CLIFFORD AND WEYL SUPERALGEBRAS AND SPINOR REPRESENTATIONS

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Abstract. We construct a family of twisted generalized Weyl algebras which includes Weyl–Clifford superalgebras and quotients of the enveloping algebras of $\mathfrak{gl}(m|n)$ and $\mathfrak{osp}(m|2n)$. We give a condition for when a canonical representation by differential operators is faithful. Lastly, we give a description of the graded support of these algebras in terms of pattern-avoiding vector compositions.

1. Introduction

Twisted generalized Weyl algebras (TGWAs) were introduced by Mazorchuk and Turowska in [20], [21] in an attempt to include a wider range of examples than Bavula’s generalized Weyl algebras (GWAs) [1]. Their structure and representations have been studied in [20], [21], [19], [24], [12], [13], [14], [15], [10]. Known examples of TGWAs include multiparameter quantized Weyl algebras [21], [12], [10], the Mickelsson–Zhelobenko step algebras associated to $(\mathfrak{gl}_{n+1}, \mathfrak{gl}_n \oplus \mathfrak{gl}_1)$ [19] and some primitive quotients of enveloping algebras [16].

In this paper we take a step further by proving that supersymmetric analogs of some classical algebras are also examples of TGWAs. Specifically, we show that Weyl–Clifford superalgebras and some quotients of the enveloping algebras of $\mathfrak{gl}(m|n)$ and $\mathfrak{osp}(m|2n)$ can be realized as twisted generalized Weyl (TGW) algebras. This suggests that much of the general representation theory from [21], [19], [12] could be applied to the study of certain families of superalgebras. In addition our new algebras provide a large supply of consistent but non-regular TGW algebras (i.e., certain elements $t_i$ are zero-divisors). This motivates future development of the theory to include such algebras.

It is also worth mentioning that, as a special case, we show that Clifford algebras can be presented as TGW algebras. This shows that TGW algebras can be finite-dimensional.

To summarize the contents of the present paper, in Section 2 we recall the definition of TGW algebras from [21] which includes certain scalars $\mu_{ij}$ that in our case will be $\pm 1$. Some known results that will be used are also stated. In Section 3
we prove that the Weyl–Clifford superalgebra from [23] can be realized as a TGW algebra.

The main object of the paper is introduced in Section 4, in which we define a family of TGW algebras $A(\gamma)^\pm$ which depend on a certain matrix $\gamma$ with integer entries. These algebras naturally come with an algebra homomorphism $\varphi$ from $A(\gamma)^\pm$ to a Clifford-Weyl algebra. This is a generalization of the construction in [16]. A sufficient condition for $\varphi_\gamma$ to be injective is given in Section 4.2. This condition is related to the graded support of the algebra $A(\gamma)^\pm$ which is combinatorially characterized in Section 4.3.

Lastly, these results are applied in Section 5 to prove that for appropriate $\gamma$, the TGW algebras $A(\gamma)^\pm$ fit into commutative diagrams involving the spinor representation $\pi$ of $U(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{gl}(m|n)$ and $\mathfrak{osp}(m|2n)$ studied by Nishiyama [23] and Coulembier [9]. As a corollary we obtain that $U(\mathfrak{g})/J$ are examples of TGW algebras for such $\mathfrak{g}$ as well as for classical Lie algebras. These results generalize previous realizations in [16]. We end with some open problems regarding exceptional types.

**Notation**

Throughout, we work over an algebraically closed field $k$ of characteristic zero. Associative algebras are assumed to have a multiplicative identity. $[a, b]$ denotes the set of integers $x$ with $a \leq x \leq b$.

2. Twisted generalized Weyl algebras

We recall the definition of TGW algebras and some of their useful properties.

**2.1. Definitions**

Let $I$ be a set.

**Definition 1 (TGW Datum).** A twisted generalized Weyl datum over $k$ with index set $I$ is a triple $(R, \sigma, t)$ where

- $R$ is an associative $k$-algebra,
- $\sigma = (\sigma_i)_{i \in I}$ is a sequence of commuting $k$-algebra automorphisms of $R$,
- $t = (t_i)_{i \in I}$ is a sequence of central elements of $R$.

Let $ZI$ denote the free abelian group on $I$, with basis denoted $\{e_i\}_{i \in I}$. For $g = \sum g_i e_i \in ZI$ put $\sigma_g = \prod \sigma_i^{g_i}$. Then $g \mapsto \sigma_g$ defines an action of $ZI$ on $R$ by $k$-algebra automorphisms.

**Definition 2 (TGW Construction).** Let

- $(R, \sigma, t)$ be a TGW datum over $k$ with index set $I$,
- $\mu$ be an $I \times I$-matrix without diagonal, $\mu = (\mu_{ij})_{i \neq j}$, with $\mu_{ij} \in k \setminus \{0\}$.

The twisted generalized Weyl construction associated to $\mu$ and $(R, \sigma, t)$, denoted $C_\mu(R, \sigma, t)$, is defined as the free $R$-ring on the set $\{X_i, Y_i \mid i \in I\}$ modulo the two-sided ideal generated by the following elements:

\[
\begin{align*}
X_i r - \sigma_i(r) X_i, & \quad Y_i r - \sigma_i^{-1}(r) Y_i, \quad \forall r \in R, i \in I, \\
Y_i X_i - t_i, & \quad X_i Y_i - \sigma_i(t_i), \quad \forall i \in I, \\
X_i Y_j - \mu_{ij} Y_j X_i, & \quad \forall i, j \in I, i \neq j.
\end{align*}
\]
The algebra $\mathcal{C}_\mu(R,\sigma,t)$ has a $\mathbb{Z}I$-gradation given by requiring $\deg X_i = e_i$, $\deg Y_i = -e_i$, $\deg r = 0$ for $r \in R$. Let $\mathcal{J}_\mu(R,\sigma,t) \subseteq \mathcal{C}_\mu(R,\sigma,t)$ be the sum of all graded ideals $J \subseteq \mathcal{C}_\mu(R,\sigma,t)$ such that $\mathcal{C}_\mu(R,\sigma,t) \cap J = \{0\}$. It is easy to see that $\mathcal{J}_\mu(R,\sigma,t)$ is the unique maximal graded ideal having zero intersection with the degree zero component.

**Definition 3 (TGW Algebra).** The twisted generalized Weyl algebra $A_\mu(R,\sigma,t)$ associated to $\mu$ and $(R,\sigma,t)$ is defined as the quotient

$$A_\mu(R,\sigma,t) := \mathcal{C}_\mu(R,\sigma,t)/\mathcal{J}_\mu(R,\sigma,t).$$

Since $\mathcal{J}_\mu(R,\sigma,t)$ is graded, $A_\mu(R,\sigma,t)$ inherits a $\mathbb{Z}I$-gradation from $\mathcal{C}_\mu(R,\sigma,t)$. The images in $A_\mu(R,\sigma,t)$ of the elements $X_i, Y_i$ will also be denoted by $X_i, Y_i$.

**Example 1.** For an index set $I$, the $I$:th Weyl algebra over $k$, $A_I = A_I(k)$ is the $k$-algebra generated by $\{x_i, \partial_i \mid i \in I\}$ subject to defining relations

$$[x_i, x_j] = [\partial_i, \partial_j] = [\partial_i, x_j] - \delta_{ij} = 0, \quad \forall i, j \in I.$$ 

There is a $k$-algebra isomorphism $A_\mu(R,\sigma,t) \to A_n$ where $\mu_{ij} = 1$ for all $i \neq j$, $R = k[u_i \mid i \in I]$, $\tau_i(u_j) = u_j - \delta_{ij}$, given by $X_i \mapsto x_i, Y_i \mapsto \partial_i, u_i \mapsto \partial_i x_i$.

**2.2. Regularity and consistency**

**Definition 4 (Reduced and monic monomials).** A monic monomial in a TGW algebra is any finite product of elements from the set $\{X_i\}_{i \in I} \cup \{Y_i\}_{i \in I}$. A reduced monomial is an element of the form $Y_{i_1} \cdots Y_{i_k} X_{j_1} \cdots X_{j_l}$ where $\{i_1, \ldots, i_k\} \cap \{j_1, \ldots, j_l\} = \emptyset$.

**Lemma 1.** [12, Lem. 3.2] $A_\mu(R,\sigma,t)$ is generated as a left (and as a right) $R$-module by the reduced monomials.

Since a TGW algebra $A_\mu(R,\sigma,t)$ is a quotient of an $R$-ring, it is an $R$-ring itself with a natural map $\rho : R \to A_\mu(R,\sigma,t)$. By Lemma 1, the degree zero component of $A_\mu(R,\sigma,t)$ (with respect to the $\mathbb{Z}I$-gradation) is equal to the image of $\rho$.

**Definition 5 (Regularity).** A TGW datum $(R,\sigma,t)$ is called regular if $t_i$ is regular (i.e., not a zero-divisor) in $R$ for all $i$.

Due to Relation (1b), the canonical map $R \to \mathcal{C}_\mu(R,\sigma,t)$ is not guaranteed to be injective, and indeed sometimes it is not [10]. It is injective if and only if the map $R \to A_\mu(R,\sigma,t)$ is injective.

**Definition 6 ($\mu$-Consistency).** A TGW datum $(R,\sigma,t)$ is $\mu$-consistent if the canonical map $\rho : R \to A_\mu(R,\sigma,t)$ is injective.

Abusing language we say that a TGW algebra $A_\mu(R,\sigma,t)$ is regular (respectively consistent) if $(R,\sigma,t)$ is regular (respectively $\mu$-consistent).

