Relation between two-point Green functions of $\mathcal{N} = 1$ SQED with $N_f$ flavors, regularized by higher derivatives, in the three-loop approximation

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Abstract

We verify the identity which relates the two-point Green functions of $\mathcal{N} = 1$ SQED with $N_f$ flavors, regularized by higher derivatives, by explicit calculations in the three-loop approximation. This identity explains why in the limit of the vanishing external momentum the two-point Green function of the gauge superfield is given by integrals of double total derivatives in the momentum space. It also allows to derive the NSVZ $\beta$-function exactly in all loops if the renormalization group functions are defined in terms of the bare coupling constant. In order to verify the considered identity we use it for constructing integrals giving the three-loop $\beta$-function starting from the two-point Green functions of the matter superfields in the two-loop approximation. Then we demonstrate that the results for these integrals coincide with the sums of the corresponding three-loop supergraphs.

Keywords: higher derivative regularization, supersymmetry, NSVZ $\beta$-function.

1 Introduction

The NSVZ $\beta$-function is a non-trivial relation between the $\beta$-function of $\mathcal{N} = 1$ supersymmetric gauge theories and the anomalous dimensions of the matter superfields. For the pure $\mathcal{N} = 1$ supersymmetric Yang-Mills theory it gives the exact in all orders expression for the $\beta$-function, which appears to be a geometric progression. The NSVZ $\beta$-function was obtained using various general arguments, namely, the structure of instanton contributions [1, 2, 3], the anomaly supermultiplet [4, 5, 6], nonrenormalization of the topological term [7]. Analogs of the NSVZ $\beta$-function were also found in theories with softly broken supersymmetry for the renormalization of the gaugino mass [8, 9, 10] and for $\mathcal{N} = (0, 2)$ deformed $(2, 2)$ two-dimensional sigma models [11].

In this paper we consider the $\mathcal{N} = 1$ supersymmetric electrodynamics (SQED) with $N_f$ flavors, for which the NSVZ $\beta$-function is given by

$$\beta(\alpha) = \frac{\alpha^2 N_f}{\pi} \left(1 - \gamma(\alpha)\right),$$

where $\gamma(\alpha)$ is the anomalous dimension of the chiral matter superfields, and $\alpha$ denotes the argument of functions in this relation. In the Abelian case the NSVZ $\beta$-function was first obtained in Refs. [12, 13]. For Abelian theories the derivation of the NSVZ relation by explicit summation of Feynman diagrams was made in [14, 15] in all orders for the RG functions defined in terms of the bare coupling constant in the case of using the higher derivative regularization.
If the RG functions are defined in terms of the renormalized coupling constant, comparing the NSVZ $\beta$-function with explicit calculations of Feynman (super)graphs one should take into account the ambiguity of choosing the subtraction scheme \cite{17}. In theories with a single coupling constant the $\beta$-function is scheme-dependent starting from the three-loop approximation. The anomalous dimension is scheme-dependent starting from the two-loop approximation. Therefore, terms proportional to $\alpha^4$ in both sides of the NSVZ relation are scheme dependent. The subtraction scheme producing the NSVZ relation (so called “NSVZ scheme”) in all orders in the Abelian case was constructed in Ref. \cite{16} by the help of the higher derivative regularization.

Nevertheless, the most popular procedure for making calculations in supersymmetric theories is the dimensional reduction \cite{18}, complemented by the DR subtraction scheme. Using this method the NSVZ relation was verified up to the four-loop approximation \cite{19, 20, 21, 22, 23} (see Ref. \cite{24} for a recent review). The scheme-independent terms proportional to $\alpha^2$ and $\alpha^3$ appeared to be the same. However, starting from the three-loop approximation the calculations made in the DR-scheme give results which do not satisfy the NSVZ relation. However, it is possible to find a finite renormalization after that the NSVZ relation is restored \cite{20, 21, 25}. The possibility of making this redefinition is nontrivial \cite{20}. The reason is that in higher loops some terms in the NSVZ relation are scheme independent. In particular, for $\mathcal{N} = 1$ SQED with $N_f$ flavors the terms proportional to the first degree of $N_f$ are scheme-independent and should satisfy NSVZ relation in all orders \cite{26}. In the non-Abelian case this is valid for terms proportional to $\text{tr} \left( C(R)^2 \right)$ in $L$ loops \cite{27}. Thus, even in higher loops the NSVZ relation imposes nontrivial constrains on the divergences. Nevertheless, (at present) the calculations made with the dimensional reduction do not allow to construct in all orders a scheme in which the NSVZ relation takes place. Moreover, the dimensional reduction (unlike the dimensional regularization \cite{28, 29, 30, 31}, which breaks supersymmetry \cite{32}) is not mathematically consistent \cite{33}. The price for removal of inconsistencies is the loss of explicit supersymmetry \cite{34}. This implies that supersymmetry can be broken by quantum corrections in higher loops \cite{35, 36}. The calculations made in \cite{35} and subsequently corrected in \cite{37} demonstrate this breaking in the three-loop approximation for $\mathcal{N} = 2$ supersymmetric Yang–Mills theory (without hypermultiplets). Most other regularizations (see, e.g., \cite{38, 39}) were usually used only for calculations in the one- and two-loop approximations, where the problems related to the scheme dependence are not essential.

It appears that a very convenient regularization for calculations of quantum corrections in supersymmetric theories is the higher covariant derivative regularization proposed by A.A.Slavnov \cite{40, 41}. Unlike the dimensional reduction, the regularization by higher covariant derivatives is consistent. For $\mathcal{N} = 1$ supersymmetric gauge theories it can be formulated in a manifestly supersymmetric way \cite{42, 43}. Thus, with this regularization supersymmetry is not broken at any stage of quantum correction calculation. The higher covariant derivative regularization can be also formulated for $\mathcal{N} = 2$ supersymmetric gauge theories \cite{44, 45}. At least in the Abelian case using of the higher covariant derivative regularization allowed to solve the long standing problem of constructing a renormalization prescription which gives the NSVZ scheme in all orders \cite{16}, see also \cite{27}. It was noted \cite{16} that one should distinguish the RG functions defined in terms of the bare coupling constant and the RG functions defined in terms of the renormalized coupling constant. If a supersymmetric gauge theory is regularized by higher derivatives, the NSVZ relation appears to be valid for the RG function defined in terms of the bare coupling constant \cite{14, 15} (see Ref. \cite{27} for a brief review). These RG functions are scheme-independent for a fixed regularization (see, e.g., \cite{16}), that is do not depend on an arbitrariness in a choice of the renormalization constants. The definition of the RG function in terms of the bare coupling constant was used in a lot of early papers devoted to the NSVZ $\beta$-function \cite{1, 2, 6, 12, 13, 16}. However, standardly, the RG functions are defined in terms of the renormalized coupling constant. Such RG functions are scheme dependent and satisfy the NSVZ relation only in the NSVZ scheme.
In case of using the DR-scheme the NSVZ relation can be obtained after a specially tuned finite renormalization in every order \cite{20,21,25}, which, however, is not so far constructed in all orders. The higher covariant derivative regularization allows to construct the NSVZ scheme in all orders by imposing the simple boundary conditions \cite{16}:

\[ Z(\alpha, x_0) = 1; \quad Z_3(\alpha, x_0) = 1, \quad (2) \]

where \( x_0 \) is an arbitrary fixed value of \( \ln \Lambda / \mu \), on the renormalization constants. They ensure that the RG functions defined in terms of the renormalized coupling constant coincide with the ones defined in terms of the bare coupling constant and satisfy the NSVZ relation in all orders.

