Small-amplitude steady water waves with critical layers: non-symmetric waves

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Abstract

The problem for two-dimensional steady water waves with vorticity is considered. Using methods of spatial dynamics, we reduce the problem to a finite dimensional Hamiltonian system. As an application, we prove the existence of non-symmetric steady water waves when the number of roots of the dispersion equation is greater than $1$.

1 Introduction

In the present paper we consider the problem for two-dimensional steady water waves with vorticity allowing critical layers. Following ideas of \cite{5} and \cite{10}, we formulate the problem as an evolutionary equation for which the role of time is played by the spatial coordinate. This approach called spatial dynamics turned to be useful for the existence theory of small-amplitude steady waves (see \cite{5, 14, 6, 11, 13}). Using center manifold reduction technique due to Mielke \cite{12}, we reduce the problem to a finite-dimensional system of ordinary differential equations. We show that this system is Hamiltonian by applying results from \cite{10} (see also \cite{4}). As an application, we find a large class of small-amplitude solutions containing non-symmetric steady waves in the case when the number of roots of the dispersion equation is greater than $1$.

2 Statement of the Problem

Let an open channel of uniform rectangular cross-section be bounded below by a horizontal rigid bottom and let water occupying the channel be bounded above by a free surface not touching the bottom. In appropriate Cartesian coordinates $(x, y)$, the bottom coincides with the $x$-axis and gravity acts in the
negative y-direction. We use the non-dimensional variables proposed by Keady and Norbury [7] (see also Appendix A in [9] for details of scaling); namely, lengths and velocities are scaled to \((Q^2/g)^{1/3}\) and \((Qg)^{1/3}\) respectively. Here \(Q\) and \(g\) are the dimensional quantities for the rate of flow and the gravity acceleration respectively, whereas \((Q^2/g)^{1/3}\) is the depth of the critical uniform stream in the irrotational case.

The steady water motion is supposed to be two-dimensional and rotational; the surface tension is neglected on the free surface of the water, where the pressure is constant. These assumptions and the fact that water is incompressible allow us to seek the velocity field in the form \((\psi_y, -\psi_x)\), where \(\psi(x, y)\) is referred to as the stream function. The vorticity distribution \(\omega\) is supposed to be a prescribed smooth function depending only on the values of \(\psi\).

We choose the frame of reference so that the velocity field is time-independent as well as the unknown free-surface profile. The latter is assumed to be the graph of \(Y = \eta(X), X \in \mathbb{R}\), where \(\eta\) is a positive continuous function, and so the longitudinal section of the water domain is \(D = \{x \in \mathbb{R}, 0 < Y < \eta(X)\}\). The following free-boundary problem for \(\psi\) and \(\eta\) which describes all kinds of waves has long been known (cf. [7]):

\[
\begin{align*}
\psi_{XX} + \psi_{YY} + \omega(\psi) &= 0, \quad (X, Y) \in D; \quad (2.1) \\
\psi(X, 0) &= 0, \quad X \in \mathbb{R}; \quad (2.2) \\
\psi(X, \eta(X)) &= 1, \quad X \in \mathbb{R}; \quad (2.3) \\
|\nabla \psi(X, \eta(X))|^2 + 2\eta(X) &= 3r, \quad X \in \mathbb{R}. \quad (2.4)
\end{align*}
\]

In condition (2.4) (Bernoulli’s equation), \(r\) is a constant considered as the problem’s parameter and referred to as Bernoulli’s constant/the total head.

### 2.1 Stream solutions

By a stream (shear-flow) solution we mean a pair \((\psi, \eta) = (u(Y), d), d = \text{const}\), solving problem \((2.1)-(2.4)\) which reduces to the following one:

\[
u'' + \omega(u) = 0 \quad \text{on} \quad (0, d), \quad u(0) = 0, \quad u(d) = 1, \quad |u'(d)|^2 + 2d = 3r. \quad (2.5)
\]

Here, the prime symbol denotes differentiation with respect to \(Y\). A detailed study of these solutions including those that describe flows with counter-currents is given in [8]. Here we present a description of such solutions borrowed from [8]. They are parametrised by \(s = u'(0)\) which satisfies the inequality

\[
s > s_0 = \sqrt{\frac{2}{\max_{0 \leq \tau \leq 1} \Omega(\tau)}}, \quad \text{where} \quad \Omega(\tau) = \int_0^\tau \omega(t)dt.
\]
In order to describe all solutions to (2.5) it is convenient to introduce the following quantities:

- Denote by $\tau_+(s)$ ($\tau_-(s)$) the smallest positive root (the largest negative root) of the equation $2\Omega(\tau) = s^2$.
- Let $U(Y; s)$, $s > 0$ be the solution of the Cauchy problem

$$U'' + \omega(U) = 0, \quad U(0; s) = 0, \quad U'(0; s) = s.$$ (2.6)

Then the maximal interval of monotonicity of the function $U$ is $(y_-(s), y_+(s))$, where

$$y_\pm(s) = \int_{\tau_\pm(0)}^{\tau_\pm(s)} \frac{d\tau}{\sqrt{s^2 - \Omega(\tau)}}.$$

We note that if $\omega(\tau_\pm(s)) = 0$ then $y_\pm(s) = \pm\infty$.

- If $s > s_0$ then

$$d(s) = \int_0^1 \frac{d\tau}{\sqrt{s^2 - \Omega(\tau)}}$$

is the smallest positive value of $d$ such that $U(d; s) = 1$. Therefore the function $U(\cdot; s)$ satisfies the first three equations in (2.5) on the interval $(0, d(s))$.

Geometrical meaning of the quantities $\tau_\pm$ is the following:

$$\tau_+ = \sup_z U(z; s) \quad \text{and} \quad \tau_- = \inf_z U(z; s).$$

Now we are in position to describe solutions to problem (2.5):

a) $(u(Y), d) = (U(Y; s), d(s))$, where $d = d_{2k}^{(+)}$, $j = 0, 1, \ldots$,

$$d_{2k}^{(+)}(s) = d(s) + 2k(y_+(s) - y_-(s)),$$

$$d_{2k+1}^{(+)}(s) = d(s) + 2(y_+(s) - d(s)) + 2k(y_+(s) - y_-(s)), \quad k = 0, 1, \ldots$$

and

$$r = R_{j}^{(+)}(s) = \frac{1}{3} \left( s^2 - 2\Omega(1) + 2d_{j}^{(+)} \right).$$

The corresponding solution $U$ has the following properties: $U''(0, s) < 0$ and $U'$ has $j$ zeros on the interval $(0, d(s))$.

b) $(u(Y), d) = (U(Y - 2y_-(s); s), d(s))$, where $d = d_{j}^{(-)}$, $j = 1, 2, \ldots$,

$$d_{j}^{(-)}(s) = d_{j}^{(+)}(s) - 2y_-(s), \quad r = R_{j}^{(-)}(s) = \frac{1}{3} \left( s^2 - 2\Omega(1) + 2d_{j}^{(-)} \right).$$

The corresponding solution $U$ satisfies: $U'(0, s) > 0$ and $U'$ has $j$ zeros inside the interval $(0, d(s))$. 

3
In what follows we assume that the vorticity function \( \omega \) depends on a parameter \( \lambda' = (\lambda_1, \ldots, \lambda_{m-1}) \in \mathbb{R}^{m-1} \), \( \omega(u) = \omega(u; \lambda') \), and that this parameter is taken from a neighborhood of some \( \lambda'_* \) and put

\[
\omega_*(p) = \omega(p; \lambda'_*),
\]

Since a stream solution \((u, d)\) depends on both the vorticity \( \omega \) (which depends on \( \lambda' \)) and \( s \), it will be convenient to use the notation \( \lambda_0 \) for the parameter \( s \) so that we have now only one vector parameter \( \lambda = (\lambda_0, \lambda') = (\lambda_0, \ldots, \lambda_{m-1}) \), which is taken from a certain neighborhood

\[
\Lambda = \{ \lambda \in \mathbb{R}^m : |\lambda - \lambda_*| < \varepsilon \}, \quad \lambda_* = (s_*, \lambda'_*),
\]

where the parameters \( \lambda'_* \) and \( s_* \) are fixed throughout the paper. Thus, both the vorticity \( \omega \) and the stream solution \((u, d)\) depend on the same parameter \( \lambda \in \Lambda \). Furthermore, we put

\[
\begin{align*}
&u_*(Y) = u(Y; \lambda_*):= U(Y; s_*), \quad d_* = d(s_*),
\end{align*}
\]

where \((u(Y; \lambda), d(\lambda))\) is the stream solution corresponding to the vorticity \( \omega(p; \lambda) \) and such that \( u' \equiv 0 \equiv \lambda_0 \). The Bernoulli constant for \( u_* \) is given by \( r_* = r(\lambda) \), where

\[
r(\lambda) = \frac{1}{3} \left( \lambda_0^3 - 2 \int_0^1 \omega(p; \lambda) dp - 2 d(\lambda) \right).
\]

In what follows we will assume that

\[
u' \neq 0, \quad d_* \neq r_*.
\]

We will always assume that the function \( \omega(p; \lambda') \) is of class \( C^{\nu+1} \) for a certain \( \nu = 1, 2, \ldots \). Since we consider only \( d_\pm(s) < \infty \) then \( \omega(\tau_\pm(s)) \neq 0 \) when it is relevant and so the functions \( \tau_\pm(s), y_\pm(s) \), which depend also on \( \lambda' \), are also \( C^{\nu+1} \) smooth for \( \lambda \in \Lambda \), provided \( \varepsilon \) is sufficiently small. Moreover all functions \( U, d, R^\pm_j \) are of the same smoothness.

