Distributed Algorithms for Finding Local Clusters Using Heat Kernel Pagerank*

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Abstract. A distributed algorithm performs local computations on pieces of input and communicates the results through given communication links. When processing a massive graph in a distributed algorithm, local outputs must be configured as a solution to a graph problem without shared memory and with few rounds of communication. In this paper we consider the problem of computing a local cluster in a massive graph in the distributed setting. Computing local clusters are of certain application-specific interests, such as detecting communities in social networks or groups of interacting proteins in biological networks. When the graph models the computer network itself, detecting local clusters can help to prevent communication bottlenecks. We give a distributed algorithm that computes a local cluster in time that depends only logarithmically on the size of the graph in the CONGEST model. In particular, when the conductance of the optimal local cluster is known, the algorithm runs in time entirely independent of the size of the graph and depends only on error bounds for approximation. We also show that the local cluster problem can be computed in the $k$-machine distributed model in sublinear time. The speedup of our local cluster algorithms is mainly due to the use of our distributed algorithm for heat kernel pagerank.

Keywords: Distributed algorithms, local cluster, sparse cut, heat kernel pagerank, heat kernel, random walk

1 Introduction

Distributed computation is an increasingly important framework as the demand for fast data analysis grows and data simultaneously becomes too large to fit in main memory. As distributed systems for large-scale graph processing such as Pregel [20], GraphLab [19], and Google’s MapReduce [11] are rapidly developing, there is a need for both theoretical and practical bounds in adapting classical graph algorithms to a modern distributed and parallel setting.

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A distributed algorithm performs local computations on pieces of input and communicates the results through given communication links. When processing a massive graph in a distributed algorithm, local outputs must be configured without shared memory and with few rounds of communication. A central problem of interest is to compute local clusters in large graphs in a distributed setting.

Computing local clusters are of certain application-specific interests, such as detecting communities in social networks [15] or groups of interacting proteins in biological networks [16]. When the graph models the computer network itself, detecting local clusters can help identify communication bottlenecks, where one set of well-connected nodes is separated from another by a small number of links. Further, being able to identify the clusters quickly prevents bottlenecks from developing as the network grows.

A local clustering algorithm computes a set of vertices in a graph with a small Cheeger ratio (or so-called conductance as defined in Section 2.2). Moreover, we ask that the algorithm use only local information. In the static setting, an important consequence of this locality constraint is running times proportional to the size of the output set, rather than the entire graph. In this paper, we present the first algorithms for computing local clusters in two distributed settings that finish in a sublinear number of rounds of communication.

A standard technique in local clustering algorithms is the so-called sweep algorithm. In a sweep, one orders the vertices of a graph according to some real-valued function defined on the vertex set and then investigates the cut set induced by each prefix of vertices in the ordering. The classical method of spectral clustering uses eigenvectors as functions for the sweep. For local clustering algorithms, the sweep functions are based on random walks [17,18,24]. In [1], the efficiency of the local clustering algorithm is due to the use of PageRank vectors as the sweep functions [4]. In this paper, the main leverage in the improved running times of our algorithms is to use the heat kernel pagerank vector for performing a sweep. In particular, we are able to exploit parallelism in our algorithm for computing the heat kernel pagerank and give a distributed random walk-based procedure which requires fewer rounds of communication and yet maintains similar approximation guarantees as previous algorithms.

In Section 2.1, we will describe two distributive models – the CONGEST model and the $k$-machine model. We demonstrate in two different distributed settings that a heat kernel pagerank distribution can be used to compute local clusters with Cheeger ratio $O(\sqrt{\phi})$ when the optimal local cluster has Cheeger ratio $\phi$. With a fast, parallel algorithm for approximating the heat kernel pagerank and efficient local computations, our algorithm works on an $n$-vertex graph in the CONGEST, or standard message passing, model with high probability in at most $O\left(\frac{\log(c^{-1})\log n}{\log \log(c^{-1})} + \frac{1}{\epsilon} \log n\right)$ rounds of communication where $\epsilon$ is an error bound for approximation. This is an improvement over the previously best-performing local clustering algorithm in [8] which uses a personalized PageRank vector and finishes in $O\left(\frac{1}{\alpha} \log^2 n + n \log n\right)$ rounds in the CONGEST model for any $0 < \alpha < 1$. We then extend our results to the $k$-machine model to show that a local cluster
can be computed in $\tilde{O} \left( \frac{\log(c^{-1})}{\epsilon^3 \log \log(c^{-1})} + \frac{1}{\epsilon^2 k^2} + \left( \frac{\log(c^{-1})}{\epsilon^2 \log(c^{-1})} + \frac{1}{\epsilon k^2} \right) \max \left\{ \frac{1}{\epsilon^3}, \Delta \right\} \right)$ rounds, where $\Delta$ is the maximum degree in the graph, with high probability.

