Summary The discussion of enhancement of superconductivity is presented in the framework of the higher-grade hybrid model of layered superconductors. The enhancement is considered from the following two points of view: (i) as a result of long-range couplings with respects to the short distance ones, and (ii) the enhancement of 3D superconductivity with respect to the 2D. Two important cases are distinguished: homogeneous bulk superconducting materials and superconducting structures composed of a finite number of layers.

1 Introduction

The higher grade hybrid model (HM) of layered superconductors has been proposed and studied in a series of papers [7]–[14]. It turned out that the model is also able to describe some phenomena which are beyond the capabilities offered by simpler phenomenologies such as the 3D anisotropic Ginzburg-Landau or Lawrence-Doniach theory [1]–[6]. As an important example one can mention the phenomenon of enhancement of superconductivity by interlayer couplings. As a result the layered superconducting structure can
exhibit better superconducting properties than the isolated layers.

The present paper is devoted to a more systematic discussion of such enhancement effects. We shall be concerned with two important cases: homogeneous bulk superconducting materials and superconducting structures composed of a finite number of layers. For the sake of simplicity we restrict ourselves to situations with no external magnetic fields.

In the simplest case the isolated atomic planes are all described by the model 2D GL with the same parameters $\alpha_0$ and $\beta$. The parameter $\alpha_0$ depends on the temperature. If $\alpha_0 > 0$, then the 2D state of such a plane is $N$ (normal), in the opposite case it is the state $S$ (superconducting). We interpret the parameter $\alpha_0$ as a measure of empirical temperature, and introduce the notation $\tau = \tau_0 + \alpha_0$. In consequence, for an isolated plane we have state $N$ for $\tau > \tau_0$ and state $S$ for $\tau < \tau_0$.

For a given material the highest temperature at which the normal state becomes unstable determines the onset of superconductivity. Instability of the normal state implies the stability of a superconducting state. The presence of interactions between planes modifies the above defined empirical temperature, which becomes a function of the coupling parameters. In the HM model there are two classes of long-range coupling constants: Josephson parameters $\gamma_q$ and proximity effect parameters $\zeta_q$, with $q \in \{1, 2, ..., K\}$. The grade $K$, expressed by an arbitrary (but specified for any particular case) integer, defines the admitted range of J-links in terms of interplanar gaps.

2 Infinite medium

According to [8], in the framework of HM the layered superconductor is considered as a one-dimensional stack of 2D layers (e.g. atomic planes) described by 2D Ginzburg-Landau theory with parameters $\alpha_0$ and $\beta$, and interlayer Josephson-type bonds (called J-links) between them. These couplings represent Josephson’s interactions as well as proximity effect.

We denote by $\psi_n$ the order parameter associated to the layer indexed by the number $n$. Its complex conjugate (c.c.) is denoted by $\psi_n^\dagger$.

The field equations then have the form

$$-rac{\hbar^2}{2m_{ab}} \nabla^2 \psi_n + \tilde{\alpha} \psi_n + \beta|\psi_n|^2 \psi_n - \frac{1}{2} \sum_q \gamma_q (\psi_{n+q} + \psi_{n-q}) = 0,$$  \hspace{1cm} (1)
where the GL temperature parameter $\alpha_0$ has been replaced by $\tilde{\alpha}$, which takes into account the influence of proximity effect couplings:

$$\tilde{\alpha} = \alpha_0 + \frac{1}{2} \sum_q \zeta_q.$$  \hspace{1cm} (2)

It is seen that $\tilde{\alpha}$ depends on $n$ for planes near the boundary (if the number of layers is finite or semi-infinite). We shall pay our attention to the case $K = 2$, i.e. the coupling between nearest and next nearest neighbours. Let us consider the ground states of HM. For the plane-uniform order parameter the field equations have the form

$$\tilde{\alpha} \psi_n + \beta |\psi_n|^2 \psi_n - \frac{1}{2} [\gamma_1 (\psi_{n+1} + \psi_{n-1}) + \gamma_2 (\psi_{n+2} + \psi_{n-2})] = 0.$$ \hspace{1cm} (3)

