A structure and energy dissipation efficiency of relativistic reconfinement shocks

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ABSTRACT
We present a semi-analytical hydrodynamical model for the structure of reconfinement shocks formed in astrophysical relativistic jets interacting with external medium. We take into account exact conservation laws, both across the shock front and in the zone of the shocked matter, and exact angular relations. Our results confirm a good accuracy of the approximate formulae derived by Komissarov & Falle. However, including the transverse pressure gradient in the shocked jet, we predict an absolute size of the shock to be about twice larger. We calculate the efficiency of the kinetic energy dissipation in the shock and show a strong dependence on both the bulk Lorentz factor and opening angle of the jet.

Key words: shock waves – galaxies: jets.

1 INTRODUCTION
Geometry (cross-sectional size, opening angle and bending) of supersonical, light jets is regulated, in general, by a complex system of oblique shocks. At certain circumstances, they take form of reconfinement shocks (Sanders 1983). Such shocks have been considered to be responsible for non-thermal activity in active galactic nucleus (AGN) radio cores (see e.g. Daly & Marscher 1988; Komissarov & Falle 1997, hereafter KF97; Stawarz et al. 2006) and, on much larger distances, in kiloparsec-scale radio knots (Komissarov & Falle 1998). They are also predicted to operate in massive X-ray binary systems (Perucho & Bosch-Ramon 2008) and in GRB collapsars (Bromberg & Levinson 2007).

A direct way to verify whether a given source can be interpreted in terms of a reconfinement shock is to determine whether location and extension of the source are consistent with a power of a jet and pressure/density of external medium. Under several simplifying assumptions analytical formulae relating these quantities were derived by Falle (1991) and, for relativistic shocks, by KF97. We have developed a semi-analytical model based on exact conservation laws and an exact dependence of a shock structure on an initial opening angle of a jet. Like in KF97, we adopt the cold jet approximation, i.e. we neglect the internal energy of the unshocked jet matter. The aim of this paper is to test the accuracy of the analytical formulae and to study the effects of including a transverse pressure gradients in the post-shock zone.

Our models are described in Section 2. They are compared with analytical results of KF97 in Section 3. We study the efficiency of energy dissipation in the reconfinement shocks in Section 4 and discuss their possible 'astrophysical appearance' in Section 5. Our main results are summarized in Section 6.

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2 DESCRIPTION OF THE RECONFINEMENT MODELS
The models we develop here are stationary, axisymmetric and purely hydrodynamical. We use a cylindrical coordinate system originating at the central source, with z-axis aligned with the jet symmetry axis and r denoting the cylindrical radius. At every point, the flow is characterized by the following parameters: rest-density ρ, pressure p and the angle between the velocity vector and the z-axis θ. We use the equation of state for the ideal gas

$$ p = (γ - 1)ε , $$

where $e$ is the internal energy density and $γ$ is the numerical coefficient, which for non-relativistic and for ultrarelativistic gases coincides with the adiabatic index with the values $5/3$ and $4/3$, respectively. For the intermediate cases and/or mixtures of non-relativistic and relativistic gases, $γ$ takes intermediate values which might somewhat differ from the respective values of the adiabatic indices (see KF97).

In a stationary flow conservation of mass, energy and momentum are expressed by the following equations:

$$ \partial_i (ρu_i) = 0 , $$

$$ \partial_i T^{iμ} = 0 , $$

where

$$ T^{μν} = uu^μu^ν + pg^{μν} $$

is the energy-momentum tensor,

$$ w = ρc^2 + e + p $$

is the enthalpy, $u^μ$ is the four-velocity (its spacial components are $u^i = Γβ$) and $g^{μν}$ is the metric tensor of signature $(-+++).$ For the purpose of this study, we assume a flat Minkowski space.
The jet is modelled as a spherically symmetric adiabatic outflow from the central source into the cone of half-opening angle $\Theta_j$. Equations (2–3) can be used to show that mass and energy fluxes of the upstream flow through any given solid angle must be conserved:

$$\frac{d}{dR} \left[R^2 (\rho u_j) \right] = 0,$$

(6)

$$\frac{d}{dR} \left[R^2 (\Gamma_j u_j) \right] = 0,$$

(7)

where $R = \sqrt{r^2 + z^2}$ is the radial distance from the central source. The quantity $\Gamma_j u_j / \rho_j$ is invariant along $R$.

