CONVERGENCE, FLUCTUATIONS AND LARGE DEVIATIONS FOR FINITE STATE MEAN FIELD GAMES VIA THE MASTER EQUATION

ALEKOS CECCHIN AND GUGLIELMO PELINO

Abstract. We show the convergence of finite state symmetric $N$-player differential games, where players control their transition rates from state to state, to a limiting dynamics given by a finite state Mean Field Game system made of two coupled forward-backward ODEs. We exploit the so-called Master Equation, which in this finite-dimensional framework is a first order PDE in the simplex of probability measures, obtaining the convergence of the feedback Nash equilibria, the value functions and the optimal trajectories. The convergence argument requires only the regularity of a solution to the Master Equation. Moreover, we employ the convergence method to prove a Central Limit Theorem and a Large Deviation Principle for the evolution of the $N$-player empirical measures. The well-posedness and regularity of solution to the Master Equation are also studied.

1. Introduction

Mean Field Games were introduced independently by Lasry and Lions [21] and by Huang et al. [18] as limit models for symmetric non-zero-sum non-cooperative $N$-player dynamic games when the number $N$ of players tends to infinity. For an introduction to the topic see for e.g. [4], [9] or [2], where the latter two deal also with mean-field type optimal control. While a wide range of different classes of Mean Field Games has been considered up to now, here we focus on finite time horizon problems with continuous time dynamics under fully symmetric cost structure and complete information, where the position of each agent belongs to a finite state space.

The relation between the $N$-player game and its limit can be studied in two opposite directions: approximation and convergence. The approximation argument consists in proving that the solutions to the Mean Field Game allow to construct approximate Nash equilibria for the prelimit game, where the error in the approximation tends to zero as $N$ goes to infinity. Convergence goes in the opposite direction: are Nash equilibria for the $N$-player game converging to solutions of the Mean Field Game when the number of players goes to infinity?

Results in the first direction are much more common and easier to obtain: for the diffusive case without jumps see for instance [18], [6], [7] and [3]. In the finite state space setting, this was achieved in [1] studying the infinitesimal generator, while in [8] an approximation result was found through a fully probabilistic approach, which allowed for less restrictive assumptions on the dynamics and the optimization costs. For the convergence case results are fewer and even more recent: while the limits of $N$-player Nash equilibria in stochastic open-loop strategies can be completely characterized (see [20] and [14] for the diffusion case), the convergence problem for Nash equilibria in feedback form with full information is more difficult. A result in this direction is given by [16] in our finite state setting, via the infinitesimal generator, but only if the time horizon is small.

A breakthrough was achieved by Cardaliaguet et al. [5] through the use of the Master Equation, again in the continuous state space case. Their convergence argument relies on having a regular
solution to the Master Equation, which in the diffusion case is a kind of infinite dimensional transport equation on the space of probability measures. As we will see in the following, it is strongly related to the Mean Field Game system: its solution provides a solution to the Mean Field Game for any initial time and initial distribution. Conditioning upon having a regular solution to the Master Equation, the crucial ingredient for proving the convergence consists in a coupling argument, in a similar fashion to the propagation of chaos property for uncontrolled systems (see [17]). Such coupling, in which independent copies of the limit process are compared to their prelimit counterparts, ultimately allows to get the desired convergence of the value functions of the N-player game to the solution of the Master Equation.

In this paper, we focus on the convergence of feedback Nash equilibria in the finite state space scenario. We follow the approach of [5], showing the convergence of the value functions of the N-player game to the solution of the Master Equation. The argument provides also the convergence of the feedback Nash equilibria and a propagation of chaos property for the associated optimal trajectories. The coupling technique necessary for the proof is the main motivation for writing the dynamics of the N players as a stochastic differential equation driven by Poisson Random measures, as in [5].

In order to motivate the present work, we introduce the equations in play at a formal level. We begin by defining the dynamics of the N-player game, given by the following system of controlled SDEs:

\begin{equation}
X_i(t) = Z_i + \int_0^t \int_{\Xi} f(X_i(s^-), \xi, \alpha^i(s, X_i(s^-))) N_i(dt, d\xi),
\end{equation}

for \(i = 1, \ldots, N\), where each \(X_i(t)\) is a process taking values in the finite space \(\Sigma = \{1, \ldots, d\}\) and we denote by \(X_i := (X_i(t), \ldots, X_N(t))\) the vector of the N processes; \(N_i\) are N i.i.d. Poisson measures on \([0, T] \times \Xi\) with \(\Xi \subset \mathbb{R}^d\) and the controls \(\alpha^i \in A \subset \mathbb{R}^d\) are only in feedback form. With a particular choice of the function \(f\), the control \(\alpha_t^i(t, x, X_i^{N,i})\) represents the rate at which player \(i\) decides to go from state \(x\) to state \(y\), when \(x \neq y\) (\(X_i^{N,i}\) denotes the states of the other \(N - 1\) players).

Under our framework, we will show that there exists a unique feedback Nash equilibrium for the N-player game, provided by the Hamilton-Jacobi-Bellman system of \(N d\) coupled ODE’s

\begin{equation}
\begin{cases}
-\frac{\partial}{\partial t} N_i^j(t, x) - \sum_{j=1}^N \alpha^i(x, \Delta v^{N,i} \cdot \Delta v^{N,j}) + H(x, \Delta v^{N,i}) = F^N,i(x), \\
v^{N,i}(T, x) = G^{N,i}(x).
\end{cases}
\end{equation}

In the above equation, the costs \(F^N,i(x) = F(x, m_N^{i,N})\) and \(G^{N,i}(x) = G(x, m_N^{i,N})\) satisfy the mean field assumption (\(m_N^{i,N}\) being the empirical measure), \(H\) is the Hamiltonian and \(\alpha^i\) its unique maximizer, and \(\Delta v^{N,i}(x) := (g(x_1, \ldots, y, \ldots, x_N) - g(x_1, \ldots, x_j, \ldots, x_N))_{y=1,\ldots,d} \in \mathbb{R}^d\) denotes the finite difference of a function \(f(x) = f(x_1, \ldots, x_N)\) with respect to its \(j\)-th entry.

The study of convergence consists in finding a limit for the \(\text{HJB}\) system as \(N\) goes to infinity: thanks to the mean field assumptions on the costs, it is easy to prove that the N-player game enjoys similar symmetry properties which make the convergence problem more tractable. More precisely, the solution \(v^{N,i}\) of the \(\text{HJB}\) system can be found in the form \(v^{N,i}(t, x) = V^N(t, x, m_N^{i,N})\). At a formal level, we can introduce the limiting equation assuming the existence of a function \(U\) such that \(v^{N,i}(t, x) \sim U(t, x, m_N^{i,N})\) for large \(N\).

In order to get an intuition of what kind of equation would this \(U\) satisfy, let us study the different components of the HJB system. First, the term

\[
\Delta^i v^{N,i}(t, x) = \left( v^{N,i} \left(t, x_i, \frac{1}{N-1}\sum_{k \neq i} \delta x_k + \frac{1}{N-1}\delta y \right) - v^{N,i} \left(t, x_i, \frac{1}{N-1}\sum_{k \neq i} \delta x_k \right) \right)_{y=1,\ldots,d}
\]

For \(j \neq i\) we should instead get

\[
\Delta^j v^{N,j}(t, x) \sim \frac{1}{N-1} D^m U(t, x_i, m_N^{i,N}, x_j),
\]
modulo terms of order $O(1/N^2)$, where a precise definition of $D^mU$, the derivative with respect to a probability measure, will be given in the next section. Then, $H(x_i, \Delta^i v^{N,i}) \to H(x, \Delta^x U)$, so that we can now study the term:

$$
\sum_{j=1,j
ot=i}^N \alpha^j(x_j, \Delta^j v^{N,j}) \cdot \Delta^j v^{N,i} \sim \frac{1}{N-1} \sum_{j=1,j
ot=i}^N \alpha^i(x_j, \Delta^x U(t, x_j, m^{N,i}_x)) \cdot D_m U(t, x_i, m^{N,i}_x, x_j)
$$

$$
\sim \int_{\Sigma} \alpha^*(y, \Delta^x U(t, y, m^{N,i}_x)) D^m U(t, x_i, m^{N,i}_x, y) dm^{N,i}_x(y)
$$

$$
\to \int_{\Sigma} D^m U \cdot \alpha^*(y, \Delta^x U) dm(y).
$$

Finally, we are able to introduce the Master Equation, that is the equation to which we would like to prove convergence

\begin{equation}
(M)
\quad \begin{cases}
- \frac{\partial}{\partial t} + H(x, \Delta^x U) - \int_{\Sigma} D^m U(t, x, m, y) \cdot \alpha^*(y, \Delta^y U(t, y, m)) dm(y) = F(x, m),
\quad U(T, x, m) = G(x, m), \quad (x, m) \in \Sigma \times P(\Sigma),
\quad t \in [0, T].
\end{cases}
\end{equation}

It is a first order PDE stated in $P(\Sigma)$, the simplex of probability measures in $\mathbb{R}^d$. A similar equation, but stated in the whole space $\mathbb{R}^d$, was studied in [23], proving the well-posedness and regularity under strong monotonicity assumptions. Here we solve it using the method developed in [5]: under slightly weaker conditions, we show that the Mean Field Game system can be seen as the characteristic curves of \(M\) and linearize it around its solution.

Here, we also study the empirical measure process of the $N$-player optimal trajectories. Indeed, the convergence obtained allows to get a Central Limit Theorem and a Large Deviation Principle for the asymptotic behavior, as $N$ tends to infinity, of such processes. The key point for proving these results is to compare the prelimit optimal trajectories with the ones in which each player chooses the control induced by the Master Equation. The fluctuations are then found by analyzing the associated infinitesimal generator, while the Large Deviation properties are derived using a result in [12]. Finally, we mention that such properties are being studied in the diffusion case, independently, via the Master Equation approach, by Lacker et al. [11].

**Structure of the paper.** The rest of the paper is organized as follows. In Section 2, we start with the notations and the definition of derivatives in the simplex. Then we present the two sets of assumptions we make use of: one for the convergence and the fluctuations results while the other, stronger, for the regularity of the Master Equation; we also show an example in which the assumptions are satisfied. Finally, we give a detailed description of both the $N$-player game and the limit model. Section 3 contains the convergence results and their proofs, while in Section 4 we employ the convergence method to derive the probabilistic properties of the empirical measure process, that is, the Large Deviation Principle and the Central Limit Theorem. Section 5 analyzes the well-posedness and regularity of the solution to the Master Equation. We conclude with Section 6 by summarizing all the main results.

2. Model and Assumptions

2.1. Notations. Here we briefly clarify the notations used throughout the paper. First of all, we are considering $\Sigma = \{1, \ldots, d\}$ to be the finite state space of any player. Let $T$ be the finite time horizon and $A$ a compact metric space, the space of control values. Furthermore, we choose $A := [\kappa, M]^d$, for $\kappa, M > 0$; this guarantees for the dynamics of all players that $P(X(t) = x) > 0$ for every $x$ in $\Sigma$ and for all times $t$. Let $\Xi := [0, M]^d$ and $\nu$ be a Radon measure on $\Xi$. Denote by

$$
P(\Sigma) := \{ m \in \mathbb{R}^d : m_j \geq 0, \ m_1 + \cdots + m_d = 1 \}
$$

the space of probability measures on $\Sigma$. Besides the euclidean distance (denoted with $\cdot, \cdot$), we may interchangeably use the Wasserstein metric $d_1$ on $P(\Sigma)$ since all metrics are equivalent. We observe that the simplex $P(\Sigma)$ is a compact and convex subset of $\mathbb{R}^d$.

In the dynamics given by equation (1), $f : \Sigma \times \Xi \times A \to \{-d, \ldots, d\}$ has to be a measurable function such that $f(x, \xi, a) \in \{1 - x, \ldots, d - x\}$. Specifically, throughout the paper we set

\begin{equation}
(2)
f(x, \xi, a) := \sum_{y \in \Sigma} (y - x) \mathbb{I}_{(0,a)}(\xi_y).
\end{equation}
We also use the notation \( u(U) \)\(^{\text{(6)}} \)

Definition 1.
We say that a function \( u \) is continuous in probability if \( u(E) := \sum_{j=1}^{d} \ell(E \cap \Xi_j) \)
for any \( E \in \mathcal{B}(\Xi) \), where \( \ell \) is the Lebesgue measure on \( \mathbb{R} \) and \( \Xi_j := \{ u \in \Xi : u_i = 0 \ \forall i \neq j \} \).

The initial datum of the \( N \)-player game is represented by \( N \) i.i.d. random variables \( Z_1, \ldots, Z_N \) with values in \( \Sigma \) and distributed as \( m_0 \in \mathcal{P}(\Sigma) \). The vector \( Z = (Z_1, \ldots, Z_N) \) is in particular exchangeable, in the sense that the joint distribution is invariant under permutations, and is assumed to be \( \mathcal{F}_0 \)-measurable, i.e. independent of the noise.

The state of player \( i \) at time \( t \) is denoted by \( X_i(t) \), with \( X_t := (X_1(t), \ldots, X_N(t)) \). The trajectories of each \( X_i \) are in \( \mathcal{D}([0,T], \Sigma) \), the space of càdlàg functions from \([0,T] \) to \( \Sigma \) endowed with the Skorokhod metric. For \( x = (x_1, \ldots, x_N) \in \Sigma^N \), denote the empirical measures

\[
m_{\Sigma}^N(t) := \frac{1}{N} \sum_{j=1}^{N} \delta_{x_j} \quad m_{\Sigma}^{N,i}(t) := \frac{1}{N - 1} \sum_{j=1, j \neq i}^{N} \delta_{x_j}.
\]

Thus, \( m_{\Sigma}^N(t) := m_{\Sigma}^N \) is the empirical measure of the system of the \( N \) players and \( m_{\Sigma}^{N,i}(t) := m_{\Sigma}^{N,i} \) is the empirical measure of all the players except the \( i \)-th. Clearly, they are \( \mathcal{P}(\Sigma) \)-valued stochastic processes. In the limiting dynamics, the empirical measure is replaced by a deterministic flow of probability measures \( m : [0,T] \to \mathcal{P}(\Sigma) \).

In choosing his/her strategy, each player minimizes the sum of three costs: a Lagrangian \( L : \Sigma \times A \to \mathbb{R} \), a running cost \( F : \Sigma \times \mathcal{P}(\Sigma) \to \mathbb{R} \) and a final cost \( G : \Sigma \times \mathcal{P}(\Sigma) \to \mathbb{R} \) (see next section for the precise definition of the \( N \)-player game). The Hamiltonian \( H \) is defined as the Legendre transform of \( L \):

\[
H(x,p) := \sup_{\alpha \in A} \{-\alpha \cdot p - L(x, \alpha)\},
\]
for \( x \in \Sigma \) and \( p \in \mathbb{R}^d \).

Given a function \( g : \Sigma \to \mathbb{R} \) we denote its first finite difference \( \Delta g(x) \in \mathbb{R}^d \) by

\[
\Delta g(x) := \begin{pmatrix}
g(1) - g(x) \\
\vdots \\
g(d) - g(x)
\end{pmatrix}.
\]

When we have a function \( g : \Sigma^N \to \mathbb{R} \) we denote with \( \Delta_j g(x) \in \mathbb{R}^d \) the first finite difference with respect to the \( j \)-th coordinate. It is useful to observe that

\[
|\Delta^y g(x)| \leq \max_{y} |\Delta^y g(x)|_y \leq 2 \max_{x} |g(x)| \leq C|g(x)|.
\]

For a function \( u : [t_0, T] \times \Sigma \to \mathbb{R} \), we denote

\[
||u|| := \sup_{t \in [t_0, T]} \max_{x \in \Sigma} |u(t, x)|.
\]

We also use the notation \( u(t) := (u_1(t), \ldots, u_d(t)) = (u(t, 1), \ldots, u(t, d)) \).

We now introduce the concept of variation with respect to a probability measure \( m \) of a function \( U(m) \).

Definition 1. We say that a function \( U : \mathcal{P}(\Sigma) \to \mathbb{R} \) is differentiable if there exists a function \( D^m U : \mathcal{P}(\Sigma) \times \Sigma \to \mathbb{R}^d \) given by

\[
[D^m U(m, y)]_z := \lim_{s \to 0^+} \frac{U(m + s(\delta_z - \delta_y)) - U(m)}{s}
\]
for \( z = 1, \ldots, d \). Moreover, we say \( U \) is \( C^1 \) if the function \( D^m U \) is continuous in \( m \).
Morally, we can think of $[D^m U(m, y)]_z$ as the (right) directional derivative of $U$ with respect to $m$ along the direction $\delta_z - \delta_y$. We also observe that $m + s(\delta_z - \delta_y)$ might be outside of the probability simplex (e.g. when we are at the boundary), in which case we consider the limit only across admissible directions. However, note that, for our purposes this is not really a problem: since in the limit $m(t)$ will be the distribution of the reference player, the bound from below for the control ensures that the boundary of the simplex will never be touched.

Together with the definition, we state an identity which will come useful in the following sections:

$[D^m U(m, y)]_z = [D^m U(m, x)]_z + [D^m U(m, y)]_z.
$

Its derivation is an immediate consequence of the linearity of the directional derivatives.

We can easily extend the above definition to the case of derivative with respect to a direction $\mu \in P_0(\Sigma)$, with

$P_0(\Sigma) := \{ \mu \in \mathbb{R}^d : \mu_1 + \cdots + \mu_d = 0 \}.$

Indeed, an element $\mu = (\mu_1, \ldots, \mu_d) \in P_0(\Sigma)$ can be rewritten as a linear combination of $\delta_z - \delta_y$ as follows

$\mu = \sum_{z \neq y} \mu_z (\delta_z - \delta_y),$

for each $y \in \Sigma$, since $\sum_{z \neq y} \mu_z (\delta_z - \delta_y) = \sum_{z \neq y} \mu_z \delta_z - \left( \sum_{z \neq y} \mu_z \right) \delta_y,$ and $\sum_{z \neq y} \mu_z = -\mu_y$.

