FINITELY GENERATED INVARIANTS OF HOPF ALGEBRAS ON FREE ASSOCIATIVE ALGEBRAS

VITOR O. FERREIRA AND LUCIA S. I. MURAKAMI

ABSTRACT. We show that the invariants of a free associative algebra of finite rank under a linear action of a finite-dimensional Hopf algebra generated by group-like and skew-primitive elements form a finitely generated algebra exactly when the action is scalar. This generalizes an analogous result for group actions by automorphisms obtained by Dicks and Formanek, and Kharchenko.

INTRODUCTION

Given a finite-dimensional vector space $V$ over a field $k$ and a finite subgroup $G$ of the group $\text{GL}(V)$ of all invertible linear operators on $V$, the action of the elements of $G$ on $V$ can be extended to the tensor algebra $T(V)$ of $V$ in a natural way. The elements of $G$ become, then, automorphisms of $T(V)$. We say that $G$ is a group of linear automorphisms of the algebra $T(V)$. Fixing a basis, say, $\{x_1, \ldots, x_r\}$, of $V$, $G$ can be regarded as a subgroup of the linear group $\text{GL}(r, k)$ and $T(V)$ becomes isomorphic to the free associative algebra $R = k\langle x_1, \ldots, x_r \rangle$.

The subalgebra $R^G$ of invariants of $R$ under the action of $G$, defined to be the set of all elements $f \in R$ such that $f^\sigma = f$, for all $\sigma \in G$, has been an object of interest for some time. In particular, questions regarding presentations for it have been addressed. Lane [11] and Kharchenko [7] have proved independently that $R^G$ is a free algebra over $k$ on a set of homogeneous elements. Somewhat later, Dicks and Formanek [3], and Kharchenko [9], again independently, showed that $R^G$ is a finitely generated algebra exactly when $G$ is a group of scalar matrices, and is, therefore, cyclic. Kharchenko’s proof has been later simplified by Dicks and this new argument appears in [2, Theorem 10.4] and in [13, Theorem 32.7].

Lie algebras of derivations of free algebras have a similar behavior with regards to constants. More precisely, given a finite-dimensional restricted Lie algebra $L$ of linear derivations of $R$, Jooste [6] and, independently, Kharchenko [8] have proved that the subalgebra of constants $R^L = \{f \in R : f^\delta = 0, \text{ for all } \delta \in L\}$ is free over $k$ on a set of homogeneous generators. It was then natural to consider the question regarding finite generation of $R^L$. It follows from the work of Koryukin [10] that exactly the same situation holds: $R^L$ is finitely generated as an algebra if and only if $L$ consists entirely of scalar derivations. In [4], the authors show that Dicks’ proof for the automorphism case can be adapted to take into account derivations.

Actions of groups by automorphisms and of Lie algebras by derivations are instances of Hopf algebra actions on rings. In fact, it was proved in [2] that a free algebra under a homogeneous action by a Hopf algebra has free invariants. In

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the present paper we address the question of finite generation of the subalgebra of invariants. We show that whenever a finite-dimensional Hopf algebra which is generated by group-like and skew-primitive elements acts in a linear fashion on a free algebra, the subalgebra of invariants is a finitely generated subalgebra if and only if the action is scalar. The proof of this fact is based on Dicks’ proof for the automorphisms case.

1. Notation

In this section we fix notation and nomenclature. Let $X$ be a nonempty set and let $F = \langle X \rangle$ denote the free monoid on $X$. Let $k$ be a field and let $R = k\langle X \rangle$ denote the free associative algebra on $X$ over $k$. The algebra $R$ is then a vector space over $k$ with basis $F$. As usual, given $f = \sum_{w \in F} \lambda_w w \in R$, with $\lambda_w \in k$, for all $w \in F$, the support of $f$ is defined to be the following subset of $F$:

$$\text{supp}(f) = \{w \in F : \lambda_w \neq 0\}.$$ 

The free algebra $R$ can be graded by the usual degree function on $R$, that is, $R = \bigoplus_{n \geq 0} R_n$, where $R_n$ stands for the linear span of all the monomials of length $n$.

