Domination number of graphs with minimum degree five*

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Abstract

We prove that for every graph $G$ on $n$ vertices and with minimum degree five, the domination number $\gamma(G)$ cannot exceed $n/3$. The proof combines an algorithmic approach and the discharging method. Using the same technique, we provide a shorter proof for the known upper bound $4n/11$ on the domination number of graphs of minimum degree four.

Keywords: Dominating set, domination number, discharging method.
AMS subject classification: 05C69

1 Introduction

In this paper we study the minimum dominating sets in graphs of given order $n$ and minimum degree $\delta$. For the case of $\delta = 5$, we improve the previous best upper bound $0.344n$ by proving that the domination number $\gamma$ is at most $n/3$. For graphs of $\delta = 4$, the relation $\gamma \leq 4n/11$ was proved by Sohn and Xudong [22] in 2009. Using a different approach, we provide a simpler proof for this theorem.

*Research supported by the Slovenian Research Agency under the project N1-0108
Standard definitions. In a simple graph $G$, the vertex set is denoted by $V(G)$ and the edge set by $E(G)$. For a vertex $v \in V(G)$, its closed neighborhood $N[v]$ contains $v$ and its neighbors. For a set $S \subseteq V(G)$, we use the analogous notation $N[S] = \bigcup_{v \in S} N[v]$. The degree of a vertex $v$ is denoted by $d(v)$, while $\delta(G)$ and $\Delta(G)$, respectively, stand for the minimum and maximum vertex degree in $G$. A set $D \subseteq V(G)$ is a dominating set if $N[D] = V(G)$. The minimum cardinality of a dominating set is the domination number $\gamma(G)$ of the graph. An earlier general survey on domination theory is [11], while two new directions were initiated recently in [6] and [5].

General upper bounds on $\gamma(G)$ in terms of the order and minimum degree. The first general upper bound on $\gamma(G)$ in terms of the order $n$ and the minimum degree $\delta$ was given by Arnautov [2] and, independently, by Payan [20]:

$$\gamma(G) \leq \frac{n}{\delta + 1} \sum_{j=1}^{\delta+1} \frac{1}{j}.$$  

(1)

We remark that a bit stronger general results were later published by Clark et al. [9] and Biró et al. [3]. On the other hand, already (1) implies the upper bound

$$\gamma(G) \leq n \left( \frac{1 + \ln(\delta + 1)}{\delta + 1} \right).$$  

(2)

It was proved by Alon [1] that (2) is asymptotically sharp when $\delta \to \infty$.

Upper bounds for graphs of small minimum degrees. There are several ways to show that $\gamma(G) \leq n/2$ holds if $\delta(G) = 1$ (see [19] for the first proof). Blank [4], and later independently McCuaig and Shepherd [18] proved that $\gamma(G) \leq 2n/5$ is true if $G$ is connected, $\delta(G) = 2$, and $n \geq 8$. For graphs $G$ with $\delta(G) = 3$, Reed [21] proved the famous result that $\gamma(G) \leq 3n/8$. He also presented a connected cubic graph on 8 vertices for which the upper bound is tight.

In the same paper [21], Reed provided the conjecture that the upper bound can be improved to $\lceil n/3 \rceil$ once the connected cubic graph has an appropriately large order. It was disproved by Kostochka and Stodolsky [14] by constructing an infinite sequence of connected cubic graphs such that all

\[1\]There are seven small graphs, the cycle $C_4$ and six graphs with $n = 7$ and $\delta = 2$, which do not satisfy $\gamma(G) \leq 2n/5$.  


of them have $\gamma(G) \geq (\frac{1}{3} + \frac{1}{69})n$. Later, in [15], the same authors proved that $\gamma(G) \leq \frac{4}{11}n = (\frac{1}{3} + \frac{1}{33})n$ holds for every connected cubic graph of order $n > 8$. However, it seems a challenging and difficult problem to close the small gap between $\frac{1}{3} + \frac{1}{69}$ and $\frac{1}{3} + \frac{1}{33}$.

For graphs of minimum degree 4, the best known upper bound is $\gamma(G) \leq \frac{4}{11}n$ that was established by Sohn and Xudong [22]. For the case of $\delta(G) = 5$, Xing, Sun, and Chen [23] proved $\gamma(G) \leq \frac{5}{14}n$ which was improved to $\gamma(G) \leq \frac{2671}{7766}n < 0.344n$ by the authors of [7]. It was also shown in [7] that for graphs of minimum degree 6, the domination number is strictly smaller than $n/3$. Note that similar upper bounds involving the girth and other parameters of the graph can be found in many papers, e.g. in [10, 12, 16, 17], while results for plane triangulations and maximal outerplanar graphs were established in [13] and [8].

Our approach. In the seminal paper [21] of Reed, the upper bound $3n/8$ was proved by considering a vertex-disjoint path cover with specific properties. Later, the same method (with updated conditions and thorough analysis) was used in [15, 22, 23] to establish results on cubic graphs and on graphs of minimum degree 4 and 5. In [7], we introduced a different algorithmic method that resulted in improvement for all cases with $5 \leq \delta \leq 50$. Here, we combine the latter approach with a discharging process. This allows us to prove that already graphs of minimum degree 5 satisfy $\gamma(G) \leq n/3$.

