ON A LINEAR NON-HOMOGENEOUS ORDINARY DIFFERENTIAL EQUATION OF THE HIGHER ORDER WHOSE COEFFICIENTS ARE REAL-VALUED SIMPLE STEP FUNCTIONS

Gogi Pantsulaia*, Khatuna Chargazia†, Givi Giorgadze‡
I. Vekua Institute of Applied Mathematics, Tbilisi State University, B. P. 0143, University St. 2, Tbilisi, Georgia
Department of Mathematics, Georgian Technical University, B. P. 0175, Kostava St. 77, Tbilisi 75, Georgia

Abstract

By using the method developed in the paper [G. Pantsulaia, G. Giorgadze, On some applications of infinite-dimensional cellular matrices, *Georg. Inter. J. Sci. Tech.*, *Nova Science Publishers*, Volume 3, Issue 1 (2011), 107-129], it is obtained a representation in an explicit form of the particular solution of the linear non-homogeneous ordinary differential equation of the higher order whose coefficients are real-valued simple functions.

2000 Mathematics Subject Classification: Primary 34Axx; Secondary 34A35, 34K06.

Key words and phrases: linear ordinary differential equation, non-homogeneous ordinary differential equation.

1. Introduction

In [4] has been obtained a representation in an explicit form of the particular solution of the linear non-homogeneous ordinary differential equation of the higher order with real-valued coefficients. The aim of the present manuscript is resolve an analogous problem for a linear non-homogeneous ordinary differential equation of the higher order when coefficients are real-valued simple step functions.

The paper is organized as follows.

In Section 2, we consider some auxiliary results obtained in the paper [4]. In Section 3, it is obtained a representation in an explicit form of the particular solution of the linear non-homogeneous ordinary differential equation of the higher order whose coefficients are real-valued simple functions. In Section 4 we present mathematical programm in MathLab for the graphical solution of the corresponding differential equation.

*E-mail address: g.pantsulaia@gtu.ge
†E-mail address: khatuna.chargazia@gmail.com
‡E-mail address: g.giorgadze@gtu.ge
2. Some auxiliary propositions

For \( n \in \mathbb{N} \), we denote by \( FD^{(n)}[-l,l[ \) a vector space of all \( n \)-times differentiable functions \( \Psi \) on \([-l,l[\) such that a series obtained by \( k \)-times differentiation term by term of the Fourier trigonometric series of \( \Psi \) pointwise converges to \( \Psi^{(k)} \) for all \( x \in [-l,l[ \) and \( 0 \leq k \leq n \).

Let \((A_n)_{0 \leq n \leq 2M}\) be a sequence of real numbers, where \( M \) is any natural number. For each \( k \geq 1 \) we put
\[
\sigma_k = \sum_{n=0}^{m} (-1)^n A_{2n} \left( \frac{k \pi}{l} \right)^{2n},
\]
\[
\omega_k = \sum_{n=0}^{m-1} (-1)^n A_{2n+1} \left( \frac{k \pi}{l} \right)^{2n+1}.
\]

**Theorem 2.1.** ([4], Theorem 3.1, p.45) For \( m \geq 1 \), let us consider an ordinary differential equation
\[
\sum_{n=0}^{2m} A_n \frac{d^n}{dx^n} \Psi = f, \tag{2.3}
\]
where
\[
f(x) = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k \cos \left( \frac{k \pi x}{l} \right) + d_k \sin \left( \frac{k \pi x}{l} \right) \in FD^{(0)}[-l,l[. \tag{2.4}
\]
and \( A_n \in \mathbb{R} \) for \( 0 \leq n \leq 2m \).

Suppose that \( A_0 \neq 0 \) and \( \sigma_k^2 + \omega_k^2 \neq 0 \) for \( k \geq 1 \), where \( \sigma_k \) and \( \omega_k \) are defined by (2.1) and (2.2), respectively.