Certainly, the main statement for constructing the NSVZ scheme is that the RG functions defined in terms of the bare coupling constant satisfy the NSVZ relation in all orders if higher derivatives are used for regularization. It is based on the observation that in this case the integrals giving the \( \beta \)-function are integrals of total derivatives \cite{47} and even double total derivatives \cite{48} in the momentum space. In the Abelian case this fact was proved exactly in all orders by two different methods \cite{14,15}, which give the same result. For a general non-Abelian \( \mathcal{N} = 1 \) supersymmetric gauge theory with chiral matter superfields factorization of integrands into total derivatives was explicitly demonstrated in the two-loop approximation \cite{49,50}. Subsequently it was verified that the corresponding integrals are also integrals of double total derivatives \cite{51,52,53}. Such a structure of integrals allows to calculate one of the momentum integrals analytically and obtain the exact NSVZ \( \beta \)-function. In this paper we would like to verify the results of Ref. \cite{15}, in which the NSVZ relation is obtained by the method based on the Schwinger–Dyson equations for \( \mathcal{N} = 1 \) SQED with \( N_f \) flavors. (The Schwinger–Dyson equations are widely used in quantum field theory, see, e.g., \cite{54,55,56}. A possibility of using them for the derivation of the NSVZ relation was first discussed in \cite{57}.) The Schwinger–Dyson equations allow to construct an equality relating two-point Green functions of the gauge superfield and of the matter superfields. This relation enables us to obtain integrals giving the \( \beta \)-function defined in terms of the bare coupling constant in every order if the integrals giving the two-point Green functions for the chiral matter superfields are known in the previous order. (The integrals for the \( \beta \)-function appear to be integrals of double total derivatives.) In this paper we verify this relation in the three-loop approximation. Namely, starting from the two-loop integrals for the two-point Green functions of the matter superfields we obtain the three-loop \( \beta \)-function in the form of integrals of double total derivatives in the momentum space. Then we compare the result with the known expressions for the sums of the three-loop supergraphs.

This paper is organized as follows: In Sect. 2 we recall basic information concerning \( \mathcal{N} = 1 \) SQED with \( N_f \) flavors regularized by higher derivatives and the relation between the two-point Green functions in this theory. In the three-loop approximation this identity is verified in Sect. 3 by explicit calculation of the supergraphs.

2 \( \mathcal{N} = 1 \) SQED with \( N_f \) flavors, regularized by higher derivatives, and the relation between its two-point Green functions

Following the paper \cite{15}, here we consider \( \mathcal{N} = 1 \) SQED with \( N_f \) flavors, which (in the massless limit) is described in terms of superfields by the following action \cite{58,59}:

\footnote{Note that in the three-loop approximation the NSVZ scheme obtained with the higher derivative regularization differs from the NSVZ scheme constructed in \cite{20}, because the expressions for the anomalous dimension are different. Certainly, these two NSVZ schemes can be related by a finite renormalization, which includes finite renormalization of the chiral matter superfields.}
\[ S = \frac{1}{4e_0} \text{Re} \int d^4x d^2\theta W^a W_a + \frac{1}{4} \sum_{\alpha=1}^{N_f} \int d^4x d^4\theta \left( \phi_\alpha^* e^{2V} \phi_\alpha + \tilde{\phi}_\alpha^* e^{-2V} \tilde{\phi}_\alpha \right), \]

where \( V \) is a real gauge superfield; \( \phi_\alpha \) and \( \tilde{\phi}_\alpha \) are chiral matter superfields. The index \( \alpha \) numerates flavors and goes from 1 to the number of flavors \( N_f \). \( e_0 \) denotes the bare coupling constant. Also we will use the notation \( \alpha_0 \equiv e^2_0/4\pi \). In the considered Abelian case the supersymmetric gauge field strength is given by

\[ W_a = \frac{1}{4} \bar{D}^2 D_a V, \]

where \( \bar{D}_a \) and \( D_a \) are the left and right supersymmetric covariant derivatives, respectively.

Calculating quantum corrections we use the background field method \[60, 61, 62\]. In the Abelian case, which is considered here, the gauge superfield is presented as a sum of the quantum field \( V \) and the background superfield \( V \):

\[ V \rightarrow V_T = V + V. \]

Moreover, it is convenient to modify the action for the chiral matter superfields by introducing an auxiliary parameter \( g \) according to the following prescription:

\[ e^{2V} \rightarrow 1 + g(e^{2V} - 1); \quad e^{-2V} \rightarrow 1 + g(e^{-2V} - 1), \]

where \( V \) is the quantum superfield. The exponents with the background gauge field remain unchanged. This implies that in each diagram the degree of \( g \) is equal to the number of vertexes containing at least one line of the quantum gauge superfield. Introducing the parameter \( g \) does not break the background gauge symmetry, but breaks the quantum gauge symmetry.

In order to regularize the theory we add to its action a term with higher derivatives and insert the Pauli–Villars determinants into the generating functional to cancel remaining one-loop divergences \[63, 64\]. The final expression for the generating functional has the following form \[15\]:

\[ Z = e^{iW} = \int D\phi D\bar{\phi} \prod_{l=1}^{n} \det(V, V, M_I)^{c_I N_I} \exp \left( iS_{\text{reg}} + iS_{\text{gf}} + iS_{\text{source}} \right) \]

\[ = \int D\mu \exp \left( iS_{\text{total}} + iS_{\text{gf}} + iS_{\text{source}} \right). \]

The masses of the Pauli–Villars fields are proportional to the parameter \( \Lambda \) in the higher derivative terms, \( M_I = a_I \Lambda \), \( a_I \) being independent of \( e_0 \). The coefficients \( c_I = (-1)^{P_I+1} \) should satisfy the conditions

\[ \sum_{l=1}^{n} c_I = 1; \quad \sum_{l=1}^{n} c_I M_I^2 = 0, \]

where \((-1)^{P_I}\) can be considered as the Grassmannian parity of the corresponding Pauli–Villars fields. Taking into account that \( c_I = \pm 1 \), the Pauli–Villars fields can be treated similarly to the usual fields. Below the usual fields will correspond to \( I = 0 \), and \( M_I=0 \). The action and the gauge fixing term in \[7\] are
\[ S_{\text{total}} = \frac{1}{4e_0} \text{Re} \int d^4 x \, d^2 \theta \, W^a R(\partial^2 / \Lambda^2) W_a + \frac{1}{4e_0} \text{Re} \int d^4 x \, d^2 \theta \, W_a W_a^a \]

\[ + \frac{1}{4} \sum_{I=0}^{N_f} \sum_{n=0}^{N_f} \int d^4 x \, d^4 \theta \, [\phi_\alpha^I e^{2V}(1 + g(e^{2V} - 1))\phi_\alpha^I + \tilde{\phi}_\alpha^I e^{-2V}(1 + g(e^{-2V} - 1))\tilde{\phi}_\alpha^I]_I \]

\[ + \sum_{I=0}^{N_f} \sum_{n=1}^{N_f} \left( \frac{1}{2} \int d^4 x \, d^2 \theta \, M_\alpha \tilde{\phi}_\alpha + \text{c.c.} \right)_I ; \]

\[ S_{\text{gf}} = -\frac{1}{64\pi^2} \int d^4 x \, d^4 \theta \left( VR(\partial^2 / \Lambda^2) D^2 \tilde{D}^2 V + V R(\partial^2 / \Lambda^2) D^2 \tilde{D}^2 V \right), \tag{9} \]

where for the background field strength we use the notation \( W_a = \tilde{D}^2 D_a V / 4 \) and \( e \) is the renormalized coupling constant. The function \( R \) is a sum of 1 corresponding to the classical action and a higher derivative term. For example, it is possible to choose \( R = 1 + \partial^{2n} / \Lambda^{2n} \). From Eq. \( \text{[9]} \) we see that the propagator of the gauge superfield (in the Euclidean space after the Wick rotation) is proportional to

\[ \frac{\epsilon_0^2}{R_k k^2} + \frac{(\epsilon_0^2 - \epsilon^2)}{16R_k k^4} \left( \tilde{D}^2 D^2 + D^2 \tilde{D}^2 \right). \tag{10} \]

