ON THE LIMITING SYSTEM IN THE SHIGESADA, KAWASAKI AND TERAMOTO MODEL WITH LARGE CROSS-DIFFUSION RATES

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Abstract. In 1979, Shigesada, Kawasaki and Teramoto [11] proposed a mathematical model with nonlinear diffusion, to study the segregation phenomenon in a two competing species community. In this paper, we discuss limiting systems of the model as the cross-diffusion rates included in the nonlinear diffusion tend to infinity. By formal calculation without rigorous proof, we obtain one limiting system which is a little different from that established in Lou and Ni [5].

1. Introduction. One main problem in population ecology is to understand the spatial distribution pattern of competing species. To theoretically investigate the problem, we firstly consider the two competing species community, and employ the nondimensional system of reaction-diffusion equations

\[
\begin{align*}
\frac{\partial w}{\partial t} &= \varepsilon \Delta w + w(1 - w - c z) \quad \text{in } \Omega_T, \\
\frac{\partial z}{\partial t} &= d\varepsilon \Delta z + z(a - b w - z) \quad \text{in } \Omega_T, \\
\frac{\partial w}{\partial \nu} &= 0, \quad \frac{\partial z}{\partial \nu} = 0 \quad \text{on } \partial \Omega_T, \\
w(x, 0) &= w_0(x), \quad z(x, 0) = z_0(x) \quad \text{in } \partial \Omega,
\end{align*}
\]

which describes the dynamics of two species’ population at the position x and the time t, where Ω is a bounded smooth domain of \(\mathbb{R}^d\) with \(d \geq 1\); \(\partial \Omega\) and \(\partial \Omega\) are the boundary and the closure, respectively, of Ω; \(\Omega_T = \Omega \times (0, T)\) and \(\partial \Omega_T = \partial \Omega \times (0, T)\) for some \(T \in (0, +\infty]\); \(\nu\) is the outward unit normal vector on \(\partial \Omega\); \(a\), \(b\) and \(c\) are positive constants. The initial values \(w_0(x)\) and \(z_0(x)\) are nonnegative smooth functions which are not identically zero. In the diffusion terms, \(d\) and \(\varepsilon\) are positive, and \(\varepsilon \Delta w\) and \(d\varepsilon \Delta z\) represent the dispersive force associated with the random movement of each species.

Kishimoto and Weinberger [2] showed that if \(\Omega\) is convex and the strong competition condition

\[
\frac{1}{c} < a < b
\]

holds true, then every nonconstant positive stationary solution of (1.1) is unstable, that is, the stable stationary solutions of (1.1) are given by \((w, z) = (1, 0)\) and \((w, z) = (0, a)\). This fact generically implies that when \(\Omega\) is convex, the stable spatially inhomogeneous pattern of two competing species can never appear under...
strong competition condition. On the other hand, Matano and Mimura [8] showed the existence of stable nonconstant positive stationary solution of (1.1) under the strong competition condition, when \( \Omega \) is suitable nonconvex domain. In ecological terms, the coexistence of two competing species crucially depends on the shape of habitat.

The spatial distribution pattern of two competing species in the nature may be realized as a result of various biological effects, for example, the shape of habitat stated above. In 1979, Shigesada, Kawasaki and Teramoto [11] proposed the reaction-diffusion system

\[
\begin{align*}
\partial_t w &= \varepsilon \Delta [(1 + \alpha z) w] + w (1 - w - c z) \quad \text{in } \Omega_T, \\
\partial_t z &= d \varepsilon \Delta [(1 + \beta w) z] + z (a - b w - z) \quad \text{in } \Omega_T, \\
\partial_n w &= 0, \quad \partial_n z = 0 \quad \text{on } \partial \Omega_T, \\
w(x, 0) &= w_0(x), \quad z(x, 0) = z_0(x) \quad \text{in } \mathbb{C} \Omega \tag{1.2}
\end{align*}
\]

in order to show that the nonlinear dispersive force has an effect on the spatial segregation pattern of two competing species. In the diffusion terms, the cross-diffusion rates \( \alpha \) and \( \beta \) are nonnegative, and the terms \( \varepsilon \alpha \Delta [w z] \) and \( d \varepsilon \beta \Delta [w z] \) mean what the gradient in the concentration of one species induces the dispersive force of another species.

