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Weighted Midpoint Hermite-Hadamard-Fejér Type Inequalities in Fractional Calculus for Harmonically Convex Functions

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Abstract: In this paper, we establish a new version of Hermite-Hadamard-Fejér type inequality for harmonically convex functions in the form of weighted fractional integral. Secondly, an integral identity and some weighted midpoint fractional Hermite-Hadamard-Fejér type integral inequalities for harmonically convex functions by involving a positive weighted symmetric functions have been obtained. As shown, all of the resulting inequalities generalize several well-known inequalities, including classical and Riemann–Liouville fractional integral inequalities.

Keywords: symmetry; weighted fractional operators; harmonically convex functions; Hermite-Hadamard-Fejér type inequality

1. Introduction

The theory of convex functions is an essential tool in various fields of pure and applied sciences. There is also a close connection between the theory of convex functions, the theory of inequalities, and fractional differential equations. At the same time, fractional differential equations are one of the most studied fields of mathematics due to their application in the real world. Many inequalities are proved for convex functions but, the most known from the related literature is Hermite-Hadamard inequality.

A function \( F : [\theta_1, \theta_2] \subset \mathbb{R} \to \mathbb{R} \) on an interval of real line is said to be convex, if for all \( \theta_1, \theta_2 \in I \) and \( \tau \in [0, 1] \), then

\[
F(\tau\theta_1 + (1-\tau)\theta_2) \leq \tau F(\theta_1) + (1-\tau)F(\theta_2). \tag{1}
\]

The Hermite-Hadamard integral inequality is a well-known inequality in the subject of convex functional analysis. It has an interesting geometric representation with numerous important applications. The extraordinary inequality states that if \( F : I \to \mathbb{R} \) is a convex mapping on the interval \( I \) of real numbers and \( \theta_1, \theta_2 \in I \) with \( \theta_1 < \theta_2 \), then

\[
F\left(\frac{\theta_1 + \theta_2}{2}\right) \leq \frac{1}{\theta_2 - \theta_1} \int_{\theta_1}^{\theta_2} F(\tau)d\tau \leq \frac{F(\theta_1) + F(\theta_2)}{2}. \tag{2}
\]

Inequality (2) was introduced by C. Hermite [1] and investigated by J. Hadamard [2] in 1893. Both inequalities hold in the inverted direction if \( F \) is concave. Many mathematicians have paid considerable attention to Hermite-Hadamard inequality due to its quality and
integrity in mathematical inequality. For significant developments, modifications, and consequences regarding the Hermite-Hadamard uniqueness property and general convex function definitions, for essential details, the interested reader would like to refer to [3–7] and references therein. Fractional calculus and applications have application areas in many different fields such as physics, chemistry and engineering, and mathematics. Applying arithmetic in classical analysis in the fractional analysis is very important in obtaining more realistic results in solving many problems. Many real dynamical systems are better characterized using non-integer order dynamic models based on fractional computation. While integer orders are a model that is not suitable for nature in classical analysis, fractional computation in which arbitrary orders are examined enables us to obtain more realistic approaches. Regarding some papers dealing with fractional integral inequalities via different types of fractional integral operators, we refer readers to [8–16].

Furthermore, Sarikaya et al. [17] generalized and reformed the Hermite-Hadamard integral inequality (2) in forms of Riemann–Liouville fractional integrals in 2013.

\[
\mathcal{F}\left(\frac{\theta_1 + \theta_2}{2}\right) \leq \frac{\Gamma(v+1)}{2(\theta_2 - \theta_1)v} \left[ I_{\theta_1}^v \mathcal{F}(\theta_2) + I_{\theta_2}^v \mathcal{F}(\theta_1) \right] \leq \frac{\mathcal{F}(\theta_1) + \mathcal{F}(\theta_2)}{2},
\]

where the function \(\mathcal{F}: [\theta_1, \theta_2] \rightarrow \mathbb{R}\) with \(0 \leq \theta_1 < \theta_2\) and \(\mathcal{F} \in L^1[\theta_1, \theta_2]\). Therefore, \(I_{\theta_1}^v\) is left-sided Riemann–Liouville fractional integrals and \(I_{\theta_2}^v\) is the right-sided Riemann–Liouville fractional integrals with order \(v > 0\), defined by [18]

\[
\begin{align*}
I_{\theta_1}^v \mathcal{F}(\tau) &= \frac{1}{\Gamma(v)} \int_{\theta_1}^{\tau} (\tau - \kappa)^{v-1} \mathcal{F}(\kappa) d\kappa, \quad \tau > \theta_1, \\
I_{\theta_2}^v \mathcal{F}(\tau) &= \frac{1}{\Gamma(v)} \int_{\tau}^{\theta_2} (\kappa - \tau)^{v-1} \mathcal{F}(\kappa) d\kappa, \quad \tau < \theta_2,
\end{align*}
\]

respectively. Here, \(\Gamma(v)\) is the Gamma function and \(I_{\theta_1}^0 \mathcal{F}(\tau) = I_{\theta_2}^0 \mathcal{F}(\tau) = \mathcal{F}(\tau)\).

Due to the use of the interval’s end \(\theta_1, \theta_2\), the inequality (3) is called endpoint Hermite-Hadamard inequality.

The midpoint Hermite-Hadamard inequality was discovered by Sarıkaya and Yıldırım [19] after expending the essential area of the integral inequalities in (2) and (3)

\[
\mathcal{F}\left(\frac{\theta_1 + \theta_2}{2}\right) \leq \frac{2^{v-1}\Gamma(v+1)}{(\theta_2 - \theta_1)v} \left[ I_{\frac{\theta_1 + \theta_2}{2}}^v \mathcal{F}(\theta_2) + I_{\frac{\theta_1 + \theta_2}{2}}^v \mathcal{F}(\theta_1) \right] \leq \frac{\mathcal{F}(\theta_1) + \mathcal{F}(\theta_2)}{2},
\]

where the function \(\mathcal{F}: [\theta_1, \theta_2] \rightarrow \mathbb{R}\) is convex and continuous. In [20], I. G. Macdonald gave the following definition.

**Definition 1.** Suppose that a function \(G: [\theta_1, \theta_2] \rightarrow [0, \infty)\) and it is symmetric with respect to \(\frac{\theta_1 + \theta_2}{2}\) if

\[G(\theta_1 + \theta_2 - \kappa) = G(\kappa), \text{ for all } \kappa \in [\theta_1, \theta_2].\]

Fejér proposed the following generalization of Hadamard inequality in 1906 (see [21]):

**Theorem 1.** Let \(\mathcal{F}: [\theta_1, \theta_2] \rightarrow \mathbb{R}\) be a convex function such that \(\theta_1 < \theta_2\). Furthermore, let \(G: [\theta_1, \theta_2] \rightarrow \mathbb{R}\) be a positive, integrable and symmetric to \(\frac{\theta_1 + \theta_2}{2}\). Then the following inequality holds:

\[
\mathcal{F}\left(\frac{\theta_1 + \theta_2}{2}\right) \int_{\theta_1}^{\theta_2} G(\tau) d\tau \leq \int_{\theta_1}^{\theta_2} \mathcal{F}(\tau)G(\tau) d\tau \leq \frac{\mathcal{F}(\theta_1) + \mathcal{F}(\theta_2)}{2} \int_{\theta_1}^{\theta_2} G(\tau) d\tau.
\]

The inequality (6) is well-known as the Fejér-Hadamard inequality in the literature. In the concept of Riemann–Liouville fractional integrals, I. Işcan [22] discovered the endpoint
version of (6), which is also the extension of (3). As a result, the final inequalities are shown as follows:

\[ \mathcal{F} \left( \frac{\theta_1 \theta_2}{\theta_1 + \theta_2} \right) \left[ \mathcal{F}^\nu_{\theta_1} \mathcal{F}(\theta_2) + \mathcal{F}^\nu_{\theta_2} (\mathcal{F})(\theta_1) \right] \leq \left[ \mathcal{F}^\nu_{\theta_1} (\mathcal{F} \mathcal{G})(\theta_2) + \mathcal{F}^\nu_{\theta_2} (\mathcal{F} \mathcal{G})(\theta_1) \right] \leq \frac{\mathcal{F}(\theta_1) + \mathcal{F}(\theta_2)}{2} \left[ \mathcal{F}^\nu_{\theta_1} \mathcal{F}(\theta_2) + \mathcal{F}^\nu_{\theta_2} \mathcal{F}(\theta_1) \right], \]  

(7)

where \( \mathcal{F} \) is convex and continuous, \( \mathcal{G} \) is symmetric and belongs to \( L^1[\theta_1, \theta_2] \), (see Definition 1).

In [23], I. Işcan gave definition of harmonically convex functions and established following Hermite-Hadamard type inequality for harmonically convex functions as follows:

**Definition 2.** Let \( I \subset \mathbb{R} \backslash \{0\} \) be an interval of nonzero real numbers. Then a function \( \mathcal{F} : I \rightarrow \mathbb{R} \) is said to be harmonically convex if

\[ \mathcal{F} \left( \frac{\theta_1 \theta_2}{\theta_1 + \theta_2} \right) \leq \tau \mathcal{F}(\theta_2) + (1 - \tau) \mathcal{F}(\theta_1), \]

holds for all \( \theta_1, \theta_2 \in I \) and \( \tau \in [0, 1] \).

In [24], Latif et al. gave the following definition.

**Definition 3.** A function \( \mathcal{F} : [\theta_1, \theta_2] \subseteq \mathbb{R} \backslash \{0\} \rightarrow \mathbb{R} \) is said to be harmonically symmetric with respect to \( 2\frac{\theta_1 \theta_2}{\theta_1 + \theta_2} \), if

\[ \mathcal{F} \left( \frac{1}{\kappa} \right) = \mathcal{F} \left( \frac{1}{\frac{\theta_1}{\theta_1} + \frac{\theta_2}{\theta_2} - \kappa} \right), \]

\( \kappa \in [\theta_1, \theta_2] \).

Işcan et al. published Hermite-inequality Hadamard’s in fractional integral type for harmonically convex functions in [22], as follows:

**Theorem 2.** Let \( \mathcal{F} : I \subset \mathbb{R} \backslash \{0\} \rightarrow \mathbb{R} \) be a harmonically convex function and \( \theta_1, \theta_2 \in I \) with \( \theta_1 < \theta_2 \). If \( \mathcal{F} \in L^1[\theta_1, \theta_2] \), then the following inequalities holds:

\[ \mathcal{F} \left( 2\frac{\theta_1 \theta_2}{\theta_1 + \theta_2} \right) \leq \frac{\theta_1 \theta_2}{\theta_2 - \theta_1} \int_{\theta_1}^{\theta_2} \frac{\mathcal{F}(\tau)}{\tau^2} \, d\tau \leq \frac{\mathcal{F}(\theta_1) + \mathcal{F}(\theta_2)}{2}. \]  

(8)

Hermite-Hadamard inequalities for harmonically convex functions were introduced in fractional integral form in [25] as follows:

**Theorem 3.** Consider a function \( \mathcal{F} : I \subset (0, \infty) \rightarrow \mathbb{R} \) such that \( \mathcal{F} \in L^1[\theta_1, \theta_2] \), where \( \theta_1, \theta_2 \in I \) with \( \theta_1 < \theta_2 \). If \( \mathcal{F} \) is a harmonically convex function on \( [\theta_1, \theta_2] \), then

\[ \mathcal{F} \left( 2\frac{\theta_1 \theta_2}{\theta_1 + \theta_2} \right) \leq \frac{\Gamma(\nu + 1)}{2} \left( \frac{\theta_2 \theta_2}{\theta_1 + \theta_2} \right)^\nu \left[ \mathcal{I}^\nu_{\theta_1} (\mathcal{F} \circ \mathcal{H}) \left( \frac{1}{\theta_2} \right) \mathcal{I}^\nu_{\theta_2} (\mathcal{F} \circ \mathcal{H}) \left( \frac{1}{\theta_1} \right) \right] \]

\[ \leq \frac{\mathcal{F}(\theta_1) + \mathcal{F}(\theta_2)}{2}, \]  

(9)

with \( \nu > 0 \) and \( \mathcal{H}(\kappa) = \frac{1}{\kappa} \) where \( \kappa = \left[ \frac{1}{\theta_2}, 1 \right] \).

