ON THE MOTIVE OF ITO–MIURA–OKAWA–UEDA CALABI–YAU THREEFOLDS

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ABSTRACT. Ito-Miura-Okawa-Ueda have constructed a pair of Calabi–Yau threefolds $X$ and $Y$ that are $L$-equivalent and derived equivalent, but not stably birational. We complete the picture by showing that $X$ and $Y$ have isomorphic Chow motives.

1. INTRODUCTION

Let $\text{Var}(k)$ denote the category of algebraic varieties over a field $k$. The Grothendieck ring $K_0(\text{Var}(k))$ encodes fundamental properties of the birational geometry of varieties. The intricacy of the ring $K_0(\text{Var}(k))$ is highlighted by the result of Borisov [2], showing that the class of the affine line $[\mathbb{A}^1]$ is a zero–divisor in $K_0(\text{Var}(k))$. Inspired by [2], Ito–Miura–Okawa–Ueda [6] exhibit a pair of Calabi–Yau threefolds $X, Y$ that are not stably birational (and so $[X] \neq [Y]$ in the Grothendieck ring), but

$$( [X] - [Y] )[\mathbb{A}^1] = 0 \text{ in } K_0(\text{Var}(k))$$

(i.e., $X$ and $Y$ are “L-equivalent”, a notion studied in [8]).

As shown by Kuznetsov [7], the threefolds $X, Y$ of [6] are derived equivalent. According to a conjecture of Orlov [10, Conjecture 1], derived equivalent smooth projective varieties should have isomorphic Chow motives. The aim of this tiny note is to check that such is indeed the case for the threefolds $X, Y$:

**Theorem** (=theorem 3.1). Let $X, Y$ be the two Calabi–Yau threefolds of [6]. Then

$$h(X) \cong h(Y) \text{ in } \mathcal{M}_{\text{rat}}.$$  

An immediate corollary is that if $k$ is a finite field, then $X$ and $Y$ share the same zeta function (corollary 4.1).

Conventions. In this note, the word variety will refer to a reduced irreducible scheme of finite type over a field $k$. For a smooth variety $X$, we will denote by $A^j(X)$ the Chow group of codimension $j$ cycles on $X$ with $\mathbb{Q}$-coefficients.

The notation $A^j_{\text{hom}}(X)$ will be used to indicate the subgroups of homologically trivial cycles. For a morphism between smooth varieties $f : X \to Y$, we will write $\Gamma_f \in A^*(X \times Y)$ for the graph of $f$, and $\Gamma_f^t \in A^*(Y \times X)$ for the transpose correspondence.

The contravariant category of Chow motives (i.e., pure motives with respect to rational equivalence as in [12], [9]) will be denoted $\mathcal{M}_{\text{rat}}$.
2. THE CALABI–YAU THREEFOLDS

**Theorem 2.1** (Ito–Miura–Okawa–Ueda [6]). Let \( k \) be an algebraically closed field of characteristic 0. There exist two Calabi–Yau threefolds \( X, Y \) over \( k \) such that

\[
[X] \neq [Y] \quad \text{in } K_0(\var(K)),
\]

but

\[
([X] - [Y])[\mathbb{A}^1] = 0 \quad \text{in } K_0(\var(K)).
\]

**Theorem 2.2** (Kuznetsov [7]). Let \( k \) be any field. The threefolds \( X, Y \) over \( k \) constructed as in [6] are derived equivalent: there is an isomorphism between the bounded derived categories of coherent sheaves

\[
D^b(X) \cong D^b(Y).
\]

In particular, if \( k = \mathbb{C} \) then there is an isomorphism of polarized Hodge structures

\[
H^3(X, \mathbb{Z}) \cong H^3(Y, \mathbb{Z}).
\]

**Proof.** The derived equivalence is [7, Theorem 5]. The isomorphism of Hodge structures is a corollary of the derived equivalence, in view of [11, Proposition 2.1 and Remark 2.3]. \( \square \)

**Remark 2.3.** The construction of the threefolds \( X, Y \) in [6] works over any field \( k \). However, the proof that \( [X] \neq [Y] \) uses the MRC fibration and is (a priori) restricted to characteristic 0. The argument of [7], on the other hand, has no characteristic 0 assumption.