### 1.1 Related Work

The idea of computing local clusters with random walks was introduced by Lovász and Simonovits in their works analyzing the isoperimetric properties of random walks on graphs \cite{Lovasz, Simonovits}. Spielman and Teng \cite{SpielmanTeng} expanded upon these ideas and gave the first nearly-linear time algorithm for local clustering, improving the original framework by sparsifying the graph. The algorithm of \cite{SpielmanTeng} finds a local cluster with Cheeger ratio $O(\sqrt{\phi \log n})$ in time $O(m(\log n/\phi)^{O(1)})$, where $m$ is the number of edges in the graph. Each of these algorithms uses the distribution of random walks of length $O(1/\phi^2)$. Andersen et al. \cite{Andersen} give a local clustering algorithm using the distribution given by a PageRank vector. Their algorithm promises a $O(\sqrt{\phi \log 1/2})$ cluster approximation and runs in time $O(m \phi \log 4 m)$. Orecchia et al. use a variant of heat kernel random walks in their randomized algorithm for computing a cut in a graph with prescribed balance constraints \cite{Orecchia}. A key subroutine in the algorithm is a procedure for computing $e^{-A}v$ for a positive semidefinite matrix $A$ and a unit vector $v$ in time $\tilde{O}(m)$. Indeed, heat kernel has proven to be an efficient and effective tool for local cluster detection \cite{Andersen, DasSarma}.

Andersen and Peres \cite{AndersenPeres} simulate a volume-biased evolving set process to find sparse cuts. Their algorithm improves the ratio between the running time of the algorithm on a given run and the volume of the output set while maintaining similar approximation guarantees as previous algorithms. Their algorithm is later improved in \cite{AroraRao}. Arora, Rao, and Vazirani \cite{AroraRaoVazirani} give a $O(\sqrt{\log n})$-approximation algorithm using semi-definite programming techniques, however it is slower than algorithms based on spectral methods and random walks.

For distributed algorithms, in \cite{DasSarma} fast random walk-based distributed algorithms are given for estimating mixing time, conductance and the spectral gap of a network. In \cite{DasSarma}, distributed algorithms are derived for computing PageRank vectors with $O(\frac{1}{\alpha} \log n)$ rounds for any $0 < \alpha < 1$ with high probability. Das Sarma et al. \cite{DasSarma} give two algorithms for computing sparse cuts in the CONGEST distributed model. The first algorithm uses random walks and is based on the analysis of \cite{SpielmanTeng}. By incorporating the results of \cite{DasSarma}, they show that the stationary distribution of a random walk of length $l$ can be computed in $O(l)$ rounds. The second algorithm in \cite{DasSarma} uses PageRank vectors and is based on the analysis of \cite{Andersen}. By using the results of \cite{DasSarma}, the authors of \cite{DasSarma} compute local clusters in $O(\frac{1}{\alpha} + n) \log n)$ rounds with standard random walks and $O(\frac{1}{\alpha} \log^2 n + n \log n)$ rounds using PageRank vectors.
2 The Setting and Our Contributions

2.1 Models of Computation

We consider two models of distributed computation – the CONGEST model and the \( k \)-machine model. In each, data is distributed across nodes (machines) of a network which may communicate over specified communication links in rounds. Memory is decentralized, and the goal is to minimize the running time by minimizing the number of rounds required for computation for an arbitrary input graph \( G \). We emphasize that local communication is taken to be free.

The CONGEST model The first model we consider is the CONGEST model. In this model, the communication links are exactly the edges of the input graph and each vertex is mapped to a dedicated machine. The CONGEST (or standard message-passing) model was introduced in [22,23] to simulate real-world bandwidth restrictions across a network.

Due to how the vertices are distributed in the network, we simplify the model by assuming the computer network is the input graph \( G = (V,E) \) on \( n = |V| \) nodes or machines and \( m = |E| \) edges or communication links. Each node has a unique \( \log n \)-bit ID. Initially each node only possesses its own ID and the IDs of each of its neighbors, and in some instances we may allow nodes some metadata about the graph (the value of \( n \), for instance). Nodes can only communicate through edges of the network and communication occurs in rounds. That is, any message sent at the beginning of round \( r \) is fully transmitted and received by the end of round \( r \). We assume that all nodes run with the same processing speed. Most importantly, we only allow \( O(\log n) \) bits to be transmitted across any edge per round.

The \( k \)-machine model The defining difference between the \( k \)-machine model and the CONGEST model is that, whereas vertices are mapped to distinct, dedicated machines in the CONGEST model, a number of vertices may be mapped to the same machine in the \( k \)-machine model. This model is meant to more accurately simulate distributed graph computation in systems such as Pregel [20] and GraphLab [19].

We consider computing over massive datasets distributed over nodes of the \( k \)-machine network. The complete data is never known by any individual machine, and there is no shared memory. Each machine executes an instance of a distributed algorithm, and the output of each machine is with respect to the data points it hosts. A solution to a full problem is then a particular configuration of the outputs of each of the machines. The model is discussed in greater detail in Section 5.

The two models are limiting and advantageous in different ways, and one is not inherently better than the other. For instance, since many vertices are mapped to a single machine in the \( k \)-machine model, there is more “local information” available since vertices sharing a machine can communicate for free.
However, since communication is restricted to the communication links in the computer network, vertex-vertex communication is somewhat less restrictive in the CONGEST model since links exactly correspond to edges. The consequences of these differences are largely observed in time complexity, and certain graph problems are more suited to one model than the other.

