REPEATED GAMES OF INCOMPLETE INFORMATION WITH LARGE SETS OF STATES

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ABSTRACT. The famous theorem of R. Aumann and M. Maschler states that the sequence of values of an \( N \)-stage zero-sum game \( \Gamma_N(\rho) \) with incomplete information on one side and prior distribution \( \rho \) converges as \( N \to \infty \), and the error term
\[
\text{err}[\Gamma_N(\rho)] = \text{val}[\Gamma_N(\rho)] - \lim_{M \to \infty} \text{val}[\Gamma_M(\rho)]
\]
is bounded by \( CN^{-\frac{1}{2}} \) if the set of states \( K \) is finite.

The paper deals with the case of infinite \( K \). It turns out that for countably-supported prior distribution \( \rho \) with heavy tails the error term can be of the order of \( N^\alpha \) with \( \alpha \in (-\frac{1}{2}, 0) \), i.e., the convergence can be anomalously slow. The maximal possible \( \alpha \) for a given \( \rho \) is determined in terms of entropy-like family of functionals.

Our approach is based on the well-known connection between the behavior of the maximal variation of measure-valued martingales and asymptotic properties of repeated games with incomplete information.

1. INTRODUCTION

Repeated zero-sum games with incomplete information on one side were introduced by R. Aumann and M. Maschler (for a comprehensive presentation of the theory of repeated games we refer to the books \[1\] and \[14\]). In such a game \( \Gamma_N(\rho) \) two players are involved in a multistage repeated interaction, but only Player 1 is completely informed of its properties, and Player 2 has an uncertainty about the actual payoffs that depend on a random \( \rho \)-distributed state \( k \in K \) chosen by Nature before the game starts; \( k \) is told to Player 1 but not to Player 2, who knows only \( \rho \). The total payoff received by Player 1 from Player 2 at the end of the game is the expected arithmetic mean of one-stage gains. Both players are rational.

As it was shown by R. Aumann and M. Maschler, the minimax values of \( \Gamma_N(\rho) \) converge as \( N \to \infty \), and the error term
\[
\text{err}[\Gamma_N(\rho)] = \text{val}[\Gamma_N(\rho)] - \lim_{M \to \infty} \text{val}[\Gamma_M(\rho)]
\]
is non-negative and is bounded from above by \( C \Psi_N(\rho) \), where \( \Psi_N(\rho) \) is the value of the following optimization problem: to maximize the expected sum of distances between consecutive values up to time \( N \) over all martingales \((\mu_n)_{n \geq 0}\) taking values in probability distributions on \( K \) with \( \mu_0 = \rho \) (the total variation distance is used). The quantity \( \Psi_N(\rho) \) is called the maximal variation and represents the maximal variability of beliefs during the process of Bayesian learning with prior \( \rho \). R. Aumann and M. Maschler proved that, if \( K \) is finite, then \( \Psi_N(\rho) \) is less than constant times \( \sqrt{N} \), and, therefore, the error term can not decrease slower than \( 1/\sqrt{N} \). In \[18\], S. Zamir showed the existence of games with the error term of the order of \( 1/\sqrt{N} \).

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Theorem: For countably infinite $K$ with heavy-tailed behaviors, the maximal variation grows like $\sqrt{N}$.

In this paper countably-supported prior distributions with heavy tails are considered. We describe asymptotic behavior of the maximal variation and the slowest possible rate of decreasing of the error term. It turns out that for heavy-tailed $\rho$ anomalous behaviors are possible. Let $\alpha_\rho(\rho)$ be such $\alpha$ that $\Psi_N(\rho)$ grows like $N^\alpha$, and let $\alpha_G(\rho)$ be the maximal $\alpha$ such that there is a game $\Gamma$ with error $\alpha(\Gamma)$ decreasing like $N^\alpha$ (the rigorous definitions of $\alpha_\rho$ and $\alpha_G$ are given in Section 3). We set

$$Z_\varepsilon(\rho) = \sum_{k \in K} \rho(\{k\}) \left[ \ln \left( \frac{1}{\rho(\{k\})} \right) \right]^{\frac{1}{2} - \varepsilon}.$$  

Our central result is the following identity

$$\alpha_\rho(\rho) = \alpha_G(\rho) + 1 = \frac{1}{2} + \varepsilon^*(\rho), \quad \text{where} \quad \varepsilon^*(\rho) = \inf \{ \varepsilon \in [0, 1/2] | Z_\varepsilon(\rho) < \infty \}.$$  

In particular, the class of $\rho$ with $\sqrt{N}$-behavior prescribed by Neyman’s condition $S(\rho) < \infty$ can be extended again, and the exact class is given by the condition of $Z_0(\rho)$ finiteness. We also discuss the case of uncountable state space $K$. The results of this paper were announced in \cite{10}. 

2. Repeated Games with Incomplete Information: Main Definitions

Here we describe an $N$-stage zero-sum repeated game $\Gamma_N(\rho)$ with incomplete information on the side of Player 2. This game is given with a 4-tuple $\Gamma = (K, I, J, A)$, a number of repetitions $N \in \mathbb{N}$, and a prior distribution $\rho \in \Delta(K)$. Here $K$ is a set of states; $I$ and $J$ are sets of actions of Player 1 and Player 2, respectively; $A : K \times I \times J \to \mathbb{R}$ is a one-stage payoff function; $\Delta(K)$ denotes the set of all probability distributions on $K$.

The game is played as follows. Before the beginning of the game Nature picks a state $k \in K$ at random with distribution $\rho$ and tells $k$ to Player 1. Player 2 knows only $\rho$. Then at each stage $n = 1, 2, \ldots, N$ players simultaneously select their actions $i_n \in I$ and $j_n \in J$ using the information they have at this stage, and these actions are publicly announced before the stage $n + 1$. A behavioral strategy $\sigma$ of Player 1 is a sequence of maps $\sigma_n : K \times (I \times J)^{n-1} \to \Delta(I)$. Player 1 randomizes his action $i_n$ according to $\sigma_n$ given $k$ and a history $(i_1, j_1, \ldots, i_{n-1}, j_{n-1})$ observed. Behavioral strategy $\tau$ of Player 2 consists of $\tau_n : (I \times J)^{n-1} \to \Delta(J)$. The prior distribution with strategies $\sigma$ and $\tau$
generate the probability measure $\mathbb{P}_{\rho,\sigma,\tau}$ on $K \times (I \times J)^N$. After the last stage Player 2 pays

$$g_N(\rho, \sigma, \tau) = \frac{1}{N} \mathbb{E}_{\rho,\sigma,\tau} \left( \sum_{n=1}^{N} A_{n,m,j_n}^k \right)$$

to Player 1 (expectation is with respect to $\mathbb{P}_{\rho,\sigma,\tau}$).

Hence, players have completely opposite goals. Player 1 aims to maximize $g_N(\rho, \sigma, \tau)$, and Player 2 wants to minimize it. The lower and upper values are given by $\text{val}[\Gamma_N(\rho)] = \sup_{\sigma} \inf_{\tau} g_N(\rho, \sigma, \tau)$ and $\text{val}[\Gamma_N(\rho)] = \inf_{\sigma} \sup_{\tau} g_N(\rho, \sigma, \tau)$, respectively. If these values coincide, the game has a value $\text{val} = \text{val} = \text{val}$. For finite $I$, $J$, and $K$ the existence of the value follows from von Neumann’s minimax theorem.

The non-revealing game $\Gamma_{NR}(\rho)$ is an auxiliary version of the one-stage game $\Gamma_1(\rho)$, where Player 1 forgets $k$, i.e., the sets of strategies of Player 1 and Player 2 can be identified with $\Delta(I)$ and $\Delta(J)$, respectively.

We denote by $\mathcal{G}(K)$ the class of all 4-tuples $\Gamma = (K, I, J, A)$ such that:
- the sets of actions $I$ and $J$ are countable;
- the norm $\|A\|_\infty = \sup_{k,i,j} |A_{k,i,j}^k|$ is finite;
- the games $\Gamma_N(\rho)$ and $\Gamma_{NR}^1(\rho)$ have values for any $\rho \in \Delta(K)$ and $N \in \mathbb{N}$.

It is natural to consider the class $\mathcal{G}(K)$ if we are going to deal with infinite $K$. The first assumption allows us to avoid measurability issues arising for uncountable $I, J$. The second assumption ensures that the anomalous asymptotic effects for infinite $K$ are caused by “infinite lack of knowledge” on the side of Player 2 and not by unbounded payoffs. The third assumption excludes some pathological situations.

### 2.1. The maximal variation and its role.

Let $K$ be countable. The sequence of measure-valued maps $\mu_n : \Omega \to \Delta(K)$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, \mathbb{P})$ is called an $(\mathcal{F}_n)_{n \geq 0}$-adapted $\Delta(K)$-valued martingale if the sequence $(\mu_n(B), \mathcal{F}_n)_{n \geq 0}$ forms a martingale for any $B \subset \mathcal{K}$.

We denote by $\mathcal{M}_{\Delta(K)}(\rho)$ the set of all $\Delta(K)$-valued martingales $\mu = (\mu_n, \mathcal{F}_n)_{n \geq 0}$ with non-random $\mu_0 = \rho$. The probability space is not fixed, i.e., more precisely, $\mathcal{M}_{\Delta(K)}(\rho)$ consists of pairs $\mu = ((\mu_n, \mathcal{F}_n)_{n \geq 0}, (\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, \mathbb{P}))$. For brevity we write $\mu \in \mathcal{M}_{\Delta(K)}(\rho)$ thus implicitly fixing the underlying probability space and we use $\mathbb{P}$ for the underlying probability measure and $\mathbb{E}$ for the expectation with respect to $\mathbb{P}$.

