THE VANISHING VISCOSITY LIMIT IN THE PRESENCE
OF A POROUS MEDIUM

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Abstract. We consider the flow of a viscous, incompressible, Newtonian fluid in a perforated domain in the plane. The domain is the exterior of a regular lattice of rigid particles. We study the simultaneous limit of vanishing particle size and distance, and of vanishing viscosity. Under suitable conditions on the particle size, particle distance, and viscosity, we prove that solutions of the Navier-Stokes system in the perforated domain converges to solutions of the Euler system, modeling inviscid, incompressible flow, in the full plane. That is, the flow is not disturbed by the porous medium and becomes inviscid in the limit. Convergence is obtained in the energy norm with explicit rates of convergence.

1. Introduction

This article concerns the vanishing viscosity limit for incompressible, Newtonian fluids in a perforated planar domain when the size and distance between obstacles vanish. By a perforated domain, we mean the exterior of a (finite) regular lattice of rigid particles. This problem can be viewed as simultaneously taking the limit of vanishing viscosity for flows in exterior domains and study the permeability of a porous medium under this flow.

From a physical point of view, it is important to study fluid flow through a porous medium in the regime of small viscosity, which model for example flow of underground water. Away from boundaries, the effect of viscosity is negligible generically for such flows, but near walls, viscous boundary layers appear where large stresses and vorticity can develop. The layer can in turn destabilize, which lead to boundary layer separation and the formation of a turbulent wake. (We refer to [31] for an introduction to the theory of boundary layers.)

Date: March 24, 2015.

2010 Mathematics Subject Classification. 35Q30,35Q31,76S05.

Key words and phrases. Navier-Stokes, Euler, porous medium, vanishing viscosity limit.

The first author is partially supported by the Agence Nationale de la Recherche, Project DYFICOLTI grant ANR-13-BS01-0003-01, and by the project Instabilities in Hydrodynamics funded by the Paris city hall (program Emergences) and the Fondation Sciences Mathématiques de Paris.

The second author was partially supported by the U.S. National Science Foundation grants DMS-1009713, DMS-1009714, and DMS-1312727.
We say that the vanishing viscosity limit holds on the time interval $[0, T]$ if the solutions of the Navier-Stokes equations converge to the solution of the Euler equations strongly in the energy norm, i.e., strongly in $L^\infty([0, T]; L^2)$. It is not known whether the vanishing viscosity limit holds in domains with boundaries, not even in two space dimensions, if no-slip boundary conditions are imposed on the velocity. The Reynolds number is a dimensionless quantity, which is inversely proportional to the viscosity. When the characteristic local length scale near the boundary becomes small at a faster rate than the viscosity, one can expect that the local Reynolds number stays of order one, preventing the formation of a strong boundary layer, as was observed in [16], where the vanishing viscosity limit was established in the case of one shrinking obstacle. Homogenizing the Navier-Stokes equations in the perforated (periodic) domain with Reynolds number below a critical value yields Darcy’s law as in the case of the Stokes system, while homogenizing the Euler equations gives non-linear filtration laws that depend on the relation between the particle size and the characteristic velocity of the problem. In the intermediate regime of high Reynolds numbers, asymptotic analysis leads to consider the Prandtl boundary layer equations, which have recently been shown to be ill-posed [10, 11], for the cell problem. (We refer to [30] and references therein, in particular [29], for a more detailed discussion. See also [3] for a recent result on stochastic homogenization.) In the case of the Darcy-Brinkman system, the equations for the boundary corrector are linear and passage to the zero-viscosity limit is possible [18, 12].

It is therefore important to identify regimes where it is possible to neglect the effects of both the viscous boundary layers as well as the porosity of the medium and use the Euler equations in the full plane. In this article, we establish the convergence of solutions to the two-dimensional Navier-Stokes equations in the perforated domains to solutions of the Euler equations in the full plane in the energy norm, when viscosity, size of the particle, and particle distance vanish in the appropriate regime: namely, particle size must be smaller than particle distance, and a certain ratio of particle size to particle distance must be bounded above by viscosity. We consider the most difficult and interesting case of no-slip boundary conditions for the Navier-Stokes problem. We confine ourselves to flows in the plane for technical reasons. However, our methods could be adapted to the more challenging study of 3D flows, at least for the case of a fixed number of obstacles (see the related discussion in [16]).

The study of 2D incompressible flows in the exterior of a single shrinking obstacle, both for viscous and inviscid fluids, was carried out in the series of papers [13, 14, 15, 16] (see also [7] for results in the periodic setting), where the case of obstacles diametrically shrinking to a point was considered. This is the geometric set up in our paper as well.

Even in the case of flows in the exterior of a finite number of shrinking obstacles at fixed positions, extending the results in [16] is not straightforward, as one needs to choose an approximation to the Euler solution which
is divergence free and satisfy the no-slip condition at the boundary. For one obstacle, it is sufficient to truncate the stream function associated to the Euler velocity. For more than one obstacle, we gain a good control on the norm of the approximation by truncating the velocity and by constructing instead an appropriate corrector to restore the divergence-free condition. When the obstacles are kept at a fixed distance from each other, our result applies to arbitrary configurations of obstacles.

When the problem is studied in a porous medium, additional difficulties arise. In fact, the permeability of the medium in the limit depends on the relation between particle distance and particle size. This relation changes drastically from the viscous to the inviscid case (cf. [1, 2] for the viscous case, and [6, 24, 25, 30] for the inviscid case). Therefore, a careful analysis of the flow in the perforated domain is needed to pass to the limit, in particular with respect to the choice of initial data for the Navier-Stokes system. In the regime considered in this paper, we take the initial data as well as the Euler correctors to be independent of viscosity. We will expand on this point further in Subsection 1.1.

We close with an outline of the paper. In the rest of the introduction we describe in detail the perforated domain and recall the fluid equations: the Navier-Stokes equations modeling viscous flows, and the Euler equations modeling inviscid flows.

We will compare the Navier-Stokes equations in the exterior of the obstacles, extended by zero, to the Euler solution in the whole plane. To this end, we will construct suitable approximations of the Euler solution in the exterior domain that satisfy the no-slip condition at the boundary and are still divergence free. In the case of one obstacle, the authors in [16] truncate the stream function, as both the stream function and the pressure are defined up to an arbitrary constant, and can then be chosen to be small on the obstacle. Smallness of the stream function and the pressure avoid a sub-optimal estimate on the gradient of the truncation. This construction does not extend to more than one obstacle. In Section 2, we present a different construction, which utilizes a direct cut-off function on the velocity and a correction to restore the divergence-free condition. This construction was introduced in [22] and is based on Bogovskii operator. Section 2 also contains the elliptic estimates that are needed to establish our main convergence result. We also recall some standard estimates for the Euler solutions.

In Section 3, we discuss the convergence of the Navier-Stokes initial data to the Euler initial data under the conditions on $\varepsilon$ (the particle size) and $d_\varepsilon$ (the distance between particles) given in Theorem 1.1, using the analysis in [6].

Finally, in Section 4, we prove convergence of the Navier-Stokes solutions $u^{\nu,\varepsilon}$ to the Euler solution $u^E$ by correcting the Euler solution thanks to the results in Section 2. In fact, we prove a more general stability result— see the statement of Theorem 4.1— analogous to the main result in [16]. The
convergence follows from this stability result and the convergence of the Navier-Stokes initial data to the Euler data, discussed in Section 3.

Throughout the paper $\nabla$ will denote the gradient of a function with respect to its argument, while $C$ will denote a generic constant that may change from line to line. We will represent a point in $\mathbb{R}^2$ by $x = (x_1, x_2)$ and denote the ball in $\mathbb{R}^2$ (disk) with center $x$ and radius $R$ by $B(x, R)$.

Acknowledgments. A. M. would like to acknowledge the hospitality and financial support of Université Paris-Diderot (Paris 7), Université Pierre et Marie Curie (Paris 6), and the financial support of Paris City Hall and the Fondation Sciences Mathématiques de Paris, to conduct part of this work.

1.1. The perforated domain. We begin by describing in detail the geometric setup. By a perforated domain we mean the complement of finitely-many inclusions arranged in a lattice. In this article, we consider the case where all the inclusions have the same shape and the lattice is regular:

$$K_{i,j}^\varepsilon := z_{i,j}^\varepsilon + \varepsilon K$$

where $K$ is a connected, simply connected compact set of $\mathbb{R}^2$,

$$K \subset [-1, 1]^2, \quad \partial K \in C^{1,1} \quad \text{and} \quad 0 \in \text{int} K.$$

The regularity of the lattice is needed only when the distance between the inclusions vanishes. If a fixed number of shrinking obstacles is considered instead, these can be arranged in an arbitrary fashion.

In (1.1), $z_{i,j}^\varepsilon$ should be chosen such that the inclusions are disjoints. We will assume that the inclusions are at least separated by a distance $2d_\varepsilon$:

$$z_{i,j}^\varepsilon := (\varepsilon + 2(i - 1)(\varepsilon + d_\varepsilon), \varepsilon + 2(j - 1)(\varepsilon + d_\varepsilon))$$

$$= (\varepsilon, \varepsilon) + 2(\varepsilon + d_\varepsilon)(i - 1, j - 1),$$

see Figure 1. We are interested in the case for which $d_\varepsilon$ goes to zero as $\varepsilon \to 0$, hence the case for which the number of inclusions goes to infinity.

In the horizontal direction, we distribute the inclusions on the unit segment, i.e. we consider

$$i = 1, \ldots, n_1^\varepsilon \quad \text{with} \quad n_1^\varepsilon = N_1^\varepsilon := \left\lfloor \frac{1 + 2d_\varepsilon}{2(\varepsilon + d_\varepsilon)} \right\rfloor$$

where $[x]$ denotes the integer part of $x$. In the vertical direction, we characterize the geometry by introducing a parameter $0 \leq \mu \leq 1$. The case $\mu = 0$ corresponds to the case in which we only have obstacles arranged on a line, whereas the case $\mu = 1$ corresponds to the case in which we have obstacles arranged in a square lattice. This configuration is achieved by considering $[N_2^\mu]$ horizontal lines of obstacles, that is,

$$j = 1, \ldots, n_2^\varepsilon \quad \text{with} \quad n_2^\varepsilon := \left\lfloor (N_2^\mu) \right\rfloor.$$

The intermediate case $0 < \mu < 1$ corresponds to a rectangular configuration.
of inclusions, such that in the limit $\varepsilon \to 0$ the density of particles becomes infinite, but the height of the rectangle vanishes. For simplicity, we carry out the analysis only in the case that all the inclusions have the same shape $\mathcal{K}$, but similar results can be established when obstacles of different shapes shrink homotetically to zero. When the number of inclusions becomes infinite, we need to assume that the possible shapes are finitely many, in order to have uniform constants in elliptic estimates (see Subsection 2.1). The main reason for studying different arrangements of inclusions as $\varepsilon$ goes to zero is that the limiting equations may change depending on the geometric configuration (this is indeed the case for the viscous homogenization of the perforated domain [1, 2]).

Throughout the paper, the fluid domain will be the exterior of the obstacles:

$$\Omega^{\varepsilon, \mu} = \Omega^{\varepsilon} := \mathbb{R}^2 \setminus \left( \bigcup_{i=1}^{n_1^\varepsilon} \bigcup_{j=1}^{n_2^\varepsilon} \mathcal{K}^{\varepsilon}_{i,j} \right).$$

For notational convenience, we suppress the dependence of the fluid domain on $\mu$.

The main purpose of this paper is to study the simultaneous limit of vanishing viscosity $\nu$ and the limit of vanishing $\varepsilon$ for the solutions of the Navier-Stokes equations in the exterior domain $\Omega^{\varepsilon}$. For different values of $\mu$, we determine a relation between $\nu$, $\varepsilon$, and $d_\varepsilon$ such that the Navier-Stokes solution in the perforated domain converges to the solution of the Euler equations in the full plane.

