Finite type invariants of knots in homology 3–spheres with respect to null LP–surgeries

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We study a theory of finite type invariants for nullhomologous knots in rational homology 3–spheres with respect to null Lagrangian-preserving surgeries. It is an analogue in the setting of the rational homology of the Garoufalidis–Rozansky theory for knots in integral homology 3–spheres. We give a partial combinatorial description of the graded space associated with our theory and determine some cases when this description is complete. For nullhomologous knots in rational homology 3–spheres with a trivial Alexander polynomial, we show that the Kricker lift of the Kontsevich integral and the Lescop equivariant invariant built from integrals in configuration spaces are universal finite type invariants for this theory; in particular, this implies that they are equivalent for such knots.

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1 Introduction

The notion of finite type invariants was first introduced independently by Goussarov and Vassiliev for the study of invariants of knots in the 3–dimensional sphere $S^3$; in this case, finite type invariants are also called Vassiliev invariants. The discovery of the Kontsevich integral, which is a universal invariant among all finite type invariants of
knots in $S^3$, revealed that this class of invariants is very prolific. It is known, for instance, that it dominates all Witten–Reshetikhin–Turaev quantum invariants. The notion of finite type invariants was adapted to the setting of 3–dimensional manifolds by Ohtsuki [19], who introduced the first examples for integral homology 3–spheres, and it has been widely developed and generalized since then. In particular, Goussarov and Habiro independently developed a theory which involves any 3–dimensional manifolds — and their knots — and which contains the Ohtsuki theory for $\mathbb{Z}$–spheres; see Garoufalidis, Goussarov and Polyak [6] and Habiro [10]. Another generalization of the Ohtsuki theory to general 3–dimensional manifolds was developed by Cochran and Melvin [5].

In general, the finite type invariants of a set of objects are defined by their polynomial behavior with respect to some elementary move. For Vassiliev invariants of knots in $S^3$, this move is the crossing change on a diagram of the knot. For 3–dimensional manifolds, the elementary move is a certain kind of surgery, for instance the Borromean surgery — a Lagrangian-preserving replacement of a genus 3 handlebody — in the Goussarov–Habiro theory.

Garoufalidis and Rozansky [8] studied the theory of finite type invariants for $\mathbb{Z}$SK–pairs, i.e. knots in integral homology 3–spheres, with respect to the so-called nullmove, which is a Borromean surgery defined on a handlebody that is nullhomologous in the complement of the knot. In this paper, we study a theory of finite type invariants for $\mathbb{Q}$SK–pairs, i.e. nullhomologous knots in rational homology 3–spheres ($\mathbb{Q}$–spheres). Our elementary move is the null Lagrangian-preserving surgery introduced by Lescop [13], which is the Lagrangian-preserving replacement of a rational homology handlebody that is nullhomologous in the complement of the knot. This latter theory can be understood as an adaptation of the Garoufalidis–Rozansky theory to the setting of the rational homology; a great part of the results in this paper are stated in both settings.

Kricker [11] constructed a rational lift of the Kontsevich integral of $\mathbb{Z}$SK–pairs. He proved with Garoufalidis [7] that his construction provides an invariant of $\mathbb{Z}$SK–pairs. This invariant takes values in a diagram space with a stronger structure than the target diagram space of the Kontsevich integral, hence it is much more structured than the Kontsevich integral, which it lifts. Garoufalidis and Kricker proved in [7] that the Kricker invariant satisfies some splitting formulas with respect to the nullmove; see also Garoufalidis and Rozansky [8]. These formulas imply in particular that the Kricker invariant is a series of finite type invariants of all degrees with respect to the nullmove.

It appears that the nullmove preserves the Blanchfield module — the Alexander module equipped with the Blanchfield form — of the $\mathbb{Z}$SK–pair. Hence the study of the
Garoufalidis–Rozansky theory of finite type invariants can be restricted to a class of
\( \mathbb{Z} \text{SK} \)-pairs with a fixed Blanchfield module. In the case of a trivial Blanchfield module,
Garoufalidis and Rozansky gave a combinatorial description of the associated graded
space. Together with the splitting formulas of Garoufalidis and Kricker, this proves
that the Kricker invariant is a universal finite type invariant of \( \mathbb{Z} \text{SK} \)-pairs with trivial
Blanchfield module with respect to the nullmove.

Another universal invariant in this context was constructed by Lescop in [12]. Lescop
proved in [13] that her invariant satisfies the same splitting formulas as the Kricker
invariant. Hence the Lescop invariant is also a universal finite type invariant of \( \mathbb{Z} \text{SK} \)-
pairs with trivial Blanchfield module with respect to the nullmove. This implies in
particular that the Lescop invariant and the Kricker invariant are equivalent for \( \mathbb{Z} \text{SK} \)-
pairs with trivial Blanchfield module. Lescop conjectured in [13] that this equivalence
holds for knots with any Blanchfield module.

The Lescop invariant is indeed defined for \( \mathbb{Q} \text{SK} \)-pairs and Lescop’s splitting formulas
are stated with respect to general null Lagrangian-preserving surgeries. In Moussard [18]
the Kricker invariant is extended to \( \mathbb{Q} \text{SK} \)-pairs and splitting formulas for this invariant
with respect to null Lagrangian-preserving surgeries are given. Hence a combinatorial
description of the graded space associated with finite type invariants of \( \mathbb{Q} \text{SK} \)-pairs with
respect to null Lagrangian-preserving surgeries would allow an explicit understanding
of the universality properties of these two invariants and provide a comparison between
them, answering the above conjecture of Lescop for general \( \mathbb{Q} \text{SK} \)-pairs.

In analogy with the integral homology setting, null Lagrangian-preserving surgeries
preserve the Blanchfield module defined over \( \mathbb{Q} \) and we study finite type invariants of
\( \mathbb{Q} \text{SK} \)-pairs with a fixed Blanchfield module. In the case of a trivial Blanchfield module,
we give a complete description of the associated graded space. This description and
the above-mentioned splitting formulas imply that the Lescop invariant and the Kricker
invariant are both universal finite type invariants of \( \mathbb{Q} \text{SK} \)-pairs with trivial Blanchfield
module, up to degree 1 invariants given by the cardinality of the first homology group
of the \( \mathbb{Q} \)-sphere. In particular, the Lescop invariant and the Kricker invariant are
equivalent for \( \mathbb{Q} \text{SK} \)-pairs with trivial Blanchfield module when the cardinality of the
first homology group of the \( \mathbb{Q} \)-sphere is fixed.

Let \((\mathfrak{A}, b)\) be any Blanchfield module with annihilator \( \delta \in \mathbb{Q}[t^{\pm 1}] \). The main goal of
this paper is to give a combinatorial description of the graded space

\[
G(\mathfrak{A}, b) = \bigoplus_{n \in \mathbb{Z}} G_n(\mathfrak{A}, b)
\]
associated with finite type invariants of \( \mathbb{Q}SK \)-pairs with Blanchfield module \((\mathfrak{A}, b)\) — precise definitions are given in the next section. The Lescop or Kricker invariant \( Z = (Z_n)_{n \in \mathbb{N}} \) is a family of finite type invariants \( Z_n \) of degree \( n \) for \( n \) even (\( Z_n \) is trivial for \( n \) odd). For \( \mathbb{Q}SK \)-pairs with Blanchfield module \((\mathfrak{A}, b)\), \( Z_n \) takes values in a space \( \mathcal{A}_n(\delta) \) of trivalent graphs with edges labeled in \( (1/\delta)\mathbb{Q}[t^\pm 1] \). The finiteness properties imply that \( Z_n \) induces a map on \( \mathcal{G}_n(\mathfrak{A}, b) \). In order to take into account the degree 1 invariants, we construct from \( Z \) an invariant \( Z_{\text{aug}} = (Z_{\text{aug}}^n)_{n \in \mathbb{N}} \) of \( \mathbb{Q}SK \)-pairs with \( Z_{\text{aug}}^n \) of degree \( n \). The invariant \( Z_{\text{aug}}^n \) takes values in a space \( \mathcal{A}_{\text{aug}}^n(\delta) \) of trivalent graphs as before, which may in addition contain isolated vertices labeled by prime integers. Again by finiteness, \( Z_{\text{aug}}^n \) induces a map on \( \mathcal{G}_n(\mathfrak{A}, b) \). This leads us to our main question.

**Question 1** Is the map \( Z_{\text{aug}}^n: \mathcal{G}_n(\mathfrak{A}, b) \to \mathcal{A}_{\text{aug}}^n(\delta) \) injective?

Injectivity of this map for any Blanchfield module \((\mathfrak{A}, b)\) is equivalent to universality of the invariant \( Z \) as a finite type invariant of \( \mathbb{Q}SK \)-pairs, up to degree 0 and 1 invariants. This would imply the equivalence of the Lescop invariant and the Kricker invariant when the Blanchfield module and the cardinality of the first homology group of the \( \mathbb{Q} \)-sphere are fixed.

To deal with Question 1, we first construct another diagram space \( \mathcal{A}_{\text{aug}}^n(\mathfrak{A}, b) \) together with a surjective map \( \varphi_n: \mathcal{A}_{\text{aug}}^n(\mathfrak{A}, b) \to \mathcal{G}_n(\mathfrak{A}, b) \). Then we compose this map with \( Z_{\text{aug}}^n \) to get a map \( \psi_n: \mathcal{A}_{\text{aug}}^n(\mathfrak{A}, b) \to \mathcal{A}_{\text{aug}}^n(\delta) \); see Figure 1.

**Figure 1**: Commutative diagram

It appears that this composed map has a simple diagrammatic description. Nevertheless, it is not easy to decide whether it is injective or not in general.

**Question 2** Is the map \( \psi_n: \mathcal{A}_{\text{aug}}^n(\mathfrak{A}, b) \to \mathcal{A}_{\text{aug}}^n(\delta) \) injective?

If Question 2 has a positive answer, then Question 1 also has, and \( \mathcal{G}_n(\mathfrak{A}, b) \) is completely described combinatorially by \( \varphi_n: \mathcal{A}_{\text{aug}}^n(\mathfrak{A}, b) \overset{\cong}{\to} \mathcal{G}_n(\mathfrak{A}, b) \).
Question 2 has a positive answer at least in the following cases, where the last two cases are treated by Audoux and Moussard in [3]:

- For a trivial Blanchfield module and any value of $n$.
- For a Blanchfield module which is a direct sum of $N$ isomorphic Blanchfield modules and $n \leq \frac{2}{3} N$.
- For a Blanchfield module of $\mathbb{Q}$–dimension 2 and $n = 2$.
- For a Blanchfield module which is a direct sum of two isomorphic Blanchfield modules of $\mathbb{Q}$–dimension 2 and of order different from $t + 1 + t^{-1}$, and $n = 2$.

In the third case, the map $\psi_n$ is not surjective, whereas in the other cases, it is an isomorphism. In particular, $Z_n^{\text{aug}}$ is not surjective in general. Moreover, for a Blanchfield module which is a direct sum of two isomorphic Blanchfield modules of $\mathbb{Q}$–dimension 2 and of order $t + 1 + t^{-1}$, and $n = 2$, Question 2 has a negative answer (see [3]), but Question 1 is open, as well as the injectivity status of $\varphi$.

The fact that Question 1 remains open while Question 2 does not have a positive answer in general leads us to the following alternatives:

- either Question 1 has a positive answer in general, in which case $G_n(\mathfrak{A}, b)$ is isomorphic to $\psi_n(\mathcal{A}_n^{\text{aug}}(\mathfrak{A}, b))$,
- or we miss some invariant to add to the augmented Lescop/Kricker invariant and the Blanchfield module to get a universal finite type invariant of $\mathbb{Q}$SK–pairs.

We also treat the Garoufalidis–Rozansky theory of finite type invariants of $\mathbb{Z}$SK–pairs in the case of a nontrivial Blanchfield module.

**Notation** Let $\mathbb{K}$ be either $\mathbb{Z}$ or $\mathbb{Q}$. A $\mathbb{K}$–sphere (resp. $\mathbb{K}$–ball, $\mathbb{K}$–torus, genus $g$ $\mathbb{K}$–handlebody) is a compact connected oriented 3–manifold with the same homology with coefficients in $\mathbb{K}$ as the standard 3–sphere (resp. 3–ball, solid torus, genus $g$ standard handlebody). A $\mathbb{K}$SK–pair $(M, K)$ is a pair made of a $\mathbb{K}$–sphere $M$ and a knot $K$ in $M$ whose homology class in $H_1(M; \mathbb{Z})$ is trivial.

**Plan of the paper** In Section 2, we introduce the necessary notions and state the main results of the paper. Section 3 is devoted to clasper calculus in the equivariant setting. We apply this calculus in Section 4 to our diagrams. This provides a surjective map from a graded diagram space to the graded space associated with $\mathbb{Z}$SK–pairs with respect to integral null Lagrangian-preserving surgeries. To get a similar map
in the case of $\mathbb{Q}$SK–pairs, we need further arguments developed in Section 5. In
Section 6, we show the universality property of the invariant $Z_{\text{aug}}$ which combines the
Lescop/Kricker invariant and the cardinality of the first homology group. In Section 7,
we answer Question 2 for a Blanchfield module which is a direct sum of $N$ isomorphic
Blanchfield modules in degree at most $\frac{2}{3}N$.

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2 Statement of the results

2.1 Filtration defined by null LP–surgeries

We first recall the definition of the Alexander module and the Blanchfield form. Let
$(M, K)$ be a $\mathbb{Q}$SK–pair. Let $T(K)$ be a tubular neighborhood of $K$. The exterior of $K$
is $X = M \setminus \text{Int}(T(K))$. Consider the projection $\pi: \pi_1(X) \to H_1(X; \mathbb{Z})/\text{torsion} \cong \mathbb{Z}$
and the covering map $p: \tilde{X} \to X$ associated with its kernel. The covering $\tilde{X}$ is the
infinite cyclic covering of $X$. The automorphism group of the covering, $\text{Aut}(\tilde{X})$, is
isomorphic to $\mathbb{Z}$. It acts on $H_1(\tilde{X}; \mathbb{Q})$. Denoting the action of a generator $\tau$ of $\text{Aut}(\tilde{X})$
as the multiplication by $\tau$, we get a structure of $\mathbb{Q}[\tau^{\pm 1}]$–module on $\mathfrak{A}(M, K) = H_1(\tilde{X}; \mathbb{Q})$. This $\mathbb{Q}[\tau^{\pm 1}]$–module is called the Alexander module of $(M, K)$. It is a
torsion $\mathbb{Q}[\tau^{\pm 1}]$–module.

On the Alexander module, the Blanchfield form, or equivariant linking pairing,

$$b: \mathfrak{A} \times \mathfrak{A} \to \frac{\mathbb{Q}(t)}{\mathbb{Q}[\tau^{\pm 1}]},$$

is defined as follows. First define the equivariant linking number of two knots. Let
$J_1$ and $J_2$ be two knots in $\tilde{X}$ such that $p(J_1) \cap p(J_2) = \emptyset$. Let $\delta \in \mathbb{Q}(t)$ be the
annihilator of $\mathfrak{A}$. There is a rational 2–chain $S$ such that $\partial S = \delta(\tau) J_1$. The equivariant
linking number of $J_1$ and $J_2$ is

$$\text{lk}_e(J_1, J_2) = \frac{1}{\delta(t)} \sum_{k \in \mathbb{Z}} \langle S, \tau^k(J_2) \rangle t^k.$$

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We say that the boundary of a genus $g$ surface. The Lagrangian $\mathcal{L}_A$ of a $\mathbb{Q}$–handlebody $A$ is the kernel of the map $i_*: H_1(\partial A; \mathbb{Q}) \to H_1(A; \mathbb{Q})$ induced by the inclusion. Two $\mathbb{Q}$–handlebodies $A$ and $B$ have $LP$–identified boundaries if $(A, B)$ is equipped with a homeomorphism $\varphi: \partial A \to \partial B$ such that $\varphi(\mathcal{L}_A) = \mathcal{L}_B$. The Lagrangian of a $\mathbb{Q}$–handlebody $A$ is indeed a Lagrangian subspace of $H_1(\partial A; \mathbb{Q})$ with respect to the intersection form.

Let $M$ be a $\mathbb{Q}$–sphere, let $A \subset M$ be a $\mathbb{Q}$–handlebody and let $B$ be a $\mathbb{Q}$–handlebody whose boundary is $LP$–identified with $\partial A$. Set $M(B/A) = (M \setminus \text{Int}(A)) \cup_{\partial A =_{\partial B}} B$. We say that the $\mathbb{Q}$–sphere $M(B/A)$ is obtained from $M$ by Lagrangian-preserving surgery, or $LP$–surgery.

Given a $\mathbb{Q}$SK–pair $(M, K)$, a $\mathbb{Q}$–handlebody null in $M \setminus K$ is a $\mathbb{Q}$–handlebody $A \subset M \setminus K$ such that the map $i_*: H_1(A; \mathbb{Q}) \to H_1(M \setminus K; \mathbb{Q})$ induced by the inclusion has a trivial image. A null $LP$–surgery on $(M, K)$ is an $LP$–surgery $(B/A)$ such that $A$ is null in $M \setminus K$. The $\mathbb{Q}$SK–pair obtained by surgery is denoted by $(M, K)(B/A)$. 

