Quasi-two-body decays $B \to DK^*(892) \to DK \pi$ in the perturbative QCD approach

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We study the quasi-two-body decays $B \to DK^*(892) \to DK \pi$ by employing the perturbative QCD approach. The two-meson distribution amplitudes $\Phi^{P\text{-wave}}_{K\pi}$ are adopted to describe the final state interactions of the kaon-pion pair in the resonance region. The resonance line shape for the $P\text{-wave} K \pi$ component $K^*(892)$ in the time-like form factor $F_{K\pi}(s)$ is parameterized by the relativistic Breit-Wigner function. For most considered decay modes, the theoretical predictions for their branching ratios are consistent with currently available experimental measurements within errors. We also discuss some ratios of the branching fractions of the concerned decay processes. More precise data from LHCb and Belle-II are expected to test our predictions.

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I. INTRODUCTION

Many three-body hadronic $B$ meson decays including $B \to DK \pi$ have been studied experimentally in recent years [1, 2]. The $B \to DK \pi$ decays have demonstrated the potential to determine the CKM angles precisely. Suggestions for the determination of the unitarity triangle angle $\gamma$ through Dalitz plot [3] analyses of the decays $B^\pm \to DK^\pm \pi^0$ and $B^0 \to DK^+ \pi^-$ were proposed in [4] and [5, 6], respectively. And the measurement has been performed by LHCb [7]. The BABAR Collaboration has presented a measurement of the weak phase $2\beta + \gamma$ from the time-dependent Dalitz plot analysis of the $B^0 \to D^+ K^0 \pi^\pm$ decays [8]. In addition, the decay modes $B \to DK \pi$ have provided rich opportunities to investigate the spectroscopy of excited charm mesons and the significant components originated from $K \pi$ system, corresponding results have been acquired from Dalitz plot analyses of $B^0 \to D^{(*)\pm} K^0 \pi^\mp$ [9], $B_s^0 \to \bar{D}^0 K^- \pi^+$ [10, 11], $B^- \to D^+ K^- \pi^-$ [12] and $B^+ \to D^+ K^+ \pi^-$ [13] decays. In the amplitude analyses of $B \to DK \pi$ decays, contributions from the $P\text{-wave} K \pi$ resonant state $K^*(892)$ were found to be the largest proportion in most cases, and the $B \to DK^*$ decays have been substantially studied in experiment by quasi-two-body approach [14–23].

On the theoretical side, the charm hadroned $B$ meson decays $B \to DK^*$ have been studied by using rather different methods in Refs. [24–32]. The $K^*$ was treated as a stable particle in the framework of two-body decays while as a resonance with the cascade decay $K^* \to K \pi$ in the three-body decays. Several approaches have been adopted to describe those three-body $B$ decays involving $K \pi$ systems. For instance, within the QCD factorization [33–35], the authors studied the CP violation and the contribution of the strong kaon-pion interactions in the three-body $B \to K \pi \pi$ decays [36, 37] where the $K^*(892)$ and $K^*_0(1430)$ resonance effects were mainly taken into account. In Ref. [38], the calculation of the localized CP violation in $B^- \to K^- \pi^+ \pi^-$ decays has been done with the $K \pi$ channels including $K^*(892), K^*_0(1430), K^*(1410), K^*(1680)$ and $K^*_2(1430)$. Using a simple model on the basis of the factorization approach, the branching ratios and direct CP violation for the charmless three-body hadronic decays $B_{(s)} \to KK \pi$ and $B_{(s)} \to K \pi \pi$ have been calculated in Refs. [39–42]. In the recent works, the $K \pi$ contributions to the decay channels $B \to \psi K \pi$ [43] and $B \to P K \pi$ [44], with $\psi = (J/\psi, \psi(2S))$ and $P = (K, \pi)$, were analyzed by employing the perturbative QCD (PQCD) factorization approach [45–48]. In addition, phenomenological studies of the processes $B^0 \to D^0 \pi^+ \pi^-$ and $B^0_s \to D^0 K^+ \pi^-$ based on results of the chiral unitary approach has been performed.

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In the following sections, $K^*$ represents $K^*(892)$ without specific reference.
in Ref. [49]. Motivated by the abundant experimental data and theoretical studies, we shall analyse the contributions of the resonance $K^*$ in the $B \to DK\pi$ decays in this work.

Theoretically, the three-body hadronic $B$ meson decays are much more complicated than the two-body cases because of their non-trivial kinematics and the different phase space distributions. While these three-body decay processes are known to be dominated by the low energy scalar, vector and tensor resonant states, which could be handled in the quasi-two-body framework by neglecting the three-body and rescattering effects [50–54]. In the quasi-two-body region of the phase space, the three final states are quasi-aligned in the rest frame of the $B$ meson and two of them almost collimate to each other, the related processes can be denoted as $B \to h_1R \to h_1h_2h_3$ where $h_1$ represents the bachelor particle and the $h_2h_3$ pair proceeds by the intermediate state $R$. In our previous works, the $S$-, $P$- and $D$-wave $\pi\pi$ ($K\pi$) resonance contributions to a series of charged or charmless three-body $B_{(s)}$ meson decays have been studied [43, 44, 53, 55–64] in the PQCD approach by introducing two-meson distribution amplitudes [65–71]. The consistency of the theoretical studies and experimental results indicates the PQCD factorization approach are applicable to the three-body and quasi-two-body hadronic $B$ meson decays. More recently, several quasi-two-body decays involving $D_0^*(2400)$ [72], $D^*(2007)^0$ and $D^*(2010)^+$ [54] as the intermediate states have been studied. And the isovector scalar resonances $a_0(980)$ and $a_0(1450)$ in the $B \to \psi(KK, \pi\eta)$ decays were presented in [73]. In this work, we will extend the previous studies to the quasi-two-body decays $B \to DK^* \to DK\pi$.