For harmonically convex functions, in [17] Chan et al. stated the Hermite-Hadamard-Fejér inequality as follows:
Theorem 4. Suppose that harmonically convex function $F : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$. If $F \in L^1(\theta_1, \theta_2)$ and $G : [\theta_1, \theta_2] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is positive, an integrable, and harmonically symmetric with respect to $\frac{2\theta_1 \theta_2}{\theta_1 + \theta_2}$, then

$$F \left( \frac{2\theta_1 \theta_2}{\theta_1 + \theta_2} \right) \int_{\theta_1}^{\theta_2} \frac{G(\tau)}{\tau^2} d\tau \leq \int_{\theta_1}^{\theta_2} \frac{F(\tau)G(\tau)}{\tau^2} d\tau \leq \frac{F(\theta_1) + F(\theta_2)}{2} \int_{\theta_1}^{\theta_2} \frac{G(\tau)}{\tau^2} d\tau,$$

and $\theta_1, \theta_2 \in I$ with $\theta_1 < \theta_2$.

In [26], İscan et al. proved Hermite-Hadamard-Fejér type inequalities for harmonically convex functions through fractional integrals:

Theorem 5. Let $F : [\theta_1, \theta_2] \to \mathbb{R}$ be a harmonically convex function and $\theta_1, \theta_2 \in I$ with $\theta_1 < \theta_2$. If $F \in L^1(\theta_1, \theta_2)$ and $G : [\theta_1, \theta_2] \to \mathbb{R}$ is positive, an integrable, and harmonically symmetric with respect to $\frac{2\theta_1 \theta_2}{\theta_1 + \theta_2}$, then

$$F \left( \frac{2\theta_1 \theta_2}{\theta_1 + \theta_2} \right) \left[ \frac{T^v_{\theta_1}}{\Gamma(1 + \frac{v}{\theta_1})} (G \circ H) \left( \frac{1}{\theta_1} \right) T^v_{\theta_2} - \left( G \circ H \right) \left( \frac{1}{\theta_1} \right) \right] \leq \left[ \frac{T^v_{\theta_1}}{\Gamma(1 + \frac{v}{\theta_1})} (F \circ H) \left( \frac{1}{\theta_1} \right) T^v_{\theta_2} + \left( F \circ H \right) \left( \frac{1}{\theta_1} \right) \right] \leq \frac{F(\theta_1) + F(\theta_2)}{2} \left[ \frac{T^v_{\theta_1}}{\Gamma(1 + \frac{v}{\theta_1})} (G \circ H) \left( \frac{1}{\theta_1} \right) T^v_{\theta_2} + \left( G \circ H \right) \left( \frac{1}{\theta_1} \right) \right],$$

with $v > 0$ and $H(\kappa) = \frac{1}{\kappa}$, $\kappa \in \left[ \frac{1}{\theta_2}, \frac{1}{\theta_1} \right]$.

In [27], Fahad et al. presented weighted fractional integrals as follows:

Definition 4. Let $(\theta_1, \theta_2) \subseteq \mathbb{R}$ and $\sigma(\tau)$ be an increasing positive and monotone function on the interval $(\theta_1, \theta_2)$ with a continuous derivative $\sigma'(\tau)$ on the open interval $(\theta_1, \theta_2)$. Then the weighted fractional integrals on the left and right sides of a function $F$ according to another function $\sigma(\tau)$ on $(\theta_1, \theta_2)$ are shown as below:

$$\left( w^{\sigma, v} F \right)(\tau) = \frac{[w(\tau)]^{-1}}{1(\tau)} \int_{\theta_1}^{\tau} \sigma'(\kappa)(\sigma(\tau) - \sigma(\kappa))^{v-1} F(\kappa) w(\kappa) d\kappa,$$

$$\left( w^{\sigma, v} F \right)(\tau) = \frac{[w(\tau)]^{-1}}{1(\tau)} \int_{\tau}^{\theta_2} \sigma'(\kappa)(\sigma(\kappa) - \sigma(\tau))^{v-1} F(\kappa) w(\kappa) d\kappa,$$

with $v > 0$ and for $[w(\tau)]^{-1} := \frac{1}{w(\tau)}$ with $w(\tau) \neq 0$.

Remark 1. Using Definition 4, we have

(i) Equation (12) can be restated in the following form: If $\sigma(\tau) = \tau$ with $w(\tau) = 1$, then the weighted fractional integrals reduce to the classical Riemann–Liouville fractional integrals.

(ii) Putting $w(\tau) = 1$, so we obtain the fractional integrals of $F$ with regard to the function $\sigma(\tau)$, for more details see [28,29]:

$$\left( T^{\sigma, v}_{\theta_1} F \right)(\tau) = \frac{1}{\Gamma(1 + \frac{v}{\theta_1})} \int_{\theta_1}^{\tau} \sigma'(\kappa)(\sigma(\tau) - \sigma(\kappa))^{v-1} F(\kappa) w(\kappa) d\kappa,$$

$$\left( T^{\sigma, v}_{\theta_2} F \right)(\tau) = \frac{1}{\Gamma(1 + \frac{v}{\theta_2})} \int_{\tau}^{\theta_2} \sigma'(\kappa)(\sigma(\kappa) - \sigma(\tau))^{v-1} F(\kappa) w(\kappa) d\kappa,$$

with $v > 0$. 

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We recall the following special functions which are known as Beta and hypergeometric function
\[
\beta(\gamma_1, \gamma_2) = \frac{\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(\gamma_1 + \gamma_2)} = \int_0^1 \tau^{\gamma_1-1}(1 - \tau)^{\gamma_2-1}d\tau, \quad \gamma_1, \gamma_2 > 0.
\]
\[
2F_1(\gamma_1, \gamma_2; \gamma_3, \zeta) = \frac{1}{\beta(\gamma_2, \gamma_3-\gamma_2)} \int_0^1 \tau^{\gamma_2-1}(1 - \tau)^{\gamma_3-\gamma_2-1}(1 - \zeta\tau)^{-\gamma_1}d\tau,
\]
for \( \gamma_3 > \gamma_2 > 0, \ |\zeta| < 1, \)
respectively, (see [18]).

The polygamma function of order \( m \) is a meromorphic function on the complex numbers \( \mathbb{C} \) defined as the \( (m+1) \)th derivative of the logarithm of the gamma function:
\[
\psi^{(m)}(\zeta) := \frac{d^m}{d\zeta^m} \ln \Gamma(\zeta).
\]
Thus
\[
\psi^{(0)}(\zeta) = \psi(\zeta) = \frac{\Gamma'(\zeta)}{\Gamma(\zeta)}
\]
holds where \( \psi(\zeta) \) is the digamma function and \( \Gamma(\zeta) \) is the gamma function.

When \( m > 0 \) and \( \text{Re} > 0 \), the integral representation of polygamma is given by
\[
\psi^{(m)}(\zeta) = (-1)^{m+1} \int_0^\infty \frac{\tau^m e^{-\zeta\tau}}{1-e^{-\tau}}d\tau
\]
\[
= -\int_0^1 \frac{\tau^{\zeta-1}}{1-\tau} (\ln \tau)^m d\tau.
\]

The generalized hypergeometric function is given by a hypergeometric series, i.e., a series for which the ratio of successive terms can be written as
\[
\frac{c_{k+1}}{c_k} = \frac{(k + \theta_1)(k + \theta_2)...(k + \theta_p)}{(k + \pi_1)(k + \pi_2)...(k + \pi_q)(k + 1)}.
\]

The resulting generalized hypergeometric function is written
\[
\sum_{k=0}^{\infty} c_k x^k = p \mathcal{F}_q \left[ \begin{array}{c} \theta_1, \theta_2, ..., \theta_p \\ \pi_1, \pi_2, ..., \pi_p \end{array} ; x \right]
\]
\[
= p \mathcal{F}_q (\theta_1, \theta_2, ..., \theta_p; \pi_1, \pi_2, ..., \pi_p; x)
\]
\[
= \sum_{k=0}^{\infty} \frac{(\theta_1)_k (\theta_2)_k ... (\theta_p)_k}{(\pi_1)_k (\pi_2)_k ... (\pi_p)_k} k! x^k
\]
where \( x \in \mathbb{C}, p \leq q, \theta_i, \pi_j \in \mathbb{C}, \pi_j \neq 0, -1, -2, ..., i = 1, 2, ..., p, j = 1, 2, ..., q \) and \( (\theta)_k \) is the Pochhammer symbol or rising factorial defined by
\[
(\theta)_k = \frac{\Gamma(\theta + k)}{\Gamma(\theta)} = \theta(\theta + 1)(\theta + 2)...(\theta + k + 1).
\]

In this article, we will use fractional weighted integrals (12) with nonnegative symmetric weighted functions in the kernel, necessary and auxiliary lemmas to study Hermite-Hadamard-Fejér type inequalities in Section 2. We shall prove our key results in Section 3, which will include new midpoint fractional Hermite-Hadamard-Fejér type integral inequalities as well as some related results. The conclusion will be presented in Section 4.
2. Auxiliary Results

Lemma 1. If $\mathcal{F} : [\theta_1, \theta_2] \rightarrow \mathbb{R}$ is integrable and harmonically symmetric with respect to $\frac{2\theta_1 \theta_2}{\kappa \theta_1 + (2 - \kappa) \theta_2}$, then

(i) for each $\kappa \in [0, 1]$, we have

$$w\left(\frac{2\theta_1 \theta_2}{\kappa \theta_2 + (2 - \kappa) \theta_1}\right) = w\left(\frac{2\theta_1 \theta_2}{\kappa \theta_1 + (2 - \kappa) \theta_2}\right).$$

(ii) For $\nu > 0$, we have

$$
\left(\mathcal{I}^{\nu, \rho}_{\sigma^{-1}\left(\frac{\theta_1 + \theta_2}{2}\right)}(w \circ \mathcal{H} \circ \sigma)\right)\left(\sigma^{-1}\left(\frac{1}{\theta_2}\right)\right) = \left(\mathcal{I}^{\nu, \rho}_{\sigma^{-1}\left(\frac{\theta_1 + \theta_2}{2}\right)}(w \circ \mathcal{H} \circ \sigma)\right)\left(\sigma^{-1}\left(\frac{1}{\theta_2}\right)\right) - 2\left(\mathcal{I}^{\nu, \rho}_{\sigma^{-1}\left(\frac{\theta_1 + \theta_2}{2}\right)}(w \circ \mathcal{H} \circ \sigma)\right)\left(\sigma^{-1}\left(\frac{1}{\theta_2}\right)\right)
$$

and $\mathcal{H}(\tau) = \frac{\tau}{\nu}, \tau \in \left[\frac{1}{\theta_2}, \frac{1}{\theta_1}\right].$

Proof.

(i) Suppose that $\tau = \frac{2\theta_1 \theta_2}{\kappa \theta_1 + (2 - \kappa) \theta_2}$ with $\tau \in [\theta_1, \theta_2]$ and $\kappa \in [0, 1]$ such that $\frac{1}{\theta_1 + \nu} - \tau = \frac{2\theta_1 \theta_2}{\kappa \theta_1 + (2 - \kappa) \theta_2}$. Then, using the hypotheses and Definition 3, we can obtain (14).

