Generalized Bour’s theorem in Minkowski 3-space

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Abstract. We obtain isometric minimal helicoidal and rotational surfaces using generalized Bour’s theorem in three dimensional Minkowski space. In addition, we show that the surfaces preserve minimality when their Gauss maps identically equal, choosing any differentiable functions on the profile curve.

1. Introduction

There are notable books for theory of surfaces in the literature, such as [2, 11, 13]. It is well known that the right helicoid (resp. catenoid) is the only ruled (resp. rotational) surface which is minimal in classical surface geometry. Focusing on the ruled (helicoid) and rotational characters, we have Bour’s theorem [1, 14].

Ikawa [7] determines pairs of surfaces by Bour’s theorem with the additional condition that they have the same Gauss map in Euclidean 3-space. Ikawa [8] also gives Bour’s theorem in Minkowski geometry. About helicoidal surfaces in Euclidean 3-space, Do Carmo and Dajczer [3], prove that there exists a two-parameter family of helicoidal surfaces isometric to a given helicoidal surface.

Some relations among the Laplace-Beltrami operator and curvatures of the helicoidal surfaces in Euclidean 3-space are shown by Güler et al [5]. In addition, they give Bour’s theorem on the Gauss map and some special examples. Güler gives Bour’s theorem on lightlike profile curve in [4]. Ji and Kim [9] prove that, a minimal helicoidal surface with Gauss curvature $K$ has an isometric minimal rotational surface if and only if $K \leq 0$ in Minkowski 3-space. On another hand, Martinez et al [12] give a complete classification of the helicoidal flat surfaces in the hyperbolic 3-space in terms of meromorphic data as well as by means of linear harmonic functions. Güler and Yaylı [6] obtain Generalized Bour’s theorem for helicoidal and rotational surfaces in three dimensional Euclidean space. This work contains extended results of Ikawa’s paper [8].

In section 2, we give some basic notions such as Lorentzian helicoidal surface and rotational surface of Minkowski geometry. We also show the Gauss’ Theorema Egregium. In section 3, we use the generalized Bour’s theorems on surfaces with spacelike or timelike axis.

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2. Preliminaries

Throughout this paper we will identify a vector \((a, b, c)\) with its transpose. In this section, we will obtain the Lorentzian rotational and helicoidal surfaces in Minkowski 3-space \(\mathbb{L}^3\).

Let \(V\) be a three dimensional Lorentz vector space with Lorentz product \(\langle \vec{x}, \vec{y} \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3\), where \(\vec{x} = (x_1, x_2, x_3)\) and \(\vec{y} = (y_1, y_2, y_3)\). A vector \(\vec{x}\) in \(V\) is called spacelike (resp. timelike) if \(\langle \vec{x}, \vec{x} \rangle > 0\) or \(\vec{x} = 0\) (resp. \(\langle \vec{x}, \vec{x} \rangle < 0\)). If \(\vec{x} \neq 0\) satisfies \(\langle \vec{x}, \vec{x} \rangle = 0\), \(\vec{x}\) is called lightlike.

Now, we define the rotational surface and helicoidal surface in \(\mathbb{L}^3\). For an open interval \(I \subset \mathbb{R}\), let \(\gamma: I \rightarrow \Pi\) be a curve in a plane \(\Pi\) in \(\mathbb{L}^3\), and let \(\ell\) be a straight line in \(\Pi\). A Lorentzian rotational surface in \(\mathbb{L}^3\) is defined as a surface rotating a curve \(\gamma\) around a line \(\ell\) (these are called the profile curve and the axis, respectively).

Suppose that, when a profile curve \(\gamma\) rotates around the axis \(\ell\), it simultaneously displaces parallel lines orthogonal to the axis \(\ell\), so that the speed of displacement is proportional to the speed of rotation. Then the resulting surface is called the Lorentzian helicoidal surface with axis \(\ell\) and pitch \(a \in \mathbb{R} \setminus \{0\}\). If the profile curve is a line perpendicular to the axis, the surface is a Lorentzian right helicoid.

We may suppose that \(\ell\) is the line spanned by the spacelike vector \((1, 0, 0)\). The orthogonal matrix which fixes the above vector is

\[
A(v) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh v & \sinh v \\ 0 & \sinh v & \cosh v \end{pmatrix}, \quad v \in \mathbb{R}.
\]

The matrix \(A\) can be found, by solving the following equations simultaneously:

\[ A\ell = \ell, \quad A^t \varepsilon A = \varepsilon, \quad \varepsilon = \text{diag}(1, 1, -1), \quad \det A = 1. \]

When the axis of rotation is \(\ell\), there is an Lorentzian transformation by which the axis is \(\ell\) transformed to the \(x\)-axis of \(\mathbb{L}^3\). Parametrization of the spacelike profile curve is given by \(\gamma(u) = (\phi(u), f(u), 0)\), where \(f(u), \phi(u): I \subset \mathbb{R} \rightarrow \mathbb{R}\) are differentiable functions for all \(u \in I\). A Lorentzian helicoidal surface in Minkowski 3-space which is spanned by the spacelike vector \((1, 0, 0)\) with pitch \(a\), as follows

\[
H(u, v) = A(v) \cdot \gamma(u) + av(1, 0, 0).
\]

When \(a = 0\), the surface is just a Lorentzian rotational surface as follows

\[
R(u, v) = (\phi(u), f(u) \cosh v, f(u) \sinh v).
\]

If the profile curve is on the \(xz\)-plane, it is given by \(\gamma(u) = (\phi(u), 0, f(u))\). So, it can be spacelike or timelike. Hence, Lorentzian rotational surface is as follows

\[
R(u, v) = (\phi(u), f(u) \sinh v, f(u) \cosh v).
\]

See [8], for details of the other Lorentzian rotational surfaces.

