COMPLEX MULTIPLICATION OF EXACTLY SOLVABLE CALABI-YAU VARIETIES

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ABSTRACT:

We propose a conceptual framework that leads to an abstract characterization for the exact solvability of Calabi-Yau varieties in terms of abelian varieties with complex multiplication. The abelian manifolds are derived from the cohomology of the Calabi-Yau manifold, and the conformal field theoretic quantities of the underlying string emerge from the number theoretic structure induced on the varieties by the complex multiplication symmetry. The geometric structure that provides a conceptual interpretation of the relation between geometry and the conformal field theory is discrete, and turns out to be given by the torsion points on the abelian varieties.

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1 Introduction

Arithmetic properties of exactly solvable Calabi-Yau varieties encode string theoretic information of their underlying conformal field theory. Results in this direction address the issue of an intrinsic geometric description of the spectrum of the conformal field theory, and a geometric derivation of the characters of the partition function. The computations that have been performed so far depend on the explicit computation of the Hasse-Weil L-function of Fermat varieties, or more generally Brieskorn-Pham type spaces. The special feature of these manifolds, first observed by Weil [1, 2] about fifty years ago, is that the cohomological L-function can be expressed in terms of number theoretic L-functions, defined by special kinds of so-called Größencharaktere, or algebraic Hecke characters. Weil’s analysis of Fermat type L-functions in terms of Jacobi-sum Größencharaktere was generalized by Yui to the class of Brieskorn-Pham L-functions [3]. It was shown in [4] that the algebraic number field that emerges from the Hasse-Weil L-function of an exactly solvable Calabi-Yau variety leads to the fusion field of the underlying conformal field theory and thereby to the quantum dimensions. It was further proven in [5] that the modular form defined by the Mellin transform of the Hasse-Weil L-function of the Fermat torus arises from the characters of the underlying conformal field theory. This establishes a connection between algebraic varieties and Kac-Moody algebras via their modular properties.

The basic ingredient of the investigations described in Refs. [4, 5] is the Hasse-Weil L-function, an object which collects information of the variety at all prime numbers, therefore providing a ‘global’ quantity that is associated to Calabi-Yau varieties. The number theoretic interpretation which leads to the physical results proceeded in a somewhat experimental way, by observing the appearance of Jacobi-sum characters in [4], and that of affine theta functions in [5]. This leaves open the question whether these results depend on the special nature of the varieties under consideration, or whether it is possible to identify an underlying conceptual framework that explains the emergence of conformal field theoretic quantities from the discrete structure of the Calabi-Yau variety. It is this problem which we address in the present paper.

In principle, the physical question raised translates into a simply stated mathematical problem: provide a theorem that states the conditions under which the geometric Hasse-Weil L-function decomposes into a product of number theoretic L-functions. If such a statement were known
one could ask whether the class of varieties that satisfies the stated conditions can be used to derive conformal field theoretic results, e.g. in the spirit of the results of [4, 5]. It turns out that this question is very difficult. In dimension one it basically is the Shimura-Taniyama conjecture, which has been recently proven in full generality by work [6] that extends the results by Wiles and Taylor in the semistable case [7].

In higher dimensions much less is known, and the problem is often summarized as the Langlands program, a set of conjectures, which might be paraphrased as the hope that certain conjectured geometric objects, called motives, lead to Hasse-Weil L-series that arise from automorphic representations [8]. At present very little is known in this direction as far as general structure theorems are concerned. There exists, however, a subclass of varieties for which useful results have been known for some time, and which turns out to be useful in the present context. In dimension one this is the class of elliptic curves with complex multiplication (CM), i.e. such curves admit a symmetry algebra that is exceptionally large. It was first shown by Deuring in the fifties [9], following a suggestion of Weil in [2], that for such tori with CM the cohomological L-function becomes a number theoretic object. More precisely, he showed that associated to the complex multiplication field of the elliptic curve are algebraic Hecke characters which describe the Hasse-Weil L-function, much like Weil's Jacobi-sum Grössencharaktere do in the case of the Fermat varieties. This provides an explicit description of the L-function for toroidal compactifications.

Complex multiplication is a group property, and it is not obvious what the most convenient physical generalization of this notion is for higher dimensional Calabi-Yau varieties. One interesting attempt in this direction was recently made by Gukov and Vafa [10], who conjectured that exactly solvable Calabi-Yau varieties can be characterized in terms of a property of the intermediate Jacobian described in [11, 12, 13] (see also [14]). In the present paper we follow a different approach, which is motivated in part by the results of [5] and [15]. In [15] our focus was on properties of black hole attractor Calabi-Yau varieties with finite fundamental group. In an interesting paper Moore [16] has shown that attractor varieties with elliptic factors are distinguished by the fact that they admit complex multiplication. The aim of [15] was to introduce a framework in which the notion of complex multiplication can be generalized to non-toroidal Calabi-Yau varieties of arbitrary dimension via abelian varieties that can be derived from the Calabi-Yau cohomology. Abelian varieties are natural higher dimensional generalizations of elliptic curves, and certain types admit complex multiplication.
The link between Calabi-Yau manifolds and abelian varieties therefore allows us to generalize
the elliptic analysis to the higher dimensional abelian case.

In the most general context, the relation between exactly solvable Calabi-Yau varieties and
complex multiplication very likely goes beyond abelian varieties and involves the theory of
motives with (potential) complex multiplication. The program of constructing a satisfactory
framework of motives is incomplete at this point, despite much effort. In this paper we
therefore focus on the simpler case of exactly solvable Calabi-Yau varieties that lead to motives
derived from abelian varieties which admit complex multiplication. We provide a conceptual
explanation of the results of [4] and thereby establish a framework that extends the analysis
of [4] to general Calabi-Yau manifolds.

The paper is organized as follows. In Sections 2 and 3 we very briefly recall the arithmetic
and number theoretic concepts that will be used in the following parts. In Section 4 we
discuss two explicit examples of Fermat type which illustrate the transition from geometry to
number theory in a concrete way. In Section 5 we discuss Deuring’s result, which provides
a number theoretic interpretation of the L-function of arbitrary elliptic curves with complex
multiplication. We use an idelic formulation of Deuring’s ideal theoretic analysis because
this allows us to clearly identify the geometric structure that provides the conceptual basis
of this result — the discrete set of torsion point on the elliptic curves. In Section 6 we review
the structure of higher dimensional abelian varieties with complex multiplication and show
how their L-functions can be expressed in terms of algebraic Hecke characters. In Section 7
we describe how one can associate abelian varieties to Calabi-Yau manifolds by tracing the
cohomology of Calabi-Yau varieties to the Jacobians of curves [15]. Section 8 contains an
example, and in Section 9 we collect some results from adelic number theory for convenience.

2 Arithmetic L-functions

2.1 The Hasse-Weil L-function

The starting point of the arithmetic analysis is the set of Weil conjectures [1], the proof of
which was completed by Deligne [17]. For algebraic varieties the Weil–Deligne result states a
number of structural properties for the congruent zeta function at a prime number $p$ defined
\[
Z(X/\mathbb{F}_p, t) \equiv \exp \left( \sum_{r \in \mathbb{N}} \# (X/\mathbb{F}_{p^r}) \frac{t^r}{r} \right). \tag{1}
\]

The motivation to arrange the numbers \( N_{p^r} = \# (X/\mathbb{F}_{p^r}) \) in this particular way, rather than a more naive generating function like \( \sum_r N_{r,p} t^r \), originates from the fact that they often show a simple behavior, as a result of which the zeta function can be shown to be a rational function. This was first shown by Artin in the 1920s for hyperelliptic function fields [18] and by Schmidt for curves of arbitrary genus [19, 20]. Further experience by Hasse, Weil, and others led to the conjecture that this phenomenon is more general, culminating in the Weil conjectures, and Deligne’s proof in the 1970s.

The part of the conjectures that is most important for the present context is that the rational factors of \( Z(X/\mathbb{F}_p, t) \)
\[
Z(X/\mathbb{F}_p, t) = \prod_{j=1}^{d} \mathcal{P}_{2j-1}^{(p)}(t), \tag{2}
\]
can be written as
\[
\mathcal{P}_{0}^{(p)}(t) = 1 - t, \quad \mathcal{P}_{2d}^{(p)}(t) = 1 - p^d t \tag{3}
\]
and for \( 1 \leq i \leq 2d - 1 \)
\[
\mathcal{P}_{i}^{(p)}(t) = \prod_{j=1}^{b_i} \left( 1 - \beta_{j}^{(i)}(p)t \right), \tag{4}
\]
with algebraic integers \( \beta_{j}^{(i)}(p) \). The degree of the polynomials \( \mathcal{P}_{i}^{(p)}(t) \) is given by the Betti numbers of the variety, \( b_i = \dim H^{i}_{\text{DeRham}}(X) \). The rationality of the zeta function was first shown by Dwork [21] by adélic methods. More details of the Weil conjectures were briefly described in [4].

We see from the rationality of the zeta function that the basic information of this quantity is parametrized by the cohomology of the variety. More precisely, one can show that the \( i \)th polynomial \( \mathcal{P}_{i}^{(p)}(t) \) is associated to the action induced by the Frobenius morphism on the \( i \)th cohomology group \( H^{i}(X) \). In order to gain insight into the arithmetic information encoded in these Frobenius actions it is useful to decompose the zeta function of the variety into pieces determined by its cohomology. This leads to the concept of a local L-function that is associated to the polynomials \( \mathcal{P}_{i}^{(p)}(t) \) via the following definition.

Let \( \mathcal{P}_{i}^{(p)}(t) \) be the polynomials determined by the rational congruent zeta function over the
field $\mathbb{F}_p$. The $i^{th}$ L-function of the variety $X$ over $\mathbb{F}_p$ then is defined via

$$L^{(i)}(X/\mathbb{F}_p, s) = \frac{1}{P_i^{(p)}(p-s)}.$$  \hfill (5)

Such L-functions are of interest for a number reasons. One of these is that often they can be modified by simple factors so that after analytic continuation they (are conjectured to) satisfy some type of functional equation.

2.2 Arithmetic via Jacobi sums

In the case of weighted projected Brieskorn-Pham varieties it is possible to provide more insight into the structure of the L-function polynomials $P_d^{(p)}(t)$. In the case of Fermat hypersurfaces it is an old result by Weil according to which the cardinalities of the variety in terms of Jacobi sums of finite fields.

**Theorem.** \cite{1} Define the number $d = (n, q - 1)$ and the set

$$A_{q,n}^s = \left\{ (\alpha_0, ..., \alpha_s) \in \mathbb{Q}_s^{s+1} \mid 0 < \alpha_i < 1, d\alpha_i = 0 \mod 1, \sum_i \alpha_i = 0 \mod 1 \right\}. \hfill (6)$$

Then the number of solutions of the projective variety

$$X_{s-1} = \left\{ (z_0 : z_1 : \cdots : z_s) \in \mathbb{P}_s \mid \sum_{i=0}^s b_i z_i^n = 0 \right\} \subset \mathbb{P}_s \hfill (7)$$

over the finite field $\mathbb{F}_q$ is given by

$$N_q(X_{s-1}) = 1 + q + q^2 + \cdots + q^{s-1} + \sum_{\alpha \in A_{q,n}^s} j(\alpha) \prod \bar{\chi}_{\alpha_i}(b_i), \hfill (8)$$

where $d = (n, q - 1)$ and

$$j_q(\alpha) = \frac{1}{q-1} \sum_{u_i \in \mathbb{F}_q \atop u_0 + \cdots + u_s = 0} \chi_{\alpha_0}(u_0) \cdots \chi_{\alpha_s}(u_s), \hfill (9)$$

with

$$\chi_{\alpha_i}(u_i) = e^{2\pi i \alpha_i u_i}. \hfill (10)$$
where \( m_i \) is determined via \( u_i = g^{m_i} \) for any generator \( g \in \mathbb{F}_q \).

