REduced BASIS METHODS BASED UPON ADAPTIVE SNAPSHOT COMPUTATIONS

KRISTINA STEIH AND KARSTEN URBAN

ABSTRACT. We use asymptotically optimal adaptive numerical solvers (here specifically a wavelet scheme) within the offline phase of the Reduced Basis Method (RBM). Those different discretizations for each parameter require also an adaptive evaluation of residual-based error estimators. The corresponding scheme is analyzed. It allows an error estimate of the RB approximation with respect to the exact solution of the parameterized partial differential equation to be treated by the RBM. We show that multiple selection of snapshots may occur and devise strategies to avoid this. Numerical experiments for stationary and in stationary problems show potential and challenges of this approach.

1. Introduction

Reduced Basis Methods (RBMs) have nowadays become a widely accepted and used tool for realtime and/or multi-query simulations of parameterized partial differential equations (PPDEs). By using an offline-online decomposition, the main idea is to use a high fidelity, detailed, but costly numerical solver offline to compute approximations to the PPDEs for certain parameter values. The selection of these ‘certain’ parameters is done by an error estimator which is efficiently computable and thus allows one to determine the ‘worst’ parameters out of a possible rich so-called training set. For those ‘bad’ parameters, the high fidelity model is used in order to determine approximations, so called snapshots. These few snapshots form the reduced basis which is then capable to produce approximations for any new parameter value extremely rapid (online). The error estimator can also be used online in order to certify this RB approximation. Both the variety of applications and the amount of recent results in RBMs go well beyond the scope of such an introduction.

The success of the RBM relies on the assumption that the high fidelity model in the offline phase is sufficiently accurate for all parameters. The same discretization is used for all snapshots and all error estimators. This may have some possible drawbacks: (1) If this high fidelity model is not accurate enough, also the RB-approximation cannot be good. (2) The other extreme is that a sufficiently accurate approximation for all possible parameters may require a high fidelity model whose dimension is too large even for an offline phase. (3) The error estimate usually
controls the difference to the high fidelity solution, not w.r.t. the exact solution of
the PPDE (with one recent exception in [27] to be discussed below).

On the other hand, there are adaptive numerical methods available, that guarantee an approximation of the exact solution of a PDE within a preselected tolerance. Such methods can be based upon finite element or wavelet discretizations, [7, 8, 20, 25]. We use such an adaptive method (we choose wavelets) for computing both snapshots and error estimators in the offline phase. This offers some features that we think are of interest, namely: (a) We use different discretizations for each parameter allowing for a minimal amount of work for any chosen parameter. (b) We can bound the RB error w.r.t. the exact solution of the PPDE.

Using adaptivity (or different discretizations) in the offline phase implies some additional sophistication of the method, at least from the conceptual point of view. The question arises under which circumstances such adaptivity might pay off. It is known e.g. from [7] that adaptive methods show faster convergence rates if the Besov regularity of the solution in a certain scale exceeds the Sobolev regularity. For the offline-RB-setting this means that the regularity of the solution with respect to the parameter is of crucial importance. If one single discretization is sufficient for equally approximating the solution \( u(\mu) \) for all possible parameters \( \mu \), then adaptivity seems not to make sense. On the other hand, if \( u(\mu) \) significantly differs depending on \( \mu \), a joint discretization may be too fine. This is e.g. the case if \( u(\mu) \) has local effects in different regions depending on the choice of \( \mu \). Our numerical examples are guided by these considerations.

The remainder of this paper is organized as follows. In Section 2, we review the main facts of the ‘classical’ Reduced Basis Method. We set the framework for PPDEs and collect those facts that are needed here. Section 3 is devoted to the use of adaptive methods for the generation of the reduced basis in the offline phase. At this point, we only require the availability of a certain adaptive solver \textsc{Solve} and do not specify which specific method is used. We have used an Adap-
tive Wavelet Galerkin Method (AWGM). Since it is not absolutely necessary to describe the precise adaptive method within the RB-framework in Section 3, we have collected a brief description of the AWGM in Appendix A. Finally, in Section 4 we describe numerical experiments for two different examples, namely heat conduction in a thermal block with several local heat sources and time-dependent convection-diffusion-reaction using a space-time variational formulation.

2. Reduced Basis Methods (RBMs)

In order to highlight differences and challenges of using adaptively computed basis functions within the Reduced Basis Method (RBM), it makes sense to briefly review ‘standard’ RBMs.

2.1. Parameterized Partial Differential Equations (PPDEs). Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain on which we consider function spaces \( \mathcal{X} = \mathcal{X}^0(\Omega), \mathcal{Y} = \mathcal{Y}^0(\Omega) \) arising from a variational formulation of a partial differential equation. Denoting by \( \mathcal{D} \subset \mathbb{R}^P \) the set of parameters, this means that we consider a differential operator \( B : \mathcal{X} \times \mathcal{D} \rightarrow \mathcal{Y} \) resp. a bounded bilinear form \( b : \mathcal{X} \times \mathcal{Y} \times \mathcal{D} \rightarrow \mathbb{R} \), where \( b(w,v;\mu) := \langle B[w;\mu],v \rangle_{\mathcal{Y}^\prime} \) for \( w \in \mathcal{X}, v \in \mathcal{Y} \) and \( \mu \in \mathcal{D} \). In particular, we assume the existence of constants \( \gamma(\mu) \leq \gamma^\text{UB} < \infty \) such that

\[
2.1 \quad b(w,v;\mu) \leq \gamma(\mu) \|w\|_{\mathcal{X}} \|v\|_{\mathcal{Y}}, \quad w \in \mathcal{X}, v \in \mathcal{Y}.
\]
For a given \( g(\mu) \in \mathcal{Y} \), the problem is then to find a \( u = u(\mu) \in \mathcal{X} \) such that \( B[u(\mu); \mu] = g(\mu) \) in \( \mathcal{Y} \), or, in variational form

\[
(2.2) \quad b(u(\mu), v; \mu) = f(v; \mu) \quad \forall v \in \mathcal{Y},
\]

where \( f(v; \mu) := \langle g(\mu), v \rangle_{\mathcal{Y}' \times \mathcal{Y}} \). It is required that a numerical solver for (2.2) is available, e.g. finite volume, finite element or wavelet methods.

We assume that (2.2) is well-posed for all \( \mu \in \mathcal{D} \), which is equivalent to the so-called \textit{Nečas condition} on \( b(\cdot, \cdot; \mu) \), \([19] [20]\), i.e. there exist \textit{inf-sup constants} \( \beta(\mu) \) and a lower bound \( \beta_{\text{LB}} \) such that

\[
(2.3) \quad \beta(\mu) = \inf_{w \in \mathcal{X}} \sup_{v \in \mathcal{Y}} \frac{b(w, v; \mu)}{\|w\|_{\mathcal{X}} \|v\|_{\mathcal{Y}}} = \inf_{w \in \mathcal{Y}} \sup_{v \in \mathcal{X}} \frac{b(w, v; \mu)}{\|w\|_{\mathcal{X}} \|v\|_{\mathcal{Y}}} \geq \beta_{\text{LB}} > 0
\]

for all \( \mu \in \mathcal{D} \).

\textbf{Remark 2.1.} (a) It is worth mentioning that (2.2) includes elliptic problems, where e.g. \( \mathcal{X} = \mathcal{Y} = H_0^1(\Omega) \) (or other boundary conditions), \( b(\cdot, \cdot; \mu) \) being coercive with constant \( \alpha(\mu) > 0 \), as well as parabolic initial value problems in space-time formulation, i.e. with the Bochner spaces \( \mathcal{X} = W_0(0, T; V) := \{ u \in L_2(0, T; V) : u_t \in L_2(0, T; V') \} \), \( \mathcal{Y} = L_2(0, T; V) \), i.e. \( \mathcal{X} \subset \mathcal{Y} \) and also time-periodic problems, see also Section 3 below.

(b) Instead of a space-time formulation for a parabolic initial value problem, one could also use a standard time-stepping scheme. There are corresponding RBMs available for such problems \([12] [13]\). In principle, our subsequent findings can be extended also to those settings, but in order to keep notations simple, we restrict ourselves to (2.2).

\subsection{2.2. Some Basics on ‘Classical’ RBMs.} Any numerical scheme for the solution of (2.2) involves a discretization of \( \mathcal{X}, \mathcal{Y} \). In a standard RB-setting these finite-dimensional discrete spaces, the so-called \textit{truth spaces}, are denoted by \( \mathcal{X}^N \subset \mathcal{X}, \mathcal{Y}^N \subset \mathcal{Y} \). Then, the Galerkin projection is considered

\[
(2.4) \quad \text{Find } u^N(\mu) \in \mathcal{X}^N : \quad b(u^N(\mu), v; \mu) = f(v; \mu) \quad \forall v \in \mathcal{Y}^N.
\]

Often, \( \mathcal{X}^N, \mathcal{Y}^N \) are spanned by \textit{local basis functions} such as finite elements or wavelets and their dimension \( N = \dim(\mathcal{X}^N) = \dim(\mathcal{Y}^N) \) is usually large, so that solving (2.4) repeatedly for many different parameters would be too costly or realtime computations would be impossible.

\textbf{Remark 2.2 (Fixed discretization).} We stress that in the standard RB setting, the spaces \( \mathcal{X}^N, \mathcal{Y}^N \) are a-priori fixed and are the same for all parameters \( \mu \in \mathcal{D} \). Moreover, it is assumed that the discretization error \( \|u(\mu) - u^N(\mu)\|_{\mathcal{X}} \) is negligibly small for all \( \mu \in \mathcal{D} \). Thus, typical RBMs view \( u^N(\mu) \) as ‘truth’, which means e.g. that all error estimates are typically w.r.t. \( u^N(\mu) \) and do not take \( u(\mu) \) into account. Just recently a first paper appeared introducing error bounds w.r.t. \( u(\mu) \) in a specific case, using techniques, however, that do not seem to be applicable in a general framework, \([27]\).  

