A new approach to Baer and dual Baer modules with some applications

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ABSTRACT. Let \( R \) be a ring. It is proved that an \( R \)-module \( M \) is Baer (resp. dual Baer) if and only if every exact sequence \( 0 \to X \to M \to Y \to 0 \) with \( Y \in \text{Cog}(M_R) \) (resp. \( X \in \text{Gen}(M_R) \)) splits. This shows that being (dual) Baer is a Morita invariant property. As more applications, the Baer condition for the \( R \)-module \( M^+ = \text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z}) \) is investigated and shown that \( R \) is a von Neumann regular ring, if \( R^+ \) is a Baer \( R \)-module. Baer modules with (weak) chain conditions are studied and determined when a Baer (resp. dual baer) module is a direct sum of mutually orthogonal prime (resp. co-prime) modules. While finitely generated dual Baer modules over commutative rings is shown to be semisimple, finitely generated Baer modules over commutative domain are studied. In particular, if \( R \) is commutative hereditary Noetherian domain then a finitely generated \( M_R \) is Baer if and only if it is projective or semisimple. Over a right duo perfect ring, it is shown that every (dual) Baer modules is semisimple.

Keywords: Baer module, character module, co-prime module, prime module, dual baer, regular ring, retractable module.

MSC(2010): Primary: 16D10; 16D40 Secondary: 13C05; 13C10.

1. Introduction

Throughout rings will have unit elements and modules will be right unitary. A ring \( R \) is said to be Baer if for every non-empty subset \( X \) of \( R \), the right annihilator \( X \) in \( R \) is of the form \( eR \) for some \( e = e^2 \in R \). Baer rings play an important role in the theory of rings of operators in functional analysis; see[17] and [4]. The concept of Baer ring was extended to modules by S.T. Rizvi and C. S. Roman in [23]. A module \( M_R \) is called Baer if for every non-empty subset \( X \) of \( \text{End}_R(M) \), the right annihilator \( X \) in \( M \) is a direct summand of \( M_R \). Baer modules and their generalizations have been studied among many other works. Every Baer module \( M \) is a D2-module (i.e., if \( M/A \) isomorphic to a direct summand of \( M \) then \( A \) is a direct summand of \( M \)), see [32], [11], [20] and [6] for recent works on the subjects. The dual notion of the Baer modules was introduced and studied in [30] where a module \( M_R \) is dual-Baer if for every non-empty subset \( X \) of \( \text{End}_R(M) \), the right annihilator \( X \) in \( M \) is a direct summand of \( M_R \). These modules are known to have the C2-property (i.e., in \( M \), every submodule isomorphic to a direct summand is a direct summand), see [1],[10], [16], [29] and [8] for some recent works on the dual Baer modules and important generalizations of them. In this paper, we first give new characterizations for a (dual) Baer module of which many are categorical; see Theorem 2.4 and Proposition 2.12. These implies that being (dual) Baer is a Morita invariant property and also co-retractable Baer (resp. retractable dual Baer) modules are semisimple; see Corollary 2.7 and Theorem 3.2. Baer (dual Baer) modules with some chain conditions (such as finite

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uniform dimension) are studied and determined when a Baer (resp. dual baer) module is a direct sum of mutually orthogonal prime (resp. co-prime) modules $\bigoplus_{i \in I} M_i$ such that $\text{Hom}_R(M_i, M_j) = 0$ ($i \neq j$) (Theorems 3.6 and 3.11). While finitely generated dual Baer modules over commutative rings is shown to be semisimple (Theorem 3.4), finitely generated Baer modules over commutative domain studied in Theorem 3.8. Among other things, it is shown that in Theorem 3.13, over a right duo perfect ring, every (dual) Baer modules is semisimple. In the last section, we apply our characterization of Baer modules to investigate conditions on $M_R$ under which the character left $R$-module $M^+$ is Baer. Any unexplained terminology and all the basic results on rings and modules that are used in the sequel can be found in [2] and [19].

2. Categorical characterization for (Dual) Baer modules

If $M_R$ is a nonzero module with $\text{End}_R(M) = S$, we shall use the following notation. If $N \leq M_R$ and $I \leq S_S$, we let $N^e = \text{Hom}_R(M,N)$ and $I^e = IM$. If $K$ and $L$ are two $R$-modules then $\text{Tr}(K,L)$ means $\sum \{f(K) \mid f : K \to L\}$ and $\text{Rej}(K,L)$ means $\bigcap \{\ker f \mid f : K \to L\}$. If $M_R$ is a module then the class of $R$-modules that are generated (resp. co-generated) by $M$ is denoted by $\text{Gen}(M_R)$ (resp. $\text{Cog}(M_R)$). If $TM_R$ is a bimodule, $A \subseteq T$, $N \subseteq M$ and $B \subseteq R$, then the right annihilator $A$ in $M$ (resp. $N$ in $R$) is canonically defined and denoted by $r_M(A)$ (resp. $r_R(N)$). The left annihilators are similarly denoted by $l_M(B)$ and $r_T(N)$. In this section, we consider a characterization stated in Theorem 2.4 and show that in addition to extracting new results from this theorem, some previous results are obtained from our theorem with simpler proof. The statements of the following Lemma have routine arguments and so we leave their proofs, but we use them later.