(ii) $w$ has symmetry characteristic, then

$$(w \circ \mathcal{H} \circ \sigma)(\kappa) = w\left(\frac{1}{\sigma(\kappa)}\right) = w\left(\frac{1}{\theta_1 + \frac{1}{\theta_2} - \sigma(\kappa)}\right), \quad \forall \kappa \in \left[\sigma^{-1}\left(\frac{1}{\theta_2}\right), \sigma^{-1}\left(\frac{1}{\theta_1}\right)\right].$$

Hence, from above and setting $\frac{1}{\sigma(\tau)} = \frac{\tau}{\theta_1 + \frac{1}{\theta_2} - \sigma(\kappa)}$, it follows that

$$
\left(\mathcal{I}^{\nu, \rho}_{\sigma^{-1}\left(\frac{\theta_1 + \theta_2}{2}\right)}(w \circ \mathcal{H} \circ \sigma)\right)\left(\sigma^{-1}\left(\frac{1}{\theta_2}\right)\right)
\begin{align*}
&= \frac{1}{\Gamma(\nu)} \int_{\tau = \frac{1}{\theta_1 + \frac{1}{\theta_2} - \sigma(\kappa)}}^{\sigma^{-1}\left(\frac{1}{\theta_2}\right)} \left(\sigma(\tau) - \frac{1}{\theta_2}\right)^{\nu-1} (w \circ \mathcal{H} \circ \sigma)(\tau) \sigma'(\tau) d\tau \\
&= \frac{1}{\Gamma(\nu)} \int_{\tau = \frac{1}{\theta_2}}^{\sigma^{-1}\left(\frac{1}{\theta_2}\right)} \left(\sigma(\tau) - \frac{1}{\theta_2}\right)^{\nu-1} w\left(\frac{1}{\theta_1 + \frac{1}{\theta_2} - \sigma(\tau)}\right) \sigma'(\tau) d\tau \\
&= \frac{1}{\Gamma(\nu)} \int_{\tau = \frac{1}{\theta_2}}^{\sigma^{-1}\left(\frac{1}{\theta_2}\right)} \left(\frac{1}{\theta_1} - \sigma(\kappa)\right)^{\nu-1} w\left(\frac{1}{\sigma'(\kappa)}\right) \left(-\sigma'(\kappa)\right) d\kappa \\
&= \frac{1}{\Gamma(\nu)} \int_{\tau = \frac{1}{\theta_2}}^{\sigma^{-1}\left(\frac{1}{\theta_2}\right)} \left(\frac{1}{\theta_1} - \sigma(\tau)\right)^{\nu-1} (w \circ \mathcal{H} \circ \sigma)(\kappa) \sigma'(\kappa) d\kappa \\
&= \left(\mathcal{I}^{\nu, \rho}_{\sigma^{-1}\left(\frac{\theta_1 + \theta_2}{2}\right)}(w \circ \mathcal{H} \circ \sigma)\right)\left(\sigma^{-1}\left(\frac{1}{\theta_2}\right)\right),
\end{align*}
$$

which brings the needed equality (15).
Theorem 6. Let \( 0 < \theta_1 < \theta_2 \), let \( \mathcal{F} : [\theta_1, \theta_2] \to \mathbb{R} \) be an \( L^1 \) harmonically convex function and \( w : [\theta_1, \theta_2] \to \mathbb{R} \) is nonnegative, an integrable, and symmetric weighted function with respect to \( \frac{2\theta_1 \theta_2}{\theta_2 - \theta_1} \). If \( \sigma \) is an increasing and positive function from \( [\theta_1, \theta_2] \) onto itself such that its derivative \( \sigma'(\tau) \) is continuous on \( (\theta_1, \theta_2) \) for \( v > 0 \), then

\[
\mathcal{F}\left( \frac{2\theta_1 \theta_2}{\theta_2 - \theta_1} \right) \left[ \left( \int_{\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} (w \circ H \circ \sigma) \left( \sigma^{-1}\left( \frac{1}{\theta_1} \right) \right) + \left( \sigma^{-1}\left( \frac{1}{\theta_2} \right) \right) \right) \right]
\times \left( \sigma^{-1}\left( \frac{1}{\theta_2} \right) \right) \leq w \left( \frac{1}{\theta_1} \right) \int_{\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} (w \circ H \circ \sigma) \left( \sigma^{-1}\left( \frac{1}{\theta_1} \right) \right) \left( \sigma^{-1}\left( \frac{1}{\theta_2} \right) \right) \leq \frac{\mathcal{F}(\theta_1) + \mathcal{F}(\theta_2)}{2}
\times \left[ \left( \int_{\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} (w \circ H \circ \sigma) \left( \sigma^{-1}\left( \frac{1}{\theta_1} \right) \right) + \left( \sigma^{-1}\left( \frac{1}{\theta_2} \right) \right) \right) \right].
\]

Proof. Since \( \mathcal{F} \) is a harmonically convex function on \( [\theta_1, \theta_2] \), we write

\[
\mathcal{F}\left( \frac{2\pi_1 \pi_2}{\pi_1 + \pi_2} \right) \leq \frac{\mathcal{F}(\pi_1) + \mathcal{F}(\pi_2)}{2}, \text{ for all } \pi_1, \pi_2 \in [\theta_1, \theta_2].
\]

Therefore, for \( \pi_1 = \frac{2\theta_1 \theta_2}{\theta_2 + (2 - \kappa) \theta_2} \) and \( \pi_2 = \frac{2\theta_1 \theta_2}{\theta_2 + (2 - \kappa) \theta_2} \), \( \kappa \in [0, 1] \), it follows

\[
2\mathcal{F}\left( \frac{2\theta_1 \theta_2}{\theta_2 + \theta_2} \right) \leq \mathcal{F}\left( \frac{2\theta_1 \theta_2}{\kappa \theta_1 + (2 - \kappa) \theta_1} \right) + \mathcal{F}\left( \frac{2\theta_1 \theta_2}{\kappa \theta_2 + (2 - \kappa) \theta_2} \right)
\]

Multiplying both sides of (17) by \( \kappa^{v-1}w\left( \frac{2\theta_1 \theta_2}{\kappa \theta_1 + (2 - \kappa) \theta_2} \right) \) and integrating the resulting inequality with respect to \( \kappa \) over \([0, 1]\), we obtain

\[
2\mathcal{F}\left( \frac{2\theta_1 \theta_2}{\theta_2 - \theta_1} \right) \int_0^1 \kappa^{v-1}w\left( \frac{2\theta_1 \theta_2}{\kappa \theta_1 + (2 - \kappa) \theta_1} \right) d\kappa
\leq \int_0^1 \kappa^{v-1} \mathcal{F}\left( \frac{2\theta_1 \theta_2}{\kappa \theta_1 + (2 - \kappa) \theta_1} \right) w\left( \frac{2\theta_1 \theta_2}{\kappa \theta_1 + (2 - \kappa) \theta_1} \right) d\kappa
\]

From the left-hand side of the inequality in (18), we use (15) to obtain

\[
\frac{1}{2} \left( \frac{2\theta_1 \theta_2}{\theta_2 - \theta_1} \right)^v \Gamma(v) \left( \int_{\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} (w \circ H \circ \sigma) \left( \sigma^{-1}\left( \frac{1}{\theta_1} \right) \right) \left( \sigma^{-1}\left( \frac{1}{\theta_2} \right) \right) \right)
\]

\[
= \left( \frac{2\theta_1 \theta_2}{\theta_2 - \theta_1} \right)^v \Gamma(v) \left( \int_{\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} (w \circ H \circ \sigma) \left( \sigma^{-1}\left( \frac{1}{\theta_1} \right) \right) \left( \sigma^{-1}\left( \frac{1}{\theta_2} \right) \right) \right)
\]

\[
= \left( \frac{2\theta_1 \theta_2}{\theta_2 - \theta_1} \right)^v \Gamma(v) \int_{\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} \left( \frac{\theta_1}{\theta_2 - \theta_1} - \sigma(\tau) \right)^{v-1} (w \circ H \circ \sigma)(\tau) \sigma'(\tau) d\tau
\]

\[
= \int_{\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} \left( \frac{2\theta_1 \theta_2}{\theta_2 - \theta_1} \right)^v \left( \frac{\theta_1}{\theta_2 - \theta_1} - \sigma(\tau) \right)^{v-1} (w \circ H \circ \sigma)(\tau) \sigma'(\tau) \frac{2\theta_1 \theta_2 d\tau}{\theta_2 - \theta_1}
\]

\[
= \int_0^1 \kappa^{v-1}w\left( \frac{2\theta_1 \theta_2}{\kappa \theta_1 + (2 - \kappa) \theta_1} \right) d\kappa.
\]
It follows that
\[
2\mathcal{F}\left(\frac{2\theta_1 \theta_2}{\theta_1 + \theta_2}\right) \int_0^{\frac{1}{y}} k^{\nu-1} w\left(\frac{2\theta_1 \theta_2}{\theta_1 + \theta_2}\right) d\kappa = \left(\frac{2\theta_1 \theta_2}{\theta_1 + \theta_2}\right)^\nu \mathcal{F}\left(\frac{2\theta_1 \theta_2}{\theta_1 + \theta_2}\right) \\
\times \left[\mathcal{I}^{\nu,\sigma}_{\frac{1}{y}}\left(w \circ H \circ \sigma\right) \left(\sigma^{-1}\left(\frac{1}{y}\right)\right) + \left(\frac{1}{y}\right)_0^{\frac{1}{y}} - \mathcal{I}^{\nu,\sigma}_{\frac{1}{y}}\left(w \circ H \circ \sigma\right) \left(\sigma^{-1}\left(\frac{1}{y}\right)\right)\right].
\]

(19)

We can demonstrate that by calculating the weighted fractional operators,
\[
w\left(\frac{1}{y}\right) \left[\mathcal{I}^{\nu,\sigma}_{\frac{1}{y}}\left(w \circ H \circ \sigma\right) \left(\sigma^{-1}\left(\frac{1}{y}\right)\right) + \left(\frac{1}{y}\right)_0^{\frac{1}{y}} - \mathcal{I}^{\nu,\sigma}_{\frac{1}{y}}\left(w \circ H \circ \sigma\right) \left(\sigma^{-1}\left(\frac{1}{y}\right)\right)\right]
\]
\[
= w\left(\frac{1}{y}\right) \left(\mathcal{I}^{\nu,\sigma}_{\frac{1}{y}}\left(w \circ H \circ \sigma\right) \left(\sigma^{-1}\left(\frac{1}{y}\right)\right) + \left(\frac{1}{y}\right)_0^{\frac{1}{y}} - \mathcal{I}^{\nu,\sigma}_{\frac{1}{y}}\left(w \circ H \circ \sigma\right) \left(\sigma^{-1}\left(\frac{1}{y}\right)\right)\right) + \frac{1}{\sigma'(y)} \left[\mathcal{I}^{\nu,\sigma}_{\frac{1}{y}}\left(w \circ H \circ \sigma\right) \left(\sigma^{-1}\left(\frac{1}{y}\right)\right) + \left(\frac{1}{y}\right)_0^{\frac{1}{y}} - \mathcal{I}^{\nu,\sigma}_{\frac{1}{y}}\left(w \circ H \circ \sigma\right) \left(\sigma^{-1}\left(\frac{1}{y}\right)\right)\right]
\]
so,
\[
\left[\left(w \circ H \circ \sigma\right)^{-1}\frac{1}{y}\right]^{-1} = \frac{1}{\left[w \circ H \circ \sigma\right^{-1}\frac{1}{y}} = \frac{1}{w'(y)} \text{ for } y = \left[\frac{1}{\theta_1}, \frac{1}{\theta_2}\right].
\]

