Speakable in quantum mechanics

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Abstract At the 1927 Como conference Bohr spoke the famous words “It is wrong to think that the task of physics is to find out how nature is. Physics concerns what we can say about nature.” However, if the Copenhagen interpretation really adheres to this motto, why then is there this nagging feeling of conflict when comparing it with realist interpretations? Surely what one can say about nature should in a certain sense be interpretation independent. In this paper I take Bohr’s motto seriously and develop a quantum logic that avoids assuming any form of realism as much as possible. To illustrate the non-triviality of this motto, a similar result is first derived for classical mechanics. It turns out that the logic for classical mechanics is a special case of the quantum logic thus derived. Some hints are provided as to how these logics are to be used in practical situations and finally, I discuss how some realist interpretations relate to these logics.

Keywords Quantum logic · Intuitionistic logic · Instrumentalism

1 Introduction

Over the last few decades, much of research in the foundations of quantum mechanics has focused on the impossibility of certain interpretations. Results such as the Kochen–Specker theorem (Kochen and Specker 1967) or the Bell inequalities (Bell 1964; Clauser et al. 1969) mainly establish what is unspeakable in quantum mechanics. Often these results are interpreted in favor of instrumentalist or Copenhagen-like interpretations of quantum mechanics. But, as it is well known, switching to an
epistemic account of physics alone isn’t sufficient to account for the counter-intuitive aspects of quantum mechanics. Furthermore, these accounts often resort to vagueness when explaining, for example, how quantum mechanics can possibly violate Bell inequalities. Somewhat notorious are for example explanations based on Bohr’s account of complementarity.\footnote{See for example Folse (1981) for how such an explanation may be ran.}

In this paper I propose a formal scheme for reasoning within epistemic approaches to physics in order to shed some light on what is speakable in quantum mechanics. After all, the more clearly one understands the epistemic part of quantum mechanics, the easier the ontological part can be investigated. To get a feeling for the problems I have in mind, consider the following result.

\textbf{Theorem 1} Suppose $\mathbb{P}$ is a probability function on a collection of sentences $S$ to the interval $[0, 1]$ that satisfies the following rules for all $A, B \in S$:

1. If $A \vdash B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$.
2. $\mathbb{P}(A \lor B) \leq \mathbb{P}(A) + \mathbb{P}(B)$.

Then, if $S$ obeys classical logic, the following inequality holds for all $A_1, A_2, B_1$ and $B_2$ in $S$:

$$\mathbb{P}(A_1 \land B_1) \leq \mathbb{P}(A_1 \land B_2) + \mathbb{P}(A_2 \land B_1) + \mathbb{P}(\neg A_2 \land \neg B_2). \quad (1)$$

\textit{Proof} The result follows from a straight-forward computation:

$$\mathbb{P}(A_1 \land B_1) = \mathbb{P}(A_1 \land B_1 \land (B_2 \lor \neg B_2))$$
$$= \mathbb{P}((A_1 \land B_1 \land B_2) \lor (A_1 \land B_1 \land \neg B_2))$$
$$\leq \mathbb{P}(A_1 \land B_1 \land B_2) + \mathbb{P}(A_1 \land B_1 \land \neg B_2)$$
$$\leq \mathbb{P}(A_1 \land B_2) + \mathbb{P}(B_1 \land \neg B_2) \quad (2)$$
$$= \mathbb{P}(A_1 \land B_2) + \mathbb{P}((B_1 \land \neg B_2 \land A_2) \lor (B_1 \land \neg B_2 \land \neg A_2))$$
$$\leq \mathbb{P}(A_1 \land B_2) + \mathbb{P}(B_1 \land \neg B_2 \land A_2) + \mathbb{P}(B_1 \land \neg B_2 \land \neg A_2)$$
$$\leq \mathbb{P}(A_1 \land B_2) + \mathbb{P}(A_2 \land B_1) + \mathbb{P}(\neg A_2 \land \neg B_2).$$

This is a Bell-type inequality of which it is well-known that it can be violated by quantum mechanics. To see this, consider a pair of entangled qubits and set

$$A_i = \left[\begin{array}{c} \sigma_{r_i}^A = \frac{1}{2} \\ \sigma_{r_i}^A = \frac{1}{2} \end{array}\right], \quad \neg A_i = \left[\begin{array}{c} \sigma_{r_i}^A = -\frac{1}{2} \\ \sigma_{r_i}^A = -\frac{1}{2} \end{array}\right],$$
$$B_i = \left[\begin{array}{c} \sigma_{r_i}^B = \frac{1}{2} \\ \sigma_{r_i}^B = \frac{1}{2} \end{array}\right] \quad \text{and} \quad \neg B_i = \left[\begin{array}{c} \sigma_{r_i}^B = -\frac{1}{2} \\ \sigma_{r_i}^B = -\frac{1}{2} \end{array}\right] \quad (3)$$

for $i = 1, 2$, where $\sigma_{r_i}^A$ is the spin along the $r_i$-axis of the qubit send to Alice, and $\sigma_{r_i}^B$ is the spin along the $r_i$-axis of the qubit send to Bob.
It is a merit of this particular Bell-type inequality that it doesn’t rely on a hidden variables framework. It is not even supposed that $P$ is a Kolmogorovian probability function. Consequently, any interpretation (realist or instrumentalist) of quantum mechanics must be able to point out which of the assumptions for the theorem are violated in quantum mechanics. More explicitly, it is part of the interpretation to explicate what is meant by the relations in (3) or to perhaps explain why these relations are devoid of meaning. For example, in certain realist interpretations these propositions will be understood as stating properties of the particles, whereas for an instrumentalist interpretation they may concern (idealized) measurement outcomes. A non-local theory may argue that a revelation of the truth value of, say, $A_1$ instantaneously causes an altering of the truth values of $B_1$ and $B_2$. From a Copenhagen perspective, one may argue that $B_1$ and $B_2$ are complementary sentences, which cannot both meaningfully occur in a single sentence, making the proof of Theorem 1 meaningless. A similar approach might be used by followers of the consistent histories approach or of the many worlds interpretation. And the list can go on.

One of the responses that I find most intriguing is that from orthodox quantum logic (Birkhoff and von Neumann 1936). In this approach, the proof fails because it uses the law of distributivity several times; the premise that $S$ obeys classical logic does not hold in quantum mechanics. The elegance of this approach is that it points to a flaw in the proof precisely at the point where the proof clashes with quantum mechanical calculations. But as far as explanations go, this approach only replaces one mystery with another. At least, as long as no explanation is given for the failure of distributivity. In particular, it raises questions about the meaning of the logical connectives “and” and “or”. It has been argued convincingly that their meaning in quantum logic should at least differ from their classical meaning (Dummett 1976). But it isn’t clear what they should be instead. From this semantic perspective, traditional quantum logic has failed in providing a framework for reasoning about quantum mechanical phenomena.

2 Intuitionistic logic

A striking example (taken from Popper (1968)) of the problem with interpreting logical connectives in quantum logic is the following. The law of excluded middle is maintained in quantum logic and therefore, for every proposition $P$ the formula $P \lor \neg P$ is always true. Consequently, for every pair of propositions $P_1$ and $P_2$ it holds that if $P_1$ is true, then $P_1 \land (P_2 \lor \neg P_2)$ is also true. However, in quantum logic there are such pairs for which $P_1$ is true, but neither $P_1 \land P_2$ nor $P_1 \land \neg P_2$ is true (due to a failure of the law of distributivity). That is, $P_1$ is a proposition that is incompatible both with $P_2$ and $\neg P_2$ and may thus be thought of as an excluded middle. Obviously, this contradiction arose because I held on to a certain interpretation of the logical connectives. But the derivation seems innocent enough for me to conclude that no satisfactory interpretation of the quantum logical connectives can be defined. Rather, I tend to agree with Popper that

the kind of change in classical logic which would fit what Birkhoff and von Neumann suggest [...] would be the rejection of the law of excluded mid-
dle [...], as proposed by Brouwer, but rejected by Birkhoff and von Neumann (Popper 1968).