Residual graph. Given a graph $G$ and a set $D \subseteq V(G)$, the residual graph $G_D$ is obtained from $G$ by assigning colors to the vertices and deleting some edges according to the following definitions:

- A vertex $v$ is **white** if $v \notin N[D]$.
- A vertex $v$ is **blue** if $v \in N[D]$ and $N[v] \not\subseteq N[D]$.
- A vertex $v$ is **red** if $N[v] \subseteq N[D]$.
- $G_D$ contains only those edges from $G$ that are incident to at least one white vertex.

In $G_D$, we refer to the set of white, blue, and red vertices, respectively, by the notations $W$, $B$, and $R$. It is clear by definitions that $D \subseteq R$ and $W \cup B \cup R = V(G)$ hold. The **white-degree** $d_W(v)$ of a vertex $v$ is the number
of its white neighbors in $G_D$. Analogously, we sometimes refer to the blue-degree $d_B(v)$ of a vertex. The maximum of white-degrees over the sets of white and blue vertices, respectively, are denoted by $\Delta_W(W)$ and $\Delta_W(B)$.

**Observation 1.** Let $G$ be a graph and $D \subseteq V(G)$. The following statements are true for the residual graph $G_D$.

(i) If $v \in W$, then $G_D$ contains all edges which are incident with $v$ in $G$ and, in particular, $N[v] \cap R = \emptyset$ and $d_W(v) + d_B(v) = d(v)$ hold.

(ii) If $v \in B$, then $d_W(v) = |W \cap N[v]| < d(v)$ and $d_B(v) = 0$.

(iii) If $v \in R$, then $v$ is an isolated vertex in $G_D$.

(iv) If $\delta(G) = d$ and $v$ is a white vertex with $d_W(v) = \ell < d$, then $d_B(v) \geq d - \ell$ holds in $G_D$.

(v) $D$ is a dominating set of $G$ if and only if $R = V(G)$ (or equivalently, $W = \emptyset$) in $G_D$.

(vi) If $D \subseteq D' \subseteq V(G)$ and a vertex $v$ is red in $G_D$, it remains red in $G_{D'}$; if $v$ is blue in $G_D$, then it is either blue or red in $G_{D'}$.

**Structure of the paper.** In the next section we prove the improved upper bound $n/3$ on the domination number of graphs with minimum degree 5. In Section 3, we consider graphs of minimum degree 4 and show an alternative proof for the theorem $\gamma \leq 4n/11$.

## 2 Graphs of minimum degree 5

**Theorem 1.** For every graph $G$ on $n$ vertices and with minimum degree 5, the domination number satisfies $\gamma(G) \leq \frac{n}{3}$.

**Proof.** Consider a graph $G$ and a subset $D$ of the vertex set $V = V(G)$. Let $W$, $B$, and $R$ denote the set of white, blue, and red vertices respectively, in the residual graph $G_D$. Further, for the sets of blue vertices that have at least 5 white neighbors, or exactly 4, 3, 2, 1 white neighbors, we use the notations $B_5$, $B_4$, $B_3$, $B_2$, and $B_1$ respectively. A vertex is a blue leaf if it belongs to $B_1$. In the proof, a residual graph $G_D$ is associated with the following value:

$$f(G_D) = 35|W| + 23|B_5| + 21|B_4| + 19|B_3| + 17|B_2| + 14|B_1|.$$

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By Observation \[ \mathbb{1}(v) \], \( f(G_D) \) equals zero if and only if \( D \) is a dominating set in \( G \). If \( G \) and \( D \) are fixed and \( A \) is a subset of \( V \setminus D \), we define

\[
s(A) = f(G_D) - f(G_{D \cup A})
\]

that is the decrease in the value of \( f \) when \( D \) is extended by the vertices of \( A \). We define the following property for \( G_D \):

**Property 1.** There exists a nonempty set \( A \subseteq V \setminus D \) such that \( s(A) \geq 105 \lvert A \rvert \).

Our goal is to prove that every graph \( G \) with \( \delta(G) = 5 \) and every \( D \subseteq V \) with \( f(G_D) > 0 \) satisfy Property 1. Once we do it, Theorem 1 will follow easily. In the continuation, we suppose that a graph \( G \) with minimum degree 5 and a set \( D \) with \( f(G_D) > 0 \) do not satisfy Property 1 and prove, by a series of claims, that this assumption leads to a contradiction.

**Claim A.** In \( G_D \), every white vertex \( v \) has at most two white neighbors, and every blue vertex \( u \) has at most three white neighbors.

**Proof.** First suppose that there is vertex \( v \in W \) with \( d_W(v) \geq 6 \). Choosing \( A = \{v\} \), the white vertex \( v \) becomes red in \( G_{D \cup A} \) that decreases \( f \) by 35. The white neighbors of \( v \) become blue or red which decreases \( f \) by at least \( 6 \cdot (35 - 23) \). Hence, we have \( s(A) \geq 35 + 72 = 107 > 105 \lvert A \rvert \) complying with Property 1. This contradicts our assumption on \( G_D \) and implies that \( \Delta_W(W) \leq 5 \).

Now, suppose that \( \Delta_W(W) = 5 \) in \( G_D \). Let \( v \) be a white vertex with \( d_W(v) = 5 \) and consider \( A = \{v\} \). In \( G_{D \cup A} \), the vertex \( v \) becomes red and its white neighbors become blue (or red). Since each neighbor \( u \) had at most 5 white neighbors in \( G_D \) and at least one of them, namely \( v \), becomes red, \( u \) may have at most 4 white neighbors in \( G_{D \cup A} \). Therefore, \( s(A) \geq 35 + 5 \cdot (35 - 21) = 105 \lvert A \rvert \) holds which is a contradiction again.

