \mu$, where $\mu : S' \to S \otimes_R S'$ is given by the push-out of $\lambda$ and $\lambda'$.

Proof. It is well known (see, e.g., [16, Lemma 6.3]) that $S \sqcup_R S' \cong S \otimes_R S'$. We keep the notation of the diagram (4.1), and set $\lambda = \mu\lambda' = \mu'\lambda$. We know from Lemma 4.5 that $\lambda$ is a ring epimorphism, and its corresponding bireflective subcategory is $\mathcal{X} = \mathcal{X} \cap \mathcal{X}'$. Applying the functor $- \otimes_R S'$ to the diagram (4.1), we obtain the following commutative diagram:

$$\begin{array}{ccc}
R \otimes_R S' & \xrightarrow{\lambda \otimes_R S'} & S \otimes_R S' \\
\downarrow & \downarrow & \downarrow \\
S' \otimes_R S' & \xrightarrow{\mu' \otimes_R S'} & S' \otimes_R S' \otimes_R S'.
\end{array}$$

Since $\lambda$ is a ring epimorphism, the multiplication map $S' \otimes_R S' \to S'$ is an isomorphism, so is the map $\lambda' \otimes_R S'$. We claim that $\mu' \otimes_R S'$ is an isomorphism as well. To this end, we will show that both $\lambda \otimes_R S'$ and $\lambda' \otimes_R S'$ are $\mathcal{X}$-reflections of $R \otimes_R S'$. Our claim will then follow from the uniqueness of reflections.
Let us consider the commutative diagram

\[
\begin{array}{ccc}
R & \xrightarrow{\lambda} & S \\
\psi_R \downarrow & & \downarrow \psi_S \\
R \otimes_R S' & \xrightarrow{\lambda \otimes_R S'} & S \otimes_R S'.
\end{array}
\]

Recall that \(\lambda\) is an \(X\)-reflection of \(R\), and \(\psi_R\) and \(\psi_S\) are \(X'\)-reflections of \(R\) and \(S\), respectively. Since \(S \otimes_R S' \in \tilde{X}\), we conclude that \(\lambda \otimes_R S'\) is an \(\tilde{X}\)-reflection. We argue similarly for \(\tilde{\lambda} \otimes_R S'\), using the commutative diagram below together with the fact that \(S \otimes_R S' \sim S \otimes_R S' \in \tilde{X}\)

\[
\begin{array}{ccc}
R & \xrightarrow{\tilde{\lambda}} & S \otimes_R S' \\
\psi_R \downarrow & & \downarrow \psi_{S \otimes_R S'} \\
R \otimes_R S' & \xrightarrow{\tilde{\lambda} \otimes_R S'} & S \otimes_R S' \otimes_R S'.
\end{array}
\]

Next, we observe that the \(X'\)-reflections \(\psi_{S'}\) and \(\psi_{S \otimes_R S'}\) are isomorphisms, because \(S'\) and \(S \otimes_R S'\) are in \(X'\). So, we have a commutative diagram with exact rows

\[
\begin{array}{ccccccc}
R \otimes_R S' & \xrightarrow{\lambda \otimes_R S'} & S \otimes_R S' & \xrightarrow{Coker \lambda \otimes_R S'} & 0 \\
\downarrow N \otimes_R S' & & \downarrow \omega \otimes_R S' & & & & \downarrow & \\
S' \otimes_R S' & \xrightarrow{\mu \otimes_R S'} & S \otimes_R S' \otimes_R S' & \xrightarrow{Coker \mu \otimes_R S'} & 0 \\
\downarrow \psi_{S'}^{-1} & & \downarrow \psi_{S \otimes_R S'}^{-1} & & & & \downarrow & \\
S' & \xrightarrow{\mu} & S \otimes_R S' & \xrightarrow{Coker \mu} & 0,
\end{array}
\]

where the isomorphisms in the first two columns induce an isomorphism \(Coker \lambda \otimes_R S' \simeq Coker \mu\).

**Proposition 4.7.** Let \(R\) be a commutative ring, and let \(T = S \oplus_R Coker \lambda\) be a silting module arising from a ring epimorphism \(\lambda: R \to S\). Assume that \(X': R \to S'\) is a ring epimorphism such that \(S \otimes_R S'\) has a pseudoflat \(X'\)-reflection over \(S'\). Then \(T \otimes_R S'\) is a silting \(S'\)-module which arises from the ring epimorphism \(\mu: S' \to S \otimes_R S'\) given by the push-out of \(\lambda\) and \(X'\).

**Proof.** By Proposition 2.8, the ring epimorphism \(\mu: S' \to S \otimes_R S'\) induces a silting \(S'\)-module \(S \otimes_R S' \oplus Coker \mu\), which by Lemma 4.6 is isomorphic to \(T \otimes_R S'\).

**Corollary 4.8.** Let \(R\) be a commutative ring, and let \(T = S \oplus_R Coker \lambda\) be a silting module arising from a ring epimorphism \(\lambda: R \to S\). Let \(X': R \to S'\) be a ring epimorphism such that the projective dimension of \(S \otimes_R S'\) over \(S'\) is at most one.

1. If \(\lambda\) is a pseudoflat ring epimorphism, then \(T \otimes_R S'\) is a minimal silting \(S'\)-module.
2. If \(\lambda\) is a universal localization, then \(T \otimes_R S'\) arises from a universal localization.

**Proof.** By Lemma 4.5, the ring epimorphism \(\mu\) is pseudoflat. Then we obtain Statement (1) by combining Lemmas 4.6 and 4.4. Statement (2) is similarly obtained by observing that \(\mu\) is a universal localization.

