Spectral Renormalization Group for the Gaussian Model

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The critical behaviour of the Gaussian model on different non-spatial networks is studied by means of the spectral renormalization group we have recently proposed. We find non-mean field behaviour for the Gaussian model on the Cayley tree and the hierarchical lattice. The results are shown to be consistent with those from exact summation and finite size scaling approaches.

I. INTRODUCTION

In a recent publication \cite{1}, we have proposed a spectral renormalization group scheme as a general approach to the study of fluctuations on arbitrary networks. Our approach is modelled on the “field theoretic” renormalization group a\`{a} Wilson \cite{2,3}. In our spectral renormalization group approach, the fluctuations of the order parameter are expanded in terms of the eigenvectors of the graph Laplacian\cite{4,5}. Elimination of the large eigenvalue fluctuations and rescaling of the effective hamiltonian yield, in the same spirit as in the Wilson renormalization group a\`{a} Wilson \cite{2,3}. In our spectral renormalization group scheme, we have also computed the critical exponents

\begin{equation}
\psi(x) = \sum_{x} [\frac{1}{2} \{ r_{\infty} \psi^{2}(x) - \psi(x) \nabla^{2} \psi(x) \} + u_{\infty} \psi^{4}(x) - h \psi(x)] .
\end{equation}

where the integral is over the volume of the system, \( r_{0} \) is proportional to the reduced temperature \( t = (T - T_{c})/T_{c} \) and \( m \) is the magnetization per spin. For comparison, exact enumeration and finite size scaling results are also presented. To double-check the consistency of our method, we have also computed the critical exponents for the Gaussian model on the square and cubic lattices.

The spectral method allows us to solve the Gaussian model exactly on arbitrary lattices. We can identify the upper critical spectral dimension on complex networks, and express critical exponents in terms of the scaling exponent, if this exists, of the spectral density of the graph Laplacian for small eigenvalues.

The study of fluctuations, and the Gaussian model in particular, on arbitrary networks has a rather long history. Hattori et al.\cite{10} have given a rigorous mathematical definition of the spectral dimension from a study of the infrared singularities of the Gaussian model and random walks for generic networks in the thermodynamic limit. Burioni and Cassi \cite{11} have shown that their results apply to very general distributions of stepping probabilities (or local temperatures).

The organization of the paper is as follows. In Section 2, we define the spectral renormalization group for the Gaussian model on a generic network. In Section 3, we implement this scenario on the Cayley tree (a regular network) and the diamond (hierarchical) lattice, as well as the square and cubic lattices. In Section 4 we present exact calculations and finite size scaling results in order to check our results. In Section 5, we provide a discussion pointing to further work.

II. THE SPECTRAL RENORMALIZATION GROUP FOR THE GAUSSIAN MODEL

The effective Ginzburg-Landau “Lagrangian” for a scalar order parameter \( \psi(x) \) is given by,

\begin{equation}
H = \int dV \{ \frac{1}{2} \{ r_{\infty} \psi^{2}(x) - \psi(x) \nabla^{2} \psi(x) \} + u_{\infty} \psi^{4}(x) - h \psi(x) \} .
\end{equation}

where the integral is over the volume of the system, \( r_{0} \) is proportional to the reduced temperature \( t = (T - T_{c})/T_{c} \) and we will assume that \( H \) is expressed in units of the thermal energy \( k_{B} T \) where \( k_{B} \) is the Boltzmann constant. The Gaussian model \cite{8} is equivalent to omitting the fourth order coupling term in Eq. (1). This model is defined only for temperatures above the critical temperature, i.e., for \( r_{0} > 0 \). Nevertheless one may formally compute the exponent \( \delta \).

For a field \( \psi(i) \), living on the nodes of an arbitrary network, the Gaussian model can be written as

\begin{equation}
H = \frac{1}{2} \sum_{ij} N \psi(i) \delta_{ij} + L_{ij} \psi(j) - h \sum_{i} \psi(i) .
\end{equation}

The usual Laplace operator appearing in the Ginzburg-Landau expansion has been replaced by the graph Laplacian \cite{4,5}, with the matrix elements,

\begin{equation}
L_{ij} = d_{i} \delta_{ij} - A_{ij} ,
\end{equation}

where \( A \) is the adjacency matrix of the network and \( d_{i} \) is the degree of the \( i \)th node. The expression in Eq. (2) is now very general, applicable to arbitrary networks, with only the requirement that the matrix \( A \) be symmetric, so that its eigenvalues are real.

