The quantum behavior of general time dependent quadratic systems linearly coupled to a bath

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Abstract

In this paper we solve for the quantum propagator of a general time dependent system quadratic in both position and momentum, linearly coupled to an infinite bath of harmonic oscillators. We work in the regime where the quantum optical master equation is valid. We map this master equation to a Schroedinger equation on Super-Hilbert space and utilize Lie Algebraic techniques to solve for the dynamics in this space. We then map back to the original Hilbert space to obtain the solution of the quantum dynamics. The Lie Algebraic techniques used are preferable to the standard Wei-Norman methods in that only coupled systems of first order ordinary differential equations and purely algebraic equations need only be solved. We look at two examples.
I. INTRODUCTION

In this note we describe the solution of the quantum optical master equation for a generally time dependent system at most quadratic in position and momentum. The coupling to the bath is taken to be a linear and the bath is comprised of harmonic oscillators. In the optical regime we can write an approximate quantum master equation \[1\]

\[
\dot{\rho} = -\frac{i}{\hbar}[H, \rho] + \Lambda \rho ,
\]

(1.1)

\[
\Lambda \rho = \frac{\gamma(\bar{n} + 1)}{2} \left\{ [a, \rho a^\dagger] + [a \rho, a^\dagger] \right\} + \frac{\gamma \bar{n}}{2} \left\{ [a^\dagger, \rho a] + [a^\dagger \rho, a] \right\} ,
\]

(1.2)

where \(\gamma\) is the strength of the coupling to the bath and \(\bar{n} = \exp(-\beta\hbar\omega) - 1\) is the equilibrium number of photons in the bath. The approximations necessary to obtain equation (1.1) are: rotating wave approximation (RWA), small system anharmonicity, Markovian evolution, weak coupling and initially no correlations between the system and bath. It is generally accepted that equation (1.1) describes well the physics near optical frequencies and forms the basis of many models studied in quantum optics \[1,2\] (and references therein).

We take as our system the Hamiltonian

\[
H/\hbar = \omega(t) a^\dagger a + f_1(t) a + \bar{f}_1(t) a^\dagger + f_2(t) a^2 + \bar{f}_2(t) a^\dagger a^\dagger ,
\]

(1.3)

where \(\omega, f_1, f_2, \bar{f}_1, \bar{f}_2\) are arbitrary functions of time. We will assume \(\omega \in \mathbb{R}\) but will not necessarily assume that \(f_1^* = \bar{f}_1\) and \(f_2^* = \bar{f}_2\). This is to accommodate the phenomenological modeling of dissipation through a non-unitary Hamiltonian \[3\]. The general form of the system (1.3) covers many specific models including the simple harmonic oscillator, the parametric oscillator, the forced harmonic oscillator and the degenerate parametric oscillator.

The isolated system also exhibits many families of coherent states (a la Perelomov \[4\]) for specific functional forms of \(\omega, f_i\) and \(\bar{f}_i\). It is of interest to study the effects of the bath on these coherent states. In particular, in the absence of the bath, the coherent state evolves solitonically under the action of a Hamiltonian which is a linear combination of the generators of the Lie group defining the coherent state \[5\].
that it follows the exact classical equations of motion forever. However, since coherent states still display quantum interference it is incorrect to consider such states as truly classical. Much work has been done recently on the decoherence effects of an external bath and the suppression of quantum interference in the system [8]. More important is the study of the effects of the bath on quantum systems which, when isolated from the bath, do not possess coherent states. Does the interaction with the bath succeed in retarding the spread of an initial wave packet? ie. can one interpret the bath as effectively measuring the system and “collapsing” the wave function of the system? Does the interaction with the bath cause a deviation from the quantal equations of motion towards the classical equations of motion? As a first step towards answering these important questions one must solve for the quantum behavior of the system coupled to the bath. We do so in the quantum optical limit for a general system linearly coupled to the bath.

The standard method used to solve equation (1.1) is to transform the master equation into a c–numbered partial differential equation for a particular quantum distribution function (QDF). However, in certain important cases (degenerate parametric oscillator in the P representation), the resulting PDE can take the form of a Fokker-Planck equation with a negative diffusion matrix. Such cases do not correspond to the usual diffusive process. To use the machinery associated with diffusive processes one must make use of the generalized P–Representation [7]. This entails doubling the degrees of freedom and “choosing a gauge” to ensure that the diffusion matrix is positive and real. However, recent numerical work using this QDF has raised questions concerning it’s validity [8]. Of course, for system non-linearities higher than quadratic one has a c–numbered PDE of order three or more. Since this does not occur in this model we will not address this here. Resolving (1.2) onto the $x, p$ basis via $a = (\omega x + ip)/\sqrt{2\hbar\omega}$ we get

$$4\hbar\omega A\rho/\gamma = -\omega^2(2n + 1)}[x, [x, \rho]] - (2n + 1)[p, [p, \rho]]$$
$$+ i\omega[p, \rho x + xp] - i\omega[x, \rho p + pp]\ ,$$

(1.4)

while resolving (1.4) onto the Glauber-Cahill family of QDFs $W(\alpha, \alpha^*, s)$, which include the
P (s=+1), Wigner (s=0) and Q (s=-1) quantum distribution functions we obtain

\[ \dot{W}(\alpha, \alpha^*, s) = -\frac{i}{\hbar} \{ [f_1 + 2\alpha f_2 + \alpha^* \omega] \partial_{\alpha^*} - [\bar{f}_1 + 2\alpha^* \bar{f}_2 + \alpha \omega] \partial_\alpha - s[f_2 - \bar{f}_2] \partial^2_{\alpha \alpha^*} \} W(\alpha, \alpha^*, s) \]

\[ + \frac{\gamma}{2} \{ \partial_\alpha \alpha + \partial_{\alpha^*} \alpha^* + (2\bar{n} + 1 - s) \partial^2_{\alpha \alpha^*} \} W(\alpha, \alpha^*, s) . \]  

(1.5)

From (1.4) we see that we now are effectively coupled to noise in both position and momentum. We now convert equation (1.1) into a Schroedinger equation on Super–Hilbert space.

