THE ENTANGLED ERGODIC THEOREM AND AN ERGODIC THEOREM FOR QUANTUM “DIAGONAL MEASURES”

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Abstract. Let $U$ be a unitary operator acting on the Hilbert space $H$, $\alpha : \{1, \ldots, 2k\} \mapsto \{1, \ldots, k\}$ a pair–partition, and finally $A_1, \ldots, A_{2k-1} \in \mathcal{B}(H)$. We show that the ergodic average

$$\frac{1}{N^k} \sum_{n_1, \ldots, n_k=0}^{N-1} U^{n_1}(1) A_1 U^{n_2}(2) \cdots U^{n_k(2k-1)} A_{2k-1} U^{n_k(2k)}$$

converges in the strong operator topology when $H$ is generated by the eigenvectors of $U$, that is when the dynamics induced by the unitary $U$ on $H$ is almost periodic. This result improves the known ones relative to the entangled ergodic theorem. We also prove the noncommutative version of the ergodic result of H. Furstenberg relative to diagonal measures. This implies that $\frac{1}{N} \sum_{n=0}^{N-1} U^n A U^n$ converges in the strong operator topology for other interesting situations where the involved unitary operator does not generate an almost periodic dynamics, and the operator $A$ is noncompact.

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1. INTRODUCTION

The investigation of ergodic properties of classical dynamical systems has a long history. As an example, we mention the well–known ergodic hypothesis (cf. e.g. [10], Section 4) which can be viewed as a justification of the microcanonical distribution in statistical mechanics. We refer the reader to [13] for a nice introduction, and the monograph [2] for the basic results and further details.

Recently, the ergodic theory of noncommutative dynamical systems has been an impetuous growth in relation to the natural applications to quantum (statistical) physics. In view to other potential applications, it is of interest to understand among the various ergodic properties, which ones survive by passing from the classical to the quantum case.
We mention the pivotal paper [12], where such a program is carried out for some basic recurrence, as well as multiple mixing properties. Notice that it is in general unclear what should be the right quantum counterpart of a classical ergodic property. For example, the reader can compare the property of the convergence to the equilibrium (i.e. ergodicity for an invariant state \( \omega \))

\[
\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \omega (B^* \alpha^n(A) B) = \omega (B^* B) \omega(A)
\]

suggested by the quantum physics, with the standard notion of ergodicity

\[
\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \omega (A \alpha^n(B)) = \omega(A) \omega(B)
\]

See [7], Proposition 1.1 for further details.

A notion which is meaningful in quantum setting is that of entangled ergodic theorem, formulated in [1] in connection with the central limit theorem for suitable sequences of elements of the group \( C^* \)-algebra of the free group \( \mathbb{F}_\infty \) on infinitely many generators.

The entangled ergodic theorem was clearly formulated in [11]. Namely, let \( U \) be a unitary operator acting on the Hilbert space \( \mathcal{H} \), and for \( m \geq k, \alpha : \{1, \ldots, m\} \mapsto \{1, \ldots, k\} \) a partition of the set \( \{1, \ldots, m\} \) in \( k \) parts. The entangled ergodic theorem concerns the convergence in the strong, or merely weak operator topology, of the multiple Cesaro mean

\[
(1.1) \quad \frac{1}{N^k} \sum_{n_1, \ldots, n_k=0}^{N-1} U^{n_{\alpha(1)}} A_1 U^{n_{\alpha(2)}} \cdots U^{n_{\alpha(m-1)}} A_{m-1} U^{n_{\alpha(m)}} ,
\]

\( A_1, \ldots, A_{m-1} \) being bounded operators acting on \( \mathcal{H} \).

Expressions like (1.1) naturally appear also in [12] relatively to the study of the multiple mixing. Namely, suppose that the dynamics of a (concrete) dynamical system is unitarily implemented by the unitary \( U \), and the vector \( \Omega \) is invariant under \( U \).\(^1\) Then firstly in [8], and more recently in [12], the behavior of the multiple correlations

\[
(1.2) \quad \frac{1}{N} \sum_{n=0}^{N-1} \omega (A_0 \text{ Ad}_U^n (A_1) \text{ Ad}_U^{2n} (A_2)) \equiv \frac{1}{N} \sum_{n=0}^{N-1} \langle U^n A_1 U^n A_2 \Omega, A_0^* \Omega \rangle
\]

has been studied in connection with the (1, 2)-multiple mixing or merely ergodicity. Notice that (1.2) is the particular case of (1.1) relative to

\(^1\)Notice that this is always the case by considering the GNS covariant representation.
the (trivial) pair–partition of two elements. Just by considering the simplest case of the partition of the empty set, the limit of the Cesaro mean in (1.1) reduces itself to the well–known mean ergodic theorem due to John von Neumann (cf. [13])

\begin{equation}
\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} U^n = E_1,
\end{equation}

$E_1$ being the selfadjoint projection onto the eigenspace of the invariant vectors for $U$.

Some applications of the entangled ergodic theorem are discussed in [6]. Apart from the other potential applications to the study of the ergodic properties of quantum dynamical systems, the entangled ergodic theorem is a fascinating self–contained mathematical problem. It is certainly true if the spectrum $\sigma(U)$ of $U$ is finite. Some very special cases for which it holds true are listed in [11]. It was shown in [5] that the entangled ergodic theorem holds true in a sufficiently general situation, that is when the operators $A_1, \ldots, A_{m-1}$ in (1.1) are compact.

