On the Schwarzian counterparts of conformal mechanics

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Abstract

It is shown that, if the energy in the Schwarzian mechanics (SM) is equal to the coupling constant in the de Alfaro-Fubini-Furlan (DAFF) model, there exists a link between these two systems. In particular, the equation of motion, $SL(2, \mathbb{R})$-symmetry transformations and the corresponding conserved charges of SM can be derived from those of the DAFF model by applying a coordinate transformation of a special type, while the general solution of the DAFF system maps to the velocity function of SM. A way to reproduce this link via the method of nonlinear realizations is presented. Schwarzian counterparts of the DAFF mechanics in Newton-Hooke (NH) spacetime as well as a higher derivative generalization of the DAFF model are discussed.

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1. Introduction

For an arbitrary function $\rho = \rho(t)$ of one variable $t$, the following expression

$$\{\rho(t), t\} = \frac{\dot{\rho}(t)}{\rho(t)} - \frac{3}{2} \left(\frac{\rho(t)}{\dot{\rho}(t)}\right)^2$$

(1)

defines the so-called Schwarzian derivative [1]-[5]. Originally, this object appeared in such areas of mathematics as differential equations, theory of conformal mapping, projective differential geometry (for a review see, e.g., [6, 7]).

The Schwarzian derivative holds invariance under the linear-fractional transformations

$$\rho'(t) = \frac{a\rho(t) + b}{c\rho(t) + d}, \quad \det A = ad - cb \neq 0.$$  

(2)

For real parameters $a$, $b$, $c$, and $d$, these transformations form an $SL^\pm(2, \mathbb{R})/\mathbb{Z}_2$ group. This property allows one to construct $SL(2, \mathbb{R})$-invariant model by using the Schwarzian derivative (1) as a Lagrangian:

$$S = \frac{1}{2} \int dt \{\rho(t), t\}.$$  

(3)

In the literature, the model (3) is referred to as the Schwarzian mechanics (SM) (see, e.g., [8]-[11]). Recently, the mechanics of such type has attracted considerable interest due to its relations to the low-energy limit of the so-called Sachdev-Ye-Kitaev model [12].

As it is well-known, the so-called de Alfaro-Fubini-Furlan (DAFF) mechanics [13, 14], which is described by the action functional

$$S = \int dt \phi(t) \left(\ddot{\phi}(t) + \frac{g^2}{\dot{\phi}^3(t)}\right),$$  

(4)

also reveals $SL(2, \mathbb{R})$-invariance. As it was noted in works [15]-[22], the action (4) takes the form

$$S = \int dt \left(\phi(t) \left(\ddot{\phi}(t) + \frac{g^2}{\dot{\phi}^3(t)}\right) + \frac{1}{2} \{\rho(t), t\} \phi^2(t)\right)$$

(5)

after applying the following change of temporal and dynamical variables

$$t \to \rho(t), \quad \phi(t) \to \sqrt{\dot{\rho}(t)} \cdot \phi(t).$$

(6)

1Here and in what follows a number of dots over a function of one variable as well as the upper superscript in braces, which is attached to the function, designate the number of derivatives with respect to this variable.

2It is so because two sets of parameters

$$(a, b, c, d) \text{ and } \left(\pm \frac{a}{\sqrt{|\det A|}}, \pm \frac{b}{\sqrt{|\det A|}}, \pm \frac{c}{\sqrt{|\det A|}}, \pm \frac{d}{\sqrt{|\det A|}}\right)$$

correspond to one and the same transformation (2).
If we formally put $\phi(t) = 1$ in (5), we arrive to the action of SM (3).

$$t \rightarrow \rho(t), \quad \phi(t) \rightarrow \sqrt{\dot{\rho}(t)}.$$  \hspace{1cm} (7)

connects actions of SM (3) and the DAFF model (4). This poses several natural questions:

1. Can such attributes of SM as the equation of motion, $SL(2, \mathbb{R})$-symmetry transformations and the corresponding conserved charges be derived from those of the DAFF model by applying the same coordinate transformation (7)?

2. A dimensionful coupling constant $g^2$ enters the DAFF mechanics (4). On the other hand, the action functional of SM (3) does not contain any dimensionful constants. What is the role of the constant $g^2$ for SM with respect to the transformation (7)?

3. Orders of differential equations that govern SM (3) and the DAFF model (4) are different. Are there any relations between the trajectories of these systems under the change of coordinates (7)?

4. As it is known, the DAFF model and SM can be obtained via the method of nonlinear realizations \[11, 14, 23\] (see also \[24\]). Can the transformation (7) be reproduced by using the same method?

5. If a function $\rho(t)$ in (7) satisfies the equation

$$\{ \rho(t), t \} = -2\Lambda, \quad \Lambda = \pm \frac{1}{R^2},$$  \hspace{1cm} (8)

the action (5) reproduces the analogue of the DAFF mechanics in Newton-Hooke (NH) spacetime with positive (upper sign) or negative (lower sign) cosmological constant $\Lambda$ \[25, 26\]. In this interpretation, $R$ is called the characteristic time. On the other hand, the well-known Niederer’s transformation, which has the form \[27\]

$$\begin{align*}
(a) & \quad \Lambda < 0 : t \rightarrow R \tan \frac{t}{R}, \quad \phi \rightarrow \frac{\phi}{\cos t/R}, \\
(b) & \quad \Lambda > 0 : t \rightarrow R \tanh \frac{t}{R}, \quad \phi \rightarrow \frac{\phi}{\cosh t/R},
\end{align*}$$  \hspace{1cm} (9)

also maps the DAFF mechanics to its NH analogue. Taking into account that the general solution of the equation (5) has three arbitrary constants of integration, it is natural to ask: does the transformation (6) provide a more general construction than Niederer’s transformation (9)?

\footnote{We may put $\phi(t) = c$, where $c$ is an arbitrary constant. Then (5) takes the form $S = \frac{c^2}{2} \int dt \{ \rho(t), t \}$. When the low energy limit of the SYK model is considered, $c^2$ place the role of a coupling constant in the theory (see, e.g., \[8\]). We thank A. Galajinsky for pointing this out to us.}
6. Do Schwarzain counterparts of other conformally invariant systems exist? For example, the DAFF mechanics in NH spacetime mentioned above also exhibits conformal invariance. What happens if we apply the coordinate transformation (7) to this system? In the work [33], a higher derivative generalization of the DAFF model has been constructed. Can the transformation (7) be generalized to the case of higher derivative DAFF mechanics?