In this paper we analyze our algorithmic techniques in the CONGEST model, and then use the Conversion Theorem of [13] to give an efficient probabilistic algorithm in the \( k \)-machine model for computing local clusters.

### 2.2 Local Clusters and Heat Kernel Pagerank

Throughout this paper, we consider a graph \( G = (V, E) \) with \( n = |V| \) and \( m = |E| \) that is connected and undirected. In this section we give some definitions that will make our problem statement and results precise.

**Personalized heat kernel pagerank** The heat kernel pagerank is so named for the heat kernel of the graph, \( H_t = e^{-tL} \), where \( L \) is the normalized graph Laplacian \( L = D^{-1/2}(D - A)D^{-1/2} \). Here \( D \) is the diagonal matrix whose entries correspond to vertex degree and \( A \) is the symmetric adjacency matrix. The heat kernel is a solution to the heat equation \( \frac{\partial u}{\partial t} = -Lu \), and thus has fundamental connections to diffusion properties of a graph. Because of its connection to random walks, for heat kernel pagerank we use a similar heat kernel matrix, \( H_t = e^{-tL} \), where \( L = I - P \). Here, \( I \) is the \( n \times n \) identity matrix and \( P = D^{-1}A \) is the transition probability matrix corresponding to the following standard random walk on the graph: at each step, move from a vertex \( v \) to a random neighbor \( u \). Then the heat kernel pagerank is defined in terms of a preference (row) vector \( f \) as \( \rho_t,f = fH_t \). When \( f \), as a row vector, is some probability distribution over the vertices, the following formulation is useful for our Monte Carlo-based approximation algorithm:

\[
\rho_t,f = fH_t = \sum_{k=0}^{\infty} e^{-t/t} fP^k.
\]

In this paper, we consider preference vectors \( f = \chi_s \) with all probability on a single vertex \( s \), called the seed, and zero probability elsewhere. This is a common starting distribution for the PageRank vector, as well, commonly referred to as a personalized PageRank (or PPR) vector. We will adapt similar terminology and refer to the vector \( \rho_{t,s} := \rho_{t,\chi_s} \) as the personalized heat kernel pagerank vector for \( s \), or simply PHKPR.

**Cheeger ratio** For a non-empty subset \( S \subset V \) of vertices in a graph, define the **volume** to be \( \text{vol}(S) = \sum_{v \in S} d_v \), where \( d_v \) is the degree of vertex \( v \). The **Cheeger ratio** of a set \( S \) is defined as \( \Phi(S) = \frac{|E(S, \bar{S})|}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}} \), where we use \( \bar{S} \) here to denote the set \( V \setminus S \), and \( E(S, \bar{S}) \) is the set of edges with one endpoint
in \( S \) and the other in \( \bar{S} \). The Cheeger ratio of a graph, then, is the minimum Cheeger ratio over all sets in the graph, \( \Phi(G) = \min_{S \subseteq V} \Phi(S) \). The Cheeger ratio provides a quantitative measure concerning graph clusters and is related to the expansion and spectral gap of a graph \(^5\).

**Local cluster and sparse cut** The sparse cut problem is to approximate the Cheeger ratio \( \Phi(G) \) of the graph. This is typically done by finding a set of vertices whose Cheeger ratio is close to \( \Phi(G) \) – that is, a set which approximates the sparsest cut in the graph. For the local clustering problem, however, we are concerned with finding a set with small Cheeger ratio within a specified subset of vertices. Alternatively, one can view this as a sparse cut problem on an induced subgraph. This Cheeger ratio is sometimes called a *local Cheeger ratio* with respect to the specified subset.

A local clustering algorithm promises the following. Given a set \( S \) of Cheeger ratio \( \phi \), many vertices in \( S \) may serve as seeds for a sweep which finds a set of Cheeger ratio close to \( \phi \).

### 2.3 Our Results

In this work we give a distributed algorithm which computes a local cluster of Cheeger ratio \( O(\sqrt{\phi}) \) with high probability, while the optimal local cluster has Cheeger ratio \( \phi \). Our algorithm finishes in \( O\left( \frac{\log(\frac{1}{\epsilon}) \log n}{\log \log (\frac{1}{\epsilon})} + \frac{1}{\epsilon} \log n \right) \) rounds in the CONGEST model (Theorem 5) where \( \epsilon \) is an error bound. Further, if \( \phi \) is known, we show how to compute a local cluster in \( O\left( \frac{\log(\frac{1}{\epsilon})}{\log \log (\frac{1}{\epsilon})} + \frac{1}{\epsilon} \right) \) rounds (Theorem 4). Our algorithm is an improvement of previous local clustering algorithms by eliminating a log factor in the performance guarantee. Further, its running time improves upon algorithms using standard and PageRank random walks. In particular, given the Cheeger ratio of an optimal local cluster, our algorithm runs in time only dependent upon the approximation error, \( \epsilon \), and is entirely independent of the input graph. The algorithms and accompanying analysis are given in Section 4.

Similar to existing local clustering algorithms, our algorithm uses a variation of random walks to compute a local cluster. However, rather than a standard random walk \(^{24} \) or a PageRank random walk with reset probabilities \(^1 \), we use the *heat kernel random walk* (see Section 3).