This remark allows us to define the derivative of $U(m)$ along the direction $\mu \in P_0(\Sigma)$ as a map $\partial \mu U : P(\Sigma) \times \Sigma \to \mathbb{R}$, defined by

$\partial \mu U(m, y) := \sum_{z \neq y} \mu_z [D^m U(m, y)]_z = \mu \cdot D^m U(m, y)$,

where the last equality comes from the fact that $[D^m U(m, y)]_y = 0$.

Finally, we note that the definition of $\partial \mu U(m, y)$ does not actually depend on $y$, i.e.

$\vartheta \mu U(m, 1) = \partial \mu U(m, 1)$

for every $y \in \Sigma$ and for this reason we will fix $y = 1$ when needed in the equations. Indeed, by means of identity (9) and the fact that $\mu \in P_0(\Sigma)$,

$\partial \mu U(m, 1) = \sum_{z = 1}^d \mu_z [D^m U(m, 1)]_z = \sum_{z = 1}^d ([D^m U(m, y)]_z + [D^m U(m, 1)]_y) \mu_z$

$= \sum_{z = 1}^d [D^m U(m, y)]_z \mu_z + [D^m U(m, 1)]_1 \sum_{z = 1}^d \mu_z = \sum_{z = 1}^d [D^m U(m, y)]_z \mu_z = \partial \mu U(m, y)$.

2.2. Assumptions. We now summarize the assumptions we make, which are different according on the results.

Because of the compactness of $A$, the continuity of $L$ with respect to its second argument is sufficient for guaranteeing the existence and finiteness of the supremum in (11) for each $(x, p)$. Moreover, we assume that there exists a unique maximizer $\alpha^*(x, p)$ in the definition of $H$ for every $(x, p)$:

$\alpha^*(x, p) := \arg \min_{\alpha \in A} \{ L(x, \alpha) + \alpha \cdot p \} = \arg \max_{\alpha \in A} \{ -L(x, \alpha) - \alpha \cdot p \}.$

With our choices for $f$ in (9) and the intensity measure $\nu$ in (9), a sufficient condition for the above assertion is given by the strict convexity of $L$ in $\alpha$ (see Lemma 3 in [9]). If $L$ is uniformly convex, such optimum $\alpha^*$ is globally Lipschitz in $p$, and whenever $H$ is differentiable it can be explicitly expressed as $\alpha^*(x, p) = -D_p H(x, p)$ (see Proposition 1 in [16] for the proof).

We will work with two sets of assumptions on $H$. We first observe that it is enough to give hypotheses for $H(x, )$ on a sufficiently big compact subset of $\mathbb{R}^d$, i.e. for $|p| \leq K$, because of the uniform boundedness of $\Delta t \nu_{N_1}$: see next section for details (Remark [1]). In what follows, the constant $K$ is fixed:

(H1) If $|p| \leq K$ then $H$ and $\alpha^*$ are Lipschitz continuous in $p$. 

We stress the fact that the above assumptions, together with the existence of a regular solution to (M), are alone sufficient for proving the convergence of the N-player game to the limiting mean-field game dynamics.

In order to establish the well-posedness and the needed regularity for the Master Equation we make use of the following additional assumptions:

(RegH) If $|p| \leq K$, $H$ is $C^2$ with respect to $p$, $H$, $D_pH$ and $D^2_{pp}H$ are Lipschitz in $p$ and its second derivative is bounded away from 0, i.e.

$$D^2_{pp}H(x, p) \geq C^{-1}. \tag{12}$$

(Mon) The cost functions $F$ and $G$ are monotone in $m$ in the Lasry-Lions sense, i.e., for every $m, m' \in P(\Sigma)$,

$$\sum_{x \in \Sigma}(F(x, m) - F(x, m'))(m(x) - m'(x)) \geq 0, \tag{13}$$

and the same holds for $G$.

(RegFG) The cost functions $F$ and $G$ are $C^1$ with respect to $m$, with $D^mF$ and $D^mG$ bounded and Lipschitz continuous. In this case (13) is equivalent to say that

$$\sum_{x} \mu(x)[D^mF(x, m, 1) \cdot \mu] \geq 0 \tag{14}$$

for any $m \in P(\Sigma)$ and $\mu \in P_0(\Sigma)$.

Observe that the assumptions on $H$ allow for quadratic Hamiltonian. As we will see, the above assumptions imply both the boundedness and Lipschitz continuity of $\Delta^2 x$ with respect to $m$ and the boundedness and Lipschitz continuity of $D^mU$ with respect to $m$. We conclude the section with an example for which all the assumptions are verified.

**Example 1.** The easiest example for the costs $F$ and $G$ would be $F(x, m) = G(x, m) = m(x)$. Slightly more in general, one can take $F(x, m) = \nabla \phi(m)(x)$, $\phi(m)$ being a convex function on $\mathbb{R}^d$.

For the choice of the Lagrangian $L$, a bit of work is needed in order to recover the regularity for $H$, since the maximization in the definition [H] of $H$ is performed only on the compact subset $A := [a, M]$.

Considering for simplicity $L(x, \alpha) \equiv L(\alpha)$, and set

$$L(\alpha) := b|\alpha - a|^2, \tag{15}$$

with $a := \frac{\epsilon M}{2}$, $(1, \ldots, 1)^T$ and $b$ a large enough constant to be chosen. The computation of $H := \sup_{\alpha \in [a, M]} \{-p \cdot \alpha - L(\alpha)\}$ for such choice of $L$ gives

$$H(p) = \frac{p^2}{2b} - a \cdot p, \tag{16}$$

for $|p| \leq \frac{bM}{2\epsilon}$, while, as one can check immediately, outside this interval $H$ is linear. While it is trivial to verify that with this choice $H$ is $C^1(\mathbb{R}^d)$ (and thus the local Lipschitz continuity assumption for $H$ is satisfied), the same does not hold for its second derivative because of the linear components. Nevertheless, the property (RegH) is satisfied whenever $|p| \leq K$, as we assumed, with the choice $b := \frac{2K}{M\epsilon}$. Finally, the Lipschitz continuity of $D_pH$ and $D^2_{pp}H$ is trivially holding because of the expression (16) for $|p| \leq K$ and the linearity outside.

### 2.3. N-Player Game

In this section we describe the N-player game in a general setting. Namely, we suppose that each individual has complete information about the states of all the other players and we do not require the players to be symmetric. Then, we show the relation between system (III) and the concept of Nash equilibria for the game through a classical Verification Theorem. To this end, neither hypotheses that guarantee the uniqueness of the Nash equilibrium are actually needed (e.g. strict convexity of the Lagrangian).

In the preliminent the dynamics are given by the system of $N$ controlled SDE

$$X_i(t) = Z_i + \int_0^t \int_{\mathbb{R}^d} f(X_i(s^-), \xi, \alpha^i(s, X_s^-))\mathcal{N}_i(ds, d\xi), \tag{17}$$

for $i = 1, \ldots, N$, where $f$ is given by (2) and $X_i = (X_1(t), \ldots, X_N(t))$. Each player is allowed to choose his control $\alpha^i$ having complete information on the state of the other players. We consider
only controls \( \alpha^N := (\alpha_1, \ldots, \alpha_N) \) in feedback form, i.e. the controls are deterministic functions of time and space \( \alpha^i : [0, T] \times \Sigma^N \rightarrow A \), \( \alpha^i = \alpha^i(t, x) \). We say that \( \alpha^i \in A \), for each \( i \) if it is a measurable function of time and denote as \( \mathcal{A}^N \) the set of feedback strategy vectors \( \alpha^N \).

We remark that the dynamics \( (17) \) is always well posed, for any choice of the control, since the state space is finite and the coefficients are then trivially Lipschitz continuous. Namely, for any \( \alpha^N \in \mathcal{A}^N \) there exists a unique strong solution to \( (17) \), in the sense that \((X_t)_{t \in [0, T]} \) is adapted to the filtration \( \mathbb{F} \) generated by the Poisson random measures.

With the choice of \( f \) in \( (2) \) and the intensity measure \( \nu \) in \( (3) \), the dynamics remain in \( \Sigma \) for any time and the feedback controls are exactly the transition rates of the continuous time Markov chains \( X_i(t) \). Indeed, one can prove that (see \( \mathbb{E} \)), for \( x \neq y \),

\[
\mathbb{P}[X_i(t + h) = y \mid X_i(t) = x] = \mathbb{E}[\alpha^i_y(t, x, X_i^{N,i})] h + o(h).
\]

So whenever \( \alpha \) denotes the rates of the Markov chain, we will set \( \alpha^i_x(x) = -\sum_{y \neq x} \alpha^i_y(x) \).

Next, we define the object of the minimization. Let \( \alpha^N = (\alpha_1, \ldots, \alpha_N) \in \mathcal{A}^N \) be a strategy vector and \( X = (X_1, \ldots, X_N) \) the corresponding solution to \( (1) \). For \( i = 1, \ldots, N \) and given functions \( F^{N,i}, G^{N,i} : \Sigma^N \rightarrow \mathbb{R} \), we associate to the \( i \)-th player the cost functional

\[
J^N_i(\alpha^N) := \mathbb{E} \left[ \int_0^T \left( L(X_i(t), \alpha^i(t, X_i)) + F^{N,i}(X_i(t)) \right) dt + G^{N,i}(X_T) \right].
\]

The optimality condition for the \( N \)-player game is given by the usual concept of Nash equilibria. For a strategy vector \( \alpha^N = (\alpha_1, \ldots, \alpha_N) \in \mathcal{A}^N \) and \( \beta \in A \), denote by \( [\alpha^{N,-i}; \beta] \) the perturbed strategy vector given by

\[
[\alpha^{N,-i}; \beta]_j := \begin{cases} 
\alpha_j, & j \neq i \\
\beta, & j = i.
\end{cases}
\]

Then, we can introduce the following

**Definition 2.** A strategy vector \( \alpha^N \) is a Nash equilibrium for the \( N \)-player game if for each \( i = 1, \ldots, N \)

\[
J^N_i(\alpha^N) = \inf_{\beta \in A} J^N_i([\alpha^{N,-i}; \beta]).
\]

We consider now the value functions of the problem \( v^{N,i}(t, x) \) and define

\[
J^N_i(t, x, \alpha^N) := \mathbb{E} \left[ \int_t^T L(X_i^t, (s), \alpha^i(s, X^t_s)) + F^{N,i}(X^t_s) \right] ds + G^{N,i}(X^T_T) \right],
\]

\[
X^t_s(x) = x_i + \int_t^s f(X^t_r(x^-), \xi, \alpha^i(r, X^t_r)) N_i(ds, d\xi) \quad s \in [t, T].
\]

We work under hypotheses that guarantee the existence of a unique maximizer \( \alpha^*(x, p) \) defined in \( (11) \). With these notations, the Hamilton-Jacobi-Bellman system associated to the above differential game is given by:

\[
\frac{\partial v^{N,i}}{\partial t} + \inf_{a \in A} \left\{ \sum_{j=1}^N \sum_{y=1}^d \alpha^*_y(x_j, \Delta^j v^{N,j}) [\Delta^j v^{N,i}(t, x)]_y + L(x_i, a) + F^{N,i}(x) \right\} = 0.
\]

Using the uniqueness of the maximizer \( \alpha^*(x, p) \), Equation \( (20) \) can be rewritten as the system (HJB) presented in the introduction

\[
\left\{ -\frac{\partial v^{N,i}}{\partial t}(t, x) - \sum_{j=1}^N \sum_{y=1}^d \alpha^*(x_j, \Delta^j v^{N,j}) \cdot \Delta^j v^{N,i}(t, x) + H(x_i, \Delta^i v^{N,i}) = F^{N,i}(x), \\
v^{N,i}(T, x) = G^{N,i}(x).
\]

This is a system of \( Nd^N \) coupled ODE’s, whose well-posedness for all \( T > 0 \) can be proved through standard ODEs techniques, because of the Lipschitz continuity of the vector fields involved in the equations.

We are now able to relate system (HJB) to the Nash equilibria for the \( N \)-player game through the following
Proposition 1 (Verification Theorem). Let \( v^{N,i}, i = 1, \ldots, N \) be a classical solution to system (HJB). Then the feedback strategy vector

\[
\alpha^{i,*}(t, x) := \alpha^*(x_i, \Delta^i v^{N,i}(t, x)) \quad i = 1, \ldots, N,
\]
defines a unique Nash equilibrium for the \( N \)-player game and

\[
v^{N,i}(t, x) = J^N_i(t, x, \alpha^{N,i}) = \inf_{\beta \in \mathcal{A}} J^N_i(t, x, [\alpha^{N,-i}, \beta]).
\]

are the value functions.

Proof. Let \( \beta \in \mathcal{A} \) be any feedback and \( X^{t,x} \) the corresponding solution to (17), given the strategy vector \( [\alpha^{N,-i}, \beta] \); denote for simplicity \( X = X^{t,x} \).

Fixing \( i = 1, \ldots, N \), Equation (20) gives

\[
\frac{\partial v^{N,i}}{\partial t} + \sum_{j=1}^{N} \sum_{y=1}^{d} \alpha^y_j(t, x_j, \Delta^j v^{N,j}(t, x_j)) |\Delta^j v^{N,i}(t, x_j)| y + L(x_i, \beta(t, x)) + F^{N,i}(x) \geq 0
\]

for any \( t, x \). Applying first Itô formula (Theorem II.5.1 in [19], p. 66) and then Lemma 3 in [8] and the above inequality, we obtain

\[
v^{N,i}(t, x) = \mathbb{E} \left[ v^{N,i}(T, X_T) - \int_t^T \frac{\partial v^{N,i}}{\partial t}(s, X_s) ds \right]
- \sum_{j=1}^{N} \mathbb{E} \int_t^T \int_\mathcal{C} \nu(ds, X_s) \nu(d\xi) ds
- \left[ v^{N,i}(T, X_T) - \int_t^T \left( \frac{\partial v^{N,i}}{\partial t}(s, X_s) + \sum_{j=1}^{N} \alpha^{j,*}(s, X_s) \cdot \Delta^j v^{N,i}(s, X_s) \right) ds \right]
\leq \mathbb{E} \left[ G^{N,i}(T, X_T) + \int_t^T (L(x_i, \beta(s, X_s)) + F^{N,i}(X_s)) ds \right]
\]

Replacing \( \beta \) by \( \alpha^{i,*} \) the inequalities become equalities. \( \square \)

Remark 1. It is important to observe that the solution \( v^{N,i} \) to (HJB) is uniformly bounded with respect to \( N \). Namely, there exists a constant \( K > 0 \) such that \( \sup_{x \in \Sigma^N} |v^{N,i}(t, x)| \leq K \), where the constant \( K \) is independent of \( N \) and \( t \). This and (5) immediately imply an analogous bound for \( |\Delta^i v^{N,i}(t, x)| \); it is for this reason that the only local regularity (assumptions (H1) and (RegH)) for \( H(x, p) \) with respect to \( p \) is enough for getting the convergence and well-posedness results.

We are interested in studying the limit of the (HJB) system as \( N \to \infty \) under symmetric properties for the \( N \)-player game. Namely, we assume that the players are all identical and indistinguishable. In practice, this symmetry is expressed through the following mean-field assumption on the costs:

\[
F^{N,i}(x) = F(x_i, m^{N,i}_x),
\]
\[
G^{N,i}(x) = G(x_i, m^{N,i}_x).
\]

An easy but crucial consequence of assumption (M-F) and the uniqueness of solution to system (HJB) is that the solution \( v^{N,i} \) of such system enjoys symmetric properties:

Proposition 2. Under the mean-field assumptions (M-F), the solution \( v^{N,i} \) to system (HJB) satisfies, for \( i = 1, \ldots, N \),

\[
v^{N,i}(t, x) = v^N(t, x_i, (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)),
\]

with \( v^N : [0, T] \times \Sigma \times \Sigma^{N-1} \to \mathbb{R} \) such that, for any \( (t, x) \in [0, T] \times \Sigma \), the function

\[
\Sigma^{N-1} \ni (y_1, \ldots, y_{N-1}) \to v^N(t, x, (y_1, \ldots, y_{N-1})
\]

is invariant under permutations of \((y_1, \ldots, y_{N-1})\).
Proof. Let \( \tilde{x} \) be defined from \( x \) after exchanging \( x_k \) with \( x_j \), for \( j \neq k \neq i \). Because of (\text{M-F}), we have that \( F^{N,i}(x) = F^{N,i}(\tilde{x}) \) and \( G^{N,i}(x) = G^{N,i}(\tilde{x}) \) and thus, by the uniqueness of solution to \( (\text{HJB}) \) we conclude \( v^{N,i}(t, x) = v^{N,i}(t, \tilde{x}) \). \( \square \)

The above proposition motivates the study of a possible convergence of system \( (\text{HJB}) \) to a limiting system, by analyzing directly the limit of the function \( v^N \).