The asymptotic of fluids outside obstacles that shrink to points is a very active area of research. For inviscid flows, Iftimie, Lopes Filho, and Nussenzveig Lopes have treated in [14] the case of one shrinking obstacle in the plane (extended to the case of one shrinking obstacle in a non simply-connected domain).
bounded domain by Lopes Filho in [26], whereas Lacave, Lopes-Filho, and Nussenzveig-Lopes have studied in [23] the case of an infinite number of shrinking obstacles. These works consider also the case where the circulation of the initial velocity around $K_{i,j}$ is non zero, but do not provide a control on the distance between the holes. In the case of zero circulation (see (1.9)), Bonnaillie-Noël, Lacave, and Masmoudi prove in [6] that, when $d_\varepsilon = \varepsilon^\alpha$, solutions of the Euler equations in the exterior domain $\Omega_\varepsilon$ defined in (1.1) converge to the Euler solution in the full plane if $\alpha < 1$ (more precisely, if $\alpha < 2 - \mu$). Hence, in the limit the ideal fluid is not perturbed by the perforated domain if $d_\varepsilon = \varepsilon^\alpha$ with $\alpha < 1$. For larger $\alpha$, Lacave and Masmoudi establish in [24] than the perforated domain becomes impermeable (e.g. when $\mu = 1$, an impermeable square appears if $\alpha > 1$). In the periodic setting (related to the case $\mu = \alpha = 1$ in the context of this paper), an homogenized limit for a modified Euler system was obtained in [25, 30].

Concerning viscous fluid with fixed viscosity $\nu$, Iftimie, Lopes-Filho, and Nussenzveig-Lopes considered again the case of one shrinking inclusion in [15] (see also [13] in dimension three). There is also an extensive literature in the homogenization framework. Following the pioneering work of Cioranescu and Murat [8] for Laplace’s equation, Allaire studied the homogenization of the Navier-Stokes equations in a perforated domain under different regimes, given in terms of relative ratios of $d_\varepsilon$ to $\varepsilon$, in [1, 2]. When $\mu = 1$ and $d_\varepsilon = 1/\sqrt{\ln \varepsilon}$, or when $\mu = 0$ and $d_\varepsilon = 1/|\ln \varepsilon|$ for the critical distance between inclusions, the limit problem is described by a filtration law of Brinkman type. If the distance is larger than this critical value, one recovers the solution of the Navier-Stokes in the full plane without any influence of the porous medium, and if the distance is smaller, one obtains Darcy’s law in the limit. We refer to [2, 6] and references therein for a more detailed discussion of this point, in particular for results on viscous flow through a sieve. We remark that the dependence of the critical distance on $\varepsilon$ needed to feel the presence of the porous medium in the limit is markedly different in the viscous and inviscid cases. For example, for distances $d_\varepsilon$ satisfying $\varepsilon^\alpha \ll d_\varepsilon \ll 1/\sqrt{\ln \varepsilon}$, the limiting system describing the homogenized viscous flow is different than the Navier-Stokes system, whereas there is no memory of the obstacles in the limit for ideal flows. As a matter of fact, we are able to treat the situation in which the distance is smaller than the critical distance (namely, $d_\varepsilon^{1/2} \sqrt{\ln \varepsilon} \to 0$) and the inclusions are distributed along a line, i.e., $\mu = 0$ in our setting, albeit with a less than optimal rate of convergence. For Navier-Stokes, this case was not addressed by Allaire in [2] (see Remark 4.2).

It is therefore natural to investigate the concurrent limit $\varepsilon, \nu \to 0$. There are also important physical motivations for studying this problem, as discussed in the introduction. (We again refer to [30] and [16] for a more in-depth discussion of the physical implications of these and our results.)
1.2. Fluids equations and initial data. We consider the flow of a viscous, incompressible, Newtonian fluid in \( \Omega^\varepsilon \) with classical no-slip boundary conditions on \( \partial \Omega^\varepsilon := \bigcup_{i,j} \partial K^\varepsilon_{ij} \). We hence let \((u^\nu\varepsilon, p^\nu\varepsilon)\) denote the solution of the Navier-Stokes equations with (kinematic) viscosity coefficient \( \nu \) in \( \Omega^\varepsilon \) and initial velocity \( u^\varepsilon_0 \):

\[
\begin{cases}
  u^\nu\varepsilon_t - \nu \Delta u^\nu\varepsilon + (u^\nu\varepsilon \cdot \nabla) u^\nu\varepsilon + \nabla p^\nu\varepsilon = 0, & \text{on } (0, +\infty) \times \Omega^\varepsilon, \\
  \text{div } u^\nu\varepsilon = 0, & \text{on } [0, +\infty) \times \Omega^\varepsilon, \\
  u^\nu\varepsilon = 0, & \text{on } (0, +\infty) \times \partial \Omega^\varepsilon, \\
  u^\nu\varepsilon(0, \cdot) = u^\varepsilon_0, & \text{on } \Omega^\varepsilon.
\end{cases}
\]

We take initial data depending only on \( \varepsilon \) and not directly on viscosity \( \nu \) for reasons that will be clear below. However, this setup can be easily relaxed. The initial data is assumed only to satisfy the no-penetration conditions forced by the impermeable boundaries. By abuse of notation, we will often refer to the velocity \( u^\nu\varepsilon \) as the Navier-Stokes solution, without mentioning the pressure \( p^\nu\varepsilon \).

We will also consider inviscid flow in the whole plane \( \mathbb{R}^2 \), and denote by \( u^E \) the solutions of the Euler equations with initial data \( u^0 \):

\[
\begin{cases}
  u^E_t + (u^E \cdot \nabla) u^E + \nabla p^E = 0, & \text{on } (0, +\infty) \times \mathbb{R}^2, \\
  \text{div } u^E = 0, & \text{on } [0, +\infty) \times \mathbb{R}^2, \\
  u^E(0, \cdot) = u^0, & \text{on } \mathbb{R}^2.
\end{cases}
\]

Our main result is a convergence result of \( u^\nu\varepsilon \) to \( u^E \) under suitable conditions on the relative strengths of \( \nu, \varepsilon, \) and \( d_\varepsilon \), assuming convergence of the respective initial data. Such a convergence can be achieved if \( u^0_0 \) is chosen to be an appropriate truncation of the Euler initial velocity \( u^0 \) that is tangent to the boundary of \( \Omega^\varepsilon \) and divergence free. \( u^0_0 \) will be close to \( u^0 \) owing to the fact that the size of the obstacles is small. A standard approach to constructing the truncated velocity is to give the initial Euler velocity \( u^0 \) in term of a fixed initial vorticity independent of the domain (as e.g. in [6, 13, 14, 15, 16, 22, 23, 24]). More precisely, let \( \omega_0 \) be a smooth function compactly supported in \( \mathbb{R}^2 \), and define \( u^0 \) by

\[
u_0(x) = K_{\mathbb{R}^2}[\omega_0](x) := \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x - y)_{\perp}}{|x - y|^2} \omega_0(y) \, dy,
\]

where \( K_{\mathbb{R}^2} \) denotes the Biot-Savart operator in the full plane. That is, \( u^0 \) is the unique solution of the problem

\[
\begin{align}
  \text{div } u^0 &= 0 \text{ in } \mathbb{R}^2, \\
  \text{curl } u^0 &= \omega_0 \text{ in } \mathbb{R}^2, \\
  \lim_{x \to \infty} |u^0(x)| &= 0.
\end{align}
\]

It is well known (see e.g. [27]) that, with this initial data, the Euler system (1.6) has a unique solution \( u^E \). The properties of such a solution will be recalled in Subsection 2.2.
For any $\varepsilon > 0$, there exists a unique vector field $u_0^\varepsilon$ defined on $\Omega^\varepsilon$ verifying:

$$\text{div} \, u_0^\varepsilon = 0 \text{ in } \Omega^\varepsilon, \quad \text{curl} \, u_0^\varepsilon = \omega_0 \text{ in } \Omega^\varepsilon, \quad u_0^\varepsilon \cdot n = 0 \text{ on } \partial \Omega^\varepsilon,$$

$$\int_{\partial K^\varepsilon_{i,j}} u_0^\varepsilon \cdot \tau \, ds = 0 \text{ for all } i, j,$$

$$\lim_{x \to \infty} |u_0^\varepsilon(x)| = 0,$$

see [19]. The convergence of $u_0^\varepsilon$ to $u_0$ will be established in Section 3 using the methods of [6]. As we will show in (3.1), the vector field $u_0^\varepsilon$ constructed in (1.9) belongs to $L^\infty \cap L^{2,\infty}(\Omega^\varepsilon)$. Hence, we can apply the result of Kozono and Yamazaki [21, Theo. 4] to conclude that the problem (1.5) has a unique, global-in-time, strong solution.

We close by observing that neither the Euler nor the Navier-Stokes solution have finite energy, nevertheless we will be able to show convergence, as $\nu, \varepsilon$ vanish, of $u_0^{\nu,\varepsilon} - u^E$ to zero in the energy norm.

### 1.3. Main result

In this work, we are interested in studying the limit of vanishing viscosity $\nu \to 0$ at the same time as the limit $\varepsilon \to 0$, in which the obstacles shrink to points. In the case of a single obstacle, it was proved in [16] that, for any given $0 < T < \infty$, there exists a constant $C_1 = C_1(K, u_0, T)$ such that, under the conditions $\varepsilon \leq C_1 \nu$, $u_0^{\nu,\varepsilon}$ converges to $u^E$ in the energy space $L^\infty((0, T), L^2(\mathbb{R}^2))$. In this result and in our main theorem, $u_0^{\nu,\varepsilon}$ and $u_0^\varepsilon$ are extended by zero to the whole plane. Unfortunately, the arguments in [16] use in a crucial way the fact that we have only one obstacle: namely, the authors utilize the stream function $\psi^E$ and choose a pressure $p^E$ for the Euler system that is zero at the center of $K$, in order to construct a corrector for the Navier-Stokes solution in the fluid domain via truncation. Taking $\psi^E$ and $p^E$ small as the obstacle shrinks is needed to balance the growth of gradients of the cut-off function. Such a choice is not possible in the case of more than one obstacle, as there is only one degree of freedom for the Euler pressure and stream function in the whole plane. Inspired from a method developed in [22], we will instead directly truncate the velocity field with good estimates on the truncation, which will allow to adapt the energy argument of [16] to establish the zero-viscosity limit.

Due to the inherent difficulty with the vanishing viscosity in bounded domains, we are only treating the case where $d_\varepsilon \geq \varepsilon$ (without loss of generality we can assume that $d_\varepsilon \leq 1$), hence the total number of inclusions satisfies

$$n_1^\varepsilon n_2^\varepsilon \leq \frac{1}{d_\varepsilon^{1+\mu}}.$$

Our main result is the following theorem.

**Theorem 1.1.** Given $\omega_0 \in C_\infty^\infty(\mathbb{R}^2)$, let $u^E$ the solution of the Euler equations (1.6) in the whole plane with initial condition $u_0$ (given in terms of $\omega_0$ in (1.7)). For any $\varepsilon, \nu > 0$, let $d^\varepsilon \geq \varepsilon$ and let $\Omega^E$ be defined in (1.1)–(1.4). Let $u_0^{\nu,\varepsilon}$ be the solution of the Navier-Stokes equations (1.5) in $\Omega^E$ with initial velocity $u_0^\varepsilon$, given by the unique solution of (1.9). Then, there exists a
constant $A$ depending only on $K$ such that if
\[ \frac{\epsilon}{d_{\epsilon}^{1(1+\mu)/2}} \leq A\nu \left\| \omega_{0} \right\|_{L^{1}\cap L^{\infty}(\mathbb{R}^{2})}, \]
then for any $T > 0$ we have
\[ \sup_{0 \leq t \leq T} \left\| u^{\nu,\epsilon} - u^{E} \right\|_{L^{2}(\Omega)} \leq B_{T}\frac{\sqrt{\nu}}{d_{\epsilon}^{(1+\mu)/2}}, \]
where $B_{T}$ is a constant depending only on $T$, $\left\| \omega_{0} \right\|_{L^{1}\cap W^{1,\infty}(\mathbb{R}^{2})}$, and $K$.

