Toward frame-like gauge invariant formulation for massive mixed symmetry bosonic fields

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Abstract

In this paper, as a first step toward frame-like gauge invariant formulation for massive mixed symmetry bosonic fields, we consider mixed tensors, corresponding to Young tableau with two rows with $k \geq 2$ boxes in the first row and only one box in the second one. We construct complete Lagrangian and gauge transformations describing massive particles in (anti) de Sitter space-time with arbitrary dimension $d \geq 4$ and investigate all possible massless and partially massless limits.

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Introduction

As is well known, in $d = 4$ dimensions for the description of arbitrary spin particles it is enough to consider completely symmetric (spin-)tensor fields only. At the same time, in dimensions greater than four in many cases like supergravity theories, superstrings and high spin theories, one has to deal with mixed symmetry (spin-)tensor fields [1, 2, 3, 4]. There are different approaches to investigation of such fields both light-cone [5, 6], as well as explicitly Lorentz covariant ones (e.g. [7, 8, 9, 10, 11, 12]). For the investigation of possible interacting theories for high spin particles as well as of gauge symmetry algebras behind them it is very convenient to use so called frame-like formulation [13, 14, 15] (see also [16, 17, 18]) which is a natural generalization of well known frame formulation of gravity in terms of veilbein $e_\mu^a$ and Lorentz connection $\omega^{ab}_\mu$.

Till now, most of the papers on frame-like formulation for mixed symmetry fields deal with massless case [19, 20, 21, 22, 23, 24] (see however [25]). The aim of this work is to start an extension of frame-like formulation to the case of massive mixed symmetry fields. Namely, we will start a construction of gauge invariant formulation for such mixed symmetry massive fields in general $(A)dS_d$ space-times with non-zero cosmological constant and arbitrary space-time dimension $d \geq 4$. There are two general approaches to gauge invariant description of massive fields. One of them use powerful BRST approach [26, 27, 28, 29, 11, 12]. Another one, which we will follow in this work, [30, 31, 32, 33, 18] (see also [9, 25, 34, 35, 36]) is a generalization to high spins of well known mechanism of spontaneous gauge symmetry breaking. In this, one starts with appropriate set of massless fields with all their gauge symmetries and obtain gauge invariant description of massive field as a smooth deformation.

One of the nice feature of gauge invariant formulation for massive fields is that it allows us effectively use all known properties of massless fields serving as building blocks. There are two different frame-like formulations for massless mixed symmetry bosonic fields. For simplicity, let us restrict ourselves with mixed symmetry tensors corresponding to Young tableau with two rows. In what follows we will denote $Y(k,l)$ a tensor $\Phi^{a_1...a_k,b_1...b_l}$ which is symmetric both on first $k$ as well as last $l$ indices, completely traceless on all indices and satisfies a constraint $\Phi^{(a_1...a_k,b_1)...b_l} = 0$, where round brackets mean symmetrization. In the first approach [20, 21, 22, 23] for the description of $Y(k,l)$ tensor ($k \neq l$) one use a one-form $e^{Y(k-1,l)}_\mu$ as a main physical field. In this, only one of two gauge symmetries is realized explicitly and such approach is very well adapted for the $(A)dS$ spaces. Another formulation [24] uses two-form $e^{Y(k-1,l-1)}_{\mu\nu}$ as a main physical field in this, both gauge symmetries are realized explicitly. Such formalism works in flat Minkowski space while deformations into $(A)dS$ space requires introduction of additional fields [37]. In this paper we will use the second formalism. As we have already seen in all cases considered previously and we will see again in this paper, gauge invariant description of massive fields always allows smooth deformation into $(A)dS$ space without introduction of any additional fields besides those that are necessary in flat Minkowski space so that restriction mentioned above will not be essential for us.

Mixed symmetry tensor fields have more gauge symmetries compared with well known case of completely symmetric tensors and, as a result, gauge invariant formulation for them requires more additional fields making construction much more involved. In this paper, as a first step toward gauge invariant frame-like formulation of mixed symmetry bosonic fields,
we consider $Y(k,1)$ tensors for arbitrary $k \geq 2$. This case turns out to be special and any-
way requires separate consideration. Indeed, in general case $Y(k,l)$, $l > 1$, auxiliary field
analogous to Lorentz connection has to be a two-form $\omega_{\mu\nu}^Y(k-1,l-1,1)$, while for the $Y(k,1)$
case one has to introduce one-form $\omega_{\mu}^Y(k-1,1,1)$ instead. Thus this case turns out to be a
natural generalization of simplest model for $Y(2,1)$ tensor constructed by us before [25].

The structure of the paper is simple. In section 1 we reproduce our results for simplest
$Y(2,1)$ tensor. Then, in section 2 we consider more complex case — $Y(3,1)$ which shows
practically all general features. At last, in section 3 we construct massive theory for general
$Y(k,1)$ tensor field. In all cases we construct complete Lagrangian and gauge transforma-
tions describing massive particles in $(A)dS_d$ spaces with arbitrary cosmological constant and
arbitrary space-time dimension $d \geq 4$ and investigate all possible massless and partially
massless limits [38, 39, 40, 32, 16].

1 Tensor $Y(2,1)$

In this case frame-like formulation requires two tensors [19]: two-form $\Phi_{\mu\nu}^a$ as a main phys-
ical field and one-form $\Omega_{\mu}^{abc}$, antisymmetric on $abc$, as analogue of Lorentz connection.
To describe correct number of physical degrees of freedom, massless Lagrangian has to be
invariant under the following gauge transformations:

$$
\delta \Phi_{\mu\nu}^a = \partial_\mu \xi_\nu^a + \eta_{\mu\nu}^a, \quad \delta \Omega_{\mu}^{abc} = \partial_\mu \eta^{abc}
$$

where $\eta^{abc}$ is completely antisymmetric. Note that $\xi$-transformations are reducible, i.e.

$$
\xi_\mu^a = \partial_\mu \chi^a \quad \Rightarrow \quad \delta \Phi_{\mu\nu}^a = 0
$$

One of the advantages of frame-like formulation is the possibility to construct an object ("torsion") out of first derivatives of main physical field $\Phi_{\mu\nu}^a$ which is invariant under $\xi$-
transformations:

$$
T_{\mu\nu\alpha}^a = \partial_\mu \Phi_{\nu\alpha}^a + \partial_\nu \Phi_{\alpha\mu}^a + \partial_\alpha \Phi_{\mu\nu}^a
$$

To find a correct form of massless Lagrangian one can use the following simple trick. Let us
consider an expression:

$$
\{ \mu \nu \alpha \beta \}_{abcd} \Omega_{\mu}^{abc} T_{\nu,\alpha \beta}^d
$$

and make a substitution $T_{\mu\alpha}^a \rightarrow \Omega_{[\mu,\nu\alpha]}^a$. We obtain:

$$
\{ \mu \nu \alpha \beta \}_{abcd} \Omega_{\mu}^{abc} T_{\nu,\alpha \beta}^d \Rightarrow \{ \mu \nu \alpha \beta \}_{abcd} \Omega_{\mu}^{abc} \Omega_{\nu,\alpha \beta}^d
$$

Thus we will look for massless Lagrangian in the form:

$$
L_0 = a_1 \{ \mu \nu \}_{ab} \Omega_{\mu}^{acd} \Omega_{\nu}^{bcd} + a_2 \{ \mu \nu \alpha \beta \}_{abcd} \Omega_{\mu}^{abcd} T_{\nu,\alpha \beta}^d
$$

