Solving Routing Problems via Important Cuts*

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Abstract. We introduce a novel approach of using important cuts which allowed us to design significantly faster fixed-parameter tractable (FPT) algorithms for the following routing problems: the Mixed Chinese Postman Problem parameterized by the number of directed edges (Gutin et al., JCSS 2017), the Minimum Shared Edges problem (MSE) parameterized by the number \( p \) of paths between two specified vertices (Fluschnik et al., JCSS 2019), and the Weighted Min Cut Prevention problem (Grüttmeier et al., WG 2021). The Minimum Vulnerability problem (MV) is a generalization of MSE (Assadi et al., Algorithmica 2014). The only known FPT algorithm for MV parameterized by \( p \) (the same parameter as for MSE) was for chordal graphs (Aoki et al., JCO 2018). We design an FPT algorithm for MV on all undirected graphs.

Keywords: important cut · parameterized algorithms · Mixed Chinese Postman Problem · Minimum Vulnerability Problem · Min Cut Prevention Problem

1 Introduction

The notion of an important cut in a graph was introduced by Marx [13] in 2006 and since then it has been used in designing fixed-parameter tractable (FPT) algorithms for many problems (cf. [9] Chapter 8). In this paper, we introduce a novel approach of using important cuts which allowed us to design significantly faster FPT algorithms for three routing problems and the first FPT algorithm for another routing problem; the four routing problems are stated and discussed below. The new approach can be used for other routing problems, e.g. see the problem stated in Section 7.

We say that the function \( f(k) \) is \textit{polynomially-single exponential} (\textit{polynomially-double exponential}, respectively) if \( f(k) = O(2^{q(k)}) \) \( (f(k) = O(2^{2^{q(k)}}), \) respectively), where \( q(k) \) is a polynomial in \( k \). The new approach is based on elimination of small-size important cuts by adding copies of edges in small-size cuts. Thus, in this paper, we allow directed, undirected and mixed graphs to have

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multiple edges, but not loops. The new approach is quite simple, but leads to
decreasing running times of FPT algorithms from polynomially-double exponential
to polynomially-single exponential with respect to the parameter. The signif-
ificant decrease in time complexity is due to using important-cut-based branching
algorithms instead of dynamic programming algorithms.

The Chinese Postman Problem (CPP) is a classic optimization problem
in graph theory and combinatorial optimization, where given an undirected,
directed or mixed weighted graph (mixed graphs may contain both directed
and undirected edges), the aim is to find a closed walk of minimum weight
which includes every edge. The undirected and directed variants of
the problem are polynomial-time solvable. However, CPP is NP-complete on
mixed graphs. The NP-completeness even holds for planar graphs with each
vertex having total degree at most 3 and all edges and arcs having weight 1 [15].
Thus, the Mixed Chinese Postman Problem (MCPP) has been studied
from the parameterized perspective. MCPP is fixed-parameter tractable (FPT)
when parameterized either by the number of undirected edges [4] or by the
number of directed edges [11]. It has also been studied with respect to various
structural parameters. For instance, MCPP is W[1]-hard when parameterized by
pathwidth, and FPT when parameterized by treedepth of the given graph [12].

The algorithm in [11] first reduces MCPP to the Balanced Chinese Post-
man Problem (BCPP) in FPT time. The authors of [11] designed a dynamic
programming algorithm to solve the BCPP problem, which is still NP-hard.
The dynamic programming algorithm uses the fact that the torso graph of
small cuts has bounded treewidth. Together the parameterized Turing reduction
from MCPP to BCPP and the dynamic programming algorithm for BCPP solve
MCPP. The resulting algorithm is rather complicated and runs in polynomially-
double exponential time. Bevern et al. [4] asked whether there is an algorithm
running in single exponential time $O^*(c^k)$ for MCPP parameterized by $k$, as it
would be more promising for the use in practice. Here $c$ is a constant, $k$ is the
number of directed edges and . Using the new approach, we design a faster FPT
algorithm to solve MCPP. While our algorithm does not answer the question of
Bevern et al., it makes a significant progress by reducing the running time from
polynomially-double exponential to polynomially-single exponential.

Given an undirected graph $G$, two vertices $s$ and $t$ in $G$, two integers $p$
and $k$, the aim of the Minimum Shared Edges problem (MSE) is to find $p$
paths between $s$ and $t$ in $G$, sharing at most $k$ edges. Here, an edge is shared
if it appears in at least two of the $p$ paths. Fluschnik et al. [9] proved that
(i) MSE is W[1]-hard when parameterized by the combination of $k$ and the
treewidth of $G$, (ii) MSE is W[2]-hard when parameterized by $k$ only, and (iii)
MSE is FPT parameterized by $p$. A generalization of MSE, called the Minimum
Vulnerability problem (MV), was introduced by Assadi et al. [2]. Given a
weighted graph $G$, two integers $r, p$, two vertices $s, t \in V(G)$, and a capacity
function $c$ on every edge, the aim of MV is to find $p$ $s - t$ paths in $G$, such
that the total weight of edges that are used in more than $r$ of the $p$ paths is
minimized. Here we also require every edge $e$ to appear in at most $c(e)$ paths.
Assadi et al. proved that MV on directed graphs admits an XP-algorithm when parametrized by \( p \). Aoki et al. [1] showed that MV on undirected graphs is NP-hard even on bipartite series-parallel graphs, but admits a pseudo-polynomial-time algorithm on graphs with bounded treewidth. In addition, Aoki et al. proved that MV is FPT on chordal graphs when parameterized by \( p \). The dynamic programming FPT algorithms in [9] and [16] for the parameter \( p \) use the fact that the torso graph of small cuts has bounded treewidth [14]. Their algorithms are polynomially-double exponential. In the present paper, we avoid using dynamic programming and manage to design a simple branching algorithm for MV on all undirected graphs which runs in polynomially-single exponential time.

In [10], Grüttemeier et al. introduced the Weighted Min Cut Prevention problem (WMCP): Given a weighted undirected graph \( G \), two vertices \( s \) and \( t \) in \( G \), an integer \( p \), an attacker wants to find an \((s,t)\)-cut of capacity at most \( p \) and a defender wants to protect edges in order to increase the capacity of any minimum \((s,t)\)-cut in \( G \) to at least \( p+1 \). The aim is to find a set \( D \) of edges with minimum total weight, such that the size of minimum \((s,t)\)-cut is at least \( p+1 \) if the edges in \( S \) are not allowed to be in the cut. While Grüttemeier et al. designed a polynomially-double exponential FPT algorithm for WMCP with respect to the parameter \( p \), our FPT algorithm is polynomially-single exponential in \( p \).

