Liouville theory and uniformization of four-punctured sphere

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Abstract

Few years ago Zamolodchikov and Zamolodchikov proposed an expression for the 4-point classical Liouville action in terms of the 3-point actions and the classical conformal block \cite{1}. In this paper we develop a method of calculating the uniformizing map and the uniformizing group from the classical Liouville action on \( n \)-punctured sphere and discuss the consequences of Zamolodchikov’s conjecture for an explicit construction of the uniformizing map and the uniformizing group for the sphere with four punctures.

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1 Introduction

The uniformization problem for \( n \)-punctured \((n \geq 3)\) spheres can be formulated as follows [2]:

\[
\text{Given } X = \mathbb{C} \setminus \{z_1, \ldots, z_{n-1}\}, \text{ find the (unique to within conjugacy) Fuchsian group } G \subset \text{PSU}(1,1) \text{ which makes } \Omega \text{ conformally equivalent to the quotient } \Delta/G \text{ of the unit disc } \Delta = \{z \in \mathbb{C} : |z| < 1\} \text{ by } G.
\]

The existence of solution to this problem, called the uniformization theorem, was first proved by Poincaré and Koebe in 1907. The universal covering map \( \lambda : \Delta \to \Delta/G \) is however explicitly known only for the thrice-punctured sphere [3] and in few very special, symmetric cases with higher number of punctures [2]. In particular an explicit construction of this map for the four-punctured sphere is a long standing and still open problem.

One possible approach, going back to Poincaré, is based on the relation of the uniformization problem to a certain Fuchs equation on \( X \). If \( \lambda : \Delta \to \Delta/G \cong X \) is the universal covering map, the inverse \( \rho = \lambda^{-1} : X \to \Delta \) is a multi-valued function with branching points \( z_j \) and with branches related by elements of the covering group \( G \subset \text{PSU}(1,1) \). One can show that the Schwarzian derivative of \( \rho \) is a holomorphic function on \( X \) of the form [2]

\[
\{\rho, z\} = \frac{1}{2} \sum_{k=1}^{n-1} \frac{1}{(z-z_k)^2} + \sum_{k=1}^{n-1} \frac{2c_k}{z-z_k}, \tag{1.1}
\]

\[
\{\rho, z\} \to \infty = \frac{1}{2z^2} + O(z^{-3}), \tag{1.2}
\]

where the accessory parameters \( c_j \) satisfy the relations

\[
\sum_{k=1}^{n-1} c_k = 0, \quad \sum_{k=1}^{n-1} (4c_kz_k + 1) = 1. \tag{1.3}
\]

It is a well known property of the Schwarzian derivative [2, 4] that the map \( \rho \) is, to within a Möbius transformation, a quotient of two linearly independent solutions of the Fuchs equation

\[
\partial_z^2 \Psi + \frac{1}{2} \{\rho, z\} \Psi = 0. \tag{1.4}
\]

This in particular means that there exists a unique to within \( \text{SU}(1,1) \) transformation fundamental system \( \{\Psi_1, \Psi_2\} \) of normalized (i.e. with the Wronskian equal to 1) solutions for which

\[
\rho = \frac{\Psi_1}{\Psi_2}. \tag{1.5}
\]

Note that any such system has to have an \( \text{SU}(1,1) \) monodromy with respect to all punctures.

Whether the Fuchs equation can be used to calculate the map \( \rho \) depends on our ability to calculate the accessory parameters and to choose an appropriate fundamental system of normalized solutions. The first problem can be easily solved for three punctures where the
Accessory parameters are completely determined by relations (1.3). It is however difficult and still unsolved for \( n > 3 \).

One can get some more insight by relating this problems to the Liouville equation on \( X \). Since PSU(1,1) is the isometry group of the Poincaré hyperbolic metric \( g_\Delta = \frac{4 \partial \rho \partial \bar{\rho}}{(1 - |\rho|^2)^2} \) on \( \Delta \) the pull back

\[
\rho^* g_\Delta = \frac{4}{(1 - |\rho|^2)^2} \left| \frac{\partial \rho}{\partial z} \right|^2 dz d\bar{z} = e^{\varphi(z, \bar{z})} dz d\bar{z}
\]

is a regular hyperbolic metric on \( X \), conformal to the standard flat metric \( dz d\bar{z} \) on \( X \subset \mathbb{C} \). Its conformal factor \( \varphi \) satisfies the Liouville equation

\[
\partial_z \partial_{\bar{z}} \varphi(z, \bar{z}) = \frac{1}{2} e^{\varphi(z, \bar{z})}
\]

and has the following asymptotic behavior at punctures

\[
\varphi(z, \bar{z}) = \begin{cases} 
-2 \log |z - z_j| - 2 \log |\log |z - z_j|| + O(1) & \text{as } z \to z_j, \\
-2 \log |z| - 2 \log |\log |z|| + O(1) & \text{as } z \to \infty.
\end{cases}
\]

It is known that there exists a unique solution to (1.7) and (1.8) \([5]\). One can show that the energy–momentum tensor \( T(z) \) of this solution is equal to one half of the Schwarzian derivative (1.1):

\[
T(z) \equiv -\frac{1}{4} (\partial \varphi)^2 + \frac{1}{2} \partial^2 \varphi = \frac{1}{2} \{\rho, z\}.
\]

This allows to calculate all accessory parameters once the classical solution \( \varphi \) is known.

The problem of selecting an appropriate fundamental system is slightly less demanding. As we shall see it is sufficient to know the next to the first two leading terms of the asymptotic of the classical solution \( \varphi \) at one arbitrary puncture.

Unfortunately, the problem to find solutions to the Liouville equation seems to be at least as hard as the problem of calculating the map \( \rho \) itself, and the reformulation does not help much on this stage. In this framework however one can consider a more general problem of spheres with \( n \) elliptic singularities characterized by real parameters \( 0 < \xi_j < 1 \). Instead of asymptotic conditions (1.8) we impose

\[
\varphi(z, \bar{z}) = \begin{cases} 
-2 (1 - \xi_j) \log |z - z_j| + O(1) & \text{as } z \to z_j, \\
-2(1 + \xi_n) \log |z - z_j| + O(1) & \text{as } z \to \infty.
\end{cases}
\]

The existence and uniqueness of \( \varphi \) was in this case proved by Picard \([6, 7]\) (see also \([8]\)). The solution can be interpreted as a conformal factor of the complete, hyperbolic metric with the Gaussian curvature \( R = -1 \) on \( X = \mathbb{C} \setminus \{z_1, \ldots, z_{n-1}\} \) and conical singularities of the opening angles \( 0 < 2\pi \xi_j < 2\pi \) at the points \( z_j \). The \( n \)-punctured sphere discussed so far corresponds to the limiting case \( \xi_j \to 0 \) for all \( j = 1, \ldots, n \).