This paper is organized as follows. In Sec. II, we give a brief introduction for the theoretical framework and perturbative calculations for the considered decays. Then, the numerical values and phenomenological analyses are given in Sec. III. Finally, the last section contains a short summary.

II. THE THEORETICAL FRAMEWORK

In the framework of the PQCD approach for the quasi-two-body decays, the nonperturbative dynamics associated with the pair of the mesons are absorbed into two-meson distribution amplitudes, then the relevant decay amplitude $A$ for the quasi-two-body decays $B \to DK^* \to DK\pi$ can be written as the convolution [65, 66]

$$A = \Phi_B \otimes H \otimes \Phi_D \otimes \Phi_{K\pi}^{P\text{-wave}},$$

(1)

where the symbol $\otimes$ means the convolution integrations over the parton momenta and the hard kernel $H$ includes the leading-order contributions. The $B$ meson ($D$ meson, $P$-wave $K\pi$ pair) distribution amplitude $\Phi_B$ ($\Phi_D, \Phi_{K\pi}^{P\text{-wave}}$) absorbs the nonperturbative dynamics in the decay processes.

A. Coordinates and wave functions

In the rest frame of the $B$ meson, we define the $B$ meson momentum $p_B$, the kaon momentum $p_1$, the $K^*$ meson momentum $p = p_1 + p_2$, and the $D$ meson momentum $p_3$ in the light-cone coordinates as

$$p_B = \frac{m_B}{\sqrt{2}}(1, 1, \mathbf{0}_T), \quad p = \frac{m_B}{\sqrt{2}}(1 - r^2, \eta, \mathbf{0}_T), \quad p_3 = \frac{m_B}{\sqrt{2}}(r^2, 1 - \eta, \mathbf{0}_T),$$

$$p_1 = \frac{m_B}{\sqrt{2}}(1 - r^2, \frac{m_B}{\sqrt{2}}(1 - \zeta \eta, \mathbf{p}_{1T}), \quad p_2 = \frac{m_B}{\sqrt{2}}(1 - \zeta)(1 - r^2), \quad p_3 = \frac{m_B}{\sqrt{2}}(\zeta, \mathbf{p}_{3T}),$$

(2)

with the mass ratio $r = m_D/m_B, m_B(D)$ is the mass of the $B(D)$ meson. The variable $\eta$ is defined as $\eta = \omega^2/[(1 - r^2)m_B^2]$ with the invariant mass squared $\omega^2 = p^2 = m^2(K\pi)$ of the kaon-pion pair and $\zeta$ is the momentum fraction for the kaon meson. The momenta of the light quarks in the $B$ meson, the $K^*$ meson and the $D$ meson are chosen as $k_B, k$ and $k_3$ respectively

$$k_B = (0, x_B p_B, \mathbf{k}_{BT}), \quad k = (z p_T, 0, \mathbf{k}_T), \quad k_3 = (0, x_3 p_3, \mathbf{k}_{3T}),$$

(3)

where the corresponding momentum fractions $x_B, z$ and $x_3$ run between zero and unity.

The $P$-wave kaon-pion distribution amplitudes are defined in the same way as in Ref. [65, 66],

$$\Phi_{K\pi}^{P\text{-wave}} = \frac{1}{\sqrt{2N_c}}[\phi_0(z, \zeta, \omega^2) + \omega \phi_4(z, \zeta, \omega^2) + \frac{\omega_1 \omega_2 - \omega_2 \omega_1}{\omega(2\zeta - 1)} \phi_1(z, \zeta, \omega^2)],$$

(4)
with the functions \[44\]

\[
\begin{align*}
\phi_0 &= \frac{3F_{K\pi}(s)}{\sqrt{2N_c}} z(1-z) \left[ 1 + a_1^{\parallel_K} 3(2z-1) + a_2^{\parallel_{K^*}} \frac{3}{2} (5(2z-1)^2 - 1) \right] P_1(2\zeta - 1), \\
\phi_s &= \frac{3F_s(s)}{2\sqrt{2N_c}} (1-2z) P_1(2\zeta - 1), \\
\phi_t &= \frac{3F_t(s)}{2\sqrt{2N_c}} (2z - 1)^2 P_1(2\zeta - 1),
\end{align*}
\]

where the Legendre polynomial \(P_1(2\zeta - 1) = 2\zeta - 1\) and the variable \(s = \omega^2 = m^2(K\pi)\). For the Gegenbauer moments, we adopt \(a_1^{\parallel_K} = 0.05 \pm 0.02\) and \(a_2^{\parallel_{K^*}} = 0.15 \pm 0.05\) determined in Ref. \[44\]. The relativistic Breit-Wigner (RBW) function is an appropriate model for narrow resonances which are well separated from any other resonant or nonresonant contributions with the same spin, and it is widely used in the experimental data analyses. Here, the time-like form factor \(F_{K\pi}(s)\) is parameterized with the RBW line shape and can be expressed as the following form \[7, 11, 12\]

