BIMEROMORPHIC AUTOMORPHISMS GROUPS OF CERTAIN CONIC BUNDLES

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Abstract. We study the groups of biholomorphic and bimeromorphic automorphisms of conic bundles over certain compact complex manifolds of algebraic dimension zero.

1. Introduction

Let $X, Y$ be compact connected manifolds endowed with a holomorphic map $p : X \to Y$. Assume that there is an analytic subset $Z \subset Y$ such that for every point $y \not\in Z$ the fiber $P_y := p^*(y)$ is reduced and isomorphic to $\mathbb{P}^1$. In this situation we call $P_y$ a general fiber and $X$ (or a triple $(X, p, Y)$) a conic bundle over $Y$. If $X$ is a projectivization $\mathbb{P}(E)$ of a rank 2 vector bundle $E$ over $Y$, we will say that $X$ is a linear conic bundle over $Y$. If $X$ is a holomorphically locally trivial fiber bundle over $Y$ with fiber $\mathbb{P}^1$ we call it a $\mathbb{P}^1$-bundle.

We study the groups $\text{Bim}(X)$ and $\text{Aut}(X)$ of bimeromorphic and biholomorphic selfmaps of $X$, respectively, in case where $X$ is a conic bundle.

We are interested in the following properties of groups.

Definition 1.1.

• A group $G$ is called bounded if the orders of its finite subgroups are bounded by a universal constant that depends only on $G$ ($[\text{Po11}, \text{Definition 2.9}]$).

• A group $G$ is called Jordan if there is a positive integer $J$ such that every finite subgroup $B$ of $G$ contains an abelian subgroup $A$ that is normal in $B$ and such that the index $[B : A] \leq J$ ($[\text{Po11}]$).

• We call a group $G$ very Jordan if there exist a commutative normal subgroup $G_0$ of $G$ and a bounded group $F$ that sit in the exact sequence

$$0 \to G_0 \to G \to F \to 0$$

(1)

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Remark 1.2. 1) Every finite group is bounded, Jordan and very Jordan.
2) Every commutative group is Jordan and very Jordan.
3) Every finitely generated commutative group is bounded.
4) A subgroup of a very Jordan group is very Jordan.

The group $\text{Aut}(Z)$ of any complex compact manifold $Z$ carries a natural structure of a complex (not necessarily connected) complex Lie group such that the action map $\text{Aut}(Z) \times Z \to Z$ is holomorphic (Theorem of Bochner-Montgomery, \cite{BM}). It is known, for example, to be Jordan if

- $Z$ is projective (Sh. Meng, D.-Q. Zheng, \cite{MZ});
- $Z$ is a normal complex Kaehler manifold (J.H. Kim, \cite{Kim}).

Moreover, the connected identity component $\text{Aut}_0(Z) \subset \text{Aut}(Z)$ of $\text{Aut}(Z)$ is Jordan for every compact complex space $Z$ (\cite[Theorem 5, Theorem 7]{Po18}).

Groups $\text{Bir}(Z)$ and $\text{Bim}(Z)$ of birational and bimeromorphic transformations, respectively, are more complicated.

In the case of a projective varieties $Z$, V.L. Popov \cite{Po11} proved that $\text{Bir}(Z)$ is Jordan if $\dim(Z) \leq 2$ and $Z$ is not birational to a product of an elliptic curve and $\mathbb{P}^1$. (The case of $Z = \mathbb{P}^2$ was done earlier by J.-P. Serre, \cite{Se09}.) Yu. Prokhorov and C. Shramov \cite{PS14} proved that $\text{Bir}(Z)$ is Jordan if either $Z$ is not uniruled or $Z = \mathbb{P}^N$ (the latter case was proven in \cite[Theorem 5]{PS14} modulo the Borisov-Alekseev-Borisov conjecture that was later established by C. Birkar \cite{Bi}).

On the other hand, $\text{Bir}(Z)$ is not Jordan if $X$ is birational to a product $A \times \mathbb{P}^n$ where $n \geq 1$ and $A$ is a positive-dimensional abelian variety \cite{Zar14}. The group $\text{Bim}(Z)$ is not Jordan for a certain class of $\mathbb{P}^1$-bundles over complex tori of positive algebraic dimension \cite{Zar19}.

We consider conic bundles $(X, p, Y)$ over a compact complex manifold $Y$ of zero algebraic dimension $a(Y)$. (Such bundles appear naturally in the classification of non-projective smooth compact Kaehler uniruled threefolds \cite{CP2}.)

Definition 1.3. We say that a compact connected complex Kaehler manifold $Y$ of positive dimension is poor if it enjoys the following properties.

- The algebraic dimension $a(Y)$ of $Y$ is 0.
- $Y$ does not contain analytic subspaces of codimension 1.
- $Y$ does not contain rational curves, i.e., it is meromorphically hyperbolic in a sense of Fujiki \cite{Fu80}.

Examples of poor manifolds are provided by complex tori $T$ with $\dim(T) \geq 2$, $a(T) = 0$, see \cite[Ch. 2, Corollary 6.4 and Example 7.4]{BL}. Indeed, a complex torus $T$ is a Kaehler manifold that does not contain rational curves. If $a(T) = 0$, it contains no analytic subsets of codimension 1 \cite[Corollary 6.4, Chapter 2]{BL}.
Remark 1.4. 1) Clearly, the complex dimension of a poor manifold is at least 2.
   2) A generic complex torus of given dimension $\geq 2$ has algebraic dimension 0 and therefore is poor.

Let $(X,p,Y)$ be a conic bundle over a poor manifold $Y$. Since $Y$ contains no rational curves, there is no non-constant holomorphic maps $\mathbb{P}^1 \to Y$. It follows that every map $f \in \text{Bim}(X)$ is $p$–fiberwise, i.e., there exists a homomorphism $\tau : \text{Bim}(X) \to \text{Aut}(Y)$ (see Lemma [2.3]) such that $p \circ f = \tau(f) \circ p$. We denote by $\text{Bim}(X)_p$, $\text{Aut}(X)_p$ the kernel of $\tau$, i.e., those $f \in \text{Bim}(X)$ (respectively, $f \in \text{Aut}(X)$) that leave every fiber $P_y := p^{-1}(y), y \in Y$ fixed.

First, we consider $\mathbb{P}^1$-bundles over $Y$ and prove

**Theorem 1.1.** Let $(X,p,Y)$ be a $\mathbb{P}^1$–bundle over a poor manifold $Y$. Then:

- $\text{Bim}(X) = \text{Aut}(X)$ (see Corollary [3.1]);
- The restriction homomorphism $\text{Aut}_p(X) \to \text{Aut}(P_y), f \to f|_{P_y}$ is a group embedding. Here $P_y = p^{-1}(y)$ for a point $y \in Y$ (Proposition [3.4]).

Assume additionally that $X \not\sim Y \times \mathbb{P}^1$. Then:

- The connected identity component $\text{Aut}_0(X)$ of the complex Lie group $\text{Aut}(X)$ is commutative (Theorem [4.1]);
- Group $\text{Aut}(X)$ is very Jordan, namely, there is a short exact sequence
  \[ 0 \to \text{Aut}_0(X) \to \text{Aut}(X) \to F \to 0, \]  
  where $F$ is a bounded group (Theorem [4.1]);
- Commutative group $\text{Aut}_0(X)$ sits in the exact sequence
  \[ 0 \to \Gamma \to \text{Aut}_0(X) \to H \to 0, \]  
  where $H$ is a complex torus and one of the following conditions holds (Proposition [4.4]):
    - $\Gamma = \{\text{id}\}$, the trivial group;
    - $\Gamma \cong \mathbb{C}_+$, the additive group of complex numbers;
    - $\Gamma \cong \mathbb{C}^*$, the multiplicative group of complex numbers.

Next, we consider flat conic bundles, i.e., conic bundles with equidimensional fibers. They may be of two different types.

**Type 1.** There is a section of projection $p : X \to Y$, i.e., there is an analytic subset $D \subset X$ of codimension 1 such that its intersection number $(D, p^*(y)) = 1$ for a point $y \in Y$. According to Lemma 3.6 of [Sh] by C. Shramov, a complex manifold $X$ of this type is bimeromorphic to a linear conic bundle $X'$ and, therefore, $\text{Bim}(X) \cong \text{Bim}(X') = \text{Aut}(X')$, thus is very Jordan.
Type 2. There is no section of projection $p : X \to Y$. If the group $\text{Bim}(X) \neq \{\text{id}\}$, then there exists a bisection of $p$, i.e., an analytic subset $D \subset X$ of codimension 1 such that the intersection number $(D, p^*(y)) = 2$ for a point $y \in Y$. The structure of finite subgroups of $\text{Bim}(X)$ is described by C. Shramov in Theorem 1.3. of [Sh]. We prove

**Theorem 1.2.** Assume that $(X, p, Y)$ is a conic bundle over a poor manifold $Y$. Assume that $X$ is not bimeromorphic to $Y \times \mathbb{P}^1$, and every component of the fiber $p^{-1}(y)$ for every $y \in Y$ has dimension 1. Then $\text{Bim}(X)$ is very Jordan (see Section 5).

The paper is organized as follows. Section 2 contains preliminary results about automorphisms of conic bundles and meromorphic groups in a sense of A. Fujiki. In Section 3 we deal with $\mathbb{P}^1$-bundles $X$ over a poor manifold $T$ and classify their nontrivial fiberwise automorphisms in terms of their fixed point sets. In Section 4 we prove the commutativeness of the identity component $\text{Aut}_0(X)$ of $\text{Aut}(X)$ for such $X$. In Section 5 we study Jordan properties of the automorphism groups of conic bundles over poor $T$ when all the fibers are one-dimensional. In Section 6 we provide a class of examples of $\mathbb{P}^1$-bundles $X$ over complex tori $T$ of algebraic dimension 0 that do not admit a section but admit a bisection that coincides with the set of fixed points of a certain fiberwise automorphism.

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2. **Preliminaries and Notation**

We assume that all complex manifolds under consideration are compact Kaehler. We use the following notation and assumptions.

**Notation and Assumptions 1**

- $\text{Bim}(X), \text{Aut}(X)$ stand for group of all bimeromorphisms and all automorphisms of a smooth complex manifold $X$, respectively.
- $\text{Aut}_0(X)$ stands for the connected identity component of $\text{Aut}(X)$ (as a complex Lie group).
- $\cong$ stands for “isomorphic groups” (or isomorphic complex Lie groups if the groups involved are the ones), and $\sim$ for biholomorphically isomorphic complex manifolds.
- $\text{id}$ stands for an identity automorphism.
- $\mathbb{P}^n(x_0, \ldots, x_n)$ stands for a complex projective space $\mathbb{P}^n$ with homogeneous coordinates $(x_0, \ldots, x_n)$.
- $\mathbb{C}_z, \overline{\mathbb{C}}_z \sim \mathbb{P}^1$ is the complex line (extended complex line, respectively) with coordinate $z$. 