If \((\frac{c_0}{2},c_1,d_1,c_2,d_2,\ldots)\) is such a sequence of real numbers that the series \( \Psi_p \), defined by
\[
\Psi_p(x) = \frac{c_0}{2A_0} + \sum_{k=1}^{\infty} \left( \frac{c_k \sigma_k - d_k \omega_k}{\sigma_k^2 + \omega_k^2} \right) \cos \left( \frac{k \pi x}{l} \right) + \left( \frac{c_k \omega_k + d_k \sigma_k}{\sigma_k^2 + \omega_k^2} \right) \sin \left( \frac{k \pi x}{l} \right), \tag{2.5}
\]
belongs to the class \( FD^{(2m)}[-l,l[ \), then \( \Psi_p \) is a particular solution of (2.3).

**Theorem 2.2.** ([4], Theorem 3.2, p.45) For \( m \geq 1 \), let us consider an ordinary differential equation (2.3), where
\[
f(x) \in C[-l,l[ \tag{2.6}
\]
and \( A_n \in \mathbb{R} \) for \( 0 \leq n \leq 2m \).

Suppose that \( A_0 \neq 0 \) and \( \sigma_k^2 + \omega_k^2 \neq 0 \) for \( k \geq 1 \), where \( \sigma_k \) and \( \omega_k \) are defined by (2.1) and (2.2), respectively. Let \((\frac{c_0}{2},c_1,d_1,c_2,d_2,\ldots)\) be Fourier coefficients of \( f \) and \((\frac{c_0}{2},c_1,d_1,c_2,d_2,\ldots)\in \ell_1 \).

Then the series \( \Psi_p \), defined by
\[
\Psi_p(x) = \frac{c_0}{2A_0} + \sum_{k=1}^{\infty} \left( \frac{c_k \sigma_k - d_k \omega_k}{\sigma_k^2 + \omega_k^2} \right) \cos \left( \frac{k \pi x}{l} \right) + \left( \frac{c_k \omega_k + d_k \sigma_k}{\sigma_k^2 + \omega_k^2} \right) \sin \left( \frac{k \pi x}{l} \right), \tag{2.7}
\]
is a particular solution of (2.3).
3. A non-homogeneous ordinary differential equation of higher order whose coefficients are continuous or real-valued step functions

Let consider a partition of \([-l,l]\) defined by
\[-l,l]=\bigcup_{s=0}^{S-1}\left\{ \frac{l(2s-S)}{S}, \frac{l(2s+2-S)}{S} \right\}

We define a differential operator
\[ L(\Psi) = \sum_{n=0}^{2m} A_n(x) \frac{d^n}{dx^n} \Psi \]
for \(\Psi \in FD^{(2m)}[-l,l]\). Notice that \(L(\Psi)\) can be rewritten as follows
\[ L(\Psi) = \bigcup_{s=0}^{S-1} Ind_{\left[\frac{l(2s-S)}{S}, \frac{l(2s+2-S)}{S}\right]} \left( \sum_{n=0}^{2m} A_n(x) \frac{d^n}{dx^n} \right) \Psi \]
for \(\Psi \in FD^{(2m)}[-l,l]\), where \(Ind\) denotes an indicator function.

For each \(S \in \mathbb{N}\) we define an operator \(L_S\) by
\[ L_S(\Psi) = \sum_{s=0}^{S-1} Ind_{\left[\frac{l(2s-S)}{S}, \frac{l(2s+2-S)}{S}\right]} \left( \sum_{n=0}^{2m} A_n(x) \frac{l(2s+1-S)}{S} \frac{d^n}{dx^n} \right) \Psi \]
for \(\Psi \in FD^{(2m)}[-l,l]\).

**Lemma 3.1.** For each \(\Psi \in FD^{(2m)}[-l,l]\) we have
\[ L(\Psi) = \lim_{S \to \infty} L_S(\Psi). \]

**Theorem 3.2.** For \(m \geq 1\), let us consider an ordinary differential equation
\[ \sum_{n=0}^{2m} A_n(x) \frac{d^n}{dx^n} \Psi = f, \quad (3.1) \]
where
\[ f(x) = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k \cos \left( \frac{k\pi x}{l} \right) + d_k \sin \left( \frac{k\pi x}{l} \right) \in FD^{(0)}[-l,l] \quad (3.2) \]
and \(A_n(x) \in C[-l,l]\) for \(0 \leq n \leq 2m\).

