A spectral approach to a constrained optimization problem for the Helmholtz equation in unbounded domains

Giulio Ciraolo Francesco Gargano Vincenzo Sciacca

January 23, 2014

Abstract

We study some convergence issues for a recent approach to the problem of transparent boundary conditions for the Helmholtz equation in unbounded domains [10]. The approach is based on the minimization on an integral functional which arises from an integral formulation of the radiation condition at infinity. In this Letter, we implement a Fourier-Chebyshev collocation method and show that this approach reduce the computational cost significantly. As a consequence, we give numerical evidence of some convergence estimates available in literature [9] and we study the robustness of the algorithm at low and mid-high frequencies.

1 Introduction

We consider a classical problem in scattering theory for the Helmholtz equation (or reduced wave equation). Let \( u : \mathbb{R}^d \to \mathbb{C} \) be the outgoing solution of

\[
\Delta u + k^2 n(x)^2 u = f, \quad x \in \mathbb{R}^d,
\]

where \( k > 0 \) is the wavenumber, \( n > 0 \) is the index of refraction of the background medium, \( f \) is a source term and \( u \) is part of the electromagnetic field. When \( n \) is constant outside some compact region, say \( n(x) \equiv 1 \) for \( x \) outside some large ball, the term outgoing solution means that we look for a solution \( u \) of (1) satisfying the following Sommerfeld radiation condition

\[
\lim_{r \to \infty} r^{d-1} \left( \frac{\partial u}{\partial r} - iku \right) = 0,
\]

uniformly, where \( r = |x| \). For more general indexes of refraction \( n \), (2) must be replaced by a more general condition like

\[
\int_{\mathbb{R}^d} \left| \nabla u(x) - ikn(x)u(x) \frac{x}{|x|} \right|^2 \frac{dx}{1 + |x|} < +\infty,
\]

where \( d = 2, 3 \).
see [27]. Condition (3) guarantees the uniqueness of the outgoing solution of (1) under very general assumptions on $n$. As an example, we mention that it applies to an index of refraction which is the sum of an angularly dependent function $n_\infty$ and some perturbation $p$, $n(x) = n_\infty(x/|x|) + p(x)$, where $p$ has a suitable decay at infinity.

A challenging issue in computational studies related to (1) is how to deal numerically with the unbounded domain. Usually, one has to introduce a (bounded) computational domain $\Omega$ and then prescribe boundary conditions on $\partial \Omega$ which approximate the problem in the whole space. As it is well-known, a large amount of work has been done on this kind of problems. The most used methods are based on local or nonlocal conditions involving $u$ ([5], [13], [18]), approximations of the Dirichlet-to-Neumann map ([21], [17], [19]), infinite element schemes ([3]), boundary element methods (see [28]) and the perfectly matched layer method ([4], [31]). For a deeper understanding of these problems and more recent developments, the interested reader can refer to the references cited in [10].

Starting from (3), in [10] we proposed a new approach to the computational study of the Helmholtz equation in unbounded domains. The idea is the following: we fix a computational bounded domain $\Omega$ and approximate the solution of (1) and (2) by the minimizer $u_\Omega$ of the following constrained optimization problem:

$$\inf \{ J_\Omega[w] : w \text{ satisfies } \Delta w + k^2 n(x)^2 w = f \text{ in } \Omega \},$$

where

$$J_\Omega[w] = \int_\Omega \frac{|\nabla w(x) - ikn(x)w(x)x|}{1 + |x|} dx.$$  \hspace{1cm} (5)

In [10] it is proved the existence and uniqueness of the minimizer $u_\Omega$. Furthermore, $u_\Omega$ converges to the outgoing solution in $H^1_{\text{loc}}$ norm as the computational domain $\Omega$ enlarges and tends to cover the whole space. More precisely, if we consider $\Omega = B_R$ (the ball or radius $R$ centered at the origin), then for any fixed $\rho > 0$ we have that

$$\|u_{B_R} - u\|_{H^1(B_\rho)} \to 0$$
as $R \to +\infty$; here, $u$ is the solution of (1) satisfying (3).

