Bending and torsion vibrations of a beam excited by a moving load

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Abstract. The linear dynamic bending- and torsion vibrations of a simply supported Bernoulli-Euler beam under a moving load with constant velocity are analysed. The main target is to derive an analytical solution of the governing partial differential equations. The deformations of the beam are computed based on the eigenmode expansion. For the computation of the solution first the method of generalized finite integral transformation is used followed by a transformation using the Laplace-Carson integral transformation. The resulting algebraic equation then is rearranged considering the boundary and initial conditions. The inverse transformation has to consider the position of the poles and after a further transformation the solution in the time domain results. The convergence and the necessary number of eigenmodes of this procedure is analysed. For the pure bending load the analysis is performed and the beam vibrations are calculated for various longitudinal speeds. For the special case of static deflections the solution is also given. The resulting deformations are plotted over time and space and the occurring phenomena are discussed. For the pure torsion deformation the Saint-Venant torsion theory is used. The described methods for solving the equations are used again. Based on the boundary and initial conditions the solution shows a similar structure like the pure bending solution. Finally the analytic solutions are compared to a numerical calculation of a finite element model which shows good agreement of the two solutions.

1. Introduction
Axially moving loads frequently are present within beam structures. Typically bridges are exposed to axially moving loads but also production machines like milling and laser engraving machines are excited by moving loads. A mechanical model is commonly using one-dimensional continua to model the beam structure where the Bernoulli-Euler-theory is applied and also the physical linearity in the form of a linear constitutive law is used. The linearized elasticity theory and a linear axis of the beam ($x$-axis) are the basic assumptions. Furthermore, it is required that the orientation of the principal axis of inertia of the cross section are the same as the $y$- und $z$-axis. The deformation due to the shear force is neglected. The homogeneity of the material and the constant cross section are assumed additionally in this work, where damping is not considered. For the considered cases of the pure bending load or pure torsion load it is assumed in this first approach that the center of gravity and the shear center of the cross section are at the same position. The consequence is the decoupling of the bending and torsional vibrations, which can be computed separately. The computation for the coupled vibrations will be discussed in some further work.
2. Beam bending deflection with transversal load

The differential equation of the deflection $w(x,t)$ of a straight beam without damping and with a constant cross section $A$ excited by an axially moving load $P(t)$ as given in Fig. 1 can be written as (see [1])

$$E J_y \frac{\partial^4}{\partial x^4} w(x,t) + \rho A \frac{\partial^2}{\partial t^2} w(x,t) = p(x,t) = \delta(x - c t) P(t)$$

The density $\rho$ of the material, the modulus of elasticity $E$ and the moment of inertia $J_y$ for bending about the $y$-axis are given. The function $\delta(x)$ represents the Dirac-Delta-function as defined in [2]. The differential equation (1) describes only vibrations in the $x$-$z$-plane. Vibrations in the perpendicular $x$-$y$-plain can be computed analogously. This linear partial differential equation of parabolic type for the independent variable $x$ and $t$ represents an initial boundary value problem. For the beam hinged on both ends as shown in Fig. 1 the geometric boundary conditions can be formulated $w(x,t) \big|_{x=0} = 0$ and $w(x,t) \big|_{x=l} = 0$. The dynamic boundary conditions are given as $\frac{\partial^2}{\partial x^2} w(x,t) \big|_{x=0} = 0$ and $\frac{\partial^2}{\partial x^2} w(x,t) \big|_{x=l} = 0$. The initial conditions are defined to be homogeneous: $w(x,t) \big|_{t=0} = 0$ and $\frac{\partial}{\partial t} w(x,t) \big|_{t=0} = 0$. The defined initial conditions describe a situation, where the beam is straight and in rest at the moment when the load is at the position $x = 0$.

In a first step the solution of the homogeneous Eq. (1) is expanded in eigenfunctions, see [3] and [4]. The solution is assumed as a product of separated functions $w(x,t) = w(x) \alpha(t)$ which transforms the partial differential equation (1) into an ordinary differential equation of fourth order in the spatial dimension $x$ and an ordinary differential equation of second order in the time domain $t$. The general solution $w(x,t)$ can be computed by superposition due to the linearity of the system and is written as the sum of the vibrations with the eigenforms $w(x,t) = \sum_j w_j(x) \alpha_j(t)$. The result are the homogeneous differential equations

$$w_j''''(x) - \frac{\rho A}{E J_y} \omega_j^2 w_j(x) = 0 \quad \text{and} \quad \dot{\alpha}_j(t) + \omega_j^2 \alpha_j(t) = 0 \quad \text{with} \quad \omega_j^2 = \frac{\lambda_j^4}{l^4} \frac{E J_y}{\rho A}.$$  