### 2.2 Dispersion equation

In what follows an important role will play a dispersion equation. It’s importance for the existence of small amplitude Stokes waves was demonstrated in \[2\] for unidirectional flows and in \[9\] for general flows. Let \((u, d)\) be a stream solution of \((2.1) - (2.4)\) with \( u'(d) \neq 0 \) for some fixed Bernoulli’s constant \( r \). The dispersion equation is defined as the following eigenvalue problem:

\[
- \varphi_{zz} - \omega'(u) \varphi = \mu \varphi \text{ on } (0, d), \quad \varphi(0) = 0, \quad \varphi_z(d) = \kappa \varphi(d).
\] (2.7)
Here
\[ \kappa = 1/k^2 - \omega(1)/k, \quad k = u'(d). \] (2.8)
The spectrum of (2.7) consists of countable set of simple real eigenvalues \( \{\mu_j\}_{j=1}^\infty \) ordered so that \( \mu_j < \mu_l \) for all \( j < l \). Furthermore, only a finite number of eigenvalues may be negative. The normalized eigenfunction corresponding to an eigenvalue \( \mu_j \) will be denoted by \( \varphi_j \). Thus, the set of all eigenfunctions \( \{\varphi_j\}_{j=1}^\infty \) forms an orthonormal basis in \( L^2(0,d) \).

Note that the Sturm-Liouville problem (2.7) depend on the triple \((\omega, u, d)\).

Thus, let \( \mu_j(\lambda) \) be the \( j \)-th eigenvalue of the problem (2.7) for the triple \((\omega(p; \lambda), u(Y; \lambda), d(\lambda))\), \( \lambda \in \Lambda \). Furthermore, we define
\[ k_* = u'_Y(d_*; \lambda_*), \quad \kappa_* = \frac{1}{k_*^2} - \frac{\omega_*(1)}{k_*}, \]
and
\[ \mu_*^j = \mu_j(\lambda_*). \]
In what follows we will assume that
\[ \mu_*^N \leq 0, \quad \mu_{N+1} > 0 \] (2.9)
for some \( N > 1 \) which will be fixed throughout the paper. Because all the eigenvalues of (2.7) are simple, then the functions \( \mu_j(\lambda) \) and \( \varphi_j(z, \lambda) \) are of class \( C^{\nu+1} \) in \( \Lambda \) and \([0, d(\lambda)] \times \Lambda \) respectively.

3 Reduction to a finite dimensional system

3.1 First order system

We start by rectifying the domain \( \mathcal{D} \). For this purpose, we change the coordinates to
\[ x = X, \quad z = Y \frac{d}{\eta(X)}, \]
while the horizontal coordinate remains unchanged. Thus, the domain \( \mathcal{D} \) transforms into the strip \( S = \mathbb{R} \times (0,d) \). Next, we introduce a new unknown function \( \Phi(x, z) \) on \( \bar{S} \) by
\[ \Phi(x, z) = \psi \left( x, \frac{z}{d} \eta(x) \right). \]
A direct calculation shows that problem (2.1)-(2.3) rewrites as
\[ \left[ \Phi_x - \frac{z \eta_x}{\eta} \Phi_z \right]_x - \frac{z \eta_x}{\eta} \left[ \Phi_x - \frac{z \eta_x}{\eta} \Phi_z \right]_z + \left( \frac{d}{\eta} \right)^2 \Phi_{zz} + \omega(\Phi) = 0; \] (3.1)
\[ \Phi(x, 0) = 0, \quad \Phi(x, d) = 1, \quad x \in \mathbb{R}; \] (3.2)
while the Bernoulli’s equation (2.4) becomes

$$\Phi^2 = \eta^2 \left( \frac{3r - 2\eta}{1 + \eta^2} \right).$$

(3.3)

The next step is to write (3.1) as a first order system. For this purpose, we introduce a new variable

$$\Psi(x, z) = \frac{\eta(x)}{\Phi(x, z)} \left[ \Phi_x(x, z) - \frac{z\eta_x(x)}{\eta(x)} \Phi_z(x, z) \right].$$

Thus, writing (3.1) in terms of $\Psi$ and $\Phi$ and using the definition of $\Psi$ to express $\Phi_x$, we obtain

$$\Phi_x = \frac{d}{\eta} \Psi + \frac{z}{\eta} \eta_x \Phi_z,$$

(3.4)

$$\Psi_x = \frac{1}{\eta} \eta_x (z \Psi) z - \frac{d}{\eta} \Phi_{zz} - \frac{\eta}{d} \omega(\Phi).$$

(3.5)

Using the identity $\eta_x = -\Psi(x, d)/\Phi_z(x, d)$ which follows from the definition of $\Psi$ by letting $z = d$, the Bernoulli’s equation (3.3) becomes

$$\Psi^2(x, d) + \Phi^2_x(x, d) = P(\eta(x)),$$

(3.6)

where the function $P(t)$ is defined by

$$P(t) = t^2[3r - 2t]/d^2.$$

If $\Psi$ and $\Phi - u$ are small then the function $\eta$ can be uniquely resolved from the equation (3.6) so that $\eta$ is close to $d$, provided $d \neq r$. Having this in mind and that $\eta_x = -\Psi(x, d)/\Phi_z(x, d)$ equations (3.4), (3.5) together with the boundary conditions (3.2) can be considered as first order system with respect to the variables $\Phi$ and $\Psi$ only. In this case the latter functions are considered continuous functions with values in $W^{2,2}_1(0, d)$ and $W^{1,2}(0, d)$ whose derivatives with respect to $x$ belong to $W^{1,2}_1(0, d)$ and $L_2(0, d)$ respectively. $W^{2,2}_1(0, d)$ consists of functions $w$ from $W^{2,2}_1(0, d)$ satisfying $w(0) = 0$ and $w(d) = \gamma$. In [10] it was shown that the system (3.4), (3.6) admits Hamiltonian structure (see also [14]).

3.2 Linearization around the stream solution

We put

$$\Phi = \bar{\Phi} + u, \quad \Psi = \bar{\Psi}, \quad \eta = \zeta + d.$$

(3.7)
Using that \( P'(d)/2k = k/d - 1/k \), equations (3.4) and (3.5) together with the boundary conditions (3.2) and (3.6) lead to

\[
\begin{align*}
\bar{\Phi}_x &= \bar{\Psi} + \frac{zu_x \zeta_x}{d} + \bar{N}_1 \\
\bar{\Psi}_x &= -\bar{\Phi}_{zz} - \frac{2\zeta \omega(u)}{d} - \omega'(u)\bar{\Phi} + \bar{N}_2 \\
\bar{\Phi}(x, 0) &= \bar{\Phi}(x, d) = \bar{\Psi}(x, 0) = 0 \\
\bar{\Phi}_z(x, d) &= \left( \frac{k}{d} - \frac{1}{k} \right) \zeta = \bar{N}_3,
\end{align*}
\]

where the nonlinear operators \( \bar{N}_j, j = 1, 2, 3 \), are given by

\[
\bar{N}_1(\bar{\Psi}, \bar{\Phi}_z; \zeta, \zeta) = -\frac{\zeta \bar{\Psi}}{d + \zeta} + z\zeta_x \left[ \frac{d\bar{\Phi}_z - \zeta u_z}{d(d + \zeta)} \right]
\]

and

\[
\begin{align*}
\bar{N}_2(\bar{\Psi}, \bar{\Psi}_z, \bar{\Phi}, \bar{\Phi}_{zz}; \zeta, \zeta_x) &= \frac{\zeta_x}{d + \zeta} (z\bar{\Psi})_z + \frac{\zeta(\bar{\Phi}_{zz} - \omega'(u)\bar{\Phi})}{d + \zeta} - \frac{\zeta^2}{d(d + \zeta)} u_{zz} \\
&\quad - \frac{\zeta + d}{d} (\omega(u + \bar{\Phi}) - \omega(u) - \omega'(u)\bar{\Phi}),
\end{align*}
\]

while the nonlinear part in the Bernoulli’s equation is defined by

\[
\bar{N}_3(\bar{\Psi}, \bar{\Phi}_z; \zeta) = \frac{1}{2k} \left[ -(\bar{\Psi}_z^2 + \bar{\Phi}_z^2)_{zz} + (P(d + \zeta) - P(d) - P'(d)\zeta) \right].
\]