**Theorem 2 ([10]).** A regular TGW algebra $A_\mu(R,\sigma,t)$ is consistent if and only if

$$\sigma_i \sigma_j(t_it_j) = \mu_{ij} \mu_{ji} \sigma_i(t_i) \sigma_j(t_j), \quad \forall i \neq j; \quad (2a)$$

$$\sigma_i \sigma_k(t_j) t_j = \sigma_i(t_j) \sigma_k(t_j), \quad \forall i \neq j \neq k \neq i. \quad (2b)$$
That relation (2a) is necessary for consistency of a regular TGW datum was known already in [20], [21]. If $(R, \sigma, t)$ is not regular, sufficient and necessary conditions for $\mu$-consistency are not known (see Problem 2). In this paper we produce many examples of consistent but non-regular TGW algebras.

Conversely, for consistent TGW algebras one can characterize regularity as follows:

**Theorem 3** ([15, Thm. 4.3]). Let $A = A_\mu(R, \sigma, t)$ be a consistent TGW algebra. Then the following are equivalent

(i) $(R, \sigma, t)$ is regular.

(ii) Each monic monomial in $A$ is non-zero and generates a free left (and right) $R$-module of rank one.

(iii) $A$ is regularly graded, i.e., for all $g \in \mathbb{Z}I$, there exists a nonzero regular element in $A_g$.

(iv) If $a \in A$ is a homogeneous element such that $bac = 0$ for some monic monomials $b, c \in A$, then $a = 0$.

2.3. Non-degeneracy of the gradation form

For a group $G$, any $G$-graded ring $A = \bigoplus_{g \in G} A_g$ can be equipped with a $\mathbb{Z}$-bilinear form $\gamma : A \times A \to A_e$ called the *gradation form*, defined by

$$\gamma(a, b) = p_e(ab)$$

where $p_e$ is the projection $A \to A_e$ along the direct sum $\bigoplus_{g \in G} A_g$, and $e \in G$ is the neutral element.

**Theorem 4** ([15, Cor. 3.3]). The ideal $I_\mu(R, \sigma, t)$ is equal to the radical of the gradation form $\gamma$ of $C_\mu(R, \sigma, t)$ (with respect to the $\mathbb{Z}I$-gradation), and thus the gradation form on $A_\mu(R, \sigma, t)$ is non-degenerate.

2.4. $R$-rings with involution

**Definition 7.** Let $R$ be a commutative ring.

(i) An *involution* on a ring $A$ is a $\mathbb{Z}$-linear map $*: A \to A, a \mapsto a^*$ satisfying $(ab)^* = b^*a^*$, $(a^*)^* = a$ for all $a, b \in A$.

(ii) An $R$-ring *with involution* is a ring $A$ equipped with a ring homomorphism $h_A : R \to A$ and an involution $*: A \to A$ such that $h(r)^* = h(r)$ for all $r \in R$.

(iii) If $A$ and $B$ are two $R$-rings with involution, then a *map of $R$-rings with involution* is a ring homomorphism $k : A \to B$ such that $k \circ h_A = h_B$ and $k(a^*) = (k(a))^*$ for all $a \in A$.

When $R$ is commutative, any TGW algebra $A = A_\mu(R, \sigma, t)$ for which $\mu_{ij} = \mu_{ji}$ for all $i, j$, can be equipped with an involution $*$ given by $X_i^* = Y_i$, $Y_i^* = X_i \forall i \in I$, $r^* = r \forall r \in R$. Together with the canonical map $\rho : R \to A$ this turns $A$ into an $R$-ring with involution. In particular we regard the Weyl algebra $A_I$ as an $R$-ring with involution in this way, where $R = k[u_i \mid i \in I]$ as in Example 1.
3. The Clifford/Weyl superalgebras

In this section let $\pm \in \{+,-\}$ and put $\mp = -\pm$. Let $p$ and $q$ be non-negative integers and put $n = p + q$. We consider supersymmetric analogs $A^{\pm}_{p/q}$ of Clifford and Weyl algebras and prove that they can be presented as TGW algebras.

3.1. Definition and properties

**Definition 8.** The Clifford/Weyl superalgebra of degree $p|q$, denoted $A^{\pm}_{p/q}$, is defined as the superalgebra with even generators $x_i, \partial_i$ ($i \in [1, p]$) and odd generators $x_i, \partial_i$ ($i \in [p+1, n]$) and relations

$$[\partial_i, x_j]_{\pm} - \delta_{ij} = [x_i, x_j]_{\pm} = [\partial_i, \partial_j]_{\pm} = 0 \quad \text{for all } i, j \in [1, n],$$

where $[-, -]_{\pm}$ denotes the super(anti-)commutator

$$[a, b]_{\pm} = ab \pm (-1)^{p(a)p(b)}ba.$$

Thus $A^{+}_{p/q}$ (respectively $A^{-}_{p/q}$) is a supersymmetric analog of the Clifford (respectively Weyl) algebra.

We will need the following result.

**Lemma 5.** The subalgebra $R$ of $A^{\pm}_{p/q}$ generated by $\{\partial_i x_i \mid i \in [1, n]\}$ is maximal commutative.

**Proof.** The algebra $A^{\pm}_{p/q}$ has a $\mathbb{Z}^n$-gradation determined by $\deg(x_i) = e_i$ and $\deg(\partial_i) = -e_i$ where $\{e_i\}_{i=1}^n$ is a $\mathbb{Z}$-basis for $\mathbb{Z}^n$. We have $[\mp \partial_i x_i, x_j]_{-} = \delta_{ij} x_j$ and $[\mp x_i \partial_i, \partial_j]_{-} = -\delta_{ij} \partial_j$. In other words, $\{[\mp x_i \partial_i, \partial_j]_{-}\}_{i=1}^n$ is a set of commuting (even) derivations on $A^{\pm}_{p/q}$ whose common eigenspaces coincide with the graded homogeneous components. Thus the centralizer of $R$ is the subalgebra $A_0$ of $A^{\pm}_{p/q}$ consisting of elements of degree $0 \in \mathbb{Z}^n$. Clearly $R \subseteq A_0$. The converse inclusion is straightforward to check using the commutation relations (3) and induction on the length of a monomial of degree zero. \qed

By the defining relations, $A^{\pm}_{p/q}$ is a graded algebra with respect to the free abelian group $\mathbb{Z}^n$. In addition $A^{\pm}_{p/q}$ has an involution $\ast$ given by $x_i^* = \partial_i, \partial_i^* = x_i$. Since $(\partial_i x_i)^* = \partial_i x_i$, $A^{\pm}_{p/q}$ is an $R$-ring with involution. Even though $A^{\pm}_{p/q}$ is not a domain in general, the following graded regularity property still holds.

**Lemma 6.** Let $a \in A^{\pm}_{p/q}$ be homogeneous of degree $g \in \mathbb{Z}^n$. If $a^* \cdot a = 0$ then $a = 0$.

**Proof.** We give a proof for $A = A^{-}_{p/q}$, the other case being analogous. Write $a = r x^{(g)}$ where $r \in R$ and $x^{(g)} = x_1^{(g_1)} \cdots x_n^{(g_n)}$ where for $s > 0$, $x_i^{(s)} = x_i^s, x_i^{(-s)} = \partial_i^s$. By reordering the indices, we may assume that the first $k$ elements of the tuple $(g_{p+1}, \ldots, g_n)$ are zero, and the rest are nonzero. Put $u_i = \partial_i x_i$. For $i > p$ we have $u_i x_i = \partial_i x_i^2 = 0$ and $u_i \partial_i = \partial_i x_i \partial_i = (1 - x_i \partial_i) \partial_i = \partial_i$. Thus we may assume that $r$ lies in the subalgebra of $R$ generated by $\{u_1, \ldots, u_{p+k}\}$. If $a \cdot a^* = 0$ then we have

$$0 = a \cdot a^* = r x^{(g)} x^{(-g)} r = r^2 cb \quad (4)$$

where
where
\[ b = x_1^{(g_1)} \cdots x_p^{(g_p)} \cdot x_p^{(-g_p)} \cdots x_1^{(-g_1)} \]
which can be written as a polynomial in \( u_i, i \leq p \), and
\[ c = x_p^{(g_{p+k+1})} \cdots x_n^{(g_n)} \cdot x_n^{(-g_n)} \cdots x_p^{(-g_{p+k+1})} \]
which can be written as a polynomial in \( u_i, i > p \). Since \( b \) is regular in \( A \), (4) implies \( r^2 c = 0 \). We have the following isomorphisms of algebras
\[ A \simeq A^-_{p|0} \otimes_k A^-_{0|q} \simeq A^-_{p|0} \otimes_k M_{2q}(k) \cong M_{2k}(A^-_{p|0}) \otimes_k M_{2q-k}(k). \]
Under this isomorphism, \( r^2 c \) is mapped to \( r^2 c \). That this is zero implies \( r^2 = 0 \).

3.2. Realization as TGW algebras
To realize \( A^\pm_{p|q} \) as TGW algebras, consider the commutative \( k \)-algebra
\[ R^\pm_{p|q} := k[u_1, u_2, \ldots, u_n]/J^\pm \]
where \( J^\pm \) is the ideal generated by \( u_i^2 - u_i \) for all \( i \) is such that \((-1)^{p(i)} = \pm 1\). There is an injective homomorphism
\[ \iota : R^\pm_{p|q} \rightarrow A^\pm_{p|q}, \]
\[ u_i \mapsto \partial_i x_i. \]
We will often use \( \iota \) to identify \( R^\pm_{p|q} \) with its image in \( A^\pm_{p|q} \). One checks that the image of \( \iota \) coincides with the degree zero subalgebra, \( (A^\pm_{p|q})_0 \), of \( A^\pm_{p|q} \) with respect to the \( \mathbb{Z}^n \)-gradation \( A^\pm_{p|q} = \bigoplus_{d \in \mathbb{Z}^n} (A^\pm_{p|q})_d \) given by \( \deg(x_i) = e_i, \deg(\partial_i) = -e_i, \forall i \in [1, n] \).