For a surface \(X(u, v)\), the coefficients of the first and second fundamental forms, the Gauss map and the other objects of Minkowski geometry are given in many books, as in [13].
3. Generalized Bour’s theorem on surfaces with spacelike or timelike axis

In this section, we show generalized Bour’s theorem for isometric surfaces with spacelike or timelike axis in three dimensional Minkowski space. See also [10] for details.

We classify a surface as “(Axis’s type, Profile curve’s type)-type” in the rest of the paper.

**Theorem 1. (Generalized Bour’s Theorem).** A helicoidal surface

\[ H(u, v) = (\phi(u) + av, f(u) \cosh v, f(u) \sinh v, \ldots) \]

is locally isometric to the rotational surface

\[ R(u, v) = \left( \begin{array}{c} \sqrt{\int \frac{(f\phi')^2 - (af')^2}{f^2 - a^2} \, du} \\ \sqrt{f^2 - a^2} \cosh \left( v - \int \frac{a\phi'}{f^2 - a^2} \, du \right) \\ \sqrt{f^2 - a^2} \sinh \left( v - \int \frac{a\phi'}{f^2 - a^2} \, du \right) \end{array} \right), \]

where \( f \) and \( \phi \) are differentiable functions, \( 0 \leq v < 2\pi \) and \( u, a \in \mathbb{R} \setminus \{0\} \).

**Proof.** We assume that the profile curve is

\[ \gamma_H(u_H) = (\phi_H(u_H), f_H(u_H), 0). \]

Since the helicoidal surface is given by rotating the profile curve \( \gamma \) around the axis \( \ell = (1, 0, 0) \) and simultaneously displacing parallel lines orthogonal to the axis \( \ell \), so that the speed of displacement is proportional to the speed of rotation, we have the following representation of the helicoidal surface

\[ H(u_H, v_H) = (\phi_H(u_H) + av_H, f_H(u_H) \cosh v_H, f_H(u_H) \sinh v_H), \]

where \( u_H, a \in \mathbb{R} \setminus \{0\} \) and \( 0 \leq v_H < 2\pi \). The line element of the helicoidal surface as above is given by

\[ ds_H^2 = (f_H'^2 + \phi_H'^2)du_H^2 + 2a\phi_H' du_H dv_H + (a^2 - f_H'^2)dv_H^2. \]

Since

\[ E_H G_H - F_H^2 = (a^2 - f_H'^2)\phi_H'^2, \]

if \( E_H G_H - F_H^2 > 0 \) (resp. \(< 0\)) then \( H(u_H, v_H) \) is spacelike (resp. timelike) surface.

Helices in \( H(u_H, v_H) \) are curves defined by \( u_H = \text{const.} \). So curves in \( H(u_H, v_H) \) that are orthogonal to helices supply the orthogonality condition as follow

\[ a\phi_H' du_H + (a^2 - f_H'^2)dv_H = 0. \]

Thus, we obtain

\[ v_H = - \int \frac{a\phi_H'}{a^2 - f_H'^2} \, du_H + c, \]

where \( c \) is constant. Hence if we put

\[ \nabla_H = v_H + \int \frac{a\phi_H'}{a^2 - f_H'^2} \, du_H, \]
then curves orthogonal to helices are given by $v_H = \text{const.}$. Substituting the equation $dv_H = d\varpi_H + \frac{a\phi_H'}{f_H^2 - a^2}du_H$ into the line element (3.3), we have

$$ds_H^2 = \left(2f_H^2 + \frac{f_H^2\phi_H'^2}{f_H^2 - a^2}\right)du_H^2 - (f_H^2 - a^2)d\varpi_H^2. \tag{3.4}$$

