Complete convergence theorem for stationary heavy tailed sequences

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Abstract

For a class of stationary regularly varying and weakly dependent multivariate time series \((X_n)\), we prove the so-called complete convergence result for the space–time point processes of the form \(N_n = \sum_{i=1}^n \delta(i/n, X_i/a_n)\). As an application of our main theorem, we give a simple proof of the invariance principle for the corresponding partial maximum process.

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1 Introduction

Point processes theory is widely recognized as a useful and elegant tool for the extremal analysis of stochastic processes. This approach is splendidly illustrated by Resnick [18] or Leadbetter and Rootzén [12]. It is well known for instance that for an iid sequence of random variables \((X_n)\), regular variation of the marginal distribution is equivalent to the so-called complete convergence result, that is to the convergence of point processes of the form

\[ N_n = \sum_{i=1}^n \delta(i/n, X_i/a_n), \]  

(1.1)
towards a suitable Poisson point process. Such a statement then yields nearly all relevant asymptotic distributional properties about the sequence \((X_n)\), cf. [12]. There exists a rich literature on extensions of this result to dependent stationary sequences in both univariate and multivariate cases, for an illustration consider for instance Davis and Resnick [8], Davis and Hsing [6], Davis and Mikosch [7], Hsing and Leadbetter [10] or Basrak et al. [3]. One of the earliest and key results
in this area was obtained by Mori [13] who showed that all possible limits for the point processes \( N_n \) have the form of a Poisson cluster process provided that the random variables \( X'_n \)'s are strongly mixing with a strictly positive extremal index. In such a case the limit can be written as

\[
N = \sum_i \sum_j \delta_{(T_i, P_i, Q_{ij})},
\]

where \( \sum_i \delta_{(T_i, P_i)} \) is a suitable Poisson process and \( (\sum_j \delta_{Q_{ij}})_i \) represents an independent iid sequence of point processes on \( \mathbb{R} \setminus \{0\} \) with the property that \( \max_j |Q_{ij}| = 1 \) for all \( i \). However, the relationship between the sequence \( (X_n) \) and the distribution of the limiting process \( N \) or its components is up to now only partly understood. Due to the dependence, for a general sequence \( (X_n) \), there is no complete convergence result which determines the shape of the limiting process \( N \) in the form suggested by Mori. It is our main goal here to give such a result for a rather wide class of weakly dependent regularly varying processes, even in the multivariate setting.

Recall that a \( d \)-dimensional random vector \( X \) is regularly varying with index \( \alpha > 0 \) if there exists a random vector \( \Theta \in \mathbb{S}^{d-1} \) such that

\[
\frac{1}{\mathbb{P}(\|X\| > x)} \mathbb{P}(\|X\| > ux, \|X\|/\|X\| \in \cdot) \Rightarrow u^{-\alpha} \mathbb{P}(\Theta \in \cdot), \tag{1.2}
\]

for every \( u > 0 \) as \( x \to \infty \), where \( \Rightarrow \) denotes the weak convergence of measures. Note that the definition does not depend on the choice of the norm, i.e. if (1.2) holds for some norm in \( \mathbb{R}^d \), it holds for all norms, with different distributions of \( \Theta \) clearly. A \( d \)-dimensional time series \( (X_n)_{n \in \mathbb{Z}} \) is regularly varying if all of the finite-dimensional vectors \( (X_k, \ldots, X_l) \), \( k, l \in \mathbb{Z} \) are regularly varying, see Davis and Hsing [6] for instance. We will consider a strictly stationary regularly varying process \( (X_n)_{n \in \mathbb{Z}} \). This means in particular, that there exists a sequence \( (a_n) \), \( a_n \to \infty \) such that

\[
n\mathbb{P}(\|X_0\| > a_n x) \to x^{-\alpha} \quad \text{for all } x > 0. \tag{1.3}
\]

According to Basrak and Segers [4], the regular variation of the stationary sequence \( (X_n) \) is equivalent to the existence of the tail process \( (Y_n)_{n \in \mathbb{Z}} \) which satisfies \( \mathbb{P}(\|Y_0\| > y) = y^{-\alpha} \) for \( y \geq 1 \) and, as \( x \to \infty \),

\[
((x^{-1} X_n)_{n \in \mathbb{Z}} \mid \|X_0\| > x) \xrightarrow{\text{fidi}} (Y_n)_{n \in \mathbb{Z}}, \tag{1.4}
\]

where \( \xrightarrow{\text{fidi}} \) denotes convergence of finite-dimensional distributions. Moreover, the so-called spectral tail process \( (\Theta_n)_{n \in \mathbb{Z}} \) defined as a sequence \( \Theta_n = Y_n/\|Y_0\| \), \( n \in \mathbb{Z} \), turns out to be independent of \( \|Y_0\| \) and satisfies