With these Jacobi sums \( j_q(\alpha_0, \ldots, \alpha_s) \) one defines the polynomials

\[
P^{(q)}_{s-1}(t) = \prod_{\alpha \in A^n_q} \left( 1 - (-1)^{s-1} j_q(\alpha_0, \ldots, \alpha_s) \prod_i \bar{\chi}_{\alpha_i}(b_i) t \right)
\]  

(11)

and the associated L-function

\[
L^{(j)}(X, s) = \prod_p \frac{1}{\mathcal{P}_j^{(p)}(p^{-s})}.
\]

(12)

A slight modification of this result is useful even in the case of weighted projective Brieskorn-Pham varieties because it can be used to compute the factor of the zeta function coming from the invariant part of the cohomology, when viewing these spaces as quotient varieties of projective spaces [3].

3 L-Functions of algebraic number fields

The surprising aspect of the Hasse-Weil L-function is that it is determined by another, a priori completely different kind of L-function that is derived not from a variety but from a number field. It is this possibility to interpret the cohomological Hasse-Weil L-function as a field theoretic L-function which establishes the connection that allows us to derive number fields \( K \) from algebraic varieties \( X \).

In the present context the type of L-function that is important is that of a Hecke L-function determined by a Hecke character, more precisely an algebraic Hecke character. Following Weil we will see that the relevant field for Fermat type varieties is the cyclotomic field extension \( \mathbb{Q}(\mu_m) \) of the rational field \( \mathbb{Q} \) by roots of unity, generated by \( \xi = e^{2\pi i/m} \) for some rational integer \( m \). It turns out that these fields fit in very nicely with the conformal field theory point of view. In order to see how this works we first describe the concept of Hecke characters and then explain how the L-function fits into this framework.

There are many different different definitions of algebraic Hecke characters, depending on the precise number theoretic framework. Originally this concept was introduced by Hecke [22] as
Größencharaktere of an arbitrary algebraic number field. In the following Deligne’s adaptation
of Weil’s Größencharaktere of type \( A_0 \) is used [23].

**Definition.** Let \( \mathcal{O}_K \subset K \) be the ring of integers of the number field \( K \), \( \mathfrak{f} \subset \mathcal{O}_K \) an integral ideal, and \( F \) a field of characteristic zero. Denote by \( \mathcal{I}_f(K) \) the set of fractional ideals of \( K \) that are prime to \( \mathfrak{f} \) and denote by \( \mathcal{I}^\mathfrak{f}_p(K) \) the principal ideals (\( \alpha \)) of \( K \) for which \( \alpha \equiv 1(\text{mod } \mathfrak{f}) \). An algebraic Hecke character modulo \( \mathfrak{f} \) is a multiplicative function \( \chi \) defined on the ideals \( \mathcal{I}^\mathfrak{f}_p(K) \) for which the following condition holds. There exists an element in the integral group ring \( \sum n_\sigma \sigma \in \mathbb{Z}[\text{Hom}(K, \overline{F})] \), where \( \overline{F} \) is the algebraic closure of \( F \), such that if \( (\alpha) \in \mathcal{I}^\mathfrak{f}_p(K) \) then

\[
\chi((\alpha)) = \prod_\sigma \sigma(\alpha)^{n_\sigma}. \quad (13)
\]

Furthermore there is an integer \( w > 0 \) such that \( n_\sigma + n_{\overline{\sigma}} = w \) for all \( \sigma \in \text{Hom}(K, \overline{F}) \). This integer \( w \) is called the weight of the character \( \chi \).

Given any such character \( \chi \) defined on the ideals of the algebraic number field \( K \) we can follow Hecke and consider a generalization of the Dirichlet series via the L-function

\[
L(\chi, s) = \prod_{\mathfrak{p} \subset \mathcal{O}_K} \frac{1}{1 - \chi(\mathfrak{p}) \mathfrak{N}\mathfrak{p}^{-s}} = \sum_{a \subset \mathcal{O}_K} \frac{\chi(a)}{\mathfrak{N}a^s}, \quad (14)
\]

where the sum runs through all the ideals. Here \( \mathfrak{N}\mathfrak{p} \) denotes the norm of the ideal \( \mathfrak{p} \), which is defined as the number of elements in \( \mathcal{O}_K/\mathfrak{p} \). The norm is a multiplicative function, hence it can be extended to all ideals via the prime ideal decomposition of a general ideal. If we can deduce from the Hasse-Weil L-function the particular Hecke character(s) involved we will be able to derive directly from the variety in an intrinsic way distinguished number field(s) \( K \).

Insight into the nature of number fields can be gained by recognizing that for certain extensions \( K \) of the rational number \( \mathbb{Q} \) the higher Legendre symbols provide the characters that enter the discussion above. Inspection then suggests that we consider the power residue symbols of cyclotomic fields \( K = \mathbb{Q}(\mu_m) \) with integer ring \( \mathcal{O}_K = \mathbb{Z}[\mu_m] \). The transition from the cyclotomic field to the finite fields is provided by the character which is determined for any algebraic integer \( x \in \mathbb{Z}[\mu_m] \) prime to \( m \) by the map

\[
\chi_\bullet(x) : \mathfrak{J}_m(\mathcal{O}_K) \rightarrow \mathbb{C}^\times, \quad (15)
\]

which is defined on ideals \( \mathfrak{p} \) prime to \( m \) by sending the prime ideal to the \( m' \)th root of unity.
for which
\[ p \mapsto \chi_p(x) = x^{N_p - 1 - m}(\text{mod } p). \] (16)

Using these characters one can define Jacobi-sums of rank \( r \) for any fixed element \( a = (a_1, ..., a_r) \) by setting
\[ J_a^{(r)}(p) = (-1)^{r+1} \sum_{\substack{u_i \in \mathcal{O}_K/p \\sum_i u_i = -1(\text{mod } p)}} \chi_p(u_1)^{a_1} \cdots \chi_p(u_r)^{a_r} \] (17)
for prime \( p \). For non-prime ideals \( a \subset \mathcal{O}_K \) the sum is generalized via prime decomposition \( a = \prod_i p_i \) and multiplicativity \( J_a(a) = \prod_i J_a(p_i) \). Hence we can interpret these Jacobi sums as a map \( J^{(r)} \) of rank \( r \)
\[ J^{(r)} : \mathcal{I}_m(\mathbb{Z}[\mu_m]) \times (\mathbb{Z}/m\mathbb{Z})^r \longrightarrow \mathbb{C}^\times, \] (18)
where \( \mathcal{I}_m \) denotes the ideals prime to \( m \). For fixed \( p \) such Jacobi sums define characters on the group \( (\mathbb{Z}/m\mathbb{Z})^r \). It can be shown that for fixed \( a \in (\mathbb{Z}/m\mathbb{Z})^r \) the Jacobi sum \( J_a^{(r)} \) evaluated at principal ideals \( (x) \) for \( x \equiv 1(\text{mod } m^r) \) is of the form \( x^{S(a)} \), where
\[ S(a) = \sum_{\ell \bmod m} \left[ \sum_{i=1}^r \left\langle \frac{\ell a_i}{m} \right\rangle \right] \sigma_{\ell}^{-1}, \] (19)
where \( <x> \) denotes the fractional part of \( x \) and \([x]\) describes the integer part of \( x \).

4 Examples

4.1 The elliptic Fermat curve

In [5] the elliptic curve defined by the plane cubic torus
\[ C_3 = \{(z_0 : z_1 : z_2) \in \mathbb{P}_2 \mid z_0^3 + z_1^3 + z_2^3 = 0\} \] (20)
was analyzed in some detail.

The zeta function (1) simplifies for curves into the form
\[ Z(X, s) = \prod_{\mathbf{Z} \text{ good prime}} \frac{\mathcal{P}(p)(p^{-s})}{(1-p^{-s})(1-p^{1-s})} = \frac{\zeta(s)\zeta(s-1)}{L_{\text{HW}}(X, s)}, \] (21)
written in terms of the Hasse-Weil L-function defined as

\[ L_{\text{HW}}(X, s) = \prod_{\mathbb{Z} \ni p \text{ good prime}} \frac{1}{P(p)(p^{-s})}, \]  

(22)
and the Riemann zeta function \( \zeta(s) = \prod_p (1 - p^{-s})^{-1} \) of the rational field \( \mathbb{Q} \).

The Hasse-Weil L-function can be determined via (1) by direct counting of the number of solutions of \( C_3/\mathbb{F}_p \) over finite extensions \([\mathbb{F}_p^r : \mathbb{F}_p]\) of the finite fields \( \mathbb{F}_p \) of prime order \( p \). This results in

\[ L_{\text{HW}}(C_3, s) = 1 - \frac{2}{4s} - \frac{1}{7s} + \frac{5}{13s} + \frac{4}{16s} - \frac{7}{19s} + \cdots, \]  

(23)
leading to the Hasse-Weil \( q \)-expansion

\[ f_{\text{HW}}(C_3, q) = q - 2q^4 - q^7 + 5q^{13} + 4q^{16} - 7q^{19} + \cdots \]  

(24)
It turns out that this is a modular form of weight 2 and modular level 27 which can be written as a product of the theta function \( \Theta(\tau) \) associated to the string function \( c(\tau) \) of the affine SU(2) Kac-Moody algebra at conformal level \( k = 1 \). More precisely, the following result was obtained.

**Theorem.** ([5]) The Mellin transform of the Hasse-Weil L-function \( L_{\text{HW}}(C_3, s) \) of the cubic elliptic curve \( C_3 \subset \mathbb{P}_2 \) is a modular form \( f_{\text{HW}}(C_3, q) \in S_2(\Gamma_0(27)) \) which factors into the product

\[ f_{\text{HW}}(C_3, q) = \Theta(q^3)\Theta(q^9). \]  

(25)
Here \( \Theta(\tau) = \eta^3(\tau)c(\tau) \) is the Hecke modular form associated to the quadratic extension \( \mathbb{Q}(\sqrt{3}) \) of the rational field \( \mathbb{Q} \), determined by the unique string function \( c(\tau) \) of the affine Kac-Moody SU(2)-algebra at conformal level \( k = 1 \).

This establishes that it is possible to derive the modularity of the underlying string theoretic conformal field theory from the geometric target space and that the Hasse-Weil L-function admits a conformal field theoretic interpretation.

The number theoretic interpretation of the Hasse-Weil L-function is best seen from the expression for the polynomials \( P^{(p)}(t) \), which completely determine the congruent zeta function and the Hasse-Weil L-function of these plane curves, in terms of the finite field Jacobi sums. For curves this reduces to

\[ P^{(p)}(t) = \prod_{\alpha \in \mathcal{A}_2^p} (1 - j_p(\alpha)t) \]  

(26)
with

\[ j_q(\alpha) = \frac{1}{q-1} \sum_{u_i \in \mathbb{F}_q \atop u_0 + u_1 + u_2 = 0} \chi_{\alpha_0}(u_0)\chi_{\alpha_1}(u_1)\chi_{\alpha_2}(u_2). \]  

(27)

Collecting values at primes is in part easier than direct counting because the cardinalities of the sets \( \mathcal{A}_2^p \) are easy to control. For the first few primes the results are collected in the following Table 1, where the zeroes follow immediately from the fact that the corresponding sets \( \mathcal{A}_2^p \) are empty.

| \( q \) | 2 | 3 | 5 | 7 | 9 | 11 | 13 |
|-----|---|---|---|---|---|----|----|
| \( j_q(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \) | 0 | 0 | 0 | 2 + 3\( \xi_3 \) | 0 | -1 + 3\( \xi_3 \) |
| \( j_q(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}) \) | 0 | 0 | 0 | 2 + 3\( \xi_3 \) | 0 | -1 + 3\( \xi_3 \) |

Table 1. Finite field Jacobi sums of the elliptic cubic curve \( C_3 \) at the lower rational primes.