3. MAIN RESULT

**Theorem 3.1.** Let \( k \) be any field, and let \( X, Y \) be the two Calabi–Yau threefolds over \( k \) constructed as in [6]. Then

\[
h(X) \cong h(Y) \quad \text{in } \mathcal{M}_{\text{rat}}.
\]

**Proof.** First, to simplify matters, let us slightly cut down the motives of \( X \) and \( Y \). It is known [6] that \( X \) and \( Y \) have Picard number 1. A routine argument gives a decomposition of the Chow motives

\[
h(X) = \mathbb{1} \oplus \mathbb{1}(1) \oplus h^3(X) \oplus \mathbb{1}(2) \oplus \mathbb{1}(3),
\]

\[
h(Y) = \mathbb{1} \oplus \mathbb{1}(1) \oplus h^3(Y) \oplus \mathbb{1}(2) \oplus \mathbb{1}(3) \quad \text{in } \mathcal{M}_{\text{rat}},
\]

where \( \mathbb{1} \) is the motive of the point \( \text{Spec}(k) \). (The gist of this “routine argument” is as follows: let \( H \in A^1(X) \) be a hyperplane section. Then

\[
\pi^2_i := c_i H^{3-i} \times H \in A^3(X \times X), \quad 0 \leq i \leq 3,
\]

defines an orthogonal set of projectors lifting the Künneth components, for appropriate \( c_i \in \mathbb{Q} \). One can then define \( \pi^3_i = \Delta_X - \sum_i \pi^2_i \in A^4(X \times X) \), and \( h^3(X) = (X, \pi^3_X, 0) \in \mathcal{M}_{\text{rat}} \), and ditto for \( Y \).)

To prove the theorem, it will thus suffice to prove an isomorphism of motives

\[
h^3(X) \cong h^3(Y) \quad \text{in } \mathcal{M}_{\text{rat}}.
\]
We observe that the above decomposition (plus the fact that $H^*(h^3(X)) = H^3(X)$ is odd-dimensional) implies equality

$$A^*(h^3(X)) = A^*_{hom}(X),$$

and similarly for $Y$.

The rest of the proof will consist in finding a correspondence $\Gamma \in A^3(X \times Y)$ inducing isomorphisms

$$\Gamma_* : A^j_{hom}(X_K) \cong A^j_{hom}(Y_K) \quad \forall j,$$

for all field extensions $K \supset k$. By the above observation, this means that $\Gamma$ induces isomorphisms

$$A^j(h^3(X)_K) \cong A^j(h^3(Y)_K) \quad \forall j,$$

which (as is well-known, cf. for instance [5, Lemma 1.1]) ensures that $\Gamma$ induces the required isomorphism of Chow motives [1].

To find the correspondence $\Gamma$, we need look no further than the construction of the threefolds $X, Y$. As explained in [6] and [7], the threefolds $X, Y$ are related via a diagram

$$\begin{array}{cccc}
D & \rightarrow & M & \leftarrow & E \\
\downarrow \pi_M & & \downarrow \rho_M & & \downarrow q \\
X & \hookrightarrow & Q & \rightarrow & F & \rightarrow & G & \leftrightarrow & Y
\end{array}$$

Here $Q$ is a smooth 5-dimensional quadric, and $G$ is a smooth intersection $G = \text{Gr}(2, V) \cap \mathbb{P}(W)$ of a Grassmannian and a linear subspace. The morphisms $\pi$ and $\rho$ are $\mathbb{P}^1$-fibrations. The morphisms $\pi_M$ and $\rho_M$ are the blow-ups with center the threefold $X$, resp. the threefold $Y$. The varieties $D, E$ are the exceptional divisors of the blow-ups.