If \( \Delta_W(W) \leq 4 \) and \( \Delta_W(B) \geq 6 \), let \( v \) be a blue vertex with \( d_W(v) \geq 6 \) and define \( A = \{v\} \) again. In \( G_D \), the vertex \( v \) belongs to \( B_5 \), while we have \( v \in R \) in \( G_{D \cup A} \) which causes a decrease of 23 in the value of \( f \). Each white neighbor \( u \) of \( v \) has at most four white neighbors in \( G_D \) and, therefore, \( u \in B_4 \cup B_3 \cup B_2 \cup B_1 \cup R \) in \( G_{D \cup A} \). Hence, we have \( s(A) \geq 23 + 6(35 - 21) = 107 > 105 \lvert A \rvert \), a contradiction to our assumption. Note that in the continuation, where we suppose \( \Delta_W(B) \leq 5 \), if a blue vertex loses \( \ell \) white neighbors in a step, it causes a decrease of at least \( 2\ell \) in the value of \( f \).
Assume that $\Delta_W(W) = 4$ and $\Delta_W(B) \leq 5$ and let $v$ be a white vertex with $d_W(v) = 4$ in $G_D$. Set $A = \{v\}$ and consider the decrease $s(A)$. As $v$ turns to be red, this contributes by $35$ to $s(A)$. The four white neighbors become blue (or red) and each of them has at most $3$ white neighbors in $G_{DUA}$. Hence, the contribution to $s(A)$ is at least $4(35-19)$. Further, we have $d_W(u) \leq 4$ for each white vertex $u$ from $N[v]$. This implies, by Observation (iv), that $u$ has at least one blue neighbor in $G_D$ the white-degree of which is smaller in $G_{DUA}$ than in $G_D$. Even if some blue vertices from $N[N[v]]$ have more than one neighbor from $N[v]$, it remains true that the sum of the white-degrees over $B \cap N[N[v]]$ decreases by at least $d_W(v) + 1 = 5$. We may conclude $s(A) \geq 35 + 4(35 - 19) + 5 \cdot 2 = 109 > 105 |A|$.

Assume that $\Delta_W(W) \leq 3$ and $\Delta_W(B) = 5$ hold in $G_D$ and $v$ is a blue vertex with $d_W(v) = 5$. Let $A = \{v\}$ and consider the decrease $s(A)$. Since $v$ belongs to $B_5$ in $G_D$ and to $R$ in $G_{DUA}$, this change contributes by $23$ to $s(A)$. The five white neighbors of $u$ become blue or red and belong to $B_3 \cup B_2 \cup B_1 \cup R$ in $G_{DUA}$. The contribution to $s(A)$ is not smaller than $5(35 - 19)$. By Observation (iv) and by $\Delta_W(W) \leq 3$, each white vertex has at least two blue neighbors in $G_D$. That is, each white neighbor has at least one blue neighbor that is different from $v$. As the five white vertices from $N(v)$ turn blue (or red) in $G_{DUA}$, the sum of the white-degrees over $B \cap (N[N[v]] \setminus \{v\})$ decreases by at least $5$. We infer that $s(A) \geq 23 + 5(35 - 19) + 5 \cdot 2 = 113 > 105 |A|$ which is a contradiction again.

The next case which we consider is $\Delta_W(W) = 3$ and $\Delta_W(B) \leq 4$. Let $v$ be a white vertex with $d_W(v) = 3$ and estimate the value of $s(A)$ for $A = \{v\}$. When $D$ is replaced by $D \cup A$, vertex $v$ is recolored red, the three white neighbors of $v$ become blue or red and belong to $B_2 \cup B_1 \cup R$ in $G_{DUA}$. Additionally, each of the three white neighbors and also $v$ itself has at least two blue neighbors. The decrease in their white-degrees contributes to $s(A)$ by at least $4 \cdot 2 \cdot 2$. Consequently, we have $s(A) \geq 35 + 3(35 - 17) + 16 = 105 |A|$ that is a contradiction.

The last case is when $\Delta_W(W) \leq 2$ and $\Delta_W(B) = 4$. We assume that $v$ is a vertex from $B_4$ in $G_D$. Let $A = \{v\}$ and observe that $v$ is recolored red and the white neighbors of $v$ belong to $B_2 \cup B_1 \cup R$ in $G_{DUA}$. Since now we have $\Delta_W(W) \leq 2$ in $G_D$, each white vertex has at least three blue neighbors. Therefore, each white neighbor of $v$ has at least two blue neighbors which are different from $v$. We conclude that $s(A) \geq 21 + 4(35 - 17) + 4 \cdot 2 \cdot 2 = 109 > 105 |A|$. This contradiction finishes the proof of Claim A. (c)
From now on we may suppose that $\Delta_W(W) \leq 2$ and $\Delta_W(B) \leq 3$ holds in the counterexample $G_D$. This implies that the graph $G_D[W]$, which is induced by the white vertices of $G_D$, contains only paths and cycles as components. Before performing a discharging, we prove some further properties of $G_D$.

**Claim B.** In $G_D[W]$, each component is a path $P_1$, $P_2$ or a cycle $C_4$, $C_5$, $C_7$ or $C_{10}$.

**Proof.** First, suppose that $P_j : v_1 \ldots v_j$ is a path component on $j \geq 3$ vertices in $G_D[W]$. Let us choose $A = \{v_2\}$. In $G_{D_0}A$ not only $v_2$ but also $v_1$ becomes red, while $v_3$ turns to be either a blue leaf or a red vertex. These changes contribute to $s(A)$ by at least $2 \cdot 35 + (35 - 14)$. By Observation $\blacksquare (iv)$, $v_1$, $v_2$, and $v_3$, respectively, have at least 4, 3, 3 blue neighbors in $G_D$. The decrease in their white-degrees contributes to $s(A)$ by at least 20. We may infer that $s(A) \geq 70 + 21 + 20 = 111 > 105 |A|$, a contradiction to our assumption.