By translating the finite field Jacobi-sums into Jacobi-sum type Hecke characters \( J_{(i,i,i)}(a) \) of the cyclotomic field \( \mathbb{Q}(\mu_3) \) one can write the geometric Hasse-Weil L-function as a number theoretic object associated to this field. Applied to the field \( \mathbb{Q}(\mu_3) \) this procedure leads to the number theoretic representation of the Hasse-Weil L-function of the plane cubic curve as

\[ L_{HW}(E, s) = L_H(J_{(1,1,1)}, s)L_H(J_{(2,2,2)}, s). \]  

(28)

4.2 The quintic threefold

Consider the Calabi-Yau variety defined by the Fermat quintic hypersurface in ordinary projective fourspace \( \mathbb{P}_4 \) defined by

\[ X = \left\{ (x_0 : \cdots : x_4) \in \mathbb{P}_4 \ \Big| \ \sum_{i=0}^{4} x_i^5 = 0 \right\}. \]  

(29)

It follows from Lefshetz’s hyperplane theorem that the cohomology below the middle dimension is inherited from the ambient space. Thus we have \( h^{1,0} = 0 = h^{0,1} \) and \( h^{1,1} = 1 \) while \( h^{2,1} = 101 \) follows from counting monomials of degree five. For the smooth Fermat quintic
the zeta function simplifies to the expression

\[ Z(X/\mathbb{F}_p, t) = \frac{\mathcal{P}_3^{(p)}(t)}{(1 - t)(1 - pt)(1 - p^2t)(1 - p^3t)}. \]  

(30)

where the numerator is given by the polynomial \( \mathcal{P}_3^{(p)}(t) = \prod_{i=1}^{204}(1 - \beta_i^{(3)}(p)t) \) which takes the form

\[ \mathcal{P}_3^{(p)}(t) = \prod_{\alpha \in \mathcal{A}} (1 - j_p(\alpha)t). \]  

(31)

This expression involves the following ingredients. Define \( \delta = (5, p - 1) \) and rational numbers \( \alpha_i \) via \( \delta \alpha_i \equiv 0 \text{ (mod 1)} \). The set \( \mathcal{A} \) is defined as

\[ \mathcal{A} = \{ \alpha = (\alpha_0, ..., \alpha_4) \mid 0 < \alpha_i < 1, \ \delta \alpha_i \equiv 0 \text{ (mod 1)}, \sum \alpha_i = 0 \text{ (mod 1)} \}. \]  

(32)

Defining the characters \( \chi_{\alpha_i} \in \hat{\mathbb{F}}_p \) in the dual of \( \mathbb{F}_p \) as \( \chi_{\alpha_i}(u_i) = \exp(2\pi i \alpha_i s_i) \) with \( u_i = g^{s_i} \) for a generating element \( g \in \mathbb{F}_p \), the factor \( j_p(\alpha) \) finally is determined as

\[ j_p(\alpha) = \frac{1}{p - 1} \sum \sum_{u_i = 0}^{4} \chi_{\alpha_i}(u_i). \]  

(33)

We thus see that the congruent zeta function leads to the Hasse-Weil L-function associated to a Calabi-Yau threefold

\[ L_{\text{HW}}(X, s) = \prod_{p \in \mathbb{P}(X)} \prod_{\alpha \in \mathcal{A}} \left( 1 - j_p(\alpha)p^s \right)^{-1}, \]  

(34)

ignoring the bad primes, which are irrelevant for our purposes.

5 L-function of elliptic curves with complex multiplication

5.1 Deuring’s decomposition

Calabi-Yau varieties of Brieskorn-Pham type provide important examples of exactly solvable string vacua, but not all exact string models which admit a geometric interpretation lead
to such spaces. The question therefore arises whether there are other varieties for which a number theoretic decomposition is possible. This issue was addressed first by Deuring for elliptic curves with complex multiplication, i.e. elliptic curves $E$ whose endomorphism algebra $\text{End}(E)$ is not restricted to multiplication by integers. He showed that a similar structure of the Hasse-Weil L-function arises with characters that are algebraic Hecke characters $\chi$ (more details on the construction of these characters will be described in the next subsection). Given any such character defined on the ideals of the algebraic number field $K$ we can consider Hecke’s L-function (14). The result of Deuring then determines the Hasse-Weil L-function of any elliptic curve with complex multiplication as a number theoretic object. Even though Deuring’s result is general for CM elliptic curve, the arguments that establish this identity depend on the behavior of the complex multiplication field $F$ relative to the field of definition $K$. The following result describes the simpler situation when the field $F$ is contained in the field $K$.

**Theorem.** ([9]). Let $E/K$ be an elliptic curve with complex multiplication by the ring of integers $\mathcal{O}_F$ of the algebraic field $F$, $\text{End}(E) \cong \mathcal{O}_F$. Assume that $F$ is contained in $K$ and let $\chi$ be the algebraic Hecke character associated to $E$. Then

$$L(E/K, s) = L(\chi, s)L(\bar{\chi}, s).$$  \hspace{1cm} (35)

This result shows that for any elliptic curve with complex multiplication the geometric Hasse-Weil L-function leads to L-functions associated to algebraic number fields. In the case of the Fermat curve this number field is the fusion field, determined by the quantum dimensions of the scaling fields of the underlying conformal field theory. Deuring’s result shows that this can be generalized to any elliptic curve with complex multiplication.

This then leads to the idea that perhaps quite generally for elliptic curves with CM the fusion field of the rational conformal field theory can be obtained via the Hecke L-function interpretation of the Hasse-Weil L-function. We therefore should ask what precisely the underlying structure is that is responsible for the existence of these characters.
5.2 Character construction from elliptic curves

Our goal in this section is to describe the conceptual origin of the Hecke characters that appear in Deuring’s result, and which turn out to provide one of the links from geometry to conformal field theory. Even though its explication is not absolutely necessary for the logic of our argument, it is worthwhile to briefly outline the key elements that facilitate the transition from the elliptic curve to the algebraic Hecke characters, because it illuminates the number theoretic nature of elliptic curves, and more generally, abelian varieties, with complex multiplication. It is this number theoretic structure that lies at the heart of the results obtained so far concerning a direct relation between exactly solvable models and geometric string vacua.

The construction is based on the action of the Artin symbols on the points of finite order of the abelian variety, and therefore combines both geometric and number theoretic aspects. While the geometric concepts are completely parallel in the 1-dimensional and the higher dimensional case, the transition from elliptic curves to more general abelian varieties introduces a number of technical complications that do not help to illuminate the essence of the argument. It is therefore useful to consider first elliptic curves. More details can be found in [24, 25, 26].

The starting point of the analysis of the action of the Artin symbol on the torsion points is the main theorem of complex multiplication. Before stating it we review some concepts. Let $E/K$ be an elliptic curve over a number field $K$ and let

$$f : \mathbb{C}/\Lambda \rightarrow E \otimes \mathbb{C}$$

$$z \mapsto (\wp(z, \Lambda), \wp'(z, \Lambda))$$

describe the analytic representation of $E \otimes \mathbb{C}$ for a lattice $\Lambda \subset \mathbb{C}$ via the Weierstrass $\wp$–function. Such a representation always exists according to the uniformization theorem for elliptic curves, and hence we obtain a polynomial description, given by the Weierstrass equation

$$E : y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda).$$

Assume that $E$ has complex multiplication by the field $F$, i.e. there exists an isomorphism

$$\theta : F \rightarrow \text{End}(E) \otimes \mathbb{Q}.\quad (38)$$

It follows that $F$ must be an imaginary quadratic field (see e.g. [27]). If an elliptic curve has CM by a field $F$ then $E$ can be constructed as $\mathbb{C}/\mathfrak{a}$, where $\mathfrak{a}$ is a $\mathbb{Z}$–lattice. More precisely,
there is a short exact sequence

\[ 0 \rightarrow \mathfrak{a} \rightarrow \mathbb{C} \rightarrow E \rightarrow 0. \]  

(39)

Denote by \( E[m] \) the set of finite order on \( E \), i.e. the kernel of the multiplication map by \( m \)

\[ E[m] = \{ z \in E \mid mz = 0 \}, \]  

(40)

and let \( E_{\text{tor}} \) denote the torsion points of \( E \), i.e. the collection of all points of arbitrary finite order. The analytic parametrization then restricts to a map, also denoted by \( f \),

\[ f : F/\mathfrak{a} \rightarrow E_{\text{tor}}. \]  

(41)

Artin symbols are particular elements in the Galois group \( \text{Gal}(F_{\text{ab}}/F) \) of the maximal abelian extension \( F_{\text{ab}} \) of \( F \). They are most efficiently described in terms of idèles \( \mathbb{A}_F^\times \) [28], which we can view as the multiplicative subgroup of the ring of adèles \( \mathbb{A}_F \) [29]. Class field theory says that there exists a homomorphism

\[ \mathbb{A}_F^\times \rightarrow \text{Gal}(F_{\text{ab}}/F) \]  

(42)

from the group of idèles onto the Galois group of the maximal abelian extension. Hence for any finite extension \( F \) and for any \( \sigma \in \text{Gal}(\mathbb{C}/F) \) there exists an idèle \( x \in \mathbb{A}_F^\times \) such that

\[ \sigma|_{F_{\text{ab}}} \in \text{Gal}(F_{\text{ab}}/F) \]  

(43)

is determined by \( x \). This element is usually denoted by \([x, F]\), and can be constructed for any finite abelian extension \( L/F \) in terms of the Artin symbol associated to the ideal \((x)\) that can be associated to the idèle by defining

\[ (x) = \prod_p p^{\text{ord}_p x_p}, \]  

(44)

where \( p \) are prime ideals in \( F \) and \( x_p \in K_p \) denotes the components of the idèle in the completion of \( K \) defined by the norm \( | \cdot |_p \) associated to \( p \). Restricted to the extension \( L \) the map \([x, F]\) then is defined as

\[ [x, F]|_L = \sigma_{(x)}. \]  

(45)

\footnote{A brief review of adèles, idèles, and class field theory is contained in the appendix.}
where for any prime \( \mathfrak{p} \) of \( L \) that divides the prime ideal \( \mathfrak{p} \) of \( F \) the Artin symbol \( \sigma_{\mathfrak{p}} \) is defined as

\[
\sigma_{\mathfrak{p}}(x) = x^{N\mathfrak{p}} \mod \mathfrak{p}.
\]  

(46)

The homomorphism (42) has a nontrivial kernel given by the connected component of the idèle class group \( C_F = \mathbb{A}_F^\times /F^\times \). Denoting this connected component by \( D_F \) then leads to an identification

\[
C_F/D_F \overset{\cong}{\longrightarrow} \text{Gal}(F_{ab}/F).
\]  

(47)

This result shows that it makes sense to ask what the action is of an ideal (class) on an elliptic curve. To answer this question we need a notion of multiplication of a fractional ideal \( \mathfrak{a} \) of \( F \) by an idèle \( x \in \mathbb{A}_F^\times \). The idea here is to use the fractional ideal \((x)\) constructed from the idèle \( x \) and then define the product via this ideal

\[
x\mathfrak{a} = (x)\mathfrak{a}, \quad x \in \mathbb{A}_F^\times, \, \mathfrak{a} \subset F.
\]  

(48)

We also need the inverse \( \mathfrak{a}^{-1} \) of an ideal \( \mathfrak{a} \) in \( F \), which can be defined as

\[
\mathfrak{a}^{-1} = \left\{ x \in F \mid x\mathfrak{a} \subset \mathcal{O}_F \right\}.
\]  

(49)

