One-loop radiative correction to the Toshev relation for neutrino oscillations in matter

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Abstract

The one-loop electroweak radiative corrections to coherent forward neutrino scattering in a medium slightly violates the tree-level universality of neutral-current contributions of three neutrino flavors to the matter potential that is relevant to neutrino oscillations in matter. We examine this small but nontrivial quantum effect by deriving the differential equations of those effective neutrino oscillation quantities with respect to the electron number density of matter. The tree-level Toshev relation $\sin 2\tilde{\theta}_{23} \sin \tilde{\delta} = \sin 2\theta_{23} \sin \delta$, which links the fundamental flavor-mixing and CP-violating parameters $(\theta_{23}, \delta)$ to their matter-corrected counterparts $(\tilde{\theta}_{23}, \tilde{\delta})$ in the standard parametrization of the $3 \times 3$ lepton flavor mixing matrix, is shown to be modified at the one-loop level. We numerically illustrate the significance of such effects for neutrino and antineutrino oscillations in dense matter.

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1 Introduction

When a neutrino beam travels in an electrically neutral and unpolarized medium, the coherent forward neutrino scattering with electrons via weak charged-current (CC) interactions and with electrons, protons and neutrons via weak neutral-current (NC) interactions can definitely modify the behaviors of neutrino oscillations— a striking phenomenon which is usually referred to as the Mikheev-Smirnov-Wolfenstein (MSW) matter effect \[\text{[1\,2\,3]}\). The effective Hamiltonian responsible for the evolution of three active neutrino flavors in matter is composed of the vacuum term and the matter potential term as follows \[\text{[4\,5]}\]:

\[
\mathcal{H}_{\text{eff}} = \frac{1}{2E} \left[ \begin{array}{ccc} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{array} \right] U \left( \begin{array}{ccc} V_\alpha & 0 & 0 \\ 0 & V_\mu & 0 \\ 0 & 0 & V_\tau \end{array} \right) \equiv \frac{1}{2E} \left[ \begin{array}{ccc} \tilde{m}_1^2 & 0 & 0 \\ 0 & \tilde{m}_2^2 & 0 \\ 0 & 0 & \tilde{m}_3^2 \end{array} \right] V^\dagger, \tag{1}\]

where \(m_i\) (for \(i = 1, 2, 3\)) and \(U\) stand respectively for the neutrino masses and the Pontecorvo-Maki-Naka-gawa-Skata (PMNS) flavor mixing matrix \[\text{[6\,7\,8]}\) in vacuum, \(\tilde{m}_i\) (for \(i = 1, 2, 3\)) and \(V\) are defined to be their effective counterparts in matter, and \(V_\alpha\) (for \(\alpha = \epsilon, \mu, \tau\)) describe the effects of coherent forward neutrino scattering with matter. At the tree level we have \(V_\epsilon = V_{\text{CC}} + V_{\text{NC}}\) and \(V_\mu = V_\tau = V_{\text{NC}}\), where \(V_{\text{CC}} = \sqrt{2}G_F N_e\) and \(V_{\text{NC}} = -G_F/\sqrt{2} \left[ (1 - 4\sin^2\theta_w) (N_e - N_p) + N_n \right]\) with \(G_F\) being the Fermi coupling constant, \(\theta_w\) being the Weinberg angle of weak interactions, and \(N_e, N_p\) and \(N_n\) being the number densities of electrons, protons and neutrons in matter. Given the fact that \(N_e = N_p\) holds for a normal medium, one is simply left with \(V_{\text{NC}} = -G_F N_n/\sqrt{2}\). Since the tree-level result of \(V_{\text{NC}}\) is universal for \(\nu_\epsilon, \nu_\mu\) and \(\nu_\tau\) neutrinos, it has no physical impact on neutrino oscillations and hence can be ignored in the standard three-flavor scheme.

The \(3 \times 3\) PMNS lepton flavor mixing matrix \(U\) in Eq. \(\text{(1)}\) can be parametrized in a way advocated by the Particle Data Group \[\text{[9]}\): \(U = P_l O_{23} O_{13} O_{12}^\dagger O_{\nu}\); namely,

\[
U = P_l \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} P_\nu = P_l \begin{pmatrix} c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta} \\ -s_{12} c_{23} - c_{12} s_{13} s_{23} e^{i\delta} & c_{12} c_{23} - s_{12} s_{13} s_{23} e^{i\delta} & c_{13} s_{23} \\ s_{12} s_{23} - c_{12} s_{13} c_{23} e^{i\delta} & -c_{12} s_{23} - s_{12} s_{13} c_{23} e^{i\delta} & c_{13} c_{23} \end{pmatrix} P_\nu, \tag{2}\]

where \(c_{ij} \equiv \cos \theta_{ij}\) and \(s_{ij} \equiv \sin \theta_{ij}\) (for \(ij = 12, 13, 23\)) with \(\theta_{ij}\) being the flavor mixing angles, \(\delta\) denotes the Dirac phase responsible for CP violation in neutrino oscillations, \(P_l\) is a diagonal phase matrix which is sensitive to rephasing the charged-lepton fields but has no physical significance, and \(P_\nu\) is another diagonal phase matrix relevant to Majorana neutrinos but has nothing to do with neutrino oscillations. We adopt the same parametrization for the effective PMNS matrix \(V\) in matter, whose Dirac phase and flavor mixing angles are denoted respectively as \(\tilde{\delta}\) and \(\tilde{\theta}_{ij}\) (for \(ij = 12, 13, 23\)). Note that \(V_\mu = V_\tau\) allows \(O_{23}, O_\delta\) and \(O_\delta^\dagger\) to commute with the diagonal matter potential term in Eq. \(\text{(1)}\), and this property leads us to the intriguing Toshev relation \[\text{[10\,11\,12]}\): \(\sin 2\tilde{\theta}_{23} \sin \tilde{\delta} = \sin 2\theta_{23} \sin \delta\), \(\tag{3}\)
which directly links the fundamental flavor-mixing and CP-violating parameters \((\theta_{23}, \delta)\) to their effective counterparts \((\tilde{\theta}_{23}, \tilde{\delta})\) in matter. Eq. (3) tells us that \(\tilde{\theta}_{23} = \pi/4\) and \(\tilde{\delta} = \pm \pi/2\) are a natural consequence of \(\theta_{23} = \pi/4\) and \(\delta = \pm \pi/2\) in the \(\mu-\tau\) reflection symmetry limit \([13, 14, 15]\).