\[
F_{K\pi}(s) = \frac{m_{K^*}^2}{(m_{K^*}^2 - s) - im_{K^*}\Gamma(s)},
\]

with the mass-dependent decay width \(\Gamma(s)\)

\[
\Gamma(s) = \Gamma_{K^*} \frac{m_{K^*}^2}{\sqrt{s}} \frac{|\vec{p}_1|^2}{|\vec{p}_0|^2} \frac{1 + (|\vec{p}_1|/r_{BW})^2}{1 + (|\vec{p}_1|/r_{BW})^2}.
\]

The \(|\vec{p}_1|\) is the momentum of one of the resonance daughters evaluated in the \(K\pi\) rest frame and \(|\vec{p}_0|\) is the value of \(|\vec{p}_1|\) when \(\sqrt{s} = m_{K^*}\). The pole mass \(m_{K^*}\) and width \(\Gamma_{K^*}\) of the resonance state \(K^*\) are chosen as \(m_{K^*} = 895.55 \pm 0.20\) (\(m_{K^*} = 891.76\pm0.25\)) MeV and \(\Gamma_{K^*} = 47.3\pm0.5\) (\(\Gamma_{K^*} = 50.3\pm0.8\)) MeV, respectively \[1\]. The parameter \(r_{BW}\) is the barrier radius which is set to 4.0 \(\text{GeV}^{-1}\) as in Ref. \[7, 11, 12\]. Following Ref. \[53\], we also assume that \(F_s(s) = f_1(s) \approx (f_{K^*}^2/f_{K^*}) F_{K\pi}(s)\) with \(f_{K^*} = 0.217 \pm 0.005\) GeV and \(f_{K^*} = 0.185 \pm 0.010\) GeV \[74\].

In this work, we use the same distribution amplitudes for the \(B\) and \(D\) meson as in Ref. \[59, 62\] where one can easily find their expressions and the relevant parameters.

### B. Analytic formulae

For the quasi-two-body decays \(B \to DK^* \to DK\pi\), the effective Hamiltonian is defined as

\[
\mathcal{H}_{eff} = \begin{cases}
\frac{G_F}{\sqrt{2}} V_{cb}^* V_{ub} [C_1(\mu) O_1(\mu) + C_2(\mu) O_2(\mu)], & \text{for } B_s \to \bar{D}_s K^* \to \bar{D}_s K\pi \text{ decays,} \\
\frac{G_F}{\sqrt{2}} V_{ub}^* V_{cq} [C_1(\mu) O_1(\mu) + C_2(\mu) O_2(\mu)], & \text{for } B_s \to D_s K^* \to D_s K\pi \text{ decays,}
\end{cases}
\]

where the Fermi coupling constant \(G_F = 1.16638 \times 10^{-5}\) \(\text{GeV}^{-2}\), \(C_{1,2}(\mu)\) denote the Wilson coefficients at the renormalization scale \(\mu\), \(O_{1,2}(\mu)\) represent the effective four quark operators and \(V_{ij}\) are the CKM matrix elements.

The typical Feynman diagrams at the leading order for the quasi-two-body decays \(B_s \to \bar{D}_s K^* \to \bar{D}_s K\pi\) (through \(b \to c\) transition) and \(B_s \to D_s K^* \to D_s K\pi\) (through \(b \to \bar{u}\) transition) are shown in Fig. 1 and 2, respectively. By making analytical evaluations for those Feynman diagrams for Fig. 1 and Fig. 2, we can obtain the total decay amplitudes of these concerned decays.

For the \(B_s \to \bar{D}_s K^* \to \bar{D}_s K\pi\) decays, their total decay amplitudes can be written explicitly in the following form

\[
A(B^+ \to \bar{D}^0 K^{++} \to \bar{D}^0 K\pi) = \frac{G_F}{\sqrt{2}} V_{cb}^* V_{us} [(C_1 + \frac{C_2}{3}) F_{eK}^{LL} + C_2 M_{eK}^{LL} + (C_1 + C_2) F_{eD}^{LL} + C_1 M_{eD}^{LL}],
\]

\[
A(B^0 \to \bar{D}^0 K^{*0} \to \bar{D}^0 K\pi) = \frac{G_F}{\sqrt{2}} V_{cb}^* V_{us} [(C_1 + \frac{C_2}{3}) F_{eK^*}^{LL} + C_2 M_{eK^*}^{LL}],
\]

\[
A(B^0 \to D^- K^{*+} \to D^- K\pi) = \frac{G_F}{\sqrt{2}} V_{cb}^* V_{us} [(C_1 + C_2) F_{eD^*}^{LL} + C_1 M_{eD^*}^{LL}],
\]
FIG. 1: The leading-order Feynman diagrams for the quasi-two-body decays $B_{(s)} \to \bar{D}(s)K^+ \to \bar{D}(s)K\pi$ where $B_{(s)} = (B^0, B^+, B^0_s)$ and $\bar{D}(s) = (\bar{D}^0, D^+, D^+_s)$. The $h_1h_2$ denotes the $K\pi$ pair and the blue ellipse represents the intermediate state $K^+$.