Setting \(\gamma_1 = \frac{2\theta_1 \theta_2}{\theta_1 + \theta_2}\) and \(\gamma_2 = \frac{2\theta_1 \theta_2 (\sigma^{-1}(y) - \frac{1}{y})}{\theta_1 + \theta_2}\), one can deduce that
\[
w\left(\frac{1}{\theta_1}\right) \left[\mathcal{I}^{\nu,\sigma}_{\frac{1}{y}}\left(w \circ H \circ \sigma\right) \left(\sigma^{-1}\left(\frac{1}{y}\right)\right) \right]
\]
\[
+ w\left(\frac{1}{\theta_2}\right) \left[\mathcal{I}^{\nu,\sigma}_{\frac{1}{y}}\left(w \circ H \circ \sigma\right) \left(\sigma^{-1}\left(\frac{1}{y}\right)\right) \right]
\]
\[
= \frac{\left(\theta_2 - \theta_1\right)^p}{\left(2\theta_1 \theta_2\right)^p \Gamma\left(\nu\right)} \int_0^1 \gamma_1^{\nu-1} F\left(\frac{2\theta_1 \theta_2}{\gamma_1 \theta_1 + (2 - \gamma_1) \theta_2}\right) w\left(\frac{2\theta_1 \theta_2}{\gamma_1 \theta_1 + (2 - \gamma_1) \theta_2}\right) d\gamma_1
\]
\[
+ \int_0^1 \gamma_2^{\nu-1} F\left(\frac{2\theta_1 \theta_2}{\gamma_2 \theta_2 + (2 - \gamma_2) \theta_1}\right) w\left(\frac{2\theta_1 \theta_2}{\gamma_2 \theta_2 + (2 - \gamma_2) \theta_1}\right) d\gamma_2
\]
\[
= \frac{\left(\theta_2 - \theta_1\right)^p}{\left(2\theta_1 \theta_2\right)^p \Gamma\left(\nu\right)} \int_0^1 k^{\nu-1} F\left(\frac{2\theta_1 \theta_2}{k \theta_1 + (2 - k) \theta_2}\right) w\left(\frac{2\theta_1 \theta_2}{k \theta_1 + (2 - k) \theta_2}\right) d\kappa
\]
\[
+ \int_0^1 k^{\nu-1} F\left(\frac{2\theta_1 \theta_2}{k \theta_2 + (2 - k) \theta_1}\right) w\left(\frac{2\theta_1 \theta_2}{k \theta_2 + (2 - k) \theta_1}\right) d\kappa
\]
As a consequence,
\[
\int_0^1 k^{\nu-1} F\left(\frac{2\theta_1 \theta_2}{k \theta_1 + (2 - k) \theta_2}\right) w\left(\frac{2\theta_1 \theta_2}{k \theta_1 + (2 - k) \theta_2}\right) d\kappa
\]
\[
+ \int_0^1 k^{\nu-1} F\left(\frac{2\theta_1 \theta_2}{k \theta_2 + (2 - k) \theta_1}\right) w\left(\frac{2\theta_1 \theta_2}{k \theta_2 + (2 - k) \theta_1}\right) d\kappa
\]
When we use (19) and (20) in (18), we obtain the following result

\[ F\left( \frac{2\theta_1\theta_2}{\theta_1 + \theta_2} \right) \left[ \left( I_{\sigma^{-1}}^{\kappa} + \left( w \circ \mathcal{H} \circ \sigma \right) \right) \left( \sigma^{-1} \left( \frac{1}{\theta_1} \right) \right) + w \left( \frac{1}{\theta_1} \right) \left( I_{\sigma^{-1}}^{\kappa} \left( \frac{\theta_1 + \theta_2}{2\theta_1\theta_2} \right) + \left( w \circ \mathcal{H} \circ \sigma \right) \right) \left( \sigma^{-1} \left( \frac{1}{\theta_2} \right) \right) \right] \leq w \left( \frac{\theta_1\theta_2}{\theta_1 + (2 - \kappa)\theta_2} \right) \left[ \left( I_{\sigma^{-1}}^{\kappa} \left( \frac{\theta_1 + \theta_2}{2\theta_1\theta_2} \right) + \left( w \circ \mathcal{H} \circ \sigma \right) \right) \left( \sigma^{-1} \left( \frac{1}{\theta_1} \right) \right) + w \left( \frac{1}{\theta_2} \right) \left( I_{\sigma^{-1}}^{\kappa} \left( \frac{\theta_1 + \theta_2}{2\theta_1\theta_2} \right) + \left( w \circ \mathcal{H} \circ \sigma \right) \right) \left( \sigma^{-1} \left( \frac{1}{\theta_2} \right) \right) \right]. \quad (21) \]

As a result, left inequality of (18) has been proven.

The second inequality of (18) can be proved using the harmonically convex function of \( F \).

\[ F\left( \frac{2\theta_1\theta_2}{\kappa \theta_1 + (2 - \kappa)\theta_2} \right) + \frac{1}{2} F\left( \frac{2\theta_1\theta_2}{\theta_1\theta_2 + (2 - \kappa)\theta_2} \right) \leq F(\theta_1) + F(\theta_2). \quad (22) \]

Multiplying both sides of (22) by \( \kappa^{\kappa-1}w\left( \frac{2\theta_1\theta_2}{\theta_1\theta_2 + (2 - \kappa)\theta_2} \right) \) and we obtain by integrating the resulting inequality in terms of \( \kappa \) on [0, 1].

\[ \int_{0}^{1} \kappa^{\kappa-1} F\left( \frac{2\theta_1\theta_2}{\kappa \theta_1 + (2 - \kappa)\theta_2} \right) w\left( \frac{2\theta_1\theta_2}{\kappa \theta_1 + (2 - \kappa)\theta_2} \right) d\kappa + \int_{0}^{1} \kappa^{\kappa-1} F\left( \frac{2\theta_1\theta_2}{\theta_1\theta_2 + (2 - \kappa)\theta_2} \right) w\left( \frac{2\theta_1\theta_2}{\theta_1\theta_2 + (2 - \kappa)\theta_2} \right) d\kappa \leq \left| F(\theta_1) + F(\theta_2) \right| \int_{0}^{1} \kappa^{\kappa-1} w\left( \frac{2\theta_1\theta_2}{\theta_1\theta_2 + (2 - \kappa)\theta_2} \right) d\kappa. \quad (23) \]

Then, using (14) and (20) in (23), we get

\[ w\left( \frac{1}{\theta_1} \right) \left[ I_{\sigma^{-1}}^{\kappa} \left( \frac{\theta_1 + \theta_2}{2\theta_1\theta_2} \right) + \left( w \circ \mathcal{H} \circ \sigma \right) \right] \left( \sigma^{-1} \left( \frac{1}{\theta_1} \right) \right) \]

\[ + w\left( \frac{1}{\theta_2} \right) \left( I_{\sigma^{-1}}^{\kappa} \left( \frac{\theta_1 + \theta_2}{2\theta_1\theta_2} \right) + \left( w \circ \mathcal{H} \circ \sigma \right) \right) \left( \sigma^{-1} \left( \frac{1}{\theta_2} \right) \right) \leq \frac{F(\theta_1) + F(\theta_2)}{2} \left[ \left( I_{\sigma^{-1}}^{\kappa} \left( \frac{\theta_1 + \theta_2}{2\theta_1\theta_2} \right) + \left( w \circ \mathcal{H} \circ \sigma \right) \right) \left( \sigma^{-1} \left( \frac{1}{\theta_1} \right) \right) + \left( \sigma^{-1} \left( \frac{\theta_1 + \theta_2}{2\theta_1\theta_2} \right) - \left( w \circ \mathcal{H} \circ \sigma \right) \right) \left( \sigma^{-1} \left( \frac{1}{\theta_2} \right) \right) \right]. \quad (24) \]

This ends our proof. \( \square \)

**Remark 2.** We can derive the following special results from Theorem 6:
(i) If $\sigma(\tau) = \tau$, then inequality (16) yields
\[
\mathcal{F}\left(\frac{2\theta_1\theta_2}{\theta_1 + \theta_2}\right) \left[ \left( T^\nu_1 \circ T^\nu_2 \right) \left( w \circ H \right) \left( \frac{1}{\theta_1 + \theta_2} \right) + \left( T^\nu_1 \circ T^\nu_2 \circ H \right) \left( \frac{1}{\theta_1 + \theta_2} \right) \right]
\leq w \left( \frac{1}{\theta_1 + \theta_2} \right) \left( w \circ H \circ \sigma \right) \left( \frac{1}{\theta_1 + \theta_2} \right) + w \left( \frac{1}{\theta_1 + \theta_2} \right) \left( w \circ H \circ \sigma \right) \left( \frac{1}{\theta_1 + \theta_2} \right)
\leq \frac{F(\theta_1 + \theta_2)}{2} \left[ \left( T^\nu_1 \circ T^\nu_2 \right) \left( w \circ H \right) \left( \frac{1}{\theta_1} \right) + \left( T^\nu_1 \circ T^\nu_2 \circ H \right) \left( \frac{1}{\theta_1} \right) \right].
\] (25)

where
\[
\left( w \circ T^\nu_1 \circ \sigma \right)(\tau) = \frac{w(\tau)}{1(\nu)} \int_{\theta_1}^{\frac{\tau}{\nu}} (\tau - \kappa)^{n-1} \mathcal{F}(\kappa) w(\kappa) d\kappa,
\]
\[(w \circ T^\nu_2 \circ \sigma)(\tau) = \frac{w(\tau)}{1(\nu)} \int_{\tau}^{\frac{\theta_2}{\nu}} (\kappa - \tau)^{n-1} \mathcal{F}(\kappa) w(\kappa) d\kappa, \quad \nu > 0.
\] (26)

(ii) If $\sigma(\tau) = \tau$ and $\nu = 1$, then inequality (16) is identical to inequality in (10).

(iii) If $\sigma(\tau) = \tau$ and $\nu = 1$, then inequality (16) is identical to inequality in (3).

(iv) If $\sigma(\tau) = \tau$ and $\nu = 1$, then inequality (16) is identical to inequality in (8).