Next we use the following change of variables (see [3], Sect.3):

\[
\hat{\Phi} = \bar{\Phi} - \frac{zu_x \zeta_x}{d}, \quad \hat{\Psi} = \bar{\Psi}.
\]

Then the relations \( \Phi(x, d) = 0 \) and \( \eta_x = -\Psi(x, d)/\Phi_z(x, d) \) imply

\[
\zeta = -(1/k)\hat{\Phi}(x, d), \quad \zeta_x = -\frac{\hat{\Psi}(x, d)}{k + \hat{\Phi}_z(x, d) - \left[ \frac{1}{d} - \frac{\omega'(1)}{k} \right] \hat{\Phi}(x, d)},
\]

for all \( x \in \mathbb{R} \). This allows to rewrite (3.8)-(3.11) as a system of the first order only in terms of the functions \( \hat{\Psi} \) and \( \hat{\Phi} \):

\[
\begin{align*}
\hat{\Phi}_x &= \hat{\Psi} + \hat{N}_1 \\
\hat{\Psi}_x &= \hat{\Phi}_{zz} - \omega'(u)\hat{\Phi} + \hat{N}_2 \\
\hat{\Phi}(x, 0) &= \hat{\Psi}(x, 0) = 0 \\
\hat{\Phi}_z(x, d) - \kappa \hat{\Phi}(x, d) &= \hat{N}_3,
\end{align*}
\]

for all \( x \in \mathbb{R} \). This allows to rewrite (3.8)-(3.11) as a system of the first order only in terms of the functions \( \hat{\Psi} \) and \( \hat{\Phi} \):

\[
\begin{align*}
\hat{\Phi}_x &= \hat{\Psi} + \hat{N}_1 \\
\hat{\Psi}_x &= \hat{\Phi}_{zz} - \omega'(u)\hat{\Phi} + \hat{N}_2 \\
\hat{\Phi}(x, 0) &= \hat{\Psi}(x, 0) = 0 \\
\hat{\Phi}_z(x, d) - \kappa \hat{\Phi}(x, d) &= \hat{N}_3,
\end{align*}
\]

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\[
\begin{align*}
\hat{\Phi}_x &= \hat{\Psi} + \hat{N}_1 \\
\hat{\Psi}_x &= \hat{\Phi}_{zz} - \omega'(u)\hat{\Phi} + \hat{N}_2 \\
\hat{\Phi}(x, 0) &= \hat{\Psi}(x, 0) = 0 \\
\hat{\Phi}_z(x, d) - \kappa \hat{\Phi}(x, d) &= \hat{N}_3,
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\[
\begin{align*}
\hat{\Phi}_x &= \hat{\Psi} + \hat{N}_1 \\
\hat{\Psi}_x &= \hat{\Phi}_{zz} - \omega'(u)\hat{\Phi} + \hat{N}_2 \\
\hat{\Phi}(x, 0) &= \hat{\Psi}(x, 0) = 0 \\
\hat{\Phi}_z(x, d) - \kappa \hat{\Phi}(x, d) &= \hat{N}_3,
\end{align*}
\]
where \( \kappa \) is given by (2.8). The nonlinear operators \( \hat{N}_j \) are naturally defined by

\[
\hat{N}_1(\hat{\Psi}, \hat{\Phi}, \hat{\Phi}_z) = -\frac{\zeta \hat{\Psi}}{d + \zeta} + \frac{z\zeta_u + z\zeta_{zz}}{d + \zeta} \frac{d\hat{\Phi}_z + z\zeta_{zz}}{d + \zeta};
\]

\[
\hat{N}_2(\hat{\Psi}, \hat{\Phi}, \hat{\Phi}_z, \hat{\Phi}_{zz}) = \frac{\zeta_x}{d + \zeta} (e\hat{\Psi}_z) + \frac{\zeta(\hat{\Phi}_{zz} - \omega'(u)\hat{\Phi} - 2\zeta z\omega'(u)u_z/d)}{d + \zeta},
\]

\[
\hat{N}_3(\hat{\Psi}, \hat{\Phi}, \hat{\Phi}_z) = \frac{1}{2k} \left[ -\left( \hat{\Psi}^2 + \left( \hat{\Phi}_z + \frac{z\zeta_{zz}}{d}\right)^2 \right)_{z=d} + (P(d + \zeta) - P(d) - P'(d)\zeta) \right].
\] (3.20)

where \( \zeta = \zeta(\hat{\Phi}) \) and \( \zeta_x = \zeta_x(\hat{\Psi}, \hat{\Phi}, \hat{\Phi}_z) \) are defined by (3.15).

Next, by using implicit function theorem we solve equation (3.19) with respect to \( \Phi_z \) and obtain

\[
\hat{\Phi}_z(x, d) - \kappa \hat{\Phi}(x, d) = F_3(\hat{\Psi}, \hat{\Phi}),
\] (3.21)

where \( F_3 \) is analytic function satisfying

\[
F_3(\xi_1, \xi_2) = O(\xi^2).
\]

Moreover, one can verify, for example by solving (3.19) with the help of fixed point iterations, that

\[
F_3(\hat{\Psi}, \hat{\Phi}) = \hat{N}_3(\hat{\Psi}, \hat{\Phi}, \kappa \hat{\Phi})(x, d) + F_{31}(\hat{\Psi}, \hat{\Phi}),
\] (3.22)

where

\[
F_{31}(\xi_1, \xi_2) = O(\xi^3).
\]

### 3.3 Spectral decomposition

Let

\[
H^2_d = \{ w \in H^2(0, d) : w(0) = 0 \}, \quad H^1_d = \{ v \in H^1(0, d) : v(0) = 0 \}.
\]

Also let

\[
X_\lambda = \{ w \in H^2_d : w_z(d) - \kappa w(d) = 0 \}.
\]

We remind that \( d \) and \( \kappa \) may depend on \( \lambda \). The space \( X_\lambda \) represents the domain of the operator \(-\partial_d^2 - \omega'(u)\) with the boundary conditions (2.7) and the standard norm in \( H^2(0, d) \) is equivalent to the graph norm.

We introduce the orthogonal projectors

\[
P\phi = \sum_{j=1}^N \alpha_j \varphi_j, \quad \alpha_j = \int_0^d \phi \varphi_j \, dz, \quad \tilde{P} = \text{id} - P.
\]

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and represent the functions $\hat{\Psi}$ and $\hat{\Phi}$ as

$$
\hat{\Phi} = P\hat{\Phi} + \tilde{P}\Phi = \sum_{j=1}^{N} \alpha_j(x)\varphi_j + \hat{\phi}, \quad \hat{\Psi} = P\hat{\Psi} + \tilde{P}\Psi = \sum_{j=1}^{N} \beta_j(x)\varphi_j + \hat{\psi}. \tag{3.23}
$$

Multiplying (3.16)-(3.17) by $\varphi_j$, $j = 1, \ldots, N$, and integrating over $(0, d)$, we obtain

$$
\alpha_j' = \beta_j + F_{1j}, \tag{3.24}
$$
$$
\beta_j' = \mu_j \alpha_j + F_{2j}; \tag{3.25}
$$

where

$$
F_{1j} = \int_{0}^{d} \hat{N}_1(\hat{\Psi}, \hat{\Phi}, \hat{\Phi}_z)\varphi_j dz,
$$
$$
F_{2j} = \int_{0}^{d} \hat{N}_2(\hat{\Psi}, \hat{\Phi}, \hat{\Phi}_z, \hat{\Phi}_{zz})\varphi_j dz - F_3(\hat{\Psi}, \hat{\Phi})(d),
$$