The first part of the present paper is devoted to prove the entangled ergodic theorem in the case when the unitary $U$ is almost periodic (i.e. when $H$ is generated by the eigenvectors of $U$) and $\alpha : \{1, \ldots, 2k\} \mapsto \{1, \ldots, k\}$ a pair–partition, without any condition on the operators $A_1, \ldots, A_{2k-1}$. This result improves those in Section 3 of [5] relative to the almost periodic case, where only very special pair–partitions were considered.

The entangled ergodic theorem is not yet available in the full generality. Then it is natural to address the problem to find other nontrivial cases for which it holds true.

Another situation of interest arises from the generalization to the noncommutative setting, of the ergodic theorem of H. Furstenberg relative to diagonal measure (cf. [8, 9]). This is precisely the argument of the second part of the present paper. Namely, we prove an ergodic theorem relative to possibly noninvariant and nonnormal states, which is the generalization of Theorem 3.1 of [9] relative to the Abelian case. This allows us to prove the following result. Let $M$ be a von Neumann algebra equipped with the adjoint action of an ergodic unitary $U$, and a standard vector $\Omega$ which is invariant under $U$. Let $M'$ be the commutant von Neumann algebra of $M$. The state defined as

\[ A \otimes B \in M \otimes M' \mapsto \langle AB\Omega, \Omega \rangle, \]
is precisely the quantum counterpart of the “diagonal measure” associated to the product state

\[ A \otimes B \in M \otimes M' \mapsto \langle A\Omega, \Omega \rangle \langle B\Omega, \Omega \rangle. \]

We show that the Cesaro mean

\[ \frac{1}{N} \sum_{n=0}^{N-1} U^n AU^n \]

converges in the strong operator topology for each \( A \in M \cup M' \).

2. TERMINOLOGY, NOTATIONS AND BASIC RESULTS

Let \( X, Y \) be linear spaces. Their algebraic tensor product is denoted by \( X \otimes Y \). If \( \mathcal{H} \) and \( \mathcal{K} \) are Hilbert spaces, the Hilbertian tensor product, that is the completion of \( \mathcal{H} \otimes \mathcal{K} \) under the norm induced by the inner product

\[ \langle x \otimes \xi, y \otimes \eta \rangle := \langle x, y \rangle \langle \xi, \eta \rangle, \]

is denoted as \( \mathcal{H} \otimes \mathcal{K} \).

Let \( \{ A_\alpha \}_{\alpha \in J} \subset \mathcal{B}(\mathcal{H}) \) be a net consisting of bounded operators acting on the Hilbert space \( \mathcal{H} \). If it converges to \( A \in \mathcal{B}(\mathcal{H}) \) in the weak operator topology, respectively strong operator topology, we write respectively

\[ \text{w} - \lim_\alpha A_\alpha = A, \quad \text{s} - \lim_\alpha A_\alpha = A. \]

Let \( U \) be a unitary operator acting on \( \mathcal{H} \). Consider the resolution of the identity \( \{ E(\Delta) : \Delta \text{ Borel subset of } \mathbb{T} \} \) of \( U \) (cf. [16], Section VII.7). Denote with an abuse of notation, \( E_z := E(\{ z \}) \). Namely, \( E_z \) is nothing but the selfadjoint projection on the eigenspace corresponding to the eigenvalue \( z \) in the unit circle \( \mathbb{T} \).

The unitary \( U \) is said to be \textit{ergodic} if the fixed–point subspace \( E_1 \mathcal{H} \) is one dimensional. By the mean ergodic theorem (1.3), it is equivalent to the existence of a unit vector \( \xi_0 \in \mathcal{H} \) such that

\[ \lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} U^n \xi = \langle \xi, \xi_0 \rangle \xi_0, \]

or equivalently,

\[ \lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \langle U^n \xi, \eta \rangle = \langle \xi, \xi_0 \rangle \langle \xi_0, \eta \rangle. \]
The unitary $U$ is said to be *weakly mixing* if there exists a unit vector $\xi_0 \in \mathcal{H}$ such that
\[
\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\langle U^n \xi, \eta \rangle - \langle \xi, \xi_0 \rangle \langle \xi_0, \eta \rangle| = 0.
\]
Of course, a weakly mixing unitary is ergodic. It is well–known that the vice–versa does not hold. Indeed, $U$ is ergodic if and only if $E_1 \mathcal{H}$ is one dimensional. It is weakly mixing if and only if in addition,
\[
\sigma_{pp}(U) = \{1\}, \quad \sigma_{pp}(U) \text{ being the pure point spectrum of } U \text{ (cf. [13])}.\]
See e.g. [12].

The unitary $U$ is said to be *almost periodic* if $\mathcal{H} = \mathcal{H}_{ap}$, $\mathcal{H}_{ap}$ being the closed subspace consisting of the vectors having relatively norm–compact orbit under $U$. It is seen in [12] that $U$ is almost periodic if and only if $\mathcal{H}$ is generated by the eigenvectors of $U$.

Define
\[
(2.1) \quad \sigma_{ap}^a(U) := \{ z \in \sigma_{pp}(U) : zw = 1 \text{ for some } w \in \sigma_{pp}(U) \}.
\]

It is immediate to verify that $\sigma_{ap}^a(U)$ is a subgroup of the unit circle $\mathbb{T}$.