For \( i, j \in [1, n] \), put
\[ \lambda_{ij} = \mp(-1)^{p(i)p(j)} \]
and for \( i \in [1, n] \), define \( \tau_i \in \text{Aut}_k(R^\pm_{p|q}) \) by
\[ \tau_i(u_j) = \begin{cases} \lambda_{ii}(u_i - 1), & \text{if } i = j, \\ u_j, & \text{otherwise}. \end{cases} \]
One checks that \( \tau_i \) preserves the relations \( u_j^2 - u_j = 0 \) for \( j \) with \((-1)^{p(j)} = \pm 1\). Let \( \tau = (\tau_i)_{i=1}^n \) and \( u = (u_i)_{i=1}^n \). Let \( \mathcal{A}_\lambda(R^\pm_{p|q}, \tau, u) \) be the corresponding TGW algebra.
Theorem 7. There is an isomorphism of $k$-algebras

$$\chi: \mathcal{A}_\lambda(R^\pm_{p|q}, \tau, u) \xrightarrow{\sim} A^\pm_{p|q},$$

$$X_i \mapsto x_i,$$

$$Y_i \mapsto \partial_i.$$  

In particular, $\mathcal{A}_\lambda(R^\pm_f, \tau, u)$ is consistent (i.e., the natural map

$$\rho: R^\pm_{p|q} \to \mathcal{A}_\lambda(R^\pm_{p|q}, \tau, u)$$

is injective).

Proof. Put $R = R^\pm_{p|q}$ and $A = A^\pm_{p|q}$ and $[\cdot, \cdot] = [\cdot, \cdot]_\pm$. The identities $x_i(\partial_j x_i) = (x_i \partial_j) x_i$, $x_i \partial_j = \lambda_{ij} \partial_j x_i$ for $i \neq j$, and $x_i \partial_i = \lambda_{ii} (\partial_i x_i + 1)$ imply that relations (1) are preserved. Thus we have a map $\mathcal{C}_\lambda(R, \tau, u) \to A$ of $R$-rings given by $X_i \mapsto x_i, Y_i \mapsto \partial_i$. Furthermore, for each $i, j \in [1, n]$, it can be checked, using Theorem 4 that $[X_i, X_j]$ and $[Y_i, Y_j]$ lie in the radical of the gradation form on $\mathcal{C}_\lambda(R, \tau, u)$. For example, if $i \neq j$ then by Lemma 1 the homogeneous component of degree $-e_i - e_j$ is equal to $RY_i Y_j + RY_j Y_i$ so by symmetry it suffices to show that $\gamma(Y_i Y_j, [X_i, X_j]_{\pm}) = 0$. For simplicity, say $i, j \leq p$ and that $\pm = -$. Then we get $\gamma(Y_i Y_j, [X_i, X_j]_{-}) = Y_i Y_j (X_i X_j - X_j X_i) = \lambda_{ij}^{-1} u_i u_j - \tau_{ij}^{-1} u_i = 0$. The other cases are checked similarly. In fact the elements $[X_i, X_j]$ and $[Y_i, Y_j]$ generate the radical. To see this, let $J'$ be the ideal of $\mathcal{C}_\lambda(R, \tau, u)$ generated by all $[X_i, X_j]$ and $[Y_i, Y_j]$. It suffices to show that $B := \mathcal{C}_\lambda(R, \tau, u)/J'$ has a non-degenerate gradation form. By Lemma 1, any nonzero homogeneous component of $B$ is a free cyclic left $R$-module. If $\pm = -$, say, then for $a = (a_1, \ldots, a_n) \in \mathbb{Z}^p \times \{-1, 0, 1\}^q$ the monomial $Z = X_1^{(a_1)} \cdots X_n^{(a_n)}$ (where $X_i^{(k)} = (X_i)^k$ for $k \geq 0$ and $X_i^{(-k)} = (Y_i)^{|k|}$ for $k < 0$) and its dual $Z^* = X_n^{(-a_n)} \cdots X_1^{(-a_1)}$ satisfy $\gamma(Z^*, Z) = Z^* Z$ which simplifies to a nonzero element of $R$. Since $B_a = RZ$ this shows that $\gamma$ is nondegenerate on $B$. Hence the commutators generate the ideal $J_\lambda(R, \tau, u)$ by Theorem 4. Since $[x_i, x_j] = [\partial_i, \partial_j] = 0$ in $A$ this shows that we have a well-defined map $\chi: \mathcal{A}_\lambda(R, \tau, u) \to A$ of $R$-rings given by $X_i \mapsto x_i, Y_i \mapsto \partial_i$. Since $x_i$ and $\partial_i$ generate $A$, the map $\chi$ is surjective. It remains to prove it is injective. Since $\chi$ is a map of $R$-rings, the following diagram is commutative:

$$\begin{array}{ccc}
\mathcal{A}_\lambda(R, \tau, u) & \xrightarrow{\chi} & A \\
\rho \downarrow & & \downarrow \\
R & \xrightarrow{\iota} & R
\end{array}$$

Since $\iota$ is injective, $\rho$ is injective. That is, $\mathcal{A}_\lambda(R, \tau, u)$ is consistent. Identifying $R$ with the images under $\rho$ and $\iota$, the map $\chi|_R$ is the identity map. Both $\mathcal{A}_\lambda(R, \tau, u)$ and $A$ are $\mathbb{Z}^n$-graded algebras and $\chi$ is a graded homomorphism. Therefore $J = \ker \chi$ is a graded ideal of $\mathcal{A}_\lambda(R, \tau, u)$. If $J \neq 0$ then, since $\mathcal{A}_\lambda(R, \tau, u)$ is a consistent TGW algebra, $J \cap R \neq 0$. However that contradicts that $\chi|_R$ is injective. Hence $J = 0$ which completes the proof that $\chi$ is an isomorphism. □
Remark 1. When \( q > 0 \), Theorem 7 implies that \( A_{p|q}^- \) is a consistent TGW algebra which is not regularly graded. Indeed, if \( j > p \), then \( u_j \) is not regular in \( R \) because \( u_j(u_j + 1) = 0 \). Thus, by Theorem 3, \( A_{\lambda}(R, \tau, u) \) is not regularly graded. Note that for non-regularly graded TGW algebras, it is not known if relations (2) are sufficient (or even necessary) for it to be consistent.

Remark 2. The algebra \( A_{0|q}^- \) is finite-dimensional (an even Clifford algebra). Hence Theorem 7 shows that TGW algebras can be finite-dimensional.

Remark 3. Theorem 7 suggests that the class of TGW algebras already contains not only quantum deformations of many algebras (see, e.g., [21, Ex. 2.2.3]), but also supersymmetric analogues of certain algebras, without modifying the definition of TGW algebras.

4. A new family of TGW algebras \( A(\gamma)^\pm \)

In this section we define a family of TGW algebras that depend on a matrix. This construction is a supersymmetric generalization of the one in [16].

4.1. Construction via monomial maps

In this section we use the Clifford/Weyl superalgebras \( A_{p|q}^\pm \) to construct new TGW algebras denoted \( A(\gamma)^\pm \). Our method is to look for maps

\[
\varphi : A_\mu(R, \sigma, t) \to A_{p|q}^\pm
\]

definitions of \( R \)-rings with involution. Here \( R = R_{p|q}^\pm \). The motivation is threefold. First it generalizes the construction from [16] which corresponds to the case \( A_{p|0}^- \). Second, the TGW algebras obtained in this way automatically come with \( \varphi \), which may be thought of as a representation by differential operators. Thirdly we show in Section 5 that certain quotients of enveloping algebras of Lie superalgebras are TGW algebras of exactly this form.

As in [16] we restrict attention to monomial embeddings

\[
\varphi(X_i) = x_1^{(\gamma_{1i})} x_2^{(\gamma_{2i})} \cdots x_n^{(\gamma_{ni})}.
\]

Here \( n = p + q \), \( \gamma_{ji} \in \mathbb{Z} \) and we use the notation

\[
x_j^{(k)} = \begin{cases} a_j^k, & k \geq 0, \\ \partial_j^{-k}, & k < 0. \end{cases}
\]

In the case of [16], under mild assumptions on \( \varphi \) the form (10) was in fact shown to be necessary. Here in our more general setting we shall be content with showing how the assumption that \( \varphi \) is a homomorphism of \( R \)-rings with involution such that (10) holds, naturally gives rise to conditions on \( \gamma_{ji} \) and also specifies the TGW datum (automorphisms \( \sigma_i \), elements \( t_i \in R \) and scalars \( \mu_{ij} \)).

First, since \( \varphi \) is supposed to be a map of rings with involution, we necessarily have

\[
\varphi(Y_i) = \varphi(X_i^*) = \varphi(X_i)^* = x_n^{(-\gamma_{ni})} x_2^{(-\gamma_{2i})} x_1^{(-\gamma_{1i})}.
\]
Second, since $\varphi$ is a map of $R$-rings and $t_i \in R$ we have

$$t_i = \varphi(t_i) = \varphi(Y_i X_i) = \varphi(Y_i) \varphi(X_i)$$

$$= x_n(-\gamma_{ni}) \cdots x_1(-\gamma_{1i}) x_1(\gamma_{1i}) x_2(\gamma_{2i}) \cdots x_n(\gamma_{ni})$$

$$= x_1(-\gamma_{1i}) x_1(\gamma_{1i}) x_2(\gamma_{2i}) \cdots x_n(\gamma_{ni}) x_n(\gamma_{ni}).$$

In the last step we used the TGW algebra realization of $A_{p|q}^\pm$ which in particular has $\tau_i(u_j) = u_j$ for $j \neq i$. To obtain an explicit formula for $t_i$ we compute $x_j(-\gamma_{ji}) x_j(\gamma_{ji})$. If $\gamma_{ji} > 0$ we have

$$x_j(-\gamma_{ji}) x_j(\gamma_{ji}) = \partial_j^{\gamma_{ji}} x_j^{\gamma_{ji}} = \tau_j^{(-\gamma_{ji}+1)}(u_j) \cdots \tau_j^{-1}(u_j) u_j.$$

Here we see that this is zero if $\lambda_{jj} = -1$ and $\gamma_{ji} > 1$ because then $\tau_j^{-1}(u_j) u_j = \tau_j^{-1}(u_j \lambda_{jj}(u_j - 1)) = 0$ due to $u_j^2 = u_j$ in $R$. To avoid this scenario (having $t_i = 0$ in a TGW algebra leads to degenerate behaviour such as $X_i = Y_i = 0$) we make our first assumption on $\gamma_{jj}$:

$$|\gamma_{ji}| \leq 1 \quad \text{for all } i, j \text{ such that } \lambda_{jj} = -1. \quad (11)$$

Under this assumption we can proceed and obtain the formula

$$x_j(-\gamma_{ji}) x_j(\gamma_{ji}) = (u_j + \gamma_{ji} - 1) \cdots (u_j + 1) u_j.$$

We used that $\tau_j(u_j) = \lambda_{jj}(u_j - 1)$, so the formula is clear when $\lambda_{jj} = 1$ while if $\lambda_{jj} = -1$ there is at most one factor (empty product is interpreted as 1.) The case $\gamma_{ji} < 0$ is handled analogously (which is why we put absolute value in (11)).

