Non-Standard KP Evolution and Quantum τ-function

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One possible way to fix partly a “canonical definition” of τ-functions beyond the conventional KP/Toda framework could be to postulate that evolution operators are always group elements. We discuss implications of this postulate for the first non-trivial case: fundamental representations of quantum groups \(SL_q(N)\). It appears that the most suited (simple) for quantum deformation framework is some non-standard formulation of KP/Toda systems. It turns out that the postulate needs to be slightly modified to take into account that no “nilpotent subgroups” exist in \(SL_q(N)\) for \(q \neq 1\). This has some definite and simple implications for \(q\)-determinant-like representations of quantum τ-functions.

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1 Introduction

 Appropriately (and broadly) generalized classical integrable hierarchies are now widely believed to describe non-perturbative effective actions of string models while their quantum counterparts should be relevant for description of the second-quantized string theory. The modest purpose of these notes is to illustrate some peculiarities on the way from classical to quantum hierarchies of the simplest – KP/Toda – type. One of the possible approaches to quantization is to make use of the group theory interpretation of hierarchies and then just substitute ordinary groups by their quantum deformations. This is the line to be followed in the present paper.

 The basic object in the theory of hierarchies is $\tau$-function – the generating functional of all the matrix elements of a given group element $g \in G$ in a given (highest weight) representation $R$:

$$\tau_R(t, l | g) = \sum_{\{m, \bar{m}\} \in R} s^R_{m, \bar{m}}(t, \bar{l}) < m | g | \bar{m} >$$

(1.1)

The choice of the functions $s^R_{m, \bar{m}}(t, \bar{l})$ is the main ambiguity in the definition of $\tau$-function and needs to be made in some clever way (not yet known in full generality). In the case of the highest weight representation $R$, it can be partly fixed by the requirement that

$$\tau_R(t, l | g) = < 0_R | U(t) g \bar{U}(\bar{l}) | 0_R >$$

(1.2)

where operators $U$ and $\bar{U}$ do not depend on $R$.

KP/Toda-type $\tau$-function arises when $G = SL(N)$ and $R$ is one of the $N - 1$ fundamental representations. All fundamental representations of $SL(N)$ are skew products of the first ($N$-dimensional) one $F \equiv F_1$: $F_n = \wedge^n F$, and thus can be also described in terms of (fermionic) intertwining operators. Entire KP/Toda hierarchy is obtained in the limit of $N \rightarrow \infty$ (with $n$ playing the role of the “zero-time”), and has also an alternative description in terms of the level $k = 1$ Kac-Moody algebras. We shall, however, concentrate on the case of the generic $N$.

Operators $U(t), \bar{U}(\bar{l})$, when restricted onto any fundamental representation $F_n$, turn into

$$U(t) = \exp \left( \sum_{k \geq 1} t_k T_+^{(k)} \right),$$

(1.3)

$$\bar{U}(\bar{l}) = \exp \left( \sum_{k \geq 1} \bar{t}_k T_-^{(k)} \right),$$

where $T_\pm^{(k)} = \sum_{\alpha \in h(\bar{a}) = k} T_{\pm \bar{a}}$ are sums of all the generators of $SL(N)$, associated with the positive/negative roots of “height” $k$ (in the first fundamental representation $F$ such $T_\pm^{(k)}$ look like $N \times N$ matrices with units at the $k$-th upper/lower diagonal and zeroes elsewhere). The crucial feature of these operators is their commutativity:

$$[T_+^{(k)} , T_+^{(l)}] = 0, \quad [T_-^{(k)} , T_-^{(l)}] = 0.$$  (1.4)

It is a peculiarity of fundamental representations of $SL(N)$ that such simple $U(t), \bar{U}(\bar{l})$ are sufficient to generate all the elements of representation (generally $U, \bar{U}$ depend on more time-variables and more mutually non-commuting generators of $G$).

Operators, defined in (1.3) have the following properties:

(i) $U, \bar{U} \in G$;

(ii) more specific, $U, \bar{U}$ belong to the “nilpotent subgroup” $NG$ of $G$. Actually $NG$ is a subgroup of Borel subgroup: $NG \subset BG \subset G$ (in $F_1$ $BG$ consists of all the upper triangular matrices with unit determinant, while matrices from $NG$
are additionally constrained to have only unit elements on the main diagonal); (iii) if comultiplication is defined so that

$$\Delta(T_{\pm \tilde{\alpha}}) = T_{\pm \tilde{\alpha}} \otimes I + I \otimes T_{\pm \tilde{\alpha}}$$

(1.5)

then

$$\Delta(U(t)) = U(t) \otimes U(t) = (U(t) \otimes I)(I \otimes U(t))$$

(1.6)

Shortly speaking, evolution operators $U$, $\bar{U}$ are just “group elements” of $NG$.\(^4\)

These properties seem to be rather appealing and it is natural to try to preserve them in any generalization. Below we consider a generalization in one particular direction – that of quantum groups: in what follows we discuss $\tau$-functions for fundamental representations of $SL_q(N)$. There are two immediate things to be taken into account. First, there is nothing similar to the operators $T^{(k)}_\pm$ for $q \neq 1$ (at least, nothing what could be defined in terms of generators without any references to a specific representation $R$). This implies that explicit expressions for $U(t)$ and $\bar{U}(\bar{t})$ should be very different from (1.3). Second, there is no reasonable notion of the nilpotent subgroup $NG_q$: only quantum deformation of the Borel subgroup $BG_q \subset G_q$ is nicely defined. Because of this one should not insist on validity of eq.(1.6): it requires some modification.

