Median Selection Subset Aggregation for Parallel Inference

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Abstract

For massive data sets, efficient computation commonly relies on distributed algorithms that store and process subsets of the data on different machines, minimizing communication costs. Our focus is on regression and classification problems involving many features. A variety of distributed algorithms have been proposed in this context, but challenges arise in defining an algorithm with low communication, theoretical guarantees and excellent practical performance in general settings. We propose a MEdian Selection Subset AGgregation Estimator (message) algorithm, which attempts to solve these problems. The algorithm applies feature selection in parallel for each subset using Lasso or another method, calculates the ‘median’ feature inclusion index, estimates coefficients for the selected features in parallel for each subset, and then averages these estimates. The algorithm is simple, involves very minimal communication, scales efficiently in both sample and feature size, and has theoretical guarantees. In particular, we show model selection consistency and coefficient estimation efficiency. Extensive experiments show excellent performance in variable selection, estimation, prediction, and computation time relative to usual competitors.

1 Introduction

The explosion in both size and velocity of data has brought new challenges to the design of statistical algorithms. Parallel inference is a promising approach for solving large scale problems. The typical procedure for parallelization partitions the full data into multiple subsets, stores subsets on different machines, and then processes subsets simultaneously. Processing on subsets in parallel can lead to two types of computational gains. The first reduces time for calculations within each iteration of optimization or sampling algorithms via faster operations; for example, in conducting linear algebra involved in calculating likelihoods or gradients [4,7]. Although such approaches can lead to substantial reductions in computational bottlenecks for big data, the amount of gain is limited by the need to communicate across computers at each iteration. It is well known that communication costs are a major factor driving the efficiency of distributed algorithms, so that it is of critical importance to limit communication. This motivates the second type of approach, which conducts computations completely independently on the different subsets, and then combines the results to obtain the final output. This limits communication to the final combining step, and may lead to simpler and much
faster algorithms. However, a major issue is how to design algorithms that are close to communication free, which can preserve or even improve the statistical accuracy relative to (much slower) algorithms applied to the entire data set simultaneously. We focus on addressing this challenge in this article.

There is a recent flurry of research in both Bayesian and frequentist settings focusing on the second approach [8-14]. Particularly relevant to our approach is the literature on methods for combining point estimators obtained in parallel for different subsets [8, 9, 13]. Mann et al. [9] suggest using averaging for combining subset estimators, and Zhang et al. [8] prove that such estimators will achieve the same error rate as the ones obtained from the full set if the number of subsets $m$ is well chosen. Minsker [13] utilizes the geometric median to combine the estimators, showing robustness and sharp concentration inequalities. These methods function well in certain scenarios, but might not be broadly useful. In practice, inference for regression and classification typically contains two important components: One is variable or feature selection and the other is parameter estimation. Current combining methods are not designed to produce good results for both tasks.

To obtain a simple and computationally efficient parallel algorithm for feature selection and coefficient estimation, we propose a new combining method, referred to as message. The detailed algorithm will be fully described in the next section. There are related methods, which were proposed with the very different goal of combining results from different imputed data sets in missing data contexts [15]. However, these methods are primarily motivated for imputation aggregation, do not improve computational time, and lack theoretical guarantees. Another related approach is the bootstrap Lasso (Bolasso) [16], which runs Lasso independently for multiple bootstrap samples, and then intersects the results to obtain the final model. Asymptotic properties are provided under fixed number of features ($p$ fixed) and the computational burden is not improved over applying Lasso to the full data set. Our message algorithm has strong justification in leading to excellent convergence properties in both feature selection and prediction, while being simple to implement and computationally highly efficient.

The article is organized as follows. In section 2, we describe message in detail. In section 3, we provide theoretical justifications and show that message can produce better results than full data inferences under certain scenarios. Section 4 evaluates the performance of message via extensive numerical experiments. Section 5 contains a discussion of possible generalizations of the new method to broader families of models and online learning. All proofs are provided in the appendix.

## 2 Parallelized framework

Consider the linear model which has $n$ observations and $p$ features,

$$Y = X\beta + \epsilon,$$

where $Y$ is an $n \times 1$ response vector, $X$ is an $n \times p$ matrix of features and $\epsilon$ is the observation error, which is assumed to have mean zero and variance $\sigma^2$. The fundamental idea for communication efficient parallel inference is to partition the data set into $m$ subsets, each of which contains a small portion of the data $n/m$. Separate analysis on each subset will then be carried out and the result will be aggregated to produce the final output.

As mentioned in the previous section, regression problems usually consist of two stages: feature selection and parameter estimation. For linear models, there is a rich literature on feature selection and we only consider two approaches. The risk inflation criterion (RIC), or more generally, the generalized information criterion (GIC) is an $l_0$-based feature selection technique for high dimensional data [17-20]. GIC attempts to solve the following optimization problem,

$$\hat{M}_\lambda = \arg \min_{M \subset \{1,2,\ldots,p\}} \|Y - X_M \beta_M\|^2_2 + \lambda |M| \sigma^2$$

for some well chosen $\lambda$. For $\lambda = 2(\log p + \log \log p)$ it corresponds to RIC [18], for $\lambda = (2 \log p + \log n)$ it corresponds to extended BIC [19] and $\lambda = \log n$ reduces to the usual BIC. Konishi and Kitagawa [18] prove the consistency of GIC for high dimensional data under some regularity conditions.
Lasso [21] is an \( l_1 \) based feature selection technique, which solves the following problem

\[
\hat{\beta} = \arg \min_\beta \frac{1}{n} \| Y - X \beta \|_2^2 + \lambda \| \beta \|_1
\]  

for some well chosen \( \lambda \). Lasso transfers the original NP hard \( l_0 \)-based optimization to a problem that can be solved in polynomial time. Zhao and Yu [22] prove the selection consistency of Lasso under the Irrepresentable condition. Based on the model selected by either GIC or Lasso, we could then apply the ordinary least square (OLS) estimator to find the coefficients.

As briefly discussed in the introduction, averaging and median aggregation approaches possess different advantages but also suffer from certain drawbacks. To carefully adapt these features to regression and classification, we propose the median selection subset aggregation (message) algorithm, which is motivated as follows.

