MINORS OF TWO-CONNECTED GRAPHS
OF LARGE PATH-WIDTH

Thanh N. Dang
and
Robin Thomas

School of Mathematics
Georgia Institute of Technology
Atlanta, Georgia 30332-0160, USA

Abstract

Let $P$ be a graph with a vertex $v$ such that $P \setminus v$ is a forest, and let $Q$ be an outerplanar graph. We prove that there exists a number $p = p(P, Q)$ such that every 2-connected graph of path-width at least $p$ has a minor isomorphic to $P$ or $Q$. This result answers a question of Seymour and implies a conjecture of Marshall and Wood.

1 Introduction

All graphs in this paper are finite and simple; that is, they have no loops or parallel edges. Paths and cycles have no “repeated” vertices or edges. A graph $H$ is a minor of a graph $G$ if we can obtain $H$ by contracting edges of a subgraph of $G$. An $H$ minor is a minor isomorphic to $H$. A tree-decomposition of a graph $G$ is a pair $(T, X)$, where $T$ is a tree and $X$ is a family $(X_t : t \in V(T))$ such that:

(W1) $\bigcup_{t \in V(T)} X_t = V(G)$, and for every edge of $G$ with ends $u$ and $v$ there exists $t \in V(T)$ such that $u, v \in X_t$, and

(W2) if $t_1, t_2, t_3 \in V(T)$ and $t_2$ lies on the path in $T$ between $t_1$ and $t_3$, then $X_{t_1} \cap X_{t_3} \subseteq X_{t_2}$.

The width of a tree-decomposition $(T, X)$ is $\max\{|X_t| - 1 : t \in V(T)\}$. The tree-width of a graph $G$ is the smallest width among all tree-decompositions of $G$. A path-decomposition of $G$ is a tree-decomposition $(T, X)$ of $G$, where $T$ is a path. We will often denote a path-decomposition as $(X_1, X_2, \ldots, X_n)$, rather than having the constituent sets indexed by the vertices of a path. The path-width of $G$ is the smallest width among all path-decompositions of $G$. Robertson and Seymour [11] proved the following:

Theorem 1.1. For every planar graph $H$ there exists an integer $n = n(H)$ such that every graph of tree-width at least $n$ has an $H$ minor.

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Robertson and Seymour [10] also proved an analogous result for path-width:

**Theorem 1.2.** For every forest $F$, there exists an integer $p = p(F)$ such that every graph of path-width at least $p$ has an $F$ minor.

Bienstock, Robertson, Seymour and the second author [1] gave a simpler proof of Theorem 1.2 and improved the value of $p$ to $|V(F)| - 1$, which is best possible, because $K_k$ has path-width $k - 1$ and does not have any forest minor on $k + 1$ vertices. A yet simpler proof of Theorem 1.2 was found by Diestel [5].

While Geelen, Gerards and Whittle [7] generalized Theorem 1.1 to representable matroids, it is not *a priori* clear what a version of Theorem 1.2 for matroids should be, because excluding a forest in matroid setting is equivalent to imposing a bound on the number of elements and has no relevance to path-width. To overcome this, Seymour [4, Open Problem 2.1] asked if there was a generalization of Theorem 1.2 for 2-connected graphs with forests replaced by the two families of graphs mentioned in the abstract. Our main result answers Seymour’s question in the affirmative:

**Theorem 1.3.** Let $P$ be a graph with a vertex $v$ such that $P \setminus v$ is a forest, and let $Q$ be an outerplanar graph. Then there exists a number $p = p(P, Q)$ such that every 2-connected graph of path-width at least $p$ has a $P$ or $Q$ minor.

Theorem 1.3 is a generalization of Theorem 1.2. To deduce Theorem 1.2 from Theorem 1.3, given a graph $G$, we may assume that $G$ is connected, because the path-width of a graph is equal to the maximum path-width of its components. We add one vertex and make it adjacent to every vertex of $G$. Then the new graph is 2-connected, and by Theorem 1.3, it has a $P$ or $Q$ minor. By choosing suitable $P$ and $Q$, we can get an $F$ minor in $G$.

Marshall and Wood [8] define $g(H)$ as the minimum number for which there exists a positive integer $p(H)$ such that every $g(H)$-connected graph with no $H$ minor has path-width at most $p(H)$. Then Theorem 1.2 implies that $g(H) = 0$ iff $H$ is a forest. There is no graph $H$ with $g(H) = 1$, because path-width of a graph $G$ is the maximum of the path-widths of its connected components. Let $A$ be the graph that consists of a cycle $a_1a_2a_4a_5a_6a_1$ and extra edges $a_1a_3, a_3a_5, a_5a_1$. Let $C_{3,2}$ be the graph consisting of two disjoint triangles. In Section 2 we prove a conjecture of Marshall and Wood [8]:

**Theorem 1.4.** A graph $H$ has no $K_4, K_{2,3}, C_{3,2}$ or $A$ minor if and only if $g(H) \leq 2$.

In Section 3 we describe a special tree-decomposition, whose existence we establish in [3]. In Section 4 we introduce “cascades”, our main tool, and prove that in any tree-decomposition with no duplicate bags of bounded width of a graph of big path-width there is an “injective” cascade of large height. In Section 5 we prove that every 2-connected graph of big path-width and bounded tree-width admits a tree-decomposition of bounded width and a cascade with linkages that are minimal. In Section 6 we analyze those minimal linkages and prove that there are essentially only two types of linkage. This is where we use the properties of tree-decompositions from Section 3. Finally, in Section 7 we convert the two types of linkage into the two families of graphs from Theorem 1.3.
2 Proof of Theorem 1.4

In this section we prove that Theorem 1.4 is implied by Theorem 1.3.

Definition Let \( h \geq 0 \) be an integer. By a binary tree of height \( h \) we mean a tree with a unique vertex \( r \) of degree two and all other vertices of degree one or three such that every vertex of degree one is at distance exactly \( h \) from \( r \). Such a tree is unique up to isomorphism and so we will speak of the binary tree of height \( h \). We denote the binary tree of height \( h \) by \( CT_h \) and we call \( r \) the root of \( CT_h \). Each vertex in \( CT_h \) with distance \( k \) from \( r \) has height \( k \). We call the vertices at distance \( h \) from \( r \) the leaves of \( CT_h \). If \( t \) belongs to the unique path in \( CT_h \) from \( r \) to a vertex \( t' \in V(T_h) \), then we say that \( t' \) is a descendant of \( t \) and that \( t \) is an ancestor of \( t' \). If, moreover, \( t \) and \( t' \) are adjacent, then we say that \( t \) is the parent of \( t' \) and that \( t' \) is a child of \( t \).

Let \( P_k \) be the graph consisting of \( CT_k \) and a separate vertex that is adjacent to every leaf of \( CT_k \).

Lemma 2.1. If a graph \( H \) has no \( K_4, C_{3,2} \), or \( A \) minor, then \( H \) has a vertex \( v \) such that \( H \setminus v \) is a forest.

Proof. We proceed by induction on \( |V(H)| \). The lemma clearly holds when \( |V(H)| = 0 \), and so we may assume that \( H \) has at least one vertex and that the lemma holds for graphs on fewer than \( |V(H)| \) vertices. If \( H \) has a vertex of degree at most one, then the lemma follows by induction by deleting such vertex. We may therefore assume that \( H \) has minimum degree at least two.

If \( H \) has a cutvertex, say \( v \), then \( v \) is as desired, for if \( C \) is a cycle in \( H \setminus v \), then \( H \setminus V(C) \) also contains a cycle (because \( H \) has minimum degree at least two), and hence \( H \) has a \( C_{3,2} \) minor, a contradiction. We may therefore assume that \( H \) is 2-connected.

We may assume that \( H \) is not a cycle, and hence it has an ear-decomposition \( H = H_0 \cup H_1 \cup \cdots \cup H_k \), where \( k \geq 1 \), \( H_0 \) is a cycle and for \( i = 1, 2, \ldots, k \) the graph \( H_i \) is a path with ends \( u_i, v_i \in V(H_0 \cup H_1 \cup \cdots \cup H_{i-1}) \) and otherwise disjoint from \( H_0 \cup H_1 \cup \cdots \cup H_{i-1} \). If \( u_1 \in \{u_i, v_i\} \) for all \( i \in \{2, 3, \ldots, k\} \), then \( u_1 \) satisfies the conclusion of the lemma, and similarly for \( v_1 \). We may therefore assume that there exist \( i, j \in \{2, 3, \ldots, k\} \) such that \( u_1 \notin \{u_i, v_i\} \) and \( v_1 \notin \{u_j, v_j\} \). It follows that \( H \) has a \( K_4, C_{3,2} \), or \( A \) minor, a contradiction. \( \square \)

Lemma 2.2. If a graph \( H \) has a vertex \( v \) such that \( H \setminus v \) is a forest, then there exists an integer \( k \) such that \( H \) is isomorphic to a minor of \( P_k \).

Proof. Let \( v \) be such that \( T := H \setminus v \) is a forest. We may assume, by replacing \( H \) by a graph with an \( H \) minor, that \( T \) is isomorphic to \( CT_t \) for some \( t \), and that \( v \) is adjacent to every vertex of \( T \). It follows that \( H \) is isomorphic to a minor of \( P_{2t} \), as desired. \( \square \)

Definition Let \( Q_1 \) be \( K_3 \). An arbitrary edge of \( Q_1 \) will be designated as base edge. For \( i \geq 2 \) the graph \( Q_i \) is constructed as follows: Now assume that \( Q_{i-1} \) has already been defined, and let \( Q_1 \) and \( Q_2 \) be two disjoint copies of \( Q_{i-1} \) with base edges \( u_1v_1 \) and \( u_2v_2 \),
Proof. This follows from Lemmas 2.1, 2.2 and 2.4.

A graph is outerplanar if it has a drawing in the plane (without crossings) such that every vertex is incident with the unbounded face. A graph is a near-triangulation if it is drawn in the plane in such a way that every face except possibly the unbounded one is bounded by a triangle.

Let $H$ and $G$ be graphs. If $G$ has an $H$ minor, then to every vertex $u$ of $H$ there corresponds a connected subgraph of $G$, called the node of $u$.

**Lemma 2.3.** Let $H$ be a 2-connected outerplanar near-triangulation with $k$ triangles. Then $H$ is isomorphic to a minor of $Q_k$. Furthermore, the minor inclusion can be chosen in such a way that for every edge $a_1a_2 \in E(H)$ incident with the unbounded face and for every $i \in \{1, 2\}$, the vertex $w_i$ belongs to the node of $a_i$, where $w_1w_2$ is the base edge of $Q_k$.

**Proof.** We proceed by induction on $k$. The lemma clearly holds when $k = 1$, and so we may assume that $H$ has at least two triangles and that the lemma holds for graphs with fewer than $k$ triangles. The edge $a_1a_2$ belongs to a unique triangle, say $a_1a_2c$. The triangle $a_1a_2c$ divides $H$ into two near-triangulations $H_1$ and $H_2$, where the edge $a_1c$ is incident with the unbounded face of $H_i$. Let $Q_1, Q_2, u_1, v_1, u_2, v_2, w_1, w_2$ be as in the definition of $Q_k$. By the induction hypothesis the graph $H_i$ is isomorphic to a minor of $Q_i$ in such a way that the vertex $u_i$ belongs to the node of $a_i$ and the vertex $v_i$ belongs to the node of $c$. It follows that $H$ is isomorphic to $Q_k$ in such a way that $w_i$ belongs to the node of $a_i$. \hfill \Box

**Lemma 2.4.** Let $H$ be a graph that has no $K_4$ or $K_{2,3}$ minor. Then there exists an integer $k$ such that $H$ is isomorphic to a minor of $Q_k$.

**Proof.** It is well-known [6, Exercise 23] that the hypotheses of the lemma imply that $H$ is outerplanar. We may assume, by replacing $H$ by a graph with an $H$ minor, that $H$ is a 2-connected outerplanar near-triangulation. The lemma now follows from Lemma 2.3. \hfill \Box

**Corollary 2.5.** Let $H$ be a graph that has no $K_4, K_{2,3}, C_{3,2}$, or $A$ minor. Then there exists an integer $k$ such that $H$ is isomorphic to a minor of $P_k$ and $H$ is isomorphic to a minor of $Q_k$.