**General Case 1. (S,S)-type, A.**

(a). Let $f_H^2 - a^2 > 0$. Setting $u_H := \int \sqrt{f_H^2 + f_H^2\phi_H'^2}du_H, \quad k_H(u_H) := \sqrt{f_H^2 - a^2}, \tag{3.4}$ becomes

$$ds_H^2 = du_H^2 - k_H^2(u_H)d\varpi_H^2. \tag{3.5}$$

The rotational surface $R(u_R, v_R) = (\phi_R(u_R), f_R(u_R)\cosh v_R, f_R(u_R)\sinh v_R)$, has the line element

$$ds_R^2 = (f_R^2 + \phi_R'^2)du_R^2 - f_R^2d\varpi_R^2. \tag{3.6}$$

Again, setting $\varpi_R := \int \sqrt{f_R^2 + \phi_R'^2}du_R, \quad k_R(\varpi_R) := f_R, \quad \varpi_R = v_R,$

then (3.6) becomes

$$ds_R^2 = du_R^2 - k_R^2(\varpi_R)d\varpi_R^2. \tag{3.7}$$

Comparing (3.5) with (3.7), if we take $\overline{u}_H = \overline{u}_R, \quad \overline{v}_H = \overline{v}_R, \quad k_H(\overline{u}_H) = k_R(\overline{u}_R),$

then we have an isometry between $H(u_H, v_H)$ and $R(u_R, v_R)$. Therefore, it follows that

$$\int \sqrt{f_H^2 + \frac{f_H^2\phi_H'^2}{f_H^2 - a^2}}du_H = \int \sqrt{f_R^2 + \phi_R'^2}du_R,$$

and we get

$$\int \phi_R' du_R = \int \sqrt{\frac{f_H^2\phi_H'^2 - a^2f_H^2}{f_H^2 - a^2}}du_H.$$

This is the end of the proof.

**Example 1. 1. A helicoidal surface (see Fig. 1 left side)**

$H(u, v) = (u^3 + av, u^2 \cosh v, u^2 \sinh v)$
is isometric to the rotational surface

\[ R(u, v) = \left( \begin{array}{c} \int \sqrt{\frac{9a^4 - 4a^2 u^2}{u^4 - a^2}} du \\
\sqrt{u^4 - a^2} \cosh \left( v - \int \frac{3au^2}{u^4 - a^2} du \right) \\
\sqrt{u^4 - a^2} \sinh \left( v - \int \frac{3au^2}{u^4 - a^2} du \right) \end{array} \right) \]

by generalized Bour’s theorem, where \( u, a \in \mathbb{R} \setminus \{0\} \) and \( 0 \leq v < 2\pi \). When \( a = 0 \), the surface has the form as follows (see Fig. 1, right side)

\( (u^3, u^2 \cosh v, u^2 \sinh v) \).

Figure 1. Left: Lorentzian helicoidal surface, right: Lorentzian rotational surface, \( \phi(u) = u^3, f(u) = u^2 \)

Next theorem called ”Theorema Egregium” (Latin: ”Remarkable Theorem”) was published by Gauss in 1827.

**Theorem 2.** (Gauss’ Theorema Egregium). The Gaussian curvature \( K \) of a 2-dimensional surface element \( f : U \to \mathbb{R}^3 \) of class depend only on the first fundamental form (and is consequently an intrinsic quantity of the surface).

The mean curvature \( H \) does not depend only on the first fundamental form. For example, the cylinder and the plane have the same fundamental form, but have \( H \neq 0 \) and \( H = 0 \), respectively. Gauss’ theorem can be stated in the language of isometries in \( \mathbb{L}^3 \):

**Theorem 3.** If \( I : M \to M^* \) is an isometry (and surfaces are locally isometric) in \( \mathbb{L}^3 \), then the Gaussssian curvatures at corresponding points are equal. That is, \( K(p) = K^*(I(p)) \) for all point \( p \) in \( M \).

So, we give an example about Theorem 1, Theorem 2 and Theorem 3.

**Example 2.** Let \( f(u) = u \) and \( \phi(u) = 0 \) in Theorem 1, then the Lorentzian right helicoid

\[ (3.8) \quad H(u, v) = (au, u \cosh v, u \sinh v) \]

is isometric to the Lorentzian catenoid

\[ (3.9) \quad R(u, v) = (ai \log (u + \sqrt{u^2 - a^2}), \sqrt{u^2 - a^2} \cosh v, \sqrt{u^2 - a^2} \sinh v) \]
where \( u, a \in \mathbb{R} \setminus \{0\} \) and \( 0 \leq v < 2\pi \). The coefficients of the first and second fundamental forms of these surfaces are

\[
E_{H(u,v)} = 1 = E_{R(u,v)}, \quad F_{H(u,v)} = 0 = F_{R(u,v)}, \quad G_{H(u,v)} = a^2 - u^2 = G_{R(u,v)},
\]

\[
L_{H(u,v)} = 0, \quad M_{H(u,v)} = -\frac{a}{\sqrt{u^2 - a^2}}, \quad N_{H(u,v)} = 0,
\]

\[
L_{R(u,v)} = -\frac{ia}{a^2 - u^2}, \quad M_{R(u,v)} = 0, \quad N_{R(u,v)} = ia.
\]

Hence, the surfaces have

\[
LN - M^2 = -\frac{a^2}{(a^2 - u^2)^2}.
\]

So, the Gaussian curvatures of the Lorentzian right helicoid and the Lorentzian catenoid are

\[
K_{H(u,v)} = -\frac{a^2}{(a^2 - u^2)^2} = K_{R(u,v)}.
\]

**Corollary 1.** If \( I : H(u,v) \rightarrow R(u,v) \) is an isometry (and surfaces are locally isometric) in \( \mathbb{L}^3 \), then the Gaussian curvatures at corresponding points are equal, i.e.

\[
K_{H(u,v)}(p) = -\frac{f^3 f' \phi'^2 + f f'' \phi'^2 - a^2 f'^4}{(f^2 \phi'^2 + f^2 f'^2 - a^2 f'^2)^2} = K_{R(u,v)}(I(p))
\]

for all point \( p \) in \( H(u,v) \).

