Quasi-local instant charges and its asymptotics at spatial infinity

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May 15, 2020

Abstract

We provide a new construction of charges (conserved quantities) for the gravity field in the (3+1) decomposition. The construction is based on (3+1) splitting of conformal Yano–Killing tensor. We obtain charges, defined on Cauchy surface, which are combined from components of Weyl tensor and conformal Killing vector. The relations between the conserved quantities and its classical ADM counterparts are revisited. Asymptotic behavior of the conserved quantities is described. The charges are analyzed for a particular choice of initial data, among others, Bowen–York spinning black hole.

1 Introduction

Conformal Yano–Killing (CYK) tensor is a generalization of conformal Killing covector to skew-symmetric $p$-forms. These geometrical objects, associated with so called hidden symmetries, significantly simplify a description of electromagnetism or weak gravitational perturbation. CYK forms also enable one to construct charges. For details see [18, 17] and [20] where a new method for constructing conformal Yano-Killing tensors in five-dimensional Anti-de Sitter space-time is presented. We give a brief survey on CYK forms in section 2.

The aim of presented research is to analyze the relation between charges, constructed from CYK two-form and Weyl tensor, see (2.8), and quasi-local instant charges, defined by (3.10) and (3.11). The investigations are mainly performed for Minkowski spacetime and conformally flat spacetimes, however, part of statements is generalized to de Sitter spacetimes. The analysis of (3+1) decomposition of CYK tensor in curved spacetimes is the main motivation for this project. For Minkowski spacetime, each CYK two-form is a linear combination of a wedge product of two conformal Killing co-vectors (CKV) or a Hodge dual of such product. This makes the (3+1) decomposition of CYK tensor simple. In particular, the choice of Cauchy surface $t = \text{const}$ is natural. In the case of curved spaces, the splitting of CYK form is much more complicated and the choice of Cauchy surface is much less intuitive. However, the CYK tensor decomposition should guarantee existence of Gaussian charges on a properly chosen surface. Deep understanding of the construction in the flat case is required for further analysis of curved spacetimes. This makes the paper valuable in the context of further research. In addition to that, the investigations of spacetimes with positive cosmological constant (see [1]) show that the charges (3.10) and (3.11) are important. In particular, mass in ADM approach is defined with the help of time translation generator which changes its causal nature when passing through cosmological horizon. Mass defined as an instant charge requires an existence of scaling generator on a three-dimensional Cauchy surface. Using scaling generator instead of time translation generator enables one to avoid interpretation problems considering mass in de Sitter spacetimes.

Moreover, we recall a classical result given by Ashtekar which states that the ADM angular momentum depends on supertranslations (for example see [3] and the references within). In other

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The citation [1] is the first paper from a series of papers published by the same authors.
words, the ADM formula for angular momentum is coordinate dependent. The supertranslation ambiguities can be removed by imposing stronger boundary conditions at spatial infinity. One of us has shown in [13] that the existence of asymptotic CYK tensor $Q_{ACYK}$ which fulfills the CYK equation, see (2.1) and (2.3), asymptotically

$$Q_{\lambda\kappa\sigma}(Q_{ACYK}) \approx r^{-1},$$

removes the supertranslational ambiguity.

The paper is organized as follows: A brief review of properties of CYK two-forms in four-dimensional spacetime and associated charges is given in section 2. Section 3 is devoted to (3+1) decomposition of CYK tensors. In particular, the theorem 3.1 about (3+1) charge splitting is one of the key points of our construction. The relations between ADM linear momentum, angular momentum and corresponding instant charges are presented in section 4. We provide definitions and discuss basic properties of gravitational charges, or quantities specified on the spatial hypersurface $\Sigma_t$ immersed in four-dimensional spacetime. The structure of quasi-local charges presented here can be used for any initial data ($\gamma_{ij}, K_{ij}$), for which three-dimensional metric $\gamma_{ij}$ is conformally flat or for the three-metric which approaches conformally flat metric at infinity. The presented construction is illustrated by a particular choice of examples in section 5. Comparison of conditions which are required for the asymptotic conservation of ADM quantities and instant charges is given in section 6. Used conventions and denotings are listed at the beginning of the paper (section 1.1).

1.1 Notation and conventions

For convenience we use index notation with Einstein summation convention. Signature of Lorentzian four-dimensional metric $g_{\mu\nu}$ is $(-, +, +, +)$. We distinguish five types of indices:

- small Greek letters $\{\alpha, \beta, \gamma, \ldots\}$, except $\theta$ and $\phi$, represent four-dimensional coordinates of spacetime,
- small Latin letters $\{i, j, k, \ldots\}$, except $x, y, z$ and $r$, run coordinates on three-dimensional spatial hypersurface $\Sigma_t$,
- capital Latin letters $\{A, B, C, \ldots\}$ represent angular coordinates on two-dimensional sphere,
- $\{x, y, z\}$ represents a set of Cartesian coordinates,
- $\{r, \theta, \phi\}$ represents a set of spherical coordinates.

The metric induced on a three-dimensional spatial hypersurface is denoted by $\gamma_{ij}$. $\eta_{\mu\nu}$ and $\eta_{AB}$ respectively mean a four-dimesional Minkowski metric and a metric on two-dimensional sphere of radius $r$, i.e. $\eta_{AB}dx^A dx^B = r^2d\theta^2 + r^2\sin^2\theta d\phi^2$. $\nabla$ and $D$ mean four-dimensional and three-dimensional covariant derivatives compatible with the metrices $g_{\mu\nu}$ and $\gamma_{ij}$ respectively. We will also use shortened, symbolic notation between indices in which semicolon (;) means a covariant derivative $\nabla$ for spacetime, vertical line ($|$) is a covariant derivative $D$ on three-dimensional hypersurfaces, and a double vertical line ($||$) means a covariant derivative on a two-dimensional sphere. Partial derivative $\partial$ is denoted by comma ($,$). For example, $A_{\mu\nu;\gamma} = \nabla_\gamma A_{\mu\nu}$. Symmetrization and antisymmetrization of indices $\alpha, \beta$ we write respectively as $(\alpha\beta)$ and $[\alpha\beta]$, we assume that both of these operations contain a numerical factor depending on the number of indices that include, in particular:

$$A_{(ij)} = \frac{1}{2}A_{ij} + \frac{1}{2}A_{ji},$$

$$A_{[ij]} = \frac{1}{2}A_{ij} - \frac{1}{2}A_{ji},$$

\[2\]The equation (1.1) means all the components of $Q_{\lambda\kappa\sigma}(Q_{ACYK})$ fall off like $r^{-1}$ or faster. $r$ is a radial coordinate which is well-defined in the asymptotically flat regime.
A_{ij} = A_{(ij)} + A_{[ij]}.

For the antisymmetric tensor, we accept the convention:
\[ \sqrt{|g|} \varepsilon^{12...n} = 1. \]

In the case of calculation related to the (3 + 1) decomposition, we will write three over tensors specified on the spatial hypersurface Σ and four for objects in four-dimensional spacetime. To simplify the notation, we omit the index in situations where it is clearly derived from the context. The geometric layout of units was adopted throughout the work c = G = 1.

Index of notation

- \( g_{\mu\nu} \) – four-dimensional metric tensor,
- \( \eta_{\mu\nu} \) – metric of Minkowski spacetime,
- \( R_{\alpha\beta\mu\nu} \) – Riemann tensor,
- \( R_{\alpha\beta} \) – Ricci tensor,
- \( R \) – curvature scalar,
- \( W_{\alpha\beta\mu\nu} \) – Weyl tensor,
- \( E_{\mu\nu} \) – electric part of Weyl tensor,
- \( B_{\mu\nu} \) – magnetic part of Weyl tensor,
- \( \Lambda \) – cosmological constant,
- \( T_{\mu\nu} \) – energy-momentum tensor,
- \( S(r) \) – two-dimensional sphere with radius r,
- \( d\Omega^2 = r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \).

The conformal Killing vectors (CKV):

- \( T_k \) – translation generators,
- \( R_k \) – rotation generators,
- \( K_k \) – generators of proper conformal transformations,
- \( S \) – scaling generator.

(3 + 1) decomposition:

- \( \Sigma \) – spatial hypersurface,
- \( N \) – lapse function,
- \( N^k \) – shift vector,
- \( \gamma_{ij} \) – Riemann metric on a hypersurface Σ,
- \( K_{ij} \) – tensor of the extrinsic curvature,
- \( K \) – trace of the tensor of the extrinsic curvature,
- \( P_{ij} \) – canonical ADM momentum,
- \( n^k \) – normalized normal vector,

\[ (A \wedge B)_a := \varepsilon_{abc}A_b^d B_{dc}. \]

2 Survey on CYK tensors and associated charges

Let \( Q_{\mu\nu} \) be a skew-symmetric tensor field (two-form) on a four-dimensional manifold \( M \), and let us denote by \( Q_{\lambda\kappa\sigma} \) a (three-index) tensor which is defined as follows:

\[ Q_{\lambda\kappa\sigma}(Q, g) := Q_{\lambda\kappa;\sigma} + Q_{\sigma\kappa;\lambda} - \frac{2}{n-1} \left( g_{\sigma\lambda} Q_{\kappa;\mu} + g_{\kappa}(Q_\sigma)^{\mu}_{\mu} \right). \tag{2.1} \]

The object \( Q \) has the following algebraic properties:

\[ Q_{\lambda\kappa\mu} g^{\lambda\mu} = 0 = Q_{\lambda\kappa\mu} g^{\lambda\kappa}, \quad Q_{\lambda\kappa\mu} = Q_{\mu\kappa\lambda}, \tag{2.2} \]

i.e. it is traceless and partially symmetric.

**Definition 2.1.** A skew-symmetric tensor \( Q_{\mu\nu} \) is a conformal Yano–Killing tensor (or simply CYK tensor) for the metric \( g \) iff

\[ Q_{\lambda\kappa\sigma}(Q, g) = 0. \tag{2.3} \]
CYK tensors are generalization of conformal Killing vectors to two-forms. The set of PDE equations in definition 2.1 is overdetermined. The solutions exist mainly for type D spacetimes. Properties and detailed information can be found in [15] and in the references within. In the context of further research is important the following property.

**Theorem 2.1** (Hodge duality). Let $g_{\mu\nu}$ be a metric of a four-dimensional differential manifold $M$. $\ast$ denotes Hodge duality. A skew-symmetric tensor $Q_{\mu\nu}$ is a CYK tensor of the metric $g_{\mu\nu}$ iff its dual $\ast Q_{\mu\nu}$ is a CYK tensor of this metric.