The $N$-term variation of a $\Delta(K)$-valued martingale $\mu$ is defined by

$$V_N(\mu) = \mathbb{E} \left( \sum_{n=0}^{N-1} \|\mu_{n+1} - \mu_n\|_{TV} \right),$$

where $\|\Phi\|_{TV}$ denotes the total variation of a signed measure $\Phi$. In other words, $V_N(\mu) = \sum_{n=0}^{N-1} \sum_{k \in \mathcal{K}} \mathbb{E} |\mu_{n+1}(\{k\}) - \mu_n(\{k\})|$. Taking supremum over all $\mu$ from $\mathcal{M}_{\Delta(K)}(\rho)$ we get the maximal variation

$$\Psi_N(\rho) = \sup_{\mu \in \mathcal{M}_{\Delta(K)}(\rho)} V_N(\mu).$$

The maximal variation is extremely important in asymptotic problems concerning repeated games with incomplete information. The fundamental estimate on the value
follows from the results of R. Aumann and M. Maschler. Denote by $u_1(\rho)$ the value of the non-revealing game $\Gamma^{1\text{NR}}(\rho)$. Let $\text{Cav}[u_1]$ be the least concave majorant of $u_1$ as a function of $\rho$, i.e., $\text{Cav}[u_1](\rho) = \sup \sum_{m=1}^{\infty} \beta_m u_1(\rho_m)$, where supremum taken is over all $M \in \mathbb{N}$, $\{\beta_m\}_{m=1,2,\ldots,M} \subset [0,1]$, and $\{\rho_m\}_{m=1,2,\ldots,M} \subset \Delta(K)$ such that $\sum_{m=1}^{M} \beta_m = 1$ and $\sum_{m=1}^{M} \beta_m \rho_m = \rho$.

**Theorem 2.1** (R. Aumann and M. Maschler). Suppose $K$ is countable and $\Gamma \in \mathcal{G}(K)$. Denote $\text{val}[\Gamma_N(\rho)] - \text{Cav}[u_1](\rho)$ by $\text{err}[\Gamma_N(\rho)]$. Then the following two-sided estimate holds:

$$
0 \leq \text{err}[\Gamma_N(\rho)] \leq \frac{||A||_{\infty}}{N} \Psi_N(\rho).
$$

Usually this result is formulated for finite $K$, $I$ and $J$, but the same proof (see [14], Theorem 2.10 p.225) works in our case.

**Remark 2.1.** Suppose Player 1 uses a behavioral strategy $\sigma$. Then Player 2 can compute the conditional distribution $\rho_n$ of $k$ after observing the history of actions $\{i_1, j_1, \ldots, i_n, j_n\}$. The reason why $\Delta(K)$-valued martingales arise is that the process $(\rho_n)_{n \geq 0}$ of Player 2 beliefs about $k$ belongs to $\mathcal{M}_{\Delta(K)}(\rho)$. Note that more general functionals similar to the variation also appear in the context of repeated games (see the paper [6] of B. De Meyer and the papers [7, 9] of F. Gensbittel). Because of the presence of other maximal variations it is more correct to call the particular one defined above “the maximal variation in the total variation norm”. For brevity we use the shorter notation.

It is well known that for countable $K$ and $\rho$ with not too heavy tails $\lim_{N \to \infty} \Psi_N(\rho)/N = 0$ (see below). Therefore, $\text{val}[\Gamma_N(\rho)]$ converges to $\text{Cav}[u_1](\rho)$. This limiting value incorporates the influence of constant strategic advantages (or disadvantages) inherited from the non-revealing game, and the error term $\text{err}[\Gamma_N(\rho)]$ reflects the impact of information asymmetry on the value. In particular, if $u_1 \equiv 0$ (in this case all the strategic asymmetries in $\Gamma_N(\rho)$ are caused by incomplete information), then the error term can be regarded as the price of information.

In [10], J.-F. Mertens and S. Zamir showed that

$$
\Psi_N(\rho) \leq \sqrt{N} \Lambda(\rho), \quad \text{where} \quad \Lambda(\rho) = \sum_{k \in K} \sqrt{\rho(\{k\})(1 - \rho(\{k\}))}
$$

(the similar but weaker estimate was derived earlier by R. Aumann and M. Maschler). Hence, the maximal variation can not grow faster than $\sqrt{N}$ if $\Lambda(\rho) < \infty$ (i.e., if $\rho$ has not too heavy tails). For $\Gamma \in \mathcal{G}(K)$ this implies $C/\sqrt{N}$ upper bound on the error term. S. Zamir) proved that the order of magnitude of this upper bound is sharp (see [18]). In other words, for any non-degenerate $\rho \in \Delta(K)$ (i.e., not concentrated at one point) there is a 4-tuple $\Gamma \in \mathcal{G}(K)$ such that $\text{err}[\Gamma_N(\rho)] \geq 1/\sqrt{N}$ for any $N$. Therefore, $1/\sqrt{N}$ is the slowest possible rate of the error term decreasing over $\mathcal{G}(K)$ for $\rho$ with $\Lambda(\rho) < \infty$. In the paper [15], A. Neyman extended the class of $\rho$ with $\sqrt{N}$-behavior of the maximal variation by obtaining the following estimate in terms of Shannon’s entropy

$$
\Psi_N(\rho) \leq \sqrt{2N S(\rho)}, \quad S(\rho) = \sum_{k \in K} \rho(\{k\}) \ln \left( \frac{1}{\rho(\{k\})} \right),
$$
where \( \ln x \) denotes the logarithm of \( x \) to the base \( e \). Indeed, if \( K = \mathbb{N} \), then \( \Lambda(\rho) \) diverges for \( \rho \) with \( \rho(\{k\}) \sim k^{-2} \) as \( k \to \infty \), but Shannon’s entropy remains finite even if \( \rho(\{k\}) \sim k^{-1}(\ln k)^{-2-\varepsilon} \) with some \( \varepsilon > 0 \).

3. The results

3.1. Countable \( K \). The exact class of \( \rho \in \Delta(K) \) with the maximal variation growing like \( \sqrt{N} \) turns out to be wider than prescribed by the condition of entropy finiteness. Consider a family of uncertainty measures of \( \rho \)

\[
Z_\varepsilon(\rho) = \sum_{k \in K} \rho(\{k\}) \left[ \ln \left( \frac{1}{\rho(\{k\})} \right) \right]^{1/2-\varepsilon}, \quad \varepsilon \leq \frac{1}{2}.
\]

The Shannon entropy corresponds to \( \varepsilon = -\frac{1}{2} \). We will see that the sharp condition for \( \sqrt{N} \)-behavior to hold is finiteness of \( Z_0(\rho) = \sum_{k \in K} \rho(\{k\}) \sqrt{\ln \left( \frac{1}{\rho(\{k\})} \right)} \). The condition \( Z_0(\rho) < \infty \) is less restrictive than entropy finiteness; if \( K = \mathbb{N} \), it holds for \( \rho \) such that \( \rho(\{k\}) \sim k^{-1}(\ln k)^{-2} \) as \( k \to \infty \) with some \( \varepsilon > 0 \), but entropy is infinite if \( \varepsilon \leq \frac{1}{2} \). Moreover, it turns out that beyond the class of \( \rho \) with finite \( Z_0(\rho) \) the maximal variation can grow anomalously fast (like \( N^{1/2+\varepsilon} \) with some \( \varepsilon \in (0, 1/2) \)), and the error term can decrease anomalously slowly.

Our main goal is to study the exponents

\[
\alpha_{\Psi}(\rho) = \limsup_{N \to \infty} \frac{\ln(\Psi_N(\rho))}{\ln N} \quad \text{and} \quad \alpha_G(\rho) = \sup_{\Gamma \in \mathcal{G}(K)} \limsup_{N \to \infty} \frac{\ln(\text{err}[\Gamma_N(\rho)])}{\ln N}
\]

as functions of \( \rho \).

The following theorem provides a one-parametric family of estimates on the maximal variation.

**Theorem 3.1.** Suppose \( K \) is countable, and \( \rho \in \Delta(K) \). Then for any \( \varepsilon \in [0, 1/2] \)

\[
(3.1) \quad \Psi_N(\rho) \leq cN^{1/2+\varepsilon}Z_\varepsilon(\rho),
\]

where \( c = \sqrt{2} \left( 1 + \frac{1}{2\ln 2} \right) < \sqrt{6} \).

The next theorem states that the estimate (3.1) after minimizing over \( \varepsilon \) gives the exact rate of growth of the maximal variation.

**Theorem 3.2.** If \( K \) is countable, and \( \rho \in \Delta(K) \) is non-degenerate, then

\[
(3.2) \quad \alpha_{\Psi}(\rho) = \frac{1}{2} + \varepsilon^*(\rho), \quad \varepsilon^*(\rho) = \inf \{ \varepsilon \in [0, 1/2] \mid Z_\varepsilon(\rho) < \infty \}.
\]

Theorem 3.1 and 3.2 are proved in Sections 4 and 5 respectively.

Note that \( Z_{1/2}(\rho) = 1 \) and, hence, \( \varepsilon^*(\rho) \) is well-defined for any \( \rho \). If tails of \( \rho \) are so heavy that \( \varepsilon^*(\rho) = \frac{1}{2} \), then a natural question about possibility of linear growth of \( \Psi_N(\rho) \) arises. The negative answer is given in Section 4 where we show that \( \frac{1}{N}\Psi_N(\rho) \to 0 \) as \( N \to \infty \) for any countably-supported \( \rho \).
From the results described in Subsection 2.1 it follows that for non-degenerate $\rho$ with $S(\rho) < \infty$ we have $\alpha_G(\rho) = \alpha_\Psi(\rho) - 1 = -1/2$. The relation between $\alpha_\Psi$ and $\alpha_G$ holds in general case.

**Theorem 3.3.** If $K$ is countable, and $\rho \in \Delta(K)$ is non-degenerate, then

$$\alpha_G(\rho) = -\frac{1}{2} + \varepsilon^*(\rho) = \alpha_\Psi(\rho) - 1. \quad (3.3)$$

This theorem is proved in Section 6 by constructing a game $\mathcal{G}_N(\rho)$ with anomalously slow decreasing of the error term.