**Proof.** This follows from Lemmas 2.1, 2.2 and 2.4. \hfill \Box

**Proof of Theorem 1.4, assuming Theorem 1.3.** To prove the “if” part notice that $P_k$ and $Q_k$ are 2-connected and have large path-width when $k$ is large, because $Q_k$ has a $CT_{k-1}$ minor. There is no vertex $v$ in $A$ such that $A \backslash v$ is acyclic. So, $A$ and $C_{3,2}$ are not minors of $P_k$ for any $k$. The graph $Q_k$ is outerplanar, so $K_4$ and $K_{2,3}$ are not minors of $Q_k$ for any positive integer $k$. This means $g(H) \geq 3$ for $H \in \{K_4, K_{2,3}, C_{3,2}, A\}$. This proves the “if” part.

To prove the “only if” part, if $H$ has no $K_4, K_{2,3}, C_{3,2}$ or $A$ minor, then by Corollary 2.5 $H$ is a minor of both $P_k$ and $Q_k$ for some $k$. Then $g(H) \leq 2$ by Theorem 1.3. \hfill \Box
3 A Special Tree-decomposition

In this section we review properties of tree-decompositions established in [3, 9, 12]. The proof of the following easy lemma can be found, for instance, in [12].

Lemma 3.1. Let \((T, Y)\) be a tree-decomposition of a graph \(G\), and let \(H\) be a connected subgraph of \(G\) such that \(V(H) \cap Y_t \neq \emptyset \neq V(H) \cap Y_{t'}\), where \(t, t' \in V(T)\). Then \(V(H) \cap Y_t \neq \emptyset\) for every \(t \in V(T)\) on the path between \(t\) and \(t'\) in \(T\).

A tree-decomposition \((T, Y)\) of a graph \(G\) is said to be linked if

(W3) for every two vertices \(t_1, t_2\) of \(T\) and every positive integer \(k\), either there are \(k\) disjoint paths in \(G\) between \(Y_{t_1}\) and \(Y_{t_2}\), or there is a vertex \(t\) of \(T\) on the path between \(t_1\) and \(t_2\) such that \(|Y_t| < k\).

It is worth noting that, by Lemma 3.1, the two alternatives in (W3) are mutually exclusive. The following is proved in [12].

Lemma 3.2. If a graph \(G\) admits a tree-decomposition of width at most \(w\), where \(w\) is some integer, then \(G\) admits a linked tree-decomposition of width at most \(w\).

Let \((T, Y)\) be a tree-decomposition of a graph \(G\), let \(t_0 \in V(T)\), and let \(B\) be a component of \(T\setminus t_0\). We say that a vertex \(v \in Y_{t_0}\) is \(B\)-tied if \(v \in Y_t\) for some \(t \in V(B)\). We say that a path \(P\) in \(G\) is \(B\)-confined if \(|V(P)| \geq 3\) and every internal vertex of \(P\) belongs to \(\bigcup_{t \in V(B)} Y_t - Y_{t_0}\). We wish to consider the following three properties of \((T, Y)\):

(W4) if \(t, t'\) are distinct vertices of \(T\), then \(Y_t \neq Y_{t'}\),

(W5) if \(t_0 \in V(T)\) and \(B\) is a component of \(T\setminus t_0\), then \(\bigcup_{t \in V(B)} Y_t - Y_{t_0} \neq \emptyset\),

(W6) if \(t_0 \in V(T)\), \(B\) is a component of \(T\setminus t_0\), and \(u, v\) are \(B\)-tied vertices in \(Y_{t_0}\), then there is a \(B\)-confined path in \(G\) between \(u\) and \(v\).

The following strengthening of Lemma 3.2 is proved in [9].

Lemma 3.3. If a graph \(G\) has a tree-decomposition of width at most \(w\), where \(w\) is some integer, then it has a tree-decomposition of width at most \(w\) satisfying (W1)-(W6).

We need one more condition, which we now introduce. Let \(T\) be a tree. If \(t, t' \in V(T)\), then by \(T[t, t']\) we denote the set of vertices belonging to the unique path in \(T\) from \(t\) to \(t'\). A triad in \(T\) is a triple \(t_1, t_2, t_3\) of vertices of \(T\) such that there exists a vertex \(t\) of \(T\), called the center, such that \(t_1, t_2, t_3\) belong to different components of \(T\setminus t\). Let \((T, W)\) be a tree-decomposition of a graph \(G\), and let \(t_1, t_2, t_3\) be a triad in \(T\). The torso of \((T, W)\) at \(t_1, t_2, t_3\) is the subgraph of \(G\) induced by the set \(\bigcup W_t\), the union taken over all vertices \(t \in V(T)\) such that either \(t \in \{t_1, t_2, t_3\}\), or for all \(i \in \{1, 2, 3\}\), \(t\) belongs to the component of \(T\setminus t_i\) containing the center of \(t_1, t_2, t_3\). We say that the
triad \( t_1, t_2, t_3 \) is \( W \)-separable if, letting \( X = W_{t_1} \cap W_{t_2} \cap W_{t_3} \), the graph obtained from the torso of \((T, W)\) at \( t_1, t_2, t_3 \) by deleting \( X \) can be partitioned into three disjoint non-null graphs \( H_1, H_2, H_3 \) in such a way that for all distinct \( i, j \in \{1, 2, 3\} \) and all \( t \in T[t_1, t_0] \), 
\[ |V(H_i) \cap W_t| \geq |V(H_i) \cap W_{t_i}| = |W_{t_i} - X|/2 \geq 1. \] 
(Let us remark that this condition implies that \( |W_{t_i}| = |W_{t_2}| = |W_{t_3}| \) and \( V(H_i) \cap W_t = \emptyset \) for \( i = 1, 2, 3 \).) The last property of a tree-decomposition \((T, W)\) that we wish to consider is

\[(W7) \text{ if } t_1, t_2, t_3 \text{ is a } W \text{-separable triad in } T \text{ with center } t, \text{ then there exists an integer } i \in \{1, 2, 3\} \text{ with } W_{t_i} \cap W_t = (W_{t_i} \cap W_{t_2} \cap W_{t_3}) \neq \emptyset. \]

The following is proven in [3].

**Theorem 3.4.** If a graph \( G \) has a tree-decomposition of width at most \( w \), where \( w \) is some integer, then it has a tree-decomposition of width at most \( w \) satisfying (W1)-(W7).

This theorem is used to prove Theorem 1.3 in Section 7.

### 4 Cascades

In this section we introduce “cascades”, our main tool. The main result of this section, Lemma 4.1, states that in any tree-decomposition with no duplicate bags of bounded width of a graph of big path-width there is an “injective” cascade of large height.

**Lemma 4.1.** Let \( p, w \) be two positive integers and let \( G \) be a graph of tree-width strictly less than \( w \) and path-width at least \( p \). Then for every tree-decomposition \((T, X)\) of \( G \) of width strictly less than \( w \), the path-width of \( T \) is at least \([p/w]\).

**Proof.** We will prove the contrapositive. Assume there exists a tree-decomposition \((T, X)\) of \( G \) of width < \( w \) such that the path-width of \( T \) is less than \([p/w]\). Because the path-width of \( T \) is less than \([p/w]\), there exists a path-decomposition \((Y_1, Y_2, ..., Y_s)\) of \( T \) with \( |Y_i| \leq [p/w] \) for all \( i \). We will construct a path-decomposition \((Z_1, Z_2, ..., Z_s)\) for \( G \) of width less than \( p \). Set \( Z_i = \bigcup_{y \in Y_i} X_y \) for every \( i \in \{1, 2, ..., s\} \). For every vertex \( v \in V(G) \), \( v \) belongs to at least one set \( X_t \) for some \( t \in V(T) \). The vertex \( t \) of the tree \( T \) must be in \( Y_l \) for some \( l \in \{1, 2, ..., s\} \), so \( v \in X_{t_l} \subseteq Z_l. \) Therefore, \( \bigcup Z_i = V(G) \). Similarly, for every edge \( uv \in E(G) \), there exists \( t \in V(T) \) such that \( u, v \in X_t \). Therefore, \( u, v \in Z_l \) for some \( l \in \{1, 2, ..., s\} \).

Now, if a vertex \( v \in V(G) \) belongs to both \( Z_a \) and \( Z_b \) for some \( a, b \in \{1, 2, ..., s\}, a < b \), we will show that \( v \in Z_c \) for all \( c \) such that \( a < c < b \). Let \( c \) be an arbitrary integer satisfying \( a < c < b \). The fact that \( v \in Z_a \) implies \( v \in X_{y_1} \) for some \( y_1 \in Y_a \). Similarly, \( v \in X_{y_2} \) for some \( y_2 \in Y_b \). Let \( H \) be the set of vertices of \( T \) on the path from \( y_1 \) to \( y_2 \). Since \( y_1 \in Y_a \) and \( y_2 \in Y_b \), \( H \cap Y_a \neq \emptyset \neq H \cap Y_b \). Hence, by Lemma 3.1 with \( H = T \) and \((T, Y)\) the path-decomposition \((Y_1, Y_2, ..., Y_s)\), we have \( H \cap Y_c \neq \emptyset \). Let \( t \in H \cap Y_c \), then \( v \in X_t \subseteq Z_c \). So \((Z_1, Z_2, ..., Z_s)\) is a path-decomposition of \( G \). Since the width of \((T, X)\) is less than \( w \), we have \(|X_y| \leq w \) for every \( y \in Y_i \), where \( i \in \{1, 2, ..., s\} \). Therefore, \(|Z_i| \leq w \cdot [p/w] \leq p \) for every \( i \in \{1, 2, ..., s\} \). Therefore, the width of \((Z_1, Z_2, ..., Z_s)\) is less than \( p \), so the path-width of \( G \) is less than \( p \), as desired. \( \square \)
Let $T, T'$ be trees. A \textit{homeomorphic embedding of $T$ into $T'$} is a mapping $\eta : V(T) \to V(T')$ such that

- $\eta$ is an injection, and
- if $tt_1, tt_2$ are edges of $T$ with a common end, and $P_t$ is the unique path in $T'$ with ends $\eta(t)$ and $\eta(t_i)$, then $P_1$ and $P_2$ are edge-disjoint.

We will write $\eta : T \leftrightarrow T'$ to denote that $\eta$ is a homeomorphic embedding of $T$ into $T'$. Since $CT_a$ has maximum degree at most three, the following lemma follows from [8, Lemma 6].

\textbf{Lemma 4.2.} \textit{Let $T$ be a forest of path-width at least $a \geq 1$. Then there exists a homeomorphic embedding $CT_{a-1} \leftrightarrow T$.}

For every integer $h \geq 1$ we will need a specific type of tree, which we will denote by $T_h$. The tree $T_h$ is obtained from $CT_h$ by subdividing every edge not incident with a vertex of degree one exactly once, and adding a new vertex $r'$ of degree one adjacent to the root $r$ of $CT_h$. The vertices of $T_h$ of degree three will be called \textit{major}, and all the other vertices will be called \textit{minor}. We say that $r$ is the \textit{major root} of $T_h$ and that $r'$ is the \textit{minor root} of $T_h$. Each major vertex at distance $2k$ from $r$ has \textit{height} $k$, and each minor vertex at distance $2k$ from $r'$ has \textit{height} $k$.

If $t$ belongs to the unique path in $T_h$ from $r'$ to a vertex $t' \in V(T_h)$, then we say that $t'$ is a \textit{descendant} of $t$ and that $t$ is an \textit{ancestor} of $t'$. If, moreover, $t$ and $t'$ are adjacent, then we say that $t$ is the \textit{parent} of $t'$ and that $t'$ is a \textit{child} of $t$. Thus every major vertex $t$ has exactly three minor neighbors. Exactly one of those neighbors is an ancestor of $t$. The other two neighbors are descendants of $t$. We will assume that one of the two descendant neighbors is designated as the \textit{left neighbor} and the other as the \textit{right neighbor}. Let $t_0, t_1, t_2$ be the parent, left neighbor and right neighbor of $t$, respectively.

We say that the ordered triple $(t_0, t_1, t_2)$ is the \textit{trinity at $t$}. In case we want to emphasize that the trinity is at $t$, we use the notation $(t_0(t), t_1(t), t_2(t))$.

Let $\eta : T \leftrightarrow T'$. We define $sp(\eta)$, the \textit{span of $\eta$}, to be the set of vertices $t \in V(T')$ that lie on the path from $\eta(t_1)$ to $\eta(t_2)$ for some vertices $t_1, t_2 \in V(T)$.

Let $s > 0$ be an integer and let $(T, X)$ be a tree-decomposition of a graph $G$. By a \textit{cascade of height $h$ and size $s$ in $(T, X)$} we mean a homeomorphic embedding $\eta : T_h \leftrightarrow T$ such that $|X_{\eta(t)}| = s$ for every minor vertex $t \in V(T_h)$ and $|X_t| \geq s$ for every $t$ in the span of $\eta$.