Suppose that \(A_0(x) = 1\) and \(\sigma_k^2(x) + \omega_k^2(x) \neq 0\) for \(x \in [-l,l]\) and \(k \geq 1\), where \(\sigma_k(x)\) and \(\omega_k(x)\) are defined by
\[ \sigma_k(x) = \sum_{n=0}^{m} (-1)^n A_{2n}(x) \left( \frac{k\pi}{l} \right)^{2n}, \quad (3.3) \]
\[ \omega_k(x) = \sum_{n=0}^{m-1} (-1)^n A_{2n+1}(x) \left( \frac{k\pi}{l} \right)^{2n+1}. \quad (3.4) \]

Suppose that the following conditions are valid:
(i) \((\frac{m}{l}, c_1, d_1, c_2, d_2, \ldots) \in \mathbb{Z};\)
(ii) There is a constant \(C > 0\) such that
\[
\left| \frac{\omega_k(x+h)}{\sigma_k^2(x+h)} + \omega_k(x+h) \right| - \left| \frac{\omega_k(x)}{\sigma_k^2(x)} + \omega_k(x) \right| \leq C|h|^2
\]
and
\[
\left| \frac{\sigma_k(x+h)}{\sigma_k^2(x+h)} + \omega_k(x+h) \right| - \left| \frac{\sigma_k(x)}{\sigma_k^2(x)} + \omega_k(x) \right| \leq C|h|^2.
\]
Then the function \(\Psi_0\), defined by
\[
\Psi_0(x) = \frac{c_0}{2} + \sum_{k=1}^{\infty} \left( \frac{c_k \sigma_k(x) - d_k \omega_k(x)}{\sigma_k^2(x) + \omega_k^2(x)} \right) \cos \left( \frac{k\pi x}{l} \right) +
\]
\[
\left( \frac{c_k \omega_k(x) + d_k \sigma_k(x)}{\sigma_k^2(x) + \omega_k^2(x)} \right) \sin \left( \frac{k\pi x}{l} \right),
\]
(3.5)
is a particular solution of (3.1).

**Proof.** We put
\[
\Psi_S(x) = \sum_{s=0}^{S-1} \text{Ind}_{\frac{s+1}{2} S_{\frac{S}{2}}} \frac{c_0}{2} +
\]
\[
\sum_{k=1}^{\infty} \left( \frac{c_k \sigma_k(\frac{(2s+1)S}{S}) - d_k \omega_k(\frac{(2s+1)S}{S})}{\sigma_k^2(\frac{(2s+1)S}{S}) + \omega_k^2(\frac{(2s+1)S}{S})} \right) \cos \left( \frac{k\pi x}{l} \right) +
\]
\[
\left( \frac{c_k \omega_k(\frac{(2s+1)S}{S}) + d_k \sigma_k(\frac{(2s+1)S}{S})}{\sigma_k^2(\frac{(2s+1)S}{S}) + \omega_k^2(\frac{(2s+1)S}{S})} \right) \sin \left( \frac{k\pi x}{l} \right),
\]
(3.6)
On the one hand, by using the result of Lemma 3.1 we have
\[
L(\lim_{S \to \infty} \Psi_S(x)) = \lim_{s \to \infty} L(\Psi_S(x)) = \lim_{S \to \infty} L_S(\Psi_S(x)) = \lim_{S \to \infty} f(x) = f(x).
\]
On the other hand we have
\[
| \lim_{S \to \infty} \Psi_S(x) - \Psi_0(x) | = \lim_{S \to \infty} | \Psi_S(x) - \Psi_0(x) | = \lim_{S \to \infty} \left| \sum_{s=0}^{S-1} \text{Ind}_{\frac{s+1}{2} S_{\frac{S}{2}}} \left( \frac{c_0}{2} - \frac{c_0}{2} \right) +
\]
\[
\sum_{k=1}^{\infty} \left( \frac{c_k \sigma_k(\frac{(2s+1)S}{S}) - d_k \omega_k(\frac{(2s+1)S}{S})}{\sigma_k^2(\frac{(2s+1)S}{S}) + \omega_k^2(\frac{(2s+1)S}{S})} \right) \cos \left( \frac{k\pi x}{l} \right) +
\]
\[
\left( \frac{c_k \omega_k(\frac{(2s+1)S}{S}) + d_k \sigma_k(\frac{(2s+1)S}{S})}{\sigma_k^2(\frac{(2s+1)S}{S}) + \omega_k^2(\frac{(2s+1)S}{S})} \right) \sin \left( \frac{k\pi x}{l} \right) \right| \leq
\]