For the beam hinged at both ends the eigenvalues are computed to $\lambda_j = j \cdot \pi$ with $j = 1, 2, 3, \ldots$ and the eigenfunctions are derived to $w_j(x) = A_j \sin \left( \frac{j \pi x}{l} \right)$. Due to the orthogonality of the eigenfunctions the solutions are decoupled for different eigenfunctions. The solution can be derived by transforming the differential equation using the method of finite integral transformation. The transformation from the time domain to the generalised coordinate is defined by (see [5])

$$W(j,t) = \int_0^l w(x,t) w_j(x) \, dx \quad j = 1, 2, 3, \ldots$$
The inverse transformation is defined by
\[ w(x, t) = \sum_{j=1}^{\infty} \frac{1}{W_j} W(j, t) w_j(x) \] with \[ W_j = \int_{0}^{l} w_j^2(x) \, dx = \frac{1}{2} \, l. \] (4)

For the transformation Eq. (1) is multiplied by the eigenfunction \( w_j \) and then it is integrated over the length of the beam. After the partial integration and considering the initial and boundary conditions it follows
\[ \ddot{W}(j, t) + \omega_j^2 W(j, t) = \frac{1}{\rho \, A} P(t) \, w_j(c \, t). \] (5)

This differential equation is solved using the Laplace-Carson-integral transformation. The transformation of the unknown function is defined by
\[ W^*(j, p) = p \int_{0}^{\infty} W(j, t) e^{-pt} \, dt \] (6)
and the corresponding inverse transformation is given by
\[ w(j, t) = \frac{1}{2\pi i} \int_{a_0-i\infty}^{a_0+i\infty} e^{tp} \frac{W^*(j, p)}{p} \, dp. \] (7)

The parameter \( p \) is a variable in the complex plain. The constant factor \( a_0 \) shows that the integration is performed at a straight line parallel to the imaginary axis which is on the right side of all singularities of the equation \( e^{pt} \frac{W^*(j, p)}{p} \). This transformation is applied to Eq. (5) and multiplied with \( e^{-pt} \), then integrated over the time \( t \) from 0 to \( \infty \) and finally multiplied with the parameter \( p \):
\[ p^2 W^*(j, p) + \omega_j^2 W^*(j, p) = \frac{1}{\rho \, A} p \int_{0}^{\infty} P(t) w_j(c \, t) e^{-p \, t} \, dt \] (8)

The inverse finite integral transformation considering the inverse transformation from Eq. (4) gives the solution for the considered case
\[ w(x, t) = \sum_{j=1}^{\infty} \left\{ \frac{1}{\rho \, A} \frac{1}{W_j} w_j(x) \int_{0}^{t} P(\tau) w_j(c \tau) \sin (\omega_j(t - \tau)) \, d\tau \right\}. \] (9)

With this result and the definition of the frequency of the excitation, which can be assigned to the axial motion
\[ \Omega_j = \frac{j \, \pi \, c}{l} \] (10)
Eq. (9) gives after the integration for the case \( \omega_j \neq \Omega_j \)
\[ w(x, t) = \sum_{j=1}^{\infty} \frac{P}{\rho \, A} \frac{w_j(x)}{W_j} \left\{ \frac{1}{\omega_j^2 - \Omega_j^2} \left[ \sin (\Omega_j \, t) - \frac{\Omega_j}{\omega_j} \sin (\omega_j \, t) \right] \right\}. \] (11)

This solution can be compared with the results given in [6], where exclusively the case \( \omega_j \neq \Omega_j \) is discussed and no case distinction was made. If the special case \( \omega_j = \Omega_k \) occurs, this index has to be excluded in the integration and the solution the solution results in
\[ w(x, t) = \sum_{j=1, \ldots, k \neq j, k+1, \ldots}^{\infty} \frac{P}{\rho \, A} \frac{w_j(x)}{W_j} \left\{ \frac{1}{\omega_j^2 - \Omega_j^2} \left[ \sin (\Omega_j \, t) - \frac{\Omega_j}{\omega_j} \sin (\omega_j \, t) \right] \right\} + \frac{P}{\rho \, A} \frac{w_k(x)}{W_k} \left[ \frac{1}{2 \, \omega_k^2} \left[ \sin (\omega_k \, t) - \omega_k \, t \cos (\omega_k \, t) \right] \right]. \] (12)
This special case is related to the integer number of the indices

\[ k = \frac{j^2 \pi}{l c} \sqrt{\frac{E J_y}{\rho A}} \in N \]  

or for a given and defined axial speed

\[ c_{j,k} = \frac{j^2 \pi}{k l} \sqrt{\frac{E J_y}{\rho A}} = \frac{l}{k} \pi \omega_j = \frac{j^2}{k} \gamma. \]  