\footnote{We always use calligraphic symbols for high-(even \( \infty \))-dimensional spaces.
\footnote{For simplicity, we assume that trial and test spaces are of the same dimension. Otherwise, one would need to use a least squares approach.}
The idea behind (standard) RBMs is the construction of low-dimensional spaces $X_N^\mathcal{Y} \subset \mathcal{X}^N$, $Y_N^\mathcal{Y} \subset \mathcal{Y}^N$, $N \ll N^\mathcal{Y}$ from so-called snapshots, i.e., solutions of \eqref{2.4} for selected parameters, i.e.
\begin{equation}
X_N^\mathcal{Y} := \text{span}\{u_N^\mathcal{Y}(\mu), i = 1, \ldots, N\} =: \text{span}\{\xi_i^\mathcal{Y}, i = 1, \ldots, N\},
\end{equation}
and $Y_N^\mathcal{Y} := \text{span}\{\xi_i^\mathcal{Y}, i = 1, \ldots, N\}$ is such that the $N$-dimensional reduced problem
\begin{equation}
\text{Find } u_N^\mathcal{Y}(\mu) \in X_N^\mathcal{Y}: \quad b(u_N^\mathcal{Y}(\mu), v; \mu) = f(v; \mu) \quad \forall v \in Y_N^\mathcal{Y}
\end{equation}
is stable. Stability in particular amounts a Ladyshenskaja-Babuška-Brezzi (LBB) condition, \cite{Brezzi, Babuska, Ciarlet}, i.e.,
\begin{equation}
\inf_{u_N \in X_N^\mathcal{Y}} \sup_{v \in Y_N^\mathcal{Y}} \frac{b(w_N, v; \mu)}{\|w_N\|_{\mathcal{X}} \|v\|_{\mathcal{Y}}} \geq \bar{\beta} > 0,
\end{equation}
with $\bar{\beta}$ independent of $N$ as $N \to \infty$. We abbreviate
\begin{equation}
S_N := \{\mu_1, \ldots, \mu_N\}
\end{equation}
as the set of parameter values corresponding to the snapshots. The system $\{\xi_i^\mathcal{Y}, i = 1, \ldots, N\}$ may arise by orthonormalization of the snapshots.

The inf-sup condition \eqref{2.3} gives rise to rigorous a posteriori error bounds, i.e. $N$-independent quantities $\Delta_N^\mathcal{Y}(\mu)$ with
\begin{equation}
\|e_N^\mathcal{Y}(\mu)\|_{\mathcal{X}} := \|u_N^\mathcal{Y}(\mu) - u_N^\mathcal{Y}(\mu)\|_{\mathcal{X}} \leq \Delta_N^\mathcal{Y}(\mu) = \frac{\|r_N^\mathcal{Y}(\cdot; \mu)\|_{\mathcal{Y}}}{\beta(\mu)},
\end{equation}
where $r_N^\mathcal{Y}(\cdot; \mu): \mathcal{Y}^N \to \mathbb{R}$ is the residual with respect to the reduced solution, i.e.
\begin{equation}
r_N^\mathcal{Y}(v; \mu) := f(v; \mu) - b(u_N^\mathcal{Y}(\mu), v; \mu), \quad \forall v \in \mathcal{Y}^N.
\end{equation}
The involved dual norms $\|r_N^\mathcal{Y}(\cdot; \mu)\|_{\mathcal{Y}'}$ are computed with the help of the Riesz representations $r_N^\mathcal{Y}(\mu) \in \mathcal{Y}^N$ solving
\begin{equation}
(r_N^\mathcal{Y}(\mu), v)_{\mathcal{Y}'} = r_N^\mathcal{Y}(v; \mu) \quad \forall v \in \mathcal{Y}^N.
\end{equation}
The efficient $N$-independent computation of $\|r_N^\mathcal{Y}(\cdot; \mu)\|_{\mathcal{Y}'} = \|\hat{r}_N^\mathcal{Y}(\mu)\|_{\mathcal{Y}}$ relies on an offline-online decomposition, see \cite{Brezzi, Babuska, Ciarlet} below. It is easily seen that this error bound is also reliable. In fact, we have for any $v \in \mathcal{Y}^N$ that
\begin{align*}
(r_N^\mathcal{Y}(\mu), v)_{\mathcal{Y}'} &= r_N^\mathcal{Y}(v; \mu) = f(v; \mu) - b(u_N^\mathcal{Y}(\mu), v; \mu) = b(u_N^\mathcal{Y}(\mu) - u_N^\mathcal{Y}(\mu), v; \mu) \\
&= b(e_N^\mathcal{Y}(\mu), v; \mu).
\end{align*}
Inserting $v = \hat{r}_N^\mathcal{Y}(\mu)$ and using the boundedness \eqref{2.1} yields
\begin{equation}
\|\hat{r}_N^\mathcal{Y}(\mu)\|_{\mathcal{Y}}^2 = b(e_N^\mathcal{Y}(\mu), \hat{r}_N^\mathcal{Y}(\mu); \mu) \leq \gamma(\mu)\|e_N^\mathcal{Y}(\mu)\|_{\mathcal{X}} \|\hat{r}_N^\mathcal{Y}(\mu)\|_{\mathcal{Y}},
\end{equation}
hence $\|\hat{r}_N^\mathcal{Y}(\mu)\|_{\mathcal{Y}} \leq \gamma(\mu)\|e_N^\mathcal{Y}(\mu)\|_{\mathcal{X}}$, so that the error estimator $\Delta_N^\mathcal{Y}(\mu) = \frac{\|\hat{r}_N^\mathcal{Y}(\mu)\|_{\mathcal{Y}}}{\beta(\mu)}$ and the error $\|e_N^\mathcal{Y}(\mu)\|_{\mathcal{X}}$ are in fact equivalent:
\begin{equation}
\|e_N^\mathcal{Y}(\mu)\|_{\mathcal{X}} \leq \Delta_N^\mathcal{Y}(\mu) \leq \frac{\gamma(\mu)}{\beta(\mu)}\|e_N^\mathcal{Y}(\mu)\|_{\mathcal{X}}.
\end{equation}
\footnote{Low-dimensional spaces are denoted by usual (non calligraphic) symbols.}
Remark 2.3. We point out (for later reference in §3.4 below) that \( u^N_N(\mu) = u^N(\mu) \) for all \( \mu \in S_N \), i.e., snapshots are reproduced by the standard RBM. In fact, we have Petrov-Galerkin orthogonality, i.e., \( b(u^N_N(\mu) - u^N_N(\mu), v_N; \mu) = 0 \) for all \( v_N \in Y^N_N \).

Since \( u^N_N(\mu) \in X^N_N \) for \( \mu \in S_N \), we have that \( e^N_N(\mu) = u^N_N(\mu) - u^N_N(\mu) \in X^N_N \) and then LBB in (2.6) yields

\[
\beta \|e^N_N(\mu)\|_X \leq \sup_{v_N \in Y^N_N} \frac{b(e^N_N(\mu), v_N; \mu)}{\|v_N\|_Y} = 0,
\]

i.e., \( u^N_N(\mu) = u^N_N(\mu) \) for all \( \mu \in S_N \). \( \square \)

2.3. Basis Construction via the Greedy Algorithm. The choice of the RB basis functions \( \zeta^N_i, i = 1, \ldots, N \), i.e. the selection of the corresponding parameter values \( \mu_1, \ldots, \mu_N \), is often done using a Greedy algorithm. The key point for the efficiency of this approach is the fact that the greedy selection is done w.r.t. the error estimator (not on the truth). Given \( \mu_1, \ldots, \mu_n, n < N \), the next parameter value \( \mu_{n+1} \) is chosen as \( \mu_{n+1} = \arg \max_{\mu \in \Xi_{\text{train}}} \Delta^N_N(\mu) \),

where \( \Xi_{\text{train}} \subset \mathcal{D} \) is a finite training set. This is also the reason why this method is also sometimes called Greedy training. Only for the chosen parameter values \( \mu_1, \ldots, \mu_N \), the (expensive) truth has to be computed. The corresponding algorithm is displayed in Algorithm 1. This shows the importance of an efficient computation of the snapshots and the error estimator \( \Delta^N_N \). Note that Algorithm 1 is a so-called weak Greedy since in line 2 we use the error estimator. Would we use the true error \( \|e^N_N(\mu)\|_X \) instead, we call it strong Greedy. If \( \|e^N_N(\mu)\|_X \leq \gamma \Delta^N_N \) for some \( \gamma > 0 \), then Algorithm 1 is called weak Greedy with parameter \( \gamma \).

Algorithm 1 \([X^N_N] = \text{Greedy}[\text{tol, } N_{\text{max}}, \Xi_{\text{train}}] \)

1: for \( N = 0, 1, \ldots, N_{\text{max}} \) do
2: Choose \( \mu^* := \arg \max_{\mu \in \Xi_{\text{train}}} \Delta^N_N(\mu) \).
3: if \( \Delta^N_N(\mu^*) < \text{tol} \) then return
4: \( N \leftarrow N + 1 \).
5: Compute 'truth' snapshot \( u^N(\mu^*) \), update basis: \( X^N_N = X^N_{N-1} \cup \{u^N(\mu^*)\} \).
6: end for

Remark 2.4. As an alternative to the Greedy algorithm one could determine \( \mu^* \) by nonlinear optimization, [7] [26]. If feasible, this approach avoids a training set.

2.4. Offline-online Decomposition. A crucial assumption for the efficiency of the RBM (in particular the efficient computation of \( u^N_N(\mu) \) and of \( \Delta^N_N(\mu) \)) is that the bilinear form and the right-hand side are affine in the parameter, i.e.

\[
(2.11) \quad b(u, v; \mu) = \sum_{q=1}^{Q_u} \theta_b^{(q)}(\mu) b^{(q)}(u, v), \quad f(v; \mu) = \sum_{q=1}^{Q_f} \theta_f^{(q)}(\mu) f^{(q)}(v).
\]

Techniques like the Empirical Interpolation Method (EIM) [2] can construct an approximation of such an affine decomposition if assumption (2.11) is not met. Decompositions as (2.11) enable an efficient offline-online decomposition of the calculations in the following sense: The parameter-independent components of the
linear system, namely \( \mathbf{B}_N^{(q)} := [b^{(q)}(\xi^N_j, \xi^N_j)]_{i,j=1,\ldots,N} \), \( q = 1, \ldots, Q_b \), and \( \mathbf{F}_N^{(q)} := [f^{(q)}(\xi^N_j)]_{j=1,\ldots,N} \), \( q = 1, \ldots, Q_f \), can be precomputed (offline) so that the assembly and solution of the reduced system \( \mathbf{B}_N(\mu)\mathbf{u}_N(\mu) = \mathbf{F}_N(\mu) \) with

\[
\text{(2.12)} \quad \mathbf{B}_N(\mu) := \sum_{q=1}^{Q_b} \theta_b^{(q)}(\mu) \mathbf{B}^{(q)} , \quad \mathbf{F}_N(\mu) := \sum_{q=1}^{Q_f} \theta_f^{(q)}(\mu) \mathbf{F}^{(q)},
\]

for a new parameter \( \mu \notin S_N \) then only involves \( N \)-dimensional matrix-vector products and can be done online (with complexity independent of \( N \)). Since \( \mathbf{B}_N(\mu) \in \mathbb{R}^{N \times N} \) is usually densely populated, the linear system \text{(2.12)} for

\[ u_N^\mu(\mu) = \sum_{i=1}^{N} u_N^{(i)}(\mu) \zeta_N^i, \quad \mathbf{u}_N(\mu) = (u_N^{(i)}(\mu))_{i=1,\ldots,N}, \]

can be solved with \( \mathcal{O}(N^3) \) operations – independent of \( N \gg N \).