According to [2] parametrization of the group elements, which admits the most straightforward quantum deformation, involves only the simple roots $\pm \tilde{\alpha}_i$:

$$\tilde{\alpha}_i = \sigma \epsilon_i \delta_i$$

$$\chi_{\alpha_i} = \chi_{\delta_i}$$

$$\tilde{\xi} = \tilde{\xi}' = \sigma$$

$$\tilde{\xi} \tilde{\xi}' = \tilde{\xi}' \tilde{\xi}$$

$$\tilde{\xi}' \chi_{\alpha_i} = \chi_{\delta_i} \tilde{\xi}$$

$$\tilde{\xi}' \chi_{\delta_i} = \chi_{\alpha_i} \tilde{\xi}$$

(1.7)

Every particular simple root $\alpha_i$ can appear several times in the product, and there are different parametrizations of group elements of such a type, depending on the choice of the set $\{s\}$ and the mapping $i(s)$ of this set into that of simple roots. Quantum deformation of such formula is especially simple because comultiplication rule is simple for the generators, associated with the simple roots:

$$\Delta(T_i) = T_i \otimes q^{-2H_i} + I \otimes T_i$$

$$\Delta(T_{-i}) = T_{-i} \otimes I + q^{2H_i} \otimes T_{-i}$$

(1.8)

For $q \neq 1$ any expression of the form (1.7) remains just the same, provided exponentials in $g_U$ and $g_L$ are understood as $q$-exponentials (in the simply-laced case, $q^{||\tilde{\alpha}_i||^2/2}$-exponentials in general), and parameters $\theta, \chi, \phi$ become non-commuting generators of the “coordinate ring” of $G_q$. Actually, they form a kind of Heisenberg algebra:

$$\theta_{s} \theta_{s'} = q^{-\tilde{\alpha}_i(s)\tilde{\alpha}_i(s')} \theta_{s} \theta_{s'}, \quad s < s'$$

$$\chi_{s} \chi_{s'} = q^{-\tilde{\alpha}_i(s)\tilde{\alpha}_i(s')} \chi_{s} \chi_{s'}, \quad s < s'$$

$$e^{\tilde{\alpha}_i \chi_{s}} = q^{\tilde{\alpha}_i \chi_{s}} e^{\tilde{\alpha}_i \chi_{s}}$$

$$e^{\tilde{\alpha}_i \theta_{s}} = q^{\tilde{\alpha}_i \theta_{s}} e^{\tilde{\alpha}_i \theta_{s}}$$

These relations imply that $\Delta(g) = g \otimes g$.

The simplest possible assumption about evolution operators would be to say that $U(t)$ is always an object of the type $g_U$, while $\bar{U}(\bar{t})$ – of the type $g_L$.

\(^4\) We refer to [1] and [2, 3] for a lengthy discussion of what we mean by “group element” in this context. In a word, this is what has been called “universal $T$-matrix” in the quantum group theory [4, 5, 6].

\(^5\) See [2] for all the notations and definitions.
However, these are no longer group elements:

\[ \Delta(gU) \neq gU \otimes gU, \quad \Delta(gL) \neq gL \otimes gL, \]

because of the lack of factors \( gD \). This is the exact meaning of the claim that there is no “nilpotent subgroup” \( NG_q \) (but \( BG_q \) does exist, since \( \Delta(gUgD) = (gUgD) \otimes (gUgD) \)). Despite this “problem” we will insist on identification of \( U \) and \( \bar{U} \) as objects of the type \( gU \) and \( gL \) respectively, and will explicitly investigate implications of the failure of (1.6) (see Conclusion where another, perhaps more attractive, option is mentioned). In fact, instead of (1.6) we will have

\[ \Delta(U(\xi)) = U_L^{(2)}(\xi) \cdot U_R^{(2)}(\xi), \]

(1.10)

where

\[ U(\xi) = \prod_s \mathcal{E}_q(\xi_s T_i(s)), \]

(1.11)

\[ U_L^{(2)} = \prod_i \mathcal{E}_q(\xi_s T_i(s) \otimes q^{-2H_i(s)}) \neq I \otimes U(\xi), \]

(1.12)

\[ U_R^{(2)} = \prod \mathcal{E}_q(\xi_s I \otimes T_i(s)) = I \otimes U(\xi) \]

and this will have simply accountable implications for determinant formulas for quantum \( \tau \)-functions.

In what follows we first discuss various interesting ways to specify the map \( i(s) \) in the case of fundamental representations. Among these there is especially simple one, \( s = 1, \ldots, r_G; \ i(s) = s \). However, it gives rise to \( U(t) \) which is different from (1.3) even in the classical case of \( q = 1 \). Therefore, we briefly describe the classical hierarchy with this non-standard evolution. Finally, we consider the corresponding quantum deformation and derive the substitute of the determinant formulas for \( \tau_n \equiv \tau_{F_n} \) in the case of \( q \neq 1 \). Let us stress that by the multiplication of the evolution operators \( U \) and \( \bar{U} \) and the group element \( g \) in the definition of \( \tau \)-function (1.2) we always understand the group multiplication law, i.e. elements of algebra \( \theta, \phi \) and \( \chi \) (1.9) in evolution operators commute with elements of the corresponding algebra in \( g \), see [2, 3]. This is very essential for the determinant formulas of section 4.

2 Group elements through simple roots: examples

We briefly discuss here three natural choices of parametrization (1.7) of the group elements.

As we already mentioned, every parametrization of this form is straightforwardly deformed to \( q \neq 1 \) [2]. The most economic way to parametrize in this way the entire group manifold of \( SL(N) \) is to take \( s = 1, \ldots, \frac{N(N-1)}{2} \) and the map

\[ i(s) : 1, 2, \ldots, r - 1, r; 1, 2, \ldots, r - 1; 1, 2, 3; 1, 2; 1; r = \text{rank } SL(N) = N - 1 \]

i.e.

\[ U(\xi) = \prod_{1 \leq i \leq N} \prod_{i < j \leq N} \exp(\xi_{ij} T_{j-i}). \]

(2.13)

This is, however, a little too much for our purposes. The orbits of \( SL(N) \) in fundamental representations can be parametrized by only \( r = N - 1 \) parameters, and the purpose is to find an adequate parametrization of such submanifolds. This is easy to do in the classical \( (q = 1) \) case, and at least, three natural possibilities will be considered in this section. However, of these three only one will be easily deformed, and it is the one with no direct relation to conventional evolution (1.3). 

3
Parametrization A. The simplest possibility is just to restrict the set \( \{s\} \) to \( s = 1, \ldots, r \) and take \( i(s) = s \), i.e. take
\[
U^{(A)}(\xi) = \prod_{i=1}^{r_0} \exp(\xi_i T_i).
\] (2.14)

This is enough to generate all the states of any fundamental representation from the corresponding vacuum (highest vector) state, but \( < 0_{F_n} | U^{(A)}(\xi) \) has little to do with \( < 0_{F_n} | U(t) \) (where \( U(t) \) is given by (1.3)). It can be better to say that identification \( < 0_{F_n} | U^{(A)}(\xi) = < 0_{F_n} | U(t) \) defines a relation \( \xi_i(t) \), which explicitly depends on \( n \).

