THE RING LEARNING WITH ERRORS PROBLEM: SPECTRAL DISTORTION

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Abstract. We answer a question posed by Y. Elias and others [8] about possible spectral distortions of algebraic numbers. We provide a closed form for the spectral distortion of certain classes of cyclotomic polynomials. Moreover, we present a bound on the spectral distortion of cyclotomic polynomials.

1. Introduction

A large fraction of lattice-based cryptographic constructions are built upon on Learning With Errors (LWE) problem or its variants learning with errors. The Learning With Errors (LWE) problem introduced by O. Regev [13], relates to solving a “noisy” linear system modulo a known integer. The “algebraically structured” variants, called RLWE [16], PLWE [13], Module-LWE [1]. As other cryptographic problems, LWE is an average-case problem which means the input instances are chosen at random from a prescribed probability distribution.

Since its introduction, the RLWE problem [13] has already been used as a building block for many cryptographic applications. It has since been used as a hardness assumption in the constructions of efficient signature schemes [18], fully-homomorphic encryption schemes [3], pseudo-random functions [2], protocols for secure multi-party computation [7], and also gives an explanation for the hardness of the NTRU cryptosystem [11].

The RLWE and PLWE problems are formulated as either “search” or “decision” problems. Let \( f(x) \in \mathbb{Z}[x] \) to be monic and irreducible of degree \( n \), \( P = \mathbb{Z}[x]/f(x) \), and \( P_q = P/qP \cong \mathbb{F}_q[x]/f(x) \) where \( q \) is a prime.

**Search PLWE Problem.** Let \( s(x) \in P_q \) be a secret. The search PLWE problem, is to discover \( s(x) \) given access to arbitrarily many independent samples of the form \( (a_i(x), b_i(x) = a_i(x)s(x) + e_i(x)) \in P_q \times P_q \), where for each \( i \), \( e_i(x) \) is chosen from a discretized Gaussian distribution of parameter \( \sigma \), and \( a_i(x) \) is uniformly random. The polynomial \( s(x) \) is the secret and the polynomials \( e_i(x) \) are the errors.

**Decision PLWE Problem.** Let \( s(x) \in P_q \) be a secret. The decision PLWE problem is to distinguish, with non-negligible advantage, between the same number of independent samples in two distributions on \( P_q \times P_q \). The first consists of samples of the form \( (a(x), b(x) = a(x)s(x) + e(x)) \) where \( e(x) \) is chosen from a discretized Gaussian distribution of parameter

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\(\sigma\), and \(a(x)\) is uniformly random. The second consists of uniformly random and independent samples from \(P_q \times P_q\).

In [6], an attack on PLWE was presented in rings \(P_q = \mathbb{F}_q[x]/(f(x))\), where \(f(1) \equiv 0 \mod q\). There are also two standard PLWE problems, quoted here from [8]. Let \(\mathbb{K}\) be number field of degree \(n\) with ring of integers \(R\). Let \(R^w\) denote the dual of \(R\), \(R^w = \{\alpha \in \mathbb{K} : \text{Tr}(\alpha x) \in \mathbb{Z} \text{ for all } x \in R\}\). The standard RLWE problems [14] for a canonical discretized Gaussian are defined as follows.

**Search RLWE Problem.** Let \(s \in R^w_q\) be a secret. The search RLWE problem is to discover \(s\) given access to arbitrarily many independent samples of the form \((a, b = as + e)\) where \(e\) is chosen from the canonical discretized Gaussian and \(a\) is uniformly random.

**Decision RLWE Problem.** Let \(s \in R_q\) be a secret. The decision RLWE problem is to distinguish with non-negligible advantage between the same number of independent samples in two distributions on \(R_q \times R^w_q\). The first consists of samples of the form \((a, b = as + e)\) where \(e\) is chosen from the canonical discretized Gaussian and \(a\) is uniformly random, and the second consists of uniformly random and independent samples from \(R_q \times R^w_q\).

In [5], [14] the authors give sufficient conditions on the ring so that the “search-to-decision” reduction for RLWE holds, and also that RLWE instances can be translated into PLWE instances, so that the RLWE decision problem can be reduced to the PLWE decision problem.

**Theorem 1.1** (Search-to-Decision Reduction for RLWE, [5], [14]). There exists a randomized, polynomial time reduction from Search-RLWE to Decision-RLWE.

We investigate the spectral distortion that occurs in the RLWE to PLWE reduction (spectral distortion), a question posed in [8]. Our results include a closed form for the spectral distortion of certain classes of polynomials, and bounds for spectral distortion and related values.

2. Preliminaries

2.1. **Learning with Errors Distributions.** The RLWE distribution is parameterized by \((K, s, q, \sigma)\), where \(K\) is a number field, \(s\) is some secret, \(q\) prime, and \(\sigma\) is the parameter for the error distribution.

**Definition 2.1** (RLWE Distribution, [8]).

For some number field \(K\), let ring \(R = \mathcal{O}_K\) be its ring of integers. Suppose \(q\) to be prime. Then, we define

\[R_q := R/qR.\]

Let \(\mathcal{U}_{R_q}\) be the uniform distribution over \(R_q\), and let \(\mathcal{G}_{\sigma, R_q}\) be the discrete Gaussian distribution centered at 0 with variance \(\sigma^2\) over \(R_q\). Let some \(s \in R_q\) be the secret. Sample \(a\) from the uniform distribution, \(a \leftarrow \mathcal{U}_{R_q}\), and the error \(e\) from the Gaussian distribution, \(e \leftarrow \mathcal{G}_{\sigma, R_q}\). Pairs of the form

\[(a, a \cdot s + e)\]
make up the RLWE distribution $\mathcal{L}_{s,G_\sigma}$ over $R_q \times R_q$. For simplicity, we let $c = a \cdot s + e$, and refer to $(a, c)$ as our sample in the future.