• For an element $m \in \text{PSL}(2, \mathbb{C})$ we define $\text{DT}(m) := \frac{\text{tr}^2(M)}{\det(M)}$ where $M \in \text{GL}(2, \mathbb{C})$ is any matrix representing $m$, $\text{tr}(M)$ and $\det(M)$ are the trace and the determinant of $M$, respectively.

• $\mathbb{C}_+$ and $\mathbb{C}^*$ stand for complex Lie groups $\mathbb{C}$ and $\mathbb{C}^*$ with additive and multiplicative group structure, respectively.

• $\dim X$, $a(X)$ are the complex and algebraic dimensions of a compact complex manifold $X$, respectively.

• Let $X, Y$ be two connected reduced analytic complex spaces. A meromorphic map $f : X \to Y$ relates to every point $x \in X$ a subset $f(x) \subset Y$ (the image of $x$) such that the following conditions are met

  1. The graph $G_f := \{(x, y) \mid y \in f(x) \subset X \times Y\}$ is connected complex analytic subspace of $X \times Y$ with $\dim(G_f) = \dim(X)$;

  2. There exist an open dense subset $X_0 \subset X$ such that $f(x)$ consists in one point for every $x \in X$.

See, for example, [Pe, Definition 1.7].

• We say that a compact complex manifold $Y$ contains no rational curves if there are no nonconstant holomorphic maps $\mathbb{P}^1 \to Y$.

• Following A. Fujiki we call a compact complex manifold meromorphically hyperbolic if it contains no rational curves, ([Fu80]).

• According to A. Fujiki [Fu78, Definition 2.1], a meromorphic structure on a complex Lie group $G$ is a compactification $G^*$ of $G$ such that

  the group multiplication $\mu : G \times G \to G$ extends to a meromorphic map $\mu^* : G^* \times G^* \to G^*$ and $\mu^*$ is holomorphic on $G^* \times G \cup G \times G^*$.

• Following A. Fujiki we say that a complex Lie group $G$ acts meromorphically on a complex manifold $Z$ if

  1. $G$ acts biholomorphically on $Z$;

  2. there is a meromorphic structure $G^*$ on $G$ such that the $G$-action $\sigma : G \times Z \to Z$ extends to a meromorphic map $\sigma^* : G^* \times Z \to Z$ (see [Fu78, Definition 2.1] for details).

It was proven in [Fu80] that if a Kaehler manifold $Y$ is meromorphically hyperbolic then

  1. every meromorphic map $f : X \to Y$ is holomorphic for any complex manifold $X$;

  2. every connected component of the set $H(Y, Y)$ of all holomorphic maps $Y \to Y$ (regarded as a certain subspace of the Douady complex analytic space $D_{Y \times Y}$) is compact;

  3. in particular, $\text{Aut}_0(Y)$ is a complex compact Lie group, that is isomorphic to a certain complex torus $\text{Tor}(Y)$ (see also [Fu78, Corollary 3.7]).
In general, let $Z$ be a complex Kaehler manifold. The group $\text{Aut}_0(Z)$ acts meromorphically on $Z$, and the analogue of the Chevalley decomposition for algebraic groups is valid for complex Lie group $\text{Aut}_0(Z)$:

$$0 \to L(Z) \to \text{Aut}_0(Z) \to \text{Tor}(Z) \to 0$$

where $L(Z)$ is meromorphically isomorphic to a linear group, and $\text{Tor}(Z)$ is a torus ([Fu78, Theorem 5.5], [Lie, Theorem 3.12], [CP, Theorem 3.28]).

If $L(Z)$ in (4) is not trivial, $Z$ contains a rational curve. Moreover, according to [Fu78, Corollary 5.10], $Z$ is bimeromorphic to a fiber space whose general fiber is $\mathbb{P}^1$.

We will use the following property of poor manifolds.

**Lemma 2.1.** Let $X, Y$ be manifolds, let $Y$ be poor and let $f : X \to Y$ be a unramified holomorphic covering. Then $X$ is also poor.

**Proof.** Indeed,

- If $\omega$ is a Kaehler form on $Y$, then $f^*\omega$ is a Kaehler form on $X$, thus $X$ is a Kaehler manifold.
- $X$ contains no rational curve $C$, otherwise $f(C)$ would be a rational curve in $Y$.
- $X$ contains no codim 1 analytic subset $Z$, otherwise $f(Z)$ would be a codim 1 analytic subset in $Y$.

Let us prove that $a(X) = 0$. Let $\Gamma$ be the (finite) group of deck transformations of $f$, let $|\Gamma| = N$, and let $\sigma_k, k \in Z$ denote the symmetric polynomials on $N$ variables:

$$\sigma_k(x_1, \ldots x_N) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq N} x_{i_1} \ldots x_{i_k}.$$

Any meromorphic function $G$ on $X$ gives rise to meromorphic functions $G_k = \sum_{\gamma \in \Gamma} \gamma^* \sigma_k(g)$, $k = 1, \ldots, N$ invariant under action of $\Gamma$. Since every $G_k$ induces a meromorphic function on $Y$, it is constant say, $c_k \in \mathbb{C}$. This implies that

$$g^n + \sum_{k=1}^{N} (-1)^k c_k g^{N-k} \equiv 0.$$

It follows that $g$ takes on at most $N$ distinct values on a certain open dense subset where all $G_k$ are defined (holomorphic). Therefore $g$ is a constant. □

A conic bundle $(X, p, Y)$ is a holomorphically locally trivial fiber bundle with general fiber $\mathbb{P}^1$ over an open dense subset $U \subset Y$. Indeed, by definition, there is an open dense subset $U \subset Y$ of points $y \in Y$ such that for $y \in U$ the fiber $P_y = p^*(y) \sim \mathbb{P}^1$. By a theorem of W. Fisher and H. Grauert ([FG]) $(p^{-1}(U), p, U)$ is a holomorphically locally trivial fiber bundle.
Definition 2.2. (Cf. [BZ18]) Let $X,Y,Z$ be three complex manifolds, $f : X \to Y, g : Z \to Y$ be holomorphic maps, and $h : X \to Z$ be a meromorphic map. We say that $h$ is $f,g$-fiberwise if there exists a holomorphic map $\tau(h) : Y \to Y$ that may be included into the following commutative diagram.

\[
\begin{array}{ccc}
X & \xrightarrow{h} & Z \\
\downarrow f & & \downarrow g \\
Y & \xrightarrow{\tau(h)} & Y 
\end{array}
\]  

(5)

Lemma 2.3. (cf. [BZ17] Lemma 3.4 for algebraic case). Assume that $Y$ is a Kähler meromorphically hyperbolic complex manifold and let $(X,p_X,Y)$ and $(Z,p_Z,Y)$ be two conic bundles over $Y$.

Then any surjective meromorphic map $f : X \to Z$ is $p_X,p_Z$-fiberwise.

Proof. Let $g_f := p_Z \circ f : X \to Y$. It is holomorphic ([Fu80, Proposition 1]). Let $U \subset Y$ be such a dense open subset of $Y$ that $(p_X^{-1}(U), p_X, U)$ is holomorphically locally trivial fiber bundle. Take a general fiber $U_p = p_X^{-1}(u), u \in U$. Since $f$ is holomorphic, image $g_f(U_p)$ may be either a point or a rational curve. Since $Y$ contains no rational curves, the restriction

\[
g_f \big|_{U_p} = g_f(u) \text{ is a constant map.} \quad (6)
\]

Since $U$ is dense and the set of points meeting the property (5) is closed, $g_f$ is constant for any $y \in Y$. Put $\tau(f)(y) := g_f \big|_{U_p}$.

For a fiber $U_p$ with $u \in U$, there exists a neighborhood $W$ of $u$ such that $V = p_X^{-1}(W) \sim W \times \mathbb{P}^1_{(x,y)}$. Then for $u \in W$ we have $\tau(f)(u) = p_Z \circ f(u, (0 : 1))$, hence is a holomorphic function on $u$. Thus, $\tau(f)$ is holomorphic on $U$, defined and continuous on $Y$.

Let $y \in Y \setminus U$ and $z = \tau(f)(y)$. Let us choose open neighbourhoods $W_y, W_z, \subset Y$ of $y, z$ respectively such that

- Both $W_y, W_z$, are biholomorphic to an open ball in $\mathbb{C}^n$ with induced coordinates $y_1, \ldots, y_n$ and $z_1, \ldots, z_n$ respectively;
- $\tau(f)(W_y) \subset W_z$.

Then the induced functions $\tau(f)^*(z_i)$ are holomorphic in $W_y \cap U$, defined and locally bounded in $W_y$ thus, by the first Riemann continuation Theorem [GR] Chapter 1, C, 3], are holomorphic in $W_y$. Hence, $\tau(f)$ is a holomorphic map.

Corollary 2.4. For a conic bundle $(X,p,Y)$ over a poor manifold $Y$ there is a group homomorphism $\tau : \Bim(X) \to \Aut(Y)$ such that $p \circ f = \tau(f) \circ p$ for every $f \in \Bim(X)$.

Remark 2.5. In particular, $\tau(\Aut_0(X))$ has the natural meromorphic structure and the homomorphism $\tau : \Aut_0(X) \to \tau(\Aut_0(X))$ is a
meromorphic map, i.e., $\tau$ is a holomorphic homomorphism of complex Lie groups (A. Fujiki, [Fu78 Lemma 2.4, 3]).

The next Proposition is similar to Lemma 3.1 of [Kim].

**Proposition 2.6.** Let $X$ be a complex Kaehler manifold and $F = \text{Aut}(X)/\text{Aut}_0(X)$. Then group $F$ is bounded.

**Proof.** There is a homomorphism

$$\phi: F \to \text{Aut}(H^2(X, \mathbb{Q})), \ f \in \text{Aut}(X) \to f^* \in \text{Aut}(H^2(X, \mathbb{Q})).$$

The image $\phi(\text{Aut}(X))$ is bounded, since it is a subgroup of a bounded (thanks to Minkowski’s theorem) group $\text{Aut}(H^2(X, \mathbb{Q})) \cong \text{GL}(b_2(X), \mathbb{Q})$ [Se16, Section 9.1]. (Here $b_2(X) = \dim_{\mathbb{Q}} H^2(X, \mathbb{Q})$ is the second Betti number of $X$). On the other hand, if $f \in \ker(\phi)$ then its action on $H^2(X, \mathbb{R})$ is trivial as well. Thus, if $\omega$ is a Kaehler form on $X$, $\overline{\omega}$ the corresponding element of $H^2(X, \mathbb{R})$ and $f \in \ker(\phi)$ then

$$f^*(\overline{\omega}) = \overline{\omega}. \quad (7)$$

The subgroup $\text{Aut}(X)_{\overline{\omega}} \subset \text{Aut}(X)$ of automorphisms meeting condition (7) has finite number of connected components ( [Fu78 Theorem 4.8], [Lie Proposition 2.2]). Clearly, $\text{Aut}_0(X)$ is a connected component of $\text{Aut}(X)_{\overline{\omega}}$ containing $id$. Thus $\ker(\phi)/\text{Aut}_0(X) \subset \text{Aut}(X)_{\overline{\omega}}/\text{Aut}_0(X)$ is a finite group. Thus, we have an exact sequence:

$$0 \to \ker(\phi)/\text{Aut}_0(X) \to (\text{Aut}(X)/\text{Aut}_0(X) = F) \to \phi(\text{Aut}(X)) \to 0.$$

The first group is finite, the last is bounded, thus $\text{Aut}(X)/\text{Aut}_0(X)$ is bounded. \hfill $\square$

Consider a $\mathbb{P}^1$-bundle over a complex Kaehler manifold $Y$ i.e., the triple $(X, p, Y)$, such that $X$ is a holomorphically locally trivial fiber bundle over $Y$ with fiber $\mathbb{P}^1$ and with the corresponding projection $p : X \to Y$. Let us fix some notation.

