An approximate solution to the Boltzmann equation for vibrated granular disks

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The behaviour of the lower order moments of the velocity distribution function for a system of inelastic granular disks driven by vertical vibrations is studied using a kinetic theory. A perturbative kinetic theory for vibro-fluidised beds was proposed by Kumaran (JFM, v. 364, 163). A scheme to generalise this theory to higher orders in the moments is presented here. With such a method it is possible to obtain an analytical solution to the moments of the distribution function up to third order.

I. INTRODUCTION

The dynamics of vibrated granular materials, its instabilities, and pattern formation are of some interest in the recent years as demonstrated by experiments of [1] and simulations of [2]. The theoretical description of such systems is complicated by the fact that it is a driven dissipative system characterised by highly inelastic collisions and hence the validity of equations of hydrodynamics is not clear at present [3]. However, it is possible to describe one idealised situation, where the dissipation due to particle collisions is small and the amplitude of wall oscillations is small compared to the mean free path, as was shown in the kinetic theories [4, 5]. Such a description might be one of the starting points where we can ascertain with some confidence the rigour of the approach used. The present work is a continuation of such an approach.

In this communication, we show that it is possible to obtain an analytical solution to the Boltzmann equation for a dilute bed of vibrated granular disks, by the method of moments, correct up to third order in the moments of the distribution function. The method of approach followed holds the same principle as followed in [4], except in the choice of the distribution function, which is done here by expanding it in the orthogonal set of Hermite polynomials. It is hoped that the analysis of this base state solution for its stability would give some clues to understanding the instabilities occurring in a vibro-fluidised bed.

In Sec. II, we present a general methodology to approximate the distribution function by expanding it in Hermite polynomials, and a procedure to solve the Boltzmann equation. We present only the important results here, the reader is referred to [4] for the details of the kinetic theory of vibro-fluidised beds. In Sec. III, we obtain an analytical solution when the formulation is restricted up to the third order in the moments. The results we obtain here are qualitatively similar to results of [4], except that we have obtained an analytical solution whereas in the latter it was an approximate series solution.

II. BASIC FORMULATION

The Boltzmann equation for the velocity distribution function, \( f(\mathbf{x}, \mathbf{u}) \), for vertically vibrated beds is [4]:

\[
\partial_t f + u_i \partial_i f - \frac{g}{\sqrt{2\pi}} \partial^2 f_{\sigma} = \frac{\partial \hat{f}}{\partial t}
\]

where the collision integral is,

\[
\frac{\partial \hat{f}}{\partial t} = \sigma \int du_2^{\ast} \mathbf{d}k \left( \mathbf{w}^{\ast} \cdot \mathbf{k} \right) \left( \hat{f}_i^1 \hat{f}_2^2 - \hat{f}_1^1 \hat{f}_2^2 \right)
\]

To obtain an approximate solution to this equation we expand the distribution function about a Maxwell distribution in some space as:

\[
f(\mathbf{x}, \mathbf{u}) = \frac{\rho}{T_0^3} \left[ 1 + A_j(\mathbf{x}) \varphi_j(\mathbf{u}) \right]
\]

where, \( f^0(\mathbf{u}) = \frac{1}{\sqrt{2\pi}} e^{\frac{u^2}{2T_0}} \) is the Maxwell distribution function with the \( T_0 \) left out of the definition for the simplifications to follow. The density \( \rho \) is also expanded about a leading order density field,

\[
\rho = \rho_0 + \rho_1
\]

The functions \( \varphi_j \) are chosen from a set of linearly independent function space. The parameters \( A_j(\mathbf{x}) \) are determined using the method of moments. In this method a set of functions, \( \psi_i(\mathbf{u}) \), equal in number to the number of unknowns are chosen and are multiplied with the Boltzmann equation, and integrated over the velocity space. This way we obtain a set of differential equations for the unknown parameters.