Next, we prove the relations among the isometric surfaces by generalized Bour’s theorem.

**Theorem 4.** Let a helicoidal and a rotational surface be isometrically related by generalized Bour’s theorem. If these two surfaces have the same Gauss map, then surfaces have the forms

\[
H(u,v) = (\phi(u) + av, f(u) \cosh v, f(u) \sinh v)
\]

and

\[
R(u,v) = \begin{pmatrix}
\sqrt{1 - a^2} \arg \cosh \left( \frac{\sqrt{f^2 - a^2}}{\sqrt{1 - a^2}} \right) \\
\sqrt{f^2 - a^2} \cosh \left( v - \int \frac{a \phi'}{f^2 - a^2} du \right) \\
\sqrt{f^2 - a^2} \sinh \left( v - \int \frac{a \phi'}{f^2 - a^2 - b^2} du \right)
\end{pmatrix}
\]

where

\[
\phi(u) = a \log \left( \sqrt{\frac{a \sqrt{f^2 - a^2 - b^2} - \sqrt{a^2 + b^2} \sqrt{f^2 - a^2}}{a \sqrt{f^2 - a^2 - b^2} + \sqrt{a^2 + b^2} \sqrt{f^2 - a^2}}} \right)
\]

\[
+ \sqrt{a^2 + b^2} \log \left( \sqrt{\frac{\sqrt{f^2 - a^2} + \sqrt{f^2 - a^2 - b^2}}{\sqrt{f^2 - a^2} - \sqrt{f^2 - a^2 - b^2}}} + c \right).
\]

\( f = f(u) \) is a differentiable function, \( u, a \in \mathbb{R} \setminus \{0\} \) and \( 0 \leq v < 2\pi \).
Proof. By virtue of the first and second fundamental forms
\[ E_H = f'^2 + \phi'^2, \quad F_H = a\phi', \quad G_H = a^2 - f^2, \]
\[ L_H = \frac{-f f'' \phi' + f f' \phi''}{\sqrt{(f^2 - a^2) f'^2 + f^2 \phi'^2}}, \quad M_H = \frac{-a f'^2}{\sqrt{(f^2 - a^2) f'^2 + f^2 \phi'^2}}, \]
\[ N_H = \frac{-f^2 \phi'}{\sqrt{(f^2 - a^2) f'^2 + f^2 \phi'^2}}, \]
the Gauss map and the mean curvature of the helicoidal surface are
\[ e_H = \frac{1}{\sqrt{(f^2 - a^2) f'^2 + f^2 \phi'^2}} \left( \begin{array}{c} f f' \\ \frac{f f'}{\sqrt{f^2 - a^2}} \sinh \left( v - \int \frac{1}{\sqrt{f^2 - a^2}} du \right) \\ \frac{f f'}{\sqrt{f^2 - a^2}} \cosh \left( v - \int \frac{1}{\sqrt{f^2 - a^2}} du \right) \end{array} \right), \]
\[ H_H = \frac{\Omega(u)}{2 \left[ (f^2 - a^2) f'^2 + f^2 \phi'^2 \right]^{3/2}}, \]
where
(3.14) \[ \Omega(u) := \left[ \left( f^2 - a^2 \right) f f'' + \left( 2a^2 - f^2 \right) f'^2 \right] f' \phi' - f^2 \phi' + \left( a^2 - f^2 \right) f f' \phi''. \]

Next, we calculate the Gauss map \( e_R \) and the mean curvature \( H_R \) of the rotational surface (3.2). Since
\[ R_u = \left( \begin{array}{c} f f' \sqrt{f^2 - a^2} \\ f f' \sinh \left( v - \int \frac{1}{\sqrt{f^2 - a^2}} du \right) \\ f f' \cosh \left( v - \int \frac{1}{\sqrt{f^2 - a^2}} du \right) \end{array} \right), \]
\[ R_v = \left( \begin{array}{c} 0 \\ \frac{\sqrt{f^2 - a^2} \sinh \left( v - \int \frac{1}{\sqrt{f^2 - a^2}} du \right)}{\sqrt{f^2 - a^2} \cosh \left( v - \int \frac{1}{\sqrt{f^2 - a^2}} du \right)} \end{array} \right), \]
the Gauss map of the rotational surface is
\[ e_R = \frac{1}{\sqrt{(f^2 - a^2) f'^2 + f^2 \phi'^2}} \left( \begin{array}{c} f f' \\ -\sqrt{(f \phi')^2 + (af')^2} \cosh \left( v - \int \frac{1}{\sqrt{f^2 - a^2}} du \right) \\ -\sqrt{(f \phi')^2 + (af')^2} \sinh \left( v - \int \frac{1}{\sqrt{f^2 - a^2}} du \right) \end{array} \right). \]

