On the Modelling of Uncertain Impulse Control for Continuous Markov Processes

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Abstract

The use of coordinate processes for the modelling of impulse control for general Markov processes typically involves the construction of a probability measure on a countable product of copies of the path space. In addition, admissibility of an impulse control policy requires that the random times of the interventions be stopping times with respect to different filtrations arising from the different component coordinate processes. When the underlying strong Markov process has continuous paths, however, a simpler model can be developed which takes the single path space as its probability space and uses the natural filtration with respect to which the intervention times must be stopping times. Moreover, this model construction allows for uncertain impulse control whereby the decision maker selects an impulse but the intervention may result in a different impulse occurring. This paper gives the construction of the probability measure on the path space for an admissible intervention policy subject to an uncertain impulse mechanism. An added feature is that when the intervention policy results in deterministic distributions for each impulse, the paths between interventions are independent and, moreover, if the same distribution is used for each impulse, then the cycles following the initial cycle are identically distributed. This paper also identifies a class of impulse policies under which the resulting controlled process is Markov. The decision to use an \((s, S)\) ordering policy in inventory management provides an example of an impulse policy for which the process is Markov and has i.i.d. cycles so a benefit of the constructed model is that one is allowed to use classical renewal arguments.

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1 Introduction

Impulse control was introduced by Bensoussan and Lions (1975) and arises naturally in a wide variety of applications such as the management of inventory, exchange rates, and financial portfolios. It occurs when the state of the system is adjusted in a discontinuous fashion and the cost includes a fixed positive charge for each such intervention.

This paper examines the construction of the mathematical model for impulse control. Intuitively, the state process evolves as a strong Markov process until the decision maker intervenes to instantly move the state to a new location (at a cost) after which the process again evolves as the strong Markov process starting from this new state. The decision maker then waits to intervene again to instantly move the process at an additional cost and these actions continue into the infinite future. The model as described can be clearly understood so some papers simply assume the existence of the impulse-controlled state process (see e.g., Richard (1977), Cadenillas and Zapatero (2000), Runggaldier and Yashudo (2018)).

For some applications one desires additional structure to the model such as independence of the evolutions of the process between interventions for some classes of intervention policies. Intuitively it is again “obvious” that such a model is possible but it is challenging to move beyond the informal description of the process to a specific mathematical model; something that has been described as a “hard problem” (see Menaldi and Robin (2017)) for which the “formal probabilistic apparatus . . . is unfortunately rather cumbersome” (Davis, 1993, p. 227).

A typical approach to defining the impulse-controlled process is to start by setting the sample space \( \Omega \) to be the path space. For example, Harrison et al. (1983) and Ormeci et al. (2008) define the uncontrolled process to be a drifted Brownian motion in \( \mathbb{R} \) while Bensoussan and Lions (1984) examines a more general strong solution to a stochastic differential equation in \( \mathbb{R}^n \). Since the fundamental evolution of the process for both of these models is continuous, these papers set \( \Omega = C_{\mathbb{R}^n}[0, \infty) \) with \( n = 1 \) in the first two papers. The authors let \( X \) denote the coordinate process, \( \mathcal{F} = \sigma(X(t) : t \geq 0) \) and use the natural filtration \( \{\mathcal{F}_t\} \) in which \( \mathcal{F}_t = \sigma(X(s) : 0 \leq s \leq t) \). The impulse policies consist of an increasing sequence of intervention times, say \( \{\tau_k\} \), and a sequence of impulse random variables \( \{\xi_k\} \). Each intervention time \( \tau_k \) must be an \( \mathcal{F}_{\tau_k} \)-stopping time and the corresponding impulse \( \xi_k \) must be \( \mathcal{F}_{\tau_k} \)-measurable.

This path space approach is also used by Robin (1978), Stettner (1983) and Lepeltier and Marchal (1984) but with more complexity in that \( \Omega = C_{\mathbb{R}^n}[0, \infty) \) so the coordinate process \( X \) includes the possibility of the process dynamics having jumps. The model, however, is built on the countable product \( \Omega = \prod_{i=0}^{\infty} \Omega \) in which the different components are used for the evolution of the state process following the different interventions. In each of these models, the intervention decisions are made subject to different filtrations \( \{\mathcal{F}_n\} \), \( n \in \mathbb{N}_0 := \{0\} \cup \mathbb{N} \), in which \( \{\mathcal{F}_n\} \) is generated by the coordinate processes in the first \( n \) components. A small but significant difference between these constructions occurs at the times when \( X \) jumps and the decision maker immediately intervenes; such a time is one of the intervention times \( \tau_k \). The models in Robin (1978) and Stettner (1983) define \( X \) over the successive half-closed intervals \( \{[\tau_k, \tau_{k+1}) : k \in \mathbb{N}_0\} \) resulting in càdlàg paths. However, the actual value to which \( X \) moves prior to the intervention which causes this intervention is never captured in this model. A difference between the models in Robin (1978) and Stettner (1983) is that the
latter paper explicitly considers the possibility of multiple interventions at the same time. Christensen (2014) adopts the construction of Stettner and tries to distinguish between the three locations $X_{\tau_n-}$, $X_{\tau_n-}$, and $X_{\tau_n}$ where the first is the left limit of $X$ at the time $\tau_n$, the second gives the location of $X$ following the natural jump of the process, if any, at time $\tau_n$ but before the impulse is applied and the last expression gives the state of $X$ after the impulse takes effect. Unfortunately, the model of Stettner (1983) defines $X$ over the half-closed intervals $\{[\tau_k, \tau_{k+1}): k \in \mathbb{N}_0\}$ so the position $X_{\tau_n-}$ is never part of the information in the natural filtration.

Lepeltier and Marchal (1984) adjusts the model by having the impulse only take effect after the time of intervention so the natural filtration of this state process observes $X$ after the jump but before the impulse. The impulse control model in Davis (1993) is quite similar to that of Lepeltier and Marchal (1984) but applies the construction to piecewise deterministic processes. One unfortunate aspect of these latter, more precise models is that the paths of the impulse-controlled process will not be càdlàg at jump times of the process which immediately bring about interventions.

Preceding the introduction of impulse control models, Ikeda et al. (1966) used a similar construction to extend a killed Markov process to infinite time so that at each time of death the process is reinitialized and in doing so, the authors define a single filtration with respect to which the rebirth times are stopping times and the resulting process is strong Markov. This addresses the issue of having multiple filtrations of information, though the single filtration is quite complex. Complete proofs of the construction are given in Meyer (1975).

An alternate approach for modelling the impulse-controlled process (see e.g., Menaldi (1980), Alvarez (2004), Øksendal and Sulem (2005), Jack and Zervos (2006), Frey and Seydel (2010), Helmes et al. (2015)) is to start with a given filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ on which the fundamental evolution of the process can be defined for each initial distribution. The impulse-controlled process is then constructed iteratively over the successive intervals $\{[\tau_k, \tau_{k+1})\}$ by pasting together a shift of the fundamental process having the required different initial positions given by the impulses. The impulse policy requires each intervention time to be an $\{\mathcal{F}_t\}$-stopping time. Since $\{\mathcal{F}_t\}$ is assumed given, it is possible that more information than that generated by the state process is included in this filtration.

Another common approach to the modelling of impulse-controlled process is to simply refer to the constructions given in one of the aforementioned papers (see e.g., Korn (1997), Menaldi and Robin (2017) and Palczewski and Stettner (2017)).

This paper develops a simple model for the impulse control of Markov processes having continuous paths which provides extra properties of the process. The path continuity implies that the left limit $X(\tau_k-)\text{ is always the state at which the impulse is applied which therefore has two important consequences. First, when the impulse occurs at time } \tau_k, the natural filtration includes the “jump from state” and, second, the resulting path is càdlàg. Both of these observations contrast with the models of Lepeltier and Marchal (1984) and Davis (1993) when the underlying process has inherent discontinuities in its paths. Their characteristic that the impulse takes effect immediately following $\tau_k$ is needed for the state from which the process jumps to be included in the natural filtration but, in general, leads to lâglâd paths.

The simplicity of our model is that it is built using the coordinate process $X$ on the filtered
space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\})\) in which \(\Omega = D_{\mathcal{E}}[0, \infty)\) with \(\mathcal{E}\) being the state space, \(\mathcal{F} = \sigma(X(s) : s \geq 0)\) and the filtration is the natural filtration generated by \(X\) so \(\mathcal{F}_t = \sigma(X(s) : 0 \leq s \leq t)\). It is shown that, for each admissible impulse control policy \((\tau, Z)\) defined below, there exists a family of probability measure \(\{P_x^{(\tau, Z)} : x \in \mathcal{E}\}\) on \((\Omega, \mathcal{F})\) under which \(X\) is the desired impulse-controlled process.

In contrast with almost all papers on impulse control previously mentioned, a special feature of the model is the inclusion of uncertainty in the application of the impulses. By this we mean that we select a control variable which, in conjunction with the state of the process at the instant before the intervention, determines a distribution on the state space to which the impulse moves the process. An example of such an uncertain impulse arises in inventory management where an order is placed but only some random fraction of the ordered amount is delivered or perhaps the entire order is delivered but some random quantity must be discarded due to manufacturing defects. A very similar model is studied in Korn (1997) for the case of a one-dimensional diffusion given as the solution to a stochastic differential equation.

Informally defining the model for the impulse-controlled state process, Korn refers to the construction of Bensoussan and Lions (1984) and indicates that this can be adapted to allow a distribution for the post-impulse location. Moreover, he assumes that the fundamental diffusion \(X\) is defined on a given filtered probability space. As indicated above, our model lives on the single space \(\Omega = D_{\mathcal{E}}[0, \infty)\) using the natural filtration of the coordinate process.

Motivated by consideration of a long-term average cost criterion, a second property of our model is that it results in the independence of the controlled process over the different intervals between interventions when the policy applies deterministic impulse distributions at the intervention times. Further, all but possibly the first of these cycles will be identically distributed when the same deterministic distribution is used. In order to have this independence, we use a countable product space similar to Robin (1978), Stettner (1983), Bensoussan and Lions (1984) and Lepeltier and Marchal (1984) but the impulse policy is initially defined on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\})\) and then carefully related to the product space. In this manner a probability measure is established on the countable product space first and the desired impulse-controlled process \(\tilde{X}\) is defined as in the other references. However, since \(\tilde{X}\) is càdlàg, its distribution on \((\Omega, \mathcal{F})\) then provides the desired measure \(P_x^{(\tau, Z)}\).

Our construction fails when the underlying process has jumps. For such processes, Lepeltier and Marchal (1984) provides the best model for the impulse controlled process. Since the process may have làglàd paths, our simpler construction would require the theory for the space of such functions similar to the one for the space of càdlàg functions.

The focus of all the impulse control papers we reviewed is on describing the controlled process and then analyzing problems of interest. Those papers which detail the construction provide a model for the process, based on having a Markov or strong Markov process for the evolution between interventions, but they do not address whether the resulting process is Markov or strong Markov. It is somewhat obvious that a general result is not possible for all policies since the admissible policies only require the intervention times to be stopping times and the intervention amounts to be measurable with respect to the associated stopped \(\sigma\)-algebras. For example, the decision maker could define a policy that only depends on the initial position of the process. The resulting controlled process cannot be Markov since the information about the initial position is not included in the \(\sigma\)-algebra generated by the state
of the process at or just prior to an intervention time. We identify a subclass of policies for which the resulting process is Markov.

The paper is organized as follows. Section 2 describes the fundamentals of the underlying Markov process, the distributions determined by the interventions which select the new states to which the process moves and defines the class of admissible nominal impulse policies. Given such an admissible policy, the existence and uniqueness of the corresponding measure on the countable product of spaces is proven in Section 3 and this is used as indicated above to obtain the measure on the space of càdlàg paths. Section 4 defines a subclass of policies under which the resulting controlled process is Markov.

2 Model Fundamentals

**Process Dynamics.** The model consists of dynamics which describe the evolution of the process in the absence of any interventions (as well as between the interventions). Let \( \mathcal{E} \) be a complete, separable metric space in which the process evolves. Since the impulse-controlled process will have at most countably many discontinuities arising from the intervention decisions, we choose to describe the model on the space \( \Omega := D_{\mathcal{E}}[0, \infty) \) of càdlàg functions. Let \( X : \Omega \to D_{\mathcal{E}}[0, \infty) \) be the coordinate process such that \( X(t, \omega) = \omega(t) \) for all \( t \geq 0 \), let \( \mathcal{F} = \sigma(X(t) : t \geq 0) \) and \( \{\mathcal{F}_t\} \) be the natural filtration. Furthermore let \( \{\mathbb{P}_x : x \in \mathcal{E}\} \) be a family of measures on \( (\Omega, \mathcal{F}) \) such that \( (\Omega, \mathcal{F}, X, \{\mathcal{F}_t\}, \{\mathbb{P}_x, x \in \mathcal{E}\}) \) is a strong Markov family in the sense of Definition 2.6.3 of Karatzas and Shreve (1988).