It is seen that there is an infinite number of axial speeds \( c_{j,k} \) where the resonance case is given. Due to the motion of the excitation load between the two bearings, the resonance case results in limited amplitudes. The solution with the expansion using the eigenfunctions is computed quite analogous to [6] and after partial integration and application of the orthogonality of the eigenfunctions the results of eqs. (11) and (12) are derived. The dimensionless solution \( \frac{w(x,t)}{w_0} \) is related to the analytic expression for the static deformation in the center of the beam \( w_0 \). In Fig. 2 the solution of the deformation of the beam is plotted as lines at selected times indicating the corresponding position of the moving load. For the considered total time interval the simulation time \( T \) is divided into 30 steps. In Fig. 2 it is shown that due to the dynamic effect of the many eigenfrequencies and variable positions of the load the superimposed vibrations do not show an evenly increasing deflection. In the center of the beam in Fig. 3 the deflection is higher than the static deflection amplitude. The analysis of the convergence behaviour of the deflection \( w(x,t) \) shows, after a division by the static deflection in the center of the beam \( w_0 \), that a convergence of fourth order is present. This means that only a few terms of the series have to be considered for an accurate solution. In Fig. 3 the fast convergence of the solution can be seen. The value of the amplitude of the second eigenfunction is 6.0 % of the amplitude of the first eigenfunction and the amplitude of the third eigenfunction is 1.2 % of the amplitude of the first eigenfunction. As a consequence the first eigenfunction describes the solution very good. The further eigenfunctions give only a small adjustment of the amplitude. A Finite-Element analysis for this case with a fine mesh needs a high effort and shows a good agreement. In order

![Figure 2. Dimensionless deflection at selected times](image)
Figure 3. Dimensionless deflection in the center of the beam

Figure 4. Dimensionless deflection in the center of the beam for various dimensionless speeds, when the load is at the position $\bar{x}_p$

to study the influence of the axial speed a dimensionless speed is defined, where the axial speed is divided by the first critical speed computed by Eq. (14). This gives the dimensionless speed formulation $\bar{c} = \frac{c}{c_{krit,1}} = \frac{l}{\pi} \sqrt{\frac{\rho A}{E J_y}}$. In Fig. 4 the relation of static and dynamic deflection in the center of the beam is shown for various dimensionless speeds of the axially moving load, when the load is at the position $\bar{x}_p = \frac{x_p}{l}$. The solid line shows the relative static deflection in the center of the beam when the load is in the center. The maximum dimensionless deflection $\bar{w} = 1.52$ occurs at a dimensionless speed of $\bar{c} = 0.35$ for a position of the load of $\bar{x}_p = 0.52$. For a speed of $\bar{c} = 0.3$ the maximum deflection of $\bar{w} = 1.41$ results for a position of the load of $\bar{x}_p = 0.46$. For a speed of $\bar{c} = 0.25$ the maximum deflection of $\bar{w} = 1.26$ is computed at a position of the load of $\bar{x}_p = 0.40$. 
3. Beam with a moving torsion load

The following analysis considers the pure torsion or Saint-Venant-torsion which means that torsion is computed without constraining warping by defining boundary and transition conditions. The beam with the load and the bearings is shown in Fig. 5. The differential equation of the distortion $\varphi_x(x,t)$ of a straight undamped beam with constant cross section excited by an axially moving moment per unit length $m_T(x,t)$ acting in the longitudinal beam axis with the shear modulus $G$, the torsion resistance $J_t$ and the torsional stiffness $G J_t$, the density $\rho$, the torsional moment of inertia $I_0$ (also called polar moment of inertia $I_p$) can be derived, see [1], [7] and [8], and is written as

$$G J_t \frac{\partial^2}{\partial x^2} \varphi_x(x,t) - \rho I_0 \frac{\partial^2}{\partial t^2} \varphi_x(x,t) = -m_T(x,t) = -\delta(x-c t) M(t) \quad (15)$$

Figure 5. Model of the beam with a torsion load.

