Craig’s Interpolation Theorem formalised and mechanised in Isabelle/HOL

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Abstract
We formalise and mechanise a constructive, proof theoretic proof of Craig’s Interpolation Theorem in Isabelle/HOL. We give all the definitions and lemma statements both formally and informally. We also transcribe informally the formal proofs. We detail the main features of our mechanisation, such as the formalisation of binding for first order formulae. We also give some applications of Craig’s Interpolation Theorem.

1. Introduction
Craig’s Interpolation Theorem is one of the main results in elementary proof theory. It is a result about FOL. Its proof is similar in style to the more famous Cut elimination theorem of Gentzen [Sza69]. In fact, the two results are intimately connected, and both are part of a general concern with “purity of methods” [Gir87].

As with Cut elimination, Craig’s Interpolation Theorem has many applications, particularly to the formalisation and mechanisation of mathematics, to the making of definitions, to the stating of lemmas, and to the general structuring of formalisations. It is primarily a result about modularity at the level of definitions and lemmas.

This work describes the first mechanised proof of Craig’s Interpolation Theorem. Why mechanise Craig’s Interpolation Theorem? Correctness is one of the main considerations. Particularly, we would like our proofs to be correctly formed (a purely syntactic condition), even if we must use our own faculties to ensure the correctness of definitions (that they conform to our informal notions). Results in proof theory are particularly appropriate for formalisation because they often involve substantial syntactic weight, which can cause typographical and real errors to creep into non-mechanised presentations.

A formal presentation also clarifies details, which in turn has pedagogic advantages. For example, the notion of variable binding and alpha conversion, which are often viewed as tricky to establish formally, are present in two places when formalising FOL. They are present when considering variable binding ∀x, ∃x in formulae. They are also present in proof terms with the notion of an eigenvariable. Much of the motivation behind the recent POPLmark challenge [ABF’05] is to assess the current state of theorem provers with regard to the mechanisation of proofs about logical systems, particularly with respect to their handling of binding. There is clearly a lot of interest in this area, and we believe our work contains contributions.

The proof of Craig’s Interpolation Theorem we mechanise here is constructive, which means that the proof contains an algorithm. For a given proof this algorithm constructs the mechanisation formulae. Thus, the proof of the theorem is simultaneously the verification of an algorithm. We believe this algorithm would be extremely hard to get right without mechanical assistance, for exactly the same reasons that it is hard to construct a correct informal proof: the details overwhelm.

In this paper, we describe the result itself, and its mechanisation in the Isabelle/HOL theorem prover. The mechanisation is presented in its entirety, save that some tactic proof scripts have been omitted. The paper should be readable with no Isabelle/HOL knowledge. By omitting the Isabelle/HOL material, a standard informal mathematical presentation is obtained. The full proof scripts can be obtained from the author’s homepage.

The mechanisation has several interesting features which we discuss after the presentation of the main result.

We briefly outline the following sections. In Sect. 2 we describe the formal syntax of Isabelle/HOL. In Sect. 3 we describe terms, and in Sect. 4 we describe formulae. In Sect. 5 we describe the system of FOL for which we prove Craig’s Interpolation Theorem. In Sect. 6 we motivate the statement of (a strong form of) Craig’s Interpolation Theorem, and in Sect. 7 we prove the theorem by induction over derivations. Throughout we give both an informal presentation, and the formal version for comparison. Our development is axiomatic. To ensure that the axioms are satisfactory, we also provide in Sect. 8 a concrete development which is conservative over the base Isabelle/HOL logic. In Sect. 9 we briefly analyse the mechanisation, and then in Sect. 10 we discuss applications of the theorem and its mechanisation. Finally, we conclude with a statement of the main contributions of this work, an examination of related work, and possibilities to extend this work in the future.

2. Isabelle/HOL Notation
In the following sections, formal results are stated in the Isabelle/HOL [PNW03] dialect of the HOL logic.

New types are introduced with the keyword typedef. New names for existing types (type aliases) are introduced with the keyword types. Type constructors are functions mapping type lists to types. Application of a type constructor is typically written postfix. For example, the type of sets over an underlying type a is a set. The type of a function with domain a and codomain b is a ⇒ b, ⇒ is an infix type constructor, which associates to the right. Lambda abstraction λx. is written λ x. The type of pairs whose first component is of type a and whose second component is of type b is a × b. The pair of x and y is written (x, y). The type of lists whose elements are of type a is list. Finite lists are written [a,b,c]. Consing an element x onto the front of the list xs is written x # xs.

A particularly important type is nat, the type of the natural numbers. Non-recursive natural number elimination, or case analysis, is written case n of 0 ⇒ a ∨ Suc n ⇒ f n.

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New constants are introduced with the keyword constdefs. A new constant is introduced by giving its name followed by :: followed by its type. Definitions are introduced with the keyword defn. A definition is written using the metaequality ≡ rather than simple HOL equality. These two keywords are combined into the single keyword constdefs.

Axioms are introduced with the keyword axioms.

Our Isabelle theory files are ASCII text files. The format of these files is described in [PNW98]. The usual logical connectives are rendered in ASCII as follows. ∨x.P x is ∀ x. P x, ∃x.P x is ∃ x. P x, A → B is A ⊢ B, A ∨ B is A ⊢ B, A ∨ B is A, and A is ~ A. A ↔ B is A = B.

A common language is that of sets. Set notation is as follows. ∈ A is a ∈ A, a ∈ A is a ∈ A. The empty set is {}. Set union A ∪ B is A ∪ B. Set intersection A ∩ B is A ∩ B. Finite sets are written {a,b,c}. A ⊆ B is A ⊆ B. The collection of the image of a function f on a set S is UNION S f.

ML-style datatypes are introduced with the keyword datatype, followed by the name of the new type, followed by constructors with the types of their arguments. The associated initial free structure with these constructors is then generated, together with various theorems about the structure. Functions can be defined by primitive recursion over the datatype. Primitive recursive functions are introduced with the keyword primrec.

3. Terms

Variables are indexed by N.

Types

var = nat

Terms are simply variables.

Types

tm = var

The extension to full first order terms is trivial. However, this obscures the development. Moreover, first order terms can be simulated using variables and relations.

4. Formulae, Occurrences

Primitive formulae P(x, y, z) are predicates P applied to a tuple of variables (x, y, z). Predicates P, Q, . . . are in reality identified by an index x ∈ N, so that primitive predicates are P0, P1, . . . Arbitrary length tuples (x, y, . . . , z) are represented by lists. Formulae A are defined inductively in the usual way from primitive formulae using additional constructors ⊥, ⊤, ∧, ∨, ¬, ∀, ∃.

Types

pred = nat

typedec form

consts

P :: pred ⇒ tm list ⇒ form

⊥ :: form

⊤ :: form

∧ :: form ⇒ form ⇒ form

∨ :: form ⇒ form ⇒ form

¬ :: form ⇒ form

∀ :: var ⇒ form ⇒ form

∃ :: var ⇒ form ⇒ form

Note that P1(x, y) is different to P0(x, y, z) but that later definitions, such as pos, neg, do not distinguish them. The usual informal solution is not to work with two predicates of the same name (index) but different arities. Alternatively a predicate could be distinguished not only by its index, but also by its arity.
\[ neg (FE\text{-inst } t (\exists a A)) = neg (\exists a A) \]

5. Sequents, Logical System

Sequents \( \Gamma \vdash \Delta \) are pairs of sets of formulae.

types \( \text{seq} = \text{form set} \times \text{form set} \)

The sets are intended to be finite. We make the restriction to finite sets of formulae when we define derivations. We write sets of formulae using \( \Gamma \), to denote (non-disjoint) set union. Thus \( \Gamma_1, \Gamma_2 = \Gamma_1 \cup \Gamma_2 \).