Notice that for \( \nu \in \mathcal{P}(\mathcal{E}) \), \( \mathbb{P}_\nu \) defined by \( \mathbb{P}_\nu(\cdot) = \int_\mathcal{E} \mathbb{P}_x(\cdot) \nu(dx) \) defines a measure on \( (\Omega, \mathcal{F}) \) under which \( X \) is the strong Markov process with \( X(0) \) having distribution \( \nu \); see, for example, Remark 2.6.8 of Karatzas and Shreve (1988). Finally we assume throughout the paper that the family \( \{\mathbb{P}_x : x \in \mathcal{E}\} \) satisfies the following support condition:

**Condition 2.1.** For each \( x \in \mathcal{E}, \{\mathbb{P}_x\} \) has its support in \( C_{\mathcal{E}}[0, \infty) \subset \Omega \). In other words, the coordinate process \( X \) is continuous.

Since for any initial distribution \( \nu \in \mathcal{P}(\mathcal{E}) \) the coordinate process \( X \) under \( \mathbb{P}_\nu \) gives the desired evolution of the process without any interventions, in the sequel we denote this process by \( X_0 \) and refer to it as the fundamental strong Markov process.

**Uncertain Impulse Mechanism.** Let \( (\mathcal{Z}, \mathfrak{I}) \) be a measurable space representing the impulse control decisions. Let \( \mathcal{Q} = \{Q_{(y,z)} : (y, z) \in \mathcal{E} \times \mathcal{Z}\} \) be a given family of probability measures on \( \mathcal{E} \) such that for each \( \Gamma \in \mathcal{B}(\mathcal{E}) \), the mapping \( (y, z) \mapsto Q_{(y,z)}(\Gamma) \) is \( \mathcal{B}(\mathcal{E}) \otimes \mathfrak{I} \)-measurable. Given the nominal intervention levels \( (y, z) \), the coordinate random variable \( V \) defined on \( (\mathcal{E}, \mathcal{B}(\mathcal{E}), Q_{(y,z)}) \) by \( V(v) = v \) for all \( v \in \mathcal{E} \) has distribution \( Q_{(y,z)} \).

**Uncertain Impulse Mechanism: Special Case.** Let \( \mathcal{Q} = \{Q_{(y,z)} : (y, z) \in \mathcal{E}^2\} \) be a given family of probability measures on \( \mathcal{E} \) such that for each \( \Gamma \in \mathcal{B}(\mathcal{E}) \), the mapping \( (y, z) \mapsto Q_{(y,z)}(\Gamma) \) is measurable with respect to \( \mathcal{B}(\mathcal{E}^2) \).

The way to view the uncertain impulses in the special case is that at a time when the decision maker intervenes, the process \( X \) is at \( y \) and the aim is to instantly move the process to \( z \), the distribution of the actual position of \( X \) following this intervention is given by \( Q_{(y,z)} \). We refer to \( z \) as the nominal impulse since this is the place to which \( X \) aims to jump by this.
intervention. Motivated by the special case, we also use the moniker of “nominal impulse” to refer to the choice of impulse control $z \in \mathcal{Z}$ in general since it determines the distribution $Q_{(y,z)}$ of the new location of $X$ following the intervention.

We now define a nominal impulse policy. In order to do so, we need to specify the filtration of information used by the decision maker to determine the nominal interventions. Let $\{\mathcal{F}_t\}$ be given by $\mathcal{F}_t = \sigma(X(s) : 0 \leq s < t)$ for $t \geq 0$. It is also important to specify the $\sigma$-algebra of information available prior to a stopping time. Let $\eta$ be an $\{\mathcal{F}_t\}$-stopping time. The $\sigma$-algebra $\mathcal{F}_\eta := \sigma(\{A \cap \{\eta > t\} : A \in \mathcal{F}_t\})$.

**Definition 2.2 (Nominal Impulse Policy).** A nominal impulse policy $(\tau, Z) = \{(\tau_k, Z_k) : k \in \mathbb{N}\}$ is a sequence of pairs in which $\{\tau_k\}$ is non-decreasing with $\tau_k \to \infty$ as $k \to \infty$ and, for each $k \in \mathbb{N}$,

- $\tau_k$ is an $\{\mathcal{F}_t\}$-stopping time and
- $Z_k$ is a $\mathcal{Z}$-valued, $\mathcal{F}_{\tau_k}$/$\mathcal{B}$-measurable random variable.

The subtle requirement that each $\tau_k$ be a stopping time relative to $\{\mathcal{F}_t\}$ prevents the decision maker from making a decision based on seeing the new location of $X$ at an intervention time. In addition we observe that ultimately each $Z_k$ will only be relevant on the set $\{\tau_k < \infty\}$ since no $k$th impulse action is taken on the set $\{\tau_k = \infty\}$.

## 3 Existence Result

Before constructing the model for the impulse-controlled process, it is helpful to state a set of necessary and sufficient conditions on a random time $\tau_k : \Omega \to [0, \infty]$ for it to be an $\{\mathcal{F}_t\}$-stopping time. These conditions play a subtle but important role in our construction of the measure on the countable product space of càdlàg paths. A similar result was first obtained in Galmarino (1963) when $\tau_k$ is an optional time; that is, when $\{\tau_k < t\}$ is $\mathcal{F}_t$-measurable for each $t \geq 0$. The analogous characterization of a random time $\tau_k$ being an $\{\mathcal{F}_t\}$-stopping time was obtained in Theorem 1.3 of Courrège and Priouret (1965). It also holds, mutatis mutandis, when $\tau_k$ is an $\{\mathcal{F}_t\}$-stopping time; we state the result in the form we use. The theorem relies on the following equivalence relation between paths $\omega_1, \omega_2 \in \Omega$: for each $t \geq 0$, $\omega_2 \overset{R^X_t}{\sim} \omega_1$ holds if and only if $\omega_2(s) = \omega_1(s)$ for all $s < t$. (Note that when $t = 0$, $\mathcal{F}_{0-} = \sigma(X(0-))$ is the $\sigma$-algebra generated by the process locations prior to any intervention at time 0 and the condition $\omega_2 \overset{R^X_t}{\sim} \omega_1$ requires $\omega_2(0-) = \omega_1(0-)$.)

**Theorem 3.1.** For a mapping $\tau_k : \Omega \to [0, \infty]$ to be a stopping time with respect to $\{\mathcal{F}_t\}$, it is necessary and sufficient that $\tau_k$ be $\mathcal{F}$-measurable and that for all $t \geq 0$, and $\omega_1, \omega_2 \in \Omega$

$$\tau_k(\omega_1) \leq t \text{ and } \omega_2 \overset{R^X_t}{\sim} \omega_1 \text{ implies } \tau_k(\omega_2) = \tau_k(\omega_1).$$

**Proof.** Theorem 1.3 of Courrège and Priouret (1965) has $\tau_k$ being an $\{\mathcal{F}_t\}$-stopping time with $R^X_t$ defined using $s \leq t$. The proof remains valid with the slight modification in the statement of the theorem.
The following corollary follows by selecting $t = \tau_k(\omega_1)$.

**Corollary 3.2** (Corollary 1 of Courrège and Priouret (1965) modified). If $\tau_k$ is an $\{\mathcal{F}_t\}-$stopping time, the relation $\omega_2(s) = \omega_1(s)$ for all $s < \tau_k(\omega_1)$ implies that $\tau_k(\omega_2) = \tau_k(\omega_1)$.

We now give the theorem which establishes the model for the uncertain impulse-controlled process corresponding to a nominal impulse policy.

**Theorem 3.3.** Let $(\tau, Z)$ be a nominal impulse policy. For each $k \in \mathbb{N}$, define the pre-impulse location $Y_k = X(\tau_k-)$ with the nominal impulse being $Z_k$ on the set $\{\tau_k < \infty\}$. Then there exists a probability measure $\mathbb{P}^{(\tau, Z)}_x$ on $(\Omega, \mathcal{F})$ under which the coordinate process $X$ satisfies the following properties:

- $X$ is the fundamental strong Markov process on the interval $[0, \tau_1)$ with $X(0) = x$ a.s.; and
- for each $k \in \mathbb{N}$, on the set $\{\tau_k < \infty\}$, $X$ is again the fundamental strong Markov process on the interval $[\tau_k, \tau_{k+1})$ with $X(\tau_k)$ having conditional distribution $Q_{(Y_k, Z_k)}$, given $(Y_k, Z_k)$.

Moreover, when the policy is such that the pre-impulse location and nominal impulse pairs $\{(y_k, z_k) : k \in \mathbb{N}\}$ form a deterministic sequence in $\mathcal{E} \times \mathcal{Z}$, then the coordinate process over the cycles $\{X(t) : \tau_k \leq t < \tau_{k+1}\}, k \in \mathbb{N}_0$, are independent under $\mathbb{P}^{(\tau, Z)}_x$. Furthermore, when all of these are the same deterministic pair $(y, z)$, the cycles for $k \in \mathbb{N}$ are identically distributed under $\mathbb{P}^{(\tau, z)}_x$.

**Proof.** In order to have a model in which the a nominal $(y, z)$ policy will result in independent cycles, we initially follow the approach of Robin (1978), Stettner (1983) and Lepeltier and Marchal (1984) by building the model on a product space from which we use the coordinate processes in the components to define the various cycles of the impulse-controlled process. However, our components have two coordinates in order to determine the position following an **uncertain** impulse. Define the sequence of measurable spaces $\{(E_k, \mathcal{G}_k) : k \in \mathbb{N}_0\}$ with

$$
(E_0, \mathcal{G}_0) = (\Omega_0, \mathcal{F}_0) := (\Omega, \mathcal{F}) \quad \text{and} \quad (E_k, \mathcal{G}_k) = (\mathcal{E}_k \times \Omega_k, \mathcal{B}(\mathcal{E}_k) \otimes \mathcal{F}_k) := (\mathcal{E} \times \Omega, \mathcal{B}(\mathcal{E}) \otimes \mathcal{F}), \quad k \in \mathbb{N},
$$

and denote an element $e_0 \in E_0$ by either $e_0$ or $\omega_0$ while for $k \in \mathbb{N}$, an element $e_k = (v_k, \omega_k)$. For each $k \in \mathbb{N}$, let $\mathcal{T}_k^\mathcal{E} = \{\emptyset, \mathcal{E}_k\}$ be the trivial $\sigma$-algebra on $\mathcal{E}_k$. We build the model by iteratively adding uncertain impulse interventions through transition functions. We will need a fixed element $\tau \in \mathcal{E}$ such that the transition function picks this initial position from one coordinate to the next when an intervention time is infinite. Also a fixed impulse control decision $\tau \in \mathcal{Z}$ is needed simply to define the choice of impulse for those paths having an infinite intervention time, even though it will never be used.

Due to its uncertainty, each additional intervention uses a transition function to pick an initial position for the measure on the path space. Thus the transitions involve two types of transition functions. For clarity, we refer to the selection of the initial position using a transition function but the addition of a new component pair uses a transition kernel.
Let \((\tau, Z)\) be a nominal impulse policy and, for simplicity of notation, define \(\tau_0 = 0\).

**No Interventions.** We begin with the measure \(\mathbb{P}_x\) on \((\Omega_0, \mathcal{F}_0)\). Notice that the coordinate process \(X_0\) on \(\Omega_0\) is the fundamental strong Markov process defined for \([0, \infty)\); it will eventually be used to define the controlled process \(\widetilde{X}\) prior to the first intervention. For notational consistency with later developments, also denote the measure \(\mathbb{P}_x\) by \(\mathbb{P}_x^{(0)}\).