We remark that in the analysis of random walks, the usual notion of approximation is total variation distance or some other vector norm based distance. However, in the approximation of PageRank or heat kernel pagerank for large graphs, the definition of approximation is quite different. Namely, we say some vector \( \hat{\rho}_{t,s} \) is an \( \epsilon \)-approximate PHKPR vector for \( \rho_{t,s} \) with a seed vertex \( s \) and diffusion parameter \( t \in R \) if:

1. \((1 - \epsilon)\rho_{t,s}(v) - \epsilon \leq \hat{\rho}_{t,s}(v) \leq (1 + \epsilon)\rho_{t,s}(v)\), and
2. for each node \( v \) with \( \hat{\rho}_{t,s}(v) = 0 \), it must be that \( \rho_{t,s}(v) \leq \epsilon \).
With the above definition of approximation, we here define the heat kernel pagerank approximation problem (or the PHKPR problem in short): given a vertex \( s \) of a graph and a diffusion parameter \( t \in \mathbb{R} \), compute values \( \hat{\rho}_{t,s}(v) \) for vertices \( v \). We give a distributed algorithm which solves the PHKPR problem and finishes after only \( O \left( \frac{\log(c^{-1})}{\log \log(c^{-1})} \right) \) rounds of communication (Theorem 2).

We extend our results to distributed \( k \)-machine model and show the existence of an algorithm which computes a local cluster over \( k \) machines in \( \tilde{O} \left( \frac{\log(c^{-1})}{c^2 \log \log(c^{-1})} + \frac{1}{c} + \left( \frac{\log(c^{-1})}{k \log \log(c^{-1})} + \frac{1}{c^2} \right) \max \left\{ \frac{1}{c}, \Delta \right\} \right) \) rounds, where \( \Delta \) is the maximum degree in the graph, with high probability (Theorem 8). We note that when hiding polylogarithmic factors, this time does not depend on the size \( n \) of the graph. We compare this to an algorithm for computing a local cluster with PageRank which will require \( \tilde{O} \left( \frac{1}{\alpha} + \frac{n}{c^2} + \left( \frac{1}{\alpha c} + \frac{n}{c^2} \right) \max \left\{ \frac{1}{c}, \Delta \right\} \right) \) rounds with high probability, which is linear in \( n \). These results are given in Section 5.

We briefly note here that local clustering algorithms can easily be extended to sparse cut algorithms. Namely, one can sample a number of random nodes in the network and perform the local clustering algorithm from each. One node in the network can store the Cheeger ratios output by each run of the algorithm and simply return the minimal Cheeger ratio as the value of the sparsest cut in the network. In [24,1], \( O(\frac{\sigma \log n}{c^2}) \) nodes are enough to compute a sparsest cut with high probability, where \( \sigma \) is the size of the cut set.

### 3 Fast Distributed Heat Kernel Pagerank Computation

The idea of the algorithm is to launch a number of random walks from the seed node in parallel, and compute the fraction of random walks which end at a node \( u \) as an estimate of the PHKPR values \( \rho_{t,s}(u) \). Recall the definition of personalized heat kernel pagerank from (1), \( \rho_{t,s} = \sum_{k=0}^{\infty} e^{-t} \frac{k}{k!} \chi_s P^k \). Then the values of this vector are exactly the stationary distribution of a heat kernel random walk: with probability \( p_k = e^{-t} \frac{k}{k!} \), take \( k \) random walk steps according to the standard random walk transition probabilities \( P \) (see Section 2.2).

To be specific, the seed node \( s \) initializes \( r \) tokens, each of which holds a random variable \( k \) corresponding to the length of its random walk. Then, in rounds, the tokens are passed to random neighbors with a count incrementor until the count reaches \( k \). At the end of the parallel random walks, each node holding tokens outputs the number of tokens it holds divided by \( r \) as an estimate for its PHKPR value. Algorithm [1] describes the full procedure.

The algorithm is based on that given in [6] in a static setting. Theorem 1 of [6] states that an \( \varepsilon \)-approximate PHKPR vector can be computed with the above procedure by setting \( r = \frac{\varepsilon}{c^2} \log n \). Further, the approximation guarantee holds when limiting the maximum length of random walks to \( K = O \left( \frac{\log(c^{-1})}{\log \log(c^{-1})} \right) \), so that each token is passed for \( \max\{k, K\} \) rounds, where \( k \) is drawn with probability \( p_k \) as described above. In the static setting, this limit keeps the running time down.
Algorithm 1 DistributedEstimatePHKPR

**input:** a network modeled by a graph $G$, a seed node $s$, a diffusion parameter $t$, an error bound $\epsilon$

**output:** estimates $\hat{\rho}_{t,s}(v)$ of PHKPR values for nodes $v$ in the network

1: seed node $s$ generates $r = \frac{16}{\epsilon^3} \log n$ tokens $t_i$

2: $K \leftarrow c \cdot \frac{\log(c^{-1})}{\log \log(c^{-1})}$ for any choice of $c \geq 1$

3: each token $t_i$ does the following: pick a value $k$ with probability $p_k = e^{-tk} k!$, then hold the counter value $k_i \leftarrow \min\{k, K\}$

4: for iterations $j = 1 \ldots K$ do

5: for every node $v$ performs the following in parallel:

6: for every token $t_i$ node $v$ currently holds do

7: if $k_i == j$ then

8: hold on to this token for the duration of the iterations

9: else

10: send $t_i$ to a random neighbor

11: end if

12: end for

13: end for

14: let $C_v$ be the number of tokens node $v$ currently holds

15: each node with $C_v > 0$ returns $C_v / r$ as an estimate for its PHKPR value $\rho_{t,s}(v)$

In contrast, the distributed algorithm DistributedEstimatePHKPR takes advantage of decentralized control to take multiple random walk steps via multiple edges at a time. That is, through parallel execution, the running time depends only on the length of random walks, whereas when running the random walks in serial, as in [6], the running time must also include the number of random walks performed. Thus, keeping $K$ small is critical in keeping the number of rounds low, and is the key to the efficiency of our local clustering algorithms.

The correctness of the algorithm follows directly from Theorem 1 in [6], and is stated here without proof. The authors additionally give empirical evidence of the correctness of the algorithm with parameters $r = \frac{16}{\epsilon^3} \log n$ and $K = \frac{2 \log(c^{-1})}{\log \log(c^{-1})}$ in an extended version of the paper [7]. They specifically demonstrate that the ranking of nodes obtained with an $\epsilon$-approximate PHKPR vector computed this way is very close to the ranking obtained with an exact vector.

Theorem 1. For any network $G$, any seed node $s \in V$, and any error bound $0 < \epsilon < 1$, the distributed algorithm DistributedEstimatePHKPR outputs an $\epsilon$-approximate PHKPR vector with probability at least $1 - \epsilon$.

The correctness of the algorithm holds for any choice of $t$, and in fact we use a particular value of $t$ in our local clustering algorithm (see Section 4). Regardless, it is clear that the running time is independent of any choice of $t$. In fact, we demonstrate in the proof of Theorem 2 that it is independent of $n$ as well.
Theorem 2. For any network $G$, any seed node $s \in V$, and any error bound $0 < \epsilon < 1$, the distributed algorithm DistributedEstimatePHKPR finishes in $O\left( \frac{\log(c^{-1})}{\log \log(c^{-1})} \right)$ rounds.

Proof. We show that there is no congestion in the network during any round of the algorithm; i.e., there are never more than $O(\log n)$ bits sent over any edge in any iteration of the random walk process. The proof then follows since each step of the random walk requires only one round of computation.

In any run of the algorithm, $\frac{16}{\epsilon^3} \log n$ tokens are created, each holding a message $k_i$ corresponding to a random walk length. The token contains no other information. In particular, no node IDs are transmitted through the tokens. Therefore passing a token involves sending a message of constant size in any iteration of the algorithm. In the worst case, every token is transmitted through a single edge in a single iteration of the algorithm. However, this is still only $O(\frac{1}{\epsilon^3} \log n)$ bits, and so meets the constraints of the model. Namely, even the worst case of sending every token over one edge can be done with a single round of communication. Therefore any random walk step requires only one round of communication, and by construction at most $O\left( \frac{\log(c^{-1})}{\log \log(c^{-1})} \right)$ random walk steps are performed in the algorithm. \hfill \Box

4 Distributed Local Cluster Detection

In this section we present a fast, distributed algorithm for the local clustering problem. The backbone of the algorithm involves investigating sets of nodes which accumulate in decreasing order of their $\hat{\rho}_{t,s}(v)/d_v$ values. The process is efficient and requires at most one linear scan of the nodes in the network (we actually show that the process can be much faster).

We describe the algorithm presently. Let $p$ be any function over the nodes of the graph, and let $\pi$ be the ordering of the nodes in decreasing order of $p(v)/d_v$. Then the majority of the work of the algorithm is investigating sufficiently many of the $n-1$ cuts $(S_j, \bar{S}_j)$ given by the first $j$ nodes in the ordering and the last $n-j$ nodes in the ordering, respectively, for $j = 1, \ldots, n-1$. However, by “sufficiently many” we indicate that we may stop investigating the cut sets when either the volume or the size of the set $S_j$ is large. Assume this point is after $j = j$. Then we choose the cut set that yields the minimum Cheeger ratio among the $j$ possible cut sets. We call this process a sweep. As such, our local clustering algorithm is a sweep of a PHKPR distribution vector.

In the static setting, this process will take $O(n \log n)$ time in general. The authors in \cite{8} give a distributed sweep algorithm that finishes in $O(n)$ rounds. We improve the analysis of \cite{8} using a PHKPR vector. The running time of our sweep algorithm is given in Lemma 1.

