Synchronization of the fractional-order chaotic system via adaptive observer

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The means to design the observer for a class of fractional-order chaotic systems is investigated. A novel Lyapunov function is proposed and a robust adaptive observer is designed to synchronize a given fractional-order chaotic system. The constructed observer could guarantee the error of state converges to zero asymptotically. Simulation results demonstrate the effectiveness and robustness of the proposed scheme.

Keywords: adaptive observer; synchronization; fractional-order nonlinear system

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1. Introduction

In the last 30 years, fractional calculus has attracted attention of many physicists and engineers. Notable contributions have been made to both the theory and applications of fractional differential equations (Bagley & Calico, 1991; Duarte & Macado, 2002; Heaviside, 1971; Ichise, Nagayanagi, & Kojima, 1971; Linares, Baillot, Oustaloup, & Ceyral, 1996; Mandelbrot & Van Ness, 1968; Oustaloup, 1995; Podlubny, 1999b; Sun, Abdelwahad, & Onaral, 1984; and references therein). Also, fractional differential equations have recently proved to be valuable tools in modeling of many physical phenomena in various fields of science and engineering.

Recently, studying fractional-order chaotic systems has become an active research field. Synchronization of fractional-order chaotic systems starts to attract increasing attention due to its potential applications in secure communication and control processing. Some approaches have been proposed to achieve chaos synchronization in fractional-order chaotic systems (Li & Deng, 2006; Li, Yu, & Luo, 2012; Liu, 2013; Lu, 2006a, 2006b; Qi, Yang, & Zhang, 2010; Wang, Wang, & Niu, 2011; Wang, Zhang, Lin, & Zhang, 2011; Wang, Zhang, & Wang, 2013; Wu, Lu, & Shen, 2009; Wu, Zhang, & Yang, 2011; Zhang & Yang, 2011a, 2011b; Zhang & Yang, 2012a, 2012b). However, the lack of the extension of the existing adaptive observers for fractional order systems is sensible.

In this paper, a novel adaptive observer is presented to solve the problem of state reconstruction for fractional-order nonlinear systems. It is shown that the proposed observer guarantees that the state estimation errors are convergent to zero. The Lyapunov approach is utilized to analyze the stability of the estimation error system. It ought to be mentioned that the proposed observer is very simple and constructive for practical applications. Moreover, utilizing fractional calculus, a new Lyapunov function is proposed for the error dynamics when the fractional-order observer is applied.

The rest of the paper is organized as follows: In Section 2, some basic concepts of fractional calculus is described and its properties are discussed. In Section 3, a novel adaptive observer is presented and a stability analysis of the fractional-order error system is given. In Section 4, the adaptive observer scheme has been tested via numerical simulations and the corresponding results are presented. Finally, some concluding remarks are drawn in Section 5.

2. Preliminaries and definitions

In this section, we introduce the definition of Reimann–Liouville fractional integration and derivative. The $\alpha$ th-order Reimann–Liouville fractional integration of function $f(t)$ with respect to $t$ and the terminal value $t_0$ is given by

$$\begin{align*}
_{0}D_{t}^{-\alpha}f(t) &= \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{1-\alpha}} \mathrm{d}\tau, \quad (1)
\end{align*}$$

and the Reimann–Liouville definition of $\alpha$ th-order fractional derivative is given by

$$\begin{align*}
_{0}D_{t}^{\alpha}f(t) &= \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{m+1-\alpha}} \mathrm{d}\tau, \quad (2)
\end{align*}$$

where $m$ is the first integer which is larger than $\alpha$, i.e. $m-1 \leq \alpha < m$ and $\Gamma$ is the Gamma function.
The material presented in the sequel is based on the aforementioned definitions of fractional differentiation and integration. Two properties of Reimann–Liouville fractional integration and derivative are given as follows (Podlubny, 1999a):

**Property 1**
\[ t_0 D_t^\beta t_0 D_t^{\beta - \alpha} f(t) = t_0 D_t^{\beta - \alpha} f(t), \quad (\alpha > 0, \beta > 0) \]  

**Property 2** (Leibniz’s rule)
\[ 0D_t^\alpha f(t) = \frac{\Gamma(1 + \alpha)}{\Gamma(1 + k)} t_0 D_t^{\alpha - k} f(t) \]
\[ = f(t) 0D_t^\alpha f(t) + \sum_{k=1}^{\infty} \frac{\Gamma(1 + \alpha)}{\Gamma(1 + k)\Gamma(1 - k + \alpha)} t_0 D_t^{\alpha - k} f(t). \]

### 3. Main results

Considering a class of fractional-order chaotic systems described by
\[ 0D_t^\alpha x = Ax + Bf(x), \quad y = Cx \]  
where \( 0 < \alpha < 1, x \in \mathbb{R}^n \) is the state vector, \( y \in \mathbb{R}^m \) is the output vector, \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times q} \) and \( C \in \mathbb{R}^{m \times n} \) are known matrices, and \( f(\cdot) \in \mathbb{R}^n \) is a nonlinear function vector.

