A note on the existence of $H$-bubbles via perturbation methods

VERONICA FELLI *

S.I.S.S.A. - Via Beirut 2-4
34014 Trieste, Italy
e-mail: felli@sissa.it

Abstract
We study the problem of existence of surfaces in $\mathbb{R}^3$ parametrized on the sphere $S^2$ with prescribed mean curvature $H$ in the perturbative case, i.e. for $H = H_0 + \varepsilon H_1$, where $H_0$ is a nonzero constant, $H_1$ is a $C^2$ function and $\varepsilon$ is a small perturbation parameter.

Key Words: $H$-surfaces, nonlinear elliptic systems, perturbative methods.
MSC classification: 53A10, 35J50, 35B20.

1 Introduction

In this paper we are interested in the existence of $H$-bubbles, namely of $S^2$-type parametric surfaces in $\mathbb{R}^3$ with prescribed mean curvature $H$. This geometrical problem is motivated by some models describing capillarity phenomena and has the following analytical formulation: given a function $H \in C^1(\mathbb{R}^3)$, find a smooth nonconstant function $\omega : \mathbb{R}^2 \to \mathbb{R}^3$ which is conformal as a map on $S^2$ and solves the problem

\[
\begin{cases}
\Delta \omega = 2H(\omega) \omega_x \wedge \omega_y, & \text{in } \mathbb{R}^2, \\
\int_{\mathbb{R}^2} |\nabla \omega|^2 < +\infty,
\end{cases}
\]

(P$_H$)

where $\omega_x = \left( \frac{\partial \omega_1}{\partial x}, \frac{\partial \omega_2}{\partial x}, \frac{\partial \omega_3}{\partial x} \right)$, $\omega_y = \left( \frac{\partial \omega_1}{\partial y}, \frac{\partial \omega_2}{\partial y}, \frac{\partial \omega_3}{\partial y} \right)$, $\Delta \omega = \omega_{xx} + \omega_{yy}$, $\nabla \omega = (\omega_x, \omega_y)$, and $\wedge$ denotes the exterior product in $\mathbb{R}^3$.

*Supported by MIUR, national project “Variational Methods and Nonlinear Differential Equations”.

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Brezis and Coron [4] proved that for constant nonzero mean curvature $H(u) \equiv H_0$ the only nonconstant solutions are spheres of radius $|H_0|^{-1}$.

While the Plateau and the Dirichlet problems has been largely studied both for $H$ constant and for $H$ nonconstant (see [3, 4, 10, 12, 13, 14, 15, 16]), problem $(P_H)$ in the case of nonconstant $H$ has been investigated only recently, see [5, 6, 7]. In [5] Caldiroli and Musina proved the existence of $H$-bubbles with minimal energy under the assumptions that $H \in C^1(\mathbb{R}^3)$ satisfies

(i) $\sup_{u \in \mathbb{R}^3} |\nabla H(u + \xi) \cdot u u| < 1$ for some $\xi \in \mathbb{R}^3$,

(ii) $H(u) \to H_\infty$ as $|u| \to \infty$ for some $H_\infty \in \mathbb{R}$,

(iii) $c_H = \inf_{u \in C^1(\mathbb{R}^3, \mathbb{R}^3)} \sup_{s > 0} \mathcal{E}_H(su) < \frac{4\pi}{3H_\infty^2}$

where $\mathcal{E}_H(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + 2 \int_{\mathbb{R}^3} Q(u) \cdot u_x \wedge u_y$ and $Q : \mathbb{R}^3 \to \mathbb{R}^3$ is any vector field such that $\text{div} \ Q = H$.

The perturbative method introduced by Ambrosetti and Badiale [1, 2] was used in [7] to treat the case in which $H$ is a small perturbation of a constant, namely

$$H(u) = H_\varepsilon(u) = H_0 + \varepsilon H_1(u),$$

where $H_0 \in \mathbb{R} \setminus \{0\}$, $H_1 \in C^2(\mathbb{R}^3)$, and $\varepsilon$ is a small real parameter. This method allows to find critical points of a functional $f_\varepsilon$ of the type $f_\varepsilon(u) = f_0(u) - \varepsilon G(u)$ in a Banach space by studying a finite dimensional problem. More precisely, if the unperturbed functional $f_0$ has a finite dimensional manifold of critical points $Z$ which satisfies a nondegeneracy condition, it is possible to prove, for $|\varepsilon|$ sufficiently small, the existence of a smooth function $\eta_\varepsilon(z) : Z \to (T_zZ)^\perp$ such that any critical point $z \in Z$ of the function

$$\Phi_\varepsilon : Z \to \mathbb{R}, \quad \Phi_\varepsilon(z) = f_\varepsilon(z + \eta_\varepsilon(z))$$

gives rise to a critical point $u_\varepsilon = z + \eta_\varepsilon(z)$ of $f_\varepsilon$, i.e. the perturbed manifold $Z_\varepsilon := \{z + \eta_\varepsilon(z) : z \in Z\}$ is a natural constraint for $f_\varepsilon$. Furthermore $\Phi_\varepsilon$ can be expanded as

$$\Phi_\varepsilon(z) = b - \varepsilon \Gamma(z) + o(\varepsilon) \quad \text{as} \quad \varepsilon \to 0 \quad (1.1)$$

where $b = f_0(z)$ and $\Gamma$ is the Melnikov function defined as the restriction of the perturbation $G$ on $Z$, namely $\Gamma = G|_Z$. For problem $(P_{H_\varepsilon})$, $\Gamma$ is given by

$$\Gamma : \mathbb{R}^3 \to \mathbb{R}, \quad \Gamma(p) = \int_{|p-q| < \frac{1}{|p_0|}} H_1(q) \, dq.$$ 

In [7] Caldiroli and Musina studied the functional $\Gamma$ giving some existence results in the perturbative setting for problem $(P_{H_\varepsilon})$. They prove that for $|\varepsilon|$ small there exists a smooth
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$H_\varepsilon$-bubble if one of the following conditions holds

1) $H_1$ has a nondegenerate stationary point and $|H_0|$ is large;
2) $\max_{p \in \partial K} H_1(p) < \max_{p \in K} H_1(p)$ or $\min_{p \in \partial K} H_1(p) > \min_{p \in K} H_1(p)$
   for some nonempty compact set $K \subset \mathbb{R}^3$ and $|H_0|$ is large;
3) $H_1 \in L^r(\mathbb{R}^3)$ for some $r \in [1,2]$.

