Off-Shell Duality in Maxwell and Born-Infeld Theories

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It is well known that the classical equations of motion of Maxwell and Born-Infeld theories are invariant under a duality symmetry acting on the field strengths. We review the implementation of the $SL(2, \mathbb{Z})$ duality in these theories as linear but non-local transformations of the potentials.

In field theory duality is often realized as symmetries of the equations of motion. It is known that the equations of motion, Bianchi identities and the energy-momentum tensor of Maxwell and Born-Infeld theories are invariant under $SO(2)$ rotations which mix the electric and magnetic fields. The action, however, is not invariant. The $SO(2)$ symmetry can be enlarged to $SL(2, \mathbb{R})$ when a dilaton and an axion are added. However, it is desirable that such symmetries could be implemented at the quantum level as symmetries of the action and partition function. In this way the symmetries will hold in any situation and not only for on-shell quantities. Off-shell symmetries must be implemented in the basic field variables (and not in the field strengths for gauge theories) either in the Lagrangian or Hamiltonian formalism. When varying the action the resulting boundary term must be local in time, giving rise to a Noether current associated to the invariance. However, the boundary term can be non-local in space provided that it has a sufficient falloff at spatial infinity. This allows the variations of the basic field variables to be non-local in space. These ideas were first explored in where the $SO(2)$ symmetry of Maxwell equations were implemented at the action level in the Hamiltonian formalism in Coulomb gauge. The same holds for the Born-Infeld theory and for gauge theories coupled to matter and gravity. These transformations leave the action invariant only in the Coulomb gauge. This could be seen as a drawback since the symmetry manifests itself only in a particular gauge. Even so, it may be quite useful. A typical example is the Chern-Simons theory in Landau gauge. In this case there appears a vector supersymmetry which can be extended to the exceptional algebra $D(2, \alpha)$. This symmetry is essential to show the renormalizability of the model.

We should point out that there is an alternative procedure to implement off-shell symmetries in the action with local transformation laws. Usually it breaks manifest Lorentz invariance and demands the introduction of more fields. For the Maxwell theory this requires a description in terms of two potentials giving rise to the Schwarz-Sen model or, alternatively, an infinite number of them. Duality manifests itself as rotations between the potentials. It is possible to show that the duality symmetry of the Schwarz-Sen model is the local form of the non-local transformations found in. Although the Schwarz-Sen model is not manifestly Lorentz covariant this symmetry can be made manifest by the inclusion of auxiliary fields and some gauge symmetry through the PST formalism. A similar situation is found for the Born-Infeld theory. It should be remarked that this situation is not exclusive of duality symmetry. Even well known symmetries, like the BRST symmetry, can be cast into a non-local form at the expense of loosing manifest Lorentz invariance.

The $SL(2, \mathbb{R})$ symmetry of the equations of motion found when a dilaton and an axion are added, manifests, at the quantum level, as an $SL(2, \mathbb{Z})$ duality of the partition function. This happens when the dilaton and the axion take their vacuum expectation value which are combined into a complex coupling constant $\tau$ with its real part being the theta term. Now the action and the partition function are functions of $\tau$ and duality manifests as modular transformations of the coupling constant. In Maxwell theory the Lagrangian partition function is found to be a modular form under $SL(2, \mathbb{Z})$ transformations of the coupling constant $\tau$. At the Hamiltonian level, the partition function is modular invariant with modular weight equal to zero. In this case duality can be implemented as a canonical transformation on the reduced phase space. This means that Gauss law holds and we are on-shell. Recently this duality was implemented off-shell as linear but non-local transformations of the potentials as we will describe below.