Let $H$ be a Hopf algebra and suppose that $H$ acts on $R$, that is, suppose that $R$ is an $H$-module algebra (see [14] or [12]). We say that the action of an element $h \in H$ on $R$ is linear if it induces a linear operator on the vector space $V = \sum_{x \in X} k x$. In other words, the action of $h$ on $R$ is linear if for every $y \in X$, there exist scalars $\eta_{xy}(h) \in k$, all but a finite number of which nonzero, such that

$$h \cdot y = \sum_{x \in X} \eta_{xy}(h) x.$$ 

Furthermore, we say that the action of $h$ on $R$ is scalar if it is linear, $\eta_{xx}(h) = \eta_{yy}(h)$, for all $x, y \in X$, and $\eta_{xy}(h) = 0$, for all $x, y \in X$ with $x \neq y$. The action of $H$ on $R$ is said to be linear if all $h \in H$ act linearly on $R$ and it is said to be scalar if it is linear and all the elements of $H$ act scalarly on $R$. If $X$ is finite, say $X = \{x_1, \ldots, x_r\}$, given $h \in H$, we often write $\eta_{ij}(h)$ for $\eta_{x_i x_j}(h)$ and write $[h]_X$ for the matrix $[\eta_{ij}(h)] \in M_r(k)$. In this case, the action of an element $h \in H$ on $R$ is scalar if it is linear and $[h]_X$ is a scalar matrix, say $[h]_X = \eta I_r$, for some $\eta \in k$, where $I_r$ stands for the $r \times r$ identity matrix. When this is the case we shall say that the action of $h$ is based on $\eta$.

If $H$ is a Hopf algebra with counit $\varepsilon$ and if $H$ acts linearly on $R$, then the subalgebra of invariants of $R$ under the action of $H$, defined by $R^H = \{f \in R : h \cdot f = \varepsilon(h)f, \text{ for all } h \in H\}$, is clearly a graded subalgebra. Hence if $f \in R^H$, then all of the homogeneous components of $f$ lie in $R^H$.

Notation for Hopf algebra theory will be the usual (see [14] or [12]), including Sweedler’s notation $\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}$, for the comultiplication $\Delta$ on an element $h \in H$.

Recall that an element $\sigma$ in a Hopf algebra $H$ is called a group-like element if $\sigma \neq 0$ and $\Delta(\sigma) = \sigma \otimes \sigma$. The set of group-like elements is linearly independent over $k$ and forms a group under multiplication. Given group-like elements $\sigma, \tau \in H$, an element $\delta \in H$ is said to be $(\sigma, \tau)$-primitive if $\Delta(\delta) = \delta \otimes \sigma + \tau \otimes \delta$. We say simply that $\delta$ is a skew-primitive element if the reference to $\sigma$ and $\tau$ is not necessary. A primitive element is just a $(1,1)$-primitive element. Finally if $\sigma$ is a group-like
element, then \( \varepsilon(\sigma) = 1 \), whereas if \( \delta \) is a skew-primitive element, then \( \varepsilon(\delta) = 0 \). (For these and other basic facts on Hopf algebras we refer to [14].)

2. Scalar actions

We start by showing that finite generation of the invariants of a linear action of a pointed Hopf algebra is a consequence of the action being scalar.

**Theorem 1.** Let \( k \) be a field and let \( H \) be a pointed Hopf \( k \)-algebra which acts linearly on the free algebra \( R = k\langle X \rangle \) on a finite set \( X \) over \( k \). If the action of \( H \) on \( R \) is scalar, then \( R^H \) is a finitely generated subalgebra.