The PLWE distribution is defined similarly; rather than the ring of integers of a number field, the distribution is defined over a polynomial ring. The PLWE distribution is parameterized by $(f, n, s, q, \sigma)$, where $f \in \mathbb{Z}[x]$ is a monic, irreducible polynomial of degree $n$, $s$ is some secret, $q$ prime, and $\sigma$ is the parameter of the error distribution.

**Definition 2.2** (PLWE Distribution, [8]).

Let $f \in \mathbb{Z}[x]$ be monic, irreducible of degree $n$. Assume that $f$ splits over $\mathbb{Z}_q := \mathbb{Z}/q\mathbb{Z}$. Then, we define

$$P := \mathbb{Z}[x]/(f(x)), P_q := P/qP.$$ 

Let $G_{\sigma,P}$ be a discretized Gaussian over $P$ spherical in the power basis of $P (1, x, x^2, \ldots, x^{n-1})$. Let $U_{P_q}$ be the uniform distribution over $P_q$, and let $G_{\sigma,P_q}$ be the discrete Gaussian distribution centered at 0 with variance $\sigma^2$ over $P_q$.

Let some $s \in P_q$ be the secret. Sample $a$ from the uniform distribution, $a \leftarrow U_{P_q}$, and the error $e$ from the Gaussian distribution, $e \leftarrow G_{\sigma,P_q}$.

Pairs of the form 

$$(a, a \cdot s + e)$$

make up the PLWE distribution $\mathcal{L}_{s,G_\sigma}$ over $P_q \times P_q$. Similarly to RLWE, we let $c = a \cdot s + e$, and refer to the samples $(a, c)$.

2.2. **Spectral Distortion.** In this section, we reference several terms commonly associated with the computation of spectral distortion.

**Definition 2.3.** Let $f$ be a monic, irreducible polynomial over $\mathbb{Z}$ of degree $n$, with some root $\alpha$, and all roots $\alpha_i$. Let $M_f$ be the Vandermonde matrix $(\alpha_i^{j-1})_{ij}$. The Minkowski embedding of the number field $K = \mathbb{Q}(\alpha)$ is a function $M : K \to \mathbb{R}^{r_1} \otimes \mathbb{C}^{2r_2}$, where every component of $M$ is a field homomorphism, $r_1$ is the number of real roots of $f$, and $2r_2$ is the number of complex roots of $f$.

Let $B$ be the unitary matrix

$$\begin{bmatrix}
I_{r_1 \times r_1} & 0 & 0 \\
0 & \frac{\sqrt{2}}{2} I_{r_2 \times r_2} & \frac{i\sqrt{2}}{2} I_{r_2 \times r_2} \\
0 & \frac{\sqrt{2}}{2} I_{r_2 \times r_2} & -\frac{i\sqrt{2}}{2} I_{r_2 \times r_2}
\end{bmatrix}$$

The columns of $B$ give an orthonormal basis under which the Minkowski space is isomorphic to $\mathbb{R}^n$ as an inner product space [4]. Note that the $\sqrt{2}$ factor ensures this $B$ is unitary. Because $B$ is unitary, $B^{-1} = B^\dagger$. 

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Remark 2.4. We note here that $B^\dagger M_f = B^{-1} M_f$ is the transpose of the real matrix

$$
\begin{bmatrix}
\sigma_1(1) & \ldots & \sigma_r(1) & \sqrt{2} \Re(\sigma_{r+1}(1)) & \ldots & \sqrt{2} \Re(\sigma_{r+2}(1)) \\
\sigma_1(\alpha) & \ldots & \sigma_r(\alpha) & \sqrt{2} \Re(\sigma_{r+1}(\alpha)) & \ldots & \sqrt{2} \Re(\sigma_{r+2}(\alpha)) \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
\sigma_1(\alpha) & \ldots & \sigma_r(\alpha) & \sqrt{2} \Re(\sigma_{r+1}(\alpha)) & \ldots & \sqrt{2} \Re(\sigma_{r+2}(\alpha))
\end{bmatrix}
$$

We have

$$(B^\dagger M_f)^\dagger (B^\dagger M_f) = M_f^\dagger B B^\dagger M_f = M_f^\dagger M_f$$

Therefore, we may implicitly compute using $B^\dagger M$ instead of $M$. We will use this fact in several of the proofs in this paper.

Because $B^\dagger M_f$ is real, $(B^\dagger M_f)^\dagger (B^\dagger M_f)$ is real, and $(B^\dagger M_f)^\dagger (B^\dagger M_f)$ is conjugate transpose symmetric, so $M_f^\dagger M$ is a real, symmetric matrix.

Definition 2.5. The spectral norm $\|M\|_2$ is the measure of the distortion between RLWE and PLWE for a specific polynomial $f$, given by the largest singular value of $M_f^\dagger M_f$ [9]. The normalized spectral norm, or spectral distortion, provides another measure of distortion that is a convenient quantity in reductions from PLWE to RLWE. The spectral distortion is defined by

$$SD(f) = \frac{\|M_f^{-1}\|_2}{|\det M_f|^{\frac{1}{n}}} = \frac{1}{\sigma_{\text{min}}(M_f)} = \frac{1}{|\det M_f|^{\frac{1}{n}}}$$

3. Cyclotomic Polynomials and Bounds on Spectral Distortion

We first consider the case that $f$ is a cyclotomic polynomial, the current class of candidates for lattice-based homomorphic encryption with ideal lattices [8]. In addition, cyclotomic polynomials tend to have a comparatively smaller spectral norm than general polynomials. In this case, the $M_f^\dagger M_f$ matrix has a convenient formula, from which its eigenvalues can be determined easily in some cases.