Expanding the field \( \psi(i) \) in terms of eigenvectors \( u_{\mu} \) of the Laplace operator, \( \psi(i) = N^{-1/2} \sum_{\mu} \psi_{\mu} u_{\mu}(i) \), the
Hamiltonian is obtained in diagonal form,

$$H = \frac{1}{2} \sum_{\mu=1}^{N} [r_{\mu} + \omega_{\mu}] \hat{\psi}_{\mu}^{2} - \hbar \hat{\psi}_{1}. \quad (4)$$

Here $\omega_{\mu}$ are the eigenvalues of $L$. The eigenvalues are ordered so that $\omega_{1} \leq \omega_{2} \leq \cdots \leq \omega_{N}$, with $\omega_{1} = 0$. We will assume the network to be connected so that $\omega_{2} > 0$.

For this system, the partition function is immediately obtained from

$$Z = \int_{-\infty}^{\infty} \prod_{\mu} d\hat{\psi}_{\mu} e^{-H}, \quad (5)$$

and the free energy is given, up to constant terms, by

$$F = \frac{1}{2} \sum_{\mu=1}^{N} \ln(r_{\mu} + \omega_{\mu}) - \hbar^{2}/r_{0}. \quad (6)$$

(Hereforth we will drop the external field term unless we are directly dealing with it.)

Note that in Eq. (4) there is a difficulty in going over from a sum (over $\mu = 1, \ldots, N$) to an integral over the eigenvalues. In general the eigenvectors $u_{\mu}$, and consequently the $\hat{\psi}_{\mu}$, no not possess, e.g., the rotational symmetries valid on periodic lattices. Therefore in general it is not justified to try to extract the renormalization factors by rewriting the Hamiltonian as

$$\frac{1}{2} \int_{0}^{\Omega} d\omega \rho(\omega) [r_{\omega} + \omega_{\mu}] \hat{\psi}_{\omega}^{2}, \quad (7)$$

where $\Omega$ is the largest eigenvalue. On the other hand, after the Gaussian integrals have been carried out, this difficulty is not there for the free energy (or its derivatives, such as the specific heat or the two-point correlation function [10]) and one may formally write,

$$F = \frac{1}{2} \int_{0}^{\Omega} d\omega \rho(\omega) \ln(r_{\omega} + \omega) \quad (8)$$

For many networks, even in the thermodynamic limit the spectral density does not become a smooth function which behaves as a power law, $\rho(\omega) \sim \omega^{\beta}$ for small $\omega$. In particular, the spectral density of the Barabasi-Albert scale free network [13] grows exponentially for small $\omega$, Ref. [11]). As in the case of the Cayley tree and the diamond hierarchical lattice, the spectral density may be nonzero only on a union of discrete sets of measure zero, such that $\rho(\omega)$ is zero almost everywhere, and one should set $\beta = 0$.

Even for regular periodic networks such as the square or cubic lattices, with periodic boundary conditions, the convergence of the numerically calculated spectral density to the thermodynamic limit is very slow, especially in the small $\omega$ region, as illustrated in Supplementary Materials [12]. For arbitrary networks where one cannot compute $\rho(\omega)$ analytically, one should be aware that its numerical determination may call for prohibitively large network sizes. Moreover, for general networks (as in the case of the random matrix [14–16]), it is not true that the first non-zero eigenvalue tends to zero with the network size $N$, and for connected networks one should then investigate the scaling behaviour, if any, of $\rho(\omega - \omega_{2})$.

We would now like to illustrate how we can implement field theoretic renormalization group ideas on a system which, besides not having an a priori known spectral density, is not embedded in a metric space, i.e., there is no concept of length. We have two possible strategies for eliminating the “small wavelength,” or large $\omega$ fluctuations from the partition function and computing the renormalization factors. In the absence of a “length like” quantity, the first method which comes to mind is to scale the number of nodes $N$ by a constant factor $B$, which is analogous to scaling the volume. The second method consists of scaling $\Omega$ by a constant $B$, in analogy with the usual renormalization group a la Wilson [2,3,8]. These two strategies are implemented below.