They prove that critical points of $\Gamma$ give rise to solutions to $(P_{H_\varepsilon})$ for $\varepsilon$ sufficiently small. Precisely the assumption that $H_0$ is large required in cases 1) and 2) ensures that if $H_1$ is not constant then $\Gamma$ is not identically constant. If we let this assumption drop, it may happen that $\Gamma$ is constant even if $H_1$ is not. This fact produces some loss of information because the first order expansion (1.1) is not sufficient to deduce the existence of critical points of $\Phi_\varepsilon$ from the existence of critical points of $\Gamma$. Instead of studying $\Gamma$ we perform a direct study of $\Phi_\varepsilon$ which allows us to prove some new results. In the first one, we assume that $H_1$ vanishes at $\infty$ and has bounded gradient, and prove the existence of a solution without branch points. Let us recall that a branch point for a solution $\omega$ to $(P_H)$ is a point where $\nabla \omega = 0$, i.e. a point where the surface parametrized by $\omega$ fails to be immersed.

**Theorem 1.1** Let $H_0 \in \mathbb{R} \setminus \{0\}$, $H_1 \in C^2(\mathbb{R}^3)$ such that

(H1) $\lim_{|p| \to \infty} H_1(p) = 0$;
(H2) $\nabla H_1 \in L^\infty(\mathbb{R}^3, \mathbb{R}^3)$.

Let $H_\varepsilon = H_0 + \varepsilon H_1$. Then for $|\varepsilon|$ sufficiently small there exists a smooth $H_\varepsilon$-bubble without branch points.

With respect to case 1) of [7] we require neither nondegeneracy of critical points of $H_1$ nor largeness of $H_0$. With respect to case 2) we do not assume that $H_0$ is large; on the other hand our assumption (H1) implies 2). Moreover we do not assume any integrability condition of type 3). With respect to the result proved in [3], we have the same kind of behavior of $H_1$ at $\infty$ (see (ii) and assumption (H1)) but we do not need any assumption of type (iii); on the other hand in [3] it is not required that the prescribed curvature is a small perturbation of a constant.

The following results give some conditions on $H_1$ in order to have two or three solutions.

**Theorem 1.2** Let $H_0 \in \mathbb{R} \setminus \{0\}$, $H_1 \in C^2(\mathbb{R}^3)$ such that $H_1$, (H2),

(H3) $\text{Hess} H_1(p)$ is positive definite for any $p \in B_{1/|H_0|}(0)$,
(H4) $H_1(p) > 0$ in $B_{1/|H_0|}(0)$,

hold. Then for $|\varepsilon|$ sufficiently small there exist at least three smooth $H_\varepsilon$-bubbles without branch points.
Remark 1.3 If we assume $(H1)$, $(H2)$, and, instead of $(H3) - (H4)$, that $H_1(0) > 0$ and $\text{Hess } H_1(0)$ is positive definite, then we can prove that for $|H_0|$ sufficiently large and $|\varepsilon|$ sufficiently small there exist at least three smooth $H_\varepsilon$-bubbles without branch points.

Theorem 1.4 Let $H_0 \in \mathbb{R} \setminus \{0\}$, $H_1 \in C^2(\mathbb{R}^3)$ such that $(H1)$ and $(H2)$ hold. Assume that there exist $p_1, p_2 \in \mathbb{R}^3$ such that

\[(H5) \quad \int_{B(p_1, 1/|H_0|)} H_1(\xi) \, d\xi > 0 \quad \text{and} \quad \int_{B(p_2, 1/|H_0|)} H_1(\xi) \, d\xi < 0.\]

Then for $|\varepsilon|$ sufficiently small there exist at least two smooth $H_\varepsilon$-bubbles without branch points.

Remark 1.5 If we assume $(H1)$, $(H2)$, and, instead of $(H5)$, that there exist $p_1, p_2 \in \mathbb{R}^3$ such that $H_1(p_1) > 0$ and $H_1(p_2) < 0$, then we can prove that for $|H_0|$ sufficiently large and $|\varepsilon|$ sufficiently small there exist at least two smooth $H_\varepsilon$-bubbles without branch points.

The present paper is organized as follows. In Section 2 we introduce some notation and recall some known facts whereas Section 3 is devoted to the proof of Theorems 1.1, 1.2, and 1.4.

Acknowledgments. The author wishes to thank Professor A. Ambrosetti and Professor R. Musina for many helpful suggestions.

2 Notation and known facts

In the sequel we will take $H_0 = 1$; this is not restrictive since we can do the change $H_1(u) = H_0 \tilde{H}_1(H_0 u)$. Hence we will always write

$$H_\varepsilon(u) = 1 + \varepsilon H(u),$$

where $H \in C^2(\mathbb{R}^3)$. Let us denote by $\omega$ the function $\omega : \mathbb{R}^2 \to S^2$ defined as

$$\omega(x, y) = (\mu(x, y)x, \mu(x, y)y, 1 - \mu(x, y)) \quad \text{where} \quad \mu(x, y) = \frac{2}{1 + x^2 + y^2}, \quad (x, y) \in \mathbb{R}^2.$$

Note that $\omega$ is a conformal parametrization of the unit sphere and solves

$$\begin{aligned}
\Delta \omega &= 2 \omega_x \wedge \omega_y \quad \text{on } \mathbb{R}^2 \\
\int_{\mathbb{R}^2} |\nabla \omega|^2 &< +\infty. \quad (2.1)
\end{aligned}$$
Problem (2.1) has in fact a family of solutions of the form $\omega \circ \phi + p$ where $p \in \mathbb{R}^3$ and $\phi$ is any conformal diffeomorphism of $\mathbb{R}^2 \cup \{\infty\}$. For $s \in (1, +\infty)$, we will set $L^s := L^s(\mathbb{S}^2, \mathbb{R}^3)$, where any map $v \in L^s$ is identified with the map $v \circ \omega : \mathbb{R}^2 \to \mathbb{R}^3$ which satisfies

$$\int_{\mathbb{R}^2} |v \circ \omega|^s \mu^2 = \int_{\mathbb{S}^2} |v|^s.$$  

We will use the same notation for $v$ and $v \circ \omega$. By $W^{1,s}$ we denote the Sobolev space $W^{1,s}(\mathbb{S}^2, \mathbb{R}^3)$ endowed (according to the above identification) with the norm

$$\|v\|_{W^{1,s}} = \left( \int_{\mathbb{R}^2} |\nabla v|^s \mu^{2-s} \right)^{1/s} + \left( \int_{\mathbb{R}^2} |v|^s \mu^2 \right)^{1/s}.$$  