**Remark 3.1.** If $K$ is countable, the game $\mathcal{G}_N(\rho)$ has countably infinite sets of actions of both players. The infiniteness of action sets turns out to be crucial for anomalous behavior of the error term. Indeed, as it was announced in the paper [15] of A. Neyman, for finite $I$ and $J$ the error term is bounded from above by $4\|A\|_\infty \sqrt{2}\#I\#J\ln 2/\sqrt{N}$ (here $\#B$ denotes the cardinality of a set $B$).

3.2. **Uncountable $K$.** If $K$ is a “good” uncountable measurable space, then one can consider repeated games with incomplete information and the maximal variation after obvious refinements of definitions.

**Remark 3.2.** A completely metrizable separable topological space $K$ equipped with its Borel sigma-field is called a Polish space (see [17], p.52). It is enough to keep in mind one of the following examples: countable sets with the discrete topology; interval $[0; 1]$ with the standard topology of the real line (or the real line itself). These are the only Polish spaces up to Borel isomorphism, i.e., up to bijection preserving the Borel structure ([17], Theorem 3.3.13 p.99). The set $\Delta(K)$ of Borel probability measures over a Polish space $K$ with the topology of weak convergence becomes Polish itself, and, hence, one can define $\Delta(K)$-valued measurable maps.

In order to consider repeated games with Polish $K$, $I$ and $J$ we assume that: the one-stage payoff function $A: K \times I \times J \to \mathbb{R}$ is measurable; behavioral strategies of Player 1 and Player 2 consist of measurable maps with values in $\Delta(I)$ and $\Delta(J)$, respectively. For a Polish space $K$ a sequence of $\Delta(K)$-valued random variables $(\mu_n)_{n \geq 0}$ is called a $\Delta(K)$-valued martingale if for any measurable $B \subset K$ the sequence $(\mu_n(B))_{n \geq 0}$ is a martingale. Up to this extension we define the maximal variation as before. For countable $K$ the new definitions are equivalent to the given above.

Note that $\rho \in \Delta(K)$ can be represented as the sum of the continuous component $\rho^c$ that has no atoms and the discrete component $\rho_d$ supported on a countable subset of $K$. If $\rho$ is purely discrete, i.e., if $\rho^c \equiv 0$, then the set of atoms can be considered as a new set of states. This reduces the problem to the case of countable $K$. In particular, the results of previous subsection can be considered in a more general framework of general Polish space $K$ and countably-supported $\rho \in \Delta(K)$.

The following theorem describes the asymptotic behavior of the maximal variation for uncountable Polish space $K$ (for example, $K = \mathbb{R}$).

**Theorem 3.4.** Suppose $K$ is a Polish space, and $\rho \in \Delta(K)$. Then

$$\Psi_N(\rho) = 2\rho^c(K)N + o(N), \quad N \to \infty. \quad (3.4)$$
Hence, if a nontrivial continuous component is presented, then the maximal variation grows linearly with \( N \). Theorem 3.3 is proved in the end of Section 6.

One can easily show that Theorem 2.1 of R. Aumann and M. Maschler holds for any 4-tuple from \( \mathcal{G}(K) \) with arbitrary Polish space \( K \). But for a prior distribution \( \rho \) with nonzero continuous component the statement becomes almost meaningless because now the upper bound on the error term has the same order as the leading term. Nevertheless, the theorem keeps giving the correct maximal order of magnitude of the error term (as it is for countable \( K \); see Theorem 3.3).

**Theorem 3.5.** Suppose \( K \) is a Polish space. Then there exists a 4-tuple \( \mathfrak{G} \in \mathcal{G}(K) \) such that for any \( \rho \in \Delta(K) \)

\[
(3.5) \quad \liminf_{N \to \infty} \text{err}[\mathfrak{G}_N(\rho)] \geq \frac{1}{2} \rho^F(K).
\]

To show this a version of the game \( \mathfrak{G}_N(\rho) \) with \( K = [0, 1] \) is described in Subsection 6.2. This game has a pathological property. The one-stage payoff function is discontinuous at every point as a function of \( k \in K \). This observation suggests that in order to avoid pathological situations for uncountable \( K \) one should consider games with some regularity of one-stage payoffs with respect to states.

4. THE MAXIMAL VARIATION: UPPER BOUNDS

Here we prove Theorem 3.1 and derive an upper bound on the maximal variation for uncountable \( K \). We begin with some auxiliary estimates.

4.1. **Scalar martingales.** Let \( X = (X_n, \mathcal{F}_n)_{n \geq 0} \) be a martingale taking values in \([0, 1]\) with non-random \( X_0 = p \). Denote by \( \mathcal{M}_{[0,1]}(p) \) the set of all such martingales. The \( N \)-term \( \text{L}^1 \)-variation of a scalar martingale \( X \in \mathcal{M}_{[0,1]}(p) \) is given by

\[
V_N(X) = \mathbb{E} \left( \sum_{n=0}^{N-1} |X_{n+1} - X_n| \right).
\]

Asymptotic behavior of the maximal \( \text{L}^1 \)-variation \( \psi_N(p) = \sup_{X \in \mathcal{M}_{[0,1]}(p)} V_N(X) \) as \( N \to \infty \) was studied by J.-F. Mertens and S. Zamir in [12].

They analyzed the limiting Bellman equation connecting \( \psi_{N+1} \) with \( \psi_N \) and obtained that

\[
(4.1) \quad \psi_N(p) = \sqrt{N} \phi(x_p)(1 + o(1)),
\]

where \( \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \) is the standard normal density and \( x_p \) is its \( p \)-quantile, i.e., \( \int_{-\infty}^{x_p} \phi(x)dx = p \). In [5], B. De Meyer derived the same asymptotics from probabilistic arguments. He introduced a new representation of the variation that reduces the initial problem to investigation of the terminal distribution of an auxiliary martingale \( S \). The normal distribution then arises from a central limit theorem applied to \( S \).

**Remark 4.1.** Let \( K \) be countable. Consider a martingale \( \mu = (\mu_n, \mathcal{F}_n)_{n \geq 0} \in \mathcal{M}_{\Delta(K)}(\rho) \). For any \( k \in K \) the scalar martingale \( \mu(\{k\}) = (\mu_n(\{k\}), \mathcal{F}_n)_{n \geq 0} \) is in \( \mathcal{M}_{[0,1]}(\rho(\{k\})) \) and \( V_N(\mu) = \sum_{k \in K} V_N(\mu(\{k\})) \). Therefore, \( \psi_N(\rho) \leq \sum_{k \in K} \psi_N(\rho(\{k\})) \). Note that the martingales \( \mu(\{k\}) \) should fulfill the constraint \( \sum_{k \in K} \mu_n(\{k\}) = 1 \) for any \( n \geq 0 \).
almost surely, and this is why the last formula is not equality. If $K$ is finite, then (4.1) implies that

$$\limsup_{N \to \infty} \frac{\Psi_N(\rho)}{\sqrt{N}} \leq \sum_{k \in K} \phi(x_{\rho(k)}).$$

Let us look at the right-hand side of (4.2) separately of the inequality itself. It can be easily checked that $\phi(x_p) \sqrt{2p\ln\frac{1}{p}} \to 1$ as $p \to +0$, and thus for countably infinite $K$ the sum converges iff $Z_0(\rho) < \infty$. This observation makes the answer to Neyman’s question about the class of $\rho$ with $\sqrt{N}$-behavior of $\Psi_N(\rho)$ rather intuitive (see Section 3). But there is an obstacle for making this reasoning completely rigorous. From [12] and [5] it can be deduced only that there is an absolute constant $C > 0$ such that

$$\psi_N(p) \sqrt{N} \leq \phi(x_p) + CN^{-q},$$

where $q = \frac{1}{2}$ in the first paper and $q = \frac{1}{4}$ in the second. The term $CN^{-q}$ at the right-hand side prevents direct derivation of inequality (4.2) in the case of infinite $K$. This also explains why Theorem 3.2 is not a direct corollary of [12] and [5] even if $\varepsilon^*(\rho) = 0$.

To overcome the mentioned difficulty we derive an estimate on $V_N(X)$ without additional term at the cost of worsening the constant.

**Proposition 4.1.** For any $p \in [0, 1]$ and $X \in \mathcal{M}_{[0,1]}(p)$ the following estimates hold:

$$V_N(X) \leq \sqrt{2Np} \sqrt{\ln \frac{1}{p} \left(1 + \frac{1}{2\ln 2}\right)};$$

(4.3)

$$V_N(X) \leq 2Np.$$ (4.4)

To infer (4.3) De Meyer’s approach is used. But instead of the central limit theorem we apply large deviation estimates.

Let $Y_n$ be a random variable equal to 1, when $X_n \geq X_{n-1}$, and -1, otherwise. The auxiliary process $S$ corresponding to a martingale $X = (X_n, \mathcal{F}_n)_{n \geq 0}$ is defined by $S_n = \sum_{m=1}^n Z_m$, where $Z_m = Y_m - \mathbb{E}(Y_m | \mathcal{F}_{m-1})$. Obviously, $S = (S_n, \mathcal{F}_n)_{n \geq 0}$ is a martingale. In [3], B. De Meyer proved that

$$V_N(X) = \mathbb{E}X_NS_N.$$ (4.5)

This identity allows to derive upper bounds on the variation from tail estimates for $S_N$. By the standard technique one can prove the following lemma.

**Lemma 4.1.** $\forall t \geq 0 \quad \mathbb{P}(|S_N| \geq t) \leq \exp\left(-\frac{t^2}{2N}\right)$.

Note that the result with worse constant follows immediately from the Bernstein or the Azuma-Hoeffding inequalities (see [2]).