One can build the theory of the KP/Toda hierarchies in terms of \( \xi \)-variables instead of conventional \( t \)-variables (see a brief discussion in s.3 below), but it can not be obtained just by change of time-variables: the whole construction looks different. For it, this new construction is immediately deformed to the case of \( q \neq 1 \): instead of (2.14) we just write
\[
U^{(A)}(\xi) = \prod_{i=1}^{r_0} \mathcal{E}_q(\xi_i T_i),
\] (2.15)
where \( \xi \)'s are non-commuting variables,
\[
\xi_i \xi_j = q^{-\delta_{ij}} \xi_j \xi_i, \quad i < j,
\] (2.16)
and it is easy to derive a quantum counterpart for any statement of the classical \((q = 1)\) theory once it is formulated for the \( \xi \)-parametrization (see s.4 for some results in the \( q \neq 1 \) case).

Parametrization B (conventional). Of course, one can insist on using the conventional \( t \)-variables, i.e. to make the identification of the group elements
\[
U^{(B)}(\xi) = \prod_s \exp(\xi_s T_{i(s)}) = U(t) = \exp \left( \sum_k t_k T_k^+ \right)
\] (2.17)
(which implies that \( < 0_{F_n} | U^{(B)}(\xi) = < 0_{F_n} | U(t) \) with some \( n \)-independent functions \( \xi_s(t) \)). The difference between the two expressions in (2.17) is that the r.h.s. contains mutually commuting combinations of (non-simple) root generators, while the l.h.s. contains only (mutually non-commuting) simple-root generators. Such reparametrization indeed exists, but the set \( \{s\} \) should contain, at least, \( \frac{N(N-1)}{2} \) elements and one can take \( i(s) \) as in (2.13) – the only thing is that now not all of the \( \xi_s \) are independent: instead they are expressed through \( r = N - 1 \) time-variables \( t_k \). For example, the \( t_1 \)-dependence of \( \xi_{ij} \) is given by
\[
\xi_{ij} = \frac{t_1}{N + i - j} + O(t_2, t_3, \ldots).
\] (2.18)

Open problem. In order to get a reasonable quantum deformation of parametrization B, one needs to reproduce the proper commutation relations
\[
\xi_{s} \xi_{s'} = q^{-\delta_{s,s'}} \xi_{s'} \xi_{s}, \quad s < s',
\] (2.19)
between the \( \frac{N(N-1)}{2} \) variables \( \xi_s \) as a corollary of some relations between \( r = N - 1 \) variables \( t_k \) (which, of course, do not need to commute when \( q \neq 1 \)). To make this possible, one should also somehow deform the relations (2.18) at \( q \neq 1 \). This is a separate problem, which we do not have immediate solution to.

Parametrization C (Miwa variables). One more option is to remain with the conventional time-variables \( t_k \), but make the (representation-independent) Miwa transform \( t_k = \frac{1}{k} \sum_a \lambda^k_a \). This Miwa parametrization is, in fact, perfectly consistent with the simple-root decomposition:
\[
U(t) = \prod_a \exp \left( \sum_{k=1}^{r_0} \frac{\lambda^k_a}{k} T_k^+ \right) = \prod_a \left( \prod_{i=1}^{r_0} e^{\lambda^k_a T_i} \right).
\] (2.20)
The set \( \{s\} \) and mapping \( i(s) \) here are not of the “most economic” type (2.13), but the general rule (1.9) of the quantum deformation is, of course, applicable.
Open problem. However, (2.13) implies the quantum formula in the form
\[
\prod_{a} \left( \prod_{i=1}^{r} E_{q} (\lambda_{ai} T_{i}) \right),
\]
where \(\lambda_{ai}\) with different \(i\) and the same \(a\) do not commute. At the same time, the constraint \(\lambda_{ai} = \lambda_{aj}\) for \(i \neq j\) is of crucial importance for the classical \((q = 1)\) formula (2.20). What is the proper deformation of this constraint remains unclear. Solution of the puzzle should probably exploit the fact that, in the classical case, the constraint selects out irreducible representations of the coordinate ring (dual algebra) of \(G\).

3 Classical \((q = 1)\) KP/Toda theory

3.1 Determinant formulas and systems of equations

Let us first consider
\[
\tau_{1} = \langle 0_{F_{1}} | U(t)g\bar{U}(\bar{t})| 0_{F_{1}} \rangle.
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\]

Note that a specific feature of \(F = F_{1}\) is
\[
< 0_{F} | U(t) = \sum_{k} P_{k}(t) < 0_{F} | T_{+}^{k} = \sum_{k} P_{k}(t) < k_{F},
\]
where the r.h.s. is reexpanded in terms of generalized Schur polynomials (the first equality in this formula defines these polynomials) and the \(N\) states of \(F = F_{1}\) are denoted by \(< k_{F}| = < 0_{F} | T_{+}^{k}, k = 0, ..., r = N - 1\). Thus,
\[
\tau_{1}(t, \bar{t})g = \sum_{k,k} P_{k}(t)P_{k}(\bar{t}) < 0_{F} | T_{+}^{k} g T_{-}^{k} | 0_{F} >=
\]
\[
= \sum_{k,k} P_{k}(t)P_{k}(\bar{t}) < k_{F} | g \bar{k}_{F} >= \sum_{k,k} P_{k}(t)g_{k,k} P_{k}(\bar{t}).
\]

One can also define
\[
\tau_{1}^{mn} \equiv \langle m_{F} | U(t)g\bar{U}(\bar{t})| m_{F} \rangle = \sum_{k,k} P_{k}(t)g_{m+k,m+k} P_{k}(\bar{t}).
\]