### 2.4. Mean Field Game and Master Equation

The limit as \( N \to \infty \) would be expectedly be characterized by a continuum of players in which the representative agent evolves according to the dynamics

\[
X(t) = Z + \int_0^t \int_{\mathbb{X}} f(X(s^-), \xi, \alpha(s, X(s^-))) \mathcal{N}(ds, d\xi), \quad t \in [0, T],
\]

where the law of the initial condition \( Z \) is \( m_0 \) and \( \mathcal{N} \) is a Poisson Random measure with intensity measure \( \nu \). The controls are feedbacks in \( \mathcal{A} \), which denotes the space of measurable functions \( \alpha : [0, T] \times \Sigma \to \mathcal{A} \). The empirical measure of the \( N \) players is replaced by a deterministic flow of probability measures \( m : [0, T] \to \mathcal{P}(\Sigma) \). The associated cost is

\[
J(\alpha, m) := \mathbb{E} \left[ \int_0^T [L(X(t), \alpha(t, X(t))) + F(X(t), m(t))] ds + G(X(T), m(T)) \right].
\]

In literature, such limiting dynamics are described by the celebrated Mean Field Game system, whose unknowns are two functions \((u, m)\). The equation in \( u \) describes the dynamics of the value function of the reference player, which optimizes his/her payoff under the influence of the collective behavior of the others, while the equation in \( m \) describes the evolution of the distribution of the players. In our discrete setting the Mean Field Game system takes the following form of a strongly coupled system of ODEs:

\[
\begin{align*}
\frac{du}{dt}(t, x) + H(x, \Delta^x u(t, x)) &= F(x, m(t)), \\
\frac{dm}{dt}(t) &= \sum_{y} m_y(t) \alpha^*(y, \Delta^y u(t, y)),
\end{align*}
\]

(MFG)

with \( \alpha^*(x, \cdot) \) as defined in (\text{MFG}) and \( u, m : [0, T] \times \Sigma \to \mathbb{R} \). From a probabilistic point of view the solution \((u, m)\) can be seen as a fixed point: starting with a flow \( m \) solve the first equation, the backward Hamilton-Jacobi-Bellman equation for \( u \), this yields a unique optimal feedback control \( \alpha \) for \( m \), then impose that the flow of the corresponding solution \( X \) to (24) is exactly \( m \), giving thus the second equation, the forward Kolmogorov-Fokker-Planck for \( m \). Hence, given a solution \((u, m)\) to (MFG), we have \( J(\alpha, m) \leq J(\beta, m) \) for any \( \beta \in \mathcal{A} \), where \( \alpha(t, x) = \alpha^*(x, \Delta^x u(t, x)) \), and \( \text{Law}(X(t)) = m(t) \) for any \( t \in [0, T] \).

As already mentioned, recently in [5] a new technique involving the use of the so-called Master Equation was introduced to get the exact relation between symmetric \( N \)-Player Differential Games and Mean Field Games. Generally speaking, the Master Equation summarizes all the information needed to get solutions to the Mean Field Game: namely, one can prove that the system (\text{MFG}) provides the characteristics curves for \( (M) \). Indeed, \( U(t_0, x, m_0) := u(t_0, x) \) solves (M), \((u, m)\) being the solution to the Mean Field Game system (\text{MFG}) starting at time \( t_0 \) up to time \( T \), with \( m(t_0) = m_0 \). Moreover, in the Introduction we already motivated heuristically the convergence result of system (\text{HJB}) to the Master Equation (\text{M}). As it will be clear from the convergence argument, all that is needed is the existence of a regular solution to (\text{M}).

To be specific on the needed regularity, we conclude this section with the definition of regular solution to (\text{M}).

**Definition 3.** A function \( U : [0, T] \times \Sigma \times \mathcal{P}(\Sigma) \to \mathbb{R} \) is said to be a classical solution to (\text{M}) if it is continuous in all its arguments, \( C^1 \) in \( t \) and \( C^1 \) in \( m \) and, for any \((t, x, m) \in [0, T] \times \Sigma \times \mathcal{P}(\Sigma) \) we have

\[
\begin{cases}
- \frac{\partial U}{\partial t} + H(x, \Delta^x U) - \int_{\Sigma} D^m U(t, x, m, y) \cdot \alpha^*(y, \Delta^y U(t, y, m)) dm(y) = F(x, m), \\
U(T, x, m) = G(x, m), \quad (x, m) \in \Sigma \times \mathcal{P}(\Sigma).
\end{cases}
\]
Theorem 1. The main result is given by the following

\[ \Delta^x U(t, x, \cdot) : P(\Sigma) \to \mathbb{R}^d \text{ is bounded and } \text{Lipschitz continuous and } D^m U(t, x, \cdot) : P(\Sigma) \to \mathbb{R}^{d \times d} \text{ is bounded.} \]

Moreover, we say that \( U \) is a regular solution to (M) if it is a classical solution and \( D^m U(t, x, \cdot) \) is also Lipschitz continuous in \( m \), uniformly in \( (t, x) \).

Let us observe that in the Master equation we could replace \( D^m U(t, x, m, y) \) by \( D^m U(t, x, m, 1) \), thanks to property (8) of the derivative.

3. The convergence argument

In this section we take for granted the well-posedness of the Master Equation (M) and focus on the study of the convergence. We give the precise statement of the convergence in terms of two theorems: the first one describes the convergence in average of the solutions of the two equations, while the second one is a propagation of chaos for the optimal trajectories.

For any \( i \in \{1, \ldots, N\} \) and \( x \in \Sigma \), set

\[ w^{N,i}(t_0, x, m_0) := \sum_{x_1=1}^d \cdots \sum_{x_{i-1}=1}^d \sum_{x_{i+1}=1}^d \cdots \sum_{x_N=1}^d U^{N,i}(t_0, x) \prod_{j \neq i} m_0(x_j), \]

where \( x = (x_1, \ldots, x_N) \), and

\[ ||w^{N,i}(t_0, \cdot, m_0) - U(t_0, \cdot, m_0)||_{L^1(m_0)} := \sum_{x=1}^d |w^{N,i}(t_0, x, m_0) - U(t_0, x, m_0)| m_0(x). \]

The main result is given by the following

**Theorem 1.** Assume (H1) and that (M) admits a unique regular solution \( U \) in the sense of Definition 1. Fix \( N \geq 1 \), \( (t_0, m_0) \in [0, T] \times P(\Sigma) \), \( x \in \Sigma^N \) and let \( (w^{N,i}) \) be the solution to (HJB). Then

\[ \frac{1}{N} \sum_{i=1}^N |w^{N,i}(t_0, x) - U(t_0, x, m_N)| \leq \frac{C}{N} \]

(26)

\[ ||w^{N,i}(t_0, \cdot, m_0) - U(t_0, \cdot, m_0)||_{L^1(m_0)} \leq \frac{C}{\sqrt{N}}. \]

In (26) and (27), the constant \( C \) does not depend on \( i, t_0, m_0 \) nor \( N \).

As stated above, the convergence can be studied also in terms of the optimal trajectories. Consider the optimal trajectories \( Y_i = (Y_i(t), \ldots, Y_i(t)) \in \mathbb{R}^d \) for the \( N \)-player game:

\[ Y_i(t) = Z_i + \int_0^t \int_{\Sigma} \sum_{y \in \Sigma} (y - Y_i(s^-)) \mathbb{1}_{[0, \alpha_i(s, Y_y^{\cdot} - ) t_0, \xi]} N_1(ds, d\xi), \quad t \in [t_0, T] \]

(28)

where \( \alpha_i(t, Y_i) \) is the optimal feedback, i.e. \( \alpha_i(t, Y_i) := [\alpha^*(y_1, \Delta^x U^{N,i}(t, y))]_y \). Moreover, let \( \tilde{X}_i = (\tilde{X}_i(t), \ldots, \tilde{X}_i(t)) \) be the i.i.d. process solution to

\[ \tilde{X}_{i,t} = Z_i + \int_0^t \int_{\Sigma} \sum_{y \in \Sigma} (y - \tilde{X}_i(s^-)) \mathbb{1}_{[0, \tilde{\alpha}_i(s, \tilde{X}_y^{\cdot} - ) t_0, \xi]} N_1(ds, d\xi), \quad t \in [t_0, T] \]

(29)

with \( \tilde{\alpha}_i(t, \tilde{X}_i) := [\alpha^*(\tilde{X}_i(t), \Delta^x U(t, \tilde{X}_i(t), \text{Law}(\tilde{X}_i(t))))]_y \). We remark that \( \text{Law}(\tilde{X}_i(t)) = m(t) \) with \( m \) the solution to the Mean Field Game.

**Theorem 2.** Under the same hypotheses of Theorem (1), for any \( N \geq 1 \) and any \( i \in \{1, \ldots, N\} \), we have

\[ \mathbb{E} \left[ \sup_{t \in [t_0, T]} |Y_i(t) - \tilde{X}_i(t)| \right] \leq CN^{-n} \]

(30)

for some constant \( C > 0 \) independent of \( t_0, m_0 \) and \( N \). In particular we obtain the Law of Large numbers

\[ \mathbb{E} \left[ \sup_{t \in [t_0, T]} |m^{\tilde{X}_i}(t) - m(t)| \right] \leq CN^{-n}. \]

(31)
3.1. Approximating the optimal trajectories. The first step in the proof of these results is to show that the projection of $U$ onto empirical measures

\[ u^{N,i}(t,x) := U(t,x_i,m_x^{N,i}) \]

satisfies the system \[ \text{HJB} \] up to a term of order $O\left(\frac{1}{N^3}\right)$. The following proposition makes rigorous the intuition we already used in the heuristic derivation of the Master Equation (M).

**Proposition 3.** Let $U$ be a regular solution to the Master Equation and $u^{N,i}(t,x)$ be defined as in (32). Then

\[ \Delta^i u^{N,i}(t,x) = \frac{1}{N-1} D^m U(t,x_i,m_x^{N,i},x_j) + O\left(\frac{1}{(N-1)^2}\right). \]

**Proof.** By definition, $[\Delta^i u^{N,i}(t,x)]_h = U(t,x_i, \frac{1}{N-1} \sum_{k \neq i,j} \delta_{x_k} + \frac{1}{N-1} \delta_{x_j}) - U(t,x_i,m_x^{N,i})$.

By standard computations we get

\[
U \left( t, x_i, \frac{1}{N-1} \sum_{k \neq i,j} \delta_{x_k} + \frac{1}{N-1} \delta_{x_j} \right) - U(t,x_i,m_x^{N,i})
\]

\[
= \int_0^{\frac{1}{N-1}} [D^m U(m_x^{N,i} + s(\delta_{x_h} - \delta_{x_j})), x_j]_h ds
\]

\[
= \int_0^{\frac{1}{N-1}} \left( [D^m U(m_x^{N,i} + s(\delta_{x_h} - \delta_{x_j})), x_j]_h + [D^m U(m_x^{N,i}, x_j)]_h - [D^m U(m_x^{N,i}, x_j)]_h \right) ds
\]

\[
= \frac{1}{N-1} [D^m U(t,x_i,m_x^{N,i},x_j)]_h + \int_0^{\frac{1}{N-1}} \left( [D^m U(m_x^{N,i} + s(\delta_{x_h} - \delta_{x_j})), x_j]_h - [D^m U(m_x^{N,i}, x_j)]_h \right) ds
\]

where the last equality is derived by exploiting the Lipschitz continuity in $m$ of $D^m U$.

For every component $h$ of $D^m U$ we have found the thesis, and thus the same holds for the whole vector. \qed

In what follows, $C$ will denote any constant independent of $i,N,t_0,m_0$ which is allowed to change from line to line. In the next proposition we show that the $u^{N,i}$’s almost solve the system \[ \text{HJB} \]:

**Proposition 4.** The functions $(u^{N,i})_{i=1,...,N}$ solve

\[
\begin{cases}
-\frac{\partial u^{N,i}}{\partial t} (t,x) = -\sum_{j=1}^N \int_{[0,T]} \alpha^j(x_j,\Delta^j u^{N,j}) \cdot \Delta^j u^{N,j} + H(x_i,\Delta^i u^{N,i}) = F^{N,i}(x) + r^{N,i}(t,x) \\
u^{N,i}(T,x) = G(x_i,m_x^{N,i}),
\end{cases}
\]

with $r^{N,i} \in C^0([0,T] \times \Sigma_N)$, $||r^{N,i}|| \leq \frac{C}{N}$.

**Proof.** We know that $U$ solves

\[
-\partial U + H(x,\Delta^x U(t,x,m)) = \int_{\Sigma} D^m U \cdot \alpha^*(y,\Delta^x U)dm(y) = F(x,m),
\]

and $U(T,x,m) = G(x,m)$. 

\section*{Finite State MFG and the Master Equation}
Computing the equation in \((t, x_i, m^N_{x_i})\) we get (we omit the \(*\) in \(\alpha^*\) for simplicity)

\[-\partial_t U(t, x_i, m^N_{x_i}) + H(x_i, \Delta^2 U(t, x_i, m^N_{x_i})) \]

\[-\int \mathcal{D}^m U(t, x_i, m^N_{x_i}, y) \cdot \alpha(y, \Delta^2 U(t, y, m^N_{x_i}) \cdot dm^N_{x_i}(y) = F(x_i, m^N_{x_i}),\]

with the correct final condition \(u^N_{x_i}(t, x) = U(T, x_i, m^N_{x_i}) = G(x_i, m^N_{x_i}) = G^N_{x_i}(x)\). By definition of empirical measure we can rewrite

\[-\partial_t U(t, x_i, m^N_{x_i}) + H(x_i, \Delta^2 U(t, x_i, m^N_{x_i})) \]

\[-\frac{1}{N-1} \sum_{j \neq i} D^m U(t, x_i, m^N_{x_i}, x_j) \cdot \alpha(x_j, \Delta^2 U(t, x_j, m^N_{x_i})) = F^N_{x_i}(x).\]

Recalling that, by Proposition 3, we want to obtain in the equation for

1) \(A\) \(\leq \sum_{j \neq i} \Delta^j u^N_{x_i}(t, x) \cdot (\Delta^j U(t, x_j, m^N_{x_i}) - \Delta^j U(t, x_j, m^N_{x_i}))\)

\[\leq C \sum_{j \neq i} ||\Delta^j u^N_{x_i}||_{\infty} d(m^N_{x_i}, m^N_{x_j})\]

\[\leq \frac{C}{N-1} \sum_{j \neq i} ||\Delta^j u^N_{x_i}|| = C ||\Delta^j u^N_{x_i}|| \leq \frac{C}{N}\]

where the last inequality is a consequence of Proposition 3 for which \(\Delta^j u^N_{x_i} = \frac{1}{N-1} \Delta_m U + O \left( \frac{1}{N^2} \right)\), and by the bound on \(||D^m U||_{\infty}\) for the solution to \([M]\). Part (B) of 1) is instead what we want to obtain in the equation for \(u^N_{x_i}\), so we leave it as it is.

For the term 2), we simply note that \(\alpha\) is bounded from above by definition, and thus the whole term 2) is also of order \(O \left( \frac{1}{N} \right)\).

The central part of the proof of the convergence is based on comparing the optimal trajectories associated to \(v^N_{x_i}\) with the ones associated to \(u^N_{x_i}\). Hence, consider the process

\[X_i(t) = Z_i + \int_0^t \int_{\Sigma} \sum_{j \neq i} (y - X_i(t^-)) 1_{[0, \alpha^*(t, x, \xi)]}(t, \xi) t^N_i dt, d\xi,\]

\[(34)\]
where \( \tilde{\alpha}_i^j(t, X_i) = [\alpha^*(X_i(t), \Delta^i u^{N,i}(t, X_i))]_y \). Observe that the processes \( X \) and \( Y \) are exchangeable. For future use, let us also recall the inequality

\[
|m^N_x - m^N_y| \leq \frac{C}{N} \sum_{i=1}^{N} |x_i - y_i|
\]

for every \( x, y \in \Sigma^N \). The result needed to prove the main theorems is the following

**Theorem 3.** With the notations introduced above, under the assumptions of Theorem 1, we have

\[
\begin{align*}
\mathbb{E} \left[ \sup_{t \in [0, T]} |Y_i(t) - X_i(t)| \right] & \leq \frac{C}{N}, \\
\mathbb{E} \left[ \sup_{t \in [0, T]} |m^N_Y(t) - m^N_X(t)| \right] & \leq \frac{C}{N}, \\
\mathbb{E} \left[ \sup_{t \in [0, T]} |\alpha^{N,i}(t, Y_i) - \nu^{N,i}(t, Y_i)|^2 + \int_0^T \left| \Delta^i \nu^{N,i}(t, Y_i) - \Delta^i \nu^{N,i}(t, Y_i) \right|^2 dt \right] & \leq \frac{C}{N^2}, \\
\frac{1}{N} \sum_{i=1}^{N} |v^{N,i}(0, Z) - u^{N,i}(0, Z)| & \leq \frac{C}{N} \mathbb{P}\text{-a.s.}
\end{align*}
\]

**Proof.** In order to prove (38), we apply the Itô Formula to the function \( \Psi(t, Y_i) = (u^{N,i}(t, Y_i) - \nu^{N,i}(t, Y_i))^2 \),

\[
d\Psi(t, Y_i) = \Psi(t, Y_i) + \sum_{j=1}^{N} \int_{\Xi} [\Psi(t, \tilde{Y}^j_i) - \Psi(t, Y_{i-})]N_j(dt, d\xi),
\]

where

\[
\tilde{Y}^j_i = \left( Y_{1,t}, \ldots, Y_{j-1,t}, Y_{j,t} + \sum_{y \in \Sigma \in \alpha^j}(y - Y_{j,t})[\Pi_{y \in \alpha^j}(\xi_y), Y_{j+1,t}, \ldots, Y_{N,t}] \right),
\]

and, as above,

\[
\alpha^j_y(t, Y_i) = [\alpha^*(Y_{j,t}, \Delta^j u^{N,j}(t, Y_i))]_y.
\]

It follows that,

\[
d\Psi(t, Y_i) = 2(u^{N,i}(t, Y_i) - \nu^{N,i}(t, Y_i))(\partial_t u^{N,i} - \partial_t \nu^{N,i})
\]

\[
+ \sum_{j=1}^{N} \int_{\Xi} [(u^{N,i}(t, \tilde{Y}^j_i) - \nu^{N,i}(t, \tilde{Y}^j_i))^2 - (u^{N,i}(t, Y_{i-}) - \nu^{N,i}(t, Y_{i-}))^2]N_j(dt, d\xi).
\]

Now, integrating on the time interval \([t, T]\) we get:

\[
[u^{N,i}(T, Y_T) - \nu^{N,i}(T, Y_T)]^2 =
\]

\[
= [u^{N,i}(t, Y_t) - \nu^{N,i}(t, Y_t)]^2 + 2 \int_t^T (u^{N,i}(s, Y_s) - \nu^{N,i}(s, Y_s))(\partial_t u^{N,i}(s, Y_s) - \partial_t \nu^{N,i}(s, Y_s))ds
\]

\[
+ \sum_{j=1}^{N} \int_t^T ds \int_{\Xi} [(u^{N,i}(s, \tilde{Y}^j_s) - \nu^{N,i}(s, \tilde{Y}^j_s))^2 - (u^{N,i}(s, Y_{s-}) - \nu^{N,i}(s, Y_{s-}))^2]N_j(ds, d\xi).
\]

