Single-Server Private Linear Transformation: The Joint Privacy Case

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Abstract—This paper introduces the problem of Private Linear Transformation (PLT) which generalizes the problems of private information retrieval and private linear computation. The PLT problem includes one or more remote server(s) storing (identical copies of) $K$ messages and a user who wants to compute $L$ independent linear combinations of a $D$-subset of messages. The objective of the user is to perform the computation by downloading minimum possible amount of information from the server(s), while protecting the identities of the $D$ messages required for the computation. In this work, we focus on the single-server setting of the PLT problem when the identities of the $D$ messages required for the computation must be protected jointly. We consider two different models, depending on whether the coefficient matrix of the required $L$ linear combinations generates a Maximum Distance Separable (MDS) code. We prove that the capacity for both models is given by $L/(K-D+L)$, where the capacity is defined as the supremum of all achievable download rates. Our converse proofs are based on linear-algebraic and information-theoretic arguments that establish connections between PLT schemes and linear codes. We also present an achievability scheme for each of the models being considered.

Index Terms—Private Information Retrieval, Private Function Computation, Information-Theoretic Privacy, Single Server, Linear Transformation, Maximum Distance Separable Codes.

I. INTRODUCTION

In this work, we introduce the problem of Private Linear Transformation (PLT). This problem includes one or more remote server(s) storing (identical copies of) a dataset consisting of $K$ messages; and a user who is interested in computing $L$ independent linear combinations of a $D$-subset of messages. The objective of the user is to recover the $L$ required linear combinations by downloading minimum possible amount of information from the server(s), while the identities of the $D$ messages required for the computation are not revealed to the server(s). The PLT problem generalizes the problems of Private Information Retrieval (PIR) [1]–[8] and Private Linear Computation (PLC) [9], [10], which have recently received a significant attention from the research community. In particular, PLT reduces to the PIR problem or the PLC problem when $L = D$ or $L = 1$, respectively. This is because in PIR, the problem is to privately retrieve a $D$-subset of messages, which is equivalent to privately computing $D$ independent linear combinations of the $D$ desired messages; and in PLC, the problem is to privately compute one linear combination of a $D$-subset of messages.

The PLT problem is motivated by the need to protect the data access patterns in several Machine Learning (ML) applications such as linear transformation for dimensionality reduction, see, e.g., [11], and training different linear regression or classification models in parallel, see, e.g., [12], [13]. For instance, consider a dataset with $N$ data samples each with $K$ attributes, represented by a $K \times N$ data matrix. Suppose there is a user who wishes to implement an ML algorithm on a subset of $D$ selected attributes, while protecting the privacy of the selected attributes. When $D$ is large, the $D$-dimensional feature space is typically mapped onto a new subspace of lower dimension, say, $L$, and the ML algorithm operates on the new $L$-dimensional subspace instead. A commonly-used technique for dimensionality reduction is linear transformation, where an $L \times D$ matrix is multiplied by the $D \times N$ submatrix of the $K \times N$ data matrix restricted to the $D$ selected attributes. This scenario matches the setup of the PLT problem, in which each message represents the $N$ data samples for one attribute, the labels of the selected attributes correspond to the identities of the messages required for the computation, and the transformation matrix is formed by the coefficient matrix of the required linear combinations.

In many practical scenarios, the dataset is stored on a single server, or multiple servers that belong to the same provider and can collude arbitrarily. Motivated by such scenarios, in this work we focus on the single-server setting of the PLT problem. A simple approach for PLT is to privately retrieve the messages required for the computation using a single-server PIR scheme, and then compute the required linear combinations locally. As shown in [14]–[21], leveraging a prior side information about the dataset, in the single-server setting, the user can retrieve a single or multiple messages privately with a much lower download cost than the trivial scheme of downloading the entire dataset. (The advantages of side information in multi-server PIR were also studied in [22]–[29].) However, when there is no side information, a PIR-based approach is extremely expensive as the entire dataset must be downloaded in order to achieve information-theoretic privacy [1]. Another approach for PLT is to privately compute the required linear combinations separately via applying a single-server PLC scheme multiple times. (The multi-server PLC problem and its extensions were studied in [9], [10], [30]–[33].) In [34], [35], it was shown that PLC can be performed more efficiently than PIR in terms of the download cost, regardless of whether the user has any side information or not. However, a PLC-based approach may still lead to an unnecessary overhead due to the redundancy in the information being downloaded. This implies the need for novel PLT schemes with optimal download rate.
Different types of privacy can be considered for PLT. In this work, we focus on the PLT problem under a strong notion of privacy, called joint privacy, which was also considered previously for PIR and PLC (see, e.g., [4], [18], [35], [36]). We refer to this problem as JPLT with Joint Privacy, or JPLT for short. The joint privacy requirement implies that the identities of all $D$ messages required for the computation must be kept private jointly. This type of privacy is of practical importance in the scenarios in which the correlation between the identities of messages required for the computation need to be kept private. For instance, the user may want to compute a linear combination of two vectors, and the server must not learn which pair of vectors were required for the computation.

In a parallel work [37], we have considered a relaxed version of joint privacy, called individual privacy, which was recently introduced for PIR and PLC (see, e.g., [16], [35]). The individual privacy condition ensures that the identity of every individual message required for the computation is kept private. In contrast to joint privacy, individual privacy finds application in the scenarios in which the correlation between the identities of the required messages does not need to be protected. For example, the dataset may contain information about individuals, and the user is required to hide information from the server on whether the data belonging to an individual was used in the computation.

Unlike the privacy requirements for the multi-server PLC problem in [9], [10] and the multi-server Private Monomial Computation problem in [38], joint and individual privacy are to protect the data access patterns, and not the values of the coefficients (or the exponents) in the required linear combination (or the required monomial function). These types of access privacy are inspired by several real-world scenarios. For example, protecting the identities of the selected attributes in the application of linear transformation for dimensionality reduction may prevent the server from learning the user’s data access patterns which, in turn, can be instrumental for hiding user’s algorithms, preferences, and objectives from the server.

A. Main Contributions

We consider two different models, referred to as Model I and Model II, for the JPLT problem. In Model I, it is assumed that the coefficient matrix of the required linear combinations is maximum distance separable (MDS), whereas in Model II, it is assumed that the coefficient matrix has full rank (but it may or may not be MDS). Model I is motivated by the scenarios in which the combination coefficients are chosen purposely to form an MDS matrix, or the coefficient matrix is randomly generated over the field of real numbers or a finite field of large size, e.g., when applying random linear transformation for dimensionality reduction [41]. Model II, on the other hand, finds application in the scenarios in which the size of the operating field is relatively small, e.g., due to the computational complexity considerations, and the number of rows ($L$) and the number of columns ($D$) of the coefficient matrix are such that $\binom{D}{L}$ is large, e.g., when a large reduction factor is required in dimensionality reduction. We refer to the JPLT problem under Model I or Model II as the JPLT-I or JPLT-II problem, respectively.

In this work, we characterize the capacity of the JPLT-I and JPLT-II problems, where the capacity of JPLT-I (or JPLT-II) is defined as the supremum of download rates over all JPLT-I (or JPLT-II) schemes. In particular, we prove that the capacity of both problems is given by $L/(K - D + L)$. This result is particularly interesting because it shows that JPLT can be performed more efficiently than applying a PIR-based or a PLC-based approach for privately computing multiple linear combinations simultaneously. For each problem, we prove the converse by using a mix of linear-algebraic and information-theoretic arguments. Our technique for proving the converse for the JPLT-II problem is more general and is applicable to the JPLT-I problem. However, this technique is based on proof-by-contradiction. On the other hand, our proof technique for the JPLT-I problem is a constructive proof which also gives insight into the design of an achievability scheme. For the JPLT-I problem, we propose an achievability scheme, termed the Specialized MDS Code protocol, which is based on the idea of extending an MDS code [3]. For the JPLT-II problem, we propose a different achievability scheme, termed the Specialized Augmented Code protocol. This scheme is based on augmenting a non-MDS code by an MDS code.

B. Notation

We denote random variables and their realizations by boldface and regular symbols, respectively. We denote sets, vectors, and matrices by roman font, and denote collections of sets, vectors, or matrices by blackboard bold roman font. For any random variables $X, Y$, we denote by $H(X)$ and $H(X|Y)$ the entropy of $X$ and the conditional entropy of $X$ given $Y$, respectively. For any integer $n \geq 1$, we denote $\{1, \ldots, n\}$ by $[n]$, and for any integers $1 < n < m$, we denote $\{n, n+1, \ldots, m\}$ by $[n : m]$. We denote the binomial coefficient $\binom{\cdot}{\cdot}$ by $\binom{\cdot}{\cdot}$.

II. Problem Setup

A. Models and Assumptions

Let $q$ be an arbitrary prime power, and let $N \geq 1$ be an arbitrary integer. Let $\mathbb{F}_q$ be a finite field of order $q$, and let $\mathbb{F}_q^N$ be the $N$-dimensional vector space over $\mathbb{F}_q$. Let $B = N \log_2 q$. Let $K, D, L \geq 1$ be integers such that $L \leq D \leq K$. We denote by $\mathcal{B}$ the set of all $D$-subsets (i.e., all subsets of size $D$) of $[K]$. Also, we denote by $\mathcal{B}_i$ the set of all $L \times D$ matrices $V$ with entries in $\mathbb{F}_q$ that are MDS, i.e., every $L \times L$ submatrix of $V$ is invertible, and denote by $\mathcal{B}_i$ the set of all $L \times D$ matrices $V$ with entries in $\mathbb{F}_q$ that have full rank, i.e., $\text{rank}(V) = L$.