The leading order density and temperature distribution were obtained in [4] for a dilute bed.

\[
\rho_0 = \frac{N \bar{q}}{T_0} e^{-g z / T_0}
\]

\[
T_0 = 4 \sqrt{\frac{2}{\pi}} \frac{\langle U^2 \rangle}{N \sigma (1 - e^2)}
\]

Here, \( N \) is the number of particles per unit width of the bed, \( g \) is the gravitational acceleration, \( \epsilon \) is the coefficient of restitution for particle-particle collisions, and \( \langle U^2 \rangle \) represents the mean square velocity of the vibrating surface. For sinusoidal forcing with characteristic velocity \( U_0 \), this is given by \( \langle U^2 \rangle = U_0^2 / 2 \).
The functions $\varphi_j$ are chosen from a set of linearly independent function space. The parameters $A_j(x)$ are determined using the method of moments. In this method a set of functions, $\psi_i(u)$, equal in number to the number of unknowns are chosen and are multiplied with the Boltzmann equation, and integrated over the velocity space. This way we obtain a set of differential equations for the unknown parameters.

a. **Non-dimensionalisation** As a simplification we use the following non-dimensionalisation, $u = u^* / \sqrt{T_0}$, $z = z^* g/T_0$. Substituting Eq. (3) in Eq. (1), multiplying by $\psi_i$ and integrating over the velocity we obtain the steady state differential equation for the moments for variation only in the vertical direction, $z$ as:

$$
\frac{g}{T_0 \sqrt{T_0}} \int du_1 \psi_i u_z f^0 \partial_z \rho (1 + A_j \varphi_j) + \frac{g \rho}{T_0 \sqrt{T_0}} \int du_1 f^0 (1 + A_j \varphi_j) \frac{\partial \psi_i}{\partial u_z} = \frac{\partial f}{\partial t}. \tag{7}
$$

Here the second term has been simplified using the divergence theorem and the condition that the distribution function vanishes for large velocities. When the collision term is integrated over the velocities, $du_1$, it can be simplified from the form in Eq. (3) to an equivalent form (see in [6], for example)

$$
\int du_1 \psi_i \frac{\partial f}{\partial t} = \frac{1}{2} \sqrt{T_0} \int du_1 du_2 d\mathbf{k} \ (w \cdot \mathbf{k}) \rho^2 
\times f^0 f^0 (1 + A_j \varphi_j_1)(1 + A_k \varphi_k_2) \Delta \psi_i. \tag{8}
$$

where, $\Delta \psi_i = [\psi_i' + \psi_i'' - \psi_i_1 - \psi_i_2]$, is the total change in $\psi_i$ due to collisions. The terms in Eq. (7) and Eq. (8) can be simplified by defining $\langle \cdot \rangle = \int du \ f^0 (\cdot)$ and $\mathcal{C}[\cdot]$ as the collision integral operator. Then we have from Eq. (7),

$$
\frac{g}{T_0 \sqrt{T_0}} \left[ \langle u_z \psi_i \rangle \partial_z \rho + \langle u_z \psi_i \varphi_j \rangle (A_j \partial_z \rho + \rho \partial_z A_j) \right] + \frac{g \rho}{T_0 \sqrt{T_0}} \left[ \langle \partial \psi_i / \partial u_z \rangle + A_j \langle \partial \psi_i / \partial u_z \varphi_j \rangle \right]
= \frac{\rho^2}{\sqrt{T_0}} \mathcal{C}[\Delta \psi_i] + A_j \mathcal{C}[\Delta \psi_i \varphi_j_1] + A_k \mathcal{C}[\Delta \psi_i \varphi_k_2]
+ A_j A_k \mathcal{C}[\Delta \psi_i \varphi_j_1 \varphi_k_2]. \tag{9}
$$

Further, dividing the above equation by $g/T_0 \sqrt{T_0}$ and defining $\rho^* \equiv \ln \rho$, we obtain the following equation for the unknown variables.

$$
S_i^0 \partial_z \rho^* + S_{ij} (A_j \partial_z \rho^* + \partial_z A_j) + C_i^0 + G_{ij} A_j = \frac{\rho \sigma T_0}{g} \left( C_{i}^{0e} + C_{i}^{0i} + C_{ij}^{1e} A_j + C_{ijk}^{2e} A_j A_k \right) \tag{10}
$$

where,

$$
S_i^0 = \langle u_z \psi_i \rangle \quad S_{ij} = \langle u_z \psi_i \varphi_j \rangle 
G_i^0 = \langle \partial \psi_i / \partial u_z \rangle 
G_{ij} = \langle \partial \psi_i / \partial u_z \varphi_j \rangle.
$$

Here, the superscript $c$ for the collision terms above indicate that the collisions are considered to be elastic.

