Enhancement of non-Gaussianity and nonclassicality of photon added displaced Fock state: A quantitative approach

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Non-Gaussian and nonclassical states and processes are already found to be important resources for performing various tasks related to quantum gravity and quantum information processing. The effect of non-Gaussianity inducing operators on the nonclassicality of quantum states has also been studied rigorously. Considering these facts, a quantitative analysis of the nonclassical and non-Gaussian features is performed here for photon added displaced Fock state, as a test case, using a set of measures like entanglement potential, Wigner Yanese skew information, Wigner logarithmic negativity and relative entropy of non-Gaussianity. It is observed that photon addition (Fock parameter) significantly increases the amount of nonclassicality and non-Gaussianity for small (large) values of the displacement parameter, which decreases both the quantum features monotonically. In this respect, the role of Fock parameter is found to be more prominent and stronger compared to photon addition. Finally, the dynamics of Wigner function under the effect of photon loss channel is used to show that only highly efficient detectors are able to detect Wigner negativity.

I. INTRODUCTION

Quantification of non-Gaussianity and nonclassicality have drawn considerable attention in the recent past. This attention is well deserved as with the advent of quantum information science, it has been realized that nonclassicality and quantum non-Gaussianity are two of the most important facets of quantum world, which lead to quantum advantage [1]. Quantification problems have been approached in various ways. However, no unique measure of nonclassicality or non-Gaussianity can be built. We begin with the definition of nonclassicality and non-Gaussianity before discussing their significance in quantum technology and the requirement to quantify these resources.

The notion of nonclassicality arises from the non-positive value of the Glauber-Sudarshan $P$ function [2, 3]

$$\rho = \int d(P(\alpha)) \left( |\alpha\rangle \langle \alpha | \right).$$

In other words, a state that cannot be represented in terms of a statistical mixture of coherent states is known as a nonclassical state. In recent years, the importance of nonclassical states has been enhanced considerably as in the domain of meteorology and information processing [4], a major demand for the enhancement of the performance of the devices appeared and it is established that the desired enhancement is possible by using quantum resources. Nonclassical states and properties are found to be essential for obtaining quantum advantage and thus, they appeared as the basic building blocks for quantum or quantum-enhanced devices having advantages over the corresponding classical versions. Here, it may be noted that the nonclassical states not only lead to the detection of gravitational waves [5], they are also found essential for quantum teleportation [6–8], quantum key distribution [9–11], quantum computation [12, 13], etc. As nonclassicality leads to quantum advantage, attempts have been made to quantify the amount of nonclassicality and the quantum advantage it can provide. Specifically, in 1987, a distance based measure was introduced by Hillery [14], but there were many computational difficulties associated with that. Following that, Mari et al. [15] gave a measure based on trace norm distance. In 1991, Lee introduced a measure for the quantification of nonclassicality known as nonclassical depth [16] (for a brief review see [17]). The studies based on these measures revealed that non-Gaussianity inducing operators, e.g., photon addition and other quantum state engineering operations [18, 19] can induce and/or enhance the amount of nonclassicality in an arbitrary quantum state.

In addition, we may note that non-Gaussian states can be defined as the states which cannot be defined in terms of a probabilistic mixture of Gaussian states [20, 21]. On the other hand, all the Gaussian states can be defined by the first two moments, which means that the mean and covariance matrix of the Gaussian states provide us their complete information. We can define a complex hull $\mathcal{G}$ with the classical probability distribution $P_{\mathcal{G}}(\lambda)$

$$\rho = \int d(P_{\mathcal{G}}(\lambda)) \left|\psi_{\mathcal{G}}(\lambda)\right\rangle \left\langle \psi_{\mathcal{G}}(\lambda) \right|$$

in the Hilbert space $\mathcal{H}$. This set contains all the Gaussian states and some non-Gaussian states. Interestingly, the non-Gaussian states obtained as the statistical mixtures of Gaussian states in the form (2) have limited applications due to its origin in classical noise [21, 22]. On the other hand, quantum non-Gaussian states $\rho$ in $\mathcal{H}$ are the states which do not belong to the complex hull
It is worth to study the quantification of quantum non-Gaussianity in such a state as these states can be used as more robust resources compared to the Gaussian states ([23–25] and references therein). Specifically, there are no-go theorems limiting the applications of Gaussian operations and Gaussian states in entanglement distillation [26], quantum error correction [27], quantum computing [23] and quantum bit commitment [28]. Use of non-Gaussian operations is known to provide advantages in computation [23], communication [29], metrology [24], etc. These set of promising applications have motivated witnesses of non-Gaussianity [20], resource theory of non-Gaussianity [25], and a set of measures of quantum non-Gaussianity ([30–32] and references therein). Further, it may be noted that traditionally quantum gravity and quantum information processing have evolved as two independent branches, but in the recent past, various significant and exciting results connecting these two fields have been reported (see [33] and references therein). Specifically, in [33], it is shown that non-Gaussianity present in matter can be used to detect the signature of quantum gravity as only quantum theory of gravity can lead to non-Gaussianity. Although, in what follows, we will discuss non-Gaussianity of field, this exciting connection between non-Gaussianity of matter and quantum gravity has also motivated us to study how the amount of non-Gaussianity varies with different state parameters of photon added displaced Fock state (PADFS).

There are a couple of reasons for choosing PADFS as the test bed for the present study on the measures of nonclassicality and non-Gaussianity. Firstly, at different limits, this state reduces to a set of well-known quantum states having a large number of applications. Further, in previous works of some of the present authors, nonclassical properties [19] and phase properties [34] of PADFS were studied. There it was observed that non-Gaussianity inducing operators significantly affect the signatures of nonclassicality viewed through different witnesses of nonclassicality.

