The Barut Second-Order Equation: Lagrangian, Dynamical Invariants and Interactions

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ABSTRACT The second-order equation in the \((1/2, 0) \oplus (0, 1/2)\) representation of the Lorentz group has been proposed by A. Barut in the 70s, ref. [1]. It permits to explain the mass splitting of leptons \((e, \mu, \tau)\). Recently, the interest has grown to this model (see, for instance, the papers by S. Kruglov [2] and J. P. Vigier et al. [3]). We continue the research deriving the equation from the first principles, finding dynamical invariants for this model, investigating the influence of potential interactions.

1 Introduction

The Barut main equation is

\[
[i \gamma_\mu \partial_\mu - \alpha^2 \frac{\partial_\mu \partial_\mu}{m} + \kappa] \Psi = 0. \tag{1.1}
\]

- It represents a theory with the conserved current that is linear in 15 generators of the 4-dimensional representation of the \(O(4, 2)\) group, \(N_{ab} = \frac{i}{2} \gamma_a \gamma_b, \gamma_a = \{\gamma_\mu, \gamma_5, i\}\).

- Instead of 4 solutions it has 8 solutions with the correct relativistic relation \(E = \pm \sqrt{\mathbf{p}^2 + m^2}\). In fact, it describes states of different masses (the second one is \(m_\mu = m_e(1 + \frac{3}{2} \alpha)\), \(\alpha\) is the fine structure constant), provided that a certain physical condition is imposed on the \(\alpha^2\) parameter (the anomalous magnetic moment should be equal to \(4\alpha/3\)).

- One can also generalize the formalism to include the third state, the \(\tau\) - lepton [1b].

- Barut has indicated at the possibility of including \(\gamma_5\) terms (e.g., \(\sim \gamma_5 \kappa\)).
2 Main Results

If we present the 4-spinor as $\Psi(p) = column(\phi_R(p), \phi_L(p))$ then Ryder states [5] that $\phi_R(0) = \phi_L(0)$. Similar argument has been given by Fautov [6]: “the matrix $B$ exists such that $Bu_\lambda(0) = u_\lambda(0)$, $B^2 = I$ for any $(2J + 1)$-component function within the Lorentz invariant theories”. The latter statement is more general than the Ryder one, because it admits

$$B = \begin{pmatrix} 0 & e^{+i\alpha} \\ e^{-i\alpha} & 0 \end{pmatrix},$$

so that $\phi_R(0) = e^{i\alpha}\phi_L(0)$. The most general form of the relation in the $(1/2, 0) \oplus (0, 1/2)$ representation has been given by Dvoeglazov [7,4a]:

$$\phi_L^h(0) = a(-1)^{\frac{J}{2}+h}e^{i(\theta_1 + \theta_2)}\Theta_{1/2}[\phi_L^{-h}(0)]^* + be^{2i\theta_2}\Xi_{1/2}[\phi_L^h(0)]^*, \quad (2.1)$$

with

$$\Theta_{1/2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_2, \quad \Xi_{1/2} = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}, \quad (2.2)$$

$\Theta_J$ is the Wigner operator for spin $J = 1/2$, $\varphi$ is the azimuthal angle $p \to 0$ of the spherical coordinate system.

Next, we use the Lorentz transformations:

$$\Lambda_{R,L} = \exp(\pm \sigma \cdot \phi/2), \quad \cosh \phi = \frac{E_p}{m}, \quad \sinh \phi = |p|/m, \quad \hat{\phi} = p/|p|. \quad (2.3)$$

Applying the boosts and the relations between spinors in the rest frame, one can obtain:

$$\phi_L^h(p) = a\left(\frac{p_0 - \sigma \cdot p}{m}\right)\phi_L^h(p) + b(-1)^{1/2+h}\Theta_{1/2}\Xi_{1/2}\phi_R^{-h}(p), \quad (2.4)$$

$$\phi_R^h(p) = a\left(\frac{p_0 + \sigma \cdot p}{m}\right)\phi_L^h(p) + b(-1)^{1/2+h}\Theta_{1/2}\Xi_{1/2}\phi_L^{-h}(p). \quad (2.5)$$