**First Intervention.** We define a transition kernel \(P_1 : E_0 \times \mathcal{G}_1 \to [0, 1]\) in two steps wherein the first step uses a transition function to select the initial distribution on \(\mathcal{E}_1\) while the second step establishes the process dynamics. Since \(E_0 = \Omega_0 = \Omega\), begin by defining the mapping \(T_0 : E_0 \to \Omega\) to be the identity mapping; thus \(T_0(\omega_0) = \omega_0\). Now define the first intervention \((\tilde{\tau}_1, \tilde{Z}_1)\) on \(E_0\) by

\[
\tilde{\tau}_1(\omega) := \tau_1(T_0(\omega_0)) = \tau_1(\omega_0)
\]

and

\[
\tilde{Z}_1(\omega) := \begin{cases} Z_1(T_0(\omega_0)) = Z_1(\omega_0), & \text{on } \{\omega_0 \in E_0 : \tilde{\tau}_1(\omega_0) < \infty\}, \\ \tau, & \text{on } \{\omega_0 \in E_0 : \tilde{\tau}_1(\omega_0) = \infty\}, \end{cases}
\]

and using \(\tilde{\tau}_1\) define the random variable \(\tilde{Y}_1 : E_0 \to \mathcal{E}_0\) by

\[
\tilde{Y}_1(\omega) := \begin{cases} X_0(\tilde{\tau}_1(\omega_0) -, \omega_0), & \text{on } \{\tilde{\tau}_1 < \infty\}, \\ \tau, & \text{on } \{\tilde{\tau}_1 = \infty\}, \end{cases}
\]

in which \(X_0\) is the coordinate process on \(\Omega_0\). (Notice that this notation is consistent when using \(\mathbb{P}_x\) on \((\Omega, \mathcal{F})\) to define the fundamental Markov process.) As with \(\tilde{Z}_1\), the definition of \(\tilde{Y}_1\) on \{\(\tilde{\tau}_1 = \infty\}\} is merely for completeness since \(\tilde{Y}_1\) will never be used in the model for the controlled process when \(\tilde{\tau}_1\) is infinite. It is now helpful to change notation by setting \(\omega_0 = e_0 \in E_0\). Also recall that \(\tau \in \mathcal{E}\) and each \(\mathcal{E}_k = \mathcal{E}\).

(1) View \(\tau \in \mathcal{E}_1\). Define the transition function \(Q_1 : E_0 \times \mathcal{B}(\mathcal{E}_1) \to [0, 1]\) such that for each \(e_0 \in E_0\) and \(G_1 \in \mathcal{B}(\mathcal{E}_1)\),

\[
Q_1(e_0, G_1) = \begin{cases} Q(\tilde{Y}_1(e_0), \tilde{Z}_1(e_0))(G_1), & \text{on } \{e_0 \in E_0 : \tilde{\tau}_1(e_0) < \infty\}, \\ \delta_\tau(G_1), & \text{on } \{e_0 \in E_0 : \tilde{\tau}_1(e_0) = \infty\}. \end{cases}
\]

Notice that on the set \{\(\tilde{\tau}_1 < \infty\}\}, for each \(e_0 \in E_0\), \(Q(\tilde{Y}_1(e_0), \tilde{Z}_1(e_0))\) is \(\mathcal{G}_0\)-measurable so \(Q_1\) is a transition function as claimed.

(2) On \((\mathcal{E}_1, \mathcal{B}(\mathcal{E}_1))\), define \(V_1(v_1) = v_1\) to be the coordinate random variable. Then on the set \{\(\tilde{\tau}_1 < \infty\}\} under the random measure \(Q(\tilde{Y}_1, \tilde{Z}_1)\) and conditional on \((\tilde{Y}_1, \tilde{Z}_1)\), \(V_1\) has distribution \(Q(\tilde{Y}_1, \tilde{Z}_1)\) whereas on \{\(\tilde{\tau}_1 = \infty\}\}, \(V_1 = \tau, Q_1(e_0, \cdot)\)-almost surely. Now recall (3.1) and let \(\mathbb{P}_{V_1}\) be the measure on \(\Omega_1\) such that the coordinate process \(X_1\) on \((\Omega_1, \mathcal{F}_1)\) is the fundamental strong Markov process defined for \([0, \infty)\) with \(\mathbb{P}_{V_1}(X_1(0) = V_1) = 1\).

Using \(Q_1\) and \(\mathbb{P}_{V_1}\), the transition kernel \(P_1 : E_0 \times \mathcal{G}_1 \to [0, 1]\) is specified by

\[
P_1(e_0, G_1 \times \Gamma_1) = \int_{G_1} \mathbb{P}_{V_1(v_1)}(\Gamma_1) Q_1(e_0, dv_1), \quad G_1 \in \mathcal{B}(\mathcal{E}_1), \Gamma_1 \in \mathcal{F}_1.
\]

\[
(3.3)
\]
To complete the modelling with the first uncertain intervention, we define the measure $\mathbb{P}_x^{(1)}$ on $(E_0 \times E_1, \mathcal{G}_0 \times \mathcal{G}_1)$ such that for each $\Gamma_0 \in \mathcal{G}_0$, $G_1 \in \mathcal{B} (\mathcal{E}_1)$, and $\Gamma_1 \in \mathcal{F}_1$, $\mathbb{P}_x^{(1)} (\Gamma_0 \times G_1 \times \Gamma_1) = \int_{\Gamma_0} P_1 (e_0, G_1 \times \Gamma_1) \mathbb{P}_x^{(0)} (de_0)$.

Notice that for $\Gamma_0 \in \mathcal{G}_0$, we have $\mathbb{P}_x^{(1)} (\Gamma_0 \times \mathcal{E}_1 \times \Omega_1) = \mathbb{P}_x^{(0)} (\Gamma_0)$.

The vector of coordinate processes and variable $(X_0, V_1, X_1)(\cdot, (\omega_0, v_1, \omega_1)) = (\omega_0(\cdot), v_1(\cdot))$ on $(E_0 \times E_1, \mathcal{G}_0 \times \mathcal{G}_1, \mathbb{P}_x^{(1)})$ then has:

- $X_0$ being the fundamental strong Markov process with $\mathbb{P}_x^{(1)} (X_0(0) = x) = 1$;
- on the set $\{\tau_1 < \infty\}$ and given $(Y_1, Z_1)$, $V_1$ has distribution $Q_{(Y_1, Z_1)}$, $X_1$ is the fundamental strong Markov process with $\mathbb{P}_x^{(1)} (X_1(0) = V_1) = 1$;
- whereas on $\{\tau_1 = \infty\}$, $V_1 = \overline{\tau}$ and $X_1$ is the fundamental strong Markov process with $X_1(0) = \overline{\tau}$.

In addition, observe that when $\tau_1 < \infty$ $\mathbb{P}_x^{(1)}$-a.s. for a deterministic $(y, z)$ nominal intervention, $Q_{(Y_1(\omega_0), Z_1(\omega_0))} = Q_{(y, z)}$ for almost all $\omega_0$ and so

$$\mathbb{P}_x^{(1)} (\Gamma_0 \times E_1 \times \Gamma_1) = \mathbb{P}_x^{(0)} (\Gamma_0) \cdot \int_{E_1} \mathbb{P}_{V_1(\omega)} (\Gamma_1) Q_{(y, z)} (dv_1)$$

establishing independence of $X_0$ and $X_1$ for such an uncertain impulse intervention.

Second Intervention. Denote points in $E_0 \times E_1$ by $e_1 = (\omega_0, v_1, \omega_1)$. The first task is to extend the definition of the initial nominal impulse $(\tau_1, \bar{Z}_1)$ to $E_0 \times E_1$. First, let $\pi_0 : E_0 \times E_1 \to E_0$ be the projection mapping. Next define

$$(\tau_1(e_1), \bar{Z}_1(e_1)) = (\tau_1(\pi_0(e_1)), \bar{Z}_1(\pi_0(e_1))) = (\tau_1(\omega_0), \bar{Z}_1(\omega_0))$$

and observe that the slight abuse of notation is allowed since $(\tau_1, \bar{Z}_1)$ is consistently defined on $E_0$ and $E_0 \times E_1$.

Now let $T_1 : E_0 \times E_1 \to \Omega$ be the mapping such that

$$\overline{\tau}_1(t) := T_1(e_1)(t) = \begin{cases} \omega_0(t), & 0 \leq t < \tau_1(\omega_0), \\ \omega_1(t - \tau_1(\omega_0)), & t \geq \tau_1(\omega_0). \end{cases}$$

Define the random time $\tau_2 : E_0 \times E_1 \to \overline{\mathbb{R}}_+$ and nominal impulse $\bar{Z}_2$ on $E_0 \times E_1$ by

$$\tau_2(e_1) = \begin{cases} \tau_2(\overline{\tau}_1) & \text{when } \tau_2(\overline{\tau}_1) < \infty, \\ \infty & \text{when } \tau_2(\overline{\tau}_1) = \infty, \end{cases}$$

$$\bar{Z}_2(e_1) = \begin{cases} Z_2(\overline{\tau}_1) & \text{when } \tau_2(e_1) < \infty, \\ \bar{\tau} & \text{when } \tau_2(e_1) = \infty. \end{cases}$$

We now make an important observation relating $\tau_1(\omega_0)$ and $\overline{\tau}_1(\omega_0)$ since $\omega_0, \overline{\tau}_1(\omega_0) \in \Omega = \Omega_0$. By definition $\overline{\tau}_1 = \omega_0$ on $[0, \tau_1(\omega_0))$. So since $\tau_1$ is an $\{\mathcal{F}_t\}$-stopping time, Corollary 3.2 shows that $\tau_1(\overline{\tau}_1) = \tau_1(\omega_0)$ and hence for each $\overline{\tau}_1 = (\overline{\tau}_1, v_1, \omega_1) \in E_0 \times \mathcal{E}_1 \times E_1$,

$$\tau_1(\overline{\tau}_1) = \tau_1(\omega_0).$$
Thus this shift from using $\omega_0$ to $\omega_1$ to define the stopping time $\tau_2$ does not affect the value of the stopping time $\tau_1$.

Using the coordinate process $X_1$ in the component space $\Omega_1$, define the mapping $\tilde{Y}_2 : E_0 \times E_1 \to \mathcal{E}_1$ by

$$
\tilde{Y}_2(e_1) = \begin{cases} 
X_1((\tilde{\tau}_2(e_1) - \tilde{\tau}_1(e_1)), \omega_1), & \text{on } \{\tilde{\tau}_2(e_1) < \infty\}, \\
\tau, & \text{on } \{\tilde{\tau}_2(e_1) = \infty\}.
\end{cases}
$$

In particular, on the set $\{\tilde{\tau}_2 < \infty\}$ we observe that $\tilde{Y}_2 \in \mathcal{E}_1$ corresponds to the value given by the pre-intervention location $\omega_1(\tilde{\tau}_2 - ) \in \mathcal{E}$ while $\tilde{Z}_2 \in \mathcal{Z}$ corresponds to the nominal impulse control $Z_2 \in \mathcal{Z}$. The definition of $\tilde{Y}_2$ on $\{\tilde{\tau}_2 = \infty\}$ is merely for completeness as $\tilde{Y}_2(e_1)$ for $e_1 \in \{\tilde{\tau}_2 = \infty\}$ will never be utilized by the controlled process.

Again, view $\tau \in \mathcal{E}_2$ and define the transition kernel $P_2 : (E_0 \times E_1) \times \mathcal{G}_2 \to [0, 1]$ using two steps:

1. Define the transition function $Q_2 : (E_0 \times E_1) \times \mathcal{B}(\mathcal{E}_2) \to [0, 1]$ such that for each $e_1 \in E_0 \times E_1$ and $G_2 \in \mathcal{B}(\mathcal{E}_2)$,

$$
Q_2(e_1, G_2) = \begin{cases} 
Q(\tilde{Y}_2(e_1), \tilde{Z}_2(e_1))(G_2), & \text{on } \{\tilde{\tau}_2 < \infty\}, \\
\delta(\tau)(G_2), & \text{on } \{\tilde{\tau}_2 = \infty\}.
\end{cases}
$$

2. Define $V_2$ to be the coordinate random variable on $(\mathcal{E}_2, \mathcal{B}(\mathcal{E}_2))$ under the random measure $Q_2$. Now let $\mathbb{P}_{V_2}$ be the measure on $\Omega_2$ such that the coordinate process $X_2$ on $(\Omega_2, \mathcal{F}_2)$ is the fundamental Markov process defined for $[0, \infty)$ with $\mathbb{P}_{V_2}(X_2(0) = V_2) = 1$.