The source term is constructed by the standard way, but it is also convenient to include into it sources for the Pauli–Villars fields. The effective action is defined by

\[ \Gamma[V, V, \phi_\alpha I, \tilde{\phi}_\alpha I] = W - S_{\text{source}}, \tag{11} \]

where the sources should be expressed in terms of fields. Terms in the effective action corresponding to the two-point functions of the background gauge superfield and the chiral matter superfields (including the Pauli-Villars ones) can be presented as

\[ \Gamma^{(2)} - S_{\text{gf}} = -\frac{1}{16\pi} \int \frac{d^4 p}{(2\pi)^4} d^4 \theta \, V(\theta, -p) \partial^2 \Pi_{1/2} V(\theta, p) d^{-1}(\alpha_0, \Lambda / p) \]

\[ + \frac{1}{4} \sum_{I=0}^{N_f} \sum_{n=1}^{N_f} \int \frac{d^4 p}{(2\pi)^4} d^4 \theta \left( \phi_\alpha^I (\theta, -p) \phi_\alpha I(\theta, p) + \phi_\alpha^I (\theta, -p) \phi_\alpha I(\theta, p) \right) G_I(\alpha_0, \Lambda / p) \]

\[ - \left( \sum_{I=0}^{N_f} \sum_{n=1}^{N_f} \int \frac{d^4 p}{(2\pi)^4} d^4 \theta \phi_\alpha I(\theta, -p) \right) \frac{D^2}{16\pi^2} \tilde{\phi}_\alpha I(\theta, p) M_I J_I(\alpha_0, \Lambda / p) + \text{c.c.} \right). \tag{12} \]

Writing explicitly expressions for the functions entering this equation in the tree approximation we can present them in the form

\[ d^{-1}(\alpha_0, \Lambda / p) = \alpha_0^{-1} + O(1); \quad G_I(\alpha_0, \Lambda / p) = 1 + O(\alpha_0); \quad J_I(\alpha_0, \Lambda / p) = 1 + O(\alpha_0). \tag{13} \]

The \( \beta \)-function defined in terms of the bare coupling constant can be easily found, if the function \( d^{-1}(\alpha_0, \Lambda / p) \) is known:

\[ \frac{d}{d \ln \Lambda} \left( d^{-1}(\alpha_0, \Lambda / p) - \alpha_0^{-1} \right) \bigg|_{\Lambda = p} = -\frac{d\alpha_0^{-1}(\alpha, \Lambda / \mu)}{d \ln \Lambda} = \frac{\beta(\alpha_0)}{\alpha_0^2}. \tag{14} \]

\[ ^2 \text{This expression is written in the Minkowski space.} \]
where \( \alpha \) is the renormalized coupling constant, \( \mu \) is the renormalization scale, and the derivative with respect to \( \ln \Lambda \) should be calculated at a fixed value of the renormalized coupling constant \( \alpha \). From this equation we see that the considered \( \beta \)-function is completely determined by the derivative of \( d^{-1}(\alpha_0, \Lambda/p) - \alpha_0^{-1} \) with respect to \( \ln \Lambda \) in the limit of the vanishing external momentum, \( p \to 0 \). This derivative can be found from the two-point Green function of the background gauge superfield after the substitution

\[
V(x, \theta) \rightarrow \theta^4 \cdot I(x) \approx \theta^4, \tag{15}
\]

where the regulator \( I(x) \) is approximately equal to 1 at finite \( x^\mu \) and tends to 0 at the large scale \( R \to \infty \). Introducing the regularized space-time volume

\[
\mathcal{V}_4 \equiv \int d^4x I^2 \sim R^4 \to \infty \tag{16}
\]

one can easily obtain \( [15] \)

\[
\frac{1}{2\pi} \mathcal{V}_4 \cdot \frac{d}{d \ln \Lambda} \left( d^{-1}(\alpha_0, \Lambda/p) - \alpha_0^{-1} \right) \bigg|_{p=0} = \frac{1}{2\pi} \mathcal{V}_4 \cdot \frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{d(\Delta \Gamma^{(2)}_V)}{d \ln \Lambda} \bigg|_{V(x, \theta) = \theta^4}, \tag{17}
\]

where we use the notation

\[
\Delta \Gamma \equiv \Gamma - \frac{1}{4e_0^2} \text{Re} \int d^4x d^2\theta W^a W_a. \tag{18}
\]

(The superscript (2) and the subscript \( V \) extract a part of \( \Delta \Gamma \) quadratic in the background gauge superfield.) The derivative with respect to \( \ln \Lambda \) should be calculated at a fixed value of the renormalized coupling constant \( \alpha \). Below for simplicity we omit the regulator \( I(x) \) in all subsequent equations.

As we already mentioned above, the integrals giving the \( \beta \)-function defined in terms of the bare coupling constant are integrals of double total derivatives in the momentum space (in these integrals the external momentum vanishes). In Ref. \( [15] \) it was shown that this fact is a consequence of the following identity relating various two-point Green function of the theory\( ^3 \)

\[
\frac{d}{d \ln \Lambda} \frac{\partial}{\partial \ln g} \left( \frac{1}{2} \int d^8x d^8y (\theta^4)_y (\theta^4)_y \frac{\delta^2 \Delta \Gamma}{\delta V_x \delta V_y} \right)
= i \frac{C(R)}{4} \frac{d}{d \ln \Lambda} \text{Tr}(\theta^4)_x \left[ y^*_\mu \left( \frac{\delta^2 \Delta \Gamma}{\delta (\phi_j)_x \delta (\phi^*_j)_y} \right)^{-1} + M^x_j \left( \frac{D^2}{8\partial^2} \right)_x \left( \frac{\delta^2 \Gamma}{\delta (\phi_k)_x \delta (\phi^*_j)_y} \right)^{-1} \right]_{y=x} \] - singularities = -singularities, \tag{19}
\]

where

\[
(y^*_\mu) \equiv x_\mu - i \bar{\theta}^a (\gamma_\mu)_a^b \theta_b, \tag{20}
\]

and

\[
[y^*_\mu, A_{xy}] \equiv (y^*_\mu)_x A_{xy} - A_{xy}(y^*_\mu)_y; \quad \text{Tr} A \equiv \int d^8x A_{xx}; \quad \int d^8x \equiv \int d^4x d^4\theta. \tag{21}
\]

\( ^3 \)The derivatives of the effective action, which are written in this equation, coincide with the corresponding derivatives of the Routhian \( \gamma \), which is used in \( [13] \).
In the momentum space the commutators with $y^\mu_\nu$ give derivatives with respect to the loop momentum. That is why from Eq. (19) we immediately obtain the statement that the integrals for the $\beta$-function defined in terms of the bare coupling constant are given by integrals of double total derivatives. In Eq. (19) we also use the notation

$$\phi_i \equiv (\phi_{\alpha I}, \tilde{\phi}_{\alpha I}), \quad \phi^i \equiv (\phi_{\alpha I}, \tilde{\phi}_{\alpha I}), \quad i = 1, \ldots, 2(n + 1)N_f. \tag{22}$$

The fields $\phi_i$ include both usual fields and the Pauli–Villars fields (which have the Grassmannian parity $(-1)^F$). In this notation

$$C(R)_{ij} = \delta_{\alpha\beta} \cdot \delta_{IJ} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{23}$$

and the mass matrix is given by

$$M^{ij} = \delta_{\alpha\beta} \cdot \delta_{IJ} \cdot \begin{pmatrix} 0 & (1)^{P_f} M_I \\ (1)^{P_f} M_I & 0 \end{pmatrix}, \tag{24}$$

where $2 \times 2$ matrix corresponds to the fields $\phi$ and $\tilde{\phi}$. The two-point Green functions of the chiral matter superfields can be easily found from Eq. (12):

$$\frac{\delta^2 \Gamma}{\delta(\phi_i)\delta(\phi^j)}_y = \delta_{\alpha\beta} \cdot \delta_{IJ} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} G_I \frac{D^2D^2_x}{16} \delta_{xy}^8; \tag{25}$$

$$\frac{\delta^2 \Gamma}{\delta(\phi_i)\delta(\phi^j)}_y = -\frac{1}{4} \delta_{\alpha\beta} \cdot \delta_{IJ} \cdot \begin{pmatrix} 0 & (1)^{P_f} M_I \\ (1)^{P_f} M_I & 0 \end{pmatrix} J_I \frac{D^2}{4(\partial^2G^2_x + M_I^2J_I^2)} \delta_{xy}^8. \tag{26}$$