Lemma 2. Let $0 < \theta_1 < \theta_2$, let $\mathcal{F} : [\theta_1, \theta_2] \to \mathbb{R}$ be a continuous with a derivative $\mathcal{F}' \in L^1[\theta_1, \theta_2]$ such that $\mathcal{F}(\tau) = \mathcal{F}\left(\frac{1}{\theta_1}\right) + \int_{\frac{1}{\theta_1}}^{\frac{1}{\theta_2}} \mathcal{F}'(\tau) d\tau$ and let $w : [\theta_1, \theta_2] \to \mathbb{R}$ be nonnegative, an integrable, positive, and symmetric weighted function with respect to $\frac{2\theta_1\theta_2}{\theta_1 + \theta_2}$. If $\sigma$ is an increasing and positive function from $[\theta_1, \theta_2]$ onto itself such that its derivative $\sigma'(\tau)$ is continuous on $(\theta_1, \theta_2)$, for $\nu > 0$, then, we have

\[
\mathcal{F}\left(\frac{2\theta_1\theta_2}{\theta_1 + \theta_2}\right) \left[ \left( T^\nu_1 \circ T^\nu_2 \circ \sigma^{-1}(\frac{\theta_1 + \theta_2}{2\theta_1}) \right) \left( w \circ H \circ \sigma \right) \left( \sigma^{-1}(\frac{1}{\theta_1}) \right) + \left( T^\nu_1 \circ T^\nu_2 \circ H \right) \left( \sigma^{-1}(\frac{\theta_1 + \theta_2}{2\theta_1}) \right) \right]
\leq w \left( \frac{1}{\theta_1} \right) \left( w \circ H \circ \sigma \circ \sigma^{-1}(\frac{1}{\theta_1}) \right) + w \left( \frac{1}{\theta_1} \right) \left( w \circ H \circ \sigma \circ \sigma^{-1}(\frac{1}{\theta_1}) \right)
\leq \frac{F(\theta_1 + \theta_2)}{2} \left[ \left( T^\nu_1 \circ T^\nu_2 \circ \sigma^{-1}(\frac{1}{\theta_1}) \right) \left( w \circ H \circ \sigma \circ \sigma^{-1}(\frac{1}{\theta_1}) \right) \left( \sigma^{-1}(\frac{1}{\theta_1}) \right) \right].
\]

\[
= \frac{1}{1(\nu)} \int_{\sigma^{-1}(\frac{1}{\theta_1})}^{\sigma^{-1}(\frac{1}{\theta_2})} \left[ \int_{\sigma^{-1}(\frac{1}{\theta_1})}^{\kappa} \sigma'(\tau) \left( \sigma'(\tau) - \frac{1}{\theta_2} \right)^{n-1} \left( w \circ H \circ \sigma \circ \sigma^{-1}(\frac{1}{\theta_1}) \right) \left( w \circ H \circ \sigma \circ \sigma^{-1}(\frac{1}{\theta_1}) \right) \left( \sigma^{-1}(\frac{1}{\theta_1}) \right) \right] d\tau.
\]

\[
\leq \frac{1}{1(\nu)} \int_{\sigma^{-1}(\frac{1}{\theta_1})}^{\sigma^{-1}(\frac{1}{\theta_2})} \left[ \int_{\kappa}^{\sigma^{-1}(\frac{1}{\theta_1})} \sigma'(\tau) \left( \sigma'(\tau) - \frac{1}{\theta_2} \right)^{n-1} \left( w \circ H \circ \sigma \circ \sigma^{-1}(\frac{1}{\theta_1}) \right) \left( w \circ H \circ \sigma \circ \sigma^{-1}(\frac{1}{\theta_1}) \right) \left( \sigma^{-1}(\frac{1}{\theta_1}) \right) \right] d\tau.
\]

Proof. Let us set

\[
\frac{1}{1(\nu)} \int_{\sigma^{-1}(\frac{1}{\theta_1})}^{\sigma^{-1}(\frac{1}{\theta_2})} \left[ \int_{\sigma^{-1}(\frac{1}{\theta_1})}^{\kappa} \sigma'(\tau) \left( \sigma'(\tau) - \frac{1}{\theta_2} \right)^{n-1} \left( w \circ H \circ \sigma \circ \sigma^{-1}(\frac{1}{\theta_1}) \right) \left( w \circ H \circ \sigma \circ \sigma^{-1}(\frac{1}{\theta_1}) \right) \left( \sigma^{-1}(\frac{1}{\theta_1}) \right) \right] d\tau.
\]

\[
- \frac{1}{1(\nu)} \int_{\sigma^{-1}(\frac{1}{\theta_1})}^{\sigma^{-1}(\frac{1}{\theta_2})} \left[ \int_{\kappa}^{\sigma^{-1}(\frac{1}{\theta_1})} \sigma'(\tau) \left( \sigma'(\tau) - \sigma^{-1}(\frac{1}{\theta_1}) \right)^{n-1} \left( w \circ H \circ \sigma \circ \sigma^{-1}(\frac{1}{\theta_1}) \right) \left( w \circ H \circ \sigma \circ \sigma^{-1}(\frac{1}{\theta_1}) \right) \left( \sigma^{-1}(\frac{1}{\theta_1}) \right) \right] d\tau.
\]
\[
\begin{align*}
\frac{1}{\Gamma(v)} \int_{\sigma^{-1}\left(\frac{1}{\theta_2}\right)}^{\psi^{-1}\left(\frac{\theta_1+\psi_2}{\theta_1}\right)} \left[ \int_{\sigma^{-1}\left(\frac{1}{\theta_1}\right)}^{\psi^{-1}\left(\frac{\theta_1+\psi_2}{\theta_1}\right)} \sigma'(\tau) \left( \sigma(\tau) - \frac{1}{\theta_1} \right) \psi^{-1} \left( w \circ \mathcal{H} \circ \sigma(\tau)d\tau \right) \right] \left( \mathcal{F}' \circ \mathcal{H} \circ \sigma \right)(k) d\tau \\
\frac{-1}{\Gamma(v)} \int_{\sigma^{-1}\left(\frac{1}{\theta_1}\right)}^{\psi^{-1}\left(\frac{\theta_1+\psi_2}{\theta_1}\right)} \left[ \int_{\sigma^{-1}\left(\frac{1}{\theta_1}\right)}^{\psi^{-1}\left(\frac{\theta_1+\psi_2}{\theta_1}\right)} \sigma'(\tau) \left( \frac{1}{\theta_1} - \sigma(\tau) \right) \psi^{-1} \left( w \circ \mathcal{H} \circ \sigma(\tau)d\tau \right) \right] \left( \mathcal{F}' \circ \mathcal{H} \circ \sigma \right)(k) d\tau \\
\end{align*}
\]
\[= Z_1 + Z_2.\]

Taking integrating by parts, applying Lemma 1, (12) and (13), we get

\[
Z_1 = \frac{1}{\Gamma(v)} \int_{\sigma^{-1}\left(\frac{1}{\theta_2}\right)}^{\psi^{-1}\left(\frac{\theta_1+\psi_2}{\theta_2}\right)} \sigma'(\tau) \left( \sigma(\tau) - \frac{1}{\theta_2} \right) \psi^{-1} \left( w \circ \mathcal{H} \circ \sigma(\tau)d\tau \right) \left( \mathcal{F} \circ \mathcal{H} \circ \sigma \right)(k) d\tau \\
- \frac{1}{\Gamma(v)} \int_{\sigma^{-1}\left(\frac{1}{\theta_2}\right)}^{\psi^{-1}\left(\frac{\theta_1+\psi_2}{\theta_2}\right)} \sigma'(\tau) \left( \sigma(\tau) - \frac{1}{\theta_2} \right) \psi^{-1} \left( w \circ \mathcal{H} \circ \sigma(\tau)d\tau \right) \left( \mathcal{F} \circ \mathcal{H} \circ \sigma \right)(k) d\tau \\
= \left( \frac{1}{\Gamma(v)} \int_{\sigma^{-1}\left(\frac{1}{\theta_2}\right)}^{\psi^{-1}\left(\frac{\theta_1+\psi_2}{\theta_2}\right)} \sigma'(\kappa) \left( \sigma(\kappa) - \frac{1}{\theta_2} \right) \psi^{-1} \left( w \circ \mathcal{H} \circ \sigma(\kappa)d\kappa \right) \left( \mathcal{F} \circ \mathcal{H} \circ \sigma \right)(\kappa) d\kappa \right) \\
- w \left( \frac{1}{\theta_2} \right) \left( \sigma^{-1}\left(\frac{\theta_1+\psi_2}{\theta_2}\right) - \mathcal{F}' \circ \mathcal{H} \circ \sigma \right)(w \circ \mathcal{H} \circ \sigma) \left( \sigma^{-1}\left(\frac{1}{\theta_2}\right) \right) \\
\]

Similarly, we get

\[
Z_2 = \frac{-1}{\Gamma(v)} \int_{\sigma^{-1}\left(\frac{1}{\theta_1}\right)}^{\psi^{-1}\left(\frac{\theta_1+\psi_2}{\theta_1}\right)} \sigma'(\tau) \left( \frac{1}{\theta_1} - \sigma(\tau) \right) \psi^{-1} \left( w \circ \mathcal{H} \circ \sigma(\tau)d\tau \right) \left( \mathcal{F} \circ \mathcal{H} \circ \sigma \right)(k) d\tau \\
- \frac{1}{\Gamma(v)} \int_{\sigma^{-1}\left(\frac{1}{\theta_1}\right)}^{\psi^{-1}\left(\frac{\theta_1+\psi_2}{\theta_1}\right)} \sigma'(\kappa) \left( \frac{1}{\theta_1} - \sigma(\kappa) \right) \psi^{-1} \left( w \circ \mathcal{H} \circ \sigma(\kappa)d\kappa \right) \left( \mathcal{F} \circ \mathcal{H} \circ \sigma \right)(k) d\kappa \\
= \left( \frac{1}{\Gamma(v)} \int_{\sigma^{-1}\left(\frac{1}{\theta_1}\right)}^{\psi^{-1}\left(\frac{\theta_1+\psi_2}{\theta_1}\right)} \sigma'(\tau) \left( \frac{1}{\theta_1} - \sigma(\tau) \right) \psi^{-1} \left( w \circ \mathcal{H} \circ \sigma(\tau)d\tau \right) \left( \mathcal{F} \circ \mathcal{H} \circ \sigma \right)(\kappa) d\kappa \right) \\
- w \left( \frac{1}{\theta_1} \right) \left(w \circ \mathcal{H} \circ \sigma \right)^{-1} \sigma^{-1}\left(\frac{1}{\theta_1}\right) \\
\times \int_{\sigma^{-1}\left(\frac{\theta_1+\psi_2}{\theta_1}\right)}^{\psi^{-1}\left(\frac{\theta_1+\psi_2}{\theta_1}\right)} \sigma'(\kappa) \left( \frac{1}{\theta_1} - \sigma(\kappa) \right) \psi^{-1} \left( w \circ \mathcal{H} \circ \sigma(\kappa)d\kappa \right) \left( \mathcal{F} \circ \mathcal{H} \circ \sigma \right)(\kappa) d\kappa.
\]
\[ Z_1 + Z_2 := \frac{\theta_1 + \theta_2}{\theta_1 + \theta_2} \left( \frac{\theta_1 + \theta_2}{\theta_1 + \theta_2} \right) + (w \circ {\mathcal H} \circ \sigma) \left( \sigma^{-1} \left( \frac{1}{\theta_1} \right) \right) \]

we achieve the intended result. \( \square \)

3. Main Results

We can conclude the following Hermite-Hadamard-Fejér inequalities with the help of Lemma 2.