This approach is remarkable because: (i) it applies to a wide range of problems and for a large variety of indexes of refraction and source functions; (ii) numerical evidence (see [10]) shows that the method does not suffer the change of geometry of the computational domain; (iii) it is suitable to be extended to other scattering problems (we have in mind the waveguide problem by using the results in [1], [7]–[12]). On the other hand, the method has a slow convergence rate: in [9], it is proved that $\|u_{B_R} - u\|_{H^1(B_\rho)} = O(R^{-1})$ as $R \to \infty$, which is a convergence rate slower than the ones of many other methods. However, as already mentioned, this approach is applicable even when the methods available in literature fail or are of difficult application.

There is another interesting aspect of this approach: the study of the scattering problem at medium and high frequencies. The main goal of this Letter is to study how the rate of convergence of the numerical scheme depends on the frequency $k$. When $k$ is large, the solution has a large amount of oscillations and the computational complexity of the problem increases. Indeed, it is well known that, for frequencies in the mid and high regime, the number of unknowns in
the finite element methods scales at least like the cube of $kt$, where $t$ is a typical
dimension of the scatterer [20]. In boundary integral methods the number of
unknowns scales at least like the square of $kt$; this growth can be sometimes
reduced by using some a priori knowledge regarding the oscillatory nature of
the solution (see [16] and reference therein). Hence, the study of new numerical
approaches at high frequencies is of great interest for the scientific community.
The numerical results presented in this Letter (for $n(x) \equiv 1$) suggest that our
approach gives consistent results for $k$ large and can be used for computing the
solution at mid-high frequencies. Since the method applies to a large variety of
indexes of refraction, the results in this paper motivate further studies for more
general settings.

In this paper we use a spectral collocation method to implement the nu-
merical simulations and calculate the minimizer of (4). By using a spectral
approach, we can reduce the computational complexity: indeed, it is expected
that the number of spectral modes needed in the computations grows linearly
with $k$. Moreover, there is a numerical evidence that the rate of convergence of
our approach improves as $k$ becomes larger, at least in the mid frequency regime.
This motivates a numerical study of the rate of convergence of the method in
dependency on the frequency $k$. The numerical studies presented in this paper
will not make use of any approximation which could be used for $k$ large (see
for instance [16]); for this reason, we believe that the results are relevant in the
case of mid-high frequency regime.

The spectral approach will be used also to implement numerical simulations
at low frequencies ($k \approx 0$). In this case, it is expected that the error norms are
worse as $k$ becomes smaller. This is confirmed by some numerical simulations
presented in this paper. However, this behavior does not appear for any choice
of the source function. Indeed, we will show numerical evidence that the error
norms (slightly) improve as $k \to 0$ when the source function $f$ has vanishing
mean value.

The paper is organized as follows. In Section 2 we recall some anal-
lytical results and we conjecture the behavior of the numerical scheme for frequen-
ties in the low, mid and high frequency regime. The study is presented in two
dimensions but it can be extended to more dimensions. In Section 3 we describe
the numerical scheme and the spectral collocation method that we used. In
Section 4 we present some numerical results that confirm our conjectures on the
convergence rates.

2 Preliminaries and general considerations

Let $u$ be the solution of (1) satisfying (3) and denote by $B_R$ the ball of radius
$R$ centered at the origin. Let $u_R$ be the solution of the following minimization
problem

$$\inf \{ J_{B_R}[w] : w \text{ satisfies } \Delta w + k^2 n(x)^2 w = f \text{ in } B_R \},$$

where $J$ is given by (5) with $\Omega = B_R$. In [10] and [9], it is proved that $u_R$
approximate $u$ in $H^1_{loc}$ norm, more precisely that, for any fixed $\rho > 0$, we have
that

$$\| u_R - u \|_{H^1(B_\rho)} \approx \frac{1}{R}, \quad \text{as } R \to +\infty,$$

as $R \to +\infty$. 

3
for \( k \) fixed. As it was noticed in [10], in (7) there is an interesting interplay between the parameters \( k, R \) and \( f \). In this Letter, we shall investigate (7) and some aspects of its dependency on these parameters. In particular, by using numerical results in the simplest case possible (\( n(x) \equiv 1 \) and \( d = 2 \)), we are interested to study how the convergence in (7) depends on the wavenumber \( k \).