It is (by construction) invariant under the $\xi$-transformations, while invariance under $\eta$-
transformations requires $a_1 = -9a_2$. We choose $a_1 = -3$, $a_2 = \frac{1}{3}$ and obtain finally:

$$
L_0 = -3 \{ \mu \nu \}_{ab} \Omega_{\mu}^{acd} \Omega_{\nu}^{bcd} + \{ \mu \nu \alpha \beta \}_{abcd} \Omega_{\mu}^{abc} \partial_\nu \Phi_{\alpha \beta}^d
$$
All things are very simple in a flat Minkowski space, but if one tries to consider a deformation of this theory into \((A)dS\) space then it turns out to be impossible \(^{37}\). Thus we turn to the massive particle and consider the most general case — massive particle in \((A)dS\) space with arbitrary cosmological constant. First of all, we have to determine which additional fields we need to construct gauge invariant formulation of such massive particle. In general, for each gauge transformation of main physical field we need appropriate Goldstone field but in most cases this Goldstone field turns out to be gauge field by itself so we need Goldstone fields of second order and so on. But for the mixed symmetry bosonic fields one has to take into account reducibility of gauge transformations. Let us illustrate this procedure on our present (simplest) case. Our main physical field \(Y(2,1)\) has two gauge transformations with parameters which are symmetric \(Y(2,0)\) and antisymmetric \(Y(1,1)\) tensors correspondingly. Thus we need two primary Goldstone fields corresponding \(Y(2,0)\) and \(Y(1,1)\). Both have their own gauge transformations with vector parameter \(Y(1,0)\), but due to reducibility of gauge transformations of the main field, we have to introduce one secondary Goldstone field \(Y(1,0)\) only. This field has its own gauge transformation with parameter \(Y(0,0)\), but due to reducibility of gauge transformations of antisymmetric second rank tensor \(Y(1,1)\), the procedure stops here. It is natural to use frame-like formulation for all fields, so we introduce four pairs of tensors: \((\Omega_{\mu}^{\ abc}, \Phi_{\mu}^{\ a})\), \((\omega_{\mu}^{\ ab}, h_{\mu}^{\ a})\), \((\Omega^{abc}, \Phi_{\mu}^{\ ab})\) and \((\omega^{ab}, h_{\mu})\).

We start with the sum of kinetic terms for all fields:

\[
\mathcal{L}_0 = -3 \left\{ \frac{\mu}{ab} \right\} \Omega_{\mu}^{\ a\ c\ d} \Omega_{\nu}^{\ b\ c\ d} + \left\{ \frac{\mu}{abc} \right\} \Omega_{\mu}^{\ a\ b\ c} D_{\nu} \Phi_{\alpha\beta}^{\ d} + 
+ \left\{ \frac{\mu}{ab} \right\} \omega_{\mu}^{\ ac\ \omega_{\nu}^{\ bc}} - \left\{ \frac{\mu\alpha}{abc} \right\} \omega_{\mu}^{\ ab\ D_{\nu} h_{\alpha}^{\ c}} - 
- \Omega_{abc}^{\ 2} + \left\{ \frac{\mu\alpha}{ab} \right\} \Omega^{\ abc} D_{\mu} \Phi_{\nu\alpha} + \omega_{ab}^{\ 2} - 2 \left\{ \frac{\mu}{ab} \right\} \omega^{ab\ D_{\mu} h_{\nu}}
\]

as well as appropriate set of initial gauge transformations:

\[
\delta_0 \Phi_{\mu}^{\ a} = D_{[\mu} \xi_{\nu]}^{\ a} + \eta_{\mu}^{\ a}, \quad \delta_0 \Omega_{\mu}^{\ abc} = D_{\mu} \eta_{\nu}^{\ abc}, \quad \delta_0 h_{\mu}^{\ a} = D_{\mu} \zeta_{\mu}^{\ a} + \chi_{\mu}^{\ a}, \\
\delta_0 \omega_{\mu}^{\ ab} = D_{\mu} \chi_{\mu}^{\ ab}, \quad \delta_0 \Phi_{\mu}^{\ \nu} = D_{[\mu} \xi_{\nu]}, \quad \delta_0 h_{\mu} = D_{\mu} \zeta
\]

where all partial derivatives are replaced by \((A)dS\) covariant ones. Here and in what follows, we will use the following convention on covariant derivatives:

\[
[D_{\mu}, D_{\nu}] \xi_{\mu}^{\ a} = -\kappa (e_{\mu}^{\ a} \xi_{\nu}^{\ a} - e_{\nu}^{\ a} \xi_{\mu}), \quad \kappa = \frac{2\Lambda}{(d-1)(d-2)}
\]

Note, that due to non-commutativity of covariant derivatives such Lagrangian is not invariant under the initial gauge transformations:

\[
\delta_0 \mathcal{L}_0 = \kappa \left\{ \frac{\mu}{ab} \right\} \left[ 3(d-3)(2\Omega_{\mu}^{\ abc} \xi_{\nu}^{\ c} + \eta_{\nu}^{\ abc} \Phi_{\mu}^{\ abc}) - (d-2)(\omega_{\mu}^{\ ab} \zeta_{\nu}^{\ a} - \chi_{\mu}^{\ ab} h_{\mu\nu}) \right]
\]

so we have to take this non-invariance into account later on. Now to proceed with the construction of gauge invariant formulation for massive particle, we have to add to the Lagrangian all possible cross terms of order \(m\) (i.e. with the coefficients having dimension of mass). Moreover, as our previous experience shows, we need to introduce cross terms for the nearest neighbours only, i.e. main field with primary Goldstone fields, primary fields with
secondary ones and so on. For the case at hands all possible such terms could be written as follows:

\[ \mathcal{L}_1 = \{ \mu\nu\rho\} \left[ a_1 \omega_{\mu}^{\ a} \Phi_{\nu}^{\ c} + a_2 \Omega_{\mu}^{\ abc} \Phi_{\nu a}^{\ c} + \{ \mu\nu\} \left[ a_3 \Omega_{\mu}^{\ abc} \Phi_{\nu c}^{\ a} + a_4 \Omega^{abc} \Phi_{\mu\nu}^{\ c} \right] + \right. 
+ \left. \{ \mu\nu\} \left[ a_5 \omega_{\mu}^{\ a} h_{\nu}^{\ c} + a_6 \omega^{ab} \Phi_{\mu\nu}^{\ b} \right] + a_7 \{ \mu\} \omega^{ab} h_{\nu}^{\ b} \right] \]

(6)

Non-invariance of these terms under the initial gauge transformations could be compensated by the following corrections to gauge transformations:

\[ \delta_1 \Phi_{\mu\nu}^{\ a} = \frac{\beta_1}{12(d-3)} e_{[\mu}^{\ a} \zeta_{\nu]} - \frac{3\alpha_1}{(d-3)} e_{[\mu}^{\ a} \chi_{\nu]}, \quad \delta_1 \Omega_{\mu}^{\ abc} = \frac{\beta_1}{6(d-3)} e_{\mu[a}^{\ \ [a} \chi_{bc]} \]

\[ \delta_1 h_{\mu}^{\ a} = \beta_1 \xi_{\mu}^{\ a} + \frac{4\rho_0}{d-2} e_{\mu}^{\ a} \zeta, \quad \delta_1 \omega_{\mu}^{ab} = -\frac{\beta_1}{2} \eta_{\mu}^{ab} \]

\[ \delta_1 \Phi_{\mu\nu} = \alpha_1 \xi_{[\mu,\nu]}, \quad \delta_1 \Omega_{\mu}^{\ abc} = -3\alpha_1 \eta_{\mu}^{abc}, \quad \delta_1 h_{\mu} = \rho_0 \zeta_{\mu} + \beta_0 \xi_{\mu}, \quad \delta_1 \omega_{\mu}^{ab} = -2\rho_0 \chi_{ab} \]

(7)

provided:

\[ a_1 = a_3 = \frac{\beta_1}{2}, \quad a_2 = a_4 = -3\alpha_1, \quad a_5 = a_7 = 4\rho_0, \quad a_6 = \beta_0 \]