The conserved quantities are constructed from the spin-2 tensor field $W_{\alpha\beta\mu\nu}$. $W_{\alpha\beta\mu\nu}$ is skew-symmetric both in the first and the second pair of indices and to both of them the Hodge star can be applied. We denote

$$W_{\ast}\mu\nu\alpha\beta = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} W_{\rho\sigma\alpha\beta}, \quad W_{\ast\mu\nu\alpha\beta} = \frac{1}{2} W_{\mu\nu\rho\sigma} \varepsilon^{\rho\sigma}_{\alpha\beta}. \quad (2.4)$$

The symmetries of spin-2 tensor provide: $\ast W = W^{\ast}$, $\ast(W) = \ast W^{\ast} = -W$. Let $W_{\mu\nu\alpha\beta}$ be a spin-2 field and $Q_{\mu\nu}$ be a CYK tensor. Let us denote by $F$ the following two-form:

$$F_{\mu\nu}(W, Q) := W_{\mu\nu\lambda\kappa} Q^{\lambda\kappa}. \quad (2.5)$$

The following formula is satisfied:

$$\nabla_{\nu} F^{\mu\nu}(W, Q) = 0. \quad (2.6)$$

Theorem 2.1 and Eq. (2.4) enable one to observe similarly

$$\ast F_{\mu\nu} = \ast W_{\mu\nu\lambda\kappa} Q^{\lambda\kappa} = W_{\mu\nu\lambda\kappa}(\ast Q)^{\lambda\kappa}, \quad \nabla_{\nu}(\ast F)^{\mu\nu}(W, Q) = 0. \quad (2.7)$$

Let $V$ be a three-volume and $\partial V$ its boundary. Formula (2.6) implies

$$\int_{\partial V} F^{\mu\nu}(W, Q) \, d\sigma_{\mu\nu} = \int_{V} \nabla_{\nu} F^{\mu\nu}(W, Q) \, d\Sigma_{\mu} = 0. \quad (2.8)$$

In this sense $Q_{\mu\nu}$ defines a charge related to the spin-2 field $W_{\mu\nu\alpha\beta}$.

## 3 \ ((3+1) decomposition of CYK tensor for Minkowski spacetime. Instant charges

The aim of this section is to present the decomposition of CYK tensor and associated conserved quantities for Minkowski spacetime.

### 3.1 Conformal Killing fields

A vector field $X$ is a Conformal Killing (Vector-)field (CKV) if it satisfies the equation:

$$\mathcal{L}_{X} g = \lambda g, \quad (3.1)$$
where $g$ is a metric tensor, and $\mathcal{L}$ is the Lie derivative.

If the Levi–Civita connection preserves $g$, the above equation becomes equivalent to:

$$\nabla_{(\mu} X_{\nu)} = \lambda g_{\mu\nu}, \quad (3.2)$$

where $\lambda$ is an arbitrary function which is related to the conformal factor. The CKV equation (3.1) holds for any dimension of spacetime. In general case function $\lambda$ depends on the field $X$, and let $n$ be a dimension of our manifold. Trace of the equation (3.2) yields:

$$2X^\mu \cdot \nu = n\lambda \Rightarrow \lambda = \frac{2}{n} X^\mu \cdot \nu. \quad (3.3)$$

CKV remains invariant under the influence of conformal transformations. More precise information is contained in the following lemma:

**Lemma 3.1.** If $X^a$ is a conformal Killing vectorfield for the metric $g_{ij}$ and function $\lambda$, than it is also a conformal Killing field for the metric $\tilde{g}_{ij} = e^{2\Omega} g_{ij}$ and function $\tilde{\lambda} = \lambda + 2\Omega X^a$.

If $\lambda = 0$, we obtain Killing field which is an isometry generator for the metric $g$. The conformally flat three-dimensional Euclidean space has ten linearly independent conformal Killing fields: three fields corresponding to translation generators $T_k$, three corresponding to rotation generators $R_k$, three conformal transformations $K_k$ and the scaling field $S$. In the Cartesian coordinate system, expressions for the fields take the following form:

$$T_k = \frac{\partial}{\partial x^k}, \quad (3.4)$$

$$R_k = \varepsilon_k^{ij} x_i \frac{\partial}{\partial x^j}, \quad (3.5)$$

$$K_k = x_k S - \frac{1}{2} r^2 \frac{\partial}{\partial x^k}, \quad (3.6)$$

$$S = x_k \frac{\partial}{\partial x^k}, \quad (3.7)$$

where $r^2 = x^2 + y^2 + z^2$. Note that the above definitions in a natural way distinguish one point — the center of the coordinate system.

### 3.2 (3+1) decomposition of CYK tensor

In [17], the following decomposition of CYK tensor for Minkowski spacetime has been proved:

**Lemma 3.2.** Each CYK tensor in Minkowski spacetime can be expressed in the following way:

$$Q = a(t) T_0 \wedge X + b(t) \ast (T_0 \wedge Y), \quad (3.8)$$

where $X$, $Y$ are (three-dimensional) conformal Killing fields; $a(t)$, $b(t)$ are quadratic polynomials of a single indeterminate $t$.

The basis of the space of solutions for the equations, given in definition 2.1 (i.e. the basis of CYK tensors) in Minkowski spacetime is twenty-dimensional. This is a maximal possible dimension of space of CYK solutions for four-dimensional spacetime. Spin-2 tensor can be splitted into well-known gravitoelectromagnetic tensors\(^7\). Let us consider how the charge (2.8) is related with instant charges (3.10), (3.11) on Cauchy surface $\Sigma$:

\[ t = \text{const.} \]

We need to introduce the following conserved quantity:

\[ E_{\alpha\beta} := W_{\alpha\mu\beta\eta} n^\mu n^\eta, \quad B_{\alpha\beta} := W_{\ast \alpha\mu\beta\eta} n^\mu n^\eta. \quad (3.9) \]
Definition 3.1 (instant charges). Consider a given spatial hypersurface $\Sigma$ equipped with a conformally flat, Riemannian metric $\gamma$. Let $A$ be a two-dimensional closed surface, embedded in $\Sigma$. We can define the following instant charges:

$$I(E, X) := \int_A E^i_j X^j dS_i,$$  \hspace{1cm} (3.10) \\
$$I(B, X) := \int_A B^i_j X^j dS_i,$$ \hspace{1cm} (3.11)

where $X^j$ is a conformal Killing vector field (CKV).

The relation between four-dimensional CYK conserved quantity and CYK charge is as follows. Using (2.7), (2.8), (3.8), and decomposition into magnetic and electric part, we can prove the following

Theorem 3.1. Let $\Sigma_t$ be a $t = \text{const}$. Cauchy surface in Minkowski spacetime. Charge $C(W, Q)$, defined by (2.8), can be decomposed at each $\Sigma_t$ surface as follows

$$C(W, Q) = \alpha(t) I(E, X) + \beta(t) I(B, Y),$$ \hspace{1cm} (3.12)

where $I(E, X)$ and $I(B, Y)$ are instant charges (3.10) and (3.11) respectively; $\alpha(t), \beta(t)$ are quadratic polynomials of a single indeterminate $t$.

The charges defined by (3.10) and (3.11) will not depend on the choice of a two-dimensional surface if the appropriate divergence is zero. They have been proposed by Ashtekar–Hansen in [2]. If we consider two two-dimensional, oriented closed surfaces $A_1, A_2$ limiting the three-dimensional volume $V$:

$$\partial V = A_1 \cup A_2,$$

then the conformal Killing vector $X^i$ and the electrical part $E^i_j$ fulfill the following equation:

$$\int_{A_1} \sqrt{\gamma} E^i_j X^j dS_i - \int_{A_2} \sqrt{\gamma} E^i_j X^j dS_i = \int_V (\sqrt{\gamma} E^i_j X^j)_i dV, \hspace{1cm} (3.13)$$

where here $\gamma := \det \gamma_{ij}$. The divergence of vector density $\sqrt{\gamma} E^i_j X^j$ can be decomposed as follows:

$$(\sqrt{\gamma} E^i_j X^j)_i = (\sqrt{\gamma} E^i_j X^j)_i = \sqrt{\gamma} E^i_{ji} X^j + \sqrt{\gamma} E^{ijj} X^j_{ji} =$$

$$= \sqrt{\gamma} E^{i}_{ji} X^j + \sqrt{\gamma} E^{ijj} X^j_{ji} = \sqrt{\gamma} E^i_{ji} X^j + \sqrt{\gamma} E^{ijj} \frac{\lambda}{2} g_{ij} =$$

$$= \sqrt{\gamma} E^i_{ji} X^j,$$

where $E_{ij}$ is symmetric and traceless, and $X^i$ is a CKV. It enables one to formulate the following

Proposition 3.1. Consider a three-dimensional Cauchy surface with a Riemannian metric $\gamma$. Let $S$ be a two-dimensional topological sphere embedded in volume $V$. If $X$ is a conformal Killing vector for metric $\gamma$ and the gravitoelectric tensor field is divergenceless on $V$, then the instant charge

$$I(E, X) = \int_S E^i_j X^j dS_i,$$

does not depend on the choice of integration surface $A \subset V$. The analogue result holds for $I(B, X)$, defined by (3.11).

3.3 Gravitoelectromagnetic tensors in terms of initial data

In [9] the electromagnetic parts of Weyl tensor are expressed in terms of initial data $(\gamma_{ij}, K_{ij})$ on the Cauchy surface $\Sigma$. The results are the following:
Theorem 3.2. If spacetime fulfills the Einstein vacuum equations with a cosmological constant:

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \quad (3.15) \]

then the electrical and magnetic parts of the Weyl tensor are expressed by the initial data \((\gamma_{ij}, K_{ij})\) on the three-dimensional spatial hypersurface \(\Sigma\) as follows:

\[ E_{ij} = -\frac{3}{2} R_{ij} - K_{ij} K^k_k + \frac{2}{3} \Lambda \gamma_{ij}, \quad (3.16) \]
\[ B_{ij} = \varepsilon^{ls} j D_s K_d, \quad (3.17) \]

where \(R_{ij}\) is a Ricci tensor of a three-dimensional metric, \(K_{ij}\) is a tensor of extrinsic curvature, and \(K = K^i_i\) is a trace of extrinsic curvature.

The proof is mainly based on (3 + 1) decomposition of Weyl tensor and geometrical identities between four-dimensional and three-dimensional curvature tensors. Similar decomposition can be done for the divergence of the electric (magnetic) part of Weyl tensor.

Theorem 3.3. If spacetime fulfills the Einstein vacuum equations with a cosmological constant, then the three-dimensional covariant divergence of the electrical part \(E\) and the magnetic part \(B\) of Weyl tensor is expressed as follows:

\[ E^i_{j|i} = (K \wedge B)_j, \quad (3.18) \]
\[ B^i_{j|i} = -(K \wedge E)_j, \quad (3.19) \]

where \(\wedge\) is an operation defined for two symmetric tensors \(A\) and \(B\) as

\[ (A \wedge B)_a := \varepsilon_{abc} A^b_d B^{dc}. \quad (3.20) \]

The theorem simplifies examination of conservancy of the charges (3.10) and (3.11). It can be especially useful in situations, where the “pure” electrical (with zero magnetic part) or “pure” magnetic (with zero electrical part) is analyzed.