But the tree-level universality of weak neutral-current contributions to \(V_\alpha\) in Eq. (1) will be slightly broken once the one-loop electroweak radiative corrections are taken into account \([16, 17]\). It is found that such quantum effects are suppressed by the factors \(G_F m_\alpha^2\) (for \(\alpha = e, \mu, \tau\)) as compared with their corresponding tree-level terms and hence negligibly small in both \(V_e\) and \(V_\mu\).

In view of \(m_e \ll m_\mu \ll m_\tau\) \([9]\), one obtains \([16, 17]\)

\[
\begin{align*}
V_e - V_\mu &= \sqrt{2} G_F N_e, \\
V_\tau - V_\mu &= -\frac{3 G_F^2 m_\mu^2}{2 \pi^2} \ln \left( \frac{m_\tau^2}{m_\mu^2} + N_p + \frac{2}{3} N_n \right),
\end{align*}
\]

in an excellent approximation, where \(m_\tau\) and \(m_W\) stand respectively for the tau-lepton and \(W\) boson masses. Taking \(N_p = N_e\) for a neutral medium and inputting \(G_F \simeq 1.1664 \times 10^{-5}\) GeV\(^{-2}\), \(m_W \simeq 80.377\) GeV and \(m_\tau \simeq 1.777\) GeV \([9]\), we arrive at

\[
\begin{align}
\tau \equiv \frac{V_\tau - V_\mu}{V_e - V_\mu} &= \frac{3 G_F m_\mu^2}{2 \sqrt{2} \pi^2} \ln \left( \frac{m_\tau^2}{m_\mu^2} + 1 - \frac{2N_n}{3N_p} \right) \\
&\simeq \begin{cases} 
5.4 \times 10^{-5} & (N_n/N_p = 1), \\
1.1 \times 10^{-4} & (N_n/N_p = 3),
\end{cases}
\end{align}
\]

where two typical numerical examples for the ratio \(N_n/N_p\) have been given for illustration. After the one-loop quantum corrections to the matter potential term are included, the effective Hamiltonian in Eq. (1) can be rewritten as

\[
\mathcal{H}_{\text{eff}} = \frac{1}{2E} \left[ U D U^\dagger + a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & r \end{pmatrix} \right] \equiv \frac{1}{2E} \tilde{V} \tilde{D} V^\dagger,
\]

where \(D \equiv \text{Diag}\{m_1^2, m_2^2, m_3^2\}\), \(\tilde{D} \equiv \text{Diag}\{\tilde{m}_1^2, \tilde{m}_2^2, \tilde{m}_3^2\}\), and \(a = 2\sqrt{2} G_F N_e E\) is referred to as the matter parameter. It becomes obvious that the \(O_{23}, O_{\delta}\) and \(O_{\delta}^\dagger\) components of \(U\) in Eq. (2) do not commute any more with the matter potential term in Eq. (6), and hence the elegant tree-level Toshev relation in Eq. (3) should have no reason to hold at the one-loop level.

Different from Ref. \([18]\), where the properties of \(|V_{\alpha i}|\) (for \(\alpha = e, \mu, \tau\) and \(i = 1, 2, 3\)) in the \(r \neq 0\) but \(a \to \infty\) limit are carefully discussed, the present paper is intended to examine to what extent the tree-level Toshev relation is modified by the one-loop radiative effects characterized by \(r \neq 0\) for finite \(a\). Although such a correction is expected to be negligible in most cases, it is likely to be significant when the neutrino beam travels in dense matter. We find that this is indeed the case. Our approach is to derive a set of differential equations of \(\Delta_{ij}, \tilde{\theta}_{ij}\) and \(\tilde{\delta}\) against the matter variable \(a\) in the \(r \neq 0\) case, from which a new Toshev-like relation between the fundamental quantities \((\theta_{23}, \delta)\) and their matter-corrected counterparts \((\tilde{\theta}_{23}, \tilde{\delta})\) is established. The running behaviors of \(\tilde{\theta}_{ij}\) (for \(ij = 12, 13, 23\)) and \(\tilde{\delta}\) with respect to \(a\), together with an appreciable breaking effect of the tree-level Toshev relation in dense matter, are numerically illustrated.
2 Analytical calculations

Eq. (11) or (6) shows that the effective Hamiltonian responsible for neutrino oscillations in matter is exactly of the same form as that in vacuum, and hence the relevant effective physical quantities evolving with the scale-like matter parameter \( a \) can automatically return to their fundamental counterparts in the \( a \rightarrow 0 \) limit. This observation reminds us that the powerful renormalization-group-equation (RGE) tool \([19, 20, 21]\) is actually applicable to describing the matter effects on neutrino masses and flavor mixing parameters (see, e.g., Refs. \([22, 23, 24, 25, 26, 27, 28]\)). Using such a RGE-like tool, we differentiate both sides of Eq. (6) with respect to \( a \) and arrive at

\[
\dot{D} + [V \dagger \dot{V}, \tilde{D}] = V \dagger \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & r \end{pmatrix} V = \begin{pmatrix} |V_{e1}|^2 & V_{e1}^* V_{e2} & V_{e1}^* V_{e3} \\ V_{e2} V_{e1} & |V_{e2}|^2 & V_{e2}^* V_{e3} \\ V_{e3} V_{e1} & V_{e3}^* V_{e2} & |V_{e3}|^2 \end{pmatrix} + r \begin{pmatrix} |V_{\tau 1}|^2 & V_{\tau 1}^* V_{\tau 2} & V_{\tau 1}^* V_{\tau 3} \\ V_{\tau 2} V_{\tau 1} & |V_{\tau 2}|^2 & V_{\tau 2}^* V_{\tau 3} \\ V_{\tau 3} V_{\tau 1} & V_{\tau 3}^* V_{\tau 2} & |V_{\tau 3}|^2 \end{pmatrix}, \tag{7}
\]