The purpose of the present work is to investigate relationships between SM and the DAFF model and to answer the questions listed above.

The paper is organized as follows. In the next section, we show that the correspondence between SM and the DAFF model with respect to (7) takes place only if the energy of SM is equal to the coupling constant of the DAFF model. We establish that in this case, the transformation (7) links the equations of motion as well as the conserved charges. While the general solution of the DAFF mechanics transforms into a velocity function of SM.

In Sect. 3, we show that some particular solutions of the equation (8), when transformed under (7), reproduce Niederer’s transformation (9). We establish that the desire to have an identical transformation in the limit $R \to \infty$ and the invariance of the DAFF model and its NH analogue under time translations unambiguously fix these particular solutions.

In Sect. 4, the equation of motion as well as a geometric description of SM associated with the DAFF model in NH spacetime are derived via the method of nonlinear realizations. The same method is also applied to reproduce the transformation (7).

In Sect. 5, we consider a higher derivative generalization of the DAFF mechanics. We modify transformation (7) so as to obtain the Schwarzian counterpart for this model. In the concluding Sect. 6 we summarize our results and discuss further possible developments.

2. Relations between the DAFF model and SM

2.1. The Schwarzian mechanics

Let us recall some basic facts about SM [3]. The dynamics of this system is governed by the following equation of motion

$$\frac{1}{2\dot{\rho}} \left( \dot{\rho}^4 - \frac{4\dot{\rho}^{(3)}}{\rho^2} + \frac{3\dot{\rho}^{(3)}}{\rho^3} \right) = \frac{1}{2\dot{\rho}} \frac{d}{dt} \{\rho(t), t\} = 0.$$  

Integration with respect to $t$ yields

$$\{\rho(t), t\} = 2\lambda,$$

where $\lambda$ is a constant.

The infinitesimal form of $SL(2, \mathbb{R})$ transformations (2) reads

$$t' = t, \quad \rho'(t') = \rho(t) + \alpha_1 + \alpha_0 \rho(t) + \alpha_1 \rho^2(t),$$

The transformations correspond to the following choice of parameters $(a, b, c, d)$ in (2): translations $\to (1, \alpha_1, 0, 1)$, dilatations $\to (e^{\alpha_0/2}, 0, 0, e^{-\alpha_0/2})$, special conformal transformations $\to (1, 0, -\alpha_1, 1)$. 

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where constants $\alpha_{-1}$, $\alpha_0$ and $\alpha_1$ are infinitesimal parameters. One more symmetry transformation of SM reflects with the invariance of the action functional (3) under time translations:

$$t' = t + \sigma, \quad \rho'(t') = \rho(t). \quad (12)$$

The generators, which correspond to the transformations (11), (12)

$$H = i \frac{\partial}{\partial t}, \quad L_{-1} = i \frac{\partial}{\partial \rho}, \quad L_0 = i \rho \frac{\partial}{\partial \rho}, \quad L_1 = i \rho^2 \frac{\partial}{\partial \rho}, \quad (13)$$

obey commutation relations of $sl(2, \mathbb{R}) \oplus \mathbb{R}$ Lie algebra

$$(a) : [H, L_n] = 0, \quad (b) : [L_n, L_m] = i(m - n)L_{n+m}. \quad (14)$$

Taking into account Eq. (10), the conserved charges associated with symmetry transformations (11) can be written as

$$\mathcal{L}_{-1} = \frac{1}{\dot{\rho}(t)} \left( \lambda + \left( \frac{\dot{\rho}(t)}{2\dot{\rho}(t)} \right)^2 \right), \quad \mathcal{L}_0 = \rho(t)\mathcal{L}_{-1} - \frac{\dot{\rho}(t)}{2\dot{\rho}(t)},$$

$$\mathcal{L}_1 = -\rho^2(t)\mathcal{L}_{-1} + 2\rho(t)\mathcal{L}_0 + \dot{\rho}(t), \quad \mathcal{H} = \mathcal{L}_{-1}\mathcal{L}_1 - \mathcal{L}_0^2 = \lambda.$$ 

Here and in what follows we designate constants of the motion by the same letters as the corresponding symmetry generators but in a calligraphic style.

Conserved charges (15) allow one to express $\dot{\rho}(t)$ in terms of these integrals of motion and $\rho(t)$:

$$\dot{\rho}(t) = \rho^2(t)\mathcal{L}_{-1} - 2\rho(t)\mathcal{L}_0 + \mathcal{L}_1 = \frac{(\rho(t)\mathcal{L}_{-1} - \mathcal{L}_0)^2 + \lambda}{\mathcal{L}_{-1}}. \quad (16)$$

As a consequence, one may readily find the general solution to the equation (10). For $\mathcal{L}_{-1} \neq 0$, this solution has the form

$$\rho(t) = \begin{cases} 
\frac{\mathcal{L}_0}{\mathcal{L}_{-1}} - \frac{\sqrt{\lambda}}{\mathcal{L}_{-1}\tan(\sqrt{\lambda}(t - C))}, & \lambda > 0, \\
\frac{\mathcal{L}_0}{\mathcal{L}_{-1}} - \frac{1}{\mathcal{L}_{-1}(t - C)}, & \lambda = 0, \\
\frac{\mathcal{L}_0}{\mathcal{L}_{-1}} - \frac{\sqrt{-\lambda}}{\mathcal{L}_{-1}\tanh(\sqrt{-\lambda}(t - C))}, & \lambda < 0,
\end{cases} \quad (17)$$

$\text{^5}$If the action functional $S = \int dt L(x_i(t), \dot{x}_i(t), ..., x_i^{(N)}(t))$ holds invariant under transformations of the form $t' = t + \delta t, \quad x'_i(t') = x_i(t) + \delta x_i$ up to a total time derivative of some function $F = F(t)$, i.e. $\delta S = \int dt \frac{dF}{dt}$, then the corresponding integral of motion can be derived by using the expression

$$L\delta t + \sum_{n=0}^{N-1} \frac{d^n}{dt^n}(\delta x_i - \dot{x}_i\delta t) \sum_{k=0}^{N-n-1} (-1)^k \frac{d^k}{dt^k} \frac{\partial L}{\partial x_i^{n+k+1}} - F.$$ 

$\text{^6}$About some systems whose Hamiltonians coincide with the $sl(2, \mathbb{R})$-Casimir operator see, e.g., [28].
where \( C \) is a constant of integration.

