On Explicit Random-Like Tournaments

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Abstract
We give a new theorem describing a relation between the quasi-random property of regular tournaments and their spectra. This provides many solutions to a constructing problem mentioned by Erdős and Moon (Can Math Bull 8(3):269–271, 1965) and Spencer (Graphs Comb 1(4):357–382, 1985).

Keywords Eigenvalues · Expander-mixing lemma · The quasi-random property · Regular tournaments

Mathematics Subject Classification 05C20

1 Introduction

A tournament is an oriented complete graph. Random tournaments $T_n$ with $n$ vertices are obtained by choosing a direction of each edge of a complete graph with $n$ vertices with probability 1/2, independently. We say that random tournaments asymptotically almost surely (a.a.s.) satisfy a property $P$ if the probability of the event that tournaments satisfy $P$ tends to 1 when $n$ goes to infinity. In graph theory, there have been many problems focusing on deterministic tournaments satisfying properties which random tournaments a.a.s satisfy; see e.g. [1, 4, 8, 10, 22].

In this paper, as such a property, we mainly focus on the quasi-random property proposed by Chung–Graham [8]. For a tournament $T$, let $\sigma$ be a bijection from the vertex set of $T$ to $\{1, 2, \ldots, n\}$. For two distinct vertices $x$ and $y$, we say an edge from $x$ to $y$ is consistent with $\sigma$ if $\sigma(x) < \sigma(y)$. We define $C(T)$ as the maximum

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number of consistent edges among all $\sigma$. Then an infinite family of tournaments $T_n$ with $n$ vertices has the quasi-random property if $C(T_n)$ is at most $(1 + o(1))n^2/4$ for sufficiently large $n$.

Our main result is to prove a new theorem (Theorem 1) describing a relation between the quasi-random property and spectra of regular tournaments $T$; more precisely, we show that $C(T)$ is bounded by the second largest eigenvalue (in absolute value) of the adjacency matrix of $T$. By this result, we can also provide many solutions to a problem, proposed by Erdős–Moon [16] and Spencer [34] (see also [1, Section 9.1]), on explicit constructions of tournaments with optimally small number of consistent edges. It is well-known that Paley tournaments have the quasi-random property (e.g. [8]). Moreover, by proving that Paley tournaments have a property stronger than the quasi-random property, Alon–Spencer [1] showed that they provide solutions to the problem by Erdős, Moon and Spencer. We note that the proof in [1] contains a part ([1, Lemma 9.1.2]) depending on the definition of Paley tournaments. Remarkably, we generalize their discussion to all regular tournaments by using a digraph-version of the expander-mixing lemma proved by Vu [37].

The rest of this paper is organized as follows. In Sect. 2, we recap the quasi-random property and introduce some related known facts. In Sect. 3, we introduce our main result and give its proof. In Sect. 4, we provide some families of regular tournaments satisfying the quasi-random property which are also solutions to the problem by Erdős, Moon and Spencer. At last, in Sect. 5, we discuss another random-like property defined as an adjacency property.

### 2 The Quasi-Random Property and Related Facts

In this section, we review the quasi-random property and some related known facts.

**Definition 1** (The quasi-random property, [8]) Let $A$ be an infinite subset of $\mathbb{N}$ and $\{T_n\}_{n \in A}$ an infinite family of tournaments $T_n$ with $n$ vertices. For each $n \in A$ and $T_n$, let $\sigma$ be a bijection from $V(T_n)$ to $\{1, 2, \ldots, n\}$. An edge $(x, y)$ of $T_n$ is said to be consistent with $\sigma$ if $\sigma(x) < \sigma(y)$. Let $C(T_n, \sigma)$ be the number of edges consistent with $\sigma$ and $C(T_n) := \max_\sigma C(T_n, \sigma)$. Then, $\{T_n\}_{n \in A}$ has the quasi-random property if it holds that