**Notation and Assumptions 2**

- $P_y$ stands for the fiber $p^{-1}(y)$.
- We call the covering $Y = \bigcup U_i$ by open subsets of $Y$ fine if there exist an isomorphism $\phi_i$ of $V_i = \pi^{-1}(U_i)$ to direct product $U_i$ and $\mathbb{P}^1(x_i, y_i)$. In other words $V_i \subset X$ stands for $p^{-1}(U_i) :$ we have an induced isomorphism $\phi_i : V_i \to U_i \times \mathbb{P}^1(x_i, y_i)$ and $(y, (x_i : y_i))$ are coordinates in $V_i$; In $(U_i \cap U_j) \times \mathbb{P}^1(x_i, y_i)$ defined is a holomorphic map $\Phi_{i,j} = (id, A_{i,j}(t)) :$

$$\begin{array}{l}
(y, (x_i : y_i)) \to (y, (x_j : y_j))
\end{array}$$

such that

- $A_{i,j} \in \text{PSL}(2, \mathbb{C})$ with representative

$$\tilde{A}_{i,j}(t) = \begin{bmatrix} a_{i,j}(t) & b_{i,j}(t) \\
 c_{i,j}(t) & d_{i,j}(t) \end{bmatrix} \in \text{GL}(2, \mathbb{C});$$
Assume that codim no analytic subsets of codimension 1, codim $S \in f$

Lemma 2.7. Let $f \in \text{Bim}(X)$. Assume that $f$ is defined at every point $x \in X \setminus \tilde{Z}$. Assume that codim $Z \geq 2$. Then $f \in \text{Aut}(X)$.

Proof. Let $\{U_i\}$ be a fine covering of $Y$. Let $g := \tau(f)$. Let $z \in \tilde{Z}$ and $W \subset U_i$ be a neighborhood of $p(z)$ such that $g(W) \subset U_j$ for some $j$.

Let $B = W \cap Z, A = W \setminus B$. For every $t \in A$ the restriction $f |_{P_t}$ is an isomorphism $P_t$ with $P_{g(t)}$ defined in corresponding coordinates by an element of $\text{PSL}(2, \mathbb{C})$. Thus, we have a holomorphic map $\psi_{W,f} : A \rightarrow \text{PSL}(2, \mathbb{C})$ such that $f(t, (x_i : y_i)) = (g(t), \psi_{W,f}(t)((x_i : y_i)))$. Since $\text{PSL}(2, \mathbb{C})$ is an affine variety, and codim $B \geq 2$, by the Levi’s continuation Theorem (see also Nar) there exists a holomorphic extension $\tilde{\psi}_{W,f} : W \rightarrow \text{PSL}(2, \mathbb{C})$. We define $\tilde{f}(t, (x_i : y_i)) = (g(t), \tilde{\psi}_{W,f}(t)((x_i : y_i)))$ in $p^{-1}(W)$. Thus we can extend $f$ holomorphically at any point $z \in \tilde{Z}$.

Since outside $\tilde{Z}$ all extensions coincide, this global extension is uniquely defined. \hfill \Box

3. $\mathbb{P}^1$-bundles over poor manifolds.

We now fix a poor complex manifold $T$ and consider a $\mathbb{P}^1$-bundle over $T$, i.e., the triple $(X, p, T)$ such that

- $T$ contains neither a rational curve nor an analytic subspace of codim 1, and algebraic dimension $a(T) = 0$;
- $X$ is a holomorphically locally trivial fiber-bundle over $T$ with fiber $\mathbb{P}^1$ and with the corresponding projection $p : X \rightarrow T$.

Corollary 3.1. $\text{Bim}(X) = \text{Aut}(X)$.

Proof. For $f \in \text{Bim}(X)$ let $\Sigma_f$ be the locus of indeterminacy of $f$ and $S_f = (p(\Sigma_f))$, (the closure of $(p(\Sigma_f))$ in $T$). Since $T$ contains no analytic subsets of codimension 1, codim $S_f \geq 2$. Moreover, $f$ is defined at points $X \setminus (p^{-1}(S_f))$. By Lemma 2.7 every $f \in \text{Bim}(X)$ and $f^{-1} \in \text{Bim}(X)$ may be holomorphically extended to $X$, hence we get $\text{Bim}(X) = \text{Aut}(X)$. \hfill \Box
Definition 3.2. A \( n \)-section \( S \) of \( p \) is a codimension 1 analytic subset \( D \subset X \) such that the intersection \( X \cap D \) consists in \( n \) distinct point for every \( t \in T \). A bisection is \( 2 \)-section. A section \( S \) of \( p \) is a \( 1 \)-section. There is a holomorphic map \( \sigma : T \to X \) such that section \( S = \sigma(T) \) and \( p \circ \sigma = id \) on \( T \). In every \( U_i \) the map \( \sigma \) is defined by a function \( \sigma_i : U_i \to V_i \) such that

\[
A_{i,j}(t) \circ \sigma_i(y) = \sigma_j(t) \tag{9}
\]

for \( t \in U_i \cap U_j \).

Lemma 3.3. Any two distinct sections of \( p \) in \( X \) are disjoint.

Proof. If a section \( S = \sigma(Y) \) intersect a section \( R = \rho(Y) \) then the intersection \( S \cap R \) is either empty or has codimension 2 in \( X \). Since none of sections contains a fiber, \( p(S \cap R) \) is either empty or has codimension 1 in \( Y \). Since \( Y \) carries no analytic subsets of codimension 1, \( p(S \cap R) = \emptyset \). \( \square \)

Recall that by \( \text{Aut}_p(X) \) we denote the kernel of a homomorphism \( \tau : \text{Bim}(X) = \text{Aut}(X) \to \text{Aut}(T) \), i.e., the subgroup of automorphisms of \( X \) that leave every fiber of \( p \) invariant.

Let \( f \in \text{Aut}(X)_p, f \neq id \). By Lemma 3.1 of [Sh] we know that the set of fixed points of \( f \) is a divisor. The following consideration shows that this divisor is either a smooth section of \( p \), or a union of two disjoint sections of \( p \), or a smooth \( 2 \)-section.

Proposition 3.4. Let \( id \neq f \in \text{Aut}(X)_p \).

1) Let \( S \) be the set of fixed points of \( f \). Then only three following cases are possible:

A. \( S = S_1 \cup S_2 \) is a union of two smooth disjoint sections of \( S_1, S_2 \), each intersecting every fiber at one point;

B. \( S \) is a smooth section of \( p \) intersecting general fiber at one point;

C. \( S \) is a smooth \( 2 \)-section intersecting every fiber at two distinct points.

2) In cases A,B conic bundle \((X, p, T)\) is linear, in case C a double cover of \( X \) is a linear conic bundle.

3) The automorphism \( f \in \text{Aut}(X)_p \) is uniquely determined by its restriction to any fiber \( P_t \) with \( t \in T \) (cf. Lemma 4.3 of [Sh]).

Proof. Let \( \{U_i\} \) be a fine covering of \( T \).

Let \( id \neq f \in \text{Aut}(X) \) be defined, as in previous item, in \( V_i \), in coordinates \((t, (x_i : y_i)) \) as \( f_i(t, (x_i : y_i)) = (t, F_i(t)(x_i : y_i)) \), where

\[
(1) \quad F_i(t)(x_i : y_i) = (f_{i,11}(t)x_i + f_{i,12}(t)y_i) : (f_{i,21}(t)x_i + f_{i,22}(t)y_i);
\]

\[
(2) \quad \tilde{F}_i(t) = \begin{bmatrix} f_{i,11}(t) & f_{i,12}(t) \\ f_{i,21}(t) & f_{i,22}(t) \end{bmatrix}
\]

represents \( F_i(t) := \psi_{U_i,f}(t) \in \text{PSL}(2, \mathbb{C}) \) (see proof of Lemma 2.7).
(3) The set of fixed points of $F_i(t)$ is an analytic subset of $X$ defined by equation
\[
(f_i,11(t)x_i + f_i,12(t)y_i) : (f_i,21(t)x_i + f_i,22(t)y_i) = (x_i : y - I), \tag{10}
\]
that is
\[
f_i,12(t)y_i^2 + (f_i,11(t) - f_i,22(t))x_i y_i - f_i,21(t)x_i^2 = 0. \tag{11}
\]
It is obviously an analytic subset of $X$. In every $U_i$ the function $TD_i(t) = TD(F_i(t)) = \frac{tr(\tilde{F}_i(t))^2}{\det(\tilde{F}_i(t))}$ is defined and holomorphic. Since $F_i(t)$ represent globally defined map $f \in Aut_p(X)$ we get (see Notation and Assumptions 2)
\[
F_j(t) \circ A_{i,j} = A_{i,j} \circ F_i(t) \tag{12}
\]
which means that
\[
\tilde{F}_j(t)\tilde{A}_{i,j} = \lambda_{i,j}(t)\tilde{A}_{i,j}\tilde{F}_i(t), \tag{13}
\]
where $\lambda_{i,j}(t) \neq 0$ in $U_i \cap U_j$. Hence,

1. $TD(t) := TD_i(t)$ for $t \in U_i$, is holomorphic and globally defined on $T$, hence constant, we denote this number $TD_F$;

2. The sets $D_f := \{\det(\tilde{F}_i) = 0\}$ and $T_f := \{\tr(\tilde{F}_i(t)) = 0\}$ are globally defined. Since $f$ is defined everywhere, $D_f = \emptyset$. Since $\text{codim} T_f = 1$, it is either empty or coincide with $T$.

3. If $\delta_f = TD_f - 4 \neq 0$ then fix a square root $A_f := \sqrt{TD_f - 4}$ and define $\lambda_f = \frac{T_f + A_f}{T_f - A_f}$ be the ratio of the eigenvalues of $\tilde{F}_i(t)$ (it does not depend on $i$). Then for every $i$ in $V_i = p^{-1}(U_i)$ one can define coordinates $(t, u_i)$. $u_i \in \mathbb{C}$, in such a way that $f(t, u_i) = (t, \lambda_f u_i)$. The set $S \cap V_i$ of fixed points of $f$ in $V_i$ is $\{u_i = 0\} \cup \{u_i = \infty\}$. Thus $S$ is an unramified double cover of $T$. Thus, it may be either union of two disjoint sections or one bisection (see cases A, C below for details).