Now we can turn to generic fundamental representation \(F_{n}\). Since
\[
< m_{1} \ldots m_{n} F_{n} | = < m_{1} F_{1} | \otimes < m_{2} F_{1} | \otimes \ldots \otimes < m_{n} F_{1} | +
\]
+antisymmetrization over \(m_{1}, \ldots, m_{n}\) =
\[
= \sum_{P} (-)^{P} < m_{P(1)} | \otimes < m_{P(2)} | \otimes \ldots \otimes < m_{P(n-1)} |\]
the vacuum (highest weight) state of \(F_{n}\) can be written as
\[
< 0_{F_{n}} | = < 0, 1, \ldots, n - 1_{F_{n}} | =
\]
\[
= \sum_{P} (-)^{P} < P(0) F_{1} | \otimes < P(1) F_{1} | \otimes \ldots \otimes < P(n - 1) F_{1} |.
\]
Taking into account that
\[ U(t)|F_n = \Delta^{n-1} U(t) = U(t)^{\otimes n}, \quad g|F_n = \Delta^{n-1}(g) = g^{\otimes n} \] one finally gets:
\[ \tau_{n+1}(t, \bar{t}) g \equiv \langle 0_{F_n} | U(t)g\tilde{U}(\bar{t})|0_{F_n} \rangle = \sum_{p, \bar{p}} (-)^p \bar{p} \prod_{k=0}^{n} \langle P(k)_F|U(t)g\tilde{U}(\bar{t})|\bar{P}(k)_F \rangle = \det \tau_{m\bar{m}} = \det \sum_{l, \bar{l}} \mathcal{P}_{l-m}(t) g_{l\bar{p}, \bar{l}p} \mathcal{P}_{l-\bar{m}}(\bar{t}) = \sum_{1\leq m_1 < m_2 < \ldots < n} \det \mathcal{P}_{m_j-i}(t) \det g_{m_j, \bar{m}_j} \det \mathcal{P}_{\bar{m}_i-j}(\bar{t}). \] (3.27)

These determinant formulas are equivalent to the KP/Toda equations. However, determinant formulas are not the simplest starting point to derive the equations (see, for example, [7]). Here we apply the simpler method [8, 1] making use of fermionic intertwining operators \( \psi_i^+ (i = 1 \ldots N) \). The key ingredient of the derivation (see [1, 3]) is the composite intertwining operator \( \Gamma = \sum_i \psi_i^+ \otimes \bar{\psi}_i^- \). In order to get a set of equations, one consider the matrix element of the interwiner identity \( \Gamma(g \otimes g) = (g \otimes g)\Gamma \) between the states \( \langle 0_{F_{n+1}}|U(t) \otimes 0_{F_{m-1}}|U(t') \rangle \) and \( \tilde{U}(\bar{t})|0_{F_n} \rangle \otimes \tilde{\tilde{U}}(\bar{t}')|0_{F_m} \rangle \).

\[ \sum_i \langle 0_{F_{n+1}}|U(t)\psi_i^+ g\tilde{U}(\bar{t})|0_{F_n} \rangle \cdot \langle 0_{F_{m-1}}|U(t')\psi_i^- g\tilde{\tilde{U}}(\bar{t}')|0_{F_m} \rangle = \sum_i \langle 0_{F_{n+1}}|U(t)g\psi_i^+ \tilde{U}(\bar{t})|0_{F_n} \rangle \cdot \langle 0_{F_{m-1}}|U(t')g\psi_i^- \tilde{\tilde{U}}(\bar{t}')|0_{F_m} \rangle. \] (3.28)

One can rewrite (3.28) through the free fermion fields \( \psi^+(z) \equiv \sum_{i=1}^{N} \psi_i^+ z^i \) and
\[ \psi^-(z) \equiv \sum_{i=1}^{N} \psi_i^- z^{N-i+1}; \]
\[ \oint_{z N+2} \frac{dz}{z} \langle 0_{F_{n+1}}|U(t)\psi^+(z)g\tilde{U}(\bar{t})|0_{F_{n-1}} \rangle \cdot \langle 0_{F_{m-1}}|U(t')\psi^-(z)g\tilde{\tilde{U}}(\bar{t}')|0_{F_m} \rangle = \oint_{z N+2} \frac{dz}{z} \langle 0_{F_{n+1}}|U(t)g\psi^+(z)\tilde{U}(\bar{t})|0_{F_{n-1}} \rangle \cdot \langle 0_{F_{m-1}}|U(t')g\psi^-(z)\tilde{\tilde{U}}(\bar{t}')|0_{F_m} \rangle. \] (3.29)

The same formulas can be written in more compact form using the following (Baker-Akhiezer) functions
\[ \Psi_{n, i}^+ \equiv \langle 0_{F_{n+1}}|\tilde{U}(t)\psi_i^+ g\tilde{U}(\bar{t})|0_{F_n} \rangle \] (3.30)

Then (3.28) can be rewritten in the form
\[ \sum_{i} \Psi_{n, i}^+ (t) \Psi_{N-i, i}^-(t') = \sum_{j} \Psi_{k+1, j}^+(t) \Psi_{l-1, j}^-(\bar{t}), \] (3.31)
where \( \Psi \) is defined analogously to (3.30) but with the fermion situated to the right of the group element \( g \).

One can also define vertex operators which generates Baker-Akhiezer functions:
\[ \sum_{i} \Psi_{k, i}^+(t) z^i \equiv \hat{X}^+(z, t) \tau_n(t), \quad \sum_{i} \Psi_{k, i}^-(t) z^{N-i+1} \equiv \hat{X}^-(z, t) \tau_{n}(t), \] (3.32)
and analogously for \( \hat{X}^\pm(z, t) \). Then, (3.28) can be also written as
\[ \oint_{z N+2} \frac{dz}{z} \hat{X}^-(z, t) \tau_n(t, \bar{t}) \hat{X}^+(z, t') \tau_{n}(t', \bar{t}') = \oint_{z N+2} \frac{dz}{z} \hat{X}^-(z, t) \tau_{n+1}(t, \bar{t}) \hat{X}^+(z, \bar{t'}) \tau_{n-1}(t', \bar{t'}), \] (3.33)

Now we apply all these formulas to the particular choices of evolution operators \( U(t), \tilde{U}(\bar{t}) \).
3.2 Conventional parametrization (B)