$$C(T_n) \leq (1 + o(1))\frac{n^2}{4}.$$

Surprisingly, Chung–Graham [8] gave some other properties which are seemingly unrelated, but actually equivalent with (1). The interested reader is referred to [8].
Consistent edges of tournaments was originally investigated by Erdős–Moon [16]. Their work was from paired comparisons (e.g. [23]). It is reasonable to find suitable rankings, that is, bijections with many consistent edges. First observe that for every tournament $T$ with $n$ vertices,

$$\frac{1}{2} \binom{n}{2} \leq C(T) \leq \binom{n}{2}.$$  

The lower bound of $C(T)$ is obtained by the following simple fact:

$$C(T, \sigma) + C(T, \sigma') = \binom{n}{2},$$  

where $\sigma'$ is the reversed ranking of $\sigma$ which is defined as $\sigma'(v) := n + 1 - \sigma(v)$ for each $v \in V(T)$. For the upper bound of $C(T)$, the equality holds if and only if $T$ is a transitive tournament. On the other hand, it is non-trivial to check the tightness of the lower bound of $C(T)$. In [16], it was proved that for any sufficiently large $n$, there exist tournaments $T$ such that $C(T) \leq (1 + o(1)) \binom{n}{2}/2$ by a probabilistic argument. Moreover Spencer [32, 33] and de la Vega [12] proved that random tournament $T_n$ a.a.s satisfies the following property which is stronger than the quasi-random property: for sufficiently large $n$,

$$C(T_n) \leq \frac{1}{2} \binom{n}{2} + O(n^3).$$  

Erdős–Moon [16] and Spencer [34] mentioned the problem on explicit constructions of tournaments $T$ such that $C(T)$ is close to the lower bound. Note that although it is remarked in [8, p. 195] that families of tournaments with the quasi-random property can be obtained from families of quasi-random graphs [9], it seems to be still non-trivial to obtain families of tournaments with optimally small number of consistent edges. At present, such a construction of tournaments $T$ giving the best known “constructive” upper bound of $C(T)$ is obtained by Alon–Spencer [1]. For a prime $p \equiv 3(\text{mod } 4)$, the Paley tournament $T_p$ is the tournament with vertex set $\mathbb{F}_p$, the finite field of order $p$, and edge set formed by all edges $(x, y)$ such that $x - y$ is a non-zero square of $\mathbb{F}_p$. In [1, Theorem 9.1.1], it was proved that

$$C(T_p) \leq \frac{1}{2} \binom{p}{2} + O(p^3 \log p).$$  

In Section 4, by applying the main theorem proved in the next section, we give some new explicit constructions of regular tournaments $T$ with $n$ vertices such that $C(T)$ is close to the lower bound.
3 Main Theorem

In this section, we prove our main theorem. We first explain some necessary terminologies to describe the main theorem. A digraph is said to be \(d\)-regular if in-degree and out-degree of each vertex are \(d\). Especially a tournament with \(n\) vertices is simply said to be regular if it is \((n - 1)/2\)-regular. The adjacency matrix \(M_D\) of a digraph \(D\) with \(n\) vertices is the \(\{0, 1\}\)-square matrix of size \(n\) whose rows and columns are indexed by the vertices of \(D\) and the \((x, y)\)-entry is equal to 1 if and only if \((x, y) \in E(D)\). A digraph \(D\) is said to be normal if \(M_D\) and its transpose \(M_D^t\) are commutative. In other word, \(D\) is normal if

\[
|\{N^+(x, y)\}| = |\{N^-(x, y)\}|
\]

for any two distinct vertices \(x\) and \(y\) where \(N^+(x, y)\) (resp. \(N^-(x, y)\)) is the set of vertices \(z\) such that \((x, z); (y, z) \in E(D)\) (resp. \((z, x); (z, y) \in E(D)\)). Notice that if \(D\) is normal, then \(M_D\) is diagonalizable. It should be noted (see also [5]) that every regular tournament \(T\) with \(n\) vertices is normal since it holds that \(M_T^t = J_n - I_n - M_T\), where \(I_n\) and \(J_n\) are the identity matrix and the all-one matrix of order \(n\), respectively.