\[
((\|X_0\|^{-1} X_n)_{n \in \mathbb{Z}} \mid \|X_0\| > x) \xrightarrow{\text{fidi}} (\Theta_n)_{n \in \mathbb{Z}}. \tag{1.5}
\]

as \( x \to \infty \). In the sequel, we assume that there exists a sequence \( (r_n) \), where \( r_n \to \infty \) and \( n/r_n \to \infty \), such that \( (X_n) \) and \( (r_n) \) satisfy the following two conditions. Denote first \( \mathcal{O} = \mathbb{R}^d \setminus \{0\} = [-\infty, \infty]^d \setminus \{0\} \).

Main assumptions

2
(A'): For every $f \in C^+_R([0,1] \times \Theta)$, denoting $k_n = \lfloor n/r_n \rfloor$, as $n \to \infty$,

$$
\mathbb{E}\left[ \exp\left\{ - \sum_{i=1}^{n} f\left( \frac{i}{n}, a_n^{-1} X_i \right) \right\} \right] - \prod_{k=1}^{k_n} \mathbb{E}\left[ \exp\left\{ - \sum_{i=1}^{r_n} f\left( \frac{kr_n}{n}, a_n^{-1} X_i \right) \right\} \right] \to 0. \tag{1.6}
$$

(AC): For every $u > 0$,

$$
\lim_{m \to \infty} \limsup_{n \to \infty} \mathbb{P}\left( \max_{m \leq |i| \leq r_n} \|X_i\| > a_n u \right) = 0. \tag{1.7}
$$

For thorough discussion of conditions (A') and (AC) we refer to [3] or Bartkiewicz et al. [1]. We observe here only that $m$-dependent multivariate sequences clearly satisfy both assumptions. More general strongly mixing sequences always satisfy condition A', which roughly speaking, allows one to break the sequence into increasing and asymptotically independent blocks. The condition AC on the other hand restricts the clustering of extremes in the sequence ($\|X_i\|$) over time. In addition to $m$-dependent sequences, it is satisfied for many other frequently used heavy tailed models, including for instance stochastic volatility or ARCH and GARCH processes, see [1] and [3].

The main object of interest in the sequel is the multivariate version of the point process in (1.1)

$$
N_n = \sum_{i=1}^{n} \delta_{(i/n, X_i/a_n)}
$$
on the state space $[0,1] \times \Theta$. Throughout, the space of point measures on a given space, $S$ say, is denoted by $M_p(S)$ and endowed by the vague topology, see Resnick [16].

Under very similar assumptions as above, Davis and Hsing in [6] analyzed the projection of the process $N_n$ on the spatial coordinate in the univariate case. They showed that the point processes $N^*_n = \sum_{i=1}^{n} \delta_{X_i/a_n}$, $n \in \mathbb{Z}$, on the space $(-\infty,0) \cup (0,\infty]$, converge in distribution to a point process which has a Poisson cluster structure which is only implicitly described. They further apply this result to determine the limiting distribution of the partial sums in the sequence ($X_n$). These results were extended to the multivariate setting by Davis and Mikosch [7]. Clearly, unlike $N_n$, the point processes $N^*_n$ contains no information about time clustering of extremes in the sequence ($X_n$).

Point processes $N_n$ were already studied by Basrak et al. in [3] in the univariate case, but the proof therein carries over to the multivariate case as observed in Basrak and Krizmanić [2] with some straightforward adjustments, to show

$$
N_n \left|_{[0,1] \times (\mathbb{R} \setminus [-u,u])^d} \Rightarrow N^{(u)} \right|_{[0,1] \times (\mathbb{R} \setminus [-u,u])^d}, \tag{1.8}
$$

as $n \to \infty$, for any threshold $u > 0$, with the limit $N^{(u)}$ unfortunately depending on that threshold. This asymptotic results can be used to deduce functional limit theorems for partial sums or partial maxima in the time series ($X_t$), see Theorem 3.4 in [3] or Proposition 4.1 below. Our goal here is to avoid restriction to various domains in (1.8), to find the correspondence of this result with
the earlier results of Davis and Hsing [6] and Davis and Mikosch [7]. Our main theorem below essentially unifies theorems on the limiting behavior of point processes given in [6, 7, 3]. Moreover, it reconciles their apparently very different statements in the framework suggested by Mori’s result [15].