Thus we have \( \delta_0 \mathcal{L}_1 + \delta_1 \mathcal{L}_0 = 0 \) and this leaves us with variations of order \( m^2 \) (taking into account non-invariance of kinetic terms due to non-commutativity of covariant derivatives) \( \delta_0 \mathcal{L}_0 + \delta_1 \mathcal{L}_1 \). In general, to compensate this non-invariance one has to introduce mass-like terms into the Lagrangian as well as appropriate corrections for gauge transformations. But in this case there are no possible mass-like terms (the only possible term \( \{ \mu\nu\} h_{\mu}^{\ a} h_{\nu}^{\ b} \) is forbidden by \( \zeta \)-invariance). Nevertheless, it turns out to be possible to achieve complete invariance without any explicit mass-like terms just by adjusting the values for our four main parameters \( \alpha_1, \beta_1, \beta_0 \) and \( \rho_0 \). We obtain:

\[ \rho_0 = \sqrt{\frac{3(d-2)}{4(d-3)}} \alpha_1, \quad \beta_0 = -\sqrt{\frac{3(d-2)}{4(d-3)}} \beta_1, \quad \beta_1^2 - 36\alpha_1^2 = -12\kappa(d-3) \]

Now we are ready to analyze results obtained. First of all, recall that there is no strict definition of what is mass in \((A)dS\) space (see e.g. discussion in [41]). Working with gauge invariant description of massive particles it is natural to define massless limit as a limit where all Goldstone fields decouple from the main gauge field. To make analyze more transparent, let us give here a Figure 1 showing the roles played by our four parameters. One can easily see that massless limit is a limit where both \( \alpha_1 \to 0 \) and \( \beta_1 \to 0 \) simultaneously. But from the last relation above it is immediately follows that such a limit is possible in flat Minkowski space \( (\kappa = 0) \) only. For non-zero values of cosmological constant one can obtain partially massless limits instead. Indeed, in \( AdS \) space \( (\kappa < 0) \) one can put \( \alpha_1 = 0 \) (and this gives \( \rho_0 = 0 \)). Then our system decomposes into two disconnected subsystems. One of them with the fields \( \Phi_{\mu\nu}^{\ a} \) and \( h_{\mu}^{\ a} \) describe partially massless theory [37] with the Lagrangian:

\[ \mathcal{L} = \mathcal{L}_0(\Phi_{\mu\nu}^{\ a}) + \mathcal{L}_0(h_{\mu}^{\ a}) + \frac{\beta_1}{2} \{ \mu\nu\rho\} \omega_{\mu}^{\ a} \Phi_{\nu\alpha}^{\ c} + \frac{\beta_1}{2} \{ \mu\nu\} \Omega_{\mu}^{\ abc} h_{\nu}^{\ c} \]

(8)
which is invariant under the following gauge transformations:

\[ \delta \Phi_{\mu\nu}^a = D_{[\mu} \xi_{\nu]}^a + \eta_{\mu\nu}^a + \frac{\beta_1}{12(d-3)} e_{[\mu}^a \xi_{\nu]}, \quad \delta \Omega_{\mu}^{abc} = D_\mu \eta^{abc} + \frac{\beta_1}{6(d-3)} e_{[\mu}^a \chi_{bc]} \]

\[ \delta h_\mu^a = D_\mu \xi^a + \chi_\mu^a + \beta_1 \xi_\mu^a, \quad \delta \omega_{\mu}^{ab} = D_\mu \chi^{ab} - \frac{\beta_1}{2} \eta_{\mu}^{ab} \]

where \( \beta_1^2 = -12\kappa(d-3) \). In this, two other fields \( \Phi_{\mu\nu}^a \) and \( h_\mu \) provide gauge invariant description of massive antisymmetric second rank tensor field. In turn, in \( dS \) space (\( \kappa > 0 \)) one can put \( \beta_1 = 0 \) (and this gives \( \beta_0 = 0 \)). In this case our system also decompose into two disconnected subsystems. One of them with the fields \( \Phi_{\mu\nu}^a \) and \( \Phi_{\mu\nu} \) gives another example of partially massless theory with the Lagrangian:

\[ \mathcal{L} = \mathcal{L}_0(\Phi_{\mu\nu}^a) + \mathcal{L}_0(\Phi_{\mu\nu}) - 3\alpha_1 \{ \mu_\alpha \}_a \Omega_\mu^{abc} \Phi_{\nu\alpha}^b - 3\alpha_1 \{ \mu_\alpha \}_{ab} \Omega^{abc} \Phi_{\mu\nu}^c \]

This is invariant under the following gauge transformations:

\[ \delta \Phi_{\mu\nu}^a = D_{[\mu} \xi_{\nu]}^a + \eta_{\mu\nu}^a - \frac{3\alpha_1}{(d-3)} e_{[\mu}^a \xi_{\nu]}, \quad \delta \Omega_{\mu}^{abc} = D_\mu \eta^{abc} \]

\[ \delta \Phi_{\mu\nu} = D_{[\mu} \xi_{\nu]} + \alpha_1 \xi_{[\mu,\nu]}, \quad \delta \Omega^{abc} = -3\alpha_1 \eta^{abc} \]

where \( 3\alpha_1^2 = \kappa(d-3) \). In this, two other fields \( h_\mu^a \) and \( h_\mu \) provide gauge invariant description of partially massless spin 2 particle \[25, 16, 18\].

2 Tensor \( Y(3,1) \)

For the description of massless field we will use natural generalization of simplest example given above. Namely, we introduce two-form \( \Phi_{\mu\nu}^{ab} \) which is symmetric and traceless on \( ab \) as a main physical field as well as auxiliary one-form \( \Omega_{\mu}^{abc,d} \) which is completely antisymmetric on \( abc \), traceless and satisfies a constraint \( \Omega_{\mu}^{[abc,d]} = 0 \). To provide correct number of physical degrees of freedom massless Lagrangian has to be invariant under the following gauge transformations:

\[ \delta \Phi_{\mu\nu}^{ab} = \partial_{[\mu} \xi_{\nu]}^{ab} + \eta_{\mu\nu}^{(a,b)}, \quad \delta \Omega_{\mu}^{abc,d} = \partial_\mu \eta^{abc,d} \]
Here $\xi_{\mu}^{ab}$ is symmetric and traceless on $ab$, while $\eta^{abc,d}$ has the same properties on local indices as $\Omega_{\mu}^{abc,d}$. Note that these gauge transformations are also reducible:

$$\xi_{\mu}^{ab} = \partial_{\mu} x^{ab} \Rightarrow \delta \Phi_{\mu}^{ab} = 0$$

To construct appropriate massless Lagrangian we will use the same trick as before. We introduce a “torsion” tensor $T_{\mu}^{\alpha\beta \gamma \delta} = \partial_{\mu} \Phi_{\alpha\beta \gamma \delta}$ which is invariant under $\xi$-transformations, consider an expression $\{\mu \nu \alpha \beta \} \Omega_{\mu}^{abc, e} T_{\nu \alpha \beta}^{d e}$ and make a substitution $T_{\mu}^{\alpha\beta \gamma \delta} \Rightarrow \Omega_{[\mu, \nu]}^{(a, b)}$. We obtain:

$$\{ \mu \nu \alpha \beta \} \Omega_{\mu}^{abc, e} T_{\nu \alpha \beta}^{d e} \Rightarrow \{ \mu \nu \alpha \beta \} \Omega_{\mu}^{abc, e} (\Omega_{\nu, \alpha \beta}^{d, e} + \Omega_{\nu, \alpha \beta}^{e, d}) \Rightarrow$$

$$\{ \mu \nu \} [3 \Omega_{\mu}^{acd, e} \Omega_{\nu}^{bed, e} + \Omega_{\mu}^{cd, e} \Omega_{\nu}^{cde, b}]$$

Thus we will look for the massless Lagrangian in the form:

$$\mathcal{L}_{0} = a_{1} \{ \mu \nu \} [3 \Omega_{\mu}^{acd, e} \Omega_{\nu}^{bed, e} + \Omega_{\mu}^{cd, e} \Omega_{\nu}^{cde, b}] + a_{2} \{ \mu \nu \alpha \beta \} \Omega_{\mu}^{abc, e} T_{\nu \alpha \beta}^{d e}$$