**Lemma 2.1.** Let $M_R$ be a nonzero module and $\text{End}_R(M) = S$. Then the following statements hold.

(i) $r_M(l_S(r_M(I))) = r_M(I)$ for all $I \subseteq S$.
(ii) $l_S(r_M(l_S(N))) = l_S(N)$ for all $N \subseteq M$.
(iii) $I^{ee} = I^e$ for all $I \leq S_S$.
(iv) $N^{ee} = N^e$ for all $N \leq M_R$.
(v) If $I \leq {}^\oplus S_S$ (resp. $I \leq {}^\oplus S_M$) then $I^e \leq {}^\oplus M_R$ (resp. $r_M(I) \leq {}^\oplus M_R$).
(vi) If $N \leq {}^\oplus M_R$ then $N^{ee} \leq {}^\oplus S_S$ and $l_S(N) \leq {}^\oplus S_S$.
(vii) $N \in \text{Gen}(M_R)$ if and only if $N = I^e$ for some $I \leq S_S$ if and only if $\text{Tr}(M,N) = N$.

The following Proposition that may be found in the literature, we record that as a consequence of Lemma 2.1 for latter uses.

**Proposition 2.2.** Let $M_R$ be a nonzero module with $\text{End}_R(M) = S$. Then the following statements hold.

(i) The module $M_R$ is Baer if and only if $r_M(I) \leq {}^\oplus M_R$ for every $I \leq S_S$.
(ii) The module $M_R$ is dual Baer if and only if $IM \leq {}^\oplus M_R$ for every $I \leq S_S$.

**Proof.** (i) ($\Rightarrow$) Let $I \leq S_S$ and $N = r_M(I)$. By Lemma 2.1(i) we have $N = r_M(l_S(N))$. Now since $M_R$ is Baer $l_S(N) \leq {}^\oplus S_S$. Thus $N \leq {}^\oplus M_R$ by Lemma 2.1(v).

($\Leftarrow$) Let $N \leq M_R$ and $I = l_S(N)$. By our assumption, $r_M(I) \leq {}^\oplus M_R$. Now by parts (vi) and (ii) of Lemma 2.1, we have $I = l_S(r_M(I)) \leq {}^\oplus S_S$. 

(ii) This is easily obtained by parts (iii)-(vi) of Lemma 2.1.

**Lemma 2.3.** Let \( N \leq M_R \) and \( \text{End}_R(M) = S \). The following statements are equivalent.

(i) \( N = r_M(I) \) for some \( I \leq_S S \).

(ii) \( N = \cap_{f \in X} \ker f \) for some non-empty \( X \subseteq S \).

(iii) \( M/N \in \text{Cog}(M_R) \).

**Proof.** (i) \( \Rightarrow \) (ii) and (ii) \( \Rightarrow \) (iii) are clear.

(iii) \( \Rightarrow \) (i). Suppose that \( \theta : M/N \to M^A \) is an injective \( R \)-homomorphism for some set \( \Lambda \). For each \( \lambda \in \Lambda \), let \( f_\lambda = \pi_\lambda \theta \) where \( \pi_\lambda \) is the canonical projection on \( M^A \). If \( X = \{ f_\lambda | \lambda \in \Lambda \} \) then it is easily seen that \( N = r_M(X) \). Clearly \( r_M(X) = r_M(I) \) where \( I = SX \).

**Theorem 2.4.** (i) Every exact sequence \( 0 \to X \to M \to Y \to 0 \) of \( R \)-modules with \( Y \in \text{Cog}(M) \) splits if and only if \( M \) is a Baer \( R \)-module.

(ii) Every exact sequence \( 0 \to X \to M \to Y \to 0 \) of \( R \)-modules with \( X \in \text{Gen}(M) \) splits if and only if \( M \) is a dual Baer \( R \)-module.

**Proof.** (i) \( (\Rightarrow) \). To show that a module \( M_R \) is Baer, we use Proposition 2.2(i). Let \( N = r_M(I) \) for some \( I \leq_S S \). By Lemma 2.3, \( M/N \in \text{Cog}(M) \). Hence the exact sequence \( 0 \to N \xrightarrow{\cdot} M \xrightarrow{\pi} M/N \to 0 \) splits by our assumption. This means \( N \leq_S M \).

\( (\Leftarrow) \). Consider that if \( 0 \to X \xrightarrow{f} M \to Y \to 0 \) is an exact sequence of \( R \)-modules with \( Y \in \text{Cog}(M_R) \), then the submodule \( \text{Im} f \) of \( M \) satisfies the equivalent conditions in the Lemma 2.3. Thus \( \text{Im} f \) is a direct summand of \( M_R \) by the Baer condition on \( M \). It follows that the exact sequence splits.

(ii) This is dual of (i) and has a similar argument. Just note that by Lemma 2.1(vii), a submodule \( N \) of \( M \) is in \( \text{Gen}(M) \) if and only if \( N = IM \) for some \( I \leq S_S \).

**Corollary 2.5.** If \( R \) is a Baer ring then \( \text{ann}_R(P) \) is a direct summand of \( R_R \) for any projective right \( R \)-module \( P \).

**Proof.** Let \( R \) be a Baer ring and \( P_R \) be projective. If \( I = \text{ann}_R(P) \) then \( R/I \in \text{Cog}(R_R) \). Hence \( I \) is a direct summand of \( R_R \) by Theorem 2.4.

By well known result from Gentile and Levy [13, Proposition 6.12], torsionfree divisible \( R \)-modules are injective when \( R \) is a semiprime right Goldie ring. hence the following result is a simple corollary of Theorem 2.4. The part (a) of below can be a generalization of [29, Proposition 4.16]. An \( R \)-module \( M \) is said to be torsionless if \( M \) is cogenerated by \( R \).