4 ADM mass, linear momentum, angular momentum and corresponding instant charges

4.1 Physical interpretation of instant charges

Let us observe the relations between traditional quantities (e.g. ADM or Komar formula) and instant charges. More precisely, we have the following table:

| KV | Charges       | CKV | 
|----|---------------|-----|
| \(I_0\) | \(I(E, S)\)  | \(S\) | (1) energy (mass) |
| \(I_k\) | \(I(B, R_k)\) | \(R_k\) | (3) linear momentum |
| \(L_{ki}\) | \(I(B, K_k)\) | \(K_k\) | (3) angular momentum |
| \(L_{0k}\) | \(I(E, K_k)\) | \(K_k\) | (3) center of mass |

Other quantities: dual mass, dual momentum, linear acceleration, and angular acceleration are usually vanishing. However, some parameters in Einstein metrics can be interpreted as topological charges, e.g. dual mass appears in Taub-NUT solution ([16], [14]) and dual momentum in Demiański metrics ([13], [10]). We discuss below the relation between ADM quantities and instant charges.

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*See the discussion below (3.14).*
4.1.1 Mass

Mass is one of the most important quantities which characterize a physical system. However, some subtleties arise and they are analyzed in section 5. We claim that the following instant charge

$$\mathcal{M} := \frac{1}{8\pi} I(E, S)$$

(4.1)

describes quasi-local mass. $S$ is CKV related with scaling generator. If we consider asymptotically flat space with two-dimensional foliation of topological spheres $S(R)$, which are parameterized by radius $R$, then the ADM mass is given by $M_{ADM} = \frac{1}{8\pi} \lim_{R \to \infty} \mathcal{M}(S(R))$.

4.1.2 Linear momentum

**Theorem 4.1.** Let $\Sigma$ be a flat, three-dimensional spatial hypersurface immersed in the space-time which satisfies Einstein vacuum equations. Assuming that the three-dimensional covariant divergence of the magnetic part disappears:

$$B^i_{j|i} = 0,$$

(4.2)

then for any two-dimensional, closed surface $A$ immersed in $\Sigma$ holds:

$$\int_A B^j_i R^i_k dS = \int_A P^i_j T^j_k dS,$$

(4.3)

where $P^i_j$ is (canonical) ADM momentum, $R_k$ is a rotation generator around the axis $k$, and $T_k$ is a translation generator along the axis $k$.

**Proof:**

The surface $\Sigma$ is assumed to be flat, so appropriate conformal Killing fields exist. First we show that an integral over any surface $A$ can be converted into an integral over a two-dimensional sphere.

The ADM momentum is expressed by the extrinsic curvature:

$$P^i_j = -K^i_{j|i} + K^i_j.$$

(4.4)

Einstein vacuum equations are satisfied, so in particular vacuum vector constraint:

$$K^i_{j|i} - K^i_{j|i} = 0 \Rightarrow P^i_{j|i} = 0.$$  

(4.5)

The divergence in the integral in the equation (4.3) can be reformulated analogically to the equation (3.14). Assuming (4.2), we obtain that the divergence of the term containing the magnetic part is zero. For the integral on the right hand side of the equation (4.3), we have:

$$\left(\sqrt{\gamma} P^i_j T^j_k\right)_i = \left(\sqrt{\gamma} P^i_j T^j_k\right)_{i|i} = \sqrt{\gamma} P^i_{j|i} T^j_k + \sqrt{\gamma} P^{(ij)}(T_k)_{(j|i)} = 0.$$ 

(4.6)

After applying the product rule, the first term in the above equation is zero from the equation (4.5), and in the second term we can add symmetrization, the translation generator fulfills the Killing equation, hence the symmetrized covariant derivative of $T$ vanishes.

We have shown that the divergences of both integrands in equation (4.3) are zero, using the Stokes theorem, the volume term has no contribution, which proves formulae (4.3). Replacing the integral on any surface $A$ by an integral on a two-dimensional sphere $S(r)$ is justified.

The left hand side of (4.3) is:

$$\int_{S(r)} B^j_i R^i_k dS = \int_{S(r)} \lambda B^j_k \phi,$$

(4.7)

9For conformally flat space, with the metric in the form $\psi \left( dR^2 + R^2 d\Omega^2 \right)$, the scaling generator has a form $S = R \partial_R$. 


where $\lambda = r^2 \sin \theta$. Using (4.3), we have
\begin{align*}
B^r \phi &= \varepsilon^{rij} K_{\phi ij} \\
&= \varepsilon^{rAB} K_{\phi A B} r^2 \varepsilon^{DC}(\cos \theta)_C \\
&= \varepsilon^{AB} K_{DA|B} r^2 \varepsilon^{DC}(\cos \theta)_C ,
\end{align*}
where the convention for two-dimensional Levi–Civita tensor is $r^2 \sin \theta \epsilon = 1$. A three-dimensional covariant derivative can be decomposed into:
\begin{equation}
K_{DA|B} = K_{DA,B} - \Gamma^m_{DB} K_{mA} - \Gamma^m_{AB} K_{Dm} = K_{DA||B} + \frac{1}{r} \eta_{AB} K_{Dr} + \frac{1}{r} \eta_{DB} K_{rA} , \tag{4.9}
\end{equation}
where we used the fact that $\Gamma^r_{AB} = -\frac{1}{r} \eta_{AB}$, true for a covariant derivative of a flat three-dimensional space, and $\eta_{AB}$ is a metric induced on a sphere with the radius $r$, it gives:
\begin{align*}
B^r \phi &= \varepsilon^{AB} K_{DA||B} r^2 \varepsilon^{DC}(\cos \theta)_C + r \varepsilon^{A}_D K_{rA} \varepsilon^{DC}(\cos \theta)_C \\
&= \varepsilon^{AB} K_{DA||B} r^2 \varepsilon^{DC}(\cos \theta)_C + \frac{1}{r} K_{r\theta} \sin \theta , \tag{4.10}
\end{align*}
where the identity $\varepsilon^{A}_D \varepsilon^{DC} = -\eta^{AC}$ holds.
\begin{align*}
\int_{S(r)} \lambda B^r \phi &= \int_{S(r)} \lambda \varepsilon^{AB} K_{DA||B} r^2 \varepsilon^{DC}(\cos \theta)_C + \int_{S(r)} \lambda^2 K_{r\theta} \sin \theta = \\
&= - \int_{S(r)} \lambda \varepsilon^{AB} K_{DA||B} r^2 \varepsilon^{DC}(\cos \theta)_C||CB + \int_{S(r)} \lambda^2 K_{r\theta} \sin \theta = \\
&= \int_{S(r)} \lambda \varepsilon^{AB} K_{DA}\varepsilon^{DC} \eta_{CB} \cos \theta + \int_{S(r)} \lambda^2 K_{r\theta} \sin \theta = \\
&= \int_{S(r)} \lambda K_{DA} \eta^{DA} \cos \theta + \int_{S(r)} \lambda^2 K_{r\theta} \sin \theta , \tag{4.11}
\end{align*}
where we have used the identity $r^2(\cos \theta)||CB = -\eta_{CB} \cos \theta$.

The field $T_z$ in the spherical coordinates:
\begin{equation}
T_z = \partial_z = \cos \theta \partial_r - \frac{\sin \theta}{r} \partial_\theta . \tag{4.12}
\end{equation}

Therefore, the right hand side of the thesis (4.3) takes the form:
\begin{align*}
&\int_{S(r)} P^i j T^j_2 dS_i = \int_{S(r)} (K^i j - \delta^i j K) T^j_2 dS_i = \\
&= \int_{S(r)} \lambda (K^r - K) \cos \theta - \lambda \frac{1}{r} \sin \theta K^r_\theta = \\
&= \int_{S(r)} -K^{AB} \eta_{AB} \lambda \cos \theta - \lambda \frac{1}{r} \sin \theta K^r_\theta = - \int_{S(r)} \lambda B^r \phi , \tag{4.13}
\end{align*}
where the last equality comes from the comparison with the right hand side of (4.11).

\subsection{Angular momentum}

\textbf{Theorem 4.2.} Let $\Sigma$ be a flat, three-dimensional spatial hypersurface immersed in the spacetime which satisfies Einstein vacuum equations. Assuming that the following charges vanish:
\begin{equation}
I(B, T_y) = I(B, T_y) = I(B, T_z) = 0 \tag{4.14}
\end{equation}
and the three-dimensional covariant divergence of the magnetic part disappears:
\begin{equation}
B^i \phi = 0 , \tag{4.15}
\end{equation}
then for any two-dimensional, closed surface $A$ immersed in $\Sigma$ holds:

$$\int_A B^i_j K^j_k dS_i = \int_A P^i_j R^j_k dS_i,$$

(4.16)

where $P^i_j$ is the ADM momentum, $K_k$ is the generator of proper conformal transformations in the direction of $k$, and $R_k$ is a rotation generator around the axis $k$.

Proof:

Analogically to the proof of theorem 4.1, it can be shown that the divergences of integrands in the thesis (4.16) disappear, that justifies the proof for integrals on two-dimensional spheres. By assumption (4.14), we can deduce that the integral from the contraction of the magnetic part with any vector with constant coefficients in the Cartesian system is zero (because $T_k = \partial_k$).

Using this observation, we have:

$$\int_{S(r)} (r) B^i_j (K^j_k + A^j) dS_i = \int_{S(r)} B^i_j K^j_k dS_i,$$

(4.17)

where $A^j$ are the coordinates of a constant vector in the Cartesian system. Any such a vector can be written in a spherical system in the following way:

$$A^i \partial_i = A^i x^i_r \partial_r + r \left( A^i x^i_r \right)^B \partial_B.$$

(4.18)

We denote $u := (A^i x^i_r)/r$, then:

$$A^i \partial_i = u \partial_r + r(u)^B \partial_B.$$

(4.19)

Now we choose $u$ to simplify the calculation of the integral (4.17). The generators of the conformal transformations in the spherical system read:

$$K_x = \frac{1}{2} r^2 \cos \phi \sin \theta \partial_r - \frac{1}{2} r \cos \phi \cos \theta \partial_\theta + \frac{1}{2} \frac{r}{\sin \theta} \partial_\phi,$$

(4.20)

$$K_y = \frac{1}{2} r^2 \sin \phi \sin \theta \partial_r - \frac{1}{2} r \sin \phi \cos \theta \partial_\theta - \frac{1}{2} \frac{r}{\sin \theta} \partial_\phi,$$

(4.21)

$$K_z = \frac{1}{2} r^2 \cos \theta \partial_r + \frac{1}{2} r \sin \theta \partial_\theta.$$

(4.22)

Let us choose the function $u$:

For $K_x$ we choose $u_x = \frac{1}{2} r^2 \cos \phi \sin \theta$.