In the cold jet approximation, $p_j$ is negligible, so $u_j \simeq \rho_j c^2$. Then $\Gamma_j$ is invariant along $R$, so from equation (7) we have $u_j \propto R^{-2}$. The total power of a jet is

$$L_j = 2\pi w_j \Gamma_j u_j c (1 - \cos \Theta_j) R^2.$$

(8)

Given $\Theta_j$, $L_j$ and $\Gamma_j$, it is now possible to calculate the flow parameters for every point within the jet.

In the interaction between the jet and external matter, a double oblique shock structure forms, but when external medium is static, it degenerates into a single shocked zone (see Fig. 1). In this scheme, the jet is bounded by the inner shock surface $r_s(z)$ and the shocked gas is bounded by the contact discontinuity $r_c(z)$. We denote the inclination angles of these surfaces by $\tan \alpha_{ic} = dr_{ic}/dz$, respectively. The flow parameters ($\Gamma$, $\rho$, $p$ and $\theta$) are marked with the following subscripts: j for the jet matter at $r_s$, c for the shocked matter at $r_c$, e for external medium at $r_e$.

The conservation laws must be satisfied across the shock surface. Equations (2–3) can be used to obtain shock jump conditions (Landau & Lifshitz 1959):

$$[\beta_j] = 0,$$

(9)

$$[\rho u_{\perp}] = 0,$$

(10)

$$[w u_{\perp} + p] = 0,$$

(11)

$$[\Gamma w u_{\perp}] = 0,$$

(12)

where $u_{\parallel}$ and $u_{\perp}$ are the tangent and normal components (with respect to the shock surface) of the local velocity field, respectively. At the shock front $r_s$, they give:

$$\beta_j \cos(\theta_j - \alpha_j) \equiv \beta_j \cos(\theta_j - \alpha_j),$$

(13)

$$u_{\perp} \rho_j \sin(\theta_j - \alpha_j) \equiv u_{\perp} \rho_j \sin(\theta_j - \alpha_j),$$

(14)

$$u_{\perp}^2 w_j \sin^2(\theta_j - \alpha_j) + p_j \equiv u_{\perp}^2 w_j \sin^2(\theta_j - \alpha_j) + p_j,$$

(15)

$$\Gamma_j u_j w_j \sin(\theta_j - \alpha_j) \equiv \Gamma_j u_j w_j \sin(\theta_j - \alpha_j).$$

(16)

For the contact discontinuity, there must be no flow through the surface, so the constraints derived from equations (2–3) are much more simple:

$$p_c = p_e,$$

(17)

$$\theta_c = \alpha_c.$$  

(18)

The purpose of our models is to find the geometrical structure and the physical conditions of the shocked zone, given the conditions in the jet and in the external medium.

### 2.1 Model 1

In our first model, we adopt an assumption made by Bromberg & Levinson (2007), that the shocked zone has no transverse structure, which means that for a given $z$: $\Gamma_c = \Gamma_e$, $\rho_c = \rho_e$, $p_c = p_e$, $\theta_c = \theta_e$. Knowing that $p_s = p_e$, we can solve equations (13–16) for the four unknowns: $\Gamma_s$, $\rho_s$, $\theta_s$, $\alpha_s$. This can be done explicitly using exact analytical formulæ (see Appendix A). We may then find $r_s$ by numerical integration over $z$.

We considered also finding the contact discontinuity surface $r_c$, by noting that $\alpha_c = \theta_c$. But when we calculated the mass flux across the shocked zone, we found that it is not conserved. Moreover, when neglecting the transverse pressure gradients, one cannot satisfy the transverse momentum balance needed to account for the curvature of the streamlines.

### 2.2 Model 2

A simple transverse structure of the shocked zone can be included in our model by assuming that parameters at opposite boundaries are independent. For a given $z$, we now have four more unknown parameters: $\Gamma_c$, $\rho_c$, $\alpha_c$, and $p_c$. Therefore, we need four additional equations that would tie the flow parameters at the shock surface to the flow parameters at the contact discontinuity.

We use conservation laws across the shocked zone introduced by Bromberg & Levinson (2007):

$$\frac{d}{dz} \left[ \int_{r_s}^{r_c} u \rho \cos \theta r \ dr \right] + \rho_j u_j n_j \frac{r_s}{\cos \alpha_s} = 0,$$

(19)

$$\frac{d}{dz} \left[ \int_{r_s}^{r_c} T \ u \ dr \right] + T_{ce} n_e \frac{r_s}{\cos \alpha_e} + T_{ce} n_c \frac{r_c}{\cos \alpha_c} = 0,$$

(20)

where $n_e$ and $n_c$ are the vectors normal to the shock surface and contact discontinuity surface, respectively, oriented outwards the shocked zone. Equation (19) describes conservation of mass, while the equation set (20) includes conservation of energy ($\mu = 0$) and of two momentum components (radial $- \mu = r$, longitudinal $- \mu = z$). Note that $T_{ce} n_e = p_e n_e \neq 0$. We can solve these equations by reducing them to a system of linear ordinary differential equations (see Appendix B).