**Proof.** By hypothesis, the action of each \( h \in H \) is based on some \( \lambda_h \in k \). Then, for \( x, y \in X \),
\[
h \cdot (xy) = \sum_{(h)} (h_{(1)} \cdot x)(h_{(2)} \cdot y) = \left( \sum_{(h)} \lambda_{h_{(1)}} \lambda_{h_{(2)}} \right) xy
\]
and, by induction on \( n \), we have, for \( x_1, x_2, \ldots, x_n \in X \),
\[
h \cdot (x_1 x_2 \ldots x_n) = \left( \sum_{(h)} \lambda_{h_{(1)}} \lambda_{h_{(2)}} \ldots \lambda_{h_{(n)}} \right) x_1 x_2 \ldots x_n.
\]
So, if \( g \in k\langle X \rangle \) is invariant and homogeneous of degree \( t \), say \( g = \sum_{i=1}^{r} \mu_i w_i \), where \( \mu_i \in k \) and \( w_i \) are monomials of degree \( t \), we have
\[
\varepsilon(h)g = h \cdot g = \sum_{i=1}^{r} \mu_i \left( \sum_{(h)} \lambda_{h_{(1)}} \ldots \lambda_{h_{(t)}} \right) w_i = \left( \sum_{(h)} \lambda_{h_{(1)}} \ldots \lambda_{h_{(t)}} \right) g.
\]
Therefore, if there exists a homogeneous invariant element of degree \( t \), then \( \varepsilon(h) = \sum_{(h)} \lambda_{h_{(1)}} \ldots \lambda_{h_{(t)}} \), for all \( h \in H \). It follows that every monomial of degree \( t \) is invariant.

If \( R^H = k \), then \( R^H \) is trivially finitely generated. Otherwise, there exist invariant monomials of degree \( \geq 1 \). Let \( t \) the least positive integer such that there exist an invariant monomial of degree \( t \). We shall show that \( R^H \) is generated by the set of all monomials of degree \( t \). Let \( m > t \) and let \( w = x_1 \ldots x_m \) be an invariant monomial of degree \( m \). Write \( m = qt + r \), with \( 0 \leq r < t \), and write \( w = w_u \), with \( u \) and \( v \) monomials of degrees \( qt \) and \( r \), respectively. Observe that \( u \) is invariant, because it is a product of \( q \) monomials of degree \( t \), all of which are invariant. Let \( \{H_n\} \) be the coradical filtration of \( H \). We shall proceed by induction on \( n \). Since \( H \) is pointed, \( H_0 = kG \), where \( G \) stands for the set of all group-like elements of \( H \). For each \( \sigma \in G \), we have
\[
w = \varepsilon(\sigma)w = \sigma \cdot w = (\sigma \cdot u)(\sigma \cdot v) = u\lambda_\sigma v = \lambda_\sigma^* v.
\]
So \( \lambda_\sigma^* = 1 \). Therefore, \( \sigma \cdot v = \varepsilon(\sigma)v \), which means that \( v \) is invariant under the action of \( H_0 \). Let \( n > 0 \) and suppose that \( v \) is invariant under the action of \( H_i \), for all \( i = 0, 1, \ldots, n-1 \). By the Taft–Wilson Theorem (see [12] Theorem 5.4.1), \( H_n \) is generated by elements \( h \) satisfying \( \Delta(h) = h \otimes \sigma + \tau \otimes h + \sum l_i \otimes s_i \), with \( \sigma, \tau \in G \) and \( l_i, s_i \in H_{n-1} \). Since \( \varepsilon(h) = \sum_{(h)} \varepsilon(h_{(1)} h_{(2)}) \), we have \( \varepsilon(h) = 2\varepsilon(h) + \sum \varepsilon(l_i s_i) \),
so, \( \sum \varepsilon(l_i s_i) = -\varepsilon(h) \). Using the fact that \( w, u \in R^H \) and \( v \) is invariant under \( H_i \), for \( i < n \), we obtain

\[
\varepsilon(h) uv = \varepsilon(h) w = h \cdot w
= (h \cdot u)(\sigma \cdot v) + (\tau \cdot u)(h \cdot v) + \sum (l_i \cdot u)(s_i \cdot v)
= \varepsilon(h) w + u(h \cdot v) + \sum \varepsilon(l_i) \varepsilon(s_i) w
= \varepsilon(h) w + u(h \cdot v) - \varepsilon(h) w
= u(h \cdot v).
\]

Hence \( h \cdot v = \varepsilon(h)v \). We conclude that \( v \) is an invariant monomial of degree \( r \). By the minimality of \( t \), it follows that \( r = 0 \) and, thus, \( m \) is a multiple of \( t \).