**Proof.** The proof is based on the exponential Chebyshev inequality. For any $\lambda > 0$ the Chebyshev inequality implies

$$\mathbb{P}(|S_N| \geq t) = \mathbb{P}(|\exp(\lambda S_N)| \geq \exp(\lambda t)) \leq \exp(-\lambda t)\mathbb{E}\exp(\lambda S_N).$$
From the martingale property it follows that
\[ \mathbb{E} \exp(\lambda S_n) = \mathbb{E} \left( \exp(\lambda S_{n-1}) \mathbb{E}(\exp(\lambda Z_n) \mid \mathcal{F}_{n-1}) \right). \]

One can easily prove that \( q e^{2\lambda (1-q)} + (1-q) e^{-2\lambda q} \leq e^{\lambda^2} \) for any \( q \in [0,1] \). Therefore, denoting \( \mathbb{P}(\{Y_n = 1\} \mid \mathcal{F}_{n-1}) \) by \( q \) we get
\[ \mathbb{E}(\exp(\lambda Z_n) \mid \mathcal{F}_{n-1}) \leq \exp(\lambda^2/2). \]

Hence, \( \mathbb{E} \exp(\lambda S_N) \leq \exp(N \lambda^2/2) \), and choosing \( \lambda = t/N \) concludes the proof. \( \square \)

We estimate \( \mathbb{E} X_N S_N \) by maximizing this expectation under constraints given by the information we have about the distributions of \( X_n \) and \( S_n \). Note that the problem of maximizing \( \mathbb{E} XY \) over all joint distributions with prescribed marginals is called the maximal covariance problem. Our proof of Lemma 4.1 is close to the proof of Theorem 6 from [6], where a solution to the maximal covariance problem is described.

**Proof of Proposition 4.1.** First we obtain the estimate (4.3) for \( 0 < p \leq \frac{1}{2} \) (if \( p = 0 \), the estimate holds trivially). Consider the auxiliary process \( S \) and denote by \( G \) the cumulative distribution function of \( S_N/\sqrt{N} \). For a monotonically increasing function \( H \) its “inverse” is defined by \( H_{\text{inv}}(z) = \sup\{t \mid H(t) \leq z\} \). De Meyer’s representation (4.3) implies
\[ V_N(X) = \sqrt{N} \int_{-\infty}^{\infty} t \mathbb{E}(X_N \mid S_N = t\sqrt{N}) dG(t) = \sqrt{N} \int_0^1 G_{\text{inv}}(z) h(z) dz, \]
where \( z = G(t) \) and \( h(z) = \mathbb{E}(X_N \mid S_N = \sqrt{N} G_{\text{inv}}(z)) \). We set \( F(t) = 1 - e^{-t^2/2} \) for \( t \geq 0 \) and \( F(t) = 0 \), otherwise. Lemma 4.1 implies \( F(t) \leq G(t) \) for all \( t \in \mathbb{R} \), and, hence, \( G_{\text{inv}}(z) \leq F_{\text{inv}}(z) \) for all \( z \in [0,1] \). Therefore, \( V_N(X) \leq \sqrt{N} \int_0^1 F_{\text{inv}}(z) h(z) dz \).

Note that \( h(z) \in [0,1] \), and \( \int_0^1 h(z) dz = p \) since \( \mathbb{E} X_N = p \). Denote by \( W(p) \) the set of all such functions. Hence, \( V_N(X) \leq \sqrt{N} \max_{f \in W(p)} \int_0^1 F_{\text{inv}}(z) f(z) dz \). The set \( W(p) \) is convex and optimization problem is linear. Therefore, the maximum is attained at a peak point of \( W(p) \). Peak points are indicator functions of subsets \( B \subset [0,1] \) of Lebesgue measure \( p \). From monotonicity of \( F_{\text{inv}} \) we get
\[ V_N(X) \leq \sqrt{N} \int_0^1 F_{\text{inv}}(z) dz = \sqrt{N} \int_{t_p}^\infty t dF(t), \]
where \( t_p = \sqrt{2 \ln \frac{1}{p}} \) is the solution of \( 1 - p = F(t_p) \). Integration by parts implies
\[ \int_{t_p}^\infty t dF(t) = t_p e^{-t_p^2/2} + \int_{t_p}^\infty e^{-t^2/2} dt. \]

The integral at the right-hand side can be estimated in the following way
\[ \int_{t_p}^\infty e^{-t^2/2} dt = \int_0^\infty e^{-t^2/2} dt - 2t_p e^{-t_p^2/2} \leq \sqrt{\pi} \int_0^\infty e^{-t^2/2} dt = \frac{1}{t_p} e^{-t_p^2/2}. \]

Taking into account that \( \left(2 \ln \frac{1}{p}\right)^{-1} \leq (2 \ln 2)^{-1} \), we get the estimate (4.3) for \( p \leq \frac{1}{2} \).

Now we consider \( p \in \left(\frac{1}{2},1\right] \). Note that if \( X \in \mathcal{M}_{[0,1]}(p) \), then \( X' \) given by \( X'_n = 1 - X_n \) belongs to \( \mathcal{M}_{[0,1]}(1-p) \), and \( V_N(X) = V_N(X') \). Put \( \varphi(p) = p \sqrt{\ln \frac{1}{p}} \). Therefore, (4.3)
follows from the already considered case by the elementary inequality \( \varphi(p) \geq \varphi(1 - p) \) for \( p \in \left[ \frac{1}{2}, 1 \right] \).

The rough estimate (4.4) immediately follows from \( |X_{n+1} - X_n| \leq X_{n+1} + X_n \) and \( \mathbb{E}X_n = p \).

4.2. \( \Delta(K) \)-valued martingales: discrete \( \rho \). Suppose \( K \) is countable. Then \( V_N(\mu) = \sum_{k \in K} V_N(\mu(\{k\})) \) for any \( \mu \in \mathcal{M}_{\Delta(K)}(\rho) \). As we will see, the appropriate upper bound from Proposition 4.1 applied to scalar martingales at the right-hand side of this identity implies Theorem 3.1. So in the proof we do not use that \( \sum_{k \in K} \mu_n(\{k\}) = 1 \) but only that the sum of expectations equals 1. One could expect that this approach gives a very rough estimate. Rather surprisingly the resulting upper bound reflects the correct order of magnitude of \( \Psi_N(\rho) \) (see Theorem 3.2).

**Proof of Theorem 3.1.** Fix arbitrary \( \mu \in \mathcal{M}_{\Delta(K)}(\rho) \), \( N \in \mathbb{N} \), and \( \varepsilon \in [0, 1/2] \) such that \( Z_\varepsilon(\rho) \) is finite. It is enough to show that \( V_N(\mu) \leq c N^{\frac{1}{2} + \varepsilon} Z_\varepsilon(\rho) \). We represent \( K \) as the union of \( K_\varepsilon = \{ k \in K \mid \ln(1/\rho(\{k\})) \leq N \} \) and \( K_\varepsilon = K \setminus K_\varepsilon \). Applying (4.3) to the contribution of \( k \in K_\varepsilon \) and taking into account the definition of \( K_\varepsilon \) we obtain

\[
\sum_{k \in K_\varepsilon} V_N(\mu(\{k\})) \leq \sqrt{2} \left( 1 + \frac{1}{2\ln 2} \right) N^{\frac{1}{2} + \varepsilon} \sum_{k \in K_\varepsilon} \rho(\{k\}) \left( \ln \frac{1}{\rho(\{k\})} \right)^{\frac{1}{2} - \varepsilon}.
\]

For \( k \in K_\varepsilon \) estimate (4.4) implies

\[
\sum_{k \in K_\varepsilon} V_N(\mu(\{k\})) \leq 2 N^{\frac{1}{2} + \varepsilon} \sum_{k \in K_\varepsilon} \rho(\{k\}) \left( \ln \frac{1}{\rho(\{k\})} \right)^{\frac{1}{2} - \varepsilon}.
\]

Since \( 2 \leq \sqrt{2} \left( 1 + \frac{1}{2\ln 2} \right) \), we get

\[
V_N(\mu) = \sum_{k \in K} V_N(\mu(\{k\})) \leq \sqrt{2} \left( 1 + \frac{1}{2\ln 2} \right) N^{\frac{1}{2} + \varepsilon} Z_\varepsilon(\rho).
\]

This completes the proof. \( \square \)

**Remark 4.2.** We claim that the growth of the maximal variation \( \Psi_N(\rho) \) is sublinear even if \( Z_\varepsilon(\rho) = \infty \) for any \( \varepsilon < \frac{1}{2} \). Indeed, let \( K_\delta \) be a subset of \( K \) such that \( \rho(K_\delta) \leq \delta \). Applying (4.3) for \( k \in K \setminus K_\delta \) and (4.4) for \( k \in K_\delta \) we get \( \Psi_N(\rho) \leq C_\delta \sqrt{N} + 2\delta N \). Since \( \delta > 0 \) is arbitrary,

\[
\lim_{N \to \infty} \frac{\Psi_N(\rho)}{N} = 0.
\]

4.3. \( \Delta(K) \)-valued martingales: \( \rho \) with nontrivial continuous component. This subsection is devoted to the case of uncountable \( K \).