Maxwell theory with a theta term is described by the following action in Minkowski space with metric $(-\ldots)$

$$S = -\frac{1}{8\pi} \int d^4x \left( \frac{4\pi}{g^2} F^\mu_\nu F^\mu_\nu + \frac{\theta}{2\pi} F_{\mu\nu}^* F_{\mu\nu} \right) ,$$

where $F^\mu_\nu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$. The Hamiltonian formulation is obtained as usual. There is a primary constraint $\Pi^0 = \frac{\delta S}{\delta A_0} = 0$ and the secondary constraint receives no contribution from the theta term, giving rise to the usual Gauss law $\partial_t \Pi^i = 0$. The Hamiltonian density is then
\[ H_M = -\frac{2\pi i}{\tau - \tilde{\tau}} \Pi^i \Pi_i - \frac{\pi + \tilde{\tau}}{\tau - \tilde{\tau}} \Pi^i B_i - i \frac{\pi \tau}{2\pi \tau - \tilde{\tau}} B^i B_i, \]  

where \( \Pi^i = \frac{\delta L}{\delta A^i} \), the magnetic field is \( B_i = \frac{1}{2} \epsilon_{ijk} F^{jk} \) and the complex coupling constant is \( \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{\theta} \). The contribution from the theta term appears in the second and third terms of Eq. (3).

The BRST charge is the same as in pure Maxwell theory since the constraint structure was not modified

\[ Q = \int d^3x \left( \partial_\tau \Pi C + \mathcal{P}_D \Pi_0 \right). \]

The ghosts obey the canonical Poisson brackets \( \{\mathcal{P}_C, \Pi\} = \{\mathcal{P}_D, D\} = -1 \) and the BRST transformations are

\[
\begin{align*}
\delta A_i &= \partial_i C, & \delta A_0 &= -\mathcal{P}_D, & \delta \Pi_0 &= 0, & \delta \Pi_i &= 0, \\
\delta C &= 0, & \delta \mathcal{P}_C &= -\partial_i \Pi^i, & \delta D &= -\Pi_0, & \delta \mathcal{P}_D &= 0.
\end{align*}
\]

The partition function is then

\[ Z(\tau) = \int D\mathcal{A}_\mu \mathcal{D}\Pi_\nu \mathcal{D}(\text{ghosts}) e^{-iS_M(\tau)}, \]

where the Maxwell effective action is

\[ S_M(\tau) = \int d^4x \left( \Pi^\mu \dot{\mathcal{A}}_\mu + \mathcal{C} \mathcal{P}_C + \mathcal{P}_D D - H_M - \{Q, \Psi\} \right), \]

and \( \Psi \) is the gauge fixing function.

In order to consider the \( SL(2,\mathbb{Z}) \) duality it is convenient to split the vector fields \( A_i \) and \( \Pi_i \) into their transversal \( A^T_i, \Pi^T_i \) and longitudinal parts \( A^L_i, \Pi^L_i \). We will also consider finite \( SL(2,\mathbb{Z}) \) transformations. We have found that the \( SL(2,\mathbb{Z}) \) transformations are given by

\[
\begin{align*}
A^T_i &= a \tilde{A}^T_i + 2\pi c \epsilon_{ijk} \frac{\partial j}{\partial Y_k} \tilde{\Pi}^{T_k}, & \Pi^T_i &= \frac{1}{|a - c\tilde{\tau}|} \tilde{\Pi}^T_i, \\
A^L_i &= |a - c\tilde{\tau}| \tilde{A}^L_i, & \Pi^L_i &= \frac{1}{|a - c\tilde{\tau}|} \tilde{\Pi}^L_i, \\
A_0 &= |a - c\tilde{\tau}| \tilde{A}_0, & \Pi_0 &= \frac{1}{|a - c\tilde{\tau}|} \tilde{\Pi}_0, \\
C &= |a - c\tilde{\tau}| \tilde{C}, & \mathcal{P}_C &= \frac{1}{|a - c\tilde{\tau}|} \tilde{\mathcal{P}}_C, \\
D &= \frac{1}{|a - c\tilde{\tau}|} \tilde{D}, & \mathcal{P}_D &= |a - c\tilde{\tau}| \tilde{\mathcal{P}}_D,
\end{align*}
\]

where \( a, b, c \) and \( d \) are integers satisfying \( ad - bc = 1 \). Notice that the transversal (or physical) part of the vectors are transformed among themselves while the gauge dependent (or non-physical) parts, which include: \( A_0, A^L_i, \Pi_0 \) and \( \Pi^L_i \) and the ghosts, transform into themselves. These transformations are local in time but non-local in space.

