Spinodal Decomposition in Finite Temperature $SU(2)$ and $SU(3)$

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After a rapid increase in temperature across the deconfinement temperature $T_d$ to temperatures $T \gg T_d$, pure gauge theories exhibit unstable long wavelength fluctuations in the approach to equilibrium. This phenomenon is analogous to spinodal decomposition observed in condensed matter physics, and also seen in models of disordered chiral condensate formation. At high temperature, the unstable modes occur only in the range $0 < k < k_c$, where $k_c$ is on the order of the Debye screening mass $m_D$. Equilibration always occurs via spinodal decomposition for $SU(2)$. For $SU(3)$ at $T$ near $T_d$, nucleation replaces spinodal decomposition as the dominant equilibration mechanism. Monte Carlo simulations of $SU(2)$ and $SU(3)$ lattice gauge theories exhibit the predicted phenomena.

For $SU(2)$, the observed value of $k_c$ is in reasonable agreement with a value predicted from previous lattice measurements of $m_D$.

1. Introduction

Much of the experimental relevance of finite temperature QCD comes from non-equilibrium situations, as in the case of heavy ion collisions. Current lattice techniques are restricted to equilibrium simulations, and cannot address directly non-equilibrium time evolution. However, there are aspects of non-equilibrium behavior that are determined by the equilibrium effective action. After a rapid change in temperature, induced by some external agency, it is the effective potential that determines the process of re-equilibration, in particular whether relaxation, nucleation or spinodal decomposition is the dominant equilibration mechanism. We have shown that a large, rapid increase of temperature across the deconfinement temperature results in spinodal decomposition in pure $SU(2)$ and $SU(3)$ gauge theories. In particular, we have found the existence of exponentially growing long wavelength modes in the approach to equilibrium, characteristic of spinodal decomposition. This behavior depends only on the features of the equilibrium effective action in an unstable region.

We consider first the generic case of a pure $SU(N)$ gauge theory in which the temperature is raised rapidly from a temperature less than $T_d$, the deconfinement temperature, to a temperature above $T_d$. We refer to such a rapid increase in system temperature as a quench, a borrowing of nomenclature from condensed matter physics. The equilibrium order parameter for the deconfinement transition is the Polyakov loop $L_F$. At temperatures $T < T_d$, the $Z(N)$ global symmetry associated with confinement is unbroken, and $\langle L_F \rangle = 0$. When the temperature is rapidly increased to $T > T_d$, $\langle L_F \rangle = 0$ is no longer the stable state of the system, and must evolve to a new equilibrium state with $\langle L_F \rangle \neq 0$, reflecting the transition to the gluon plasma phase.

2. Effective Potential

The instability of $\langle L_F \rangle = 0$ at high temperatures is seen directly from the one-loop finite temperature effective potential for the Polyakov loop, as derived by Gross, Pisarski and Yaffe and Weiss. For simplicity, consider the case of $SU(2)$. It is convenient to parametrize the Polyakov loop as

$$L_F = \cos \left[ \pi \left( 1 - \frac{\psi}{2} \right) \right],$$

where $-1 \leq \psi \leq 1$. The effective potential at one loop for gauge bosons in a constant Polyakov...
The effective action takes the form
\[ V(\psi) = -\frac{\pi^2 T^4}{15} + \frac{\pi^2 T^4}{12} (1 - \psi^2)^2. \]  
(2)

The one loop result dominates the effective potential for \( T \gg \Lambda_{QCD} \) due to asymptotic freedom. The first term is the standard black body result, obtained when \( \psi = 1 \). The use of the variable \( \psi \) makes the \( Z(2) \) symmetry of the potential under \( \psi \to -\psi \) manifest. Note that the equilibrium value of \( \psi \) is \( \pm 1 \), corresponding to \( L_F = \pm 1 \); \( \psi = 0 \), corresponding to \( L_F = 0 \), is a maximum of \( V(\psi) \).

Our picture of the quenching process is that the system is initially in a state where \( \psi \) is equal to zero at some temperature below \( T_d \). When the system is quickly raised to a new temperature \( T > T_d \), the system is still in the state with \( \psi = 0 \). However, the system is unstable, and must eventually find its way to either \( \psi = +1 \) or \( \psi = -1 \). Because we quench into a region of the phase diagram where \( V''(\psi) < 0 \), the system will decay to the equilibrium state via spinodal decomposition.

