A Design of Space-Time Block Codes
Achieving Full Diversity with Partial
Interference Cancellation Group Decoding

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Abstract

A partial interference cancellation (PIC) group decoding based space-time block code (STBC) design criterion was recently proposed by Guo and Xia, where the decoding complexity and the code rate trade-off is dealt when the full diversity is achieved. In this paper, a systematic design of STBC is proposed for any number of transmit antennas that can obtain full diversity when a PIC group decoding (with a particular grouping scheme) is applied at receiver. The proposed STBC are designed from multiple diagonal layers and each layer is composed of a fixed number of coded symbols which are encoded from a cyclotomic lattice design. With the PIC group decoding and an appropriate grouping scheme for the decoding, the proposed STBC are shown to obtain the same diversity gain as the ML decoding, but have a much less decoding complexity compared to the ML decoding. Moreover, the code rate of the proposed STBC can be up to full, i.e., \( M \) symbols per channel use for an MIMO system with \( M \) transmit antennas when the codeword length is sufficiently large. Some code design examples are given from the systematic code design approach. Simulation results show that the newly proposed STBC can obtain full diversity over Rayleigh fading channels and outperform some existing codes given the same bandwidth efficiency.

Index Terms

Diversity technique, space-time block codes, linear receiver, partial interference cancellation, group decoding.

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I. Introduction

Multiple-input multiple-output (MIMO) systems have been under active consideration in the last decade because of their potential for achieving higher data rate and providing more reliable reception performance compared with traditional single-antenna systems for wireless communications. Space-time (ST) coding is a bandwidth-efficient transmission technique that can improve the reliability of data transmission in MIMO wireless systems [1], [2]. It encodes a data stream across different transmit antennas and time slots, so that multiple redundant copies of the data stream can be transmitted through independent fading channels. Orthogonal space-time block coding (OSTBC) is one of the most attractive ST coding approaches because the special structure of orthogonality guarantees a full diversity and a simple (linear) maximum-likelihood (ML) decoding. The first OSTBC design was proposed by Alamouti in [1] for two transmit antennas and was then extended by Tarokh et al. in [2] for any number of transmit antennas. A class of OSTBC from complex design with the code rate of $1/2$ was also given by Tarokh et al. in [2]. Later, systematic constructions of complex OSTBC of rates $(k + 1)/(2k)$ for $M = 2k - 1$ or $M = 2k$ transmit antennas for any positive integer $k$ were proposed in [3]–[5]. However, the OSTBC has a low code rate not more than $3/4$ for more than two transmit antennas [6].

To enhance the transmission rate of the STBC, various STBC design approaches were proposed such as quasi-OSTBC [7]–[9], [11]–[13], [15]–[18] and algebraic number theory based STBC [19]–[27]. The quasi-OSTBC increases the code rate by relaxing the orthogonality condition on the code matrix, which was originally proposed in [7], [8], and [9], independently. The main idea of the quasi-OSTBC is to keep the orthogonality among the column pairs of the code matrix. Due to the group orthogonality, the ML decoding is performed pair-wise or group-wise with an increased complexity compared to the single-symbol decoding. Note the original quasi-OSTBC proposed in [7]–[9] do not have full diversity property. This problem was later dealt with by a constellation rotation of information symbols in [10]–[12]. By judiciously choosing a rotation angle for any given signal constellations on square lattices or equal-literal triangular lattices [12], the quasi-OSTBC can achieve both the maximum diversity and maximum coding gain. Later, in [14]–[16], quasi-OSTBC was studied in the sense of minimum decoding complexity, i.e., a real pair-wise symbols decoding. In [16]–[18], the pair-wise decoding was generalized to a general group-wise decoding. The decoding for these codes is the ML decoding and their rates
are basically limited by that of OSTBC. The algebraic number theory based STBC are designed mainly based on the ML decoding that may have high complexity and even though some near-ML decoder, such as sphere decoder [28] can be used, the expected decoding complexity is still dominated by polynomial terms of a number of symbols which are jointly detected [29].

To reduce the large decoding complexity of the high rate STBC aforementioned, several fast-decodable STBC were recently proposed [30] [31]. The STBC proposed in [30] achieves a high rate and a reduced decoding complexity at the cost of loss of full diversity. The fast-decoable STBC in [31] can obtain full rate, full diversity and the reduced ML decoding complexity, but the code design is limited to $2 \times 2$ and $4 \times 2$ MIMO transmissions only. Another new perspective of reducing the decoding complexity was recently considered in [33] and [34] to resort to conventional linear receivers such as zero-forcing (ZF) receiver or minimum mean square error (MMSE) receiver instead of the ML receiver to collect the full diversity. The outage and diversity of linear receivers in flat-fading MIMO channels were studied in [32], but no explicit code design was given to achieve the full diversity when the linear receivers are used. Based on the new STBC design criterion for MIMO systems with linear receivers, Toeplitz STBC [33] and overlapped-Alamouti codes [34] were proposed and shown to achieve the full diversity with the linear receivers. However, it is shown in [34] that the code rate of STBC achieving full diversity with linear receivers is upper bounded by one. Later, Guo and Xia proposed a partial interference cancellation (PIC) group decoding scheme [35] which can be viewed as an intermediate decoding approach between the ML receiver and the ZF receiver by trading a simple single-symbol decoding complexity for a high code rate larger than one symbol per channel use. Moreover, in [35] an STBC design criterion was derived to achieve full diversity when the PIC group decoding is applied at the receiver. A few code design examples were presented in [35], but a general design of STBC achieving full diversity with the PIC group decoding remains an open problem.

In this paper, we propose a systematic design of STBC which can achieve full diversity with the PIC group decoding for any number of transmit antennas. The proposed STBC have a structure of multiple diagonal layers and for each diagonal layer there are exactly $M$ coded symbols embedded, being equal to the number of transmit antennas, which are obtained from a cyclotomic lattice design. Indeed, each diagonal layer of the coded symbols can be viewed as the conventional rate-one diagonal STBC [36], [37]. The code rate of the proposed STBC can
be from one to $M$ symbols per channel use by adjusting the codeword length, i.e., embedding different number of layers in the code matrix. If only one layer is embedded in the code matrix, the proposed STBC is equivalent to the conventional diagonal STBC and the code rate is one. If a sufficiently large number of layers are embedded in the code matrix, the code rate can be almost equal to $M$ symbols per channel use. However, the complexity of the PIC group decoding is irrespective of the code rate and keeps a similar complexity as the code with one layer only. In other words, the complexity of the proposed STBC with the PIC group decoding is similar to the complexity of the ML decoding of $M$ independent information symbols jointly. For the PIC group decoding of the proposed STBC, we propose a grouping scheme in which every $M$ neighboring columns of the equivalent channel matrix are clustered into one group. Some code design examples are shown. Simulation results demonstrate that the proposed STBC with the PIC group decoding can obtain the full diversity.