Using the coefficients of the second fundamental form of \( R(u,v) \)
\[ L_R = \frac{a^2 \left( f^2 - a^2 \right) f'^4 + \left( -f f'' + a^2 f^3 f'' + a^4 f^2 \right) \phi'^2}{\left( f^2 - a^2 \right)^{3/2} \sqrt{(f^2 - a^2) f'^2 + f^2 \phi'^2}}, \]
\[ M_R = \frac{a \phi' \sqrt{f^2 \phi'^2 - a^2 f'^2}}{\sqrt{f^2 - a^2 \left( f^2 - a^2 \right) f'^2 + f^2 \phi'^2}} \]
\[
N_R = -\frac{\sqrt{f^2 - a^2} \sqrt{f^2 \phi'^2 - a^2 f'^2}}{\sqrt{(f^2 - a^2) f'^2 + f^2 \phi'^2}},
\]
by the straight calculation, the mean curvature of the rotational surface is

\[
H_R = \frac{-f^2 \phi' \Omega(u)}{2 \left[(f^2 - a^2) f'^2 + f^2 \phi'^2\right]^{3/2} \sqrt{f^2 - a^2} \sqrt{f^2 \phi'^2 - a^2 f'^2}},
\]
where \(\Omega(u)\) is the function in (3.14).

Now, suppose that the Gauss map \(e_H\) is identically equal to \(e_R\). Comparing (3.12) and (3.15), we have

\[
a f' = \sqrt{(f \phi')^2 - (a f')^2} \sinh \left(v - \int \frac{a \phi'}{f^2 - a^2} du\right),
\]
\[
f \phi' = \sqrt{(f \phi')^2 - (a f')^2} \cosh \left(v - \int \frac{a \phi'}{f^2 - a^2} du\right).
\]

Differentiating equation (3.17), and using (3.18) we get

\[
[(f^2 - a^2) f'' + (2a^2 - f^2) f'^2] \phi' - f^2 \phi'^3 + (a^2 - f^2) f f' \phi'' = 0.
\]

This equation means \(\Omega(u) = 0\) in (3.13) and (3.16). So, helicoidal surface and the rotational surface have zero mean curvature. Then, it follows that

\[
b \arg \cosh \left(\frac{\sqrt{f^2 - a^2}}{b}\right) = \int \sqrt{\frac{(f \phi')^2 - (a f')^2}{f^2 - a^2}} \, du.
\]
Differentiating this equation, we get

\[
\phi' = \frac{a^2 + b^2 \sqrt{f^2 - a^2 f'}}{f \sqrt{f^2 - a^2 - b^2}}.
\]
To solve this differential equation, if we take

\[
t := \frac{f^2 - a^2}{f^2 - a^2 - b^2}, \quad r := \sqrt{a^2 + b^2},
\]
it follows that

\[
\phi = -b^2 r \int \frac{t^2}{(r^2 t^2 - a^2)(t^2 - 1)} \, dt.
\]
Solving this integral we get the function \(\phi(u)\) in the theorem. This completes the proof.

(b). \(a^2 f'^2 / (f'^2 + \phi'^2) < f^2 < a^2 f'^2\):
Taking

\[
\pi_H := \int \frac{f_H^2 \phi_H'^2}{a^2 - f_H^2} - f_H^2 du_H, \quad k_H(\pi_H) := \sqrt{a^2 - f_H^2},
\]
from (3.4) we get

\[
ds_H^2 = -du_H^2 + k_H(\pi_H) dv_H^2.
\]
An (S,T)-type rotational surface has the form

\[
R(u_H, v_H) = (\phi_H(u_H), f_H(u_H) \sinh v_H, f_H(u_H) \cosh v_H).
\]
Hence, we have the theorem as follows.
THEOREM 5. A helicoidal surface 
\[(\phi(u) + av, f(u) \sinh v, f(u) \cosh v)\]
is locally isometric to the rotational surface 
\[\left( \begin{array}{c} -\int \sqrt{\frac{(a')^2 - (f')^2}{a^2 - f^2}} du \\ \sqrt{a^2 - f^2} \sinh \left( v + \int \frac{a'}{a^2 - f^2} du \right) \\ \sqrt{a^2 - f^2} \cosh \left( v + \int \frac{a'}{a^2 - f^2} du \right) \end{array} \right). \]