The following is our main theorem.

**Theorem 1** Let \(T\) be a regular tournament with \(n\) vertices. Suppose that the adjacency matrix \(M_T\) of \(T\) has eigenvalues such that \((n - 1)/2 = \lambda_1, \lambda_2, \ldots, \lambda_n\). Let

\[
\lambda(T) := \max_{2 \leq i \leq n} |\lambda_i|.
\]

Then,

\[
C(T) \leq \frac{1}{2} \binom{n}{2} + \lambda(T) \cdot n \log_2(2n).
\]

**Remark 1** Theorem 1 implies that every infinite family of regular tournaments \(T_n\) with \(n\) vertices such that \(\lambda(T_n) = o(n / \log n)\) has the quasi-random property. It should be remarked that Kalyanasundaram–Shapira [22] shows a better result; a proof of Lemma 2.3 and the first concluding remark in [22] imply that an infinite family of regular tournaments \(T_n\) with \(n\) vertices has the quasi-random property if and only if it satisfies that \(\lambda(T_n) = o(n)\). (In [22], the authors considered the eigenvalues of the \(\{0, \pm1\}\)-matrix \(2M_{T_n} - J_n + I_n\), but these eigenvalues can be directly computed from ones of \(M_{T_n}\).)

On the other hand, Theorem 1 not only gives a spectral condition for the quasi-random property, but also implies that estimating eigenvalues of \(M_{T_n}\) provides better upper bounds of \(C(T_n)\) than the bound (1); for example, if an infinite family of regular tournaments \(T_n\) with \(n\) vertices satisfies \(\lambda(T_n) = o(n / \log n)\), then Theorem 1 implies that

\[
C(T_n) \leq \binom{n}{2} / 2 + o(n^2),
\]

which is much better than (1). Considering (3), Theorem 1 provides a spectral condition for a property, which random tournaments a.a.s. satisfy, stronger than the quasi-random property.

In the proof of Theorem 1, we use the following expander-mixing lemma for normal regular digraphs proved by Vu [37].
Lemma 1 (Expander-mixing lemma, [37]) Let $D$ be a normal $d$-regular digraph with $n$ vertices. Suppose that $\mu_1, \ldots, \mu_n$ are eigenvalues of $M_D$ and let $\lambda(D) := \max_{2 \leq i \leq n} |\mu_i|$. For two disjoint subsets $A, B \subseteq V(D)$

$$e(A, B) := |\{(a, b) \in E(D) \mid a \in A, b \in B\}|.$$

Then for every pair of two disjoint subsets $A, B \subseteq V(D)$, it holds that

$$\left| e(A, B) - \frac{d}{n} \cdot |A| \cdot |B| \right| \leq \lambda(D) \sqrt{|A| \cdot |B|}.$$

From this lemma, we can easily obtain the following corollary.

Corollary 1 Let $D$ be a normal $d$-regular digraph with $n$ vertices. Then for every pair of two disjoint subsets $A, B \subseteq V(D)$,

$$|e(A, B) - e(B, A)| \leq 2\lambda(D) \sqrt{|A| \cdot |B|}.$$

Proof From the triangle inequality, we see that

$$|e(A, B) - e(B, A)| = \left| \left( e(A, B) - \frac{d}{n} \cdot |A| \cdot |B| \right) - \left( e(B, A) - \frac{d}{n} \cdot |B| \cdot |A| \right) \right|$$

$$\leq \left| e(A, B) - \frac{d}{n} \cdot |A| \cdot |B| \right| + \left| e(B, A) - \frac{d}{n} \cdot |B| \cdot |A| \right|.$$

Thus, by Lemma 1, we get the corollary.

By Corollary 1, we get the following lemma which follows by combining Corollary 1 and the argument in [1, pp. 150–151] to prove the bound (4) for Paley tournaments.