### A. Scaling the number of nodes

One obvious choice is to simply scale the total number of nodes, or the “volume” of the network. This means integrating out the degrees of freedom with larger values of $\omega$, so that we keep the first $N/B$ eigenvalues in the effective Hamiltonian. (Picking the scale factor $B$ in keeping with the overall symmetries of the system is convenient: if no such obvious scale symmetry is available, $B = N/n$, with $n$ integer, eliminates spurious points from the scaling plots.) We get

$$H_B = \frac{1}{2} \sum_{\mu=1}^{(N/B)} [r_{\mu} + \omega_{\mu}] \hat{\psi}_{\mu}^{2} - \hbar \hat{\psi}_{1}. \quad (9)$$

Restoring the Hamiltonian to its full range calls for rescaling factors to be inserted, viz.,

$$H' = \frac{1}{2} \sum_{\mu=1}^{N} [r_{\mu} B^{-\phi_{1}} + B^{-\phi_{1}-\phi_{2}}] (\hat{\psi}'_{\mu})^{2} - \hbar \hat{\psi}'_{1}. \quad (10)$$

where $\hat{\psi}' = z \hat{\psi}$, and $z$ is the so called “wave function renormalization.” [2,8]

We define the rescaling factors $\sigma_{1}^{Y}$ and $\sigma_{2}^{Y}$ as

$$\sigma_{1}^{Y}(B) \equiv \sum_{\mu=1}^{N} (N/B)_{1}^{-\phi_{1}} = B^{\phi_{1}} \quad (11)$$

and

$$\sigma_{1}^{Y}(B) \sigma_{2}^{Y}(B) \equiv \sum_{\mu=1}^{N} (N/B)_{1}^{-\phi_{1}+\phi_{2}} = B^{\phi_{1}+\phi_{2}}. \quad (12)$$

where clearly $\phi_{1} = 1$, while $\sum_{\mu=1}^{N} \omega = N \sigma$. 

To find $z$ we require the coefficient of the Laplace term to remain fixed, and get $(\sigma_1^V \sigma_2^V)^{-1/2} = 1$, which yields $z = (\sigma_1^V \sigma_2^V)^{-1/2}$.

The renormalization of the reduced temperature is then given by,

$$ r' = (\sigma_1^V)^{-1} z^2 r_0 = \sigma_2^V r_0 = B^{\phi_2} r_0 . $$

The external field term rescales as $h' \varphi_1' = h \varphi_1$, so that

$$ h' = zh' . $$

The Kadanoff scaling relations for the renormalized free energy per node are

$$ f(t, h) = B^{-1} f'(B^Y t, B^Y h) . $$

From Eqs. (13-14), we find $Y^V_t = \phi_2$ and $Y^V_h = (1+\phi_2)/2$. Finally

$$ f(t, 0) \sim t^{2-\alpha} , $$

yields the specific heat critical exponent

$$ \alpha = 2 - \frac{1}{\phi_2} . $$

Setting $t = 0$ in Eq. (15), we similarly obtain, the magnetic field critical exponent,

$$ \delta = \frac{1 + \phi_2}{1 - \phi_2} . $$

**B. Scaling the maximum eigenvalue**

An alternative strategy for eliminating degrees of freedom with large $\omega$ is to eliminate all degrees of freedom with $\omega \geq \Omega/B$, where $B$ is again an arbitrary scale factor. In this case, we define the scaling factors $\sigma_1^\Omega$ and $\sigma_2^\Omega$ as

$$ \sigma_1^\Omega(B) \equiv \frac{N}{\sum_{\mu=1}^N} = B^{p_1} , $$

where

$$ \mu_B = \sup \{ \mu \in [1, N] : \omega_\mu < \Omega/B \} , $$

and $p_1$ is now a non-trivial scaling exponent, with $N/N' = B^{p_1}$. We also have,

$$ \sigma_1^\Omega(B) \sigma_2^\Omega(B) \equiv \frac{N \omega}{\sum_{\mu=1}^N \omega} = B^{p_1+p_2} . $$

Using $\sigma_1^\Omega$ and $\sigma_2^\Omega$ to rescale the truncated Hamiltonian, one can derive in a way completely analogous to Eqs. (10-11) that the recursion relation for the reduced temperature is given by, $r' = \sigma_1^\Omega r_0$. Going over to an integral representation of the sums (notwithstanding the caveat mentioned above) is sufficient to show from simple power counting that $p_2 = 1$.

