Translation-invariant quasi-free states for fermionic systems and the BCS approximation

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Abstract

We study translation-invariant quasi-free states for a system of fermions with two-particle interactions. The associated energy functional is similar to the BCS functional but includes also direct and exchange energies. We show that for suitable short-range interactions, these latter terms only lead to a renormalization of the chemical potential, with the usual properties of the BCS functional left unchanged. Our analysis thus represents a rigorous justification of part of the BCS approximation. We give bounds on the critical temperature below which the system displays superfluidity.

1 Introduction and Main Results

The BCS theory \cite{BCS} was introduced in 1957 to describe superconductivity, and was later extended to the context of superfluidity \cite{BFM, S} as a microscopic description of fermionic gases with local pair interactions at low temperatures. It can be deduced from quantum physics in three steps. One restricts the allowed states of the system to quasi-free states, assumes translation-invariance and $SU(2)$ rotation invariance, and finally dismisses the direct and exchange terms in the energy. With these approximations, the resulting BCS functional depends, besides the temperature $T$ and the chemical potential $\mu$, on the interaction potential $V$, the momentum distribution $\gamma$ and the Cooper pair wave function $\alpha$. A non-vanishing $\alpha$ implies a macroscopic coherence of the particles involved, i.e., the formation of a condensate of Cooper pairs. This motivates the characterization of a superfluid phase by the existence of a minimizer of the BCS functional for which $\alpha \neq 0$.

A rigorous treatment of the BCS functional was presented in \cite{EF, FIS, Hainzl}, where the question was addressed for which interaction potentials $V$ and at which temperatures $T$ a superfluid phase exists. In the present work, we focus on the question to what extent it is justifiable to dismiss the direct and exchange terms in the energy. A heuristic justification was given in \cite{BFM, Hainzl}, where it was argued that as long as the range of the interaction potential is suitably small, the only effect of the direct and exchange terms is to renormalize the chemical potential.

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In this paper we derive a gap equation for the extended theory with direct and exchange terms and investigate the existence of non-trivial solutions for general interaction potentials. We give a rigorous justification for dismissing the two terms for potentials whose range $\ell$ is short compared to the scattering length $a$ and the Fermi wave length $\frac{2\pi}{\sqrt{3\mu}}$. The potentials are required to have a suitable repulsive core to assure stability of the system. We show that, for small enough $\ell$, the system still can be described by the conventional BCS equation if the chemical potential is renormalized appropriately. In the limit $\ell \to 0$, the spectral gap function $\Delta_{\ell}(p)$ converges to a constant function and we recover the BCS equation in its form found in the physics literature.

While we do not prove that for fixed, finite $\ell$ there exists a critical temperature $T_c$ such that superfluidity occurs if and only if $T < T_c$, we find bounds $T^+_{\ell}$ and $T^-_{\ell}$ such that $T < T^-_{\ell}$ implies superfluidity and $T > T^+_{\ell}$ excludes superfluidity. Moreover, in the limit $\ell \to 0$ the two bounds converge to the same temperature, $\lim_{\ell \to 0} T^-_{\ell} = \lim_{\ell \to 0} T^+_{\ell}$, which can be determined by the usual BCS gap equation. The situation is illustrated in the following sketch.

![Sketch](image)

We note that similar models as the one considered in our paper are sometimes referred to as Bogoliubov-Hartree-Fock theory and have been studied previously mainly with Newtonian interactions, modeling stars, and without the restriction to translation-invariant states [3, 17, 10]. The proof of existence of a minimizer in [17] turns out to be surprisingly difficult and even more strikingly, the appearance of pairing is still open. It was confirmed numerically for the Newton model and also for models with short range interaction in [18]. Hence the present work represents the first proof of existence of pairing in a translation-invariant Bogoliubov-Hartree-Fock model in the continuum. For the Hubbard model at half filling this was shown earlier in [4].

1.1 The Model

We consider a gas of spin 1/2 fermions in the thermodynamic limit at temperature $T \geq 0$ and chemical potential $\mu \in \mathbb{R}$. The particles interact via a local two-body potential which we denote by $V$. We assume $V$ to be reflection-symmetric, i.e., $V(-x) = V(x)$. The state of the system is described by two functions $\hat{\gamma} : \mathbb{R}^3 \to \mathbb{R}_+$ and $\hat{\alpha} : \mathbb{R}^3 \to \mathbb{C}$, with $\hat{\alpha}(p) = \hat{\alpha}(-p)$, which are conveniently combined into a $2 \times 2$ matrix

$$
\Gamma(p) = \begin{pmatrix}
\hat{\gamma}(p) & \hat{\alpha}(p) \\
\hat{\alpha}(p) & 1 - \hat{\gamma}(-p)
\end{pmatrix},
$$

required to satisfy $0 \leq \Gamma \leq 1_{\mathbb{C}^2}$ at every point $p \in \mathbb{R}^3$. The function $\hat{\gamma}$ is interpreted as the momentum distribution of the gas, while $\hat{\alpha}$ (the inverse Fourier transform of $\hat{\alpha}$) is the Cooper pair wave function. Note that there are no spin variables in $\Gamma$; the full, spin dependent Cooper pair wave function is the product of $\alpha(x - y)$ with an antisymmetric spin singlet.

The BCS-HF functional $F^V_{\Gamma}$, whose infimum over all states $\Gamma$ describes the negative of the pressure
of the system, is given as

\[ F_V^T(\Gamma) = \int_{\mathbb{R}^3} (p^2 - \mu) \hat{\gamma}(p) \, d^3p + \int_{\mathbb{R}^3} |\alpha(x)|^2 V(x) \, d^3x - TS(\Gamma) \]

\[ - \int_{\mathbb{R}^3} |\gamma(x)|^2 V(x) \, d^3x + 2\gamma(0)^2 \int_{\mathbb{R}^3} V(x) \, d^3x, \]

where

\[ S(\Gamma) = - \int_{\mathbb{R}^3} \text{Tr}_{C^2} (\Gamma(p) \ln \Gamma(p)) \, d^3p \]

is the entropy of the state \( \Gamma \). Here, \( \gamma \) and \( \alpha \) denote the inverse Fourier transforms of \( \hat{\gamma} \) and \( \hat{\alpha} \), respectively. The last two terms in (1.2) are referred to as the exchange term and the direct term, respectively. The functional (1.2) can be obtained by restricting the many-body problem on Fock space to translation-invariant and spin-rotation invariant quasi-free states, see [9, Appendix A] and [4]. The factor 2 in the last term in (1.2) originates from two possible orientations of the particle spin.

A normal state \( \Gamma_0 \) is a minimizer of the functional (1.2) restricted to states with \( \alpha = 0 \). Any such minimizer can easily be shown to be of the form

\[ \hat{\gamma}_0(p) = \frac{1}{1 + e^{\varepsilon(p) - \mu}}, \]

where we denote, for general \( \gamma \),

\[ \varepsilon^\gamma(p) = p^2 - \frac{2}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \left( \hat{V}(p-q) - \hat{V}(0) \right) \hat{\gamma}(q) \, d^3q, \]

\[ \tilde{\mu}^\gamma = \mu - \frac{2}{(2\pi)^{3/2}} \hat{V}(0) \int_{\mathbb{R}^3} \hat{\gamma}(p) \, d^3p. \]

In the absence of the exchange term the normal state would be unique, but this is not necessarily the case here. The system is said to be in a superfluid phase if and only if the minimum of \( F_V^T \) is not attained at a normal state, and we call a normal state \( \Gamma_0 \) unstable in this case.

### 1.2 Main Results

Our first goal is to characterize the existence of a superfluid phase for a large class of interaction potentials \( V \). We first find sufficient conditions on \( V \) for (1.2) to have a minimizer. These conditions are stated in the following proposition.

**Proposition 1** (Existence of minimizers). Let \( \mu \in \mathbb{R} \), \( 0 \leq T < \infty \), and let \( V \in L^1(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3) \) be real-valued with \( ||\hat{V}||_{\infty} \leq 2\hat{V}(0) \). Then \( F_V^T \) is bounded from below and attains a minimizer \( (\gamma, \alpha) \) on

\[ D = \left\{ \Gamma \text{ of the form (1.1)} \mid \gamma \in L^1(\mathbb{R}^3, (1+p^2) \, d^3p), \alpha \in H^1(\mathbb{R}^3, d^3x), 0 \leq \Gamma \leq 1_{C^2} \right\}. \]

Moreover, the function

\[ \Delta(p) = \frac{2}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \hat{V}(p-q) \hat{\alpha}(q) \, d^3q \]

satisfies the BCS gap equation

\[ \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \hat{V}(p-q) \frac{\Delta(q)}{K_{\gamma^*}^\alpha(q)} \, d^3q = -\Delta(p) \]
In (1.6) we have introduced the notation
\[ K_{T,\mu}^{\gamma,\Delta}(p) = \frac{E_{0,\mu}^{\gamma,\Delta}(p)}{\tanh \left( \frac{E_{0,\mu}^{\gamma,\Delta}(p)}{2T} \right)}, \]
(1.7)
\[ E_{\mu}^{\gamma,\Delta}(p) = \sqrt{(\varepsilon(p) - \bar{\mu})^2 + |\Delta(p)|^2}, \]
(1.8)
with \( \varepsilon \) and \( \bar{\mu} \) defined in (1.3) and (1.4), respectively. For \( T = 0 \), (1.7) is interpreted as \( K_{0,\mu}^{\gamma,\Delta}(p) = E_{\mu}^{\gamma,\Delta}(p) \).

We note that the BCS gap equation (1.6) can equivalently be written as
\[ (K_{T,\mu}^{\gamma,\Delta} + V)\hat{\alpha} = 0, \]
where \( K_{T,\mu}^{\gamma,\Delta} \) is interpreted as a multiplication operator in Fourier space, and \( V \) as multiplication operator in configuration space. This form of the equation will turn out to be useful later on.

Proposition 1 shows that the condition \( \|V\|_\infty \leq 2\hat{V}(0) \) is sufficient for stability of the system. The simplicity of this criterion is due to the restriction to translation-invariant quasi-free states. Without imposing translation-invariance, the question of stability is much more subtle. Note that \( \mathcal{F}_V \) is not bounded from below for negative \( V \), in contrast to the BCS model (where the direct and exchange terms are neglected).

Proposition 1 gives no information on whether \( \Delta \neq 0 \). A sufficient condition for this to happen is given in the following theorem.

**Theorem 1** (Existence of a superfluid phase). Let \( \mu \in \mathbb{R}, 0 \leq T < \infty \), and let \( V \in L^1(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3) \) be real-valued with \( \|V\|_\infty \leq 2\hat{V}(0) \). Let \( \Gamma_0 = (\gamma_0, 0) \) be a normal state and recall the definition of \( K_{T,\mu}^{\gamma_0,0}(p) \) in (1.7), (1.8).

(i) If \( \inf \text{spec}(K_{T,\mu}^{\gamma_0,0} + V) < 0 \), then \( \Gamma_0 \) is unstable, i.e., \( \inf_{\Gamma \in \mathcal{D}} \mathcal{F}_V(\Gamma) < \mathcal{F}_V(\Gamma_0) \).

(ii) If \( \Gamma_0 \) is unstable, then there exist \( (\gamma, \alpha) \in \mathcal{D} \), with \( \alpha \neq 0 \), such that \( \Delta \) defined in (1.5) solves the BCS gap equation (1.6).

The theorem follows from the following arguments. The operator \( K_{T,\mu}^{\gamma_0,0} + V \) naturally appears when looking at the second derivative of \( t \mapsto \mathcal{F}_V(\Gamma_0 + t\Gamma) \) at \( t = 0 \). If it has a negative eigenvalues, the second derivative is negative for suitable \( \Gamma \), hence \( \Gamma_0 \) is unstable. On the other hand, an unstable normal state implies the existence of a minimizer with \( \alpha \neq 0 \), which satisfies the Euler-Lagrange equations for \( \mathcal{F}_V \), resulting in (1.6) according to Proposition 1. The details are given in Section 2.1.

**Remark 1.** In the usual BCS model, where the direct and exchange terms are neglected, the existence of a non-trivial solution to \((K_{T,\mu}^{\gamma_0,0} + V)\hat{\alpha} = 0\) implies the existence of a negative eigenvalue of \( K_{T,\mu}^{\gamma_0,0} + V \) \[ \text{[Theorem 1]}. \] This follows from the fact that \( K_{T,\mu}^{\gamma_0,0} \) is monotone in \( \Delta \), i.e., \( K_{T,\mu}^{\gamma_0,0}(p) > K_{T,\mu}^{\gamma_0,0}(\hat{V}(0)) \) for \( \Delta \neq 0 \). In particular, the system is superfluid if and only if the operator \( K_{T,\mu}^{\gamma_0,0} + V \) has a negative eigenvalue. Since this operator is monotone in \( T \), the equation
\[ \inf \text{spec}(K_{T,\mu}^{\gamma_0,0} + V) = 0 \]
determines the critical temperature. In the model considered here, where the direct and exchange terms are not neglected, the situation is more complicated. Due to the additional dependence of \( K_{T,\mu}^{\gamma,\Delta} \) on \( \gamma \), we can no longer conclude that \( K_{T,\mu}^{\gamma,\Delta}(p) > K_{T,\mu}^{\gamma_0,0}(p) \). But by Theorem 1 the solution \( T \) of
\[ \inf \text{spec}(K_{T,\mu}^{\gamma_0,0} + V) = 0 \] still remains a lower bound for the critical temperature.
Our main result concerns the case of short-range interaction potentials $V$, where we can recover monotonicity in $\Delta$, and hence conclude that (1.9) indeed defines the correct critical temperature. More precisely, we shall consider a sequence of potentials $\{V^\ell\}_{\ell>0}$ with $\ell \to 0$, which satisfies the following assumptions.

**Assumption 1.** (A1) $V^\ell \in L^1 \cap L^2$

(A2) the range of $V^\ell$ is at most $\ell$, i.e., $\text{supp } V^\ell \subseteq B_\ell(0)$

(A3) the scattering length $a(V^\ell)$ is negative and does not vanish as $\ell \to 0$, i.e., $\lim_{\ell \to 0} a(V^\ell) = a < 0$

(A4) $\limsup_{\ell \to 0} \|V^\ell\|_1 < \infty$

(A5) $\hat{V^\ell}(0) > 0$ and $\lim_{\ell \to 0} \hat{V^\ell}(0) = \mathcal{V} \geq 0$

(A6) $\|\hat{V^\ell}\|_\infty \leq 2\hat{V^\ell}(0)$

(A7) for small $\ell$, $\|V^\ell\|_2 \leq C_1 \ell^{-N}$ for some $C_1 > 0$ and $N \in \mathbb{N}$

(A8) $\exists 0 < b < 1$ such that $\inf \text{spec}(p^2 + \frac{V^\ell}{2} - |p|^b) > C_2 > -\infty$ holds independently of $\ell$

(A9) the operator $1 + \frac{V^\ell}{2} \frac{1}{p^2} |V^\ell|^{1/2}$ is invertible, and has an eigenvalue $e_\ell$ of order $\ell$, with corresponding eigenvector $\phi_\ell$. Moreover, $(1 + \frac{V^\ell}{2} \frac{1}{p^2} |V^\ell|^{1/2})^{-1}(1 - P_\ell)$ is uniformly bounded in $\ell$, where $\quad P_\ell = \langle J_\ell \phi_\ell | \phi_\ell \rangle \frac{1}{\|\phi_\ell\|}$ and $J_\ell = \text{sgn}(V^\ell)$

(A10) the eigenvector $\phi_\ell$ satisfies $|\langle \phi_\ell | \text{sgn}(V^\ell) \phi_\ell \rangle|^{-1}(\|V^\ell\|_2^2)^{-1} \leq O(\ell^{1/2})$ for small $\ell$.