The final formula for $t_i$ is

$$t_i = u_{1i} u_{2i} \cdots u_{ni},$$

$$u_{ji} = \begin{cases} (u_j + \gamma_{ji} - 1) \cdots (u_j + 1) u_j, & \gamma_{ji} > 0, \\ 1, & \gamma_{ji} = 0, \\ (u_j - |\gamma_{ji}|) \cdots (u_j - 2)(u_j - 1), & \gamma_{ji} < 0. \end{cases} \quad (12)$$

Similarly $\sigma_i$ can be deduced as follows. We have

$$\varphi(X_i u_j) = \varphi(\sigma_i(u_j) X_i).$$

Since $\varphi$ is a homomorphism of $R$-rings, we have

$$\varphi(X_i) u_j = \sigma_i(u_j) \varphi(X_i).$$

Substituting (10) we immediately obtain the sufficient condition

$$\sigma_i = \tau_1^\gamma_{1i} \tau_2^\gamma_{2i} \cdots \tau_n^\gamma_{ni}. \quad (13)$$
What remains is to ensure that for \( i \neq j \),

\[
X_i Y_j = \mu_{ij} Y_j X_i
\]

holds for appropriate scalars \( \mu_{ij} \), under suitable assumptions on \( \gamma_{kl} \). We have

\[
\varphi(X_i)\varphi(Y_j) = x_1^{(\gamma_{i1})} \cdots x_n^{(\gamma_{in})} \cdot x_1^{(-\gamma_{j1})} \cdots x_n^{(-\gamma_{jn})}.
\]

First we observe that if there exists \( k \in \{1, 2, \ldots, n\} \) with \( \lambda_{kk} = -1 \) and \( \gamma_{ki}\gamma_{kj} < 0 \) then \( \varphi(X_i)\varphi(Y_j) = 0 = \varphi(Y_j)\varphi(X_i) \). If no such \( k \) exists we want to move all factors on the right \( x_l^{(-\gamma_{ij})} \) to the left of all factors \( x_k^{(\gamma_{ki})} \). The only problem is when \( k = l \). A natural assumption for it to be possible is that actually \( \gamma_{ki}\gamma_{kj} \leq 0 \), because then the two factors are either both powers of \( x_k \) or both powers of \( x_l \).

To summarize, we make the following second assumption on \( \gamma_{ji} \):

\[ \forall i \neq j : \text{Either } \gamma_{ki}\gamma_{kj} < 0 \text{ for some } k \text{ with } \lambda_{kk} = -1, \text{ or } \gamma_{ki}\gamma_{kj} \leq 0 \text{ for all } k. \]

Under this assumption we then have for all \( k, l \):

\[
x_k^{(\gamma_{ki})} x_l^{(\gamma_{lj})} = \lambda_{kl}^{\gamma_{ki}\gamma_{lj}} x_l^{(\gamma_{lj})} x_k^{(\gamma_{ki})}.
\]

Thus we finally obtain that

\[
\varphi(X_i)\varphi(Y_j) = \mu_{ij} \varphi(Y_j)\varphi(X_i)
\]

holds, provided

\[
\mu_{ij} = \prod_{1 \leq k, l \leq n} \lambda_{kl}^{\gamma_{ki}\gamma_{lj}}.
\]

Using that \( \lambda_{kl} = (\mp 1)(-1)^{p(k)p(l)} \) this can be written

\[
\mu_{ij} = \mu_{ij}^x = (\mp 1)^{p'(i)p'(j)} \cdot (-1)^{p(i)p(j)},
\]

where the parities are defined by

\[
p(i) = \sum_{k=1}^{n} \bar{\gamma}_{ki} p(k), \quad (15)
\]

\[
p'(i) = \sum_{k=1}^{n} \bar{\gamma}_{ki}
\]

\((\bar{x} \in \mathbb{Z}/2\mathbb{Z} \text{ is the image of } x \in \mathbb{Z} \text{ under the canonical projection}).

Note that (15) expresses that the matrix \( \gamma \), when regarded as a \( \mathbb{Z} \)-module map \( \mathbb{Z}^m \to \mathbb{Z}^n \), is an even map, with respect to the parity \( p(a_1, \ldots, a_n) = \sum_k \bar{a}_k p(k) \).

**Theorem 8.** Let \( p, q, m \) be non-negative integers, put \( n = p + q \). Let \( \gamma = (\gamma_{ji}) \) be a \( n \times m \)-matrix with integer entries satisfying the following two conditions:
(i) $|\gamma_{ji}| \leq 1$ whenever $\lambda_{ii} = -1$,
(ii) $\forall i \neq j$: either $\gamma_{ki} \gamma_{kj} < 0$ for some $k$ with $\lambda_{kk} = -1$, or $\gamma_{ki} \gamma_{kj} \leq 0$ for all $k$.

Then there exist a TGW algebra $A(\gamma)^{\pm} = A_\mu(R, \sigma, t)$ with index set $[1, m]$, and a homomorphism of $R$-rings with involution

$$\varphi : A_\mu(R, \sigma, t) \to A_{p|q}^{\pm}.$$  \hfill (17)

The homomorphism is uniquely determined by the condition

$$\varphi(X_i) = x_1^{(\gamma_{1i})} x_2^{(\gamma_{2i})} \cdots x_n^{(\gamma_{ni})},$$

and the TGW algebra is given by the following data:

$$R = R_{p|q}^{\pm} = \mathbb{k}[u_1, \ldots, u_n]/(u_i^2 - u_i \mid \lambda_{ii} = -1),$$  \hfill (18)

where $\lambda_{ij} = \mp(-1)^{p(i)p(j)}$ and $t = (t_1, \ldots, t_m)$ where

$$t_i = u_{1i} u_{2i} \cdots u_{ni} \quad \text{and} \quad u_{ji} = \begin{cases} (u_j + \gamma_{ji} - 1) \cdots (u_j + 1) u_j, & \gamma_{ji} > 0, \\ 1, & \gamma_{ji} = 0, \\ (u_j - \gamma_{ji}) \cdots (u_j - 2)(u_j - 1), & \gamma_{ji} < 0. \end{cases}$$  \hfill (19)

Lastly, $\sigma = (\sigma_1, \ldots, \sigma_m)$, where

$$\sigma_i = \tau_1^{\gamma_{1i}} \tau_2^{\gamma_{2i}} \cdots \tau_n^{\gamma_{ni}}$$  \hfill (20)

where

$$\tau_i(u_j) = \begin{cases} \lambda_{ii}(u_i - 1), & \text{if } i = j, \\ u_j, & \text{otherwise}, \end{cases}$$  \hfill (21)

and $\mu = (\mu_{ij})_{1 \leq i, j \leq m}$ where

$$\mu_{ij} = (\mp 1)^{p'(i)p'(j)} \cdot (-1)^{p(i)p(j)}$$

where $p(i)$ and $p'(i)$ were defined in (15)–(16).

**Proof.** The discussion preceding the theorem proves that there exists a homomorphism of $R$-rings with involution

$$\varphi' : \mathcal{C} = \mathcal{C}_\mu(R, \sigma, t) \to A_{p|q}^{\pm}.$$  \hfill (16)

All that remains is to show that $\varphi'(\mathcal{J}) = 0$ where $\mathcal{J}$ is the unique maximal $\mathbb{Z}^m$-graded ideal trivially intersecting the degree zero component of $\mathcal{C}$. If $a$ is a homogeneous element of $\mathcal{J}$ then $a^+ \cdot a = 0$ hence, $\varphi'(a)^+ \cdot \varphi'(a) = 0$. By Lemma 6, it follows that $\varphi(a) = 0$. \hfill $\square$
Remark 4. Theorem 8 provides a large family of consistent non-regular TGW algebras.

Remark 5. Let \( p = 3, q = 2, m = 4 \) and

\[
\gamma = \begin{bmatrix}
1 & -1 & 1 \\
-1 & 1 & -1 \\
1 & 1 & 1
\end{bmatrix}.
\]

(The dashed line separates even from odd rows.) The corresponding TGW algebra \( \mathcal{A}(\gamma)^- \) is a quotient of \( U(\mathfrak{gl}(3|2)) \) (see Section 5).

4.2. Injectivity of \( \varphi \)

We prove a theorem which gives equivalent conditions for \( \varphi \) defined in (17) to be injective. This result will be used in Section 5.

Lemma 9 (Weak injectivity of \( \varphi \)). If \( g \in \mathbb{Z}^m \) and \( a \in \mathcal{A}(\gamma)^\pm_g, a \neq 0 \), then \( \varphi(a) \neq 0 \).