Let us discuss the case when the energy \( \mathcal{H} = \lambda \) of the system (3) is positive. Then according to the expression for conserved charge \( \mathcal{L}_{-1} \) in (15), all solutions of the equation (10) can be divided into two disjoint parts. The first part is related to the positive-definite velocity functions \( \dot{\rho}(t) \) (for negative \( \mathcal{L}_{-1} \)) while the second one corresponds to the negative-definite \( \dot{\rho}(t) \) (for negative \( \mathcal{L}_{-1} \)). Each member of any part can be mapped into some instance of another by reflection

\[
\rho(t) \rightarrow -\rho(t). \tag{18}
\]

For this discrete symmetry of SM, the \( SL(2, \mathbb{R}) \)-conserved charges in (15) are transformed as follows

\[
\mathcal{L}_n \rightarrow (-1)^n \mathcal{L}_n. \tag{19}
\]

Analogous change of \( sl(2, \mathbb{R}) \)-generators does not change the structure relations of the algebra (14), i.e. it is an automorphism of this algebra.

It is natural to suggest that the coordinate transformation (7) is appropriate for dealing with solutions of SM which correspond to positive-definite velocity functions. While solutions with \( \dot{\rho}(t) < 0 \) can be treated with the aid of the change

\[
t \rightarrow -\rho(t), \quad \phi(t) \rightarrow \sqrt{-\dot{\rho}(t)}, \tag{20}
\]

which can be obtained from (7) by applying (18).

### 2.2. A link between the DAFF model and SM

As is known [13], \( sl(2, \mathbb{R}) \)-symmetry transformations, which leave invariant the DAFF mechanics (1), have the form

(a) : \( t' = t + \alpha_{-1} + \alpha_0 t + \alpha_1 t^2 \), \hspace{1cm} (b) : \( \phi'(t') = \phi(t) + \frac{\alpha_0}{2} \phi(t) + \alpha_1 t \phi(t) \). \tag{21}

For these symmetries, the Noether theorem yields the following constants of the motion

\[
\mathcal{L}_{-1} = \dot{\phi}^2(t) + \frac{g^2}{\phi^2(t)}, \quad \mathcal{L}_0 = t\mathcal{L}_{-1} - \dot{\phi}(t)\dot{\phi}(t), \quad \mathcal{L}_1 = -t^2\mathcal{L}_{-1} + 2t\mathcal{L}_0 + \phi^2(t). \tag{22}
\]

The integral of motion associated with the Casimir operator of \( sl(2, \mathbb{R}) \) is equal to the coupling constant \( g^2 \):

\[
\mathcal{H} = \mathcal{L}_{-1}\mathcal{L}_1 - \mathcal{L}_0^2 = g^2. \tag{23}
\]

So, the conserved charges (22) are functionally dependent.
The equation of motion of the DAFF mechanics (4)

\[ \phi^3(t)\ddot{\phi}(t) = g^2 \]  

(24)
can be solved by using the integrals of motion (22). Indeed, by using the identity

\[ \phi^2(t) = t^2\mathcal{L}_{-1} - 2t\mathcal{L}_0 + \mathcal{L}_1 \]

and by taking into account the relation (23), one readily obtains

\[ \phi^2(t) = \frac{(t\mathcal{L}_{-1} - \mathcal{L}_0)^2 + g^2}{\mathcal{L}_{-1}}. \]  

(25)

Let us discuss, in which way the symmetry transformations (21), the equation of motion (24), the conserved charges (22) and the solution (25) are transformed under (7) for \( \dot{\rho}(t) > 0 \) (or (20)). Firstly, it can be verified that (21a) and (21b) are mapped into (11)

\[ \dot{\rho}'(t) = \dot{\rho}(t) + \alpha_0\dot{\rho}(t) + 2\alpha_1\rho(t)\dot{\rho}(t), \]

respectively. The latter is a consequence of (11).

Secondly, the equation of motion of the DAFF model (24) is linked to the equation of SM (10) only if \( \lambda = g^2 \). This means that the correspondence between equations of motion exists only if the energy of SM is equal to the coupling constant in the DAFF model. The same restriction must be satisfied so that the conserved charges of SM (15) can be derived from those of the DAFF model (22).

Thirdly, the general solution of the DAFF model (25) is mapped to the expression for the velocity function (16) of SM. So, when the correspondence (7), (20) is considered, constants \( \mathcal{L}_{-1}, \mathcal{L}_0 \) and \( g^2 \) of the DAFF system fix three constants of integration in the general solution of SM (17). While the fourth constant \( C \) has no DAFF prototype. At least, for this reason, the coordinate transformations (7), (20) do not provide a one-to-one correspondence between the DAFF model and SM: one DAFF trajectory corresponds to infinitely many trajectories in SM.

3. DAFF mechanics in NH spacetime and its Schwarzian counterpart

3.1. Niederer’s transformation for the DAFF model from SM

Let us consider the action functional

\[ S = \int dt \phi(t) \left( \ddot{\phi}(t) + \frac{g^2}{\phi^3(t)} - \Lambda \phi(t) \right), \]  

(26)

where \( \Lambda \) is defined in (3). This action functional describes the analogue of the DAFF model (1) in Newton-Hooke (NH) space-time with a universal cosmological attraction (\( \Lambda < 0 \)) or
repulsion \((\Lambda > 0)\) \cite{25, 26}. As is well known, the action (26) can be derived from (4) by applying the so-called Niederer transformation \cite{27}. For the case of a negative cosmological constant, this transformation is given by \cite{9, 8}.