Here we use the notation $\text{sgn}(V) = \{ 1, V \geq 0 \}$ and $V^{1/2}(x) = \text{sgn}(V)|V(x)|^{1/2}$. As discussed in [12], the scattering length of a real-valued potential $V \in L^1(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3)$ is given by

$$a(V) = \frac{1}{4\pi} \left\langle |V|^{1/2} \left| \frac{1}{1+|V|^{1/2}} V^{1/2} \right| V^{1/2} \right\rangle.$$  \hspace{1cm} (1.10)

Assumptions (A6)–(A10) are to some extent technical and are needed, among other things, to guarantee that $\mathcal{F}^\ell_V$ is bounded from below uniformly in $\ell$. Our main results presumably hold for a larger class of potentials with less restrictive assumptions, but to avoid additional complications in the proofs we do not aim here for the greatest possible generality. Assumption (A10) implies, in particular, that $V^\ell$ converges to a point interaction as $\ell \to 0$, and we refer to [6] for a general study of point interactions arising as limits of short-range potentials of the form considered here.

**Remark 2.** As an example for such a sequence of short-range potentials $V^\ell$ we have the following picture in mind: 

![Diagram](image-url)
The attractive part allows to adjust the scattering length. The repulsive core is needed to guarantee stability, and can be used to adjust the $L^1$ norm. If its range is small compared to the range of the attractive part, i.e., $\epsilon_\ell \ll \ell$, the scattering length is essentially unaffected by the repulsive core. In Appendix A we construct an explicit example of such a sequence, satisfying all the assumptions (A1)–(A10). As $\ell \to 0$, it approximates a contact potential, defined via suitable selfadjoint extensions of $-\Delta$ on $\mathbb{R}^3 \setminus \{0\}$. Functions in its domain are known to diverge as $|x|^{-1}$ for small $x$, hence decay like $p^{-2}$ for large $|p|$. This suggests the validity of (AS) for $b < 1$. Assumption (A9) is easy to show in case $V_\ell$ is uniformly bounded in $L^{3/2}$ (in which case $\hat{V}_\ell(0) = O(\ell)$) but much harder to prove if $\lim_{\ell \to 0} \hat{V}_\ell(0) > 0$. It is possible to generalize (A9) and allow finitely many eigenvalues of $\mathbb{1} + V_\ell^{1/2} \frac{1}{|p|} V_\ell^{1/2}$ of order $\ell$. For simplicity we restrict to the case of only one eigenvalue of order $\ell$, however.

For the remainder of this section, we assume that the sequence $V_\ell$ satisfies (A1)–(A10). We shall use the notation

$$\tilde{\mu}^{\gamma_\ell} = \mu - \frac{2}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \gamma_\ell(p) \, d^3p$$

in analogy to (1.4).

**Theorem 2** (Effective Gap equation). Let $T \geq 0$, $\mu \in \mathbb{R}$, and let $(\hat{\gamma}_\ell, \hat{\alpha}_\ell)$ be a minimizer of $\mathcal{F}^V_\ell$ with corresponding $\Delta_\ell = 2(2\pi)^{-3/2} \hat{V}_\ell * \hat{\alpha}_\ell$. Then there exist $\Delta \geq 0$ and $\hat{\gamma} : \mathbb{R}^3 \to \mathbb{R}_+$ such that $|\Delta_\ell(p)| \to \Delta$ pointwise, $\hat{\gamma}_\ell(p) \to \hat{\gamma}(p)$ pointwise and $\tilde{\mu}^{\gamma_\ell} \to \tilde{\mu}$ as $\ell \to 0$, satisfying

$$\tilde{\mu} = \mu - \frac{2V}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \hat{\gamma}(p) \, d^3p$$

$$\hat{\gamma}(p) = \frac{1}{2} - \frac{p^2 - \tilde{\mu}}{2K_{T,\tilde{\mu}}(p)}.$$

(1.11)

If $\Delta_\ell \neq 0$ for a subsequence of $\ell$’s going to zero, then, in addition,

$$\frac{1}{4\pi a} = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \left( \frac{1}{K_{T,\tilde{\mu}}^\Delta} - \frac{1}{p^2} \right) \, d^3p.$$

(1.12)

Recall that, according to our definitions (1.3)–(1.8),

$$K_{T,\tilde{\mu}}^0(\Delta)(p) = \frac{E_0^{\Delta}(p)}{\tanh \left( \frac{E_0^{\Delta}(p)}{2T} \right)}; \quad E_\tilde{\mu}^\Delta(p) = \sqrt{(p^2 - \tilde{\mu})^2 + |\Delta|^2}.$$

**Remark 3.** If we consider potentials such that $\hat{V}_\ell(0) \to 0$, we obtain at the same time that $\tilde{\mu}^{\gamma_\ell} \to \mu$ and consequently (1.12) becomes

$$\frac{1}{4\pi a} = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \left( \frac{1}{K_{T,\mu}^\Delta} - \frac{1}{p^2} \right) \, d^3p.$$

(1.13)

Equation (1.13) is the form of the BCS gap equation one finds in the literature, see for instance [15].

The effective gap equation (1.12) suggests to define the critical temperature of the system via the solution of (1.12) for $\Delta = 0$, in which case $\hat{\gamma}$ is given by $(1 + e^{\frac{2\pi k}{\lambda}})^{-1}$. 

6
Definition 1 (Critical temperature / renormalized chemical potential). Let $\mu > 0$. The critical temperature $T_c$ and the renormalized chemical potential $\tilde{\mu}$ in the limit of a contact potential with scattering length $a < 0$ and $\lim_{\ell \to 0} \hat{V}_\ell(0) = V \geq 0$ are implicitly given by the set of equations

\[
-\frac{1}{4\pi a} = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \left( \frac{\tanh \left( \frac{p^2 + \tilde{\mu}}{2T_c} \right)}{p^2 - \tilde{\mu}} - \frac{1}{p^2} \right) d^3p, \\
\tilde{\mu} = \mu - \frac{2V}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \frac{1}{1 + e^{\frac{p^2}{\gamma} - \tilde{\mu}}} d^3p.
\] (1.14)

We will show existence and uniqueness of $T_c$ and $\tilde{\mu}$ in Appendix B. Note that it is essential that $\mu > 0$. If $\mu \leq 0$, then $\tilde{\mu} \leq 0$ and hence the right side of the first equation in (1.14) is always non-positive, hence there is no solutions for $a < 0$. In other words, $T_c = 0$ for $\mu \leq 0$.

Remark 4. In [13], the behavior of the first integral on the right side of (1.14) as $T_c \to 0$ was examined. This allows one to deduce the asymptotic behavior of $T_c$ as $a$ tends to zero, which equals

\[ T_c = \tilde{\mu} \left( \frac{8}{\pi} e^{\gamma - 2} + o(1) \right) e^{\pi a}, \]

with $\gamma \approx 0.577$ denoting Euler’s constant. Similarly, one can study the asymptotic behavior as $\mu \to 0$.

Although this definition for $T_c$ is only valid in the limit $\ell \to 0$, it serves to make statements about upper and lower bounds on the critical temperature for small (but non-zero) $\ell$, as sketched in the figure on page 2.

Theorem 3 (Bounds on critical temperature). Let $\mu \in \mathbb{R}$, $T \geq 0$ and let $(\gamma_0^\ell, 0)$ be a normal state for $F_{V_\ell}^\gamma$.

(i) For $T < T_c$, there exists an $\ell_0(T) > 0$ such that for $\ell < \ell_0(T)$, $\inf \text{spec}(K_{T,\mu}^{\gamma_0^\ell} + V_\ell) < 0$. Consequently, the system is superfluid.

(ii) For $T > T_c$, there exists an $\ell_0(T) > 0$ such that for $\ell < \ell_0(T)$, $F_{V_\ell}^{\gamma_0^\ell}$ is minimized by a normal state. I.e., the system is not superfluid.

Theorem 3 shows that Definition 1 is indeed the correct definition of the critical temperature in the limit $\ell \to 0$. In addition, it also shows that in this limit there is actually equivalence of statements (i) and (ii) in Theorem 1. In particular, one recovers the linear criterion for the existence of a superfluid phase valid in the usual BCS model, as discussed in Remark 1.

2 Proofs

2.1 General Potentials

In this section we prove Proposition 1 and Theorem 1. As a first step we show that $F_{V_\ell}^{\gamma_0}$ is bounded from below and has a minimizer.

Lemma 1. Let $V \in L^1 \cap L^{3/2}$, $\hat{V}(0) \geq 0$ and $\hat{V}(p) \leq 2\hat{V}(0)$ for all $p \in \mathbb{R}^3$. Then $F_{V_\ell}^{\gamma_0}$ is bounded from below and there exist a minimizer $\Gamma$ of $F_{V_\ell}^{\gamma_0}(\Gamma)$.
Proof. The case without direct and exchange term was treated in [9, Proposition 2]. The Hartree-Fock part of the functional $F_{VT}$ gives the additional contribution

$$-\int_{\mathbb{R}^3} |\gamma(x)|^2 V(x) \, d^3x + 2\gamma(0)^2 \int_{\mathbb{R}^3} V(x) \, d^3x = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \gamma(2\hat{V}(0) - \hat{V}) \ast \hat{\gamma} \, d^3p,$$

which is non-negative because of our assumption $\hat{V}(p) \leq 2\hat{V}(0)$. Hence the same lower bound as in the case without direct and exchange term applies.

To show the existence of a minimizer, it remains to check the weak lower semicontinuity of $F_{VT}$ in $L^q(\mathbb{R}^3) \times H^1(\mathbb{R}^3, d^3x)$ (note that $\hat{\gamma} \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$). The exchange term $\gamma \mapsto \int_{\mathbb{R}^3} V(x) |\gamma(x)|^2 \, d^3x$ is actually weakly continuous on $H^1(\mathbb{R}^3)$, see, e.g., [19, Thm. 11.4]. Since also $\lim_{n\to\infty} \int_{\mathbb{R}^3} \hat{\gamma}_n \, d^3p \geq \int_{\mathbb{R}^3} \hat{\gamma} \, d^3p$, the direct term is weakly lower semicontinuous. In the proof of [9, Proposition 2] it was shown that all other terms in $F_{VT}$ are weakly lower semicontinuous as well. As a consequence, a minimizing sequence will actually converge to a minimizer.

Lemma 2. The Euler-Lagrange equations for a minimizer $(\gamma, \alpha)$ of $F_{VT}$ are of the form

$$\hat{\gamma}(p) = \frac{1}{2} - \frac{\varepsilon \gamma(p) - \bar{\mu} \gamma(p)}{2K_{\gamma,\mu}(p)}$$

(2.1)

$$\hat{\alpha}(p) = \frac{1}{2} \Delta(p) \tanh \left( \frac{E_{\gamma,\Delta}(p)}{2\Delta} \right),$$

(2.2)

where we used the abbreviations introduced in (1.3) – (1.8). In particular, the BCS gap equation (1.6) holds.

Proof. The proof works similar to [9]. We sketch here an alternative, more concise derivation, restricting our attention to $T > 0$ for simplicity. A minimizer $\Gamma = (\gamma, \alpha)$ of $F_{VT}$ fulfills the inequality

$$0 \leq \frac{d}{dt} \bigg|_{t=0} F_{VT}(\Gamma + t(\tilde{\Gamma} - \Gamma))$$

(2.3)

for arbitrary $\tilde{\Gamma} \in \mathcal{D}$. Here we may assume that $\Gamma$ stays away from 0 and 1 by arguing as in [9, Proof of Lemma 1]. A simple calculation using

$$S(\Gamma) = -\int_{\mathbb{R}^3} \text{Tr}_{\mathbb{C}^2} \Gamma \ln \Gamma \, d^3p = -\frac{1}{2} \int_{\mathbb{R}^3} \text{Tr}_{\mathbb{C}^2} \left( \Gamma \ln(\Gamma) + (1 - \Gamma) \ln(1 - \Gamma) \right) \, d^3p$$

shows that

$$\frac{d}{dt} \bigg|_{t=0} F_{VT}(\Gamma + t(\tilde{\Gamma} - \Gamma)) = \frac{1}{2} \int_{\mathbb{R}^3} \text{Tr}_{\mathbb{C}^2} \left( H_{\Delta}(\tilde{\Gamma} - \Gamma) \right) \, d^3p + T(\tilde{\Gamma} - \Gamma) \ln \left( \frac{\Gamma}{\tilde{\Gamma} - \Gamma} \right) \right] \, d^3p,$$

with

$$H_{\Delta} = \begin{pmatrix} \varepsilon \gamma - \bar{\mu} \gamma & \Delta \\ \Delta & -\varepsilon \gamma + \bar{\mu} \gamma \end{pmatrix},$$

using the definition

$$\Delta = 2(2\pi)^{-3/2} \hat{V} \ast \hat{\alpha}.$$
Separating the terms containing no \( \tilde{\Gamma} \) and moving them to the left side in (2.3), we obtain
\[
\int_{\mathbb{R}^3} \text{Tr}_{C^2} \left( H_\Delta (\Gamma - (0 \ 0 \ 1)) + T \Gamma \ln \left( \frac{\Gamma}{1 - \Gamma} \right) \right) \, d^3 p \leq \int_{\mathbb{R}^3} \text{Tr}_{C^2} \left( H_\Delta (\tilde{\Gamma} - (0 \ 0 \ 1)) + T \tilde{\Gamma} \ln \left( \frac{\Gamma}{1 - \Gamma} \right) \right) \, d^3 p.
\]
Note that \( \int_{\mathbb{R}^3} \text{Tr}_{C^2} \left( H_\Delta \Gamma \right) \, d^3 p \) is not finite but \( \int_{\mathbb{R}^3} \text{Tr}_{C^2} \left( H_\Delta (\tilde{\Gamma} - (0 \ 0 \ 1)) \right) \, d^3 p \) is. Since \( \tilde{\Gamma} \) was arbitrary, \( \Gamma \) also minimizes the linear functional \( \tilde{\Gamma} \mapsto \int_{\mathbb{R}^3} \text{Tr}_{C^2} \left( H_\Delta \left( \tilde{\Gamma} - (0 \ 0 \ 1) \right) \right) \, d^3 p \), whose Euler-Lagrange equation is of the simple form
\[
0 = H_\Delta + T \ln \left( \frac{\Gamma}{1 - \Gamma} \right),
\]
which is equivalent to
\[
\Gamma = \frac{1}{1 + e^{\frac{1}{T} H_\Delta}}.
\]
This in turn implies (2.1) and (2.2). Indeed, \( H_\Delta = |E_{\mu}^{\gamma, \Delta}|^2 I_{C^2} \) and, therefore,
\[
e^{\frac{1}{T} H_\Delta} = \cosh \left( \frac{1}{T} E_{\mu}^{\gamma, \Delta} \right) I_{C^2} + \frac{1}{E_{\mu}^{\gamma, \Delta}} \sinh \left( \frac{1}{T} E_{\mu}^{\gamma, \Delta} \right) H_\Delta.
\]
With the relations
\[
1 + \cosh(x) = \frac{\sinh(x)}{\tanh(x/2)},
\]
\[
1 - \cosh(x) = -\tanh(x/2) \sinh(x),
\]
we see that
\[
(1 + e^{\frac{1}{T} H_\Delta}) H_\Delta = -K_{T, \mu}^{\gamma, \Delta} (1 - e^{\frac{1}{T} H_\Delta}), \quad \text{where} \quad K_{T, \mu}^{\gamma, \Delta} = \frac{E_{\mu}^{\gamma, \Delta}}{\tanh \left( \frac{1}{2T} E_{\mu}^{\gamma, \Delta} \right)}.
\]
Consequently,
\[
\Gamma = \frac{1}{1 + e^{\frac{1}{T} H_\Delta}} = \frac{1}{2} + \frac{1}{2} \frac{1 - e^{\frac{1}{T} H_\Delta}}{1 + e^{\frac{1}{T} H_\Delta}} = \frac{1}{2} - \frac{1}{2K_{T, \mu}^{\gamma, \Delta}} H_\Delta = \left( \frac{1}{2} - \frac{\varepsilon - \mu^{\gamma}}{2K_{T, \mu}^{\gamma, \Delta}} \right) - \frac{\Delta}{2K_{T, \mu}^{\gamma, \Delta}} \left( \frac{1}{2} + \frac{\varepsilon - \mu^{\gamma}}{2K_{T, \mu}^{\gamma, \Delta}} \right) \right).
\]

**Proof of Proposition 1.** This is an immediate consequence of Lemmas 1 and 2.

**Proof of Theorem 1.** The proof works exactly as the analog steps in [9] Proof of Theorem 1. To see (i) note that \( \langle \hat{\alpha}, (K_{T, \mu}^{\gamma, 0} + V) \hat{\alpha} \rangle \) is the second derivative of \( F_{\mu}^{\gamma, 0} \) with respect to \( \alpha \) at \( \Gamma = \Gamma_0 \). For (ii), we use the fact that the gap equation is a combination of the Euler-Lagrange equation (2.2) of the functional and the definition of \( \Delta \).
2.2 Sequence of Short-Range Potentials

In the following we consider a sequence of potentials \( V_\ell \) satisfying the assumptions (A1)–(A10) in Assumption [1]. Since \( V_\ell \) converges to a contact potential, Lemma [1] is not sufficient to prove that \( \mathcal{F}_T^{V_\ell} \) is uniformly bounded from below. To this aim, we have to use more subtle estimates involving bounds on the relative entropy obtained in [8], and we heavily rely on assumption (A8).