If we now think of \( E \) in terms of its analytic representation \( \mathbb{C}/\mathfrak{a} \) with \( \mathfrak{a} \subset \mathcal{O}_F \), then for any non-zero \( \mathfrak{b} \subset \mathcal{O}_F \) we can ask what the action of the ideal (class) does on the elliptic curve, i.e. we can consider \( E' = \mathbb{C}/\mathfrak{b}^{-1}\mathfrak{a} \) (here the appearance of the inverse of \( \mathfrak{b} \) is conventional). If, on the other hand, we think of \( E \) as defined by a polynomial then we can ask how the transformation of the coefficients in the polynomial affects the curve. This leads to the notion of the \( \sigma \)–transform \( E^\sigma \) of an elliptic curve \( E \) for automorphisms of \( \mathbb{C} \), which is defined via an action of the map on the coefficients of the curve. In the case of the Weierstrass representation (37) the \( \sigma \)–transform is described by the curve

\[
E^\sigma : \quad y^2 = 4x^3 - g_2^\sigma x - g_3^\sigma.
\]  

(50)

The main result in the theory of complex multiplication for elliptic curves answers the question what the relation is between these two geometric transformations of an elliptic curve, and what this relation implies for the torsion points.
**Theorem.** Let $E$ be an elliptic curve described by an isomorphism

$$ f : C / a \xrightarrow{\approx} E(C), \quad (51) $$

where $a$ is a fractional ideal in $F$. Assume that $E$ admits complex multiplication by the number field $F$, $\text{End}(E) \cong \mathcal{O}_F$, and let $\sigma \in \text{Gal}(C/F) = \text{Aut}_F(C)$ and $x \in \mathbb{A}_F^\times$ be such that

$$ \sigma|_{F_{ab}} = [x,F]|_{F_{ab}}. \quad (52) $$

Then there exists an analytic isomorphism

$$ f' : C / x^{-1}a \xrightarrow{\approx} E(C)^\sigma \quad (53) $$

such that the following diagram is commutative

$$ \begin{array}{ccc}
F/a & \xrightarrow{x^{-1}} & F/x^{-1}a \\
\downarrow f & & \downarrow f' \\
E(C) & \xrightarrow{\sigma} & E(C)^\sigma.
\end{array} \quad (54) $$

Put slightly different, we can take any element $v \in F/a$, consider the corresponding torsion point $f(v)$, and ask what the action of the automorphism $\sigma$ does. This means that we want to know what $f(v)^{[x,F]}$ is. The diagram (54) then says that the image under the Artin symbol is given by

$$ f(a)^{[x,F]} = f'(x^{-1}a), \quad \forall a \in F/a, \ x \in \mathbb{A}_F^\times. \quad (55) $$

This result by A. Robert is foundational because it implies the highlights of the classical theory of complex multiplication, such as the construction of the Hilbert class field, as well as the maximal abelian extension of quadratic imaginary fields (see e.g. [26, 25]). A discussion of these standard results in a physical context can be found in [16, 15]. The main ingredient of the proof of this theorem is the Kronecker congruence relation in a form given first by Hasse [30]. It states that for an extension $L/F$ the Artin symbol (46) of a prime $L \supset \mathfrak{P}\mid p$ acts on the $j$–invariant of the elliptic curve as

$$ \sigma_{\mathfrak{P}}(j(a)) = j(p^{-1}a). \quad (56) $$

This leads to the rhs of the diagram because the $j$–invariant is an isomorphism invariant for elliptic curves over algebraically closed fields.
The construction of the algebraic Hecke character is now a two-step procedure. The first ingredient is a map \( \alpha \) constructed as follows. For any finite extension \( L/F \) an idèlic norm map

\[
N^L_F : \mathbb{A}^\times_L \longrightarrow \mathbb{A}^\times_F,
\]

(57)
can be defined by specifying what the \( v \)th component is of the image idèlé, where \( v \) runs through the finite primes as well as the infinite primes, which are associated to the embeddings of the number field. For \( x \in \mathbb{A}^\times_K \) one sets

\[
(N^L_F x)_v = \prod_{w|v} N^L_{F_w} x_w,
\]

(58)

where \( L_w \) and \( F_v \) are completions of the fields \( L \) and \( F \) at the primes \( w \) and \( v \) respectively (see the appendix for more details on adèles). Let further \( F^\times \) denote the invertible elements of \( F \).

**Theorem.** Let \( E/K \) be an elliptic curve with parametrization

\[
f : \mathbb{C}/a \longrightarrow E(\mathbb{C}),
\]

(59)

where \( a \subset F \) is a fractional ideal. Assume that \( E \) has complex multiplication by the ring of integers \( \mathcal{O}_F \) of \( F \), and that \( F \subset K \). Then \( K(E_{tor}) \) is abelian over \( K \). Let further \( x \in \mathbb{A}^\times_K \) be an idèle of \( K \), and \( y = N_{K/F} x \). Then there exists a unique \( \alpha = \alpha_{E/K}(x) \in F^\times \) with the following properties:

1) \( \alpha \mathcal{O}_F = (y) \), where \( (y) \subset F \) is the ideal of \( y \).
2) For any fractional ideal \( a \subset F \) and any analytic isomorphism \( f \) the following diagram is commutative

\[
\begin{array}{ccc}
F/a & \xrightarrow{\alpha y^{-1}} & F/a \\
\downarrow f & & \downarrow f \\
E(K_{ab}) & \xrightarrow{[x,K]} & E(K_{ab})
\end{array}
\]

(60)

This result leads to a map \( \alpha : \mathbb{A}^\times_K \longrightarrow F^\times \) which does not yet define an algebraic Hecke character because it is not trivial on the units. But it can be modified in a way such that an appropriate character emerges.

**Theorem.** Let \( E/K \) be an elliptic curve with complex multiplication by the ring of integers
\( \mathcal{O}_F \) of \( F \) and assume \( F \subset K \). Let

\[
\alpha_{E/K} : \mathbb{A}_K^\times \to F^\times
\]

be the map described above. For any idèl \( x \in \mathbb{A}_F^\times \), let \( x_\infty \in \mathbb{C}^\times \) be the component of \( x \) corresponding to the unique archimedean absolute value of \( F \). Define a map

\[
\psi_{E/K} : \mathbb{A}_K^\times \to \mathbb{C}^\times
x \mapsto \alpha_{E/K}(x)N_F^K(x^{-1})_\infty.
\]

Then the following hold:

1) \( \psi_{E/K} \) is a Größencharakter.

2) Let \( \mathfrak{P} \) be a prime of \( K \). Then \( \psi_{E/K} \) is unramified at \( \mathfrak{P} \) if and only if \( E \) has good reduction at \( \mathfrak{P} \). (A Größencharakter \( \psi : \mathbb{A}_K^\times \to \mathbb{C}^\times \) is said to be unramified at \( \mathfrak{P} \) if \( \psi(\mathcal{O}_\mathfrak{P}^\times) = 1 \).)

It is the L-function \( L(\psi_{E/K}, s) \) of this algebraic Hecke character \( \psi_{E/K} \) which provides the number theoretic description of the Hasse-Weil L-function described above in eq. (35) (it can be shown that the idèlic version of the Hecke characters reduces to Deligne’s formulation described in the first part of this section [31]).

In summary, the important conceptual point here is that the arithmetic structure of the torsion points of elliptic curves with complex multiplication carries the essential information which turns the geometric L-function into a number theoretic object. It is this number theoretic structure which allows the translation of geometric properties into conformal field theoretic objects (we will return to this aspect further below).

In the formulation above we have made the simplifying assumption that the complex multiplication field \( F \) is contained in the field of definition \( K \). A similar analysis goes through when this assumption is dropped, as has been shown by Deuring in the 1950s in his important sequence of papers [9], in which he developed the algebraic approach to complex multiplication used here (see also [32]). In particular the first paper in this series remains an illuminating reference about the relation between these different L-functions up to finitely many primes.

The generalization to higher dimensional abelian varieties is possible, but complicated by the fact that the number theoretic structure arises from the so-called reflex field of the complex multiplication field \( F \), as will become clear further below.
6 Abelian varieties

6.1 General definition

We first review some relevant concepts in the context of abelian varieties. An abelian variety over some number field \( K \) is a smooth, geometrically connected, projective variety, which is also an algebraic group with the group law \( A \times A \rightarrow A \) defined over \( K \). A concrete way to construct such manifolds is via complex tori \( \mathbb{C}^n/\Lambda \) with respect to some lattice \( \Lambda \subset \mathbb{C}^n \), or, put differently, via an exact sequence

\[
0 \rightarrow \Lambda \rightarrow \mathbb{C}^n \xrightarrow{f} A \rightarrow 0,
\]

where \( f \) is a holomorphic map. The lattice \( \Lambda \) is not necessarily integral and admits a Riemann form, which is defined as an \( \mathbb{R} \)-bilinear form \( \langle , \rangle \) on \( \mathbb{C}^n \) such that the following hold:

1. \( \langle x, y \rangle \) takes integral values for all \( x, y \in \Lambda \);
2. \( \langle x, y \rangle = - \langle y, x \rangle \);
3. \( \langle x, iy \rangle \) is a symmetric and positive definite form in \( x, y \).

The result then is that a complex torus \( \mathbb{C}^n/\Lambda \) has the structure of an abelian variety if and only if there exists a non-degenerate Riemann form on \( \mathbb{C}^n/\Lambda \).

6.2 Abelian varieties of CM type

A special class of abelian varieties are those of complex multiplication (CM) type. These are varieties which admit automorphism groups that are larger than those of general abelian manifolds. The reason why CM type varieties are special is because certain number theoretic questions can be addressed in a systematic fashion for this class. The first to discover this was Weil [2] in the context of Fermat type hypersurfaces. The fact that this relation can be traced to the property of CM for abelian varieties was first shown by Deuring in the context of elliptic curves, following a suggestion by Weil. This was later generalized conditionally to higher dimensions by Taniyama and Shimura [33, 34], Serre and Tate [35], and Shimura [26, 36].

Consider a number field \( F \) over the rational numbers \( \mathbb{Q} \) and denote by \([ F : \mathbb{Q} ]\) the degree of the field \( F \) over \( \mathbb{Q} \), i.e. the dimension of \( F \) over the subfield \( \mathbb{Q} \). An abelian variety \( A \)
of dimension $n$ is called a CM-variety if there exists an algebraic number field $F$ of degree $[F : \mathbb{Q}] = 2n$ over the rational numbers $\mathbb{Q}$ which can be embedded into the endomorphism algebra $\text{End}(A) \otimes \mathbb{Q}$ of the variety. More precisely, a CM-variety is a triplet $(A, \theta, F)$, where
\begin{equation}
\theta : F \rightarrow \text{End}(A) \otimes \mathbb{Q}
\end{equation}
describes the embedding of $F$. It follows from this that the field $F$ necessarily is a CM field, i.e. a totally imaginary quadratic extension of a totally real field. The important ingredient here is that the restriction to $\theta(F) \subset \text{End}(A) \otimes \mathbb{Q}$ is equivalent to the direct sum of $n$ isomorphisms $\varphi_1, \ldots, \varphi_n \in \text{Iso}(F, \mathbb{C})$ such that $\text{Iso}(F, \mathbb{C}) = \{\varphi_1, \ldots, \varphi_n, \rho \varphi_1, \ldots, \rho \varphi_n\}$, where $\rho$ denotes complex conjugation. These considerations suggest calling the pair $(F, \{\varphi_i\})$ a CM-type. In principle we can think of the CM type as an abstract representation defined by some matrix
\begin{equation}
\Phi(a) = \begin{pmatrix}
a \varphi_1 \\
\vdots \\
a \varphi_n
\end{pmatrix}, \quad \text{for } a \in F,
\end{equation}
but in the present context $(F, \Phi = \{\varphi_i\})$ describes the CM-type of a CM-variety $(A, \theta, F)$.