The sweep involves two phases. In Phase 1, the goal is for each node to know its place in the ordering $\pi$. Each node can compute their own $\hat{\rho}_{t,s}(v)/d_v$ value locally, and we use $O\left( \frac{1}{\epsilon} \right)$ rounds to ensure each node knows the $\pi$ values of all
Algorithm 2 DistributedLocalCluster

input: a network modeled by a graph $G$, a seed node $s$, a target cluster size $\sigma$, a target cluster volume $\varsigma$, an optimal Cheeger ratio $\phi$, an error bound $\epsilon$

output: a set of nodes $S$ with $\Phi(S) \in O(\sqrt{\phi})$

1: $t \leftarrow \phi^{-1} \log(\frac{2\varsigma}{1 - \epsilon} + 2\epsilon\sigma)$
2: compute PHKPR values $\hat{\rho}_{t,s}(v)$ with DistributedEstimatePHKPR($G$, $s$, $t$, $\epsilon$)
3: every node $v$ with a non-zero PHKPR value estimate sends $\hat{\rho}_{t,s}(v)/d_v$ to every other node with a non-zero PHKPR value estimate \(\triangleright\) Phase 1
4: let $\pi$ be the ordering of nodes in decreasing order of $\hat{\rho}_{t,s}(v)/d_v$ \(\triangleright\) Phase 1
5: compute Cheeger ratios of each of the cut sets with a call of the Distributed Sweep Algorithm and output the cut set of minimum Cheeger ratio \(\triangleright\) Phase 2

other nodes (see the proof of Lemma 1). In Phase 2, we use the decentralized sweep of $S$ described presently:

**Distributed Sweep Algorithm.** Let $N$ denote the number of nodes with a non-zero estimated PHKPR value after running the algorithm DistributedEstimatePHKPR. Assume each node knows its position in the ordering $\pi$ after Phase 1. We will refer to nodes by their place in the ordering. Define $S_j$ to be the cut set of the first $j$ nodes in the ordering. Then computing the Cheeger ratio of each cut set $S_j$ involves a computation of the volume of the set as well as $|E(S_j, \bar{S}_j)|$.

Define the following:
- $L^*_j$ is the number of neighbors of node $j$ in $S_{j-1}$, and
- $R^*_j$ is the number of neighbors of node $j$ in $\bar{S}_j$.

Then the Cheeger ratio of each cut set can be computed locally by:

\[
\circ |E(S_j, \bar{S}_j)| = |E(S_{j-1}, \bar{S}_{j-1})| - L^*_j + R^*_j, \text{ with } |E(S_1, \bar{S}_1)| = d_1 \quad (2)
\]
\[
\circ \text{vol}(S_j) = \text{vol}(S_{j-1}) + L^*_j + R^*_j, \text{ with } \text{vol}(S_1) = d_1. \quad (3)
\]

We now show that a sweep can be performed in $O(N)$ rounds. Each node knows the IDs of its neighbors and after Phase 1 each node knows the place of every other node in the ordering $\pi$. Therefore, each node can compute locally if a neighbor is in $S_{j-1}$ or $\bar{S}_1$, and so $L^*_j$ and $R^*_j$ can be computed locally for each node $j$. Each node can then prepare an $O(\log n)$-bit message of the form $(\text{ID}, L^*_j, R^*_j)$. Each of the $N$ messages of this form can then be sent to the first node in the ordering using the upcasting algorithm (described in the proof of Lemma 1 using the $\pi$ ordering as node rank. We note that the $N$ nodes in the ordering are necessarily in a connected component of the network, and so the upcasting procedure can be performed in $O(N)$ rounds. Finally, once the first node in the ordering is in possession of the ordering $\pi$, and the values of $(L^*_j, R^*_j)$ for every node in the ordering, it may iteratively compute $\Phi(S_j)$ locally using the rules (2) and (3). Thus, this node can output the minimum Cheeger ratio $\phi^*$ as well as the $j^*$ such that $\Phi(S_{j^*}) = \phi^*$ after $O(N)$ rounds.
Lemma 1. Performing Phases 1 and 2 of a distributed sweep takes $O(\frac{1}{\epsilon})$ rounds.

Proof. First we describe how to send $N O(\log n)$-sized messages to a single node in $O(N)$ rounds of communication. For this we can use the upcasting algorithm of [23] (as described in [8]). We first construct a priority BFS tree of the $N$ nodes with non-zero PHKPR value. We emphasize again that these nodes are necessarily in a connected component of the network, and it is shown in [23] that such a BFS tree can be constructed in $O(N)$ time. Each node in the tree then upcasts its message to the root node through the edges of the tree.

In Phase 1, the nodes need to be sorted according to their (non-zero) $\pi$ values. In this case, the nodes use their $\hat{\rho}_{t,s}(v)/d_v$ value as their rank so that the node with the highest $\hat{\rho}_{t,s}(v)/d_v$ value is the root of the tree. Then each node upcasts its $\hat{\rho}_{t,s}(v)/d_v$ value to the root through the edges of the tree. The root node locally sorts these values and then floods all the $\pi$ values to the nodes through tree edges. The upcast and flooding process take $O(N)$ rounds to reach each of the nodes in the tree.

Phase 2 consists of the Distributed Sweep Algorithm, where the first node in the ordering computes the Cheeger ratio of $S_j$ for each node $j$ in the ordering. In order to send each of the (ID, $L_j^\pi$, $R_j^\pi$) messages to the first node of the ordering we again upcast through the edges of a priority BFS tree, however in this round we use $\pi$ values as node rank. The root node is then able to locally compute Cheeger ratios and output the cutset of minimum Cheeger ratio after $O(N)$ rounds for upcasting.