This paper is organized as follows. A system model of ST transmission over MIMO channels with the PIC group decoding is introduced in Section II. In Section III a systematic design of high rate full diversity STBC with the PIC group decoding is proposed. Its asymptotic full-rate is derived and full diversity is shown. Several code design examples are given in Section IV. Simulation results are presented in Section V. Finally, in Section VI we draw our conclusions.

**Notations:** Column vectors (matrices) are denoted by boldface lower (upper) case letters. Superscripts $^t$ and $^H$ stand for transpose and conjugate transpose, respectively. $\mathbb{C}$ denotes the field of complex numbers. $I_n$ denotes the $n \times n$ identity matrix, and $0_{m \times n}$ denotes the $m \times n$ matrix all of whose elements are 0. vec$(X)$ is the vectorization of matrix $X$ by stacking the columns of $X$ on top each other.

**II. System Model and PIC Group Decoding**

In this section, we first briefly describe the system model and then describe the PIC group decoding proposed in [35].

**A. System Model**

We consider a MIMO transmission with $M$ transmit antennas and $N$ receive antennas over block fading channels. The received signal matrix $Y \in \mathbb{C}^{T \times N}$ is

$$Y = \sqrt{\frac{\rho}{\mu}} X H + W,$$  (1)
where $X \in \mathbb{C}^{T \times M}$ is the codeword matrix, transmitted over $T$ time slots, $W \in \mathbb{C}^{T \times N}$ is a noise matrix with independent and identically distributed (i.i.d.) entries being circularly symmetric complex Gaussian distributed $\mathcal{CN}(0, 1)$, $H \in \mathbb{C}^{M \times N}$ is the channel matrix whose entries are also i.i.d. with the distribution $\mathcal{CN}(0, 1)$, $\rho$ denotes the average signal-to-noise ratio (SNR) per receive antenna and $\mu$ is the normalization factor to ensure that the average energy of the coded symbols transmitting from all antennas during one symbol period is 1. The realization of $H$ is assumed to be known at the receiver, but not known at the transmitter. Therefore, the signal power is allocated uniformly across the transmit antennas.

**Definition 1 (Code Rate):** Let $L$ be the number of independent information symbols $\{s_l\}, l = 1, \cdots, L$ per codeword $X$, selected from a complex constellation $\mathcal{A}$. The code rate of the STBC is defined as $R = \frac{L}{T}$ symbols per channel use. If $L = TM$, the STBC is said to have full rate, i.e., $R = M$ symbols per channel use.

In this paper, we consider that information symbols $\{s_l\}, l = 1, \cdots, L$ are coded by linear dispersion STBC as

$$X = \sum_{l=1}^{L} A_l s_l, \quad (2)$$

where $A_l \in \mathbb{C}^{T \times M}$ is the linear STBC matrix. If $L = TM$, the codeword $X$ has full rate.

To decode the transmitted sequence $s$ at the receiver, we need to extract $s$ from $X$. This can be done by as follows. By substituting (2) into (1), we get

$$Y = \sqrt{\frac{\rho}{\mu}} \sum_{l=1}^{L} A_l H s_l + W. \quad (3)$$

Then, by taking vectorization of the matrix $Y$ we have

$$y \triangleq \text{vec}(Y) = \sqrt{\frac{\rho}{\mu}} \sum_{l=1}^{L} \text{vec} (A_l H) s_l + \text{vec}(W)$$

$$= \sqrt{\frac{\rho}{\mu}} H s + w, \quad (4)$$

where $y \in \mathbb{C}^{TN \times 1}$, $w \in \mathbb{C}^{TN \times 1}$, $s = [ s_1 \ s_2 \ \cdots \ s_L ]^t$, and $H \in \mathbb{C}^{TN \times L}$ is an equivalent channel matrix,

$$H = \begin{bmatrix} g_1 & g_2 & \cdots & g_L \end{bmatrix} \quad (5)$$
with the \( l \)-th column \( \mathbf{g}_l = \text{vec} ( \mathbf{A}_l \mathbf{H} ) \), \( l = 1, 2, \cdots, L \).

For a ZF receiver, the estimate \( \hat{s}^{ZF} \) of the transmitted symbol sequence \( s \) is,

\[
\hat{s}^{ZF} = \arg \min_{s \in \mathcal{A}_L} \left\| \mathbf{Q}^{ZF} \mathbf{y} - s \right\|^2,
\]  

(6)

where \( \mathbf{Q}^{ZF} = \sqrt{\frac{\rho}{\mu}} (\mathbf{H}^H \mathbf{H})^{-1} \mathbf{H}^H \). Equivalently, it can be written as the single-symbol decoding as follows,

\[
\hat{s}_l^{ZF} = \arg \min_{s_l \in \mathcal{A}} \left\| \mathbf{Q}^{ZF}_{l,:} \mathbf{y} - s_l \right\|^2, \quad l = 1, 2, \cdots, L, \tag{7}
\]

where \( \mathbf{Q}^{ZF}_{l,:} \) denotes the \( l \)-th row of \( \mathbf{Q}^{ZF} \).

For an ML receiver, the estimate of \( \hat{s}^{ML} \) that achieves the minimum of the squared Frobenius norm is given by

\[
\hat{s}^{ML} = \arg \min_{s \in \mathcal{A}_L} \left\| \mathbf{y} - \sqrt{\frac{\rho}{\mu}} \mathbf{H} s \right\|^2. \tag{8}
\]

In the ML decoding, computations of squared Frobenius norms for all possible codewords are needed and therefore result in prohibitively huge computational complexity when the length of the information symbols vector to be decoded is large. In the following, we give a metric to evaluate the computational complexity of the ML decoding, which is the same as the one shown in [31, Definition 2].

**Definition 2 (Decoding Complexity):** The decoding complexity \( \mathcal{O} \) is defined as the number of squared Frobenius norms \( \| \cdot \|^2 \) that should be computed in the decoding process.

With the above definition, we have the following two remarks.

**Remark 1:** The decoding complexity of the ZF detection is \( \mathcal{O} = L \cdot |\mathcal{A}| \), i.e., \( L \) times of the cardinality of the signal constellation. It is equivalent to the single-symbol decoding complexity.

**Remark 2:** The decoding complexity of the ML detection is \( \mathcal{O} = |\mathcal{A}|^L \), i.e., the complexity of the full exhaustive search of all \( L \) information symbols drawn from the constellation \( \mathcal{A} \).

We next describe the PIC group decoding studied in [35].