Therefore \( R^H \) is generated by the set of all monomials of degree \( t \) and, since \( X \) is a finite set, \( R^H \) is finitely generated. \( \square \)

3. Finitely generated invariants

In the proof of [13, Theorem 32.7] and of the main theorem of [14], essential use of the fact that there exist invariants and constants, respectively, containing monomials with arbitrary initial segments in their support is made. In each case, an appropriate power of a standard polynomial is shown to possess such a property. This combinatorial ingredient is no longer at our disposal in the Hopf algebra context, for the action of the symmetric group does not commute with the action of skew-derivations. Nevertheless, we can resort to a substitute construction resulting from the lemma below.

We start by introducing some notation. For each positive integer \( n \), let \( c_n(Y, Z) \) be the following element of the algebra \( \mathbb{k}[Y, Z] \) of commutative polynomials over the field \( \mathbb{k} \) in the indeterminates \( Y \) and \( Z \),

\[
c_n(Y, Z) = \sum_{i=0}^{n-1} Y^{n-1-i} Z^i.
\]

Note that \( c_n(Y, Z)(Y - Z) = Y^n - Z^n \).

In what follows, \( X \) will denote a nonempty set, \( R = \mathbb{k}(X) \) the free algebra on \( X \) over \( \mathbb{k} \), and \( H \) a Hopf \( \mathbb{k} \)-algebra which acts linearly on \( R \). Given \( w \in \mathcal{F} \), we shall write \( w\mathcal{F} \) for the set of monomials in \( \mathcal{F} \) that are left divisible by \( w \), that is,

\[
w\mathcal{F} = \{wu : u \in \mathcal{F}\}.
\]

**Lemma 2.** Suppose that \( X \) is finite, say \( X = \{x_1, \ldots, x_r\} \), let \( \sigma, \tau \in H \) be group-like elements and let \( \delta \in H \) be a \((\sigma, \tau)\)-primitive element. Suppose that \( \sigma \) and \( \tau \) act scalarly on \( R \), based on \( \eta \) and \( \mu \), respectively, and that \([\delta]_X \) is in Jordan normal form. Then for each positive integer \( n \) and each \( i = 1, \ldots, r \), there exists a nonzero \( f \in R \) satisfying

1. \( \delta \cdot f = c_n(\eta, \mu) g \), for some \( g \in R \), and
2. \( \text{supp}(f) \cap x_i \mathcal{F} \neq \emptyset \).