Theorem 3.1. Let $n = p_1^{k_1} \cdots p_{\omega(n)}^{k_{\omega(n)}}$, for primes $p_i$ and $k_i \in \mathbb{N}$. Then, the $M_f^\dagger M_f$ matrix is of the following form:

$$(M_f^\dagger M_f)_{ij} = \begin{cases} 
\varphi(n) & \text{if } i = j \\
0 & \text{if } \frac{n}{\varphi(n)} \nmid i - j \\
(-1)^{(\omega(n)+\omega(d)} \left( \frac{n}{\varphi(n)} \right) \varphi \left( \frac{n}{\varphi(n)} \right) & \text{if } \frac{n}{\varphi(n)} \mid i - j
\end{cases}$$

where $d = \gcd \left( \frac{i-j}{n/\varphi(n)}, n \right)$

Proof. Let $c_1, \ldots, c_{\varphi(n)}$ be the integers coprime to $n$, up to $n$. Then, we label the roots of $f$, the primitive $n$-th roots of unity, as $\zeta_n^{c_1}, \ldots, \zeta_n^{c_{\varphi(n)}}$. By properties of $n$-th roots of unity, we know that $\zeta_n^{c_i}$ and $\zeta_n^{c_{\varphi(n)+1-i}}$ are complex conjugates.
Then, we note that the $j$-th row of $M^†$ looks like

\[
\left[ \sqrt{2} \Re (\zeta_{n}^{j}) \cdots \sqrt{2} \Re (\zeta_{n}^{j\varphi(n)/2}) \sqrt{2} \Im (\zeta_{n}^{j}) \cdots \sqrt{2} \Im (\zeta_{n}^{j\varphi(n)/2}) \right]
\]

where $\Re(\zeta_{n}^{i}) = \cos(2\pi c_{l}/n)$ and $\Im(\zeta_{n}^{i}) = \sin(2\pi c_{l}/n)$.

\[
(M_{f}^{†} M_{f})_{ij} = 2 \sum_{l=1}^{\varphi(n)/2} \left( \cos(2\pi c_{l}/n) \cos(2\pi j c_{l}/n) + \sin(2\pi c_{l}/n) \sin(2\pi j c_{l}/n) \right)
\]

\[
= 2 \sum_{l=1}^{\varphi(n)/2} \cos(2\pi c_{l}(i-j)/n) = 2 \sum_{l=1}^{\varphi(n)/2} \Re \zeta_{n}^{l(i-j)}
\]

\[
= \sum_{l=1}^{\varphi(n)/2} (\Re \zeta_{n}^{l(i-j)c_{l}} + \Re \zeta_{n}^{l(i-j)c_{l}}) = \sum_{l=1}^{\varphi(n)} \Re \zeta_{n}^{l(i-j)c_{l}}
\]

Let $g_{l}$ iterate through the $n - \varphi(n)$ integers not coprime to $n$. If $i - j = 0$, then we see that $(M_{f}^{†} M_{f})_{ij} = \varphi(n)$. If $i - j \neq 0$, then we have

\[
\sum_{l=1}^{\varphi(n)} \zeta_{n}^{l(i-j)c_{l}} = \sum_{l=1}^{n - \varphi(n)} \zeta_{n}^{l(i-j)c_{l}} = \sum_{l=0}^{n-1} \zeta_{n}^{l(i-j)d_{l}} = 0 \implies (M_{f}^{†} M_{f})_{ij} = -\Re \sum_{g_{l}} \zeta_{n}^{l(i-j)c_{l}}
\]

The next part of the proof uses inclusion-exclusion on the prime factors of $n$ to count all roots with a nontrivial common factor to $n$ (or all roots not coprime to $n$). Let $p_{1}, \ldots, p_{\omega(n)}$ be the prime factors of $n$ where $\omega(n)$ denotes the number of all distinct prime factors of $n$. For the last term, there is just one possible set of $\omega(n)$ unique prime factors.

\[
- \sum_{g_{l}} \zeta_{n}^{l(i-j)c_{l}} = - \sum_{k=1}^{\omega(n)} \sum_{t=0}^{\omega(n)/p_{k} - 1} \zeta_{n}^{l(i-j)c_{p_{k}}} + \sum_{k<l} \sum_{t=0}^{\omega(n)/p_{k} p_{l} - 1} \zeta_{n}^{l(i-j)c_{p_{k} p_{l}}} + \ldots + (-1)^{\omega(n)} \sum_{t=0}^{n/(\text{rad}(n)) - 1} \zeta_{n}^{l(i-j)c_{p_{\text{rad}(n)}}}
\]

\[
= \sum_{k=1}^{\omega(n)} (-1)^{k} \sum_{p_{l_{1}} \cdots < p_{l_{k}}} \zeta_{n}^{l(i-j)c_{p_{l_{1}} \cdots p_{l_{k}}}}
\]

We observe

\[
\sum_{t=0}^{n/p_{l_{1}} - 1} \zeta_{n}^{l(i-j)c_{p_{l_{1}} \cdots p_{l_{k}}}} = \sum_{t=0}^{n/p_{l_{1}} - 1} \zeta_{n}^{l(i-j)c_{p_{l_{1}} \cdots p_{l_{k}}}}
\]

\[
\begin{pmatrix}
\frac{n}{\text{rad}(n)} & \frac{n}{\text{rad}(n)} - 1
\end{pmatrix}
\]

\[
\frac{n}{\text{rad}(n)} \zeta_{n}^{l(i-j)c_{p_{l_{1}} \cdots p_{l_{k}}}} = \frac{n}{\text{rad}(n)} \zeta_{n}^{l(i-j)c_{p_{l_{1}} \cdots p_{l_{k}}}}
\]

\[
\frac{n}{\text{rad}(n)} \zeta_{n}^{l(i-j)c_{p_{l_{1}} \cdots p_{l_{k}}}} = \frac{n}{\text{rad}(n)} \zeta_{n}^{l(i-j)c_{p_{l_{1}} \cdots p_{l_{k}}}}
\]