II. SUPER–HILBERT SPACE AND LIE ALGEBRAIC SOLUTIONS

We initially follow a technique developed by Barnett and Knight in association with the applications of Thermofield theory to quantum optics [9]. In this theory, the density operator is transformed into a state vector in an expanded Hilbert space (Super-Hilbert space) and operations on operators become super-operators in the Super-Hilbert space. The transformation between Hilbert space and Super-Hilbert space is accomplished through the state

\[ |I\rangle \equiv \sum_N |N, N\rangle \equiv \sum_N |N\rangle_1 \times |N\rangle_2 , \]

(2.1)

and through defining \(|\rho\rangle = \rho |I\rangle\). One can show \(a|I\rangle = \bar{a}^\dagger |I\rangle\) and \(a^\dagger |I\rangle = \bar{a} |I\rangle\) where the \(a\)’s are the annihilation operators in Hilbert space 1 (the original space) and the \(\bar{a}\)’s are the annihilation operators in Hilbert space 2. Using (2.1), equation (1.1) becomes a Schroedinger like equation on the Super-Hilbert space

\[ \frac{d}{dt} |\rho\rangle = -i \tilde{H} |\rho\rangle , \]

(2.2)

where

\[ -i \tilde{H} = -i \{ \omega [a^\dagger a - \bar{a}^\dagger \bar{a}] + f_1 [a - \bar{a}^\dagger] + \bar{f}_1 [a^\dagger - \bar{a}] + f_2 [a^2 - \bar{a}^\dagger] + \bar{f}_2 [a^\dagger 2 - \bar{a}^\dagger 2] \}

\[ + \frac{\gamma}{2} (\bar{n} + 1) [2a\bar{a} - a^\dagger a - \bar{a}^\dagger \bar{a}] + \frac{\gamma}{2} \bar{n} [2a^\dagger \bar{a}^\dagger - aa^\dagger - \bar{a}\bar{a}^\dagger] . \]

(2.3)
The “Super-Hamiltonian”, \( \tilde{H} \) seems at first glance totally intractable, however, upon inspection one can rewrite (2.2) as

\[
\frac{d}{dt} U(t) = -iH(t) U(t) \quad |\rho(t)\rangle = U(t)|\rho(0)\rangle ,
\]

(2.4)

\[
H(t) = \sum_{i=0}^{N} h_i(t) N_i ,
\]

(2.5)

where the \( N_i \) are the generators of an \( N \) dimensional Lie algebra \( \mathcal{L} \). We therefore have access to powerful Lie Algebraic techniques to solve (2.4) for the propagator \( U(t) \).

The algebra \( \mathcal{L} \) can be identified as the two photon subgroup of \( \text{Sp}(6,\mathbb{R}) \). This subgroup is a 15 dimensional, semisimple Lie algebra, possessing a five dimensional ideal \( \mathcal{I} \), \( \{N_0, N_1, N_2, N_3, N_4\} \) and compact \( U(2) \) subgroup \( \{N_8, N_9, N_{13}, N_{14}\} \). It also contains the single photon algebra \( h_4 \) and Weyl–Heisenberg algebra \( h_3 \) for both sets of oscillators. This algebra has been studied in detail by Gilmore and Yuan \[10\]. The description of the generators \( N_i \) and coefficient functions \( h_i(t) \) are given in Table I and Table II.

The standard Lie Algebraic method used to solve equations such as (2.4) is the Wei–Norman method \[11\]. In this method one assumes a particular form for the propagator as an ordered product of exponentials of the generators

\[
U(t) = \prod_{i=0}^{N} \exp(g_i(t) N_i) ,
\]

(2.6)

One then substitutes this ansatz into (2.4) to obtain differential equations relating \( g_i \) to \( h_i \). However, the resulting equations are highly non-linear and coupled. The complexity of these equations and the effort needed to obtain them grows rapidly with the dimension of the Lie group and proves prohibitive for semisimple Lie groups with dimensions greater than six. Even if one succeeds in solving the coupled ODEs for the \( g_i \)'s the global validity of the chosen ansatz is not guaranteed and must be tested through examination of the \( g_i \)'s \[12\]. Finally, the whole procedure is highly sensitive to the particular ordering of generators chosen in the ansatz for \( U(t) \). We will, instead, adopt a method discovered by Fernandez \[13\], which can reduce the problem to the solution of a coupled set of linear ODEs and
algebraic equations. Key to this approach is the existence of a suitably large proper ideal $\mathcal{I}$. One effectively solves for the dynamics of the ideal and “lifts” this information via Baker–Campbell–Hausdorff (BCH) identities to determine the dynamics of the entire algebra.