For $K_y$ we choose $u_y = \frac{1}{2} r^2 \sin \phi \sin \theta$.

For $K_z$ we choose $u_z = \frac{1}{2} r^2 \cos \theta$.

We will denote new “corrected” vectors by $\tilde{K}_k := K_k + (A_k)^i \partial_i$, and we obtain:

$$\tilde{K}_x = r^2 \cos \phi \sin \theta \partial_r = r^2 \frac{\partial r}{\partial x} \partial_r,$$

(4.23)

$$\tilde{K}_y = r^2 \sin \phi \sin \theta \partial_r = r^2 \frac{\partial r}{\partial y} \partial_r,$$

(4.24)

$$\tilde{K}_z = r^2 \cos \theta \partial_r = r^2 \frac{\partial r}{\partial z} \partial_r.$$

(4.25)

The vectors $\tilde{K}_k$ now have only the component in the radial direction.

Without loss of generality we consider (4.16) for $k$ equal to $z$. Coordinate system that the $z$ axis is turned in the direction which is invariant under the rotation generator $R_k$ always can be chosen.

In the spherical system, the metric on $\Sigma$ takes the form:

$$\gamma_{ij} = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2,$$

(4.26)
and the metric induced on the spheres:

\[ \eta_{AB} = r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \]  \hfill (4.27)

Note that the rotation generator field around the z axis such that \( \mathcal{R}_z = \partial_\phi \) can be written as:

\[ \partial_\phi = r^2 \varepsilon^{AB} (\cos \theta)_B \partial_A, \]  \hfill (4.28)

where \( \varepsilon^{AB} \) is an antisymmetric tensor on the sphere, by definition:

\[ \sqrt{\eta} \varepsilon^\theta \phi = r^2 \sin \theta \varepsilon^\theta \phi = 1. \]  \hfill (4.29)

We denote: \( \lambda = \sqrt{\gamma} \) and reformulate the right hand side of (4.16):

\[ \int_{S(r)} P^{ij} \mathcal{R}^j_z dS_i = - \int_{S(r)} \lambda K^r_A r^2 \varepsilon^{AB} (\cos \theta)_B = \int_{S(r)} r^2 (\lambda K^r_A \varepsilon^{AB})_{||B} \cos \theta = \int_{S(r)} r^2 \lambda \varepsilon^{AB} K^{rA}_{||B} \cos \theta. \]  \hfill (4.30)

For the left hand side of the thesis:

\[ \int_{S(r)} B^{ij} \mathcal{K}^j_z dS_i = \int_{S(r)} \lambda B^r^j \mathcal{K}^j_z = \int_{S(r)} \lambda \varepsilon^{lsr} K^j_{jl} \mathcal{K}^j_z = \int_{S(r)} \lambda \varepsilon^{ABr} K^{rA}_{||B} r^2 \cos \theta = \int_{S(r)} r^2 \lambda \varepsilon^{AB} K^{rA}_{||B} \cos \theta. \]  \hfill (4.31)

By comparing the formulae (4.30) and (4.31) we get the thesis. \( \square \)

5 Examples

We apply the concepts introduced in section 3 to the analysis of conserved quantities for particular choices of initial data. The aim of this section is to perform a detailed discussion of two of the most important parameters which characterize a black hole solution: mass and angular momentum. We compare our results with classical ADM approach and present methods of obtaining quasi-local quantities (like quasi-local mass).

First two examples contain initial data on particularly chosen surfaces in Schwarzschild–de Sitter spacetime. We will consider two families of foliations with spatial hypersurfaces \( \Sigma \). The first of them, denoted \( \Sigma^t \) corresponds to the surfaces of constant time (i.e. the coordinate appearing in the standard form of Schwarzschild metric) \( t = \text{const.} \), the second (denoted \( \Sigma^p \)) will be a hypersurface foliation with a flat inner geometry. Next example belongs to Bowen–York initial data type. We discuss charges for spinning black hole.

5.1 The constant time hypersurfaces

Consider the Kottler metric with the standard variables \( (t, r, \theta, \phi) \), whose linear element is given by the formula:

\[ ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \]  \hfill (5.1)

where \( f(r) = 1 - \frac{2m}{r} - \frac{s^2}{r^2} \), and we assume that \( f(r) > 0 \). Parameter \( \frac{s^2}{r^2} \) is a scaled cosmological constant \( \frac{s^2}{r^2} = \frac{\Lambda}{3} \), where \( s = \pm 1 \). The foliation with spatial hypersurfaces \( \Sigma_t : t = \text{const.} \) is examined. Three-dimensional Riemannian metric induced on \( \Sigma_t \) has the following form:

\[ ds^2_3 = \frac{dr^2}{f(r)} + r^2 d\Omega. \]  \hfill (5.2)
The Cotton tensor for the metric (5.2) vanishes, hence (in three dimensions) it is a necessary and sufficient condition for conformal flatness. Using one-dimensional coordinate transformation

$$\log R = \int \frac{dr}{r \sqrt{1 - \frac{2m}{r} - \frac{s}{r^2}}}.$$  (5.3)

we can transform (5.2) into explicitly conformally flat form

$$ds^2 = \frac{r^2 (\log R)}{R^2} \left[ dR^2 + R^2 d\Omega \right].$$  (5.4)

$I(E, S)$ is our conserved quantity which has an interpretation as a mass. $S$ is a CKV associated with conformal rescaling. In $(R, \theta, \phi)$ coordinates, in which the metric is conformally flat, $S$ has the form: $S = R \partial_R$. Transforming into $(r, \theta, \phi)$ coordinates, we have

$$S = r \sqrt{f(r)} \partial_r.$$  (5.5)

The chosen data is time-symmetric. It means that the second fundamental form is identically equal to zero:

$$K_{ij} = 0.$$  (5.6)

The theorems 3.2 and 3.3 give

$$E_{ij} = -\frac{3}{3} R_{ij} + \frac{2}{3} \Lambda \gamma_{ij},$$

$$B_{ij} = 0,$$

$$E_{ij}^i = 0.$$  (5.7)  (5.8)  (5.9)

According to equation (3.14) and the comment below, the result (5.9) guarantees that the quasi-local mass does not depend on the choice of integration surface\footnote{The two-dimensional surface has to be homotopic to a round sphere – an orbit of rotational symmetry of the Schwarzschild–de Sitter solution. In particular, it is the same for all $r = \text{const.}$ surfaces.}. The non-zero components of electric part of Weyl tensor do not depend on cosmological constant

$$E_{rr} = \frac{2m}{r^3},$$

$$E_{\theta \theta} = -\frac{m}{r^3},$$

$$E_{\phi \phi} = -\frac{m}{r^3}.$$  (5.10)  (5.11)  (5.12)

The equations (5.5) and (5.10) enable one to calculate the quasi local mass $M := \frac{1}{8\pi} I(E, S)$:

$$M = \frac{1}{8\pi} \int_{S(r)} E''_{r} X'' \sqrt{\gamma} d\theta d\phi = \frac{1}{8\pi} \int_{S(r)} \frac{2m}{r^4} r \sqrt{f(r)} \frac{1}{\sqrt{f(r)}} r^2 \sin \theta d\theta d\phi = m.$$  (5.13)

The obtained result is comparable with the ADM type quasi-local mass. We will use formulas from [7] to calculate ADM type quasi-local mass relative to any vector field $X$ and any reference space. Let $\gamma_{ij}$ denote the metric at $\Sigma_s$, the reference spacetime $\beta_{ij}$ is the de Sitter spacetime. Assuming the above, the ADM type quasi-local mass is expressed by the formula:

$$M_{\text{ADM}} = \frac{1}{16\pi} \int_{S(r)} \left( \mathbb{U}^{r} + \mathbb{V}^{r} \right),$$  (5.14)

where:

$$\mathbb{U}^{i}(V) = 2 \sqrt{\text{det} \gamma} \left[ V_{\gamma^{[i}} \gamma^{j]} \mathcal{D}_{j} \gamma_{kl} + D^{i} V_{\gamma^{jk} e_{jk}} \right],$$  (5.15)
\[ \mathcal{V}^l(Y) = 2 \sqrt{\det \gamma} \left[ (P^l_k - \bar{P}^l_k)Y^k - \frac{1}{2} Y^l \bar{P}^{mn} e_{mn} + \frac{1}{2} Y^k \bar{P}^l_k \beta^{mn} e_{mn} \right], \quad (5.16) \]

\[ e_{ij} := \gamma_{ij} - \beta_{ij}, \quad (5.17) \]

\[ P^{lk} := \gamma^{lk} \text{tr} K - \overline{K}^{lk}, \quad \text{tr} \gamma K := \gamma^{lk} \overline{K}^{lk}, \quad (5.18) \]

\[ \overline{P}^{lk} := \beta^{lk} \overline{\text{tr} \beta K} - \overline{K}^{lk}, \quad \text{tr} \beta \overline{K} := \beta^{lk} \overline{K}^{lk}. \quad (5.19) \]

Form of the Hamiltonian field \( X \):

\[ X = V n^\mu \partial_\mu + Y^k \partial_k = V N^\partial_0 + \left( Y^k - \frac{V}{N} N^k \right) \partial_k. \quad (5.20) \]

For mass (energy), we use \( X = \partial_0 \), so \( V = N \) and \( Y^k = N^k \). The metric \( \gamma \) is given by the equation (5.2). The reference frame is simply given by

\[ \beta_{ij} = \text{diag} \left( \frac{1}{1 - \frac{2m}{r} - \frac{\Lambda}{r^2}}, r^2, r^2 \sin^2 \theta \right), \quad (5.21) \]

hence:

\[ e_{ij} = \text{diag} \left( \frac{1}{1 - \frac{2m}{r} - \frac{\Lambda}{r^2}} - \frac{1}{1 - \frac{2m}{r} - \frac{\Lambda}{r^2}}, 0, 0 \right). \quad (5.22) \]

The extrinsic curvature is zero, therefore \( P^k_l = \bar{P}^k_l = 0 \Rightarrow \mathcal{V}^l(Y) = 0 \).

The expression for ADM type quasi-local mass is reduced to:

\[ M_{ADM} = \frac{1}{8\pi} \int_{S(r)} \sqrt{\det \gamma} \left[ N \gamma^{[k} \gamma^{j]l} \bar{D}_j \gamma_{kl} + D^{[r} N \gamma^{j]k} e_{jk} \right], \quad (5.23) \]

which finally gives

\[ M_{ADM} = m. \quad (5.24) \]

Thus, the ADM type quasi-local mass and the “electromagnetic” quasi-local mass \( M \) are equal and they do not depend on the cosmological constant \( \Lambda = \frac{3\Lambda}{r^2} \).