As was mentioned in the Introduction, the action functional \((3)\) coincides with \((26)\) if the function \(\rho(t)\) in \((5)\) satisfies the equation \((8)\). According to \((17)\), this restriction can be met if

\[
\rho(t) = \frac{L_0}{L_{-1}} - \frac{1}{RL_{-1}} \cot \left( \frac{t - C}{R} \right).
\]

This solution results in the following coordinate transformations

\[
t \rightarrow \pm \frac{L_0}{L_{-1}} + \frac{1}{RL_{-1}} \cot \left( \frac{t - C}{R} \right), \quad \phi(t) \rightarrow \frac{\phi(t)}{\sqrt{\pm L_{-1}R \sin (t - C)/R}},
\]

after taking into account \((7)\) (upper sign) and \((20)\) (lower sign). At first sight, this transformation may be more general than Niederer’s one \((9)\). But the presence of the constants \(L_0/L_{-1}\) and \(C\) can be associated with the invariance of the DAFF model \((1)\) and its NH analogue \((20)\) under time translations. Because of this, we may put

\[
L_0 = 0, \quad C = -\frac{\pi R}{2} \quad \Rightarrow \quad t \rightarrow \pm \frac{1}{RL_{-1}} \tan \left( \frac{t}{R} \right), \quad \phi(t) \rightarrow \frac{\phi(t)}{\sqrt{\pm L_{-1}R \cos t/R}},
\]

without loss generality, while the constant \(L_{-1}\) is fixed if we require that the transformation is identical in the limit \(R \rightarrow \infty\):

\[
L_{-1} = \pm \frac{1}{R^2}.
\]

Then we arrive to the Niederer’s transformation \((9)\).

Relations between the general solution of SM \((17)\) and Niederer’s transformation for \(\Lambda > 0\) \((9)\) can be established from the analysis above for \(\Lambda < 0\) by implementing the formal change of characteristic time

\[
R \rightarrow iR.
\]

In what follows a consideration of the DAFF mechanics in NH spacetime will be restricted by the discussion of the case of a negative cosmological constant only. While the case of a positive cosmological constant can be treated by implementing the same change of the characteristic time \((29)\).

\(^7\)About another transformations which connect the systems \((1)\) and \((20)\) see also \cite{29, 30}.

\(^8\)When \(g^2 = 0\), the transformation \((9)\) relates the motion of a free particle to a half-period of harmonic oscillator.
3.2. The DAFF mechanics in NH spacetime

Let us consider the model (26) for the case of a negative cosmological constant \[13\]

\[ S = \int dt \phi(t) \left( \ddot{\phi}(t) + \frac{g^2}{\phi^3(t)} + \frac{\phi(t)}{R^2} \right). \] \hspace{1cm} (30)

The equation of motion of this system is given by

\[ \ddot{\phi}(t) = \frac{g^2}{\phi^3(t)} - \frac{\phi(t)}{R^2}. \] \hspace{1cm} (31)

The model (30) holds invariance under symmetry transformations

\[ t' = t + \alpha_1 + \frac{R}{2} \sin \frac{2t}{R} \alpha_0 + R^2 \sin^2 \frac{t}{R} \alpha_1, \] \hspace{1cm} (32)

\[ \phi'(t') = \phi(t) + \frac{1}{2} \cos \frac{2t}{R} \phi(t) \alpha_0 + \frac{R}{2} \sin \frac{2t}{R} \phi(t) \alpha_1, \] \hspace{1cm} (33)

whose generators read

\[ L_{-1} = i \frac{\partial}{\partial t}, \quad L_0 = i \frac{R}{2} \sin \frac{2t}{R} \frac{\partial}{\partial t} + i \frac{1}{2} \cos \frac{2t}{R} \frac{\partial}{\partial \phi}, \quad L_1 = i R^2 \sin^2 \frac{t}{R} \frac{\partial}{\partial t} + i \frac{R}{2} \sin \frac{2t}{R} \frac{\partial}{\partial \phi}. \] \hspace{1cm} (34)

These generators form \(sl(2, \mathbb{R})\) Lie algebra with commutation relations

\[ [L_{-1}, L_0] = i \left( L_{-1} - \frac{2}{R^2} L_1 \right), \quad [L_{-1}, L_1] = 2i L_0, \quad [L_0, L_1] = i L_1. \] \hspace{1cm} (35)

It is straightforward to verify that these relations can be obtained from (35b) by implementing the linear change of the basis

\[ L_{-1} \rightarrow L_{-1} + \frac{1}{R^2} L_1. \] \hspace{1cm} (36)

The Noether theorem yields the following conserved charges associated with \(sl(2, \mathbb{R})\) symmetry transformations (32)

\[ \mathcal{L}_{-1} = \dot{\phi}^2(t) + \frac{g^2}{\phi^2(t)} - \frac{\phi^2(t)}{R^2}, \quad \mathcal{L}_0 = \frac{R}{2} \sin \frac{2t}{R} \mathcal{L}_{-1} - \phi(t) \dot{\phi}(t) \cos \frac{2t}{R} - \frac{\phi^2(t)}{R} \sin \frac{2t}{R}, \]

\[ \mathcal{L}_1 = R^2 \sin^2 \frac{t}{R} \mathcal{L}_{-1} - \phi(t) \dot{\phi}(t) R \sin \frac{2t}{R} + \phi^2(t) \cos \frac{2t}{R}. \] \hspace{1cm} (37)

Taking into account the identity

\[ \phi^2(t) = \mathcal{L}_{-1} R^2 \sin^2 \frac{t}{R} - \mathcal{L}_0 R \sin \frac{2t}{R} + \mathcal{L}_1 \cos \frac{2t}{R}. \]
and the expression for the integral of motion which corresponds to the $sl(2,\mathbb{R})$-Casimir operator

$$\mathcal{H} = \mathcal{L}_1 - \mathcal{L}_0^2 - \frac{\mathcal{L}_1^2}{R^2} = g^2,$$

one obtains the general solution to the equation (31)

$$\phi^2(t) = \left(\mathcal{L}_1 \cos \frac{t}{R} - \mathcal{L}_0 R \sin \frac{t}{R}\right)^2 + g^2 R^2 \sin^2 \frac{t}{R}. \quad (38)$$

As was mentioned above, the action functional (30) can be obtained from the action of the DAFF mechanics (4) by applying Niederer’s transformation (9a). By the same argument, the equation of motion (31), the symmetry transformations (32) and the associated integrals of motion (37) of the model (26) can be obtained from corresponding expressions of the DAFF mechanics (4).

### 3.3. SM associated with the DAFF model in NH spacetime

Applying the coordinate transformation (7) to the action functional (30) gives

$$S = \frac{1}{2} \int dt \left( \{\rho(t),t\} + \frac{2\dot{\rho}^2(t)}{R^2} \right). \quad (39)$$

The dynamics of this model is governed by the equation of motion

$$\frac{1}{2\dot{\rho}} \frac{d}{dt} \left( \{\rho(t),t\} + \frac{2\dot{\rho}^2(t)}{R^2} \right) = 0. \quad (40)$$

Integration with respect to time yields

$$\{\rho(t),t\} + \frac{2\dot{\rho}^2(t)}{R^2} = 2\lambda, \quad (41)$$

where $\lambda$ is a constant of integration.