Motivated by the above facts, we set ourselves the task to quantify the nonclassicality and quantum non-Gaussianity of PADFS using some measures, namely linear entropy potential, skew information based measure, relative entropy based measure of non-Gaussianity and Wigner logarithmic negativity. Here, it is also worth stressing that the skew information based measure and Wigner logarithmic negativity fail to detect nonclassicality in some quantum states and are thus categorized as quantifiers of nonclassicality. However, our study on the skew information based measure is restricted to the pure states, where we can use it as a reliable measure of nonclassicality. Highly efficient detectors are required to detect Wigner negativity. This motivated us to quantify the nonclassicality and non-Gaussianity based on Wigner negativity over the photon loss channel, which also models inefficient real detectors used in experiments.

The rest of the paper is organized as follows. In Section II, the properties of PADFS and its limiting cases are discussed. In the next section, we report nonclassicality measures based on entanglement potential, skew information based measure, Wigner logarithmic negativity, and non-Gaussianity based measure. A comparative study of measures of nonclassicality and non-Gaussianity is performed in Section IV. In Section V, the dynamics of the Wigner function and Wigner logarithmic negativity for PADFS over photon loss channel are reported. Finally, the paper is concluded in Section VI.

II. PHOTON ADDED DISPLACED FOCK STATE AND ITS WIGNER FUNCTION

Displaced Fock state (DFS) is analytically expressed as \( D(\alpha) \ket{n} \), where \( D(\alpha) \) is displacement operator and \( \ket{n} \) is the Fock state. It has been studied very rigorously in the past [35–40] because of the fact that this state is nonclassical but its special case (displaced vacuum state which is known as coherent state and is obtained for \( n = 0 \)) is classical. In the present work, the state of interest is PADFS [19] which is obtained by the application of the creation operator (non-Gaussianity inducing operator) on the DFS [35]. In the previous works, some of the present authors have reported the nonclassical and phase properties of PADFS through various witnesses [19, 34]. However, neither non-Gaussianity was discussed nor the nonclassicality present in PADFS was quantified. This will be attempted here, but before we do that, it will be apt to introduce the state of our interest and to specifically state the motivation for this particular choice of state.
Figure 2. (Color Online) Wigner function for the single PADFS with Fock parameter \( n = 1 \) and displacement parameter \( \alpha = 0.5 \). In the contour plot at the bottom, the dark part represents the negative region whereas the contour lines indicate the positive region of the Wigner function.

The analytic expression for \( k \) photon added DFS can be given as

\[
|\psi(k, n, \alpha)\rangle = \sum_{m=0}^{\infty} C_m(n, k, \alpha) |m + k\rangle ,
\]

where

\[
C_m(n, k, \alpha) = N \exp \left[-\frac{|\alpha|^2}{2} \right] \frac{|\alpha|^m}{m!} L_{n-k}^m \left(|\alpha|^2 \right)
\]

with normalization factor

\[
N = \left( \sum_{m=0}^{\infty} \exp \left[-\frac{|\alpha|^2}{2} \right] \frac{|\alpha|^m}{m!} L_{n-k}^m \left(|\alpha|^2 \right) \right)^{-\frac{1}{2}} .
\]

Here, \( n \) is the Fock parameter, \( L_n^m \) is an associated Laguerre polynomial, and \( \alpha \) is the displacement parameter. In principle, PADFS, described as above, can be generated experimentally, and the schematic diagram of the setup to be used to generate PADFS was reported by us in our previous work [19]. Apart from the experimental realizability, the PADFS is of interest because in different limits, it reduces to different important and useful quantum states (including vacuum, Fock, Coherent, photon added coherent states, etc.). In Fig. 1, the limiting cases of PADFS are explicitly shown. In what follows, we obtain the signature of nonclassicality and non-Gaussianity present in PADFS using Wigner function as that can reveal signatures of both the features (i.e., non-Gaussianity and nonclassicality).

One can define Wigner function [41] of a quantum state with density matrix \( \rho \) in the coherent state representation as

\[
W(\gamma, \gamma^*) = \frac{2}{n!\pi} \exp \left[-2|\gamma|^2\right] \int d^2\lambda \langle -\lambda |\rho|\lambda \rangle \exp \left[-2(\gamma^* \lambda - \gamma \lambda^*)\right] .
\]

We have calculated (see Appendix VI) the analytical expression for Wigner function of PADFS as

\[
W(\gamma, \gamma^*) = \frac{2|N|^2}{n!\pi} \exp \left[-2|\gamma - \alpha|^2\right] \left[ \sum_{t=0}^{n} S_{tt} P_{tt} + 2\text{Re} \sum_{t=1}^{n} \sum_{t'=0}^{t-1} S_{t't} P_{t't'} \right] ,
\]

where

\[
P_{t't} = (-1)^{t+t'} (k+t')!\eta^{t-t'} L_{k+t'}^{t-t'} \left(|\eta|^2 \right)
\]

with \( \eta = -\alpha + 2\gamma \) and

\[
S_{t't} = \binom{n}{t'} \binom{n}{t} \alpha^{t'}(n-t') \alpha^{n-t}
\]
are polynomials depending upon photon addition \((k)\) and Fock parameter \((n)\), respectively.