Subtracting the sum of equations (3.24) and (3.25) multiplied by $\varphi_j$ from (3.16) and (3.17) respectively, we obtain

$$
\tilde{\phi}_x = \tilde{\psi} + \tilde{P}(\hat{N}_1), \tag{3.26}
$$
$$
\tilde{\psi}_x = -\tilde{\phi}_{zz} - \omega'(u)\tilde{\phi} + \tilde{P}(\hat{N}_2) + \sum_{j=1}^{N} F_3(\hat{\Psi}, \hat{\Phi})(d)\varphi_j(d)\varphi_j. \tag{3.27}
$$

The boundary conditions (3.18) and (3.21) take the form

$$
\tilde{\phi}(x, 0) = \tilde{\psi}(x, 0) = 0, \quad \tilde{\phi}_z(x, d) - \kappa\tilde{\phi}(x, d) = F_3(\hat{\Psi}, \hat{\Phi}). \tag{3.28}
$$

In what follows we are going to apply a center manifold reduction theorem due to Mielke and for this purpose we need to fix the domain of the operators in (3.26)-(3.28) independently of the parameter $\lambda \in \Lambda$. To reduce the problem to the fixed strip $\mathbb{R} \times (0, d_*)$, we apply the following change of variables:

$$
y = d_* z/d \quad \text{and} \quad \tilde{\phi}(x, y) = \tilde{\phi}(x, \frac{dy}{d_*}), \quad \tilde{\psi}(x, y) = \tilde{\psi}(x, \frac{dy}{d_*}). \tag{3.29}
$$

We use the same notations for the functions $\tilde{\phi}$ and $\tilde{\psi}$ in new and old coordinates and hope that this will not cause misunderstanding. Then the system (3.26)-(3.28) transforms to

$$
\tilde{\phi}_x = \tilde{\psi} + \tilde{P}(\hat{N}_1),
$$
$$
\tilde{\psi}_x = -d_*^2 \tilde{\phi}_{yy} - \omega'(u)\tilde{\phi} + \tilde{P}(\hat{N}_2) + \sum_{j=1}^{N} F_3(\hat{\Psi}, \hat{\Phi})(d)\varphi_j(d)\varphi_j. \tag{3.30}
$$
\[ \tilde{\phi}(x,0) = \tilde{\psi}(x,0) = 0, \quad \tilde{\phi}_y(x,d_*) = \frac{d\kappa}{d_*} \tilde{\phi}(x,d_*) = \frac{d}{d_*} F_{3j}(\hat{\Psi},\hat{\Phi}). \quad (3.31) \]

Moreover, the projector \( P \) in new coordinates becomes
\[
P u(y) = P_{\lambda} u(y) = \begin{bmatrix} \frac{d}{d_*} \sum_{j=1}^{N} \int_{0}^{d_*} u(\tau) \varphi_j(\tau d_*/d) \, d\tau \end{bmatrix} \varphi_j(y d_*/d) \quad (3.32)
\]
and the variables \( \tilde{\phi}_z \) and \( \tilde{\phi}_{zz} \) must be replaced by \( d_* \tilde{\phi}_y/d \) and \( d_*^2 \tilde{\phi}_y/d^2 \) respectively.

### 3.4 Main result

We put
\[ \mathcal{H}_n^\alpha = \{ \tilde{\psi} \in H_{d_*}^n : P_{\lambda} \tilde{\psi} = 0 \}, \quad n = 0, 1, 2. \]

**Theorem 3.1.** There exists neighborhoods \( W, W_2 \) and \( W_1 \) of the origins in \( \mathbb{R}^{2N}, \mathcal{H}_2^\alpha \) and in \( \mathcal{H}_1^\alpha \) respectively and smooth vector functions
\[ h : W \times \Lambda \to W_2, \quad g : W \times \Lambda \to W_1 \]
with the following properties:

1. Functions \( h \) and \( g \) are of the class \( C^\nu \) provided the vorticity function \( \omega(y;\lambda') \) is of the class \( C^{\nu+1} \). Moreover,
\[ ||h; H_{d_*}^2|| + ||g; H_{d_*}^1|| = O(|\alpha|^2 + |\beta|^2). \]

2. We introduce the system
\[
\begin{align*}
\alpha'_j &= \beta_j + f_{1j}(\alpha, \beta; \lambda) \\
\beta'_j &= \mu_j^2 \alpha_j + f_{2j}(\alpha, \beta; \lambda), \quad j = 1, \ldots, N. \quad (3.33)
\end{align*}
\]
Here
\[ f_{1j}(\alpha, \beta; \lambda) = F_{1j}(\hat{\Psi}, \hat{\Phi}, \hat{\Phi}_z) \text{ and } f_{2j}(\alpha, \beta; \lambda) = F_{2j}(\hat{\Psi}, \hat{\Phi}, \hat{\Phi}_z, \hat{\Phi}_{zz}), \]
where \( F_{1,j} \) and \( F_{2,j} \) are the same as in system \( (3.24), (3.25) \) and
\[
\hat{\Phi}(\alpha, \beta)(z) = \sum_{j=1}^{N} \alpha_j \varphi_j(z) + h(zd_*/d; \alpha, \beta), \quad \hat{\Psi}(\alpha, \beta)(z) = \sum_{j=1}^{N} \beta_j \varphi_j(z) + g(zd_*/d; \alpha, \beta). \quad (3.34)
\]
We do not write here dependence on $\lambda$ but certainly, $h$, $g$, $\varphi_j$ and $\hat{\Phi}$, $\hat{\Psi}$ depend also on $\lambda$. Let also

$$M^\lambda = \{\Phi(\alpha, \beta), \hat{\Psi}(\alpha, \beta) : (\alpha, \beta) \in W\}, \quad \Phi(\alpha, \beta)(z) = u(z) - \frac{zu_z(z)\hat{\Phi}(d)}{kd} + \hat{\Phi}(z).$$

Then

(i) $M^\lambda \subset W^2_1(0, d) \times W^1_1(0, d)$ is a locally invariant manifold of (3.4)-(3.5): through every point in $M^\lambda$ there passes a unique solution of (3.4)-(3.5) that remains on $M^\lambda$ as long as $(h, g)$ remains in $W^2_0 \times W^1_0$;

(ii) every bounded solution $(\Phi, \Psi)$ of (3.4)-(3.5) for which $(\alpha, \beta) \in W$ and $(h, g) \in W_2 \times W_1$ lies completely in $M^\lambda$, provided the norm $\|(\Phi - u, \Psi)\|$ in the space $W^2_0(0, d) \times W^1_0(0, d)$ is small;

(iii) every solution $(\alpha, \beta) : (a, b) \rightarrow W$ of the reduced system (3.33) generates a solution

$$\Phi(x, z) = u(z) - \frac{zu_z(z)\hat{\Phi}(x, d)}{kd} + \hat{\Phi}(x, z), \quad \hat{\Phi}(x, z) = \sum_{j=1}^N \alpha_j(x)\varphi_j(z) + h(zd_z/d; \alpha(x), \beta(x))$$

and

$$\Psi(x, z) = \sum_{j=1}^N \beta_j(x)\varphi_j(z) + g(zd_z/d; \alpha(x), \beta(x))$$

of the full problem (3.4)-(3.5) satisfying the boundary condition (3.2).

If we introduce the functions

$$f_{1j}^0 = \int_0^d \hat{N}_1(\psi, \phi, \phi_z)\varphi_j dz,$$

and

$$f_{2j}^0 = \int_0^d \hat{N}_2(\psi, \phi, \phi_z, \phi_{zz})\varphi_j dz - \hat{N}_3(\psi, \phi, \kappa\phi)(d)\varphi_j(d),$$

where

$$\phi(z) = \sum_{j=1}^N \alpha_j\varphi_j(z), \quad \psi(z) = \sum_{j=1}^N \beta_j\varphi_j(z), \quad (3.35)$$

then

$$f_{1j}(\alpha, \beta; \lambda) = f_{1j}^0 + f_{1j}^1, \quad f_{2j}(\alpha, \beta; \lambda) = f_{2j}^0 + f_{2j}^1$$

where

$$|f_{1j}^1| + |f_{2j}^1| = O(|\alpha|^3 + |\beta|^3).$$

The proof of this theorem is given in the next three sections.
3.5 Change of variables

The proof of the theorem is based on the application of a reduction theorem due to Mielke. However, we can not apply Mielke’s result directly to the system \((3.30), (3.26), (3.27)\) because of the nonlinear boundary condition \((3.31)\). We overcome this difficulty by passing to a new variables for which all boundary conditions are homogeneous.