The paper is organized as follows. The next section provides further terminology and notation in graph theory and parameterized algorithmics. Section 3 discusses important cuts and their elimination via adding copies of existing edges. In particular, the section contains Lemma 3 which is a key result for our approach to solving routing problems. Sections 4, 5 and 6 are devoted to designing polynomially-single exponential FPT algorithms for MCPP, MV and WMCP, respectively. We conclude the paper in Section 7.

### 2 Further Terminology and Notation

For two positive integers \( p \) and \( q \) with \( q < p \), \([q, p]\) denotes the set \( \{q, q+1, \ldots, p\} \) and \([p]\) denotes the set \([1, p]\). We use \( \mathbb{N}_{>0} \) and \( \mathbb{N} \) to denote the set of positive and non-negative integers respectively.

In this paper, we allow multiple edges in undirected, directed, and mixed graphs, but no loops are allowed. A mixed graph may have both directed and undirected edges. Henceforth, we will often call a directed edge an arc and an undirected edge an edge. To avoid confusion, we denote an edge between two vertices \( u, v \) as \( uv \), and an arc from \( u \) to \( v \) as \( \overrightarrow{uv} \). For a graph \( G \), the \textit{multiplicity} \( \mu_G(\overrightarrow{uv}) \) (\( \mu_G(uv) \), respectively) of an arc \( \overrightarrow{uv} \) (an edge \( uv \), respectively) is the number of arcs (edges, respectively) of the form \( \overrightarrow{uv} \) (\( uv \), respectively) in \( G \).

\textit{Multiplication of an edge or arc} \( e \) in \( G \) is the addition of \( t \geq 0 \) copies of \( e \) to \( G \), thus increasing \( \mu(e) \) by \( t \). A graph \( H \) is a \textit{multiplication} of \( G \) if \( H \) can be obtained from \( G \) by multiplication of edges and arcs in \( G \). For a mixed graph \( G \), let a \textit{multi-orientation} \( D \) be a digraph derived from \( G \) by replacement of each edge \( uv \) in \( G \) with one or two of the arcs \( \overrightarrow{uv} \) and \( \overrightarrow{vu} \) and then multiplication of all arcs. If \( D \) is a multi-orientation of \( G \) and \( \mu_D(\overrightarrow{uv}) + \mu_D(\overrightarrow{vu}) = \mu_G(\overrightarrow{uv}) + \mu_G(uv) + \mu_G(\overrightarrow{vu}) \)
for each $u, v \in V$ of $G$, then $D$ is an orientation. If $D$ is an orientation of an undirected graph $G$, then $G$ is the undirected version of $D$. A digraph $G$ is connected if its undirected version is connected. All walks in directed and mixed graphs traverse arcs in their directions. A digraph $G$ is strongly connected if for any two vertices $u$ and $v$ of $G$, there is a path from $u$ to $v$.

For a directed graph $D = (V, A)$ and $v \in V(D)$, $d^+_D(v)$ and $d^-_D(v)$ denote the out-degree and in-degree of $v$ in $D$, respectively. Let $t : V \to Z$ be a function and $V^+_t = \{ u \in V, t(u) > 0 \}$, $V^-_t = \{ u \in V, t(u) < 0 \}$. A vertex $u$ in $D$ is $t$-balanced if $d^+_D(u) - d^-_D(u) = t(u)$. The directed graph $D$ is $t$-balanced if every vertex is $t$-balanced. A function $t : V \to Z$ is zero-sum if $\sum_{v \in V(D)} t(v) = 0$. Note that if $D$ is $t$-balanced then $t$ is zero-sum. A directed graph is Eulerian if it contains an Eulerian trail i.e., a closed trail which contains every arc exactly once and every vertex at least once. Recall that a digraph is Eulerian if and only if it is connected and every vertex has equal in-degree and out-degree.

For an undirected graph $G = (V, E)$, and two vertex sets $X, Y \subseteq V(G)$, a path between two vertices $u \in X$ and $v \in Y$ is called a $u - v$ path, also an $X - Y$ path. An $(X, Y)$-cut is an edge set $S \subseteq E$, such that $G - S$ contains no $X - Y$ path. When $X = \{ x \}$ and $Y = \{ y \}$, we speak of an $(x, y)$-cut. A set $S$ of edges is a minimal $(X, Y)$-cut if there is no $X - Y$ path in $G - S$ and no proper subset of $S$ is an $(X, Y)$-cut. For a vertex set $R$, $\delta(R)$ denotes the set of edges with exactly one endpoint in $R$. Note that $\delta(R)$ is an $(R, V \setminus R)$-cut. It is well-known [6] that an $(X, Y)$-cut $S$ can be expressed as $S = \delta(R)$, for some vertex set $R$ with $X \subseteq R$ and $R \cap Y = \emptyset$. Given an undirected graph $G = (V, E)$, two vertices $s, t \in V$, and an integer $p$, a $(p, s, t)$-routing is a set of $p s - t$ paths. An edge is called shared if it is contained in at least two of the paths in the routing.

**Parameterized Complexity.** A parameterized problem is a set of instances $(I, k)$, where $I \in \Sigma^*$ for a finite alphabet $\Sigma$, and $k \in \mathbb{N}$ is the parameter. A parameterized problem $Q$ is fixed-parameter tractable (FPT) if there exists an algorithm that on input $(I, k)$ decides whether $(I, k)$ is a yes-instance of $Q$ in $f(k)|I|^{O(1)}$ time, where $f$ is a computable function independent of $|I|$. In parameterized algorithmics, the expression $f(k)|I|^{O(1)}$ is often replaced by $O^*(f(k))$ i.e., unlike $O$, $O^*$ hides not only constants but also polynomial factors. A parameterized Turing reduction from a parameterized problem $P_1$ to a parameterized problem $P_2$ is an FPT algorithm $A$ that decides $P_1$ and has an oracle to $P_2$ such that there is a computable function $g : \mathbb{N} \to \mathbb{N}$ for which all oracle queries posed by $A$ of whether $(I', k') \in P_2$, $k' \leq g(k)$.