The adaptive observer for system (5) is constructed as follows:
\[ 0D_t^\alpha \hat{x} = A\hat{x} + Bf(\hat{x}) + \frac{1}{2}kB(y - C\hat{x}). \]

To study the synchronization between systems (5) and (6), some necessary assumptions must be made as follows:

**Assumption 1** The nonlinear function vector \( f(x) \) satisfies Lipschitz conditions:
\[ ||f(x) - f(\hat{x})|| < l_f ||x - \hat{x}|| = l_f ||y||, \]
where \( l_f \) is a Lipschitz constant, \( ||\cdot|| \) denotes 2-norm.

**Remark 1** Based on the boundedness of chaotic systems, Assumption 1 is reasonable.

**Assumption 2** Suppose that the pair \((A, C)\) is observable and the pair \((A, B)\) is controllable. Further, there exists a constant vector \( L \in \mathbb{R}^{n \times m} \) to make \((A - LC)T = -Q\), \( B^T = C \), where \( P = P^T \) and \( Q = Q^T \) are two positive definite matrices.

Let the error signals be
\[ e = x - \hat{x}, \quad \tilde{k} = k^* - k, \]
where \( k^* \) is a constant to be determined. From systems (5) and (6), the error system is as follows:
\[ 0D_t^\alpha e = (A - LC)e + B(f(x) - f(\hat{x})) - \frac{1}{2}kBCE + LCE. \]

It is obvious that the synchronization between systems (5) and (6) is achieved if and only if the error system (8) is asymptotically stable at the origin.

Now, we give our main result.

**Theorem 1** Suppose that Assumptions 1 and 2 hold, then the response system (6) can asymptotically synchronize the drive system (5) if the following conditions are satisfied:

The adaptation law \( k \) is chosen as
\[ \dot{k} = c_k \| y - C\hat{x} \|^2, \]
where \( c_k \) is a positive number to be chosen suitably.

**Proof** From Assumption 2, there exist two positive definite matrices \( P = P^T \) and \( Q = Q^T \) such that the following equation holds:
\[ (A - LC)^T P + P(A - LC) = -Q, \quad B^T = C. \]

Now, consider a Lyapunov candidate function
\[ V = 2_0D_t^{\alpha - 1}e^TPe + \frac{1}{2c_k}(k^* - k)^2. \]

By using (7), (9), and (10) and the Properties 1 and 2, the derivative of \( V \) with respect to \( t \) is given by
\[ \dot{V} = 2_0D_t^{\alpha - 1}e^TPe - (k^* - k)\dot{k}/c_k \]
\[ = e^TP_0D_t^{\alpha}e + (0D_t^{\alpha}e)^TPe + y - (k^* - k) \| y - C\hat{x} \|^2 \]
\[ = e^T[(A - LC)^TP + P(A - LC)]e + 2e^TPB[f(x) - f(\hat{x})] \]
\[ + 2e^TPEC + y - ke^TPBCE - (k^* - k)\|CE\|^2 \]
\[ = e^T[(A - LC)^TP + P(A - LC)]e + 2e^TCE + y - (k^* - k) \| CE \|^2 \]
\[ = e^T[(A - LC)^TP + P(A - LC)]e + 2e^TCE + y - (k^* - k) \| CE \|^2 \]
\[ + 2e^TPEC + y - k^* \| CE \|^2, \]
where \( D_t^{\alpha} = d/dt, \gamma = 2 \sum_{k=1}^{\infty}(1 + \alpha)/\Gamma(1 + k) \Gamma(1 - k + \alpha), \Gamma(1 - k + \alpha)/(\alpha/\Gamma(1 + k)) \) for \( \alpha > 1 \).