Also the error estimate can be computed online-efficient (independent of \( N \)). This can again be achieved by using \text{(2.11)} as follows: The problems

\[
\text{(2.13a)} \quad \text{Find } \hat{\xi}_N^i \in \mathcal{Y}^N : \quad (\hat{\xi}_N^i, v)_{\mathcal{Y}} = b^{(q)}(\xi^N_N, v) \quad \forall v \in \mathcal{Y}^N ;
\]

\[
\text{(2.13b)} \quad \text{Find } \hat{\xi}_N^{i'} \in \mathcal{Y}^N : \quad (\hat{\xi}_N^{i'}, v)_{\mathcal{Y}} = f^{(q')}(v) \quad \forall v \in \mathcal{Y}^N ;
\]

are solved offline (with complexity \( \mathcal{O}(N) \)) for all \( q = 1, \ldots, Q_b \), \( n = 1, \ldots, N \), \( q' = 1, \ldots, Q_f \) and the inner products

\[
\left[ (\hat{\xi}_N^i, \hat{\xi}_N^{i'})_{\mathcal{Y}} \right]_{q,q'=1}^{Q_f, Q_b} , \quad \left[ (\hat{\xi}_N^i, \hat{\xi}_N^{i'}, \tilde{x}_N^{i,i'})_{\mathcal{Y}} \right]_{q,q'=1}^{Q_f, Q_b} , \quad \left[ \tilde{\xi}_N^{i,i'} \right]_{q,q'=1}^{Q_f, Q_b} , \quad i, i' = 1, \ldots, N, \]

are computed and stored. During the online phase, the parameter-dependent norm \( ||r_N^\mu(:; \mu)||_{\mathcal{Y}^N} = ||\hat{r}_N^\mu(\mu)||_{\mathcal{Y}^N} \) with

\[
\hat{r}_N^\mu(\mu) = \sum_{q=1}^{Q_b} \theta_b^{(q)}(\mu) \hat{\xi}_N^i + \sum_{i=1}^{N} u_N^{(i)}(\mu) \sum_{q=1}^{Q_b} \theta_b^{(q)}(\mu) \hat{\xi}_N^{i,i'}
\]

can be computed using only \( N \)-dependent matrix-vector products and simple function evaluations for any reduced solution \( u_N^\mu(\mu) \). These offline/online-techniques are incorporated into the Greedy scheme in a straightforward manner.

3. Adaptive Reduced Basis Generation

In this section, we describe those issues that arise when avoiding fixed truth spaces \( \mathcal{X}^N \) and \( \mathcal{Y}^N \) and use adaptive methods instead. We assume that we have the following routine \textsc{Solve} at our disposal. In Appendix [A] below, we detail one possibility to realize \textsc{Solve} by an Adaptive Wavelet Galerkin Method (AWGM), but one could also use other schemes with the above properties such as adaptive finite element methods, see e.g. [20] for an overview.

\textsc{Solve}: \([A, b, \varepsilon] \rightarrow x^\varepsilon\): Approximation of \( x = A^{-1}b \) with \( \|x - x^\varepsilon\|_{\mathcal{X}} \leq \varepsilon \) and linear complexity.
3.1. Adaptive Snapshot Computation. We assume that we have an (adaptive) numerical solver SOLVE at our disposal that computes so-called \( \varepsilon \)-exact approximations \( u^\varepsilon(\mu) \) of \( u(\mu) \in \mathcal{X} \), i.e. that constructs parameter-dependent discrete spaces \( \mathcal{X}_\mu^\varepsilon, \mathcal{Y}_\mu^\varepsilon \) of (possibly) arbitrary finite (from an RB point of view 'large', but in a certain sense minimal) dimension \( \mathcal{N}(\mu, \varepsilon) \) with

\[
\begin{align*}
(3.1) & \quad u^\varepsilon(\mu) \in \mathcal{X}_\mu^\varepsilon : \quad b(u^\varepsilon(\mu), v; \mu) = f(v; \mu) \quad \forall \, v \in \mathcal{Y}_\mu^\varepsilon \\
(3.2) & \quad \| u(\mu) - u^\varepsilon(\mu) \|_X \leq \varepsilon.
\end{align*}
\]

The lack of common truth spaces for all parameters necessitates a re-interpretation of some RB ingredients which we will describe now. The reduced space is now spanned by approximate snapshots computed during the offline training phase, i.e.

\[(3.3) & \quad \mathcal{X}_N^\varepsilon := \text{span}\{u^\varepsilon(\mu_i), i = 1, \ldots, N\} =: \text{span}\{\zeta_i^\varepsilon, i = 1, \ldots, N\},
\]

and the reduced solution \( u_N^\varepsilon(\mu) \in \mathcal{X}_N^\varepsilon \) is the Petrov-Galerkin projection onto this space (and the corresponding reduced LBB-stable test space \( Y_N^\varepsilon \)).

The adaptive setting now also allows us to consider the error with respect to the exact solution in \( \mathcal{X} \), i.e.

\[
(3.4) & \quad e_N^\varepsilon(\mu) := u(\mu) - u_N^\varepsilon(\mu),
\]

and not (only) the error w.r.t. a fixed and a priori given truth discretization. In fact, using standard arguments as above yields

\[
(3.5) & \quad \| e_N^\varepsilon(\mu) \|_X \leq \Delta_N^\varepsilon(\mu) = \frac{\| r_N^\varepsilon(\cdot; \mu) \|_{Y'}}{\beta(\mu)} \leq \frac{\gamma(\mu)}{\beta(\mu)} \| e_N^\varepsilon(\mu) \|_X,
\]

with the residual defined as \( r_N^\varepsilon(v; \mu) := f(v; \mu) - b(u_N^\varepsilon(\mu), v; \mu) \) for \( v \in \mathcal{Y} \). This means that \( \Delta_N^\varepsilon(\mu) \) is a surrogate for the true error \( \| e_N^\varepsilon(\mu) \|_X \). Note, however, that this is an infinite-dimensional object, namely on \( \mathcal{Y} \).

On the other hand, in this setting, the truth spaces \( \mathcal{X}_\mu^\varepsilon, \mathcal{Y}_\mu^\varepsilon \) corresponding to \( u^\varepsilon(\mu) \) are not known a priori, so that neither the set of involved local basis functions for representing \( u^\varepsilon(\mu) \) (with parameter-dependent dimension \( \mathcal{N}(\mu, \varepsilon) \)) nor the Riesz representation of the corresponding residual \( r_N^\varepsilon(\cdot; \mu) \) can be determined without computing \( u^\varepsilon(\mu) \) itself.

Remark 3.1. The above formulated adaptive framework can also be interpreted as using different finite element meshes for different \( \mu \in \mathcal{D} \) in the snapshot generation.

3.2. Adaptive Computation of the Error Estimator. Now we consider the computation of the error estimator which consists of the constant \( \beta(\mu) \) and the dual norm of the residual. We aim at using adaptivity for approximating both. As for the residual, the norm of \( r_N^\varepsilon(\cdot; \mu) \) would require computations on the infinite-dimensional space \( \mathcal{Y} \). Hence, it seems natural also to use an adaptive method for computing the Riesz representations which are given in the infinite-dimensional setting as

\[
(3.6) & \quad \text{Find } r_N^\varepsilon(\mu) \in \mathcal{Y} : \quad (r_N^\varepsilon(\mu), v)_\mathcal{Y} = r_N^\varepsilon(v; \mu) \quad \forall \, v \in \mathcal{Y}.
\]

This is the reason why we need to approximate \( \Delta_N^\varepsilon \) by some \( \Delta_{N,\delta} \), where we need to choose the tolerance \( \delta > 0 \) appropriately. We start by analyzing in which sense we need to approximate the residual or, slightly more general, an error estimator which is not completely computable.
Lemma 3.2. Assume $\Delta_N \geq 0$ is an equivalent bound for an error $e_N \geq 0$, i.e., there are absolute constants $0 < \xi^e \leq \overline{C}^e < \infty$ with
\[
\xi^e e_N \leq \Delta_N \leq \overline{C}^e e_N.
\]
Let $\delta \in [0, 1)$ and $\Delta_{N, \delta}$ be an approximation of $\Delta_N$ satisfying
\[
|\Delta_N - \Delta_{N, \delta}| \leq \delta \cdot \left\{ \begin{array}{ll}
\Delta_{N, \delta}, & \text{if } \Delta_N > 0, \\
0, & \text{if } \Delta_N = 0.
\end{array} \right.
\]
Then, the approximate error estimator $\Delta_{N, \delta}$ is equivalent to the error in the sense
\[
\frac{\xi^e}{1 + \delta} e_N \leq \Delta_{N, \delta} \leq \overline{C}^e e_N.
\]

Proof. If $\Delta_N = 0$, then $e_N = 0$ and $\Delta_N = \Delta_{N, \delta} = 0$, thus (3.9) holds. Now let $\Delta_N \neq 0$, then $(1 - \delta)\Delta_{N, \delta} \leq \Delta_N \leq \overline{C}^e e_N$, hence the upper inequality of (3.9). On the other hand, we have $(1 + \delta)\Delta_{N, \delta} \geq \Delta_N \geq \xi^e e_N$, the lower bound in (3.9). \hfill \Box