**Remark 1.1.**

i) In the case of a fixed number of obstacles shrinking homotetically to points (arbitrary located), the above theorem is a direct extension of the results in [16], namely, under the condition $\epsilon \leq \tilde{A}\nu$ (with $\tilde{A}$ depending on the distance and the number of obstacles and on $\left\| \omega_{0} \right\|_{L^{1}\cap L^{\infty}(\mathbb{R}^{2})}$) we obtain the convergence of the Navier-Stokes solution to the Euler solution with a rate of convergence of order $\sqrt{\nu}$.

ii) The fact that our Euler correctors (see Proposition 2.4) are $\epsilon$-dependent, but not $\nu$-dependent, can be seen as a consequence of the fact that the local Reynolds number, built in terms of the characteristic size of the obstacles, stays of order one as $\nu$ and $\epsilon$ go to zero in the regime of Theorem 1.1. In this situation, the boundary layers are negligible, as already observed in [16]. In fact, the proof of Theorem 1.1 is similar to the proof of Kato’s criterion, which implies the vanishing viscosity limit [17, 33, 35].

iii) The additional factor $\frac{1}{d_{\epsilon}^{1(1+\mu)/2}}$ in (1.11) can be viewed as an effect of the homogenization of the porous medium. Using standard correctors as for the Laplace problem (see [11, 25, 32]), one would obtain a smaller bound of the form $\frac{1}{d_{\epsilon}^{1(1+\mu)/2}} \sqrt{\left| \ln \epsilon \right|}$. The limit of $d_{\epsilon}^{(1+\mu)/2} \sqrt{\ln \epsilon}$ determines precisely when the so-called “strange term” in the homogenization appears. However, this improvement is less relevant from an application standpoint, as discussed next, and would require a significant amount of additional technical work (cf. the discussion in Remark 4.2).

iv) Refined consequences of our main result arise by considering specific regimes for the ratio of particle distance to particle size, as discussed at the end of Subsection 1.1. For instance, when $\mu = 1$ (inclusions are uniformly distributed in a square) and $d_{\epsilon} = 1/\sqrt{\ln \epsilon}$, the critical distance for the homogenization problem, if the viscosity $\nu$ satisfies $C\sqrt{\ln \epsilon} \leq \nu \ll 1/|\ln \epsilon|$ — for example, $\nu = \epsilon^{\alpha}$ for some $\alpha \in (0, 1)$ — our result gives convergence $u^{\nu,\epsilon} \to u^{E}$, whereas if viscosity is kept fixed as $\epsilon \to 0$ $u^{\nu,\epsilon}$ does not converge to the Navier-Stokes solutions in the full plane, as recalled already. Similar conclusions can be drawn for smaller inter-particle distances, namely the theorem implies the convergence to $u^{E}$ if $\epsilon \ll d_{\epsilon}^{3}$, as in the case...
\[ d_\varepsilon = \varepsilon^\alpha \text{ with } \alpha < 1/3. \] When \( \mu = 0 \), the critical distance between inclusions for the homogenization problem \[2\] is \( d_\varepsilon = 1/|\ln \varepsilon| \), whereas in our result the limit holds when \( \varepsilon \ll d_\varepsilon^{3/2} \), as for the case \( d_\varepsilon = \varepsilon^\alpha \) with \( \alpha < 2/3 \).

2. Preliminaries

We begin by presenting some basic elliptic estimates and estimates on the Euler solution that will be used throughout the article.

2.1. Basic elliptic estimates. For the energy estimate in Section 4, we will need to approximate a divergence-free vector field \( u \) defined in \( \mathbb{R}^2 \) by divergence vector fields \( u_\varepsilon \) verifying the Dirichlet boundary condition on \( \partial \Omega^\varepsilon \). To this end, we introduce an appropriate cut-off function \( \phi_\varepsilon \) as follows.

As we consider the case where \( d_\varepsilon \geq \varepsilon \), we deduce from (1.1)-(1.3) that

\[ K^\varepsilon_{i,j} \subset z^\varepsilon_{i,j} + \varepsilon [-1, 1]^2 \quad \forall (i,j) \]

and that

\[ (z^\varepsilon_{i,j} + \varepsilon (-2, 2)^2) \cap (z^\varepsilon_{p,q} + \varepsilon (-2, 2)^2) = \emptyset \quad \forall (i,j) \neq (p,q). \]

We let \( \phi \geq 0 \) be a smooth, cut-off function such that:

\[ \begin{cases} \phi(x) = 1, & |x|_\infty \leq 3/2, \\ \phi(x) = 0, & |x|_\infty \geq 2, \end{cases} \]

where \( |x|_\infty = \max_i |x_i| \), and set

\[ \phi_\varepsilon(x) = 1 - \sum_{i=1}^{n_1^\varepsilon} \sum_{j=1}^{n_2^\varepsilon} \phi \left( \frac{x - z^\varepsilon_{i,j}}{\varepsilon} \right), \quad x \in \mathbb{R}^2. \]

Then, \( 0 \leq \phi_\varepsilon \leq 1 \) in \( \mathbb{R}^2 \), \( \phi_\varepsilon \equiv 1 \) away from the obstacles and \( \phi_\varepsilon \equiv 0 \) on a small neighborhood of the obstacles. More precisely, denoting

\[ A_\varepsilon := \bigcup_{i,j} \left( z^\varepsilon_{i,j} + \varepsilon (-2, 2)^2 \setminus \varepsilon K^\varepsilon \right), \]

we observe that \( \text{Supp}(1 - \phi_\varepsilon) \subset A_\varepsilon \). As we have the bound (1.10) on the number of obstacles \( n_1^\varepsilon n_2^\varepsilon \), we can easily estimate the Lebesgue measure of \( A_\varepsilon \) and conclude that

\[ \|1 - \phi_\varepsilon\|_{L^p} + \varepsilon \|\nabla \phi_\varepsilon\|_{L^p} \leq C_p \frac{\varepsilon^{2/p}}{d_\varepsilon^{(1+p)/p}}, \quad 1 \leq p \leq \infty, \]

with \( C_p \) a constant independent of \( \varepsilon \).

As \( \phi_\varepsilon u \) is not divergence free, the stream function \( \psi \), satisfying \( u = \nabla^\perp \psi \), can be truncated instead as in [13, 14, 15, 16], that is, one sets \( u^\varepsilon := \nabla^\perp (\phi_\varepsilon \psi) \), which is divergence free and vanishes at the boundary. In this case, one needs to compensate the growth of the gradient of \( \phi_\varepsilon \) as the obstacles shrink, which is of order \( 1/\varepsilon \). For a single inclusion shrinking to a point, which we identify with the origin, one can choose \( \psi \) such that \( \psi(0) = 0 \) and
then $u^\varepsilon = \phi^\varepsilon u + \psi \nabla^\perp \phi^\varepsilon = \mathcal{O}(1)$. Of course, such a procedure cannot be applied if we consider more than one obstacle or if the obstacle is shrinking to a curve instead to a point. Here we present an alternate way to truncate divergence-free vector fields so that they have support in the complement of several disjoint obstacles. The following method was used in \cite{22} to treat the case of an obstacle shrinking to a curve and it is related to the Bogovskiǐ operator (see \cite{9}).

For $1 \leq p < \infty$ and for any $\varepsilon > 0$ fixed, we define an equivalent norm $\| \cdot \|_{W^{1,p}_\varepsilon}$ in the Sobolev space $W^{1,p}$ by

$$
(2.6) \quad \| f \|_{W_{\varepsilon}^{1,p}} := \left( \frac{1}{\varepsilon^p} \| f \|_{L^p(A_\varepsilon)}^p + \| \nabla f \|_{L^p(A_\varepsilon)}^p \right)^{1/p}.
$$

**Lemma 2.1.** Given $1 < p < \infty$, there exists a constant $\tilde{C}_p$ depending only on $p$, such that the following holds: given any divergence-free vector field $u(t, \cdot) \in L^\infty(\mathbb{R}^2)$ for all $t \in [0, \infty)$ and given any $\varepsilon > 0$, the problem

$$
\text{div } h^\varepsilon = \nabla \phi^\varepsilon \cdot u, \quad \text{on } [0, \infty) \times A_\varepsilon,
$$

has a solution $h^\varepsilon(t, \cdot) \in W^{1,p}_0(A_\varepsilon)$ satisfying

$$
\| h^\varepsilon(t, \cdot) \|_{W^{1,p}_\varepsilon} \leq \tilde{C}_p \| \nabla \phi^\varepsilon \cdot u(t, \cdot) \|_{L^p(A_\varepsilon)} \quad \forall t \in [0, \infty),
$$

$$
\| h^\varepsilon(t, \cdot) - h^\varepsilon(s, \cdot) \|_{W^{1,p}_\varepsilon} \leq \tilde{C}_p \| \nabla \phi^\varepsilon \cdot (u(t, \cdot) - u(s, \cdot)) \|_{L^p(A_\varepsilon)} \quad \forall t, s \in [0, \infty).
$$

Moreover, if $\frac{\partial u(t, \cdot)}{\partial t} \in L^p(A_\varepsilon)$ then

$$
\left\| \frac{\partial h^\varepsilon(t, \cdot)}{\partial t} \right\|_{W^{1,p}_\varepsilon} \leq \tilde{C}_p \left\| \nabla \phi^\varepsilon \cdot \frac{\partial u(t, \cdot)}{\partial t} \right\|_{L^p(A_\varepsilon)}.
$$

Later on in Proposition 2.4, we will employ this Lemma to construct a suitable corrector to the Euler solution that is supported away from the obstacles. Informally, extending $h^\varepsilon$ by zero, we readily verify that $u^\varepsilon := \phi^\varepsilon u - h^\varepsilon$ is divergence free, agrees with $u$ away from the obstacles, and the $L^p$ norm of $u^\varepsilon$ is of order $\| u \|_{L^p} + \varepsilon \| \nabla \phi^\varepsilon \|_{L^p} \| u \|_{L^\infty}$ (cutting the stream function $\psi$ would instead give a bound of order $\| u \|_{L^p} + \| \nabla \phi^\varepsilon \|_{L^p} \| \psi \|_{L^\infty}$).

**Proof of Lemma 2.1.** We start by introducing the bounded open set:

$$
(2.7) \quad U := (-2, 2)^2 \setminus \mathcal{K}.
$$

As $U$ satisfies the cone condition (see e.g. \cite{9} Rem. III.3.4 for the definition), Theorem III.3.1 and Exercise III.3.6 (solvable thanks to Remark III.3.3) in \cite{9} state that there exists $\tilde{C}_p$ with the following property. Given any function $f(t, x)$ such that $f(t, \cdot) \in L^p(U)$, and $\int_U f(t, \cdot) = 0$ for all $t$, there exists a solution $h(t, \cdot) \in W^{1,p}_0(U)$ of the problem:

$$
(2.8) \quad \text{div } h = f, \quad \text{on } [0, \infty) \times U,
$$

where $\text{div}$ is the divergence operator.
such that
\[
\| h(t, \cdot) \|_{W^{1, p}(U)} \leq \tilde{C}_p \| f(t, \cdot) \|_{L^p(U)} \quad \forall t \in [0, \infty),
\]
\[
\| h(t, \cdot) - h(s, \cdot) \|_{W^{1, p}(U)} \leq \tilde{C}_p \| f(t, \cdot) - f(s, \cdot) \|_{L^p(U)} \quad \forall t, s \in [0, \infty).
\]
Moreover, if \( \frac{\partial f(t, \cdot)}{\partial t} \in L^p(U) \) then
\[
\left\| \frac{\partial h(t, \cdot)}{\partial t} \right\|_{W^{1, p}(U)} \leq \tilde{C}_p \left\| \frac{\partial f(t, \cdot)}{\partial t} \right\|_{L^p(U)}.
\]
Solutions are non unique, and the main difficulty addressed in [9] is how to obtain a solution \( h \) vanishing at the boundary and verifying the estimates with constant \( \tilde{C}_p \) depending only on \( U \) and \( p \). The existence of such a solution follows from an explicit representation formula due to Bogovskiǐ [41.5].

We next utilize this result to construct a solution \( h_\varepsilon \) on the domain
\[
A_\varepsilon = \bigcup_{i,j} \left( z_{i,j}^\varepsilon + \varepsilon U \right).
\]
Let \( u \) be a given vector field on \( \mathbb{R}^2 \) such that \( u(t, \cdot) \in L^\infty(\mathbb{R}^2) \) for any \( t \in [0, \infty) \). For any \( i = 1, \ldots, n_1^\varepsilon \) and \( j = 1, \ldots, n_2^\varepsilon \) fixed, the function:
\[
f_{i,j}(t, x) := \varepsilon (\nabla \phi^\varepsilon \cdot u)(t, z_{i,j}^\varepsilon + \varepsilon x)
\]
is defined on the set \( U \), is bounded (hence it belongs to \( L^p(U) \)), and satisfies for each \( t \):
\[
\int_U f_{i,j}(t, \cdot) = \frac{1}{\varepsilon} \int_{z_{i,j}^\varepsilon + \varepsilon U} \text{div}(\phi^\varepsilon u)(t, y) \, dy = \frac{1}{\varepsilon} \int_{z_{i,j}^\varepsilon + \varepsilon U} \phi^\varepsilon u \cdot n \, ds
\]
\[
= \frac{1}{\varepsilon} \int_{z_{i,j}^\varepsilon + \varepsilon \partial(-2,2)^2} u \cdot n \, ds = \frac{1}{\varepsilon} \int_{z_{i,j}^\varepsilon + \varepsilon \partial(-2,2)^2} \text{div} u = 0,
\]
where we have used twice that \( u \) is divergence free on \( \mathbb{R}^2 \), that \( \phi^\varepsilon \equiv 1 \) on \( z_{i,j}^\varepsilon + \varepsilon \partial(-2,2)^2 \), and that \( \phi^\varepsilon \equiv 0 \) on \( z_{i,j}^\varepsilon + \varepsilon \partial K \).