The existence of nonclassical and non-Gaussian features in the state chosen here can be studied in various ways. Notice that for \(k = n = 0\) the Wigner function (7) has Gaussian form (as it corresponds to the Wigner function of coherent state). The polynomial in the product of this Gaussian function (i.e., \(S_{t1}P_{t1}\)) has a degree more than one for \(k \neq 0\) and/or \(n \neq 0\). For example, in the Wigner function of photon added coherent state for \(k \neq 0 = n\), we have \(S_{t1} = 1\) and \(P_{t1} = (-1)^k k!L_k \left(|\eta|^2\right)\) responsible for the non-Gaussian nature of the corresponding state. Similarly, in the case of DFS, \(P_{t1} = 1\) and \(S_{t1}\) is polynomial of degree more than one for \(n > 0\). Further, the non-zero value of displacement parameter can influence the non-Gaussian behavior iff either \(k \neq 0\) or \(n \neq 0\). Thus, one can easily conclude qualitatively that the polynomial in the product with a Gaussian factor due to photon addition and non-zero Fock parameter leads to non-Gaussianity.

One may further study the nonclassical and non-Gaussian behavior of PADFS for different choice of state parameters by plotting the Wigner function. Specifically, the negative values of Wigner function indicate the presence of nonclassicality. On top of that, according to Hudson’s theorem [42, 43], a pure quantum state having positive Wigner function is always Gaussian. This infers that the negative values of Wigner function (7) witness the non-Gaussianity in the state. Keeping this point into consideration, we have obtained the Wigner function shown in Fig. 2, which clearly reveals the nonclassical and non-Gaussian behavior. Especially, the existence of both these features is clearly observed through the negative values of the Wigner function. Note that the Wigner function only witnesses the presence of nonclassicality and non-Gaussianity, but does not help us to quantify the amount of nonclassicality and non-Gaussianity. In what follows, we quantify both these features using different measures of nonclassicality and non-Gaussianity.

III. MEASURES OF NONCLASSICALITY AND NON-GAUSSIANITY

Photon addition is a frequently used operation in quantum state engineering and it is often used to induce non-Gaussianity. Consequently, it is usually considered as a non-Gaussianity inducing operator. Further, it can also be used to introduce nonclassicality by hole burning [44, 45]. Here, we aim to study the variation of the amount of nonclassicality and non-Gaussianity present in PADFS with the state parameters (i.e., the number of photons added \((k)\), the Fock \((n)\) and displacement \((\alpha)\) parameters). The relevance of the study underlies in the fact that such a study may help us to identify the suitable state parameters for an engineered quantum state which is to be used to perform a quantum computation, communication or metrology task that requires nonclassical and/or non-Gaussian states. For the quantification of nonclassicality, the closed form analytical expressions of an entanglement potential (to be referred to as linear entropy potential), skew information based measure, and Wigner logarithmic negativity are obtained in this section. Further, Wigner logarithmic negativity and relative entropy of non-Gaussianity are computed as measures of non-Gaussianity. In what follows, we would briefly introduce these measures and explicitly show how the amount of nonclassicality and non-Gaussianity quantified by these measures vary with the state parameters.

A. Entanglement potential: Linear entropy potential

To begin the discussion on the measures of nonclassicality, we must note that there are several nonclassicality measures (e.g., nonclassical depth [16], distance based measures [14]), but each has its own limitation(s) (for a quick review see [17]). In 2005, Asboth [46] introduced a new measure of nonclassicality, which quantifies the single-mode nonclassicality using the fact that if a single-mode nonclassical (classical) state is inserted from an input port of a beamsplitter (BS) and vacuum state is inserted from the other port then the output two-mode state must be entangled (separable). Consequently, a measure of entanglement can be used to indirectly measure the nonclassicality of the input single-mode state (the state other than the vacuum state inserted in the BS). However, there exist many quantitative measures of entanglement and any of those can be used to measure the nonclassicality of the input single-mode state. When a measure of entanglement is used to measure the single mode nonclassicality using this approach, corresponding entanglement measure is referred to as the entanglement potential in analogy with the terminology used by Asboth. Specifically, if we use concurrence (linear entropy) to measure the entanglement of the post-BS two-mode output state in an effort to quantify the nonclassicality of the single-mode input state, then the nonclassicality measure is called concurrence potential (linear entropy potential) [17]. In what follows, we quantify the nonclassicality of PADFS using Asboth’s approach considering linear entropy potential as the entanglement potential.

We know that the linear entropy for a bipartite state \(\rho_{AB}\) is defined in terms of the purity of the reduced subsystem as

\[
L_E = 1 - \text{Tr} \left( \rho_B^2 \right),
\]

where \(\rho_B\) can be obtained by taking partial trace over the subsystem \(A\). Thus, \(L_E = 0 \) (1) for a separable (maximally entangled) state and a non-zero value for an entangled state in general [47, 48]. To compute it, we would require the post-BS state \(\rho_{AB}\) that will originate if PADFS and vacuum states are mixed at a BS. To begin with, we may note that the post-BS transformation can be understood in terms of Hamiltonian \(H = -\hbar \theta (a^\dagger b + ab^\dagger)\), where \(a^\dagger, b^\dagger\) are the creation operators and \(a\) and \(b\) are annihilation
operators for the two input modes. Also, we have chosen $\theta = \pi/4$ here for a symmetric-BS parameter. The expression for the two mode state having Fock state at one port and vacuum at another port can be written as

$$ |n\rangle \otimes |0\rangle = |n, 0\rangle \xrightarrow{\text{BS}} \frac{1}{2^{n/2}} \sum_{j=0}^{n} \sqrt{n!} |j, n-j\rangle. \quad (9) $$