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Let $\mathcal{F}_0$ be the rational vector space generated by all $\mathbb{Q}$SK–pairs up to orientation-preserving homeomorphism. Let $\mathcal{F}_n$ be the subspace of $\mathcal{F}_0$ generated by the

$$\left[ (M, K); \left( \frac{B_i}{A_i} \right)_{1 \leq i \leq n} \right] = \sum_{I \subset \{1, \ldots, n\}} (-1)^{|I|} (M, K) \left( \left( \frac{B_i}{A_i} \right)_{i \in I} \right)$$

for all $\mathbb{Q}$SK–pairs $(M, K)$ and all families of $\mathbb{Q}$–handlebodies $(A_i, B_i)_{1 \leq i \leq n}$, where the $A_i$ are null in $M \setminus K$ and disjoint, and each $\partial B_i$ is LP–identified with the corresponding $\partial A_i$. Here and in all the article, $| \cdot |$ stands for the cardinality. Since $\mathcal{F}_{n+1} \subset \mathcal{F}_n$, this defines a filtration.

**Definition 2.1** A $\mathbb{Q}$–linear map $\lambda: \mathcal{F}_0 \to \mathbb{Q}$ is a finite type invariant of degree at most $n$ of $\mathbb{Q}$SK–pairs with respect to null LP–surgeries if $\lambda(\mathcal{F}_{n+1}) = 0$.

**Theorem 2.2** [17, Theorem 1.14] A null LP–surgery induces a canonical isomorphism between the Blanchfield modules of the involved $\mathbb{Q}$SK–pairs. Conversely, for any isomorphism $\zeta$ from the Blanchfield module of a $\mathbb{Q}$SK–pair $(M, K)$ to the Blanchfield module of a $\mathbb{Q}$SK–pair $(M', K')$, there is a finite sequence of null LP–surgeries from $(M, K)$ to $(M', K')$ which induces the composition of $\zeta$ by the multiplication by a power of $t$.

This result provides a splitting of the filtration $(\mathcal{F}_n)_{n \in \mathbb{N}}$, as follows. For an isomorphism class $(\mathfrak{A}, b)$ of Blanchfield modules, let $\mathcal{P}(\mathfrak{A}, b)$ be the set of all $\mathbb{Q}$SK–pairs, up to orientation-preserving homeomorphism, whose Blanchfield modules are isomorphic to $(\mathfrak{A}, b)$. Let $\mathcal{F}_0(\mathfrak{A}, b)$ be the subspace of $\mathcal{F}_0$ generated by the $\mathbb{Q}$SK–pairs $(M, K) \in \mathcal{P}(\mathfrak{A}, b)$. Let $(\mathcal{F}_n(\mathfrak{A}, b))_{n \in \mathbb{N}}$ be the filtration defined on $\mathcal{F}_0(\mathfrak{A}, b)$ by null LP–surgeries. Then, for $n \in \mathbb{N}$, $\mathcal{F}_n$ is the direct sum over all isomorphism classes $(\mathfrak{A}, b)$ of Blanchfield modules of the $\mathcal{F}_n(\mathfrak{A}, b)$. Set

$$\mathcal{G}_n(\mathfrak{A}, b) = \mathcal{F}_n(\mathfrak{A}, b)/\mathcal{F}_{n+1}(\mathfrak{A}, b) \quad \text{and} \quad \mathcal{G}(\mathfrak{A}, b) = \bigoplus_{n \in \mathbb{N}} \mathcal{G}_n(\mathfrak{A}, b).$$

We wish to describe the graded space $\mathcal{G}(\mathfrak{A}, b)$. By Theorem 2.2, $\mathcal{G}_0(\mathfrak{A}, b) \cong \mathbb{Q}$. In Section 5, as a consequence of Theorem 2.7, we prove:

**Theorem 2.3** Let $(\mathfrak{A}, b)$ be a Blanchfield module. Let $(M, K) \in \mathcal{P}(\mathfrak{A}, b)$. For any prime integer $p$, let $B_p$ be a $\mathbb{Q}$–ball such that $H_1(B_p; \mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$. Then

$$\mathcal{G}_1(\mathfrak{A}, b) = \bigoplus_{p \text{ prime}} \mathbb{Q} \left[ (M, K); \frac{B_p}{B^3} \right],$$

where $B^3$ is any standard 3–ball in $M \setminus K$. 

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2.2 Borromean surgeries

Let us define a specific type of LP–surgery.

The *standard Y–graph* is the graph $\Gamma_0 \subset \mathbb{R}^2$ represented in Figure 2. The looped edges of $\Gamma_0$ are the *leaves*. The vertex incident to three different edges is the *internal vertex*. To $\Gamma_0$ is associated a regular neighborhood $\Sigma(\Gamma_0)$ of $\Gamma_0$ in the plane. The surface $\Sigma(\Gamma_0)$ is oriented with the usual convention. This induces an orientation of the leaves and an orientation of the internal vertex, ie a cyclic order of the three edges. Consider a 3–manifold $M$ and an embedding $h: \Sigma(\Gamma_0) \to M$. The image $\Gamma$ of $\Gamma_0$ is a *Y–graph*, endowed with its *associated surface* $\Sigma(\Gamma) = h(\Sigma(\Gamma_0))$. The Y–graph $\Gamma$ is equipped with the framing induced by $\Sigma(\Gamma)$. A *Y–link* in a 3–manifold is a collection of disjoint Y–graphs.

Let $\Gamma$ be a Y–graph in a 3–manifold $M$. Let $\Sigma(\Gamma)$ be its associated surface. In $\Sigma \times [-1, 1]$, associate with $\Gamma$ the six-component link $L$ represented in Figure 3. The *Borromean surgery on* $\Gamma$ is the surgery along the framed link $L$. The surgered manifold is denoted by $M(\Gamma)$. As proved by Matveev in [14], a Borromean surgery can be

![Figure 2: The standard Y–graph](image)

![Figure 3: Y–graph and associated surgery link](image)
realized by cutting a genus 3 handlebody (a regular neighborhood of the Y–graph) and regluing it in another way, which preserves the Lagrangian. If \((M, K)\) is a \(\mathbb{Q}SK\)–pair and if \(\Gamma\) is an \(n\)–component Y–link, null in \(M \setminus K\), then \([\langle M, K \rangle; \Gamma] \in \mathcal{F}_0\) denotes the bracket defined by the \(n\) disjoint null LP–surgeries on the components of \(\Gamma\).

For \(n \geq 0\), let \(G_n^b(\mathfrak{A}, b)\) be the subspace of \(G_n(\mathfrak{A}, b)\) generated by the classes of the brackets defined by null Borromean surgeries. The following result is a consequence of Proposition 2.6 and Lemma 2.5.

**Proposition 2.4** For any Blanchfield module \((\mathfrak{A}, b)\) and any \(n \geq 0\), \(G_n^{b}(\mathfrak{A}, b) = 0\).

### 2.3 Spaces of diagrams

Fix a Blanchfield module \((\mathfrak{A}, b)\). Let \(\delta \in \mathbb{Q}(t)\) be the annihilator of \(\mathfrak{A}\). An \((\mathfrak{A}, b)\)–colored diagram \(D\) is a unitrivalent graph without strut \(\left(\right)\), with the following data:

- Trivalent vertices are oriented, where an *orientation of a trivalent vertex* is a cyclic order of the three half-edges that meet at this vertex.
- Edges are oriented and colored by \(\mathbb{Q}[t^{\pm 1}]\).
- Univalent vertices are colored by \(\mathfrak{A}\).
- For all \(v \neq v'\) in the set \(V\) of univalent vertices of \(D\), a rational fraction \(f_{vv'}^D(t) \in (1/\delta(t))\mathbb{Q}[t^{\pm 1}]\) is fixed such that \(f_{vv'}^D(t) \mod \mathbb{Q}[t^{\pm 1}] = b(\gamma, \gamma')\), where \(\gamma\) (resp. \(\gamma'\)) is the coloring of \(v\) (resp. \(v'\)), with \(f_{vv'}^D(t) = f_{vv'}^D(t^{-1})\).

In the pictures, the orientation of trivalent vertices is given by \(\underleftarrow{\to} \leftarrow \to\). When it does not seem to cause confusion, we write \(f_{vv'}\) for \(f_{vv'}^D\). The *degree* of a colored diagram is the number of trivalent vertices of its underlying graph. The unique degree 0 diagram is the empty diagram. For \(n \geq 0\), set

\[
\widetilde{\mathcal{A}}_n(\mathfrak{A}, b) = \frac{\mathbb{Q}\langle(\mathfrak{A}, b)\text{–colored diagrams of degree } n \rangle}{\mathbb{Q}\langle\text{AS, IHX, LE, OR, Hol, LV, EV, LD} \rangle},
\]

where the relations AS (antisymmetry), IHX, LE (linearity for edges), OR (orientation reversal), Hol (holonomy), LV (linearity for vertices), EV (edge-vertex) and LD (linking difference) are as described in Figure 4.

The automorphism group \(\text{Aut}(\mathfrak{A}, b)\) of the Blanchfield module \((\mathfrak{A}, b)\) acts on \(\widetilde{\mathcal{A}}_n(\mathfrak{A}, b)\) by acting on the colorings of all the univalent vertices of a diagram simultaneously.
Denote by $\text{Aut}$ the relation which identifies two diagrams obtained from one another by the action of an element of $\text{Aut}(\mathfrak{A}, b)$. Set

$$A_n(\mathfrak{A}, b) = \widetilde{A}_n(\mathfrak{A}, b)/\langle \text{Aut} \rangle \quad \text{and} \quad A(\mathfrak{A}, b) = \bigoplus_{n \in \mathbb{N}} A_n(\mathfrak{A}, b).$$

Since the opposite of the identity is an automorphism of $(\mathfrak{A}, b)$, we have:

**Lemma 2.5**  For all $n \geq 0$, $A_{2n+1}(\mathfrak{A}, b) = 0$.

In Section 4, we prove (see Proposition 4.5):

**Proposition 2.6**  Fix a Blanchfield module $(\mathfrak{A}, b)$. For all $n \geq 0$, there is a canonical surjective $\mathbb{Q}$–linear map

$$\varphi_n: A_n(\mathfrak{A}, b) \to \mathcal{G}^b_n(\mathfrak{A}, b).$$

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To get a similar surjective map onto $G_n(\mathfrak{A}, b)$, we need more general diagrams. An $(\mathfrak{A}, b)$–augmented diagram is the union of an $(\mathfrak{A}, b)$–colored diagram (its Jacobi part) and of finitely many isolated vertices colored by prime integers. The degree of an $(\mathfrak{A}, b)$–augmented diagram is the number of its vertices of valence 0 or 3. Set

$$A_{\text{aug}}^n(\mathfrak{A}, b) = \frac{\mathbb{Q} \langle (\mathfrak{A}, b)\text{-augmented diagrams of degree } n \rangle}{\mathbb{Q} \langle \text{AS, IHX, LE, OR, Hol, LV, EV, LD, Aut} \rangle} \quad \text{for } n \geq 0,$$

$$A_{\text{aug}}(\mathfrak{A}, b) = \bigoplus_{n \in \mathbb{N}} A_{\text{aug}}^n(\mathfrak{A}, b).$$

In Section 5, we prove:

**Theorem 2.7**  Fix a Blanchfield module $(\mathfrak{A}, b)$. For all $n \geq 0$, there is a canonical surjective $\mathbb{Q}$–linear map

$$\varphi_n: A_{\text{aug}}^n(\mathfrak{A}, b) \rightarrow G_n(\mathfrak{A}, b).$$

We will see in the next subsection that this map is an isomorphism when the Blanchfield module $(\mathfrak{A}, b)$ is trivial.

### 2.4 The Lescop invariant and the Kricker invariant

In order to introduce the Kricker invariant of [7] and the Lescop invariant of [12], we first define the graded space $A(\delta)$ where they take values and we relate it to the graded space $A(\mathfrak{A}, b)$.

Let $\delta \in \mathbb{Q}[t^{\pm 1}]$. A $\delta$–colored diagram is a trivalent graph whose vertices are oriented and whose edges are oriented and colored by $(1/\delta(t))\mathbb{Q}[t^{\pm 1}]$. The degree of a $\delta$–colored diagram is the number of its vertices. Set

$$A_n(\delta) = \frac{\mathbb{Q} \langle \delta\text{-colored diagrams of degree } n \rangle}{\mathbb{Q} \langle \text{AS, IHX, LE, OR, Hol, Hol'} \rangle},$$

where the relations AS, IHX, LE, OR, Hol are represented in Figure 4 and the relation Hol’ is represented in Figure 5. Here the relations LE, OR and Hol hold with edges labeled in $(1/\delta(t))\mathbb{Q}[t^{\pm 1}]$. Note that in the case of $A_n(\mathfrak{A}, b)$, the relation Hol’ is induced by the relations Hol, EV and LD. Since any trivalent graph has an even number of vertices, we have $A_{2n+1}(\delta) = 0$ for all $n \geq 0$.

To an $(\mathfrak{A}, b)$–colored diagram $D$ of degree $n$, we associate a $\delta$–colored diagram $\overline{\psi}_n(D)$. Let $V$ be the set of univalent vertices of $D$. A pairing of $V$ is an involution of $V$
with no fixed point. Let \( p \) be the set of pairings of \( V \). Fix \( p \in p \). Define a \( \delta \)-colored diagram \( p(D) \) in the following way. If \( v \in V \) and \( v' = p(v) \), replace in \( D \) the vertices \( v \) and \( v' \), and their adjacent edges, by a colored edge, as indicated in Figure 6. Now set

\[
\tilde{\psi}_n(D) = \sum_{p \in p} p(D).
\]

Note that \( \tilde{\psi}_n(D) = 0 \) when the number of univalent vertices is odd. We obtain a \( \mathbb{Q} \)-linear map \( \tilde{\psi}_n \) from the rational vector space freely generated by the \( (A, b) \)-colored diagrams of degree \( n \) to \( A_n(\delta) \). One easily checks that \( \tilde{\psi}_n \) induces a map

\[
\psi_n: A_n(A, b) \to A_n(\delta).
\]

The disjoint union of diagrams defines on \( A(\delta) = \bigoplus_{n \in \mathbb{N}} A_n(\delta) \) a multiplicative operation, which endows it with a graded algebra structure. Denote by \( \exp \) the exponential map with respect to this multiplication.

The following result asserts the existence and the properties of an invariant \( Z \), which may be either the Lescop invariant or the Kricker invariant. Although it is not known whether they are equal or not, they both satisfy the properties of the theorem. In the sequel, we will refer to “the invariant \( Z \”).

**Theorem 2.8** [12; 13; 11; 7; 18] There is an invariant \( Z = (Z_n)_{n \in \mathbb{N}} \) of \( \mathbb{Q} \)SK–pairs with the following properties:
If \((M, K)\) is a \(\mathbb{Q}SK\)--pair with Blanchfield module \((\mathcal{A}, b)\), then \(Z_n(M, K) \in A_n(\delta)\), where \(\delta\) is the annihilator of \(\mathcal{A}\).

Fix a Blanchfield module \((\mathcal{A}, b)\). Let \(\delta\) be the annihilator of \(\mathcal{A}\). The \(\mathbb{Q}\)--linear extension of \(Z_n: \mathcal{P}(\mathcal{A}, b) \to A_n(\delta)\) to \(\mathcal{F}_0(\mathcal{A}, b)\) vanishes on \(\mathcal{F}_{n+1}(\mathcal{A}, b)\) and \(Z_n \circ \varphi_n = \psi_n\), where \(\varphi_n: A_n(\mathcal{A}, b) \to \mathcal{G}_n^b(\mathcal{A}, b)\) is the surjection of Proposition 2.6.

Let \(p_n: A_n(\delta) \to A_n(\delta)\) be the map which sends a connected diagram to itself and nonconnected diagrams to 0. Set \(Z_n^c = p_n \circ Z_n\) and \(Z^c = \sum_{n \geq 0} Z_n^c\). Then \(Z^c\) is additive under connected sum and \(Z = \exp_c(Z^c)\).

We will detail the second statement of this theorem in Section 4. Note that, in particular, if the map \(\psi_n\) is injective, then the map \(\varphi_n\) is an isomorphism.

