Gauge invariant cutoff QED

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Abstract

A hidden generalized gauge symmetry of a cutoff QED is used to show the renormalizability of QED. In particular, it is shown that the corresponding Ward identities are valid all along the renormalization group flow. The exact renormalization group flow equation corresponding to the effective action of a cutoff $\lambda \phi^4$ theory is also derived. Generalization to any gauge group is indicated.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Diagrammatic proof of the renormalizability of QED in the Bogoliubov–Parasiuk–Hepp–Zimmermann (BPHZ) context is a lengthy derivation \cite{1}. In 1984, Polchinski presented a simple proof using a version of the Wilsonian renormalization group equation (RGE) for the $\lambda \phi^4$ theory \cite{2}. Although Polchinski’s proof of renormalizability could, in principle, be extended to QED, the manifest violation of gauge invariance in his momentum cutoff formulation was an obstacle to a straightforward proof. This has been extensively studied and is by now a well-understood derivation of renormalizability of QED \cite{3, 4} (for recent reviews see \cite{5} and the literature therein). However, it remains an untidy procedure involving either complicated cutoff insertions or additional labor for the proof of Ward identities.

In this work, we present a simple extension of Polchinski’s proof of renormalizability to QED using a hidden generalized gauge invariance of the cutoff formulation, thus circumventing the need for explicit verification of the Ward identities. In section 2, we introduce our slightly modified cutoff procedure of Polchinski and apply it to the $\lambda \phi^4$ theory. The cutoff procedure we use is by simply multiplying the fields by an appropriate cutoff function in the momentum representation, as explained in section 2. The new point-wise product in the momentum space defines a ‘deformed’ nonlocal product in the coordinate space. For the reasons that will be explained below and which are originally indicated in \cite{6}, the cutoff function is to be taken as a sequence of analytic functions which converge to a sharp cutoff. This procedure has its origin in Kogut and Wilson’s \cite{7} ‘incomplete integration’, with which Wilson’s exact RGE was first formulated. Wetterich introduced a similar concept of ‘average field’ in the discussion of RGE \cite{8}, which was a precursor to his well-known presentation of exact average RGE \cite{5, 9–11}. We, however, arrived at this cutoff procedure from an entirely different angle of translationally invariant noncommutative gauge theory \cite{14, 15}, which led us to the symmetry of the cutoff effective action for gauge theories, explained in section 3. Here, using the new deformed product of functions, we present the generalized (deformed) cutoff gauge symmetry of the cutoff QED and indicate its renormalizability. Based on the idea proposed in the present paper, Lizzi and Vitale \cite{6} showed recently that the new ‘deformed’ gauge symmetry, defined by a ‘deformed’ product of fields, leads to a new cocommutative Hopf algebra with ‘deformed’ costructures. They argue that in order to preserve the associativity of the new deformed product of functions, the above-mentioned cutoff function is to be analytic. Being analytic, however, the new product can be interpreted as a simple redefinition of fields, which is isomorphic to a point-wise product and therefore physically trivial. To circumvent this problem, they propose the cutoff function to be a sequence of analytic functions, which converge to a sharp cutoff, as indicated above. Moreover, using a rigorous mathematical construction of the new deformed Hopf algebra, they explicitly show that the map between this deformed Hopf algebra and the standard (undeformed) one, which is to be
compatible with the suggested field redefinition, does not satisfy the coalgebra and the Hopf algebra homomorphisms. Thus, the new deformed gauge symmetry is inequivalent to the standard one and defines a bona fide new symmetry. In order to show that the proposed cutoff QED is renormalizable without destroying the (deformed) cutoff gauge invariance, the Ward–Takahashi identities of the cutoff theory are to be verified. This will be done in section 3. A derivation of the deformed Ward–Takahashi identities will be presented in appendix A for the sake of completeness. As it turns out, the new deformed symmetry is preserved along the renormalization group flow, for every fixed ultraviolet (UV) cutoff. This was indeed as expected, because as was pointed out in [6], although the sharp cutoff cannot define a deformed associative product, the theory with a sharp cutoff, being a limit of a Hopf gauge invariant theory, exhibits the same symmetries of the theory defined with the deformed product of fields.

Let us note that the cutoff procedure we use and the consequent symmetry can be easily generalized to the gauge theory with any group and matter field, including $SU(N)$, as the noncommutative structure from which it is derived in section 3 can be extended trivially to these cases. The point being that the ‘noncommutative geometric’ structure we will be using is in fact Abelian. For simplicity of presentation, we will restrict ourselves to the case of $U(1)$ gauge group, QED. In [6], the case of the $SU(N)$ gauge group is studied.

In section 4, we will use the idea of field deformation and the method developed in [9, 10] to derive the corresponding exact RG flow equation for the effective average action of $\lambda \phi^4$ theory including an appropriate infrared (IR) cutoff function (see also appendix B for a detailed proof). We will show that the deformed flow equation is different from the usual RG exact flow equation derived in [9, 10]. The main reason for this difference is the multiplicative nature of our cutoff function, in contrast to the additive IR cutoff introduced in [9, 10]. Modifying each field with such a multiplicative cutoff, the cutoff function appears not only in the kinetic term, but also in the interacting part of the classical cutoff action. It is interesting to generalize this procedure to gauge theories and to explore the possible practical consequences of the new exact average RG for the effective average action as well as the hidden generalized gauge symmetry along the flow, pointed out in this paper. The origin of this gauge symmetry will, nevertheless, be discussed in section 5.

2. Modified cutoff regularization

In this section we redo Polchinski’s proof of $\lambda \phi^4$ theory [2] by a slightly different cutoff procedure. The real scalar theory is defined by the action

$$S = \int d^4 x \left[ -\frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} \lambda \phi^4 \right]. \quad (2.1)$$

and the momentum cutoff $\Lambda_0$ in the Euclidean space is regularized by Polchinski via introduction of a momentum cutoff $K_{\Lambda_0}(p)$ into the propagator,

$$\frac{K_{\Lambda_0}(p)}{p^2 + m^2}. \quad (2.2)$$

The main property of the function $K_{\Lambda_0}(p)$ is that it has a value equal to 1 for $p^2 < \Lambda_0^2$ and decreases rapidly for $p^2 \rightarrow \infty$. An example is

$$K_{\Lambda_0}(p) = \begin{cases} 
1, & p^2 \leq \frac{\Lambda_0^2}{4}, \\
\exp \left[ \left( 1 - \frac{p^2}{\Lambda_0^2} \right)^{-1} \exp \left( 4 - \frac{p^2}{\Lambda_0^2} \right)^{-1} \right], & \frac{\Lambda_0^2}{4} < p^2 < \Lambda_0^2, \\
0, & p^2 \geq \Lambda_0^2.
\end{cases}$$

(2.3)

The cutoff function $K_{\Lambda_0}(p)$ effectively cuts off the momentum integral in all loops, rendering them UV finite in perturbation theory. The introduction of a cutoff function in the propagators and consequently in all loops is an efficient procedure for implementing Wilson’s renormalization group flow, as the cutoff momentum $p$ can now be lowered all the way down to zero in the path integral for the effective action. A remarkable aspect of the introduction of the cutoff function in the loop integrals is that loop integrals can now be estimated easily and the cutoff independence of Green’s functions be demonstrated in a few steps in marked contrast to the lengthy BPHZ proof of renormalizability.

In this work, we propose to modify this procedure by extending the cutoff to all the terms in the action, and in fact to all fields. In our formulation, we replace the field $\tilde{\psi}(p)$ by their cutoff counterpart

$$h_\Lambda(p)\tilde{\psi}(p)$$

in momentum space. Comparing to (2.1), $K_\Lambda$ is to be identified with $h_\Lambda^{-2}$. This is therefore a straightforward implementation of Wilson’s cutoff procedure executed directly on the fields rather than on the path integrals.