The corresponding inverse functions are constructed, e.g., in [15]:

$$\left(\frac{\delta^2 \Gamma}{\delta(\phi_i)\delta(\phi^j)}_y\right)^{-1} = (1)^{P_f} \delta_{\alpha\beta} \cdot \delta_{IJ} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{G_I D^2}{4(\partial^2G^2_x + M_I^2J_I^2)} \delta_{xy}^8; \tag{27}$$

$$\left(\frac{\delta^2 \Gamma}{\delta(\phi_i)\delta(\phi^j)}_y\right)^{-1} = -\delta_{\alpha\beta} \cdot \delta_{IJ} \cdot \begin{pmatrix} 0 & (1)^{P_f} M_I \\ (1)^{P_f} M_I & 0 \end{pmatrix} \frac{J_I D^2}{4(\partial^2G^2_x + M_I^2J_I^2)} \delta_{xy}^8.$$

The term ”singularities” in Eq. (19) denotes the sum of singular contributions, which appear due to the identity

$$\left[x^\mu \frac{\partial}{\partial x^\mu}\right] \left[-i \frac{\partial}{\partial p_\mu} \right] = -2\pi^2 \delta^4(p_E) = -2\pi^2 i \delta^4(p) = -2\pi^2 i \delta^4(\partial). \tag{27}$$

According to Refs. [14, 15] the sum of these singular contributions gives the NSVZ $\beta$ relation for the RG functions defined in terms of the bare coupling constant in all orders of the perturbation theory (in the case of using the considered version of the higher derivative regularization).

3 Three-loop verification of the relation between the two-point Green functions

3.1 Double total derivatives in the three-loop approximation

In order to verify Eq. (19) it is necessary to calculate the two-point Green function of the background gauge superfield assuming that $g \neq 1$. In the three-loop approximation this Green function is given by the sum of diagrams presented in Fig. 1 to which one should attach two
external lines of the background gauge superfield. As an example, in Fig. 2 we present two-loop diagrams which correspond to the two-loop diagram (1) in Fig. 1. Each vertex containing the line of the quantum gauge superfield gives the factor $g$, and each closed loop of the matter superfield gives the factor $N_f$. The overall factors for all diagrams are also written in Fig. 1. Also in this figure below each graph we present the corresponding diagrams contributing to the two-point Green functions of the chiral matter superfields. These diagrams are obtained by cutting matter lines in the considered graph.

\[ \text{(1)} \quad \text{(2)} \quad \text{(3)} \quad \text{(4)} \quad \text{(5)} \quad \text{(6)} \quad \text{(7)} \]

\[ g^2 N_f \quad g^3 N_f \quad g^3 N_f^2 \quad g^4 N_f \quad g^4 N_f^2 \quad g N_f \]

Figure 1: Two- and three-loop graphs. Diagrams which contribute to the $\beta$-function can be obtained from them by attaching two external lines of the gauge superfield. Below we present factors corresponding to each of these diagrams and the corresponding diagrams contributing to the two-point Green function of the matter superfield.

\[ \text{Figure 2: Diagrams giving the two-point Green function of the background gauge superfield in the two-loop approximation. They correspond to the diagram (1) in Fig. 1.} \]

For the case $N_f = 1$, $g = 1$ all diagrams giving the three-loop $\beta$-function were calculated in Ref. [47]. Using the results of Ref. [47] one can find expressions for all diagrams in Fig. 1. As we discussed above, it is convenient to substitute the background gauge superfield corresponding to the external lines with $\theta^4$. Below we collect contributions of all diagrams to the derivative of the expression (17) with respect to $\ln g$ in the Euclidean space (after the Wick rotation). Note that all terms proportional to $e_0^2$, which are not essential in the considered approximation, are omitted. The result has the following form:

\[
\frac{\partial}{\partial \ln g} \frac{d}{d \ln \Lambda} \text{diagram (1)} = \mathcal{V}_4 \cdot 4g^2 N_f \frac{d}{d \ln \Lambda} \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 R_k} \times \frac{\partial}{\partial q^\mu} \frac{\partial}{\partial k^\mu} \sum_{l=0}^{n} (-1)^{P_l} \left( \frac{e_0^2}{(q^2 + M_f^2)(q + k)^2 + M_f^2} \right)^l - \frac{2(e_0^2 - e^2)}{k^2 (q^2 + M_f^2)} \right] \quad (28)
\]

The one-loop graph is not included, because the one-loop approximation in this approach should be considered separately.
\[
\frac{\partial}{\partial \ln g} \frac{d}{d \ln \Lambda} \quad \text{diagram (2)} = -\mathcal{V}_4 \cdot 24g^3 N_f \frac{d}{d \ln \Lambda} \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} e_0^4 \frac{1}{(q^2 + M_f^2) ((q + k)^2 + M_f^2) ((q + l)^2 + M_f^2)}; \tag{29}
\]

\[
\frac{\partial}{\partial \ln g} \frac{d}{d \ln \Lambda} \quad \text{diagram (3)} = -\mathcal{V}_4 \cdot 24g^3 N_f^2 \frac{d}{d \ln \Lambda} \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} e_0^4 \frac{1}{(p^2 + M_f^2)(q^2 + M_f^2) ((q + k)^2 + M_f^2)}; \tag{30}
\]

\[
\frac{\partial}{\partial \ln g} \frac{d}{d \ln \Lambda} \quad \text{diagram (4)} = \mathcal{V}_4 \cdot 16g^3 N_f \frac{d}{d \ln \Lambda} \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} e_0^4 \frac{1}{(q^2 - M_f^2)(q^2 + M_f^2) ((q + k)^2 + M_f^2) ((q + l)^2 + M_f^2)}; \tag{31}
\]

\[
\frac{\partial}{\partial \ln g} \frac{d}{d \ln \Lambda} \quad \text{diagram (5)} = \mathcal{V}_4 \cdot 8g^4 N_f \frac{d}{d \ln \Lambda} \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} e_0^4 \frac{1}{(q^2 + M_f^2) ((q + k)^2 + M_f^2) ((q + l)^2 + M_f^2)}; \tag{32}
\]

\[
\frac{\partial}{\partial \ln g} \frac{d}{d \ln \Lambda} \quad \text{diagram (6)} = -16\mathcal{V}_4 \cdot g^4 N_f^2 \frac{d}{d \ln \Lambda} \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} e_0^4 \frac{k^2 - (q + k)^2 - q^2 - (l + k)^2 - l^2 - 2M_f^2 - 2M_f^2}{(q^2 + M_f^2)((q + k)^2 + M_f^2)((q + l)^2 + M_f^2)}; \tag{33}
\]

\[
\frac{\partial}{\partial \ln g} \frac{d}{d \ln \Lambda} \quad \text{diagram (7)} = 4\mathcal{V}_4 \cdot g N_f \frac{d}{d \ln \Lambda} \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \frac{e_0^4 - e^2}{k^4 R_k} \frac{1}{(q^2 + M_f^2)}; \tag{34}
\]

We see that each graph is given by an integral of double total derivatives, as it was argued in [38].