We now prove that no cycle of length $3k$ occurs in $G_D[W]$. Assuming that a cycle $C_{3k} : v_1 \ldots v_3k v_1$ exists, all vertices of it can be dominated by the $k$-element set $A = \{v_3, v_6, \ldots, v_{3k}\}$. Then, in $G_{D_0}A$, all the $3k$ vertices are red and, by Observation $\blacksquare (iv)$, the sum of the white-degrees of the blue neighbors decreases by at least $3 \cdot 3k$. Consequently, we get the contradiction $w(A) \geq 35 \cdot 3k + 2 \cdot 9k = 123k > 105 |A|$.

Similarly, if we suppose the existence of a cycle $C_{3k+2} : v_1 \ldots v_{3k+2} v_1$ with $k \geq 2$ and define $A = \{v_3, v_6, \ldots, v_{3k}, v_{3k+2}\}$, the set $A$ dominates all vertices. Since $k \geq 2$, the relation $s(A) \geq 35 \cdot (3k + 2) + 2 \cdot 3 \cdot (3k + 2) = 123k + 82 > 105(k + 1) = 105 |A|$ clearly holds and gives the contradiction.

In the last case, consider a cycle $C_{3k+1} : v_1 \ldots v_{3k+1} v_1$ with $k \geq 4$ and set $A = \{v_3, v_6, \ldots, v_{3k}, v_{3k+1}\}$. In $G_{D_0}A$, every vertex from the cycle is red and, as before, one can prove that $s(A) \geq 35 \cdot (3k + 1) + 2 \cdot 3 \cdot (3k + 1) = 123k + 41 > 105(k + 1) = 105 |A|$. This contradiction finishes the proof of Claim B. $\square$

For $i = 0, 1, 2$, we will use the notation $W_i$ for the set of white vertices having exactly $i$ white neighbors in $G_D$. Note that $W_0$ consists of the vertices of the components of $G_D[W]$ which are isomorphic to $P_1$, while $W_1$ and $W_2$, respectively, contain the vertices from the $P_2$-components and the cycles of $G_D[W]$.

**Claim C.** No vertex from $B_3$ is adjacent to a vertex from $W_0$ in $G_D$. 


Proof. In contrary, suppose that a vertex \( v \in B_3 \) has a neighbor \( u \) from \( W_0 \). Let \( A = \{v\} \) and denote by \( u_1 \) and \( u_2 \) the further two white neighbors of \( v \). In \( G_{D\cup A} \), we have \( v, u \in R \) and \( u_1, u_2 \in B_2 \cup B_1 \cup R \). This contributes to \( s(A) \) by at least \( 19 + 35 + 2(35 - 17) = 90 \). By Observation \( iv \), the neighbors \( u, u_1 \) and \( u_2 \) have, respectively, at least 4, 2, 2 blue neighbors which are different from \( v \). As follows, \( s(A) \geq 90 + 2 \cdot 8 = 106 > 105 |A| \) must be true but this contradicts our assumption on \( G_D \). \( \square \)

We call a vertex from \( B_2 \) special, if it is adjacent to a vertex from \( W_0 \).

Claim D. No special vertex is adjacent to two vertices from \( W_0 \).

Proof. Suppose that a vertex \( v \in B_2 \) is adjacent to two vertices, say \( u_1 \) and \( u_2 \) from \( W_0 \). Then, we set \( A = \{v\} \) and observe that all the three vertices \( v, u_1 \) and \( u_2 \) are red in \( G_{D\cup A} \). By Claim C, all the blue neighbors of \( u_1 \) and \( u_2 \) are from \( B_2 \cup B_1 \) in \( G_D \) and, therefore, when the white-degree of these neighbors decreases by \( \ell \), the value of \( f \) falls by at least \( (17 - 14) \ell = 3\ell \). Since, by Observation \( ii \), each of \( u_1 \) and \( u_2 \) has at least four blue neighbors, we have \( s(A) \geq 17 + 2 \cdot 35 + 3 \cdot 8 = 111 > 105 |A| \). This contradiction proves the claim. \( \square \)

Claim E. No special vertex is adjacent to a vertex from a \( C_4 \) or \( C_7 \).

Proof. Suppose first that a special vertex \( v \in B_2 \) is adjacent to \( u_1 \) which is from a 4-cycle component \( C_4 \): \( u_1u_2u_3u_4u_1 \) in \( G_D \). The other neighbor of \( v \) is \( u_0 \) which is from \( W_0 \). Let \( A = \{v, u_3\} \) and observe that all the six vertices \( v, u_0, u_1, u_2, u_3 \) and \( u_4 \) are red in \( G_{D\cup A} \). In \( G_D \), the white vertex \( u_0 \) has at least four blue neighbors which are different from \( v \) and, by Claim C, each of them belongs to \( B_2 \cup B_1 \); \( u_1 \) has at least two neighbors from \( (B_3 \cup B_2 \cup B_1) \setminus \{v\} \); each of \( u_2, u_3 \) and \( u_4 \) has at least three neighbors from \( (B_3 \cup B_2 \cup B_1) \setminus \{v\} \). Therefore, \( s(A) \geq 17 + 5 \cdot 35 + 4 \cdot 3 + 11 \cdot 2 = 226 > 105 |A| \), a contradiction.

