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Sufficient Conditions for Labelled 0–1 Laws

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If \( F(x) = e^{G(x)} \), where \( F(x) = \sum f(n)x^n \) and \( G(x) = \sum g(n)x^n \), with \( 0 \leq g(n) = O\left(n^\theta n!/n!\right) \), \( \theta \in (0, 1) \), and \( \gcd (n : g(n) > 0) = 1 \), then \( f(n) = O(f(n-1)) \). This gives an answer to Compton’s request in Question 8.3 [Compton 1987] for an “easily verifiable sufficient condition” to show that an adequate class of structures has a labelled first-order 0–1 law, namely it suffices to show that the labelled component count function is \( O\left(n^\theta n!/n!\right) \) for some \( \theta \in (0, 1) \). It also provides the means to recursively construct an adequate class of structures with a labelled 0–1 law but not an unlabelled 0–1 law, answering Compton’s Question 8.4.

Keywords: ratio test, labelled structure, zero-one law

1 Introduction

Exponentiating a power series can have the effect of smoothing out the behavior of the coefficients. In this paper we look at conditions on the growth of the coefficients of \( G(x) = \sum g(n)x^n \), which ensure that \( f(n-1)/f(n) \rightarrow \infty \), where \( F(x) = e^{G(x)} \). One application of this result is to 0-1 laws, where we find, see Theorem 7, that if the labelled component count function for an adequate class of structures is \( O(n^\theta n!/n!) \) for some \( \theta \in (0, 1) \) then the class has a labelled monadic second-order 0-1 law.

Useful notation will be \( f(n) \prec g(n) \) for \( f(n) \) eventually less than \( g(n) \) and \( f(n) \in \mathbb{R} \), for \( f(n-1)/f(n) \rightarrow \infty \); the notation \( \mathbb{R} \) stands for the ratio test.

2 The Coefficients of \( e^{\text{poly}} \)

Proposition 1 Given

\[
G(x) := g(1)x + \cdots + g(d)x^d, \quad g(i) \geq 0, \quad g(d) > 0,
\]

with \( \gcd (j \leq d : g(j) > 0) = 1 \)

\[F(x) := \sum_{n \geq 0} f(n)x^n = e^{G(x)},\]

the function \( F(x) \) is Hayman-admissible. Thus

\[
f(n) \sim \frac{F(r_n)}{r_n^{1/2} \sqrt{2\pi B(r_n)}}, \quad (1)
\]
where $r_n$ is the unique positive solution to
\[ x \cdot G'(x) = n, \]
and $B(x) := x^2G''(x) + xG'(x)$.

**Proof:** Theorem X of Hayman \[5\] shows that $F(x)$ is Hayman-admissible. Then the rest of the claim is an immediate consequence of Corollary II of \[5\] where the saddle-point method is applied to find the asymptotics of the coefficients of an admissible function. \[\square\]

**Corollary 2** For $F(x)$, $G(x)$ as in the above proposition,

(a) $f(n) \in \text{RT}_\infty$.

(b) $f(n) = \exp \left( - \frac{n \log n}{d} (1 + o(1)) \right)$.

**Proof:** Item (a) follows immediately from Corollary IV of Hayman \[5\].

For item (b) one uses $r_nG'(r_n) = n$ to obtain:

\[
\left( \frac{n}{cdg(d)} \right)^{1/d} \leq r_n \leq \left( \frac{n}{dg(d)} \right)^{1/d} \text{ for } c > 1
\]

\[
r_n = (1 + o(1))\left( \frac{n}{dg(d)} \right)^{1/d}
\]

\[
r_n^n = (1 + o(1))^n \left( \frac{n}{dg(d)} \right)^{n/d}
\]

\[
B(r_n) = (1 + o(1))d^2g(d)\left( \frac{n}{dg(d)} \right) = (1 + o(1))dn
\]

\[
G(r_n) = (1 + o(1))g(d)r_n^d = (1 + o(1))\frac{n}{d}
\]

\[
F(r_n) = \exp \left( \frac{n}{d} (1 + o(1)) \right).
\]

Apply these results to (1). \[\square\]

### 3 Some Technical Lemmas

Now we drop the assumption that $G(x)$ is a polynomial, but keep the requirement

\[ \gcd (n : g(n) > 0) = 1. \]

(2)

This implies that $f(n) \succ 0$.