In order to take into account the whole quotient \(\mathcal{G}_n(\mathcal{A}, b)\), we extend the invariant \(Z\). Define a \(\delta\)--augmented diagram as the disjoint union of a \(\delta\)--colored diagram with finitely many isolated vertices colored by prime integers. The degree of such a diagram is the number of its vertices. Set

\[
A_n^{\text{aug}}(\delta) = \frac{\mathbb{Q}\langle \text{\(\delta\)--augmented diagrams of degree } n \rangle}{\mathbb{Q}\langle \text{AS, IHX, LE, OR, Hol, Hol'} \rangle}.
\]

The map \(\psi_n\) naturally extends to a map \(\psi_n: A_n^{\text{aug}}(\mathcal{A}, b) \to A_n^{\text{aug}}(\delta)\) preserving the isolated vertices. We now define an invariant \(Z^{\text{aug}} = (Z_n^{\text{aug}})_{n \in \mathbb{N}}\) of \(\mathbb{Q}SK\)--pairs such that the \(\mathbb{Q}\)--linear extension of \(Z_n^{\text{aug}}\) to \(\mathcal{F}_0(\mathcal{A}, b)\) takes values in \(A_n^{\text{aug}}(\delta)\), from which the invariant \(Z\) is recovered by forgetting the isolated vertices. For a prime integer \(p\), define an invariant \(\rho_p\) by \(\rho_p(M, K) = -v_p(|H_1(M; \mathbb{Z})|)\cdot p\), where \(v_p\) is the \(p\)--adic valuation. Once again, the disjoint union makes \(A_n^{\text{aug}}(\delta) = \bigoplus_{n \in \mathbb{N}} A_n^{\text{aug}}(\delta)\) a graded algebra. Set

\[
Z^{\text{aug}} = Z \cup \exp_c \left( \sum_{p\text{ prime}} \rho_p \right).
\]

In Section 6, we prove:

**Theorem 2.9** Fix a Blanchfield module \((\mathcal{A}, b)\), and let \(\delta\) be the annihilator of \(\mathcal{A}\). Consider the surjection \(\varphi_n: A_n^{\text{aug}}(\mathcal{A}, b) \to \mathcal{G}_n(\mathcal{A}, b)\) of Theorem 2.7 and the map \(\psi_n: A_n^{\text{aug}}(\mathcal{A}, b) \to A_n^{\text{aug}}(\delta)\). Then the \(\mathbb{Q}\)--linear extension of \(Z_n^{\text{aug}}: \mathcal{P}(\mathcal{A}, b) \to A_n^{\text{aug}}(\delta)\) to \(\mathcal{F}_0(\mathcal{A}, b)\) vanishes on \(\mathcal{F}_{n+1}(\mathcal{A}, b)\) and \(Z_n^{\text{aug}} \circ \varphi_n = \psi_n\).

Let \(\mathcal{A}_0\) be the trivial Blanchfield module. The relations LV and LD allow us to express the elements of \(A_n^{\text{aug}}(\mathcal{A}_0)\) without diagrams with univalent vertices. It follows that this

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Diagram space has a simpler presentation as

\[ A_{\text{aug}}^n(\mathfrak{A}_0) = \mathbb{Q}\langle \text{augmented diagrams of degree } n \rangle / \mathbb{Q}\langle \text{AS, IHX, LE, OR, Hol, Hol'} \rangle, \]

where an augmented diagram is the disjoint union of a trivalent part — a trivalent graph whose vertices are oriented and whose edges are oriented and colored by \( \mathbb{Q}[t^{\pm 1}] \) — and a finite number of isolated vertices colored by prime integers. The degree of an augmented diagram is the number of its vertices. The space \( A_{\text{aug}}(\mathfrak{A}_0) \) admits a similar description without isolated vertices; the corresponding graded space \( A_{\text{aug}}(\mathfrak{A}_0) / A_{\text{aug}}(1) \) is an isomorphism. Hence Theorems 2.7 and 2.9 imply the next results.

**Theorem 2.10** We have a graded space isomorphism \( G(\mathfrak{A}_0) \cong A_{\text{aug}}(\mathfrak{A}_0) \).

**Theorem 2.11** Let \( Z_{\text{Les}} = (Z_{n,\text{Les}})_{n \in \mathbb{N}} \) and \( Z_{\text{Kri}} = (Z_{n,\text{Kri}})_{n \in \mathbb{N}} \) denote the Lescop equivariant invariant and the Kricker invariant, respectively. Let \( (M, K) \) and \( (N, J) \) be \( \mathbb{Q}\text{SK} \)–pairs with trivial Blanchfield module, such that \( H_1(M; \mathbb{Z}) \) and \( H_1(N; \mathbb{Z}) \) have the same cardinality. Then, for any \( n \in \mathbb{N} \), \( Z_{k,\text{Les}}(M, K) = Z_{k,\text{Les}}(N, J) \) for all \( k \leq n \) if and only if \( Z_{k,\text{Kri}}(M, K) = Z_{k,\text{Kri}}(N, J) \) for all \( k \leq n \).

**Proof** Let \( Z = (Z_n)_{n \in \mathbb{N}} \) be the Lescop or Kricker invariant. Since \( H_1(M; \mathbb{Z}) \) and \( H_1(N; \mathbb{Z}) \) have the same cardinality, the assertion “\( Z_k(M, K) = Z_k(N, J) \) for all \( k \leq n \)” is equivalent to “\( Z_{k,\text{aug}}(M, K) = Z_{k,\text{aug}}(N, J) \) for all \( k \leq n \)”. Since the \( Z_{k,\text{aug}} \colon G_k(\mathfrak{A}_0) \to A_{k,\text{aug}}(\mathfrak{A}_0) \) are isomorphisms, this last assertion is equivalent to \( (M, K) = (N, J) \in \mathcal{F}_{n+1}(\mathfrak{A}_0) \). \( \square \)

In general, note that “the map \( \psi_n \colon A_{\text{aug}}^n(\mathfrak{A}, b) \to A_{\text{aug}}^n(\delta) \) is injective” is equivalent to “the map \( \psi_k \colon A_k(\mathfrak{A}, b) \to A_k(\delta) \) is injective for all \( k \leq n \)”. Hence we focus on the study of injectivity of the map \( \psi_n \colon A_n(\mathfrak{A}, b) \to A_n(\delta) \).

**2.5 About the map \( \psi_n \colon A_n(\mathfrak{A}, b) \to A_n(\delta) \) and perspectives**

We now state a result about the injectivity of the map \( \psi_n \colon A_n(\mathfrak{A}, b) \to A_n(\delta) \) for \( n \) even.

In Section 7, we prove:
Theorem 2.12 Let $n$ be an even positive integer and $N \geq \frac{3}{2}n$. Fix a Blanchfield module $(\mathfrak{A}, b)$. Let $\delta$ be the annihilator of $\mathfrak{A}$. Define the Blanchfield module $(\bar{\mathfrak{A}}, \bar{b})$ as the direct sum of $N$ copies of $(\mathfrak{A}, b)$. Then the map $\bar{\psi}_n: A_n(\bar{\mathfrak{A}}, \bar{b}) \to A_n(\delta)$ is an isomorphism.

This result provides a rewriting of the map $\psi_n$ in the general case. We have a natural map $\iota_n: A_n(\mathfrak{A}, b) \to A_n(\bar{\mathfrak{A}}, \bar{b})$ defined on a diagram by interpreting the labels of its univalent vertices as elements of the first copy of $(\mathfrak{A}, b)$ in $(\bar{\mathfrak{A}}, \bar{b})$. The following diagram commutes:

\[
\begin{array}{ccc}
A_n(\bar{\mathfrak{A}}, \bar{b}) & \xrightarrow{\psi_n} & A_n(\delta) \\
\downarrow{\iota_n} & & \downarrow{\bar{\psi}_n} \\
A_n(\mathfrak{A}, b) & & 
\end{array}
\]

We mention here results from [3] about the map $\psi_2: A_2(\mathfrak{A}, b) \to A_2(\delta)$ for small Alexander modules.

Proposition 2.13 [3] If $\dim_{\mathbb{Q}}(\mathfrak{A}) = 2$, then $\psi_2$ is injective but not surjective.

Proposition 2.14 [3] If $\mathfrak{A}$ is the direct sum of two isomorphic Blanchfield modules of $\mathbb{Q}$–dimension 2 with annihilator $\delta$, then $\psi_2$ is injective if and only if $\delta \neq t + 1 + t^{-1}$. In this case, it is an isomorphism.

Perspectives As mentioned in the introduction, our main goal in this paper is to study Question 1 in order to determine if the Lescop/Kricker invariant $Z$ is a universal finite type invariant of $\mathbb{Q}$SK–pairs up to degree 0 and 1 invariants. Theorem 2.12 provides the following rewriting of this question.

We have a map $A_n((\mathfrak{A}, b)^{\oplus k}) \to A_n((\mathfrak{A}, b)^{\oplus k+1})$ defined by viewing the labels of the univalent vertices in the direct sum of the first $k$ copies of $(\mathfrak{A}, b)$ in $(\mathfrak{A}, b)^{\oplus k+1}$. We also have a map $C_n: G_n^b((\mathfrak{A}, b)^{\oplus k}) \to G_n^b((\mathfrak{A}, b)^{\oplus k+1})$ induced by the connected sum with a fixed $\mathbb{Q}$SK–pair $(M, K) \in \mathcal{P}(\mathfrak{A}, b)$. Using Theorem 2.2, one can check that the map $C_n$ is independent of the fixed pair $(M, K)$. These maps provide the following commutative diagram for any integer $N$ such that $N \geq \frac{3}{2}n$, where the vertical arrows are the maps $\varphi_n$ and $Z_n$:
It follows that the map \( Z_n: G_n^b((\mathbb{A}, b)^{\oplus k}) \to A_n(\delta) \) is injective for all \( k \) if and only if \( C_n: G_n^b((\mathbb{A}, b)^{\oplus k}) \to G_n^b((\mathbb{A}, b)^{\oplus k+1}) \) is injective for all \( k \). This assertion is true for all \((\mathbb{A}, b)\) and all \( n \) if the space of finite type invariants of \( \mathbb{Q}SK\)-pairs is generated as an algebra by degree 0 invariants and invariants that are additive under connected sum.

### 2.6 The case of knots in \( \mathbb{Z} \)-spheres

A great part of the results stated up to this point have an equivalent in the case of \( \mathbb{Z}SK\)-pairs. In this subsection, we adapt the definitions and state the results in this case.

Given a \( \mathbb{Z}SK\)-pair \((M, K)\) and the infinite cyclic covering \( \tilde{X} \) of the exterior of \( K \) in \( M \), define the integral Alexander module of \((M, K)\) as the \( \mathbb{Z}[t^{\pm 1}]\)-module \( \mathbb{A}_Z(M, K) = H_1(\tilde{X}; \mathbb{Z}) \) and the Blanchfield form \( b_Z \) on this module. The integral Alexander module of a \( \mathbb{Z}SK\)-pair \((M, K)\) endowed with its Blanchfield form is its integral Blanchfield module denoted by \((\mathbb{A}_Z, b_Z)(M, K)\). In the sequel, by an integral Blanchfield module, we mean a pair \((\mathbb{A}_Z, b_Z)\) which can be realized as the integral Blanchfield module of a \( \mathbb{Z}SK\)-pair.

Replacing \( \mathbb{Q} \) by \( \mathbb{Z} \) in the definitions of Section 2.1, define integral Lagrangians, integral LP–surgeries and integral null LP–surgeries. Note that a Borromean surgery is an integral LP–surgery.

For diagram spaces, we have to adapt the relation \( \text{Aut} \). Given \( (\mathbb{A}_Z, b_Z) \), set \( (\mathbb{A}, b) = \mathbb{Q} \otimes (\mathbb{A}_Z, b_Z) \). Define the relation \( \text{Aut} Z \) on \((\mathbb{A}, b)\)-colored diagrams as the relation \( \text{Aut} \) restricted to the action of the automorphisms in \( \text{Aut}(\mathbb{A}, b) \) that are induced by automorphisms of the \( \mathbb{Z}[t^{\pm 1}]\)-module \((\mathbb{A}_Z, b_Z)\). Set

\[
A_n^Z(\mathbb{A}_Z, b_Z) = \mathcal{A}_n(\mathbb{A}, b)/\langle \text{Aut} Z \rangle \quad \text{and} \quad A^Z(\mathbb{A}_Z, b_Z) = \bigoplus_{n \in \mathbb{N}} A_n^Z(\mathbb{A}_Z, b_Z).
\]

Since the opposite of the identity is an automorphism of \((\mathbb{A}_Z, b_Z)\), we have:
Lemma 2.15  For all $n \geq 0$, $A_{2n+1}^Z(\mathfrak{A}_Z, bZ) = 0$.

The filtration $(F_n)_{n \in \mathbb{N}}$ of Section 2.1 generalizes the following filtration introduced by Garoufalidis and Rozansky in [8]. Let $F_0^Z$ be the rational vector space generated by all $\mathbb{Z}$SK–pairs, up to orientation-preserving homeomorphism. Define a filtration $(F_n^Z)_{n \in \mathbb{N}}$ of $F_0^Z$ by means of null Borromean surgeries.

Remark  Habegger [9, Theorem 2.5] and Auclair and Lescop [2, Lemma 4.11] proved that two $\mathbb{Z}$–handlebodies whose boundaries are LP–identified can be obtained from one another by a finite sequence of Borromean surgeries. Therefore, the filtration defined on $F_0^Z$ by integral null LP–surgery is equal to the filtration $(F_n^Z)_{n \in \mathbb{N}}$.

Theorem 2.16  [17, Theorem 1.15] An integral null LP–surgery induces a canonical isomorphism between the integral Blanchfield modules of the involved $\mathbb{Z}$SK–pairs. Conversely, for any isomorphism $\xi$ from the integral Blanchfield module of a $\mathbb{Z}$SK–pair $(M, K)$ to the integral Blanchfield module of a $\mathbb{Z}$SK–pair $(M', K')$, there is a finite sequence of integral null LP–surgery from $(M, K)$ to $(M', K')$ which induces the composition of $\xi$ with multiplication by a power of $t$.

This result provides a splitting of the filtration $(F_n^Z)_{n \in \mathbb{N}}$ as the direct sum of filtrations $(F_n^Z(\mathfrak{A}_Z, bZ))_{n \in \mathbb{N}}$ of subspaces $F_0^Z(\mathfrak{A}_Z, bZ)$ of $F_0^Z$, where $(\mathfrak{A}_Z, bZ)$ runs along all isomorphism classes of integral Blanchfield modules. Set $G_n^Z(\mathfrak{A}_Z, bZ) = F_n^Z(\mathfrak{A}_Z, bZ)/F_{n+1}^Z(\mathfrak{A}_Z, bZ)$ and $G_n^Z(\mathfrak{A}_Z, bZ) = \bigoplus_{n \in \mathbb{N}} G_n^Z(\mathfrak{A}_Z, bZ)$. Theorem 2.16 implies $G_0^Z(\mathfrak{A}_Z, bZ) = \mathbb{Q}$. In [8], Garoufalidis and Rozansky identified the graded space $G^Z(\mathfrak{A}_0)$, where $\mathfrak{A}_0$ is the trivial Blanchfield module, with the graded space $A^Z(\mathfrak{A}_0)$. Theorem 2.10 generalizes this result.

Proposition 4.6 implies:

Theorem 2.17  Fix an integral Blanchfield module $(\mathfrak{A}_Z, bZ)$. For all $n \geq 0$, there is a canonical surjective $\mathbb{Q}$–linear map

$$\varphi_n^Z: A_n^Z(\mathfrak{A}_Z, bZ) \rightarrow G_n^Z(\mathfrak{A}_Z, bZ).$$

Corollary 2.18  Fix an integral Blanchfield module $(\mathfrak{A}_Z, bZ)$ and an integer $n \geq 0$. Then $G_{2n+1}^Z(\mathfrak{A}_Z, bZ) = 0$.

As in Section 2.4, we have a map $\psi_n^Z: A_n^Z(\mathfrak{A}_Z, bZ) \rightarrow A_n(\delta)$, where $\delta$ is the annihilator of $\mathfrak{A} = \mathbb{Q} \otimes \mathfrak{A}_Z$. Theorem 2.8 implies that the degree $n$ part of the invariant $Z$ provides a $\mathbb{Q}$–linear map $Z_n: F_0^Z(\mathfrak{A}_Z, bZ) \rightarrow A_n(\delta)$ such that $Z_n \circ \varphi_n^Z = \psi_n^Z$. 

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Set $b = \text{id}_Q \otimes \text{id}_Z$. We have a natural projection $A_*^Z(\mathfrak{A}_Z, b_Z) \to A_*(\mathfrak{A}, b)$. The map $\psi_n^Z$ is the composition of the map $\psi_n$ with this projection. Hence we could adapt Theorem 2.12 and get a surjective map $\psi_n^Z$, but we would not get injectivity, which is what we are mostly interested in.

3 Equivariant clasper calculus

For a $\mathbb{Q}$SK–pair $(M, K)$, let $\mathcal{F}_0^b(M, K)$ be the rational vector space generated by all the $\mathbb{Q}$SK–pairs that can be obtained from $(M, K)$ by a finite sequence of null Borromean surgeries, up to orientation-preserving homeomorphism. For $n > 0$, let $\mathcal{F}_n^b(M, K)$ be the subspace of $\mathcal{F}_0^b(M, K)$ generated by the $[(M, K); \Gamma]$ for all $m$–component null Y–links with $m \geq n$.

Lemma 3.1 [6, Lemma 2.2] Let $\Gamma$ be a Y–graph in a 3–manifold $V$ which has a 0–framed leaf that bounds a disk in $V$ whose interior does not meet $\Gamma$. Then $V(\Gamma) \cong V$.