Averaging of sparse regression models leads to an inflated number of features having non-zero coefficients, and hence is not appropriate for model aggregation when feature selection is of interest. When conducting Bayesian variable selection, the median probability model has been recommended as selecting the single model that produces the best approximation to model-averaged predictions under some simplifying assumptions [23]. The median probability model includes those features having inclusion probabilities greater than 1/2. We can apply this notion to subset-based inference under some simplifying assumptions [23]. The median probability model includes those features having inclusion probabilities greater than 1/2. We can apply this notion to subset-based inference by including features that are included in a majority of the subset-specific analyses, leading to selecting the ‘median model’. Let \( \gamma^{(i)} = (\gamma_1^{(i)}, \ldots, \gamma_p^{(i)}) \) denote a vector of feature inclusion indicators for the \( i^{th} \) subset, with \( \gamma_j^{(i)} = 1 \) if feature \( j \) is included so that the coefficient \( \beta_j \) on this feature is non-zero, with \( \gamma_j^{(i)} = 0 \) otherwise. The inclusion indicator vector for the median model \( M_\gamma \), can be obtained by

\[
\gamma = \arg \min_{\gamma \in \{0,1\}^p} \sum_{i=1}^m \| \gamma - \gamma^{(i)} \|_1,
\]

or equivalently,

\[
\gamma_j = \text{median}\{\gamma_j^{(i)}, i = 1, 2, \ldots, m\} \text{ for } j = 1, 2, \ldots, p.
\]

If we apply Lasso or GIC to the full data set, in the presence of heavy-tailed observation errors, the estimated feature inclusion indicator vector will converge to the true inclusion vector at a polynomial rate. It is shown in the next section that the convergence rate of the inclusion vector for the median model can be improved to be exponential, leading to substantial gains in not only computational time but also feature selection performance. The intuition for this gain is that in the heavy-tailed case, a proportion of the subsets will contain outliers having a sizable influence on feature selection. By taking the median, we obtain a central model that is not so influenced by these outliers, and hence can concentrate more rapidly. As large data sets typically contain outliers and data contamination, this is a substantial practical advantage in terms of performance even putting aside the computational gain. After feature selection, we obtain estimates of the coefficients for each selected feature by averaging the coefficient estimates from each subset, following the spirit of [8]. The message algorithm (described in Algorithm 1) only requires each machine to pass the feature indicators to a central computer, which (essentially instantaneously) calculates the median model, passes back the corresponding indicator vector to the individual computers, which then pass back coefficient estimates for averaging. The communication costs are negligible.

3 Theory

In this section, we provide theoretical justification for the message algorithm in the linear model case. The theory is easily generalized to a much wider range of models and estimation techniques, as will be discussed in the last section.

Throughout the paper we will assume \( X = (x_1, \ldots, x_p) \) is an \( n \times p \) feature matrix, \( s = |S| \) is the number of non-zero coefficients and \( \lambda(A) \) is the eigenvalue for matrix \( A \). Before we proceed to the theorems, we enumerate several conditions that are required for establishing the theory. We assume there exist constants \( V_1, V_2 > 0 \) such that
but for simplicity we rely on this version. To ameliorate possible concerns on how realistic these conditions are, we provide further justifications via Theorem 5 and 6 in the appendix.

**Algorithm 1 Message algorithm**

**Initialization:**
1: Input \((Y, X), n, p,\) and \(m\)
2: \# \(n\) is the sample size, \(p\) is the number of features and \(m\) is the number of subsets
3: Randomly partition \((Y, X)\) into \(m\) subsets \((Y^{(i)}, X^{(i)})\) and distribute them on \(m\) machines.

**Iteration:**
4: for \(i = 1\) to \(m\) do
5: \(\gamma^{(i)} = \min_{M_{\gamma}} \text{loss}(Y^{(i)}, X^{(i)})\) # \(\gamma^{(i)}\) is the estimated model via Lasso or GIC
6: # Gather all subset models \(\gamma^{(i)}\) to obtain the median model \(M_{\gamma}\)
7: for \(j = 1\) to \(p\) do
8: \(\gamma_j = \text{median}\{\gamma_j^{(i)}, i = 1, 2, \cdots, m\}\)
9: # Redistribute the estimated model \(M_{\gamma}\) to all subsets
10: for \(i = 1\) to \(m\) do
11: \(\beta^{(i)} = (X_{\gamma}^{(i)^T} X_{\gamma}^{(i)})^{-1} X_{\gamma}^{(i)^T} Y_{\gamma}^{(i)}\) # Estimate the coefficients
12: # Gather all subset estimations \(\beta^{(i)}\)
13: \(\bar{\beta} = \sum_{i=1}^{m} \beta^{(i)}/m\)
14: 15: return \(\bar{\beta}, \gamma\)

A.1 Consistency condition for estimation.
- \(\frac{1}{n} \sum_{i=1}^{n} x_i^T x_i \leq V_1\) for \(i = 1, 2, \cdots, p\)
- \(\lambda_{\min}(\frac{1}{n} X_{\pi}^T X_{\pi}) \geq V_2\)

A.2 Conditions on \(\epsilon, |S|, \) and \(\beta\)
- \(E(\epsilon^{2k}) < \infty\) for some \(k > 0\)
- \(s = |S| \leq c_1 n^\epsilon\) for some \(0 \leq \epsilon < 1\)
- \(\min_{i \in S} |\beta_i| \geq c_0 n^{-\alpha} \) for some \(0 < \tau \leq 1\)

A.3 (Lasso) The strong irrepresentable condition.
- Assuming \(X_S\) and \(X_{S^C}\) are the features having non-zero and zero coefficients, respectively, there exists some positive constant vector \(\eta\) such that
  \[|X_{S^C}^T X_S (X_S^T X_S)^{-1} \text{sign}(\beta_S)| \leq 1 - \eta\]

A.4 (Generalized information criterion, GIC) The sparse Riesz condition.
- There exist constants \(\kappa \geq 0\) and \(c > 0\) such that \(\rho > cn^{-\alpha}\), where
  \[\rho = \inf_{|S| \leq c_0} \lambda_{\min}(X_{\pi}^T X_{\pi})\]

A.1 is the usual consistency condition for regression. A.2 restricts the behaviors of the three key terms and is crucial for model selection. These are both usual assumptions. See [19,20,22]. A.3 and A.4 are specific conditions for model selection consistency for Lasso/GIC. As noted in [22], A.3 is almost sufficient and necessary for sign consistency. A.4 could be relaxed slightly as shown in [19], but for simplicity we rely on this version. To ameliorate possible concerns on how realistic these conditions are, we provide further justifications via Theorems 5 and 6 in the appendix.