in which the overhead dot denotes the derivative of a matter-corrected quantity, and the square brackets represents the commutator of two matrices (i.e., \([A, B] \equiv AB - BA\)). Then the diagonal and off-diagonal parts of Eq. (7) lead us to

\[
\tilde{\Delta}_{ij} = |V_{ei}|^2 - |V_{ej}|^2 + r \left( |V_{\tau i}|^2 - |V_{\tau j}|^2 \right), \tag{8}
\]

and

\[
\sum_{\alpha} V_{\alpha i}^* \dot{V}_{\alpha j} = \frac{V_{e1}^* V_{e2} + r V_{\tau 1}^* V_{\tau 2}}{\tilde{\Delta}_{ji}}, \tag{9}
\]

where \( \tilde{\Delta}_{ij} \equiv \tilde{m}_i^2 - \tilde{m}_j^2 \) and \( i \neq j \) (for \( i, j = 1, 2, 3 \)). Thanks to the unitarity of \( V \), we simply have

\[
\sum_{i} \left( V_{\alpha i}^* \dot{V}_{\beta i} + \dot{V}_{\alpha i} V_{\beta i} \right) = \sum_{\alpha} \left( V_{\alpha i}^* \dot{V}_{\alpha j} + \dot{V}_{\alpha i} V_{\alpha j} \right) = 0, \tag{10}
\]

no matter whether \( \alpha = \beta \) (or \( i = j \)) holds or not for the first (or second) equalities. Now let us consider the orthogonality relation

\[
\sum_{j \neq i} V_{\alpha j}^* V_{\beta j} = \delta_{\alpha \beta} - V_{\alpha i} V_{\beta i}, \tag{11}
\]

multiply both of its sides with \( \dot{V}_{\alpha i} \) and sum over the flavor index \( \alpha \). Then we obtain

\[
\dot{V}_{\beta i} = \sum_{\alpha} \dot{V}_{\alpha i} V_{\alpha i}^* V_{\beta i} + \sum_{j \neq i} \left( V_{e1} V_{e2} + r V_{\tau 1} V_{\tau 2} \right) \frac{V_{\alpha j}^* V_{\beta j}}{\Delta_{ij}}, \tag{12}
\]

with the help of Eq. (9). This expression allows us to calculate the differentials of nine squared moduli \( X_{\alpha i} \equiv |V_{\alpha i}|^2 \) against the matter parameter \( a \) as follows:

\[
\dot{X}_{\alpha i} = \dot{V}_{\alpha i}^* V_{\alpha i} + V_{\alpha i}^* \dot{V}_{\alpha i} = 2 \sum_{j \neq i} \frac{\text{Re}(V_{e1} V_{\alpha j} V_{e2}^* V_{\alpha i}) + r \text{Re}(V_{\tau 1} V_{\alpha j} V_{\tau 2} V_{\alpha i})}{\Delta_{ij}}. \tag{13}
\]
If the one-loop contribution characterized by $r$ is switched off, Eqs. (8) — (13) will reproduce the previous tree-level results obtained in Ref. [24].

Besides the rephasing invariants $X_{\alpha i}$ in matter, the strength of leptonic CP violation in neutrino oscillations is measured by the effective Jarlskog invariant $\tilde{J}$ defined through [29]

$$\text{Im}(V_{\alpha i} V_{\beta j}^* V_{\alpha j}^* V_{\beta i}^*) = \tilde{J} \sum_{\gamma} \sum_{k} \epsilon_{\alpha \beta \gamma} \epsilon_{ijk}, \quad (14)$$

where $\epsilon_{\alpha \beta \gamma}$ and $\epsilon_{ijk}$ are the three-dimensional Levi-Civita symbols with the Greek and Latin subscripts running respectively over $(e, \mu, \tau)$ and $(1, 2, 3)$. The relationship of $\tilde{J}$ with its fundamental counterpart $J$ in vacuum, which is defined in the same manner as $\tilde{J}$ in Eq. (14), is known as the Naumov relation [30, 31, 32].

$$\frac{\tilde{J}}{J} = \frac{\Delta_{21} \Delta_{31} \Delta_{32}}{\Delta_{21} \Delta_{31} \Delta_{32}}, \quad (15)$$

where $\Delta_{ij} \equiv m_i^2 - m_j^2$ (for $i, j = 1, 2, 3$). Our next step is to show that this tree-level relation holds at the one-loop level by starting from $\tilde{J} = \text{Im}(V_{\tau 1} V_{\tau 2}^* V_{\tau 1}^*)$. With the help of Eq. (12), we find

$$\dot{\tilde{J}} = \text{Im}(\dot{V}_{\tau 1} V_{\tau 2}^* V_{\tau 1}^*) + \text{Im}(V_{\tau 1} V_{\tau 2}^* V_{\tau 1}^*) + \text{Im}(V_{\tau 1} \dot{V}_{\tau 2}^* V_{\tau 1}^*) + \text{Im}(V_{\tau 1} V_{\tau 2} \dot{V}_{\tau 1}^*)$$

$$= \tilde{J} \sum_{j > i} \frac{|V_{ei}|^2 - |V_{ej}|^2}{\Delta_{ji}} + r \left( |V_{\tau 1}|^2 - |V_{\tau j}|^2 \right). \quad (16)$$

Combining this equation with Eq. (8), we simply arrive at

$$\frac{\text{d}}{\text{d}a} \ln \left( \tilde{J} \Delta_{21} \Delta_{31} \Delta_{32} \right) = \frac{\dot{\tilde{J}}}{\tilde{J}} + \frac{\dot{\Delta}_{21}}{\Delta_{21}} + \frac{\dot{\Delta}_{31}}{\Delta_{31}} + \frac{\dot{\Delta}_{32}}{\Delta_{32}} = 0. \quad (17)$$

This result indicates that the combination $\tilde{J} \Delta_{21} \Delta_{31} \Delta_{32}$ is actually independent of the matter parameter $a$, and hence it is just equal to $J \Delta_{21} \Delta_{31} \Delta_{32}$ at $a = 0$ (i.e., in vacuum); namely, Eq. (15) is valid. We conclude that the intriguing Naumov relation in Eq. (15) does hold at the one-loop level with $r \neq 0$, simply because in this case the matter potential in Eq. (16) remains diagonal.