**B. PIC Group Decoding**

Define index set \( \mathcal{I} \) as

\[
\mathcal{I} = \{1, 2, \cdots, L\},
\]
where \( L \) is the number of information symbols in \( s \). We then partition \( \mathcal{I} \) into \( P \) groups:
\[
\mathcal{I}_p = \{I_{p,1}, I_{p,2}, \ldots, I_{p,l_p}\}, \quad p = 1, 2, \ldots, P,
\]
where \( l_p \) is the cardinality of the subset \( \mathcal{I}_p \). We call \( \mathcal{I} = \{\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_P\} \) a grouping scheme. For such a grouping scheme, we have
\[
\mathcal{I} = \bigcup_{p=1}^{P} \mathcal{I}_p, \quad \text{and} \quad \sum_{p=1}^{P} l_p = L.
\]
Define
\[
s_p = \begin{bmatrix} s_{I_{p,1}} & s_{I_{p,2}} & \cdots & s_{I_{p,l_p}} \end{bmatrix}^t, \quad p = 1, \ldots, P. \tag{9}
\]
\[
G_p = \begin{bmatrix} g_{I_{p,1}} & g_{I_{p,2}} & \cdots & g_{I_{p,l_p}} \end{bmatrix}, \quad p = 1, \ldots, P. \tag{10}
\]
With these notations, (4) can be written as
\[
y = \sqrt{\frac{\rho}{\mu}} \sum_{p=1}^{P} G_p s_p + w. \tag{11}
\]
Suppose we want to decode the symbols embedded in the group \( s_p \). The PIC group decoding first implements linear interference cancellation with a suitable choice of matrix \( Q_p \) in order to completely eliminate the interferences from other groups \([35]\), i.e., \( Q_p G_q = 0, \forall q \neq p \) and \( q = 1, 2, \ldots, P \). Then, we have
\[
z_p \triangleq \sqrt{\frac{\rho}{\mu}} Q_p y = \sqrt{\frac{\rho}{\mu}} Q_p G_p s_p + Q_p w, \quad p = 1, 2, \ldots, P, \tag{12}
\]
where the interference cancellation matrix \( Q_p \) can be chosen as follows \([35]\),
\[
Q_p = I_{TN} - G_p^c H G_p^c \begin{pmatrix} G_p^c \end{pmatrix}^H, \quad p = 1, 2, \ldots, P, \tag{13}
\]
when the following matrix has full column rank:
\[
G_p^c = \begin{bmatrix} G_1 & \cdots & G_{p-1} & G_{p+1} & \cdots & G_P \end{bmatrix}. \tag{14}
\]
Afterwards, the symbols in the group \( s_p \) are decoded with the ML decoding algorithm as follows,
\[
\hat{s}_p = \arg \min_{s_p \in A^p} \left\| z_p - \sqrt{\frac{\rho}{\mu}} Q_p G_p s_p \right\|^2. \tag{15}
\]
The above PIC group decoding is connected to some of the known decodings as in the following remarks.

**Remark 3 (ML and PIC Group Decoding):** For one special case of \( P = 1 \), the grouping scheme is \( \mathcal{I} = \{ \mathcal{I}_1 \} \) with \( \mathcal{I}_1 = \mathcal{I} \). From (13), we have \( Q_p = I_{TN} \). Then, the PIC group decoding is equivalent to the ML decoding where all information symbols are jointly decoded.

**Remark 4 (ZF and PIC Group Decoding):** For the special case of \( P = L \), the grouping scheme is \( \mathcal{I} = \{ \mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_L \} = \{ \{1\}, \{2\}, \ldots, \{L\} \} \), i.e., every single symbol is regarded as one group. Then, the PIC group decoding is equivalent to the ZF decoding where every single symbol is separated from all the other symbols and then decoded.

**Remark 5 (ZF, ML and PIC Group Decoding):** The PIC group decoding with \( 1 \leq P \leq L \) can be viewed as an intermediate decoding approach between the ML decoding and the ZF decoding. Alternatively, the ML decoding and the ZF decoding can both be regarded as the special cases of the PIC group decoding corresponding to \( P = 1 \) and \( P = L \), respectively.

**Remark 6 (PIC Group Decoding Complexity):** For the PIC group decoding, the following two steps are needed: the group zero-forcing to cancel the interferences coming from all the other groups as shown in (12) and the group ML decoding to jointly decode the symbols in one group as shown in (15). Therefore, the decoding complexity of the PIC group decoding should reside in the above two steps. Note that the interference cancellation process shown in (12) mainly involves with linear matrix computations, whose computational complexity is small compared to the ML decoding for an exhaustive search of all candidate symbols. Therefore, to evaluate the decoding complexity of the PIC group decoding, we mainly focus on the computational complexity of the ML decoding within the PIC group decoding algorithm. According to Definition 2, the ML decoding complexity in the PIC group decoding algorithm is \( O = \sum_{p=1}^{P} |\mathcal{A}|^p \). It can be seen that the PIC group decoding provides a flexible decoding complexity which can be from the ZF decoding complexity \( L|\mathcal{A}| \) to the ML decoding complexity \( |\mathcal{A}|^L \).

The performance of a decoding algorithm for a wireless communication system is related to the diversity order. If the average probability of a detection error for communication over a
fading channel usually behaves as:

\[ P_e(\text{SNR}) \leq c \cdot \text{SNR}^{-G_d} \]

where \( c \) is a constant, \( G_d \) is called the diversity order of the system. For an MIMO communication system, the maximum diversity order is \( MN \), i.e., the product of the number of transmit antennas and the number of receiver antennas. In order to optimize the reception performance of the MIMO system, a full diversity is usually pursued which can be achieved by a proper signal transmission scheme or data format (e.g., STBC). In [2], the “rank-and-determinant criterion” of STBC design was proposed to maximize both the diversity gain \( G_d \) and the coding gain \( \frac{1}{c} \) of the MIMO system with an ML decoding. Recently, in [35] an STBC design criterion was derived to achieve full diversity when the PIC group decoding is used at the receiver. In the following, we cite the main result of the STBC design criterion proposed in [35].

**Proposition 1:** [35, Theorem 1] [Full-Diversity Criterion under PIC Group Decoding]

For an STBC \( X \) with the PIC group decoding, the full diversity is achieved when

1) the code \( X \) satisfies the full rank criterion, i.e., it achieves full diversity when the ML receiver is used; and

2) \( G_1, G_2, \cdots, G_P \) are linearly independent vector groups for any \( H \neq 0 \).

In [35], the STBC achieving full diversity with PIC group decoding were proposed for 2 and 4 transmit antennas. However, a systematic code design of the full-diversity STBC with PIC group decoding remains an open problem.

### III. Full-Diversity STBC with PIC Group Decoding

In this section, we propose a systematic design of full-diversity STBC with PIC group decoding. Then, we prove that the proposed codes can obtain full diversity with ML decoding and PIC group decoding, respectively.

