Classification of mixed high-dimensional multipartite systems

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We present an inequality that classifies mixed multipartite systems of an arbitrary dimension with respect to separability and positivity of partial transpose properties. This inequality gives a way to experimentally classify the observed state of multipartite systems of an arbitrary dimension. The inequality also implies that a sufficient condition for a density operator to have no positive partial transpose with respect to any subsystem is that the fidelity to a generalized Greenberger-Horne-Zeilinger state [A. Cabello, Phys. Rev. A 63, 022104 (2001)] is larger than 1/2 for mixed multipartite systems of an arbitrary dimension.

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I. INTRODUCTION

In the quantum information processing, entanglement is a very important ingredient. Moreover, understanding about the nature of entanglement we can approach the core of a quantum world. Werner proposed the class of separable (classically correlated) states and showed that there exists some inseparable state which can admit local hidden variable model for two-particle systems [1]. Peres presented a necessary condition for separability, namely, the partial transpose of a density operator is a positive operator [2]. The partial transpose of an operator on Hilbert space \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) is defined by

\[
\left( \sum_i A_i^1 \otimes A_i^2 \right)_{\text{T}_1} = \sum_i A_i^1 T \otimes A_i^2, \tag{1}
\]

where the superscript \( T \) denotes transposition in the given basis. Horodecki et al. showed that the condition is also sufficient as to composite quantum systems on \( \mathcal{H} = C^2 \otimes C^2 \) and \( \mathcal{H} = C^2 \otimes C^3 \). Dür et al. formulated criteria that determine to which class, with respect to separability and distillability properties, a density operator of multiqubit systems belongs [4, 5]. For mixed high-dimensional multipartite systems, the criteria that determine whether a density operator is separable or not with respect to a subsystem become much more complicated (see, for example, Refs. [4, 5]).

II. INEQUALITY FOR CLASSIFYING MULTIPARTICLE SYSTEMS

Consider a partition of \( n \)-particle system \( \mathbf{N}_n = \{1, 2, \ldots, n\} \) into \( k \) nonempty and disjoint subsets \( \alpha_1, \ldots, \alpha_k \), where \( \sum_{i=1}^k |\alpha_i| = n \), to which we refer as a \( k \)-partite split of the system [6]. Let us now consider the density operators \( W \) on \( \mathcal{H} = \otimes_{i=1}^n \mathcal{H}_i \), where \( \mathcal{H}_j \) represents the Hilbert space with respect to particle \( j \).

A density operator \( W \) may be called \( k \)-separable with respect to a partition \( \alpha_1, \ldots, \alpha_k \) iff it can be written as

\[
W = \sum_l p_l \left( \otimes_{i=1}^k W_{l_i}^{\alpha_i} \right), \quad \left( p_l \geq 0, \sum_l p_l = 1 \right), \tag{2}
\]

where \( W_{l_i}^{\alpha_i}, \forall l \) are the density operators on the partial Hilbert space \( \otimes_{j \in \alpha_i} \mathcal{H}_j \).

Considering any subset \( \alpha \subset \mathbf{N}_n \) and the positive operators \( X \) acting on \( \mathcal{H} \), let \( X^{T_\alpha} \) denote the partial transpose of all sites belonging to \( \alpha \). Let \( \mathcal{P} \) denote a family of sets, which consists of all unions of \( \alpha_1, \ldots, \alpha_k \) together with the empty set, so that \( \mathcal{P} \) has \( 2^k \) elements. A positive operator \( X \) may be called \( k \)-positive partial transpose (\( k \)-PPT) with respect to this specific partition iff \( X^{T_\alpha} \geq 0 \) for all \( \alpha \in \mathcal{P} \).

Clearly if a density operator \( W \) is not \( k \)-PPT with respect to a specific partition, the state \( W \) should not be \( k \)-separable with respect to the specific partition.

For each particle \( j \), we assume there are \( d_j \) orthonormal states as a certain basis set each, which are denoted by \( \{|b_j^1\}, |b_j^2\}, \ldots, |b_{d_j}^j\rangle \), \( j \in \mathbf{N}_n \), when we represent \( W \) in the matrix form. We choose two arbitrary orthonormal
states, which are included in each basis of all particles. We denote the two states as \{0\}, \{1\}. That is to say, 
\[ |0\rangle, |1\rangle \in \{|b_1\rangle, |b_2\rangle, \ldots, |b_{d_{\nu}}\rangle\}, \forall j \in N_n. \]  
(3)

Let us consider a subspace \(\mathcal{H}^2\), so that \(\mathcal{H}^2\) is a subspace supported by \(\{0\}, \{1\}\)^{\otimes n}, and hence \(\text{Dim}(\mathcal{H}^2) = 2^n\), where \(\text{Dim}(\mathcal{H}) = d = \prod_{j=1}^{n} d_j\).