FIG. 2: The leading-order Feynman diagrams for the quasi-two-body decays $B_{(s)} \to D_{(s)}K^0 \to D_{(s)}K\pi$ where $B_{(s)} = (B^0, B^+, B^0_s)$ and $D_{(s)} = (D^0, D^+, D^+_s)$. The $h_1h_2$ denotes the $K\pi$ pair and the blue ellipse represents the intermediate state $K^+$.

\[
\mathcal{A}(B^0 \to D_s^- K^{++} \to D_s^- K\pi) = \frac{G_F}{\sqrt{2}} V_{cb}^* V_{ud} [(C_1 + \frac{C_2}{3}) F_{eK^+} + C_2 M_{eK^+}] ,
\]
(14)

\[
\mathcal{A}(B^0_s \to \bar{D}^0 K^{+0} \to \bar{D}^0 K\pi) = \frac{G_F}{\sqrt{2}} V_{cb}^* V_{ud} [(C_1 + \frac{C_2}{3}) F_{eK^+} + C_2 M_{eK^+}] ,
\]
(15)

\[
\mathcal{A}(B^0 \to D_s^- K^{++} \to D_s^- K\pi) = \frac{G_F}{\sqrt{2}} V_{cb}^* V_{ud} [(C_1 + \frac{C_2}{3}) F_{eK^+} + C_1 M_{eK^+} + (C_1 + \frac{C_2}{3}) F_{aK^+} + C_2 M_{aK^+}] ,
\]
(16)

while the total decay amplitudes for $B_{(s)} \to D_{(s)}K^0 \to D_{(s)}K\pi$ decays can be written as

\[
\mathcal{A}(B^+ \to D^0 K^{+0} \to D^0 K\pi) = \frac{G_F}{\sqrt{2}} V_{ab}^* V_{cs} [(C_1 + \frac{C_2}{3}) F_{eK^+} + C_2 M_{eK^+} + (C_1 + \frac{C_2}{3}) F_{aK^+} + C_2 M_{aK^+}] ,
\]
(17)

\[
\mathcal{A}(B^+ \to D^+ K^{*0} \to D^+ K\pi) = \frac{G_F}{\sqrt{2}} V_{ab}^* V_{cs} [(C_1 + \frac{C_2}{3}) F_{eK^+} + C_2 M_{eK^+} + (C_1 + \frac{C_2}{3}) F_{aK^+} + C_2 M_{aK^+}] ,
\]
(18)
\[ A(B^+ \rightarrow D_s^+ K^{*0} \rightarrow D_s^+ K \pi) = \frac{G_F}{\sqrt{2}} V_{ub}^* V_{cd} [(C_1 + C_2) F_{eK}^{LL} + C_1 M_{eK}^{LL}] , \] (19)

\[ A(B^0 \rightarrow D^0 K^{*0} \rightarrow D^0 K \pi) = \frac{G_F}{\sqrt{2}} V_{ub}^* V_{cs} [(C_1 + C_2) F_{eK}^{LL} + C_2 M_{eK}^{LL}] , \] (20)

\[ A(B^0 \rightarrow D_s^+ K^{*-} \rightarrow D^+ K \pi) = \frac{G_F}{\sqrt{2}} V_{ub}^* V_{cd} [(C_1 + C_2) F_{eK}^{LL} + C_2 M_{eK}^{LL}] , \] (21)

\[ A(B^0 \rightarrow D^0 K^{*-} \rightarrow D^0 K \pi) = \frac{G_F}{\sqrt{2}} V_{ub}^* V_{cs} [(C_1 + C_2) F_{eK}^{LL} + C_2 M_{eK}^{LL}] , \] (22)

\[ A(B^0 \rightarrow D^+ K^{*-} \rightarrow D^+ K \pi) = \frac{G_F}{\sqrt{2}} V_{ub}^* V_{cd} [(C_1 + C_2) F_{eK}^{LL} + C_1 M_{eK}^{LL}] , \] (23)

\[ A(B^0 \rightarrow D_s^+ K^{*-} \rightarrow D_s^+ K \pi) = \frac{G_F}{\sqrt{2}} V_{ub}^* V_{cd} [(C_1 + C_2) F_{eK}^{LL} + C_1 M_{eK}^{LL}] , \] (24)

where the individual amplitude \( F_{eK}^{LL} \), \( M_{eK}^{LL} \), \( F_{eK}^{LL} \), \( M_{eK}^{LL} \), and \( M_{eK}^{LL} \) are the amplitudes from different sub-diagrams in Fig. 1 and Fig. 2. Since the \( P \)-wave kaon-pion distribution amplitudes in Eq. (4) have the same Lorentz structure as that of two-pion ones in Ref. [53, 59], the concerned expressions of those individual amplitudes in Ref. [59] can be employed in this work directly by replacing the distribution amplitudes \((\phi_0, \phi_\pi, \phi_\pi)\) of the \( \pi\pi \) system with the corresponding twists of the \( K\pi \) ones in Eq. (5)-(7). The parameter \( c \) in the Eq. (33) of [59] is adopted to be 0.4 in this work according to the Refs. [75, 76].