It is possible to prescribe the CM structure and construct an abelian variety with that given structure by constructing a diagram of the following form
\begin{equation}
\begin{array}{cccccccc}
0 & \rightarrow & a & \rightarrow & F_\mathbb{R} & \rightarrow & F_\mathbb{R}/a & \rightarrow & 0 \\
\downarrow & & \downarrow u & & \downarrow & & & \\
0 & \rightarrow & \Lambda & \rightarrow & \mathbb{C}^n & \rightarrow & A & \rightarrow & 0,
\end{array}
\end{equation}
where $u$ is the map
\begin{equation}
u : F_\mathbb{R} \rightarrow \mathbb{C}^n
\end{equation}
da $a \mapsto \begin{pmatrix} a \varphi_1 \\ \vdots \\ a \varphi_n \end{pmatrix},
\end{equation}
defined as an $\mathbb{R}$-linear extension on $F$, and $a$ is the preimage of $u$ of the lattice $\Lambda$. The abelian variety is thereby obtained as the quotient $F_\mathbb{R}/a$ of $F_\mathbb{R} = F \otimes \rho \mathbb{R}$, with $\rho$ denoting complex conjugation, by an ideal in $F$, with a complex structure determined by $u$, and an embedding $\theta : F \rightarrow \text{End}(A) \otimes \mathbb{Q}$ given by (65).

Concrete examples of these concepts which have been discussed in [4] in the context of the Calabi-Yau/conformal field theory relation are varieties which have complex multiplication by
a cyclotomic field $F = \mathbb{Q}(\mu_n)$, where $\mu_n$ denotes the cyclic group generated by a nontrivial $n$'th root of unity $\xi_n$. The degree of $\mathbb{Q}(\mu_n)$ is given by $[\mathbb{Q}(\mu_n) : \mathbb{Q}] = \phi(n)$, where $\phi(n) = \# \{ m \in \mathbb{N} \mid m < n, \gcd(m, n) = 1 \}$ is the Euler function. Hence the abelian varieties encountered have complex dimension $\dim A = \phi(n)/2$.

The simplest examples of abelian CM varieties are elliptic curves with complex multiplication, as discussed above. More interestingly for the present context, they occur in the context of higher dimensional Calabi-Yau varieties via the Shioda-Katsura decomposition of the cohomology. Further below we briefly review the reduction of the cohomology of the Brieskorn-Pham varieties to that generated by curves and then analyze the structure of the resulting weighted curve Jacobians.

### 6.3 L-Function of abelian varieties with complex multiplication

In this section we describe the generalization of the conceptual framework underlying the number theoretic interpretation of the zeta function of abelian varieties with complex multiplication. Our goal is to detail the underlying structure that explains this phenomenon. As in the case of elliptic curves the main objects that provide the transition from the discrete geometry of the variety to number theory are the torsion points on the abelian variety, i.e. the points in the kernel of a multiplication map $n : A \to A$, analogous to the corresponding map on the elliptic curves.

The general concept of a geometric L-function is derived from the reduction of a variety over discrete fields $\mathbb{F}_q$ of order $q$. One way to think about this structure is by considering the fields $\mathbb{F}_q$ as residue fields $\mathcal{O}_K/p$, generated by prime ideals $p$ in the ring of integers $\mathcal{O}_K$ of some algebraic number field $K$. Denoting the residue field produced by $p$ as $K(p)$ and the reduced variety by $X(p)$ one can define the local zeta function as

$$Z(X, p, s) := Z(X(p)/K(p), t = Np^{-s}).$$

By combining these local zeta functions for all prime ideals one obtains the global zeta function

$$Z(X/K, s) = \prod_{p \subset \mathcal{O}_K} Z(X, p, s)$$

of the variety $X$ defined over the number field $K$. 

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When the variety has complex multiplication with respect to some number field $F$ the zeta function admits a number theoretic interpretation which generalizes the results of Deuring for elliptic curves with complex multiplication. Associated to the field $F$ are again Größencharaktere $\chi_i$, $i = 1, \ldots, n$ which lead to Hecke $L$-functions $L(\chi_i, s)$. The zeta function of the abelian variety with complex multiplication then is described by these Hecke $L$-functions.

**Theorem.** (see [36]) Let the abelian CM-variety $(A, \theta, F)$ be defined over an algebraic number field $K$ of finite degree. Then the zeta function of $A$ over $K$ coincides exactly with the product

$$\prod_{i=1}^{n} L(\chi_i, s)L(\overline{\chi_i}, s), \quad (70)$$

where the $\chi_i$ are Größencharaktere, and $\overline{\chi_i}$ is the complex conjugate of $\chi_i$.

This result was first shown in a conditional formulation by Taniyama and Shimura, and Serre and Tate. It shows that if we could recover abelian varieties from Calabi-Yau manifolds then we could generalize to higher dimensions the ideas about the CY/CFT relation formulated in the previous section for elliptic curves.

### 6.4 Character construction from abelian varieties

The character construction from higher dimensional abelian varieties differs somewhat from that of elliptic curves because of the emergence of the reflex type, denoted here by $(\hat{F}, \hat{\Phi} = \{\hat{\varphi}_i\})$ of the complex multiplication type $(F, \Phi = \{\varphi_i\})$. This reflex field is defined by adjoining to $F$ all traces determined by the CM type of $F$, i.e. $\hat{F} = F(\{\sum_i x^{\varphi_i} \mid x \in F\})$. To define the reflex type $\hat{\Phi}$ consider a Galois extension $L/\mathbb{Q}$ over the rationals that contains the CM field $F$. Denote by $S$ the subset of all those elements of the Galois group $\text{Gal}(L/\mathbb{Q})$ of $L$ that induce some $\varphi_i$ on $F$ and define further

$$S^{-1} = \{\sigma^{-1} \mid \sigma \in S\}$$

$$H = \{\gamma \in \text{Gal}(L/\mathbb{Q}) \mid \gamma S^{-1} = S^{-1}\}. \quad (71)$$

Then the reflex type of $(F, \Phi)$ is completed by defining the maps

$$\hat{\varphi}_i : \hat{F} \longrightarrow \mathbb{C} \quad (72)$$

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as those that are obtained from $S^{-1}$. In the one dimensional case the discussion simplifies because one has $\hat{F} = F$.

Consider an abelian variety $A/K$ of dimension $n$ defined over a number field $K$ with complex multiplication, i.e. with an embedding $\theta : F \rightarrow \text{End}(A) \otimes \mathbb{Q}$. Denote this structure by $(A/K, \theta, F)$ with type $(F, \Phi = \{\phi_i\}_{i=1,\ldots,n})$. The appearance of $\hat{F}$ leads to a modification of the norm map that appears in the elliptic construction of the character. To simplify the discussion we assume that $\hat{F} \subset K$. We can therefore consider the id\`elic norm map $N_{\hat{F}}^F : \mathbb{A}_F^\times \rightarrow \mathbb{A}_{\hat{F}}^\times$, defined earlier, and compose it with the determinant map

$$\delta : \mathbb{A}_F^\times \rightarrow \mathbb{A}_{\hat{F}}^\times,$$  \hspace{1cm} (73)

defined as the continuous extension of the determinant of the reflex type

$$\delta(x) = \det \hat{\Phi}(x), \quad \forall x \in \hat{F}^\times.$$  \hspace{1cm} (74)

The composition $g := \delta \circ N_{\hat{F}}^F$ of the norm map and the determinant map provides a map from the $K$–id\`èles to the $F$–id\`èles

$$g : \mathbb{A}_K^\times \rightarrow \mathbb{A}_{\hat{F}}^\times,$$  \hspace{1cm} (75)

generalizing the norm map that we encountered in the elliptic case.

The construction of the character is based again on the same idea as in the one dimensional case, the action of the id\`elic Artin symbol $[x, K]$ for $x \in \mathbb{A}_K^\times$ on the torsion points and the map $g$ enters this construction in much the same way that the norm map itself did in the earlier discussion. The main result of the theory of complex multiplication in the case of abelian varieties that are relevant to us can be summarized as follows.

\textbf{Theorem.} Let $(F, \Phi)$ be a CM-type, $(\hat{F}, \hat{\Phi})$ its reflex, and $\mathfrak{a}$ a lattice in $F$. Let further $(A, \theta)$ be of type $(F, \Phi)$, $u : F_\mathbb{R} \rightarrow \mathbb{C}^n$ the map in the diagram (66), and $f$ the map that defines $A$ via

$$0 \rightarrow \Lambda \rightarrow \mathbb{C}^n \xrightarrow{f} A \rightarrow 0.$$  \hspace{1cm} (76)

Further let $\sigma \in \text{Aut}(\mathbb{C}/\hat{F})$, $x \in \mathbb{A}_F^\times$ be an id\`ele of the reflex field such that

$$\sigma|_{\hat{F}_{ab}} = [x, \hat{F}],$$  \hspace{1cm} (77)

\footnote{This condition generalizes the assumption in the elliptic case that the CM field is contained in the field of definition.}
and \( g = \delta \circ N^K \) the map defined by the idèlic extension of the determinant map. Then there is an exact sequence

\[
0 \longrightarrow u(g(x)^{-1}a) \longrightarrow \mathbb{C}^n \longrightarrow A^\sigma \longrightarrow 0
\]

(78)
such that

\[
f(u(v)) = f'(u(g(x)^{-1}v)) \quad \forall v \in F/\mathfrak{a},
\]

(79)
i.e. there exists a commutative diagram

\[
\begin{array}{ccc}
F/\mathfrak{a} & \xrightarrow{g(x)^{-1}} & F/g(x)^{-1}\mathfrak{a} \\
\omega \downarrow & & \downarrow \omega' \\
A_{\text{tor}} & \xrightarrow{\sigma} & A_{\text{tor}}^\sigma
\end{array}
\]

(80)

where \( \omega = f \circ u \) and \( \omega' = f' \circ u \).

The construction of the algebraic Hecke character associated to the torsion points of \( A \) is now again achieved by constructing an idèlic map \( \alpha \) of \( K \) in the following way.

**Theorem.** For the map \( \omega : F_\mathbb{R} \longrightarrow A \) defined by \( \omega = f \circ u \) with \( F_\mathbb{R} = F \otimes_\rho \mathbb{R} \) and \( \rho \) denotes complex conjugation, there exists a map

\[
\alpha : \mathbb{A}^\times_K \longrightarrow F^\times
\]

(81)
such that

\[
\omega(v)^{[x,K]} = \omega(\alpha(x)g(x)^{-1}v) \quad \forall x \in \mathbb{A}^\times_K, \; v \in F/\mathfrak{a},
\]

(82)
which is determined uniquely by the following properties

\[
\begin{align*}
\alpha(x)g(x)^{-1}a &= a \\
\alpha(x)\alpha(x)^\rho &= N(x),
\end{align*}
\]

(83)
where \( (x) \) is the ideal associated to \( x \). Furthermore the kernel of \( \alpha \) is open in the idèles.

As in the elliptic case we can now define characters \( \psi_i \) on the idèles by picking appropriate components from the map that describes the Artin symbol on the elements of \( F/\mathfrak{a} \). More precisely, define

\[
\psi_i(x) = (\alpha(x)g(x)^{-1})_{\infty}, \quad i = 1, \ldots, n,
\]

(84)
via the infinite primes of the complex multiplication field \( F \).
7 Abelian varieties from Brieskorn-Pham type hypersurfaces

In this section we review our construction of abelian varieties with complex multiplication from Calabi-Yau varieties [15]. The idea is to first reduce the intermediate cohomology of the Calabi-Yau via the Shioda-Katsura construction to the cohomology spanned by curves embedded in the manifold and then to use the results of Faddeev, Gross, Rohrlich, and others to decompose the Jacobian varieties derived from these curves to find factors that admit complex multiplication.

7.1 The Shioda-Katsura decomposition

The decomposition of the intermediate cohomology of projective hypersurfaces was first described by Shioda and Katsura [37] and Deligne [38]. Their analysis can be generalized to weighted hypersurfaces, in particular the class of Brieskorn-Pham varieties, perhaps the simplest class of exactly solvable Calabi-Yau manifolds. This generalization works because the cohomology $H^3(X)$ for these varieties decomposes into the monomial part and the part coming from the resolution. The monomial part of the intermediate cohomology can easily be obtained from the cohomology of a projective hypersurface of the same degree by realizing the weighted projective space as a quotient variety with respect to a product of discrete groups determined by the weights of the coordinates.