4. If $\delta_f = TD_f - 4 = 0$ then $\tilde{F}_i(t)$ is proportional to a unipotent matrix and for every $i$ in $V_i = p^{-1}(U_i)$ one can define coordinates $(t, w_i)$, $w_i \in \mathbb{C}$, in such a way that $f(t, w_i) = (t, (w_i + a_i(t))$ where $a_i(t)$ are holomorphic functions in $U_i$. The set $S$ of fixed points in $V_i$ is thus union of the section $\{w_i = \infty\}$ and $R_f = \cup \{a_i(t) = 0\} = \{t \in T \mid f \mid_{R_f} = \text{id}\}$ (see case B below for details).

In other words, for every $t \in T$ the map $F_i(t)$ in $P_i$ is either identical, or has two fixed points, or has one fixed point. If $\delta_f - 4 \neq 0$ then $F_i(t)$ defines a smooth analytic subset $S$ of $X$ and $p^{-1}(t) \cap S$ contains precisely 2 distinct points for any $t \in T$. Therefore, $S$ is either an unramified smooth double cover of $T$ or a union of two smooth disjoint sections of $p$. 
If $TD_f - 4 = 0$ then (11) defines a smooth section of $p$ over the complement to an analytic subset $R_f$ of $T$ or is identically valid on $X$.

Thus globally, we have three cases.

**Case A.** Map $f \in \text{Aut}(X)_p$ defines two disjoint sections $S_1$ and $S_2$ of projection $p$. Let $z_i = \frac{y_i}{x_i} \in \mathbb{C}$, and $S_1 \cap U_i = \{(t, z_i = a_i(t))\}$, $S_2 \cap U_i = \{(t, z_i = b_i(t))\}$. Since $F_i(t) = \psi_{U_i,f}(t)$ depend on $t$ holomorphically, $a_i(t), b_i(t)$ are meromorphic functions in $U_i$. Since $S_1 \cap S_2 = \emptyset, a_i(t) - b_i(t) \neq 0$.

The holomorphic coordinate change in $V_i$ introduced in item(1) is:

$$(t, z_i) \rightarrow (t, \frac{z_i - a_i(t)}{z_i - b_i(t)}) = (t, u_i).$$

Since both sections are globally defined and $f$-invariant, we have

$$(t, u_j) = \Phi_{i,j}(t, u_i) = (t, \mu_{i,j}u_i).$$

Since $u_j = \mu_{i,j}u_i = \mu_{k,j}u_k$ in $U_i \cap U_j \cap U_k$, we have $\mu_{i,k} = \mu_{j,k}\mu_{i,j}$, that is we have a cocycle. It defines a line bundle $L_f$ on $T$ with transition functions $\mu_{i,j}$ such that $X \setminus S_2$ is a total body of a $L_f$ and $S_1$ is the zero section of $L_f$.

Moreover,

$$f(t, u_i) = (t, \lambda_fu_i), \quad \lambda_f \neq 0.$$  

Thus, our $f \in \text{Aut}(X)_p$ defines two sections and a number $\lambda_f \in \mathbb{C}^\ast$. We will say that $f$ has type $A$ with Data $(S_1, S_2)$ (an ordered pair). The maps having the same Data differ only by the coefficient $\lambda_f \in \mathbb{C}^\ast$. It follows that an automorphism of type $A$ is uniquely defined by its restriction to any fiber $P_t, t \in T$ (cf. Lemma 4.3 of $[Sh]$).

In this case $X = \mathbb{P}(E)$ where $E$ is a rank two decomposable vector bundle on $T$.

**Case B.** $\delta_f = TD_f - 4 = 0, f \neq id$. In this case (11) has the set of solutions $S = \{2y_if_{i,12} + x_i(f_{i,11} - f_{i,22}) = 0\} \subset X$ of fixed points of $f$, and $f$ has precisely one fixed point in general fiber $P_t$.

Therefore $S$ is a section of $p$ over a set $\bar{T} := T \setminus R_f$ where $R_f = \{t \in T : f \mid_{P_t} = id\} \subset T$. In $U_i$ the set $R_f$ is defined by conditions

$$f_{i,12}(t) = f_{i,21}(t) = 0, \quad f_{i,11}(t) = f_{i,22}(t).$$

(14)

It follows that $R_f$ is an analytic subset of $T$, hence codim $R_f \geq 2$. Note that $p^{-1}(R_f) \subset S$. Consider the function

$$g_i(t) = \frac{f_{i,22}(t) - f_{i,11}(t)}{f_{i,21}(t)} = \frac{2f_{i,12}(t)}{f_{i,11}(t) - f_{i,22}(t)}$$

(the equality follows from $\delta_f = 0$). Function $g_i$ is meromorphic in $U_i \setminus R_f$. Since codim $R_f \geq 2$, by the Levi’s Theorem ($[Le]$) it may be extended to a meromorphic function to $U_i$. 

Define \( w_i = \frac{\nu_i}{x_i + y_i g(t)} \in \overline{\mathbb{C}} \) (as in item (2)). The direct computation shows that \( f(t, w_i) = w_i + a_i(t) \), where \( a_i = \frac{2\nu_i g(t)}{\text{tr}(F_i(t))} \). Since \( \delta_f = 0 \) the denominator never vanishes, thus \( a_i(t) \) is a holomorphic function in \( U_i \).

The set \( \{a_i(t) = 0\} = R_f \cap U_i \) has codimension 1, which is impossible if \( R_f \neq \emptyset \). It follows that \( R_f = \emptyset \). Thus, \( f \mid_{P_t} \neq \text{id} \) for any \( t \in T \) and \( a_i(t) \) does not vanish in \( U_i \).

Since \( S = \{w_i = \infty\} \) is globally defined, we have \( w_j = A_{i,j}(w_i) = \nu_{i,j}w_i + \tau_{i,j} \), where \( \nu_{i,j} \) and \( \tau_{i,j} \) are holomorphic in \( U_i \cap U_j \). Since \( f \) is globally defined

\[
\nu_{i,j}(w_i + a_i(t)) + \tau_{i,j} = (\nu_{i,j}w_i + \tau_{i,j}) + a_j(t),
\]

we have

- \( \{\nu_{i,j}\} \) do not vanish in \( U_i \cap U_j \) and form cocycle, thus define a line bundle \( \mathcal{L}_f \).
- \( \{a_i(t)\} \) is the section of \( \mathcal{L}_f \).

Since a nontrivial line bundle on \( T \) has no nonzero sections, either \( a_i(t) \equiv 0 \) and \( f = \text{id} \), or \( \mathcal{L}_f \) is trivial and we have a global holomorphic, hence constant function \( a_i(t) \equiv a_f \). We say that \( f \) has type \( B \) with Data \( S \). The maps having the same Data differ only by the summand \( a_f \in \mathbb{C} \).

It follows that an automorphism of type \( B \) is uniquely defined by its restriction to a fiber \( P_t \) for every \( t \in T \).

If there are no other sections (see Proposition 1.4), then \( X = \mathbb{P}(E) \) where \( E \) is a rank two non-decomposable vector bundle on \( T \).

**Case C.** The set \( S \subset X \) of the fixed points of \( f \) is a smooth unramified double cover of \( T \). Consider \( \tilde{X} := \tilde{X}_f := S \times_T X = \{(s, x) \in S \times X \subset X \times X : p(s) = p(x)\} \). We denote the restriction of \( p \) onto \( S \) by the same letter, \( p_X \) and \( \tilde{p} \) stand for the restrictions onto \( \tilde{X} \) of natural projections \( S \times X \to X \) and \( S \times X \to S \) respectively, \( \text{inv} : S \to S \) is an involution (the only non-trivial deck transformation for \( \tilde{p} \)).

We have

a) The following diagram commutes

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{p_X} & X \\
\tilde{p} \downarrow & & \downarrow p \\
S \subset X & \xrightarrow{p} & T
\end{array}
\]  
\quad (15)

b) \( p_X : \tilde{X} \to X \) is a unramified double cover of \( X \);

c) Every fiber \( \tilde{p}^{-1}(s), s \in S \) is isomorphic to \( P_{p(s)} = p^{-1}(p(s)) \sim \mathbb{P}^1 \);

d) \( \mathbb{P}^1 \)-bundle \( \tilde{X} \) over \( S \) has two sections: \( S_+ := S_+(f) := \{(s, s) \in \tilde{X}, \ s \in S \subset X\} \) and \( S_- := S_-(f) := \{(s, \text{inv}(s)) \in \tilde{X}, \ s \in S \subset X\} \). They are mapped onto \( S \) isomorphically by \( p_X \).
e) Every section \( N = \{ t, \sigma(t) \} \) of \( p \) in \( X \) induces a section \( \tilde{N} := \{ (s, \sigma(p(s))) \} \) of \( \tilde{p} \) in \( \tilde{X} \). We have \( p_X(\tilde{N}) = N \) is a section of \( p \) thus \( \tilde{N} \) cannot coincide \( S_+ \) or \( S_- \).

f) Every \( h \in \text{Aut}(X)_p \) induces \( \tilde{h} \in \text{Aut}(\tilde{X})_{\tilde{p}} \) defined by \( \tilde{h}(s, x) = (s, h(x)) \). In particular, for \( \tilde{f} \) the points of \( S_+ \) and \( S_- \) are fixed, hence, \( \tilde{f} \) is of type A. This defines a group embedding \( h \rightarrow \tilde{h} \) of \( \text{Aut}(X)_p \) to \( \text{Aut}(\tilde{X})_{\tilde{p}} \). Since a map \( \tilde{f} \) is uniquely defined by its restriction to a fiber, the homomorphism \( h \rightarrow \tilde{h} \) is an embedding of groups. Moreover, it follows that \( \tilde{f} \) and, hence, \( f \) is uniquely defined by the restrictions to fibers \( \tilde{P}_s = \tilde{p}^{-1}(s) \) and on fiber \( P_t = p^{-1}(t) \), respectively.

g) Involution \( s \rightarrow \text{inv}(s) \) may be extended from \( S \) to \( \tilde{X} \) by \( \text{inv}(s, x) = (\text{inv}(s), x) \);

h) \( S \) is a poor manifold by Lemma 2.1.

We will call such an \( f \) an automorphism of type \( C \) defined by data \( S \). The maps having the same Data differ only by the coefficient \( \lambda_{\tilde{f}} \in \mathbb{C}^\ast \).

In this case \( \tilde{X} \sim \mathbb{P}(E) \) where \( E \) is a rank two decomposable vector bundle on \( S \). It is an étale double cover of \( X \). \( \square \)

Remark 3.5. Let us formulate a byproduct of the proof of the Proposition. Assume that \((V, p, U)\) is a \( \mathbb{P}^1 \)–bundle over a complex (not necessary compact) manifold, let \( f \in \text{Aut}_p(V) \). Then

- the function \( \text{TD}(u) \) is globally defined;
- If \( \text{TD}(u) = \text{const} \neq 4 \) on \( U \) then the set of fixed points of \( f \) is an unramified (may be reducible) double cover of \( U \);
- If \( \text{TD}(u) \equiv 4 \) on \( U \), then the set of fixed points of \( f \) is a section of \( p \).