4
\[
\lim_{S \to \infty} \sum_{s=0}^{S-1} \sup_{x \in \left[\frac{(-2s-1)}{S}, \frac{(-2s-2)}{S}\right]} \left\{ \sum_{k=1}^{\infty} \left[ \frac{c_k \sigma_k\left(\frac{l(2s+1)-s}{S}\right)}{\sigma_k^2\left(\frac{l(2s+1)-s}{S}\right)} - d_k \omega_k\left(\frac{l(2s+1)-s}{S}\right) \right] - \frac{c_k \sigma_k(x) - d_k \omega_k(x)}{\sigma_k^2(x) + \omega_k^2(x)} \right\} + \\
\left| \frac{c_k \omega_k\left(\frac{l(2s+1)-s}{S}\right)}{\sigma_k^2\left(\frac{l(2s+1)-s}{S}\right)} + d_k \sigma_k\left(\frac{l(2s+1)-s}{S}\right) - \frac{c_k \omega_k(x) + d_k \sigma_k(x)}{\sigma_k^2(x) + \omega_k^2(x)} \right| = \\
\lim_{S \to \infty} \sum_{s=0}^{S-1} \sup_{x \in \left[\frac{(-2s-1)}{S}, \frac{(-2s-2)}{S}\right]} \left\{ \sum_{k=1}^{\infty} \left[ \frac{c_k \sigma_k\left(\frac{l(2s+1)-s}{S}\right)}{\sigma_k^2\left(\frac{l(2s+1)-s}{S}\right)} + \omega_k\left(\frac{l(2s+1)-s}{S}\right) \right] - \frac{\sigma_k(x)}{\sigma_k^2(x) + \omega_k^2(x)} \right\} + \\
\left| \frac{d_k\left(\frac{l(2s+1)-s}{S}\right)}{\sigma_k^2\left(\frac{l(2s+1)-s}{S}\right)} + \omega_k\left(\frac{l(2s+1)-s}{S}\right) - \frac{\sigma_k(x)}{\sigma_k^2(x) + \omega_k^2(x)} \right| + \\
\left| \frac{\omega_k\left(\frac{l(2s+1)-s}{S}\right)}{\sigma_k^2\left(\frac{l(2s+1)-s}{S}\right)} + \omega_k\left(\frac{l(2s+1)-s}{S}\right) - \frac{\omega_k(x)}{\sigma_k^2(x) + \omega_k^2(x)} \right| \right\} \leq \\
\lim_{S \to \infty} \sum_{s=0}^{S-1} \sup_{x \in \left[\frac{(-2s-1)}{S}, \frac{(-2s-2)}{S}\right]} \left\{ \sum_{k=1}^{\infty} 2(|c_k| + |d_k|) \frac{4C^2}{S^2} \right\} \leq \\
\lim_{S \to \infty} \frac{8C^2}{S} \sum_{k=1}^{\infty} \left( |c_k| + |d_k| \right) = 0.
\]

\[\square\]

**Remark 3.3.** Theorem 3.2 is a generalization of Theorem 2.2. Indeed, Theorem 2.2 is a simple consequence of Theorem 3.2, when \( A_n(x) = \text{const} \) for \( 0 \leq n \leq 2m \), because in that cases all conditions of Theorem 3.2 are fulfilled.