In the original Polchinski’s formulation, the cutoff independence of the theory is

$$\frac{\partial Z}{\partial \Lambda} = 0,$$ with $\frac{\partial}{\partial \Lambda} = \Lambda \frac{\partial}{\partial \Lambda}$, \quad (2.5)

and the running effective action is

$$S_{\text{ef}} = \int \frac{d^4 p}{(2\pi)^4} \left[ -\frac{1}{2} \tilde{\psi}(p) \frac{p^2 + m^2}{K_\Lambda(p)} \tilde{\psi}(-p) + L_{\text{ef}}(\tilde{\psi}, \Lambda) \right].$$

(2.6)

Equation (2.5) then determines the running effective interaction Lagrangian $L_{\text{ef}}$ at the $\Lambda$ scale, which now satisfies the functional differential equation

$$\frac{\partial L_{\text{ef}}}{\partial t} = -\frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 + m^2} \frac{\partial K_\Lambda(p)}{\partial t} \times \left[ \frac{\partial L_{\text{ef}}}{\partial \tilde{\psi}(p)} \frac{\partial \tilde{\psi}(p)}{\partial \tilde{\psi}(-p)} + \frac{\partial^2 L_{\text{ef}}}{\partial \tilde{\psi}(p) \partial \tilde{\psi}(-p)} \tilde{\psi}(-p) \right].$$

(2.7)

Our formulation does not yet differ from Polchinski’s as the functional $L_{\text{ef}}$ is not yet specified. The difference shows up
when expanding \( L_\alpha \) in a series in \( \tilde{\psi} \).

\[
L_\alpha(\tilde{\psi}, \Lambda) = \sum_{n=1}^{\infty} \frac{1}{(2n)!} \int \frac{d^4 p_1 \ldots d^4 p_{2n}}{(2\pi)^{8n-4}} L_{2n}(p_1, \ldots, p_{2n}; \Lambda) \\
\times \delta \left( \sum_i p_i \right) \tilde{\psi}(p_1) \ldots \tilde{\psi}(p_{2n}).
\] (2.8)

In Polchinski’s formulation the renormalization flow equation (2.7) reduces to a set of equations for the coefficient functions \( L_{2n}(p_1, \ldots, p_{2n}; \Lambda) \), whose solution would give the effective action at scale \( \Lambda \).

\[
\left( \frac{\partial}{\partial t} + 4 - 2n \right) L_{2n}(p_1, \ldots, p_{2n}; \Lambda) \\
= -\sum_{\ell=1}^{n} \left\{ Q_\Lambda(P, m) L_{2\ell}(p_1, \ldots, p_{2\ell-1}; \Lambda) \\
\times L_{2n+2-2\ell}(p_2, \ldots, p_{2n}, -P; \Lambda) + \text{permutation} \right\} \\
- \frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} L_{2n+2}(p_1, \ldots, p_{2n}, P, -P; \Lambda) Q_\Lambda(P, m) \right\}.
\] (2.9)

Here, \( P = \sum_{i=1}^{2n-1} p_i, \) and

\[
Q_\Lambda(P, m) = \frac{1}{P^2 + m^2} \Lambda^2 \frac{\partial K_\Lambda(P)}{\partial t}.
\] (2.10)

A convenient rescaling \( L_{2n} \rightarrow \Lambda^{4-2n} L_{2n} \) has been inserted into (2.9).

In our formulation, however, the expansion in (2.8) has to be replaced by

\[
L_\alpha(\tilde{\psi}, \Lambda) = \sum_{n=1}^{\infty} \frac{1}{(2n)!} \int \frac{d^4 p_1 \ldots d^4 p_{2n}}{(2\pi)^{8n-4}} L'_{2n}(p_1, \ldots, p_{2n}; \Lambda) \\
\times \delta \left( \sum_i p_i \right) h_\Lambda(p_1) \tilde{\psi}(p_1) \ldots h_\Lambda(p_{2n}) \tilde{\psi}(p_{2n}),
\] (2.11)

as every momentum space field \( \tilde{\psi}(p) \) is rescaled and cut off by the cutoff function \( h_\Lambda(p) \) from (2.4). It may seem that the subsequent steps in renormalizability may get complicated, but this is not the case. In fact, the resulting renormalization group flow equations for the function \( h_\Lambda(p_1) \ldots h_\Lambda(p_{2n}) L'_{2n}(p_1, \ldots, p_{2n}; \Lambda) \) would now be of exactly the same form as (2.9) for the coefficient functions, where the function \( K_\Lambda(p) \) of Polchinski has to be replaced by \( h_\Lambda^{-1}(p) \), see [2] for more details. The advantage of Polchinski’s approach to renormalization is that the insertion of a cutoff function in the integrals allows him to estimate the coefficient functions \( L_{2n}(p_1, \ldots, p_{2n}) \) through functional analytic methods and put bounds on them; and then to prove the cutoff independence of Green’s functions in perturbation theory, through a series of lemmas. Note that in all this the function \( K_\Lambda \) appears, as in (2.7), only in its derivative form \( \partial_\Lambda K_\Lambda \), which does not affect the estimate arguments which involve various norms of functions. We will not go through the entire analysis, but only point out the validity of the procedure in our approach in the cases where there is ground for doubt.

The first instance that our approach may invalidate Polchinski’s result is when a bound on \( L'_{2n}(p_1, \ldots, p_{2n}) \) is obtained through (2.9) using the bounds

\[
\int \frac{d^4 p}{(2\pi)^4} |Q_\Lambda(p, m)| < C \Lambda^4
\] (2.12)

and

\[
\max \left| \frac{\partial^2}{\partial p^2} Q_\Lambda(p, m) \right| < D_\Lambda \Lambda^{-n}.
\] (2.13)

Here, \( C \) and \( D_\Lambda \) are appropriate constants. The result

\[
\max |L'_{2n}(p_1, \ldots, p_{2n}; \Lambda)| \leq P_{2n} \left( \ln \frac{\Lambda_0}{\Lambda} \right), \quad \text{for } r+1-n > 0,
\]

\[
= 0, \quad \text{for } r+1-n < 0,
\] (2.14)

and similar bounds are central to the proof of renormalizability. In (2.14), \( L'_{2n} \) is the \( r \)th term in the perturbative expansion of \( L_{2n} \), and \( P_{2n} \) is polynomials of order \( 2r - n \).

In our case, (2.9) involves \( h_\Lambda(p_1) \ldots h_\Lambda(p_{2n}) L_{2n} \) rather than \( L_{2n} \). One must make sure that the appearance of the \( h_\Lambda \) does not ruin the bounds in Polchinski and therefore ruin the arguments on renormalizability of the theory. We have identified the function \( K_\Lambda \) of (2.2) with \( h_\Lambda^2 \) of (2.4), as mentioned above. Therefore if \( K_\Lambda \) should behave as in Polchinski’s formulation, i.e. go to zero when \( p^2 \) approaches \( \Lambda^2 \), and vanish for \( p^2 > \Lambda^2 \), then \( h_\Lambda \) should become large as \( p^2 \) approaches \( \Lambda^2 \). Of course, we set \( h_\Lambda(p^2) = 0 \) for \( p^2 > \Lambda^2 \). Therefore in the left-hand side (lhs) of (2.9), after inserting \( h_\Lambda \ldots h_\Lambda L_{2n} \) for \( L_{2n} \), as the \( h_\Lambda \) are larger than one in their range of definition, they can be dropped in the ensuing inequality. On the right-hand side (rhs) of (2.9), we now use the functional norm with an appropriate weight to kill off the value of \( h_\Lambda > 1 \) for \( p^2 \rightarrow \Lambda^2 \), leading to the desired inequality (2.14), this time for \( L_{2n} \). Let us note that the above choice for \( K_\Lambda \) is not a unique one. It is easy to show that Polchinski’s proof is also correct for \( K_\Lambda \), becoming large as \( p^2 \) approaches \( \Lambda^2 \). In the next section, we will use this second alternative, and define, as in [6], the cutoff function \( h_\Lambda(p) \) as a sequence of analytic functions satisfying

\[
h_{\epsilon, \Lambda}(p) \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0
\]

\[
\Theta_\Lambda(p) = \begin{cases} 0 & \text{for } p^2 \geq \Lambda^2, \\ 1 & \text{for } p^2 < \Lambda^2. \end{cases}
\] (2.15)

Here, \( \Theta_\Lambda(p) \) is a sharp (UV) cutoff\(^5\). This property is necessary to avoid the interpretation of the deformation (2.4) to be just a redefinition of fields in the momentum space [6] (for more mathematical details, see the discussions at the end of the next section).

### 3. Gauge invariance of cutoff QED and Ward identities

Polchinski’s procedure was applied to QED very early and provided a simple proof of renormalizability of the theory [3]. In QED there are now two propagators to cut off that of the electron and that of the photon. This can be done with

\(^4\) See the footnote on page 280 of Polchinski [2].

\(^5\) The subscript \( \epsilon \) on \( h_{\epsilon, \Lambda}(p) \) will be omitted in the rest of this paper.
the same cutoff function. The RGE is similarly derived and estimates for the coefficient functions and bounds on Green’s functions obtained. The situation is then a straightforward extension of the \( \lambda \phi^4 \) theory. There is only one significant hurdle to overcome that has engaged the authors of [3, 4] ever since and is also the subject of the present work, namely the question of gauge invariance.

The problem is that Polchinski’s approach, and in fact any approach involving a momentum cutoff, inherently violates gauge invariance: Gauge invariance is a statement about the behavior of gauge fields in a space–time point, involving all momenta. Thus, at any finite cutoff scale \( \Lambda \), the flow equation and its solutions are not gauge invariant. However, it was proved that the final IR point of the flow \( \Lambda \to 0 \), the expressions for the quantum effective action and Green’s functions are indeed invariant [4]. There were also nontrivial modifications of the cutoff procedure which were not gauge invariant all along the flow, but only at its end points. These formulations have been extensively pursued in the application of the exact renormalization group in such areas as QCD (see [5] for recent reviews) and gravity [13]. In all these works derivation of the modified Ward–Takahashi identities is the essential complication.