**Theorem 7.** Suppose that all the conditions of Lemma 2 and \(|{\mathcal F}|\) is harmonically convex on \([\theta_1, \theta_2]\) and \(\sigma\) is an increasing and positive function from \([\theta_1, \theta_2]\) onto itself such that its derivative \(\sigma'(\tau)\) is continuous on \((\theta_1, \theta_2)\), for \(v > 0\), then we have

\[
|Z_1 + Z_2| \leq \left| \frac{\theta_1 + \theta_2}{\theta_1 + \theta_2} \right|_{(\theta_1, \theta_2)} \left[ \left| W_1(\theta_1, \theta_2) \right| \left( \frac{\theta_1 + \theta_2}{\theta_1 + \theta_2} \right) + \left| W_2(\theta_1, \theta_2) \right| \left( \frac{\theta_1 + \theta_2}{\theta_1 + \theta_2} \right) \right]
\]

\[
+ \left| \frac{\theta_1 + \theta_2}{\theta_1 + \theta_2} \right|_{(\theta_1, \theta_2)} \left[ \left| W_3(\theta_1, \theta_2) \right| \left( \frac{\theta_1 + \theta_2}{\theta_1 + \theta_2} \right) + \left| W_4(\theta_1, \theta_2) \right| \left( \frac{\theta_1 + \theta_2}{\theta_1 + \theta_2} \right) \right]
\]

\[
\leq \left| \frac{\theta_1 + \theta_2}{\theta_1 + \theta_2} \right|_{(\theta_1, \theta_2)} \left( \left| W_1(\theta_1, \theta_2) + W_3(\theta_1, \theta_2) \right| \left( \sigma^{-1} \left( \frac{1}{\theta_1} \right) \right) + \left| W_2(\theta_1, \theta_2) + W_4(\theta_1, \theta_2) \right| \left( \sigma^{-1} \left( \frac{1}{\theta_2} \right) \right) \right),
\]

where \(W_1(\theta_1, \theta_2), W_2(\theta_1, \theta_2), W_3(\theta_1, \theta_2),\) and \(W_4(\theta_1, \theta_2)\) are defined as follows:

\[
W_1(\theta_1, \theta_2) = \frac{2^{-v}(\theta_1 + \theta_2)^{-1}}{(\theta_1 + \theta_2)^2} \left( -v, -v, 1 - v; \frac{2\theta_2}{\theta_1 + \theta_2} \right) v
\]

\[
+ \frac{2^{-v-1}(\theta_1 - \theta_2)}{\theta_2(v + 1)} - \pi \theta_1 \theta_2 \left( \frac{1}{\theta_2} \right)^{v-1} \csc(\pi v),
\]
\[
\mathbb{W}_2(\theta_1, \theta_2) = -\frac{2^{-\nu-1}(\theta_1 \theta_2)^{-\nu} \left( v \left( 2^{\nu+1}(v+1)\pi \theta_2 \theta_1^{v+2} \csc(\pi v) + (\theta_2 - \theta_1)^{v+1} \right) - 2(v+1)\theta_1^2 \theta_2 (\theta_1 + \theta_2)^v \right)_2 F_1 \left( -v, -v; 1 - v; \frac{2\theta_2}{\sigma_1 + \sigma_2} \right)}{\theta_1 v(v+1)},
\]

\[
\mathbb{W}_3(\theta_1, \theta_2) = 2^{-\nu-1}(\theta_2 - \theta_1) \left( \frac{1}{\sigma_1} - \frac{1}{\sigma_2} \right)^v \frac{\theta_1 \theta_2 F_1 \left( 1, v + 1; v + 2; \frac{\theta_2 - \theta_1}{\theta_2} \right) - 1}{\theta_2 (v+1)}
\]

and

\[
\mathbb{W}_4(\theta_1, \theta_2) = 2^{-\nu-1}(\theta_1 - \theta_2) \left( \frac{1}{\sigma_1} - \frac{1}{\sigma_2} \right)^v \frac{\theta_2^2 F_1 \left( 1, v + 1; v + 2; \frac{\theta_2 - \theta_1}{\theta_2} \right) - 1}{\theta_1 (v+1)}
\]

**Proof.** Using the Lemma 2 as well as properties of the modulus and the harmonically convex function of \( |F| \), we get

\[
|Z_1 + Z_2| = \left| \frac{1}{\Gamma(v)} \int^{\kappa}_{\sigma_1^{-1}(\frac{1}{\sigma_2})} \int_{\sigma_1^{-1}(\frac{1}{\sigma_2})}^{\kappa} \sigma'(\tau) \left( \sigma(\tau) - \frac{1}{\theta_2} \right)^{-\nu-1} (w \circ H \circ \sigma)(\tau) d\tau \right| (F' \circ H \circ \sigma)(\kappa) \sigma'(\kappa) d\kappa
\]

\[
- \frac{1}{\Gamma(v)} \int^{\sigma_1^{-1}(\frac{1}{\sigma_2})}_{\sigma_1^{-1}(\frac{1}{\sigma_2})} \left[ \int_{\sigma_1^{-1}(\frac{1}{\sigma_2})}^{\kappa} \sigma'(\tau) \left( \frac{1}{\theta_1} - \sigma(\tau) \right)^{-\nu-1} (w \circ H \circ \sigma)(\tau) d\tau \right] (F' \circ H \circ \sigma)(\kappa) \sigma'(\kappa) d\kappa
\]

\[
\leq \frac{1}{\Gamma(v)} \int_{\sigma_1^{-1}(\frac{1}{\sigma_2})}^{\kappa} \sigma'(\tau) \left( \frac{1}{\theta_1} - \sigma(\tau) \right)^{-\nu-1} (w \circ H \circ \sigma)(\tau) d\tau \left| (F' \circ H \circ \sigma)(\kappa) \sigma'(\kappa) \right| d\kappa
\]

\[
+ \frac{1}{\Gamma(v)} \int_{\sigma_1^{-1}(\frac{1}{\sigma_2})}^{\kappa} \sigma'(\tau) \left( \frac{1}{\theta_1} - \sigma(\tau) \right)^{-\nu-1} (w \circ H \circ \sigma)(\tau) d\tau \left| (F' \circ H \circ \sigma)(\kappa) \sigma'(\kappa) \right| d\kappa.
\]

Since \( |F'| \) is harmonic-convex on \( [\theta_1, \theta_2] \), where \( \kappa \in \left[ \sigma^{-1}(\frac{1}{\sigma_1}), \sigma^{-1}(\frac{1}{\sigma_2}) \right] \)

\[
\left| (F' \circ H \circ \sigma)(\kappa) \right| = \left| F' \left( \frac{\theta_1 (\theta_2 - \sigma(\kappa))}{\sigma(\kappa)(\theta_2 - \theta_1)} \frac{1}{\theta_1} + \frac{\theta_2 (\sigma(\kappa) - \theta_1)}{\sigma(\kappa)(\theta_2 - \theta_1)} \frac{1}{\theta_2} \right) \right|
\]

\[
\leq \frac{\theta_1 (\theta_2 - \sigma(\kappa))}{\sigma(\kappa)(\theta_2 - \theta_1)} \left| (F' \circ H)(\theta_1) \right| + \frac{\theta_2 (\sigma(\kappa) - \theta_1)}{\sigma(\kappa)(\theta_2 - \theta_1)} \left| (F' \circ H)(\theta_2) \right|
\]

Consequently, we obtain

\[
|Z_1 + Z_2| \leq \left| \frac{|w|}{(\theta_2 - \theta_1)\Gamma(v)} \int_{\sigma_1^{-1}(\frac{1}{\sigma_2})}^{\kappa} \sigma'(\tau) \left( \sigma(\tau) - \frac{1}{\theta_2} \right)^{-\nu-1} d\tau \right|
\]

\[
\times \left[ \frac{\theta_1 (\theta_2 - \sigma(\kappa))}{\sigma(\kappa)} \left| (F' \circ H)(\theta_1) \right| + \frac{\theta_2 (\sigma(\kappa) - \theta_1)}{\sigma(\kappa)} \left| (F' \circ H)(\theta_2) \right| \right] \sigma'(\kappa) d\kappa
\]

\[
+ \left| \frac{|w|}{(\theta_2 - \theta_1)\Gamma(v)} \int_{\sigma_1^{-1}(\frac{1}{\sigma_2})}^{\kappa} \sigma'(\tau) \left( \frac{1}{\theta_1} - \sigma(\tau) \right)^{-\nu-1} d\tau \right|
\]

\[
\times \left[ \frac{\theta_1 (\theta_2 - \sigma(\kappa))}{\sigma(\kappa)} \left| (F' \circ H)(\theta_1) \right| + \frac{\theta_2 (\sigma(\kappa) - \theta_1)}{\sigma(\kappa)} \left| (F' \circ H)(\theta_2) \right| \right] \sigma'(\kappa) d\kappa
\]

where

\[
\int_{\kappa}^{\sigma_1^{-1}(\frac{1}{\sigma_2})} \sigma'(\tau) \left( \frac{1}{\theta_1} - \sigma(\tau) \right)^{\nu-1} d\tau = \frac{1}{\nu} \left( \frac{1}{\sigma_1} - \sigma(\kappa) \right)^\nu
\]
and
\[ \int_{r^{-1}\left(\frac{1}{\theta_2}\right)}^{\kappa} \sigma'(\tau) \left(\sigma(\tau) - \frac{1}{\theta_2}\right)^{v-1} d\tau = \left(\sigma(\kappa) - \frac{1}{\theta_2}\right)^v. \]

Using the above calculations, we obtain the following integral

\[ |Z_1 + Z_2| \leq \frac{||w|||\frac{1}{\theta_2-\theta_1}\Gamma'(v)|}{|\theta_2-\theta_1|\Gamma(v)} \int_{r^{-1}\left(\frac{1}{\theta_2}\right)}^{\kappa} \left[ \frac{\theta_1 \left(\frac{1}{\theta_2} - \frac{1}{\theta_2}\right)^v (\theta_2 - \sigma(\kappa))}{\sigma(\kappa)} \right] \left(\mathcal{F}^\circ \mathcal{H}\right)(\theta_1) \]

\[ + \frac{\theta_2 \left(\frac{1}{\theta_2} - \frac{1}{\theta_2}\right)^v (\sigma(\kappa) - \theta_1)}{\sigma(\kappa)} \left(\mathcal{F}^\circ \mathcal{H}\right)(\theta_2) \]

\[ + \bar{w} \left(\frac{1}{\theta_2} - \sigma(\kappa)\right)^v (\sigma(\kappa) - \theta_1) \left(\mathcal{F}^\circ \mathcal{H}\right)(\theta_2) \]

\[ \left(\mathcal{F}^\circ \mathcal{H}\right)(\theta_2) \] (32)

We can obtain the necessary result (28) by doing basic integral calculations based on inequality (32).

**Theorem 8.** Suppose that all the conditions of Lemma 2 and $|\mathcal{F}'|^q$ is harmonically convex on $[\theta_1, \theta_2]$ with $q \geq 1$ and $\sigma$ is an increasing and positive function from $[\theta_1, \theta_2]$ onto itself such that its derivative $\sigma'(\tau)$ is continuous on $(\theta_1, \theta_2)$, for $v > 0$, then we have

\[ |Z_1 + Z_2| \leq \left(\frac{(\theta_2 - \theta_1)^v + 1}{(2\theta_1\theta_2)^v(v+1)}\right)^{1-\frac{1}{q}} \]

\[ \times \left[ \frac{||w|||\frac{1}{\theta_2-\theta_1}\Gamma'(v)|}{|\theta_2-\theta_1|\Gamma(v)} \left(\mathcal{W}_1(\theta_1, \theta_2)\right|\left(\mathcal{F}^\circ \mathcal{H}\right)(\theta_1)\right)^q \right] \]

\[ + \left(\frac{\theta_1 \left(\frac{1}{\theta_2} - \frac{1}{\theta_2}\right)^v (\theta_2 - \sigma(\kappa))}{\sigma(\kappa)} \left(\mathcal{F}^\circ \mathcal{H}\right)(\theta_1)\right)^q \]

\[ \left(\mathcal{F}^\circ \mathcal{H}\right)(\theta_2) \] (33)

where $\mathcal{W}_1(\theta_1, \theta_2)$, $\mathcal{W}_2(\theta_1, \theta_2)$, $\mathcal{W}_3(\theta_1, \theta_2)$, and $\mathcal{W}_4(\theta_1, \theta_2)$ are defined in Theorem 7.