Let 
\[
\tilde{P}_\lambda = I - P_\lambda, \quad \text{where} \quad P_\lambda \text{ is given by (3.32)}.
\]
If \(\lambda = \lambda_*\) and hence \(d = d_*\) then we will use the notation 
\[
P_* = P_{\lambda_*} \quad \text{and} \quad \tilde{P}_* = \tilde{P}_{\lambda_*}.
\]
By a new change of variable we reduce the last boundary condition in \((3.31)\) to a homogeneous one (without nonlinear term), which is independent of the parameter \(\lambda\). The change of variable
\[
H_2^\lambda \times H_2^\lambda \ni (\tilde{\varphi}, \tilde{\psi}) \rightarrow (w, v) \in H_2^{\lambda_*} \times H_2^{\lambda_*},
\]
is the following (compare with [1], [3]):
\[
v = \tilde{P}_* \tilde{\psi}, \quad w = \tilde{P}_* \tilde{\Phi} \quad \text{(3.36)}
\]
\[
\tilde{\Phi} = \phi + \tilde{\Phi} \int_y^{d_*} \frac{d}{d_*} F_3(\tilde{\Psi}, \tilde{\Phi})(\tau) + \frac{\kappa d - \kappa_* d_*}{d_*} \tilde{\Phi}(\tau) \, d\tau. \quad \text{(3.37)}
\]
Here
\[
\tilde{\Phi} = \phi + \tilde{\phi}, \quad \tilde{\Psi} = \psi + \tilde{\psi}, \quad \text{and} \quad \phi = \sum_{j=1}^{N} \alpha_j \phi_j, \quad \psi = \sum_{j=1}^{N} \beta_j \phi_j.
\]
One can verify directly that the function \(\tilde{\Phi}\), and hence \(w\) satisfies the boundary condition
\[
\tilde{\Phi}_y(d_*) - \kappa_* \tilde{\Phi} = 0 \quad \text{(3.38)}
\]
if \(\tilde{\phi}\) satisfies the last boundary condition in \((3.31)\). Our aim is to invert relations \((3.36)\) and \((3.37)\) and to express \(\tilde{\psi}\) and \(\tilde{\phi}\) through \(v, w\) and \(\alpha, \beta, \lambda\). This can be done directly by means of the implicit function theorem, however we are also interested in explicit formulas giving the leading terms of the approximation. First, we find the inverse to the operator
\[
H_2^{\lambda_*} \ni \tilde{\psi} \rightarrow \tilde{P}_* \tilde{\psi} = v \in H_2^{\lambda_*}.
\]
We are looking for it in the form
\[
\tilde{\psi} = \tilde{P}_\lambda (I + S_\lambda)v, \quad \text{(3.39)}
\]
where \(S_\lambda : H_2^{\lambda_*} \rightarrow H_2^{\lambda_*}\). Then substituting \((3.39)\) in \((3.36)\), we find
\[
v = \tilde{P}_* \tilde{\psi} = (I - \tilde{P}_*(P_\lambda - P_*))(I + S_\lambda)v.
\]
Therefore, we express
\[ S_\lambda v = \bar{P}_*(\mathcal{P}_\lambda - \mathcal{P}_*)(I + S_\lambda)v. \]

Note that the operator \( I - \bar{P}_*(\mathcal{P}_\lambda - \mathcal{P}_*) \) is invertible, provided \( \lambda \) is close to \( \lambda_* \).

Thus, we can resolve \( S_\lambda v \) from the last identity, which gives
\[ S_\lambda v = (I - \bar{P}_*(\mathcal{P}_\lambda - \mathcal{P}_*))^{-1} \bar{P}_*(\mathcal{P}_\lambda - \mathcal{P}_*)v. \]

Thus we can resolve the first relations in (3.36):
\[ \bar{\psi} = \bar{P}_\lambda \left( I + (I - \bar{P}_*(\mathcal{P}_\lambda - \mathcal{P}_*))^{-1} \bar{P}_*(\mathcal{P}_\lambda - \mathcal{P}_*) \right) v. \] (3.40)

The operator \( S_\lambda : \mathcal{H}^{2}_\lambda \rightarrow \mathcal{H}^{2}_\lambda \) is continuous with the norm of the order \( O(|\lambda - \lambda_*|^2) \).

Solving the second equation in (3.36), we have
\[ \bar{\Phi} = \bar{P}_\lambda \left( I + (I - \bar{P}_*(\mathcal{P}_\lambda - \mathcal{P}_*))^{-1} \bar{P}_*(\mathcal{P}_\lambda - \mathcal{P}_*) \right) w. \] (3.41)

We can also write relations (3.40) and (3.41) as
\[ \bar{\psi} = v + M(\lambda)v \quad \text{and} \quad \bar{\Phi} = w + M(\lambda)w, \] (3.42)

where
\[ M(\lambda) = (I - \bar{P}_\lambda(I - \bar{P}_*(\mathcal{P}_\lambda - \mathcal{P}_*))^{-1} \bar{P}_*)(\mathcal{P}_* - \mathcal{P}_\lambda). \]

The operator function \( M \) is \((\nu + 1)\) times differentiable and of order \( O(|\lambda - \lambda_*|^2) \).

Let us solve equation (3.37) with respect to \( \bar{\phi} \). We are looking for the solution in the form
\[ \bar{\phi} = \Phi + R, \quad R = R(\Phi, v, \alpha, \beta, \lambda), \]

where \( R \) is a nonlinear operator defined in a neighborhood of the origin in \( \mathcal{H}^{2}_\lambda \).

Substituting this in to (3.37), we get
\[ R = -\bar{P}_* \int_y^d \frac{y}{d_s} \left( \frac{d}{d_s} F_3(\bar{\Psi}, \bar{\Phi})(\tau) + \frac{\kappa d - \kappa_* d_s}{d_s} (\Phi(\tau) + R) \right) d\tau. \] (3.43)

Here one must put
\[ \bar{\Psi} = \psi + \mathcal{P}_\lambda(I + S)v, \quad \bar{\Phi} = \phi + \Phi + R. \]

Applying the fixed point theorem to equation (3.43) we find \( R \) as the function of \( (\Phi, v, \alpha, \beta, \lambda) \). Moreover, if the function \( F_3 \) is \( C^{\nu+1} \) smooth with respect to these arguments the same is true for the function \( R \). Furthermore, if \( \bar{\phi} \) satisfies the relation (3.21) then the function \( \bar{\Phi} \) satisfies (3.38).
Thus, we obtain
\[ \tilde{\psi} = v + M(\lambda)v \text{ and } \tilde{\phi} = w + M(\lambda)w + R(v, w; \alpha, \beta, \lambda), \]  
(3.44)
where \( R \) is \( C^{\nu+1} \) function in a neighborhood of \( (0, 0, 0, \lambda_*) \) in the space \( \mathcal{H}^1_{\lambda_*} \times \mathcal{H}^2_{\lambda_*} \times \mathbb{R}^N \times \mathbb{R}^N \times \Lambda \) with values in \( \mathcal{H}^2_{\lambda_*} \). We can represent also \( R \) as \( R_1 + Q \), where \( Q \)
\[ Q = -\bar{\mathcal{P}}_* \int_y^{d_*} \frac{y(\kappa d - \kappa_* d_*)}{d_*^2} (\bar{\Phi}(\tau) + Q) d\tau \]  
(3.45)
and \( R_1 \) solves the equation
\[ R_1 = -\bar{\mathcal{P}}_* \int_y^{d_*} \frac{y}{d_*} \left( \frac{d}{d_*} F_3(\bar{\Psi}, \bar{\Phi})(\tau) + \frac{\kappa d - \kappa_* d_*}{d_*} R_1 \right) d\tau. \]  
(3.46)
Then \( Q = Q(w; \lambda) \) is a linear operator with respect to \( w \) satisfying
\[ ||Q(w; \lambda); H^2_{d_*}|| \leq C|\lambda - \lambda_*| ||w; H^1_{d_*}|| \]
and \( R_1(v, w; \alpha, \beta, \lambda) \) is \( C^{\nu+1} \) function mapping a neighborhood of \( (0, 0, 0, \lambda_*) \) to \( \mathcal{H}^2_{\lambda_*} \) and
\[ ||R_1(v, w; \alpha, \beta, \lambda); H^1_{d_*}|| = O \left( ||v; H^1_{d_*}||^2 + ||w; H^2_{d_*}||^2 + |\alpha|^2 + |\beta|^2 \right). \]
The representation (3.44) takes the form
\[ \tilde{\psi} = v + M(\lambda)v \text{ and } \tilde{\phi} = w + M(\lambda)w + Q(w; \lambda) + R_1(v, w; \alpha, \beta, \lambda). \]  
(3.47)
After this change of variables the problem (3.30)-(3.31) becomes
\[ w_x = v + \bar{\mathcal{P}}_* f_1(w, v, \alpha, \beta, \lambda) \]
\[ v_x = -w_{yy} - \omega'(u)w + \bar{\mathcal{P}}_* (Q_1(w; \lambda) + f_2(w, v, \alpha, \beta, \lambda)) \]  
(3.48)
and
\[ w(x, 0) = v(x, 0) = 0, \quad w_y(x, d_*) - \kappa_* w(x, d_*) = 0. \]  
(3.49)
Here \( Q_1(w; \lambda) \) is a linear operator with respect to \( w \) satisfying
\[ ||Q_1(w; \lambda); H^2_{d_*}|| \leq C|\lambda - \lambda_*| ||w; H^2_{d_*}||. \]
To describe required properties of \( f_1 \) and \( f_2 \) let
\[ B^1_\varepsilon = \{ v \in \mathcal{H}^1_{\lambda_*} : ||v; H^1_{d_*}|| \leq \varepsilon \}, \quad B^2_\varepsilon = \{ w \in \mathcal{H}^2_{\lambda_*} : ||w; H^2_{d_*}|| \leq \varepsilon \} \]
and