The notion of accessory parameters can be introduced in terms of the energy momentum tensor of the solution \( \varphi \). In the present case it takes the form \([4]\):

\[
T(z) = \sum_{j=1}^{n-1} \left[ \frac{\delta_j}{(z - z_j)^2} + \frac{c_j}{z - z_j} \right],
\]

2
$$T(z) \overset{z \to \infty}{= \delta_n z^2 + O(z^{-3})},$$

where $\delta_j = \frac{1-\xi^2_j}{4}$, $j = 1, \ldots, n$ are classical conformal weights. The multi-valued function $\rho: X \to \Delta$ is still of interest although it is not longer an inverse to the universal covering of $X$. It can be used as before in formula (1.6) to construct solutions to the Liouville equation with asymptotic behavior (1.10).

The ideas described above are classic and most of them were already pursued by Poincaré. For almost a century the problem of accessory parameters had reminded unsolved. An essentially new insight was brought in by the so called Polyakov conjecture in 1982 [9]. It states that the (properly defined and normalized) Liouville action functional evaluated on the classical solution $\varphi_{cl}(z, \bar{z})$ is a generating function for the accessory parameters:

$$c_j = -\frac{\partial S^{(cl)}(\delta_i; z_i)}{\partial z_j}.$$  \hfill (1.12)

This formula was derived within path integral approach to the quantum Liouville theory as the quasi-classical limit of the conformal Ward identity [10]. In the case of the parabolic singularities on $n$-punctured Riemann sphere a rigorous proof based on the theory of quasi-conformal mappings was given by Zograf and Takhtajan [11]. Other proofs, valid both in the case of parabolic and general elliptic singularities, were proposed in [12] and [4].

The next significant step was done by Zamolodchikov and Zamolodchikov [1]. Analyzing the classical limit of the four-point function of the quantum Liouville theory they argued that the classical Liouville action for four elliptic (parabolic) singularities can be expressed in terms of the classical Liouville action for three singularities and some special function called the classical conformal block. Recently this conjecture has been successfully tested by symbolic and numerical calculations in [13]. It should be stressed however that it is still far from being rigorously proved. The basic problem is the classical block itself which is so far accessible only via term by term calculation of the classical limit of the quantum conformal block.

The aim of the present paper is to analyze to what extend the Zamolodchikovs conjecture can provide an explicit construction of the uniformization of four-punctured sphere. Our motivation is to get a better insight into a geometric content of this conjecture and to develop a theoretical framework for its new numerical tests. The results indicate that the classical conformal block plays a central role in the problem and certainly deserves further investigations.

The content of the paper is as follows. In Section 2 we analyze the problem of selecting an appropriate pair of solutions to the Fuchs equation in the case of elliptic weights. It is shown that all the information required is encoded in the derivatives of the classical Liouville action with respect to the parameters $\xi_j$. The case of parabolic singularities is obtained by taking an appropriate limit. The main result is that in order to calculate the map $\rho$ from the Fuchs equation it is sufficient to know the classical Liouville action as a function of $z_j$'s and $\xi_j$'s.
In Section 3 we analyze the problem of calculating monodromies of the Fuchs equation once the accessory parameters are known. Only the case of 4 punctures is considered. For the standard locations \(0, x, 1, \infty\) we develop systematic expansions of monodromy matrices at \(0, x, 1\) in terms of power expansions in \(x\) and \(1 - x\). The results of Sections 2 and 3 are general and are independent of the form in which the classical Liouville action is available.

In Section 4 the Zamolodchikovs conjecture is formulated and some schemes of calculation of the classical Liouville action and accessory parameters are developed. They lead to very efficient methods of numerical calculations. It should be stressed however that many steps in their derivation still require sound mathematical proofs. Concluding this section we discuss some open problems and possible extensions of our work.

## 2 Map \(\rho : X \rightarrow \Delta\)

In the case of the sphere with \(n\) elliptic singularities with weights \(\delta_j = \frac{1 - \xi_j^2}{4}\) we define the functional

\[
S_L[\delta; \phi] = \frac{1}{4\pi} \lim_{\epsilon \to 0} S_L^\epsilon[\delta; \phi],
\]

\[
S_L^\epsilon[\delta; \phi] = \int_{X_\epsilon} d^2z \left[ |\partial \phi|^2 + e^\phi \right] + \sum_{j=1}^{n-1} \left(1 - \xi_j \right) \int_{|z-z_j|<\epsilon} |dz| \kappa_{zz} \phi + \left(1 + \xi_n \right) \int_{|z|=\frac{1}{\epsilon}} |dz| \kappa_{z\bar{z}} \phi - 2\pi \sum_{j=1}^{n-1} (1 - \xi_j)^2 \log \epsilon - 2\pi (1 + \xi_n)^2 \log \epsilon,
\]

where \(X_\epsilon = \mathbb{C} \setminus \bigcup_{j=1}^{n} \{|z-z_j|<\epsilon\} \cup \{|z|>\frac{1}{\epsilon}\}\).

The classical Liouville action \(S^{(cl)}(\delta; z_i)\) is then defined as [4, 12]

\[
S^{(cl)}(\delta; z_i) = S_L[\delta; \varphi_{cl}],
\]

where \(\varphi_{cl}(z, \bar{z})\) is the unique solution to (1.7) and (1.10).

Once the classical action is known one can use the Polyakov conjecture (1.12) to calculate the accessory parameters and write down the corresponding Fuchs equation

\[
\partial_z^2 \Psi + T \Psi = 0,
\]

with the energy momentum tensor \(T\) given by (1.11). Our aim in this section is to select a fundamental system of solutions to this equation such that their quotient yields a multi-valued function \(\rho : X \rightarrow \Delta\) with SU(1,1) monodromy at each \(z_j\) and regular for all \(z \in X\).