### 5.2 Flat hypersurfaces

As in the previous section, we start with the Kottler metric:

\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \quad (5.25) \]

for \( f = 1 - \frac{2m}{r} - \frac{\Lambda}{r^2}, \ f > 0, \ m \geq 0 \).

We will examine the foliation with the hypersurfaces \( \Sigma_t \), on which induced three-dimensional metric is flat. Using the following time transformation:

\[ t_p := t - \int \frac{\sqrt{1 - f(r)}}{f(r)} \, dr, \quad (5.26) \]

we obtain the metric in the Painleve–Gullstrand form:

\[ ds^2 = -f(r)dt_p^2 - 2\sqrt{1 - f(r)}dr dt_p + dr^2 + r^2 d\Omega^2. \quad (5.27) \]
The lapse $N$ and the shift vector $N^i$:

$$N = 1,$$

$$N_r = -\sqrt{\frac{2m}{r} + \frac{s}{r^2} r^2}, \quad N_\theta = 0, \quad N_\phi = 0,$$ (5.28)

enable one to calculate the extrinsic curvature from the formula

$$K_{ij} = \frac{1}{2N} (N_{ij} + N_{j|i} - \partial_t g_{ij}).$$ (5.30)

Non-zero components of the extrinsic curvature are:

$$K_{rr} = \frac{m - \frac{s}{r^3} r^3}{\sqrt{2mr^3 + \frac{s}{r^2} r^6}},$$ (5.31)

$$K_{\theta\theta} = -\sqrt{2mr^3 + \frac{s}{r^2} r^6},$$ (5.32)

$$K_{\phi\phi} = -\sqrt{2mr^3 + \frac{s}{r^2} r^6} \sin^2 \theta.$$ (5.33)

The trace of extrinsic curvature is equal to:

$$K = K_{ij} \gamma^{ij} = -\frac{3(m + \frac{s}{r^3} r^3)}{\sqrt{2mr^3 + \frac{s}{r^2} r^6}}.$$ (5.34)

The electrical part of Weyl tensor, (3.16), reduces to

$$E_{ij} = -KK_{ij} + K_{ik}K_{kj} + \frac{2}{3} \Lambda \gamma_{ij} = -KK_{ij} + K_{ik}K_{kj} + \frac{2}{3} \frac{s}{r^2} \gamma_{ij}.$$ (5.35)

We obtain the following non-zero components of the electrical part of Weyl tensor:

$$E_{rr} = \frac{2m}{r^3},$$ (5.36)

$$E_{\theta\theta} = -\frac{m}{r},$$ (5.37)

$$E_{\phi\phi} = -\frac{m}{r} \sin^2 \theta.$$ (5.38)

The magnetic part is zero.

Because the metric induced on $\Sigma_p$ is flat, we have a full set of ten conformal Killing vectors. Note that vanishing magnetic part causes that the three-dimensional covariant divergence of the electrical part vanishes (theorem 3.3), and therefore the “electric” charges do not depend on the two-dimensional integration surface. The charge responsible for the mass:

$$M = \frac{1}{8\pi} \int_{S(r)} E_{ij} S^i S^j dS_i = \frac{1}{8\pi} \int_{S(r)} E^r r^3 \sin \theta d\theta d\phi = m.$$ (5.39)

We compare the obtained result with the ADM type quasi-local mass. As in the previous subsection, we define:

$$e_{ij} := \gamma_{ij} - \beta_{ij},$$ (5.40)

$$\Pi^j(V) = 2\sqrt{|\det \gamma|} \left[ V^l \gamma_{lj} \gamma^j \gamma^i D_l \gamma_{ki} + D^i V^j \gamma^{jk} e_{jk} \right],$$ (5.41)
\[ \forall^l(Y) = 2\sqrt{\det \gamma \left( (P^l_k - \bar{P}^l_k) Y^k - \frac{1}{2} Y^l \bar{P}^{mn} e_{mn} + \frac{1}{2} Y^k \bar{P}^l_k \beta^{mn} e_{mn} \right)}, \quad (5.42) \]

\[ X = V n^\mu \partial_\mu + Y^k \partial_k = \frac{V}{N} \partial_0 + \left( Y^k - \frac{V}{N} N^k \right) \partial_k. \quad (5.43) \]

We obtain the mass, therefore \( X = \partial_0 \Rightarrow V = N \) and \( Y^k = N^k \). \( \forall^l \) is zero, because in the case of foliations by flat hypersurfaces \( \gamma_{ij} = \beta_{ij} \), where \( e_{ij} = 0 \), and in the first part the covariant derivative \( \bar{D} \) can be converted to \( D \).

The non-zero components of the canonical ADM momentum \( (P^i_j = \delta^i_j K - K^i_j) \) are:

\[ P^r_r = -\frac{2\sqrt{2m + \frac{s}{r^3}}}{r^{3/2}}, \quad (5.44) \]

\[ P^\theta_\theta = -\frac{m + \frac{2s}{r^3}}{r^{3/2}\sqrt{2m + \frac{s}{r^3}}}, \quad (5.45) \]

\[ P^\phi_\phi = -\frac{m + \frac{2s}{r^3}}{r^{3/2}\sqrt{2m + \frac{s}{r^3}}}. \quad (5.46) \]

The background ADM momentum \( \bar{P}^i_j \) can be obtained from the equations (5.44)–(5.46) by setting \( m = 0 \), hence, for \( s = +1 \), we get:

\[ \bar{P}^r_r = \bar{P}^\theta_\theta = \bar{P}^\phi_\phi = -\frac{2}{l}. \quad (5.47) \]

The only non-omitting element in the expression for ADM type quasi-local mass is:

\[ M_{\text{ADM}} = \frac{1}{16\pi} \int_{S(r)} 2\sqrt{\gamma} (P^r_r - \bar{P}^r_r) N^r d\theta d\phi. \quad (5.48) \]

Explicitly,

\[ M_{\text{ADM}} = 2m + \frac{r^3}{l^2} \sqrt{\frac{r^3}{l^2} \left( 2m + \frac{r^3}{l^2} \right)}. \quad (5.49) \]

Note that for Schwarzschild spacetime (\( \Lambda = 0, \ l \to \infty \)) we get \( M_{\text{ADM}} = 2m \), which is twice as much as the mass parameter \( m \) in the Kottler metric. This is caused by too slow fall off in \( r \) of the ADM momentum. In the case of foliation with surfaces of flat internal geometry (Painleve–Gullstrand foliation), the canonical ADM momentum tensor behaves like \( r^{-3/2} \). The usually assumed assumption for the canonical ADM momentum (see classical results in section 6.1) is the fall off like \( r^{-3/2-\varepsilon} \), where \( \varepsilon \) is strictly positive. The result (5.39) shows that the “electromagnetic” mass (at least in this case) has better properties, because after dividing by the normalizing factor \( 8\pi \) accurately reproduces the parameter \( m \) occurring in the metric. It is independent of the radius \( r \) and the cosmological constant \( \Lambda \).

### 5.3 Bowen–York initial data

In this section we discuss an initial data which is originally done by Bowen and York [4] with the help of conformal methods. However, we use an approach to the data which is given in [8]. Physically relevant initial data \( (\gamma_{ij}, K_{ij}) \) and conformally related data \( \bar{h}_{ij}, \bar{K}^{ij} \) is described by the relations

\[ \gamma_{ij} = \varphi^4 \bar{h}_{ij}, \quad (5.50) \]

\[ K^{ij} = \varphi^{-10} \bar{K}^{ij}. \quad (5.51) \]
Moreover, $\tilde{K}_{ij}$ is a symmetric, trace-free tensor which fulfills

$$D_i \tilde{K}^{ij} = 0,$$

(5.52)

where $D_a$ is the covariant derivative with respect to $\tilde{h}_{ij}$. The Hamiltonian constraint implies the following equation for the conformal factor $\varphi$:

$$L_{\tilde{h}} \varphi = -\frac{\tilde{K}^{ij} \tilde{K}_{ij}}{8 \varphi^2},$$

(5.53)

where $L_{\tilde{h}} = D^i D_i - \tilde{R}/8$, $\tilde{R}$ is the Ricci scalar of the metric $\tilde{h}_{ij}$ and the indices are moved with $\tilde{h}_{ij}$. If the conditions (5.52) and (5.53) hold then the physically relevant initial data $(\gamma_{ij}, K_{ij})$ satisfy vacuum constraint equations without cosmological constant.

Remarkable simplifications on (5.52) and (5.53) occur, when $\tilde{h}_{ij}$ has a Killing vector $\eta^a$. We will assume that $\eta^a$ is hypersurface orthogonal and we define $\eta$ by $\eta := \eta^a \eta^b \tilde{h}_{ij}$. Firstly we analyze the momentum constraint (5.52). Consider the following vector field

$$S^i = \frac{1}{\eta} \epsilon^{ijk} \eta_j D_k \omega,$$

(5.54)

where $\mathcal{L}_{\eta}$ is the Lie derivative with respect $\eta^a$ and $\epsilon_{ijk}$ is the volume element of $\tilde{h}_{ij}$. The scalar function $\omega$ is arbitrary. In particular it does not depend on the metric $\tilde{h}_{ij}$. It follows that $S^a$ satisfies

$$\mathcal{L}_{\eta} S^i = 0, \quad S^j \eta_j = 0, \quad D_k S^k = 0.$$  

(5.55)

Using the Killing equation $D_i (\eta_j) = 0$, the fact that $\eta^a$ is hypersurface orthogonal and equations (5.54) we conclude that the tensor

$$\tilde{K}_{ij} = \frac{2}{\eta} S^{(i} \eta^{j)}$$

(5.56)

is trace-free and satisfies (5.52). The square of $\tilde{K}_{ij}$ can be written in terms of $\omega$:

$$\tilde{K}_{ij} \tilde{K}_{ij} = \frac{2 D_k \omega D^k \omega}{\eta^2}.$$  

(5.57)

### 5.4 Example: Bowen–York spinning black hole

This example originally has been given in [4]. It is a Bowen–York type initial date with a Killing symmetry. The data is characterized by a flat conformal three-metric

$$\tilde{h} = dR^2 + R^2 d\theta^2 + R^2 \sin^2 \theta d\phi^2$$

(5.58)

with the following generating function $\omega$

$$\omega_{BY} = J(3 \cos \theta - \cos^3 \theta).$$

(5.59)

The Killing vector used in data construction is

$$\eta = \frac{\partial}{\partial \phi}.$$  

(5.60)

Using the procedure described in section 5.3, we obtain that the only non-vanishing component of $\tilde{K}_{ij}$ is the following

$$\tilde{K}_{R \phi} = \frac{-3 J \sin^2 \theta}{R^2}.$$  

(5.61)

\footnote{It satisfies $D_i \eta_j = -\eta_i D_j \ln \eta$.}
Let us observe that the corresponding ADM momentum tensor density defined as follows:

\[
\sqrt{\det \gamma_{kl}} P^i_j := \sqrt{\det \gamma_{kl}} (\delta^i_j K - K^i_j)
\]

has the only one non-vanishing component \(\sqrt{\det \gamma_{ij}} P^R_\varphi = 3J \sin^3 \theta\) which gives quasi-locally the ADM angular momentum \(J = \frac{1}{8\pi} \int_{S(R)} \sqrt{\det \gamma_{ij}} P^R_\varphi\) because of the symmetry \(\eta = \frac{\partial}{\partial \varphi}\).