Choose a positive integer $L \geq 2$ sufficiently large so

\[ n > L \Rightarrow [x^n] \exp \left( g(1)x + \cdots + g(L)x^L \right) > 0. \]

(3)
Given $\ell > L$ with $g(\ell) > 0$ let

\[
G_0(x) := \sum_{n \geq 1} g_0(n) x^n := \sum_{1 \leq n \leq \ell} g(n) x^n
\]
\[
F_0(x) := \sum_{n \geq 0} f_0(n) x^n := \exp(G_0(x))
\]
\[
G_1(x) := \sum_{n \geq 1} g_1(n) x^n := \sum_{n \geq \ell + 1} g(n) x^n
\]
\[
F_1(x) := \sum_{n \geq 0} f_1(n) x^n := \exp(G_1(x)).
\] (4)

**Lemma 3** Suppose $r \geq -1$ is such that

\[
ge(n) = O\left(f_0(n + r)\right).
\] (5)

Then

\[
f_1(n) = O\left(f(n + r)\right).
\]

**Proof:** In view of (3) and (5) we can choose $C_r$ such that

\[
ge(n) \leq C_r f_0(n + r) \quad \text{for } n + r \geq L + 1.
\] (6)

Differentiating (4) gives

\[
f_1(n) = \sum_{j = \ell + 1}^{n} j g(j) \cdot f_1(n - j)
\]
\[
\leq C_r \sum_{j = \ell + 1}^{n} f_0(j + r) \cdot f_1(n - j) \quad \text{by (6)}
\]
\[
\leq C_r \sum_{j = 0}^{n + r} f_0(j) \cdot f_1(n + r - j)
\]
\[
= C_r f(n + r),
\]

the last line following from $F(x) = F_0(x) \cdot F_1(x)$. \[\square\]

**Lemma 4** Suppose for every integer $r \geq -1$

\[
ge(n) = O\left(f_0(n + r)\right).
\]

Then $f(n - 1)/f(n) \to \infty$.

**Proof:** Since $f_0(n) \in RT_\infty$ by Corollary\[2\] there is a monotone decreasing function $\varepsilon(n)$ such that for any sufficiently large $M$ we have $\varepsilon(n) > f_0(n)/f_0(n - 1)$ for $n \geq M$, and $\varepsilon(n) \to 0$ as $n \to \infty$.\[\square\]
Thus
\[ f(n) = \sum_{0 \leq j \leq n} f_0(j) f_1(n - j) \]
\[ = \sum_{0 \leq j \leq M-1} f_0(j) f_1(n - j) + \sum_{M \leq j \leq n} f_0(j) f_1(n - j) \]
\[ \leq o(f(n-1)) + \varepsilon(M) \sum_{M \leq j \leq n} f_0(j-1) f_1(n-j) \]
by Lemma 3 and the choice of \( \varepsilon \)
\[ \leq o(f(n-1)) + \varepsilon(M) f(n-1). \]

Thus
\[ \limsup_{n \to \infty} \frac{f(n)}{f(n-1)} \leq \varepsilon(M), \]
and as \( M \) can be arbitrarily large it follows that
\[ \lim_{n \to \infty} \frac{f(n)}{f(n-1)} = 0. \]

\section{4 Main Result}

We are now in a position to prove the main result, making use of
\[ n! = \exp \left( n \log n \cdot (1 + o(1)) \right), \]
which follows from Stirling’s result.

