Miklós Laczkovich (Budapest) asked if there exists a Haussdorff (or even normal) space in which every subset is Borel yet it is not meager. The motivation of the last condition is that under $\text{MA}_\kappa$ every subspace of the reals of cardinality $\kappa$ has the property that all subsets are $F_\sigma$ however Martin’s axiom also implies that these subsets are meager. Here we answer Laczkovich’ question. I thank Peter Komjath – the existence of this paper owes much to him.

**Theorem.** The following are equiconsistent.

1. There exists a measurable cardinal.
2. There is a non-meager $T_1$ space with no isolated points in which every subset is Borel.
3. There is a non-meager $T_4$ space with no isolated points in which every subset is the union of an open and a closed set.

**Proof.** Assume first that $\kappa$ is measurable in the model $V$. Add $\kappa$ Cohen reals, that is, force with the partial ordering $\text{Add}(\omega, \kappa)$. Our model will be $V[G]$ where $G \subseteq \text{Add}(\omega, \kappa)$ is generic. We first observe that in $V[G]$ there is a $\kappa$-complete ideal on $\kappa$ such that the complete Boolean algebra $P(\kappa)/I$ is isomorphic to the Boolean algebra of the complete closure of $\text{Add}(\omega, j(\kappa))$ where $j : V \to M$ is the corresponding elementary embedding. Indeed we let $X \in I$ if and only if $1 \forces \kappa \not\in j(\tau)$ for some $\tau$ satisfying $X = \tau^G$, that is, $\tau$ is a name for $X \subseteq \kappa$. Moreover, the mapping $X \mapsto [\kappa \in j(\tau)]$ is an isomorphism between $P(\kappa)/I$ and the regular Boolean algebra of $\text{Add}(\omega, j(\kappa) \setminus \kappa)$ (where $\tau$ is a name for $X$). Notice that $|j(\kappa)| = 2^\kappa$.

We observe that this Boolean algebra has the following properties. There are $2^\kappa$ subsets $\{A_\alpha : \alpha < 2^\kappa\}$ which are independent mod $I$, that is, if $s$ is a function from a finite subset of $\kappa$ into $\{0, 1\}$ then the intersection

$$B_s \overset{\text{def}}{=} \bigcap_{\alpha \in \text{Dom}(s)} A_\alpha^{s(\alpha)}$$

is not in $I$ (here $A^1 = A$ and $A^0 = \kappa \setminus A$). Moreover, if $A \subseteq \kappa$ then there are countably many pairwise contradictory functions $s_0, s_1, \ldots$ as above, such that

$$A/I = B_{s_0}/I \lor B_{s_1}/I \lor \ldots,$$

that is, $A$ can be written as $B_{s_0} \cup B_{s_1} \cup \ldots$ add-and-take-away a set in $I$.

By cardinality assumptions we can assume that for every pair $(X, Y)$ of disjoint members of $I$ there is some $\alpha < 2^\kappa$ with $X \subseteq A_\alpha$, $Y \subseteq \kappa \setminus A_\alpha$.

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We define a topology on $\kappa$ by declaring the system
\[ \{ A_\alpha \setminus Z, A^1 \setminus Z : \alpha < 2^\kappa, Z \in I \} \]
a subbasis, or, what is the same, the collection of all sets of the form $B_s \setminus Z$ (where $Z \in I$) a basis.

We prove the following statements on the space.

**Claim.** The space has the following properties.
1. Every set of the form $B_s$ is clopen, every set in $I$ is closed.
2. Every meager set is in $I$.
3. Every set is the union of an open and a closed set.
4. The closure of $B_s \setminus Z$ is $B_s$.
5. The space is $T_4$.

**Proof.**
1. Straightforward.
2. Every set not in $I$ contains a subset of the form $B_s \setminus Z$ (by one of the properties of the Boolean algebra mentioned above), which is open, so every nowhere dense, therefore every meager set is in $I$.
3. If $A \subseteq \kappa$ then $A/I$ can be written as $A/I = B_{s_0}/I \vee B_{s_1}/I \vee \ldots$ and then clearly
\[ A = \left( (B_{s_0} \setminus Z_0) \cup (B_{s_1} \setminus Z_1) \cup \ldots \right) \cup Z \]
for some sets $Z_0, Z_1, \ldots, Z$ in $I$. But this is a decomposition into the union of an open and a closed set.
4. Clear.
5. Assume we are given the disjoint closed sets $F$ and $F'$. They can be written as
\[ F = (B_{s_0} \setminus Z_0) \cup (B_{s_1} \setminus Z_1) \cup \ldots \cup Z \]
and
\[ F' = (B_{s_0}' \setminus Z_0') \cup (B_{s_1}' \setminus Z_1') \cup \ldots \cup Z'. \]
As $F$ and $F'$ are closed, using 4., we can assume that
\[ Z_0 = Z_1 = \ldots = Z_0' = Z_1' = \ldots = \emptyset. \]
Set $G = B_{s_0} \cup B_{s_1} \cup \ldots$, $G' = B_{s_0}' \cup B_{s_1}' \cup \ldots$, then $F = G \cup Z$, $F' = G' \cup Z'$ and these four sets are pairwise disjoint. It suffices to separate each of the pairs $(G, G')$, $(G, Z')$, $(G', Z)$, and $(Z, Z')$. There is no problem with the first case, as $G, G'$ are open. For the last case we use our assumption that some $A_\alpha$ separates $Z$ and $Z'$. For the second, we can assume that $G$ is non empty hence $B_{s_0}$ is well defined and disjoint to $Z'$, now choose $\alpha < \kappa$ such that $Z'$ is a subset of $A_\alpha$, and so $G, A_\alpha \setminus B_{s_0}$ is a pair of disjoint open sets as required. Lastly the third case is similar to the second.

We have proved $(1) \rightarrow (3)$, and $(3) \rightarrow (2)$ is trivial; lastly for $(2) \rightarrow (1)$ assume that $(X, T)$ is a non-meager $T_1$ space with no isolated points in which every subset is Borel. Let $\{ G_\alpha : \alpha < \tau \}$ be a maximal system of disjoint, nonempty, meager open sets. Such a system exists by Zorn’s lemma. Set $Y = \bigcup \{ G_\alpha : \alpha < \tau \}$. Clearly, $Y$ is meager. As the boundary of the open $Y$ is nowhere dense, we get that even the closure of $Y$ is meager. Then the nonempty subspace $Z = X - \overline{Y}$ has the property that no nonempty open set is meager and every subset is Borel. If $I$
is the meager ideal on $Z$ then every subset is equal to some open set mod $I$. We claim that $I$ is precipitous on $Z$ which implies that in some inner model there is a measurable cardinal (see [1], [2]).

For this, assume that $W^0, W^1, \ldots$ is a refining sequence of mod $I$ partitions. That is, every $W^n$ is a maximal system of $I$-almost disjoint open sets, and if $A$ is a member of some $W^{n+1}$ then there is some member of $W^n$ which includes $A$ mod $I$.

We try to find a member $A_n \in W^n$ such that $\bigcap\{A_n : n < \omega\}$ is nonempty. To this, observe that the intersection of two members in $W^n$ is a meagre open set, hence is the empty set. Therefore, $W^n$ is actually a decomposition of $Z \setminus Z_0$ into the union of disjoint open sets where $Z_n$ is a meager set. Pick an element in $Z \setminus \bigcup\{Z_n : n < \omega\}$ then it is in some member of $W^n$ for every $n$ and we are done.

References

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