Consider a server that stores $K$ messages $X_1, \ldots, X_K$, where $X_i \in \mathbb{F}_q^N$ for $i \in [K]$ is a row-vector of length $N$. Let

1A $k \times n$ matrix is said to be MDS if it generates an $[n, k]$ MDS code.
2A direct application of Schwartz-Zippel lemma [39], [40] shows that a matrix whose entries are randomly chosen from a sufficiently large field is MDS with high probability.
3Extending a code is performed by adding new columns to the generator matrix of the code.
4Augmenting a code is performed by adding new rows to the generator matrix of the code.
X \triangleq [X_1^T, \ldots, X_K^T]^T. \text{ Note that } X \text{ is a matrix of size } K \times N. \text{ For every } S \subset [K], \text{ we denote } X_S \text{ the submatrix of } X \text{ restricted to its rows indexed by } S, \text{ i.e., } X_S = [X_{i_1}^T, \ldots, X_{i_S}^T]^T, \text{ where } S = \{i_1, \ldots, i_S\}. \text{ Note that } X_S \text{ is a matrix of size } |S| \times N, \text{ where } |S| \text{ denotes the size of } S. \text{ Consider a user who wishes to compute } L \text{ linear combinations of } D \text{ messages, namely, } v_1X_W, \ldots, v_LX_W, \text{ where } W \in \mathbb{W} \text{ is the index set of the } D \text{ messages required for the computation, and } v_l \text{ for each } l \in [L] \text{ is a row-vector of length } D \text{ with entries in } \mathbb{F}_q, \text{ denoting the coefficient vector of the } l\text{th required linear combination.} \text{ We represent the collection of the required linear combinations in the matrix form as } Z^{[W,V]} \triangleq VX_W = UX, \text{ where } V = [v_1^T, \ldots, v_L^T]^T \text{ is an } L \times D \text{ matrix with entries in } \mathbb{F}_q, \text{ denoting the coefficient matrix pertaining to the required linear combinations, and } U \text{ is an } L \times K \text{ matrix such that the submatrix of } U \text{ restricted to the columns indexed by } W \text{ is equal to } V, \text{ and the rest of the columns of } U \text{ are all-zero.} \text{ Note that } Z^{[W,V]} \text{ is a matrix of size } L \times N \text{ with entries in } \mathbb{F}_q. \text{ We refer to } Z^{[W,V]} \text{ as the demand, } W \text{ as the support of the demand, } V \text{ as the coefficient matrix of the demand, } U \text{ as the global coefficient matrix of the demand, } D \text{ as the support size of the demand, and } L \text{ as the dimension of the demand.} \text{ In this work, we consider two different models:} 

- Model I: \text{ } v_lX_W's \text{ are } L \text{ MDS-coded linear combinations of the } D \text{ messages indexed by } W, \text{ i.e., } V \in \mathbb{V}_I. 

- Model II: \text{ } v_lX_W's \text{ are } L \text{ linearly independent (but not necessarily MDS-coded) linear combinations of the } D \text{ messages indexed by } W, \text{ i.e., } V \in \mathbb{V}_II. 

Throughout, we make the following assumptions: 

1) X_1, \ldots, X_K \text{ are independently and uniformly distributed over } \mathbb{F}_q^N. \text{ Thus, } H(X) = KB, \text{ and } H(X_3) = |S|B \text{ for every } S \subset [K]. \text{ Moreover, } H(Z^{[W,V]}) = LB \text{ for Model I and Model II.} 

2) W, V, X \text{ are independent random variables.} 

3) W \text{ is distributed uniformly over } \mathbb{W}. 

4) V \text{ is distributed uniformly over } \mathbb{W}_I \text{ or } \mathbb{W}_II \text{ for Model I or Model II, respectively.} 

5) The demand's support size } D \text{ and dimension } L, \text{ and the distributions of } W \text{ and } V \text{ are initially known by the server, whereas the realizations } W \text{ and } V \text{ are initially unknown to the server.} 

B. Privacy and Recoverability Conditions 

Given } W \text{ and } V, \text{ the user generates a query } Q^{[W,V]}, \text{ simply denoted by } Q, \text{ and sends it to the server. For simplicity, we denote } Q^{[W,V]} \text{ by } Q. \text{ The query } Q \text{ is a deterministic or stochastic function of } W, V. \text{ In the case of a deterministic query, } H(Q^{[W,V]}) = 0, \text{ and in the case of a stochastic query, } H(Q^{[W,V], V, R}) = 0, \text{ where } R \text{ is a random key generated by the user (independently from } W, V, X), \text{ and unknown to the server.} 

Given the query } Q, \text{ every } D\text{-subset of message indices must be equally likely to be the demand’s support } W, \text{ i.e., for every } W \in \mathbb{W}, \text{ it must hold that } 

\Pr(W = \tilde{W}|Q = Q) = \Pr(W = \tilde{W}) = 1/C_{K,D}. 

\text{ We refer to this condition as the joint privacy condition.} 

\text{ Upon receiving the query } Q, \text{ the server generates an answer } A^{[W,V]}, \text{ simply denoted by } A, \text{ and sends it back to the user. For simplicity, we denote } A^{[W,V]} \text{ by } A. \text{ The answer } A \text{ is a deterministic function of } Q \text{ and } X. \text{ That is, } H(A|Q, X) = 0. \text{ The answer } A, \text{ the query } Q, \text{ and the realizations } W, V \text{ must collectively enable the user to retrieve the demand } Z^{[W,V]}, \text{ i.e.,} 

H(Z|A, Q, W, V) = 0, 

\text{ where } Z^{[W,V]} \text{ is denoted by } Z. \text{ We refer to this condition as the recoverability condition.} 

C. Problem Statement 

The problem is to design a protocol for generating a query } Q^{[W,V]} \text{ and the corresponding answer } A^{[W,V]} \text{ for any given } W \text{ and } V \text{ such that the joint privacy and recoverability conditions are satisfied. We refer to this problem as single-server Private Linear Transformation (PLT) with Joint Privacy, or JPLT for short. The JPLT problem under Model I (or Model II) is referred to as the JPLT-I (or JPLT-II) problem, and a protocol for JPLT-I (or JPLT-II) is referred to as a JPLT-I (or JPLT-II) protocol. A protocol is called linear if the server’s answer to the user’s query consists only of linear combinations of the messages; otherwise, the protocol is called non-linear.} 

\text{ We measure the efficiency of a JPLT-I or JPLT-II protocol by its rate—defined as the ratio of the entropy of the demand (i.e., } H(Z) = LB) \text{ to the entropy of the answer (i.e., } H(A)). \text{ We define the capacity of JPLT-I or JPLT-II as the supremum of rates over all JPLT-I or JPLT-II protocols, respectively. In this work, our goal is to characterize the capacity of these settings in terms of } K, D, L. \text{ Note that the capacity may also depend on the field size } q \text{ in general. Notwithstanding, in this work we are interested in characterizing the supremum of rates over all protocols and all } q. 

III. MAIN RESULTS 

In this section, we present our main results. Theorems \ref{thm:main} \text{ and } \ref{thm:main2} \text{ characterize the capacity of JPLT-I and JPLT-II, respectively. The proofs are given in Sections \ref{sec:main1} \text{ and } \ref{sec:main2} \text{ respectively.} 

\textbf{Theorem 1. For the JPLT-I setting with } K \text{ messages, demand’s support size } D, \text{ and demand’s dimension } L, \text{ the capacity is given by } L/(K - D + L).} 

\text{ The proof of converse is based on a mix of linear-algebraic and information-theoretic arguments. A key ingredient of the proof is the result of Lemma \ref{lem:main} \text{ which follows from the joint privacy and recoverability conditions for Model I. The converse bound naturally serves as an upper bound on the rate of any JPLT-I protocol. We prove the achievability by designing a linear JPLT-I protocol, termed the Specialized MDS Code protocol, that achieves the converse bound. This protocol generalizes those in [18] \text{ and } [34] \text{ for single-server PIR and PLC with joint privacy (when the user has no prior side information about the content of the messages available at the server), and is based on the idea of extending the MDS code generated by the coefficient matrix of the demand.} 

\text{ Our converse bounds hold for any } q, \text{ and our achievability schemes achieve these converse bounds when } q \text{ is sufficiently large, depending on } K, D, L.}
Theorem 2. For the JPLT-II setting with $K$ messages, demand’s support size $D$, and demand’s dimension $L$, the capacity is given by $L/(K-D+L)$.

We prove the converse for the JPLT-II problem by relying on the result of Lemma 4 which follows from the joint privacy and recoverability conditions for Model II. The proof is by the way of contradiction, and is also applicable to the JPLT-I problem. That said, for the JPLT-I problem we present a different converse proof based on construction, which also gives insight into the design of an achievability scheme. Note that our constructive proof technique does not extend to the JPLT-II problem. This is because the construction we propose in the proof relies on the fact that MDS matrices do not contain any all-zero columns. This condition, however, does not always hold for full (row-) rank matrices.

To prove the achievability result, we propose a linear JPLT-II protocol, termed the Specialized Augmented Code protocol, that achieves the converse bound. This protocol is based on the idea of augmenting the global coefficient matrix of the demand by an MDS code. The main difference between our achievability schemes for JPLT-I and JPLT-II is that unlike the Specialized MDS Code protocol, the Specialized Augmented Code protocol does not necessarily generate an MDS code.