This equation can be used to obtain post-BS state $\rho_{AB} = \langle \phi \rangle_{AB} |\phi\rangle_{AB}$ that originates on the insertion of PADFS and vacuum state from the input ports of a BS with

$$ |\phi\rangle_{AB} = |\psi(k, n, \alpha)\rangle \otimes |0\rangle \xrightarrow{\text{BS}} \sum_{m=0}^{\infty} C_m \langle n, k, \alpha | \sum_{k_1=0}^{m+k} \sqrt{\left(\frac{1}{\sqrt{2}}\right)^{k+m+1} |j, m+k-k_1\rangle |j, m + k - k_1\rangle. \quad (10) $$

In order to calculate the expression for linear entropy, we obtain partial trace over the subsystem $A$ in the post-BS state $\rho_{AB}$ and use Eq. (8) to yield an analytic expression for linear entropy as

$$ L_E = 1 - \sum_{m, m', r=0} \sum_{j=0}^{m+k} C_m \langle j, m, \alpha | C^*_{m'} \langle n, k, \alpha | C_r \langle n, k, \alpha | C^*_{m+r-m'} \langle n, k, \alpha |$$

$$ \times \sum_{k_1=0}^{m+k} \sqrt{\left\langle m+k \right\rangle \sqrt{\left\langle m'+k \right\rangle \sqrt{\left\langle r+k \right\rangle \sqrt{\left\langle m'-r+k \right\rangle \sqrt{\left\langle r-m'+k \right\rangle \sqrt{\left\langle 1 \right\rangle \sqrt{\left\langle m+r+2k \right\rangle}}}}}} \quad (11) $$

We illustrate the variation of the amount of nonclassicality with respect to the state parameters in Fig. 3 which quantifies the nonclassicality of PADFS through linear entropy potential. Nonclassicality decreases with increase in the displacement parameter as for $\alpha = 0$ the state reduces to Fock state, the most nonclassical state. The value of the displacement parameter is significant in controlling the outcome of photon addition. Specifically, Fig. 3 (a) shows that up to a certain value of displacement parameter $\alpha$, nonclassicality increases with photon addition while an opposite behavior is observed after a certain value of $\alpha$. In what follows, we will refer to this particular value of $\alpha$ as $\alpha_{\text{inversion}}$ as, at this value of $\alpha$, the amount of nonclassicality based ordering of different PADFSs get inverted. Specifically, if we add an increasing order of single photon added, two photon added and three photon added DFS (with the same initial Fock parameter), based on the amount of nonclassicality quantified by linear entropy potential, then from Fig. 3 (a) we can see that for $\alpha < \alpha_{\text{inversion}}$ the ordered sequence is single photon added DFS, two photon added DFS, three photon added DFS, but the sequence is opposite for $\alpha > \alpha_{\text{inversion}}$. Interestingly, $\alpha_{\text{inversion}}$ depends on all the state parameters, such as $L_{E}^{k=1} = L_{E}^{k=2}$ for $\alpha_{\text{inversion}} = 0.45$ while $L_{E}^{k=2} = L_{E}^{k=3}$ for $\alpha_{\text{inversion}} = 0.42$ when Fock parameter is $n = 1$. Further, from Fig. 3 (b), it is easy to observe that with an increase in the Fock parameter the nonclassicality is found to be increased, and the performance is better than in the case of photon addition. In other words, photon addition is an effective tool for enhancing nonclassicality for the small values of $\alpha$ while higher values of the Fock parameter are more beneficial at larger values of the displacement parameter. This can be attributed to the origin of nonclassicality from corresponding operations as photon addition burns a hole in the photon number distribution, which influences more on the state with the small values of $\alpha$ due to larger probability amplitude for vacuum in the Fock basis.

### B. Skew information based measure

This skew information based measure of nonclassicality was introduced by Shunlong Luo et al. [49] in 2019 and is based on Wigner Yanase skew information [50]. In the case of a pure state $\rho$, this is defined as

$$ N(\rho) = \frac{1}{2} \hat{a} \hat{a}^\dagger + \langle \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a} \rangle \langle \hat{a} \rangle. \quad (12) $$
In the case of linear entanglement potential and skew information based measure. Specifically, any increase in the displacement parameter enhances nonclassicality and non-Gaussianity, whereas with increase in photon addition the Wigner logarithmic negativity increases (decreases) for small (large) displacement parameter (cf. Fig. 4). Among the photon addition and Fock parameter, the latter performs better than the former in enhancing nonclassicality and non-Gaussianity. Although, a Gaussian operation, namely displacement operation, is not expected to increase non-Gaussianity of DFS but it plays very significant role in determining the non-Gaussianity of PADFS.

**C. Wigner logarithmic negativity**

We have already mentioned in Section II that the negative values of Wigner function are signatures of nonclassicality. This motivated to use the volume of the negative part of the Wigner function as a quantifier of nonclassicality [51]. Recently, further extending this idea, a measure of nonclassicality, named Wigner logarithmic negativity, is introduced as [25]

\[
\mathcal{W} = \log_2 \left( \int d^2\gamma |W(\gamma)| \right),
\]

where the integration is performed over whole phase space. Interestingly, in the resource theory of quantum information, it has been noted that \(\mathcal{W}\) also quantifies the amount of non-Gaussianity present in the state as the negative values of Wigner function (cf. Fig. 2) also witness the non-Gaussianity of PADFS.