THEOREM 6. In Theorem 5, if these surfaces have the same Gauss map, then the surfaces have the forms follow 
\[\left( \begin{array}{c} \sqrt{1 - a^2} \arg \sinh \left( \frac{\sqrt{1 - a^2}}{1 - a^2} \right) \\ \sqrt{a^2 - f^2} \sinh \left( v + \int \frac{a'}{a^2 - f^2} du \right) \\ \sqrt{a^2 - f^2} \cosh \left( v + \int \frac{a'}{a^2 - f^2} du \right) \end{array} \right), \]
where 
\[\phi(u) = a \log \left( \frac{\sqrt{a^2 + b^2 - f^2} + \sqrt{a^2 + b^2 \sqrt{a^2 - f^2}}}{\sqrt{a^2 + b^2 - f^2} - \sqrt{a^2 + b^2 \sqrt{a^2 - f^2}}} \right) + \sqrt{a^2 + b^2} \log \left( \frac{\sqrt{a^2 - f^2 - \sqrt{a^2 + b^2 - f^2}}}{\sqrt{a^2 - f^2 + \sqrt{a^2 + b^2 - f^2}}} \right) + c, \]
\(0 < f^2 < a^2 f'^2/(f'^2 + \phi'^2)\).
(3.4) reduces to \(ds^2 = -du^2 + k^2(u)dv^2\), and the rotational surface has the form 
\[R(u, v) = (\phi_R(u), f_R(u) \sinh v, f_R(u) \cosh v). \]

So, we have the theorem as follows.

THEOREM 7. A helicoidal surface 
\[(\phi(u) + av, f(u) \sinh v, f(u) \cosh v)\]
is locally isometric to the rotational surface 
\[\left( \begin{array}{c} -\int \sqrt{\frac{(a')^2 - (f')^2}{a^2 - f^2}} du \\ \sqrt{a^2 - f^2} \sinh \left( v + \int \frac{a'}{a^2 - f^2} du \right) \\ \sqrt{a^2 - f^2} \cosh \left( v + \int \frac{a'}{a^2 - f^2} du \right) \end{array} \right). \]

THEOREM 8. In Theorem 7, if these surfaces have the same Gauss map, then the surfaces have the forms 
\[\left( \begin{array}{c} \sqrt{a^2 + 1} \arg \sinh \left( \frac{u}{\sqrt{a^2 + 1}} \right) \\ \sqrt{a^2 - f^2} \sinh \left( v + \int \frac{a'}{a^2 - f^2} du \right) \\ \sqrt{a^2 - f^2} \cosh \left( v + \int \frac{a'}{a^2 - f^2} du \right) \end{array} \right), \]
where

\[
\phi(u) = a \sqrt{b^2 - a^2} \log \left( \sqrt{\frac{a \sqrt{f^2 + b^2 - a^2} - \sqrt{b^2 - a^2} \sqrt{a^2 - f^2}}{a \sqrt{f^2 + b^2 - a^2} + \sqrt{b^2 - a^2} \sqrt{a^2 - f^2}}} + \sqrt{b^2 - a^2} \arctan \left( \frac{a^2 - f^2}{f^2 + b^2 - a^2} \right) + c, \right.
\]

**Corollary 2.** Let \( f(u) = u, \phi(u) = 0 \) in generalized Bour's theorem. Our findings agree with the results of the isometric surfaces in [8].

Next, we focus on the other cases as (S,S)-type, B, (S,T), (T,S) and (T,T)-types without proofs, since the techniques of the proofs similar.

**General Case 2. (S,S)-type, B.**

Let us assume that the profile curve is on the \( xz \)-plane.

**Theorem 9.** A helicoidal surface

\[
(\phi(u) + av, f(u) \sin v, f(u) \cosh v)
\]

is locally isometric to the rotational surface

\[
\begin{pmatrix}
\int \sqrt{\frac{\left(\frac{f^2}{f^2 + a^2}\right)^2}{f^2 + a^2}} \, du \\
\sqrt{f^2 + a^2} \sinh \left( v + \int \frac{a' \phi}{f^2 + a^2} \, du \right) \\
\sqrt{f^2 + a^2} \cosh \left( v + \int \frac{a' \phi}{f^2 + a^2} \, du \right)
\end{pmatrix}
\]

**Theorem 10.** In Theorem 9, if these surfaces have the same Gauss map, then the surfaces have the forms

(a) \( f^2 < a^2 f'^2 / (\phi'^2 - f'^2) \), timelike surface, \( ds^2 = -du^2 + k^2(u)dv^2 \):

\[
\begin{pmatrix}
\phi(u) + av \\
f(u) \sin v \\
f(u) \cosh v
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\sqrt{1 + a^2} \arcsin \left( \frac{\sqrt{f^2 + a^2}}{1 + a^2} \right) \\
\sqrt{f^2 + a^2} \sinh \left( v + \int \frac{a' \phi}{f^2 + a^2} \, du \right) \\
\sqrt{f^2 + a^2} \cosh \left( v + \int \frac{a' \phi}{f^2 + a^2} \, du \right)
\end{pmatrix}
\]

where

\[
\phi(u) = a \log \left( \sqrt{\frac{a \sqrt{f^2 + a^2 - b^2} - \sqrt{a^2 - b^2} \sqrt{f^2 + a^2}}{a \sqrt{f^2 + a^2 - b^2} + \sqrt{a^2 - b^2} \sqrt{f^2 + a^2}}} \right)
\]