#### A. Encoding Technique

Our proposed space-time code \( C \), i.e., \( X \) in (1), is of size \( T \times M \) (for any given \( T, M \) and \( T \geq M \)) and will be transmitted from \( M \) antennas over \( T \) time slots. Let \( P = T - M + 1 \). The symbol stream \( \{s_l\}, l = 1, \cdots, L \) (composed of \( L = MP \) complex symbols chosen from
QAM constellation and then scaled by $1/\sqrt{E[|s_l|^2]}$ is first parsed into $M \times 1$ symbol vectors $s_p$ ($p = 1, 2, \cdots, P$). Each symbol vector is linearly precoded by an $M \times M$ matrix $\Theta$, which is a chosen constellation rotation matrix. Next, the $M \times 1$ vector $\Theta s_p$ is used to form the space-time code matrix $C$, in which the $p$-th descending diagonal from left to right is the diagonal form of $\Theta s_p$.

The resulting transmitted code matrix $C$ is given by

$$C = \begin{bmatrix}
X_{1,1} & 0 & \cdots & 0 \\
X_{2,1} & X_{1,2} & \ddots & \vdots \\
\vdots & X_{2,2} & \ddots & 0 \\
0 & X_{P,2} & \cdots & X_{2,M} \\
\vdots & 0 & \ddots & \vdots \\
0 & \vdots & \cdots & X_{P,M}
\end{bmatrix},$$  \hspace{1cm} (16)

where the $p$-th descending diagonal from left to right, denoted by $X_p = \begin{bmatrix} X_{p,1} & X_{p,2} & \cdots & X_{p,M} \end{bmatrix}^t$ is given by

$$X_p = \Theta s_p, \quad p = 1, 2, \cdots, P$$  \hspace{1cm} (17)

and the $M \times 1$ information symbol vector $s_p$ is given by

$$s_p = \begin{bmatrix} s_{(p-1)M+1} & s_{(p-1)M+2} & \cdots & s_{pM} \end{bmatrix}^t,$$  \hspace{1cm} (18)

for $p = 1, 2, \cdots, P$.

**Proposition 2:** The proposed STBC in (16) has asymptotically full rate.

**Proof:** In the codeword in (16), a total number of $MP$ independent information symbols are encoded into the codeword $C$, which is then transmitted from $M$ antennas over $T$ time slots. The code rate of transmission is therefore

$$R = \frac{MP}{T} = \frac{M(T - M + 1)}{T} = M \left(1 - \frac{M - 1}{T}\right).$$  \hspace{1cm} (19)

For a very large block length $T$, it can be seen that the rate $R$ of the proposed ST coding scheme approaches $M$ symbols per channel use, i.e. the full rate.
B. Choice of Rotation Matrix \( \Theta \)

In [37], the rotation matrix \( \Theta \) was designed for diagonal STBC to achieve the full diversity gain and the optimal diversity product. With the optimal cyclotomic lattices design for \( M \) transmit antennas, from [37, Table I] we can get a set of integers \((m, n)\) and let \( K = mn \). Then, the optimal lattice \( \Theta \) is given by [37, Eq. (16)]

\[
\Theta = \begin{bmatrix}
\zeta_K & \zeta_K^2 & \cdots & \zeta_K^M \\
\zeta_K^{1+n_2m} & \zeta_K^{2(1+n_2m)} & \cdots & \zeta_K^{M(1+n_2m)} \\
\vdots & \vdots & \ddots & \vdots \\
\zeta_K^{1+n_Mm} & \zeta_K^{2(1+n_Mm)} & \cdots & \zeta_K^{M(1+n_Mm)}
\end{bmatrix},
\]

(20)

where \( \zeta_K = \exp(j2\pi/K) \) with \( j = \sqrt{-1} \) and \( n_2, n_3, \cdots, n_M \) are distinct integers such that \( 1 + n_i m \) and \( K \) are co-prime for any \( 2 \leq i \leq M \).

**Example 1:** For 4 transmit antennas we can choose \( m = 3, n = 5 \) and \( K = 15 \) according to [37, Table I]. Then, in order to ensure that \( 1 + n_i m \) and \( K \) are co-prime for any \( 2 \leq i \leq M \) we can obtain \( n_2 = 1, n_3 = 2, n_4 = 4 \). When \( m = 3 \), the signal constellation is located on the equal literal triangular lattice. When \( m = 4 \), \( n \) can be 4 and \( n_i \) can be \( 0, 1, 2, 3 \), and in this case the signal constellation is located on the square lattice.

**Example 2:** For 5 transmit antennas we can select \( m = n = 5 \) and \( K = 25 \). Then, \( n_2 = 1, n_3 = 2, n_4 = 3, n_5 = 4 \).

The cyclotomic design of the matrix \( \Theta \) is vital for the design of the algebraic STBC. In the following, we show some properties of the matrix \( \Theta \) that will be used later for our design.

**Property 1:** [37] The diagonal cyclotomic ST code \( \Omega \) defined by \( \Omega = \left\{ \text{diag} \left[ X_1 \ X_2 \ \cdots \ X_M \right] \right\} \) achieves full diversity under ML decoding, where \( \left[ X_1 \ X_2 \ \cdots \ X_M \right] = \Theta \left[ s_1 \ s_2 \ \cdots \ s_M \right]^t \) and \( \Theta \) is given by (20).

**Property 2:** Every entry of the matrix \( \Theta \) in (20) is non-zero.

This property is obvious from (20).
C. Achieve Full Diversity with ML Decoding

We show the main result of the proposed STBC when an ML decoding is used at the receiver, as follows.

**Theorem 1 (Full Diversity with ML Decoding):** Consider a MIMO transmission with \( M \) transmit antennas and \( N \) receive antennas over block fading channels. The STBC \( C \) as described in (16) achieves full diversity under the ML decoding.

**Proof:** In order to prove that the ST code \( C \) in (16) can obtain full diversity under ML decoding, it is sufficient to prove that \( \Delta_C = C - \hat{C} \) achieves full rank for any distinct pair of ST codewords \( C \) and \( \hat{C} \).

For any pair of distinct codewords \( C \) and \( \hat{C} \), there exists at least one index \( p \) (\( 1 \leq p \leq P \)) such that \( X_p - \hat{X}_p \neq 0 \), where \( X_p \) and \( \hat{X}_p \) are related to \( s_p \) and \( \hat{s}_p \) from (17), respectively. Let \( p \) denote the minimum index of vectors satisfying \( X_p - \hat{X}_p \neq 0 \). Then, for any index \( q \) with \( q < p \), it must have \( X_q - \hat{X}_q = 0 \). Define \( \tilde{X} = X - \hat{X} \) as the difference between symbols \( X \) and \( \hat{X} \). Then, from (16) \( \Delta_C \) can be expressed as

\[
\Delta_C = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ddots & \ddots & \vdots \\
\tilde{X}_{p,1} & 0 & \ddots & \vdots \\
\vdots & \tilde{X}_{p,2} & \ddots & \vdots \\
\tilde{X}_{P,1} & \ddots & \ddots & 0 \\
0 & \tilde{X}_{P,2} & \ddots & \tilde{X}_{p,M} \\
\vdots & 0 & \ddots & \vdots \\
0 & \ddots & \ddots & \tilde{X}_{P,M}
\end{bmatrix},
\]

where \( \tilde{X}_{p,m} \neq 0 \) for \( m = 1, 2, \cdots, M \). This is because for \( X_p - \hat{X}_p \neq 0 \), it exists \( s_p - \hat{s}_p \neq 0 \). Due to the suitably chosen constellation rotation matrix \( \Theta \) in (20), \( X_p - \hat{X}_p \) must have nonzero entries for any \( s_p \neq \hat{s}_p \). Then, the matrix \( \Delta_C \) has full rank.