Remark 3.3. (a) We apply Lemma 3.2 to $e_N = \|e_N^\mu(u)\|_X = \|u(\mu) - u_X^\mu(\mu)\|_X$ and $\Delta_N = \Delta_N^\mu(\mu) = \|\mu_{\mu}^\mu(\mu)\|_{Y^\mu}$ so that (3.8) means
\[
\|\mu_{\mu}^\mu(\mu)\|_Y - \|\mu_{\mu}^\mu(\mu)\|_Y \leq \delta \|\mu_{\mu}^\mu(\mu)\|_Y.
\]
Obviously, (3.10) requires an adaptive method with a relative error bound.
(b) The right-hand side of the relative error criterion (3.8) only depends on the online computable quantity $\|\mu_{\mu}^\mu(\mu)\|_Y$. Thus, (3.8) can be verified a posteriori for all $\mu \in \mathcal{D}$.
(c) The explicit knowledge of the equivalence constants yields the equivalence
\[
\frac{1}{1 + \delta} \|e_N^\mu(\mu)\|_X \leq \Delta_N^\mu(\mu) \leq \frac{1}{1 - \delta} \frac{\gamma(\mu)}{\beta(\mu)} \|e_N^\mu(\mu)\|_X,
\]
i.e., also the computable error $\Delta_N^\mu(\mu)$ can be used as a surrogate for the true error $\|e_N^\mu(\mu)\|_X$. Of course, we have to take into account that the amount of work required to compute $\Delta_N^\mu(\mu)$ grows as $\delta \to 0$. Finally, in the Greedy training phase, the efficiency of an error estimator is more important than its rigor, as the main requirement on the surrogate $\Delta_N^\mu(\mu)$ is a correct choice of the next snapshot parameter in terms of the maximization. \hfill \Box

We have seen that SOLVE provides us with an adaptive scheme for an absolute error. It is not hard to see how to use it to derive a relative error tolerance.

Lemma 3.4. Let $a > 0$ and $(b_\delta)_{\delta > 0}$ be a sequence with $|a - b_\delta| \leq \delta$. Moreover, let $\rho \in (0, 1)$ be given. Then, choosing $\delta^*(\rho) < \rho_2$ yields $|a - b_{\delta^*(\rho)}| \leq \rho |b_{\delta^*(\rho)}|$. Proof. Since $\rho < 1$ we have $|a - b_{\delta^*(\rho)}| < \frac{a}{2}$, thus $b_{\delta^*(\rho)} \in \left(\frac{a}{2}, \frac{3a}{2}\right)$, in particular $b_{\delta^*(\rho)} > \frac{a}{2} > 0$. Then, $|a - b_{\delta^*(\rho)}| \leq \delta^*(\rho) < \rho \frac{a}{2} < \rho |b_{\delta^*(\rho)}| = \rho |b_{\delta^*(\rho)}|$, which proves the claim. \hfill \Box

3.3. Adaptive Greedy Algorithm. Now we have all ingredients at hand to formulate a fully adaptive version of the Greedy algorithm in Algorithm 2. The adaptive computations take place in line 2 concerning the error estimator and in line 8 for the snapshot. If Algorithm 2 terminates, we get $\Delta_N^\mu(\mu) < \text{tol}$ for all $\mu \in \mathcal{Z}_{\text{train}}$, hence
\[
\max_{\mu \in \mathcal{Z}_{\text{train}}} \|e_N^\mu(\mu)\|_X \leq (1 + \delta)\text{tol}.
\]
Fixing some $\delta \in (0, 1)$, one has to use $\tilde{\text{tol}} = \frac{\text{tol}}{1 + \delta}$ in line 3 in order to reach a desired tolerance $\text{tol}$ for the true error on $\Xi_{\text{train}}$. Of course the quality of $\Xi_{\text{train}}$ is also important for controlling the error for all $\mu \in D$. Again, we mention nonlinear optimization as an alternative to the Greedy scheme, [5, 26].

Algorithm 2 $[X_{\varepsilon, N}^{\delta}] = \text{AdaptGreedy}[^{\tilde{\text{tol}}}, N_{\max}, \varepsilon, \delta, \Xi_{\text{train}}]$

1: for $N = 0, 1, \ldots, N_{\max}$ do
2: Choose $\mu^* := \text{argmax}_{\mu \in \Xi_{\text{train}}} \Delta_{\varepsilon, N}^{\delta}(\mu)$.
3: if $\Delta_{\varepsilon, N}^{\delta}(\mu^*) < \tilde{\text{tol}}$ then return
4: $N \leftarrow N + 1$.
5: Compute snapshot $u_{\varepsilon, N}^*(\mu^*)$.
6: Update reduced basis: $X_{\varepsilon, N, \delta} = X_{\varepsilon, N-1, \delta} \cup \{u_{\varepsilon}^*(\mu^*)\}$.
7: end for

It seems natural that the tolerance $\varepsilon > 0$ bounds the reduction error $e_{\varepsilon, N}(\mu) = u(\mu) - u_{\varepsilon, N}^*(\mu)$ from below in the sense that the error cannot be smaller than the accuracy of the snapshot approximations. A result from [3] shows that this can lead to a stalling of the Greedy training at a certain level. As usual, the benchmark for the Greedy algorithm is the Kolmogorov $n$-width for some $\Sigma \subset X$, i.e.,

$$d_n(\Sigma) := \inf_{\dim(X_n) = n} \sup_{f \in \Sigma} \min_{g \in X_n} \|f - g\|_X.$$

Theorem 3.5 ([3]). Let $\mathcal{M}(\mu) := \{u(\mu) : \mu \in D\}$ be compact and suppose that $d_0(\mathcal{M}(\mu)) \leq M$, $d_n(\mathcal{M}(\mu)) \leq M n^{-\theta}$ for some $M, \theta > 0$. Then, the approximation $X_{\varepsilon, N, \delta} = \text{AdaptGreedy}[^{\tilde{\text{tol}}}, N_{\max}, \varepsilon, \delta, D]$ satisfies

$$\sup_{\sigma \in \mathcal{M}(\mu)} \min_{g \in X_{\varepsilon, N, \delta}} \|\sigma - g\|_X \leq C \max\{M n^{-\theta}, \varepsilon\}, \quad \gamma := \frac{1 - \delta \varepsilon^c}{1 + \delta \frac{\varepsilon^c}{C}}$$

with a constant $C = C(\theta, \gamma)$.

This result tells us in fact that $\varepsilon$ sets the lower bound for the error. On the other hand, this result seems to answer all possible open questions for adaptive RBMs since it states that the convergence is almost optimal as compared to the Kolmogorov $n$-width. However, we have observed in several numerical tests (see also Section 4 below) some problems in the sense that $\text{AdaptGreedy}$ does not converge at all since the snapshot selection in line 2 often re-produces previously chosen parameters which ends in an infinite loop. First note that Theorem 3.5 is related to $\Xi_{\text{train}} = D$, which –at least numerically– is often an unrealistic scenario. Still, our numerical observation seems to contradict Theorem 3.5. As we will describe below this is not the case. Moreover, the subsequent investigations will shed some light to the practical relevance of the above convergence result.

3.4. (Non-)Reproduction of Snapshots. As we have pointed out in Remark 2.3 on a fixed truth discretization, we have that $\Delta_{\varepsilon, N}^{\delta}(\mu) = 0$ (up to numerical influences) for all $\mu \in S_N$, i.e. the error bound vanishes on the set of snapshot parameters, as all snapshots $u^N(\mu) \in X^N, \mu \in S_N$, can be reconstructed exactly from the basis functions and the Riesz representation for the error estimator is only
based upon $X^N$, $Y^N$. As we will explain now this is not the case in the adaptive framework. The reason is that $u^\circ(\mu) \in X^N_\mu$ but $u^\circ(\mu) \not\in X^*_\mu$. In fact, we only have
\begin{equation}
 u^*_N(\mu) \in \bigcup_{\hat{\mu} \in S_N} X^*_\hat{\mu} =: X^{\varepsilon,S_N},
\end{equation}

Hence, the argument using Petrov-Galerkin orthogonality as in Remark 2.3 fails. In fact, note that $e_N^*(\mu) = u(\mu) - u^*_N(\mu)$ is the error w.r.t. the unknown solution $u(\mu)$, whereas $e_N^*(\mu) = u^N(\mu) - u^*_N(\mu)$ involves the ‘truth’ solution, which is in principle computable (up to numerical precision). This is important since $u^N(\mu)$ is used in the ‘classical’ case as a snapshot, whereas $u(\mu)$ cannot be used in the adaptive setting but only an approximation $u^\circ(\mu)$. However, $b(u^\circ(\mu) - u^*_N(\mu), v^*_N; \mu)$ will in general not vanish! This means that –as opposed to the ‘classical’ RBM– snapshots are not reproduced in the adaptive setting. Reproduction of RB basis functions is not a consequence of the fact that the RB spaces are spanned by those functions but a consequence of the Petrov-Galerkin orthogonality.

Of course, one could use $X^{\varepsilon,S_N}$ defined in (3.12) as a joint common truth space. However, if the discretizations for different $\mu$ are significantly different, this would be far too costly.

Hence, we face a reproduction error, which is now investigate little further.

**Proposition 3.6.** Let $b(\cdot, \cdot; \mu) : X \times X \to \mathbb{R}$ be coercive with constant $\alpha(\mu) > 0$ for any $\mu \in \mathcal{D}$. Then $\|u(\mu) - u^*_N(\mu)\|_X \leq \frac{\gamma(\mu)}{\alpha(\mu)} \varepsilon$ for all $\mu \in S_N$.