\[ B_\varepsilon(a; M) = \{ X \in \mathbb{R}^M : |X - a| \leq \varepsilon \}. \]

Then for sufficiently small \( \varepsilon \)

\[ f_1 : B_\varepsilon^2 \times B_\varepsilon^1 \times B_\varepsilon(0; 2N) \times B_\varepsilon(\lambda_0; m) \rightarrow H^1_{d_*}, \]
\[ f_2, Q : B_\varepsilon^2 \times B_\varepsilon^1 \times B_\varepsilon(0; 2N) \times B_\varepsilon(\lambda_0; m) \rightarrow L^2(0, d_*) \]

are \( C^{\nu+1} \) times continuously differentiable maps, satisfying

\[ ||f_1 : H^1_{d_*}|| + ||f_2 : L^2(0, d_*)|| = O(||w : H^2_{d_*}||^2 + ||w : H^1_{d_*}||^2 + |\alpha|^2 + |\beta|^2). \]

3.6 Mielke’s reduction theorem

Let \( X \) be a Hilbert space which is represented as a product \( X = X_1 \times X_2 \) of two Hilbert spaces \( X_1 \) (finite dimensional) and \( X_2 \) (infinite dimensional) with the norms \( || \cdot || \) and \( || \cdot ||_2 \) respectively. Consider the following system of differential equations

\[ \dot{x}_1 = A(\lambda)x_1 + f_1(x_1, x_2, \lambda), \]
\[ \dot{x}_2 = Bx_2 + Q(x_1, x_2, \lambda)x_2 + f_2(x_1, x_2, \lambda), \]

where \( A(\lambda) \) is a linear operator in \( X_1 \) and \( B : \mathcal{D} \subset X_2 \rightarrow X_2 \) is a closed linear operator. We will consider \( \mathcal{D} \) as a Hilbert space supplied with the graph norm \( ||x_2: \mathcal{D}|| = (||x_2||^2 + ||Bx_2||^2)^{1/2} \). Here \( \lambda \) is a parameter located in a neighborhood \( \Lambda' \subset \mathbb{R}^m \) of a point \( \lambda_0 \);

\[ f_1 : \mathcal{U}'_1 \times \mathcal{U}'_2 \times \Lambda' \rightarrow X_1, \quad f_2 : \mathcal{U}'_1 \times \mathcal{U}'_2 \times \Lambda' \rightarrow X_2 \]
\[ Q : \mathcal{U}'_1 \times \mathcal{U}'_2 \times \Lambda' \rightarrow X_2 \]

are continuously differentiable functions. Here \( \mathcal{U}'_1 \) and \( \mathcal{U}'_2 \) are neighborhoods of the origin in the spaces \( X_1 \) and \( \mathcal{D} \) respectively.

We assume that

(A1) the operator function \( A(\lambda) \) is \((\nu + 1)\) times continuously differentiable with respect to \( \lambda \) and the spectrum of \( A(\lambda_0) \) lies on the imaginary axis;

(A2) the operator \( B : \mathcal{D} \rightarrow X_2 \) is continuous operator and for all \( \xi \in \mathbb{R} \) the operator \( B - i\xi \) is invertible and

\[ ||(B - i\xi)^{-1}|| \leq \frac{C}{1 + |\xi|} \]

for some constant \( C \) independent of \( \xi \);
(A3) the function (3.52) are \((\nu + 1)\) times continuously differentiable and its derivatives are uniformly bounded on the corresponding definition domains. Moreover,
\[
\|f_1(x_1, x_2, \lambda)\| = O(||x_1||^2 + ||x_2 ; D||^2), \quad \|f_2(x_1, x_2, \lambda)\|_2 = O(||x_1||^2 + ||x_2 ; D||^2)
\]
\[
\|Q(x_1, x_2, \lambda)\|_2 = O(||x_1|| + ||x_2 ; D|| + |\lambda - \lambda_0|)
\]
(3.54)
for \((x_1, x_2, \lambda) \in U_1' \times U_2' \times \Lambda\).

Then there exist neighborhoods \(U_1 \subset U_1', \ D_2 \subset D_2', \ \Lambda \subset \Lambda'\)

of 0, 0 and \(\lambda_0\) respectively and a reduction function
\[
h : U_1 \times \Lambda \to D
\]
(3.55)
with the following properties: the function (3.55) is \(k\) times continuously differentiable and its derivatives are bounded and uniformly continuous on \(U_1 \times \Lambda\) and
\[
h(x_1, \lambda) = O(||x_1||^2) \quad \text{for all } \lambda \in \Lambda.
\]
(3.56)
The graph
\[
M^\Lambda_\lambda = \{(x_1, h(x_1, \lambda) \in U_1 \times D_2 : x_1 \in U_1\}
\]
is a center manifold for (3.50), (3.51), which means that:

(1) every small bounded solution of (3.50), (3.51) with \(x_1(t) \in U_1\) and \(x_2(t) \in D_2\) lies completely in \(M^\Lambda_\lambda\);

(2) every solution \(x_1(t), t \in \mathbb{R}\), of the reduced equation
\[
\dot{x}_1 = A(\lambda)x_1 + f_1(x_1, h(x_1, \lambda), \lambda)
\]
generates a solution \((x_1(t), x_2(t))\), \(x_2(t) = h(x_1(t), \lambda)\) of the equation (3.50), (3.51).

This theorem is taken basically from [12], see also [11], [13], [14], [5]. The only small difference is that we split the right-hand side in (3.50), (3.51) into linear parts with respect to \(x_1\) and \(x_2\) and quadratically depending on \(x_1\) and \(x_2\). This allows us to write more explicit estimate (3.56) for the reduction function \(h\). The proof of this improvement is quite straightforward and we omit it.
3.7 Proof of Theorem 3.1

We will apply Mielke’s theorem to the problem (3.24), (3.25), (3.48), (3.49). In order to do this we choose

\[ X_1 = \mathbb{R}^{2N}, \quad X_2 = \mathbb{H}_1^{\lambda^*} \times \mathbb{H}_0^{\lambda^*}, \quad D = \mathbb{H}_2^{\lambda^*} \times \mathbb{H}_1^{\lambda^*}. \]

The property (A1) follows from (2.9).