The full rankness of \( \Delta_C \) can be examined similar to that for the Toeplitz code (or delay diversity code) [33] by checking if the columns of \( \Delta_C \) are linearly independent. Specifically, we establish \( \Delta_C \bar{\alpha} = 0 \) with \( \bar{\alpha} = [\alpha_1 \ \alpha_2 \ \cdots \ \alpha_M]^t \). First, we examine the \( p \)-th equation
in $\Delta C\vec{\alpha} = 0$ and get $\alpha_1 \vec{X}_{p,1} = 0$. Because $\vec{X}_{p,m} \neq 0$ for $m = 1, 2, \ldots, M$, $\alpha_1 = 0$. Then, we examine the $(p+1)$-th equation and get $\alpha_2 \vec{X}_{p,2} = 0$. Immediately, $\alpha_2 = 0$. Likewise, we examine the $(p+2)$-th equation until $(p+M-1)$-th equation, and we can get $\alpha_1 = \alpha_2 = \cdots = \alpha_M = 0$. Therefore, all columns of $\Delta C$ are linearly independent and $\Delta C$ has full rank.

This property will be used in next section in the proof of the full diversity property under the PIC group decoding.

D. Achieve Full Diversity with PIC Group Decoding

We show the main result of the proposed STBC when a PIC group decoding with a particular grouping scheme is used at the receiver, as follows.

**Theorem 2 (Full Diversity with PIC Group Decoding):** Consider a MIMO transmission with $M$ transmit antennas and $N$ receive antenna over block fading channels. The STBC $C$ as described in (16) is used at the transmitter. The equivalent channel matrix is $H \in \mathbb{C}^{TN \times MP}$. If the received signal is decoded using the PIC group decoding with the grouping scheme $\mathcal{I} = \{\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_P\}$ where $\mathcal{I}_p = \{(p-1)M+1, (p-1)M+2, \ldots, pM\}$ for $p = 1, 2, \ldots, P$, i.e., the size of each group is equal to the number of transmit antennas $M$, then the code $C$ achieves the full diversity.

In order to prove Theorem 2, let us first introduce the following lemmas.

**Lemma 1:** Consider the system as described in Theorem 2 with $N = 1$ and the STBC $C$ as given by (16),

1) the equivalent channel matrix $H \in \mathbb{C}^{T \times MP}$ can be expressed as

$$H = \begin{bmatrix} G_1 & G_2 & \cdots & G_P \end{bmatrix}$$

(22)

where

$$G_p = \begin{bmatrix} 0_{(p-1) \times M} \\ \text{diag}(h) \Theta \\ 0_{(P-p) \times M} \end{bmatrix}, \quad p = 1, 2, \ldots, P,$$

(23)

2) $G_1, G_2, \ldots, G_P$ are linearly independent vector groups as long as $h \neq 0$, where $h = H$.

A proof of Lemma 1 is given in Appendix I.
Lemma 2: Consider the system as described in Theorem 2 and the STBC $C$ as given by (16). For the equivalent channel matrix $H \in \mathbb{C}^{TN \times MP}$, $G_1, G_2, \cdots, G_P$ are linearly independent vector groups for $h \neq 0$ when $N > 1$ if and only if $G_1, G_2, \cdots, G_P$ are linearly independent vector groups for $h \neq 0$ when $N = 1$.

The proof of Lemma 2 is straightforward and is also the same as what is mentioned in [35].

Proof of Theorem 2:

As shown in Proposition 7 a codeword $C$ with PIC group decoding can obtain the full diversity if

1) $C$ achieves the full diversity with the ML receiver, and

2) $G_1, G_2, \cdots, G_P$ are linearly independent vector groups as long as $h \neq 0$.

For the proposed code $C$ in (16), the first condition is satisfied as shown in Theorem 1. The second condition is satisfied as shown in Lemma 1 for $N = 1$ and Lemma 2 for $N > 1$, respectively. Therefore, the code $C$ in (16) can obtain full diversity with the PIC group decoding provided that the grouping scheme is $\mathcal{I} = \{\mathcal{I}_1, \mathcal{I}_2, \cdots, \mathcal{I}_P\}$ where $\mathcal{I}_p = \{(p-1)M + 1, (p-1)M + 2, \cdots, pM\}$ for $p = 1, 2, \cdots, P$. It completes the proof.

Corollary 1: The decoding complexity of the PIC group decoding of the proposed STBC with the grouping scheme as described in Theorem 2 is $O = P \cdot |A|^M$.

Remark 7: The decoding complexity of the proposed STBC with the PIC group decoding is equivalent to the ML decoding of $M$ independent information symbols jointly. As shown in (19), the code rate of the proposed STBC in (16) for a given $M$ can be increased by embedding larger number of groups in the codeword, i.e., increasing the value of $P$. It is noteworthy to mention that the increase of the code rate does not result in the increase of the decoding complexity.

IV. CODE DESIGN EXAMPLES

In this section, we show a few code design examples. We denote $C_{M,T,P}$ the code constructed by (16) for given parameters: $M$ the number of transmit antennas, $T$ the block length of the code, and $P$ the number of groups to be decoded in the PIC group decoding. For notational brevity, in the following we only show the equivalent channel of the proposed codes for MISO systems.
A. For Two Transmit Antennas

Consider a code for 2 transmit antennas with 3 time slots. According to the code structure (16), we have

$$C_{2,3,2} = \begin{bmatrix} X_{1,1} \\ X_{2,1} & X_{1,2} \\ X_{2,2} \end{bmatrix},$$

(24)

where $[ X_{1,1} \ X_{1,2} ]^t = \Theta [ s_1 \ s_2 ]^t$ and $[ X_{2,1} \ X_{2,2} ]^t = \Theta [ s_3 \ s_4 ]^t$. The constellation rotation matrix $\Theta$ can be chosen as

$$\Theta = \begin{bmatrix} \gamma & \delta \\ -\delta & \gamma \end{bmatrix},$$

where $\gamma = \cos \theta$ and $\delta = \sin \theta$ with $\theta = 1.02$ [35].