Now taking $f = B^{-p_1} f'(t', h')$, with $t' = B^{\phi_1} t$ and $h' = B^{\phi_2} h$, one finds that $Y^\Omega_t = p_2 = 1$ and $Y^\Omega_h = (1+p_1)/2$. From Eq. (16) one gets,

$$ \alpha = 2 - \frac{p_1}{p_2} = 2 - 1 , $$

and similarly,

$$ \delta = \frac{p_1 + p_2}{p_1 - p_2} = \frac{p_1 + 1}{p_1 - 1} . $$

FIG. 1: (Color online) The degeneracies, $\tau_n$, of the distinct eigenvalues $\omega^{(n)}$ for the Cayley tree (with branching number $b = 3$), drawn for $r = 9$ generations. The smallest nonzero eigenvalue tends to zero as $b^{-r}$. We do not display $\omega_1 = 0$. The degeneracies fall on a straight line with unit slope (shown in red) in this log-log plot. Nevertheless, the spectral density is zero almost everywhere, and the exponent $\beta$, defined via $\rho(\omega) \sim \omega^\beta$, is equal to zero. See text.

If a continuous spectral density obtains in the thermodynamic limit, it is easy to show that the exponents $p_1$ and $\phi_2$ are related to $\beta$ via $p_1 = 1 + \beta$ and $\phi_2 = 1/(1 + \beta)$. From $\beta \geq 0$, we are ensured that $p_1 \geq 1$ and $\phi_2 \leq 1$, and finally, $\alpha = 1 - \beta$ and $\delta = (2 + \beta)/\beta$.

**III. SPECTRAL RG FOR SOME DETERMINISTIC NETWORKS**

In this section we present numerical and semi-analytical results for the spectral renormalization group for the Cayley tree and the diamond lattice, as well as the square and cubic lattices for comparison. We display the scaling behaviour for the “volume” renormalization, as described in Section 2.A; the analogous graphs for the eigenvalue renormalization (Section 2.B) are provided in the Supplementary Material [12].
A. Gaussian model on the Cayley tree

An inspection of Fig. 1 shows that the spectral density of the graph Laplacian for the Cayley tree can be written as

\[ \rho(\omega) = \sum_{n=1}^{r} \tau_n \delta(\omega - \omega^{(n)}) , \]

where \( \omega^{(n)} \) and \( \tau_n = \tau(\omega^{(n)}) \propto \omega^{(n)} \) are the nth distinct eigenvalue and its degeneracy. In the interval \( 0 < \omega < \omega^* \), where \( \omega^* \) is the value at which \( \tau(\omega) \) is maximum, we see that, for branching number \( b \), \( \tau_n = b^{n+1} - b^n + 1 \) and \( \omega^{(n)} \approx a_n b^{-(r-n+1)} \), where \( a_n \) does not depend strongly on \( n \), at least for small \( n \). The number of eigenvalues within this interval is approximately \( \approx b^{r+1}/2 \sim N_r/2 \).

(Closed form expressions for the eigenvalues of the graph Laplacian of the Cayley tree are known [13, 19]; however, to our knowledge, explicit solutions for all the eigenvalues are not available.)

From a knowledge of the structure of the discrete eigenvalue spectrum, one may directly compute the value of the rescaling factors. Choosing \( B = b^k \equiv B_k \), we have, for the rescaling factors,

\[ \sigma^Y_1(B_k) = \frac{N}{\sum_{n=0}^{r-1-k} \tau_n} = B^{\phi_1} , \]

and

\[ \sigma^Y_1(B_k)\sigma^Y_2(B_k) = \frac{N\omega}{\sum_{n=0}^{r-1-k} \tau_n \omega^{(n)}} \propto B^{\phi_1+\phi_2} , \]

where we have made the approximation, \( \omega^{(n)} \propto \text{const.} b^{-(r-n+1)} \). Doing the sums for \( r \gg 1 \), \( 1 \ll k < r \), (i.e. in the small \( \omega \) region), we find \( \phi_1 = \phi_2 = 1 \).

The scaling behavior of \( \sigma^Y_1 \) and \( \sigma^Y_2 \), obtained numerically, are shown in Fig. 2 confirming our semi-analytical result.