**Proof.** Using the Lemma 2 as well as properties of power mean inequality and the harmonically convex function of $|\mathcal{F}'|^q$, we get
\[ |Z_1 + Z_2| \]

\[
\leq \frac{1}{\Gamma(v)} \int_{\sigma^{-1}(\frac{1}{\theta_1})}^{\kappa} \int_{\sigma^{-1}(\frac{1}{\theta_2})}^{\kappa} \sigma'(\tau) \left( \sigma(\tau) - \frac{1}{\theta_2} \right)^{v-1} (w \circ H \circ \sigma)(\tau) d\tau \left| \left( F' \circ H \circ \sigma \right)(\kappa) \right| \sigma'(\kappa) d\kappa \\
+ \frac{1}{\Gamma(v)} \int_{\sigma^{-1}(\frac{1}{\theta_1})}^{\kappa} \int_{\sigma^{-1}(\frac{1}{\theta_2})}^{\kappa} \sigma'(\tau) \left( \frac{1}{\theta_1} - \sigma(\tau) \right)^{v-1} (w \circ H \circ \sigma)(\tau) d\tau \left| \left( F' \circ H \circ \sigma \right)(\kappa) \right| \sigma'(\kappa) d\kappa \\
\leq \frac{1}{\Gamma(v)} \left( \int_{\sigma^{-1}(\frac{1}{\theta_1})}^{\kappa} \int_{\sigma^{-1}(\frac{1}{\theta_2})}^{\kappa} \sigma'(\tau) \left( \sigma(\tau) - \frac{1}{\theta_2} \right)^{v-1} (w \circ H \circ \sigma)(\tau) d\tau \right| \left| \left( F' \circ H \circ \sigma \right)(\kappa) \right| \sigma'(\kappa) d\kappa \right)^{1-\frac{1}{q}} \\
\times \left( \int_{\sigma^{-1}(\frac{1}{\theta_1})}^{\kappa} \int_{\sigma^{-1}(\frac{1}{\theta_2})}^{\kappa} \sigma'(\tau) \left( \frac{1}{\theta_1} - \sigma(\tau) \right)^{v-1} (w \circ H \circ \sigma)(\tau) d\tau \right| \left| \left( F' \circ H \circ \sigma \right)(\kappa) \right| \sigma'(\kappa) d\kappa \right)^{\frac{1}{q}}. 
\]

Since \( |F'| \) is harmonic-convex on \( [\theta_1, \theta_2] \), where \( \kappa \in \left[ \sigma^{-1}(\frac{1}{\theta_1}), \sigma^{-1}(\frac{1}{\theta_2}) \right] \)

\[
|\left( F' \circ H \circ \sigma \right)(\kappa)| = \left| F' \left( \frac{\theta_1 (\theta_2 - \sigma(\kappa))}{\sigma(\kappa)(\theta_2 - \theta_1)} \right) \frac{1}{\theta_1} + \left( \frac{\theta_2 (\sigma(\kappa) - \theta_1)}{\sigma(\kappa)(\theta_2 - \theta_1)} \right) \frac{1}{\theta_2} \right| \\
\leq \frac{\theta_1 (\theta_2 - \sigma(\kappa))}{\sigma(\kappa)(\theta_2 - \theta_1)} \left( F' \circ H \right)(\theta_1) + \frac{\theta_2 (\sigma(\kappa) - \theta_1)}{\sigma(\kappa)(\theta_2 - \theta_1)} \left( F' \circ H \right)(\theta_2). 
\]

As a result, we get

\[
|Z_1 + Z_2| \leq \frac{||w|| \left[ \int_{\sigma^{-1}(\frac{1}{\theta_1})}^{\kappa} \int_{\sigma^{-1}(\frac{1}{\theta_2})}^{\kappa} \sigma'(\tau) \left( \sigma(\tau) - \frac{1}{\theta_2} \right)^{v-1} d\tau \right| \sigma'(\kappa) d\kappa \right]^{1-\frac{1}{q}} \\
\times \left( \int_{\sigma^{-1}(\frac{1}{\theta_1})}^{\kappa} \int_{\sigma^{-1}(\frac{1}{\theta_2})}^{\kappa} \sigma'(\tau) \left( \sigma(\tau) - \frac{1}{\theta_2} \right)^{v-1} d\tau \right| \left| \left( F' \circ H \circ \sigma \right)(\kappa) \right| \sigma'(\kappa) d\kappa \right)^{\frac{1}{q}} \\
+ \frac{||w|| \left[ \int_{\sigma^{-1}(\frac{1}{\theta_1})}^{\kappa} \int_{\sigma^{-1}(\frac{1}{\theta_2})}^{\kappa} \sigma'(\tau) \left( \frac{1}{\theta_1} - \sigma(\tau) \right)^{v-1} d\tau \right| \sigma'(\kappa) d\kappa \right]^{1-\frac{1}{q}} \\
\times \left( \int_{\sigma^{-1}(\frac{1}{\theta_1})}^{\kappa} \int_{\sigma^{-1}(\frac{1}{\theta_2})}^{\kappa} \sigma'(\tau) \left( \frac{1}{\theta_1} - \sigma(\tau) \right)^{v-1} d\tau \right| \left| \left( F' \circ H \circ \sigma \right)(\kappa) \right| \sigma'(\kappa) d\kappa \right)^{\frac{1}{q}} \\
\leq \frac{||w|| \left[ \int_{\sigma^{-1}(\frac{1}{\theta_1})}^{\kappa} \int_{\sigma^{-1}(\frac{1}{\theta_2})}^{\kappa} \sigma'(\tau) \left( \sigma(\tau) - \frac{1}{\theta_2} \right)^{v-1} d\tau \right| \left( F' \circ H \circ \sigma \right)(\theta_1)}{\theta_1 - \theta_2} \\
\times \frac{\theta_2 (\sigma(\kappa) - \theta_1)}{\sigma(\kappa)} \left( F' \circ H \right)(\theta_2)^{\frac{1}{q}} \\
+ \frac{\theta_2 (\sigma(\kappa) - \theta_1)}{\sigma(\kappa)} \left( F' \circ H \right)(\theta_2)^{\frac{1}{q}} \sigma'(\kappa) d\kappa \right)^{\frac{1}{q}}.
\]
\[+ \frac{|w|}{\left| \frac{\eta_1 + \eta_2}{2} \right| \Gamma(v)} \left( \int_{\sigma_{1}^{-1}(\eta_1)}^{\sigma_{1}^{-1}(\eta_2)} \left( \frac{1}{\theta_1} - \sigma(\tau) \right)^{v-1} \left( \sigma(\tau) \right) d\tau \right) \frac{1}{\sigma(\tau)} \left( \frac{1}{\theta_1} - \sigma(\tau) \right)^{v-1} \int_{\sigma_{1}^{-1}(\eta_1)}^{\sigma_{1}^{-1}(\eta_2)} \sigma'(\tau) d\tau \right) \left( \frac{1}{\theta_1} - \sigma(\tau) \right)^{v-1} \int_{\sigma_{1}^{-1}(\eta_1)}^{\sigma_{1}^{-1}(\eta_2)} \sigma'(\tau) d\tau \right) \left( \frac{1}{\theta_1} - \sigma(\tau) \right)^{v-1} \int_{\sigma_{1}^{-1}(\eta_1)}^{\sigma_{1}^{-1}(\eta_2)} \sigma'(\tau) d\tau \right), \]

where it is obvious that

\[\int_{\sigma_{1}^{-1}(\eta_1)}^{\sigma_{1}^{-1}(\eta_2)} \sigma'(\tau) \left( \frac{1}{\theta_1} - \sigma(\tau) \right)^{v-1} \int_{\sigma_{1}^{-1}(\eta_1)}^{\sigma_{1}^{-1}(\eta_2)} \sigma'(\tau) d\tau \right) \leq \frac{\left| w \right|}{\left| \frac{\eta_1 + \eta_2}{2} \right| \Gamma(v)} \left( \frac{(\theta_2 - \theta_1)^{v+1}}{(2\theta_1\theta_2)^{v+1}v(v+1)} \right) \left| \sigma'(\tau) \right| \left( \int_{\sigma_{1}^{-1}(\eta_1)}^{\sigma_{1}^{-1}(\eta_2)} \sigma'(\tau) d\tau \right) \left( \frac{1}{\theta_1} - \sigma(\tau) \right)^{v-1} \int_{\sigma_{1}^{-1}(\eta_1)}^{\sigma_{1}^{-1}(\eta_2)} \sigma'(\tau) d\tau \right) \left( \frac{1}{\theta_1} - \sigma(\tau) \right)^{v-1} \int_{\sigma_{1}^{-1}(\eta_1)}^{\sigma_{1}^{-1}(\eta_2)} \sigma'(\tau) d\tau \right) \left( \frac{1}{\theta_1} - \sigma(\tau) \right)^{v-1} \int_{\sigma_{1}^{-1}(\eta_1)}^{\sigma_{1}^{-1}(\eta_2)} \sigma'(\tau) d\tau \right), \]

We can obtain the following integral after inserting (37) into (36)

\[|Z_1 + Z_2| \leq \frac{\left| w \right|}{\left| \frac{\eta_1 + \eta_2}{2} \right| \Gamma(v)} \left( \frac{(\theta_2 - \theta_1)^{v+1}}{(2\theta_1\theta_2)^{v+1}v(v+1)} \right) \left| \sigma'(\tau) \right| \left( \int_{\sigma_{1}^{-1}(\eta_1)}^{\sigma_{1}^{-1}(\eta_2)} \sigma'(\tau) d\tau \right) \left( \frac{1}{\theta_1} - \sigma(\tau) \right)^{v-1} \int_{\sigma_{1}^{-1}(\eta_1)}^{\sigma_{1}^{-1}(\eta_2)} \sigma'(\tau) d\tau \right) \left( \frac{1}{\theta_1} - \sigma(\tau) \right)^{v-1} \int_{\sigma_{1}^{-1}(\eta_1)}^{\sigma_{1}^{-1}(\eta_2)} \sigma'(\tau) d\tau \right) \left( \frac{1}{\theta_1} - \sigma(\tau) \right)^{v-1} \int_{\sigma_{1}^{-1}(\eta_1)}^{\sigma_{1}^{-1}(\eta_2)} \sigma'(\tau) d\tau \right), \]

We can obtain the necessary result (33) by doing basic integral calculations based on inequality (38). \(\square\)

**Theorem 9.** Suppose that all the conditions of Lemma 2 and \(|F'|^q|\) is harmonically convex on \([\theta_1, \theta_2]\] with \(q > 1\) and \(\sigma\) is an increasing and positive function from \([\theta_1, \theta_2]\) onto itself such that its derivative \(\sigma'(\tau)\) is continuous on \((\theta_1, \theta_2)\), for \(v > 0\), then we have

\[|Z_1 + Z_2| \leq \left( \frac{(\theta_2 - \theta_1)^{v+1}}{(2\theta_1\theta_2)^{v+1}v(v+1)} \right)^{\frac{1}{q}} \left[ \left| w \right| \left| \frac{\eta_1 + \eta_2}{2} \right| \Gamma(v) \left( X_1(\theta_1, \theta_2; \sigma(\tau)) \right) \left( \left| F(\theta_1) \right|^q + X_2(\theta_1, \theta_2; \sigma(\tau)) \right) \left( \left| F(\theta_2) \right|^q \right) \right]^{\frac{1}{q}} \]

\[+ \left| w \right| \left| \frac{\eta_1 + \eta_2}{2} \right| \Gamma(v) \left( X_3(\theta_1, \theta_2; \sigma(\tau)) \right) \left( \left| F(\theta_1) \right|^q + X_2(\theta_1, \theta_2; \sigma(\tau)) \right) \left( \left| F(\theta_2) \right|^q \right) \right]^{\frac{1}{q}},\]
\[
\begin{aligned}
&\leq \left( \frac{(\theta_2 - \theta_1)^{p-1}}{(2\theta_1\theta_2)^{p-1} p\nu(p\nu + 1)} \right)^{\frac{1}{p}} \left( \frac{\|v\|_{\frac{1}{p\nu+1}, \infty}}{\|v - \theta_1\|_{\frac{1}{p\nu+1}, \infty}} \right)^{\frac{1}{p}} \left\{ \left( (\mathcal{X}_1(\theta_1, \theta_2; \sigma(\nu)) + \mathcal{X}_3(\theta_1, \theta_2; \sigma(\nu))) \|(F' \circ \mathcal{H})(\theta_1)\|^{\frac{q}{p}} \right)^{\frac{1}{q}} + \left( (\mathcal{X}_2(\theta_1, \theta_2; \sigma(\nu)) \mathcal{X}_4(\theta_1, \theta_2; \sigma(\nu))) \|(F' \circ \mathcal{H})(\theta_2)\|^{\frac{q}{p}} \right)^{\frac{1}{q}} \right\}, \tag{39}
\end{aligned}
\]