To this end let us first observe that (1.7) and (1.9) imply that \(e^{-\frac{2\varphi_{cl}}{2}}\) is a real solution to the Fuchs equations (2.1) and its complex conjugate. It can be therefore expressed as a bilinear combination of any fundamental system and its complex conjugate. In order to fix the freedom related to the SU(1,1) transformations let us choose a normalized system with
diagonal monodromy at an arbitrarily chosen singular point $z_j$:

$$
\Psi^{(j)}_{\xi,\pm}(z) = \frac{A_{j}^{\pm 1}}{\sqrt{\xi_j}} (z - z_j)^{\frac{1 + \xi_j}{2}} (1 + \mathcal{O}(z - z_j)), \quad A_j \in \mathbb{R}.
$$

(2.2)

It follows from reality, positivity and single valuedness of $e^{-\varphi(z,\bar{z})}$ on $X$ that the parameter $A_j$ can be adjusted such that

$$
e^{-\varphi(z,\bar{z})} = \frac{D_j}{2} \left[ \left| \Psi^{(j)}_{\xi,-}(z) \right|^2 - \left| \Psi^{(j)}_{\xi,+}(z) \right|^2 \right],
$$

where $D_j$ is a positive constant. Although the formula above is derived by considering a small neighborhood of $z_j$, it holds for all $z \in X$. Using it one easily derive all the required properties of the map

$$
\rho_j(z) = \frac{\Psi^{(j)}_{\xi,+}(z)}{\Psi^{(j)}_{\xi,-}(z)}.
$$

One can in particular apply the formula (1.6) to construct the hyperbolic metric with Gaussian curvature $-1$ on $X$. As this metric coincides with $e^{\varphi(z,\bar{z})} dz d\bar{z}$, the constant $D_j$ has to be equal to 1 and

$$
e^{-\varphi(z,\bar{z})} = \frac{1}{2} \left[ \left| \Psi^{(j)}_{\xi,-}(z) \right|^2 - \left| \Psi^{(j)}_{\xi,+}(z) \right|^2 \right].
$$

(2.3)

Analyzing the limit $z \to z_j$ one finds that $A_j^2 = \frac{1}{2\xi_j} e^{\frac{1}{2} f_j}$, where $f_j$ is the next to the leading term in the $\varphi_{cl}(z,\bar{z})$ asymptotic at $z \to z_j$. On the other hand

$$
\frac{\partial S^{(cl)}(\delta_i; z_i)}{\partial \xi_j} = \lim_{\epsilon \to 0} \left[ (1 - \xi_j) \log \epsilon - \frac{1}{4\pi} \int_{|z - z_j| = \epsilon} |dz| \kappa_z \phi \right] = \frac{1}{2} f_j,
$$

(2.4)

and consequently

$$
A_j^2 = \frac{1}{2\xi_j} \exp \left\{ \frac{\partial}{\partial \xi_j} S^{(cl)}(\delta_i; z_i) \right\}.
$$

(2.5)

This equation, along with the Polyakov conjecture, shows that that the classical Liouville action contains all the information needed to calculate the map $\rho : X \to \Delta$ from the Fuchs equation.

To study the parabolic limit $\xi_j \to 0$ we shall first define the pair $\tilde{\Psi}^{(j)}_{\xi,\pm}(z)$, related to $\Psi^{(j)}_{\xi,\pm}(z)$ through the SU(1,1) transformation,

$$
\begin{pmatrix}
\tilde{\Psi}^{(j)}_{\xi,+}(z) \\
\tilde{\Psi}^{(j)}_{\xi,-}(z)
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
\xi_j^{1/2} + \xi_j^{-1/2} & \xi_j^{1/2} - \xi_j^{-1/2} \\
\xi_j^{1/2} - \xi_j^{-1/2} & \xi_j^{1/2} + \xi_j^{-1/2}
\end{pmatrix}
\begin{pmatrix}
\Psi^{(j)}_{\xi,+}(z) \\
\Psi^{(j)}_{\xi,-}(z)
\end{pmatrix}.
$$

From (2.5) it can be shown in the general case (and will be illustrated by the explicit calculation in the next section) that

$$
A_j = 1 + \xi_j a_j + \mathcal{O}(\xi_j^2) = e^{\xi_j a_j} + \mathcal{O}(\xi_j^2),
$$

(2.6)
with $a_j$ independent of $\xi_j$. From this fact and (2.2) it follows that the limit
\[
\begin{pmatrix}
\Psi_+^{(j)}(z) \\
\Psi_-^{(j)}(z)
\end{pmatrix} = \lim_{\xi_j \to 0} \begin{pmatrix}
\tilde{\Psi}_+^{(j)}(z) \\
\tilde{\Psi}_-^{(j)}(z)
\end{pmatrix}
\tag{2.7}
\]
exists.

The inverse of the map
\[
\rho^{(j)}(z) = \frac{\Psi^{(j)}_+(z)}{\Psi^{(j)}_+(z)}
\tag{2.8}
\]
is the universal covering map of the punctured sphere $X$ by the Poincaré disc $\Delta$. In particular, as
\[
\lim_{z \to z_j} \frac{\Psi^{(j)}_-(z)}{\Psi^{(j)}_+(z)} = 1,
\]
the puncture is mapped onto the point on the boundary of the Poincaré disc.

### 3 Monodromy matrices

In this section we shall consider the Riemann sphere with four punctures at the standard locations $z = 0, x, 1$ and $\infty$. In this case
\[
T(z) = \frac{1}{4z^2} + \frac{1}{4(z-x)^2} + \frac{1}{4(z-1)^2} + \frac{c_1(x)}{z} + \frac{c_2(x)}{z-x} + \frac{c_3(x)}{z-1},
\]
and the relations (1.3) can be written in the form
\[
c_1(x) = \frac{1}{2} - (1-x)c_2(x), \quad c_3(x) = -\frac{1}{2} - xc_2(x). \tag{3.1}
\]

Our aim is to calculate in this case the monodromies of the fundamental systems of solutions to the Fuchs equation (1.4) constructed in the previous section.