\textbf{Theorem 5} Suppose \( F(x) = \exp(G(x)) \) with \( F(x) = \sum_{n \geq 0} f(n) x^n \), \( G(x) = \sum_{n \geq 1} g(n) x^n \), and \( f(n), g(n) \geq 0 \). Suppose also that \( \gcd(n : g(n) > 0) = 1 \) and that for some \( \theta \in (0, 1) \)
\[ g(n) = O\left( n^{\theta n} / n! \right). \]
Then
\[ f(n) \in \text{RT}_\infty. \]

\textbf{Proof:} From Corollary 2 for any integer \( r \geq -1 \) and any \( \theta \in (0, 1) \), by choosing \( \ell > L \) such that \( 1/\ell < 1 - \theta \), we have
\[ f_0(n + r) = \exp \left( - \frac{(n + r) \log(n + r)}{\ell} (1 + o(1)) \right) \]
\[ = \exp \left( - \frac{n \log n}{\ell} (1 + o(1)) \right) \]
\[ \geq \frac{n^{\theta n}}{(n-1)!}. \]
Thus \( ng(n) = O(f_0(n+r)) \). The Theorem then follows from Lemma 4.
5  Best Possible Result

The main result is in a natural sense the best possible.

**Proposition 6** Suppose $t(n) \geq 0$ with $\gcd \left( n : t(n) > 0 \right) = 1$ is such that for every $\theta \in (0, 1)$

$$t(n) \neq O\left(n^{\theta n} / n!\right).$$

Then there is a sequence $g(n) \geq 0$ with $\gcd \left( n : g(n) > 0 \right) = 1$ and $g(n) \leq t(n)$ but $f(n) \notin \mathbb{RT}_\infty$, where one has $F(x) = \exp(G(x))$.

**Proof:** For $\theta \in (0, 1)$ let

$$S(\theta) = \{ n \geq 1 : t(n) > n^{\theta n} / n! \}.$$

Then $S(\theta)$ is an infinite set.

Let $M$ be such that $\gcd \left( n \leq M : t(n) > 0 \right) = 1$, and let

$$g_1(n) := \begin{cases} t(n) & \text{if } n \leq M \\ 0 & \text{if } n > M \end{cases}$$

$$G_1(x) := \sum g_1(n)x^n$$

$$d_1 := \deg(G_1(x))$$

$$F_1(x) := e^{G_1(x)}.$$

For $m \geq 2$ we give a recursive procedure to define polynomials $G_m(x)$; then letting

$$d_m := \deg(G_m(x))$$

$$F_m(x) := e^{G_m(x)},$$

by Proposition 6

$$f_m(n) = \exp \left( - \frac{n \log n}{d_m} (1 + o(1)) \right).$$

To define $G_{m+1}(x)$, having defined $G_m(x)$, let

$$h_m(n) := \frac{1}{n!} n^{(1-1/2d_m)n}.$$

Then

$$\frac{h_m(n)}{f_m(n-1)} \to \infty \quad \text{as } n \to \infty.$$

Thus we can choose an integer $d_{m+1} > d_m$ such that

$$d_{m+1} \in S \left( 1 - \frac{1}{2d_m} \right)$$

$$h_m(d_{m+1}) > f_m(d_{m+1} - 1).$$
This ensures that \( h_m(d_{m+1}) \leq t(d_{m+1}) \). Let 
\[
G_{m+1} := G_m(x) + h_m(d_{m+1})x^{d_{m+1}}.
\]

Then 
\[
\frac{f_{m+1}(d_{m+1})}{f_{m+1}(d_{m+1} - 1)} \geq \frac{h_m(d_{m+1})}{f_m(d_{m+1} - 1)} > 1.
\]

Now let \( G(x) \) be the nonnegative power series defined by the sequence of polynomials \( G_m(x) \); and let 
\( F(x) = e^{G(x)} \). Then \( g(n) \leq t(n) \) but \( f(n) \notin \mathcal{RT}_\infty \) as
\[
\frac{f(d_{m+1})}{f(d_{m+1} - 1)} = \frac{f_{m+1}(d_{m+1})}{f_{m+1}(d_{m+1} - 1)} > 1.
\]

\[\Box\]

6 Application to 0–1 laws

A class \( \mathcal{K} \) of finite relational structures is adequate if it is closed under disjoint union and the extraction of components. One can view the structures as being unlabelled with the component count function \( p_U(n) \) and the total count function \( a_U(n) \), both counting up to isomorphism. The corresponding ordinary generating series are
\[
P_U(x) := \sum_{n \geq 1} p_U(n)x^n, \quad A_U(x) := \sum_{n \geq 0} a_U(n)x^n
\]
connected by the fundamental equation
\[
A_U(x) = \prod_{j \geq 1} (1 - x^j)^{-p_U(j)}. \tag{7}
\]