In [10], it was noticed that the convergence is expected to worsen as \( k \) becomes smaller (See Remark 3.1 in [10]), that is

\[
\lim_{k \to 0^+} \| u_R - u \|_{H^1(B_\rho)} = +\infty, \quad R \text{ fixed.} \tag{8}
\]

It is not surprising that there is a behaviour like (8) for \( k \) small. Indeed, when \( k = 0 \) the Helmholtz equation reduces to the Laplace equation and (3) (or the Sommerfeld radiation condition (2)) is not appropriate to guarantee that the scattering problem is well-posed. The first two goals of this Letter are: (i) a deeper study of the rate of blow-up in (8) and (ii) to understand if there are some sources for which there is no blow-up of \( \| u_R - u \|_{H^1(B_\rho)} \) as \( k \to 0^+ \). Such issues are motivated by the following considerations. Remark 3.1 in [10] and the numerical simulations presented in this Letter suggest that the blow-up rate is approximately \( 1/k \). Such a guess is confirmed by the following argument: it is known that the resolvent associated to (1)–(2) with \( n(x) \equiv 1 \) is not analytic at \( k = 0 \) (see [29]). Moreover, it is expected that the behavior of the resolvent improves when it applies to functions with vanishing mean value. As we will show by numerical simulations, such behavior of the resolvent is abided by our numerical scheme.

Now, let us discuss the behavior of the algorithm in the high-frequency regime. Following Remark 3.1 in [10], it is conjectured that the convergence improves for \( k \) large:

\[
\lim_{k \to \infty} \| u_R - u \|_{H^1(B_\rho)} = 0, \quad R \text{ fixed.} \tag{9}
\]

We do not know the exact rate of the convergence in (9): a deeper study of this issue is the latter goal of this Letter.

To achieve this goal, it is needed to study the Helmholtz equation numerically for large values of \( k \). Due to the large number of oscillations, this is a nontrivial issue. We notice that if the constrained optimization problem (5) is implemented by using finite element or difference methods, then the computational complexity of the problem increases as the cube of \( k \). Such a growth can be mitigated by (9) as \( k \) becomes large, but the overall computational complexity will still increase as the square of \( k \), at least.

For this reason, it is reasonable to use spectral methods to implement the numerical algorithm. Indeed, it is expected that the number \( N \) of spectral modes needed to implement the problem depends linearly on the number of oscillations of the solution. A qualitative analysis of the problem suggests that the number of relevant oscillations of the solution is comparable to the one of \( e^{ikr} \) for \( k \) large, which implies that the number of oscillations increases linearly with \( k \). Hence, \( N \approx k \) and we conjecture that the computational complexity of the numerical algorithm is stable for \( k \) large.

This heuristic argument motivates the numerical studies in the following sections. In particular, we shall implement a spectral collocation method which permits to reduce the computational complexity and then to test the method for large and small wavenumbers.
3 Numerical scheme

We use a spectral Fourier-Chebyshev collocation method to implement the constrained optimization problem (4). We notice that, thanks to the collocation method, the accuracy of a Fourier-Chebyshev-type scheme is equivalent to the one of a Fourier-Jacobi method when one uses (roughly) the same order of polynomials [22, 25, 32, 33, 23, 6].

As already mentioned, we implement the minimization problem for \( \Omega = B_R \) in \( \mathbb{R}^2 \), a disk of radius \( R \) centered at the origin which is parameterized in terms of the usual polar coordinates \((\rho, \theta)\). We discretize the disk by using a periodic Fourier grid in the angular variable \( \theta \) and Fourier points \( \theta_i = \frac{(i+1) \cdot 2\pi}{M_\theta+1} \), \( i = 0, \ldots, M_\theta \), and we use a Chebyshev grid in the radial variable \( \rho \), with Gauss-Lobatto points \( \rho_j = R \cos\left(\frac{j \cdot \pi}{M_\rho-1}\right) \), \( j = 0, \ldots, M_\rho - 1 \). To have the well-posedness of the numerical scheme at \( \rho = 0 \), we follow [14, 15, 30] and consider the symmetry condition \( u(\rho, \theta) = u(-\rho, (\theta + \pi) \mod 2\pi) \) in \((\rho, \theta)\)-space.

