HYPERBOLIC STRUCTURES ON 3-MANIFOLDS, II:
SURFACE GROUPS AND 3-MANIFOLDS WHICH FIBER OVER THE CIRCLE

WILLIAM P. THURSTON

Abstract. The main result (0.1) of this paper is that every atoroidal three-manifold that fibers over the circle has a hyperbolic structure. Consequently, every fibered three-manifold admits a geometric decomposition. The main tool for constructing hyperbolic structures on fibered three-manifolds is the double limit theorem (4.1), which is of interest for its own sake and lays out general conditions under which sequences of quasi-Fuchsian groups have algebraically convergent subsequences. The main tool in proving the double limit theorem is an analysis of the geometry of hyperbolic manifolds that are homotopy equivalent to a surface. This analysis is also of interest in its own right.

This eprint is based on the August 1986 version of this preprint, which was submitted, refereed, and accepted for publication; for reasons that are hard to fathom, I never returned a corrected version to the journal. I apologize for my long neglect of its publication, and I want to thank the referee for detailed comments which have been incorporated into the present eprint. No other significant changes have been made, except conversion to \LaTeX, which has resulted in changes in numbering. The 1986 preprint was in turn a revision of a 1981 preprint; the various versions were fairly widely circulated in the early 1980’s, and the results became widely known and used.

Too many developments have intervened to be easily summarized, except for pointers particularly to the works of McMullen [McM96] and Otal [Ota96] that give alternative proofs for the main results of this paper and contain other interesting material as well.

Figure 1. This is a portion of the limiting sphere-filling curve for a fiber of the punctured torus bundle over $S^1$ with gluing map $R^4L$, where $R$ is a right-handed Dehn twist about a $(1,0)$-curve and $L$ is a left-handed Dehn twist around a $(0,1)$-curve.

Date: August 1986 preprint → Jan 1998 eprint.
This project has been supported by the NSF, currently #DMS-9704135.
0. Introduction

This is the second in a series of papers dealing with the conjecture that all compact 3-manifolds admit canonical decompositions into geometric pieces. The main purpose of the current paper is to prove

**Theorem 0.1** (Mapping torus hyperbolic). Let $M^3$ be a compact 3-manifold (possibly with boundary) which fibers over $S^1$, and whose fiber is a compact surface of negative Euler characteristic.

Then the interior of $M$ either

(i): has a complete $\mathbb{H}^2 \times \mathbb{R}$ structure of finite volume, and can be described as a Seifert fibration over some hyperbolic 2-orbifold,

(ii): contains an embedded incompressible torus not isotopic to a boundary component, and splits along this torus into two simpler three-manifolds, or

(iii): (generic case) has a complete hyperbolic structure of finite volume.

Cases (i) and (ii) are not mutually exclusive, but (iii) excludes the other two cases.

If $M$ fibers over the circle and has non-empty boundary, then the boundary is a union of tori and (in the non-orientable case) Klein bottles.

A statement equivalent to the main theorem is that a 3-manifold admits a hyperbolic structure if and only if it is homotopically atoroidal. A 3-manifold is **homotopically atoroidal** if every map of a torus into the manifold which is injective on fundamental groups is homotopic to the boundary. This is not quite the same as the condition that every embedded incompressible torus is isotopic to the boundary; a manifold with the latter property is **geometrically atoroidal**. The product of a three-punctured sphere with a circle is an example which is geometrically atoroidal, but not homotopically atoroidal. The complement of an open regular neighborhood of any torus knot is another example.

There are other results of independent interest in this paper. In particular, **Theorem 4.1** is a strong general existence theorem for limits of surface groups acting in hyperbolic three-space, and it is the main ingredient in the proof of Theorem 0.1. After that proof is complete, further results are proven concerning limits of surface groups. For example, **Theorem 7.2** shows how to construct infinitely generated groups as geometric limits of surface groups of a fixed genus. It illustrates the fact that the **measured lamination** is not precisely the right structure for controlling limits of Kleinian groups.

This paper depends directly on Theorem 5.7 of [Thu86], but it is otherwise independent of [Thu86]. Certain information about geodesic laminations, homeomorphisms of surfaces and limits of Kleinian groups will be assumed. This information is summarized in [Thu78] and [Thu80]. The proofs can be gleaned from chapters 8 and 9 of [Thu79] together with [Fet al.79], but an exposition will also be given in parts V and VI of this series.

**Theorem 0.1** was proven for the case when the fiber is a torus minus an open disk by Troels Jørgensen.

The proof of Theorem 0.1 is considerably different from the proof of the related result for other Haken manifolds. It is essentially a proof concerning surface groups.

The existence of a hyperbolic structure on any 3-manifold which fibers over the circle is paradoxical: in the universal covering space of such a 3-manifold, the covering space of a fiber is necessarily a uniformly bounded distance from any image of itself by a covering transformation. This would at first seem to be inconsistent with hyperbolic geometry: two distinct hyperbolic planes, for instance, cannot have a uniformly...
bounded separation. Two horospheres can have uniformly bounded separation, but they have the intrinsic geometry of the Euclidean plane, which is not possible for the universal covering space of a fiber.

A striking image is obtained by adjoining $S^2_{\infty}$, the sphere at infinity for hyperbolic space. The action of the fundamental group of a hyperbolic 3-manifold of finite volume is minimal on $S^2_{\infty}$, that is, it admits no closed proper invariant subsets. The closure of the universal covering of any fiber, intersected with $S^2_{\infty}$, is a closed invariant subset, since the fundamental group of a fiber is normal. Consequently, the closure of the cover of any fiber contains the entire sphere at $\infty$! (See figures 1 and 2.)

The universal covering space of the interior of any compact surface of negative Euler characteristic has a canonical compactification as a disk. Here is a more delicate fact about the fibers ([CT85]; see also [Fen92], [Min94], and [Thu97] for generalizations of this theorem):

**Figure 2.** An approximation to a sphere filling curve which arises from the fiber of a hyperbolic three-manifold which fibers over the circle. The three-manifold in this case is the complement of the figure eight knot, and the fiber is the punctured torus bounded by the figure eight knot.

**Theorem 0.2** (Sphere filling curve; Cannon and Thurston). Let $M^3$ be a hyperbolic 3-manifold which fibers over $S^1$ with fiber $F$. Then the map $i: \tilde{F} \to \tilde{M}$ extends continuously to a map

$$\tilde{i} : D^2 \to D^3.$$ 

The boundary of $D^2$ thus gives a sphere filling curve, or Peano curve, on $S^2_{\infty}$.

A second purpose of this paper is to develop some of the general theory of surface groups, especially those which share with the fibers of fibered hyperbolic 3-manifolds the property that their limit set is the entire sphere. The result of Cannon and Thurston above is not known in the general case.

Theorem 0.1 will finally be an easy corollary of a general result (Theorem 4.1) which constructs limiting surface group actions. Precise statements along with their proofs will be given in §4 after some background material is reviewed in §1 and §2 and a key technical theorem, 3.3, is proven in §3. Theorem 0.1 will be derived in §5 from the result of §4.
After the proof of the main theorem, there are two more sections, which take up about half the length of this paper. These two sections, §6 and §7, continue with the analysis of the limiting behaviour of surface group actions. Some of this information will be used later in this series, but the real thrust of this material is toward the goal of a complete understanding of finitely generated Kleinian groups.

The first examples of hyperbolic 3-manifolds which fiber over a circle were constructed and recognized by Troels Jørgensen through his deep study of limits of quasi-Fuchsian actions of the fundamental group of the punctured torus acting on hyperbolic 3-space. See [Jør77] for a description of some examples. An earlier attempt I made, before I met Jørgensen, to wrestle with the question of the existence even of a Riemannian metric of negative curvature on any such manifold, led me to really look at homeomorphisms of surfaces and develop a geometric theory of their classification: see [Thu84], a detailed exposition of which appears in Fathi, Laudenbach, Poénaru et al. [Fet al.79].

I would like to thank Troels Jørgensen for opening up this subject, and for sharing with me his many visions and insights. I would also like to thank Dennis Sullivan for his relentless attack on the original unwritten, soon-to-be-forgotten proof of Theorem 0.1, from which he cut out parasitical cusps and other undesirable elements. See his Bourbaki seminar exposition [Su80] for the first written account of Theorem 0.1. Much of the further refinement of the current version also was inspired by conversations with Sullivan.

An earlier version of this paper was circulated in preprint form in 1981. The outline of the proof of the main theorem is substantially the same as in the 1981 version, but the execution has been streamlined, especially in the proof in §3 of Theorem 2. The last section of the current version, §7, is entirely new, and its main result is new; §6 also contains new material. The current version is essentially the same as a version from 1986 that was refereed and accepted for publication. I dropped the ball and did not correct and return a final copy to the journal. This is the 1986 version, translated to \[\text{TEX}\], modified according to the referee’s comments, but with little change otherwise.

1. **Quasi-Fuchsian groups**

We recall some notation from [Thu86]. Let $M^n$ be any oriented manifold and $P^{n-1} \subset \partial M$ be any submanifold such that the fundamental group of each component of $P$ contains an abelian subgroup of finite index. Then the set $H(M, P)$ (or simply $H(M)$ when $P = \emptyset$) is the set of complete hyperbolic $n$-manifolds, equipped with a homotopy equivalence $f : M \to N$ which sends $P$ into horoball neighborhoods of cusps of $N$. Two such objects are equivalent if there is an orientation-preserving isometry between them in the homotopy class which makes the diagram commute. There are three significant topologies on $H(M, P)$: the algebraic topology $AH(M, P)$, the geometric topology $GH(M, P)$, and the quasi-isometric topology $QH(M, P)$.

The maps

$$QH(M, P) \to GH(M, P) \to AH(M, P)$$

are continuous.

In the present paper we shall be dealing mainly with the case that $(M, P) = (\bar{S} \times I, \partial \bar{S} \times I)$ where $\bar{S}$ is a compact surface, and $S$ will denote its interior. An element $N \in H(\bar{S} \times I, \partial \bar{S} \times I)$ is Fuchsian if its limit set (that is, the limit set of the group of covering transformations of $\mathbb{H}^3$ over $N$) is a geometrical circle. An equivalent condition is that $N$ contains a totally geodesic oriented surface homeomorphic to $S$. The subset of Fuchsian elements $F(S) \subset H(\bar{S} \times I, \partial \bar{S} \times I)$ is thus identified with $H(\bar{S}, \partial \bar{S})$: the three topologies restricted to this set agree, and are the same as the Teichmüller space $T(S)$. The classical uniformization theorem further identifies Teichmüller space with the space of conformal classes of Riemannian metrics of divergence type on $S$, that is, metrics for which every harmonic function is a constant.

An element $N \in AH(\bar{S} \times I, \partial \bar{S} \times I)$ is quasi-Fuchsian if $N$ is quasi-isometric to a Fuchsian manifold. This is equivalent to the condition that the limit set for $N$ is homeomorphic to a circle. The three topologies also agree on the set $QF(S)$ consisting of quasi-Fuchsian elements of $AH(\bar{S} \times I, \partial \bar{S} \times I)$.

There is a fourth definition of a topology on $H(M, P)$, namely the quasiconformal topology $Q^H(M, P)$. A quasiconformal map between two metric spaces $X$ and $Y$ is a map $f : X \to Y$ for which there exists a constant $K$ such that for all $x \in X$,

$$\lim_{r \to 0} \sup_{x' \in S_r(x)} \frac{\sup_{x' \in S_r(x)} d(f(x'), f(x))}{\inf_{x' \in S_r(x)} d(f(x), f(x'))} \leq K,$$

where $S_r(x)$ denotes the sphere of radius $r$ about $x$. 


If $f$ is a quasiconformal map and if $K_1$ is a constant which works in the above inequality for almost all $x$ on $S^2$, then $f$ is a $K_1$-quasiconformal map.

An $\epsilon$-neighborhood of $N \in H(M, P)$ in the quasiconformal deformation space consists of those $N' \in H(M, P)$ such that the two actions of $\pi_1(M)$ on $S^2_\infty$ are conjugate by an orientation preserving $\epsilon$-quasiconformal homeomorphism of $S^2_\infty$. The topologies $QH(M, P)$ and $QHF(M, P)$ are homeomorphic, by a result proved independently by Tukia [TV82], [Tuk85] and myself (cf. [Hsu79], chapter 11 for a partial discussion). The current paper does not depend logically on the homeomorphism of $QH(M, P)$ with $QHF(M, P)$; we will work with $QH(M, P)$. Nonetheless, to get a good intuitive feeling it is important to think about the quasi-isometric structure of hyperbolic manifolds.

When $M$ is a complete hyperbolic manifold, let $QH_0(M)$ denote the component of $M$ in $QH(M_0, P)$, where $M_0$ is $M$ minus horoball neighborhoods of its cusps and $P$ is its boundary. We recall the fundamental deformation theorem for Kleinian groups; the strong version as stated here is due to Sullivan [Sul83]. For any Kleinian group $\Gamma$, we denote its limit set by $L\Gamma$. The deformation theorem for Kleinian groups; the strong version as stated here is due to Sullivan [Sul83]. For any Kleinian group $\Gamma$, we denote its limit set by $L\Gamma$ and its domain of discontinuity by $D\Gamma$.

**Theorem 1.1** (Quasiconformal deformations; Ahlfors, Bers, . . ., Mostow, . . ., Sullivan). Let $M$ be any complete hyperbolic 3-manifold with finitely generated fundamental group. Suppose that every component of the domain of discontinuity of $\pi_1(M)$ is simply connected. The conformal invariant of the quotient of the domain of discontinuity defines a homeomorphism

$$conf: QH_0(M) \rightarrow D_{\pi_1(M)}/\pi_1(M).$$

Theorem 1 gives a canonical isomorphism of $QF(S)$ with $T(S) \times T(S)$ since the quotient of the domain of discontinuity for a quasi-Fuchsian group has two components, each homeomorphic to $S$. If $g, h \in T(S)$, we denote as $qf(g, h)$ the group determined (up to conjugacy) by $g$ and $h$ in this parametrization. Inversely, if $\Gamma$ is a quasi-Fuchsian group, we denote the two conformal structures on the quotient surfaces of the domain of discontinuity by $c_1(\Gamma), c_2(\Gamma) \in T(S)$. The two components are distinguished by the orientation induced from $S^2_\infty$: the first has positive and the second negative orientation. (The semantic convenience of orientation as a distinction between the two halves is one reason we are sticking for now with oriented manifolds.)

It is easy to see that the subspace $F(S) \subset AH(S \times I, \partial S \times I)$ is closed. Note that $F(S)$ is the diagonal in the product structure for $QF(S)$. The whole of $QF(S)$, on the other hand, is not closed. For example,

**Theorem 1.2** (Bers slice). The closure in $AH(S \times I, \partial S \times I)$ of any slice $x \times F(int S)$ of the product structure for $QF(S)$ is compact.

In this paper, we will find algebraic limits for sequences of quasi-Fuchsian groups which are tending to $\infty$ not just in one factor, but in both factors. We will prove that such limits exists, provided the two coordinates go to $\infty$ in directions far enough apart (thereby avoiding, for example, a sequence of different “markings” of a fixed group).

The analysis will involve the geometry of the quotient manifold, so we need to relate the conformal structure at infinity to the geometry of the interior. We will now give two such relations; either of these is sufficient as a starting point for the rest of the paper.

For any element $\gamma \in \pi_1(S)$ and any $N \in QF(S)$, let $l(\gamma)$ denote the length of the closed geodesic in $N$ homotopic to $\gamma$. Let $length_{+\infty}$ and $length_{-\infty}$ denote the lengths of the closed geodesics homotopic to $\gamma$ on the two quotient surfaces at infinity, using their Poincaré metrics.

**Proposition 1.3** (Poincaré length bounds hyperbolic length [Ber70]).

$$\frac{1}{l(\gamma)} \geq \frac{1}{2} \left( \frac{1}{\text{length}_{+\infty}(\gamma)} + \frac{1}{\text{length}_{-\infty}(\gamma)} \right)$$

and in particular

$$l(\gamma) < 2 \text{length}_{+\infty}(\gamma).$$

**Remark.** Note that the first inequality becomes an equation when $N$ is Fuchsian. A sequence of examples can be constructed to show that the constant, 2, in the second inequality is sharp. The idea is to “bend” a Fuchsian group along a closed geodesic whose length is near zero, with the bending angle near $\pi$. See Theorem 1 or 2 for constructions which shows that there are no inequalities of this form in the opposite sense.
Proof of 1.3. Let $D_+$ and $D_-$ denote the positive and negative components of the domain of discontinuity. Then $D_+ / \gamma$ and $D_- / \gamma$ are cylinders. Calculation (say in the upper half-plane) shows these cylinders are conformally constructed from rectangles whose dimensions are $\pi \times \text{length}_{\gamma}(\gamma)$ and $\pi \times \text{length}_{-\gamma}(\gamma)$, by isometrically gluing the sides of height $\pi$. These two annuli fit inside the torus $(S^2 - L_{\gamma}) / \gamma$. This torus is conformally constructed from a $2\pi \times l(\gamma)$-rectangle by isometrically gluing first the two sides of length $l(\gamma)$, then the two circles of length $2\pi$ of the resulting cylinder (with an arbitrary twist). A standard extremal length argument (see for example [AS60]) gives the proposition.