where \( \mathcal{X}_1(\theta_1, \theta_2; \sigma(\nu)), \mathcal{X}_2(\theta_1, \theta_2; \sigma(\nu)), \mathcal{X}_3(\theta_1, \theta_2; \sigma(\nu)), \) and \( \mathcal{X}_4(\theta_1, \theta_2; \sigma(\nu)), \) are defined as follows:

\[
\begin{aligned}
\mathcal{X}_1(\theta_1, \theta_2; \sigma(\nu)) &= \frac{1}{2} \left( \frac{\theta_1}{\theta_2} - 1 \right) + \theta_1 \theta_2 \left( \ln \left( \frac{\theta_1 + \theta_2}{\theta_1 \theta_2} \right) - \ln \left( \frac{2}{\theta_2} \right) \right), \\
\mathcal{X}_2(\theta_1, \theta_2; \sigma(\nu)) &= \frac{1}{2} \left( \frac{\theta_2}{\theta_1} - 2 \theta_1 \theta_2 \left( \ln \left( \frac{\theta_1 + \theta_2}{\theta_1 \theta_2} \right) - \ln \left( \frac{2}{\theta_1} \right) \right) - 1 \right), \\
\mathcal{X}_3(\theta_1, \theta_2; \sigma(\nu)) &= \frac{1}{2} \left( \frac{\theta_1}{\theta_2} - 1 \right) + \theta_1 \theta_2 \left( - \ln \left( \frac{\theta_1 + \theta_2}{\theta_1 \theta_2} \right) + \ln \left( \frac{1}{\theta_1} \right) + \ln 2 \right) \\
\mathcal{X}_4(\theta_1, \theta_2; \sigma(\nu)) &= \frac{1}{2} \left( \frac{\theta_2}{\theta_1} - 2 \theta_1 \theta_2 \left( - \ln \left( \frac{\theta_1 + \theta_2}{\theta_1 \theta_2} \right) - \ln \theta_1 + \ln 2 \right) - 1 \right).
\end{aligned}
\]

**Proof.** Using the Lemma 2 as well as properties of well-known Hölder’s inequality and the harmonically convex function of \(|F'\|^{\frac{q}{p}}\), we have

\[
\begin{aligned}
|Z_1 + Z_2| &\leq \frac{1}{\Gamma(\nu)} \int_{\sigma^{-1} \left( \frac{1}{\theta_1} \right)}^{\sigma^{-1} \left( \frac{1}{\theta_2} \right)} \int_{\sigma^{-1} \left( \frac{1}{\theta_2} \right)}^{\kappa} \left| \sigma'(\tau) \left( \sigma(\tau) - \frac{1}{\theta_2} \right)^{p-1} (w \circ \mathcal{H} \circ \sigma)(\tau) \right| \left| (F' \circ \mathcal{H} \circ \sigma)(\kappa) \right| \left| \sigma'(\kappa) \right| d\kappa \\
&\quad + \frac{1}{\Gamma(\nu)} \int_{\sigma^{-1} \left( \frac{1}{\theta_2} \right)}^{\sigma^{-1} \left( \frac{1}{\theta_1} \right)} \int_{\frac{1}{\theta_2}}^{\kappa} \left| \sigma'(\tau) \left( \frac{1}{\theta_1} - \sigma(\tau) \right)^{p-1} (w \circ \mathcal{H} \circ \sigma)(\tau) \right| \left| (F' \circ \mathcal{H} \circ \sigma)(\kappa) \right| \left| \sigma'(\kappa) \right| d\kappa \\
&\leq \frac{1}{\Gamma(\nu)} \left( \int_{\sigma^{-1} \left( \frac{1}{\theta_1} \right)}^{\sigma^{-1} \left( \frac{1}{\theta_2} \right)} \int_{\sigma^{-1} \left( \frac{1}{\theta_2} \right)}^{\kappa} \left| \sigma'(\tau) \left( \sigma(\tau) - \frac{1}{\theta_2} \right)^{p-1} (w \circ \mathcal{H} \circ \sigma)(\tau) \right| ^{\frac{p}{q}} \left| \sigma'(\kappa) \right| ^{\frac{q}{p}} d\kappa \right)^{\frac{1}{q}} \\
&\quad \times \left( \int_{\sigma^{-1} \left( \frac{1}{\theta_1} \right)}^{\sigma^{-1} \left( \frac{1}{\theta_2} \right)} \left| (F' \circ \mathcal{H} \circ \sigma)(\kappa) \right| ^{q} \left| \sigma'(\kappa) \right| d\kappa \right)^{\frac{1}{q}} \\
&\quad + \frac{1}{\Gamma(\nu)} \left( \int_{\sigma^{-1} \left( \frac{1}{\theta_2} \right)}^{\sigma^{-1} \left( \frac{1}{\theta_1} \right)} \int_{\frac{1}{\theta_2}}^{\kappa} \left| \sigma'(\tau) \left( \frac{1}{\theta_1} - \sigma(\tau) \right)^{p-1} (w \circ \mathcal{H} \circ \sigma)(\tau) \right| ^{\frac{p}{q}} \left| \sigma'(\kappa) \right| ^{\frac{q}{p}} d\kappa \right)^{\frac{1}{q}} \\
&\quad \times \left( \int_{\sigma^{-1} \left( \frac{1}{\theta_2} \right)}^{\sigma^{-1} \left( \frac{1}{\theta_1} \right)} \left| (F' \circ \mathcal{H} \circ \sigma)(\kappa) \right| ^{q} \left| \sigma'(\kappa) \right| d\kappa \right)^{\frac{1}{q}}.
\end{aligned}
\tag{40}
\]

Since \( |F'| \) is harmonic-convex on \([\theta_1, \theta_2]\), where \( \kappa \in \left[ \sigma^{-1} \left( \frac{1}{\theta_2} \right), \sigma^{-1} \left( \frac{1}{\theta_1} \right) \right] \)

\[
\begin{aligned}
|F' \circ \mathcal{H} \circ \sigma(\kappa)| &= \left| F' \left( \frac{\theta_1 (\theta_2 - \sigma(\nu))}{\sigma(\nu)(\theta_2 - \theta_1)} \right) + \frac{\theta_2 (\sigma(\nu) - \theta_1)}{\sigma(\nu)(\theta_2 - \theta_1)} \right| \\
&\quad \leq \frac{\theta_1 (\theta_2 - \sigma(\nu))}{\sigma(\nu)(\theta_2 - \theta_1)} \left| (F' \circ \mathcal{H})(\theta_1) \right| + \frac{\theta_2 (\sigma(\nu) - \theta_1)}{\sigma(\nu)(\theta_2 - \theta_1)} \left| (F' \circ \mathcal{H})(\theta_2) \right|.
\end{aligned}
\tag{41}
\]

As a result, we get
\[ |Z_1 + Z_2| \leq \frac{||w||_{\frac{1}{\theta_2} + \frac{\nu}{2} \Gamma(\nu)}}{\left(\frac{\nu}{2} \right) \Gamma(\nu)} \left( \int_{\sigma^{-1}(\frac{1}{\theta_2})}^{\sigma^{-1}(\frac{1}{\theta_1})} \left| \int_{\sigma^{-1}(\frac{1}{\theta_2})}^{\sigma^{-1}(\frac{1}{\theta_1})} \theta' \left( \frac{1}{\theta_1} - \sigma(\tau) \right)^{\nu-1} d\tau \right| \sigma'(\tau) d\tau \right)^{\frac{1}{\nu}} \]

\[ \times \left( \int_{\sigma^{-1}(\frac{1}{\theta_2})}^{\sigma^{-1}(\frac{1}{\theta_1})} \left| ((\mathcal{F} \circ \mathcal{H})(\theta_1))|^{q} \sigma'(\tau) d\tau \right| \sigma'(\tau) d\tau \right)^{\frac{1}{q}} \]

\[ + \frac{||w||_{\frac{1}{\theta_2} + \frac{\nu}{2} \Gamma(\nu)}}{\left(\frac{\nu}{2} \right) \Gamma(\nu)} \left( \int_{\sigma^{-1}(\frac{1}{\theta_2})}^{\sigma^{-1}(\frac{1}{\theta_1})} \left| \int_{\sigma^{-1}(\frac{1}{\theta_2})}^{\sigma^{-1}(\frac{1}{\theta_1})} \theta' \left( \frac{1}{\theta_1} - \sigma(\tau) \right)^{\nu-1} d\tau \right| \sigma'(\tau) d\tau \right)^{\frac{1}{\nu}} \]

\[ \times \left( \int_{\sigma^{-1}(\frac{1}{\theta_2})}^{\sigma^{-1}(\frac{1}{\theta_1})} \left| ((\mathcal{F} \circ \mathcal{H})(\theta_1))|^{q} \sigma'(\tau) d\tau \right| \sigma'(\tau) d\tau \right)^{\frac{1}{q}} \]

where it is obvious

\[ \int_{\sigma^{-1}(\frac{1}{\theta_2})}^{\sigma^{-1}(\frac{1}{\theta_1})} \left| \int_{\sigma^{-1}(\frac{1}{\theta_2})}^{\sigma^{-1}(\frac{1}{\theta_1})} \theta' \left( \frac{1}{\theta_1} - \sigma(\tau) \right)^{\nu-1} d\tau \right| \sigma'(\tau) d\tau = \frac{(\theta_2 - \theta_1)^{\nu+1}}{(2\theta_1\theta_2)^{\nu+1} \nu(\nu+1)}. \]

We obtain the following integral after inserting (43) into (42):

\[ |Z_1 + Z_2| \leq \frac{||w||_{\frac{1}{\theta_2} + \frac{\nu}{2} \Gamma(\nu)}}{\left(\frac{\nu}{2} \right) \Gamma(\nu)} \left( \int_{\sigma^{-1}(\frac{1}{\theta_2})}^{\sigma^{-1}(\frac{1}{\theta_1})} \left| \int_{\sigma^{-1}(\frac{1}{\theta_2})}^{\sigma^{-1}(\frac{1}{\theta_1})} \theta' \left( \frac{1}{\theta_1} - \sigma(\tau) \right)^{\nu-1} d\tau \right| \sigma'(\tau) d\tau \right)^{\frac{1}{\nu}} \]

\[ \times \left( \int_{\sigma^{-1}(\frac{1}{\theta_2})}^{\sigma^{-1}(\frac{1}{\theta_1})} \left| ((\mathcal{F} \circ \mathcal{H})(\theta_1))|^{q} \sigma'(\tau) d\tau \right| \sigma'(\tau) d\tau \right)^{\frac{1}{q}} \]

We can obtain the necessary result (39) by doing basic integral calculations based on inequality (44).
4. Conclusions

In this paper, inequalities of the Hermite-Hadamard-Fejér type for harmonically convex functions in fractional integral forms are given in this study. Using weighted fractional integrals with positive weighted symmetric function kernels, an integral identity and various midpoint fractional Hermite-Hadamard-Fejér type integral inequalities for harmonically convex functions are also found.

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