**Lemma 3.** There exists \( C_1 > 0 \), independent of \( \ell \), such that

\[
\mathcal{F}_T^{V_\ell}(\Gamma) \geq -C_1 + \frac{1}{2} \int_{\mathbb{R}^3} (1 + p^2)(\hat{\gamma} - \hat{\gamma}_0)^2 \, d^3p + \frac{1}{2} \int_{\mathbb{R}^3} |p| \hat{\alpha}^2 \, d^3p,
\]

where we denote \( \hat{\gamma}_0(p) = \frac{1}{1 + e(p^2 - \mu)/T} \).

**Proof.** We rewrite \( \mathcal{F}_T^{V_\ell}(\Gamma) \) as

\[
\mathcal{F}_T^{V_\ell}(\Gamma) = \frac{1}{2} \int_{\mathbb{R}^3} \text{Tr}_{C^2} \left( H_0(\Gamma - (0, 0, 0)) \right) \, d^3p + \int_{\mathbb{R}^3} V_\ell(x)|\alpha(x)|^2 \, d^3x
- T S(\Gamma) + \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \left( (2\hat{V}_\ell(0) - \hat{V}_\ell) \ast \hat{\gamma} \right)(p) \hat{\gamma}(p) \, d^3p,
\]

where \( \Gamma = \Gamma(\gamma, \alpha) \) and

\[
H_0 = \begin{pmatrix}
  p^2 - \mu & 0 \\
  0 & -(p^2 - \mu)
\end{pmatrix}.
\]

Since \( \hat{\gamma}(p) \geq 0 \) and, by assumption (A6) \( 2\hat{V}_\ell(0) - \hat{V}_\ell(p) \geq 0 \), the combination of direct plus exchange term is non-negative and it suffices to find a lower bound for

\[
\mathcal{F}_T^{V_\ell}(\Gamma) = \frac{1}{2} \int_{\mathbb{R}^3} \text{Tr}_{C^2} \left( H_0(\Gamma - (0, 0, 0)) \right) \, d^3p + \int_{\mathbb{R}^3} V_\ell(x)|\alpha(x)|^2 \, d^3x - T S(\Gamma).
\]

We compare \( \mathcal{F}_T^{V_\ell}(\Gamma) \) to the value \( \mathcal{F}_T^{V_\ell}(\Gamma_0) \), where \( \Gamma_0 = \frac{1}{1 + e^{p_0/T}} \). Their difference equals

\[
\mathcal{F}_T^{V_\ell}(\Gamma) - \mathcal{F}_T^{V_\ell}(\Gamma_0) = \frac{1}{2} \int_{\mathbb{R}^3} \text{Tr}_{C^2} \left( H_0(\Gamma - \Gamma_0) \right) \, d^3p - T \left( S(\Gamma) - S(\Gamma_0) \right)
+ \int_{\mathbb{R}^3} V_\ell(x)|\alpha(x)|^2 \, d^3x.
\]

Using \( H_0 + T \ln\left( \frac{p_0}{1 - \mu} \right) = 0 \) in the trace and performing some simple algebraic transformations, we may write

\[
\mathcal{F}_T^{V_\ell}(\Gamma) - \mathcal{F}_T^{V_\ell}(\Gamma_0) = \frac{T}{2} \mathcal{H}(\Gamma, \Gamma_0) + \int_{\mathbb{R}^3} V_\ell(x)|\alpha(x)|^2 \, d^3x,
\]

where

\[
\mathcal{H}(\Gamma, \Gamma_0) = \int_{\mathbb{R}^3} \text{Tr}_{C^2} \left[ \Gamma(\ln(\Gamma) - \ln(\Gamma_0)) + (1 - \Gamma)(\ln(1 - \Gamma) - \ln(1 - \Gamma_0)) \right] \, d^3p
\]

denotes the relative entropy of \( \Gamma \) and \( \Gamma_0 \). Lemma 3 in [3], which is an extension of Theorem 1 in [11], implies the lower bound

\[
\frac{T}{2} \mathcal{H}(\Gamma, \Gamma_0) \geq \frac{1}{2} \int_{\mathbb{R}^3} \text{Tr}_{C^2} \left[ \frac{H_0}{\tanh(H_0/2T)}(\Gamma - \Gamma_0)^2 \right] \, d^3p
= \int_{\mathbb{R}^3} K_0^{0,0}(p)(|\hat{\gamma}(p) - \hat{\gamma}_0(p)|^2 + |\hat{\alpha}(p)|^2) \, d^3p.
\]
Hence we obtain
\[ \tilde{F}_T^V(\Gamma) - \tilde{F}_T^V(\Gamma_0) \geq \left\langle \alpha \left| K_{T,\mu}^{0,0} + V_\ell \right| \alpha \right\rangle + \int_{\mathbb{R}^3} K_{T,\mu}^{0,0}(\hat{\gamma}(p) - \hat{\gamma}_0(p))^2 d^3p. \]

In both terms, we can use \( K_{T,\mu}^{0,0} \geq p^2 - \mu \), therefore
\[ \tilde{F}_T^V(\Gamma) - \tilde{F}_T^V(\Gamma_0) \geq \left\langle \alpha \left| p^2 + V_\ell - \mu \right| \alpha \right\rangle + \frac{1}{2} \int_{\mathbb{R}^3} (1 + p^2)(\hat{\gamma} - \hat{\gamma}_0)^2 d^3p + \int_{\mathbb{R}^3} \left( \frac{p^2}{2} - \mu - \frac{1}{2} \right)(\hat{\gamma} - \hat{\gamma}_0)^2 d^3p. \]

Using \( (\hat{\gamma} - \hat{\gamma}_0)^2 \leq 1 \), we can bound
\[ \int_{\mathbb{R}^3} \left( \frac{p^2}{2} - \mu - \frac{1}{2} \right)(\hat{\gamma} - \hat{\gamma}_0)^2 d^3p \geq - \int_{\mathbb{R}^3} \left[ \frac{p^2}{2} - \mu - \frac{1}{2} \right] d^3p, \]
where \( |t|_\mu = \max\{0, -t\} \) denote the negative part of a real number \( t \). By assumption \([A8]\), inf \( \text{spec}(p^2 + V_\ell - |p|^b) \) is bounded by some number \( C \) independent of \( \ell \). Thus
\[ \int_{\mathbb{R}^3} \left( \frac{p^2}{2} + V_\ell - \mu \right)|\hat{\alpha}|^2 d^3p \geq \int_{\mathbb{R}^3} (|p|^b + C - \mu)|\hat{\alpha}|^2 d^3p = \frac{1}{2} \int_{\mathbb{R}^3} |p|^b|\hat{\alpha}|^2 d^3p + \int_{\mathbb{R}^3} \left( \frac{|p|^b}{2} + C - \mu \right)|\hat{\alpha}|^2 d^3p. \]

With \( |\hat{\alpha}|^2 \leq 1 \) conclude
\[ \int_{\mathbb{R}^3} \left( \frac{|p|^b}{2} + C - \mu \right)|\hat{\alpha}|^2 d^3p \geq - \int_{\mathbb{R}^3} \left[ \frac{|p|^b}{2} - \mu + C \right] d^3p. \]

Our final lower bound is thus
\[ F_T^V(\Gamma) \geq \tilde{F}_T^V(\Gamma) \geq \tilde{F}_T^V(\Gamma_0) - C_1 \int_{\mathbb{R}^3} \left( 1 + p^2 \right)(\hat{\gamma} - \hat{\gamma}_0)^2 d^3p + \frac{1}{2} \int_{\mathbb{R}^3} |p|^b|\hat{\alpha}|^2 d^3p, \]
with
\[ C_1 = - \tilde{F}_T^V(\Gamma_0) + \int_{\mathbb{R}^3} \left[ \frac{p^2}{2} - \mu - \frac{1}{2} \right] d^3p + \int_{\mathbb{R}^3} \left[ \frac{|p|^b}{2} - \mu + C \right] d^3p. \]

Since \( \tilde{F}_T^V(\Gamma_0) \) does not depend on \( \ell \) (the off-diagonal entries of \( \Gamma_0 \) being 0) this concludes the proof.

**Lemma 4.** If \((\gamma_\ell, \alpha_\ell)\) is a minimizer of \( F_T^V \), then \( \int_{\mathbb{R}^3} \hat{\gamma}_\ell(p)|p|^b d^3p \) is uniformly bounded in \( \ell \).

**Proof.** To simplify notation, we leave out the index \( \ell \). A minimizer \((\gamma, \alpha)\) of \( F_T^V \) satisfies the Euler-Lagrange equation \([2.1]\). Using the abbreviation
\[ K_{T,\mu}^{\gamma,\Delta} = \frac{E_{\mu,\gamma}^{\Delta}(p)}{\tanh \left( \frac{E_{\mu,\gamma}^{\Delta}(p)}{2T} \right)}, \]
we may express \([2.1]\) in the form
\[ \hat{\gamma} = \frac{1}{2} - \frac{1}{2} \varepsilon\gamma - \frac{1}{2} K_{T,\mu}^{\gamma,\Delta}. \]
Adding and subtracting $\frac{1}{2} E_{\gamma, \Delta}^\gamma = \frac{1}{2} \tanh \left( \frac{E_{\gamma, \Delta}^\gamma (p)}{2T} \right)$, we may write
\begin{equation}
\begin{aligned}
\hat{\gamma} &= \frac{1}{2} \left( 1 - \tanh \left( \frac{E_{\gamma, \Delta}^\gamma (p)}{2T} \right) \right) + \frac{1}{2} \frac{E_{\gamma, \Delta}^\gamma (p)}{K_{T, \mu}^\gamma \\
&= \frac{1}{1 + e^{\frac{E_{\gamma, \Delta}^\gamma (p)}{2T}}} + \frac{1}{2} \frac{|\Delta|^2}{E_{\gamma, \Delta}^\gamma (p) + (\varepsilon^\gamma - \tilde{\mu}^\gamma) K_{T, \mu}^\gamma}\end{aligned}
\end{equation}

Using the Euler-Lagrange equation $\Delta = 2K_{T, \mu}^\gamma \hat{\alpha}$ for $\alpha$, we obtain
\begin{equation}
\hat{\gamma} = \frac{1}{1 + e^{\frac{E_{\gamma, \Delta}^\gamma (p)}{2T}}} + 2 \frac{|\hat{\alpha}|^2 K_{T, \mu}^\gamma}{E_{\gamma, \Delta}^\gamma (p) + (\varepsilon^\gamma - \tilde{\mu}^\gamma)}.
\end{equation}

Assumption [A6] implies that $\varepsilon^\gamma - \tilde{\mu}^\gamma \geq p^2 - \mu$. In particular, the contribution of the first term is bounded by
\[\int_{\mathbb{R}^3} \frac{1}{1 + e^{\frac{E_{\gamma, \Delta}^\gamma (p)}{2T}}} |p|^b \, d^3p \leq \int_{\mathbb{R}^3} \frac{1}{1 + e^{\frac{E_{\gamma, \Delta}^\gamma (p)}{2T}}} |p|^b \, d^3p\]
which is independent of $\ell$. To treat the second term, we split the domain of integration $\mathbb{R}^3$ into two disjoint sets and show that the integral is uniformly bounded on each subset. On the set $B = \{ |p| \tanh(\frac{E_{\gamma, \Delta}^\gamma (p)}{2T}) \geq \frac{2}{3} \}$ we have that $\tanh(\frac{E_{\gamma, \Delta}^\gamma (p)}{2T}) \geq \frac{2}{3}$ and $\varepsilon^\gamma - \tilde{\mu}^\gamma \geq 0$. This implies that
\[\frac{|\hat{\alpha}|^2 K_{T, \mu}^\gamma}{E_{\gamma, \Delta}^\gamma (p) + (\varepsilon^\gamma - \tilde{\mu}^\gamma)} \leq \frac{3}{2} |\hat{\alpha}|^2,
\]
whose integral over $B$ is bounded uniformly in $\ell$ by (2.5), even after multiplication by $|p|^b$. The complement $B^c = \{ |p| \tanh(\frac{E_{\gamma, \Delta}^\gamma (p)}{2T}) \leq \frac{2}{3} \}$ of $B$ is compact and thus also $\int_{B^c} \hat{\gamma}(p)|p|^b \, d^3p$ is trivially bounded, because $0 \leq \hat{\gamma} \leq 1$.

In the following lemma we show that, as $\ell \to 0$, pointwise limits for the main quantities exist. In the case of $\Delta_\ell$, observe that $\Delta_\ell(x) = 2V_\ell(x)\alpha_\ell(x)$ is supported in $|x| \leq \ell$. Heuristically, if the norm $||\Delta_\ell||_1$ stays finite, $\Delta_\ell$ should converge to a $\delta$ distribution and its Fourier transform $\Delta_\ell$ to a constant function. While we do not show that $||\Delta_\ell||_1$ stays finite, we can use assumption [A7] to at least show that it cannot increase too fast as $\ell \to 0$, which will turn out to be sufficient. The pointwise convergence $\gamma_\ell(p) \to \gamma(p)$ then follows from Lemma 4 together with the Euler-Lagrange equation (2.1) for $\gamma_\ell$.