A leading order equation is obtained by considering the $A_j \varphi_j$ as perturbations to the Maxwell distribution and by considering elastic collision in the collision integral. Thereby we obtain by setting $A_j = 0$,

$$
S_i^0 \partial_z \rho^* + G_i^0 = \frac{\rho \sigma T_0}{g} C_{i}^{0e}. \tag{11}
$$

With, $\psi_i = u_z$, the leading order density variation is given by $\partial \rho^* = -1 \partial z + c \rho$, where $\rho^* = N g/T_0$ is the density at $z = 0$. Kumaran [6] had obtained the values of $\rho^*$ and $T_0$ using a balance of the leading order source and dissipation, for low densities are given in Eqs. (5) and (6). A high density correction to these values was obtained in [6] in the leading order. In the present analysis we restrict ourselves to the low density limit. To obtain a first order balance in this limit, we neglect the quadratic term $A_j A_k$, and subtract out the leading order equation for low densities.

$$
S_{ij} \partial_z A_j = \left[ S_{ij} - G_{ij} + \frac{\epsilon^z}{\epsilon} C_{ij}^{1e} \right] A_j + \frac{\epsilon^z}{\epsilon} C_{ij}^{0i}. \tag{12}
$$

Here we have used $1/\epsilon = \rho^* \sigma T_0 / g = N \sigma$. Eqs. (12) is a set of coupled linear non-autonomous first order ordinary differential equations in the variables $A_j$. If we incorporate the perturbation to density, Eq. (8), in Eq. (9), then the above equation for the first order quantities reads:

$$
S_{ij} \partial_z A_j + S_i^0 \partial_z \rho_1 = \left[ S_{ij} - G_{ij} + \frac{\epsilon^z}{\epsilon} C_{ij}^{1e} \right] A_j 
+ \left[ S_i^0 - G_i^0 + \frac{2 \epsilon^z}{\epsilon} C_{i}^{0e} \right] \rho_1 + \frac{\epsilon^z}{\epsilon} C_{i}^{0i} \tag{13}
$$

A. **Boundary conditions**

The boundary conditions for the above equations may be obtained by the following method. A balance of the value of a moment $\phi_i$ is considered when it collides with a wall moving with a velocity $U$. The change in the value of a moment due
to the collision is given by the relation:
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} du_x^* du_x f(u) \phi_i(u) = \int_{-\infty}^{\infty} du_x^* \int_{-\infty}^{\infty} du_x f(u) \phi_i(u)
\]
\[
+ \int_{-\infty}^{\infty} du_x^* \int_{-\infty}^{\infty} du_x f(u) \left[ \phi_i(u') - \phi_i(u) \right]
\]
(14)
where the primed variable denotes the velocity of the particle after a collision. Simplifying the above equation we get,
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} du_x^* du_x f(u) \phi_i(u) = \int_{-\infty}^{\infty} du_x^* \int_{-\infty}^{\infty} du_x f(u) \phi_i(u)
\]
(15)