### 3 Important Cuts and Elimination of Cuts

Marx [13] introduced the notion of an important cut, which allowed researchers to solve many parameterized problems [6]. In this section, $G = (V, E)$ is an undirected graph, $X, Y \subseteq V$ and $p$ a is positive integer.

**Definition 1.** [6] Let $X, Y \subseteq V$. A minimal $(X, Y)$-cut $\delta(R)$ is important if there is no $(X, Y)$-cut $\delta(R')$ with $R \subset R'$ and $|\delta(R')| \leq |\delta(R)|$. 

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Note: The above text contains a summary of concepts from a parameterized complexity viewpoint, focusing on important cuts and the elimination of cuts. The definitions and theorems are adapted from the referenced sources to provide a concise explanation of the key ideas. The context of the text is designed to help readers understand the foundational aspects of parameterized complexity, particularly in relation to cuts in graphs. The text is structured to ensure clarity and coherence, with specific references to key definitions and theorems to guide further exploration of the topic.
A cut of $G$ is *small* if its size is smaller than $p$. The next lemma follows directly from the definition of important cut. It implies that in order to eliminate small $(X,Y)$-cuts, it suffices to eliminate important small $(X,Y)$-cuts.

**Lemma 1.** If there is a small $(X,Y)$-cut, then there is an important small $(X,Y)$-cut.

**Lemma 2.** Let $S$ be an important small $(X,Y)$-cut in $G$. If no edge in $S$ is multiplicated, then no matter how many copies of edges in $E \setminus S$ are added, $S$ is still a small $(X,Y)$-cut.

*Proof.* Since $S$ is an important $(X,Y)$-cut, there is no $X - Y$ path in $G - S$. Thus, no matter how many copies of edges in $G - S$ are added into $G$, there is no $X - Y$ path in $G - S$. Therefore $S$ is still of size less than $p$. □

By Lemma 2 in order to eliminate small $(X,Y)$-cuts, some edges in every important small $(X,Y)$-cut must be multiplicated. The following lemma is a key result in our process of elimination of small cuts. The lemma holds by the argument in the first paragraph after [6, Proposition 8.8] and [6, Theorem 8.5].

**Lemma 3.** There is a unique important $(X,Y)$-cut of minimum size and it can be found in polynomial time.

Lemmas 1 and 3 imply the following:

**Lemma 4.** Multiplication of any edge in the minimum-size important $(X,Y)$-cut of $G$ increases the size of minimum important $(X,Y)$-cut in $G$.

*Proof.* By Lemmas 1 and 3 there is a unique important $(X,Y)$-cut $F$ of minimum size. By Definition 1 an important $(X,Y)$-cut is a minimal $(X,Y)$-cut. Thus, for any $e \in F$, $F \setminus \{e\}$ is not a $(X,Y)$-cut of $G$. Let $G'$ be the graph obtained from $G$ by adding $t \geq 1$ copies of $e$. Then $F$ is not an $(X,Y)$-cut of $G'$. Thus, the size of minimum important $(X,Y)$-cut in $G'$ is greater than that in $G$. □

### 4 Solving $k$-arc Chinese Postman Problem

Let us first give a formal definition of the **Mixed Chinese Postman Problem** parameterized by the number of arcs.

**$k$-arc Chinese Postman Problem ($k$-arc CPP)**

*Instance:* A strongly connected mixed graph $G = (V, E \cup A)$, with vertex set $V$, edge set $E$ and arc set $A$, and a weight function $w : E \cup A \to \mathbb{N}$. Parameter: $k = |A|$.

*Output:* A minimum weight Eulerian multi-orientation of $G$. 
In the output we do not require a minimum weight Eulerian trail as it can be found in polynomial time from a minimum weight Eulerian multi-orientation of $G$. An Eulerian multi-orientation of $G$ is called a solution to $k$-arc CPP. A solution to $k$-arc CPP is optimal if it is of minimum weight among all solutions.

Gutin et al. [11] solved $k$-arc CPP by using a Turing parameterized reduction to Balanced Chinese Postman Problem (BCPP) stated in Section 4.1. We will use a Turing parameterized reduction to a modification of BCPP, Bounded BCPP, also stated in Section 4.1.

4.1 Reducing MCPP to Bounded BCPP and Parity Condition

For the definitions of a zero-sum function, $V^+_t$ and $V^-_t$, see Section 2.

| Balanced Chinese Postman Problem (BCPP) |
|----------------------------------------|
| **Instance:** A weighted undirected graph $G = (V, E)$, a weight function $w : E \rightarrow \mathbb{N}$, and a zero-sum function $t : V \rightarrow \mathbb{Z}$. |
| **Parameter:** $p = \sum_{v \in V^+_t} t(v)$. |
| **Output:** A minimum weight multiplication $H$ of $G$ which has a $t$-balanced orientation. |

| Bounded Balanced Chinese Postman Problem (Bounded BCPP) |
|--------------------------------------------------------|
| **Instance:** A weighted undirected graph $G = (V, E)$, a weight function $w : E \rightarrow \mathbb{N}$, a capacity function $c : E \rightarrow \mathbb{N}_{>0}$, and a zero-sum function $t : V \rightarrow \mathbb{Z}$. |
| **Parameter:** $p = \sum_{v \in V^+_t} t(v)$ and $q = \max\{c(e), e \in E\}$. |
| **Output:** A minimum weight multiplication $H$ of $G$ which has a $t$-balanced orientation and $\mu_H(e) \leq c(e)$ holds for every edge $e \in E$. |

A multiplication $H$ of $G$ is called a solution to Bounded BCPP if it has a $t$-balanced orientation and $\mu_H(e) \leq c(e)$ holds for every edge $e \in E$. A solution to Bounded BCPP is optimal if it is of minimum weight among all solutions.

Let us recall a key result in the parameterized Turing reduction from $k$-arc CPP to BCPP given in [11]. Henceforth, $\kappa = \lfloor k^2/2 + 2k \rfloor$.

**Lemma 5.** [11] Let $G = (V, E \cup A)$ be a mixed graph, and let $k = |A|$. Then for any optimal solution $D$ to $k$-arc CPP on $G$ with minimal number of arcs, we have that $\sum_{u \in A} \mu_D(\overline{uv}) \leq \kappa$.

