A GENERALIZATION OF THE BARBAN-DAVENPORT-HALBERSTAM THEOREM TO NUMBER FIELDS
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ETHAN SMITH

Abstract. For a fixed number field $K$, we consider the mean square error in estimating the number of primes with norm congruent to $a$ modulo $q$ by the Chebotarëv Density Theorem when averaging over all $q \leq Q$ and all appropriate $a$. Using a large sieve inequality, we obtain an upper bound similar to the Barban-Davenport-Halberstam Theorem.

1. Introduction

The mean square error in Dirichlet’s Theorem for primes in arithmetic progressions was first studied by Barban [1] and by Davenport and Halberstam [3, 4]. Bounds such as the following are usually referred to as the Barban-Davenport-Halberstam Theorem, although this particular refinement is attributed to Gallagher. Let

$$\psi(x; q, a) := \sum_{p^m \leq x, \ p^m \equiv a \pmod{q}} \log p.$$ 

Then, for fixed $M > 0$,

$$\sum_{q \leq Q} \sum_{(a, q) = 1} (\psi(x; q, a) - \frac{x}{\varphi(q)})^2 \ll xQ \log x \quad (1)$$

if $x(\log x)^{-M} \leq Q \leq x$. See [5, p. 169]. Here $\varphi$ is the Euler totient function. The sum on the left may be viewed as the mean square error in the Chebotarëv Density Theorem when averaging over cyclotomic extensions of $\mathbb{Q}$.

The inequality in (1) was later refined by Montgomery [12] and Hooley [8], both of whom gave asymptotic formulae valid for various ranges of $Q$. See also [2, Theorem 1]. Montgomery’s method is based on a result of Lavrik [10] on the distribution of twin primes, while Hooley’s method relies on the large sieve. For recent work concerning such asymptotics, see [11].

Results of this type have also been generalized to number fields. Wilson considered error sums over prime ideals falling into a given class of the narrow ideal class group in [13]. While in [7], Hinz considered sums of principal prime ideals given by a generator which is congruent to a given algebraic integer modulo an integral ideal and whose conjugates fall into a designated range.

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Let $K$ be a fixed algebraic number field which is normal over $\mathbb{Q}$. In this paper, we consider the mean square error in the Chebotarëv Density Theorem when averaging over cyclotomic extensions of $K$. That is, we consider sums of the form

$$\psi_K(x; q, a) := \sum_{Np^m \leq x, \quad Np^m \equiv a \pmod{q}} \log Np.$$ 

Here the sum is over powers of prime ideals of $K$, and there is no restriction to principal primes.

For each positive integer $q$, we let $A_q := K \cap \mathbb{Q}(\zeta_q)$, so $A_q$ is an Abelian (possibly trivial) extension of $\mathbb{Q}$. We have a natural composition of maps:

$$\text{Gal}(K(\zeta_q)/K) \rightarrow \text{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q}) \rightarrow (\mathbb{Z}/q\mathbb{Z})^*;$$

and in fact, $\text{Gal}(K(\zeta_q)/K) \cong \text{Gal}(\mathbb{Q}(\zeta_q)/A_q)$. We let $G_q$ denote the image of composition of maps in (2). Then, in particular, $G_q \cong \text{Gal}(K(\zeta_q)/K)$. See the diagram below.

Define $\varphi_K(q) := |G_q|$. By the Chebotarëv Density Theorem, for each $a \in G_q$,

$$\psi_K(x; q, a) = \sum_{Np^m \leq x, \quad Np^m \equiv a \pmod{q}} \log Np \sim \frac{x}{\varphi_K(q)}.$$ 

**Theorem 1.** For a fixed $M > 0$,

$$\sum_{q \leq Q} \sum_{a \in G_q} \left( \psi_K(x; q, a) - \frac{x}{\varphi_K(q)} \right)^2 \ll xQ \log x$$

if $x(\log x)^{-M} \leq Q \leq x$.

**Remark 1.** Note that the above agrees with (1) when $K = \mathbb{Q}$. The method of proof is essentially an adaptation of the proof of (1) given in [5, pp. 169-171], the main idea being an application of the large sieve.

**Remark 2.** We say nothing about the constant implied by the symbol $\ll$ in the present paper. However, the author has recently adapted the methods of Hooley [8] to refine the result into an asymptotic formula. This work is the subject of a forthcoming paper.
Remark 3. The above result is unconditional and gives a better bound than the Grand Riemann Hypothesis (GRH). See Section 4 for comparison with GRH.