As is known, $sl(2,\mathbb{R})\oplus\mathbb{R}$-symmetry transformations, which leave the model (39) invariant, read

$$t' = t + \sigma, \quad \rho'(t') = \rho(t) + \alpha_{-1} + \frac{R}{2} \sin \frac{2\rho(t)}{R} \alpha_0 + R^2 \sin^2 \frac{\rho(t)}{R} \alpha_1. \quad (42)$$

Taking into account the equation (41), the corresponding integrals of motion can be written as follows:

$$\mathcal{L}_1 = \frac{1}{\rho} \left( \lambda + \left(\frac{\ddot{\rho}}{2\dot{\rho}}\right)^2 + \frac{\dot{\rho}^2}{R^2} \right), \quad \mathcal{L}_0 = \frac{R}{2} \sin \frac{2\rho}{R} \left( \mathcal{L}_1 - \frac{2\dot{\rho}}{R^2} \right) - \frac{1}{2} \cos \frac{2\rho}{R} \frac{\dot{\rho}}{\dot{\rho}},$$

$$\mathcal{H} = \mathcal{L}_1 - \mathcal{L}_0^2 - \frac{\mathcal{L}_1^2}{R^2} = \lambda, \quad \mathcal{L}_1 = R^2 \sin^2 \frac{\rho}{R} \left( \mathcal{L}_1 - \frac{2\dot{\rho}}{R^2} \right) - \frac{R}{2} \sin \frac{2\rho}{R} \dot{\rho} + \dot{\rho}. \quad (43)$$
These constants of the motion allow one to determine $\dot{\rho}$ by purely algebraic means

$$\dot{\rho} = \frac{(L_1 \cos \frac{\rho}{R} - L_0 R \sin \frac{\rho}{R})^2 + \lambda R^2 \sin^2 \frac{\rho}{R}}{L_1}. \quad (44)$$

Then one readily obtains the dynamics of $\rho(t)$

$$\rho(t) = \begin{cases} 
R \arctan \frac{L_1}{L_0 R - \sqrt{\lambda R} \tan (\sqrt{\lambda} t - C)}, & \lambda > 0, \\
R \arctan \frac{L_1(t - C)}{R(1 + L_0(t - C))}, & \lambda = 0, \\
R \arctan \frac{L_1}{L_0 R - \sqrt{-\lambda R} \tanh (\sqrt{-\lambda} t + C)}, & \lambda < 0, \end{cases}$$

where $C$ is a constant of integration.

As is known, the model (39) can be obtained from SM (3) by applying the coordinate transformation of the form

$$\rho \to R \tan \frac{\rho}{R}. \quad (45)$$

In this sense, (39) can be viewed as an NH analogue of (3). On the other hand, it can be verified that, if the energy of the system (39) is equal to the coupling constant $g^2$ in the model (30), expressions (41), (42), (43), (44) can be derived from those of the DAFF mechanics in the NH spacetime (30) with the aid of the change (7). In this sense, the model (39) can be viewed as the Schwarzian counterpart of the system (30).

4. Geometric approach to SM

4.1. Maurer-Cartan one-forms and the equation of motion of SM

Following Ref. [11], let us consider the Lie algebra $sl(2, \mathbb{R}) \oplus \mathbb{R}$ which involves the generators $H, L_{-1}, L_0, \text{ and } L_1$. The structure relations of the algebra involve (14a) and (35). To obtain the equation of motion of SM (39), one considers the coset space

$$G(t, \rho, s, u) = e^{itH} e^{i\rho L_{-1}} e^{isL_1} e^{iuL_0}, \quad (46)$$

parameterized by coordinates $t, \rho, s, \text{ and } u$. Left multiplication by an element $G(\sigma, \alpha_{-1}, \alpha_1, \alpha_0)$ with infinitesimal parameters $\sigma, \alpha_{-1}, \alpha_0, \text{ and } \alpha_1$ determines the action of the group on the coset space (46):

$$G(\sigma, \alpha_{-1}, \alpha_1, \alpha_0) \cdot G(t, \rho, s, u) = G(t + \delta t, \rho + \delta \rho, s + \delta s, u + \delta u),$$

where

$$G(t, \rho, s, u) = e^{itH} e^{i\rho L_{-1}} e^{isL_1} e^{iuL_0}, \quad (46)$$

is the coset space parameterized by coordinates $t, \rho, s, \text{ and } u$. Left multiplication by an element $G(\sigma, \alpha_{-1}, \alpha_1, \alpha_0)$ with infinitesimal parameters $\sigma, \alpha_{-1}, \alpha_0, \text{ and } \alpha_1$ determines the action of the group on the coset space (46):

$$G(\sigma, \alpha_{-1}, \alpha_1, \alpha_0) \cdot G(t, \rho, s, u) = G(t + \delta t, \rho + \delta \rho, s + \delta s, u + \delta u),$$

where $\delta t, \delta \rho, \delta s, \text{ and } \delta u$ are infinitesimal changes.
where
\[ \delta t = \sigma, \quad \delta \rho = \alpha_{-1} + \frac{R}{2} \sin 2\rho \alpha_0 + R^2 \sin^2 \frac{\rho}{R} \alpha_1, \quad \delta u = \cos \frac{2\rho}{R} \alpha_0 + R \sin \frac{2\rho}{R} \alpha_1, \]
\[ \delta s = -\left( s \cos \frac{2\rho}{R} + \frac{1}{R} \sin \frac{2\rho}{R} \right) \alpha_0 + \left( \cos \frac{2\rho}{R} - sR \sin \frac{2\rho}{R} \right) \alpha_1. \]
(47)

Then let us construct the Maurer-Cartan one-forms:
\[ G^{-1}dG = i(\omega_H H + \omega_{-1} L_{-1} + \omega_0 L_0 + \omega_1 L_1), \]
(48)

where
\[ \omega_H = dt, \quad \omega_{-1} = e^{-u} d\rho, \quad \omega_0 = du - 2s d\rho, \quad \omega_1 = e^u (ds + s^2 d\rho) + \frac{2 \sinh u}{R^2} d\rho. \]
(49)

At the next step, let us put \( \rho = \rho(t), u = u(t), s = s(t) \) and impose restrictions
\[ \omega_n = \nu_n \omega_H, \quad n = -1, 0, 1, \]
(50)

where \( \nu_n \) are constants\(^9\). Two constraints allow one to eliminate the fields \( u = u(t) \) and \( s = s(t) \) from the consideration:
\[ \omega_{-1} = \nu_{-1} \omega_H \quad \Rightarrow \quad e^{-u} = \frac{\nu_{-1}}{\rho}; \]
\[ \omega_0 = \nu_0 \omega_H \quad \Rightarrow \quad s = \frac{\dot{\rho}}{2\rho^2} - \frac{\nu_0}{2\rho}. \]
(51)

The third restriction yields the equation (41) with
\[ \lambda = \nu_{-1} \nu_1 + \frac{\nu_{-1}^2}{R^2} - \frac{\nu_0^2}{4}. \]
(52)

Finally, (47) yields the symmetry transformations of SM (42).