**Proposition 4.2.** Suppose \( K \) is a Polish space, and \( \rho \in \Delta(K) \) is decomposed as \( \rho^a + \rho^d \). Then

\[
\Psi_N(\rho) \leq 2 \rho^a(K) N + \Psi_N(\rho^d),
\]

where \( \rho^{d+} = \rho^d + (1 - \rho^a(K)) \delta_{k_0} \), and \( \delta_{k_0} \) is the Dirac \( \delta \)-measure concentrated at a point \( k_0 \in K \) such that \( \rho^d(\{k_0\}) = 0 \).
Therefore, \( u_B \) (for sets of zero measure and is denoted by supp \( \rho \)). The main feature of \( \nu \) (5.1) is that if \( \rho \) is purely continuous, then \( \Psi_N(\rho^d) \) is denoted by \( \Phi(B) \). The minimal closed support of \( \rho \) is called dyadic. The set of all dyadic martingales from \( \mathcal{M}_{\Delta(K)}(\rho) \) is denoted by \( \mathcal{M}_{\Delta(K)}^d(\rho) \). Denote by \( 1_B \) the indicator function of a set \( B \). For two finite signed measures \( \Phi, \Phi' \), let \( \frac{d\Phi}{d\Phi'} = f \). Let \( u^M \) be the uniform distribution over \( 2^M \)-element subset of \( K \). In [13], the following dyadic martingale \( \nu^M = (\nu^M_n, F^2_n, n \geq 0) \in \mathcal{M}_{\Delta(K)}(u^M) \) was constructed. It starts from \( u^M \) and evolves according to

\[
\nu_n^M = 2(1_{B_n, \omega_n} + 1_{K \setminus B_n}(1 - \omega_n))\nu^M_{n-1}
\]

for any \( n = 1, 2, \ldots, M \); here \( B_n \subset K \) is a predictable \( (F^2_{n-1}) \)-measurable) random subset such that \( \nu^M_{n-1}(B_n) = 1/2 \). For \( n \geq M \) the martingale \( \nu^M \) is constant, i.e., \( \nu_n^M = \nu_M^M \). The main feature of \( \nu^M \) is that \( V_N(\nu^M) = N \) for \( N \leq M \), i.e., the variation of \( \nu^M \) grows linearly if \( N \) is not too large.

5.1. Martingale of dyadic splittings for \( \rho \in \Delta(\mathbb{R}) \). A dyadic martingale \( \Upsilon^\rho \) described here extends the construction of \( \nu^M \) to an arbitrary prior distribution \( \rho \) over the real line.

The minimal closed support of \( \rho \in \Delta(\mathbb{R}) \) is the complement to the union of open sets of zero measure and is denoted by supp \( \rho \). Consider a pair of transformations \( \pi_i : \Delta(\mathbb{R}) \to \Delta(\mathbb{R}), \ i = 0, 1 \), such that for any \( \rho \in \Delta(\mathbb{R}) \)

\[
\rho = \frac{1}{2}\pi_0[\rho] + \frac{1}{2}\pi_1[\rho], \ \ \text{and} \ \ \supp\pi_0[\rho] \leq \supp\pi_1[\rho]
\]

(for \( B_1, B_2 \subset \mathbb{R} \) we write \( B_1 \leq B_2 \) if \( \forall x_1 \in B_1, \forall x_2 \in B_2 \ x_1 \leq x_2 \)).
Such transformations exist, and one can show that they are uniquely determined by \((5.1)\). Let us describe them explicitly. A number \(m_\rho = \min \{ x \in \mathbb{R} \mid \rho((\infty, x]) \geq \frac{1}{2} \} \) is the least median of \(\rho\). Define a function \(\lambda_\rho : \mathbb{R} \to [0, 2] \) by the following conditions: \(\lambda_\rho(x) = 2\) for \(x < m_\rho\); \(\lambda_\rho(x) = 0\) for \(x > m_\rho\); and \(\int_{\mathbb{R}} \lambda_\rho(x) d\rho(x) = 1\). Then we set \(\pi_0[\rho] = \lambda_\rho\rho\) and \(\pi_1[\rho] = 2\rho - \pi_0[\rho]\). Roughly speaking, \(\pi_0[\rho]\) and \(\pi_1[\rho]\) are the normalized restrictions of \(\rho\) to the left from the median and to the right, respectively.

We define the process \(\Upsilon^\rho\)
\[
\Upsilon^\rho_n = \pi_{\omega_n}[\Upsilon^\rho_{n-1}] \quad \text{and} \quad \Upsilon^\rho_0 = \rho.
\]
The properties \((5.1)\) of \(\pi_i\) immediately imply that \(\Upsilon^\rho = (\Upsilon^\rho_n, F^2_n)_{n \geq 0}\) is a dyadic martingale from \(M_{\Delta(K)}(\rho)\). If \(\rho = u^M\), then \(\Upsilon^\rho\) and \(\nu^M\) coincide.

Remark 5.1. There is a more compact alternative description of \(\Upsilon^\rho, \rho \in \Delta(\mathbb{R})\). Denote by \(F\) the cumulative distribution function corresponding to \(\rho\). Let \(z\) be a random number uniformly distributed over \([0, 1]\). Hence, \(k = sup\{ F(t) \leq z \} \) is \(\rho\)-distributed. Consider a binary representation \(z = 0.\omega_1\omega_2\omega_3...\) Then \(\{\omega_n\}_{n=1}^\infty\) generate the so-called standard dyadic filtration \((F^2_n)_{n \geq 0}\) of \([0, 1]\). Finally, we define \(\Upsilon^\rho_n\) as the conditional distribution of \(k\) given \(F^2_n\), i.e., \(\Upsilon^\rho_n(B) = \mathbb{P}(\{ k \in B \} \mid F^2_n)\) for any Borel \(B \subset \mathbb{R}\).

5.2. Countably-supported prior distributions.

Proposition 5.1. Suppose \(K\) is countable and \(\rho \in \Delta(K)\) is such that \(Z_\varepsilon(\rho) = \infty\) for some \(\varepsilon \in [0, \frac{1}{2}]\). Then there exists a dyadic \(\Delta(K)\)-valued martingale \(\mu \in M_{\Delta(K)}(\rho)\) such that for any \(\gamma < \varepsilon\)
\[
limsup_{N \to \infty} \frac{V_N(\mu)}{N^{\frac{1}{2} + \gamma}} = \infty.
\]

Without loss of generality, \(K = \mathbb{N}\), and \(\rho\) is monotonically decreasing, i.e., \(\rho(\{ k + 1 \})\) is greater than \(\rho(\{ k \})\) for all \(k \in \mathbb{N}\) (indeed, starting from general \(K\) we can enumerate its elements in a suitable way). We identify probabilities on \(\mathbb{N}\) and probabilities on \(\mathbb{R}\) supported on \(\mathbb{N}\). The objective is to show that in this case we can take \(\mu = \Upsilon^\rho\). Several lemmas precede the proof.

Denote \(V_N(\Upsilon^\rho)\) by \(Q_N(\rho)\). We need a lower estimate on \(Q_N(\rho)\), but the situation differs from considered in A. Neyman’s paper [13]. For the martingale \(\nu^M\) the two possible values of \(\nu^M_{n+1}\) given \(\nu^M_n\) are mutually singular. This is why each stage \(n \leq M\) contributes 1 to the variation of \(\nu^M\). But not every probability distribution can be represented as the arithmetic mean of two mutually singular probability distributions (for example, Bernoulli distribution with success probability different from 1/2). In particular, the two values of \(\Upsilon^\rho_{n+1}\) given \(\Upsilon^\rho_n\) (\(\pi_0[\Upsilon_n]\) and \(\pi_1[\Upsilon_n]\)) are not mutually singular in general case, and this makes the problem more complicated. We overcome this difficulty by observing that they are “almost mutually singular” if all the atoms of \(\Upsilon^\rho_n\) are small enough. Therefore, we should control the heaviest atom of \(\Upsilon^\rho_n\). This is why the condition \(Z_\varepsilon(\rho) = \infty\) arises for the lower bound on \(Q_N(\rho)\) to be of the order of \(N^{\frac{1}{2} + \varepsilon}\). We set \(H(\rho) = \max_k \rho(\{ k \})\).

Lemma 5.1. \(1 - H(\rho) \leq Q_1(\rho) \leq 1\) for any \(\rho \in \Delta(\mathbb{R})\).
Proof. By the definition $Q_1(\rho) = \frac{1}{2} ||\pi_0[\rho] - \rho||_{TV} + \frac{1}{2} ||\pi_1[\rho] - \rho||_{TV}$. Using the property (5.1) we get $Q_1(\rho) = \frac{1}{2} ||\pi_0[\rho] - \pi_1[\rho]||_{TV}$. Note that $\pi_0[\rho]$ and $\pi_1[\rho]$ can have a joint atom at $k = m_\rho$. “The worst case” is $\pi_0[\rho](\{m_\rho\}) = \pi_1[\rho](\{m_\rho\})$. This implies the desired lower bound $Q_1(\rho) \geq 1 - \rho(\{m_\rho\}) \geq 1 - H(\rho)$. The upper bound corresponds to “the best case” of mutually singular $\pi_0[\rho]$ and $\pi_1[\rho]$. \qed

From the recurrent structure of $\Upsilon^\rho$ we immediately get the following identity.

Lemma 5.2. $Q_{M+N}(\rho) = Q_M(\rho) + \mathbb{E}Q_N(\Upsilon^\rho_M)$ for any $\rho \in \Delta(\mathbb{R})$ and $M, N \in \mathbb{N}$.

The next estimate follows from Lemmas 5.1 and 5.2.

Lemma 5.3. $N - \mathbb{E}\left(\sum_{n=1}^{N-1} H(\Upsilon_n^\rho)\right) \leq Q_N(\rho) \leq N$.

For any $d = (d_1, d_2, \ldots, d_M) \in \{0, 1\}^M$ we denote the composition $\pi_{d_M} \circ \pi_{d_{M-1}} \circ \cdots \circ \pi_{d_1}$ by $\pi(d)$. Let $N(\rho, d)$ be the integer part of $-\log_2 H(\pi(d)[\rho])$.

Lemma 5.4. $Q_{M+N(\rho, d)}(\rho) \geq 2^{-M}(N(\rho, d) - 1) \quad \forall M \in \mathbb{N}, \ d \in \{0, 1\}^M$.