There exist solutions with constant amplitude and difference of phase between adjacent atomic planes; they can be obtained by the ansatz $\psi_n = C e^{i n \theta}$, which gives the relation

$$C^2 = -\alpha^*/\beta.$$ \hspace{1cm} (4)

The condition of vanishing Josephson current reads

$$\gamma_1 \sin \theta + 2 \gamma_2 \sin 2 \theta = 0.$$ \hspace{1cm} (5)

Solving (5) with respect to $\theta$ we obtain 3 classes of ground states, fulfilling (4) with $\alpha^*$ expressed by coupling parameters:

- **uniform**: $\psi_n = C$, $\alpha^* = \alpha_0 + \zeta_1 + \zeta_2 - \gamma_1 - \gamma_2$,
- **alternating**: $\psi_n = (-1)^n C$, $\alpha^* = \alpha_0 + \zeta_1 + \zeta_2 + \gamma_1 - \gamma_2$,
- **phase modulated**: $\psi_n = C e^{i n \theta}$, $\theta = \arccos(-\frac{2 \gamma_2}{\gamma_1})$.

The phase modulated solutions exist if the parameters $\gamma_1$ and $\gamma_2$ satisfy the relation $|\gamma_1| \leq 4 |\gamma_2|$. Then $\alpha^*$ is expressed by the coupling constants according to the formula

$$\alpha^* = \alpha_0 + \zeta_1 + \zeta_2 + \gamma_2 (1 + \frac{\gamma_1^2}{8 \gamma_2^2}).$$ \hspace{1cm} (6)

The special symbols in definitions of classes of solutions refer to the regions at the plane $(\gamma_1, \gamma_2)$ in which the solutions are stable:

- **uniform**: $\gamma_1 > 0, \gamma_1 + 4 \gamma_2 > 0$.
The same symbols will be used to mark the corresponding sectors in the plot below,

\[ \oplus : \gamma_1 < 0, \gamma_1 - 4\gamma_2 < 0, \]  
\[ \ominus : \gamma_2 < 0, 4\gamma_2 < \gamma_1 < -4\gamma_2. \]

which illustrates the enhancement of superconductivity. As a measure of enhancement we have used the quantity \( \Delta \alpha = \alpha^* - \tilde{\alpha} \) considered a function of the coupling angle \( \varphi \) defined as the polar angle in the plane \((\gamma_1, \gamma_2)\). The second polar co-ordinate will be denoted by \( \gamma \): \( \gamma_1 = \gamma \cos \varphi, \gamma_2 = \gamma \sin \varphi \).

The numerical values have been computed for \( \gamma = 1 \). The enhancement reaches maximum when \( \Delta \alpha \) has minimum. We shall denote the minimum value of \( \Delta \alpha \) by \( M \). For the individual classes of solutions we have

\[ \ominus : \Delta \alpha = \sqrt{2}\gamma \sin(\varphi - \frac{3}{4}\pi), -\varphi_0 \leq \varphi \leq \pi - \varphi_0, M = \Delta \alpha(\varphi/4) = -\sqrt{2}\gamma, \]  
\[ \oplus : \Delta \alpha = \sqrt{2}\gamma \sin(\varphi + \frac{3}{4}\pi), \varphi_0 \leq \varphi \leq \pi + \varphi_0, M = \Delta \alpha(\frac{3}{4}\pi) = -\sqrt{2}\gamma, \]  
\[ \ominus : \Delta \alpha = \gamma(1 + \frac{1}{8}\cot^2 \varphi) \sin \varphi, -\pi + \varphi_0 \leq \varphi \leq -\varphi_0, M = \Delta \alpha(-\frac{\pi}{2}) = -\gamma. \]

In the above formulae the notation \( \varphi_0 = \arctan(\frac{1}{4}) \) have been used.

It follows from the formulae defining the solutions, that for appropriate values
of coupling parameters one can obtain negative value of $\alpha^*$ with positive $\alpha_0$. Even more, $\alpha^*$ can be negative while $\tilde{\alpha}$ remains positive. For such materials the out-of-plane superconductivity appears in spite of the fact that all the layers remain in overcritical in-plane states.