2. Preliminaries and Intermediate Estimates

We will use lower case Roman letters for rational integers and Fraktur letters for ideals of the number field $K$. In particular, $p$ will always denote a rational prime and $p$ will always denote a prime ideal in $\mathcal{O}_K$, the ring of integers of $K$. We also let $g(K/Q;p)$ and $f(K/Q;p)$ denote the number of primes of $K$ lying above $p$ and the degree of any prime of $K$ lying above $p$, respectively. Note that $f(K/Q;p)$ is well-defined since $K$ is normal over $\mathbb{Q}$.

Let $\mathcal{X}(q)$ denote the character group modulo $q$, $\mathcal{X}^*(q)$ the characters which are primitive modulo $q$, and let $G_q^\perp$ denote the subgroup of characters that are trivial on $G_q$. Then the character group $\hat{G}_q$ is isomorphic to $\mathcal{X}(q)/G_q^\perp$, and the number of such characters is $\phi_K(q) = |G_q| = \varphi(q)/|G_q^\perp|$. As usual, we denote the trivial character of the group $\mathcal{X}(q)$ by $\chi_0$.

For any Hecke character $\xi$ on the ideals of $\mathcal{O}_K$, we define

$$\psi_K(x, \xi) := \sum_{Na \leq x} \xi(a) \Lambda_K(a);$$

and for each character $\chi \in \mathcal{X}(q)$, we define

$$\psi'_K(x, \chi \circ N) := \begin{cases} \psi_K(x, \chi \circ N), & \chi \not\equiv \chi_0 \pmod{G_q^\perp}, \\ \psi_K(x, \chi \circ N) - x, & \chi \equiv \chi_0 \pmod{G_q^\perp}. \end{cases}$$

Here, $\Lambda_K$ is the von Mangoldt function defined on the ideals of $\mathcal{O}_K$, i.e.,

$$\Lambda_K(a) := \begin{cases} \log Np, & a = p^m, \\ 0, & \text{otherwise}. \end{cases}$$

Lemma 1.

$$\sum_{q \leq Q} \frac{\varphi(q)}{\varphi(q)} \sum_{\chi \in \mathcal{X}^*(q)} |\psi_K(x, \chi \circ N)|^2 \ll (x + Q^2)x \log x.$$ 

Proof. For $n \in \mathbb{N}$, we define

$$D_K(n) := \# \{p^m \triangleleft \mathcal{O}_K : Np^m = n\};$$

$$\Lambda_K(n) := \begin{cases} \log p^{f(K/Q;p)}, & n = p^k, \\ 0, & \text{otherwise}. \end{cases}$$

Now, note that

$$\psi_K(x, \chi \circ N) = \sum_{Na \leq x} \chi(Na) \Lambda_K(a) = \sum_{n \leq x} \chi(n) D_K(n) \Lambda_K^*(n).$$
We apply the large sieve in the form of Theorem 4 in chapter 27 of [3] to see that

\[
\sum_{q \leq Q} \phi(q) \sum_{\chi \in A^*(q)} |\psi_K(x, \chi \circ N)|^2 \ll (x + Q^2) \sum_{n \leq x} (D_K(n) \Lambda^*_K(n))^2
\]

\[
= (x + Q^2) \sum_{p^k \leq x} g(K/Q; p) D_K(p^k) \Lambda^*_K(p^k)^2
\]

\[
\ll (x + Q^2) \sum_{Np^m \leq x} \Lambda_K(p^m)^2
\]

\[
\ll (x + Q^2) x \log x.
\]

Lemma 2. If \(\xi_1\) and \(\xi_2\) are Hecke characters modulo \(q_1\) and \(q_2\) respectively, and if \(\xi_1\) induces \(\xi_2\), then

\[
\psi_K(x, \xi_2) = \psi_K(x, \xi_1) + O \left( (\log qx)^2 \right),
\]

where \((q) = q_2 \cap \mathbb{Z}\).

Proof.

\[
|\psi_K(x, \xi_1) - \psi_K(x, \xi_2)| = \left| \sum_{Np^m \leq x, \ (p,q) > 1} \xi_1(p^m) \log Np \right| \leq \sum_{p^k \leq x, \ (p,q) > 1} D_K(p^k) f(K/Q; p) \log p
\]

\[
= \sum_{p | q} \sum_{k=1, f(K/Q; p) | k} g(K/Q; p) f(K/Q; p) \log p
\]

\[
\ll \sum_{p | q} \left\lfloor \frac{\log x}{\log p} \right\rfloor \log p \ll (\log qx)^2.
\]

Lemma 3. If \(\chi\) is a character modulo \(q \leq (\log x)^{M+1}\), then there exists a positive constant \(C\) (depending on \(M\)) such that

\[
\psi'_K(x, \chi \circ N) \ll x \exp \left\{ -C \sqrt{\log x} \right\}.
\]