Let $\Pi_{r} p_{l_{r}} = \frac{\text{rad}(n)}{\Pi_{l_{1}}} p_{l_{1}}$ be the complement set of $\omega(n) - k$ primes where $\text{rad}(n)$ denotes the product of all distinct prime factors of $n$. Then,
\[
\begin{align*}
&= \sum_{k=1}^{\omega(n)} (-1)^k \sum_{p_{l_1} < \cdots < p_{l_k}} n / \prod_{l} p_{l_k}^{-1} \sum_{t=0}^{n/\prod_{l} p_{l_k}} \zeta_n^{(i-j)t} \prod_{l} p_{l_k} \\
&= \frac{n}{\text{rad}(n)} \sum_{k=1}^{\omega(n)} (-1)^k \sum_{p_{l_1} < \cdots < p_{l_{\omega(n)-k}}} \begin{cases} 
\prod_{l} p_{l_k} & \text{if } \frac{n}{\text{rad}(n)} \mid (i-j) \\
0 & \text{if } \frac{n}{\text{rad}(n)} \nmid (i-j) 
\end{cases} \\
\end{align*}
\]

We see that if \( \frac{n}{\text{rad}(n)} \mid i-j \), then \( n / (i-j) \mid \text{rad}(n) \), and the above summations are all zero. If \( \frac{n}{\text{rad}(n)} \nmid i-j \), then we can factor out \( n / \text{rad}(n) \mid i-j \) from our cases to get

\[
M^\dagger M_{ij} = \frac{n}{\text{rad}(n)} \sum_{k=1}^{\omega(n)} (-1)^k \sum_{p_{l_1} < \cdots < p_{\omega(n)-k}} \begin{cases} 
\prod_{l} p_{l_k} & \text{if } \frac{\prod_{l} p_{l_k}}{n/\text{rad}(n)} \mid (i-j) \\
0 & \text{if } \frac{n}{\text{rad}(n)} \nmid (i-j) 
\end{cases}
\]

Note that since \( \frac{n}{\text{rad}(n)} \mid i-j \), then \( n / (i-j) \mid \text{rad}(n) \), and \( \zeta_n^{(i-j)t} \mid \text{rad}(n) = 1 \). So, the last term of our summation is

\[
(-1)^{\omega(n)} \sum_{t=0}^{n/(\text{rad}(n))} \zeta_n^{(i-j)t} = (-1)^{\omega(n)} \frac{n}{\text{rad}(n)}
\]

If there are no primes \( p_l \) such that \( p \mid \frac{i-j}{n/\text{rad}(n)} \), then all of the other summations are zero, and \( M^\dagger M = (-1)^{\omega(n)} \frac{n}{\text{rad}(n)} \). Otherwise, let \( d = \gcd(\frac{i-j}{n/\text{rad}(n)}, n) \). There exist \( k = \omega(d) \) primes \( q_1, \ldots, q_k \) that do divide \( (i-j)/(n/\text{rad}(n)) \) and \( n \).

Let \( S = q_1, q_2, \ldots, q_k \) be the set of all such primes. Since \( \forall q \in S, q \mid (i-j)/(n/\text{rad}(n)) \), we know that for any subset \( S_1 \subset S \), \( \prod_{q \in S_1} (i-j)/(n/\text{rad}(n)) \).

Moreover, if any product contains primes \( p \) such that \( p \notin S \), then that product cannot divide \( (i-j)/(n/\text{rad}(n)) \), as \( p \nmid (i-j)/(n/\text{rad}(n)) \).

Thus, every nonzero term in our summation corresponds exactly to the product of elements in \( S_1, \forall S_1 \subset S \), and we can rewrite our expression as below.
Let \( c = (-1)^{\omega(n) - \omega(d)} \). We can factor the summation as follows:

\[
(M_f^\dagger M_f)_{ij} = c \cdot \frac{n}{\text{rad}(n)} \left( q_1 \cdots q_k - \sum_{q_1 \cdots q_{k-1} \in S} q_{f_1} \cdots q_{f_{k-1}} + \cdots + (-1)^{k-1} \sum_{q \in S} q + (-1)^k \right)
\]

\[
= c \cdot \frac{n}{\text{rad}(n)} (q_k - 1) \left( q_1 \cdots q_{k-1} - \sum_{q_1 \cdots q_{k-2} \in S \setminus q_k} q_{f_1} \cdots q_{f_{k-2}} + \cdots + (-1)^{k-1} \right)
\]

\[
= c \left( \frac{n}{\text{rad}(n)} \right) (q_k - 1)(q_{k-1} - 1) \cdots (q_2 - 1)(q_1 - 1) = c \left( \frac{n}{\text{rad}(n)} \right) \prod_{q \in S} \varphi(q)
\]

\[
= c \left( \frac{n}{\text{rad}(n)} \right) \varphi \left( \text{rad} \left( \gcd \left( \frac{i-j}{n/\text{rad}(n), n} \right) \right) \right)
\]

We get the desired result

\[
(M_f^\dagger M_f)_{ij} = (-1)^{\omega(n) - \omega(d)} \left( \frac{n}{\text{rad}(n)} \right) \varphi(\text{rad}(d))
\]

\[\square\]

**Corollary 3.2.** Let \( f = \Phi_n \) be \( n \)th cyclotomic polynomial. The \( M_f^\dagger M_f \) matrix for \( f \) is of the form:

\[
M_f^\dagger M_f = \left( \frac{n}{\text{rad}(n)} \right) M_{\Phi_n}^\dagger M_{\Phi_n} \otimes I_{\frac{n}{\text{rad}(n)}}
\]

**Remark 3.3.** Let the eigenvalues of \( M_{\Phi_n}^\dagger M_{\Phi_n} \) be \( \lambda_1, \ldots, \lambda_{\varphi(\text{rad}(n))} \). This implies that the eigenvalues of \( M_n^\dagger M_n \) are \( \frac{n}{\text{rad}(n)} \lambda_1, \ldots, \frac{n}{\text{rad}(n)} \lambda_{\varphi(\text{rad}(n))} \) with multiplicity \( \frac{n}{\text{rad}(n)} \). In particular, for a prime \( p \), \( M_n^\dagger M_n = pI_{\varphi(p)} - 1_{\varphi(p)} \). Also, in particular, for any number \( n \) with prime factor \( p \), \( M_{\Phi_n}^\dagger M_{\Phi_n} = pM_{\Phi_n} \otimes I_p \).