We say that \( (a_k)_{0 \leq k \leq s} \) is partition of \([-l,l]\) if \(-l = a_0 < a_1 < \cdots < a_{s-1} < a_s = l\).

We say that a real-valued function \( f \) on \([-l,l]\) is simple function if there exists a partition \( (a_k)_{0 \leq k \leq s} \) of \([-l,l]\) and a sequence of real numbers \((A_k)_{1 \leq k \leq s}\) such that

\[f(x) = \sum_{k=1}^{s} A_k \text{Ind}_{[a_{k-1},a_k]}(x)\]

for \( x \in [-l,l] \).

We have the following proposition.

**Theorem 3.4.** Suppose that \((A_n(x))_{0 \leq n \leq 2m}\) is a sequence of real-valued simple step functions on \([-l,l]\), i.e., for every \( n \) \((0 \leq n \leq 2m)\) there exists a partition \((a_k^{(n)})_{0 \leq k \leq s_n}\) of \([-l,l]\) and a sequence of real numbers \((A_k^{(n)})_{1 \leq k \leq s_n}\) such that

\[A_n(x) = \sum_{k=1}^{s_n} A_k^{(n)} \text{Ind}_{[a_{k-1}^{(n)},a_k^{(n)}]}(x)\]
for $x \in [-l, l]$.

Suppose that $A_0(x)$ does not remain a zero value on $[-l, l]$ and $\sigma_k^2(x) + \omega_k^2(x) \neq 0$ for $x \in [-l, l]$ and $k \geq 1$, where $\sigma_k(x)$ and $\omega_k(x)$ are defined by (3.3) and (3.4). Suppose also that Fourier coefficients of the function $f$ standing in the right side of the equation (3.1) satisfy the following condition $(\frac{c}{p}, c, 0, 1, 2, \ldots, d) \in E_1$.

Then the function $\Psi_0$, defined by

$$
\Psi_0(x) = \frac{c_0}{2A_0(x)} + \sum_{k=1}^{\infty} \left( \frac{c_k \sigma_k^2(x) - d_k \omega_k^2(x)}{\sigma_k^2(x) + \omega_k^2(x)} \right) \cos \left( \frac{k\pi x}{l} \right) + \left( \frac{c_k \omega_k(x) + d_k \sigma_k(x)}{\sigma_k^2(x) + \omega_k^2(x)} \right) \sin \left( \frac{k\pi x}{l} \right),
$$

(3.7)

for $x \in [-l, l]$, satisfies (3.1)–(3.2) at each point of the set

$$
(-l, l) \setminus \cup_{0 \leq n \leq 2m} \{ a_1^{(n)}, a_2^{(n)}, \ldots, a_{r_n-1}^{(n)} \}.
$$

Proof. If $x_0 \in (-l, l) \setminus G := \cup_{0 \leq n \leq 2m} \{ a_1^{(n)}, a_2^{(n)}, \ldots, a_{r_n-1}^{(n)} \}$, then by virtue of the openness of the $G$ there exists a positive real number $r > 0$ such that $(x_0 - r, x_0 + r) \subseteq G$. It is obvious that $(A_n(x))$ is constant on $(x_0 - r, x_0 + r)$ for $0 \leq n \leq 2m$. We set $A_n := A_n(x_0)$ for $0 \leq n \leq 2m$.