Remark: Let us notice that the vector constraint (4.5), maximal surface condition \(K = 0\) and CVF lead to conserved charge in a similar way as in Definition 3.1. However, for BY data almost all CVFs give vanishing integrals except vectorfield \(\partial_\varphi\) corresponding to angular momentum \(J\).

Equations (5.53) and (5.57) enable one to obtain the conformal factor \(\varphi\) in terms of the derivatives of \(\omega\):

\[
\Delta \varphi = -\frac{(\partial_\theta \omega)^2}{4R^6 \sin^2 \theta \varphi^7},
\]

where \(\Delta\) is the flat Laplacian corresponding to Euclidean metric (5.58). The following boundary conditions are used:

\[
\lim_{R \to \infty} \varphi = 1, \quad \lim_{R \to 0} R \varphi = \frac{M}{2},
\]

where \(M\) is a positive constant called bare mass. Below we give an asymptotic behavior of the solution for large \(R\). The first few terms of the asymptotic series for the conformal factor \(\varphi\) are the following

\[
\varphi_{BY} = 1 + \frac{M}{2R} + \frac{p(\theta)}{R^4} + O \left( \frac{1}{R^5} \right),
\]

where

\[
p(\theta) = \frac{J^2}{8} \left( 3 \cos^2 \theta^2 - 1 \right) - \frac{J^2}{8}.
\]

The charges (3.10) and (3.11) require gravitoelectromagnetic tensor density of the considered solution. We denote

\[
E^i_j = \sqrt{\det \gamma_{kl}} E^i_j \quad B^i_j = \sqrt{\det \gamma_{kl}} B^i_j
\]

The non-vanishing components of electric part of Weyl tensor density are:

\[
E^R_R = \frac{2M \sin \theta}{R} + \frac{J^2 \sin \theta \left( 21 \cos^2 \theta - 11 \right)}{2R^4} + O \left( \frac{1}{R^5} \right),
\]

\[
E^R_\theta = \frac{15J^2}{2R^3} \left( \cos \theta - \cos^3 \theta \right) + O \left( \frac{1}{R^4} \right),
\]

\[
E^\theta_\theta = -\frac{M \sin \theta}{R} - \frac{J^2 \sin \theta \left( 3 \cos^2 \theta + 2 \right)}{2R^4} + O \left( \frac{1}{R^5} \right),
\]

\[
E^\phi_\phi = -\frac{M \sin \theta}{R} - \frac{J^2 \sin \theta \left( 18 \cos^2 \theta - 13 \right)}{2R^4} + O \left( \frac{1}{R^3} \right).
\]

The quasi-local mass\(^{12}\) is calculated for \(R = \text{const. surfaces. It reads}

\[
M = \frac{1}{8\pi} \int_{S(R)} E^R_R X^R \sqrt{\det \gamma_{ij}} \, d\theta d\phi = M - \frac{J^2}{R^3} + O \left( \frac{1}{R^4} \right).
\]

\(^{12}\)We recall the conformal Killing vector field, corresponding to the mass, is simply \(X = R \partial_R\).
It is convenient to rewrite the formula (3.17) in the form

\[ B^i_j = \varepsilon^i_j \left[ \partial_t \left( \sqrt{\det \gamma_{ab} K^i} \right) - \sqrt{\det \gamma_{ab}} \left( K^i, \Gamma^p_{sp} - \Gamma^i_{sq} K^q \right) \right], \quad (5.73) \]

and note that \( \sqrt{\det \gamma_{ab} K^i} \) does not depend on conformal factor \( \varphi_{BY} \). The gravitomagnetic tensor density is relatively simple:

\[
\begin{align*}
B^R_R &= \frac{6 J \sin \theta \left( \varphi_{BY} \cos \theta - 2 \sin \theta \partial \varphi_{BY} \right)}{\varphi_{BY, R}^3 R^2} \\
&= \frac{6 J \sin \theta \cos \theta}{R^2 \varphi_{BY}^3} \left[ 1 + \frac{M}{2R} + O \left( \frac{1}{R^4} \right) \right] \\
&= \frac{6 J \sin \theta \cos \theta}{R^2} \left[ 1 + \frac{M}{2R} \right]^{-2} \left[ 1 + O \left( \frac{1}{R^4} \right) \right] \\
&= \frac{6 J \sin \theta \cos \theta}{R^2} \left[ 1 - \frac{M}{R} + \frac{3M^2}{4R^2} - \frac{M^3}{2R^3} + O \left( \frac{1}{R^4} \right) \right],
\end{align*}
\]

\[
B^R_\theta = \frac{3 J \sin^2 \theta \left( 2R \partial R \varphi_{BY} + \varphi_{BY} \right)}{\varphi_{BY, R}^3 R^2} \\
= \frac{3 J \sin^2 \theta}{R^2 \varphi_{BY}^3} \left[ 1 - \frac{M}{2R} + O \left( \frac{1}{R^4} \right) \right] \\
= \frac{3 J \sin^2 \theta}{R^2} \left[ 1 - \frac{2M}{R} + \frac{9M^2}{4R^2} - \frac{2M^3}{R^3} + O \left( \frac{1}{R^4} \right) \right],
\]

\[
B^\phi_\theta = - \frac{3 J \sin \theta \cos \theta}{\varphi_{BY}^3 R^2} \\
= - \frac{3 J \sin \theta \cos \theta}{R^2 \varphi_{BY}^3} \left[ 1 + \frac{M}{2R} + O \left( \frac{1}{R^4} \right) \right] \\
= - \frac{3 J \sin \theta \cos \theta}{R^2} \left[ 1 + \frac{M}{2R} \right]^{-2} \left[ 1 + O \left( \frac{1}{R^4} \right) \right] \\
= - \frac{3 J \sin \theta \cos \theta}{R^2} \left[ 1 - \frac{M}{R} + \frac{3M^2}{4R^2} - \frac{M^3}{2R^3} + O \left( \frac{1}{R^4} \right) \right].
\]

Angular momentum \( I(B, K_z) \) is generated by conformal acceleration in \( z \) axis:

\[
K_z = \frac{1}{2} R^2 \cos \theta \partial_R + \frac{1}{2} R \sin \theta \partial \theta.
\]

The integrand for instant angular momentum on the sphere \( R = \text{const.} \) reads

\[
\begin{align*}
B^R_R K_z^i &= B^R_R K_z^i + B^R_\theta K_z^\theta \\
&= \frac{3 J \sin^3 \theta \varphi_{BY}^3 (R \varphi_{BY})}{2 \varphi_{BY}^3} + \frac{3 J \sin \theta}{\varphi_{BY}^3} \left( 1 - 3 \cos^2 \theta \right) + \frac{3 J}{\varphi_{BY}^3} \left( \sin^2 \theta \cos(\theta) \right) \\
&= \frac{3 J \varphi_{BY}^3}{\varphi_{BY}^3} \left( \sin^2 \theta \cos(\theta) \right) + \frac{3 J \sin \theta}{\varphi_{BY}^3} \left[ \sin^2 \theta + \frac{1}{2} \left( 1 + \frac{M}{2R} \right) \left( 1 - 3 \cos^2 \theta \right) \\
&+ \frac{J^2}{R^4} \left( 9 \cos^4 \theta - 21 \cos^2 \theta + 10 \right) \right] + O \left( \frac{1}{R^4} \right),
\end{align*}
\]

\[
(5.89)
\]
The angular momentum is finally given by
\[ \frac{1}{8\pi} I(\mathcal{B}, K_z) = \frac{1}{8\pi} \int_{\partial \mathcal{R}} \left( B^R R K^R_z + B^R_\theta K^\theta_z \right) d\theta d\phi \]

\[ = J \left( 1 - \frac{3 M}{2 R} + \frac{3 M^2}{2 R^2} - \frac{5 M^3}{4 R^3} \right) + O \left( \frac{1}{R^4} \right) \] (5.90)

The equation (5.89) suggests that the instant angular momentum can be corrected by multiplicative factor
\[ \frac{1}{8\pi} \tilde{I}(\mathcal{B}, K_z) = \frac{1}{8\pi} \int_{\partial \mathcal{R}} \varphi^3_{\text{BY}} \left( B^R R K^R_z + B^R_\theta K^\theta_z \right) d\theta d\phi \]

\[ = J + \frac{9}{10} J^3 + O \left( \frac{1}{R^5} \right) \] (5.91)

The above correction is compatible with the right-hand side of equation (3.19) in Theorem 3.3. More precisely, for BY data multiplying equation (3.19) by \( \varphi^3_{\text{BY}} \) we can kill the term \( K \wedge E \) up to higher order in \( 1/R \).

6 Asymptotic charges

6.1 Brief review of classical and recent results

The formulation presented below follows the results obtained by Chruściel in [5]. Let us assume that \( \Sigma \) is a spacelike hypersurface extending up to infinity in an asymptotically flat spacetime, where “asymptotic flatness” is to be understood as follows: outside a world tube there exists a coordinate system such that
\[ g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}, \] (6.1)

where \( \tilde{g}_{\mu\nu} \) is the Minkowski metric, and \( h_{\mu\nu} \) satisfies
\[ |h_{\mu\nu}| \leq C/r^\alpha, \quad |h_{\mu\nu,\sigma}| \leq C/r^{\alpha+1}, \] (6.2)

for some \( \alpha \) to be specified later.

We will perform asymptotic analysis in the case of a fixed Cauchy hypersurface \( \Sigma \). Before giving the precise statement of the theorems, it is useful to introduce first some terminology. Suppose one is given a pair \( (g, \Phi) \), where

1. \( g \) is a Riemannian metric on a three-dimensional manifold \( N \) which is diffeomorphic to \( \mathbb{R}^3 \setminus B(R) \), where \( B(R) \) is a closed ball \( (N \text{ can be thought of as one of (possible many) “ends” of } \Sigma) \).

2. \( \Phi \) is a coordinate system in the complement of a compact set \( K \) of \( N \) such that in local coordinates \( \Phi^i(p) = x^i \) the metric takes the following form:
\[ g_{ij} = \delta_{ij} + k_{ij}, \] (6.3)

and \( k_{ij} \) satisfies
\[ \forall i,j,k,x \quad |k_{ij}(x)| \leq C/(r + 1)^\alpha, \quad |\partial k_{ij}/\partial x^k(x)| \leq C/(r + 1)^{\alpha+1} \] (6.4)

for some constant \( C \in \mathbb{R} \) and \( r = (\sum_i (x^i)^2)^{1/2} \). Such a pair \( (g, \Phi) \) will be called \( \alpha \)-admissible.