This evolution leads to the standard KP/Toda hierarchy. One makes use of expressions (1.3). It gives

\[ < 0_F|U(t) = \leq 0_F|\exp \left( \sum_k t_k T_k^+ \right) = \sum_k P_k(t) < k_F, \]

with the orthodox Schur polynomials \( P_k(t) \) defined by \( \exp \left( \sum_k t_k z^k \right) = \sum_k P_k(t) z^k \). The main peculiarity of this evolution is the property

\[ \tau_1^m = \frac{\partial}{\partial t_m} \frac{\partial}{\partial t_1} \tau_1 = \left( \frac{\partial}{\partial t_1} \right)^m \tau_1. \] (3.34)

Determinant formula (3.27) in this parametrization

\[ \tau_n(t,\bar{t}) = \sum_{1 < m_1 < m_2 < \ldots} \left[ \det_{ji} P_{m_j-i}(t) \det_{ji} g_{m_j-i} \right] \]

leads to equations which are nothing but bilinear Plücker relations [9], while (3.35) demonstrates explicitly that \( \tau \)-function is spanned by the particular Plücker coordinates – Schur functions \( \xi_\beta = \det_{ij} P_{m_j-i}(t) \), where \( \beta \) labels Young tables – see [10]).

In order to obtain the equations in parametrization B one should note that, in this case, vertex operators (3.32) are

\[ \hat{X}^+(z,t) = \Pr \left[ e^{i\xi(t,z)} \Pr \left[ z^n e^{-i\xi(z\bar{z},\bar{t})} \right] \right], \]

\[ \hat{X}^-(z,t) = \Pr \left[ e^{-\xi(t,z)} \Pr \left[ z^{N-n+1} e^{i\xi(z\bar{z},\bar{t})} \right] \right], \]

\[ \xi(z,t) = \sum_i z^i \bar{t}_i, \quad \bar{t}_k = \frac{1}{k} \partial_{t_k} \]

(and similarly for the right vacuum state), where \( \Pr[f(z)] \) projects onto the polynomial part of function \( f(z) \) and \( \Pr[l][f(z)] \) projects onto the polynomial part of the degree \( l \).

One can explicitly write equations (3.33) in the integral form, which can be easily transformed to an infinite set of differential equations by expanding in powers of time differences \( t_i - t'_i \) etc. [8]. As \( N \to \infty \), these equations give rise to the standard Toda Lattice hierarchy.

In particular, the simplest equation, which contains only derivatives with respect to the first times, is

\[ \frac{\partial \tau_n(t,\bar{t})}{\partial t_1} - \frac{\partial \tau_n(t,\bar{t})}{\partial t_1} \tau_n(t,\bar{t}) = \tau_{n+1}(t,\bar{t}) \tau_{n-1}(t,\bar{t}). \] (3.37)

Equation of such a type can be also obtained relatively easy from the determinant formula [7].

3.3 On KP/Toda hierarchy in parametrization A

Now we consider the same conventional hierarchy with a different evolution A. Our purpose is to demonstrate that in this parametrization the main features of the hierarchy are preserved – there are determinant formulas and a hierarchy of differential equations.

From now on we denote for brevity \( \hat{U}(\xi) \equiv \hat{U}(A)(\xi) \) and the corresponding \( \tau \)-function will be \( \hat{\tau}(\xi,\bar{\eta}|g) \).\footnote{KP/Toda system itself arises in the limit \( N \to \infty \), and \( n \) plays the role of the "zero-time" \( T_0 \) of the Toda system. In fact, most of equations and their interesting properties are essentially independent of \( N \).} This function is a linear in each time-variable \( \xi \), hence, it satisfies simpler determinant formulas and simpler hierarchy of equations. Indeed, (3.22) turns into:

\[ \hat{\tau}_1(\xi,\bar{\eta}|g) = \leq 0_F|\hat{U}(\xi)g\hat{U}(\xi)|0_F = \sum_{k,k \geq 0} \frac{s_k \bar{s}_k}{k|g|k} > \] (3.38)
where \( s_k = \xi_1 \xi_2 \ldots \xi_k, \ s_0 = 1, \) while (3.23) is substituted by:

\[
\hat{r}_1^m = \frac{1}{s_m s_{\bar{m}}} \sum_{k \geq m \geq k} \frac{\partial}{\partial \log s_k} \frac{\partial}{\partial \log s_{\bar{k}}} r_1(\xi, g) = \frac{1}{s_m s_{m-1} \partial \xi_m} \partial \xi_m r_1(\xi, g).
\]

Thus,

\[
\hat{r}_{n+1} = \det_{0 \leq m, m \leq n} \hat{r}_1^m = \left( \prod_{m=1}^n s_m s_{\bar{m}} \right)^{-1} \det_{(m, \bar{m})} \left( \sum_{k \geq m \geq k} s_k s_{\bar{k}} < k|g| \bar{k} > \right) = \frac{1}{s_n s_n} \sum_{k, k \geq n} s_k s_{\bar{k}} \det_{0 \leq m, m \leq n-1} \left( g_{mn} g_{\bar{m}n} \right) = \frac{1}{s_n s_n} \sum_{k, k \geq n} s_k s_{\bar{k}} D^{(n)}_k.
\]

One can compare determinant representations (3.35) and (3.40) to get the connection between different coordinates \( t \) and \( \xi \). One can see that this is of the type \( s_k \sim \) some functions of \( P_j(t) \). \( \text{7} \)

Equations for the \( \tau \)-function in parameterization A can be easily derived. Indeed, it is straightforward to find the Baker-Akhiezer functions (3.30) and substitute this into equations (3.31):

\[
\Psi_{n+1}^{+}(\xi) = \psi_n + \frac{s_{n+k}}{s_n} \left( \tau_n(\xi) - \xi_n \frac{\partial \tau_n(\xi)}{\partial \xi_n} \right);
\]

\[
\Psi_{n+1}^{-}(\xi) = \psi_n - \frac{s_{n-1}}{s_{n-1}} \left( \tau_n(\xi) - \xi_n \frac{\partial \tau_n(\xi)}{\partial \xi_n} \right) + \frac{s_{n-1}}{s_{k-2}} \frac{\partial \tau_n(\xi)}{\partial \xi_{k-1}} \text{ for } k > n, \quad (3.41)
\]

\[
\Psi_{n-1}^{-}(\xi) = \xi_{n-1} \tau_n(\xi), \ \Psi_{n}^{-}(\xi) = 0 \text{ for } k < n - 1.
\]

As for the values of \( \Psi_{n+k}(\xi) \) for \( k < n \), they are constants which can be hardly expressed as an action of a differential operator on \( \tau_n(\xi) \). This means that the relation (3.31), where manifest expressions for \( \hat{\psi} \), analogous to (3.41), can be easily written down, does not lead to differential equations when \( k \) and \( l \) are arbitrarily chosen. \( \text{7} \), however, one chooses \( k \leq l - 1 \), because of multiple cancelations due to (3.41), (3.31) is almost a differential equation. It can be easily transformed to a differential equation by putting \( \xi_{n-1} = 0 \) (see (3.41)).