Let $\alpha : \{1, \ldots, 2k\} \mapsto \{1, \ldots, k\}$ be a pair–partition of the set $\{1, \ldots, k\}$. It is shown in Proposition 2.3 of [5], that the net
\[
\left\{ \sum_{z_1, \ldots, z_k \in F} E_{z_{\alpha(1)}} A_1 E_{z_{\alpha(2)}} \cdots E_{z_{\alpha(2k-1)}} A_{2k-1} E_{z_{\alpha(2k)}} : F \subset \sigma_{ap}^a(U) \text{ finite subsets} \right\} \subset \mathcal{B}(\mathcal{H})
\]
converges in the weak operator topology to a bounded operator written symbolically as
\[
(2.2) \quad S_{\alpha;A_1,\ldots,A_{2k-1}} = \sum_{z_1, \ldots, z_k \in \sigma_{ap}^a(U)} E_{z_{\alpha(1)}} A_1 E_{z_{\alpha(2)}} \cdots E_{z_{\alpha(2k-1)}} A_{2k-1} E_{z_{\alpha(2k)}}.
\]

More precisely, the pairs $z_{\alpha(i)}$ are alternatively $z_j$ and $\bar{z}_j$ whenever $\alpha(i) = j$, and finally the sum is understood as the limit in the weak operator topology of the above mentioned net obtained by considering all the finite truncations of the r.h.s. of (2.2).\(^2\)

\(^2\)If for example, $\alpha$ is the pair–partition $\{1, 2, 1, 2\}$ of four elements,
\[
S_{\alpha;A,B,C} = \sum_{z,w \in \sigma_{ap}^a(U)} E_z A E_w B E_z C E_w.
\]
Fix a pair–partition $\beta : \{1, \ldots, 2k + 2\} \mapsto \{1, \ldots, k + 1\}$. Let $k_{\beta} \in \{1, \ldots, 2k + 1\}$ be the first element of the pair $\beta^{-1}(\{k + 1\})$, and $\alpha_{\beta}$ the pair–partition of $\{1, \ldots, 2k\}$ obtained by deleting $\beta^{-1}(\{k + 1\})$ from $\{1, \ldots, 2k + 2\}$, and $k + 1$ from $\{1, \ldots, k + 1\}$. Notice that, if $x \in H$ is an eigenvector of $U$ with eigenvalue $z_0$, then we obtain

$$S_{\beta; A_1, \ldots, A_{2k + 1}} x = \sum_{n=0}^{N-1} \omega(A\alpha^{n}(B)) = \omega(A)\omega(B).$$

It is said to be weakly mixing if

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \left| \omega(A\alpha^{n}(B)) - \omega(A)\omega(B) \right| = 0$$

for each $A, B \in \mathcal{A}$.

For a (discrete) $C^*$-dynamical system we mean a triplet $(\mathcal{A}, \alpha, \omega)$ consisting of a $C^*$-algebra $\mathcal{A}$, an automorphism $\alpha$ of $\mathcal{A}$, and a state $\omega \in \mathcal{S}(\mathcal{A})$ invariant under the action of $\alpha$.

A $C^*$-dynamical system $(\mathcal{A}, \alpha, \omega)$ is said to be ergodic if for each $A, B \in \mathcal{A}$,

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \omega(A\alpha^{n}(B)) = \omega(A)\omega(B).$$

Let $(H, \pi, U, \Omega)$ be the GNS covariant representation (cf. [15], Section I.9) canonically associated to the dynamical system under consideration. Then $(\mathcal{A}, \alpha, \omega)$ is ergodic (respectively weakly mixing) if and only if $U$ is ergodic (respectively weakly mixing), see e.g. [12].

Let $s(\omega)$ be the support of $\omega$ in the bidual $\mathcal{A}^{**}$. Then $s(\omega) \in Z(\mathcal{A}^{**})$ if and only if $\Omega$ is separating for $\pi(\mathcal{A})''$, $Z(\mathcal{A}^{**})$ being the centre of $\mathcal{A}^{**}$ (see e.g. [14], Section 10.17).

Denote $M := \pi(\mathcal{A})''$, and with an abuse of notation, $\alpha := \text{Ad}_U$ the adjoint action of $U$ on $\mathcal{B}(H)$. The commutant von Neumann algebra is $M' \equiv \pi(\mathcal{A})'$. For $z$ in $\mathbb{T}$ denote

$$M_z = \{A \in M : \alpha(A) = zA\}, \quad (M')_z = \{B \in M' : \alpha(B) = zB\}.$$

The following results are probably known to the experts. We provide their proof for the convenience of the reader.

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Notice that this definition of ergodicity for an invariant state differs from the standard one. Indeed, an invariant state $\varphi \in \mathcal{S}(\mathcal{A})$ is said to be ergodic if it is extremal among all the states of $\mathcal{A}$ which are invariant under the action of $\alpha$. It can be shown that if $\varphi$ is asymptotically Abelian (or for an even and graded–asymptotically Abelian state $\varphi$, when $\mathcal{A}$ is a $\mathbb{Z}_2$–graded $C^*$-algebra), both definitions coincide. See e.g. [4], Section 3 for further details.
Proposition 2.1. Let the $C^*$–dynamical system $(\mathcal{A}, \alpha, \omega)$ be such that $s(\omega)$ is central. Then, with the previous notations,

$$M_z\Omega = (M')_z\Omega = E_z\mathcal{H},$$

and we can choose an orthonormal basis $\{u_{\alpha z}\}_{\alpha z \in I_z} \subset M_z\Omega$ (equivalently $\{v_{\beta z}\}_{\beta z \in J_z} \subset (M')_z\Omega$) for $E_z\mathcal{H}$.

In addition, $\sigma_{pp}(U) = \sigma_{pp}(U)^{-1}$, and if $z \in \sigma_{pp}(U)$,

$$\left\{ \frac{(A^z_{\alpha z})^*\Omega}{\sqrt{\omega(A^z_{\alpha z}(A^z_{\alpha z})^*)}} \right\}_{\alpha z \in I_z}$$

is an orthonormal basis for $E_z\mathcal{H}$ whenever $\{A^z_{\alpha z}\}_{\alpha z \in I_z}$ is an orthonormal basis for $E_z\mathcal{H}$.