If $s'$ is the conjugate exponent of $s$, i.e. $\frac{1}{s} + \frac{1}{s'} = 1$, the duality product between $W^{1,s}$ and $W^{1,s'}$ is given by

$$\langle v, \varphi \rangle = \int_{\mathbb{R}^2} \nabla v \cdot \nabla \varphi + \int_{\mathbb{R}^2} v \cdot \varphi \mu^2$$ for any $v \in W^{1,s}$ and $\varphi \in W^{1,s'}$.

Let $Q$ be any smooth vector field on $\mathbb{R}^3$ such that $\text{div} \ Q = H$. The energy functional associated to problem

$$\begin{aligned}
\begin{cases}
\Delta u = 2(1 + \varepsilon H(u)) u_x \wedge u_y, & \text{in } \mathbb{R}^2, \\
\int_{\mathbb{R}^2} |\nabla u|^2 < +\infty,
\end{cases}
\end{aligned} \quad (P_\varepsilon)$$

is given by

$$\mathcal{E}_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 + 2V_1(u) + 2\varepsilon V_H(u), \quad u \in W^{1,3},$$

where

$$V_H(u) = \int_{\mathbb{R}^2} Q(u) \cdot u_x \wedge u_y$$

has the meaning of an algebraic volume enclosed by the surface parametrized by $u$ with weight $H$ (it is independent of the choice of $Q$); in particular

$$V_1(u) = \frac{1}{3} \int_{\mathbb{R}^2} u \cdot u_x \wedge u_y.$$  

In [7], Caldiroli and Musina studied some regularity properties of $V_H$ on the space $W^{1,3}$. In particular they proved the following properties.

a) For $H \in C^1(\mathbb{R}^3)$, the functional $V_H$ is of class $C^1$ on $W^{1,3}$ and the Fréchet differential of $V_H$ at $u \in W^{1,3}$ is given by

$$dV_H(u)\varphi = \int_{\mathbb{R}^2} H(u) \varphi \cdot u_x \wedge u_y$$ for any $\varphi \in W^{1,3}$ \quad (2.2)
and admits a unique continuous and linear extension on $W^{1,3/2}$ defined by (2.2). Moreover for every $u \in W^{1,3}$ there exists $\mathcal{V}'(u) \in W^{1,3}$ such that

$$\langle \mathcal{V}'(u), \varphi \rangle = \int_{\mathbb{R}^2} H(u) \varphi \cdot u_x \wedge u_y \quad \text{for any } \varphi \in W^{1,3/2}. \quad (2.3)$$

b) For $H \in C^2(\mathbb{R}^3)$, the map $\mathcal{V}' : W^{1,3} \to W^{1,3}$ is of class $C^1$ and

$$\langle \mathcal{V}'(u), \eta, \varphi \rangle = \int_{\mathbb{R}^2} H(u) \varphi \cdot (\eta_x \wedge u_y + u_x \wedge \eta_y) + \int_{\mathbb{R}^2} (\nabla H(u) \cdot \eta) \varphi \cdot (u_x \wedge u_y)$$

for any $u, \eta \in W^{1,3}$ and $\varphi \in W^{1,3/2}$. \quad (2.4)

Hence for all $u \in W^{1,3}$, $\mathcal{E}'(u) \in W^{1,3}$ and for any $\varphi \in W^{1,3/2}$

$$\langle \mathcal{E}'(u), \varphi \rangle = \int_{\mathbb{R}^2} \nabla u \cdot \nabla \varphi + 2 \int_{\mathbb{R}^2} \varphi \cdot u_x \wedge u_y + 2 \varepsilon \int_{\mathbb{R}^2} H(u) \varphi \cdot u_x \wedge u_y.$$

As remarked in [7], critical points of $\mathcal{E}$ in $W^{1,3}$ give rise to bounded weak solutions to $(P_\varepsilon)$ and hence by the regularity theory for $H$-systems (see [9]) to classical conformal solutions which are $C^{3,\alpha}$ as maps on $\mathbb{S}^2$.

The unperturbed problem, i.e. $(P_\varepsilon)$ for $\varepsilon = 0$, has a 9-dimensional manifold of solutions given by

$$Z = \{ R\omega \circ L_{l,\xi} + p : R \in SO(3), l > 0, \xi \in \mathbb{R}^2, p \in \mathbb{R}^3 \}$$

where $L_{l,\xi} z = l(z - \xi)$ (see [11]). In [11] the nondegeneracy condition $T_u Z = \ker \mathcal{E}_0''(u)$ for any $u \in Z$ (where $T_u Z$ denotes the tangent space of $Z$ at $u$) is proved (see also [8]).

As observed in [7], in performing the finite dimensional reduction, the dependence on the 6-dimensional conformal group can be neglected since any $H$-system is conformally invariant. Hence we look for critical points of $\mathcal{E}$ constrained on a three-dimensional manifold $Z_\varepsilon$ just depending on the translation variable $p \in \mathbb{R}^3$.

### 3 Proof of Theorem 1.1

We start by constructing a perturbed manifold which is a natural constraint for $\mathcal{E}$.