$(\theta_1 = \theta_2 = 0, \ p_0 = E_p = \sqrt{p^2 + m^2})$. In the Dirac form we have:

$$[a\frac{\hat{p}}{m} - 1]u_h(p) + ib(-1)^{\frac{J}{2}+h}\mathcal{C}u_h^*(p) = 0, \quad (2.6)$$

where $\mathcal{C} = \begin{pmatrix} 0 & i\Theta_{1/2} \\ -i\Theta_{1/2} & 0 \end{pmatrix}$, the charge conjugate operator. In the QFT form we must introduce the creation/annihilation operators. Let $b_\downarrow = -ia_\uparrow$, $b_\uparrow = +ia_\downarrow$, then

$$[a\frac{i\gamma^\mu \partial_\mu}{m} + b\mathcal{C} - 1]\Psi(x^\nu) = 0. \quad (2.7)$$
If one applies the unitary transformation to the Majorana representation [8]

\[
U = \frac{1}{2} \begin{pmatrix}
1 - i\Theta_{1/2} & 1 + i\Theta_{1/2} \\
-1 - i\Theta_{1/2} & 1 - i\Theta_{1/2}
\end{pmatrix}, \quad UCKU^{-1} = -K, \tag{2.8}
\]

then \(\gamma\)-matrices become to be pure imaginary, and the equations are pure real.

\[
\begin{bmatrix}
\frac{i\hat{\theta}}{m} - b - 1 \\
\frac{i\hat{\theta}}{m} + b - 1
\end{bmatrix} \Psi_1 = 0, \tag{2.9}
\]

\[
\begin{bmatrix}
\frac{i\hat{\theta}}{m} - b - 1 \\
\frac{i\hat{\theta}}{m} + b - 1
\end{bmatrix} \Psi_2 = 0, \tag{2.10}
\]

where \(\Psi = \Psi_1 + i\Psi_2\). It appears as if the real and imaginary parts of the field have different masses. Finally, for superpositions \(\phi = \Psi_1 + \Psi_2\), \(\chi = \Psi_1 - \Psi_2\), multiplying by \(b \neq 0\) we have:

\[
[2a\frac{i\gamma^\mu \partial_\mu}{m} + a^2 \partial^\mu \partial_\mu + b^2 - 1] \frac{\phi(x')}{\chi(x')} = 0, \tag{2.11}
\]

If we put \(a/2m \rightarrow \alpha_2, \frac{1-b^2}{2a} m \rightarrow \kappa\) we recover the Barut equation.

How can we get the third lepton state? See the refs. [1b,4b]:

\[
M_\tau = M_\mu \frac{3}{2} - \alpha^{-1} n^4 M_e = M_e + \frac{3}{2} \alpha^{-1} 14 M_e + \frac{3}{2} \alpha^{-1} 24 M_e = 1786.08 \text{ MeV}. \tag{2.12}
\]

The physical origin was claimed by Barut to be in the magnetic self-interaction of the electron (the factor \(n^4\) appears due to the Bohr-Sommerfeld rule for the charge moving in circular orbits in the field of a fixed magnetic dipole \(\mu\)). One can start from (2.6), but, as opposed to the abovementioned consideration, one can write the coordinate-space equation in the form:

\[
[a\frac{i\gamma^\mu \partial_\mu}{m} + b_1 CK - 1] \Psi(x') + b_2 \gamma^5 CK \tilde{\Psi}(x') = 0, \tag{2.13}
\]

with \(\Psi^{MR} = \Psi_1 + i\Psi_2, \quad \tilde{\Psi}^{MR} = \Psi_3 + i\Psi_4\). As a result,

\[
(a\frac{i\gamma^\mu \partial_\mu}{m} - 1) \phi - b_1 \chi + ib_2 \gamma^5 \tilde{\phi} = 0, \tag{2.14}
\]

\[
(a\frac{i\gamma^\mu \partial_\mu}{m} - 1) \chi - b_1 \phi - ib_2 \gamma^5 \tilde{\chi} = 0. \tag{2.15}
\]

The operator \(\tilde{\Psi}\) may be linear-dependent on the states included in the \(\Psi\). Let us apply the most simple form \(\Psi_1 = -i\gamma^5 \Psi_4, \quad \Psi_2 = +i\gamma^5 \Psi_3\). Then, one can recover the 3rd order Barut-like equation [4b]:

\[
[i\gamma^\mu \partial_\mu - m \frac{1 \pm b_1 \pm b_2}{a}][i\gamma^\nu \partial_\nu + a \frac{1}{2m} \partial^\nu \partial_\nu + m \frac{b_2^2 - 1}{2a}]\Psi_{1,2} = 0. \tag{2.16}
\]
It is simply the product of 3 Dirac equations with different masses. Thus, we have three mass states.