From the argument discussed by Dür et al.\[14,15\], we can see that applying local operations, an arbitrary positive operator of \(n\) spin-1/2 systems can be transformed into one of the members of a family of positive operators as 
\[
\rho_n = \sum_{\sigma=\pm} \lambda_0^\sigma \langle \Psi_0^\sigma | \Psi_0^\sigma \rangle + \sum_{j=1}^{2^{(n-1)}-1} \lambda_j (|\Psi_j^+\rangle\langle \Psi_j^+| + |\Psi_j^-\rangle\langle \Psi_j^-|),
\]
(4)
where \(|\Psi_j^\pm\rangle\rangle represent the orthonormal GHZ basis \[11\] by 
\[
|\Psi_j^\pm\rangle = \frac{1}{\sqrt{2}}(|j\rangle|0\rangle \pm |2^{n-1} - j - 1\rangle|1\rangle),
\]
(5)
where \(j = j_1 j_2 \ldots j_{n-1}\) is understood in binary notation.

If the suitable local unitary transformations proposed by Dür et al.\[14,15\], by which are denoted \(\{U_\zeta\}\), are applied on a density operator \(W\), i.e., 
\[
W \rightarrow \sum_{\zeta} p_\zeta(U_\zeta \otimes I)W(U_\zeta^\dagger \otimes I),
\]
(6)
where \(I\) represents the identity operator for the \((d-2^n)\)-dimensional space, then a suboperator of the density operator \(W\) on the subspace \(\mathcal{H}^2\) can be transformed into the positive operator \(\rho_n\), where the normalization condition \(\text{Tr}[W] = 1\) leads to \(\text{Tr}[\mathcal{H}^2]p_n \leq 1\).

The values of the positive coefficients of \(\rho_n\), i.e., all \(\lambda\) are kept unchanged during the transformation procedure. Therefore, one finds 
\[
\lambda_0^\pm = \text{Tr}[W(|\Psi_0^\pm\rangle\langle \Psi_0^\pm| \otimes 0)],
\]
\[
2\lambda_j = \text{Tr}[W\{(|\Psi_j^+\rangle\langle \Psi_j^+| + |\Psi_j^-\rangle\langle \Psi_j^-|) \otimes 0\}],
\]
(7)
where \(0\) represents the null operator for the \((d-2^n)\)-dimensional space. Let \(\Delta\) be \(\lambda_0^+ - \lambda_0^-\) and suppose that \(\Delta = \lambda_0^+ - \lambda_0^- \geq 0\).

Let \(\beta\) be a subset \(\beta \subset N_n\) and \(l(\beta)\) be the integer \(l_1 \cdots l_n\) in binary notation with \(l_m = 1\) for \(m \in \beta\) and \(l_m = 0\) otherwise, and let \(j(\beta)\) be the integer binary-represented by \(l_1 \cdots l_{n-1}\). We define a function 
\[
g : \beta \mapsto g(\beta) \in \{0\} \cup N_{2^{(n-1)}-1},
\]
(8)
as follows: 
\[
g(\beta) = \begin{cases} 
  j(\beta), & l(\beta) \equiv 0 \pmod{2}, \\
  2^{n-1} - j(\beta) - 1, & l(\beta) \equiv 1 \pmod{2}.
\end{cases}
\]
(9)
The function \(g\) is two-to-one.

From the result obtained by Dür et al.\[14,15\], it is easy to see that 
\[
\rho_n^T \geq 0 \text{ iff } \Delta \leq 2\lambda_g(\beta).
\]
(10)
[We cannot change the relation (10)] even if the normalization condition with respect to the positive operator \(\rho_n\) is modified as we have mentioned above.] Hence, a positive operator \(\rho_n\) is \(k\)-PPT with respect to a specific partition if \(\Delta \leq 2\lambda_g(\alpha)\) for all \(\alpha \in \mathcal{P}\). Even though \(\mathcal{P}\) has \(k\) elements, we can see that the number of the really necessary elements of \(\mathcal{P}\) is \(2^{k-1} - 1\) because it does not matter whether we transpose \(\alpha\) or its complement and \(\mathcal{P}\) contains the empty set. We now define a set of integers \(\tau\) by means of the two-to-one function \(g\) as follows. Let \(\tau'\) be a set of integers \(\{g(\alpha) | \alpha \in \mathcal{P}\}\), i.e., the image of the family of sets \(\mathcal{P}\), and define \(\tau\) by a difference set \(\tau'\setminus\{0\}\) so that \(|\tau| = 2^{(k-1)} - 1\). At this stage, we can show the following.

**Proposition.** Let \(W\) be the density operators on \(\mathcal{H} = \bigotimes_{j=1}^{n} \mathcal{H}_j\) and let \(\mathcal{Y}(k)\) be an operator \(|\Psi_0^+\rangle\langle \Psi_0^+| - (1 - 2^{-k})|\Psi_0^+\rangle\langle \Psi_0^-| \otimes 0\). Under the conditions that 
\[
\text{Tr}[W\{(|\Psi_0^+\rangle\langle \Psi_0^+| - |\Psi_0^-\rangle\langle \Psi_0^-|) \otimes 0\}] \geq 0
\]
and that \(W\) is \(k\)-PPT with respect to a specific partition, the maximum value of \(\text{Tr}[W\mathcal{Y}(k)]\) over \(W\) is \(2^{1-k}\).