Lemma 3.6. If \( \text{Aut}(X)_p \neq \{ \text{id} \} \) then \( X \) is a Kaehler manifold.

Proof. Let \( f \neq \text{id}, f \in \text{Aut}(X)_p \). Then either \( X \) or its étale double cover \( \tilde{X} \) is \( \mathbb{P}(E) \) where \( E \) is a rank two vector bundle over a Kaehler manifold \( T \). In the first case \( X \) is Kaehler according to ([Vo], Proposition 3.18).

In the latter case \( \tilde{X} \) is Kaehler. Let \( \Omega \) be a Kaehler form on \( \tilde{X} \) and let \( \Omega = \tilde{\Omega} + \text{inv}^\ast(\tilde{\Omega}) \). It is a closed Kaehler form and \( \Omega \) that is invariant under involution \( \text{inv} \), hence may be pushed down onto \( X \). \( \square \)

Corollary 3.7. \( \text{Bim}(X) = \text{Aut}(X) \) is Jordan.

Proof. Corollary follows from the result of Jin Hong Kim, [Kim]. \( \square \)

4. The Connected Identity Component of \( \text{Aut}(X) \).

In this section we continue to consider a variety \( X \) that is a \( \mathbb{P}^1 \)–bundle over a poor manifold \( T \).

Lemma 4.1. Assume that \( f \in \text{Aut}(X)_p \), \( f \neq \text{id} \) and \( f \) is of type \( C \) defined by Data (bisection) \( S \). If \( \tilde{X} := \tilde{X}_f \sim S \times \mathbb{P}^1 \) then there is
a section $S_1 \subset X$ of $p$. Moreover, $\text{Aut}(X)_p$ has an abelian subgroup
$\Gamma \cong \mathbb{C}^*$ of index at most 2.

Proof. Assume that a non-identity $f \in \text{Aut}(X)_p$ is of type $C$ with

Data $S$. Assume that $X \sim S \times \mathbb{P}^1$. Let $z : S \times \mathbb{P}^1 \to \mathbb{P}^1 \sim \mathbb{C}$ be the

natural projection. Since $S_+ = \{(s, s)\}$ and $S_{-} = \{(s, \text{inv}(s))\}$ have
algebraic dimension 0, the rational function $z$ is constant along these
sections. We may assume that $z = 0$ on $S_+ = \{(s, s)\}$ and $z = \infty$ on
$S_{-} = \{(s, \text{inv}(s))\}$.

Take $s \in S$, let $t = p(s)$, and let $P_t = p^{-1}(t) \subset X$. The points $s$
and $\text{inv}(s)$ are the intersection points of $P_t$ with $S$ in $X$. The preimage
$p^{-1}_X(P_t)$ in $\tilde{X}$ consists of two disjoint components: $\tilde{P}_s = \tilde{p}^{-1}(s) = s \times P_t$
and $P_{\text{inv}(s)} = \tilde{p}^{-1}(\text{inv}(s)) = \text{inv}(s) \times P_t$. In the fiber $s \times P_t$ we have:

\[ z(s, s) = 0, \quad z(s, \text{inv}(s)) = \infty, \quad (16) \]

and in fiber $\text{inv}(s) \times P_t$ we have

\[ z(\text{inv}(s), \text{inv}(s)) = 0, \quad z(\text{inv}(s), s) = \infty. \quad (17) \]

Consider the following diagram

\[
\begin{array}{ccc}
P_t & \xrightarrow{(s, \text{id})} & s \times P_t & \xrightarrow{z} & \mathbb{C} \\
\downarrow \text{id} & & \downarrow & & \downarrow \alpha \\
P_t & \xrightarrow{\text{inv}(s), \text{id}} & \text{inv}(s) \times P_t & \xrightarrow{z} & \mathbb{C}
\end{array}
\]

Here automorphism $\alpha$ of $\mathbb{C}$ is defined in such a way that the diagram
is commutative. From (16) and (17) we get that $\alpha(0) = \infty, \alpha(\infty) = 0$.
Hence $\alpha(z) = \frac{z}{\nu}$, for some $\nu \neq 0$. We may assume that $\nu = 1$. (Indeed,
choose a $\sqrt{\nu}$ and divide $z$ by it).

Consider an automorphism $h \in \text{Aut}(X)_p$. Let $\tilde{h}$ be its pullback to
$\text{Aut}(\tilde{X})_p$ defined by $\tilde{h}(s, x) = (s, h(x))$. Since $\tilde{X}$ is a direct product,
and $\tilde{h}$ is $\tilde{p}$ invariant, we have

\[ \tilde{h}(s, z) = (s, z'), \text{ where } z' = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C}. \quad (19) \]

from (18), (16), and (17) we get $\alpha(z') = \frac{1}{z'}, \frac{az + b}{cz + d} = \frac{cz + d}{az + b}$.

There are two types of possible $\tilde{h}$ with this property: for every $(a : b) \in \mathbb{P}^1, a^2 - b^2 \neq 0,$

\[ z' = \frac{az + b}{bz + a} = \tilde{h}_{a,b}(z), \quad (20) \]

and

\[ z' = \frac{az + b}{bz + a} = -\tilde{h}_{a,-b}(z) = \tilde{h}_{a,-b}(-z). \quad (21) \]
Note that the only nontrivial automorphism leaving $z = 0, z = \infty$ invariant is $-\tilde{h}_{a,b}$, which is the lift $\tilde{f}$ of $f$. The transformations $h_{a,b}$ form an abelian group $\Gamma$ with

$$h_{a,b}h_{\alpha,\beta} = h_{c,d}, \quad c = a\alpha + b\beta, \quad d = a\beta + b\alpha.$$ 

The transformations $-h_{a,b}$ form $-h(1 : 0)\Gamma$. Thus $\Gamma$ has obviously index 2 in $\text{Aut}(X)_p$. Note that the sets $\{z = 1\}$ and $\{z = -1\}$ are the fixed points of all the maps $\tilde{h}_{a,b}$ if $b \neq 0$. Moreover, they are invariant under deck transformation $(s, z) \rightarrow (\text{inv}(s), 1/z)$. Thus their images provide two sections of conic bundle $p : X \rightarrow T$. Hence, in this case $X = \mathbb{P}(E)$ for some decomposable rank two vector bundle $E$ over $T$. If we change coordinates $w = \frac{z}{s}$ then $w' = \frac{\tilde{z}}{s'} = \tilde{h}_{a,b}(w) = w_{a,b} := w\mu_{a,b}$ and $\tilde{h}_{a,b}(\tilde{h}_{a,\beta}w)$ corresponds to $\mu_{a,b}\mu_{\alpha,\beta}$. The condition $a^2 - b^2 \neq 0$, provides $\mu \neq 0, \infty$. Thus, $\Gamma \cong \mathbb{C}^*$ as a complex Lie group.

\[\Box\]

Remark 4.2. In coordinates $w$ we have $\tilde{h}_{a,b}(w) = \frac{1}{w\mu_{a,b}}$.

Corollary 4.3. If $X$ admits a non-identity automorphism of type C and $\tilde{X}_f \not\sim S \times \mathbb{P}^1$ then conic bundle $p : X \rightarrow T$ does not admit a section. In particular, it does not admit automorphisms of type A or B.

\[\text{Proof.}\] Indeed an automorphism of type A or B would provide a section thus $\tilde{X}_f$ would admit three disjoint sections, thus $\tilde{X}_f$ would be the direct product $S \times \mathbb{P}^1$. \[\Box\]

Proposition 4.4. Assume that $\text{Aut}(X)_p \neq \{\text{id}\}$. Then the following cases are possible:

- $X \sim T \times \mathbb{P}^1$;
- $\text{Aut}(X)_p$ is commutative and is isomorphic as a complex Lie group to $\mathbb{C}^+$;
- $\text{Aut}(X)_p$ contains a closed commutative complex Lie subgroup $\Gamma \subset \mathbb{C}^*$ with $[\text{Aut}(X)_p : \Gamma] \leq 2$;
- $\text{Aut}(X)_p$ is finite.

Remark 4.5. C. Shramov has proven in [Sh] that if $p$ does not admit a section, than every finite subgroup of $\text{Aut}(X)_p$ has order at most 4. Thus in last case of Proposition 4.4 if $\text{Aut}(X)_p$ is finite, then the order $|\text{Aut}(X)_p| \leq 4$.

\[\text{Proof.}\] Recall that $\text{Aut}(X)_p$ is a complex Lie group ([BM]).

Consider Cases. 

AB. Assume that $f, h \in \text{Aut}(X)_p$, $f$ is of type A with Data $S_1, S_2$, and $h$ is of type B with Data $S_3$. Than among $S_1, S_2, S_3, h(S_1), h(S_2)$, there are at least three different disjoint sections. ($h$ is translation, thus if $S_1 \neq S_3$ then $S_1 \neq h(S_1) \neq h(h(S_1)))$. Thus $X = T \times \mathbb{P}^1$. 

AC, BC. Assume that \( f, h \in \text{Aut}(X)_p \), \( f \) is of type \( A \) or \( B \) with Data containing section \( S_1 \), and \( h \) is of type \( C \) with Data \( S_3 \). Then \( \tilde{X}_h \) is \( P(E) \) where \( E \) is a decomposable vector bundle. The corresponding line bundle \( \mathcal{L}_h \) has non-zero section \( \tilde{p}^{-1}(S_1) \) thus is trivial. It follows that \( \tilde{X}_h \sim S \times \mathbb{P}^1 \) (in notation of Lemma 4.1). According to Lemma 4.1 there is a subgroup \( \Gamma \cong \mathbb{C}^* \) with index at most 2 in \( \text{Aut}(X)_p \).

AA. Assume that \( f, h \in \text{Aut}(X)_p \), both of type \( A \) with Data \( (S_1, S_2) \) and \( (S_3, S_4) \) respectively. Assume that \( X \) admits no automorphisms of type \( B \) or \( C \). Recall that \( X \setminus S_2 \) is the total body of the line bundle \( \mathcal{L}_f \).

If \( S_1 \cup S_2 \neq S_3 \cup S_4 \) then \( \mathcal{L}_f \) has at least three disjoint sections, thus \( \mathcal{L}_f \) is trivial and \( X = T \times \mathbb{P}^1 \). If \( S_1 \cup S_2 = S_3 \cup S_4 \), then take subgroup \( \Gamma \) consisting of all those maps in \( \text{Aut}(X)_p \) that leave each section fixed. They all have the same Data and have type \( A \). Thus \( \Gamma \) as a complex Lie group that is isomorphic to a subgroup of \( \text{Aut}(\mathbb{C}^*) \) leaving punctures fixed, hence is either \( \mathbb{C}^* \) or finite. Evidently, \( [\text{Aut}(X)_p : \Gamma] \leq 2 \).

