Involute, minimal, outer and increasingly trapped surfaces

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Eight different re-namings of trapped surfaces are proposed, of three basic types, each intended as potential stability conditions. M inimal trapped surfaces are strictly minimal with respect to the dual expansion vector. Outer trapped surfaces have positivity of a certain curvature, related to surface gravity. Increasingly (future, respectively past) trapped surfaces generate surfaces which are more trapped in a (future, respectively past) causal variation, with three types: in any such causal variation, along the expansion vector, and in some such causal variation. This suggests a definition of doubly outer trapped surface involving two independent curvatures. This in turn suggests a definition of involute trapped surface. Adding a weaker condition, the eight conditions form an interwoven hierarchy, with four independent relations which assume the null energy condition, and another holding in a special case of symmetric curvature.

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I. INTRODUCTION

Trapped surfaces, as originally defined by Penrose [1], play an important role in gravitational physics, both for black holes and in cosmology. Such surfaces were crucial in the singularity theorems of Penrose and Hawking [1,2,3]. More recent years have seen the development of a local dynamical theory of black holes in terms of minimal surfaces, a limit of trapped surfaces, including laws of black-hole dynamics involving physical quantities such as mass and surface gravity [4,5,6,7,8,9,10].

On the other hand, trapped surfaces as simply defined can have some very peculiar properties, such as the boundary of the region of trapped surfaces not being composed of minimal surfaces [11,12,13]. As has been argued in a preliminary article [14], this suggests re-naming trapped surfaces in some natural way. Penrose already showed that the surface should be compact [1]. The most obvious other re-naming is to positive Gaussian curvature, which implies spherical topology if orientable, as will be assumed henceforth, but it does not suffice to resolve the above issue [15]. It seems that a stability condition is required, which one might expect to be an inequality of one differential order higher than that of trapped itself, as for minimal surfaces [11,12,13].

The preliminary article on spherical symmetry proposed seven such re-namings, of essentially three types [14]. In general cases, comparison of the definitions using the Einstein equation motivates an additional condition. The conditions are of some interest in themselves and turn out to be related: they form an interwoven hierarchy, with some direct relations and some relations which assume the Einstein equation, or more exactly, just the null energy condition (NEC).

The article is organized as follows. Section II explains how any spatial surface in space-time is extremal in some sense along direction, and defines minimal trapped surfaces. Section III defines outer trapped surfaces and shows that the condition is implied by minimal trapped. Section IV defines increasingly trapped surfaces and shows that the condition is implied by outer trapped. Section V defines doubly outer trapped surfaces, and shows that the condition in implies both outer trapped and a stricter type of increasingly trapped. Section VI defines involute trapped surfaces, and shows that the condition in implies both minimal trapped and doubly outer trapped. A laxer condition of involute trapped in implies a laxer condition of increasingly trapped only in cases where outer trapped and doubly outer trapped are equivalent. The required geometric quantities, such as expansion vector $H$, dual expansion vector $H^\perp$ and curvatures $K$ and $K^\perp$ related to surface gravity, are introduced as required, with concurrent comparisons to spherical symmetry, where the physical meanings are clearer. Section VII concludes with a hierarchy diagram, which one might well consult.
II. MINIMAL TRAPPED SURFACES

The objects of study are spatial surfaces $S$ embedded in a given space-time. Here one may note a classification of such surfaces by Senovilla [20]. Of concern are normal vectors, with $L$ denoting the Lie derivative along. This induces a norm all-form $L$ derivative denoted by

$$L(\cdot) = L : \quad (1)$$

Note that while the norm al space need not be integrable, there is a norm al vector space at each point, and therefore a 1-form space denoted by their action on such vectors. The expansion 1-form is denoted by

$$\omega = L(\omega) \quad (2)$$

where $\omega$ is the Hodge operator induced on $S$ by the space-time metric $g$, i.e. $\omega$ is the area form and its logarithm is norm al derivative. The expansion vector (a.k.a. mean-curvature vector) is

$$H = g^{i}(\cdot) ; \quad (3)$$

There is also a Hodge operator in the norm al space, e.g. is the dual expansion 1-form. This induces a duality operation on norm al vectors by

$$g(\cdot) = g(\cdot) \quad (4)$$

or equivalently

$$g(\cdot) = 0 ; \quad g(\cdot) = g(\cdot) ; \quad (5)$$

In particular, there is the dual expansion vector $H$. In spherical symmetry, one has $H = 2k = r$ and $H = 2k = r$, where $r$ is the area radius and $k$ is the Kodama vector [14, 21, 22, 23].