Let us reveal the connections with other models. For instance, in refs. [3, 9] the following equation has been studied:

\[
[(i\partial - eA)(i\partial - eA) - m^2]\Psi =
\]

\[
= \left[(i\partial_\mu - eA_\mu)(i\partial^\mu - eA^\mu) - \frac{1}{2}i\sigma^{\mu\nu}F_{\mu\nu} - m^2\right]\Psi = 0 \quad (2.17)
\]

for the 4-component spinor \(\Psi\). This is the Feynman-Gell-Mann equation.

In the free case we have the Lagrangian (see Eq. (9) of ref. [3c]):

\[
L_0 = (i\bar{\Psi}\partial\Psi) - m^2\bar{\Psi}\Psi . \quad (2.18)
\]

We can note:

- The Barut equation is the sum of the Dirac equation and the Feynman-Gell-Mann equation.
- Recently, it was suggested to associate an analogue of Eq. (2.18) with the dark matter [10], provided that \(\Psi\) is composed of the self/anti-self charge conjugate spinors, and it has the dimension \([\text{energy}]^1\) in \(c = \hbar = 1\). The interaction Lagrangian is \(L^I \sim g\bar{\Psi}\Psi\phi^2\).
- The term \(\sim \bar{\Psi}\sigma^{\mu\nu}\Psi F_{\mu\nu}\) will affect the photon propagation, and non-local terms will appear in higher orders.
- However, it was shown in [3b,c] that a) the Mott cross-section formula (which represents the Coulomb scattering up to the order \(\sim e^2\)) is still valid; b) the hydrogen spectrum is not much disturbed; if the electromagnetic field is weak the corrections are small.
- The solutions are the eigenstates of \(\gamma^5\) operator.
- In general, \(J_0\) is not the positive-defined quantity, since the general solution \(\Psi = a\Psi_+ + b\Psi_-\), where \(|i\gamma^\mu\partial_\mu \pm m|\Psi_\pm = 0\), see also [11].

The most general conserved current of the Barut-like theories is

\[
J_\mu = \alpha_1\gamma_\mu + \alpha_2\rho_\mu + \alpha_3\sigma_{\mu\nu}q^\nu . \quad (2.19)
\]

Let us try the Lagrangian:

\[
L = L_{Dirac} + L_{add} , \quad (2.20)
\]

\[
L_{Dirac} = \alpha_1[\bar{\Psi}\gamma^\mu(\partial_\mu \Psi) - (\partial_\mu \bar{\Psi})\gamma^\mu \Psi] - \alpha_4\bar{\Psi}\Psi , \quad (2.21)
\]

\[
L_{add} = \alpha_2(\partial_\mu \bar{\Psi})(\partial^\mu \Psi) + \alpha_3\bar{\Psi}\sigma^{\mu\nu}\partial_\nu \Psi . \quad (2.22)
\]
Then, the equation follows:

$$[2\alpha_1\gamma^\mu \partial_\mu - \alpha_2 \partial_\mu \partial^\mu - \alpha_4] \Psi = 0, \quad (2.23)$$

and its Dirac-conjugate:

$$\bar{\Psi}[2\alpha_1\gamma^\mu \partial_\mu + \alpha_2 \partial_\mu \partial^\mu + \alpha_4] = 0. \quad (2.24)$$

The derivatives acts to the left in the second equation. Thus, we have the Dirac equation when \( \alpha_1 = \frac{1}{2} \), \( \alpha_2 = 0 \), and the Barut equation when \( \alpha_2 = \frac{1}{m} \frac{2\alpha_3/3}{1 + 4\alpha_3/3} \).