**Theorem 1. (GIC)** Assume each subset satisfies A.1, A.2 and A.4, and \(p \leq n^\alpha\) for some \(\alpha < k(\tau - \eta)\), where \(\eta = \max\{\epsilon/k, 2\kappa\}\). If \(r < \tau, 2\kappa < \tau\) and \(\lambda\) in \([1]\) are chosen so that \(\lambda = c_0/(\sigma^2(n/m)^{\tau-\kappa})\) for some \(c_0 < cc_2/2\), then there exists some constant \(C_0\) such that for \(n \geq (2C_0 p) (k^{r-kn})^{-1}\) and \(m = [(4C_0)^r (k^{r-kn})^{-1} n/p (k^{r-kn})^{-1}]\), the selected model \(M_{\gamma}\) follows,

\[P(M_{\gamma} = M_{S}) \geq 1 - \exp \left\{ - \frac{n^{1-\alpha}/(k^{r-kn})}{24(4C_0)^r (k^{r-kn})^{-1}} \right\},\]

and the mean square error of the aggregated estimator follows,

\[E \|\bar{\beta} - \beta\|^2 \leq \frac{\sigma^2 V_2^{-1}s}{n} + \exp \left\{ - \frac{n^{1-\alpha}/(k^{r-kn})}{24(4C_0)^r (k^{r-kn})^{-1}} \right\} \left(1 + 2C_0^{-1}sV_1\|\beta\|^2 + C'_0^{-1}\sigma^2\right),\]

where \(C'_0 = \min_i \lambda_{\min}(X_{\gamma}^{(i)^T} X_{\gamma}^{(i)}/\nu_i)\).
Theorem 2. (Lasso) Assume each subset satisfies A.1, A.2 and A.3, and \( p \leq n^\alpha \) for some \( \alpha < k(\tau - \iota) \). If \( \iota < \tau \) and \( \lambda \) in (2) are chosen so that
\[
\lambda = c_0(n/m)^{1 - \alpha/(k\tau - k\iota)}
\]
for some \( c_0 < c_1V_2/c_2 \), then there exists some constant \( C_0 \) such that for \( n \geq (2C_0p)^{(k\tau - k\iota)^{-1}} \) and \( m = \lfloor (4C_0)^{(k\tau - k\iota)^{-1}} \cdot \sqrt{n/p} - 1 \rfloor \), the selected model \( M_{\gamma} \) follows
\[
P(M_{\gamma} = M_S) \geq 1 - \exp \left\{ - \frac{n^{1 - \alpha/(k\tau - k\iota)}}{24(4C_0)^{(k\tau - k\iota)}} \right\},
\]
and with the same \( C_0' \) defined in Theorem 1, we have
\[
E\| \hat{\beta} - \beta \|_2^2 \leq \frac{\sigma^2V_2^{-1}s}{n} + \exp \left\{ - \frac{n^{1 - \alpha/(k\tau - k\iota)}}{24(4C_0)^{(k\tau - k\iota)}} \right\} \left( 1 + 2C_0'^{-1}sV_1 \right) \| \beta \|_2^2 + C_0'^{-1} \sigma^2.
\]

The above two theorems boost the model consistency property from the original polynomial rate \([20, 22]\) to an exponential rate for heavy-tailed errors. In addition, the mean square error, as shown in the above equation, preserves almost the same convergence rate as if the full data is employed and the true model is known. Therefore, we expect a similar or better performance of message with a significantly lower computation load. Detailed comparisons are demonstrated in Section 4.

4 Experiments

This section assesses the performance of the message algorithm via extensive examples, comparing the results to

- Full data inference. (denoted as “full data”)
- Subset averaging. Partition and average the estimates obtained on all subsets. (denoted as “averaging”)
- Subset median. Partition and take the geometric median of the estimates obtained on all subsets (denoted as “median”)
- Bolasso. Run Lasso on multiple bootstrap samples and intersect to select model. Then estimate the coefficients based on the selected model. (denoted as “Bolasso”)

The Lasso part of all algorithms will be implemented by the “glmnet” package \([24]\). (We did not use ADMM \([25]\) for Lasso as its actual performance might suffer from certain drawbacks \([6]\) and is reported to be slower than “glmnet” \([26]\).)

4.1 Synthetic data sets

We use the linear model and the logistic model for \((p; s) = (1000; 3)\) or \((10,000; 3)\) with different sample size \( n \) and different partition number \( m \) to evaluate the performance. The feature vector is drawn from a multivariate normal distribution with correlation \( \rho = 0 \) or 0.5. Coefficients \( \beta \) are chosen as,
\[
\beta_i \sim (-1)^{\text{ber}(0.4)}(8 \log n/\sqrt{n} + |N(0, 1)|), i \in S
\]

Since GIC is intractable to implement (NP hard), we combine it with Lasso for variable selection: Implement Lasso for a set of different \( \lambda \)'s and determine the optimal one via GIC. The concrete setup of models are as follows,

Case 1 Linear model with \( \epsilon \sim N(0, 2^2) \).
Case 2 Linear model with \( \epsilon \sim t(0, df = 3) \).
Case 3 Logistic model.

For \( p = 1,000 \), we simulate 200 data sets for each case, and vary the sample size from 2000 to 10,000. For each case, the subset size is fixed to 400, so the number of subsets will be changing from 5 to 25. In the experiment, we record the mean square error for \( \hat{\beta} \), probability of selecting the true model and computational time, and plot them in Fig 9 - 14. For \( p = 10,000 \), we simulate 50
Figure 1: Results for case 1 with $\rho = 0$.

Figure 2: Results for case 1 with $\rho = 0.5$.

Figure 3: Results for case 2 with $\rho = 0$.

Figure 4: Results for case 2 with $\rho = 0.5$. 
data sets for each case, and let the sample size range from 20,000 to 50,000 with subset size fixed to 2000. Results for $p = 10,000$ are provided in appendix.

It is clear that message had excellent performance in all of the simulation cases, with low MSE, high probability of selecting the true model, and low computational time. The other subset-based methods we considered had similar computational times and also had computational burdens that effectively did not increase with sample size, while the full data analysis and bootstrap Lasso approach both were substantially slower than the subset methods, with the gap increasing linearly in sample size. In terms of MSE, the averaging and median approaches both had dramatically worse performance than message in every case, while bootstrap Lasso was competitive (MSEs were same order of magnitude with message ranging from effectively identical to having a small but significant advantage), with both message and bootstrap Lasso clearly outperforming the full data approach. In terms of feature selection performance, averaging had by far the worst performance, followed by the full data approach, which was substantially worse than bootstrap Lasso, median and message, with no clear winner among these three methods. Overall message clearly had by far the best combination of low MSE, accurate model selection and fast computation.

4.2 Individual household electric power consumption

This data set contains measurements of electric power consumption for every household with a one-minute sampling rate [27]. The data have been collected over a period of almost 4 years and contain 2,075,259 measurements. There are 8 predictors, which are converted to 74 predictors due to re-coding of the categorical variables (date and time). We use the first 2,000,000 samples as the training set and the remaining 75,259 for testing the prediction accuracy. The data are partitioned into 200 subsets for parallel inference. We plot the prediction accuracy (mean square error for test samples) against time for full data, message, averaging and median method in Fig 7. Bolasso is excluded as it did not produce meaningful results within the time span.