For brevity, for the remaining part of the proof we set \( u^j := u^{N,i}(t, Y_i) \) and \( v^j := \nu^{N,i}(t, Y_i) \). Next, we take the conditional expectation on the initial data \( Z \), i.e. \( \mathbb{E}^Z = \mathbb{E}[\cdot | Y_i = Z] \); notice that we are allowed to condition on such event since it has positive probability, thanks to the bound from
below of the jump rates. Applying again Lemma 3 of [8], we obtain

\[ E[Z[(u^i_t - v^i_t)^2]] = E[Z[(u^i_t - v^i_t)^2]] + 2E[Z \left[ \int_t^T (u^i_s - v^i_s)(\partial_t u^i_s - \partial_t v^i_s)ds \right] + \sum_{j=1}^{N} E[Z \left[ \int_t^T ds \alpha^i(s, Y_s) \cdot \Delta^j[(u^i_s - v^i_s)^2] \right]. \]

Let us first study the term \( E[Z \left[ \int_t^T (u^i_s - v^i_s)(\partial_t u^i_s - \partial_t v^i_s)ds \right]. \) Applying equations (33) and (HJB) we get:

\[ E[Z \left[ \int_t^T (u^i_s - v^i_s)(\partial_t u^i_s - \partial_t v^i_s)ds \right] = E[Z \left[ \int_t^T ds(u^i_s - v^i_s) \left\{ \sum_{j=1}^{N} \left( -\alpha^j(Y_{j,s}, \Delta^j u^i_s) \cdot \Delta^j u^i_s + \alpha^j(Y_{j,s}, \Delta^j v^i_s) \cdot \Delta^j v^i_s + \alpha^j \cdot \Delta^j u^i_s - \alpha^j \cdot \Delta^j v^i_s \right) \right. \right. \right. \]

\[ + H(Y_{i,s}, \Delta^j u^i_s) + H(Y_{i,s}, \Delta^j v^i_s) - r_{N,i}(s, Y_s) \right\]. \]

Recall that \( \alpha^j(\Delta^j u^i_s) = \Delta \alpha^j. \) Note that we also added and subtracted \( \alpha^j \cdot \Delta^j u^i_s \) in the last line so that we can use the lipschitz properties of \( H, D_p H \) and the bound on \( r_{N,i} \) to get the correct estimates (as in [5]). Specifically, we can rewrite

\[ E[Z \left[ \int_t^T (u^i_s - v^i_s)(\partial_t u^i_s - \partial_t v^i_s)ds \right] = E[Z \left[ \int_t^T ds(u^i_s - v^i_s) \left\{ \sum_{j=1, j \neq i}^{N} \left( (\alpha^j - \Delta \alpha^j) \cdot \Delta^j u^i_s - \alpha^j \cdot (\Delta^j u^i_s - \Delta^j v^i_s) \right) + \right. \right. \right. \]

\[ - H(Y_{i,s}, \Delta^j u^i_s) + H(Y_{i,s}, \Delta^j v^i_s) - r_{N,i}(s, Y_s) \right]\].

Putting things together,

\[ E[Z[(u^i_t - v^i_t)^2]] = E[Z[(u^i_t - v^i_t)^2]] + 2E[Z \left[ \int_t^T (u^i_s - v^i_s)(\partial_t u^i_s - \partial_t v^i_s)ds \right] + \sum_{j=1}^{N} E[Z \left[ \int_t^T ds \alpha^i(s, Y_s) \cdot \Delta^j[(u^i_s - v^i_s)^2] \right] \]

\[ = E[Z[(u^i_t - v^i_t)^2]] + 2E[Z \left[ \int_t^T ds(u^i_s - v^i_s) \left\{ \sum_{j=1, j \neq i}^{N} \left( (\alpha^j - \Delta \alpha^j) \cdot \Delta^j u^i_s - \alpha^j \cdot (\Delta^j u^i_s - \Delta^j v^i_s) \right) + \right. \right. \right. \]

\[ - H(Y_{i,s}, \Delta^j u^i_s) + H(Y_{i,s}, \Delta^j v^i_s) - r_{N,i}(s, Y_s) \right]\] + E[Z \left[ \int_t^T ds \alpha^i(s, Y_s) \cdot \Delta^j[(u^i_s - v^i_s)^2] \right] \]

\[ = E[Z[(u^i_t - v^i_t)^2]] + 2E[Z \left[ \int_t^T ds(u^i_s - v^i_s) \left\{ \sum_{j=1}^{N} ((\alpha^j - \Delta \alpha^j) \cdot \Delta^j u^i_s - \alpha^j \cdot (\Delta^j u^i_s - \Delta^j v^i_s)) \right) \right. \]

\[ + \sum_{j \neq i} \frac{1}{2} \alpha^j \cdot \Delta^j[(u^i_s - v^i_s)^2]ds + \int_t^T (u^i_s - v^i_s)(-H(Y_{i,s}, \Delta^j u^i_s) + H(Y_{i,s}, \Delta^j v^i_s) - r_{N,i}(s, Y_s))ds \]

\[ + E[Z \left[ \int_t^T ds \alpha^i(s, Y_s) \cdot \Delta^j[(u^i_s - v^i_s)^2] \right]. \]
On the other hand, observing that $\Delta^j[(u^i - v^i)^2] = \Delta^j(u^i - v^i) \times (\Delta^j(u^i - v^i) + 2(1(u^i - v^i)))$ (× being the element by element product between vectors and $1 = (1, \ldots, 1)^T$), the expression

$$
\mathbb{E}^Z \left[ \int_t^T (u^i_s - v^i_s) \left\{ \sum_{j=1,j \neq i}^N (-2\alpha^j \cdot (\Delta^j u^i_s - \Delta^j v^i_s)) \right\} ds + \int_t^T \sum_{j=1,j \neq i}^N (\alpha^j \cdot \Delta^j[(u^i_s - v^i_s)^2]) ds \right]
$$

can be simplified as follows

$$
\mathbb{E}^Z \left[ \int_t^T ds(u^i_s - v^i_s) \left\{ \sum_{j=1,j \neq i}^N (-2\alpha^j \cdot (\Delta^j u^i_s - \Delta^j v^i_s)) \right\} + \int_t^T \sum_{j=1,j \neq i}^N (\alpha^j \cdot \Delta^j[(u^i_s - v^i_s)^2]) ds \right]
$$

$$
= \sum_{j=1,j \neq i}^N \mathbb{E}^Z \left[ \int_t^T ds (-2\alpha^j \cdot (u^i_s - v^i_s)(\Delta^j u^i_s - \Delta^j v^i_s)) + \alpha^j \cdot (\Delta^j(u^i_s - v^i_s) \times (\Delta^j(u^i_s - v^i_s) + 2(1(u^i_s - v^i_s)))) \right]
$$

$$
= \sum_{j=1,j \neq i}^N \mathbb{E}^Z \left[ \int_t^T \alpha^j \cdot (\Delta^j(u^i_s - v^i_s))^2 ds \right].
$$

Thus, we have found

$$
0 = \mathbb{E}^Z[(u^i_T - v^i_T)^2]
$$

$$
= \mathbb{E}^Z[(u^i_0 - v^i_0)^2] + 2 \mathbb{E}^Z \left[ \int_t^T ds(u^i_s - v^i_s) \left\{ \sum_{j=1,j \neq i}^N ((\alpha^j - \bar{\alpha}^j) \cdot \Delta^j u^i_s) - H(Y_{i,s}, \Delta^j v^i_s) \right\} + \mathbb{E}^Z \left[ \int_t^T ds \alpha^i(s, Y_s) \cdot \Delta^i[(u^i_s - v^i_s)^2] \right]
$$

$$
+ \sum_{j=1,j \neq i}^N \mathbb{E}^Z \left[ \int_t^T \alpha^j \cdot (\Delta^j(u^i_s - v^i_s))^2 ds \right].
$$

Now, we use again the expression for $\Delta^i[(u^i_s - v^i_s)^2)$,

$$
\mathbb{E}^Z \left[ \int_t^T ds \alpha^i(s, Y_s) \cdot \Delta^i[(u^i_s - v^i_s)^2] \right]
$$

$$
= \mathbb{E}^Z \left[ \int_t^T ds \alpha^i(s, Y_s) \cdot (\Delta^i(u^i_s - v^i_s))^2 \right]
$$

$$
+ \mathbb{E}^Z \left[ \int_t^T ds \alpha^i(s, Y_s) \cdot (\Delta^i(u^i_s - v^i_s) \times 2(1(u^i_s - v^i_s))) \right],
$$

so that we can rewrite the previous as

$$
\mathbb{E}^Z[(u^i_0 - u^i_T)^2] + \sum_{j=1}^N \mathbb{E}^Z \left[ \int_t^T \alpha^j \cdot (\Delta^j(u^i_s - v^i_s))^2 ds \right]
$$

$$
= -2 \mathbb{E}^Z \left[ \int_t^T ds(u^i_s - v^i_s) \left\{ \sum_{j=1,j \neq i}^N ((\alpha^j - \bar{\alpha}^j) \cdot \Delta^j u^i_s) - H(Y_{i,s}, \Delta^j v^i_s) + H(Y_{i,s}, \Delta^j v^i_s) - r^{N,i}(s, Y_s) \right\} \right]
$$

$$
- \mathbb{E}^Z \left[ \int_t^T ds \alpha^i(s, Y_s) \cdot (\Delta^i(u^i_s - v^i_s) \times 2(1(u^i_s - v^i_s))) \right].
$$
Recalling that $\alpha^j \geq 0$ since it is a vector of rates of transitions, we can estimate the left hand side as

$$\mathbb{E}^Z[(u^i_t - v^j_t)^2] + \sum_{j=1}^{N} \mathbb{E}^Z\left[ \int_t^T \alpha^i \cdot (\Delta^j(u^i_t - v^j_t))^2 ds \right]$$

$$\leq 2\mathbb{E}^Z\left[ \int_t^T ds |u^i_s - v^j_s| \left( \sum_{j \neq i}^N \left( |\alpha^j - \tilde{\alpha}^j| \cdot \Delta^i u^i_s \right) + |H(Y_{i,s}, \Delta^i v^i_s) - H(Y_{i,s}, \Delta^j v^j_s)| + |r^{N,i}(s, Y_s)| \right) \right]$$

$$+ 2\mathbb{E}^Z\left[ \int_t^T ds |u^i_s - v^j_s| |\alpha^i(s, Y_s) \cdot \Delta^j(u^i_s - v^j_s)| \right].$$

This also implies, erasing the terms with $j \neq i$ in the left hand side,

$$\mathbb{E}^Z[(u^i_t - v^j_t)^2] + \mathbb{E}^Z\left[ \int_t^T \alpha^i \cdot (\Delta^i(u^i_t - v^j_t))^2 ds \right]$$

$$\leq 2\mathbb{E}^Z\left[ \int_t^T ds |u^i_s - v^j_s| \left( \sum_{j \neq i}^N \left( |\alpha^j - \tilde{\alpha}^j| \cdot \Delta^i u^i_s \right) + |H(Y_{i,s}, \Delta^i v^i_s) - H(Y_{i,s}, \Delta^j v^j_s)| + |r^{N,i}(s, Y_s)| \right) \right]$$

$$+ 2\mathbb{E}^Z\left[ \int_t^T ds |u^i_s - v^j_s| |\Delta^i(u^i_s - v^j_s)| \right].$$

For the boundedness of $\alpha^i$ from below and above (recall that the admissible controls $\alpha$ are such that $\alpha \in A = [\kappa, \bar{M}]$) we get

$$\mathbb{E}^Z[(u^i_t - v^j_t)^2] + \kappa \mathbb{E}^Z\left[ \int_t^T |\Delta^i(u^i_s - v^j_s)|^2 ds \right]$$

$$\leq 2\mathbb{E}^Z\left[ \int_t^T ds |u^i_s - v^j_s| \left( \sum_{j \neq i}^N \left( |\alpha^j - \tilde{\alpha}^j| \cdot \Delta^i u^i_s \right) + |H(Y_{i,s}, \Delta^i v^i_s) - H(Y_{i,s}, \Delta^j v^j_s)| + |r^{N,i}(s, Y_s)| \right) \right]$$

$$+ 2C\mathbb{E}^Z\left[ \int_t^T ds |u^i_s - v^j_s| |\Delta^i(u^i_s - v^j_s)| \right].$$

We now use the Lipschitz continuity of $H, \alpha^j, \tilde{\alpha}^j$ (assumption (H1)) and the bounds on $|r^{N,i}|_{\infty} \leq \frac{\kappa}{2}$ and $||\Delta^j u^i|| \leq \frac{1}{2} ||D^mU|| \leq \frac{\kappa}{2}$ proved in Proposition 3 and 11 to obtain

$$\mathbb{E}^Z[(u^i_t - v^j_t)^2] + \kappa \mathbb{E}^Z\left[ \int_t^T |\Delta^i(u^i_s - v^j_s)|^2 ds \right]$$

$$\leq 2\mathbb{E}^Z\left[ \int_t^T |u^i_s - v^j_s| \left( \frac{C}{N} \sum_{j=1,j \neq i}^N |\Delta^j u^i_s - \Delta^j v^j_s| + C |\Delta^i(v^j_s - u^i_s)| + \frac{C}{N} \right) ds \right]$$

$$+ 2C\mathbb{E}^Z\left[ \int_t^T ds |u^i_s - v^j_s| |\Delta^i(u^i_s - v^j_s)| \right]$$

$$\leq \frac{C}{N} \mathbb{E}^Z\left[ \int_t^T |u^i_s - v^j_s| ds \right] + \frac{C}{N} \sum_{j \neq i} \int_t^T |u^i_s - v^j_s| |\Delta^j(u^i_s - v^j_s)| ds$$

$$+ C\mathbb{E}^Z\left[ \int_t^T ds |u^i_s - v^j_s| |\Delta^i(u^i_s - v^j_s)| ds \right].$$
By the convexity inequality $ab \leq rac{a^2}{2} + \frac{b^2}{2}$ we can further estimate the right hand side to get
\[
\mathbb{E}^Z[(u_i^t - v_i^t)^2] + \frac{\kappa}{2} \mathbb{E}^Z \left[ \int_0^T |\Delta^i(u_s^t - v_s^t)|^2 \, ds \right] \leq \frac{C}{N^2} + \frac{\kappa}{2N} \sum_{j=1}^N \mathbb{E}^Z \left[ \int_0^T |\Delta^j(u_s^t - v_s^t)|^2 \, ds \right].
\]
By Gronwall’s Lemma, we can write
\[
\text{(40) } \sup_{t \in [0,T]} \mathbb{E}^Z[(u_i^t - v_i^t)^2] \leq \frac{C}{N^2} + \frac{\kappa}{2N} \sum_{j=1}^N \mathbb{E}^Z \left[ \int_0^T |\Delta^j(u_s^t - v_s^t)|^2 \, ds \right].
\]
Taking the expectation and using the exchangeability of the processes $(Y_{j,t})_{j=1,...,N}$ we obtain (39).

In order to derive (39), we consider (40) in $t = 0$ and average over $i = 1, \ldots, N$, so that we get
\[
\frac{1}{N} \sum_{i=1}^N \mathbb{E}^Z[u^N,i(0, Z) - v^N,i(0, Z)]^2 \leq \frac{C}{N^2},
\]
which immediately implies (39) almost surely.

We now estimate the difference $X_i - Y_i$. Thanks to equations (34) and (28) and the Lipschitz continuity in $x$ and $\alpha$ of the dynamics $f$ (see Lemma 2 in [4]), we obtain
\[
\mathbb{E} \left[ \sup_{s \in [0,t]} |X_{i,s} - Y_{i,s}| ds \right] \leq C \mathbb{E} \left[ \int_0^t \alpha^*(X_{i,s}, \Delta^i u^{N,i}(X_s)) - \alpha^*(Y_{i,s}, \Delta^i v^{N,i}(Y_s)) \, ds \right] + C \mathbb{E} \left[ \int_0^t |X_{i,s} - Y_{i,s}| ds \right]
\]
\[
\leq C \mathbb{E} \left[ \int_0^t |X_{i,s} - Y_{i,s}| ds \right] + C \mathbb{E} \left[ \int_0^t \left( \alpha^*(X_{i,s}, \Delta^i u^{N,i}(X_s)) - \alpha^*(Y_{i,s}, \Delta^i v^{N,i}(Y_s)) \right) \, ds \right]
\]
\[
+ C \mathbb{E} \left[ \int_0^t \left( \Delta^s u^{N,i}(X_s) - \Delta^s v^{N,i}(Y_s) \right) \, ds \right]
\]
\[
\leq \frac{C}{N} + C \mathbb{E} \left[ \int_0^t \sup_{r \in [0,s]} |X_{i,r} - Y_{i,r}| \, ds \right] + C \mathbb{E} \left[ \int_0^t \sup_{r \in [0,s]} \left| m^{N,i}_X - m^{N,i}_Y \right| \, ds \right]
\]
where we applied (33) and the Lipschitz continuity of $\Delta^s U$ in the last inequality. Then inequality (55) and the exchangeability of $(X,Y)$ yield
\[
\mathbb{E} \left[ \sup_{s \in [0,t]} |X_{i,s} - Y_{i,s}| ds \right] \leq \frac{C}{N} + C \mathbb{E} \left[ \int_0^t \sup_{r \in [0,s]} |X_{i,r} - Y_{i,r}| \, ds \right]
\]
and by Gronwall’s inequality we get (36). Finally (36), applying again (35), gives (37). \qed

3.2. Proofs of the main results. We are now in the position to prove the main results.

Proof. (Theorem 7)
For Equation (26) we just compute (39) for $Z$ uniformly distributed on $\Sigma$; this yields
\[
\frac{1}{N} \sum_{i=1}^N |U(t_0, x_i, m^{N,i}_x) - v^{N,i}(t_0, x)| \leq \frac{C}{N}
\]
Then, we can replace $U(t_0, x_i, m^{N,i}_x)$ with $U(t_0, x, m^N_x)$ using the Lipschitz continuity of $U$ with respect to $m$, the additional error term being of order $1/N$. 

\[\]
For (27), we compute
\[
||u^{N,i}(t_0, \cdot, m_0) - U(t_0, \cdot, m_0)||_{L^1(m_0)} = \\
= \sum_{x_1=1}^d |u^{N,i}(t_0, x, m_0) - U(t_0, x, m_0)|m_0(x) \\
= \sum_{x_1=1}^d \left| \sum_{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N = 1}^d v^{N,i}(t, \mathbf{x}) \prod_{j \neq i} m_0(x_j) - U(t, x, m_0) \right| m_0(x_i) \\
= \sum_{x_1=1}^d \left| \sum_{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N = 1}^d v^{N,i}(t, \mathbf{x}) \prod_{j \neq i} m_0(x_j) - u^{N,i}(t, \mathbf{x}) \prod_{j = 1}^N m_0(x_j) \\
+ u^{N,i}(t, \mathbf{x}) \prod_{j = 1}^N m_0(x_j) - U(t, x, m_0) \right| m_0(x_i)
\]
where in the last inequality the initial data \( Z = (Z_1, \ldots, Z_N) \) are distributed as \( m_0 \).