For the \( B \rightarrow D K^{*} \rightarrow D K \pi \) decays, the differential decay rate can be described as

\[ \frac{d\mathcal{B}}{d\omega^2} = \tau_B \frac{|\vec{p}_1||\vec{p}_3|}{64\pi^3 m_B^3} |A|^2 , \] (25)

where \( \tau_B \) is the mean lifetime of \( B \) meson, the kinematic variables \( |\vec{p}_1| \) and \( |\vec{p}_3| \) denote the magnitudes of the \( K \) and \( D \) momenta in the center-of-mass frame of the kaon-pion pair,

\[ |\vec{p}_1| = \frac{1}{2} \sqrt{[(m_K^2 - m_D^2)^2 - 2(m_K^2 + m_D^2)w^2 + w^4]/w^2}, \]

\[ |\vec{p}_3| = \frac{1}{2} \sqrt{[(m_B^2 - m_D^2)^2 - 2(m_B^2 + m_D^2)w^2 + w^4]/w^2} . \] (26)

### III. NUMERICAL RESULTS AND DISCUSSIONS

The adopted input parameters in our numerical calculations are summarized as following (the masses, decay constants and QCD scale are in units of GeV) [1]:

\[
\Lambda_{f=4}^{(f=4)} = 0.25, \quad m_{B^+} = 5.279, \quad m_{B_0} = 5.280, \quad m_{B_s} = 5.367, \quad m_{D^\pm} = 1.870, \\
m_{D^0/D^+} = 1.865, \quad m_{D_s^0} = 1.968, \quad m_{B^0} = 4.8, \quad m_c = 1.275, \quad m_{D^+} = 0.140, \\
m_{D^0} = 0.135, \quad m_{K^\pm} = 0.494, \quad m_{K^0/K^0} = 0.949, \quad f_B = 0.19, \quad f_{B_s} = 0.236, \\
f_D = 0.2119, \quad f_{D_s} = 0.249, \quad \tau_{B^+} = 1.638 \text{ ps}, \quad \tau_{B_0} = 1.520 \text{ ps}, \quad \tau_{B_s} = 1.509 \text{ ps}. \] (27)

For the Wolfenstein parameters \((A, \lambda, \bar{\rho}, \bar{\eta})\) of the CKM mixing matrix, we use \( A = 0.836 \pm 0.015, \quad \lambda = 0.22453 \pm 0.00044, \quad \bar{\rho} = 0.122^{+0.018}_{-0.017}, \quad \bar{\eta} = 0.355^{+0.012}_{-0.011} [1]. \)

By using the decay amplitudes as given in Eq. (11-24) and the differential branching ratio in Eq. (25), integrating over the full invariant mass spectrum for the individual resonant components, we obtain the branching ratios for the quasi-two-body decays \( B_{(s)} \rightarrow D_{(s)} K^{*} \rightarrow D_{(s)} K \pi \) and \( B_{(s)} \rightarrow D_{(s)} K^{*} \rightarrow D_{(s)} K \pi \), and list the numerical results in Table 1 and Table II. The
first error of these PQCD predictions comes from the \(B(B_s)\) meson shape parameter uncertainty \(\omega_B = 0.40 \pm 0.04 \ (\omega_B = 0.50 \pm 0.05)\) GeV, the following two errors are from the Gegenbauer coefficients in the kaon-pion distribution amplitudes: \(a_1^{1K^*} = 0.05 \pm 0.02, a_2^{1K^*} = 0.15 \pm 0.05\) and the last one is induced by \(C_D = 0.5 \pm 0.1\) \((C_D = 0.4 \pm 0.1)\) for \(D(D_s)\) meson wave function. The errors come from the uncertainties of the parameters, for instance, the Wolfenstein parameters, the pole mass \(m_{K^*}\) and width \(\Gamma_{K^*}\), are very small and have been neglected. Although the three-body \(B\) meson decay offers an ideal ground to study the distribution of \(CP\) asymmetry, there are no direct \(CP\) violations for these decays in this work since only the tree diagrams contribute to the considered decay processes.

### Table I: The PQCD predictions for the branching ratios of the \(B(s) \rightarrow \bar{D}(s)K^* \rightarrow \bar{D}(s)K\pi\) decays in the quasi-two-body framework
together with experimental data.

| Mode | Unit | \(B_{th}\) | \(B_{exp}\) |
|------|------|------------|------------|
| \(B^+ \rightarrow \bar{D}^0 K^{*+} \rightarrow \bar{D}^0 K\pi\) | \(10^{-4}\) | \(5.38 \pm 0.75 \pm 0.04 \pm 0.12 \pm 0.41\) | BABAR[15]: \(5.29 \pm 0.30 \pm 0.34\) |
| \(B^0 \rightarrow D^0 K^{*0} \rightarrow D^0 K\pi\) | \(10^{-5}\) | \(9.33 \pm 0.33 \pm 0.07 \pm 0.18 \pm 0.05\) | Belle[16]: \(14.8 \pm 1.5\) |
| \(B^0 \rightarrow D^- K^{*+} \rightarrow D^- K\pi\) | \(10^{-4}\) | \(3.63 \pm 1.16 \pm 0.04 \pm 0.02 \pm 0.35\) | BABAR[9]: \(4.6 \pm 0.6 \pm 0.5\) |
| \(B^0 \rightarrow D_s^+ K^{*0} \rightarrow D_s^+ K\pi\) | \(10^{-4}\) | \(1.0 \pm 0.18 \pm 0.03 \pm 0.07 \pm 0.15\) | BABAR[19]: \(0.35 \pm 0.10\) |
| \(B_s^0 \rightarrow \bar{D}^0 K^{*0} \rightarrow \bar{D}^0 K\pi\) | \(10^{-4}\) | \(2.33 \pm 0.94 \pm 0.31 \pm 0.60 \pm 0.25\) | LHCb[20]: \(4.72 \pm 1.07 \pm 0.48 \pm 0.37 \pm 0.74\) |
| \(B_s^0 \rightarrow D_s^+ K^{*+} \rightarrow D_s^+ K\pi\) | \(10^{-4}\) | \(2.06 \pm 0.74 \pm 0.02 \pm 0.01 \pm 0.17\) | LHCb[11]: \(4.29 \pm 0.09 \pm 0.11 \pm 0.14 \pm 0.63\) |