Lemma 2 Let $T$ be a regular tournament with $n$ vertices and let $\sigma$ be a bijection from $V(T)$ to $\{1, 2, \ldots, n\}$. Then

$$C(T, \sigma) - C(T, \sigma') \leq 2\lambda(T) \cdot n \log_2(2n).$$

Proof Fix a bijection $\sigma$. Let $r$ be the smallest integer such that $n \leq 2^r$. Let $n = a_1 + a_2$, where $a_1 := \lfloor n/2 \rfloor$ and $a_2 := \lceil n/2 \rceil$. Consider a partition of $V(T)$, say $A_1$ and $A_2$, such that $A_1$ is the set of “highly ranked” $a_1$ vertices in $\sigma$ and $A_2$ is the remaining $a_2$ vertices. Since $a_1, a_2 \leq 2^{r-1}$, it follows from Corollary 1 that

$$e(A_1, A_2) - e(A_2, A_1) \leq 2\lambda(T) \sqrt{a_1 a_2} \leq 2\lambda(T) \cdot 2^{r-1}.$$

Next, let $a_1 = a_{11} + a_{12}$, $a_{11} := \lfloor a_1/2 \rfloor$ and $a_{12} := \lceil a_1/2 \rceil$, and similarly for $a_2 = a_{21} + a_{22}$. As above, divide $A_1$ into two subsets, say $A_{11}$ and $A_{12}$, where $A_{11}$ is the set of “highly ranked” $a_{11}$ vertices of $A_1$ in $\sigma$ and $A_{12}$ is the remaining $a_{12}$...
vertices of $A_1$. For $a_{11}$ and $a_{22}$, two subsets $A_{21}$ and $A_{22}$ of $A_2$ are defined in the same way as $A_{11}$, $A_{12}$. Since $a_{11}, a_{12}, a_{21}, a_{22} \leq 2^{r-2}$, it then follows from Corollary 1 that
\[
e(A_{11}, A_{12}) - e(A_{12}, A_{11}) + e(A_{21}, A_{22}) - e(A_{22}, A_{21}) \\
\leq 2\lambda(T)\sqrt{a_{11}a_{12}} + 2\lambda(T)\sqrt{a_{21}a_{22}} \leq 2 \cdot 2\lambda(T) \cdot 2^{r-2}.
\]
Then iterate such estimation from the first to the $r$th step. In the $i$th step, $V(T)$ is partitioned into $2^i$ subsets, say $A_{k1}$ and $A_{k2}$ ($e \in \{1, 2\}^i$), such that each $A_{kj}$ ($j = 1, 2$) contains at most $2^{r-i}$ vertices which are consecutive in $\sigma$. It follows from Corollary 1 that
\[
\sum_{k \in \{1, 2\}^{i-1}} \{e(A_{k1}, A_{k2}) - e(A_{k2}, A_{k1})\} \leq 2^{i-1} \cdot 2\lambda(T) \cdot 2^{r-i} = 2\lambda(T) \cdot 2^{r-1}.
\]
On the other hand, it turns out from the construction of partitions that
\[
\sum_{1 \leq i \leq r} \sum_{k \in \{1, 2\}^{i-1}} \{e(A_{k1}, A_{k2}) - e(A_{k2}, A_{k1})\} = C(T, \sigma) - C(T, \sigma').
\]
Thus by combining (5) and (6), it follows that
\[
C(T, \sigma) - C(T, \sigma') \leq r \cdot 2\lambda(T) \cdot 2^{r-1} \leq 2\lambda(T) \cdot n \log_2(2n).
\]
\[\square\]