The warping is not restricted and the corresponding axial deformation of the cross section is assumed to be free at the end. The influence of warping can be neglected although the distributed torsion load is not constant. The linear partial differential equation of hyperbolic type in the variables $x$ and $t$ represents an initial boundary value problem. For the beam in Fig. 5 the rotations are fixed at the bearings and the geometric boundary conditions are (homogeneous rotation at the boundary) $\varphi_x(x,t)|_{x=0} = 0$ and $\varphi_x(x,t)|_{x=l} = 0$. The initial values for the beam are $\varphi_x(x,t)|_{t=0} = 0$ and $\frac{\partial}{\partial t} \varphi_x(x,t)|_{t=0} = 0$. Again analogous to the beam bending for the corresponding eigenvalue problem a product of separated solution is assumed for the rotation angle $\varphi_x(x,t) = \sum_j \varphi_j(x) \beta_j(t)$. The partial differential equations (15) are decoupled by the Bernoulli separation product into two ordinary differential equations of second order in the spatial dimension $x$ and the time domain $t$:

$$\varphi''_j(x) + \frac{\rho I_0}{G J_t} \omega^2_j \varphi_j(x) = 0 \quad \text{and} \quad \beta_j(t) + \omega^2_j \beta_j(t) = 0 \quad \text{with} \quad \omega^2_j = \frac{\lambda^2_j}{l^2} \frac{G J_t}{\rho I_0}. \quad (16)$$

Considering the boundary conditions the eigenvalues are $\lambda_j = j \pi$ with $n = 1, 2, 3, \ldots$ and the computed eigenfunctions are $\varphi_j(x) = \sin \left( \frac{j \pi x}{l} \right)$. Due to the considered boundary conditions these eigenfunctions of the torsion vibrations are the same as that for the bending vibrations. For the torsion loaded beam the same finite integral transformation is used and the same expansion of the solution in eigenfunctions is performed so that the solution of the torsional vibrations result in the case $\omega_j \neq \Omega_j$ to

$$\varphi_x(x,t) = \sum_{j=1}^{\infty} \left\{ \frac{M}{\rho I_0} \varphi_j(x) \frac{1}{\omega^2_j - \Omega^2_j} \left[ \sin (\Omega_j t) - \sin (\omega_j t) \frac{\Omega_j}{\omega_j} \right] \right\} \quad (17)$$
For the special case $\omega_j = \Omega_k$ it follows analogous as before

$$
\varphi_x(x, t) = \sum_{j=1, \ldots, k-1, k+1, \ldots}^{\infty} \left\{ \frac{M}{\rho I_0} \frac{\varphi_j(x)}{\Phi_j} \frac{1}{\omega_j^2 - \Omega_j^2} \left[ \sin (\Omega_j t) - \sin (\omega_j t) \frac{\Omega_j}{\omega_j} \right] \right\}
+ \frac{M}{\rho I_0} \frac{\varphi_k(x)}{\Phi_k} \frac{1}{2 \omega_k^2} \left[ \sin (\omega_k t) - \omega_k t \cos (\omega_k t) \right]
$$

(18)

This special case occurs only for integer indices $k = j c_k$ or for defined axial speeds $c_{2,j,k} = \frac{j}{k} \sqrt{\frac{G J_t}{\rho I_0}} \sqrt{\frac{G J_t}{\rho I_0}}$. Again the solution is shown in Fig. 6 in a dimensionless form, where the distortion is divided by the static distortion in the center of the beam $\varphi_0$ when the load position is also in the center. The lines are the distortions at selected times drawn with the position of the moving moment. A simulation time of $T$ is divided into 30 steps. This result shows again a good correlation with a Finite-Element simulation.

Figure 6. Dimensionless distortion angle at selected times.

In Fig. 7 the convergence of the solution can be seen. A comparison shows that the convergence in this case is only of second order and hence much slower. The maximum amplitude of the second eigenfunction is 25 % of the amplitude of the first eigenfunction and the amplitude of the third eigenfunction has a value of 11.1 %. Now the axial speed has a small influence on the convergence and only for high variations the influence can be detected. For the selected case it is seen that the dynamic amplitude is less than the amplitude of the static solution of the distortion. Defining the dimensionless axial speed $\bar{c} = \frac{c}{c_{krit,1}} = \frac{\sqrt{\rho I_0}}{G J_t}$ analogous as before the influence of the axial speed of the torsion load to the dimensionless distortion is shown in Fig. 8. The solid line is the relative static distortion for a position of the load in the center of the beam. It can be seen that the maximum of the amplitude of the distortion is less than that for the bending solution.

4. Conclusion

The bending and the torsion vibrations have been computed for an axially moving transversal and torsion load. For the selected boundary conditions the eigenfunctions are the same and
closed form solutions have been derived. It was shown in the computed results that the convergence is better for the bending deflection. A Finite-Element simulation requires a high effort and shows very good agreement.

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