### Table II: The PQCD predictions for the branching ratios of the CKM suppressed \(B(s) \rightarrow D(s)K^* \rightarrow D(s)K\pi\) decays in the quasi-two-body framework
together with experimental data.

| Mode | Unit | \(B_{th}\) | \(B_{exp}\) |
|------|------|------------|------------|
| \(B^+ \rightarrow \bar{D}^0 K^{*+} \rightarrow \bar{D}^0 K\pi\) | \(10^{-6}\) | \(4.49 \pm 1.72 \pm 0.32 \pm 0.58 \pm 0.01\) | BABAR[22]: \(< 30\) at 90\% C.L. |
| \(B^+ \rightarrow D^0 K^{*0} \rightarrow D^0 K\pi\) | \(10^{-7}\) | \(6.27 \pm 0.58 \pm 0.14 \pm 0.04 \pm 0.39\) | LHCb[23]: \(< 18\) at 90\% C.L. |
| \(B^0 \rightarrow D_s^+ K^{*+} \rightarrow D_s^+ K\pi\) | \(10^{-3}\) | \(3.86 \pm 0.23 \pm 0.01 \pm 0.33 \pm 0.25\) | LHCb[23]: \(< 4.9(6.1)\) at 90(95)\% C.L. |
| \(B^0 \rightarrow D^0 K^{*0} \rightarrow D^0 K\pi\) | \(10^{-3}\) | \(3.25 \pm 1.06 \pm 0.18 \pm 0.36 \pm 0.08\) | Belle[16]: \(< 18\) at 90\% C.L. |
| \(B_s^0 \rightarrow D_s^0 K^{*0} \rightarrow D_s^0 K\pi\) | \(10^{-5}\) | \(3.06 \pm 0.42 \pm 0.31 \pm 0.27 \pm 0.15\) | BABAR[18]: \(< 11\) at 90\% C.L. |
| \(B_s^0 \rightarrow D_s^+ K^{*+} \rightarrow D_s^+ K\pi\) | \(10^{-3}\) | \(1.36 \pm 0.43 \pm 0.01 \pm 0.19 \pm 0.07\) | |
| \(B_s^0 \rightarrow D_s^0 K^{*0} \rightarrow D_s^0 K\pi\) | \(10^{-3}\) | \(1.15 \pm 0.36 \pm 0.05 \pm 0.22 \pm 0.12\) | |
| \(B^0 \rightarrow D^+ K^{*+} \rightarrow D^+ K\pi\) | \(10^{-6}\) | \(2.06 \pm 0.74 \pm 0.02 \pm 0.01 \pm 0.01\) | |
| \(B^0 \rightarrow D^0 K^{*0} \rightarrow D^0 K\pi\) | \(10^{-5}\) | \(4.79 \pm 1.82 \pm 0.07 \pm 0.12 \pm 0.01\) | |

From the calculations and the numerical results as listed in Table I and Table II, one can find the following points:

1. By assuming the \(B(K^* \rightarrow K\pi) \approx 100\\%\) [1] and accepting a simple relation between the branching ratio of the same kind of decay evaluated in the quasi-two-body and the two-body framework, it is easy to have

\[
B(B \rightarrow D K^* \rightarrow D K\pi) = B(B \rightarrow D K^*) \cdot B(K^* \rightarrow K\pi) \approx B(B \rightarrow D K^*).
\]

One can extract the PQCD predictions for the decay rates of the related quasi-two-body decays from the results in Table I and Table II. Take the quasi-two-body decay \(B^+ \rightarrow \bar{D}^0 K^{*+} \rightarrow \bar{D}^0 K^+ \pi^0\) for example, the relation between \(B(B^+ \rightarrow \bar{D}^0 K^{*+} \rightarrow \bar{D}^0 K^+ \pi^0)\) and \(B(B^0 \rightarrow D^0 K^{*+})\) can be described as

\[
B(B^+ \rightarrow \bar{D}^0 K^{*+} \rightarrow \bar{D}^0 K^+ \pi^0) = B(B^+ \rightarrow \bar{D}^0 K^{*+}) \cdot B(K^{*+} \rightarrow K^+ \pi^0),
\]

(29)
where the isospin relation $\mathcal{B}(K^+ \rightarrow K^+\pi^0) = \frac{1}{4}$. Combining with the central value of $\mathcal{B}(B^+ \rightarrow \bar{D}^0 K^+ \rightarrow \bar{D}^0 K\pi) = 5.38 \times 10^{-4}$ in Table I, one can obtain the PQCD prediction for $\mathcal{B}(B^+ \rightarrow D^0 K^{*+} \rightarrow D^0 K^+\pi^0) = 1.79 \times 10^{-4}$ easily.