To this end we shall split $T(z)$ onto the “free” term $T_0(z)$ and the “interaction” $-V(z)$,
\[
T(z) = T_0(z) - V(z).
\]

$T_0(z)$ contains by construction all terms in $T(z)$ singular at $z = 0$, term with a second order pole at $z = x$ plus a “correction” enforcing the behavior $\sim z^{-2}$ at the infinity,
\[
T_0(z) = \frac{1}{4z^2} + \frac{1}{4(z-x)^2} + \frac{c_1(x)}{z} - \frac{c_1(x)}{z-x} = \frac{1}{4z^2} + \frac{1}{4(z-x)^2} - \frac{xc_1(x)}{z(z-x)}
\equiv \frac{1}{4z^2} + \frac{1}{4(z-x)^2} - \frac{1 + \nu^2(x)}{4z(z-x)}
\]
with
\[
\nu^2(x) = 4xc_1(x) - 1 = -4x(1-x)c_2(x) + x - (1-x), \tag{3.2}
\]
and
\[
V(z) = -\frac{(1-x)c_3(x)}{(z-x)(z-1)} - \frac{1}{4(z-1)^2}.
\]
Solutions to the “free” equation

\[ \partial_z^2 f(z) + T_0(z)f(z) = 0 \]

are then expressible through the hypergeometric functions with the well-known monodromy around \( z = 0, x \), while the solutions (and monodromies) of the “full” equation

\[ \partial_z^2 \Psi(z) + T_0(z)\Psi(z) = V(z)\Psi(z) \quad (3.3) \]

can be obtained by perturbation theory with a \( n \)-th order correction proportional to \( x^n \).

One then repeats the calculation above with \( x \to 1 - x \). Since this problem is related through the global conformal transformation \( z \to w = 1 - z \) to our original problem with the points 0 and 1 exchanged (and opposite orientation of the \( z \) and \( w \) planes) it is clear that the monodromy matrix around \( w = 0 \) yields the monodromy matrix around \( z = 1 \). Note however that this time the parameter in the perturbative expansion is \( 1 - x \). Our method thus yields all three monodromy matrices\(^3\) only for those \( x \) for which both \( x \) and \( 1 - x \) are small enough.

Let

\[ T_0^{(\xi, \epsilon)}(z) = \frac{1 - \epsilon^2}{4z^2} + \frac{1 - \xi^2}{4(z - x)^2} + \frac{\xi^2 + \epsilon^2 - x^2 - 1}{4z(z - x)}. \quad (3.4) \]

We can choose the normalized basis in the space of solutions to the equation

\[ \partial^2 f(z) + T_0^{(\xi, \epsilon)}(z)f(z) = 0 \]

in the form of the pair of functions with the diagonal monodromy matrix around \( z = 0 \),

\[
\begin{align*}
\phi_{\pm}^{(\xi, \epsilon)}(z) &= \sqrt{\frac{x}{\epsilon}} \left( \frac{1 - z}{x} \right)^{\frac{1 + \xi}{2}} \left( 1 - \frac{z}{x} \right)^{\frac{1 + \xi}{2}} \text{$_2$F$_1$}
\left( 1 + \xi + \nu \pm \epsilon, 1 + \xi - \nu \pm \epsilon; 1 \pm \epsilon, \frac{z}{x} \right),
\end{align*}
\]

or in the form of the pair of functions with the diagonal monodromy matrix around \( z = x \),

\[
\begin{align*}
\psi_{\pm}^{(\xi, \epsilon)}(z) &= e^{\pm \xi a} \sqrt{x} \left( \frac{x}{\xi} \right)^{\frac{1 + \xi}{2}} \left( 1 - \frac{z}{x} \right)^{\frac{1 + \xi}{2}} \text{$_2$F$_1$}
\left( 1 \pm \xi + \nu + \epsilon, 1 \pm \xi - \nu + \epsilon; 1 \pm \xi, 1 - \frac{z}{x} \right).
\end{align*}
\]

These two pairs of functions are obviously expressible through each other,

\[
\begin{pmatrix}
\psi_{\pm}^{(\xi, \epsilon)}(z) \\
\psi_{\pm}^{(\xi, \epsilon)}(z)
\end{pmatrix}
= C^{(\xi, \epsilon)}
\begin{pmatrix}
\phi_{\pm}^{(\xi, \epsilon)}(z) \\
\phi_{\pm}^{(\xi, \epsilon)}(z)
\end{pmatrix}
\]

with [14]

\[
C^{(\xi, \epsilon)} = \sqrt{\frac{x}{\xi}} \left( \begin{array}{cc}
eg^a e^{-\epsilon \gamma(-\epsilon)\Gamma(1+\xi)} & \Gamma(-\epsilon)\Gamma(1+\xi) \\
\Gamma(-\epsilon)\Gamma(1-\xi) & \Gamma(1+\xi) e^{\epsilon \gamma(1-\epsilon)\Gamma(1-\xi)}
\end{array} \right)
\]

To take the limit \( \epsilon \to 0 \) we define, similarly as in (2.7),

\[
\begin{pmatrix}
\phi_+^{(\xi)}(z) \\
\phi_-^{(\xi)}(z)
\end{pmatrix}
= \lim_{\epsilon \to 0} B_{\epsilon}
\begin{pmatrix}
\phi_+^{(\xi, \epsilon)}(z) \\
\phi_-^{(\xi, \epsilon)}(z)
\end{pmatrix}
\]

\[3\]The monodromy matrix around the puncture \( z = \infty \) is equal to the inverse of the product of the remaining three matrices.
and
\[ C^{(\xi)} = \lim_{\epsilon \to 0} C^{(\xi, \epsilon)} \cdot B_\epsilon^{-1} \]
with
\[ B_\epsilon = \frac{1}{2} \begin{pmatrix} \sqrt{\epsilon} + \frac{1}{\sqrt{\epsilon}} & \sqrt{\epsilon} - \frac{1}{\sqrt{\epsilon}} \\ \sqrt{\epsilon} - \frac{1}{\sqrt{\epsilon}} & \sqrt{\epsilon} + \frac{1}{\sqrt{\epsilon}} \end{pmatrix}. \]

For \( z \to x \)
\[ \phi^{(\xi)}_\pm (z) = \sqrt{z} \left( 1 \pm \frac{1}{2} \log \frac{z}{x} \right) + o(z), \]
and the monodromy matrix of this pair around \( z = 0 \) (for \( z \to e^{2\pi i}z \)) is
\[ M^{(z=0)}_{\phi^{(\xi)}} = \begin{pmatrix} 1 + \frac{i\pi}{2} & \frac{i\pi}{2} \\ -\frac{i\pi}{2} & 1 - \frac{i\pi}{2} \end{pmatrix}. \]