### 3.2 Terms proportional to \( e_0^2 \) in Eq. (19)

From the above equations we see that expressions for all graphs in Fig. 11 are integrals of double total derivatives. Eq. (19) allows to construct these integrals of double total derivatives without calculating the three-loop diagrams. In order to use this equation one should calculate the two-point Green functions of the chiral matter superfields in the previous (two-loop) approximation. Let us first demonstrate this for the first (two-loop) diagram in Fig. 11. Cutting the line of the matter superfield we obtain the corresponding (one-loop) diagram for the two-point
Green functions of the matter superfields, which is also presented in Fig. 1. Calculating this diagram we obtain its contributions to the functions $G_I$ and $J_I$ (in the Euclidean space after the Wick rotation, as functions of the Euclidean momentum $q_\mu$):

$$\Delta G_I^{(1)} = -g^2 \int \frac{d^4k}{(2\pi)^4} \frac{2}{k^2 R_k} \left( \frac{e_0^2}{((q+k)^2 + M_I^2)} - \frac{e_0^2 - e^2}{k^2 ((q+k)^2 + M_I^2)} \right); \quad (35)$$

$$\Delta J_I^{(1)} = g^2 \int \frac{d^4k}{(2\pi)^4} \frac{4(e_0^2 - e^2)q^2}{k^4 R_k ((q+k)^2 + M_I^2)}. \quad (36)$$

The difference $e_0^2 - e^2$ is evidently proportional to $e_0^4$. Therefore, the contributions proportional to $e_0^2 - e^2$ are essential only in the next order. This implies that in the one-loop approximation

$$G_I(\alpha_0, \Lambda/q) = 1 - \int \frac{d^4k}{(2\pi)^4} \frac{2e_0^2g^2}{k^2 R_k ((q+k)^2 + M_I^2)} + O(e_0^4);$$

$$J_I(\alpha_0, \Lambda/q) = 1 + O(e_0^4). \quad (37)$$

Then we construct the inverse Green functions entering Eq. (19) using Eq. (26). In the momentum representation they are proportional to

$$\left(\frac{\delta^2 \Gamma}{\delta (\phi_i)_y \delta (\phi^i)_y}\right)^{-1} \sim \frac{G_I}{4(q^2 G_I^2 + M_I^2 J_I^2)}$$

$$= \frac{1}{4(q^2 + M_I^2)} \left(1 + \frac{q^2 - M_I^2}{q^2 + M_I^2}\right) \int \frac{d^4k}{(2\pi)^4} \frac{2e_0^2g^2}{k^2 R_k ((q+k)^2 + M_I^2)} + O(e_0^4);$$

$$\left(\frac{\delta^2 \Gamma}{\delta (\phi_i)_y \delta (\phi^i)_y}\right)^{-1} \sim \frac{M_I}{q^2 + M_I^2} \left(1 + \frac{q^2 - M_I^2}{q^2 + M_I^2}\right) \int \frac{d^4k}{(2\pi)^4} \frac{4e_0^2g^2}{k^2 R_k ((q+k)^2 + M_I^2)} + O(e_0^4). \quad (38)$$

These functions are needed for calculating the expression\footnote{The integral over $d^4\theta$ comes from Tr in Eq. (19).}

$$\frac{i}{4} C(R)^2 \int d^4\theta_x (\theta^4)_x \left(\left(\frac{\delta^2 \Gamma}{\delta (\phi_j)_y \delta (\phi^j)_y}\right)^{-1} + M^{ik} \left(\frac{D^2}{8\theta^2}\right)_x \left(\frac{\delta^2 \Gamma}{\delta (\phi_k)_y \delta (\phi_j)_y}\right)^{-1} + M^{jk} \left(\frac{D^2}{8\theta^2}\right)_x \right) \right|_{\theta_y = \theta_x} = 2N_f \sum_{l=0}^n (-1)^{P_l} \int \frac{d^4q}{(2\pi)^4} \left(\frac{1}{q^2 + M_I^2} + \frac{M_I^2}{q^2 + M_I^2}\right) \left(\frac{2e_0^2g^2}{2\pi)^4} \frac{2e_0^2g^2}{2\pi)^4} \frac{2e_0^2g^2}{2\pi)^4} \exp \left(iq_{\mu}(x^\mu - y^\mu)\right) + O(e_0^4) \right)$$

$$= 4N_f \sum_{l=0}^n (-1)^{P_l} \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + M_I^2} \int \frac{d^4k}{(2\pi)^4} \frac{e_0^2g^2}{k^2 R_k ((q+k)^2 + M_I^2)} \exp \left(iq_{\mu}(x^\mu - y^\mu)\right) + O(e_0^4), \quad (39)$$

where $x^\mu$ and $y^\mu$ denote Euclidean coordinates. Deriving the last equality we take into account that one of the terms is proportional to
\[ \sum_{l=0}^{n} (-1)^{F_l} = 1 - \sum_{l=1}^{n} c_l = 0. \] (40)

At the next step it is necessary to calculate the commutators with \( y^\mu_\mu \). Taking into account that for the trace of the commutator always vanishes, we can replace commutators with \( y^\mu_\mu \) by the commutators with \( x_\mu \). These commutators in the momentum representation give

\[ [x_\mu, [x^\mu, \ldots]]_M = -[x_\mu, [x^\mu, \ldots]]_E \rightarrow \frac{\partial}{\partial q^\mu} \frac{\partial}{\partial q_\mu}, \] (41)

which acts on the exponent. Then we integrate by parts and calculate the remaining part of the trace by setting \( y = x \) and integrating over \( d^4x \). This gives

\[
\frac{d}{d \ln \Lambda} \frac{d}{d \ln g} \left( \frac{1}{2} \int d^8x d^8y (\theta^4)_x (\theta^4)_y \frac{\delta^2 \Delta \Gamma}{\delta V_x \delta V_y} \right) = 4V_4N_f \sum_{l=0}^{n} (-1)^{F_l} \frac{d}{d \ln \Lambda} \times \int \frac{d^4q}{(2\pi)^4} \frac{\partial}{\partial q^\mu} \frac{\partial}{\partial q_\mu} \left( \frac{1}{(q^2 + M_f^2)} \right) \int \frac{d^4k}{(2\pi)^4} k^2 R_k \left( (q + k)^2 + M_f^2 \right) + O(e_0^4). \] (42)

(Certainly, the presence of the regulator \( L^2(x) \) is assumed, so that the integration over \( d^4x \) gives \( V_4 \rightarrow \infty \).) The expression in the right hand side of this equation coincides with the result of explicit two-loop calculations, which is given by the first term in Eq. (28). Thus, Eq. (19) has been verified in the two-loop approximation.

The integrand in Eq. (42) contains singularities due to the identity

\[ \frac{\partial}{\partial q^\mu} \frac{\partial}{\partial q_\mu} \frac{1}{q^2} = -4\pi^2 \delta^4(q). \] (43)

If we define the integrals by the standard way, then we should surround the singularities by small spheres and take into account the corresponding surface integrals. As a consequence, the integral in Eq. (42) corresponds to the trace of commutator minus singularities. This can be illustrated by the following simple example. If \( f(q^2) \) is a noningular function which rapidly decreases at the infinity, then the vanishing trace of commutator corresponds, e.g., to the object

\[
0 = \int \frac{d^4q}{(2\pi)^4} \frac{\partial}{\partial q^\mu} \left( \frac{q^\mu}{q^2} f(q^2) \right) = \int \frac{d^4q}{(2\pi)^4} \left( \frac{\partial}{\partial q^\mu} \frac{q^\mu}{q^2} \right) f(q^2) + \frac{q^\mu}{q^2} \frac{\partial}{\partial q^\mu} f(q^2) = \int \frac{d^4q}{(2\pi)^4} 2\pi^2 \delta(q^2) f(q^2) = \frac{1}{8\pi^2} f(0) + \int \frac{d^4q}{(2\pi)^4} \frac{\partial}{\partial q^\mu} \left( \frac{q^\mu}{q^2} f(q^2) \right), \] (44)

where the last integral is defined by the standard way.