But if one is willing to drop the law of excluded middle, this has to be done with care. For some propositions the law may be maintained and I will call such propositions *decidable*. As an example, consider a measurement of $\sigma^A_{ri}$. In that case one knows that either $A_i$ or $\neg A_i$ will be true. The decidability of a proposition thus depends on the context in which the proposition is formulated. But in general, decidability in one context can’t be expected to hold in another context. Of course this is already well-known. Feynman formulated this difficult nature as follows for the two-slit experiment:

What we must say (to avoid making wrong predictions) is the following. If one looks at the holes or, more accurately, if one has a piece of apparatus which is capable of determining whether the electrons go through hole 1 or hole 2, then one *can* say that it goes either through hole 1 or hole 2. *But*, when one does *not* try to tell which way the electron goes, when there is nothing in the experiment to disturb the electrons, then one may *not* say that an electron goes either through hole 1 or hole 2. If one does say that, and starts to make any deductions from the statement, he will make errors in the analysis. This is the logical tightrope on which we must walk if we wish to describe nature successfully (Feynman et al. 1963, pp. 37–39).

The aim is now to provide a logical framework in which this ‘logical tightrope’ has a natural place.

Some readers may get the feeling they are being tricked into the use of intuitionistic logic so I would like to explain that in the present case the use of this kind of logic is not that strange at all. Classical logic relies on the idea that there is a fact of the matter as to which every sentence is either true or false. Intuitionistic logic focuses more on the epistemic view, interpreting ‘true’ as ‘knowing it to be true’ (or ‘having a proof of’). The truth of a negation is then read as ‘knowing it to be false’. An example in mathematics is the sentence

$$\forall x \in \mathbb{R} : x \geq 0 \text{ or } x \leq 0.$$  \hspace{1cm} (4)

Classically this is a true sentence, but intuitionists emphasize that real numbers can be constructed for which one simply does not know whether $x \geq 0$ or $x \leq 0$ is the case, and so they reject this sentence. Platonists often argue that a sentence like (4) is true by virtue of the existence of the set of real numbers in a Platonic realm. In physics intuitionistic logic then seems appropriate if one recognizes that one does not know what the ontology of the system under investigation is.

These arguments are not likely to convince a true classical logician who may argue that negation is just not defined correctly here. In that sense, the negation of “knowing it to be true” should be “not knowing it to be true” (which is what one does in formal epistemic logic). Many more propositions would have to be added to make the logic classical and most of them are rather dull. For example, in the mathematical case one would now have propositions about not having a proof of $x \leq 0$ for some $x \in \mathbb{R}$. In physics one would have to add propositions about not performing measurements.
this perspective, the choice for intuitionistic logic in this text is based on simplicity. So no claim for the necessity of intuitionistic logic is being made here. In particular, I disagree with the idea that logic may be empirical (Putnam 1969).

3 The epistemic approach

In an epistemic approach one has to have some notion of how to avoid ontological assumptions. This is non-trivial because often in discussions the ontology of a theory is directly coupled to the structure of that theory. For example, it is not uncommon to simply identify the state space of the theory with the set of possible ontological states. So not assuming realism involves not making this identification. In Sect. 4 it will become clear that this actually makes a difference.

As a further restriction, I will not assume that measurements reveal real properties of the system under consideration. Consequently, I take ‘measurement’ as a primitive concept in this paper. It is my opinion that this is unproblematic as long as one focuses only on the epistemic part of a theory. With this I mean that measurement as it is used in this paper is coupled with the idealized2 empirical predictions the theory makes.

A consequence of this approach is noted by Bell:

When one forgets the role of the apparatus, as the word ‘measurement’ makes all too likely, one despairs of ordinary logic – hence ‘quantum logic’. When one remembers the role of the apparatus, ordinary logic is just fine (Bell 1990, p. 34).

That I find myself defining an intuitionistic quantum logic may thus be seen as a consequence of taking ‘measurement’ as a primitive concept. However, I would like to go further than Bell and state that, if one finds an ontology for the apparatus in terms of the theory, classical logic seems a necessity; reality forces the law of excluded middle upon us. In particular, I am skeptic about the generalized notion of reality proposed in the topos theoretic approach in Döring and Isham (2011) or the ‘quantum numbers’ approach in Corbett and Durt (2009). At the opposite end, it may be clear that classical logic is by no means sufficient for a clear ontological description (Baltag and Smets 2011).

But setting the focus to measurements alone is not enough. The conceptually most difficult part in the derivation of an epistemic logic for a physical theory is to let go of any ontological prejudices. To be careful, I develop my logic ‘bottom up’, starting with certain elementary propositions, and then extending to more complex propositions by adding logical connectives. It is the consensus that “elementary” propositions in physical theories are of the form $A \in \Delta$ where $A$ is some observable and $\Delta$ is some subset of the set of all possible measurement outcomes for $A$ (usually taken to be $\mathbb{R}^n$ or some subset thereof). Oddly enough there is no consensus on what the proposition $A \in \Delta$ stands for exactly (Isham 1995). In the traditional viewpoint observables correspond to certain elements of a physical reality and, as such, have a definite value at

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2 There is always a gap between the actual (unsharp) measurement outcomes in the lab and the ideal (sharp) predictions made by the theory. It is not my aim to bridge this gap in this paper.
all times. The proposition $A \in \Delta$ is then taken to be a proposition about the definite value of the element of physical reality corresponding to $A$; $A$ has a value in $\Delta$. It is an ontological proposition.

For the sake of clarity I propose a new interpretation of $A \in \Delta$ and have it stand for “I measure $A$ and the result lies in $\Delta$”. This proposal is inspired by Peres’ credo that “unperformed experiments have no result”\(^3\); any informative proposition considering an observable $A$ must also consider the measurement of this observable. To emphasize this new interpretation I also introduce a new notation and the standard formula $A \in \Delta$ will now be $M_A(\Delta)$. In this reading one already finds a discrepancy with standard logics for physical theories for in general the disjunction $M_A(\Delta) \lor M_A(\Delta^c)$ (where $\Delta^c$ denotes the complement of $\Delta$) will no longer be true. If no experiment is performed to measure $A$, nothing can be said about the outcome of that measurement.

It should be noted that this does not directly entail a violation of the law of excluded middle. Such an entailment would require that $M_A(\Delta^c)$ is understood as the negation of $M_A(\Delta)$. But the introduction of logical connectives must be postponed until after the elementary propositions are fully laid down. In the end it will turn out that in the logics I will derive the negation of $M_A(\Delta)$ will be weaker than $M_A(\Delta^c)$, but still too strong for the law of excluded middle to hold.

From the above considerations one may expect that for any $\Delta$, $M_A(\Delta)$ can only have a truth value if $A$ is measured. However, for the sake of simplicity I’d like to make one exception, namely, the case where $\Delta$ is the empty set $\emptyset$. In that case I take it that $M_A(\Delta)$ is simply false. For this point I am adhering to the inverse of Peres’ credo, namely: “performed experiments have a result”.