In the Euclidean metrics the dynamical invariants are

$$J_\mu = -i \sum_i \left[ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Psi^i)} \Psi^i - \bar{\Psi}^i \frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\Psi}^i)} \right], \quad (2.25)$$

$$T_{\mu\nu} = - \sum_i \left[ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Psi^i)} \partial_\nu \Psi^i + \partial_\nu \bar{\Psi}^i \frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\Psi}^i)} \right] + \mathcal{L} \delta_{\mu\nu}, \quad (2.26)$$

$$S_{\mu\nu,\lambda} = -i \sum_{ij} \left[ \frac{\partial \mathcal{L}}{\partial (\partial_\lambda \Psi^i)} N^\Psi_{\mu\nu} \Psi^j + \bar{\Psi}^i N_{\mu\nu,ij} \bar{\Psi} \frac{\partial \mathcal{L}}{\partial (\partial_\lambda \bar{\Psi}^j)} \right]. \quad (2.27)$$

\( N^\Psi_{\mu\nu} \) are the Lorentz group generators.

Then, the energy-momentum tensor is

$$T_{\mu\nu} = -\alpha_3 \left[ \bar{\Psi} \gamma_\mu \partial_\nu \Psi - \bar{\Psi} \gamma_\nu \partial_\mu \Psi \right] - \alpha_2 \left[ \bar{\Psi} \gamma_\mu \partial_\nu \Psi + \partial_\nu \bar{\Psi} \partial_\mu \Psi \right] -$$

$$- \alpha_2 \left[ \partial_\mu \bar{\Psi} \sigma_{\alpha\beta} \partial_\nu \Psi + \partial_\nu \bar{\Psi} \sigma_{\mu\alpha} \partial_\beta \Psi \right] + \alpha_1 \left( \bar{\Psi} \gamma_\mu \partial_\nu \Psi - \partial_\nu \bar{\Psi} \gamma_\mu \Psi \right) +$$

$$+ \alpha_3 \partial_\lambda \bar{\Psi} \partial_\alpha \Psi + \sigma_{\alpha\beta} \partial_\gamma \partial_\nu \bar{\Psi} \partial_\mu \Psi + \Psi \delta_{\mu\nu} \right]. \quad (2.28)$$

Hence, the Hamiltonian \( \hat{\mathcal{H}} = -i \mathcal{P}_4 = - \int T_{\mu\nu} d^3 \mathbf{x} \) is

$$\hat{\mathcal{H}} = \int d^3 \mathbf{x} \left[ \alpha_1 \left[ \bar{\Psi} \gamma_\mu \partial_\nu \Psi - \bar{\Psi} \gamma_\nu \partial_\mu \Psi \right] + \alpha_2 \left[ \bar{\Psi} \gamma_\mu \partial_\nu \Psi - \partial_\nu \bar{\Psi} \gamma_\mu \Psi \right] -$$

$$- \alpha_3 \left[ \partial_\mu \bar{\Psi} \sigma_{\alpha\beta} \partial_\nu \Psi - \partial_\nu \bar{\Psi} \sigma_{\alpha\beta} \partial_\mu \Psi \right] - \alpha_4 \bar{\Psi} \Psi \right]. \quad (2.29)$$

The 4-current is

$$J_\mu = -i \left\{ 2\alpha_1 \bar{\Psi} \gamma_\mu \Psi + \alpha_2 \left[ (\partial_\mu \Psi) \bar{\Psi} - \bar{\Psi} (\partial_\mu \Psi) \right] + \alpha_3 \left[ \partial_\lambda \bar{\Psi} \sigma_{\mu\lambda} \Psi - \Psi \sigma_{\mu\lambda} \partial_\lambda \Psi \right] \right\} \quad (2.30)$$

Hence, the charge operator \( \hat{Q} = -i \int J_\mu d^3 \mathbf{x} \) is

$$\hat{Q} = - \int \left\{ 2\alpha_1 \bar{\Psi} \gamma_\mu \Psi + \alpha_2 \left[ (\partial_\mu \Psi) \bar{\Psi} - \bar{\Psi} (\partial_\mu \Psi) \right] + \alpha_3 \left[ \partial_\lambda \bar{\Psi} \sigma_{\mu\lambda} \Psi - \Psi \sigma_{\mu\lambda} \partial_\lambda \Psi \right] \right\} d^3 \mathbf{x} \quad (2.31)$$