**Proof.** Fix an \( i = 1, \ldots, r \) and let \( J \) be the Jordan block of \([\delta]_X \) containing the eigenvalue \( \lambda \) on the \( i \)-th diagonal entry of \([\delta]_X \). Let \( s \) be the positive integer such that the last occurrence of \( \lambda \) in \( J \) lies on the \( s \)-th diagonal entry of \([\delta]_X \). The figure below might help illustrate the choices of \( J \) and \( s \).
Then the element
\[ f = \sum_{j_1 + \cdots + j_n = i + (n-1)s} x_{j_1} \cdots x_{j_n} \]
satisfies both \( \text{(1)} \) and \( \text{(2)} \). Indeed, it is easily checked that
\[
\delta \cdot (x_{j_1} x_{j_2} \cdots x_{j_n}) = \lambda c_n(\eta, \mu) x_{j_1} x_{j_2} \cdots x_{j_n} \\
+ \eta^{n-1} x_{j_1+1} x_{j_2} \cdots x_{j_n} \\
+ \eta^{n-2} \mu x_{j_1} x_{j_2+1} \cdots x_{j_n} \\
+ \cdots \\
+ \eta \mu^{n-2} x_{j_1} \cdots x_{j_{n-1}+1} x_{j_n} \\
+ \mu^{n-1} x_{j_1} \cdots x_{j_{n-1}} x_{j_n+1},
\]
where each term in which there is an occurrence of \( x_{s+1} \) should be interpreted as being zero. Therefore, \( \delta \cdot f = \lambda c_n(\eta, \mu) f + f' \), where \( f' \) is an element of \( R \) with
\[
\text{supp}(f') \subseteq \{ x_{k_1}, \ldots, x_{k_n} : k_1 + \cdots + k_n = (i + 1) + (n-1)s \\
\text{and } k_q \leq s, \text{ for all } q = 1, \ldots, n \}.
\]
Note that the restrictions \( k_1 + \cdots + k_n = (i + 1) + (n-1)s \) and \( k_q \leq s \), for all \( q = 1, \ldots, n \), for the elements \( x_{k_1} \cdots x_{k_n} \) in \( \text{supp}(f') \) imply \( k_q \geq 2 \), for all \( q = 1, \ldots, n \). Therefore, the element \( x_{k_1} \cdots x_{k_n} \) occurs in the supports of the image of the action of \( \delta \) on \( x_{k_1-1} x_{k_2} \cdots x_{k_n} \), on \( x_{k_1} x_{k_2-1} \cdots x_{k_n} \), and on \( x_{k_1} x_{k_2} \cdots x_{k_n-1} \) with coefficients \( \eta^{n-1}, \eta^{n-2} \mu, \ldots, \) and \( \mu^{n-1} \), respectively. Hence, \( f' = c_n(\eta, \mu) f'' \), for some \( f'' \in R \). It follows that \( \delta \cdot f = c_n(\eta, \mu)(\lambda f + f'') \). Thus, \( f \) satisfies \( \text{(1)} \). For \( \text{(2)} \), note that, by definition of \( f \), \( x_{i} x_{s} \cdots x_{s} \in \text{supp}(f) \). ∎

For the next result, we need the following definition by Koryukin. Given non-negative integers \( i, j, k \), let \( \tau_{ijk} \) be the linear operator of \( R_{i+j+k} \) which, on monomials \( u \in R_i, v \in R_j \), and \( w \in R_k \), satisfies \( \tau_{ijk}(uvw) = uvw \). A graded subalgebra \( S \)

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1The authors thank Jair Donadelli Jr. for this simple argument.
of $R$ is called an **algebra with inserts** if $\tau_{ijk}(S_{i+j}S_k) \subseteq S_{i+j+k}$, for all non-negative $i,j,k$. The next result is due to Koryukin. We include a proof for the reader’s convenience.

**Proposition 3** ([10, Lemma 1.6]). The subalgebra of invariants $R^H$ is an algebra with inserts.

**Proof.** Let $i,j,k$ be non-negative integers, let $u$ and $v$ be monomials of degrees $i$ and $j$, respectively, and let $g \in R^H_k$. Then for all $h \in H$, we have

$$h \cdot \tau_{ijk}(uvg) = h \cdot (uvg) = \sum_{(h)} (h(1) \cdot u)(h(2) \cdot g)(h(3) \cdot v)$$

$$= \sum_{(h)} (h(1) \cdot u)(h(2)g)(h(3) \cdot v)$$

$$= \sum_{(h)} (h(1) \cdot u)(h(2) \cdot v) = \tau_{ijk}((h \cdot (uv))g),$$

since the action is linear. Hence, by the linearity of $\tau_{ijk}$, we have, for all $f \in R_{i+j}$ and $g \in R^H_k$,

$$h \cdot \tau_{ijk}(fg) = \tau_{ijk}((h \cdot f)g).$$

Therefore, if $f \in R^H_{i+j}$ and $g \in R^H$ then $\tau_{ijk}(fg) \in R^H$. □

**Corollary 4.** Let $x \in X$. If there exists $f \in R^H$ with $\text{supp}(f) \cap xF \neq \emptyset$ then, for each positive integer $k$, there exists $\bar{f} \in R^H$ such that $\text{supp}(\bar{f}) \cap xkF \neq \emptyset$.