**Lemma 3.1** Assume $H \in C^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and $\nabla H \in L^\infty(\mathbb{R}^3, \mathbb{R}^3)$. Then there exist $\varepsilon_0 > 0$, $C_1 > 0$, and a $C^1$ map $\eta : (-\varepsilon_0, \varepsilon_0) \times \mathbb{R}^3 \to W^{1,3}$ such that for any $p \in \mathbb{R}^3$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$

$$\mathcal{E}'(\omega + p + \eta(\varepsilon, p)) \in T_\omega Z, \quad (3.1)$$

$$\eta(\varepsilon, p) \in (T_\omega Z)^\perp, \quad (3.2)$$

$$\int_{\mathbb{S}^2} \eta(\varepsilon, p) = 0, \quad (3.3)$$

$$\|\eta(\varepsilon, p)\|_{W^{1,3}} \leq C_1 |\varepsilon|. \quad (3.4)$$
Moreover if we assume that the limit of $H$ at $\infty$ exists and

$$\lim_{|p| \to \infty} H(p) = 0 \quad (3.5)$$

we have that $\eta(\varepsilon, p)$ converges to 0 in $W^{1,3}$ as $|p| \to \infty$ uniformly with respect to $|\varepsilon| < \varepsilon_0$.

**Proof.** Let us define the map

$$F = (F_1, F_2) : \mathbb{R} \times \mathbb{R}^3 \times W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3 \to W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3$$

$$\langle F_1(\varepsilon, p, \eta, l, \alpha), \varphi \rangle = \langle E'_\varepsilon(\omega + p + \eta), \varphi \rangle - \sum_{i=1}^{6} l_i \int_{\mathbb{R}^2} \nabla \varphi \cdot \nabla \tau_i + \alpha \cdot \int_{S^2} \varphi, \quad \forall \varphi \in W^{1,3/2}$$

$$F_2(\varepsilon, p, \eta, l, \alpha) = \left( \int_{\mathbb{R}^2} \nabla \eta \cdot \nabla \tau_1, \ldots, \int_{\mathbb{R}^2} \nabla \eta \cdot \nabla \tau_6, \int_{S^2} \eta \right)$$

where $\tau_1, \ldots, \tau_6$ are chosen in $T_{\omega}Z$ such that

$$\int_{\mathbb{R}^2} \nabla \tau_i \cdot \nabla \tau_j = \delta_{ij} \quad \text{and} \quad \int_{S^2} \tau_i = 0 \quad i, j = 1, \ldots, 6$$

so that $T_{\omega}Z$ is spanned by $\tau_1, \ldots, \tau_6, e_1, e_2, e_3$. It has been proved by Caldiroli and Musina \[7\] that $F$ is of class $C^1$ and that the linear continuous operator

$$\mathcal{L} : W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3 \to W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3$$

$$\mathcal{L} = \frac{\partial F}{\partial (\eta, l, \alpha)}(0, p, 0, 0, 0)$$

i.e.

$$\langle \mathcal{L}_1(v, \mu, \beta), \varphi \rangle = \langle E''_0(\omega) \cdot v, \varphi \rangle - \sum_{i=1}^{6} \mu_i \int_{\mathbb{R}^2} \nabla \varphi \cdot \tau_i - \beta \int_{S^2} \varphi, \quad \forall \varphi \in W^{1,3/2}$$

$$\mathcal{L}_2(v, \mu, \beta) = \left( \int_{\mathbb{R}^2} \nabla v \cdot \nabla \tau_1, \ldots, \int_{\mathbb{R}^2} \nabla v \cdot \nabla \tau_6, \int_{S^2} v \right)$$

is invertible. Caldiroli and Musina applied the Implicit Function Theorem to solve the equation $F(\varepsilon, p, \eta, l, \alpha) = 0$ locally with respect to the variables $\varepsilon, p$, thus finding a $C^1$-function $\eta$ on a neighborhood $(-\varepsilon_0, \varepsilon_0) \times B_R \subset \mathbb{R} \times \mathbb{R}^3$ satisfying (3.1), (3.2), and (3.3). We will use instead the Contraction Mapping Theorem, which allows to prove the existence of such a function $\eta$ globally on $\mathbb{R}^3$, thanks to the fact that the operator $\mathcal{L}$ does not depend on $p$ and hence it is invertible uniformly with respect to $p \in \mathbb{R}^3$.

We have that $F(\varepsilon, p, \eta, l, \alpha) = 0$ if and only if $(\eta, l, \alpha)$ is a fixed point of the map $T_{\varepsilon,p}$ defined as

$$T_{\varepsilon,p}(\eta, l, \alpha) = -\mathcal{L}^{-1} F(\varepsilon, p, \eta, l, \alpha) + (\eta, l, \alpha).$$
To prove the existence of $\eta$ satisfying (3.1), (3.2), and (3.3), it is enough to prove that $T_{\varepsilon,p}$ is a contraction in some ball $B_p(0)$ with $p = p(\varepsilon) > 0$ independent of $p$, whereas the regularity of $\eta(\varepsilon, p)$ follows from the Implicit Function Theorem.

We have that if $\|\eta\|_{W^{1,3}} \leq \rho$

$$
\|T_{\varepsilon,p}(\eta, l, \alpha)\|_{W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3}
\leq C_2\|F(\varepsilon, p, \eta, l, \alpha) - \mathcal{L}(\eta, l, \alpha)\|_{W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3}
\leq C_2\|\mathcal{E}_\varepsilon'(\omega + p + \eta) - \mathcal{E}_0''(\omega)\eta\|_{W^{1,3}}
\leq C_2\left(\int_0^1 \|\mathcal{E}_0''(\omega + t\eta) - \mathcal{E}_0''(\omega)\|_{W^{1,3}} dt\right)\|\eta\|_{W^{1,3}}
\leq C_2\rho \sup_{\|\eta\|_{W^{1,3}} \leq \rho} \|\mathcal{E}_0''(\omega + \eta) - \mathcal{E}_0''(\omega)\|_{W^{1,3}} + 2C_2\|\mathcal{E}_0''(\omega + \eta)\|_{W^{1,3}}
\leq C_2\rho \sup_{\|\eta\|_{W^{1,3}} \leq \rho} \|\mathcal{E}_0''(\omega + \eta) - \mathcal{E}_0''(\omega)\|_{W^{1,3}} + 2C_2\|\mathcal{E}_0''(\omega + \eta)\|_{W^{1,3}}
\leq C_2\rho \sup_{\|\eta\|_{W^{1,3}} \leq \rho} \|\mathcal{E}_0''(\omega + \eta) - \mathcal{E}_0''(\omega)\|_{W^{1,3}} + 2C_2\|\mathcal{E}_0''(\omega + \eta)\|_{W^{1,3}}
\leq C_2\rho \sup_{\|\eta\|_{W^{1,3}} \leq \rho} \|\mathcal{E}_0''(\omega + \eta) - \mathcal{E}_0''(\omega)\|_{W^{1,3}} + 2C_2\|\mathcal{E}_0''(\omega + \eta)\|_{W^{1,3}}
\leq \sup_{\|\eta\|_{W^{1,3}} \leq 3\rho} \|\mathcal{E}_0''(\omega + \eta) - \mathcal{E}_0''(\omega)\|_{W^{1,3}} + 2C_2\|\mathcal{E}_0''(\omega + \eta)\|_{W^{1,3}}.
$$