Proof. The fact that $M_z\Omega$ is dense in $E_z\mathcal{H}$ follows from Proposition 3.2 of [12]. Then, by exchanging the role between $M$ and $M'$, $(M')_z\Omega$ is also dense in $E_z\mathcal{H}$. By taking into account that the adjoint action of $U$ on $M$ (or equivalently on $M'$) is an automorphism, if $A \in M_z$ is nonnull, $A^*$ is a nonnull element of $M_z$, that is $z \in \sigma_{pp}(U)$ implies $\bar{z} \in \sigma_{pp}(U)$.

For $z \in \sigma_{pp}(U)$, choose an orthonormal basis $\{u_{\alpha z}\}_{\alpha z \in I_z} \subset M_z\Omega$ for $M_z\Omega$ which always exists by Proposition 3.2 of [12]. Then $\omega(A_{\alpha z}^*B_{\beta z}) = 0$ for some nonnull numbers $\alpha, \beta$. Indeed, $A_{\alpha z}^*B_{\beta z}\Omega$ is invariant under $U$.

Thus by ergodicity, $A^*B\Omega = \alpha\Omega$, and by the fact that $\Omega$ is separating,
$A^*B = \alpha I$. In addition, suppose that $\alpha = 0$. As $AA^*$ is a nonnull multiple, say $c$, of the identity, we have $AA^*B = 0$, which means $B = 0$, a contradiction. At the same way, we verify $BA^* \neq 0$. Now, $\alpha^{-1}A^*$ and $\beta^{-1}A^*$ are left and right inverses of $B$. This means that $B$ is invertible and $B^{-1} = \alpha^{-1}A^*$. At the same way, $A$ is invertible too. Moreover, $AB^{-1} = \alpha^{-1}AA^* = \alpha^{-1}cI$. This means $A = \alpha^{-1}cB$, that is $A$ is a multiple of $B$. In addition, in this situation $AA^* = cI$ means that $A$ is a multiple of the unitary $A/\sqrt{c}$. If $z,w \in \sigma_{pp}(U)$, let $V_z \in M_z$, $V_w \in M_w$ be the corresponding unitaries.

Then $V_zV_w \in M_{zw}$ is nonnull, that is $zw \in \sigma_{pp}(U)$. □

3. THE ENTANGLED ERGODIC THEOREM IN THE ALMOST PERIODIC CASE

The present section is devoted to the almost periodic situation, without any restriction relative to the operators appearing in (1.1), and the pair–partition $\alpha : \{1, \ldots, 2k\} \mapsto \{1, \ldots, k\}$. In this way, we improve the results in Section 3 of [5] where only very special pair–partitions were considered. We start by recalling for the reader convenience the known results relative to the entangled ergodic theorem.

Let $U$ be a unitary operator acting on the Hilbert space $\mathcal{H}$, and for $m \geq k$, $\alpha : \{1, \ldots, m\} \mapsto \{1, \ldots, k\}$ a partition of the set $\{1, \ldots, m\}$. It was shown in Theorem 2.6 of [5] that the multiple Cesaro mean in (1.1) converges in the weak operator topology when $A_1, \ldots, A_{m-1} \in \mathcal{K}(\mathcal{H})$, $\mathcal{K}(\mathcal{H})$ being the algebra of all the compact operators acting on $\mathcal{H}$. In the case of a pair–partition $\alpha : \{1, \ldots, 2k\} \mapsto \{1, \ldots, k\}$, we have (cf. [5], Theorem 2.5),

$$\lim_{N \to +\infty} \left\{ \frac{1}{N^k} \sum_{n_1, \ldots, n_k=0}^{N-1} U^{\alpha(1)} A_1 U^{\alpha(2)} \cdots U^{\alpha(2k-1)} A_{2k-1} U^{\alpha(2k)} \right\}$$

$$= S_{\alpha;A_1, \ldots, A_{2k-1}},$$

where $A_1, \ldots, A_{m-1} \in \mathcal{K}(\mathcal{H})$, and $S_{\alpha;A_1, \ldots, A_{2k-1}}$ is given in (2.2). After passing to the finite rank operators, the Cesaro Mean in (1.1) disentangles, and the proof follows by Lebesgue dominated convergence theorem.

In order to treat the almost periodic case, from now on we suppose in the present section that $\mathcal{H}$ is generated by the eigenvectors of $U$.

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A partition $\alpha : \{1, \ldots, m\} \mapsto \{1, \ldots, k\}$ of the set made of $m$ elements in $k$ parts is nothing but a surjective map, the parts of $\{1, \ldots, m\}$ being the preimages $\{\alpha^{-1}(\{j\})\}_{j=1}^k$. 

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The proof of the following result relies upon the mean ergodic theorem (1.3), by showing that one can reduce oneself to the dense subspace algebraically generated by the eigenvectors of $U$.