We employ a standard multiple conclusion sequent calculus, see Fig. 5. Formulæ in the conclusion of a rule are retained in the premises. Exchange does not apply because we are working with sets of formulæ. Similarly contraction. Weakening is actually admissible, but we include it as an explicit rule because it makes the proofs more elegant. For the weakening rules, it is important to recognise that \( A \) may appear in \( \Gamma, \Delta \).

The logical system describes the construction of a derivation. A derivation is a tree where each node is an instance of a rule.

datatype \( \text{deriv} \) = \( \text{Init seq} \)
\[ \lor \text{L seq} \]
\[ \lor \text{R seq} \]
\[ \land \text{seq deriv} \]
\[ \land \text{seq deriv deriv} \]
\[ \land \text{seq deriv} \]
\[ \lor \text{seq deriv} \]
\[ \lor \text{L seq deriv} \]
\[ \lor \text{R seq deriv} \]
\[ \lor \text{L seq deriv deriv} \]
\[ \lor \text{R seq deriv} \]
\[ \exists \text{L seq deriv} \]
\[ \exists \text{R seq deriv} \]
\[ \forall \text{L seq deriv} \]
\[ \forall \text{R seq deriv} \]

The first argument to each derivation constructor indicates the root sequent of the derivation formed using the constructor. The additional arguments provide auxiliary information necessary to determine the rule. For example, in the case of \( \lor \text{L} \), we must give the formulæ \( A \) and \( B \) where \( A \lor B \) is the formula we are analysing, and we must also provide a subderivation of the premise of the rule. The exact requirements are explicitly stated when we define \( \text{is-deriv} \).

The root of a derivation is straightforward.

consts \( \text{root} : : \text{deriv} \Rightarrow \text{seq} \)

primrec

\[ \text{root} (\text{Init } \Gamma \Delta) = \Gamma \Delta \]
\[ \text{root} (\lor \text{L } \Gamma \Delta) = \Gamma \Delta \]
\[ \text{root} (\lor \text{R } \Gamma \Delta) = \Gamma \Delta \]
\[ \text{root} (\land \text{R } \Gamma \Delta d) = \Gamma \Delta \]
\[ \text{root} (\lor \text{L } \Gamma \Delta d) = \Gamma \Delta \]
\[ \text{root} (\lor \text{R } \Gamma \Delta d) = \Gamma \Delta \]
\[ \text{root} (\lor \text{L } \Gamma \Delta d \text{ dr}) = \Gamma \Delta \]
\[ \text{root} (\lor \text{R } \Gamma \Delta d \text{ dr}) = \Gamma \Delta \]
\[ \text{root} (\exists \text{L } \Gamma \Delta d) = \Gamma \Delta \]
\[ \text{root} (\exists \text{R } \Gamma \Delta d) = \Gamma \Delta \]
\[ \text{root} (\forall \text{L } \Gamma \Delta d) = \Gamma \Delta \]
\[ \text{root} (\forall \text{R } \Gamma \Delta d) = \Gamma \Delta \]

We use a predicate to pick out wellformed derivations.

consts \( \text{is-deriv} : : \text{deriv} \Rightarrow \text{bool} \)

primrec

\[ \text{is-deriv} (\text{Init } \Gamma \Delta) = (\text{let } (\Gamma, \Delta) = \Gamma \Delta \text{ in } \text{finite } \Gamma) \]
\[ \land \text{finite } \Delta \]
\[ \land (\exists A, A \in \Gamma) \]
\[ \land A \in \Delta) \)
\[ \text{is-deriv} (\lor \text{L } \Gamma \Delta) = (\text{let } (\Gamma, \Delta) = \Gamma \Delta \text{ in } \text{finite } \Gamma) \]
\[ \land \text{finite } \Delta \]
\[ \land (A \in \Gamma) \)
\[ \text{is-deriv} (\lor \text{R } \Gamma \Delta d) = (\text{let } (\Gamma, \Delta) = \Gamma \Delta \text{ in } \text{finite } \Gamma) \]
\[ \land \text{finite } \Delta \]
\[ \land (\exists A, A \in \Gamma, A B \in \Gamma) \]
\[ \land \text{is-deriv } d \]
\[ \land \text{root } d = (\{A, B \} \cup \{\Gamma, \Delta\}) \)
\[ \text{is-deriv} (\lor \text{R } \Gamma \Delta d \text{ dr}) = (\text{let } (\Gamma, \Delta) = \Gamma \Delta \text{ in } \text{finite } \Gamma) \]
\[ \land \text{finite } \Delta \]
\[ \land (\exists A, A \in \Gamma, \forall A B \in \Gamma) \]
\[ \land \text{is-deriv } d \]
\[ \land \text{root } d = (\{A \} \cup \{\Gamma, \Delta\}) \)
\[ \text{is-deriv} (\land \text{L } \Gamma \Delta d) = (\text{let } (\Gamma, \Delta) = \Gamma \Delta \text{ in } \text{finite } \Gamma) \]
\[ \land \text{finite } \Delta \]
\[ \land (\exists C, C \in \Gamma, \forall A B \in \Gamma) \]
\[ \land \text{is-deriv } d \]
\[ \land \text{root } d = (\{C \} \cup \{\Gamma, \Delta\}) \)
\[ \text{is-deriv} (\land \text{R } \Gamma \Delta d) = (\text{let } (\Gamma, \Delta) = \Gamma \Delta \text{ in } \text{finite } \Gamma) \]
\[ \land \text{finite } \Delta \]
\[ \land (\exists C, C \in \Gamma, \forall A B \in \Gamma) \]
\[ \land \text{is-deriv } d \]
\[ \land \text{root } d = (\{C \} \cup \Gamma, \Delta) \)
\[ \text{is-deriv} (\forall \text{L } \Gamma \Delta d) = (\text{let } (\Gamma, \Delta) = \Gamma \Delta \text{ in } \text{finite } \Gamma) \]
\[ \land \text{finite } \Delta \]
\[ \land (\exists A, A \in \Gamma, \forall A B \in \Gamma) \]
\[ \land \text{is-deriv } d \]
\[ \land \text{root } d = ((\{\text{Fall-inst } t (\forall a A) \} \cup \Gamma, \Delta) \))
rules for a multiple conclusion sequent calculus

\[
\begin{align*}
\text{Init} & : A, \Gamma \vdash \Delta, A \\
\bot, \Gamma & \vdash \Delta \quad \bot L \\
A, B, A \land B, \Gamma & \vdash \Delta \quad \land L \\
A \land B, \Gamma & \vdash \Delta \\
\neg A, \Gamma & \vdash \Delta \\
\neg A, \Gamma & \vdash \Delta \quad \neg L \\
A[t], \forall x. A[x], \Gamma & \vdash \Delta \\
\forall x. A[x], \Gamma & \vdash \Delta \quad \forall L \\
A[a], \exists x. A[x], \Gamma & \vdash \Delta \\
\exists x. A[x], \Gamma & \vdash \Delta \quad \exists L \\
\Gamma & \vdash \Delta \\
\Gamma & \vdash \Delta \quad W L \\
A, \Gamma & \vdash \Delta \\
\end{align*}
\]

\[
\begin{align*}
\text{\lor R} & : \Gamma \vdash \Delta, A \lor B, A \quad \lor R \\
\Gamma & \vdash \Delta, A \lor B \\
\neg A, \Gamma & \vdash \Delta \\
\neg A, \Gamma & \vdash \Delta \quad \neg R \\
\Gamma & \vdash \Delta, \forall x. A[x], A[a] \\
\forall x. A[x], \Gamma & \vdash \Delta \quad \forall R \\
\Gamma & \vdash \Delta, \exists x. A[x] \\
\Gamma & \vdash \Delta, \exists x. A[x] \quad \exists R \\
\Gamma & \vdash \Delta \\
\Gamma & \vdash \Delta \quad W R
\end{align*}
\]

\[\forall R, \exists L : a \text{ not free in the conclusion of the rule.}\]