One can easily check that the number of independent equations obtained in this way is sufficient to determine \( \tau \)-function in full.

4 Quantum (\( q \neq 1 \)) case

4.1 \( q \)-Determinant-like representation

In this section we demonstrate how the technique developed in the previous sections is deformed to the quantum case and, in particular, obtain \( q \)-determinant-like representation analogous to (3.27). We also demonstrate that in parametrization A relation (3.39) expressing \( \tau_1^m \) through \( \tau_1 \) derivations is
still correct for \( q \neq 1 \), with all the derivatives replaced by difference operators.

In this subsection we present the statements valid for any \( U(\xi) \) of the form (1.11), without reference to particular parameterization. \(^8\)

As a result of the absence of diagonal factor \( g_8 \) co-product (3.26) is replaced by the following comultiplication rule

\[
\Delta^{n-1}(U\{T_i\}) = \prod_{m=1}^{n} U^{(m)}
\]  

(4.42)

where

\[
U^{(m)} = U \left\{ I \otimes \ldots \otimes I \otimes T_i \otimes q^{-2H_i} \otimes \ldots \otimes q^{-2H_i} \right\}
\]  

(4.43)

(\( T_i \) appears at the \( m \)-th place in the tensor product). Similarly

\[
\tilde{U}^{(m)} = \tilde{U} \left\{ q^{2H_i} \otimes \ldots \otimes q^{2H_i} \otimes T_{-i} \otimes I \otimes \ldots \otimes I \right\}.
\]  

(4.44)

Let

\[
H_i[j_{F_i}] = h_{i,j} - h_{j,F_i} > 0, \quad j_{F_i} = h_{i,j} < h_{F_i} |
\]

(in fact for \( SL(N) \) \( 2h_{i,i-1} = +1, 2h_{i,i} = -1, \) all the rest are vanishing). Then

\[
\tau_n(j_1\ldots j_n) = \sum_{m=0}^{n-1} \tau_1^{P(m)P'(m)} \left( \xi_s \xi_t \right)^2 \sum_{l,m=0}^{n-1} h_{i(l),i(l)}
\]  

(4.45)

In order to get a \( q \)-determinant-like counterpart of (3.27), one should replace antisymmetrization by \( q \)-antisymmetrization in eqs. (3.24)-(3.25), since, in quantum case, fundamental representations are described by \( q \)-antisymmetrized vectors (see s.5.2 of [1] for more details). We define \( q \)-antisymmetrization as a sum over all permutations,

\[
\left\{ [1, \ldots, k]_q \right\} = \sum_{P} (-q)^{deg P} (P(1), \ldots, P(k)),
\]  

(4.46)

where

\[
deg P = \# \text{ of inversions in } P.
\]  

(4.47)

Then, \( q \)-antisymmetrizing (4.45) with \( j_k = k-1, \ j_k = k-1 \), one finally gets

\[
\tau_n(\xi, \xi|g) = \sum_{P,P'} (-q)^{deg P + deg P'} \times \prod_{m=0}^{n-1} \tau_1^{P(m)P'(m)} \left( \xi_s \xi_t \right)^2 \sum_{l,m=0}^{n-1} h_{i(l),i(l)}
\]  

(4.48)

This would be just a \( q \)-determinant\(^9\); be there no the \( q \)-factors, which twist the time variables in (4.48).

To make this expression more transparent, let us consider the simplest example of the second fundamental representation. Denote through \( \{u\} \) and \( \{v\} \) the

\[^8\text{Actually, we only require that } U(\xi) \text{ is an element from } NG_q \text{ and is expressed only through the generators associated with simple positive roots: } U(\xi) = U\{\xi_i T_i\}. \text{ Formula (1.11) is a possible but not the unique realization of these requirements.}

[^9\text{Let us note that the relevant } q \text{-determinant is defined as [1]}
\]

\[
\text{det}_q A \sim A_{[1]}^{[1]} \ldots A_{[n]}^{[n]} = \sum_{P,P'} (-q)^{deg P + deg P'} \prod_a A_{P(a)}^{P'(a)}
\]  

(4.49)

This is not necessarily the same as \( A_{[1]}^{[1]} \ldots A_{[n]}^{[n]} \). It is the same only provided by special commutation relations of the matrix elements \( A_{ij}^{kl} \).
subsets of \{s\} such that \(i(s) = 1\) and \(i(s) = 2\) respectively. Then
\[
\tau_2 = \tau_1^{00}(\{q\xi_0\}, \{q^{-1}\xi_0\}, \xi_0; \{\xi_0\}, \{\xi_0\}, \xi_0)\tau_1^{11}(\{\xi_0\}, \{\xi_0\}, \xi_0; \{q\xi_0\}, \{\xi_0\}, \xi_0) -
- q\tau_1^{01}(\{q\xi_0\}, \{q^{-1}\xi_0\}, \xi_0; \{\xi_0\}, \{\xi_0\}, \xi_0)\tau_1^{10}(\{\xi_0\}, \{\xi_0\}, \xi_0; \{q^{-1}\xi_0\}, \{q\xi_0\}, \xi_0) -
- q\tau_1^{10}(\{q^{-1}\xi_0\}, \{\xi_0\}, \xi_0; \{q\xi_0\}, \{\xi_0\}, \xi_0)\tau_1^{01}(\{\xi_0\}, \{\xi_0\}, \xi_0; \{q\xi_0\}, \{\xi_0\}, \xi_0) +
+ q^2\tau_1^{11}(\{q^{-1}\xi_0\}, \{\xi_0\}, \xi_0; \{\xi_0\}, \{\xi_0\}, \xi_0)\tau_1^{00}(\{\xi_0\}, \{\xi_0\}, \xi_0; \{q^{-1}\xi_0\}, \{q\xi_0\}, \xi_0).
\] (4.50)
\(\xi_s\) here denotes all the time variables with \(i(s) > 2\). Let us note that \(q\)-factors in all these expressions can be reproduced by action of the operators
\[
M_j^\pm : M_j^\pm \xi_s = q^{\pm \delta_{j,i(s)}} \xi_s,
\]
\[
\bar{M}_j^\pm : \bar{M}_j^\pm \bar{\xi}_s = q^{\pm \delta_{j,i(s)}} \bar{\xi}_s.
\]
Now we briefly discuss the set of equations satisfied by quantum \(\tau\)-function.