Proof. Suppose \( a \neq 0 \). Then, by the non-degeneracy of the gradation form of a TGW algebra, \( ba \neq 0 \) for some \( b \in \mathcal{A}(\gamma)^\pm_g \). Applying \( \varphi \) we get \( \varphi(ba) \neq 0 \) since \( \varphi|_{R_E} \) is injective. Hence \( \varphi(b)\varphi(a) \neq 0 \), so in particular \( \varphi(a) \neq 0 \). \( \Box \)

Let \( * : A^\pm_{p|q} \rightarrow A^\pm_{p|q}, \ a \mapsto a^* \), be the unique \( \mathbb{k} \)-linear map satisfying \((a^*)^* = a, (ab)^* = b^*a^* \) for all \( a,b \in A^\pm_{p|q} \), and \( x^*_i = \partial_i \) for all \( i \).

Lemma 10. Let \( \gamma \) be a matrix satisfying the conditions of Theorem 8 and let \( \mathcal{A}(\gamma)^\pm \) be the corresponding TGW algebra. Let \( a \in \mathcal{A}(\gamma)^\pm \) be a homogeneous element of degree \( g \in \mathbb{Z}^m \). If \( a^* \cdot a = 0 \) then \( a = 0 \).

Proof. Suppose \( a \neq 0 \). By Lemma 9, \( \varphi(a) \neq 0 \). So, by Lemma 6, \( \varphi(a)^* \cdot \varphi(a) \neq 0 \). Since \( \varphi \) is a map of rings with involution, \( \varphi(a^* \cdot a) \neq 0 \). Hence \( a^* \cdot a \neq 0 \). \( \Box \)

Remark 6. If \( \lambda_{ii} = 1 \) for all \( i \) then \( R^\pm_{p|q} \) defined in (18) is a domain. Then, by [10, Prop. 2.9], \( \mathcal{A}(\gamma)^\pm \) is also a domain. Hence Lemma 10 holds trivially in this case.

For a \( \mathbb{Z}I \)-graded algebra \( A = \bigoplus_{g \in \mathbb{Z}I} A_g \) we define the (graded) support of \( A \) to be \( \text{Supp}(A) := \{ g \in \mathbb{Z}I \mid A_g \neq \{0\} \} \).

Lemma 11. Let \( \mathcal{A}(\gamma)^\pm \) be a TGW algebra as constructed in Theorem 8. Let \( S^\pm \subseteq \mathbb{Z}^m \) be the support of \( \mathcal{A}(\gamma)^\pm \). Then, regarding \( \gamma \) as a \( \mathbb{Z} \)-linear map from \( \mathbb{Z}^m \) to \( \mathbb{Z}^n \) we have

\[
\gamma(S^+) \subseteq \{-1,0,1\}^p \times \mathbb{Z}^q,
\]

\[
\gamma(S^-) \subseteq \mathbb{Z}^p \times \{-1,0,1\}^q.
\]

Proof. We consider the case \( S^- \). The other case is analogous. Let \( g \in S^- \). Since any TGW algebra is generated as a left \( R \) module by the reduced monomials (Lemma 1), there exist sequences \((i_1,i_2,\ldots,i_k)\) and \((j_1,j_2,\ldots,j_l)\) of elements from \( \{1,2,\ldots,m\} \) with \( \{i_1,i_2,\ldots,i_k\} \cap \{j_1,j_2,\ldots,j_l\} = \emptyset \) such that

\[
a = Y_{i_1}Y_{i_2}\cdots Y_{i_k} \cdot X_{j_1}X_{j_2}\cdots X_{j_l}.
\]
is a nonzero element in $A(\gamma)^{-}$. By Lemma 9, $\varphi(a) \neq 0$. We have

$$\varphi(a) = \prod_{r=1}^{m} x_{r}^{-\gamma_{r1}} \cdots x_{r}^{-\gamma_{rk}} x_{r}^{\gamma_{rj1}} \cdots x_{r}^{\gamma_{rj1}}.$$  

For $r > p$, a product of the form

$$x_{r}^{-\gamma_{r1}} \cdots x_{r}^{-\gamma_{rk}} x_{r}^{\alpha_{rj1}} \cdots x_{r}^{\gamma_{rj1}}$$

can only be nonzero if the factors $x_{r}^{(\beta)}$ alternate between $x_{r}$ and $\partial_{r}$ (ignoring factors where $\beta = 0$). In particular, the number of $x_{r}$’s must differ from the number of $\partial_{r}$’s by at most one. $\square$

To prove that homomorphisms from TGW algebras are injective, the following result is useful.

**Theorem 12** ([15, Thm. 3.6]). If $A = A_{\mu}(R, \sigma, t)$ is consistent, then the centralizer $C_{A}(R)$ of $R$ in $A$ is an essential subalgebra of $A$, in the sense that $J \cap C_{A}(R) \neq \{0\}$ for any nonzero ideal $J$ of $A$.

**Theorem 13.** Let $\gamma$ be a matrix as in Theorem 8 and $A = A(\gamma)^{\pm}$ be the corresponding TGW algebra. Put $R = R_{p|q}$. The following statements are equivalent.

(i) $R$ is a maximal commutative subalgebra of $A$.

(ii) If $g \in \text{Supp}(A)$ is such that $\sigma_{g} := \prod_{i=1}^{m} \sigma_{g}^{\gamma_{i}} = \text{Id}_{R}$, then $g = 0$.

(iii) Put

$$\mathbb{Z}_{-}^{p|q} = \mathbb{Z}^{p} \times (\mathbb{Z}/2\mathbb{Z})^{q}, \quad \mathbb{Z}_{+}^{p|q} = (\mathbb{Z}/2\mathbb{Z})^{p} \times \mathbb{Z}^{q}.$$  

Then the composition

$$\text{Supp}(A) \to \mathbb{Z}^{m} \xrightarrow{\gamma} \mathbb{Z}^{n} = \mathbb{Z}^{p} \times \mathbb{Z}^{q} \xrightarrow{P} \mathbb{Z}_{+}^{p|q}$$

is injective (the first map is inclusion and the last is canonical projection).

(iv) The restriction of $\gamma: \mathbb{Z}^{m} \to \mathbb{Z}^{n}$ to Supp$(A)$ is injective.

(v) The map $\varphi$ defined in (17) is injective.

**Proof.** (i)⇒(ii): Suppose $g \in \text{Supp}(A)$ with $\sigma_{g} = \text{Id}_{R}$. Then for any $a \in A_{g}$ and $r \in R$ we have $ar = \sigma_{g}(r)a = ra$ which means that $A_{g} \subseteq C_{A}(R)$. But $C_{A}(R) = R$ by (i). Thus, since $A_{g} \neq \{0\}$, this means that $g$ must be $0$ and $A_{g} = R$.

(ii)⇒(iii): Suppose $P \circ \gamma(g) = 0$ in $\mathbb{Z}_{+}^{p|q}$ for some $g \in \text{Supp}(A)$. Then $\sigma_{g} = \prod_{r=1}^{m} \gamma^{(\gamma)(r)-} = \text{Id}_{R}$ because $\tau_{r}^{2} = \text{Id}_{R}$ for $r > p$ when $\pm = -$ and for $r \leq p$ when $\pm = +$. By (ii) this implies $g = 0$.

(iii)⇒(i): For simplicity we assume $\pm = -$. The other case is symmetric. Suppose $a \in C_{A}(R)$, $a \neq 0$. Since $C_{A}(R)$ is a graded subalgebra of $A$ we may without loss of generality suppose there exists $g \in \mathbb{Z}^{m}$ such that $a \in A_{g} \cap C_{A}(R)$. Since $a \neq 0$, this implies $g \in \text{Supp}(A)$. For all $r \in R$ we have $(\sigma_{g}(r) - r)a = ar - ra = 0$. Taking $r = u_{j}$ we get

$$0 = (\sigma_{g}(u_{j}) - u_{j})a = (\tau_{j}^{\gamma(g)})(u_{j}) - u_{j})a = \begin{cases} -\gamma(g)_{j}a, & j \leq p, \\ 0, & j > p, \gamma(g)_{j} = 0 \text{ in } \mathbb{Z}/2\mathbb{Z}, \\ (1 - 2u_{j})a, & j > p, \gamma(g)_{j} = 1 + 2\mathbb{Z}. \end{cases}$$
Since $a \neq 0$, we get $\gamma(g)_j = 0$ for all $j \leq p$. Suppose $j > p$ and $\gamma(g)_j = 1 + 2\mathbb{Z}$, then $0 = u_j(1 - 2u_j)a = -u_ja$ since $u_j^2 = u_j$. Combining this with $(1 - 2u_j)a = 0$ we get $a = 0$, a contradiction. Therefore, for $j > p$ we must have $\gamma(g)_j = 0$ in $\mathbb{Z}/2\mathbb{Z}$. This proves that $\gamma(g) = 0$ in $\mathbb{Z}^p \oplus (\mathbb{Z}/2\mathbb{Z})^q$.

(iii) $\Rightarrow$ (iv): Trivial.

(iv) $\Rightarrow$ (iii): Suppose $P \circ \gamma(g) = 0$ for some $g \in \text{Supp}(A)$. By Lemma 11 we get $\gamma(g) = 0$ so by (iv), $g = 0$.

(i) $\Rightarrow$ (v): Let $K = \ker(\varphi)$. If $K \neq \{0\}$, then by Theorem 12, $K \cap C_A(R) \neq \{0\}$. By (i), $C_A(R) = R$. Hence $K \cap R \neq \{0\}$. But by Theorem 8, $\varphi$ is a map of $R$-rings with involution and thus in particular $\varphi|_R = \text{Id}_R$ (where we used the injective maps $\rho$ and $\iota$ to identify $R$ with its image in $A$ and $A_E(k)$ respectively). This contradiction shows that $K = \{0\}$.