For any element of the proper ideal $N_i \in \mathcal{I} \ (i \in \{0, \ldots, 4\})$ we have $U^{-1}\mathcal{I}U \in \mathcal{I}$ where $U$ is the propagator in (2.4). In particular

$$N_i(t) \equiv U^{-1}(t)N_i U(t) = \sum_{j=0}^{4} u_{ij}(t) N_j \ , \quad i \in \{0, \ldots, 4\} \ .$$

(2.7)

It is easy to show that the time dependent coefficients $u_{ij}(t)$ obey

$$\dot{u} = i\mathcal{H} \cdot u \ , \quad \mathcal{H}_{ij}(t) = \sum_{m=1}^{N} h_m(t) C_{mj}^{k} \ , \quad u_{ij}(t = 0) = \delta_{ij} \ ,$$

(2.8)

where $C_{ij}^{k}$ are the structure functions of $\mathcal{L}$. To solve for $u_{ij}(t)$ we must solve the linear set of coupled ordinary differential equations (2.8). By transforming to a new set of generators for $\mathcal{L}$ we can simplify the structure of (2.8) greatly. We will denote this new set by $\{W_i\}$. This change of basis decouples (2.8) into pairs of coupled linear first order differential equations and also greatly simplifies the BCH disentangling for the relations between the $u_{ij}$’s and the coefficient functions $g_i(t)$ appearing in (2.6). To obtain the time dependent coefficients $g_i(t)$ appearing in (2.6) we use BCH disentangling identities to compute the functionals $F_j(g_i)$ appearing in

$$U^{-1}(t)N_i U(t) = \sum_{j=0}^{4} F_j(g_i) N_j \ .$$

(2.9)

The $F_j$’s are algebraic functions of the $g_i$’s. Equating the coefficients of $N_j$ in (2.7) and (2.9) for a particular $N_i$ yields algebraic relations between the $g_i$’s and the $u_{ij}$’s. To check the global validity of the particular ansatz chosen one must only determine when the algebraic relations become degenerate.

Before we solve for the dynamics in the extended Super–Hilbert space we show first how to recover the dynamics of the density operator in the original Hilbert space. Resolving the density operator into number states gives

$$\rho = \sum_{nm} \rho_{nm} |n\rangle\langle m| \ , \quad \rho_{nm} = \langle n|\rho|m\rangle \ .$$

(2.10)
We obtain the density state vector in the extended Hilbert space through

$$|\rho\rangle \equiv \rho |I\rangle = \sum_{nm} \rho_{nm} |n\rangle \times |m\rangle = \sum_{nm} \rho_{nm} |n, m\rangle . \quad (2.11)$$

Thus, the original density operator may be recovered through

$$\rho = \sum_{nm} \langle m, n| \rho |n\rangle \langle m | . \quad (2.12)$$

Through similar steps one can also recover the components of the original density operator in the coherent state basis. We resolve unity in the coherent state basis through

$$I = \frac{1}{\pi} \int d^2 \alpha |\alpha\rangle \langle \alpha|$$

and using the identity

$$\sum_{n=0}^{\infty} \langle \beta|n\rangle |n\rangle = |\beta^*\rangle , \quad (2.13)$$

we obtain

$$|\rho\rangle = \frac{1}{\pi^2} \int d^2 \alpha d^2 \beta \rho (\alpha^*, \beta) e^{-|(\alpha|+|\beta|)/2} |\alpha, \beta^*\rangle , \quad (2.14)$$

where we have set $|\alpha, \beta\rangle \equiv |\alpha\rangle \times |\beta\rangle$ and $\rho (\alpha^*, \beta) = \langle \alpha|\rho|\beta\rangle$. One can easily show

$$\langle \alpha|\rho|\beta\rangle = \langle \beta^*, \alpha|\rho\rangle , \quad (2.15)$$

as expected. From (2.15) we can obtain the Q distribution function

$$Q(\alpha^*, \alpha, t) = \langle \alpha|\rho(t)|\alpha\rangle = \langle \alpha^*, \alpha|\rho(t)\rangle . \quad (2.16)$$

However, in the extended Hilbert space, $|\rho(t)\rangle = U(t)|\rho(0)\rangle$. Using this with equations (2.14) and (2.16) we finally obtain

$$Q(\alpha^*, \alpha, t) =$$

$$\frac{1}{\pi^2} \int d^2 \alpha d^2 \beta \rho_0 (\tilde{\alpha}^*, \tilde{\beta}) e^{-|(\alpha|+|\beta|)/2} \langle \tilde{\alpha}^*, \tilde{\alpha}|U(t)|\alpha, \beta^*\rangle . \quad (2.17)$$

In the examples treated in this paper the coherent state basis proves more useful than the number basis. Thus to calculate the Q function it will be necessary to compute

$$\langle \tilde{\alpha}^*, \tilde{\alpha}|U(t)|\alpha, \beta^*\rangle , \quad (2.18)$$
for various propagators $U(t)$. In the systems treated, $U$ will be at most quadratic in the annihilation and creation operators of the two Hilbert spaces. Writing

$$|\alpha^*, \beta\rangle = e^{-|\alpha|/2-|\beta|/2} e^{\alpha a^\dagger+\beta^* \tilde{a}^\dagger} |0,0\rangle,$$

(2.19)
equation (2.18) becomes $\langle 0,0|\tilde{U}(t)|0,0\rangle$ where

$$\tilde{U}(t) = e^{-|\tilde{\alpha}|-(|\alpha|+|\beta|)/2} e^{\tilde{\alpha}^* a+\tilde{\alpha} \tilde{a}} U(t) e^{\alpha a^\dagger+\beta^* \tilde{a}^\dagger}.$$

(2.20)

To evaluate the above we use BCH identities to normal order the operator $\tilde{U}$ as

$$\tilde{U}(t) = e^{R_{ij} a^i_{\dagger} a^j_{\dagger} + c_i a^i_{\dagger}} e^{D_{ij} (\alpha_{\dagger} \alpha_{\dagger}/2) + \eta I} e^{L_{ij} \alpha_{\dagger} \alpha_{\dagger} + \lambda_{ij} a_{\dagger} a_{\dagger}},$$

(2.21)

where the index $i$ labels the two Hilbert spaces. The vacuum expectation value of $\tilde{U}$ becomes trivial and yields

$$\langle 0,0|\tilde{U}(t)|0,0\rangle = \exp(D_{11} + D_{22}/2 + \eta).$$

(2.22)