Proof. As a Hecke character on the ideals of \(O_K\), \(\chi \circ N\) may not be primitive modulo \(qO_K\). Let \(\xi = \xi_\chi\) be the primitive Hecke character which induces \(\chi \circ N\), and let \(f_\chi\) be its conductor. Write \(s = \sigma + it\). By [9, Theorem 5.35], there exists an effective constant \(c_0 > 0\) such that the Hecke L-function \(L(s, \xi) := \sum_{N \leq x} \xi(a)(Na)^{-s}\) has at most one zero in the region

\[
\sigma > 1 - \frac{c_0}{[K : \mathbb{Q}] \log (|d_K| N|t| + 3)),
\]

where \(d_K\) denotes the discriminant of the number field. Further, if such a zero exists, it is real and simple. In the case that such a zero exists, we call it an “exceptional zero” and
denote it by \( \beta_\xi \). Thus, by [9, Theorem 5.13], there exists \( c_1 > 0 \) such that

\[
\psi_K(x, \xi) = \delta_\xi x - \frac{x^{\beta_\xi}}{\beta_\xi} + O \left( x \exp \left\{ \frac{-c_1 \log x}{\sqrt{\log x + \log N_{f_\chi}}} \right\} (\log(xN_{f_\chi}))^4 \right),
\]

where

\[
\delta_\xi = \begin{cases} 
1, & \xi \text{ trivial,} \\
0, & \text{otherwise,}
\end{cases}
\]

and the term \( x^{\beta_\xi}/\beta_\xi \) is omitted if the \( L \)-function \( L(s, \xi) \) has no exceptional zero in the region (3). Now, since \( f_\chi qO_K \) and \( q \leq (\log x)^{M+1} \), we have the following bound on the error term:

\[
x \exp \left\{ \frac{-c_1 \log x}{\sqrt{\log x + \log N_{f_\chi}}} \right\} (\log(xN_{f_\chi}))^4 \ll x \exp \left\{ -c_2 \sqrt{\log x} \right\}
\]

for some positive constant \( c_2 \).

By [6, Theorem 3.3.2], we see that for every \( \epsilon > 0 \), there exists a constant \( c_\epsilon > 0 \) such that if \( \beta_\xi \) is an exceptional zero for \( L(s, \xi) \), then

\[
\beta_\xi < 1 - \frac{c_\epsilon}{(N_{f_\chi})^\epsilon} \leq 1 - \frac{c_\epsilon}{q[K:Q]\epsilon}.
\]

Thus,

\[
x^{\beta_\xi} < x \exp \left\{ -c_\epsilon (\log x)q^{-[K:Q]\epsilon} \right\} < x \exp \left\{ -c_\epsilon (\log x)^{1/2} \right\}
\]

upon choosing \( \epsilon \) so that \( [K:Q]\epsilon = (2M + 2)^{-1} \). Whence, for \( q \leq (\log x)^{M+1} \), there exists \( C > 0 \) such that

\[
\psi_K(x, \xi) = \delta_\xi x + O \left( x \exp \left\{ -C \sqrt{\log x} \right\} \right).
\]

Therefore, by Lemma 2

\[
\psi'_K(x, \chi \circ N) \ll x \exp \left\{ -C \sqrt{\log x} \right\}
\]

for \( q \leq (\log x)^{M+1} \). \( \square \)

3. PROOF OF MAIN THEOREM

For \( a \in G_q \), we define the error term \( E_K(x; q, a) := \psi_K(x; q, a) - \frac{x}{\varphi_K(q)} \), and note that

\[
E_K(x; q, a) = \frac{1}{\varphi_K(q)} \sum_{\chi \in G_q} \bar{\chi}(a) \psi'_K(x, \chi \circ N).
\]
Now we form the square of the Euclidean norm and sum over all \( a \in G_q \) to see

\[
\sum_{a \in G_q} |E_K(x; q, a)|^2 = \frac{1}{\varphi_K(q)^2} \sum_{a \in G_q} \left| \sum_{\chi \in \hat{G}_q} \overline{\tilde{\chi}_1(a)} \tilde{\chi}_2(a) \psi'_K(x, \chi_1 \circ N) \psi'_K(x, \chi_2 \circ N) \right|^2
\]

\[
= \frac{1}{\varphi_K(q)^2} \sum_{\chi \in \hat{G}_q} |\psi'_K(x, \chi \circ N)|^2
\]

\[
= \frac{1}{\varphi(q)} \sum_{\chi \in \hat{X}(q)} |\psi'_K(x, \chi_1 \circ N)|^2.
\]