If the mean free path of the particles is large compared to the amplitude of vibration, then we can make an assumption that the wall is stationary at one position, then the above equation can be further simplified by averaging over different probable velocities of the bottom wall, which, in the case of a sine wave oscillation is,
\[
P(U) dU = \frac{1}{\pi (U_0^2 - U^2)}
\]
(16)
if we write \( U = U_0 \sin \theta \), and \( \epsilon \equiv U_0^2 / T_0 \), then the averaged equation in terms of the nondimensional quantities for the balance of \( \phi_i \), after making the substitution for the distribution function at \( z = 0 \), will be,
\[
A_j \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\epsilon} \sin \theta \int_{-\infty}^{\infty} du_x \int_{-\infty}^{\infty} du_x f^0 \phi'_i \varphi_j - A_j \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\epsilon} \sin \theta \int_{-\infty}^{\infty} du_x \int_{-\infty}^{\infty} du_x f^0 \phi_i \varphi_j
\]
\[
= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\epsilon} \sin \theta \int_{-\infty}^{\infty} du_x \int_{-\infty}^{\infty} du_x f^0 \phi_i - \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\epsilon} \sin \theta \int_{-\infty}^{\infty} du_x \int_{-\infty}^{\infty} du_x f^0 \phi'_i
\]
(17)

or simply as,
\[
E_{ij} A_j(0) = B_i
\]
(18)

Solving these simultaneous equations we obtain conditions satisfied by \( A_j \) at \( z = 0 \).

### III. THIRD-ORDER MOMENT FORMULATION

We now take the specific case of the third order formulation. A third order formulation is the lowest order in which we can include the anisotropy in the distribution function. We now briefly explain the choice of the moments \( \varphi_j \). The distribution function satisfies the following criteria. (i) the time averaged vertical flux is zero, \( \langle u_z \rangle = 0 \), (ii) the distribution function is normalised to unity, (iii) it is symmetric in the horizontal velocities. It is convenient to choose the lower order moments from a multi-dimensional Hermite polynomials for the following reasons. These polynomials form a linearly independent orthogonal basis, and the resulting equations are more convenient than a linearly independent set. In addition two of the above conditions will be automatically satisfied by the distribution function by setting the corresponding the corresponding \( A_j \) to zero. Note that since the leading order distribution function \( f^0 \) already satisfies the unit normalisation condition, we would require that \( A_j \langle \varphi_j \rangle = 0 \). The flux condition requires that \( A_j \langle u_z \varphi_j \rangle = 0 \). Since 1 and \( u_z \) are two of the linearly independent functions in the orthogonal Hermite polynomials, setting the coefficients of these two will ensure that the above two conditions will be satisfied at all orders of the polynomials. Thus the choice of the \( \varphi_j \) becomes simple by incorporating the constraints on the distribution function directly.

The set of multi-dimensional orthogonal polynomials can be obtained from the recurrence relation for the Hermite polynomials,
\[
H^{n+1}(x) = x H^n(x) - n H^{n-1}(x)
\]
(19)
with \( H^0(x) = 1 \), \( H^1(x) = x \)

and the multidimensional set is then given by,
\[
H(u_x, u_z) = H^m(u_x) H^n(u_z) \quad \text{even } m, \text{ all } n.
\]
(20)

A symmetric distribution in the horizontal velocity can be ensured by taking only even powers of \( u_x \), i.e., even \( m \) in the above expression. The above polynomials are can be normalised to unity and the factor is \( 1/(m!n!) \).

In the case of the third order approximation we obtain the
following polynomials,
\[
\varphi_j = \{ -1 + u_2^2, -3u_z + u_3^2 - 1 + u_2^2, -u_z + u_2^2 u_z \},
\]
(21)
in which the first two members viz., 1 and \(u_z\) have been omitted for reasons discussed above. We choose the same set for the moment generating functions \(\psi_i\) in Eq. (3) and the functions for the boundary conditions, \(\phi_i\) in Eq. (18). This way we can get the same order of representation in the moment equations as well as in the boundary conditions.