Because pure \( SU(2) \) gauge theory has a second-order deconfining transition, spinodal decomposition will occur after quenching to any temperature \( T > T_d \). The situation is more subtle for \( SU(3) \), where the deconfinement phase transition is first order. This implies the existence of a metastable confined phase for some range of temperatures above \( T_d \), which in turn implies that nucleation is the dominant equilibration mechanism just above \( T_d \). Perturbation theory gives no hint of this behavior, and the one-loop effective potential is unstable at \( L_F = 0 \). For temperatures sufficiently large that the one-loop effective potential for \( L \) is reliable, spinodal decomposition will occur.

3. Langevin Model

In order to study the dynamics of this transition, we use the effective action of Bhattacharyya combined with Langevin dynamics.\[ \text{For } SU(2), \text{ the effective action takes the form} \]
\[ S_{\text{eff}}[\psi] = \int \! d^3 \! x \left[ \frac{\pi^2 T}{2g^2} (\nabla \psi)^2 + V(\psi)/T \right]. \]  
(3)

The equilibrium distribution will be \( \exp \left[ -S_{\text{eff}}[\psi] \right] \). We postulate Langevin dynamics of the form
\[ \frac{\partial \psi(x, \tau)}{\partial \tau} = -\frac{\delta S_{\text{eff}}[\psi]}{\delta \psi(x, \tau)} + \eta(x, t) \]  
(4)
where the white noise \( \eta \) is normalized to
\[ \langle \eta(x, \tau) \eta(x', \tau') \rangle = 2\Gamma^3 (x - x') \delta(\tau - \tau'). \]  
(5)

The late-time relaxational behavior of \( \psi \) is controlled by the Debye screening mass \( m_D \), given by \( m_D^2 = 2g^2 T^2 / 3 \). Near equilibrium, any effects of initial conditions decay exponentially as \( \exp \left[ -2\pi^2 \Gamma x (k^2 + m_D^2) \right] \). However, for early times, the initial conditions contribute to \( \langle \bar{\psi}(k, \tau) \bar{\psi}(-k, \tau) \rangle \) a term of the form
\[ \bar{\psi}(k, 0) \bar{\psi}(-k, 0) \exp \left[ \frac{2\pi^2 \Gamma T}{g^2} (k^2 - k_c^2) \right] \]  
(6)
where \( k_c^2 = g^2 T^2 / 3 = m_D^2 / 2 \). Modes with \( k < k_c \) are initially not damped but grow exponentially, with the \( k = 0 \) mode growing the fastest. This is a characteristic feature of spinodal decomposition with a non-conserved order parameter.\[ \text{4. Monte Carlo Results} \]

We have verified the existence of spinodal decomposition with simulations of rapidly quenched pure \( SU(2) \) gauge theory. Lattices of size \( 32^3 \times 4 \) and \( 64^3 \times 4 \) were equilibrated at \( \beta = 2.0 \), below the deconfinement transition at \( \beta_d = 2.2986 \pm 0.0006 \). The coupling constant was increased instantaneously to \( \beta = 3.0 \), and the approach to equilibrium monitored via the Polyakov loop and other observables. The heat bath algorithm was used both for equilibration at low temperature before the quench and for subsequent dynamical evolution after the quench. While this time evolution is not the true time evolution of the nonequilibrium quantum field theory, features such as spinodal decomposition which depend only on the equilibrium effective action will occur with any local updating algorithm which converges to the equilibrium distribution. The abrupt change in \( \beta \) is a potential cause of concern with this procedure, since the lattice spacing, and hence the
physical volume, changes in all directions when $\beta$ is changed. However, the large spatial sizes used should mitigate this effect.

Figure 1. $S(k, \tau)$ versus Monte Carlo time.