Therefore, we can apply Galdi's results recalled above with \( f \) replaced by \( f_{i,j}(t, x) \) to conclude that there exists a function \( h_{i,j} \) on \([0, \infty) \times U \) such that \( h_{i,j}(t, \cdot) \in W^{1, p}_0(U) \) is a solution of (2.8) with \( f = f_{i,j} \) and satisfies (2.9)-(2.10). Finally, we extend \( h_{i,j} \) by zero on \( \mathbb{R}^2 \) and we define:
\[
h^\varepsilon(t, x) := \sum_{i,j} h_{i,j} \left( t, \frac{x - z_{i,j}^\varepsilon}{\varepsilon} \right).
\]
To finish the proof, we show that \( h^\varepsilon \) has all the properties stated in Lemma 2.1. By definition, \( h^\varepsilon(t, \cdot) \in W^{1, p}_0(A_\varepsilon) \) for all \( t \) and we have
\[
\text{div} h^\varepsilon(t, x) = \frac{1}{\varepsilon} \sum_{i,j} (\text{div} h_{i,j}) \left( t, \frac{x - z_{i,j}^\varepsilon}{\varepsilon} \right) = \frac{1}{\varepsilon} \sum_{i,j} f_{i,j} \left( t, \frac{x - z_{i,j}^\varepsilon}{\varepsilon} \right) \mathbbm{1}_{z_{i,j}^\varepsilon + \varepsilon U}
\]
\[
= \nabla \phi^\varepsilon \cdot u(t, x),
\]
using that the obstacles are separated and \( \text{Supp} \nabla \phi^\varepsilon \subset \bigcup \{ z_{i,j}^\varepsilon + \varepsilon U \} \). In the formula above, \( 1_{z_{i,j}^\varepsilon + \varepsilon U} \) denotes the characteristic function of the set \( z_{i,j}^\varepsilon + \varepsilon U \). Next, for any \( t \), we compute

\[
\| h^\varepsilon(t, \cdot) \|_{W^{1,p}_\varepsilon}^p = \frac{1}{\varepsilon^p} \sum_{i,j} \int_{z_{i,j}^\varepsilon + \varepsilon U} \| h_{i,j}^\varepsilon(t, \frac{x - z_{i,j}^\varepsilon}{\varepsilon}) \|^p \, dx + \sum_{i,j} \int_{z_{i,j}^\varepsilon + \varepsilon U} \left| \frac{1}{\varepsilon} \nabla h_{i,j}^\varepsilon(t, \frac{x - z_{i,j}^\varepsilon}{\varepsilon}) \right|^p \, dx
\]

\[
= \varepsilon^{-p} \sum_{i,j} \| h_{i,j}^\varepsilon(t, \cdot) \|_{W^{1,p}(U)}^p \leq \varepsilon^{-p} \tilde{C}_p \sum_{i,j} \| f_{i,j}^\varepsilon(t, \cdot) \|_{L^p(U)}^p
\]

\[
\leq \varepsilon^{-p} \tilde{C}_p \sum_{i,j} \int_U |\nabla \phi^\varepsilon \cdot u|^p(t, z_{i,j}^\varepsilon + \varepsilon x) \, dx
\]

\[
\leq \tilde{C}_p \| \nabla \phi^\varepsilon \cdot u(t, \cdot) \|_{L^p(A_\varepsilon)}^p.
\]

A similar calculation applies to derive the bounds on \( \| h^\varepsilon(t, \cdot) - h^\varepsilon(s, \cdot) \|_{W^{1,p}_\varepsilon} \), and \( \| \partial_t h^\varepsilon(t, \cdot) \|_{W^{1,p}_\varepsilon} \).

\[\square\]

**Remark 2.1.** In the proof of Lemma 2.1 above, we have used several times the fact that the obstacles are well separated (see Equation (2.1)). Therefore, our analysis can be applied only to the case \( d^\varepsilon \geq C\varepsilon \) for some \( C \) independent of \( \varepsilon \). For distances \( d^\varepsilon \ll \varepsilon \), we would need to understand the behavior of \( \tilde{C}_p \) on domains of the form \((-1,1)^2 \setminus \rho K\) as \( \rho \to 1^-\).

In the same way, changing variables in the classical Poincaré inequality, we obtain the following result. The proof can be found in [16, Lem. 3]. We sketch it here for the reader’s convenience.

**Lemma 2.2.** Let \( \chi \) be a smooth cutoff function such that \( \chi \equiv 1 \) in \( B(0,3) \) and \( \chi \equiv 0 \) in \( B(0,4)^c \). There exists a constant \( K_1 \) depending only on \( K \) such that for any \( u \) verifying

\[
\chi u \in H^1_0(\Omega^\varepsilon),
\]

then

\[
\| u \|_{L^2(A_\varepsilon)} \leq \varepsilon K_1 \| \nabla u \|_{L^2(A_\varepsilon)},
\]

where \( A_\varepsilon \) is given in Equation (2.4).

**Proof.** We note that \( A_\varepsilon \subset B(0,1) \) for all \( \varepsilon \leq 1 \).

By the usual Poincaré inequality,

\[
\| u \|_{L^2(U)} \leq K_1 \| \nabla u \|_{L^2(U)},
\]
for all \( u \) such that \( \chi u \in H^1_0(B(0,4) \setminus K) \), where \( U \) is defined in (2.7). Then, from the definition of \( A_\varepsilon \) given in (2.4), it easily follow that

\[
\|u\|_{L^2(A_\varepsilon)}^2 = \sum_{i,j} \int_{\varepsilon z_{i,j} + \varepsilon U} |u(x)|^2 \, dx = \sum_{i,j} \int_U |u(\varepsilon z_{i,j} + \varepsilon y)|^2 \, dy \\
\leq \sum_{i,j} \varepsilon^2 K_1^2 \int_U |\varepsilon \nabla u(\varepsilon z_{i,j} + \varepsilon y)|^2 \, dy = \varepsilon^2 K_1^2 \|\nabla u\|_{L^2(A_\varepsilon)}^2,
\]

which ends the proof.

We finish this subsection with a Sobolev-type embedding.

**Lemma 2.3.** Let \( \chi \) be a cutoff function as in Lemma 2.2, and let \( A_\varepsilon \) again be given in (2.4). Then, there exists a constant \( K_2 \) depending only on \( K \) such that for any \( u \) verifying \( \chi u \in H^1_0(\Omega) \), it holds

\[
\|u\|_{L^4(U)} \leq \varepsilon K_2 \|\nabla u\|_{L^2(U)}.
\]

**Proof.** Bringing together the Sobolev embedding of \( H^1(\Omega) \) in \( L^4(\Omega) \) and the Poincaré inequality, we have

\[
\|u\|_{L^4(U)} \leq K \|u\|_{H^1(\Omega)} \leq KK_2 \|\nabla u\|_{L^2(U)},
\]

for all \( u \) such that \( \chi u \in H^1_0(\Omega) \).

Hence, a change of variables gives

\[
\|u\|_{L^4(A_\varepsilon)}^2 = \left( \sum_{i,j} \int_{\varepsilon z_{i,j} + \varepsilon U} |u(x)|^4 \, dx \right)^{1/2} \leq \sum_{i,j} \left( \int_{\varepsilon z_{i,j} + \varepsilon U} |u(x)|^4 \, dx \right)^{1/2} \\
\leq \sum_{i,j} \left( \int_U |u(\varepsilon z_{i,j} + \varepsilon y)|^4 \, \varepsilon^2 dy \right)^{1/2} \\
\leq \sum_{i,j} KK_1 \varepsilon \int_U |\varepsilon \nabla u(\varepsilon z_{i,j} + \varepsilon y)|^2 \, dy = \varepsilon K_2^2 \|\nabla u\|_{L^2(A_\varepsilon)}^2.
\]

\( \square \)

### 2.2. Basic estimates on the Euler equations.

We end these preliminaries by collecting some results on the Euler solution \( u^E \) which will be used mainly in Section 4. Except for Proposition 2.4, the statements presented here are well known and complete proofs can be found, for example, in [27].

As mentioned in the Introduction, we give the initial velocity as \( u_0 = K_{\mathbb{R}^2}[\omega_0] \), where \( K_{\mathbb{R}^2} \) is the Biot-Savart kernel defined in (1.7) and the initial vorticity \( \omega_0 \in C_c^{\infty}(\mathbb{R}^2) \). Hence, by a result of McGrath [28] there is a unique global strong solution \( u^E \) of the Euler equations (1.6) in the full plane.

The vorticity \( \omega := \text{curl} u^E \) verifies in a weak sense the transport equation:

\[
\partial_t \omega + u^E \cdot \nabla \omega = 0,
\]

which allows us to deduce the conservation of the \( L^p \) norm of the vorticity:

\[
\|\omega(t, \cdot)\|_{L^p(\mathbb{R}^2)} = \|\omega_0\|_{L^p(\mathbb{R}^2)}, \quad \forall t > 0, \quad 1 \leq p \leq \infty.
\]
By a classical estimate on the Biot-Savart kernel, we obtain that $u^E$ is uniformly bounded:

\[(2.12) \quad \| u^E \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^2)} \leq C \| \omega \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^2)}^{1/2} \| \omega \|_{L^1(\mathbb{R}^2)}^{1/2} \leq C \| \omega_0 \|_{L^1 \cap L^\infty}.
\]

and by the Calderón-Zygmund inequality, we infer that for all $p \in (1, \infty)$:

\[(2.13) \quad \| \nabla u^E \|_{L^\infty(\mathbb{R}^+ \times L^p(\mathbb{R}^2))} \leq C_p \| \omega \|_{L^\infty(\mathbb{R}^+ \times L^p(\mathbb{R}^2))} \leq C_p \| \omega_0 \|_{L^1 \cap L^\infty}.
\]

Hence, the nonlinear term $u^E \cdot \nabla u^E$ belongs to $L^\infty(\mathbb{R}^+ ; L^2 \cap L^4(\mathbb{R}^2))$. Since $p^E$ is a solution of $\Delta p^E = \text{div}(u^E \cdot \nabla u^E)$, up to choosing $p^E$ with zero mean value on $B(0,2)$, it belongs to $L^\infty(\mathbb{R}^+ ; H^1 \cap W^{1,4}(B(0,2)))$ and we deduce from elliptic estimates and the Sobolev embedding that

\[(2.14) \quad p^E \in L^\infty(\mathbb{R}^+ \times B(0,2)).
\]

These estimates also imply that $u^E_t$ is in $L^\infty(\mathbb{R}^+ ; L^4(B(0,2)))$.

Since the Calderón-Zygmund inequality is not true for $p = \infty$, there is no bound on $\| \nabla u^E(t, \cdot) \|_{L^\infty(\mathbb{R}^2)}$ uniformly in time. However, the following well-known estimate \cite{36} holds:

\[(2.15) \quad \| \nabla u^E(t, \cdot) \|_{L^\infty(\mathbb{R}^2)} \leq C_0 e^{C_0 t} \quad \forall t \geq 0,
\]

where $C_0$ depends on $\| \omega_0 \|_{W^{1,\infty}}$. (See \cite{20} for a discussion of results concerning the sharpness of the double exponential growth of $\| \nabla \omega \|_{L^\infty}$.)

In this paper, we do not investigate the minimal regularity of the initial data needed for our result to hold, as the vanishing viscosity limit is a singular problem even for smooth data. In particular, it is known that (2.15) holds under much weaker conditions (e.g. $\omega_0$ in the Besov space $B^2_{p,1}$ \cite{34}).