**Remark 3.4.** Note that \( M_{\Phi_n}^\dagger M_{\Phi_n} \) forms a symmetric Toeplitz matrix. \[\square\]

We can also describe the \( M_{\Phi_n}^\dagger M_{\Phi_n} \) matrix’s construction as follows:

- Let \( t = p_1 \cdots p_s \) be a squarefree integer. Then the matrix \( M_f^\dagger M \) for \( \Phi_t \) is given by the symmetric Toeplitz matrix generated by the vector \( v \), where \( v \) is constructed as follows:
  - (1) Let \( v \) be a constant vector of value \((-1)^s\) of length \( \varphi(t) \), indexed by \( i \) from 0 to \( \varphi(t) - 1 \).
  - (2) For all \( i \), if \( p_j \) divides \( i \), then let \( v[i] \leftarrow -\varphi(p_j) \cdot v[i] \).
- Let \( n = p_1^{k_1} \cdots p_s^{k_s} \) be an arbitrary integer and \( L \) be the Toeplitz matrix of \( s \) as constructed above. Then the matrix \( M_f^\dagger M_f \) for \( n \) is given by \( \frac{n}{\text{rad}(n)} L \otimes I_{\frac{n}{\text{rad}(n)}} \) where \( I_q \) is the identity matrix of size \( q \).
- Equivalently, the matrix for \( n \) can be given by

\[
\left( \frac{n}{s} \right) \left( \sum_{i=1}^{s} \left[ \begin{array}{c} 1_{\varphi(s)} \\ 0_{[\varphi(s)/p]-1} \\ 0_{[\varphi(s)/p]} \end{array} \right] \right) \ast \left( p \ast 0_{[\varphi(s)/p]} \otimes I_p \right) \ast \left( \begin{array}{c} 1_{\varphi(s)} \\ 0_{[\varphi(s)/p]} \end{array} \right) \otimes I_{n/s}
\]

---

1A Toeplitz matrix, or a diagonal-constant matrix, is a matrix \( A \) such that \( A_{i,j} = A_{i+1,j+1} \).
where \( \odot \) denotes the Hadamard, or entrywise, product.

**Example 3.5.** For \( f = \Phi_{15} \), we have a symmetric Toeplitz matrix

\[
\begin{bmatrix}
8 & 1 & 1 & -2 & 1 & -4 & -2 & 1 \\
1 & 8 & 1 & 1 & -2 & 1 & -4 & -2 \\
1 & 1 & 8 & 1 & 1 & -2 & 1 & -4 \\
-2 & 1 & 1 & 8 & 1 & 1 & -2 & 1 \\
1 & -2 & 1 & 1 & 8 & 1 & 1 & -2 \\
-4 & 1 & -2 & 1 & 1 & 8 & 1 & 1 \\
-2 & -4 & 1 & -2 & 1 & 1 & 8 & 1 \\
1 & -2 & -4 & 1 & -2 & 1 & 1 & 8
\end{bmatrix}
\]

We can use this rich structure to derive more specific properties of spectral distortion for cyclotomic polynomials. The following theorem shows that the spectral distortion of the \( n \)th cyclotomic polynomial depends only on the radical of \( n \).

**Corollary 3.6.**

\[
\text{SD}(\Phi_n) = \text{SD}(\Phi_{\text{rad} \ n})
\]

**Proof.** Let \( n \geq 1 \). Let \( p \) be a prime that divides \( n \). We show \( \text{SD}(\Phi_n) = \text{SD}(\Phi_{np}) \). For cyclotomic polynomials, \( |\text{Disc}(\Phi_n)| = \prod_{p|n}(p^{\varphi(n)/p-1}) \).

\[
\det(M_{\Phi_{np}})^{1/\varphi(np)} = \sqrt{\frac{(np)^{\varphi(np)}}{\prod_{p|n}(p^{\varphi(n)/p-1})}} = \sqrt{\frac{np}{\prod_{p|n}(p^{1/(p-1)})}}
\]

\[
\det(M_{\Phi_{n}})^{1/\varphi(n)} = \sqrt{\frac{n^{\varphi(n)}}{\prod_{p|n}(p^{\varphi(n)/p-1})}} = \sqrt{\frac{n}{\prod_{p|n}(p^{1/(p-1)})}}
\]

\[\implies \det(M_{\Phi_{np}})^{1/\varphi(np)} = \sqrt{p} \det(M_{\Phi_{n}})^{1/\varphi(n)}\]

We see in Theorem 3.1 that the largest eigenvalue of \( M_{\Phi_{np}} \) increases by a factor of \( p \), so \( \|M_{\Phi_{np}}\| = \sqrt{p}\|M_{\Phi_{n}}\| \). Thus, we have

\[
\text{SD}(\Phi_{np}) = \frac{\sqrt{p} \det(M_{\Phi_{n}})^{\varphi(n)}}{\sigma_{\text{min}}(M_{\Phi_{np}})} = \frac{\sqrt{p} \det(M_{\Phi_{n}})^{\varphi(n)}}{\sqrt{p} \cdot \sigma_{\text{min}}(M_{\Phi_{n}})} = \text{SD}(\Phi_{n})
\]

\[\square\]

**Theorem 3.7.** The eigenvalues of \( M_{\Phi_{p}}^\dagger M_{\Phi_{p}} \) for prime \( p \) are 1 with multiplicity 1 and \( p \) with multiplicity \( p - 2 \).