**Theorem 3.1.** Let $U$ be an almost periodic unitary operator acting on the Hilbert space $\mathcal{H}$. Then for each pair–partition $\alpha : \{1, \ldots, 2k\} \mapsto \{1, \ldots, k\}$, and $A_1, \ldots, A_{2k-1} \in \mathcal{B}(\mathcal{H})$,

$$
\lim_{N \to +\infty} \left\{ \frac{1}{N^k} \sum_{n_1, \ldots, n_k=0}^{N-1} U^{n_\alpha(1)} A_1 U^{n_\alpha(2)} \ldots U^{n_\alpha(2k-1)} A_{2k-1} U^{n_\alpha(2k)} \right\} = S_{\alpha; A_1, \ldots, A_{2k-1}}.
$$

(3.1)

**Proof.** We treat the case of the partition $\{1, 2, 1, 3, 2, 3\}$, the general case follows analogously. Fix $\varepsilon > 0$, and suppose that $A, B, C, D, F \in \mathcal{B}(\mathcal{H})$ have norm one. Let $I_\varepsilon$ be such that

$$
\| x - \sum_{\sigma \in I_\varepsilon} E_\sigma x \| < \varepsilon.
$$

For each $\sigma \in I_\varepsilon$, let $I_\varepsilon(\sigma)$ be such that

$$
\| F E_\sigma x - \sum_{\tau \in I_\varepsilon(\sigma)} E_\tau F E_\sigma x \| < \frac{\varepsilon}{|I_\varepsilon|}.
$$

By taking into account the mean ergodic theorem (1.3), choose $N_\varepsilon$ such that

$$
\left\| \left( \frac{1}{N} \sum_{n=0}^{N-1} (\sigma U)^n - E_\sigma \right) DE_\sigma F E_\sigma x \right\| < \frac{\varepsilon}{\sum_{\sigma \in I_\varepsilon} |I_\varepsilon(\sigma)|},
$$

whenever $N > N_\varepsilon$ and $\sigma \in I_\varepsilon$, $\tau \in I_\varepsilon(\sigma)$. Finally, for each $\sigma \in I_\varepsilon$, $\tau \in I_\varepsilon(\sigma)$, let $I_\varepsilon(\sigma, \tau)$ be such that

$$
\left\| C E_\sigma DE_\tau F E_\sigma x - \sum_{\rho \in I_\varepsilon(\sigma, \tau)} E_\rho C E_\sigma DE_\tau F E_\sigma x \right\| < \frac{\varepsilon}{\sum_{\sigma \in I_\varepsilon} |I_\varepsilon(\sigma)|}.
$$
We obtain by (2.3),
\[
\left\| \frac{1}{N^3} \sum_{k,m,n=0}^{N-1} U^k A U^m B U^n C U^m D U^m F U^n x - S_{\alpha;A,B,C,D,F} x \right\| \\
\leq 5 \varepsilon + \sum_{\sigma \in I_\varepsilon} \sum_{\tau \in I_\varepsilon(\sigma)} \sum_{\rho \in I_\varepsilon(\sigma,\tau)} \left\| \left( \frac{1}{N} \sum_{k=0}^{N-1} (\rho U)^k \right) A \left( \frac{1}{N} \sum_{m=0}^{N-1} (\tau U)^m \right) \right\| \\
\times B E_\rho C E_\sigma D E_\tau F E_\sigma x - E_\beta A E_\sigma B E_\rho C E_\sigma D E_\tau F E_\sigma x .
\]
Taking the limsup on both sides, we obtain the assertion by the mean ergodic theorem (1.3), by taking into account the fact that \( \varepsilon > 0 \) is arbitrary. \( \square \)

4. AN ERGODIC THEOREM FOR NON INVARIANT STATES

The present section concerns the generalization to the quantum case of a well–known classical ergodic theorem due to H. Furstenberg, for non invariant measures (cf. [8, 9]). Such a theorem has a natural application to the noncommutative case of “diagonal measures”.

We start with a \( C^* \)-dynamical system \((\mathcal{A}, \alpha, \varphi)\), whose GNS covariant representation is denoted as \((\mathcal{H}_\varphi, \pi_\varphi, U, \Phi)\). Consider another state \( \omega \in \mathcal{S}(\mathcal{A}) \).

Let \( E_1 \) be the selfadjoint projection onto the invariant vectors for the unitary \( U \). Suppose that \( \pi_\varphi(\mathcal{B}) \Phi \cap E_1 \mathcal{H}_\varphi \) is dense in \( E_1 \mathcal{H}_\varphi \).

Definition 4.1. The state \( \omega \) is said to be generic for \((\mathcal{A}, \alpha, \varphi)\) w.r.t. \( \mathcal{B} \) if for each \( B \in \mathcal{B} \),
\[
\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \omega(\alpha^n(B)) = \varphi(B) .
\]

Let \( E_1 \) be the selfadjoint projection onto the invariant vectors for the unitary \( U \). Suppose that \( \pi_\varphi(\mathcal{B}) \Phi \cap E_1 \mathcal{H}_\varphi \) is dense in \( E_1 \mathcal{H}_\varphi \).

Lemma 4.2. Under the above conditions,
\[
\pi_\varphi(B) \Phi \in \pi_\varphi(\mathcal{B}) \Phi \cap E_1 \mathcal{H}_\varphi \implies \pi_\omega(B) \Omega \in \mathcal{H}_\omega
\]
uniquely defines a partial isometry \( V : \mathcal{H}_\varphi \mapsto \mathcal{H}_\omega \) such that \( V^* V = E_1 \mathcal{H}_\varphi \).
Proof. It is enough to show that, under our assumptions, the map in (4.1) is isometric. We get for each $B \in \mathcal{B}$ invariant under the action of $\alpha$, first by taking into account the invariance of $B$, and then the genericity of $\omega$,

$$\|\pi_\omega(B)\Omega\|^2 \equiv \omega(B^*B) \equiv \frac{1}{N} \sum_{n=0}^{N-1} \omega(\alpha^n(B^*B))$$

$$\equiv \lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \omega(\alpha^n(B^*B)) \equiv \varphi(B^*B) \equiv \|\pi_\varphi(B)\Phi\|^2.$$  

□

For $A \in \mathcal{B}(\mathcal{H})$, denote $|A| := (A^*A)^{1/2}$. We need also the following technical results.