Let us integrate the equality (42) over \( \ln g \) from \( g = 0 \) to \( g = 1 \). For \( g = 1 \) the theory coincides with \( N = 1 \) SQED with \( N_f \) flavors. The case \( g = 0 \) corresponds to the theory without quantum gauge superfield. Therefore, the two-point Green function of the background gauge superfield in this case is contributed only by the one-loop diagram. Thus, using Eq. (17) we obtain

\[
\frac{\beta(\alpha_0)}{\alpha_0^2} - \frac{\beta_{1\text{-loop}}(\alpha_0)}{\alpha_0^2} = 4\pi N_f \frac{d}{d \ln \Lambda} \int \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{e_0^2}{k^2 R_k} \frac{\partial}{\partial q^\mu} \frac{\partial}{\partial q_\mu} \times \left( \frac{1}{q^2(q + k)^2} \right) \int \frac{d^4q}{(2\pi)^4} \frac{\partial}{\partial q^\mu} \frac{\partial}{\partial q_\mu} \left( \frac{1}{(q^2 + M_f^2) ((q + k)^2 + M_f^2)} \right) + O(e_0^4). \] (45)

11
This integral does not vanish due to the singularities of the integrand. Surrounding singular points \( q^2 = 0 \) and \( q^2 = -k^2 \) by small spheres and calculating integrals of total derivatives taking into account all surface integrals we easily find

\[
\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{\beta_{1\text{-loop}}(\alpha_0)}{\alpha_0^2} + \frac{2}{\pi} N_f \frac{d}{d \ln \Lambda} \int \frac{d^4k}{(2\pi)^4} \frac{e_{\alpha_0}^2}{k^4R_k} + O(\epsilon_0^4) = \frac{\beta_{1\text{-loop}}(\alpha_0)}{\alpha_0^2} + \frac{\alpha_0 N_f}{\pi^2} + O(\alpha_0^2). \quad (46)
\]

Substituting the expression for the one-loop \( \beta \)-function of the considered theory we reobtain the well-known result (see, e.g., [25, 35]) for the two-loop \( \beta \)-function

\[
\beta(\alpha_0) = \frac{N_f \alpha_0^2}{\pi} \left( 1 + \frac{\alpha_0}{\pi} + O(\alpha_0^2) \right). \quad (47)
\]

### 3.3 Terms proportional to \( \epsilon_0^4 \) in Eq. (19)

Verification of Eq. (19) in the next order is made similarly. However, in the three-loop approximation one encounters some subtleties, which are absent in the previous order. Let us sequentially discuss application of Eq. (19) to finding expressions for the three-loop diagrams presented in Fig. 1.

1. We will start with the graph (1) in Fig. 1. The corresponding contributions to the functions \( G_I \) and \( J_I \) are given by Eqs. (35) and (36). Terms which are essential in the considered approximation are proportional to \( \epsilon_0^2 - \epsilon^2 \):

\[
\Delta G_I^{(1)} \leftarrow g^2 \int \frac{d^4k}{(2\pi)^4} \frac{2(e_{\alpha_0}^2 - \epsilon^2)(q+k)^2 + g^2}{k^4R_k((q+k)^2 + M_f^2)};
\]

\[
\Delta J_I^{(1)} \leftarrow g^2 \int \frac{d^4k}{(2\pi)^4} \frac{4(e_{\alpha_0}^2 - \epsilon^2)q^2}{k^4R_k((q+k)^2 + M_f^2)}. \quad (48)
\]

Repeating the calculation similarly to the one made in the previous section, we obtain

\[
\frac{d}{d \ln \Lambda} \frac{\partial}{\partial \ln g} \left( \frac{1}{2} \int d^8x d^8y (\theta_4)_x(\theta_4)_y \frac{\delta^2 \Delta \Gamma}{\delta V_x \delta V_y} \right) \leftarrow -4V_4 \cdot g^2 N_f \frac{d}{d \ln \Lambda} \int \frac{d^4q}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} \frac{\epsilon_{\alpha_0}^2 - \epsilon^2}{k^4R_k^2} \frac{\partial}{\partial q^\mu} \frac{\partial}{\partial q_{\mu}} \sum_{i=0}^{n} (-1)^{P_i} \frac{(k+q)^2 + q^2 + 2M_f^2}{(q^2 + M_f^2)(q+k)^2 + M_f^2}. \quad (49)
\]

This contribution coincides with the second term in Eq. (28) after some simple transformations. (The first term was obtained earlier in the previous section.)

2. Diagrams corresponding to the second graph in Fig. 1 give the following contributions to the functions \( G_I \) and \( J_I \):

\[
\Delta G_I^{(2)} = g^3 \int \frac{d^4k}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} \frac{4e_{\alpha_0}^4}{k^2R_k^2R_l} \left( \frac{1}{((q+k)^2 + M_f^2)(q+l)^2 + M_f^2)} + \frac{2}{((q+k)^2 + M_f^2)(q+k+l)^2 + M_f^2)} \right);
\]

\[
\Delta J_I^{(2)} = 0. \quad (50)
\]

Repeating the above calculations in this case we obtain the contribution (29).
3. Diagram (3) in Fig. 1 contains two closed loops of the matter superfields. Constructing the corresponding diagrams which contribute to the two-point function of the matter superfields it is necessary to take into account a possibility of cutting each of these loops. The diagrams obtained as a result of this procedure are presented in the third column of Fig. 1. They give

\[ \Delta G_I^{(3)} = g^3 N_f \int \frac{d^4k}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} \frac{4e_0^4}{k^4 R_k^2} \left( \frac{2}{((q + k)^2 + M_f^2)} \sum_{j=0}^{n} (-1)^{p_j} \frac{1}{l^2 + M_j^2} + \sum_{j=0}^{n} (-1)^{p_j} \frac{1}{(l^2 + M_j^2) ((l + k)^2 + M_j^2)} \right) ; \]

\[ \Delta J_I^{(3)} = 0. \] (51)

Using of Eq. (19) for diagrams with more than one loop of the matter superfields should be made very carefully. The accurate analysis of the calculations made in Ref. [15] shows that in this case it is necessary to add terms with insertions of the double total derivatives into every closed matter loop. For the considered graph this implies that

\[
\frac{d}{d \ln \Lambda} \frac{\partial}{\partial \ln \Lambda} \left( \frac{1}{2} \int d^8x d^8y (\theta^4)_x (\theta^4)_y \frac{\delta^2 \Delta \Gamma}{\delta V_x \delta V_y} \right) \leftarrow -8V_4 \cdot N_f^2 \sum_{l=0}^{n} (-1)^{p_2} \frac{d}{d \ln \Lambda} \int \frac{d^4q}{(2\pi)^4} \times \left\{ \frac{\partial}{\partial q^\mu} \frac{\partial}{\partial q'^\nu} \left( \frac{1}{(q^2 + M_f^2)} \int \frac{d^4k}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} c_0^4 g^4 \left[ \frac{2}{((q + k)^2 + M_f^2)} \sum_{j=0}^{n} (-1)^{p_j} \frac{1}{l^2 + M_j^2} \right] + \frac{1}{(q^2 + M_f^2)} \int \frac{d^4k}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} c_0^4 g^4 \left[ \frac{2}{((q + k)^2 + M_f^2)} \sum_{j=0}^{n} (-1)^{p_j} \frac{1}{l^2 + M_j^2} + \sum_{j=0}^{n} (-1)^{p_j} \frac{1}{(l^2 + M_j^2) ((l + k)^2 + M_j^2)} \right] \right\}. \] (52)

This expression is given by the sum of four terms. The first two terms (containing the derivatives with respect to \( q^\mu \)) are obtained exactly as in the previous section. Two last terms (with the derivatives with respect to \( l^\mu \)) are obtained by inserting the double total derivatives into the closed loop which corresponds to the momentum \( l^\mu \).