To illustrate details of the performance, we split the time line into two parts: the early stage shows how all algorithms adapt to a low prediction error and a later stage captures more subtle performance of faster algorithms (full set inference excluded due to the scale). It can be seen that message dominates other algorithms in both speed and accuracy.
4.3 HIGGS classification

The HIGGS data have been produced using Monte Carlo simulations from a particle physics model \[28\]. They contain 27 predictors that are of interest to physicists wanting to distinguish between two classes of particles. The sample size is 11,000,000. We use the first 10,000,000 samples for training a logistic model and the rest to test the classification accuracy. The training set is partitioned into 1,000 subsets for parallel inference. The classification accuracy (probability of correctly predicting the class of test samples) against computational time is plotted in Fig 8 (Bolasso excluded for the same reason as above).

Message adapts to the prediction bound quickly. Although the classification results are not as good as the benchmarks listed in \[28\] (due to the choice of a simple parametric logistic model), our new algorithm achieves the best performance subject to the constraints of the model class.

5 Discussion and conclusion

In this paper, we proposed a flexible and efficient message algorithm for regression and classification with feature selection. Message essentially eliminates the computational burden attributable to communication among machines, and is as efficient as other simple subset aggregation methods. By selecting the median model, message can achieve better accuracy even than feature selection on the full data, resulting in an improvement also in MSE performance. Extensive simulation experiments show outstanding performance relative to competitors in terms of computation, feature selection and prediction.

Although the theory described in Section 3 is mainly concerned with linear models, the algorithm is applicable in fairly wide situations. Geometric median is a topological concept, which allows the median model to be obtained in any normed model space. The properties of the median model result from independence of the subsets and weak consistency on each subset. Once these two conditions are satisfied, the property shown in Section 3 can be transferred to essentially any model space. The follow-up averaging step has been proven to be consistent for all M estimators with a proper choice of the partition number \[8\].
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Appendix A: Proof of Theorem 1 and Theorem 2

In this section, we prove the following stronger version of Theorem[1] and Theorem[2].

**Theorem 3.** (GIC) Assume each subset satisfies A.1, A.2 and A.4, and \( p \leq n^\alpha \) for some \( \alpha < k(\tau - \eta) \), where \( \eta = \max(i/k, 2k) \). If \( \gamma < \tau, 2k < \tau \) and \( \lambda \) are chosen so that \( \lambda = c_0 / \sigma^2(n/m)^{1-k} \) for some \( c_0 < cc_2/2 \), then there exists some constant \( C_0 \) such that for \( n \geq (2C_0p)^{(k+\eta-1)} \), any \( \delta \in (0, 1/2) \) and \( m \) \( \in \{ (\frac{1}{c_0})^{(k+\eta-1)}n/p^{(k+\eta-1)} \} \), the selected model \( M_G \) follows,

\[
P(M_G = M_S) \geq 1 - \exp \left\{ -\frac{(1/2 - \delta)^2}{2(1-\delta)} \left( \frac{\delta}{C_0} \right)^{(k+\eta-1)} n^{1-\alpha(k+\eta-1)} \right\},
\]

and the mean square error of the aggregated estimator follows,

\[
E[\|\hat{\beta} - \beta\|^2] \leq \frac{\sigma^2 V_{21}}{n} + \exp \left\{ -\frac{(1/2 - \delta)^2}{2(1-\delta)} \left( \frac{\delta}{C_0} \right)^{(k+\eta-1)} n^{1-\alpha(k+\eta-1)} \right\} \left( 1 + 2C_0^{-1}sV_1 \|\beta\|^2 + C_0^{-1}\sigma^2 \right),
\]

where \( C_0 = \min_\lambda \lambda_{\text{min}}(X^{(i)T}X^{(i)}/n_i) \).

**Theorem 4.** (Lasso) Assume each subset satisfies A.1, A.2 and A.3, and \( p \leq n^\alpha \) for some \( \alpha < k(\tau - \eta) \). If \( \gamma < \tau \) and \( \lambda \) are chosen so that \( \lambda = c_0(n/m)^{1-k} \) for some \( c_0 < c_1V_2/c_2 \), then there exists some constant \( C_0 \) such that for \( n \geq (2C_0p)^{(k+\eta-1)} \), any \( \delta \in (0, 1/2) \) and \( m \) \( \in \{ (\frac{1}{c_0})^{(k+\eta-1)}n/p^{(k+\eta-1)} \} \), the selected model \( M_G \) follows,

\[
P(M_G = M_S) \geq 1 - \exp \left\{ -\frac{(1/2 - \delta)^2}{2(1-\delta)} \left( \frac{\delta}{C_0} \right)^{(k+\eta-1)} n^{1-\alpha(k+\eta-1)} \right\},
\]

and with the same \( C_0 \) defined in Theorem[3] we have

\[
E[\|\hat{\beta} - \beta\|^2] \leq \frac{\sigma^2 V_{21}}{n} + \exp \left\{ -\frac{(1/2 - \delta)^2}{2(1-\delta)} \left( \frac{\delta}{C_0} \right)^{(k+\eta-1)} n^{1-\alpha(k+\eta-1)} \right\} \left( 1 + 2C_0^{-1}sV_1 \|\beta\|^2 + C_0^{-1}\sigma^2 \right).
\]

Fixing \( \delta = 1/4 \) gives exactly the Theorem[1] and Theorem[2] in the article. The above two theorems can be implied from the following three lemmas.

**Lemma 1.** (Median model for Lasso) Assume each subset satisfies A.1, A.2 and A.3. If \( \gamma < \tau \) and \( \lambda \) is chosen so that \( \lambda = c_0(n/m)^{1-k} \) for some \( c_0 < c_1V_2/c_2 \), then there exists some constant \( C_0 \) such that for \( n \geq (2C_0p)^{(k+\eta-1)} \) and any \( \delta \in (0, 1/2) \), the selected median model \( M_G \) satisfies

\[
P(M_G = M_S) \geq 1 - \exp \left\{ -\frac{(1/2 - \delta)^2}{2(1-\delta)} \left( \frac{\delta}{C_0} \right)^{(k+\eta-1)} n^{1-\alpha(k+\eta-1)} \right\},
\]

where \( \delta \) determines the number of subsets \( m \)

\[ m = \left\lfloor \left( \frac{\delta}{C_0} \right)^{k-1/(\tau-\eta)}n/p^{(k+\eta-1)} \right\rfloor \]

and constant \( C_0 \) is defined in the proof.