The transition kernel $P_2 : (E_0 \times E_1) \times \mathcal{G}_2 \to [0, 1]$ is specified by

$$
P_2(e_1, G_2 \times \Gamma_2) = \int_{G_2} \mathbb{P}_{V_2}(\tau_2)Q_2(e_1, d\nu_2), \quad e_1 \in E_0 \times E_1, G_2 \in \mathcal{B}(\mathcal{E}_2), \Gamma_2 \in \mathcal{F}_2. \quad (3.5)
$$

**Induction Step.** We now wish to define a transition kernel $P_{k+1} : \prod_{i=0}^{k} E_i \times \mathcal{G}_{k+1} \to [0, 1]$ for $k \geq 1$. To simplify notation, let $e_k = (\omega_0, v_1, \omega_1, \ldots, v_k, \omega_k) \in \prod_{i=0}^{k} E_i$ and set $\pi_{k-1} : \prod_{i=0}^{k} E_i \to \prod_{i=0}^{k-1} E_i$ to be the projection mapping. Similarly as in the second intervention, for $i = 1, \ldots, k$, extend the definitions of the nominal impulses $(\tilde{\tau}_i, \tilde{Z}_i)$ to $\prod_{i=0}^{k} E_i$ by setting $(\tilde{\tau}_i(e_k), \tilde{Z}_i(e_k)) = (\tilde{\tau}_i(\pi_{k-1}(e_k)), \tilde{Z}_i(\pi_{k-1}(e_k)))$. Also, define $\tilde{\tau}_0 = 0$.

Now let $T_k : \prod_{i=0}^{k} E_i \to \Omega$ be the mapping defined by

$$
\omega_k(t) := T_k(e_k)(t) = \begin{cases} 
\omega_i(t - \tilde{\tau}_i(e_k)), & \tilde{\tau}_i(e_k) \leq t < \tilde{\tau}_{i+1}(e_k), \quad i = 0, 1, \ldots, k - 1, \\
\omega_k(t - \tilde{\tau}_k(e_k)), & t \geq \tilde{\tau}_k(e_k).
\end{cases}
$$

Recall each $\tau_k$ is an $\{\mathcal{F}_t\}$-stopping time. As before by Corollary 3.2, for $e_k \in \prod_{i=0}^{k} E_i$, the fact that $\omega_k = \omega_0$ on $[0, \tau_1(\omega_0))$ implies that $\tau_1(\omega_k) = \tau_1(\omega_0)$. Similarly since $\omega_k = \omega_i$ on $[0, \tau_{i+1}(\omega_i))$ for $i = 1, \ldots, k - 1$, it follows that $\tau_{i+1}(\omega_k) = \tau_{i+1}(\omega_i)$ for $i = 1, \ldots, k - 1$. Hence shifting to $\omega_k$ for the definition of $\tilde{\tau}_{k+1}$ does not affect the previous intervention times.
Now define the nominal impulse \((\tilde{\tau}_{k+1}, \tilde{Z}_{k+1})\) on \(\prod_{i=0}^{k} E_i\) by

\[
\tilde{\tau}_{k+1}(e_k) = \begin{cases} 
\tau_{k+1}(\omega_k) & \text{when } \tilde{\tau}_k(e_k) < \infty, \\
\infty & \text{when } \tilde{\tau}_k(e_k) = \infty,
\end{cases}
\]

\[
\tilde{Z}_{k+1}(e_k) = \begin{cases} 
Z_{k+1}(\omega_k) & \text{when } \tilde{\tau}_k(e_k) < \infty, \\
\tau & \text{when } \tilde{\tau}_k(e_k) = \infty.
\end{cases}
\]

Define the mapping \(\tilde{Y}_{k+1} : \prod_{i=0}^{k} E_i \to \mathcal{E}_k\) using the coordinate process \(X_k\) by

\[
\tilde{Y}_{k+1}(e_k) = \begin{cases} 
X_k(\tilde{\tau}_{k+1}(e_k) - \tilde{\tau}(e_k) - \omega_k), & \text{on } \{e_k : \tilde{\tau}_{k+1}(e_k) < \infty\}, \\
\tau, & \text{on } \{e_k : \tilde{\tau}_{k+1}(e_k) = \infty\},
\end{cases}
\]

so that on the set \(\{\tilde{\tau}_{k+1} < \infty\}\), \(\tilde{Y}_{k+1} \in \mathcal{E}_k\) corresponds to \(\omega_k(\tau_{k+1} -) \in \mathcal{E}\) and \(\tilde{Z}_{k+1} \in \mathcal{Z}\) corresponds to \(Z_{k+1}(\omega_k) \in \mathcal{Z}\).

Define the transition kernel \(P_{k+1} : (\prod_{j=0}^{k} E_j) \times \mathcal{G}_{k+1} \to [0,1]\) by first determining the element \(v_{k+1}\) of \(\mathcal{E}_{k+1}\) and then the element \(\omega_{k+1} \in \Omega_{k+1}\) using the following two steps.

1. Again view \(\tau \in \mathcal{E}_{k+1}\) and define the transition function \(Q_{k+1} : (\prod_{j=0}^{k} E_j) \times \mathcal{B}(\mathcal{E}_{k+1}) \to [0,1]\) such that for each \(e_k \in \prod_{j=0}^{k} E_k\) and \(G_{k+1} \in \mathcal{B}(\mathcal{E}_{k+1})\),

\[
Q_{k+1}(e_k, G_{k+1}) = \begin{cases} 
Q(\tilde{Y}_{k+1}(e_k), \tilde{Z}_{k+1}(e_k))(G_{k+1}), & \text{on } \{\tilde{\tau}_{k+1} < \infty\}, \\
\delta_{\tilde{\tau}}(G_{k+1}), & \text{on } \{\tilde{\tau}_{k+1} = \infty\}.
\end{cases}
\]

2. Define \(V_{k+1}\) to be the coordinate random variable on \((\mathcal{E}_{k+1}, \mathcal{B}(\mathcal{E}_{k+1}))\) under the random measure \(Q_{k+1}\). Now let \(\mathbb{P}_{V_{k+1}}\) be the measure on \(\Omega_{k+1}\) such that the coordinate process \(X_{k+1}\) on \((\Omega_{k+1}, \mathcal{F}_{k+1})\) is the fundamental Markov process with \(X_{k+1}(0) = V_{k+1}\) almost surely.

The transition kernel \(P_{k+1} : (\prod_{j=0}^{k} E_j) \times \mathcal{G}_{k+1} \to [0,1]\) is specified for \(e_k \in \prod_{j=0}^{k} E_j\), \(G_{k+1} \in \mathcal{B}(\mathcal{E}_{k+1})\), and \(\Gamma_{k+1} \in \mathcal{F}_{k+1}\) by

\[
P_{k+1}(e_k, G_{k+1} \times \Gamma_{k+1}) = \int_{\mathcal{G}_{k+1}} \mathbb{P}_{V_{k+1}}(v_{k+1})(\Gamma_{k+1}) Q(\tilde{Y}_{k+1}(e_k), \tilde{Z}_{k+1}(e_k))(dv_{k+1}).
\]

Finally define the measure \(\mathbb{P}^{(k+1)}_x\) on \((\prod_{j=0}^{k+1} E_j, \mathcal{G}^{(k+1)}_x)\) such that for each \(\tilde{\Gamma}_k \in (\otimes_{j=0}^{k} \mathcal{G}_j)\), \(G_{k+1} \in \mathcal{B}(\mathcal{E})_{k+1}\), and \(\Gamma_{k+1} \in \mathcal{F}_{k+1}\),

\[
\mathbb{P}^{(k+1)}(\tilde{\Gamma}_k \times G_{k+1} \times \Gamma_{k+1}) = \int_{\tilde{\Gamma}_k} P_{k+1}(e_k, G_{k+1} \times \Gamma_{k+1}) \mathbb{P}^{(k)}(de_k).
\]

Again taking \(G_{k+1} = \mathcal{E}_{k+1}\) and \(\Gamma_{k+1} = \Omega_{k+1}\), we see that for each \(\tilde{\Gamma}_k \in \prod_{j=0}^{k} \mathcal{G}_j\), \(\mathbb{P}^{(k+1)}_x(\tilde{\Gamma}_k \times \mathcal{E}_{k+1} \times \Omega_{k+1}) = \mathbb{P}^{(k)}_x(\tilde{\Gamma}_k)\). Moreover for \(j = 1, 2, \ldots, k+1\), the coordinate process \(X_j\) on \(\Omega_j\) is the fundamental Markov process and given \((\tilde{Y}_j, \tilde{Z}_j)\) on \(\{\tau_j < \infty\}\), the coordinate random variable \(V_j\) has distribution \(Q(\tilde{Y}_j, \tilde{Z}_j)\) and \(X_j(0) = V_j Q(\tilde{Y}_j, \tilde{Z}_j)\) a.s. while \(X_j(0) = \tau\) on \(\{\tau_j = \infty\}\). These processes are independent when \(\tau_{k+1} < \infty\) \(\mathbb{P}^{(k+1)}_x\) a.s. and each \((\tilde{Y}_j, \tilde{Z}_j) = (y_j, z_j)\)
is deterministic. Furthermore, for a fixed \((y, z)\) policy, the processes \(X_1, \ldots, X_{k+1}\) (not \(X_0\)) are identically distributed when \(\tau_{k+1}\) is finite \(\mathbb{P}_x^{(k+1)}\) almost surely.

This iterative process defines a sequence of transition kernels \(\{P_k; k \in \mathbb{N}\}\) (see (3.7)) which meets the conditions of the Ionescu Tulcea Extension Theorem (see Proposition V.1.1 (page 162) of Neveu (1965)). As a result, there exists a measure \(\mathbb{P}_x^{(\infty)}\) on \((\tilde{\Omega}, \tilde{\mathcal{G}})\) := \((\prod_{j=0}^{\infty} E_j, \otimes_{j=0}^{\infty} G_j)\) for which each coordinate process \(X_k\) on \(\Omega_k\) is the fundamental strong Markov process with \(X_0(0) = x\) a.s. and for \(k \geq 1\), \(X_k(0) = V_{k+1}\) \(\mathbb{P}_x^{(\infty)}\)-almost surely. Moreover, under \(\mathbb{P}_x^{(\infty)}\), when each \(\tau_k\) is finite a.s. and the nominal intervention location and impulse pairs \(\{(y_k, z_k)\}\) are deterministic, the coordinate processes \(\{X_k : k = 0, 1, 2, \ldots\}\) are independent. In addition when a single deterministic \((y, z)\) is used for which each \(\tau_k\) is finite a.s., the processes \(\{X_k : k = 1, 2, 3, \ldots\}\) are identically distributed.

For any nominal impulse policy \((\tau, Z)\) on \((\Omega, \mathcal{G})\), the above construction defines a measure \(\mathbb{P}_x^{(\infty)}\) on \((\tilde{\Omega}, \tilde{\mathcal{G}})\) for which each component coordinate process \(X_k\) is the fundamental strong Markov process. Notice that the random times \(\tau_k\) are well-defined on \(\tilde{\Omega}\) using the projection of \(\tilde{\Omega}\) onto \(\prod_{j=0}^{k} E_j\) for each \(k \in \mathbb{N}\). This probability space can be used to define the uncertain-intervention-controlled process. Set \(\tau_0 = 0\) and define the process \(\tilde{X}\) on \((\tilde{\Omega}, \tilde{\mathcal{G}}, \mathbb{P}_x^{(\infty)})\) by

\[
\tilde{X}(t) = X_k(t - \tau_k) \quad \text{for} \quad \tau_k \leq t < \tau_{k+1}, \quad k = 0, 1, 2, \ldots.
\]

Observe that for any \(\tilde{\omega} \in \tilde{\Omega}\) for which \(\tau_{k+1}(\tilde{\omega}) = \infty\) but \(\tau_k(\tilde{\omega}) < \infty\), exactly \(k\) interventions occur and the process \(\tilde{X}\) only relies on the coordinate processes \(X_0, \ldots, X_k\) for its definition.

Finally, view \(\tilde{X} : \tilde{\Omega} \rightarrow \tilde{\Omega}\). Define \(\mathbb{P}_x^{(\tau, Z)} := \mathbb{P}_x^{(\infty)} \tilde{X}^{-1}\) to be the distribution of \(\tilde{X}\) on \(\Omega\). Of particular note is the fact that the construction of the nominal interventions \((\tau, \tilde{Z}) = \{((\tau_k, \tilde{Z}_k) : k \in \mathbb{N}\}\) is such that they correspond to the original nominal impulse policy \((\tau, Z) = \{\tau_k, Z_k) : k \in \mathbb{N}\}\). The desired model for the state process under the nominal impulse control policy \((\tau, Y)\) is therefore given by the coordinate process \(X\) on \((\Omega, \mathcal{F}, \mathbb{P}_x^{(\tau, Z)})\).

\[\square\]

**Remark 3.4.** Let \(\omega \in \Omega\) be in the image of the mapping \(\tilde{X}\); that is, suppose there exists some \(\tilde{\omega} \in \tilde{\Omega}\) such that \(\omega = \tilde{X}(\tilde{\omega})\). Recall that \(X_k(t, \tilde{\omega}) = \omega_k(t)\) for each \(t \geq 0\) is the coordinate process on \(\Omega_k\) of the \(k\)th component \(E_k\). We wish to recover the coordinate processes \(\{X_k(\cdot, \tilde{\omega}) = \omega_k : k \in \mathbb{N}_0\}\) in \(\tilde{\Omega}\) for the period of time which they contribute to \(\omega\).