In the following proposition, we utilize Lemma \ref{lem:2.1} to construct a corrector to the truncated velocity $\phi^E u^E$ to meet the no-slip boundary conditions and the divergence-free condition, which will be employed for the energy estimate in Section 4

**Proposition 2.4.** Let $\omega_0 \in C^c_\infty(\mathbb{R}^2)$ and let $u^E$ be the solution of the Euler equations \eqref{1.6} with initial condition $u_0$, given in terms of $\omega_0$ in \eqref{1.7}. For any $\varepsilon > 0$, let $A_\varepsilon$ be defined as in \eqref{2.3}, \eqref{2.4}. Then, there exists $h^\varepsilon \in L^\infty(\mathbb{R}^+, W^{1,4}_0(A_\varepsilon))$ such that, after extending $h^\varepsilon$ in $\mathbb{R}^2$ by zero, and setting

\[u^\varepsilon := \phi^E u^E - h^\varepsilon,
\]

we have that $u^\varepsilon$ is divergence free and identically zero on $\partial \Omega^\varepsilon$. Moreover, there exists a constant $K_\varepsilon$, dependent on $u^E$ but independent of $\varepsilon$, such that for all $\varepsilon > 0$:

1. $\| h^\varepsilon \|_{L^\infty(\mathbb{R}^+ ; L^4(\mathbb{R}^2))} \leq K_\varepsilon E \frac{\sqrt{\varepsilon}}{d(1+\mu)^{1/4}}$;
2. $\| \partial \varepsilon \|_{L^\infty(\mathbb{R}^+ ; L^2(\mathbb{R}^2))} \leq K_\varepsilon \frac{\sqrt{\varepsilon}}{d(1+\mu)^{1/4}}$;
3. $\| \nabla h^\varepsilon \|_{L^\infty(\mathbb{R}^+ ; L^2(\mathbb{R}^2))} \leq K_\varepsilon \frac{1}{d(1+\mu)^{1/2}}$;
4. $\| u^E - u^\varepsilon \|_{L^\infty(\mathbb{R}^+ ; L^4(\mathbb{R}^2))} \leq K_\varepsilon \frac{\sqrt{\varepsilon}}{d(1+\mu)^{1/4}}$. 
Proof. The main idea is to use Lemma 2.1 with $p = 4$. As $u^E$ is divergence free and uniformly bounded, there exists $h^\varepsilon \in W^{1,4}_0(A_\varepsilon)$ such that $\text{div} \, h^\varepsilon = \nabla \phi^\varepsilon \cdot u^E$, so that
\[
\text{div} \, u^\varepsilon = \nabla \phi^\varepsilon \cdot u^E + \phi^\varepsilon \text{div} \, u^E - \text{div} \, h^\varepsilon \equiv 0.
\]
Using (2.5) in conjunction with (2.12), the estimates in Lemma 2.1 give:
\[
\| h^\varepsilon \|_{L^\infty(L^4)} \leq \tilde{C}_4 \varepsilon \| \nabla \phi^\varepsilon \cdot u^E \|_{L^\infty(L^4)} \leq \tilde{C}_4 C_4 \frac{\varepsilon^{1/2}}{d_\varepsilon^{(1+\mu)/4}} \| u^E \|_{L^\infty},
\]
which proves (1). Since $u^E$ belongs to $L^\infty(L^4(B(0,2)))$, we also obtain (2) by Hölder’s inequality:
\[
\| \partial_t h^\varepsilon \|_{L^\infty(L^2)} \leq C_4 \frac{\varepsilon^{1/2}}{d_\varepsilon^{(1+\mu)/4}} \| \partial_t h^\varepsilon \|_{L^\infty(L^4)} \\
\leq C_4 \frac{\varepsilon^{1/2}}{d_\varepsilon^{(1+\mu)/4}} \tilde{C}_4 \varepsilon \| \nabla \phi^\varepsilon \|_{L^\infty} \| u^E_t \|_{L^\infty(L^4(Supp \nabla \phi^\varepsilon))}.
\]
We prove (3) in a similar way:
\[
\| \nabla h^\varepsilon \|_{L^\infty(L^2)} \leq C_4 \frac{\varepsilon^{1/2}}{d_\varepsilon^{(1+\mu)/4}} \tilde{C}_4 \varepsilon \| \nabla \phi^\varepsilon \cdot u^E \|_{L^\infty(L^4)} \leq \frac{C}{d_\varepsilon^{(1+\mu)/2}} \| u^E \|_{L^\infty}.
\]
Finally, to establish (4), we observe that
\[
\| u^E - u^\varepsilon \|_{L^4} \leq \| (1 - \phi^\varepsilon) u^E \|_{L^4} + \| h^\varepsilon \|_{L^4} \leq C \frac{\varepsilon^{1/2}}{d_\varepsilon^{(1+\mu)/4}},
\]
which concludes the proof. \(\square\)

Remark 2.2. Since $u^E_\varepsilon$ is bounded in $L^\infty(L^p)$ for any $p < \infty$, Lemma 2.1 with $p = 2$ also gives that
\[
\| \partial_t h^\varepsilon \|_{L^\infty(L^2)} \leq \tilde{C}_2 \varepsilon \| \nabla \phi^\varepsilon \cdot u^E_t \|_{L^\infty(L^2)} \leq C \frac{\varepsilon^{2/q}}{d_\varepsilon^{(1+\mu)/q}} \| u^E_t \|_{L^\infty(L^p)},
\]
where $q = \frac{2p}{p-2} > 2$ can be chosen as close as we want of 2.

Nevertheless, for the sequel, we only need the case $q = 4$ (the case corresponding to the bound (2) in Proposition 2.4 above).

Remark 2.3. As we did not use (2.15) in the previous proof, we can give a more precise dependence of the constant $K_E$ on the geometry and the Euler solutions (which are both fixed throughout). In fact, $K_E$ depends only on $K$ and $\| \omega_0 \|_{L^1 \cap L^\infty}$. In particular, the bounds (1), (3), (4) in Proposition 2.4 are linear in $\| u^E \|_{L^\infty}$ and $\| u^E \|_{L^\infty}$. Hence, in these estimates $K_E$ is of the form $K \| \omega_0 \|_{L^1 \cap L^\infty}$, where $K$ depends only on $K$. 


3. Convergence of the initial velocity

In this section, we discuss the convergence of the Navier-Stokes initial data (taken dependent on $\varepsilon$ only) to the Euler initial data in the energy norm, as $\varepsilon$ goes to zero.

Let $\omega_0 \in C_c^\infty(\mathbb{R}^2)$, then for any $\varepsilon > 0$ there exists a unique vector field $u_0^\varepsilon$, which is solution of (1.9) (see e.g. [19] for a proof). We choose $R$ so that $\text{Supp} \omega_0 \cup \partial \Omega^\varepsilon \subset B(0, R)$. The function

$$
\psi : z = x_1 + ix_2 \mapsto (u_0^\varepsilon)_1(x_1, x_2) - i(u_0^\varepsilon)_2(x_1, x_2)
$$

verifies the Cauchy-Riemann equations on $B(0, R)^c$. Hence, $\psi$ is holomorphic and admits a Laurent series decomposition $\psi(z) = \sum_{k=1}^{\infty} c_k z^k$, which allows us to conclude that $u_0^\varepsilon \in L^\infty \cap L^2, \infty(B(0, R)^c)$ (where $L^2, \infty$ denotes the Marcinkiewicz weak $L^2$-space). By elliptic regularity, $u_0^\varepsilon$ is bounded in $B(0, R) \cap \Omega^\varepsilon$, implying that

$$
(3.1) \quad u_0^\varepsilon \in L^\infty \cap L^2, \infty(\Omega^\varepsilon).
$$

This regularity of the initial data is sufficient to apply the result of Kozono and Yamazaki [21], yielding existence and uniqueness of global solution to the Navier-Stokes equations (1.5).

The goal of this section is to prove that $u_0^\varepsilon$ converges in $L^2$ to $u_0$ defined in (1.7). More precisely, we prove the following:

**Proposition 3.1.** Let $\Omega^\varepsilon$ be defined in (1.1)-(1.4) with $d_\varepsilon \geq \varepsilon$ and let $\omega_0 \in C_c^\infty(\mathbb{R}^2)$. Then, there exists a constant $C$, which depends only on $K$, such that

$$
\|u_0^\varepsilon - u_0\|_{L^2(\Omega^\varepsilon)} \leq C\|\omega_0\|_{L^1 \cap L^\infty} \frac{\varepsilon|\ln \varepsilon|}{d_\varepsilon^{(1+\mu)/2}},
$$

where $u_0^\varepsilon$ is the unique solution of (1.9) and $u_0$ the unique solution of (1.8).

This proposition follows from the analysis developed in [6]. There, the authors have looked for the best condition on $\varepsilon$ and $d_\varepsilon$ ensuring convergence of $u_0^\varepsilon$ to $u_0$ in $L^2(\Omega^\varepsilon)$, but unfortunately, they did not explicitly state the rate of convergence. To obtain the rate, we now briefly review the results in [6]. For brevity, we will denote $M_0 := \|\omega_0\|_{L^1 \cap L^\infty}$.

3.1. Correction and decomposition. We recall that $u_0$ has an explicit formula in terms of $\omega_0$ via the Biot-Savart kernel (Equation (1.7)). However, there is no such formula available for $u_0^\varepsilon$. In addition, $u_0$ is not tangent to $\partial \Omega^\varepsilon$ and hence, it is necessary to introduce an explicit corrector, which is given in terms of:

1. a cutoff function $\phi_{i,j}^\varepsilon$, equal to 1 close to $K_{i,j}^\varepsilon$:

$$
\phi_{i,j}^\varepsilon(x) := \phi \left( \frac{x - z_{i,j}^\varepsilon}{\varepsilon} \right) \quad (\text{with } \phi \text{ given in (2.2))};
$$

2. a biholomorphism $\mathcal{T} : K^\varepsilon \to \mathbb{R}^2 \setminus B(0, 1)$ such that $\mathcal{T}(z) = \beta z + h(z)$ for some $\beta \in \mathbb{R}^+$ and $h$ a bounded holomorphic function.
Then, the correction is defined by
\[ v^\varepsilon := \nabla^\perp \psi^\varepsilon, \]
where
\[ \psi^\varepsilon(x) := \frac{1}{2\pi} \int_{\Omega^\varepsilon} \ln |x - y|\omega_0(y) dy \]
\[ \quad - \frac{1}{2\pi} \sum_{i,j} \phi^\varepsilon_{i,j}(x) \int_{\Omega^\varepsilon} \ln \frac{\beta|x - y|}{\varepsilon|T^\varepsilon_{i,j}(x) - T^\varepsilon_{i,j}(y)|} \omega_0(y) dy \]
\[ \quad + \frac{1}{2\pi} \sum_{i,j} \phi^\varepsilon_{i,j}(x) \int_{\Omega^\varepsilon} \ln \frac{|T^\varepsilon_{i,j}(x)|}{|T^\varepsilon_{i,j}(x) - T^\varepsilon_{i,j}(y)|^2} \omega_0(y) dy, \]
with
\[ T^\varepsilon_{i,j}(x) := T \left( \frac{x - \varepsilon z_{i,j}}{\varepsilon} \right). \]

Above, we have denoted by \( y^\varepsilon = \frac{y}{|y|} \) the conjugate point to \( y \) across the unit circle in \( \mathbb{R}^2 \). This formula is related to the Biot-Savart law outside one obstacle, and we can check that \( v^\varepsilon \) verifies the following properties:
\[ \text{div } v^\varepsilon = 0 \text{ in } \Omega^\varepsilon, \quad v^\varepsilon \cdot n = 0 \text{ on } \partial \Omega^\varepsilon, \]
\[ \int_{\partial K^\varepsilon_{i,j}} v^\varepsilon \cdot \tau ds = 0 \text{ for all } i, j, \quad \lim_{x \to \infty} |v^\varepsilon(x)| = 0. \]

Then, we decompose \( u_0 - v^\varepsilon \) as
\[ u_0 - v^\varepsilon = \sum_{k=1}^{4} w^\varepsilon_k, \]
where
\[ w^\varepsilon_1(x) = \frac{1}{2\pi} \sum_{i,j} \nabla \phi^\varepsilon_{i,j}(x) \int_{\Omega^\varepsilon} \ln \frac{\beta|x - y|}{\varepsilon|T^\varepsilon_{i,j}(x) - T^\varepsilon_{i,j}(y)|} \omega_0(y) dy, \]
\[ w^\varepsilon_2(x) = \frac{1}{2\pi} \sum_{i,j} \nabla \phi^\varepsilon_{i,j}(x) \int_{\Omega^\varepsilon} \ln \frac{|T^\varepsilon_{i,j}(x) - T^\varepsilon_{i,j}(y)^*|}{|T^\varepsilon_{i,j}(x)|} \omega_0(y) dy, \]
\[ w^\varepsilon_3(x) = \frac{1}{2\pi} \sum_{i,j} \phi^\varepsilon_{i,j}(x) \int_{\Omega^\varepsilon} \left( \frac{x - y}{|x - y|^2} - (D\nabla T^\varepsilon_{i,j} \nabla T^\varepsilon_{i,j}) (T^\varepsilon_{i,j} - T^\varepsilon_{i,j}(y))^\perp \right) \omega_0(y) dy, \]
\[ w^\varepsilon_4(x) = \frac{1}{2\pi} \sum_{i,j} \phi^\varepsilon_{i,j}(x) (D\nabla T^\varepsilon_{i,j} \nabla T^\varepsilon_{i,j}) (T^\varepsilon_{i,j} - T^\varepsilon_{i,j}(y))^\perp \int_{\Omega^\varepsilon} \left( \frac{T^\varepsilon_{i,j} - T^\varepsilon_{i,j}(y)}{|T^\varepsilon_{i,j} - T^\varepsilon_{i,j}(y)|^2} - \frac{T^\varepsilon_{i,j}}{|T^\varepsilon_{i,j}|^2} \right) \omega_0(y) dy. \]

In \[6\] and in \[23\], the explicit formula for \( w^\varepsilon_k \) is utilized to prove that \( v^\varepsilon \) converges to \( u_0 \) as \( \varepsilon \to 0 \). In \[6\], a convergence analysis was also performed for \( d_\varepsilon \ll \varepsilon \), which complicates the choice of the cutoff function. In our case where \( d_\varepsilon \geq \varepsilon \), the cutoff \( \phi^\varepsilon_{i,j} \) defined above is sufficient to apply the estimates.
in [6]. We recall these estimates, adapted to the set-up of this paper, in the
remainder of the section.