The integration in Eq. (14) is performed numerically using Wigner function (7) to study the effect of different parameters on the Wigner logarithmic negativity of PADFS. The Wigner logarithmic negativity shows similar behavior with all the parameters as that for linear entanglement potential and skew information based measure. Specifically, any increase in the displacement (Fock) parameter only reduces (increases) nonclassicality and non-Gaussianity, whereas with increase in photon addition the Wigner logarithmic negativity increases (decreases) for small (large) displacement parameter (cf. Fig. 3). Among the photon addition and Fock parameter, the latter performs better than the former in enhancing nonclassicality and non-Gaussianity. Although, a Gaussian operation, namely displacement operation, is not expected to increase non-Gaussianity of DFS but it plays very significant role in determining the non-Gaussianity of PADFS.
where the relative entropy
\[ S(\rho) = \text{Tr} [\rho \log \rho - \log \Lambda] \]
where
\[ \log \Lambda = \sum_{m,n,k,\alpha} \sigma_{\alpha} \sigma_{\beta} \]
we aim to re-investigate this feature by quantifying non-Gaussianity using relative entropy of non-Gaussianity.

we have quantified non-Gaussianity using Wigner logarithmic negativity and observed that in the case of PADFS, variation of
\[ \delta \] non-Gaussianity with the state parameters of interest is analogous to that of nonclassicality as quantified through different measures. This is justified as the set of Wigner negative states is a subset of non-Gaussian states \[ \Delta \]. In what follows, we aim to re-investigate this feature by quantifying non-Gaussianity using relative entropy of non-Gaussianity.

1. Relative entropy based measure of non-Gaussianity

The relative entropy of a non-Gaussian state with the set of all Gaussian states is defined as
\[ \delta [\rho] = S (\rho \| \tau_G) , \]
where the relative entropy \[ S (\rho || \Lambda) = \text{Tr} [\rho (\log \rho - \log \Lambda)] \], and the reference mixed Gaussian state \[ \tau_G \] is chosen to have the same first and second moments as that of \( \rho \). However, in the case of pure states, \[ S (\rho = |\psi \rangle \langle \psi |) = 0 \] and we can define \[ \delta [|\psi \rangle] = S (\tau_G) \], von Neumann entropy of the reference state.

Further, we know that a Gaussian state is fully characterized by its first two moments and thus its covariance matrix. Specifically, the covariance matrix \( \sigma \) can be written as [25]
\[ \sigma = \begin{bmatrix} \sigma_{qq} & \sigma_{qp} \\ \sigma_{qp} & \sigma_{pp} \end{bmatrix}, \]
(15)
where \( \sigma_{uv} = (\langle uv + vu \rangle - 2 \langle u \rangle \langle v \rangle) \) for position \( q = \frac{a + a^*}{\sqrt{2}} \) and momentum \( p = \frac{a - a^*}{i\sqrt{2}} \). All the elements of covariance matrix \( \sigma \) for PADFS are obtained as
\[ \begin{align*}
\sigma_{qq} &= \sum_{m=0}^{\infty} \text{Re} \left ( C_{m+2} (n, k, \alpha) C_{m}^* (n, k, \alpha) \right ) \sqrt{(m+k+1)(m+k+2)} \\
& \pm 2 |C_m (n, k, \alpha)|^2 (m+k)(m+k-1) - 1 \\
& \pm \sum_{m=0}^{\infty} (C_{m+1} (n, k, \alpha) C_m^* (n, k, \alpha) \pm \text{c.c.})^2 (m+k+1) 
\end{align*} \]
and
\[ \sigma_{qp} = \sum_{m=0}^{\infty} \text{Im} \left ( C_{m+2} (n, k, \alpha) C_{m}^* (n, k, \alpha) \right ) \sqrt{(m+k+1)(m+k+2)} \\
- \sum_{m=0}^{\infty} \text{Im} \left ( C_{m+1}^2 (n, k, \alpha) C_{m}^* (n, k, \alpha) \right ) (m+k+1), \]
where \( \text{Re}(z) \) and \( \text{Im}(z) \) correspond to the real and imaginary parts of the complex number \( z \). Thus, relative entropy of non-Gaussianity in terms of the covariance matrix reduces to
\[ \delta [|\psi \rangle] = S (\tau_G) = h (\text{det} (\sqrt{\sigma})) , \]
(16)
where \( h (z) = \frac{\log_2 (\frac{z + 1}{z - 1}) - \frac{\log_2 (\frac{z + 1}{z - 1})}{2}}{2} \).

It may be noted that the variation of the amount of non-Gaussianity present in a single photon added coherent state (PACS) has already been studied using different measures of non-Gaussianity (see Refs. [20, 55, 56]). Interestingly, for \( k = 1, n = 0, \)

Figure 5. (Color Online) Variation of Wigner logarithmic negativity with displacement parameter (a) for different values of photon addition \( (k) \) and \( n = 1 \) and (b) for different values of Fock parameter \( (n) \) and single photon addition \( k = 1 \).

D. Measure of non-Gaussianity

A number of criteria to quantify non-Gaussianity have been reported in literature [21, 25, 52–54]. In the previous section, we have quantified non-Gaussianity using Wigner logarithmic negativity and observed that in the case of PADFS, variation of the amount of non-Gaussianity with the state parameters of interest is analogous to that of nonclassicality as quantified through different measures. This is justified as the set of Wigner negative states is a subset of non-Gaussian states \[ \Delta \]. In what follows, we aim to re-investigate this feature by quantifying non-Gaussianity using relative entropy of non-Gaussianity.