### 3 Geometrical Properties of Reconfinement Shocks

The crucial characteristic of the reconfinement shocks is their lengthscale, which may be estimated observationally. KF97 provided simple analytic formulæ, in which they connect geometrical properties of the shock surface to physical parameters, such as the external pressure $p_e$, the total power of a jet $L_j$ and its bulk Lorentz factor $\Gamma_j$. They assumed that: the pre-shock plasma is cold...
\[ p_s = \mu \gamma p_e c^2 \sin^2(\theta_\gamma - \alpha), \]  
with \( \mu = 17/24 \).

Below, we rewrite their results, using slightly different notation. We launch the jet from the distance \( z = z_0 \). Let the external pressure profile be \( p_s(z) = p_0(z/z_0)^{-\eta} \) (we expect \( \eta \geq 0 \)). The shock surface should satisfy a boundary condition \( r_s(z_0) = z_0 \tan(\Theta_1) \), where \( \Theta_1 \) is the jet half-opening angle. Then the shock surface equation is

\[ r_s(z) = \left[ 1 - \frac{\delta}{\delta + \eta} \left( \frac{z}{z_0} \right)^{\frac{1}{\delta + \eta}} \right] \Theta_1 z, \]

where \( \delta = 1 - \eta/2 \), and

\[ \Lambda = \sqrt{\frac{\mu \gamma L_j}{\pi \rho_0 c}} \]

is a characteristic lengthscale.\(^1\) The reconfinement is found for \( \eta < 2(1 + z_0/\Lambda) \) at

\[ z_r = z_0 \left( 1 + \frac{\Lambda}{z_0} \right)^{1/\delta}. \]

The maximum width of unshocked jet

\[ r_m = \frac{z_0^2}{\Lambda} \left( \frac{z_r}{1 + \delta z_0} \right)^{1+\delta} \Theta_1 \]

is achieved at

\[ z_m = \frac{z_r}{(1 + \delta)^{1/\delta}}. \]

The aspect ratio of the jet is given by

\[ \frac{r_m}{z_r} = \frac{\delta + \frac{\delta}{\delta + 1} \Theta_1}{1 + \delta \Theta_1}. \]

The shock surface inclination at \( z_0 \) is

\[ \Theta_0 = \Theta_1 \left( 1 - \frac{z_0}{\Lambda} \right). \]

The half-closing angle (equal to minus the shock inclination at \( z_r \)) is

\[ \Theta_\gamma = \Theta_1 \left( \delta + \frac{z_0}{\Lambda} \right). \]

For the case of uniform external pressure (\( \eta = 0, \delta = 1 \)) and a jet originating close to the central source (\( z_0 \ll \Lambda \)), we find the shock to be parabolic, with very simple characteristics: \( z_r \approx \Lambda, r_m/z_r \approx \Theta_1/4, z_m \approx z_r/2, \Theta_0 \approx \Theta_1 \approx \Theta_\gamma \). We have tested these relations in our two models, the results are shown on Figs 2–5, as a function of half-opening angle \( \Theta_1 \) for a fixed \( \Gamma_1 \). Other parameters used were \( L_j = 10^{36} \erg \s^{-1}, p_0 = 10^{-2} \dyne \cm \), and \( z_0 = 10^{15} \cm \). The characteristic length for these parameters is \( \Lambda = 2.74 \times 10^{18} \cm = 0.89 \pc \).

A very good agreement between the results of Model 1 and the analytical formulae results from the same value of the pressure behind the shock (\( p_s = p_e \)). Deviations for \( \Gamma_1 \Theta_1 > 1 \) reflect the small angle approximation employed in analytical formulae. Small

\(^1\) Note that in equation (23) we have \( \Lambda \propto L_j^{1/2} \beta_1^{1/2} \), while models of non-relativistic jets predict \( \Lambda \propto L_j^{1/2} \beta_1^{-1/2} \) (see equation 1 in Komissarov 1994 and reference therein). The reason for the difference is that the term \( L_j \) includes the flux of the rest energy, \( M_\gamma c^2 \), while \( L_c \) does not.