In this section, we will show that our version of the introduction of the momentum cutoff in the theory ensures persistence of gauge invariance in the form of a generalized deformed symmetry of the cutoff QED and derive the resultant deformed Ward–Takahashi identities in a standard manner. We start from the classical action of QED\(^6\),

\[
S_{\text{QED}} = \int d^4x \left\{ \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi - e \bar{\psi} \gamma^\mu A_\mu \psi - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{1}{2 \xi} (\partial_\mu A^\mu)^2 \right\}.
\]

(3.1)

with the cutoff function \( h_\Lambda (p) \) inserted on each field in the momentum space

\[ \bar{\psi} (p) \to h_\Lambda (p) \bar{\psi} (p) \quad \text{and} \quad A_\mu (p) \to h_\Lambda (p) A_\mu (p). \] (3.2)

We arrive at the deformed cutoff (effective) action

\[
S_{\text{h}_\Lambda} = \int d^4k \left[ \frac{1}{4} g_{\mu \nu} (k) \partial_\mu \psi^\dagger \partial_\nu \psi + \frac{1}{2 \xi} \partial_\mu A^\mu \right] + \int d^4p \left[ h_\Lambda^2 (p) \bar{\psi} (p) \left( \gamma_\mu p^\mu - m \right) \psi (p) - \frac{1}{4} \left( \partial_\mu A^\mu \right)^2 \right] + \int d^4p d^4q \int d^4\ell \left\{ \bar{\psi} (\ell) \psi (\ell + q - p) \right\} \times \Lambda_\mu (\ell) \psi (q) \delta^4 (\ell + q - p).
\]

(3.3)

Here, the cutoff function \( h_\Lambda (p) \) is to be analytic and has to converge to a sharp (UV) cutoff function, as is expressed in (2.15). Moreover, it has to satisfy \( h_\Lambda (-p) = h_\Lambda (p) \). The reason for this specific choice will be explained in what follows. But before doing this, let us consider the effective action (3.3). As it turns out, it has a symmetry which is the generalization of gauge symmetry of \( S_{\text{QED}} \). Whereas the gauge symmetry of \( S_{\text{QED}} \) is

\[
\psi (x) \to e^{i \epsilon (x)} \psi (x), \quad \bar{\psi} (x) \to e^{-i \epsilon (x)} \bar{\psi} (x),
\]

(3.4)

the symmetry of \( S_{\text{h}_\Lambda} \) is similarly defined but now involves \( h_\Lambda \). To introduce this new (deformed) cutoff gauge symmetry of \( S_{\text{h}_\Lambda} \), let us first note that when in momentum space two functions \( \bar{\psi} (p) \) and \( h_\Lambda (p) \) are point-wise multiplied, their corresponding functions in the configuration space, \( \psi (x) \) and \( h_\Lambda (x) \), are multiplied via convolution

\[
h_\Lambda (p) \bar{\psi} (p) \to \tilde{h}_\Lambda (x) \psi (x),
\]

(3.5)

where convolution of two functions \( f (x) \) and \( g (x) \) is defined by

\[
(f \circ g) (x) = \int d^4 y \, f (x - y) g (y).
\]

(3.6)

The above-mentioned deformed gauge symmetry transformation of \( S_{\text{h}_\Lambda} \) is then given by

\[
\psi (x) \to \left( \tilde{h}_\Lambda \circ g \right) \left( \tilde{h}_\Lambda \circ \psi \right).
\]

(3.7)

Here, \( g (x) \) is the generalization of \( e^{i \epsilon (x)} \) defined by

\[
g (x) = 1 + i e \epsilon (x) + \frac{1}{2!} \left( \tilde{h}_\Lambda \circ \left( i e \epsilon \right) \right) \left( \tilde{h}_\Lambda \circ \left( i e \epsilon \right) \right) + \cdots.
\]

(3.8)

The transformation of \( A_\mu (x) \) is

\[
A_\mu (x) \to A_\mu (x) + \left( \tilde{h}_\Lambda \circ \left( \tilde{h}_\Lambda \circ (\partial_\mu g^{-1}) \right) \right).
\]

(3.9)

These strange-looking transformations come from a simple generalized noncommutative geometric construction, related to the translationally invariant noncommutative star-product, introduced originally in [14]. To understand the origin of the above deformed gauge transformations (3.7)–(3.9), we review, in what follows, this generalized noncommutative field theory.

Let us start by defining the generalized translationally invariant noncommutative star-product from [14, 15], as a generalization of the usual \( C^* \)-algebra of point-wise multiplication algebra of functions

\[
(f \star g) (x) = \int \frac{d^4p}{(2 \pi)^4} \frac{d^4q}{(2 \pi)^4} \frac{d^4\ell}{(2 \pi)^4} \mathcal{K}(p, q) \tilde{f}(p - q) \tilde{g}(q).
\]

(3.10)

The point-wise multiplication is the special case of \( \mathcal{K} = 1 \). Associativity of the algebra is the main constraint on the function \( \mathcal{K} \). It is

\[
\mathcal{K}(p, q) \mathcal{K}(q, r) = \mathcal{K}(p, r) \mathcal{K}(p - r, q - r).
\]

(3.11)

It was shown in [15] that the following expression is a solution of (3.11):

\[
\mathcal{K}(p, q) = h^{-1} (p) h(q) h(p - q) e^{\Omega(p, q)}
\]

(3.12)
with
\[ \Omega(p, q) = \theta_{\mu\nu} p^{\mu} q^{\nu} + \eta(q) - \eta(p) + \eta(p - q). \] (3.13)
Here, \( \theta_{\mu\nu} \) is an antisymmetric constant matrix, and \( h(p) \) and \( \eta(p) \) are arbitrary real even and odd functions, respectively. Later, \( h(p) \) will be identified with the cutoff function \( h_\Lambda(p) \) satisfying the properties (2.15) and converging to a sharp cutoff function \( \Theta_\Lambda(p) \). It is readily seen that the algebra is noncommutative for \( \theta \neq 0 \), and is commutative when \( \theta = 0 \). When \( \theta = 0 \) and \( \eta = 0 \), the new star-product in the momentum space involves multiplications of the functions of the algebra by the fixed function \( h(p) \)
\[ (f \star g)(x) = \int \frac{d^d p}{(2\pi)^d} \frac{1}{h(p)} e^{ipx} \int \frac{d^d q}{(2\pi)^d} [h(p - q)] f(p - q) \times f(p) \eta(q) \eta(g)(q). \] (3.14)
Thus
\[ h(p) (f \star g)(p) = \int \frac{d^d p}{(2\pi)^d} [h(p - q)] f(p - q) \eta(q) \eta(g)(q)] = [(h \tilde{f} \circ h \tilde{g})](p). \] (3.15)
where the convolution of two functions \( f \) and \( g \) in the coordinate space is defined in (3.6). Using (3.15), the product (3.14) is defined as
\[ (f \star g)(x) = (\tilde{h}^{-1} \circ (h \circ f))(x). \] (3.16)
Here, we have also used the relation
\[ (f \tilde{g}) = (\tilde{f} \circ \tilde{g}). \] (3.17)
It must be mentioned that as long as \( h \) is a smooth one-to-one map, the new star-algebra is an isomorphism of the algebra with the \( C^* \)-algebra. However, there is nothing to forbid the function \( h \) to be singular. In fact, we have used such a singular \( h \) to cut off the momentum of the field theory. From now on, we will identify \( h \) with \( h_\Lambda \), satisfying (2.15) and converging to a sharp cutoff function \( \Theta_\Lambda(p) \) [6]. In our case the function \( h_\Lambda \) effectively cuts down the domain of the function space. The inverse function \( h_\Lambda^{-1} \) appearing above should be understood in this context. Indeed, we do not need to worry about the appearance of \( h_\Lambda^{-1} \) in the definition (3.10). Using \( h_\Lambda(p = 0) = 1 \) in (3.14), with \( h \) identified with \( h_\Lambda \), we have
\[ \int d^d x (f \star g)(x) = \int \frac{d^d q}{(2\pi)^d} [h_\Lambda(-q)] f(q) \eta(g)(q)] = \int \frac{d^d q}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} \tilde{g}(p + q) [h_\Lambda(p)] f(p) \eta(g)(q)]. \] (3.18)
Hence, as it turns out, the integrated form of the star-product of two functions, \( f \) and \( g \), in the coordinate space can be understood as modifying the Fourier transformed of these functions with a ‘cutoff function’ \( h_\Lambda \), i.e.
\[ \tilde{f}(p) \rightarrow h_\Lambda(p) \tilde{f}(p), \quad \text{and} \quad \tilde{g}(p) \rightarrow h_\Lambda(p) \tilde{g}(p). \]
Using (3.18), it is easy to show that the effective action of cutoff QED from (3.3) can be given in terms of the translationally invariant star-product (3.10) with the specific choice of \( \theta = 0 = \eta \) in (3.13),
\[ S_{h_\Lambda} = \int d^d x \left\{ \tilde{\psi} (\gamma_\mu \partial_\mu m) \psi - e \tilde{\psi} \gamma_\mu A_\mu \psi \right\} \right. - \frac{1}{4} \frac{F^{\mu\nu}}{\xi} \left. F_{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\nu ) \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \r...
for the generating function $W[J_\mu, \chi, \tilde{\chi}]$ of the connected Green function, and
\[
-\frac{1}{\xi^2} p^\mu p_\mu h_\Lambda(p)A^\mu(p) - p_\mu \frac{\delta \Gamma}{\delta A_\mu(-p)}
+ eh_\Lambda(p) \int \frac{d^4q}{(2\pi)^4} \left[ h_\Lambda^{-1}(q)h_\Lambda(q-p)\tilde{\psi}(q-p) - \frac{\delta \Gamma}{\delta \tilde{\psi}(q)} \right]
+ h_\Lambda(q)h_\Lambda^{-1}(q-p)\frac{\delta \Gamma}{\delta \tilde{\psi}(q-p)}\tilde{\psi}(q) = 0,
\]
(3.23)
for the one-particle irreducible (1PI) vertex functions $\Gamma[A_\mu, \tilde{\psi}, \psi]$. It is known that $W[J_\mu, \chi, \tilde{\chi}]$ and $\Gamma[A_\mu, \tilde{\psi}, \psi]$ are related through the Legendre transformation
\[
\Gamma[A_\mu, \tilde{\psi}, \psi] = W[\tilde{J}_\mu, \tilde{\chi}, \tilde{\chi}] - \int \frac{d^4p}{(2\pi)^4} h_\Lambda(p) \left[ \tilde{\chi}(p)\tilde{\psi}(p) + \tilde{\psi}(p)\tilde{\chi}(p) + \tilde{J}_\mu(p)\tilde{A}^\mu(-p) \right],
\]
(3.24)
with respect to the sources $(\tilde{J}_\mu, \tilde{\chi}, \tilde{\chi})$ corresponding to the fields $(\tilde{A}_\mu, \tilde{\psi}, \tilde{\psi})$ in momentum space. The Ward identities (3.22) and (3.23) correspond to the usual Ward identities of QED with identical physical significance establishing the role of our generalized 'cutoff gauge invariance' in the renormalization group flow and the renormalization program of QED. They also guarantee the invariance of the full quantum action under the (deformed) cutoff gauge invariance (3.7)–(3.9).