We proceed to derive the differential equations of $\tilde{\theta}_{ij}$ (for $ij = 12, 13, 23$) and $\tilde{\delta}$ with respect to $a$ in the standard parametrization of $V$ which is exactly parallel to the one of $U$ in Eq. (2). Namely, $V$ can be expressed as $V = \tilde{P}_l \tilde{O}_{23} \tilde{O}_\delta \tilde{O}_{13} \tilde{O}_\delta \tilde{O}_{12} \tilde{P}_\nu$. Note that $\tilde{P}_\nu$ disappears in the product $V \tilde{D} V^\dagger$, and thus it does not affect the effective Hamiltonian $\mathcal{H}_{\text{eff}}$ in Eq. (1) or Eq. (6). Without loss of generality, here we simply ignore $\tilde{P}_\nu$ and take $\tilde{P}_l = \text{Diag} \{ e^{i \phi_e}, e^{i \phi_\mu}, 1 \}$. In this case we have $V^\dagger \dot{V} = U^\dagger (\tilde{P}_l^\dagger \tilde{P}_l) U' + U' \dot{U}'$ with $U' \equiv \tilde{O}_{23} \tilde{O}_\delta \tilde{O}_{13} \tilde{O}_\delta \tilde{O}_{12}$, from which we obtain

$$\sum_{\alpha} U_{\alpha i} U_{\alpha j} + i \left[ \phi_e U_{\alpha i} U_{\alpha j} + \phi_\mu U_{\alpha i} U_{\alpha j} \right] = \frac{U_{\alpha i} U_{\alpha j} + r U_{\alpha i} U_{\alpha j}}{\Delta_{ij}}, \quad (18)$$

where $i \neq j$ is required. The diagonal part of this equation allows us to reproduce Eq. (8), and its off-diagonal part leads us to the following differential equations for the three effective flavor
mixing angles and the effective CP-violating phase:

\[
\dot{\theta}_{12} = \dot{c}_{12}\dot{s}_{12}\left(\frac{c_{13}^2}{\Delta_{21}} - \frac{s_{13}^2}{\Delta_{31}}\right) + r \left(\frac{c_{12}^2 - s_{12}^2}{\Delta_{21}} s_{13}\dot{c}_{23}\dot{s}_{23}\dot{c}_{4} + c_{12}\dot{s}_{12}\left(s_{13}^2\dot{c}_{23}^2 - s_{23}^2\right)\frac{\dot{\Delta}_{21}}{\Delta_{31}}\right).
\]

\[
\dot{\theta}_{13} = \dot{c}_{13}\dot{s}_{13}\left(\frac{c_{12}^2}{\Delta_{31}} + \frac{s_{12}^2}{\Delta_{32}}\right) - r\dot{c}_{13}\dot{c}_{23}\left(\frac{s_{13}\dot{c}_{23}}{\Delta_{32}} + \frac{\dot{c}_{12}}{\Delta_{31}}\frac{s_{12}\dot{s}_{23} - \dot{c}_{12}s_{13}\dot{c}_{23}}{\Delta_{32}}\right),
\]

\[
\dot{\theta}_{23} = \dot{c}_{12}s_{12}\dot{s}_{13}\dot{c}_{4}\dot{\Delta}_{21} - r\dot{c}_{23}\left(\frac{s_{12}\dot{s}_{23}}{\Delta_{31}} + \frac{c_{12}}{\Delta_{32}} - \frac{\dot{c}_{12}s_{12}\dot{s}_{23} - \dot{c}_{12}s_{13}\dot{c}_{23}^3\dot{\Delta}_{21}}{\Delta_{31}\Delta_{32}}\right),
\]

and

\[
\dot{\bar{\delta}} = -\frac{\dot{c}_{12}s_{12}\dot{s}_{13}\left(c_{12}^2 - s_{12}^2\right)}{\dot{c}_{23}\dot{s}_{23}\Delta_{31}\Delta_{32}}\frac{\dot{\Delta}_{21}}{\dot{c}_{23}\dot{c}_{33}\dot{\Delta}_{31}\dot{\Delta}_{32}} \frac{\dot{s}_{13}\dot{s}_{23} - \dot{s}_{12}\dot{s}_{13}\dot{c}_{23}^2\dot{s}_{23}\dot{\Delta}_{21}}{\dot{c}_{12}\dot{s}_{12}\Delta_{21}} - \frac{s_{13}^3\dot{s}_{23}^3 - c_{12}\dot{s}_{12}\dot{s}_{23}^3 - \dot{c}_{12}s_{12}\dot{s}_{23}^3}{\dot{s}_{12}\dot{\Delta}_{31}\Delta_{32}} \frac{\dot{\Delta}_{21}}{\dot{c}_{23}\dot{c}_{33}\dot{\Delta}_{31}\dot{\Delta}_{32}},
\]

where \(\bar{c}_{ij} \equiv \cos \bar{\theta}_{ij}, \bar{s}_{ij} \equiv \sin \bar{\theta}_{ij}, \bar{c}_\delta \equiv \cos \bar{\delta}\) and \(\bar{s}_\delta \equiv \sin \bar{\delta}\) have been defined. In comparison, the two unphysical phases \(\phi_e\) and \(\phi_\mu\) evolve with \(a\) as follows:

\[
\dot{\phi}_e = -\frac{\dot{c}_{12}s_{12}\dot{s}_{13}\dot{c}_{23}\dot{s}_{23}\dot{\Delta}_{21}}{\dot{s}_{23}\Delta_{31}\Delta_{32}} - r\dot{s}_{13}\dot{c}_{23}\dot{s}_{23}\frac{\dot{s}_{23}}{\dot{c}_{12}\dot{s}_{12}\Delta_{21}} - \frac{s_{13}^3\dot{s}_{23}^3 + \dot{c}_{12}s_{12}\dot{s}_{23}^3}{\dot{s}_{12}\dot{\Delta}_{31}\Delta_{32}} \frac{\dot{\Delta}_{21}}{\dot{s}_{23}\Delta_{31}\Delta_{32}},
\]

\[
\dot{\phi}_\mu = -\frac{\dot{c}_{12}s_{12}\dot{s}_{13}\dot{s}_{23}\dot{\Delta}_{21}}{\dot{c}_{23}\dot{s}_{23}\Delta_{31}\Delta_{32}} + r\left(\frac{\dot{c}_{12}s_{12}\dot{s}_{13}\dot{c}_{23}^3\dot{\Delta}_{21}}{\dot{s}_{23}\Delta_{31}\Delta_{32}}\right),
\]