### 4.2. A geometric description of SM

Following Ref. [14], let us consider the matrix representation of the \( sl(2,\mathbb{R}) \) algebra in NH basis \(^{33}\)
\[ L_{-1} = \begin{pmatrix} 0 & 1 \\ 1/R^2 & 0 \end{pmatrix}, \quad L_0 = i/2 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \]

\(^9\)In the literature conditions of such type are also referred to as the inverse Higgs constraints [31, 32].
Let us construct the 3d metric
\[ ds^2 = -Tr (L_n \omega_n)^2 = 2\omega_{-1} \omega_1 - \frac{\omega_0^2}{2} + \frac{2\omega_1^2}{R^2} = 2d\rho(ds + sdu) - \frac{1}{2} du^2 + \frac{2}{R^2} d\rho^2, \] (53)
where the Maurer-Cartan one forms \( \omega_n \) are given in (49). It describes an Einsteinian space-time because the corresponding Ricci tensor is equal to the metric.

The geodesic equations, which correspond to (53), read
\[
\begin{align*}
(\text{a}) : \ddot{\rho}(t) - \dot{\rho}(t) \dot{u}(t) &= 0, \\
(\text{b}) : \ddot{u}(t) - 2\dot{\rho}(t)\dot{s}(t) - 2s(t)\dot{\rho}(t)\dot{u}(t) &= 0, \\
(\text{c}) : \ddot{s}(t) + 2s(t)\dot{\rho}(t)\dot{s}(t) + 2 \left( s^2(t) + \frac{1}{R^2} \right) \dot{\rho}(t)\dot{s}(t) + \dot{s}(t)\dot{u}(t) &= 0,
\end{align*}
\] (54)
where a proper time is denoted by \( t \). Equations (54a) and (54b) allow one to express \( \dot{u}(t) \) and \( s(t) \) in terms of \( \dot{\rho}(t) \) and \( \ddot{\rho}(t) \):
\[
\dot{u}(t) = \frac{\ddot{\rho}(t)}{\dot{\rho}(t)}, \quad s = \frac{\dot{s}(t)}{2\dot{\rho}^2} - \frac{\nu_0}{2\dot{\rho}},
\]
where \( \nu_0 \) is a constant of integration, while Eq. (54b) leads to (41).

On the other hand, the equation (54a) after one integration yields
\[
\dot{\rho}(t) = \mu_1 e^{u(t)},
\] (55)
where \( \mu_1 \) is an arbitrary constant. This relation enables one to rewrite (54b) and (54c) as follows:
\[
\begin{align*}
(\text{a}) : \ddot{u}(t) - 2\mu_1 e^{u(t)}\dot{s}(t) - 2\mu_1 s e^{u(t)}\dot{u}(t) &= 0, \\
(\text{b}) : \ddot{s}(t) + 2\mu_1 e^{u(t)}s(t)\dot{s}(t) + 2\mu_1 e^{u(t)}\dot{u}(t) \left( s^2(t) + \frac{1}{R^2} \right) + \dot{s}(t)\dot{u}(t) &= 0.
\end{align*}
\] (56)

After one integration, Eq. (56a) takes the form
\[
\dot{u}(t) - 2\mu_1 e^{u(t)}s(t) = \mu_2 \Rightarrow s(t) = \frac{1}{2\mu_1} e^{-u(t)} \dot{u}(t) - \frac{\mu_2}{2\mu_1} e^{-u(t)},
\] (57)
where \( \mu_2 \) is also an arbitrary constant. By taking \( \rho \) as an independent variable instead of \( t \), one rewrites (57) in the following way:
\[
s(\rho) = \frac{1}{2} \frac{du(\rho)}{d\rho} - \frac{\mu_2}{2\mu_1} e^{-u(\rho)}. 
\]

At the same time (56b) gives
\[
\frac{1}{2} \frac{d^2u(\rho)}{d\rho^2} + \frac{1}{4} \left( \frac{du(\rho)}{d\rho} \right)^2 + \frac{1}{R^2} = \frac{\lambda}{\mu_1^2} e^{-2u(\rho)}. 
\] (58)
Introducing the new variable
\[ \phi(\rho) = \sqrt{|\dot{\rho}(t)|} = \sqrt{|\mu_1| e^{u(\rho)/2}} \]  \hfill (59)
one can rewrite the equation \[ (58) \] as follows:
\[ \frac{d^2 \phi(\rho)}{d\rho^2} = \frac{\lambda}{\phi^3(\rho)} - \frac{\phi(\rho)}{R^2}. \]  \hfill (60)

The relation \[ (59) \] and the change of independent variable \( t \rightarrow \rho \) reproduce the transformation \[ (7) \].

5. Schwarzian counterparts of higher derivative DAFF mechanics

In Ref. [33] it was shown that the model, which is described by the action functional
\[ S = \int dt \left( \lambda_{ij} \phi_i \phi_j^{(2l+1)} + \frac{2g^{2l+1}}{\phi_i \phi_j^{1/2l}} \right), \quad \lambda_{ij} = \begin{cases} \delta_{ij} & i, j = 1, 2, \ldots, d, \text{ for half-integer } l, \\ \epsilon_{ij} & i, j = 1, 2, \text{ for integer } l, \end{cases} \]  \hfill (61)
where \( \epsilon_{ij} \) is the Levi-Civitá symbol, exhibits \( \text{sl}(2, \mathbb{R}) \) symmetry. The case of \( l = 1/2 \) corresponds to the \( d \)-dimensional DAFF model, while the action \[ (61) \] for other values of \( l \) can be viewed as \( (2l + 1) \)-order derivative generalization of the DAFF model \[ (4) \]. The Lagrangian formulation for the odd-order DAFF mechanics is known only for \( d = 2 \).