In the following, we use the definition
\[m_{\gamma_\ell, \Delta_\ell}(T) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \left( \frac{1}{K_{T, \mu}^{\gamma_\ell, \Delta_\ell} - \frac{1}{p^2}} \right) \, d^3p.
\]

**Lemma 5.** Let $(\gamma_\ell, \alpha_\ell)$ be a sequence of minimizers of $F_{T, \mu}^{\gamma_\ell}$ and $\Delta_\ell = 2(2\pi)^{-3/2} \hat{V}_\ell \ast \hat{\alpha}_\ell$. Then there are subsequences of $\gamma_\ell$ and $\alpha_\ell$, which we continue to denote by $\gamma_\ell$ and $\alpha_\ell$, and $\gamma \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, $\Delta \in \mathbb{R}_+$, such that

(i) $|\Delta_\ell(p)|$ converges pointwise to the constant function $\Delta$ as $\ell \to 0$,

(ii) $\lim_{\ell \to 0} \int_{\mathbb{R}^3} \hat{\gamma_\ell} \, d^3p = \int_{\mathbb{R}^3} \hat{\gamma} \, d^3p$,
The fact that 
\[ \lim_{\ell \to 0} \tilde{\pi}^\gamma = \mu^\gamma, \text{ where } \tilde{\pi}^\gamma = \mu - 2(2\pi)^{-3/2} \int_{\mathbb{R}^3} \tilde{\gamma}(p) \, d^3p, \]

(iv) \( \varepsilon^\gamma(p) \to p^2 \text{ pointwise as } \ell \to 0, \)

(v) \( \gamma_\ell(p) \to \gamma(p) \text{ pointwise as } \ell \to 0, \) and Eq. (1.11) is satisfied for \( \gamma, \tilde{\mu}^\gamma, \Delta, \)

(vi) \( \lim_{\ell \to 0} m^\gamma_{\mu} \Delta_T(T) = m^\gamma_{\mu}(T) = m^{0, \Delta}(T). \)

We shall see later that it is not necessary to restrict to a subsequence, the result holds in fact for the whole sequence.

**Proof.** (i) Lemma 3 and Assumption (A7) imply that, with \( \tilde{\Delta}_\ell = 2V_\ell \alpha_\ell, \)

\[ \|\tilde{\Delta}_\ell\|_1 \leq 2\|V_\ell\|_2\|\alpha_\ell\|_2 \leq C\ell^{-N}. \]  \( (2.8) \)

The fact that \( \tilde{\Delta}_\ell \) is compactly supported in \( B_\ell(0) \) will allow us to argue that a suitable subsequence of \( \Delta_\ell(p) \) converges to a polynomial in \( p. \) Furthermore, the fact that \( \alpha_\ell = -2(K_{T,\mu}^{\gamma, \Delta_T})^{-1} \Delta_T \) is uniformly bounded in \( L^2 \) forces the polynomial to be a constant.

We denote by

\[ P_{\ell,N}(p) = \frac{1}{(2\pi)^{3/2}} \sum_{j=0}^{N} (-i)^j j! \sum_{i_1, \ldots, i_j = 1}^{3} c_{i_1, \ldots, i_j}^{(j)} p_{i_1} \cdots p_{i_j} \]

the \( N \)-th order Taylor approximation of \( \Delta_\ell(p) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} \Delta_\ell(x)e^{-ip\cdot x} \, d^3x \) at \( p = 0, \) with coefficients given by

\[ c_{i_1, \ldots, i_j}^{(j)} = \int_{\mathbb{R}^3} \Delta_\ell(x) x_{i_1} \cdots x_{i_j} \, d^3x. \]

Using that \( \tilde{\Delta}_\ell \) is supported in \( B_\ell(0) \) we may estimate the remainder term as

\[ |\Delta_\ell(p) - P_{\ell,N}(p)| = \frac{1}{(2\pi)^{3/2}} \left| \int_{\mathbb{R}^3} \Delta_\ell(x) \left( e^{-ip\cdot x} - \sum_{j=0}^{N} (-i\cdot x)^j/j! \right) \, d^3x \right| \leq \frac{\ell^{N+1}}{(2\pi)^{3/2}} \|\tilde{\Delta}_\ell\|_1|p|^{N+1} e^{\ell|p|}, \]

which goes to zero pointwise for \( \ell \to 0 \) by (2.8).

Now let \( \bar{c}_\ell = \max_{0 \leq j \leq N} \max_{1 \leq i_1, \ldots, i_j \leq 3} \{|c_{i_1, \ldots, i_j}^{(j)}|\}. \) We want to show that \( \bar{c} = \limsup_{\ell \to 0} \bar{c}_\ell < \infty. \) If \( \bar{c} = 0, \) we are done. If not, then there is a subsequence of \( P_{\ell,N}(p)/\bar{c}_\ell \) which converges pointwise to some polynomial \( P(p) \) of degree \( n \leq N. \) We now use the uniform boundedness of \( 2\|\alpha_\ell\|_2 = \|\Delta_\ell / K_{T,\mu}^{\gamma, \Delta_T}\|_2 \)

to conclude that \( P(p) \) cannot be a polynomial of degree \( n \geq 1, \) and that \( \bar{c} \) is finite. We first rewrite \( 2\alpha_\ell \) as

\[ \frac{\Delta_\ell}{K_{T,\mu}^{\gamma, \Delta_T}} = \frac{\Delta_\ell}{E_{\mu}^{\gamma, \Delta_T}} \tanh \left( E_{\mu}^{\gamma, \Delta_T} / (2T) \right) = \frac{\Delta_\ell}{E_{\mu}^{\gamma, \Delta_T}} - \frac{\Delta_\ell}{E_{\mu}^{\gamma, \Delta_T}} \frac{2}{1 + \exp(E_{\mu}^{\gamma, \Delta_T} / T)}. \]

Using \( E_{\mu}^{\gamma, \Delta_T} \geq \varepsilon^\gamma - \tilde{\mu}^\gamma \geq p^2 - \mu \) and \( |\Delta_\ell| \leq E_{\mu}^{\gamma, \Delta_T}, \) it is easy to see that the \( L^2 \) norm of the second summand on the right side is uniformly bounded in \( \ell. \) Furthermore, by assumption (A6) we have

\[ \varepsilon^\gamma - \tilde{\mu}^\gamma \leq p^2 + \nu \text{ for } \nu = -\mu + \frac{6}{(2\pi)^{3/2}} \sup_{\ell > 0} \|\hat{V}_\ell(0)\|, \]

which is finite due to assumption (A5) and Lemma 4. In particular,

\[ E_{\mu}^{\gamma, \Delta_T} \leq \sqrt{(p^2 + \nu)^2 + |\Delta_\ell|^2}. \]
Recall that $\Delta_\ell(p)/\bar{c}_\ell$ converges pointwise to $P(p)$, and that $\bar{c} = \limsup_{\ell \to 0} \bar{c}_\ell$. Assume, for the moment, that $\bar{c} < \infty$. Then, by dominated convergence,

$$\limsup_{\ell \to 0} \int_{|p| \leq R} \frac{|\Delta_\ell|^2}{(p^2 + \nu)^2 + |\Delta_\ell|^2} \, d^3p = \int_{|p| \leq R} \frac{|\bar{c} P(p)|^2}{(p^2 + \nu)^2 + |\bar{c} P(p)|^2} \, d^3p$$  \hspace{1cm} (2.9)

for any $R > 0$. If $\bar{c} = \infty$, the same holds, with the integrand replaced by 1. In particular, if either $\bar{c} = \infty$ or $P$ is a polynomial of degree $n \geq 1$, the right side of (2.9) diverges as $R \to \infty$, contradicting the uniform boundedness of $\Delta_\ell/E_{\mu_\ell}^{\gamma_\ell, \Delta_\ell}$ in $L^2(\mathbb{R}^3)$. We thus conclude that $n = 0$ and $\bar{c} < \infty$, i.e., $\lim_{\ell \to 0} \Delta_\ell(p) = \bar{c}$ for a suitable subsequence.

(ii) The uniform bound (2.5) for $F_{\mu_\ell}^{\gamma_\ell}$ implies that $\hat{\gamma}_\ell$ is uniformly bounded in $L^2$. Thus, there is a subsequence which converges weakly to some $\hat{\gamma}$ in $L^2$. For that subsequence, we have for arbitrary $R > 0$

$$\lim_{\ell \to 0} \int_{B_R(0)} \hat{\gamma}_\ell(0) \, d^3p = \int_{B_R(0)} \hat{\gamma}(0) \, d^3p.$$  \hspace{1cm} (2.10)

In particular,

$$\lim_{\ell \to 0} \int_{\mathbb{R}^3} \hat{\gamma}_\ell \, d^3p \geq \int_{\mathbb{R}^3} \hat{\gamma} \, d^3p.$$

Therefore, $\lim_{\ell \to 0} \int_{\mathbb{R}^3} \hat{\gamma}_\ell \, d^3p = \int_{\mathbb{R}^3} \hat{\gamma} \, d^3p + \delta$ for an appropriate $\delta \geq 0$. Then

$$\lim_{\ell \to 0} \int_{|p| \geq R} \hat{\gamma}_\ell(p) |p|^b \, d^3p \geq R^b \lim_{\ell \to 0} \int_{|p| \geq R} \hat{\gamma}_\ell \, d^3p = R^b \lim_{\ell \to 0} \left[ \int_{\mathbb{R}^3} \hat{\gamma}_\ell \, d^3p - \int_{|p| \leq R} \hat{\gamma}_\ell \, d^3p \right] \geq \delta R^b.$$

Since $R$ can be arbitrarily large and the left side is bounded, $\delta$ has to be 0.

(iii) This follows immediately from part (ii) together with assumption (A5).

(iv) Let $D_\ell(p) = \varepsilon_\ell^\gamma(p) - p^2$. We compute

$$\begin{align*}
|D_\ell(p)| &= 2(2\pi)^{-3/2} |(\hat{V}_\ell - \hat{V}(0)) \ast \hat{\gamma}_\ell| \\
&\leq \frac{2}{(2\pi)^3} \int_{\mathbb{R}^3} d^3k \int_{\mathbb{R}^3} d^3x |V_\ell(x)(e^{-i(p-k)x} - 1) \hat{\gamma}_\ell(k) |\\
&\leq \frac{2||\hat{V}_\ell||_1}{(2\pi)^3} \int_{\mathbb{R}^3} \hat{\gamma}_\ell(k) \sup_{|x| \leq \ell} |e^{-i(p-k)x} - 1| \, d^3k \\
&\leq \frac{2||\hat{V}_\ell||_1}{(2\pi)^3} \ell^b (||\hat{\gamma}_\ell||_1 |p|^b + ||\hat{\gamma}_\ell||_1 |p|^b),
\end{align*}$$  \hspace{1cm} (2.11)

where we applied the fact that $|e^{it} - 1| \leq |t|^b$ for $t \in \mathbb{R}$ and $0 \leq b \leq 1$, as well as $|p - k|^b \leq |p|^b + |k|^b$.

By Lemma 4, $||\hat{\gamma}_\ell||_1 |p|^b$ is uniformly bounded in $\ell$, hence this concludes the proof.

(v) Recall the Euler-Lagrange equation (2.1) for $\hat{\gamma}_\ell$, which states that

$$\hat{\gamma}_\ell = \frac{1}{2} \frac{\varepsilon_\ell^\gamma(p) - \mu \gamma_\ell}{K_{\mu_\ell}^{\gamma_\ell, \Delta_\ell}(p)}.$$  \hspace{1cm} (2.12)

We have just shown that the right side converges pointwise to

$$\hat{\gamma}(p) = \frac{1}{2} \frac{p^2 - \mu \gamma}{K_{\mu_\ell}^{\gamma_\ell, \Delta_\ell}(p)}.$$  \hspace{1cm} (2.12)

Since $\hat{\gamma}$ is the weak limit of $\hat{\gamma}_\ell$, it has to agree with the pointwise limit $\tilde{\gamma}$, i.e., $\tilde{\gamma} = \hat{\gamma}$ almost everywhere. Therefore $\gamma$ satisfies Eq. (1.11).
(vi) We have already shown that the integrand converges pointwise. We want to use dominated convergence to show that also the integrals converge. For this purpose, we rewrite the integrand in \( m_{\mu,\Delta}^\gamma(T) \) in terms of \( \gamma^\ell \). With \( \xi(x) = \frac{x}{e^x - 1} \), we have
\[
\frac{1}{K_{\gamma,\Delta}^T} - \frac{1}{p^2} = \frac{\mu^\gamma - K_{\gamma,\Delta}^T p^2}{K_{\gamma,\Delta}^T p^2} + \frac{\mu^\gamma - \tilde{\mu}^\gamma - 2T\xi(e^{\gamma^\ell} / T)}{K_{\gamma,\Delta}^T p^2}.
\]
By comparing the first summand with the right side of (2.6), i.e.,
\[
\gamma^\ell = \frac{1}{1 + e^{\gamma^\ell / T}} + \frac{1}{2} \frac{E_{\mu,\gamma^\ell}^T - \gamma^\ell}{K_{\gamma,\Delta}^T p^2},
\]
we see that
\[
\gamma^\ell = \frac{1}{1 + e^{\gamma^\ell / T}} + \frac{1}{2} \frac{E_{\mu,\gamma^\ell}^T - \gamma^\ell}{K_{\gamma,\Delta}^T p^2}.
\]

We can now argue as above to show that, by dominated convergence, the integrals of all summands on the right side except for \(-2 \frac{2}{p^2} \mu^\gamma \) converge to their corresponding expressions with \( \gamma^\ell \) replaced by its limit \( \gamma \) and \( \Delta^\ell \) replaced by \( \Delta \). Indeed, assumption \( (A6) \) implies \( \gamma^\ell - \tilde{\mu}^\gamma \geq p^2 - \mu \) and thus
\[
E_{\mu,\gamma^\ell}^T \geq |\gamma^\ell - \tilde{\mu}^\gamma| \geq \gamma^\ell - \tilde{\mu}^\gamma \geq p^2 - \mu.
\]
For this reason,
\[
\frac{2}{p^2} \frac{1}{1 + e^{\gamma^\ell / T}} \leq \frac{2}{p^2} \frac{1}{1 + e^{\gamma^\ell / T}}.
\]
Moreover, the function
\[
\kappa_c(x) = \begin{cases} \frac{x}{\tanh(x)}, & x \geq 0 \\ 1, & x \leq 0 \end{cases}
\]
is monotone increasing, so
\[
K_{\gamma,\Delta}^T = 2T \kappa_c(\frac{E_{\mu,\gamma^\ell}^T}{2T}) \geq 2T \kappa_c(\frac{p^2 - \mu}{2T}).
\]
Together with \( \xi(x) \leq 1 \) for \( x \geq 0 \) and the bound (2.11) on \(|p^2 - \gamma^\ell(p)|\), this implies the statement.

Finally, we can argue as in (ii) above to conclude that \( \lim_{\gamma^\ell \to 0} \int_{\mathbb{R}^3} \frac{2}{p^2} \, d^3p = \int_{\mathbb{R}^3} \frac{2}{p^2} \, d^3p \), and hence obtain the desired result
\[
\lim_{\gamma^\ell \to 0} \int_{\mathbb{R}^3} \left( \frac{1}{K_{\gamma,\Delta}^T} - \frac{1}{p^2} \right) \, d^3p = \int_{\mathbb{R}^3} \left( \frac{1}{K_{\gamma,\Delta}^T} - \frac{1}{p^2} \right) \, d^3p,
\]
where we used that the limit \( \gamma \) also satisfies a suitable Euler-Lagrange equation, as shown in (v), and hence satisfies an identity as in (2.13) as well.

With the aid of Lemma 5, we can now give the
Proof of Theorem 2. The convergence of $|\Delta_v(p)|$, $\tilde{\mu}^{\gamma_\ell}$ and $\tilde{\gamma}_\ell(p)$ follows immediately from Lemma 5 at least for a suitable subsequence. To prove the validity of (1.12), we follow a similar strategy as in [12, Lemma 1]. From Theorem 1 we know that

$$(K_{T,\mu}^{\gamma_\ell,\Delta_\ell} + V_\ell)\alpha_\ell = 0, \quad \text{with } \alpha_\ell \in H^1(\mathbb{R}^3),$$

and we assume that $\alpha_\ell$ is not identically zero. According to the Birman-Schwinger principle, $K_{T,\mu}^{\gamma_\ell,\Delta_\ell} + V_\ell$ has 0 as eigenvalue if and only if

$$V_\ell^{1/2} \frac{1}{K_{T,\mu}^{\gamma_\ell,\Delta_\ell}} |V_\ell|^{1/2}$$

has $-1$ as an eigenvalue.