To consider the totality of all elementary propositions $E_{PT}$ within a theory $T$, one has to classify all possible measurements and all possible outcomes for each of these measurements. Depending on the theory, this set will already have some logical structure stemming from the laws within the theory. Possible measurements are usually identified with observables within the theory. In this text I will simply equate the two, but use the standard grammar. E.g. I use formulations like “if the observable $A$ is measured then...”. To compare elementary propositions, I will make extensive use of the following assumption:

**LMR (Law-Measurement Relation)** If $A_1$ and $A_2$ are two observables in a theory $T$ that can be measured together according to that theory, and if $f$ is a function such that whenever $A_1$ and $A_2$ are measured together the outcomes $a_1$ and $a_2$ satisfy $a_1 = f(a_2)$ (i.e. $f$ represents a law within $T$), then a measurement of $A_2$ alone also counts as a measurement of $A_1$ with outcome $f(a_2)$.

This is the main rule that I will use to convert $E_{PT}$ into a partial ordered set $S_T$. Then, depending on the structure of the theory (classical or quantum mechanics), I introduce logical connectives to embed this partial ordered set into a lattice $L_T$. It turns out that both for classical and for quantum mechanics the resulting lattice is a proper Heyting algebra (i.e. one in which not every proposition is decidable).

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\(^3\) This is the title of Peres’ article (Peres 1978) in which he argues against the use of counterfactual reasoning. This opinion is also reflected in Peres (1984, 2002).
4 Classical mechanics

4.1 Elementary propositions

In classical mechanics, every observable \( A \) is identified with a function \( f_A \) on a phase space \( \Omega \) (assumed to be a measure space with \( \sigma \)-algebra \( \mathcal{F}(\Omega) \)) taking values in \( V_A \subset \mathbb{R} \), the set of possible measurement outcomes for \( A \) (endowed with the Lebesgue measure). For the total of observables I take a set \( \text{Obs} \) that is isomorphic to the set of all measurable functions from \( \Omega \) to (subsets of) \( \mathbb{R} \):

\[
\text{Obs} \simeq \{ f_A : \Omega \to V_A ; \ A \in \text{Obs} \} = \{ f : \Omega \to V \subset \mathbb{R} ; \ f \text{ is measurable} \}.
\] (5)

This choice may come of as too strict; it is conceivable that not every function corresponds to an observable. However, it is only for the sake of keeping the arguments in this text clear that I make this assumption. It may be relaxed while keeping the general gist of an epistemic logic for classical mechanics. The set of elementary propositions for classical mechanics is thus given by

\[
E_{PCM} := \{ M_A(\Delta) ; \ A \in \text{Obs}, \Delta \subset V_A \text{ measurable} \}.
\] (6)

Now every observable \( A \) generates a partition of the set \( \Omega \) (i.e. a collection of non-empty disjoint subsets whose union equals \( \Omega \)) by

\[
P_A := \left\{ f_A^{-1}(x) ; \ x \in V_A \right\}.
\] (7)

The set of all partitions generated by \( \text{Obs} \)

\[
L_{\text{Obs}} := \{ P_A ; \ A \in \text{Obs} \},
\] (8)

will play an important role. It is turned into a lattice by the definition

\[
P_{A_1} \leq P_{A_2} \iff \forall U_1 \in P_{A_1}, \exists U_2 \in P_{A_2} : U_1 \subset U_2,
\] (9)

with join and meet given by

\[
P_{A_1} \wedge P_{A_2} = \left\{ U_1 \cap U_2 ; \ U_1 \in P_{A_1}, U_2 \in P_{A_2} \right\},
\] (10a)

\[
P_{A_1} \vee P_{A_2} = \bigwedge \left\{ P_A ; \ A \in \text{Obs}, P_{A_1} \leq P_A, P_{A_2} \leq P_A \right\}.
\] (10b)

One may check that the operations (10a) and (10b) again correspond to elements of \( L_{\text{Obs}} \). For example, \( P_{A_1} \wedge P_{A_2} \) corresponds to the partition generated by any observable \( A \) for which there is an invertible function \( f : V_{A_1} \times V_{A_2} \to V_A \) such that \( f \circ (f_{A_1} \times f_{A_2}) = f_A \). The bottom element of this lattice is the partition of \( \Omega \) in singleton sets \( \{ \{ \omega \} ; \ \omega \in \Omega \} \) and the top element of \( L_{\text{Obs}} \) is given by the partition \( \{ \Omega \} \).
There is a direct physical significance to this lattice. Note that $P_{A_1} \leq P_{A_2}$ iff there is a surjective function $f : V_{A_1} \to V_{A_2}$ such that

$$f \circ f_{A_1} = f_{A_2}. \quad (11)$$

By LMR this means that if $P_{A_1} \leq P_{A_2}$, a measurement of $A_1$ is also a measurement of $A_2$. From LMR it also follows that a measurement of $A_3$ counts as a measurement of $A_1$ and $A_2$ whenever $P_{A_3} = P_{A_1} \wedge P_{A_2}$. And in the case that $P_{A_3} = P_{A_1} \vee P_{A_2}$, then both a measurement of $A_1$ counts as a measurement of $A_3$ and a measurement of $A_2$ counts as one for $A_3$.

Up until now the focus has been on the structure of $L_{\text{Obs}}$, but obviously the lattice of subsets of the outcome sets also plays an important role. Combining these two structures leads to the following demand for the preorder on $EPCM$:

**Lemma 1**

$$\left( P_{A_1} \leq P_{A_2} \text{ and } f_{A_1}^{-1}(\Delta_1) \subset f_{A_2}^{-1}(\Delta_2) \right) \implies M_{A_1}(\Delta_1) \leq M_{A_2}(\Delta_2). \quad (12)$$

**Proof** Firstly, a sufficient condition for two propositions $M_{A_1}(\Delta_1)$ and $M_{A_2}(\Delta_2)$ to be logically equivalent is $P_{A_1} = P_{A_2}$ and $f_{A_1}^{-1}(\Delta_1) = f_{A_2}^{-1}(\Delta_2)$. Indeed, in this case measurements of $A_1$ and $A_2$ are virtually the same measurements, and if the outcome sets $\Delta_1$ and $\Delta_2$ are in correspondence, the two propositions $M_{A_1}(\Delta_1)$ and $M_{A_2}(\Delta_2)$ imply each other.

Consider now the situation where two observables $A_1$ and $A_2$ satisfy the relation $P_{A_1} \leq P_{A_2}$ and that $f_{A_1}^{-1}(\Delta_1) = f_{A_2}^{-1}(\Delta_2)$. Since a measurement of $A_1$ is also a measurement of $A_2$, it follows that one can conclude $M_{A_2}(\Delta_2)$ from the proposition $M_{A_1}(\Delta_1)$. So thus far it has been established that a preorder $\leq$ on $EPCM$ should satisfy

$$\left( P_{A_1} \leq P_{A_2} \text{ and } f_{A_1}^{-1}(\Delta_1) = f_{A_2}^{-1}(\Delta_2) \right) \implies M_{A_1}(\Delta_1) \leq M_{A_2}(\Delta_2). \quad (13)$$

On the other hand, if one keeps the observable $A$ fixed, it is clear that one should have that if $\Delta \subset \Delta' \subset V_A$ then

$$M_A(\Delta) \leq M_A(\Delta'). \quad (14)$$

So this establishes what the preorder on $EPCM$ should be when restricting to a single observable. Now it is time to combine these results.