At this stage some remarks on the properties of the cutoff function $h_\Lambda(p)$ are in order. First, let us note that $h_\Lambda(p)$ is to be an analytic function. Otherwise, the algebra defined by the translationally invariant star-product (3.10) is not associative (see (3.11) for the associativity condition of our deformed product). On the other hand, taking $h_\Lambda(p)$ as an arbitrary analytic function without requiring that it converges to a sharp cutoff function may lead to the interpretation that the proposed deformation (3.2) of fields is just a simple field redefinition, which is isomorphic to a point-wise product and therefore physically trivial [6]. To bypass this apparent discrepancy, the cutoff function $h_\Lambda(p)$ is to be chosen as a sequence of analytical functions which converge to a sharp UV cutoff $\Theta_\Lambda(p)$ (see (2.15)). This was recently suggested by Lizzi and Vitale in [6]. Based on the ideas proposed in the present paper, they show that the new deformed product of fields leads to a new cocommutative Hopf algebra with deformed costructure. Using a rigorous mathematical construction, they also show that taking a cutoff function $h_\Lambda(p)$ that satisfies (2.15) guarantees that the deformed Hopf algebra is inequivalent to the standard (undeformed) Hopf algebra and that the new (deformed) cutoff gauge invariance, (3.7)–(3.9), is indeed an authentic new gauge symmetry. Note that the fact that $h_\Lambda(p)$ converges to $\Theta_\Lambda(p)$ guarantees the invariance of the cutoff action (3.3) under the new (deformed) cutoff gauge invariance at each step of the limiting procedure. The question of whether this symmetry is destroyed by renormalization is negated by the explicit proof of Ward–Takahashi’s identities, which seems to arise from the standard Ward identities by a simple redefinition of fields à la (3.2).

4. The exact RG flow equation for the effective average action of cutoff $\lambda \phi^4$ theory

In a separate development an alternative RGE was derived for the effective average action [5, 9–11] to be defined below. The idea was to add an IR cutoff term
\[
\Delta S_\lambda[\phi] = \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} R_k(q)\bar{\psi}(-q)\psi(q)
\]
(4.1)
to the classical action (2.1) in the Euclidean space, and to modify in this way the standard effective action of the theory. In (4.1), the IR cutoff $R_k$ satisfies the following properties:
\[
R_k(q) \to 0 \quad \text{for} \quad k \to 0,
\]
\[
R_k(q) \to \infty \quad \text{for} \quad k \to \Lambda \text{ or } k \to \infty.
\]
(4.2)
An example of $R_k(q)$, which is also used in [11], is
\[
R_k(q) \sim \frac{q^2}{e^{q^2} - 1},
\]
(4.3)
which behaves as $R_k(q) \sim k^2$ for fluctuations with small momenta $q^2 \ll k^2$ and vanishes for $q^2 \gg k^2$. Adding $\Delta S_\lambda[\phi]$ to the classical action and integrating over all fluctuations to derive the effective action of the theory will induce automatically an effective mass $\sim k$ to those Fourier modes of $\bar{\psi}(q)$ with small momenta $q^2 \ll k^2$, prohibiting them from contributing to the effective average action of the theory, $\Gamma_k$. The resulting effective average action $\Gamma_k[\phi]$ will depend on the scale $k$ and satisfies the RG flow equation (see, e.g., [5, 9–11] for a rigorous derivation of (4.4))
\[
\frac{\partial}{\partial t} \Gamma_k[\phi] = \frac{1}{2} \text{Tr} \left[ G_k^{(2)} \tilde{\delta}_k R_k \right],
\]
(4.4)
where $\phi \equiv \langle \phi \rangle$ and $\tilde{\delta}_k \equiv k \delta_k$. The trace involves an integration over momenta. Moreover, $G_k^{(2)}$ is the full connected two-point Green function satisfying
\[
G_k^{(2)} = [\Gamma_k^{(2)} + R_k]^{-1}.
\]
(4.5)
Here, $\Gamma_k^{(2)}$ is the exact 1PI two-point vertex function, arising from the variation of the effective average action $\Gamma_k[\phi]$ two times with respect to $\phi$. By definition, the effective average action interpolates between the classical action, $\Gamma_k \approx S_0$, and the full effective action $\Gamma = \lim_{k \to 0} \Gamma_k$ [11]. In $\Gamma_k$, $\Lambda$ is a natural cutoff that characterizes the theory. The diagrammatic representation of (4.4) is presented in figure 1.

In this section, we will present the exact flow equation of the effective average action of a cutoff $\lambda \phi^4$ theory with our cutoff procedure of (2.4). Our goal is to compare the final form of the corresponding RG flow equation with (4.4). Applications of the new RG flow equation and the consequences of the new hidden gauge invariance, pointed out in section 3, will be presented elsewhere [12]. To derive the above-mentioned flow equation, let us start by considering the action (2.1) in the Euclidean space and to replace all fields $\bar{\psi}(q)$ in momentum space by $h_\lambda(q)\bar{\psi}(q)$. In contrast to the UV cutoff function $h_\lambda(p)$, which satisfies (2.15), the IR cutoff function $h_k(q)$ is considered to be a sequence of analytic
\[ \partial_t \Gamma_k[\phi] = \frac{1}{2} \]

**Figure 1.** Diagramatic representation of the exact RG flow equation (4.4). The thick black line represents the full connected two-point Green function in the presence of the additional IR cutoff \( R_k(q) \), i.e. \( G^{(2)}_k = [1 + R_k]^{-1} \). The filled green box represents the insertion of a factor \( \partial_t R_k \).

cutoff functions, converging to a sharp IR cutoff \( \Theta_{k,\Lambda}(q) \),

\[ h_{k,\Lambda}(q) \xrightarrow{\epsilon \to 0} \Theta_{k,\Lambda}(q) = \begin{cases} 1 & \text{for } k^2 < q^2 < \Lambda^2, \\ 0 & \text{for } q^2 \leq k^2. \end{cases} \quad (4.6) \]

Here, \( \Lambda \) is an arbitrary UV cutoff that cuts the UV modes with \( q > \Lambda \). Thus, in contrast to Wetterich’s method, where the IR modes, with momenta smaller than the IR cutoff \( k \), are ‘screened in a mass-like fashion’, \( m \sim k \) [5], as described above, in our case the IR modes are excluded from the theory via an analytic IR cutoff function \( h_k(q) \), satisfying (4.6). Let us note that our cutoff procedure is similar to the standard blocking procedure leading to the well-known Wegner–Houghton RGE [16] (see [17] for a review). The difference is that instead of the standard sharp IR cutoff, the IR cutoff function \( h_k(q) \) has to be an analytic function that converges to the sharp IR cutoff \( \Theta_k(q) \). This is indeed necessary, because otherwise a deformed theory with a sharp cutoff defines a deformed product that does not satisfy the desired associativity condition (3.11) (see our explanations in the previous section).