Since then, many mathematicians have tried to study the system (1.2) from various viewpoints (for example, see Jünger [1], Ni [10], and their references). Lou and Ni [4] obtained useful a priori estimates for any solution, and sufficient conditions for the existence/nonexistence of nonconstant positive solution. Moreover they in the subsequent paper [5] characterized the asymptotic behavior of solutions as \( \alpha \to +\infty \), and established the following two kinds of limiting systems:

**Proposition 1** (Theorem 1.4 in [5]). Suppose that \( \ell \leq 3 \), \( a \neq b \) and \( a \neq 1/c \) are satisfied, and that \( a/d \) is not equal to every eigenvalue of \( -\Delta \) with homogeneous Neumann boundary condition on \( \partial \Omega \). Let \( \{(w_n, z_n)\} \) be any sequence of nonconstant positive solutions of (1.2) with \( \alpha = \alpha_n \to +\infty \). Then there exists a small \( \delta_1 = \delta_1(a, b, c, \varepsilon, d) > 0 \) such that if \( \beta \leq \delta_1 \), then as \( \alpha_n \to +\infty \), by passing to a subsequence if necessary, either \( (w_n, z_n) \) converges uniformly to \( (w, Z) \), where \( (w, Z) \) is a positive solution of

\[
\begin{align*}
\varepsilon \Delta [(1 + Z) w] + (1 - w) w &= 0, \quad w > 0 \quad \text{in } \Omega, \\
d \varepsilon \Delta [(1 + \beta w) Z] + (a - b w) Z &= 0, \quad Z > 0 \quad \text{in } \Omega, \\
\partial_n w &= 0, \quad \partial_n Z = 0 \quad \text{on } \partial \Omega, 
\end{align*}
\tag{1.3}
\]

or \( (w_n, z_n) \) converges uniformly to \( (\tau/z, z) \), where \( \tau \) is positive, and \( (\tau, z) \) satisfies

\[
\begin{align*}
d \varepsilon \Delta z + (a - z) z - b \tau &= 0, \quad z > 0 \quad \text{in } \Omega, \\
\partial_n z &= 0 \quad \text{on } \partial \Omega, \\
\int_{\Omega} \frac{1}{z} \left( 1 - \frac{\tau}{z} - c z \right) \, dx &= 0.
\end{align*}
\tag{1.4}
\]

Since the appearance of the above proposition, there are many studies on the qualitative property of positive solution for (1.3) and (1.4) in various ranges of parameter (for instance, refer to Lou, Ni and Yotsutani [6, 7], and Mori, Suzuki and Yotsutani [9] for (1.4), and Kuto [3] for (1.3)). From these studies, we can theoretically understand the global bifurcation structure of positive stationary solutions for (1.2). We note here that the above proposition is valid when \( \beta \) is not so large.
Figure 1 and Figure 2 show the density distribution of numerical example for (1.2). Figure 1 says that a periodic solution with spatial segregation appears when $\alpha$ and $\beta$ are large. Figure 2 is represented as the dynamics of $(u, v)(x, t)$ with

\[ u(x, t) = \frac{(1 + \alpha z(x, t)) w(x, t)}{\alpha}, \quad v(x, t) = \alpha z(x, t) - \beta w(x, t) \]

for the numerical solution $(w, z)(x, t)$ of (1.2) shown in Figure 1, and shows that the $u$-component is a almost constant function in $x \in \Omega$ for each $t > 0$ when $\alpha$ and $\beta$ are large.

**Figure 1.** Density distribution of radially symmetric solution $(w, z)(x, t)$ for (1.2) with $\Omega = \{ x \in \mathbb{R}^2 \mid |x| < \pi \}$ for the case where $a = 1.04$, $b = 1.1$, $c = 1.1$, $d = 15.0$, $\varepsilon = 0.005$, $\alpha = 1200.0$ and $\beta = 2400.0$. The horizontal axis and the vertical axis indicate the distance $r = |x|$ and the time $t$, respectively.

In this paper, motivated by Proposition 1 and numerical examples, we address the following problem: When $a$, $b$, $c$, $d$ and $\varepsilon$ are suitably fixed, what kinds of limiting systems describing the dynamics of (1.2) can we derive as $\alpha \to +\infty$ and $\beta \to +\infty$? To study the problem, we assume that $\beta = \beta(\alpha)$ is a continuous function in $\alpha \geq 0$ satisfying the condition (A) stated in the next section, and we shall discuss limiting systems in (1.2) as $\alpha \to +\infty$.