According to this discretization, we define the following complex-valued vectors

\[
v_{j+(M_\theta+1)i} = u(\rho_j, \theta_i), \\
\phi_{h+(M_\rho+1)i} = f(\rho_h, \theta_i),
\]

for \( i = 0, \ldots, M_\theta \), \( j = 0, \ldots, M_\rho - 1 \) and \( h = 1, \ldots, M_\rho - 1 \). We stress that the minimizer of Problem 4 has to satisfy the Helmholtz equation only in the interior points of \( \Omega \). For this reason, we do not need to evaluate \( f \) at boundary points.

We implement the constrain \( \Delta v + k^2 v = f \) in (4) by discretizing the Laplacian by the well-known differential matrix for spectral collocation methods (see [6, 15, 30]). Hence, the discretized Helmholtz equation can be seen as an algebraic constrain of the form

\[
Av = \phi, \quad (10)
\]

where the rectangular matrix \( A \) has dimension \((M_\theta+1) \cdot (M_\rho-1) \times (M_\theta+1) \cdot M_\rho\).

We notice that, since we are imposing the Helmholtz equation only at the interior points of the grid, \( A \) is a rectangular matrix and the \( M_\theta + 1 \) boundary values of \( v \) are the degrees of freedom of the problem.

By using Gauss quadrature formulas as integration rule in the functional (5), the discretized functional can be written as a quadratic form

\[
J_{B_R}[v] = \frac{1}{2} v^T H v, \quad (11)
\]

where the matrix \( H \) has dimension \((M_\theta+1) \cdot M_\rho \times (M_\theta+1) \cdot M_\rho\).

Hence, the discretized problem can be stated as follows: to find a complex-valued vector \( v \) of size \((M_\theta+1) \cdot M_\rho\) which minimizes the problem

\[
\min \frac{1}{2} v^T H v \text{ such that } Av = \phi. \quad (12)
\]

As it is well-known, constrained optimization problems involve a set of Lagrange multipliers \( \lambda \). By standard optimization theory, we have that the minimizer of (12) is given by the solution of the following (sparse) algebraic linear system:
\[
\begin{pmatrix}
A & 0 \\
H & A^T
\end{pmatrix}
\begin{pmatrix}
v \\
\lambda
\end{pmatrix}
= 
\begin{pmatrix}
\phi \\
0
\end{pmatrix},
\tag{13}
\]
where the vector of Lagrange multipliers \( \lambda \) has dimension \( (M_\theta + 1) \cdot (M_\rho - 1) \).

We compute the solution of (13) by a least square iterative method (choosing a tolerance of \( 10^{-10} \)). To test our numerical scheme, we consider problem (1) and (3) with \( n(x) = 1 \) (notice that in this case (3) is equivalent to (2)). In this case, the exact solution \( u \) is given by

\[
u(x) = \frac{i}{4} \int_{\mathbb{R}^2} H_0^{(1)}(k|x - y|) f(y) dy,
\]
where \( H_0^{(1)}(\cdot) \) is the zeroth-order Hankel function of the first kind. As source term in (1), we consider

\[
f(x) = -e^{-\sigma|x|^2},
\tag{14}
\]
with the choice of \( \sigma = 30 \).

We tested the convergence properties of the numerical scheme by evaluating several error norms \( (L^2, H^1 \text{ and } L^\infty \text{ as in [10]}) \) and observed the same order of errors as in [10]. However, thanks to the spectral approach, we can use less grid points. We checked that the numerical solution does not depend on the spatial resolution that we choose (both in the angular and radial variables). For instance, by fixing \( R = 4, k = 1 \) and \( M_\rho = 100 \), the errors remain at order \( 10^{-3} \) for values of \( M_\theta \) from 10 to 80. Viceversa, analogous results can be observed if we fix \( M_\theta = 20 \) and varying \( M_\rho \) from 50 to 600. It results that we obtain satisfying numerical results by using a not too high resolution and, as a consequence, we can perform numerical simulations for large values of \( R \) and \( k \).