For projective varieties

$$X_d^n = \{(z_0, \ldots, z_{n+1}) \in \mathbb{P}_{n+1} \mid z_0^d + \cdots + z_{n+1}^d = 0\} \subset \mathbb{P}_{n+1}$$

(85)

the intermediate cohomology can be determined by lower-dimensional varieties in combination with Tate twists by reconstructing the higher dimensional variety $X_d^n$ of degree $d$ and dimension $n$ in terms of lower dimensional varieties $X_d^r$ and $X_d^s$ of the same degree with $n = r + s$. Briefly, this works as follows. The decomposition of $X_d^n$ is given as

$$X_d^{r+s} \cong B_{Z_1, Z_2}\left((\pi_Y^{-1}(X_d^r \times X_d^s))/\mu_d\right),$$

(86)

which involves the following ingredients.
(1) \( \pi_Y^{-1}(X_d^r \times X_d^s) \) denotes the blow-up of \( X_d^r \times X_d^s \) along the subvariety
\[
Y = X_d^{r-1} \times X_d^{s-1} \subset X_d^r \times X_d^s.
\] (87)

The variety \( Y \) is determined by the fact that the initial map which establishes the relation between the three varieties \( X_d^{r+s}, X_d^r, X_d^s \) is defined on the ambient spaces as
\[
((x_0, \ldots, x_{r+1}), (y_0, \ldots, y_{s+1}) \mapsto (x_0y_{s+1}, \ldots, x_{r}y_{s+1}, x_{r+1}y_0, \ldots, x_{r+1}y_s).
\] (88)

This map is not defined on the subvariety \( Y \).

(2) \( \pi_Y^{-1}(X_d^r \times X_d^s)/\mu_d \) denotes the quotient of the blow-up \( \pi_Y^{-1}(X_d^r \times X_d^s) \) with respect to the action of
\[
\mu_d \ni \xi : ((x_0, \ldots, x_r, x_{r+1}), (y_0, \ldots, y_s, y_{s+1}) \mapsto ((x_0, \ldots, x_r, \xi x_{r+1}), (y_0, \ldots, y_s, \xi y_{s+1})).
\]

(3) \( B_{Z_1, Z_2} \left( (\pi_Y^{-1}(X_d^r \times X_d^s)) /\mu_d \right) \) denotes the blow-down in \( \pi_Y^{-1}(X_d^r \times X_d^s)/\mu_d \) of the two subvarieties
\[
Z_1 = \mathbb{P}_r \times X_d^{s-1}, \quad Z_2 = X_d^{r-1} \times \mathbb{P}_s.
\]

This construction leads to an iterative decomposition of the cohomology which takes the following form. Denote the Tate twist by
\[
H^i(X) (j) := H^i(X) \otimes W^\otimes j
\] (89)

with \( W = H^2(\mathbb{P}_1) \) and let \( X_d^{r+s} \) be a Fermat variety of degree \( d \) and dimension \( r + s \). Then
\[
H^{r+s}(X_d^{r+s}) \oplus \sum_{j=1}^r H^{r+s-2j}(X_d^{r-1}) (j) \oplus \sum_{k=1}^s H^{r+s-2k}(X_d^{s-1}) (k)
\]
\[
\cong H^{r+s}(X_d^r \times X_d^s)^{\mu_d} \oplus H^{r+s-2}(X_d^{r-1} \times X_d^{s-1}) (1).
\] (90)

This allows us to trace the cohomology of higher dimensional varieties to that of curves.

Weighted projective hypersurfaces can be viewed as resolved quotients of hypersurfaces embedded in ordinary projective space. The resulting cohomology has two components, the invariant part coming from the projection of the quotient, and the resolution part. As described in [39], the only singular sets on arbitrary weighted hypersurface Calabi-Yau threefolds are either points or curves. The resolution of singular points contributes to the even cohomology group \( H^2(X) \) of the variety, but does not contribute to the middle-dimensional cohomology group.
$H^3(X)$. Hence we need to be concerned only with the resolution of curves (see e.g. [40]). This can be described for general CY hypersurface threefolds as follows. If a discrete symmetry group $\mathbb{Z}/n\mathbb{Z}$ of order $n$ acting on the threefold leaves invariant a curve then the normal bundle has fibres $\mathbb{C}_2$ and the discrete group induces an action on these fibres which can be described by a matrix

$$\begin{pmatrix}
\alpha^{mq} & 0 \\
0 & \alpha^m
\end{pmatrix},$$

(91)

where $\alpha$ is an $n$'th root of unity and $(q,n)$ have no common divisor. The quotient $\mathbb{C}_2/(\mathbb{Z}/n\mathbb{Z})$ by this action has an isolated singularity which can be described as the singular set of the surface in $\mathbb{C}_3$ given by the equation

$$S = \{(z_1, z_2, z_3) \in \mathbb{C}_3 \mid z_3^n = z_1 z_2^{n-q}\}.$$

(92)

The resolution of such a singularity is completely determined by the type $(n,q)$ of the action by computing the continued fraction of $\frac{n}{q}$

$$\frac{n}{q} = b_1 - \frac{1}{b_2 - \frac{1}{\ddots - \frac{1}{b_s}}} \equiv [b_1, ..., b_s].$$

(93)

The numbers $b_i$ specify completely the plumbing process that replaces the singularity and in particular determine the additional generator to the cohomology $H^*(X)$ because the number of $\mathbb{P}_1$s introduced in this process is precisely the number of steps needed in the evaluation of $\frac{n}{q} = [b_1, ..., b_s]$. This can be traced to the fact that the singularity is resolved by a bundle which is constructed out of $s + 1$ patches with $s$ transition functions that are specified by the numbers $b_i$. Each of these gluing steps introduces a sphere, which in turn supports a $(1,1)$-form. The intersection properties of these 2-spheres are described by Hirzebruch-Jung trees, which for a $\mathbb{Z}/n\mathbb{Z}$ action is just an $SU(n + 1)$ Dynkin diagram, while the numbers $b_i$ describe the intersection numbers. We see from this that the resolution of a curve of genus $g$ thus introduces $s$ additional generators to the second cohomology group $H^2(X)$, and $g \times s$ generators to the intermediate cohomology $H^3(X)$.

Hence we see that the cohomology of weighted hypersurfaces is determined completely by the cohomology of curves. Because the Jacobian variety is the basic geometric invariant of a smooth projective curve this says that for weighted hypersurfaces the main cohomological structure is carried by their embedded curves.
7.2 Cohomology of weighted curves

For smooth algebraic curves $C$ of genus $g$ the de Rham cohomology group $H^1_{\text{dR}}(C)$ decomposes (over the complex number field $\mathbb{C}$) as

$$H^1_{\text{dR}}(C) \cong H^0(C, \Omega^1) \oplus H^1(C, O).$$

(94)

The Jacobian $J(C)$ of a curve $C$ of genus $g$ can be identified with

$$J(C) = \mathbb{C}^g/\Lambda,$$

(95)

where $\Lambda$ is the period lattice

$$\Lambda := \left\{ \left( \ldots, \int_a^p \omega_i, \ldots \right) \mid a \in H_1(C, \mathbb{Z}), \ \omega_i \in H^0(C, \Omega^1) \right\},$$

(96)

where the $\omega_i$ form a basis. Given a fixed point $p_0 \in C$ on the curve there is a canonical map from the curve to the Jacobian, called the Abel-Jacobi map

$$\Psi : C \longrightarrow J(C),$$

(97)

defined as

$$p \mapsto \left( \ldots, \int_{p_0}^p \omega_i, \ldots \right) \mod \Lambda.$$

(98)

We are interested in curves of Brieskorn-Pham type, i.e. curves of the form

$$C_d = \left\{ x^d + y^a + z^b = 0 \right\} \in \mathbb{P}_{(1,k,\ell)}[d],$$

(99)

such that $a = d/k$ and $b = d/\ell$ are positive rational integers. Without loss of generality we can assume that $(k, \ell) = 1$. The genus of these curves is given by

$$g(C_d) = \frac{1}{2}(2 - \chi) = \frac{(d - k)(d - \ell) + (k\ell - d)}{2k\ell}.$$  

(100)

For non-degenerate curves in the configurations $\mathbb{P}_{(1,k,\ell)}[d]$ the set of forms

$$H^1_{\text{dR}}(\mathbb{P}_{(1,k,\ell)}[d]) = \left\{ \omega_{r,s,t} = y^{s-1}z^{t-d/\ell}dy \mid r + ks + \ell t = 0 \mod d, \begin{cases} 1 \leq r \leq d - 1, \\ 1 \leq s \leq \frac{d}{k} - 1, \\ 1 \leq t \leq \frac{d}{\ell} - 1 \end{cases} \right\}$$

(101)

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defines a basis for the de Rham cohomology group $H^1_{dR}(C_d)$ whose Hodge split is given by
\[
H^0(C_d, \Omega^1_C) = \{ \omega_{r,s,t} \mid r + ks + \ell t = d \}
\]
\[
H^1(C_d, O_C) = \{ \omega_{r,s,t} \mid r + ks + \ell t = 2d \}.
\]

In order to show this we view the weighted projective space as the quotient of projective space with respect to the actions $Z_k : [0 1 0]$ and $Z_\ell : [0 0 1]$, where we use the abbreviation $Z_k = \mathbb{Z}/k\mathbb{Z}$ and for any group $\mathbb{Z}_r$ the notation $[a, b, c]$ indicates the action
\[
[a, b, c] : (x, y, z) \mapsto (\gamma^a x, \gamma^b y, \gamma^c z),
\]
where $\gamma$ is a generator of the group. This allows us to view the weighted curve as the quotient of a projective Fermat type curve
\[
\mathbb{P}_{(1,k,\ell)}[d] = \mathbb{P}[d]/\mathbb{Z}_k \times \mathbb{Z}_\ell : \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

These weighted curves are smooth and hence their cohomology is determined by considering those forms on the projective curve $\mathbb{P}[d]$ which are invariant with respect to the group actions. A basis for $H^1_{dR}(\mathbb{P}[d])$ is given by the set of forms
\[
H^1_{dR}(\mathbb{P}[d]) = \{ \omega_{r,s,t} = y^{s-1}z^{t-d}dy \mid 0 < r, s, t < d, \quad r + s + t = 0 \ (mod \ d), \quad r, s, t \in \mathbb{N} \}.
\]

Denote the generator of the $Z_k$ action by $\alpha$ and consider the induced action on $\omega_{r,s,t}$
\[
Z_k : \omega_{r,s,t} \mapsto \alpha^s \omega_{r,s,t}.
\]
It follows that the only forms that descend to the quotient with respect to $Z_k$ are those for which $s = 0 (mod \ k)$. Similarly we denote by $\beta$ the generator of the action $Z_\ell$ and consider the induced action on the forms $\omega_{r,s,t}$
\[
Z_\ell : \omega_{r,s,t} \mapsto \beta^{t-d} \omega_{r,s,t}.
\]
Again we see that the only forms that descend to the quotient are those for which $t = 0 (mod \ \ell)$.

### 7.3 Abelian varieties from weighted Jacobians

Jacobian varieties in general are not abelian varieties with complex multiplication. The question we can ask, however, is whether the Jacobians of the curves that determine the cohomology of the Calabi-Yau varieties can be decomposed such that the individual factors admit
complex multiplication by an order of a number field. In this section we show that this is indeed the case and therefore we can define the complex multiplication type of a Calabi-Yau variety in terms of the CM types induced by the Jacobians of its curves.