In particular, if \( p \leq n^\alpha \) for some \( \alpha < k(\tau - \eta) \) then

\[
P(M_G = M_S) \geq 1 - \exp \left\{ -\frac{(1/2 - \delta)^2}{2(1-\delta)} \left( \frac{\delta}{C_0} \right)^{(k+\eta-1)} n^{1-\alpha(k+\eta-1)} \right\}.
\]

**Lemma 2.** (Median model for GIC) Assume each subset satisfies A.1, A.2 and A.4. Let \( \eta = \max(i/k, 2k) \). If \( \gamma < \tau, 2k < \tau \) and \( \lambda \) is chosen so that \( \lambda = c_0 / \sigma^2(n/m)^{1-k} \) for some \( c_0 < cc_2/2 \), then there exists some constant \( C_0 \) such that for any \( \delta \in (0, 1/2) \) and \( n \geq (2C_0p)^{(k+\eta-1)} \), the selected median model \( M_G \) satisfies

\[
P(M_G = M_S) \geq 1 - \exp \left\{ -\frac{(1/2 - \delta)^2}{2(1-\delta)} \left( \frac{\delta}{C_0} \right)^{(k+\eta-1)} n^{1-\alpha(k+\eta-1)} \right\}.
\]
where $\delta$ determines the number of subsets $m$

$$m = \lfloor (\frac{\delta}{C_0})^{(k_\tau-k_1)^{-1}}n/p^{(k_\tau-k_1)^{-1}} \rfloor.$$ 

In particular, when $p \leq n^\alpha$ for some $\alpha < k(\tau - \eta)$ we have

$$P(M_\gamma = M_S) \geq 1 - \exp \left\{ - \frac{(1/2 - \delta)^2}{2(1 - \delta)} \left( \frac{\delta}{C_0} \right)^{(k_\tau-k_n)^{-1}} n^{1-\alpha(k_\tau-k_n)^{-1}} \right\}.$$ 

Recall that the design matrix for the true model $M_S$ is assumed to be positive-definite (Assumption A.1) for all subsets. It is therefore reasonable to ensure the selected model $M_\gamma$ possess the same property, and thus we have,

**Lemma 3. (MSE for averaging)** Assume $\hat{\beta}_i$ is the OLS estimator obtained from each subset based on the selected model $\gamma$, then the averaged estimator $\hat{\beta}$ has the mean square error,

$$E\left[ \| \hat{\beta} - \beta \|_2^2 \mid M_\gamma = M_S \right] \leq \frac{\sigma^2 V_2^{-1}s}{n}.$$ 

and

$$E\left[ \| \hat{\beta} - \beta \|_2^2 \mid M_\gamma \neq M_S \right] \leq (1 + 2C_0^{-1}sV_1)\| \beta \|_2^2 + C_0^{-1}\sigma^2.$$ 

where $C_0^i = \min_i \lambda_{min}(X^{(i)}_Y X^{(i)}_Y/n_i).$

**Proof of Lemma [1]**

**Proof.** Following the proof of Theorem 3 in [22], we have the following result: for the $i^{th}$ subset, the selected model $M_{\gamma(i)}$ follows that,

$$P(M_{\gamma(i)} = M_S) \geq 1 - C_1(\frac{2}{c_1})^{2k}n_i^{-k_{\tau+i}} - C_2\frac{4^k \sigma m_i^k}{\lambda^{2k}},$$ 

where $C_1, C_2$ are constants:

$$C_1 = \frac{c_1(2k - 1)!!E(\epsilon^{2k})}{V_2}, \quad C_2 = (2k - 1)!!V_1 E(\epsilon^{2k})$$

and $n_i = n/m$. If $\lambda$ is chosen to be $c_0(n/m)^{\frac{z+1}{z}}$, then the above result can be updated as

$$P(M_{\gamma(i)} = M_S) \geq 1 - C_1(\frac{2}{c_1})^{2k}n_i^{-k_{\tau+i}} - C_2(2/c_0)^{2k}m_i^{-k_{\tau+i}},$$

$$\geq 1 - C_0m_i^{-k_{\tau+i}},$$

where $C_0$ equals to

$$C_0 = C_1(\frac{2}{c_1})^{2k} + C_2(2/c_0)^{2k}.$$ 

For any fixed $m$ if the sample size $n$ satisfies

$$n \geq m(2C_0p)^{\frac{1}{1-\alpha}},$$

then we have $P(M_{\gamma(i)} = M_S) > 1/2$ on each subset. Recall the definition for the median model,

$$\gamma = \min_{\gamma \in \{0,1\}} \sum_{i=1}^{m} \| \gamma - \gamma^{(i)} \|_1.$$ 

Notice that as long as half subsets select the correct model, i.e., $card(\{i : M_{\gamma(i)} = M_S\}) \geq m/2$, we will have $M_{\gamma} = M_S$. Therefore, letting $S_{cor} = \sum_{i=1}^{m} I_{M_{\gamma(i)} = M_S}$, where $I_A$ is the indicator function for $A$, we have

$$P(M_{\gamma} = M_S) = P(S_{cor} \geq \lceil m/2 \rceil).$$
In particular, when \( \delta \in \delta \) for any \( \delta \), we can apply the Chernoff inequality (13) or Proposition A.6.1 of [29] to obtain that,

\[
P(M_\gamma = M_S) \geq 1 - \exp \left\{ - \frac{(1/2 - C_0 p/n)^{k \tau + i/2}}{2(1 - C_0 p/n)^{k \tau + i/2} - m} \right\}.
\]

(7)

Equivalently, for any \( n \geq (2C_0 p)^{1/(k \tau - \eta)} \), if we choose \( m = \lfloor (\frac{\delta}{C_0})^{1/(\tau - \eta)} \cdot n/p^{k \tau - 1/(\tau - \eta)} \rfloor \) for any \( \delta \in (0, 1/2) \), we have

\[
P(M_\gamma = M_S) \geq 1 - \exp \left\{ - \frac{(1/2 - \delta)^2}{2(1 - \delta)} \left( \frac{\delta}{C_0} \right)^{(k \tau - k) - 1} n^{1 - \alpha (k \tau - k) - 1} \right\}.
\]

In particular, when \( p \leq n^\alpha \) for any \( \alpha < k(\tau - \eta) \) we have

\[
P(M_\gamma = M_S) \geq 1 - \exp \left\{ - \frac{(1/2 - \delta)^2}{2(1 - \delta)} \left( \frac{\delta}{C_0} \right)^{(k \tau - k) - 1} n^{1 - \alpha (k \tau - k) - 1} \right\}.
\]

(8)

\[\square\]

Proof of Lemma 2

Proof. The proof is essentially the same as Lemma 1. Following the proof of Theorem 2 in [20], we will have the initial result on each subset,

\[
P(M_\gamma(\cdot) = M_S) \geq 1 - C_1 \left( \frac{2}{c_2} \right)^{2k} n_i^{-k \tau + i} - C_2 \left( \frac{2}{\sigma} \right)^{2k} \left( \frac{p}{\rho \lambda} \right)^{k},
\]

(9)

where \( n_i = n/m \) and

\[
C_1 = \frac{c_1 (2k - 1)!}{V_2} E^2, \quad C_2 = (2k - 1)! V_1 E^2.
\]