**Proof.** By \( 3.1 \) \( M_{\Phi_{p}}^\dagger M_{\Phi_{p}} \) is a circulant matrix with row entries \( c_0 = p - 1, c_1 = \cdots = c_{p-2} = -1 \). By well-known properties of circulant matrix eigenvalues, for \( 0 \leq j < p - 2 \), the
eigenvalues of $M^\dagger_\Phi \Phi M^\dagger_\Phi$ are of the form
\[
\lambda_j = c_0 + \sum_{k=1}^{p-2} c_{p-1-k}\zeta^{jk}
\]
\[
= (p - 1) - \sum_{k=1}^{p-2} \zeta^{jk}
\]
If $j = 0$, then
\[
(p - 1) - \sum_{k=1}^{p-2} \zeta^{jk} = (p - 1) - \sum_{k=1}^{p-2} 1 = (p - 1) - (p - 2) = 1
\]
For the other $p - 2$ cases, $j \neq 0$, and
\[
(p - 1) - \sum_{k=1}^{p-2} \zeta^{jk} = (p - 1) + \zeta^0 - \sum_{k=1}^{p-2} \zeta^{jk} = (p - 1) + 1 - \sum_{k=0}^{p-2} \zeta^{jk} = (p - 1) + 1 - 0 = p
\]

**Corollary 3.8.** For prime $p$,
\[
SD(\Phi_p) = p^{\frac{p-2}{2(p-1)}}
\]

**Proof.** For cyclotomic polynomials, $|\text{Disc}(\Phi_n)| = \prod_{p|n}(\frac{p^{\phi(n)}}{p-1})$.
\[
\det(M_p)^{1/(p-1)} = \sqrt[p-1]{\frac{(p-1)^{1/(p-1)}}{p^{1/(p-1)}}} = \sqrt[p-1]{\frac{p^{1/(p-1)}}{p}} = p^{\frac{p-2}{p-1}}
\]
We know that $\text{Det}(M^{-1}) = (\text{Det}(M)^{1/(p-1)})^{-1} = p^{-\frac{p-2}{2(p-1)}}$. We know also from 3.7 that the smallest eigenvalue of $M^\dagger_\Phi \Phi M^\dagger_\Phi$ for prime $p$ is 1. So,
\[
\|M^{-1}_\Phi\| = \frac{1}{\sigma_{\min}(M^\dagger_\Phi)} = 1
\]
\[
SD(\Phi_n) = \frac{\|M^{-1}_\Phi\|_2}{|\det(M^{-1}_\Phi)|^{1/(p-1)}} = \frac{1}{p^{\frac{p-2}{2(p-1)}}} = p^{\frac{p-2}{2(p-1)}}
\]

**Lemma 3.9.** The $M^\dagger_\Phi M^\dagger_\Phi$ matrix for $f = \Phi_{2n}$, $2 \not| n$, is of the form:
\[
(M^\dagger_\Phi \Phi M^\dagger_\Phi)_{ij} = (-1)^{i+j}(M^\dagger_\Phi \Phi M^\dagger_\Phi)_{ij}
\]

**Proof.** Note that since $2 \not| n$, $\phi(2n) = 2(1 - \frac{1}{2})\phi(n) = \phi(n)$, and $\frac{2n}{\text{rad}(2n)} = \frac{2n}{2\text{rad}(n)} = \frac{n}{\text{rad}(n)}$.
We need to check each case given in 3.1.

**Case 1:** $i = j$
In this case,

\[
    (M_{\Phi_{2n}}^\dagger M_{\Phi_{2n}})_{ij} = \phi(2n) = 2 \left(1 - \frac{1}{2}\right) \phi(n) = \phi(n)
\]

\[
    = (M_{\Phi_{n}}^\dagger M_{\Phi_{n}})_{ij} = (-1)^{i+j} (M_{\Phi_{n}}^\dagger M_{\Phi_{n}})_{ij}
\]
as \(2 \mid (i + j)\).

**Case 2:** \(\frac{2n}{\text{rad}(2n)} \mid (i - j)\)

Since \(\frac{2n}{\text{rad}(2n)} = \frac{n}{\text{rad}(n)}\), then \(\frac{n}{\text{rad}(n)} \mid (i - j)\), and

\[
    (M_{\Phi_{2n}}^\dagger M_{\Phi_{2n}})_{ij} = 0 = (-1)^{i+j} (M_{\Phi_{n}}^\dagger M_{\Phi_{n}})_{ij}
\]

**Case 3:** \(\frac{2n}{\text{rad}(2n)} \mid (i - j)\)

Recall that \(\omega(n)\) is the number of distinct prime factors of \(n\). Note that \(\omega(2n) = \omega(n) + 1\), as \(2 \nmid n\).

Consider when \(2 \mid (i - j)\). Then, \(2 \mid \frac{i-j}{n/\text{rad}(n)}\), and

\[
    \gcd \left( \frac{i-j}{2n/\text{rad}(2n)}, 2n \right) = \gcd \left( \frac{i-j}{n/\text{rad}(n)}, 2n \right) = \gcd \left( \frac{i-j}{n/\text{rad}(n)}, n \right)
\]
so \(d_{2n} = d_n\). Thus,

\[
    (M_{\Phi_{2n}}^\dagger M_{\Phi_{2n}})_{ij} = (-1)^{s_n + \omega(d_n) + 1} \left(\frac{n}{\text{rad}(n)}\right) \phi(\text{rad}(d_n))
\]

\[
    = - (M_{\Phi_{n}}^\dagger M_{\Phi_{n}})_{ij} = (-1)^{i+j} (M_{\Phi_{n}}^\dagger M_{\Phi_{n}})_{ij}
\]

Consider now when \(2 \mid (i - j)\). Then, \(2 \mid \frac{i-j}{n/\text{rad}(n)}\), and

\[
    \gcd \left( \frac{i-j}{2n/\text{rad}(2n)}, 2n \right) = 2 \gcd \left( \frac{i-j}{n/\text{rad}(n)}, n \right)
\]
so \(d_{2n} = 2d_n\), and \(\omega(d_{2n}) = \omega(d_{2n}) + 1\). Thus,

\[
    (M_{\Phi_{2n}}^\dagger M_{\Phi_{2n}})_{ij} = (-1)^{s_n + \omega(d_n) + 1 + 1} \left(\frac{n}{\text{rad}(n)}\right) \phi(\text{rad}(d_n))
\]

\[
    = (M_{\Phi_{n}}^\dagger M_{\Phi_{n}})_{ij} = (-1)^{i+j} (M_{\Phi_{n}}^\dagger M_{\Phi_{n}})_{ij}
\]

\[\square\]