This Lagrangian is (by construction) invariant under $\xi$-transformations, while invariance under $\eta$-transformations requires $a_{1} = -3a_{2}$. We choose $a_{1} = 1$, $a_{2} = -\frac{4}{3}$ and finally obtain:

$$\mathcal{L}_{0}(\Phi_{\mu}^{ab}) = \{ \mu \nu \} [3 \Omega_{\mu}^{acd, e} \Omega_{\nu}^{bed, e} + \Omega_{\mu}^{cd, e} \Omega_{\nu}^{cde, b}] - \{ \mu \nu \alpha \beta \} \Omega_{\mu}^{abc, e} \partial_{\nu} \Phi_{\alpha \beta}^{d e} \quad (13)$$

As in the previous case, it is not possible to deform this massless Lagrangian into $(A)dS$ space without introduction of additional fields. Thus we turn to the general case — massive particle in $(A)dS$ space with arbitrary cosmological constant. First of all, we have to determine the set of additional fields which is necessary for gauge invariant description of such massive particle. Our main gauge field $Y(3, 1)$ has two gauge transformations (combined into one $\xi_{\nu}^{ab}$ transformation in frame-like formalism) with parameters corresponding to $Y(2, 1)$ and $Y(3, 0)$. Recall that these transformations are reducible with the reducibility parameter $Y(2, 0)$. Thus we have to introduce two primary Goldstone fields $- Y(2, 1)$ and $Y(3, 0)$. The first one also has two gauge transformations with parameters $Y(1, 1)$ and $Y(2, 0)$ with the reducibility $Y(1, 0)$, while the second field has one gauge transformation $Y(2, 0)$ only. Taking into account reducibility of main field gauge transformations it is enough to introduce two secondary fields $Y(1, 1)$ and $Y(2, 0)$ only. Both have gauge transformations with parameters $Y(1, 0)$, but due to reducibility of gauge transformations for $Y(2, 1)$ field it is enough to introduce one additional field $Y(1, 0)$. It has its own gauge transformation $Y(0, 0)$, but due to reducibility of gauge transformations for $Y(1, 1)$ field, the procedure stops here. Thus we need six fields $- Y(l, 1), Y(l, 0), 1 \leq l \leq 3$.

Again we will use frame-like formalism for the description of all fields and introduce six pairs: $(\Omega_{\mu}^{abc, d}, \Phi_{\mu}^{ab})$, $(\omega_{\mu}^{a, bc}, h_{\mu}^{ab})$, $(\Omega_{\mu}^{abc}, \Phi_{\mu}^{a})$, $(\omega_{\mu}^{ab}, h_{\mu}^{a})$, $(\Omega^{abc}, \Phi_{\mu})$ and $(\omega^{ab}, h_{\mu})$. Note, that here and in what follows we use the same conventions for the frame-like formulation of $Y(k, 0)$ fields as in [18]. We start with the sum of kinetic terms for all six fields:

$$\mathcal{L}_{0} = \{ \mu \nu \} [3 \Omega_{\mu}^{acd, e} \Omega_{\nu}^{bed, e} + \Omega_{\mu}^{cd, e} \Omega_{\nu}^{cde, b}] - \{ \mu \nu \alpha \beta \} \Omega_{\mu}^{abc, e} D_{\nu} \Phi_{\alpha \beta}^{d e} -$$
covariant derivatives this Lagrangian is not invariant under the initial gauge transformations:

\[ \Omega_{\mu}^{abc} D_{\nu} \Phi_{\alpha} + \frac{1}{2} \omega_{ab}^{2} = - \{ \mu_{ab}^{\nu} \} \omega_{\nu}^{ab} \partial_{\mu} h_{\nu} \]

as well as with appropriate set of initial gauge transformations:

\[ \delta_{0} \Phi_{\mu \nu}^{ab} = D_{\mu} \xi_{\nu}^{ab} + \eta_{\mu \nu}^{(a,b)}, \quad \delta_{0} \Omega_{\mu}^{abc,d} = D_{\mu} \eta^{abc,d}, \quad \delta_{0} \Omega_{\mu}^{abc,d} = D_{\mu} \chi^{abc,d}, \quad \delta_{0} \Omega_{\mu}^{abc,d} = D_{\mu} \xi^{abc,d} \]

where all derivatives are now (A)dS covariant ones. As usual, due to non-commutativity of covariant derivatives this Lagrangian is not invariant under the initial gauge transformations:

\[ \delta_{0} L_{0} = -3\kappa(d - 2) \{ \mu_{ab}^{\nu} \} (2 \Omega_{\mu}^{abc,d} \xi_{\nu}^{cd} - \eta^{abc,d} \Phi_{\mu \nu}^{cd}) + 3\kappa(d - 1)(\omega_{\mu}^{a,b} \xi_{\nu}^{ab} - \chi_{\mu}^{a,b} h_{\nu}^{ab}) \]

but we will take this non-invariance into account later on.

To construct gauge invariant description of massive particles we proceed by adding cross terms of order \( m \) (i.e. terms with the coefficients with dimension of mass) to the Lagrangian. As we have already noted, one has to introduce such cross terms for the nearest neighbours only, i.e. main gauge field with primary ones, primary with secondary and so on. To simplify the presentation we consider these terms step by step.

\( \Phi_{\mu \nu}^{ab} \Leftrightarrow \Phi_{\mu \nu}^{a}, h_{\nu}^{ab} \). In this case additional terms to the Lagrangian could be written in the following form:

\[ L_{1} = \{ \mu_{ab}^{\nu} \} \{ a_{1} \Omega_{\mu}^{abc,d} \Phi_{\nu \alpha}^{d} + a_{2} \Phi_{\mu \nu}^{ad} \Omega_{\alpha}^{bcd} + a_{3} \Phi_{\mu \nu}^{ad} \omega_{\alpha}^{b,cd} \} + \{ \mu_{ab}^{\nu} \} a_{4} \Omega_{\mu}^{abc,d} h_{\nu}^{cd} \]

As usual, their non-invariance under the initial gauge transformations could be compensated by appropriate corrections to gauge transformations:

\[ \delta_{1} \Phi_{\mu \nu}^{ab} = - \frac{4\alpha_{2}}{d - 2} [e_{\mu}^{[a} \xi_{\nu] b]} + \frac{2}{d} g^{ab} \xi_{\mu,\nu}] + \frac{\beta_{2}}{6(d - 2)} e_{\mu}^{[a} \xi_{\nu] b]} + \frac{\beta_{2}}{3(d - 3)} [e_{\mu}^{[a} \chi_{\nu] b]} - \frac{4}{d - 2} g^{[a} \chi_{\nu] b]} \]

\[ \delta_{1} \Omega_{\mu}^{abc,d} = - \frac{\alpha_{2}}{d} [3 e_{\mu}^{d} \eta^{abc} + e_{\mu}^{[a} \eta^{bc]d} - \frac{4}{d - 2} g^{[a} \eta^{bc]d} \mu] + \frac{\beta_{2}}{3(d - 3)} [e_{\mu}^{[a} \chi_{\nu] b]} - \frac{1}{d - 2} g^{[a} \chi_{\nu] b]} \]

\[ \delta_{1} \Phi_{\mu \nu}^{a} = \alpha_{2} \xi_{\mu,\nu}] a, \quad \delta_{1} \Omega_{\mu}^{abc} = -4\alpha_{2} \eta^{abc} \mu, \]

\[ \delta_{1} \omega_{\mu}^{a, bc} = -\frac{\beta_{2}}{2} \eta_{\mu}^{a(b,c)} \]

provided \( a_{1} = 4\alpha_{2}, a_{2} = 3\alpha_{2}, a_{3} = a_{4} = -\beta_{2} \).
\( \Phi_{\mu \nu}^a, h_\mu^{ab} \leftrightarrow \Phi_{\mu \nu}, h_\mu^a \). Now additional terms to the Lagrangian have the form:

\[
\Delta L_1 = \{ \mu \nu \alpha \} \left[ a_5 \delta \omega_\mu^a a \Phi_{\nu a}^c + a_6 \Omega_\mu^a a a_7 a_8 \Phi_{\nu a}^c + \{ \mu \nu \} \left[ a_7 \Omega_\mu^a a h_\nu^c + a_8 \Omega_\mu^a a a_9 \right] \right] + \{ \mu \nu \} \left[ a_9 \omega_\mu^a a h_\nu^c + a_10 h_\mu^a a \omega_\nu^c \right]
\]