Figure 2. The ratio of reconfinement position \( z_r \) to the characteristic length \( \Lambda \) as a function of \( \Gamma_1 \Theta_1 \). Results for Model 1 (dashed lines) and Model 2 (solid lines) are shown for different equations of state of the shocked matter: \( \gamma_s = 4/3 \) (black lines) and \( \gamma_s = 5/3 \) (grey lines).

Figure 3. The aspect ratio of unshocked jet \( r_m/z_r \), divided by the value \( \Theta_1/4 \) predicted by KP97, as a function of \( \Gamma_1 \Theta_1 \). The linestyles are the same as in Fig. 2.

Figure 4. The ratio of the maximum jet width position \( z_m \) to the reconfinement position \( z_r \) as a function of \( \Gamma_1 \Theta_1 \). The linestyles are the same as in Fig. 2.

but systematic deviations of \( z_r \) from \( \Lambda \) result from approximate pressure balance equation.

In Model 2, the pressure behind the shock is systematically lower than \( p_e \), but it cannot be fitted to a single power-law function of \( z \). This results in longer reconfinement structures (by a factor of about 2.2). We have found that for small and intermediate half-opening angles: \( r_m/z_r < \Theta_1/4, z_m < z_r/2 \) and, accordingly, \( \Theta_0 < \Theta_\gamma \).
Figure 5. The ratio of the half-closing angle $\theta_c$ to the half-opening angle $\theta_1$ as a function of $\Gamma_1\theta_1$. The linestyles are the same as in Fig. 2.

large angles the deviations from analytical predictions are more pronounced. Nevertheless, the effects of independent values for the $p_s$ are not particularly strong. The analytical formulae are still very useful within the order of magnitude accuracy.

4 ENERGY DISSIPATION

The kinetic energy flux through the shock front is dissipated with efficiency

$$\epsilon_{\text{diss}} = \frac{F_{\text{kin}(j)} - F_{\text{kin}(s)}}{F_{\text{kin}(j)}},$$

(30)

where

$$F_{\text{kin}} = \rho c^2 u_\perp (\Gamma - 1)$$

(31)

and $u_\perp$ is the four-velocity component normal to the shock front. Combining equations (30), (31) and (14) gives

$$\epsilon_{\text{diss}} = \frac{\Gamma_j - \Gamma_s}{\Gamma_j - 1}.$$ 

(32)

As averaged over the entire shock front area, the efficiency of energy dissipation is found to strongly depend on the product $\Gamma_j\theta_1$. Results for both models with a fixed $\Gamma_j = 10$ are shown in Fig. 6. We find that the averaged efficiency is very similar in both models and is insensitive to the value of $\gamma_s$. It approximately scales like $\epsilon_{\text{diss}} \sim 0.06(\Gamma_j \theta_1)^2$ for $\Gamma_j \theta_1 < 1$, but its increase slows down at $\Gamma_j \theta_1 > 1$.

In order to determine whether $\epsilon_{\text{diss}}$ is truly a function of $\Gamma_j\theta_1$, in Fig. 7 we present the results for Model 2 with $\gamma_s = 4/3$ and different values of $\Gamma_j$. We find little discrepancy between the curves, which implies that it is a well-defined dependence.

We have investigated the $z$ profiles of the dissipated energy flux. In Fig. 8, we show the results for both models, with $\Gamma_j = 10$ and $\theta_1 = 5^\circ$. Although the reconfinement position $z_r$ is more than twice large in Model 2, as compared to Model 1, the total amount of dissipated energy is very similar. The dissipated energy profiles have a well-defined maximum, which we denote as $z_{\text{diss,max}}$. The ratio of $z_{\text{diss,max}}$ to $z_r$ is shown in Fig. 9. It is larger in Model 1, but larger than $1/2$ in both models for $\Gamma_j \theta_1 < 1$. It decreases strongly with increasing $\Gamma_j \theta_1$ for $\Gamma_j \theta_1 > 1$. 

Figure 6. Dissipation efficiency $\epsilon_{\text{diss}}$ as a function of $\Gamma_j\theta_1$, calculated for $\Gamma_j = 10$. The linestyles are the same as in Fig. 2.

Figure 7. Dissipation efficiency $\epsilon_{\text{diss}}$ as a function of $\Gamma_j\theta_1$, calculated for Model 2 with $\gamma_s = 4/3$. Line colour indicates the value of $\Gamma_j$: 5 (light grey), 10 (grey), 20 (dark grey) and 40 (black).