(3.6)

where $C_2 = \|\mathcal{L}^{-1}\|_{\mathcal{L}(W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3)}$. For $(\eta_1, l_1, \alpha_1), (\eta_2, l_2, \alpha_2) \in B_p(0) \subset W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3$ we have

$$
\frac{\|T_{\varepsilon,p}(\eta_1, l_1, \alpha_1) - T_{\varepsilon,p}(\eta_2, l_2, \alpha_2)\|_{W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3}}{C_2\|\eta_1 - \eta_2\|_{W^{1,3}}}
\leq \frac{\|\mathcal{E}_\varepsilon'(\omega + p + \eta_1) - \mathcal{E}_\varepsilon'(\omega + p + \eta_2) - \mathcal{E}_0''(\omega)(\eta_1 - \eta_2)\|_{W^{1,3}}}{C_2\|\eta_1 - \eta_2\|_{W^{1,3}}}
\leq \int_0^1 \|\mathcal{E}_0''(\omega + \eta_2 + t(\eta_1 - \eta_2)) - \mathcal{E}_0''(\omega)\|_{W^{1,3}} dt
\leq \int_0^1 \|\mathcal{E}_0''(\omega + \eta_2 + t(\eta_1 - \eta_2)) - \mathcal{E}_0''(\omega)\|_{W^{1,3}} dt
+ 2\|\mathcal{E}_0''(\omega + \eta_2 + t(\eta_1 - \eta_2))\|_{W^{1,3}} dt
\leq \sup_{\|\eta\|_{W^{1,3}} \leq 3\rho} \|\mathcal{E}_0''(\omega + \eta) - \mathcal{E}_0''(\omega)\|_{W^{1,3}} + 2\|\mathcal{E}_0''(\omega + \eta)\|_{W^{1,3}}.
$$

From (2.3), (2.4), and the Hölder inequality it follows that there exists a positive constant $C_3$ such that for any $\eta \in W^{1,3}, p \in \mathbb{R}^3$

$$
\|\mathcal{V}_H'(\omega + p + \eta)\|_{W^{1,3}} \leq C_3\left(\int_{\mathbb{R}^2} |H(\omega + p + \eta)|^{3/2} |\nabla \omega|^2 \mu^{-1}\right)^{2/3} + \|\eta\|_{W^{1,3}}^2.
$$

(3.7)

$$
\|\mathcal{V}_H''(\omega + p + \eta)\|_{W^{1,3}} \leq C_3\left[\left(\int_{\mathbb{R}^2} |H(\omega + p + \eta)|^2 |\nabla (\omega + \eta)|^2\right)^{1/2}
+ \left(\int_{\mathbb{R}^2} |D H(\omega + p + \eta)|^{3/2} |\nabla (\omega + \eta)|^3 \mu^{-1}\right)^{2/3}\right].
$$

(3.8)
Choosing $\rho_0 > 0$ such that
\[
C_2 \sup_{\|\eta\|_{W^{1,3}} \leq 3\rho_0} \|\mathcal{E}_0''(\omega + \eta) - \mathcal{E}_0''(\omega)\|_{W^{1,3/2}} < \frac{1}{2}
\]
and $\varepsilon_0 > 0$ such that
\[
8C_2C_3\varepsilon_0 \|H\|_{L^\infty(\mathbb{R}^3)} \|\omega\|^2_{W^{1,3}} < \min\left\{1, \rho_0, \frac{1}{8C_2C_3\varepsilon_0}\right\}, \tag{3.9}
\]
\[
\sup_{\|\eta\|_{W^{1,3}} \leq \rho_0, p \in \mathbb{R}^3} \|\nabla H(\omega + p + \eta)\|_{W^{1,3}} < \frac{\rho_0}{6\varepsilon_0 C_2}, \tag{3.10}
\]
\[
\sup_{\|\eta\|_{W^{1,3}} \leq 3\rho_0, p \in \mathbb{R}^3} \|\nabla'' H(\omega + p + \eta)\|_{W^{1,3/2}} < \frac{1}{8\varepsilon_0 C_2}, \tag{3.11}
\]
we obtain that $T_{\varepsilon,p}$ maps the ball $B_{\rho_0}(0)$ into itself for any $|\varepsilon| < \varepsilon_0$, $p \in \mathbb{R}^3$, and is a contraction there. Hence it has a unique fixed point $(\eta(\varepsilon, p), l(\varepsilon, p), \alpha(\varepsilon, p)) \in B_{\rho_0}(0)$. From (3.4) we have that the following property holds

\[
(*) \quad T_{\varepsilon,p} \text{ maps a ball } B_{\rho}(0) \subset W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3 \text{ into itself whenever } \rho \leq \rho_0 \text{ and } \rho > 4|\varepsilon|C_2 \sup_{\|\eta\|_{W^{1,3}} \leq \rho} \|\nabla H(\omega + p + \eta)\|_{W^{1,3}}.
\]

In particular let us set
\[
\rho_{\varepsilon} = 5|\varepsilon|C_2 \sup_{\|\eta\|_{W^{1,3}} \leq \rho_0, p \in \mathbb{R}^3} \|\nabla H(\omega + p + \eta)\|_{W^{1,3}}. \tag{3.12}
\]