(18)

To compensate their non-invariance under the initial gauge transformations we introduce the following corrections to gauge transformations:

\[
\delta_1 \Phi_{\mu \nu}^a = \frac{\beta_1}{12(d-3)} e_\mu^a \xi_\nu - \frac{3\alpha_1}{d-3} e_\mu^a \zeta_\nu, \quad \delta_1 \Omega_\mu^a a = \frac{\beta_2}{6(d-3)} e_\mu^a [a, \chi^c]
\]

\[
\delta_1 h_\mu^{ab} = \frac{\rho_1}{d-1} \left[ e_\mu^a \xi_\nu - \frac{2}{d} g_\mu^a \zeta_\mu \right]
\]

\[
\delta_1 \omega_\mu^{a,bc} = \frac{\rho_1}{d} \left[ \chi (e_\mu^a \xi_\nu + \frac{1}{d-1} (2 g_\mu^a \chi_\mu^a - g_\mu^a (\chi_\mu^a)) \right]
\]

\[
\delta_1 h_\mu^a = \beta_1 \xi_\mu^a + \rho_1 \zeta_\mu, \quad \delta_1 \omega_\mu^{ab} = -\frac{\beta_1}{2} \eta_\mu^{ab} + \rho_1 \chi [a, b]_\mu
\]

\[
\delta_1 \Phi_{\mu \nu} = \alpha_1 \xi_\mu \xi_\nu, \quad \delta_1 \Omega_{\mu \nu}^{ab} = -3 \alpha_1 \eta_{\mu \nu}^{ab}
\]

(19)

provided \( a_5 = a_7 = \beta_1/2, a_6 = a_8 = -3 \alpha_1, a_9 = a_{10} = -2 \rho_1 \).

\( \Phi_{\mu \nu}, h_\mu^a \leftrightarrow \Phi_{\mu \nu}, h_\mu^a \). Finally, we add to the Lagrangian terms (we already familiar with):

\[
\Delta L_1 = \{ \mu \nu \} \left[ a_{11} \delta \omega_\mu^{ab} h_\nu + a_{12} \omega_\mu^{ab} \Phi_{\mu \nu} \right] + a_{13} \omega_\mu^{ab} h_{ab}
\]

(20)

and corresponding corrections to gauge transformations:

\[
\delta h_\mu^a = \frac{2 \rho_0}{d-2} e_\mu^a \zeta_\mu, \quad \delta h_\mu = \rho_0 \zeta_\mu + \beta_0 \zeta_\mu, \quad \delta \omega_{\mu \nu}^{ab} = -2 \rho_0 \omega_{\mu \nu}^{ab}
\]

(21)

where \( a_{11} = a_{13} = 2 \rho_0, a_{12} = 2 \beta_0/2 \).

Collecting all pieces together, we obtain complete set of cross terms:

\[
L_1 = \{ \mu \nu \alpha \} \left[ 4 \alpha \Omega_\mu a \omega_\nu^{b,c} \Phi_{\nu a}^d + 3 \beta \Phi_{\mu \nu}a a \Omega_{b}^{d, e} \omega_{\nu}^{c} - \beta_2 \Phi_{\mu \nu} a \omega_{\nu}^{b,c} \right] - \beta_2 \{ \mu \nu \} \left[ \Omega_{\nu}^{a,b,c} \omega_{\nu}^{d} \right] + \{ \mu \nu \} \left[ \beta_2 \Phi_{\mu \nu} a \omega_{\nu}^{b,c} \right] + \{ \mu \nu \} \left[ \beta_1 \Phi_{\mu \nu} a \omega_{\nu}^{b,c} \right] + \{ \mu \nu \} \left[ \beta_1 \Phi_{\mu \nu} a \omega_{\nu}^{b,c} \right] + \{ \mu \nu \} \left[ 2 \rho_0 \omega_{\mu \nu}^{a,b} h_{ab} \right]
\]

(22)

Now, as we have achieved cancellation of all variations of order \( m \delta_0 L_1 + \delta_1 L_0 \), we have to take care on variations of order \( m^2 \) (including contribution from kinetic terms due to non-commutativity of covariant derivatives) \( \delta_0 L_0 + \delta_1 L_1 \). As in the previous case, there are no any explicit mass-like terms allowed here, but complete invariance of the Lagrangian could be achieved just by adjusting the values of remaining free parameters \( \alpha_{1,2}, \beta_{0,1,2} \) and \( \rho_{0,1} \):

\[
\beta_1 = -2 \sqrt{ \frac{d-1}{d-2} } \beta_2, \quad \beta_0 = - \sqrt{ \frac{6(d-1)}{d-3} } \beta_2, \quad \rho_1 = 2 \sqrt{ \frac{d-1}{d-2} } \alpha_2, \quad \rho_0 = \sqrt{ \frac{3(d-2)}{2(d-3)} } \alpha_1
\]

\[
24 \alpha_2^2 - \beta_2^2 = 6(d-2) \kappa, \quad 12(d+1) \alpha_2^2 - 3d \alpha_1^2 = d(d+1) \kappa
\]
Figure 2: General massive theory for $Y(3,1)$ tensor

The role that each of the parameters plays could be easily seen from the Figure 2. Now we are ready to analyze the results obtained. First of all, note that the massless limit (i.e decoupling of main gauge fields from all others) requires $\alpha_2 = \beta_2 = 0$. As the first of last two relations clearly shows this is possible in flat Minkowski space ($\kappa = 0$) only. In this, for non-zero values of cosmological constant there exists a number of partially massless limits.

In $dS$ space ($\kappa > 0$) one can put $\beta_2 = 0$ (and this simultaneously gives $\beta_1 = \beta_0 = 0$). In this complete system decompose into two disconnected subsystems, as shown on the Figure 3. One of them, with the fields $\Phi_{\mu\nu ab}$, $\Phi_{\mu\nu a}$ and $\Phi_{\mu\nu}$ gives new example of partially massless

Figure 3: Partially massless limit in $dS$ space

theory with the Lagrangian:

$$\mathcal{L} = \mathcal{L}_0(\Phi_{\mu\nu ab}) + \mathcal{L}_0(\Phi_{\mu\nu a}) + \mathcal{L}_0(\Phi_{\mu\nu}) +$$

$$+ \alpha_2 \{ \mu \nu a \} \left[ 4 \Omega_{\mu abc} \Phi_{\nu a} d + 3 \Phi_{\mu \nu ad} \Omega_{\alphabcd} \right] -$$

$$- 3 \alpha_1 \{ \nu a \} \Omega_{a \mu \nu} \Phi_{\mu a} + \{ \mu a \} \Omega_{abc} \Phi_{\mu \nu c}$$

(23)

where $4\alpha_2^2 = (d - 2)\kappa$, $3d\alpha_1^2 = 2(d + 1)(d - 3)\kappa$, which is invariant under the following gauge transformations (for simplicity we reproduce here gauge transformations for physical fields only):

$$\delta \Phi_{\mu\nu ab} = D_{[\mu} \xi_{\nu]} ab + \eta_{\mu\nu} (a, b) - \frac{4\alpha_2}{d-2} e_{[\mu} a \xi_{\nu]} b$$

$$\delta \Phi_{\mu\nu a} = D_{[\mu} \xi_{\nu]} a + \eta_{\mu\nu} a + \alpha_2 \xi_{[\mu, \nu]} a - \frac{3\alpha_1}{d-3} e_{[\mu} a \xi_{\nu]}$$

(24)

$$\delta \Phi_{\mu\nu} = D_{[\mu} \xi_{\nu]} + \alpha_1 \xi_{[\mu, \nu]}$$

At the same time, three other fields $h_{\mu ab}$, $h_{\mu a}$ and $h_{\mu}$ give gauge invariant description of partially massless spin 3 particle [16, 18].
One more example of partially massless theory appears if one put $\alpha_1 = 0$ (and hence $\rho_0 = 0$). In this, complete system also decompose into two disconnected subsystems as shown on Figure 4.

$$\Phi_{\mu \nu ab} \xrightarrow{\alpha_2} \Phi_{\mu \nu} \xrightarrow{\beta_2} h_{\mu}^{ab} \xrightarrow{\rho_1} h_{\mu} \quad \text{and} \quad \Phi_{\mu \nu a} \xrightarrow{\beta_1} h_{\mu}^a \xrightarrow{\beta_0} h_{\mu}$$

Figure 4: Non-unitary partially massless theory

Note, however that in this case we obtain $12\alpha_2^2 = d\kappa$, $\beta_2^2 = -4(d-3)\kappa$, so that such theory (as is often to be the case) "lives" inside unitary forbidden region.