Thus Phase 1 requires $O(N)$ rounds for upcasting and flooding values. Phase 2 requires $O(N)$ rounds for upcasting values necessary for locally computing Cheeger ratios. Since we compute an $\epsilon$-approximate PHKPR vector as our distribution, we know that $N$ is no more than $O(\frac{1}{\epsilon})$. This is because we assume $\sum_{v \in V} \rho_{t,s}(v) = 1$, and so no more than $\frac{1}{\epsilon}$ vertices can have values at least $\epsilon$. Thus the full sweep takes $O(\frac{1}{\epsilon})$ rounds.

We note here that the time required for the sweep may be reduced if there are size or volume restraints for the cut set. In this case, an alternative distributed sweep algorithm may be utilized. As usual, we refer to each node by their place in the ordering $\pi$. Node 1 begins the sweep by sending $\text{vol}(S_1),|E(S_1,\bar{S}_1)|$ to node 2. Then nodes $j = 2, \ldots, N$ iteratively compute $\Phi(S_j)$ using the values of $\text{vol}(S_{j-1}),|E(S_{j-1},\bar{S}_{j-1})|,L_j^\pi$ and $R_j^\pi$, and then subsequently sending $\text{vol}(S_j),|E(S_j,\bar{S}_j)|$ to the next node $j+1$ in the ordering. Additionally, each node can send the minimum Cheeger ratio $\phi^*$ computed thus far as well as the $j^*$ such that $\Phi(S_{j^*}) = \phi^*$. Thus the last node in the ordering can output $S_{j^*}$. In this algorithm, each iteration $j$ will require $d$ rounds of communication, were $d$ is the shortest path distance between nodes $j-1$ and $j$. However, no two nodes will ever be at a distance of greater than $O(\frac{\log(\epsilon^{-1})}{\log \log(\epsilon^{-1})})$ steps by construction. In this way, the first $j$ Cheeger ratios can be computed in $O(j\frac{\log(\epsilon^{-1})}{\log \log(\epsilon^{-1})})$ rounds.

If size or volume restraints are placed on the cluster, we may stop the sweep at node $\frac{j}{2}$ when the size or volume of $S_{\frac{j}{2}}$ is greater than some specified value. We
output the set $S_j^*$ for that iteration, and this process requires $O(j \log \log(\epsilon^{-1}))$ rounds.

The algorithm DistributedLocalCluster (Algorithm 2) is a complete description of our distributed local clustering algorithm. The correctness of the algorithm follows directly from [6] and we omit the proof here.

**Theorem 3.** For any network $G$, suppose there is a set of Cheeger ratio $\phi$. Then at least half of the vertices in $S$ can serve as the seed $s$ so that for any error bound $0 < \epsilon < 1$, the algorithm DistributedLocalCluster will find a set of Cheeger ratio $O(\sqrt{\phi})$ with probability at least $1 - \epsilon$.

**Theorem 4.** For any network $G$, any seed node $s \in V$, and any error bound $0 < \epsilon < 1$, the algorithm DistributedLocalCluster finishes in $O\left(\frac{\log(\epsilon^{-1})}{\log \log(\epsilon^{-1})} + \frac{1}{\epsilon} \right)$ rounds.

**Proof.** The only distributed computations are those for computing approximate PHKPR values (line 2) and Phase 1 (lines 3 and 4) and Phase 2 (line 5) of the distributed sweep. Computing PHKPR values takes $O\left(\frac{\log(\epsilon^{-1})}{\log \log(\epsilon^{-1})}\right)$ rounds by Theorem 2, and Phases 1 and 2 together take $O\left(\frac{1}{\epsilon}\right)$ rounds by Lemma 1. Thus the running time follows.

One possible concern with the algorithm DistributedLocalCluster is that one cannot guarantee knowing the value of $\phi$ ahead of time for any particular node $s$. Therefore a true local clustering algorithm should be able to proceed without this information. This can be achieved by “testing” a few values of $\phi$ (and fixing some reasonable values for $\sigma$ and $\varsigma$). Namely, begin with $\phi = 1/2$ and run the algorithm above. If the output cut set $S$ satisfies $\Phi(S) \in O(\sqrt{\phi})$, we are done. If not, halve the value of $\phi$ and continue. There are $O(\log n)$ such guesses, and we have arrived at the following.

**Theorem 5.** For any network $G$, any node $s$, and any error bound $0 < \epsilon < 1$, there is a distributed algorithm that computes a set $S$ with Cheeger ratio within a quadratic of the optimal which finishes in $O\left(\frac{\log(\epsilon^{-1})}{\log \log(\epsilon^{-1})} + \frac{1}{\epsilon} \log n\right)$ rounds.

In particular, when ignoring polylogarithmic factors, the running time is $\tilde{O}\left(\frac{\log(\epsilon^{-1})}{\log \log(\epsilon^{-1})} + \frac{1}{\epsilon} \right)$.

## 5 Computing Local Clusters in the $k$-Machine Model

In this section we consider a graph on $n$ vertices which is distributed across $k$ nodes in a computer network. This is the $k$-machine model introduced in Section 2.1.

