Trapped states and bound states of a soliton in a well

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Abstract

The nature of the interaction of a soliton with an attractive well is elucidated using a model of two interacting point particles. The model explains the existence of trapped states at positive kinetic energy, as well as reflection by an attractive impurity. The transition from a trapped soliton state to a bound state is studied. Bound states of the soliton in a well are also found.

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Topological solitons arise as nontrivial solutions in field theories with non-linear interactions. These solutions are stable against dispersion. Topology enters through the absolute conservation of a topological charge, or winding number.

It is for this reason they become so important in the description of phenomena like, optical self-focusing, magnetic flux in Josephson junctions[1] or even the very existence of stable elementary particles, such as the skyrmion[2, 3], as a model of hadrons.

Interactions of solitons with external agents become extremely important. These interactions allow us to test the validity of such models in real situations.

In a previous work[7] the interaction of a soliton in one space dimension with finite size impurities was investigated.

In the works of Kivshar et al.[4] (see also ref. [5, 6]), it was found that the soliton displays unique phenomena when it interacts with an external impurity. The existence of trapped solutions for positive energy or, bound states in the continuum, is a very distinctive effect for the soliton in interaction with an attractive well.

We can understand the origin of impurity interactions of a soliton by looking at the impurity as a nontrivial medium in which the soliton propagates. An easy way to visualize these interactions consists in introducing a nontrivial metric for the relevant spacetime. The metric carries the information of the medium characteristics.

Consider for example a 1+1 dimensional scalar field theory supporting topological solitons in flat space, immersed in a background determined by the metric $g_{\mu\nu}$. The standard manner of coupling the scalar field to the metric is

$$\mathcal{L} = \sqrt{g} \left[ g^{\mu\nu} \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - U(\phi) \right]$$

where $g$ is the determinant of the metric, and $U$ is the self-interaction
that enables the existence of the soliton. For a weak potential we have:

\[
\begin{align*}
    g_{00} & \approx 1 + V(x) \\
    g_{11} & = -1 \\
    g_{-11} & = g_{1-1} = 0
\end{align*}
\] (2)

Where \( V(x) \) is the external space dependent potential. The equation of motion of the soliton in the background space becomes

\[
\frac{\partial^2 \phi}{\partial t^2} - \sqrt{g} \frac{\partial}{\partial x} \left[ \sqrt{g} \frac{\partial \phi}{\partial x} \right] + g_{00} \frac{\partial U}{\partial \phi} = 0.
\] (3)

This equation is identical, for slowly varying potentials, to the equation of motion of a soliton interacting with an impurity \( V(x) \). Impurity interactions are therefore acceptable couplings of a soliton to an external potential. It is also the only way to couple the soliton without spoiling the topological boundary conditions.

The interaction of a soliton with an attractive impurity shows, however, some puzzling effects. A soliton can be trapped in it, when it impinges onto the well with positive kinetic energy. Energy conservation demands that the soliton fluctuates and distorts in trapped states inside the well. Even more counterintuitive is the fact that the soliton can be reflected by the well.

Neither of these effects are possible for classical point particles. The difference must obviously be due to the extended character of the soliton. We should then be able to reproduce these effects with a classical model for an extended object.

The simplest extended object one can envisage consists of two classical point particles connected by a massless spring. A repulsive force between them is also needed to prevent their collapse to zero size.
Consider such a system where each particle interacts also with an external attractive well. The classical nonrelativistic one-dimensional lagrangian for the system of equal masses $m_1 = m_2 = 1$ becomes:

$$
\mathcal{L}_{sys} = \frac{\dot{x}_1^2}{2} + \frac{\dot{x}_2^2}{2} - k \frac{(x_1 - x_2)^2}{2} - \frac{\alpha}{|x_1 - x_2|^n} + V(x_1) + V(x_2) \quad (4)
$$

For the potential well we take

$$
V(x) = A e^{-\beta x^2} \quad (5)
$$

We prepare the two-particle system at rest at a large distance far away from the well with an initial speed $v$. The equilibrium interparticle separation is $x_0^{n+2} = \frac{n \alpha}{k}$.

The equations of motion are not solvable analytically. Using the numerical algorithm used in ref. [7] we can find the outcome of the scattering events as a function of the initial speed.

Figure 1 exemplifies the results for the choice of parameters $k = 1$, $\alpha = 1$, $n = 2$, $A = 2$, $\beta = 1$.

Quite unexpectedly, it is found that the system behaves exactly like the soliton.

The system can be trapped, reflected or transmitted through the well by changing the initial speed.

When the system is trapped, it oscillates with a null average speed, the kinetic energy stored in the vibrational and deformation modes.

Minute changes of the initial speed around a value leading to a trapped state, may generate reflection or transmission events.

The effects are independent of the functional dependence of the interactions and external potential, and of the values of the parameters. It looks as if the behavior is universal.

In figure 1 we used a grid for $v$ of $dv = .001$. Using a finer grid, each region of reflection-transmission unfolds to more islands of trapping, reflection and transmission.
Finer and finer grids show more and more structure.

Figure 2 shows a detailed expansion of the velocity range around $v=.12$ with a grid $dv=.0002$. The system is chaotic, an infinitesimal change in the initial speed produces diverging results.