A surface is trapped if $H$ is temporal, or equivalently if $H$ is spatial. Assuming a time-orientable space-time, the surface is future (respectively past) trapped if $H$ is future (respectively past) temporal.

Now

$$H = 0 \quad (6)$$

which expresses that any surface is extremal in the $H$ direction. In the special case $H = 0$, the surface is extremal in any norm al direction, but otherwise $H$ gives the unique such norm al direction. Thus a trapped surface is equivalently defined as a surface which is extremal in a unique spatial norm al direction. Then it is natural to ask whether the surface is not merely extremal but minimal.

Definition 1. A (strictly) minimal trapped surface is a trapped surface for which, for small variation,

$$Q = H^{a}H^{b}r_{ba} > 0 \quad (7)$$

where $r$ is the covariant derivative operator of $g$. Note that minimality itself requires only a non-strict inequality, but the strict sign will turn out to be convenient. In expressions with indices, there is no need to distinguish from $H$, or from $H$, but the distinction will be maintained here.

To avoid cumbersome calculations with connection components, it is useful to rewrite the minimality condition as follows. First note that

$$Q = H^{a}H^{b}r_{ba} \quad (8)$$

due to orthogonality of $H$ and $H$. For any 1-form, one has

$$b_{r_{ba}} = b_{r_{a b}} + 2 b_{r_{[a b]}}$$

$$= \frac{1}{2} (dg^{i}(\cdot)_{a} + b^{d}(\cdot)_{ba}) \quad (9)$$

where $d$ is the exterior derivative, so that one can do computationally straightforward calculations using exterior calculus. Since the concern here is with norm al vectors and 1-forms, one can write

$$2Q = H^{a}(dg^{1}(\cdot)_{a}) + 2H^{a}H^{b}(d_{b})_{a} \quad (10)$$

where $d$ is henceforth the norm al exterior derivative.
III. OUTER TRAPPED SURFACES

In spherical symmetry, outer trapped spheres were defined by $T > 0$, where $T$ is surface gravity \[14,23,24\]. The relevant object in general turns out to be

\[ K = \int d^3 x \sqrt{g} \left( \frac{16}{27} \right) \]

where $d$ is the normal codi- ecal divisor, $\partial$ the sign convention. For want of a better term, $K$ will be called the surface curvature. In spherical symmetry, one ends $K = 4 = r$. This object has appeared before: a previous definition \[25\] of quasi-local surface gravity was

\[ K = \frac{1}{16} \int \frac{R}{R} \, d^3 x \]

where $R = P \frac{\partial}{\partial A} 4$ is the area radius, i.e. area

\[ A = \frac{1}{4} 4 \frac{R}{R} \]

This enters a quasi-local rst law for trapping horizons \[25\] involving the Hawking mass \[24\].

Definition 2. An outer trapped surface is a trapped surface for which, for some variation,

\[ K > 0 \]

Lemma 1. Assuming the Einstein equation with units $G = 1$,

\[ 2Q = g^{11} \left( \frac{1}{16} H + w \right) \]

where

\[ w = \frac{1}{2} \text{tr} (T + ) \]

is an energy density, where the trace is in the normal space, and is an effective energy tensor for gravitational radiation which has appeared in various contexts \[8,27,28,29,30,31,32,33,34,35,36\].