Remark 1. In [34], it was shown that the rate $1/(K-D+1)$ is achievable for single-server PLC with joint privacy when the user has no prior side information about the messages available at the server. The optimality of this rate, however, was not shown. The results of Theorems 1 and 2 for $L = 1$ prove the optimality of this rate. For $L = D$, the JPLT-I and JPLT-II problems are equivalent to the problem of single-server PIR without any prior side information when joint privacy is required. As was shown in [18], an optimal solution for this problem is to download the entire dataset. This is consistent with the results of Theorems 1 and 2 for $L = D$.

Remark 2. The results of Theorems 1 and 2 show that JPLT-I and JPLT-II, collectively referred to as JPLT, can be performed more efficiently than using either of the following PIR-based and PLC-based approaches: (i) retrieving the messages required for the user’s computation using a single-server multi-message PIR scheme that achieves joint privacy [18], and then computing the required linear combinations locally, or (ii) computing each of the required linear combinations separately via applying a single-server PLC scheme that achieves joint privacy [34]. Note that the optimal rate for the PIR-based or PLC-based scheme is $L/K$ or $1/(K-D+1)$, respectively, whereas an optimal JPLT protocol achieves the rate $L/(K-D+L)$. Fig. 1 depicts the download rate of an optimal JPLT protocol, the PIR-based scheme, and the PLC-based scheme, for different values of $D \in \{10, 20, \ldots, 1000\}$, where $K = 1000$, and $L/D = 0.6$ (left plot) or $L/D = 0.4$ (right plot). As can be seen in Fig. 1 for a fixed ratio $L/D$, the advantage of an optimal JPLT protocol over the PIR-based scheme is more pronounced as $D$ increases. For instance, for $L/D = 0.4$, the rate of an optimal JPLT protocol is about 15% and 30% more than that of the PIR-based scheme for $D = 250$ and $D = 500$, respectively. It can also be seen in Fig. 1 that when the ratio $L/D$ is fixed, the gap between the rate of an optimal JPLT protocol and the rate of the PLC-based scheme increases as $D$ increases up to a threshold very close to $K$; and beyond this threshold, the gap decreases rapidly as $D$ increases up to $K$. In addition, a comparison of the left and right plots in Fig. 1 shows that for a fixed value of $D$, the smaller is the ratio $L/D$, the more is the advantage of an optimal JPLT protocol over the best of the other two schemes. For instance, for $D = 250$, the rate of an optimal JPLT protocol is about 10% and 15% more than that of the PIR-based scheme for $L/D = 0.6$ and $L/D = 0.4$, respectively.

IV. LINEAR JPLT PROTOCOLS AND LINEAR CODES

While any linear or non-linear JPLT protocol must satisfy the joint privacy and recoverability conditions, for linear JPLT protocols these conditions can be translated into the language of linear codes as discussed below.
In the following, we refer to a JPLT-I or JPLT-II protocol, simply as a JPLT protocol, and denote both V_I and V_{II} for Model I and V_{II} for Model II by V for the ease of notation.

Let w ≜ [W] and v ≜ [V], and let \{W_k\}_k∈[w] and \{V_j\}_j∈[v] be an arbitrary ordering of all elements in W and V, respectively. Consider an arbitrary linear JPLT protocol. For any instance (W_k,V_j) for k ∈ [w] and j ∈ [v], the protocol can be specified by an ensemble of n (= n(k,l)) distinct linear codes \(c_{k,l}^i\) of length K, for some integer n, and their respective probabilities \(p_{k,l}^i > 0\). More specifically, for each h ∈ [n], \(c_{k,l}^h\) is chosen with probability \(p_{k,l}^h\) as the corresponding code for the instance (W_k,V_j), i.e., the code corresponding to the coefficient matrix of the linear combinations that constitute the answer A[W_k,V_j] to the query Q[W_k,V_j]. Note that \(\sum_{h=1}^{n} p_{k,l}^h = 1\).

Below, we introduce the notion of \((k,l)\)-feasibility, which we will use to restate the joint privacy and recoverability conditions in the terminology of linear codes. For any k,l, we say that a linear code \(c\) of length K is \((k,l)\)-feasible if \(c\) contains a collection \(C\) of L codewords whose support is a subset of W_k, and the code generated by C, when punctured at the coordinates indexed by W_k, is identical to the code generated by V_j. Note that, for satisfying the recoverability condition, it is necessary and sufficient that for any k,l,h, the code \(c_{k,l}^h\) is \((k,l)\)-feasible. Note that \(\{c_{k,l}^h\}\) is a multiset in general because \(c_{k,l}^h\)'s are not necessarily distinct. Let m be the number of distinct elements in \(\{c_{k,l}^h\}\), and let \(e_1, \ldots, e_m\) be the distinct elements in \(\{c_{k,l}^h\}\). For any k ∈ [w] and j ∈ [v], let \(q_{k,j}\) be the sum of probabilities \(p_{k,l}^h\) over all l,h such that \(c_{k,l}^h\) is \((k,l)\)-feasible, and \(c_{k,l}^h\) and \(e_j\) are identical. For any j ∈ [v], let \(r_j\) be the sum of probabilities \(p_{k,l}^h\) over all k,l,h such that \(c_{k,l}^h\) and \(e_j\) are identical. Note that \(q_{k,j}/r_j\) is the conditional probability that the message index set W_k is the demand’s support, given that \(e_j\) is the code corresponding to the answer. It should be obvious that \(q_{k,j}/r_j > 0\) for all k,j is a necessary condition for joint privacy. Note that this condition is only necessary, and not sufficient. A necessary and sufficient condition for joint privacy is that for any j ∈ [v], \(q_{k,j} = q_j\) for all k ∈ [w], for some \(q_j > 0\).

For any k,l, let \(d_{k,l}\) be the expected value of the dimension of a randomly chosen code from the ensemble \(\{c_{k,l}^1, \ldots, c_{k,l}^m\}\) for the instance (W_k,V_j), according to the probability distribution \(\{p_{k,l}^1, \ldots, p_{k,l}^m\}\). Let \(d_{\text{avg}}\) be the average of \(d_{k,l}\)’s over all k,l. It should be obvious that the rate of a linear JPLT protocol is equal to \(1/d_{\text{avg}}\). Maximizing the rate of a linear JPLT protocol is then equivalent to minimizing \(d_{\text{avg}}\), subject to the aforementioned necessary and sufficient conditions for joint privacy and recoverability.

V. PROOF OF THEOREM II

We prove the converse in Section V.B and present the achievability scheme in Section V.B.

A. Converse Proof

The following result is useful in the proof of converse for the JPLT-I problem.

**Lemma 1.** Given any JPLT-I protocol, for any \(\tilde{W} \in \mathbb{W}\), there must exist \(\tilde{V} \in \mathbb{V}_1\) such that

\[
H(\mathbb{Z}^{W,V}_1|A, Q) = 0.
\]

**Proof:** The proof is by the way of contradiction. Consider an arbitrary JPLT-I protocol. Let Q and A be the query and the corresponding answer generated by this protocol for an arbitrary instance (W,V). Suppose that there does not exist \(\tilde{V} \in \mathbb{V}_1\) such that \(H(\mathbb{Z}^{W,V}_1|A, Q) = 0\). This implies that \(\mathbb{W} \neq \mathbb{W}, \mathbb{V}\), given that \(Q = Q\) (otherwise, if \(\mathbb{W} = \mathbb{W}, \mathbb{V}\), the recoverability condition is not satisfied). Thus, \(\text{Pr}(W = \tilde{W}|Q = Q) = 0\). This is, however, a contradiction because by the joint privacy condition, \(\text{Pr}(\tilde{W} = W|Q = Q) = 1/C_{K,D} \neq 0\).

When considering linear protocols, the result of Lemma 1 is equivalent to the necessary (but not sufficient) condition for joint privacy in Section IV. In contrast to this result which is more information theoretic and more instrumental in the proofs, the necessary and sufficient condition for joint privacy in Section IV is more combinatorial and harder to analyze. Moreover, the necessary and sufficient condition for joint privacy in Section IV is specific to linear protocols; whereas Lemma 1 applies also to non-linear protocols.

**Lemma 2.** The rate of any JPLT-I protocol for K messages, demand’s support size D, and demand’s dimension L is upper bounded by \(L/(K - D + L)\).

**Proof:** Consider an arbitrary JPLT-I protocol that generates a query-answer pair \(Q^{\mathbb{W},V}_1, A^{\mathbb{W},V}_1\) for any given (W,V). For simplifying the notation, we denote the random variables \(Q^{\mathbb{W},V}_1\) and \(A^{\mathbb{W},V}_1\) by Q and A, respectively. To show that the rate is upper bounded by \(L/(K - D + L)\), we need to show that \(H(A) \geq (K - D + L)B\), where \(B = N\log_2 q\) is the entropy of a uniformly distributed message over \(\mathbb{F}_q^N\).