One thus gets a monodromy matrix of the pair \( \psi^{(\xi)}_\pm = \lim_{\epsilon \to 0} \psi^{(\xi, \epsilon)}_\pm \) around \( z = 0 \) in the form
\[ M^{(z=0)}_{\psi^{(\xi)}} = C^{(\xi)} \cdot M^{(z=0)}_{\phi^{(\xi)}} \cdot \left( C^{(\xi)} \right)^{-1}. \]

The limit \( \xi \to 0 \) can be taken as in (2.7),
\[ \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \lim_{\xi \to 0} B_\xi \cdot \begin{pmatrix} \psi^{(\xi)}_+ \\ \psi^{(\xi)}_- \end{pmatrix} \]
with
\[ B_\xi = \frac{1}{2} \begin{pmatrix} \sqrt{\xi} + \frac{1}{\sqrt{\xi}} & \sqrt{\xi} - \frac{1}{\sqrt{\xi}} \\ \sqrt{\xi} - \frac{1}{\sqrt{\xi}} & \sqrt{\xi} + \frac{1}{\sqrt{\xi}} \end{pmatrix}. \]

We have
\[ \psi_+(z) = \sqrt{x - z} (1 + a) + \frac{1}{2} \sqrt{x - z} \log \left( 1 - \frac{z}{x} \right) + o(z - x), \]
\[ \psi_-(z) = \sqrt{x - z} (1 - a) - \frac{1}{2} \sqrt{x - z} \log \left( 1 - \frac{z}{x} \right) + o(z - x). \]

The monodromy matrix of this pair around \( z = x \) (in the counterclockwise direction, \( (z-x) \to e^{2\pi i} (z-x) \)) thus reads
\[ M^{(z=x)}_{\psi} = \begin{pmatrix} 1 - \frac{i\pi}{2} & -\frac{i\pi}{2} \\ \frac{i\pi}{2} & 1 + \frac{i\pi}{2} \end{pmatrix}, \]
and the monodromy matrix of the \( \psi_\pm \) pair around \( z = 0 \) (in the counterclockwise direction, \( z \to e^{2\pi i}z \)) can be calculated as
\[ M^{(z=0)}_{\psi} = \lim_{\xi \to 0} B_\xi \cdot M^{(z=0)}_{\psi^{(\xi)}} \cdot B_\xi^{-1} \]
\[ = \frac{i}{2\pi} \begin{pmatrix} 2i\pi + (\beta^2(x) - 4) \cos^2 \frac{\pi \nu(x)}{2} \\ -(\beta(x) + 2) \cos^2 \frac{\pi \nu(x)}{2} \end{pmatrix} \begin{pmatrix} (\beta(x) - 2)^2 \cos^2 \frac{\pi \nu(x)}{2} & 2i\pi - (\beta^2(x) - 4) \cos^2 \frac{\pi \nu(x)}{2} \end{pmatrix}. \]

Here
\[ \beta(x) \equiv \psi_0 \left( \frac{1 - \nu(x)}{2} \right) + \psi_0 \left( \frac{1 + \nu(x)}{2} \right) + 2\gamma - 2a(x), \]
ψ₀(\(z\)) = \(\frac{d}{dz} \log \Gamma(z)\) is the digamma function and \(\gamma_\text{E}\) denotes the Euler-Mascheroni constant.

Monodromy matrix around \(z = 1\) can now be obtained (paying attention to the opposite orientation of the planes \(z\) and \(w = 1 - z\)) by repeating the calculation above with \(x\) substituted by \(1 - x\). As these calculations are fairly straightforward we refrain from writing them down explicitly.

We shall now turn to the power-like corrections. Define

\[
(\hat{\mathcal{G}} f)(z) = \int_x^\infty dz' \left( \psi_+(z)\psi_-(z') - \psi_-(z)\psi_+(z') \right) f(z')
\]

with \(\psi_\pm\) given by (3.6). Since

\[
(\partial^2 + T_0) \hat{\mathcal{G}} = 1,
\]

we can rewrite (3.3) in the form of an integral equation

\[
\Psi_\pm = \psi_\pm + \hat{\mathcal{G}} V \Psi_\pm
\]

with a (formal) solution

\[
\Psi_\pm = \left( 1 - \hat{\mathcal{G}} V \right)^{-1} \psi_\pm.
\]

From (3.8) one can read off the \(n\)-th order correction to the functions \(\psi_\pm(z)\),

\[
\delta^{(n)} \psi_\pm(z) = \int_x^\infty dz_1 \cdots \int_x^\infty dz_n \mathcal{G}(z, z_n) V(z_n) \cdots \mathcal{G}(z_n-1, z_{n-1}) V(z_{n-1}) \cdots \mathcal{G}(z_2, z_1) V(z_1) \psi_\pm(z_1).
\]

Let us discuss in some details the case \(n = 1\),

\[
\delta^{(1)} \psi_\pm(z) = \psi_+(z) \int_x^\infty dz' \psi_-(z') V(z') \psi_\pm(z') - \psi_-(z) \int_x^\infty dz' \psi_+(z') V(z') \psi_\pm(z').
\]

The functions \(\psi_\pm(z') V(z') \psi_\pm(z')\) have integrable (logarithmic) singularities for \(z \to x\) and \(z \to 0\). Consequently,

\[
\delta^{(1)} \psi_\pm(z) = o(z - x) \quad \text{for} \quad z \to x,
\]

which means that the monodromy matrix of the \(\psi_\pm\) pair remains unchanged. For \(z \to 0\) the leading correction takes the form

\[
\delta^{(1)} \psi_\pm(z) = \alpha^{(1)}_{+,+} \psi_+(z) + \alpha^{(1)}_{+, -} \psi_-(z) + o(z),
\]

with

\[
\alpha^{(1)}_{+,+} = \int_x^0 dz' \psi_+(z') V(z') \psi_+(z').
\]