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Further, we know that a Gaussian state is fully characterized by its first two moments and thus its covariance matrix. Specifically, the covariance matrix \( \sigma \) can be written as [25]
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where \( \sigma_{uv} = (\langle uv + vu \rangle - 2 \langle u \rangle \langle v \rangle) \) for position \( q = \frac{a + a^*}{\sqrt{2}} \) and momentum \( p = \frac{a - a^*}{i\sqrt{2}} \). All the elements of covariance matrix \( \sigma \) for PADFS are obtained as
\[ \begin{align*}
\sigma_{qq} &= \sum_{m=0}^{\infty} \text{Re} \left ( C_{m+2} (n, k, \alpha) C_{m}^* (n, k, \alpha) \right ) \sqrt{(m+k+1)(m+k+2)} \\
& \pm 2 |C_m (n, k, \alpha)|^2 (m+k)(m+k-1) - 1 \\
& \pm \sum_{m=0}^{\infty} (C_{m+1} (n, k, \alpha) C_m^* (n, k, \alpha) \pm \text{c.c.})^2 (m+k+1) 
\end{align*} \]
and
\[ \sigma_{qp} = \sum_{m=0}^{\infty} \text{Im} \left ( C_{m+2} (n, k, \alpha) C_{m}^* (n, k, \alpha) \right ) \sqrt{(m+k+1)(m+k+2)} \\
- \sum_{m=0}^{\infty} \text{Im} \left ( C_{m+1}^2 (n, k, \alpha) C_m^* (n, k, \alpha) \right ) (m+k+1), \]
where \( \text{Re}(z) \) and \( \text{Im}(z) \) correspond to the real and imaginary parts of the complex number \( z \). Thus, relative entropy of non-Gaussianity in terms of the covariance matrix reduces to
\[ \delta [|\psi \rangle] = S (\tau_G) = h (\text{det} (\sqrt{\sigma})) , \]
(16)
where \( h (z) = \frac{\log_2 (\frac{z + 1}{z - 1}) - \frac{\log_2 (\frac{z + 1}{z - 1})}{2}}{2} \).

It may be noted that the variation of the amount of non-Gaussianity present in a single photon added coherent state (PACS) has already been studied using different measures of non-Gaussianity (see Refs. [20, 55, 56]). Interestingly, for \( k = 1, n = 0, \)
Figure 6. (Color Online) Non-Gaussianity as a function of $\alpha$ (a) for variation in photon addition with $n = 1$, (b) with respect to different Fock parameter for $k = 1$.

Figure 7. (Color Online): Parametric plot as a function of Wigner logarithmic negativity for various measures ($N_c$) for PADS (solid lines) with $k = 1$, $n = 1$, PACS (dashed lines) with $k = 1$, and solid circle (square) representing the values of different measures for single photon (two photon) Fock state $|1\rangle$ ($|2\rangle$).

PADFS reduces to PACS, and we observe that in this specific case, variation of non-Gaussianity reported here in a more general scenario provides similar pattern as reported in the earlier works (cf. Fig. 2 of [55]) where it was observed that non-Gaussianity reduces with $\alpha$. For a coherent state, $\alpha^2$ (if $\alpha$ is considered to be real) is the average photon number. Similarly, for PACS average photon number is $\frac{\alpha^4 + 3\alpha^2 + 1}{\alpha^2 + 1}$ and a much more complex expression of average photon number as a function of $\alpha$ for PADFS is reported in [19]. In all these cases, average photon number is found to increase with $\alpha$. For a relatively large value of $\alpha$, average photon number is high. Consequently, addition of one or two photons or equivalently application of non-Gaussianity inducing creation operator for once or twice does not have much impact on non-Gaussianity or nonclassicality. This intuitive logic explains the nature of plots (initially decreasing with $\alpha$ and gradually approaching a line parallel to the horizontal axis) observed here and also in earlier studies. Further, for $\alpha \rightarrow 0$, PADFS reduces to a Fock state $|m\rangle = |k + n\rangle$. Figs. 5 and 6 clearly show that non-Gaussianity of Fock state $|m\rangle$ increases with $m$ which is also consistent with the earlier results. For example, in a recent work [55], non-Gaussianity has been quantified using an uncertainty relation and it is found that non-Gaussianity of Fock state $|m\rangle$ is $m$. Thus, addition of photon to a Fock state will always increase non-Gaussianity, but the same is not true in general for a superposition of Fock states.

**IV. A COMPARATIVE STUDY OF MEASURES OF NONCLASSICALITY AND NON-GAUSSIANITY**

So far we have studied nonclassicality and non-Gaussianity independently with a quantitative approach, but noted that Wigner function can be used to witness both the features. Further, photon addition in a Gaussian state (coherent state) is known to be significant in inducing both the features. More recently comparison of qualitative behavior of the variation of the amount of non-Gaussianity quantified using two measures, namely relative entropy based measure and Wigner logarithmic negativity, is shown using a parametric plot [25]. Here, we show the amount of nonclassicality and non-Gaussianity obtained by different measures through a parametric plot as function of the amount of Wigner logarithmic negativity. Specifically, the amount of nonclassicality quantified through linear entropy potential and skew information based measure as well as that of non-Gaussianity measured using relative entropy based measure as function of the amount of Wigner logarithmic negativity for PADFS (solid lines), PACS (dashed lines) and Fock state is illustrated in Fig. 7. One can verify that the amount of nonclassicality and non-Gaussianity in PACS varies linearly between that of the values for the single photon and vacuum. On the other hand, due to non-zero Fock parameter in PADFS, the amount of nonclassicality and non-Gaussianity in PADFS can be observed to vary approximately
between that of the values for the single-photon and two-photon Fock states. The slope of the line representing the variation of linear entropy potential is the least as the entanglement based measure is upper bounded to unity while the rest of the measures are not bounded.