**Figure 1.** Rules for a Multiple Conclusion Sequent Calculus

\[is-deriv (\forall R \Gamma \Delta d) = (\]

let \((\Gamma, \Delta) = \Gamma \Delta \text{ in}\)

finite \(\Gamma\)

\[\land \text{ finite } \Delta\]

\[\land (\exists \ a \ A, \forall \ a \ A \in \Delta)\]

\[\land a \notin \text{ UNION } (\Gamma \cup \Delta) \text{ (set o fv)}\]

\[\land \text{ is-deriv } d\]

\[\land \text{ root } d = (\Gamma, \Delta \cup \{A\})\]

\[is-deriv (\exists L \Gamma \Delta d) = (\]

let \((\Gamma, \Delta) = \Gamma \Delta \text{ in}\)

finite \(\Gamma\)

\[\land \text{ finite } \Delta\]

\[\land (\exists \ a \ A, \exists \ a \ A \in \Delta)\]

\[\land \text{ a } \notin \text{ UNION } (\Gamma \cup \Delta) \text{ (set o fv)}\]

\[\land \text{ is-deriv } d\]

\[\land \text{ root } d = (\Gamma, \Delta \cup \{A\})\]

\[is-deriv (\exists R \Gamma \Delta d) = (\]

let \((\Gamma, \Delta) = \Gamma \Delta \text{ in}\)

finite \(\Gamma\)

\[\land \text{ finite } \Delta\]

\[\land \text{ is-deriv } d\]

\[\land \text{ root } d = (\Gamma, \Delta)\]

\[\land \Delta = (\{A\} \cup \Gamma, \Delta)\]

\[is-deriv (\forall L \Gamma \Delta d) = (\]

let \((\Gamma, \Delta) = \Gamma \Delta \text{ in}\)

finite \(\Gamma\)

\[\land \text{ finite } \Delta\]

\[\land \text{ is-deriv } d\]

\[\land \text{ root } d = (\Gamma, \Delta)\]

\[\land \Delta = (\{A\} \cup \Gamma, \Delta)\]

\[is-deriv (\exists R \Gamma \Delta d) = (\]

let \((\Gamma, \Delta) = \Gamma \Delta \text{ in}\)

finite \(\Gamma\)

\[\land \text{ finite } \Delta\]

\[\land \text{ is-deriv } d\]

\[\land \text{ root } d = (\Gamma, \Delta)\]

\[\land \Delta = (\{A\} \cup \Gamma, \Delta)\]

\[\land \text{ finite } \Gamma\]

6. **Statement of Craig’s Interpolation Theorem**

**Theorem 6.1.** (Craig’s Interpolation Theorem) If

\[\Gamma \vdash \Delta\]

then there exists a formula \(C\) such that

\[\Gamma \vdash C \quad \text{and} \quad C \vdash \Delta\]

and moreover such that

- Any predicate that occurs positively in \(C\) occurs positively in \(\Gamma\) and in \(\Delta\).
- Any predicate that occurs negatively in \(C\) occurs negatively in \(\Gamma\) and in \(\Delta\).

**Lemma craig:**

\[
\forall d \Gamma \Delta, \quad is-deriv d \\
\land \text{ root } d = (\Gamma, \Delta) \\
\quad \rightarrow \quad (\exists C. \quad (\exists dl. \text{ is-deriv } dl \land \text{ root } dl = (\Gamma, \{C\})) \\
\land (\exists dr. \text{ is-deriv } dr \land \text{ root } dr = (\{C\}, \Delta)) \\
\land (\text{pos } C \subseteq (\text{UNION } \Gamma \text{ pos}) \cap (\text{UNION } \Delta \text{ pos})) \\
\land (\text{neg } C \subseteq (\text{UNION } \Gamma \text{ neg}) \cap (\text{UNION } \Delta \text{ neg})))
\]
Craig’s interpolation theorem is almost provable directly by structural induction over the derivation. For example, consider the case where the derivation ends in $\Gamma \vdash A \land R$.

$$\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B \quad \Gamma \vdash \Delta$$

We have that $A \land B \in \Delta$. Using the induction hypothesis twice, we obtain a $C''$ such that

$$\Gamma \vdash C'' \quad \text{and} \quad C'' \vdash A \land B$$

and a $C''$ such that

$$\Gamma \vdash C' \quad \text{and} \quad C' \vdash \Delta, A$$

because, for instance, $C'$ may contain a positive occurrence of $A$ (also occurring in $\Gamma$), whereas $A$ may not appear in $\Delta$, so that $C$ may not contain a positive occurrence of $A$. Thus the direct approach to proving Craig’s Interpolation Theorem breaks down for polarity altering connectives.

The solution is to prove a stronger theorem. It is clear that the problem lies with the polarity altering connectives such as $\neg$. It is reasonably easy to motivate a split sequent $\Gamma, \Delta \vdash \Delta_1, \Delta_2$. A goal sequent $\Gamma \vdash \Delta$ is obtained by taking $\Gamma_1 = \Gamma, \Delta_2 = \{\}$. From $\Gamma_2, \Delta_1$ are used to keep track of the polarity changes occurring in rules such as $\neg L$.

**Theorem 6.2.** *(Strengthened Interpolation Theorem)* If

$$\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$$

then there exists a formula $C$ such that

$$\Gamma_1 \vdash \Delta_1, C \quad \text{and} \quad C, \Gamma_2 \vdash \Delta_2$$

and moreover such that

- Any predicate that occurs positively in $C$ occurs positively$^5$ in $\Gamma_1, \neg \Delta_1$ and positively in $\neg \Gamma_2, \Delta_2$. \\
- Any predicate that occurs negatively in $C$ occurs negatively in $\Gamma_1, \neg \Delta_1$ and negatively in $\neg \Gamma_2, \Delta_2$.

**Lemma Craig**$^4$:

\forall d \Gamma_1, \Gamma_2 \Delta_1, \Delta_2. \\
\text{is-deriv} d \\
\& \text{root} d = (\Gamma_1 \cup \Gamma_2, \Delta_1 \cup \Delta_2) \rightarrow \quad \exists C. \\
(\exists dl. \text{is-deriv} dl \& \text{root} dl = (\Gamma_1, \Delta_1 \cup \{C\})) \\
\land (\exists dr. \text{is-deriv} dr \& \text{root} dr = (\{C\} \cup \Gamma_2, \Delta_2)) \\
\land (\text{pos} C \subseteq (\text{UNION} \Gamma_1 \text{pos}) \cup (\text{UNION} \Delta_1 \text{neg})) \\
\land (\text{pos} C \subseteq (\text{UNION} \Gamma_2 \text{pos}) \cup (\text{UNION} \Delta_2 \text{pos})) \\
\land (\text{neg} C \subseteq (\text{UNION} \Gamma_1 \text{neg}) \cup (\text{UNION} \Delta_1 \text{pos})) \\
\land (\neg \text{neg} C \subseteq (\text{UNION} \Gamma_2 \text{pos}) \cup (\text{UNION} \Delta_2 \text{neg})))

The actual induction is a structural induction over the derivation of $\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$. It is easiest to state this as an induction over the size of the derivation.