For $m \geq 1$, let us consider an ordinary differential equation

$$
\sum_{n=0}^{2m} A_n \frac{d^n}{dx^n} \Psi = f.
$$

(3.8)

Note that for (3.8) all conditions of Theorem 2.2 are fulfilled. Hence the series $\Psi_p$, defined by

$$
\Psi_p(x) = \frac{c_0}{2A_0} + \sum_{k=1}^{\infty} \left( \frac{c_k \sigma_k - d_k \omega_k}{\sigma_k^2 + \omega_k^2} \right) \cos \left( \frac{k\pi x}{l} \right) + \left( \frac{c_k \omega_k + d_k \sigma_k}{\sigma_k^2 + \omega_k^2} \right) \sin \left( \frac{k\pi x}{l} \right),
$$

(3.9)

is a particular solution of (3.8), where $\sigma_k$ and $\omega_k$ are defined by (2.1) and (2.2), respectively.

Notice that $\Psi_p$ defined by (3.9) coincides with $\psi_0$ defined by (3.7) at all point $x \in (x_0 - r, x_0 + r)$. Similarly, the equation (3.8) with (3.2) coincides with the equation (3.1) with (3.2) at all point $x \in (x_0 - r, x_0 + r)$. Hence $\psi_0$ defined by (3.7) satisfies (3.1)–(3.2) at each point of the set $(x_0 - r, x_0 + r)$, in particular, at point $x_0$. Since $x_0 \in (-l, l) \setminus G$ was taken arbitrary, we end the proof of Theorem 3.4.

4. On a graphical solution of the linear non-homogeneous ordinary differential equation of the higher order whose coefficients are real-valued simple step functions

Let consider the linear non-homogeneous ordinary differential equation of the 22-th order

$$
\Psi(x) + A_2(x) \frac{d^2}{dx^2} \Psi(x) + A_5(x) \frac{d^5}{dx^5} \Psi(x) + A_{14}(x) \frac{d^{14}}{dx^{14}} \Psi(x) + A_{20}(x) \frac{d^{20}}{dx^{20}} \Psi(x) +
$$
\[ A_{22}(x) \frac{d^{22}}{dx^{22}} \Psi(x) = 1 + 2 \cos(x), \quad (4.1) \]

where

\[ A_2(x) = -0.001 \times Ind_{[-\pi, -\pi/2]}(x) - 0.002 \times Ind_{[-\pi/2, 0]}(x) - 0.001 \times Ind_{[0, \pi/2]}(x) - 0.002 \times Ind_{[\pi/2, \pi]}(x), \]

\[ A_5(x) = 0.01 \times Ind_{[-\pi, -\pi/2]}(x) - 0.01 \times Ind_{[-\pi/2, 0]}(x) + 0.002 \times Ind_{[0, \pi/2]}(x) - 0.002 \times Ind_{[\pi/2, \pi]}(x), \]

\[ A_{14}(x) = 0.1 \times Ind_{[-\pi, -\pi/2]}(x) - 0.1 \times Ind_{[-\pi/2, 0]}(x) - 0.4 \times Ind_{[0, \pi/2]}(x) + 0.007 \times Ind_{[\pi/2, \pi]}(x), \]

\[ A_{20}(x) = -0.01 \times Ind_{[-\pi, -\pi/2]}(x) + 0.01 \times Ind_{[-\pi/2, 0]}(x) + 0.002 \times Ind_{[0, \pi/2]}(x) - 0.22 \times Ind_{[\pi/2, \pi]}(x), \]

\[ A_{22}(x) = 0.001 \times Ind_{[-\pi, -\pi/2]}(x) - 0.001 \times Ind_{[-\pi/2, 0]}(x) + 0.0003 \times Ind_{[0, \pi/2]}(x) - 0.0003 \times Ind_{[\pi/2, \pi]}(x). \]

**Definition 4.1** We say that \( g \in FD^{(22)}([-\pi, \pi] \setminus G) \) (\( G := \{-\pi, -\pi/2, 0, \pi/2, \pi\} \)) if

\[ g(x) = g_1(x) \times Ind_{[-\pi, -\pi/2]}(x) + g_2(x) \times Ind_{[-\pi/2, 0]}(x) + g_3(x) \times Ind_{[0, \pi/2]}(x) + g_4(x) \times Ind_{[\pi/2, \pi]}(x) \quad (4.2) \]

for some \( g_1, g_2, g_3, g_4 \in FD^{(22)}([-\pi, \pi]). \)