**Lemma 4.3.** Let $A_1, \ldots, A_n$ be bounded operators acting on the Hilbert space $\mathcal{H}$. Then

$$\left| \frac{1}{n} \sum_{k=1}^{n} A_k \right|^2 \leq \frac{1}{n} \sum_{k=1}^{n} |A_k|^2.$$  

Proof. The proof easily follows if one verifies $A^*B + B^*A \leq A^*A + B^*B$. But,

$$0 \leq (A - B)^*(A - B) = A^*A + B^*B - (A^*B + B^*A).$$  

□

**Lemma 4.4.** Let $\{a_k\}_{k \in \mathbb{N}} \subset \mathbb{C}$ be a bounded sequence. Then for each fixed $M$,

$$\lim_{N \to +\infty} \left( \frac{1}{MN} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} a_{m+n} - \frac{1}{N} \sum_{n=0}^{N-1} a_n \right)$$

at a rate depending on $\sup_k |a_k|$.

Proof. The proof follows by taking into account

$$\left| \frac{1}{MN} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} a_{m+n} - \frac{1}{N} \sum_{n=0}^{N-1} a_n \right| \leq \frac{(M-1)(M+2)}{MN} \sup_k |a_k|.$$  

□

The following theorem is nothing but the announced generalization of Theorem 4.14 of [9] (see also [8]) to the quantum case.
Theorem 4.5. Let $\omega$ be generic for $(\mathfrak{A}, \alpha, \varphi)$ w.r.t. a $\ast$--subalgebra $\mathfrak{B}$ which is globally stable under the action of $\alpha$, and satisfies
\[ \pi_\varphi(\mathfrak{B}) \Phi \cap E_1 \mathcal{H}_\varphi = E_1 \mathcal{H}_\varphi. \]
Then for each $B \in \mathfrak{B}$,
\[ \lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \pi_\omega(\alpha^n(B)) \Omega = V(\pi_\varphi(B) \Phi), \]
where $V : \mathcal{H}_\varphi \ni \mathcal{H}_\omega$ is the partial isometry given in Lemma 4.2.

Proof. Let $B \in \mathfrak{B}$ and $\varepsilon > 0$ be given. Choose a $\alpha$--invariant $B_\varepsilon \in \mathfrak{B}$ such that $\| (E_1 \pi_\varphi(B) - \pi_\varphi(B_\varepsilon)) \Phi \| \leq \varepsilon$. By the mean ergodic theorem (1.3),
\[ \frac{1}{M} \sum_{m=0}^{M-1} \pi_\varphi(\alpha^m(B - B_\varepsilon)) \Phi \to (E_1 \pi_\varphi(B) - \pi_\varphi(B_\varepsilon)) \Phi. \]

Thus, for $M$ sufficiently large,
\[ \varphi \left( \left| \frac{1}{M} \sum_{m=0}^{M-1} \alpha^m(B - B_\varepsilon) \right|^2 \right) < \varepsilon^2. \]

Denote
\[ \Gamma := \frac{1}{M} \sum_{m=0}^{M-1} \alpha^m(B - B_\varepsilon). \]

By hypothesis, $\Gamma^* \Gamma \in \mathfrak{B}$, and as $\omega$ is generic w.r.t. $\mathfrak{B}$,
\[ \frac{1}{N} \sum_{n=0}^{N-1}\omega(\alpha^n(\Gamma^* \Gamma)) \to \varphi(\Gamma^* \Gamma). \]

So, for each $N$ sufficiently large,
\[ \frac{1}{N} \sum_{n=0}^{N-1} \omega(\alpha^n(\Gamma^* \Gamma)) < \varepsilon^2. \]

By applying Lemma 4.3, we have for a fixed $M$ sufficiently large and each $N$ sufficiently large,
\[ \omega \left( \left| \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \alpha^{n+m}(B - B_\varepsilon) \right|^2 \right) < \varepsilon^2. \]

By taking into account Lemma 4.4, (4.2) becomes
\[ \omega \left( \left| \frac{1}{N} \sum_{n=0}^{N-1} \alpha^n(B - B_\varepsilon) \right|^2 \right) < \varepsilon^2. \]
which means
\[ \left| \frac{1}{N} \sum_{n=0}^{N-1} \pi_\omega(\alpha^n(B) - B_\varepsilon)\Omega \right| < \varepsilon \]
for each large \( N \). Thus, we obtain for each sufficiently large \( N \),
\[ \left| \frac{1}{N} \sum_{n=0}^{N-1} \pi_\omega(\alpha^n(B))\Omega - V E_1 \pi_\varphi(B)\Phi \right| \leq \left| \frac{1}{N} \sum_{n=0}^{N-1} \pi_\omega(\alpha^n(B) - B_\varepsilon)\Omega \right|
+ \left| V(\pi_\varphi(B_\varepsilon) - E_1 \pi_\varphi(B))\Phi \right| < \varepsilon + \left| (\pi_\varphi(B_\varepsilon) - E_1 \pi_\varphi(B))\Phi \right| < 2\varepsilon. \]

5. THE CASE OF “DIAGONAL MEASURES”

The present section is devoted to the natural generalization to the quantum case of the celebrated result due to H. Furstenberg relative to the diagonal measures (cf. [8], see also [9], Section 4.4).

We start with a \( C^* \)-dynamical system \((\mathfrak{A}, \alpha, \omega)\), together with its GNS covariant representation \((\mathcal{H}, \pi, U, \Omega)\). Denote \( M := \pi(\mathfrak{A})'' \), the von Neumann algebra acting on \( \mathcal{H} \) generated by the representation \( \pi \). The commutant von Neumann algebra is denoted as \( M' \). Suppose further that the support \( s(\omega) \) in \( \mathfrak{A}^{**} \) is central.