\textbf{Lemma 4.3.} \textit{For any positive integer $h$ and nonnegative integers $a, k$, the following holds. Let $m = (a+2)h + a$. Let $(T, X)$ be a tree-decomposition of a graph $G$ and let $\phi : CT_m \leftrightarrow T$ be a homeomorphic embedding such that $|X_t| \geq k$ for all $t \in sp(\phi)$. If for every $t \in V(CT_m)$ at height $l \leq m - a$ there exist a descendant $t'$ of $t$ at height $l + a$ and a vertex $r \in T[\phi(t), \phi(t')]$ such that $|X_r| = k$, then there exists a cascade $\eta$ of height $h$ and size $k$ in $(T, X)$.}

\textit{Proof.} By hypothesis there exist a vertex $x_0 \in V(CT_m)$ at height $a$ and a vertex $u_0 \in V(T)$ on the path from the image under $\phi$ of the root of $CT_m$ to $\phi(x_0)$ such that $|X_{u_0}| = k$. Let
$x$ be a child of $x_0$, and let $x_1$ and $x_2$ be the children of $x$. By hypothesis there exist, for $i = 1, 2$, a vertex $y_i \in V(CT_m)$ at height $2a + 2$ that is a descendant of $x_i$ and a vertex $u_i \in T[\phi(x_i), \phi(y_i)]$ such that $|X_{u_i}| = k$. Let $r$ be the major root of $T_1$, and let $(t_0, t_1, t_2)$ be its trinity. We define $\eta_1 : T_1 \rightarrow T$ by $\eta_1(t_i) = u_i$ for $i = 0, 1, 2$ and $\eta_1(r) = \phi(x)$. Then $\eta_1$ is a cascade of height one and size $k$ in $(T, X)$. If $h = 1$, then $\eta_1$ is as desired, and so we may assume that $h > 1$.

Assume now that for some positive integer $l < h$ we have constructed a cascade $\eta_l : T_l \rightarrow T$ of height $l$ and size $k$ in $(T, X)$ such that for every leaf $t_0$ of $T_l$ other than the minor root there exists a vertex $x_0 \in V(CT_m)$ at height $(a + 2)l + a$ such that the image under $\eta_l$ of every vertex on the path in $T_l$ from the minor root to $t_0$ belongs to the path in $T$ from the image under $\phi$ of the root of $CT_m$ to $\phi(x_0)$. Our objective is to extend $\eta_l$ to a cascade $\eta_{l+1}$ of height $l + 1$ and size $k$ in $(T, X)$ with the same property. To that end let $\eta_{l+1}(t) = \eta_l(t)$ for all $t \in V(T_l)$, let $t_0$ be a leaf of $T_l$ other than the minor root and let $x_0$ be as earlier in the paragraph. Let $x$ be a child of $x_0$, and let $x_1$ and $x_2$ be the children of $x$. By hypothesis there exist, for $i = 1, 2$, a vertex $y_i \in V(CT_m)$ at height $(a + 2)(l + 1) + a$ that is a descendant of $x_i$ and a vertex $u_i \in T[\phi(x_i), \phi(y_i)]$ such that $|X_{u_i}| = k$. Let $r$ be the child of $t_0$ in $T_{l+1}$, and let $(t_0, t_1, t_2)$ be its trinity. We define $\eta_{l+1}(t_i) = u_i$ for $i = 1, 2$ and $\eta_{l+1}(r) = \phi(x)$. This completes the definition of $\eta_{l+1}$.

Now $\eta_h$ is as desired.

Lemma 4.4. For any two positive integers $h$ and $w$, there exists a positive integer $p = p(h, w)$ such that if $G$ is a graph of path-width at least $p$, then in any tree-decomposition of $G$ of width less than $w$, there exists a cascade of height $h$.

Proof. Let $a_{w+1} = 0$, and for $k = w, w - 1, \ldots, 0$ let $a_k = (a_{k+1} + 2)h + a_{k+1}$, and let $p = w(a_0 + 1)$. We claim that $p$ satisfies the conclusion of the lemma. To see that let $(T, X)$ be a tree-decomposition of $G$ of width less than $w$. Let $k \in \{0, 1, \ldots, w + 1\}$ be the maximum integer such that there exists a homeomorphic embedding $\phi : CT_{a_k} \rightarrow T$ satisfying $|X_t| \geq k$ for all $t \in sp(\phi)$. Such an integer exists, because $k = 0$ satisfies those requirements by Lemmas 4.1 and 4.2, and it satisfies $k \leq w$, because the width of $(T, X)$ is less than $w$. The maximality of $k$ implies that for the integers $h, k$ and $a_{k+1}$ the hypothesis of Lemma 4.3 is satisfied. Thus the lemma follows from Lemma 4.3.

Let $(T, X)$ be a tree-decomposition of a graph $G$, and let $\eta : T_h \rightarrow T$ be a cascade of height $h$ and size $s$ in $(T, X)$. We say that $\eta$ is injective if there exists $I \subseteq V(G)$ such that $|I| < s$ and $X_{\eta(t)} \cap X_{\eta(t')} = I$ for every two distinct vertices $t, t' \in V(T_h)$. We call this set $I$ the common intersection set of $\eta$.

Lemma 4.5. Let $a, b, s, w$ be positive integers and let $k$ be a nonnegative integer. Let $(T, X)$ be a tree-decomposition of a graph $G$ of width strictly less than $w$. Let $h = 2(a + 2)w + 2) + b$. If there is a cascade $\eta$ of height $h$ and size $s$ in $(T, X)$ such that $|\bigcap_{t \in V(T_h)} X_{\eta(t)}| \geq k$, then either there is a cascade $\eta'$ of height $a$ and size $s$ in $(T, X)$ such that $|\bigcap_{t \in V(T_h)} X_{\eta'(t)}| \geq k + 1$ or there is an injective cascade $\eta'$ of height $b$, size $s + k$ and common intersection set of size $k$ in $(T, X)$.

Proof. We may assume that
(*) there does not exist a cascade \( \eta' \) of height \( a \) and size \( s + k \) in \((T, X)\) such that \(|\nabla_{\eta'}| \geq k + 1\).

Let \( F = \bigcap_{t \in V(T_h)} X_{\eta(t)} \). By (*), \(|F| = k\). We claim the following.

**Claim 4.5.1.** For every vertex \( t \in V(T_h) \) at height \( l \leq h - (a + 2)w \) and every \( u \in X_{\eta(t)} - F \) there exists a descendant \( t' \in V(T_h) \) of \( t \) at height at most \( l + a + 2 \) such that \( u \not\in X_{\eta(t')} \).

To prove the claim let \( u \in X_{\eta(t)} - F \). By (*) in the subtree of \( T_h \) consisting of \( t \) and its descendants there is a vertex \( t' \) of height at most \( l + a + 2 \) such that \( u \not\in X_{\eta(t')} \). This proves the claim.

We use the previous claim to deduce the following generalization.

**Claim 4.5.2.** For every vertex \( t \in V(T_h) \) at height \( l \leq h - (a + 2)w \) there exists a descendant \( t' \in V(T) \) of \( t \) at height at most \( l + (a + 2)w \) such that \( X_{\eta(t)} \cap X_{\eta(t')} = F \).

To prove the claim let \( X_{\eta(t)} \backslash F = \{ u_1, u_2, \ldots, u_p \} \), where \( p \leq w \). By Claim 4.5.1 there exists a descendant \( t_1 \in V(T) \) of \( t \) at height at most \( l + a + 2 \) such that \( u_1 \not\in X_{\eta(t')} \). By another application of Claim 4.5.1 there exists a descendant \( t_2 \in V(T) \) of \( t_1 \) at height at most \( l + 2(a + 2) \) such that \( u_2 \not\in X_{\eta(t')} \). By (W2) \( u_1 \not\in X_{\eta(t')} \). By continuing to argue in the same way we finally arrive at a vertex \( t_p \) that is a descendant of \( t \) at height at most \( l + (a + 2)p \) such that \( X_{\eta(t)} \cap X_{\eta(t_p)} = F \). Thus \( t_p \) is as desired. This proves the claim.

Let \( x_0 \in V(T_h) \) be the minor root of \( T_h \). By Claim 4.5.2 and (W2) there exists a major vertex \( x \in V(T) \) at height at most \( (a + 2)w + 1 \) such that \( X_{\eta(x_0)} \cap X_{\eta(x)} = F \). Let \( y_1 \) and \( y_2 \) be the children of \( x \). By Claim 4.5.2 and (W2) there exists, for \( i = 1, 2 \), a minor vertex \( x_i \in V(T_h) \) at height at most \( 2(a + 2)w + 2 \) that is a descendant of \( y_i \) and such that \( X_{\eta(x_i)} \cap X_{\eta(x)} = F \). Let \( r \) be the major root of \( T_1 \), and let \( (t_0, t_1, t_2) \) be its trinity. We define \( \eta_i : T_i \rightarrow T \) by \( \eta_i(t_i) = \eta(x_i) \) for \( i = 0, 1, 2 \) and \( \eta_1(r) = \eta(x) \). Then \( \eta_1 \) is an injective cascade of height one and size \( s + k \) in \((T, X)\) with common intersection set \( F \).

If \( b = 1 \), then \( \eta_1 \) is as desired, and so we may assume that \( b > 1 \).

Assume now that for some positive integer \( l < b \) we have constructed an injective cascade \( \eta_l : T_l \rightarrow T \) of height \( l \) and size \( s + k \) with common intersection set \( F \) in \((T, X)\) such that for every leaf \( t_0 \) of \( T_l \) other than the minor root there exists a vertex \( x_0 \in V(T_h) \) at height \( (2(a + 2)w + 2)l \) such that the image under \( \eta_l \) of every vertex on the path in \( T_l \) from the minor root to \( t_0 \) belongs to the path in \( T \) from the image under \( \eta \) of the root of \( T_h \) to \( \eta(x_0) \). Our objective is to extend \( \eta_l \) to an injective cascade \( \eta_{l+1} \) of height \( l + 1 \), size \( s + k \), and common intersection set \( F \) in \((T, X)\) with the same property. To that end let \( \eta_{l+1}(t) = \eta_l(t) \) for all \( t \in V(T_l) \), let \( t_0 \) be a leaf of \( T_1 \) other than the minor root, and let \( x_0 \) be as earlier in the paragraph. By Claim 4.5.2 and (W2) there exists a descendant \( x \) of \( x_0 \) at height at most \((2(a + 2)w + 2)l + (a + 2)w + 1 \) such that \( x \) is major and \( X_{\eta_l(t_0)} \cap X_{\eta(x)} = F \). Let \( y_1 \) and \( y_2 \) be the children of \( x \). By Claim 4.5.2 and (W2) there exists, for \( i = 1, 2 \), a minor vertex \( x_i \in V(T_h) \) at height at most \((2(a + 2)w + 2)(l + 1) \) that is a descendant of \( y_i \) and such that \( X_{\eta(x_i)} \cap X_{\eta(x)} = F \). Let \( r \) be the child of \( t_0 \) in \( T_{l+1} \), and let \((t_0, t_1, t_2)\) be its trinity. We define \( \eta_{l+1}(t_i) = \eta(x_i) \) for \( i = 1, 2 \) and \( \eta_{l+1}(r) = \eta(x) \). This completes the definition of \( \eta_{l+1} \).
Now \( \eta_0 \) is as desired. \( \square \)

**Lemma 4.6.** For any two positive integers \( h \) and \( w \), there exists a positive integer \( p = p(h, w) \) such that if \( G \) is a graph of tree-width less than \( w \) and path-width at least \( p \), then in any tree-decomposition \( (T, X) \) of \( G \) that has width less than \( w \) and satisfies (W4), there is an injective cascade of height \( h \).

**Proof.** Let \( a_w = 0 \), and for \( k = w - 1, \ldots, 0 \) let \( a_k = (2(a_{k+1} + 2)w + 2)h \). Let \( p \) be the integer in Lemma 4.4 for input integers \( a_0 \) and \( w \). We claim that \( p \) satisfies the conclusion of the lemma. To see that let \( (T, X) \) be a tree-decomposition of \( G \) of width less than \( w \) satisfying (W4). By Lemma 4.4, there exists a cascade \( \eta \) of height \( a_0 \) in \( (T, X) \). Let \( k \in \{0, 1, \ldots, w\} \) be the maximum integer such that there exists a cascade \( \eta' : T_{a_k} \hookrightarrow T \) satisfying \( |\bigcap_{t \in V(T_{a_k})} X_{\eta'(t)}| \geq k \). Such an integer exists, because \( k = 0 \) satisfies those requirements and \( k < w \) because of (W4) and because the width of \( (T, X) \) is less than \( w \). The maximality of \( k \) implies that there does not exist a cascade \( \eta'' : T_{a_{k+1}} \hookrightarrow T \) satisfying \( |\bigcap_{t \in V(T_{a_{k+1}})} X_{\eta''(t)}| \geq k + 1 \). Thus the lemma follows from Lemma 4.5. \( \square \)

## 5 Ordered Cascades

The main result of this section, Theorem 5.3, states that every 2-connected graph of big path-width and bounded tree-width admits a tree-decomposition of bounded width and a cascade with linkages that are minimal.