From the other hand, in $AdS$ space ($\kappa < 0$) one can put $\alpha_2 = 0$ (and hence $\rho_1 = 0$). In this, decomposition into two subsystems looks as follows (Figure 5):

$$\Phi_{\mu \nu ab} \xrightarrow{\alpha_1} \Phi_{\mu \nu} \xrightarrow{\beta_2} h_{\mu}^{ab} \xrightarrow{\rho_0} h_{\mu} \quad \text{and} \quad \Phi_{\mu \nu a} \xrightarrow{\beta_1} h_{\mu}^a \xrightarrow{\beta_0} h_{\mu}$$

Figure 5: Partially massless limit in $AdS$ space

Thus we obtain one more example of partially massless theory with two fields $\Phi_{\mu \nu ab}$ and $h_{\mu}^{ab}$. The Lagrangian:

$$\mathcal{L} = \mathcal{L}_0(\Phi_{\mu \nu ab}) + \mathcal{L}_0(h_{\mu}^{ab}) - \beta_2 \left\{ \frac{\mu \nu \lambda}{abc} \right\} \Phi_{\mu \nu} \omega_{\alpha b,cd} - \beta_2 \left\{ \frac{\mu \nu}{ab} \right\} \Omega_{\mu \nu}^{abc,d} h_{\nu}^{cd}$$

where $\beta_2^2 = -6(d-2)\kappa$, is invariant under the following gauge transformations:

$$\delta \Phi_{\mu \nu ab} = D_{[\mu} \xi_{\nu]}^{ab} + \eta_{\mu \nu}^{(a,b)} + \frac{\beta_2}{6(d-2)} c_{[a} \xi_{\nu]}^{b)}, \quad \delta h_{\mu}^{ab} = D_{\mu} \xi^{ab} + \chi_{\mu}^{ab} + \beta_2 \xi_{\mu}^{ab}$$

In this, four remaining fields $\Phi_{\mu \nu}^a$, $h_{\mu}^a$, $\Phi_{\mu \nu}$ and $h_{\mu}$ just gives the same gauge invariant massive theory as in the previous section.

3 Tensor $Y(k, 1)$

For the description of massless particles we introduce main physical field — two-form $\Phi_{\mu \nu}^{a_1...a_{k-1}} = \Phi_{\mu \nu}^{(k-1)}$ (here and in what follows we will use the same condensed notations for tensor objects as in [18]) which is completely symmetric and traceless on local indices and auxiliary
one-form $\Omega^{\mu abc}(k-2)$ which is completely antisymmetric on $abc$, traceless on all local indices and satisfies a constraint $\Omega^{\mu [abc,d]}(k-3) = 0$. To have correct number of physical degrees of freedom massless Lagrangian has to be invariant under the following gauge transformations:

$$\delta \Phi_{\mu \nu}^{(k-1)} = \partial_{[\mu} \xi_{\nu]}^{(k-1)} + \eta_{\mu \nu}^{(1,k-2)}, \quad \delta \Omega^{\mu abc,(k-2)} = \partial_{\mu} \eta^{abc,(k-2)}$$

(27)

where properties of parameters $\xi$ and $\eta$ correspond to that of $\Phi_{\mu \nu}$ and $\Omega_{\mu}$. To find appropriate Lagrangian, we introduce a tensor $T_{\mu \nu \alpha \beta}^{(k-1)} = \partial_{[\mu} \Phi_{\nu \alpha \beta]}^{(k-1)}$, which is invariant under $\xi$-transformations, consider an expression $\{ \mu \nu \alpha \beta \}_a^{abc, (k-2)}$ and make a substitution $T_{\mu \nu \alpha \beta}^{(k-1)} \rightarrow \Omega_{[\mu \nu \alpha \beta]}^{(k-1)}(1,k-2)$. We obtain:

$$\{ \mu \nu \alpha \beta \}_a^{abc, (k-2)} = \partial_{\mu} \eta^{abc,(k-2)}$$

$$\Rightarrow \{ \mu \nu \alpha \beta \}_a^{abc, (k-2)}$$

Then we will look for massless Lagrangian in the form:

$$L_{0} = a_1 \{ \mu \nu \}_a^{abc, (k-2)} \Omega^{\mu abc, (k-2)} + \eta_{\mu \nu}^{(1,k-2)} \Rightarrow \{ \mu \nu \alpha \beta \}_a^{abc, (k-2)}$$

$$\eta_{\mu \nu}^{(1,k-2)}$$

It is by construction invariant under the $\xi$-transformations, while invariance under the $\eta$-transformations requires $a_1 = -3a_2$. We choose $a_1 = (-1)^{k-1}, a_2 = -(-1)^{k-1}/3$ and obtain finally:

$$(-1)^{k-1}L_{0} = \{ \mu \nu \}_a^{abc, (k-2)} \Omega^{\mu abc, (k-2)} + \eta_{\mu \nu}^{(1,k-2)} \Rightarrow \{ \mu \nu \alpha \beta \}_a^{abc, (k-2)}$$

$$\eta_{\mu \nu}^{(1,k-2)}$$

(28)

As in the previous cases, it is not possible to deform this massless theory into $(A)dS$ space without introduction of additional fields, so we will turn to the general case — massive particle in $(A)dS$ space with arbitrary cosmological constant. Our first task — to determine the set of additional fields which are necessary for gauge invariant description of such massive particle. Our main gauge field $Y(k, 1)$ has two gauge transformations with parameters $Y(k-1,1)$ and $Y(k,0)$ with the reducibility $Y(k-1,0)$, thus we need two primary Goldstone fields $Y(k-1,1)$ and $Y(k,0)$. The first one has two own gauge transformations with parameters $Y(k-2,1)$ and $Y(k-1,0)$ with reducibility $Y(k-2,0)$, while the second one has one gauge transformation with parameter $Y(k-1,0)$ only. So we need two secondary fields $Y(k-2,1)$ and $Y(k-1,0)$ and so on. As in the previous cases, this procedure stops at vector field $Y(1,0)$, thus we totally have to introduce fields $Y(l,1)$ and $Y(l,0)$ with $1 \leq l \leq k$.