**Lemma 3.10.** Let \(A\) be a matrix. The matrix \((-1)^{i+j} A_{ij}\) has the same eigenvalues as \(A\).

**Proof.** The eigenvalues of \(A\) are defined by the characteristic equation \(\det(\lambda I - A)\).
By the Leibniz formula for determinants,
\[
\det(\lambda I - ((-1)^{i+j} A_{ij})) = \sum_\sigma (-1)^\sigma \prod_i (\lambda I - (-1)^{i+\sigma(i)} A_{i\sigma(i)})
\]

Taking out the identity permutation, we have
\[
\prod_i (\lambda I - A_{ii}) + \sum_{\sigma/i} (-1)^\sigma \prod_i (-1)^{i+\sigma(i)} A_{i\sigma(i)}
\]

Because
\[
\prod (-1)^{i+\sigma(i)} = \prod (-1)^i \prod (-1)^{\sigma(i)} = (-1)^{\varphi(n)/2} = 1
\]

We have
\[
\prod_i (\lambda I - A_{ii}) + \sum_{\sigma/i} (-1)^\sigma \prod_i A_{i\sigma(i)} = \det(\lambda I - A)
\]

\[\square\]

**Theorem 3.11.** Let \( n \in \mathbb{N} \) be odd. The eigenvalues of \( M_{\Phi_{2n}}^\dagger M_{\Phi_{2n}} \) are the same as the eigenvalues of \( M_{\Phi_n}^\dagger M_{\Phi_n} \).

**Proof.** From Lemma 3.9 we know that \( (M_{\Phi_{2n}}^\dagger M_{\Phi_{2n}})_{ij} = (-1)^{i+j}(M_{\Phi_n}^\dagger M_{\Phi_n})_{ij} \). The proof then follows directly from the above lemma 3.10. \[\square\]

**Corollary 3.12.** For odd \( n \),
\[
\text{SD}(\Phi_{2n}) = \text{SD}(\Phi_n)
\]

**Proof.** First we look at the denominator, \( \det(M_{\Phi_n})^{1/\varphi(n)} \):
\[
\det(M_{\Phi_n})^{1/\varphi(n)} = \sqrt{\frac{n^{\varphi(n)}}{\prod_{p\mid n}(p^{\varphi(n)/p-1})}} = \sqrt{\frac{n}{\prod_{p\mid n}(p^{1/(p-1)})}}
\]
\[
\det(M_{\Phi_{2n}})^{1/\varphi(2n)} = \sqrt{\frac{(2n)^{\varphi(2n)}}{\prod_{p\mid (2n)}(p^{\varphi(2n)/p-1})}} = \sqrt{\frac{2n}{\prod_{p\mid (2n)}(p^{1/(p-1)})}}
\]
\[
= \sqrt{\frac{2n}{2(\prod_{p\mid n}(p^{1/(p-1)})}}} = \sqrt{\frac{n}{\prod_{p\mid n}(p^{1/(p-1)})}}
\]
\[
\Rightarrow \det(M_{\Phi_n})^{1/\varphi(n)} = \det(M_{\Phi_{2n}})^{1/\varphi(2n)}
\]

From Theorem 3.11 we know that the eigenvalues of \( M_{\Phi_{2n}}^\dagger M \) are the same as those of \( M_{\Phi_n}^\dagger M \), and therefore the spectral norm for \( 2n \) and \( n \) are the same. It follows that \( \text{SD}(\Phi_{2n}) = \text{SD}(\Phi_n) \). \[\square\]

3.1. **Non-Cyclotomic Polynomials.** We now turn to results that encompass non-cyclotomic polynomials.
Theorem 3.13. Let $h(x)$ be a monic, irreducible polynomial over $\mathbb{Z}$. Let $f(x) = h(x^k)$. Let $\alpha_t$ be the roots of $h(x)$.

\[
(M_f^\dagger M_f)_{ij} = \begin{cases} 
k \left( \sum_{\text{real } \alpha_t} \alpha_t^{(i+j)/k} + \sum_{\text{non-real } \alpha_t} \alpha_t^{i/k} \overline{\alpha_t^{j/k}} \right) & \text{if } k \mid i - j \\
0 & \text{if } k \nmid i - j \end{cases}
\]

Proof.

\[
(M_f^\dagger M_f)_{ij} = \sum_{\alpha_t} \sum_{s=0}^{k-1} \left( \zeta_k^s \alpha_t^{1/k} \right)^i \overline{\left( \zeta_k^s \alpha_t^{1/k} \right)^j} = \sum_{\alpha_t} \alpha_t^{i/k} \overline{\alpha_t^{j/k}} \sum_{s=0}^{k-1} \zeta_k^{s(i-j)}
\]

\[
= \begin{cases} 
k \sum_{\alpha_t} \alpha_t^{i/k} \overline{\alpha_t^{j/k}} & \text{if } i - j = 0 \mod k \\
0 & \text{if } i - j \neq 0 \mod k \end{cases}
\]

If $i = j \mod k$, then

\[
(M_f^\dagger M_f)_{ij} = k \left( \sum_{\text{real } \alpha_t} \alpha_t^{(i+j)/k} + \sum_{\text{non-real } \alpha_t} \alpha_t^{i/k} \overline{\alpha_t^{j/k}} \right)
\]

\[\square\]

Corollary 3.14. Let $s = i \mod k$.