**Lemma Craig**$^5$:

\forall n. \forall d. \text{size} d = n \rightarrow \\
(\forall \Gamma_1, \Gamma_2 \Delta_1, \Delta_2. \\
\text{is-deriv} d \\
\& \text{root} d = (\Gamma_1 \cup \Gamma_2, \Delta_1 \cup \Delta_2)) \\
\rightarrow \exists C. \\
(\exists dl. \text{is-deriv} dl \& \text{root} dl = (\Gamma_1, \Delta_1 \cup \{C\})) \\
\land (\exists dr. \text{is-deriv} dr \& \text{root} dr = (\{C\} \cup \Gamma_2, \Delta_2)) \\
\land (\text{pos} C \subseteq (\text{UNION} \Gamma_1 \text{pos}) \cup (\text{UNION} \Delta_1 \text{neg})) \\
\land (\text{pos} C \subseteq (\text{UNION} \Gamma_2 \text{pos}) \cup (\text{UNION} \Delta_2 \text{pos})) \\
\land (\text{neg} C \subseteq (\text{UNION} \Gamma_1 \text{neg}) \cup (\text{UNION} \Delta_1 \text{pos})) \\
\land (\neg \text{neg} C \subseteq (\text{UNION} \Gamma_2 \text{pos}) \cup (\text{UNION} \Delta_2 \text{neg})))

**Corollary 6.3.** *(Craig’s Interpolation Theorem)*

**Proof.** The original formulation of Craig’s Interpolation Theorem follows immediately from Thm. 6.2 by taking $(\Gamma, \{\}, \{\}, \Delta) = (\Gamma_1, \Gamma_2, \Delta_1, \Delta_2)$.

$^3$“positively in $\Gamma_1, \neg \Delta_1$” means positively in $\Gamma_1$ or negatively in $\Delta_1$ etc.
7. Proof of Craig’s Interpolation Theorem

We aim to prove the strengthened form of Craig’s Interpolation Theorem, Thm. 6.2. We induct over the size of the derivation \(d\) of \(\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2\), so that we can use the induction hypothesis for all derivations of smaller size. The body of the proof proceeds by a case analysis on the last constructor of the given derivation.

In the following cases, apart from the Init case, we do not check the conditions regarding positive and negative occurrences in the interpolation formula. These conditions are straightforward to verify.

7.1 Case Init

We give a formal Isar rendition of the case \(d\) ends in rule Init. There are four subcases. We give a full rendition of the first subcase. The 3 remaining subcases are very similar. We provide the explicit \(C\) for these cases but suppress the mundane proofs. The full details of the remaining subcases can be found in the mechanised theory script.

Lemma assumes: \(a\) is-deriv and \(b\): root \(d = (\Gamma_1 \cup \Gamma_2, \Delta_1 \cup \Delta_2)\) and \(c\): \(d = \text{Init} \Gamma \Delta\).

shows \(\exists C. (\exists dl \text{ is-deriv } dl \land \text{root } dl = (\Gamma_1, \Delta_1 \cup \{C\})) \land (\exists dr \text{ is-deriv } dr \land \text{root } dr = (\{C\} \cup \Gamma_2, \Delta_2))\)

\(\land (\text{pos } C \subseteq (\text{UNION } \Gamma_1 \text{ pos}) \lor (\text{UNION } \Delta_1 \text{ neg})\)

\(\land (\text{neg } C \subseteq (\text{UNION } \Gamma_2 \text{ neg}) \lor (\text{UNION } \Delta_2 \text{ pos}))\)

\(\land (\text{neg } C \subseteq (\text{UNION } \Gamma_1 \text{ neg}) \lor (\text{UNION } \Delta_1 \text{ pos}))\)

\((\text{is } \exists C. \left(\forall P. \neg C\right))\)

Proof.

\(\begin{align*}
\text{from } a, b, c \text{ obtain } A \text{ where } & (A \in \Gamma_1 \land A \in \Delta_1) \lor (A \in \Gamma_2 \land A \in \Delta_2) \\
\text{thus } & \exists \text{thesis} \\
\text{proof (elim disj)} \\
\text{assume } & A \in \Gamma_1 \land A \in \Delta_1 \\
\text{have } & \exists P \exists L \\
\text{proof (intro conj)} \\
\text{show } & (\exists dl \text{ is-deriv } dl \land \text{root } dl = (\Gamma_1, \Delta_1 \cup \{L\})) \\
\text{proof} & \\
\text{let } & dl = \text{Init} (\Gamma_1, \Delta_1 \cup \{L\}) \\
\text{show } & \text{is-deriv } dl \land \text{root } dl = (\Gamma_1, \Delta_1 \cup \{L\}) \text{ by (force! simp add: Let-def)} \\
\text{qed} &
\end{align*}\)

7.2 Case \(\land L\)

We have

\[
\frac{A, B, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2 \quad \land L}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}
\]

From well-formedness of the derivation, we have that \(A \land B \in \Gamma_1, \Gamma_2\). There are two subcases, \(A \land B \in \Gamma_1\) or \(A \land B \in \Gamma_2\).

- Case \(A \land B \in \Gamma_1\). The I.H. applied to \(d\) gives \(C', \text{dl, dr such that} \)

\[
\frac{\text{dl}}{\Gamma_1 \vdash \Delta_1, C'} \quad \text{and} \quad \frac{\text{dr}}{C', \Gamma_2 \vdash \Delta_2}
\]

Take \(C = C'\). Then

\[
\frac{\text{dl}}{\Delta L (\Gamma_1, \Delta_1 \cup \{C\}) \text{ dl}}
\]

\(\text{dr}\) is already a witness for \(C', \Gamma_2 \vdash \Delta_2\).

- Case \(A \land B \in \Gamma_2\). The I.H. applied to \(d\) gives \(C', \text{dl, dr such that} \)

\[
\frac{\text{dl}}{\Gamma_1 \vdash \Delta_1, C'} \quad \text{and} \quad \frac{\text{dr}}{C', A, B, \Gamma_2 \vdash \Delta_2}
\]

Take \(C = C'\). Then

\[
\frac{\text{dr}}{C', A, B, \Gamma_2 \vdash \Delta_2 \quad \land L}
\]

The formal proof witness is

\[
\Delta L (\{C\} \cup \Gamma_2, \Delta_2) \text{ dr}
\]

\(\text{dl}\) is already a witness for \(\Gamma_1 \vdash \Delta_1, C'\).

7.3 Case \(\lor R\)

We have

\[
\frac{\text{dl} \quad \text{dr}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, A \quad \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, B \quad \lor R}
\]

From well-formedness of the derivation, we have that \(A \land B \in \Delta_1, \Delta_2\). There are two subcases, \(A \land B \in \Delta_1\) or \(A \land B \in \Delta_2\).
• Case $A \land B \in \Delta_1$. The I.H. applied to $dl$ gives $C', dll, dlr$ such that

$$dll \quad dlr$$

$$\Gamma_1 \vdash \Delta_1, A, C' \quad \text{and} \quad C', \Gamma_2 \vdash \Delta_2$$

The I.H. applied to $dr$ gives $C'', drl, drr$ such that

$$drl \quad drr$$

$$\Gamma_1 \vdash \Delta_1, B, C'' \quad \text{and} \quad C'', \Gamma_2 \vdash \Delta_2$$

Take $C = C' \lor C''$. Then

$$dll \quad dlr$$

$$\Gamma_1 \vdash \Delta_1, A, C' \quad \text{and} \quad C', \Gamma_2 \vdash \Delta_2$$

The formal proof witness is

$$dll \quad dlr$$

$$\Gamma_1 \vdash \Delta_1, C' \quad \text{and} \quad C', \Gamma_2 \vdash \Delta_2, B$$

Take $C = C' \land C''$. Then

$$dll \quad drr$$

$$\Gamma_1 \vdash \Delta_1, C' \quad \text{and} \quad C', \Gamma_2 \vdash \Delta_2, B$$

The formal proof witness is

$$dll \quad drr$$

$$\Gamma_1 \vdash \Delta_1, C' \quad \text{and} \quad C', \Gamma_2 \vdash \Delta_2, C'' \land \Gamma \vdash \Delta_2$$

Similarly

$$dlr \quad drr$$

$$\Gamma_1 \vdash \Delta_1, C' \quad \text{and} \quad C', \Gamma_2 \vdash \Delta_2, A$$

The formal proof witness is

$$dlr \quad drr$$

$$\Gamma_1 \vdash \Delta_1, C' \quad \text{and} \quad C', \Gamma_2 \vdash \Delta_2, B$$

Symmetric to $\land R$.