Suppose $P_{A_1} \leq P_{A_2}$ and let $f$ be as in (11). It then follows that $\Delta_1 \subset f(\Delta_2)$. Applying subsequently (14) and then (13) leads to the result:

$$M_{A_1}(\Delta_1) \leq M_{A_1}(f(\Delta_2)) \leq M_{A_2}(\Delta_2). \quad (15)$$

This lemma could be derived relying fully on LMR. However, LMR only works in one direction and so the preorder on $EPCM$ could only be investigated in one direction. To also obtain results in the other direction one has to consider the question...
if, when \( P_{A_1} \leq P_{A_2} \), a measurement of \( A_2 \) furnishes any information about \( A_1 \). In the standard approach to logics for classical mechanics the answer is definitely yes (e.g. (Isham 1995, §4.3)). The proposition \( M_{A_2}(\Delta_2) \) would then imply \( M_{A_1}(f^{-1}(\Delta_2)) \). This conclusion is based on the doctrine that a measurement of \( A_2 \) reveals information about the state \( \omega \in \Omega \) in which the system finds itself, and that in all the possible states in which \( M_{A_2}(\Delta_2) \) is true, the proposition \( M_{A_1}(f^{-1}(\Delta_2)) \) is also true. Here one interprets \( M_A(\Delta) \) as a proposition about a property of the system. However, this use of terminology is no longer acceptable if one abandons the realist interpretation of the state of a system. In fact, it may be clear by now that if one focuses on the concept of measurement, the proposition \( M_{A_1}(V_{A_1}) \) is a stronger one than \( M_{A_2}(V_{A_2}) \) if \( P_{A_1} \leq P_{A_2} \).

So in this light it seems natural to just define \( \leq \) by replacing ‘\( \implies \)’ by ‘\( \iff \)’ in (12). I would say this is almost correct. There are actually situations in which the order structure of \( L_{Obs} \) doesn’t matter, and that is when the elementary proposition itself is a contradiction: \( M_A(\emptyset) \). Remember that this is a consequence of the idealization that performed experiments have a result. With this idealization, the preorder on \( EM_{CM} \) becomes

\[
M_{A_1}(\Delta_1) \leq M_{A_2}(\Delta_2) \iff \\
\left( P_{A_1} \leq P_{A_2} \text{ and } f_{A_1}^{-1}(\Delta_1) \subset f_{A_2}^{-1}(\Delta_2) \right) \text{ or } \Delta_1 = \emptyset.
\]  

(16)

This preorder leads to the non-trivial equivalence relation

\[
M_{A_1}(\Delta_1) \sim M_{A_2}(\Delta_2) \iff \\
M_{A_1}(\Delta_1) \leq M_{A_2}(\Delta_2) \text{ and } M_{A_2}(\Delta_2) \leq M_{A_1}(\Delta_1),
\]

(17)

and the set of equivalence classes can now be characterized by the set

\[
S_{CM} := \{(P, U) : P \in L_{Obs}, \emptyset \subsetneq U \subset \Omega, P \leq \{U, U^c\}\} \cup \{\bot\},
\]

(18)

where \( \bot \) corresponds with the equivalence class \( \{M_A(\emptyset) : A \in \text{Obs}\} \). The inherited partial order on \( S_{CM} \) takes the form

\[
(P_1, U_1) \leq (P_2, U_2) \iff P_1 \leq P_2 \text{ and } U_1 \subset U_2,
\]

(19)

and of course \( \bot \leq (P, U) \) for all \((P, U)\). It should be noted that already \( S_{CM} \) has a much richer structure than the standard logic for classical mechanics; there is no way to associate every \((P, U)\) with a subset of the state space in a consistent way without ‘forgetting about \( P \)’.
4.2 A Heyting algebra for classical mechanics

It is now time to introduce logical connectives that can operate on the elementary propositions which will then result to the logic \( L_{CM} \) for classical mechanics. Consider two observables \( A_1, A_2 \in \mathcal{O}_{bs} \). Then for any pair of sets \( \Delta_1, \Delta_2 \) the proposition \( M_{A_1}(\Delta_1) \land M_{A_2}(\Delta_2) \) is read as "I measure \( A_1 \) and the result lies in \( \Delta_1 \) and I measure \( A_2 \) and the result lies in \( \Delta_2 \)". If the use of the second ‘and’ in this sentence is in any sense similar to the uses of the other ‘and’-s (a natural requirement), this may also be read as "I measure \( A_1 \) and \( A_2 \) and the result of the first lies in \( \Delta_1 \) and the result of the second in \( \Delta_2 \)". By assumption there is an observable \( A_3 \) whose measurement counts as a measurement of both \( A_1 \) and \( A_2 \) and that satisfies \( P_{A_3} = P_{A_1} \land P_{A_2} \). The above sentence is then equivalent to the sentence "I measure \( A_3 \) and \( f_1 \) applied to the result lies in \( \Delta_1 \) and \( f_2 \) applied to the result lies in \( \Delta_2 \)", where \( A_1 = f_1(A_3) \) and \( A_2 = f_2(A_3) \). But this is just the same as saying that "I measure \( A_3 \) and the result lies in \( f^{-1}_1(\Delta_1) \cap f^{-1}_2(\Delta_2) \)", which again corresponds to an elementary proposition. In conclusion, in \( L_{CM} \) it should at least hold that

\[
(P_1, U_1) \land (P_2, U_2) = \begin{cases} (P_1 \land P_2, U_1 \cap U_2), & U_1 \cap U_2 \neq \emptyset \\ \bot, & \text{else.} \end{cases} \tag{20}
\]

So conjunctions are pretty much what one would expect. Disjunctions on the other hand are more difficult because they cannot be imbedded within the partial ordered set of elementary propositions. At least, not without running into interpretational difficulties. There is of course the option to define disjunction as the least upper bound given the definition of conjunction:

\[
(P_{A_1}, U_1) \lor (P_{A_2}, U_2) := \bigwedge \left\{ (P_A, U) \in S_{CM} ; \begin{array}{c} (P_{A_1}, U_1) \leq (P_A, U), \\ (P_{A_2}, U_2) \leq (P_A, U) \end{array} \right\} \\
= (P_{A_1} \lor P_{A_2}, U_1 \cup U_2). \tag{21}
\]

But it seems inappropriate to identify a measurement of \( A_1 \) or \( A_2 \) with a measurement of neither. The present interpretation demands something stronger. On the other hand however it is also impossible to identify the disjunction with a joint measurement of the two observables. This is too strong a demand and in conflict with the partial order on \( S_{CM} \) (i.e. logically inconsistent). Rather, the aim is to broaden the lattice such that

\[
(P_{A_1} \land P_{A_2}, U_1 \cup U_2) < (P_{A_1}, U_1) \lor (P_{A_2}, U_2) < (P_{A_1} \lor P_{A_2}, U_1 \cup U_2). \tag{22}
\]

That is, the disjunction of two elementary propositions is simply no longer an elementary proposition. As far as I know, this is a novelty for classical mechanics.