### III. SOLVING THE DYNAMICS

In Table III we show the relation between $N_i$ and $W_i$ and in Table IV we give the commutation table for the ideal $I$. From the $\{W_i\}$ commutation table we can construct the “Hamiltonian” $H$ in (2.8)

$$H_{ij} = \begin{bmatrix}
0, & 0, & 0, & 0, & 0, & 0, \\
-2h_4, & -h_8/4 - h_9/2, & -2h_6, & 0, & 0, \\
-2h_3, & -2h_5, & -h_8/4 + h_9/2, & 0, & 0, \\
2h_2, & 2h_7, & 0, & h_8/4 - h_9/2, & 2h_6, \\
2h_1, & 0, & 2h_7, & 2h_5, & h_8/4 + h_9/2
\end{bmatrix}.$$

(3.1)

We see that the degree of coupling within (2.8) is related to the number of nonzero entries in each row of $H_{ij}$. This is in turn related to the number of different $W_i$’s appearing in each row of the commutation table of the ideal $I$. From the initial condition $u_{ij}(t=0) = \delta_{ij}$ and the structure of $H_{ij}$ we immediately obtain
\[ u_{i0} = \delta_{i0}, \quad u_{13} = u_{23} = u_{14} = u_{24} = 0 \quad \forall t. \quad (3.2) \]

Thus the set (2.8) of ODE's for \( u_{1i} \) and \( u_{2i} \) can be re-cast as

\[ \dot{X}_1 = iAX_1 + iB, \quad (3.3) \]
\[ \dot{X}_2 = iCX_2 + iD, \quad (3.4) \]

where

\[ X_1 \equiv \begin{pmatrix} u_{10} & u_{11} & u_{12} \\ u_{20} & u_{21} & u_{22} \end{pmatrix}, \quad (3.5) \]
\[ X_2 \equiv \begin{pmatrix} u_{33} & u_{34} & u_{31} & u_{32} & u_{30} \\ u_{43} & u_{44} & u_{41} & u_{42} & u_{40} \end{pmatrix}, \quad (3.6) \]

\[ A \equiv \begin{pmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} \\ \mathcal{H}_{21} & \mathcal{H}_{22} \end{pmatrix} = \begin{pmatrix} -i\gamma/2 - \omega, & 2\bar{f}_2 \\ -2f_2, & -i\gamma/2 + \omega \end{pmatrix}, \quad (3.7) \]
\[ C \equiv \begin{pmatrix} \mathcal{H}_{33} & \mathcal{H}_{34} \\ \mathcal{H}_{21} & \mathcal{H}_{44} \end{pmatrix} = \begin{pmatrix} -i\gamma/2 - \omega, & 2\bar{f}_2 \\ -2f_2, & i\gamma/2 + \omega \end{pmatrix}, \quad (3.8) \]
\[ B \equiv \begin{pmatrix} \mathcal{H}_{10} & 0 & 0 \\ \mathcal{H}_{20} & 0 & 0 \end{pmatrix}, \quad (3.9) \]
\[ D \equiv \begin{pmatrix} 0 & 0 & \mathcal{H}_{31}u_{11} & \mathcal{H}_{31}u_{12} & \mathcal{H}_{30} \\ 0 & 0 & \mathcal{H}_{42}u_{21} & \mathcal{H}_{42}u_{22} & \mathcal{H}_{40} \end{pmatrix}. \quad (3.10) \]

From (2.8), the initial value of \( X_1 \) and \( X_2 \) are

\[ X_1(t = 0) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (3.11) \]
\[ X_2(t = 0) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad (3.12) \]
However, from Table III, \( h_4 = h_3 = 0 \) and thus \( B \) vanishes identically. Equation (3.3) then gives

\[ u_{10} = u_{20} = 0 \quad \forall t \, . \quad (3.13) \]

Note, however, that equation (3.3) must already be solved to correctly specify \( D \) in (3.4). The solution of equations (3.3, 3.4) together with (3.2) completely determine the dynamical evolution of the \( u_{ij}(t) \).

To finally compute the propagator we must calculate the algebraic relations between the \( g_i(t) \) and \( u_{ij}(t) \). We must evaluate the action of the group on each of the elements of the ideal, that is, we must calculate

\[ \exp(-g_i \text{ ad } W_i) W_j \equiv e^{-g_i W_i} W_j e^{g_i W_i} , \quad (3.14) \]

for \( i \in (0, \ldots, 14) \), \( j \in (0, \ldots, 4) \). This is achieved using Baker–Campbell–Hausdorff disentangling identities for Lie Groups. The relevant identities are given in Appendix 1. Tables \( \mathbb{V} \) and \( \mathbb{VI} \) summarize the action of the group on the elements of the ideal. The particular set of algebraic relations obtained will depend sensitively on the choice of ordering of the generators in the ansatz for \( U(t) \). We choose the ordering

\[ U(t) = \prod_i \exp(g_i(t) W_i) \, , \quad (3.15) \]

where \( \prod_i \) is the product ordered from the left in the sequence 1, 2, 13, 5, 3, 11, 7, 8, 9, 14, 10, 6, 12, 4, 0. With this ordering the six conditions (3.2, 3.13) give

\[ g_4 = g_3 = g_{14} = 0 \, , \quad g_{10} = -2g_{12} \, , \quad g_{11} = -2g_{13} \, . \quad (3.16) \]

The very complicated disentangling of \( W_{14} \) is thus avoided as \( g_{14} = 0 \) with this choice of ordering. Working through the BCH identities with a symbolic manipulator we finally arrive at the relations in Table \( \mathbb{VII} \).