We decompose $V_\ell^{1/2} \frac{1}{K_{T,\mu}^{\gamma_\ell,\Delta_\ell}} |V_\ell|^{1/2}$ as

$$V_\ell^{1/2} \frac{1}{K_{T,\mu}^{\gamma_\ell,\Delta_\ell}} |V_\ell|^{1/2} = V_\ell^{1/2} \frac{1}{p^2} |V_\ell|^{1/2} + m_{\mu,T,\ell}(T) |V_\ell|^{1/2} \langle|\ell| + A_{\mu,T,\ell},$$

where

$$A_{\mu,T,\ell} = V_\ell^{1/2} \left( \frac{1}{K_{T,\mu}^{\gamma_\ell,\Delta_\ell}} - \frac{1}{p^2} \right) |V_\ell|^{1/2} - m_{\mu,T,\ell}(T) |V_\ell|^{1/2} \langle|\ell|.$$

By assumption (A9), $1 + V_\ell^{1/2} \frac{1}{p^2} |V_\ell|^{1/2}$ is invertible. Hence we can write

$$1 + V_\ell^{1/2} \frac{1}{K_{T,\mu}^{\gamma_\ell,\Delta_\ell}} |V_\ell|^{1/2} = \left(1 + V_\ell^{1/2} \frac{1}{p^2} |V_\ell|^{1/2}\right) \times$$

$$\times \left(1 + \frac{1}{1 + V_\ell^{1/2} \frac{1}{p^2} |V_\ell|^{1/2}} \left( m_{\mu,T,\ell}(T) |V_\ell|^{1/2} \langle|\ell| + A_{T,\mu,\ell} \right) \right),$$

and conclude that the operator

$$\frac{1}{1 + V_\ell^{1/2} \frac{1}{p^2} |V_\ell|^{1/2}} \left( m_{\mu,T,\ell}(T) |V_\ell|^{1/2} \langle|\ell| + A_{T,\mu,\ell} \right)$$

has an eigenvalue $-1$.

We are going to show below that

$$\lim_{\ell \to 0} \left\| \frac{1}{1 + V_\ell^{1/2} \frac{1}{p^2} |V_\ell|^{1/2}} A_{\mu,T,\ell} \right\| = 0.$$ (2.15)

As a consequence, $1 + (1 + V_\ell^{1/2} \frac{1}{p^2} |V_\ell|^{1/2})^{-1} A_{\mu,T,\ell}$ is invertible for small $\ell$, and we can argue as above to conclude that the rank one operator

$$m_{\mu,T,\ell}(T) \left(1 + \frac{1}{1 + V_\ell^{1/2} \frac{1}{p^2} |V_\ell|^{1/2}} A_{\mu,T,\ell} \right)^{-1} \frac{1}{1 + V_\ell^{1/2} \frac{1}{p^2} |V_\ell|^{1/2}} |V_\ell|^{1/2} \langle|\ell|$$

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has an eigenvalue $-1$, i.e.,

$$-1 = m_\mu^{\gamma_\ell,\Delta_\ell}(T) \left\langle |V_\ell|^{1/2} \right| \left( 1 + \frac{1}{1 + V_\ell^{1/2} \frac{1}{p^2} |V_\ell|^{1/2}} A_{\mu,T,\ell} \right)^{-1} \frac{1}{1 + V_\ell^{1/2} \frac{1}{p^2} |V_\ell|^{1/2}} |V_\ell^{1/2} \right| \cdot (2.16) \right.$$

With the aid of $(1.10)$ and the resolvent identity, we can rewrite $(2.16)$ as

$$4\pi a(V_\ell) + \frac{1}{m_\mu^{\gamma_\ell,\Delta_\ell}(T)} = \left\langle |V_\ell|^{1/2} \right| \left( 1 + \frac{1}{1 + V_\ell^{1/2} \frac{1}{p^2} |V_\ell|^{1/2}} A_{\mu,T,\ell} \right)^{-1} \frac{1}{1 + V_\ell^{1/2} \frac{1}{p^2} |V_\ell|^{1/2}} |V_\ell^{1/2} \right| \cdot (2.17)$$

We are going to show below that the term on the right side of $(2.17)$ goes to zero as $\ell \to 0$ and, as a consequence,

$$\lim_{\ell \to 0} m_\mu^{\gamma_\ell,\Delta_\ell}(T) = -\lim_{\ell \to 0} \frac{1}{4\pi a(V_\ell)} = -\frac{1}{4\pi a} \cdot (2.18)$$

On the other hand, by Lemma $5$ there is a subsequence of $(\gamma_\ell, \alpha_\ell)$ such that

$$\lim_{\ell \to 0} m_\mu^{\gamma_\ell,\Delta_\ell}(T) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \left( \frac{1}{K_{\ell,\Delta}} - \frac{1}{p^2} \right) \text{d}^3 p,$$

where $\Delta$ is the pointwise limit of $|\Delta_\ell(p)|$ and $\tilde{\mu}$ is the limit of $\tilde{\mu}^{\gamma_\ell}$. This shows $(1.12)$, at least for a subsequence.

It remains to show $(2.15)$ and $(2.18)$. We start with the decomposition

$$\frac{1}{1 + V_\ell^{1/2} \frac{1}{p^2} |V_\ell|^{1/2}} = \frac{1}{\sigma_\ell} P_\ell + \frac{1}{1 + V_\ell^{1/2} \frac{1}{p^2} |V_\ell|^{1/2}} (1 - P_\ell), \cdot (2.19)$$

where the second summand is uniformly bounded by assumption $(A9)$. The integral kernel of $A_{\mu,T,\ell}$ is given by

$$A_{\mu,T,\ell}(x,y) = \frac{V_\ell(x)^{\frac{1}{2}} |V_\ell(y)|^{\frac{1}{2}}}{(2\pi)^3} \int_{\mathbb{R}^3} \left( \frac{1}{K_{\ell,\Delta}} - \frac{1}{p^2} \right) (e^{-i(x-y)p} - 1) \text{d}^3 p, \cdot (2.20)$$

which can be estimated as

$$|A_{\mu,T,\ell}(x,y)| \leq \frac{|V_\ell(x)|^{\frac{1}{2}} |V_\ell(y)|^{\frac{1}{2}}}{(2\pi)^3} \int_{\mathbb{R}^3} \left| \frac{1}{K_{\ell,\Delta}} - \frac{1}{p^2} \right| (|x - y| |p|)^q \text{d}^3 p \cdot (2.21)$$

for any $0 \leq q \leq 1$. In the proof of Lemma $5 (vi)$ we found that the integral

$$\int_{\mathbb{R}^3} \left| \frac{1}{K_{\ell,\Delta}} - \frac{1}{p^2} \right| |p|^q \text{d}^3 p$$

is uniformly bounded in $\ell$ for $q < 1$. With the aid of Assumption $(A1)$ we can thus bound the Hilbert-Schmidt norm of $A_{\mu,T,\ell}$ as

$$\|A_{\mu,T,\ell}\|_2 \leq \text{const} \ell^q \|V_\ell\|_1. \cdot (2.22)$$

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In particular, because of Assumption (A4),
\[
\left\| \frac{1}{1 + V_{\ell}^{1/2} \| V_{\ell} \|^{1/2}} (1 - P_{\ell}) A_{\mu,T,\ell} \right\| \leq O(\ell^{q})
\]
for small $\ell$. It remains to show that the contribution of the first summand in $\text{(2.19)}$ to the norm in question vanishes as well. We have
\[
\| P_{\ell} A_{\mu,T,\ell} \| = \frac{\| A_{\mu,T,\ell}^{*} J_{\ell} \phi_{\ell} \|}{\| J_{\ell} \phi_{\ell} \|}.
\]
By (2.21),
\[
\left| A_{\mu,T,\ell}^{*} J_{\ell} (x) \right| \leq C \ell^{q} |V(x)|^{1/2} \int_{\mathbb{R}^{3}} |V(y)|^{1/2} |\phi_{\ell}(y)| \, d^{3}y,
\]
and hence
\[
\| P_{\ell} A_{\mu,T,\ell} \| \leq \text{const} \ell^{q} \frac{1}{\| J_{\ell} \phi_{\ell} \|} \left\| V_{\ell} \right\|^{1/2} \| \phi_{\ell} \| \left\| V_{\ell} \right\|^{1/2}.
\]
By (A10) we know that $\frac{\| V_{\ell} \|^{1/2} |\phi_{\ell}|}{\| J_{\ell} \phi_{\ell} \|} \leq O(\ell^{1/2})$. Since $c_{\ell} = O(\ell)$ by assumption, we arrive at
\[
\left\| \frac{1}{1 + V_{\ell}^{1/2} \| V_{\ell} \|^{1/2}} A_{\mu,T,\ell} \right\| \leq O(\ell^{q-1/2}),
\]
which vanishes by choosing $1/2 < q < 1$.

To show (2.18), i.e., that the term on the right side of (2.17) vanishes as $\ell \to 0$, we can again use the decomposition (2.19) to argue that
\[
\left\| \left( 1 + \frac{1}{1 + V_{\ell}^{1/2} \| V_{\ell} \|^{1/2}} A_{\mu,T,\ell} \right)^{-1} \frac{1}{1 + V_{\ell}^{1/2} \| V_{\ell} \|^{1/2}} V_{\ell}^{1/2} \right\| \leq O(\ell^{-1/2}),
\]
where we used (2.15) as well as Assumptions (A4), (A9) and (A10). Moreover,
\[
\left\| A_{\mu,T,\ell}^{*} \frac{1}{1 + V_{\ell}^{1/2} \| V_{\ell} \|^{1/2}} V_{\ell}^{1/2} \right\| \leq O(\ell^{q}) + \frac{1}{c_{\ell}} \| P_{\ell} A_{\mu,T,\ell} \| \left\| P_{\ell}^{*} \right\| \left\| V_{\ell} \right\|^{1/2} \leq O(\ell^{q})
\]
using (2.23). The last term in (2.17) thus is of order $\ell^{q-1/2}$, and vanishes as $\ell \to 0$ for any $1/2 < q < 1$. This proves (2.18).

As a last step, we show that the limit points for $\tilde{\mu}^{\gamma_{c}}$ and $|\Delta_{c}(p)|$, and thus also of $\tilde{\gamma}(p)$, are unique. We use the fact that the limit points solve the two implicit equations (1.11) and (1.12), i.e.,
\[
F(\tilde{\mu}, \Delta) = 0, \quad G(\tilde{\mu}, \Delta) = 0,
\]
where
\[
F(\nu, \Delta) = \nu - \mu + \frac{V}{(2\pi)^{3/2}} \int_{\mathbb{R}^{3}} \left( 1 - \frac{\gamma_{c}^{2} - \nu}{K_{\Delta,T,\nu}} \right) \, d^{3}p,
\]
\[
G(\nu, \Delta) = \frac{1}{4\pi a} + \frac{1}{(2\pi)^{3}} \int_{\mathbb{R}^{3}} \left( \frac{1}{K_{\Delta,T,\nu}} - \frac{1}{p^{2}} \right) \, d^{3}p.
\]
It is straightforward to check that
\[ \partial_\nu F > 0 \quad \partial_\nu G > 0 \]
\[ \partial_\Delta F > 0 \quad \partial_\Delta G < 0 \]
(compare with similar computations in Appendix B). Hence the set where \( F \) vanishes defines a strictly decreasing curve \( \mathbb{R}_+ \to \mathbb{R} \), while the analogous curve for the zero-set of \( G \) is strictly increasing. Consequently, they can intersect at most once.

This proves uniqueness under the assumptions that \( \Delta_\ell \neq 0 \) for a sequence of \( \ell \)'s going to zero. In the opposite case, \( \Delta_\ell = 0 \) for \( \ell \) small enough, hence \( \Delta = 0 \). The uniqueness in this case follows as above, looking at the equation \( F(\tilde{\mu}, 0) = 0 \). This completes the proof of Theorem 2.

Remark 5. In case \( K^{\gamma_\ell, \Delta_\ell}_{T, \mu} \) is reflection-symmetric in \( p \), one can show that the bound (2.22) holds also for \( q = 1 \). Indeed, in this case only the symmetric part of \( e^{-i(x-y) \cdot p} - 1 \) contributes to the integral kernel of \( A_{\mu, T, \ell} \), and hence
\[
A_{\mu, T, \ell}(x, y) = \frac{V_{\ell}^1(x)|V_{\ell}^1(y)}{(2\pi)^3} \int_{\mathbb{R}^3} \left( \frac{1}{K_{T, \mu}^{\gamma_\ell, \Delta_\ell}} - \frac{1}{p^2} \right) \left( \cos ((x - y) \cdot p) - 1 \right) d^3p.
\]
Again using (2.13), we may write
\[
\left| \frac{1}{K_{T, \mu}^{\gamma_\ell, \Delta_\ell}} - \frac{1}{p^2} \right| \leq \text{const} \frac{1}{1 + p^4} + R_\ell(p),
\]
such that
\[
\int_{\mathbb{R}^3} |pR_\ell(p)| d^3p
\]
is uniformly bounded in \( \ell \). Since
\[
\int_{\mathbb{R}^3} \frac{1 - \cos (p \cdot (x - y))}{1 + p^4} d^3p = \sqrt{2}\pi^2 \left[ 1 - e^{-\frac{|x-y|}{\sqrt{2}}} \frac{\sin \left( \frac{|x-y|}{\sqrt{2}} \right)}{|x-y|/\sqrt{2}} \right] \leq \pi^2 |x - y|,
\]
we get
\[
\|A_{\mu, T, \ell}\|_2 \leq \text{const} \left[ \int_{\mathbb{R}^3} |V_{\ell}(x)||V_{\ell}(y)||x - y|^2 d^3x d^3y \right]^{1/2} \leq O(\ell)
\]
in this case.

2.3 Critical Temperature

In this section we will prove Theorem 3. We start with the following observation.

Lemma 6. Let \( \mu > 0 \), \( T < T_c \), and let \( (\gamma_\ell^0, 0) \) be a family of normal states for \( \mathcal{F}^{V_\ell}_{T} \). Then
\[
\liminf_{\ell \to 0} m^{\gamma_\ell^0, 0}_{\mu}(T) > -\frac{1}{4\pi a}. \tag{2.26}
\]
Proof. By mimicking the proof of Lemma 5, we observe that (for a suitable subsequence)
\[
\lim_{\ell \to 0} m_{\mu,\gamma}^{0,0}(T) = m_{\mu,\gamma}^{0,0}(T)
\]
where \(\tilde{m}_{\mu,\gamma} = \mu - 2(2\pi)^{-3/2} V \int_{\mathbb{R}^3} \hat{\gamma}(p) \, dp\) and \(\hat{\gamma}(p) = (1 + e^{(p^2 - \tilde{m}_{\mu,\gamma})/T})^{-1}\). It is shown in Appendix A that \(m_{\mu,\gamma}^{0,0}(T)\) is a strictly decreasing function of \(T\). At \(T = T_c\), it equals \(-1/(4\pi\alpha)\) according to Definition 4, hence \(m_{\mu,\gamma}^{0,0}(T) > -1/(4\pi\alpha)\) for \(T < T_c\).

The first part of Theorem 3 then follows from the following lemma.

Lemma 7. Let \((\gamma_0,0)\) be a normal state of \(\mathcal{F}_{T^\ell}\). Assume that \(\lim_{\ell \to 0} m_{\mu,\gamma}^{0,0}(T) > -\frac{1}{4\pi\alpha}\). Then, for small enough \(\ell\), the linear operator \(K_{T,\mu}^{0,0} + V_\ell\) has at least one negative eigenvalue.