Many of the phenomena related to chaotic behavior may be identified in the system. Scaling and bifurcation are evident and perhaps even fractal structure. (This issue will be taken up in another work)

It is now safe to claim that the unexpected behavior of a soliton interacting with an attractive well may be traced back to its extended nature. If we consider each $\phi(x)$ as a classical pointlike object we will find interactions
between neighboring particles of attractive and repulsive character. The basic attractive interaction is provided by the space derivative of the soliton lagrangian and a piece of the self-interaction potential, while the repulsive interaction is provided by the latter only.
We now focus on the fate of a trapped soliton state. Consider the kink lagrangian

\[ \mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} \Lambda \left( \phi^2 - \frac{m^2}{\lambda} \right)^2 \]  

(6)

Here

\[ \Lambda = \lambda + V(x) \]  

(7)

\( \lambda \) being a constant, and \( V(x) \) the impurity potential[7]

\[ V(x) = h \cosh^{-2} \left( a \left( x - x_c \right) \right) \]  

(8)

Independently of the choice of parameters it is found that trapped states decay. The soliton radiates energy and consequently the amplitude of the oscillations decreases. The trapped states become asymptotically bound states.

Figure 3 shows the amplitude of the oscillation of the soliton, namely, the value of the field at the center of the well, as a function of time. Here we used \( m = 1, \lambda = 1, h = -3, a = 2, x_1 = 3 \). The soliton impinges from the left. The initial location of the center of the soliton is chosen to be far enough from the well at \( x = -3 \), with an initial speed \( v = .025 \).

In order to visualize the decay and emission of radiation we extended the x-axis to \(-140 \leq x \leq 140\) with a grid of \( dx = 0.1 \). This coordinate span allows for radiation to propagate for a long distance away from the trapping zone without being reflected.

When the soliton reaches the well, it oscillates and starts to emit radiation. The emission of radiation damps the oscillations. After a certain time, and due to the finite extent of the x-axis, radiation reflects back from the boundaries and reaches the soliton. The soliton subsequently absorbs the radiation and its amplitude starts to increase. The time taken for radiation to return to the soliton is the travel time for the fastest 'mesons' of the theory.

The dispersion relation for the radiated mesons can be extracted from the expansion of the scalar field around the soliton solution. Using \( \lambda = m = 1 \) we find \( \omega^2 = k^2 + 2 \). The velocity of the mesons is bounded by
$u_{\text{max}} = \left(\frac{\omega}{k}\right)_{\text{max}} = 1$. This is clearly observed in figure 3. The reabsorption of radiation starts after the first mesons arrive back from the boundaries to the well. The distance between the well and the boundary is 140, therefore $t_{\text{absorption}} = 280/u_{\text{max}} = 280$

The frequency of the oscillations of the soliton in a trapped state may be estimated analytically. Using an expansion of the potential in eq. (8) around the bottom of the well $V(x) \approx -V_0 + \epsilon y^2$, $y = x - x_c$ and an ansatz appropriate for small oscillations of the soliton around the center of the well $\phi \approx (y + \delta y^3/2) \sin(\omega(t - t_0))$ we find $\omega^2 = 2 \mu$.

With $\mu \approx \pm \sqrt{\frac{1}{3}\epsilon + \frac{9}{10}(V_0 - 1)}$. (The positive solution has to be chosen)
The formula compares reasonably well with the leading frequency of oscillation of the soliton inferred from a Fourier analysis of the amplitude of the field at the center of the well. However, the fluctuation of the soliton in the well is anharmonic.

Another way to observe the decay of a trapped state to a bound state consists in adding a dissipative force of the form $\gamma \frac{\partial \phi}{\partial t}$. This force cannot be derived from a Hamiltonian, but, it can arise from the interaction to a bath. Inserting this term in the soliton equation of motion yields the results depicted in figure 4, where we took the same set of parameters as those of the radiation run of figure 3, but with a friction coefficient $\gamma = .1$. 
Attenuation is the dominant effect in this case. The soliton loses its energy by dissipation instead of radiating it. Other choices of parameters may lead to a mixture of both processes. It is, however, evident that trapped states will eventually become bound states.

Hence, there should exist static bound state solutions of the soliton in the well. We found those solutions, by integrating the static equations of motion starting from the center of the well. There appears to be only a single bound state for each choice of well depth and width. Two bound state solutions are depicted in figure 5. The soliton is markedly modified by the potential. The total energy of the soliton may even become negative, as for the soliton depicted with the dashed curve in the figure. For this case, the binding energy exceeds the free soliton mass.

We have also found static bound state solutions for a soliton located off-center from the well. Those solutions exist for the soliton located anywhere on the x-axis.

Several questions arise from the present work and they will be addressed in future works. One of the intriguing conclusions we can draw is that simple classical extended objects may have unsuspected behavior, like trapping, reflection from an attractive potential, chaotic behavior, etc. Turning the process backwards: an extended object, may it be a soliton or a classical assembly of bound particles, in a trapped state, can suddenly be freed from it provided some random interaction causes the reversal of the process of trapping, a process reminiscent of the decay of metastable states in quantum mechanics.
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**Figure Captions**

**Fig. 1**: Final velocity of the two-particle system as a function of the initial velocity for the parameters $k = 1$, $\alpha = 1$, $n = 2$, $A = 2$, $\beta = 1$ with a velocity grid $dv = .001$.

**Fig. 2**: Same as figure 1, but with finer velocity grid $dv = .0002$.

**Fig. 3**: Amplitude of the oscillation of the soliton in a trapped state as a function of time. Soliton parameters: $m = 1$, $\lambda = 1$, impurity parameters: $h = -3$, $a = 2$, $x_c = 3$.

**Fig. 4**: Same as figure 3 but including attenuation. Friction coefficient $\gamma = .1$.

**Fig. 5**: Bound state soliton solution in a well with parameters $h = -0.05$ and $a = 0.12$ (solid line), and $h = -5$, $w = 1.2$ (dashed line).