Let \( (T, X) \) be a tree-decomposition of a graph \( G \), and let \( \eta \) be an injective cascade in \( (T, X) \) with common intersection set \( I \). Assume the size of \( \eta \) is \(|I| + s\). Then we say \( \eta \) is ordered if for every minor vertex \( t \in V(T_h) \) there exists a bijection \( \xi_t : \{1, 2, \ldots, s\} \rightarrow X_{\eta(t)} - I \) such that for every major vertex \( t_0 \) with trinity \( (t_1, t_2, t_3) \), there exist \( s \) disjoint paths \( P_1, P_2, \ldots, P_s \) in \( G \setminus I \) such that the path \( P_i \) has ends \( \xi_{t_1}(i) \) and \( \xi_{t_2}(i) \), and there exists \( s \) disjoint paths \( Q_1, Q_2, \ldots, Q_s \) in \( G \setminus I \) such that the path \( Q_i \) has ends \( \xi_{t_1}(i) \) and \( \xi_{t_3}(i) \). In that case we say that \( \eta \) is an ordered cascade with orderings \( \xi_t \). We say that the set of paths \( P_1, P_2, \ldots, P_s \) is a left \( t_0 \)-linkage with respect to \( \eta \), and that the set of paths \( Q_1, Q_2, \ldots, Q_s \) is a right \( t_0 \)-linkage with respect to \( \eta \).

We will need to fix a left and a right \( t_0 \)-linkage for every major vertex \( t_0 \in V(T_h) \); when we do so we will indicate that by saying that \( \eta \) is an ordered cascade in \( (T, X) \) with orderings \( \xi_t \) and specified linkages, and we will refer to the specified linkages as the left specified \( t_0 \)-linkage and the right specified \( t_0 \)-linkage. We will denote the left specified \( t_0 \)-linkage by \( P_1(t_0), P_2(t_0), \ldots, P_s(t_0) \) and the right specified \( t_0 \)-linkage by \( Q_1(t_0), Q_2(t_0), \ldots, Q_s(t_0) \). We say that the specified \( t_0 \)-linkages are minimal if for every set of disjoint paths \( P_1, P_2, \ldots, P_s \) in \( G \setminus I \) from \( X_{\eta(t_1)} - I \) to \( X_{\eta(t_2)} - I \) such that \( \xi_{t_1}(i) \) is an end of \( P_i \) (let the other end be \( p_i \)) and every set of disjoint paths \( Q_1, Q_2, \ldots, Q_s \) in \( G \setminus I \) from \( X_{\eta(t_1)} - I \) to \( X_{\eta(t_3)} - I \) such that \( \xi_{t_3}(i) \) is an end of \( Q_i \) (let the other end be \( q_i \)) we have

\[
\left| E \left( \bigcup_{i \in \{1, 2, \ldots, s\}} (x_i P_i \cup x_i Q_i q_i) \right) \right| \geq E \left( \bigcup_{i \in \{1, 2, \ldots, s\}} (y_i P_i(t_0) \xi_{t_2}(i) \cup y_i Q_i(t_0) \xi_{t_3}(i)) \right),
\]

(1)

where the unions are taken over \( i \in \{1, 2, \ldots, s\} \), \( x_i \) is the first vertex from \( \xi_{t_1}(i) \) that \( P_i \) departs from \( Q_i \), and \( y_i \) is the first vertex from \( \xi_{t_3}(i) \) that \( P_i(t_0) \) departs from \( Q_i(t_0) \).
**Lemma 5.1.** Let \( h \) and \( s \) be two positive integers, and let \( \eta : T_h \hookrightarrow T \) be an injective cascade of height \( h \) and size \( s \) in a linked tree-decomposition \((T, X)\) of a graph \( G \). Then the cascade \( \eta \) can be turned into an ordered cascade with specified \( t_0 \)-linkages that are minimal for every major vertex \( t_0 \in V(T_h) \).

**Proof.** Let \( s' := s - |I| \). To show that \( \eta \) can be made ordered let \( r \) be the minor root of \( T_h \), let \( \xi_r : \{1, 2, \ldots, s'\} \to X_{\eta(r)} - I \) be arbitrary, assume that for some integer \( l \in \{0, 1, \ldots, h - 1\} \) we have already constructed \( \xi_t : \{1, 2, \ldots, s'\} \to X_{\eta(t)} - I \) for all minor vertices \( t \in V(T_h) \) at height at most \( l \), let \( t \in V(T_h) \) be a minor vertex at height exactly \( l \), let \( t_0 \) be its child, and let \((t, t_1, t_2)\) be the trinity at \( t_0 \). By condition (W3) there exist \( s' \) disjoint paths \( P_1, P_2, \ldots, P_{s'} \) in \( G \setminus I \) from \( X_{\eta(t)} - I \) to \( X_{\eta(t_1)} - I \) and \( s' \) disjoint paths \( Q_1, Q_2, \ldots, Q_{s'} \) in \( G \setminus I \) from \( X_{\eta(t)} - I \) to \( X_{\eta(t_2)} - I \). We may assume that \( \xi_x(i) \) is an end of \( P_i \) and \( Q_i \) and we define \( \xi_{t_1}(i) \) and \( \xi_{t_2}(i) \) to be their other ends, respectively. We may also assume that these paths satisfy the minimality condition (1). It follows that \( \eta \) is an ordered cascade with orderings \( \xi_t \) and specified \( t_0 \)-linkages that are minimal for every major vertex \( t_0 \in V(T_h) \).

Let \( h, h' \) be integers. We say that a homeomorphic embedding \( \gamma : T_{h'} \hookrightarrow T_h \) is monotone if

- \( t \) is a major vertex of \( T_{h'} \) with trinity \((t_1, t_2, t_3)\), then \( \gamma(t_2) \) is the left neighbor of \( \gamma(t) \) and \( \gamma(t_3) \) is the right neighbor of \( \gamma(t) \), and

- the image under \( \gamma \) of the minor root of \( T_{h'} \) is the minor root of \( T_h \).

**Lemma 5.2.** For every two integers \( a \geq 1 \) and \( k \geq 1 \) there exists an integer \( h = h(a, k) \) such that the following holds. Color the major vertices of \( T_h \) using \( k \) colors. Then there exists a monotone homeomorphic embedding \( \eta : T_a \hookrightarrow T_h \) such that the major vertices of \( T_a \) map to major vertices of the same color in \( T_h \).

**Proof.** Let \( c \) be one of the colors. We will prove by induction on \( k \) and subject to that by induction on \( b \) that there is a function \( h = g(a, b, k) \) such that there is either a monotone homeomorphic embedding \( \eta : T_a \hookrightarrow T_h \) such that the major vertices of \( T_a \) map to major vertices of the same color in \( T_h \), or a monotone homeomorphic embedding \( \eta : T_b \hookrightarrow T_h \) such that the major vertices of \( T_b \) map to major vertices of color \( c \) in \( T_h \). In fact, we will show that \( g(a, b, 1) = a, g(a, 1, k+1) \leq g(a, a, k) \), and \( g(a, b+1, k+1) \leq g(a, b, k+1) + g(a, a, k) \).

The assertion holds for \( k = 1 \) by letting \( h = a \) and letting \( \eta \) be the identity mapping. Assume the statement is true for some \( k \geq 1 \), let the major vertices of \( T_h \) be colored using \( k+1 \) colors, and let \( c \) be one of the colors. If \( b = 1 \), then if \( T_h \) has a major vertex colored \( c \), then the second alternative holds; otherwise at most \( k \) colors are used and the assertion follows by induction on \( k \).

We may therefore assume that the assertion holds for some integer \( b \geq 1 \) and we must prove it for \( b+1 \). To that end we may assume that \( T_h \) has a major vertex \( t_0 \) colored \( c \) at height at most \( g(a, a, k) \), for otherwise the assertion follows by induction on \( k \). Let the trinity at \( t_0 \) be \((t_1, t_2, t_3)\). For \( i = 2, 3 \) let \( R_i \) be the subtree of \( T_h \) with minor root \( t_i \). If for some \( i \in \{2, 3\} \) there exists a monotone homeomorphic embedding \( T_a \hookrightarrow R_i \) such
that the major vertices of $T_a$ map to major vertices of the same color in $T_h$, then the statement holds. We may therefore assume that for $i \in \{2, 3\}$ there exists a monotone homeomorphic embedding $\eta_i : T_b^i \hookrightarrow R_i$ such that the major vertices of $T_b^i$ map to major vertices of color $c$, the major root of $T_{b+1}$ is $r_0$, the trinity at $r_0$ is $(r_1, r_2, r_3)$ and $T_b^i$ is the subtree of $T_{b+1} - \{r_0, r_1\}$ with minor root $r_i$. Let $\eta : T_{b+1} \hookrightarrow T_h$ be defined by $\eta(t) = \eta_i(t)$ for $t \in V(T_b^i)$, $\eta(r_0) = t_0$ and $\eta(r_1)$ is defined to be the minor root of $T_h$. Then $\eta : T_{b+1} \hookrightarrow T_h$ is as desired. This proves the existence of the function $g(a, b, k)$.

Now $h(a, k) = g(a, a, k)$ is as desired.

Let $G$ be a graph, let $v \in V(G)$ and for $i = 1, 2, 3$ let $P_i$ be a path in $G$ with ends $v$ and $v_i$ such that the paths $P_1, P_2, P_3$ are pairwise disjoint, except for $v$. Assume that at least two of the paths $P_i$ have length at least one. We say that $P_1 \cup P_2 \cup P_3$ is a tripod with center $v$ and feet $v_1, v_2, v_3$.

Let $(T, X)$ be a tree-decomposition of a graph $G$, and let $\eta : T_h \hookrightarrow T$ be an injective cascade in $(T, X)$ with common intersection set $I$. Let $t_0 \in V(T_h)$ be a major vertex, and let $(t_1, t_2, t_3)$ be the trinity at $t_0$. We define the $\eta$-torso at $t_0$ as the subgraph of $G$ induced by $\bigcup X_t - I$, where the union is taken over all $t$ in $V(T)$ such that the unique path in $T$ from $t$ to $\eta(t_0)$ does not contain $\eta(t_1), \eta(t_2)$, or $\eta(t_3)$ as an internal vertex.

Let $s > 0$ be an integer. Let $(T, X)$ be a tree-decomposition of a graph $G$, let $\eta : T_h \hookrightarrow T$ be an ordered cascade in $(T, X)$ of size $|I| + s$ and with orderings $\xi_t$, where $I$ is the common intersection set of $\eta$. Let $t_0 \in V(T_h)$ be a major vertex, let $(t_1, t_2, t_3)$ be the trinity at $t_0$, let $G'$ be the $\eta$-torso at $t_0$, and let $i, j \in \{1, 2, \ldots, s\}$ be distinct. We say that $t_0$ has property $A_{ij}$ in $\eta$ if there exist disjoint tripod $L_i, L_j$ in $G'$ such that for each $m \in \{i, j\}$ the tripod $L_m$ has feet $\xi_{t_1}(m), \xi_{t_2}(m_2), \xi_{t_3}(m_3)$ for some $m_2, m_3 \in \{i, j\}$.

We say that $t_0$ has property $B_{ij}$ in $\eta$ if there exist vertices $v_{x,y}$ for all $x \in \{i, j\}, y \in \{1, 2, 3\}$, and tripods $L_i, L_j$ in $G'$ with centers $c_i, c_j$ such that

- for each $y \in \{1, 2, 3\}, \{v_{i,y}, v_{j,y}\} = \{\xi_{t_y}(i), \xi_{t_y}(j)\}$
- for each $m \in \{i, j\}, L_m$ has feet $v_{m,1}, v_{m,2}, v_{m,3}$
- $L_i \cap L_j = c_iL_{i,v_{i,3}} \cap c_jL_{j,v_{j,2}}$ and it is a path that does not contain $c_i, c_j$.