By (33), the first term in (42) is of order \( 1/N \). For the second term we further estimate, using again the Lipschitz continuity of \( U \) with respect to \( m \),
\[
\sum_{x_1, \ldots, x_N = 1}^d |u^{N,i}(t, \mathbf{x}) - U(t, x_i, m_0)| \prod_{j = 1}^N m_0(x_j) \\
= \sum_{x_1, \ldots, x_N = 1}^d |U(t, x_i, m^{N,i}_x) - U(t, x_i, m_0)| \prod_{j = 1}^N m_0(x_j) \\
\leq C \mathbb{E} \left[ d_1(m^{N,i}_x, m_0) \right] \leq \frac{C}{\sqrt{N}},
\]
where for the last inequality we used that \( \mathbb{E} \left[ d_1(m^{N}_Z, m_0) \right] \leq \frac{C}{\sqrt{N}} \), thanks to [15], where \( Z := (Z_1, \ldots, Z_N) \), the \( Z_i \)'s are i.i.d. initial data, \( m_0 \)-distributed, \( d_1 \) is the 1-Wasserstein distance and \( m^{N}_Z \) is the corresponding empirical measure. Overall, we have bounded (42) by a term of order \( 1/N \), and thus part (27) is also proved.

Finally, we get to the proof of the propagation of chaos (Theorem 2). Recall that the \( Y_i \)'s are the optimal processes, i.e. the solutions to system (28), the \( X_{i,t} \) are the processes associated to the functions \( u^{N,i} \), i.e. they solve system (33), while the \( \tilde{X}_{i,t} \) are the decoupled limit processes (they solve system (29)) to which we would like to prove convergence. First, we need the following lemma, whose proof can be found for example in [24]:

**Lemma 1.** Let \( \tilde{X}_t = (\tilde{X}_{i,t})_{i\in\{1,\ldots,N\}} \) be \( N \) i.i.d. random variables with values in a one dimensional space, with law \( m_t \). Then
\[
\mathbb{E} \left[ \sup_{t \in [t_0, T]} d_1(m^N_{\tilde{X}_t}, m_t) \right] \leq C N^{-1/9}.
\]

Notice that the supremum is taken inside the mean, giving thus a slow convergence of order \( N^{-1/9} \), while if the supremum were taken outside the convergence would be of order \( N^{-1/2} \) by [15].

**Proof.** *(Theorem 3)*
The assertion of the theorem is proved if we show that
\[
\mathbb{E} \left[ \sup_{t \in [t_0, T]} |X_{i,t} - \tilde{X}_{i,t}| \right] \leq C N^{-1/9}.
\]

Indeed, by the triangle inequality and (30) in Theorem 3 we can estimate
\[ E \left( \sup_{t \in [t_0, T]} |Y_{i,t} - \tilde{X}_{i,t}| \right) \leq E \left( \sup_{t \in [t_0, T]} |Y_{i,t} - \tilde{X}_{i,t}| \right) + E \left( \sup_{t \in [t_0, T]} |X_{i,t} - \tilde{X}_{i,t}| \right) \leq C(N^{-1} + N^{-1/9}). \]

We are then left to prove (44). As in the proof of (30), we have
\[ \rho(t) := E \left( \sup_{s \in [t_0, t]} |X_{i,s} - \tilde{X}_{i,s}| \right) \leq E \left( \int_{t_0}^{t} |\alpha^*(X_{i,s}, \Delta^x U(s, \tilde{X}_{i,s}, m_s)) - \alpha^*(\tilde{X}_{i,s}, \Delta^x U(s, \tilde{X}_{i,s}, m_s))| \, ds \right) \]
\[ \leq E \left( \int_{t_0}^{t} \left| \alpha^*(X_{i,s}, \Delta^x U(r, \tilde{X}_{i,s}, m_{X_{i,s}})) - \alpha^*(X_{i,s}, \Delta^x U(s, \tilde{X}_{i,s}, m_{X_{i,s}})) \right| \, ds \right) \]
\[ + \int_{t_0}^{t} |\alpha^*(X_{i,s}, \Delta^x U(r, \tilde{X}_{i,s}, m_{X_{i,s}})) - \alpha^*(\tilde{X}_{i,s}, \Delta^x U(s, \tilde{X}_{i,s}, m_s))| \, ds. \]

By the Lipschitz continuity of the optimal controls, and of \( \Delta^x U \), we can write
\[ \rho(t) \leq C \int_{t_0}^{t} E \left[ |X_{i,s} - \tilde{X}_{i,s}| + d_1(m_{X_{i,s}}^N, m_{X_{i,s}}^{N,i}) + d_1(m_{X_{i,s}}^{N,i}, m_{X_{i,s}}) \right] \, ds \]
\[ \leq C \int_{t_0}^{t} E \left[ |X_{i,s} - \tilde{X}_{i,s}| + \frac{1}{N-1} \sum_{j \neq i} |X_{j,s} - \tilde{X}_{j,s}| + d_1(m_{X_{i,s}}^{N,i}, m_{X_{i,s}}) \right] \, ds. \]

Using (43) of Lemma 1 and the exchangeability of the processes, we obtain
\[ \rho(t) \leq C \int_{t_0}^{t} \left( E \left( \sup_{r \in [t_0, t]} |X_{i,r} - \tilde{X}_{i,r}| \right) + \frac{1}{N-1} \sum_{j \neq i} E \left( \sup_{r \in [t_0, t]} |X_{j,r} - \tilde{X}_{j,r}| \right) \right) \, ds \]
\[ + CE \left( \sup_{r \in [t_0, T]} d_1(m_{X_{i,t}}^{N,i}, m_{X_{i,t}}) \right) \]
\[ \leq C \int_{t_0}^{t} \rho(s) \, ds + CN^{-1/9}, \]
which, by Gronwall’s Lemma, ends the proof of (30). Finally (41) follows from (30) and (43), using also (39).

4. Fluctuations and Large Deviations

The convergence results, Theorem 1 and 2, allow to derive immediately a Large Deviation Principle for the empirical measure process of the \( N \)-player game, when choosing the optimal control. First of all, we recall from Proposition 2 that, for any \( i \), the value function \( v^{N,i} \) of player \( i \) in the \( N \)-player game is invariant under permutation of \( (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \). This is equivalent to say that the value functions can be viewed as functions of the empirical measure of the system, i.e. there exists a map \( V^{N} : [0, T] \times \Sigma \times P(\Sigma) \) such that
\[ v^{N,i}(t, x) = V^{N}(t, x_i, m^N_{x_i}) \]
for any \( i = 1, \ldots, N \), \( t \in [0, T] \) and \( x \in \Sigma^N \).

4.1. Dynamics of the empirical measure process. We consider then trajectories of the empirical measure process of the optimal evolution \( Y \) (defined in (28)) of the \( N \)-player game. If the system is in \( x \) at time \( t \), then the rate at which player \( i \) goes from \( x_i \) to \( y \) is given, via the optimal control, by
\[ \alpha^y_{i}(x_i, \Delta^y N(t, x_i, m^N_{x_i})) := \Gamma^N_{x_i, y}(t, m^N_{x_i}), \]
i.e. by a function \( \Gamma^{N} \) which depends only on the empirical measure \( m^N_{x} \) and on the number of players \( N \).
Thus the empirical measure of the system \((m^N(t))_{t \in [0,T]}, m^N_1 := m^N_1(t) = \frac{1}{N} \sum_{i=1}^N \delta_{X_i,t}\), evolves as a (time-inhomogeneous) Markov process on \([0,T]\), with values in \(S_N := P(\Sigma) \cap \mathbb{R}^d \). The number of players in state \(x\) when the empirical measure is \(m\) is \(Nm_x\). Hence the jump rate of \(m^N\) in the direction \(\frac{1}{N}(\delta_y - \delta_x)\) at time \(t\) is \(Nm_x \Gamma_{x,y}^N(t,m)\). Therefore the generator of the time-inhomogeneous Markov process \(m^N\) is given, at time \(t\), by

\[
\mathcal{L}^N g(m) := N \sum_{x,y \in \Sigma} m_x \Gamma_{x,y}^N(t,m) \left[ g\left( m + \frac{1}{N}(\delta_y - \delta_x) \right) - g(m) \right],
\]

for any \(g : S_N \rightarrow \mathbb{R}\).

Theorem 2 implies that the empirical measures converge in law (on the space \(D([0,T], P(\Sigma))\)) to the deterministic flow of measures \(m\) which is the unique solution of the Mean Field Game system, whose dynamics is given by the KFP ODE

\[
\begin{cases}
\frac{d}{dt} m(t) = \Gamma(t,m(t))^T m(t) \\
m(0) = m_0,
\end{cases}
\]

where \(\Gamma\) is the matrix defined by

\[
\Gamma_{x,y}(t,m) := \alpha^*_y(x, \Delta x U(t,x,m))
\]

and \(U\) is the solution to the Master Equation. Viewing \(m(t)\) as a Markov process, its infinitesimal generator is given, at time \(t\), by

\[
\mathcal{L}_t g(m) := \sum_{x,y \in \Sigma} m_x \Gamma_{x,y}(t,m)[D^m g(m,x)]_y
\]

for any \(g : P(\Sigma) \rightarrow \mathbb{R}\). Thanks to (32), the generator can be equivalently written as

\[
\mathcal{L}_t g(m) := \sum_{x} m_x \Gamma_{x,y}(t,m)[D^m g(m,1)]_y = m^T \Gamma(t,m) D^m g(m,1).
\]

We will also consider the empirical measure of the process \(X\) defined in (34), in which each player chooses the same control \(\Gamma_{x,y}\) independent of \(N\). We denote by \(\eta^N_i := \sum_{i=1}^N \delta_{X_i(t)}\) the empirical measure process of \(X\), whose generator is

\[
\mathcal{M}^N_i g(m) := N \sum_{x,y \in \Sigma} m_x \Gamma_{x,y}(t,m) \left[ g\left( m + \frac{1}{N}(\delta_y - \delta_x) \right) - g(m) \right].
\]

### 4.2. Central Limit Theorem.

A natural refinement of the Law of Large Numbers (31) consists in studying the fluctuations around the limit, that is the asymptotic distribution of \(m^N_1 - m_1\). This can be done through a functional Central Limit Theorem: we define the fluctuation flow

\[
\rho^N(t) := \sqrt{N}(m^N_1 - m_1)_{t \in [0,T]}
\]

and study its asymptotic behavior when \(N\) tends to infinity, following a classical weak convergence-type approach based on uniform convergence of the generator of the fluctuation flow \(\rho^N\) to a limiting generator of a diffusion process to be determined (see for e.g. [9], [10] for reference). Before stating the theorem we observe that the process \(\rho^N\) has values in \(P_0(\Sigma)\), which in the following we treat as a subset of \(\mathbb{R}^d\).

**Theorem 4** (Central Limit Theorem). Let \(U\) be a regular solution to the Master Equation and assume (37). Then the fluctuation flow \(\rho^N\) in (53) converges as \(N \rightarrow \infty\), in the sense of weak convergence of stochastic processes, to a limiting process \(\rho^\infty\) which is the solution of the following SDE

\[
\begin{cases}
\frac{d\rho}{dt} = \Gamma(t,m(t))^T \rho(t) + b(t,m(t)) dt + \sigma(t,m(t)) dB_t, \\
\rho_0 = \bar{\rho},
\end{cases}
\]

where \(\bar{\rho}\) is the limit of \(\rho^N\) in distribution, \(B\) is a standard \(d\)-dimensional Brownian motion, \(\Gamma\) is the transition rate matrix in \((19)\), \(b(t,\mu) \in \mathbb{R}^d\) is defined by

\[
b(t,\mu)_y := \sum_{x \in \Sigma} (m_t)_x [D^m \Gamma_{x,y}(t,m_t,1) \cdot \mu],
\]
for $y \in \Sigma$, $\mu \in P_0(\Sigma)$, and $\sigma \in \mathbb{R}^{d \times d}$ is given by the relations
\begin{equation}
(\sigma^2)_{x,y}(t,m) = -(m_x \Gamma_{x,y}(t,m) + m_y \Gamma_{y,x}(t,m)), \quad \text{for } x \neq y,
\end{equation}
\begin{equation}
(\sigma^2)_{x,x}(t,m) = \sum_{y \neq x} (m_y \Gamma_{y,x}(t,m) + m_x \Gamma_{x,y}(t,m)).
\end{equation}

In particular the matrix $\sigma^2$ is the opposite of the generator of a Markov chain, is symmetric and positive semidefinite with one null eigenvalue, and the same properties hold for $\sigma$, meaning that $\mu \in P_0(\Sigma)$ for any $t$.

Proof. The key observation is that we can reduce ourselves to study the asymptotics of the fluctuation flow
\begin{equation}
\mu_t^N := \sqrt{N}(\eta_t^N - \eta_t^0),
\end{equation}
which is more standard since $\eta_t^N$, whose generator $\mathcal{M}$ is defined in (52), is the empirical measure of an uncontrolled system of $N$ mean-field interacting particles. Indeed, by (57) we have that $\sqrt{N}(m_t^N - \eta_t^0)$ tends to 0 almost surely as $N$ goes to infinity.

Thus, it remains to prove the convergence in law of $\mu_t^N$ to $\mu_t^0$. The convergence of $\mu_t^N$ (and $\rho_t^N$) to the initial condition $\bar{\rho}$ follows from the Central Limit Theorem for the i.i.d. sequence of initial conditions $Z_i$ in system (22) and (28). Then, we compute the generator of (58) for $t \geq 0$. Now we note that $\mu_t^N$ is obtained from $\eta_t^N$ through a time dependent, linear invertible transformation $T_t: S_N \to P_0(\Sigma) \subset \mathbb{R}^d$, defined by
\begin{equation}
T_t(\partial) := \sqrt{N}(\partial - \mu_t),
\end{equation}
with inverse $T_t^{-1}(\mu) := m_t + \frac{\mu}{\sqrt{N}}$. Thus, the generator $\mathcal{H}_t^N$ of (58) can be written as
\begin{equation}
\mathcal{H}_t^N g(\mu) = \mathcal{M}_t^N[\mu] g(T_t^{-1}(\mu)) + \frac{\partial}{\partial t} [g \circ T_t](T_t^{-1}(\mu)),
\end{equation}
for a $g : P_0(\Sigma) \to \mathbb{R}$ regular and with compact support (we can extend the definition of $g$ to a smooth function in the whole space $\mathbb{R}^d$, so that the usual derivatives are well defined). We have
\begin{align*}
\frac{\partial}{\partial t} [g \circ T_t](T_t^{-1}(\mu)) &= -\sqrt{N} \nabla^\mu g(\mu) \cdot \dot{m}_t = -\sqrt{N} \nabla^\mu g(\mu) \cdot \left( \Gamma(t, m_t)^T m_t \right) \\
&= -\sqrt{N} \sum_{x,y \in \Sigma} \frac{\partial}{\partial \mu_y} g(\mu) \Gamma_{x,y}(t,m_t)(m_t)_x.
\end{align*}
where the second equality follows from the KFP equation for $m_t$. For the remaining part in the expression (59) we have
\begin{align*}
\mathcal{M}_t^N[\mu] g(T_t^{-1}(\mu)) &= N \sum_{x,y \in \Sigma} \left( m_t + \frac{\mu}{\sqrt{N}} \right) \Gamma_{x,y} \left( t, m_t + \frac{\mu}{\sqrt{N}} \right) \times \\
&\quad \left\{ \left[ g \circ T_t \right] \left( m_t + \frac{\mu}{\sqrt{N}} + \frac{1}{N} (\delta_y - \delta_x) \right) - \left[ g \circ T_t \right] \left( m_t + \frac{\mu}{\sqrt{N}} \right) \right\} \\
&= N \sum_{x,y \in \Sigma} \left( m_t + \frac{\mu}{\sqrt{N}} \right) \Gamma_{x,y} \left( t, m_t + \frac{\mu}{\sqrt{N}} \right) \left\{ g \left( \mu + \frac{1}{\sqrt{N}} (\delta_y - \delta_x) \right) - g(\mu) \right\}.
\end{align*}
Thus, we have found
\begin{align*}
\mathcal{H}_t^N g(\mu) &= N \sum_{x,y \in \Sigma} \left( m_t + \frac{\mu}{\sqrt{N}} \right) \Gamma_{x,y} \left( t, m_t + \frac{\mu}{\sqrt{N}} \right) \left\{ g \left( \mu + \frac{1}{\sqrt{N}} (\delta_y - \delta_x) \right) - g(\mu) \right\} \\
&\quad - \sqrt{N} \sum_{x,y \in \Sigma} \frac{\partial}{\partial \mu_y} g(\mu) \Gamma_{x,y}(t,m_t)(m_t)_x.
\end{align*}
In order to perform a Taylor expansion of the generator, we develop the term
\begin{align*}
g \left( \mu + \frac{1}{\sqrt{N}} (\delta_y - \delta_x) \right) - g(\mu) &= \frac{1}{\sqrt{N}} \nabla^\mu g(\mu) \cdot (\delta_y - \delta_x) + \frac{1}{2N} (\delta_y - \delta_x)^T D^2_{\mu \mu} g(\mu) (\delta_y - \delta_x) + O \left( \frac{1}{N^{3/2}} \right).
\end{align*}
Substituting, we get
\[
\mathcal{H}_t^N g(\mu) = \sqrt{N} \sum_{x, y \in \Sigma} \left( m_t + \frac{\mu}{\sqrt{N}} \right) \Gamma_{x,y} \left( t, m_t + \frac{\mu}{\sqrt{N}} \right) \nabla_\mu g(\mu) \cdot (\delta_y - \delta_x) \\
+ \frac{1}{2} \sum_{x, y \in \Sigma} \left( m_t + \frac{\mu}{\sqrt{N}} \right) \Gamma_{x,y} \left( t, m_t + \frac{\mu}{\sqrt{N}} \right) \left( \delta_y - \delta_x \right)^T D_{\mu\mu}^2 g(\mu) (\delta_y - \delta_x) \\
- \sqrt{N} \sum_{x, y \in \Sigma} \frac{\partial}{\partial \mu_y} g(\mu) \Gamma_{x,y}(t, m_t)(m_t)_x + O \left( \frac{1}{\sqrt{N}} \right).
\]