In the $k$-machine model, we consider a network of $k > 1$ distinct machines that are pairwise interconnected by bidirectional point-to-point communication
links. Each machine executes an instance of a distributed algorithm. The computation advances in rounds where, in each round, machines can exchange messages through their communication links. We again assume that each link has a bandwidth of \( O(\log n) \) meaning that \( O(\log n) \) bits may be transmitted through a link in any round. We also assume no shared memory and no other means of communication between nodes. When we say an algorithm solves a problem in \( x \) rounds, we mean that \( x \) is the maximum number of rounds until termination of the algorithm, over all \( n \)-node, \( m \)-edge graphs \( G \).

In this model we are solving massive graph problems in which the vertices of the graph are distributed among the \( k \) machines. We assume \( n \geq k \) (typically \( n \gg k \)). Initially the entire graph is not known by a single machine but rather partitioned among the \( k \) machines in a “balanced” fashion so that the nodes and/or edges are partitioned approximately evenly among the machines. There are several ways of partitioning vertices, and we will consider a random partition, where vertices and incident edges are randomly assigned to machines. Formally, each vertex \( v \) of \( G \) is assigned independently and randomly to one of the \( k \) machines, which we call the home machine of \( v \). The home machine of \( v \) thereafter knows the ID of \( v \) as well as the IDs and home machines of neighbors of \( v \).

In the remainder of this section we prove the existence of efficient algorithms for solving the PHKPR and local cluster problems in the \( k \)-machine model. Our main tool is the Conversion Theorem of [13].

Define \( M \) as the message complexity, the worst case number of messages sent in total during a run of the algorithm. Also define \( C \) as the communication degree complexity, or the maximum number of messages sent or received by any node in any round of the algorithm. Then we use as a key tool the Conversion Theorem as restated below.

**Theorem 6 (Conversion Theorem [13]).** Suppose there is an algorithm \( A_C \) that solves problem \( P \) in the CONGEST model for any \( n \)-node graph \( G \) with probability at least \( 1 - \epsilon \) in time \( T_C(n) \). Further, let \( A_C \) use message complexity \( M \) and communication degree complexity \( C \). Then there exists an algorithm \( A_k \) that solves \( P \) for any \( n \)-node graph \( G \) with probability at least \( 1 - \epsilon \) in the \( k \)-machine model in \( \tilde{O} \left( \frac{M}{k^2} + \frac{T_C(n)C}{k} \right) \) rounds with high probability.

In the forthcoming theorems, by “high probability” we mean with probability at least \( 1 - 1/n \).

We note that the proof of the Conversion Theorem is constructive, describing precisely how an algorithm \( A_k \) in the \( k \)-machine model simulates the algorithm \( A_C \) in the CONGEST model. We omit the simulation here but encourage the reader to refer to the proof for implementation details.

By Theorem 2, we know that PHKPR values can be estimated with \( \epsilon \)-accuracy in \( O \left( \frac{\log(c^{-1})}{\log \log(c^{-1})} \right) \) rounds. A total of \( O \left( \frac{1}{c^3 \log n} \right) \) messages are generated and propagated for at most \( O \left( \frac{\log(c^{-1})}{\log \log(c^{-1})} \right) \) random walk steps, for a total of \( O \left( \frac{\log(c^{-1}) \log n}{c^3 \log \log(c^{-1})} \right) \) messages sent during a run of the algorithm. In the
first random walk step, each of the $O\left(\frac{1}{\epsilon} \log n\right)$ messages may be passed to a neighbor of the seed node, so the message complexity is $O\left(\frac{1}{\epsilon} \log n\right)$. Therefore we arrive at the following.

**Theorem 7.** There exists an algorithm that solves the PHKPR problem for any $n$-node graph in the $k$-machine model with probability at least $1 - \epsilon$ and runs in $\tilde{O}\left(\frac{\log(n)}{\log(\log(\epsilon^{-1}))}\left(\frac{1}{k} + 1\right)\right)$ rounds with high probability.

By Theorem 5 a local cluster about any seed node can be computed in $O\left(\frac{\log(n)}{\log(\log(\epsilon^{-1}))} \log n\right)$ rounds. The message complexity for the PHKPR phase is $O\left(\frac{\log(n)}{\log(\log(\epsilon^{-1}))} \log n\right)$ and for the sweep phase is $O\left(\frac{1}{\epsilon} \log n\right)$, for a total message complexity of $O\left(\frac{\log(n)}{\log(\log(\epsilon^{-1}))} \log n + \frac{1}{\epsilon} \log n\right)$. The communication degree complexity is $O\left(\frac{1}{\epsilon} \log n\right)$ for the PHKPR phase (as above), and $O(\Delta)$, where $\Delta$ is the maximum degree in the graph, for the sweep phase. Thus the communication degree complexity for the algorithm is the maximum of these two. We therefore have the following result for the $k$-machine model.

**Theorem 8.** There exists an algorithm that computes a local cluster for any $n$-node graph in the $k$-machine model with probability at least $1 - \epsilon$ and runs in $\tilde{O}\left(\frac{\log(n)}{\log(\log(\epsilon^{-1}))} + \frac{1}{\epsilon} \log(n) + \left(\frac{\log(n)}{\log(\log(\epsilon^{-1}))} + \frac{1}{\epsilon} \log(n)\right) \max\left\{\frac{1}{\epsilon}, \Delta\right\}\right)$ rounds, where $\Delta$ is the maximum degree in the graph, with high probability.

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