The above lemma shows that the arcs in $A$ have a bounded total multiplicity in every optimal solution to $k$-arc CPP on $G$ with minimal number of arcs. Next, we describe a parameterized Turing reduction from $k$-arc CPP to Bounded BCPP.
Arc Removal Reduction. Let \((G = (V, A \cup E), w)\) be an instance of \(k\)-arc CPP. Let \(D\) be an optimal solution to \(k\)-arc CPP on \((G, w)\) with minimal number of arcs. By Lemma 6, we have that \(\sum_{\overrightarrow{uv} \in A} \mu_D(\overrightarrow{uv}) \leq \kappa\). Let \(G' = (V, E)\) be the graph obtained from \(G\) by deleting all the arcs, and let \(w' = w\) restricted to \(E\). Let \(\phi : A \rightarrow [\kappa]\) be a guess (function) such that \(\sum_{\overrightarrow{uv} \in A} \phi(\overrightarrow{uv}) \leq \kappa\). Here \(\phi\) is a guess on the multiplicities in \(D\) for the arcs in \(A\). Let \(t_\phi : V \rightarrow [-\kappa, \kappa]\) be a function such that \(t_\phi(v) = \sum_{\overrightarrow{uv} \in A} \phi(\overrightarrow{uv}) - \sum_{\overrightarrow{vu} \in A} \phi(\overrightarrow{vu})\) holds for every vertex \(v \in V\). Observe that any multi-orientation of \(G'\) is \(t_\phi\)-balanced plus \(\phi(\overrightarrow{uv})\) copies of each arc \(\overrightarrow{uv} \in A\) would give an Eulerian multi-orientation of \(G\). In other words, \(\phi\) produces a zero-sum function \(t_\phi : V \rightarrow \mathbb{Z}\) for every vertex in \(G\). As the total multiplicities of all the arcs in \(A\) is upper bounded by \(\kappa\), there are at most \(\binom{\kappa}{2}\) different guesses of \(\phi\). After each guess, \(k\)-arc CPP can be parameterized Turing reduced to BCPP according to the following theorem.

**Theorem 1.**\(^{11}\) Suppose there is an algorithm that finds the optimal solution to an instance of BCPP on \((G', w', t')\) with parameter \(p\) in time \(f(p)|V|^{O(1)}\). Then there exists an algorithm which finds the optimal solution to an instance of \(k\)-arc CPP on \((G = (V, A \cup E), w)\) with parameter \(k\), which runs in time \(\left(\frac{\kappa}{k}\right) f(\kappa)|V|^{O(1)}\). Thus, if BCPP is FPT, then so is \(k\)-arc CPP.

Therefore, to prove the fixed-parameter tractability of \(k\)-arc CPP, it suffices to design an FPT algorithm for BCPP, because \(p \leq \kappa\). In \(^{11}\), MCPP was first reduced to BCPP with parameter \(p \leq \kappa\). Then BCPP was solved utilizing the notion of a graph torso. The torso graph was used as a tool to construct a tree decomposition of the original graph, which does not have bounded width, but has enough other structural restrictions to make a dynamic programming algorithm fixed-parameter tractable. The running time of the algorithm is not explicitly given, but it is polynomially-double exponential with respect to \(k\).

In the rest of the section, we will use BOUNDED BCPP instead of BCPP and show how to solve it using our novel approach. Note that to solve BOUNDED BCPP, it suffices to multiplicate edges with minimum total weight, such that there is an orientation of the resulting graph in which every vertex is \(t\)-balanced.

The following two lemmas bound the multiplicity of every edge in an optimal solution to BOUNDED BCPP with minimal number of edges.

**Lemma 6.** Let \(G = (V, E \cup A)\) be a mixed graph, \(k = |A| \geq 1\), \(w, v \in E\), and \(D\) an optimal solution to \(k\)-arc CPP on \(G\) with minimal number of arcs. Then we have that

1. if \(\mu_D(\overrightarrow{uv}) + \mu_D(\overrightarrow{vu}) \geq 3\), then \(\mu_D(\overrightarrow{uv}) = 0\) or \(\mu_D(\overrightarrow{vu}) = 0\);
2. \(\mu_D(\overrightarrow{uv}) \leq 2k\).

**Proof.** Firstly, suppose that \(\mu_D(\overrightarrow{uv}) \geq 1, \mu_D(\overrightarrow{vu}) \geq 1\) and \(\mu_D(\overrightarrow{uv}) + \mu_D(\overrightarrow{vu}) \geq 3\). Let \(D'\) be the graph obtained from \(D\) by deleting one copy of both \(\overrightarrow{uv}\) and \(\overrightarrow{vu}\). Marx et al. \(^{14}\) Remark 2.14] observed that the treewidth of the torso graph has to be bounded by an exponential function. Therefore, the dynamic programming algorithm of \(^{11}\) based on tree decomposition is at least double exponential.
\[ \overrightarrow{vu}. \text{ Then } D' \text{ is still Eulerian and thus also an optimal solution to } k\text{-arc CPP on } G, \text{ with less arcs, a contradiction. Therefore, if } \mu_D(\overrightarrow{uv}) + \mu_D(\overrightarrow{vu}) \geq 3, \text{ then } \mu_D(\overrightarrow{vu}) = 0 \text{ or } \mu_D(\overrightarrow{uv}) = 0. \]

Secondly, suppose \( \mu_D(\overrightarrow{vu}) \geq 3 \) and \( \mu_D(\overrightarrow{uv}) = 0 \). We will prove that \( \mu_D(\overrightarrow{vu}) \leq 2|A| = 2k \). Let \( A = A_1 \cup A_2 \) where \( A_1 = \{ x \overrightarrow{y} : \overrightarrow{xy} \in A \text{ and } \mu_D(\overrightarrow{xy}) \geq 3 \} \) and \( A_2 = A \setminus A_1 \). Since \( D \) is Eulerian, \( d_D^-(z) = d_D^+(z) \) holds for every vertex \( z \in V(D) \). Therefore, there are \( \mu_D(\overrightarrow{vu}) \) arc-disjoint directed cycles in \( D \), each containing one copy of \( \overrightarrow{uv} \). We claim that each such cycle must contain at least one copy of some arc from \( A_2 \). Otherwise, there is a cycle \( C \) containing \( \overrightarrow{uv} \) that consists of arcs from \( A_1 \) and arcs corresponding to edges in \( E \). Construct a directed graph \( D' \) as follows: remove from \( D \) two copies of \( \overrightarrow{uv} \), two copies of each arc from \( A_1 \) that appears in \( C \), and reverse the arcs in \( C \) that correspond to edges in \( E \setminus \{ uv \} \). Observe that \( D' \) is still an Eulerian multi-orientation of \( G \), and so \( D' \) is also a solution. However, \( D' \) is a solution with smaller weight or fewer arcs than \( D \), contradicting the choice of \( D \). Thus the claim holds and we have \( \mu_D(\overrightarrow{vu}) \leq 2|A_2| \leq 2|A| = 2k \). \( \square \)