**Proof.** Let $\mu \in S_N$. Then $u^\circ(\mu) \in X^*_N$ and we can use Galerkin orthogonality, i.e., $b(u(\mu) - u^*_N(\mu), v^*_N; \mu) = 0$ for all $v^*_N \in X^*_N$. We use this for $v^*_N = u^\circ(\mu) - u^*_N(\mu) \in X^*_N$ to derive
\begin{align*}
\alpha(\mu)\|u(\mu) - u^*_N(\mu)\|^2_X &\leq b(u(\mu) - u^*_N(\mu), u(\mu) - u^*_N(\mu); \mu) \\
&= b(u(\mu) - u^*_N(\mu), u(\mu) - u^\circ(\mu); \mu) \\
&\leq \gamma(\mu)\|u(\mu) - u^*_N(\mu)\|_X\|u(\mu) - u^\circ(\mu)\|_X \\
&\leq \varepsilon \gamma(\mu)\|u(\mu) - u^*_N(\mu)\|_X,
\end{align*}
which proves the claim. \hfill \Box

This simple estimate explains both the accordance with Theorem 3.5 and also our observations of non-converging Greedy training loops. In fact, the constant $\frac{\gamma(\mu)}{\alpha(\mu)}$ above coincides with the constant $C$ in Theorem 3.5. As a consequence, we get $\mu^* := \text{arg max}_{\bar{\mu} \in \Xi_{\text{train}}} \Delta^\varepsilon_{N+1,\delta}(\bar{\mu}) \in S_N$ if $\varepsilon \frac{\gamma(\mu^*)}{\alpha(\mu^*)} \geq \text{max}_{\bar{\mu} \in \Xi_{\text{train}} \setminus S_N} \Delta^\varepsilon_{N+1,\delta}(\bar{\mu})$. The first idea is to replace line 2 in Algorithm 2 by

Choose $\mu^* := \text{arg max}_{\mu \in \Xi_{\text{train}} \setminus S_N} \Delta^\varepsilon_{N+\delta}(\mu)$.

As already mentioned, in the ‘classical’ RBM, we have $\Delta^N_N(\mu) = 0$ for all $\mu \in S_N$. In turns, this ensures that $\mu_{N+1} = \text{arg max}_{\mu \in \Xi_{\text{train}}} \Delta^N_{N,\delta}(\mu) \not\in S_N$, so that an updating is possible and no snapshot parameter is chosen twice.

3.5. **Adaptive Offline-online Decomposition.** As in the ‘standard’ RBM, we assume an affine decomposition of $b(\cdot, \cdot; \mu)$ and $f(\cdot; \mu)$ as in (2.11).
3.5.1. Computation of the RB Solution. It is readily seen that $u_h^\varepsilon(\mu)$ can be computed online-efficient like in the standard case. In fact, the matrix and right-hand side terms $B_{\varepsilon,N}^{(q)} := [b^{(q)}(u^r(\mu_i), u^r(\mu_j))]_{i,j=1,\ldots,N}$, $q = 1, \ldots, Q_b$, and $F_{\varepsilon,N}^{(q)} := [f^{(q)}(u^r(\mu_j))]_{j=1,\ldots,N}$ can be computed offline. Then, $B_{\varepsilon,N}(\mu)$ and $F_{\varepsilon,N}(\mu)$ are formed as in the classical case and the solution $u_{\varepsilon,N}(\mu)$ of the linear system corresponding to \ref{eq:2.12} yields the RB solution $u_h^\varepsilon(\mu)$.

3.5.2. Error Estimator. In principle we could follow the same path as above and solve the analogue to \ref{eq:2.13}, i.e.,

\begin{align}
\text{(3.13a) } & \text{Find } \hat{b}^{q,n} \in \mathcal{Y} : \quad (\hat{b}^{q,n}, v)_\mathcal{Y} = b^{(q)}(\zeta_n^\varepsilon, v) \quad \forall v \in \mathcal{Y}, \\
\text{(3.13b) } & \text{Find } \hat{f}^q \in \mathcal{Y} : \quad (\hat{f}^q, v)_\mathcal{Y} = f^{(q)}(v) \quad \forall v \in \mathcal{Y}.
\end{align}

Note that \ref{eq:3.13a} and \ref{eq:3.13b} are again variational problems on the infinite-dimensional space $\mathcal{Y}$. Using corresponding adaptive schemes yield (for given tolerances $\delta_{br,n}, \delta_{fr} > 0$) approximations $\hat{b}^{q,n}_\delta$, $\hat{f}^q_\delta$ satisfying

$$||\hat{b}^{q,n}_\delta - \hat{b}^{q,n}||_{\mathcal{Y}} < \delta_{br,n}, \quad ||\hat{f}^q_\delta - \hat{f}^q||_{\mathcal{Y}} < \delta_{fr}.$$  

As in \ref{eq:2.14} we could compute the required inner products and would obtain an approximation, say $\Delta_{N,\delta}^{\varepsilon,aff}(\mu)$ of $\Delta_{N}^\varepsilon(\mu)$. However, we need to verify the relative accuracy in \ref{eq:3.8}. This can be done as follows. Denoting by $\delta_{fr}$, $\delta_{br,n}$ the accuracies of $\hat{f}^q_\delta$ and $\hat{b}^{q,n}_\delta$, respectively, it holds that

$$||\hat{f}_N(\mu) - \hat{f}^\varepsilon_N(\mu)||_{\mathcal{Y}}^2 = \left|\sum_{q=1}^{Q_f} \theta_{f}^{(q)}(\mu) [\hat{f}^q - \hat{f}^q_\delta] + \sum_{n=0}^{N} u_{N}^{(n)}(\mu) \sum_{q=1}^{Q_b} \theta_{b}^{(q)}(\mu) [\hat{b}^{q,n}_\delta - \hat{b}^{q,n}] \right|^2_{\mathcal{Y}} \leq \sum_{q,q'=1}^{Q_f} \left( \theta_{f}^{(q)}(\mu) \theta_{f}^{(q')}(\mu) \right) \delta_{fr} \delta_{fr'} + 2 \sum_{n=0}^{N} \sum_{q=1}^{Q_b} \sum_{q'=1}^{Q_b} \left( u_{N}^{(n)}(\mu) \theta_{b}^{(q)}(\mu) \theta_{b}^{(q')}(\mu) \right) \delta_{fr} \delta_{br,n} \delta_{br',n'} \leq \delta_{aff}(\mu)^2,$$

Using similar adaptive approximations of the involved constants (e.g. $\beta(\mu)$) then leads to $\Delta_{N,\delta}^{\varepsilon,aff}(\mu)$ satisfying an absolute error bound

$$|\Delta_{N}^\varepsilon(\mu) - \Delta_{N,\delta}^{\varepsilon,aff}(\mu)| \leq \delta_{aff}(\mu).$$

Note that we can arrange $\delta_{aff}(\mu) > 0$ to be as small as we wish by using sufficiently small tolerances $\delta_{fr}$, $\delta_{br,n}$ — at the expense of possibly high numerical cost. It is however not possible to do this completely offline, since $\delta_{aff}(\mu)$ is $\mu$-dependent. This means that we possibly need to update the adaptive approximations during the online-phase. On the other hand, we can then use Lemma \ref{lem:3.4} which leads us from an absolute accuracy, i.e., $|\Delta_{N}^\varepsilon(\mu) - \Delta_{N,\delta}^{\varepsilon,aff}(\mu)| \leq \delta_{aff}(\mu)$ to a relative one, i.e., $|\Delta_{N}^\varepsilon(\mu) - \Delta_{N,\delta}^{\varepsilon,aff}(\mu)| \leq \rho \Delta_{N,\delta}^{\varepsilon,aff}(\mu)$ for a given relative accuracy $\rho \in (0, 1)$ and a suitably chosen $\delta^*(\rho) \in (0, 1)$. Thus, we verify the validity of \ref{eq:3.8} a posteriori as follows: Since we can compute $\delta_{aff}(\mu)$, we check if $\delta_{aff}(\mu) \leq \rho \Delta_{N,\delta}^{\varepsilon,aff}(\mu)$. If not, we decrease the involved tolerances. Due to the convergence of the used adaptive schemes, this approach in fact converges and results in an approximate estimate.
\[ \Delta_{N,\delta}^{\varepsilon}(\mu) \] which is equivalent to \( \|e_N^{\varepsilon}(\mu)\|_X \) as in (3.11). However, the above estimate of \( \|\tilde{r}_N(\mu) - \tilde{r}_{N,\delta}(\mu)\|_Y \) in terms of \( \delta_{\text{aff}}(\mu) \) is rather crude so that we expect non-optimal numerical performance — as we shall also see in our numerical experiments below. If we store the adaptive approximation (i.e., the ‘active’ wavelet coefficients) of each Riesz representation, we only need to update those discretizations if the choice of \( \mu \) requires additional accuracy.

4. Numerical Experiments

4.1. Data for the Experiments. We describe the data for our two examples.

4.1.1. Elliptic Equation. We consider heat conduction in a 2D thermal block \( \Omega = (0, 1)^2 \) consisting of two subdomains \( \Omega_0 = [0.5, 1] \times [0, 1], \Omega_1 = [0, 0.5] \times [0, 1], \) with different conductivities \( \mu_0 = 1, \mu_1 \in [0.01, 100] \). [21] The heat influx is modeled as a constant local source on different parts \( \tilde{\Omega}_i, i = 1, \ldots, 9, \) of the domain, where the current location depends on a (discrete) parameter \( \mu_2 \in \{1, \ldots, 9\} \), see Figure 1. We impose homogeneous Dirichlet boundary conditions on \( \Gamma_D := \partial \Omega \cap \{x = 0 \lor x = 1\} \) and homogeneous Neumann conditions on \( \Gamma_N := \partial \Omega \cap \{y = 0 \lor y = 1\} \). The variational formulation then reads: Find \( u \in X := H_{D}^1(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\} \) such that

\[
\int_{\Omega_0} \nabla u \cdot \nabla v + \mu_1 \int_{\Omega_1} \nabla u \cdot \nabla v = (f(\mu_2), v)_{L_2(\Omega)} \quad \forall v \in X, \quad f(\mu_2) := \sum_{i=1}^{9} \delta_{\mu_2,i} 1_{\tilde{\Omega}_i}.
\]

We employ a multitree-based AWGM (see Appendix A below) with a tensor basis consisting of bi-orthogonal B-spline wavelets from [10] of order \( d_x = m_x = 2^m \) and \( L_2(0, 1) \)-orthonormal (multi-)wavelets as in [22] of order \( d_y = m_y = 2 \), with homogeneous boundary conditions. In Figure 2, we display snapshots corresponding to two different parameter values. We also indicate the support centers of the adaptively chosen wavelets which shows that an adaptive discretization is in fact useful for this example.