Below we present the program in MathLab which gives the graphical solution of the differential equation (4.1) in the class \( FD^{(22)}([-\pi, \pi] \setminus G). \)

\[ A_1 = [0, -0.001, 0, 0, 0.01, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -0.01, 0, 0, 0.001]; \]
\[ A_2 = [0, -0.002, 0, 0, -0.01, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.001]; \]
\[ A_3 = [0, -0.001, 0, 0, 0.002, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.002, 0, 0.0003]; \]
\[ A_4 = [0, -0.002, 0, 0, -0.002, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.007, 0, 0, 0, 0, 0, 0.22, 0, -0.0003]; \]
\[ C = [2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]; \]
\[ D = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]; \]
\[ C_0 = 2; A_{10} = 1; A_{20} = 1; A_{30} = 1; A_{40} = 1; \]
\[ x = 1 : 20; \]
\[ S_1 = A_{10}; S_2 = A_{20}; S_3 = A_{30}; S_4 = A_{40}; \]

**for** \( k = 1 : 11 \)

\[ S_1 = S_1 + (-1)(k) \ast A_1(2 \ast k) \ast x.(2 \ast k); \]
\[ S_2 = S_2 + (-1)(k) \ast A_2(2 \ast k) \ast x.(2 \ast k); \]
\[ S_3 = S_3 + (-1)(k) \ast A_3(2 \ast k) \ast x.(2 \ast k); \]
\[ S_4 = S_4 + (-1)(k) \ast A_4(2 \ast k) \ast x.(2 \ast k); \]

**end**

\[ O_1 = A_1(1); O_2 = A_2(1); O_3 = A_3(1); O_4 = A_4(1); \]

**for** \( k = 1 : 10 \)
\[ O1 = O1 + (-1)k \times A1(2k + 1) \times (2k + 1); \]
\[ O2 = O2 + (-1)k \times A2(2k + 1) \times (2k + 1); \]
\[ O3 = O3 + (-1)k \times A3(2k + 1) \times (2k + 1); \]
\[ O4 = O4 + (-1)k \times A4(2k + 1) \times (2k + 1); \]

end

\[ x1 = (-pi) : (pi/100) : (-pi/2); \]
\[ y1 = C0/(2 \times A10); \]

for \( n = 1 : 20 \)
\[ y1 = y1 + \cos(n \times x1) \times (C(n) \times S1(n) - D(n) \times O1(n))/(S1(n)2 + O1(n)2) + \]
\[ \sin(n \times x1) \times (C(n) \times O1(n) + D(n) \times S1(n))/(S1(n)2 + O1(n)2); \]

end

\[ x2 = (-pi/2) : (pi/100) : 0; \]
\[ y2 = C0/(2 \times A20); \]

for \( n = 1 : 20 \)
\[ y2 = y2 + \cos(n \times x2) \times (C(n) \times S2(n) - D(n) \times O2(n))/(S2(n)2 + O2(n)2) + \]
\[ \sin(n \times x2) \times (C(n) \times O2(n) + D(n) \times S2(n))/(S2(n)2 + O2(n)2); \]

end

\[ x3 = 0 : (pi/100) : (pi/2); \]
\[ y3 = C0/(2 \times A30); \]

for \( n = 1 : 20 \)
\[ y3 = y3 + \cos(n \times x3) \times (C(n) \times S3(n) - D(n) \times O3(n))/(S3(n)2 + O3(n)2) + \]
\[ \sin(n \times x3) \times (C(n) \times O3(n) + D(n) \times S3(n))/(S3(n)2 + O3(n)2); \]

end

\[ x4 = (pi/2) : (pi/100) : pi; \]
\[ y4 = C0/(2 \times A40); \]