The Hamiltonian density Eq. (2) is not invariant under Eqs. (7-8). It transforms as

\[
H_M = \tilde{H}_M - \frac{2\pi i}{\tilde{\tau} - \tilde{\tau}} \frac{2(a - |a - c\tilde{\tau}| - c(\tilde{\tau} + \tilde{\tau})}{|a - c\tilde{\tau}|} \tilde{\Pi}^T_i \tilde{\Pi}^L_i \\
- \frac{i}{\tilde{\tau} - \tilde{\tau}} \frac{(a - |a - c\tilde{\tau}|)(\tilde{\tau} + \tilde{\tau}) - 2c\tilde{\tau} \tilde{\tau}}{|a - c\tilde{\tau}|} \tilde{\Pi}^L_i \tilde{B}_i,
\]

and upon integration the extra terms give rise to surface contributions. The kinetic terms in the effective action Eq. (3) are also invariant up to surface terms. Hence, the Hamiltonian is modular invariant up to surface terms.

The gauge fixing term in Eq. (4) requires some care but it can be proved to be BRST invariant [1]. We then conclude that the effective action is modular invariant for any gauge choice. The Jacobian of the transformations Eqs. (7) can...
be computed and it is found to be equal to one. Therefore, they can be regarded as a canonical transformation. Hence the path integral measure is also invariant. As a consequence, the partition function is modular invariant. It should be stressed that the partition function in the Lagrangian formalism is not modular invariant under duality, rather it transforms as a modular form \[\Pi\]. However, the phase space partition function is modular invariant \[\Pi^2\] \[17, 18\]. It can also be shown that Eqs.\[3\] reduce to the familiar duality transformations of the classical equations of motion.

It is well known that the Born-Infeld theory has an \(SO(2)\) symmetry in its classical equations of motion which can be extended to \(SL(2,\mathbb{R})\) if an axion and a dilaton are added \[4\]. If we consider just the axion and dilaton vacuum expectation values we get a Born-Infeld theory with a theta term. The action is

\[S = \int d^4x \left(1 - \frac{\theta}{16\pi^2} F^{\mu\nu} F_{\mu\nu} - \sqrt{1 + \frac{1}{g^2} F^{\mu\nu} F_{\mu\nu} - \frac{1}{4g^4}(F^{\mu\nu} F_{\mu\nu})^2}\right). \tag{10}\]

In the weak field limit it reduces to Maxwell theory with a theta term Eq.\[1\]. There is also a dimensionful constant in the square root in Eq.\[12\]. Either non-linear terms must be introduced in Eqs.\[7\] or something else must be modified.

It must be noted that both the Maxwell and the Born-Infeld Hamiltonian densities can be rewritten in terms of a complex vector field

\[P_i = \Pi_i + \frac{\tau}{2\pi} B_i, \tag{13}\]

We find that

\[H_M = -\frac{2\pi i}{\tau - \bar{\tau}} P_i \bar{P}_i, \tag{14}\]

and

\[H_{BI} = \sqrt{1 - \frac{4\pi^2}{\tau - \bar{\tau}} P_i \bar{P}_i - \frac{4\pi^2}{(\tau - \bar{\tau})^2}(P \times \bar{P})^2 - 1}, \tag{15}\]

where the overline denotes complex conjugation. The vector \(P_i\) transforms under duality as

\[P_i = \frac{1}{a - c\tau} \left(\bar{\Pi}_i^T + \frac{a - c\tau}{|a - c\tau|} \bar{\Pi}_i^T + \frac{\bar{\tau}}{2\pi} B_i\right), \tag{16}\]

while

\[\frac{1}{\tau - \bar{\tau}} = \frac{|a - c\tau|^2}{\bar{\tau} - \tau}. \tag{17}\]

This explains why the Maxwell Hamiltonian is not invariant. The longitudinal and transversal parts of \(\bar{\Pi}_i\) do not combine themselves back into \(\bar{\Pi}_i\) so that \(P_i\) is not a modular form. If instead of \(|a - c\tau|\) in the denominator of the \(\Pi_i^T\) term in Eq.\[14\] we had just \(a - c\tau\) we could recover \(\bar{P}_i\). But taking out the modulus in the transformations Eqs.\[16\] is not consistent because all fields are real. On the other side if we could change only the transformation for \(\Pi_i^T\) that
would do the job. It is then necessary that \( \Pi^I_i \) possess an imaginary part. For consistency \( A_0, \Pi_0, A^I_i \) and the ghosts must have an imaginary part as well.