With the replacement \( \tilde{\phi}(q) \to h_k(q)\tilde{\phi}(q) \), the modified cutoff action of a \( \lambda \phi^4 \) theory reads

\[ S_k[\phi] = \frac{1}{2} \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \tilde{\mathcal{H}}^{(2)}_k(q_1, q_2) \tilde{\phi}(q_1) \tilde{\phi}(q_2) + \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_4}{(2\pi)^4} \tilde{\mathcal{H}}^{(4)}_k(q_1, ..., q_4) \tilde{\phi}(q_1) \tilde{\phi}(q_2) \tilde{\phi}(q_3) \tilde{\phi}(q_4), \quad (4.7) \]

where the cutoff functions \( \tilde{\mathcal{H}}^{(2)}_k \) and \( \tilde{\mathcal{H}}^{(4)}_k \) are defined by

\[ \tilde{\mathcal{H}}^{(2)}_k(q_1, q_2) \equiv h_k(q_1)h_k(q_2)(q_1^2 + m^2)\delta(q_1 + q_2), \]

\[ \tilde{\mathcal{H}}^{(4)}_k(q_1, q_2, q_3, q_4) \equiv \frac{\lambda}{4!} h_k(q_1)h_k(q_2)h_k(q_3)h_k(q_4) \times \delta(q_1 + q_2 + q_3 + q_4). \quad (4.8) \]

In appendix B, we will follow the method described in [9] and will derive the corresponding exact RG flow equation to the effective average action arising from (4.7). We will show that the RG flow equation of the effective average action of the cutoff \( \lambda \phi^4 \) theory is given by (see also (B.11))

\[ \frac{\partial \Gamma_k[\phi]}{\partial t} = \frac{1}{2} \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{d^4q_3}{(2\pi)^4} \left[ G^{(2)}_k(q_1, q_2) \right] \]

\[ + \tilde{\phi}(q_1)\tilde{\phi}(q_2) + \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_4}{(2\pi)^4} \left[ G^{(4)}_k(q_1, q_2, q_3, q_4) + 6G^{(2)}_k(q_1, q_2)G^{(2)}_k(q_3, q_4) \right] \times \left[ \tilde{\phi}(q_1)\tilde{\phi}(q_2)\tilde{\phi}(q_3)\tilde{\phi}(q_4) + 4G^{(3)}_k(q_1, q_2, q_3)\tilde{\phi}(q_4) + G^{(4)}_k(q_1, ..., q_4) \right], \quad (4.9) \]

where the full two-point Green functions \( G^{(2)}_k(p, q) \) are to be replaced by \( G^{(2)}_k(q) = \tilde{G}^{(2)}_k(q)\delta(p - q) \) with \( \tilde{G}^{(2)}_k(q) = [\Gamma^{(2)}_k(q)]^{-1} \). For the three- and four-point Green functions, \( \tilde{G}^{(3)}_k \) and \( \tilde{G}^{(4)}_k \) in the remaining terms of (4.9), they will be replaced by

\[ G^{(3)}_k(q_1, q_2, q_3) = -[\Gamma^{(2)}_k(q_1)]^{-1}[\Gamma^{(2)}_k(q_2)]^{-1}[\Gamma^{(2)}_k(q_3)]^{-1} \times \Gamma^{(3)}_k(q_1, q_2, q_3), \]

\[ G^{(4)}_k(q_1, ..., q_4) = -[\Gamma^{(2)}_k(q_1)]^{-1}...[\Gamma^{(2)}_k(q_3)]^{-1} \times \Gamma^{(4)}_k(q_1, q_2, q_3, q_4) + 3[\Gamma^{(2)}_k(q_1)]^{-1}...[\Gamma^{(2)}_k(q_3)]^{-1} \times \int \frac{d^4\ell}{(2\pi)^4} [\Gamma^{(2)}_k(\ell)]^{-1}\Gamma^{(3)}_k(q_1, q_2, \ell)\Gamma^{(3)}_k(\ell, q_3, q_4). \quad (4.10) \]

The graphical representation of (4.9) is demonstrated in figure 2. At this stage a couple of remarks are in order. First, let us note that in the cutoff \( \lambda \phi^4 \) theory, the relations between the n-point Green functions \( \Gamma^{(n)}_k \) and n-point vertex functions \( \Gamma^{(n)} \) are not directly affected by the cutoff function \( h_k(q) \) (for a proof, see appendix B). This is in contrast to, e.g., (4.5), where the ordinary relation between the two-point Green function and 1PI two-point vertex function is modified with an additional term including the cutoff function \( R_k(q) \). As it turns out, this is because of the multiplicativity of \( h_k(q) \), in contrast to the additive nature of \( \Delta S_k \) from (4.1), consisting of the IR cutoff function \( R_k(q) \). The second point concerns the appearance of new additional contributions in (4.9) compared to (4.4). This is because in the cutoff \( \lambda \phi^4 \) theory, \( h_k(q) \) appears not only in the kinetic part of the classical cutoff action, as in the standard derivation of (4.4), but also in the interaction part of the classical action, as all the new contributions are proportional to \( \partial_t \tilde{\mathcal{H}}^{(4)}_k \), with \( \tilde{\mathcal{H}}^{(4)}_k \) from (4.8), appearing in the interaction part of \( \Delta S_k \) from (4.7). It would be interesting to explore the practical consequences of these new terms in the RG flow equation (4.9). This will be done elsewhere [12]. Let us also note that the procedure leading to (4.9) can be easily generalized to Abelian and non-Abelian gauge theories. As we have shown in the previous section, a new hidden gauge symmetry associated with the cutoff procedure used in this paper exists, which guarantees the gauge invariance along the flow equation. This is in contrast to the situation of the standard Wetterich’s exact RGE, where...
the manifest gauge invariance is lost, because the regulator is not manifestly gauge invariant [5].

5. Conclusions

In the first part of this work we presented the procedure for introducing a momentum cutoff in a field theory by directly cutting off the momentum on each field via a sequence of analytic UV cutoff functions, \( h_\Lambda(p) \), that converge appropriately to a sharp UV cutoff \( \Theta_\Lambda(p) \) from (2.15). For QED the resulting exact renormalization group flow equation was shown to respect a generalized ‘cutoff gauge invariance’ which ensures renormalizability and unitarity without the need for an explicit calculation using the modified Ward–Takahashi identities.

We need to emphasize that although the gauge symmetry found is motivated from a noncommutative geometric setup, this symmetry is an inherent symmetry of the ordinary QED cutoff in momentum, in the spirit of Polchinski’s procedure. Our cutoff procedure may also be applied to non-Abelian gauge theories in a similar manner and the resultant exact renormalization group flow be used for the calculation of various nonperturbative quantities in QCD [5] and in gravity [13].

There is a subtle point that we would like to emphasize in conclusion and that is the singular nature of our cutoff function \( h_\Lambda(p) \). As stated earlier, this function is quite general, subject only to the restriction that it be equal to unity for momenta smaller than \( \Lambda \) and vanishing rapidly above it. Strictly speaking, the cutoff function \( h_\Lambda(p) \) is to be chosen as a sequence of analytic functions that converge to a sharp UV cutoff function \( \Theta_\Lambda(p) \) defined in (2.15). This was also recently indicated in [6]. Let us note again that \( h_\Lambda(p) \) is to be smooth, as a sharp cutoff function violates the associativity of our translationally invariant star-algebra. Moreover, the symmetries and the correct Ward–Takahashi’s identities for each fixed \( \Lambda \) are only guaranteed when \( h_\Lambda \) satisfies (2.15). In [6], it is also shown that the new deformed symmetry, proposed in the present paper, results in a new Hopf algebra, which is mathematically inequivalent to the undeformed one. This guarantees that the new (deformed) cutoff gauge invariance is a new genuine symmetry. This also resolves the puzzle that multiplication of the fields by a function \( h_\Lambda \), if it was smooth, would simply be a field redefinition of the theory and therefore physically trivial.