Now because \( \lambda = \frac{\gamma}{\sigma^2 (m/n)^{\tau - \kappa}} \) we update the above equation to

\[
P(M_\gamma(\cdot) = M_S) \geq 1 - C_1 \left( \frac{2}{c_2} \right)^{2k} n_i^{-k \tau + i} - C_2 \left( \frac{2}{c_0 c_3} \right)^{k} p^{-k(\tau - 2\kappa)}
\]

(10)

\[
\geq 1 - C_0 p n_i^{-k(\tau - \eta)},
\]

(11)

where \( \eta = \max \{ i/k, 2\kappa \} \) and

\[
C_0 = \frac{c_1 (2k - 1)!}{V_2} E^2 + C_2 \left( \frac{2}{c_0 c_3} \right)^{k}.
\]

With exactly the same argument as in Lemma 1, once the sample size exceeds \( (2C_0 p)^{1/(\tau - \eta)} \), then for any \( \delta \in (0, 1/2) \) and \( m = \lfloor (\frac{\delta}{C_0})^{1/(\tau - \eta)} \cdot n/p^{k \tau - 1/(\tau - \eta)} \rfloor \), we have

\[
P(M_\gamma = M_S) \geq 1 - \exp \left\{ - \frac{(1/2 - \delta)^2}{2(1 - \delta)} \left( \frac{\delta}{C_0} \right)^{(k \tau - k \eta) - 1} n^{1 - \alpha (k \tau - k \eta) - 1} \right\}.
\]

(12)

In particular, when \( p \leq n^\alpha \) for some \( \alpha < k(\tau - \eta) \) we have

\[
P(M_\gamma = M_S) \geq 1 - \exp \left\{ - \frac{(1/2 - \delta)^2}{2(1 - \delta)} \left( \frac{\delta}{C_0} \right)^{(k \tau - k \eta) - 1} n^{1 - \alpha (k \tau - k \eta) - 1} \right\}.
\]

(13)

\[\square\]

Proof of Lemma 3

Proof. To simplify the notation, let \( X \) denote the selected feature matrix \( X_\gamma \). Now if the selected model is correct, the error of OLS estimator can be described in the following form,

\[
\hat{\beta} - \beta = (X^T X)^{-1} X^T \epsilon.
\]

(14)
Hence the error of averaged estimator is,
\[ \hat{\beta} - \beta = \sum_{i=1}^{m} (X(i)^T X(i))^{-1} X(i)^T \epsilon(i) / m. \] (15)

Because \( E \epsilon^2 = \sigma^2 \) we have
\[ E\|\hat{\beta} - \beta\|^2 = \frac{\sigma^2}{m^2} \sum_{i=1}^{m} tr[(X(i)^T X(i))^{-1}]. \] (16)

As the smallest eigenvalue for each subset feature matrix is lower bounded by \( V_2 \), i.e.,
\[ \lambda_{\min}(X(i)^T X(i)/n_i) \geq V_2, \]
we have
\[ E\|\hat{\beta} - \beta\|^2 \leq \frac{\sigma^2}{m^2n_i} \sum_{i=1}^{m} n_i \lambda_{\min}(X(i)^T X(i)/n_i) \]
\[ = \frac{\sigma^2 V_2^{-1} s}{n}. \] (17)

However, if the model is incorrect, we can bound the incorrect estimators in the following way. For each subset,
\[ \|\hat{\beta} - \beta\|^2 = \|\hat{\beta} - \beta\|^2 + \|\beta - \beta\|^2 \leq \|\hat{\beta} - \beta\|^2 + \|\beta - \beta\|^2. \] (18)

Now to quantifiy the first term, we first notice that
\[ (X\hat{\beta}_i - X\beta_i)/\sqrt{\bar{m}_i} = (X(X^T X)^{-1} X^T Y - X\beta_i)/\sqrt{\bar{m}_i}, \]
and therefore
\[ C_0 n \|\hat{\beta} - \beta\|^2 \leq \|X(X^T X)^{-1} X^T Y - X\beta_i\| \leq 2\|X\beta_i\|^2. \] (19)

Taking expectation on both sides we have,
\[ E\|\hat{\beta}_i - \beta\|^2 \leq C_0^{-1} (E \|X\|^2/n_i + \|X\beta\|^2)/n_i \]
\[ \leq C_0^{-1} (\|X\beta\|^2/n_i + \sigma^2 + \|X\beta\|^2)/n_i \]
\[ \leq C_0^{-1} (2\|\beta\|^2\lambda_{\max}(X^T X/n_i) + \sigma^2) \]
\[ \leq C_0^{-1} (2sV_1 \|\beta\|^2 + \sigma^2). \] (20)

Therefore, we have
\[ E\|\hat{\beta} - \beta\|^2 \leq (1 + 2C_0^{-1}sV_1) \|\beta\|^2 + C_0^{-1}\sigma^2. \] (21)

The above bound holds for all subset estimators \( \beta^{(i)} \), it should also hold for their average \( \hat{\beta} \), i.e.,
\[ E\|\hat{\beta} - \beta\|^2 \leq (1 + 2C_0^{-1}sV_1) \|\beta\|^2 + C_0^{-1}\sigma^2. \] (22)

\[ \square \]

Appendix B: Assumption justification

It is important to carefully assess whether the conditions assumed to obtain theoretical guarantees can be satisfied in applications. There is typically no comprehensive answer to this question, as the answer usually varies across application areas. Nonetheless, we provide some discussion below to provide some insights, while being limited by the complexity of the question.

In following paragraphs, we attempt to justify A.1, A.3 and A.4 with examples and theorems. The main reason to leave A.2 alone is because A.2 is an assumption on basic model structure that is routine in the high-dimensional literature. See Zhao and Yu (2006) and Kim et al. (2012).

The discussion is divided into two parts. In the first part, we consider the case where features or predictors are independent. In the second part, we will address the correlated case. Because we can always standardize feature matrix \( X \) prior to any analysis, it will be convenient to assume \( x_{ij} \) having mean 0 and variance 1. For independent features, we have the following result.
Theorem 5. If the entries of the \( n \times p \) feature matrix \( X \) are i.i.d random variables with finite \( 4w \)th moments for some integer \( w > 0 \), then A.1, A.3 and A.4 will hold for all \( m \) subsets with probability,

\[
P(A.1, A.3 \text{ and } A.4 \text{ hold for all subsets}) \geq 1 - O\left(\frac{m^{2w} (2s - 1)^{2w}p^2}{n^{2w-1}}\right),
\]

where \( s \) is the number of non-zero coefficients.

Alternatively, for a given \( \delta_0 > 0 \), if the sample size \( n \) satisfies that

\[
n \geq m(2s - 1)\left(9^w (2w - 1)!M_1 (2s - 1)mp^2 \delta_0^{-1}\right)^{\frac{1}{2w-1}},
\]

where \( M_1 \) is some constant, then with probability at least \( 1 - \delta_0 \), all subsets satisfy A.1, A.3 and A.4.