(v) $\Rightarrow$ (i): If $a \in C_A(R)$ then $\varphi(a) \in C_{A_{p|q}^\pm}(R)$ which equals $R$ by Lemma 5. By (v) this implies $a \in R$. \hfill $\square$

**Example 2.** Let $p, q$ be non-negative integers and $n = p + q > 0$. Consider the matrices

$$
\alpha = \begin{bmatrix}
1 & -1 & 1 \\
-1 & 1 & -1 \\
\vdots & \ddots & \ddots \\
-1 & 1 & -1 \\
\end{bmatrix}, \quad \beta = \begin{bmatrix}
1 & -1 & 1 \\
-1 & 1 & -1 \\
\vdots & \ddots & \ddots \\
-1 & 1 & -1 \\
\end{bmatrix},
$$

$$
\gamma = \begin{bmatrix}
1 & -1 & 1 \\
-1 & 1 & -1 \\
\vdots & \ddots & \ddots \\
-1 & 1 & 2 \\
\end{bmatrix}.
$$

These are $n \times m$ matrices (where $m = n - 1$ in the case of $\alpha$ and $m = n$ for $\beta, \gamma$) and define $\mathbb{Z}$-linear maps $\mathbb{Z}^m \rightarrow \mathbb{Z}^n$. In each case the top $p$ rows are defined to be even and the remaining $q$ rows are odd. It is easy to see that these maps are injective, hence by Theorem 13(iv) $\Rightarrow$ (v), the homomorphism $\varphi : A(\zeta)^\pm \rightarrow A_{p|q}^\pm$ is injective for $\zeta = \alpha, \beta, \gamma$.

### 4.3. A description of the graded support of $A(\gamma)^-$

Although sufficient for the application to Lie superalgebras, the characterization in Theorem 13 of the injectivity of the map (17) is not completely satisfactory because we lack a good description of the support of $A(\gamma)^\pm$. In this section we give a combinatorial description of the support of $A(\gamma)^-$ in terms of certain pattern-avoiding vector compositions of the columns of $\gamma$. A similar analysis applies to $A(\gamma)^+$. This allows us to compute the support in the certain cases. In addition, it shows that this is a non-trivial problem for a general (non-regular) TGW algebra.

Put $W = \mathbb{Z}^d$. A $d$-dimensional vector composition of $w \in W$ is a tuple $c = (c_1, c_2, \ldots, c_\ell) \in W^\ell$ such that $c_1 + c_2 + \cdots + c_\ell = w$. The non-negative integer
\( \ell \) is the length of \( c \). The \( c_i \) are called the parts of the composition \( c \). A given vector \( u \in W \) appears with multiplicity \( m \) (in \( c \)) if \( c_j = u \) for exactly \( m \) choices of \( j \in \{1,4\} \).

Example 3. \( \begin{bmatrix} 1 & 3 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} \) is a 3-dimensional vector composition of \( \begin{bmatrix} 7 \\ 0 \end{bmatrix} \).

Theorem 14. Let \( A = A(\gamma)^{-} \) be a TGW algebra constructed as in Theorem 8. The following are equivalent for \( g \in \mathbb{Z}^m \):

(i) \( g \in \text{Supp}(A) \).
(ii) There exists an \( n \)-dimensional vector composition of \( \gamma(g) \) of length \( |g| = \sum_{i \in V} |g_i| \) such that

(a) each part is of the form \( \text{sgn}(g_i)\gamma(e_i) \) for \( i \in V \) which appears with multiplicity \( |g_i| \),
(b) for each \( r > p \) the sequence \( (\text{sgn}(g_{i_1})\gamma_{r_{i_1}}, \ldots, \text{sgn}(g_{i_{|g|}})\gamma_{r_{i_{|g|}}}) \) contains no consecutive subsequence of the form

\[
(1,0,\ldots,0,1) \quad \text{or} \quad (-1,0,\ldots,0,-1)
\]

where there are zero or more 0’s.

Proof. By Lemma 1, \( g \in \text{Supp}(A) \) if and only if \( A_g \) contains a reduced monomial \( a = Z_{i_1}Z_{i_2}\cdots Z_{i_{|g|}} \) (where each \( Z_{i_k} \in \bigcup_{j \in V} \{X_j, Y_j\} \)) such that \( a \neq 0 \), which by Lemma 9 is equivalent to \( \varphi(a) \neq 0 \). Put \( \varepsilon_k = \text{sgn}(g_{i_k}) \). We have

\[
\varphi(a) = \varphi(Z_{i_1})\cdots \varphi(Z_{i_{|g|}}) = \pm \prod_{r \in E} x_r^{(\varepsilon_1\gamma_{r_{i_1}})\cdots x_r^{(\varepsilon_{|g|}\gamma_{r_{i_{|g|}}})}}
\]

which is nonzero if and only if property (b) in the theorem holds. \( \square \)

Example 4. If \( q = 0 \) then \( \text{Supp}(A(\gamma)^{-}) = \mathbb{Z}^m \) because condition (b) is void.

Example 5. Let \( m = 3, p = 1, q = 2 \) and \( \gamma = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix} \). Example 3 shows that \( 1,2,1 \) belongs to the graded support of the TGW algebra \( A(\gamma)^{-} \). On the other hand \( 2,1,0 \) does not, because there is no vector composition of length 3 with two parts equal to \( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \) and one part equal to \( \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \) which avoids the pattern \( (1,0,\ldots,0,1) \) in the second row.

Example 6. Let \( m = 2, p = 0, q = 1 \) and \( \gamma = \begin{bmatrix} 1 & -1 \end{bmatrix} \). Then

\[
\text{Supp}(A(\gamma)) = \{(g_1, g_2) \in \mathbb{Z}^2 \mid |g_1 - g_2| \leq 1\}.
\]

Example 7. Let \( m = 2, p = 0, q = 2, \gamma = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \), then

\[
\text{Supp}(A(\gamma)) = \{(0,0), (0,1), (1,0), (1,1), (1,2)\}.
\]
5. Relation to $\mathfrak{gl}(m|n)$ and $\mathfrak{osp}(m|2n)$

Irreducible completely pointed weight modules have been classified and realized by differential operators in the case of simple finite-dimensional complex Lie algebras $\mathfrak{g}$ in [6], [4] and over $U_q(\mathfrak{sl}_n)$ in [11]. In [9, Sect. 6], Coulombier classified all irreducible completely pointed highest weight modules over the orthosymplectic Lie superalgebras $\mathfrak{osp}(m|2n)$, and realized them by differential operators on supersymmetric Grassmann algebras. See also [23] for a uniform treatment of spinor representations of orthosymplectic Lie superalgebras. In this section we show that, analogously to the Lie algebra case [16], the realization of $\mathfrak{osp}(m|2n)$ by differential operators factors through a corresponding twisted generalized Weyl algebra of the form $\mathcal{A}(\alpha)$.

Recall that the Lie superalgebra $\mathfrak{gl}(m|n)$ is the Lie superalgebra of all linear transformations of $(m|n)$-dimensional vector superspace, and $\mathfrak{osp}(m|2n)$ is the subalgebra of $\mathfrak{gl}(m|2n)$ preserving a non-degenerate even symmetric bilinear form on an $(m|2n)$-dimensional vector superspace or, equivalently, the subalgebra of $\mathfrak{gl}(2n|m)$ preserving a non-degenerate even skew-symmetric bilinear form on an $(2n|m)$-dimensional vector superspace. The even part of $\mathfrak{osp}(m|2n)$ is the direct sum $\mathfrak{so}(m) \oplus \mathfrak{sp}(2n)$. The Lie superalgebras $\mathfrak{gl}(m|n)$ and $\mathfrak{osp}(m|2n)$ are Kac–Moody superalgebras and can be described by Chevally generators and relations; see [17], as follows. Let $p, q$ be nonnegative integers, $n = p+q > 0$. The Chevalley generators of $\mathfrak{gl}(p|q)$ are $e_1, \ldots, e_{n-1}, h_1, \ldots, h_n, f_1, \ldots, f_{n-1}$, with the convention that $e_p, f_p$ are odd and all other generators are even. They satisfy the relations

$$[h_i, h_j] = 0, \quad [h_i, e_j] = \delta_{i,j}e_j - \delta_{i,j+1}e_j, \quad [h_i, f_j] = -\delta_{i,j}f_j + \delta_{i,j+1}f_j,$$

$$[e_i, f_j] = \delta_{i,j}(h_i - (-1)^{i+j}h_{i+1}).$$

The Lie superalgebra $\mathfrak{gl}(p|q)$ is the quotient of the infinite-dimensional Lie algebra with the above relations by the maximal ideal which intersects trivially the Cartan subalgebra generated by $h_1, \ldots, h_n$. The Chevalley generators of $\mathfrak{osp}(2p+1|2q)$ are obtained from those for $\mathfrak{gl}(p|q)$ by adding odd generators $e_n, f_n$ and relations

$$[h_i, e_n] = \delta_{i,n}e_n, \quad [h_i, f_n] = -\delta_{i,n}f_n, \quad [e_n, f_n] = h_n,$$

$$[e_i, f_n] = [e_n, f_i] = 0 \text{ if } n \neq i.$$ 

The Chevalley generators of $\mathfrak{osp}(2p|2q)$ are obtained from those for $\mathfrak{gl}(p|q)$ by adding even generators $e_n^2, f_n^2$. From the above description it is not difficult to see that we have an embedding of Lie superalgebras

$$\mathfrak{gl}(p|q) \subset \mathfrak{osp}(2p|2q) \subset \mathfrak{osp}(2p+1|2q).$$

5.1. Weyl superalgebra and $\mathfrak{osp}(2p|2q)$

Let $V$ be a vector superspace equipped with even skew-symmetric form $\omega : V \times V \to \mathfrak{k}$. We define the Weyl superalgebra $W(V, \omega)$ as the quotient of the tensor superalgebra $T(V)$ by the relations

$$v \otimes w - (-1)^{p(v)p(w)}w \otimes v = \omega(v, w).$$
Lemma 15. Let $g$ denote the span of the elements of the form $vw + (-1)^{p(v)p(w)}wv$ for all $v, w \in V$. Then $g$ is closed under the supercommutator and the adjoint action of $g$ on $V$ preserves the form $\omega$.