Recall Arc Removal Reduction as it is used in the following:

**Lemma 7.** Let \( (G = (V, E \cup A), w) \) be an instance of MCPP, and \( \phi : A \to [k] \) be a guess on the multiplicities of the arcs in \( A \) in the optimal solution with minimal number of arcs. Let \( G' = (V, E) \) together with \( t_\phi : V \to Z \) be the resulting instance of BCPP. If \( \phi \) is a guessing that corresponds to an optimal solution to \( k\text{-arc CPP on } G \) with minimal number of arcs, then for any optimal solution \( M \) to BCPP on \( G' \) with minimal number of edges, we have that \( \mu_M(uv) \leq 2k \) for every edge \( uv \in E \).

**Proof.** Let \( M \) be an optimal solution of BCPP on \( G' \) with minimal number of edges. If \( \phi \) is a guess that corresponds to an optimal solution to \( k\text{-arc CPP on } G \) with minimal number of arcs, then by Lemma 5 \( \sum_{\overrightarrow{uv} \in A} \phi(\overrightarrow{uv}) \leq k \). Then \( M \) has an orientation which together with the multipliciated arcs according to \( \phi \) gives an optimal solution \( D \) of \( k\text{-arc CPP on } G \) with minimum number of arcs. Suppose there is an edge \( uv \in E \), such that \( \mu_M(uv) > 2k \). By Lemma 6 we can find a solution with less arcs. It follows that \( \phi \) is not a guess that corresponds to the optimal solution \( D \) or \( M \) is not an optimal solution to CPP on \( G' \) with minimal number of edges. \( \square \)

It follows from Lemma 7 that there is an optimal solution to BOUNDED BCPP in which every edge has multiplicity at most \( 2k \), if the guess \( \phi \) corresponds to an optimal solution with minimal number of arcs. For a given instance of BOUNDED BCPP, if there is no solution in which every edge has multiplicity at most \( 2k \), then the guess \( \phi \) is a “bad” one.

**Definition 2.** Let \( (G = (V, E), w, t) \) be an instance of BOUNDED BCPP, in which \( t : V \to Z \) is the zero-sum function. A \((V^+_t, V^-_t)\)-cut with size smaller than \( p \) is called a small \( t \)-cut, where \( p = \sum_{v \in V^+_t} t(v) \).
Definition 2 is motivated by Lemma 8 from [11] which proved that in every solution to BCPP, there is no small $t$-cut.

**Lemma 8.** [11] Let $G$ be an undirected graph and let $(G, w, t)$ be an instance of BOUNDED BCPP. Then $(G, w, t)$ has an $t$-balanced orientation if and only if $G$ has no small $t$-cut and for every vertex $v$ of $G$, $d_G(v) - t(v)$ is even. Furthermore, such an orientation can be found in polynomial time.

Thus, we have the following:

**Theorem 2.** A multiplication $H$ of $G$ is an optimal solution of BOUNDED BCPP if and only if $H$ is of minimum weight among multiplications $H'$ of $G$ which have no small $t$-cuts and $d_{H'}(v) - t(v)$ is even for every vertex $v$ of $G$.

The condition that $d_{H'}(v) - t(v)$ is even for every vertex $v$ of $G$ will be called the parity condition. The rest of the section, we will first give necessary results for construction of feasible multiplications of $G$ with no small $t$-cuts, then we will deal with the parity condition, and finally using the above we will design Algorithms 1 and 2, which together solve BOUNDED BCPP.

Given a graph $G = (V, E)$ and a vertex set $X \subseteq V$, an $X$-Join is an edge set $J_X \subseteq E$ such that $|J_X \cup E(v)|$ is odd if and only if $v \in X$, where $E(v)$ is the set of edges incident to $v$. In other words, an $X$-Join is a set of edges that only changes the degree parities of vertices in $X$. Given an edge weighted graph $G = (V, E)$ and a vertex set $X \subseteq V$, the **Minimum Weight X-Join Problem** asks to find an $X$-Join of minimum weight. It can be solved in the follow way. Construct an auxiliary graph $F$ with vertex set $X \subseteq V$, an $X$-Join is an edge set $J_X \subseteq E$ such that $|J_X \cup E(v)|$ is odd if and only if $v \in X$, where $E(v)$ is the set of edges incident to $v$. In other words, an $X$-Join is a set of edges that only changes the degree parities of vertices in $X$. Given an edge weighted graph $G = (V, E)$ and a vertex set $X \subseteq V$, the **Minimum Weight X-Join Problem** asks to find an $X$-Join of minimum weight. It can be solved in the follow way. Construct an auxiliary graph $F$ with vertex set $X$. For any two vertices $u, v \in X$, add an edge $uv$ with weight $d_G(u, v)$, which is the distance between $u$ and $v$ in $G$. Then compute a minimum weight perfect matching $M$ of $F$. Let $P_{uv}$ be the shortest path between $u$ and $v$, for each edge $uv \in M$. Then edges in $\bigcup_{uv \in M} E(P_{uv})$ form an optimal solution to the **Minimum Weight X-Join** problem in $G$. This leads to the following:

**Lemma 9.** [7] Minimum Weight X-Join can be solved in time $O(n^3)$.

### 4.2 Algorithm ABBCP

Results of Sections 3 and 4.1 allow us to design an algorithm for solving BOUNDED BCPP. For better readability, our algorithm called **Algorithm ABBCP** is partitioned into Algorithms 1 and 2. The phrase of multiplying edges to eliminate small $t$-cut is implemented in Algorithm 1. The output of Algorithm 1 is then used in Algorithm 2 to solve BOUNDED BCPP.

**Theorem 3.** BOUNDED BCPP parameterized by $p$ and $q$ can be solved in time $O^*(q^p)$.

**Proof.** Algorithm ABBCP runs Algorithm 1 and then Algorithm 2. We will first show the correctness of Algorithm ABBCP and then prove that the running time of Algorithm ABBCP is $O^*(q^p)$. 