In the second part of the paper, we used an analytic IR cutoff function, \( h_k(q) \), that converges, as its UV counterpart, to a sharp IR cutoff function, \( \Theta_k(q) \) defined in (4.6). For this scale-dependent cutoff function, \( h_k \), the RG flow of the effective average action is derived and is shown to be different from the standard flow equation from [5, 9–11]. In [12], we will generalize the method leading to the exact RG equation of cutoff \( \lambda \phi^4 \) theory to gauge theories and will explore the practical consequences of the new generalized gauge symmetry together with the effect of new terms appearing in the RG flow equation (4.9) of the effective average action corresponding to these cutoff gauge theories.

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Appendix A. The proof of Ward identities

To prove the Ward identities of cutoff QED, we start with the generating functional for the full Green functions

\[
Z[J_\mu, \chi, \overline{\chi}] = \int D\psi D\overline{\psi} DA_\mu e^{S_{\text{tot}}},
\]

(A.1)

with \( S_{\text{tot}} = S_{\text{QED}} + S_{\text{source}} \). Here, \( S_{\text{QED}} \) is given by (3.19), where the ghost terms can be ignored, and \( S_{\text{source}} \) by

\[
S_{\text{source}} = \int d^4x \left( \overline{\chi} \psi + \overline{\psi} \chi + J_\mu \star A^\mu \right).
\]

(A.2)

Varying \( Z[J_\mu, \chi, \overline{\chi}] \) in (A.1) with respect to the star-gauge transformation (3.20) and replacing the star-products with the expression on the rhs of (3.18) to introduce the cutoff function

\[
\partial_t \Gamma_k[\phi] = \frac{1}{\Lambda^2} \frac{\partial \mathcal{H}_k^{(2)}}{\partial t} + \frac{1}{2} - 4 - 3 + 6 - 4 - 3 + 4 + 3
\]

Figure 2. Diagramatic representation of the RG flow equation of the corresponding effective average action for cutoff \( \lambda \phi^4 \) theory (4.9). The thick black line represents full two-point Green’s functions \( G_k^{(2)} = [G_k^{(1)}]^{-1} \). The filled red boxes and the small red circles represent the insertion of a factor \( \delta_k \mathcal{H}_k^{(2)} \) and \( \delta_k \mathcal{H}_k^{(4)} \), respectively. The thin lines connected to \( \otimes \) denote the background field \( \phi = \langle \phi \rangle \). The big gray and black circles are 1PI three- as well as four-point vertex functions, \( \Gamma_k^{(3)} \) as well as \( \Gamma_k^{(4)} \), respectively.
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\( h_\Lambda \), we arrive at

\[
\left\{ - \frac{1}{\xi} p^2 p^\mu h_\Lambda^2(p) \tilde{A}_\mu(p) + p^\mu h_\Lambda(p) \tilde{J}_\mu(p) \\
- e h_\Lambda(p) \int \frac{d^4 q}{(2\pi)^4} \left[ h_\Lambda(p - q) \tilde{\psi}(q - p) h_\Lambda(q) \tilde{x}(q) \\
- h_\Lambda(p - q) \tilde{x}(q) h_\Lambda(q) \tilde{\psi}(q) \right]\right\} Z = 0. \tag{A.3}
\]

Using the relations\(^{10}\)

\[
\tilde{\psi}(p) = h_\Lambda^{-2}(p) \frac{\delta Z}{i \delta \tilde{x}(p)}; \quad \tilde{\psi}(p) = - h_\Lambda^{-2}(p) \frac{\delta Z}{i \delta \tilde{\psi}(p)},
\]

\[
\tilde{A}_\mu(p) = h_\Lambda^{-2}(p) \frac{\delta Z}{i \delta J_\mu(-p)}, \tag{A.4}
\]

and replacing \( Z[J_\mu, \chi, \tilde{\chi}] \) by \( Z = e^{W} \), with \( W[J_\mu, \chi, \tilde{\chi}] \) the generating function of connected Green’s functions, we arrive at

\[
- \frac{1}{\xi} p^2 p_\mu \frac{\delta W}{\delta J_\mu(-p)} + h_\Lambda^{2}(p) p_\mu \tilde{J}_\mu(p) \\
+ e h_\Lambda(p) \int \frac{d^4 q}{(2\pi)^4} \left[ h_\Lambda(q) h_\Lambda^{-1}(q - p) \frac{\delta W}{\delta \tilde{\psi}(q - p)} \tilde{x}(q) \\
+ h_\Lambda^{-1}(q) h_\Lambda(q - p) \frac{\delta W}{\delta \tilde{x}(q)} \tilde{\psi}(q) \right] = 0. \tag{A.5}
\]

To derive (A.4), \( h_\Lambda(p) = h_\Lambda(-p) \) is used. To arrive at the Ward identity in terms of \( [A_\mu, \psi, \tilde{\psi}] \), the generating functional for 1PI Green’s function, we use the Legendre transformation (3.24) leading to

\[
\frac{\delta W}{\delta J_\mu(-p)} = h_\Lambda^2(p) \tilde{A}_\mu(p), \quad \frac{\delta W}{\delta \tilde{x}(p)} = -h_\Lambda^2(p) \tilde{\psi}(p),
\]

\[
\frac{\delta W}{\delta \tilde{\psi}(p)} = h_\Lambda^2(p) \tilde{\psi}(p), \quad \tilde{J}_\mu(p) = -h_\Lambda^{-2}(p) \frac{\delta \Gamma}{\delta A^{\mu}(-p)}, \tag{A.6}
\]

\[
\tilde{x}(p) = -h_\Lambda^{-2}(p) \frac{\delta \Gamma}{\delta \tilde{\psi}(p)}, \quad \tilde{\chi}(p) = h_\Lambda^{-2}(p) \frac{\delta \Gamma}{\delta \tilde{x}(p)}. \tag{A.7}
\]

Plugging these relations in (A.5), we arrive at

\[
- \frac{1}{\xi} p^2 p_\mu h_\Lambda^2(p) A^{\mu}(p - p_\mu \frac{\delta \Gamma}{\delta A^{\mu}(-p)} + e h_\Lambda(p) \\
\times \int \frac{d^4 q}{(2\pi)^4} \left[ h_\Lambda^{-1}(q) h_\Lambda(q - p) \tilde{\psi}(q - p) \frac{\delta \Gamma}{\delta \tilde{\psi}(q)} \\
+ h_\Lambda(q) h_\Lambda^{-1}(q - p) \frac{\delta \Gamma}{\delta \tilde{\psi}(q)} \tilde{\psi}(q) \right] = 0. \tag{A.7}
\]

As a first example on the application of (A.5), let us differentiate it with respect to \( J_\mu(p') \) and set eventually \( J_\mu = \tilde{x} = \tilde{\chi} = 0 \). We arrive at the Ward identity for the full photon propagator of the cutoff QED in momentum space, \( \tilde{D}_A^{\mu
u} \),

\[
i \xi p^2 p_\mu h_\Lambda^2(p) \tilde{D}_A^{\mu
u}(p) = p^\nu. \tag{A.8}
\]

Here, we have used (A.6) to obtain first

\[
\frac{\delta W}{\delta J_\mu(-p) \delta \tilde{J}_\nu(p)} \bigg|_{j_\mu = \tilde{\chi} = \tilde{\psi} = 0} = h_\Lambda^4(p) (\tilde{A}^{\mu}(p) \tilde{A}^{\nu}(-p)), \tag{A.9}
\]

and defined the full cutoff-dependent photon propagator \( \tilde{D}_A^{\mu
u}(p) \) by

\[
i \tilde{D}_A^{\mu
u}(p) \delta(p - p') \equiv (\tilde{A}^{\mu}(p) \tilde{A}^{\nu}(-p')). \tag{A.10}
\]

Equation (A.8) is, in particular, satisfied by the tree level photon propagator of cutoff QED [15]

\[
\tilde{D}_A^{\mu
u}(p) \bigg|_{\text{tree-level}} = - \frac{i h^2}{p^2} \left( g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right). \tag{A.11}
\]