In general, let \(\hat{\theta}_k : \Omega \rightarrow \Omega\) denote the shift operator such that \(\omega(s) \mapsto \omega(t + s)\) for \(s \geq 0\); note this operator cuts off the path \{\(\omega(r) : 0 \leq r < t\)\} from \(\omega\). Similarly, for a stopping time \(\eta\), define the random shift operator \(\hat{\theta}_\eta : \Omega \rightarrow \Omega\) by \(\omega(s) \mapsto \omega(\eta(\omega) + s)\) for \(s \geq 0\).

For each \(k\), using the random shift operator \(\hat{\theta}_{\tau_k}\), we have

\[
\omega_k(t) = [\hat{\theta}_{\tau_k}(\omega)](t) \quad \text{for} \quad 0 \leq t < \tau_{k+1} - \tau_k.
\]

**Remark 3.5.** The definition of an admissible nominal impulse policy in Definition 2.2 requires the sequence of stopping times to go to \(\infty\) surely. This condition is required in order that the mapping \(\tilde{X}\) have càdlàg paths. The construction of the measure \(\mathbb{P}_x^{(\infty)}\) on \((\tilde{\Omega}, \tilde{\mathcal{F}})\) still holds when \(\lim_{k \rightarrow \infty} \tau_k = T < \infty\) but the mapping \(\tilde{X}\) then has infinitely many interventions in a finite time interval and is not defined at or beyond \(T\).
infinite cost to such a path over the interval $[0, T)$. If a policy $(\tau, Y)$ were to result in positive $\tilde{\mathbb{P}}_x(\infty)$-probability for the set of paths with finite limits, the cost for such a policy would be infinite and therefore it would not be optimal. The restriction on admissibility of the intervention policy can then be relaxed to require $\tau_k \to \infty$ $\tilde{\mathbb{P}}_x(\infty)$-a.s. as $k \to \infty$. The mapping $\tilde{X} : \tilde{\Omega} \to \Omega$ would need to be arbitrarily defined for $\tilde{\omega} \in \tilde{\Omega}$ having such a path but would still provide a construction of the measure $\mathbb{P}_x^{(\tau, Z)}$ on $(\Omega, \mathcal{F})$. A limitation of this extension is that whether or not a policy is admissible is only determined after the construction of the measure $\tilde{\mathbb{P}}_x(\infty)$. Nevertheless, this relaxation of the definition of admissibility allows for the construction of policies where the intervention is triggered by the process hitting a value or set since some paths can result in cycles having summable cycle lengths but such paths will occupy a null set as long as the expected cycle lengths are not summable.

Remark 3.6. One limitation of this and the other impulse models concerns how they handle multiple interventions at the same time. Consider an intervention at time $\tau_k$ for an inventory model which receives only a small fraction of the ordered amount so that the manager wants to immediately place a second order. In our model, the second order at time $\tau_k$ is based on the left-hand limit $X(\tau_k -)$ which does not include the delivered amount. None of the product space models, in fact, facilitate simultaneous interventions with full information about the results of the earlier interventions since the controlled process can have only one value at each time. Multiple simultaneous interventions are accommodated but not with the preferred amount of information if one solely relies on the natural filtration when making interventions.

4 Markov Nominal Policies

We now turn to the question of when the impulse-controlled process is Markov. Our construction of the process in Theorem 3.3 relies on defining transition kernels which select a new process for the next coordinate given the previous coordinates and employs the Ionescu Tulcea Extension Theorem to obtain the existence of a unique measure on the infinite-product space. This construction may be viewed as a chain whose states consist of paths of the fundamental Markov process. Neveu (1965) gives a proof of the extension theorem (see (Neveu, 1965, Theorem V.1.1)) and then in Proposition V.2.1 (see p. 168) establishes the Markov property of the chain resulting from the successive applications of the transition kernel. A key aspect of the model which yields the Markov property in Proposition V.2.1 is that the transition function only depends on the current state, not the entire past as in Theorem V.1.1.

For our construction of the model, the transition kernels are defined on the product space so, as mentioned previously, both the “current state” and the “next state” are pairs of initial positions and paths of the foundational Markov process. These paths are then pasted together at the appropriate stopping times to form the controlled process $\tilde{X}$. Intuitively, to mimic Proposition V.2.1 of Neveu (1965) to establish the Markov property of the process, at the minimum the intervention decision will need to be based solely on the process during the current cycle.

The subclass of Markov nominal impulse policies consists of those policies for which future decisions are independent of the past, given the present. For each “present time”
s ≥ 0, we therefore need to define the filtration of future information \( \{ \mathcal{F}_t^q : t \geq s \} \) in which \( \mathcal{F}_s^q = \sigma(X(u) : s \leq u \leq t) \). Note that when \( s = 0 \), \( \{ \mathcal{F}_t^0 \} = \{ \mathcal{F}_t \} \) is the natural filtration. We will also need the filtration generated after an intervention; that is, define \( \mathcal{F}_{\tau_k+u}^q = \sigma(X(\tau_k + r) : 0 \leq r \leq u) \) for \( u \geq 0 \) and each \( k \). One of the challenges in the analysis of the Markov property is relating “clock times” to their corresponding times in whichever cycles they occur. Throughout this section, \( s \) and \( t \) will represent clock times from the beginning whereas \( u \) will represent the time in the current cycle.

**Definition 4.1** (Markov Nominal Impulse Policy). Set \( \tau_0 = 0 \). A Markov nominal impulse policy is a sequence of pairs \((\tau, Z) = \{(\tau_k, Z_k) : k \in \mathbb{N}\} \) in which for each \( k \in \mathbb{N} \), \( \tau_k = \sum_{i=1}^k \sigma_k \) where

(a) \( \sigma_k \) is an \( \{ \mathcal{F}_{\tau_k-}^{\tau_{k-1}+} \} \)-stopping time satisfying the property that for each \( u \geq 0 \),

\[
\omega \in \{\sigma_k > u\} \implies \sigma_k(\omega) = \sigma_k(\theta_{\sigma_k^{-1}+u}(\omega)) + u; \tag{4.1}
\]

and

(b) \( Z_k = \sigma(X(\tau_k-))/\mathcal{F}_{\tau_k-} \)-measurable.

Additionally it is required that \( \tau_k \to \infty \) as \( k \to \infty \).

**Remark 4.2.** In the general theory of Markov processes, the condition (4.1) says that \( \sigma_k \) is a terminal time. For more information and examples on terminal times, we refer the reader to Meyer (1975) and Chung and Walsh (2005).

The first task is to show that the class of Markov nominal impulse policies consists of nominal impulse policies. The only condition to verify is that each \( \tau_k \) is an \( \{ \mathcal{F}_{\tau_-} \} \)-stopping time.

**Proposition 4.3.** For each \( k \), let \( \sigma_k \) and \( \tau_k \) be as in Definition 4.1. Then each \( \tau_k \) is an \( \{ \mathcal{F}_{\tau_-} \} \)-stopping time.

**Proof.** The proof follows an induction argument. First observe that \( \tau_1 = \sigma_1 \) and \( \mathcal{F}_{\tau_0-}^{\tau_{0+}} = \mathcal{F}_{\tau_-} \). Thus we have the result for \( k = 1 \).

Now consider \( k = 2 \) and note that \( \tau_1 = \sigma_1 \) so these may be used interchangeably. Since \( \tau_2 = \tau_1 + \sigma_2 \), we have

\[
\{\omega : \tau_2(\omega) \leq t\} = \{\omega : \tau_1(\omega) + \sigma_2(\omega) \leq t\} = \bigcup_{q \in \mathbb{Q}, 0 \leq q < t} (\{\omega : \tau_1(\omega) \leq q\} \cap \{\omega : \sigma_2(\omega) \leq t - q\}).
\]

By Definition 4.1, \( \{\tau_1 \leq q\} \in \mathcal{F}_{\tau_-} \subset \mathcal{F}_{\tau_-} \) and \( \{\sigma_2 \leq t - q\} \in \mathcal{F}_{\tau_1+}^{\tau_1+q} \). We claim that the \( \{\sigma_1 \leq q\} \)-trace of \( \mathcal{F}_{\tau_1+}^{\tau_1+q} \) is contained in \( \mathcal{F}_{\tau_-} \). To see this, consider a generating set of the form \( A = \{X_{\tau_1+u_1} \in B_1, \ldots, X_{\tau_1+u_n} \in B_n\} \) in which \( n \in \mathbb{N} \), \( 0 \leq u_1 < \cdots < u_n < t - q \) and \( B_1, \ldots, B_n \in \mathcal{B}(\mathcal{E}) \). Then

\[
\{\tau_1 \leq q\} \cap A = \{\omega : \tau_1(\omega) \leq q, X_{\tau_1(\omega)+u_1} \in B_1, \ldots, X_{\tau_1(\omega)+u_n} \in B_n\}
\]
and it follows that the times \( \tau_i(\omega) + u_i < q + t - q = t \) for each \( i \), establishing the claim.

The argument for \( \tau_{k+1} \) being an \( \{\mathcal{F}_{t-}\} \)-stopping is the same as is seen by writing

\[
\{ \omega : \tau_{k+1}(\omega) \leq t \} = \bigcup_{q \in \mathbb{Q}, 0 \leq q < t} (\{ \omega : \tau_k(\omega) \leq q \} \cap \{ \omega : \sigma_{k+1}(\omega) \leq t - q \})
\]

and arguing as above that the \( \{\tau_k \leq q\} \)-trace of \( \mathcal{F}_{\tau_{k+1}^q} \) is contained in \( \mathcal{F}_{t-} \). \qedhere

Definition 4.1(a) requires that the intervention time only be dependent on the current cycle. In addition, condition (4.1) implies that the stopping time \( \sigma_k \) is Markov in the sense that if it has not occurred by time \( u \) within the cycle, then the time until it occurs consists of the known elapsed time \( u \) plus the stopping time for the shifted path that only looks to the future. Definition 4.1(b) requires the nominal impulse to only depend on the position of the process just prior to the intervention.

Our goal is to show the existence of a unique measure \( \mathbb{P}_0^{(x,Z)} \) on \((\Omega, \mathcal{F})\) corresponding to each Markov nominal impulse policy \((\tau, Y)\) and initial distribution \( \nu \) under which the coordinate process \( X \) satisfies the properties from the conclusions of Theorem 3.3 and \( X \) is a Markov process. As in Theorem 3.3, the measure \( \mathbb{P}_0^{(x,Z)} \) is a distribution on \( \Omega \) on \( \mathcal{F} \) defined on a (countable product) space \((\tilde{\Omega}, \tilde{\mathcal{G}}, \tilde{\mathbb{P}}^{(\infty)})\); the next theorem proves the existence of the measure \( \tilde{\mathbb{P}}^{(\infty)} \). It suffices to consider \( \nu = \delta_{(x)} \).

**Theorem 4.4** (Simplified Construction of \( \mathbb{P}_0^{(x,Z)} \)). Let \((\tau, Z)\) be a Markov nominal impulse policy on \((\Omega, \mathcal{F})\) and let \( X \) denote the coordinate process. For each \( k \in \mathbb{N} \), define the pre-impulse location \( Y_k = X(\tau_k) \) with the nominal impulse being \( Z_k \) on the set \( \{\tau_k < \infty\} \). Then for each \( x \in \mathcal{E} \), there exists a probability measure \( \mathbb{P}_x^{(x,Z)} \) defined on \((\Omega, \mathcal{F})\) under which the coordinate process \( X \) satisfies the following properties:

(a) \( X \) is the fundamental strong Markov process on the interval \([0, \tau_1]\) with \( X(0) = x \) a.s.;

(b) for each \( k \in \mathbb{N} \), on the set \( \{\tau_k < \infty\} \), \( X \) is again the fundamental strong Markov process on the interval \([\tau_k, \tau_{k+1}]\) with \( X(\tau_k) \) having conditional distribution \( Q_{(Y_k,Z_k)} \), given \((Y_k,Z_k)\); and

(c) \( X \) is a Markov process.