4 Numerical Results

In this section we study some convergence properties for the optimization problem (12). In particular, we show a numerical evidence for the convergence rate estimates (7), and we study how the numerical algorithm behaves at low and mid/high frequencies.

We present the numerical results for two sources \( f_1 \) and \( f_2 \):

\[
f_1(x) = -e^{-\sigma|x|^2},
\tag{15}
\]
\[
f_2(x) = e^{-\sigma|x - p_k|^2} - e^{-\sigma|x - p_k - 0|^2},
\tag{16}
\]
where \( p_k = (\pm 0.25, 0) \) and with \( \sigma = 30 \). The computational domain is the disk \( B_R \), where \( R \) will be specified from time to time. We performed much more simulations with different sources and we observed that the numerical results are analogous to the one for \( f_1 \) and \( f_2 \); for this reason, they are not reported in this Letter.

4.1 Dependence on \( R \)

In this subsection we give numerical evidence of the estimate (7). We fix \( \rho = 1 \) and compute the numerical solution for several values of \( R \), which range from \( R = 4 \) to \( R = 32 \). For a fixed \( R \), the solution is obtained by taking \( M_\theta = \)
To obtain the estimate (7), we need to interpolate the solution $u_R$ of problem (12) - which is defined in the ball $B_R$ - in $B_1$. The interpolation is performed by using the spectral Fourier-Chebyshev expansion of the solutions: firstly, we compute the Fourier-Chebyshev modes by using a discrete Fourier-Chebyshev transform of the solution $u_R$ in $B_R$, and then we evaluate the Fourier-Chebyshev expansion in $B_1$, which is discretized by using $M_{\theta}^{B_1} = 20$ and $M_{\rho}^{B_1} = 100$ grid points in angular and radial variables respectively.

In Figures 1 and 2 we show the error norms in $B_1$ in the log-log scale for $f = f_1$ and $f = f_2$, respectively. We notice that the rate of decay is proportional to $1/R$, which perfectly agrees with (7).

### 4.2 Dependence on $k$

In this subsection we study numerically how the optimization problem behaves for small and large values of $k$. We fix the computational domain $B_R$ with $R = 4$, and we compute the numerical solution for several values of $k$, starting from $k = 1/32$ and up to $k = 96$. The resolution is fixed with $M_{\theta} = 20$ and $M_{\rho} = 800$.

In Fig.3 and Fig.4 we show the values of the $L^2$ and $H^1$ error norms for $f = f_1$ and $f = f_2$, respectively.

We notice that there is a different behavior in the two cases at low frequencies ($k$ small). For $f = f_1$, the error norms deteriorates as $k$ becomes smaller, while for $f = f_2$ they are stable (slightly improve). As already mentioned in Section 2, such a different behavior depends on the mean values of the source function: if it has non-vanishing mean value, then the convergence deteriorates, while it is robust if the source has vanishing mean value.
When $k$ becomes larger, Figures 3 and 4 show that the error norms improve in both cases. The numerical simulations suggest that the convergence improves as $k^{-\alpha}$, where $\alpha \approx 3$ for $f_1$ and $\alpha \approx 2.1$ for $f_2$. This behavior is remarkable because, for $k$ large, the numerical algorithm is not only stable but it improves in the mid-high frequency regime.

5 Conclusions

We considered a Fourier-Chebyshev collocation method for studying a constrained optimization problem which is related to a new approach to the problem of transparent boundary conditions for the Helmholtz equation in unbounded domains (see [10]). We gave numerical evidence of an estimate available in literature [9] which gives the rate of convergence of the numerical scheme at a fixed frequency $k$. We studied numerically the problem at low and mid-high frequencies and show that the numerical scheme is very robust in the mid-high regime. In the low frequency regime, we observed that the numerical results deteriorates for source functions with non-vanishing mean value, while it is robust when the source function has vanishing mean value.