---

The meaning of the parameters \( d \) and \( m \) are explained in Appendix A below.
4.1.2. **Parabolic Periodic Space-Time Equation.** As a second example, we consider the time-periodic convection-diffusion-reaction (CDR) equation

\[
\begin{aligned}
  u_t - u_{xx} + \mu_1 \beta(x) u_x + \mu_2 u &= \cos(2\pi t) & \text{on } \Omega = (0, 1), \\
  u(t, 0) &= u(t, 1) & \text{for all } t \in [0, T], \\
  u(0, x) &= u(T, x) = 0 & \text{on } \Omega,
\end{aligned}
\]

with coefficient function \( \beta(x) = 0.5 - x \). Defining \( V := H^1_0(\Omega), \ H^1_{\text{per}}(0, T) := \{ v \in H^1(0, T) : v(0) = v(T) \} \), we define the spaces \( \mathcal{Y} := L_2(0, T; V) \) and \( \mathcal{X} := L_2(0, T; V) \cap H^1_{\text{per}}(0, T; V') \), i.e.,

\[
\mathcal{X} = \{ v \in L_2(0, T; V) : v_t \in L_2(0, T; V'), \ v(0) = v(T) \} \text{ in } H^1,
\]

where \( \mathcal{X} \) is equipped with the norm \( \| v \|_{\mathcal{X}}^2 := \| v \|^2_{L_2(0, T; V)} + \| v_t \|^2_{L_2(0, T; V')} \), \( v \in \mathcal{X} \). Note that \( v(0), v(T) \) are well-defined due to \( H^1(0, T) \subset C([0, T]) \) and \( \{ v \in L_2(0, T; V) : v_t \in L_2(0, T; V') \} \subset C(0, T; H) \), e.g. [9]. We obtain the variational problem:

\[
(4.1) \quad \text{Find } u \in \mathcal{X} : \quad b(u, v; \mu) = f(v) \quad \forall v \in \mathcal{Y}, \ \mu = (\mu_1, \mu_2),
\]

with forms \( b(\cdot, \cdot; \mu) : \mathcal{X} \times \mathcal{Y} \times D \to \mathbb{R}, f(\cdot) : \mathcal{Y} \to \mathbb{R} \) defined by

\[
(4.3) \quad b(u, v; \mu) := \int_0^T [\langle v(t), u_t(t) \rangle_{V \times V'} + a(u(t), v(t); \mu)] dt,
\]

\[
f(v) := \int_0^T \cos(2\pi t) \langle v(t), 1 \rangle_{V \times V'} dt \text{ and } a(\phi, \eta; \mu) = (\phi_x, \eta_x)_{L_2(\Omega)} + \mu_1 (\beta \phi_x, \eta)_{L_2(\Omega)} + \mu_2 (\phi, \eta)_{L_2(\Omega)}.
\]

As bases we use space-time tensor functions: In time, we use a collection of bi-orthogonal B-spline wavelets on \( \mathbb{R} \) of order \( d_t = m_t = 2 \), periodized onto \([0, T]\), [25].

The spatial basis is chosen as bi-orthogonal B-spline wavelets of order \( d_x = m_x = 2 \) with homogeneous boundary conditions from [10].

The test basis is a tensor product of the above mentioned linear B-spline wavelets with 2 vanishing moments from [10] with homogenous boundary conditions in the univariate spatial basis.

In this example, the snapshots have different temporal evolutions. Since time is a ‘normal’ variable in a space-time variational formulation, this means that different discretizations for the snapshots in space-time pay off. This justifies adaptivity.

4.2. **Adaptive Greedy Training.** We start by investigating the snapshot selection using the Greedy method in Algorithm 2 of [1]. The results for both examples are shown in Figure 3. In both cases the error is computed w.r.t. a previously chosen fine training set \( \Xi_{\text{train}} \). We also consider the strong Greedy method, i.e., we use the error \( u^*_N(\mu) \) instead of the estimate \( \Delta_N(\mu) \) (or its approximation \( \Delta_{N,\delta}(\mu) \)) for the
snapshot selection. This is done by computing a sufficiently accurate approximation to each $u(\mu)$. As we see, there is a plateau due to the multiple selection of snapshots as described in §3.4 above. We stress that the multiple selection is not a numerical issue. In the same figure, we compare the strong Greedy with the direct computation of $\Delta_{N, \delta}(\mu)$, meaning that the dual of the residual $\|r_{\delta N}(\mu)\|_Y = \|\tilde{r}_{\delta N}(\mu)\|_Y$ is computed by adaptively solving (3.6) without using an affine decomposition as in §3.5.2. Of course, the computation is not online-efficient but guarantees the required relative accuracy in Lemma 3.2. This method will be called “Direct”.

In both parts of Figure 3, we compare “Direct” for different tolerances $\varepsilon$ used for the snapshot accuracy. We have used $\delta = 0.99$ in order to minimize computational cost for the error estimator. As expected, the plateau appears later for increased accuracies. The comparison in Figure 3(b) is interesting. At certain stages ($N = 2, 3$) the error for “Direct” is smaller than for the strong Greedy. The reason is that different snapshots are selected and “Direct” appears to select “better” ones at these stages.

4.3. Computation of the Error Estimator. Next, we test the computation of the error estimator $\Delta^{\varepsilon, \text{aff}}_{N, \delta}(\mu)$ using the affine decomposition as described in §3.5.2 which will be termed “Affine”. To this end, we fix the accuracies $\delta_{iN,n}, \delta_{f n} > 0$ in the adaptive computations in (3.13) for the Riesz representations of right-hand side and bilinear form terms and combine them to an error indicator (it may not be an estimator). We do not check if $\delta_{\text{aff}}(\mu)$ is below the desired tolerance, so that we only have a heuristics. The results are shown in Figure 4. For comparison purpose, we also indicate the strong Greedy error from Figure 3.

First of all, we see (until the plateau is reached) that the efficiency of this heuristics is of the order of 10, which seems to be a good value. Again, the plateau is lower for better accuracies of the snapshots. Finally, we check the variant of the Greedy that prevents us from multiple snapshot selection by excluding parameters in $S_N$ from the determination of $\mu_{N+1}$. As we see, this approach gives quite good results even in comparison to the strong Greedy which sets the benchmark.

4.4. Adjusting Accuracies. Theorem 3.5 indicates that the accuracy for all selected snapshots should be chosen equal. Intuitively this seems clear as the weakest link determines the stability of the whole chain. On the other hand, this might be computationally suboptimal, in particular since we do not have the true error
available but only an estimate at hand. This is the motivation for updating snapshots by lowering the corresponding approximation tolerance if an already chosen parameter is selected again. This leads us to the following strategy:

(S) Given $\varepsilon^{(0)}$, set for some pre-specified reduction factor $\varrho_u \in (0, 1)$

$$
\varepsilon^{(n+1)} := \begin{cases} 
\varrho_u \cdot \varepsilon^{(n)} & \text{if } \mu^* \in S_n, \\
\varepsilon^{(n)} & \text{else.} 
\end{cases}
$$

In Figure 5 we have depicted the results for both examples using the direct computation of $\Delta_{N,\delta}^\varepsilon(\mu)$ starting with $\varepsilon^{(0)} = 0.005$. We compare with the previous results in Figure 4 without updating the snapshots. As we can see, we can in fact avoid the plateau using the reduction factor $\varrho_u = 0.1$. Next, also in Figure 5, we perform the same comparison with the heuristic approximation of $\Delta_{N,\delta}^\varepsilon(\mu)$ using the affine decomposition. Even though the update strategy (S) for the snapshots with $\varrho_u = 0.1$ gives rise to an improvement, we again obtain a plateau. The comparison with the Greedy scheme excluding multiple selection shows, however, that there is room for improvements. Recalling that the heuristic approximation of $\Delta_{N,\delta}^\varepsilon(\mu)$ is based upon offline computed adaptive approximations of $\hat{q}_{p,b}^n$ and $\hat{f}_q^n$ in (3.13), it seems natural to update these quantities using a strategy similar to (S), i.e.,

(E) Starting with $\delta_f^{(0)} = \delta_{\hat{f}}^{(0)}$ reduce both tolerances by factors $\varrho_{\hat{f}}, \varrho_{\hat{b}} \in (0, 1)$ if a parameter is re-chosen, i.e.

$$
\delta_f^{(n+1)} := \begin{cases} 
\varrho_{\hat{f}} \cdot \delta_f^{(n)} & \text{if } \mu^* \in S_n, \\
\delta_f^{(n)} & \text{else,} 
\end{cases} \quad \delta_{\hat{b}}^{(n+1)} := \begin{cases} 
\varrho_{\hat{b}} \cdot \delta_{\hat{b}}^{(n)} & \text{if } \mu^* \in S_n, \\
\delta_{\hat{b}}^{(n)} & \text{else.} 
\end{cases}
$$

As shown in Figure 6 we obtain a significant improvement indicating that this is a meaningful approach. In fact, an error estimator which is not sufficiently accurate may not correctly reflect the error behavior w.r.t. different parameter values.

Finally, we investigate the potential for adaptivity. In Table 1 we compare the number of active wavelet coefficients, i.e., the size of the adaptive approximations. The huge numbers in the first two rows correspond to the Greedy results in Figure 6. The total number of wavelets for all computations are shown. The quite moderate number in the last row corresponds to the direct (and certified) approximation of $\Delta_N^\varepsilon(\mu)$. We show the average over all $\mu \in \Xi_{\text{train}}$. These results indicate that the use of the rigorous estimate might be more efficient even though it is theoretically...
not online efficient. The offline cost of the heuristics is huge. The reason is the poor estimate in §3.5.2 in the following sense: We need to fix the error tolerances for the Riesz representations offline. Then, we combine these terms to an error indicator whose accuracy we can only check a posteriori, i.e., for a given parameter $\mu \in \mathcal{D}$. Since the estimate for this verification is currently too crude, we need to update the parameter-independent Riesz representations leading to extremely (and overly) fine discretizations. Obviously, there is room and a definite need for improvements. Another issue is that the adaptive offline computation of the Riesz representations is updated if they are not sufficiently accurate for a given $\mu \in \mathcal{D}$. This means that the required accuracies are determined by the ‘worst case’ parameter $\mu$. We will investigate this further in the near future.

| $N$  | 1    | 2    | 3    | 4    | 5    | 6    |
|------|------|------|------|------|------|------|
| Affine, no update | 41 127 | 59 705 | 62 648 | 74 459 | 74 529 | 76 082 |
| Affine, update (S) & (E) | 52 004 | 70 010 | 72 812 | 84 409 | 52 192 | 76 590 |
| Direct (mean over $\Xi_{\text{train}}$) | 175 | 183 | 200 | 251 | 509 | 684 |

Table 1. Example 2 – Number of active wavelets for the computation of the Riesz representations $\hat{r}_{N,\beta}(\mu)$. 