In view of (3.10) and (3.12), we have that for any $|\varepsilon| < \varepsilon_0$ and for any $p \in \mathbb{R}^3$
\[
\rho_{\varepsilon} \leq \rho_0 \quad \text{and} \quad \rho_{\varepsilon} > 4|\varepsilon|C_2 \sup_{\|\eta\|_{W^{1,3}} \leq \rho_{\varepsilon}} \|\nabla H(\omega + p + \eta)\|_{W^{1,3}}
\]

so that, due to (*), $T_{\varepsilon,p}$ maps $B_{\rho_{\varepsilon}}(0)$ into itself. From the uniqueness of the fixed point we have that for any $|\varepsilon| < \varepsilon_0$ and $p \in \mathbb{R}^3$
\[
\|\eta(\varepsilon, p), l(\varepsilon, p), \alpha(\varepsilon, p)\|_{W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3} \leq \rho_{\varepsilon} \leq C_1|\varepsilon| \tag{3.13}
\]
for some positive constant $C_1$ independent of $p$ and hence $\|\eta(\varepsilon, p)\|_{W^{1,3}} \leq \rho_{\varepsilon} \leq C_1|\varepsilon|$ thus proving (3.4). Assume now (3.3) and set for any $p \in \mathbb{R}^3$
\[
\rho_{p} = 8C_2C_3\varepsilon_0 \left( \int_{\mathbb{R}^2} \sup_{|q-p| \leq 1+C_0} |H(q)|^{3/2}|\nabla \omega|^3 \mu^{-1} \right)^{2/3}
\]
where $C_0$ is a positive constant such that $\|u\|_{L^\infty} \leq C_0\|u\|_{W^{1.3}}$ for any $u \in W^{1.3}$. From (3.9) we have that

$$\rho_p < \min \left\{ 1, \rho_0, \frac{1}{8C_2C_3\varepsilon_0} \right\}.$$  

Hence, due to (3.7), we have that for $|\varepsilon| < \varepsilon_0$ and $\|\eta\|_{W^{1,3}} \leq \rho_p$

$$4|\varepsilon|C_2\|Y_H'(\omega + p + \eta)\|_{W^{1,3}} \leq 4\varepsilon_0C_2C_3\left( \int_{\mathbb{R}^2} \sup_{|q-p| \leq 1 + C_0} |H(q)|^{3/2}|\nabla \omega|^3 \mu^{-1} \right)^{2/3} + 4\varepsilon_0C_2C_3\rho_p^2 < \rho_p.$$  

From $(\ast)$ and the uniqueness of the fixed point, we deduce that $\|\eta(\varepsilon, p)\|_{W^{1,3}} \leq \rho_p$ for any $|\varepsilon| < \varepsilon_0$ and $p \in \mathbb{R}^3$. On the other hand, since $H$ vanishes at $\infty$, by the definition of $\rho_p$ we have that $\rho_p \to 0$ as $|p| \to \infty$, hence

$$\lim_{|p| \to \infty} \eta(\varepsilon, p) = 0 \text{ in } W^{1,3} \text{ uniformly for } |\varepsilon| < \varepsilon_0.$$  

The proof of Lemma 3.1 is now complete. \hfill \Box

**Remark 3.2** The map $\eta$ given in Lemma 3.1 satisfies

$$\langle \mathcal{E}'_\varepsilon(\omega + p + \eta(\varepsilon, p)), \varphi \rangle - \sum_{i=1}^6 l_i(\varepsilon, p) \int_{\mathbb{R}^2} \nabla \varphi \cdot \nabla \tau_i + \alpha(\varepsilon, p) \cdot \int_{S^2} \varphi, \quad \forall \varphi \in W^{1,3/2}$$

where $(\eta(\varepsilon, p), l(\varepsilon, p), \alpha(\varepsilon, p)) \in \overline{B_{\rho_\varepsilon}(0)} \subset W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3$ being $\rho_\varepsilon$ given in (3.12), hence

$$\int_{\mathbb{R}^2} \nabla(\omega + \eta(\varepsilon, p)) \cdot \nabla \varphi + 2 \int_{\mathbb{R}^2} \varphi \cdot (\omega + \eta(\varepsilon, p))_x \wedge (\omega + \eta(\varepsilon, p))_y$$

$$+ 2\varepsilon \int_{\mathbb{R}^2} H(\omega + p + \eta(\varepsilon, p)) \varphi \cdot (\omega + \eta(\varepsilon, p))_x \wedge (\omega + \eta(\varepsilon, p))_y$$

$$= \sum_{i=1}^6 l_i(\varepsilon, p) \int_{\mathbb{R}^2} \nabla \varphi \cdot \nabla \tau_i - \alpha(\varepsilon, p) \cdot \int_{S^2} \varphi, \quad \forall \varphi \in W^{1,3/2},$$

i.e. $\eta(\varepsilon, p)$ satisfies the equation

$$\Delta \eta(\varepsilon, p) = F(\varepsilon, p)$$

where

$$F(\varepsilon, p) = 2(\omega + \eta(\varepsilon, p))_x \wedge (\omega + \eta(\varepsilon, p))_y - 2\omega_x \wedge \omega_y + l(\varepsilon, p) \cdot \Delta \tau - \alpha(\varepsilon, p) \mu^2$$

$$+ 2\varepsilon H(\omega + p + \eta(\varepsilon, p))(\omega + \eta(\varepsilon, p))_x \wedge (\omega + \eta(\varepsilon, p))_y \text{ in } \mathbb{R}^2.$$
Since \( F(\varepsilon, p) \in L^{3/2} \) and, in view of (3.4) and (3.13), \( F(\varepsilon, p) \to 0 \) in \( L^{3/2} \) as \( \varepsilon \to 0 \) uniformly with respect to \( p \), by regularity we have that
\[
\eta(\varepsilon, p) \in W^{2,3/2} \quad \text{and} \quad \eta(\varepsilon, p) \to 0 \quad \text{in} \quad W^{2,3/2}
\]
hence, by Sobolev embeddings, \( F(\varepsilon, p) \in L^3 \) and \( F(\varepsilon, p) \to 0 \) in \( L^3 \) as \( \varepsilon \to 0 \) uniformly with respect to \( p \). Again by regularity
\[
\eta(\varepsilon, p) \in W^{2,3} \quad \text{and} \quad \eta(\varepsilon, p) \to 0 \quad \text{in} \quad W^{2,3}
\]
hence \( \eta(\varepsilon, p) \in C^{1,1/3} \) and
\[
\eta(\varepsilon, p) \to 0 \quad \text{in} \quad C^{1,1/3} \quad \text{as} \quad \varepsilon \to 0 \quad \text{uniformly with respect to} \quad p. \quad (3.14)
\]
For any \( \varepsilon \in (-\varepsilon_0, \varepsilon_0) \), let us define the perturbed manifold
\[
Z_\varepsilon := \{ \omega + p + \eta(\varepsilon, p) : \ p \in \mathbb{R}^3 \}.
\]
From [2], we have that \( Z_\varepsilon \) is a natural constraint for \( \mathcal{E}_\varepsilon \), namely any critical point \( p \in \mathbb{R}^3 \) of the functional
\[
\Phi_\varepsilon : \mathbb{R}^3 \to \mathbb{R}, \quad \Phi_\varepsilon(p) = \mathcal{E}_\varepsilon(\omega + p + \eta(\varepsilon, p))
\]
gives rise to a critical point \( u_\varepsilon = \omega + p + \eta(\varepsilon, p) \) of \( \mathcal{E}_\varepsilon \).