**Corollary 2.6.** (a) If \( R \) is a semiprime right Goldie (resp. right hereditary Noetherian) ring \( R \), every torsionfree injective (resp. injective) module \( M_R \) is dual Baer.

(b) If \( R \) is a ring such that \( R^{(n)}_R \) is extending (e.g., \( R \) is right self-injective) then every finitely generated non-singular \( R \)-module \( M_R \) is Baer.

(c) If \( R \) is a right hereditary right Noetherian prime ring then every finitely generated torsionless \( R \)-module \( M \) is Baer.
Proof. Let $0 \to X \to M \to Y \to 0$ be an exact sequence of $R$-modules with $X \in \text{Gen}(M)$ (resp. $Y \in \text{Cog}(M)$). For (a) (resp. (b)), our assumptions on $R$ imply that $X_R$ is injective (resp. $Y_R$ is projective). For (c), note that $Y$ is also torsionless and so by [22, Proposition 3.4.3], $Y_R$ must be projective. Hence, in any cases, the exact sequence splits and the result holds by Theorem 2.4.

Corollary 2.7. Being Baer (dual Baer) module is a Morita invariant property.

Proof. This is obtained by Theorem 2.4 and the fact that the category equivalences preserve exact sequences and direct (co) products [2, Proposition 21.6(3) and (5)].

Corollary 2.8. A ring $R$ is Baer if and only if every cyclic torsionless right (left) $R$-module is projective. Furthermore, every $n$-generated torsionless right(left) $R$-module is projective if and only if $M_n(R)$ is a Baer ring.

Proof. Since a cyclic $R$-module $R/I$ is projective if and only if the exact sequence $0 \to I \to R \to R/I \to 0$ splits, the result is an application of Theorem 2.4 for $M = R$. The last statement is now obtained by the fact that the standard Morita equivalent between $R$ and $M_n(R)$ corresponds $n$-generated $R$-modules to cyclic $M_n(R)$-modules.

Corollary 2.9. [23, Theorem 2.17] and [30, Corollary 2.5]. A direct summand of a Baer (resp. dual Baer) module is a Baer (resp. dual Baer) module.

Proof. This is obtained by Theorem 2.4 and the fact that an exact sequence $0 \to X \xrightarrow{f} N \xrightarrow{g} Y \to 0$ splits if the exact sequence $0 \to X \oplus L \xrightarrow{f \oplus L} N \oplus L \xrightarrow{\beta} Y \to 0$ with $\beta(n, l) = g(n)$ splits. In fact, if there exists $N \oplus L \xrightarrow{\alpha} X \oplus L$ such that $\alpha(f \oplus 1_L) = 1_{X \oplus L}$, then we have $hf = 1_X$ where $h : N \to X$ with $h(n) = \pi \alpha(n, 0)$ and $\pi : X \oplus L \to X$ is the natural projection.

Lemma 2.10. Let $M = \bigoplus_{i \in I} M_i$ (an index set) such that $\text{Hom}_R(M_i, M_j) = 0$ ($i \neq j$). If $0 \to X \xrightarrow{f} M \xrightarrow{g} Y \to 0$ is an exact sequence of $R$-modules. For each $i$, replace $\iota_i(M_i)$ with $M_i$ and let $g_i = g|_{M_i}$, $K_i = \ker g_i$, $X_i = f^{-1}(K_i)$ and $f_i = f|_{X_i}$. Then we have:

(a) For each $i$, the sequence $0 \to X_i \xrightarrow{f_i} M_i \xrightarrow{g_i} g(M_i) \to 0$ is exact.
(b) If $Y \in \text{Cog}(M)$ then for each $i$, $g(M_i) \in \text{Cog}(M_i)$ and $g(M) = \bigoplus_{i \in I} g(M_i)$.
(c) If $X \in \text{Gen}(M)$ then for each $i$, $X_i \in \text{Gen}(M_i)$ and $X = \bigoplus_{i \in I} X_i$.
(d) If $N \oplus L = M$ then $N = \bigoplus_{i \in I} (N \cap M_i)$.

Proof. (a) It has a routine argument.

(b) Let $Y \xrightarrow{g} \bigoplus_{i \in I} (M_i)^{\Lambda}$ then by hypothesis, $\theta g(M_i) \subseteq (M_i)^{\Lambda}$ for each $i \in I$. It follows that $g(M) = \bigoplus_{i \in I} g(M_i)$ and $g(M_i) \in \text{Cog}(M_i)$.

(c) Clearly, $\{X_i\}_{i \in I}$ are $R$-linear independent. Let $\bigoplus_{i \in I} (M_i)^{\Lambda} \xrightarrow{\alpha} X$ be a surjective $R$-homomorphism. By hypothesis $f \alpha \psi_i((M_i)^{\Lambda}) \subseteq M_i$ where $\psi_i : M_i^{(\Lambda)} \to \bigoplus_{i \in I} (M_i)^{\Lambda}$ is the natural $R$-monomorphism. If $x \in X$, then there are $u_i \in M_i^{(\Lambda)}$ ($i = 1, \ldots, n$) such that $\alpha(u_i) = x$. Since $f \alpha(u_i) \in M_i$, we have $f \alpha(u_i) \in K_i$. Thus $x \in \sum_i X_i$. It follows that $X = \sum_i X_i$ and so $X_i \in \text{Gen}(M_i)$.