A subset $A$ of a complete Riemannian manifold is convex if every geodesic arc with endpoints in $A$ is contained in $A$. Note that this definition depends strongly on the topology of the ambient manifold: for instance a closed manifold has no proper convex subsets. (This follows from the fact that the geodesic flow for such a manifold is recurrence). Clearly the intersection of any collection of convex subsets is convex. Any non-empty convex set must contain a possibly broken geodesic in every homotopy class. From this it follows that a convex set in a complete hyperbolic manifold contains every closed geodesic, since the broken geodesics in homotopy classes $\alpha^n$, from no matter what basepoint, wind arbitrarily near the closed geodesic in the free homotopy class of $\alpha$, if such a closed geodesic exists.

It follows that any complete hyperbolic manifold $N$ whose fundamental group contains at least one hyperbolic element has a minimal non-empty subset which is convex; this set, $C(N)$, is called the convex core of $N$. $C(N)$ is a codimension 0 submanifold except in degenerate circumstances, when it may be a submanifold of any lower dimension, possibly with boundary.

Figure 3. This is an approximate drawing of the convex hull of the limit set of a quasi-Fuchsian group. The group is a punctured torus group, that is, it is generated by two elements whose commutator is parabolic. This picture is in true perspective as viewed by an observer on the sphere at infinity of hyperbolic space. It may also be thought of as a more standard picture of the projective ball model for hyperbolic space, where the ball is fairly large compared to the field of vision, and the limit set has been transformed by Möbius transformation so that it fits within the frame.

An alternate description of $C(N)$ for a hyperbolic manifold is that it is the quotient of the convex hull of $L_{\pi_1(N)}$ in the projective ball model for hyperbolic space intersected with hyperbolic space and quotiented by the action of $\pi_1(N)$. When $N$ is 3-dimensional and $C(N)$ is non-degenerate, the boundary of $C(N)$ is a developable surface: it is a hyperbolic surface, homeomorphic to the quotient of the domain of discontinuity, and isometrically embedded in $N$. Since $\partial C(N)$ is often not a $C^1$ submanifold, we cannot take the definition of an isometric embedding from standard differential geometry, and we should clarify what definition we use: A definition

\footnote{We use the term ‘hyperbolic’ here in its inclusive meaning, to include the loxodromic case.}
which serves well is that an embedding is isometric if every geodesic in the surface is mapped to a rectifiable path of the same arc-length. See \cite{Flm79}, chapter 8, for more background.

**Proposition 1.4** (Bounded distortion to infinity; Sullivan). There is a constant \(1 < K < \infty\) such that for every complete hyperbolic 3-manifold \(N\) and any incompressible component \(S\) of \(\partial C(N)\), there is a \(K\)-quasi-isometry (in the correct homotopy class) from \(S\) to the corresponding surface at \(\infty\) equipped with its Poincaré metric.

See \cite{Su1} for a brief proof, or \cite{PM0} for a detailed writeup which produces a concrete value for \(K\). The reasonable conjecture seems to be that the best \(K\) is 2, but it is hard to find an angle for proving a sharp constant.

## 2. Geodesic laminations and pseudo-Anosov homeomorphisms

In this section we will review some background concerning the geometry of hyperbolic surfaces near \(\infty\) in Teichmüller space, and the effect of a homeomorphism of a surface on its geometry. For details, the reader is referred to \cite{Fet al.79}, \cite{CB88}, and chapters 8 and 9 of \cite{Flm79}.

We will use geodesic laminations on surfaces, objects which are generalizations of simple closed curves in much the same way that real numbers are generalizations of the rational numbers. In fact, on the torus, a simple closed curve is described by its slope, which is a rational number, while the slope of a geodesic lamination is a real number.

An alternative structure which serves much the same purpose is the measured foliation. Another very closely related object is the quadratic differential on a Riemann surface. These notions are also closely related to (and partly inspired by) the work of Nielsen on homeomorphisms of surfaces. See \cite{Nie86a} and \cite{Nie86b}, and \cite{HT85} or \cite{Gil82} for a discussion of Nielsen’s work and its relation to mine.

Let \(S\) be any complete hyperbolic surface of finite area. A **geodesic lamination** \(\lambda\) on \(S\) is a closed subset of \(S\) which is the disjoint union of simple geodesics. These geodesics are called the **leaves** of \(\lambda\). They may be either infinite simple geodesics, or simple closed geodesics. One way to think of a geodesic lamination is to pass to \(\tilde{\lambda}\), a Hausdorff limit of \(\lambda\), which is the disjoint union of simple geodesics. These geodesics are called the **leaves** of \(\tilde{\lambda}\). A geodesic lamination is defined by a subset of the projectivized tangent bundle of a surface, namely the tangent spaces to the leaves of the lamination. The Hausdorff limit of the tangent spaces of the leaves of a family of geodesic laminations itself comes from a geodesic lamination; we call this lamination the Hausdorff limit of the family.

Every geodesic lamination on a hyperbolic surface is a Hausdorff limit of a sequence of geodesic laminations which have only finitely many leaves. By forming limits of geodesic laminations with only one leaf, it is not hard to see that if \(S\) is more complicated than a 3-punctured sphere, it has uncountably many geodesic laminations.

A geodesic lamination on a hyperbolic surface always has zero Lebesgue measure on \(S\) (in contrast to the situation on a torus).

A **transverse measure** for a geodesic lamination is a measure on \(G_\lambda\) invariant by \(\pi_1(S)\). Described directly in terms of \(S\), a transverse measure assigns a measure to each curve \(\alpha\) in \(S\) transverse to \(\lambda\), in such a way that for any two curves \(\alpha\) and \(\beta\) and any homeomorphism \(f: \alpha \to \beta\) which takes \(l \cap \alpha\) to \(l \cap \beta\) for every leaf \(l\) of \(\lambda\), \(f\) preserves the measure. One thinks of the measure as measuring the quantity of leaves crossed by the \(\alpha\). A geodesic lamination equipped with a transverse measure of full support is a **measured lamination**.

The set of measured laminations has a topology, coming from the weak topology on measures on the Moebius band \(M\). Denote this space, including the empty lamination with the trivial measure, by \(\text{ML}(S)\).

The subspace of compactly supported laminations is denoted by \(\text{ML}_0(S)\).

**Theorem 2.1** (ML Euclidean). \(\text{ML}_0(S)\) is homeomorphic to Euclidean space of dimension equal to that of \(T(S)\).
Two non-trivial measured laminations $\mu_1$ and $\mu_2$ are projectively equivalent if their underlying laminations agree, and one measure is a constant times the other. The space of projective classes of measured laminations is denoted $\text{PL}(S)$, and the projective classes of compactly supported measured laminations is $\text{PL}_0(S)$. As expected from Theorem 2.1, $\text{PL}_0$ is a sphere. Each simple closed curve defines a point in $\text{PL}_0(S)$. These points are dense.

Laminations and measured laminations can easily be transferred from one hyperbolic surface to any homeomorphic surface by means of the set $\mathcal{G}_\lambda$, using the fact that $S^1_\infty$ is a topological invariant of a surface. Thus, the spaces $\text{ML}(S)$, $\text{PL}(S)$ and variations really depend only on the topological surface $S$.

The notion of length of a simple closed geodesic extends readily to a continuous function

$$\text{length} : T(S) \times \text{ML}_0(S) \to \mathbb{R}.$$  

One way to define this extension is to define the length $\text{length}_S(\mu)$ of a measured lamination $\mu$ on a hyperbolic surface $S$ to be the total mass of the “product” of transverse measure with 1-dimensional Lebesgue measure of the leaves of $\lambda$. More precisely, the product measure is defined by its rule of integration, which is an iterated integral: in any small coordinate patch, first integrate along the leaves of $\lambda$ with respect to Lebesgue measure, then with respect to the transverse measure. When a simple closed curve is given its transverse counting measure, this definition agrees with the length for the curve.

The geometric intersection number $i(\alpha, \beta)$ of two simple closed geodesics $\alpha$ and $\beta$ is the total number of intersection points, unless $\alpha$ and $\beta$ coincide, in which case $i(\alpha, \beta) = 0$. This also extends to a continuous function

$$i : \text{ML}(S) \times \text{ML}(S) \to \mathbb{R}.$$  

The definition for $i(\mu_1, \mu_2)$ on $S$ is defined as the total mass of a measure $\mu_1 \times \mu_2$ on $S$, where $\mu_1 \times \mu_2$ is the product of the two transverse measures in any small open set where the laminations are transverse to each other, and zero on any leaves which the two laminations have in common. Transverse intersections are automatically confined to a compact subset of $S$, so this intersection number is always finite. It depends only on the topological surface $S$.

**Theorem 2.2** (Laminations compactify Teichmuller space). The union $\overline{T(S)} = T(S) \cup \text{PL}_0(S)$ has a natural topology homeomorphic to a disk.

In this topology, a sequence $\{g_i\}$ of hyperbolic structures in $F(S)$ converges to a lamination $\mu \in \text{PL}_0(S)$ if and only if there is a sequence $\{\mu_i\} \to \infty$ of measured laminations converging projectively to $\mu$ such that for all $\mu' \in \text{ML}_0(S)$ for which $i(\mu', \mu) \neq 0$,

$$\lim_{i \to \infty} \frac{\text{length}_{g_i}(\mu')}{i(\mu_i, \mu')} = 1.$$  

Furthermore, $\text{length}_{g_i}(\mu_i) \to \infty$, but $\text{length}_{g_i}(\mu_i)$ remains bounded. Moreover, there is a constant $C$ such that

$$i(\mu', \mu_i) \leq \text{length}_{g_i}(\mu') \leq i(\mu', \mu_i) + C \text{length}_{g_i}(\mu').$$  

In other words, intersection number with an appropriate lamination is quite a good approximation to hyperbolic length near infinity in Teichmüller space. It is not always the case that for a measured lamination $\overline{\gamma}$ in the projective class of $\mu$ length $g_i(\overline{\gamma})$ itself goes to zero. For example, consider a sequence of metrics obtained by the sequence of $i$th powers of Dehn twists about a single geodesic $\gamma$. These metrics converge to $\gamma$, yet $\gamma$ has constant length. The convergence of the $g_i$ in this case is “tangential” to the boundary of Teichmüller space, so that a sequence of $\mu_i$ satisfying the conditions converge to $\gamma$ rather slowly in $\text{PL}_0(S)$.

Without the conditions in the last two paragraphs of the theorem, the sequence of measured laminations $i\text{length}_{g_i}(\gamma)$ would serve.

If $\beta$ is any simple closed curve which intersects $\gamma$, then a certain constant multiple of the sequence of images of $\beta$ by the $i$th power of the inverse Dehn twist will satisfy the conditions for $\mu_i$ except for the last paragraph.

The laminations $\mu_i$ are constructed in $\text{Thua}$, in the course of development of cataclysm coordinates. This is quite related to the proof of Theorem 3.3 as well; in cataclysm coordinates, given a lamination $\lambda$ and a metric $g$, a measured foliation $\mathcal{F}$ is constructed such that the length along $\lambda$ in $g$ exactly agrees with its transverse measure. The first stage of the polygonal approximations made in the proof of 3.3 show that the measured lamination $\nu$ defined by $\mathcal{F}$ serves well to estimate length.
Note that if the $\mu_i$ are normalized to converge to $\mu$ as measured laminations, rather than simply projective
laminations, then $\text{length}_{g_i}(\mu_i) \to 0$.

Pairs of laminations are often important in the theory of surfaces. A pair $(\mu, \nu)$ of laminations is called $\text{binding}$ if

1. $(a)$: they have no leaves in common, and
2. $(b)$: for each component $U$ of the complement of the union of $\mu$ and $\nu$, the metric completion of $U$ is either a compact polygon, a polygon with one ideal vertex which tends to a cusp, or a punctured polygon, where the puncture is a cusp of the surface.

**Proposition 2.3** (Lamination crosses binding). A pair of laminations $(\mu, \nu)$ on a surface $S$ is binding if and only if every simple geodesic on $S$ has at least one transverse intersection with a leaf of $\mu$ or a leaf of $\nu$.

A pair of compactly supported measured laminations $(\mu, \nu)$ is binding if and only if for every compactly supported measured lamination $\lambda$,

$$i(\lambda, \mu) + i(\lambda, \nu) > 0$$

**Proof of 2.3.** If $(\mu, \nu)$ is binding, then every leaf of $\mu$ crosses at least one leaf of $\nu$ (and $\text{vice versa}$). For suppose, on the contrary, that $(\mu, \nu)$ is a pair of laminations such that $\mu$ has a leaf $l$ which doesn’t meet $\nu$. Then the closure of $l$ is a sublamination $\lambda$ of $\mu$; if $\lambda$ meets $\nu$, then the two laminations have leaves in common, so the pair is not binding. Otherwise, since $\nu$ is closed, $\lambda$ has a neighborhood not meeting $\nu$. This violates (b), so the pair is not binding.

If $(\mu, \nu)$ is binding, and if $g$ is any simple geodesic on the surface, then if $g$ intersects some component of the complement of the union of $\mu$ and $\nu$, it is clear that $g$ intersects one of the two laminations transversely, by (b). The only other possibility is that $g$ is a leaf of $\mu$ or of $\nu$, in which case it intersects the other lamination transversely, by the preceding paragraph.

On the other hand, suppose that $(\mu, \nu)$ is a pair of laminations such that every simple geodesic intersects one or the other transversely. Then, in particular, they have no leaves in common. If any region of the complement is not simply-connected, then its fundamental group must be parabolic, otherwise it would contain a simple closed geodesic. The metric completion of a region of the complement has boundary made up of segments of leaves of $\mu$ and of $\nu$, so it is polygonal with possibly some missing vertices. There can be no more than one missing vertex, for otherwise, there would be a simple geodesic connecting one ideal vertex to the other. The neighborhood of any ideal vertex must tend toward a cusp of $S$, because if it recurred in compact subsets of $S$, one could form a limit, and construct a new simple geodesic with no transverse intersections with $\mu$ or $\nu$. If the boundary has any missing vertices, then the region is simply-connected, or again there would be a simple geodesic tending to the ideal vertex at either end, looping around the fundamental group.

The only remaining possibilities are those mentioned in (b). This establishes the claim of the first paragraph of the proposition.

Now let us suppose that $\mu$ and $\nu$ are compactly supported measured laminations. From the previous condition, it follows that if $\lambda$ is a non-trivial measured lamination such that $i(\lambda, \mu) = i(\lambda, \nu) = 0$, then $(\mu, \nu)$ is not binding.

To establish the converse, recall (for instance, from Proposition 5.4 of [Thu86]) that every measured lamination is the finite union of minimal sublaminations. Therefore, if $\mu$ and $\nu$ have any leaves in common, the set of common leaves constitutes a finite union of minimal sets of both. It inherits a transverse measure from $\mu$, to give it the structure of a measured lamination $\lambda$ for which $i(\lambda, \mu) = i(\lambda, \nu) = 0$.

If a region of the complement of $\mu \cup \nu$ violates condition (b), there are two possibilities. If the region has a fundamental group not consisting of parabolic elements, there is a measured lamination $\lambda$ supported on a simple closed geodesic and not intersecting $\mu$ or $\nu$. If the boundary of the region has an ideal vertex, it cannot tend to a cusp, since $\mu$ and $\nu$ have compact support. Therefore, its limit set gives a minimal set in $\mu$ or $\nu$, or both. It inherits a transverse measure from $\mu$ or from $\nu$, thereby defining a measured lamination having intersection number 0 with $\mu$ and $\nu$.

For later use, we need:
Proposition 2.4 (Binding confinement). Let \( \mu_1 \) and \( \mu_2 \) be any two measured laminations which have no leaves in common, and for which \( S \setminus (\mu_1 \cup \mu_2) \) consists of pieces which are simply connected or neighborhoods of cusps.

Then for any constant \( C > 0 \), the set of hyperbolic structures \( g \) on \( S \) for which

\[
\text{length}_g(\mu_1) \leq C
\]

and

\[
\text{length}_g(\mu - 2) \leq C
\]

is compact.

Proof of 2.4. Two such laminations \( \mu_1 \) and \( \mu_2 \) have the property that for any measured lamination \( \nu \),

\[
i(\mu_1, \nu) > 0 \text{ or } i(\mu_2, \nu) > 0.
\]

Applying this to the case that \( \nu \) ranges over a sequence \( \mu_i \) from 2.2, we see that no sequence of metrics for which both \( \mu_1 \) and \( \mu_2 \) have bounded lengths can approach the boundary of Teichmüller space.

For a different, directly geometric proof of a more general proposition, see Theorem 6.3. Here is another fact for future reference:

A homeomorphism \( \phi : S \to S \) is reducible if \( \phi \) permutes some finite system of disjoint simple closed geodesics. The terminology is based on the fact that by cutting \( S \) along such a system of curves, one can reduce the study of \( \phi \) to the study of a homeomorphism of a simpler surface. This is analogous to the notion of a reducible three-manifold, or (closer to home) a torus-reducible three-manifold.

Theorem 2.5 (Classification of surface homeomorphisms). Every homeomorphism \( \phi' \) of \( S \) is isotopic to a homeomorphism \( \phi \) which either

(i): has finite order
(ii): is reducible, or
(iii): does not satisfy (i) or (ii), and preserves a unique pair of projective classes of measured laminations, \( \mu_s \) and \( \mu_u \).

In the case (iii), for any point \( x \in \overline{T(S)} \), if \( x \neq \mu_s \) and \( x \neq \mu_u \), then

\[
\lim_{n \to +\infty} \phi^n(x) = \mu_s
\]

and

\[
\lim_{n \to -\infty} \phi^n(x) = \mu_u.
\]

There is a constant \( \lambda > 1 \) such that \( \phi \) multiplies the transverse measure for \( \mu_u \) by \( \lambda \) and the transverse measure for \( \mu_s \) by \( 1/\lambda \).