Although \(\phi_e\) and \(\phi_\mu\) have no physical significance, they cannot be ignored in deriving the differential equations of those physical parameters in Eqs. (19) and (20) so as to keep the self-consistency of our calculations. Two immediate comments on the evolution of \(\bar{\delta}\) and \(\bar{\theta}_{23}\) with \(a\) are in order.

- Eq. (20) tells us that \(\dot{\bar{\delta}} \propto \sin \bar{\delta}\) holds at the one-loop level. This interesting proportionality is fully consistent with \(\dot{\bar{\delta}} = \dot{c}_{12}\dot{s}_{12}\dot{c}_{13}\dot{s}_{13}\dot{c}_{23}\dot{s}_{23}\dot{\delta} \propto J = c_{12}\dot{s}_{12}\dot{c}_{13}\dot{s}_{13}\dot{c}_{23}\dot{s}_{23}\dot{\delta}\) revealed by the one-loop Naumov relation as shown in Eq. (17), and it implies that a nontrivial effective CP-violating phase cannot be generated from the matter-induced correction to \(\delta = 0\) (or \(\pi\)).

- \(\dot{\bar{\theta}}_{23} \propto \cos \bar{\delta}\) and \(\dot{\bar{\theta}} \propto \cos 2\bar{\theta}_{23}\) hold in the \(r = 0\) case, so \(\bar{\theta}_{23} = \pi/4\) and \(\bar{\delta} = \pm\pi/2\) in vacuum (i.e., \(a = 0\)) will automatically assure \(\bar{\theta}_{23} = \pi/4\) and \(\bar{\delta} = \pm\pi/2\) to hold in matter. In other words, the \(\mu-\tau\) reflection symmetry of \(U\) is respected by the matter effect at the tree level, but it will be slightly broken at the one-loop level (i.e., \(r \neq 0\)). So the Toshev relation is not expected to exactly hold at the one-loop level either.

The explicit running behaviors of \(\bar{\delta}\) and \(\bar{\theta}_{ij}\) (for \(ij = 12, 13, 23\)) with respect to the matter parameter \(a\) will be numerically illustrated in the next section.
Using the last two equations obtained in Eq. (20), we may easily examine to what extent the Toshev relation is broken at the one-loop level. We find

\[
\frac{d}{da} \ln \left( \sin 2\tilde{\theta}_{23} \sin \tilde{\delta} \right) = \left( 2 \cot 2\tilde{\theta}_{23} \right) \dot{\tilde{\theta}}_{23} + \left( \cot \tilde{\delta} \right) \dot{\tilde{\delta}} = -r\kappa ,
\]

where

\[
\kappa = \tilde{s}_{13} \tilde{c}_{23} \tilde{s}_{23} \tilde{c}_{13} = \frac{\tilde{c}_{12} \tilde{s}_{12} \tilde{s}_{13} (\tilde{c}_{23}^2 - \tilde{s}_{23}^2) + \tilde{s}_{12} \tilde{c}_{23} \tilde{s}_{23} \tilde{c}_{13} (\tilde{c}_{12}^2 - \tilde{s}_{12}^2)}{\tilde{c}_{12} \tilde{s}_{12} \tilde{s}_{13} \tilde{c}_{23} \tilde{s}_{23} \tilde{c}_{13} \tilde{c}_{12} \tilde{s}_{12} \tilde{s}_{13} \tilde{c}_{23} \tilde{s}_{23} \tilde{c}_{13}}
\]

is a complicated nonlinear function of \( a \). Although it is almost impossible to solve Eq. (22) in an exact analytical way, a formal solution to this equation can be expressed as

\[
R \equiv \frac{\sin 2\tilde{\theta}_{23} \sin \tilde{\delta}}{\sin 2\tilde{\theta}_{23} \sin \tilde{\delta}} = \exp \left[ -r \int_0^a \kappa da \right].
\]

It is obvious that \( R = 1 \) exactly holds either in the \( a = 0 \) case (i.e., in vacuum) or in the \( r = 0 \) case (i.e., at the tree level of coherent forward neutrino scattering with matter). Note that \( a \) is proportional to \( N_e \) but \( r \) is closely associated with \( N_n/N_p \), so a significant deviation of \( R \) from one is still possible for a small value of \( r \) provided \( a \) is large enough.

So far we have only considered the case of a neutrino beam traveling in matter. When an antineutrino beam is taken into account, the PMNS matrix \( U \) and the matter parameter \( a \) in Eq. (6) should be replaced respectively with \( U^* \) and \( -a \) such that \( a \) itself is universal for both neutrinos and antineutrinos. In this case it is \( V^* \) that effectively describes antineutrino oscillations in a medium. As a result, the right-hand sides of Eqs. (8) and (19) should be multiplied by a negative sign but Eq. (20) keeps unchanged when they are applied to calculating the effective antineutrino oscillation parameters in matter.