Since \( B^T = C \), and \( (0D_t^{\alpha}e)^TP_0D_t^{\alpha - k}e = (0D_t^{\alpha}e)^TPB^m(B^T)^{-1}0D_t^{\alpha - k}e \) we suppose
\[ \gamma \leq 2c_\gamma \| e \| \| CE \| \leq \frac{c_\gamma^2}{\epsilon_\gamma} \| CE \|^2 + \epsilon_\gamma \| e \|^2, \]
in which \( c_\gamma \) is a positive constant, and \( \epsilon_\gamma \) is a suitable positive constant.
By Assumption 1, we get the following inequalities:
\[
2e^T C^T [f(x) - f(\hat{x})] \leq 2\epsilon \| x - \hat{x} \| + \epsilon \| e \|^2,
\]
\[
\frac{1}{\epsilon}\| x - \hat{x} \| + \epsilon \| e \|^2 \leq \frac{1}{\epsilon} \| e \|^2 + \epsilon \| e \|^2.
\] (13)

Notice that
\[
2e^T PLCe = 2e^T PLB^T Pe \leq 2 \| L^T Pe \| \cdot \| B^m Pe \| \xrightarrow{\epsilon \epsilon_1} \frac{1}{\epsilon_1} \| B^T Pe \|^2 + \epsilon \| e \|^2 + \epsilon l\max(PPL^T P) \| e \|^2,
\]
where \( \epsilon_1 \) and \( \epsilon_2 \) are two suitable positive constants.

Then, we get
\[
\dot{V} \leq e^T [(A - LC)^T P + P(A - LC)] + (\epsilon \epsilon_1 + \epsilon \lambda\max(PPLL^T P)) e \| e \|^2.
\]
\[
\dot{V} \leq e^T [(A - LC)^T P + P(A - LC)] + (\epsilon \epsilon_1 + \epsilon \lambda\max(PPLL^T P)) e \| e \|^2.
\]

From the above inequality, \( \epsilon \epsilon_1, \epsilon \lambda_m \), and \( \epsilon_1 \) can be chosen to be small enough such that \( \dot{V} < 0 \) for all nonzero error. Therefore, the error system (7) is asymptotically stable at the origin, i.e. the response system (6) can asymptotically synchronize the drive system.

4. Simulation

In this section, in order to show the effectiveness of the proposed scheme in the preceding section, a numerical example on the fractional-order chaotic Chen system will be provided.

The fractional-order chaotic Chen system (Li & Chen, 2004)
\[
\begin{align*}
D^\alpha x_1 &= a(x_2 - x_1) \\
D^\alpha x_2 &= d x_1 - x_1 x_3 + c x_2 \\
D^\alpha x_3 &= x_1 x_2 - b x_3.
\end{align*}
\] (14)

When \( \alpha = 0.93, a = 35, b = 3, c = 28, \) and \( d = -7, \) the fractional-order Chen system behaves chaotically. Figure 1 shows the chaotic trajectory of (14). Rewrite the fractional-order chaotic Chen system as follows:
\[
D^\alpha_t x = A x + B f(x),
\] (15)

where
\[
A = \begin{pmatrix} -35 & 35 & 0 \\ -7 & 28 & 0 \\ 0 & 0 & -3 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad f(x) = \begin{pmatrix} -x_1 x_3 \\ x_1 x_2 \end{pmatrix}.
\]

The output vector is \( y = C x. \) Matrix \( C \) is chosen
\[
C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

Clearly, the pair \( (A, C) \) is observable. Let system (15) be a master system. Applying the results in Section 3, an adaptive observer is designed as follows:
\[
D^\alpha_t \hat{x} = A \hat{x} + B f(\hat{x}) + u,
\] (16)
\[
u = \frac{1}{2} k b (y - C \hat{x}), \quad \hat{k} = c_k \| y - C \hat{x} \|^2.
\] (17)
To confirm the validity of the above conclusion, we give numerical simulation with the following choices of the initial conditions: \( x(0) = (0.2, 0.5, 6)^\top, \dot{x}(0) = (1, -2, 3)^\top, \) and \( k(0) = 0, \alpha = 0.93. \) When an adaptive part \( u \) (17) is added into the system at the time of 5th, the error \( e = x - \dot{x} \) curves are shown in Figure 2. It is found that the adaptive part can quickly render the two systems synchronization.

5. Conclusions

In this paper, the synchronization problem for fractional-order chaotic systems is investigated. An adaptive observer-based slave system is designed to synchronize a given chaotic master system. Based on the Lyapunov stability theorem, the global synchronization between the master and slave systems is ensured. Simulation results demonstrate the effectiveness and robustness of the proposed scheme.

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