BB. Assume that \( f, h \in \text{Aut}(X)_p \), \( f \) is of type \( B \) with Data \( S \) and \( h \) is of type \( B \) with Data \( R \). Assume that \( S \neq R \). Then \( S, R, f(R) \) are three disjoint sections and \( X = T \times \mathbb{P}^1 \).

If \( \text{Aut}(X)_p \) consists of the maps of types \( B \) with the same Data, then \( \text{Aut}(X)_p \) is a complex Lie group isomorphic to a subgroup of \( \text{Aut}(\mathbb{C}) \) and consisting of translations only, thus is either \( \mathbb{C}_+ \) or trivial.

CC. Assume that \( f, h \in \text{Aut}(X)_p \), \( f \) is of type \( C \) with Data \( S \) and \( h \) is of type \( C \) with Data \( R \). Then \( \text{Aut}(X)_p \) embeds into a subgroup \( \tilde{G} \) of \( \text{Aut}(\tilde{X}_f)_p \), containing an automorphism \( \tilde{f} \) of type \( A \). It follows from the previous cases that the following options are possible:

- \( \text{Aut}(\tilde{X}_f)_p \) is finite, hence \( \text{Aut}(X)_p \) is finite as well.
- \( \tilde{X}_f \sim S \times \mathbb{P}^1 \) (in notation of Lemma 4.1). According to Lemma 4.1 there is a subgroup \( \Gamma \cong \mathbb{C}^* \) with index at most 2 in \( \text{Aut}(X)_p \).
- \( \text{Aut}(\tilde{X}_f)_p \) contains a subgroup \( \Gamma \cong \mathbb{C}^* \) of index at most 2. The subgroup \( \tilde{G} \cap \Gamma \) is either finite or whole \( \Gamma \), since both are complex Lie groups. It is evident that \( [\tilde{G} : \tilde{G} \cap \Gamma] \leq 2 \).

\( \square \)

Lemma 4.6. Consider a short exact sequence of connected complex Lie groups:

\[ 0 \to A \xrightarrow{i} B \xrightarrow{j} D \to 0. \]

Here \( i \) is a closed embedding and \( j \) is surjective holomorphic. Assume that \( D \) is a complex torus and \( A \) is isomorphic either to \( \mathbb{C}_+ \) or to \( \mathbb{C}^* \). Then \( B \) is commutative.

Proof. Step 1. First let us proof that \( A \) is a central subgroup in \( B \). Take any element \( b \in B \). Define a holomorphic map \( \phi_b : A \to A \), \( \phi_b(a) = bab^{-1} \in A \) for an element \( a \in A \). Since it depends holomorphically on \( b \), we have a holomorphic map \( \xi : B \to \text{Aut}(A), b \to \phi_b. \)
Since \( A \) is commutative, for every \( a \in A \) we have \( \phi_{ab} = \phi_b \). Thus there is a well defined map \( \psi \) fitting into the following commutative diagram

\[
\begin{array}{c}
B \\
\downarrow \psi \\
\xi \\
D \\
\end{array}
\]

The map \( \psi = \xi \circ j^{-1} \) is defined at every point of \( D \). It is holomorphic (see, for example, [OV], § 3). Actually, it follows from the implicit function theorem. It says that for every point \( d \in D \), and every \( b \in J_d = j^{-1}(d) \) there exist

- a neighbourhood \( U_d \) and \( V_b \) of \( d \) and \( b \) respectively such that \( j(V_b) = U_d \);
- a section \( \sigma : U_d \rightarrow V_b \) of \( j \) such that \( b = \sigma(d) \) and \( j \circ \sigma = id \);
- \( \psi \mid_{U_d} = \xi \circ \sigma \) thus is holomorphic.

Since \( D \) is a complex torus, and \( \text{Aut}(A) \) is either \( \mathbb{C}^* \) (if \( A = \mathbb{C}^+ \)) or \( \{ -id, id \} \) (if \( A = \mathbb{C}^* \)), we have \( \psi(D) \) is \( \{ id \} \). It follows that \( A \) is a central subgroup of \( B \).

**Step 2.** Let us now show that \( B \) is commutative. Consider a holomorphic map \( \text{com} : B \times B \rightarrow A \) defined by \( \text{com}(x, y) = xyx^{-1}y^{-1} \).

Since \( A \) is a central subgroup of \( B \), similarly to **Step 1** we get a holomorphic map \( D \times D \rightarrow A \). It has to be constant since \( D \) is a torus and \( A \) is either \( \mathbb{C}_+ \) or \( \mathbb{C}^* \).

**Theorem 4.1.** Let \( X \) be a \( \mathbb{P}^1 \)-bundle over a poor manifold \( T \) and \( X \not\sim T \times \mathbb{P}^1 \). Then \( \text{Aut}_0(X) \), the connected identity component of \( \text{Aut}(X) \), is commutative and the quotient \( \text{Aut}(X)/\text{Aut}_0(X) \) is a bounded group.

**Proof.** From [14], applied to \( X \) and \( T \) and from Lemma 2.4 we have the following commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & L(X) & \rightarrow & \text{Aut}_0(X) & \rightarrow & \text{Tor}(X) & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \tau \downarrow & & \\
0 & \rightarrow & 0 & \rightarrow & \text{Aut}_0(T) & \cong & \text{Tor}(Y) & \rightarrow & 0
\end{array}
\]

We will identify a torus with the group of its translations. Denote by \( H = \tau(\text{Aut}_0(X)) \). This is an image of a torus, containing identity, thus a subtorus of \( \text{Tor}(Y) \). Let \( G = \tau^{-1}(H) \).

One has the following short exact sequence of complex Lie groups.

\[
0 \rightarrow (\text{Aut}(X)_p \cap \text{Aut}_0(X)) \rightarrow \text{Aut}_0(X) \rightarrow H \rightarrow 0. \tag{22}
\]

According to Lemma 1.3 three cases are possible.

**Case 1.** \( \text{Aut}(X)_p \) is finite. Then \( (\text{Aut}(X)_p \cap \text{Aut}_0(X)) \) is finite as well, hence \( \text{Aut}_0(X) \rightarrow H \) is a finite surjective holomorphic homomorphism of Lie groups, thus an unramified finite covering ([OV], §4.3). It follows that \( \text{Aut}(X)_0 \) is a torus, hence commutative.
**Case 2.** $\text{Aut}(X)_p \cong \mathbb{C}_+$ or $\text{Aut}(X)_p \cong \mathbb{C}^*$. In this case $G$ is connected since every fiber of the homomorphism $G \to H$ is connected. Thus $G = \text{Aut}_0(X)$. Moreover the exact sequence (22) and Lemma 4.6 provide that group $G$ is commutative.

**Case 3.** $\text{Aut}(X)_p$ has a closed subgroup $\Gamma \cong \mathbb{C}^*$ of index 2. In this case every fiber of the map $\tau : G \to H$ has two connected components and the same is valid for $G$ as well: $G = \text{Aut}_0(X) \cup B$, where $B$ is the second connected component of $G$. Therefore $\text{Aut}_0(X)$ may be included into the following short exact sequence:

$$0 \to \Gamma \to \text{Aut}_0(X) \to H \to 0.$$  

By Lemma 4.6, we have $\text{Aut}_0(X)$ is commutative.

The group $F := \text{Aut}(X)/\text{Aut}_0(X)$ is bounded accorded to Proposition 2.6. □

5. **Equidimensional conic bundles.**

We start this section with the following Theorem of C. Shramov.

**Theorem 5.1.** (Sh) Let $X$ and $Y$ be complex manifolds. Let $f : X \to Y$ be a proper holomorphic map whose fibers are one-dimensional and whose typical fiber is isomorphic to $\mathbb{P}^1$. Suppose that there is a divisor $D$ on $X$ such that the intersection number of $D$ with a fiber of $f$ equals 1. Then $X$ is bimeromorphic to a projectivization of a rank 2 vector bundle on $Y$.

**Corollary 5.1.** Let $(X, p, T)$ be a conic bundle over a poor manifold $T$. Assume that $X$ is not bimeromorphic to $T \times \mathbb{P}^1$, and the dimension of every component of every fiber of $p$ is 1. If $X$ contains a divisor intersecting general fiber at one point, then $\text{Bim}(X)$ is very Jordan.

**Proof.** According to Theorem 5.1, $X$ is bimeromorphic to a linear $\mathbb{P}^1$-bundle, i.e., a holomorphic locally trivial (but, by assumption, nontrivial) $\mathbb{P}^1$-bundle $(Y, \psi, T)$. Thus $\text{Bim}(X) \sim \text{Bim}(Y) \sim \text{Aut}(Y)$ has the needed property according to Theorem 4.1. □

**Proposition 5.2.** Let $(X, p, T)$ be a conic bundle over a poor manifold $T$. Assume that the dimension of every component of every fiber of $p$ is 1. Assume that $X$ contains no divisor intersecting general fiber at one point and $\text{Bim}(X)_p \neq \{\text{id}\}$. Then $X$ is bimeromorphic to a $\mathbb{P}^1$-bundle $Y$ over $T$.

**Proof.** Let $\text{id} \neq f \in \text{Bim}(X)_p$. Let $U$ be an open dense subset of $T$ such that $(p^{-1}(U), p, U)$ is a $\mathbb{P}^1$-bundle. Let $\Sigma$ be the closure of the set all points of $t \in T$ such that either $f$ is not defined at some point of $p^{-1}(t)$ or $t \not\in U$. Since $T$ is poor, $\text{codim} \Sigma \geq 2$. According to Remark 3.5 the function $TD(t)$ is defined and holomorphic in $U$, thus, by the Levi continuation Theorem [Le] may be extended to a holomorphic function
on $T$. Hence, it is a constant. It follows from Remark 3.5 that the set $V$ of fixed points of the restriction of $f$ onto $\mathbb{P}^1$—bundle $(p^{-1}(U), p, U)$ is unramified double cover of $U$ or a section of $p$ over $U$.

We define the set of fixed points of $f$ as the projection $D_f$ onto the first factor of the intersection of the graph of $f$ with the diagonal in $X \times X$. By [Sh, Lemma 3.1], $D_f$ is a closed analytic subset of $X$.

We may assume that $D_f$ has an irreducible component $D_1$ such that $D_1 \cap P_u = p_1^{-1}(u)$ consist in two distinct points for $u \in U$ (otherwise the result follows readily from Theorem 5.1). Let $p_1 := p \mid_{D_1}: D_1 \to T$ be the restriction of $p$ onto $D_1$. Clearly, $V = p_1^{-1}(U)$ is a dense open subset of $D_1$ and, under $p_1$, is a unramified double cover of $U$.

We proceed in several steps.

**Step 1.** We built a complex manifold $\tilde{D}$ such that

- $\tilde{D}$ is bimeromorphic to $D_1$, we denote the corresponding bimeromorphic map by $\nu$;
- $\tilde{D}$ contains an open dense subset $\tilde{V}$ such that $\nu: \tilde{V} \to V$ is a biholomorphic isomorphism;
- the induced map $\mu := p_1 \circ \nu \tilde{D} \to T$ is unramified double covering;
- there is a holomorphic involution $s: \tilde{D} \to \tilde{D}$ such that the quotient $\tilde{D}/s = T$.