We say that $t_0$ has property $C_{ij}$ in $\eta$ if there exist three pairwise disjoint paths $R_i, R_j, R_{i,j}$ and a path $R$ in $G'$ such that the ends of $R_i$ are $\xi_{t_1}(i)$ and $\xi_{t_2}(i)$, the ends of $R_j$ are $\xi_{t_1}(j)$ and $\xi_{t_3}(j)$, the ends of $R_{i,j}$ are $\xi_{t_2}(j)$ and $\xi_{t_3}(i)$, and $R$ is internally disjoint from $R_i, R_j, R_{i,j}$ and connects two of these three paths. We will denote these paths as $R_i(t_0), R_j(t_0), R_{i,j}(t_0), R(t_0)$ when we want to emphasize they are in the torso at the major vertex $t_0$.

We say that the path $P_i$ of a left or right $t_0$-linkage is confined if it is a subgraph of the $\eta$-torso at $t_0$.

Now let $\eta : T_h \hookrightarrow T$ be an ordered cascade in $(T, X)$ with orderings $\xi_t$ and specified linkages. Let $t_0 \in V(T_h)$ be a major vertex with trinity $(t_1, t_2, t_3)$, and let $P_1, P_2, \ldots, P_s$ be the left specified $t_0$-linkage. We define $A_{t_0}$ to be the set of integers $i \in \{1, 2, \ldots, s\}$ such that the path $P_i$ is confined, and we define $B_{t_0}$ in the same way but using the right specified $t_0$-linkage instead. Define $C_{t_0}$ as the set of all triples $(i, l, m)$ such that
\( i \in \{1, 2, \ldots, s\} \), the path \( P_i \) is not confined and when following \( P_i \) from \( \xi_{t_1}(i) \), it exits the \( \eta \)-torso at \( t_0 \) for the first time at \( \xi_{t_2}(l) \) and re-enters the \( \eta \)-torso at \( t_0 \) for the last time at \( \xi_{t_3}(m) \). Let \( D_{t_0} \) be defined similarly, but using the right \( t_0 \)-linkage instead. We call the sets \( A_{t_0}, B_{t_0}, C_{t_0} \) and \( D_{t_0} \) the confinement sets for \( \eta \) at \( t_0 \) with respect to the specified linkages.

Let \( A_{t_0} \) and \( B_{t_0} \) be the confinement sets for \( \eta \) at \( t_0 \). We say that \( t_0 \) has property \( C \) in \( \eta \) if \( s \) is even, \( A_{t_0} \) and \( B_{t_0} \) are disjoint and both have size \( s/2 \), and there exist disjoint paths \( R_1, R_2, \ldots, R_{3s/2} \) in \( G' \) in such a way that

- each \( R_i \) is a subpath of both the left specified \( t_0 \)-linkage and the right specified \( t_0 \)-linkage,
- for \( i \in A_{t_0} \), the path \( R_i \) has ends \( \xi_{t_1}(i) \) and \( \xi_{t_2}(i) \),
- for \( i \in B_{t_0} \), the path \( R_i \) has ends \( \xi_{t_1}(i) \) and \( \xi_{t_3}(i) \), and
- for \( i = s + 1, s + 2, \ldots, 3s/2 \), the path \( R_i \) has one end \( \xi_{t_2}(k) \) and the other and \( \xi_{t_3}(l) \) for some \( k \in B_{t_0} \) and \( l \in A_{t_0} \).

Let \((T, X)\) be a tree-decomposition of a graph \( G \), let \( \eta: T_h \hookrightarrow T \) be a cascade in \((T, X)\) and let \( \gamma: T_{h'} \hookrightarrow T_h \) be a monotone homeomorphic embedding. Then the composite mapping \( \eta':=\eta \circ \gamma: T_{h'} \hookrightarrow T \) is a cascade in \((T, X)\) of height \( h' \), and we will call it a subcascade of \( \eta \).

**Lemma 5.3.** Let \((T, X)\) be a tree-decomposition of a graph \( G \), let \( \eta: T_h \hookrightarrow T \) be an ordered cascade in \((T, X)\) with orderings \( \xi_t \), specified linkages and common intersection set \( I \), let \( \gamma: T_{h'} \hookrightarrow T_h \) be a monotone homeomorphic embedding, and let \( \eta':=\eta \circ \gamma: T_{h'} \hookrightarrow T \) be a subcascade of \( \eta \) of height \( h' \). Then for every major vertex \( t_0 \in V(T_{h'}) \)

(i) \( \eta' \) is an ordered cascade with orderings \( \xi_{t_0}(t) \) and common intersection set \( I \),

(ii) if the vertex \( \gamma(t_0) \) has property \( A_{ij} \) \((B_{ij}, C_{ij}, \text{resp.})\) in \( \eta \), then \( t_0 \) has property \( A_{ij} \) \((B_{ij}, C_{ij}, \text{resp.})\) in \( \eta' \).

Furthermore, the specified linkages for \( \eta' \) may be chosen in such a way that

(iii) \( (A_{t_0}, B_{t_0}, C_{t_0}, D_{t_0}) = (A_{\gamma(t_0)}, B_{\gamma(t_0)}, C_{\gamma(t_0)}, D_{\gamma(t_0)}) \),

(iv) the vertex \( t_0 \) has property \( C \) in \( \eta' \) if and only if \( \gamma(t_0) \) has property \( C \) in \( \eta \), and

(v) if the specified linkages for \( \eta \) are minimal, then the specified linkages for \( \eta' \) are minimal.

**Proof.** For each major vertex \( t \in V(T_{h'}) \) or \( t \in V(T_h) \) we denote its trinity by \((t_1(t), t_2(t), t_3(t))\). Assume \( t_0 \) is a major vertex of \( T_{h'} \). Let \( v_0 = \gamma(t_1(t_0)), v_1, \ldots, v_k = t_1(\gamma(t_0)) \) be the minor vertices on \( T_h[v_{0}, v_k] \). Let \( U \) be the union of the left (or right) linkage from \( X_{\gamma(v_i)} - I \) to \( X_{\gamma(v_{i+1})} - I \) for all \( i \in \{0, 1, \ldots, k - 1\} \) depending on whether \( v_{i+1} \) is a left (or right) neighbor of its parent. Let \( P \) be the left specified \( \gamma(t_0) \)-linkage and \( Q \) be the right
specified $\gamma(t_0)$-linkage. Then $U \cup P$ is a left $t_0$-linkage and $U \cup Q$ is a right $t_0$-linkage. We designate $U \cup P$ to be the left specified $t_0$-linkage and $U \cup Q$ to be the right specified $t_0$-linkage. It is easy to see that this choice satisfies the conclusion of the lemma.

Let $(T, X)$ be a tree-decomposition of a graph $G$, and let $\eta$ be an ordered cascade with specified linkages in $(T, X)$ of height $h$ and size $|I|+s$, where $I$ is the common intersection set. We say that $\eta$ is regular if there exist sets $A, B \subseteq \{1, 2, \ldots, s\}$, and sets $C$ and $D$ such that the confinement sets $A_{t_0}, B_{t_0}, C_{t_0}$ and $D_{t_0}$ satisfy $A_{t_0} = A$, $B_{t_0} = B$, $C_{t_0} = C$ and $D_{t_0} = D$ for every major vertex $t_0 \in V(T_h)$.

**Lemma 5.4.** For every two positive integers $a$ and $s$ there exists a positive integer $h = h(a, s)$ such that the following holds. Let $(T, X)$ be a linked tree-decomposition of a graph $G$. If there exists an injective cascade $\eta$ of height $h$ in $(T, X)$, then there exists a regular cascade $\eta' : T_a \rightarrow T$ of height $a$ in $(T, X)$ with specified $t_0$-linkages that are minimal for every major vertex $t_0 \in V(T_a)$ such that $\eta'$ has the same size and common intersection set as $\eta$.

**Proof.** Let $\eta$ be an injective cascade of size $|I|+s$ and height $h$ in $(T, X)$, where we will specify $h$ in a moment. By Lemma 5.1, $\eta$ can be turned into an ordered cascade with specified $t_0$-linkages that are minimal for every major vertex $t_0 \in V(T_h)$. For every major vertex $t_0 \in V(T_h)$, the number of possible quadruples $(A_{t_0}, B_{t_0}, C_{t_0}, D_{t_0})$ is a finite number $k = k(s)$ that depends only on $s$.

Consider each choice of $(A_{t_0}, B_{t_0}, C_{t_0}, D_{t_0})$ as a color; then by Lemma 5.2, there exists a positive integer $h = h(a, k)$ such that there exists a monotone homeomorphic embedding $\gamma : T_a \rightarrow T_h$ such that the quadruple $(A_{\gamma(t)}, B_{\gamma(t)}, C_{\gamma(t)}, D_{\gamma(t)})$ for $\eta$ is the same for every $t \in V(T_a)$. Now, let $\eta' = \eta \circ \gamma : T_a \rightarrow T$. Then $\eta'$ is as desired by Lemma 5.3.

The following is the main result of this section.

**Theorem 5.5.** For any two positive integers $a$ and $w$, there exists a positive integer $p = p(a, w)$ such that the following holds. Let $G$ be a 2-connected graph of tree-width less than $w$ and path-width at least $p$. Then $G$ has a tree-decomposition $(T, X)$ such that:

- $(T, X)$ has width less than $w$,
- $(T, X)$ satisfies (W1)–(W7), and
- for some $s$, where $2 \leq s \leq w$, there exists a regular cascade $\eta : T_a \rightarrow T$ of height $a$ and size $s$ in $(T, X)$ with specified $t_0$-linkages that are minimal for every major vertex $t_0 \in V(T_a)$.

**Proof.** Given positive integers $a$ and $w$ let $h$ be as in Lemma 5.4 and let $p = p(h, w)$ be as in Lemma 4.6. We claim that $p$ satisfies the conclusion of the theorem. To see that let $G$ be a graph of tree-width less than $w$ and path-width at least $p$. By Theorem 3.4, $G$ admits a tree-decomposition $(T, X)$ of width less than $w$ satisfying (W1)–(W7). By Lemma 4.6 there is an injective cascade of height $h$ in $(T, X)$. Let $s$ be the size of this cascade, then $s \leq w$. If $G$ is 2-connected, then $s \geq 2$. The last conclusion of the theorem follows from Lemma 5.4.

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6 Taming Linkages

Lemma 6.6, the main result of this section, states that there are essentially only two types of linkage.

Let \( s > 0 \) be an integer. Let \((T, X)\) be a tree-decomposition of a graph \( G \), let \( \eta : T_h \rightarrow T \) be an ordered cascade in \((T, X)\) of size \( |I| + s \) and with orderings \( \xi_t \), where \( I \) is the common intersection set of \( \eta \). Let \( t_0 \in V(T_h) \) be a major vertex, let \((t_1, t_2, t_3)\) be the trinity at \( t_0 \), let \( G' \) be the \( \eta \)-torso at \( t_0 \), and let \( i, j \in \{1, 2, \ldots, s\} \) be distinct. We say that \( t_0 \) has property \( AB_{ij} \) in \( \eta \) if there exist disjoint paths \( L_i, L_j \) and disjoint paths \( R_i, R_j \) in \( G' \) such that the two ends of \( L_m \) are \( \xi_{t_i}(m) \) and \( \xi_{t_j}(m) \) for each \( m \in \{i, j\} \) and the two ends of \( R_m \) are \( \xi_{t_i}(m) \) and \( \xi_{t_j}(m) \) for each \( m \in \{i, j\} \).

If \( P \) is a path and \( u, v \in V(P) \), then by \( uPv \) we denote the subpath of \( P \) with ends \( u \) and \( v \).

**Lemma 6.1.** Let \((T, X)\) be a tree-decomposition of a graph \( G \). Let \( \eta : T_1 \rightarrow T \) be an ordered cascade in \((T, X)\) with orderings \( \xi_t \) of height one and size \( s + |I| \), where \( I \) is the common intersection set. Let \( t_0 \) be the major vertex in \( T_1 \), and let \( i, j \in \{1, 2, \ldots, s\} \) be distinct. If \( t_0 \) has property \( AB_{ij} \) in \( \eta \), then \( t_0 \) has either property \( A_{ij} \) or property \( B_{ij} \) in \( \eta \).

**Proof.** Let \((t_1, t_2, t_3)\) be the trinity at \( t_0 \). Let \( G' \) be the \( \eta \)-torso at \( t_0 \). Since \( t_0 \) has property \( AB_{ij} \) in \( \eta \), there exist disjoint paths \( L_i, L_j \) and disjoint paths \( R_i, R_j \) in \( G' \) such that two endpoints of \( L_m \) are \( \xi_{t_i}(m) \) and \( \xi_{t_j}(m) \) for each \( m \in \{i, j\} \), and two endpoints of \( R_m \) are \( \xi_{t_i}(m) \) and \( \xi_{t_j}(m) \) for each \( m \in \{i, j\} \).

