We implement the Minimal Geometric Deformation method to extend exterior solutions to anisotropic domains by imposing a regularity condition in the TolmanOppenheimerVolkoff equation of the decoupling sector. We obtain that the decoupling function can be formally obtained in terms of an integral involving the $g^{rr}$ component of the metric of the seed solution. As a particular example, we implement the method to extend the Schwarzschild exterior and obtain that the solution satisfies the dominant energy conditions out the event horizon.

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I. INTRODUCTION

In the searching of new solutions of the Einstein field equations we can try to extend well known solutions by adding extra sources and then we need to interpret such an extra matter sector. Regardingly, given the non–linearity of the Einstein field equations, it is well known that this strategy complicates the system of differential equations involved. In this respect, the first simple, systematic and direct way to decoupling gravitational sources has been introduced in the framework of General Relativity [1, 2] by using the the so–called Minimal Geometric Deformation (MGD) method [1–58]. Given the simplicity of the method, it has opened a range of new possibilities to obtain solutions of Einstein equations. The method can be described as follows. Suppose that the energy momentum tensor corresponds to the superposition of a perfect fluid and some scalar, vector, tensor or any other field. Now, after introducing a geometric deformation on the $g^{rr}$ component of a line element describing a static and spherically symmetric space–time, the original equations become in a system of two decoupled equations; one for each source involved. The advantage of this methodology lies in the fact that, contrary to what is usually believed, the extended solution is simply a superposition of the solution obtained for each system separately. The method has been successfully implemented to obtain anisotropic like–Tolman IV solutions [1, 19], anisotropic like–Tolman VII solutions [46], models for neutron stars [47], among others. Otherwise, in the context of modified theories of gravitation, the method has been used to obtain solutions in $f(G)$ gravity [26], Lovelock [43], $f(R, T)$ [41], Rastall[50] and recently black holes and interior solutions in the context of braneworld [49].

In other contexts, MGD has been used to extend black holes (BH) in $3 + 1$ and $2 + 1$ dimensional space–times [22, 33, 35, 52]. In [22] it was found the first extension of a Schwarzschild BH. The result was BH solutions with extra critical points and in some cases naked singularities. Regardingly, the solution with a naked singularity can be considered as an exterior solution for central objects with radius greater than the critical point. Otherwise, in [52], MGD was applied to extend a Reissner-Nordstr¨ om background. It was found that after an elaborated analysis of the free parameters the extended solution is free of extra singularities and satisfies all the energy conditions.

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It is worth mentioning that in all the extensions of exterior solutions above mentioned, the strategy is to impose a suitable equation of state in the decoupling sector as the well known barotropic, polytropic or linear combinations of the component of the energy–momentum tensor. In this work we propose an alternative strategy in the sense that we shall impose regularity conditions on the extra anisotropy induced by the decoupling sector in the framework of MGD in order to ensure an acceptable behaviour of the TolmanOppenheimerVolkoff (TOV) equation. As we shall see later, we base our study in the pioneering works of Bowers and Liang [60] and Cosenza et al [61, 62] where the condition is applied to avoid the apparition of singularities in the anisotropy function in the context of interior stellars. It is worth mentioning that the same strategy has been implemented to extend interior solutions in Ref. [56].

This work is organized as follows. In the next section we review the main aspects on MGD and then we study the regularity condition on the decoupling sector. In section III, we study the regularity condition in the anisotropic sector and illustrate the method using the Schwarzschild solutions as a seed. Finally, the last section is devoted to final remarks. Throughout the article we shall employ relativistic geometrized units where $G = c = 1$ thus the overall constant $\kappa^2$ is equal to $8\pi$ and the mostly positive signature $\{-,+,+\}$. 