B. Gaussian model on the diamond lattice

We next consider the hierarchical lattice generated by iteratively replacing each edge of a rhombus by yet another rhombus, sometimes also known as the diamond lattice. In Fig. 3 for a lattice obtained from \( r \) iterations, we see \( r \) different families of eigenvalues, each corresponding to the \( r-1 \) different sets of nodes with degrees \( d_q \sim 2^k \), \( k = 0, \ldots, r - 1 \). In Fig. 4, \( \sigma^Y_1 \) and \( \sigma^Y_2 \) are calculated numerically, again with a set of scale factors \( B(k) = 2^k \) respecting the discrete scaling symmetry of the lattice. The exponents are found to be \( \phi_1 = 1 \), \( \phi_2 = 1.01 \pm 0.03 \).

C. Square and cubic lattices

For completeness, we have also computed the spectral densities of the square and cubic lattices, and their rescaling factors. The Laplace eigenvalues for the square and cubic lattices are analytically given by

\[ \omega_q = 4 \sum_{j=1}^{d} \sin^2 \left( \frac{q_j}{2} \right) \]

where we have indexed the eigenvalues by the wave vector, the lattice spacing is unity, \( d \) is the Euclidean dimension and \( q_j = (\pi n_j N^{-1/d}) \) are the Cartesian components of \( q \). In the limit of small \( q = ||q|| \), \( \omega_q \propto q^2 \). Then the spectral density is \( \rho(\omega) \propto \omega^\beta \) with \( \beta = d/2 - 1 \).
and $\ln \sigma_2$. The scale factors have been chosen as $B(k) = 2^k$ and $k = 1, \ldots, r - 1$, for a total number of generations $r = 7$. Once again, $\phi_1 = 1$ and $\phi_2 = 1.01 \pm 0.03$ which gives $\alpha = 1.01 \pm 0.03$.

### Table I: The critical exponents $\alpha$ and $\delta$ obtained from the Spectral Renormalization Group for the Gaussian model on spatial and non-spatial networks from scaling the number of nodes $N$, see Eqs. [17, 18]. The exponents we find for the square and cubic lattices are consistent with the Gaussian values, $\alpha = (4 - d)/2$ and $\delta = (d + 2)/(d - 2)$. Moreover, replacing $d$ by the spectral dimension, $d = 2\beta + 2$, reproduces our semi-analytic results for the Cayley and diamond lattices, see text and Figs. (1,3). The positions of the first nonzero Laplace eigenvalues of the graph Laplacian for all the different lattices are shown in Fig. (4). The critical scaling behaviour of the specific heat critical exponent has been chosen as $B(k) = 2^k$ and $k = 1, \ldots, r - 1$, for a total number of generations $r = 7$. Once again, $\phi_1 = 1$ and $\phi_2 = 1.01 \pm 0.03$ which gives $\alpha = 1.01 \pm 0.03$.

| Model   | $\alpha$   | $\delta$ | $\beta$ | $\alpha_{\text{Gauss}}$ | $\delta_{\text{Gauss}}$ |
|----------|------------|----------|---------|-------------------|-----------------|
| Square   | 0.996 ± 0.003 | $\infty$ | 0.00 ± 0.02 | 1 | $\infty$ |
| Cubic    | 0.5 ± 0.1 | 4.9 ± 0.9 | 0.5 ± 0.1 | 1/2 | 5 |
| Cayley3  | 1.03 ± 0.04 | $\infty$ | 0 | 1 | $\infty$ |
| Cayley5  | 1.06 ± 0.09 | $\infty$ | 0 | 1 | $\infty$ |
| Diamond  | 1.01 ± 0.03 | $\infty$ | 0 | 1 | $\infty$ |

The numerical results for the non-trivial exponents are $\beta = 0.00 \pm 0.02$, $p_1 = 1.00 \pm 0.01$, and independently, $\phi_2 = 0.996 \pm 0.003$. For the cubic lattice we find $\beta = 0.5 \pm 0.1$, $p_1 = 1.49 \pm 0.02$, and independently, $\phi_2 = 0.66 \pm 0.05$.

The plots of the spectral density and rescaling factors are provided in Supplementary Materials [12]. It is instructive to compare the accuracy obtainable from the rescaling factors as opposed to the spectral densities themselves, which converge very slowly, in the small $\omega$ region, to the asymptotic form of their spectral densities.