In order to verify the property (A2) in the Mielke’s theorem let us consider the problem

\[
\begin{align*}
    i\xi w - v &= f, \quad (3.57) \\
    i\xi v + w_{zz} + \omega' \phi(u)w &= g, \quad (3.58)
\end{align*}
\]

where \((f, g) \in X_2\) and \((v, w) \in D\). We are looking for solution to problem (3.57), (3.58) in the form

\[ w = \sum_{j=N+1}^{\infty} a_j \phi_j, \quad v = \sum_{j=N+1}^{\infty} b_j \phi_j. \]

We represent also

\[ f = \sum_{j=N+1}^{\infty} f_j \phi_j, \quad g = \sum_{j=N+1}^{\infty} g_j \phi_j. \]

Then

\[ b_j = -\frac{\mu_j}{\mu_j + \xi^2} \left( f_j + \frac{i\xi}{\mu_j} g_j \right), \quad a_j = \frac{-i\xi}{\mu_j + \xi^2} f_j - \frac{1}{\mu_j + \xi^2} g_j \]

Since the norms \(||w; H^1||\) and \(||v; L^2||\) are equivalent to the norms

\[ \left( \sum_{j=N+1}^{\infty} \mu_j a_j^2 \right)^{1/2} \quad \text{and} \quad \left( \sum_{j=N+1}^{\infty} b_j^2 \right)^{1/2} \]

respectively, we verify that

\[ ||w; H^1||^2 + ||v; L^2||^2 \leq \frac{C}{1 + \xi^2} \left( ||f; H^1||^2 + ||g; L^2||^2 \right). \quad (3.59) \]

(A3) We require that the vorticity function \(\omega(y; \lambda)\) is of class \(C^{\nu+1}\), which implies \((\nu + 1)\) times continuous differentiability of the functions \(\hat{N}_1, \hat{N}_2\) and \(\hat{N}_3\). The relations (3.54) are true for the functions \(N_j, j = 1, 2, 3\), and they preserve after changes of variables.

Now the application of Mielke’s theorem gives the existence of the reduction vector function

\[ (h, g) : W \times \Lambda \to W_2 \times W_1, \]
which reduces the problem (3.24), (3.25), (3.48), (3.49) to the fine dimensional system (3.33). Now returning to system (3.4), (3.5) with boundary conditions (3.2) with the help of transformations (3.36), (3.37), (3.29), (3.23), (3.14) and (3.7) we complete the proof of our theorem.

3.8 Hamiltonian structure of the reduced system

Let \( W^{2,2}_2(0, d) \) be the space introduced at the end of Sect. 3.1. For each pair \((\Phi, \Psi) \in W^{2,2}_2(0, d) \times W^{1,2}_1(0, d)\) we define the symplectic form

\[
\varpi(\Phi, \Psi)(\Phi_1, \Psi_1; \Phi_2, \Psi_2) = \int_0^d (\Psi_2 \Phi_1 - \Phi_2 \Psi_1) \, dz
\]

\[+ \frac{d^2}{3n_0}(r - \eta_0) \left\{ [\Psi \Phi_2 + \Phi \Phi_2] z = d \int_0^d z(\Phi_1 \Psi + \Phi_2 \Psi_1) \, dz \right.\]

\[ - [\Psi \Phi_1 + \Phi \Phi_1] z = d \int_0^d z(\Phi_2 \Psi + \Phi_2 \Psi_2) \, dz,\]

where \((\Phi_k, \Psi_k) \in W^{2,2}_2(0, d) \times W^{1,2}_1(0, d)\). We introduce also the flow force invariant as

\[
S(\Phi, \Psi) = \left[ \frac{3}{2} r + \Omega(1) \right] \eta - \frac{\eta^2}{2} - \int_0^d \left[ \frac{d}{2\eta}(\Psi^2 - \Phi^2) + \frac{\eta}{d} \Omega(\Psi) \right] \, dz.
\]

It was proved in [10] that

\[
\varpi(\Phi, \Psi)(\delta \Phi, \delta \Psi; \Phi_x, \Psi_x) = dS(\delta \Phi, \delta \Psi)
\]

(3.60)

is a Hamiltonian form of the system (3.4), (3.5) with boundary conditions (3.2), where the function \(\eta\) is found from (3.6) and \(\eta_x = -\Psi(x, d)/\Phi_z(x, d)\).

Now we use the change of variables \((3.7), (3.14), (3.34), (3.23), (3.29)\) and \((3.36)\), and reduce our analysis to the center manifold from Theorem 3.1 i.e.

\[
\Phi = u(z) + \sum_{j=1}^N \alpha_j \varphi_j - \frac{zu}{kd} \left( \sum_{j=1}^N \alpha_j \varphi_j + h(\alpha, \beta) \right) + h(\alpha, \beta)
\]

(3.61)

and

\[
\Psi = \sum_{j=1}^N \beta_j \varphi_j + g(\alpha, \beta),
\]

(3.62)

where \(h\) and \(g\) are functions of order \(O(|\alpha|^2 + |\beta|^2)\). Our aim is to reduce the Hamiltonian form (3.60) of system (3.4), (3.5) to a Hamiltonian form of the system (3.33) on the center manifold given by (3.61), (3.62). The tangent vectors we take in the form

\[
\Phi_1 = \sum \tilde{\alpha}_j \varphi_j(z) - \frac{zu}{kd} \sum \tilde{\alpha}_j \varphi_j(d) + \delta_1
\]
\[ \Psi_1 = \sum \dot{\alpha}_j \varphi_j (z) + \delta_2 \]

where the remainders \( \delta_j \) consist of the lower order terms with respect to \(|\alpha|^2 + |\beta|^2\) and \( \hat{\alpha} \) and \( \hat{\beta} \) are vectors in \( \mathbb{R}^N \). The corresponding symplectic form is

\[
\vartheta(\hat{\alpha}^1, \hat{\beta}^1; \hat{\alpha}^2, \hat{\beta}^2) = \int_0^d \left( \sum \hat{\alpha}^2_j \varphi_j \left( \frac{zu_z}{kd} \sum \hat{\alpha}^1_j \varphi_j (d) \right) \right) dz
- \int_0^d \left( \sum \hat{\alpha}^1_j \varphi_j \left( \frac{zu_z}{kd} \sum \hat{\alpha}^2_j \varphi_j (d) \right) \right) dz
- \frac{k}{3(r-d)} \left\{ \left( \sum \hat{\alpha}^1_j \varphi_j' (d) - \frac{du_z (d) + uz_z (d)}{kd} \sum \hat{\alpha}^1_j \varphi_j (d) \right) \int_0^d zu_z \sum \hat{\beta}^2_j \varphi_j dz \right. \\
\left. - \left( \sum \hat{\alpha}^2_j \varphi_j' (d) - \frac{du_z (d) + uz_z (d)}{kd} \sum \hat{\alpha}^2_j \varphi_j (d) \right) \int_0^d zu_z \sum \hat{\beta}^1_j \varphi_j dz \right\} + \vartheta_\delta(\hat{\alpha}^1, \hat{\beta}^1; \hat{\alpha}^2, \hat{\beta}^2).
\]

Here we used that \( \Phi_z = u_z + \ldots, \eta_0 = d + \ldots \) and \( u_z (d) = k \). Using additionally that \( k^2 = 3r - 2d, u_{zz} (d) = o(1) \) and \( k\kappa = k^{-1} - o(1) \), we derive from (3.63) that

\[
\vartheta(\hat{\alpha}^1, \hat{\beta}^1; \hat{\alpha}^2, \hat{\beta}^2) = \hat{\alpha}^1 \cdot \hat{\beta}^2 - \hat{\alpha}^2 \cdot \hat{\beta}^1 + \vartheta_\delta(\hat{\alpha}^1, \hat{\beta}^1; \hat{\alpha}^2, \hat{\beta}^2).
\]

So we obtain a small perturbation of the standard symplectic form. If we introduce the reduced force flow invariant by

\[
s(\alpha, \beta) = S(\Phi, \Psi),
\]

where \( \Phi \) and \( \Psi \) are given by (3.61) and (3.62) respectively then

\[
\vartheta(\hat{\alpha}, \hat{\beta}; \alpha_x, \beta_x) = ds(\hat{\alpha}, \hat{\beta}),
\]

which coincides with the our reduced system. Using (3.64), we can write the reduced system (3.33) in the form

\[
\frac{d}{dx} (\alpha, \beta)^T = (J + \Upsilon(\alpha, \beta)) [\nabla s]^T,
\]

where the \( 2N \times 2N \) matrix \( J + \Upsilon(\alpha, \beta) \) is symplectic, \( J \) is the standard symplectic matrix, i.e. \( J(\alpha, \beta)^T = (-\beta, \alpha)^T \) and \( \Upsilon(\alpha, \beta) = O(|\alpha| + |\beta|) \). Here \( T \) stands for the transposition of a vector.