Moreover, when the policy is such that the pre-impulse location and nominal impulse pairs \( \{(y_k, z_k) : k \in \mathbb{N}\} \) are deterministic, then the coordinate process over the cycles \( \{X(t) : \tau_k \leq t < \tau_{k+1}\}, k \in \mathbb{N}_0 \), are independent under \( \mathbb{P}_x^{(x,Z)} \). Furthermore, when all of these are the same deterministic pair \((y, z)\), the cycles for \( k \in \mathbb{N} \) are identically distributed under \( \mathbb{P}_x^{(x,Z)} \).

**Proof.** Let \((\tau, Z)\) be a Markov nominal impulse policy on \((\Omega, \mathcal{F})\). A careful examination of the definition of \( P_{k+1} \) in (3.7) shows that \( P_{k+1} \) only depends on the path \( \omega_k \), not on \( e_{k-1} \). The existence and uniqueness of the measures \( \{\mathbb{P}_x^{(x,Z)}\} \) for which the coordinate process \( X \) on \((\Omega, \mathcal{F}, \mathbb{P}_x^{(x,Z)})\) models the impulses follows from Theorem 3.3.

To distinguish the measure and process on the product space \((\tilde{\Omega}, \tilde{\mathcal{G}})\) arising from a Markov nominal impulse policy from that for a general nominal impulse policy, we designate the measures as \( \{\tilde{\mathbb{P}}^{(x)}\} \) and the corresponding controlled process satisfying (3.9) by \( \tilde{X} \).
It remains to prove that (c) holds, namely that the coordinate process \( X \) defined on 
\((\Omega, \mathcal{F}, \mathbb{P}_x)\) is a Markov process. This result will follow from the fact that the process \( \tilde{X} \) on 
\((\tilde{\Omega}, \tilde{\mathcal{F}}, \mathbb{P}_x^{(\infty)})\) is a Markov process (proven in Theorem 4.6). To see this, view the coordinate 
process \( X \) on \((\Omega, \mathcal{F}, \mathbb{P}_x^{(r,\infty)})\) as an \( \mathcal{C}^{0,\infty} \)-valued process and note that by construction, \( X \) 
under \( \mathbb{P}_x^{(r,\infty)} \) has the same finite-dimensional distributions as \( \tilde{X} \) under \( \mathbb{P}_x^{(\infty)} \). The proof of 
Theorem 1.3 of Friedman (1975) (see p. 22 for the statement and pp. 31,32 for a sketch of 
the proof) now applies to establish that \( X \) is a Markov process. \( \Box \)

**Remark 4.5.** The structure of the transition kernels \( \{P_k\} \) satisfies the hypotheses of Propo-
sition V.2.1 of Neveu (1965). As a result, these kernels define a discrete-time Markov chain 
whose states are the pairs \((v, X)\) of initial position and fundamental Markov process. Notice 
that the Markov process \( X \) is defined for all time and that the Markov nominal impulse 
policy determines the times at which the transitions occur (as well as the distribution of 
the new state).

**Theorem 4.6.** Let \((\tau, Y)\) be a Markov nominal impulse policy and \( \nu \in \mathcal{P}(\mathcal{E}) \). Let \( \{\mathbb{P}_x^{(\infty)} : 
x \in \mathcal{E}\} \) be the probability measures on \((\tilde{\Omega}, \tilde{\mathcal{F}})\) given by the construction in Theorem 4.4, 
\[ \mathbb{P}_\nu^{(\infty)} = \int \mathbb{P}_x^{(\infty)} \nu(dx) \] and \( \tilde{X} \) be defined by (3.9). Then \( \tilde{X} \) is a Markov process under the 
measure \( \mathbb{P}_\nu^{(\infty)} \).

**Proof.** \( \tilde{X} \) is Markov if for each \( 0 \leq s < t \) and \( B \in \mathcal{B}(\mathcal{E}) \), 
\[ \mathbb{P}_\nu^{(\infty)}(\tilde{X}(t) \in B|\tilde{\mathcal{F}}_s) = \mathbb{P}_\nu^{(\infty)}(\tilde{X}(t) \in B|\tilde{X}(s)) \] 
(4.2) 
in which \( \tilde{\mathcal{F}}_s = \sigma(\tilde{X}(u) : 0 \leq u \leq s) \). Arbitrarily fix \( 0 \leq s < t \) and \( B \in \mathcal{B}(\mathcal{E}) \). The argument 
involves conditioning on the cycle in which \( \tilde{X} \) is evolving at time \( s \). For each \( j \in \mathbb{N}_0 \), the 
definition of \( \tilde{\sigma}_{j+1} \) on \( \tilde{\Omega} \) may also be written as 
\[ \tilde{\sigma}_{j+1}(\tilde{\omega}) = \sigma_{j+1}(T_j \circ \Pi_j(\tilde{\omega})) \], 
noting that \( \tilde{\sigma}_{j+1} \) also satisfies (4.1), and we have defined \( \tilde{\tau}_0 = 0 \) and \( \tilde{\tau}_j = \sum_{i=1}^j \tilde{\sigma}_i \) with the 
additional observation that \( \tilde{\tau}_j \) only depends on the paths \((\omega_0, \ldots, \omega_{j-1})\). Then the left-hand 
side of (4.2) can be decomposed as 
\[ \mathbb{P}_\nu^{(\infty)}(\tilde{X}(t) \in B|\tilde{\mathcal{F}}_s) = \sum_{j=0}^{\infty} \mathbb{E}_{\nu}^{(\infty)} |I_{\{\tilde{\tau}_j \leq s < \tilde{\tau}_{j+1}\}} I_{\{\tilde{X}(t) \in B\}}|\tilde{\mathcal{F}}_s| \]. 
(4.3) 

We therefore examine the right-hand side term for the event \( \{\tilde{\tau}_j \leq s < \tilde{\tau}_{j+1}\} \) for any \( j \in \mathbb{N}_0 \).

To prove (4.2) on this event, we use the definition of conditional expectation along with 
the trace \( \sigma \)-algebra of \( \tilde{\mathcal{F}}_s \) on \( \{\tilde{\tau}_j \leq s < \tilde{\tau}_{j+1}\} \). Simplify notation by setting \( \omega_j = (\omega_0, \ldots, \omega_j) \) 
and define \( \Pi_j : \tilde{\Omega} \to \prod_{i=0}^j \Omega_i \) to be the projection mapping. Observe that the event 
\[ \{\tilde{\tau}_j \leq s < \tilde{\tau}_{j+1}\} = \Pi_j^{-1}\{\omega_j : \tilde{\tau}_j(\omega_{j-1}) \leq s < \tilde{\tau}_{j+1}(\omega_j)\} \] 
\[ = \Pi_{j-1}^{-1}\{\omega_{j-1} : \tilde{\tau}_{j-1}(\omega_{j-1}) \leq s\} \cap \Pi_j^{-1}\{\omega_j : \sigma_{j+1}(\omega_j) > s - \tilde{\tau}_j(\omega_{j-1})\} \]
so this event only depends on the paths in the components $\Omega_0 \times \cdots \times \Omega_j$. For ease of reading, we slightly abuse notation by writing the right-hand side as

$$\{\omega_{j-1} : \hat{\tau}_{j-1}(\omega_{j-1}) \leq s\} \cap \{\omega_j : \hat{\sigma}_{j+1}(\omega_j) > s - \hat{\tau}_j(\omega_{j-1})\}.$$

The trace $\sigma$-algebra of $\hat{F}_s$ on $\{\hat{\tau}_j \leq s < \hat{\tau}_{j+1}\}$ is therefore determined solely by $\omega_j = (\omega_0, \ldots, \omega_j)$. It is sufficient to consider events $A \in \hat{F}_s$ in which the trace event has the form

$$\{\hat{\tau}_j \leq s < \hat{\tau}_{j+1}\} \cap A = \{\hat{\tau}_j \leq s < \hat{\tau}_{j+1}\} \cap (A_{j-1} \cap \Gamma_j)$$

$$= (\{\omega_{j-1} : \hat{\tau}_j(\omega_{j-1}) \leq s\} \cap A_{j-1}) \cap (\{\omega_j : \hat{\sigma}_{j+1}(\omega_j) > s - \hat{\tau}_j(\omega_{j-1})\} \cap \Gamma_j)$$

with $A_{j-1} \in \hat{F}_{\hat{\tau}_{j-1}}$ and $\Gamma_j \in \hat{F}_{\hat{\tau}_j(\omega_{j-1})} := \sigma(\hat{X}(u) : \hat{\tau}_j(\omega_{j-1}) \leq u \leq s)$. Notice that $\{\omega_{j-1} : \hat{\tau}_j(\omega_{j-1}) \leq s\} \cap A_{j-1} \in \prod_{i=0}^{j-1} \mathcal{F}_i$. Also observe that for each fixed $\omega_{j-1}$, the $\{\hat{\tau}_j \leq s < \hat{\tau}_{j+1}\}$-trace $\sigma$-algebra of $\hat{F}_{\hat{\tau}_j(\omega_{j-1})}$ is the $\sigma$-algebra generated by the $j$th coordinate process $X_j$ for the elapsed time in the $j$th cycle; that is, restricted to the event $\{\hat{\tau}_j \leq s < \hat{\tau}_{j+1}\}$,

$$\hat{F}_{\hat{\tau}_j(\omega_{j-1})} = \sigma(\hat{X}(r) : \hat{\tau}_j(\omega_{j-1}) \leq r \leq s)$$

$$= \sigma(X_j(r) : 0 \leq r \leq s - \hat{\tau}_j(\omega_{j-1})) =: \mathcal{F}^{(j)}_{\hat{\tau}_j(\omega_{j-1})}.$$

Thus for fixed $\omega_{j-1}$ with $\hat{\tau}_j(\omega_{j-1}) \leq s$, the second compound event $\{\omega_j : \hat{\sigma}_{j+1}(\omega_j) > s - \hat{\tau}_j(\omega_{j-1})\} \cap \Gamma_j \in \mathcal{F}^{(j)}_{\hat{\tau}_j(\omega_{j-1})}$.

Referring to the decomposition (4.3), we claim that $I_{\{\hat{\tau}_j \leq s < \hat{\tau}_{j+1}\}} \mathbb{E}^{\hat{\nu}} I_{\{\hat{X}(t) \in B\}}[\hat{X}(s)]$ is a version of $\mathbb{E}^{\hat{\nu}} I_{\{\hat{\tau}_j \leq s < \hat{\tau}_{j+1}\}} I_{\{\hat{X}(t) \in B\}}[\hat{X}(s)]$. To show this, we will establish

$$\mathbb{E}^{\hat{\nu}} I_{\{\hat{\tau}_j \leq s < \hat{\tau}_{j+1}\}} I_{\{\hat{X}(t) \in B\}}[\hat{X}(s)] = \mathbb{E}^{\hat{\nu}} \left[I_{\Lambda} I_{\{\hat{\tau}_j \leq s < \hat{\tau}_{j+1}\}} \mathbb{E}^{\hat{\nu}} I_{\{\hat{X}(t) \in B\}}[\hat{X}(s)]\right]. \tag{4.4}$$

The argument conditions on whether $t$ is in the same cycle as $s$ or whether it is in a later cycle.