**Proof.** Let \(\rho\) be the suboperator of \(W\) on the subspace \(\mathcal{H}^2\). If the density operator \(W\) is positive, then any suboperator on the subspace, which is supported by a subbasis set that supports the original Hilbert space \(\mathcal{H}\), is positive \[12\]. Therefore, we find \(W^T \geq 0 \Rightarrow \rho^T \geq 0\). Since the transformation procedure is performed by local operations, we cannot change the positivity of partial transpose, i.e., \(\rho^T \geq 0 \Rightarrow \rho_n^T \geq 0\). Hence if \(W\) is \(k\)-PPT, then \(\rho_n\) should be \(k\)-PPT. So the following holds:

\[
\Delta \leq 2\lambda_1, \forall i \in \tau,
\]
(11)
where \(\tau\) is the set of integers \(\{i_1, i_2, \ldots, i_{2^{(k-1)}-1}\}\), which has been defined as mentioned above. The relation (11) implies 
\[
\Delta \leq 2 \min\{\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_{2^{(k-1)}-1}}\}.
\]
(12)
On the other hand, according to the following relations 
\[
\text{Tr}[\mathcal{H}^2(\rho_n)] \leq 1 \Leftrightarrow 2 \sum_{j=1}^{2^{(n-1)}-1} \lambda_j + \lambda_0^+ + \lambda_0^- \leq 1,
\]
(13)
we have
\[
\sum_{j \in \tau} 2 \lambda_j \leq 1 - \lambda_0^+ - \lambda_0^-.
\]
(14)
This leads to the following relation:
\[
2 \min\{\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_{2^{(k-1)}-1}}\} \leq \frac{1 - \lambda_0^+ - \lambda_0^-}{2^{(k-1)} - 1}.
\]
(15)
Hence we get
\[ \Delta \leq 1 - \frac{\lambda_+^k - \lambda_0^-}{2^{(k-1)}} \iff \Delta \leq 2^{1-k} - 2^{1-k} \lambda_0^- , \] (16)
and, from Eq. 7,
\[ \text{Tr}[W \hat{\Upsilon}(k)] \leq 2^{1-k}. \] (17)
The equality of the relation (17) holds when \( \text{Tr}_{H^2}[\rho_n] = 1 \)
and \( \lambda_i = \frac{1-\lambda_+^k - \lambda_0^-}{2^{(k-1)}} \forall i \in \tau \). Q.E.D.

Dür et al. showed that \( \rho_n \) is \( k \) separable with respect to a partition \( \alpha_1, \ldots, \alpha_k \) iff \( \rho_n^{\alpha_i} \geq 0 \) for all \( \alpha \in \mathcal{P} \) in the case that \( \text{Tr}_{H^2}[\rho_n] = 1 \). Hence it is easy to see that the inequality (17) and the equality of the relation (17) can hold when we assume that \( W \) is \( k \) separable with respect to the specific partition.

In the case that \( \Delta = \lambda_+^k - \lambda_0^- < 0 \), we exchange \( \lambda_+^k \) and \( \lambda_0^- \). The above arguments then go in the same way. We have chosen the two orthonormal states as \( \{|0\rangle, |1\rangle\} \).

In fact, how to represent relation (17) can be represented as
\[ \text{Tr}[W \hat{\Upsilon}(k)] \leq 2^{1-k}, \forall\{|0\rangle, |1\rangle\}. \] (18)

If the following is held:
\[ \text{Tr}[W \hat{\Upsilon}(k)] > 2^{1-k}, \exists\{|0\rangle, |1\rangle\}, \] (19)
we confirm that a given state \( W \) cannot belong to \( k' \)-PPT state class, and \( W \) cannot belong to \( k' \) separable state class (\( k' \geq k \)).

For \( k = 2 \), the operator \( \hat{\Upsilon}(k) \) becomes the projection operator \( |\Psi_0^\perp\rangle\langle \Psi_0^\perp| \oplus 0 \), and hence the relation (19) is written by
\[ \text{Tr}[W(|\Psi_0^\perp\rangle\langle \Psi_0^\perp| \oplus 0)] > 1/2, \exists\{|0\rangle, |1\rangle\}. \] (20)

Therefore, for mixed multipartite systems of an arbitrary dimension we obtain a sufficient condition for a density operator to have no positive partial transpose with respect to any subsystem, which is that the fidelity to the quantum state belonging to the generalized GHZ state class that was discussed by Cabello is larger than 1/2.

### III. CONCLUSION

In conclusion, we have presented an inequality that classifies mixed multipartite systems of an arbitrary dimension with respect to separability and positivity of partial transpose. The inequality enables experimentally feasible classification of the observed state of mixed multipartite systems of an arbitrary dimension. The inequality also implies that a sufficient condition for a density operator to have no positive partial transpose with respect to any subsystem is that the fidelity to a generalized GHZ state is larger than 1/2 for mixed multipartite systems of an arbitrary dimension.

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