**II. MINIMAL GEOMETRIC DEFORMATION**

In this section we shall review the main aspects about MGD in a self content way. Let us consider the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa^2T^{(\text{tot})}_{\mu\nu},$$

where we propose that the energy–momentum tensor, $T^{(\text{tot})}_{\mu\nu}$, can be decomposed as

$$T^{(\text{tot})}_{\mu\nu} = T_{\mu\nu} + \alpha\theta_{\mu\nu},$$

Now, let us consider $T_{\mu\nu} = \text{diag}(-\rho, p_r, p_\perp, p_\perp)$ and $\theta_{\mu\nu} = \text{diag}(-\rho^\theta, p_r^\theta, p_\perp^\theta, p_\perp^\theta)$ as two anisotropic sources. Note that, since the Einstein tensor is divergence free, the total energy momentum tensor $T^{(\text{tot})}_{\mu\nu}$ satisfies

$$\nabla_{\mu}T^{(\text{tot})\mu\nu} = 0.$$ 

In what follows, we shall consider spherically symmetric space–times with line element parametrized as

$$ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2d\Omega^2,$$

where $\nu$ and $\lambda$ are functions of the radial coordinate $r$ only. Now, considering Eq. (4) as a solution of the Einstein equations, we obtain

$$\kappa^2 \tilde{\rho} = \frac{1}{r^2} + e^{-\lambda} \left( \frac{\nu'}{r} - \frac{1}{r^2} \right),$$

$$\kappa^2 \tilde{p}_r = \frac{1}{r^2} + e^{-\lambda} \left( \frac{\nu'}{r} + \frac{1}{r^2} \right),$$

$$\kappa^2 \tilde{p}_\perp = \frac{e^{-\lambda}}{4} \left( \nu'' - \nu' \lambda' + 2\nu'' + 2\nu' - \lambda' \right),$$

where the primes denote derivation respect to the radial coordinate and we have defined

$$\tilde{\rho} = \rho + \alpha p_r^\theta,$$

$$\tilde{p}_r = p_r + \alpha p_r^\theta,$$

$$\tilde{p}_\perp = p_r + \alpha p_\perp^\theta.$$

Notice that the decomposition (2) seems to be a naive separation of the constituents of the matter content in the sense that, given the non–linearity of Einstein’s equations, the resulting system will not correspond to two set of equations, one for each source involved. However, contrary to which is broadly believed, the decoupling of the equations can be successfully obtained in the framework of MGD. The method works because besides the separation of the total energy momentum tensor, we introduce a geometric deformation in the metric functions given by

$$\nu = \xi + \alpha g,$$

$$e^{-\lambda} = \mu + \alpha f,$$
where \( \{g, f\} \) are the so–called decoupling functions and \( \alpha \) is the decoupling parameter. Although a general treatment considering deformation in both components of the metric is possible (see Ref. [2]), in this work we shall develop the particular case \( g = 0 \) and \( f \neq 0 \), namely, we only deform the \( g^{rr} \) component of the metric which corresponds to the minimal deformation we can perform. After that, we obtain two sets of differential equations: the set

\[
\begin{align*}
\kappa^2 \rho &= \frac{1 - r \mu' - \mu}{r^2} \\
\kappa^2 p_r &= \frac{r \mu' + \mu - 1}{r^2} \\
\kappa^2 p_t &= \frac{\mu' (rv' + 2) + \mu (2rv'' + rv'^2 + 2v')}{4r}
\end{align*}
\]

with

\[
\nabla_{\mu} T_{\nu}^{\mu} = p' + \frac{\nu'}{2} (\rho + p) - \frac{2}{r} (p_\perp^\theta - p_r^\theta) = 0,
\]

corresponding the the source \( T_{\mu\nu} \) and

\[
\begin{align*}
\kappa^2 \rho^\theta &= \frac{rf' + f}{r^2} \\
\kappa^2 p_r^\theta &= \frac{rfv' + f}{r^2} \\
\kappa^2 p_\perp^\theta &= \frac{f' (rv' + 2) + f (2rv'' + rv'^2 + 2v')}{4r}
\end{align*}
\]