As a second example, let us differentiate (A.7) with respect to \( \tilde{\psi}(-\ell) \) and \( \tilde{\psi}(k) \) and set eventually \( \tilde{\mu} = \tilde{\psi} = \tilde{\psi} = 0 \). Using (A.6), we arrive at

\[
\frac{1}{h_\Lambda(p) h_\Lambda(\ell + p) h_\Lambda(\ell)} p_\mu \tilde{D}_A^{\mu}(\ell - p, -\ell; -p) \\
= e \left[ h_\Lambda^{-2}(\ell + p) \tilde{S}_A^{-1}(\ell + p) - h_\Lambda^{-2}(\ell) \tilde{S}_A^{-1}(\ell) \right]. \tag{A.12}
\]

where the 1PI three-point vertex function

\[
\tilde{\Gamma}_A^{\mu}(k, \ell; -p) \delta(k + p + \ell) \equiv \frac{\delta \Gamma}{\delta \tilde{\psi}(k) \delta \tilde{\psi}(-\ell) \delta \tilde{A}_\mu(-p)} \bigg|_{\tilde{\psi} = \tilde{\chi} = 0}, \tag{A.13}
\]

as well as the fermionic 1PI two-point function at finite cutoff \( \Lambda \)

\[
\tilde{S}_A^{-1}(\ell) \delta(k + \ell) \equiv \frac{\delta \Gamma}{\delta \tilde{\psi}(k) \delta \tilde{\psi}(-\ell)} \bigg|_{\tilde{\psi} = \tilde{\chi} = 0}, \tag{A.14}
\]

are introduced. Taking the limit \( p \to 0 \) in (A.12) and using \( h_\Lambda(0) = 1 \), we arrive at the standard relation

\[
\tilde{\Gamma}_A^{\mu}(\ell, -\ell; 0) = e \frac{\delta \tilde{S}_A^{-1}(\ell)}{\delta \tilde{\mu}}. \tag{A.15}
\]

Assuming that \( h_\Lambda(p) \) is a nearly constant function for \(|p| < \Lambda \), and using\(^{11}\)

\[
\tilde{S}_A^{-1}(\ell) = h_\Lambda^2(\ell) \tilde{S}_A^{-1}(\ell),
\]

\[
\tilde{\Gamma}_A^{\mu}(k, \ell; p) = h_\Lambda(k) h_\Lambda(\ell) h_\Lambda(p) \tilde{\Gamma}_A^{\mu}(k, \ell; p), \tag{A.16}
\]

we obtain

\[
\tilde{\Gamma}_A^{\mu}(\ell, -\ell; 0) = e \frac{\delta \tilde{S}_A^{-1}(\ell)}{\delta \tilde{\mu}}, \tag{A.17}
\]

where \( \tilde{S}_A \) and \( \tilde{\Gamma}_A^{\mu} \) are the 1PI two- and three-point vertex functions of QED in the \( \Lambda \to \infty \) limit. Assuming at this stage that (A.17) is also valid for renormalized Green’s functions \( \tilde{I}^{\mu
u} = Z_2^{-1} Z_3^{-1/2} \tilde{I}_\mu^{\nu} \) and \( \tilde{S}_A^{-1} = Z_2^{-1} \tilde{S}_A^{-1} \), as well as for renormalized coupling \( e_\Lambda \equiv Z_1 Z_2^{-1/2} e \), with \( Z_1 \), \( Z_2 \) and \( Z_3 \) the renormalization constants corresponding to the vertex

\(^{11}\) Relations (A.16) are shown to be valid at one-loop level (see [15] for more details).
function, fermion and photon propagators, respectively, we arrive at $Z_1 = Z_2$.

Appendix B. The exact RG flow equation of cutoff $\lambda \phi^4$ theory

Let us start by considering the bare action of $\lambda \phi^4$ theory in Euclidean space (2.1). As we have explained in section 4, each field $\phi(q)$ shall be replaced by $h_k(q)\phi(q)$, where $k$ is the renormalization scale. The modified classical action is then given by (4.7) with the cutoff functions given in (4.8). The corresponding generating functional of this cutoff theory then reads

$$Z_k[J] = \int D\phi \exp \left( -S_k[\phi] + \int \frac{d^4q}{(2\pi)^4} h_k^2(q)J(-q)\phi(q) \right),$$  
(B.1)

where $h_k(-q) = h_k(q)$ is assumed. The Legendre transformation between $W_k[J] \equiv \ln Z_k[J]$, the generating functional of the connected Green function, and the 1PI effective average action, $\Gamma_k[\phi]$, is given by

$$\Gamma_k[\phi] = -W_k[J] + \int \frac{d^4q}{(2\pi)^4} h_k^2(q)J(-q)\phi(q).$$  
(B.2)

where $\phi \equiv \langle \phi \rangle$, and

$$\frac{\delta W_k[J]}{\delta J(-q)} = h_k^2(q)\phi(q).$$  
(B.3)

It is the purpose of this appendix to derive the scale dependence of $\Gamma_k[\phi]$. To do this, we will follow the method described in [5, 9, 11]. First, we differentiate $\Gamma_k[\phi]$ from (B.2) with respect to $k$ and arrive at

$$\frac{\partial \Gamma_k[\phi]}{\partial k} = -\frac{\partial W_k[J]}{\partial k} + 2 \int \frac{d^4q}{(2\pi)^4} h_k(q)\frac{\partial h_k(q)}{\partial k}J(-q)\phi(q)$$

$$= \left\{ \frac{\partial S_k[\phi]}{\partial k} \right\},$$  
(B.4)

where (B.3) and the standard notation

$$\langle O[\phi] \rangle = Z_k^{-1}[J] \int D\phi O[\phi]$$

$$\times \exp \left( -S_k[\phi] + \int \frac{d^4q}{(2\pi)^4} h_k^2(q)J(-q)\phi(q) \right)$$  
(B.5)

are used. Plugging now (4.7) in (B.4), we obtain

$$\frac{\partial \Gamma_k[\phi]}{\partial k} = \left\{ \frac{\partial S_k[\phi]}{\partial k} \right\}$$

$$= \frac{1}{2} \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{\partial Z_k^2(q_1, q_2)}{\partial k} \langle \hat{\phi}(q_1)\hat{\phi}(q_2) \rangle$$

$$+ \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{\partial Z_k^4(q_1, \ldots, q_4)}{\partial k} \times \langle \hat{\phi}(q_1)\hat{\phi}(q_2)\hat{\phi}(q_3)\hat{\phi}(q_4) \rangle.$$  
(B.6)

At this stage we shall replace the two- and four-point Green functions appearing on the rhs of (B.6) by a combination of connected and disconnected Green’s functions. To do this, let us vary $W_k[J] = \ln Z_k[J]$ two times with respect to $J$ to obtain

$$\frac{\delta^2 W_k[J]}{\delta J(-q_1)\delta J(-q_2)} = -\frac{1}{Z_k^2[J]} \frac{\delta Z_k[J]}{\delta J(-q_1)} \frac{\delta Z_k[J]}{\delta J(-q_2)}$$

$$+ \frac{1}{Z_k[J]} \frac{\delta^2 Z_k[J]}{\delta J(-q_1)\delta J(-q_2)}$$

$$= h_k^2(q_1)h_k^2(q_2)(-\langle \hat{\phi}(q_1)\hat{\phi}(q_2) \rangle + \langle \hat{\phi}(q_1)\hat{\phi}(q_2) \rangle).$$  
(B.7)

Here, the definition of $Z_k[J]$ from (B.1) is used. Defining then the connected $n$-point Green functions as

$$\frac{\delta^n W_k[J]}{\delta J(-q_1)\ldots J(-q_n)} = h_k^2(q_1)\ldots h_k^2(q_n)G_k^{(n)}(q_1, \ldots, q_n),$$  
(B.8)

and plugging the corresponding relation for $n = 2$ on the lhs of (B.7), we obtain the standard relation

$$\langle \hat{\phi}(q_1)\hat{\phi}(q_2) \rangle = G_k^{(2)}(q_1, q_2) + \hat{\phi}(q_1)\hat{\phi}(q_2).$$  
(B.9)

A similar relation exists also between the connected four-point Green function $G_k^{(4)}(q_1, \ldots, q_4)$ and $\langle \hat{\phi}(q_1)\hat{\phi}(q_2)\hat{\phi}(q_3)\hat{\phi}(q_4) \rangle$ appearing on the rhs of (B.6). It is given by

$$\langle \hat{\phi}(q_1)\hat{\phi}(q_2)\hat{\phi}(q_3)\hat{\phi}(q_4) \rangle = G_k^{(4)}(q_1, q_2, q_3, q_4)$$

$$+ G_k^{(3)}(q_1, q_2, q_3)\hat{\phi}(q_4) + G_k^{(3)}(q_1, q_3, q_4)\hat{\phi}(q_2) + G_k^{(3)}(q_1, q_2, q_4)\hat{\phi}(q_3) + G_k^{(3)}(q_1, q_3, q_2)\hat{\phi}(q_4) + G_k^{(3)}(q_1, q_4, q_3)\hat{\phi}(q_2) + G_k^{(3)}(q_1, q_2, q_4)\hat{\phi}(q_3)$$

$$+ G_k^{(2)}(q_1, q_2)\hat{\phi}(q_3)\hat{\phi}(q_4) + G_k^{(2)}(q_1, q_3)\hat{\phi}(q_2)\hat{\phi}(q_4) + G_k^{(2)}(q_1, q_4)\hat{\phi}(q_2)\hat{\phi}(q_3) + G_k^{(2)}(q_2, q_3)\hat{\phi}(q_1)\hat{\phi}(q_4) + G_k^{(2)}(q_2, q_4)\hat{\phi}(q_1)\hat{\phi}(q_3) + G_k^{(2)}(q_3, q_4)\hat{\phi}(q_1)\hat{\phi}(q_2)$$

$$+ G_k^{(2)}(q_1, q_2)G_k^{(2)}(q_3, q_4) + G_k^{(2)}(q_1, q_3)G_k^{(2)}(q_2, q_4) + G_k^{(2)}(q_1, q_4)G_k^{(2)}(q_2, q_3)$$

$$+ G_k^{(2)}(q_2, q_3)G_k^{(2)}(q_1, q_4) + G_k^{(2)}(q_2, q_4)G_k^{(2)}(q_1, q_3) + G_k^{(2)}(q_3, q_4)G_k^{(2)}(q_1, q_2) + \hat{\phi}(q_1)\hat{\phi}(q_2)\hat{\phi}(q_3)\hat{\phi}(q_4).$$  
(B.10)