Let us restate the remaining boundary conditions (6.2) in the ADM language:
\[ \forall i, j, x \quad |N_i(x)| \leq C/(r + 1)^\alpha, \quad |N^i(x)| \leq C/(r + 1)^\alpha, \quad |N_i(x)| \leq C/(r + 1)^{\alpha+1}, \quad |P_{ij}(x)| \leq C/(r + 1)^{\alpha+1}, \quad |N^{ij}(x)| \leq C/(r + 1)^{\alpha+1}. \] (6.5)

In [5], Chruściel proved the following theorems:
Theorem 6.1. Suppose that

1. $(g, \Phi)$ is $\alpha$–admissible, with $\alpha > 1/2$,
2. the conditions \[\text{(6.3)}\] are satisfied,
3. $(g, P)$ satisfy the constraint equations, with integrable sources.

Let $S(R)$ be any one-parameter family of differentiable spheres, such that $r(S(R)) = \min_{x \in S(R)} r(x)$ tends to infinity, as $R$ does. Define

$$M(g, \Phi) = \lim_{R \to \infty} \frac{1}{16\pi} \int_{S(R)} (g_{k,i} - g_{i,k}) dS_k,$$

$$P_i(g, \Phi) = \lim_{R \to \infty} \frac{1}{8\pi} \int_{S(R)} P_{ij} dS_j$$

(\text{these integrals have to be calculated in the local $\alpha$–admissible coordinates $\Phi^i(p) = x^i$}). $M$ and $P_i$ are finite, independent upon the particular family of spheres $S(R)$ chosen, provided $r(S(R))$ tends to infinity as $R$ does.

Lemma 6.1. Let $(g, \Phi_1)$ and $(g, \Phi_2)$ be $\alpha_1$ and $\alpha_2$–admissible, respectively, with any $\alpha_a > 0$. Let $\Phi_1 \circ \Phi_2^{-1} : \mathbb{R}^3 \setminus K_2 \to \mathbb{R}^3 \setminus K_1$ be a twice differentiable diffeomorphism, for some compact sets $K_1$ and $K_2 \subset \mathbb{R}^3$. Then, in local coordinates

$$\Phi_1^i(p) = x^i, \quad \Phi_2^i(p) = y^i,$$

the diffeomorphisms $\Phi_1 \circ \Phi_2^{-1}$ and $\Phi_2 \circ \Phi_1^{-1}$ take the form

$$x^i(y) = \omega^i_j y^j + \eta^i(y), \quad y^i(x) = (\omega^{-1})^i_j x^j + \zeta^i(x),$$

$\zeta^i$ and $\eta^i$ satisfy, for some constant $C \in \mathbb{R}$,

$$|\zeta^i(x)| \leq C(r(x) + 1)^{1-\alpha}, \quad |\zeta^i_j(x)| \leq C(r(x) + 1)^{-\alpha},$$
$$|\eta^i(y)| \leq C(r(y) + 1)^{1-\alpha}, \quad |\eta^i_j(y)| \leq C(r(y) + 1)^{-\alpha},$$
$$r(x) = \left(\sum (x^i)^2\right)^{1/2}, \quad r(y) = \left(\sum (y^i)^2\right)^{1/2},$$

with $\alpha = \min(\alpha_1, \alpha_2)$, $\omega^i_j$ is an $O(3)$ matrix, and $r^0$ is to be understood as $\ln r$.

Theorem 6.2. Let $(g, \Phi_a), a = 1, 2$, satisfy the hypotheses of theorem \[\text{(6.1)}\] and lemma \[\text{(6.1)}\]. Then

1. $M(g, \Phi_1) = M(g, \Phi_2),$
2. $P_i(g, \Phi_1) = \omega^i_j P_j(g, \Phi_2)$, where $\omega \in O(3)$, given by lemma \[\text{(6.1)}\].

It seems unavoidable that a limiting process is involved in the definitions in theorem \[\text{(6.1)}\].

But finding expressions that do not depend on the first derivatives but on rather more geometric quantities is an old question that has attracted the attention of many authors. It was suggested by Ashtekar and Hansen [2], see also Chruściel [6], that the mass could be rather defined from the Ricci tensor and a conformal Killing field of the Euclidean space. We now recall the alternative definition of asymptotic invariant via the Ricci tensor:

Definition 6.1. Let $X$ be the radial vector field $S = r \partial_r$ in the chosen chart at infinity. Then, we define the Ricci version of the mass of $M$ by

$$m_R(g) = -\frac{1}{8\pi} \lim_{r \to \infty} \int_{S_r} \left( R^k l - \frac{3}{2} R \delta^k l \right) S^l dS_k.$$

Equality between the two definitions, as well as a similar identity for the center of mass, has then been proved rigorously by Huang using a density theorem [12], see also a simplified proof given by Herzlich in [11].
6.2 Existence of asymptotic $I(E, X)$ and $I(B, X)$

In the section 3 the definition of instant charges has been given. The conservation laws for instant charges have been provided in proposition 3.1. We generalize the concepts by considering objects which fulfill the assumptions from the proposition only in the asymptotic regime. For convenience, we define a symmetric tensor which describes deviation of a vector field $Y$ from being CKV:

$$V_{ij}(Y) := Y_{(ij)} - \frac{1}{3} Y^k |_{k\gamma ij}.$$  \hspace{1cm} (6.8)

If $X$ is a conformal Killing vector for $\gamma$ then $V_{ij}(X, \gamma) = 0$. With the help of $V$, the definition of asymptotic conformal Killing vector field reads:

**Definition 6.2. (asymptotic CKV)** Vector field $X$ will be called asymptotic conformal Killing vector field iff

$$\lim_{r \to \infty} V_{ij}(X) = 0,$$

for all $i, j$. The limit $\lim_{r \to \infty}$ is understood as the spatial infinity regime.

It enables one to define asymptotic instant charges:

**Definition 6.3 (asymptotic instant charges).** Consider a given spatial hypersurface $\Sigma$ equipped with an asymptotically flat\textsuperscript{13} Riemannian metric $\gamma$. Let $A(r) \subset \Sigma$ be an one-parameter family of two-dimensional closed surfaces such that $\lim_{r \to \infty} A(r)$ represents a topological sphere at spatial infinity. We can define the following asymptotic instant charges:

$$I_{as}(E, X) := \lim_{r \to \infty} \int_{S(r)} E^i_j X^j dS_i,$$  \hspace{1cm} (6.9)

$$I_{as}(B, X) := \lim_{r \to \infty} \int_{S(r)} B^i_j X^j dS_i,$$  \hspace{1cm} (6.10)

where $X^j$ is an asymptotic conformal Killing vector field (see definition 6.2).

Let us recall the reasoning which leads to conservation laws for instant charges. See the equation (3.13) and comments nearby. Analogically, the boundary integrals, (6.9) and (6.10), would be well-defined if the divergences, $(\sqrt{\gamma}E^{ij}X_j)_i$ and $(\sqrt{\gamma}B^{ij}X_j)_i$ respectively, will be integrable. We have

$$\sqrt{\gamma}E^{ij}X_j)_i = \sqrt{\gamma}E^{ij}(X_{(ij)} - \frac{1}{3} Y^k |_{k\gamma ij}) + \sqrt{\gamma}E^i_j X^j = \sqrt{\gamma}E^{ij}V_{ij}(X) + \sqrt{\gamma}X^j(K \wedge B)_j,$$  \hspace{1cm} (6.11)

where (3.18) is used. Analogical calculations for $(\sqrt{\gamma}B^{ij}X_j)_i$ give

$$\sqrt{\gamma}B^{ij}(X)_i = \sqrt{\gamma}B^{ij}V_{ij}(X) - \sqrt{\gamma}X^j(K \wedge E)_j.$$  \hspace{1cm} (6.12)

The asymptotic instant charges will be finite if the appropriate divergences are integrable at infinity, i.e.

$$E^{ij}V_{ij}(X) + X^j(K \wedge B)_j = O(r^{-3-\varepsilon}),$$  \hspace{1cm} (6.13)

$$B^{ij}V_{ij}(X) - X^j(K \wedge E)_j = O(r^{-3-\varepsilon}),$$  \hspace{1cm} (6.14)

where $\varepsilon > 0$.

Note that the basic conformal Killing vectors have different asymptotic relative to $r$, using (3.4)–(3.7) we have:

\textsuperscript{13}We assume there exists a coordinate chart such that the metric tends to the Euclidean metric at spatial infinity.
\( T_k = O(1), \)
\( R_k = O(r), \)
\( S = O(r), \)
\( K_k = O(r^2). \)

Generators of proper conformal transformations behave like \( r^2 \), therefore charges defined with \( K_k \) (e.g. angular momentum) impose the strongest conditions on the asymptotic in the equations (6.13)–(6.14). For convenience, we can rewrite the asymptotic of CKV in more compact form
\[
X = O(r^q), \quad q \in \{0, 1, 2\}.
\]

The conditions (6.13) and (6.14) can be interpreted in various ways. For example, if we assume that \( X \) is a CKV (\( V_{ij}(X) = 0 \)) and the metric is asymptotically flat, i.e. the conformal factor of the metric tends to unity at spatial infinity, then the conditions (6.13) and (6.14) are equivalent to
\[
(K \wedge E)_j = O(r^{-3-q-\varepsilon}),
\]
\[
(K \wedge B)_j = O(r^{-3-q-\varepsilon}),
\]
where \( q \) depends on the type of charge\(^{14} \) and associated CKV (6.15).

We can also consider surfaces \( \Sigma \) equipped with a metric which is no longer conformally flat but only asymptotically conformally flat.

**Theorem 6.3.** Let the metric on the hypersurface \( \Sigma \) be in the form\(^{15} \),
\[
\gamma_{ij} = \left(1 + \frac{M}{2r}\right)^4 (\eta_{ij} + h_{ij}),
\]
where \( \eta_{ij} \) is a three-dimensional Euclidean metric. We assume the following behavior of presented objects at \( r \to \infty \):
\[
h_{ij} = O(r^d), \quad (\Gamma_{\eta + h})^a_{ij} = O(r^c), \quad K_{ij} = O(r^k),
\]
\[
E_{ij} = O(r^e), \quad B_{ij} = O(r^b), \quad X = O(r^q), \quad q \in \{0, 1, 2\},
\]
where \( c < d < 1 \) and \( X \) is a CKV for \( \eta \) defined by (6.15) and the comments nearby.
If
\[
e + c + q < -3, \quad e + q + d < -3, \quad c + k + b < -3,
\]
then the asymptotic charge
\[
I_{as}(E, X) = \lim_{r \to \infty} \int_{S(r)} E^i_j X^j dS_i,
\]
is integrable. Analogically, the following conditions
\[
b + c + q < -3, \quad b + q + d < -3, \quad c + k + e < -3,
\]
guarantee that the asymptotic instant charge
\[
I_{as}(B, X) = \lim_{r \to \infty} \int_{S(r)} B^i_j X^j dS_i
\]
is finite.