for \( \theta_{\mu\nu} \). Note that whenever \( p_r^\theta \neq p_\perp^\theta \), the decoupling matter, \( \theta_{\mu\nu} \), can induce an extra local anisotropy in the system or allows to isotropize the solution as illustrated in [51]. Otherwise, the conservation equation \( \nabla_{\mu} \theta_{\nu}^{\mu} = 0 \) leads to

\[
(p_\perp^\theta)' + \frac{\nu'}{2} (\rho^\theta + p_\perp^\theta) - \frac{2}{r} (p_\perp^\theta - p_r^\theta) = 0,
\]

which is a linear combination of Eqs. (17), (18) and (19). In this sense, there is not exchange of energy–momentum tensor between the fluid \( T_{\mu\nu} \) and the \( \theta \)–sector and henceforth the interaction is purely gravitational. What is more, given the way the sources interact, the \( \theta \)–sector can be thought as dark matter or any other source which nature is unknown. It is noticeable that, although Eqs. (17), (18) and (19) do not correspond to Einstein’s equations (there is a missing \( 1/r^2 \) term), Eq. (20) corresponds to the classical TOV equation.

To proceed, we need to provide \( \{\nu, \mu\} \) (sourced by \( \{\rho, p_r, p_\perp\} \)) in order to use (17), (18) and (19) to obtain \( f \) after choosing suitable conditions on the anisotropic source \( \theta_{\mu\nu} \). For example, when the method is employed to extend interior solutions embedded in a Schwarzschild vacuum, the continuity of the second fundamental form is ensure when the so called mimic constraint for the radial pressure is imposed [1],

\[
-p_\perp^\theta = p,
\]

in the interior of the star. Remarkably, this condition yield an algebraic equation for \( f \) such that, in principle, any isotropic solution can be extended with this constraint. Another possibility is to use the mimic constraint for the density which leads to a differential equation for \( f \) which can be solved in some situations (see for example [19]). In the case of exterior solutions the use of mimic constraints do not have any sense (or there is not any fundamental physical reason to consider them, at least). In this respect the strategy consist in to impose some equations of state for the decoupling sector. For example, as commented previously, in Refs. [22, 52] it was assumed barotropic, polytropic and isotropic equations of state. An alternative strategy consist in to assume realistic equations of states which only have sense when we are studying interior solutions. However, there is not any fundamental reason in to consider some special numerical equation of state for the decoupling sector in the case of BH solutions. The other possibility is to impose suitable conditions, as for example, regularity in the TOV equation of the decoupling sector as we shall illustrate in the next section.

\section{Regularity Condition in the Decoupling Sector}

As it is well known, the study of interior isotropic solutions require the setting of extra information either an equation of state or a geometrical constraint in order to close the system. Similarly, when an anisotropy is introduced we need
to impose an extra condition which usually corresponds to a constraint on the anisotropy function. In the pioneering works of Bowers–Liang [60] and Cosenza et. al. [61, 62] this condition is implemented in such a manner that the TOV equation remains free of singularities, namely

$$\tilde{p}_\perp - \tilde{p}_r = c\tilde{f}(\hat{\rho}, r)(\hat{\rho} + \tilde{p}_r)r^n,$$  \hspace{1cm} (22)

where $c$ and $n$ are constants and $\tilde{f}(\hat{\rho}, r)$ is, in principle, an arbitrary function which allows to regularize the TOV. In this work we adapt this strategy to deal with exterior solutions in the context of MGD as follows. First, note that as $T_{\mu\nu}$ corresponds to the matter sector of a well known solution, non extra condition is needed to be imposed. Otherwise, the $\theta$–sector must be constrained in order to close the system so that we propose

$$p^\theta_\perp - p^\theta_r = c\tilde{f}(\theta_1, r)(\rho^\theta + p^\theta_r)r^n.$$  \hspace{1cm} (23)

Now, as $\tilde{f}$ remains as an arbitrary function we propose the Cosenza–Herrera–Esculpi–Witten ansatz which reads

$$\tilde{f}(\hat{\rho}, r) = r^{1-n
u'}/2.$$  \hspace{1cm} (24)

At this point a couple of comments are in order. First, it is worth noticing that the same strategy was successfully implemented to extend the Tolman IV solution in Ref. [56]. Second, note that regularity conditions imposed on the TOV equation associated to the decoupling sector is not a mandatory requirement. However, there are not reasons to assume that the $\theta$–sector suffers from the same pathologies that the original background we are extending by MGD.