### IV. COMPARISON WITH CONVENTIONAL METHODS

#### A. Exact Enumeration

The specific heat can be calculated to leading order explicitly, via

$$c_h \simeq k_B \frac{T^2}{2} \sum_{\mu=1}^{N} \frac{1}{(r_0 + \omega_{\mu})^2}.$$  \hspace{1cm} (28)

Specific heat exact summation results for the Cayley tree, the diamond lattice, and square and cubic lattices, are shown in Fig. (5). The critical scaling behaviour of $c_h$ is obtained for $r_0$, between the first nonzero Laplace eigenvalue and the van Hove singularity in the Laplacian spectral density (which falls near unity), and yields the specific heat critical exponent $\alpha$.

#### B. Finite Size Scaling for non-spatial lattices

In order to eliminate possible errors due to the finiteness of the lattices considered, a finite size scaling (FSS) analysis adapted to non-spatial lattices, was also performed on each lattice. The relevant effective “field” in this case is taken to be $N^{-1}$, in place of the linear scale $\frac{1}{N}$, as is customary for spatial lattices.
The specific heat is found to scale as

\[ \frac{c_h(t, N^{-1})}{N^{2Y-1}} = \begin{cases} \text{const.}, & x < 1 \\ x^{1-2Y}, & x > 1. \end{cases} \] (29)

where \( x = t^{1/Y} N \) and it can easily be shown that \( \alpha = (2Y-1)/Y \). For spatial lattices in \( d \) dimensions, one finds \( Y^{-1} = d\nu \), where \( \nu \) is the correlation length exponent. Here we only display the scaling behaviour for the Cayley tree for \( b = 3 \) (see Fig. 6).

![FIG. 6: (Color online) Finite Size Scaling for the Cayley tree with branching number \( b = 3 \). Five different sized lattices are used. The number of generations on the tree is \( r \), with \( N \sim b^{r+1} \). The collapse is obtained for \( Y = 1 \), yielding \( \alpha = 1 \) for the specific heat exponent. (See Text).](image)

![FIG. 7: (Color online) The crossover function \( F(N, t) \) for the specific heat of the Cayley tree, see Eq. (31).](image)

The finite size scaling behavior for the Cayley tree may, in fact, be deduced from a knowledge of the values and degeneracies of the eigenvalues of the Laplacian, which have already been discussed in Section 3. The leading singularity in the specific heat is given by

\[ c_h = \frac{1}{N^r} \sum_{n=0}^{r-1} \frac{\tau_n}{[t + \omega(n)]^2}. \] (30)

Substituting for \( \tau_n \) and \( \omega(n) \), where \( b^r \sim N_r \), one gets, after multiplying and dividing the RHS by \( N_r^2 \) and simplifying,

\[ \frac{c_h}{N_r} \simeq F(N_r t), \] (31)

yielding \( Y = 1 \), and therefore \( \alpha = 1 \). We have calculated the scaling function \( F(N_r t) \) numerically. The collapse of the curves for different network sizes is displayed in Fig. 7.

V. CONCLUSIONS AND DISCUSSION

In summary, using the spectral renormalization group method, we have calculated the specific heat and external field critical exponents \( \alpha \) and \( \delta \) for the Gaussian model on the square, cubic, diamond lattices and the Cayley tree. We have checked our results by using exact summation of the leading term in the specific heat, and the finite size scaling method.

Numerical results for the scaling behavior of the spectral density \( \beta \), as well as the critical exponents \( \alpha \) and \( \delta \) for the different spatial and non-spatial lattices considered in this paper are given in Table I.

Our results for the square and cubic lattices are consistent with the exactly known (non-mean field) values of \( \alpha \) and \( \delta \) for the Gaussian model in two and three dimensions, namely, \( \alpha = (4 - d)/2 \) and \( \delta = (d + 2)/(d - 2) \), respectively, where \( d \) is the Euclidean dimension. Both the Cayley tree and the diamond lattice have discrete spectra, i.e., their spectral density is nonzero only on a set of measure zero, so that \( \beta \), the scaling exponent of their spectral density is zero. The spectral dimension \( d \), which is given by \( \beta = d/2 - 1 \) [10] is therefore equal to 2. We find Gaussian (non-mean field) exponents for both these networks, equal in value to the Gaussian model exponents in two-dimensional Euclidean space.