**Proof of Theorem 1** The theorem is a direct consequence of the Eq. (2) and Lemma 2.

**Remark 2** It should be noted that for every regular tournament $T$ with $n$ vertices, $\lambda(T) \cdot n \log_2(2n)$ cannot be less than $\sqrt{n^3 + n \log_2(2n)}/2$. In fact, for every such tournament $T$, it holds that
\[
\lambda(T) \geq \frac{\sqrt{n + 1}}{2}.
\]
Indeed, for every strongly-connected normal $d$-regular digraph $D$ with $n$ vertices, it holds that
\[
d = E(D) = \text{Tr}(M_DM_D^T) = \sum_{i=1}^{n} |\lambda_i|^2 \leq d^2 + (n - 1)\lambda(D)^2,
\]
which follows from the hand shaking lemma and the Perron–Frobenius theorem (see e.g. [36]). The idea of the above inequality can be found in [24, p. 217]. Now it suffices to show that every regular tournament $T$ with $n$ vertices is strongly connected. To that end, we thank a reviewer for pointing out the following nice elementary proof. If $v$ is a fixed vertex of $T$ and $V$ is the set of vertices $u$ such that there exists a directed path from $v$ to $u$, then the subtournament $T_V$ induced by $V$ has $|V|(n - 1)/2$ edges since each vertex in $V$ must have out-degree $(n - 1)/2$ in $T_V$. 

\[\diamond Springer\]
4 Examples of Quasi-Random Regular Tournaments

In this section, we give some infinite families of regular tournaments $T_n$ with $n$ vertices such that $\lambda(T_n) = o(n/\log n)$. As will be shown below, we can construct such families for almost all positive integers $n$.

First we consider the following tournaments constructed from finite fields which are variants of cyclotomic tournaments (see e.g. [27] and reference therein). Let $m$ be a positive even integer and $p \equiv m + 1 \pmod{2m}$ be a prime. Note that there exist infinitely many such primes by the Dirichlet’s theorem on arithmetic progressions and the fact that $m + 1$ and $2m$ are coprime when $m$ is even. Recall that $\mathbb{F}_p$ denotes the finite field of order $p$. Let $g$ be a primitive root of $\mathbb{F}_p$. For even $m$, the multiplicative group of $\mathbb{F}_p$, which is denoted by $\mathbb{F}_p^*$, is divided into $m$ cosets $S_0, S_1, \ldots, S_{m-1}$ where $S_i := \{g^t \mid t \equiv i \pmod{m}\}$ for each $0 \leq i \leq m - 1$. Note that $S_j = -S_i$ if $j \equiv -i \pmod{m}$.

**Definition 2** Let $i := \{i_1, i_2, \ldots, i_{m/2}\}$ with $i_j \in \{0, 1, \ldots, m\}$ for each $1 \leq j \leq m/2$ such that $S_i := S_{i_1} \cup \cdots \cup S_{i_{m/2}}$ and $\mathbb{F}_p^* \setminus S_i = -S_i$. Then the tournament $T_p^m(S_i)$ is defined as follows:

$$V(T_p^m(S_i)) := \mathbb{F}_p,$$

$$E(T_p^m(S_i)) := \{(x, y) \in \mathbb{F}_p^2 \mid x - y \in S_i\}.$$

This is a direct generalization of Paley tournament since $T_p^m(S_i)$ is exactly $T_p$ in the case of $m = 2$. Moreover from the definition, it is not so hard to see that $T_p^m(S_i)$ is a regular tournament with $p$ vertices.

Now we obtain the following corollary.

**Corollary 2** Let $m$ be a fixed positive even integer and $p \equiv m + 1 \pmod{2m}$ be any sufficiently large prime. Let $T_p^m(S_i)$ be the regular tournament in Definition 2. Then

$$C(T_p^m(S_i)) \leq \frac{1}{2} \binom{p}{2} + O(p^{3/2} \log p).$$

**Proof** By Lemma 2, it suffices to prove that

$$\lambda(T_p^m(S_i)) \leq \frac{m\sqrt{p}}{2}. \quad (8)$$

First, by a simple calculation, it can be shown that the set of eigenvalue of $M_{T_p^m(S_i)}$ is
where an additive character of $\mathbb{F}_p$ is a homomorphism from the additive group of $\mathbb{F}_p$ to the multiplicative group of $\mathbb{C}$. Notice that $\lambda_{\psi} = |S_i| = (p - 1)/2$ if a character $\psi$ is trivial, that is, a constant function. Since $S_i = g^i S_0$ for each $1 \leq i \leq m - 1$, we see that

$$\sum_{s \in S_i} \psi(s) = \sum_{s \in g^i S_0} \psi(s) = \sum_{s \in S_0} \psi(g^i s).$$