It is helpful to recall the measure of $\hat{P}^{(k+1)}_x$ of (3.8) on finitely many coordinates, now denoted by $\hat{P}^{(k+1)}_{\hat{x}}$, and observe that this is the restriction of $\hat{P}^{(\infty)}_x$ to $\left(\prod_{i=0}^{k+1} E_j, \prod_{i=0}^{k+1} G_i\right)$. We again set $\hat{P}^{(k+1)}_x = \int \hat{P}^{(k+1)}_{\hat{x}}(\nu)dx$. Then for $n > j - 1$, the measures $\hat{P}^{(n)}_\nu$ and $\hat{P}^{(j-1)}_\nu$ are related by

$$\hat{P}^{(n)}_\nu \left(\prod_{i=0}^{n} F_i\right) = \int_{\prod_{i=0}^{n} F_i} \cdots \int_{F_n} P_n(e_{n-1}, de_{n}) \cdots P_j(e_{j-1}, de_j) \hat{P}^{(j-1)}_\nu(de_{j-1}).$$

Case (i): $\{\hat{\tau}_j \leq s < \hat{\tau}_{j+1}\}$. We begin by examining the case in which $\hat{\tau}_j \leq t < \hat{\tau}_{j+1}$ so both $s$ and $t$ are in the $j$th cycle (and recall $s < t$). Consider again (4.4). Since $s < t$, notice that $\{\hat{\tau}_j \leq s\} \subset \{\hat{\tau}_j \leq t\}$ and, on the event $\{\hat{\tau}_j \leq s\}$, $\{\hat{\tau}_{j+1} > t\} \subset \{\hat{\tau}_{j+1} > s - \hat{\tau}_j\}$ in which $s - \hat{\tau}_j$ gives the elapsed time $u$ in the $j$th cycle. We now apply (4.1) of Definition 4.1(a) but care must be taken to properly account for the difference between clock time and cycle time. Property (4.1) shifts the path $\omega = X(\omega)$ defined on $(\Omega, \mathcal{F}, \hat{P}^{(\infty)}_{\hat{x}})$ by the clock time $\tau_j + u$ in which the shift of $\tau_j$ eliminates the parts of the path of $\hat{X}$ defined on
given by the coordinates \((\omega_0, \ldots, \omega_{j-1})\) in (3.9). Therefore applying this Markov stopping time property at \(u = s - \tilde{\tau}_j(\omega_{j-1})\) results in

\[
\{\omega_{j-1} : \tilde{\tau}_j(\omega_{j-1}) \leq s \} \cap \{\omega_j : \tilde{\tau}_{j+1}(\omega_j) > t\} = \{\omega_{j-1} : \tilde{\tau}_j(\omega_{j-1}) \leq s \} \cap \{\omega_j : \sigma_{j+1}(t_{\tilde{\tau}_j}(\omega_{j-1})) + s - \tilde{\tau}_j(\omega_{j-1}) > t - \tilde{\tau}_j(\omega_{j-1})\} = \{\omega_{j-1} : \tilde{\tau}_j(\omega_{j-1}) \leq s \} \cap \{\omega_j : \sigma_{j+1}(t_{\tilde{\tau}_j}(\omega_{j-1})) > t - s\}.
\]

The importance of this observation is that for fixed \(\omega_{j-1}\) with \(\tilde{\tau}_j(\omega_{j-1}) \leq s\), the latter event only depends on the \(\omega_j\) component from time \(s - \tilde{\tau}_j(\omega_{j-1})\) onward; that is, for fixed \(\omega_{j-1}\) with \(\tilde{\tau}_j(\omega_{j-1}) \leq s\),

\[
\{\omega_j = (\omega_{j-1}, \nu_j, \omega_j) : \sigma_{j+1}(t_{\tilde{\tau}_j}(\omega_{j-1})) > t - s\} \in \sigma(X_j(u) : u \geq s - \tilde{\tau}_j(\omega_{j-1})).
\]

In addition, this latter event has a natural interpretation for such \(\omega_{j-1}\): for \(\tilde{\tau}_j\) to exceed \(t - \tilde{\tau}_j(\omega_{j-1})\) given that it exceeds \(s - \tilde{\tau}_j(\omega_{j-1})\), it is required that \(\sigma_{j+1}\) of the shifted path exceeds \(t - s\). Now simplify notation by setting \(F_{j-1} = A_{j-1} \cap \{\tilde{\tau}_j \leq s\}\). Since \(s < t\), for the first equality below \(I_{\{\tilde{\tau}_j \leq s < \tilde{\tau}_{j+1}\}} I_{\{\tilde{\tau}_j \leq t < \tilde{\tau}_{j+1}\}} = I_{\{\tilde{\tau}_j \leq s < \tilde{\tau}_{j+1}\}} I_{\{t < \tilde{\tau}_{j+1}\}}\) and for the fifth equality below \(I_{\{\tilde{\tau}_j \leq s\}} I_{\{\tilde{\tau}_j \leq t\}} = I_{\{\tilde{\tau}_j \leq s\}} I_{\{\tilde{\tau}_j \leq t\}}\). Then using the Markov property of the coordinate process \(X_j\) in the fourth equality below,

\[
\begin{align*}
\mathbb{E}_\nu^{\infty}\left[IA_{\{\tilde{\tau}_j \leq s < \tilde{\tau}_{j+1}\}} I_{\{X_j(t) \in B\}} I_{\{\tilde{\tau}_j \leq t < \tilde{\tau}_{j+1}\}}\right] &= \mathbb{E}_\nu^{\infty}\left[IA_{\{\tilde{\tau}_j \leq s < \tilde{\tau}_{j+1}\}} I_{\{X_j(t) \in B\}} I_{\{t < \tilde{\tau}_{j+1}\}}\right] \\
&= \int_{F_{j-1}} \int_{E_j} \mathbb{E}_{\nu}^{\infty}(\nu_j) \left[I_{\nu_j}(\omega_j) I_{\{\tilde{\tau}_{j+1}(\omega_j) > s - \tilde{\tau}_j(\omega_{j-1})\}} I_{\{X_j(t - \tilde{\tau}_j(\omega_{j-1})) \in B\}} \right. \\
&\quad \cdot I_{\{\tilde{\tau}_j > t - s\}} \left[Q_{\tilde{\tau}_j(\omega_{j-1})}(\omega_j, \tilde{\tau}_j(\omega_{j-1})) \right] Q_{\tilde{\tau}_j(\omega_{j-1})}(\omega_j) \left(\nu_j\right) \hat{P}_{\nu}^{(j-1)}(d\nu_j) \mathbb{E}_{\nu}^{\infty}(\nu_j) \left[I_{\{t - \tilde{\tau}_j(\omega_{j-1}) \in B\}}\right] \\
&= \int_{F_{j-1}} \int_{E_j} \mathbb{E}_{\nu}^{\infty}(\nu_j) \left[I_{\nu_j}(\omega_j) I_{\{\tilde{\tau}_{j+1}(\omega_j) > s - \tilde{\tau}_j(\omega_{j-1})\}} \mathbb{E}_{\nu}^{\infty}(\nu_j) \left[I_{\{X_j(t - \tilde{\tau}_j(\omega_{j-1})) \in B\}}\right] \\
&\quad \cdot I_{\{\tilde{\tau}_j > t - s\}} \left[Q_{\tilde{\tau}_j(\omega_{j-1})}(\omega_j, \tilde{\tau}_j(\omega_{j-1})) \right] Q_{\tilde{\tau}_j(\omega_{j-1})}(\omega_j) \left(\nu_j\right) \hat{P}_{\nu}^{(j-1)}(d\nu_j) \mathbb{E}_{\nu}^{\infty}(\nu_j) \left[I_{\{t - \tilde{\tau}_j(\omega_{j-1}) \in B\}}\right] \\
&= \int_{F_{j-1}} \int_{E_j} \mathbb{E}_{\nu}^{\infty}(\nu_j) \left[I_{\nu_j}(\omega_j) I_{\{\tilde{\tau}_{j+1}(\omega_j) > s - \tilde{\tau}_j(\omega_{j-1})\}} \mathbb{E}_{\nu}^{\infty}(\nu_j) \left[I_{\{X_j(t - \tilde{\tau}_j(\omega_{j-1})) \in B\}}\right] \\
&\quad \cdot I_{\{\tilde{\tau}_j > t - s\}} \left[Q_{\tilde{\tau}_j(\omega_{j-1})}(\omega_j, \tilde{\tau}_j(\omega_{j-1})) \right] Q_{\tilde{\tau}_j(\omega_{j-1})}(\omega_j) \left(\nu_j\right) \hat{P}_{\nu}^{(j-1)}(d\nu_j) \mathbb{E}_{\nu}^{\infty}(\nu_j) \left[I_{\{t - \tilde{\tau}_j(\omega_{j-1}) \in B\}}\right] \\
&= \mathbb{E}_\nu^{\infty}\left[IA_{\{\tilde{\tau}_j \leq s \}} I_{\{X_j(t - \tilde{\tau}_j) \in B\}} I_{\{\tilde{\tau}_j \leq t\}} I_{\{\tilde{\tau}_j > t - s\}} | X_j(s - \tilde{\tau}_j)\right] \\
&= \mathbb{E}_\nu^{\infty}\left[IA_{\{\tilde{\tau}_j \leq s \}} I_{\{X_j(t - \tilde{\tau}_j) \in B\}} I_{\{\tilde{\tau}_j \leq t\}} | X_j(s - \tilde{\tau}_j)\right].
\end{align*}
\]
By Lemma 4.7 following this proof, when restricted to the event \( \{ \hat{\tau}_j \leq s < \hat{\tau}_{j+1} \} \), the conditional expectation with respect to \( X_j(s - \hat{\tau}_j) \) under \( \mathbb{P}_{V_j} \) is a version of the conditional expectation with respect to \( X_j(s - \hat{\tau}_j) \) under \( \hat{\mathbb{P}}_{\nu}^{(\infty)} \):

\[
I_{\{ \hat{\tau}_j \leq s < \hat{\tau}_{j+1} \}} \mathbb{E}_{V_j}^{\mathbb{P}_{V_j}} \{ I_{X_j(t - \hat{\tau}_j) \in B} I_{\{ \hat{\tau}_j \leq t < \hat{\tau}_{j+1} \}} | X_j(s - \hat{\tau}_j) \} = I_{\{ \hat{\tau}_j \leq s < \hat{\tau}_{j+1} \}} \mathbb{E}_{\nu}^{\hat{\mathbb{P}}_{\nu}} \{ I_{\{ \hat{X}(t) \in B \}} I_{\{ \hat{\tau}_j \leq t < \hat{\tau}_{j+1} \}} | X_j(s - \hat{\tau}_j) \}.
\]

Using this identity in (4.5) results in

\[
\mathbb{E}_{\nu}^{\hat{\mathbb{P}}_{\nu}} \left[ I_{A} I_{\{ \hat{\tau}_j \leq s < \hat{\tau}_{j+1} \}} I_{\{ \hat{X}(t) \in B \}} I_{\{ \hat{\tau}_j \leq t < \hat{\tau}_{j+1} \}} \right] = \mathbb{E}_{\nu}^{\hat{\mathbb{P}}_{\nu}} \left[ I_{A} I_{\{ \hat{\tau}_j \leq s < \hat{\tau}_{j+1} \}} \mathbb{E}_{\nu}^{\hat{\mathbb{P}}_{\nu}} \{ I_{\{ \hat{X}(t) \in B \}} I_{\{ \hat{\tau}_j \leq t < \hat{\tau}_{j+1} \}} | X_j(s - \hat{\tau}_j) \} \right] = \mathbb{E}_{\nu}^{\hat{\mathbb{P}}_{\nu}} \left[ I_{A} I_{\{ \hat{\tau}_j \leq s < \hat{\tau}_{j+1} \}} \mathbb{E}_{\nu}^{\hat{\mathbb{P}}_{\nu}} \{ I_{\{ \hat{X}(t) \in B \}} I_{\{ \hat{\tau}_j \leq t < \hat{\tau}_{j+1} \}} | \hat{X}(s) \} \right];
\]

(4.6)

the last equality is true since the trace \( \sigma \)-algebra of \( \hat{X}(s) \) on the event \( \{ \hat{\tau}_j \leq s < \hat{\tau}_{j+1} \} \) is the \( \sigma \)-algebra \( \sigma(X_j(s - \hat{\tau}_j)) \).