Now, we note that
\[
\sum_{x, y \in \Sigma} \left( m_t + \frac{\mu}{\sqrt{N}} \right) \Gamma_{x,y} \left( t, m_t + \frac{\mu}{\sqrt{N}} \right) \nabla_\mu g(\mu) \cdot (\delta_y - \delta_x) = \sum_{x, y \in \Sigma} \left( m_t + \frac{\mu}{\sqrt{N}} \right) \Gamma_{x,y} \left( t, m_t + \frac{\mu}{\sqrt{N}} \right) \frac{\partial}{\partial \mu_y} g(\mu),
\]
because \( \sum_y \Gamma_{x,y} = 0 \). This property allows us to rewrite
\[
\mathcal{H}_t^N g(\mu) = \sum_{x, y \in \Sigma} \mu_x \Gamma_{x,y} \left( t, m_t + \frac{\mu}{\sqrt{N}} \right) \frac{\partial}{\partial \mu_y} g(\mu) \\
+ \sqrt{N} \sum_{x, y \in \Sigma} (m_t)_x \frac{\partial}{\partial \mu_y} g(\mu) \left[ \Gamma_{x,y} \left( t, m_t + \frac{\mu}{\sqrt{N}} \right) - \Gamma_{x,y}(t, m_t) \right] \\
+ \frac{1}{2} \sum_{x, y \in \Sigma} \left( m_t + \frac{\mu}{\sqrt{N}} \right) \Gamma_{x,y} \left( t, m_t + \frac{\mu}{\sqrt{N}} \right) \left( \delta_y - \delta_x \right)^T D_{\mu\mu}^2 g(\mu) (\delta_y - \delta_x) + O \left( \frac{1}{\sqrt{N}} \right).
\]

Then, using the Lipschitz continuity of \( \Gamma \) as we did in Proposition 3, we linearize the term
\[
\Gamma_{x,y} \left( t, m_t + \frac{\mu}{\sqrt{N}} \right) - \Gamma_{x,y}(t, m_t) = \frac{1}{\sqrt{N}} D^m \Gamma_{x,y}(t, m_t, 1) \cdot \mu + O \left( \frac{1}{N} \right).
\]

We thus deduce
\[
\lim_{N \to \infty} \sup_{t \in [0,T]} \sup_{\mu \in \mathcal{P}_1(\Sigma)} |\mathcal{H}_t^N g(\mu) - \mathcal{H}_t g(\mu)| = 0,
\]
where
\[
(60) \quad \mathcal{H}_t g(\mu) := \sum_{x, y \in \Sigma} \mu_x \Gamma_{x,y}(t, m_t) \frac{\partial}{\partial \mu_y} g(\mu) + \sum_{x, y \in \Sigma} (m_t)_x [D^m \Gamma_{x,y}(t, m_t, 1) \cdot \mu] \frac{\partial}{\partial \mu_y} g(\mu) \\
+ \frac{1}{2} \sum_{x, y \in \Sigma} (m_t)_x \Gamma_{x,y}(t, m_t) (\delta_y - \delta_x)^T D_{\mu\mu}^2 g(\mu) (\delta_y - \delta_x).
\]

The proof is then completed if we show that the generator \( (60) \) is associated to the SDE \( (54) \).

The drift component can be immediately identified, since
\[
\sum_{x, y \in \Sigma} \mu_x \Gamma_{x,y}(t, m_t) \frac{\partial}{\partial \mu_y} g(\mu) = \Gamma(t, m_t)^T \mu \cdot \nabla_\mu g(\mu),
\]
and
\[
\sum_{x, y \in \Sigma} (m_t)_x [D^m \Gamma_{x,y}(t, m_t, 1) \cdot \mu] \frac{\partial}{\partial \mu_y} g(\mu) = b(t, \mu) \cdot \nabla_\mu g(\mu).
\]

For the diffusion component, we first note that, for \( x, y \in \Sigma \)
\[
(\delta_y - \delta_x)^T D_{\mu\mu}^2 g(\mu) (\delta_y - \delta_x) = \frac{\partial^2}{\partial \mu_y \mu_y} g(\mu) + \frac{\partial^2}{\partial \mu_x \mu_x} g(\mu) - \frac{\partial^2}{\partial \mu_x \mu_y} g(\mu) - \frac{\partial^2}{\partial \mu_y \mu_x} g(\mu),
\]

so that
\[
\frac{1}{2} \sum_{x,y \in \Sigma} (\delta_y - \delta_x)^T D_{\mu \mu} g(\mu)(\delta_y - \delta_x)m_x \Gamma_{x,y,}
\]
\[= \frac{1}{2} \sum_{x,y \in \Sigma} \left[ \frac{\partial^2}{\partial \mu_y \partial \mu_x} g(\mu) + \frac{\partial^2}{\partial \mu_x \partial \mu_y} g(\mu) - \frac{\partial^2}{\partial \mu_x \partial \mu_y} g(\mu) - \frac{\partial^2}{\partial \mu_y \partial \mu_x} g(\mu) \right] m_x \Gamma_{x,y},
\]
which is equal to
\[
\frac{1}{2} Tr(\sigma^2 D_{\mu \mu} g(\mu)) = \frac{1}{2} \sum_{x,y \in \Sigma} (\sigma^2)_{x,y} \frac{\partial^2}{\partial \mu_x \partial \mu_y} g(\mu),
\]
if we define $(\sigma^2)_{x,y} \in \Sigma$ by the relations \((60)\) and \((67)\).

Finally, we observe that the limiting process $\rho_t$ defined in \((63)\) takes values in $P_0(\Sigma)$, as required. Indeed, by diagonalizing $\sigma^2$ (which is symmetric and such that its rows sum to 0) we can get that all the eigenvectors, besides the constant one relative to the null eigenvalue, have components which sum to 0 (by orthogonality). The same properties hold for the square root matrix $\sigma$, so that equation \((63)\) preserves the space $P_0(\Sigma)$.

4.3. Large Deviation Principle. We state the large deviation result, which is a sample path LDP on $D([0, T], P(\Sigma))$. To define the rate function, we first introduce the local rate function $\lambda : \mathbb{R} \to [0, +\infty]$,

\[
\lambda(r) := \begin{cases} 
    r \log r - r + 1 & r > 0, \\
    1 & r = 0, \\
    +\infty & r < 0.
\end{cases}
\]

For $t \in [0, T]$, $m \in P(\Sigma)$ and $\mu \in P_0(\Sigma)$, define

\[
\Lambda(t, m, \mu) := \inf \left\{ \sum_{x,y \in \Sigma} m_x \Gamma_{x,y}(t, m) \lambda \left( \frac{q_{x,y}}{\Gamma_{x,y}(t, m)} \right) : q_{x,y} \geq 0, \sum_{x,y \in \Sigma} q_{x,y}(\delta_y - \delta_x) = \mu \quad \forall x, y \right\}.
\]

and set, for $\gamma : [0, T] \to P(\Sigma)$,

\[
I(\gamma) := \begin{cases} 
    \int_0^T \Lambda(t, \gamma(t), \dot{\gamma}(t)) dt & \text{if } \gamma \text{ is absolutely continuous and } \gamma(0) = m_0 \\
    +\infty & \text{otherwise.}
\end{cases}
\]

We are now able to state the large deviation principle. We equip $D([0, T], P(\Sigma))$ with the Skorokhod $J_1$-topology and denote by $B(D([0, T], P(\Sigma)))$ the associated Borel $\sigma$-algebra.

**Theorem 5** (Large Deviation Principle). Assume that there exists a unique regular solution $U$ to the master equation and suppose that the claim \((25)\) of Theorem 1 holds. Also, assume that the initial conditions $(m_0^N)_{N \in \mathbb{N}}$ are deterministic and $\lim_N m_0^N = m_0$. Then the sequence of empirical measure processes $(m^N)_{N \in \mathbb{N}}$ satisfies the sample path large deviation principle on $D([0, T], P(\Sigma))$ with the (good) rate function $I$. Specifically,

(i) if $E \in B(D([0, T], P(\Sigma)))$ is closed then

\[
\limsup_N \frac{1}{N} \log \mathbb{P}(m^N \in E) \leq - \inf_{\gamma \in E} \{I(\gamma)\}
\]

(ii) if $E \in B(D([0, T], P(\Sigma)))$ is open then

\[
\liminf_N \frac{1}{N} \log \mathbb{P}(m^N \in E) \geq - \inf_{\gamma \in E} \{I(\gamma)\}
\]

(iii) For any $M < +\infty$ the set

\[
\{ \gamma \in D([0, T], P(\Sigma)) : I(\gamma) \leq M \}
\]

is compact.
We remark that the initial conditions are assumed to be deterministic only for simplicity, otherwise there would be another term in the rate function $I$. Before proving the Theorem, let us give another characterization of $I$. For $m \in P(\Sigma)$ and $\theta \in \mathbb{R}^d$, define
\begin{equation}
\Psi(t, m, \theta) := \sum_{x,y} m_{x} \Gamma_{x,y}(t, m) \left[ e^{\theta (\delta_x - \delta_y)} - 1 \right]
\end{equation}
and let $\Lambda^0$ be the Legendre transform of $\Psi$:
\begin{equation}
\Lambda^0(t, m, \mu) = \sup_{\theta \in \mathbb{R}^d} [\theta \cdot \mu - \Psi(t, m, \theta)].
\end{equation}
Define $I^0$ as in (63) but with $\Lambda$ replaced by $\Lambda^0$. Via a standard result in convex analysis, Proposition 6.2 in [12] shows that $\Lambda = \Lambda^0$ and then $I = I^0$.

Several authors studied large deviation properties of mean field interacting processes similar to ours. However, most of them deal with the case in which the prelimit jump rates, $m_N \Gamma$, are constant and equal to the limiting rates $m \Gamma$; see e.g. [22], [20] and [25]. We mention that in this latter paper, as in many others, it is also assumed that the jump rates of the prelimit process are bounded below and away from 0; this does not apply to our case, since the number of agents in a state $x$ could be 0, implying that $m_N \Gamma_{x,y}$ might also be 0.

To prove the claim, we apply the results in [12] to our knowledge, it is the first paper which proves a Large Deviation Principle considering the jump rates of any player depending on $N$ (and deals also with systems with simultaneous jumps). Theorem 3.4.1 in [27] shows, however, the exponential equivalence of the processes $m_N$ and the processes $\eta_N$ given by (3) in which the jump rates of the prelimit system $m_N \Gamma_N$ are replaced by $m \Gamma$, which does not depend on $N$; the proof uses a coupling of the two Markov chains. These results are derived assuming the following properties:

1. the dynamics of any agent is ergodic and the jump rates are uniformly bounded;
2. for each $x, y \in \Sigma$, the limiting jump rates $\Gamma_{x,y}$ are Lipschitz continuous in $m$;
3. for each $x, y \in \Sigma$, given any sequence $m_N \in S_N$ such that $\lim_N m_N = m$,
\begin{equation}
\lim N \sup_{0 \leq t \leq T} |m_N \Gamma_N(t, m_N) - m \Gamma_{x,y}(t, m)| = 0.
\end{equation}

Property (1) holds in our model since the jump rates of any player belong to $[\kappa, M]$, while (2) is true because of the regularity of the solution $U$ to the Master Equation.

**Proof.** (Theorem 5)

The fact that $I$ is a good rate function, i.e condition (iii), is proved for instance in Theorem 1.1 of [13]. Due to Theorem 3.9 in [12], in order to prove the claims (i) and (ii) it is enough to show (68). Actually [12] studies time homogeneous Markov processes, but their results are still true if the convergence (68) is uniform over time.

Assume then (68) and let $x, y \in \Sigma$, $m_N = m_N \in S_N$, $\dot{x} = (x_1, \ldots, x_N) \in \Sigma^N$ and $m_N \rightarrow m$.

Hence
\begin{align*}
|m_N \Gamma_N(t, m_N) - m \Gamma_{x,y}(t, m)| &\leq |m_N \Gamma_N(t, m_N) - m_N \Gamma_{x,y}(t, m_N)| \\
&+ |m_N \Gamma_{x,y}(t, m_N) - m \Gamma_{x,y}(t, m)|
\end{align*}

The first term goes to zero, uniformly over time, thanks to (68):

\begin{align*}
A &= \frac{1}{N} \sum_{i=1}^{N} \mathbb{I}_{\{x_i = x\}} \alpha_i^*(x_i, \Delta^N U(t, x_i, m_N)) - \frac{1}{N} \sum_{i=1}^{N} \mathbb{I}_{\{x_i = x\}} \alpha_i^*(x_i, \Delta^N U(t, x_i, m_N)) \\
&\leq C \frac{1}{N} \sum_{i=1}^{N} |\Delta^N U(t, x_i, m_N) - \Delta^N U(t, x_i, m^N_N)| \\
&\leq C \sup \dot{x} \in \Sigma^N \frac{1}{N} \sum_{i=1}^{N} |\dot{y}^N(t, \dot{x}) - U(t, x_i, m_N)| \leq C.
\end{align*}
While $B$ converges to 0, uniformly over $t$, by the regularity of $U$:

$$B = |[m^N_x] - m_x(x, \Delta^2 U(t, x, m_x))| - m_x(x, \Delta^2 U(t, x, m_x))|$$

$$\leq |m^N_x - m_x| + C|m_x||\Delta^2 U(t, x, m_x) - \Delta^2 U(t, x, m)|$$

$$\leq C|m^N_x - m_x| \to 0.$$  

5. The Master Equation: well-posedness and regularity

In this section we study the well-posedness of equation (M) under the assumptions of monotonicity and regularity for $F, G, H$ we already introduced (Mon), (RegFG), (RegH). A preliminary remark is that, thanks to Proposition 1 in [10], if $H$ is differentiable (and this is indeed the case of our assumptions) then

$$\alpha^*_x(y, p) = -\frac{\partial}{\partial p_x} H(y, p).$$

For this reason, we will in the following use $\alpha^*$ interchangeably with $-D_p H$.

**Theorem 6.** Assume (Mon), (RegFG) and (RegH). Then there exists a unique classical solution to (M) in the sense of Definition 3. Moreover it is regular.

The method of proof follows from the renowned method of characteristics, which consists in proving that

$$U(t_0, x, m_0) := u(t_0, x)$$

solves (M), $u$ being the solution of the Mean Field Game system (MFG). First of all, in order to perform the computations, we have to prove the regularity in $m$ of the function $U(t_0, x, m)$ defined above. In particular, we have to show that $D^m U$ exists and it is bounded. For this, we follow the strategy shown in [5] - which is developed in infinite dimension - adapting it to our discrete setting. The idea consists in studying the well-posedness and regularity properties of the linearized version of the system (MFG), whose solution will end up coinciding with $D^m U \cdot \mu_0$, for all possible directions $\mu_0 \in P_0(\Sigma)$.

5.1. Estimates on the Mean Field Game system. We start by proving the well-posedness of the system (MFG)

$$\begin{cases}
-\frac{\partial}{\partial t} u(t, x) + H(x, \Delta^2 u(t, x)) = F(x, m(t)), \\
\frac{\partial}{\partial t} m_x(t) = \sum y m_y(t) \alpha^*_x(y, \Delta^2 u(t, y)), \\
u(T, x) = G(x, m(T)), \\
m_x(t_0) = m_x(0),
\end{cases}$$

and a useful a priori estimate on its solution $(u, m)$. The existence of solutions follows from a standard fixed point argument: see Proposition 4 of [10]. Let us remark that any flow of measures $m$ lies in the space

$$\{m \in C([t_0, T], \mathcal{P}(\Sigma)) : |m(t) - m(s)| \leq 2\nu(U)\sqrt{d}[t - s]\},$$

which is a compact and convex subset of the space of continuous functions, endowed with the uniform norm (Lemma 4 of [8]). On the other hand the uniqueness of solution, under our assumptions, is a consequence of the following a priori estimates. Before stating the proposition, recall the notation $|u| := \sup_{x \in [t_0, T]} \max_{x \in \Sigma} |g(u, x)|$.