We may choose \( L_i, L_j \) and \( R_i, R_j \) such that \(|E(L_i) \cup E(L_j) \cup E(R_i) \cup E(R_j)|\) is as small as possible.

Let \( x_k = \xi_{t_i}(k) \) and \( z_k = \xi_{t_j}(k) \) for \( k \in \{i, j\} \). Starting from \( z_i \), let \( a \) be the first vertex where \( R_i \) meets \( L_i \) \( \cup \) \( L_j \), and starting from \( z_j \), let \( b \) be the first vertex where \( R_j \) meets \( L_i \) \( \cup \) \( L_j \). If \( a \) and \( b \) are not on the same path (one on \( L_i \) and the other on \( L_j \) ), then by considering \( L_i, L_j \) and the parts of \( R_i \) and \( R_j \) from \( z_i \) to \( a \) and from \( z_j \) to \( b \) we see that \( t_0 \) has property \( A_{ij} \) in \( \eta \).

If \( a \) and \( b \) are on the same path, then we may assume they are on \( L_i \). We may also assume that \( a \in L_i[y_k, b] \). Then following \( R_i \) from \( a \) away from \( z_i \), the paths \( R_i \) and \( L_i \) eventually split; let \( c \) be the vertex where the split occurs. In other words, \( c \) is such that \( aL_i \cap aR_i \) is a path and its length is maximum. Let \( d \) be the first vertex on \( cR_i \cap (L_i \cup L_j) \) \( - \) \( \{c\} \) when traveling on \( R_i \) from \( c \) to \( x_i \). If \( d \in V(L_i) \), then by replacing \( cL_id \) by \( cR_id \) we obtain a contradiction to the choice of \( L_i, L_j, R_i, R_j \). Thus \( d \in V(L_j) \).

Now \( L_i, L_j \) and the paths \( z_iR_id \) and \( z_jR_jb \) show that \( t_0 \) has property \( B_{ij} \) in \( \eta \).

Let \((T, X)\) be a tree-decomposition of a graph \( G \) and let \( \eta : T_h \rightarrow T \) be an injective cascade in \((T, X)\) of height \( h \) and size \( |I| + s \), where \( I \) is the common intersection set.

Let \( v \) be a vertex of \( T_h \) and let \( Y \) consist of \( \eta(v) \) and the vertex-sets of all components of \( T \setminus \eta(v) \) that do not contain the image under \( \eta \) of the minor root of \( T_h \). Let \( H \) be the subgraph of \( G \) induced by \( \bigcup_{t \in Y} X_t - I \). We will call \( H \) the outer graph at \( v \).

**Lemma 6.2.** Let \((T, X)\) be a tree-decomposition satisfying \((W6)\) of a graph \( G \) and let \( \eta : T_h \rightarrow T \) be an ordered cascade in \((T, X)\) of height \( h \) and size \( |I| + s \), where \( I \) is the
common intersection set. Let \( v \) be a minor vertex of \( T_h \) at height at most \( h - 1 \), let \( H \) be the outer graph at \( v \), and let \( x, y \in \mathcal{X}_{\eta(v)} \). Then there exists a path of length at least two with ends \( x \) and \( y \) and every internal vertex in \( V(H) - \mathcal{X}_{\eta(v)} \).

Proof. Let \( v_0 \) be the child of \( v \), let \( v_1 \) be a child of \( v_0 \), and let \( B \) be the component of \( T - \eta(v) \) that contains \( \eta(v_1) \). We show that \( x \) is \( B \)-tied. This is obvious if \( x \in I \), and so we may assume that \( x \notin I \). Since \( \eta \) is ordered, there exist \( s \) disjoint paths from \( \mathcal{X}_{\eta(v)} - I \) to \( \mathcal{X}_{\eta(v_1)} - I \) in \( G \setminus I \). It follows that each of the paths uses exactly one vertex of \( \mathcal{X}_{\eta(v)} - I \), and that vertex is its end. Let \( P \) be the one of those paths that ends in \( x \), and let \( x' \) be the neighbor of \( x \) in \( P \). The vertex \( x' \) exists, because \( \mathcal{X}_{\eta(v)} \cap \mathcal{X}_{\eta(v_1)} = I \). By (W1) there exists a vertex \( t \in V(T) \) such that \( x, x' \in X_t \). Since \( P - x \) is disjoint from \( \mathcal{X}_{\eta(v)} \), it follows from Lemma 3.1 applied to the path \( P - x \) and vertices \( t \) and \( \eta(v_1) \) of \( T \) that \( t \in V(B) \). Thus \( x \) is \( B \)-tied and the same argument shows that so is \( y \). Hence the lemma follows from (W6).

We will refer to a path as in Lemma 6.2 as a W6-path.

Let \( h, h' \) be integers. We say that a homeomorphic embedding \( \gamma : T_{h'} \hookrightarrow T_h \) is weakly monotone if for every two vertices \( t, t' \in V(T_{h'}) \)

- if \( t' \) is a descendant of \( t \) in \( T_{h'} \), then the vertex \( \gamma(t') \) is a descendant of \( \gamma(t) \) in \( T_h \)
- if \( t \) is a minor vertex of \( T_{h'} \), then the vertex \( \gamma(t) \) is minor in \( T_h \).

Let \((T, X)\) be a tree-decomposition of a graph \( G \), let \( \eta : T_h \hookrightarrow T \) be a cascade in \((T, X)\) and let \( \gamma : T_{h'} \hookrightarrow T_h \) be a weakly monotone homeomorphic embedding. Then the composite mapping \( \eta' := \eta \circ \gamma : T_{h'} \hookrightarrow T \) is a cascade in \((T, X)\) of height \( h' \), and we will call it a weak subcascade of \( \eta \).

Lemma 6.3. Let \( s \geq 2 \) be an integer, let \((T, X)\) be a tree-decomposition of a graph \( G \) satisfying (W6), and let \( \eta : T_5 \hookrightarrow T \) be a regular cascade in \((T, X)\) of height five and size \( |I| + s \) with specified linkages that are minimal, where \( I \) is the common intersection set of \( \eta \). Then either there exists a weak subcascade \( \eta' : T_1 \hookrightarrow T \) of \( \eta \) of height one such that the unique major vertex of \( T_1 \) has property \( A_{ij} \) or \( B_{ij} \) in \( \eta' \) for some distinct integers \( i, j \in \{1, 2, \ldots, s\} \), or the major root of \( T_5 \) has property \( C \) in \( \eta \).

Proof. We will either construct a weakly monotone homeomorphic embedding \( \gamma : T_1 \hookrightarrow T_5 \) such that in \( \eta' = \eta \circ \gamma \) the major root of \( T_1 \) will have property \( A_{ij} \) for some distinct \( i, j \in \{1, 2, \ldots, s\} \), or establish that the major root of \( T_5 \) has property \( C \) in \( \eta \). By Lemma 6.1 this will suffice.

Since \( \eta \) is regular, there exist sets \( A, B, C, D \) as in the definition of regular cascade. Let \( t_0 \) be the unique major vertex of \( T_1 \) and let \((t_1, t_2, t_3)\) be its trinity. Let \( u_0 \) be the major root of \( T_5 \) and let \((v_1, v_2, v_3)\) be its trinity. Let \( u_1 \) be the major vertex of \( T_5 \) of height one that is adjacent to \( v_3 \) and let \((v_3, v_4, v_5)\) be its trinity. Let us recall that for a major vertex \( u \) of \( T_5 \) we denote the paths in the specified left \( u \)-linkage by \( P_i(u) \) and the paths in the specified right \( u \)-linkage by \( Q_i(u) \). If there exist two distinct integers \( i, j \in A \cap B \), then the paths \( P_i(u_0), P_j(u_0), Q_i(u_0), Q_j(u_0) \) show that \( u_0 \) has property \( AB_{ij} \) in \( \eta \). Let \( \gamma : T_1 \hookrightarrow T_5 \)
be the homeomorphic embedding that maps $t_0, t_1, t_2, t_3$ to $u_0, v_1, v_2, v_5$, respectively. Then $\eta' = \eta \circ \gamma$ is as desired. We may therefore assume that $|A \cap B| \leq 1$.

For $i \in \{1, 2, \ldots, s\} - A$ the path $P_i(u_0)$ exits and re-enters the $\eta$-torso at $u_0$, and it does so through two distinct vertices of $X_{\eta(v_1)}$. But $|X_{\eta(v_1)} - I| = s$, and hence $|A| \geq s/2$. Similarly $|B| \geq s/2$. By symmetry we may assume that $|B| \geq |A|$. It follows that $|A| = \lfloor s/2 \rfloor$, and hence for $i \in \{1, 2, \ldots, s\} - A$ and every major vertex $w$ of $T_5$ the path $P_i(w)$ exits and re-enters the $\eta$-torso at $w$ exactly once. The set $C$ includes an element of the form $(i, l, m)$, which means that the vertices $\xi_{w_1}(i), \xi_{w_3}(l), \xi_{w_3}(m), \xi_{w_2}(i)$ appear on the path $P_i(w)$ in the order listed. Let $l_i := l, m_i := m, x_i(w) := \xi_{w_3}(l), y_i(w) := \xi_{w_3}(m), X_i(w) := \xi_{w_1}(i)P_i(w)x_i(w)$ and $Y_i(w) := y_i(w)P_i(w)\xi_{w_2}(i)$. Thus $X_i(w)$ and $Y_i(w)$ are subpaths of the $\eta$-torso at $w$. We distinguish two main cases.

**Main case 1:** $|A \cap B| = 1$. Let $j$ be the unique element of $A \cap B$. We claim that $B - A \neq \emptyset$. To prove the claim suppose for a contradiction that $B \subseteq A$. Thus $|B| = 1$, and since $|B| \geq |A|$ we have $|A| = 1$, and hence $s = 2$. We may assume, for the duration of this paragraph, that $A = B = \{1\}$. The paths $P_1(u_0), X_2(u_0), Y_2(u_0)$ are pairwise disjoint, because they are subgraphs of the specified left $u_0$-linkage. The path $Q_2(u_0)$ is unconfined, and hence it has a subpath $R$ joining $\xi_{v_2}(1)$ and $\xi_{v_2}(2)$ in the outer graph at $v_2$. It follows that $P_1(u_0) \cup R \cup Y_2(u_0)$ and $X_2(u_0)$ are disjoint paths from $X_{\eta(v_1)}$ to $X_{\eta(v_2)}$, and it follows from the minimality of the specified $u_0$-linkage that they form the specified right $u_0$-linkage, contrary to $1 \in A$. This proves the claim that $B - A \neq \emptyset$, and so we may select an element $i \in B - A$.

![Figure 1: First case of the construction of the path $R$.](image)

Let us assume as a case that either $l_i \in A$ or $l_i \notin B$. In this case we let $\gamma$ map $t_0, t_1, t_2, t_3$ to $u_0, v_1, v_2, v_5$, respectively, and we will prove that $t_0$ has property $AB_{ij}$ in $\eta'$. To that end we need to construct two pairs of disjoint paths. The first pair is $Q_i(u_0) \cup Q_i(u_1)$ and $Q_j(u_0) \cup Q_j(u_1)$. The second pair will consist of $P_j(u_0)$ and another path from $\xi_{v_1}(i)$ to $\xi_{v_2}(i)$ which is a subgraph of a walk that we are about to construct. It will consist of $X_i(u_0) \cup Y_i(u_0)$ and a walk $R$ in the outer graph of $v_3$ with ends $x_i(u_0)$ and $y_i(u_0)$. To construct the walk $R$ we will construct paths $R_1, R_2$ and a walk $R_3$, whose union will contain the desired walk $R$. If $l_i \in A$, then we let $R_1 := P_i(u_1)$. If $l_i \notin B$, then the path $Q_{l_i}(u_3)$ is unconfined, and hence includes a subpath $R_1$ from $x_i(u_0)$ to $X_{\eta(v_3)}$ that is a subgraph of the $\eta$-torso at $u_1$. We need to distinguish two subcases depending on whether $m_i \in B$. Assume first that $m_i \notin B$ and refer to Figure 1. Then similarly as above the
path $Q_{m_i}(u_i)$ is unconfined, and hence includes a subpath $R_3$ from $y_i(u_0)$ to $X_{y_i(u_3)}$ that is a subgraph of the $\eta$-torso at $u_1$, and we let $R_2$ be a W6-path in the outer graph at $v_4$ joining the ends of $R_1$ and $R_3$ in $X_{y_i(u_4)}$. This completes the subcase $m_i \notin B$, and so we may assume that $m_i \in B$. In this subcase we define $R_3 := Y_i(u_1) \cup Q_{m_i}(u_1)$ and we define $R_2$ as above. See Figure 2. This completes the case that either $l_i \in A$ or $l_i \notin B$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{Figure2}
\caption{Second case of the construction of the path $R$.}
\end{figure}

Next we consider the case $l_i \in B$ and $m_i \notin A - B$. We proceed similarly as in the previous paragraph, but with these exceptions: the homeomorphic embedding $\gamma$ will map $t_3$ to $v_4$, rather than $v_5$, the first pair of disjoint paths will now be $Q_i(u_0) \cup P_i(u_1)$ and $Q_j(u_0) \cup P_j(u_1)$, and for the second pair we define $R_1 = Q_i(u_1)$, $R_3 = X_{m_i}(u_1)$ if $m_i \notin A$ and $R_3 = Q_{m_i}(u_1)$ if $m_i \in B$, and $R_2$ will be a W6-path in the outer graph of $v_5$ joining the ends of $R_1$ and $R_3$.