Lemma 3.2 [6, Theorem 3.1; 1, Lemma 5.1.1] Let $\Gamma_0, \Gamma_1$ and $\Gamma_2$ be the Y–graphs drawn in a genus 4 handlebody in Figure 7. Assume this handlebody is embedded in a 3–manifold $V$. Then $V(\Gamma_0) \cong V(\Gamma_1 \cup \Gamma_2)$.

![Figure 7: Topological equivalence for edge sliding](image)

| Lemma 3.3 | Let $\Gamma$ be an $n$–component Y–link which is null in $M \setminus K$. Let $J$ be a framed knot which is rationally nullhomologous in $M \setminus K$ and disjoint from $\Gamma$. Let $\Gamma'$ be obtained from $\Gamma$ by sliding an edge of $\Gamma$ along $J$ (see Figure 8). Then $[(M, K); \Gamma] = [(M, K); \Gamma'] \mod \mathcal{F}_n^b(M, K)$.

Proof | Let $\Gamma'_0$ be the component of $\Gamma'$ that contains the slid edge and let $\Gamma_0$ be the corresponding component of $\Gamma$. By Lemma 3.2, the surgery on $\Gamma'_0$ is equivalent to the simultaneous surgeries on $\Gamma_0$ and on a null Y–graph $\hat{\Gamma}_0$ which has a leaf.
which is a meridian of a leaf of $\Gamma_0$. It follows that $[(M, K); \Gamma] - [(M, K); \Gamma'] = [(M, K); \Gamma \cup \Gamma_0] \in \mathcal{F}^{b}_{n+1}(M, K)$.

In particular, the above lemma shows that the class of $[(M, K); \Gamma] \mod \mathcal{F}^{b}_{n+1}(M, K)$ is invariant under full twists of the edges.

**Lemma 3.4** [6, Theorem 3.1] *Let $\Gamma_0$, $\Gamma_1$, $\Gamma_2$ be the Y–graphs drawn in a genus 4 handlebody in Figure 9. Assume this handlebody is embedded in a 3–manifold $V$. Then $V(\Gamma_0) \approx V(\Gamma_1 \cup \Gamma_2)$.*

**Lemma 3.5** *Let $\Gamma$ be an $n$–component Y–link null in $M \setminus K$. Let $\ell$ be a leaf of $\Gamma$. Let $\gamma$ be a framed arc starting at the vertex incident to $\ell$ and ending in another point of $\ell$, embedded in $M \setminus K$ as the core of a band glued to the associated surface of $\Gamma$ as shown in Figure 10. The arc $\gamma$ splits the leaf $\ell$ into two leaves $\ell'$ and $\ell''$. Denote by $\Gamma''$ (resp. $\Gamma'''$) the Y–link obtained from $\Gamma$ by replacing the leaf $\ell$ by $\ell'$ (resp. $\ell''$). If $\ell'$ and $\ell''$ are rationally nullhomologous in $M \setminus K$, then $\Gamma'$ and $\Gamma'''$ are null Y–links and $[(M, K); \Gamma] = [(M, K); \Gamma'] + [(M, K); \Gamma'''] \mod \mathcal{F}^{b}_{n+1}(M, K)$.*

**Proof** *Let $\Gamma_0$ (resp. $\Gamma_0'$, $\Gamma_0''$) be the component of $\Gamma$ (resp. $\Gamma'$, $\Gamma''$) that contains the leaf $\ell$ (resp. $\ell'$, $\ell''$). By Lemma 3.4, the surgery on $\Gamma_0$ is equivalent to simultaneous surgeries on $\Gamma_0''$ and on a null Y–graph $\widehat{\Gamma}_0'$ obtained from $\Gamma_0'$ by sliding an edge*
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Figure 10: Cutting a leaf

along $\ell''$. Set $\hat{\Gamma}' = (\Gamma \setminus \Gamma_0) \cup \hat{\Gamma}_0$. We have $[(M, K); \hat{\Gamma}'] + [(M, K); \Gamma''] - [(M, K); \Gamma] = [(M, K); (\Gamma \setminus \Gamma_0) \cup \hat{\Gamma}_0 \cup \Gamma''] \in F^b_{n+1}(M, K)$. Conclude with Lemma 3.3.

The next lemma is a consequence of [6, Lemma 4.8].

**Lemma 3.6** Let $\Gamma$ be an $n$–component Y–link null in $M \setminus K$. If a leaf $\ell$ of $\Gamma$ bounds a disk in $(M \setminus K) \setminus (\Gamma \setminus \ell)$ and has framing 1 (ie the linking number of $\ell$ with its parallel induced by the framing of $\Gamma$ is 1) then $[(M, K); \Gamma] = 0 \mod F^b_{n+1}(M, K)$.

The above two lemmas imply that the class of $[(M, K); \Gamma] \mod F^b_{n+1}(M, K)$ does not depend on the framing of the leaves of $\Gamma$.

**Lemma 3.7** Let $\Gamma$ be an $n$–component Y–link null in $M \setminus K$. Let $\ell$ be a leaf of $\Gamma$. Assume $\Gamma \setminus \ell$ is fixed. Then $[(M, K); \Gamma] \mod F^b_{n+1}(M, K)$ only depends on the homotopy class of $\ell$ in $(M \setminus K) \setminus (\Gamma \setminus \ell)$.

**Proof** If the leaf $\ell$ is modified by an isotopy in $(M \setminus K) \setminus (\Gamma \setminus \ell)$, then the homeomorphism class of $(M, K)(\Gamma)$ is preserved. If the leaf $\ell$ crosses itself during a homotopy, apply Lemma 3.5, as shown in Figure 11, and conclude that $[(M, K); \Gamma] \mod F^b_{n+1}(M, K)$ is unchanged by applying Lemma 3.1.

**Lemma 3.8** Let $\Gamma$ be an $n$–component Y–link null in $M \setminus K$. Let $\ell$ be a leaf of $\Gamma$. Let $\Gamma'$ be an $n$–component null Y–link such that $\Gamma' \setminus \ell'$ coincides with $\Gamma \setminus \ell$, where $\ell'$ is a leaf of $\Gamma'$. Let $\Gamma \setminus \ell$ be the preimage of $\Gamma \setminus \ell$ in the infinite cyclic covering $\tilde{X}$ associated with $(M, K)$. Let $\tilde{\ell}$ and $\tilde{\ell}'$ be lifts of $\ell$ and $\ell'$, respectively, with the same basepoint. If $\ell$ and $\ell'$ are homotopic in $M \setminus K$ and $\tilde{\ell}$ and $\tilde{\ell}'$ are rationally homologous in $\tilde{X} \setminus (\Gamma \setminus \ell)$, then $[(M, K); \Gamma] = [(M, K); \Gamma'] \mod F^b_{n+1}(M, K)$.

**Proof** Consider a homotopy from $\ell$ to $\ell'$ in $M \setminus K$. Thanks to Lemma 3.7, it suffices to treat the case when the leaf crosses some edges or leaves of $\Gamma \setminus \ell$ during the homotopy.
As shown in Figure 12, Lemma 3.5 implies that the bracket $[(M, K); \Gamma]$ has the bracket $[(M, K); \hat{\Gamma}]$ added to it, where $\hat{\Gamma}$ is the null Y–link obtained from $\Gamma$ by adding the cutting arc to the edge adjacent to $\ell$, and by replacing $\ell$ by a meridian of the crossed edge or leaf. In the case of a meridian of an edge, Lemmas 3.1 and 3.3 show that the added bracket vanishes.

Fix a leaf $\ell_0$ of $\Gamma \setminus \ell$. Let $[(M, K); \hat{\Gamma}_i]$, for $i \in I$, be the brackets added during the homotopy when the leaf $\ell$ crosses the leaf $\ell_0$. In each $\hat{\Gamma}_i$, pull the basepoint of the leaf replacing the leaf $\ell$ onto the initial basepoint of $\ell$. Let $\ell_i$ be the obtained leaf. Let $\tilde{\ell}_i$ be the lift of $\ell_i$ which has the same basepoint as $\tilde{\ell}$. Let $Y$ be the complement in $\tilde{X}$ of the preimage of $\ell_0$. In $H_1(Y; \mathbb{Q})$, we have $\tilde{\ell} = \sum_{i \in I} \tilde{\ell}_i + \tilde{\ell}'$. Since $\tilde{\ell}$ and $\tilde{\ell}'$ are homologous in $\tilde{X} \setminus (\hat{\Gamma} \setminus \ell)$, this implies that $\sum_{i \in I} \text{lk}_e(\tilde{\ell}_i, \ell_0) = 0$, where $\tilde{\ell}_0$ is a lift of $\ell_0$. By construction of the $\tilde{\ell}_i$, each $\text{lk}_e(\tilde{\ell}_i, \ell_0)$ is equal to $\pm t^k$ for some $k \in \mathbb{Z}$. Thanks to Lemmas 3.1, 3.3 and 3.5, it follows that the $\hat{\Gamma}_i$ can be grouped by pairs with opposite corresponding brackets. Hence $[(M, K); \Gamma] = [(M, K); \Gamma'] \mod \mathcal{F}_{n+1}^b(M, K)$.

**Lemma 3.9** Let $\Gamma$ be an $n$–component Y–link null in $M \setminus K$. Let $\ell$ be a leaf of $\Gamma$. Let $\hat{\Gamma} \setminus \ell$ be the preimage of $\Gamma \setminus \ell$ in the infinite cyclic covering $\tilde{X}$ associated with

![Diagram](image-url)
(M, K). Let \( \tilde{\ell} \) be a lift of \( \ell \). If \( \tilde{\ell} \) is trivial in \( H_1(\widetilde{\Omega} \setminus (\Gamma \setminus \ell); \mathbb{Q}) \), then \( [(M, K); \Gamma] = 0 \mod \mathcal{F}_n^{b+1}(M, K) \).

**Proof** Since \( \tilde{\ell} \) has a multiple which is trivial in \( H_1(\widetilde{\Omega}; \mathbb{Z}) \), Lemma 3.5 allows us to assume \( \tilde{\ell} \) itself is trivial in \( H_1(\widetilde{\Omega}; \mathbb{Z}) \). Hence \( \tilde{\ell} \) is a product of commutators of loops in \( \widetilde{\Omega} \). It follows that \( \ell \) is homotopic to \( \prod_{i \in I} \left[ \alpha_i, \beta_i \right] \) in \( M \setminus K \), where \( I \) is a finite set and the \( \alpha_i \) and \( \beta_i \) satisfy \( \operatorname{lk}(\alpha_i, K) = 0 \) and \( \operatorname{lk}(\beta_i, K) = 0 \). Construct a surface \( \Sigma \) in \( (M \setminus K) \setminus \Gamma \) whose handles are bands around the \( \alpha_i \) and \( \beta_i \), so that \( \partial \Sigma \) is homotopic to \( \ell \) in \( M \setminus K \). Let \( \Gamma' \) be the Y–link obtained from \( \Gamma \) by replacing \( \ell \) by \( \partial \Sigma \). Note that the lifts of \( \partial \Sigma \) are nullhomologous in \( \widetilde{\Omega} \). Hence, by Lemma 3.8, \( [(M, K); \Gamma] = [(M, K); \Gamma'] \mod \mathcal{F}_n^{b+1}(M, K) \).

Let us prove that \( [(M, K); \Gamma'] = 0 \mod \mathcal{F}_n^{b+1}(M, K) \). Apply Lemma 3.5 to cut the leaf \( \partial \Sigma \) into leaves \( \alpha_i, \beta_i, \alpha_i^{-1}, \beta_i^{-1} \). Apply it again to reglue each leaf \( \alpha_i \) with the corresponding leaf \( \alpha_i^{-1} \) and each leaf \( \beta_i \) with the corresponding leaf \( \beta_i^{-1} \). The obtained Y–links all have a leaf which is homotopically trivial in the complement of \( K \) and of the complement of the leaf in the Y–link. Then the result follows from Lemma 3.7.

**Lemma 3.10** Let \( \Gamma \) be an \( n \)–component Y–link null in \( M \setminus K \). Let \( \ell \) be a leaf of \( \Gamma \). Let \( \Gamma \setminus \ell \) be the preimage of \( \Gamma \setminus \ell \) in the infinite cyclic covering \( \widetilde{\Omega} \) associated with \( (M, K) \). Let \( \tilde{\ell} \) be a lift of \( \ell \). Fix \( \Gamma \setminus \ell \). Then the class of \( [(M, K); \Gamma] \mod \mathcal{F}_n^{b+1}(M, K) \) only depends on the class of \( \tilde{\ell} \) in \( H_1(\widetilde{\Omega} \setminus (\Gamma \setminus \ell); \mathbb{Q}) \), and this dependence is \( \mathbb{Q} \)–linear.

**Proof** Let \( \Gamma' \) be a null Y–link which has a leaf \( \ell' \) such that \( \Gamma' \setminus \ell' \) coincides with \( \Gamma \setminus \ell \), and \( \tilde{\ell}' \) is homologous to \( \tilde{\ell} \) in \( \widetilde{\Omega} \setminus (\Gamma \setminus \ell) \), where \( \tilde{\ell}' \) is the lift of \( \ell' \) which has the same basepoint as \( \tilde{\ell} \). Construct another null Y–link \( \Gamma^b \) by replacing the leaf \( \ell \) by \( \ell - \ell' \) in \( \Gamma \); see Figure 13. By Lemma 3.9, \( [(M, K); \Gamma^b] = 0 \mod \mathcal{F}_n^{b+1}(M, K) \). Thus Lemma 3.5 implies \( [(M, K); \Gamma] = [(M, K); \Gamma'] \mod \mathcal{F}_n^{b+1}(M, K) \). Linearity follows from Lemma 3.5.

**4 Colored diagrams and Y–links**

In this section, we apply clasper calculus to obtain the maps from diagram spaces to graded quotients of Proposition 2.6 and Theorem 2.17.

Fix a Blanchfield module \((\mathfrak{A}, b)\). An \((\mathfrak{A}, b)\)–colored diagram is an elementary \((\mathfrak{A}, b)\)–colored diagram if its edges that connect two trivalent vertices are colored by powers.
of $t$ and its edges adjacent to univalent vertices are colored by 1. Below, we associate a null Y–link with some elementary diagrams that generate $\mathcal{A}_n(\mathfrak{A}, b)$. Let $(M, K) \in \mathcal{P}(\mathfrak{A}, b)$. Let $\xi: (\mathfrak{A}, b) \to (\mathfrak{A}, b)(M, K)$ be an isomorphism. Let $m(K)$ be a meridian of $K$.

Let $D$ be an elementary diagram. An embedding of $D$ in $M \setminus K$ is \textit{admissible} if the following conditions are satisfied:

- The vertices of $D$ are embedded in some ball $B \subset M \setminus K$.
- Consider an edge colored by $t^k$. The homology class in $H_1(M \setminus K; \mathbb{Z})$ of the closed curve obtained by connecting the extremities of this edge by a path in $B$ is $k \cdot m(K)$.

Such an embedding always exists. It suffices to embed the diagram in $B$, and to let each edge colored by $t^k$ turn $k$ times around $K$. To an admissible embedding of an elementary diagram, we wish to associate a null Y–link.

Let $\Gamma$ be a Y–graph, null in $M \setminus K$. Let $p$ be the internal vertex of $\Gamma$. Let $\ell$ be a leaf of $\Gamma$. The curve $\hat{\ell}$ drawn in Figure 14 is the \textit{extension of $\ell$ in $\Gamma$}.

Let $D$ be an elementary diagram, equipped with an admissible embedding in $M \setminus K$. Equip $D$ with the framing induced by an immersion in the plane which induces the fixed orientation of the trivalent vertices. If an edge connects two trivalent vertices, insert a little Hopf link in this edge, as shown in Figure 15. At each univalent vertex $v$,
Let $D$ be the ball in the definition of the admissible embedding of $D$. Let $B$ be a lift of $B$ in the infinite cyclic covering $\widetilde{X}$ of the exterior of $K$ in $M$. For $v \in V$, let $\gamma_v$ be the coloring of $v$, let $\tilde{\ell}_v$ be the extension of $\ell_v$ in $\Gamma$ and let $\tilde{\ell}_v$ be the lift of $\ell_v$ in $\widetilde{X}$ defined by lifting the basepoint in $\tilde{B}$. The null Y–link $\Gamma$ is a realization of $D$ in $(M, K)$ with respect to $\xi$ if the following conditions are satisfied:

- $\tilde{\ell}_v$ is homologous to $\xi(\gamma_v)$ for all $v \in V$.
- $\text{lk}_\varepsilon(\tilde{\ell}_v, \tilde{\ell}_{v'}) = f_{vv'}$ for all $(v, v') \in V^2$.

If such a realization exists, the elementary diagram $D$ is $\xi$–realizable.

**Lemma 4.1** Let $(M, K) \in \mathcal{P}(\mathfrak{A}, b)$. Let $\tilde{\xi} : (\mathfrak{A}, b) \to (\mathfrak{A}, b)(M, K)$ be an isomorphism. Let $D \in \mathcal{A}(\mathfrak{A}, b)$ be an elementary diagram of degree $n > 0$, $\xi$–realizable. Let $\Gamma$ be a realization of $D$ in $(M, K)$ with respect to $\xi$. Then the class of $[(M, K); \Gamma]$ mod $\mathcal{F}^b_{n+1}(M, K)$ does not depend on the realization of $D$.