(d) By hypothesis each $M_i$ is a fully invariant submodule of $M_R$. Hence for each $i$ we have $M_i = (N \cap M_i) \oplus (L \cap M_i)$. Thus $M = \bigoplus_{i \in I} (N \cap M_i) \oplus \bigoplus_{i \in I} (L \cap M_i) = N \oplus L$. It
follows that \( N = \bigoplus_i (N \cap M_i) \).

**Corollary 2.11.** Let \( M = \bigoplus_{i \in I} M_i \) (I an index set) such that \( \text{Hom}_R(M_i, M_j) = 0 \) \( (i \neq j) \).

(a) [24, Proposition 3.20]. If every \( M_i \) is a Baer \( R \)-module then \( M_R \) is Baer.

(b) If every \( M_i \) is a dual Baer \( R \)-module then \( M_R \) is dual Baer.

**Proof.** These are obtained by Lemma 2.10 and Theorem 2.4.

The Theorem 2.4 and Lemma 2.1(vii) show that a module \( M_R \) is dual Baer if and only if \( \text{Tr}(M, X) \leq M_R \) for every \( X \leq M_R \). Clearly, \( \text{Tr}(M, X) \leq X \). Thus if \( M_R \) is dual Baer then a submodule \( X \leq M_R \) contains a non-zero direct summand of \( M_R \) if and only if \( \text{Hom}_R(M, X) \neq 0 \). Modules \( M_R \) in which \( \text{Hom}_R(M, N) \neq 0 \) for every \( 0 \neq N \leq M_R \) is called retractable [18]. Dually, an \( R \)-module \( M_R \) is called co-retractable if \( \text{Hom}_R(M/N, M) \neq 0 \) for every \( N < M_R \). In the next section, we shall study the retractable and co-retractable conditions for (dual) Baer modules. We introduce the dual notation for \( \text{Tr}(M, N) \) where \( N \leq M \). By \( \text{Rej}^{-1}(N) \), we mean \( \pi^{-1}(\text{Rej}(M/N, M)) \) where \( \pi : M \to M/N \) is the canonical epimorphism. Clearly \( N \leq \text{Rej}^{-1}(N) \), and \( N = \text{Rej}^{-1}(N) \) if and only if \( M/N \in \text{Cog}(M) \). We record the following characterization for Baer (dual Baer) modules and use that in the next section.

**Proposition 2.12.** (i) The following conditions are equivalent for a module \( M_R \).

(a) \( M_R \) is Baer.

(b) For every nonempty set \( \Lambda \) and every \( R \)-homomorphism \( f : M \to M^\Lambda \), the inverse image \( f^{-1}(D) \) is a direct summand of \( M \) where \( D \) is a direct summand of \( M^\Lambda \).

(c) For every nonempty set \( \Lambda \) and every \( R \)-homomorphism \( f : M \to M^\Lambda \), \( \ker f \) is a direct summand of \( M_R \).

(d) For every \( N \leq M_R \), \( \text{Rej}^{-1}(N) \leq M_R \).

(ii) The following conditions are equivalent for a module \( M_R \).

(a) \( M_R \) is dual Baer.

(b) For every nonempty set \( \Lambda \), every \( R \)-homomorphism \( f : M^{(\Lambda)} \to M \) preserves direct summands.

(c) For every nonempty set \( \Lambda \), every \( R \)-homomorphism \( f : M^{(\Lambda)} \to M \), \( \text{Im} f \) is a direct summand of \( M_R \).

(d) For every \( N \leq M_R \), \( \text{Tr}(M, N) \) \( \leq M_R \).

**Proof.** We proof (i). (a) \( \Rightarrow \) (b). Let \( M_R \) be Baer and \( f : M \to M^\Lambda \) be an \( R \)-homomorphism for some \( \Lambda \). If \( D \) is a direct summand of \( M^\Lambda \) and \( N = f^{-1}(D) \) then we have the natural monomorphism \( M/N \to M/D \). It follows that \( M/N \in \text{Cog}(M) \). Hence \( N \) is a direct summand of \( M \) by Theorem 2.4.

(b) \( \Rightarrow \) (c). is clear.

(c) \( \Rightarrow \) (a). Let \( \{f_i\}_i \in \text{End}_R(M) \). Consider the \( R \)-homomorphism \( f : M \to M^\Lambda \) with \( f(m) = \{f_i(m)\}_i \). Then by our assumption \( f^{-1}(0) \) is a direct summand of \( M_R \). Thus \( \cap_i \ker f_i \) is a direct summand of \( M \), proving that \( M_R \) is Baer.

The equivalence (d) \( \Leftrightarrow \) (a), follows by Theorem 2.4 and the above notes.
3. (Dual) Baer module with retractable or co-retractable condition

We are now going to study the retractable (co-retractable) condition for Baer and dual Baer modules and then we shall give some applications of our results.

**Lemma 3.1.** Let $M_R$ be a nonzero module.

(i) If $M_R$ is co-retractable then $\text{Rej}^{-1}(N)/N \ll M/N$ for every proper $N \leq M_R$.

(ii) If $M_R$ is retractable then $\text{Tr}(M, N) \leq \text{ess } N$ for every nonzero $N \leq M_R$.