### 3 Numerical illustration

To illustrate how the four effective flavor mixing parameters \( (\tilde{\theta}_{12}, \tilde{\theta}_{13}, \tilde{\theta}_{23}, \tilde{\delta}) \) and the three effective neutrino mass-squared differences \( (\tilde{\Delta}_{21}, \tilde{\Delta}_{31}, \tilde{\Delta}_{32}) \) evolve with the matter parameter \( a \) at the one-loop level, we typically choose \( r \approx 5.4 \times 10^{-5} \) for a neutral medium with \( N_n = N_p = N_e \) and \( r \approx 1.1 \times 10^{-4} \) for a more dense medium with \( N_n = 3N_p = 3N_e \) as estimated in Eq. (5). The RGE-like differential equations obtained in Eqs. (8), (19) and (20) will be used for doing the numerical calculations. We input the following best-fit values of the six independent neutrino

\[O\text{-if} a \text{ is sufficiently large, however, there will be no good reason to neglect the electroweak radiative correction to the tree-level matter potential of } H_{\text{eff}} \text{ in Eq. (9) because it is the term } ar \text{ that takes effect. This observation implies that some previous discussions about the “asymptotic” behaviors of } |V_{\alpha i}| \text{ (for } \alpha = e, \mu, \tau \text{ and } i = 1, 2, 3) \text{ or } (\tilde{\theta}_{12}, \tilde{\theta}_{13}, \tilde{\theta}_{23}, \tilde{\delta}) \text{ in the } a \to \infty \text{ limit but at the tree level (see, e.g., Refs. [33, 34]) are likely problematic and even wrong. The present work can therefore provide an important clarification in this regard.}]}^{1}
oscillation parameters extracted from current experimental data in vacuum $^{35}$: (1) as for the normal mass ordering (NMO) of three active neutrinos (i.e., $m_1 < m_2 < m_3$), $\theta_{12} \simeq 33.45^\circ$, $\theta_{13} \simeq 8.62^\circ$, $\theta_{23} \simeq 42.1^\circ$, $\delta \simeq 230^\circ$, $\Delta_{21} \simeq 7.42 \times 10^{-5}$ eV$^2$ and $\Delta_{31} \simeq 2.510 \times 10^{-3}$ eV$^2$; (2) as for the inverted mass ordering (IMO) of three active neutrinos (i.e., $m_3 < m_1 < m_2$), $\theta_{12} \simeq 33.45^\circ$, $\theta_{13} \simeq 8.61^\circ$, $\theta_{23} \simeq 49.0^\circ$, $\delta \simeq 278^\circ$, $\Delta_{21} \simeq 7.42 \times 10^{-5}$ eV$^2$ and $\Delta_{32} = -2.490 \times 10^{-3}$ eV$^2$. Our results are illustrated by Figs. 1–8, in which both the NMO and IMO cases for both neutrino and antineutrino oscillations have been taken into account $^2$. Some brief discussions are in order.

(1) The NMO case for neutrino oscillations (Figs. 1 and 2). Switching off the one-loop radiative correction to the matter potential of $\mathcal{H}_{\text{eff}}$ (i.e., taking $r = 0$), we have confirmed that our numerical results for $\tilde{\theta}_{ij}$, $\tilde{\delta}$ and $\tilde{\Delta}_{ji}$ (for $ij = 12, 13, 23$) are fully compatible with those obtained in Ref. $^{24}$, where all the salient features of these effective neutrino oscillation parameters evolving with $a$ have been understood and interpreted with the help of their corresponding differential equations.

$^2$We have plotted these figures by allowing the matter parameter $a$ to change from $10^{-7}$ eV$^2$ to $10^4$ eV$^2$, such that the behaviors of relevant neutrino oscillation parameters evolving with $a$ at the one-loop level can be fully exhibited. It is therefore reasonable to assume that these two “endpoints” of $a$ are numerically equivalent to the vacuum limit ($a \to 0$) and the dense matter limit ($a \to \infty$) to reveal the “asymptotic” values of $\tilde{\theta}_{12}$, $\tilde{\theta}_{13}$, $\tilde{\theta}_{23}$ and $\tilde{\delta}$. 

Figure 1: An illustration of the four effective flavor-mixing and CP-violating parameters evolving with the matter parameter $a$ in the NMO case for neutrino oscillations.

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![Figure 1](image-url)
Figure 2: An illustration of the three effective neutrino mass-squared differences evolving with the matter parameter $a$ in the NMO case for neutrino oscillations.

Here we do not repeat the same discussions but focus on the new effect caused by $r \neq 0$. One can see that the quantum effect characterized by $r \gtrsim 5.4 \times 10^{-5}$ becomes important for $\tilde{\theta}_{12}$ and $\tilde{\delta}$ when $a \gtrsim 0.1$ eV$^2$ holds in a dense medium. The reason is that both $\dot{\tilde{\theta}}_{12}$ and $\dot{\tilde{\delta}}$ contain the terms proportional to $r/\tilde{\Delta}_{21}$, but $\tilde{\Delta}_{21}$ itself is not significantly enhanced by the matter effect until $a \gtrsim 10$ eV$^2$. Such a delayed matter-induced enhancement of $\tilde{\Delta}_{21}$ implies that $\tilde{\theta}_{12}$ and $\tilde{\delta}$ will finally approach their corresponding “fixed points” when $a$ is much larger, as can be seen in Fig. 1. In comparison, $\tilde{\theta}_{13}$ and $\tilde{\theta}_{23}$ are essentially insensitive to $r \neq 0$ because they depend only upon $r/\tilde{\Delta}_{31}$ and $r/\tilde{\Delta}_{32}$ but $\tilde{\Delta}_{31}$ and $\tilde{\Delta}_{32}$ themselves increase rapidly when $a \gtrsim 10^{-2}$ eV$^2$ holds.