Proof. Note that 

$$vw + (-1)^{p(v)p(w)}wv = 2vw - \omega(v, w)$$

and

$$[vw, u] = v[w, u] + (-1)^{p(w)p(u)}[v, u]w = \omega(w, u)v + (-1)^{p(w)p(u)}\omega(v, u)w.$$ 

The super Jacobi identity ensures that $\omega$ is ad$_{vw}$-invariant. Indeed,

$$\omega([vw, u_1], u_2) + (-1)^{p(vw)p(u_1)}\omega(u_1, [vw, u_2]) = [[vw, u_1], u_2] + (-1)^{p(vw)p(u_1)}[u_1, [vw, u_2]] = [vw, [u_1, u_2]] = 0.$$

Finally, $g$ is closed under supercommutator as

$$[vw, xz] = [vw, x]z + (-1)^{p(vw)p(x)}x[vw, z] = [vw, x]z + (-1)^{p(vw)p(xz)}[vw, z]x.$$

$$\square$$

Corollary 16. If $\omega$ is non-degenerate then $g$ constructed in the previous lemma is isomorphic to $osp(r|s)$ where $r = \dim V_1$ and $s = \dim V_0$.

Let us assume that the $\omega$ is non-degenerate and both $r$ and $s$ are even. Set $r = 2p$, $s = 2q$ and $n = p + q$. Choose basis $x_1, \ldots, x_n, y_1, \ldots, y_n$ in $V$ such that

$$\omega(x_i, x_j) = \omega(y_i, y_j) = 0, \quad \omega(y_i, x_j) = \delta_{i,j}.$$ 

The parity is defined by

$$p(x_i) = p(y_i) = \begin{cases} 1 & \text{if } i \leq p \\ 0 & \text{if } i > p. \end{cases}$$

In this case the Weyl algebra is isomorphic to $A^\pm_{q|p}$ since the defining relations are

$$x_ix_j - (-1)^{p(i)p(j)}x_jx_i = y_iy_j - (-1)^{p(i)p(j)}y_jy_i = 0,$$

$$y_ix_j - (-1)^{p(i)p(j)}x_jy_i = \delta_{i,j}.$$ 

Let $g = gl(p|q)$, or $osp(2p|2q)$ and identify $\mathbb{Z}^m$ with the root lattice of $g$ with basis consisting of the distinguished simple roots of $g$. Let $\zeta : \mathbb{Z}^m \rightarrow \mathbb{Z}^n$ be the $\mathbb{Z}$-linear maps given by the matrices

$$\begin{bmatrix} 1 \\ -1 & 1 \\ -1 & & 1 \\ & \ddots & 1 \\ & & -1 \\ & & & & \ddots & 1 \\ & & & & & -1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 & 1 \\ & -1 & 1 \\ & & -1 \\ & & & \ddots & 1 \\ & & & & -1 & 2 \end{bmatrix}$$

respectively. Let $A^\pm_{q|p}$ be the Weyl superalgebra.
**Theorem 17.** Let $p, q$ be nonnegative integers, $n = p + q > 0$. Let $\mathfrak{g} = \mathfrak{gl}(p|q)$, or $\mathfrak{osp}(2p|2q)$ and let $\zeta$ be as above. Then there is a commutative triangle of associative algebras with involution

\[
\begin{array}{c}
U(\mathfrak{g}) \\
\pi \\
\psi
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\downarrow \\
\varphi
\end{array} \quad \begin{array}{c}
\mathcal{A}_q \ominus p \\
\mathcal{A}(\zeta) \\
\end{array}
\]  

where $\varphi$ is given by Theorem 8, $\psi(e_i) = X_i$, $\psi(f_i) = Y_i$, $\psi(h_{ii}) = \lambda_{ii}(u_i - 1)$, and

\[
\pi(e_i) = \begin{cases} x_i \partial_{i+1}, & i < n, \\ x_n^2, & i = n, \end{cases} \quad \pi(f_i) = \pi(e_i)^* \quad \pi(h_i) = x_i \partial_i + (-1)^{p(i)} \frac{1}{2}.
\]

**Proof.** First, the existence of $\pi$ follows from Corollary 16. We need to check that $\tilde{\pi}(j) = 0$. This follows immediately from the fact that $\tilde{\pi}(h)$ is the self-centralizing subalgebra of $\tilde{\pi}(\mathfrak{g})$. Therefore we have a map $\tilde{\pi} : \mathfrak{g} \to A_q \ominus p$ which extends to the homomorphism $\pi : U(\mathfrak{g}) \to A_q \ominus p$ of associative algebras. By Theorem 13, $\varphi$ is injective. Moreover, the image of $\varphi$ coincides with the image of $\pi$. This immediately proves the existence of a unique map $\psi$ such that the diagram commutes. \hfill $\square$

### 5.2. Clifford superalgebra and $\mathfrak{osp}(2p + 1|2q)$

Let $V$ be a vector superspace equipped with even symmetric form $\beta : V \times V \to \mathbb{k}$. We define the Clifford superalgebra $\text{Cliff}(V, \beta)$ as the quotient of the tensor superalgebra $T(V)$ by the relations

$$v \otimes w + (-1)^{p(v)p(w)} w \otimes v = \beta(v, w).$$

Note that $\text{Cliff}(V, \beta)$ is finite-dimensional if $V$ is purely even. As any associative superalgebra $\text{Cliff}(V, \beta)$ has the associated Lie superalgebra structure defined by $[x, y] = xy - (-1)^{p(x)p(y)}yx$. Let $\mathfrak{g}$ denote the Lie subalgebra of $\text{Cliff}(V, \beta)$ generated by $V$.

**Lemma 18.** We have the decomposition $\mathfrak{g} = V \oplus [V, V]$ such that $[[V, V], V] \subset V$. As a vector space $[V, V]$ is isomorphic to $\Lambda^2 V$ and concides with the span of $2vw - \beta(v, w)$ for all $v, w \in V$.

**Proof.** First, we compute the commutator

$$[v, w] = vw - (-1)^{p(v)p(w)} wv = 2vw - \beta(v, w).$$

Next we compute the commutator between $[v, w]$ and $u$ using super Leibniz identity

$$[u, [v, w]] = 2[u, vw] = 2([u, v]w + (-1)^{p(u)p(v)} v[u, w])$$

$$= 2(2uvw - \beta(u, v)w + (-1)^{p(u)p(v)} 2vuw - (-1)^{p(u)p(v)} \beta(u, w)v).$$

Using $vu = (-1)^{p(u)p(v)} uv + \beta(v, u)$ and the symmetry of $\beta$ we obtain

$$[u, [v, w]] = 2(\beta(u, v)w - (-1)^{p(u)p(v)} \beta(u, w)v).$$

Hence we have obtained $[[V, V], V] \subset V$ and by Jacobi identity $[[V, V], [V, V]] \subset [V, V]$. \hfill $\square$
We concentrate on the case when $\beta$ is non-degenerate and $\dim V = (2p|2q)$, let $n = p + q$ and choose a basis $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n$ such that

$$\beta(\xi_i, \xi_j) = \beta(\eta_i, \eta_j) = 0, \quad \beta(\eta_i, \xi_j) = \delta_{i,j}. $$

The parity is defined by

$$p(\xi_i) = p(\eta_i) = \begin{cases} 0 & \text{if } i \leq p, \\ 1 & \text{if } i > p. \end{cases}$$

The corresponding Clifford superalgebra is isomorphic to $A^+_{p|q}$. The defining relations are

$$\xi_i \xi_j + (-1)^{p(i)p(j)} \xi_j \xi_i = \eta_i \eta_j + (-1)^{p(i)p(j)} \eta_j \eta_i = 0,$$

$$\eta_i \xi_j + (-1)^{p(i)p(j)} \xi_j \eta_i = \delta_{ij}. $$

**Lemma 19.** The Lie subsuperalgebra of $A^+_{p|q}$ generated by $\xi_i, \eta_i$ for $i = 1, \ldots, n$ is isomorphic to $\mathfrak{osp}(2p+1|2q)$.

**Proof.** In notations of Lemma 18, consider the adjoint action of $[V, V]$ on $V$. The Leibniz rule implies that the form $\beta$ is invariant under this action. Hence $[V, V]$ is isomorphic to $\mathfrak{osp}(2p, 2q)$ and $V$ is its natural representation. Since obviously $V \oplus [V, V]$ is simple, it must be isomorphic to $\mathfrak{osp}(2p+1|2q)$. $\square$

**Corollary 20.** There exist homomorphisms of associative superalgebras

$$\pi_1 : U(\mathfrak{osp}(2p|2q)) \to A^+_{p|q} \quad \text{and} \quad \pi_2 : U(\mathfrak{osp}(2p+1|2q)) \to A^+_{p|q}. $$

Let $q \neq 0$. Let us assume that $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ are the Chevalley generators of $\mathfrak{osp}(2p+1|2q)$ such that $e_p, f_p, e_n, f_n$ are odd and all other generators are even. Then we have

$$\pi_2(e_i) = \begin{cases} \xi_i \eta_{i+1} & \text{if } i < n, \\ \xi_n & \text{if } i = n, \end{cases} \quad \pi_2(f_i) = \begin{cases} \xi_{i+1} \eta_i & \text{if } i < n, \\ \eta_n & \text{if } i = n, \end{cases} $$

and $\pi_1$ is obtained from $\pi_2$ by restriction.