In quantum case, one should distinguish between the right and left intertwiners:
\(\Phi_{\pm,R} : \bigotimes_{n=1}^i \xi_0 \Rightarrow \bigotimes_{n=1}^{i+1} \xi_0\) and \(\Phi_{\pm,L} : \bigotimes_{n=1}^i \xi_0 \otimes \bigotimes_{n=1}^{i+1} \xi_0\). These operators \(\Phi_{\pm,R,L}\) can be expressed through the classical intertwining operators (fermions):
\[
\Phi_{\pm,R} = q^{-\sum_{j=1}^{i-1} v_j^+ v_j^-} \psi_i^\pm, \quad \Phi_{\pm,L} = q^{\sum_{j=1}^{i-1} v_j^+ v_j^-} \psi_i^\pm.
\] (4.51)

In analogy with the classical case, one can consider the operator \(\Gamma = \sum_i \Phi_{i,L} \otimes \Phi_{i,-R}\) that commutes with \(q \otimes g\). Then, introducing vertex operators, or Baker-Akhiezer functions as averages of quantum intertwining operators (properly labeled by indices \(L\) and \(R\)), one obtains equations (3.28)-(3.33) with the properly defined entries. Technically, vertex operators can be calculated with the help of equation (4.51). In the next subsection we show how it works in the concrete case of parameterization A.

### 4.2 Parameterization A

Now we apply formulas of the previous subsection to the case of parameterization A. In fact, most of expressions from subsection 3.3 remain almost the same in the quantum case. In particular,
\[
\tau_1(\xi, \bar{\xi}|\bar{g}g) < 0_{F_1} |\bar{U}(\xi)g\bar{U}(\bar{\xi})|0_{F_1} > = \sum_{k,k \geq 0} s_k \bar{s}_k < k|g|k >
\] (4.52)
where again \(s_k = \xi_1 \xi_2 \ldots \xi_k,\ s_0 = 1\), while \(\bar{s}_k = \bar{\xi}_1 \ldots \bar{\xi}_k,\ \bar{s}_0 = 1\) and
\[
\hat{\tau}_1^{m,n}(\xi, \bar{\xi}|g) = s_m^{-1} \left( \sum_{k \geq m} s_k \bar{s}_k < k|g|k > \right) s_m^{-1} =
\] (4.53)
\[
= s_m^{-1} (D_{\xi_m}^{-1} D_{\bar{\xi}_m}^{-1} \tau_1(\xi, \bar{\xi}|g) S_m^{-1}).
\]

Here\(^10\) \(D_{\xi} f(\xi) = \frac{1}{q} \frac{M^{2i-1}}{q^{-}\xi} f(\xi),\ \bar{D}_{\bar{\xi}} f(\xi) = \frac{1}{q} \frac{M^{2i-1}}{q^{-}\bar{\xi}} f(\bar{\xi})\). Then, one can express \(\tau_n\) through \(\tau_1\) manifestly using formulas (4.48) and (4.53).

Equation (4.48) remains just the same. In our example (4.50) of the second fundamental representation each set \(\{u\}\) and \(\{v\}\) consists of the single element:
\([u] = \{s = 1\}, [v] = \{s = 2\}\). Then
\[
\tau_2 = \tau_1^{00}(q\xi_1, q^{-1}\xi_2, \xi_i; \xi_1, \bar{\xi}_2, \bar{\xi}_i)\tau_1^{11}(\xi_1, \bar{\xi}_2, \xi_i; q\xi_1, \bar{\xi}_2, \bar{\xi}_i) -
- q\tau_1^{01}(q\xi_1, q^{-1}\xi_2, \xi_i; \xi_1, \bar{\xi}_2, \bar{\xi}_i)\tau_1^{10}(\xi_1, \bar{\xi}_2, \xi_i; q^{-1}\xi_1, q\xi_2, \bar{\xi}_i) -
- q\tau_1^{10}(q^{-1}\xi_1, \xi_2, \xi_i; \xi_1, \bar{\xi}_2, \bar{\xi}_i)\tau_1^{01}(\xi_1, \bar{\xi}_2, \xi_i; q\xi_1, \bar{\xi}_2, \bar{\xi}_i) +
+ q^2\tau_1^{11}(q^{-1}\xi_1, \xi_2, \xi_i; \xi_1, \bar{\xi}_2, \bar{\xi}_i)\tau_1^{00}(\xi_1, \bar{\xi}_2, \xi_i; q^{-1}\xi_1, q\xi_2, \bar{\xi}_i).
\] (4.54)