(b) \( f^2 > a^2 f'^2 / (\phi'^2 - f'^2) \), spacelike surface, \( ds^2 = -du^2 + k^2(u)dv^2 \):

\[
\begin{pmatrix}
\phi(u) + av \\
f(u) \sin v \\
f(u) \cosh v
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\sqrt{1 + a^2} \arcsin \left( \frac{\sqrt{f^2 + a^2}}{1 + a^2} \right) \\
\sqrt{f^2 + a^2} \sinh \left( v + \int \frac{a' \phi}{f^2 + a^2} \, du \right) \\
\sqrt{f^2 + a^2} \cosh \left( v + \int \frac{a' \phi}{f^2 + a^2} \, du \right)
\end{pmatrix}
\]
where
\[
\phi(u) = a \log \left( \frac{a \sqrt{f^2 + a^2 - b^2} - \sqrt{a^2 - b^2} \sqrt{f^2 + a^2}}{a \sqrt{f^2 + a^2} - b^2 + \sqrt{a^2 - b^2} \sqrt{f^2 + a^2}} \right) \\
+ \sqrt{a^2 - b^2} \log \left( \frac{\sqrt{f^2 + a^2} + \sqrt{f^2 + a^2 - b^2}}{\sqrt{f^2 + a^2} - \sqrt{f^2 + a^2 - b^2}} \right) + c,
\]

**General Case 3. (S,T)-type.**
In this case, surfaces are timelike and \( ds^2 = -du^2 + k^2(u)dv^2 \).

**Theorem 11.** A helicoidal surface
\[
(\phi(u) + av, f(u) \sin v, f(u) \cosh v)
\]
is locally isometric to the rotational surface
\[
\left( \begin{array}{c}
\sqrt{f^2 + a^2} \sinh \left( v + \int \frac{a'}{f^2 + a^2} du \right) \\
\sqrt{f^2 + a^2} \cosh \left( v + \int \frac{a'}{f^2 + a^2} du \right)
\end{array} \right)
\]

**Theorem 12.** In Theorem 11, if these surfaces have the same Gauss map, then the surfaces have the forms
\[
\left( \begin{array}{c}
\phi(u) + av \\
f(u) \sin v \\
f(u) \cosh v
\end{array} \right) \quad \text{and} \quad \left( \begin{array}{c}
\sqrt{1 - a^2} \arg \sinh \left( \frac{\sqrt{f^2 + a^2}}{\sqrt{1 - a^2}} \right) \\
\sqrt{f^2 + a^2} \sinh \left( v + \int \frac{a'}{f^2 + a^2} du \right) \\
\sqrt{f^2 + a^2} \cosh \left( v + \int \frac{a'}{f^2 + a^2} du \right)
\end{array} \right),
\]

where
\[
\phi(u) = a \log \left( \frac{a \sqrt{f^2 + a^2 + b^2} - \sqrt{a^2 + b^2} \sqrt{f^2 + a^2}}{a \sqrt{f^2 + a^2} + b^2 + \sqrt{a^2 + b^2} \sqrt{f^2 + a^2}} \right) \\
+ \sqrt{a^2 - b^2} \log \left( \frac{\sqrt{f^2 + a^2 + b^2} + \sqrt{f^2 + a^2 + b^2}}{\sqrt{f^2 + a^2} - \sqrt{f^2 + a^2 + b^2}} \right) + c,
\]

**General Case 4. (T,S)-type.**

**Theorem 13.** A helicoidal surface
\[
(f(u) \cos v, f(u) \sin v, \phi(u) + av)
\]
is locally isometric to the rotational surface
\[
\left( \begin{array}{c}
\sqrt{f^2 - a^2} \cos \left( v - \int \frac{a'}{f^2 - a^2} du \right) \\
\sqrt{f^2 - a^2} \sin \left( v - \int \frac{a'}{f^2 - a^2} du \right) \\
\int \frac{(f')^2 - (a')^2}{f^2 - a^2} du
\end{array} \right), \quad a^2 < f^2
\]
or

\[
\begin{pmatrix}
-\int \sqrt{1 - \frac{f^2(1-(\phi')^2)}{a^2 - f^2}} \, du \\
\sqrt{a^2 - f^2} \cosh \left( v + \int \frac{a\phi'}{a^2 - f^2} \, du \right) \\
\sqrt{a^2 - f^2} \sinh \left( v + \int \frac{a\phi'}{a^2 - f^2} \, du \right)
\end{pmatrix}, \quad f^2 < a^2.
\]

When \( f^2 < a^2 \) the surfaces have different axes and the Gauss maps are not same.