Let $\mathfrak{g} = \mathfrak{gl}(p|q), \mathfrak{osp}(2p|2q)$ or $\mathfrak{osp}(2p+1|2q)$ and identify $\mathbb{Z}^m$ with the root lattice of $\mathfrak{g}$ with basis consisting of the distinguished simple roots of $\mathfrak{g}$. Let $\zeta : \mathbb{Z}^m \to \mathbb{Z}^n$ be the $\mathbb{Z}$-linear maps given by the matrices

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ -1 & 1 & \cdots & 1 \\ \vdots & -1 & \ddots & \vdots \\ -1 & \vdots & \cdots & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & \cdots & 1 \\ -1 & 1 & \cdots & 1 \\ \vdots & -1 & \ddots & \vdots \\ -1 & \vdots & \cdots & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & \cdots & 1 \\ -1 & 1 & \cdots & 1 \\ \vdots & -1 & \ddots & \vdots \\ -1 & \vdots & \cdots & 1 \end{bmatrix}$$

(23)

respectively. Let $A^+_{p|q} = A_I$ be the Weyl algebra with index superset $I$, $I_0 = [1, p], I_1 = [p + 1, p + q]$. 
Theorem 21. Let $p, q$ be nonnegative integers, $n = p + q > 0$. Let $\mathfrak{g} = \mathfrak{gl}(p|q)$, $\mathfrak{osp}(2p|2q)$ or $\mathfrak{osp}(2p + 1|2q)$. Then there is a commutative triangle of associative algebras with involution

\[
\begin{array}{ccc}
U(\mathfrak{g}) & \xrightarrow{\pi} & A^+_{p|q} \\
\psi & & \varphi \\
& A(\zeta)^+ & 
\end{array}
\]

(24)

where $\varphi$ is given by Theorem 8, $\psi(e_i) = X_i$, $\psi(f_i) = Y_i$, $\psi(h_{ii}) = \lambda_{ii}(u_i - 1)$, and

\[
\pi(e_i) = \begin{cases} 
  x_i \partial_{i+1}, & i < n, \\
  x_n, & i = n, \mathfrak{g} = \mathfrak{osp}(2q + 1|2p), \\
  x_n^2, & i = n, \mathfrak{g} = \mathfrak{osp}(2q|2p), 
\end{cases}
\]

\[
\pi(f_i) = \pi(e_i)^*, \quad \pi(h_i) = x_i \partial_i - (-1)^{p(i)} \frac{1}{2}.
\]

The proof is similar to Theorem 17 and we leave it to the reader.

5.3. On $A^+_{p|q}$ versus $A^-_{q|p}$

If we disregard $\mathbb{Z}_2$-grading, then we have an isomorphism of associative algebras $A^\pm_{p|0} \simeq A^\mp_{0|p}$. We suspect that $A^+_{p|q}$ and $A^-_{q|p}$ are not isomorphic in general. Note also that $A^-_{p|q}$ is isomorphic to the tensor product $M_{2q} \otimes (A^-_{p|0})$, while $A^+_{q|p}$ is isomorphic to the supertensor product $M_{2q} \otimes (A^+_{0|p})$. However, we do have the following result.

Corollary 22. Consider the sublattice

\[
\Gamma = \{ (a_1, \ldots, a_n) | a_1 + \cdots + a_n \in 2\mathbb{Z} \}
\]

in $\mathbb{Z}^n$. Let $C^\pm_{p|q}$ denote the subsuperalgebra of elements of $A^\pm_{p|q}$ with the support in $\Gamma$. Then $C^+_{p|q}$ and $C^-_{q|p}$ are isomorphic superalgebras.

Proof. Theorems 17 and 21 provide the homomorphisms from $U(\mathfrak{osp}(2p|2q))$ to $A^-_{q|p}$ and $A^+_{p|q}$ respectively. It follows from formulas defining these isomorphisms that $C^-_{q|p}$ and $C^+_{p|q}$ are respective images. Consider the modules

\[
M^- := A^-_{q|p} \otimes_{\mathbb{Z}[\partial_1, \ldots, \partial_n]} \mathbb{k}, \quad M^+ := A^+_{p|q} \otimes_{\mathbb{Z}[\eta_1, \ldots, \eta_n]} \mathbb{k},
\]

and let

\[
N^- = C^-_{q|p} (1 \otimes 1), \quad N^+ = C^+_{p|q} (1 \otimes 1).
\]

Note that $N^\pm$ is a simple module over $C^+_{q|p}$ and $C^-_{q|p}$, respectively, hence both $N^+$ and $N^-$ are simple $U(\mathfrak{osp}(2p|2q))$-modules. Furthermore if $v = 1 \otimes 1$, then

\[
f_i v = 0, \quad h_i v = -(-1)^{p(i)} v.
\]

Thus both $N^+$ and $N^-$ are simple lowest weight modules with the same lowest weight. Thus, $N^+$ and $N^-$ are isomorphic, therefore they have the same annihilator $J \subset U(\mathfrak{osp}(2p|2q))$ and we obtain

\[
C^+_{p|q} \simeq U(\mathfrak{osp}(2p|2q))/J \simeq C^-_{q|p}.
\]
5.4. Consequence for classical Lie algebras

Taking $q = 0$ in Theorem 21 we immediately get the following result.

**Corollary 23.** For $\mathfrak{g} = \mathfrak{sl}_n, \mathfrak{so}_{2n+1},$ or $\mathfrak{so}_{2n},$ there is a corresponding $\gamma$ and a commutative triangle of associative algebras with involution

$$
\begin{array}{c}
U(\mathfrak{g}) \\
\downarrow \psi \\
A(\gamma)^+ \\
\downarrow \varphi \\
A_n^+ \\
\end{array}
$$

We can now prove that further primitive quotients of enveloping algebras of classical Lie algebras are examples of TGWAs. This extends previous results by the authors [16], where a condition for $U(\mathfrak{g})/J$ to be a not-necessarily abelian TGW algebra (i.e., we allowed $\sigma_i \sigma_j \neq \sigma_j \sigma_i$) was given.

**Theorem 24.** If $\mathfrak{g} = \mathfrak{so}_{2n}, \mathfrak{so}_{2n+1}$ or $\mathfrak{sp}_{2n}$ and $M$ be a finite-dimensional completely pointed simple $\mathfrak{g}$-module and let $J = \text{Ann}_{U(\mathfrak{g})} M$. Then $U(\mathfrak{g})/J$ is graded isomorphic to a TGWA of the form $A(\gamma)^+$. The same is true for any fundamental representation of $\mathfrak{sl}_n$.

**Proof.** The problem is to show that we can choose $\sigma_i$ so that the group $G$ generated by $\sigma_i$ is abelian.

If $\mathfrak{g} = \mathfrak{so}_{2n}$ or $\mathfrak{so}_{2n+1}$ and $M$ is a spinor representation, then $U(\mathfrak{g})/J$ is isomorphic to a subalgebra in the Clifford algebra with abelian $G$ as follows from Corollary 23.

Let $\mathfrak{g} = \mathfrak{sl}_n$. Consider the embedding $\mathfrak{sl}_n \subset \mathfrak{so}_{2n+1}$ induced by the embedding of the corresponding Dynkin diagrams. The restriction of the spinor representation to $\mathfrak{sl}_n$ contains all fundamental representations. Let $\gamma$ be the rightmost matrix in (23) and consider the subalgebra in $\mathfrak{C} \subset A(\gamma)^+$ generated by $X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_{n-1}$. Let $I = \text{Ann}_{\mathfrak{C}} M$ and $\mathfrak{B} = \mathfrak{C}/I \simeq \text{End}(M)$. Then $\mathfrak{B}$ is a direct summand in the semisimple algebra $\mathfrak{C}$. Hence $\sigma_i$ for $i = 1, \ldots, n-1$ preserve $\mathfrak{B} \cap R$ and the statement follows.

Let $\Gamma$ denote the set of weights of $M$. Note that $\sigma_i$ must permute projectors $E_\beta$, hence it is defined by a permutation of $\Gamma$.

Let $M$ be the standard representation of $\mathfrak{sp}_{2n}$. Then $\Gamma = \{\pm \varepsilon_1\}$. Let $\sigma_1 = \sigma_2 = \cdots = \sigma_{n-1}$ be defined by the permutation $\kappa = (\varepsilon_1, \ldots, \varepsilon_n) (-\varepsilon_n, \ldots, -\varepsilon_1)$ and $\sigma_n$ be defined by the permutation $\tau = (\varepsilon_1, -\varepsilon_1) \cdots (\varepsilon_n, -\varepsilon_n)$.

If $\mathfrak{g} = \mathfrak{so}_{2n}$ and $M$ is the standard representation, then we choose $\sigma_1 = \cdots = \sigma_{n-1}$ as in the previous case and let $\sigma_n$ be given by the permutation $\kappa \tau$.

Finally, if $\mathfrak{g} = \mathfrak{so}_{2n+1}$ and $M$ is the standard representation, then $\Gamma = \{\pm \varepsilon_1, 0\}$ and we define $\sigma_1 = \cdots = \sigma_n$ by the permutation $(\varepsilon_1, \ldots, \varepsilon_n, 0, -\varepsilon_n, \ldots, -\varepsilon_1).$ \hfill $\Box$

6. Open problems

**Problem 1.** For a simple Lie algebra $\mathfrak{g}$, list all finite-dimensional irreducible $\mathfrak{g}$-modules $M$ for which there exists a graded isomorphism between $U(\mathfrak{g})/\text{Ann}_{U(\mathfrak{g})} M$ and a TGW algebra (equivalently, for which there is a choice of commuting $\sigma_i$).
We believe none of the non-fundamental representations of $\mathfrak{sl}_n$ for $n > 2$ are in this list. The remaining cases to consider are the 27-dimensional representation of $E_6$ and 56-dimensional representation of $E_7$.

Problem 2. Find necessary and sufficient conditions for a not necessarily regular TGW algebra $A_\mu(R, \sigma, t)$ to be consistent, generalizing the main result of [10].

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