\(^10\)There is an ambiguity in the choice of these operators as \(\tau\)-function is a linear function of times, and therefore, any linear operator which makes unity from \(\xi\) is suitable. We fix them to act naturally on the corresponding \(q\)-exponentials in (1.7).
This expression can be written in a more compact form with the help of operators
\[
\begin{align*}
\mathcal{D}_i^L &\equiv M_1^- D_i \otimes I, \quad \mathcal{D}_i^R \equiv M_1^+ M_2^- \otimes D_i, \\
\bar{\mathcal{D}}_i^L &\equiv \bar{D}_i \otimes M_2^- M_1^+ \otimes \bar{D}_i, \\
\bar{\mathcal{D}}_i^R &\equiv I \otimes M_1^- \bar{D}_i.
\end{align*}
\]
Indeed,
\[
\tau_2 = (\mathcal{D}_i^R \mathcal{D}_i^L - q \mathcal{D}_i^L \mathcal{D}_i^R - q \mathcal{D}_i^R \mathcal{D}_i^L + q^2 \mathcal{D}_i^L \mathcal{D}_i^L) \tau_1 \otimes \tau_1 = (\mathcal{D}_i^L - q \mathcal{D}_i^L) \cdot (\mathcal{D}_i^R - q \mathcal{D}_i^L) \tau_1 \otimes \tau_1.
\]
These operators satisfy the following commutation relations (like the algebra of \(\theta\) and \(\chi\) in (1.9)):
\[
\begin{align*}
\mathcal{D}_i^L \mathcal{D}_j^R &= q \mathcal{D}_j^R \mathcal{D}_i^L, \\
\bar{\mathcal{D}}_i^L \bar{\mathcal{D}}_j^R &= q \bar{\mathcal{D}}_j^R \bar{\mathcal{D}}_i^L.
\end{align*}
\]
These formulas can be rewritten in a more “invariant” form in terms of operators
\[
\begin{align*}
\mathcal{D}_i^L &\equiv D_i \otimes I, \quad \mathcal{D}_i^R \equiv \prod_j M_j^{-\alpha_i \bar{\alpha}_j} \otimes D_i, \\
\bar{\mathcal{D}}_i^L &\equiv \bar{D}_i \otimes \prod_j \bar{M}_j^{-\bar{\alpha}_i \alpha_j}, \quad \bar{\mathcal{D}}_i^R \equiv I \otimes \bar{D}_i,
\end{align*}
\]
which commute as
\[
\begin{align*}
\mathcal{D}_i^L \mathcal{D}_j^R &= q^{\alpha_i \bar{\alpha}_j} \mathcal{D}_j^R \mathcal{D}_i^L, \\
\bar{\mathcal{D}}_i^L \bar{\mathcal{D}}_j^R &= q^{\bar{\alpha}_i \alpha_j} \bar{\mathcal{D}}_j^R \bar{\mathcal{D}}_i^L.
\end{align*}
\]
Then,
\[
\tau_2 = M_1^- \otimes M_1^+ (\mathcal{D}_i^R - q \mathcal{D}_i^L) \cdot (\bar{\mathcal{D}}_i^R - q \bar{\mathcal{D}}_i^L) \tau_1 \otimes \tau_1.
\]
Baker-Akhiezer functions for the \(\tau\)-function in parameterization A are given by the following expressions
\[
\begin{align*}
\Psi_{n+k}^{+} (\xi) &= q^{n+1} s_{n+k} M_{n+1}^+ \cdots M_{n+k-1}^+ \left(\tau_n (\xi) - \xi D_n \tau_n (\xi)\right), \\
\Psi_{n}^{+} (\xi) &= q^n - q^{-n} D_n \tau_n (\xi), \\
\Psi_{n-k}^{-} (\xi) &= -q^{n-1} \xi D_{n-1} \Psi_{n-k}^{-} (\xi) = q^{-n} \tau_n (\xi), \\
\Psi_{n-k}^{-} (\xi) &= q^{n-1} \xi D_{n-1} \Psi_{n-k}^{-} (\xi), \\
\Psi_{n-k}^{-} (\xi) &= q^{-n} \xi D_{n-1} \Psi_{n-k}^{-} (\xi) = 0 \text{ for } k < n - 1.
\end{align*}
\]
Substituting these expressions to formula (3.31), one obtains the set of equations which is a quantum counterpart of KP/Toda hierarchy in parameterization A.

5 Conclusion

In the paper we described the general way to construct quantum deformations of the determinant representations of KP/Toda \(\tau\)-functions. For this, we did not need any concrete form of the time evolution but only the suggestion that the evolution is described by (quasi-)group elements.

We observed that, for \(q \neq 1\), the determinant representations turn into \(q\)-determinant-like ones. Moreover, we do not get just \(q\)-determinants only because the evolution operator in quantum case is not a group element. This happens, because no nilpotent subgroup \(NG_q\) exists in the quantum group. To avoid this problem, one could begin from a slightly different parametrization of classical \(\tau\)-function, such that the evolution operator lies in Borel (not just nilpotent) subgroup \(BG_q\). In the classical limit, additional Cartan generators can be removed by redefinition of the element \(g\) labeling the point of the Grass-
mannian, but, in the quantum case, the Cartan part of the evolution would essentially change the result: evolution operator is now a group element (for $BG_q$) and, thus, additional twistings of times disappear from formula (4.48), and so defined quantum $\tau$-functions are just the $q$-determinants. In order to fulfill this program, one still needs to find an appropriate parametrization of (a set of) the group elements of $BG_q$ by exactly $r_G$ ”time-variables”. In this paper we followed another way: evolution has been taken to lie in $NG_q$, and we explicitly described the corrections to naive $q$-deformed formulas, which originated from the fact that $NG_q$ is not a subgroup.

Another problem, which remains beyond the scope of this paper, is deformation of the standard KP/Toda evolutions $B$ and $C$. We mentioned in section 2 that this problem is equivalent to restricting possible representations of algebra (1.9) of functions $(\theta, \chi, \phi)$ in such a way that the number of independent variables is appropriately reduced. Unfortunately, we are not aware of explicit solutions to these constraints neither in the $B$, nor in $C$ case.

The third problem, which deserves to be mentioned here, concerns interpretation of integrable hierarchies in terms of the Grassmannian. In fact, in this paper we derived hierarchies of integrable equations making use of intertwining operators (fermions). These latter ones give a natural definition of $q$-deformed fermions, each fermion being doubled due to the difference between right and left intertwiners in quantum group. We observed that, in order to derive the equations, one needed bilinear combinations mixing the right and the left intertwiners$^{11}$. On the other hand, in order to properly describe the $q$-Grassmannian, one might need combinations of all intertwiners including those with only left, or only right ones. In this case, in order to obtain an adequate description of quantum hierarchies, one needs a sort of a doubled $q$-Grassmannian.

The details of description of quantum hierarchies in terms of quantum intertwining operators ($q$-fermions) and related problems will be presented in [11].

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