**Proof.** We prove this fact by induction on $k$. We can assume that $f$ is homogeneous, for the action of $H$ is linear. Let $d$ be the degree of $f$ and let $xw \in \text{supp}(f)$. For $k = 1$, there is nothing to prove. Suppose $k > 1$ and assume that there exists a homogeneous $\bar{f} \in R^H$ of degree $t$, with $\text{supp}(\bar{f}) \cap x^{k-1}F \neq \emptyset$, say $x^{k-1}\bar{w} \in \text{supp}(\bar{f})$. Then $\bar{f} = \tau_{k-1,t-k+1,d}(\bar{f}f)$ is invariant, by the previous lemma, and $x^kw\bar{w} \in \text{supp}(\bar{f})$. □

The following is a well known fact. It follows, for instance, from [12, Lemma 5.5.1].

**Proposition 5.** A Hopf algebra which is generated by group-like and skew-primitive elements is pointed.

Finally, we shall make use of the following Galois correspondence.

**Correspondence Theorem** ([5, Theorem 1.2]). Let $k$ be a field, let $X$ be a set with $|X| > 1$, and let $R = k\langle X \rangle$ be the free algebra on $X$ over $k$. Let $H$ be a finite-dimensional pointed Hopf $k$-algebra which acts faithfully and linearly on $R$. Then there exists an inclusion-inverting one-to-one correspondence between the set of all free subalgebras of $R$ containing $R^H$ and the set of all right coideal subalgebras of $H$.

We are ready to state and prove the main result of the paper.

**Theorem 6.** Let $k$ be a field, let $X$ be a nonempty set, and let $R = k\langle X \rangle$ be the free algebra on $X$ over $k$. Let $H$ be a finite-dimensional Hopf $k$-algebra which acts faithfully and linearly on $R$. Suppose that $H$ is generated by group-like and skew-primitive elements. Then the subalgebra of invariants $R^H$ is finitely generated if and only if $X$ is finite and the action of $H$ on $R$ is scalar.
Proof. If $X$ is finite and the action of $H$ on $R$ is scalar then $R^H$ is finitely generated by Theorem 1. Conversely, suppose $R^H$ is finitely generated. Then there exist $x_1, \ldots, x_n \in X$ such that $R^H \subseteq k(x_1, \ldots, x_n)$. If $X$ were infinite there would exist an infinite chain
\[
R^H \subseteq k(x_1, \ldots, x_n) \subsetneq k(x_1, \ldots, x_n, x_{n+1}) \subsetneq \ldots
\]
of free subalgebras of $R$ containing $R^H$. By the Correspondence Theorem, there would also exist an infinite chain of subalgebras of $H$, which is impossible, since $H$ is finite-dimensional. Therefore, $X$ is finite.

If $X$ contains only a single element, then any linear action is scalar. Thus we can assume that $|X| > 1$, say $X = \{x_1, \ldots, x_r\}$. By Proposition 1.1 $H$ is pointed, so its coradical is $H_0 = kG$, where $G$ is the set of all group-like elements of $H$. By the Correspondence Theorem, $R_0 = R$ is a free subalgebra of $R$ containing $R^H$ and, by the same argument as above, $R^H$ is finitely generated. Since $H$ is finite-dimensional, $G$ is finite. So, by [3, Theorem 5.3], the action of $H_0$ on $R$ is scalar.