2. **Limiting systems.** We set $w = (w, z)$ and $u = (u, v)$, and define the map $\varphi(w) = (\varphi_u, \varphi_v)(w)$ by

\[ \varphi_u(w) = (1 + \alpha z) w, \quad \varphi_v(w) = (1 + \beta w) z. \]

Since it follows from

\[ \det \partial_w \varphi(w) = \det \begin{pmatrix} 1 + \alpha z & \alpha w \\ \beta z & 1 + \beta w \end{pmatrix} = 1 + \beta w + \alpha z > 0 \]

for any $w \in \mathbb{R}^2_+$ that $u = \varphi(w)$ is a bijection from $\text{Cl} \mathbb{R}^2_+$ to $\text{Cl} \mathbb{R}^2_+$, we find out that the inverse map $w = \Phi(u) = (\Phi_w, \Phi_z)(u)$ of $u = \varphi(w)$ exists. Actually, by simple
calculation, we can represent $\Phi_w(u)$ and $\Phi_z(u)$ as

$$
\Phi_w(u) = \frac{\hat{\Phi}_w(\beta u, \alpha v)}{\beta}, \quad \Phi_z(u) = \frac{\hat{\Phi}_z(\beta u, \alpha v)}{\alpha},
$$

respectively, where $U = (U, V)$,

$$
\hat{\Phi}_w(U) = \frac{\sqrt{(U - V - 1)^2 + 4U + U - V - 1}}{2},
\hat{\Phi}_z(U) = \frac{\sqrt{(U - V - 1)^2 + 4U - U + V - 1}}{2}.
$$

We should remark that

$$
U = (1 + \hat{\Phi}_z(U)) \hat{\Phi}_w(U), \quad V = (1 + \hat{\Phi}_w(U)) \hat{\Phi}_z(U)
$$

are satisfied.

Setting $D = \text{diag}(1, d)$, $f(u) = (f, g)(u)$,

$$
f(u) = (1 - \Phi_w(u) - c \Phi_z(u)) \Phi_w(u),
g(u) = (a - b \Phi_w(u) - \Phi_z(u)) \Phi_z(u),
$$

and employing the change of variables $u = \varphi(w)$, we represent (1.2) as

$$
\frac{\partial_u \Phi(u) \partial_t u}{\varepsilon D \Delta u + f(u)} \text{ in } \Omega_T
$$

with the Neumann boundary condition. Hereafter, we assume that $\beta = \beta(\alpha)$ is a nonnegative continuous function in $\alpha \geq 0$, and satisfies

$$
\beta(\alpha) = \kappa \alpha + \omega + o(1) \quad \text{as } \alpha \to +\infty
$$

(A)

with constants $\kappa$ and $\omega$. 
2.1. **Case where** $\kappa = 0$ **and** $\omega = 0$. Suppose the case where $U = u/\alpha$ and $V = v$ are positive and bounded as $\alpha \to +\infty$. After simple calculations, as $\alpha \to +\infty$, we have

$$
\Phi_u(\alpha U, V) = \frac{U}{V} + o(1), \quad \Phi_z(\alpha U, V) = V + o(1),
$$

and then we obtain

$$
f(\alpha U, V) = \left(1 - \frac{U}{V} - cV\right) \frac{U}{V} + o(1),
$$

$$
g(\alpha U, V) = \left(a - \frac{bU}{V^*} - V\right) V + o(1).
$$

From the Neumann boundary condition and

$$
\alpha \in \Delta U = \partial_t [\Phi_u(\alpha U, V)] - f(\alpha U, V) = \partial_t \left[\frac{U}{V}\right] - \left(1 - \frac{U}{V} - cV\right) \frac{U}{V} + o(1)
$$

as $\alpha \to +\infty$, we see that $U = U(x, t)$ as $\alpha \to +\infty$ converges to a constant function $U^* = U^*(t)$ in $x \in \Omega$ for each $t > 0$, and $U^*(t)$ satisfies

$$
\partial_t [U^* \mathcal{A}(V^*)] = \mathcal{F}(U^*, V^*), \quad t > 0,
$$

where $V^* = V^*(x, t) = \lim_{\alpha \to +\infty} V(x, t)$ in $(x, t) \in \Omega_T$, and

$$
\mathcal{A}(V^*) = \int_{\Omega} \frac{1}{V^*(x, t)} dx, \quad \mathcal{F}(U^*, V^*) = \int_{\Omega} \left(1 - \frac{U^*}{V^*} - cV^*\right) \frac{U^*}{V^*} dx.
$$