Let us now rename the summation indexes and integration variables in the second and fourth terms by making the replacements \( I \leftrightarrow J \) and \( q \leftrightarrow l \). Then we obtain the expression (50), which was found earlier by explicit summation of three-loop diagrams with two external lines of the background gauge superfield.

4. Cutting the matter line in diagram (4) we can obtain a diagram which is not 1PI. The functions \( G_I \) and \( J_I \) are evidently contributed only by the 1PI diagrams, because these functions are constructed from the effective action. In order to construct all relevant 1PI diagrams in the considered case it is also necessary to include 1PI diagrams which are obtained after two cuts of the matter line. This procedure gives the one-loop diagram presented in Fig. 1 in the fourth column. Therefore, calculating the considered contribution we should take into account both the one-loop and two-loop diagrams presented in this column. They correspond to

\[ \Delta G_I^{(4)} = -g^2 \int \frac{d^4k}{(2\pi)^4} \frac{2e_0^2}{k^2 R_k ((q + k)^2 + M_f^2)} \]  

\[ -g^4 \int \frac{d^4k}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} \frac{4e_0^4 ((q + k)^2 - M_f^2)}{k^2 R_k l^2 R_l ((q + k)^2 + M_f^2)^2 ((q + k + l)^2 + M_f^2)} ; \]
\[ \Delta J_i^{(4)} = 0. \] (53)

Then we calculate the same functions as in Eq. (38), but keep the terms proportional to \( g^4 \), which are evidently contributed by both terms in Eq. (53):

\[
\begin{align*}
\frac{G_I}{4(q^2 G_I^2 + M_I^2 J_I^2)} & \leftarrow \frac{q^2 (q^2 - 3M_I^2)}{(q^2 + M_I^2)^3} \int \frac{d^3k}{(2\pi)^3} \frac{d^4l}{(2\pi)^4} k^2 R_k l^2 R_l \left((q + k)^2 + M_I^2\right) \left((q + l)^2 + M_I^2\right) \frac{g^4 \epsilon_0^4}{I} \\
+ \frac{q^2 - M_I^2}{(q^2 + M_I^2)^2} \int \frac{d^3k}{(2\pi)^3} \frac{d^4l}{(2\pi)^4} g^4 \epsilon_0^4 \left((q + k)^2 - M_I^2\right) \\
+ \frac{q^2 G_I^2 + M_I^2 J_I^2}{(q^2 + M_I^2)^2} \leftarrow \frac{q^2 (3q^2 - M_I^2) M_I}{(q^2 + M_I^2)^3} \int \frac{d^3k}{(2\pi)^3} \frac{d^4l}{(2\pi)^4} 4g^4 \epsilon_0^4 \left((q + k)^2 + M_I^2\right) \left((q + l)^2 + M_I^2\right) \\
+ \frac{q^2 M_I}{(q^2 + M_I^2)^2} \int \frac{d^3k}{(2\pi)^3} \frac{d^4l}{(2\pi)^4} 8g^4 \epsilon_0^4 \left((q + k)^2 - M_I^2\right). 
\end{align*}
\]

(54)

Using these expressions we proceed similarly to the previous section and find the contribution

\[
\frac{d}{d\ln \Lambda} \frac{\partial}{\partial \ln g} \left( \frac{1}{2} \int d^8 x \frac{d^8 y}{(\theta^+ x \theta^+)} \frac{\delta^2 \Delta \Gamma}{\delta V_x \delta V_y} \right) \leftarrow 8N_f \sum_{\ell=0}^n (-1)^{\ell} P_\ell \frac{d}{d\ln \Lambda}
\times \int \frac{d^4q}{(2\pi)^4} \frac{\partial}{\partial q^\mu} \left( \frac{q^2 - M_I^2}{(q^2 + M_I^2)^2} \right) \int \frac{d^4k}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} \frac{\epsilon_0^4 g^4}{I} \left((q + k)^2 - M_I^2\right) \\
+ \frac{1}{(q^2 + M_I^2)^2} \int \frac{d^4k}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} k^2 R_k l^2 R_l \left((q + k)^2 + M_I^2\right) \left((q + l)^2 + M_I^2\right) 
\]

Making the substitution \( q^\mu \rightarrow q^\mu - k^\mu \) in the last integral we obtain Eq. (31).

5. The fifth graph in Fig. II is analyzed completely similarly to the two-loop case. The only corresponding diagram contributing to the two-point Green functions of the matter superfields gives

\[
\Delta G_I^{(5)} = -g^4 \int \frac{d^4k}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} \frac{4e_0^4}{I} \left((2q + k + l)^2 + M_I^2\right) \left((q + k)^2 + M_I^2\right) \left((q + l)^2 + M_I^2\right)
\]

\[
\Delta J_I^{(5)} = -g^4 \int \frac{d^4k}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} \frac{4e_0^4 q^2}{I} \left((2q + k + l)^2 + M_I^2\right) \left((q + k)^2 + M_I^2\right) \left((q + l)^2 + M_I^2\right).
\]

(56)

Repeating the transformations of the previous section we obtain Eq. (32).

6. The diagram (6) also contains two matter loops. The corresponding contributions to the two-point Green functions of the matter superfields are given by a single diagram presented in the sixth column of Fig. II. This diagram gives the following contributions to the functions \( G_I \) and \( J_I \):

\[
\Delta G_I^{(6)} = g^4 N_f \int \frac{d^4k}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} \frac{4e_0^4}{I} \left((k^2 - (k + q)^2 - q^2)/(l^2 + M_I^2)\right) \left((k + l)^2 + M_I^2\right) \\
\times \sum_{J=0}^n (-1)^{P_J} \left( \frac{k^2 - (k + q)^2 - q^2}{(l^2 + M_I^2)\left((k + l)^2 + M_I^2\right)} \right) - \frac{2}{l^2 + M_I^2};
\]

(57)
\[
\Delta J_1^{(6)} = -g^4 N_f \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} \frac{8e_0^4 q^2}{k^4 R_k^2 ((q + k)^2 + M_f^2)} \times \sum_{j=0}^{n} (-1)^{P_j} \frac{1}{(l^2 + M_f^2) ((k + l)^2 + M_f^2)}. \tag{57}
\]

Using them we can construct the contribution to the \(\beta\)-function defined in terms of the bare coupling constant according to the prescriptions given above. It is important that we should also add a term in which double total derivatives are inserted into the closed loop. The result has the following form:

\[
\frac{d}{d\ln \Lambda} \frac{\partial}{\partial \ln g} \left( \frac{1}{2} \int d^8 x d^8 y (\theta^i)_x(\theta^i)_y \frac{\delta^2 \Delta \Gamma}{\delta V_x \delta V_y} \right) \leftarrow -8\lambda_4 \cdot g^4 N_f \frac{d}{d\ln \Lambda} \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} \frac{e_0^4}{k^4 R_k^2} \times \left( \frac{\partial}{\partial q^i} \frac{\partial}{\partial q_\mu} + \frac{\partial}{\partial l^2} \frac{\partial}{\partial l^\mu} \right) \sum_{l, J=0}^{n} (-1)^{P_l+P_J} \frac{k^2 - (k + q)^2 - q^2 - (l + k)^2 - l^2 - 2M_f^2 - 2M_f^2}{(l^2 + M_f^2) (q + k)^2 + M_f^2)} \left( (q + k)^2 + M_f^2) \right). \tag{58}
\]

(The term with the derivatives with respect to \(l^\mu\) corresponds to insertion of double total derivatives into the closed loop of the matter superfields.) It is easy to see that (after an appropriate change of variables) this expression coincides with (33).