Proof. It is convenient to use a dual-null form, describing null hypersurfaces generated from the surface in the null normal directions. This will not be described in detail here, instead referring to \[37\] and, in almost the same notation as here, \[33\]. In terms of null coordinates $x$ and the corresponding null normal vectors $l$, and with the shorthand notation $L = l$, one has

\[ e^{2T} = g^{11} (dx^+ ; dx^-) \]

\[ l = e^{2T} g^{11} (dx^-) \]

\[ = dx^- + dx \]

\[ = dx^- + \frac{1}{2} \, ^2L \, (dx^-) \]

\[ H = e^{2T} (dx^- + L + L^-) \]

\[ H = e^{2T} (dx^- + L + L^-) \]

Then

\[ g^{11} (x) = 2 e^{2L} \]

\[ d = (L^- + L^+) dx^+ \]

\[ d = e^{2T} (L^- + L^+) \]

\[ K = e^{2T} (L^- + L^+ + + ) \]

Then \[30\] expands as

\[ Q = e^{2T} (L^- + L^+) (e^{2T} + ) 2e^{4T} + (L^- + L^+) \]

\[ = e^{2T} 2 L^- (e^{2T} + ) + L^+ (e^{2T} ) e^{4T} + (L^- + L^+) \]

\[ = e^{2T} 2 L^- (e^{2T} + ) + 2 e^{2T} + + L^+ (e^{2T} ) + 2 e^{2T} 2 e^{4T} + (L^- + L^+ + + ) \]
where the last step has added and subtracted terms in $e^{4' \ 
\frac{1}{2} \ 2}$. The reason for this comes from the null-null components of the Einstein equation, which can be written as [31,37]:
\[ e^{2'' \ T} (e^{2''}) + \frac{1}{2} \ 2 = 8 \ T \ \frac{1}{2} \ jj \ jj \]
where $jj$ are the null shears and $jj$ their norms with respect to the induced metric on the surface. Now the effective energy tensor has null-null components [27,28,29,30,31,32,33,34,35,36]
\[ = jj \ jj = 32 \]
and will cancel out. Thus
\[ Q = 8 \ e^{4' \ 2} (T_{++} + \ldots) + \frac{1}{2} (T + \ldots) \ e^{4'} + (L_{+} + L_{\ldots} + \ldots); \]
Finally one has
\[ w = e^{2' (T_{++} + \ldots)} \]
\[ = e^{2'} (T_{++} + \ldots)dx^+ + \ldots (T + \ldots)dx \]
which implies
\[ H = e^{4' \ 2} (T_{++} + \ldots) + \frac{1}{2} (T + \ldots); \]
Then (15) follows by collecting terms.
Proposition 1. NEC and minimal trapped implies outer trapped.
Proof. NEC $H = 0$, as is most easily seen from $T = 0$, $0$ and the above equation. For a trapped surface, $g^{1}(\ ; \ ) < 0$, then inspect signs in [4], [14], and [15].
In spherical symmetry, one finds $Q = 8k_2 \ k_2 \ k_2 = 8$, $g^{1}(\ ; \ ) = 4(1 - 2m = r) = r^2$ and $H = 4k = r^2$ in terms of the mass function $m$ and energy $ux$ [14,22,23,27,38].

IV. INCREASINGLY TRAPPED SURFACES

Noting that $g^{1}(\ ; \ ) = g^{1}(\ ; \ )$ vanishes for marginal surfaces and is positive for trapped surfaces, it can be taken as a measure of how trapped a surface is. The idea then is to ask whether it is increasing to the future (respectively past) for a future (respectively past) trapped surface.
Definition 3. An increasingly trapped surface is a trapped surface for which, for some variation,
\[ H \ dg^{1}(\ ; \ ) < 0; \]
Lemma 2. Assuming the Einstein equation,
\[ H \ dg^{1}(\ ; \ ) = 16 \ H \ g^{1}(\ ; \ ) (K \ g^{1}(\ ; \ )); \]
The proof is a special case of the one given in the next section.
Proposition 2. NEC and outer trapped implies increasingly trapped.
Proof. For a trapped surface, $g^{1}(\ ; \ ) < 0$, and NEC $H = 0$ as before, then inspect signs in [4], [36]. This motivates the following.
Definition 4. An anyhow increasingly trapped surface is a future (respectively past) trapped surface for which, for all variations along a future (respectively past) causal normal vector,
\[ dg^{1}(\ ; \ ) < 0; \]
Definition 5. A somehow increasingly trapped surface is a future (respectively past) trapped surface for which, for some variation along a future (respectively past) causal normal vector,
\[ dg^{1}(\ ; \ ) < 0; \]
Clearly anyhow increasingly trapped implies increasingly trapped, which implies somehow increasingly trapped.
V. Doubly Outer Trapped Surfaces