Let us now calculate the leading terms for the Hamiltonian \( s(\alpha, \beta) \). Let the functions \( \Phi, \Psi \) and \( \zeta \) be defined by (3.61), (3.62) and (3.15) respectively. We
start with
\[ S(\Psi, \Phi) = \left[ \frac{3}{2} r + \Omega(1) \right] (d + \zeta) - \frac{d^2 + 2d\zeta + \zeta^2}{2} + I, \]

where
\[ I = \text{q.p.} \left[ - \int_0^d \left( \frac{d}{2\eta}(\Psi^2 - \Phi^2_z) + \frac{\eta}{d} \Omega(\Phi) \right) dz \right]. \]

Here "q.p.\[\cdot\]" stands for the quadratic part of the expression inside brackets.

Thus, we have
\[ I = \text{q.p.} \left[ - \int_0^d \frac{d}{2\eta}(\Psi^2 - \Phi^2_z) + \frac{\eta}{d} \Omega(\Phi) \right] dz = \text{q.p.}[I_1 + I_2 + I_3]. \]

Let us evaluate \( I_2 \):
\[ q.p.[I_2] = \left[ \int_0^d \frac{1}{2} \left( 1 - \frac{\zeta d}{d^2} \right) \left( \Phi_z + u_z + \frac{u_z}{d} \zeta + \frac{z u_{zz} z \zeta}{d} \right)^2 \right] = \]
\[ = \int_0^d \left[ \Phi_z^2 + u_z^2 + \frac{z^2 u_{zz} z^2}{d^2} + 2\Phi_z u_z + \frac{2\Phi_z z u_{zz} z \zeta}{d} + \frac{u_z^2 \zeta d}{d} + \frac{2u_z u_{zz} z \zeta}{d} \right] dz. \]

Integrating by parts in the last integral, we find
\[ q.p.[I_2] = \frac{u_z^2(d)\zeta}{2} + \frac{1}{2} \int_0^d \left[ \Phi_z^2 + u_z^2 + \frac{z^2 u_{zz} z^2}{d^2} + 2\Phi_z u_z + \frac{2\Phi_z z u_{zz} z \zeta}{d} \right] dz. \]

Now we calculate \( I_3 \):
\[ I_3 = - \int_0^d \left( 1 + \frac{\zeta d}{d^2} \right) \left( \Omega(u) + \omega(u) \left[ \Phi + \frac{z u_z}{d} \zeta \right] + \frac{\omega'(u)}{2} \left[ \Phi + \frac{z u_z \zeta}{d} \right]^2 \right). \]

Thus,
\[ q.p.[I_3] = - \int_0^d \left[ \Omega(u) + \omega(u) \Phi + \frac{\omega(u) z u_z}{d} \zeta + \frac{\omega'(u) \Phi^2}{2} + \frac{\omega'(u) z^2 u_{zz} \zeta^2}{2d^2} \right] \]
\[ - \int_0^d \left[ \frac{\omega'(u) \Phi z u_z}{d} + \frac{\zeta \Omega(u)}{d} + \frac{\zeta \omega(u) \Phi}{d} + u_z \zeta \omega(u) \right] dz. \]

After integration by parts, we get
\[ q.p.[I_3] = -\Omega(1)\zeta - \frac{\omega(1) k \zeta^2}{2} - \frac{1}{2} \int_0^d \frac{z^2 u_{zz} \zeta^2}{d^2} - \]

\[ - \int_0^d \left[ \Phi_z^2 + u_z^2 + \frac{z^2 u_{zz} z^2}{d^2} + 2\Phi_z u_z + \frac{2\Phi_z z u_{zz} z \zeta}{d} + \frac{u_z^2 \zeta d}{d} + \frac{2u_z u_{zz} z \zeta}{d} \right] dz. \]
\[
\int_0^d \left[ \Omega(u) + \omega(u)\Phi + \frac{\omega'(u)\Phi^2}{2} + \frac{\omega'(u)\Phi z u_z \zeta}{d} + \frac{\zeta \omega(u)\Phi}{d} \right].
\]

Now combining the calculations, we obtain
\[
S - S_0 = 3r\zeta - d\zeta - \frac{\zeta^2}{2} - \frac{\omega(1)k\zeta^2}{2} + \frac{k^2\zeta}{2} - \frac{1}{2} \int_0^d \Phi^2 +
\]
\[
\frac{1}{2} \int_0^d \left[ \Phi_z^2 + 2\Phi_z u_z \Phi - \frac{2\omega(u)\Phi - \omega'(u)\Phi^2}{d} - \frac{2\omega'(u)\Phi z u_z \zeta}{d} - \frac{2\zeta \omega(u)\Phi}{d} \right].
\]

Here \(S_0\) stands for the flow force calculated for the stream solution \(u\). After integration by parts, we find
\[
S - S_0 = 3r\zeta - d\zeta - \frac{\zeta^2}{2} - \frac{\omega(1)k\zeta^2}{2} + \frac{k^2\zeta}{2} + k\Phi(d) - \omega(1)\Phi(d)\zeta
\]
\[
- \frac{1}{2} \int_0^d \Phi^2 + \frac{1}{2} \int_0^d \left[ \Phi_z^2 - \omega'(u)\Phi \right].
\]

Taking into account the relation \(\zeta = -\Phi(d)/k\), we arrive at
\[
2(S - S_0) = \kappa[\Phi(d)]^2 - \frac{1}{2} \int_0^d \Phi^2 + \frac{1}{2} \int_0^d \left[ \Phi_z^2 - \omega'(u)\Phi \right] + o(\Phi^2 + \zeta^2 + \Phi_z^2 + \Phi^2).
\]

Taking into account the definition of \(\tilde{\Psi}, \tilde{\Phi}\) and using the following identity
\[
\kappa \alpha_j \phi_j(d) \alpha_i \phi_i(d) - \int_0^d [\alpha_j \phi'_j \alpha_i \phi'_i - \omega'(u)\alpha_j \phi_j \alpha_i \phi_i] = \begin{cases} 0 & \text{if } i \neq j \\ (\mu_j) \alpha_i \alpha_j & \text{if } i = j \end{cases},
\]

which follows from (2.7), we conclude
\[
2s(\alpha, \beta) = 2S_0 - \sum_j \beta_j^2 - \sum_j (\mu_j) \alpha_j^2 + o(|\alpha|^2 + |\beta|^2).
\]

### 4 Non-symmetric steady waves

Our goal is a proof of the existence of non-symmetric steady waves which is given by

**Theorem 4.1.** Assume that all the eigenvalues \(\mu_j^*\), \(j = 1, ..., N\) are strictly negative. Then the following statements are true:

(i) there exists an open neighbourhood \(W_\ast\) of the origin in \(\mathbb{R}^{2N}\) such that the reduced system ([3.33]) for \(\lambda = \lambda_\ast\) possesses a unique solution \((\alpha, \beta)\) for any initial data \((\alpha(0), \beta(0)) \in W_\ast\);
(ii) let $W_*^{sym}$ be the set of all $(p,q) \in W_*$ such that there exists a solution $(\alpha,\beta)$ which is symmetric around some vertical line $x = x_0$ and such that $(p,q) = (\alpha(t), \beta(t))$ for some $t \in \mathbb{R}$. Then the Hausdorff dimension of $W_*^{sym}$ is less or equal than $N + 1$.

Thus, if $2N > N + 1$, then the most of solutions in $W_*$ are not symmetric.

Proof. As the first step, we prove that the system (3.33) possesses a unique solution for any sufficiently small initial data. This follows from the fact that $s(\alpha, \beta)$ is a definite first integral of system (3.65) (which coincides with (3.33)) together with the application of Theorem 3.1.

To prove the second part of the statement, we consider the mapping

$$
\mathcal{G} : W_* \times \mathbb{R} \to W_*
$$

defined by

$$
\mathcal{G}(p,q;x) = (\alpha(x), \beta(x)),
$$

where $(\alpha, \beta)$ stands for solution with the initial data $(p,q) \in W_*$. It is easy to show that $\mathcal{G} \in C^2(W_* \times [-T,T])$ for any $T \in \mathbb{R}$, which implies that $\mathcal{G}$ is lipschitz on every compact subset of $W_* \times \mathbb{R}$. Let $W_0 = \{(p,q) \in W_* : q = 0\}$. Then the Hausdorff dimension of the image $\mathcal{G}(W_0 \times \mathbb{R})$ is less or equal than the dimension of $W_0 \times \mathbb{R}$ which is $N + 1$. To finish the proof it is left to note that $W_*^{sym} = \mathcal{G}(W_0 \times \mathbb{R})$.}

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