Bradde et al. [17] have diagonalized the Ising model by expanding the fluctuations in terms of the eigenvectors of the matrix of connection probabilities (which reduces to the adjacency matrix in the deterministic case) and analyzing the problem in terms of the scaling form of the spectral distribution of this probability matrix, and defining the analogue of an upper critical dimension (a Ginzburg criterion) in terms of this scaling exponent.

The present paper is complementary to the approach of Bradde et al., in the sense that we concentrate on the spectral distribution of the graph Laplacian, in the region of small eigenvalues (i.e., the infrared singularity). Both the Cayley tree and the diamond lattice are embeddable.
in two dimensions. However we treat them as non-spatial networks.

Where a power-law behavior does obtain for the spectral density, one may to show that the upper critical value for $\beta$ is unity. Replacing the sum in Eq. (28) by an integral,

$$c_h \propto \int_0^\Omega \frac{d\omega \omega^\beta}{(r_0 + \omega)^2},$$

we see that for $\beta \geq 1$, the “ultraviolet” singularity cancels the critical divergence and one gets the mean field result, $\alpha = 0$ identically. Substituting $\beta = 1$ in $p_1 = 1 + \beta$, and using Eqs. (22, 23), yields the mean field values $(0, 3)$ for $(\alpha, \delta)$, as expected.

Goldenfeld [8] discusses the anomaly of obtaining non-classical (non-mean field) values for the critical exponents of the Gaussian model, which is based on a Landau expansion (see Eq. [1]) and points out that the anomaly can be understood in terms of the dangerous irrelevant field $u_0$. We may repeat the argument in the present case, using the scaling relation in Eq. (15) with a third scaling field, $u$, so that $f = B^{-x} f'(B^{\beta x} t, B^{\gamma x} h, B^{\delta x} u)$, yielding

$$m(0, h, u) = h^{-Y_h(x) + x}/Y_h(x) M(u h^{-Y_u(x)/Y_h(x)}),$$

for the magnetization on the critical isotherm. Here $x = 1$ or $x = p_1 = 1 + \beta$ depending on whether we scale the volume (Section 2.A) or the maximum eigenvalue (Section 2.B), respectively. We will require that $Y_h \propto 1 - \beta$. The Landau expansion gives $m(0, h, u) \propto (h/u)^{1/3}$; therefore one takes [8] the scaling function $M(v) \sim v^{-1/3}$ in the limit of small $h$. Eq. (33) then gives

$$-1 + \frac{x}{Y_h(x)} + \frac{1}{3} \frac{Y_u(x)}{Y_h(x)} = \frac{1}{\delta}.$$  

For $x = 1 + \beta$, one has $Y_h = (2 + \beta)/2, Y_u = 1 - \beta$. For $x = 1, Y_h = (2 + \beta)/(2(1 + \beta))$ and one must take $Y_u = (1 - \beta)/(1 + \beta)$. In either case, $\beta$ cancels out of the final result, yielding the mean field value for $\delta$.

One may similarly show that keeping $u_0$ in the calculation and using the Landau expansion for $h = 0$ to get $m \propto \sqrt{-r_0/u_0}$ gives the order parameter exponent $\beta_m$ to be $1/2$, from which one may derive $\alpha = 0$ using the scaling relation $\beta_m(1 + \delta) = 2 - \alpha$.

In a forthcoming publication we will extend our results to $\psi^4$ theory, where we find, as in the Euclidean case, that to first order in the $\psi^4$ coupling constant, the Gaussian fixed point is stable. Higher order calculations are possible using a perturbation expansion in both the coupling constant and $1 - \beta \equiv \epsilon$. Further work is in progress to apply the spectral renormalization group to networks with annealed and quenched disorder.

VI. SUPPLEMENTARY MATERIAL

Plots for the numerical calculation of the exponents $p_1, p_2$ for the Cayley tree and the diamond lattice, the spectral density and the exponent $\beta$ for the square lattice of size $N = 4 \times 10^4$ and the cubic lattice of size $N = (35)^3$, and the exponents $\phi_1, \phi_2, p_1$ and $p_2$ for the rescaling factors, together with a summary of the critical exponents obtained, are included in the Supplementary Materials [12].

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