One can also view the structures as being labelled (in all possible ways) with the count functions \( p_L(n) \) for the connected members of \( \mathcal{K} \), and \( a_L(n) \) for all members of \( \mathcal{K} \). The corresponding exponential generating series are
\[
P_L(x) := \sum_{n \geq 1} p_L(n)x^n/n!, \quad A_L(x) := \sum_{n \geq 0} a_L(n)x^n/n!
\]
connected by the fundamental equation
\[
A_L(x) = e^{P_L(x)}. \tag{8}
\]

All references to Compton in this section are to the two papers \[3\] and \[4\].
### 6.1 Unlabelled 0–1 Laws for Adequate Classes

Let $\mathcal{K}$ be an adequate class with unlabelled count functions and ordinary generating functions as described above. Compton showed that if the radius of convergence $\rho_U$ of $A_U(x)$ is positive then $\mathcal{K}$ has an unlabelled 0–1 law if

$$\frac{a_U(n-1)}{a_U(n)} \rightarrow 1 \text{ as } n \rightarrow \infty.$$  

$\mathcal{K}$ is finitely generated if $r = \sum p_U(n) < \infty$, that is, there are only finitely many connected structures in $\mathcal{K}$. In the finitely generated case the asymptotics for the coefficients $a_U(n)$ have long been known to have the simple polynomial form

$$a_U(n) \sim C n^{r-1}$$  \hspace{1cm} (9)

provided $\gcd\{n : p_U(n) > 0\} = 1$. Item (ii) leads to the fact that $a_U(n) \in \text{RT}_1$, and hence to an unlabelled 0–1 law. In addition to using this result, Compton notes that the work of Bateman and Erdős [11] shows that if $p_U(n) \in \{0, 1\}$, for all $n$, then one has $a_U(n) \in \text{RT}_1$.

Both of these results were subsumed in the powerful result of Bell [2] which says that if $p_U(n)$ is polynomially bounded, that is, there is a $c$ such that $p_U(n) = O(n^c)$, then $a_U(n) \in \text{RT}_1$.

### 6.2 Labelled 0–1 Laws

Compton shows that if $\rho_L$, the radius of convergence of $A_L(x)$, is positive, then $\mathcal{K}$ has a labelled 0–1 law if

$$\frac{a_L(n-k)/(n-k)!}{a_L(n)/n!} \rightarrow \infty \text{ whenever } \rho_L(k) > 0.$$  \hspace{1cm} (10)

In particular it suffices to show that $a_L(n)/n! \in \text{RT}_\infty$.

Compton’s method to show that a given adequate class of finite relational structures $\mathcal{K}$ has a labelled 0–1 law is to show that its exponential generating function $A_L(x) = \sum a_L(n)x^n/n!$ is Hayman-admissible with an infinite radius of convergence. This guarantees that $a_L(n)/n! \in \text{RT}_\infty$ [5, Corollary IV]. However, as Compton notes, showing that $A_L(x)$ is Hayman-admissible can be quite a challenge.

Question 8.3 of [3] first asks if, in the unlabelled case, the result of Bateman and Erdős, namely $p_U(n) \in \{0, 1\}$ implies $a_U(n) \in \text{RT}_1$, can be extended to the much more general statement that $p_U(n) = O(n^k)$ implies $a_U(n) \in \text{RT}_1$, yielding an unlabelled 0–1 law. As mentioned earlier, this was proved to be true by Bell. The second part of Question 8.3 asks if there is a simple sufficient condition along similar lines for the labelled case. We can now answer this in the affirmative with a result that is an excellent parallel to Bell’s result for unlabelled structures.

**Theorem 7** If $\mathcal{K}$ is an adequate class of structures with

$$p_L(n) = O\left(n^{\theta n}\right) \text{ for some } \theta \in (0, 1)$$

then $a_L(n)/n! \in \text{RT}_\infty$, and consequently $\mathcal{K}$ has a labelled monadic second-order 0–1 law.