Let us now turn to the dual situation. We will see that minimality of cosilting modules is often preserved by extensions along arbitrary ring epimorphisms.

**Proposition 4.9** (See [13, Remark 2.3]). Let \(\lambda: R \to S\) be a ring homomorphism, and let \(C\) be a cosilting module with respect to an injective copresentation \(\omega\). Then \(Hom_R(S, C)\) is a cosilting \(S\)-module with respect to \(Hom_R(S, \omega)\) if and only if \(Hom_R(S, C)\), viewed as an \(R\)-module, lies in \(CogenC\).

**Proposition 4.10.** Let \(R\) be a commutative ring and \(C\) be a minimal cosilting module arising from a ring epimorphism \(\lambda: R \to S\). Assume that \(X': R \to S'\) is a ring epimorphism such that \((S \otimes_R S')^+\) has a precosilting copresentation over \(S'\). Then \(Hom_R(S', C)\) is a minimal cosilting \(S'\)-module.
Proof. The push-out of \( \lambda \) and \( \lambda' \) in (4.1) gives rise to a ring epimorphism \( \mu : S' \to S \otimes_R S' \) as in Theorem 2.10(i). Hence \( (S \otimes_R S')^+ \oplus \text{Ker} \mu^+ \) is a minimal cosilting \( S' \)-module. On the other hand, we infer from the shape of \( C = S^+ \oplus \text{Ker} \lambda^+ \) that

\[
\text{Hom}_R(S', C) = \text{Hom}_R(S', S^+) \oplus \text{Hom}_R(S', \text{Ker} \lambda^+) \cong \text{Hom}_R(S', S^+) \oplus \text{Hom}_R(S', (\text{Coker} \lambda)^+) \\
\cong (S \otimes_R S')^+ \oplus (\text{Coker} \lambda \otimes_R S')^+.
\]

Now recall from Lemma 4.6 that \( \text{Coker} \lambda \otimes_R S' \cong \text{Coker} \mu \). We deduce that

\[
\text{Hom}_R(S', C) \cong (S \otimes_R S')^+ \oplus \text{Ker} \mu^+,
\]

which concludes the proof. \( \square \)

Corollary 4.11. Let \( R \) be a commutative noetherian ring, or a commutative ring of weak global dimension at most one. Then all the minimal cosilting modules extend to minimal cosilting modules along any pseudoflat ring epimorphism.

Proof. Assume that \( R \) is commutative and noetherian. Recall from [4, Proposition 4.5] that every pseudoflat ring epimorphism \( R \to S \) is flat. By Remark 2.11, every minimal cosilting module \( C \) arises from a flat ring epimorphism \( \lambda : R \to S \). If \( \lambda' : R \to S' \) is a flat ring epimorphism, then \( S' \) is again commutative and noetherian by [23, Corollary 1.2 and Proposition 1.6]. We infer from Lemma 4.5 that the push-out of \( \lambda \) and \( \lambda' \) in (4.1) gives rise to a flat ring epimorphism \( \mu : S' \to S \otimes_R S' \). Then \( (S \otimes_R S')^+ \) has a precosilting copresentation over \( S' \) by Remark 2.11, and the claim follows from Proposition 4.10.

Now assume that \( R \) is a commutative ring of weak global dimension at most one. Then every pseudoflat ring epimorphism \( R \to S \) is homological, and \( S \) has the weak global dimension at most one, because the functors \( \text{Tor}_i^R \) and \( \text{Tor}_i^S \) agree on \( S \)-modules for all \( i \geq 1 \) by [18, Theorem 4.4]. So, we can proceed as above. By Remark 2.11, every minimal cosilting module \( C \) arises from a homological ring epimorphism \( \lambda : R \to S \). The push-out of \( \lambda \) with a homological ring epimorphism \( \lambda' : R \to S' \) yields a homological ring epimorphism \( \mu : S' \to S \otimes_R S' \). Then \( (S \otimes_R S')^+ \) has a precosilting copresentation over \( S' \), and the claim follows from Proposition 4.10. \( \square \)

Recall that a ring \( R \) is said to be semihereditary if every finitely generated right ideal is projective. Moreover, a cosilting module is said to be of cofinite type if it is equivalent to the dual \( T^+ \) of a silting module \( T \).

Corollary 4.12. Over a commutative ring, every minimal cosilting module arising from a universal localization extends to a minimal cosilting module along any ring epimorphism.

In particular, over a commutative (semi)hereditary ring, every cosilting module (of cofinite type) extends to a minimal cosilting module along any ring epimorphism.

Proof. Let \( C \) be a minimal cosilting module arising from a universal localization \( \lambda : R \to S \), and let \( \lambda' : R \to S' \) be an arbitrary ring epimorphism. We know from Lemma 4.5 that the push-out of \( \lambda \) and \( \lambda' \) in (4.1) gives rise to a universal localization \( \mu : S' \to S \otimes_R S' \). Then \( \mu \) is a flat ring epimorphism by [4, Corollary 4.4]. By Remark 2.11, this implies that \( (S \otimes_R S')^+ \) has a precosilting copresentation over \( S' \), and the claim follows from Proposition 4.10.

Now assume that \( R \) is semihereditary. As observed in [3, the paragraph after Proposition 6.5], every cosilting module of cofinite type is a minimal cosilting module arising from a universal localization. The claim then follows from the first statement. Finally, if \( R \) is hereditary, all the cosilting modules are of cofinite type by [3, Theorem 3.11]. \( \square \)

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