The argumentation is similar if we suppose that a special vertex \( v \) is adjacent to \( u_0 \) from \( W_0 \) and to a vertex \( u_1 \) from the 7-cycle \( u_1 \ldots u_7 u_1 \). Here we set \( A = \{v, u_3, u_6\} \) and observe that \( s(A) \geq 17 + 8 \cdot 35 + 4 \cdot 3 + 20 \cdot 2 = 349 > 105 |A| \) that contradicts our assumption on \( G_D \). \( \square \)

Claim F. If \( v_1 \) and \( v_2 \) are two adjacent vertices from \( W_1 \), then at most one of them may have a special blue neighbor.

Proof. Assume to the contrary that \( v_1 \) is adjacent to the special vertex \( u_1 \), and \( v_2 \) is adjacent to the special vertex \( u_2 \). Denote the other neighbors of
and $u_2$ by $x_1$ and $x_2$, respectively. Hence, $v_1, v_2 \in W_1$, $u_1, u_2 \in B_2$ and $x_1, x_2 \in W_0$ hold in $G_D$. Consider the set $A = \{u_1, u_2\}$ and observe that all the six vertices become red in $G_{D \cup A}$. Further, for $i = 1, 2$, vertex $x_i$ has at least four neighbors from $(B_2 \cup B_1) \setminus \{u_i\}$ and $v_i$ has at least three neighbors from $(B_3 \cup B_2 \cup B_1) \setminus \{u_i\}$. Thus, $s(A) \geq 2 \cdot 17 + 4 \cdot 35 + 8 \cdot 3 + 6 \cdot 2 = 210 = 105 |A|$ and this contradiction proves the claim. ($\square$)

Having Claims A-F in hand, we are ready to prove that every $G_D$ (where $D$ is not a dominating set) satisfies Property 1. The last step of this proof is based on a discharging.

Discharging. First, we assign charges to the (non-red) vertices of $G_D$ so that every white vertex gets 35, and every vertex from $B_3$, $B_2$, and $B_1$ gets 19, 17, and 14, respectively. Note that the sum of the charges equals $f(G_D)$. Then, every blue vertex, except the special ones, distributes its charge equally among the white neighbors. The exact rules are the following:

- Every vertex from $B_3$ gives $19/3$ to each white neighbor.
- Every non-special vertex from $B_2$ gives $17/2$ to each white neighbor.
- Every special vertex gives 14 to its neighbor from $W_0$, and gives 3 to the other neighbor.
- Every vertex from $B_1$ gives 14 to its neighbor.

After the discharging, every vertex from a $P_1$-component of $G_D$ has a charge of at least $35 + 5 \cdot 14 = 105$. By Claim F, every $P_2$-component has at least four non-special blue neighbors and, therefore, its charge is at least $2 \cdot 35 + 4 \cdot 3 + 4 \cdot 19/3 = 321/3$. By Claim E, every $C_4$-component has at least $4 \cdot 35 + 12 \cdot 19/3 = 216$ and every $C_7$-component has at least $7 \cdot 35 + 21 \cdot 19/3 = 378$ as a charge. Finally, every $C_5$-component has $5 \cdot 35 + 15 \cdot 3 = 220$, and every $C_{10}$-component has $10 \cdot 35 + 30 \cdot 3 = 440$ after the discharging. Let the number of $P_1$, $P_2$, $C_4$, $C_5$, $C_7$, and $C_{10}$-components of $G_D[W]$ be denoted by $p_1$, $p_2$, $c_4$, $c_5$, $c_7$, and $c_{10}$, respectively, and let $A$ be a minimum dominating set in $G_D[W]$. Then,

$$|A| = p_1 + p_2 + 2 c_4 + 2 c_5 + 3 c_7 + 4 c_{10}.$$
As $D \cup A$ is a dominating set in the graph $G$, we have $f(G_{D \cup A}) = 0$. Thus, $s(A) = f(G_D)$, and the discharging shows the following lower bound:

$$s(A) = f(G_D) \geq 105 p_1 + \frac{321}{3} p_2 + 216 c_4 + 220 c_5 + 378 c_7 + 440 c_{10}$$

$$\geq 105 (p_1 + p_2 + 2 c_4 + 2 c_5 + 3 c_7 + 4 c_{10}) = 105 |A|.$$  

As it contradicts our assumption on $G_D$, we infer that every graph $G$ with minimum degree 5 and every $D \subseteq V(G)$ with $f(G_D) > 0$ satisfy Property 1.

To finish the proof of Theorem 1, we first observe that $f(G_\emptyset) = 35 n$. Then, by Property 1, there exists a nonempty set $A_1$ such that $f(G_{A_1}) \leq f(G_\emptyset) - 105 |A_1|$. Applying this iteratively, at the end we obtain a dominating set $D = A_1 \cup \cdots \cup A_j$ such that

$$f(G_D) = 0 \leq f(G_\emptyset) - 105 |D| = 35 n - 105 |D|,$$

and we may conclude

$$\gamma(G) \leq |D| \leq \frac{35 n}{105} = \frac{n}{3}.$$  

In a graph $G$, a set $X \subseteq V(G)$ is a 2-packing, if any two different vertices from $X$ are at a distance of at least 3. The proof of Theorem 1 directly corresponds to an algorithm that outputs a dominating set of cardinality at most $n/3$. If $G$ is 5-regular and $X$ is a 2-packing in it, we may start the algorithmic process with choosing the vertices of $X$ one by one. Hence, we conclude the following.