Proof. With the aid of the Birman-Schwinger principle, we will attribute the existence of an eigenvalue of \(K_{T,\mu}^{0,0} + V_\ell\) below the essential spectrum to a solution of a certain implicit equation. We then show the existence of such a solution, which proves the existence of a negative eigenvalue.

Note that the infimum of the essential spectrum of \(K_{T,\mu}^{0,0} + V_\ell\) is \(2T\). Let \(e < 2T\). According to the Birman-Schwinger principle, \(K_{T,\mu}^{0,0} + V_\ell\) has an eigenvalue \(e\) if and only if
\[
V_\ell^{1/2} \frac{1}{K_{T,\mu}^{0,0} - e} [V_\ell]^{1/2}
\]
has an eigenvalue \(-1\). As in the proof of Theorem 2, we decompose the operator (2.27) as
\[
V_\ell^{1/2} \frac{1}{K_{T,\mu}^{0,0} - e} [V_\ell]^{1/2} = V_\ell^{1/2} \frac{1}{p^2} |V_\ell|^{1/2} + m_{\mu,\gamma}^{0,0}(T) |V_\ell|^{1/2} + m_{\mu,\gamma}^{0,0}(T) |V_\ell|^{1/2} + A_{\mu,T,\ell,e}
\]
where
\[
m_{\mu,\gamma}^{0,0}(T) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \left( \frac{1}{K_{T,\mu}^{0,0} - e} - \frac{1}{p^2} \right) \, dp.
\]

We claim that the remainder \(A_{\mu,T,\ell,e}\) is bounded above by \(O(|e|)\) in Hilbert-Schmidt norm, for any \(0 \leq q < 1\), uniformly in \(e\) for \(e \leq 0\). This will follow from the same estimates as in the proof of Theorem 2 if we can show that
\[
\int_{\mathbb{R}^3} |p|^q \left( \frac{1}{K_{T,\mu}^{0,0} - e} - \frac{1}{K_{T,\mu}^{0,0}} \right) \, dp
\]
is uniformly bounded in \(\ell\) for \(0 \leq q < 1\). But since
\[
\frac{1}{K_{T,\mu}^{0,0} - e} - \frac{1}{K_{T,\mu}^{0,0}} = \frac{e}{K_{T,\mu}^{0,0} (K_{T,\mu}^{0,0} - e)} \leq \frac{e}{(2T)^2 \kappa_c (\mu^2 - \mu) - \frac{e}{2T}},
\]
with \(\kappa_c\) defined in (2.14), this is indeed the case.

Again, the operator
\[
1 + V_\ell^{1/2} \frac{1}{p^2} |V_\ell|^{1/2} + A_{\mu,T,\ell,e} = \left( 1 + V_\ell^{1/2} \frac{1}{p^2} |V_\ell|^{1/2} \right) \left( 1 + \frac{1}{1 + V_\ell^{1/2} \frac{1}{p^2} |V_\ell|^{1/2} A_{\mu,T,\ell,e}} \right)
\]

This completes the proof.

This follows from the same arguments as in (2.24)–(2.25), in fact. Recall that, by assumption, implies the existence of a negative eigenvalue of \( \ell \). We conclude from Eqs. (1.11) and (1.12) that

\[
\ell(e) \rightarrow \infty \quad \text{as} \quad e \rightarrow 0,
\]

with the same argument as in the proof of Theorem 2. We conclude that \( K_{T,\mu}^{\gamma,0} + V_\ell \) has an eigenvalue \( e \) if and only if the rank one operator

\[
m_{\mu,e}(T) \left( 1 + V_\ell^{1/2} \frac{1}{p^2} |V_\ell|^{1/2} + A_{\mu,T,\ell,e} \right)^{-1} |V_\ell|^{1/2} \left( |V_\ell|^{1/2} \right)
\]

has an eigenvalue \(-1\), i.e., if

\[
\tilde{a}_{\ell,e} = \left\langle |V_\ell|^{1/2} \left( 1 + V_\ell^{1/2} \frac{1}{p^2} |V_\ell|^{1/2} + A_{\mu,T,\ell,e} \right)^{-1} V_\ell^{1/2} \right\rangle = -\frac{1}{m_{\mu,e}(T)}. \quad (2.30)
\]

We claim that, for small enough \( \ell \), the implicit equation \((2.30)\) has a solution \( e < 0 \), which implies the existence of a negative eigenvalue of \( K_{T,\mu}^{\gamma,0} + V_\ell \). We first argue that \( \lim_{\ell \to 0} \tilde{a}_{\ell,e} = 4\pi a \). This follows from the same arguments as in (2.24)–(2.25), in fact. Recall that, by assumption, \( \lim_{\ell \to 0} m_{\mu,e}(T) > -\frac{1}{4\pi a} \). Moreover, the integral \( m_{\mu,e}(T) \) is monotone increasing in \( e \) and \( e \mapsto m_{\mu,e}(T) \) maps \((-\infty, 0]\) onto the interval \((-\infty, m_{\mu,e}(T)]\). Since \( \tilde{a}_{\ell,e} \) depends continuously on \( e \), there has to be a solution \( e < 0 \) to \((2.30)\) for small enough \( \ell \), and thus \( K_{T,\mu}^{\gamma,0} + V_\ell \) must have a negative eigenvalue. This completes the proof.

We now give the

**Proof of Theorem 3.** Part (i) follows immediately from Lemmas 6 and 7. To prove part (ii), we argue by contradiction. Suppose that \( T > T_c \) and that there does not exist an \( \ell_0(T) \) such that for \( \ell < \ell_0(T) \) all minimizers of \( F^{T,\gamma}_{T,\mu} \) are normal. Then there exists a sequence of \( \ell \)'s going to zero and corresponding minimizers \( (\gamma_\ell, \alpha_\ell) \) with \( \alpha_\ell \neq 0 \) and thus, by Theorem 2 Eqs. (1.11) and (1.12) hold in the limit \( \ell \to 0 \). We claim that these equations do not have a solution for \( T > T_c \), thus providing the desired contradiction.

At the end of the proof of Theorem 2 we have already argued that the right side of (1.12) is monotone decreasing in \( \Delta \) and increasing in \( \tilde{\mu} \). Moreover, \( \tilde{\mu} \) is decreasing in \( \Delta \). In particular, we conclude from Eqs. (1.11) and (1.12) that

\[
-\frac{1}{4\pi a} \leq \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \left( \frac{\tanh \left( \frac{x^2 - \tilde{\mu}}{2T} \right)}{p^2 - \tilde{\mu}} - \frac{1}{p^2} \right) d^3p
\]

with \( \tilde{\mu} \) given by

\[
\tilde{\mu} = \mu - \frac{2V}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \frac{1}{1 + e^{x^2 - \mu}} d^3p.
\]

According to our analysis in Appendix 3, this implies \( T \leq T_c \), however.

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A Example for a Sequence of Short-Range Potentials

In dimension $d = 3$, contact potentials are realized by a one-parameter family $-\Delta_a$ of self-adjoint extensions of the Laplacian $-\Delta_{C_c^\infty(\mathbb{R}^3(0))}$, indexed by the scattering length $a$. Moreover, $-\Delta_a$ can be obtained as a norm resolvent limit of short-range Hamiltonians of the form $-\Delta + V_\ell$. This is presented in [2] in the case of $0 < \lim_{\ell \to 0} \|V_\ell\|_{3/2} < \infty$, and was extended in [6] to cases where $0 < \lim_{\ell \to 0} \|V_\ell\|_1 < \infty$. In this Appendix, we use an approach similar to [6] to construct a sequence of potentials $V_\ell$ converging to a contact potential. In particular, we are interested in the case where the scattering length $a(V_\ell)$ converges to a negative value $a < 0$, and where all the assumptions in Assumption [I] are satisfied.

A.1 Example 1

As a first example, we follow [1] chap I.1.2-4. We start with an arbitrary potential $V \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$, such that

1. $p^2 + V(x) \geq 0$, and $V$ has a simple zero-energy resonance, i.e., there is a simple eigenvector $\phi \in L^2(\mathbb{R}^3)$ with $(V^{1/2} \frac{1}{p^2} |V|^{1/2} + 1)\phi = 0$, and $\psi(x) = \frac{1}{p^2} |V|^{1/2}\phi \in L^2_{loc}(\mathbb{R}^3)$,

2. $\|\hat{V}(p)\| \leq 2\hat{V}(0)$.

Define $V_\ell(x) = \lambda(\ell)\ell^{-2}V(\frac{x}{\ell})$, where $\lambda(0) = 1$, $\lambda < 1$ for all $\ell > 0$ and $1 - \lambda(\ell) = O(\ell)$. The important point of this scaling is the following. Denote by $U_\ell$ the unitary scaling operator $(U_\ell\varphi)(x) = \ell^{-3/2}\varphi(\frac{x}{\ell})$. By a simple calculation one obtains the relation

$$U_\ell V^{1/2} \frac{1}{p^2} |V|^{1/2} U_\ell^{-1} = \frac{1}{\lambda(\ell)} V^{1/2} \frac{1}{p^2} |V|^{1/2},$$

such that, with $\phi_\ell = U_\ell\phi$,

$$V^{1/2} \frac{1}{p^2} |V|^{1/2} \phi_\ell = \lambda(\ell) U_\ell V^{1/2} \frac{1}{p^2} |V|^{1/2} \phi = -\lambda(\ell) \phi_\ell. \quad (A.1)$$

This shows that the lowest eigenvalue of $1 + V^{1/2} \frac{1}{p^2} |V|^{1/2}$ is $1 - \lambda(\ell) = O(\ell)$.

Moreover, by construction, $\hat{V}_\ell(p) = \ell\lambda(\ell)\hat{V}(\ell p)$, $\|V_\ell\|_{3/2} = \lambda(\ell)\|V\|_{3/2}$, and the 1-norm can be bounded as $\|V_\ell\|_1 \leq (\frac{3}{2}\pi)^{1/3}\ell\lambda(\ell)\|V\|_{3/2}$, hence [A1], [A2] and [A4]–[A7] hold.

The validity of Assumption [A8] is a consequence of the following general fact.

Lemma 8. If $\|V\|_{3/2}$ is uniformly bounded, assumptions [A1] and [A9] imply assumption [A8].

Proof. We look for $C > 0$ such that $p^2 + V_\ell - |p|^b + C$ is non-negative for all $\ell > 0$. By the Birman-Schwinger principle, this is the case if and only if

$$1 + V^{1/2} \frac{1}{p^2} - |p|^b + C + E |V|^{1/2} = 1 + J_\ell X_\ell^{C+E} + R^E_\ell$$

is invertible for all $E > 0$. Here $J_\ell = \{ -1, V_\ell \geq 0 \}$, $X_\ell^E = |V|^{1/2} \frac{1}{p^2+|p|^b+|V|^{1/2}}$ and

$$R^E_\ell = V^{1/2} \frac{1}{p^2} - |p|^b + C + E |V|^{1/2} - V^{1/2} \frac{1}{p^2+|p|^b+|V|^{1/2}} |V|^{1/2}.$$
By expanding in a Neumann series, we see that $1 + J_{\ell}X_{\ell}^{C+E} + R_{\ell}^E$ has a bounded inverse provided that
\[
\| (1 + J_{\ell}X_{\ell}^{C+E})^{-1} \| \| R_{\ell}^E \| < 1. \tag{A.2}
\]
We first examine $\| (1 + J_{\ell}X_{\ell}^{E})^{-1} \|$. We have
\[
\frac{1}{1 + J_{\ell}X_{\ell}^{E}} = 1 - J_{\ell}(X_{\ell}^{E})^{1/2} \frac{1}{1 + (X_{\ell}^{E})^{1/2} J_{\ell}(X_{\ell}^{E})^{1/2}},
\]
and thus
\[
\left\| \frac{1}{1 + J_{\ell}X_{\ell}^{C+E}} \right\| \leq 1 + \| X_{\ell}^{E} \| \left\| \frac{1}{1 + (X_{\ell}^{E})^{1/2} J_{\ell}(X_{\ell}^{E})^{1/2}} \right\|.
\]
Using the fact that $(4\pi|x - y|)^{-1}e^{-\sqrt{E}|x-y|}$ is the integral kernel of the operator $\frac{1}{p^2 + E}$ for $E \geq 0$, the Hardy-Littlewood-Sobolev inequality [19 Thm. 4.3] implies that $\| X_{\ell}^{E} \| \leq \| X_{\ell}^{0} \|_2 \leq c_2 \| V_{\ell} \|_{3/2}$. Moreover, $\| (1 + (X_{\ell}^{E})^{1/2} J_{\ell}(X_{\ell}^{E})^{1/2})^{-1} \|$ is the inverse of the eigenvalue of $1 + (X_{\ell}^{E})^{1/2} J_{\ell}(X_{\ell}^{E})^{1/2}$ with smallest modulus, and this latter operator is isospectral to $1 + J_{\ell}X_{\ell}^{E}$. We conclude that $\| (1 + (X_{\ell}^{E})^{1/2} J_{\ell}(X_{\ell}^{E})^{1/2})^{-1} \| \leq e_{\ell}(E)^{-1}$, where $e_{\ell}(E)$ is the smallest eigenvalue of $1 + J_{\ell}X_{\ell}^{E}$. The latter is bigger than $e_{\ell}(0)$, which is of order $O(\ell)$ by assumption (A9). This shows that there is a constant $c_1 > 0$ such that
\[
\left\| \frac{1}{1 + J_{\ell}X_{\ell}^{C+E}} \right\| \leq c_1 \ell^{-1}
\]
for small $\ell$.

It remains to bound the operator $R_{\ell}^E$, whose integral kernel is given by
\[
R_{\ell}^E(x, y) = \frac{1}{(2\pi)^3} V_{\ell}(x)^{1/2} |V_{\ell}(y)|^{1/2} \int_{\mathbb{R}^3} \frac{1}{(p^2 - |p|^b + C + E)} \left( \frac{1}{p^2} - \frac{1}{p^2} + C + E \right) e^{-ip(y-x)} \, dp.
\]
The trace norm $\| R_{\ell}^E \|_1$ equals
\[
\| R_{\ell}^E \|_1 = \frac{4\pi}{(2\pi)^3} \| V_{\ell} \|_1 \int_0^{\infty} \frac{p^b}{p^2 - p^b + C + E} \, dp.
\]
By dominated convergence, the integral tends to 0 as $C \to \infty$. By Hölder’s inequality, $\| V_{\ell} \|_1 \leq O(\ell)$, so there exists a $C$ such that $\| R_{\ell}^E \| < c_1 \ell$. This shows (A.2).