The set of propositions increases immensely with this step, for it now also includes all propositions of the form

\[
\bigvee_{A \in O} (P_A, U_A), \quad O \subseteq \mathcal{O}_{bs}, (P_A, U_A) \in S_{CM}. \tag{23}
\]
Fortunately, conjunctions of such propositions can then again be defined by postulating that the lattice should be distributive. The equality
\[
\left( \bigvee_{A_1 \in O_1} (P_{A_1}, U_{A_1}) \right) \land \left( \bigvee_{A_2 \in O_2} (P_{A_2}, U_{A_2}) \right) = \bigvee_{A_1 \in O_1, A_2 \in O_2} \left( (P_{A_1}, U_{A_1}) \land (P_{A_2}, U_{A_2}) \right)
\]  
(24)

will then simply be taken as the definition of the left-hand side. The exact consequences of the step are captured in the following theorem:

**Theorem 2** The epistemic approach to classical mechanics leads to the logic

\[
L_{CM} := \left\{ S : \mathcal{L}_{Obs} \rightarrow \mathcal{F}(\Omega) ; \begin{array}{l}
(P, S(P)) \in S_{CM} \text{ or } S(P) = \emptyset \text{ and } \\
S(P_1) \subseteq S(P_2) \text{ whenever } P_1 \geq P_2
\end{array} \right\},
\]  
(25)

where \( \mathcal{F}(\Omega) \) denotes the set of all measurable subsets of \( \Omega \). The partial order is given by

\[
S_1 \leq S_2 \iff S_1(P) \subseteq S_2(P) \forall P \in \mathcal{L}_{Obs}.
\]  
(26)

\( L_{CM} \) is a complete Heyting algebra with disjunction, conjunction and implication given by

\[
\left( \bigvee_{i \in I} S_i \right)(P) = \bigcup_{i \in I} S_i(P),
\]  
(27a)

\[
\left( \bigwedge_{i \in I} S_i \right)(P) = \bigcap_{i \in I} S_i(P),
\]  
(27b)

\[
S_1 \rightarrow S_2 := \bigvee \{ S \in L_{CM} ; S \land S_1 \leq S_2 \}.
\]  
(27c)

The partial ordered set of (equivalence classes of) elementary propositions \( S_{CM} \) is embedded in this algebra through

\[
i : S_{CM} \rightarrow L_{CM}, \ i : (P, \emptyset) \mapsto S(P, \emptyset), \ S_{(P, U)}(P') := \begin{cases} U, & P' \leq P \\ \emptyset, & \text{else}. \end{cases}
\]  
(28)

**Proof** First, introduce the notation \((P, \emptyset)\) as just another way of writing \( \bot \). Then every proposition of the form (23) can be written as a disjunction over all observables simply by taking disjunctions with \((P_A, \emptyset)\) for all \( A \notin O \). This has the advantage that every proposition can be written as a function \( S : \mathcal{L}_{Obs} \rightarrow \mathcal{F}(\Omega) \) with the
restriction that \((P_A, S(P_A)) \in SCM\) for all \(A\). This function can then be identified with a proposition in the following way:\(^4\)

\[
S \simeq \bigvee_{P \in L_{O\beta}} (P, S(P)).
\]

Clearly the set

\[
F := \{S : L_{O\beta} \to \mathcal{P}(\Omega) ; (P, S(P)) \in SCM \text{ or } S(P) = \emptyset\}
\]

with its interpretation (29) is rich enough to incorporate all propositions, but it also contains a lot of logically equivalent propositions.

To show that \(LCM\) is the appropriate subset of \(F\), consider an elementary proposition \((P, U)\). This proposition can be identified with several functions in \(F\). Indeed, any \(S \in F\) will do as long as \(S(P) = U\) and \((P', S(P')) \leq (P, U)\) for all \(P' \in L_{O\beta}\). Note that this necessitates that \(S(P') = \emptyset\) whenever neither \(P \leq P'\) nor \(P' \leq P\). That is, as long as \(S\) represents the disjunction of \((P, U)\) with sentences that are all stronger than \((P, U)\), then the meaning of \(S\) is equivalent to \((P, U)\). A special procedure for constructing \(S\) is by setting for each \(P' \in O\beta\) \(S(P') = \emptyset\).

The disjunctions in the form of (23) can now officially be defined in terms of these functions:

\[
\left( \bigvee_{A \in O} S_{(P_A, U_A)} \right)(P) := \bigcup_{A \in O} (S_{(P_A, U_A)}(P)).
\]

The set of all functions of this form is precisely \(LCM\). Indeed, it is straightforward to check that the construction (31) always leads to an element of \(LCM\). Conversely, every element of \(LCM\) is of this form since every \(S \in LCM\) satisfies

\[
S = \bigvee_{P \in L_{O\beta}} S_{(P, S(P))}.
\]

That is, every element of \(LCM\) is associated with a disjunction of (equivalence classes of) elementary propositions. I leave it to the reader to prove that \(LCM\) is indeed a Heyting algebra.

In order for \(LCM\) to be a proper extension of \(SCM\) it remains to be shown that the embedding function \(i\) respects the partial order and preserves arbitrary meets. This is indeed the case:

\(^4\) The introduction of these functions is purely for mathematical convenience. One may also formally introduce the set of objects of the form of (23) and then introduce disjunctions that are consistent with this notation.
\[(P_1, U_1) \leq (P_2, U_2) \iff S_{(P_1, U_1)} \leq S_{(P_2, U_2)},\]
\[S_{(P_1, U_1) \land (P_2, U_2)} = S_{(P_1, U_1)} \land S_{(P_2, U_2)}.\]

\(\Box\)

It takes some time to appreciate the complexity of \(L_{CM}\). Obviously, assuming the standard ontology for classical mechanics is much more convenient in everyday life. But now imagine (if you can) a person unaware of any realist interpretation of classical mechanics. Then his/her logic for reasoning in classical mechanics is likely to resemble \(L_{CM}\). In fact, this is the situation we find ourselves in with respect to quantum mechanics, with the difference of not having a formal logic \(L_{QM}\). So what is again the advantage of such a logic? First of all, it makes clearer what is speakable in quantum mechanics independent of any realist interpretation. And secondly, realist interpretations can be compared with respect to their simplification of \(L_{QM}\). It makes clear the explaining role of the interpretation. For example, consider meeting the aforementioned person. At first you are confused about his/her form of reasoning. But then you recognize the form of \(L_{CM}\) and you see that the reasoning is correct, but just very cumbersome. Of course, you will propose your ontological view and explain that propositions like \(S_{(P_1, U)}\) and \(S_{(P_2, U)}\) actually are equivalent. And then this person can reflect about whether or not to accept this interpretation. Are the philosophical consequences of the interpretation satisfactory or not? I will return to this discussion in Sect. 6.

5 Quantum mechanics

5.1 Elementary propositions

For the definition of \(L_{QM}\) I will follow an approach similar to the one for classical mechanics. But of course, since it is a different theory, differences in the logic will emerge. In quantum mechanics a system is associated with a \(C^*\)-algebra \(\mathcal{C}\) and every observable \(A\) is associated with a self-adjoint operator \(\hat{A}\) within this algebra. For convenience I will assume conversely that every self-adjoint operator is associated with an observable. In Sect. 6 it will be shown that this assumption can be relaxed. Also, for mathematical convenience, I will only consider finite-dimensional \(C^*\)-algebras. More general cases can be studied, but the necessary mathematical care involved would soon blur the discussion. Furthermore, despite the fact that many quantum mechanical systems can only be described in the infinite dimensional case, the finite-dimensional situation is rich enough to incorporate those systems that are discussed most in the foundations: spin-systems.

The set of possible measurement outcomes for an observable \(A\) is given by the spectrum \(\sigma(\hat{A})\) of the operator \(\hat{A}\). The set of elementary propositions for quantum mechanics is thus given by

\[EP_{QM} := \left\{ M_A(\Delta) : \hat{A} = \hat{A}^* \land \Delta \subset \sigma(\hat{A}) \right\}. \]
As with the classical case, both elements of the pair that constitute an elementary proposition bring along a mathematical structure. This then imposes a preorder on $E_{QPM}$. The correct preorder depends on the interpretation of the elementary propositions.