**Figure 5.** Updating tolerances, direct computation of error estimator.

**Figure 6.** Updating tolerances, affine decomposition, no check.
APPENDIX A. Adaptive Wavelet Galerkin Methods (AWGMs)

To obtain an adaptive approximation for the snapshots $u^N(\mu)$ as well as the error estimators $\Delta^N_N(\mu)$ we employ adaptive wavelet Galerkin methods (AWGMs) that have first been introduced in [7, 8] for stationary problems and extended to space-time variational parabolic problems in [23]. Here, we used multitree-based versions developed in [14, 15, 16, 17], which we briefly review. Let $A: X \rightarrow Y'$ be a linear differential (or integral) operator which may or may not depend on $\mu \in D$. Given some $b \in Y'$, we look for $x \in X$ such that

$$A x = b \quad \text{in} \quad Y'. \quad \tag{A.1}$$

A.1. Equivalent Bi-infinite Matrix-Vector Problem. Variational equations of the form (A.1) can be reformulated as equivalent $\ell_2$-problems by considering Riesz bases of the Hilbert spaces $X, Y$. We call $Y := \{y_i : i \in \mathbb{N}\} \subset Z$ a Riesz basis for a separable Hilbert space $Z$ if its linear span is dense in $Z$ and if there exist $c, C > 0$ such that

$$c \|v\|_{\ell_2(\mathbb{N})} \leq \|v\|_Z \leq C \|v\|_{\ell_2(\mathbb{N})} \quad \forall v = (v_i)_{i \in \mathbb{N}} \in \ell_2(\mathbb{N}), \ v = \sum_{i=1}^{\infty} v_i y_i. \quad \tag{A.2}$$

For $X, Y$, we denote these Riesz wavelet bases by

$$\Psi^X := \{\psi^X_\lambda : \lambda \in \tilde{J}\} \subset X, \quad \Psi^Y := \{\tilde{\psi}^Y_\lambda : \lambda \in \bar{J}\} \subset Y, \quad \tag{A.3}$$

for countable index sets $\tilde{J}, \bar{J}$. Such bases can be constructed by first building univariate wavelet bases $\Psi = \{\psi_\lambda : \lambda \in J\}$ for $L_2(0, 1)$ that are sufficiently smooth to constitute (after a proper normalization) also Riesz bases for a whole range of Sobolev spaces $H^s(0, 1)$, $s \in (-\hat{\gamma}, \hat{\gamma})$, where $\gamma, \hat{\gamma} > 0$ depend on the choice of the wavelets, cf. [25]. Typically the index takes the form $\lambda = (j, k)$, where $|\lambda| := j$ denotes the level (e.g. $\text{supp } \psi_\lambda \sim 2^{-|\lambda|}$) and $k$ the location in $(0, 1)$, e.g. the center of its support. We consider piecewise polynomial wavelets of order $d$ (degree plus one). Wavelets are oscillating (“small waves”) which is reflected by their degree $m$ of vanishing moments, i.e., $\int_0^1 x^r \psi_\lambda(x) \; dx = 0$ for all $|\lambda| > 0$ and all $0 \leq r \leq m - 1$, where $|\lambda| = 0$ denotes the coarsest level, $0 = \min_{\lambda \in J} |\lambda|$. Those functions are no ‘true’ wavelets but e.g. splines (scaling functions). The above mentioned constants $\gamma$ and $\hat{\gamma}$ are determined by $d, m$ and $\tilde{d}, \tilde{m}$, which are the corresponding parameters of the dual wavelet basis $\tilde{\psi} = \{\tilde{\psi}_\lambda : \lambda \in \bar{J}\}$ with $\int_0^1 \tilde{\psi}_\lambda(x) \psi_\mu(x) \; dx = \delta_{\lambda, \mu}$ for all $\lambda, \mu \in J$ with $|\lambda|, |\mu| > 0$.

Tensorization of the univariate functions then allows for appropriate bases in higher dimensions as well as for a vast range of Bochner spaces arising in the formulation of parabolic PDEs, see e.g. [23]. Constructions for more complicated domains $\Omega$ are also available.

Then, we equivalently formulate (A.1) as the discrete, but infinite-dimensional equation

$$\text{Find } x \in \ell_2(\tilde{J}) : \quad Ax = b, \quad b \in \ell_2(\bar{J}), \quad \tag{A.4}$$

where $A := (\Psi^Y, A(\Psi^X)), \ b = [b(\tilde{\psi}^Y_\lambda)]_{\lambda \in \bar{J}}$ and $x$ are the coefficients of the (unique) expansion $x = x^T \Psi^X$. 

A.2. Adaptive Methods and Nonlinear Approximation. In order to approximately solve the infinite-dimensional equation (A.4), AWGMs iteratively construct a sequence of nested finite index sets $\hat{\Lambda}_k \subseteq \mathcal{J}$, to which (A.4) is restricted. Considering (just for ease of presentation) a linear self-adjoint operator $A:\mathcal{X} \to \mathcal{X}$ and $\tilde{\Psi} = \tilde{\Psi}^\mathcal{X} = \tilde{\Psi}^\mathcal{Y}$, in each iteration the finite-dimensional problem

$$\text{(A.5)} \quad \text{Find } x_{\hat{\Lambda}_k} \in \ell_2(\hat{\Lambda}_k): \hat{\Lambda}_k A x_{\hat{\Lambda}_k} = b_{\hat{\Lambda}_k}, \quad b_{\hat{\Lambda}_k} \in \ell_2(\hat{\Lambda}_k),$$

is solved, where for $\Lambda \subset \mathcal{J}$, $v|\Lambda := v|_{\ell_2(\Lambda)}$ denotes the restriction of $v \in \ell_2(\mathcal{J})$ to $\ell_2(\Lambda)$ and $\Lambda A := (A E_\Lambda)|\Lambda$ with trivial embedding $E : \ell_2(\Lambda) \to \ell_2(\mathcal{J})$ the restriction of $A$ in both rows and columns.

The extension of $\Lambda_k$ to $\Lambda_{k+1}$ is then based on the residual $r^k := b - A x_{\Lambda_k}$ and its norm $\|r^k\|_{\ell_2(\mathcal{J})}$ which forms an equivalent error estimator, since

$$\text{(A.6)} \quad \|A\|^{-1} \|r^k\|_{\ell_2(\mathcal{J})} \leq \|x - x_{\Lambda_k}\|_{\ell_2(\mathcal{J})} \leq \|A\| \|r^k\|_{\ell_2(\mathcal{J})}.$$ 

Note that $r^k$ is supported on the infinite-dimensional set $\mathcal{J}$ even if $x_{\Lambda_k}$ is finitely supported. Hence we have to use appropriate approximation methods for the residual evaluation in order to arrive at an implementable AGWM, see [A.3] below.

The next index set is obtained by a so-called bulk-chasing: Choose $\Lambda_{k+1} \supset \Lambda_k$ as the smallest index set such that $\|r^k_{\Lambda_{k+1}}\|_{\ell_2(\Lambda_{k+1})} \geq c \|r^k\|_{\ell_2(\mathcal{J})}$ for some $0 < c < 1$. This implies that the indices of the largest residual coefficients are added to $\Lambda_k$ and the adaptive index set is steered into the direction of the largest error.

Under appropriate assumptions on the exactness and computational cost of the solution of (A.5), the approximation of $r^k$ and the implementation of the bulk chasing process, a quasi-optimality result is known. In order to formulate it, we introduce the nonlinear approximation class

$$\text{(A.7)} \quad \mathcal{A}^s := \{ v \in \ell_2(\mathcal{J}) : \|v\|_{\mathcal{A}^s} := \sup_{\varepsilon > 0} \varepsilon \left( \min_{N \in \mathbb{N}_0} \{ N \in \mathbb{N}_0 : \|v - v_N\|_{\ell_2(\mathcal{J})} \leq \varepsilon \} \right)^s < \infty \}$$

with $v_N$ being the best $N$-term approximation on $v$, consisting of the $N$ largest coefficients in modulus of $v$.

**Theorem A.1 (cf. [11] [24]).** There exist implementable routines and parameters such that the (approximate) computations of $x_{\Lambda_k}$, $r^k$ and $\Lambda_{k+1}$ can be performed with controllable tolerances and computational cost: If the AWGM is terminated when $\|r^k_{\Lambda_k}\|_{\ell_2(\Lambda_k)} \leq \varepsilon / \|A\|^{-1}$, the output $x := x_{\Lambda_k}$ satisfies $\|x - x_{\Lambda_k}\|_{\ell_2(\mathcal{J})} \leq \varepsilon$. If, moreover, $x \in \mathcal{A}^s$ for some $s > 0$, it holds for $N_k := \#\Lambda_k$ that

$$\text{(A.8)} \quad \|x - x_{\Lambda_k}\|_{\ell_2(\mathcal{J})} \leq C \|x\|_{\mathcal{A}^s}^{1/s} N_k^{-s} \quad \text{and} \quad \#\supp x \leq C \varepsilon^{-1/s} \|x\|_{\mathcal{A}^s}^{1/s}.$$ 

If $s$ is small enough, the computation of $x_{\Lambda_k}$ can be realized with a computational cost that is bounded by an absolute multiple of $\varepsilon^{-1/s} \|x\|_{\mathcal{A}^s}^{1/s}$, i.e., linear complexity. □

Theorem A.1 states that AWGMs are quasi-optimal in the sense that the optimal convergence rate for best $N$-term approximations of $x$ can be realized up to some constant within linear computational complexity. These techniques can be extended to problems that are neither symmetric nor positive-definite by considering the normal equations $A^T A x = A^T b$. This includes Petrov-Galerkin problems as they arise e.g. in space-time formulations of parabolic PDEs, even if the wavelet bases $\Psi^\mathcal{X}$, $\Psi^\mathcal{Y}$ for $\mathcal{X}$ and $\mathcal{Y}$ differ not only in scaling but are even obtained from different sets of wavelets [6, 15].
A.3. Multitree-based Implementations. Several different implementations of quasi-optimal AWGMs have been proposed. The algorithms in [7] use a thresholding step in order to retrieve the optimal computational complexity in Theorem [A.1] which in the case of [8] is combined with an inexact Richardson iteration on the infinite-dimensional equation (A.4). In [11] a residual approximation method is employed that does not require thresholding and can thus be proven to be more efficient. However, like the afore-mentioned algorithms it relies on the application of a so-called APPLY routine in order to approximate the arising infinite-dimensional matrix-vector products \( Av \in \ell_2(\mathcal{J}) \). Such routines are based on wavelet compression schemes, require certain characteristics of the wavelet bases as well as compressibility results for the operator \( A \) and are in general quantitatively demanding. For these reasons, we employ multitree-based matrix-vector product evaluations in the solution of (A.5) and the approximation of the residual \( r^k \), as proposed in [16, 17]. That is, we restrict the index sets \( \Lambda_k \) to multitrees in the sense of the following definition.