5.1. Schwarzian version of one-dimensional third-order DAFF mechanics

The action functional of the third-order DAFF mechanics reads
\[ S = \int dt \left( \epsilon_{ij} \phi_i \phi_j^{(2l+1)} + \frac{2g^3}{\sqrt{\phi_i \phi_j}} \right). \]  \hfill (62)
By analogy with \[ (6) \], let us consider the following coordinate transformation:
\[ t \rightarrow \rho(t), \quad \phi_i(t) \rightarrow \dot{\rho}(t) \phi_i(t), \]  \hfill (63)
application of which to \[ (62) \] results in
\[ S = \int dt \left( \epsilon_{ij} \phi_i \phi_j^{(2l+1)} + \frac{2g^3}{\sqrt{\phi_i \phi_j}} + 2 \{ \rho(t), t \} \epsilon_{ij} \phi_i \phi_j \right). \]  \hfill (64)

\footnote{As it was shown in Ref. [34], the model \[ (61) \] with vanishing potential function reveals the so-called \( l \)-conformal Galilei symmetry \[ [35, 36] \].}
If the function $\rho(t)$ obeys the equation (8), this action describes the third-order Pais-Uhlenbeck (PU) oscillator \cite{37, 38} in the presence of a potential function which preserves $sl(2, \mathbb{R})$-symmetry. It is straightforward to verify that Niederer’s transformation \cite{36}-\cite{40}

\[ t \to R \tan \frac{t}{R}, \quad \phi \to \frac{\phi}{\cos^2 \frac{t}{R}}. \]

can be obtained from the solution of SM (17) in the same manner as it was made in subsection 3.1.

If we put $\phi_i(t) = c_i$ in (64), where $c_i$ is a constant vector, then this action functional vanishes. Therefore one cannot obtain the Schwarzian counterpart for the two-dimensional third-order DAFF mechanics. Yet, let us consider a one-dimensional non-Lagrangian analogue of the model (62). The corresponding equation of motion reads

\[ \ddot{\phi}(t) = \frac{g^3}{\dot{\phi}^2(t)}. \] (65)

This equation takes the form

\[ \frac{d}{dt} \{ \rho(t), t \} = g^3, \] (66)

after applying the transformation

\[ t \to \rho(t), \quad \phi(t) \to \dot{\rho}(t). \]

In this sense, (66) can be viewed as the Schwarzian counterpart of (65).

**5.2. Schwarzian version of the fourth-order DAFF mechanics**

Let us consider the one-dimensional fourth-order DAFF mechanics, which corresponds to the case of $l = 3/2$ and $d = 1$ in (61)

\[ S = \int dt \left( \phi(t) \phi^{(4)}(t) + \frac{3g^4}{(\phi(t))^{2/3}} \right). \] (67)

This model is governed by the following equation of motion

\[ \phi^{(4)}(t) = \frac{g^4}{(\phi(t))^{5/3}}. \] (68)

The invariance of the action functional (67) under $sl(2, \mathbb{R})$-transformations

\[ t' = t + \alpha_{-1} + \alpha_0 t + \alpha_1 t^2, \quad \phi'(t') = \phi(t) + \frac{3}{2} \alpha_0 \phi(t) + 3t \alpha_1 \phi(t), \] (69)
yields the following conserved charges
\[ L_{-1} = 2\dot{\phi}(t)\ddot{\phi}(t) - \dot{\phi}^2(t) + \frac{3g^4}{\sqrt{(\phi(t))^2}}, \]
\[ L_0 = tL_{-1} - 3\dot{\phi}(t)\ddot{\phi}(t) + \dot{\phi}(t)\ddot{\phi}(t), \]
\[ L_1 = -t^2L_{-1} + 2tL_0 + 6\dot{\phi}(t)\ddot{\phi}(t) - 4\dot{\phi}^2(t). \]

By analogy with (7), let us consider the transformation
\[ t \rightarrow \rho(t), \quad \phi(t) \rightarrow \sqrt{\dot{\rho}^3(t)}, \]
where we assume that \( \dot{\rho}(t) > 0 \). Then the action (67) takes the form
\[ S = \frac{9}{4} \int dt \left\{ \rho(t), t \right\}^2. \]

The dynamics of this system is described by the equation of motion
\[ \frac{1}{\rho(t)} \frac{d}{dt} \left( \frac{3}{2} \frac{d^2}{dt^2} \left\{ \rho(t), t \right\} + \frac{9}{4} \left\{ \rho(t), t \right\}^2 \right) = 0 \Rightarrow \frac{d^2}{dt^2} \left\{ \rho(t), t \right\} + \frac{3}{2} \left\{ \rho(t), t \right\}^2 = \frac{2\lambda}{3}, \]
where \( \lambda \) is a constant of integration.

It is evident that the model (72) holds invariance under \( sl(2, \mathbb{R}) \)-transformations (11). The same transformations are derived from (69) by applying the change (71). It can also be verified that if one sets
\[ \lambda = g^4, \]
the integrals of motion associated with \( SL(2, \mathbb{R}) \)-symmetry in the model (72) can be derived from (70) with the aid of (71).

When the condition (74) is satisfied, (73) takes the form
\[ \ddot{y}(t) - 6y^2(t) = -\frac{g^4}{6}, \]
where
\[ y(t) = -\frac{1}{4} \left\{ \rho(t), t \right\}. \]

The order of this equation can be reduced by implementing the change
\[ \dot{y}(t) = p(y(t)) \Rightarrow p(y) \frac{dp(y)}{dy} = 6y^2 = -\frac{g^4}{6}. \]

After one integration one obtains
\[ \dot{y}^2(t) = 4y^3(t) - \frac{g^4}{3}y(t) + C_1, \]
where \( C_1 \) is a constant of integration. The solution of this equation is the Weierstrass elliptic function \( y(t) = \wp(t + C_2; g^4/3; C_1) \) (see, e.g., [41]). So, (73) leads to

\[
\{ \rho(t), t \} = -4\wp \left( t + C_2; g^4/3; C_1 \right),
\]

integration of which is problematic.