The relations are analytic except when \( u_{11} = 0 \) or \( u_{33} = 0 \). Thus the ansatz (3.13) is global if both \( u_{11} \neq 0 \) and \( u_{33} \neq 0 \) for all \( t \). Except for \( g_0(t) \), the propagator is formally
determined. To solve for \( g_0(t) \) however, is not trivial. It cannot be determined through BCH identities as \( W_0 \) commutes with every element of the group. Instead we must substitute (3.15) into the “Schroedinger” equation \( H = i\dot{U}U^{-1} \) and from this determine \( g_0 \). In [13] the resulting relation (just below equation (20) in [13]) is quite simple. However, for the much larger algebra \( \text{Sp}(6,\mathbb{r}) \) the resulting relation can be quite complicated. To compute the above “Schroedinger” equation and obtain the general relation for \( g_0 \) it is expedient to use the faithful matrix representation of \( \text{Sp}(6,\mathbb{r}) \) given by Gilmore and Yuan [10]. We will not give the general formula but will compute it case by case.

IV. EXAMPLES

In this section we apply the above methods to solve two model systems. The first is the simple harmonic oscillator. We set \( \omega(t) \equiv \omega \) and \( f_1 = f_2 = \bar{f}_1 = \bar{f}_2 = 0 \). We compute the matrix \( H_{ij} \) in (2.8) and construct the matrices \( A, B \) and \( C \). We can easily solve the coupled sets of ODEs for \( X_1 \) and \( X_2 \) to find the non-vanishing \( u_{ij} \) to be

\[
\begin{align*}
u_{11} &= u_{22}^* = u_{33}^{-1} = u_{44}^{-1} = e^{(-\gamma/2+i\omega)t}, \\
u_{31} &= -2(2\bar{n} + 1)e^{-i\omega t} \sinh \frac{\gamma t}{2}, \\
u_{42} &= e^{2i\omega t} u_{31}. \quad (4.1)
\end{align*}
\]

The resulting non-zero \( g \)’s are

\[
\begin{align*}
g_7 &= (2\bar{n} + 1)e^{-\gamma/2t} \sinh \frac{\gamma t}{2}, \quad g_8 = 2\gamma t, \quad g_9 = -2i\omega t. \quad (4.3)
\end{align*}
\]

The propagator on the extended Hilbert space is \( U(t) = e^{g_7W_7}e^{g_8W_8}e^{g_9W_9}e^{g_9W_0}. \) This may be normal ordered in terms of \( \text{SU}(1,1) \) generators [13],

\[
\begin{align*}
K_+ &= a^\dagger \tilde{a}^\dagger, \\
K_- &= a\tilde{a}, \\
K_3 &= (a^\dagger a + \tilde{a}\tilde{a})/2.
\end{align*}
\]
\[ K_0 = a^\dagger a - \bar{a}^\dagger \bar{a}, \]
to give

\[ U(t) = \exp(xK_+) \exp(yK_3 + g_9/2K_0) \exp(xK_-) \exp(g_0W_0) \]

where

\[ x = \frac{g_7 e^{g_8/4} - \sinh g_8/4}{g_7 e^{g_8/4} + \cosh g_8/4}, \quad (4.4) \]
\[ y = -2 \ln \left[ g_7 e^{g_8/4} + \cosh g_8/4 \right], \quad (4.5) \]
\[ z = \frac{g_7 e^{g_8/4} + \sinh g_8/4}{g_7 e^{g_8/4} + \cosh g_8/4}. \quad (4.6) \]

To obtain \( g_0 \) we compute \( \mathcal{H} = i\dot{U}U^{-1} \) to find \( \dot{g}_0 = \gamma/2 \). With \( U \) normal ordered and the identity \[2\],

\[ [\exp(-x a^\dagger a)]_{\text{normal}} = \sum_{l=0}^{\infty} \frac{(e^{-x} - 1)^l}{l!} a^\dagger a^l, \]

it is a simple matter to compute \[2.18\],

\[ \langle \tilde{\beta}^*, \tilde{\alpha}|U(t)\langle \alpha, \beta^* \rangle = \frac{1}{\kappa} \langle \tilde{\alpha}|\tilde{\beta}\rangle \langle \beta|\alpha \rangle \exp \left[ -\frac{1}{\kappa} (\tilde{\alpha} - \beta \chi)^*(\tilde{\beta} - \alpha \chi) \right], \quad (4.7) \]

where \( \kappa \equiv 1 + \bar{n}(1 - e^{-\gamma t}) \) and \( \chi \equiv \exp(-\gamma/2 + i\omega)t \). This result is well known and corresponds to the off-diagonal elements of the density matrix in the coherent state basis in the presence of a bath where \( \rho(t = 0) = |\alpha\rangle \langle \beta| \).

For a second example we take \( \omega(t) = \omega_0(1 + A \sin \eta t), f_1 = i e^{i\omega_0 t}D/2, \tilde{f}_1 = f_1^* \) and \( f_2 = \tilde{f}_2 = 0 \). This corresponds to a variable frequency oscillator with a driving force. One again proceeds as before, however, the ODE’s are slightly more complicated due to the forcing term. The non-vanishing \( g \)’s are

\[ g_1 = \frac{D/2(\exp(i\omega_0 t) - \exp(-\gamma_+ t - \epsilon(\cos \eta \eta - 1)))}{\gamma/2 - i\delta \omega + \frac{\epsilon}{\eta} \sin \eta t}, \quad (4.8) \]
\[ g_2 = \frac{D/2(\exp(-i\omega_0 t) - \exp(-\gamma_+ t + \epsilon(\cos \eta - 1)t))}{\gamma/2 + i\delta \omega - \frac{\epsilon}{\eta} \sin \eta t}, \quad (4.9) \]
\[ g_7 = (2n + 1)e^{-\gamma t/2} \sinh \frac{\gamma t}{2}, \]  
\[ g_8 = 2\gamma t, \]  
\[ g_9 = -2i\omega_0 t + 2\epsilon(\cos \eta t - 1), \]  
where \( \gamma^\pm = \gamma/2 \pm i\omega_0, \) \( \delta \omega = \omega_0 - \omega_d \) and \( \epsilon = i\omega_0 A/\eta. \) The propagator on the extended Hilbert-space is now

\[ U(t) = e^{g_1 W_1} e^{g_2 W_2} e^{g_3 W_3} e^{g_4 W_4} e^{g_5 W_5} e^{g_6 W_6}. \]