**Corollary 1.** If $X$ is a 2-packing in a 5-regular graph $G$, then $X$ can be extended to a dominating set $D$ of cardinality at most $n/3$.

### 3 Graphs of minimum degree 4

In this section, we apply the previous approach for graphs of minimum degree four and get a shorter alternative proof for the following theorem which was first proved by Sohn and Xudong [22] in 2009.
Theorem 2. For every graph $G$ on $n$ vertices and with minimum degree 4, the domination number satisfies $\gamma(G) \leq \frac{4n}{11}$.

Proof. Consider a graph $G$ of minimum degree 4 and let $D$ be a subset of $V = V(G)$. Let $W$, $B$, and $R$ denote the set of white, blue, and red vertices in $G_D$. The set of blue vertices that have at least 4 white neighbors is denoted by $B_4$ while, for $i = 1, 2, 3$, $B_i$ stands for the set of blue vertices that have exactly $i$ white neighbors. In the proof, a residual graph $G_D$ is associated with the following value:

$$g(G_D) = 16|W| + 10|B_4| + 9|B_3| + 8|B_2| + 7|B_1|.$$ 

For a set $A \subseteq V \setminus D$, we use the notation

$$s(A) = g(G_D) - g(G_{D\cup A})$$

and define the following property for $G_D$:

Property 2. There exists a nonempty set $A \subseteq V \setminus D$ such that $s(A) \geq 44|A|$. 

We now suppose for a contradiction that a residual graph $G_D$ with $\delta(G) = 4$ and $g(G_D) > 0$ does not satisfy Property 2. We prove several claims for $G_D$ and then get the final contradiction via performing a discharging.

Claim G. $\Delta_W(W) \leq 2$ and $\Delta_W(B) \leq 3$ hold.

Proof. All the following cases can be excluded:

- **Case 1.** $\Delta_W(W) \geq 5$
  Choose a white vertex $v$ with $d_W(v) \geq 5$ and let $A = \{v\}$. In $G_{D\cup A}$, the white vertex $v$ becomes red and its white neighbors become blue or red. This gives $s(A) \geq 16 + 5 \cdot (16 - 10) = 46 > 44 |A|$ which contradicts our assumption that $G_D$ does not satisfy Property 2.

- **Case 2.** $\Delta_W(W) = 4$
  Consider a white vertex $v$ with $d_W(v) = 4$ and set $A = \{v\}$. In $G_{D\cup A}$, the vertex $v$ becomes red and its white neighbors become blue or red. Since each white neighbor $u$ had at most four white neighbors in $G_D$, $u$ may have at most three white neighbors in $G_{D\cup A}$. Therefore, $s(A) \geq 16 + 4 \cdot (16 - 9) = 44 |A|$, a contradiction.
Case 3. $\Delta_W(W) \leq 3$ and $\Delta_W(B) \geq 5$

Let $v$ be a blue vertex with $d_W(v) \geq 5$ and define $A = \{v\}$ again. In $G_D$, the vertex $v$ belongs to $B_4$, while we have $v \in R$ in $G_{D \cup A}$. Further, since $\Delta_W(W) \leq 3$, each white neighbor $u$ of $v$ has at most three white neighbors in $G_D$ and $u \in B_3 \cup B_2 \cup B_1 \cup R$ in $G_{D \cup A}$. As follows, $s(A) \geq 10 + 5(16 - 9) = 45 > 44|A|$ that is a contradiction to our assumption.

Case 4. $\Delta_W(W) = 3$ and $\Delta_W(B) \leq 4$

First remark that, by the condition $\Delta_W(B) \leq 4$, if a blue vertex loses $\ell$ white neighbors in a step, then $g(G_D)$ decreases by at least $\ell$. Select a white vertex $v$ with $d_W(v) = 3$ and let $A = \{v\}$. In $G_{D \cup A}$, vertex $v$ becomes red and its three white neighbors become blue or red having at most 2 white neighbors. By Observation 1 (iv), each of $v$ and its white neighbors has at least one blue neighbor in $G_D$. Thus, we get $s(A) \geq 16 + 3(16 - 8) + 4 \cdot 1 = 44|A|$ which is a contradiction.

Case 5. $\Delta_W(W) \leq 2$ and $\Delta_W(B) = 4$

Here, we choose a vertex $v$ from $B_4$ and define $A = \{v\}$. First, observe that $v$ belongs to $B_4$ in $G_D$ and to $R$ in $G_{D \cup A}$. In $G_D$, $v$ has four white neighbors which become blue or red and belong to $B_2 \cup B_1 \cup R$ in $G_{D \cup A}$. By Observation 1 (iv) and by $\Delta_W(W) \leq 2$, each white neighbor has at least one blue neighbor that is different from $v$. Therefore, $s(A) \geq 10 + 4(16 - 8) + 4 \cdot 1 = 46 > 44|A|$ that is a contradiction again. This finishes the proof of the claim. ($\blacksquare$)

In the continuation, we suppose that $\Delta_W(W) \leq 2$ and $\Delta_W(B) \leq 3$ hold in the counterexample $G_D$ and, therefore, the graph $G_D[W]$, which is induced by the white vertices of $G_D$, consists of components which are paths and cycles. We prove some further properties for $G_D$.

Claim H. In $G_D[W]$, each component is a path $P_1$, $P_2$ or a cycle $C_4$ or $C_7$.