Since $H$ is generated by $G$ and skew-primitive elements, it remains to show that each skew-primitive element acts scalarly on $R$. We start by remarking that it can be assumed that $k$ is algebraically closed, for if $\overline{k}$ denotes the algebraic closure of $k$, then the action of $H$ on $R$ induces an action of $\overline{H} = \overline{k} \otimes_k H$ on $\overline{R} = \overline{k} \otimes_k R$ such that $\overline{R}^H = \overline{k} \otimes_k R^H$ is finitely generated.

Let $\delta \in H$ be a $(\sigma, \tau)$-primitive element, where $\sigma, \tau$ are group-like elements with actions based on $\eta$ and $\mu$, respectively. Consider the subalgebra $H(\delta)$ of $H$ generated by $\{\delta, \sigma, \tau\}$. Then $H(\delta)$ is clearly a Hopf subalgebra of $H$ and, by the Correspondence Theorem, the subalgebra of invariants $R^{H(\delta)}$ of $R$ under $H(\delta)$ is finitely generated. Now, under our assumption that $k$ is algebraically closed, we can further assume that $[\delta]_X$ is in Jordan normal form, since any basis for the vector space $V = \sum_{x \in X} kx$ is a set of free generators of $R$ giving rise to the same grading.

We shall start by showing that $[\delta]_X$ is a diagonal matrix. Suppose otherwise. After reordering the basis, if necessary, we would have $\delta \cdot x_1 = \lambda x_1 + x_2$ and $\delta \cdot x_2 = \lambda x_2 + x_3$, where $\lambda, \zeta \in k$, $\zeta = 0$ or $\zeta = 1$, and $x_1, x_2 \notin \text{supp}(\delta \cdot x_i)$, for $i \geq 3$. Let $A$ be the set of all monomials different from 1 which occur in the support of the elements of a finite set of generators for $R^{H(\delta)}$. We claim that there is an $m \geq 1$ such that $x_1^m \in A$. In fact, it is sufficient to show that there exists $\phi \in R^{H(\delta)}$ satisfying
\[
(*) \quad \text{supp}(\phi) \cap x_1 F \neq \emptyset
\]
Indeed, having such an invariant $\phi$, using Corollary 2.3 we can produce $\tilde{\phi} = R^{H(\delta)}$ with $\text{supp}(\tilde{\phi}) \cap x_1^k F \neq \emptyset$, say $x_1^k w \in \text{supp}(\tilde{\phi})$, where $k$ is an integer greater than the degrees of the elements of $A$. Since $x_1^k w$ is a product of elements of $A$ we must have $x_1^m \in A$, for some $m \geq 1$.

In order to exhibit such an invariant element, let $f \in R$ be the element obtained in Lemma 2.2 with $i = 1$ and $n = |G|$. We have $\delta \cdot f = c_0(\eta, \mu) g$, for some $g \in R$, and $\text{supp}(f) \cap x_1 F \neq \emptyset$. Since $\sigma, \tau \in G$, we have $\sigma^n = \tau^n = 1$ and, therefore, $\eta^n = \mu^n = 1$. If, on the other hand, $\eta \neq \mu$, it follows that $c_0(\eta, \mu) = 0$; hence $\delta \cdot f = 0 = \varepsilon(h) f$, that is, $f$ is invariant under $\delta$. Since $f$ is homogeneous of degree $n$ it also follows that $\sigma \cdot f = \tau \cdot f = f$. Thus $f \in R^{H(\delta)}$. So $\phi = f$ satisfies (1). On the other hand, if $\eta = \mu$, then $\sigma = \tau$, for the action of $H$ on $R$ is faithful. In this case $\sigma^{-1} \delta$ is a primitive element. Therefore, the characteristic of $k$ must be positive,
otherwise, the subalgebra of $H$ generated by $\sigma^{-1}\delta$ would be infinite-dimensional. Since $c_n(\eta, \eta) = n\eta^{n-1}$, we have $\delta \cdot f^k = kn\eta^{n+k-2}\delta^k$, for every $k \geq 1$. If $p$ is the characteristic of $k$, then $\delta \cdot f^p = 0 = (\delta h) f^p$, that is, $f^p$ is invariant under $\delta$. Again, because it is homogeneous of degree $pn$, $f^p$ is also invariant under $\sigma$ and $\tau$. So, a fortiori, $f^p \in R^H(\delta)$. In this case $\phi = f^p$ is an invariant element satisfying (2).