The moments of the distribution can be obtained by substituting the expression (21) in Eq. (3), which are:
\[
\langle u_i^2 \rangle = (1 + 2A_3), \quad \langle u_z^3 \rangle = (1 + 2A_1), \quad \langle u_z^2 u_z \rangle = 2A_4, \quad \langle u_z^3 \rangle = 6A_2.
\]
(22)

IV. SOLUTION

With the moments considered in Eq. (21), the set of relations in Eq. (12) for \(A_i\) can be written as
\[
\partial_z A_i = L_{i,j}^0 A_j + L_{i,j}^1 A_j e^{-z} + b_i^1 e^{-z},
\]
(23)
where,
\[
L_{i,j}^0 = S_{ik}^{-1} (S_{kj} - G_{kj}), \quad L_{i,j}^1 = \frac{1}{\epsilon_t} S_{ik}^{-1} C_{kj}^{1e}, \quad b_i^1 = \frac{1}{\epsilon_t} S_{ik}^{-1} C_{ki}^{0i},
\]
and by considering a moment \(\psi_i = u_z\) in Eq. (13) we have for the density correction,
\[
\partial_z \rho_1 + 2 \partial_z A_1 = 2A_1,
\]
(24)
In the case of the third order approximation, we have
\[
L_{i,j}^0 = \{ \{ 0, 0, 0, 0 \}, \{ 0, 1, 0, 0 \}, \{ 0, 0, 0, 0 \}, \{ 0, 0, 0, 1 \} \}
\]
\[
L_{i,j}^1 = \frac{\pi}{\epsilon_t} \{ \{ 0, -\frac{3}{2}, 0, \frac{1}{2} \}, \{ -\frac{3}{2}, 0, \frac{1}{2}, 0 \}, \{ 0, \frac{3}{2}, 0, -\frac{1}{2} \}, \{ 0, 0, 1, 0 \} \}
\]
\[
b_i^1 = -\frac{(1-e^{-z})\sqrt{\pi}}{6\epsilon_t} \{ 0, 1, 0, 3 \}
\]
We note here that the \(A_i\) are independent of the density correction \(\rho_1\) in the first order approximation These equations can be rearranged into a single fourth order equation in \(A_1\), which is easily accomplished through a symbolic routine:
\[
A_1^{(4)}(z) = -4A_1^{(3)}(z) + \left( \frac{4\pi}{\epsilon_t} e^{-2z} - 4 \right) A_1''(z)
- \frac{\pi^2}{\epsilon_t^2} (1 - e^{-2}) e^{-4z}
\]
(25)
With this simplification, the other variables can be written down in terms of \(A_1\) and its derivatives as
\[
A_2(z) = \frac{(-1 + e^{-z})}{24} \left( \frac{\varphi_i}{\sqrt{\pi}} e^{-z} \right)^{-1}
+ \left( -A_1'(z) + \frac{A_1''(z)}{6} \right) \left( \frac{\varphi_i}{\sqrt{\pi}} e^{-z} \right)^3
+ \left( \frac{A_1''(z)}{24} - \frac{A_1^{(4)}(z)}{4} \right) \left( \frac{\varphi_i}{\sqrt{\pi}} e^{-z} \right)^3
\]
(26a)
\[
A_3(z) = A_1(z) - A_1''(z) \left( \frac{\varphi_i}{\sqrt{\pi}} e^{-z} \right)^2
\]
(26b)
\[
A_4(z) = \frac{(-1 + e^{-z})}{8} \left( \frac{\varphi_i}{\sqrt{\pi}} e^{-z} \right)^{-1}
+ \left( -A_1'(z) + \frac{A_1''(z)}{2} \right) \left( \frac{\varphi_i}{\sqrt{\pi}} e^{-z} \right)^3
+ \frac{1}{2} \left( \frac{A_1''(z)}{4} - \frac{A_1^{(4)}(z)}{4} \right) \left( \frac{\varphi_i}{\sqrt{\pi}} e^{-z} \right)^3
\]
(26c)
To solve Eq. (25), we can make a reduction in order by the following substitution
\[
A''(z) = y(z)
\]
(27)
With the transformation \(x = e^{-z}\), we obtain from Eq. (25). (Note: here \(x\) is just a transformation variable and not the cartesian co-ordinate)
\[
\dot{y} - 3\frac{\dot{x}}{x} - \frac{1}{x^2} (c x^2 - 4) y = D x^2
\]
(28)
where a dot accent denotes \(\partial_x\), and the constants are \(c \equiv 4\pi/\epsilon_t^2, D \equiv -\pi(1 - e^{-2})/\epsilon_t^2\). Further with
\[
w(x) = y(x)/x^2,
\]
(29)
we get,
\[
\ddot{w} + \frac{\dot{w}}{x} - c w = D,
\]
(30)
which is a modified Bessel equation of zeroth order. The solution to the homogeneous equation is given by:
\[
w_h(x) = c_1 I_0(\sqrt{c} x) + c_2 K_0(\sqrt{c} x)
\]
(31)
A particular solution to the inhomogeneous equation can be obtained by variation of parameters. The Wronskian of the above solutions can be written down from standard references,
\[
\begin{vmatrix}
I_0(\sqrt{c} x) & K_0(\sqrt{c} x) \\
I_0(\sqrt{c} x) & K_0(\sqrt{c} x)
\end{vmatrix} = -\frac{1}{x}
\]
(32)
The particular solution so obtained is given by:
\[
w_p(x) = -\frac{D x}{\sqrt{c}} \left[ I_0(\sqrt{c} x) K_1(\sqrt{c} x) + K_0(\sqrt{c} x) I_1(\sqrt{c} x) \right].
\]
(33)
The most general solution to the inhomogeneous equation is then
\[ w(x) = w_h + w_p(x). \] (34)