(9)

Since $S_0$ is the set of non-zero $m$th power elements and each non-zero $m$-th power residue appears exactly $m$ times in the sequence $(\lambda^m)_{\lambda \in \mathbb{F}_p}$,

$$\sum_{s \in S_0} \psi(g^i s) = \frac{1}{m} \sum_{s \in \mathbb{F}_p^*} \psi(g^i s).$$

(10)

At last, we use the following known estimation (see e.g. [29, p. 44]):

$$\left| \sum_{s \in \mathbb{F}_p} \psi(a\lambda^m) \right| \leq (m - 1) \sqrt{p},$$

(11)

for any $a \neq 0$ and any non-trivial additive character $\psi$. By combining (9), (10) and (11),

$$\lambda(T^m_p(S_i)) \leq \frac{m}{2} \cdot \frac{1}{m} \cdot \{(m - 1) \sqrt{p} + 1\} = \frac{(m - 1) \sqrt{p} + 1}{2} \leq \frac{m \sqrt{p}}{2}.$$

The second example is doubly regular tournament which has been extensively studied in algebraic combinatorics and related areas (e.g. [25]). As explained in [7, 19], doubly regular tournaments form a specific class of directed strongly regular graphs [14] and normally regular digraphs [19]. Also doubly regular tournaments are distance-regular digraphs [11] with diameter 2 and girth 3.

**Definition 3** Let $n$ be a positive integer such that $n \equiv 3 \pmod{4}$. A tournament $T$ with $n$ vertices is called a doubly regular tournament if $T$ is a regular tournament such that for any distinct two vertices $x$ and $y$, $N^+(x,y) = N^-(x,y) = (n - 3)/4$.

Let $DRT_n$ denote a doubly regular tournament with $n$ vertices. Although the following result has already been proved in the present author’s paper [26] by using a slightly different version of Lemma 2, we give a spectral proof to show that this also can be obtained as a consequence of Theorem 1.

**Corollary 3** For any sufficiently large positive integer $n \equiv 3 \pmod{4}$,
Proof. By Lemma 2, it suffices to apply the following known fact which also shows that the inequality (7) is tight.

\[ \lambda(DRT_n) = \sqrt{\frac{n+1}{2}}. \]

We give a proof for the reader’s convenience. Let \( M := M_{DRT_n} \). Then by the definition, it holds that

\[ MM^t = \frac{n+1}{4} I_n + \frac{n-3}{4} J_n. \]

Since \( M + M^t = J_n - I_n \), we obtain the following equation.

\[ M^2 + M + \frac{n+1}{4} I_n - \frac{n+1}{4} J_n = O. \]

Since \( DRT_n \) is regular, we see that \((n-1)/2\) is an eigenvalue of \( M \) and a corresponding eigenvector is the all-one eigenvector \( \mathbf{1} \). Since \( DRT_n \) is normal, each eigenvalue \( \theta \) except for \((n-1)/2\) has an eigenvector \( v \) which is orthogonal to \( \mathbf{1} \). Thus,

\[ \left( \theta^2 + \theta + \frac{n+1}{4} \right) v = \theta. \]

Since \( v \neq 0 \), we get

\[ \left( \theta^2 + \theta + \frac{n+1}{4} \right) = 0, \]

completing the proof. \( \square \)

Remark 3. We remark that Corollary 3 is a generalization of the bound (4) because Paley tournaments are also doubly-regular tournaments. For other infinite families of non-isomorphic doubly regular tournaments, see e.g. [21, 35]. As shown in, for example, [18, 25], there are some known constructions of infinite families of doubly regular tournaments such that the number of vertices is non-prime (and non-prime power). Especially, constructions of complex codebooks in [18] provide \( DRT_n \) for every integer \( n \) such that each prime factor \( f \) of \( n \) is the form of \( f \equiv 3(\text{mod} \ 4) \).