The monodromy matrix of the \(\psi_\pm + \delta^{(1)} \psi_\pm\) pair is therefore given by

\[
M_\psi^{(0)} (\psi_\pm + \delta^{(1)} \psi_\pm) = \left( 1 + \alpha^{(1)}_\psi \right) M_\psi^{(0)} \left( 1 + \alpha^{(1)}_\psi \right)^{-1}.
\]
Notice further that \( \frac{1}{\sqrt{x}} \psi_\pm(x \zeta) \) does not depend on \( x \) and \( V(xz) = O(x^{-1}) \), so that (with \( z' = x \zeta \))

\[
\alpha_{\pm, \pm}^{(1)} = x \int_1^0 d\zeta \left( \frac{1}{\sqrt{x}} \psi_\pm(x \zeta) \right) xV(x \zeta) \left( \frac{1}{\sqrt{x}} \psi_\pm(x \zeta) \right) = O(x).
\]

Similarly one gets

\[
\delta^{(n)} \psi_\pm(z) = o(z - x) \quad \text{for} \quad z \to x,
\]

and

\[
\delta^{(n)} \psi_\pm(z) = \alpha_{\pm, +}^{(n)} \cdot \psi_+(z) + \alpha_{\pm, -}^{(n)} \cdot \psi_-(z) + o(z) \quad \text{for} \quad z \to 0,
\]

with

\[
\alpha_{\pm, \pm}^{(n)} = O(x^n).
\]

Up to this order the monodromy of the pair \( \Psi_\pm \) around \( z = 0 \) is therefore given by

\[
M^{(0)}_\Psi = \left( 1 + \sum_{k=1}^n \alpha^{(k)} \right) M^{(0)}_\psi \left( 1 + \sum_{k=1}^n \alpha^{(k)} \right)^{-1}.
\]

Let us remark that although the matrices \( \alpha^{(k)} \) are difficult to evaluate analytically, their numerical calculation is rather straightforward.

## 4 Zamolodchikovs conjecture

The 4-point function of the DOZZ theory with the operator insertions at \( z_1 = 0, z_3 = 1, z_4 = \infty \) and \( z_2 = x \), can be expressed as an integral of s-channel conformal blocks and DOZZ couplings over the continuous spectrum of the theory. In the semiclassical limit the integrand can be written in terms of 3-point classical Liouville actions and the classical block [1],

\[
\left\langle V_4(\infty, \infty)V_3(1, 1)V_2(x, \bar{x})V_1(0, 0) \right\rangle \sim \int dp \, e^{-Q^2 S(\delta; x; \delta)}
\]

where \( \delta = \frac{1}{4} + p^2 \) and

\[
S(\delta; x; \delta) = S^{(cl)}(\delta_1, \delta_3, \delta) + S^{(cl)}(\delta, \delta_2, \delta_1) - f_\delta \left[ \frac{\delta_3 \delta_2}{\delta_4 \delta_1} \right](x) - \bar{f}_\delta \left[ \frac{\delta_3 \delta_2}{\delta_4 \delta_1} \right](\bar{x}).
\]

The 3-point classical Liouville action with a parabolic \( \delta_1 = \frac{1}{4} \), an elliptic \( \delta_2 = \frac{1}{4} (1 - \xi^2) \), and a hyperbolic weight \( \delta = \frac{1}{4} + p^2 \), reads [1, 15]

\[
S^{(cl)}(\delta, \delta_2, \delta_1) = -(1 - \xi) \log 2 + 2F \left( \frac{1 - \xi}{2} + ip \right) + 2F \left( \frac{1 - \xi}{2} - ip \right) - F(\xi) + H(2ip) + \pi |p| + \text{const},
\]

(4.3)
where

$$F(x) = \int_{1/2}^x dy \log \frac{\Gamma(y)}{\Gamma(1-y)}, \quad H(x) = \int_0^x dy \log \frac{\Gamma(-y)}{\Gamma(y)}.$$  

$f_\delta \left[ \delta, \delta_2 \right] (x)$ is the classical conformal block [1] (or the “classical action” of [16, 17]), defined as the semiclassical asymptotic

$$\mathcal{F}_{1+6Q^2, \Delta} \left[ \Delta_2, \Delta_1 \right] (x) \sim \exp \left( Q^2 f_\delta \left[ \delta, \delta_2 \right] (x) \right)$$  \hspace{1cm} (4.4)

of the BPZ conformal block [18].

In the classical limit $Q^2 \to \infty$ the integral on the r.h.s. of relation (4.1) is dominated by its saddle point value. One thus gets

$$S^{(cl)} (\delta_i; x) = S(\delta_i; x; \delta_s)$$

$$= S^{(cl)} (\delta_4, \delta_3, \delta_s) + S^{(cl)} (\delta_s, \delta_2, \delta_1) - f_{\delta_s} \left[ \delta, \delta_2 \right] (x) - \tilde{f}_{\delta_s} \left[ \delta_4, \delta_1 \right] (x).$$

(4.5)

where $\delta_s = \frac{1}{4} + p_s^2(x)$ and the saddle point momentum $p_s(x)$ is determined by the equation

$$\frac{\partial}{\partial p} S (\delta_i; x; \frac{1}{4} + p^2) \big|_{p=p_s} = 0.$$  \hspace{1cm} (4.6)

Since the semiclassical limit should be independent of the choice of the channel in the factorization of the DOZZ 4-point function the Zamolodchikovs conjecture (4.5) yields three different expressions for the 4-point classical Liouville action. The corresponding consistency equations (classical bootstrap equations) has been numerically verified for punctures [13] and for punctures and one and two elliptic singularities [19].

Taking into account the classical geometry corresponding to hyperbolic weights [20] one may expect that the saddle point momentum $p_s(x)$ in the s-channel is related to the length $\ell_s(x)$ of the closed geodesic separating the “initial” $z = 0, x$ from the “final” $z = 1, \infty$ singularities:

$$\ell_s(x) = 4\pi p_s(x).$$  \hspace{1cm} (4.7)

In the case of punctures this conjecture has strong numerical support [13].