Case (iii) is both most common and most interesting. The isotopy class of \( \phi \) in case (iii) is called pseudo-Anosov. There is a more geometric form for \( \phi \), in this case, if the measured laminations are replaced by measured foliations. When this is done, \( \phi \) becomes almost an Anosov homeomorphism: it is Anosov in the complement of a finite collection of points.

From any homeomorphism \( \phi \) of a surface \( S \) to itself, a 3-manifold called the mapping torus of \( \phi \) can be constructed by first forming the product \( S \times I \), and then gluing \( S \times \{1\} \) to \( S \times \{0\} \) using \( \phi \). The resulting 3-manifold \( M_\phi \) fibers over \( S^1 \), and \( \phi \) is called the monodromy of the fibration of \( M_\phi \).

Actually, only the isotopy class of \( \phi \) is determined by the fibering \( M_\phi \to S^1 \), and the isotopy class of \( \phi \) determines \( M_\phi \) up to homeomorphisms which commute with \( M_\phi \to S^1 \).

Proposition 2.6 (Non-pseudo-Anosov mapping torus). If \( \phi \) is a homeomorphism of a compact surface \( \tilde{S} \) of negative Euler characteristic, then

(i): \( \phi \) is isotopic to a homeomorphism of finite order if and only if the interior of \( M_\phi \) admits a complete \( \mathbb{H}^2 \times \mathbb{R} \) structure of finite volume
(ii): \( \phi \) is isotopic to a reducible homeomorphism if and only if \( M_\phi \) admits an embedded incompressible torus not isotopic to \( \partial M \), and
(iii): otherwise \( \phi \) is pseudo-Anosov.
Proof of 2.4. If $\phi$ is isotopic to a homeomorphism of finite order, then there is a hyperbolic structure on $S$ which is invariant by $\phi$ up to isotopy. From this, a $\mathbb{H}^2 \times \mathbb{R}$ structure is constructed on $M_\phi$.

If the interior of $M_\phi$ admits a $\mathbb{H}^2 \times \mathbb{R}$ structure, that structure defines on $M_\phi$ a codimension two transversely hyperbolic foliation. It was shown in [Thu79] that such a foliation either describes a Seifert fibration, or the manifold fibers over the circle with fiber a torus. Clearly the former case obtains. Consider the projection of the fiber $\bar{S}$ of the foliation over $S^4$ to the base of the Seifert fibration. There is an induced map from the fundamental group of $\bar{S}$ to the fundamental group of the base orbifold. The image is a normal subgroup. The quotient group must be finite, since there is no finitely generated normal subgroup of the fundamental group of a hyperbolic 2-orbifold with quotient group $\mathbb{Z}$. The map is also injective on the level of fundamental group. This implies it is homotopic to a covering map over the base orbifold. Then $\phi$ is isotopic to a deck transformation, so it has finite order.

If $\phi$ is isotopic to a reducible homeomorphism, then the trace of a reducing curve, dragged around by $\phi$ until it comes back to itself, with the same orientation, gives an incompressible torus.

Conversely, if there is an incompressible torus, consider the projection of the torus to the circle. It can be perturbed to have Morse-type singularities. A standard 3-manifold argument shows that the singularities can all be eliminated by an isotopy of the torus, so that the torus becomes transverse to each fiber. The intersection with one of the fibers defines a family of reducing circles for some homeomorphism isotopic to $\phi$.

3. An estimate for the shapes of certain pleated surfaces

Let $\bar{S}$ be a compact surface and $N \in \text{AH}(\bar{S} \times I, \partial \bar{S} \times I)$ be a hyperbolic 3-manifold. For purposes of reference, fix a complete hyperbolic structure of finite area on $S$. Most non-trivial simple closed curves (except parabolics) are realized as closed geodesics in $N$. Similarly, most geodesic laminations on $S$ have realizations in $N$. In particular, the realizable laminations contain an open dense subset of $\text{PL}(S)$. This theory is developed in chapters 8 and 9 of [Thu79]. In the special case $N \in \text{QF}(S)$, all geodesic laminations on $S$ are realized in $N$.

The quantity length$_N(\mu)$ of a realizable measured lamination $\mu$ in a hyperbolic 3-manifold $N$ is defined similarly to the length of a lamination on a surface, as the total mass of a “product” measure formed from 1-dimensional Lebesgue measure along the leaves with transverse invariant measure. To put it another way, in a local coordinate system, the transverse invariant measure is a measure on the space of leaves of $\mu$; the integral of a continuous function $f$ with respect to the product measure is defined by first integrating $f$ with respect to Lebesgue measure on each local leaf, then integrating with respect to the transverse measure. A further complication is that $\mu$ is only mapped into $N$, but not necessarily embedded: one should do the computation in the domain, by pulling back Lebesgue measure from $N$.

Proposition 3.1 (Length continuous). The length of laminations is a continuous function on the set $R \subset \text{PL}(S) \times \text{AH}(\bar{S} \times I, \partial \bar{S} \times I)$ of realizable laminations.

The details of this and other basic facts about measured laminations in two and three dimensions will be proven in part VI of the series.

A geodesic lamination $\lambda$ of a surface $S$ is maximal if it is maximal among all geodesic laminations, which is equivalent to the condition that each region of $S - \lambda$ is isometric to the interior of an ideal triangle. Every lamination is contained in a maximal lamination, although a lamination may be maximal among measured laminations without being maximal in the current sense: to extend it to a maximal lamination, one must often make do with a lamination admitting no transverse measure of full support. Every surface admits maximal laminations which have only finitely many leaves. See for example Figure 2.1 of [Thu80] for a way to construct laminations by “spinning” triangulations.

If $\partial \bar{S} \neq \emptyset$, the situation can be simplified by allowing no closed leaves on $S$. Then $\lambda$ becomes the 1-skeleton of an “ideal triangulation” of $S$.

A lamination $\lambda$ with a finite number of leaves is realizable provided all its closed leaves are realizable. If $\lambda$ is maximal, then it determines a pleated surface

$$f_\lambda : P_\lambda \rightarrow N,$$

where $P_\lambda$ is a complete hyperbolic structure on $S$ and $f_\lambda$ is an isometric map of $P_\lambda$ into $N$ which folds or pleats, at most, along $\lambda$. 

If $\lambda$ is a maximal lamination with only finitely many leaves and $\mu$ is a measured lamination of compact support, we will define a quantity 

$$a(\lambda, \mu)$$

which in some sense measures the complexity of $\mu$ relative to $\lambda$. We will define $a(\lambda, \mu)$ first in the case that $\mu$ is a simple closed curve. If $\mu$ is a leaf of $\lambda$, then we define $a(\lambda, \mu) = 0$. Otherwise, there is at most a countable set of intersections of the two laminations. If $x$ and $y$ are two intersection points with no intervening intersections along $\mu$, then the leaves of $\lambda$ through $x$ and $y$ are two sides of an ideal triangle of $\lambda$ (since $\lambda$ is maximal, by hypothesis). These leaves are therefore asymptotic on either one side of $\mu$, or the other. If $x$, $y$, and $z$ are three successive intersection points along $\mu$, and if the leaves of $\lambda$ through $x$ and $y$ are asymptotic on the opposite side of $\mu$ to the leaves of through $y$ and $z$, then let us call $y$ a boundary intersection.

![Figure 4. The boundary intersections of a geodesic $\mu$ with a lamination $\lambda$ are marked in the universal cover, $\mathbb{H}^2$. Even where leaves of $\lambda$ accumulate, the boundary intersections are isolated.](image)

It can also happen that a point $y$ of intersection of $\lambda$ and $\mu$ is an accumulation point of leaves of $\lambda$. In such a case, the leaf of $\lambda$ through $y$ is a closed leaf, and the leaves of $\lambda$ spiral to the closed leaf on both sides. We will call such an intersection point a boundary intersection if the directions of spiraling on the two sides are different. Note that both kinds of boundary intersection are isolated, and so there is a finite number of boundary intersections in all. We define $a(\lambda, \mu)$ (in the case that $\mu$ is a simple closed curve) to be the total number of boundary intersections of $\lambda$ and $\mu$. In other words, $a(\lambda, \mu)$ measures the total number of times the direction of asymptoticity of the leaves of $\lambda$ changes as you go around $\mu$.

When $\mu$ is a general measured lamination, the situation is much the same. Designate a point $y$ of transverse intersection of $\lambda$ and $\mu$ a boundary intersection if the direction of asymptoticity of the leaves of $\lambda$ changes at $y$. Then compute $a(\lambda, \mu)$ as the total $\mu$-transverse measure of the set of boundary intersection points.

**Proposition 3.2 (Continuity of alternation number).** For a fixed maximal finite lamination $\lambda$, the alternation number $a(\lambda, \mu)$ is a finite-valued, continuous function of the measured lamination $\mu$.

**Proof of 3.2.** Associated with a measured lamination $\mu$ is a measure $M(\mu)$ on the projective bundle $\mathbb{P}(S)$ of $S$, defined as the product of 1-dimensional Lebesgue measure with the transverse measure. We will construct a continuous function $C(\lambda)$ such that for all $\mu$,

$$a(\lambda, \mu) = \int_{\mathbb{P}(S)} C(\lambda) dM(\mu).$$

This implies that $a(\lambda, \mu)$ is continous in $\mu$.

The idea for constructing $C(\lambda)$ is that instead of counting boundary intersections directly, we can spread the contribution out so that it becomes an integral over a larger portion of $\mathbb{P}(S)$. 
Every isolated leaf \( l \) of \( \lambda \), lifted to the universal cover of the surface, separates two ideal triangles. The union of the two triangles forms an ideal quadrilateral. If a geodesic \( m \) crosses \( l \), it is a boundary intersection iff \( m \) passes through opposite sides of this quadrilateral.

Choose a continuous function on the unit interval with integral 1 which is 0 at the two endpoints of the interval. By scaling, this transfers to a function whose integral is 1 on an arbitrary interval.

For each quadrilateral obtained by removing an isolated leaf of \( \lambda \) in the universal cover, and for each geodesic \( m \) which intersects opposite sides of the quadrilateral, scale this function to the intersection of \( m \) with the quadrilateral, and lift to \( \mathbb{P}(\mathbb{H}^2) \). Define a function in \( \mathbb{P}(\mathbb{H}^2) \) by adding up over all quadrilaterals and all geodesics. For any point of \( \mathbb{P}(\mathbb{H}^2) \), there are at most 3 contributions, so this gives a well-defined function on \( \mathbb{P}(\mathbb{H}^2) \). It is continuous, because when a sequence of geodesics \( m \) which cross opposite sides of a quadrilateral converge to one that doesn’t, the length goes to \( \infty \) so the function tends to 0. This function is invariant by deck transformations, so it gives a continuous function on \( S \).

Similarly, if \( \alpha \) is a closed geodesic of \( \lambda \) for which the spiraling is in opposite directions on its two sides, we add a contribution for each geodesic \( m \) which crosses \( \alpha \), supported on the intersection of \( m \) with an \( \epsilon \) neighborhood of \( \alpha \). The result, after adding all the contributions, is a continuous function \( C(\lambda) \), with the desired properties.

The following theorem gives an estimate of how efficiently certain homotopy classes can be represented on a pleated surface with finite pleating locus:

**Theorem 3.3** (Efficiency of pleated surfaces). Let \( \bar{S} \) be a fixed compact surface, possibly with boundary. For any \( \epsilon > 0 \) there is a constant \( C < \infty \) such that the following holds:

Let \( \lambda \) be any finite maximal lamination on \( S \).

Let \( N \) be any element of \( \text{AH}(\bar{S} \times I, \partial \bar{S} \times I) \). Suppose that no closed leaf of \( \lambda \) has length less than \( \epsilon \) in \( N \), so that in particular, no such leaf is parabolic, and consequently a surface \( f_\lambda : P_\lambda \to N \) exists which is pleated along \( \lambda \).

Let \( \mu \in \text{ML}_0(S) \) be any compactly-supported measured lamination which is realizable in \( N \).

Then

\[
\text{length}_N(\mu) \leq \text{length}_{P_\lambda}(\mu) \leq \text{length}_N(\mu) + Ca(\lambda, \mu).
\]

This inequality does not generalize to an inequality for lengths on more general pleated surfaces. The finite combinatorial complexity of \( \lambda \) is crucial.

When \( \mu \) is a not necessarily realizable lamination in \( \text{ML}_0(S) \), define \( \text{length}_N(\mu) \) to be the lim inf of lengths of nearby realizable laminations. If \( \mu \) has only one component, then it follows from [Thu73] or [Bon86] that \( \text{length}_N(\mu) = 0 \).

**Corollary 3.4** (Efficiency for unrealizable laminations). Theorem 3.3 works for arbitrary \( \mu \in \text{ML}_0(S) \) if length\(_N(\mu) \) is replaced by \( \text{length}_N(\mu) \).

**Proof of 3.4.** Apply the theorem to an appropriate sequence of realizable laminations converging to \( \mu \). \( \square \)

**Proof of Theorem 3.3.** In view of proposition 3.1, it suffices to prove 3.3 in the case that \( \mu \) is a simple closed curve, since simple closed curves are dense among realizable laminations and the inequality is homogeneous in \( \mu \).

The idea of the proof is to represent a simple closed curve \( \mu \) by a polygonal path on \( P_\lambda \) whose number of sides is \( O(a(\lambda, \mu)) \) and which follows along the leaves of \( \lambda \) except for \( O(a(\lambda, \mu)) \) of its length. By an application of the uniform injectivity theorem [Thu84, Thm. 5.7], one finds that this polygonal path on \( P_\lambda \), when mapped to \( N \), cannot “double back” very much, so that it cannot be too inefficient.

From now on in the proof, \( \mu \) will be a simple closed geodesic on \( P_\lambda \). We shall homotope \( \mu \) to a polygonal curve.

We may assume that \( \mu \) is not a leaf of \( \lambda \), for in that case there is nothing to prove: its length in \( N \) equals its length in the pleated surface.

We may also assume that \( \mu \) is not a geodesic shorter than \( \epsilon \) on \( S \), for if it is not a leaf of \( \lambda \), then \( a(\lambda, \mu) \geq 2 \), and the inequality is trivial.
Otherwise, \( a(\lambda, \mu) > 0 \), and there is a chain of length \( a(\lambda, \mu) \) consisting of the leaves of \( \lambda \) on which there are boundary intersections with \( \mu \). Successive leaves in the chain are asymptotic, and if the asymptotic pairs of ends of leaves are replaced by short jumps, one obtains a curve homotopic to \( \mu \).

It is easiest at this point to describe the picture in the universal cover of the surface, \( \mathbb{H}^2 \). Let \( R \) be a strip of constant width, say \( .5 \), about the geodesic \( \tilde{\mu} \). For concreteness, we can consider the picture in the Poincaré disk model, with \( \tilde{\mu} \) being the horizontal diameter of the disk.

We have an infinite chain \( g_i \ [i = -\infty \ldots \infty] \) of geodesics connecting the two endpoints of \( \tilde{\mu} \), with endpoints of successive geodesics meeting at \( S_\infty^1 \). Each \( g_i \) crosses \( R \). Let \( a_i \) be the lower endpoint of \( R \cap g_i \), and \( b_i \) the upper endpoint. Let \( A_i \) and \( B_i \) be the (hyperbolic) perpendicular projections of \( a_i \) and \( b_i \) to \( \tilde{\mu} \). Then \( A_i < A_{i+1}, B_i < B_{i+1} \), and \( |A_{i+1} - B_i| \) is bounded by an \( a \) priori constant.

If \( B_i > A_{i+1} \), let \( x_{i+1} \) be the point of \( g_{i+1} \) whose perpendicular projection to \( \tilde{\mu} \) is \( B_i \); otherwise, let \( x_{i+1} = a_{i+1} \).

\[ \text{Figure 5.} \] An arbitrary simple closed geodesic \( \mu \) on a pleated surface can be approximated by a polygonal path with \( 2a(\lambda, \mu) \) sides, which follows leaves of \( \lambda \) except on \( O(a(\lambda, \mu)) \) of its length, and whose length exceeds that of \( \mu \) only by \( O(a(\lambda, \mu)) \). The path simply follows appropriate segments of the sequence of asymptotic geodesics given by its boundary intersections with \( \lambda \). The possibly long segments of this polygonal path which follow leaves of \( \lambda \) are mapped efficiently in any \( \lambda \)-pleated surface. By making further adjustments to avoid problems with the thin parts of the surface, the polygonal path can be forced to map efficiently as a whole on a \( \lambda \)-pleated surface, up to an additive constant which is a bounded multiple of \( a(\lambda, \mu) \).

Define a polygonal path \( \tilde{m} \subset \mathbb{H}^2 \) to consist of the the geodesic segments

\[ \ldots \ x_i | b_i \ b_i x_{i+1} \ldots \]

This polygonal path consists of intervals of the leaves \( g_i \), interspersed with jumps of bounded length from one leaf to another. The projection to \( P_\lambda \) is a polygonal curve \( m \) whose length exceeds the length of \( \mu \) only by a constant times \( a(\lambda, \mu) \). Furthermore, \( m \) lies on leaves of \( \lambda \) except for a portion of its length which is a constant times \( a(\lambda, \mu) \).