**Proof** If the ball $B$ and its lift $\tilde{B}$ are fixed, then the result follows from Lemmas 3.3 and 3.10. Fix the ball $B$ and consider another lift $\tilde{B}' = \tau^k(\tilde{B})$ of $B$, where $\tau$ is the automorphism of $\widetilde{X}$ which induces the action of $t$ and $k \in \mathbb{Z}$. A realization of $D$ with respect to $\tilde{B}'$ can be obtained from $\Gamma$ by letting the internal vertex of each Y–graph in $\Gamma$ turn $k$ times around $K$, and come back into $B$, by an isotopy of $(M, K, \Gamma)$. This does not change the result of the surgeries on these Y–graphs, hence this does not modify the bracket $[(M, K); \Gamma]$. Now consider two balls $B_1$ and $B_2$ in $M \setminus K$. If $B_1 \subset B_2$, a realization of $D$ with respect to $B_1$ is a realization of $D$ with respect to $B_2$. If $B_1 \cap B_2 \neq \emptyset$, there is a ball $B_3 \subset B_1 \cap B_2$. If $B_1 \cap B_2 = \emptyset$, there is a ball $B_3 \supset B_1 \cup B_2$. Hence the class of the bracket $[(M, K); \Gamma]$ does not depend on the chosen ball $B$. \hfill $\square$

In the sequel, if $D$ is a $\xi$–realizable elementary diagram, $[(M, K); D]_\xi$ denotes the class of $[(M, K); \Gamma]$ mod $\mathcal{F}^b_{n+1}(M, K)$.

Let $D$ be any elementary diagram. Let $V$ be the set of all univalent vertices of $D$. For any family of rational numbers $(q_v)_{v \in V}$, define an elementary diagram $D' = (q_v)_{v \in V} \cdot D$.
from $D$ in the following way. Keep the same graph and the same colorings of the edges. For $v \in V$, multiply the coloring of $v$ by $q_v$. For $v \neq v' \in V$, set $f_{vv'}^{D} = q_v q_{v'} f_{vv'}^{D}$.

**Lemma 4.2** Let $(M, K) \in \mathcal{P}(\mathfrak{A}, b)$. Let $\xi: (\mathfrak{A}, b) \to (\mathfrak{A}, b)(M, K)$ be an isomorphism. Let $D$ be any elementary diagram. Let $V$ be the set of all univalent vertices of $D$. Then there exists a family of positive integers $(s_v)_{v \in V}$ such that $(s_v)_{v \in V} \cdot D$ is $\xi$–realizable.

**Proof** Let $\tilde{X}$ be the infinite cyclic covering associated with $(M, K)$. Since any homology class in $\mathfrak{A}$ has a multiple which can be represented by a knot in $\tilde{X}$, we can assume that the color $\gamma_v$ of each vertex $v$ in $V$ can be represented by a knot in $\tilde{X}$. From $D$, define as above a Y–link $\Gamma$, null in $M \setminus K$, with leaves $\ell_v$ which satisfy the condition that $\ell_v$ is homologous to $\xi(\gamma_v)$ for all $v \in V$. For $v \neq v' \in V$, set $P_{vv'} = \text{lk}(\ell_v, \ell_{v'}) - f_{vv'}$. We can assume that $P_{vv'} \in \mathbb{Z}[t^\pm]$ for all $v \neq v' \in V$. Add well-chosen meridians of $\ell_v$ to $\ell_{v'}$ to get $P_{vv'} = 0$. \[\square\]

**Lemma 4.3** Let $(M, K) \in \mathcal{P}(\mathfrak{A}, b)$. Let $\xi: (\mathfrak{A}, b) \to (\mathfrak{A}, b)(M, K)$ be an isomorphism. Let $D$ be an elementary $(\mathfrak{A}, b)$–colored diagram. Let $V$ be the set of all univalent vertices of $D$. Let $(s_v)_{v \in V}$ and $(s'_v)_{v \in V}$ be families of integers such that $(s_v)_{v \in V} \cdot D$ and $(s'_v)_{v \in V} \cdot D$ are $\xi$–realizable. Then

$$\prod_{v \in V} s'_v [(M, K); (s_v)_{v \in V} \cdot D][\xi] = \prod_{v \in V} s_v [(M, K); (s'_v)_{v \in V} \cdot D][\xi].$$

**Proof** Let $\Gamma$ be a realization of $(s_v)_{v \in V} \cdot (s'_v)_{v \in V} \cdot D$ in $(M, K)$ with respect to $\xi$. By Lemma 3.10, $[(M, K); \Gamma]$ is equal to both sides of the equality. \[\square\]

Let $D$ be an elementary $(\mathfrak{A}, b)$–colored diagram. Let $V$ be the set of all univalent vertices of $D$. The above result allows us to define

$$[(M, K); D][\xi] = \prod_{v \in V} \frac{1}{s_v} [(M, K); (s_v)_{v \in V} \cdot D][\xi] \in \mathcal{G}_n^b(M, K),$$

where $(s_v)_{v \in V}$ is any family of integers such that $(s_v)_{v \in V} \cdot D$ is $\xi$–realizable.

**Lemma 4.4** Let $D$ be an elementary $(\mathfrak{A}, b)$–colored diagram. Let $(M, K)$ and $(M', K')$ be QS–pairs in $\mathcal{P}(\mathfrak{A}, b)$. Fix isomorphisms $\xi: (\mathfrak{A}, b) \to (\mathfrak{A}, b)(M, K)$ and $\xi': (\mathfrak{A}, b) \to (\mathfrak{A}, b)(M', K')$. Then $[(M', K')}; D][\xi'] = [(M, K); D][\xi] \mod \mathcal{F}_{n+1}(\mathfrak{A}, b)$.
Proposition 4.5 Fix a Blanchfield module \((\mathfrak{A}, b)\). Let \(n > 0\). There is a canonical, \(\mathbb{Q}\)–linear and surjective map \(\varphi_n\) : \(A_n(\mathfrak{A}, b) \to \mathcal{S}_n^b(\mathfrak{A}, b)\), given by \(D \mapsto [D]\) for any elementary diagram \(D\).

Proof Let \(\mathcal{D}_n\) be the rational vector space freely generated by the \((\mathfrak{A}, b)\)–colored diagrams of degree \(n\). If \(D\) is an elementary \((\mathfrak{A}, b)\)–colored diagram, set \(\varphi_n(D) = [D]\). Define \(\varphi_n(D)\) for any \((\mathfrak{A}, b)\)–colored diagram \(D\) so that the obtained \(\mathbb{Q}\)–linear map \(\varphi_n : \mathcal{D}_n \to \mathcal{S}_n^b(\mathfrak{A}, b)\) satisfies the relations LE and EV. Let us check that \(\varphi_n\) satisfies the relations AS, IHX, OR, Hol, LV, LD and Aut. OR is trivial. LV follows from Lemma 3.10. Hol is obtained by letting the corresponding vertex of a realization of \(D\) turn around the knot \(K\). AS and IHX respectively follow from [6, Corollary 4.6] and [6, Lemma 4.10]. Aut follows from Lemma 4.4. For the relation LD, it suffices to prove that \(\varphi_n(D) = \varphi_n(D') + \varphi_n(D_0)\), where \(D, D'\) and \(D_0\) are elementary diagrams which are identical except for the part drawn in Figure 16. Note that the edges adjacent to \(v_1\) and \(v_2\) are colored by \(1\). Since the diagram \(D_0\) and the diagram \(D'\) drawn in Figure 17 can be realized by the same null \(Y\)–link, we have \(\varphi_n(D_0) = \varphi_n(D'_0)\). To

\[
\begin{align*}
\varphi_n(D) &= \varphi_n(D') + \varphi_n(D_0) \\
&= f^D_{v_1v_2} + f^D'_{v_1v_2} + tk \\
\end{align*}
\]

Figure 16: The diagrams \(D, D'\) and \(D_0\), where \(\gamma_1, \gamma_2 \in \mathfrak{A}\) and \(k \in \mathbb{Z}\)
see that \( \tilde{\varphi}_n(D'_0) = \tilde{\varphi}_n(D''_0) \), apply the relation LV at the vertex \( v_2 \) to obtain \( \tilde{\varphi}_n(D'_0) = \tilde{\varphi}_n(D'_0) + \tilde{\varphi}_n(D_{00}) \), then apply the relation LV at the vertex \( v_1 \) to obtain \( \tilde{\varphi}_n(D_{00}) = 0 \). Apply the relation LV again at the vertex \( v_1 \) to get \( \tilde{\varphi}_n(D) = \tilde{\varphi}_n(D') + \tilde{\varphi}_n(D''_0) \).

Finally, the map \( \tilde{\varphi}_n \) induces a canonical \( \mathbb{Q} \)-linear map \( \varphi_n : A_n(\mathfrak{A}, b) \to \mathcal{O}_n^b(\mathfrak{A}, b) \).

For \( (M, K) \in \mathcal{P}(\mathfrak{A}, b) \), any \( n \)-component \( Y \)-link null in \( M \setminus K \) is a realization of an elementary \( (\mathfrak{A}, b) \)-colored diagram, which is the disjoint union of \( n \) diagrams of degree 1. Hence \( \varphi_n \) is surjective.

Now that the map \( \varphi_n \) is well-defined, we can prove the second point of Theorem 2.8.

**Proof of the second statement of Theorem 2.8**

Take an \( (\mathfrak{A}, b) \)-colored diagram of degree \( n \). Let \( \Gamma = \Gamma_1 \sqcup \cdots \sqcup \Gamma_n \) be a realization of \( D \) in some \( \mathbb{Q} \text{SK} \)-pair \( (M, K) \in \mathcal{P}(\mathfrak{A}, b) \). For each \( i \in \{1, \ldots, n\} \), fix a lift \( \tilde{\Gamma}_i \) of \( \Gamma_i \) in the infinite cyclic covering \( \tilde{X} \) of \( M \setminus K \), and represent it schematically as

\[
\ell_3 \quad \ell_2
\]

\[
\ell_1 \quad G_i
\]

where \( \ell_1, \ell_2, \ell_3 \) are the leaves of \( \tilde{\Gamma}_i \). By [13, Theorem 1.1] for the Lescop invariant and [18, Theorem 1.1] for the Kricker invariant, the image by \( Z \) of the bracket \( [(M, K); \Gamma] \) is, modulo \( \mathcal{F}_{n+1}(\delta) \), the sum of all diagrams obtained from \( G = \bigsqcup_{1 \leq i \leq n} G_i \) by pairwise gluing all univalent vertices as follows:

\[
\ell \quad \ell' \quad \text{lk}_e(\ell, \ell')
\]
When an edge of $D$ joins two trivalent vertices, then the corresponding two univalent vertices in $G$ are labeled by curves $\ell$ and $\ell'$ such that the equivariant linking of $\ell$ is 0 with any curve labeling a vertex of $G$ other than $\ell'$, and vice versa. Moreover, relevant choices of the lifts of the $\Gamma_i$ ensure that $\text{lk}_{e}(\ell, \ell') = 1$. Finally, modulo $\mathcal{F}_{n+1}(\delta)$, we have $Z([(M, K); \Gamma]) = \psi_n(D)$. Hence $Z_k([(M, K); \Gamma]) = 0$ if $k < n$ and $Z_n \circ \phi_n(D) = \psi_n(D)$. 

In the setting of $\mathbb{Z}$SK–pairs, all the results of Section 3 apply since we work modulo $\mathcal{F}^b_{n+1}(M, K)$. All the results of the current section apply as well. In Lemma 4.4, note that we use Theorem 2.16 instead of Theorem 2.2. We finally get a similar result to the above proposition.

**Proposition 4.6** Fix an integral Blanchfield module $(\mathbb{A}_\mathbb{Z}, b_\mathbb{Z})$. Let $n > 0$. There is a canonical, $\mathbb{Q}$–linear and surjective map $\phi_n^\mathbb{Z} : A_n^\mathbb{Z}(\mathbb{A}_\mathbb{Z}, b_\mathbb{Z}) \to \mathcal{G}_n^\mathbb{Z}(\mathbb{A}_\mathbb{Z}, b_\mathbb{Z})$, given by $D \mapsto [(M, K); D]\xi$ for any elementary diagram $D$, where $(M, K)$ is any $\mathbb{Z}$SK–pair with Blanchfield module $(\mathbb{A}_\mathbb{Z}, b_\mathbb{Z})$ and $\xi : (\mathbb{A}_\mathbb{Z}, b_\mathbb{Z}) \to (\mathbb{A}_\mathbb{Z}, b_\mathbb{Z})(M, K)$ is any isomorphism.

Fix $(\mathbb{A}_\mathbb{Z}, b_\mathbb{Z})$ and set $(\mathbb{A}, b) = (\mathbb{Q} \otimes \mathbb{Z} \mathbb{A}_\mathbb{Z}, \text{Id}_{\mathbb{Q}} \otimes b_\mathbb{Z})$. The corresponding map $\phi_n$ satisfies $\phi_n \circ p_n = \omega_n \circ \phi_n^\mathbb{Z}$, where $p_n : A_n^\mathbb{Z}(\mathbb{A}, b) \to A_n(\mathbb{A}, b)$ is the natural projection and $\omega_n : \mathcal{G}_n^\mathbb{Z}(\mathbb{A}_\mathbb{Z}, b_\mathbb{Z}) \to \mathcal{G}_n^b(\mathbb{A}, b)$ is the map induced by the inclusion $\mathcal{F}^\mathbb{Z}_n(\mathbb{A}_\mathbb{Z}, b_\mathbb{Z}) \hookrightarrow \mathcal{F}_n(\mathbb{A}, b)$.

## 5 The surjective map $\phi_n : \mathcal{A}_n^\text{aug}(\mathbb{A}, b) \to \mathcal{G}_n(\mathbb{A}, b)$

In this section, we prove Theorems 2.7 and 2.3.

Fix a Blanchfield module $(\mathbb{A}, b)$. Let $(M, K)$ be a $\mathbb{Q}$SK–pair in $\mathcal{P}(\mathbb{A}, b)$. Let $\xi : (\mathbb{A}, b) \to (\mathbb{A}, b)(M, K)$ be an isomorphism. Let $n > 0$. Let $D$ be an $(\mathbb{A}, b)$–augmented diagram of degree $n$ whose Jacobi part $D_J$ is elementary. With an isolated vertex colored by a prime integer $p$, we associate a surgery $B_p/B^3$, where $B_p$ is a fixed $\mathbb{Q}$–ball such that $|H_1(B_p; \mathbb{Z})| = p$. Hence, if $D_J$ is $\xi$–realizable with a realization of $D_J$, we associate a family of $n$ disjoint null LP–surgeries.

**Lemma 5.1** Let $(M, K) \in \mathcal{P}(\mathbb{A}, b)$. Let $\xi : (\mathbb{A}, b) \to (\mathbb{A}, b)(M, K)$ be an isomorphism. Let $n > 0$. Let $D$ be an $(\mathbb{A}, b)$–augmented diagram whose Jacobi part $D_J$ is elementary. Let $(p_i)_{1 \leq i \leq n-k}$ be the labels of the isolated vertices of $D$. If $D_J$ is
\( \xi \)-realizable, let \( \Gamma \) be a realization of \( D_J \) in \( (M, K) \) with respect to \( \xi \). Then

- \( [(M, K); D]_{\xi} := [(M, K); (B_{pi}/B^3)_{1 \leq i \leq n-k}, \Gamma] \in \mathcal{G}_n(\mathfrak{A}, b) \) does not depend on \( (M, K) \), on \( \xi \) or on the realization \( \Gamma \) of \( D_J \).

If \( D_J \) is any elementary diagram, set \( [D] = \prod_{v \in V} (1/s_v) [(M, K); (s_v)_{v \in V} \cdot D]_{\xi} \), where \( (s_v)_{v \in V} \) is a family of integers such that \( (s_v)_{v \in V} \cdot D_J \) is \( \xi \)-realizable and \( (s_v)_{v \in V} \cdot D \) is the disjoint union of \( (s_v)_{v \in V} \cdot D_J \) with the 0-valent part of \( D \). Then

- \( [D] \in \mathcal{G}_n(\mathfrak{A}, b) \) does not depend on \( (s_v)_{v \in V} (M, K) \) or \( \xi \).