Let \(T \triangleq K - D + 1\). For each \(i \in [T]\), let \(W_i \triangleq \{i, i+1, \ldots, i+D-1\}\). Note that \(W_1, \ldots, W_T \in \mathbb{W}\). By Lemma 1 there exists \(V_j \in \mathbb{V}_1\) for \(i \in [T]\) such that \(H(Z_i|A, Q) = 0\), where \(Z_i \triangleq Z_{[W_i,V_j]}\). (Note that \(V_j\) is an MDS matrix.) This readily implies that \(H(Z_1, \ldots, Z_T|A, Q) = 0\) since \(H(Z_1, \ldots, Z_T|A, Q) \leq \sum_{i=1}^T H(Z_i|A, Q) = 0\). Thus,

\[
H(A) \geq H(A|Q) + H(Z_1, \ldots, Z_T|Q, A) \geq H(Z_1, \ldots, Z_T|Q) + H(A|Q, Z_1, \ldots, Z_T) \geq H(Z_1, \ldots, Z_T),
\]

where (1) holds because \(H(Z_1, \ldots, Z_T|A, Q) = 0\), as shown earlier; (2) follows from the chain rule of conditional entropy; and (3) holds because (i) \(Z_i\)'s are independent from Q, noting that \(Z_i\)'s only depend on \(X\) and Q is independent of \(X\); and (ii) \(H(A|Q, Z_1, \ldots, Z_T) \geq 0\).
To lower bound $H(Z_1, \ldots, Z_T)$, we proceed as follows. By the chain rule of entropy, we have

$$H(Z_1, \ldots, Z_T) = H(Z_1) + \sum_{i=2}^T H(Z_i | Z_1, \ldots, Z_{i-1}).$$  \hspace{1cm} (4)$$

Let $Z_{i,j}, i \leq j$ be the $L$ rows of the matrix $Z_i$, i.e., $Z_{i,j} \triangleq v_{i,j}x_{W_i}$, where $v_{i,j}$ is the $l$th row of $V_i$. Note that $Z_i$ consists of $L$ row-vectors $Z_{i,1}, \ldots, Z_{i,L}$, and these vectors are independent because their corresponding coefficient vectors $v_{i,1}, \ldots, v_{i,L}$ are linearly independent. Moreover, $Z_{i,1}, \ldots, Z_{i,L}$ are uniform over $F_q^L$, i.e., $H(Z_{i,j}) = B$ for $i \in [L]$. Thus, $H(Z_i) = H(Z_{i,1}, \ldots, Z_{i,L}) = LB$, particularly, $H(Z_i) = LB$. Note, also, that there exists some $l \in [L]$ such that $Z_{i,l}$ is dependent on $X_{i+D-1}$, i.e., the coefficient of $X_{i+D-1}$ in the linear combination $Z_{i,l}$ is nonzero. Otherwise, $V_i$ contains an all-zero column, which contradicts with the fact that $V_i$ is MDS. Moreover, there does not exist any $l \in [L]$ such that $Z_{i,j}$ for any $j < l$ depends on $X_{i+D-1}$ (by construction of $W_1, \ldots, W_i$). This implies that there exists at least one row-vector, namely, $Z_{i,j}$, that is independent of the row-vectors pertaining to $Z_{i,1}, \ldots, Z_{i-1}$. This further implies that $H(Z_i | Z_{i,1}, \ldots, Z_{i-1}) \geq H(Z_{i,j}) = B$, and consequently, $\sum_{i=2}^T H(Z_i | Z_{1,1}, \ldots, Z_{i-1}) \geq (T-1)B$. From [4], it then follows that

$$H(Z_1, \ldots, Z_T) \geq LB + (T-1)B = (K - D + L)B.$$  \hspace{1cm} (5)

Combining [3] and [5], we have $H(A) \geq (K - D + L)B$. \hfill \Box

B. Achievability Scheme

In this section, we present a JPLT-I protocol, termed the Specialized MDS Code protocol, which is capacity-achieving for sufficiently large $q$—depending on the parameters $K, D, L$. An illustrative example of this protocol can be found in Appendix $\Box$

The Specialized MDS Code protocol consists of three steps as described below.

**Step 1:** Given the demand’s support $W \in \mathbb{W}$ and the demand’s coefficient matrix $V = \left[v_1^T, \ldots, v_T^T\right]^T \in \mathbb{V}_T$, the user constructs a query $Q^{[W, V]}$ in the form of a MDS code, such that the user’s query, i.e., the matrix $G$, and the server’s corresponding answer $A^{[W, V]}$, i.e., the matrix $Y = GX$, satisfy the recoverability and joint privacy conditions.

To satisfy the joint privacy condition, it is required that, for any index set $W \in \mathbb{W}$, the code generated by the matrix $G$ contains $L$ codewords whose support are some subsets of $W$, and the coordinates of these codewords (indexed by $W$) form an MDS matrix $\tilde{V} \in \mathbb{V}_L$. By the properties of MDS codes [42], it is easy to verify that the generator matrix of any $[K, K-D+L]$ MDS code satisfies this requirement. However, not any such generator matrix is guaranteed to satisfy the recoverability condition. For satisfying the recoverability condition, it is required that, as a generator matrix, generates a code that contains $L$ codewords with the support $W$, and the coordinates of these codewords (indexed by $W$) must conform to the coefficient matrix $V$. To construct a matrix $G$ that satisfies these requirements, the user proceeds as follows.

First, the user constructs the parity-check matrix $\Lambda$ of the $[D, L]$ MDS code generated by $V$. Since $V$ is an MDS matrix, then $\Lambda$ generates a $[D, D-L]$ MDS code. The user then constructs a $(D - L) \times K$ matrix $H$ that satisfies the following two conditions: (i) the matrix $H$ contains $\Lambda$ as a submatrix, and (ii) the matrix $H$ is MDS. Since $\Lambda$ is an MDS matrix, constructing $H$ reduces to extending the $[D, D-L]$ MDS code generated by $\Lambda$ to a $[K, D-L]$ MDS code. (An application of Schwartz-Zippel lemma shows that such an extension is feasible so long as $q$ is sufficiently large.) The user then constructs a matrix $H$ by permuting the columns of $H$ arbitrarily such that $\Lambda$ is the submatrix of $H$ restricted to the columns indexed by $W$. For simplicity, we also denote $H$ by $H$. Next, the user constructs a $(K - D + L) \times K$ matrix $G$ that generates the MDS code defined by the parity-check matrix $H$. (Since $H$ generates a $[K, D-L]$ MDS code, $H$ is the parity-check matrix of a $[K, K-D+L]$ MDS code.) The user then sends $G$ as the query $Q^{[W, V]}$ to the server.

**Step 2:** Given the query $Q^{[W, V]}$, i.e., the matrix $G$, the server computes the $(K - D + L) \times N$ matrix $Y \triangleq GX$, and sends $Y$ as the answer $A^{[W, V]}$ back to the user.

**Step 3:** Upon receiving the answer $A^{[W, V]}$, i.e., the matrix $Y$, the user constructs a matrix $[\tilde{G}, \tilde{Y}]$ by performing row operations on the augmented matrix $[G, Y]$, so as to zero out the submatrix formed by the first $L$ rows and the columns indexed by $[K] \setminus W$. Since the submatrix of $[\tilde{G}, \tilde{Y}]$ formed by the first $L$ rows and the columns indexed by $W$ (or $[K] \setminus W$) is equal to the matrix $V$ (or an all-zero matrix), the $l$th row of the demand matrix $Z^{[W, V]}$, i.e., $v_{l}x_{W_l}$ for $l \in [L]$, can be recovered from the $l$th row of the matrix $\tilde{Y}$.

In the following, we provide a more explicit description of the Specialized MDS Code protocol for the cases in which the coefficient matrix $V$ generates a GRS code. We refer to this protocol as the Specialized GRS Code protocol. Note that this protocol is applicable for any field size $q \geq K$.

**Step 1:** Suppose that the entry $(i, j) \in V$ is given by $V_{i,j} \triangleq v_j \omega_j^{-1}$, where $v_1, \ldots, v_D$ be distinct elements from $\mathbb{F}_q \setminus \{0\}$, and $\omega_1, \ldots, \omega_D$ be distinct elements from $\mathbb{F}_q$. The parameters $v_1, \ldots, v_D$ and $\omega_1, \ldots, \omega_D$ are the multipliers and the evaluation points of the GRS code generated by $V$, respectively. Since the dual of a GRS code is also a GRS code [42], the parity-check matrix $\Lambda$ of the GRS code generated by $V$ is a $(D - L) \times D$ matrix whose entry $(i, j)$ is given by $\Lambda_{i,j} \triangleq \lambda_j \omega_j^{-1}$, where

$$\lambda_j \triangleq v_j^{-1} \prod_{k \in [D]\setminus\{j\}} (\omega_j - \omega_k)^{-1}.$$.

Note that $\lambda_1, \ldots, \lambda_D$ be nonzero. Extending the $(D - L) \times D$ matrix $\Lambda$ to a $(D - L) \times K$ matrix $H$—satisfying the conditions (i) and (ii)—is performed as follows.