Proof. Since $H(\Upsilon_n^\rho) \leq 2^n H(\rho)$, from Lemma 6.3 it follows that $Q_N(\rho) \geq N - H(\rho)2^N$. By Lemma 5.2 we get $Q_{M+N}(\rho) \geq \mathbb{P}(\{\Upsilon_M^\rho = \pi(d)[\rho]\})Q_N(\pi(d)[\rho])$, and the definition of $\Upsilon^\rho$ implies $\mathbb{P}(\{\Upsilon_M^\rho = \pi(d)[\rho]\}) = 2^{-M}$.

\qed

Proposition 5.1 is proved by selecting an appropriate sequence $d^{(M)} \in \{0, 1\}^M$, $M \in \mathbb{N}$, and then applying Lemma 5.4.

Proof of Proposition 5.7. As it is mentioned above, we can assume that $K = \mathbb{N}$ and that $\rho \in \Delta(\mathbb{N}) \subset \Delta(\mathbb{R})$ is monotonically decreasing. We set $d^{(M)} = (1, 1, \ldots, 1, 0)$ and denote $N(\rho, d^{(M)})$ by $N_M$. By Lemma 5.4 it is enough to show that

$$\limsup_{M \to \infty} \frac{2^{-M}N_M}{(N_M + M)^{\frac{1}{2} + \gamma}} = \infty.$$

Moreover, it is sufficient to check only that

$$\limsup_{M \to \infty} 2^{-M} (N_M)^{\frac{1}{2} - \gamma} = \infty.$$ (5.3)

Indeed, for any sequence $M_L \to \infty$ such that $2^{-M_L} (N_{ML})^{\frac{1}{2} - \gamma} \to \infty$ we also have $M_L + N_{ML} = N_{ML}(1 + o(1))$. We put $\rho_M = \pi(d^{(M)})[\rho] = \pi_0(\pi_1)^{M-1}[\rho]$. From the monotonicity of $\rho$ we get $\rho(\{k\}) \geq 2^{-M-1}H(\rho_{M+1})$ for any $k \in \text{supp} \rho_M$. Since $\rho$ can be represented as $\sum_{M=1}^{\infty} 2^{-M} \rho_M$, the condition $Z_c(\rho) = \infty$ implies the divergence of the sum

$$\sum_{M=1}^{\infty} 2^{-M} \left(\frac{\ln 2^{M+1}}{H(\rho_{M+1})}\right)^{\frac{1}{2} - \varepsilon}.$$

As $(\ln ab)^c \leq (2\ln b)^c + (2\ln a)^c$ for positive $c$ and $a, b \geq 1$, we see that the sum $\sum_{M=1}^{\infty} 2^{-M} (N_M)^{\frac{1}{2} - \varepsilon}$ also diverges. This implies (5.3) and concludes the proof. \qed

Define the maximal variation of dyadic martingales by $\Psi^2_N(\rho) = \sup_{\mu \in M^2(\mathbb{N})} V_N(\mu)$. 


Proposition 5.2. Suppose \( K \) is countable. Then for any non-degenerate \( \rho \in \Delta(K) \) and \( \gamma < \varepsilon^*(\rho) \) we have

\[
(5.4) \quad \limsup_{N \to \infty} \frac{\Psi_N^2(\rho)}{N^2+\gamma} = \infty.
\]

Proof. If \( \varepsilon^*(\rho) > 0 \), then \( Z_\varepsilon(\rho) = \infty \) for any \( \varepsilon \) between \( \gamma \) and \( \varepsilon^*(\rho) \), and, therefore, the statement follows from Proposition 5.1. Now assume that \( \varepsilon^*(\rho) = 0 \). It is enough to show that there is a sequence of martingales \( \mu^{(N)} \) from \( \mathcal{M}_{\Delta(K)}(\rho) \) such that \( \limsup_{N \to \infty} V_N(\mu^{(N)})/\sqrt{N} > 0 \). Let \( k^* \) be the heaviest atom of \( \rho \). Denote \( \rho(\{k^*\}) \) by \( \rho \). It follows from the results of B. De Meyer (see [5], the end of Section 1) that for any \( p \in (0,1) \) there is a sequence of scalar dyadic martingales \( X^{(N)} \in \mathcal{M}_{[0,1]}(p) \) such that \( V_N(X^{(N)})/\sqrt{N} \to \phi(x_p) > 0 \), where \( \phi(x_p) \) is the normal density at its \( p \)-quantile. Finally we set \( \mu^{(N)}_n = (1_{\{k^*\}}x^{(N)} + 1_{K\setminus\{k^*\}}(1 - X^{(N)}))/\rho \). Thus \( V_N(\mu^{(N)})/\sqrt{N} \to 2\phi(x_p) > 0 \). \( \square \)

Now we pass to the proof of Theorem 3.2.

Proof of Theorem 3.2. From Theorem 3.1 it follows that \( \alpha_m(\rho) \leq \frac{1}{2} + \varepsilon^*(\rho) \). Since \( \Psi_N(\rho) \geq \Psi_N^2(\rho) \), Proposition 5.2 implies the reverse inequality. \( \square \)

5.3. Prior distributions with nontrivial continuous component. In this subsection Theorem 3.4 is proved. We begin with a lemma describing the case with no atoms.

Lemma 5.5. Let \( \rho \in \Delta(\mathbb{R}) \) be purely continuous. Then \( \Psi_N(\rho) = 2N \).

Proof. By Proposition 4.2 it is enough to show that \( \Psi_N(\rho) \geq 2N \). For \( D \in \mathbb{N} \) define a martingale \( \Upsilon^\rho, D \) by \( \Upsilon^\rho_n = Y^\rho_n \). For this martingale all \( 2^D \) possible values of \( \Upsilon^\rho_{n+1} \) given \( \Upsilon^\rho_n \) are mutually singular. The same reasoning as in Lemma 5.1 leads to \( V_N(\Upsilon^\rho, D) = 2N(1 - 2^{-D}) \). Therefore, \( \Psi_N(\rho) \geq 2N(1 - 2^{-D}) \). Since \( D \) is arbitrary, the proof is completed. \( \square \)

Proof of Theorem 3.4. The maximal variation does not depend on geometry of \( K \), and so the measurable structure is important. Since all uncountable Polish spaces are Borel isomorphic, it is enough to consider a particular one, and we choose \( K = \mathbb{R} \).

Let us show that \( \Psi_N(\rho) \geq 2N\rho^*(\mathbb{R}) \) for \( \rho \in \Delta(\mathbb{R}) \). If \( \rho \) is discrete, this inequality is trivial. Otherwise we set \( \rho' = \frac{1}{\rho^*(\mathbb{R})}\rho \in \Delta(\mathbb{R}) \). Note that for any martingale \( \mu' \in \mathcal{M}_{\Delta(\mathbb{R})}(\rho') \) the martingale \( \mu \) given by \( \mu_n = \rho^*(\mathbb{R})\mu'_n + \rho \) belongs to \( \mathcal{M}_{\Delta(\mathbb{R})}(\rho) \), and \( V_N(\mu) = \rho^*(\mathbb{R})V_N(\mu') \). Therefore, \( \Psi_N(\rho) \geq \rho^*(\mathbb{R})\Psi_N(\rho') \), and the required inequality follows from Lemma 5.4. Proposition 4.2 gives the upper bound. Together with Remark 4.3 this concludes the proof. \( \square \)

Remark 5.2. Consider the dyadic case. If \( \rho = \rho^{\varepsilon} \), then \( H(\Upsilon^\rho_n) = 0 \), and Lemma 5.3 implies \( \Psi_N^2(\rho) \geq N \) (in fact, one can show that in this case \( \Psi_N^2(\rho) = N \)). Hence, by the same reasoning as in the proof of Theorem 3.4 we get \( \Psi_N^2(\rho) \geq N\rho^*(K) \) for arbitrary \( \rho \in \Delta(K) \).
6. ANOMALOUS BEHAVIOR OF ERROR TERM

In this section we prove Theorem 3.3 that describes the slowest possible rate of error term decreasing.

Recall that $\Psi_N^2(\rho)$ is the maximal variation of dyadic martingales (see Section 5).

**Proposition 6.1.** If $K$ is countable, then there exists a 4-tuple $G$ from $\mathcal{G}(K)$ such that

\[
err[\mathcal{G}_N(\rho)] \geq \frac{1}{2N} \Psi_N^2(\rho) \quad \text{for any } \rho \in \Delta(K) \text{ and } N \in \mathbb{N}.
\]

Theorem 3.3 immediately follows from this proposition.

**Proof of Theorem 3.3.** From Theorem 2.1 we get $\alpha_G(\rho) \leq \alpha\Psi(\rho) - 1$, and Theorem 3.2 implies $\alpha\Psi(\rho) = \frac{1}{2} + \varepsilon^*(\rho)$. From Proposition 6.1 it follows that $\alpha_G(\rho) \geq -1 + \limsup_{N \to \infty} \frac{\ln \Psi_N^2(\rho)}{\ln N}$, and by Proposition 5.2 this limit is greater than $\frac{1}{2} + \varepsilon^*(\rho)$. \qed

To prove Proposition 6.1 we construct the repeated game $\mathcal{G}_N(\rho)$ explicitly. The proof is divided into several lemmas, where we check the properties of $\mathcal{G}_N(\rho)$.

6.1. **The game $\mathcal{G}_N(\rho)$ and its properties.** The construction is based on two ingredients. One of them is the well-known repeated game $G_N(p)$ with incomplete information introduced by S. Zamir in [18]. This game has two-element set of states $\{0, 1\}$ and stage payoffs given by the following matrices (Player 1 is the row-chooser)

\[
a^1 = \begin{pmatrix} 3 & -1 \\ -3 & 1 \end{pmatrix} \quad \text{and} \quad a^0 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.
\]

As it was shown by J.-F. Mertens and S. Zamir in [11], $\text{err}[G_N(p)] = \frac{1}{2N} \Psi_N(p)(1 + o(1))$ as $N \to \infty$ for any $p \in \Delta(\{0, 1\})$. Hence, this game exhibits $1/\sqrt{N}$-behavior of the error term, i.e., the slowest possible rate of decreasing over $\mathcal{G}(\{0, 1\})$. Another ingredient is a special structure that

1. allows informed player to choose what (state-dependent) game he wants to play at a current stage from a sufficiently large amount of alternatives,
2. informs Player 2 of this choice.