The code rate of the code is $4/3$. In fact, this code is equivalent to the one proposed in [35, Section VI - Example 1].

The equivalent channel of the code $C_{2,3,2}$ is given by

$$H = \begin{bmatrix} \gamma h_1 & \delta h_1 \\ -\delta h_2 & \gamma h_2 & \gamma h_1 & \delta h_1 \\ -\delta h_2 & \gamma h_2 \end{bmatrix}.$$  

(25)

The grouping scheme for the PIC group decoding is $I_1 = \{1, 2\}$ and $I_2 = \{3, 4\}$. It can be seen that $G_1$ and $G_2$ are linearly independent. Then, the code can obtain full diversity with the PIC group decoding.

B. For Four Transmit Antennas

Consider 6 time slots. We can design the code according to (16) as follows,

$$C_{4,6,3} = \begin{bmatrix} X_{1,1} \\ X_{2,1} & X_{1,2} \\ X_{3,1} & X_{2,2} & X_{1,3} \\ X_{3,2} & X_{2,3} & X_{1,4} \\ X_{3,3} & X_{2,4} \\ X_{3,4} \end{bmatrix},$$

(26)
where
\[
\begin{bmatrix}
X_{1,1} & X_{1,2} & X_{1,3} & X_{1,4}
\end{bmatrix}^t = \Theta \begin{bmatrix}
s_1 & s_2 & s_3 & s_4
\end{bmatrix}^t,
\]
\[
\begin{bmatrix}
X_{2,1} & X_{2,2} & X_{2,3} & X_{2,4}
\end{bmatrix}^t = \Theta \begin{bmatrix}
s_5 & s_6 & s_7 & s_8
\end{bmatrix}^t,
\]
\[
\begin{bmatrix}
X_{3,1} & X_{3,2} & X_{3,3} & X_{3,4}
\end{bmatrix}^t = \Theta \begin{bmatrix}
s_9 & s_{10} & s_{11} & s_{12}
\end{bmatrix}^t.
\]

The code rate of this code is 2 which is higher than that of the code proposed in [35, Section VI - Example 2] in the same case. The equivalent channel of the code \( C_{4,6,3} \) can be written as
\[
\mathcal{H} = \begin{bmatrix}
G_1 & G_2 & G_3
\end{bmatrix},
\]
where
\[
G_1 = \begin{bmatrix}
B & 0_{1 \times 4} & 0_{1 \times 4}
\end{bmatrix},
G_2 = \begin{bmatrix}
0_{1 \times 4} & B & 0_{1 \times 4}
\end{bmatrix},
G_3 = \begin{bmatrix}
0_{1 \times 4} & 0_{1 \times 4} & B
\end{bmatrix},
\]
and
\[
B = \begin{bmatrix}
h_{1,1} \theta_{1,1} & h_{1,1} \theta_{1,2} & h_{1,1} \theta_{1,3} & h_{1,1} \theta_{1,4} \\
h_{2,1} \theta_{2,1} & h_{2,1} \theta_{2,2} & h_{2,1} \theta_{2,3} & h_{2,1} \theta_{2,4} \\
h_{3,1} \theta_{3,1} & h_{3,1} \theta_{3,2} & h_{3,1} \theta_{3,3} & h_{3,1} \theta_{3,4} \\
h_{4,1} \theta_{4,1} & h_{4,1} \theta_{4,2} & h_{4,1} \theta_{4,3} & h_{4,1} \theta_{4,4}
\end{bmatrix},
\]
with \( \theta_{i,j} \) being the \((i, j)\)-th entry of the matrix \( \Theta \) for \( i, j = 1, 2, 3, 4 \). The grouping scheme for the PIC group decoding is \( \mathcal{I}_1 = \{1, 2, 3, 4\}, \mathcal{I}_2 = \{5, 6, 7, 8\} \) and \( \mathcal{I}_3 = \{9, 10, 11, 12\} \). It can be seen that the groups \( G_1, G_2 \) and \( G_3 \) are linearly independent to each other. Then, the code can obtain full diversity with the PIC group decoding.

Moreover, we can also design the code for \( T = 6 \) as follows,
\[
C_{4,6,2} = \begin{bmatrix}
X_{1,1} & 0 & X_{1,2} \\
X_{2,1} & 0 & X_{1,3} \\
X_{2,2} & 0 & X_{1,4} \\
X_{2,3} & 0 & X_{2,4}
\end{bmatrix}.
\]
The code rate of the code is 4/3 which has the same rate as the one proposed in [35, Section VI - Example 2]. In fact, the code \( C_{4,6,2} \) is a special case of the code \( C_{4,6,3} \) in (26) when no symbols are transmitted on the second diagonal layer from left to right. The equivalent channel of the code \( C_{4,6,2} \) is

\[
\mathcal{H} = \begin{bmatrix} \begin{bmatrix} B \\ 0_{2 \times 4} \end{bmatrix} & 0_{2 \times 4} \\ \end{bmatrix},
\]

(30)

where \( B \) is given by (28).

For given \( T = 5 \), the code can be designed as follows,

\[
C_{4,5,2} = \begin{bmatrix}
X_{1,1} & X_{1,2} \\
X_{2,1} & X_{1,2} \\
X_{2,2} & X_{1,3} \\
X_{2,3} & X_{1,4} \\
X_{2,4} & X_{1,5}
\end{bmatrix},
\]

(31)

This code has a code rate of 8/5 and two groups to be decoded. The equivalent channel of the code \( C_{4,5,2} \) is

\[
\mathcal{H} = \begin{bmatrix} \begin{bmatrix} B \\ 0_{1 \times 4} \end{bmatrix} & 0_{1 \times 4} \\ \end{bmatrix},
\]

(32)

where \( B \) is given by (28).

C. For Five Transmit Antennas

For given \( T = 6 \) and \( P = 2 \), the code is designed as follows,

\[
C_{5,6,2} = \begin{bmatrix}
X_{1,1} & X_{1,2} \\
X_{2,1} & X_{1,2} \\
X_{2,2} & X_{1,3} \\
X_{2,3} & X_{1,4} \\
X_{2,4} & X_{1,5} \\
X_{2,5} & X_{1,5}
\end{bmatrix},
\]

(33)

The code rate of the code \( C_{5,6,2} \) is 5/3. The equivalent channel is

\[
\mathcal{H} = \begin{bmatrix} \begin{bmatrix} \text{diag}(h) \Theta_5 \\ 0_{1 \times 5} \end{bmatrix} & 0_{1 \times 5} \\ \end{bmatrix},
\]

(34)
where $\Theta_5$ is the rotation matrix of size $5 \times 5$.

The grouping scheme for the PIC group decoding of $C_{5,6,2}$ is $\mathcal{I}_1 = \{1, 2, 3, 4, 5\}$ and $\mathcal{I}_2 = \{6, 7, 8, 9, 10\}$. It can be seen that the groups $G_1$ and $G_2$ are linearly independent to each other. Then, the code $C_{5,6,2}$ can obtain full diversity with the PIC group decoding.