3.2. Estimates on \( w_1^\varepsilon \) and \( w_3^\varepsilon \). Following the proofs of Proposition 3.3
and Proposition 3.4 in [6], we first tackle estimates for \( w_1^\varepsilon \) and \( w_3^\varepsilon \). When
\( \mathcal{K} = \overline{B(0,1)} \), \( T = \text{Id} \) (so \( \beta = 1 \)) and \( w_1^\varepsilon = w_3^\varepsilon \equiv 0 \). In the general case, the
main idea is to use that
\[
T_{i,j}^\varepsilon (x) = T\left(\frac{x - z_{i,j}^\varepsilon}{\varepsilon}\right) \sim \beta \frac{x - z_{i,j}^\varepsilon}{\varepsilon} = \beta \text{Id}\left(\frac{x - z_{i,j}^\varepsilon}{\varepsilon}\right).
\]

In [6, Proposition 3.3], it was proved that
\[
\left\| \int_{\Omega^\varepsilon} \ln \frac{\beta|x-y|}{\varepsilon|T_{i,j}^\varepsilon(x) - T_{i,j}^\varepsilon(y)|} \omega_0(y) dy \right\|_{L^\infty(\Omega^\varepsilon)} \leq CM_0 \varepsilon |\ln \varepsilon|,
\]
where \( C \) depends only on \( \mathcal{K} \). Using that the supports of \( \nabla \phi_{i,j}^\varepsilon \) are disjoint
for distinct \( i, j \) (see (2.1)), we deduce that:
\[
\|w_3^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq CM_0 \varepsilon |\ln \varepsilon| \frac{\varepsilon}{d_\varepsilon^{(1+\mu)/2}} = CM_0 \varepsilon |\ln \varepsilon| \frac{\varepsilon}{d_\varepsilon^{(1+\mu)/2}}.
\]

In [6, Proposition 3.4], it was also shown that, for all \( i, j \) and \( x \in \Omega^\varepsilon \), it holds
\[
\left| \int_{\Omega^\varepsilon} \left( \frac{(x - y)^+}{|x-y|^2} - (D T_{i,j}^\varepsilon)^\top (T_{i,j}^\varepsilon(x) - T_{i,j}^\varepsilon(y)^+ \frac{(T_{i,j}^\varepsilon(x) - T_{i,j}^\varepsilon(y))}{|T_{i,j}^\varepsilon(x) - T_{i,j}^\varepsilon(y)|^2}) \omega_0(y) dy \right) \right|
\leq CM_0 \left( \varepsilon^{1/2} + \frac{\varepsilon}{|x - z_{i,j}^\varepsilon|} \right),
\]
where \( C \) depends only on \( \mathcal{K} \). As there exists \( \delta > 0 \) such that \( B(0, \delta) \subset \mathcal{K} \),
\( \text{Supp} \phi_{i,j}^\varepsilon \subset B(z_{i,j}^\varepsilon, 2\sqrt{2} \varepsilon) \setminus B(z_{i,j}^\varepsilon, \delta \varepsilon) \), and we can estimate the \( L^2 \) norm of \( w_3^\varepsilon \) as follows:
\[
\|w_3^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq CM_0 \varepsilon^{1/2} \frac{\varepsilon}{d_\varepsilon^{(1+\mu)/2}} + CM_0 \varepsilon \left( \frac{1}{d_\varepsilon^{(1+\mu)/2}} \int_{B(0,2\sqrt{2} \varepsilon) \setminus B(0,\delta \varepsilon)} \frac{dx}{|x|^2} \right)^{1/2}
\leq CM_0 \varepsilon |\ln \varepsilon|^{1/2} \frac{\varepsilon}{d_\varepsilon^{(1+\mu)/2}}.
\]

3.3. Estimates on \( w_2^\varepsilon \) and \( w_4^\varepsilon \). When \( \mathcal{K} = \overline{B(0,1)} \), we have \( T_{i,j}^\varepsilon(y)^* =
\frac{\varepsilon^2 y - z_{i,j}^\varepsilon}{|y - z_{i,j}^\varepsilon|^2} \). We observe that, even though \( |y - z_{i,j}^\varepsilon| \) can be of order \( \varepsilon \) (this
is, in fact, the case if \( y \in \partial B(z_{i,j}^\varepsilon, \varepsilon) \)), \( T_{i,j}^\varepsilon(y)^* \) is still small compared to \( T_{i,j}^\varepsilon(x) \),
since \( |T_{i,j}^\varepsilon(x)| \geq 1 \). Hence, \( w_2^\varepsilon \) and \( w_4^\varepsilon \) should be small.

More precisely, in [6, Proposition 3.5], it was proved that, for all \( i, j \),
\[
\left\| \int_{\Omega^\varepsilon} \ln \frac{|T_{i,j}^\varepsilon(x) - T_{i,j}^\varepsilon(y)^*|}{|T_{i,j}^\varepsilon(x)|} \omega_0(y) dy \right\|_{L^\infty(\text{Supp} \phi_{i,j}^\varepsilon)} \leq CM_0 \varepsilon.
\]
where $C$ depends only on $K$. Therefore, we find that
\[ \|w_k^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq CM_0^2 \frac{\varepsilon}{\varepsilon} \|\nabla \phi\|_{L^\infty} \varepsilon = CM_0 \frac{\varepsilon}{d_0^{(1+\mu)/2}}. \]

Concerning the estimate on $w_4^\varepsilon$, the proof of Proposition 3.6 in [6] shows that, for all $i,j$ and $x \in \Omega^\varepsilon$,
\[ \left| (D^T_{i,j})^T(x) \int_{\Omega^\varepsilon} \left( \frac{T_{i,j}^\varepsilon(x) - T_{i,j}^\varepsilon(y)^*}{|T_{i,j}^\varepsilon(x) - T_{i,j}^\varepsilon(y)^*|^2} \right)^\perp \omega_0(y) dy \right| \leq CM_0 \left( \varepsilon + \frac{\varepsilon}{\varepsilon|T_{i,j}^\varepsilon(x)|} \right), \]
where $C$ depends only on $K$. Therefore, after the change of variable $z = \varepsilon T_{i,j}^\varepsilon(x)$, we can estimate the $L^2$ norm in the same way as for $w_3^\varepsilon$ (see [6] for details), and we conclude that
\[ \|w_4^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq CM_0 \frac{\varepsilon}{d_0^{(1+\mu)/2}}. \]

### 3.4. End of the proof of Proposition 3.1
Collecting the estimates on $w_k^\varepsilon$ for $k = 1, \ldots, 4$ gives that for any $\varepsilon > 0$, there exists a constant $C > 0$ such that
\[ \|u_0 - v^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq CM_0 \frac{\varepsilon |\ln \varepsilon|^{1/2}}{d_0^{(1+\mu)/2}}. \]

Finally, we observe that $u_0^\varepsilon - v^\varepsilon$ is the Leray projection of $u_0 - v^\varepsilon$ (see Proposition 2.4). Hence, the $L^2$-orthogonality of this projection implies
\[ \|u_0^\varepsilon - v^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq \|u_0 - v^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq CM_0 \frac{\varepsilon |\ln \varepsilon|}{d_0^{(1+\mu)/2}}. \]

We conclude the proof of Proposition 3.1 by the triangle inequality.

### 4. Energy estimate

In this section, we establish convergence of the solution $u^{\mu\varepsilon}$ of the Navier-Stokes equations in $\Omega^\varepsilon$ to the solution $u^E$ of the Euler equations in the whole plane, provided $\varepsilon/d_0^{(1+\mu)/2} \leq A \nu$ for some given positive constant $A$. The proof is performed by an energy estimate on the difference $u^{\mu\varepsilon} - u^\varepsilon$, where $u^\varepsilon$ vanishes on $\partial \Omega^E$ and is close to $u^E$ (see Proposition 2.4). This strategy is related to the proof of Kato’s criterion (see [17, 33, 35]) and was used in [16] to treat the case of one shrinking obstacle.

Heuristically, it is possible to pass to the zero-viscosity limit even in the presence of the boundary layer due to the no-slip boundary condition, since under the conditions of the theorem below the contribution to the energy from the Euler solution restricted to the obstacles is negligible. Nevertheless, in the case of infinitely-many obstacles a dependence on $d_\varepsilon$, which reduces
the expected rate of convergence of $\sqrt{\nu}$, valid for one shrinking obstacle, appears in the convergence estimate (4.2). This dependence can be interpreted as a “ghost” of the perforated domain.

**Theorem 4.1.** Let $u^E$ be the solution of the Euler equations (1.6) on $[0, \infty) \times \mathbb{R}^2$ with initial condition $u_0$ (given in 1.7) in terms of an initial vorticity $\omega_0 \in C^\infty_c(\mathbb{R}^2)$. Let $u^{\nu, \varepsilon}$ be the solution of the Navier-Stokes problem (1.5) on $[0, \infty) \times \Omega^\varepsilon$ (where $\Omega^\varepsilon$ is defined in (1.1) - (1.4) with $d^\varepsilon \geq \varepsilon$) with initial velocity $u^{\nu, \varepsilon}_0 \in L^\infty \cap L^2(\Omega^\varepsilon)$, satisfying the divergence-free condition and no-penetration boundary condition, and such that $u^{\nu, \varepsilon}_0 - u_0 \in L^2(\Omega^\varepsilon)$. Then, there exists a constant $A$, depending only on $K$, such that if

$$
(4.1) \quad \frac{\varepsilon}{d^\varepsilon(1+\mu)/2} \leq \|\omega_0\|_{L^1 \cap L^\infty},
$$

then for any $0 < T < \infty$,

$$
(4.2) \quad \sup_{0 \leq t \leq T} \|u^{\nu, \varepsilon} - u^E\|_{L^2(\Omega^\varepsilon)} \leq B_T \left( \frac{\sqrt{\nu}}{d^\varepsilon(1+\mu)/2} + \|u^{\nu, \varepsilon}_0 - u_0\|_{L^2(\Omega^\varepsilon)} \right),
$$

where $B_T$ is a constant depending only on $T$, $\|\omega_0\|_{L^1 \cap \dot{W}^{1, \infty}}$, and $K$.

We recall that $\Omega^\varepsilon$ depends on the configuration of the obstacles, which are arranged according to the parameter $\mu \in [0, 1]$.

**Remark 4.1.** Before giving the proof of the theorem, we list two immediate corollaries:

(a) By choosing as initial data for Navier-Stokes $u^{\nu, \varepsilon}_0 = u_0^\varepsilon$, where $u_0^\varepsilon$ solves (1.9), Proposition 3.1 and Theorem 4.1 immediately give that

$$
(4.1) \quad \frac{\varepsilon}{d^\varepsilon(1+\mu)/2} \leq \|\omega_0\|_{L^1 \cap L^\infty},
$$

then for any $0 < T < \infty$,

$$
(4.2) \quad \sup_{0 \leq t \leq T} \|u^{\nu, \varepsilon} - u^E\|_{L^2(\Omega^\varepsilon)} \leq B_T \left( \frac{\sqrt{\nu}}{d^\varepsilon(1+\mu)/2} + \|u^{\nu, \varepsilon}_0 - u_0\|_{L^2(\Omega^\varepsilon)} \right),
$$

where

$$
\varepsilon |\ln \varepsilon| \leq C \sqrt{\nu} \leq C \sqrt{\nu}.
$$

This estimate then implies Theorem 1.1.