7.4 Case $\lor L$

7.5 Case $\lor R$

Symmetric to $\lor L$.

7.6 Case $\neg L$

We have

$$d$$

$$\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, A \quad \neg L$$

From wellformedness of the derivation, we have that $\neg A \in \Gamma_1, \Gamma_2$. There are two subcases, $\neg A \in \Gamma_1$ or $\neg A \in \Gamma_2$.  

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• Case \( \neg A \in \Gamma_1 \). Then the I.H. applied to \( d \) gives \( C', dl, dr \) such that

\[
\begin{align*}
& dl \\
& \Gamma_1 \vdash \Delta_1, A, C' \\
& dr \\
& \Gamma_1 \vdash \Delta_1, C'
\end{align*}
\]

Take \( C = C' \). Then

\[
\begin{align*}
& dl \\
& A[t], \Gamma_1 \vdash \Delta_1, C' \\
& \Gamma_1 \vdash \Delta_1, C' \\
& \forall L
\end{align*}
\]

The formal proof witness is

\[
\forall L (\Gamma_1, \Delta_1 \cup \{C'\}) \, dl
\]

\( dr \) is already a witness for \( C', \Gamma_2 \vdash \Delta_2 \).

• Case \( \neg A \in \Gamma_2 \). Then the I.H. applied to \( d \) gives \( C', dl, dr \) such that

\[
\begin{align*}
& dl \\
& \Gamma_1 \vdash \Delta_1, C' \\
& dr \\
& C', \Gamma_2 \vdash \Delta_2, A
\end{align*}
\]

Take \( C = C' \). Then

\[
\begin{align*}
& dr \\
& C', \Gamma_2 \vdash \Delta_2, A \\
& C', \Gamma_2 \vdash \Delta_2 \\
& \forall L
\end{align*}
\]

The formal proof witness is

\[
\forall L ((C') \cup \Gamma_2, \Delta_2) \, dr
\]

\( dl \) is already a witness for \( \Gamma_1 \vdash \Delta_1, C' \).

7.7 Case \( \neg R \)

Symmetric to \( \neg L \).

7.8 Case \( \forall L \)

We have

\[
\begin{align*}
& dl \\
& A[t], \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2 \\
& \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2 \\
& \forall L
\end{align*}
\]

From wellformedness of the derivation, we have that \( \forall x.A[x] \in \Gamma_1, \Gamma_2 \). There are two subcases, \( \forall x.A[x] \in \Gamma_1 \) or \( \forall x.A[x] \in \Gamma_2 \).

• Case \( \forall x.A[x] \in \Gamma_1 \). Then the I.H. applied to \( d \) gives \( C'[a], dl, dr \) such that

\[
\begin{align*}
& dl \\
& A[t], \Gamma_1 \vdash \Delta_1, C' \\
& \Gamma_1 \vdash \Delta_1, C' \\
& \forall L
\end{align*}
\]

The formal proof witness is

\[
\forall L (\Gamma_1, \Delta_1 \cup \{C'[a]\}) \, dl
\]

\( dr \) is already a witness for \( C', \Gamma_2 \vdash \Delta_2 \).

• Case \( \forall x.A[x] \in \Gamma_2 \). Then the I.H. applied to \( d \) gives \( C'[a], dl, dr \) such that

\[
\begin{align*}
& dl \\
& A[t], \Gamma_1 \vdash \Delta_1, C', C'[a] \\
& \Gamma_1 \vdash \Delta_1, C', C'[a] \\
& \forall R
\end{align*}
\]

The formal proof witness is

\[
\forall L (\Gamma_1, \Delta_1 \cup \{C'[a]\}) \, dl
\]

\( dr \) is already a witness for \( C'[a], \Gamma_2 \vdash \Delta_2 \).

Take \( C = C'. \) Then

\[
\begin{align*}
& dl \\
& A[t], \Gamma_1 \vdash \Delta_1, C', C'[a] \\
& \Gamma_1 \vdash \Delta_1, C', C'[a] \\
& \forall R
\end{align*}
\]

The formal proof witness is
Let \( dll' = \text{WR}(\Gamma_1, \Delta_1 \cup \{A, C, \exists \ a \ C'\}) \) \( dl \) in \( \forall R(\Gamma_1, \Delta_1 \cup \{\exists \ a \ C'\}) \) \( dl' \) in

\[
\forall R \left( \Gamma_1 \Delta_1 \cup \{\exists a C'\} \right) dl'' \]

Similarly

\[
dr \quad \exists x. C'[x], \Gamma_2 \vdash \Delta_2 \quad WL \]

\[
\exists x. C'[x], \Gamma_2 \vdash \Delta_2 \quad \exists L
\]

The formal proof witness is

\[
let \ dr' = WL \left( \{\exists a C'\} \cup \{C'\} \cup \Gamma_2, \Delta_2 \right) \ dr \in \quad WL \left( \{\exists a C'\} \cup \Gamma_2, \Delta_2 \right) \ dr''
\]

- Case \( \forall x. A[x] \in \Delta_2 \). Then the I.H. applied to \( d \) gives \( C'[a] \), \( dl \), \( dr \) such that

\[
dl \quad \Gamma_1 \vdash \Delta_1, C'[a] \quad and \quad C'[a], \Gamma_2 \vdash \Delta_2, A[a]
\]

Take \( C = \forall x. C'[x] \). Then

\[
dl \quad \Gamma_1 \vdash \Delta_1, C'[a] \quad \forall R \quad \Gamma_1 \vdash \Delta_1, C'[a], \forall x. C'[x]
\]

The formal proof witness is

\[
let \ dl'' = WR(\Gamma_1, \Delta_1 \cup \{C' \forall x \ a C'\}) dl \ in \quad \forall R(\Gamma_1, \Delta_1 \cup \{\forall x a C'\}) dl''
\]

Similarly

\[
dr \quad \forall x. C'[x], \Gamma_2 \vdash \Delta_2, A[a] \quad WL \quad \forall L \quad \forall x. C'[x], \Gamma_2 \vdash \Delta_2, A[a]
\]

The formal proof witness is

\[
let \ dr' = WL \left( \{\forall x a C', C'\} \cup \Gamma_2, \Delta_2 \cup \{A\} \right) dr \ in \quad \forall R \left( \{\forall x a C'\} \cup \Gamma_2, \Delta_2 \right) dr''
\]

\[
7.10 \quad \text{Case } \exists L
\]

Symmetric to \( \forall R \).

\[
7.11 \quad \text{Case } \exists R
\]

Symmetric to \( \forall L \).

\[
7.12 \quad \text{Case } WL
\]

We have

\[
dl \quad \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2 \quad WL
\]

From wellformedness of the derivation we have that \( A, \Gamma = \Gamma_1, \Gamma_2 \). The I.H. applied to \( d \) gives \( C', dl, dr \) such that

\[
dl \quad \Gamma \cap \Gamma_1 \vdash \Delta_1, C' \quad and \quad C', \Gamma \cap \Gamma_2 \vdash \Delta_2
\]

Take \( C = C' \). There are four subcases.