for \( n = 1 : 20 \)
\[ y4 = y4 + \cos(n \times x4) \times (C(n) \times S4(n) - D(n) \times O4(n))/(S4(n)2 + O4(n)2) + \]
\[ \sin(n \times x4) \times (C(n) \times O4(n) + D(n) \times S4(n))/(S4(n)2 + O4(n)2); \]

end

for \( i = 1 : 20 \)
\[ if O1(i)2 + S1(i)2 = 0; O2(i)2 + S2(i)2 = 0; O3(i)2 + S3(i)2 = 0; O3(i)2 + S3(i)2 = 0; \]
\[ plot(x1,y1,x2,y2,x3,y3,x4,y4) \]
\[ else error(‘the ordinary differential equation has no solution or has infinitely many \]
\[ solutions’ in the class \( FD^{(22)}([-\pi, \pi] \setminus G) \)) \]
\[ end \]

end

On Figure 1, the graphical solution of the differential equation (4.1) is presented.

Remark 4.1 Notice that for each natural number \( M > 1 \), one can easily modify this \( \) 
program in MathLab for obtaining a graphical solution of the differential equation (3.1)-
(3.2) in \( FD^{(2M)}([-l, l] \setminus G) \) whose coefficients \( (A_n(x))_{0 \leq n \leq 2M} \) are real-valued \( \) 
simple step functions on \( [-l, l] \), \( f \) is a trigonometric polynomial on \( [-l, l] \) and \( G \) is the partition of \( \) 
the interval \( [-l, l] \) defined by the family \( (A_n(x))_{0 \leq n \leq 2M} \).
Remark 4.2 Since each constant \( c \) admits the following evident representation

\[
c = c \times \text{Ind}_{[-\pi, -\pi/2]}(x) + c \times \text{Ind}_{[-\pi/2, 0]}(x) + c \times \text{Ind}_{[0, \pi/2]}(x) + c \times \text{Ind}_{[\pi/2, \pi]}(x),
\]

we can use above mentioned program for a solution of the differential equation (2.3)-(2.4) with constant coefficients.

On Figure 2, the graphical solution of the linear non-homogeneous ordinary differential equation of the of the second order with real-valued constant coefficients

\[
\Psi(x) - \frac{d^2}{dx^2} \Psi(x) = 1/2 + \cos(x),
\]

is presented, which has been obtained by entering in the above mentioned program of the following data:

- \( A1 = [0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] \);
- \( A2 = [0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] \);
- \( A3 = [0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] \);
- \( A4 = [0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] \);
- \( C = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] \);
- \( D = [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] \);
- \( C0 = 1; A10 = 1; A20 = 1; A30 = 1; A40 = 1; \)
Remark 4.1. The approach of Theorem 3.4 used for a solution of (3.1)-(3.2) with real-valued simple step functions \( (A_n(x))_{0 \leq n \leq 2M} \) can be used in such a case when the corresponding coefficients are continuous functions on \([-l, l]\). If we will approximate these coefficients by real-valued simple step functions, then it is natural to wait that under some "nice restrictions" on these coefficients the solution obtained by Theorem 3.4, will be a "good approximation" of the corresponding solution.

References

[1] Linear differential equation, [http://en.wikipedia.org/wiki/Linear_differential_equation](http://en.wikipedia.org/wiki/Linear_differential_equation)

[2] G. Birkhoff, G. Rota, *Ordinary Differential Equations*, New York: John Wiley and Sons, Inc., 1978.

[3] J.C. Robinson, *An Introduction to Ordinary Differential Equations*, Cambridge, UK.: Cambridge University Press, 2004.

[4] G. Pantsulaia, G. Giorgadze, On some applications of infinite-dimensional cellular matrices, *Georg. Inter. J. Sci. Tech.*, *Nova Science Publishers*, Volume 3, Issue 1 (2011), 107-129.
[5] A. Stanoyevitch, *Introduction to MATLAB® with numerical preliminaries*, Wiley-Interscience [John Wiley & Sons], Hoboken, NJ, 2005. x+331 pp.