Algorithm 1: Branching algorithm for eliminating small \( t \)-cuts.

**Input:** An edge weighted undirected graph \( G = (V, E) \), with edge weights \( \omega : E \rightarrow \mathbb{N} \), a multiplicity upper bound \( c : E \rightarrow \mathbb{N}_{>0} \), and a zero-sum function \( t : V(G) \rightarrow \mathbb{Z} \), such that \( \sum_{v \in V(G)} t(v) = 0 \).

**Parameters:** \( p = \sum_{u \in V^+} t(u) \) and \( q = \max \{c(e), e \in E\} \).

**Output:** A set of multiplications of \( G \), in which every graph contains no small \( t \)-cut and satisfies the multiplicity upper bound.

Set \( S = \{G\} \) and \( S' = \emptyset \).

if \( S = \emptyset \) then

return \( S' \).

else

foreach \( H \in S \) do

if \( H \) has no small \( t \)-cut then

Move \( H \) from \( S \) to \( S' \).

else

find the minimum important \((V^+_1, V^-_1)\)-cut \( F \) of \( H \);

foreach edge \( e \) in \( F \) and \( r \in \lfloor c(e) - 1 \rfloor \) do

Let \( H_{e,r} \) be obtained by adding \( r \) copies of \( e \) into \( H \).

Add \( H_{e,r} \) into \( S \).

Delete \( H \) from \( S \).

end

end

end

end

end

It follows from Lemma 8, Theorem 2 and the lemmas of Section 3 that all graphs in \( S' \) together with the edges in their minimum weight \( X \)-Joins are solutions to BOUNDED BCPP. Thus, it suffices to prove that they contain an optimal solution. Let \( G_0 = (V, E) \) be a graph and let \( H \) be an optimal solution to BOUNDED BCPP on \( G_0 \). We prove that there is a graph \( R \in S' \) such that \( w(R) = w(H) \). Let \( F_1 \) be the minimum important \((V^+_1, V^-_1)\)-cut in \( G_0 \) and let \( e_1 \) be an edge in \( F_1 \). Let \( G_1 \) be the graph obtained from \( G \) by multiplying \( e_1 \) in \( F_1 \) such that \( \mu_{G_1}(e_1) = \mu_H(e_1) \). Note that \( H \) is an optimal solution, therefore \( \mu_H(e_1) \leq c(e_1) \). It follows that \( G_1 \) will be generated by Algorithm 1 and put into \( S \).

Let \( F_2 \) be the minimum important \((V^+_2, V^-_2)\)-cut in \( G_1 \) and let \( e_2 \) be an edge in \( F_2 \). Let \( G_2 \) be the graph obtained from \( G_1 \) by multiplying \( e_2 \) in \( F_2 \) such that \( \mu_{G_2}(e_2) = \mu_H(e_2) \). Note that \( G_2 \) will be generated by Algorithm 1 and put into \( S \). Algorithm 1 will repeat this process till it obtains a graph \( G_i \) which does not have small \( t \)-cuts anymore. Note that Algorithm 1 will add \( G_i \) to \( S' \). Observe that \( \mu_{G_i}(e_j) = \mu_H(e_j) \) for \( j \in [i] \). Hence, \( G_i \) is a subgraph of \( H \) that contains no small \( t \)-cut. Algorithm 2 will add a minimum weight \( X \)-Join \( J \) to \( G_i \) resulting in a graph \( \hat{G} \). Suppose that \( \hat{G} \neq H \). Observe that \( H \) can be obtained from \( G_i \) by adding an \( X \)-Join \( E(H) \setminus E(G_i) \), as \( H \) has a \( t \)-balanced orientation. Since \( H \) is optimal, \( w(\hat{G}) \geq w(H) \). However, since \( J \) is a minimum weight \( X \)-Join, \( w(\hat{G}) \leq w(H) \). Thus, \( w(\hat{G}) = w(H) \), \( w_{rec} = w(H) \) and Algorithm 2 will return an optimal solution \( R \). Therefore, Algorithm ABBCP is correct.
Algorithm 2: Branching algorithm for BOUNCED BCPP

**Input:** An edge weighted undirected graph \( G = (V, E) \), with edge weight \( \omega : E \rightarrow \mathbb{N} \), a multiplicity upper bound \( c : E \rightarrow \mathbb{N}_{>0} \), a zero-sum function \( t : V(G) \rightarrow \mathbb{Z} \), such that \( \sum_{v \in V(G)} t(v) = 0 \), and \( S' \), the output of Algorithm 1 on \( G \).

**Output:** An optimal solution of Bounded BCPP on \( (G, \omega, c, t) \).

Set \( w_{rec} = \infty \).

foreach \( H \in S' \) do

Compute the minimum weight \( X \)-Join \( J_H \) of \( H \), where \( X = \{ u \in V, d_H(u) - t(u) \text{ is odd} \} \).

Add the edges of \( J_H \) into \( H \).

if \( w(H) < w_{rec} \) then

Set \( w_{rec} = w(H) \) and \( B = H \).

Return \( B \).

Now we will estimate the running time of Algorithm ABBCP. Let \( F \) be the unique minimum important \((V^+_H, V^-_H)\)-cut of \( G \). Note that \( F \) can be found in polynomial time by Lemma 3. Suppose that \( |F| < p \). Then Algorithm 1 adds graphs to \( S \) (it also moves some of them to \( S' \)); we call it Iteration 1 of Algorithm 1. Since \( \mu_H(e) \leq q \) for all \( e \in E \), the number of added graphs in Iteration 1 is less than \( q^p \). In Iteration 2, Algorithm 1 checks whether \( S \) has graphs with small \( t \)-cuts and eliminates the minimum important \( t \)-cuts from such graphs, Algorithm 1 adds new graphs to \( S \) (again some of them are moved to \( S' \)). Similarly, Algorithm 1 carries out consequent iterations till \( S = \emptyset \). There will be less than \( p \) iterations as the minimum size of a small \( t \)-cut in graphs of \( S \) will strictly increase in each iteration. Thus, the number of total graphs added to \( S \) in one of the iterations is at most \( (q^p)^p = q^{p^2} \). Therefore, \(|S'| \leq q^{p^2}\) and the running time of Algorithm 1 is \( O^*(q^{p^2}) \). As we can solve the MINIMUM WEIGHT \( X \)-JOIN PROBLEM in polynomial time, the running time of Algorithm 2 is \( O^*(q^{p^2}) \). Therefore, Algorithm ABBCP runs in time \( O^*(q^{p^2}) \).