Importance of this structure comes from the intuition that, if Player 1 has enough choices of different stage games, then at a current stage he can emphasize any particular part of uninformed player’s lack of knowledge and uniformly benefit from it. Building of games with such a structure from “elementary” blocks was considered by A. Neyman in [15].

We start with informal description of the game $\mathcal{G}_N(\rho)$ corresponding to the 4-tuple $\mathcal{G}$. As usual, at the beginning of the game Nature chooses a $\rho$-distributed random element $k$ from the set of states $K$. Player 1 is informed of $k$ and Player 2 is not. It is convenient to represent each stage $n = 1, 2, ...N$ of the game $\mathcal{G}_N(\rho)$ as being played in two steps:

1. at the first step Player 1 selects a finite subset $X_n \subset K$, and Player 2 selects a finite subset $Y_n \subset K$, then $X_n \cap Y_n$ is told Player 2;
(2) at the second step they play the matrix game $a^1$ if $k \in X_n$, and they play $a^0$, otherwise.

That is, at the second step Player 1 selects a row $x_n \in \{1,2\}$ using information he has, and Player 2 selects a column $y_n \in \{1,2\}$. Before the next stage the selected actions $i_n = (X_n, x_n)$ and $j_n = (Y_n, y_n)$ are announced. The total gain of Player 1 is the expected arithmetic mean of per-step gains.

**Remark 6.1.** It is reasonable for Player 2 to select $Y_n$ as big as possible. But we forbid him to choose infinite subsets in order to truncate his set of actions $J$ and, therefore, to make it countable for countably infinite $K$.

Now we define $\mathcal{G}$ formally. The set of states is $K$. The action set $I$ of Player 1 consists of pairs $(X, x)$, where $X$ is a finite subset of $K$, and $x$ is an element of $\{1,2\}$. The set of actions $J$ of Player 2 consists of pairs $(Y, y)$, where $Y$ is a finite subset of $K$, and $y$ is a function from $2^Y$ to $\{1,2\}$. The one-stage payoff function $A$ is given by

$$A_{i,j}^{k}(X,x),(Y,y) = a_{x,y}(X\cap Y).$$

First we prove a version of Proposition 6.1 for finite $K$. For this purpose we pass from games to the martingale optimization problem similar to the maximal variation one. This reduction was introduced by B. De Meyer in [6] to analyze a market model based on repeated games with incomplete information. His approach was extended to general repeated games with incomplete information having finite $K$ and Polish $I$ and $J$ by F. Gensbittel in [7].

**Theorem 6.1** (F. Gensbittel). Consider a 4-tuple $\Gamma = (K, I, J, A)$ with finite $K$, Polish $I$ and $J$, and bounded $A$. Define an auxiliary 4-tuple $\Gamma^\Delta$ with the set of states $K^\Delta = \Delta(K)$, action sets $I$ and $J$, and one-stage payoff function given by the formula $A_{i,j}^{k} = \sum_{k \in K} A_{i,j}^{k} \rho(\{k\})$ for any $\rho \in K^\Delta$. Then for any $\rho \in \Delta(K)$ and $N \in \mathbb{N}$

$$\text{val}[\Gamma_N(\rho)] = \sup_{(\mu_n, F_n)_{n \geq 0} \in M_{\Delta(K)}(\rho)} \mathbb{E} \left( \frac{1}{N} \sum_{n=1}^{N} \text{val}[\Gamma_1^\Delta(\Lambda_n)] \right),$$

where $\Lambda_n \in \Delta(K^\Delta)$ is the conditional distribution of $\mu_n$ given $F_{n-1}$.

**Remark 6.2.** The game $\Gamma_N^\Delta(\Lambda)$ with $\Lambda \in \Delta(K^\Delta)$ can be interpreted as a partial information extension of $\Gamma_N(\rho)$, i.e., Player 1 does not know the state $k$ exactly but receives a noisy signal $M$ with partial information about it. So $\Lambda$ represents the distribution of informed player’s beliefs after observing $M$. Note that the non-revealing game $G_1^{\text{NR}}(\rho)$ can be clearly identified with $\Gamma_1^\Delta(\delta_\rho)$, where $\delta_\rho$ is the Dirac $\delta$-measure at $\rho$.

Consider the partial-information extension of $G_1(p)$. So $p = (p(\{0\}), p(\{1\}))$ is distributed according to $\lambda \in \Delta(\Delta(\{0,1\}))$. Let $\pi$ be the distribution of $p(\{1\})$. Then $G_1^\Delta(\lambda)$ is equivalent to the following game. Nature picks a random $\pi$-distributed state $s = p(\{1\}) \in [0,1]$ and tells it to Player 1. Further Player 1 and Player 2 play the following matrix game

$$a^s = sa^1 + (1-s)a^0 = \begin{pmatrix} s & s-2 \\ -(s+2) & -(s-2) \end{pmatrix}.$$

**Lemma 6.1.** $\text{val}[G_1^\Delta(\lambda)] = \min_{m \in [0,1]} \mathbb{E}[s - m]$. 
Proof. If \( s \) takes only finite number of different values, then the game \( G^A_1(\lambda) \) has a value by the minimax theorem. In general case the existence of the value can be checked by standard approximation arguments. Let us compute \( \text{val}[G^A_1(\lambda)] \). Assume that Player 2 uses a mixed strategy \( (t, 1-t) \), i.e., selects the left column with probability \( t \) and the right one with probability \( 1-t \). Then the optimal reply of Player 1 is to select the row that gives him a positive gain (depending on \( s \)). Hence, his average gain is \( E|s+(4t-2)|. \) Therefore, \( \text{val}[G^A_1(\lambda)] = \min_{t\in[0,1]} E|s+(4t-2)| = \min_{m\in[0,1]} E|s-m|. \)

Remark 6.3. If \( p \) takes only two different values \( p^0 \) and \( p^1 \) equally likely, i.e., \( \lambda = \frac{1}{2}\delta_{\rho^0(\{0\})},\rho^0(\{1\})) + \frac{1}{2}\delta_{\rho^1(\{0\}),\rho^1(\{1\})}) \), then

\[
\text{val}[G^A_1(\lambda)] = \frac{1}{2}E\|p - E\rho\|_{TV}.
\]

Lemma 6.2. The statement of Proposition 6.1 holds for finite \( K \).

Proof. We claim that the constructed 4-tuple \( \mathfrak{G} \) fits the requirements of Proposition 6.1. Indeed, since \( K \) is finite, the action sets \( I \) and \( J \) are also finite, and so the games \( \mathfrak{G}_N(\rho) \) and \( \mathfrak{G}^\mathrm{NR}_1(\rho) \) have values by the minimax theorem. Therefore, the 4-tuple \( \mathfrak{G} \) belongs to \( \mathcal{G}(K) \).

From Lemma 6.1 and Remark 6.2 it follows that \( u_G(p) = 0 \) for any \( p \). Let us check that \( u_\emptyset \) is also identically zero, too. If at the first step of \( \mathfrak{G}^\mathrm{NR}_1(\rho) \) Player 1 selects an arbitrary \( X_1 \subset K \), then at the second step both players know only that the probability to play \( a^1 \) is \( \rho(X_1) \) (note that Player 2 can take \( Y_1 = K \) because \( K \) is finite). Therefore, the game at the second step is in fact \( G^\mathrm{NR}_1(\rho) \) with \( p = (\rho(K \setminus X_1), \rho(X_1)) \). Playing optimally in \( G^\mathrm{NR}_1(\rho) \) both players guarantee to get at least 0. Therefore, \( u_\emptyset(\rho) = 0 \) and \( \text{val}[\mathfrak{G}_N(\rho)] = \text{err}[\Gamma_N(\rho)] \) for any \( \rho \).

From Theorem 6.1 it follows that the estimate (6.1) can be proved by checking that for any \( \mu \in \mathcal{M}^2_{\Delta(K)}(\rho) \)

\[
\text{val}[\mathfrak{G}^A_1(\Lambda_1)] \geq \frac{1}{2}E\|\mu_1 - E\mu_1\|_{TV},
\]

where \( \Lambda_1 \) is the distribution of \( \mu_1 \). Recall that for any dyadic martingale \( \mu \) the random measure \( \mu_1 \) takes only two different values, say, \( \rho^0 \) and \( \rho^1 \) equally likely. Consider the following strategy of Player 1 in \( \mathfrak{G}^A_1(\Lambda_1) \). At the first step he selects \( X_1 = K_{<s} \), where \( K_{<s} = \{k \in K \mid \rho^0(\{k\}) < \rho^1(\{k\})\} \). Then at the second step the payoff matrix is \( a^s \), where \( s \) equals \( \rho^0(K_{<s}) \) or \( \rho^1(K_{<s}) \) with probability \( \frac{1}{2} \). Player 2 does not know \( s \), but Player 1 does. In other words, at the second step players face the game \( G^A_1(\lambda) \) with \( \lambda = \frac{1}{2}\delta_{\rho^0} + \frac{1}{2}\delta_{\rho^1}, \) where \( p^i = (\rho^i(K \setminus K_{<s}), \rho^i(K_{<s})) \), \( i = 0, 1 \). By (6.3) the optimal behavior at the second step gives Player 1 at least \( \frac{1}{2}E\|p - E\rho\|_{TV} = \frac{1}{2}E\|\mu_1 - E\mu_1\|_{TV} \). This concludes the proof.

To ensure that the result remains valid also for countably infinite \( K \) we use finite approximations. For this purpose we should check that \( \Psi^2_N(\rho) \) depends on \( \rho \) in a regular way.