Next, we will make a homotopy of \( m \) to a new curve \( n \) with similar properties to \( m \), but which also has the property that none of the jumps between leaves of \( \lambda \) occur in the thin part of \( P_\lambda \). We may assume that \( \mu \) is not a short geodesic The curve \( m \) enters only those thin parts of \( P_\lambda \) which are neighborhoods of short geodesics. Since, by hypothesis, no closed geodesics of \( \lambda \) are short, each leaf of \( \lambda \) which enters such a thin part exits on the opposite side. All leaves of \( \lambda \) which enter the thin part remain fairly close together and roughly parallel for the entire intersection with the thin part.

For any interval of \( m \) which crosses a thin part of \( P_\lambda \), we can inductively replace the first segment on a leaf in the thin part and the first jump in the thin part by a jump just outside the thin part, and a longer segment on a leaf. Each such step increases the length of the curve by at most a constant. The total number of steps is at most \( a(\lambda, \mu) \), so in the end we obtain a polygonal curve \( n \) with the desired properties.

The image \( P_\lambda(n) \) of \( n \) in the three-manifold \( N \) is probably not polygonal, but we can homotope it to a polygonal path \( p \) by homotoping the image of each jump segment of \( n \) to its geodesic in \( N \). The length of
Let \( \mu' \) be the geodesics of \( N \) homotopic to \( P_\lambda(\mu) \). Construct a pleated annulus \( A \) representing the homotopy from \( p \) to \( \mu' \) by spinning around \( \mu' \); concretely, for each side of \( p \), form a triangle based on that side with third “vertex” spiralling infinitely around \( \mu' \). Form \( A \) by gluing all these triangles together, and completing with \( \mu' \). The area of each triangle is less than \( \pi \), so the area of \( A \) is less than \( 2\pi a(\lambda, \mu) \).

In the intrinsic hyperbolic metric of \( A \), the portion of the \( p \) boundary component of \( A \) which admits a regular neighborhood of width \( \epsilon \) in \( A \) is bounded by \( \text{area}(A)/\epsilon \), which is less than a constant times \( a(\lambda, \mu) \).

Any remaining length of \( p \) consists of segments on leaves of \( \lambda \) in \( N \) which run close and nearly parallel to some other portion of the boundary of \( A \).

We claim that the nearby portion of the boundary of \( A \) is part of \( \mu' \), not part of \( p \), except possibly for a total length bounded by a constant times \( a(\lambda, \mu) \).

For any portion of \( p \) which is not in \( P_\lambda(\mu) \), this will follow fairly directly from Theorem 5.7 of [Thu86]. In fact, if there is any short arc \( \alpha \) on \( A \) connecting two segments of \( p \) on leaves of \( \lambda \) but not in \( P_\lambda(\mu) \), \( \alpha \) would be homotopic to a short arc on \( S \), by the uniform injectivity theorem. From the picture on \( A \), \( \alpha \) would also be homotopic to an interval of \( p \) and of \( n \), so it gives a way to shorten \( n \). This is possible at most for a part of \( n \) of length less than a constant times \( a(\lambda, \mu) \).

The polygonal curve \( n \) on \( P_\lambda \) has the property that if there is any interval \( I \) of \( n \) with endpoints in \( P_\lambda(\mu) \) which is homotopic rel endpoints to \( P_\lambda(\mu) \), then this interval is contained on a single leaf of \( \lambda \). The map from components of the thin set of \( P_\lambda \) to the thin set of \( N \) is injective. Therefore, two distinct segments of \( p \) which are in the thin part of \( P_\lambda \) cannot be close together on \( A \).

Therefore, all but a constant times \( a(\lambda, \mu) \) of the length of \( p \) runs close to \( \mu' \) on \( A \). This implies that

\[
\text{length}_{P_\lambda}(\mu) \leq \text{length}(p) \leq \text{length}_N(\mu) + Ca(\lambda, \mu),
\]

as desired.

Theorem 3.3 is false without the stipulation that closed leaves of \( \lambda \) have length at least \( \epsilon \) in \( P_\lambda \), or equivalently, in \( N \). It is possible to construct examples where there is a pleated surfaces which has a nearly
180° fold along a very short closed leaf of \( \lambda \). Any geodesic of \( P_\lambda \) which intersects such a fold can be shortened considerably in \( N \).

The limiting case of complete inefficiency is for surface groups which have an accidental parabolic. Take \( \lambda \) to be a lamination having a closed leaf whose homotopy class is parabolic. Then there is no actual pleated surface \( P_\lambda \), but there is still a type of “pleated surface with nodes”, which goes off to \( \infty \) at the parabolic curve. Any homotopy class passing through the parabolic curve has infinite length on \( P_\lambda \), but finite length in \( N \).

Here is a slightly stronger version of the theorem, which might be useful sometime:

**Theorem 3.5** (Restricted efficiency of pleated surfaces). Let \( \bar{S} \) be a fixed compact surface, possibly with boundary. For any \( \epsilon > 0 \) there is a constant \( C < \infty \) such that the following holds:

Let \( \lambda \) be any finite maximal lamination on \( S \).

Let \( N \) be any element of \( \text{AH} (\bar{S} \times I, \partial \bar{S} \times I) \). Let \( \lambda_1 \) be the measured lamination which is the sum of the closed leaves of \( \lambda \) of length less than \( \epsilon \) in \( N \), and let \( \lambda_2 \) be the union of closed leaves of length zero in \( N \), that is, closed leaves which are parabolics.

Then there is a pleated surface \( f_\lambda : P_\lambda \to N \) pleated along \( \lambda \setminus \lambda_2 \), where \( P_\lambda \) is a complete hyperbolic structure on \( S - \lambda_2 \). For any \( \mu \in \text{ML}_0(S) \) such that \( i(\mu, \lambda_1) = 0 \),

\[
\text{length}_N(\mu) \leq \text{length}_{P_\lambda}(\mu) \leq \text{length}_N(\mu) + Ca(\lambda, \mu).
\]

**Proof of 3.5.** The proof of 3.3 works also for this statement. The absence of short geodesics of \( \lambda \) was used only in the construction of a polygonal approximation to the curve \( \mu \), such that no jumps take place inside \( N_{\text{thin}} \). The difficulty is averted if \( \mu \) does not enter the components of \( P_\lambda \) thinning short closed leaves of \( \lambda \). But any simple closed geodesic remains entirely outside such a thin neighborhood, unless it crosses it.

**Remark.** In fact, both Theorem 3.3 and the more general version, Theorem 3.5, can be extended to curves on the surface which are not simple, and to more general measured laminations tangent to the geodesic flow in \( \mathbb{P}(S) \), which do not project to simple laminations on \( S \). The alternation number makes just as much sense, and the polygonal approximations work equally well. A non-simple geodesic may enter the neighborhood of a short geodesic or a neighborhood of a cusp, wind around many times, and then exit through the boundary component of the thin set where it entered. In this case, one can again push all the jumps between leaves of \( \lambda \) in a polygonal approximation out of the thin set, one by one, adding only a bounded multiple of the alternation number.

4. Existence of double limits of quasi-Fuchsian groups

Using the results of 3.1, we can now prove a good existence theorem for limits of surface group actions in \( \mathbb{H}^3 \). In particular, the theorem will show the existence of many doubly degenerate groups, that is, surface groups whose limit sets constitute the entire sphere. We will construct such groups as limits of quasi-Fuchsian groups where the conformal structures on the two components of the domain of discontinuity go to infinity in different directions.

**Theorem 4.1** (Double limit theorem). Let \( \mu \) and \( \mu' \) be any two laminations in \( \text{ML}_0(S) \) satisfying the condition that for all \( \nu \in \text{ML}_0(S) \), \( i(\mu, \nu) + i(\mu', \nu) > 0 \).

Then for any sequence \( \{(g_i, h_i)\} \) in \( T(S) \times T(S) \) converging to \( (\mu, \mu') \) in \( \bar{T}(S) \times \bar{T}(S) \), the sequence of quasi-Fuchsian groups

\[
\{qf(g_i, h_i)\}
\]

has a subsequence whose associated manifolds \( \mathbb{H}^3 / qf(g_i, h_i) \) converge algebraically to a point \( N_\infty \in \text{AH}(\bar{S} \times I, \partial \bar{S} \times I) \).

**Proof of 4.1.** First we will prove the theorem in the case that \( \bar{S} \) has non-trivial boundary, since this lacks some of the complications.

In this case, we can choose a lamination \( \lambda \) of \( S \) which is the 1-skeleton of an ideal triangulation. Represent the projective classes of \( \mu \) and \( \mu' \) by measured laminations of the same names.

By Theorem 2.2 there are sequences of measured laminations \( \mu_i \to \mu \) and \( \mu_i' \to \mu' \) such that

\[
\text{length}_{\mu_i}(\mu_i) \to 0
\]
and

$$\text{length}_{h_i}(\mu'_i) \to 0.$$  

Let \( N_i \) be the quotient manifold of hyperbolic space by \( qf(g_i, h_i) \). From either 1.3 or 1.4, it follows that length \( N_i, (\mu_i) \to 0 \) and length \( N_i, (\mu'_i) \to 0 \). Let \( P_{\lambda,i} \) denote the \( \lambda \)-pleated surface in \( N_i \). We deduce from 3.3 that the lengths of \( \mu_i \) and \( \mu'_i \) on \( P_{\lambda,i} \) remain bounded as \( i \to \infty \). From 2.4 it follows that the hyperbolic structures of \( P_{\lambda,i} \) remain in a bounded portion of Teichmüller space. Therefore, generators of the group \( qf(g_i, h_i) \) remain bounded, since they move a base point of \( P_{\lambda,i} \) not further in \( N_i \) than in \( P_{\lambda,i} \). The theorem for the case that \( \partial \bar{S} \) is not empty follows.

To prove the theorem when \( S \) is a closed surface, we will show that for any infinite sequence \( \{N_i\} \) in \( \text{AH}(\bar{S} \times I, \partial \bar{S} \times I) \), we can always choose the lamination \( \lambda \) so that for an infinite subsequence the length of closed leaves of \( \lambda \) remains bounded above zero.

Since \( \lambda \) can be chosen to have only one closed leaf, which can be any simple geodesic on \( S \), we only need to show that not all lengths of simple geodesics tend to zero.

**Proposition 4.2** (Four not all small). Suppose that \( A \) and \( B \) are isometries of \( \mathbb{H}^3 \) which generate a discrete, non-elementary group. Then one of the four isometries \( A, B, AB \) or \( AB^{-1} \) is not conjugate to an isometry near the identity.

This can be proven by looking at \( \text{SL}(2, \mathbb{C}) \), which is the double cover of the group of isometries of \( \mathbb{H}^3 \) and studying traces. It is hard to keep track of the geometry, however, in studying traces. Instead, we will prove the proposition using pleated surfaces.

**Proof of 4.2.** We can map the universal cover of the three-punctured sphere \( M \) into \( \mathbb{H}^3 \), to represent the homorphism \( \pi_1(M) \to G(A, B) \). The three-punctured sphere has an ideal triangulation with two triangles, such that the three corners of each triangle go to different punctures. Choose an orientation for the axes of \( A, B \) and \( C = AB \) if these transformations are not parabolic. We can define an equivariant map from the universal cover of this triangulation to \( \mathbb{H}^3 \) by mapping each corner of a triangle to the positive endpoint of the corresponding axis in \( \mathbb{H}^3 \), or to the parabolic fixed point if it has no axis. The three corners of any triangle map to three distinct points, for otherwise the group would be reducible, hence elementary. This gives us a pleated surface, homeomorphic to a three-punctured sphere (but geometrically, more likely the interior of a pair of pants), in the quotient manifold or quotient orbifold.

The group can now be conveniently described in terms of the local geometry of the pleated surface. To each edge is associated a complex number, which may be thought of as the edge invariant for the tetrahedron formed by the four vertices of the two triangles which join at that edge. An edge invariant of 1 means that the one triangle is obtained from the other by a 180° rotation about the edge; an edge invariant of \(-1\) means that the two triangles are the same. Denote the edge invariant for the edge going from boundary component \( A \) to boundary component \( B \) by \( c \), and so on.

Let \( \alpha, \beta \) and \( \gamma \) be the complex translation invariants for \( A, B \) and \( C \); thus, \( \log |\alpha| \) is the translation distance of \( A \) (along its axis), and \( \arg(\alpha) \) is the rotation angle, etc. Then, by arranging an axis so its positive endpoint is at infinity in the upper half-space model, it is easy to see that

\[ bc = \alpha, \ ca = \beta, \text{ and } ab = \gamma. \]

If \( A, B \) and \( C \) are conjugate to isometries near the identity, then \( \alpha, \beta \) and \( \gamma \) are near 1. There are algebraically two possibilities for \( a, b \) and \( c \); they can all be near 1, or all be near \(-1\). But if they were all near \(-1\), then the two triangles across any edge would nearly coincide, and a coordinate system in \( \mathbb{H}^3 \) could be chosen so that \( A, B \) and \( C \) were simultaneously near the identity; this is impossible for a discrete, non-elementary group. Therefore, if \( \alpha, \beta, \) and \( \gamma \) are near 1, the edge invariants are near 1. This means that the triangles are nearly flat across the edges, so the group action is near that of a standard three-punctured sphere. But on a hyperbolic three-punctured sphere, \( AB^{-1} \) is not near the identity.

**Corollary 4.3** (Four curves not all short). There is an \( \epsilon \) such that for any \( N \in \text{AH}(\bar{S} \times I, \partial \bar{S} \times I) \), and any two simple closed curves \( \alpha \) and \( \beta \) on \( S \) such that \( i(\alpha, \beta) = 1 \), then the length in \( N \) of \( \alpha, \beta \) or one of the two simple curves obtained by cutting and pasting \( \alpha \) and \( \beta \) at their intersection point is greater than \( \epsilon \).
Proof of 4.4. In the fundamental group of $S$ based at the point of intersection of $\alpha$ and $\beta$, the homotopy classes of the four simple curves have the form $A, B, AB$ and $AB^{-1}$.

We claim that if a simple curve on $S$ has a short length in $N$, then it also has a small angle of rotation, so that its complex translation invariant is near 1. To establish the claim, construct a pleated surface containing a short closed geodesic $g$. From the thick-thin decomposition of $N$, we know that $g$ has a regular neighborhood in $N$ whose diameter goes to infinity as the length of $g$ goes to zero. If $g$ has rotation angle $\theta$, then the shortest curve homotopic to $g$ on the boundary of the solid torus of radius $R$ has length at least $\theta \sinh R$. In the covering of $N$ with fundamental group generated by $g$, $S$ is divided by $g$ into two semi-infinite cylinders, one of which intersects this torus in a cycle homologous to $\theta$. If $g$ has a canonical regular neighborhood of radius $R$, the area of the intersection of the pleated surface with this neighborhood is therefore greater than $\int_0^R \theta \sinh t \, dt = \theta(\cosh R - 1)$. Since the area of the pleated surface is bounded, if $R$ is large, then $\theta$ must be small.

Applying proposition 4.2 to the four elements of $\pi_1(S)$, we conclude that they are not all conjugate to isometries near the identity, and therefore their lengths in $N$ are not all small.

The proof of theorem 4.1 can now be completed in the case of a closed surface. Suppose we are given a sequence $\{(g_i, h_i)\} \in T(S) \times T(S)$ whose limit is $(\mu, \mu')$ as in the statement of the theorem. We choose two curves $\alpha$ and $\beta$ on $S$ intersecting in one point, and an infinite subsequence such that one of the four curves from the corollary, call it $\lambda_0$, always has length greater than $\epsilon$ in $qf(g_i, h_i)$. Extend $\lambda_0$ to a maximal lamination $\lambda$ by adding a finite number of leaves spiraling at both ends around $\lambda_0$, and consider the pleated surface $P_{\lambda,i}$ in the quotient manifold $N_i$.

As before, there are measured laminations $\mu_i \to \mu$ and $\mu'_i \to \mu'$ such that $\text{length}_{N_i}(\mu_i) \to 0$ and $\text{length}_{N_i}(\mu'_i) \to 0$. From 3.3 we deduce that the length of $\mu_i$ and $\mu'_i$ on $P_{\lambda,i}$ remain bounded as $i \to \infty$. From 2.4 it follows that the hyperbolic structures of $P_{\lambda,i}$ remain in a bounded portion of Teichmüller space, and hence that some subsequence of the $N_i$ converges algebraically to a limit $N_\infty \in \text{AH}(S \times I, \partial S \times I)$.

For easier reference, we collect in one place the main ingredients of this proof:

**Theorem 4.4 (Surfaces approximate three-manifolds).** For every surface $S$ equipped for reference with a complete hyperbolic structure $h$ of finite area, there is a constant $C$ such that for any hyperbolic three-manifold $N \in \text{AH}(S \times I, \partial S \times I)$, there is a complete hyperbolic structure of finite area $g$ on $S$ and an isometric embedding $f_\lambda$ of $S$ as a pleated surface in $N$, such that for all laminations $\mu \in ML_0(S)$,

$$\text{length}_h(\mu) \leq \text{length}_g(\mu) \leq \text{length}_h(\mu) + C\text{length}_h(\mu).$$

**Proof of 4.4.** Apply 4.3 to conclude that there are four specific laminations $\lambda_i$ such that in any element $N \in \text{AH}(S \times I, \partial S \times I)$, at least one of the four satisfies the hypotheses of 3.3. The alternation number $a(\lambda_i, \mu)$ is bounded by some multiple of $\text{length}_h(\mu)$, for each $i$; hence, the inequality.