We again find \( g_0 = \gamma/2. \) To compute the result analogous to (4.7) we normal order the \( W_7, W_8 \) and \( W_9 \) exponentials and insert a resolution of unity in the Super–Hilbert space between \( e^{g_2 W_2} \) and \( e^{xK^+}. \) To complete the calculation we perform the coherent state integral over the introduced partition of unity using

\[ \int d^2 \alpha \frac{\pi}{e^{-A^*\alpha + B\alpha + C^*}} = \frac{1}{A} e^{BC}. \]

The final result is

\[ \langle \tilde{\beta}^*, \tilde{\alpha} | U(t) | \alpha, \beta^* \rangle = \]

\[ \frac{1}{\kappa} \langle \tilde{\alpha} | \tilde{\beta} \rangle \langle \beta | \alpha \rangle \exp \left[ -\frac{1}{\kappa} (g_1 + \tilde{\alpha} - \beta \chi)^*(g_2 + \tilde{\beta} - \alpha \chi) \right], \]  

where

\[ \ln \chi = -\gamma t/2 + g_9/2 = -\gamma t/2 - i\omega_0(t - \frac{A}{\eta}(\cos \eta t - 1)) \]
.

In the latter model \( u_{11} = \exp(\gamma - \epsilon[\cos \eta t - 1]) \) and \( u_{33} = \exp(-\gamma + \epsilon[\cos \eta t - 1]). \) Since \( |u_{11}| \neq 0 \) and \( u_{33} \neq 0 \) for all \( \epsilon \) and finite \( t \) the resulting propagator in both models is global. This method can thus treat complicated time dependencies. It can also be applied to very particular nonlinear systems [16].

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APPENDIX A: BCH IDENTITIES

In this Appendix we derive the necessary BCH identities to disentangle the action of the propagator \( U(t) \) in (3.13) on the elements \( N_i \) of the ideal \( \mathcal{I}. \)
From \( [17] \) we have the expansion
\[
e^{xA} B e^{-xB} = B + x [A, B] + \frac{x^2}{2!} [A, [A, B]] + \cdots ,
\] (A1)
where \( x \) is a c–number and \( A \) and \( B \) are elements of a Lie group. For the case where the commutator of \( A \) and \( B \) is a c–number we are left with the first two terms in the expansion (A1). For the simplest non–trivial case where the commutation of \( A \) with \( B \) closes onto \( B \) itself, ie., \([A, B] = mB\) one can show
\[
e^{x \text{ad} A}B = B e^{mx} .
\] (A2)
For the more complicated example of \( \exp(-g_{14} \text{ad} W_{14}) W_i \) the commutation does not close on the first iteration. Here we use Wilkox’s method of parameter differentiation [18].

Letting
\[
G(x) \equiv e^{xA} B e^{-xA} ,
\] (A3)
with
\[
[A, B] = m_1 B + m_2 C , \quad [A, C] = m_3 B + m_4 C ,
\] (A4)
we can obtain
\[
G'(x) \equiv \frac{dG}{dx} = [A, G(x)] .
\] (A5)
We now assume that \( G(x) \) is of the form \( G(x) = a(x)A + b(x)B + c(x)C \). With the initial condition \( G(x = 0) = B \) we see that \( a(x = 0) = c(x = 0) = 0 \) and \( b(x = 0) = 1 \). Substituting this ansatz for \( G(x) \) into (A5) we obtain differential equations for \( a(x) \), \( b(x) \), and \( c(x) \). Immediately we get \( a(0) = 0 \) and the coupled set
\[
\begin{pmatrix} b \\ c \end{pmatrix} ' = \begin{pmatrix} m_1 & m_3 \\ m_2 & m_4 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} .
\] (A6)
For the particular example \( \exp(-g_{14} \text{ad} W_{14}) W_1 \) the values for the \( m_i \) can be read off Table IV and yield \( m_1 = -1/2, m_2 = -1, m_3 = 2, m_4 = 1/2 \). Inserting these values into (A6) and solving with the above initial conditions, gives
\[ e^{-g_{14}\text{ad}W_{14}} W_1 = \left[ \cos \frac{g_{14}}{\chi} + \frac{\chi}{2} \sin \frac{g_{14}}{\chi} \right] W_1 + \left[ \chi \sin \frac{g_{14}}{\chi} \right] W_3, \quad (A7) \]

where \( \chi = 2/\sqrt{7} \). The results of the action of \( W_{14} \) on the elements of the ideal are given in Table VI.
REFERENCES

[1] C. W. Gardiner, *Quantum Noise* (Springer–Verlag, Berlin, 1991).

[2] W. H. Louisell, *Quantum Statistical Properties of Radiation* (Wiley–Interscience, New York, 1990).

[3] A. T. G. Dattoli and R. Mignani, Phys. Rev. A **42**, 1467 (1990).

[4] A. M. Perelomov, *Generalised Coherent States and their Applications* (Springer–Verlag, Berlin, 1986).

[5] C. C. Gerry, Phys. Rev. A **31**, 2721 (1985).

[6] W. H. Zurek, Physics Today **Oct**, 36 (1991).

[7] P. D. Drummond and C. W. Gardiner, J. Phys. A **13**, 2353 (1980).