It was shown by Faddeev [41] that the Jacobian variety $J(C_d)$ of Fermat curves $C_d \subset \mathbb{P}_2$ splits into a product of abelian factors $A_{\mathcal{O}_i}$

$$J(C_d) \cong \prod_{\mathcal{O}_i \in \mathcal{I}/(\mathbb{Z}/d\mathbb{Z})^\times} A_{\mathcal{O}_i},$$

where the set $\mathcal{I}$ provides a parametrization of the cohomology of $C_d$, and the sets $\mathcal{O}_i$ are orbits in $\mathcal{I}$ of the multiplicative subgroup $(\mathbb{Z}/d\mathbb{Z})^\times$ of the group $\mathbb{Z}/d\mathbb{Z}$. More precisely it was shown that there is an isogeny

$$i : J(C_d) \longrightarrow \prod_{\mathcal{O}_i \in \mathcal{I}/(\mathbb{Z}/d\mathbb{Z})^\times} A_{\mathcal{O}_i},$$

where an isogeny $i : A \rightarrow B$ between abelian varieties is defined to be a surjective homomorphism with finite kernel.

In the parametrization used in the previous subsection $\mathcal{I}$ is the set of triplets $(r, s, t)$ in (105) and the periods of the Fermat curve have been computed by Rohrlich [44] to be

$$\int_{A \cup B^\times} \omega_{r, s, t} = \frac{1}{d} B \left( \frac{s}{d}, \frac{t}{d} \right) (1 - \xi^r)(1 - \xi^t) \xi^{js+kt},$$

where $\xi$ is a primitive $d$–th root of unity, and

$$B(u, v) = \int_0^1 t^{u-1}(1 - v)^{v-1}dt$$

is the classical beta function. $\mathcal{A}, \mathcal{B}$ are the two automorphism generators

$$\mathcal{A}(1, y, z) = (1, \xi y, z)$$
$$\mathcal{B}(1, y, z) = (1, y, \xi z)$$

and $\kappa$ is the generator of $H_1(C_d)$ as a cyclic module over $\mathbb{Z}[\mathcal{A}, \mathcal{B}]$. The period lattice of the Fermat curve therefore is the span of

$$\left( \ldots, \xi^{js+kt}(1 - \xi^r)(1 - \xi^s) \frac{1}{d} B \left( \frac{r}{d}, \frac{s}{d} \right), \ldots \right)_{1 \leq r, s, t \leq d-1, \ r+s+t=d}, \ \forall 0 \leq j, k \leq d - 1.$$

The abelian factor $A_{[(r, s, t)]}$ associated to the orbit $\mathcal{O}_{r, s, t} = [(r, s, t)]$ can be obtained as the quotient

$$A_{[(r, s, t)]} = \mathbb{C}^{\phi(d_0)/2} / \Lambda_{r, s, t},$$

More accessible are the references [42], [43], [44] on the subject.
where \( d_0 = d / \gcd(r, s, t) \) and the lattice \( \Lambda_{r,s,t} \) is generated by elements of the form
\[
\sigma_a(z)(1 - \xi^{as})(1 - \xi^{at}) \frac{1}{d} B \left( \frac{<as>}{d}, \frac{<at>}{d} \right),
\]
where \( z \in \mathbb{Z}[\mu_{d_0}] \), \( \sigma_a \in \Gal(\mathbb{Q}(\mu_{d_0})/\mathbb{Q}) \) runs through subgroups of the Galois group of the cyclotomic field \( \mathbb{Q}(\mu_{d_0}) \) and \( <x> \) is the smallest integer \( 0 \leq x < 1 \) congruent to \( x \mod d \).

Alternatively, the abelian variety \( A_{d}^{r,s,t} \) can be constructed in a more geometric way as follows. Consider the orbifold of the Fermat curve \( \mathcal{C}_d \) with respect to the group defined as
\[
G_{r,s,t}^d = \{ (\xi_1, \xi_2, \xi_3) \in \mu_d^3 | \xi_1^r\xi_2^s\xi_3^t = 1 \}.
\]
The quotient \( \mathcal{C}_d / G_{r,s,t}^d \) can be described algebraically via projections
\[
T_d^{r,s,t}: \mathcal{C}_d \rightarrow \mathcal{C}_d^{r,s,t}
\]
\[
(x, y) \mapsto (x^d, x^ry^s) =: (u, v),
\]
which map \( \mathcal{C}_d \) into the curves
\[
\mathcal{C}_d^{r,s,t} = \{ v^d = u^r(1 - u)^s \}.
\]
For prime degrees the abelian varieties \( A_{d}^{r,s,t} \) can be defined simply as the Jacobians \( J(\mathcal{C}_d^{r,s,t}) \) of the projections \( \mathcal{C}_d^{r,s,t} \). When \( d \) has nontrivial divisors \( m|d \), this definition must be modified as follows. Consider the projected Fermat curves
\[
\mathcal{C}_d \rightarrow \mathcal{C}_m
\]
\[
(x, y) \mapsto (\bar{x}, \bar{y}) := \left( \frac{x^d}{m}, \frac{y^d}{m} \right),
\]
whose Jacobians can be embedded as \( e: J(\mathcal{C}_m) \rightarrow J(\mathcal{C}_d) \). Composing the projection \( T_d^{r,s,t} \) as
\[
J(\mathcal{C}_m) \xrightarrow{e} J(\mathcal{C}_d) \xrightarrow{T_d^{r,s,t}} J(\mathcal{C}_d^{r,s,t})
\]
for all proper divisors \( m|d \) leads to a collection of subvarieties \( \cup_{m|d} T_d^{r,s,t}(e(J(\mathcal{C}_m))) \). The abelian variety of interest then is defined as
\[
A_{d}^{r,s,t} = J(\mathcal{C}_d^{r,s,t}) / \cup_{m|d} T_d^{r,s,t}(e(J(\mathcal{C}_m))).
\]
The abelian varieties \( A_{d}^{r,s,t} \) are not necessarily simple but it can happen that they in turn can be factored. This question can be analyzed via a criterion of Shimura-Taniyama, described in
Applied to the $A_d^{r,s,t}$ discussed here the Shimura-Taniyama criterion involves computing for each set $H_d^{r,s,t}$ defined as

$$H_d^{r,s,t} := \{ a \in (\mathbb{Z}/d\mathbb{Z})^\times \mid <ar> + <aks> + <a\ell t> = d \}$$ (122)

another set $W_d^{r,s,t}$ defined as

$$W_d^{r,s,t} = \{ a \in (\mathbb{Z}/d\mathbb{Z})^\times \mid aH_d^{r,s,t} = H_d^{r,s,t} \}.$$ (123)

If the order $|W_d^{r,s,t}|$ of $W_d^{r,s,t}$ is unity then the abelian variety $A_d^{r,s,t}$ is simple, otherwise it splits into $|W_d^{r,s,t}|$ factors [45].

We adapt this discussion to the weighted case. Denote the index set of triples $(r,s,t)$ parametrizing the one-forms of the weighted curves $C_d \in \mathbb{P}_{(1,k,\ell)[d]}$ again by $\mathcal{I}$. The cyclic group $(\mathbb{Z}/d\mathbb{Z})^\times$ again acts on $\mathcal{I}$ and produces a set of orbits

$$O_{r,s,t} = [(r,s,t)] \in \mathcal{I}/(\mathbb{Z}/d\mathbb{Z})^\times.$$ (124)

Each of these orbits leads to an abelian variety $A_{[(r,s,t)]}$ of dimension

$$\dim A_{[(r,s,t)]} = \frac{1}{2}\varphi(d_0),$$ (125)

where $\varphi$ is the Euler function $\varphi(n) = \#\{m \mid (m,n) = 1\}$, and complex multiplication with respect to the field $F_{[(r,s,t)]} = \mathbb{Q}(\mu_{d_0})$, where $d_0 = d/\gcd(r,ks,\ell t)$. This leads to an isogeny

$$i : J(C_d) \rightarrow \prod_{[(r,s,t)] \in \mathcal{I}/(\mathbb{Z}/d\mathbb{Z})^\times} A_{[(r,s,t)]}.$$ (126)

The complex multiplication type of the abelian factors $A_{r,s,t}$ of the Jacobian $J(C)$ can be identified with the set $H_d^{r,s,t}$ via a homomorphism from $H_d^{r,s,t}$ to the Galois group. More precisely, the CM type is determined by the subgroup $G_d^{r,s,t}$ of the Galois group of the cyclotomic field that is parametrized by $H_d^{r,s,t}$

$$G_d^{r,s,t} = \{ \sigma_a \in \text{Gal}(\mathbb{Q}(\mu_{d_0})/\mathbb{Q}) \mid a \in H_d^{r,s,t} \}$$ (127)

by considering

$$(F, \{\phi_a\}) = (\mathbb{Q}(\mu_{d_0}), \{\sigma_a \mid \sigma_a \in G_d^{r,s,t}\}).$$ (128)
8 The Fermat quintic threefold

8.1 CM type

Consider the projective threefold embedded in projective 4–space and defined by

\[ X_5 = \{(z_0 : z_1 : \ldots : z_5) \in \mathbb{P}_4 \mid z_0^5 + \cdots + z_4^5 = 0\}. \quad (129) \]

We can split \( d = 3 = 1 + 2 = r + s \) and apply the Shioda-Katsura construction to obtain the decompositions

\[ H^3(X_5) \oplus H^1(C_5)(1) \cong H^3(C_5 \times S_5)^\mu_5 \oplus H^1(X_0^5 \times C_d)(1) \quad (130) \]

and

\[ H^2(S_5) \cong H^2(C_5 \times C_5)^\mu_5 \oplus d(d - 2)H^2(\mathbb{P}_1) \quad (131) \]

in terms of the cohomology groups of the Fermat curve

\[ C_5 = \{x^5 + y^5 + z^5 = 0\} \subset \mathbb{P}_2 \quad (132) \]

and the Fermat surface \( S_5 \).

From this we see that the basic building block of the cohomology decomposition is given by the plane projective curve \( C_5 \) which has genus \( g(C_5) = 6 \). The index set \( \mathcal{I} \)

\[ \mathcal{I} = \{(1, 1, 3), (1, 3, 1), (3, 1, 1), (1, 2, 2), (2, 1, 2), (2, 2, 1); \]
\[ (2, 4, 4), (4, 2, 4), (4, 4, 2), (3, 3, 4), (3, 4, 3), (4, 3, 3)\} \]

parametrizes a basis of the first cohomology group of \( C_5 \), which can be written as

\[ H^1_{dR}(C_5) = \{\omega_{r,s,t} = x^{r-1}y^{s-5}dx \mid (r, s, t) \in \mathcal{I}\}. \quad (133) \]

The action of \((\mathbb{Z}/5\mathbb{Z})^\times\) leads to the orbits

\[ O_{1,1,3} = \{(1, 1, 3), (2, 2, 1), (3, 3, 4), (4, 4, 2)\} \]
\[ O_{1,3,1} = \{(1, 3, 1), (2, 1, 2), (3, 4, 3), (4, 2, 4)\} \]
\[ O_{3,1,1} = \{(3, 1, 1), (1, 2, 2), (3, 4, 3), (4, 4, 2)\} \quad (134) \]
Hence the Jacobian decomposes into a product of three abelian varieties

\[ J(C_5) = \prod_{\mathcal{O}_{r,s,t} \in \mathbb{Z}/(\mathbb{Z}/5\mathbb{Z})^\times} A_{r,s,t} \]

\[ = A_{1,1,3} \times A_{1,3,1} \times A_{3,1,1}, \quad (135) \]

each of dimension \( \varphi(5)/2 = 2 \), which arise from the Jacobians of the genus two curves

\[ C_{5}^{1,1,3} = \{ v^5 - u(1 - u) = 0 \} \]
\[ C_{5}^{1,3,1} = \{ v^5 - u(1 - u)^3 = 0 \} \]
\[ C_{5}^{3,1,1} = \{ v^5 - u^3(1 - u) = 0 \}, \quad (136) \]

obtained via the maps \( T_{5}^{r,s,t} \).

In order to check the simplicity of the abelian factors we can use the criterion of Shimura-Taniyama, described above. Computing the sets \( W_{5}^{r,s,t} \) for any of the triplets \((r,s,t)\) shows that the order of these groups is unity, hence all three factors are in fact simple.