Therefore assume that $l_i \in B - A$ and $m_i \in A - B$ for every $i \in B - A$. Let $u_2$ be the major vertex of $T_5$ at height two whose trinity includes $v_5$ and assume its trinity is $(v_5, v_6, v_7)$. Let $u_3$ be the major vertex of $T_5$ at height three whose trinity includes $v_7$ and assume its trinity is $(v_7, v_8, v_9)$. Let $\gamma$ map $t_0, t_1, t_2, t_3$ to $u_0, v_1, v_2, v_8$, respectively. Then $t_0$ also has property $AB_{ij}$ in $\eta'$. To see that the first pair of disjoint paths is $Q_i(u_0) \cup Q_i(u_1) \cup Q_j(u_2) \cup P_j(u_3)$ and $Q_j(u_0) \cup Q_j(u_1) \cup Q_j(u_2) \cup P_j(u_3)$. The first path of the second pair is $P_j(u_0)$. Let $R_1 = Y_i(u_0) \cup Q_{m_i}(u_1) \cup P_{m_i}(u_2)$, $R_2 = R_j(u_2) \cup Q_j(u_2) \cup Q_j(u_3)$, and $R_3 = X_i(u_0) \cup Q_i(u_1) \cup X_j(u_2) \cup X_j(u_3)$. Then the second path of the second pair is a path from $\xi_m(i)$ to $\xi_m(i)$ that is a subgraph of $R_1 \cup R_2 \cup R_3 \cup R_4 \cup R_5$, where $R_4$ is a W6-path in the outer graph of $v_6$ joining the ends of $R_1$ and $R_2$, and $R_5$ is a W6-path in the outer graph of $v_9$ joining the ends of $R_2$ and $R_3$. See Figure 3. This completes main case 1.

**Main case 2:** $A \cap B = \emptyset$. It follows that $s$ is even and $|A| = |B| = s/2$. Assume as a case that for some integer $i \in B$ either $l_i, m_i \in A$ or $l_i, m_i \in B$. But the integers $l_i, m_i$ are pairwise distinct, and so if $l_i, m_i \in A$, then there exists $j \in B$ such that $l_j, m_j \in B$, and similarly if $l_i, m_i \in B$. We may therefore assume that $l_i, m_i \in A$ and $l_j, m_j \in B$ for some distinct $i, j \in B$. Let us recall that $u_4$ is the child of $v_5$ and $(v_5, v_6, v_7)$ is its trinity. We let $\gamma$ map $t_0, t_1, t_2, t_3$ to $u_0, v_1, v_2, v_6$, respectively, and we will prove that $t_0$ has property $AB_{ij}$ in $\eta'$. To that end we need to construct two pairs of disjoint paths. The first pair is $Q_i(u_0) \cap Q_i(u_1) \cap P_i(u_2)$ and $Q_j(u_0) \cap Q_j(u_1) \cap P_j(u_2)$. The first path of the second
Figure 3: Second pair when $l_i \in B - A$ and and $m_i \in A - B$.

The second pair will consist of the union of $X_i(u_0)$ with a subpath of $Q_{l_i}(u_1)$ from $X_{\eta(v_3)}$ to $X_{\eta(v_4)}$, and $Y_i(u_0)$ with a subpath of $Q_{m_i}(u_1)$ from $X_{\eta(v_3)}$ to $X_{\eta(v_4)}$, and a suitable W6-path in the outer graph of $v_4$ joining their ends, and the second path will consist of the union of $X_j(u_0) \cup Q_{l_j}(u_1) \cup Q_{l_j}(u_2)$ and $Y_j(u_0) \cup Q_{m_j}(u_1) \cup Q_{m_j}(u_2)$ and a suitable W6-path in the outer a graph of $v_7$ joining their ends. See Figure 4. This completes the case that for some integer $i \in B$ either $l_i, m_i \in A$ or $l_i, m_i \in B$.

Figure 4: Second pair when $l_i, m_i \in A$ and $l_j, m_j \in B$ for some distinct $i, j \in B$.

We may therefore assume that for every $i \in B$ one of $l_i, m_i$ belongs to $A$ and the other belongs to $B$. Let us recall that for every $i \in B$ a subpath of $P_i(u_0)$ joins $\xi_{v_3}(l_i)$ to $\xi_{v_3}(m_i)$ in the outer graph at $v_3$ and is disjoint from the $\eta$-torso at $u_0$, except for its ends. Let $J$ be the union of these subpaths; then $J$ is a linkage from $\{\xi_{v_3}(i) : i \in A\}$ to $\{\xi_{v_3}(i) : i \in B\}$. For $i \in B$ the path $Q_i(u_0)$ is a subgraph of the $\eta$-torso at $u_0$. For
\(i \in A\) the intersection of the path \(Q_i(u_0)\) with the \(\eta\)-torso at \(u_0\) consists of two paths, one from \(X_{\eta(v_1)}\) to \(X_{\eta(v_2)}\), and the other from \(X_{\eta(v_2)}\) to \(X_{\eta(v_3)}\). Let \(L\) denote the union of these subpaths over all \(i \in A\). It follows that \(J \cup L \cup \bigcup_{i \in B} Q_i(u_0)\) is a linkage from \(X_{\eta(v_1)}\) to \(X_{\eta(v_2)}\), and so by the minimality of the specified \(u_0\)-linkages, it is equal to the specified left \(u_0\)-linkage. It follows that \(u_0\) has property \(C\) in \(\eta\).

**Lemma 6.4.** Let \((T, X)\) be a tree-decomposition of a graph \(G\) satisfying (W6) and (W7). If there exists a regular cascade \(\eta : T_3 \rightarrow T\) with orderings \(\xi_i\) in which every major vertex has property \(C\), then there is a weak subcascade \(\eta'\) of \(\eta\) of height one such that the major vertex in \(\eta'\) has property \(C_{ij}\) for some \(i, j\).

**Proof.** Let the common confinement sets for \(\eta\) be \(A, B, C, D\). For a major vertex \(w \in V(T_3)\) with trinity \((v_1, v_2, v_3)\) there are disjoint paths in the \(\eta\)-torso at \(w\) as in the definition of property \(C\). For \(a \in A\) and \(b \in B\) let \(R_a(w)\) denote the path with ends \(\xi_{v_1}(a)\) and \(\xi_{v_2}(a)\), let \(R_b(w)\) denote the path with ends \(\xi_{v_1}(b)\) and \(\xi_{v_3}(b)\), and let \(R_{ab}(w)\) denote the path with ends \(\xi_{v_1}(b)\) and \(\xi_{v_3}(a)\).

Assume the major root of \(T_3\) is \(u_0\) and its trinity is \((v_1, v_2, v_3)\), and let \(I\) be the common intersection set of \(\eta\). Then \(\eta(v_1), \eta(v_2), \eta(v_3)\) is a triad in \(T\) with center \(\eta(u_0)\) and for all \(i \in \{1, 2, 3\}\) we have \(X_{\eta(v_i)} \cap X_{\eta(u_0)} = I = X_{\eta(v_1)} \cap X_{\eta(v_2)} \cap X_{\eta(v_3)}\), and hence the triad is not \(X\)-separable by (W7). Thus by Lemma 6.1 there is a path \(R(u_0)\) connecting two of the three sets of disjoint paths in the \(\eta\)-torso at \(u_0\). Assume without loss of generality that one end of \(R(u_0)\) is in a path \(R_i(u_0)\), where \(i \in A\). Then the other end of \(R(u_0)\) is either in a path \(R_j(u_0)\), where \(j \in B\) or in a path \(R_{aj}(u_0)\), where \(j \in B\) and \(a \in A\). In the former case we define \(a \in A\) to be such that \(R_{aj}(u_0)\) is a path in the family.

Let the major root of \(T_1\) be \(t_0\) and its trinity be \((t_1, t_2, t_3)\). Let \(\gamma(t_0) = u_0, \gamma(t_1) = v_1, \gamma(t_2) = v_2\). Let the major vertex that is the child of \(v_3\) be \(u_1\), and the trinity of \(u_1\) be \((v_3, v_4, v_5)\). Let \(\gamma(t_3) = v_5\). We will prove that \(t_0\) has property \(C_{ij}\) in \(\eta' = \eta \circ \gamma\). Let \(b \in B\) be such that \(R_{ab}(u_1)\) is a member of the family of the disjoint paths in the \(\eta\)-torso at \(u_1\) as in the definition of property \(C\). By Lemma 6.2 there exists a W6-path \(P\) in the outer graph at \(v_4\) joining \(\xi_{v_1}(a)\) and \(\xi_{v_3}(b)\). By considering the paths \(R_a(u_0), R_j(u_0) \cup R_j(u_1), R_{aj}(u_0) \cup R_{ai}(u_1) \cup P \cup R_{ab}(u_1)\) and \(R(u_0)\) we find that \(t_0\) has property \(C_{ij}\) in \(\eta'\), as desired.

**Lemma 6.5.** Let \(s \geq 2\) be an integer and let \((T, X)\) be a tree-decomposition of a graph \(G\) satisfying (W6). Let \(\eta : T_3 \rightarrow T\) be an ordered cascade in \((T, X)\) of height three and size \(|I| + s\) with orderings \(\xi_i\) and common intersection set \(I\) such that every major vertex of \(T_3\) has property \(C_{ij}\) for some distinct \(i, j \in \{1, 2, \ldots, s\}\). Then there exists a weak subcascade \(\eta' : T_1 \rightarrow T\) of \(\eta\) of height one such that the unique major vertex of \(T_1\) has property \(B_{ij}\) in \(\eta'\).

**Proof.** Assume that the three major vertices at height zero and one of \(T_3\) are \(u_0, u_1, u_2\). Let the trinity at \(u_0\) be \((v_1, v_2, v_3)\), the trinity at \(u_1\) be \((v_2, v_4, v_5)\), and the trinity at \(u_2\) be \((v_3, v_6, v_7)\). Assume the major vertex of \(T_1\) is \(t_0\), and its trinity is \((t_1, t_2, t_3)\). For a major vertex \(w \in V(T_3)\) let \(R_i(w), R_j(w), R_{ij}(w)\) and \(R(w)\) be as in the definition of property \(C_{ij}\).
We need to find a weakly monotone homeomorphic embedding \( \gamma : T_1 \hookrightarrow T_3 \) such that \( \eta' = \eta \circ \gamma \) satisfies the requirement. Set \( \gamma(t_0) = u_0 \) and \( \gamma(t_1) = v_1 \). Our choice for \( \gamma(t_2) \) will be \( v_4 \) or \( v_5 \), depending on which two of the three paths \( R_i(u_1), R_j(u_1), R_{ij}(u_1) \) in the torso at \( u_1 \) the path \( R(u_1) \) is connecting. If \( R(u_1) \) is between \( R_i(u_1) \) and \( R_j(u_1) \), then choose either \( v_4 \) or \( v_5 \) for \( \gamma(t_2) \). If \( R(u_1) \) is between \( R_i(u_1) \) and \( R_{ij}(u_1) \), then set \( \gamma(t_2) = v_4 \), and if it is between \( R_j(u_1) \) and \( R_{ij}(u_1) \), then set \( \gamma(t_2) = v_5 \). Do this similarly for \( \gamma(t_3) \). Then \( \eta' = \eta \circ \gamma \) will satisfy the requirement. In fact, we will prove this for the case when \( R(u_1) \) is between \( R_i(u_1) \) and \( R_{ij}(u_1) \) and \( R(u_2) \) is between \( R_j(u_2) \) and \( R_{ij}(u_2) \). See Figure 5. The other five cases are similar.

![Diagram](image)

Figure 5: The case when \( R(u_1) \) is between \( R_i(u_1) \) and \( R_{ij}(u_1) \) and \( R(u_2) \) is between \( R_j(u_2) \) and \( R_{ij}(u_2) \).