7. The last (two-loop) graph (7) in Fig. 1 also gives a contribution proportional to \(e_0^2 - e^2 \sim e_0^4\). The corresponding functions \(G_I\) and \(J_I\) are given by

\[
\Delta G_I^{(7)} = -g \int \frac{d^4 k}{(2\pi)^4} \frac{2(e_0^2 - e^2)}{k^4 R_k}; \quad \Delta J_I^{(7)} = 0. \tag{59}
\]

Starting from these expressions by the usual way we obtain the contribution (34):

\[
\frac{d}{d\ln \Lambda} \frac{\partial}{\partial \ln g} \left( \frac{1}{2} \int d^8 x d^8 y (\theta^i)_x(\theta^i)_y \frac{\delta^2 \Delta \Gamma}{\delta V_x \delta V_y} \right) \leftarrow 4\lambda_4 \cdot g N_f \frac{d}{d\ln \Lambda} \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} \frac{e_0^4}{k^4 R_k^2} \left( \frac{\partial}{\partial q^i} \frac{\partial}{\partial q_\mu} \sum_{l=0}^{n} (-1)^{P_l} \frac{1}{(q^2 + M_f^2)} \right) \times \sum_{I, J=0}^{n} (-1)^{P_l+P_J} \frac{k^2 - (k + q)^2 - q^2 - (l + k)^2 - l^2 - 2M_f^2 - 2M_f^2}{(l^2 + M_f^2) (q + k)^2 + M_f^2)} \left( (q + k)^2 + M_f^2) \right). \tag{60}
\]

Thus, we have obtained the correct expressions for all graphs presented in Fig. 1 without explicit calculation of the three-loop diagrams with external lines of the background gauge superfields. This confirms the correctness of Eq. (19) in the considered approximation.

In order to complete the calculation we derive the NSVZ relation between the three-loop \(\beta\)-function and the two-loop anomalous dimension defined in terms of the bare coupling constant of the matter superfields. Taking a sum of Eqs. (28) – (34) and performing the integration over \(\ln g\) from \(g = 0\) to \(g = 1\) after simple transformations we obtain

\[
\frac{\beta(a_0)}{a_0^2} - \frac{\beta_1\text{-loop}(a_0)}{a_0^2} = 4\pi N_f \sum_{l=0}^{n} (-1)^{P_l} \frac{d}{d\ln \Lambda} \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} \frac{e_0^2}{k^2 R_k \partial q^i \partial q_\mu (q^2 + M_f^2)} \times \left\{ \frac{1}{((q + k)^2 + M_f^2)} + \frac{1}{((q + l)^2 + M_f^2)} \right\} \times \left\{ 1 + \frac{d^4 l}{(2\pi)^4} \frac{e_0^2}{l^2 R_l} \left( - \frac{4}{((q + l)^2 + M_f^2)} + \frac{2(q^2 - M_f^2)}{(q^2 + M_f^2) ((q + l)^2 + M_f^2)} \right) \right\} + O(e_0^8). \tag{61}
\]
Then we calculate the integrals of double total derivatives taking into account that they do not vanish only in the massless case, which corresponds to \( I = 0 \). The result is given by the following well-defined integrals:

\[
\frac{\beta(\alpha_0)}{\alpha_0^2} - \frac{\beta_{1\text{-loop}}(\alpha_0)}{\alpha_0^2} = \frac{N_f}{\pi} \frac{d}{d \ln \Lambda} \left\{ \int \frac{d^4k}{(2\pi)^4} \frac{2e_0^2}{k^2R_k} + \int \frac{d^4l}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \frac{4e_0^4}{k^2l^2R_k R_l} \left( -\frac{2}{k^2(k+l)^2} \right) \right\} + \frac{1}{k^2l^2} \left( -N_f \sum_{j=0}^n (-1)^P_j \frac{1}{((l+M_j^2) ((l+k)^2 + M_j^2)} \right) - q^2 \int \frac{d^4k}{(2\pi)^4} \frac{2e_0^4}{k^2l^2R_k (q+k)^2} - \frac{q^2}{e_0^2} \frac{1}{e_0^2} + 2N_f \frac{1}{k^2l^2R_k} \right) + O(e_0^5) \right) \right \} + O(e_0^5). \tag{62}
\]

It is important that the derivative with respect to \( \ln \Lambda \) (which removes infrared divergences) should be calculated at a fixed value of the renormalized coupling constant \( \alpha \).

From the other side, taking the sum of all contributions to the function \( G_{I=0} \equiv G \) and setting \( g = 1 \) we obtain

\[
G(\alpha_0, \Lambda/q) = 1 - \int \frac{d^4k}{(2\pi)^4} \frac{2e_0^2}{k^2R_k (q+k)^2} + \int \frac{d^4l}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \frac{4e_0^4}{k^2l^2R_k R_l} \left( \frac{1}{(q+k)^2(q+l)^2} \right) + \frac{1}{(q+k)^2(q+k+l)^2} - \frac{(2q+k+l)^2}{(q+k)^2(q+l)^2(q+k+l)^2} \right) + N_f \int \frac{d^4k}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} \frac{4e_0^4}{k^2l^2R_k (q+k)^2} \right) \left\{ \frac{1}{e_0^2} - \frac{1}{e_0^2} + 2N_f \frac{1}{k^2l^2R_k} \right) \right) + \right\} + O(e_0^5). \tag{63}
\]

Except for the last term proportional to \( q^2 \) all integrals here do not contain infrared divergences.

The last term vanishes on shell and is finite in the ultraviolet region (in the limit \( \Lambda \to \infty \)), because the expression in the brackets is finite. This term evidently does not contribute to the two-loop anomalous dimension, because it vanishes after differentiation with respect to \( \ln \Lambda \).

That is why it can be omitted (as it was done, e.g., in [66]). The anomalous dimension defined in terms of the bare coupling constant is given by [66]

\[
\gamma(\alpha_0) = \frac{d \ln \Lambda}{d \ln \Lambda} = \frac{d \ln \Lambda}{d \ln \Lambda} \left|_{\alpha = 0} \right. = \frac{d}{d \ln \Lambda} \left\{ \int \frac{d^4k}{(2\pi)^4} \frac{2e_0^2}{k^2R_k} + \int \frac{d^4l}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \frac{4e_0^4}{k^2l^2R_k R_l} \left( \frac{1}{k^2(k+l)^2} - \frac{1}{2k^2l^2} \right) + N_f \int \frac{d^4k}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} \frac{4e_0^4}{k^2l^2R_k (q+k)^2} \right\} \left\{ \frac{1}{(l^2+M_j^2) ((l+k)^2 + M_j^2)} \right) + O(e_0^5) \right\} \right \} + O(e_0^5). \tag{64}
\]

This expression is well-defined, because due to the differentiation with respect to \( \ln \Lambda \) (which should be made before the integrations) there are no infrared divergences. Comparing Eqs. (62) and (63) we obtain the relation

\[
\frac{\beta(\alpha_0)}{\alpha_0^2} - \frac{\beta_{1\text{-loop}}(\alpha_0)}{\alpha_0^2} = -\frac{N_f}{\pi} \gamma(\alpha_0), \tag{65}
\]

which gives Eq. (1) with the argument \( \alpha_0 \) after substituting the well-known expression for the one-loop \( \beta \)-function of the considered theory.
4 Conclusion

The NSVZ relation in supersymmetric theories is obtained because the $\beta$-function defined in terms of the bare coupling constant is given by integrals of (double) total derivatives. In the Abelian case this statement was proved in [14, 15] exactly in all orders. The proof made in Ref. [14] is based on a special identity relating the two-point Green function of the gauge superfield and the two-point Green functions of the chiral matter superfields. This relation allows to construct integrals for the $\beta$-function if only the two-point Green functions for the matter superfields are known. In this paper the identity between the Green functions is verified at the three-loop level by explicit calculations of the Feynman graphs. Starting from the expressions for the Green functions of the chiral matter superfields in the two-loop approximation we construct three-loop integrals for the $\beta$-function defined in terms of the bare coupling constant. Comparing the result with the sum of the corresponding supergraphs we see the agreement. This calculation confirms the results obtained in [15] exactly in all orders, which, in particular, allow to derive the NSVZ $\beta$-function (for the RG functions defined in terms of the bare coupling constant) in the case of using the higher derivative regularization.

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