Figure 1 shows the Fourier transform of the connected Polyakov two point function $S(k, \tau)$ for low values of the wave number as a function of Monte Carlo time for the same simulation. Note the early exponential rise in these modes, followed by a sharp disappearance as the Polyakov loop reaches its equilibrium value, characteristic of spinodal decomposition. Only the low momentum modes exhibit this growth; above $k_c$ no such growth occurs. Although the general behavior of $S(k, \tau)$ is the same for each run, many details are run dependent. In this particular run, the $k/T = 0.68$ mode achieves a larger amplitude than $k/T = 0.56$, which is atypical. In some runs, there is clear evidence for mode-mode coupling, reflecting the nonlinearity of the system. For each run, we have estimated the rate of growth of each low-momentum mode by fitting $\log(S(k, \tau))$ to a straight line in $\tau$ for early times. We can extract $k_c$ as the value where the growth rate is zero. From equation (6), the growth rate of each line is proportional to $k_c^2 - k^2$ for the linearized theory, but this may not represent the true time evolution of the simulation. In any case, the growth rates measured in each run are highly dependent on initial conditions. In figure 2, we plot these growth rates versus $k^2/T^2$ for the same run used in figure 1. The error bars are naive estimates of the error for each growth rate for this particular run. The x intercept provides an estimate of $k_c^2$ for each run. Using multiple $64^3 \times 4$ runs at $\beta = 3.0$, we have estimated $k_c/T$ to be $1.14 \pm 0.02$. The principle errors in this estimate come from the sensitivity of individual modes to initial conditions and the discrete character of $k$ on the lattice. We can compare this with lattice measurements of the Debye screening length, assuming the one-loop relation

$$
\frac{k_c}{T} = \frac{m_D(T)}{\sqrt{2T}}
$$

holds in general. Using the results of Heller et al. [8] for $m_D(T)$, we obtain $k_c/T = 1.35(5)$. We consider this to be reasonable agreement, given the many uncertainties involved.

It is natural to ask what happens when rapid cooling takes place, as might occur in the late stages of the expansion of a quark-gluon plasma, or in the early universe. At low temperatures, we expect that $V(\psi)$ has a single minimum and $V''(\psi) > 0$ everywhere. Thus theory predicts the absence of unstable modes, and that relaxational processes should dominate the approach to equilibrium. Simulations of such a cooling process, in which $32^3 \times 4$ lattice configurations equilibrated at $\beta = 3.0$ are suddenly cooled to $\beta = 2.0$, shows no sign of spinodal decomposition.
We have begun studies of the more physically relevant case of $SU(3)$. Since nucleation dominates at temperatures just above $T_d$, it is necessary to go to higher temperatures to observe spinodal decomposition. Working with $32^3 \times 4$ lattice configurations equilibrated at $\beta = 5.5$, we performed quenches to various values of $\beta > \beta_d \approx 5.69$. For values of $\beta$ close to $\beta_d$, we believe we see evidence for metastability. We have observed exponentially growing low wavelength modes for quenches to $\beta \geq 5.80$. In figure 3, we show the evolution of the three lowest non-zero modes as a function of simulation time for a quench from $\beta = 5.5$ to $\beta = 5.92$. Note the similarity to the behavior seen in $SU(2)$. A systematic investigation is underway, emphasizing $k_c$ as well as the limit of metastability.

5. Conclusions

Analytical and simulation results both indicate the relevance of spinodal decomposition in the equilibration of a gluon plasma after a rapid quench to high temperature. In the case of $SU(N)$ gauge theories, we have shown that the physical parameter $k_c$ which controls domain growth can be determined from simulations. In the case of $SU(3)$, determination of $k_c$ as a function of $\beta$ will allow us to define the limit of metastability as the point $k_c(\beta_m) = 0$. Note that small values of $k_c$ require correspondingly large lattice sizes to see non-zero exponentially growing modes.

Just as the formation of a DCC\cite{9} may lead to enhanced production of low-momentum pions, spinodal decomposition could lead to enhanced production of low-momentum gluons in the early stages of plasma formation. The characteristic scale for such a phenomenon would be $gT$.

Full QCD has quarks as well as gluons, and the phase structure is different. It would be very interesting to study the behavior of both the Polyakov loop and the chiral condensate after abrupt changes in temperature. The Polyakov loop can be studied fairly easily if large equilibrated unquenched lattice field configurations are available. However, the chiral condensate requires a reliable estimator for the local condensate as a function of space and simulation time.

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