The proof will be provided in next section. Theorem 5 requires \( m = o(n) \), which seems to conflict with Theorem 1 and 2 in the article where \( m \) is assumed to be \( O(n) \). This is, however, caused by the choice of \( \delta \) (see the stronger version of Theorem 1 provided in the first section). \( \delta \) is fixed at 1/4 in the article for simplicity, leading to the conclusion of \( m = O(n) \). With a different choice of \( \delta \) satisfying \( \delta = o(n) \), Theorem 3 along with Theorem 5 can be satisfied simultaneously. The same argument can be applied to Theorem 4 introduced in the next part as well.

Next, we consider the case when features are correlated. For data sets with correlated features, pre-processing such as preconditioning might be required to satisfy some of the conditions. Due to the complexity of the problem, we restrict our attention to data sets following elliptical distributions and under high dimensional setting (\( p > n \)). Real world data commonly follow elliptical distributions approximately, with density proportional to \( g(x^T \Sigma^{-1} x) \) for some non-negative function \( g(\cdot) \). The Multivariate Gaussian is a special case with \( g(z) = \exp(-z/2) \). Following the spirit of Jia and Rohe (2012), we make use of \( (X^TX/p)^{-1/2} (XX^T/p) \) as preconditioning matrix and then use results from Wang and Leng (2014) to show A.1, A.3 and A.4 hold with high probability. Thus, we have the following result.

Theorem 6. Assume \( p > n \) and define \( \tilde{X} = (X^TX/p)^{-1/2} X \) and \( \tilde{Y} = (X^TX/p)^{-1/2} Y \). If each row of feature matrix \( X \) are i.i.d samples drawn from an elliptical distribution with covariance \( \Sigma \) and the condition number of \( \Sigma \) satisfies that \( \text{cond}(\Sigma) < M_2 \) for some \( M_2 > 0 \), then for any \( M > 0 \) there exist some \( M_3, M_4 > 0 \) such that A.1, A.3 and A.4 hold for all subsets with probability,

\[
P(A.1, A.3 \text{ and } A.4 \text{ hold for all subsets}) \geq 1 - O\left(m^p \exp\left(-\frac{Mn}{2m \log n}\right)\right),
\]

if \( n \geq \exp\left(4M_3 M_4 (2s - 1)^2\right) \).

Alternatively, for any \( \delta_0 > 0 \), if the sample size satisfies that

\[
n \geq \max\left\{O\left(2m(\log m + 2 \log p - \log \delta_0)/M\right), \exp\left(4M_3 M_4 (2s - 1)^2\right)\right\},
\]

then with probability at least \( 1 - \delta_0 \), A.1, A.3 and A.4 hold for all subsets.

The proof is also provided in next section.

Proof of Theorem 5

Proof. Let \( C = \frac{1}{n} X^T X \). We divide the proof into four parts. In Part I and II, we examine the magnitude of \( C_{ij} \) and \( C_{ii} \) and give the probability that A.3 and the first part of A.1 hold for a single data set. In Part III, we give the probability that A.4 and the second part of A.1 hold on a single data set. We generalize the result to multiple data sets in Part IV.

Part 1. For \( C_{ii} \) we have

\[
EC_{ii} = E\|x_i\|^2/n = \frac{1}{n} \sum_{j=1}^{n} E x_{ij}^2 = 1,
\]
and

\[ E[C_{ii} - 1]^{2w} = \frac{1}{n^{2w}} E \sum_{j=1}^{n} (x_{ij}^2 - 1)^{2w} \leq \frac{(2w - 1)!!E[x_{12}^{2w} - 1]^{2w}}{n^{2w-1}}, \]

where the last inequality follows the proof of Theorem 3 in [22]. Because \( E|x_{12}^{4w}| < \infty \), it is clear that \( E|x_{12}^{2w} - 1|^{2w} = \sum_{j=0}^{\infty} (-1)^j C_{2j} E|x_{12}^{4w-2j}| \) is also a finite value, which will be denoted by \( M_0 \). Now by the Chebyshev’s inequality we have for any \( t > 0 \),

\[ P(|C_{ii} - 1| > t) < \frac{(2w - 1)!!M_0}{n^{2w-1}t^{2w}}. \]

Therefore, by taking the union bound over \( i = 1, 2, \ldots, p \) we have,

\[ P\left( \max_{i \neq k} |C_{ik} - 1| > t \right) < \frac{(2w - 1)!!M_1p^2}{n^{2w-1}t^{2w}}. \]

Recall the definition of \( C_{ii} \), [24] immediately implies that the first part of A.1 will hold with probability at least \( 1 - O\left( \frac{p}{n^{2w-1}} \right) \) for a single data set.

**Part II.** Following the same argument, we can establish the same inequality for \( C_{ik} \) where the only difference is the mean,

\[ EC_{ik} = Ex_i^T x_k/n = \frac{1}{n} \sum_{j=1}^{n} Ex_{ij}x_{jk} = 0. \]

From Chebyshev’s inequality we have for any \( t > 0 \),

\[ P(|C_{ik}| > t) < \frac{(2w - 1)!!M_1}{n^{2w-1}t^{2w}}, \]

where \( M_1 = \max \{ E|x_{12}^{2w}|, M_0 \} \) is a constant. Taking union bound over all off-diagonal terms we have,

\[ P\left( \max_{i \neq k} |C_{ik}| > t \right) < \frac{(2w - 1)!!M_1p^2}{n^{2w-1}t^{2w}}. \]

With [24] and [25], we can quantify the sample correlation between \( x_i \) and \( x_k \), which is \( C_{ik}/\sqrt{C_{ii}C_{kk}} \). Taking \( t = (6s - 3)^{-1} \) for both inequalities, we have

\[ P\left( \max_{i \neq k} |\text{cor}(x_i, x_k)| > \frac{1}{4s - 2} \right) < 2(2w - 1)!!9^wM_1(2s - 1)2w/p^2 \]

With Corollary 2 in [22], the above result essentially states that A.3 will hold with probability at least \( 1 - O\left( \frac{p^2}{n^{2w-1}} \right) \) for a single machine.