In the following section we show an interesting preliminary result about the structure of extreme clusters in the process \((X_n)\). Then, in Section 3, we prove a general theorem about point process convergence for regularly varying time series. In Section 4, we apply this theorem to show the invariance principle for the so-called maximal process in the space \(D[0, 1]\). A corresponding theorem for iid sequences is well known and can be found in Resnick [16]. For nonnegative stationary regularly varying sequences, it was recently proved by Krizmanić [11]. Our version of this result includes other regularly varying sequences, and since it relies on our main theorem, the proof is straightforward and relatively simple. We also exhibit a simple technique to avoid problems at the left tail which are typically a source of frustration in this sort of limiting theorems.

2 Preliminaries

It was shown in [4] that under the main assumptions above, there exists

\[
\theta = \mathbb{P} \left( \sup_{i \leq -1} \|Y_i\| \leq 1 \right) = \lim_{r \to \infty} \lim_{x \to \infty} \mathbb{P} (M_r \leq x \|X_0\| > x > 0), \tag{2.1}
\]

where \(M_r = \max\{\|X_k\| : k = i, \ldots, r\}\). The number \(\theta > 0\) represents the extremal index of the random nonnegative sequence \((\|X_i\|)\).

Following [4], we define an auxiliary sequence \((Z_j)_{j \in \mathbb{Z}}\) as a sequence of random variables distributed as \((Y_j)_{j \in \mathbb{Z}}\) conditionally on the event \(\{\sup_{i \leq -1} \|Y_i\| \leq 1\}\). More precisely,

\[
\mathcal{L} \left( \sum_{j \in \mathbb{Z}} \delta Z_j \right) = \mathcal{L} \left( \sum_{i \in \mathbb{Z}} \delta Y_i \bigg| \sup_{i \leq -1} \|Y_i\| \leq 1 \right), \tag{2.2}
\]

where \(\mathcal{L}(N)\) denotes the distribution of the point process \(N\). It was shown in Theorem 4.3. in [4] that

\[
\mathcal{L} \left( \sum_{i=1}^{r_n} \delta_{(a_n u)^{-1} X_i} \bigg| M_{r_n} > a_n u \right) \Rightarrow \sum_{j \in \mathbb{Z}} \delta Z_j. \tag{2.3}
\]

Expressing the weak convergence of point processes using Laplace functionals, (2.3) is equivalent to

\[
\mathbb{E} \left( e^{-\sum_{i=1}^{r_n} f((a_n u)^{-1} X_i)} \bigg| M_{r_n} > a_n u \right) \to \mathbb{E} \left[ e^{-\sum_{j \in \mathbb{Z}} f(Y_j)} \bigg| \sup_{i \leq -1} \|Y_i\| \leq 1 \right] \tag{2.4}
\]

for all \(u \in (0, \infty)\) and \(f \in C^+_K(\mathbb{O})\).
Observe that by $\mathcal{AC}$, $\|Y_n\| \rightarrow 0$ with probability 1 as $|n| \rightarrow \infty$, see Proposition 4.2 in [4]. Therefore the same holds for $Z_n$'s in (2.2). In particular, the random variable

$$L_Z = \sup_{j \in \mathbb{Z}} \|Z_j\|$$

is a.s. finite, and clearly not smaller than 1. To determine the distribution of $L_Z$, observe that from Proposition 4.2 in [4] it follows that

$$k_n \mathbb{P}(M_{r_n} > a_n u) \rightarrow \theta u^{-\alpha}. \quad (2.5)$$

Now for $v \geq 1$, we have by (2.3)

$$\mathbb{P}(L_Z > v) = 1 - \mathbb{P}\left(\sum_j \delta_{\|Z_j\|((v, \infty))} = 0\right)$$

$$= 1 - \lim_{n \rightarrow \infty} \mathbb{P}\left(\sum_{i=1}^{r_n} \delta_{(a_n u - 1)\|X_i\|((v, \infty))} = 0 \mid M_{r_n} > a_n u\right)$$

$$= \lim_{n \rightarrow \infty} \mathbb{P}(M_{r_n} > a_n u \mid M_{r_n} > a_n u) = \lim_{n \rightarrow \infty} \frac{k_n \mathbb{P}(M_{r_n} > a_n u v)}{k_n \mathbb{P}(M_{r_n} > a_n u)}.$$

Therefore,

$$\mathbb{P}(L_Z > v) = v^{-\alpha}. \quad (2.6)$$

For $(Z_j)$ as in (2.2), we define a new sequence $(Q_j)_{j \in \mathbb{Z}}$ by

$$Q_j = Z_j / L_Z, \quad j \in \mathbb{Z}.$$

We will show the independence between the point process $\sum_j \delta_{Q_j}$ and the random variable $L_Z$, which might not be entirely surprising in the view of the independence between $\|Y_0\|$ and the spectral tail process in (1.5). However, this result seems to be new.