**Proof.** We only prove (i). Let $N < M_R$ and $K = \text{Rej}^{-1}(N)$. Clearly $N \leq K$. Suppose that $K/N + L/N = M/N$. If $L \neq M$, since $M_R$ is co-retractable, there exists nonzero homomorphism $g : M/L \to M$. Consider the natural epimorphism $p : M/N \to M/L$, then $0 \neq gp \in \text{Hom}_R(M/N, M)$ and $gp(L/N) = 0$. On the other hand, by the definition of $K$, we have $gp(K/N) = 0$. It follows that $gp(M/N) = 0$, a contradiction. Thus $L = M$ and we are done.

The equivalences (i) $\iff$ (ii) and (i) $\iff$ (iii) of below are dual of each others. The equivalence (i) $\iff$ (iii) is appeared in [?], corollary 2.19.

**Theorem 3.2.** The following statements are equivalent for a module $M_R$.

(i) $M_R$ is semisimple.

(ii) $M_R$ is co-retractable and Baer.

(iii) $M_R$ is retractable and dual Baer.

**Proof.** We need to show that (ii) or (iii) $\Rightarrow$ (i). Let $M_R$ be co-retractable and Baer (the other case is similar). If $N \leq M_R$ and $K = \text{Rej}^{-1}(N)$ then $K \leq \oplus M_R$ by Proposition ?? . This shows that $K/N \leq \oplus M/N$. On the other hand, $K/N \ll M/N$ by Lemma 3.1. Hence $N = K$. Therefore, every submodule of $M$ is a direct of $M$, as desired.

**Corollary 3.3.** Let $M_R$ be a non-zero quasi-projective module. Then the $R$-module $M/J(M)$ is dual Baer if and only if it is a semisimple $R$-module.

**Proof.** By [12, 3.4] any quasi-projective module with zero Jacobson radical is retractable. Hence the result is obtained by Theorem 3.2.

In [28], rings over which all nonzero modules are retractable are studied. Hence over such rings (e.g., commutative semi-Artinian rings) dual Baer modules are precisely semisimple modules.

**Theorem 3.4.** Let $R$ be a ring Morita invariant to a commutative ring.

(i) A finitely generated module $R$-module is dual Baer if and only if it is semisimple.

(ii) If $R$ is semi-Artinian then dual Baer $R$-modules are precisely semisimple $R$-modules.

**Proof.** By Corollary 2.7, we can suppose that $R$ is a commutative ring. Thus by [15, Theorem 2.7] every finitely generated $R$-module is retractable. Also if $R$ is semi-Artinian then all nonzero $R$-modules are retractable [15, Theorem 2.8]. Thus the result is obtained by Theorem 3.2.
If \(X\) and \(Y\) are \(R\)-modules then it is well known that \(\text{Rej}(X,Y)\) is a fully invariant submodule of \(X\) and \(X/\text{Rej}(X,Y)\) lies in \(\text{Cog}(Y)\). Thus if \(M_R\) is a Baer module with no non-trivial fully invariant direct summand, then Theorem 2.4 shows that for any \(0 \neq N \leq M\) we have \(\text{Hom}_R(M,N) = 0\) or \(M \in \text{Cog}(N)\). In [14, Theorem 2.5], the triangulating dimension \((\tau \text{dim}(M))\) was defined for a module \(M_R\) as follows: \(\text{Sup}\{k \in \mathbb{N} \mid M = \bigoplus_{i=1}^k M_i\ \text{with} \ M_i \neq 0 \text{and} \ \text{Hom}_R(M_i,M_j) = 0 \text{for any} \ i \neq j\}\) and it was shown that \(\tau \text{dim}(M_R) < \infty\) if and only if \(M = \bigoplus_i M_i\) where \(\text{Hom}_R(M_i,M_j) = 0 \ (i < j)\) and each \(M_i\) has no non-trivial fully invariant direct summand \((\text{i.e.,} \ \tau \text{dim}(M_i) = 1)\) if and only if \(M\) has ascending and descending chain conditions on fully invariant direct summands if and only if \(\text{End}_R(M)\) has a generalized triangular matrix representation. In [7, Proposition 2.16] Baer rings with a generalized triangular matrix representation are studied. Below we study Baer modules with finite \(\tau\)dimension and give some applications that are generalizations of earlier results in the literature. Note that Noetherian condition \(\Rightarrow \) finite uniform dimension \(\Rightarrow\) ascending (descending) chain condition on direct summands \(\Rightarrow\) finite \(\tau\)dimension. A modules that is cogenerated by each of its nonzero submodule, is called prime in [5] and \(*\)prime in [25]). It is easy to verify that every prime module has no non-trivial fully invariant direct summand).

The following theorem shows that the study of Baer modules with finite \(\tau\)dimension reduces to the study of such modules when they are prime. Recall that two \(R\)-module \(X\) and \(Y\) are called orthogonal to each other, if they do not contain nonzero isomorphic submodules.

**Lemma 3.5.** (a) Let \(M_R\) be a nonzero retractable Baer module. Then either \(M_R\) is prime or there is a decomposition \(M = N \oplus K\) such that \(N\) is a non-trivial fully invariant direct summand of \(M_R\).

(b) If \(M = M_1 \oplus M_2\) is a Baer \(R\)-module and \(M_i\) is prime \(R\)-module \((i = 1,2)\) such that \(\text{Hom}_R(M_1,M_2) = 0\), then \(\text{Hom}_R(M_2,M_1) = 0\) and \(M_1\) and \(M_2\) are orthogonal to each other.

**Proof.** (a) Let \(M_R\) is not a prime module. Thus there exists a non-trivial submodule \(X\) of \(M\) such that \(\text{Rej}(M,X) =: N\) is nonzero. Clearly \(N\) is a fully invariant of \(M\) such that \(M/N \in \text{Cog}(M)\). Since \(M_R\) is retractable, \(N \neq M\). Thus \(N\) is a non-trivial fully invariant submodule of \(M_R\) by Theorem 2.4.