Let \( \mathfrak{M} := M \otimes_{\text{max}} M' \) be the completion of the algebraic tensor product \( \mathfrak{M} := M \otimes M' \) w.r.t. the maximal \( C^* \)-norm (cf. [15], Section IV.4). It is easily seen that on \( \mathfrak{M} \) the following two states are automatically well-defined. The first one is the canonical product state
\[ \varphi(A \otimes B) := \langle A\Omega, \Omega \rangle \langle B\Omega, \Omega \rangle, \quad A \in M, \ B \in M'. \]
The second one is uniquely defined by
\[ \psi(A \otimes B) := \langle AB\Omega, \Omega \rangle, \quad A \in M, \ B \in M'. \]

The state \( \varphi \) can be considered the (quantum analogue of the) “diagonal measure” of the “measure” \( \varphi \).

On \( \mathfrak{M} \) is also uniquely defined the automorphism
\[ \gamma := \text{Ad}_U \otimes \text{Ad}_{U^2} , \]
see [15], Proposition IV.4.7. Of course, \((\mathfrak{M}, \gamma, \varphi)\) is a \( C^* \)-dynamical system whose GNS covariant representation is precisely \((\mathcal{H} \otimes \mathcal{H}, \text{id} \otimes \text{id}, U \otimes U^2, \Omega \otimes \Omega)\). Denote \( E_1 \) the selfadjoint projection onto the invariant vectors under \( U \otimes U^2 \). Notice that the \( * \)-subalgebra \( \mathfrak{N} \) is globally stable under the action of \( \gamma \).

In addition, again by Proposition IV.4.7 of [15],
\[ \sigma(A \otimes B) := AB , \quad A \in M, \ B \in M'. \]
uniquely defines a representation of \( \mathcal{M} \) on \( \mathcal{H} \) such that \((\mathcal{H}, \sigma, \Omega)\) is precisely the GNS representation of the state \( \psi \).\(^5\)

**Proposition 5.1.** Suppose that \((\mathfrak{A}, \alpha, \omega)\) is ergodic. Then the state \( \psi \in \mathcal{S}(\mathcal{M}) \) is generic for \((\mathcal{M}, \gamma, \varphi)\) w.r.t. \( \mathfrak{N} \).

**Proof.** Let \( A \in M \), \( B \in M' \). Then by the mean ergodic theorem (1.3),

\[
\frac{1}{N} \sum_{n=0}^{N-1} \psi(\gamma^n(A \otimes B)) = \frac{1}{N} \sum_{n=0}^{N-1} \langle AU^n B \Omega, \Omega \rangle \\
\equiv \left\langle A \left( \frac{1}{N} \sum_{n=0}^{N-1} U^n \right) B \Omega, \Omega \right\rangle \\
\rightarrow \langle A \Omega, \Omega \rangle \langle B \Omega, \Omega \rangle \equiv \varphi(A \otimes B) .
\]

\( \square \)

**Theorem 5.2.** Let \((\mathfrak{A}, \alpha, \omega)\) be an ergodic \( C^* \)-dynamical system such that its support \( c(\omega) \) in \( \mathfrak{A}^{**} \) is central. Then with the above notations, the following assertions hold true.

(i) Let \( \sum_j A_j \otimes B_j \) be the generic element of \( \mathfrak{N} \). The map

\[
\sum_j A_j \Omega \otimes B_j \Omega \in \mathfrak{N} \Omega \cap \mathcal{E}_1 \mathcal{H} \otimes \mathcal{H} \mapsto \sum_j A_j B_j \Omega \in \mathcal{H}
\]

uniquely defines a partial isometry \( V : \mathcal{H} \otimes \mathcal{H} \mapsto \mathcal{H} \) such that \( V^* V = \mathcal{E}_1 \mathcal{H} \otimes \mathcal{H} \).

(ii) For \( A \in M \), \( B \in M' \),

\[
\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} U^n A U^n B \Omega = V (A \Omega \otimes B \Omega) .
\]

**Proof.** Define \( \Sigma := \{(z, w) \in \sigma_{pp}(U) \times \sigma_{pp}(U) : zw^2 = 1\} \).

Then by Lemma 4.18 of [9],

\[
\mathcal{E}_1 = \bigoplus_{s \in \Sigma} E_{z_s}^U \otimes E_{w_s}^U ,
\]

\( E_{z_s}^U \) being the selfadjoint projection onto the eigenspace of \( U \) corresponding to the eigenvalue \( z \). As \( U \) is ergodic, by Proposition 2.2 \( E_{z_s}^U \mathcal{H} \) is one dimensional, and \( E_{z_s}^U \mathcal{H} \) and \( E_{w_s}^U \mathcal{H} \) are generated by \( V_{z_s} \Omega \), \( W_{w_s} \Omega \), where \( V_{z_s} \) and \( W_{w_s} \) are unitaries of \( M_{z_s} \), \( (M')_{w_s} \) respectively. Thus,

\( ^5\)Notice that, even if it is enough for our purpose to consider \( M \otimes_{\text{max}} M' \), all these properties hold true for \( M \otimes_{\text{bin}} M' \), the latter being the completion of \( M \otimes M' \) with the binormal \( C^* \)-norm (cf. [3]).
$E_z \mathcal{H} \otimes E_w \mathcal{H}$ is one dimensional, and it is generated by $V_z \Omega \otimes W_w \Omega$. This means that $\Omega \mathcal{H} \cap E_1 \mathcal{H} \otimes \mathcal{H}$ is dense in $E_1 \mathcal{H} \otimes \mathcal{H}$.