**Proof** Take \( (M', K') \in \mathcal{P}(\mathfrak{A}, b) \) and an isomorphism \( \xi': (\mathfrak{A}, b) \to (\mathfrak{A}, b)(M', K') \) such that \( D_J \) is \( \xi' \)-realizable. Let \( \Gamma' \) be a realization of \( D_J \) with respect to \( \xi' \). By Proposition 4.5,

\[
[(M', K'); \Gamma'] = [(M, K); \Gamma] \mod \mathcal{F}_{k+1}(\mathfrak{A}, b).
\]

Let \( p \) be a prime integer. Let \( M_p = B^3 \cup_{\partial B^3} B_p \). In the equality in \( \mathcal{F}_0(\mathfrak{A}, b) \) corresponding to the above relation, make a connected sum of each \( \mathbb{Q}SK \)-pair with \( M_p \). Then subtract the new equality from the original one, to obtain

\[
\left( (M', K'); \frac{B_p}{B^3}, \Gamma' \right) = \left( (M, K); \frac{B_p}{B^3}, \Gamma \right) \mod \mathcal{F}_{k+2}(\mathfrak{A}, b).
\]

Applying this process \( n-k \) times, we get

\[
\left( (M', K'); \left( \frac{B_{pi}}{B^3} \right)_{1 \leq i \leq n-k}, \Gamma' \right) = \left( (M, K); \left( \frac{B_{pi}}{B^3} \right)_{1 \leq i \leq n-k}, \Gamma \right) \mod \mathcal{F}_{n+1}(\mathfrak{A}, b).
\]

If \( D_J \) is any elementary diagram, use Lemma 3.10, as in Lemma 4.3, to prove that

\[
[(M, K); D]_{\xi} = \prod_{v \in V} (1/s_v) [(M, K); (s_v)_{v \in V} \cdot D]_{\xi} \]

does not depend on the family of integers \( (s_v)_{v \in V} \) such that \( (s_v)_{v \in V} \cdot D_J \) is \( \xi \)-realizable. Conclude with the first assertion of the lemma.

The above result implies that the map \( \varphi_n: A_n(\mathfrak{A}, b) \to \mathcal{G}_n(\mathfrak{A}, b) \) extends to a canonical \( \mathbb{Q} \)-linear map \( \varphi_n: A_n^{\text{aug}}(\mathfrak{A}, b) \to \mathcal{G}_n(\mathfrak{A}, b) \) defined by \( \varphi_n(D) = [D] \) for any diagram \( D \in A_n^{\text{aug}}(\mathfrak{A}, b) \) whose Jacobi part is elementary. To prove Theorem 2.7, it remains to show that the map \( \varphi_n: A_n^{\text{aug}}(\mathfrak{A}, b) \to \mathcal{G}_n(\mathfrak{A}, b) \) is surjective. We first recall results from [15] and give consequences of them.

**Definition 5.2** Let \( d \) be a positive integer. A \( d \)-torus is a \( \mathbb{Q} \)-torus \( T_d \) such that

- \( H_1(\partial T_d; \mathbb{Z}) = \mathbb{Z} \alpha \oplus \mathbb{Z} \beta \), with algebraic intersection number \( \langle \alpha, \beta \rangle = 1 \);
Finite type invariants of knots in homology 3–spheres with respect to null LP–surgeries

$d\alpha = 0$ in $H_1(T_d; \mathbb{Z})$;

$\beta = d\gamma$ in $H_1(T_d; \mathbb{Z})$, where $\gamma$ is a curve in $T_d$;

$H_1(T_d; \mathbb{Z}) = (\mathbb{Z}/d\mathbb{Z})\alpha \oplus \mathbb{Z}\gamma$.

**Definition 5.3** An *elementary surgery* is an LP–surgery among the following ones:

1. Connected sum (genus 0).
2. LP–replacement of a standard torus by a $d$–torus (genus 1).
3. Borromean surgery (genus 3).

The next result generalizes the similar result of Habegger [9] and Auclair and Lescop [2] for $\mathbb{Z}$–handlebodies and Borromean surgeries.

**Theorem 5.4** [15, Theorem 1.15] If $A$ and $B$ are two $\mathbb{Q}$–handlebodies with LP–identified boundaries, then $B$ can be obtained from $A$ by a finite sequence of elementary surgeries and their inverses in the interior of the $\mathbb{Q}$–handlebodies.

**Corollary 5.5** The space $\mathcal{F}_n(\mathfrak{A}, b)$ is generated by the $[(M, K); (E_i'/E_i)_{1\leq i\leq n}]$ defined by a $\mathbb{Q}$SK–pair $(M, K) \in \mathcal{P}(\mathfrak{A}, b)$ and elementary null LP–surgeries $(E_i'/E_i)$.

**Proof** Consider $[(M, K); (A_i'/A_i)_{1\leq i\leq n}] \in \mathcal{F}_n(\mathfrak{A}, b)$. By Theorem 5.4, for each $i$, $A_i$ and $A_i'$ can be obtained from one another by a finite sequence of elementary surgeries or their inverses. Write $A_1' = A_1(E_1'/E_1) \cdots (E_k'/E_k)$. For $0 \leq j \leq k$, set $B_j = A_1(E_1'/E_1) \cdots (E_j'/E_j)$. Then

$$\left[(M, K); \left(\frac{A_i'}{A_i}\right)_{1\leq i\leq n}\right] = \sum_{j=1}^{k} \left[(M, K); \left(\frac{B_{j-1}}{B_0}\right); \left(\frac{E_j'}{E_j}\right) \cdot \left(\frac{A_i'}{A_i}\right)_{2\leq i\leq n}\right].$$

Decompose each surgery $(A_i'/A_i)$ in this way and conclude with

$$\left[(M, K); \left(\frac{E_j'}{E_j}\right) \cdot \left(\frac{A_i'}{A_i}\right)_{2\leq i\leq n}\right] = -\left[(M, K); \left(\frac{E_j'}{E_j}\right) \cdot \frac{E_i'}{E_i} \cdot \left(\frac{A_i'}{A_i}\right)_{2\leq i\leq n}\right].$$

Let $\mathcal{F}^0_{\mathbb{Q}s}$ be the rational vector space generated by all $\mathbb{Q}$–spheres up to orientation-preserving homeomorphism. Let $(\mathcal{F}^n_{\mathbb{Q}s})_{n\in\mathbb{N}}$ be the filtration of $\mathcal{F}^0_{\mathbb{Q}s}$ defined by LP–surgeries, as before Definition 2.1. Let $\mathcal{G}^n_{\mathbb{Q}s} = \mathcal{F}^n_{\mathbb{Q}s}/\mathcal{F}^{n+1}_{\mathbb{Q}s}$ be the associated quotients.
Lemma 5.6  [15, Proposition 1.8] For each prime integer \( p \), let \( B_p \) be a \( \mathbb{Q} \)–ball such that \( H_1(B_p; \mathbb{Z}) \cong \mathbb{Z} / p\mathbb{Z} \). Then

\[
\mathcal{G}_1^{Q_s} = \bigoplus_{p \text{ prime}} \mathbb{Q} \left[ S^3; \frac{B_p}{B^3} \right].
\]

Lemma 5.7  For each prime \( p \), let \( B_p \) be a \( \mathbb{Q} \)–ball such that \( H_1(B_p; \mathbb{Z}) \cong \mathbb{Z} / p\mathbb{Z} \). Let \( (M, K) \) be a \( \mathbb{Q} \text{SK} \)–pair in \( \mathcal{P}(\mathcal{A}, b) \). Let \( B \) be a \( \mathbb{Q} \)–ball. Let \((A'_i/A_i)_{1 \leq i < n}\) be disjoint null LP–surgeries in \((M, K)\). Then

\[
\left[ (M, K); \frac{B_p}{B^3}, \left( \frac{A'_i}{A_i} \right)_{1 \leq i < n} \right]
\]

is a rational linear combination of the

\[
\left[ (M, K); \frac{B_p}{B^3}, \left( \frac{A'_i}{A_i} \right)_{1 \leq i < n} \right]
\]

and elements of \( \mathcal{F}_{n+1}(\mathcal{A}, b) \).

Proof  By Lemma 5.6, there is a relation

\[
\left[ S^3; \frac{B}{B^3} \right] = \sum_{p \text{ prime}} a_p \left[ S^3; \frac{B_p}{B^3} \right] + \sum_{j \in J} b_j \left[ N_j; C_j, D_j \right],
\]

where \( J \) is a finite set, the \( a_p \) and \( b_j \) are rational numbers, the \( a_p \) are all trivial except for a finite number and \( [N_j; C_j', D_j'] \in \mathcal{F}_2^{Q_s} \) for \( j \in J \). For \( I \subset \{1, \ldots, n-1\} \), make the connected sum of each \( \mathbb{Q} \)–sphere in the relation with \( M((A'_i/A_i)_{i \in I}) \) to obtain

\[
\left[ (M, K)\left( \left( \frac{A'_i}{A_i} \right)_{i \in I} \right); \frac{B}{B^3} \right] = \sum_{p \text{ prime}} a_p \left[ (M, K)\left( \left( \frac{A'_i}{A_i} \right)_{i \in I} \right); \frac{B_p}{B^3} \right] + \sum_{j \in J} b_j \left[ (M \# N_j, K)\left( \frac{A'_i}{A_i} \right)_{i \in I}; \frac{C_j'}{C_j}, \frac{D_j'}{D_j} \right].
\]

Summing these equalities for all \( I \subset \{1, \ldots, n-1\} \), with appropriate signs, we get

\[
\left[ (M, K); \frac{B_p}{B^3}, \left( \frac{A'_i}{A_i} \right)_{1 \leq i < n} \right] = \sum_{p \text{ prime}} a_p \left[ (M, K); \frac{B_p}{B^3}, \left( \frac{A'_i}{A_i} \right)_{1 \leq i < n} \right] + \sum_{j \in J} b_j \left[ (M \# N_j, K); \frac{C_j'}{C_j}, \frac{D_j'}{D_j}, \left( \frac{A'_i}{A_i} \right)_{1 \leq i < n} \right].
\]

This concludes the proof. \( \Box \)
Corollary 5.8  Let \((M, K) \in \mathcal{P}(\mathcal{A}, b)\). Let \((E'_i / E_i)_{1 \leq i \leq n}\) be null elementary surgeries of genus 0 or 3. Then
\[
\left[ (M, K) ; \left( \frac{E'_i}{E_i} \right)_{1 \leq i \leq n} \right] \in \varphi_n(\mathcal{A}_{\text{aug}}(\mathcal{A}, b)).
\]

Proof  Thanks to Lemma 5.7, it suffices to treat the case when the genus 0 surgeries are surgeries of type \((B / B^3)\) for a \(\mathbb{Q}\)–ball \(B\) such that \(|H_1(B; \mathbb{Z})|\) is a prime integer. In this case, the considered bracket is the image of a diagram given as the disjoint union of 0–valent vertices and of \((\mathcal{A}, b)\)–colored diagrams of degree 1.

To conclude the proof of Theorem 2.7, we need the next result about degree 1 invariants of framed \(\mathbb{Q}\)–tori, i.e. \(\mathbb{Q}\)–tori equipped with an oriented longitude. Note that any two framed \(\mathbb{Q}\)–tori have a canonical LP–identification of their boundaries, which identifies the fixed longitudes. LP–surgeries are well-defined on framed \(\mathbb{Q}\)–tori and we have an associated notion of finite type invariants.

Lemma 5.9  \([15, \text{Corollary 5.10}]\)  For any prime integer \(p\), let \(M_p\) be a \(\mathbb{Q}\)–sphere such that \(H_1(M_p; \mathbb{Z}) \cong \mathbb{Z} / p\mathbb{Z}\). Let \(T_0\) be a framed standard torus. If \(\mu\) is a degree 1 invariant of framed \(\mathbb{Q}\)–tori such that \(\mu(T_0) = 0\) and \(\mu(T_0 \# M_p) = 0\) for any prime integer \(p\), then \(\mu = 0\).

Proof of Theorem 2.7  Take \(\lambda \in (\mathcal{F}_n(\mathcal{A}, b))^*\) such that \(\lambda(\mathcal{F}_{n+1}(\mathcal{A}, b)) = 0\). Assume that \(\lambda(\varphi_n(\mathcal{A}_{\text{aug}}(\mathcal{A}, b))) = 0\). In order to prove that \(\varphi_n\) is onto, it is enough to prove that \(\lambda = 0\). Thanks to Corollary 5.5, it suffices to prove that \(\lambda\) vanishes on the brackets defined by elementary surgeries. For elementary surgeries of genus 0 and 3, this follows from Corollary 5.8.

Consider a bracket
\[
\left[ (M, K) ; \left( \frac{T_{d_i}}{T_i} \right)_{1 \leq i \leq k} , \left( \frac{E'_i}{E_i} \right)_{1 \leq i \leq n - k} \right],
\]
where \((M, K) \in \mathcal{P}(\mathcal{A}, b)\), the \(T_i\) are standard tori null in \(M \setminus K\), the \(T_{d_i}\) are \(d_i\)–tori for some positive integers \(d_i\), and the \((E'_i / E_i)\) are null elementary surgeries of genus 0 or 3. By induction on \(k\), we will prove that \(\lambda\) vanishes on this bracket. We have treated the case \(k = 0\). Assume \(k > 0\). Fix a parallel of \(T_1\). If \(T\) is a framed \(\mathbb{Q}\)–torus, set
\[
\overline{\lambda}(T) = \lambda \left( \left[ (M, K) ; \left( \frac{T_{d_i}}{T_i} \right)_{2 \leq i \leq k} , \left( \frac{E'_i}{E_i} \right)_{1 \leq i \leq n - k} \right] \right).
\]
where the LP–identification $\partial T \cong \partial T_1$ identifies the preferred parallels. Then $\overline{\lambda}$ is a degree 1 invariant of framed $\mathbb{Q}$–tori:

$$
\overline{\lambda}\left(\left[\begin{array}{c} T_1: B_1, B_2 \\ A_1, A_2 \end{array}\right]\right) = -\lambda\left(\left[\begin{array}{c} (M, K) \left(\frac{T_1}{T}\right); B_1, B_2, \left(\frac{T_i}{T_1}\right)_{2 \leq i \leq k}, \left(\frac{E_i'}{E_i}\right)_{1 \leq i \leq n-k} \end{array}\right]\right) = 0.
$$

Moreover, we have $\overline{\lambda}(T_1) = 0$ and, by induction, $\overline{\lambda}(T_1(B_p/B^3)) = 0$. Thus, by Lemma 5.9, $\overline{\lambda} = 0$.

**Proof of Theorem 2.3** Theorem 2.7 provides a surjective map $\varphi_1: \mathcal{A}_{\text{aug}}(\mathfrak{G}, b) \to \mathcal{G}_1(\mathfrak{G}, b)$. Thanks to Lemma 2.5, we have $\mathcal{A}_{\text{aug}}(\mathfrak{G}, b) = \bigoplus_{p \text{ prime}} \mathbb{Q} \ast_p$. Hence $\mathcal{G}_1(\mathfrak{G}, b)$ is generated by the images of the diagrams $\bullet_p$, which are the brackets $[(M, K); B_p/B^3]$ for all prime integers $p$, with any $(M, K) \in \mathcal{P}(\mathfrak{G}, b)$.

For any prime integer $p$, define a $\mathbb{Q}$–linear map $\nu_p: \mathcal{F}_0 \to \mathbb{Q}$ by setting $\nu_p(M, K) = v_p(|H_1(M; \mathbb{Z})|)$ for all $\mathbb{Q}$SK–pairs $(M, K)$, where $v_p$ denotes the $p$–adic valuation. By [15, Proposition 1.9], the $\nu_p$ are degree 1 invariants of $\mathbb{Q}$–spheres, hence they are also degree 1 invariants of $\mathbb{Q}$SK–pairs. This implies that the family

$$
\left[\begin{array}{c} (M, K); B_p/B^3 \end{array}\right]_{p \text{ prime}}
$$

is free in $\mathcal{G}_1(\mathfrak{G}, b)$.

**6 Extension of the Lescop/Kricker invariant**

In this section, we prove Theorem 2.9.

Given two invariants $\lambda_1$ and $\lambda_2$ of $\mathbb{Q}$SK–pairs, define their product on any $\mathbb{Q}$SK–pair $(M, K)$ by $(\lambda_1 \lambda_2)(M, K) = \lambda_1(M, K) \lambda_2(M, K)$ and extend to $\mathcal{F}_0$ by linearity. The following lemma is classical and holds for any objects and any invariants with values in some ring; see for instance [15, Lemma 6.2].

**Lemma 6.1** The following relation holds:

$$
\left(\prod_{j=1}^n \lambda_j\right)\left[\begin{array}{c} (M, K); \left(\frac{B_i}{A_i}\right)_{i \in I} \end{array}\right] = \sum_{\emptyset = J_0 \subset \ldots \subset J_n = I} \prod_{j=1}^n \lambda_j \left[\begin{array}{c} (M, K)\left(\left(\frac{B_i}{A_i}\right)_{i \in J_{j-1}}\right); \left(\frac{B_i}{A_i}\right)_{i \in J_j \setminus J_{j-1}} \end{array}\right].
$$
This lemma implies in particular that a product of finite type invariants is a finite type invariant whose degree is at most the sum of the degrees of the factors.

**Proof of Theorem 2.9** We begin with a preliminary remark about the invariant $Z$. It follows from the last point in Theorem 2.8 that $Z_n \circ \varphi_n$ vanishes on diagrams that contain isolated vertices. Now, the degree $n$ part of $Z_{\text{aug}}$ is given by

$$Z_{\text{aug}}^n = \sum_{k=0}^{n} \sum_{p_1 < \cdots < p_s} \sum_{t_1 + \cdots + t_s = n-k \atop t_i > 0} Z_k \sqcup \left( \bigcup_{i=1}^{s} \frac{1}{t_i!} (\rho_{p_i})^{t_i} \right).$$

That $Z_{\text{aug}}^n$ vanishes on $\mathcal{F}_{n+1}$ follows from Lemma 6.1.