In this case, our choice is \( \gamma(t_0) = u_0, \gamma(t_1) = v_1, \gamma(t_2) = v_4, \gamma(t_3) = v_7 \). Assume the two endpoints of \( R(u_1) \) are \( x \) and \( y \) and the two endpoints of \( R(u_2) \) are \( w \) and \( z \). By Lemma 6.2, there exists a W6-path \( P_1 \) between \( \xi_{v_5}(i) \) and \( \xi_{v_5}(j) \) in the outer graph at \( v_5 \) and a W6-path \( P_2 \) between \( \xi_{v_6}(i) \) and \( \xi_{v_6}(j) \) in the outer graph at \( v_6 \). Now let

\[
P = yR_{ij}(u_1)\xi_{v_5}(i) \cup P_1 \cup R_j(u_1) \cup R_{ij}(u_0) \cup R_i(u_2) \cup P_2 \cup \xi_{v_6}(j)R_{ij}(u_2)w,
\]

\[
L_i = R_i(u_0) \cup R_i(u_1) \cup R(u_1) \cup P \cup wR_{ij}(u_2)\xi_{v_7}(i)
\]

and

\[
L_j = R_j(u_0) \cup R_j(u_2) \cup R(u_2) \cup P \cup yR_{ij}(u_1)\xi_{v_6}(j).
\]

The tripods \( L_i \) and \( L_j \) show that the major vertex of \( \eta' = \eta \circ \gamma : T_1 \hookrightarrow T \) has property \( B_{ij} \).

**Lemma 6.6.** For every positive integers \( h' \) and \( w \geq 2 \) there exists a positive integer \( h = h(h', w) \) such that the following holds. Let \( s \) be a positive integer such that \( 2 \leq s \leq w \). Let \( (T, X) \) be a tree-decomposition of a graph \( G \) of width less than \( w \) and satisfying \( (W6) \) and \( (W7) \). Assume there exists a regular cascade \( \eta : T_h \hookrightarrow T \) of size \( |I| + s \) with specified linkages that are minimal, where \( I \) is its common intersection set. Then there exist distinct integers \( i, j \in \{1, 2, \ldots, s\} \) and a weak subcascade \( \eta' : T_{h'} \hookrightarrow T \) of \( \eta \) of height \( h' \) such that

- every major vertex of \( T_{h'} \) has property \( A_{ij} \) in \( \eta' \), or

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• every major vertex of $T_h$ has property $B_{ij}$ in $\eta$.

Proof. Let $h(a, k)$ be the function of Lemma 5.2, let $a_3 = 3h'$, $a_2 = h(a_3, 2^{\binom{k}{2}})$, $a_1 = 5a_2$ and $h = h(a_1, 2)$. Consider having property $C$ or not having property $C$ as colors, then by Lemma 5.2 there exists a monotone homeomorphic embedding $\gamma : T_{a_1} \hookrightarrow T_h$ such that either $\gamma(t)$ has property $C$ in $\eta$ for every major vertex $t \in V(T_{a_1})$ or $\gamma(t)$ does not have property $C$ in $\eta$ for every major vertex $t \in V(T_{a_1})$. By Lemma 5.3, $\eta_1 = \eta \circ \gamma : T_{a_1} \hookrightarrow T$ is still a regular cascade with specified linkages that are minimal. Also, either $t$ has property $C$ in $\eta_1$ for every major vertex $t \in V(T_{a_1})$ or $t$ does not have property $C$ in $\eta_1$ for every major vertex $t \in V(T_{a_1})$.

If $t$ has property $C$ in $\eta_1$ for every major vertex $t \in V(T_{a_1})$, then by Lemma 6.4 there exists a weak subcascade $\eta_2$ of $\eta_1$ of height $a_2$ such that every major vertex of $T_{a_2}$ has property $C_{ij}$ in $\eta_2$ for some distinct $i, j \in \{1, 2, \ldots, s\}$. Consider each choice of pair $i, j$ as a color; then by Lemma 5.2 there exists a monotone homeomorphic embedding $\gamma_1 : T_{a_2} \hookrightarrow T_{a_3}$ such that for some distinct $i, j \in \{1, 2, \ldots, s\}$, $\gamma_1(t)$ has property $C_{ij}$ in $\eta_2$ for every major vertex $t \in V(T_{a_3})$. Let $\eta_3 = \eta_2 \circ \gamma_1$. Then by Lemma 5.3 this implies $t$ has property $C_{ij}$ in $\eta_3$ for every major vertex $t \in V(T_{a_3})$. Then by Lemma 6.5 there exists a weak subcascade $\eta_4 : h' \hookrightarrow a_3$ of $\eta_3$ such that every major vertex of $T_{h'}$ has property $B_{ij}$ in $\eta_4$. Hence $\eta_4$ is as desired.

If $t$ does not have property $C$ in $\eta_1$ for every major vertex $t \in V(T_{a_1})$, then by Lemma 6.3 there exists a weak subcascade $\eta_2$ of $\eta_1$ of height $a_2$ such that every major vertex of $T_{a_2}$ has property $A_{ij}$ or $B_{ij}$ for some distinct $i, j \in \{1, 2, \ldots, s\}$. Consider each property $A_{ij}$ or $B_{ij}$ as a color; then by Lemma 5.2 there exists a monotone homeomorphic embedding $\gamma_1 : T_{a_2} \hookrightarrow T_{a_3}$ such that for some distinct $i, j \in \{1, 2, \ldots, s\}$, either $\gamma_1(t)$ has property $A_{ij}$ in $\eta_2$ for every major vertex $t \in V(T_{a_3})$ or $\gamma_1(t)$ has property $B_{ij}$ in $\eta_2$ for every major vertex $t \in V(T_{a_3})$. Let $\eta_3 = \eta_2 \circ \gamma_1$. Then $t$ has property $A_{ij}$ in $\eta_3$ for every major vertex $t \in V(T_{h'})$ or $t$ has property $B_{ij}$ in $\eta_3$ for every major vertex $t \in V(T_{h'})$ by Lemma 5.3. Hence $\eta_3$ is as desired. \qed

7 Proof of Theorem 1.3

By Lemmas 2.2 and 2.4 Theorem 1.3 is equivalent to the following theorem.

Theorem 7.1. For any positive integer $k$, there exists a positive integer $p = p(k)$ such that for every 2-connected graph $G$, if $G$ has path-width at least $p$, then $G$ has a minor isomorphic to $P_k$ or $Q_k$.

We need the following lemma.

Lemma 7.2. Let $(T, X)$ be a tree-decomposition of a graph $G$, let $\eta : T_h \hookrightarrow T$ be an ordered cascade in $(T, X)$ with orderings $\xi_t$ of height $h$ and size $s + 1$, where $I$ is the common intersection set, and let $i, j \in \{1, 2, \ldots, s\}$ be distinct and such that every major vertex of $T_h$ has property $B_{ij}$ in $\eta$. Let $t$ be the minor root of $T_h$, and let $w_1 w_2$ be the base edge of $Q_h$. Then $G$ has a minor isomorphic to $Q_h - w_1 w_2$ in such a way $\xi_t(i)$ belongs to the node of $w_1$ and $\xi_t(j)$ belongs to the node of $w_2$. 22
Proof. We proceed by induction on \(h\). Let \(t_0\) be the major root of \(T_h\), let \((t_1, t_2, t_3)\) be its trinity, and let \(L_i\) and \(L_j\) be the tripods in the \(\eta_j\)-torso at \(t_0\) as in the definition of property \(B_{ij}\). The graph \(L_i \cup L_j\) contains a path \(P\) joining \(\xi_{t_1}(i)\) to \(\xi_{t_1}(j)\), which shows that the lemma holds for \(h = 1\).

We may therefore assume that \(h > 1\) and that the lemma holds for \(h-1\). For \(k \in \{2, 3\}\) let \(R_k\) be the subtree of \(T_h\) rooted at \(t_k\), let \(\eta_k\) be the restriction of \(\eta\) to \(R_k\), and let \(G_k\) be the subgraph of \(G\) induced by \(\bigcup \{X_r : r \in sp(\eta_k)\}\). By the induction hypothesis applied to \(\eta_k\) and \(G_k\), the graph \(G_k\) has a minor isomorphic to \(Q_{h-1} - u_1u_2\) in such a way \(\xi_{t_k}(i)\) belongs to the node of \(u_1\) and \(\xi_{t_k}(j)\) belongs to the node of \(u_2\), where \(u_1u_2\) is the base edge of \(Q_{h-1}\). By using these two minors, the path \(P\) and the rest of the triads \(L_i\) and \(L_j\) we find that \(G\) has the desired minor.

We deduce Theorem 7.1 from the following lemma.

Lemma 7.3. Let \(k\) and \(w\) be positive integers. There exists a number \(p = p(k, w)\) such that for every 2-connected graph \(G\), if \(G\) has tree-width less than \(w\) and path-width at least \(p\), then \(G\) has a minor isomorphic to \(P_k\) or \(Q_k\).

Proof. Let \(h' = 2k + 1\), let \(h = h(h', w)\) be the number as in Lemma 6.6 and let \(p\) be as in Theorem 5.5 applied to \(a = h\) and \(w\). We claim that \(p\) satisfies the conclusion of the lemma. By Theorem 5.5 there exists a tree-decomposition \((T, X)\) of \(G\) such that:

- \((T, X)\) has width less than \(w\),
- \((T, X)\) satisfies (W1)–(W7), and
- for some \(s\), where \(2 \leq s \leq w\), there exists a regular cascade \(\eta : T_h \rightarrow T\) of height \(h\) and size \(s\) in \((T, X)\) with specified \(t_0\)-linkages that are minimal for every major vertex \(t_0 \in V(T_h)\).

Let \(I\) be the common intersection set of \(\eta_j\), let \(\xi_\ell\) be the orderings, and let \(s_1 = s - |I|\). Then \(s_1 \geq 1\) by the definition of injective cascade.

Assume first that \(s_1 = 1\). Since \(s \geq 2\), it follows that \(I \neq \emptyset\). Let \(x \in I\). Let \(R\) be the union of the left and right specified \(t\)-linkage with respect to \(\eta\), over all major vertices \(t \in V(T_h)\) at height at most \(h - 2\). The minimality of the specified linkages implies that \(R\) has a subtree isomorphic to a subdivision of \(CT_{\lfloor (h-1)/2 \rfloor}\). Let \(t\) be a minor vertex of \(T_h\) at height \(h - 1\). By Lemma 6.2 there exists a W6-path with ends \(\xi_\ell(1)\) and \(x\) and every internal vertex in the outer graph at \(t\). The union of \(R\) and these W6-paths shows that \(G\) has a \(P_k\) minor, as desired.

We may therefore assume that \(s_1 \geq 2\). By Lemma 6.6 there exist distinct integers \(i, j \in \{1, 2, \ldots, s\}\) and a subcascade \(\eta' : T_{h'} \rightarrow T\) of \(\eta\) of height \(h'\) such that

- every major vertex of \(T_{h'}\) has property \(A_{ij}\) in \(\eta'\), or
- every major vertex of \(T_{h'}\) has property \(B_{ij}\) in \(\eta'\)
Assume next that every major vertex of $T_{h'}$ has property $A_{ij}$ in $\eta'$, and let $R$ be the union of the corresponding tripods, over all major vertices $t \in V(T_{h'})$ at height at most $h' - 2$. It follows that $R$ is the union of two disjoint trees, each containing a subtree isomorphic to $CT_{(h'-1)/2}$. Let $t$ be a minor vertex of $T_{h'}$ at height $h' - 1$. By Lemma 6.2 there exists a W6-path with ends $\xi_t(i)$ and $\xi_t(j)$ in the outer graph at $t$. By contracting one of the trees comprising $R$ and by considering these W6-paths we deduce that $G$ has a $P_k$ minor, as desired.

We may therefore assume that every major vertex of $T_{h'}$ has property $B_{ij}$ in $\eta'$. It follows from Lemma 7.2 that $G$ has a minor isomorphic to $Q_{h'-1}$, as desired. □

Proof of Theorem 7.1. Let a positive integer $k$ be given. By Theorem 1.1 there exists an integer $w$ such that every graph of tree-width at least $w$ has a minor isomorphic to $P_k$. Let $p = p(k, w)$ be as in Lemma 7.3. We claim that $p$ satisfies the conclusion of the theorem. Indeed, let $G$ be a 2-connected graph of path-width at least $p$. By Theorem 1.1 if $G$ has tree-width at least $w$, then $G$ has a minor isomorphic to $P_k$, as desired. We may therefore assume that the tree-width of $G$ is less than $w$. By Lemma 7.3 $G$ has a minor isomorphic to $P_k$ or $Q_k$, as desired. □

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