**Proposition 2.1** Assume that a regularly varying stationary sequence $(X_n)$ with corresponding sequence $(r_n)$ satisfies conditions $\mathcal{A}'$ and $\mathcal{AC}$. Then

$$\left(\sum_{i=1}^{r_n} \delta_{M_{r_n}^{-1}X_i, \frac{M_{r_n}}{a_n u} \mid M_{r_n} > a_n u}\right) \Rightarrow \left(\sum_j \delta_{Q_j, L_Z}\right). \quad (2.7)$$

Moreover, $L_Z$ and $\sum_j \delta_{Q_j}$ on the right hand side are independent.

**Proof.** For $m = \sum_i \delta_{x_i} \in M_p(\mathbb{D})$ denote by $x_m$ the largest norm of any point in $m$, i.e. $x_m = \sup_i \|x_i\|$ and define the mapping $\phi$ by

$$\phi: m \mapsto (m, x_m).$$

5
As observed in [6] such a mapping is continuous. Hence, applying $\phi$ to (2.3), we obtain

\[
\left( \sum_{i=1}^{r_n} \delta_{(a_nu)^{-1}X_i, \frac{M_{r_n}}{a_nu}} \bigg| M_{r_n} > a_nu \right) \Rightarrow \left( \sum_j \delta_{Z_j, L_Z} \right). \tag{2.8}
\]

Consider for $\nu \in M_p$ and $b \in (0, \infty)$, the mapping

\[
\psi : (\nu, b) \mapsto \nu b \in M_p,
\]

where $\nu_b(\cdot) = \nu(b^{-1} \cdot)$. Mapping $\psi$ is again continuous by Proposition 3.18 in Resnick [17] for instance. Hence, (2.8) implies (2.7) by the continuous mapping theorem.

To show the independence between $L_Z$ and $\sum_j \delta Q_j$, it suffices to show

\[
\mathbb{E} \left[ \exp \left( - \sum_j f_1(Q_j) \right) \mathbb{1}_{\{L_Z > v\}} \right] = \mathbb{E} \left[ \exp \left( - \sum_j f_1(Q_j) \right) \right] P(L_Z > v), \tag{2.9}
\]

for an arbitrary function $f_1 \in C^+_K(\mathbb{D})$ and $v \geq 1$. By (2.7), the left-hand side of (2.9) is the limit of

\[
\mathbb{E} \left[ \exp \left( - \sum_{i=1}^{r_n} f_1(X_i/M_{r_n}) \right) \mathbb{1}_{\{(au_n)^{-1}M_{r_n} > v\}} \bigg| M_{r_n} > a_n \right],
\]

which further equals

\[
\mathbb{E} \left[ \exp \left( - \sum_{i=1}^{r_n} f_1(X_i/M_{r_n}) \right) \bigg| M_{r_n} > a_n \right] \frac{\mathbb{P}(M_{r_n} > a_n v)}{\mathbb{P}(M_{r_n} > a_n)}.
\]

By (2.7) and the continuous mapping theorem, the first term above tends to $\mathbb{E} \left[ \exp \left( - \sum_j f_1(Q_j) \right) \right]$ as $n \to \infty$. By (2.5), the second term tends to $\mathbb{P}(L_Z > v) = v^{-\alpha}$, which implies (2.9).

\section{Main theorem}

**Theorem 3.1** Let $(X_n)_{n \in \mathbb{Z}}$ be a stationary series of jointly regularly varying random vectors with index $\alpha$, satisfying conditions $A'$ and $AC$ for a certain sequence $(r_n)$. Then

\[
N_n = \sum_{i=1}^{n} \delta_{(i/n, X_i/a_n)} \Rightarrow N = \sum_i \sum_j \delta_{(T_i, R_i, \eta_{ij})}, \tag{3.1}
\]

where

i) $\sum_i \delta_{(T_i, R_i)}$ is a Poisson process on $[0, 1] \times (0, \infty)$ with intensity measure $\text{Leb} \times \nu$ where $\nu(dy) = \theta \alpha y^{-\alpha - 1} dy$ for $y > 0$. 