- Case \( A \in \Gamma_1, A \notin \Gamma_2 \). Then

\[
dl \quad \Gamma_1 \vdash \Delta_1, C' \quad \forall R \quad \Gamma_1 \vdash \Delta_1, C'
\]

The formal proof witness is

\[
WL(\Gamma_1, \Delta_1 \cup \{C'\}) dl
\]

Similarly

\[
dr \quad \forall x. C'[x], \Gamma_2 \vdash \Delta_2 \quad WL \quad \forall L \quad \forall x. C'[x], \Gamma_2 \vdash \Delta_2
\]

The formal proof witness is

\[
let \ dl'' = WL(\Gamma_1, \Delta_1 \cup \{C'\}) dl \ in \quad \forall R(\Gamma_1, \Delta_1 \cup \{\forall x a C'\}) dl''
\]

- Case \( A \notin \Gamma_1, A \in \Gamma_2 \). Then

\[
dl \quad \Gamma_1 \vdash \Delta_1, A \quad \forall L \quad \Gamma_1 \vdash \Delta_1, A
\]

The formal proof witness is

\[
WL(\Gamma_1, \Delta_1 \cup \{C'\}) dl
\]

\( dr \) is already a witness for \( C', \Gamma_2 \vdash \Delta_2 \).

- Case \( A \notin \Gamma_1, A \notin \Gamma_2 \). Symmetric to previous case.

- Case \( A \notin \Gamma_1, A \notin \Gamma_2 \). Contradiction.

\[
7.13 \quad \text{Case } WR
\]

Symmetric to \( WL \).
8. Concrete Development of Formulae

The development described in the previous sections is axiomatic. We also provide a fully conservative definition based on a de Bruijn representation of binding for formulae. Formulae are defined as follows.

datatype form = P pred (tm list)
  ∨ |
  ∨ |
  ∨ ∃ form form
  ∨ ∃ form form
  ∨ ∃ form
  ∨ FAll form
  ∨ FEx form

Substitution is defined as usual.

consts fsubst :: (var ⇒ tm) ⇒ form ⇒ form
primrec
fsubst s (P i tms) = P i (map s tms)
fsubst s (|A| B) = |B| (fsubst s A)
fsubst s (|A| B) = (fsubst s A) (fsubst s B)
fsubst s (FAll A) = (∃ s A) = (fsubst s A)
fsubst s (FEx A) = (∀ s A) = (fsubst s A)
fsubst s (FAll A) = (λ s. A v of 0 ⇒ 0 ∨ Suc n ⇒ Suc s n))
in FAll (fsubst s A))
fsubst s (FEx A) = (λ s. A v of 0 ⇒ 0 ∨ Suc n ⇒ Suc s n))
in FEx (fsubst s A))

The axiomatic formulae constructors are defined concretely as follows.

consts
∀ var ⇒ form ⇒ form
∃ var ⇒ form ⇒ form
defs
∀ a A ≡ FAll (fsubst (λ v. if a then 0 else Suc v) A)
∃ a A ≡ FEx (fsubst (λ v. if a then 0 else Suc v) A)

Instantiation of quantified formulae is defined as follows.

consts
FAll-inst tm ⇒ form ⇒ form
FEx-inst tm ⇒ form ⇒ form
primrec
FAll-inst t (FAll A) = fsubst (λ v. case v of 0 ⇒ t ∨ Suc n ⇒ n A)
FEx-inst t (FEx A) = fsubst (λ v. case v of 0 ⇒ t ∨ Suc n ⇒ n A)

Positive and negative occurrences are defined in a mutually recursive fashion.

consts
posneg :: form ⇒ pred set ⇒ pred set
primrec
posneg (P i vs) = (∥ i ∥)
posneg |i| = (∥ i ∥)
posneg (|f| g) = let (fp fn) = posneg f in
let (gp gn) = posneg g in
(f ∪ fp, fn ∪ gn)
posneg (|f| g) = let (fp fn) = posneg f in
let (gp gn) = posneg g in
(f ∪ fp, fn ∪ gn)
posneg (|f| g) = let (p n) = posneg f in
(posneg (FAll f) = posneg f
(posneg (FEx f) = posneg f

constdefs neg :: form ⇒ pred set
neg ≡ sn d o posneg

Free variables are defined using an auxiliary function.

consts preSuc :: nat list ⇒ nat list
primrec
preSuc [] = []
preSuc (a#list) = (case a of 0 ⇒ preSuc list ∨ Suc n ⇒ n#(preSuc list))
consts fv :: form ⇒ var list
primrec
fv (P i tms) = tms
fv |i| = []
fv (|A| B) = (fv A) @ (fv B)
fv (|A| B) = (fv A) @ (fv B)
fv (∅ A) = fv A
fv (FAll A) = preSuc (fv A)
fv (FEx A) = preSuc (fv A)

All properties which we previously asserted axiomatically are proved for the corresponding concrete development. The main proof of Craig’s Interpolation Theorem can run happily using either the axiomatic development or the concrete development.

9. Analysis

9.1 Formal v. Informal

In the preceeding sections we have given an informal account of a formal mechanised proof. We have omitted numerous checks from the informal proof. For example:

- We noted already the omission of the checks on the polarity of predicates appearing in the interpolation formula.
- We omitted checking wellformedness of intermediate derivations which are used as witnesses in the proof.
- We omitted cases where symmetry is sufficient to allow the reader to reconstruct the proof from a previous case.
- We omitted eigenvariable checks in V R, ∃L cases.

Suffice it to say, including these details would have substantially increased the size of the informal presentation. Never-the-less, the informal presentation is by no means short.

The formal, mechanised version can be significantly shorter than an informal presentation because much of the proof can be relegated to automation. However, the formal proof is certainly less readable.

Ideally one would like the formal and the informal presentation to inhabit the same document. Ideally the formal terms should be typeset as informal practice. For example, derivations used in the proof should be typeset as such, not just quoted as HOL terms. Although Isabelle possesses some facilities in this area, improvements can certainly be made.

9.2 Mechanisation Statistics

Our abstract development of formulae consists of 95 lines (including whitespace), of which none are tactic lines, and our concrete development contains 210 lines, of which 51 are tactic lines.

Our main mechanised theory file contains 410 tactic lines. Each case in the main proof requires us to prove about 10 different subgoals, and each subgoal corresponds roughly to a single line of tactic script. We have 5 connectives or quantifiers, 10 corresponding left and right rules, and 2 subcases per rule, giving 20 cases in total. In addition, there are 4 cases for the Init rule, and 4 cases each for the two I’ rules, giving a total of 20 + 4 + (4 + 4) = 32 cases in all.
At approximately 10 lines per case, this gives rise to approximately 320 tactic lines, with the rest related to setting up outside inductions, and the derivation of the weak form of Craig’s Interpolation Theorem. The total line count is under 1000 lines, and this includes many whitespace lines, lemmas that reproduce in Isar what previously was conducted using tactics, and lines whose sole purpose is to re-quote formal witnesses so that they can be included in this informal presentation.

The point is simply that this development is extremely short.

9.3 Aims of the Mechanisation

In this section, we discuss what we tried to achieve with the mechanisation. Some of these achievements are far from obvious even when replaying the mechanised text step by step.