Let $W = \{i_1, \ldots, i_D\}$ and $[K] \setminus W = \{i_{D+1}, \ldots, i_K\}$, and let $\pi$ be a permutation on $[K]$ such that $\pi(j) = i_j$. Let $\lambda_{D+1}, \ldots, \lambda_K$ be $K - D$ elements chosen randomly (with replacement) from $\mathbb{F}_q \setminus \{0\}$, and let $\omega_{D+1}, \ldots, \omega_D$ be $K - D$ elements chosen randomly (without replacement) from $\mathbb{F}_q \setminus \{\omega_1, \ldots, \omega_D\}$. For every $j \in [D]$, let the $\pi(j)$th column of $H$ be the $j$th column of $\Lambda$, and for every $j \in [K] \setminus [D]$,
let the $\pi(j)$th column of $H$ be $[\lambda_j, \lambda_j \omega_j, \ldots, \lambda_j \omega_j^{p-D-1}]^\top$.
Since $H$ is the parity-check matrix of a $[K, K-D+L]$ GRS code, the generator matrix of this code, $G$, can be constructed by taking the $\pi(j)$th column of $G$ to be $[\alpha_j, \alpha_j \omega_j, \ldots, \alpha_j \omega_j^{p-D-1}]^\top$, where

$$
\alpha_j \triangleq \lambda_j^{-1} \prod_{k \in [K] \setminus \{j\}} (\omega_j - \omega_k)^{-1}.
$$

The parameters $\{\alpha_j\}_{j \in [K]}$ and $\{\omega_j\}_{j \in [K]}$ are the multipliers and the evaluation points of the GRS code generated by $G$, respectively. The user then sends the matrix $G$ to the server.

**Step 2:** Given the matrix $G$, the server computes the matrix $Y = GX$, where the $i$th row of $Y = [Y_1^\top, \ldots, Y_{K-D+L}^\top]^\top$ is given by

$$
Y_i \triangleq \sum_{j=1}^{K} \alpha_j \omega_j^{-i} X_j
$$
and sends $Y$ back to the user.

**Step 3:** Given the matrix $Y$, the user recovers the demand matrix $Z^{[W,V]}$ as follows. First, the user constructs $L$ polynomials $f_1(x), \ldots, f_L(x)$, where

$$
f_l(x) \triangleq x^{D-1} \prod_{j=D+1}^{K} (x - \omega_j).
$$

For each $l \in [L]$, let $c_l \triangleq [c_{l,1}, \ldots, c_{l,K-D-1}]^\top$, where $c_{l,j}$ is the coefficient of the monomial $x^{D-1}$ in the polynomial expansion of $f_l(x)$. The user then recovers the $i$th row of the demand matrix $Z^{[W,V]}$, namely, $v_i X_W$, by computing $c_i^\top Y$.

**Proposition 1** (Symmetry Property of MDS Codes). Given any $[n,k]$ MDS code, for any $S \subseteq [n]$ such that $|S| \geq n-k+1$, the code space contains a unique $(|S| - n + k)$-dimensional subspace indexed by $S$, and any basis of this subspace (restricted to the coordinates indexed by $S$) forms an MDS matrix.

**Proof:** Consider an arbitrary $[n,k]$ MDS code $\mathcal{C}$. Let $d \triangleq n-k+1$ be the minimum distance of $\mathcal{C}$. By the properties of MDS codes [42], for any $d$-subset $T \subseteq [n]$, the code $\mathcal{C}$ has a codeword whose support is $T$. Consider an arbitrary $S \subseteq [n]$ such that $|S| \geq d$. Let $s \triangleq |S|$, and $S \triangleq \{i_1, \ldots, i_s\}$. Let $m \triangleq s - d + 1$. Note that $m \leq s$. For each $i \in [m]$, let $S_i \triangleq \{i_1, \ldots, i_{j+d-1}\}$, and let $c_i$ be a codeword of $\mathcal{C}$ whose support is $S_i$. Note that $c_i$’s are row-vectors of length $n$. Consider an $m \times n$ matrix $C$ whose $i$th row is $c_i$, i.e., $C \triangleq [c_1, \ldots, c_m]^\top$. Note that the $i_1+d-1$th entry of the $i$th row of $C$ is nonzero for each $i \in [m]$, and the $i_{j+d-1}$th entry of the $j$th row of $C$ is zero for any $j < i$. This readily implies that $\text{rank}(C) = m$. Thus, the row space of $C$, i.e., the (linear) span of the codewords $c_{1}, \ldots, c_{m}$, is an $m$-dimensional subspace on the coordinates indexed by $S = \cup_{i=1}^{m} S_{i}$. Note that $m = s - d + 1 = s - (n-k+1) + 1 = s - n + k$. This proves that the code space contains an $(s - n + k)$-dimensional subspace on the coordinates indexed by $S$.

Next, we show that any basis of the subspace spanned by the rows of $C$ (restricted to the coordinates indexed by $S$) forms an MDS matrix. Consider an arbitrary basis of this subspace.

The matrix formed by this basis can be written as $RC$ for some $m \times m$ invertible matrix $R$. Let $\tilde{C}$ be an $m \times s$ submatrix of $C$ formed by the columns indexed by $S$. We need to show that $RC$ is an MDS matrix. If $\tilde{C}$ is an MDS matrix, any $m \times m$ submatrix of $\tilde{C}$, and consequently, any $m \times m$ submatrix of $RC$, is invertible, and hence, $RC$ is an MDS matrix. Thus, it suffices to show that $\tilde{C}$ is an MDS matrix. Consider the $[s,m]$ code $\tilde{C}$ generated by $\tilde{C}$. The minimum distance of $\tilde{C}$ is at most $s - m + 1 = n - k + 1 (= d)$. The weight of the codewords of $\tilde{C}$ corresponding to the rows of $\tilde{C}$ is $d$. Moreover, any other (nonzero) codeword of $\tilde{C}$ is a linear combination of the rows of $\tilde{C}$, and has a weight at least $d$. (If $\tilde{C}$ has a codeword of weight less than $d$, then $C$ must have a codeword of weight less than $d$, which is a contradiction since the minimum distance of $\tilde{C}$ is $d$.) Thus, the minimum distance of $\tilde{C}$ is $d (= s - m + 1)$, and $\tilde{C}$ is an $[s,m]$ MDS code.

Now, we prove the uniqueness by the way of contradiction. Suppose that the code space contains two distinct subspaces on the coordinates indexed by $S$. For $i \in \{1,2\}$, let $M_i$ be an $m \times n$ matrix formed by an arbitrary basis of the $i$th subspace. Consider the matrix $M = [M_1^\top, M_2^\top]^\top$. Note that the rows of $M$ are codewords of $\tilde{C}$. Obviously, $\tilde{C} \triangleq \text{rank}(M) > m$. This is because $\text{rank}(M_1) = m$, and there exists at least one row in $M_2$ that is linearly independent of the rows of $M_1$. Let $M$ be an $m \times n$ matrix formed by an arbitrary basis of the row space of $M$. Note that $\text{rank}(M) = \tilde{m}$. By performing Gaussian-Jordan elimination on a properly chosen column-permutation of $M$, we can obtain a matrix of the form $[I, P, 0]$, where $I$ is an $\tilde{m} \times \tilde{m}$ identity matrix, $P$ is an $\tilde{m} \times (n - \tilde{m})$ matrix, and $0$ is an $\tilde{m} \times (n - s)$ all-zero matrix. Note that the row space of $[I, P, 0]$ is the same as the row space of $M$ which is itself the same as the row space of $\tilde{C}$, and hence, the rows of $[I, P, 0]$ are codewords of $\tilde{C}$. Fix an arbitrary $i \in [\tilde{m}]$. Consider the codeword corresponding to the $i$th row of $[I, P, 0]$. The weight of this codeword is at most $s - \tilde{m} + 1$, because there is only one nonzero coordinate within the first $\tilde{m}$ coordinates, and there are at most $s - \tilde{m}$ nonzero coordinates within the last $n - \tilde{m}$ coordinates. Thus, the minimum distance of $\tilde{C}$ is at most $s - \tilde{m} + 1$ which is strictly less than $s - m + 1 = s - (s - n + k) + 1 = n - k + 1 = d$ since $\tilde{m} > m$. This is a contradiction because the minimum distance of $\tilde{C}$ is $d$. ⊓⊔

**Lemma 3.** The Specialized MDS Code protocol is a JPLT-I protocol, and achieves the rate $L/(K-D+L)$.

**Proof:** Since the answer $Y = GX$ is a matrix with $K-D+L$ rows, and the rows of this matrix are linearly independent coded combinations of the messages $X_1, \ldots, X_K$ (noting that the matrix $G$ has full rank), the entropy of the answer is given by $(K-D+L)B$, where $B$ is the entropy of a message. Thus, the rate of this protocol is $L/(K-D+L)$.

Next, we prove that the joint privacy condition is satisfied. Note that the matrix $G$ generates a $[K, K-D+L]$ MDS code with minimum distance $D-L+1$. By the symmetry property of MDS codes (Proposition 1), the row space of $G$ contains a unique $L$-dimensional subspace on every $D$-subset of coordinates. Note that each of these $L$-dimensional subspaces (corresponding to a distinct $D$-subset of coordinates) is equally likely to be the subspace spanned by the rows
of the demand’s global coefficient matrix, from the server’s perspective. Combining these arguments, given the matrix $G$, every $D$-subset of message indices is equally likely to be the demand’s support. This completes the proof of joint privacy.

The recoverability follows readily from the construction. Let $U$ be the global coefficient matrix of the demand. We need to show that the rows of $U$ are $L$-codewords of the code generated by $G$. Since $H$ is the parity-check matrix of the code generated by $G$, this is equivalent to showing that $UH$ is an all-zero matrix. This can be shown as follows. Firstly, the submatrix of $UH$ restricted to the columns indexed by $W$ is equal to $VA^T$, and $VA^T$ is an all-zero matrix because $A$ is the parity-check matrix of the code generated by $V$. Secondly, the submatrix of $UH$ formed by the columns indexed by $[K] \setminus W$ is an all-zero matrix because the submatrix of $U$ restricted to these columns is an all-zero matrix. Thus, $UH$ is an all-zero matrix. This completes the proof of recoverability. ∎

VI. PROOF OF THEOREM 2

The proof of converse is given in Section VI-A and the achievability scheme is presented in Section VI-B.