**Case (ii):** \( \{ \hat{\tau}_j \leq s < \hat{\tau}_{j+1} \leq t \} \). For each \( j \in \mathbb{N}_0 \), we claim that there exists a measurable function \( \phi_j \) such that for \( A \in \prod_{i=j+1}^{n} \mathcal{G}_i \),

\[
\hat{\mathbb{P}}_{\nu}(A|\hat{\mathcal{F}}_{\hat{\tau}_{j+1}-})(\hat{\omega}) = \phi_j(\omega_j(\hat{\sigma}_{j+1}-))
\]

(4.7)

where \( \omega_j = \Pi_j(\hat{\omega}) \) is the projection of \( \hat{\omega} \) to its path \( \omega_j \) in component \( \Omega_j \). To see this, note that for each finite rectangle \( A = \prod_{i=j+1}^{n} F_i \in \prod_{i=j+1}^{n} \mathcal{G}_i \) with \( F_i = G_i \times \Gamma_i \),

\[
\hat{\mathbb{P}}_{\nu}(A|\hat{\mathcal{F}}_{\hat{\tau}_{j+1}-})(\hat{\omega}) = \int_{F_{j+1}} \int_{F_{j+2}} \cdots \int_{F_{n}} P_n(e_{n-1}, de_n) \cdots P_{j+2}(e_{j+1}, de_{j+2}) P_{j+1}(e_j, de_{j+1})
\]

\[
= \int_{G_{j+1}} \int_{\Gamma_{j+1}} \int_{F_{j+2}} \cdots \int_{F_{n}} P_n(e_{n-1}, de_n) \cdots P_{j+2}(e_{j+1}, de_{j+2})
\]

\[
\mathbb{P}_{V_j}(v_j+1)(d\omega_{j+1}) Q_{\hat{\tau}_{j+1}(\omega_j(\hat{\sigma}_{j+1}-)), \hat{\tau}_{j+1}(\omega_j(\hat{\sigma}_{j+1}-))}(d\nu_{j+1}).
\]

Consequently for any \( A \in \hat{\mathcal{F}}_s \), we have

\[
\mathbb{E}_{\nu}^{\hat{\mathbb{P}}_{\nu}} \left[ I_{A} I_{\{ \hat{\tau}_j \leq s < \hat{\tau}_{j+1} \}} I_{\{ \hat{X}(t) \in B \}} I_{\{ t \geq \hat{\tau}_{j+1} \}} \right] = \mathbb{E}_{\nu}^{\hat{\mathbb{P}}_{\nu}} \left[ I_{A} I_{\{ \hat{\tau}_j \leq s < \hat{\tau}_{j+1} \}} \mathbb{E}_{\nu}^{\hat{\mathbb{P}}_{\nu}} \{ I_{\{ \hat{X}(t) \in B \}} I_{\{ t \geq \hat{\tau}_{j+1} \}} | \hat{\mathcal{F}}_{\hat{\tau}_{j+1}-} \} \right]
\]

\[
= \mathbb{E}_{\nu}^{\hat{\mathbb{P}}_{\nu}} \left[ I_{A} I_{\{ \hat{\tau}_j \leq s < \hat{\tau}_{j+1} \}} \phi_j(\omega_j(\hat{\sigma}_{j+1}-)) \right].
\]

(4.8)

Next we show a companion result to (4.6), namely that

\[
\mathbb{E}_{\nu}^{\hat{\mathbb{P}}_{\nu}} \left[ I_{A} I_{\{ \hat{\tau}_j \leq s < \hat{\tau}_{j+1} \}} I_{\{ \hat{X}(t) \in B \}} I_{\{ t \geq \hat{\tau}_{j+1} \}} \right] = \mathbb{E}_{\nu}^{\hat{\mathbb{P}}_{\nu}} \left[ I_{A} I_{\{ \hat{\tau}_j \leq s < \hat{\tau}_{j+1} \}} \mathbb{E}_{\nu}^{\hat{\mathbb{P}}_{\nu}} \{ I_{\{ \hat{X}(t) \in B \}} I_{\{ t \geq \hat{\tau}_{j+1} \}} | X_j(s - \hat{\tau}_j) \} \right].
\]

As before, on the set \( \{ \hat{\tau}_j \leq s < \hat{\tau}_{j+1} \} \), it is enough to consider \( A \in \hat{\mathcal{F}}_s \) of the form \( A_{j-1} \cap \Gamma_j \), in which \( A_{j-1} \in \hat{\mathcal{F}}_{\hat{\tau}_j-} \) and \( \Gamma_j \in \hat{\mathcal{F}}_{\hat{\tau}_j} = \mathcal{F}_{s-\hat{\tau}_j}^{(j)} \). Then using (4.8) we compute

\[
\mathbb{E}_{\nu}^{\hat{\mathbb{P}}_{\nu}} \left[ I_{A} I_{\{ \hat{\tau}_j \leq s < \hat{\tau}_{j+1} \}} I_{\{ \hat{X}(t) \in B \}} I_{\{ t \geq \hat{\tau}_{j+1} \}} \right] = \int_{A_{j-1} \cap \{ \hat{\tau}_j \leq s \}} \int_{\mathcal{E}_j} \mathbb{P}_{V_j}(v_j) \{ I_{\{ \sigma_{j+1}(\omega) > s - \hat{\tau}_j(\omega_j-1) \}} \}
\]

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Remark 4.8. The identities (4.10) and (4.11) can be understood as saying that the \( \{ \tilde{\tau}_j \leq s < \tilde{\tau}_{j+1} \} \)-trace conditional expectation with respect to \( \sigma(X_j(s - \tilde{\tau}_j)) \) relative to \( \mathbb{P}_{V_j} \) is a version of the similar trace conditional expectation under \( \hat{\mathbb{P}}_v^{(\infty)} \).
of the similar trace conditional expectation relative to \( \hat{\mathbb{P}}^{(\infty)}_\nu \). By the \( \{ \hat{T}_j \leq s < \hat{T}_{j+1} \} \)-trace conditional expectation with respect to \( \sigma(X_j(s - \hat{T}_j)) \) we mean the conditional expectation with respect to the \( \{ \hat{T}_j \leq s < \hat{T}_{j+1} \} \)-trace of \( \sigma(X_j(s - \hat{T}_j)) \) defined to be
\[
\{ A \cap \{ T_j \leq s < T_{j+1} \} : A \in \sigma(X_j(s - T_j)) \}.
\]

**Proof.** For any \( \tilde{\omega} \in \tilde{\Omega} \) with \( \hat{T}_j(\omega_{j-1}) \leq s \), pick \( \Gamma \in \sigma(X_j(s - \hat{T}_j)) \). To establish (4.10), using the definition of conditional expectation it is enough to show that
\[
\mathbb{E}^{\hat{\mathbb{P}}^{(\infty)}_\nu} \left[ I_T I_{\{ \hat{T}_j \leq s < \hat{T}_{j+1} \}} I_{\{ \hat{X}(t) \in B \}} I_{\{ \hat{T}_j \leq t < \hat{T}_{j+1} \}} \right] = \mathbb{E}^{\hat{\mathbb{P}}^{(\infty)}_\nu} \left[ I_T I_{\{ \hat{T}_j \leq s < \hat{T}_{j+1} \}} I_{\{ \hat{X}(t) \in B \}} I_{\{ \hat{T}_j \leq t < \hat{T}_{j+1} \}} | X_j(s - \hat{T}_j) \right].
\]

To this end, observe first that \( \{ \hat{T}_j(\omega_{j-1}) \leq s \} \subset \{ \hat{T}_j(\omega_{j-1}) \leq t \} \) (which is used in the fourth equality below) and that \( \{ \hat{T}_j(\omega_{j-1}) \leq t \} \) is constant relative to the integration with respect to both \( \Gamma \) and the expectation with respect to \( \mathbb{P}_{V_j(\nu)} \) in the fifth equality below as well as with respect to the \( \mathbb{P}_{V_j(\nu)} \) conditional expectation since \( \sigma(X_j(s - \hat{T}(\omega_j))) \) only depends on the \( j \)th coordinate process. Thus
\[
\mathbb{E}^{\hat{\mathbb{P}}^{(\infty)}_\nu} \left[ I_T I_{\{ \hat{T}_j \leq s < \hat{T}_{j+1} \}} I_{\{ \hat{X}(t) \in B \}} I_{\{ \hat{T}_j \leq t < \hat{T}_{j+1} \}} \right] = \int_{\{ \hat{T}_j \leq s \}} \int_{\mathcal{E}_j} \mathbb{E}^{\mathbb{P}_{V_j(\nu)}} \left[ I_T I_{\{ \hat{X}(t) \in B \}} I_{\{ \hat{T}_j \leq t < \hat{T}_{j+1} \}} \right] Q_{V_j(\omega_j, \hat{T}_j)}(\nu) \mathbb{P}_{V_j(\nu)}(d\nu)(de_{j-1})
\]
\[
= \int_{\{ \hat{T}_j \leq s \}} \int_{\mathcal{E}_j} \mathbb{E}^{\mathbb{P}_{V_j(\nu)}} \left[ I_T I_{\{ \hat{X}(t) \in B \}} I_{\{ \hat{T}_j \leq t < \hat{T}_{j+1} \}} \right] Q_{V_j(\omega_j, \hat{T}_j)}(\nu) \mathbb{P}_{V_j(\nu)}(d\nu)(de_{j-1})
\]
\[
= \int_{\{ \hat{T}_j \leq s \}} \int_{\mathcal{E}_j} \mathbb{E}^{\mathbb{P}_{V_j(\nu)}} \left[ I_T I_{\{ \hat{X}(t) \in B \}} I_{\{ \hat{T}_j \leq t < \hat{T}_{j+1} \}} \right] Q_{V_j(\omega_j, \hat{T}_j)}(\nu) \mathbb{P}_{V_j(\nu)}(d\nu)(de_{j-1})
\]
\[
= \int_{\{ \hat{T}_j \leq s \}} \int_{\mathcal{E}_j} \mathbb{E}^{\mathbb{P}_{V_j(\nu)}} \left[ I_T I_{\{ \hat{X}(t) \in B \}} I_{\{ \hat{T}_j \leq t < \hat{T}_{j+1} \}} \right] Q_{V_j(\omega_j, \hat{T}_j)}(\nu) \mathbb{P}_{V_j(\nu)}(d\nu)(de_{j-1})
\]
\[
= \mathbb{E}^{\hat{\mathbb{P}}^{(\infty)}_\nu} \left[ I_T I_{\{ \hat{T}_j \leq s < \hat{T}_{j+1} \}} \mathbb{E}^{\mathbb{P}_{V_j(\nu)}} \left[ I_{\{ \hat{X}(t) \in B \}} I_{\{ \hat{T}_j \leq t < \hat{T}_{j+1} \}} \right] \right].
\]

This identity shows that the \( \{ \hat{T}_j \leq s < \hat{T}_{j+1} \} \)-trace conditional expectation of the product \( I_{\{ X_j(t, \omega_{j-1}) \in B \}} I_{\{ \hat{T}_j \leq t < \hat{T}_{j+1} \}} \) with respect to \( \sigma(X_j(s - \hat{T}_j)) \) relative to \( \mathbb{P}_{V_j(\nu)} \) is a version of the same trace conditional expectation relative to \( \hat{\mathbb{P}}^{(\infty)}_\nu \) as claimed in (4.10).
Turning to (4.11), the proof is more straightforward and follows by using $\phi_j(\omega_j(\tilde{\sigma}_{j+1}-))$ in place of $I_{\{X_j(t-\tilde{\tau}_j(e_{j-1})) \in B\}}I_{\{\tilde{\tau}_j(e_{j-1}) \leq t\}}$ in the previous argument:
\[
\mathbb{E}^{\hat{\nu}}\left[I_{\{\tilde{\tau}_j \leq s < \tilde{\tau}_{j+1}\}}\phi_j(\omega_j(\tilde{\sigma}_{j+1}-))\right]
= \int_{\{\tilde{\tau}_j \leq s\}} \int_{\mathcal{E}_j} \mathbb{E}^{\nu_j(e_j)}\left[I_{\{\tilde{\sigma}_{j+1}(\omega_j) > s - \tilde{\tau}_j(\omega_{j-1})\}}\phi_j(\omega_j(\tilde{\sigma}_{j+1}-))\right]
\quad \cdot Q_{Y_j(\omega_{j-1}(\tilde{\sigma}_{j-1})),Z_j(\omega_{j-1}(\tilde{\sigma}_{j-1}))(dv_j)\hat{P}_\nu^{(j-1)}(de_{j-1})}
= \int_{\{\tilde{\tau}_j \leq s\}} \int_{\mathcal{E}_j} \mathbb{E}^{\nu_j(e_j)}\left[I_{\{\tilde{\sigma}_{j+1}(\omega_j) > s - \tilde{\tau}_j(\omega_{j-1})\}}\phi_j(\omega_j(\tilde{\sigma}_{j+1}-))\right]
\quad \cdot \mathbb{E}^{\nu_j(e_j)}\left[I_{\{\tilde{\sigma}_{j+1}(\omega_j) > s - \tilde{\tau}_j(\omega_{j-1})\}}\phi_j(\omega_j(\tilde{\sigma}_{j+1}-))\right]
\quad \cdot Q_{Y_j(\omega_{j-1}(\tilde{\sigma}_{j-1})),Z_j(\omega_{j-1}(\tilde{\sigma}_{j-1}))(dv_j)\hat{P}_\nu^{(j-1)}(de_{j-1})}
= \mathbb{E}^{\hat{\nu}}\left[I_{\{\tilde{\tau}_j \leq s < \tilde{\tau}_{j+1}\}}\phi_j(\omega_j(\tilde{\sigma}_{j+1}-))X_j(s - \tilde{\tau}_j(\omega_{j-1}))\right].
\]
Again, the $\{\tilde{\tau}_j \leq s < \tilde{\tau}_{j+1}\}$-trace conditional expectation of $\phi_j(\omega_j(\tilde{\sigma}_{j+1}-))$ with respect to $\sigma(X_j(s - \tilde{\tau}_j(\omega_{j-1})))$ relative to $\hat{P}_\nu$ is a version of the same trace conditional expectation relative to $\hat{P}^{(\infty)}$ as claimed in (4.10).

Having identified a class of policies for which the controlled process can be shown to be Markov, a natural question would be to determine conditions under which the process would also be strong Markov. We leave this issue for further study.

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