By

$$
\partial_t [\Phi_u(\alpha U, V)] = d \varepsilon \Delta V + g(\alpha U, V),
$$

we have the following limiting system:

$$
\begin{align*}
\partial_t [U^* \mathcal{A}(V^*)] &= \mathcal{F}(U^*, V^*) \quad \text{in } t > 0, \\
\partial_t V^* &= d \varepsilon \Delta V^* + \left(a - \frac{bU^*}{V^*} - V^*\right) V^* \quad \text{in } \Omega_T, \\
\partial_t V^* &= 0 \quad \text{on } \partial\Omega_T.
\end{align*}
$$

The stationary problem of (2.2) is the same as (1.4) established by Lou and Ni [5], and the positive solution of (2.2) corresponds to the bounded positive solution of (1.2) as $\alpha \to +\infty$, because

$$
w = \Phi_u(\alpha U, V) = \frac{U^*}{V^*} + o(1), \quad z = \Phi_z(\alpha U, V) = V^* + o(1)
$$

are satisfied as $\alpha \to +\infty$.

2.2. **Case where** $\kappa = 0$ **and** $\omega \geq 0$. Suppose the case where $U = u$ and $V = \alpha v$ are positive and bounded as $\alpha \to +\infty$. We put

$$
U^* = U^*(x, t) = \lim_{\alpha \to +\infty} U(x, t), \quad V^* = V^*(x, t) = \lim_{\alpha \to +\infty} V(x, t),
$$

$$
W^* = W^*(x, t) = \lim_{\alpha \to +\infty} \Phi_u \left(\frac{U(x, t)}{\alpha}\right),
$$

$$
Z^* = Z^*(x, t) = \lim_{\alpha \to +\infty} \left\{\alpha \Phi_z \left(\frac{V(x, t)}{\alpha}\right)\right\}.
$$

After simple calculations, as $\alpha \to +\infty$, we obtain

$$
W^* = \hat{\Phi}_u(\omega U^*, V^*), \quad Z^* = \hat{\Phi}_z(\omega U^*, V^*)
$$
for the case where \( \omega > 0 \), and

\[
W^* = \frac{U^*}{1 + V^*}, \quad Z^* = V^*
\]

for the case where \( \omega = 0 \). Hence we find out that

\[
U^* = (1 + Z^*) W^*, \quad V^* = (1 + \omega W^*) Z^*
\]

are satisfied for any \( \omega \geq 0 \). Since we have

\[
\partial_t \Phi_w \left( U, \frac{V}{\alpha} \right) = \varepsilon \Delta U + \left\{ 1 - \Phi_w \left( U, \frac{V}{\alpha} \right) \right\} \Phi_w \left( U, \frac{V}{\alpha} \right) + o(1),
\]

\[
\partial_t \alpha \Phi_z \left( U, \frac{V}{\alpha} \right) = d \varepsilon \Delta V + \left\{ a - b \Phi_w \left( U, \frac{V}{\alpha} \right) \right\} \Phi_z \left( U, \frac{V}{\alpha} \right) + o(1)
\]

as \( \alpha \to +\infty \), we obtain the following limiting system:

\[
\begin{aligned}
&\partial_t W^* = \varepsilon \Delta \left[(1 + Z^*) W^* \right] + (1 - W^*) W^* \quad \text{in } \Omega_T, \\
&\partial_t Z^* = d \varepsilon \Delta \left[(1 + \omega W^*) Z^* \right] + (a - b W^*) Z^* \quad \text{in } \Omega_T, \\
&\partial_t W^* = 0, \quad \partial_t Z^* = 0 \quad \text{on } \partial\Omega_T.
\end{aligned}
\]

The stationary problem of (2.3) is the same as (1.3) established by Lou and Ni [5], and the positive solution of (2.3) corresponds to the bounded positive solution of (1.2) such that the \( z \)-component is close to 0 as \( \alpha \to +\infty \), because

\[
w = \Phi_w \left( U, \frac{V}{\alpha} \right) = W^* + o(1), \quad z = \Phi_z \left( U, \frac{V}{\alpha} \right) = Z^* \frac{(1 + o(1))}{\alpha}
\]

are satisfied as \( \alpha \to +\infty \).