Proof. Assume that there is a path component $P_j: v_1 \ldots v_j$ of order $j \geq 3$ in $G_D[W]$. We set $A = \{v_2\}$ and observe that both $v_1$ and $v_2$ become red and $v_3$ belongs to $B_1 \cup R$ in $G_{D \cup A}$. This contributes to $s(A)$ by at least $2 \cdot 16 + (16 - 7)$. By Observation 1 (iv), $v_1$, $v_2$, and $v_3$, respectively, have at least 3, 2, 2 blue neighbors in $G_D$. The decrease in their white-degrees contributes to $s(A)$ by at least $7 \cdot 1$. Then, we get $s(A) \geq 32 + 9 + 7 = 48 > 44|A|$, a contradiction.

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Now, assume that a cycle \( C_{3k} : v_1 \ldots v_{3k}v_1 \) exists in \( G_D[W] \) and set \( A = \{v_3, v_6, \ldots, v_{3k}\} \). In \( G_{D\cup A} \), all the 3k vertices of the cycle are recolored red and, by Observation 2(ii), the sum of the white-degrees of the blue vertices decreases by at least \( 2 \cdot 3k \). Consequently, we get the contradiction \( w(A) \geq 16 \cdot 3k + 6k = 54k > 44 |A| \). A similar argumentation can be given if the cycle is \( C_{3k+2} : v_1 \ldots v_{3k+2}v_1 \), where \( k \geq 1 \), and \( A = \{v_3, v_6, \ldots, v_{3k}, v_{3k+2}\} \). Here, \( |A| = k+1 \) and we get \( s(A) \geq 16 \cdot (3k+2) + 2 \cdot (3k+2) = 54k + 36 > 44k + 44 = 44 |A| \) that is a contradiction. For the case when the cycle is of order 3k+1, we suppose \( k \geq 3 \) and obtain a contradiction as follows. Let \( C_{3k+1} : v_1 \ldots v_{3k+1}v_1 \) and let \( A \) be the \((k+1)\)-element dominating set \( \{v_3, v_6, \ldots, v_{3k}, v_{3k+1}\} \). We get \( s(A) \geq 16 \cdot (3k+1) + 2 \cdot (3k+1) = 54k + 18 > 44k + 44 = 44 |A| \) since \( k \geq 3 \) is supposed. This finishes the proof of Claim H. (□)

Claim I. No vertex from \( B_3 \) is adjacent to any vertices from \( W_0 \) in \( G_D \).

Proof. Assume for a contradiction that a vertex \( v \in B_3 \) has a neighbor \( u_0 \) from \( W_0 \). Let \( A = \{v\} \) and denote by \( u_1 \) and \( u_2 \) the further two white neighbors of \( v \). In \( G_{D\cup A} \), \( v, u_0 \in R \) and \( u_1, u_2 \in B_2 \cup B_1 \cup R \). This change contributes to \( s(A) \) by at least \( 9 + 16 + 2(16-8) = 41 \). By Observation 1(iv), the neighbors \( u_0, u_1 \) and \( u_2 \) have, respectively, at least 3, 1, 1 blue neighbors which are different from \( v \). Therefore, \( s(A) \geq 41 + 5 \cdot 1 = 46 > 44 |A| \) should be true but this contradicts our assumption on \( G_D \). (□)

As follows, the vertices from \( W_0 \) may be adjacent only to some vertices from \( B_2 \cup B_1 \). We call a vertex from \( B_2 \) special, if it is adjacent to a vertex from \( W_0 \).

Claim J. No special vertex is adjacent to two vertices from \( W_0 \).

Proof. Suppose that a vertex \( v \in B_2 \) is adjacent to two vertices, say \( u_1 \) and \( u_2 \) from \( W_0 \). We set \( A = \{v\} \) and observe that all the three vertices \( v, u_1 \) and \( u_2 \) are red in \( G_{D\cup A} \). By Observation 1(iv), each of \( u_1 \) and \( u_2 \) has at least three blue neighbors different from \( v \). This yields \( s(A) \geq 8 + 2 \cdot 16 + 6 \cdot 1 = 46 > 44 |A| \) that contradicts our assumption on \( G_D \). (□)

Claim K. No special vertex is adjacent to a vertex from a \( C_4 \) or \( C_7 \).

Proof. If a special vertex \( v \) is adjacent to a vertex \( u_0 \) from \( W_0 \) and to a vertex \( u_1 \) from a 4-cycle component \( C_4 : u_1u_2u_3u_4u_1 \) of \( G_D[W] \), then we set \( A = \{v, u_3\} \) and observe that \( v, u_0, u_1, u_2, u_3 \) and \( u_4 \) turn red in \( G_{D\cup A} \). In \( G_D \), the vertices \( u_0, u_1, u_2, u_3 \) and \( u_4 \), respectively, have at least 3, 1, 2, 2,
2 neighbors from \((B_3 \cup B_2 \cup B_1) \setminus \{v\}\). Thus, \(s(A) \geq 8 + 5 \cdot 16 + 10 \cdot 1 = 98 > 44 |A|\), a contradiction. Similarly, if we suppose that a special vertex \(v\) is adjacent to \(u_0\) from \(W_0\) and to a vertex \(u_1\) from the 7-cycle \(u_1 \ldots u_7u_1\), we set \(A = \{v, u_3, u_6\}\) and conclude that \(s(A) \geq 8 + 8 \cdot 16 + 16 \cdot 1 = 152 > 44 |A|\) that contradicts our assumption on \(G_D\). \(\Diamond\)

**Claim L.** If \(v_1\) and \(v_2\) are two adjacent vertices from \(W_1\), then at most one of them may have a special blue neighbor.