(b) If \(f : M_2 \to M_1\) is nonzero then by the Baer condition on \(M\), we must have \(M_2 \simeq \ker f \oplus \text{Im} f\). Now since \(M_1\) is prime, it lies in \(\text{Cog}(\text{Im} f)\). Hence \(\text{Hom}_R(M_1,M_2) \neq 0\), contradiction.

**Theorem 3.6.** Let \(M_R\) be a non-zero Baer module. Then \(M = \bigoplus_{i=1}^n M_i\) such that each \(M_i\) is a prime \(R\)-module and \(M_i\)'s are mutually orthogonal with \(\text{Hom}_R(M_i,M_j) = 0 \ (i \neq j)\) if and only if \(\tau \text{dim}(M_R) < \infty\) and every direct summand of \(M_R\) is retractable.

**Proof.** \((\Leftarrow)\). This obtained by [14, Theorem 2.5] and Lemma 3.5.

\((\Rightarrow)\). Let \(0 \neq K \leq N \leq M = N \oplus N'\). We shall show that \(\text{Hom}_R(N,K) \neq 0\). By induction, we can suppose that \(N \cap M_i \neq 0\) for all \(i\). Also by Lemma 2.10(d), we have \(N = \bigoplus_{i=1}^n (N \cap M_i)\). On the other hand, we may suppose that \(K \cap M_j \neq 0\) for some \(j\). Thus the prime condition on \(M_j\) implies that \(\text{Hom}_R(N \cap M_j,K) \neq 0\), proving that \(\text{Hom}_R(N,K) \neq 0\).
Following [25]) a module $M_R$ is called \textit{compressible} if $M$ can be embedded in every non-zero its submodule. $R$-modules $X$ and $Y$ are said to be \textit{sub-isomorphic} if $X$ can be embedded in $Y$ and vice versa.

\textbf{Proposition 3.7.} Let $M_R$ be a non-zero Baer module with ascending (descending) chain condition on direct summands. Then $M = \bigoplus_{i=1}^{n} M_i$ such that $\text{Hom}_R(M_i, M_j) = 0$ $(i \neq j)$ and each $M_i$ is a finite direct sum of indecomposable compressible $R$-modules that are mutually sub-isomorphic if and only if every direct summand of $M_R$ is retractable.

\textbf{Proof.} $(\Rightarrow)$. It is easy to show that each $M_i$ is a prime. Hence every direct summand of $M_R$ is retractable by Theorem 3.6.

$(\Leftarrow)$. By [19, Proposition 6.59], $\tau\text{dim}(M_R)$ is finite. Hence by Theorem 3.6, we may suppose that $M_R$ is a prime module. Thus $\text{Hom}_R(X, Y) \neq 0$ for every non-zero submodules $X$ and $Y$ of $M_R$. On the other hand, by [2, proposition 10.14], $M = \bigoplus_{i=1}^{n} P_i$ is a finite direct sum of indecomposable submodules. Since now $P_i \oplus P_j$ is Baer for all $i, j$, every $f : P_i \to P_j$ is one to one. It follows that $M$ must be a finite direct sum of indecomposable compressible $R$-modules that are mutually sub-isomorphic.

As we stated in the proof of Theorem 3.4, if $R$ is commutative and $M_R$ is finitely generated then every direct summand of $M_R$ is retractable. Thus the following result can be a generalization of [23, Proposition 2.19].

\textbf{Theorem 3.8.} Let $R$ be a ring Morita invariant to a commutative domain. Then every finitely generated Baer $R$-module $M$ with a finite $\tau$-dimension is either torsion or a finite direct sum of uniform right ideals of $R$.

\textbf{Proof.} First note that $M_R$ is singular if and only if there exists an exact sequence $0 \to B \to A \to M \to 0$ such that the map $B \to A$ is an essential monomorphism. Since (essential) monomorphisms and co-kernels are preserved under Morita equivalences, we may suppose by Corollary 2.7 that $R$ is a commutative domain. Thus every prime $R$-module is torsion or torsionless (note that if $M_R$ is not torsion then $M$ contains isomorphically a nonzero (right) ideal of $R$). Also since $R$ is assumed to be a commutative domain, the uniform dimension of $R$ is finite. It follows that every torsionfree $R$-module is faithful with finite uniform dimension. On the other hand, by [27, Theorem 3.14], if $M_R$ is finitely generated and faithful then $\text{Hom}_R(M, X) \neq 0$ for every nonzero $X_R$. Now we can apply Theorem 3.6 to deduce that $M_R$ is either torsion or faithful torsionfree $R$-module with finite uniform dimension. If $M_R$ is faithful torsionfree $R$-module with finite uniform dimension then Proposition 3.7 shows that $M$ is a finite direct sum of uniform right ideals of $R$.

\textbf{Corollary 3.9.} Let $R$ be a ring Morita invariant to a commutative hereditary Noetherian domain and $M_R$ is finitely generated. Then $M_R$ is a Baer $R$-module $M$ if and only if it is semisimple or projective.

\textbf{Proof.} The sufficiency follows from Corollary 2.6. For the necessity, note that every singular $R$-module has nonzero socle by [22, Proposition 5.4.5]. Hence the result is obtained by Proposition 3.7 and Theorem 3.8.