---

(i) Given a logic $\mathcal{L}$, $\mathcal{K}$ has an unlabelled $\mathcal{L}$ 0–1 law means that for any $\mathcal{L}$ sentence $\varphi$, the probability that $\varphi$ holds in $\mathcal{K}$ will be either 0 or 1. In [3] Compton worked with first-order logic, in [4] with monadic second-order logic. In both papers he simply used the phrases “unlabelled 0–1 law” and “labelled 0–1 law”.

(ii) This result is usually known as Schur’s Theorem [6, 3.15.2]. One can easily find the asymptotics using a partial fraction decomposition of the right side of (7). The labelled case with finitely many components is more difficult—we needed to invoke Hayman’s treatise [5] just to obtain the asymptotics for $\log a_L(n)/n!$ (see Corollary 2).
**Proof:** This is an immediate consequence of Theorem 5 and Compton’s proof that $a_L(n)/n! \in RT_\infty$ guarantees such a 0–1 law.

Now we list the examples of classes $K$ which Compton shows have a labelled 0–1 law, giving $p_L(n)$ in each case. It is trivial to check in each case that $p_L(n) = O\left(n^{n/2}\right)$, thus the 0–1 law in each case follows from our Theorem 7.

(a) 7.1 Unary Predicates $p_L(n) = 0$ for $n > 1$.

(b) 7.12 Forests of Rooted Trees of Height 1 $p_L(n) = n$.

(c) 7.15 Only Finitely Many Components $p_L(n)$ is eventually 0.

(d) 7.16 Equivalence Relations $p_L(n) = 1$.

(e) 7.17 Partitions with a Selection Subset $p_L(n) = 2^n - 1$.

We can now augment this list by, in each case, coloring the members of $K$ by a fixed set of $r$ colors in all possible ways. This will increase the original $p_L(n)$ by a factor of at most $r^n$. This will still give $p_L(n) = O\left(n^{n/2}\right)$. Furthermore, in each of these colored cases let $\mathcal{P}$ be any subset of the connected members, and let $K$ be the closure of $\mathcal{P}$ under disjoint union. Each such $K$ has a labelled 0–1 law.

Another application of Theorem 7 is to answer Question 4 of [3] by exhibiting an adequate class $K$ such that $p_L(n) = O\left(n^{3n/4}\right)$, hence there is a labelled 0–1 law for $K$; but also such that $\rho_U \in (0, 1)$, so $K$ does not have an unlabelled 0–1 law.

Let the components of $K$ be the one-element tree $T_1$ along with rooted trees $T_{3n}$ of size $3n$ and height $n$ consisting of a chain $C_n$ of $n$ nodes, with an antichain $L_{2n}$ of $2n$ nodes (the leaves of the tree) below the least member of the chain; and the chain $C_n$ is two-colored while the remaining nodes are uncolored. One can visualize these as brooms with 2-colored handles, see Figure 1.

The number of unlabelled components is given by $p_U(1) = 1$, $p_U(3n) = 2^n$. Thus the radius of convergence of the ordinary generating function of $K$ is $\rho_U = \sqrt[3]{2}$. Since this is positive and not 1 it follows from Theorem 5.9(ii) of [3] that $K$ does not have an unlabelled 0–1 law.

![Fig. 1: Brooms with two-colored handles](image-url)
For the number \( p_L(3n) \) of labelled components of size 3\( n \):

\[
p_L(3n) \leq 2^n \binom{3n}{n} n! \\
\leq 2^n (3n)^n \exp\left(n \log n \cdot (1 + o(1))\right) \\
= \exp\left(2n \log n \cdot (1 + o(1))\right) \\
= (3n)^{(2/3)(3n)}(1+o(1)) \\
= O\left((3n)^{(3/4)(3n)}\right).
\]

Thus \( p_L(n) = O(n^{3n/4}) \), so \( a_L(n)/n! \in \text{RT}_\infty \) by Theorem 7, showing that \( \mathcal{K} \) has a labelled 0–1 law.
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