The classical limit of the quantum conformal block is up to now the only method of calculating the classical conformal block\(^4\). An efficient recursive method of calculating coefficients of the expansion of the quantum block in powers of $x$ were developed by Al. Zamolodchikov [16]. Using this method and taking the limit $Q \to \infty$ one can calculate term by term the coefficients of the power expansion

$$f_\delta \left[ \delta, \delta_2 \right] (x) = (\delta - \delta_1 - \delta_2) \log x + \sum_{n=1}^\infty x^n f_n \left[ \delta, \delta_2 \right] (x).$$

(4.8)

\(^4\)Let us note that it is by no means obvious that the quantum conformal block for “heavy” weights does have the asymptotic of the form assumed in (4.4). This is however very well supported by symbolic calculations for first few terms in a number of cases.
In the case of three punctures and one elliptic singularity the first few terms of this expansion read
\[
q_{\frac{1}{4}+p^2} \left[ \frac{1}{4}; \frac{1-\xi^2}{4} \right] (x) = \left( p^2 - \frac{1-\xi^2}{4} \right) \log x + \left( \frac{1-\xi^2}{8} + \frac{p^2}{2} \right) x \tag{4.9}
\]
\[+ \left( \frac{9 (1-\xi^2)}{128} + \frac{13 p^2}{64} + \frac{(1-\xi^2)^2}{1024 (1+p^2)} \right) x^2 + \mathcal{O} \left( x^3 \right). \]

The limitation of formulae (4.8) and (4.9) is that the power series involved are supposed to converge only for \(|x| < 1\). A more convenient representation of the conformal block was developed by Al. Zamolodchikov in [17], where he proposed to regard the block as a function of the variable
\[
q(x) = e^{-\frac{\pi K(1-x)}{K(x)}}, \quad K(x) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-xt^2)}}.
\]

In terms of \(q\) the classical conformal block reads:
\[
f_{\delta_1, \delta_2}^{\delta_3} [x, 1] (x) = \left( \frac{1}{4} - \delta_1 - \delta_2 \right) \log x + \left( \frac{1}{4} - \delta_2 - \delta_3 \right) \log (1-x) \tag{4.10}
\]
\[+ \left( \frac{1}{4} - 2(\delta_1 + \delta_2 + \delta_3 + \delta_4) \right) \log \left( \frac{2\pi K(x)}{x} \right)
\]
\[+ \left( \delta - \frac{1}{4} \right) \log 16 - (\delta - \frac{1}{4}) \pi \frac{K(1-x)}{K(x)} + h_{\delta_1, \delta_2}^{\delta_3, \delta_4} (q), \]
where
\[
h_{\delta_1, \delta_2}^{\delta_3, \delta_4} (q) = \sum_{n=1}^{\infty} (16q)^n h_{\delta_1, \delta_2}^{\delta_3, \delta_4} (q) \tag{4.11}
\]
is supposed to converge uniformly on each subset \(\{q : |q| < e^{-t} < 1\}\).

The coefficients of power series (4.11) can be determined term by term using Zamolodchikov's recursive method to calculate coefficients in the \(q\)-expansion of the quantum block [17] and then taking the limit \(Q \to \infty\). This for instance yields:
\[
h_{\frac{1}{4}+p^2} \left[ \frac{1}{4}; \frac{1-\xi^2}{4} \right] (q) = \left( \frac{1-\xi^2}{4} \right)^2 \left( \frac{1-\xi^2}{8} + \frac{p^2}{2} \right) x^2 \tag{4.12}
\]
\[+ \left( \frac{9 (1-\xi^2)}{128} + \frac{13 p^2}{64} + \frac{(1-\xi^2)^2}{1024 (1+p^2)} \right) x^2 + \mathcal{O} \left( q^6 \right). \]

In the case of four punctures the saddle point equation determining \(p_s(x)\) reads
\[
0 = \frac{\partial}{\partial p} S \left( \frac{1}{4}; x; \frac{1}{4} + p^2 \right) = 2\pi + 4i \log \frac{\Gamma \left( \frac{1}{4} + ip \right) \Gamma (-2ip)}{\Gamma \left( \frac{1}{4} - ip \right) \Gamma (2ip)} - 2\Re \frac{\partial}{\partial p} f_{\frac{1}{2}+p^2} \left[ \frac{1}{4}; x \right] (x), \tag{4.13}
\]
and the classical Liouville action is given by
\[
S^{(cl)} \left( \frac{1}{4}; x \right) = S \left( \frac{1}{4}; x; \frac{1}{4} + p_s^2 (x) \right). \tag{4.14}
\]
Using (4.2), (4.3) and (4.9) one gets:

\[
c_2(x) = -\frac{\partial S^{(c)}(\frac{1}{4}; x)}{\partial x} - \frac{\partial S(\frac{1}{4}; x; \frac{1}{4} + p^2)}{\partial p} \left. \frac{\partial p_s(x)}{\partial x} - \frac{\partial S(\frac{1}{4}; x; \frac{1}{4} + p^2)}{\partial x} \right|_{p=p_s(x)}
\]

\[
= -\frac{\partial S(\frac{1}{4}; x; \frac{1}{4} + p^2)}{\partial x} \bigg|_{p=p_s(x)} = \frac{\partial}{\partial x} f_{\frac{1}{4} + p^2} \left( \frac{1}{4} + \frac{1}{4} \right) \bigg|_{p=p_s(x)}
\]

\[
= \frac{4p_s^2(x) - 1}{4x} + \frac{1}{8} (4p_s^2(x) + 1) + \left[ \frac{9}{2} + 13p_s^2(x) + \frac{1}{8} \frac{1}{1 + p_s^2(x)} \right] \frac{x}{32} + \mathcal{O}(x^2)
\]

or, employing the \(q\) expansion of the classical block,

\[
c_2(x) = -\frac{1}{4x(1-x)} \left[ 1 + \frac{7}{4} \left( E(x) + x - 1 \right) \right] - \frac{\pi^2}{4x(1-x)K^2(x)} \left[ p_s^2(x) + \frac{1}{1 + p_s^2(x)} \frac{q^2}{2} \right]
\]

\[
+ \left( \frac{15}{1 + p_s^2(x)} + \frac{81}{4 + p_s^2(x)} + \frac{3}{4 + p_s^2(x)} - \frac{4}{(1 + p_s^2(x))^2} \right) \frac{q^4}{32} + \mathcal{O}(q^6) \right]
\]

where \(E(x)\) denotes the complete elliptic integral of the second kind. Equations (4.15), (4.16) are simple consequences of the Zamolodchikovs conjecture. They provide a new relation between the accessory parameters, the classical conformal block and the geodesic length function \(\ell_s(x)\) which is certainly worth further investigations.