**Proposition 3.3** Assume \( H \in C^2(\mathbb{R}^3) \), \( \nabla H \in L^\infty(\mathbb{R}^3, \mathbb{R}^3) \), and \( \lim_{|p| \to \infty} H(p) = 0 \). Then for any \( |\varepsilon| < \varepsilon_0 \)
\[
\lim_{|p| \to \infty} \Phi_\varepsilon(p) = \text{const} = \mathcal{E}_0(\omega).
\]

**Proof.** We have that
\[
\Phi_\varepsilon(p) = \mathcal{E}_\varepsilon(\omega + p + \eta(\varepsilon, p)) = \mathcal{E}_0(\omega + p + \eta(\varepsilon, p)) + 2\varepsilon \mathcal{V}_H(\omega + p + \eta(\varepsilon, p))
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \omega|^2 + \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \eta(\varepsilon, p)|^2 + \int_{\mathbb{R}^2} \nabla \omega \cdot \nabla \eta(\varepsilon, p)
\]
\[
+ \frac{2}{3} \int_{\mathbb{R}^2} (\omega + p + \eta(\varepsilon, p)) \cdot (\omega + \eta(\varepsilon, p))_x \wedge (\omega + \eta(\varepsilon, p))_y
\]
\[
+ 2\varepsilon \left[ \mathcal{V}_H(\omega + p) + \langle \mathcal{V}'_H(\omega + p), \eta(\varepsilon, p) \rangle + o(\|\eta(\varepsilon, p)\|_{W^{1,3}}) \right]
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \omega|^2 + \frac{2}{3} \int_{\mathbb{R}^2} \omega \cdot \omega_x \wedge \omega_y + \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \eta(\varepsilon, p)|^2 + \int_{\mathbb{R}^2} \nabla \omega \cdot \nabla \eta(\varepsilon, p)
\]
\[
+ \frac{2}{3} \int_{\mathbb{R}^2} \omega \cdot (\omega_x \wedge \eta(\varepsilon, p)_y + \eta(\varepsilon, p)_x \wedge \omega_y)
\]
\[
+ \frac{2}{3} \int_{\mathbb{R}^2} \omega \cdot \eta(\varepsilon, p)_x \wedge \eta(\varepsilon, p)_y + \frac{2}{3} \int_{\mathbb{R}^2} \eta(\varepsilon, p) \cdot (\omega + \eta(\varepsilon, p))_x \wedge (\omega + \eta(\varepsilon, p))_y
\]
\[
+ 2\varepsilon \mathcal{V}_H(\omega + p) + 2\varepsilon \langle \mathcal{V}'_H(\omega + p), \eta(\varepsilon, p) \rangle + 2\varepsilon o(\|\eta(\varepsilon, p)\|_{W^{1,3}})
\]
\[
(3.15)
\]
where we have used the fact that
\[ \int_{\mathbb{R}^2} p \cdot u_x \wedge u_y = 0 \quad \forall \, p \in \mathbb{R}^3, \, u \in W^{1,3}, \]
(see [7], Lemma A.3). Notice that from Lemma 3.1 we have that
\[ \int_{\mathbb{R}^2} |\nabla \eta(\varepsilon, p)|^2 \leq \sqrt{4\pi} \|\eta(\varepsilon, p)\|_{W^{1,3}}^2 \xrightarrow{|p| \to \infty} 0, \]  \tag{3.16}
and, by the Hölder inequality and Lemma 3.1
\[ \left| \int_{\mathbb{R}^2} \nabla \omega \cdot \nabla \eta(\varepsilon, p) \right| \leq \sqrt{4\pi} \left( \int_{\mathbb{R}^2} |\nabla \omega|^2 \right)^{1/2} \|\eta(\varepsilon, p)\|_{W^{1,3}} \xrightarrow{|p| \to \infty} 0, \]  \tag{3.17}
and, by the Hölder inequality and Lemma 3.1
\[ \left| \int_{\mathbb{R}^2} \omega \cdot (\omega_x \wedge \eta(\varepsilon, p)y + \eta(\varepsilon, p)x \wedge \omega_y) \right| \leq 2\|\omega\|^2_{W^{1,3}} \|\eta(\varepsilon, p)\|_{W^{1,3}} \xrightarrow{|p| \to \infty} 0, \]  \tag{3.18}
\[ \left| \int_{\mathbb{R}^2} \omega \cdot (\eta(\varepsilon, p)x \wedge \eta(\varepsilon, p)y) \right| \leq \|\omega\|_{W^{1,3}} \|\eta(\varepsilon, p)\|^2_{W^{1,3}} \xrightarrow{|p| \to \infty} 0, \]  \tag{3.19}
\[ \left| \int_{\mathbb{R}^2} \eta(\varepsilon, p) \cdot (\omega + \eta(\varepsilon, p))_x \wedge (\omega + \eta(\varepsilon, p))_y \right| \leq \|\omega + \eta(\varepsilon, p)\|^2_{W^{1,3}} \|\eta(\varepsilon, p)\|_{W^{1,3}} \xrightarrow{|p| \to \infty} 0. \]  \tag{3.20}
Moreover the Gauss-Green Theorem yields
\[ \mathcal{V}_H(\omega + p) = - \int_{B_1} H(\xi + p) \, d\xi \]
so that by the Dominated Convergence Theorem we have that
\[ \lim_{|p| \to \infty} \mathcal{V}_H(\omega + p) = 0. \]  \tag{3.21}
From (2.3), Hölder inequality, and Lemma 3.1 we have that
\[ |\langle \mathcal{V}'_H(\omega + p), \eta(\varepsilon, p) \rangle| = \left| \int_{\mathbb{R}^2} H(\omega + p) \eta(\varepsilon, p) \cdot \omega_x \wedge \omega_y \right| \leq \|H\|_{L^\infty(\mathbb{R}^3)} \|\omega\|^2_{W^{1,3}} \|\eta(\varepsilon, p)\|_{W^{1,3}} \xrightarrow{|p| \to \infty} 0. \]  \tag{3.22}
From (3.15) - (3.22), it follows that
\[ \lim_{|p| \to \infty} \Phi_\varepsilon(p) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \omega|^2 + \frac{2}{3} \int_{\mathbb{R}^2} \omega \cdot \omega_x \wedge \omega_y = \mathcal{E}_0(\omega). \]
The proposition is thereby proved.
Proof of Theorem 1.1. As already observed at the beginning of Section 2, it is not restrictive to take $H_0 = 1$. From Proposition 3.3 it follows that for $|\varepsilon| < \varepsilon_0$ either $\Phi_\varepsilon$ is constant (and hence we have infinitely many critical points) or it has a global maximum or minimum point. In any case $\Phi_\varepsilon$ has a critical point. Since $Z_\varepsilon$ is a natural constraint for $\mathcal{E}_\varepsilon$, we deduce the existence of a critical point of $\mathcal{E}_\varepsilon$ for $|\varepsilon| < \varepsilon_0$ and hence of a solution to $(P_\varepsilon)$. The $H_\varepsilon$-bubble $\omega_\varepsilon$ found in this way is of the form $\omega = p^\varepsilon + \eta(p^\varepsilon)$ for some $p^\varepsilon \in \mathbb{R}^3$ where $\eta$ is as in Lemma 3.1. Remark 3.2 yields that $\omega_\varepsilon$ is closed in $C^{1,1/3}(S^2, \mathbb{R}^3)$-norm to the manifold $\{\omega + p : p \in \mathbb{R}^3\}$ for $\varepsilon$ small. Since $\omega$ has no branch points, we deduce that $\omega_\varepsilon$ has no branch points.