Let us compute $Z_{\text{aug}}^n \circ \varphi_n$. Let $D$ be an $(\mathcal{A},b)$–augmented diagram of degree $n$. Write $D$ as the disjoint union of its Jacobi part $D_J$ and its 0–valent part $D_*$. Apply Lemma 6.1, noting that for a term in the right-hand side of the obtained equality to be nontrivial,

- each bracket must have exactly the order of the corresponding invariant,
- each invariant $\rho_p$ must be evaluated on a bracket associated with the diagram $\bullet_p$, and
- the invariant $Z_k$ must be evaluated on a bracket associated with a diagram without isolated vertices.

It follows that $Z_{\text{aug}}^n \circ \varphi_n(D) = (Z_k \circ \varphi_k(D_J)) \sqcup D_* = \psi_k(D_J) \sqcup D_* = \psi_n(D). \quad \square$

**7 Inverse of the map $\tilde{\psi}_n$**

In this section, we prove Theorem 2.12. To this end, we construct the inverse of the map $\tilde{\psi}_n$. The rough idea is to open the edges of a given $\delta$–colored diagram, inserting univalent vertices whose fixed equivariant linking is the label of the initial edge. We need some preliminaries.

**Proposition 7.1** Fix a Blanchfield module $(\mathcal{A}, b)$. Assume $\mathcal{A}$ is a direct sum $\mathcal{A} = \mathcal{A}' \oplus \mathcal{A}''$, orthogonal with respect to the Blanchfield form. Let $D$ and $D'$ be $(\mathcal{A}, b)$–colored diagrams which differ only by the labels of their univalent vertices; ie $D$ and $D'$ have the same underlying graph, with a common set $V$ of univalent vertices, the same orientations and edges labels, and the same linkings between the univalent vertices. Further assume that
there are two vertices \( v \) and \( w \) in \( V \) whose labels in \( D \) and \( D' \) are elements of \( A' \);

- for all other vertices in \( V \), the labels in \( D \) and \( D' \) are equal and are elements of \( A'' \);
- for any \( u \in V \) different from \( v \) and \( w \), we have \( f_{uv} = 0 \) and \( f_{uw} = 0 \) for \( D \) and \( D' \).

Then \( D \) and \( D' \) are equal in \( A_n(\mathfrak{A}, b) \), where \( n \) is the degree of \( D \) and \( D' \).

We first prove a few lemmas in the setting of the proposition. In the following, we denote by

\[
\begin{array}{ccc}
\gamma & \longrightarrow & D \\
\downarrow & & \downarrow f \\
\eta & \longrightarrow & \eta
\end{array}
\]

the diagram identical to \( D \) except for the labelings of \( v \) and \( w \), which are \( \gamma \) and \( \eta \) respectively, and the linking \( f_{v,w} \), which is equal to \( f \).

We will use the structure of the Blanchfield module recalled in the next theorem. The dual of a polynomial \( P(t) \in \mathbb{Q}[t^{\pm 1}] \) is the polynomial \( \overline{P}(t) = P(t^{-1}) \). The polynomial \( P \) is symmetric if \( \overline{P}(t) = at^k P(t) \) for some \( a \in \mathbb{Q} \) and \( k \in \mathbb{Z} \).

**Theorem 7.2** [16, Proposition 1.2 and Theorem 1.3] The Blanchfield module \( (\mathfrak{A}, b) \) of a \( \mathbb{Q}SK \)–pair is an orthogonal direct sum of

- cyclic submodules

\[
\mathbb{Q}[t^{\pm 1}] / (\pi^n) \gamma,
\]

where \( n \) is a positive integer, \( \pi \) is either a symmetric prime polynomial with \( \pi(\pm 1) \neq 0 \), or \( (t + 2 + t^{-1}) \), or a product of two dual nonsymmetric prime polynomials, and \( b(\gamma, \gamma) = P / \pi^n \) for some polynomial \( P \) which is symmetric and prime to \( \pi \); and

- submodules

\[
\frac{\mathbb{Q}[t^{\pm 1}]}{(t + 1)^m} \rho \oplus \frac{\mathbb{Q}[t^{\pm 1}]}{(t + 1)^m} \rho',
\]

where \( m \) is an odd positive integer, \( b(\rho, \rho) = 0 \), \( b(\rho', \rho') = 0 \) and \( b(\rho, \rho') = 1/(t + 1)^m \).

**Lemma 7.3** Assume \( \mathfrak{A}' = \mathfrak{A}_1 \oplus^\perp \mathfrak{A}_2 \). If \( \gamma \in \mathfrak{A}_1 \) and \( \eta \in \mathfrak{A}_2 \), then

\[
\begin{array}{ccc}
\gamma & \longrightarrow & D \\
\downarrow & & \downarrow 0 \\
\eta & \longrightarrow & \eta
\end{array}
\]

\( = 0 \).
Proof Apply the Aut relation with the automorphism of \((\mathfrak{A}, b)\) given by the opposite of the identity on \(\mathfrak{A}_1\) and the identity on \(\mathfrak{A}_2 \oplus \mathfrak{A}'\).

**Corollary 7.4** Assume \(\mathfrak{A}'\) is the orthogonal direct sum of submodules \(\mathfrak{A}_i\) for \(i = 1, \ldots, k\). Let \(\gamma, \eta \in \mathfrak{A}'\). Write \(\gamma = \sum_{i=1}^{k} \gamma_i\) and \(\eta = \sum_{i=1}^{k} \eta_i\), with \(\gamma_i, \eta_i \in \mathfrak{A}_i\). Then

\[
\gamma \xrightarrow{D} \eta = \sum_{i=1}^{k} \gamma_i \xrightarrow{D} f_i \eta_i
\]

for all families of rational fractions \(f_i\) such that \(b(\gamma_i, \eta_i) = f_i \mod \mathbb{Q}[t^{\pm 1}]\) and \(\sum_{i=1}^{k} f_i = f\).

**Lemma 7.5** If \(\gamma, \eta \in \mathfrak{A}'\) and \(P \in \mathbb{Q}[t^{\pm 1}]\), then

\[
P \gamma \xrightarrow{D} \eta = \gamma \xrightarrow{D} \eta
\]

**Proof** In the case where \(P\) is a power of \(t\), apply the Aut relation with the automorphism of \((\mathfrak{A}, b)\) given by multiplication by some power of \(t\) on \(\mathfrak{A}'\) and identity on \(\mathfrak{A}''\). Conclude with the LV relation.

**Corollary 7.6** Assume

\[
\mathfrak{A}' = \mathbb{Q}[t^{\pm 1}] / (\pi)
\]

Then

\[
D = \gamma \xrightarrow{D} f \xrightarrow{P} \eta
\]

for some \(P \in \mathbb{Q}[t^{\pm 1}]\), with \(f = f^{D}_{vw}\).

**Lemma 7.7** Assume

\[
\mathfrak{A}' = \frac{\mathbb{Q}[t^{\pm 1}]}{((t + 1)^m)\rho} \oplus \frac{\mathbb{Q}[t^{\pm 1}]}{((t + 1)^m)\rho'}
\]

with \(m\) odd, \(b(\rho, \rho) = 0\), \(b(\rho', \rho') = 0\) and \(b(\rho, \rho') = 1/(t + 1)^m\). Then

\[
D = \rho \xrightarrow{D} f \xrightarrow{Q} \rho
\]

for some \(Q \in \mathbb{Q}[t^{\pm 1}]\), with \(f = f^{D}_{vw}\).

**Proof** Write

\[
D = v \xrightarrow{D} f \xrightarrow{v'}
\]
with \( v = A\rho + B\rho' \) and \( v' = A'\rho + B'\rho' \). Applying the Aut relation with the automorphism given by \( \rho \mapsto 2\rho \), \( \rho' \mapsto \frac{1}{2}\rho' \) and identity on \( \mathcal{A}'' \), we see that the diagrams

\[
A\rho \; \overset{D}{\longrightarrow} \; A'\rho \quad \text{and} \quad B\rho' \; \overset{D}{\longrightarrow} \; B'\rho'
\]

are trivial. Hence we can decompose \( D \) as

\[
D = A\rho \; \overset{D}{\longrightarrow} \; B'\rho' + B\rho' \; \overset{D}{\longrightarrow} \; A'\rho.
\]

Now the automorphism given by \( \rho \mapsto \rho' \), \( \rho' \mapsto i^m\rho \) and identity on \( \mathcal{A}'' \) gives

\[
B\rho' \; \overset{D}{\longrightarrow} \; A'\rho = Bt^m\rho \; \overset{D}{\longrightarrow} \; A'\rho'.
\]

Thanks to Lemma 7.5, we get

\[
D = \rho \; \overset{D}{\longrightarrow} \; P\rho'
\]

with \( P = AB' + Bt^{-m}A' \).

Proof of Proposition 7.1 For \( \pi \in \mathbb{Q}[t^{\pm 1}] \), the \( \pi \)-component of a \( \mathbb{Q}[t^{\pm 1}] \)-module is the submodule of its elements of order some power of \( \pi \). Any Blanchfield module is the direct sum of its \( \pi \)-components, where \( \pi \) runs through all prime symmetric polynomials (including \( t + 1 \)) and all products of two dual prime nonsymmetric polynomials. Thanks to Corollary 7.4, we can assume that \( \mathcal{A}' \) is reduced to one \( \pi \)-component.

First case \( (\pi(-1) \neq 0) \) The module \( \mathcal{A}' \) can be written as an orthogonal direct sum

\[
\mathcal{A}' = \bigoplus_{i=1}^{p} \mathbb{Q}[t^{\pm 1}]_{\gamma_i} / (\pi n_i)
\]

with \( b(\gamma_i, \gamma_i) = P_i / \pi^{n_i} \) for some symmetric polynomial \( P_i \) prime to \( \pi \), and

\[
n := n_1 = \cdots = n_q > n_{q+1} \geq \cdots \geq n_p.
\]

Replacing \( \gamma_1 \) by some rational multiple if necessary, we can assume that \( \sum_{i=1}^{q} P_i \) is prime to \( \pi \). Set \( \gamma = \sum_{i=1}^{p} \gamma_i \). Then \( b(\gamma, \gamma) = P / \pi^n \) with \( P \) symmetric and prime to \( \pi \). It follows that the submodule \( \langle \gamma \rangle \) of \( \mathcal{A}' \) generated by \( \gamma \) has a trivial intersection with its orthogonal \( \langle \gamma \rangle^\perp \), thus

\[
\mathcal{A}' = \langle \gamma \rangle \oplus \langle \gamma \rangle^\perp.
\]
By Corollaries 7.4 and 7.6, we can decompose $D$ as

$$D = \sum_{i=1}^{p} \gamma_i \xrightarrow{f_i} \alpha_i$$

for some polynomials $Q_i$. Corollary 7.4 gives

$$D = \gamma \xrightarrow{f} \eta$$

with $\eta = \sum_{i=1}^{p} Q_i \gamma_i$ and $f = \sum_{i=1}^{p} f_i$. Write $\eta = A\gamma + \mu$ with $\mu \in \langle \gamma \rangle$. Since

$$\gamma \xrightarrow{f} \mu = 0$$

by Lemma 7.3, we get

$$D = \gamma \xrightarrow{f} A\gamma.$$

Similarly,

$$D' = \gamma \xrightarrow{f} B\gamma.$$

The condition on $f$ implies $AP/\pi^n = (BP/\pi^n) \mod \mathbb{Q}[t^{\pm 1}]$, thus $A = B \mod \pi^n$ and $A\gamma = B\gamma$.

**Second case** ($\pi = t + 1$) In this case, the decomposition of $\mathfrak{A}'$ may involve noncyclic submodules. We have $\mathfrak{A}' = \mathfrak{A}_1 \oplus \mathfrak{A}_2$, where

$$\mathfrak{A}_1 = \left( \bigoplus_{i=1}^{p} \frac{\mathbb{Q}[t^{\pm 1}]}{(t + 2 + t^{-1})n_i \gamma_i} \right)$$

and

$$\mathfrak{A}_2 = \left( \bigoplus_{j=1}^{k} \left( \frac{\mathbb{Q}[t^{\pm 1}]}{(t + 1)m_j \rho_j} \right) \right).$$

with $b(\gamma_i, \gamma_i) = P_i/(t + 2 + t^{-1})^{n_i}$, $P_i(-1) \neq 0$, $b(\rho_j, \rho_j) = 0$, $b(\rho_j', \rho_j') = 0$, $b(\rho_j, \rho_j') = 1/(t + 1)m_j$, $n_1 = \cdots = n_q > n_{q+1} \geq \cdots \geq n_p$ and $m_1 \geq \cdots \geq m_k$ with $m_j$ odd. We can assume $\sum_{i=1}^{q} P_i$ is prime to $(t + 1)$. Set $\gamma = \sum_{i=1}^{p} \gamma_i$ and $\rho = \sum_{j=1}^{k} \rho_j$.

Proceeding as in the first case, applications of Corollaries 7.4 and 7.6 and Lemma 7.7 give

$$D = \gamma \xrightarrow{f} \alpha + \rho \xrightarrow{f} \beta$$

with $\alpha \in \mathfrak{A}_1$ and $\beta \in \mathfrak{A}_2$. Finally,

$$D = (\gamma + \rho) \xrightarrow{f} \eta.$$
with $\eta \in \mathfrak{A}'$ and $f = f_{vw}^D$. Similarly,

$$D' = (\gamma + \rho) \xymatrix{ & D \ar[r]^f & \eta'}$$

with $\eta' \in \mathfrak{A}'$.

First assume $2n_1 > m_1$. We have

$$b(\gamma + \rho, \gamma + \rho) = \sum_{i=1}^{p} \frac{P_i}{(t + 2 + t^{-1})^{n_1}} = \frac{P}{(t + 2 + t^{-1})^{n_1}}$$

with $P(-1) \neq 0$. We get $\mathfrak{A}' = (\gamma + \rho) \oplus (\gamma + \rho) \perp$ and we conclude as in the first case.

Now assume $m_1 > 2n_1$. It is easily checked that $(\gamma + \rho, \rho'_1) \cap (\gamma + \rho, \rho'_1) \perp = 0$. Hence $\mathfrak{A}' = (\gamma + \rho, \rho'_1) \oplus (\gamma + \rho, \rho'_1) \perp$, and we can assume $\eta, \eta' \in (\gamma + \rho, \rho'_1)$. By Theorem 7.2, there is a basis $(\mu, \mu')$ of $(\gamma + \rho, \rho'_1)$ such that $b(\mu, \mu) = 0$, $b(\mu', \mu') = 0$ and $b(\mu, \mu') = 1/(t + 1)^{m_1}$. By Lemma 7.7, we have

$$D = \xymatrix{ \mu \ar[r]^f & A \mu'} \quad \text{and} \quad D' = \xymatrix{ \mu \ar[r]^f & B \mu'}.$$

Since the linking $f$ is the same, we get $A = B \mod (t + 1)^{m_1}$ and $A \mu' = B \mu'$. □

Let us fix some notation. Let $n$ be an even positive integer and $N \geq \frac{1}{2}n$. For $i = 1, \ldots, N$, let $(\mathfrak{A}_i, b_i)$ be a copy of $(\mathfrak{A}, b)$ and fix an isomorphism $\xi_i: (\mathfrak{A}, b) \cong (\mathfrak{A}_i, b_i)$. Let $(\mathfrak{A}, b)$ be the orthogonal direct sum of the $(\mathfrak{A}_i, b_i)$. Define permutation automorphisms $\xi_{ij}$ of $(\mathfrak{A}, b)$ by $\xi_j \circ \xi_i^{-1}$ on $\mathfrak{A}_i$, $\xi_i \circ \xi_j^{-1}$ on $\mathfrak{A}_j$ and identity on the other $\mathfrak{A}_i$. Given a diagram $D$ with set of univalent vertices $V$, denote by $D((\gamma_v)_{v \in V}, (f_{vw})_{v \neq w \in V})$ the diagram obtained from $D$ by replacing the label of the vertex $v$ by $\gamma_v$ and the linking between $v$ and $w$ by $f_{vw}$. If all the linkings are the same as in $D$, we drop this part of the notation.

**Definition 7.8** An $(\mathfrak{A}, b)$–colored diagram $D$ is **distributed** if there are a decomposition of the set of univalent vertices of $D$ as $V = \bigsqcup_{i=1}^{\lfloor V/2 \rfloor} \{v_i, w_i\}$ and indices $\ell_i$ with $\ell_i \neq \ell_j$ if $i \neq j$ such that the labels of $v_i$ and $w_i$ are elements of $\mathfrak{A}_{\ell_i}$ for all $i$ and the linkings between vertices in different pairs are trivial.