**Proposition 5.** Assume (Mon), (RegFG) and (RegH). Let $(u_1, m_1)$ and $(u_2, m_2)$ be two solutions to (MFG) with initial conditions $m_1(t_0) = m_1^0$ and $m_2(t_0) = m_2^0$. Then

$$||u_1 - u_2|| \leq C|m_0^1 - m_0^2|$$

$$||m_1 - m_2|| \leq C|m_0^1 - m_0^2|.$$
We apply the monotonicity of \( F \) two lines. In fact, recalling that \( \alpha \)

\[
\begin{align*}
\text{Step 1. Use of Monotonicity. The couple } (u, m) \text{ solves}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial}{\partial t} u(t, x) + H(x, \Delta^x u_1(t, x)) - H(x, \Delta^x u_2(t, x)) &= F(x, m_1(t)) - F(x, m_2(t)) \\
\frac{\partial}{\partial x} m(t, x) &= \sum_y [m_1(t, y) \alpha^*_y(y, \Delta^y u_1(t, y)) - m_2(t, y) \alpha^*_y(y, \Delta^y u_2(t, y))] \\
u(T, x) &= G(x, m_1(T)) - G(x, m_2(T)) \\
m(0, x) &= m_0^1 - m_0^2,
\end{align*}
\]

Since \( \sum_x m(x) u(x) = \sum_x m(x) \frac{\partial}{\partial x} (x) + \sum_x \frac{\partial}{\partial x} (x) u(x) \), integrating over \([0, T] \) we have

\[
\begin{align*}
\sum_x [m(T, x) u(T, x) - m(0, x) u(0, x)] &= \int_0^T \sum_x [H(x, \Delta^x u_1) - H(x, \Delta^x u_2) - F(x, m_1) + F(x, m_2)] (m_1(x) - m_2(x)) \, dt \\
&+ \int_0^T \sum_x \sum_y [m_1(y) \alpha^*_y(y, \Delta^y u_1) + m_2(y) \alpha^*_y(y, \Delta^y u_2)] (u_1(x) - u_2(x)) \, dt
\end{align*}
\]

Using the fact that \( \sum_x \alpha^*_y(y) = 0 \) we can rewrite

\[
\begin{align*}
\sum_x [G(x, m_1) - G(x, m_2)](m_1(x) - m_2(x)) + \int_0^T \sum_x [F(x, m_1) - F(x, m_2)] (m_1(x) - m_2(x)) \, dt \\
= \sum_x (m_1^0(x) - m_2^0(x))(u_1(0, x) - u_2(0, x)) \\
+ \int_0^T \sum_x [(H(x, \Delta^x u_1) - H(x, \Delta^x u_2))(m_1(x) - m_2(x)) \\
+ \Delta^x u \cdot (m_1(x) \alpha^*(x, \Delta^x u_1) - m_2(x) \alpha^*(x, \Delta^x u_2))] \, dt
\end{align*}
\]

We apply the monotonicity of \( F \) and \( G \) to the first line and the uniform convexity of \( H \) to the last two lines. In fact, recalling that \( \alpha^*_y(x, p) = -\frac{\partial}{\partial p} H(x, p) \), by \( \text{(RegH)} \) we have that, for each \( x \)

\[
\begin{align*}
H(x, \Delta^x u_1) - H(x, \Delta^x u_2) - \Delta^x u \cdot \frac{\partial}{\partial p} H(x, \Delta^x u_1) &\leq -C^{-1} |\Delta^x u|^2 \\
H(x, \Delta^x u_2) - H(x, \Delta^x u_1) + \Delta^x u \cdot \frac{\partial}{\partial p} H(x, \Delta^x u_2) &\leq -C^{-1} |\Delta^x u|^2.
\end{align*}
\]

Hence we obtain

\[
\begin{align*}
\int_0^T \sum_x |\Delta^x u|^2 (m_1(x) + m_2(x)) \, dt &\leq C(m_0^1 - m_0^2) \cdot (u_1(0) - u_2(0)) \tag{74}
\end{align*}
\]

\[
\text{Step 2. Estimate on Kolmogorov-Fokker-Planck equation. Integrating the second equation in } \tag{74} \text{ over } [0, t] \text{ we get}
\]

\[
m(t, x) = m(0, x) + \int_0^t \sum_y [m_1(s, y) \alpha^*_y(y, \Delta^y u_1(s, y)) - m_2(s, y) \alpha^*_y(y, \Delta^y u_2(s, y))] \, ds
\]

Thus the boundedness and Lipschitz continuity of the rates give

\[
\max_x |m(t, x)| \leq C|m_0^1 - m_0^2| + C \int_0^t \max_x |m(s, x)| ds + C \int_0^t \sum_x |\Delta^x u(s, x)| m_1(s, x) ds
\]

and hence, by Gronwall’s Lemma,

\[
||m|| \leq C|m_0^1 - m_0^2| + C \int_0^T \sqrt{\sum_x |\Delta^x u(s, x)|^2 m_1(x)} dt.
\]

This, together with inequality \( \text{(76)} \), yields

\[
||m|| \leq C|m_0^1 - m_0^2| + |m_0^1 - m_0^2|^{1/2} ||u||^{1/2}
\]

\[
\text{Proof. Without loss of generality, let us set } t_0 = 0. \text{ Let } u := u_1 - u_2 \text{ and } m := m_1 - m_2. \text{ This proof is carried out in three steps.}
\]

\[
\begin{align*}
\text{Step 1. Use of Monotonicity. The couple } (u, m) \text{ solves}
\end{align*}
\]
Step 3. Estimate on Hamilton-Jacobi-Bellman equation Integrating the first equation in (73) over \([t, T]\) we get

\[
    u(t, x) = G(x, m_1(T)) - G(x, m_2(T)) + \int_t^T \left[ F(x, m_1) - F(x, m_2) + H(x, \Delta^x u_2) - H(x, \Delta^x u_1) \right] ds.
\]

Using the Lipschitz continuity of \(F, G, H\) and the bound \(\max_x |\Delta^x u(x)| \leq C \max_x |u(x)|\) we obtain

\[
    \max_x |u(t, x)| \leq C|m_1(T) - m_2(T)| + C \int_t^T |m_1(s) - m_2(s)| ds + C \int_t^T \max_x |u(s, x)| ds
\]

and then Gronwall’s Lemma gives

\[
    \|u\| \leq C\|m\|.
\]

This bound (78) and estimate (77) yield claim (73). Again (78) finally proves claim (72). \(\Box\)

5.2. Linearized MFG system and regularity. For proving Theorem 4 we need to introduce the linearized version of system (MFG) around its solutions and then prove that it provides the derivative of \(u(t_0, x)\) with respect to the initial condition \(m_0\).

As a preliminar step, we study a related linear system of ODE’s, which will come useful several times.

In the next result we prove the well-posedness of system (79) together with useful a priori estimates on its solution.

**Lemma 2.** If (RegFG) holds then the equation

\[
    \begin{align*}
        -\frac{\partial}{\partial t} z(t, x) - \alpha^*(x, \Delta^x u) \cdot \Delta^x z(t, x) &= D^n F(x, m(t), 1) \cdot \rho(t) + b(t, x) \\
        z(T, x) &= D^n G(x, m(T), 1) \cdot \rho(T) + z_T(x)
    \end{align*}
\]

has a unique solution for each final condition \(z_T(x)\) and satisfies

\[
    ||z|| \leq C \max_x |z_T(x)| + ||\rho|| + ||b||.
\]

**Proof.** The well-posedness of the equation is immediate from classical ODE’s theory. Integrating over the time interval \([t, T]\) and using that

\[
    \alpha^*(x, \Delta^x u) \cdot \Delta^x z(t, x) = \sum_y \alpha^*_y(x, \Delta^x u) z_y(t),
\]

we find

\[
    z(t, x) - z(T, x) - \int_t^T \sum_y \alpha^*_y(x, \Delta^x u) z_y(s) = \int_t^T D^n F \cdot \rho(t) + \int_t^T b(s, x) ds.
\]

Substituting the expression for \(z(T, x)\), and using the bound on the control and on the derivatives of \(F\) and \(G\) we can estimate

\[
    \max_x |z(t, x)| \leq \max_x |z_T(x)| + C \max_x |\rho(T, x)|
    + C \int_t^T \max_x |z(t, x)| ds + C \int_t^T \max_x |\rho(t, x)| ds + \int_t^T \max_x |b(t, x)| ds.
\]

and thus, applying Gronwall’s Lemma and taking the supremum on \(t\) we get (81). \(\Box\)

In the next result we prove the well-posedness of system (79) together with useful a priori estimates on its solution.
Proposition 6. Assume (RegH), (Mon) and (RegFG). Then for any (measurable) $b,c,z_T$, the linear system (79) has a unique solution $(z,\rho) \in C^1([0,T],\mathbb{R}^d \times P_0(\Sigma))$. Moreover it satisfies

\begin{align}
\|z\| &\leq C(|z_T| + \|b\| + \|c\| + |\rho_0|) \\
\|\rho\| &\leq C(|z_T| + \|b\| + \|c\| + |\rho_0|).
\end{align}

**Proof.** Without loss of generality we assume $t_0 = 0$. We use a fixed-point argument to prove the existence of a solution to (79). Uniqueness will be then implied by estimates (82) and (83), thanks to the linearity of the system.

We define the map $T : C^0([0,T]; P_0(\Sigma)) \to C^0([0,T]; P_0(\Sigma))$ as follows: for a fixed $\rho \in C^0([0,T]; P_0(\Sigma))$ we consider the solution $z = z(\rho)$ to equation (80), and define $T(\rho)$ to be the solution of the second equation in (79) with $z = z(\rho)$. In order to prove the existence of a fixed point of $T$, which is clearly a solution to (79), we apply Leray-Schauder Fixed Point Theorem. We remark the fact that more standard fixed point theorems are not applicable to this situation since we can not assume that $\rho$ belongs to a compact subspace of $C^0([0,T]; P_0(\Sigma))$. First of all, we note that $C^0([0,T]; P_0(\Sigma))$ is convex and that the map $T$ is trivially continuous (for the linearity of the system). Moreover, using the equation for $\rho$ in system (79), it is easy to see that $T$ is a compact map, i.e. it sends bounded sets of $C^0([0,T]; P_0(\Sigma))$ into bounded sets of $C^1([0,T]; P_0(\Sigma))$. Thus, to apply Leray-Schauder Theorem it remains to prove that the set $\{\rho : \rho = \lambda T(\rho)\}$ is bounded in $C^0([0,T]; P_0(\Sigma))$.

Let us fix a $\rho$ such that $\rho = \lambda T(\rho)$. Then the couple $(z,\rho)$ solves

\[
\begin{aligned}
-\frac{d}{dt}z(t,x) - \alpha^*(x,\Delta^2 u) \cdot \Delta z(t,x) &= \lambda(D^m F(x,m(t),1) \cdot \rho(t) + b(t,x)) \\
\frac{d}{dt}\rho(t,x) &= \sum_y \rho_y \alpha^*_y(y,\Delta^y u) + \lambda \left( \sum_y m_y(t) D\rho \alpha^*_y(y,\Delta^y u) \cdot \Delta^y z + c(t,x) \right) \\
z(T,x) &= \lambda(D^m G(x,m(T),1) \cdot \rho(T) + z_T(x)) \\
\rho(t_0,\cdot) &= \rho_0.
\end{aligned}
\]

First, we note that we can restrict to $\lambda > 0$, since otherwise $\rho = 0$. Therefore, we can use the equations (for brevity we omit the dependence of $\alpha^*$ on the second variable) to get

\[
\frac{d}{dt} \sum_x z(t,x) \rho_x(t) = -\lambda \sum_x \rho(t,x) [D^m F(x,m(t),1) \cdot \rho(t) + b(t,x)] \\
- \sum_{x,y} \rho_x(t) \alpha^*_y(x) [z(t,y) - z(t,x)] + \sum_{x,y} \rho_y(t) \alpha^*_y(y) z_x(t) \\
+ \lambda \sum_{x,y} m_y D\rho \alpha^*_y(y) \cdot \Delta^y z + c(t,x) \sum_z z(t,x).
\]

The second line is 0, using the fact that $\sum_x \rho_x(t) = 0$ and changing $x$ and $y$ in the second double sum. Integrating over $[0,T]$ and using the expression for $z(T,x)$ we obtain

\[
\lambda \sum_x \rho_x(T) [D^m G(x,m(T),1) \cdot \rho(t) + z_T(x)] - \lambda z(0) \cdot \rho_0 \\
= - \lambda \int_0^T \sum_x \rho_x(t) [D^m F(x,m(t),1) \cdot \rho(t) + b(t,x)] dt \\
+ \lambda \int_0^T \sum_{x,y} m_y D\rho \alpha^*_y(y) \cdot \Delta^y (z(t,x) - z(t,y)) dt \\
+ \lambda \int_0^T \sum_x c(t,x) z(t,x) dt - \lambda \int_0^T \rho(t,x) D^m G(x,m(T),1) \cdot \rho(T) dt,
\]

where in the third term of the sum we have also used that $\sum_{x,y} [m_y D\rho \alpha^*_y(y) \cdot \Delta^y z](t,y) = 0$. 
Dividing by \( \lambda > 0 \) and bringing the terms with \( F \) and \( G \) on the left hand side, together with the term in \( m \) and \( D_\rho \alpha^* \), we can rewrite

\[
- \int_0^T \sum_{x,y} m_y \Delta^y z D_\rho \alpha^*_x(y) \cdot \Delta^y z dt + \int_0^T \sum_x \rho(t,x) [D^m F(x,m(t),1) \cdot \rho(t)] dt \\
+ \sum_x \rho(T,x) D^m G(x,m(T),1) \cdot \rho(T) \\
= - \sum_x z_T(x) \rho(T,x) + \sum_x z(0,x) \rho_0(x) - \int_0^T \sum_x \rho(t,x) b(t,x) dt + \int_0^T \sum_x c(t,x) z(t,x) dt.
\]

We now observe that, by (Mon) and (RegFG), we have

\[
\sum_x \rho(t,x) [D^m F(x,m(t),1) \cdot \rho(t)] \geq 0,
\]

(84)

\[
\sum_x \rho(T,x) D^m G(x,m(T),1) \cdot \rho(T) \geq 0.
\]

Further assumption (12) yields

\[
- \int_0^T \sum_{x,y} m_y \Delta^y z D_\rho \alpha^*_x(y) \cdot \Delta^y z \geq C^{-1} \int_0^T \sum_x m_x |\Delta^x z|^2 dt,
\]

so that we can estimate the previous equality by

\[
C^{-1} \int_0^T \sum_x m_x |\Delta^x z|^2 dt \leq |z_T \cdot \rho(T)| + |z(0) \cdot \rho_0| + \int_0^T |c(t) \cdot z(t)| dt + \int_0^T |\rho(t) \cdot b(t)| dt
\]

(86)

On the other hand, by the equation for \( \rho \) we have

\[
\rho(t,x) = \rho_0(x) + \int_0^t \sum_y \rho(y,s) \alpha^*_x(y) ds + \int_0^t \left[ \sum_y m_y D_\rho \alpha^*_x(y) \cdot \Delta^y z + c(x) \right] ds,
\]

and thus

\[
|\rho(t,x)| \leq |\rho_0(x)| + M \int_0^t \sum_y |\rho_y| ds + C \int_0^t \left[ \sum_y m_y |\Delta^y z| + |c(x)| \right] ds,
\]

so that, by Gronwall’s Lemma and taking the sum for \( x \in \Sigma \) and the sup over \( t \in [0,T] \),

\[
||\rho|| \leq C ||\rho_0|| + C \int_0^T \sum_x \sqrt{m_x} \sqrt{\sum \Delta^x z^2} dt + C ||c||
\]

\[
\leq C ||\rho_0|| + C \int_0^T \sqrt{\sum \left( \sqrt{m_x} \right)^2} \sqrt{\sum m_x |\Delta^x z|^2} dt + C ||c||
\]

\[
= C ||\rho_0|| + C \int_0^T \sqrt{\sum m_x |\Delta^x z|^2} dt + C ||c||
\]

\[
\leq C ||\rho_0|| + C \int_0^T \sum m_x |\Delta^x z|^2 dt + C ||c||.
\]

Now we use the estimate (85) on \( \sum x m_x |\Delta^x z|^2 \) that we found above to get

\[
||\rho|| \leq C \|c\| + C ||\rho_0|| + C \left( ||\rho_0|| |z(0)| + |z_T||\rho(T)| + \int_0^T |c(t)||z(t)| dt + \int_0^T |\rho(t)||b(t)| \right)^{1/2}
\]

\[
\leq C \|c\| + C ||\rho_0|| + C \left( |z(0)|^{1/2} ||\rho_0||^{1/2} + |z_T|^{1/2} ||\rho(T)||^{1/2} + |c|^{1/2} ||z||^{1/2} + ||\rho||^{1/2} ||b||^{1/2} \right).
\]
We can further estimate the right hand side using the bound \(|z| \leq C(|z_T| + |\rho| + |b|)| of Lemma 2

\[|\rho| \leq C(|c| + |\rho_0|) + C \left[ |z_T|^{1/2} |\rho(T)|^{1/2} + (|c|^{1/2} + |\rho_0|^{1/2})(|z_T|^{1/2} + |\rho|^{1/2} + |b|^{1/2}) + |\rho|^{1/2} |b|^{1/2} \right].\]

Using the inequality \(AB \leq \varepsilon A^2 + \frac{1}{\varepsilon} B^2\) for \(A, B > 0\) we obtain

\[||\rho|| \leq C(||c|| + |z_T| + |b| + |\rho_0|) + \frac{1}{2} ||\rho||,\]

which implies \((83)\). Then \((82)\) follows from Lemma 2.

Given the solution \((u, m)\) to the system \((\text{MFG})\), with initial condition \(m_0\) for \(m\) and final condition \(G\) for \(u\), we introduce the linearized system:

\[
\begin{aligned}
\left(\text{LIN}\right) \quad v(t, x) &= D^m G(x, m(T), 1) - \mu(T), \\
\rho(t, x) &= m(t, x), \quad \mu(t, x) = \mu_0 \in P_0(\Sigma).
\end{aligned}
\]

We observe that in the RHS of the first equation \(D^m F(x, m(t), 1) \cdot \mu(t) = D^m F(x, m(t), j) \cdot \mu(t)\) for every \(j \in \Sigma\), using identity \((82)\) and the fact that \(\mu(t) \in \mathcal{P}(\Sigma)\) for every \(t\) (i.e. identity \((10)\)). For this reason we just fixed the choice to \(D^m F(x, m(t), 1)\) and \(D^m G(x, m(T), 1)\) in system \((\text{LIN})\).