The Eq. (27) can now be solved in terms of the independent variable, by further reduction in order, by writing it,
\[ x (x \tilde{A}_1(x) + \dot{A}_1(x)) = y(x), \] (35)
as:
\[ \dot{B}(x) = w(x) - \frac{B(x)}{x} \] (36)
where,
\[ \dot{A}_1(x) = B(x). \] (37)
The solutions to these first order equations can be easily written as
\[ B(x) = \frac{1}{x} \left[ \int_0^x dx' w(x') x' + c_3 \right] \] (38)
and
\[ A_1(x) = \int_0^x x'' \left[ \int_0^{x'} dx'' w(x'') x'' + c_3 \right] + c_4. \] (39)
The correction to the density is then from Eq. (24),
\[ \rho_1(x) = -2 \int_0^x dx' \left( -\frac{A_1(x')}{x'} + B(x') \right) + c_5. \] (40)
The derivatives of \( A_1(z) \) can be written down as
\[ A_1'(z) = -x \tilde{A}_1(x) = -x B(x) \] (41a)
\[ A_1''(z) = -x^2 w(x) \] (41b)
\[ A_1'''(z) = -2x^2 w(x) - x^3 \dot{w}(x) \] (41c)
These are used to evaluate the other unknown functions in Eqs. (26).

A. Boundary conditions

Some of the constants of integration can be directly eliminated by requiring that the functions to take finite values for \( x = 0 \) (or equivalently, \( z \to \infty \)). Consider one of the linearly independent solutions of \( w_h(x) \) in Eq. (33), \( c_2 K_0(\sqrt{c} x) \), which when integrated through Eqs. (38) and (39) gives a term in \( A_1(x) \): \( c_2 K_0(\sqrt{c} x) / c \). Since \( K_0(\sqrt{c} x) \) is singular as \( x \to 0 \), we require \( c_2 = 0 \). Similarly we require that \( c_3 = 0 \), as this will lead to a singular term in \( A_1(x) \): \( c_3 \ln x \).