Combining Lemma 7 and Theorems 1 and 3, we obtain the following:

**Theorem 4.** There is an algorithm that solves the \( k \)-arc CPP in time \( O^*(kO(k^4)) \).

**Proof.** By Lemma 7 and Theorem 3 the special case of BOUNDED BCPP used to solve \( k \)-arc CPP can be solved in time \( O^*(q^{p^2}) \), where \( q \leq 2k \) and \( p \leq \kappa = O(k^2) \). Thus, \( O^*(q^{p^2}) = O^*((2k)^{k^4}) \). Then by Theorem 1 we can solve \( k \)-arc CPP in time \( O^*((\binom{k}{p})(2k)^{k^4}) = O^*(kO(k^4)) \).

5 Minimum Vulnerability and Minimum Shared Edges

In this section, we solve the MINIMUM VULNERABILITY problem and the MINIMUM SHARED EDGES problem.
A \((p, s, t)\)-routing in \(G\) is a solution of MV if no edge \(e\) participates in more than \(c(e)\) paths. A solution is optimal if the sum of the weights of edges that participate in more than \(r\) of the \(p\) paths is minimal.

**Minimum Vulnerability (MV)**

**Input:** Graph \(G = (V, E, \omega, c)\), two vertices \(s, t \in V\), edge weight \(\omega : E \to \mathbb{N}_{>0}\), edge capacities \(c : E \to \mathbb{N}_{>0}\), and two integers \(r, p \in \mathbb{N}\).

**Parameter:** \(p\).

**Output:** Find a \((p, s, t)\)-routing in \(G\) with the minimum possible sum of the weights of edges that participate in more than \(r\) of the \(p\) paths subject to the constraint that no edge \(e\) participates in more than \(c(e)\) paths.

**Minimum Shared Edges (MSE)**

**Input:** A graph \(G = (V, E)\), two vertices \(s, t \in V\), and two integers \(k, p \in \mathbb{N}\).

**Parameter:** \(p\).

**Question:** A \((p, s, t)\)-routing in \(G\) in which at most \(k\) edges are shared.

Minimum Shared Edges is NP-complete, even for graphs with maximum degree at most 5 \([9]\). In the edge-weighted version, the aim is to find at most \(k\) edges with minimum total weight such that there is a \((p, s, t)\)-routing with only these edges shared. Minimum Shared Edges is polynomial-time solvable when \(k = 0\), but NP-hard for general values of \(k\) in \([9]\). MSE is \(W[1]\)-hard when parameterized by the treewidth of the graph and \(k\) \([9]\). On the positive side, it is FPT parameterized by \(p\) and can be solved in time \(O^*(2^p \cdot p^3 \cdot \text{poly}(p))\) \([9]\).

Our Algorithm 3 is a simple FPT algorithm for solving Minimum Vulnerability parameterized by \(p\). The algorithm appears to be the first FPT algorithm for MV and it is even faster than the FPT algorithm of \([9]\) for MSE.

**Theorem 5.** Minimum Vulnerability can be solved in time \(O^*(p^p)\).

**Proof.** We first show that Algorithm 3 solves Minimum Vulnerability.

Let \(G^*\) be an optimal solution of Minimum Vulnerability such that if \(e \in E(G^*)\) is of multiplicity larger than \(r\) then \(\mu_{G^*}(e) = c(e)\). (We may make this assumption due to the definition of the weight of a solution.) Observe that there is no \((s, t)\)-cut of size smaller than \(p\) in \(G^*\).

Let \(\omega'(G^*) = \sum_{e \in E(G^*) \mid \mu_{G^*}(e) > r} \omega(e)\). Then the total weight of edges that appear in more than \(r\) of the \(p\) \(s-t\) paths in \(G^*\) is \(\omega'(G^*)\). Indeed, otherwise there is an edge \(e \in E(G^*)\) with \(\mu_{G^*}(e) > r\) that appears in at most \(r\) of the \(p\) \(s-t\) paths and hence deleting one copy of \(e\) from \(G^*\) allows for a \((p, s, t)\)-routing of a smaller weight, contradicting the choice of \(G^*\).

We show that there is a graph \(H\) in the final \(S'\) of Algorithm 3 with \(\omega'(H) \leq \omega'(G^*)\) that contains no \((s, t)\)-cut smaller than \(p\).

Since \(G^*\) is a multiplication of \(G\), we have \(\mu_G(e) \leq \mu_{G^*}(e)\) for every \(e \in E(G)\). Let \(F_1\) be the minimum important \((s, t)\)-cut in \(G\). Let \(e_1\) be an edge in \(F_1\) such
Algorithm 3: Branching algorithm for Minimum Vulnerability.

Input: Graph $G = (V, E, \omega, c)$, two vertices $s, t \in V$, edge costs $\omega : E \to \mathbb{N}_{>0}$, edge capacities $c : E \to \mathbb{N}_{>0}$, and two integers $r, p \in \mathbb{N}$.

Parameters: $p$.

Output: An optimal solution, if it exists.

Let $Q$ be the multiplication of $G$ such that $\mu_Q(e) = c(e)$.

if $Q$ has an $(s, t)$-cut smaller than $p$ then
    Return “No solution.”

Multiplicate every edge $e \in E$ such that its multiplicity becomes $\min\{c(e), r\}$. Set $S = \{G\}$ and $S' = \emptyset$.

if $S = \emptyset$ then
    For each $H \in S'$, let $\omega'(H) = \sum_{e \in E(H), \omega_H(e) > r} \omega(e)$. Let $H^*$ be the graph in $S'$ that achieves the minimum $\omega'(H)$ among all graphs in $S'$. Find a $(p, s, t)$-routing of $H^*$ in which no edge is shared. Convert $F^*$ into a $(p, s, t)$-routing of $G$ and return it.

else
    foreach $H \in S$ do
        if $H$ has no $(s, t)$-cut smaller than $p$ then
            Move $H$ from $S$ to $S'$.
        else
            Find the minimum important $(s, t)$-cut $F_H$ of $H$;
            foreach edge $e$ in $F_H$ with $r < c(e)$ do
                Let $H_e$ be obtained by adding $c(e) - r$ copies of $e$ into $H$.
                Add $H_e$ into $S$.
                Delete $H$ from $S$.
        fi
    od
fi

that $\mu_{G'}(e_1) > \mu_G(e_1)$. Let $G_1$ be the graph obtained from $G$ by multiplicating $e_1$ such that $\mu_{G_1}(e_1) = c(e_1)$. Note that $G_1$ will be put into $S$ if it contains an $(s, t)$-cut of size smaller than $p$.