Next we show that validity of (A9). Let $J = \{ \frac{1}{2}, -\frac{1}{2}, 0 \}, X = |V|^{1/2} \frac{1}{p^2} |V|^{1/2}$ and $P = \frac{1}{d\phi|\phi|^{2}d\phi|\phi|} \langle J\phi| \phi \rangle$, the projection onto the eigenfunction $\phi$ corresponding to the zero eigenvalue of $1 + V^{1/2} \frac{1}{p^2} |V|^{1/2} = 1 + JX$. Using the unitary scaling operator $U_{\ell}$ we also introduce the scaled versions $\phi_{\ell} = U_{\ell} \phi, J_{\ell} = U_{\ell} J U_{\ell}^{-1}$ and $P_{\ell} = U_{\ell} P U_{\ell}^{-1}$, and let $X_{\ell} = \lambda(\ell) U_{\ell} X U_{\ell}^{-1}$. Then $\langle J_{\ell} \phi_{\ell}| \phi_{\ell} \rangle = \langle J\phi| \phi \rangle = -\langle X \phi| \phi \rangle < 0$ does not vanish, since $X = |V|^{1/2} \frac{1}{p^2} |V|^{1/2}$ is a positive operator whose kernel does not contain $\phi$. Note that $[JX, P] = 0$, which follows from
\[
(1 + JX)P = 0
\]
and
\[
P(1 + JX) = JP^*J(1 + JX) = JP^*(1 + JX)^*J = J((1 + JX)P)^*J = 0.
\]
We decompose $1 + J_{\ell}X_{\ell}$ as
\[
1 + J_{\ell}X_{\ell} = (1 + J_{\ell}X_{\ell})P_{\ell} + (1 + J_{\ell}X_{\ell})(1 - P_{\ell})
\]
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and examine the two parts separately. For the first summand note that since $JX\phi = -\phi$,

$$(1 + J\ell X\ell)P_\ell = (1 - \lambda(\ell))P_\ell.$$  

Next, we study the operator $J\ell X\ell (1 - P_\ell)$. The operator

$$T = 1 + JX(1 - P) = 1 + JX + P$$

has no zero eigenvalue. Indeed, if $T\psi = 0$, then

$$0 = (1 + JX + P)(P + (1 - P))\psi = P\psi + (1 - P)(1 + JX)\psi,$$

where we used that $P$ commutes with $1 + JX$. Projecting onto $P$ and $1 - P$, respectively, yields

$$0 = P\psi, \quad 0 = (1 - P)(1 + JX)\psi = (1 + JX)\psi,$$

which constrains $\psi$ to be 0.

Due to the compactness of $P + JX$, eigenvalues of $T$ can only accumulate at 1, and hence $T$ has a bounded inverse $T^{-1}$. Now $J\ell X\ell = \lambda(\ell)U_\ell JXU_\ell^{-1}$, and we have the decomposition

$$U_\ell^{-1}(1 + J\ell X\ell)U_\ell = 1 + \lambda(\ell)JX = (1 - \lambda(\ell))P + (1 + \lambda(\ell)(T - 1))(1 - P)$$

with inverse

$$U_\ell^{-1} \frac{1}{1 + J\ell X\ell} U_\ell = \frac{1}{1 - \lambda(\ell)}P + \frac{1}{1 - \lambda(\ell)} + \frac{1}{1 - \lambda(\ell)}T(1 - P).$$

This shows that

$$\frac{1}{1 + J\ell X\ell}(1 - P_\ell) = U_\ell^{-1} \frac{1}{1 - \lambda(\ell)} + \frac{1}{1 - \lambda(\ell)}T(1 - P_\ell)U_\ell^{-1}.$$

Since 0, and thus also a neighborhood of 0, is not in the spectrum of $T$, and $\lambda(\ell) \to 1$ as $\ell \to 0$, we conclude that $\frac{1}{1 - \lambda(\ell)}$ is uniformly bounded for small $\ell$. This yields the uniform boundedness of $\frac{1}{1 + J\ell X\ell}(1 - P_\ell)$.

In order to prove (A3) we decompose $\frac{1}{1 + J\ell X\ell}$ as

$$1 + J\ell X\ell = \frac{1}{1 - \lambda(\ell)}P_\ell + \frac{1}{1 + J\ell X\ell}(1 - P_\ell) = \frac{1}{1 - \lambda(\ell)}P_\ell + \frac{1}{1 + J\ell X\ell}(1 - P_\ell).$$

We have just shown that the second summand is uniformly bounded in $\ell$. This allows us to calculate the limit $\ell \to 0$ of the scattering length $a(V_\ell)$, which equals

$$4\pi a(V_\ell) = \left\langle |V_\ell|^{1/2} \left| \frac{1}{1 + V_\ell^{1/2} P V_\ell^{1/2}} V_\ell^{1/2} \right| \right\rangle$$

$$= \frac{1}{1 - \lambda(\ell)} \left\langle |V_\ell|^{1/2} P V_\ell^{1/2} \right\rangle + \left\langle |V_\ell|^{1/2} \left| \frac{1}{1 + J\ell X\ell}(1 - P_\ell) V_\ell^{1/2} \right| \right\rangle.$$

Using the uniform boundedness of the second summand together with the fact that $||V_\ell||_1 \to 0$ as $\ell \to 0$, we see that the second summand vanishes in the limit $\ell \to 0$. Therefore

$$\lim_{\ell \to 0} 4\pi a(V_\ell) = \lim_{\ell \to 0} \frac{1}{1 - \lambda(\ell)} \left\langle |V_\ell|^{1/2} (P_\ell \phi) \right\rangle^2$$

$$= \lim_{\ell \to 0} \sqrt{\lambda(\ell)} \frac{\ell}{1 - \lambda(\ell)} \left\langle |V_\ell|^{1/2} (P_\ell \phi) \right\rangle^2$$

$$= \frac{1}{\lambda'(0)} \left\langle |V_\ell|^{1/2} (P_\ell \phi) \right\rangle^2.$$
We are left with demonstrating (A10). This is immediate, since
\[ \langle |V_\ell|^{1/2} |\phi_\ell \rangle = \sqrt{\lambda(\ell)} \ell^{1/2} \langle |V|^{1/2} |\phi \rangle \]
and
\[ \langle \phi_\ell | \text{sgn}(V_\ell) \phi_\ell \rangle = \langle \phi | \text{sgn}(V) \phi \rangle. \]

A.2 Example 2

We consider a sequence of potentials as suggested in [15], of the form
\[ V_\ell = V_\ell^+ - V_\ell^-, \quad V_\ell^+(x) = (k^+_\ell)^2 \chi_{(|x|<\epsilon_\ell)}(x), \quad k^+_\ell = k^+ \epsilon_\ell^{-3/2}, \]
\[ V_\ell^-(x) = (k^-_\ell)^2 \chi_{(|x|<\epsilon_\ell)}(x), \quad k^-_\ell = \frac{\omega}{\ell-\epsilon_\ell}, \]
with \( \omega > 0 \), \( k^+ > 0 \) and \( 0 < \epsilon_\ell < c \ell^2 \) with \( c < 2\omega/\pi \). The function \( \chi_A(x) \) denotes the characteristic function of the set \( A \). (See the sketch on page 5.) We shall show that this sequence of potentials satisfies assumptions (A1), (A2), (A4), (A5) and (A7) are in fact obvious, and \( \mathcal{V} = \lim_{\ell \to 0} \hat{V}_\ell(0) = \sqrt{2/\pi}(k^+)^2/3. \)

(A3) To calculate the scattering length \( a(V_\ell) \), we have to find the solution \( \psi_\ell \) of \(-\Delta \psi_\ell + V_\ell \psi_\ell = 0\), with \( \lim_{|x| \to \infty} \psi_\ell(x) = 1 \). The scattering length then appears in the asymptotics
\[ \psi_\ell(x) \approx 1 - \frac{a(V_\ell)}{|x|} \]
for large \( |x| \). To solve the zero-energy scattering equation, we write \( \psi_\ell(x) = \frac{u_\ell(|x|)}{|x|} \) with \( u_\ell(0) = 0 \). Then \( u_\ell \) solves the equation
\[ -u''_\ell + V_\ell u_\ell = 0. \]

For \( r \geq \ell \) the function \( u_\ell \) is of the form \( u_\ell(r) = c_1 r + c_2 \). The normalization at infinity requires \( c_1 = 1 \), and \( u_\ell \) automatically has the desired asymptotics with \( a(V_\ell) = -c_2 \).

In our example, the equation we have to solve is
\[
\begin{align*}
-u''_\ell(r) + (k^+_\ell)^2 u_\ell(r) &= 0, & 0 \leq r \leq \epsilon_\ell \\
-u''_\ell(r) - (k^-_\ell)^2 u_\ell(r) &= 0, & \epsilon_\ell \leq r \leq \ell \\
-u''_\ell(r) &= 0, & r \geq \ell,
\end{align*}
\]
with the solution
\[ u_\ell(r) = \begin{cases} A \sinh(k^+_\ell r), & 0 \leq r \leq \epsilon_\ell \\ B_1 \cos(k^-_\ell r) + B_2 \sin(k^-_\ell r), & \epsilon_\ell \leq r \leq \ell \\ r - a(V_\ell), & r \geq \ell. \end{cases} \]

Continuity of \( u_\ell \) and \( u'_\ell \) then requires
\[ A \sinh(k^+_\ell \epsilon_\ell) = B_1 \cos(k^-_\ell \epsilon_\ell) + B_2 \sin(k^-_\ell \epsilon_\ell) \]
\[ Ak^+_\ell \cosh(k^+_\ell \epsilon_\ell) = -B_1 k^-_\ell \sin(k^-_\ell \epsilon_\ell) + B_2 k^-_\ell \cos(k^-_\ell \epsilon_\ell) \]
and

\[ B_1 \cos(k_1^\ell \ell) + B_2 \sin(k_1^\ell \ell) = \ell - a(V_\ell) \]

\[-B_1 k_1^\ell \sin(k_1^\ell \ell) + B_2 k_1^\ell \cos(k_1^\ell \ell) = 1.\]

Solving for \( a(V_\ell) \) yields

\[ a(V_\ell) = \ell - \frac{1}{k_1^\ell} \frac{k_1^\ell \tan(k_1^\ell (\ell - \epsilon_\ell)) + k_1^\ell \tan(k_1^\ell \epsilon_\ell)}{k_1^\ell k_1^- - k_1^-} \tan(k_1^\ell (\ell - \epsilon_\ell)) \tan(k_1^\ell \epsilon_\ell). \quad (A.4) \]

By Eq. (A.3), \((\ell - \epsilon_\ell)k_1^- = \frac{\pi}{2} - \ell \omega \) and \( k_1^\ell \epsilon_\ell = k_1^\ell \epsilon_\ell^{-1/2} \). Since we assume that \( \epsilon_\ell = O(\ell^2) \), we thus obtain as expression for the scattering length in the limit \( \ell \to 0 \)

\[ \lim_{\ell \to 0} a(V_\ell) = - \lim_{\ell \to 0} \tan \left( \frac{\pi}{2} - \ell \omega \right) = - \frac{2}{\pi \omega}. \]

This shows the validity of (A3).

**(A6)** To verify assumption (A6) we have to compute the Fourier transform of \( V_\ell \), which equals

\[ \hat{V}_\ell(p) = \sqrt{\frac{2}{\pi}} k_1^\ell \left( (k_1^\ell)^2 + (k_1^-)^2 \right) \omega \left( |p| \epsilon_\ell - (k_1^-)^2 \ell^3 \omega \right), \]

with \( \omega(x) = \frac{1}{2\pi} \left( \sin(x) - x \cos(x) \right) \). Since \( |\omega(p)| \leq 1/3 \), one readily checks that \( |\hat{V}_\ell(p)| \leq 2|\hat{V}_\ell(0)| \) for \( \ell \) small enough.

**(A8)** Our next goal is to verify assumption (A8). Let \( U(x) = \frac{\pi^2}{4} \chi_{|x| \leq 1}(x) \) and set \( U_\ell(x) = \ell^{-2}U(x/\ell) \). For \( \lambda(\ell) = \left( 1 - \frac{2\pi^2}{\ell^2} \right)^2 \), the potential \( W_\ell(x) = \lambda(\ell)U_\ell(x) \) agrees with \( V_\ell^- \) on its support, so obviously \(-W_\ell(x) \leq -V_\ell^- \leq V_\ell(x) \) holds. The function \( U_\ell(x) \) is chosen such that \( p^2 - U_\ell(x) \) has a zero energy resonance. Indeed,

\[ \psi(x) = \begin{cases} \sin(\frac{\pi}{2} |x|)/|x|, & |x| \leq 1 \\ 1/|x|, & |x| \geq 1 \end{cases} \in L^2_{\text{loc}}(\mathbb{R}^3) \setminus L^2(\mathbb{R}^3) \]

is a generalized eigenfunction of \( p^2 - U \) and \( \psi_\ell(x) = \psi(x/\ell) \) is a generalized eigenfunction of \( p^2 - U_\ell \).

Therefore, \( U_\ell^{1/2} V_\ell^{-1/2} \) has the eigenvector \( U_\ell^{1/2} \psi_\ell \in L^2(\mathbb{R}^3) \) to the eigenvalue 1.

Note that our condition on \( \epsilon_\ell \) implies that \( \lambda(\ell) < 1 - c\ell \) for some constant \( c > 0 \) and small enough \( \ell \). Since \( V_\ell^- \leq W_\ell \), the largest eigenvalue of \( (V_\ell^-)^{1/2} U_\ell^{1/2} (V_\ell^-)^{1/2} \) is smaller or equal to \( \lambda(\ell) \), i.e.,

\[ \left\| (V_\ell^-)^{1/2} U_\ell^{1/2} (V_\ell^-)^{1/2} \right\| \leq \lambda(\ell) \leq 1 - c\ell. \quad (A.5) \]

Now choose \( C > 0 \) such that \( p^2 - |p|^b + C > 0 \) and define the operator

\[ R_\ell = (V_\ell^-)^{1/2} \frac{1}{p^2 - |p|^b} + C (V_\ell^-)^{1/2} - (V_\ell^-)^{1/2} \frac{1}{p^2 + C} = C (V_\ell^-)^{1/2} \].

Its trace norm equals

\[ \|R_\ell\|_1 = \frac{1}{2\pi^2} \left| V_\ell^- \right| \int \left[ \frac{p^2}{p^2 + C} - \frac{p^b}{p^b + C} \right] dp \],

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which tends to zero as $C \to \infty$ by monotone convergence. Since $\|V^-\|_1 = O(\ell)$, there is a $C$ such that $\|R_\ell\| < c\ell$, proving that

$$
\left\| \left( V^- \right)^{1/2} \frac{1}{p^2 - |p|^b + C} \left( V^- \right)^{1/2} \right\| \leq \left\| \left( V^- \right)^{1/2} \frac{1}{p^2 + C} \left( V^- \right)^{1/2} \right\| + \|R_\ell\|
$$

$$
< \left\| \left( V^- \right)^{1/2} \frac{1}{p^2} \left( V^- \right)^{1/2} \right\| + c\ell \leq 1,
$$

where we have used (A.5) in the last step. By the Birman-Schwinger principle, this shows that $p^2 - V^- - |p|^b + C \geq 0$, and hence also $p^2 + V_0 - |p|^b + C \geq 0$.

(A9) Note that $V^{1/2}_\ell \frac{1}{p^2} |V|^{1/2}$ has an eigenvalue $-\lambda^{-1} \neq 0$ if and only if $p^2 + \lambda V_\ell$ as a zero-energy resonance. Equivalently, the scattering length $a(\lambda V_\ell)$ diverges. According to our calculation (A.4), this happens for $\lambda > 0$ either satisfying

$$
k^+_\ell = k^-_\ell \tan \left( \sqrt{\lambda k^-_\ell (\ell - \epsilon_\ell)} \right) \tanh(\sqrt{\lambda k^+_\ell \epsilon_\ell})
$$

or $\sqrt{\lambda} k^-_\ell (\ell - \epsilon_\ell) = m\pi/2$ for odd integer $m$. The smallest $\lambda$ satisfying either of these equations is $\lambda = 1 + 4\ell \omega/\pi + O(\ell^2)$, hence the smallest eigenvalue of $1 + V^{1/2}_\ell \frac{1}{p^2} |V|^{1/2}$ is

$$
e_\ell = 4\ell \omega/\pi + O(\ell^2).
$$

We are left with showing that

$$
\left( 1 + V^{1/2}_\ell \frac{1}{p^2} |V|^{1/2} \right)^{-1} (1 - P_\ell)
$$

is uniformly bounded in $\ell$. This follows directly from [6, Consequence 1]. For the sake of completeness we repeat the argument here.