**Theorem 14.** In Theorem 13, if these surfaces have the same Gauss map, then the surfaces have the forms

(a). \( f^2 > a^2 f^2 / (\phi'^2 - f^2) \), spacelike surface, \( ds^2 = du^2 + k^2(u)dv^2 \):

\[
\begin{pmatrix}
\phi(u) + av \\
f(u) \sinh v \\
f(u) \cosh v
\end{pmatrix}
\text{ and }
\begin{pmatrix}
\sqrt{f^2 - a^2} \cos \left( v + \int \frac{a\phi'}{f^2 - a^2} \, du \right) \\
\sqrt{f^2 - a^2} \sin \left( v + \int \frac{a\phi'}{f^2 - a^2} \, du \right) \\
\sqrt{1 + a^2 \arcsinh \left( \frac{\sqrt{f^2 - a^2}}{\sqrt{1+a^2}} \right)}
\end{pmatrix},
\]

where

\[
\phi(u) = -a \arctan \left( \frac{\sqrt{b^2 - a^2} \sqrt{f^2 - a^2}}{a \sqrt{f^2 + b^2 - a^2}} \right) + \sqrt{b^2 - a^2} \log \left( \frac{\sqrt{f^2 - a^2} - \sqrt{f^2 + b^2 - a^2}}{\sqrt{f^2 - a^2} + \sqrt{f^2 + b^2 - a^2}} \right) + c,
\]

(b). \( a^2 f^2 < f^2 < a^2 f^2 / (\phi'^2 - f^2) \), timelike surface, \( ds^2 = -du^2 + k^2(u)dv^2 \):

\[
\begin{pmatrix}
f(u) \cos v \\
f(u) \sin v \\
\phi(u) + av
\end{pmatrix}
\text{ and }
\begin{pmatrix}
\sqrt{f^2 - a^2} \cos \left( v - \int \frac{a\phi'}{f^2 - a^2} \, du \right) \\
\sqrt{f^2 - a^2} \sin \left( v - \int \frac{a\phi'}{f^2 - a^2} \, du \right) \\
\sqrt{1 - a^2 \arcsin \left( \frac{\sqrt{f^2 - a^2}}{\sqrt{1-a^2}} \right)}
\end{pmatrix},
\]

where

\[
\phi(u) = a \log \left( \frac{a \sqrt{f^2 - a^2} - b^2 - \sqrt{a^2 + b^2} \sqrt{f^2 - a^2}}{a \sqrt{f^2 - a^2} - b^2 + \sqrt{a^2 + b^2} \sqrt{f^2 - a^2}} \right) + \sqrt{a^2 - b^2} \log \left( \frac{\sqrt{f^2 - a^2} - \sqrt{f^2 - a^2 - b^2}}{\sqrt{f^2 - a^2} + \sqrt{f^2 - a^2 - b^2}} \right) + c,
\]

**General Case 5.** \((T,T)\)-type.

**Theorem 15.** A helicoidal surface

\((f(u) \cos v, f(u) \sin v, \phi(u) + av)\)
is locally isometric to the rotational surface

\[
\begin{pmatrix}
\sqrt{f^2 - a^2} \cos \left( v - \int \frac{a \phi'}{f^2 - a^2} du \right) \\
\sqrt{f^2 - a^2} \sin \left( v - \int \frac{a \phi'}{f^2 - a^2} du \right) \\
\int \sqrt{\frac{(f')^2 - (af')^2}{f^2 - a^2}} du
\end{pmatrix}, \quad a^2 < f^2
\]

or

\[
\begin{pmatrix}
\sqrt{a^2 - f^2} \cosh \left( v + \int \frac{a \phi'}{f^2 - a^2} du \right) \\
\sqrt{a^2 - f^2} \sinh \left( v + \int \frac{a \phi'}{f^2 - a^2} du \right) \\
\int \frac{1}{1 + \frac{1}{a^2 - f^2} \frac{(f')^2 - 1}{a^2 - f^2}} du
\end{pmatrix}, \quad f^2 < a^2.
\]

The surfaces are timelike, \( ds^2 = -du^2 + k^2(u)dv^2 \), where \( a^2 < f^2 \) (\( ds^2 = du^2 - k^2(u)dv^2 \), where \( f^2 < a^2 \)). The case \( f^2 < a^2 \) surfaces have different axes and the Gauss maps are not same.

**Theorem 16.** In Theorem 15, if these surfaces have the same Gauss map, then the surfaces have the forms

\[
(\phi(u) + av, f(u) \cos v, f(u) \sin v)
\]

and

\[
\begin{pmatrix}
\phi(u) + av \\
f(u) \cos v \\
f(u) \sin v
\end{pmatrix}
\]

\[
\begin{pmatrix}
\sqrt{f^2 - a^2} \cos \left( v - \int \frac{a \phi'}{f^2 - a^2} du \right) \\
\sqrt{f^2 - a^2} \sin \left( v - \int \frac{a \phi'}{f^2 - a^2} du \right) \\
\sqrt{1 + a^2} \arctan \left( \frac{\sqrt{f^2 - a^2}}{\sqrt{1 + a^2}} \right)
\end{pmatrix},
\]

where \( a^2 < f^2 \),

\[
\phi(u) = -a \arctan \left( \frac{\sqrt{b^2 - a^2} \sqrt{f^2 - a^2}}{a \sqrt{f^2 + b^2 - a^2}} \right)
\]

\[- \sqrt{b^2 - a^2} \log \left( \frac{\sqrt{f^2 - a^2} + \sqrt{f^2 + b^2 - a^2}}{\sqrt{f^2 - a^2} - \sqrt{f^2 + b^2 - a^2}} \right) + c.
\]

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