Let us start with the sum of kinetic terms for all these fields:

$$L_{0} = \sum_{l=2}^{k} L_{0}^{(l-1)}(\Phi_{\mu \nu}^{(l-1)}) - \Omega^{abc}_{\mu \nu} + \{ \mu \nu \alpha \}_a^{abc, (k-2)} \Omega^{\mu abc} D_{\mu} \Phi_{\nu \alpha} +$$

$$+ \sum_{l=2}^{k} L_{0}^{(l-1)}(h_{\mu}^{(l-1)}) + \frac{1}{2} \omega_{ab}^{2} - \{ \mu \nu \}_a^{abc} \omega_{ab} D_{\mu} h_{\nu}$$

(29)
as well as appropriate set of initial gauge transformations:

\[
\begin{align*}
\delta \Phi_{\mu\nu}^{(l)} &= D[\mu] \xi^{(l)} + \eta_{\mu\nu}^{(1, l-1)}, \\
\delta \Omega_{\mu ab, (l-1)} &= D\mu \eta^{abc, (l-1)}, \\
\delta \Phi_{\mu\nu} &= D[\mu] \xi_{\mu\nu}^{(l)}
\end{align*}
\]

where all derivatives are now (A)dS covariant ones. Due to non-commutativity of covariant derivatives this Lagrangian is not invariant under the initial gauge transformations:

\[
\delta_0 \mathcal{L}_0 = \sum_{l=2}^{k} (-1)^l \kappa \{ \mu \nu \} \{ 3(d + l - 4)(-2 \Omega_{\mu ab, (l-1)} \xi_{\mu}^{(l-1)} + \eta^{abc, (l-1)} \Phi_{\mu\nu}^{(l-1)}) + 2(d + l - 3)(\omega_{\mu ab, (l-1)} \xi_{\mu}^{(l-1)} - \frac{1}{l} \chi^{(l)} h_{\mu}^{(l)}) \}
\]

but we will take this non-invariance into account later on.

To proceed with the construction of gauge invariant description of massive particle, we have to add to the Lagrangian cross terms of order \( m \) (i.e. with the coefficients having dimension of mass). As we have already noted above, one has to introduce such cross terms for the nearest neighbours only (i.e. main field with primary ones, primary with secondary and so on). For the case at hands, this means introduction cross terms between pairs \( Y(l + 1, 1), Y(l + 2, 0) \) and \( Y(l, 1), Y(l + 1, 0) \). Moreover, due to symmetry and tracelessness properties of the fields, there exists such terms for three possible cases \( Y(l + 1, 1) \Leftrightarrow Y(l, 1), Y(l + 1, 1) \Leftrightarrow Y(l + 1, 0), Y(l + 2, 0) \Leftrightarrow Y(l + 1, 0) \) only. We consider these three possibilities in turn.

\[
\Omega_{\mu ab, (l-1)}, \Phi_{\mu\nu}^{(l-1)} \Leftrightarrow \Omega_{\mu ab, (l-2)}, \Phi_{\mu\nu}^{(l-1)}.
\]

Here additional terms to the Lagrangian could be written as follows:

\[
\mathcal{L}_1 = (-1)^l \{ ab \} \{ a_1 \Omega_{\mu ab, (l-1)} \Phi_{\nu\alpha}^{(l-1)} + a_2 \Omega_{\mu ab, (l-2)} \Phi_{\nu\alpha}^{cd(l-2)} \}
\]

Their non-invariance under the initial gauge transformations could be compensated by the following corrections to gauge transformations:

\[
\begin{align*}
\delta_1 \Phi_{\mu\nu}^{(l)} &= -\frac{(l + 2)\alpha_l}{(l - 1)(d + l - 4)} [e_{[\mu}^{(1)} \xi_{\nu]}^{(l - 1)} + \frac{2}{d + 2l - 4} \xi_{[\mu, \nu]}^{(l - 2) g^{12}}] \\
\delta_1 \Omega_{\mu ab, (l-1)} &= -\frac{\alpha_l}{(l - 1)d} [3 \eta^{abc, (l-2)} e_{\mu}^{(1)} + e_{[a}^{(1)} \eta^{bc] (l-2) - Tr} ] \\
\delta_1 \Phi_{\mu\nu}^{(l-1)} &= a_1 \xi_{[\mu, \nu]}^{(l - 1)} - \frac{(l + 2)\alpha_l}{l - 1} \eta_{abc, (l-2)}^{(l-1)} \\
\delta_1 \Omega_{\mu ab, (l-2)} &= -\frac{(l - 1)}{l + 1} \eta_{abc, (l-2)}^{(l-1)}
\end{align*}
\]

provided:

\[
a_{1l} = \frac{(l + 2)}{(l - 1)} \alpha_l, \quad a_{2l} = 3 \alpha_l
\]
\[ \Omega^{abc,(l-1)}, \Phi_{\mu
u}^{(l)} \leftrightarrow \omega^{a,(l)}, h^{(l)}. \] This time additional terms to the Lagrangian have the form:

\[ \mathcal{L}_1 = (-1)^l \left[ a_{3l} \{ \mu \nu \} \Omega^{\mu \nu \rho \sigma (l-1)} h^{\rho \sigma (l-1)} + a_{4l} \{ \rho \sigma \} \omega^{\rho \sigma \rho \sigma (l-1)} \Phi_{\mu \nu}^{(l-1)} \right] \] (33)

and their non-invariance under the initial gauge transformations could be compensated by:

\[ \begin{align*}
\delta_1 \Phi_{\mu \nu}^{(l)} &= \frac{\beta_l}{6(d+l-4)} e_{[\mu}^{(l-1)} \xi_{\nu]} \\
\delta_1 \Omega^{\mu \nu \rho \sigma (l-1)} &= \frac{\beta_l}{3(d-3)} \left[ e_{[a,b,c]^{(l-1)}} - Tr \right] \\
\delta_1 h^{(l)} &= \beta_l \xi^{(l)}, \quad \delta_1 \omega^{(l)} = -\frac{\beta_l}{2} \eta^{(l)}
\end{align*} \] (34)

provided \( a_{3l} = a_{4l} = -\beta_l \).

\[ \omega^{a,(l+1)}, h^{(l+1)} \leftrightarrow \omega^{a,(l)}, h^{(l)} \]. This case (that has already been considered in [18]) requires additional terms to the Lagrangian in the form:

\[ \mathcal{L}_1 = (-1)^l \left[ a_{5l} \omega^{a,(l)} h^{(l)} + a_{6l} \omega^{a,(l)} h^{b(l)} \right] \] (35)

as well as the following corrections to gauge transformations:

\[ \begin{align*}
\delta_1 h^{(l+1)} &= \frac{(l+1)\rho_l}{l(d+l-2)} \left[ e_{[\mu}^{(l)} \xi_{\nu]} - Tr \right], \quad \delta_1 \omega^{(l+1)} = \frac{(l+1)\rho_l}{l(d+l-1)} \left[ \eta^{(l)} e_{[\mu}^{(l)} \right. \\
\delta_1 h^{(l)} &= \rho_l \xi^{(l)}, \quad \delta_1 \omega^{(l)} = \frac{\rho_l}{l} \left[ \eta^{(l)} + (l+1) \eta^{(l)} - Tr \right]
\end{align*} \] (36)

where \( a_{5l} = a_{6l} = \frac{2(l+1)}{l-1} \rho_l \).

Collecting all pieces together, we obtain finally:

\[ \mathcal{L}_1 = \sum_{l=2}^{k-1} \left[ \frac{l+2}{l-1} \Omega^{\mu \nu \rho \sigma (l-1)} \Phi_{\mu \nu}^{(l-1)} + 3 \Omega^{\mu \nu \rho \sigma (l-2)} \Phi_{\mu \nu}^{(l-2)} \right] \]

\[ - \beta_l \left[ \left\{ \mu \nu \right\} \Omega^{\mu \nu \rho \sigma (l-1)} h^{\rho \sigma (l-1)} + \{ \mu \nu \} \omega^{\mu \nu \rho \sigma (l-1)} \Phi_{\mu \nu}^{(l-1)} \right] + \]

\[ + \frac{2(l+1)}{l} \rho_l \left[ \left\{ \mu \nu \right\} \omega^{\rho \sigma (l)} h^{\rho \sigma (l)} + \omega^{\rho \sigma (l)} h^{\rho \sigma (l)} \right] - \left\{ \mu \nu \right\} \omega^{\rho \sigma (l)} h^{\rho \sigma (l)} + \omega^{\rho \sigma (l)} h^{\rho \sigma (l)} \right] - \left\{ \mu \nu \right\} \omega^{\rho \sigma (l)} h^{\rho \sigma (l)} + \omega^{\rho \sigma (l)} h^{\rho \sigma (l)} \right] - \left\{ \mu \nu \right\} \omega^{\rho \sigma (l)} h^{\rho \sigma (l)} + \omega^{\rho \sigma (l)} h^{\rho \sigma (l)} \right]
\] (37)