(b) In the case of an arbitrary, but fixed and independent of $\varepsilon$, number of shrinking obstacles, we recover the rate of convergence of $\sqrt{\nu}$ obtained for one obstacle in [16].

**Proof of Theorem 4.1.** With $u^\varepsilon$ defined in Proposition 2.4 we introduce

$$
W^{\nu, \varepsilon} := u^{\nu, \varepsilon} - u^\varepsilon,
$$

which is divergence free and verifies the Dirichlet boundary condition on $\partial \Omega^\varepsilon$. We observe that $u^\varepsilon$ satisfies:

$$
u u^\varepsilon_t = \phi^\varepsilon u^\varepsilon_t - h^\varepsilon_t = \phi^\varepsilon (-u^E \nabla u^E - \nabla p^E) - h^\varepsilon_t.
$$

Hence, $W^{\nu, \varepsilon}$ satisfies:

$$
W^{\nu, \varepsilon}_t - \nu \Delta W^{\nu, \varepsilon} = - \nabla p^{\nu, \varepsilon} + \nu \Delta u^\varepsilon - (u^{\nu, \varepsilon} \cdot \nabla) u^{\nu, \varepsilon} + \phi^\varepsilon ((u^E \cdot \nabla) u^E + \nabla p^E) + h^\varepsilon_t.
$$
Multiplying this equation by $W^{\nu,\varepsilon}$ and integrating by parts, we obtain:

\[ \frac{1}{2} \frac{d}{dt} \| W^{\nu,\varepsilon} \|^2_{L^2(\Omega^\varepsilon)} + \nu \| \nabla W^{\nu,\varepsilon} \|^2_{L^2(\Omega^\varepsilon)} = - \nu \int_{\Omega^\varepsilon} \nabla W^{\nu,\varepsilon} : \nabla u^\varepsilon \, dx - \int_{\Omega^\varepsilon} W^{\nu,\varepsilon} \cdot [(u^{\nu,\varepsilon} \cdot \nabla) u^{\nu,\varepsilon}] \, dx \]

\[ + \int_{\Omega^\varepsilon} W^{\nu,\varepsilon} \cdot [\phi^\varepsilon (u^E \cdot \nabla) u^E] \, dx + \int_{\Omega^\varepsilon} W^{\nu,\varepsilon} \cdot \phi^\varepsilon \nabla p^E \]

\[ + \int_{\Omega^\varepsilon} W^{\nu,\varepsilon} \cdot h^\varepsilon \, dx. \]  

(4.3)

Under the assumption on the initial data, $u^{\nu,\varepsilon}$ and $u^\varepsilon$ have enough regularity to justify the integration by parts and, hence, the energy identity.

We group the trilinear terms in (4.3) together and tackle each of the other terms separately. We start with the trilinear terms:

\[ \mathcal{I} := - \int_{\Omega^\varepsilon} W^{\nu,\varepsilon} \cdot [(u^{\nu,\varepsilon} \cdot \nabla) u^{\nu,\varepsilon}] \, dx, \]

which we rewrite, owing to $u^{\nu,\varepsilon} = W^{\nu,\varepsilon} + u^\varepsilon$ and the divergence-free condition on $W^{\nu,\varepsilon}$, as

\[ \mathcal{I} = - \int_{\Omega^\varepsilon} W^{\nu,\varepsilon} \cdot [(W^{\nu,\varepsilon} \cdot \nabla) u^\varepsilon] \, dx - \int_{\Omega^\varepsilon} W^{\nu,\varepsilon} \cdot [(u^\varepsilon \cdot \nabla) u^\varepsilon] \, dx \]

\[ + \int_{\Omega^\varepsilon} W^{\nu,\varepsilon} \cdot [\phi^\varepsilon (u^E \cdot \nabla) u^E] \, dx. \]

We will utilize that $u^\varepsilon$ is close to both $\phi^\varepsilon u^E$ and $u^E$ to control the last two terms. Hence, we add and subtract $\int_{\Omega^\varepsilon} W^{\nu,\varepsilon} \cdot [(u^\varepsilon \cdot \nabla) u^E] \, dx$:

\[ \mathcal{I} = - \int_{\Omega^\varepsilon} W^{\nu,\varepsilon} \cdot [(W^{\nu,\varepsilon} \cdot \nabla) u^\varepsilon] \, dx - \int_{\Omega^\varepsilon} W^{\nu,\varepsilon} \cdot [(u^\varepsilon \cdot \nabla) (u^\varepsilon - u^E)] \, dx \]

\[ + \int_{\Omega^\varepsilon} W^{\nu,\varepsilon} \cdot [((\phi^\varepsilon u^E - u^\varepsilon) \cdot \nabla) u^E] \, dx \]

\[ = - \int_{\Omega^\varepsilon} W^{\nu,\varepsilon} \cdot [(W^{\nu,\varepsilon} \cdot \nabla) u^\varepsilon] \, dx + \int_{\Omega^\varepsilon} (u^E - u^E) \cdot [(u^\varepsilon \cdot \nabla) W^{\nu,\varepsilon}] \, dx \]

\[ + \int_{\Omega^\varepsilon} W^{\nu,\varepsilon} \cdot [(h^\varepsilon \cdot \nabla) u^E] \, dx, \]

(4.4)

where we have integrated by parts in the second term and have used that $\phi^\varepsilon u^E - u^\varepsilon = h^\varepsilon$.

We bound each of the three integrals in (4.4) above separately. We set:

\[ \mathcal{I}_1 := - \int_{\Omega^\varepsilon} W^{\nu,\varepsilon} \cdot [(W^{\nu,\varepsilon} \cdot \nabla) u^\varepsilon] \, dx. \]

We have

\[ \nabla u^\varepsilon = \phi^\varepsilon \nabla u^E + \nabla \phi^\varepsilon \otimes u^E - \nabla h^\varepsilon, \]
with the last two terms supported on $A_\varepsilon$ (defined in (2.4)). Hence, using estimates (2.5) on $\nabla \phi^\varepsilon$, (2.12)-(2.15) on $u^E$, and estimate (3) in Proposition 2.4 on $\nabla h^\varepsilon$, we have thanks to Lemma 2.3:

\[
|I_1| \leq \|W^{\nu,\varepsilon}\|_{L^2(\Omega^\varepsilon)}^2 \|\nabla u^E\|_{L^\infty} + \|W^{\nu,\varepsilon}\|_{L^4(A_\varepsilon)}^2 \|u^E\|_{L^\infty} \|\nabla \phi^\varepsilon\|_{L^2}
\]
\[
+ \|W^{\nu,\varepsilon}\|_{L^4(A_\varepsilon)}^2 \|\nabla h^\varepsilon\|_{L^2}
\]
\[
(4.5) \quad \leq C_0 e^{C_1 \varepsilon} \|W^{\nu,\varepsilon}\|_{L^2(\Omega^\varepsilon)}^2 + K_3 \frac{\varepsilon}{d_\varepsilon^{(1+\mu)/2}} \|\nabla W^{\nu,\varepsilon}\|_{L^2(\Omega^\varepsilon)}^2,
\]

where $K_3$ depends only on $\omega_0$ and $\mathcal{K}$. From the discussion in Remark 2.3, $K_3$ can be taken of the form $K_3 = \tilde{K}_3 \|\omega_0\|_{L^1 \cap L^\infty}$, where $\tilde{K}_3$ depends only on $\mathcal{K}$. For

\[
I_2 := \int_{\Omega^\varepsilon} (u^\varepsilon - u^E) \cdot [(u^\varepsilon \cdot \nabla) W^{\nu,\varepsilon}] \, dx,
\]

we note that $\text{supp}(u^\varepsilon - u^E) \subset A_\varepsilon$. Then, Hölder’s and Cauchy’s inequalities, together with estimates (2.5) on $1 - \phi^\varepsilon$, (2.12) on $u^E$, and estimates (1) and (4) in Proposition 2.4 give that

\[
|I_2| \leq \|\nabla W^{\nu,\varepsilon}\|_{L^2(\Omega^\varepsilon)} \|u^\varepsilon - u^E\|_{L^4} \|u^\varepsilon\|_{L^4(A_\varepsilon)} \|W^{\nu,\varepsilon}\|_{L^2(\Omega^\varepsilon)}^2
\]
\[
\leq \|\nabla W^{\nu,\varepsilon}\|_{L^2(\Omega^\varepsilon)} K_F \frac{\varepsilon}{d_\varepsilon^{(1+\mu)/4}} \left( \|u^E\|_{L^\infty} \frac{C_4 \varepsilon}{d_\varepsilon^{(1+\mu)/4}} + K_F \frac{\varepsilon}{d_\varepsilon^{(1+\mu)/4}} \right)
\]
\[
(4.6) \quad \leq \frac{\nu}{8} \|\nabla W^{\nu,\varepsilon}\|_{L^2(\Omega^\varepsilon)}^2 + K_4 \frac{\varepsilon^2}{\nu d_\varepsilon^{1+\mu}},
\]

where $K_4$ depends only on $\omega_0$ and $\mathcal{K}$.

Lastly, we bound

\[
I_3 := \int_{\Omega^\varepsilon} W^{\nu,\varepsilon} \cdot [(h^\varepsilon \cdot \nabla) u^E] \, dx.
\]

Using that $\text{supp} h^\varepsilon \subset A_\varepsilon$, we apply Hölder’s and Cauchy’s inequalities again, together with estimates (1) of Proposition 2.4 for $h^\varepsilon$, (2.13) for $\nabla u^E$, and Lemma 2.3 to obtain:

\[
|I_3| \leq \|W^{\nu,\varepsilon}\|_{L^4(A_\varepsilon)} \|h^\varepsilon\|_{L^4} \|\nabla u^E\|_{L^2} \leq K_5 \frac{\varepsilon}{d_\varepsilon^{(1+\mu)/4}} \|\nabla W^{\nu,\varepsilon}\|_{L^2(\Omega^\varepsilon)}
\]
\[
(4.7) \quad \leq \frac{\nu}{8} \|\nabla W^{\nu,\varepsilon}\|_{L^2(\Omega^\varepsilon)}^2 + K_5 \frac{\varepsilon^2}{\nu d_\varepsilon^{1+\mu}},
\]

where $K_5$ depends only on $\omega_0$ and $\mathcal{K}$.

From (4.4) and (4.5), (4.6), (4.7), it follows that

\[
|I| \leq \left( \frac{\nu}{4} + K_4 \frac{\varepsilon}{d_\varepsilon^{(1+\mu)/2}} \right) \|\nabla W^{\nu,\varepsilon}\|_{L^2(\Omega^\varepsilon)}^2 + C_0 e^{C_1 \varepsilon} \|W^{\nu,\varepsilon}\|_{L^2(\Omega^\varepsilon)}^2
\]
\[
+ K_6 \frac{\varepsilon^2}{\nu d_\varepsilon^{1+\mu}},
\]

where $K_6$ depends only on $\omega_0$ and $\mathcal{K}$. 

We now turn to the remaining integral terms in (4.3). We begin with
\[ J := -\nu \int_{\Omega^\varepsilon} \nabla W^{\nu,\varepsilon} : \nabla u^\varepsilon \, dx. \]
Applying Cauchy-Schwartz followed by Cauchy's inequality, owing to estimate (3) in Proposition 2.4 for \( \nabla h^\varepsilon \), (2.5) for \( \nabla \phi^\varepsilon \), and (2.12)-(2.13) for \( u^E \), gives:
\[
|J| \leq \nu/2 \| \nabla W^{\nu,\varepsilon} \|_{L^2(\Omega^\varepsilon)} \sqrt{2\nu} \| \nabla u^\varepsilon \|_{L^2} \\
\leq \frac{\nu}{4} \| \nabla W^{\nu,\varepsilon} \|_{L^2(\Omega^\varepsilon)}^2 + \nu \left( \| u^E \|_{L^\infty} \frac{C_2}{d_\varepsilon^{(1+\mu)/2}} + \| \nabla u^E \|_{L^2} + \frac{K_E}{d_\varepsilon^{(1+\mu)/2}} \right)^2 \\
(4.9) \leq \frac{\nu}{4} \| \nabla W^{\nu,\varepsilon} \|_{L^2(\Omega^\varepsilon)}^2 + K_7 \frac{\nu}{d_\varepsilon^{1+\mu}},
\]
where \( K_7 \) depends only on \( \omega_0 \) and \( K \).