**Proof.** Assume to the contrary that \(v_1u_1, v_2u_2 \in E(G)\) such that \(u_1\) and \(u_2\) are special vertices in \(G_D\), and let \(x_1\) and \(x_2\) be the further white neighbors of \(u_1\) and \(u_2\). Hence, we have \(v_1, v_2 \in W_1, u_1, u_2 \in B_2\), and \(x_1, x_2 \in W_0\) in \(G_D\). Consider the set \(A = \{u_1, u_2\}\) and observe that all the six vertices \(v_1, v_2, u_1, u_2, x_1, x_2\) become red in \(G_D \cup A\). For \(i = 1, 2\), by Claim I and Observation \(\Box\) \((iv)\), the vertex \(x_i\) has at least three neighbors from \((B_2 \cup B_1) \setminus \{v\}\) and \(v_i\) has at least two neighbors from \((B_3 \cup B_2 \cup B_1) \setminus \{v\}\). This implies the contradiction \(s(A) \geq 2 \cdot 8 + 4 \cdot 16 + 10 \cdot 1 = 90 > 44 |A|\). \(\Box\)

**Discharging.** Applying Claims G–L, we now perform a discharging and prove that \(G_D\) satisfies Property 1. We assign charges to the (non-red) vertices of \(G_D\) so that every white vertex gets 16, and every vertex from \(B_3, B_2\), and \(B_1\) gets 9, 8, and 7, respectively. We remark that the sum of these charges equals \(g(G_D)\). Then, every blue vertex, except the special ones, distributes its charge equally among the white neighbors as follows:

- Every vertex from \(B_3\) gives 3 to each white neighbor.
- Every non-special vertex from \(B_2\) gives 4 to each white neighbor.
- Every special vertex gives 7 to its neighbor from \(W_0\), and gives 1 to the other neighbor.
- Every vertex from \(B_1\) gives 7 to its neighbor.

After the discharging, every vertex from a \(P_1\)-component of \(G_D[W]\) has a charge of at least \(16 + 4 \cdot 7 = 44\). By Claim L, every \(P_2\)-component has at least three non-special blue neighbors and, therefore, its charge is at least \(2 \cdot 16 + 3 \cdot 1 + 3 \cdot 3 = 44\). By Claim K, every \(C_4\)-component has at least \(4 \cdot 16 + 8 \cdot 3 = 88\) and every \(C_7\)-component has at least \(7 \cdot 16 + 14 \cdot 3 = 154\) as a charge. Let the number of \(P_1\), \(P_2\), \(C_4\), and \(C_7\)-components of \(G[W]\)
be denoted by $p_1$, $p_2$, $c_4$, and $c_7$, respectively, and let $A$ be a minimum dominating set in $G[W]$. Then,
\[ |A| = p_1 + p_2 + 2c_4 + 3c_7. \]
As $D \cup A$ is a dominating set in the graph $G$, we have $g(G_{D\cup A}) = 0$. Thus, $s(A) = g(G_D)$, and the discharging proves the following lower bound:
\[ s(A) = g(G_D) \geq 44 p_1 + 44 p_2 + 88 c_4 + 154 c_7 \]
\[ \geq 44 (p_1 + p_2 + 2 c_4 + 3 c_7) = 44 |A|. \]
As it contradicts our assumption on $G_D$, we infer that every graph $G$ with minimum degree 4 and every $D \subseteq V(G)$ with $g(G_D) > 0$ satisfy Property 2.

To prove Theorem 2, we observe that $g(G_\emptyset) = 16n$ and, by Property 2, there exists a set $A_1$ such that $g(G_{A_1}) \leq g(G_\emptyset) - 44 |A_1|$. As $G_{A_1}$ also satisfies Property 2, we may continue the process if $g(G_{A_1}) > 0$, and at the end we obtain a dominating set $D = A_1 \cup \cdots \cup A_j$ such that
\[ g(G_D) = 0 \leq g(G_\emptyset) - 44 |D| = 16 n - 44 |D|. \]
Consequently,
\[ \gamma(G) \leq |D| \leq \frac{16 n}{44} = \frac{4}{11} n \]
holds for every graph $G$ of minimum degree 4.

\section{Concluding remarks}

Theorem 1 shows that $\gamma(G) \leq n/3$ holds for every graph with minimum degree at least 5. However, I do not believe that this upper bound is tight over the class of graphs with $\delta(G) \geq 5$. Examples with $\gamma/n > 1/4$ can possibly be found among larger graphs via computer search or large constructions, but it seems that $\delta(G) \geq 5$ and $n \leq 12$ together implies $\gamma(G) \leq n/4$ that is quite far from the proved $n/3$-upper bound.

Unfortunately, Theorem 2 does not seem sharp either. However, here we have 4-regular examples where the quotient $\gamma/n$ equals 1/3 that is relatively close to the proved upper bound 4/11. The smallest such 4-regular graph is $G = K_6 - M$ that is obtained from the complete graph $K_6$ by the deletion of a perfect matching. Then, we have $\gamma(G) = 2 = n/3$. One may guess that this is the sharp upper bound for graphs of minimum degree 4 or, at least, it is true under the following stronger condition:
Conjecture 1. There exists a constant $n_0$ such that for every connected 4-regular graph $G$ of order $n > n_0$, we have $\gamma(G) \leq \frac{n}{3}$.

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