Figure 8. Profiles of dissipated energy flux produced at the shock surface, calculated for $\Gamma_j = 10$ and $\theta_1 = 5^\circ$. The linestyles are the same as in Fig. 2.

Figure 9. Ratio of the location of maximum of dissipated energy $z_{\text{diss,max}}$ to the reconfinement position $z_r$ as a function of $\Gamma_j\theta_1$. The linestyles are the same as in Fig. 2.
Noting that the energy flux $F_e = w \Gamma u_e = (\rho c^2 + \gamma e) \Gamma u_e$, is conserved across the shock front (see equation 16), one can find that, for $\gamma_1 \sim \gamma$, the efficiency of the internal energy production is

$$\epsilon_e = \frac{F_{\epsilon,1} - F_{\epsilon,0}}{F_{\text{kin},1}} \sim \frac{1}{\gamma} \epsilon_{\text{diss}}. \quad (33)$$

A fraction of this energy is tapped by particles accelerated to relativistic energies and lost by non-thermal radiation. Such processes, if efficient, may significantly affect the shock structure.

It should be noted that in the case of particle acceleration with a broad energy distribution, most relativistic electrons may lose energy very efficiently even if the average energy dissipation efficiency is low. It means that, independently of the total energetics, the emissivity of such electrons will be maximized very close to the shock front and its spatial distribution will match the distribution of the energy dissipation.

5 ASTROPHYSICAL APPEARANCE

As theoretical analyses and numerical simulations demonstrate, formation of reconfinement shocks is accompanied by formation of reflection shocks (Sanders 1983; KP97). Furthermore, depending on a distribution of pressure or density of external medium, reconfinement and reflected shocks can form more or less abundant sequences of reconfinement shocks. Their radiative appearance is commonly modelled by assuming proportionality of the emissivity to the gas pressure (e.g. Gómez 2002; KP97). This leads to the predictions that most of the non-thermal radiation is produced around the reflection shocks. However, proportionality of the emissivity to the pressure is not what should be expected, if the efficiency of particle acceleration scales with the efficiency of energy dissipation. The latter is maximized at the shock fronts and, therefore, radiation emitted by most relativistic electrons will match geometrical structure of the shock fronts rather than the volume distribution of the pressure in the post-shock flows. Of course, ‘the shock front radiation’ is likely to be accompanied by emission from the entire post-shock volume, by both slowly cooling lower energy electrons and by electrons accelerated in turbulent plasma in the second order Fermi process. Specific geometrical and kinematical structures of reconfinement shocks are expected to be reflected in polarization properties, provided that magnetic fields are dominated by the shock compressed random field (Laing 1980; Cawthorne & Cobb 1990). This may explain the polarization electric vectors in the radio knots of AGN kiloparsec scale jets perpendicular to the jet direction (Bridle et al. 1994).

6 CONCLUSIONS

(i) Semi-analytical models were developed to calculate the structure of reconfinement shocks based on exact conservation laws and exact angular relations. The approximate analytical formulae of KP97, that describe a shape of the reconfinement shock and its dependence on the power of a jet and the pressure of external medium, were confirmed with a very good accuracy, even for $\Gamma_j \theta_j$ up to a few. The absolute size of the structure is found to be larger by a factor of about 2, when including the transverse pressure gradient in the post-shock flow.

(ii) The efficiency of energy dissipation in the relativistic reconfinement shocks scales approximately as $(\Gamma_j \theta_j)^2$ for $\Gamma_j \theta_j < 1$ and reaches about 6 per cent at $\Gamma_j \theta_j = 1$. For both models, with or without the transversal pressure gradient, the efficiency is very similar and for a given value of $\Gamma_j \theta_j$ practically does not depend on the bulk Lorentz factor.