Namely, we consider the normalization $\tilde{D}$ of the Stein factorization of the map $p_1$. Since every point $v \in V$ is the connected component of the fiber $p_1^{-1}(p_1(v)) \in D_1$ (and there are two such components), there is an open dense subset $\tilde{V} \subset \tilde{D}$ that is mapped by $\nu$ onto $V$ one-to-one.

We have

$$
\begin{align*}
\tilde{D} &\xrightarrow{\nu} D_1 \xrightarrow{p_1} T \\
\cup &\cup \\
\tilde{V} &\xrightarrow{\nu} V \xrightarrow{p_1} U
\end{align*}
$$

Recall that for each $d \in \tilde{D}$ we write $\nu(d)$ for the image of $\nu$ in $D$. If $\nu$ is holomorphic at $d$ then the set $\nu(d)$ is a singleton. Map $\mu := p_1 \circ \nu \tilde{D} \to T$ is holomorphic, because $T$ is *poor*. It is finite, because the map from Stein factorization of $p_1$ to $T$ is finite by definition, and so is the normalization map. This map is an unramified double covering over $U$. It follows that ramification locus $R \subset \tilde{D}$ of $\mu$ is contained in $\tilde{D} \setminus \tilde{V}$ and its image $\mu(R)$ is contained in a closed analytic subset $\Sigma \subset T$. 


hence has codimension at least 2. But the image of the ramification locus has pure codimension 1 ([DG, Section 1, 9], [Pe, Theorem 1.6]). It follows that $\mu$ is an unramified and, hence, $\tilde{D}$ is a manifold. We define involution $s : \tilde{D} \to \tilde{D}$ as (the only non-trivial) deck transformation of $\mu$ and the first step is done.

**Step 2.** We build a manifold $\tilde{X}$ equipped with holomorphic maps $\tilde{p} : \tilde{X} \to \tilde{D}$ and $\tilde{\pi} : \tilde{X} \to X$ such that

- $\tilde{\pi} : \tilde{X} \to X$ is an unramified double cover of $X$;
- the involution $s : \tilde{D} \to \tilde{D}$ may be lifted to an involution $\tilde{s}$ on $\tilde{X} : \tilde{s}(x,d) = (x,s(d))$;
- the quotient $\tilde{X}/\tilde{s} = X$.

We take $\tilde{X} = X \times_T \tilde{D} = \{(x, \tilde{d}) \mid \mu(\tilde{d}) = p(x)\} \subset \tilde{D} \times X$ and let $\pi, \tilde{p}$, be the natural projections on the first and the second factors respectively.

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\pi} & X \\
\tilde{p} \downarrow & & \downarrow p \\
\tilde{D} & \xrightarrow{\mu} & T
\end{array}
\]

(1) Take $x \in X$, let $t = p(x)$. Then $\mu^{-1}(t)$ contains two points: $d$ and $s(d)$ in $\tilde{D}$. Then

$$\tilde{p}^{-1}(\mu^{-1}(t)) = \pi^{-1}(x) = \{(x, d), (x, s(d))\}$$

consists in two distinct points. Thus $\pi : \tilde{x} \to x$ is indeed an unramified double cover.

(2) The involution $s : \tilde{D} \to \tilde{D}$ may be lifted to an involution $\tilde{s}$ on $\tilde{X} : \tilde{s}(x,d) = (x,s(d))$;

(3) from (1) and (2) we get $\tilde{X}/\tilde{s} = X$.

(4) Take $d \in \tilde{D}$ with $\mu(d) = t \in T$. We have

$$\tilde{P}_d = \tilde{p}^{-1}(d) = \{(x, d) : x \in P_t\} \sim \mathbb{P}^1,$$

thus $(\tilde{X}, \tilde{p}, \tilde{D})$ is an equidimensional conic bundle.

(5) Consider $S = D_1 \times_T \tilde{D} \subset \tilde{X}$. Then for every fixed $d \in \tilde{D}$ we have

$$\tilde{p}^{-1}(d) \cap S = \{\nu(d), d\}.$$ 

If $d \in \tilde{V}$ then $\nu$ is holomorphic at $d$ and $\nu(d)$ consists of one point. Thus, $S$ is a section over $\tilde{V}$.

**Step 2** is done.

**Step 3.** We build a $\mathbb{P}^1$–bundle $(\tilde{Y}, \tilde{q}, \tilde{D})$ such that

- there is a bimeromorphic map $\Phi : \tilde{Y} \to \tilde{X}$.
- there is an involution $\sigma : \tilde{Y} \to \tilde{Y}$ without fixed points;
• the following diagrams are commutative:

\[
\begin{array}{cccc}
\tilde{X} & \xrightarrow{\Phi} & \tilde{Y} \\
\downarrow{\tilde{\sigma}} & & \downarrow{\tilde{\sigma}} \\
\tilde{D} & \xrightarrow{id} & \tilde{D} \\
\end{array}
\quad \begin{array}{cccc}
\tilde{X} & \xrightarrow{\tilde{s}} & \tilde{X} \\
\downarrow{\Phi} & & \downarrow{\Phi} \\
\tilde{D} & \xrightarrow{id} & \tilde{D} \\
\end{array}
\quad \begin{array}{cccc}
\tilde{Y} & \xrightarrow{\sigma} & \tilde{Y} \\
\downarrow{\tilde{q}} & & \downarrow{\tilde{q}} \\
\tilde{D} & \xrightarrow{s} & \tilde{D} \\
\end{array}
\quad \begin{array}{cccc}
\tilde{X} & \xrightarrow{\tilde{\Phi}} & \tilde{Y} \\
\downarrow{\tilde{\sigma}} & & \downarrow{\tilde{\sigma}} \\
\tilde{D} & \xrightarrow{id} & \tilde{D} \\
\end{array}
\] (24)

Indeed, by Theorem 5.1 there exists a \(\mathbb{P}^1\)-bundle \((\tilde{Y}, \tilde{q}, \tilde{D})\) and a bimeromorphic map \(\Phi : \tilde{Y} \to \tilde{X}\). The first diagram is commutative due to Lemma 2.2. Since \(\Phi\) is bimeromorphic, defined is bimeromorphic map \(\sigma : \tilde{Y} \to \tilde{Y}, \sigma = \Phi^{-1} \circ \tilde{s} \circ \Phi\), that fits into the third diagram. By Corollary 3.1 \(\sigma \in \text{Aut}(\tilde{Y})\). Map \(\sigma\) is an involution since \(\sigma^2 = \Phi^{-1} \circ \tilde{s} \circ \Phi \circ \Phi^{-1} \circ \tilde{s} \circ \Phi = id\). The second diagram is valid, because by construction we have the following commutative diagram:

\[
\begin{array}{cccc}
\tilde{Y} & \xrightarrow{\Phi} & \tilde{X} & \xrightarrow{\Phi^{-1}} & \tilde{Y} \\
\downarrow{\tilde{q}} & & \downarrow{\tilde{p}} & & \downarrow{\tilde{q}} \\
\tilde{D} & \xrightarrow{id} & \tilde{D} & \xrightarrow{s} & \tilde{D} \\
\end{array}
\] (25)

There are no fixed points for \(\sigma\), because \(s\) has no fixed points. Step 3 is done. Now, since \(\sigma\) has no fixed points, one may consider manifold \(Y = \tilde{Y}/\sigma\) with projection \(\pi_Y : \tilde{Y} \to Y\). From (24) we get that there is a bimeromorphic map \(\phi : Y \to X\), defined by commutativity of the diagram

\[
\begin{array}{cccc}
\tilde{X} & \xrightarrow{\pi} & \tilde{X}/s = X \\
\downarrow{\tilde{\Phi}} & & \downarrow{\phi} \\
\tilde{Y} & \xrightarrow{\pi_Y} & \tilde{Y}/\sigma = Y \\
\end{array}
\] (26)

**Corollary 5.3.** Let \((X, p, T)\) be a conic bundle over a poor manifold \(T\). Assume that the dimension of every component of every fiber of \(p\) is 1. Assume that \(X\) contains no divisor intersecting general fiber at one point and \(\text{Bim}(X)_p \neq \{id\}\). Then \(\text{Bim}(X)\) is very Jordan.

**Proof.** According to Proposition 5.1 conic bundle \(X\) is bimeromorphic to a \(\mathbb{P}^1\)-bundle \((Y, \psi, T)\). If \(Y \sim T \times \mathbb{P}^1\), then both \(Y\) and \(X\) would contain a divisor intersecting general fiber at one point. Since this is not a case, \(Y \not\sim T \times \mathbb{P}^1\). Thus \(\text{Bim}(X) \cong \text{Bim}(Y) \cong \text{Aut}(Y)\) has the needed property according to Theorem 4.1 \(\square\)

**Lemma 5.4.** Let \((X, p, T)\) be a conic bundle over a poor manifold \(T\). Assume that \(\text{Bim}(X)_p = \{id\}\). Then \(\text{Bim}(X)\) is very Jordan.
Proof. In this case we have a group embedding $\tau : \text{Bim}(X) \to \text{Aut}(T)$. Since $\text{Aut}_0(T)$ is a torus, and $\text{Aut}(T)/\text{Aut}_0(T)$ is bounded (Proposition 2.6), $\text{Aut}(T)$ is very Jordan. Since a subgroup of a very Jordan group is very Jordan, $\text{Bim}(X)$ is very Jordan. \hfill \Box

Now Theorem 1.2 follows from the combination of Corollary 5.1, Corollary 5.3, and Lemma 5.4.

6. Examples of $\mathbb{P}^1$-bundles without sections.

If $S$ is a complex manifold then we write $\mathbb{1}_S$ for the trivial line bundle $S \times \mathbb{C}$ over $S$.

In this section we construct a $\mathbb{P}^1$-bundle $(X, p, T)$ such that

- $T$ is a complex torus with $\dim(T) = n \geq 2$, $a(T) = 0$;
- projection $p : X \to T$ has no section, i.e., there is no a divisor $\Delta \subset X$ that meets every fiber $P_t$ at a single point;
- $\text{Aut}(X)_p$ contains no automorphisms of type $A$ or $B$;
- $\text{Aut}(X)_p$ contains an automorphism of type $C$;
- there exists a bisection of $p$, i.e., a divisor $D \subset X$ that intersects every fiber $P_t$ at two points.

We use the fact that the sections of a $\mathbb{P}^1$-bundle over a torus $T$ with $a(T) = 0$ do not intersect, thus our example is impossible with $\dim(T) = 1$.

Let $S$ be a torus with $\dim(S) = n \geq 2$, $a(S) = 0$.

Let $\mathcal{L}$ be a nontrivial holomorphic line bundle over $S$ such that

1. $\mathcal{L} \in \text{Pic}_0(S)$;
2. $\mathcal{L} \otimes^2 = \mathbb{1}_S$.