The assertions follow from Proposition 5.1 and Theorem 4.5, by taking into account that $\mathfrak{H}$ is left globally invariant by $\text{Ad}_U \otimes \text{Ad}_{U^2}$. □

The following results are a direct consequence of the previous one.

**Corollary 5.3.** Under the hypotheses of Theorem 5.2,

\begin{equation}
\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} U^n AU^n = V (A \Omega \otimes \cdot),
\end{equation}

for each $A \in M \bigcup M'$.

**Proof.** By a standard density argument, if $A \in M$, the proof follows from Theorem 5.2, by taking into account that $\Omega$ is cyclic from $M'$. The proof for $A \in M'$ follows by exchanging the role between $M$ and $M'$.

By taking into account Proposition 2.1 and Proposition 2.2, it is straightforward to verify that (5.1) for $A \in M \bigcup M'$ coincides with (3.1) when $U$ is almost periodic and ergodic.\(^6\) Namely,

\[ E_z = \langle \cdot, V_z \Omega \rangle V_z \Omega = \langle \cdot, W_z \Omega \rangle W_z \Omega \]

for unitaries $V_z, W_z$ in $M_z, (M')_z$ respectively. Then

\[ = \sum_{w \in \sigma_{pp}(U)} E_w AE_w \xi = \sum_{w \in \sigma_{pp}(U)} \sum_{\{z \in \sigma_{pp}(U) : zw^2 = 1\}} E_{zw} AE_w \xi \]

\[ = \sum_{\{z, w \in \sigma_{pp}(U) : zw^2 = 1\}} E_{zw} AE_w \xi \]

\[ = \sum_{\{z, w \in \sigma_{pp}(U) : zw^2 = 1\}} \langle \xi, W_w \Omega \rangle \langle AW_w \Omega, V_z W_w \Omega \rangle V_z W_w \Omega \]

\[ = \sum_{\{z, w \in \sigma_{pp}(U) : zw^2 = 1\}} \langle A \Omega, V_z \Omega \rangle \langle \xi, W_w \Omega \rangle V_z W_w \Omega \]

\[ = V (A \Omega \otimes \xi). \]

**Corollary 5.4.** Under the hypotheses of Theorem 5.2,

\[ \lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \omega(A_0 \alpha^n(A_1)\alpha^{2n}(A_2)) \]

\[ = \langle V (\pi(A_1) \Omega \otimes \pi(A_2) \Omega), \pi(A_0) \Omega \rangle. \]

\(^6\)Notice that in this situation, $\sigma_{pp}(U) = \sigma_{pp}^a(U)$, with $\sigma_{pp}^a(U)$ given in (2.1), and $\sigma(U) = \sigma_{pp}(U)$.
Proof. A simple application of Corollary 5.3. □

Suggested by the Abelian situation (cf [9], pag. 96), one can ask for the convergence of the Cesaro mean

\[
\frac{1}{N} \sum_{n=0}^{N-1} \pi(a^{nm_1}(A_1)a^{nm_2}(A_2))\Omega
\]

(5.2)

for the other cases with fixed \(0 < m_1 < m_2\). Starting from

\[
\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \langle AU^{n(m_2-m_1)}B\Omega,\Omega \rangle
\]

(5.3)

whenever \(A \in \pi(\mathfrak{A})''\), \(B \in \pi(\mathfrak{A})'\), in order to apply Theorem 4.5 one firstly demand if (5.3) uniquely defines a state on \(\pi(\mathfrak{A})'' \otimes_{\text{max}} \pi(\mathfrak{A})'\). Such a state will be necessarily invariant under the action of \(\text{Ad}_{U^{m_1}} \otimes \text{Ad}_{U^{m_2}}\). This is certainly true when \(#\{z \in \sigma_{pp}(U) : z^{m_2-m_1} = 1\}\) is finite.

After verifying the remaining hypotheses of Theorem 4.5, one might argue that the Cesaro mean in (5.2) converges, at least when the subspace consisting of all the invariant vectors for \(U^{m_2-m_1}\) is finite dimensional. We end with the simplest case of weakly mixing dynamical systems. Then we have an alternative proof of (a weaker result than) Theorem 1.3 of [12], following the line of Theorem 5.2.

Proposition 5.5. Let \((\mathfrak{A},\alpha,\omega)\) be a weakly mixing \(C^*\)-dynamical system such that its support \(\sigma(\omega)\) in \(\mathfrak{A}^{**}\) is central, and \(0 < m_1 < m_2\) natural numbers.

Then with the above notations,

\[
\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} U^{nm_1}AU^{nm_2} = \langle A\Omega,\Omega \rangle \langle \cdot,\Omega \rangle \Omega,
\]

for each \(A \in M \cup M'\).

Proof. We apply Theorem 4.5 by considering

\[
\gamma := \text{Ad}_{U^{m_1}} \otimes \text{Ad}_{U^{m_2}}.
\]

Indeed, \(E_1\mathcal{H} \otimes \mathcal{H} = \mathbb{C}\Omega \otimes \Omega\) where \(E_1\) is the selfadjoint projection onto the invariant vectors under \(U^{m_1} \otimes U^{m_2}\). In addition, if \(A \in M\),
\[ B \in M', \]

\[
\frac{1}{N} \sum_{n=0}^{N-1} \psi(\gamma^n (A \otimes B)) = \frac{1}{N} \sum_{n=0}^{N-1} \langle AU_{n(m_2-m_1)}B\Omega, \Omega \rangle \\
\rightarrow \langle A\Omega, \Omega \rangle \langle B\Omega, \Omega \rangle \equiv \varphi(A \otimes B).
\]

□

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