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APPENDIX A: SOLVING THE SHOCK JUMP EQUATIONS

We show here a method by which the parameters of matter behind the shock may be determined in an exact analytical manner from equations (13–16). First, we express the angular parameters with non-angular ones. From equation (13) we find an expression for $\alpha_s$, which is also a differential equation for the shock surface:

$$\tan \alpha_s = \frac{\alpha_s}{\rho_s} = -\beta_s \cos \theta_s - \beta_c \cos \theta_j \beta_s \sin \theta_s - \beta_c \sin \theta_j. \quad (A1)$$

From equations (14) and (A1), we find the deflection angle of velocity field:

$$\cos(\theta_s - \theta_j) = \frac{\Gamma_s \rho_i \beta_s^2 + \Gamma_i \rho_i \beta_i^2}{(\Gamma_i \rho_i + \Gamma_s \rho_s) \beta_s \beta_i}. \quad (A2)$$

Now we find two more equations for two unknown parameters: $\Gamma_s$, $\rho_i$:

$$\frac{w_s}{w_j} = \frac{\Gamma_j \rho_s}{\Gamma_i \rho_i} \quad \quad (A3)$$

$$\left(\Gamma_s^2 - 1\right) \rho_s w_j = (p_j - p_s) \Gamma_s (\Gamma_i \rho_i + \Gamma_j \rho_j). \quad (A4)$$

Equation (A3) is the result of dividing equation (16) by equation (14). Equation (A4) is derived from equation (15) by eliminating trigonometric functions using equations (13–14) and eliminating $w_s$ using equation (A3). Using the equation of state for the shocked matter (and choosing the value of $\gamma_j$), we finally find from equations (A3) and (A4) a quadratic equation for $\Gamma_j$:

$$\left[\gamma_s p_j (w_j - p_j + p_s) \Gamma_s^2 + \left(\gamma_j - 1\right)(p_j - p_s) \rho_j \rho_s^2 \right] \Gamma_j \Gamma_s + \left[-w_j (\gamma_s - 1) p_j + p_s \right] \Gamma_j^2 = 0. \quad (A5)$$
Analysing the parameters of this equation, we know that there is always only one positive solution. During our calculations, we set an alert for unphysical $\Gamma_s < 1$, but it never triggered. Finding $\Gamma_s$, we calculate $\rho_s$ from equation (A4) and then we find the angular parameters: $\theta_s$ from equation (A2) and $\alpha_s$ from equation (A1).

**APPENDIX B: SOLVING THE EQUATIONS FOR CONSERVATION LAWS ACROSS THE SHOCKED ZONE**

The normal vectors $n_s$ and $n_c$ are given explicitly by

$$n_s = -\cos \alpha_s e_r + \sin \alpha_s e_z,$$

$$n_c = \cos \alpha_c e_r - \sin \alpha_c e_z.$$  \hspace{1cm} (B1)

Equations (19–20) may be expanded into

$$\frac{d}{dz} \left[ \int_{r_s}^{r_c} (u^2 w \cos \theta) r \, dr \right] = (u_s^2 w_s \cos \theta_s) \frac{r_s}{\cos \alpha_s},$$  \hspace{1cm} (B3)

$$\frac{d}{dz} \left[ \int_{r_s}^{r_c} \Gamma w \cos \theta r \, dr \right] = (\Gamma_s u_s w_s \sin \delta_s) \frac{r_s}{\cos \alpha_s},$$  \hspace{1cm} (B4)

$$\frac{d}{dz} \left[ \int_{r_s}^{r_c} u^2 w \sin \theta \cos \theta r \, dr \right] = (u_s^2 w_s \sin \delta_s \sin \theta_s + p_s \cos \alpha_s) \frac{r_s}{\cos \alpha_s} - p_s r_c \tan \alpha_s,$$  \hspace{1cm} (B5)

To perform the integrals, we have to describe the flow parameters between $r_s$ and $r_c$ as functions of $r$. We note that the expressions to be integrated are linear functions of $\rho$ and $p$ (since the enthalpy $w$ is also their linear function) and non-linear functions of $\Gamma$ and $\theta$. We decompose the integrated functions into $f(r) = g(\Gamma(r), \theta(r)) h(r) r$, where $h(r)$ is one of $\rho(r), p(r)$ or $w(r)$. We assume that $h(r)$ is linear:

$$h(r) = h_s + \frac{h_c - h_s}{r_c - r_s} (r - r_s).$$  \hspace{1cm} (B7)

The integrals are approximated with

$$\int_{r_s}^{r_c} f \, dr \simeq \frac{g(\Gamma_s, \theta_s) + g(\Gamma_c, \theta_c) \int_{r_s}^{r_c} h(r) r \, dr}{2}.$$  \hspace{1cm} (B8)

Substituting these formulae to equations (B3–B6), we obtain a system of four differential equations for eight variables: $\Gamma_s, \rho_s, \theta_s, \alpha_s, p_s, \Gamma_c, \rho_c$ and $\alpha_c$. The system is then closed by including differential forms of equations (13–16).