For each \( \chi \in \hat{X}(q) \), we let \( \chi^* \) denote the primitive character which induces \( \chi \). By Lemma \( \text{[2]} \) we have \( \psi'_K(x, \chi \circ N) = \psi'_K(x, \chi^* \circ N) + O((\log qx)^2) \). Hence, summing over \( q \leq Q \) and exchanging each character for its primitive version, we have

\[
\sum_{q \leq Q} \sum_{a \in G_q} E_K(x; q, a)^2 \ll \sum_{q \leq Q} (\log qx)^4 + \sum_{q \leq Q} \frac{1}{\varphi(q)} \sum_{\chi \in \hat{X}(q)} |\psi'_K(x, \chi^* \circ N)|^2.
\]

The first term on the right is clearly smaller than \( xQ \log x \), so we concentrate on the second. Now,

\[
\sum_{q \leq Q} \frac{1}{\varphi(q)} \sum_{\chi \in \hat{X}(q)} |\psi'_K(x, \chi^* \circ N)|^2 = \sum_{q \leq Q} \sum_{\chi \in \hat{X}(q)} |\psi'_K(x, \chi \circ N)|^2 \sum_{k \leq Q/q} \frac{1}{\varphi(kq)}
\]

\[
\ll \sum_{q \leq Q} \frac{1}{\varphi(q)} \left( \log \frac{2Q}{q} \right) \sum_{\chi \in \hat{X}(q)} |\psi'_K(x, \chi \circ N)|^2 \quad (4)
\]

since \( \sum_{k \leq Q/q} 1/\varphi(kq) \ll \varphi(q)^{-1} \log(2Q/q) \). See \[3\] p. 170]. The proof will be complete once we show that \( (4) \) is smaller than \( xQ \log x \) for \( Q \) in the specified range.

As with the proof of \( (1) \) in \[3\] pp. 169-171], we consider large and small \( q \) separately. We start with the large values. Since \( \psi'_K(x, \chi \circ N) \ll \psi_K(x, \chi \circ N) \), by Lemma \( \text{[4]} \) we have

\[
\sum_{U < q \leq 2U} \frac{U}{\varphi(q)} \sum_{\chi \in \hat{X}(q)} |\psi'_K(x, \chi \circ N)|^2 \ll (x + U^2)x \log x,
\]

which implies

\[
\sum_{U \leq q \leq 2U} \frac{1}{\varphi(q)} \left( \log \frac{2Q}{q} \right) \sum_{\chi \in \hat{X}(q)} |\psi'_K(x, \chi \circ N)|^2 \ll (xU^{-1} + U)x \log x \left( \log \frac{2Q}{U} \right)
\]
for $1 \leq 2U \leq Q$. Summing over $U = Q2^{-k}$, we have

$$
\sum_{Q_1 < q \leq Q} \frac{1}{\varphi(q)} \left( \log \frac{2Q}{q} \right) \sum_{\chi \in \chi^*(q)} |\psi'_{K}(x, \chi \circ N)|^2 \ll x \log x \sum_{k=1}^{\left[ \frac{\log(Q/Q_1)}{\log 2} \right]} (x2^kQ^{-1} + Q2^{-k}) 
\ll x^2Q_1^{-1}(\log x) \log Q + xQ \log x 
\ll xQ \log x
$$

if $x((\log x)^{-M} \leq Q \leq x$ and $Q_1 = \left( (\log x)^{M+1} \right)$. We now turn to the small values of $q$. Applying Lemma 3, we have

$$
\sum_{q \leq Q_1} \frac{1}{\varphi(q)} \left( \log \frac{2Q}{q} \right) \sum_{\chi \in \chi^*(q)} |\psi'_{K}(x, \chi \circ N)|^2 \ll Q_1(\log Q) \left( x \exp \left\{ -c\sqrt{\log x} \right\} \right)^2 
\ll x^2(\log x)^{-M} \ll xQ \log x.
$$

Combining (5) and (6), the theorem follows.

4. Comparison with GRH

Using the bound on the analytic conductor of the $L$-function $L(s, \chi \circ N)$ given in [9, p. 129], GRH implies

$$
\sum_{q \leq Q} \sum_{a \in G_q} E_{K}(x; q, a)^2 = \sum_{q \leq Q} \frac{1}{\varphi(q)} \sum_{\chi \in \chi(q)} |\psi'_{K}(x, \chi \circ N)|^2 
\ll (\sqrt{x}(\log x)^2)^2 \sum_{q \leq Q} \frac{1}{\varphi(q)} \sum_{\chi \in \chi(q)} 1 
= xQ(\log x)^4.
$$

See [9] Theorem 5.15 for this implication of GRH.

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Department of Mathematical Sciences, Clemson University, Box 340975 Clemson, SC 29634-0975

E-mail address: ethans@math.clemson.edu

URL: www.math.clemson.edu/~ethans