The C*-algebraic approach to quantum mechanics is particularly convenient for applying LMR. For every observable $A$ there is a unique Abelian sub-algebra $\mathcal{A} \subset C$ in which all self-adjoint elements are functions of the operator associated with $A$, and every function of $\hat{A}$ is an element of $\mathcal{A}$. Conversely, every Abelian sub-algebra of $C$ is of this form. The set of all Abelian sub-algebras is denoted $\mathfrak{A}(C)$. Furthermore, quantum mechanics predicts that if $\hat{A}_2 = f(\hat{A}_1)$ for some function $^f$, then $A_2$ and $A_1$ can be measured simultaneously and the measurement outcomes will obey the same functional relationship. More specifically, it follows by LMR that if $A_2 \subset A_1$, then a measurement of $A_1$ is also a measurement of $A_2$.

Like the partitions of $\Omega$ in the classical case, the Abelian sub-algebras of $C$ form a partial ordered set by taking set inclusion as partial order. It should be noted that this poset is not a lattice. A meet is still defined by taking the intersection, but in general, for a pair of Abelian algebras $A_1$ and $A_2$, there is no Abelian algebra containing both. In fact, this is only the case if the algebras commute: $[A_1, A_2] = 0$. That is, every element of $A_1$ commutes with every element of $A_2$. It is important to note that the partial order is reversed in comparison with the classical case. That is, for two observables $A_1$ and $A_2$ with $A_2 = f(A_1)$ one has $P_{A_1} \leq P_{A_2}$ in the classical case, and $A_2 \subset A_1$ in the quantum case.

In quantum mechanics the set of possible measurement outcomes $V_A$ coincides with the spectrum of the operator associated with $A$. Furthermore, every subset $\Delta$ of $V_A$ can be associated with a projection operator $\mu_{\hat{A}}(\Delta) \in \mathcal{A}$, where $\mu_{\hat{A}}$ is the spectral measure for $\hat{A}$. All projection operators in $\mathcal{A}$ are of this form, and I will denote the set of all projection operators in $\mathcal{A}$ by $\mathcal{P}(\mathcal{A})$. The analogy with the classical case thus far is summarized as follows:

$$f_A : \Omega \to V_A \leftrightarrow \hat{A} \in C, \quad \hat{A} = \hat{A}^*; \quad (35)$$

$$P_A \in L_{\text{obs}} \leftrightarrow A \in \mathfrak{A}(C); \quad (36)$$

$$\{ U \subset \Omega \; ; \; P_A \leq \{ U, U^c \} \} \leftrightarrow \mathcal{P}(\mathcal{A}). \quad (37)$$

In orthodox quantum logic, the partial order structure on the set of observables is completely ignored. There, an elementary proposition $M_{A_1}(\Delta_1)$ is simply identified with the projection operator $\mu_{\hat{A}_1}(\Delta_1)$. The underlying thought is most likely that propositions concern statements about the actual state of the system. In the Hilbert space formalism, the proposition $M_{A_1}(\Delta_1)$ is then identified with the statement that the state lies in the linear space spanned by $\mu_{\hat{A}_1}(\Delta_1)$. But although that interpretation may be consistent for classical mechanics, in quantum mechanics it is controversial. In fact, there aren’t many people who think this line of ontological reasoning leads to satisfactory results. That is the tale of quantum logic (Maudlin 2005) (see also Stairs (1983) for a critical exposition of the compatibility of quantum logic with realism).

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5 Formally, the function of an operator isn’t well-defined for arbitrary $f$, but it is for a huge class of functions such as Borel functions. This is all well-explained in von Neumann (1955).
The solution is to take the notion of measurement (and thus the structure of the set $\mathcal{A}(C)$) seriously, leading to the definition

$$M_{A_1}(\Delta_1) \sim M_{A_2}(\Delta_2) \iff A_1 = A_2 \quad \text{and} \quad \hat{\mu}_{A_1}(\Delta_1) = \hat{\mu}_{A_2}(\Delta_2), \quad (38)$$

where it is assumed that $\Delta_1$ and $\Delta_2$ are not empty. Propositions of the form $M_A(\emptyset)$ are again identified with contradiction. The set of (equivalence classes of) elementary propositions is characterized by the set

$$S_{QM} := \{(A, P) : A \in \mathcal{A}(C), P \in \mathcal{P}(A), P \neq 0\} \cup \{\perp\}, \quad (39)$$

with injection $M_A(\Delta) \mapsto (A, \hat{\mu}_{A}(\Delta))$. The partial order is defined analogously to the classical case:

$$(A_1, P_1) \leq (A_2, P_2) \iff (A_2 \subset A_1 \text{ and } P_1 \leq P_2) \quad \text{or} \quad P_1 = 0, \quad (40)$$

where all elements of the form $(A, 0)$ are considered equivalent and equal to $\perp$.

5.2 A Heyting algebra for quantum mechanics

Like in the classical case, it should be investigated how disjunctions and conjunctions of elementary propositions behave. But whereas in the classical case it was unproblematic to understand $M_{A_1}(\Delta_1) \land M_{A_2}(\Delta_2)$ as a proposition about a joint measurement of $A_1$ and $A_2$, in the quantum case joint measurements play a special role. When the corresponding operators $\hat{A}_1$ and $\hat{A}_2$ commute joint measurements of $A_1$ and $A_2$ are considered unproblematic. But when the corresponding operators do not commute uncertainty relations arise. The consensus is that in such a case a limitation arises to the precision with which both observables can be measured. Explicating how this works requires going beyond the orthodox quantum theory, and that lies beyond the scope of this paper. In any case it seems one cannot speak of the idealized joint measurement of two observables within orthodox quantum mechanics. The solution I propose is quite simple; if $[\hat{A}_1, \hat{A}_2] = 0$, then the conjunction can be understood in the classical way, if $[\hat{A}_1, \hat{A}_2] \neq 0$ the conjunction expresses a contradiction.\footnote{Admittedly, this assumption is quite crude. However, in my opinion it is a leap forward from orthodox quantum logic which doesn’t take into account the notion of incompatible observables at all. Indeed, it may well be the case that $\mu_{\hat{A}_1}(\Delta_1) = \mu_{\hat{A}_2}(\Delta_2)$ even when $A_1$ and $A_2$ do not commute. It seems hardly appropriate that two propositions can be considered equal even if they involve mutually exclusive measurements.} In other words:

$$(A_1, P_1) \land (A_2, P_2) = \begin{cases} (A_1 \lor A_2, P_1 \land P_2), & [A_1, A_2] = 0 \\ \perp, & \text{else,} \end{cases} \quad (41)$$

where $A_1 \lor A_2$ is the smallest Abelian sub-algebra that has both $A_1$ and $A_2$ as a subset.
For disjunctions one faces the same difficulties as for the classical case, and the set of propositions has to be expanded. The approach is entirely similar to the classical case, and so is the result:

**Theorem 3** The epistemic approach to quantum mechanics leads to the logic

\[ L_{QM} := \left\{ S: \mathfrak{A}(C) \to \mathcal{P}(C); \begin{array}{lcl} S(A) & \in & \mathcal{P}(A) \text{ and} \\ S(A_1) & \leq & S(A_2) \text{ whenever } A_1 \subset A_2 \end{array} \right\} \]  

with partial order given by

\[ S_1 \leq S_2 \iff S_1(A) \leq S_2(A) \quad \forall A \in \mathfrak{A}(C), \]  

where the partial order on the right-hand side is the standard partial order for projection operators. \( L_{QM} \) is a complete Heyting algebra with disjunction, conjunction and implication given by

\[ \bigvee_{i \in I} S_i(A) = \bigvee_{i \in I} S_i(A), \]  

\[ \bigwedge_{i \in I} S_i(A) = \bigwedge_{i \in I} S_i(A), \]  

\[ S_1 \to S_2 = \bigvee \left\{ S \in L_{QM}; S \wedge S_1 \leq S_2 \right\}. \]

The lattice operations on the right-hand side are the standard ones for projection operators. The partial ordered set of (equivalence classes of) elementary propositions \( S_{CM} \) is embedded in this algebra through

\[ i: S_{QM} \to L_{QM}, \quad i: (A, P) \mapsto S_{(A, P)}, \quad S_{(A, P)}(A') := \begin{cases} P, & A \subset A' \\ 0, & \text{else.} \end{cases} \]  

**Proof** Because of the partial order structure on \( S_{QM} \), a proposition \((A, P)\) should be equivalent to the disjunction of all propositions \((A', P')\) with \(A \subset A'\) and \(P' \leq P\). For all other \(A'\) one can just take the disjunction with \((A', 0)\). In other words, if one takes for each \(A' \in \mathfrak{A}(C)\) the largest \(P'\) such that \((A', P') \leq (A, P)\) one finds the requirement that

\[ (A, P) = \bigvee_{A' \in \mathfrak{A}(C)} (A', S_{(A, P)}(A')). \]
with $S_{(A, p)}$ defined in (45). Secondly, the disjunction should satisfy the relation

$$(A_1, P_1) \lor (A_2, P_2) = \left( \bigvee_{A' \in \mathfrak{A}(C)} (A', S_{(A_1, P_1)}(A')) \right) \lor \left( \bigvee_{A'' \in \mathfrak{A}(C)} (A'', S_{(A_2, P_2)}(A')) \right).$$

But for fixed $A'$ the disjunction has a clear meaning for in natural language the sentence “I measure $A$ and the result lies in $\Delta_1$, or I measure $A$ and the result lies in $\Delta_2$” easily translates to “I measure $A$ and the result lies in $\Delta_1$ or $\Delta_2$”. In other words, for each $A'$

$$((A', S_{(A_1, P_1)}(A')) \lor (A', S_{(A_2, P_2)}(A'))) = (A', S_{(A_1, P_1)}(A') \lor S_{(A_2, P_2)}(A')).$$

The disjunction of two elementary propositions can thus again be identified with a function in $L_{QM}$. Taking arbitrary disjunctions in this way one obtains the entire set $L_{QM}$ and, these disjunctions coincide with the join on $L_{QM}$. The meet on $L_{QM}$ is also consistent with the interpretation of conjunction since

$$S_{(A_1, P_1) \land (A_2, P_2)} = S_{(A_1, P_1)} \land S_{(A_2, P_2)}.$$  

Again, I leave the proof that $L_{QM}$ is a Heyting algebra to the reader.

6 Discussion

The Heyting algebra $L_{QM}$ is not a new logic for quantum mechanics but has actually been proposed earlier in Caspers et al. (2009) and studied further in Heunen et al. (2011). Although it was studied there to some extent from a mathematical point of view, no philosophical derivation was given as to why this should be the correct logic for describing quantum systems. The only attempt I found at interpreting $L_{QM}$ was the line

Each element of $L_{QM}$ corresponds to a “Bohrified” proposition, in the sense that to each classical context $[A \in \mathfrak{A}(C)]$ it associates a yes-no question (i.e. an element of the Boolean lattice $[P(A)]$ of projections in $[A]$), rather than being a single projection as in standard quantum logic (Caspers et al. 2009, p. 732)

In the present article, a more specific interpretation has been given, and I have shown that this interpretation is not only consistent with this logic, but I have also derived this logic from the interpretation. In other words, whereas Heunen et al. (2011) found a
mathematical (topos-theoretical) motivation for the use of $L_{QM}$ in quantum mechanics, this article provides a more direct philosophical motivation. A “Bohrified” proposition may now be understood as a proposition written as a disjunction of elementary propositions. However, the elementary propositions as I introduced them (namely, functions of the form $S_{(A,P)}$) play no role in the article of Caspers et al. It is an open question if the interpretation given here is the only one consistent with $L_{QM}$.

The classical case results as a special case of the quantum case. To see this, consider the set $C_0(\Omega_1)$ of all complex valued functions on $\Omega$ that vanish at infinity. This set is an (Abelian) C*-algebra and so $L_{QM}$ can be constructed for it. As it turns out, the lattice thus obtained is precisely the lattice $L_{CM}$. The details of this analogy are left to the reader.\(^7\)

When comparing the lattices $L_{CM}$ and $L_{QM}$, it is rather surprising that, despite their similarity, the first allows a simple modification if the assumption of realism is made. However, for the second it is unclear if this can be done (and many consider it to be impossible). At least it is known that the method for classical mechanics cannot be used for quantum mechanics. This is a consequence of the Kochen–Specker theorem, which implies that no state space can be defined in which every state dictates the values for all observables in a way consistent with the laws (i.e. algebraic relations) of the theory. Realist interpretations of quantum mechanics can avoid this result by re-defining the notion of all observables. The most well-known way to do this is to expand the set of observables by introducing a notion of contextuality. Another method is Bell’s approach to Bohmian mechanics (Bohm 1952; Bell 1982), which circumvents the problem by assuming that every proposition $M_A(\Delta)$ is in fact of the form $M_X(\Delta')$ where $X$ is a special observable denoting the position of some particle (for example, the pointer on a measuring apparatus). Alternatively, one may restrict oneself to a particular subset of all self-adjoint operators which is done explicitly in so-called MKC-models for quantum mechanics (Meyer 1999; Kent 1999; Clifton and Kent 2001; Hermens 2011). Another unorthodox approach is the qr-numbers approach (Corbett and Durt 2009) according to which the proposition $M_A(\Delta)$ contains an idealization of the concept of measurement, and the ontology is better described by assigning definite values to observables in the form of qr-numbers rather than ordinary real numbers. In fact, any advocate of a specific realist interpretation of quantum mechanics will argue that $L_{QM}$ is a very cumbersome logic for reasoning, but many will differ in pointing out which aspect is precisely cumbersome.

In connection with the above discussion it should be noted that the assumption to associate the set of all (equivalence classes of) observables with the set of all Abelian subalgebras can be relaxed. Instead one may take a subset $\mathfrak{A}_{O6}(C) \subset \mathfrak{A}(C)$ and use

$$L'_{QM} := \left\{ S : \mathfrak{A}_{O6}(C) \rightarrow \mathcal{P}(C) : S(A) \in \mathcal{P}(A) \text{ and } S(A_1) \leq S(A_2) \text{ whenever } A_1 \subset A_2 \right\}. \quad (50)$$

A prerequisite in this case is that (41) is still defined, i.e. for every pair of commuting algebras in $\mathfrak{A}_{O6}(C)$ there is a smallest element of $\mathfrak{A}_{O6}(C)$ of which they are both a

\(^7\) It may be useful to note that $C_0(\Omega)$ need not be a finite-dimensional C*-algebra. In that case the classical case presents itself as a generalization of the quantum case. This strengthens the idea that $L_{QM}$ and its interpretation can be generalized to encompass more general infinite-dimensional cases.
subalgebra. So interpretations that attack assumptions on the set of observables can actually be considered within this framework.