6
\[ \left( \sum_j \delta_n_{ij} \right)_i \text{ is an i.i.d. sequence of point processes in } \mathbb{O} \text{ independent of } \sum_i \delta(T_i, P_i) \text{ and with common distribution equal to the distribution of } \sum_j \delta_{Q_j}. \]

**Proof.** To show (4.2), it is sufficient to show that

\[ \mathbb{E} e^{-N_n(f)} \to \mathbb{E} e^{-N(f)}, \]

for an arbitrary \( f \in C^+_K([0, 1] \times \mathbb{O}). \) Observe that for any such function \( f \) there is a constant \( u > 0 \), such that the support of \( f \) is a subset of \([0, 1] \times \mathbb{O}_u\). By (1.8), i.e. by the multivariate expansion of Theorem 2.3 in [3], we know that

\[ \mathbb{E} e^{-N_n(f)} \to \exp \left[ - \int_0^1 \left( 1 - \mathbb{E} e^{-\sum_j f(t, v Z_j)} \right) \theta u^{-\alpha} dt \right] \quad (3.2) \]

Consequently, it suffices to show that the right hand side above corresponds to \( \mathbb{E} e^{-N(f)} \) for any such function \( f \) and corresponding \( u > 0 \).

As in Theorem 2.3 in [3], one can write

\[ \mathbb{E} e^{-N(f)} = \mathbb{E} \exp \left\{ - \sum_i \sum_j f(T_i, P_i \eta_{ij}) \right\} \]

\[ = \mathbb{E} \left[ \prod_i \mathbb{E} \left\{ \exp \left\{ - \sum_j f(T_i, P_i \eta_{ij}) \right\} \right\} \left| (T_k, P_k)_k \right) \right\} \]

\[ = \mathbb{E} \left[ \prod_i \mathbb{E} \left[ \exp \left\{ - \sum_j f(T_i, P_i \eta_{ij}) \right\} \right| (T_k, P_k) k \right), \]

where the last equation follows from Lemma 3.10 in [16], since \( \sum_i \delta(T_i, P_i) \) and \( \left( \sum_j \delta_{n_{ij}} \right)_i \) on the right-hand side of (4.2) are independent. Define \( h(t, v) = \mathbb{E} \exp \left\{ - \sum_j f(t, v Q_j) \right\}. \) We have

\[ \mathbb{E} e^{-N(f)} = \mathbb{E} \exp \left( \sum_i \ln h(T_i, P_i) \right). \]

The right-hand side is the Laplace functional of \( \sum_i \delta(T_i, P_i) \) evaluated at \(- \ln h\). Since \( \sum_i \delta(T_i, P_i) \sim \text{PRM}(\text{Leb} \times \nu) \) on \([0, 1] \times (0, \infty)\), the Laplace functional of this process has a very special form, see Resnick [16].

\[ \mathbb{E} e^{-N(f)} = \exp \left[ - \int_{[0, 1] \times (0, \infty)} \left( 1 - h(t, v) \right) (\text{Leb} \times \nu)(dt, dv) \right] \]

\[ = \exp \left[ - \int_0^1 \int_0^\infty \left( 1 - \mathbb{E} e^{-\sum_j f(t, v Q_j)} \right) \theta \alpha v^{-\alpha - 1} dv dt \right]. \quad (3.3) \]
Consider now the right hand side in (3.2). Using \( Z_j = L Z Q_j \), the distribution of \( L Z \) calculated in (2.6) and independence shown in Proposition 2.1, we have

\[
\exp \left[ - \int_0^1 \left( 1 - E e^{-\sum_j f(t,u Z_j)} \right) \theta u^{-\alpha} dt \right] = \exp \left[ - \int_0^1 \int_1^\infty \left( 1 - E e^{-\sum_j f(t,u Q_j)} \right) \alpha u^{-\alpha-1} du \theta u^{-\alpha} dt \right] = \exp \left[ - \int_0^1 \int_0^\infty \left( 1 - E e^{-\sum_j f(t,v Q_j)} \right) \alpha v^{-\alpha-1} dv \theta dt \right] = \exp \left[ - \int_0^1 \int_0^\infty \left( 1 - E e^{-\sum_j f(t,v Q_j)} \right) \theta v^{-\alpha} dv dt \right] = E e^{-N(f)},
\]

where the second equality follows using the change of variable \( v = ul \), and the last equality follows from the fact that \( \sup_j \|Q_j\| = 1 \) and \( f(t,x) = 0 \) for \( x < u \).

As we discussed in the introduction, one consequence of the theorem above is the functional limit theorem for partial sums of the univariate sequence \((X_t)\), see Theorem 3.4 in [3]. In the multivariate case, under the conditions of Theorem 3.1, it was shown by Davis and Mikosch [7] that partial sums \( S_n = X_1 + \cdots + X_n, \ n \geq 1 \), satisfy

\[
\frac{S_n}{a_n} \xrightarrow{d} \xi_{\alpha}, \quad (3.4)
\]

for \( \alpha \in (0, 1) \) and an \( \alpha \)-stable random vector \( \xi_{\alpha} \), actually the same holds for \( \alpha \in [1, 2) \) if one assumes for instance that \( X_i \)'s have a symmetric distribution and that for any \( \delta > 0 \) the following standard technical assumption holds \( \lim_{x \to 0} \limsup_{n \to \infty} P(\|a_n^{-1} \sum_{i=1}^n X_i\| \leq \epsilon a_n) > \delta = 0 \), see [7] again.