**Proposition 7.9** The space $A_n(\mathfrak{A}, b)$ is generated by distributed $(\mathfrak{A}, b)$–colored diagrams.
Proof Let $D$ be an $(\overline{\mathbb{A}}, \overline{b})$–colored diagram of degree $n$. First note that $D$ has $n$ trivalent vertices and each univalent vertex is related to a trivalent vertex by an edge since we avoid struts, hence $D$ has at most $3n$ univalent vertices. We shall prove that $D$ is a linear combination of distributed diagrams. Thanks to the LV relation, we can assume that all labels of univalent vertices of $D$ are elements of the $\mathbb{A}_i$. Thanks to the LD and LV relations, we can assume that all univalent vertices have nontrivial labels and the linking $f_{vw}$ is trivial if $v$ and $w$ are labeled in different $\mathbb{A}_i$. If $D$ has an odd number of univalent vertices labeled in some $\mathbb{A}_i$, application of the automorphism given by opposite identity on $\mathbb{A}_i$ and identity on the other $\mathbb{A}_j$ shows it is trivial. Assume $D$ has an even number of univalent vertices labeled in each $\mathbb{A}_i$. Let $i$ be an index such that the number of univalent vertices of $D$ labeled in $\mathbb{A}_i$ is maximal; denote this number by $2s$. If $s > 1$, there is an $\mathbb{A}_j$ that contains no labels of univalent vertices of $D$. Consider the following automorphism $\chi_{ij}$ of $(\overline{\mathbb{A}}, \overline{b})$:

$$\chi_{ij}(\gamma) := \begin{cases} x\gamma + y\xi_i \circ \xi_i^{-1}(\gamma) & \text{if } \gamma \in \mathbb{A}_i, \\ y\xi_i \circ \xi_j^{-1}(\gamma) - x\gamma & \text{if } \gamma \in \mathbb{A}_j, \\ \gamma & \text{if } \gamma \in \mathbb{A}_\ell \text{ with } \ell \neq i, j, \end{cases}$$

where $x$ and $y$ are positive rational numbers such that $x^2 + y^2 = 1$. Apply the Aut relation with $\chi_{ij}$ to $D$ and use the LV relation to express $D = D((\gamma_v)_{v \in V})$ as the sum of $x^{2s}D$, $y^{2s}D((\xi_{ij}(\gamma_v))_{v \in V})$ and a linear combination $C$ of diagrams with strictly fewer than $2s$ vertices in $\mathbb{A}_i$ and in $\mathbb{A}_j$. Now $D$ and $D((\xi_{ij}(\gamma_v))_{v \in V})$ are equal thanks to the Aut relation with $\xi_{ij}$. It follows that $D$ is a rational multiple of $C$. Conclude by iterating.

Remark In the case of $ZSK$–pairs, Proposition 7.9 is the point in this section that does not work. Indeed, this proposition uses automorphisms $\chi_{ij}$ whose definition is based on rational numbers $x$ and $y$ that are not integers. Thus it is not clear whether such isomorphisms are induced by isomorphisms of the underlying integral Blanchfield module. For instance, consider the integral Blanchfield module $(\mathbb{A}_Z, \mathbb{b}_Z)$ defined by

$$\mathbb{A}_Z = \mathbb{Z}[t^{\pm 1}] \frac{1}{(\delta)} \gamma \oplus \mathbb{Z}[t^{\pm 1}] \frac{1}{(\delta)} \eta \quad \text{with } \delta(t) = t - 1 + t^{-1} \text{ and } \mathbb{b}_Z(\gamma, \gamma) = \mathbb{b}_Z(\eta, \eta).$$

Then any isomorphism of $(\mathbb{A}_Z, \mathbb{b}_Z)$ preserves the given direct sum decomposition. Indeed, an isomorphism of $(\mathbb{A}_Z, \mathbb{b}_Z)$ has the form

$$\begin{align*}
\gamma &\mapsto P\gamma + Q\eta, \\
\eta &\mapsto R\gamma + S\eta.
\end{align*}$$
with \( P\overline{P} + Q\overline{Q} = 1, \overline{R}R + S\overline{S} = 1 \) and \( P\overline{R} + Q\overline{S} = 0 \), where the polynomials are considered in \( \mathbb{Z}[t^{\pm 1}]/(\delta) \). Since \( \delta \) has degree 2, one can write \( P(t) = at + b \) and \( Q(t) = ct + d \) with \( a, b, c, d \in \mathbb{Z} \). This gives
\[
P\overline{P} + Q\overline{Q} = a^2 + b^2 + ab + c^2 + d^2 + cd = \frac{1}{2}((a+b)^2 + a^2 + b^2 + (c+d)^2 + c^2 + d^2).
\]
If \( PQ \neq 0 \), then \( a \neq 0 \) or \( b \neq 0 \), and \( c \neq 0 \) or \( d \neq 0 \). It follows that \( P\overline{P} + Q\overline{Q} \geq 2 \), contradicting the first condition on \( P \) and \( Q \). Hence \( PQ = 0 \) and the conditions on the polynomials \( P, Q, R \) and \( S \) give \( P = S = 0 \) or \( Q = R = 0 \).

Recall that the map \( \iota_n \colon A_n(\mathcal{A}, b) \to A_n(\overline{\mathcal{A}}, \overline{b}) \) is defined on diagrams by
\[
\iota_n(D((\gamma_v)_{v \in V})) = D((\xi_1(\gamma_v))_{v \in V}).
\]

**Proposition 7.10** If \( D \) is an \((\mathcal{A}, b)\)–colored diagram of degree \( n \) with an even number of univalent vertices, then
\[
\iota_n(D((\gamma_v)_{v \in V}, (f_{vw})_{v \neq w \in V})) = \frac{1}{s!} \sum_{\sigma \in \Upsilon} D((\xi_{\sigma(v)}(\gamma_v))_{v \in V}, (\delta_{\sigma(v)}(\gamma_w)f_{vw})_{v \neq w \in V}),
\]
where \( s = \frac{1}{2}|V| \) and \( \Upsilon = \{\sigma \colon V \to \{1, \ldots, s\} \mid |\sigma^{-1}(i)| = 2 \text{ for all } i = 1, \ldots, s\} \).

**Proof** We apply the method of the previous proposition with precise computations. We indeed prove a slightly more general result. Consider an \((\overline{\mathcal{A}}, \overline{b})\)–colored diagram \( D = D((\gamma_v)_{v \in V \cup W}, (f_{vw})_{v \neq w \in V \cup W}) \) with \( |V| = 2s \), \( \gamma_v \in \mathcal{A}_1 \) if \( v \in V \), \( \gamma_w \in \mathcal{A}_i \) with \( i > s \) if \( w \in W \), and \( f_{vw} = 0 \) if \( v \in V \) and \( w \in W \). We prove by induction on \( s \) that in \( A_n(\overline{\mathcal{A}}, \overline{b}) \),
\[
D = \frac{1}{s!} \sum_{\sigma \in \Upsilon} D((\xi_{\sigma(v)}(\gamma_v))_{v \in V} \cup (\gamma_w)_{w \in W}, (\delta_{\sigma(v)}(\gamma_w)f_{vw})_{v \neq w \in V} \cup (f_{vw})_{v \neq w \in W}),
\]
where the unindicated linkings are trivial. We will use that our formulas remain valid when permuting the indices of the \( \mathcal{A}_i \), without mentioning it.

The result is trivial if \( s = 1 \). Take \( s > 1 \). Applying the Aut relation with \( \chi_{12} \) to \( D \), we get
\[
D = \sum_{k=0}^{s} \sum_{\substack{V=V_1 \cup V_2 \mid |V_1|=2k \mid |V_2|=2s-2k}} \chi_{2k} y^{2(s-k)} D((\xi_{11}(\gamma_v))_{v \in V_1} \cup (\xi_{12}(\gamma_v))_{v \in V_2} \cup (\gamma_w)_{w \in W})
\]
with, for the diagram in the right-hand side, the linking \( \delta_{\sigma(v)}(\gamma_w)f_{vw} \) if \( v \neq w \) are both in \( V_1 \) or both in \( V_2 \), \( f_{vw} \) if \( v \neq w \) are both in \( W \) and 0 otherwise. Now apply
the induction hypothesis twice with \( V_1 \) and \( V_2 \) instead of \( V \) to obtain

\[
(1 - x^{2s} - y^{2s}) D = \\
\sum_{k=1}^{s-1} \sum_{V=V_1 \cup V_2 \mid |V_1|=2k} \frac{x^{2k} y^{2(s-k)}}{k!(s-k)!} \sum_{\sigma \in T_1, \nu \in T_2} D((\xi_1 \sigma(v)(\gamma_v))_{v \in V_1} \cup (\xi_1 \nu(v)(\gamma_v))_{v \in V_2} \cup (\gamma_w)_{w \in W})
\]

with the required linkings, where \( \Upsilon_1 \) (resp. \( \Upsilon_2 \)) is defined as \( \Upsilon \) with \( V_1 \) and \( \{1, \ldots, k\} \) (resp. \( V_2 \) and \( \{k+1, \ldots, s\} \)) instead of \( V \) and \( \{1, \ldots, s\} \). To conclude, note that each diagram in the right-hand side occurs once for each value of \( k \).

**Proof of Theorem 2.12** Define the inverse \( \Phi_n \) of the map \( \overline{\psi}_n : A_n(\overline{\mathbb{A}}, \overline{b}) \to A_n(\delta) \) in the following way. Given a \( \delta \)-colored diagram \( D \) of degree \( n \), denote by \( e_i \), \( i = 1, \ldots, k \), its edges whose labels are nonpolynomial. “Open” each such edge \( e_i \) as represented in Figure 18, label the created vertices \( v \) and \( w \) with some \( v \) and \( w \) in \( \mathbb{A}_i \) such that \( b(\gamma_v, \gamma_w) = f \mod \mathbb{Q}[t^{\pm 1}] \), and fix the linking \( f_{vw} = f \). Such \( \gamma_v \) and \( \gamma_w \) always exist: note that \( \gamma_v \) can be chosen to have order \( \delta \), the annihilator of \( \mathbb{A}_i \), then use the nondegeneracy of the Blanchfield form and the fact that the denominator of \( f \) has to divide \( \delta \). Fix the other linkings to \( 0 \) so that we obtain a distributed diagram \( \Phi_n(D) \). It does not depend on the numbering of the edges of \( D \) thanks to the \( \text{Aut} \) relation in \( A_n(\overline{\mathbb{A}}, \overline{b}) \) with the permutation automorphisms \( \xi_{ij} \). It is also independent of the choice of labels \( \gamma_v, \gamma_w \in \mathbb{A}_i \) by Proposition 7.1. Note that these independence arguments imply that any distributed diagram in \( A_n(\overline{\mathbb{A}}, \overline{b}) \) is a \( \Phi_n(D) \).

We have to check that the relations defining \( A_n(\delta) \) are respected. It is immediate for \( \text{AS} \) and \( \text{IHX} \). \( \text{OR} \) follows from the rule \( f_{vw}(t) = f_{vw}(t^{-1}) \) on linkings. \( \text{Hol} \) and \( \text{Hol}' \) are recovered via \( \text{Hol} \) and \( \text{EV} \). \( \text{LE} \) follows from \( \text{LE} \) when the involved edges have polynomial labels, from \( \text{LD} \) when one of the involved edges has a polynomial label, and from \( \text{LV} \) when the involved edges have nonpolynomial labels. In this latter case, note that one can open this edge with the same label on one univalent vertex for the three diagrams. Finally, we have a well-defined map \( \Phi_n : A_n(\delta) \to A_n(\overline{\mathbb{A}}, \overline{b}) \) satisfying, by construction, \( \overline{\psi}_n \circ \Phi_n = \text{id}_{A_n(\delta)} \). Now \( \Phi_n \) is surjective by Proposition 7.9. Thus \( \overline{\psi}_n \) and \( \Phi_n \) are inverse isomorphisms.
We end with a few results that derive from the above argument and prove useful in the study of the structure of the diagram space $A_n(\mathfrak{A}, b)$, as it appears in [3]. The first one gives a simplified presentation of $A_n(\mathfrak{A}, b)$.

**Proposition 7.11** Keeping notation as fixed before Definition 7.8, we have

$$A_n(\mathfrak{A}, b) \cong \frac{\mathbb{Q}\langle \text{degree } n \text{ distributed } (\mathfrak{A}, b)\text{-colored diagrams } \rangle}{\mathbb{Q}\langle \text{AS, IHX, Hol, OR, LE, LV, EV, LD, Aut}_{\text{res}} \rangle},$$

where the relation $\text{Aut}_{\text{res}}$ is the Aut relation restricted to the following automorphisms of $(\mathfrak{A}, b)$: permutation automorphisms $\xi_{ij}$, and automorphisms fixing one $\mathfrak{A}_i$ setwise and the others pointwise. Moreover, if $(\mathfrak{A}, b)$ is cyclic, we can further restrict the Aut relation to permutation automorphisms $\xi_{ij}$, and multiplication by $t$ or $-1$ on one $\mathfrak{A}_i$ and identity on the others.

**Proof** We can see that the space defined by the given presentation is isomorphic to $A_n(\delta)$ as we did for $A_n(\mathfrak{A}, b)$ in the proof of Theorem 2.12. At the level of generators, the proof of Theorem 2.12 only uses distributed diagrams and at the level of relations, one has to check that the proof of Proposition 7.1 only uses the Aut relation with the allowed automorphisms.

In order to study $A_n(\mathfrak{A}, b)$, it is natural and helpful to consider the filtration induced by the number of univalent vertices. For $k = 0, \ldots, 3n$, let $A_n^{(k)}(\mathfrak{A}, b)$ be the subspace of $A_n(\mathfrak{A}, b)$ generated by diagrams with at most $k$ univalent vertices, and set

$$\hat{A}_n^{(k)}(\mathfrak{A}, b) = \frac{\mathbb{Q}\langle (\mathfrak{A}, b)\text{-colored diagrams of degree } n \text{ with at most } k \text{ univalent vertices } \rangle}{\mathbb{Q}\langle \text{AS, IHX, OR, Hol, LV, EV, LD, Aut} \rangle}.$$

Similarly, let $A_n^{(k)}(\delta)$ be the subspace of $A_n(\delta)$ generated by diagrams with at most $\frac{1}{2}k$ edges with a nonpolynomial label, and set

$$\hat{A}_n^{(k)}(\delta) = \frac{\mathbb{Q}\langle \delta\text{-colored diagrams of degree } n \text{ with at most } \frac{1}{2}k \text{ edges with a nonpolynomial label } \rangle}{\mathbb{Q}\langle \text{AS, IHX, OR, Hol, Hol'} \rangle}.$$

Recall that all these diagram spaces are trivial when $n$ is odd. Moreover, the number of trivalent vertices and the number of univalent vertices in a unitrivalent graph have the same parity. So we are only interested in cases where $n$ and $k$ are even. Define a map $\hat{\psi}_n^{(k)}: \hat{A}_n^{(k)}(\mathfrak{A}, b) \to \hat{A}_n^{(k)}(\delta)$ via pairings of vertices, as $\psi_n$ was defined in Section 2.4.
Proposition 7.12  Let $n$, $k$ and $K$ be integers such that $0 \leq k \leq 3n$ and $k \leq 2K$. Then:

- The isomorphism $\overline{\psi}_n: \mathcal{A}_n(\overline{\mathfrak{A}}, \overline{b}) \to \mathcal{A}_n(\delta)$ induces an isomorphism $\mathcal{A}_n^{(k)}(\overline{\mathfrak{A}}, \overline{b}) \cong \mathcal{A}_n^{(k)}(\delta)$.
- The map $\hat{\psi}_n^{(k)}: \hat{\mathcal{A}}_n^{(k)}((\mathfrak{A}, b)\oplus K) \to \hat{\mathcal{A}}_n^{(k)}(\delta)$ is an isomorphism.
- The space $\hat{\mathcal{A}}_n^{(k)}((\mathfrak{A}, b)\oplus K)$ admits the presentation
  \[
  \hat{\mathcal{A}}_n^{(k)}((\mathfrak{A}, b)\oplus K) \cong \frac{\mathbb{Q}\{\text{degree } n \text{ distributed } ((\mathfrak{A}, b)\oplus K)\text{–colored diagrams with at most } k \text{ univalent vertices}\}}{\mathbb{Q}\{\text{AS, IHX, Hol, OR, LE, LV, EV, LD, Aut}_{\text{res}}\}}.
  \]

Proof  The first point is due to the fact that $\overline{\psi}_n$ identifies the $(\mathfrak{A}, b)$–colored diagrams that have at most $k$ univalent vertices with the $\delta$–colored diagrams that have at most $\frac{1}{2}k$ edges with a nonpolynomial label. The last two points follow from the same argument as Theorem 2.12 and Proposition 7.11.

A Blanchfield module $(\mathfrak{A}, b)$ for which $\hat{\mathcal{A}}_2^{(4)}(\overline{\mathfrak{A}}, \overline{b}) \not\cong \hat{\mathcal{A}}_2^{(4)}(\overline{\mathfrak{A}}, \overline{b})$ is given explicitly in [3], and this provides the example with a negative answer to Question 2 mentioned in the introduction.

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