Finally, the spin tensor is

$$S_{\mu\nu,\lambda} = - \frac{i}{2} \left\{ \alpha_1 \left[ \bar{\Psi} \sigma_{\mu\lambda} \Psi + \Psi \sigma_{\mu\lambda} \gamma_\lambda \Psi \right] + \alpha_2 \left[ \partial_\lambda \bar{\Psi} \sigma_{\mu\lambda} \Psi - \Psi \sigma_{\mu\lambda} \partial_\lambda \Psi \right] +$$

$$+ \alpha_3 \left[ \partial_\lambda \bar{\Psi} \sigma_{\mu\lambda} \Psi - \Psi \sigma_{\mu\lambda} \partial_\lambda \Psi \right] \right\}. \quad (2.32)$$
In the quantum case the corresponding field operators are written:

\[ \Psi(x^\mu) = \sum_h \int \frac{d^3p}{(2\pi)^3} [u_h(p)a_h(p)e^{+ip \cdot x} + v_h(p)b_h^\dagger(p)e^{-ip \cdot x}], \tag{2.33} \]

\[ \bar{\Psi}(x^\mu) = \sum_h \int \frac{d^3p}{(2\pi)^3} [\bar{u}_h(p)a_h^\dagger(p)e^{-ip \cdot x} + \bar{v}_h(p)b_h(p)e^{+ip \cdot x}], \tag{2.34} \]

The 4-spinor normalization is

\[ \bar{u}_h u_{h'} = \delta_{hh'}, \quad \bar{v}_h v_{h'} = -\delta_{hh'}. \tag{2.35} \]

The commutation relations are

\[ [a_h(p), a_{h'}^\dagger(k)]_+ = (2\pi)^3 \frac{m}{p_4} \delta^{(3)}(p-k)\delta_{hh'}, \tag{2.36} \]

\[ [b_h(p), b_{h'}^\dagger(k)]_+ = (2\pi)^3 \frac{m}{p_4} \delta^{(3)}(p-k)\delta_{hh'}, \tag{2.37} \]

with all other being equal to zero. The dimensions of the \( \Psi, \bar{\Psi} \) are as usual, \( [\text{energy}]^{3/2} \). Hence, the second-quantized Hamiltonian is written

\[ \hat{H} = -\sum_h \int \frac{d^3p}{(2\pi)^3} \frac{2E_p^2}{m} [\alpha_1 + m\alpha_2] : [a_h^\dagger a_h - b_h b_{h'}^\dagger] : . \tag{2.38} \]

(Remember that \( \alpha_1 \sim \frac{i}{2} \), the commutation relations may give another \( i \), so the contribution of the first term to eigenvalues will be real. But if \( \alpha_2 \) is real, the contribution of the second term may be imaginary). The charge is

\[ \hat{Q} = -\sum_{hh'} \int \frac{d^3p}{(2\pi)^3} \frac{2E_p}{m} [\alpha_1 + m\alpha_2] \delta_{hh'} - i\alpha_3 \bar{u}_h \sigma_{ij} p_i u_{h'} : [a_{h'}^\dagger a_h - b_{h'} b_h^\dagger] : . \tag{2.39} \]

However, due to \( [\Lambda_{R,L}, \sigma \cdot p] = 0 \) the last term with \( \alpha_3 \) does not contribute.

3 Conclusions

The conclusions are:

- We obtained the Barut-like equations of the 2nd order and 3rd order in derivatives. The Majorana representation has been used.

- We obtained dynamical invariants for the free Barut field on the classical and quantum level.
We found relations with other models (such as the Feynman-Gell-Mann equation).

As a result of analysis of dynamical invariants, we can state that at the free level the term $\sim \alpha_3 \partial_\mu \bar{\Psi}_\sigma \sigma_{\mu\nu} \partial_\nu \Psi$ in the Lagrangian does not contribute.

However, the interaction terms $\sim \alpha_3 \bar{\Psi}_\sigma \sigma_{\mu\nu} \partial_\nu \Psi A_\mu$ will contribute when we construct the Feynman diagrams and the $S$-matrix. In the curved space (the 4-momentum Lobachevsky space) the influence of such terms has been investigated in the Skachkov works [12]. Briefly, the contribution will be such as if the 4-potential were interact with some “renormalized” spin. Perhaps, this explains, why did Barut use the classical anomalous magnetic moment $g \sim 4\alpha/3$ instead of $\frac{\alpha}{\pi}$.

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