Finally, we consider the last two integral terms in (4.3). We set:
\[ H_1 := \int_{\Omega^\varepsilon} \phi^\varepsilon W^{\nu,\varepsilon} : \nabla p^E \, dx = -\int_{\Omega^\varepsilon} p^E W^{\nu,\varepsilon} : \nabla \phi^\varepsilon \, dx. \]
Consequently, by Hölder’s and Cauchy’s inequalities, and the Poincaré’s inequality (2.11), using that \( \nabla \phi^\varepsilon \) is supported on \( A_\varepsilon \) together with estimates (2.5) for \( \nabla \phi^\varepsilon \) and (2.14) for \( p^E \), we have
\[
|H_1| \leq \| W^{\nu,\varepsilon} \|_{L^2(A_\varepsilon)} \| \nabla \phi^\varepsilon \|_{L^2(A_\varepsilon)} \| p^E \|_{L^\infty(A_\varepsilon)} \\
\leq \varepsilon K_1 \| W^{\nu,\varepsilon} \|_{L^2(A_\varepsilon)} \frac{C_2}{d_\varepsilon^{(1+\mu)/2}} \| p^E \|_{L^\infty(A_\varepsilon)} \\
(4.10) \leq \frac{\nu}{8} \| \nabla W^{\nu,\varepsilon} \|_{L^2(A_\varepsilon)}^2 + K_8 \frac{\varepsilon^2}{\nu d_\varepsilon^{1+\mu}},
\]
where \( K_8 \) depends only on \( \omega_0 \) and \( K \).

We tackle the last integral term
\[ H_2 := -\int_{\Omega^\varepsilon} W^{\nu,\varepsilon} : h^\varepsilon \, dx. \]
As \( h^\varepsilon \) is supported on \( A_\varepsilon \), a similar application of the Cauchy-Schwartz, Cauchy’s inequalities, and the Poincaré’s inequality (2.11), together with estimate (2) in Proposition 2.4 on \( h^\varepsilon \), gives:
\[
|H_2| \leq \| W^{\nu,\varepsilon} \|_{L^2(A_\varepsilon)} \| h^\varepsilon \|_{L^2(A_\varepsilon)} \leq \varepsilon K_1 \| \nabla W^{\nu,\varepsilon} \|_{L^2(A_\varepsilon)} K E \frac{\sqrt{\varepsilon}}{d_\varepsilon^{(1+\mu)/4}} \\
(4.11) \leq \frac{\nu}{8} \| \nabla W^{\nu,\varepsilon} \|_{L^2(A_\varepsilon)}^2 + K_9 \frac{\varepsilon^3}{\nu d_\varepsilon^{(1+\mu)/2}},
\]
where \( K_9 \) depends only on \( \omega_0 \) and \( K \).
Putting together the bounds (4.8) obtained for $\mathcal{I}$, (4.9) for $\mathcal{J}$, and (4.10)-(4.11) for $\mathcal{H}$, yields the following a priori energy estimate:

$$
\frac{1}{2} \frac{d}{dt} \|W^{\nu,\varepsilon}\|_{L^2(\Omega^\varepsilon)}^2 + \nu \|\nabla W^{\nu,\varepsilon}\|_{L^2(\Omega^\varepsilon)}^2 \leq \left( \frac{3\nu}{4} + K_3 \frac{\varepsilon}{d_\varepsilon^{(1+\mu)/2}} \right) \|\nabla W^{\nu,\varepsilon}\|_{L^2(\Omega^\varepsilon)}^2 + C_0 e^{C_0 t} \|W^{\nu,\varepsilon}\|_{L^2(\Omega^\varepsilon)}^2 + \frac{\varepsilon^2}{\nu d_\varepsilon^{1+\mu}} \left( K_6 + K_8 + K_9 \varepsilon^3 d_\varepsilon^{(1+\mu)/2} \right) + K_{10} \nu.
$$

We will apply Grönwall’s Lemma to the previous inequality. We therefore need the coefficient of $\|\nabla W^{\nu,\varepsilon}\|_{L^2}^2$ to be strictly less than $\nu$, which forces the following relation between $\nu$, $\varepsilon$ and $d_\varepsilon$:

$$
\frac{\varepsilon}{d_\varepsilon^{(1+\mu)/2}} \leq \tilde{A} \nu,
$$

where $A = 1/(4K_3)$. Recalling that $K_3 = \tilde{K}_3 \|\omega_0\|_{L^1 \cap L^\infty}$, we obtain the compatibility condition (4.1) with $A = 1/(4\tilde{K}_3)$. This condition also implies that

$$
\frac{\varepsilon^2}{\nu d_\varepsilon^{1+\mu}} \left( K_6 + K_8 + K_9 \varepsilon^3 d_\varepsilon^{(1+\mu)/2} \right) \leq K_{10} \nu,
$$

where $K_{10}$ depends only on $\omega_0$ and $\mathcal{K}$.

For any given $0 < T < \infty$, we next define

$$
K_T := C_0 e^{C_0 T},
$$

with $C_0$ as in (2.15). Then, by Grönwall’s Lemma we have that, for any $t \in [0, T]$,

$$
\|W^{\nu,\varepsilon}(t, \cdot)\|_{L^2(\Omega^\varepsilon)}^2 \leq \left[ \|W^{\nu,\varepsilon}(0, \cdot)\|_{L^2(\Omega^\varepsilon)}^2 + \frac{K_{10} \nu}{K_T} + \frac{K_7 \nu}{K_T d_\varepsilon^{1+\mu}} \right] e^{K_T t},
$$

which implies

$$
\|W^{\nu,\varepsilon}(t, \cdot)\|_{L^2(\Omega^\varepsilon)} \leq \left[ \|W^{\nu,\varepsilon}(0, \cdot)\|_{L^2(\Omega^\varepsilon)} + \sqrt{\frac{K_{10} \nu}{K_T}} + \sqrt{\frac{K_7 \nu}{K_T d_\varepsilon^{1+\mu}}} \right] e^{K_T t}.
$$

To conclude the proof, we apply the triangle inequality to obtain

$$
\|(u^{\nu,\varepsilon} - u^E)(t, \cdot)\|_{L^2(\Omega^\varepsilon)} \leq \|W^{\nu,\varepsilon}(t, \cdot)\|_{L^2(\Omega^\varepsilon)} + \|(u^{\varepsilon} - u^E)(t, \cdot)\|_{L^2(\Omega^\varepsilon)},
$$

and

$$
\|W^{\nu,\varepsilon}(0, \cdot)\|_{L^2(\Omega^\varepsilon)} \leq \|u^{\nu,\varepsilon}_0 - u_0\|_{L^2(\Omega^\varepsilon)} + \|u^{\varepsilon}(0, \cdot) - u_0\|_{L^2(\Omega^\varepsilon)}.
$$

It then follows from (2.5), (2.12), and point (1) in Proposition 2.4 that

$$
\|(u^{\varepsilon} - u^E)(t, \cdot)\|_{L^2(\Omega^\varepsilon)} \leq \|u^E\|_{L^\infty} \|1 - \phi^\varepsilon\|_{L^2} + \|h^\varepsilon\|_{L^4(A_\varepsilon)} 1_{L^4(A_\varepsilon)} \leq C \frac{\varepsilon}{d_\varepsilon^{(1+\mu)/2}} \leq C' \nu.
$$

for all $t \in [0, \infty)$, using also condition (4.12). Lastly, we combine (4.13)-(4.14) and note that
\[ \nu, \sqrt{\nu} \leq C \frac{\nu}{d^2(1+\mu)/2}, \]
for some $C > 0$, as $\nu$ and $d_\varepsilon$ are both bounded above. Estimate (4.2) then follows.

\[ \square \]

**Remark 4.2.** The dependence on $d_\varepsilon$ in estimate (4.2) comes from the contribution of $J$, more precisely from the terms:
\[ \nu \int_{A_\varepsilon} \nabla W^{\nu,\varepsilon} : \nabla \phi^{\varepsilon} \otimes u^E \, dx \quad \text{and} \quad \nu \int_{A_\varepsilon} \nabla W^{\nu,\varepsilon} : \nabla h^{\varepsilon} \, dx, \]
Without these terms, we would establish a rate of convergence of order $\sqrt{\nu}$ instead to $\sqrt{\nu} d(1+\mu)/2 \varepsilon$, as in the case of a fixed number of inclusions. Even though these integrals are taken on $A_\varepsilon$, the measure of which is small, the $L^2$-norm of $\nabla \tilde{\phi}^{\varepsilon}$ is unbounded (2.5), and hence we are not able to prove these terms are of the same order as the other terms in the energy estimate.

To improve the energy estimate, we can ask whether there exists a better choice for $\phi^{\varepsilon}$ than our construction (2.2)-(2.3). Namely, we look for a $\tilde{\phi}^{\varepsilon}$ that vanishes in $B(0,d_\varepsilon)^c$, is equal to 1 in $B(0,\varepsilon)$, and such that $\|\nabla \tilde{\phi}^{\varepsilon}\|_{L^2}$ is the smallest possible. This optimization problem has a unique solution, which is the solution of
\[ \Delta \tilde{\phi}^{\varepsilon} = 0, \quad \tilde{\phi}^{\varepsilon}|_{\partial B(0,\varepsilon)} = 1, \quad \tilde{\phi}^{\varepsilon}|_{\partial B(0,d_\varepsilon)} = 0. \]
This solution has an explicit expression of the form:
\[ \tilde{\phi}^{\varepsilon}(x) = \frac{\ln(|x|)}{\ln(d_\varepsilon)}. \]
Defining
\[ \phi^{\varepsilon}(x) := 1 - \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \tilde{\phi}^{\varepsilon}(x - z^{\varepsilon}_{i,j}), \]
it follows for $d_\varepsilon \gg \varepsilon$,
\[ \|\nabla \phi^{\varepsilon}\|_{L^2} \leq C \sqrt{\frac{n_1^n n_2^n}{|\ln \varepsilon|}} \leq \frac{C}{d^{(1+\mu)/2} \varepsilon \sqrt{|\ln \varepsilon|}}, \]
which is a slight improvement over (2.5). We cannot expect a rate better than $\frac{\nu}{d^{(1+\mu)/2} \sqrt{|\ln \varepsilon|}}$ in (4.2), because, at $\nu$ fixed, it is well known that a “strange term” appears when
\[ d^{(1+\mu)/2} \sqrt{|\ln \varepsilon|} \to C > 0. \]
(See [1] for the case $\mu = 1$ and [2] for the case $\mu = 0$.)
We note that for \( \mu = 0 \), the situation in which the distance is smaller than the critical distance (namely, \( d_1^{1/2} \sqrt{|\ln \varepsilon|} \to 0 \)) was not treated in [2] at fixed viscosity. This case was considered only when the inclusions are distributed in both directions (related to the case \( \mu = 1 \)) filling a bounded domain [1]. Indeed, this set up implies a uniform Poincaré estimate. Allaire investigated the limit of \( u^{\nu,\varepsilon}/(d_\varepsilon \sqrt{|\ln \varepsilon|}) \), which in turn implies a limit for the term:

\[
\int_{A_\varepsilon} |\nabla u^{\nu,\varepsilon}||\nabla \phi^\varepsilon| \, dx = \int_{A_\varepsilon} \left| \frac{\nabla u^{\nu,\varepsilon}}{d_\varepsilon \sqrt{|\ln \varepsilon|}} \right| \, d_\varepsilon \sqrt{|\ln \varepsilon|} |\nabla \phi^\varepsilon| \, dx.
\]

Therefore, guided by the optimal results for the Laplace and Stokes equations, it would be natural to consider \( \phi^\varepsilon \) defined as in (4.15) rather than in (2.3). However, the more natural construction of the cut-off function leads to a series of technical complications. Besides the adaptation of (2.5), we have used several times elliptic estimates on a domain of the form \( U = (-2,2)^2 \setminus \mathcal{K} \) and the fact that \( A_\varepsilon = \bigcup (z_{i,j}^\varepsilon + \varepsilon U) \). On the other hand, dilating \( B(0,d_\varepsilon) \setminus B(0,\varepsilon) \) by a factor of \( \varepsilon \) results in a set the size of which grows unboundedly as \( \varepsilon \to 0 \). To adapt the approach of Section [2] to the improved setting, we would therefore have to:

- estimate the Bogovskiǐ constant \( \tilde{C}_p \) in Lemma 2.1 in terms of \( \varepsilon \), which should be uniformly bounded thanks to [9, Theorem III.3.1];
- estimate the Poincaré constant \( K_1 \) in Lemma 2.2;
- estimate the embedding constant \( K \) in the proof of Lemma 2.3.

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