Remark 4. By the definition of \( DRT_n \), \( n \) must be a positive integer of the form \( n \equiv 3(\text{mod} \ 4) \). On the other hand, as an analogue of \( DRT_n \) for integers \( n \) of the form \( n \equiv 1(\text{mod} \ 4) \), Savchenko [27] introduced the notion of a nearly-doubly-regular tournament \( CNDR_n \) with \( n \) vertices which is a certain regular tournament with exactly four eigenvalues distinct to \((n-1)/2\) with multiplicity \((n-1)/4\).
According to [27], it holds that $\lambda(CNDR_n) = (\sqrt{n} + 1)/2$. Thus if there exists a $CNDR_n$ for infinitely many $n \equiv 1(\mod 4)$, then it holds that

$$C(CNDR_n) \leq \frac{1}{2} \binom{n}{2} + O(n^3 \log n).$$

It is conjectured in [27] (see also [28]) that there exists a $CNDR_n$ for every $n \equiv 1(\mod 4)$. Interestingly, Savchenko [27] also found examples of $CNDR_p$ for primes $p = 5, 13, 29, 53, 173, 229, 293$ and 733 from the class of $T_p^4(S_{(0,1)})$ in the first example, and thus the inequality (8) can be improved for these examples. (It is shown in [27] that for every prime $p \equiv 5(\mod 8)$, $T_p^4(S_{(0,1)})$ has exactly four eigenvalues distinct to $(p - 1)/2$ with multiplicity $(p - 1)/4$.) It would be interesting to prove or disprove the existence of infinitely many primes $p \equiv 5(\mod 8)$ such that the tournament $T_p^4(S_{(0,1)})$ is in the class of $CNDR_p$.

The third example is based on a construction of pseudo-random graphs due to Shparlinski [30]. For related facts on elliptic curves, see [30, Section 2.1]. For a prime $p$, let $n \in [p + 1 - 2\sqrt{p}, p + 1 + 2\sqrt{p}]$ be an odd integer. It is known (e.g. [6, 13]) that there exists an elliptic curve $E$ over $\mathbb{F}_p$ such that the number of $\mathbb{F}_p$-rational points of $E$ is $n$. It is also known (e.g. [31]) that all $\mathbb{F}_p$-rational points of $E$ form an abelian group $G$ of order $n$ under an operation $\oplus$. Let $0_G$ be the identity of $G$. For an element $s \in G$ and a subset $S \subseteq G$, the inverse of $s$ is denoted by $\ominus s$ and let $\ominus S := \{ \ominus s \mid s \in S \}$.

**Definition 4** Let $S \subseteq G$ be a subset such that $S \cup \ominus S \cup \{0_G\} = G$ and $|S| = (n - 1)/2$. Then the tournament $T_{p,n}(S)$ is defined as follows.

$$V(T_{p,n}(S)) := G,$$

$$E(T_{p,n}(S)) := \{(x, y) \in G^2 \mid x \ominus y \in S\}.$$  

By the definition, $T_{p,n}(S)$ is a regular tournament with $n$ vertices.

**Corollary 4** For any sufficiently large prime $p$ and an odd integer $n \in [p + 1 - 2\sqrt{p}, p + 1 + 2\sqrt{p}]$, there exists a subset $S \subseteq G$ with the conditions in Definition 4 such that

$$C(T_{p,n}(S)) \leq \frac{1}{2} \binom{n}{2} + O(n^3 \log^2 n).$$

The following lemma in [30] plays a key role to prove the corollary. Recall that a character of $G$ is a homomorphism from $G$ to the multiplicative group of $\mathbb{C}$. A character of $G$ is said to be trivial if it is a constant function.
Lemma 3 [30] For any sufficiently large prime $p$ and an odd integer $n \in [p + 1 - 2\sqrt{p}, p + 1 + 2\sqrt{p}]$, there exists a subset $S \subset G$ with the conditions in Definition 4 such that

$$\max_x \left| \sum_{s \in S} \chi(s) \right| = O(\sqrt{n}\log n),$$

where $\chi$ runs over all non-trivial characters of $G$.