V. Simulation Results

In this section, simulation results of the proposed STBC with the PIC group decoding over Rayleigh fading channels are presented. We first show bit error rate (BER) performance of the codes proposed in this paper for four transmit antennas and compare them to the one proposed in [35]. Specifically, we consider three STBC for four transmit antennas proposed in this paper, i.e., $C_{4,6,3}$ in (26), $C_{4,6,2}$ in (29) and $C_{4,5,2}$ in (31), and then compare them with Guo-Xia’s code given in [35, Section VI - Example 2]. In order to make a fair performance comparison, we keep the same bandwidth efficiency of 8 bps/Hz. Thus, we use 16QAM for the code $C_{4,6,3}$ (the code rate is $R = 2$) and 64QAM for both codes $C_{4,6,2}$ (the code rate is $R = \frac{4}{3}$) and Guo-Xia’s code (the code rate is $R = \frac{4}{3}$). For the code $C_{4,5,2}$, because it has a code rate $R = \frac{8}{5}$ we use 64QAM and thus its bandwidth efficiency is 9.6 bps/Hz higher than the other three codes. Since we use square QAM, in the rotation matrix $\Theta$ we use $m = 4$ in Example 1 of Section III.

From Fig. 1 it can be seen that all the codes can obtain the full diversity at high SNR. In particular, the code $C_{4,6,3}$ in (26) achieves the best BER performance among all the simulated codes and 2 dB better than Guo-Xia’s code. This is attributed to more information symbols embedded in the code $C_{4,6,3}$ than the other codes and its code rate is the highest. It can be observed that the code $C_{4,6,2}$ has very similar performance to Guo-Xia’s code. Moreover, the code $C_{4,5,2}$ has 1 dB loss compared to Guo-Xia’s code. This is because it has a higher bandwidth efficiency than Guo-Xia’s code.

We also consider the simulation of the proposed code $C_{5,6,2}$ in (33) with the PIC group decoding for five transmit antennas. Fig. 2 shows the BER performance of the code $C_{5,6,2}$ for three, four and five receive antennas, respectively. It demonstrates that an increase of the number of receive antennas results in a larger diversity gain as illustrated by the slope of the BER curves.

VI. Conclusion

In this paper, a systematic design of STBC that can achieve full diversity with PIC group decoding was proposed. The proposed STBC are constructed with multiple diagonal layers and
each layer is composed of a fixed number of coded symbols equal to $M$, i.e., the number of the transmit antennas. The code rate of the proposed STBC is varied in accordance with the number of layers embedded in the codeword. For the PIC group decoding, a grouping scheme was proposed to cluster every $M$ neighboring columns of the equivalent channel matrix into one group. With the proposed STBC and the PIC group decoding in MIMO systems, it was proved that full diversity can be achieved. Moreover, the proposed STBC has asymptotical full rate with a large block length, i.e., $M$ symbols per channel use for $M$ transmit antennas. A few examples of code design were given. Simulation results demonstrated the superior performance of the proposed codes compared to some existing codes. It should be mentioned that the decoding complexity of the proposed STBC is equivalent to the ML decoding of $M$ independent information symbols jointly.

**APPENDIX I - PROOF OF LEMMA 1**

**A. Proof of Lemma 1.1**

**Proof:** Define the $T \times M$ matrix $C_p$ ($p = 1, 2, \ldots, P$) as follows,

$$C_p = \binom{0_{(p-1)\times M}}{\text{diag}(X_p)} \binom{0_{(P-p)\times M}}{, \ p = 1, 2, \ldots, P,}$$

(35)

where $P = T - M + 1$ and $X_p = \begin{bmatrix} X_{p,1} & X_{p,2} & \cdots & X_{p,M} \end{bmatrix}^t$ is given by (17). Then, (16) can be written as

$$C = \sum_{p=1}^{P} C_p.$$  

(36)

For MISO systems, we have $H = h = [h_1, h_2, \ldots, h_M]^t$ with $h_m$ ($m = 1, 2, \ldots, M$) being the channel gain from the $m$-th transmit antenna to the receiver. Using (36), we can express (1) as

$$y = \sqrt{\frac{\rho}{\mu}} \sum_{p=1}^{P} C_p h + w = \sqrt{\frac{2}{\mu}} \sum_{p=1}^{P} H_p X_p + w,$$

(37)
where \( y \in \mathbb{C}^{T \times 1} \), \( w \in \mathbb{C}^{T \times 1} \) and \( H_p \in \mathbb{C}^{T \times M} \) is given by

\[
H_p = \begin{bmatrix}
0_{(p-1) \times M} \\
\text{diag}(h) \\
0_{(P-p) \times M}
\end{bmatrix}, \quad p = 1, 2, \ldots, P. \tag{38}
\]

Using (17), we can further write (37) as

\[
y = \sqrt{\frac{\rho}{\mu}} \sum_{p=1}^{P} H_p \Theta s_p + w
= \sqrt{\frac{\rho}{\mu}} H s + w, \tag{39}
\]

where the equivalent channel matrix \( H \in \mathbb{C}^{T \times MP} \) is given by

\[
H = \begin{bmatrix}
(H_1 \Theta) & (H_2 \Theta) & \cdots & (H_P \Theta)
\end{bmatrix} \tag{40}
\]

and \( s = [ s^t_1 \ s^t_2 \ \cdots \ s^t_P ]^t \).

Let \( G_p = H_p \Theta \) for \( p = 1, 2, \ldots, P \). Using (38), we get

\[
G_p = \begin{bmatrix}
0_{(p-1) \times M} \\
\text{diag}(h) \Theta \\
0_{(P-p) \times M}
\end{bmatrix}, \quad p = 1, 2, \ldots, P. \tag{41}
\]

Then, (40) can be written as

\[
H = \begin{bmatrix}
G_1 & G_2 & \cdots & G_P
\end{bmatrix}. \tag{42}
\]

B. Proof of Lemma 1.2

Proof: Next, we shall prove that \( G_1, G_2, \ldots, G_P \) are linearly independent vector groups as long as \( h \neq 0 \). To do so, we may need the following definitions.

Definition 3: \([35]\). Let \( V = \{ v_i \in \mathbb{C}^n, i = 0, 1, \ldots, k-1 \} \) be a set of vectors. Vector \( v_k \) is said to be independent of \( V \) if for any \( a_i \in \mathbb{C} \), \( i = 0, 1, \ldots, k-1 \),

\[
v_k - \sum_{i=0}^{k-1} a_i v_i \neq 0.
\]
Definition 4: [35]. Let $\mathcal{V}_0, \mathcal{V}_1, \cdots, \mathcal{V}_{n-1}, \mathcal{V}_n$ be $n+1$ groups of vectors. Vector group $\mathcal{V}_n$ is said to be independent of $\mathcal{V}_0, \mathcal{V}_1, \cdots, \mathcal{V}_{n-1}$ if every vector in $\mathcal{V}_n$ is independent of $\bigcup_{i=0}^{n-1} \mathcal{V}_i$.