Now let us explain why $\tilde{\theta}_{12} \simeq 51.1^\circ$ holds as the one-loop fixed point of $\tilde{\theta}_{12}$ in the $a \to \infty$ limit. One the one hand, we see $\tilde{\theta}_{13} \to 90^\circ$ and $\tilde{\delta} \to 360^\circ$ (or equivalently, $\tilde{\delta} \to 0^\circ$) in this limit, and thus the dominant term of $\dot{\tilde{\theta}}_{12}$ in Eq. (19) can be simplified to

$$\dot{\tilde{\theta}}_{12} \simeq \frac{r \sin 2(\tilde{\theta}_{12} + \tilde{\theta}_{23})}{2\tilde{\Delta}_{21}} \to 0$$

(25)

when $\tilde{\theta}_{12}$ approaches its fix point, implying $\tilde{\theta}_{12} + \tilde{\theta}_{23} \simeq 90^\circ$ in this special case. On the other hand, the asymptotic value of $\tilde{\theta}_{23}$ can be determined from the tree-level formula

$$\tilde{t}_{23} \simeq |t_{23} + \frac{e^{i\delta}}{\Delta_{21}} \frac{\Delta_{21}}{\Delta_{31}} \frac{c_{12}^2 s_{12}^2}{s_{13} c_{23}}|$$

(26)

with $t_{23} \equiv \tan \theta_{23}$ and $\tilde{t}_{23} \equiv \tan \tilde{\theta}_{23}$ in the approximation of $\Delta_{21} \ll \Delta_{31} \ll a$ [11, 36], because the one-loop radiative correction to $\tilde{\theta}_{23}$ is negligibly small. We obtain $\tilde{\theta}_{23} \simeq 38.9^\circ$ from Eq. (26), and thus arrive at $\tilde{\theta}_{12} \simeq 90^\circ - \tilde{\theta}_{23} \simeq 51.1^\circ$ as $a \to \infty$. This one-loop result is remarkably different from the tree-level one obtained in Ref. [24], telling us why the one-loop radiative corrections to the matter potential of $H_{\text{eff}}$ must be taken into account for neutrino oscillations in dense matter.

(2) The IMO case for neutrino oscillations (Figs. 3 and 4). The situation is quite different in this case. Fig. 3 shows that $\tilde{\theta}_{12}$ and $\tilde{\theta}_{13}$ are insensitive to the one-loop correction characterized by $r \neq 0$ and have reached their corresponding asymptotic values $90^\circ$ and $0^\circ$ as $a \simeq 0.1$ eV$^2$ [21], but $\tilde{\theta}_{23}$ and $\tilde{\delta}$ are very sensitive to this quantum effect when $a \gtrsim 0.1$ eV$^2$. Note that $|\tilde{\Delta}_{32}| \simeq |\tilde{\Delta}_{21}| \gg |\tilde{\Delta}_{31}|$.
holds for $a \gtrsim 0.1 \text{eV}^2$, as shown in Fig. 4, and thus the evolution of $\tilde{\theta}_{23}$ and $\tilde{\delta}$ with $a \gtrsim 0.1 \text{eV}^2$ is dominated by the terms proportional to $r/\tilde{\Delta}_{31}$ in Eq. (19). To be explicit,

$$\dot{\tilde{\theta}}_{23} \approx -\frac{r \sin 2\tilde{\theta}_{23}}{2\tilde{\Delta}_{31}} \to 0$$ (27)
in this region, leading to the fixed point $\tilde{\theta}_{23} \approx 90^\circ$. In comparison, it is not straightforward to analytically understand the running behavior of $\tilde{\delta}$ nearby its asymptotic value $\tilde{\delta} \approx 349.4^\circ$, as $\tilde{c}_{12} \to 0$ and $\tilde{s}_{13} \to 0$ appear in the denominators of three terms of $\tilde{\delta}$ given by Eq. (19) although their divergent effects are not only compensated by $\tilde{c}_{23} \to 0$ but also suppressed respectively by $\tilde{s}_{13} \to 0$ and $\tilde{c}_{12} \to 0$ in the corresponding numerators. Of course, $\tilde{J} \to 0$ holds even in the limit of $\tilde{\delta} \to 349.4^\circ$ because its vanishing is simultaneously governed by $\tilde{c}_{12} \to 0$, $\tilde{s}_{13} \to 0$ and $\tilde{c}_{23} \to 0$.

(3) The NMO case for antineutrino oscillations (Figs. 5 and 6). As for an antineutrino beam travelling in matter, the right-hand sides of Eqs. (8) and (19) need to be multiplied by a negative sign, but Eq. (20) keeps unchanged. Note that $\tilde{\Delta}_{31} \approx \tilde{\Delta}_{21} \gg \tilde{\Delta}_{32}$ when $\tilde{a} \gtrsim 0.1$ eV$^2$ holds, in which case both $\tilde{\theta}_{12}$ and $\tilde{\theta}_{13}$ approach zero as they are insensitive to the one-loop radiative correction described by $r \neq 0$. In comparison, $\tilde{\theta}_{23}$ and $\tilde{\delta}$ are sensitive to this quantum effect for $\tilde{a} \gtrsim 0.1$ eV$^2$. The asymptotic value $\tilde{\theta}_{23} \approx 90^\circ$ in the $\tilde{a} \to \infty$ limit can therefore be understood from

$$ \dot{\theta}_{23} \approx \frac{r \sin 2\tilde{\theta}_{23}}{2\tilde{\Delta}_{32}} \to 0$$

which is quite similar to Eq. (27). Like the IMO case for neutrino oscillations, here an analytical understanding of the asymptotic value $\tilde{\delta} \approx 188.4^\circ$ is not straightforward either.
Figure 6: An illustration of the three effective neutrino mass-squared differences evolving with the matter parameter $a$ in the NMO case for antineutrino oscillations.

Figure 7: An illustration of the four effective flavor-mixing and CP-violating parameters evolving with the matter parameter $a$ in the IMO case for antineutrino oscillations.

(4) The IMO case for antineutrino oscillations (Figs. 7 and 8). In this case we find that $|\tilde{\Delta}_{32}| \sim |\tilde{\Delta}_{31}| \gg \tilde{\Delta}_{21}$ holds for $a \gtrsim 0.1$ eV$^2$, and the effective flavor mixing angles $\tilde{\theta}_{13}$ and $\tilde{\theta}_{23}$ are insensitive to the one-loop quantum effect because they depend only upon $r/\tilde{\Delta}_{31}$ and $r/\tilde{\Delta}_{32}$ which are strongly suppressed in dense matter. In comparison, $\tilde{\theta}_{12}$ and $\tilde{\delta}$ become sensitive to $r \neq 0$. 
when \( a \gtrsim 0.1 \text{ eV}^2 \) holds, and they approach their respective fixed points \( \tilde{\theta}_{12} \simeq 48.2^\circ \) and \( \tilde{\delta} \simeq 180^\circ \) for \( a \gtrsim 10^3 \text{ eV}^2 \). To understand why \( \tilde{\theta}_{12} \) assumes such an asymptotic value, let us take a look at the dominant term of \( \dot{\tilde{\theta}}_{12} \) in Eq. (19) by multiplying its right-hand side with a negative sign and inputting \( \tilde{\theta}_{13} \simeq 90^\circ \) and \( \tilde{\delta} \simeq 180^\circ \) in the \( a \to \infty \) limit,

\[
\dot{\tilde{\theta}}_{12} \simeq -\frac{r \sin 2(\tilde{\theta}_{12} - \tilde{\theta}_{23})}{2\Delta_{21}} \to 0 .
\] (29)