Of course, an equivalent statement would be obtained if $\text{length}_h(\mu)$ were replaced by wordlength with respect to some fixed set of generators.

5. Hyperbolic structures for mapping tori

We will now prove the main theorem, 0.1 which we restate:

**Theorem 0.1.** Let $M^3$ be a compact 3-manifold (possibly with boundary) which fibers over $S^1$, and whose fiber is a compact surface of negative Euler characteristic.

Then the interior of $M$ either

(i): has a complete $\mathbb{H}^2 \times \mathbb{R}$ structure of finite volume, and can be described as a Seifert fibration over some hyperbolic 2-orbifold,

(ii): contains an embedded incompressible torus not isotopic to a boundary component, and splits along this torus into two simpler three-manifolds, or

(iii): (generic case) has a complete hyperbolic structure of finite volume.

Cases (i) and (ii) are not mutually exclusive, but (iii) excludes the other two cases.
Proof of 0.4. Let $M^3$ be a 3-manifold which fibers over the circle, and let $\phi$ be its monodromy. By 2.6, either $M_\phi$ has a complete, finite volume $\mathbb{H}^2 \times \mathbb{R}$ structure, is torus-reducible, or $\phi$ is pseudo-Anosov. What remains to be proven is that if $\phi$ is pseudo-Anosov, then the interior of $M_\phi$ admits a complete hyperbolic structure of finite volume.

Choose an arbitrary element $g \in T(S)$, and consider the sequence of quasi-Fuchsian groups $\Gamma_i = qf(\phi^{-n}g, \phi^{n}g)$. By 2.3, the sequence $\{\phi^{-n}g, \phi^{n}g\}$ tends to the limit $(\mu_s, \mu_u)$ in $\overline{T(S)} \times \overline{T(S)}$, where $\mu_s$ and $\mu_u$ are the stable and unstable laminations of $\phi$. This pair of laminations satisfies the hypothesis of 1.1, so there is a subsequence of the $\Gamma_i$ which converges to a group $\Gamma_\infty$.

Note that all the hyperbolic surfaces $\phi^n(g)$ are isometric: they differ only by their parametrization, or “marking”. Furthermore, $\phi$ acts geometrically as the same map on each of the surfaces, since it commutes with any iterate of itself. In particular, $\phi$ has a fixed quasi-isometric constant on each of the hyperbolic surfaces $\phi^n(g)$.

If we lift these actions of $\phi$ to the two components of the domain of discontinuity for $\Gamma_i$, we obtain a homeomorphism $h_i$ of the sphere, equivariant twisted by $\phi$ with respect to $\Gamma_i$. (That is, $h(\gamma(x)) = \phi(\gamma)(h(x)))$. This homeomorphism is quasiconformal (by general principles), and its quasiconformal constant is bounded uniformly over $i$, since it is uniformly bounded on the domain of discontinuity, which has full measure: in fact, the best constant is a function only of the pseudo-Anosov constant.

The groups $\Gamma_i$ are really only defined up to conjugacy by Moebius transformations. To pin them down as actual groups, pick three non-commuting hyperbolic elements, and arrange that the attracting fixed points of these three elements are at 0, 1 and $\infty$ on $S^2_\infty$. The actual groups now converge, and the points $\phi(0)$, $\phi(1)$, and $\phi(\infty)$ also converge to three distinct fixed points of the limits of the images of the three elements by $\phi$. (In the limit, the elements conceivably might not be hyperbolic, but that is not a difficulty).

The sequence of maps $h_i$ is uniformly quasiconformal, and converges on the three points 0, 1 and $\infty$ to three distinct limits. Therefore, the sequence is equicontinuous, and there is a convergent subsequence. The limit is a quasiconformal homeomorphism $h_\infty$ which is equivariant twisted by $\phi$ with respect to $\Gamma_\infty$.

We claim that the limit set of $\Gamma_\infty$ is all of $S^2_\infty$. One proof of that fact goes as follows. By the Ahlfors finite area theorem, the quotient of $D_{\Gamma_\infty}$ by $\Gamma_\infty$ is conformally equivalent to a (not necessarily connected) hyperbolic surface $A$ of finite area. The homeomorphism $h_\infty$ descends to a homeomorphism of this quotient surface. If there were a component of $A$ which had a cusp, there would be a finite number of cusps, which were permuted by $h_\infty$; this would contradict the fact that $\phi$ is pseudo-Anosov. The other possibility for a component of $A$ is that it is a finite sheeted covering space of $S$. If such existed, then the conformal structure induced on a finite sheeted cover of one component of the domain of discontinuity would converge; that is absurd.

Now we can apply theorem 1.1 to conclude that the quasiconformal homeomorphism $h_\infty$ actually induces a conformal automorphism of $\Gamma_\infty$ to itself. This defines a representation of $\pi_1(N)$ in the group of conformal homeomorphisms of $S^2_\infty$, where $\pi_1(S)$ acts as $\Gamma_\infty$ and $\phi$ acts as $h_\infty$. We claim that the group is discrete and faithful. To see this, consider the construction in terms of quotient manifolds. First form the quotient $N_\infty$ of $\mathbb{H}^3$ by $\Gamma_\infty$: it is homotopy equivalent to the surface $S$. The conformal map $h_\infty$ extends to an isometry of hyperbolic space, which descends to an action as an isometry on $N_\infty$. But the group of isometries of any hyperbolic manifold with non-elementary fundamental group is easily seen to be discrete, so therefore the action of $\pi_1(N)$ is discrete.

By a theorem of Stallings [Sta62], the quotient hyperbolic manifold is in fact homeomorphic to $N$. 

6. ON LIMITS AND LIMITING BEHAVIOR OF SURFACE GROUPS

We have seen that if a sequence $\{(g_i, h_i)\}$ in $T(S) \times T(S)$ converges to a point at infinity $(\mu, \mu')$ far from the diagonal, then the corresponding sequence of quasi-Fuchsian groups has at least a subsequence that converges algebraically. On the other hand, the quasi-Fuchsian groups corresponding to a diagonal sequence $(g_i, g_i)$ which tends to infinity can never converge.

Let us reformulate this by describing a compactification of $\text{AH}(\overline{S} \times I, \partial \overline{S} \times I)$ which adjoins abstract limits to all non-convergent sequences.

A point $N \in \text{AH}(\overline{S} \times I, \partial \overline{S} \times I)$ gives rise to a continuous length function

$$\text{length}_N = l_N : \text{ML}_0 \to \mathbb{R}$$
which we can think of as defining a point \(l^*(N)\) in the vector space \(R^{ML_0}\). To consider how ratios of lengths behave in hyperbolic manifolds, we project \(R^{ML_0}\) into its associated projective space \(P(R^{ML_0})\), and write

\[P l^* : AH(\hat{S} \times I, \partial \hat{S} \times I) \to P(R^{ML_0})\]

for the composition. Let \(P(R^{ML_0})\) have the quotient topology of the compact-open topology on \(R^{ML_0}\).

**Theorem 6.1** (Closure compact). The closure of the image of \(P l^*\) is compact. The added set \(\overline{\text{Image } P l^* - \text{Image } P l^*}\) is the same as for Teichmüller space: it is \(PL_0\), as it is embedded in \(P(R^{ML_0})\) via geometric intersection number.

In other words, for any sequence of hyperbolic three-manifolds \(N_i\) in \(AH(\hat{S} \times I, \partial \hat{S} \times I)\), either there is an algebraically convergent subsequence, or there is a subsequence such that the ratios of lengths of simple closed curves (and laminations) all converge, and there is a sequence of Fuchsian groups with the same limiting ratios!

**Proof of 6.1.** The proof is a corollary of what we have already done. Apply Theorem 4.4 to find a uniformly efficient pleated surface in each member of the sequence of quotient manifolds \(N_i\), that is, satisfying

\[\text{length}_{N_i}(\mu) \leq \text{length}_{P_\lambda}(\mu) \leq \text{length}_{N_i}(\mu) + Ca(\lambda, \mu).\]

If some of the lengths tend to infinity, the deviation bounded by \(Ca(\lambda, \mu)\) becomes irrelevant in projective space. There is either a subsequence that converges algebraically, or, a subsequence such that the sequence of hyperbolic structures on \(S\) defined by \(P_\lambda\) converge to a point in \(PL_0\). The sequence of length functions defined by \(N_i\) then converge to the image point in \(P(R^{ML_0})\).

The limiting behavior of the length functions in \(P(R^{ML_0})\) is only part of the story. Sometimes one needs to understand infinite nonconvergent sequences of Fuchsian or quasi-Fuchsian groups in a more refined way.

**Theorem 6.2** (Converge on subsurface). Let \(\{N_i\}\) be any sequence of elements of \(AH(\hat{S} \times I, \partial \hat{S} \times I)\). Then there exists a subsequence \(N_{i_{c(j)}}\) and a possibly empty, possibly disconnected subsurface \(S'\) of \(S\) with incompressible boundary, such that

(a) For each component \(S'_i\) of \(S'\), the sequence of representations of \(\pi_1(S'_i)\) in \(\text{Isom}\, \mathbb{H}^3\) coming from \(N_{i_{c(j)}}\) converges up to conjugacy.

(b) If \(\Gamma\) is any nontrivial subgroup of \(\pi_1(S)\) such that for some subsequence of indices \(\{c(d(j))\}\) of \(\{c(j)\}\), its sequence of representations converges up to conjugacy, then \(\Gamma\) is conjugate to a subgroup of \(\pi_1(S'_i)\) for some \(i\).

The subsequence \(\{c(j)\}\) is a sequence of maximal convergence, and the subsurface \(S'\) is the surface of maximal convergence associated to that subsequence. There may be various subsequences of maximal convergence contained in a given sequence, associated with different subsurfaces.

**Proof of 6.2.** This theorem is reduced to the Fuchsian case, by an application of 3.3 and 4.3.

To prove the theorem in the Fuchsian case, that is, for a sequence of hyperbolic structures on \(S\), we proceed as follows.

To any collection \(A\) of homotopy classes of closed curves on a surface is associated a subsurface \(\text{ls}(A)\), unique up to isotopy, which is the least surface with incompressible boundary that contains \(A\). It is characterized by the properties that

(a) Every \(\alpha \in A\) is isotopic to a curve in \(\text{ls}(A)\)

(b) No component of \(\text{ls}(A)\) is an annulus isotopic into another component,

(c) For any other subsurface \(R\) of \(S\) satisfying (a), \(\text{ls}(A)\) is isotopic to a subsurface of \(R\), and

(d) \(\partial \text{ls}(A)\) is incompressible.

Here is one construction for \(\text{ls}(A)\). Choose a hyperbolic structure on \(S\), and represent each element \(\alpha \in A\) by its geodesic, or a small horocycle in the case that the element is parabolic. Now fill in each component of the complement which is a disk. Also, fill in, one by one, any region of the complement which is an annulus one of whose boundary components is a component of the set so far constructed. Thicken the result, so that it is a subsurface. It is easy to verify that it has the desired properties.
We claim that if we have a sequence of hyperbolic structures of on a surface \( S \) and if the lengths of all homotopy classes of curves in a collection \( A \) remain bounded, then the sequence of representations of the fundamental group of any component of the surface \( \text{ls}(A) \) converges.

The basic observation is that two simple closed geodesics \( \alpha \) and \( \beta \) intersect in a combinatorial pattern which depends only on their homotopy classes. If the lengths of \( \alpha \) and \( \beta \) remain bounded in a sequence of hyperbolic structures, then \( \alpha \cup \beta \) forms a 1-skeleton for the surface \( \text{ls}(\{\alpha, \beta\}) \), so the representations of the fundamental group of that entire subsurface remain bounded, and a subsequence can be chosen so that the sequence of representations converge.

Unfortunately, the combinatorial pattern of the intersection of three or more simple closed geodesics is not invariant: it depends on the hyperbolic metric. However, an inductive argument can be used to show that if \( A \) is any collection of simple closed curves on a surface whose lengths all remain bounded, then the sequence of representations of the fundamental group of any component of \( \text{ls}(A) \) converges. In fact, if \( S_1 \) is the least subsurface for some finite subcollection \( A_1 \), and if \( \alpha \) is an additional curve, then it is possible to find a collection \( A_0 \) of disjoint simple curves in \( \text{ls}(A_1) \) such that \( \text{ls}(\{A_0\} \cup \alpha) = \text{ls}(\{A_1\} \cup \alpha) \). Note that \( A_0 \) need not be a subset of \( A \). The geodesics in the \( \{A_0\} \cup \alpha \) provide a bounded 1-skeleton for this surface.

This takes care of collections of simple curves, but not of general curves. For any finite collection \( F \) of homotopy classes of loops on a surface, there is a finite-sheeted covering of the surface where the induced covers of each loop in the collection are embedded. This is a consequence of a theorem of Peter Scott [Sc078], although this particular case is easier than the general theorem. This implies that the sequence of representations of a subgroup of finite index in the fundamental group of any component of \( \text{ls}(F) \) remains bounded if the lengths of elements of \( F \) remain bounded, so a subsequence can be chosen so that these representations converge. But hyperbolic transformations of \( \mathbb{H}^2 \) are uniquely divisible, so the representation of a subgroup of finite index determines the representation of the full group.

Theorem 6.3 is related to proposition 2.4, and they can be combined into a single statement which generalizes 4.1. If \( \mu \) and \( \nu \) are geodesic laminations, let us say that \((\mu, \nu)\) binds a subsurface \( R \subset S \) if \( \partial R \) is incompressible in \( S \), each component of \( R \) has negative Euler characteristic, and \((\mu, \nu)\) binds \( R \) equipped with any complete hyperbolic metric of finite area.

In this situation, if \( \mu \subset \mu_1 \) and \( \nu \subset \nu_1 \), we will also say that \((\mu_1, \nu_1)\) also bind \( S \). The purpose of this is to allow for components of one lamination which do not intersect the other.

**Theorem 6.3 (Laminations bind subsurface).** If \( N_i \in \Lambda \mathcal{H}(\bar{S} \times I, \partial \bar{S} \times I) \) is a sequence of hyperbolic manifolds and \( \mu_i \rightarrow \mu, \nu_i \rightarrow \nu \in \text{ML}_0 \) are sequences of measured laminations such that \( \text{length}_{N_i}(\mu_i) \) and \( \text{length}_{N_i}(\nu_i) \) remain bounded, then if \((\mu, \nu)\) bind a subsurface \( R \subset S \), there is a subsequence such that the sequence of representations of the fundamental group of \( R \) converges.

**Proof of 6.3.** As before, we need only prove the theorem in the Fuchsian case, since under the hypotheses we can construct a pleated surface on which both \( \mu \) and \( \nu \) remain bounded in length.

We will exploit the fact that the combinatorial pattern of intersections of \( \mu \) with \( \nu \) does not depend on a hyperbolic structure on \( S \). We will phrase the proof for the case that \( \mu \) and \( \nu \) have no measure concentrated on individual leaves. The general case works in the same way, but it is awkward to word the proof to apply to all cases at once.

Let \( \mu_0 \) and \( \nu_0 \) be components of \( \mu \) and \( \nu \), respectively, which actually intersect each other. It suffices to construct a subsequence which works for the surface which \( R_0 = (\mu_0, \nu_0) \) binds, in view of 5.2.

Let \( x \) be a point in \( \mu_0 \cap \nu_0 \). A small open and closed neighborhood \( V \) of \( x \) in the intersection can be formed which has the structure of the product of the local leaf space for \( \mu \) with the local leaf space for \( \nu \). \( V \) is contained in a quadrilateral \( Q \) on \( S \) whose sides are leaves of \( \mu \) and \( \nu \).

Every leaf in \( \mu_0 \) and in \( \nu_0 \) eventually intersects \( Q \), in each direction. The leaves of \( \mu_0 \) and of \( \nu_0 \) minus \( Q \) therefore group into a finite number of bands which remain roughly parallel. Each of the bands has a positive transverse measure. We will work with extended bands, where both ends are extended across \( Q \).

Consider the set of hyperbolic structures on \( S \) for which the length of \( \mu_0 \) and the length of \( \nu_0 \) both are less than a constant \( C \). Then the average length of a leaf in one of the extended bands is less than \( 2C \) divided by the transverse measure of the band — a uniform constant. Select one leaf from each extended band whose length does not exceed the average length.

The union of the selected leaves forms a 1-skeleton for the subsurface \( R_0 \). The combinatorial type outside of \( Q \) is fixed; \( Q \) is simply connected, and inside \( Q \), any two points can be connected by a path of bounded
length. This proves that the sequence of representations of $\pi_1(R_0)$ has a convergent subsequence, under the hypothesis that the lengths of $\mu$ and $\nu$ remain bounded.

This proof generalizes without difficulty to the case where $\mu_i \to \mu$ and $\nu_i \to \nu$, with the lengths of $\mu_i$ and $\nu_i$ bounded by a constant $C$. To do this, consider a neighborhood in $S$ slightly bigger than $Q$, but still intersecting the same leaves of $\mu$. For each band of leaves of $\mu$ there is a nearby band of nearly parallel leaves of $\mu_i$, whose total transverse measure cannot decrease suddenly. There may also be new bands of small measure hitting $Q$, but we don’t need to use them. As before, we can construct a 1-skeleton for $R_0$, made up from leaves of $\mu_i$ and $\nu_i$.

The proof is completed by letting $\mu_0$ and $\nu_0$ range over all components of $\mu$ and $\nu$ and applying \ref{thm:6.5} (An alternative would be to use several quadrilaterals).