Let $Y$ be the total body of $\mathcal{L}$ and $q : Y \to S$ the corresponding surjective holomorphic map. Consider the rank two vector bundle $E := \mathcal{L} \oplus \mathbb{1}_S$ on $S$ and let $\overline{Y} = \mathbb{P}(E)$ be the projectivization of $E$ and $\overline{q} : \overline{Y} \to S$ the holomorphic extension of $q$ to $\overline{Y}$. The holomorphic map $\overline{q}$ has precisely two sections, namely, $D_0$ that is the zero section of $\mathcal{L}$ and $D_\infty = \overline{Y} \setminus Y$. Since $\mathcal{L}$ is a nontrivial line bundle and $a(S) = 0$, there are no other sections of $\overline{q}$.

We may describe $Y$ in the following way (see [BL, Ch. 1, Sect. 2]). Let $S = V/\Gamma$, where $V = \mathbb{C}^n$ is a vector space, $n = \dim(S)$ and $\Gamma$ is a discrete lattice of rank $2n$. Then there exists a nontrivial group homomorphism

$$\xi : \Gamma \to \{\pm 1\} \subset \mathbb{C}^*$$

such that $Y$ is the quotient $(V \times \mathbb{C})/\Gamma$ with respect to the action of group $\Gamma$ on $V \times \mathbb{C}$ by automorphisms

$$g_\gamma : (v, z) = (v + \gamma, \xi(\gamma)z) \quad \forall \gamma \in \Gamma, (v, z) \in V \times \mathbb{C}.$$  (27)
We may extend the action of \( \Gamma \) to \( V \times \mathbb{P}^1 = V \times \mathbb{C}_z \) by the same formula (27) and get \( \overline{Y} = (V \times \mathbb{C}_z)/\Gamma \).

Let us consider the following three holomorphic automorphisms of \( \overline{Y} \).

1) The line bundles \( \mathcal{L} \) and \( \mathcal{L}^{-1} \) are isomorphic. Hence there is a holomorphic involution map \( I_L : \overline{Y} \to \overline{Y} \) such that \( I_L(D_0) = D_\infty \) and \( I_L \circ I_L = id \). Automorphism \( I_L \) may be included into the commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{I_L} & Y \\
\pi \downarrow & & \pi \downarrow \\
S & \xrightarrow{id} & S 
\end{array}
\] (28)

In order to define \( I_L \) explicitly, let us consider a holomorphic involution \( \tilde{I}_L : V \times \overline{\mathbb{C}}_z \to V \times \overline{\mathbb{C}}_z \), \((v,z) \mapsto (v,\frac{1}{z})\). (29)

We have for all \( \gamma \in \Gamma \)
\[
g_\gamma \circ \tilde{I}_L(v,z) = (v + \gamma, \xi(\gamma) \cdot \frac{1}{z}),
\]
\[
\tilde{I}_L \circ g_\gamma(v,z) = (v + \gamma, \frac{1}{\xi(\gamma)z}) = g_\gamma \circ \tilde{I}_L(v,z),
\]
since \( \xi(\gamma)^2 = 1 \). In other words, \( \tilde{I}_L \) commutes with the action of \( \Gamma \) and therefore descends to the holomorphic involution of \((V \times \overline{\mathbb{C}}_z)/\Gamma = \overline{Y} \) and this involution is our \( I_L \).

2) Let us choose \( \gamma_0 \in \Gamma \) such that
\[
\gamma_0 \notin 2\Gamma, \xi(\gamma_0) = 1.
\] (30)
(Such a \( \gamma_0 \) does exist, since the rank of \( \Gamma \) is greater than 1.) Let us put
\[
v_0 = \frac{\gamma_0}{2} \in \frac{1}{2} \Gamma \subset V
\]
and consider the order 2 point
\[
P = v_0 + \Gamma \in V/\Gamma = S.
\]
Then the translation map
\[
T_P : S \to S, \ s \mapsto s + P
\]
is a holomorphic involution on \( S : T_P^2 = id \). Since \( \mathcal{L} \in \text{Pic}^0(S) \), translation \( T_P \) induces a holomorphic involution \( I_P : \overline{Y} \to \overline{Y} \) [Zar19] that “lifts” \( T_P \) and leaves \( D_0 \) and \( D_\infty \) invariant. The automorphism \( I_P \) may be included in the commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{I_P} & Y \\
\pi \downarrow & & \pi \downarrow \\
S & \xrightarrow{T_P} & S 
\end{array}
\] (31)
In order to describe $I_P$ explicitly, let us consider a holomorphic automorphism
\[ \tilde{I}_P : V \times \bar{C}_z \to V \times \bar{C}_z, \ (v, z) = (v + v_0, z). \] (32)

Clearly,
\[ \tilde{I}_P^2 = g_{\gamma_0} \]
(recall that $\xi(\gamma_0) = 1$). For all $\gamma \in \Gamma$
\[ g_\gamma \circ \tilde{I}_P(v, z) = (v + v_0 + \gamma, \xi(\gamma)z), \]
\[ \tilde{I}_P \circ g_\gamma(v, z) = (v + \gamma + v_0, \xi(\gamma)z) = (v + v_0 + \gamma, \xi(\gamma)z), \]
i.e., $\tilde{I}_P$ and $g_\gamma$ do commute. This implies that $\tilde{I}_P$ descends to the holomorphic involution of $(V \times \bar{C}_z)/\Gamma = \bar{Y}$, and this involution is our $I_P$.  

3) Let $\tilde{h} \in \Aut(\bar{Y})$ be the holomorphic involution that acts as multiplication by $-1$ in every fiber of $L$. (In notation of [Zar19] $\tilde{h} = \mult(-1)$.) In $\bar{Y} = (V \times \bar{C}_z)/\Gamma$ map $\tilde{h}$ is induced by the holomorphic involution
\[ \tilde{h} : V \times \bar{C}_z \to V \times \bar{C}_z, \ (v, z) = (v, -z), \] (33)
which commutes with all $g_\gamma$. Indeed, for all $\gamma \in \Gamma$
\[ g_\gamma \circ \tilde{h}(v, z) = (v + \gamma, \xi(\gamma)(-z)) = (v + \gamma, -\xi(\gamma)z), \]
\[ \tilde{h} \circ g_\gamma(v, z) = (v + \gamma, -\xi(\gamma)z) = g_\gamma \circ \tilde{h}(v, z). \]

Let us show that $I_L$, $I_Y$ and $\tilde{h}$ commute. It suffices to check that $\tilde{I}_L$, $\tilde{I}_Y$ and $\tilde{h}$ commute, which is an immediate corollary of the following direct computations.
\[ \tilde{I}_L \circ \tilde{h}(v, z) = (v, \frac{1}{-z}) = (v, -\frac{1}{z}) = \tilde{h} \circ \tilde{I}_L(v, z), \]
\[ \tilde{I}_L \circ \tilde{I}_P(v, z) = (v + v_0, \frac{1}{z}) = \tilde{I}_P \circ \tilde{I}_L(v, z), \]
\[ \tilde{h} \circ \tilde{I}_P(v, z) = (v + v_0, -z) = \tilde{I}_P(v, -z) = \tilde{I}_P \circ \tilde{h}(v, z). \]

Let us put now
\[ \inv := I_P \circ I_L : \bar{Y} \to \bar{Y}. \]

Then:
1) $\inv^2 = id$;
2) $\inv \circ \tilde{h} = \tilde{h} \circ \inv$;
3) $\mathcal{F} \circ \inv = T_P \circ \mathcal{F}$;
4) $T_P$ has no fixed points, thus $\inv$ has no fixed points;
5) $\inv(D_0) = D_\infty$;
6) If $d_1, d_2 \in D_0$ then $\inv(d_1) \neq d_2$. 

Let $X$ be the quotient of $\bar{Y}$ by the action of the order 2 group \{id, inv\}, and $\pi_Y : \bar{Y} \to X$ be the corresponding quotient map. Let $T$ be the quotient of $S$ by action of the order 2 group \{id, $T_P$\}, and $\pi_S : S \to T$ be the corresponding quotient map. Then $X, T$ enjoy the following properties.

- For any $x \in X$ there are precisely two points $y, \text{inv}(y)$ in $\pi_Y^{-1}(x)$.
- For any $t \in T$ there are precisely two points $s, T_P(s)$ in $\pi_S^{-1}(t)$.
- Both $\pi_Y : \bar{Y} \to X$ and $\pi_S : S \to T$ are double unramified coverings.
- $X$ is a smooth complex manifold (by (4)).
- $T$ is a complex torus with $\text{a}(T) = 0, \dim(T) = \dim(S) \geq 2$.
- It follows from (3) that there is a holomorphic map $p : X \to T$ such that the following diagram commutes:

$$\begin{array}{ccc}
\bar{Y} & \xrightarrow{\pi_Y} & X \\
\downarrow & & \downarrow p \\
S & \xrightarrow{\pi_S} & T
\end{array}$$

(34)

- If $\pi_S(s) = t \in T$ then $p^{-1}(t) \sim \pi_Y^{-1}(s) \sim \mathbb{P}^1$.
- It follows from (2) that there is a holomorphic map (pushdown) $h : X \to X$ such that the diagram commutes:

$$\begin{array}{ccc}
\bar{Y} & \xrightarrow{\pi_Y} & \bar{Y} \\
\downarrow & & \downarrow \pi_Y \\
X & \xrightarrow{h} & X
\end{array}$$

(35)

- Thanks to (5), we have $p_Y(D_0) = p_Y(D_\infty) := D$;
- Thanks to (6), the restriction $p \big|_D : D \to T$ is a double covering.

It follows that $X$ is a $\mathbb{P}^1$-bundle over $T$, $D$ is a bisection of $p$ and $h$ is a nontrivial automorphism in $\text{Aut}(X)_p$ of order 2, whose set of fixed points coincides with $D$.

Assume that $p$ has a section $\sigma : T \to X$. Let $\Sigma := \sigma(T) \subset X$ and $\Delta := \pi_Y^{-1}(\Sigma)$.

As we have already seen, both maps $\pi_S$ and $\pi_Y$ are double unramified covers. For every point $t \in T$ there are precisely two distinct points $s_i$ and $\text{inv}(s_i)$ in $\pi_S^{-1}(t)$, and there are precisely two distinct points in $\pi_Y^{-1}(\sigma(t))$, say, $y_i$ and $\text{inv}(y_i)$. One of them is mapped by $\pi_Y$ to $s$, another to $\text{inv}(s)$. It follows that for every $s \in S$ there is precisely one point in $\Delta \cap \pi_Y^{-1}(s)$. Hence, $\Delta$ is a section of $\pi_Y$, which is impossible, since $\pi_Y$ has no sections except $D_0$ and $D_\infty$. Moreover, $\Delta$ cannot coincide with $D$ since $\Delta$ is a section and $D$ is not.

Hence, $p$ has no sections and there are no automorphisms of type $A$ and $C$ in $\text{Aut}(X)_p$. 
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