In order to calculate \(a_2 \equiv a\) for four punctures one can replace one puncture by an elliptic singularity and then take the limit

\[
a_2 = \lim_{\xi \to 0} \frac{1}{\xi} \log A^2.
\]

Using (2.5), (4.2), (4.3) and (4.9) one obtains

\[
a_2(x) = \lim_{\xi \to 0} \left\{ \frac{1}{\xi} \log \frac{\Gamma(\frac{1+x}{4} - ip_s(x)) \Gamma(\frac{1+x}{4} + ip_s(x))}{\Gamma(\frac{1-x}{4} - ip_s(x)) \Gamma(\frac{1-x}{4} + ip_s(x))} - 2 \frac{\partial}{\partial x} f_{\frac{1}{4} + p^2} \left( \frac{1}{4} + \frac{1}{4} \right) \bigg|_{p=p_s(x)} \right\}
\]

\[
= \psi_0 \left( \frac{1}{2} + ip_s(x) \right) + \psi_0 \left( \frac{1}{2} - ip_s(x) \right) - \frac{2}{\xi} \frac{\partial}{\partial \xi} f_{\frac{1}{4} + p^2} \left( \frac{1}{4} + \frac{1}{4} \right) \bigg|_{p=p_s(x)} \xi \to 0 \right)
\]

\[
= \psi_0 \left( \frac{1}{2} + ip_s(x) \right) + \psi_0 \left( \frac{1}{2} - ip_s(x) \right) - \frac{1}{2} \log x \bar{x} + \frac{\Re x}{2} + \left[ 36 + \frac{1}{1 + p_s^2(x)} \right] \frac{\Re x^2}{27} + \mathcal{O}(x^3)
\]

or, using (4.10) and (4.12):

\[
a_2(x) = \psi_0 \left( \frac{1}{2} + ip_s(x) \right) + \psi_0 \left( \frac{1}{2} - ip_s(x) \right)
\]

\[
- \frac{1}{2} \log x \bar{x} - \frac{1}{2} \log |1 - x|^2 - \log \left( \frac{2}{\pi} K(x) \right)^2 + \frac{2}{1 + p_s^2(x)} \Re q^2
\]

\[
+ \left[ \frac{45}{4 + p_s^2(x)} + \frac{3}{1 + p_s^2(x)} + \frac{3}{(1 + p_s^2(x))^2} - \frac{4}{(1 + p_s^2(x))^3} \right] \frac{\Re q^4}{8} + \mathcal{O}(q^6).
\]

Note that in the formulae above one only needs the saddle point momentum \(p_s(x)\) for four punctures \((\xi = 0)\).
Let us finally turn to the problem of determining the saddle point momentum \( p_s(x) \). As was discussed in [13] \( p_s(x) \) can be determined numerically, using the \( q \) expansion of the classical block, with an essentially arbitrary high precision everywhere but at small vicinities of the singular points \( x = 1 \) and \( x = \infty \). On the other hand, the problem of analytic determination of the saddle point momentum still remains to be solved and only partial results are available.

Both geometrical arguments and the form of (4.9) indicate that for \( x \to 0 \) the solution of (4.13) should also tend to zero. For \( p \to 0 \)

\[
4i \log \frac{\Gamma^2 \left( \frac{1}{2} + ip \right) \Gamma(-2ip)}{\Gamma^2 \left( \frac{1}{2} - ip \right) \Gamma(2ip)} = -4\pi + 32p \log 2 + 16 \sum_{k=1}^{\infty} \frac{(-1)^k \zeta(2k+1)}{2k+1} \left(2^{2k} - 1\right) p^{2k+1}
\]

(4.19)

so that, up to the leading terms, the saddle point equation (4.13) takes the form

\[
-\pi + 16p \log 2 - p \log xx = 0
\]

(4.20)

and, to this order,

\[
p_s(x) \sim \varepsilon(x) \equiv \frac{\pi}{-\log xx + 16 \log 2}.
\]

(4.21)

We can now solve (4.13) by iteration, in the form of a double series expansion in \( \Re x \) and \( \varepsilon(x) \) (since \( x = \exp \left( \frac{1}{1/(\log x)} \right) \), the powers of \( x \) can be viewed as “non–perturbative” corrections to the \( \varepsilon(x) \) series). For instance, keeping in (4.13) terms up to \( p^3 \) and \( x^2 \) we get

\[
p_s = \frac{\pi}{-\log xx + 16 \log 2 - \Re x - \frac{207}{512} \Re x^2} + \frac{8\zeta(3)}{\pi} \frac{1}{256} \Re x^2
\]

(4.22)

\[
= \frac{\varepsilon(x) + 8\zeta(3)}{\pi} \varepsilon^4(x) + \left( \frac{1}{\pi} \varepsilon^2(x) + \frac{32\zeta(3)}{\pi^2} \varepsilon^5(x) \right) \Re x + \frac{\varepsilon^2(x)}{\pi^2} (\Re x)^2 + \left( \frac{207}{512\pi} \varepsilon^2(x) + \frac{1}{256\pi} \varepsilon^4(x) + \frac{207\zeta(3)}{16\pi^2} \varepsilon^5(x) \right) \Re x^2 + \mathcal{O} \left( \varepsilon^6(x), x^3 \varepsilon(x) \right).
\]

As in the case of the accessory parameters, one can also work out the formula for the saddle point momentum involving the \( q \) expansion of the classical conformal block.

For a sufficiently small \( x \) the r.h.s. of (4.22) agrees with the numerically calculated saddle point momentum and is well within the known analytic bounds on \( \frac{\ell_s(x)}{x^2} \) [2, 13]. However, to determine the radius of convergence of the series in (4.22) or to give an estimate on the omitted terms one would need to know the classical conformal block exactly.

As it was discussed in the previous section in order to determine the monodromy matrix at \( x = 1 \) one needs a power expansion of \( p_s(x) \) at this point. In other words, an analytic
continuation of the classical block from the vicinity of \( x = 0 \) to \( x = 1 \) is required. One possible approach to this problem is to consider the classical limit of the braiding relation for the quantum BPZ block (for conformal blocks corresponding to degenerated fields this calculation is quite straightforward).

We believe that a better understanding of the classical conformal block and the geodesic length function may provide an essentially new insight into the problem of finding an analytic expressions for the map \( \rho \) and the uniformizing group. Further studies of these structures are definitely worth pursuing.

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