We give some more examples illustrating how the double limit theorem can be applied to construct Kleinian groups.

**Theorem 6.4** (Accidental parabolics). Let $\alpha$ and $\beta$ be collections of disjoint simple closed geodesics on $S$. If $\alpha \cap \beta = \emptyset$, then there is a sequence of elements of $\text{QF}(S)$ converging to an element of
\[
\text{AH}(S \times I, \partial S \times I \cup \text{Nhbd}(\alpha) \times \{0\} \cup \text{Nhbd}(\beta) \times \{1\}) \subset \text{AH}(\bar{S} \times I, \partial \bar{S} \times I).
\]

**Proof of 6.4.** Construct a sequence $\{g_i\}$ of hyperbolic structures on $S$ such that it is a sequence of maximal convergence associated with the surface $A = S - \text{Nhbd}(\alpha)$.

Construct a similar sequence $\{h_i\}$ for $\beta$, converging on the surface $B = S - \text{Nhbd}(\alpha)$.

Now apply \ref{thm:6.2} to $\text{QF}(g_i, h_i)$. The subsurface of maximal convergence must contain both $A$ and $B$, so it must be all of $S$. Thus there is a subsequence converging to a limiting group $\Gamma$.

Each of the curves in $\alpha$ and $\beta$ is parabolic, so the manifold is in $\text{AH}(S \times I, \partial S \times I \cup \text{Nhbd}(\alpha) \times \{0\} \cup \text{Nhbd}(\beta) \times \{1\})$.

Here is a variation of this construction, where $\alpha$ and $\beta$ are laminations which do not necessarily bind $S$, or even a subsurface of $S$. The statement is more complicated because not every measured lamination whose length in $N_i$ remains bounded is contained in the subsurface of convergence.

**Proposition 6.5** (Convergence on non-binding laminations). Let $\lambda_i \ [i = 1, 2]$ be measured laminations on $S$. For any complementary region of $\lambda_i$ and any ideal polygonal curve on its boundary which is homotopically non-trivial, suppose that there is a closed leaf of $\lambda_i$ in its homotopy class.

Then if $\lambda_1$ has no leaves in common with $\lambda_2$, there is a sequence of elements in $\text{QF}(S)$ converging to an element of $N \in \text{AH}(\bar{S} \times I, \partial \bar{S} \times I)$ in which $\lambda_1$ and $\lambda_2$ are unrealizable, and
\[
\text{length}_N(\lambda_1) = \text{length}_N(\lambda_2) = 0.
\]

**Proof of 6.5.** Let $A_i \subset S$ be the possibly empty subsurface formed as the union of all components of $S - \lambda_i$ whose boundary consists of closed geodesics. Let $g^i_j \ [i = 1, 2]$ be a sequence of maximal convergence of hyperbolic structures on $S$ such that $A_i$ is its subsurface of maximal convergence, and the length of $\lambda_i$ tends to zero.

Form a subsequence of maximal convergence for the sequence of quasi-Fuchsian groups $\text{QF}(g^1_j, g^2_j)$.

Let $\mu_i \subset \lambda_i$ be the union of components which intersect $\lambda_i$ where $j \neq i$. Then $(\mu_1, \mu_2)$ bind a certain subsurface $S_0$ of $S$. The subsurface of maximal convergence must contain $A_1$, $A_2$, and $S_0$. The only possibility is all of $S$.

\[\square\]

7. Infinitely generated geometric limits

In this section, we will describe a construction for a different kind of limit for surface groups, of a type first discovered by Troels Jørgensen — geometric limits of sequences of quasi-Fuchsian manifolds — whose fundamental groups are not finitely generated.

One reason for studying geometric limits, as well as algebraic limits, is that only the geometric limit captures the quasi-isometric information of the approximating manifolds. So far, there are three known invariants which can distinguish Kleinian groups (at least those groups which are not free products): the topology, the conformal structure on the quotient of the domain of discontinuity, and the ending lamination
(defined in Ch. 8 and 9 of Fls79, and now known to exist in a general setting, by the main result of Bon86). If we can precisely analyze the quasi-isometric structure of surface groups, we can probably resolve the question of whether these three invariants are sufficient to determine a group. The most difficulty seems to come from surface groups which have arbitrarily short geodesics — that is precisely the situation which we will deal with in this section.

Let $S$ be a hyperbolic surface of finite area. We will study finite and infinite collections $C$ of disjoint non-trivial simple curves on levels in $S \times \mathbb{R}$, $C = \{\gamma_i = \alpha_i \times \{t_i\}\}$, up to isotopy through similar systems of curves. We assume that no two curves of $C$ are isotopic. We also assume that $C$ is closed, so that the set of levels involved is discrete. Given such a collection $C$, define $M_C = S \times \mathbb{R} - \bigcup C$.

There is a partial ordering on $C$, defined by $\gamma_i \prec \gamma_j$ if $t_i < t_j$, and if further, this inequality remains true after an arbitrary isotopy. Clearly, if $i(\alpha_i, \alpha_j)$ is not zero, then the two curves $\gamma_i$ and $\gamma_j$ are comparable, and ordered according to $t_i$ and $t_j$. Furthermore, if there is a finite sequence $\{c\}(k)$ such that $i(\alpha(c(k)), \alpha(c(k + 1))) > 0$, and $t_{c(k)} < t_{c(k+1)}$, then the elements of the sequence $\gamma_{c(k)}$ are in ascending linear order.

Conversely, if there is no such chain connecting two elements of $C$ (in either order), then they are incomparable. Any set of mutually incomparable elements is isotopic to a single level. The maximum size of such a set is bounded as a function of the topology of $S$. On the other hand, it might happen that some element of $C$ is incomparable to an infinite collection of other elements.

If $\gamma_1 \prec \gamma_2$, and if there is no intervening element $\gamma_3$ such that $\gamma_1 \prec \gamma_3 \prec \gamma_2$, then $\gamma_2$ is a successor of $\gamma_1$. The partial ordering is generated by the successor relation, as its transitive closure. The successors of an element are incomparable, so the number is bounded as a function of the topology of the surface. The partial ordering can be represented by a directed acyclic graph, with an edge joining each element to each of its successors.

A cut of $C$ is a partition of $C$ into two parts $L$ and $U$, such that

1: no element of $L$ is greater than any element of $U$,
2: every linearly ordered subset of $L$ contains a maximum, and
3: every linearly ordered subset of $U$ contains a minimum.

The cuts themselves have a partial order, with $(L, U) \prec (L', U')$ if $L$ is a proper subset of $L'$.

**Proposition 7.1** (Surfaces cut). For every cut $K = (L, U)$ of $C$, there is an isotopy of $C$ so that $L$ consists of the curves below $S \times \{0\}$, and $U$ consists of the curves above $S \times \{0\}$. There is associated to $K$ an isotopy class $S_K$ of surfaces in $M_C$ which separates $L$ from $U$, and is isotopic in $S \times \mathbb{R}$ to $S$. Conversely, every isotopy class of embeddings of $S$ which project homeomorphically under the map $M_C \to S$ defines a cut.

**Proof of 7.1.** There can originally be only be a finite number of elements of $L$ above $S \times 0$: if there were an infinite number, then there would be an infinite sequence which is physically ascending, therefore non-descending in the partial order. But the property that the maximal size of pairwise incomparable set is bounded implies that every infinite non-descending sequence contains an infinite ascending subsequence.

Similarly, there are only a finite number of elements of $U$ below $S \times 0$.

We can repeatedly move the physically lowest element of $L$ which is above $S \times 0$, until there are none left, because the elements of $U$ cannot interfere. Then we can do the same process for $U$, proving the proposition.

After the isotopy of $C$, $S \times 0$ serves as the surface $S_K$. Because all isotopies of the system of curves can be arranged so that they involve only vertical motion, the isotopy class of $S_K$ is well-defined.

Note that the partial ordering on cuts corresponds to the partial ordering on surfaces: if $A$ and $B$ are two surfaces, $A \prec B$ if $B$ can be isotoped below $A$, but $A$ and $B$ are not isotopic.

**Theorem 7.2** (Drill holes). Let $C$ be a collection of curves in $S \times \mathbb{R}$ as above. Suppose that $C$ has the property that for every non-trivial simple closed curve $\beta$ on $S$, if $C$ is unbounded above then the set of $\alpha \times t \in C$ such that $i(\alpha, \beta) > 0$ is unbounded above, and if $C$ is unbounded below then the set of $\alpha \times t \in C$ such that $i(\alpha, \beta) > 0$ is unbounded below.

Then there is a sequence of hyperbolic manifolds in $QF(S)$ whose geometric limit is a manifold $N$ homeomorphic to $M_C = S \times \mathbb{R} - \bigcup C$. In particular, if $C$ is infinite, $\pi_1(N)$ is not finitely generated.

**Remarks.** A similar analysis would seem to work if the hypothesis on $C$ were dropped, but the picture is somewhat different. In the present situation, the new cusps in the geometric limit are all $\mathbb{Z} \times \mathbb{Z}$-cusps; in the more general situation, there would also be new $\mathbb{Z}$-cusps.
The proof of this theorem is probably of more interest than the statement; it gives techniques for finding when there are short geodesics in a hyperbolic manifold with the fundamental group of a surface. Eventually, it would be nice to have a method so that, given two conformal structures at infinity, and given any measured lamination on the surface, we can say, to within some bounded factor, what is the length of the lamination in the resulting quasi-Fuchsian manifold. A prerequisite is probably a better understanding of the geometry of the mapping class group of a surface, and of the various geometries of Teichmüller space.

There is, in fact, a cheap proof of the current theorem at least for many families of curves $C$. Here, in outline, is how it goes. First, construct by some device a hyperbolic manifold homeomorphic to $M_C$. For example, suppose that $C$ can be isotoped to the form that on each level where a curve in $C$ occurs, there are enough other elements to fill up the surface. Then we can construct a geometrically finite surface group corresponding to the region between any two such levels, such that the curves above and below are parabolic, by citing 6.4. All the stabilizers of domains of discontinuity are three-punctured spheres, which implies that the convex core of these surface groups has totally geodesic boundary, each component of which is isometric to a standard three-punctured sphere. Glue together three-punctured spheres, as appropriate, to obtain the limit.

Once a limit manifold is constructed, it is possible to work backwards, at least in a case such as the one described, with the aid of the theory of hyperbolic Dehn filling. Throw away all the pieces which have been assembled, except for a finite number in the middle; then hyperbolic Dehn filling can be performed, to obtain a quasi-Fuchsian group.

Alternatively, one can use the general existence of hyperbolic structures on Haken manifolds to obtain the manifolds with finitely many stages (to be proven in part IV of this series), and use the deformation theory for three-manifolds which have cylinders, (to be presented in part III of this series) to conclude that as more and more stages are added, there is a geometric limit homeomorphic to $M_C$. Each of the finite stages is a limit of quasi-Fuchsian groups, using hyperbolic Dehn filling, and one can diagonalize to show that $M_C$ itself is a geometric limit.

Rather than taking an external approach such as this, we will give an internal proof: we will begin with the quasi-Fuchsian groups, and find enough ways to analyze their quasi-isometric geometry to understand the nature of the limit.

Proof of 7.4. The construction, in its geometric form, goes as follows.

Index the elements of $C$ by an interval of positive and negative integers including 0, so that the the physical height of the curves is monotone in the indices.

For any curve $\gamma$ in $S \times \mathbb{R}$ which lies on a level, let $A_\gamma$ be an annular neighborhood of $\gamma$ which lies on the same level. If we slit open $S \times \mathbb{R}$ along $A_\gamma$, and then reglue the two sides of the cut by a power of a Dehn twist, we obtain a new manifold which is still homeomorphic to $S \times \mathbb{R}$. However, there are two natural isotopy classes of homeomorphisms to the original space: there is one homeomorphism which is the identity below the level of $\gamma$, and another which is the identity above that level. This construction is a special case of Dehn surgery along $\gamma$.

Suppose we perform such a construction, using various powers $q(i)$ of Dehn twists (perhaps sometimes the 0th power), on all the elements of our set $C$. We obtain a new manifold $M_q$. For each cut $K$, there is an isotopy class of homeomorphisms $h(q, K)$ with the original model so that $SK$ maps as the identity. A pair of cuts $K$ and $K'$ determine an isotopy class of homeomorphisms $\phi(q, K, K')$ from $S$ to itself, which give the comparison: $h(q, K') \circ \phi(q, K, K') \simeq h(q, K)$.

We will arrange a sequence of quasi-Fuchsian manifolds so that the conformal structures on the top and bottom are constant up to diffeomorphism. Choose a conformal structure, and think of it as being on $S \times +\infty$ and $S \times -\infty$. We will choose the powers $q$ so that all but a finite number are 0. Perform Dehn surgery to obtain $M_q$. The homeomorphisms $h(q, K)$ all agree for $K$ sufficiently high; define this as $h(q, +\infty)$. Define $h(q, -\infty)$, similarly, and also extend the definition of $\phi(q, K, K')$ to allow $K$ or $K'$ to be $\pm \infty$. If $K$ is an arbitrary cut, then a quasi-Fuchsian group $\Gamma(q, K)$ is defined by pushing the conformal structures at the two ends of $M_q$ to $S \times I$ via the homeomorphism $h(q, K)$: that is,

$$\Gamma(q, K) = qf(\phi(q, +\infty, K)(g), \phi(q, -\infty, K)(g))$$.
The groups for the various cuts are all isomorphic, of course. Another way to think of this construction is that we have defined a sequence of representations
\[
\rho_q : \pi_1(M_C) \to \text{Isom}(\mathbb{H}^3),
\]
non-faithful with a quasi-Fuchsian image, and the parametrization by \(\pi_1(S)\) associated with the cut \(K\) is induced by the embedding of \(S\) in \(M_C\) as \(S_K\).

We will use sequences of powers \(q_i(i)\) which are 0 outside a finite interval of integers \([-n, m]\), and which are large within that interval. Given such a choice, what do the conformal structures at \(+\infty\) look like from the perspective of a surface \(S_K\)?

**Proposition 7.3** (Big twists). Let \(\alpha\) be a measured lamination which is an integer linear combination of simple closed curves, and let \(D\) be a diffeomorphism which is the composition of either \(\tau_{\gamma^n}\) or \(\tau_{\gamma^{-n}}\) where \(\gamma\) ranges over the components of \(\alpha\) and \(n\) is the weight of \(\gamma\).

Write \(\alpha = \sum \alpha(i)\), where \(\alpha(i)\) ranges over the components of \(\alpha\). If \(\lambda\) is any measured lamination, then
\[
\lim_{q \to \infty} \frac{1}{q} D^q(\lambda) = \sum i(\lambda, \alpha(i))\alpha(i)
\]
in \(\text{ML}_0(S)\).

If \(g\) is any hyperbolic structure on \(S\), then
\[
\lim_{q \to \infty} \frac{1}{q} D^q(g) = \sum \text{length}_g(\alpha_i)
\]
in \(\overline{T(S)}\).

**Proof of 7.3.** If \(\lambda\) is a simple closed curve, then the result of \(D^q\) applied to \(\lambda\) is to modify \(\lambda\) so that it winds \(n(i)q\) times around the support of \(\alpha(i)\), each time it intersects. Clearly the limit is as stated.

The case of \(\gamma\) could be proven in a similar way with the aid of train tracks, but it can also be logically derived, as follows, from the case of simple closed curves. Any measured lamination is determined by its intersection numbers with simple closed curves. To show that we have the right formula for a general lamination \(\lambda\), it suffices to show that the intersection numbers with simple closed curves have the correct limits.

For any measured lamination \(\lambda\), write \(A(\lambda) = \sum i(\lambda, \alpha(i))\alpha(i)\), the expected answer. We have
\[
i(\lambda, A(\lambda')) = i(A(\lambda), \lambda'),
\]
and also
\[
i(D^q(\lambda), \lambda') = i(\lambda, D^{-q}\lambda').
\]
Letting \(\lambda'\) range over simple closed curves, these equations imply that \(A(\lambda)\) has the correct limiting intersection with all simple closed curves, so it is the correct limit.

Similarly if \(g\) is a hyperbolic structure, define \(A(g) = \sum \text{length}_g(\alpha_i)\alpha_i\). Again, we have the symmetry \(i(A(g), \lambda) = \text{length}_g(A(\lambda))\). As before, this proves that \(A(g)\) is the right answer for the limit of \(D^q(g)\). \(\square\)

**Proposition 7.4** (Compose big twists). Let \(A = \{\alpha(i)|i = 1 \ldots n\}\) be a sequence of simple closed curves on \(S\).

If \(\mu \in \text{ML}_0\) is a measured lamination such that each of its components has positive intersection number with at least one element of \(A\), then for sufficiently large positive or negative integers \(\{q(i)|i = 1, \ldots, n\}\), the image of \(\mu\) by the composition
\[
(\tau_{\alpha(1)}^{q(1)} \circ \tau_{\alpha(2)}^{q(2)} \circ \ldots \circ \tau_{\alpha(n)}^{q(n)})(\mu)
\]
is near the simplex in \(\text{PL}_0(S)\) generated by those curves \(\alpha_i\) which do not intersect any earlier element of the sequence.