[8] A. M. Smith and C. W. Gardiner, Phys. Rev. A **39**, 3511 (1989).

[9] S. M. Barnett and P. L. Knight, J. Opt. Soc. Am. B. **2**, 467 (1985).

[10] R. Gilmore and J. Yuan, J. Chem. Phys. **91**, 917 (1989), note: there is a sign error in Table 1 line 2. It should read $M_{i,-j} + M_{j,-i}$.

[11] J. Wei and E. Norman, J. Math. Phys. **4**, 575 (1963).

[12] J. Wei and E. Norman, Proc. Am. Math. Soc. **15**, 327 (1963).

[13] F. M. Fernandez, Phys. Rev. A **40**, 41 (1989).

[14] M. O. S. M. Hillery, R. F. O’Connell and E. P. Wigner, Phy. Rep. **106**, 121 (1984).

[15] K. Wodkiewicz and J. Eberly, J. Opt. Soc. Am. B **2**, 485 (1985).

[16] S. Chaturvedi and V. Srinivasan, J. Mod. Opt. **38**, 777 (1991).

[17] J. C. G. G. Dattoli and A. Torre, Riv. Nuovo Cimento **11**, 1 (1988).

[18] R. M. Wilkox, J. Math. Phys. **8**, 962 (1967).
TABLES

TABLE I. We give the realisation of the two-photon subalgebra of $\text{Sp}(6,r)$ in terms of creation and annihilation operators $a^\dagger, \tilde{a}^\dagger, a, \tilde{a}$.

| $N_0$ | $N_1$ | $N_2$ | $N_3$ | $N_4$ | $N_5$ | $N_6$ | $N_7$ | $N_8$ | $N_9$ | $N_{10}$ | $N_{11}$ | $N_{12}$ | $N_{13}$ | $N_{14}$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-----------|-----------|-----------|-----------|-----------|
| 1     | $a$   | $\tilde{a}$ | $a^\dagger$ | $\tilde{a}^\dagger$ | $a\tilde{a}$ | $a^2$ | $\tilde{a}^2$ | $a^\dagger \tilde{a}$ | $a\tilde{a}^\dagger$ | $a^\dagger 2$ | $\tilde{a}^\dagger 2$ | $a^\dagger a + 1/2$ | $\tilde{a}^\dagger + 1/2$ |

TABLE II. Table of coefficients $h_i$, appearing in the ‘Super-Hamiltonian’ \[2,3\].

| $h_0$ | $h_1$ | $h_2$ | $h_3$ | $h_4$ | $h_5$ | $h_6$ | $h_7$ | $h_8$ | $h_9$ | $h_{10}$ | $h_{11}$ | $h_{12}$ | $h_{13}$ | $h_{14}$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-----------|-----------|-----------|-----------|-----------|
| $\gamma/2$ | $f_3$ | $-f_1^*$ | $f_1^*$ | $-f_3$ | $i\gamma(n+1)$ | $f_2$ | $-f_2^*$ | 0 | 0 | $f_2^*$ | $-f_2$ | $i\gamma \tilde{n}$ | $\omega - i\gamma \tilde{n} - i\gamma/2$ | $-\omega - i\gamma \tilde{n} - i\gamma/2$ |
TABLE III. We give the relation between the new generators \( W_i \) and the old generators \( N_i \). Also given are the new coefficient functions appearing in (2.3) and the corresponding two-photon operator.

| \( W_i \) | \( N_i \) | \( \hat{\mathbf{h}}_i \) | Operator                                      |
|-----------|-----------|-----------------|-----------------------------------------------|
| \( W_0 \) | \( N_0 \) | \( \gamma/2 \)  | 1                                             |
| \( W_1 \) | \( N_1 - N_4 \) | \( f_1 \)       | \( a - \hat{a}^\dagger \)                     |
| \( W_2 \) | \( N_2 - N_3 \) | \( -f_1^* \)    | \( \hat{a} - a^\dagger \)                     |
| \( W_3 \) | \( N_1 + N_4 \) | 0               | \( a + \hat{a}^\dagger \)                     |
| \( W_4 \) | \( N_2 + N_3 \) | 0               | \( \hat{a} + a^\dagger \)                     |
| \( W_5 \) | \( N_6 - N_{11} \) | \( f_2 \)       | \( a^2 - a^\dagger \)                         |
| \( W_6 \) | \( N_7 - N_{10} \) | \( -f_2^* \)    | \( \hat{a}^\dagger - a^\dagger \)             |
| \( W_7 \) | \( N_5 + N_{12} - (N_{13} + N_{14}) \) | \( i\gamma(2\hat{a} + 1)/2 \) | \( a\hat{a} + a^\dagger\hat{a}^\dagger - (a^\dagger a + \hat{a}^\dagger\hat{a} + 1) \) |
| \( W_8 \) | \( (N_5 - N_{12})/4 \) | \( 2i\gamma \)  | \( (a\hat{a} - a^\dagger\hat{a}^\dagger)/4 \)   |
| \( W_9 \) | \( (N_{13} - N_{14})/2 \) | \( 2\omega \)   | \( (a^\dagger a - \hat{a}^\dagger\hat{a})/2 \) |
| \( W_{10} \) | \( N_8 \) | 0               | \( a^\dagger\hat{a} \)                        |
| \( W_{11} \) | \( N_9 \) | 0               | \( \hat{a}^\dagger a \)                       |
| \( W_{12} \) | \( N_7 + N_{10} \) | 0               | \( \hat{a}^2 + a^\dagger \)                   |
| \( W_{13} \) | \( N_6 + N_{11} \) | 0               | \( a^2 + \hat{a}^\dagger \)                   |
| \( W_{14} \) | \( 2N_5 + N_{12} - (N_{13} + N_{14})/2 \) | 0               | \( 2a\hat{a} + a^\dagger\hat{a}^\dagger - (a^\dagger a + \hat{a}^\dagger\hat{a} + 1)/2 \) |