As observed by Mikosch and Wintenberger [13], under condition that

\[
E \left( \sum_{j \geq 1} \|Q_j\| \right)^\alpha < \infty,
\]

which indeed always holds in the case \( \alpha < 1 \) or if \( (X_n) \) is \( m \)-dependent, the spectral measure \( \Gamma_{\alpha} \) of the stable random vector \( \xi_{\alpha} \) in (3.4) can be characterized as follows

\[
\int_{\mathbb{S}^{d-1}} \langle t, s \rangle^\alpha \Gamma_{\alpha}(ds) = \theta \frac{\alpha}{2 - \alpha} E \left[ \left( \sum_{j \geq 1} \langle t, Q_j \rangle \right)^\alpha \right]_+ = \theta \frac{\alpha}{2 - \alpha} E \left[ \left( \sum_{j \geq 1} \langle t, Y_j / \sup_j \|Y_j\| \rangle \right)^\alpha \right]_+ \left( \sup_{i \leq -1} \|Y_i\| \leq 1 \right),
\]

where \( t \in \mathbb{S}^{d-1} \), \( \langle t, s \rangle \) denotes the inner product on \( \mathbb{S}^{d-1} \) and \( \| \cdot \| \) denotes the Euclidean norm. For a definition of the spectral measure of a stable random vector, see [19, Section 2.3]. This can be used in certain cases to give an alternative representation of the so-called cluster index.
$b(t) \in S^{d-1}$ studied by Mikosch and Wintenberger in [13, 14]. Indeed, for a regularly varying $(X_n)$ which is Markov chain under conditions of [13, Theorem 4.1]

$$b(t) = \frac{1 - \alpha}{\Gamma(2 - \alpha) \cos(\pi \alpha/2)} \int_{S^{d-1}} \langle t, s \rangle \alpha \Gamma(\alpha) (ds).$$

As observed in [13, 14], the cluster index $b(t)$ plays the key role in the description of large deviations for partial sums of regularly varying sequences.

A more direct consequence of Theorem 3.1 is the functional limit theorem for the partial maxima of a univariate sequence considered in the following section.

### 4 Invariance principle for maximal process

Denote by $M'_n = \max\{X_1, \ldots, X_n\}, n \geq 1$. Following Resnick [16] or Embrechts et al. [9] we consider the following continuous time càdlàg process

$$Y_n(t) = \begin{cases} \frac{M'_{\lfloor nt \rfloor}}{a_n} & t \geq 1/n, \\ \frac{X_1}{a_n} & t < 1/n, \end{cases}$$

indexed over the segment $[0, 1]$. It is actually customary to exclude $t = 0$ to avoid technical problems, see [16]. In our approach, such issues are easily avoided, and therefore we include $t = 0$. From the practical perspective, it seems sufficient to consider $t \leq 1$. Extending the main result below to $t \in [0, \infty)$ remains a technical, but relatively straightforward task, see Chapter 3 in Billingsley [5].

Recall that the extremal process generated by an extreme value distribution function $G$ ($G$-extremal process, for short) is a continuous time stochastic process with finite dimensional distributions $G_{s_1, \ldots, s_k}$ satisfying

$$G_{s_1, \ldots, s_k}(x_1, \ldots, x_k) = G^{s_1}(\wedge_{i=1}^k x_i) G^{s_2 - s_1}(\wedge_{i=2}^k x_i) \cdots G^{s_k - s_{k-1}}(x_k),$$

for all choices of $k \geq 1$, $0 < s_1 < \cdots < s_k, x_i \in \mathbb{R}, i = 1, \ldots, k$, see Resnick [16]. The processes $Y_n$ introduced above are clearly random elements in the space of real valued càdlàg functions $D[0, 1]$. We will show that they converge weakly to a particular extremal process with respect to Skorohod’s $M_1$ topology on $D[0, 1]$. We refer to Whitt [20] for the definition and discussion of various topologies in that space, see also [3].

In the proposition below we make a small technical assumption about the right tail of the marginal distribution of $X_t$’s, i.e. we suppose that it satisfies $\liminf_{n \to \infty} n \mathbb{P}(X_0 > a_n) > 0$. This implies that

$$n \mathbb{P}(X_0 > a_n x) \to px^{-\alpha} \quad \text{for all } x > 0,
\quad (4.1)$$

and some constant $p \in (0, 1]$. 