2.3. Case where \( \kappa > 0 \). Suppose the case where

\[
U = \frac{u}{\alpha}, \quad V = \frac{\alpha v - \beta(\alpha) u}{\alpha}
\]

are positive and bounded as \( \alpha \to +\infty \). We set

\[
\Psi_w(U) = \Phi_w(\alpha U, V + \beta(\alpha) U), \quad F(U) = f(\alpha U, V + \beta(\alpha) U),
\]

\[
\Psi_z(U) = \Phi_z(\alpha U, V + \beta(\alpha) U), \quad G(U) = g(\alpha U, V + \beta(\alpha) U),
\]

\[
\hat{\Psi}(U) = \Psi_z(U) - \frac{\beta(\alpha) d}{\alpha} \Psi_w(U), \quad H(U) = G(U) - \frac{\beta(\alpha) d}{\alpha} F(U).
\]

Defining \( P^*(U) = \lim_{\alpha \to +\infty} P(U) \) for each function \( P(U) \) depending on \( \alpha \), we easily obtain

\[
\Psi_w(U) = \frac{\sqrt{\kappa^2 + 4 \kappa U}}{2 \kappa} - V, \quad F^*(U) = \frac{(\kappa + V) \left( \sqrt{\kappa^2 + 4 \kappa U} - V \right)}{2 \kappa^2} - \frac{1 + c \kappa}{\kappa} U,
\]

\[
\Psi_z(U) = \frac{\sqrt{\kappa^2 + 4 \kappa U} + V}{2}, \quad G^*(U) = \frac{(a - V) \left( \sqrt{\kappa^2 + 4 \kappa U} + V \right)}{2} - (b + \kappa) U,
\]

\[
\hat{\Psi}^*(U) = \Psi_z(U) - d \kappa \Psi_w(U), \quad H^*(U) = G^*(U) - d \kappa F^*(U).
\]

At this time, we represent (1.2) as

\[
\partial_t \Psi_w(U) = \alpha \varepsilon \Delta U + F(U) \quad \text{in } \Omega_T,
\]

\[
\partial_t \Psi_z(U) = d \varepsilon \Delta V + \beta(\alpha) d \varepsilon \Delta U + G(U) \quad \text{in } \Omega_T
\]

with the Neumann boundary condition. From the above equations, we have

\[
\partial_t \hat{\Psi}(U) = d \varepsilon \Delta V + H(U) \quad \text{in } \Omega_T.
\]
From the Neumann boundary condition and
\[ \alpha \varepsilon \Delta U = \partial_t [\Psi_w(U)] - F^*(U) + o(1) \quad \text{as } \alpha \to +\infty, \]
we see that the solution of (2.4) as \( \alpha \to +\infty \) tends to a constant function in \( x \in \Omega \) for each \( t > 0 \). We denote \( U^* = U^*(t) \) and \( V^* = V^*(x, t) \) by the limit functions of \( U = U(x, t) \) and \( V = V(x, t) \), respectively, as \( \alpha \to +\infty \). By
\[ \partial_t \Psi_w(U) = \frac{1}{\sqrt{V^2 + 4\kappa U}}, \quad \partial_t \Psi_w(U) = -\frac{\Psi_w^*(U)}{\sqrt{V^2 + 4\kappa U}}, \]
we derive the equation
\[ A_u(U^*) \partial_t U^* + A_v(U^*) = F(U^*), \quad t > 0 \]
from (2.4) as \( \alpha \to +\infty \), where \( U^* = (U^*, V^*) \),
\[ A_u(U^*) = \int_\Omega \partial_t \Psi_w(U) \, dx, \quad A_v(U^*) = \int_\Omega \partial_t V^* \, dx, \]
\[ F(U^*) = \int_\Omega F^*(U^*) \, dx. \]
Moreover the limiting equation of (2.5) as \( \alpha \to +\infty \) becomes
\[ B_u(U^*) \partial_t U^* + B_v(U^*) \partial_t V^* = d\varepsilon \Delta V^* + H^*(U^*) \quad \text{in } \Omega_T, \]
where
\[ B_u(U^*) = \partial_t \Psi_w^*(U) = \frac{(1 - d)\kappa}{\sqrt{V^2 + 4\kappa U}}, \]
\[ B_v(U^*) = \partial_t \Psi_w^*(U) = \frac{(\Psi_w^*(U) + d\kappa \Psi_w^*(U))}{\sqrt{V^2 + 4\kappa U}}. \]
Summarizing the above argument, we have the following limit system:
\[ \begin{cases} 
A_u(U^*) \partial_t U^* + A_v(U^*) = F(U^*) & \text{in } t > 0, \\
B_u(U^*) \partial_t U^* + B_v(U^*) \partial_t V^* = d\varepsilon \Delta V^* + H^*(U^*) & \text{in } \Omega_T, \\
\partial_t V^* = 0 & \text{on } \partial\Omega_T. 
\end{cases} \quad (2.6) \]
Since
\[ \Psi_w^*(U) = \frac{2U}{\sqrt{V^2 + 4\kappa U} + V} = \frac{U}{V} + o(1), \]
\[ \Psi_z^*(U) = \frac{\sqrt{V^2 + 4\kappa U} + V}{2} = V + o(1) \]
are satisfied as \( \kappa \to 0 \), we should note here that (2.6) becomes (2.2) as \( \kappa \to 0 \), which implies that (2.6) is an extension of (2.2).