Now, it is clear that the replacement of (17), (18) and (19) in (23), leads to a differential equation for $f$ where the only required information is the metric function $\nu$, which in the context of MGD is common for the three sectors involved. Indeed, from Eq. (23) and after imposing the Consenza–Herrera–Esculpi–Witten anisotropy we obtain

$$((2c + 1)\nu'' + 2)f' + \left(r(1 - 2c)\nu' - 2 + 2\nu'' - \frac{4}{r}\right)f = 0,$$  \hspace{1cm} (25)

which can be formally solved to obtain

$$f = c_1 e^{\int \frac{\nu'((2c-1)\nu' + 2 - 2\nu'' + 4)}{u(2c + 1)\nu' + 2)} du}.$$  \hspace{1cm} (26)

It is worth noticing that finding an analytical solution of the above integral will depend on the particular form of the metric function $\nu$. In this work, we shall consider the Schwarzschild black hole as seed solutions of the method to illustrate the method described before.

### A. Schwarzschild solution

In this particular case the metric of the seed corresponds to

$$e^{\nu} = e^{-\lambda} = 1 - \frac{2M}{r}.$$  \hspace{1cm} (27)

Next, replacing (27) in (26) we obtain

$$f = c_1 \left(1 - \frac{2M}{r}\right)^2 (2c - 1)M + r)^2,$$  \hspace{1cm} (28)

which determines the decoupling sector. Note that $c_1$ should be a constant with dimensions of inverse of length squared.

To complete the MGD program, the rest of the section is devoted to obtain the total like–Schwarzschild exterior solution. From Eq. (12), the $g^{rr}$ component of the metric reads

$$e^{-\lambda} = \left(1 - \frac{2M}{r}\right)^2 (2c - 1)M + r)^2.$$  \hspace{1cm} (29)

Note that, as usual, the solution maintains the same horizon, $r_H = 2M$. Even more, non extra horizons appear, i.e, $e^{-\lambda} \neq 0$ for every $r \neq r_H$, whenever the equation

$$1 + \alpha c_1 (2c - 1)M + r)^2 = 0,$$  \hspace{1cm} (30)
leads to either $r < 0$ or $r \in \mathbb{C}$. A straightforward computation reveals that Eq. (30) leads to

\begin{align}
r_1 &= \frac{-2i\sqrt{c_1 \alpha} + 2c_1 M \alpha - 4c_1 M \alpha}{2c_1 \alpha} \\
r_2 &= \frac{2i\sqrt{c_1 \alpha} + 2c_1 M \alpha - 4c_1 M \alpha}{2c_1 \alpha}
\end{align}

A first possibility is to choose $\alpha c_1 > 0$ which entails $r \in \mathbb{C}$. The second possibility is to choose $\alpha c_1 < 0$ from where Eqs. (31) and (32) read

\begin{align}
r_1 &= \frac{-2\sqrt{|c_1 \alpha|} - 2|c_1 \alpha|M + 4c|c_1 \alpha|M}{2|c_1 \alpha|} \\
r_2 &= \frac{-2\sqrt{|c_1 \alpha|} - 2|c_1 \alpha|M + 4c|c_1 \alpha|M}{2|c_1 \alpha|}
\end{align}