On the other hand, any function \( \rho = \rho(t) \) which satisfies

\[
\{ \rho(t), t \} = \frac{2g^2}{3},
\]

is a particular solution to (73). Therefore according to (17) we have the following class of particular solutions to the equation (73)

\[
\rho(t) = \frac{A_0}{A_{-1}} - \frac{\tilde{g}}{A_{-1} \tan(\tilde{g}(t - C))},
\]

where \( A_{-1} \) and \( A_0 \) are arbitrary constants and \( \tilde{g} = g/\sqrt{3} \). The velocity function, related to this class of solutions, reads

\[
\dot{\rho}(t) = \frac{(\rho(t)A_{-1} - A_0)^2 + \tilde{g}^2}{A_{-1}}.
\]

Taking into account the coordinate transformation (71), we immediately obtain a class of particular solutions to (68)

\[
\phi^2(t) = \left( \frac{(tA_{-1} - A_0)^2 + \tilde{g}^2}{A_{-1}} \right)^3.
\]

Note that \( \lambda \) in (9) has the meaning of the energy of the system (72). So, the correspondence between (72) and (67) exists only if the energy of the model (72) coincides with the coupling constant in (67).

5.3. Schwarzian analogues of one-dimensional higher derivative DAFF dynamics

The equation of motion, which corresponds to one-dimensional \((2l + 1)\)-order derivative DAFF mechanics, has the form

\[
\phi^{(2l+1)}(t) = \frac{g^{2l+1}}{\phi^{(l+1)/l}(t)}.
\]

To obtain Schwarzian counterparts of this equation, let us introduce the transformation

\[
t \rightarrow \rho(t), \quad \phi(t) \rightarrow (\dot{\rho}(t))^l.
\]
The equation, which arises as the result of applying (77) to (76), can be represented in the form
\[ l \cdot S_{2l+1}(\rho) = g^{2l+1}, \]  
(78)
where \( S_{2l+1} \) are defined by the following recurrent relation
\[
S_2(\rho) = \{\rho(t), t\}, \\
S_{2l+1}(\rho) = \frac{d}{dt}S_{2l}(\rho) + \frac{1}{4} \sum_{k=2}^{2l-1} \frac{(2l+1)!}{k!(2l-k+1)!} S_k(\rho)S_{2l-k+1}(\rho), \quad l \geq 1. 
\]  
(79)

The formal symbol \( \sum_{k=2}^{1} f(k) \), which appears for \( l = 1 \), is assumed to be equal to zero. The similar Schwarzian structures appeared in [42]-[44].

Eq. (78) for \( l = 1 \) and \( l = 3/2 \) reproduces (66) and (73), respectively, while, for example, for \( l = 2, l = 5/2, \) and \( l = 3 \) we have
\[
\ddot{S}_2(\rho) + 8S_2(\rho)\dot{S}_2(\rho) = \frac{g^5}{2}; \\
S_2^{(4)}(\rho) + \frac{31}{2}S_2(\rho)S_2(\rho) + 13S_2^2(\rho) + \frac{45}{4}S_2^3(\rho) = \frac{2g^6}{5}; \\
S_2^{(5)}(\rho) + 26S_2(\rho)S_2^{(3)}(\rho) + 59S_2(\rho)\dot{S}_2(\rho) + 144S_2^2(\rho)\dot{S}_2(\rho) = \frac{g^7}{3}.
\]

NH counterpart of (76) can be obtained by applying Niederer’s transformation
\[
\prod_{k=0}^{l-1/2} \left( \frac{d^2}{dt^2} + \frac{(2k+1)^2}{R^2} \right) \phi(t) = \frac{g^{2l+1}}{\phi^{(l+1)/l}(t)}, \quad \text{for half-integer } l, \\
\prod_{k=1}^{l} \left( \frac{d^2}{dt^2} + \frac{(2k)^2}{R^2} \right) \phi(t) = \frac{g^{2l+1}}{\phi^{(l+1)/l}(t)}, \quad \text{for integer } l.
\]

These equations describe the conformally invariant \((2l+1)\)-order PU oscillator [45]-[48] in the external field which preserves conformal symmetry. By applying the transformation (77), we obtain the Schwarzian analogues which read as in (78) with
\[
S_2(\rho) = \{\rho(t), t\} + \frac{2\rho^2(t)}{R^2},
\]
while \( S_{2l+1} \) for \( l > 1/2 \) are given by the same recurrent relation (79).

It should be mentioned that for any half-integer \( l \) the equation (78) has a class of particular solutions of the form (75) with
\[
\bar{g} = \frac{g}{\sqrt{(2l)!!}}.
\]
As a consequence, the function

\[ \phi^2(t) = \left( \frac{(tA_1 - A_0)^2 + \tilde{g}^2}{A_1} \right)^{2l} \]

represents a class of particular solutions to the corresponding DAFF equation (76).

6. Conclusion

To summarize, in this work a relationship between the DAFF model and SM was elucidated. It was demonstrated that:

1) the equation of motion, \( SL(2, \mathbb{R}) \)-symmetry transformations, and corresponding conserved charges of SM can be derived from those of the DAFF model by applying the coordinate transformations (7);

2) the correspondence between the models exists if the coupling constant \( g^2 \) in the DAFF model is equal to the energy of SM;

3) the link (7) maps the solution of the DAFF model to the velocity function of SM. As a result, one DAFF trajectory corresponds to infinitely many solutions of SM;

4) the coordinate transformation (7) can be reproduced within the method of nonlinear realizations;

5) the link (7), accompanied by the solution of SM, provides a more general construction (28) than Niederer’s transformation (9). But additional freedom in (28) can be fixed by taking into account the invariance of the DAFF model and its NH counterpart under time translations and by imposing the requirement that the transformation is identical in the limit \( R \rightarrow \infty \);

6) the Schwarzian counterpart of the DAFF mechanics in the NH spacetime can be derived by applying the same coordinate transformation (7). The link (7) can be generalized to derive Schwarzian counterparts for one-dimensional higher derivative DAFF mechanics. This link allowed us to obtain a two-parametric set of particular solutions for the even-order DAFF mechanics.

Turning to further developments, it would be interesting to investigate the relationships between the DAFF model and SM within the Hamiltonian formalism. Taking into account the results in [49], it is of interest to construct possible Schwarzian counterparts of supersymmetric conformal mechanics. In particular, it is worth studying whether the analysis in
this work can be generalized to the case of the $\mathcal{N} = 1$ higher derivative mechanical systems obtained in [50]. These issues will be studied elsewhere.

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