9
Proposition 4.1 Let \((X_n)_{n \in \mathbb{Z}}\) be a stationary sequence of jointly regularly varying random variables with index \(\alpha\), satisfying conditions \(\mathcal{A}'\) and \(\mathcal{AC}\). Assume that \(\liminf_{n \to \infty} n \mathbb{P}(X_0 > a_n) > 0\), then

\[
Y_n \Rightarrow \xi, \quad (4.2)
\]

where \(\xi(t), t > 0\) is a \(G\)-extremal process for the nonstandard Fréchet distribution function \(G(x) = e^{-\kappa x^{-\alpha}}, x \geq 0\), for some \(\kappa > 0\), and the convergence takes place in \(D[0,1]\) endowed with the Skorohod's \(M_1\) topology.

**Proof.** Consider the functional \(T^+: M_p([0,1] \times \Omega) \to D[0,1]\) on the space of Radon point measures given by

\[
T^+(m)(t) = \sup_{t_i \leq t} 0, \quad (4.3)
\]

where \(m = \sum_i \delta_{(t_i,j_i)}\) denotes an arbitrary Radon point measure on the space \([0,1] \times \mathbb{E}\), where we set for convenience \(\sup \emptyset = 0\).

Denote \(M'_p = \{m \in M_p : m([0,s] \times (0,\infty)) > 0 \text{ for all } s > 0\}\) and \(M''_p = \{m \in M_p : m(\{0,1\} \times (0,\infty]) = 0\}\). We will show that \(T^+\) is continuous on the set \(M'_p \cap M''_p\). Assume, \(m_n \Rightarrow m \in M'_p \cap M''_p\).

Because, \(m\) is a Radon point measure, the set of times \(t\) for which \(m(\{t\} \times \Omega) = 0\) is dense in \([0,1]\). For all such \(t\)'s

\[
T^+ m_n(t) \to T^+ m(t),
\]

as \(n \to \infty\) by Proposition 3.13 in [16]. Observe that \(T^+ m\) is a nondecreasing function for any \(m\), therefore an application of Corollary 12.5.1 in Whitt [20] yields the convergence of \(T^+ m_n \to T^+ m\) in \(D[0,1]\) endowed with \(M_1\) topology.

Observe now that the limiting point process \(N\) in Theorem 3.1 lies in \(M'_p \cap M''_p\) a.s. Hence for \(N_n\) in the same theorem, an application of the continuous mapping theorem yields the weak convergence in

\[
T^+ N_n \Rightarrow T^+ N,
\]

as \(n \to \infty\).

By Slutsky argument, to show that \(Y_n \Rightarrow T^+ N\), it is sufficient to prove that

\[
d_{M_1}(Y_n, T^+ N_n) \overset{P}{\to} 0,
\]

where \(d_{M_1}\) denotes the \(M_1\) metric on the space \(D[0,1]\), see Whitt [20]. This follows from the fact that metric \(d_{M_1}\) is weaker than the uniform metric \(d_{\infty}\), and the following obvious limit

\[
d_{\infty}(Y_n, T^+ N_n) = \frac{|X_1|}{a_n} \mathbb{P}_{\{X_1 < 0\}} \overset{a.s.}{\to} 0,
\]

as \(n \to \infty\).

Observe that, by definition, \(N([0,t] \times (0,\infty]) = 0\) implies \(T^+ N(t) = 0\) for a fixed \(t \geq 0\). Therefore

\[
T^+ N(t) = \sup_{T_i \leq t} P_i \sup_{T_i \leq t} \eta_{ij} = \sup_{T_i \leq t} P_i U_i, \quad (4.4)
\]
denoting $U_i = \sup_j \eta_{ij} \lor 0$. Note further that $U_i$'s form an iid sequence of random variables on the interval $[0,1]$. By (4.1), $\mathbb{P}(U_i \in (0,1)) > 0$. Moreover, sequence $(U_i)$ is independent of the PRM $\sum_i \delta_{T_i,P_i}$. It is straightforward to check that $\sum_i \delta_{T_i,P_i,U_i}$ is PRM with mean measure $\text{Leb} \times \nu'$ where $\nu'(dy) = \theta \mathbb{E}U^\alpha y^{\alpha-1}dy$ for $y > 0$, cf. propositions 3.7 and 3.8 in Resnick [16]. It follows, by proposition 5.4.4 in Embrechts et al. [9] for instance, that $T^+ N(t)$, $t > 0$ in (4.4) is a $G$-extremal process for

$$G(u) = \mathbb{P}(T^+ N(1) \leq u) = \mathbb{P} \left( \sum_i \delta_{P_i,U_i} (u,\infty] = 0 \right) = \exp \left( -\theta \mathbb{E} U^\alpha u^{-\alpha} \right) ,$$

for any $u > 0$.