The expressions in Eq. (33) cannot be integrated to obtain a closed form expression. Nevertheless, they can be easily evaluated numerically by converting the expressions Eq. (38) and (39) to a definite integral, \( \int_0^x dx' \) added with some constant. The choice of the lower limit, \( x_0 \), can, in general, be arbitrary to suit the convenience of matching the given boundary conditions; but such a choice, say \( \int_0^{x_0} dx' \), for the integral in Eq. (38) will lead to singularities in the expression for \( A_1(x) \) in Eq. (39), similar to those obtained by the constant \( c_3 \) term. To do away with this singularity, we choose \( x_0 = 0 \), expand the integral in Eq. (38) in Taylor’s series, and subtract out the source of the singularity (which essentially is the constant of integration, i.e., the value of the integral evaluated at the lower limit). In expanding \( w(x) \) about \( x = 0 \), we note from Eq. (34) that \( w(0) = 0 \), therefore,
\[ B(x) = \frac{1}{x} \int_0^x dx' w(x') x' = \frac{1}{x} \int_0^x dx' \left( w_1 x' + w_2 x'^2 + \ldots \right) x' = \left( \frac{w_1}{2} x^2 + \frac{w_2}{3} x^3 + \ldots \right), \]
where \( w_1, w_2, \ldots \) are constants coming from the Taylor’s expansion. Such an expansion is possible as the series converges for \( 0 \leq x < 1 \). The above expression identically vanishes at \( x = 0 \), therefore we do not have to explicitly subtract out any singularity during numerical computation of the definite integral. Such problems of subtracting out the singularity, however, does not arise in the integral of Eq. (39) and any arbitrary point, \( x_0 \), can simply be chosen for the numerical integration, keeping in mind the boundary conditions.

We are now left with two arbitrary constants, \( c_1 \) and \( c_4 \). These are evaluated using boundary conditions in a manner similar to those used in \( q \). For the sake of convenience we choose the lower limit for the definite integral in Eq. (39) to be \( x_0 = 1 \), then \( c_4 \) will simply be equal to the value of \( A_1(x = 1) \). It can be seen from Eq. (22b) that \( A_1 \) is directly proportional to the moment of the distribution function \( \langle u_z^2 \rangle \), whose value at \( z = 0 \) can be obtained by considering it as a vertical flux of momentum because of collisions with the wall. Assuming that the particles collide with the wall having a leading distribution to be a Maxwellian, the flux of momentum along the vertical direction is given by,
\[ \langle u_z^2 \rangle|_{z=0} = 1 + \frac{\epsilon}{2}. \] (42)
substituting in Eq. (22b) we have, \( A_1(z=1) = \epsilon / 4 \), therefore,
\[ c_4 = \frac{\epsilon}{4}. \] (43)
Since, momentum is transferred only in the vertical direction we have from Eqs. (22), \( A_2|_{z=1} = 0 \) and \( A_4|_{z=1} = 0 \). The constant, \( c_1 \), can now be evaluated from Eqs. (27), (29) and (34).
\[ c_1 = \frac{1}{I_0(\sqrt{c})} \left( A_1''(z) - w_p(x) \right)|_{z=0 \text{ or } x=1}, \] (44)
The constant \( A_1''(z)|_{z=0} \) can be solved for from Eqs. (24)
\[ A_1''(z)|_{z=0} = \frac{\pi}{\epsilon^2} \left( A_1 + A_3 \right)|_{z=0} = \frac{\pi \epsilon}{4 \epsilon f} \] (45)
The other constant $c_5$, is obtained by considering mass balance for the density correction $\rho_1$. Since the total mass of the bed is balanced in the leading order density profile $\rho_0$, the balance for the correction to the density is given by

$$
\int_0^\infty dz' \rho_0 \rho_1 = 0,
$$

or,

$$
\int_0^1 dx \rho_1 = 0.
$$

We note here that we have not strictly incorporated the boundary conditions as discussed in Sec. II. This is because of a need to satisfy more strong boundary conditions of non-diverging solutions for $z \to \infty$. The equations of the sort of Eq. (13) would be useful in obtaining, for example, a series solution by numerical methods where we expand the solution in decaying functions such as Laguerre polynomials.

Furthermore, the moments related to the horizontal direction are not exactly satisfied, i.e., the boundary conditions related to $A_3$ and $A_4$. While the value of $A_3$ is not satisfied independently, but only partly in Eq. (44). $A_4$ is never used. These can be used only in a higher order polynomial approximation. Strictly therefore, only $A_1$ is satisfied exactly.