2.4. Estimate of positive stationary solution. In this subsection, we consider the estimate of positive solution for the stationary problem of (2.1). Setting
\[ \mathcal{N}_w = \{ (w, z) \in \mathbb{ClR}^2_+ \mid 1 - w - cz = 0 \}, \]
\[ \mathcal{N}_z = \{ (w, z) \in \mathbb{ClR}^2_+ \mid a - bw - z = 0 \}, \]
Figure 3. Density distribution of solution $U^* = (U^*, V^*) (x, t)$ for (2.6), where $a$, $b$, $c$, $d$ and $\varepsilon$ are the same as in Figure 1.

Figure 4. Density distribution of $(W^*, Z^*) = (\Psi_w(U^*), \Psi_z(U^*))$ with $U^* = U^*(x, t)$ shown in Figure 3

we can easily have the estimates

$$
\hat{M}_{uw}(\alpha) \equiv \max_{(w, z) \in \mathcal{N}_w} \{(1 + \alpha z) w\} = \hat{M} \left( \frac{\alpha}{c} \right),
$$

$$
\hat{M}_{uz}(\alpha) \equiv \max_{(w, z) \in \mathcal{N}_z} \{(1 + \alpha z) w\} = a \hat{M} \left( \frac{a \alpha}{b} \right),
$$

$$
\hat{M}_{vw}(\beta) \equiv \max_{(w, z) \in \mathcal{N}_v} \{(1 + \beta w) z\} = \hat{M} \left( \frac{\beta}{c} \right),
$$