Let $F_2$ be the minimum important $(s, t)$-cut in $G_1$ and let $e_2$ be an edge in $F_2$ such that $\mu_{G'}(e_2) > \mu_G(e_2)$. Let $G_2$ be the graph obtained from $G_1$ by multiplicating $e_2$ such that $\mu_{G_2}(e_2) = c(e_2)$. Note that $G_2$ will be generated by Algorithm 3 and put into $S$. Algorithm 3 will repeat this process till it obtains a graph $G_i$, which has no $(s, t)$-cut of size smaller than $p$ anymore. Note that Algorithm 3 will add $G_i$ to $S'$. Observe that $\omega(G_i) \leq \omega(G^*)$, as $\mu_{G_i}(e) > r$ only if $\mu_{G'}(e) > r$ for every edge $e \in E(G)$. Moreover, $G_i$ contains a $(p, s, t)$-routing. The minimum weight $(p, s, t)$-routing in $G_i$ has weight at most $\omega(G_i) \leq \omega(G^*)$. Therefore, Algorithm 3 outputs an optimal solution.

Note that there is only one iteration in Algorithm 3 which branches among $|F| < p$ choices. Each branching increases the size of minimum important $(s, t)$-cut, till there is no $(s, t)$-cut of size smaller than $p$. It follows that Algorithm 3 solves the Minimum Vulnerability problem in time $O^*(p^p)$. □
Algorithm 4: Branching algorithm for Weighted Min \((s, t)\)-Cut Prevention.

**Input:** Graph \(G = (V, E, \omega, c)\), two vertices \(s, t \in V\), edge weight \(\omega : E \to \mathbb{N}_{>0}\), edge capacities \(c : E \to \mathbb{N}_{>0}\), and two integers \(d, p \in \mathbb{N}\).

**Parameter:** \(p\).

**Output:** An optimal solution.

Let \(Q\) be the multiplication of \(G\) such that \(\mu_Q(e) = c(e)\).

if \(Q\) has an \((s, t)\)-cut smaller than \(p\) then
  Return “No solution.”

Set \(S = \{G\}\) and \(S' = \emptyset\).

if \(S = \emptyset\) then
  For each graph \(H \in S'\), let \(\omega'(H) = \sum_{e \in E, \mu_H(e) > p} \omega(e)\). Let \(H^*\) be the graph in \(S'\) that achieves the minimum \(\omega'(H)\) among all graphs in \(S'\).
  Return \(D = \{e \in E, \mu_{H^*}(e) > p\}\).

else
  foreach \(H \in S\) do
    if \(H\) has no \((s, t)\)-cut of capacity at most \(p\) then
      Move \(H\) from \(S\) to \(S'\).
    else
      Find the minimum important \((s, t)\)-cut \(F_H\) of \(H\);
      foreach edge \(e\) in \(F_H\) do
        Let \(H_e\) be obtained by adding \(p\) copies of \(e\) into \(H\).
        Add \(H_e\) into \(S\).
        Delete \(H\) from \(S\).
  Recall that Minimum Shared Edges is a special case of Minimum Vulnerability that asks for the existence of a solution with weight at most \(k\), in which \(r = 1\), \(\omega(e) = 1\) and \(c(e) = p\) for every edge \(e \in E(G)\). Thus, Algorithm 3 can be used to solve Minimum Shared Edges.

**Corollary 1.** Minimum Shared Edges can be solved in time \(O^*(p^p)\).

6 Small Cut Prevention

In \[10\], Grüttemeier et al. introduced the following problem: an attacker wants to find an \((s, t)\)-cut of capacity at most \(p\) and a defender wants to protect some edges in order to increase the capacity of any minimum \((s, t)\)-cut in \(G\) to at least \(p + 1\). The formal definition of the problem is as follows.

**Weighted Min \((s,t)\)-Cut Prevention (WMCP)**

**Input:** A graph \(G = (V, E)\), two vertices \(s, t \in V\), a cost function \(\omega : E \to \mathbb{N}_{>0}\), a capacity function \(c : E \to \mathbb{N}_{>0}\), and an integers \(p\).

**Question:** Find a set \(D \subseteq E\) with minimal weight \(\omega(D) := \sum_{e \in D} \omega(e)\) such that for every \((s, t)\)-cut \(A \subseteq (E \setminus D)\) in \(G\) we have \(c(A) := \sum_{e \in A} c(e) > p\).
A set $D \subseteq E$ is called a **solution** to WMCP if for every $(s, t)$-cut $A \subseteq (E \setminus D)$ in $G$ we have $c(A) := \sum_{e \in A} c(e) > p$. A solution $D$ is **optimal** if its weight $\omega(D)$ is minimal.

Note Grüttemeier et al. [10] considered a decision version of the problem where given an integer $d$ the aim is to find a solution $D$ with $\omega(D) \leq d$. Clearly, an optimal solution to the optimization version allows us to immediately solve the decision version.

\textbf{Min $(s, t)$-Cut Prevention (MCP)} is the special case when only unit capacity and cost are considered. Both MCP and WMCP are shown to be NP-complete even on subcubic graphs and FPT when parameterized by $p$ [10]. The running time of the algorithm in [10] is polynomially-double exponential.

Algorithm 4 solves WMCP faster.

**Theorem 6.** The Weighted Min $(s, t)$-Cut Prevention problem can be solved in time $O^*(p^p)$.

We can prove Theorem 6 similarly to Theorem 5 and thus we omit the proof.

7 **Conclusion**

Our new approach can be used to solve the following Multi-Connect problem: given an edge-weighted undirected graph $G$, $t$ pairs of vertices $(u_i, v_i)$ in $G$, and $t$ positive integers $r_1, r_2, \ldots, r_t$. We are asked to multiplicate a set of edges with minimum total weights, such that there are at least $r_i$ edge-disjoint paths between $u_i$ and $v_i$, for each $i \in [t]$. Using our approach, it is easy to design an FPT algorithm for the Multi-Connect problem, parameterized by $\max_{i \in [t]} r_i$. 
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