Now, to enforce $r_1, r_2 < 0$ we need to impose

\begin{align}
2\sqrt{|c_1 \alpha|} - 2|c_1 \alpha|M + 4c|c_1 \alpha|M > 0 \quad (35) \\
-2\sqrt{|c_1 \alpha|} - 2|c_1 \alpha|M + 4c|c_1 \alpha|M > 0,
\end{align}

from where we obtain $c > \frac{1}{2}$ and

\begin{equation}
|c_1 \alpha| > \frac{1}{M^2 - 4cM^2 + 4c^2 M^2}.
\end{equation}

It is worth noticing that, the imposition of regularity on the TOV by the Cosenza–Herrera–Escalepi–Witten anisotropy (see Eqs. (23) and (24)), leads to an extended Schwarzschild solution that maintain the same singular point at $r = 0$ and the apparition of extra singularities is impossible for every value of the constants involve. Indeed, the Ricci, $R$, Ricci squared $\mathcal{R}$ and Kretschmann scalar $K$ read

\begin{equation}
R = -\frac{\alpha c_1 \left(2 \left(2 - 5c + 2c^2\right) M^2 + (8c - 7) Mr + 3r^2\right)}{r^2},
\end{equation}

\begin{equation}
\mathcal{R} = \frac{4\alpha^2 c_1^2}{r^4} \left[(2c - 1) M + r\right]^2 \left[(2c^2 - 6c + 7) M^2 (4c - 9) Mr + 3r^2\right],
\end{equation}

\begin{equation}
K = 2\alpha \left(2c - 1\right) c_1 M^3 r \left(20 + 27c_1 \alpha r^2 + 8\alpha^2 c_1 r^2 - 4c \left(1 + 7c_1 r^2\right)\right) \\
+ 3\alpha^2 c_1^2 r^6 + 16\alpha^2 \left(c - 1\right) c_1^2 M^5 r^5 \\
+ 2\alpha \left(1 - 2c^2\right)^2 c_1 M^4 \left(12 + \alpha \left(2c^2 - 14c + 29\right) c_1 M^2\right) \\
+ M^2 \left(12 - 8\alpha \left(c - 2\right) c_1 r^2 + \alpha^2 \left(32c^2 - 72c + 37\right) c_1^2 r^4\right) \\
+ \frac{4}{r^6} \left[12\alpha^2 \left(1 - 2c\right)^4 c_1^2 M^6 - 8\alpha^2 \left(c - 5\right) \left(2c - 1\right)^3 c_1^2 M^5 r\right].
\end{equation}

At this point it should be noted that when $\alpha$ is zero the equations (38) and (39) representing the Ricci scalar and the square of the Ricci tensor are vanishing while from (40) the original Kretschmann invariant corresponding to the original Schwarzschild solution is recovered.

From now on, we shall focus our attention in the study of energy conditions of the matter sector sustaining the extended solution. From (5), (6) and (7) we obtain

\begin{align}
\hat{\rho} &= -\frac{\alpha c_1 \left((2c - 1) M + r\right) \left((2c - 5) M + 3r\right)}{8\pi r^2} \\
\hat{p}_r &= \frac{\alpha c_1 \left((2c - 1) M + r\right)^2}{8\pi r^2} \\
\hat{p}_\perp &= -\frac{\alpha c_1 \left(M - r\right) \left((2c - 1) M + r\right)}{8\pi r^2}.
\end{align}
Note that in order to ensure the positivity of the density energy, \( \rho \), the condition \( \alpha c_1 \) previously obtained must be discarded. Besides, in order to avoid real roots of \( \dot{\rho} \) we need to restrict the parameter \( c \) even more, namely, \( c > \frac{1}{2} \). Let us now consider the energy conditions. The null energy condition states that

\[
\dot{\rho} + \rho_r \geq 0 \tag{44}
\]
\[
\dot{\rho} + \rho_\perp \geq 0. \tag{45}
\]

Now, from Eqs. (41), (42) and (43) we obtain

\[
\dot{\rho} + \rho_r = \frac{2c_1 \alpha (2M - r)((2c - 1)M + r)}{8\pi r^2} \geq 0 \tag{46}
\]
\[
\dot{\rho} + \rho_\perp = -c_1 \alpha ((c - 2)M + r)((2c - 1)M + r) \geq 0. \tag{47}
\]

It is noticeable that expression (47) is satisfied everywhere when we restrict \( c > 2 \). However, (46) can be satisfied for \( r > 2M \) only, i.e., out the event horizon. Now, the above condition combined with the constraint for \( \dot{\rho} > 0 \) corresponds to the weak energy condition (WEC). In this sense we say that our solution satisfies the WEC out the event horizon.