For the complex multiplication type we find from

\[ H_{5}^{1,1,3} = \{ a \in (\mathbb{Z}/5\mathbb{Z})^\times = \{ 1, 2, 3, 4 \} \mid < a > + < a > + < 3a > = 5 \} = \{ 1, 2 \} \quad (137) \]

that \( G_{5}^{1,1,3} = \{ \sigma_1, \sigma_2 \} \) and therefore the complex multiplication type of \( A_{1,1,3} \) is given by

\[ (\mathbb{Q}(\mu_5), \{ \varphi \} = \{ \sigma_1, \sigma_2 \}). \quad (138) \]

The remaining factors are described in the same way.

More explicitly, we can use the maps \( T_{5}^{r,s,t} \) to express the differentials of \( C_d \) invariant under the action of \( G_{5}^{r,s,t} \) in terms of the \((u,v)\) coordinates of \( C_{5}^{r,s,t} \) and observe their transformation behavior under the map

\[ (u, v) \mapsto (u, \xi_5 v). \quad (139) \]

### 8.2 Fusion field and quantum dimensions

The field of complex multiplication derived for the quintic is given by the cyclotomic field \( \mathbb{Q}(\mu_5) \) and embedded in this field is the real subfield \( \mathbb{Q}(\sqrt{5}) \), generated by the elements \((\xi_5 + \xi_5^{-1})\). To
compare this to the number field determined by the string we briefly recall some facts about the corresponding Gepner model [46, 47].

The underlying exactly solvable model of the quintic threefold is determined by the affine Kac-Moody algebra SU(2) at conformal level \(k = 2\). The central charge \(c(k) = 3k/(k + 2)\) at level \(k\) then leads to \(c = 9/5\), leading to a product of five models to make a theory of total charge \(c = 9\). The physical spectrum of this model is constructed from world sheet operators of the individual SU(2) factors with the anomalous dimensions

\[
\Delta_j^{(k)} = \frac{j(j + 2)}{4(k + 2)}, \quad j = 0, ..., k,
\]

leading in the case \(k = 3\) to \(\Delta_j^{(3)} \in \left\{0, \frac{3}{20}, \frac{2}{5}, \frac{3}{4}\right\}\).

These anomalous dimensions can be mapped into the quantum dimensions \(Q_{ij}\) via the Rogers dilogarithm

\[
L(z) = Li_2(z) + \frac{1}{2} \log(z) \log(1 - z),
\]

where \(Li_2\) is Euler’s classical dilogarithm

\[
Li_2(z) = \sum_{n \in \mathbb{N}} \frac{z^n}{n^2},
\]

via the relation[48, 49, 50]

\[
\frac{1}{L(1)} \sum_{i=1}^{k} L \left( \frac{1}{Q_{ij}} \right) = \frac{3k}{k + 2} - 24\Delta_j^{(k)} + 6j.
\]

Here the \(Q_{ij}\) are defined as

\[
Q_{ij} = \frac{S_{ij}}{S_{0j}},
\]

where the

\[
S_{ij} = \sqrt{\frac{2}{k + 2}} \sin \left( \frac{(i + 1)(j + 1)\pi}{k + 2} \right), \quad 0 \leq i, j \leq k
\]

describe the modular behavior of the SU(2) affine characters

\[
\chi_i \left(-\frac{1}{\tau}, \frac{u}{\tau}\right) = e^{\pi i ku^2/2} \sum_j S_{ij} \chi_j(\tau, u).
\]

Applying this map to the theory at conformal level three leads to the quantum dimensions \(Q_i = Q_{i0}\)

\[
Q_i (SU(2)_3) \in \left\{1, \frac{1}{2}(1 + \sqrt{5})\right\} \subset \mathbb{Q}(\sqrt{5}).
\]
9 Appendix: Some number theory

The adelic language is a useful tool in the analysis of torsion points of abelian varieties. The construction of this group proceeds by first considering \( p \)-adic number fields and then pasting these together in such a way that one obtains something that is still manageable.

9.1 \( p \)-adic numbers

For rational primes \( p \) \( p \)-adic number fields are constructed much like the real number field \( \mathbb{R} \) as completion of the rational number field \( \mathbb{Q} \) with respect to norms \( | \cdot |_p \), which are defined for each rational prime \( p \) by noting that each rational number \( x \in \mathbb{Q} \) can be written as \( x = \frac{m}{n}p^r \) for some \( r \in \mathbb{N} \) and \( m, n \in \mathbb{Z} \). The exponent \( r \) is called the order \( \text{ord}_p x = r \) of \( x \) at \( p \), and one writes \( |x|_p = p^{\text{ord}_p x} \). In order to introduce a common language one calls the norms for the reals and the complex numbers the norms of the 'infinite primes' (for no particularly good reason). More precisely, consider an algebraic number field \( K \) with degree \([K : \mathbb{Q}] = r_1 + 2r_2\), its real embeddings \( \{\sigma_i\}_{i=1,...,r_1} \), and its complex embeddings \( \{\sigma_{r_1+i}, \overline{\sigma}_{r_1+i}\}_{i=1,...,r_2} \). Then for each real \( \sigma_i \) one defines a valuation on \( K \) via

\[
|x|_{\sigma_i} = |x^{\sigma_i}|,
\]

(148)

where \( x^{\sigma_i} \) denotes the action of \( \sigma_i \) on \( x \in K \). For the complex embeddings define the valuation as

\[
|x|_{\sigma_{r_1+i}} = x^{\sigma_i} \overline{x}^{\overline{\sigma_i}}.
\]

(149)

These valuations satisfy the archimedean inequality and are therefore called archimedean. They define a metric with respect to which \( K \) is a topological field, and one can consider the completions, denoted by \( K_{\infty,i} \), of \( K \) with respect to these valuations. This proceeds in complete analogy to the construction of the real number field as the completion of the rational number field with respect to the standard norm.

This construction can be generalized to other number fields via valuations derived from the finite primes, i.e. prime ideals \( p \subset \mathcal{O}_K \subset K \), where \( \mathcal{O}_K \) denotes the ring of algebraic integers in \( K \). Define for any \( x \in K \) the valuation \( | \cdot |_p \) as

\[
|x|_p = Np^{-\text{ord}_p x},
\]

(150)
where \( Np = \#(\mathcal{O}_K/p) \) is the norm of the ideal \( p \), and \( \text{ord}_p x \) is the power to which the ideal \( p \) occurs in the factorization of the principal ideal \( (x) \). Each such valuation again defines a metric on \( K \), \( d_p = |x - y|_p \), and one can again consider the completion with respect to this metric, denoted by \( K_p \). The multiplicative subgroup is denoted by \( K_p^\times \), and the ring of integers in this field is defined by

\[
\mathcal{O}_p = \{ y \in K_p \mid |y|_p \leq 1 \},
\]

while its group of units is given as

\[
\mathcal{O}_p^\times = \{ y \in K_p \mid |y|_p = 1 \}.
\]

### 9.2 Adèles and idèles

It is often useful in number theory to consider an embedding of a number field of degree \( n = r_1 + 2r_2 \) into a vector space spanned by its completions with respect to the standard metric ( [4] contains an application in a physical context). Given the completions of \( K \) with respect to finite and infinite primes, it makes sense to try and put them all together into a single large product \( \prod_p K_p \) that combines all of them. It turns out that this object is too large, and that it is more useful to define the ring of adèles \( \mathbb{A}_K \) of the field \( K \) to be the subset defined by a restricted product, consisting of all elements \( x = (x_p) \) such that \( x_p \) is a \( p \)-adic integer for all but finitely many non-archimedean valuations \( | \cdot |_p \)

\[
\mathbb{A}_K = \{(x_p) \mid x_p \in K_p \forall p, \ x \in \mathcal{O}_p \text{ for all but finitely many } p \}.
\]

Put somewhat differently, consider a set \( S \) of primes that contains the infinite ones. Then we can construct the ring

\[
\mathbb{A}_K^S = \prod_{p \notin S} \mathcal{O}_p \prod_{q \in S} K_q,
\]

where the multiplication is understood to be component wise. For \( S \subset S' \) there is an injection \( \mathbb{A}_S \rightarrow \mathbb{A}_{S'} \) and one can define

\[
\mathbb{A}_K = \lim_S \mathbb{A}_K^S.
\]

Embedded within the adèles is the multiplicative subgroup \( \mathbb{A}_K^\times \) called the idèles. These were originally introduced by Chevalley [28], independently of the later notion of adèles [29], and
can be described in a way completely analogous to the adèles as

\[ A_K^\times = \{(x_p) \mid x_p \in K_p^\times \forall p, \ x \in \mathcal{O}_p^\times \text{ for all but finitely many } p\} \]  \hspace{1cm} (156)

Addition and multiplication in the adèles and idèles are understood to be component wise.

If \( K/F \) is a finite extension then one can define an idèlic norm map

\[ N^K_F : A^K \rightarrow A_F^\times \]  \hspace{1cm} (157)

which is determined by specifying what the \( v \)-th component is of the image idèle, where \( v \) describes either the finite or infinite primes. For \( x \in A^K \) one sets

\[ (N^K_F x)_v = \prod_{w|v} N^K_{w,v} x_w, \]  \hspace{1cm} (158)

where the product is over all \( w \) which contain \( v \), and \( K_w, F_v \) denote the completions of \( K \) and \( F \) with respect to the norms defined by \( w \) and \( v \) respectively.

### 9.3 Class field theory

Class field theory deals with the behavior of primes in number fields \( K \) when \( K \) is embedded in finite extensions \( L/K \). For simple fields such as quadratic extensions of the rationals this behavior is controlled by the Legendre symbol. For more complicated fields the Artin symbol provides an appropriate generalization.

Consider a finite number field extension \( L/K \) that is Galois and an ideal \( \mathfrak{p} \subset \mathcal{O}_L \) in the ring of integers \( \mathcal{O}_L \) of \( L \) that divides the prime ideal \( \mathfrak{p} \subset \mathcal{O}_K \). The Artin symbol is then defined as the element \( \sigma_{\mathfrak{p}} \) of the Galois group \( \text{Gal}(L/K) \), often denoted by \((\mathfrak{p}, L/K)\), or \((\frac{L/K}{\mathfrak{p}})\), for which

\[ \sigma_{\mathfrak{p}}(x) = x^{Np} \mod \mathfrak{p}. \]  \hspace{1cm} (159)

It is a multiplicative function on the primes

\[ \left(\frac{L/K}{\mathfrak{p}}\right) : \mathcal{I}(L) \rightarrow \text{Gal}(L/K) \]  \hspace{1cm} (160)

and therefore can be defined for any ideal \( \mathfrak{a} \subset \mathcal{O}_L \) via the prime decomposition \( \mathfrak{a} = \prod \mathfrak{p}_i \) as

\[ \left(\frac{L/K}{\mathfrak{a}}\right) = \prod_i \left(\frac{L/K}{\mathfrak{p}_i}\right). \]  \hspace{1cm} (161)
When the extension $L/K$ is abelian the Artin symbol is independent of the prime $\mathfrak{p}$ chosen above $p$ and depends only on $p$.

The idèlic formulation of class field theory aims at an efficient discussion of all abelian extensions at the same time.

**Theorem.** Let $K$ be a number field and $K_{ab}$ be the maximal abelian extension of $K$. There exists a unique continuous homomorphism, called the reciprocity map

$$\mathbb{A}_K^\times \to \text{Gal}(K_{ab}/K)$$

$$x \mapsto [x,K], \quad (162)$$

with the property that if $L/K$ is a finite abelian extension, $x \in \mathbb{A}_K^\times$ an idèle whose ideal $(x)$ is not divisible by any primes that ramify in $L$, then

1) $[x,K]_L = ((x), L/K)$ is the Artin symbol of the ideal $(x)$.
2) The reciprocity map is compatible with the norm map: if $L$ is a finite abelian extension of $K$ then

$$[x,L]_{K_{ab}} = [N^L_K x, K], \quad \forall x \in \mathbb{A}_L^\times. \quad (163)$$

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