\(^{14}\) We remind that the physical interpretation of the charges is given in section 4.1.

\(^{15}\) The metric in this form corresponds to slightly disturbed Schwarzschild spacetime, cf. formula (5.3) — Schwarzschild metric expressed in isotropic variables).
Proof:
The aim of the proof is to show that at \( r \to \infty \) divergences given by equations (6.11), (6.12) are integrable at infinity. It leads to asymptotic given by (6.13) and (6.14). For the metric \( \eta \), (6.13)–(6.14) are exact solutions of CKV equation \((V_{ij} = 0)\). Asymptotic of such CKV is given by (6.15). Let us examine expressions containing tensor \( V_{ij} \). The tensor \( V \) behaves with respect to the conformal transformation as follows:

\[
V_{ij}(\Omega^4 g, X) = \Omega^4 V_{ij}(g, X),
\]

where \( g \) is any metric. In our case \( \Omega = 1 + \frac{M}{r^2} \), hence the leading term in the conformal factor \( \Omega \) is \( O(1) \). We assume that the field \( X \) is a conformal Killing vector for a flat, three-dimensional metric \( \eta \), thus the following equation is satisfied:

\[
X^{(i,j)} - \frac{1}{3} X^{k} h_{ij} = 0.
\]

Hence,

\[
V(\eta + h, X) = -\Gamma_{mij}^{m} X_m - \frac{1}{3} X_k^{k} h_{ij} \Omega^4 \eta_{ij}.
\]

The Christoffel symbols appearing in the above formula come from the metric \( \eta + h \), we get

\[
V(\Omega^4(\eta + h), X) = \Omega^4 V(\eta + h, X) = O(r^v),
\]

where

\[
v = \max(c + q, q + h - 1, q + h) = \max(c + q, q + h).
\]

It leads to

\[
E^{ij} V_{ij}(g, X) = O(r^c) \cdot O(r^v) = O(r^{c+v}),
\]

\[
B^{ij} V_{ij}(g, X) = O(r^b) \cdot O(r^v) = O(r^{b+v}).
\]

Next, we check the asymptotic of the terms containing the extrinsic curvature:

\[
X^{j}(K \wedge B)_{j} = O(r^c) \cdot O(r^k) \cdot O(r^b) = O(r^{c+k+b}),
\]

\[
X^{j}(K \wedge E)_{j} = O(r^c) \cdot O(r^k) \cdot O(r^e) = O(r^{c+k+e}).
\]

Comparing (6.13), (6.29) and (6.31), we observe that asymptotic charge

\[
I_{as}(E, X) = \lim_{r \to \infty} \int_{S(r)} E^{ij} X^{j} dS_i,
\]

will be integrable if \( \max(e + c + q, e + q + h, c + k + b) < 3 \).

With the help of (6.13), (6.30) and (6.32), for the charge

\[
I_{as}(B, X) = \lim_{r \to \infty} \int_{S(r)} B^{ij} X^{j} dS_i,
\]

we can formulate similar condition \( \max(b + c + q, b + q + h, c + k + e) < 3 \).

\[\square\]

Remark: Let us observe that we can relax the assumption (6.25) (i.e. \( X \) is an exact CKV). More precisely, the above proof holds also when we substitute (6.25) by weaker condition:

\[
V_{ij}(X) = O(r^p),
\]

where \( p \leq 0 \) will be specified below. If we extend (6.20) by the condition

\[
e + p < -3,
\]

then the asymptotic charge (6.21) is integrable for the asymptotic CKV (6.35). The similar condition

\[
b + p < -3
\]

enables one to generalize the theorem for the charge (6.23).
Conclusions

We have provided relations between four-dimensional charges (2.8), constructed with the help of conformal Yano–Killing tensor, and the instant charges (definition 3.1) after (3+1) decomposition. The construction presented here enables one to calculate twenty local charges in terms of initial data ($\gamma_{ij}, K_{ij}$) on a conformally flat spatial hypersurface $\Sigma$ immersed in a spacetime satisfying the Einstein vacuum equations with a cosmological constant.

In the future, we plan to perform the analysis of (3+1) decomposition of CYK tensor in curved spacetimes. For Minkowski spacetime, each CYK two-form is a linear combination of a wedge product of two conformal Killing co-vectors (CKV) or a Hodge dual of such product. This makes the (3+1) decomposition of CYK tensor simple. In particular, the choice of Cauchy surface $t = \text{const.}$ is natural. In the case of curved spaces, the splitting of CYK form is much more complicated and the choice of Cauchy surface is much less intuitive. However, the CYK tensor decomposition should guarantee existence of Gaussian charges on a properly chosen surface. Deep understanding of the construction in the flat case is required for further analysis of curved spacetimes. This makes the paper valuable in the context of further research.

In addition, we have proved theorems explaining the relation between linear momentum and angular momentum (defined as the contractions of the magnetic part of Weyl tensor with appropriate conformal Killing vectors), and the traditional ADM linear and angular momentum. The analysis performed for the Kottler spacetime showed that the mass defined as the contraction of the electrical part of the Weyl tensor with the scaling generator may have better properties than the ADM mass. In the proposed example of foliation with surfaces with flat internal geometry, the mass calculated with conformal Killing fields was not dependent on radius or cosmological constant (as opposed to ADM mass).

The thesis formulated in the last section shows that in certain specific cases we can use the proposed method to define asymptotic charges (even if the metric on spatial surfaces is not conformally flat).

Acknowledgments

This work was supported in part by Narodowe Centrum Nauki (Poland) under Grant No. 2016/21/B/ST1/00940.

This material is based upon work supported by the Swedish Research Council under grant no. 2016-06596 while one of the authors (JJ) was in residence at Institut Mittag-Leffler in Djursholm, Sweden during the Research Program: General Relativity, Geometry and Analysis: beyond the first 100 years after Einstein, 02 September - 13 December 2019.

A Lower order corrections for Ricci mass

We propose a correction to the Ricci mass, introduced in Definition 6.1, which is proportional to the curvature scalar

$$\tilde{m}_R = -\frac{1}{8\pi} \int_{S(r)} \left( R^i - \frac{1}{2} \delta^k_i \delta^3 R \right) S^i dS_i - \frac{1}{8\pi} \int_{S(r)} \left( \beta + \frac{1}{2} \right) \delta^i_j \delta^3 R S^i dS_i. \quad (A.1)$$

Using CKV equation and contracted Bianchi identity,

$$S_{(ij)} = \lambda \gamma_{ij}, \quad (A.2)$$

$$\left( R^{kl} - \frac{1}{2} \gamma^{kl} \frac{3}{2} \right) |_k = 0, \quad (A.3)$$

we have

$$\left( R^{kl} S_l + \beta S^k \right) |_k = \lambda \bar{R} + \frac{1}{2} R_{\bar{k}} S^k + \beta n \lambda \bar{R} + \beta R_{\bar{k}} S^k. \quad (A.4)$$

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Assuming $\Phi = p(\theta) r^\alpha$, we have

$$
\left( R^{kl} S_l + \beta S^k R \right)_{|k} = \lambda R + \frac{1}{2} a R + \beta n \lambda R + \beta a R
$$

$$
= \lambda R + \frac{1}{2} a R + \beta(\lambda n + a)
$$

(A.5)

In particular, the corrected Ricci mass for Bowen–York spinning black hole has better asymptotics if we choose $\lambda = 1$, $n = 3$, $a = -6$. It leads to

$$
\beta = -\frac{2}{3}
$$

(A.6)

Using constraint equation with $K = 0$, $R - K^{kl} K_{kl} = 0$, the following quantity is almost conserved up to controlled term:

$$
-\frac{1}{8\pi} \int_{S(R)} \left( \frac{3}{R} R_j - \frac{2}{3} K^{kl} K_{kl} \delta^R_j \right) S^j \sqrt{\det \gamma} = m - \frac{4J^2}{R^3} + \frac{2J^2}{3 R^3} + O \left( \frac{1}{R^4} \right)
$$

$$
= m + O \left( \frac{1}{R^4} \right)
$$

(A.7)

The correction enables one to get rid of unwanted term proportional to square of angular momentum. Similar mechanism was observed for the Hamiltonian of asymptotically Kerr spacetimes [19].

Analogical correction can be implemented for instant mass. For convenience, we introduce a tensor

$$
A^i_j := -K^{im} K_{mj} + \frac{1}{2} K^{im} K_{lm} \delta^j_l + KK^j_l - \frac{1}{2} K^2 \delta^j_l
$$

(A.8)

The divergence of gravitoelectric tensor (3.18) may be corrected as follows

$$
E^i_j = (K \wedge B)_j =
$$

$$
\left( E^i_j - K^{im} K_{mj} + \frac{1}{2} K^{im} K_{lm} \delta^j_l + KK^j_l - \frac{1}{2} K^2 \delta^j_l \right)_{|i} =
$$

$$
\left( E^k_l + A^k_l \right)_{|k} = 0
$$

(A.9)

We denote $A := A^m_m$. [A.10] enables one to obtain a candidate for corrected electromagnetic mass in the following way

$$
\left( E^k_l + A^k_l - \frac{1}{3} A \delta^k_l + \frac{1}{3} A \delta^k_l \right) S^l = 0
$$

$$
\left[ \left( E^k_l + A^k_l - \frac{1}{3} A \delta^k_l \right) S^l \right]_{|k} = -\frac{1}{3} A_{,l} S^l
$$

(A.10)

For Bowen–York spinning black hole, we have the following asymptotic of $A = O \left( R^{-6} \right)$. It gives

$$
2(S^l A)_{|l} - S^l A = \frac{1}{R^6} S^l \left( R^6 A \right)_{,l} + 2(S^l A)_{,l} - 3)A = O \left( \frac{1}{R^7} \right)
$$

(A.11)

The equation (A.12) leads to the following modification of (A.11)

$$
\left[ \left( E^k_l + A^k_l - \frac{1}{3} A \delta^k_l \right) S^l \right]_{|k} + \frac{2}{3} (S^l A)_{|l} = O \left( \frac{1}{R^7} \right)
$$

(A.12)

Finally, the corrected instant mass for Bowen–York spinning black hole is given by

$$
\tilde{I} := \frac{1}{8\pi} \int_{S(R)} \left( E^R_l + A^R_l + \frac{1}{3} A \delta^R_l \right) S^l \sqrt{\det \gamma}
$$

(A.13)

In the case of Bowen-York spinning black hole ($\Lambda = 0$), the correction leads to that the integrand of a corrected instant mass is equal to an integrand of corrected Ricci mass.
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