So we start with the non-physical sector \( A_0, \Pi_0, A^I_i, \Pi^I_i \) and the ghosts all described by complex fields. Since the number of ghosts has also doubled the number of physical degrees of freedom is still the same. The vectors \( A_i \) and \( \Pi_i \) are now complex with their transversal part taken to be real while their longitudinal parts are taken to be complex.

The effective action is now

\[
S_{BI} = \int d^4x \left( \frac{1}{2} \Pi^\mu \dot{A}_\mu + \frac{1}{2} \Pi^\nu \dot{A}_\nu + \frac{1}{2} \overline{C} T \dot{C} + \frac{1}{2} \overline{C} C + \frac{1}{2} \dot{P}_D D + \frac{1}{2} \dot{P}_D \overline{D} - H_{BI} - \{Q, \Psi\} \right),
\]

(18)

The Hamiltonian density has the same form as in Eq.(13) with \( P_i \) defined by Eq.(13) but with complex fields instead of real fields. The integrand in the square root in Eq.(15) is real.

The BRST charge is now

\[
Q = \frac{1}{2} \int d^3x \left( \partial_i \Pi^i C + \partial_i \Pi^i \overline{C} + \partial_i \overline{D} \Pi_0 + \partial_i \Pi_0 \overline{D} \right),
\]

(19)

so that \( Q \) is real. The BRST transformations are modified in a straightforward way. The gauge fixing fermion reads now

\[
\Psi = \frac{1}{2} \int d^3x \left( \chi \overline{D} + \overline{\chi} D + A_0 \overline{\otimes} C + A_0 \overline{\otimes} \overline{C} \right),
\]

(20)

and is also real. It can be shown that this theory is equivalent to the original Born-Infeld theory [1].

Then the \( SL(2, \mathbb{Z}) \) duality transformations are now

\[
A^I_i = a A^I_i + 2\pi \epsilon_{ijk} \frac{\partial^j}{\partial^k} \Pi^T_k, \quad \Pi^I_i = \frac{1}{a - c\tau} \Pi^I_i, \quad A^0 = (a - c\tau) \dot{A}^0, \quad \Pi^0 = \frac{1}{a - c\tau} \Pi^0, \quad C = (a - c\tau) C, \quad \overline{P}_C = \frac{1}{a - c\tau} \overline{P}_C, \quad \overline{D} = \frac{1}{a - c\tau} \overline{D}, \quad P_D = (a - c\tau) P_D.
\]

(21)

The unphysical sector is composed of modular forms. The vector \( P_i \) is also a modular form. It transforms as

\[
P_i = \frac{1}{a - c\tau} \tilde{P}_i,
\]

(22)

so that the Maxwell Hamiltonian is modular invariant with no surface terms being generated. The Born-Infeld Hamiltonian is also modular invariant. It is easy to show that the kinetic terms in the effective action Eq.(18) are also invariant up to surface terms. The BRST charge Eq.(14) is also invariant. It can also be shown that the gauge fixing term in Eq.(18) is also modular invariant so that the effective action Eq.(15) is modular invariant. Finally we can show that the duality transformations Eqs.[21] have a unity Jacobian so that the partition function is modular invariant. Also, the duality transformations Eqs.[22] reduce to the usual duality transformations of the classical equations of motion.

We have shown how it is possible to generalize the \( SL(2, \mathbb{R}) \) symmetry of the equations of motion, for Maxwell and Born-Infeld theories, to an off-shell duality. For the Maxwell theory we found that the Hamiltonian \( H_M \) is modular invariant up to a surface term. In the Born-Infeld case it was necessary to consider the longitudinal part of the fields as complex fields. Then the Born-Infeld Hamiltonian \( H_{BI} \) is strictly modular invariant with no boundary terms being generated by the transformation. Of course, we could consider Maxwell theory with the longitudinal part of the fields being complex as well. In this case the Hamiltonian would be modular invariant without any boundary term. However there is no clear interpretation for the complex longitudinal fields introduced in these theories.

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