Similarly, if \(g\) is any element of \(T(S)\), then for sufficiently large integers \(\{q(i)|i = 1, \ldots, n\}\), the image of \(g\) by the composition
\[
(\tau_{\alpha(1)}^{q(1)} \circ \tau_{\alpha(2)}^{q(2)} \circ \ldots \circ \tau_{\alpha(n)}^{q(n)})(\mu)
\]
is near the simplex in \(\text{PL}_0(S)\) (considered as the boundary of Teichmüller space) generated by those curves \(\alpha_i\) which do not intersect any earlier element of the sequence.
Note that the first elements of any such subsequences of maximal length must have zero intersection number with all earlier elements in \( A \). Therefore, the first elements of any pair of maximal subsequences have zero intersection number, so that a linear combination of them makes sense as a measured lamination, and gives a well-defined projective class. The set of all linear combinations of them map to a simplex in \( \text{PL}_0(S) \).

**Proof of [7.4].** How can we measure distance from the simplex \( \Delta(\Gamma) \) spanned by a collection \( \Gamma \) of disjoint simple closed curves?

Define \( K(\Gamma) \) to be the set of measured laminations such that for every \( \gamma \in \Gamma \), \( i(\lambda, \gamma) = 0 \). An element \( [\mu] \in \text{PL}_0(S) \) is in \( \Delta(\Gamma) \) if and only if \( i(\lambda, \mu) = 0 \) for all \( \lambda \in K(\Gamma) \).

We can use a normalized version of this to define neighborhoods of \( \Delta(\Gamma) \). That is, for any lamination \( \lambda \in K(\Gamma) \), the function \( n_\lambda(\mu) = i(\lambda, \mu)/\text{length}(\lambda) \) depends only on the class \( [\mu] \in \text{PL}_0(S) \). A neighborhood for \( \Delta(\Gamma) \) is defined by picking a finite collection of such \( \lambda \), and requiring that the functions \( n_\lambda \) have value less than \( \epsilon \).

The proof works by induction. Let \( \mu \) be as hypothesized. We start with the last \( i \) such that \( i(\alpha_i, \mu) > 0 \). The image of \( \mu \) by a large power of this Dehn twist is near \( \alpha_i \), by [7.3].

Define \( \Gamma_j \) to be the set of \( \alpha_k \) where \( j \leq k \), and for no \( j \leq k \) is \( i(\alpha_i, \alpha_k) > 0 \). Assume by induction on \( i - j \) that the image \( \mu_j \) of \( \mu \) by the \( j \) through \( n \) terms of the composition is near \( \Delta(\Gamma_j) \). Consider the result of applying the \( q - 1 \)st power of the Dehn twist about \( \alpha_{j-1} \) to \( \mu_j \). If the intersection number of \( \mu_j \) with \( \alpha_{j-1} \) is not too small compared with the length of \( \mu_j \), then according to [7.3] the image will be close to \( \alpha_{j-1} \).

Otherwise, \( \mu_j \) is already near the simplex \( \Gamma_{j-1} \). The length of a lamination can be estimated to within a constant factor as the sum of the intersection numbers with any finite collection of simple closed curves \( C \) such that \( \text{ls}(C) = S \). Choose \( C \) so that it has at least one element which intersects any component of \( \Gamma_{j-1} \) but none of the others. Then, if \( \mu_j \) is close to the subsimplex of \( \Gamma_{j-1} \) opposite \( \alpha_j \), we see that Dehn twists about \( \alpha_j \) cannot diminish the length of \( \mu_j \) beyond a bounded factor, since these twists do not affect certain of the intersection numbers which at the beginning contribute a significant fraction of the estimate of the length of \( \mu_j \). Dehn twists about \( \alpha_j \) do not affect the intersection numbers with \( \lambda \in K(\Gamma) \), so they cannot increase the functions \( n_\lambda \) beyond a bounded factor. We conclude that \( \mu_{j-1} \) is near \( \Delta(\Gamma_{j-1}) \).

The proof of the assertion for a metric \( g \) is similar, but slightly simpler, since we can start the induction at the last term in the composition. The initial step follows form [7.3]. One can measure a neighborhood of a lamination in compactified Teichmüller space in the analogous way. The easiest way to deal with the normalization is to choose a set of curves whose least subsurface is all of \( S \), and normalize a metric by multiplying with a constant which makes the sum of their lengths equal to 1.

A more elegant way is to extend the function \( \text{length}_g(\lambda) \) to a function \( \text{length}_h(h) \), where \( h \in T(S) \), with the aid of “random geodesic” for \( g \) and for \( h \). The length of the \( h \)-random geodesic in the metric \( g \) is the desired function. This quantity is also the intersection number between the random geodesics in the \( h \) metric and the \( g \) metric, so it is symmetric in \( g \) and \( h \). See [Thur] for details.

The subsequent steps are identical to the case for a lamination, since the image of \( g \) is near a lamination.

The first phase of the proof of Theorem [7.2] will be to show that there is a subsequence such that \( \rho_k \) converges.

Before proceeding in general, it is worth describing how this phase of the proof goes in some relatively simple cases.

First consider the case that \( S \) is a punctured torus. Then \( < \) is actually a linear ordering. Consider the surface at height \( n + 0.5 \) — label this cut \( K_n \). By [7.4], the conformal structure of the top component of the domain of discontinuity for \( \Gamma(q_j, K_n) \) converges to \( \alpha(n + 1) \), provided we choose \( q_j \) appropriately, and the conformal structure on the bottom component converges to \( \alpha(n) \). The double limit theorem applies, yielding a subsequence such that the representations restricted to \( S_{K_n} \) converge. By a diagonal process, we reach a subsequence so that these groups converge, for all \( n \).

For any two consective integers, the image of the fundamental group of the two surfaces in \( \text{MC} \) intersect on the fundamental group of a thrice-punctured torus. This implies that the fundamental group of their amalgamation also converges (since the intersection is non-elementary). Inductively, it follows that the representations of the entire fundamental group of \( \text{MC} \) converges.
Another special case which is fairly easy to handle is the case of an arbitrary surface $S$ when the sequence of curves $C$ is either finite or semi-infinite. Suppose, for example, that the index set is the positive integers. Let $m$ be the set of minimal elements of $<\infty$. They form a system of disjoint curves on $S$. Let $R_1 \subset M_C$ be a surface which is above all curves in $m$, and below all others. For appropriate power functions $q$, the conformal structure at $+\infty$ will appear as some convex combination of curves in a collection $\Gamma$ which are minimal in $C - m$ from the point of view of $S_1$. Let $m_1 \subset m$ be the subset of $m$ which elements of $\Gamma$ intersect. Let $S_1$ be a surface which is above the $m_i$, but below all other curves in $C$. For each element $\beta$ of $m_1$, there is a simple closed curve on $S$ which intersects $\beta$ but none of the other curves in $m_1$. From the point of view of $S_1$, this curve appears as a lamination close to $\beta$. Forming some convex combination of such curves on $S$, we see that there is a lamination which has modest length on $S$, and when transformed to $S_1$ is near the sum of the elements of $m_1$. An application of 5.3 shows that the subsurface filled up by $\Gamma \cup m$ has bounded geometry. Since the geometry of $S_1 - m_1$ also remains bounded, the geometry of $S_1$ remains bounded.

This argument can be repeated inductively, using $S_1$ in place of $S$, to prove that in this circumstance, all of $\pi_1(M_C)$ converges.

This argument could be extended to the case that $C$ is doubly infinite, provided one could find at least one surface whose geometry remained bounded. For instance, if there is at least one cut in $C$ such that any maximal curve below the cut together with any minimal curve above the cut bind the entire surface, that serves as a good starting point. Examples where this happens are easy to arrange. Unfortunately, it is also easy to arrange examples where it is difficult to get started. The trouble is that laminations or hyperbolic structures at $\pm\infty$, when transformed to a surface buried in the middle of $M_C$, tend to be concentrated near a single curve, rather than to have a positive coefficient for laminations near each of the possible curves. Thus, it is hard to control things so that a lamination which is short at $+\infty$ is guaranteed to have a positive intersection number with a lamination which is short at $-\infty$.

Note also that this special case feeds into the remark at the beginning of the proof of 7.2. It is not hard to show that if $C$ is finite, but large enough to bind the surface, then $AH(M_C)$ is compact, with an argument similar to the one we have just seen. Therefore, as $C$ increases through the finite subsets of a bi-infinite example, the representations remain bounded, so we could arrange for them to converge to a limit. We could then diagonalize, to prove the theorem; this method, however, would lose any information concerning rates of convergence.

So that we can gracefully handle the general case, we will make a more careful study of laminations and combinations of laminations which can have moderate length on hyperbolic structures on $S$.

First, let $\alpha$ be a simple closed curve on $S$. We will study laminations which have a small intersection number with $\alpha$, or in other words, are near $K(\alpha)$. The condition $i(\alpha, \lambda) = 0$ is a codimension one condition on $\lambda - K(\alpha)$ has dimension one less than the dimension of $ML_0(S)$. The concrete application we will make of this is that if $\mu \in K(\alpha)$ has a positive weight on $\alpha$ itself, then a small neighborhood of $\mu$ is subdivided into three types of laminations:

1. laminations in $K(\alpha)$, having a positive weight on $\alpha$
2. laminations $\nu$ rightward of $K(\alpha)$, such that $i(\nu, \alpha) > 0$, and leaves of $\nu$ spiral toward the right as they approach $\alpha$, and
3. laminations $\nu$ leftward of $K(\alpha)$, such that $i(\nu, \alpha) > 0$, and leaves of $\nu$ spiral toward the left as they approach $\alpha$.

This subdivision can be recognized with the aid of a short arc $\alpha$ transverse to $\alpha$. The return map for leaves of laminations $\nu$ near $\mu$ gives the information: if leaves return a bit to the left of where they started, then one is in the leftward portion of the neighborhood. Note that the definitions depend only on the orientation of $S$, not on an orientation for $\alpha$.

**Proposition 7.5** (Centrist constriction). Let $\alpha$ be a simple closed geodesic on $S$, and let $\mu, \mu' \in K(\alpha)$ be elements having weight 1 on $\alpha$. There are neighborhoods $U$ and $U'$ of $\mu$ and $\mu'$ such that for any hyperbolic structure $g$ on $S$, if there is a lamination $\nu$ in $U$ rightward of $K(\alpha)$ and a lamination $\nu'$ in $U$ leftward of $K(\alpha)$ such that $\text{length}_g(\nu) < A$ and $\text{length}_g(\nu') < A$, then there is another lamination $\lambda \in K(\alpha)$ having weight 1 on $\alpha$, such that

$$\text{length}_g(\lambda) < 1.1A$$
Proof of [7.4]. The lamination \( \lambda \) will come from the “complex geodesic” determined by \( \nu \) and \( \nu' \). The laminations \( \mu \) and \( \mu' \) are really only used to control the slope of leaves of \( \nu \) and \( \nu' \) near \( \alpha \). We want the slopes to be small. If \( \nu \) and \( \nu' \) have any leaves in common, throw out the union of all such leaves, to obtain new laminations whose leaves are still nearly tangent to \( \alpha \). Also throw out any leaves which do not meet the other lamination.

Once the leaves of \( \nu \) and \( \nu' \) are transverse, there is a construction for laminations which are weighted combinations of \( \nu \) and \( \nu' \). This can be done with the help of weighted train tracks. Construct a foliation \( \mathcal{F} \) in a neighborhood of \( \nu \cup \nu' \) which is transverse to both, using the condition that if local coordinates are chosen so that the leaves of \( \nu \) are horizontal and leaves of \( \nu' \) are vertical, the leaves of \( \mathcal{F} \) run from the lower left to the upper right. In a neighborhood of \( \alpha \), \( \mathcal{F} \) can be taken perpendicular to \( \alpha \). If the support neighborhood for \( \mathcal{F} \) is trimmed down far enough, then all its leaves are intervals and bounded in length. The intervals fall into a finite number of isotopy classes within the support neighborhood. The quotient space of the neighborhood by the leaves of \( \mathcal{F} \) is a train track — a good quality train track (in that the complementary regions have the right types) if the neighborhood is trimmed carefully enough so that each leaf of \( \mathcal{F} \) meets a leaf of \( \nu \) or of \( \nu' \). Each of \( \nu \) and \( \nu' \) are carried on the resulting train track with positive weights. The lamination \( \lambda \) is obtained by taking the \((s,t)\) convex combination of the weights for \( \nu \) and the weights for \( \nu' \).

Clearly, the length of \( \lambda(s,t) \) does not exceed \( s \times \text{length of } \nu \) plus \( t \times \text{length of } \nu' \). Choose the ratio of \( s \) and \( t \) so that the leftward shifting and the rightward shifting of the return map to \( \alpha \) exactly cancel. The resulting lamination has a positive weight on \( \alpha \). Its weight can be estimated by looking at the total transverse measure of intersection with \( \alpha \). This weight is approximately \( s + t \), but the formula is not precise. Multiplying by a constant near 1, we obtain the lamination \( \lambda \) as claimed.

Corollary 7.6 (Curves twisted tight). Let \( C \) be a family of curves in \( S \times \mathbb{R} \), as above, and \( q \) as above an appropriate choice of Dehn surgeries. Let \( K < K' \) be cuts which differ by only one curve \( \gamma = \alpha \times t \). Suppose that \( \nu \) is a lamination which has moderate length on the top domain of discontinuity from the point of view of \( S_K \), and \( \nu' \) is a lamination which has moderate length on the bottom domain of discontinuity from the point of view of \( S_{K'} \). If \( \nu \) and \( \nu' \) are close to laminations in \( K(\alpha) \) with definite components of \( \alpha \), then the length of \( \alpha \) in the quotient three-manifolds of \( \Gamma(q,K) \) and \( \Gamma(q,K') \) is moderate.

Proof of [7.4]. The lamination \( \nu \) has the property that a certain composition \( \phi \) of high powers of Dehn structures about a sequence of curves, beginning with \( \alpha \), sends it to a bounded lamination. Let us use the convention that the \( \tau_\alpha \) twists toward the left, so that a high positive power of \( \tau_\alpha \) sends most laminations to leftward laminations near \( K(\alpha) \). If the power of \( \tau_\alpha \) which occurs as the first term in \( \phi \) is positive, then \( \nu \) must be rightward of \( K(\alpha) \), by [7.4] and its proof — each step in the proof either maintains the lamination roughly unchanged, or drastically increases its length. The only chance for \( \nu \) to transform to something of moderate length is for it to reduce in length at the first step, which will only happen if it is a rightward lamination.

Similarly, if the power of \( \tau_\alpha \) which occurs as the first term in \( \phi \) is positive, then the power of \( \tau_\alpha \) which occurs as the first term in the analogous diffeomorphism \( \phi' \) which sends \( \nu' \) to a lamination with moderate length is negative, and \( \nu' \) must be a leftward lamination.

The two laminations \( \nu \) and \( \nu' \) are on different surfaces, differing by a change of parametrization of the form \( \tau_\alpha^N \). There is a compromise parametrization, obtained by applying the \( \lfloor N/2 \rfloor \) power of \( \tau_\alpha \) to one of the two. When \( \nu \) and \( \nu' \) are transformed to the compromise surface, they are near laminations with roughly half the size of their approximate components of \( \alpha \), and the images are still on opposite sides of \( K(\alpha) \). If we represent this surface by an efficient pleated surface, we obtain both leftward and rightward laminations on the same surface, so we can apply [7.4] to conclude that on this surface, \( \alpha \) itself has moderate length.

Proposition 7.7 (Minor detours). Let \( h \) be a hyperbolic structure fixed on \( S \) for reference, and let \( \alpha \) be a simple closed geodesic on \( S \).

For every \( \epsilon \) there is a \( \delta \) such that if \( g \) is a hyperbolic structure on \( S \) such that

\[
\text{length}_g(\alpha) < A \text{length}_h(\alpha),
\]

and if \( \mu \in \text{ML}_0(S) \) is a measured lamination normalized to have length 1 such that

\[
i(\mu, \alpha) < \delta
\]


and
\[ \text{length}_g(\mu) < B, \]
then \( \mu \) is near a lamination \( \mu' \in \text{ML}_0(S) \) of length 1 such that
\[ \text{length}_g(\mu') < (1 + \epsilon)(A + B) \]
and
\[ i(\mu', \alpha) = 0. \]

Proof of 7.7. We will modify \( \mu \) by cutting the strands of \( \mu \) which intersect \( \alpha \), and joining the resulting ends by paths which run near intervals of \( \alpha \).

We begin by performing the positive or negative power \( P \) of \( \tau_\alpha \) which reduces the length of \( \mu \) as much as possible. At the very end we will compensate for this unwinding around \( \alpha \) by adding a component \( i(\alpha, \mu)|P|\alpha \) to obtain the final result.

Let \( \mu_1 \) be the result of this first unwinding step. If \( \mu_1 \) has a very small length, we can proceed immediately to the final step; a multiple of \( \alpha \) is then sufficiently close to \( \mu \).

Otherwise, proceed as follows. Some components of \( \mu_1 \) may intersect \( \alpha \), and others may not. Since every component of a measured lamination is minimal, every leaf of the components which do intersect \( \alpha \) intersects in a bi-infinite sequence. The leaves when cut by \( \alpha \) group into a finite number of bands of isotopic and parallel arcs, where the number is bounded by the genus of the surface. The angles at which the leaves meet \( \alpha \) are bounded away from zero (depending on the geometry of \( h \)).