V. WIGNER FUNCTION OF P ADFS EVOLVING UNDER PHOTON LOSS CHANNEL

An interaction between the quantum system and its surrounding induces quantum to classical transition. Therefore, the observed nonclassical and non-Gaussian features are expected to decay due to the evolution of P ADFS under lossy channel. Specifically, the temporal evolution of a quantum state $\rho$ over lossy channel can be studied using LGKS master equation\(^5\)

$$\frac{\partial \rho}{\partial t} = \kappa \left( 2 a \rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a \right),$$  \hspace{1cm} (17)

where $\kappa$ is the rate of decay. Analogously, the time evolution of Wigner function at time $t$ in terms of initial Wigner function of the state evolving under lossy channel\(^5\) can be defined as

$$W(\zeta,t) = \frac{2}{\pi} \int \frac{d^2\gamma}{\pi} \exp \left[ -\frac{2}{T} |\zeta - \gamma e^{-\kappa t}|^2 \right] W(\gamma,0),$$ \hspace{1cm} (18)

where $T = 1 - \exp (-2\kappa t)$. The time evolution of Wigner function (18) models dissipation due to interaction with a vacuum reservoir as well as inefficient detectors with efficiency $\eta = 1 - T$.

We can clearly observe that the Wigner function at non-zero time $t$ is obtained as the convolution of the initial Wigner function. Using the description of Wigner function in an ideal scenario, a compact expression of Wigner function can be obtained. One can easily notice from Eq. (18) that both non-Gaussianity and nonclassicality of P ADFS cannot increase due to its temporal evolution over a photon loss channel. This is clearly illustrated through Fig. 8 where we can observe the negative region shrinking with increasing values of rescaled time $\kappa t$ for a single photon added and single Fock parameter P ADFS with displacement parameter $\alpha = 0.5$. In other words, the area of dark color in the contour plots representing the negative region is decreasing as we are increasing $\kappa t$ implying that the quantum features are diminishing. Further, as expected from Eq. (18) the quantum state loses its nonclassical character as $t \to \infty$, and the Wigner function becomes positive (as it reduces to that of the vacuum). In fact, the averaging involved in the dynamics of Wigner function smoothes its behavior which in turn reduces its non-Gaussian features as well. For instance, the peak at the origin of the phase space can be observed to be decreasing with time in Fig. 8. This also shows that the present results can be used to study a non-Gaussianity witness based on the value of the Wigner function at the origin of the phase space\(^2\). However, it is worth mentioning here that as the Wigner function fails to detect all the nonclassicality (i.e., Wigner function is positive for squeezed states) and non-Gaussianity (as non-Gaussian state with positive Wigner function is reported in\(^5\)), it would be appropriate to refer to the observed features under the effect of photon loss channel as the lower bound on the nonclassicality and quantum non-Gaussianity in the state. In other words, the nonclassicality and non-Gaussianity in the P ADFS measured using realistic inefficient detector may be more than that reflected through the quantifier based on the Wigner negativity, which we referred here as lower bound.

To observe the above finding in a quantitative manner, the dynamics of the Wigner logarithmic negativity can be studied. Interestingly, time evolution of the Wigner logarithmic negativity (cf. Fig. 9) for the P ADFS evolving under photon loss channel
shows similar behavior as the variation of Wigner logarithmic negativity in noiseless case with the displacement parameter (see Fig. 5). This can be attributed to the fact that the photon loss channel is a Gaussian channel and thus non-Gaussianity cannot increase due to the application of this map. However, unlike Fig. 5, the nonclassicality and non-Gaussianity disappear beyond a threshold value of $\kappa t$. Interestingly, this threshold value is closely associated with a measure of nonclassicality proposed by Lee [16].

VI. CONCLUSION

Quantification of nonclassicality and quantum non-Gaussianity as well as the study of the relation between them are problems of interest in their own merit. These problems are addressed here by considering PADFSs as a test case. Specifically, PADFS is chosen as it can be reduced to various quantum states having important applications in different limits. By construction, PADFSs are pure quantum states and Figs. 5 and 6 clearly show that different choices of $n$ and $k$ lead to different quantum states having different amount of non-Gaussianity. In this work, apart from comparing non-Gaussianity, we have also used linear entropy potential, Wigner logarithmic negativity and Wigner Yanase skew information as measures of nonclassicality to compare the amount of nonclassicality present in a set of different non-Gaussian pure states. Here, it will be appropriate to note that the choice of nonclassicality measures is important as all the known measures of nonclassicality cannot be used to compare the amount of nonclassicality present in two non-Gaussian pure states. Specifically, nonclassical depth, a well-known and rigorously studied measure of nonclassicality introduced by Lee in [16] and subsequently used by many authors (see [17, 60] and references therein) cannot be used for this purpose as nonclassical depth is 1 for all non-Gaussian pure states [61]. Here, it will be apt to note that Lee used the fact that Gaussian noise can deplete nonclassicality (as we observed in the dynamics of Wigner function under photon loss channel) and considered the minimum amount of noise needed to destroy nonclassicality as a measure of nonclassicality which he referred to as nonclassical depth. Interestingly, noise can change the nature of non-Gaussianity as discussed in [20] as well as it can induce (classical) non-Gaussianity in the Gaussian states due to phase fluctuations [22]. The present study shows that quantum non-Gaussianity and nonclassicality witnessed through Wigner negativity (and quantified using Wigner logarithmic negativity) declines due to photon loss channels, which also infers that the reported features can only be observed with highly efficient detectors. However, we cannot discard from the present study that the useful quantum non-Gaussianity (with Wigner negativity) of PADFS is not eventually transformed into a non-Gaussian state having positive Wigner function that cannot be used for quantum computing [62]. Further, squeezing is expected to play an important role in combating depleting Wigner negativity in view of some recent results [63].