We now consider the dual of Theorem 3.6. Following [31], a module $M$ is called \textit{co-prime} whenever $M/N$ generates $M$ for any proper submodule $N < M_R$. Furthermore, we
say that $M_R$ is co-compressible if $M_R$ is a homomorphic image of every non-zero its factor. Clearly, co-compressible modules are co-prime. For the dual notion of sub-isomorphic, by $X \epi Y$ the $R$-modules $X$ and $Y$ are called epi-invariant if $X$ is a homomorphic image of $Y$ and vice versa.

**Lemma 3.10.** (a) Let $M_R$ be a nonzero co-retractable dual Baer module. Then either $M_R$ is co-prime or there is a decomposition $M = N \oplus K$ such that $N$ is a non-trivial fully invariant direct summand of $M_R$. (b) If $M = M_1 \oplus M_2$ is a dual Baer $R$-module and $M_i$ is co-prime $R$-module $(i = 1, 2)$ such that $\text{Hom}_R(M_1, M_2) = 0$, then $\text{Hom}_R(M_2, M_1) = 0$. (c) Let $M = \bigoplus_{i \in I} C_i$ such that for any $i \in I$, the $R$-module $C_i$ has no non-trivial fully invariant direct summand. If $V$ is a non-zero fully invariant direct summand of $M_R$ then there is non-empty subset $J$ of $I$ such that $V = \bigoplus_{j \in J} C_j$ and $\text{Hom}_R(V, C_i) = 0$ for any $i \in I \setminus J$.

**Proof.** (a) and (b) are dual of Lemma 3.5 and obtained from the dual arguments. (c) Since $V$ is fully invariant submodule of $M_R$, we must have $V = \bigoplus_{i \in I} V \cap C_i$, also if $M = V \oplus V'$ then $\text{Hom}_R(V, V') = 0$. On the other hand, for every $i \in I$, it is easy to verify that $V \cap C_i$ is a fully invariant direct summand of $C_i$. Now let $J = \{i \in I \mid V \cap C_i \neq 0\}$.

**Theorem 3.11.** Let $M_R$ be a non-zero dual Baer module. Consider the following conditions.

(a) Every direct summand of $M_R$ is co-retractable. (b) $M = \bigoplus_{i \in I} M_i$ is a direct sum of co-prime $R$-modules with $\text{Hom}_R(M_i, M_j) = 0$ $(i \neq j)$ such that each $M_i$ is a direct sum of indecomposable co-compressible modules that are mutually epi-invariant.

Then (a) implies (b) and the converse is true if $I$ is a finite set.

**Proof.** (a)⇒(b). By [30, corollary 2.6], $M = \bigoplus_{i \in I} C_i$ is a direct sum of indecomposable modules. Also every indecomposable co-retractable dual Baer module $X$ is a co-compressible module. To see this suppose that there is a proper submodule $X$ of $X$. Since $X$ is assumed to be co-retractable, there exits non-zero $f : X/Y \to X$. Now $\text{Im}(f)$ lies in $\text{Gen}(X)$ and so must be a direct summand of $X$ by the dual Baer condition. It follows that $f$ is surjective, proving that $X$ is co-compressible. Now for any $i \in I$, let $M_i = \bigoplus\{C_j \mid C_i \epi C_j\}$. By Lemma 3.10(c), $M_i$ has no non-zero fully invariant direct summand and hence by part (a) of Lemma 3.10, $M_i$ must be a co-prime $R$-module. On the other hand, if $\text{Hom}_R(C_i, C_j) \neq 0 (i \neq j)$ then $C_i \epi C_j$. It follows that $\text{Hom}_R(M_i, M_j) = 0 (i \neq j)$.

For the converse, let $I$ be a finite set with $|I| = n$, $N \oplus L = M$ and $K < N_R$. We shall show that $\text{Hom}_R(N/K, N) \neq 0$. Let $V = K \oplus L$. Then $N/K \simeq M/V$ and we will show that $\text{Hom}_R(M/V, N) \neq 0$. In fact, it is enough to show that there exists $i \in I$ such that $M_i \not\subseteq V$ and $M_i \in \text{Gen}(M/V)$. Because by hypothesis $M_i$ is a fully invariant submodule of $M_R$ and so we have $M_i = (N \cap M_i) \oplus (L \cap M_i)$ with $N \cap M_i \neq 0$. Now let $W = M_i + V$. If $W = M$ then $M_i \not\subseteq V$ and $M/V \simeq M_i/(V \cap M_i)$. Hence the co-prime condition on $M_i$ implies that $M_i \in \text{Gen}(M/V)$, as desired. If $W \neq M$, let $J = \{i \in I \mid M_i \subseteq W\}$ and $U = \bigoplus_{j \in J} M_j$. Then $W/U$ is a proper submodule of $M/U \simeq \bigoplus_{i \not\in J} M_i$ which is co-retractable by the induction assumption. Thus $\text{Hom}_R(M/W, C) \neq 0$ for some $i \not\in J$. It follows $\text{Hom}_R(M/W, C) \neq 0$ such that $C$ is an indecomposable module appeared in
the decomposition of $M_i$. Now the dual Baer condition on $M_R$ implies that $C$ is a homomorphic image of $M/W$. Since indecomposable modules in the decomposition of $M_i$ are mutually epi-invariant, we have $M_i \in \text{Gen}(M/W) \subseteq \text{Gen}(M/V)$. The proof is now completed.