The lengths of arcs in a given band differ by less than \( 2 \text{length}_h(\alpha) \). If \( \delta \) is small compared to the length of \( \mu_1 \), then the average length in \( h \) for bands constituting most of the total transverse measure of \( \mu_1 \) along \( \alpha \) is large. We assume that \( \delta \) is sufficiently small, and we begin the process of forming \( \mu' \) by cutting all the leaves of \( \mu_1 \) along \( \alpha \), retracting their ends a bit so that they end on the boundary of a thin annular neighborhood of \( \alpha \), and throwing all bands whose length in \( h \) is less than some fairly large constant.

We will describe the construction of \( \mu' \) by constructing a weighted train track which carries it. Choose one representative arc from each band, and mark it with the total transverse measure for the band. Now isotop all the arcs so as to minimize their length. The result of the isotopy will be a family of arcs which are orthogonal to \( \alpha \). The isotopy moves the endpoints a bounded distance, since the angles were bounded at the beginning. Consequently, most of the length of each of the arcs moves a rather short distance.

The endpoints of the arcs divide each side of the annulus into a bounded number of intervals, so the longest interval has length bounded above zero. Join the ends of the arcs which bound the longest interval by an arc inside the annulus, and assign it a weight which is the minimum of the weights for the two arcs. If two weights were different, then extend the arc with the larger weight, assigning the difference to the extension. Now make an isotopy to retract the added portion out of the annulus, until only the one possible endpoint touches the boundary of the annulus. Continue inductively. At the last step, there may be only one arc touching a boundary component of the annulus; in that case, add a loop which goes completely around the boundary of the annulus, with half the weight of that arc.

Each trajectory on the weighted train track we have constructed is homotopic to a polygonal path, with right angle bends, with side lengths (in \( h \)) alternately bounded below by a constant greater than zero, and bounded below by a fairly large constant. Such a path is homotopic to a geodesic which is close to the original polygon, along most of the lengths of the long segments. This means that the weighted train track we have constructed determines a lamination \( \mu_2 \) which is near \( \mu_1 \), and such that \( \text{length}_h(\mu_2) \) is close to \( \text{length}_h(\mu_1) \).

On the other hand, we can represent the train track for \( \mu_2 \) in \( g \) by using leaves of \( \mu_1 \), joining them by short arcs near \( \alpha \). The total length of the resulting weighted train track is less than \( B + i(\alpha, \mu)A\text{length}_h(\alpha) \).

Now add \( i(\alpha, \mu)|P|\alpha \) to \( \mu_2 \), to obtain \( \mu' \). The resulting lamination is close to \( \mu \), and the length of its image in \( h \) satisfies the inequality asserted in the proposition.

We return now to the proof of 7.2. We will focus on an arbitrary element \( \gamma \in C \), and show that its length remains bounded.

Start with the cut \( G \) whose lower part is the set of all curves \( \beta \prec \gamma \) together with \( \gamma \) itself. It follows from the hypotheses of the theorem that this in fact defines a cut. The lower half of \( G \) has a unique maximal element, namely \( \gamma \), so the lower conformal structure from the point of view of \( S_G \) is near \( \gamma \), by 7.4. Let \( \mu_0 \) a lamination near \( \gamma \) which has very small length in this conformal structure, guaranteed by Theorem 2.2.
We will construct an increasing sequence $K_i$ of cuts for which $\gamma$ is a maximal element in the lower part, and such that there is a lamination $\mu_i$ on $K_i$ which is either short in the bottom conformal structure, or at least has bounded length in $N$, and which has a substantial approximate component of $\gamma$. We will continue until we reach a cut where the upper conformal structure is near a lamination having positive intersection number with $\gamma$. There is a greatest cut $H$ for which $\gamma$ is a maximal element in the lower part: its upper part consists of the curves $\beta$ such that $\gamma \prec \beta$. If we ever reach $H$, we will be done, since all minimal elements of the upper part of $H$ intersect $\gamma$.

Set $K_0 = G$. When $K_i$ has been defined, let $\lambda_i$ be a lamination whose length is near 0 in the upper conformal structure from the point of view of $R = S_{K_i}$. Let $\beta_i$ be a linear combination of simple closed curves on $R$ isotopic to minimal elements above $R$ which closely approximates $\lambda_i$, and let $\delta_i$ be the curve with the largest weight.

If $i(\delta_i, \gamma) > 0$, then stop, construct an efficient pleated surface representing $R$, and apply 7.3 to conclude that $\gamma$ has bounded length in $N$.

Otherwise, define $K_{i+1}$ by moving $\delta_i$ from the upper part to the lower part, and abbreviate $Q = S_{K_{i+1}}$.

Let $\nu$ be $\mu_i$ pulled back in the homotopy class of $R$ rather than $Q$. There are two cases.

First, it may happen that $\nu$ has a substantial approximate component of $\delta_i$. In that case, observe that $\nu$ and $\delta_i$ must be on opposite sides of $K(\delta_i)$, as in proposition 7.6. Therefore, $\delta_i$ has bounded length in $N$. Let $R$ be represented efficiently by some pleated surface, so that $\mu_i$ has bounded length on $R$. Since $\delta_i$ also has bounded length, we can apply 7.7 to find another lamination $\mu'$ near $\mu_i$ which has bounded length on this pleated surface. Define $\mu_{i+1}$ to be $\mu$ transported to $Q$. We may as well assume that the component of $\delta_i$ in $\mu_{i+1}$ is zero, since it is negligibly small anyway.

The other case is that $\nu$ does not have a substantial component of $\delta_i$. In that case, we can take $\mu_{i+1} = \nu$. The domain length (on $S$) of $\mu_i$ and of $\mu_{i+1}$ is approximately the same, and in the range, the laminations are exactly the same.

In the end, we must eventually arrive at a point where $i(\delta_i, \gamma) > 0$. This shows that the length of $\gamma$ is bounded.

It is worth commenting here on the constants involved in this induction, because it may be useful someday for the development of more precise analysis of the quasi-isometric geometry of surface groups.

How does the ratio $B$ of length of $\mu_i$ in $N$ to that in $S$ progress? The ratio starts out near 0. If the power of the Dehn twist about $\gamma$ is reasonably large, the component of $\delta_i\mu_i$ in $\mu_i$ is negligible in the current situation, so the constant $(1 + \epsilon)(A + B)$ could be replaced by $(1 + \epsilon)B$ in a step of the first type.

We lose an additive constant $a(\lambda, \mu_i)$ when we apply 7.3. However, for the purposes of this proof, we could suppose that $\gamma$ does not have a short length (even though it does). In that case, we could take $\lambda$ always to have a closed leaf isotopic to $\gamma$ — this would mean that $a(\lambda, \mu_0)$ at least is quite small. A similar trick works at successive stages — if $\mu_i$ has approximate components on curves which are not short, choose $\lambda$ to contain closed curves in those homotopy classes, otherwise use 7.7 to eliminate those approximate components.

Continuing with the proof of 7.3, we take a subsequence of the representations of $\pi_1(M_C)$ such that restricted to any surface $S_K$, the subsequence is maximally convergent. What are the subsurfaces of maximal convergence? An argument similar to the previous argument, for the case that the sequence of curve is finite or semi-infinite, shows that representations converge on the entire surface $S_K$, as follows. The subsurface $R$ of convergence for $S_K$ must contain curves in the isotopy classes of at least the set $m$ of all minimal elements above $S_K$ and the set $M$ of all maximal elements below. Let $\delta$ be a minimal element greater than the set $m$, and let $m_1 \subset m$ be those elements of $m$ which are less than $\delta$. Consider the surface $S_{K'}$, where $K'$ is obtained from $K$ by moving $m_1$ to the lower part. The subsurface $R'$ of convergence for $S_{K'}$ contains $R$ cut by $m_1$, and also contains $\delta$. We can continue upward in this way, until at some point we arrive at a surface whose subsurface of convergence is the entire surface. As we have already seen, this implies that the entire fundamental group of $M_C$ converges.

The second phase of the proof of 7.2 is to prove that the geometric limit of the manifolds is, in fact, homeomorphic to $M_C$.

We claim first that the limit representation of $\pi_1(M_C)$ is faithful. It is a general fact that if the limit a sequence of representations of a group $G$ is non-faithful, and if $G$ is not almost abelian, then any element $g$ which is represented trivially in the limit must be represented trivially for all but a finite number of representations in the sequence. We know exactly the kernel of the $i$th representation: it is the same as
the kernel of the homomorphism \( \pi_1(M_C) \) to \( \pi_1(S) \). It is easy to verify (e.g., by estimating the hyperbolic geodesic on \( S \) representing the image of a loop in \( M_C \)) that the shortest word in the kernel has length which goes to infinity with \( i \), so the limit representation indeed is faithful.

Let \( P \) be the geometric limit manifold, and let \( N \) be the quotient of \( \mathbb{H}^3 \) by the limiting representation of \( \pi_1(M_C) \). By general principles, there is a covering map \( N \to P \). We must show that the covering is trivial, and that \( N \) is indeed homeomorphic to \( M_C \).

The covering cannot be finite to one: this can be seen with the aid of the unique divisibility property for \( \pi_1(S) \) and \( \pi_1(M_C) \), that if any element is expressible as a \( k \)th power, it is expressible in only one way as a \( k \)th power. Furthermore, if an infinite number of the images of \( \beta \in \pi_1(M_C) \) in \( \pi_1(S) \) are divisible by \( k \), then \( \beta \) itself is divisible by \( k \). Therefore, if \( \alpha \in \pi_1(P) \) has \( k \)th power in \( \pi_1(N) \), it follows that \( \alpha \in \pi_1(N) \).

The other possibility might be that the covering is infinite to one. This possibility we will rule out by a geometric argument. First, by Ch. 8 and 9 of [Thu79], the injectivity radius is bounded above throughout \( N \). This implies that in any infinite-sheeted covering projection, cusps are identified only with cusps. The projection of each cusp to \( P \) is one-to-one, since it is finite-to-one.

Let \( \Gamma \subset C \) be an independent set of maximal size. Construct a pleated surface \( P \), obtained by pinching all elements of \( \Gamma \) to their cusps.

Consider first the case that \( C \) is bi-infinite. The space \( B \) of 2-cycles in \( N \), up to chains of bounded volume, is isomorphic to \( \mathbb{Z} \), and any surface \( S_K \) represents a generator.

If the covering \( N \to P \) is infinite to one, then each cusp is a member of an infinite sequence of cusps identified to one cusp in \( P \). There are pleated surfaces representing a generator of \( B \) passing through each of the cusps, by Ch. 8 and 9 of [Thu79]. If some of these surfaces have \( \varepsilon \)-thin parts which are isotopic to cusps of \( N \), we can modify them by actually pinching such short curves to their cusps. But all surfaces of this type are contained in a subset of \( P \) of bounded volume and finitely generated fundamental group, again by Ch. 8 and 9 of [Thu79], since \( N \) and hence \( P \) has no \( \mathbb{Z} \)-cusps. (Any two points on a pleated surface are connected by a path whose total length, excluding its intersection with the thin part of the surface, is \( \textit{a priori} \) bounded. The set of points in \( P \) accessible by such paths has bounded volume and finitely generated fundamental group). It follows that some of the pleated surfaces are mapped to isotopic pleated surfaces in \( P \), and that the total algebraic volume of the isotopy is bounded. Combining this isotopy with the image of a chain in \( N \) whose boundary is the difference of the cycles represented by the two pleated surfaces, we obtain a 3-cycle of finite but non-zero volume in \( P \to P \) has finite volume.

However, the geometric limit of hyperbolic manifolds of infinite volume never has finite volume — a limit manifold is homeomorphic to the approximating manifolds in its thick part, which implies that any approximants also would have finite volume.

Therefore, the covering \( N \to P \) is trivial, and the geometric limit is \( N \approx M_C \), as claimed.

If the sequence is not bi-infinite, we can argue with the domain of discontinuity. There is say a lowest surface or a highest surface \( Q \). In the sequence of representations for \( \pi_1(Q) \), the conformal structure on the bottom domain of discontinuity is constant. In the limit, this bottom domain of discontinuity is still present. It embeds in the domain of discontinuity for the geometric limit. Thus there is a covering map \( D_{\pi_1(N)/\pi_1(P)} \to D_{\pi_1(P)/\pi_1(P)} \). Since both domain and range are hyperbolic surfaces of finite area, such a covering map can only be finite-to-one, so as we have already seen, it must be one-to-one.

We have a homotopy equivalence \( f : M_C \to N \approx P \). Why is it homotopic to a homeomorphism? First, we can easily homotope until \( f \) is a homeomorphism near the cusps. For each surface cut \( S_K \), the fundamental group is represented as a free product amalgamated along \( \pi_1(S)_K \), so by standard three-manifold topology, we can find an incompressible surface whose fundamental group is contained in \( \pi_1(S)_K \) and which separates \( N \) into two pieces with the expected fundamental groups, and which meets the cusps of \( N \) in the expected way. The only possibility, for homological reasons, is that the surface has the full fundamental group of \( S_K \), and so it is homeomorphic to \( S_K \). We can continue, by the standard technique, to conclude that \( f \) is homotopic to a homeomorphism \( M_C \approx N \).

References

[A860] Ahlfors and Sario. \textit{Riemann Surfaces}. Princeton University Press, 1960.
[Ber70] Lipman Bers. On boundaries of Teichmüller spaces and Kleinian groups: I. \\ \textit{Annals of Math.}, pages 570–600, 1970.
[Bon86] Francis Bonahon. Bouts des variétés hyperboliques de dimension 3. \\ \textit{Annals of Math.}, 124:71–158, 1986.
[CB88] Andrew J. Casson and Steven A. Bleiler. \textit{Automorphisms of surfaces after Nielsen and Thurston}, volume 9 of \\ \textit{London Mathematical Society Student Texts}. Cambridge University Press, Cambridge, 1988.
[CTS85] James W. Cannon and William P. Thurston. Group invariant peano curves. *Preprint*, 1985.
[EM98] D.B.A Epstein and A. Marden. Convex hulls in hyperbolic space, a theorem of Sullivan, and measured pleated surfaces. In *Proceedings of Durham conference on Three-manifolds and Kleinian Groups*. Cambridge University Press, 1987.
[Fen92] Sérgio R. Fenley. Asymptotic properties of depth one foliations in hyperbolic 3-manifolds. *J. Differential Geom.*, 36(2):269–313, 1992.
[Fet al.79] Laudenbach Fathi and Poenaru *et al.* Travaux de Thurston sur les surfaces. *Astérisque*, 66–67:1–284, 1979.
[Gil82] Jane Gilman. Determining Thurston classes using Nielsen types. *Trans. Amer. Math. Soc.*, 272(2):669–675, 1982.
[HT85] Michael Handel and William P. Thurston. New proofs of some results of Nielsen. *Adv. in Math.*, 56(2):173–191, 1985.
[Jør77] Troels Jørgensen. Compact 3-manifolds of constant negative curvature fibering over the circle. *Ann. of Math.*, 106:61–72, 1977.
[McM96] Curtis T. McMullen. *Renormalization and 3-manifolds which fiber over the circle*, volume 142 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1996.
[Min94] Yair N. Minsky. On rigidity, limit sets, and end invariants of hyperbolic 3-manifolds. *J. Amer. Math. Soc.*, 7(3):539–588, 1994.
[Nie86a] Jakob Nielsen. *Jakob Nielsen: collected mathematical papers. Vol. 1*. Contemporary Mathematicians. Birkhäuser Boston Inc., Boston, Mass., 1986. Edited and with a preface by Vagn Lundsgaard Hansen.
[Nie86b] Jakob Nielsen. *Jakob Nielsen: collected mathematical papers. Vol. 2*. Contemporary Mathematicians. Birkhäuser Boston Inc., Boston, Mass., 1986. Edited and with a preface by Vagn Lundsgaard Hansen.
[Ota96] Jean-Pierre Otal. Le théorème d’hyperbolisation pour les variétés fibrées de dimension 3. *Astérisque*, (235):x+159, 1996.
[Sco78] Peter Scott. Subgroups of surface groups are almost geometric. *J. London Math. Soc. (2)*, 17(3):555–565, 1978.
[Sta62] John Stallings. On fibering certain 3-manifolds, pages 95–100. Prentice Hall, 1962.
[Sul80] Dennis P. Sullivan. Travaux de Thurston sur les groupes quasifuchsiens et les variétés hyperboliques de dimension 3 fibrées sur $S^1$. *Sém. Bourbaki*, 554, 1980.
[Sul83] Dennis Sullivan. Conformal dynamical systems. In *Geometric dynamics (Rio de Janeiro, 1981)*, volume 1007 of *Lecture Notes in Math.*, pages 725–752. Springer, Berlin, 1983.
[Thu] William P. Thurston. Minimal stretch maps between surfaces. math.GT/9801039.
[Thu97] William P. Thurston. On the geometry and dynamics of homeomorphisms of surfaces. *Preprint*.
[Thu86] William P. Thurston. *Geometry and Topology of Three-Manifolds*. Princeton lecture notes, 1979, http://www.msri.org/gt3m.
[Thu86] William P. Thurston. Hyperbolic structures on 3-manifolds, I: Deformation of acylindrical manifolds. *Annals of Math.*, 124:203–246, 1986, math.GT/9801019.
[Thu97] William P. Thurston. Three-manifolds, foliations and circles, I, 1997, math.GT/9712268.
[Tuk85] Pekka Tukia. Quasiconformal extension of quasisymmetric mappings compatible with a Möbius group. *Acta Math.*, 154(3-4):153–193, 1985.
[TV82] P. Tukia and J. Väisälä. Quasiconformal extension from dimension $n$ to $n + 1$. *Ann. of Math. (2)*, 115(2):331–348, 1982.