Plugging (B.9) and (B.10) in (B.6) and using the symmetry of $H_k^{(2)}(q_1, q_2)$ and $H_k^{(4)}(q_1, \ldots, q_4)$ under permutation of $q_i, i = 1, \ldots, 4$, we arrive at the flow equation of $\Gamma_k[\phi]$ in terms of the connected $n = 1, \ldots, 4$-point Green functions

$$\frac{\partial \Gamma_k[\phi]}{\partial t} = \frac{1}{2} \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{\partial H_k^{(2)}(q_1, q_2)}{\partial k}$$

$$\times [\hat{\phi}(q_1)\hat{\phi}(q_2) + \hat{\phi}(q_2)\hat{\phi}(q_1)]$$

$$+ \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{\partial H_k^{(4)}(q_1, \ldots, q_4)}{\partial k}$$

$$\times [\hat{\phi}(q_1)\hat{\phi}(q_2)\hat{\phi}(q_3)\hat{\phi}(q_4) + 3G_k^{(2)}(q_1, q_2)G_k^{(2)}(q_3, q_4)$$

$$+ 6G_k^{(2)}(q_1, q_2)G_k^{(2)}(q_3, q_4) + 4G_k^{(3)}(q_1, q_2, q_3)\hat{\phi}(q_4) + G_k^{(4)}(q_1, \ldots, q_4)].$$  
(B.11)
This flow equation is to be compared with the standard flow equation \((4.4)\), where only a term similar to the first term on the rhs of \((B.11)\) appears. The appearance of additional terms in \((B.11)\), including the contributions of \(G_k^{(n)}(q_1, \ldots, q_n)\), \(n = 1, \ldots, 4\), is, in particular, a consequence of the replacement of \(\phi(q)\) by \(h_k(q)\phi(q)\) in the interaction term of the original classical action, in contrast to the standard procedure \([5, 9–11]\).

In a last step, we shall use the relations between the connected \(n\)-point Green function, \(G_k^{(n)}(q_1, \ldots, q_n)\), defined in \((B.8)\) and the IPI \(n\)-point vertex function, \(\Gamma_k^{(n)}(q_1, \ldots, q_n)\), defined by

\[
\Gamma_k^{(n)}(q_1, \ldots, q_n) \equiv \frac{\delta^2 \Gamma_k[\phi]}{\delta \phi(q_1) \ldots \delta \phi(q_n)},
\]

\((B.12)\)

to replace \(G_k^{(n)}(q_1, \ldots, q_n)\), \(n = 1, \ldots, 4\), in \((B.11)\) by the corresponding expressions in terms of \(\Gamma_k^{(n)}(q_1, \ldots, q_n)\). To do this, let us first consider the relation

\[
\int \frac{d^4\ell_1}{(2\pi)^4} \frac{\delta \tilde{J}(-q_1)}{\delta \phi(\ell_1)} = \delta(q_1 - q_2).
\]

\((B.13)\)

It can easily be shown that

\[
\frac{\delta \tilde{J}(-q_1)}{\delta \phi(\ell_1)} = h_k^{-2}(q_1)\Gamma_k^{(2)}(q_1, \ell_1), \quad \text{and}
\]

\[
\frac{\delta \tilde{\phi}(\ell_1)}{\delta \delta J(-q_2)} = h_k^2(q_2)G_k^{(2)}(\ell_1, q_2).
\]

\((B.14)\)

Plugging \((B.14)\) in \((B.13)\), and using \(G_k^{(2)}(\ell_1, q_2) = G_k^{(2)}(q_2)\delta(\ell_1 - q_2)\) as well as \(\Gamma_k^{(2)}(q_1, \ell_1) = \Gamma_k^{(2)}(q_1)\delta(\ell_1 - \ell_1)\), we arrive, after integrating over \(\ell_1\), at

\[
G_k^{(2)}(q)\Gamma_k^{(2)}(q) = 1.
\]

\((B.15)\)

This is the standard relation between \(G_k^{(2)}(q)\) and \(\Gamma_k^{(2)}(q)\). In contrast to \((4.5)\), the cutoff function \(h_k(q)\) does not appear in \((B.15)\). This is because of the multiplicative nature of the cutoff function \(h_k(q)\). A similar relation can also be derived between \(G_k^{(3)}(q)\) and \(\Gamma_k^{(3)}\). It is simplify given by

\[
G_k^{(3)}(q_1, q_2, q_3) = -\{\Gamma_k^{(2)}(q_1)\}^{-1}\{\Gamma_k^{(2)}(q_2)\}^{-1}\{\Gamma_k^{(2)}(q_3)\}^{-1}
\times \Gamma_k^{(3)}(q_1, q_2, q_3),
\]

\((B.16)\)

which is derived by differentiating \((B.13)\) with respect to \(\tilde{J}(-q_1)\), and plugging \((B.14)\) as well as

\[
\frac{\delta^2 \tilde{J}(-q_1)}{\delta \tilde{\phi}(\ell_1) \delta \tilde{J}(-q_3)} = h_k^{-2}(q_1)h_k^2(q_3) \int \frac{d^4\ell_2}{(2\pi)^4}
\]

\[
\times G_k^{(2)}(\ell_2, q_3)\Gamma_k^{(3)}(q_1, \ell_1, \ell_2),
\]

\[
\frac{\delta^2 \tilde{\phi}(\ell_1)}{\delta \delta J(-q_2) \delta \delta J(-q_3)} = h_k^2(q_2)h_k^2(q_3)G_k^{(3)}(\ell_1, q_2, q_3)
\]

\((B.17)\)

in the resulting expression. Similarly, to derive the relation between \(G_k^{(4)}(q)\) and \(\Gamma_k^{(4)}(q)\), we differentiate \((B.13)\) with respect to \(\tilde{J}(-q_3)\) and \(\tilde{J}(-q_4)\). Plugging \((B.14), (B.17)\) and

\[
\frac{\delta^3 \tilde{J}(-q_1)}{\delta \delta \phi(\ell_1) \delta \tilde{J}(-q_3) \delta \tilde{J}(-q_4)} = h_k^{-2}(q_1)h_k^2(q_3)h_k^2(q_4)
\]

\[
\times \int \frac{d^4\ell_2}{(2\pi)^4}G_k^{(2)}(\ell_2, q_3)\Gamma_k^{(3)}(q_1, \ell_1, \ell_2)
\]

\[
+ \int \frac{d^4\ell_2}{(2\pi)^4} \frac{d^4\ell_4}{(2\pi)^4}G_k^{(2)}(\ell_2, q_3)G_k^{(2)}(\ell_4, q_4)\Gamma_k^{(4)}(q_1, \ell_1, \ell_2, \ell_3)
\]

\[
\frac{\delta^3 \tilde{\phi}(\ell_1)}{\delta \delta \delta J(-q_2) \delta \delta \delta J(-q_3) \delta \delta \delta J(-q_4)} = h_k^2(q_2)h_k^2(q_3)h_k^2(q_4)
\]

\[
\times G_k^{(4)}(\ell_1, q_2, q_3, q_4)
\]

\((B.18)\)

in the resulting expression, we arrive after some algebra at

\[
G_k^{(4)}(q_1, \ldots, q_4) = -\{\Gamma_k^{(2)}(q_1)\}^{-1}\{\Gamma_k^{(2)}(q_2)\}^{-1}\{\Gamma_k^{(2)}(q_3)\}^{-1}
\times \left\{\Gamma_k^{(3)}(q_1, q_2, q_3) - \int \frac{d^4\ell}{(2\pi)^4}G_k^{(2)}(\ell)\right\}^{-1}
\times \Gamma_k^{(3)}(q_1, q_2, q_3)
\]

\((B.19)\)

where \((B.15)\) and \((B.16)\) are also used. Plugging \((B.15), (B.16)\) and \((B.19)\) in the flow equation \((B.11)\) and using the symmetry of the cutoff function \(h_k^2(q_1, \ldots, q_4)\) under permutation of \(q_i, i = 1, \ldots, 4\), we arrive at \((4.9)\).

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