TABLE IV. The commutation table for the ideal \( \mathcal{I} = \{W_0, W_1, W_2, W_3, W_4\} \) with the whole two-photon group \( \mathcal{L} \). The entries are \([W_i, W_j]\).

| \( W_i \backslash W_j \) | \( W_1 \) | \( W_2 \) | \( W_3 \) | \( W_4 \) | \( W_5 \) | \( W_6 \) | \( W_7 \) | \( W_8 \) | \( W_9 \) | \( W_{10} \) | \( W_{11} \) | \( W_{12} \) | \( W_{13} \) | \( W_{14} \) |
|------------------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| \( W_1 \)              | 0       | 0       | 0       | 2W_0    | 0       | 2W_2    | 0       | W_1/4   | W_1/2   | W_4     | 0       | 2W_4    | 0       | W_1/2 + W_3 |
| \( W_2 \)              | 0       | 0       | 2W_0    | 0       | 2W_1    | 0       | 0       | W_2/4   | -W_2/2  | 0       | W_3     | 0       | 2W_3    | W_2/2 + W_4 |
| \( W_3 \)              | 0       | -2W_0   | 0       | 0       | 0       | -2W_4   | -2W_1  | -W_3/4  | W_3/2   | W_2     | 0       | -2W_2   | 0       | -2W_1 - W_3/2 |
| \( W_4 \)              | -2W_0   | 0       | 0       | 0       | -2W_3   | 0       | -2W_2  | -W_4/4  | -W_4/2  | W_1     | 0       | -2W_1   | -2W_2  | -W_4/2      |
TABLE V. Table of the action of the group element $W_i$ on an element $W_j$ of the ideal $\mathcal{I}$, i.e. each entry shows $\exp(-g_i W_i) W_j \exp(g_i W_i)$.

| $W_i \setminus W_j$ | $W_1$        | $W_2$        | $W_3$        | $W_4$        |
|---------------------|--------------|--------------|--------------|--------------|
| $W_1$               | $W_1$        | $W_2$        | $W_3$        | $W_4 - 2g_1$ |
| $W_2$               | $W_1$        | $W_2$        | $W_3 - 2g_2$ | $W_4$        |
| $W_3$               | $W_1$        | $W_2 + 2g_3$ | $W_3$        | $W_4$        |
| $W_4$               | $W_1 + 2g_4$ | $W_2$        | $W_3$        | $W_4$        |
| $W_5$               | $W_1$        | $W_2 + 2g_5 W_1$ | $W_3$        | $W_4 - 2g_5 W_3$ |
| $W_6$               | $W_1 + 2g_6 W_2$ | $W_2$        | $W_3 - 2g_6 W_4$ | $W_4$        |
| $W_7$               | $W_1$        | $W_2$        | $W_3 - 2g_7 W_1$ | $W_4 - 2g_7 W_2$ |
| $W_8$               | $W_1 e^{g_8/4}$ | $W_2 e^{g_8/4}$ | $W_3 e^{-g_8/4}$ | $W_4 e^{-g_8/4}$ |
| $W_9$               | $W_1 e^{g_9/2}$ | $W_2 e^{-g_9/2}$ | $W_3 e^{g_9/2}$ | $W_4 e^{-g_9/2}$ |
| $W_{10}$            | $W_1 + g_{10} W_4$ | $W_2$        | $W_3 + g_{10} W_2$ | $W_4$        |
| $W_{11}$            | $W_1$        | $W_2 + g_{11} W_3$ | $W_3$        | $W_4 + g_{11} W_1$ |
| $W_{12}$            | $W_1 + 2g_{12} W_4$ | $W_2$        | $W_3 - 2g_{12} W_2$ | $W_4$        |
| $W_{13}$            | $W_1$        | $W_2 + 2g_{13} W_3$ | $W_3$        | $W_4 - 2g_{13} W_1$ |

TABLE VI. Table of the action of $W_{14}$ on the elements of the ideal $\mathcal{I}$. We have set $\tan \mu_1 = 7^{-1/2}$ and $\mu_2 = g_{14}/\alpha$ where $\alpha = 2/7^{1/2}$. See the Appendix for more details.

|           | $W_{14}$                   |
|-----------|-----------------------------|
| $W_1$     | $\alpha(\sqrt{2} \cos(\mu_1 - \mu_2) W_1 + \sin \mu_2 W_3)$ |
| $W_2$     | $\alpha(\sqrt{2} \cos(\mu_1 - \mu_2) W_2 + \sin \mu_2 W_4)$ |
| $W_3$     | $\alpha(\sqrt{2} \cos(\mu_1 + \mu_2) W_3 - 2 \sin \mu_2 W_1)$ |
| $W_4$     | $\alpha(\sqrt{2} \cos(\mu_1 + \mu_2) W_4 - 2 \sin \mu_2 W_2)$ |
| $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ | $g_7$ | $g_8$ | $g_9$ | $g_{10}$ | $g_{11}$ | $g_{12}$ | $g_{13}$ |
|------|------|------|------|------|------|------|------|------|-------|-------|-------|-------|
| $-u_{40}/2$ | $-u_{30}/2$ | $0$ | $0$ | $u_{21}/(2u_{11})$ | $u_{12}/(2u_{11})$ | $-u_{31}/(2u_{11})$ | $2 \ln(u_{11}/u_{33})$ | $\ln(u_{11}u_{33})$ | $-2g_{12}$ | $-2g_{13}$ | $-u_{32} - u_{31}u_{12}/u_{11}/(4u_{33})$ | $-u_{41} + u_{21}u_{31}/u_{11}/(4u_{11})$ |