**Definition A.2.** (i) For a univariate uniformly local, piecewise polynomial wavelet basis \( \Psi = \{ \psi_\lambda : \lambda \in \mathcal{J} \} \), a set \( \Lambda \subset \mathcal{J} \) is called a tree if for any \( \lambda \in \Lambda \) with \( |\lambda| \geq 0 \) it holds that \( \text{supp} \psi_\lambda \subset \bigcup_{\mu \in \Lambda, |\mu| = |\lambda| - 1} \text{supp} \psi_\mu \). (ii) An index set \( \Lambda \in \mathcal{J} \) belonging to a tensor product wavelet basis \( \Psi = \{ \psi_\lambda : \lambda \in \mathcal{J} \} \) is called a multitree if for all \( i \in \{0, \ldots, n\} \) and all indices \( \mu_j \in \mathcal{J}^{(i)} \) for \( j \neq i \), the index set

\[
(\text{A.9}) \quad \Lambda^{(i)} := \{ \lambda_i \in \mathcal{J}^{(i)} : (\mu_0, \ldots, \mu_{i-1}, \lambda_i, \mu_{i+1}, \ldots, \mu_n) \in \Lambda \} \subset \mathcal{J}^{(i)}
\]

is either the empty set or a tree.

The restriction to such index sets preserves the quasi-optimality of the AWGM in the constrained approximation class \( A^s_{\text{mtree}} \) defined w.r.t. \( \|v\|_{A^s_{\text{mtree}}} := \sup_{\varepsilon > 0} \varepsilon \cdot \left[ \min_{\mathcal{N} \in \mathbb{N}_0} \| v - v_{\mathcal{N}} \|_{\ell_2(\mathcal{J})} \leq \varepsilon \wedge \text{supp} v_{\mathcal{N}} \text{ is a multitree} \right] \) and allows a computationally very efficient evaluation of finite-dimensional matrix-vector products:

**Theorem A.3 ([17] Theorem 3.1]).** Let \( A \) be a linear differential operator with polynomial coefficients and let \( \tilde{\Lambda} \subset \mathcal{J} \), \( \hat{\Lambda} \in \mathcal{J} \) be multitrees. Then, for any \( v_{\tilde{\Lambda}} \in \ell_2(\tilde{\Lambda}) \), the product \( \tilde{\Lambda} A_{\tilde{\Lambda}} v_{\tilde{\Lambda}} \) can be computed in \( O(\#\tilde{\Lambda} + \#\hat{\Lambda}) \) operations.

Moreover, we obtain the following approximation result for the residual:

**Theorem A.4 ([18]).** Let \( 0 < \omega < 1 \), let \( A \) be a differential operator with polynomial coefficients and let \( \mathbf{x} \in A^s_{\text{mtree}} \) for some \( s > 0 \). Then, for all finite multitrees \( \Lambda \subset \mathcal{J} \) and all \( w_{\hat{\Lambda}} \in \ell_2(\hat{\Lambda}) \), there exists a multitree \( \tilde{\Xi}(\Lambda, \omega) \subset \mathcal{J} \) such that for \( r := b_{\Xi} - \varepsilon A_{\hat{\Lambda}} w_{\hat{\Lambda}} \), it holds that \( \#\tilde{\Xi} \leq C \#\hat{\Lambda} + \| r \|_{\ell_2(\mathcal{J})}^{-1/s} \) and

\[
(\text{A.10}) \quad \| (b - A w_{\hat{\Lambda}}) - r \|_{\ell_2(\mathcal{J})} \leq \omega \| r \|_{\ell_2(\mathcal{J})}.
\]

Thus, the computational cost for the residual approximation is of the order \( O(\#\tilde{\Lambda} + \| r \|_{\ell_2(\mathcal{J})}^{-1/s}) \) if the right hand side coefficients \( b_{\Xi} \) can be computed efficiently. Explicit constructions of \( \tilde{\Xi} \) are discussed in [16] and [15], where the multitree-based AWGM is extended to the normal equations. In particular, such AWGM satisfies the conditions posed for the routine SOLVE in Section [3]. We used AWGM for all adaptive computations (snapshots, Riesz representations, error estimates).
References

[1] I. Babuška. Error-bounds for finite element method. Numer. Math., 16:322–333, 1970/1971.
[2] M. Barrault, Y. Maday, N.C. Nguyen, and A.T. Patera. An empirical interpolation method: application to efficient reduced-basis discretization of partial differential equations. C. R. Acad. Sci. Paris, Ser. I, 339(9):667 – 672, 2004.
[3] P. Binev, A. Cohen, W. Dahmen, R. DeVore, G. Petrova, and P. Wojtaszczyk. Convergence rates for greedy algorithms in reduced basis methods. SIAM Journal on Mathematical Analysis, 43(3):1457–1472, 2011.
[4] F. Brezzi. On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers. Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge, 8(R-2):129–151, 1974.
[5] T. Bui-Thanh, K. Willcox, and O. Ghattas. Model reduction for large-scale systems with high-dimensional parametric input space. SIAM J. Sci. Comput., 30(6):3270–3288, 2008.
[6] N. G. Chegini and R. P. Stevenson. Adaptive wavelet schemes for parabolic problems: sparse matrices and numerical results. SIAM J. Numer. Anal., 49(1):182–212, 2011.
[7] A. Cohen, W. Dahmen, and R. DeVore. Adaptive wavelet methods for elliptic operator equations: convergence rates. Math. Comput., 70(233):27–75, 2001.
[8] A. Cohen, W. Dahmen, and R. DeVore. Adaptive wavelet methods II - beyond the elliptic case. Found. Comput. Math., 2:203–245, 2002.
[9] R. Dautray and J.L. Lions. Mathematical analysis and numerical methods for science and technology. Evolution Problems I, volume 5. Springer-Verlag, Berlin, 1992.
[10] T. Dijkema. Adaptive tensor product wavelet methods for solving PDEs. PhD thesis, Universiteit Utrecht, 2009.
[11] T. Gantumur, H. Harbrecht, and R. P. Stevenson. An optimal adaptive wavelet method without coarsening of the iterands. Math. Comput., 76(258):615–629, 2007.
[12] M.A. Grepl and A.T. Patera. A posteriori error bounds for reduced-basis approximations of parameterized parabolic partial differential equations. ESAIM: Mathematical Modelling and Numerical Analysis, 39(1):157–181, 2005.
[13] B. Haasdonk and M. Ohlberger. Reduced basis method for finite volume approximations of parameterized linear evolution equations. ESAIM: Mathematical Modelling and Numerical Analysis, 42:277–302, 2008.
[14] S. Kestler. On the adaptive tensor product wavelet Galerkin method with applications in finance. PhD thesis, University of Ulm, 2013.
[15] S. Kestler, K. Steih, and K. Urban. An efficient space-time wavelet Galerkin method for time-periodic parabolic partial differential equations. submitted, 2013.
[16] S. Kestler and R. P. Stevenson. An efficient approximate residual evaluation in the adaptive tensor product wavelet method. J. Sci. Comp., 2013.
[17] S. Kestler and R. P. Stevenson. Fast evaluation of system matrices w.r.t. multi-tree collections of tensor product refinable basis functions. accepted for publication in J. Comput. Appl. Math., 2013.
[18] O. A. Ladyshenskaya. Funktionalanalytische Untersuchungen der Navier-Stokesschen Gleichungen. In deutscher Sprache herausgegeben von Joachim Förste. Mathematische Lehrbücher und Monographien, II. Abteilung, Mathematische Monographien, Band XIX. Akademie-Verlag, Berlin, 1965.
[19] J. Nečas. Sur une méthode pour résoudre les équations aux dérivées partielles du type elliptique, voisine de la variationnelle. Ann. Scuola Norm. Sup. Pisa (3), 16:305–326, 1962.
[20] R.H. Nochetto, K.G. Siebert, and A. Veeser. Theory of adaptive finite element methods: an introduction. In Multiscale, nonlinear and adaptive approximation, pages 409–542. Springer, Berlin, 2009.
[21] G. Rozza, D.B.P. Huynh, and A.T. Patera. Reduced basis approximation and a posteriori error estimation for affinely parametrized elliptic coercive partial differential equations. Archives of Computational Methods in Engineering, 15(3):229–275, 2008.
[22] A. Rupp. High Dimensional Wavelet Methods for Structures Financial Products. PhD thesis, University of Ulm, (2013).
[23] C. Schwab and R. P. Stevenson. Space-time adaptive wavelet methods for parabolic evolution problems. Math. Comput., 78(267):1293–1318, 2009.
[24] R. P. Stevenson. Adaptive wavelet methods for solving operator equations: An overview. In R. DeVore and A. Kunoth, editors, Multiscale, Nonlinear and Adaptive Approximation, pages 543–598. Springer (Berlin), 2009.

[25] K. Urban. Wavelet methods for elliptic partial differential equations. Oxford University Press, 2009.

[26] K Urban, S. Volkwein, and O. Zeeb. Greedy sampling using nonlinear optimization. In A. Quarteroni and G. Rozza, editors, Reduced Order Methods for modeling and computational reduction, pages 137–157. Springer Switzerland, 2014.

[27] M. Yano. A Reduced Basis Method with Exact-Solution Certificates for Symmetric Coercive Equations. Technical Report 64, MIT (USA), MechE, 2013. [http://augustine.mit.edu/methodology/methodology_technical_papers.html](http://augustine.mit.edu/methodology/methodology_technical_papers.html).

Kristina Steih, Institute for Numerical Mathematics, University of Ulm, Helmholtzstrasse 20, D-89069 Ulm, Germany
E-mail address: kristina.steih@uni-ulm.de

Karsten Urban, Institute for Numerical Mathematics, University of Ulm, Helmholtzstrasse 20, D-89069 Ulm, Germany
E-mail address: karsten.urban@uni-ulm.de