To prove Theorems 1.2 and 1.4 we need the following expansion for $\Phi_\varepsilon$ (see [7])

$$\Phi_\varepsilon(p) = \mathcal{E}_0(\omega) - 2\varepsilon \Gamma(p) + O(\varepsilon^2) \quad \text{as } \varepsilon \to 0 \text{ uniformly in } p \in \mathbb{R}^3. \quad (3.23)$$

Proof of Theorem 1.2. Let $\varepsilon > 0$ small. Assumption (H4) implies that $\Gamma(0) > 0$ and hence from (3.23) we have that for $\varepsilon$ small $\Phi_\varepsilon(0) < \mathcal{E}_0(\omega)$, whereas from assumption (H3) we have that $\text{Hess} \, \Gamma(0)$ is positive definite so that $\Gamma$ has a strict local minimum in 0 and hence from (3.23) $\Phi_\varepsilon$ has a strict local maximum in $B_r(0)$ for some $r > 0$ such that $\Phi_\varepsilon(p) < \Phi_\varepsilon(0) - c_\varepsilon < \mathcal{E}_0(\omega)$ for $|p| = r$, where $c_\varepsilon$ is some positive constant depending on $\varepsilon$. In particular $\Phi_\varepsilon$ has a mountain pass geometry. Moreover by Theorem 1.1 $\Phi_\varepsilon(p) \to \mathcal{E}_0(\omega)$ as $|p| \to \infty$, and so $\Phi_\varepsilon$ must have a global minimum point. If the minimum point and the mountain pass point coincide then $\Phi_\varepsilon$ has infinitely many critical points. Otherwise $\Phi_\varepsilon$ has at least three critical points: a local maximum point, a global minimum point, and a mountain pass. As a consequence $(P_\varepsilon)$ has at least three solutions provided $\varepsilon$ is sufficiently small.

As observed in Remark 1.3 if $H_1(0) > 0$ and Hess $H_1(0)$ is positive definite, by continuity we have that for $H_0$ sufficiently large $\Gamma(0) > 0$ and Hess $\Gamma(0)$ is positive definite, so that we can still prove the existence of three solutions arguing as above.

Proof of Theorem 1.4. Assumption (H5) implies that $\Gamma(p_1) > 0$ and $\Gamma(p_2) < 0$. Since $\Phi_\varepsilon(p) = \mathcal{E}_0(\omega) + 2\varepsilon \left( - \Gamma(p) + o(1) \right)$ as $\varepsilon \to 0$, we have for $\varepsilon$ sufficiently small

$$\Phi_\varepsilon(p_1) < \mathcal{E}_0(\omega) \quad \text{and} \quad \Phi_\varepsilon(p_2) > \mathcal{E}_0(\omega)$$

if $\varepsilon > 0$ and the inverse inequalities if $\varepsilon < 0$. Since by Theorem 1.1 $\Phi_\varepsilon(p) \to \mathcal{E}_0(\omega)$ as $|p| \to \infty$, we conclude that $\Phi_\varepsilon$ must have a global maximum point and a global minimum point in $\mathbb{R}^3$. Since $Z_\varepsilon$ is a natural constraint for $\mathcal{E}_\varepsilon$, we deduce the existence of two critical points of $\mathcal{E}_\varepsilon$ for $|\varepsilon|$ sufficiently small and hence of two solutions to $(P_\varepsilon)$.

As observed in Remark 1.5 if $H_1(p_1) > 0$ and $H_1(p_2) < 0$, by continuity we have that for $H_0$ sufficiently large $\Gamma(p_1) > 0$ and $\Gamma(p_2) < 0$, so that we can still prove the existence of two solutions arguing as above.
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