A. Converse Proof

The converse proof for the JPLT-II problem relies on the following result.

**Lemma 4.** Given any JPLT-II protocol, for any $W \in \mathbb{W}$, there must exist $V \in \mathbb{V}_{II}$ such that

$$H(Z_{WV}|A, Q) = 0.$$  

**Proof:** The result follows from the same argument as in the proof of Lemma 1 except where $V_I$ is replaced by $V_{II}$. ∎

**Lemma 5.** The rate of any JPLT-II protocol for $K$ messages, demand’s support size $D$, and demand’s dimension $L$ is upper bounded by $L/(K - D + L)$.

**Proof:** Consider an arbitrary JPLT-II protocol that generates a pair $(Q_{WV}, A_{WV})$ for any given $(W, V)$. We denote $Q_{WV}$ and $A_{WV}$ by $Q$ and $A$, respectively. To show the rate upper bound, we need to show that $H(A) \geq (K - D + L)B$, where $B = N \log_2 q$ is the entropy of a uniformly distributed message over $\mathbb{F}_q^N$. Let $T \triangleq C_{K,D}$. Consider an arbitrary ordering of all elements in $\mathbb{W}$, say, $W_1, \ldots, W_T$, where $W_i$’s are distinct $D$-subsets of $\mathbb{K}$. By Lemma 4, there exist $n_i \geq 1$ matrices $V_1^i, \ldots, V_{n_i}^i \in \mathbb{V}_{II}$ for $i \in \{T\}$, each of rank $L$, such that $H(Z_i^j|A, Q) = 0$ for $j \in [n_i]$, where $Z_i^j \triangleq Z_{W_iV_i^j}$. Thus, $H(Z_1^1, \ldots, Z_1^{n_1}, \ldots, Z_T^1, \ldots, Z_T^{n_T}|A, Q) = 0$. Similarly as in (1)-(3), we can then show that

$$H(A) \geq H(Z_1^1, \ldots, Z_1^{n_1}, \ldots, Z_T^1, \ldots, Z_T^{n_T}).$$  

(6)

In the following, we lower bound the right hand-side of (6).

For any $i \in \{T\}$ and $j \in [n_i]$, let $U_i^j$ be the global coefficient matrix of the potential demand $Z_i^j$, i.e., the submatrix of $U_i$ formed by the columns indexed by $W_i$ is equal to $V_i^j$, and the rest of the columns of $U_i^j$ are all-zero. Note that the rank of $U_i^j$ is $L$. This is simply because $U_i^j$ has $L$ rows, and it contains the matrix $V_i^j$ of rank $L$ as a submatrix. Consider the $TL \times K$ matrix formed by vertically concatenating the $L \times K$ matrices $U_1^1, \ldots, U_1^{n_1}, \ldots, U_T^1, \ldots, U_T^{n_T}$. Choose an arbitrary basis of the row space of this $TL \times K$ matrix, and let $M$ be a matrix formed by the chosen basis. It is easy to see that each row of $Z_i^j$ can be written as a linear combination of the rows of the $TL \times N$ matrix $MX$, where $X = [X_1^1, \ldots, X_1^{n_1}]^T$ is the $K \times N$ matrix of messages. Since $X_1^1, \ldots, X_K$ are independently and uniformly distributed over $\mathbb{F}_q^N$, we have

$$H(Z_1^1, \ldots, Z_1^{n_1}, \ldots, Z_T^1, \ldots, Z_T^{n_T}) = \text{rank}(M) \times B.$$  

(7)

Combining (6) and (7), we have $H(A) \geq \text{rank}(M) \times B$. Recall that we need show that $H(A) \geq (K - D + L)B$. Thus, it suffices to show that $\text{rank}(M) \geq K - D + L$. We prove this by the way of contradiction.

Let $m \triangleq \text{rank}(M)$. Note that $m \geq L$. This is because by the recoverability condition, the row space of $M$ must contain the rows of the matrix $U$, where $U$ is the global coefficient matrix of the user’s demand $Z = UX$, and the rank of $U$ is $L$. Suppose that $m < K - D + L$. Choose an arbitrary basis of the row space of $M$, and let $\hat{M}$ be an $m \times K$ matrix formed by this basis. By performing Gauss-Jordan elimination on a properly chosen column-permutation of $\hat{M}$, we can obtain a matrix of the form $[I, P]$, where $I$ is an $m \times m$ identity matrix, and $P$ is an $m \times (K - m)$ matrix. By the construction of $[I, P]$, the rows of $U_i^j$ must be in the row space of $[I, P]$. Without loss of generality, assume that $[I, P]$ is obtained by performing elimination on $\hat{M}$ (instead of a column-permutation of $\hat{M}$).

Next, we prove that $P$ is an all-zero matrix. Let $S \triangleq C_{K-m,D-L}$, and let $W_1, \ldots, W_S$ be all $(D - L)$-subsets of $[m + 1 : K]$. Without loss of generality, assume that $W_i = [L] \cup \hat{W}_i$ for $i \in S$. Since $m \geq L$, $\hat{W}_i$ and $W_i$ are disjoint, and $|W_i| = L + (D - L) = D$. For arbitrary $i \in S$ and $j \in [n_i]$, consider the matrix $U_i^j$, and the submatrix of $[I, P]$ formed by the first $L$ rows, denoted by $[\hat{I}, 0, \hat{P}]$, where $\hat{I}$ is an $L \times L$ identity matrix, $0$ is an $L \times (m - L)$ all-zero matrix, and $\hat{P}$ is an $L \times (K - m)$ matrix. Note that $W_i$ does not contain any index in $[L + 1 : m]$, and any row of $[I, P]$ with an index in $[L + 1 : m]$, i.e., any row of $[I, P]$ that is not included in $[\hat{I}, 0, \hat{P}]$, has a nonzero entry at a distinct column with an index in $[L + 1 : m]$. Thus, the rows of $U_i^j$ must be in the row space of $[\hat{I}, 0, \hat{P}]$. Recall that $U_i^j$ and $[\hat{I}, 0, \hat{P}]$ have $L$ rows. Since $W_i = [L] \cup \hat{W}_i$ and $W_i$ is a $(D - L)$-subset of $[m + 1 : K]$, the rows of $U_i^j$ lie in the row space of $[\hat{I}, 0, \hat{P}]$ iff the submatrix of $\hat{P}$ formed by the columns indexed by $[m + 1 : K] \setminus \hat{W}_i$ is an all-zero matrix. Using the same argument for all $i \in S$, it follows that $\hat{P}$ is an all-zero matrix. Re-defining $W_1, \ldots, W_S$ by replacing $[L]$ with different $L$-subsets of $[m]$, and repeating the same arguments as above, it follows that $P$ is an all-zero matrix.

Now, we can simply arrive at a contradiction. Recall that by assumption $m < K - D + L$, or equivalently, $K - m > D - L$, and $P$ has $K - m$ columns. Without loss of generality, assume that $W_1 = [L - 1] \cup [m + 1 : m + D - L + 1]$. Note that $|W_1| = D$. Consider the matrix $U_1^j$. Recall that $\text{rank}(U_1^j) = L$, and the rows of $U_1^j$ must lie in the row space of $[\hat{I}, P]$, or particularly, in the row space of the submatrix of $[I, P]$
formed by the first \( L - 1 \) rows (all rows of \( U_1^\top \) are linearly independent of the last \( m - L + 1 \) rows of \( [L^\top P^\top] \)). The rank of this submatrix is \( L - 1 \), and this is a contradiction because \( \text{rank}(U_1) = L \). Thus, \( m \geq K - D + L \).

### B. Achievability Scheme

In this section, we present a JPLT-II protocol, termed the Specialized Augmented Code protocol, which is capacity-achieving for any \( q \geq K \). An illustrative example of this protocol can be found in Appendix B.

The Specialized Augmented Code protocol consists of three steps as described below.

**Step 1:** Given \( W \in \mathbb{W} \) and \( V = [v_1^\top, \ldots, v_q^\top]^\top \in \mathbb{W}_q \), the user constructs a \((K - D + L) \times K\) matrix \( G \), and sends \( G \) as the query \( Q(W,V) \) to the server. To construct \( G \), the user first constructs the global coefficient matrix \( U \) from \( V \). Next, the user constructs a \((K - D + L) \times K\) matrix \( \tilde{G} \) by vertically concatenating the \( L \times K \) matrix \( U \) and an arbitrary \((K - D) \times K\) MDS matrix \( M \)—generated independently from \( W \) and \( V \). (For any \( q \geq K \), the matrix \( M \) can be constructed as the generator matrix of a \([K, K - D]\) GRS code over \( \mathbb{F}_2 \), with arbitrary nonzero multipliers and distinct evaluation points.) That is, \( \tilde{G} = [U^\top, M^\top]^\top \). Observe that the code generated by \( \tilde{G} \) is the result of augmenting the code generated by \( U \) with the codewords of the MDS code generated by \( M \). The user then constructs the matrix \( G \) by multiplying the \( \tilde{G} \) by a randomly generated \((K - D + L) \times (K - D + L)\) invertible matrix \( R \), i.e., \( G = R \tilde{G} \). Note that the matrix \( G \) does not necessarily generate an MDS code, and this protocol may not serve as a JPLT-I protocol in general.