By the proof, the constant $\kappa$ in Proposition 4.1 equals

$$\kappa = \theta \mathbb{E} U^\alpha ,$$

where $U \overset{d}{=} \sup_j \eta_{ij} \lor 0$. In the iid case, the extremal index $\theta = 1$, while $\sum_j \delta_{\eta_{ij}} \overset{d}{=} \delta_Q$ where $\mathbb{P}(Q = 1) = 1 - \mathbb{P}(Q = -1) = p$. Therefore, $\kappa = p$ as known from Proposition 4.20 in Resnick [16]. In the case of nonnegative random variables $p = 1$ and $\sup \eta_{ij} = 1$ a.s. Therefore $\kappa = \theta$ in this case, cf. Krizmanić [11].

Remark 1 Observe that more commonly used $J_1$ topology is not applicable in our setting, because of the clustering of extremes. For an illustration of the problem, consider the point measures $m_n = \delta_{1/2-1/n,1/2} + \delta_{1/2,1} \overset{v}{\rightarrow} m = \delta_{1/2,1/2} + \delta_{1/2,1}$ for $n \to \infty$, and note that for $T^+ m_n$ does not converge to $T^+ m$ in $J_1$ topology.

In the case $\theta = 1$, the limiting point process $N$ is a Poisson random measure, and the convergence in the theorem above holds in the standard $J_1$ topology. The proof only has to be adapted to show that $T^+$ is an a.s. continuous functional with respect to the distribution of such a process. Such a result was already stated in Remark 2 of Mori [15].

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References

[1] Bartkiewicz, K., Jakubowski, A., Mikosch, T., Winterberger, O.: Stable limits for sums of dependent infinite variance random variables. Probab. Theory Relat. Fields 3-4, 337–372 (2011)
[2] Basrak, B., Krizmani´c, D.: A multivariate functional limit theorem in weak m1 topology. J. Theoret. Probab. 28, 119–136 (2015)

[3] Basrak, B., Krizmani´c, D., Segers, J.: A functional limit theorem for dependent sequences with infinite variance stable limits. Ann. Probab. 40, 2008–2033 (2012)

[4] Basrak, B., Segers, J.: Regularly varying multivariate time series. Stoch. Process. Appl. 119, 1055–1080 (2009)

[5] Billingsley, P.: Convergence of Probability Measures, 2nd edition. Wiley, New York (1999)

[6] Davis, R.A., Hsing, T.: Point processes and partial sum convergence for weakly dependent random variables with infinite variance. Ann. Probab. 23, 879–917 (1995)

[7] Davis, R.A., Mikosch, T.: The sample autocorrelation function of heavy-tailed processes with application to ARCH. Ann. Statist. 26, 2049–2080 (1998)

[8] Davis, R.A., Resnick, S.I.: Limit theory for moving averages of random variables with regularly varying tail probabilities. Ann. Probab. 13, 179–195 (1985)

[9] Embrechts, P., Kluppelberg, C., Mikosch, T.: Modelling Extremal Events for Insurance and Finance. Springer-Verlag, Berlin (1997)

[10] Hsing, T., Leadbetter, M.R.: On the excursion random measure of stationary processes. Ann. Probab. 26, 710–742 (1998)

[11] Krizmani´c, D.: Weak convergence of partial maxima processes in the m1 topology. Extremes 17, 447–465 (2014)

[12] Leadbetter, M.R., Rootzén, H.: Extremal theory for stochastic processes. Ann. Probab. 16, 431–478 (1988)

[13] Mikosch, T., Wintenberger, O.: The cluster index of regularly varying sequences with applications to limit theory for functions of multivariate markov chains. Probab. Th. Rel. Fields 159, 157–196 (2014)

[14] Mikosch, T., Wintenberger, O.: A large deviations approach to limit theory for heavy-tailed time series. Probab. Th. Rel. Fields pp. 1–37 (2015)

[15] Mori, T.: Limit distributions of two-dimensional point processes generated by strong-mixing sequences. Yokohama Math. J. 25, 155–168 (1977)

[16] Resnick, S.I.: Extreme Values, Regular Variation, and Point Processes. Springer, New York (1987)

[17] Resnick, S.I.: Heavy-Tail Phenomena: Probabilistic and Statistical modelling. Springer, New York (2007)
[18] Resnick, S.I.: Extreme Values, Regular Variation, and Point Processes. Springer, New York (2008)

[19] Samorodnitsky, G., Taqqu, M.: Stable Non-Gaussian Random Processes. CRC Press (1994)

[20] Whitt, W.: Stochastic-Process Limits. Springer-Verlag, New York (2002)