3 Finite structures

According to [14], one considers the structure of 3 GL layers coupled by J-links of first and second grade. The order parameters will be denoted by $\psi_{-1}$, $\psi_0$ and $\psi_1$. Their real and imaginary parts will be denoted by $a$ and $b$ with appropriate indices. The field equations have the form

\begin{align}
(\tilde{\alpha}_1 + \beta |\psi_1|^2)\psi_1 - \frac{1}{2} [\gamma_1 \psi_0 + \gamma_2 \psi_{-1}] &= 0, \\
(\tilde{\alpha}_0 + \beta |\psi_0|^2)\psi_0 - \frac{1}{2} \gamma_1 (\psi_1 + \psi_{-1}) &= 0, \\
(\tilde{\alpha}_{-1} + \beta |\psi_{-1}|^2)\psi_{-1} - \frac{1}{2} [\gamma_1 \psi_0 + \gamma_2 \psi_1] &= 0,
\end{align}

where

$$\tilde{\alpha}_0 = \alpha_0 + \zeta_1.$$  

and

$$\tilde{\alpha}_1 = \tilde{\alpha}_{-1} = \alpha_0 + \frac{1}{2}(\zeta_1 + \zeta_2) = \tilde{\alpha}_0 - \frac{1}{2} \delta,$$

with

$$\delta = \zeta_1 - \zeta_2.$$  

From the form of equations it follows that any solution is gauge-equivalent to a triplet $\psi_{-1}$, $\psi_0$, $\psi_1$, for which

$$b_0 = 0, \quad b_1 = -b_{-1}, \quad b_1 [\gamma_1 a_0 + \gamma_2 (a_1 + a_{-1})] = 0.$$  

For any real solution either $a_1 = a_{-1}$, or

$$\alpha^*_1 + \beta (a_1^2 + a_1 a_{-1} + a_{-1}^2) = 0,$$

where

$$\alpha^*_1 = \tilde{\alpha}_0 + \frac{1}{2} (\gamma_2 - \delta).$$
The set of nontrivial solutions to the system (13-15) is partitioned into four disjoint classes, two of which are specially interesting from the point of view of the onset and enhancement of superconductivity. Namely the class (A) containing solutions fulfilling conditions

\[ b_1 = 0, \quad a_0 = 0, \quad a_1 = -a_{-1} \neq 0, \] (22)

and the class (C) with solutions fulfilling conditions

\[ b_1 = 0, \quad a_0 \neq 0, \quad a_1 = a_{-1}. \] (23)

The solutions of the class (A) may be explicitly written as

\[ a_1^2 = -\frac{\alpha_1^*}{\beta}, \] (24)

so that (taking into account the positiveness of \( \beta \)) the necessary and sufficient condition for existence of mode A is

\[ 2\tilde{\alpha}_0 + \gamma_2 - \delta < 0. \] (25)

For the class (C) the system of equations (13-15) may be transformed into the form

\[ a_0 = \frac{2}{\gamma_1}(\alpha_1^* - \gamma_2 + \beta a_1^2)a_1, \] (26)

\[ a_1 = \frac{1}{\gamma_1}(\tilde{\alpha}_0 + \beta a_0^2)a_0, \] (27)

where \( \alpha_1^* \) is given by the eqn. (21). Introducing

\[ x = \tilde{\alpha}_0 + \beta a_0^2, \] (28)

one can express the necessary conditions for the existence of mode C:

\[ \beta a_0^2 = x - \tilde{\alpha}_0 > 0 \] (29)

and

\[ \beta a_1^2 = \gamma_1^2 \frac{1}{x} - \alpha_1^* - \gamma_2 > 0. \] (30)
Further
\[ \frac{x^3}{\gamma_1^2} = \frac{\gamma_2^2 - 2\tilde{\alpha}_0 + (\delta + \gamma_2)}{2(x - \tilde{\alpha}_0)}. \]  
(31)

Hence, geometrically, the solutions from class (C) are determined by the points of intersection of two curves:
\[ y = \frac{2}{\gamma_1^2} x^3, \]  
(32)

and
\[ y = \frac{\gamma_2^2 - (2\tilde{\alpha}_0 - \delta - \gamma_2)x}{x - \tilde{\alpha}_0}. \]  
(33)