In short, nonclassicality and non-Gaussianity present in the PADFS is quantified, which shows that both photon addition and Fock parameter enhance these quantum features where the former (latter) is a more effective tool at small (large) displacement parameter. In contrast, displacement parameter is observed to reduce the quantum features. In view of Hudson’s theorem and resource theory of non-Gaussianity based on Wigner negativity [25], the Gaussian operations are free operations and thus, displacement operation and photon loss channels are not expected to enhance non-Gaussianity and/or Wigner negativity. Although, in case of pure PADFS, the Wigner negativity captures all the quantum non-Gaussianity but cannot discard conclusively the feasibility of quantum non-Gaussianity in PADFS measured through inefficient detectors (or evolved through lossy channels) with the positive Wigner function. Similarly, skew information based and linear entropy potential measures show that Wigner function succeeds in detecting all the nonclassicality of the pure PADFS.

Results reported here seem to be experimentally realizable, as PACS, which is a special case of PADFS has not only been prepared experimentally by Bellini et al. [64], non-Gaussianity present in this state is also measured experimentally [56]. Further, in [34], a schematic diagram of the setup that can generate PADFS is already reported. Keeping the above and the recent progresses in quantum state engineering and quantum information in mind, we conclude this article with an expectation that PADFSs will soon be produced in laboratories and will be used for performing quantum optical, meteorological and computing tasks.
PM and AP acknowledge the support from DRDO, India project no. ANURAG/MMG/CARS/2018-19/071. KT acknowledges the financial support from the Operational Programme Research, Development and Education - European Regional Development Fund project no. CZ.02.1.01/0.0/0.0/16_019/0000754 of the Ministry of Education, Youth and Sports of the Czech Republic.

ACKNOWLEDGMENT

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It is straightforward to show that

\[
\langle -\lambda | \rho | \lambda \rangle = |N|^2 \frac{(-1)^{n+k} \lambda^k \lambda^k}{n!} (\lambda^* + \alpha^*)^n (\lambda - \alpha)^n \exp \left[ (\alpha^* \lambda - \lambda^* \alpha) - \left( |\lambda|^2 + |\alpha|^2 \right) \right].
\]  
(19)

Substituting it in (6) and writing

\[
(\lambda^* + \alpha^*)^n = \sum_{t=0}^{n} \binom{n}{t} (\alpha^*)^{n-t} (\lambda^*)^t,
\]

\[
(\lambda - \alpha)^n = \sum_{t'=0}^{n} \binom{n}{t'} (-\alpha)^{n-t'} (\lambda)^{t'}.
\]

\(W(\gamma, \gamma^*)\) can be written in the form

\[
\text{APPENDIX A: DETAILS OF WIGNER FUNCTION}
\]
\[ W(\gamma, \gamma^*) = \frac{2|N|^2}{\pi} \exp \left[ 2|\gamma|^2 - |\alpha|^2 \right] \frac{(i-1)n+k}{n!} \sum_{t=0}^{n} \sum_{t'=0}^{n} \binom{n}{t} \binom{n}{t'} (\alpha^*)^{n-t} (-\alpha)^{n-t'} \times \left( -\frac{\partial}{\partial \gamma} \right)^{k+t} \left( \frac{\partial}{\partial \gamma^*} \right)^{k+t'} \int \frac{d^2\lambda}{\pi} \exp \left[ -|\lambda|^2 - \eta^* \lambda + \eta \lambda^* \right], \]  

where \( \eta = 2\gamma - \alpha, \frac{\partial}{\partial \alpha} \equiv -\frac{\partial}{\partial \eta}, \frac{\partial}{\partial \alpha^*} \equiv -\frac{\partial}{\partial \eta^*}. \) Using the formula

\[ \int \frac{d^2z}{\pi} \exp \left[ \zeta |z|^2 + \xi z + \eta z^* \right] = \frac{1}{\sqrt{\zeta^2}} \exp \left[ -\frac{\xi \eta}{\zeta^2} \right], \]

the integration over \( \lambda \) yields \( \exp [-\eta^* \eta]. \) Further, setting \( k + t = m', k + t' = n' \), we use the formula

\[ \left( \frac{\partial}{\partial \eta} \right)^{m'} \left( \frac{\partial}{\partial \eta^*} \right)^{n'} \exp [-\eta^* \eta] = \left\{ \begin{array}{ll}
(1)^{n'} m'! \exp [-\eta^* \eta] L_{m'-m'}^{n'} (\eta \eta^*) & m' \leq n', \\
(1)^{m'} n'! \exp [-\eta^* \eta] L_{m'-n'}^{m'} (\eta \eta^*) & n' \leq m'.
\end{array} \right. \]  

(21)

Substituting in (20), the double sum can be re-ordered to give the final expression (7) for the Wigner function \( W(\gamma, \gamma^*). \)