**Lemma 3.2.** Let $Q$ and $G$ be as above. We have

$$A^i_{hom}(Q) = A^i_{hom}(G) = 0 \quad \forall i.$$

**Proof.** It is well-known that a 5-dimensional quadric $Q$ has trivial Chow groups. (Indeed, [3] Corollary 2.3) gives that $A^i_{hom}(Q) = 0$ for $i \geq 3$. The Bloch–Srinivas argument [1], combined with the fact that $H^3(Q) = 0$, then implies that $A^i_{hom}(Q) = 0.$

As $\pi : F \rightarrow Q$ is a $\mathbb{P}^1$-fibration, it follows that the variety $F$ has trivial Chow groups. But $\rho : F \rightarrow G$ is a $\mathbb{P}^1$-fibration, and so $G$ also has trivial Chow groups. \qed

The blow-up formula, combined with lemma [3.2], gives isomorphisms

$$i_*p^* : A^i_{hom}(X) \cong A^{i+1}_{hom}(M),$$

$$j_*q^* : A^i_{hom}(Y) \cong A^{i+1}_{hom}(M).$$
What’s more, the inverse isomorphisms are induced by a correspondence: the compositions

\[ A_i^{\hom}(X) \xrightarrow{i^*p^*} A_{i+1}^{\hom}(M) \xrightarrow{p^*i^*} A_i^{\hom}(X), \]
\[ A_i^{\hom}(Y) \xrightarrow{j^*q^*} A_{i+1}^{\hom}(M) \xrightarrow{q^*j^*} A_i^{\hom}(Y), \]
\[ A_{i+1}^{\hom}(M) \xrightarrow{p^*i^*} A_i^{\hom}(X) \xrightarrow{i^*p^*} A_{i+1}^{\hom}(M), \]
\[ A_{i+1}^{\hom}(M) \xrightarrow{q^*j^*} A_i^{\hom}(Y) \xrightarrow{j^*q^*} A_{i+1}^{\hom}(M), \]

are all equal to the identity [13, Theorem 5.3].

This suggests how to find a correspondence \( \Gamma \) doing the job. Let us define

\[ \Gamma := \Gamma_q \circ \Gamma_j \circ \Gamma_i \circ \Gamma_p \text{ in } A^3(X \times Y). \]

Then we have (by the above) that

\[ \Gamma^* \Gamma_* = \text{id}: A_i^{\hom}(X) \to A_i^{\hom}(X), \]
\[ \Gamma_* \Gamma^* = \text{id}: A_i^{\hom}(Y) \to A_i^{\hom}(Y) \]

for all \( i \), and so there are isomorphisms

\[ \Gamma_*: A_i^{\hom}(X) \to A_i^{\hom}(Y) \quad \forall i. \]

Given a field extension \( K \supset k \), the threefolds \( X_K, Y_K \) are related via a blow-up diagram as above, and so the same reasoning as above shows that there are isomorphisms

\[ \Gamma_*: A_i^{\hom}(X_K) \to A_i^{\hom}(Y_K) \quad \forall i. \]

We have now established that \( \Gamma \) verifies (2), which clinches the proof.

\[ \square \]

4. A COROLLARY

**Corollary 4.1.** Let \( k \) be a finite field, and let \( X, Y \) be the Calabi–Yau threefolds over \( k \) constructed as in [6]. Then \( X \) and \( Y \) have the same zeta function.

**Proof.** The zeta function can be expressed (via the Lefschetz fixed point theorem) in terms of the action of Frobenius on \( \ell \)-adic étale cohomology, hence depends only on the motive. \( \square \)

**Remark 4.2.** Corollary 4.1 can also be deduced from [4], where it is proven that derived equivalent varieties of dimension 3 have the same zeta function. The above proof (avoiding recourse to [7] and [4]) is more straightforward.

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