$$
\hat{M}_{vz}(\beta) \equiv \max_{(w, z) \in \mathcal{N}_z} \{(1 + \beta w) z\} = a \hat{M} \left( \frac{b \beta}{a} \right),
$$
where
\[ \hat{M}(x) = \begin{cases} 
1 & \text{if } 0 \leq x \leq 1, \\
\frac{(x+1)^2}{4x} & \text{if } x > 1.
\end{cases} \]
Moreover, by setting
\[ \hat{M}_u(\alpha) = \max(\hat{M}_w(\alpha), \hat{M}_z(\alpha)), \quad \hat{M}_v(\beta) = \max(\hat{M}_w(\beta), \hat{M}_z(\beta)), \]
we obtain
\[ \lim_{\alpha \to +\infty} \frac{\hat{M}_u(\alpha)}{\alpha} = \max \left\{ \frac{1}{4c}, \frac{a^2}{4b} \right\}, \quad \lim_{\beta \to +\infty} \frac{\hat{M}_v(\beta)}{\beta} = \max \left\{ \frac{1}{4c}, \frac{b}{4} \right\}. \]
By the definition of \( f(u) \), we have
\[ f(u) < 0, \quad g(u) < 0 \quad \text{for any } u \in \mathbb{R}^2 \setminus \mathcal{R}(\alpha, \beta), \]
where \( \mathcal{R}(\alpha, \beta) = [0, \hat{M}_u(\alpha)] \times [0, \hat{M}_v(\beta)] \). From the maximum principle, we obtain
\[ u(x) \in \mathcal{R}(\alpha, \beta) \quad \text{on } x \in \text{Cl}\Omega \quad (2.7) \]
for any nonnegative stationary solution \( u(x) \) of (2.1), which implies that \( u(x)/\alpha \) is uniformly bounded on \((x, \alpha, \beta) \in (\text{Cl}\Omega) \times \mathbb{R}^2_+ \).
Let \( \omega^\alpha(x) = (w^\alpha, v^\alpha)(x) \) be a positive stationary solution of (2.1) for the case where \( \kappa = 0 \) and \( \omega = 0 \). Since \( \hat{M}_u(\beta(\alpha)) \to 1 \) satisfies as \( \alpha \to +\infty \), it follows from (2.7) that
\[ U^\alpha(x) = \frac{w^\alpha(x)}{\alpha}, \quad V^\alpha(x) = v^\alpha(x) \]
are uniformly bounded on \((x, \alpha) \in (\text{Cl}\Omega) \times \mathbb{R}^+ \). By passing to a subsequence if necessary, we see that
\[ U^\ast(x) = \lim_{\alpha \to +\infty} (U^\alpha, V^\alpha)(x) \]
is a nonnegative stationary solution of (2.2).
Let \( w^\alpha(x) = (w^\alpha, z^\alpha)(x) = \Phi(u^\alpha(x)) \), we can easily check that \( w^\alpha(x) \) is a positive stationary solution of (1.2), and
\[ \beta(\alpha) w^\alpha(x) - \alpha v^\alpha(x) = \beta(\alpha) w^\alpha(x) - \alpha z^\alpha(x) \]
holds true for each \( x \in \text{Cl}\Omega \) and \( \alpha \in \mathbb{R}^+ \). Putting \( U^\alpha(x) = (U^\alpha, V^\alpha)(x) \) with
\[ U^\alpha(x) = \frac{w^\alpha(x)}{\alpha}, \quad V^\alpha(x) = \frac{\alpha v^\alpha(x) - \beta(\alpha) w^\alpha(x)}{\alpha}, \]
we see from (2.7) that \( U^\alpha(x) \) is uniformly bounded on \((x, \alpha) \in (\text{Cl}\Omega) \times \mathbb{R}^+ \). Since
\[ w^\alpha(x) = \Psi_w(U^\alpha(x)) = \Psi_w(U^\alpha(x)) + o(1), \]
\[ z^\alpha(x) = \Psi_z(U^\alpha(x)) = \Psi_z(U^\alpha(x)) + o(1), \]
\[ U^\alpha(x) = \frac{(1 + \alpha z^\alpha(x)) w^\alpha(x)}{\alpha} = w^\alpha(x) z^\alpha(x) (1 + o(1)), \]
\[ V^\alpha(x) = \frac{\alpha z^\alpha(x) - \beta(\alpha) w^\alpha(x)}{\alpha} = (z^\alpha(x) - \kappa w^\alpha(x))(1 + o(1)) \]
are satisfied as \( \alpha \to +\infty \), we find out that \( U^\ast(x) \) is uniformly bounded on \((x, \alpha) \in (\text{Cl}\Omega) \times \mathbb{R}^+ \) if and only if \( w^\ast(x) \) is uniformly bounded on \((x, \alpha) \in (\text{Cl}\Omega) \times \mathbb{R}^+ \).
3. **Concluding remarks.** In this paper, we discussed the limiting system of (1.2) when the cross diffusion rate $\beta = \beta(\alpha)$ has the asymptotic behavior (A) as $\alpha \to +\infty$. The limiting system (2.6) obtained in this paper is a little different from that established in Lou and Ni [5]. Although the numerical examples show the validity of the limiting system (2.6), we can not give the rigorous proof on the validity because the uniform boundedness of nonnegative solution for the limiting system (2.6) is not obtained. Moreover we have not yet studied the property of positive solution for the limiting system (2.6). Since the limiting system (2.6) has just been obtained, many problems such as the above remain.

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