So we arrive at \( \tilde{\theta}_{12} \simeq \tilde{\theta}_{23} \simeq 48.2^\circ \) at the fixed points, where \( \tilde{\theta}_{23} \simeq 48.2^\circ \) can be obtained from Eq. (26) as a good approximation or. Of course, these two asymptotic values can also be read off directly from our numerical results shown in Fig. 7.

To numerically illustrate to what extent the tree-level Toshev relation can be modified by the one-loop radiative correction in a dense medium with either \( r \simeq 5.4 \times 10^{-5} \) (i.e., \( N_n = N_p = N_e \)) or \( r \simeq 1.1 \times 10^{-4} \) (i.e., \( N_n = 3N_p = 3N_e \)), we plot the evolution of \( R \) defined by Eq. (24) with respect to the matter parameter \( a \) in Fig. 9 for both neutrino and antineutrino oscillations in both the NMO and IMO cases. It is clear that \( R \simeq 1 \) is an excellent approximation for \( a \lesssim 0.1 \text{ eV}^2 \), but the quantum effect becomes significant as \( a \) is much larger. In particular, \( R \to 0 \) when \( a \gtrsim 10^3 \text{ eV}^2 \) holds for all the four cases, as a straightforward consequence of either \( \tilde{\delta} \to 0^\circ \) (or \( 180^\circ \)) or \( \tilde{\theta}_{23} \to 90^\circ \). Note that there is a peak in the curve of \( R \) when \( a \) is close to a few \( \text{eV}^2 \) and up to about \( 10 \text{ eV}^2 \) in the NMO case for neutrino oscillations or in the IMO case for antineutrino oscillations. In either of these two scenarios \( \tilde{\theta}_{23} \) almost keeps unchanged but \( \tilde{\delta} \) crosses its threshold value \( \tilde{\delta} = 270^\circ \) as can be seen in Fig. 1 or Fig. 7, which allows \( R \propto \sin \tilde{\delta} = -1 \) to have a local peak.

As an interesting by-product, the ratio of the effective Jarlskog invariant \( \tilde{J} \) in matter to the fundamental Jarlskog invariant \( J \) in vacuum is numerically calculated and its evolution with the matter parameter \( a \) is illustrated in Fig. 10 for both the NMO and IMO cases and for both neutrino and antineutrino oscillations. Now that the Naumov relation in Eq. (15) keeps valid at the one-loop level, our result is certainly insensitive to \( r \neq 0 \) and thus consistent with those obtained previously at the tree level (see, e.g., Refs. [24, 37]). Note that one of the three effective neutrino mass-squared differences \( \tilde{\Delta}_{ji} \) (for \( ji = 21, 31, 32 \)) is actually sensitive to \( r \neq 0 \) when \( a \gtrsim 10 \text{ eV}^2 \).
Figure 9: An illustration of the one-loop radiative correction to the tree-level Toshev relation in dense matter, where $R \equiv \frac{\sin 2\tilde{\theta}_{23} \sin \tilde{\delta}}{\sin 2\theta_{23} \sin \delta}$ has been defined in Eq. (24) and its deviation from one signifies the quantum effect.

holds, as one can see from Figs. 2, 4, 6 and 8. But this effect does not appreciably manifest itself in Fig. 10, simply because $\tilde{J}$ has already approaches zero for $a \gtrsim 0.1 \text{eV}^2$.

4 Summary

Although the one-loop electroweak radiative corrections to coherent forward neutrino scattering in a medium may slightly break the tree-level universality of neutral-current contributions of three active neutrino flavors to the matter potential term of the effective Hamiltonian that is responsible for neutrino oscillations in matter, they have not attracted much attention because their effects on those currently available experiments of solar, atmospheric, reactor and accelerator neutrino oscillations are negligibly small. However, such quantum corrections are expected to be significant when studying neutrino oscillations in dense matter. In this paper we have examined this kind of small but nontrivial quantum effect by deriving the RGE-like differential equations for the relevant effective neutrino oscillation quantities with respect to the matter parameter $a$. The tree-level Toshev relation $\sin 2\tilde{\theta}_{23} \sin \tilde{\delta} = \sin 2\theta_{23} \sin \delta$, which links the genuine flavor-mixing and CP-violating parameters $(\theta_{23}, \delta)$ in vacuum to their effective counterparts $(\tilde{\theta}_{23}, \tilde{\delta})$ in matter, is found to be modified at the one-loop level. In comparison, we have shown that the tree-level Naumov relation $\tilde{J}\Delta_{21}\Delta_{31}\Delta_{32} = J\Delta_{21}\Delta_{31}\Delta_{32}$ keeps valid at the one-loop level. A numerical illustration of the significance of such a one-loop quantum effect has also been presented for neutrino and
Figure 10: An illustration of the one-loop radiative correction to the effective Jarlskog invariant $\tilde{J}$, which is normalized by its fundamental counterpart $J$, in dense matter.

We emphasize that the Toshev relation may provide a simple but instructive way to test matter effects on flavor mixing and CP violation in neutrino (or antineutrino) oscillations in the upcoming precision measurement era, especially when both the neutrino and antineutrino beams can be prepared and studied in a single experiment.

It is no doubt that our analysis made above is conceptually important and may even find some useful applications in exploring the phenomena of neutrino and antineutrino oscillations in a dense matter environment. On the other hand, this work provides a new example which supports the second Weinberg’s law of progress in theoretical physics, namely “Do not trust arguments based on the lowest order of perturbation theory” [38].

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