**Part III.** For the second part of A.1 and A.4, we might need to quantify the minimum value of \( v^TCv \) for any vector \( |v|_2 = 1 \) with support \( |\text{supp}\{v\}| \leq s \). Here \( \text{supp}\{a\} \) stands for all non-zero coordinates of vector \( a \). Let \( S \) be index set for non-zero coefficients. Noticing that

\[ \lambda_{\min}\left( \frac{1}{n} X_i^T X_S \right) = \min_{\|v\|=1, \text{supp}\{v\}=S} v^T C \geq \min_{\|v\|=1, |\text{supp}\{v\}| \leq s} v^T C, \]

and

\[ \inf_{|\pi| \leq s} \lambda_{\min}\left( \frac{1}{n} X_{\pi}^T X_{\pi} \right) = \inf_{|\pi| \leq s} \min_{\|v\|=1, \text{supp}\{v\}=\pi} v^T C = \min_{\|v\|=1, |\text{supp}\{v\}| \leq s} v^T C. \]

Thus, evaluating \( \min_{\|v\|=1, |\text{supp}\{v\}| \leq s} v^T C \) solely is adequate. In fact, for any vector \( v \) with \( |\text{supp}\{v\}| \leq s \) we have,

\[ v^T C = \sum_{i \in \text{supp}\{v\}} C_{ii}v_i^2 + \sum_{i \neq k \in \text{supp}\{v\}} C_{ik}v_iv_k \]

\[ \geq \min_{i \in \{1, 2, \ldots, p\}} C_{ii} - \left( (s^2 - s) \max_{i \neq k \in \{1, 2, \ldots, p\}} C_{ik}^{(S)^2} \right)^{1/2} \]

(26)
The second step is an application of Cauchy-Schwarz inequality and the fact that if $\|v\|_2 = 1$ then $\sum_{i\neq k} v_i^2 v_k^2 < 1$. Combining (26) with (24) and (25), and taking $t = (2s + 2)^{-1}$ we have

$$\min_{\|v\|=1, |\supp(v)| \leq s} v^T C v \geq 1 - \frac{1}{2s + 2} - \frac{s}{2s + 2} = \frac{1}{2},$$

with probability at least

$$1 - \frac{2^{2w+1}(2w - 1)!M_1(s + 1)^{2w}p^2}{n^{2w-1}}.$$

**Part IV.** Consequently, A.1, A.3 and A.4 will hold for data set on a single machine with probability,

$$P(A.1, A.3 \text{ and } A.4) \geq 1 - O\left\{ \frac{(2s - 1)^{2w}p^2}{n^{2w-1}} \right\}.$$ 

Now if we have $m$ subsets, each with sample size $n/m$, then the probability that all subsets satisfy A.1, A.3 and A.4 follows,

$$P(A.1, A.3 \text{ and } A.4 \text{ hold for all}) \geq 1 - O\left\{ \frac{m^{2w}(2s - 1)^{2w}p^2}{n^{2w-1}} \right\}.$$ 

Alternatively, for a given $\delta_0 > 0$ and the number of subsets $m$, if the sample size $n$ satisfies that

$$n \geq m(2s - 1)\left\{ 9^w(2w - 1)!M_1(2s - 1)m^2p^2\delta_0^{-1} \right\} \frac{1}{1 - \delta_0},$$

then with probability at least $1 - \delta_0$, all subsets satisfy A.1, A.3 and A.4.

**Proof of Theorem 6**

**Proof.** The proof procedure is essentially the same as Theorem 5. We begin by looking at data set on a single machine. Let $C = X^T X/n = p/n \cdot X^T (XX^T)^{-1} X$. From Lemma 4 and Lemma 5 of [30] we have, for any $M > 0$ there exists some constant $M_3, M_4 > 0$ such that,

$$P\left( \max_{i \in \{1, 2, \ldots, p\}} C_{ii} > M_3 \right) \text{ or } P\left( \min_{i \in \{1, 2, \ldots, p\}} C_{ii} < M_3^{-1} \right) < 4p \cdot e^{-Mn}, \quad (27)$$

and

$$P\left( \max_{i \neq k \in \{1, 2, \ldots, p\}} C_{ik} > \frac{M_4}{\sqrt{\log n}} \right) \leq O\left\{ p^2 \cdot \exp\left( -\frac{Mn}{2 \log n} \right) \right\}. \quad (28)$$

Inequality (27) implies the first part of A.1. For A.3, we can use (27) and (28) to bound the sample correlation $\text{cor}(x_i, x_k) = C_{ik}/\sqrt{C_{ii}C_{kk}}$ as follows,

$$P\left( \max_{i \neq k \in \{1, 2, \ldots, p\}} |\text{cor}(x_i, x_k)| > \frac{M_3M_4}{\sqrt{\log n}} \right) \leq O\left\{ p^2 \cdot \exp\left( -\frac{Mn}{2 \log n} \right) \right\}.$$ 

Therefore, to satisfy A.3 only requires

$$n \geq \exp\left( 4M_3M_4(2s - 1)^2 \right). \quad (29)$$

As $s$ is assumed to be small, (29) will not be a big threat to the sample size. For A.4 and the second part of A.1, we continue to apply the same strategy in the proof of Theorem 3 (Part III). Using (26) we have,

$$\min_{\|v\|=1, |\supp(v)| \leq s} v^T C v \geq M_3^{-1} - \frac{M_4s}{\sqrt{\log n}}$$

with probability $1 - O\left\{ p^2 \cdot \exp\left( -\frac{Mn}{2 \log n} \right) \right\}$. To satisfy A.4 and the second part of A.1, we just need $n$ to be greater than $\exp(M_3M_4s^2)$, which is already true if (29) holds.
Consequently, for a single machine if (29) is satisfied, A.1, A.3 and A.4 hold with probability,

\[
\Pr(A.1, A.3 \text{ and } A.4) \geq 1 - O\left(\frac{p^2 \cdot \exp\left(-\frac{Mn}{2 \log n}\right)}{\exp\left(-\frac{Mn}{2 \log n}\right)}\right).
\]

Now for \( m \) subsets, each with sample size \( n/m \), the probability that A.1, A.3 and A.4 hold for all subsets follows,

\[
\Pr(A.1, A.3 \text{ and } A.4 \text{ hold for all subsets}) \geq 1 - O\left(\frac{mp^2 \cdot \exp\left(-\frac{Mn}{2m \log n}\right)}{\exp\left(-\frac{Mn}{2m \log n}\right)}\right).
\]

Alternatively, for any \( \delta_0 > 0 \), if

\[
n \geq \max \left\{ O\left(2m(\log m + 2 \log p - \log \delta_0)/M\right), \exp\left(4M_3M_4(2s - 1)^2\right)\right\}
\]

then A.1, A.3 and A.4 hold for all subsets with probability at least \( 1 - \delta_0 \).

**Appendix C: Results for** \( p = 10,000 \)

we simulate 50 data set for each case, and let the sample size range from 20,000 to 50,000 with subset size fixed to 2,000. Bolasso is not implemented as the computation cost is too expensive. The results are plotted in Fig.9-14.
Figure 11: Results for case 2 with $\rho = 0$.

Figure 12: Results for case 2 with $\rho = 0.5$.

Figure 13: Results for case 3 with $\rho = 0$.

Figure 14: Results for case 3 with $\rho = 0.5$. 