Now take a homogeneous $z \in R^H(\delta)$ of degree $m$ with $x^m_1 \in \text{supp}(z)$ and write $z = x^m_1 + \nu x^{m-1}_1 x_2 + \bar{z}$, with $\nu \in k$ and $x^m_1, x^{m-1}_1 x_2 \not\in \text{supp}(\bar{z})$. Then
\[
\delta \cdot z = \lambda c_m(\eta, \mu)x^m_1 + \mu^{-1}x^{m-1}_1 x_2 \\
+ \nu \lambda c_{m-1}(\eta, \mu)x^{m-1}_1 x_2 + \nu \lambda \mu^{-1}x^{m-1}_1 x_2 + \bar{z}
\]
where $x^m_1, x^{m-1}_1 x_2 \not\in \text{supp}(\bar{z})$. Since $\delta \cdot z = 0$, by comparing coefficients, we get $\lambda c_m(\eta, \mu) = 0$ and $0 = \nu \lambda(\eta c_{m-1}(\eta, \mu) + \mu^{-1}) + \mu^{-1} = \nu \lambda c_m(\eta, \mu) + \mu^{-1}$, which implies $\mu = 0$, a contradiction. Therefore $[\delta]_x$ must be a diagonal matrix. So, there exist $\lambda_1, \ldots, \lambda_r \in k$ such that $\delta \cdot x_i = \lambda_i x_i$, for $i = 1, \ldots, r$.

In order to show that all these $\lambda_i$ are equal, we observe that there exists $w \in F$ such that $wx_i \in A$ for every $i = 1, \ldots, r$, otherwise we would be able to construct a monomial $\bar{w}$ of degree $k$ (where $k$ was chosen to be an integer greater than the degrees of all the elements of $A$) whose initial segments would all lie outside $A$. However this could not happen, since, for each $i = 1, \ldots, r$, there exists $f_i \in R^H(\delta)$ with $x_i \bar{w} \cap \text{supp}(f_i) \neq \emptyset$ (this can be proved in the same way as we have done for $i = 1$ using Lemma 2). But because $R^H(\delta)$ is an algebra with inserts we would eventually be able to produce an $f \in R^H(\delta)$ with $\bar{w} \cap \text{supp}(f) \neq \emptyset$. Now this would imply that some initial segment of $\bar{w}$ should be an element of $A$; a contradiction. Write $w = x_i x_{i_1} \ldots x_{i_t}$. So $\delta \cdot w = \xi w$, where $\xi = \sum_{j=1}^{t} \lambda_{i_j} \eta^{t-j} \mu^{j-1}$. Therefore $0 = \delta \cdot (wx_i) = \xi \eta \mu^{t(t-1)}$, for every $\lambda_i = -\mu^{t-1} \eta \mu^{t-1}$. Hence, the action of $\delta$ is scalar.

We believe the hypothesis on $H$ being generated by group-like and skew-primitive elements not to be too restrictive, for a great number of finite-dimensional pointed Hopf algebras do have this property. In fact, it has been conjecture in [1, 1.4] that all finite-dimensional pointed Hopf algebras over an algebraically closed field of characteristic 0 are generated as algebras by group-like and skew-primitive elements. In view of this fact it would not seem unreasonable to believe that Theorem [4] holds for arbitrary finite-dimensional pointed Hopf algebras, but we do not have a proof for this fact.

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Department of Mathematics - IME, University of São Paulo, Caixa Postal 66281, São Paulo, SP, 05311-970, Brazil.
E-mail address: vofer@ime.usp.br

Department of Mathematics - IME, University of São Paulo, Caixa Postal 66281, São Paulo, SP, 05311-970, Brazil.
E-mail address: ikemoto@ime.usp.br