But how cumbersome is the logic in practice? The lattice $L_{QM}$ is claimed to provide a consistent way to reason about quantum mechanical propositions. In most practical cases however, one only considers a finite set of possible measurements and it then seems appropriate to use propositions that essentially talk only about these measurements. For example, one may ask the question “what is the appropriate logic if I know I will measure $A$?” The natural approach is to say that in the case of an actual measurement some of the propositions in $L_{QM}$ may be considered equivalent:

$$S_1 \sim_A S_2 \iff S_1(A) = S_2(A).$$

(51)

It is like taking the conjunction of every proposition with “I measure $A$”; $S_i$ may be a disjunction over many possible measurements, but since I know I will measure $A$, only that part is of interest to me. It turns out that the set $L_{QM}/\sim_A$ is the Boolean lattice of projection operators in $A$. In other words: given the measurement of $A$, every proposition about outcomes of the measurement of $A$ becomes decidable (and the measurement outcome will make the decision so to speak). So in practice, the intuitionistic logic $L_{QM}$ behaves classically. Note that this result is similar to the case of orthodox quantum logic.

In general, a measurement of an observable $A$ may not be a complete measurement, as there may be observables of which $A$ is a function. So actually a measurement of $A$ only gives certainty about the observables $A'$ with $A' \subseteq A$, but not about the observables $A''$ with $A \subseteq A''$. From this perspective, one should introduce the more subtle equivalence relation

$$S_1 \sim'_A S_2 \iff S_1(A') = S_2(A') \ \forall A' \supset A.$$  

(52)

If $A$ is a maximal observable, this relation coincides with the equivalence relation $\sim_A$, but in general it gives rise to a more refined conditional logic:

$$L_{QM}/\sim'_A = \left\{ S : \uparrow A \rightarrow \mathcal{P}(C) : \begin{array}{c}
S(A') \in \mathcal{P}(A') \\
S(A_1) \leq S(A_2) \text{ whenever } A_1 \subseteq A_2
\end{array}\right\},$$

(53)

where $\uparrow A = \{A' \in \mathfrak{A}(C) : A' \supset A\}$. This is again a Heyting algebra (with partial order, join and meet defined analogously to the standard case). The decidable elements in this algebra (i.e. the propositions $S$ for which $S \vee \neg S = \top$) are given by the equivalence classes $[S(A,P)]$ with $P \in \mathcal{P}(A)$ i.e. those that correspond with the elementary propositions about measurements of $A$. Thus the logic $L_{QM}/\sim'_A$ is the logic in which propositions concerning measurements of $A$ are decidable, but other propositions aren’t. So the context-dependency of the decidability of propositions as discussed on page 3268 has been captured by the logic $L_{QM}$. Consequently, Feynman’s ‘logical tightrope’ has become much less of a hand-waving argument.

The equivalence relation $\sim'_A$ was defined by appealing to an uncertainty about what a full measurement of the system would be, but it can also be seen as appealing to a certain notion of locality. Indeed, given a measurement of $A$, one remains uncertain.
about any other measurement made on the system possibly at some distance from where the measurement of $A$ was performed. Symmetrically, the person performing the distant measurement remains uncertain about the measurement of $A$ (and about whether or not it is performed). As a consequence, people who are studying the same system at distant locations will in general use different logics describing the system.\(^8\)

As an example, consider the standard EPR-Bohm situation where Alice can choose between two possible measurements $A_1$ and $A_2$, whilst Bob can choose between two possible measurements $B_1$ and $B_2$. Suppose both know from each other that these are the only possible measurements to choose from. Let $A_iB_j$ denote the algebra generated by $A_i$ and $B_j$. If Alice chooses to measure $A_i$ the appropriate logic for her will be

$$L_{A_i} = \left\{ S : \{A_i, A_iB_1, A_iB_2\} \rightarrow \mathcal{P}(C) ; S(A)_{i} \in \mathcal{P}(A) \text{ for } A \in \{A_i, A_iB_1, A_iB_2\} \right\}.$$

Symmetrically, Bob will use the logic

$$L_{B_i} = \left\{ S : \{B_i, A_1B_i, A_2B_i\} \rightarrow \mathcal{P}(C) ; S(B)_{i} \in \mathcal{P}(B) \text{ for } B \in \{B_i, A_1B_i, A_2B_i\} \right\}.$$

To properly discuss the connection of these logics with Bell-type inequalities, it is necessary to introduce a notion of probability. It is beyond the scope of this article to investigate what all consistent possibilities would be for the new quantum logic. However, I do want to sketch some ideas. It seems natural that for Alice, using the logic $L_{A_i}$, the probabilities provided by the formalism of quantum mechanics that are relevant to her are those that are assigned to the decidable propositions in her logic, i.e. those about the possible outcomes for the measurement of $A_i$. That is, she will assign probabilities to those propositions that are decidable for her. But for other propositions, the situation is not clear-cut. The proof of Theorem 1 makes clear what is at stake. In order for the proof to make sense, one must be able to assign a probability to the proposition $A_1 \land B_1 \land (B_2 \lor \neg B_2)$. Although one can identify the proposition with an element in $L_{A_1}$, the probability Alice should assign to it is unknown and may well depend on an underlying ontological model. However, the structure of $L_{A_1}$ does make clear that it may well be a lower value than that assigned to $A_1 \land B_1$. Note that this is not necessarily an artifact of $L_{QM}$ being intuitionistic, but rather is due the fact that $\neg B_2$ is not represented in $L_{QM}$ as the negation of $B_2$. That is, in general one has

$$S_{(A, P)} \leq \neg S_{(A, P)},$$

and no equality. Either way, this is the domain where one is to seek a ‘peaceful coexistence’ between quantum mechanics and some form of locality.

A more recent attempt at finding such a coexistence is delivered by the consistent histories approach (Griffiths 2011). In this approach one only looks at partial logics constructed with equivalence relations of the form (51). Such a partial logic is

\(^8\) Obviously, this discussion relies on some notions not explicitly specified within the framework of $L_{QM}$ such as the splitting of a system in localized subsystems or the notion of localization for that matter. I trust that it causes no problems for the present discussion to consider these notions as primitive.
called a framework, and one postulates that all reasoning must be performed within one single framework. It follows roughly from this postulate that no Bell inequality can be derived, since every such inequality involves more than one framework. Now according to Griffiths, the choice of a framework is a pure epistemological act; it does not influence the system under consideration. Therefore, the propositions within such a framework are purely epistemic too, for if they were ontological, a change of framework would influence the system. For example, a sentence about a property of the system could shift from being true to being meaningless. It then seems to me that the consistent histories approach is not capable of properly describing an ontology for quantum systems, which makes the entire discussion of locality (which is, to be sure, an ontological concept) quite meaningless. Indeed, the logic suggested by Griffiths is quite reminiscent of what I have proposed in this paper, with the difference that $L_{QM}$ acts on a framework-transcending level; each sentence in $L_{QM}$ is a disjunction of sentences, each formulated within a single framework.

In conclusion, I don’t believe that the problems in the foundations of quantum mechanics could vanish by introducing the ‘correct’ logic. Neither do I believe that $L_{QM}$ should be conceived as the correct logic. However, I do think that logic can play an important role in carefully investigating the philosophical problems we face in quantum mechanics. A careful distinction between epistemic and ontological assumptions is mandatory for this, and I suggest that the logic $L_{QM}$ may help in making this distinction more clear.

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