**Step 2:** Given the query \( Q(W,V) \), i.e., the matrix \( G \), the server computes the \((K - D + L) \times N\) matrix \( \hat{Y} = GX \), and sends \( \hat{Y} \) as the answer \( A(W,V) \) back to the user.

**Step 3:** Upon receiving the answer \( A(W,V) \), i.e., the matrix \( \hat{Y} \), the user computes the \((K - D + L) \times N\) matrix \( \tilde{U} = R^{-1} \hat{Y} \), and recovers the \( l \)-th row of the demand matrix \( \tilde{Z}(W,V) \), i.e., \( v_l^\top \tilde{W} \), for \( l \in [L] \), from the \( l \)-th row of \( \tilde{Y} \).

**Lemma 6.** The Specialized Augmented Code protocol is a JPLT-II protocol, and achieves the rate \( L/(K - D + L) \).

**Proof:** Similar to Lemma 3 to prove that the rate of this protocol is \( L/(K - D + L) \), it suffices to show that the \((K - D + L) \times K\) matrix \( G \) has full rank, i.e., \( \text{rank}(G) = K - D + L \). Since \( G = R \tilde{G} \) and \( R \) is invertible, we need to show that \( \text{rank}(\tilde{G}) = K - D + L \). Recall that \( \tilde{G} = [U^\top, M^\top]^\top \). Each row of \( U \) has at most \( D \) nonzero entries. However, the row space of \( M \) does not contain any row-vector with less than \( D + 1 \) nonzero entries. This is because \( M \) generates a \([K, K - D]\) MDS code with the minimum distance \( K - (K - D) + 1 = D + 1 \). By these arguments, the rows of \( U \) do not lie in the row space of \( M \), and hence, \( \text{rank}(\tilde{G}) = \text{rank}(M) + \text{rank}(U) \). Obviously, \( \text{rank}(M) = K - D \) because \( M \) is a \((K - D) \times K\) MDS matrix, and \( \text{rank}(U) = L \) because \( U \) contains \( V \) as a submatrix, and \( \text{rank}(V) = L \) (by assumption). Thus, \( \text{rank}(\tilde{G}) = K - D + L \), as was to be shown.

Next, we prove that the joint privacy condition is satisfied. To this end, we show that, for any \( \hat{W} \in \mathbb{W} \), the row space of \( \hat{G} \), or equivalently, the row space of \( \hat{G} \), contains a unique \( L \)-dimensional subspace on the coordinates indexed by \( \hat{W} \). Without loss of generality, assume that \( \hat{W} = [K - D + 1 : K] \). We can rewrite the matrix \( \hat{G} \) as

\[
\hat{G} = \begin{bmatrix} U_1 & U_2 \\ M_1 & M_2 \end{bmatrix},
\]

where \( U_1 \) (or \( M_1 \)) is an \( L \times (K - D) \) submatrix of \( U \) (or \( M \)) formed by the columns indexed by \([K] \setminus \hat{W} \), and \( U_2 \) (or \( M_2 \)) is an \( L \times D \) submatrix of \( U \) (or \( M \)) formed by the columns indexed by \( \hat{W} \). Since \( M_1 \) is a \((K - D) \times (K - D)\) submatrix of \( M \), and \( M \) is a \((K - D) \times K\) MDS matrix, the row space of \( M_1 \) is a \((K - D)\)-dimensional subspace on the \( K - D \) coordinates indexed by \([K] \setminus \hat{W} \). This implies that each row of \( U_1 \) can be written as a unique linear combination of the rows of \( M_1 \). Thus, by performing Gauss-Jordan elimination on the matrix \( \hat{G} \), we can obtain a \((K - D + L) \times K\) matrix \( \hat{G} \) given by

\[
\hat{G} = \begin{bmatrix} 0 & \hat{U} \\ 1 & M \end{bmatrix},
\]

where \( 0 \) is an \( L \times (K - D) \) all-zero matrix, \( I \) is a \((K - D) \times (K - D)\) identity matrix, \( \hat{U} \) is an \( L \times D \) matrix, and \( M \) is a \((K - D) \times D\) matrix. Note that \( \text{rank}(\hat{G}) = \text{rank}(\hat{G}) = K - D + L \), and \( \text{rank}(\hat{U}) = (K - D) + \text{rank}(\hat{U}) \). Thus, \( \text{rank}(\hat{U}) = L \). This implies that the row space of the \( L \times K \) matrix \([0, \hat{U}] \) is an \( L \)-dimensional subspace on the coordinates indexed by \([K] \setminus \hat{W} \). Moreover, this subspace is unique because the \((K - D) \times K\) matrix \([L, M] \) is MDS. Note, also, that from the perspective of the server, each of these \( L \)-dimensional subspaces (corresponding to a distinct \( \hat{W} \in \mathbb{W} \)) is equally likely to be the subspace spanned by the rows of the demand’s global coefficient matrix \( U \). Thus, given the query (i.e., the matrix \( G \)), every \( \hat{W} \in \mathbb{W} \) is equally likely to be the demand’s support. This completes the proof of joint privacy.

The proof of recoverability is straightforward. By Step 3 of the protocol, \( \hat{Y} = R^{-1} \hat{Y} \). Rewriting \( Y \) as \( GX = R \hat{G} X \), it follows that \( \hat{Y} = \hat{G} X = [(UX)^\top, (MX)^\top]^\top \). This shows that the \( L \times N \) submatrix of \( \hat{Y} \) formed by the first \( L \) rows is equal to the demand matrix \( UX = VXW \).

### VII. Conclusion and Future Work

In this work, we introduced the problem of Private Linear Transformation (PLT) which generalizes the Private Information Retrieval (PIR) and Private Linear Computation (PLC) problems. The PLT problem includes a dataset that is stored on a single (or multiple) remote server(s), and a user who wishes to compute multiple linear combinations of a subset of items belonging to the dataset. The goal is to perform the computation such that the total amount of information downloaded is minimized, while the identities of items required for the computation are kept private.

We focused on the single-server setting of the PLT problem with joint privacy guarantees, referred to as the JPLT problem. The notion of joint privacy ensures that the identities of all items required for the computation are protected jointly.
We considered two different models, depending on whether the coefficient matrix of the required linear combinations is MDS. For each model, we characterized the capacity, where the capacity is defined as the supremum of all achievable download rates. In addition, we presented a capacity-achieving scheme for each of the models being considered.

There remain several open problems—closely related to the JPLT problem. Below we list a few of these problems.

1) It was recently shown that, as compared to the single-server setting, PIR and PLC can be performed much more efficiently (in terms of the download rate) when there are multiple servers that store identical copies or coded versions of the dataset, see, e.g., [2], [4], [9], [30], [31], [43]– [45], [45], [46]. Motivated by these results, an important direction for research is to characterize the capacity of multi-server PLT with joint privacy guarantees.

2) Establishing the fundamental limits of (single-server or multi-server) PLT with joint privacy in the presence of a prior side information is another direction for future work. This is motivated by the recent developments in PIR and PLC with side information, see, e.g., [14]– [28].

3) Many machine learning and cloud computing algorithms require computing non-linear functions on a subset of dataset. For instance, evaluating polynomials on a subset of training samples finds application in distributed stochastic gradient descent for linear regression [47]. The need for protecting the data access privacy in such scenarios motivates the problem of designing efficient privacy-preserving schemes for non-linear function computation.

APPENDIX

A. An Example of the Specialized MDS Code Protocol

Consider a scenario in which the server has $K = 10$ messages $X_1, \ldots, X_{10} \in \mathbb{F}_1^{11}$ for an arbitrary integer $N \geq 1$, and the user wishes to compute $L = 2$ linear combinations of $D = 5$ messages $X_2, X_4, X_5, X_7, X_8$, say, $Z_1 = X_2 + 3X_4 + 2X_5 + X_7 + 6X_8$, and $Z_2 = 3X_2 + 10X_4 + 7X_5 + 4X_7 + 8X_8$. For this example, $W = \{2, 4, 5, 7, 8\}$, and

$$V = \begin{bmatrix} 1 & 3 & 2 & 1 & 6 \\ 3 & 10 & 7 & 4 & 8 \end{bmatrix}.$$ 

It is easy to verify that $V$ generates a $[5, 2]$ GRS code with the multipliers $\{v_1, \ldots, v_5\} = \{1, 3, 2, 1, 6\}$ and the evaluation points $\{w_1, \ldots, w_5\} = \{3, 7, 9, 4, 5\}$. Then, the user obtains the parity-check matrix $\Lambda$ of the code generated by $V$ as $\Lambda = \begin{bmatrix} 3 & 10 & 8 & 8 & 7 \\ 9 & 4 & 6 & 10 & 2 \\ 5 & 6 & 10 & 7 & 10 \end{bmatrix}$. Note that $\Lambda$ generates a $[5, 3]$ MDS code with the multipliers $\{\lambda_1, \ldots, \lambda_5\} = \{3, 10, 8, 8, 7\}$ and the evaluation points $\{w_1, \ldots, w_5\} = \{3, 7, 9, 4, 5\}$.