\[
(M_f^\dagger M_f)_{ij} = \begin{cases} 
k(M_h^\dagger M_h)_{i',j', h(0)^{s/k}} & \text{if } k \mid i - j \\
0 & \text{if } k \nmid i - j \end{cases}
\]

and

\[
M_f^\dagger M_f = M_h^\dagger M_h \otimes 
\begin{bmatrix}
h(0)^{0/k} & 0 & 0 & \cdots \\
0 & h(0)^{1/k} & 0 & 0 & \cdots \\
0 & 0 & \ddots \\
\vdots & 0 & h(0)^{k-1/k} & h(0)^{0/k} & h(0)^{1/k} & \cdots \\
& & & & & h(0)^{k-1/k}
\end{bmatrix}
\]
Proof. When \( i \equiv j \mod k \), we have \( i = i'k + s \) and \( j = j'k + s \) for some \( s \leq k, i', j' \in \mathbb{Z} \). Then,

\[
k \left( \sum_{\alpha \in \mathbb{R}} \alpha^{((i'k+s)+(j'k+s))/k} + \sum_{\alpha \not\in \mathbb{R}} \alpha^{((i'k+s)+j'k+s)/k} \right) = k \left( \sum_{\alpha \in \mathbb{R}} \alpha^{(i'+j'+2s)/k} + \sum_{\alpha \not\in \mathbb{R}} \alpha^{s/k} \alpha^{i' \alpha} \right)
\]

If \( h(x) \) is a quadratic polynomial with negative discriminant, then \( |\alpha|^{2s/k} = h(0)^{s/k} \), so we have the listed results.

\[ \square \]

**Corollary 3.15.** Let \( h(x) = x^2 + bx + c \) have negative discriminant.

\[
M_h^iM_h = \begin{bmatrix} 2 & -b \\ -b & 2c \end{bmatrix}
\]

\( M_h^iM_h \) has characteristic polynomial

\[
(\lambda - 2)(\lambda - c) - b^2 = \lambda^2 - (2 + 2c)\lambda - b^2
\]

And eigenvalues

\[ 1 + c \pm \sqrt{b^2 + c^2 + 2c + 1} \]

Therefore, we can calculate the eigenvalues and therefore spectral norm of \( M_f \) for all \( f(x) = h(x^k) \).

### 3.2. Bounds on Spectral Distortion

In [12], Hong and Pan derive a lower bound on the smallest singular value of general matrices \( A \):

\[
\sigma_{\min}(A) \geq \left( \frac{n-1}{n} \right)^{(n-1)/2} |\det(A)| \max \left\{ \frac{r_{\min}(A)}{\prod_{i=0}^{n} r_i(A)}, \frac{c_{\min}(A)}{\prod_{i=0}^{n} c_i(A)} \right\}
\]

where \( r_i \) is the \( L^2 \) norm of the \( i \)th row, and \( c_i \) is the \( L^2 \) norm of the \( i \)th column.

We use this lower bound to create an upper bound for general spectral distortion:

**Theorem 3.16.** Let \( r_i \) be the \( L^2 \) norm of the \( i \)th row of \( M_f \), and \( c_i \) be the \( L^2 \) norm of the \( i \)th column of \( M_f \). For a polynomial \( f \) of degree \( n \),

\[
\text{SD}(f) \leq \left( \frac{n}{n-1} \right)^{(n-1)/2} |\det(M_f)|^{\frac{1}{n}} \max \left\{ \frac{r_{\min}(M_f)}{\prod_{i=0}^{n} r_i(M_f)}, \frac{c_{\min}(M_f)}{\prod_{i=0}^{n} c_i(M_f)} \right\}
\]

Proof.

\[
\text{SD}(f) = \frac{\|M_f^{-1}\|_2}{|\det(M_f)|^{\frac{1}{n}}} = \frac{1}{\sigma_{\min}(M_f)} = \frac{|\det M_f|^\frac{1}{n}}{\sigma_{\min}(M_f)}
\]

\[
\text{SD}(f) \leq \frac{|\det M_f|^\frac{1}{n}}{\left( \frac{n-1}{n} \right)^{(n-1)/2} |\det(M_f)|^{\frac{1}{n}} \prod_{i=0}^{n} (M_f)} = \left( \frac{n}{n-1} \right)^{(n-1)/2} |\det(M_f)|^{\frac{1}{n}} \prod_{i=0}^{n} \frac{1}{r_{\min}(M_f)}
\]
\( \implies \text{SD}(f) \leq \left( \frac{n}{n-1} \right)^{(n-1)/2} |\det(M_f)| \prod_{i=0}^{n-1} r_i(M_f) / \min r_{\text{min}}(M_f) \)

Similarly, in [17], Yu and Gu presented another lower bound on the minimum singular value based on the Frobenius norm. With the Frobenius norm defined as

\( \|A\|_F^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \)

The minimum singular value of matrix \( A \) is bounded as follows:

\[ \sigma_{\text{min}}(A) \geq |\det A| \left( \frac{n-1}{\|A\|_F^2} \right)^{n-1/2} \]

We use this now to propose another bound on spectral distortion.

**Theorem 3.17.** For a polynomial \( f \) of degree \( n \),

\[ \text{SD}(f) \leq \left( \frac{\|M_f^2\|}{n-1} \right)^{(n-1)/2} |\det M_f|^{1-n} \]

**Proof.**

\[ \text{SD}(f) = \frac{\|M_f^{-1}\|_F}{|\det M_f|^{1/n}} = \frac{\frac{1}{\sigma_{\text{min}}(M_f)}}{|\det M_f|^{1/n}} = \frac{|\det M_f|^{1/n}}{\sigma_{\text{min}}(M_f)} \]

\[ \text{SD}(f) \leq \frac{|\det M_f|^{1/n}}{\left( \frac{n-1}{\|M_f^2\|_F} \right)^{n-1/2} |\det M_f|} \]

This implies

\[ \text{SD}(f) \leq \left( \frac{\|M_f^2\|}{n-1} \right)^{(n-1)/2} |\det M_f|^{1-n} \]

\( \square \)

4. Conclusion

In this paper, we showed that the \( M_f^\dagger M_f \) matrix from which the spectral distortion is derived has a convenient formula with special properties for the case of a cyclotomic polynomial \( f \). Moreover, we derived mild generalizations of these properties for non-cyclotomic polynomials. Finally, we found bounds on the eigenvalues of this matrix for the general case, as well as bounds on the spectral distortion in the cyclotomic case.

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