As for the corrections to gauge transformations, we once again restrict ourselves with the transformations for physical fields only:

\[ \begin{align*}
\delta_1 \Phi_{\mu \nu}^{(l)} &= \alpha_{l+1} \xi_{[\mu \nu,]}^{(l)} - \frac{(l+2)\alpha_l}{l-1} \left[ e_{[\mu}^{(l-1)} \xi_{\nu]} - Tr \right] + \frac{\beta_l}{6(d+l-4)} e_{[\mu}^{(l-1)} \xi_{\nu]} \\
\delta_1 \Phi_{\mu \nu}^{a} &= \alpha_{l} \xi_{[\mu \nu,]}^{a} - \frac{3\alpha_l}{d-3} e_{[\mu}^{a} \xi_{\nu]} + \frac{\beta_l}{6(d-3)} e_{[\mu}^{a} \xi_{\nu]}, \quad \delta_1 \Phi_{\mu \nu} = \alpha_{l} \xi_{[\mu \nu,]} \\
\delta_1 h^{(l)} &= \beta_l \xi^{(l)} + \rho_l \xi^{(l)} + \frac{l\rho_{l-1}}{(l-1)(d+l-3)} \left[ e_{[\mu}^{(l-1)} \xi_{\nu]} - Tr \right] \\
\delta_1 h^{a} &= \beta_l \xi^{a} + \rho_l \xi^{a} + \frac{\rho_{l-1}}{d-2} e_{[\mu}^{a} \xi_{\nu]}, \quad \delta_1 h^{a} = \beta_l \xi^{a} + \rho_l \xi^{a}
\end{align*} \] (38)
Now, having achieved cancellation of all variations of order \( m \) \( \delta_0 \mathcal{L}_1 + \delta_1 \mathcal{L}_0 = 0 \), we have to take care on variations of order \( m^2 \) (including contribution of kinetic terms due to non-commutativity of covariant derivatives) \( \delta_0 \mathcal{L}_0 + \delta_1 \mathcal{L}_1 \). As in the previous cases, complete invariance of the Lagrangian could be achieved without introduction of any explicit mass-like terms into the Lagrangian (and appropriate corrections to gauge transformations). Indeed, rather long calculations give four relations:

\[
\alpha_l \beta_{l-1} = -\beta_l \rho_{l-1}
\]

\[
(l + 1)(d + l - 3)\beta_l \rho_l = -(l + 3)(d + l - 2)\alpha_{l+1} \beta_{l+1}
\]

\[
-\frac{6(l + 3)(d + l - 4)(d + 2l)}{l(d + l - 3)(d + 2l - 2)} \alpha_{l+1}^2 + \frac{6(l + 2)}{l - 1} \alpha_l^2 - \beta_l^2 = 6 \kappa(d + l - 4)
\]

\[
\frac{d + l - 3}{3(d + l - 4)} \beta_l^2 + \frac{2(l + 1)(d + l - 3)(d + 2l)}{l(d + l - 2)(d + 2l - 2)} \rho_l^2 - \frac{2l}{l - 1} \rho_{l-1}^2 = -2 \kappa(d + l - 3)
\]

To solve these relations we proceed as follows. From the first one we get:

\[
\rho_l = -\frac{\beta_l}{\beta_{l+1}} \alpha_{l+1}
\]

Putting this relation into the second one, we obtain recurrent relation on parameters \( \beta \):

\[
\beta_l^2 = \frac{(l + 3)(d + l - 2)}{(l + 1)(d + l - 3)} \beta_{l+1}^2
\]

This allows us to express all parameters \( \beta_l \) in terms of \( \beta_{k-1} \):

\[
\beta_l^2 = \frac{k(k + 1)(d + k - 4)}{(l + 1)(l + 2)(d + l - 3)} \beta_{k-1}^2
\]

In this, one can show that fourth equation is equivalent to third one. When all parameters \( \beta \) are known, the third equation becomes recurrent relation on parameters \( \alpha \) and this allows us (taking into account that \( \alpha_k = 0 \)) to express all \( \alpha_l \) in terms of \( \alpha_{k-1} \). Let us introduce a notation \( M^2 = \frac{k(k+1)}{k-2} \alpha_{k-1}^2 \), then the expression for \( \alpha_l \) could be written as follows:

\[
\alpha_l^2 = \frac{(l - 1)(d + k + l - 3)}{(l + 1)(l + 2)(d + 2l - 2)} [M^2 - (k - l - 1)(d + k + l - 4) \kappa]
\]

Thus we are managed to express all parameters in terms of two main ones \( \beta_{k-1} \) and \( M \) (or \( \alpha_{k-1} \)), in this the following relation must hold:

\[
6M^2 - k \beta_{k-1}^2 = 6 \kappa(d + k - 5) \kappa
\]

Now we are ready to analyze the results obtained. In complete theory we have three sets of parameters \( \alpha, \beta \) and \( \rho \) and the roles they play could be easily seen from the Figure 6.

First of all note, that massless limit (that requires \( M \to 0 \) and \( \beta_{k-1} \to 0 \) simultaneously) is indeed possible in flat Minkowski space only, while for non-zero values of cosmological constant we can obtain a number of partially massless theories. In \( AdS \) space \( (\kappa < 0) \) one
can put $\alpha_{k-1} = 0$ (and this gives $\rho_{k-2} = 0$), in this two fields $\Phi_{\mu \nu}^{(k-1)}$ and $h_{\mu}^{(k-1)}$ decouple and describe partially massless theory with the Lagrangian (Figure 7):

$$\mathcal{L} = \mathcal{L}_0(\Phi_{\mu \nu}^{(k-1)}, h_{\mu}^{(k-1)}) + (-1)^k \beta_{k-1}[\{\mu^\nu\} \Omega_{\mu ab}^{(k-2)} h_{\nu}^{c(k-2)} + \{^\mu a\}_b^{ab} \omega_{\mu}^{a,b(k-2)} \Phi_{\nu a}^{c(k-2)}]$$

(39)

which is invariant under the following gauge transformations:

$$\delta \Phi_{\mu \nu}^{(k-1)} = D_{[\mu} \xi_{\nu]}^{(k-1)} + \eta_{\mu \nu}^{(1,k-2)} + \frac{\beta_{k-1}}{6(d+k-5)} c_{[\mu}^{(1)} \xi_{\nu]}^{k-2})$$

$$\delta h_{\mu}^{(k-1)} = D_{\mu} \xi_{\mu}^{(k-1)} + \chi_{\mu}^{(k-1)} + \beta_{k-1} \xi_{\mu}^{(k-1)}$$

(40)

At the same time, all other fields just give gauge invariant description of massive $\Phi_{\mu \nu}^{(k-2)}$ tensor.

On the other hand, in $dS$ space ($\kappa > 0$) one can put $\beta_{k-1} = 0$ (and this results in $\beta_l = 0$ for all $l$). In this case complete system decompose into two disconnected subsystems (Figure 8). One subsystem with the fields $\Phi_{\mu \nu}^{(l)}, 0 \leq l \leq k-1$ gives new example of partially massless theory, while the partially massless theory described by the second subsystem $h_{\mu}^{(k)}, 0 \leq l \leq k-1$ is already known [16, 18]. Besides, a number of (non-unitary) partially massless theories appears then one put one of the $\alpha_l = 0$ (and hence $\rho_{l-1} = 0$). In this, complete system also decompose into two disconnected subsystems. One of them gives partially massless theory with the fields $\Phi_{\mu \nu}^{(n)}, h_{\mu}^{(n)}, l \leq n \leq k-1$ (Figure 9), while the rest of fields just give massive theory for the $\Phi_{\mu \nu}^{(l-1)}$ tensor.
Figure 8: Partially massless limit in $dS$ space

Figure 9: Example of non-unitary partially massless theory

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