The existence and uniqueness of solution \((v, \mu) \in C^1([0, T], R^d \times P_0(\Sigma))\) is ensured by Proposition 6. The aim is to show that the solution \((v, \mu)\) to system \((\text{LIN})\) satisfies

\[v(t_0, x) = D^m U(t_0, x, m_0, 1) \cdot \mu_0.\]

This proves that the solution \(U\) defined via \((71)\) is differentiable with respect to \(m_0\) in any direction \(\mu_0\), with derivative given by \((77)\), and also that \(D^m U\) is continuous in \(m\). This last assertion is implied by the following Theorem.

**Theorem 7.** Assume \((\text{RegH}), (\text{Mon})\) and \((\text{RegFG})\). Let \((u, m)\) and \((\hat{u}, \hat{m})\) be the solutions to \((\text{MFG})\) respectively starting from \((t_0, m_0)\) and \((t_0, \hat{m}_0)\). Let \((v, \mu)\) be the solution to \((\text{LIN})\) starting from \((t_0, \mu_0)\), with \(\mu_0 := \hat{m}_0 - m_0\). Then

\[||\hat{u}(t, \cdot) - u(t, \cdot) - v(t, \cdot)|| + ||\hat{m}(t, \cdot) - m(t, \cdot) - \mu(t, \cdot)|| \leq C|m_0 - \hat{m}_0|^2.\]

**Proof.** Set \(z := \hat{u} - u + v\) and \(\rho := \hat{m} - m - \mu\), they solve \((79)\)

\[
\begin{aligned}
-\frac{\partial}{\partial t} z(t, x) &= \alpha^*(x, \Delta^x u(t, x)) \cdot \Delta^x z(t, x) = D^m F(x, m(t), 1) \cdot \rho(t) + b(t, x), \\
\frac{\partial}{\partial t} \rho(t, x) &= \sum_y \rho_y(\alpha^*_x(y, \Delta^y u(t, x)) + \sum_y m_y(t)D_y \alpha^*_x(y, \Delta^y u(t, x)) \cdot \Delta^y z(t, x) + c(t, x), \\
z(t, x) &= D^m G(x, m(T), 1) \cdot \rho(T) + z_T(x), \\
\rho(t_0, \cdot) &= 0,
\end{aligned}
\]

with

\[
\begin{aligned}
b(t, x) &:= A(t, x) + B(t, x), \\
A(t, x) &:= - \int_0^1 [D_y H(x, DU + s(\Delta^x \hat{u} - \Delta^x u)) - D_y H(x, DU)] \cdot (\Delta^x \hat{u} - \Delta^x u) ds, \\
B(t, x) &:= \int_0^1 [D^m F(x, m + s(\hat{m} - m), 1) - D^m F(x, m, 1)] \cdot (\hat{m} - m) ds, \\
c(t, x) &:= \sum_y (\hat{m}_y - m_y)D_y \alpha^*_x(y, \Delta^y u) \cdot (\Delta^y \hat{u} - \Delta^y u) \\
&+ \sum_y \hat{m}_y \int_0^1 [D_y \alpha^*_x(y, \Delta^y u + s(\Delta^x \hat{u} - \Delta^x u)) - D_y \alpha^*_x(y, \Delta^y u)] \cdot (\Delta^y \hat{u} - \Delta^y u) ds, \\
z_T(x) &:= \int_0^1 [D^m G(x, m(T) + s(\hat{m}(T) - m(T)), 1) - D^m G(x, m(T), 1)] \cdot (\hat{m}(T) - m(T)) ds.
\end{aligned}
\]
Using the assumptions, namely the Lipschitz continuity of $D_pH$, $D^2_pH$, $D^mF$ and $D^mG$, and the bound $\max_x |\Delta^u u| \leq C|u|$, we estimate
\[
|b| \leq |A| + |B|,
|A| \leq C|\bar{u} - u|^2,
|B| \leq C|\bar{m} - m|^2,
|z_T| \leq C|\bar{m}(T) - m(T)|^2,
|c| \leq C|\bar{m} - m| \cdot |\bar{u} - u| + C|\bar{u} - u|^2.
\]
Applying (82) and (83) to the above system and finally (72) and (73), we obtain
\[
\text{For the first term we have}
\]
Using the assumptions, namely the Lipschitz continuity of $D_pH$, $D^2_pH$, $D^mF$ and $D^mG$, and the bound $\max_x |\Delta^u u| \leq C|u|$, we estimate
\[
|b| \leq |A| + |B|,
|A| \leq C|\bar{u} - u|^2,
|B| \leq C|\bar{m} - m|^2,
|z_T| \leq C|\bar{m}(T) - m(T)|^2,
|c| \leq C|\bar{m} - m| \cdot |\bar{u} - u| + C|\bar{u} - u|^2.
\]
Applying (82) and (83) to the above system and finally (72) and (73), we obtain
\[
|z| + |\rho| \leq C(|z_T| + ||b|| + ||c||)
\leq C (||\bar{u} - u||^2 + ||\bar{m} - m||^2 + ||\bar{m} - m|| \cdot ||\bar{u} - u||)
\leq C|\bar{m}_0 - \bar{m}_0|^2.
\]
\[\text{□}\]

5.3. Proof of Theorem 6. We are finally in the position to prove the main Theorem of this Section.

5.3.1. Existence. Let $U$ be the function defined in (74), i.e. $U(t_0, x, m_0) := u(t_0, m_0)$. We have shown in the above Theorem 7 that $U$ is $C^1$ in $m$, while $C^1$ in $t$ is clear. We compute the limit, as $h$ tends to 0, of
\[
\frac{U(t_0 + h, x, m_0) - U(t_0, x, m_0)}{h} = \frac{U(t_0 + h, x, m(t_0 + h)) - U(t_0 + h, x, m(t_0 + h))}{h} + \frac{U(t_0 + h, x, m(t_0 + h)) - U(t_0, x, m_0)}{h}.
\]
For the first term we have
\[
U(t_0 + h, x, m(t_0 + h)) - U(t_0, x, m(t_0))
= [m_s := m(t_0) + s(m(t_0 + h) - m(t_0)), \text{ for each } j]
= \int_0^1 \frac{\partial}{\partial (m(t_0 + h) - m(t_0))} U(t_0 + h, x, m_s, j)ds
= \int_0^1 D^mU(t, x, m_s, j) \cdot (m(t_0 + h) - m(t_0))ds
= \int_0^1 ds \int_{t_0}^{t_0+h} D^mU(t, x, m_s, j) \cdot \left( \sum_{k=1}^d m_k(t) \alpha^*(k, \Delta^k u(t)) \right) dt
= \int_0^1 ds \int_{t_0}^{t_0+h} \sum_{k=1}^d \sum_{z=1}^{d} m_k(t) [D^mU(t, x, m_s, j)]_z \alpha^*_z(k, \Delta^k u(t)) dt.
\]
Using equation (8), we obtain
\[
U(t_0 + h, x, m(t_0 + h)) - U(t_0, x, m(t_0))
= \int_0^1 ds \int_{t_0}^{t_0+h} \sum_{k=1}^d \sum_{z=1}^{d} m_k(t) [D^mU(t, x, m_s, k)]_z \alpha^*_z(k, \Delta^k u(t)) dt
+ \int_0^1 ds \int_{t_0}^{t_0+h} \sum_{k=1}^d \sum_{z=1}^{d} m_k(t) [D^mU(t, x, m_s, j)]_k \alpha^*_z(k, \Delta^k u(t)) dt
= \int_0^1 ds \int_{t_0}^{t_0+h} \sum_{k=1}^d \sum_{z=1}^{d} m_k(t) [D^mU(t, x, m_s, k)]_z \alpha^*_z(k, \Delta^k u(t)) dt.
where the last equality follows from
\[
\sum_{z=1}^{d} \sum_{k=1}^{d} m_k(t) \left[D^m U(t, x, m_j, j) \right]_k \alpha^*_z(k) = 0,
\]

since \( \sum_{z=1}^{d} \alpha^*_z = 0 \) because \( \alpha^*_z(k) = -\sum_{z \neq k} \alpha^*_z(k) \).

Summarizing, we have found that,
\[
U(t_0 + h, x, m(t_0 + h)) - U(t_0 + h, x, m(t_0)) = \int_0^1 ds \int_{t_0}^{t_0 + h} D^m U(t, x, m_s, y) \cdot \alpha^*(y, \Delta^y u(t)) dt dm(y).
\]

Formally, dividing by \( h \) and letting \( h \to 0 \) we get
\[
\lim_{h \to 0} \frac{U(t_0 + h, x, m(t_0 + h)) - U(t_0 + h, x, m(t_0))}{h} = \int_{\Sigma} D^m U(t_0, x, m_0, y) \cdot \alpha^*(y, \Delta^y u(t_0)) dm(y)
\]

The second term (for \( h > 0 \)) is instead
\[
U(t_0 + h, x, m(t_0 + h)) - U(t_0, x, m_0) = u_x(t_0 + h) - u_x(t_0) = h \frac{d}{dt} u_x(t_0) + o(h),
\]

and thus
\[
\lim_{h \to 0^+} \frac{U(t_0 + h, x, m(t_0 + h)) - U(t_0, x, m_0)}{h} = \frac{d}{dt} u_x(t_0).
\]

Finally, we can rewrite (59) after taking the limit \( h \to 0 \) to obtain
\[
\partial_t U(t_0, x, m_0) = -\int_{\Sigma} D^m U(t_0, x, m_0, y) \cdot \alpha^*(y, \Delta^y U(t_0, y, m_0)) dm(y)
\]
\[
+ \frac{d}{dt} u_x(t_0) = [\text{using the equation for } u]
\]
\[
= -\int_{\Sigma} D^m U(t_0, x, m_0, y) \cdot \alpha^*(y, \Delta^y U(t_0, y, m_0)) dm(y)
\]
\[
+ H(x, \Delta^x U(t_0, x, m_0)) - F(x, m_0),
\]

and thus
\[
-\partial_t U(t_0, x, m_0) + H(x, \Delta^x U(t_0, x, m_0)) - \int_{\Sigma} D^m U(t_0, x, m_0, y) \cdot \alpha^*(y, \Delta^y U) dm(y) = F(x, m_0),
\]

which is exactly (59) computed in \((t_0, m_0)\).

5.3.2. Uniqueness. Let us consider another regular solution \( V \) of (59). Since \( ||D^m V||_{\infty} \leq C \), we know that \( V \) is Lipschitz with respect to \( m \), and so is \( \Delta^y V \). From this remark and the Lipschitz continuity of \( \alpha^* \) with respect to \( p \), it follows that the equation
\[
\begin{cases}
\frac{d}{dt} \tilde{m}(t) = \sum_j \tilde{m}(t) \alpha^*(j, \Delta^j V(t, j, \tilde{m}(t))) \\
\tilde{m}(t_0) = m_0
\end{cases}
\]

has a unique solution in \([t_0, T]\).
If we now set \( \tilde{u}(t) := V(t, x, \tilde{m}(t)) \), we can compute (using for e.g. \( D^m V(\cdot, \cdot, 1) \))

\[
\frac{d}{dt} \tilde{u}(t) = \partial_t V(t, x, \tilde{m}(t)) + D^m V(t, x, \tilde{m}(t), 1) \cdot \frac{d}{dt} \tilde{m}(t)
\]

= [using the equation for \( \tilde{m} \)]

\[
= \partial_t V(t, x, \tilde{m}(t)) + D^m V(t, x, \tilde{m}(t), 1) \cdot \left( \sum_j \tilde{m}^j(t) \alpha^*(j, \Delta^j V(t, j, \tilde{m}(t))) \right)
\]

= [using identity (5) on \( D^m V(\cdot, \cdot, 1) \)]

\[
= \partial_t V(t, x, \tilde{m}(t)) + \int \sum_j D^m V(t, x, \tilde{m}(t), y) \cdot \alpha^*(y, \Delta^y V(t, y, \tilde{m}(t)) \cdot d\tilde{m}(t)(dy)
\]

= [using the equation for \( V \)]

\[
= H(x, \Delta^x V(t, x, \tilde{m})) - F(x, \tilde{m}) = H(x, \Delta^x \tilde{u}(t)) - F(x, \tilde{m}),
\]

and thus the pair \((\tilde{u}(t), \tilde{m}(t))\) satisfies

\[
\begin{cases}
- \frac{d}{dt} \tilde{u}(t, x) + H(x, \Delta^x \tilde{u}(t)) = F(x, \tilde{m}), \\
\frac{d}{dt} \tilde{m}(t) = \sum_j \tilde{m}^j(t) \alpha^j(y, \Delta^y \tilde{u}), \\
\tilde{u}(T, x) = V(T, x, \tilde{m}(T)) = G(x, \tilde{m}(T)), \\
\tilde{m}(t_0) = m_0.
\end{cases}
\]

Namely, \((\tilde{u}, \tilde{m})\) solves the system (MFG), whose solution is unique thanks to Proposition 5, so that we can conclude \( V(t_0, x, m_0) = \tilde{U}(t_0, x, m_0) \) for each \((t_0, x, m_0)\), and thus the uniqueness of \( M \) follows.

5.3.3. Regularity. It remains to prove that the unique classical solution defined via (71) is regular, in the sense of Definition 3 i.e. that \( D^m U \) is Lipschitz continuous with respect to \( m \), uniformly in \( t, x \).

So let \((u_1, m_1)\) and \((u_2, m_2)\) be two solution to (MFG) with initial conditions \( m_1(t_0) = m_0 \) and \( m_2(t_0) = m_0 \), respectively. Let also \((v_1, \mu_1)\) and \((v_2, \mu_2)\) be the associated solutions to (LIN) with \( \mu_1(t_0) = \mu_2(t_0) = \mu_0 \). Recall from equation (57) that \( v_1(t_0, x) = D^m U(t_0, x, m_0, 1) \cdot \mu_0 \) and \( v_2(t_0, x) = D^m U(t_0, x, m_0, 1) \cdot \mu_0 \), thus we have to estimate the norm \( ||v_1 - v_2|| \).

Set \( z := v_1 - v_2 \) and \( \rho := \mu_1 - \mu_2 \), they solve the linear system (79) with \( \rho_0 = 0 \) and rests

\[
b(t, x) := [D^m F(x, m_1, 1) - D^m F(x, m_2, 1)] \cdot \mu_2 + \alpha^*(x, \Delta^x u_1) - \alpha^*(x, \Delta^x u_2) \cdot \Delta^x v_2
\]

\[
c(t, x) := \sum y \mu_{2,y} [\alpha^*(y, \Delta^y u_1) - \alpha^*(y, \Delta^y u_2)]
\]

\[
+ \sum y [m_{1,y} D_y \alpha^*(y, \Delta^y u_1) - m_{2,y} D_y \alpha^*(y, \Delta^y u_2)] \cdot \Delta^x v_2
\]

\[
z_T(x) := [D^m G(x, m_1(T), 1) - D^m G(x, m_2(t), 1)] \cdot \mu_2.
\]

Using the Lipschitz continuity of \( D_y H, D_{yy} H, D^m F \) and \( D^m G \), applying the bounds (52) to \( v_2 \) and (53) to \( \mu_2 \) and also (72) and (73), we can estimate

\[
||b|| \leq C||m_1 - m_2|| + ||v_2|| + C||u_1 - u_2|| \cdot ||v_2|| \leq C|m_1 - m_2| \cdot |\mu_0|
\]

\[
||c|| \leq C||m_1 - m_2|| \cdot ||v_2|| + C||u_1 - u_2|| \cdot ||v_2|| \leq C|m_1 - m_2| \cdot |\mu_0|
\]

\[
||z_T|| \leq C||m_1 - m_2|| \cdot ||v_2|| \leq C|m_1 - m_2| \cdot |\mu_0|.
\]

Then (52) gives

\[
||z|| \leq C(||b|| + ||c|| + ||z_T||) \leq C|m_1 - m_2| \cdot |\mu_0|.
\]
which, since $z(t_0, x) = (D^m U(t_0, x, m^1_0, 1) - D^m U(t_0, x, m^2_0, 1)) \cdot \mu_0$, yields
\[
\max_x |D^m U(t_0, x, m^1_0, 1) - D^m U(t_0, x, m^2_0, 1)| \\
\leq C \max_x \sup_{\mu_0 \in P_0(\Sigma)} \left| \frac{(D^m U(t_0, x, m^1_0, 1) - D^m U(t_0, x, m^2_0, 1)) \cdot \mu_0}{|\mu_0|} \right| \\
\leq C|m^1_0 - m^2_0|.
\]

6. Conclusions

Let us summarize the results we have obtained. The two set of assumptions are given in Section 2.2 and verified in Example 2.1.

(1) If (H1) holds and there exists a regular solution $U$ to the Master Equation (M), in the sense of Definition 3 then the value functions of the $N$-player game converge to $U$ (Theorem 1) and the optimal trajectories (28) satisfy a propagation of chaos property, i.e, they converge to the limiting i.i.d solution to (29) (Theorem 2);

(2) Under the assumptions for convergence, the empirical measure process (47) associated with the optimal trajectories satisfies a Central Limit Theorem (Theorem 4) and a Large Deviation Principle with rate function $I$ in (50) (Theorem 5);

(3) Assuming (RegH), (Mon) and (RegFG), there exists a unique classical solution to (M) and it is also regular in the sense of Definition 3.

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(A. Cecchin and G. Pelino)

DEPARTMENT OF MATHEMATICS “TULLIO LEVI CIVITA”

UNIVERSITY OF PADUA

VIA TRIESTE 63, 35121 PADOVA, ITALY

E-mail address, A. Cecchin: alexos.cecchin@math.unipd.it

E-mail address, G. Pelino: guglielmo.pelino@math.unipd.it