V. DISCUSSION

The results obtained here are qualitatively the same as obtained in \[4\]. At the time of the previous work, the only results available were some experimental measurements of \[7\] and only a qualitative comparison was possible to ascertain the validity of the theory. We have now compared the results of with a numerical Event Driven (ED) simulation of vibrated hard disks. The ED simulation was done with periodic boundaries and with an approximate representation of the bottom wall \[8\]. The following figures show that the theory is indeed good in the limit of its validity. The figures also show a comparison with the approximate series solution which was obtained in \[5\]. In most cases the series solution provides a good approximate.

The parameters chosen for the simulation are in correspondence with the limit of validity of the theory: $\epsilon \ll 1$, and $N\sigma \sim O(1)$, (see \[3, 8\], for a discussion). There is a small negative correction to the density near the bottom wall, as shown in Fig. 1, due to the energy flow near the bottom wall, after which it falls of exponentially as the leading order density.

One important difference, in the formulation of the density, between the present and the previous work \[8\] is the following. In the previous work a correction to the distribution function due to variations in the distribution function over distances comparable to the particle diameter were described by using a small parameter, $\epsilon c_2$. It was shown in a later work \[8\] that this correction is essentially equivalent to the high density correction obtained on the lines of Enskog correction to dense gases, therefore it has been omitted in the present consideration of dilute bed.

Due to the anisotropic nature of the energy input to the vibro-fluidised bed, the lower order moments of the distribution clearly show the anisotropy. Figs. 2 and 3 show the horizontal and vertical temperature profiles, while Fig. 4 shows the vertical flow of energy. Anisotropies were also observed by us in deep bed simulations of disks which had the wave-like surface patterns \[4\]; although the nature of anisotropy was more pronounced even in the shape of the distribution function itself (the vertical distribution had double peaks and the horizontal distribution had single peak and exponential tails). Could the presence of anisotropy be an important feature giving rise to an instability in one direction? A stability analysis of the solution from the present analysis model might help resolve this. The usual models based on hydrodynamic equations do not take into account this anisotropy.

As pointed out earlier, the density correction considered here is different from the one used in the previous work, in that the high density correction is not considered in first order in the distribution function. This results in a better prediction of the vertical temperature Fig. 5 and flux of energy Fig. 6 even when the density prediction is not expected to be good.

To conclude, we have shown that (a) It is possible to obtain an analytical solution to the Boltzmann equation for vibro fluidised bed in the low density limit correct up to the third order in the moments of the distribution function. (b) The qualitative nature of results obtained here are similar to those obtained in \[4\]. (c) The correction to the distribution function...
due to spatial variation of the order of a particle diameter was neglected here as this is turns out to be a correction to a higher order in density \cite{8}. With this it is seen that the density still shows a negative correction at the bottom wall due to the energy flow. (d) The boundary conditions are overspecified in the problem and we chose to satisfy exactly, only the ones involved in the momentum transfer the vertical direction. (e) Even with this restricted choice, the theoretical values for the different moments compare reasonably well with the simulation particularly in the anisotropy exhibited. (f) This gives us some confidence to explore further with the stability of the solution, and with an higher order approximation to the distribution function if required.

FIG. 2: Horizontal temperature profile for $N \sigma = 3$, $\epsilon = 0.3$. A magnitude comparison with Fig. 3 shows the degree of anisotropy in a vibrated bed.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{Horizontal temperature profile for $N \sigma = 3$, $\epsilon = 0.3$. A magnitude comparison with Fig. 3 shows the degree of anisotropy in a vibrated bed.}
\end{figure}

\[\langle u^2 \rangle \]\n
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FIG. 3: Vertical temperature profile for $N\sigma = 3$, $\epsilon = 0.3$. The theoretical values are closer to the simulation here, than for the horizontal temperature because of the boundary conditions imposed.

FIG. 4: Flux of energy in the vertical direction for $N\sigma = 3$, $\epsilon = 0.3$. Here again, the theoretical values compare well in order of magnitude, because of boundary conditions are imposed in this direction.
FIG. 5: Vertical temperature profile prediction is good even when the density is high ($\nu \sim 0.3i$).

FIG. 6: Flux of energy prediction is good even when the density is high ($\nu \sim 0.3$).