• Clear, Correct and Complete We hope our presentation is clear. Existing presentations are lacking in this area. For example, Girard in [Gir7] rephrases the induction statement halfway through the proof, whereas we have been careful to state our theorems precisely. Moreover, because we have formalised the proofs, many details that were murky have been uncovered. A particular area of concern is the informal tradition of requiring that the analysed formula appear explicitly in the conclusion of a rule. We believe the resulting proofs are often hard to read. For example, Girard follows the tradition, but the individual cases must introduce extra variables \( \Gamma_1, \ldots, \) which are later constrained such that e.g. either \( \Gamma_1' = \Gamma_1, A \) or \( \Gamma_1' = \Gamma_1' \). This doubling of the number of variables in play makes the proofs harder to follow.

One of the aims of mechanisation is to ensure that the proofs are impeccable. Existing presentations are deficient in this regard. For example, Girard’s presentation contains numerous typographical mistakes. Perhaps more worryingly, Girard dismisses the structural cases as trivial, and omits the proofs. However, our experience was that the structural rules, \( WL, WR \), combined with sequents that are (pairs of) \( sets \) of formulae, were the hardest to get right. We hope their inclusion here will clarify what otherwise might have remained a murky part of the proof. Certainly we have addressed all relevant cases, so that our presentation is complete.

Correctness of the proofs ultimately rests on the foundation of the theorem prover in which the mechanisation has taken place. Isabelle/HOL is a fully expansive theorem prover, whose kernel is small and has often been certified by experts. It is extremely unlikely that Isabelle would incorrectly assert that a theorem had been proven.

For correctness, one also requires that the definitions correspond to the related informal notions. We have tried to ensure that this is the case in two ways. We have used concrete mathematical structures which directly correspond to the intuitive notions wherever possible, rather than resorting to sophisticated techniques such as HOAS. Our derivations are concrete objects. Our sets of formulae are indeed sets. We have provided a standard presentation of first order formulae based on de Bruijn indices. Since much other work has been conducted with de Bruijn indices, they are fairly well understood, so that it should be easy to convince oneself of the correctness of our concrete presentation. On the other hand, we do not want our definitions to be over concrete, and so introduce unnecessary complexities. For example, we do not want our proofs to take advantage of properties that are present only for one particular implementation of formulae. For this reason, we have also isolated the weakest possible properties required in our proofs. For example, our axiomatic presentation of first order formulae, which involves variable binding, is extremely weak. For our particular proof of Craig’s Interpolation Theorem, these properties cannot be made weaker. These properties should be satisfied by any reasonable concrete implementation of first order formulae. Of course, we tied the two presentations of formulae together by proving that the axiomatic properties we require are satisfied by the de Bruijn representation.

It is still the case that the informal presentation in this paper, which is written by hand, may contain typos and other errors. Until the mechanised text becomes primary, this is inevitable. We have attempted to prevent errors creeping in by explicitly quoting the formal witnesses in the informal text. However, errors may still arise. The mechanised text does not suffer from these problems. Against this, even our informal presentation surely contains less errors and typos than appear in standard presentations. We hope that our presentation becomes definitive.

• Appropriate use of Automation To formally prove Craig’s Interpolation Theorem without automation would be a very lengthy task. We have used automation extensively to keep the formal mechanised proof to a small size. On the other hand, the only parts of the tactic script that are really essential are the initial use of induction over the size of the derivation, and the witnesses used to instantiate quantifiers. Thus, the proof could be made considerably smaller, i.e. the proof could simply be a call to automation with the existential witnesses supplied as a hint. However, we also wanted to preserve the structure of the proof, so that although the proof could be automated in one or two lines of tactic script, we prefer to sketch out the main case splits and match reasonably high-level subgoals to tactic lines in the mechanised proof.

• Elegance, Simplicity Our mechanisation is succinct. Our proofs are the weakest and most direct that we could manage. Usually there is some trade off in this area. Weakest proofs are typically those arrived at using \( \text{Cut} \) free proofs, and minimal strengthening of induction statements. However, it is sometimes the case that one can strengthen the induction statement in many ways, perhaps so that it is much stronger than required, but such that it is syntactically simpler than the minimal strengthening. The only possible place where we have strengthened an induction statement is in the statement of the strong form of Craig’s Interpolation Theorem, and this is a standard strengthening which we felt it would be unwise to deviate from. Moreover, we did not see much scope for a syntactically simpler version. Other than this, our proofs are \( \text{Cut} \) free, and as weak as they can be. This is what gives rise to the very weak axiomatisation of the properties of first order formulae.

For us, elegance is closely tied to syntactic properties of proofs and definitions. Thus, \( \text{Cut} \) free proofs are inherently elegant because, for example, they proceed without detour, direct from assumptions to conclusions. In addition, we have strived to keep our definitions simple and elementary. Simplicity aids understanding. Our aim in this is that the reader should never at any point feel that the development is not completely straightforward and elementary.

As an example of how we achieve simplicity, what is not so obvious from the informal and formal mechanised presentation of the result is the extent to which we have played around with various definitions to allow the mechanisation to be as clear and

\[ \text{This is not quite true, since in order to make the mechanisation as slick as possible, we have used equality rather than alpha equivalence. A mechanisation based on named variables and alpha equivalence would have to quotient the type of formulae by alpha in order to satisfy our axioms.} \]
straightforward as possible. For example, the two $W$ rules are actually admissible. However, since they are used extensively to form the derivation witnesses required in the proof, we would have to prove them admissible if we omitted them from our basic system. This in turn would involve a separate inductive proof to show the well known substitutivity property of eigenvariables in proofs. This would be a considerable detour, whilst we prefer our mechanisation to remain focused solely on the proof of Craig’s Interpolation Theorem. Craig’s Interpolation Theorem is essentially a structural theorem, so a detour into eigenvariable properties would be out of place and detract from the essence of the proof. For these reasons, we include the $W$ rules explicitly. The cost is that we must treat these cases in the proof. However, these cases are intrinsically interesting, as the hardest cases, and are required in other presentations, so that including these cases explicitly is a double gain.

- **Modularity**

In order to support different implementations of formulae, in this particular case the axiomatic version and the version based on de Bruijn notation, we have modularised our development. This consists of two related tasks.

- Identifying the weakest properties of formulae that are required in the main proof of Craig’s Interpolation Theorem.
- Identifying a minimal common language that all implementations of formulae have.

To find the weakest properties of formulae, one typically develops a Cut free proof, and examines the leaves of the proof to identify those that are provable solely in the language of formulae. To identify a minimal common language one examines the formulae constructs that appear in the main proof and tries to eliminate as much as possible.

In fact, these two activities are linked: one cannot conduct a proof, or even state the theorem, without some notion of what a formula is. On the other hand, the statement of the theorem may involve references to formulae constructs that are really redundant, yet their presence in the theorem statement forces their use throughout the proof.

For example, the notion of substitution which appears in the de Bruijn presentation, is present in some form in all concrete representations. It is therefore part of the common language. However, its absence from our axiomatic presentation of formulae indicates that it is not a necessary notion in order to prove Craig’s Interpolation Theorem. Whilst conducting early versions of the proofs, we began to suspect that substitution could be eliminated from the common language we were using for formulae, and we worked to bring this about. This is related to our previous comments on including the weakening rules explicitly.

We discuss these issues further in the section on applications of Craig’s Interpolation Theorem, Sect. 10.

10. Applications

In the introduction we claimed that Craig’s Interpolation Theorem has many applications. In fact, it is the kind of result that becomes part of one’s way of thinking about mechanisation, such as its diverse applications.

Let us immediately repeat that Craig’s Interpolation Theorem has a constructive proof, which is to say, it is an algorithm that transforms proofs and furnishes the interpolation formula. We have not expressed it as a deterministic algorithm, because the proof is essentially non-deterministic, so that determining it would be inelegant. Never-the-less it would be simple to write a primitive recursive function which produced the interpolant.