Next, the user extends the $3 \times 5$ matrix $\Lambda$ to a $3 \times 10$ matrix $H$ that satisfies the conditions (i) and (ii) specified in Step 1 of the Specialized GRS Code protocol. Suppose the user randomly chooses 6 additional multipliers $\{\lambda_6, \ldots, \lambda_{10}\} = \{3, 5, 1, 1, 4\}$ from $\mathbb{F}_{11} \setminus \{0\}$, and 6 additional evaluation points $\{\omega_6, \ldots, \omega_{10}\} = \{6, 1, 10, 2, 8\}$ from $\mathbb{F}_{11} \setminus \{\omega_1, \ldots, \omega_5\}$. Followed by constructing a permutation $\pi$ as described in Step 1 of the Specialized GRS Code protocol, say, $\{\pi(1), \ldots, \pi(10)\} = \{2, 4, 5, 7, 8, 1, 3, 6, 9, 10\}$, the user constructs the extended matrix $H$ as $H = \begin{bmatrix} 3 & 5 & 10 & 8 & 1 & 8 & 7 & 1 & 4 \\ 9 & 5 & 5 & 6 & 10 & 7 & 10 & 4 & 3 \end{bmatrix}$, where the columns of $H$ indexed by $\pi(1), \pi(2), \pi(3), \pi(4), \pi(5)$ (i.e., the columns $2, 4, 5, 7, 8$) correspond to the columns $1, 2, 3, 4, 5$ of $\Lambda$, respectively, and the columns of $H$ indexed by $\pi(6), \pi(7), \pi(8), \pi(9), \pi(10)$ (i.e., the columns $1, 3, 6, 9, 10$) correspond to the columns of the generator matrix of a $[5, 3]$ GRS code with the multipliers $\{\lambda_6, \ldots, \lambda_{10}\}$ and the evaluation points $\{w_6, w_7, w_8, w_9, w_{10}\}$. That is, the $(i)$th column of $H$ for $i \in \{6, \ldots, 10\}$ is given by $\{\lambda_i, \lambda_i w_i, \lambda_i w_i^2\}^T$. Since $H$ generates a $[10, 3]$ GRS code with the multipliers $\alpha_\delta = 9$, $\alpha_7 = 10$, $\alpha_7 = 2$, $\alpha_3 = 7$, $\alpha_3 = 3$, $\alpha_9 = 1$, $\alpha_4 = 5$, $\alpha_4 = 4$, $\alpha_9 = 9$, $\alpha_{10} = 9$ and the evaluation points $\omega_6 = 6$, $\omega_1 = 3$, $\omega_7 = 1$, $\omega_2 = 7$, $\omega_3 = 9$, $\omega_8 = 10$, $\omega_4 = 4$, $\omega_5 = 5$, $\omega_2 = 2$, $\omega_{10} = 8$. (The process of computing $\alpha_i$’s is explained in Step 1 of the Specialized GRS Code protocol.) The user then obtains the generator matrix $G$ of this code, $G = \begin{bmatrix} 9 & 10 & 2 & 7 & 3 & 1 & 5 & 4 & 9 & 9 \\ 10 & 8 & 2 & 5 & 5 & 10 & 9 & 7 & 6 \\ 5 & 2 & 2 & 2 & 1 & 1 & 3 & 1 & 3 & 4 \\ 8 & 6 & 2 & 3 & 9 & 10 & 1 & 5 & 6 & 10 \\ 4 & 7 & 2 & 10 & 4 & 1 & 4 & 3 & 1 & 3 \\ 2 & 10 & 2 & 4 & 3 & 10 & 5 & 4 & 2 & 2 \\ 1 & 8 & 2 & 6 & 5 & 1 & 9 & 9 & 4 & 5 \end{bmatrix}$. Then, the user sends the matrix $G$ as the query to the server. The server then computes the matrix $Y = GX$, and sends it back to the user. Next, the user constructs two polynomials $f_1(x) = (x - \omega_6)(x - \omega_7)(x - \omega_8)(x - \omega_9)(x - \omega_{10}) = (x - 6)(x - 1)(x - 10)(x - 2)(x - 8)$, and $f_2(x) = x f_1(x)$. The coefficient vectors of the polynomials $f_1(x)$ and $f_2(x)$ are given by $c_1 = [8, 1, 8, 9, 6, 1, 0]^T$ and $c_2 = [0, 8, 1, 8, 9, 6, 1]^T$, respectively. The user then recovers their demand, i.e., $Z_1$ and $Z_2$, by computing $Z_1 = c_1^T Y = X_2 + 3X_4 + 2X_5 + X_7 + 6X_8$, $Z_2 = c_2^T Y = 3X_2 + 10X_4 + 7X_5 + 4X_7 + 8X_8$. For this example, the rate of the Specialized MDS Code protocol is $L/(K - D + L) = 2/7$, whereas the rate of a PIR-based scheme or a PLC-based scheme is $L/K = 2/10$ or $1/(K - D) = 1/5$, respectively.
Consider a scenario in which the server has $K = 10$ messages $X_1, \ldots, X_{10} \in \mathbb{F}_3^{10}$ for an arbitrary integer $N \geq 1$, and the user wants to compute $L = 2$ linear combinations of $D = 5$ messages $X_2, X_4, X_5, X_7, X_8$, say, $Z_1 = 3X_2 + X_4 + 6X_5 + 2X_7 + 6X_8$ and $Z_2 = 10X_2 + 4X_4 + 8X_5 + 7X_7 + 9X_8$. For this example, $W = \{2, 4, 5, 7, 8\}$, and

$$V = \begin{bmatrix} 3 & 1 & 6 & 2 & 6 \\ 10 & 4 & 8 & 7 & 9 \end{bmatrix}.$$ 

Note that $V$ has full rank, but it is not MDS. First, the user constructs the demand’s global coefficient matrix $U$ as

$$U = \begin{bmatrix} 0 & 3 & 0 & 1 & 6 & 0 & 2 & 6 & 0 & 0 \\ 0 & 10 & 0 & 4 & 8 & 0 & 7 & 9 & 0 & 0 \end{bmatrix}.$$ 

Next, the user generates an arbitrary $5 \times 10$ MDS matrix $M$, independently from $W$ and $V$. For this example, suppose the matrix $M$ is given by

$$M = \begin{bmatrix} 2 & 1 & 4 & 7 & 9 & 1 & 10 & 5 & 4 & 3 \\ 6 & 5 & 3 & 5 & 3 & 6 & 10 & 6 & 10 & 6 \\ 7 & 3 & 5 & 2 & 1 & 3 & 10 & 5 & 3 & 1 \\ 10 & 4 & 1 & 3 & 4 & 7 & 10 & 6 & 2 & 2 \\ 8 & 9 & 9 & 10 & 5 & 9 & 10 & 5 & 5 & 4 \end{bmatrix}.$$ 

The user then constructs a $7 \times 10$ matrix $\hat{G}$ by vertically concatenating the matrices $U$ and $M$, i.e., $\hat{G} = [U^\top, M^\top]^\top$.

$$\hat{G} = \begin{bmatrix} 0 & 3 & 0 & 1 & 6 & 0 & 2 & 6 & 0 & 0 \\ 0 & 10 & 0 & 4 & 8 & 0 & 7 & 9 & 0 & 0 \\ 2 & 1 & 4 & 7 & 9 & 1 & 10 & 5 & 4 & 3 \\ 6 & 5 & 3 & 5 & 3 & 6 & 10 & 6 & 10 & 6 \\ 7 & 3 & 5 & 2 & 1 & 3 & 10 & 5 & 3 & 1 \\ 10 & 4 & 1 & 3 & 4 & 7 & 10 & 6 & 2 & 2 \\ 8 & 9 & 9 & 10 & 5 & 9 & 10 & 5 & 5 & 4 \end{bmatrix}.$$ 

Then, the user randomly generates a $7 \times 7$ invertible matrix $R$, and constructs a $7 \times 10$ matrix $G = RG$. For this example, suppose that $G$ is given by

$$G = \begin{bmatrix} 7 & 10 & 7 & 7 & 9 & 1 & 10 & 0 & 10 & 10 \\ 4 & 2 & 0 & 9 & 7 & 6 & 6 & 0 & 8 & 7 \\ 7 & 2 & 6 & 7 & 10 & 2 & 9 & 4 & 8 & 4 \\ 8 & 10 & 10 & 3 & 7 & 2 & 5 & 6 & 4 & 7 \\ 1 & 1 & 3 & 2 & 0 & 2 & 8 & 5 & 3 & 3 \\ 7 & 2 & 9 & 6 & 9 & 5 & 5 & 8 & 6 & 9 \\ 7 & 10 & 8 & 7 & 3 & 2 & 5 & 10 & 6 & 3 \end{bmatrix}.$$ 

Next, the user sends $G$ to the server. Given $G$, the server computes the matrix $Y = GX$, where $X = \{X_1^\top, \ldots, X_{10}^\top\}^\top$, and sends $Y$ back to the user. Given the matrix $Y$, the user recovers their demand matrix $[Z_1^\top, Z_2^\top]^\top = VXW = UX$ from the matrix formed by the first 2 rows of the matrix $Y = R^{-1}Y = (R^{-1}G)X = \hat{G}X = [(UX)^\top, (MX)^\top]^\top$.

For this example, the rate of the Specialized Augmented Code protocol is $L/(K - D + L) = 2/7$, whereas the rate of a PIR-based scheme or a PLC-based scheme is $L/K = 2/10$ or $1/(K - D) = 1/5$, respectively.