Lemma 6.3. If \( K \) is countable, then \( \Psi^2_N(\rho) \) is a \( 2N \)-Lipschitz function of \( \rho \) in the total variation norm, i.e.,

\[
|\Psi^2_N(\rho) - \Psi^2_N(\rho')| \leq 2N\|\rho - \rho'\|_{TV} \quad \forall \rho, \rho' \in \Delta(K).
\]
Remark 6.4. The same proof shows that $\Psi_N(\rho)$ is also $2N$-Lipschitz.

Proof. It is enough to show that for any $\rho \neq \rho'$ and any martingale $\mu \in \mathcal{M}_d^2(K)(\rho)$ there exists $\mu' \in \mathcal{M}_d^2(K)(\rho')$ such that $|V_N(\mu) - V_N(\mu')| \leq 2N\|\rho - \rho'\|_{\text{TV}}$. Let $\theta_n$ be a positive measure on $K_1 = \{k \in K \mid \rho'(\{k\}) < \rho(\{k\})\}$ such that $\theta_n(\{k\}) = \frac{\rho'(\{k\}) - \rho(\{k\})}{\rho(\{k\})}\mu_n(\{k\})$ for $k \in K_1$. Define $\mu'$ on one-element sets by

$$
\mu'_n(\{k\}) = \mu_n(\{k\}) + \begin{cases} 
-\theta_n(\{k\}), & k \in K_1 \\
\frac{\rho'(\{k\}) - \rho(\{k\})}{\rho(\{k\})}\theta_n(K_1), & k \in K \setminus K_1.
\end{cases}
$$

Then it is easy to check that $\mu' \in \mathcal{M}_d^2(K)(\rho')$, and $E\|\mu_n - \mu'_n\|_{\text{TV}} = \|\rho - \rho'\|_{\text{TV}}$ for any $n$. Thus the result follows from the triangle inequality.

The following lemma is well known (see [14], p.219 and p.222) in the case of finite $K, I$ and $J$.

**Lemma 6.4.** Suppose that in a 4-tuple $\Gamma = (K, I, J, A)$ the sets $K$, $I$, and $J$ are arbitrary Polish spaces and $A$ is bounded. Then $\text{val}[\Gamma^{NR}_N(\rho)], \text{val}[\Gamma^{NR}_N(\rho)], \text{val}[\Gamma_N(\rho)],$ and $\text{val}[\Gamma_N(\rho)]$ are $|A|_{\text{\infty}}$-Lipschitz functions of $\rho$ in the total variation norm.

Proof. The idea of the proof comes from the paper [7] of F. Gensbittel (Proposition 2.1). It is enough to show that $|g_N(\rho, \sigma, \tau) - g_N(\rho', \sigma, \tau)| \leq |A|_\infty \|\rho - \rho'\|_{\text{TV}}$ for any behavioral strategies $\sigma$ and $\tau$ and for any $\rho, \rho' \in \Delta(K)$ (recall that $g_N(\rho, \sigma, \tau)$ is defined by (2.1)). The quantity $q_N(k_0, \sigma, \tau) = \frac{1}{N}E_{\rho, \sigma, \tau}(\sum_{n=1}^{\infty} A_{n,\rho_0}^{k_0} \mid k = k_0)$ can be regarded as unconditional expectation with respect to the probability measure over $\{k_0\} \times (I \times J)^N$ generated by $\sigma$ and $\tau$, and, hence, $q$ does not depend on the prior distribution $\rho$. Therefore,

$$
g_N(\rho, \sigma, \tau) - g_N(\rho', \sigma, \tau) = \int_K (d\rho(k) - d\rho'(k))q_N(k, \sigma, \tau).
$$

Since $q$ is bounded by $|A|_\infty$ in absolute value, the proof is completed.

Now we turn to the proof of Proposition [6.1].

Proof of Proposition [6.1]. The case of finite $K$ is considered in Lemma [6.2] and, hence, we assume that $K$ is countably infinite and analyze $\mathcal{G}$ in this case. Note that $I$ and $J$ are also countably infinite for such $K$.

We claim that for any $\rho \in \Delta(K)$ the games $\mathcal{G}_1^{NR}(\rho)$ and $\mathcal{G}_N(\rho)$ have values and that [6.1] holds. Indeed, consider a sequence $\rho_n$ of finitely-supported probabilities converging to $\rho$ in the total variation norm. By Lemmas [6.4] and [6.3] if we already know the result for all $\rho_n$, then we also get it for $\rho$.

Therefore, the problem is reduced to the case of $\rho$ supported on a finite subset of $K$, say $K'$. We write $K\mathcal{G}_N(\rho)$ to indicate the set of states explicitly. Then there are two different games $K\mathcal{G}_N(\rho)$ and $K\mathcal{G}_N(\rho)$. In contrast to the second game, the fist one has infinite sets of actions. In $K\mathcal{G}_N(\rho)$ both players know that $k$ belongs to $K'$ almost surely. Then selecting $X_n \cap K'$ instead of $X_n$ leads to the same game at the second step, and also choosing $Y_n \cap K'$ instead of $Y_n$ does not affect the information Player 2 receives. Hence, if players are restricted to use in $K\mathcal{G}_N(\rho)$ only the strategies inherited from $K'\mathcal{G}_N(\rho)$, then they get the same guaranteed payoffs as without any restrictions.
Therefore, by Lemma 6.2 \( \val[K] \mathcal{G}_N(\rho) = \overline{\val[K]} \mathcal{G}_N(\rho) = \val[K] \mathcal{G}_N(\rho) \geq \frac{1}{2N} \Psi_N^2(\rho) \) and \( \val[K] \mathcal{G}_N^{NR}(\rho) = \overline{\val[K]} \mathcal{G}_N^{NR}(\rho) = 0 \), which concludes the proof. \( \square \)

6.2. Extension to uncountable \( K \). Here we briefly describe a version of the game \( \mathfrak{G}_N(\rho) \) for \( K = [0, 1] \). Let \((\mathcal{F}_n^2)_{n \geq 0}\) be the standard dyadic filtration of \([0, 1]\) with the Lebesgue measure (see Remark 5.1). Denote by \( X \) the union of \( \mathcal{F}_n^2 \) over all \( n \). Each stage \( n \) of \( \mathfrak{G}_N(\rho) \) consists of two steps as it was for countable \( K \). But now at the first step Player 1 selects \( X_n \subseteq X \), and Player 2 selects \( M_n \in \mathbb{N} \). Before the second step that proceeds as before the smallest set \( Y_n \subseteq \mathcal{F}_M^2 \) containing \( X_n \) is told to Player 2.

**Proof of Theorem 3.3** The scheme of the proof is the following. Analyzing the properties of \( \mathfrak{G} \) one can show that Proposition 6.1 remains valid for \( K = [0, 1] \) (or even for any uncountable Polish space). Then Remark 5.2 implies the result. \( \square \)

**Remark 6.5**. The game constructed has a pathological property: for any \( k, k' \in [0, 1] \) such that \( k \neq k' \) we have \( \sup_{i,j} |A_{k}^{i,j} - A_{k'}^{i,j}| = 1 \). In other words, the payoff function is discontinuous at every point of \([0, 1]\). This suggests that the non-decreasing error term is a kind of pathology.

7. Concluding remarks

A well-known problem in the theory of repeated games with incomplete information is to show that after proper normalization the error term and the maximal variation converge as \( N \to \infty \). Even for finite \( K \) this problem is not solved in full generality (the main existing results can be found in \([3, 4, 5, 6, 8, 11, 12, 13]\)).

Of course, this convergence problem has a counterpart concerning anomalous behavior for heavy-tailed prior distributions \( \rho \) over countable \( K \). Even for logarithmic asymptotics we do not know the existence of the limits. For example, one can put \( \liminf \) instead of \( \limsup \) in the definitions of \( \alpha_{\Psi}(\rho) \) and \( \alpha_{\mathcal{G}}(\rho) \) without changing the statements of Theorems 3.2 and 3.3.

We guess that the answer to this question is “yes”. Hence, \( \Psi_N(\rho) = N^{\alpha_{\Psi}(\rho) + o(1)} \) as \( N \to \infty \) for non-degenerate \( \rho \). This leads to a more delicate question about convergence of \( \Psi_N(\rho)/N^{\alpha_{\Psi}(\rho)} \) for large \( N \). We expect that this limit may fail to exist for general \( \rho \) such that \( \alpha_{\Psi}(\rho) > 1/2 \). The reason is that the exponent \( \alpha_{\Psi} \) is not constant when \( \rho \) ranges over the class of such heavy-tailed distributions, and this suggests the possibility of “intermediate” behaviors (for example, \( N^{o(1)} \) can contain logarithmic growth). Our conjecture is that the proper normalization of \( \Psi_N(\rho) \) should be given in terms of the “distribution function” \( F(n) = \max_{B \subseteq K, \#B \leq n} \rho(B) \), and, hence, should precisely reflect tail asymptotics of \( \rho \).

Note that the anomalous behavior has the following universality property: if the exponent \( \alpha_{\Psi}(\rho) \) is greater than \( 1/2 \), then the asymptotic behavior of \( \Psi_N(\rho) \) is determined only by the tails of \( \rho \). Indeed, the contribution of any finite \( B \subset K \) to the maximal variation is of the order of \( \sqrt{N} \) and, hence, can be neglected. This explains why further investigation of heavy-tailed setting can be even simpler than investigation of the classical setting with finite \( K \): for finite \( K \) one should take into account the contribution of each atom of \( \rho \).
Another question for the further study is to characterize the slowest speed of error term decreasing for games with uncountable $K$. To avoid pathological situations in this case (as in the end of Section 6) some regularity assumptions on one-stage payoffs should be made. Therefore, the problem is to find the impact of regularity on behavior of the error term. An approach allowing to take regularity into account in upper estimates on the error term is developed by F. Gensbittel in [9]. The main ingredient of this approach is the maximal variation, where instead of the total variation norm the Kantorovich (Wasserstein) metric is used.

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