In spherical symmetry, outer trapped implies anyhow increasingly trapped \[14\], but this does not hold in general. Instead, a stricter version of outer trapped has this property, as follows. Introduce two more surface curvatures:

\[ K_{(1)} = \frac{d}{d^2} g^1(\ ; \ ) \] (39)

where ( ) indicates a label rather than an index. Then \( 2K = K_{(1)} + K_{(1)} \).

Definition 6. A doubly outer trapped surface is a trapped surface for which, for all variations,

\[ K_{(1)} > 0; \quad K_{(1)} > 0; \] (40)

Clearly doubly outer trapped implies outer trapped.

Lemma 3. Assuming the Einstein equations,\[ \frac{1}{2} g^1(\ ; \ ) = 16 + g^1(\ ; \ ) \quad (K_2 \frac{1}{2} ( + )K) ; \] (41)

Proof. First:

\[ \frac{d}{d^2} = (L_v L_v) dx^v \] (42)
\[ d = e^{2v} (L_v L_v) \] (43)
\[ K_{(1)} = e^{2v} (2L_v + \ ; \ ) ; \] (44)

Then

\[ \frac{1}{2} g^1(\ ; \ ) = (L_v + L_v) e^{2v} \]
\[ = (L_v + L_v) e^{2v} + \]
\[ = (L_v + L_v) e^{2v} + \]
\[ = (L_v + L_v) e^{2v} + \]
\[ + e^{2v} \]
\[ + e^{2v} \]
\[ + (L_v + \frac{1}{2} + ) + (L_v + \frac{1}{2} + ) \] (45)

where terms have been added and subtracted in the last step in order to use the null-null Einstein equations \[29\] again. Then

\[ \frac{1}{2} g^1(\ ; \ ) = 8 e^{2v} \]
\[ + e^{2v} \]
\[ + (L_v + \frac{1}{2} + ) + (L_v + \frac{1}{2} + ) \] (46)

where more terms have been added and subtracted. The terms in can be written invariantly using

\[ = 2 dx ; \] (47)

Finally \[33\] gives

\[ = e^{2v} \]
\[ + (T_v + \ + \ ) + (T_v + \ ) \] (48)

and \[41\] follows.

Proposition 3. NEC and doubly outer trapped implies anyhow increasingly trapped. Proof. For a trapped surface, \( g^1(\ ; \ ) < 0 \), while for in the appropriate causal quadrant, \( < 0 \), \( 0 \) and \( 0 \), so \( \) \( 0 \), then NEC \( 0 \), as is most easily seen from the above equation. The \[34\] in \[43\] implies \( \frac{1}{2} g^1(\ ; \ ) < 0 \) for all such .

Taking \( = H \) proves Lemma 2. The proof also makes clear that outer trapped generally does not imply anyhow increasingly trapped.
VI. INVOLUTE TRAPPED SURFACES

Given the above result, doubly outer trapped seems to be a natural condition, so one might ask whether it is implied by minimal trapped. The answer is negative, but if the latter is needed further, such a result can be obtained. The form motivates the following.

Definition 7. An involute trapped surface is a future (respectively past) trapped surface for which, for all variations along a future (respectively past) causal normal vector,

$$Q = aH^bR_{b}(j) > 0:$$

Clearly involute trapped implies minimal trapped. Involute means curved or curled inwards, as of a leaf, in this case referring to the curvature of the surface in the space-time. The Latin root involutere means to envelop or conceal, as also appropriate in the context of black holes.