(2) Compare our numerical results with the experimental data in Table I, one can see that:

- The PQCD prediction for the branching ratio of the decay $B^+ \rightarrow \bar{D}^0 K^+ \rightarrow \bar{D}^0 K\pi$ agrees well with the value $(5.29 \pm 0.30 \pm 0.34) \times 10^{-4}$ [15] measured by the BABAR detector at the PEP-II B Factory. In Ref. [9], the branching ratio of $B^0 \rightarrow D^- K^{*+}$ decay was measured to be $(4.6 \pm 0.6 \pm 0.5) \times 10^{-4}$ and the $K^*$ resonant fraction $\frac{\mathcal{B}(B^0 \rightarrow D^- K^{*+})}{\mathcal{B}(B^0 \rightarrow D^- K^{*+})} = 0.63 \pm 0.08 \pm 0.04$ has been also obtained. $\mathcal{B}(B^0 \rightarrow D^- K^{*+} \rightarrow D^- K\pi) = (3.63 \pm 0.45 \pm 0.27 \pm 0.25 \pm 0.35) \times 10^{-4}$ predicted in this work agrees with the result within the measurement uncertainties. We also have the PQCD prediction $\frac{\mathcal{B}(B^0 \rightarrow D^- K^{*+})}{\mathcal{B}(B^0 \rightarrow D^- K^{*+})} \approx 0.5$ by taking $\mathcal{B}(B^0 \rightarrow D^- K^{*+}) = (4.9 \pm 0.7 \pm 0.5) \times 10^{-4}$ from Ref. [9].

- For the decay $B^0 \rightarrow \bar{D}^0 K^{*0} \rightarrow \bar{D}^0 K\pi$, the central value of the branching fraction predicted by PQCD is less than 50% of the available experimental measurements but agrees with the PQCD prediction in the ordinary two-body framework in [28] within errors. The prediction $\mathcal{B}(B_s^0 \rightarrow \bar{D}^0 K^{*0} \rightarrow \bar{D}^0 K\pi) = (2.33 \pm 0.94 \pm 0.31 \pm 0.60 \pm 0.25 \pm 0.35) \times 10^{-4}$ in this work is more close to $(3.3 \pm 0.3 \pm 0.1 \pm 0.3 \pm 0.5) \times 10^{-4}$ in Ref. [21] than other two results from LHCb [11, 20]. Furthermore, in Ref. [20], the measurement of the ratio of branching fractions for the decays $B_s^0 \rightarrow \bar{D}^0 K^{*0}$ and $B^0 \rightarrow D^0 L^0$ was found to be

$$\frac{\mathcal{B}(B_s^0 \rightarrow \bar{D}^0 K^{*0})}{\mathcal{B}(B^0 \rightarrow D^0 L^0)} = 1.48 \pm 0.34 \pm 0.15 \pm 0.12,$$

while a similar ratio between $\mathcal{B}(B_s^0 \rightarrow \bar{D}^0 K^{*0})$ and $\mathcal{B}(B^0 \rightarrow \bar{D}^0 K^{*0})$ was measured in Ref. [21]

$$\frac{\mathcal{B}(B_s^0 \rightarrow \bar{D}^0 K^{*0})}{\mathcal{B}(B^0 \rightarrow \bar{D}^0 K^{*0})} = 7.8 \pm 0.7 \pm 0.3 \pm 0.6.$$  

Utilize the PQCD predictions in Table I and $\mathcal{B}(B^0 \rightarrow \bar{D}^0 L^0) = 1.39 \times 10^{-4}$ taken from our previous work in Ref. [59], we estimate the ratio of branching fractions $\frac{\mathcal{B}(B_s^0 \rightarrow \bar{D}^0 K^{*0})}{\mathcal{B}(B^0 \rightarrow \bar{D}^0 L^0)} = 1.68$, in agreement with the Date, while our prediction $\frac{\mathcal{B}(B_s^0 \rightarrow \bar{D}^0 K^{*0})}{\mathcal{B}(B^0 \rightarrow \bar{D}^0 K^{*0})} = 12.07$ is a bit larger than the value announced by LHCb [21] after taking errors into consideration. If we accept $\mathcal{B}(B_s^0 \rightarrow \bar{D}^0 K^{*0}) = 4.29 \times 10^{-4}$ [11] $(4.72 \times 10^{-4}$ [20]) and $\mathcal{B}(B^0 \rightarrow \bar{D}^0 K^{*0}) = 4.0 \times 10^{-5}$ in Ref. [18], our result for the ratio $\frac{\mathcal{B}(B_s^0 \rightarrow \bar{D}^0 K^{*0})}{\mathcal{B}(B^0 \rightarrow \bar{D}^0 K^{*0})}$ is still consistent with the data.