The dominant energy condition (DEC) states that

\[
\dot{\rho} - |\rho_r| \geq 0 \tag{48}
\]
\[
\dot{\rho} - |\rho_\perp| \geq 0. \tag{49}
\]

From Eq. (42) it can be seen that \( \rho_r \) is negative everywhere, so that Eq. (48) yield \( \dot{\rho} + \rho_r \) which coincides with (46). In other words, the requirement given by (48) is satisfied out the event horizon only. Regarding Eq. (49) the situation is subtler. Indeed, Eq. (43) reveals that \( \rho_\perp > 0 \) for \( r < M \) an \( \rho_\perp < 0 \) for \( r > M \). In this manner, for \( r < M \), Eq. (49) reads

\[
\dot{\rho} - \rho_\perp = -\frac{\alpha c_1 ((2c - 1)M + r)((c - 3)M + 2r)}{4\pi r^2}, \tag{50}
\]

which is positive when \( c > 3 \). Similarly, for \( r > M \) we have that (49) reduces to \( \dot{\rho} + \rho_\perp > 0 \), which coincides with (47) and it is fulfilled everywhere for \( c > 2 \). Finally we study the strong energy condition (SEC) that states

\[
\dot{\rho} + \rho_r + 2\rho_\perp \geq 0, \tag{51}
\]

together with Eqs. (46) and (47). As we discussed previously, (46) and (47) are satisfied out the horizon. Now we shall explore the remaining condition. From Eqs. (41), (42) and (43) we have

\[
\dot{\rho} + \rho_r + 2\rho_\perp = \frac{\alpha c_1 M ((2c - 1)M + r)}{4\pi r^2}, \tag{52}
\]

which is negative everywhere for the accepted values of \( c \). In summary, we have found that the conditions of regularity of the TOV equation of the anisotropic sector leads to a extension of the Schwarzschild exterior which satisfies the DEC out the horizon but violates the SEC everywhere. In this respect, the solution could be used as the exterior geometry of a central self gravitating object with radius \( R > r_H \). The previous discussion about the energy conditions is corroborated in Fig. 1 which displays the behaviour of these constrains on the components of the energy-momentum tensor versus the radial coordinate \( r \). It is worth mentioning that to ensure a positive energy-density \( \rho \) the integration constant \( c_1 \) and the dimensionless constant \( \alpha \) must be opposite in sign.

IV. FINAL REMARKS

In this work we have implemented the Minimal Geometric Deformation method to construct an extension of exterior solutions by imposing a regularity condition in the TolmanOppenheimerVolkoff equation. More precisely, we followed the pioneering work of Bowers and Liang to propose a particular form of the anisotropy function on the decoupling sector of the solution. The main result of the paper is that the decoupling function, \( f \), admits a formal solution in terms of an integral involving the metric potential associated to the \( g^{\tau \tau} \) component of the metric. Given this result, we conclude that it is possible in principle to extend any exterior solution with the methodology here developed. However, obtaining an analytical solution for \( f \) will depend on the particular form of the seed. To illustrate the
method we used the Schwarzschild exterior as the seed and obtain that, after a careful analysis of the free parameters of the solution, the new anisotropic background satisfies the dominant energy condition out the event horizon so that it should be considered as the exterior of a central object with mass greater than the horizon of the solution. We would like to conclude this work by pointing out that the exterior solution here obtained would allow to study classical tests of General Relativity as the precession of Mercury and the bending of light. Indeed, this study would allow to obtain physical bound on the decoupling parameter $\alpha$, for example. Besides, the nature of the $\theta$ source can be explored in terms of scalar, vector or tensor fields after comparing the results with the Lagrangian $L_x$ appearing in the general treatment of the problem. However, this and other aspects lie out of the scope of this article and we leave this discussions for future works.

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