For the details of a construction of such a subset $S$, see [30].

Proof of Corollary 4 Let $S$ be a subset satisfying the conditions in Lemma 3. By Lemma 2, it suffices to prove that

$$\hat{\lambda}(T_{p,n}(S)) = O(\sqrt{n}\log n).$$

As in the proof of Corollary 2, one can see that the set of eigenvalues of $M_{T_{p,n}(S)}$ is

$$\left\{ \mu_\chi := \sum_{s \in S} \chi(s) \mid \chi \text{ is a character of } G \right\},$$

where $(\chi(x))_{x \in G}$ is an eigenvector corresponding to $\mu_\chi$ (see also [36, Problem 31F]). Notice that $\mu_\chi = |S| = (n-1)/2$ if $\chi$ is trivial. Thus Lemma 3 implies the desired estimation of $\hat{\lambda}(T_{p,n}(S))$.

Remark 5 It is worth noting that as shown in [30], almost all positive integers are in the interval $[p + 1 - 2\sqrt{p}, p + 1 + 2\sqrt{p}]$ for some prime $p$. Indeed, it holds [30] that

$$\lim_{N \to \infty} \frac{|\{n \leq N \mid \exists \text{ prime } p \text{ s.t. } n \text{ is odd and } n \in [p + 1 - 2\sqrt{p}, p + 1 + 2\sqrt{p}]\}|}{\left[ \frac{N}{2} \right]} = 1.$$ 

Thus the third example provides an infinite family of regular tournaments $T_n$ with $n$ vertices and small $\hat{\lambda}(T_n)$ for almost all positive integers $n$.

5 The Shütte Problem for Tournaments

At last, in this section, we focus on another random-like property.

Definition 5 Let $k$ be a positive integer. A tournament $T$ has the property $S_k$ if for every $A \subset V(T)$ of size $k$, there exists a vertex $z \not\in A$ directing to all members of $A$.

The Shütte problem asks the existence of tournaments satisfying this property for given $k \geq 1$ (see [15, 20]). As shown by Erdős [15], random tournaments a.a.s. satisfy $S_k$ for any $k \geq 1$. On the other hand, the problem of explicit constructions has been considered in graph theory. For example, Graham–Spencer [17] showed that the Paley tournament $T_p$ satisfies $S_k$ if $p > k^22^{2k-2}$ for each $k \geq 1$. From the digraphs constructed in [3], we can also construct tournaments satisfying $S_k$ for every $k$ by adding some edges. It would be interesting to provide more explicit constructions of
tournaments satisfying both of the quasi-random property and $S_k$. We add the following construction to the list; the following proposition and Corollary 2 show that for each $k \geq 1$ and even $m$, the tournament $T_p^m(S_i)$ in Definition 2 forms a family of regular tournaments with both of the quasi-random property and $S_k$.

**Proposition 1** Let $m$ be an even positive integer. Then for every $k \geq 1$, there exists a prime $p_m(k)$ such that for every prime $p > p_m(k)$, the tournament $T_p^m(S_i)$ has the property $S_k$.

Proposition 1 is proved by a direct generalization of the discussion in [2, 17], so we omit the proof here. Moreover, it is not so hard to prove that $T_p^m(S_i)$ has the existentially closed property (see e.g. [4]).

We also note that doubly regular tournaments constructed in [35] satisfy both of the quasi-random property and $S_2$, which follows from Corollary 3 and the corollary in [35, p. 277].

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**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest.

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