From the equations (26-27) and (28) it follows that for \( x = \tilde{\alpha}_0 \) we have the zero-solution, hence the transition to the normal state. The \( \tilde{\alpha}_0 \) fulfills the equation
\[ (2\tilde{\alpha}_0 - \delta - \gamma_2)\tilde{\alpha}_0 - \gamma_1^2 = 0, \]  
(34)

which always has two real roots. We shall denote them by
\[ \alpha_{01} = \frac{1}{4}(\delta + \gamma_2 - \sqrt{(\delta + \gamma_2)^2 + 8\gamma_1^2}), \]  
(35)

and
\[ \alpha_{02} = \frac{1}{4}(\delta + \gamma_2 + \sqrt{(\delta + \gamma_2)^2 + 8\gamma_1^2}). \]  
(36)

The class (C) is empty if
\[ \tilde{\alpha}_0 > \alpha_{02}. \]  
(37)

When \( \tilde{\alpha}_0 < \alpha_{01} \), there exist two solutions.

When \( \alpha_{01} < \tilde{\alpha}_0 < \frac{1}{2}(\delta + \gamma_2) \) or \( \frac{1}{2}(\delta + \gamma_2) < \tilde{\alpha}_0 < \alpha_{02} \), there is one solution.

The temperature relation (37) is one of the set of conditions for stability of the normal state. From the analysis of the second variation of energy of the normal state it follows that the remaining two conditions have the form \( \tilde{\alpha}_0 > 0 \) and \( \tilde{\alpha}_0 > \frac{1}{2}(\delta - |\gamma_2|) \).

When the normal mode becomes unstable, either mode A (onset A) or mode C (onset C) can appear, depending on the position in the material plane \((\gamma_2, \delta)\). The plane is divided into two regions of different onset by the curve
\[ \delta = \gamma_2 - \frac{\gamma_1^2}{\gamma_2}, \quad \gamma_2 < 0. \]  
(38)
The following combinations of the material parameters define two empiric temperatures

\[ \tau_A = \tau_0 - \zeta_1 + \frac{1}{2}(\delta - \gamma_2), \]  

(39)

and

\[ \tau_C = \tau_0 - \zeta_1 + \alpha_{02}, \]  

(40)

where \( \alpha_{02} \) is given by the eqn. (36). As long as \( \tau > \max(\tau_A, \tau_C) \), the normal state is stable. Below this limit a stable superconducting mode appears. If \( \tau_A > \tau_C \), the mode is A; in the opposite case – it is the mode C. The quantity \( \tau_0 \) describes the onset for isolated GL planes: if \( \tau > \tau_0 \) then the planes are in normal state, if \( \tau < \tau_0 \) then the state is superconducting.

Let us now examine the increment of the onset temperatures \( \tau_A \) and \( \tau_C \) due to the long distance couplings. Denoting the onset temperatures for the first grade material by \( \tau_{A0} \) and \( \tau_{C0} \), one can calculate

\[ \tau_A - \tau_{A0} = -\frac{1}{2}(\zeta_2 + \gamma_2), \]  

(41)

and

\[ \tau_C - \tau_{C0} = -\frac{1}{4}[\sqrt{(\zeta_1 - (\zeta_2 - \gamma_2))^2 + 8\gamma_2^2} - \sqrt{\zeta_1^2 + 8\gamma_2^2} + (\zeta_2 - \gamma_2)]. \]  

(42)

Hence the mode A superconductivity is enhanced provided that \( \zeta_2 + \gamma_2 < 0 \). On the other hand, the mode C superconductivity is enhanced provided that \( \zeta_2 - \gamma_2 > 0 \).

It follows from the formulae (39) and (40) that for appropriate values of parameters \( \zeta_1, \zeta_2 \) and \( \gamma_2 \), the onset temperatures \( \tau_A \) and/or \( \tau_C \) may become greater than the 2D critical temperature \( \tau_0 \). For such a material the structure may be superconducting even if it is composed of planes which after separation are overcritical.
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