- The branching fraction for the pure annihilation decay $B^0 \rightarrow D^- K^{*+}$: $\mathcal{B} = (0.35^{+0.10}_{-0.09} \pm 0.04) \times 10^{-4}$ selected from BABAR [19] is smaller than the values $1.05 \times 10^{-4}$ obtained from this work and $1.82 \times 10^{-4}$ acquired from the PQCD framework for two-body decays [28]. More theoretical studies and experimental measurements are needed to improve the estimation for the pure annihilation decays.

(3) For the branching ratios of eight CKM suppressed $B(s) \rightarrow D(s) D^* K^+ \rightarrow D(s) K\pi$ decays as listed in Table II, the PQCD predictions are in the order of $10^{-5}$ to $10^{-6}$ and we have some comments as follows:

- Since no enough significant signals have been observed, there were hardly any specific data for the branching fractions of those decays but the upper limits. As shown in Table II, the experimental collaborations have determined the upper limits for three of the considered decays at 90(95)% confidence level and it is easy to see that all the PQCD predictions for the branching ratios are consistent with the corresponding experimental ones.

- In Ref. [13], the ratio of branching fractions was measured to be

$$\frac{\mathcal{B}(B^+ \rightarrow D^+ K^{*0} \rightarrow D^+ K^+\pi^-)}{\mathcal{B}(B^+ \rightarrow D^+ K^{*+})} < 0.0044(0.0055),$$

at 90(95)% confidence level. By adopting the measured value $\mathcal{B}(B^+ \rightarrow D^- K^{*+}) = (7.31 \pm 0.19 \pm 0.22 \pm 0.39) \times 10^{-5}$ [12], the isospin relation $\mathcal{B}(K^{*0} \rightarrow K^{*+}) = \frac{1}{3}$, and our result $\mathcal{B}(B^+ \rightarrow D^+ K^{*0}) = (6.27^{+0.58+0.14}_{-0.79-0.10-0.04-0.35}) \times 10^{-7}$, we estimate the ratio $\frac{\mathcal{B}(B^+ \rightarrow D^+ K^{*0} \rightarrow D^+ K^+\pi^-)}{\mathcal{B}(B^+ \rightarrow D^+ K^{*+})} \approx 0.0057$. It is to be tested further since the upper limit for the branching fraction of $B^+ \rightarrow D^+ K^{*0} \rightarrow D K\pi$ given by Ref. [13] is much less than that in the Ref. [22] and Ref. [23].
We suggest more studies for those CKM suppressed $B(s) \to D(\bar{s}) K^* \to D(\bar{s}) K \pi$ decays in which the decay mode $B^0_s \to D^*_s K^{*-} \to D^*_s K \pi$ has a large branching ratio, and could be measured in the LHCb and Belle-II experiments.

Different from the fixed kinematics of the two-body $B$ meson decays, the decay amplitudes of the quasi-two-body $B$ meson decays show a strong dependence on the $K \pi$ invariant mass $\omega$. In Fig. 3, we plot the differential decay branching ratio of the decay mode $B^+ \to D^0 K^{*+} \to D^0 K \pi$ versus the invariant mass $\omega$ in the range of $[m_K + m_\pi, 2 GeV]$. The main portion of the branching ratio lies obviously in the region around the pole mass of the resonant state which presented as a narrow peak in the plot, the contributions from the energy region $\omega > 1.2$ GeV can be omitted safely.

IV. SUMMARY

Motivated by the abundant experimental data, we studied the contributions of the $P$-wave resonant states $K^*$ to the decays $B(s) \to D(\bar{s}) K \pi$ and the CKM suppressed decays $B(s) \to D(\bar{s}) K \pi$ by employing the PQCD factorization approach. The final-state interactions between the $K \pi$ pair are factorized into the two-meson distribution amplitudes in which the resonant line shape for the resonance $K^*$ in the time-like form factor $F_{K\pi}$ is described by the RBW function.

By the numerical evaluations and the phenomenological analyses, we found the following points:

1. By adopting the $B(K^* \to K \pi) \approx 100 \%$, one can obtain $B(B \to D K^* \to D K \pi) \approx B(B \to D K^*)$ easily, and it provides us a new way to study those two-body $B$ meson decays in the framework for the quasi-two-body cases.

2. The PQCD predictions for the branching fractions of the decay processes $B(s) \to D(\bar{s}) K^* \to D(\bar{s}) K \pi$ do agree with the experimental data, except for the cases of the color-suppressed decay $B^0 \to D^0 K^{*0} \to D^0 K \pi$ and the pure annihilation decay $B^0 \to D^- K^{*+} \to D^- K \pi$. More precise data from the LHCb and the Belle-II experiments can help us to test our predictions and to improve the theoretical framework itself.

3. For the CKM suppressed $B(s) \to D(\bar{s}) K^* \to D(\bar{s}) K \pi$ decays, all the PQCD predictions for the branching ratios are consistent with currently available experimental measurements. Experimentally, the upper limit for the branching fraction of $B^+ \to D^+ K^{*0} \to D \pi K \pi$ given by Ref. [13] is much less than that in the Ref. [22] and Ref. [23], and it is to be verified further.

4. Unlike the fixed kinematics of the two-body $B$ meson decays, the decay amplitudes of the quasi-two-body $B$ meson decays have a strong dependence on the $K \pi$ invariant mass $\omega$ and the main portion lies in the region around the pole mass of the resonant state.

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