Definition 5: [35]. Let $\mathcal{V}_0, \mathcal{V}_1, \cdots, \mathcal{V}_{n-1}, \mathcal{V}_n$ be $n+1$ groups of vectors. The vector groups $\mathcal{V}_0, \mathcal{V}_1, \cdots, \mathcal{V}_{n-1}, \mathcal{V}_n$ are said to be linearly independent if for $0 \leq k \leq n$, $\mathcal{V}_k$ is independent of the remaining vector groups $\mathcal{V}_0, \mathcal{V}_1, \cdots, \mathcal{V}_{k-1}, \mathcal{V}_{k+1}, \cdots, \mathcal{V}_n$.

Our proof will be given in the following two steps:

1) Prove that $G_1$ is independent of $G_2, \cdots, G_P$ for any $h \neq 0$.

2) Prove that $G_q$ is independent of the vector groups $G_{q+1}, \cdots, G_P$ for any $h \neq 0$, $q = 2, 3, \cdots, P-1$.

Step 1 - $G_1$ is independent of $G_2, \cdots, G_P$ for any $h \neq 0$.

For any $h \neq 0$, we can find a minimal index $\sigma$ ($1 \leq \sigma \leq M$) such that $h_\sigma \neq 0$. That is, $h_1 = \cdots = h_{\sigma-1} = 0$.

Let $g_{1,m}$ be the $m$-th ($1 \leq m \leq M$) column of the matrix $G_1$. In order to prove that $G_1$ is independent of $G_2, \cdots, G_P$, from Definition 3 we see it is equivalent to prove that the vector $g_{1,m}$ is independent of $\bigcup_{p=2}^{P} G_p$ for all $m = 1, 2, \cdots, M$. Further, using Definition 3 it is equivalent to prove that

$$g_{1,m} - \sum_{p=2}^{P} G_p \bar{\beta}_p \neq 0, \quad \forall m, m = 1, 2, \cdots, M,$$  \hspace{1cm} (43)

where $\bar{\beta}_p$ denotes an $M \times 1$ vector for $p = 2, \cdots, P$. Equivalently, (43) can be expressed as

$$ag_{1,m} - \sum_{p=2}^{P} G_p a \bar{\beta}_p \neq 0, \quad \forall m, m = 1, 2, \cdots, M,$$  \hspace{1cm} (44)

where $a$ is a constant and $a \neq 0$.

In order to prove (44), we can use proof by contradiction. That is, we assume that $ag_{1,m} - \sum_{p=2}^{P} G_p a \bar{\beta}_p = 0, \quad \forall m, m = 1, 2, \cdots, M$, and $a \neq 0$. Then, we examine the $\sigma$-th equation (from top to bottom) of (44) and get $ag_{1,m}^{(1)} = 0$ for $1 \leq m \leq M$, where $g_{1,m}^{(1)}$ denotes the $(\sigma,m)$-th entry of the matrix $G_1$. This is because all top $\sigma$ rows of $G_2, \cdots, G_P$ are all zeros when $h_1 = \cdots = h_{\sigma-1} = 0$ as seen from (23). Again from (23), we have $g_{1,m}^{(1)} = h_\sigma \theta_{\sigma,m}$, where $\theta_{\sigma,m}$ denotes the $(\sigma,m)$-th entry of the matrix $\Theta$. For given $h_\sigma \neq 0$ and $\theta_{\sigma,m} \neq 0$ for all
Step 2 - $G_q$ is independent of the vector groups $G_{q+1}, \cdots, G_P$ for any $h \neq 0, q = 2, 3, \cdots, P-1$.

For any $h \neq 0$, we can have a minimal index $\sigma$ ($1 \leq \sigma \leq M$) such that $h_\sigma \neq 0$. That is, $h_1 = \cdots = h_{\sigma-1} = 0$.

Following the similar way in the proof of Step 1, we can prove Step 2 as follows.

Let $g_{q,m}$ denote the $m$-th ($1 \leq m \leq M$) column of $G_q$, $q = 2, 3, \cdots, P-1$. In order to prove that $G_q$ is independent of $G_{q+1}, \cdots, G_P$, from Definition 2 it is equivalent to prove that the vector $g_{q,m}$ is independent of $\bigcup_{p=q+1}^{P} G_p$ for all $m = 1, 2, \cdots, M$. Further, using Definition 3 it is equivalent to prove that

$$ag_{q,m} - \sum_{p=q+1}^{P} G_p a \bar{\beta}_p \neq 0, \quad \forall m, m = 1, 2, \cdots, M,$$

where $\bar{\beta}_p$ denotes an $M \times 1$ vector for $p = q+1, \cdots, P$. Equivalently, (45) can be expressed as

$$a g_{q,m} - \sum_{p=q+1}^{P} G_p a \bar{\beta}_p \neq 0, \quad \forall m, m = 1, 2, \cdots, M,$$

(46)

where $a$ is a constant and $a \neq 0$.

In order to prove (46), we can use proof by contradiction. We assume that $ag_{q,m} - \sum_{p=q+1}^{P} G_p a \bar{\beta}_p = 0, \quad \forall m, m = 1, 2, \cdots, M$, and $a \neq 0$. Then, we examine the $(\sigma + q - 1)$-th equation (from top to bottom) of (46) and get $ag_{\sigma+q-1,m} = 0$ for $1 \leq m \leq M$, where $g_{\sigma+q-1,m}^{(q)}$ denotes the $(\sigma + q - 1, m)$-th entry of the matrix $G_q$. This is because the top $(\sigma + q - 1)$ rows of $G_{q+1}, \cdots, G_P$ are all zeros when $h_1 = \cdots = h_{\sigma-1} = 0$ as seen from (23). Again from (23), we have $g_{\sigma+q-1,m}^{(q)} = h_\sigma \theta_{\sigma,m}$ where $\theta_{\sigma,m}$ denotes the $(\sigma, m)$-th entry of the matrix $\Theta$. For given $h_\sigma \neq 0$ and $\theta_{\sigma,m} \neq 0$ for all $m = 1, 2, \cdots, M$, we then get $a = 0$. This contradicts with the assumption $a \neq 0$. Therefore, (46) holds and Step 2 is proved.

To summarize Step 1 and Step 2, we prove that $G_1, G_2, \cdots, G_P$ are linearly independent vector groups for any $h \neq 0$.

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Fig. 1. BER performance of various codes.
Fig. 2. BER performance of the proposed code for 5 transmit antennas.