As another example, Craig’s Interpolation Theorem can be used in automatic proof search. Suppose we have two (disjunct) languages (set of predicates which may appear in formulae) $L_1, L_2$. We wish to prove

$$\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$$

where $\Gamma_i, \Delta_i$ is expressed in language $L_i$. By the strengthened interpolation theorem, Thm. 12, we can find a formula $C$ such that

$$\Gamma_1 \vdash \Delta_1, C \quad \text{and} \quad C, \Gamma_2 \vdash \Delta_2$$

and moreover such that all predicates appearing in $C$ appear also in $\Gamma_1, \Delta_1$ and in $\Gamma_2, \Delta_2$. But since $L_1, L_2$ are disjoint, $C$ can only be $\bot$ or $\top$. So

$$\Gamma_1 \vdash \Delta_1 \quad \text{or} \quad \Gamma_2 \vdash \Delta_2$$

We can then call our automation separately and in parallel on these two subproblems. In this way we have reduced the search space considerably, with nothing but syntactic considerations. This is an example of "purity of methods". As another example of purity of methods, if a sequent expressed in the language of $L_1$ is provable, it is provable without taking a detour via $L_2$, which is direct from the subformula property of Cut free derivations. Clearly this is extremely useful when restricting automation which would otherwise wander off into the extensive libraries of modern theorem provers in its search for a proof of some specific statement in a clearly defined sublanguage.

Let us now consider a more subtle use of Craig’s Interpolation Theorem. Suppose we wish to conduct a mechanisation that uses some form of variable abstraction and binding. For example, in our mechanisation we wish to have a representation of first order formulae. We might wish to use our favourite representation, say, de Bruijn. The basic type of abstraction provided by de Bruijn representations binds the free 0th variable, as is evident in the datatype of de Bruijn formulae.

```plaintext
datatype form =
    | FAll form
    | FEx form
```

Whilst this may make sense for a representation where variables are numbers, it makes no sense for a named representation say, where there is no inherent notion of order on the variables.

Craig’s Interpolation Theorem suggests that we should pay close attention to the language we use to state theorems. For example, let $F$ represent the axioms for our representation of formulae, and $T$ the main theorem we wish to prove. Then we can find a $C$ such that

$$F \vdash C \quad \text{and} \quad C \vdash T$$

This $C$ is the interface between the subtheory generated by $F$ and our main theory in which we prove $T$. If we now replace $F$ with some other implementation of formulae, $F'$, we would have to rephrase $T$ in terms of this new implementation as $T'$, the lemmas $C$ exported by our theory of formulae would change to $C'$, and much additional reworking of proofs would result. To remedy this, we should express $T$ using formulae constructs that are found in every implementation. In this case, Craig’s Interpolation Theorem assures us that $C$ must be expressed also in this shared language,

5 Or conjunctions, disjunctions, negations of $\top, \bot$… which can be simplified to $\bot$ or $\top$. 
and the only rework that is required when changing formulae representations is in the proof of $F' \vdash C$.

Returning to our example, if we used the de Bruijn representation directly, our phrasing of Craig’s Interpolation Theorem would include de Bruijn constructs, and our mechanisation would include much that was specific to de Bruijn representations.

For this reason, we avoid the basic de Bruijn abstraction, and work instead with a named abstraction, even though named abstraction is not a given for our underlying de Bruijn implementation. We do not unnecessarily bias our development towards named implementations either—rather than instantiate a quantifier $\forall x. A$ as $[t/x]A$, we have an operation of “instantiation on the top most quantifier”. $\mathsf{Fail}{\mathsf{inst}}\ t\ (\forall x.\ A)$. Our axiomatic presentation (which is nothing more than the separate clauses of the interpolant $C$) certainly hides the de Bruijn specific constructs. The advantage is that we could later substitute some new implementation of formulæ (named, bound/free) without any additional work in the main theory, though we would of course have to prove our axioms (the clauses of our interpolant $C$) were satisfied in this new implementation. In this way we have used Craig’s Interpolation Theorem in the mechanisation of the proof of the theorem itself!

This approach can also be used to refactor existing theories, since Craig’s Interpolation Theorem transforms existing proofs.

In my thesis [Rid05] I suggest other ways in which Craig’s Interpolation Theorem can shape a mechanisation.

11. Conclusion

We presented the first complete mechanisation of Craig’s Interpolation Theorem. We also talked about some aspects of the mechanisation, and some of the applications of the theorem to mechanised reasoning. In the main text, we have indicated where the contributions of the paper lie, and we briefly recap some of these here.

- Clear, correct and complete formal presentation of Craig’s Interpolation Theorem.
- We have worked hard to isolate the minimal properties we require during the proof. For example, we present a very weak axiomatisation of first order formulæ. For another example, we phrase the logical system in such a way that we avoid a detour through the eigenvariable properties of derivations.
- Complete rendition of mechanised version, save that some proof scripts have been omitted.
- Discussion of the application of Craig’s Interpolation Theorem to mechanisation and automation.
- Particularly, we described our development of first order formulæ with their notion of binding, and how we obtained such a weak axiomatisation.

There is some related work. In [Bou96], the author develops a partial proof of Craig’s Interpolation Theorem in Coq. This is based on a single propositional connective, NAND. As the author admits, the intent was to extend the work to the usual formulæ, but unfortunately this was never attempted. This work is certainly considerably more involved than that presented here. Moreover, the importance of these results usually does not lie in the result itself, but in the details of the proof: if one understands the details, one can adapt the proof and use variants of the result in one’s own work. Thus, the restriction to a rather unusual connective is indeed a real restriction, since one has to work much harder to translate the usual formulæ one meets during proof into NAND form. Furthermore, the proof is sufficiently complicated that much of the beauty of Craig’s Interpolation Theorem has been lost in the details of formalisation.

Further afield, there is much formalised proof theory. Let us briefly mention the work of Pfenning on formalised Cut elimination $\mathsf{Pte}\mathsf{mC}$ in Twelf, which is inspirational. A more sophisticated development is that of strong normalisation for System F by Altenkirch in LEGO [Alt93].

References

[ABF+05] Brian E. Aydemir, Aaron Bohannon, Matthew Fairbairn, J. Nathan Foster, Benjamin C. Pierce, Peter Sewell, Dimitrios Vytiniotis, Geoffrey Washburn, Stephanie Weirich, and Steve Zdancewic. Mechanized metatheory for the masses: The POPLmark challenge. In International Conference on Theorem Proving in Higher Order Logics (TPHOLs), August 2005.

[Alt93] Thorsten Altenkirch. A formalization of the strong normalization proof for System F in LEGO. In J.F. Groote M. Bezem, editor, Typed Lambda Calculi and Applications, LNCS 664, pages 13 – 28, 1993.

[Bou96] S. Boulme. A proof of Craig’s Interpolation Theorem in Coq, 1996.

[Gir87] Jean-Yves Girard. Proof Theory and Logical Complexity, Volume 1. bibliopolis, 1987.

[Pfe00] Frank Pfenning. Structural cut elimination I. intuitionistic and classical logic. Information and Computation, 157(1/2):84–141, March 2000.

[PNW03] Larry Paulson, Tobias Nipkow, and Markus Wenzel. The Isabelle distribution, 2003. http://www.cl.cam.ac.uk/Research/HVG/Isabelle.

[Rid05] Tom Ridge. Enhancing the expressivity and automation of an interactive theorem prover in order to verify multicast protocols. PhD thesis, LFCS, Edinburgh University, 2005.

[Sza69] M. E. Szabo, editor. The Collected Papers of Gerhard Gentzen. North-Holland Publishing Co., Amsterdam, 1969.

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