First, recall that $\phi_\ell$ denotes the eigenvector of $1 + V^{1/2}_\ell \frac{1}{p^2} |V|^{1/2}$ to its smallest eigenvalue $e_\ell$, and $J_\ell = \{ -1, V_{\ell} < 0 \}$. We also introduce the notation $X_\ell = |V|^{1/2}_\ell \frac{1}{p^2} |V|^{|1/2}_\ell$ and $X_\ell^\pm = |V^\pm|^{1/2}_\ell \frac{1}{p^2} |V^\pm|^{|1/2}_\ell$.

We now pick some $\psi \in L^2(\mathbb{R}^3)$ and set

$$
\varphi = (1 + V^{1/2}_\ell \frac{1}{p^2} |V|^{1/2})^{-1} (1 - P_\ell) \psi = \frac{1}{1 + J_\ell X_\ell} (1 - P_\ell) \psi = \frac{1}{J_\ell + X_\ell} J_\ell (1 - P_\ell) \psi. \quad (A.6)
$$

Below we are going to show that there exists a constant $c > 0$ such that for small enough $\ell$

$$
\langle \varphi | (1 - X^-_\ell) \varphi \rangle \geq c \|\varphi\|^2_{L^2}. \quad (A.7)
$$

In order to utilize this inequality we need the following lemma, which already appeared in [6, Lemma 1].

**Lemma 9.** Let $V = V_+ - V_-$, where $V_-, V_+ \geq 0$ have disjoint support. Denote $J = \{ -1, V_{\ell} < 0 \}$, $X = |V|^{1/2}_\ell \frac{1}{p^2} |V|^{1/2}$ and $X_\pm = V^{1/2}_\ell \frac{1}{p^2} V^{1/2}_\pm$. Then for any $\phi \in L^2(\mathbb{R}^3)$, we have

$$
\sqrt{2} \|\phi\| \|(J + X) \phi\| \geq \langle \phi | (X_+ - 1 - X_-) \phi \rangle. \quad (A.8)
$$
Proof. Decompose $\phi = \phi_+ + \phi_-$, such that $\text{supp}(\phi_-) \subseteq \text{supp}(V_-)$ and $\text{supp}(\phi_+) \cap \text{supp}(V_-) = \emptyset$. By applying the Cauchy-Schwarz inequality, we have

\begin{align*}
    \|(J + X)\phi\|\|\phi_+\| &\geq \Re\langle(1 + X_+ + X_-)\phi_+\rangle + \Re\|\phi_+\|V_+^{1/2} \frac{1}{\rho^2} V_-^{1/2} \phi_-\|
    \|(J + X)\phi\|\|\phi_-\| &\geq \Re\langle(1 + X_+ + X_-)\phi_-\rangle - \Re\|\phi_+\|V_+^{1/2} \frac{1}{\rho^2} V_-^{1/2} \phi_-\|
\end{align*}

We add the two inequalities and obtain

\begin{align*}
    \|(J + X)\phi\| (\|\phi_+\| + \|\phi_-\|) &\geq \langle(1 + X_+ + X_-)\phi_+\rangle + \langle(1 + X_+ + X_-)\phi_-\rangle = \langle\phi|(X_+ + 1 - X_-)\phi\rangle.
\end{align*}

Finally, we use that $\|\phi_+\| + \|\phi_-\| \leq \sqrt{2}\|\phi\|$, which completes the proof.

In combination with Lemma 9, the inequality (A.7) immediately yields

\begin{align*}
    \sqrt{2}\|\phi\|\|J_\ell X_\ell\| &\geq \langle\phi|(1 - X_\ell^-)\phi\rangle \geq c\|\phi\|^2,
\end{align*}

which further implies that

\begin{align*}
    \|\psi\| &\geq \|J_\ell(1 - P_\ell)\psi\| = \|(J_\ell + X_\ell)\varphi\| \geq \frac{c}{\sqrt{2}} \|\varphi\| = \frac{c}{\sqrt{2}} \|(1 + J_\ell X_\ell)^{-1}(1 - P_\ell)\psi\|
\end{align*}

proving uniform boundedness of $(1 + V_+^{1/2} \frac{1}{\rho^2} V_-^{1/2})^{-1}(1 - P_\ell)$.

It remains to show the inequality (A.7). To this aim we denote by $\phi^-_\ell$ the eigenvector corresponding to the smallest eigenvalue $\epsilon^-_\ell > 0$ of $1 - X^-_\ell$ and by $P_{\phi^-_\ell}$ the orthogonal projection onto $\phi^-_\ell$. The Birman-Schwinger operator $X^-_\ell$ corresponding to the potential $V^-_\ell$ has only one eigenvalue close to 1. All other eigenvalues are separated from 1 by a gap of order one. Hence there exists $c_1 > 0$ such that

\begin{align*}
    (1 - X^-_\ell)(1 - P_{\phi^-_\ell}) \geq c_1
\end{align*}

and, therefore,

\begin{align*}
    \langle\varphi|(1 - X^-_\ell)\varphi\rangle \geq c_1 \langle\varphi|(1 - P_{\phi^-_\ell})\varphi\rangle + \epsilon^-_\ell \langle\varphi|P_{\phi^-_\ell}\varphi\rangle \\
    = c_1 \|\varphi\|^2 + (\epsilon^-_\ell - c_1) \langle\varphi|P_{\phi^-_\ell}\varphi\rangle.
\end{align*}

With $P_{J_\ell\phi_\ell} = |J_\ell\phi_\ell\rangle\langle J_\ell\phi_\ell|$ being the orthogonal projection onto $J_\ell\phi_\ell$ we can write

\begin{align*}
    \varphi = (1 - P_{J_\ell\phi_\ell})\varphi,
\end{align*}

simply for the reason that, because of (A.6) and the fact that $P_\ell$ commutes with $B_\ell$, $P_{J_\ell\phi_\ell}\varphi = P_{J_\ell\phi_\ell}(1 + J_\ell X_\ell)^{-1}(1 - P_\ell)\psi = P_{J_\ell\phi_\ell}(1 - P_\ell)(1 + J_\ell X_\ell)^{-1}\psi = 0$.

Consequently,

\begin{align*}
    \|\varphi\|P_{\phi^-_\ell}\varphi\rangle \| = \|\varphi\|(1 - P_{J_\ell\phi_\ell})P_{\phi^-_\ell}\varphi\rangle \| \leq \|\varphi\|\|2\| \|(1 - P_{J_\ell\phi_\ell})P_{\phi^-_\ell}\| \| \\
    = \|\varphi\|\|2\| \|(1 - P_{J_\ell\phi_\ell})P_{\phi^-_\ell}\| \|^2 = \|\varphi\|\|2\| \|(1 - P_{\phi^-_\ell})J_\ell\phi_\ell\| \|^2.
\end{align*}

To estimate $\|(1 - P_{\phi^-_\ell})J_\ell\phi_\ell\|$, we apply Lemma 9 to $\phi_\ell$ and obtain

\begin{align*}
    \sqrt{2}\epsilon_\ell = \sqrt{2}\|(J_\ell + X_\ell)\phi_\ell\| \geq \langle\phi_\ell|(1 - X^-_\ell)\phi_\ell\rangle = \langle J_\ell\phi_\ell|(1 - X^-_\ell)J_\ell\phi_\ell\rangle \\
    = \epsilon^-_\ell |\langle J_\ell\phi_\ell|\phi^-_\ell\rangle|^2 + \|(1 - P_{\phi^-_\ell})J_\ell\phi_\ell|(1 - X^-_\ell)(1 - P_{\phi^-_\ell})J_\ell\phi_\ell\rangle \|^2 \geq c_1 \|(1 - P_{\phi^-_\ell})J_\ell\phi_\ell\| \|^2.
\end{align*}

This shows that $\|(1 - P_{\phi^-_\ell})J_\ell\phi_\ell\| = O(\ell^{1/2})$ and consequently (A.7) holds for small enough $\ell$. 

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By construction, $1 - (V^e_\ell)^{1/2} \frac{1}{p^2} (V^e_\ell)^{1/2}$ has no negative eigenvalues. By applying Lemma 12 to $\phi_\ell$, we obtain

$$\left\langle \phi_\ell \left| (V^e_\ell)^{1/2} \frac{1}{p^2} (V^e_\ell)^{1/2} \phi_\ell \right| \right\rangle \leq \sqrt{2}\epsilon_\ell \quad \text{and} \quad \left\langle \phi_\ell \left| \left(1 - (V^e_\ell)^{1/2} \frac{1}{p^2} (V^e_\ell)^{1/2}\right) \phi_\ell \right| \right\rangle \leq \sqrt{2}\epsilon_\ell. \quad (A.9)$$

We claim that this implies

$$\lim_{\ell \to 0} \langle J_\ell \phi_\ell | \phi_\ell \rangle = -1.$$

Indeed,

$$\langle J_\ell + X_\ell | \phi_\ell \rangle = \epsilon_\ell J_\ell \phi_\ell$$

and thus

$$(1 - \epsilon_\ell)\langle J_\ell \phi_\ell | \phi_\ell \rangle = -\langle \phi_\ell | X_\ell \phi_\ell \rangle = -\langle \phi_\ell | X^+_\ell \phi_\ell \rangle - \langle \phi_\ell | X^-_\ell \phi_\ell \rangle = -\langle \phi_\ell | (V^-_\ell)^{1/2} \frac{1}{p^2} (V^+_\ell)^{1/2} \phi_\ell \rangle - \langle \phi_\ell | (V^+_\ell)^{1/2} \frac{1}{p^2} (V^-_\ell)^{1/2} \phi_\ell \rangle.$$

Adding 1 on both sides yields

$$(1 - \epsilon_\ell)\langle (1 + J_\ell) \phi_\ell | \phi_\ell \rangle + \epsilon_\ell = \langle \phi_\ell | X^+_\ell \phi_\ell \rangle + \langle \phi_\ell | (1 - X^-_\ell) \phi_\ell \rangle - \langle \phi_\ell | (V^-_\ell)^{1/2} \frac{1}{p^2} (V^+_\ell)^{1/2} \phi_\ell \rangle - \langle \phi_\ell | (V^+_\ell)^{1/2} \frac{1}{p^2} (V^-_\ell)^{1/2} \phi_\ell \rangle.$$

By taking the absolute value, applying the Cauchy-Schwarz inequality and using $\langle \phi_\ell | (V^-_\ell)^{1/2} \frac{1}{p^2} (V^+_\ell)^{1/2} \phi_\ell \rangle - \langle \phi_\ell | (V^+_\ell)^{1/2} \frac{1}{p^2} (V^-_\ell)^{1/2} \phi_\ell \rangle = O(\epsilon_\ell^{1/2})$, we obtain

$$\langle (1 + J_\ell) \phi_\ell | \phi_\ell \rangle \leq \frac{1}{1 - \epsilon_\ell} \left( 1 + 2\epsilon_\ell + 2\sqrt{\langle \phi_\ell | X^+_\ell \phi_\ell \rangle \langle \phi_\ell | X^-_\ell \phi_\ell \rangle} \right) = O(\epsilon_\ell^{1/2}). \quad (A.10)$$

Finally, to bound $\langle |V^e_\ell|^{1/2} | \phi_\ell \rangle$, we note that $\langle |V^-_\ell|^{1/2} | \phi_\ell \rangle \leq \|V^\ell\|^{1/2} O(\ell^{1/2})$. For the analogous bound with $V^e_\ell$ replaced by $V^e_\ell^+$, we can again employ Lemma 12, which implies that $\sqrt{2}\epsilon_\ell \geq \langle \phi^+_\ell | X^+_\ell + 1 | \phi^+_\ell \rangle \geq \|\phi^+_\ell\|^2_2$ (where $\phi^+_\ell = \frac{1}{2}(1 + J_\ell | \phi_\ell \rangle$, hence $\langle |V^e_\ell|^{1/2} | \phi_\ell \rangle \leq \|V^e_\ell\|^{1/2} \|\phi^+_\ell\|_2 \leq O(\ell^{1/2})$. This completes the proof.

## B The Definition of $T_c$

In this appendix we shall show that the equations (1.14) define $T_c$ and $\bar{\mu}$ uniquely. To start, let $F : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^2$ be defined by its components

$$F_1(\nu, T) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \left( \frac{\tanh \left( \frac{p^2 - \nu}{2T} \right)}{p^2} - \frac{1}{p^2} \right) \, d^3p \quad (B.1)$$

and

$$F_2(\nu, T) = \nu + \frac{2\nu}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \frac{1}{1 + e^{\frac{p^2 - \nu}{2T}}} \, d^3p. \quad (B.2)$$

We clearly have $\partial F_1 / \partial T < 0$ and $\partial F_2 / \partial \nu > 0$ (since $V \geq 0$ by assumption). By dominated convergence, we may interchange the derivative with the integral and compute

$$\frac{\partial F_1}{\partial \nu} = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \kappa \left( \frac{p^2 - \nu}{2T} \right) \kappa'(\frac{p^2 - \nu}{2T}) \, d^3p, \quad (B.3)$$
where \( \kappa(x) = x / \tanh(x) \). If \( \nu \leq 0 \), this is positive, since \( \kappa'(t) \geq 0 \) for \( t \geq 0 \). If \( \nu > 0 \), on the other hand, we can integrate out the angular coordinates and change variables to \( \pm t = p^2 - \nu \), respectively, to obtain

\[
\frac{\partial F_1}{\partial \nu} = \frac{1}{4\pi^2} \int_0^\infty \frac{\kappa'(t)}{\kappa^2(\frac{t}{\sqrt{\nu}})} \frac{\nu + t}{\sqrt{\nu - t}} \, dt - \frac{1}{4\pi^2} \int_0^\nu \frac{\kappa'(t)}{\kappa^2(\frac{t}{\sqrt{\nu}})} \frac{\nu - t}{\sqrt{\nu - t}} \, dt. \tag{B.4}
\]

Since \( \sqrt{\nu + t} > \sqrt{\nu - t} \), it is clear that this sum is positive, i.e., \( \partial F_1/\partial \nu > 0 \).

We proceed similarly to show that \( \partial F_2/\partial T > 0 \). We have

\[
\frac{\partial F_2}{\partial T} = \frac{\nu}{2T^2} \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \frac{p^2 - \nu}{\cosh^2 \left( \frac{\sqrt{\nu} p}{T} \right)} \, d^3 p = \frac{\nu}{2T^2 \sqrt{2\pi}} \left( \int_0^\infty \frac{t \sqrt{\nu + t}}{\cosh^2 \left( \frac{\sqrt{\nu} t}{T} \right)} \, dt - \int_0^\nu \frac{t \sqrt{\nu - t}}{\cosh^2 \left( \frac{\sqrt{\nu} t}{T} \right)} \, dt \right) > 0. \tag{B.5}
\]

In particular, the Jacobian determinant of \( F \) is strictly positive.

For fixed \( T \), we have \( \lim_{\nu \to -\infty} F_2(\nu, T) = -\infty \) and \( \lim_{\nu \to \infty} F_2(\nu, T) = \infty \). Hence there is a unique solution \( \nu_T \) of the equation \( F_2(\nu, T) = \mu \), for any \( \mu \in \mathbb{R} \), and \( \nu_T \) is decreasing in \( T \). Moreover, the function \( T' \mapsto F_1(\nu_T, T) \) is strictly decreasing, and hence the equation \( F_1(\nu_T, T) = \lambda \) has a unique solution for \( \lambda = -1/(4\pi a) \), hence a strictly decreasing function of \( \lambda \) for \( a > 0 \).

For \( \mu \leq 0 \), one checks that \( \lim_{T \to 0} F_1(\nu_T, T) \leq 0 \), hence \( T_c = 0 \). For \( \mu > 0 \), however, \( \lim_{T \to 0} F_1(\nu_T, T) = \infty \), hence \( T_c > 0 \) for any \( a < 0 \).

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