**Proposition 3.12.** Let $M$ be a non-zero dual Baer $R$-module such that every direct summand of $M_R$ is co-retractable. Then $M = M_1 \oplus M_2$ where $M_1$ is singular and $M_2$ is a nonsingular semisimple module.

**Proof.** If $X$ is co-prime and $N$ is a proper essential submodule of $X$, then $X/N$ and hence $X$ must be singular. Therefore every co-prime module $X$ is either singular or semisimple (as semisimple modules have no proper essential submodules). The proof is now completed by Theorem 3.11.

**Theorem 3.13.** Let $R$ be a ring Morita invariant to a right duo perfect ring, then the following condition are equivalent.

(i) $M_R$ is Baer.

(ii) $M_R$ is dual Baer.

(iii) $M_R$ is semisimple.

**Proof.** By Corollary 2.7, we suppose that $R$ is a right duo and a perfect ring. Then it is well known that every non-zero $R$-module has a non-zero maximal submodule. It follows that co-compressible $R$-modules are simple. Therefore, by Theorems 3.2 and 3.11, it is enough to show that every non-zero $R$-module is co-retractable. Now let $N$ be a proper submodule of a non-zero module $M_R$. Since $R$ is right perfect then the non-zero module $M/N$ has a maximal submodule $K/N$. On the other hand, by [3, Theorem 2.14] $M_R$ is a Kasch module. Hence the simple module $M/K$ can be embedded in $M_R$. It follows that $\text{Hom}(M/N, M)$ is nonzero, as desired.

4. **Further applications**

As another application of the Theorem 2.4, we conclude the paper with a result to show that if $\text{Hom}_R(M, Q/Z)$ is Baer as a left $R$-module then $M_R$ has a condition close to the dual Baer condition. A submodule $N$ of a module $M_R$ is called pure if for any left $R$-module $A$, the homomorphism $i \otimes 1_A$ is one to one where $i : N \to M$ is the inclusion map and $1_A : A \to A$ is the identity map. The exact sequence $0 \to X \xrightarrow{f} M \to Y \to 0$ is then called pure exact if $\text{Im} f$ is a pure submodule of $M$. Clearly, every direct summand of $M_R$ is a pure submodule of $M$. Hence, in view of Proposition ??, we may consider the condition weaker than the dual Baer condition for a module $M$: $\text{Tr}(M, X)$ is a pure submodule $M_R$ for any $X \leq M_R$. For any right $R$-module $M$, the character left $R$-module $\text{Hom}_R(M, Q/Z)$ is denoted by $M^+$. First we recall some facts on purity.

**Proposition 4.1.** Let $M$ be a non-zero $R$-module.

(i) The exact sequence $0 \to N \to M$ of $R$-modules is pure exact if and only if the sequence $M^+ \to N^+ \to 0$ splits.

(ii) Let $M_R$ be flat. An exact sequence $0 \to X \to M \to Y \to 0$ of $R$-modules is pure if and only if $Y_R$ is flat.
**Proof.** (i) This follows from [9, Proposition 5.3.8]
(ii) It is true by [19, Corollary 4.86].

**Theorem 4.2.** Let $M_R$ be a non-zero module with $\text{End}_R(M) = S$. If the left $R$-module $M^+$ is Baer then $\text{Tr}(M, X)$ is a pure submodule $M_R$ for any $X \leq M_R$. If further, $M_R$ is flat then $M/N$ is a flat $R$-module for any $N \in \text{Gen}(M)$.

**Proof.** Suppose that the left $R$-module $M^+$ is Baer. Let $X \leq M_R$ and $\text{Tr}(M, X) = N$. Then $N \in \text{Gen}(M)$. Consider the exact sequence $0 \to N \to M \to M/N \to 0$ in $\text{Mod}-R$. Note that since $N \in \text{Gen}(M)$, we have $N^+ \in Cog(M^+)$. Thus we obtain the exact sequence $0 \to (M/N)^+ \to M^+ \to N^+ \to 0$ in $\text{R-Mod}$ with $N^+ \in Cog(M^+)$ by [19, Proposition 4.8]. Now by the Baer condition on $M^+$ and Theorem 2.4, the sequence $M^+ \to N^+ \to 0$ splits. Hence $N$ is a pure submodule of $M_R$ by Proposition 4.1(i). The last statement is now obtained by Proposition 4.1(ii).

**Corollary 4.3.** Let $M$ be a non-zero $R$-module. (i) If $M_R$ is self-generator and the left $R$-module $M^+$ is Baer then all submodules of $M_R$ are pure.
(ii) If $M_R$ is a generator for $\text{Mod}-R$, then the left $R$-module $M^+$ is Baer if and only if it is a semisimple left $R$-module.

**Proof.** (i) This is an immediate corollary of Theorem 4.2.
(ii) Just note that for every left $R$-module $L$, we have $L$ can be embedded in the left $R$-module $L^{++}$. Hence, if $M_R$ is a generator for $\text{Mod}-R$, then $M^+$ is a co-generator for $\text{R-Mod}$. Thus the result is obtained by Theorem 2.4.

A ring $R$ is called **von Neumann regular** if for every $a \in R$ there is $b \in R$ such that $a = aba$. It is well known that $R$ is a von Neumann regular ring if and only if all cyclic (left) right $R$-module are flat if and only if every right (left) ideal is a pure in $R_R$ ($R_R$).

**Corollary 4.4.** If the left $R$-module $R^+$ is Baer then $R$ is a von Neumann regular ring.

**Proof.** It is obtained by Corollary 4.3 and the above notes.

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