Lemma 4. Assuming the Einstein equation,

$$2Q = 16 + \frac{i}{2} (\kappa^1 + \frac{i}{2} (\kappa^1 \kappa)$$

Proof. Using the identity again,

$$Q = \frac{1}{2} a^2 (dg_1(\gamma))_a + aH^b(d \gamma_b):$$

The second term expands as

$$aH^b(d \gamma_b) = e^{2^{(1)}(\gamma^1 + \gamma^1) + \kappa^1 \kappa^1(L_+ + L_+)}$$

while the first term has been calculated above. Adding, one finds that two contributions from the second term cancel, leaving

$$Q = 8 e^{2^{(1)}(\gamma^1 + \gamma^1) + \kappa^1 \kappa^1(L_+ + L_+)}$$

Collecting terms yields.

Proposition 4. NEC and involute trapped implies doubly outer trapped.

Proof. As before, 0 and NEC 0. Considering the cases = 1, both K(1) must be positive.

The proof also makes clear that minimal trapped generally does not imply doubly outer trapped.

Definition 8. An involute trapped surface is a future (respectively past) trapped surface for which, for some variation along a future (respectively past) causal normal vector,

$$Q = aH^bR_{b}(j) > 0:$$

Clearly minimal trapped implies somehow involute trapped. One might ask whether somehow involute trapped implies somehow increasingly trapped. The answer is negative, except for a special case.

Proposition 5. NEC and somehow involute trapped implies somehow increasingly trapped if K(1) = K(1).

Proof. From one gets

$$K(1) + K(1) < 0:$$

However, to get increasingly trapped instead generally requires

$$K(1) + K(1) < 0:$$

Thus it works if K(1) = K(1).
VII. REMARKS

Before concluding, one issue should be addressed. Concerning marginal surfaces, say with \( \kappa = 0 \), the author has previously proposed defining an outer trapping horizon by \( L_{\kappa} < 0 \), as this naturally expresses that the vanishing expansion decreases in an inward direction, and implies some basic properties expected of black holes, such as spherical topology, adrenality or one-way traversability, and a second law, assuming either the dominant energy condition or just NEC [4, 9, 31, 32]. This is equivalent to \( K_{+} > 0 \). Conversely, for a trapping horizon with \( \kappa = 0 \), one would use \( K_{\kappa} > 0 \). On the other hand, for a trapped surface, there is no reason to break the symmetry between \( K_{\kappa} \), so the natural conditions are either \( K > 0 \) or both \( K_{\kappa} > 0 \). Thus if one wants a unified treatment of both trapped and marginal surfaces, and surfaces which are partly trapped and partly marginal, in the context of black holes, it suggests the doubly outer condition for all. Thus if the motivating conjecture turns out to be true, that is, the boundary of some suitably refined trapped region is a trapping horizon, one might further conjecture that it is a doubly outer trapping horizon, i.e. both \( K_{\kappa} > 0 \).

In summary, the hierarchy of trapped surfaces is illustrated as follows:

\[
\begin{align*}
&\text{involute) doubly outer) anyhow increasingly} \\
&+\quad \text{NEC} + \quad \text{NEC} + \\
&\text{minimal) outer) increasingly} \\
&+\quad \text{NEC} \quad \text{NEC} + \\
&\text{somehow involute) somehow increasingly} \\
&\text{NEC, } K_{+} = K_{\kappa} \quad (57)
\end{align*}
\]

where the vertical implications are straightforward, while the horizontal implications require NEC, and in the last case, the symmetry where the curvatures \( K_{\kappa} \) are equal, or equivalently where outer trapped planes doubly outer trapped, in which case the hierarchy collapses to a strict hierarchy as in spherical symmetry [14]. Otherwise, to borrow language of warp and weft from weaving, the threads are respectively geometrical warp and physical weft.

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