Our numerical studies were performed for a constant index of refraction because in this case we have an analytic expression for the exact solution, and we can compare it with the numerical one. However, the method applies to a large variety of indexes of refraction and we believe that the numerical studies presented in this Letter give a hint on the convergence properties of the algorithm also in more general settings.
Figure 3: The values of the error norms at several values of $k$ for $f = f_1$. Here, $R = 4$, $M_\theta = 20$, $M_\rho = 800$.

Figure 4: The values of the error norms at several values of $k$ for $f = f_2$. Here, $R = 4$, $M_\theta = 20$, $M_\rho = 800$. 
References

[1] O. Alexandrov and G. Ciraolo, Wave propagation in a 3-D optical waveguide. Math. Models Methods Appl. Sci. (M3AS), 14 (2004), no.6, 819–852.

[2] C. Atamian, Q.V. Dinh, R. Glowinski, J. He, J. Periaux Control approach to fictitious domain methods: Applications to fluid dynamics and electromagnetics. Proc. Fourth Int. Conf. on Domain Decomposition Methods for Partial Differential Equations, SIAM, Philadelphia (1991), pp. 275–309.

[3] P. Bettess, Infinite elements, Int. J. Numer. Methods Engrg., 11 (1977), 53–64.

[4] J.-P. Bérenger, A perfectly matched layer for the absorption of electromagnetic waves, J. Comput. Phys. 114 (1994), 185–200.

[5] A. Bayliss and E. Turkel, Radiation boundary conditions for wave-like equations, Comm. Pure Appl. Math., 33 (1980), 707–725.

[6] J.P. Boyd, Chebyshev and Fourier Spectral Methods, Dover, 2001.

[7] G. Ciraolo, A method of variation of boundaries for waveguide grating couplers, Applicable Analysis, 87 (2008), 1019–1040.

[8] G. Ciraolo, A radiation condition for the 2-D Helmholtz equation in stratified media, Comm. Part. Diff. Eq., 34 (2009), 1592–1606.

[9] G. Ciraolo, Helmholtz equation in unbounded domains: some convergence results for a constrained optimization problem. Preprint [arXiv:1307.1260].

[10] G. Ciraolo, F. Gargano, V. Sciacca, A computational method for the Helmholtz equation in unbounded domains based on the minimization of an integral functional, Journal of Computational Physics 246 (2013), 78–95.

[11] G. Ciraolo and R. Magnanini. Analytical results for 2-D non-rectilinear waveguides based on the Green’s function. Math. Methods Appl. Sci., 31 (2008), no.13, 1587–1606.

[12] G. Ciraolo and R. Magnanini, A radiation condition for uniqueness in a wave propagation problem for 2-D open waveguides, Math. Methods Appl. Sci., 32 (2009), 1183–1206.

[13] B. Engquist and A. Majda, Radiation boundary conditions for acoustic and elastic wave calculations, Comm. Pure Appl. Math., 32 (1979), 314–358.

[14] B. Fornberg, A pseudospectral approach for polar and spherical geometries, SIAM J. Sci. Comp. 16 (1995), 1071–1081.

[15] B. Fornberg, A Practical Guide to Pseudospectral Methods, Cambridge University Press, 1996.

[16] E. Giladi, Asymptotically derived boundary elements for the Helmholtz equation in high frequencies, J. Comput. Appl. Math. 198 (2007), no. 1, 52–74.
[17] D. Givoli and J.B. Keller, *A finite element method for large domains*, Comput. Methods Appl. Mech. Engrg., 76 (1989), 41–66.

[18] D. Givoli, *High-order local non-reflecting boundary conditions: a review*, Wave motion, 39 (2004), 319–326.

[19] M.J. Grote and J.B. Keller, *On nonreflecting boundary conditions*, J. Comput. Phys., 122 (1995), 231–243.

[20] F. Ihlenburg, I. Babuska, *Finite element solution of the Helmholtz equation with high wavenumber part i: the h-version of the fem*, Comput. Math. Appl. 30 (1995) 9–37.

[21] J.B. Keller and D. Givoli, *Exact nonreflecting boundary conditions*, J. Comput. Phys., 82 (1989), 172–192.

[22] P. W. Livermore, C. A. Jones, S. J. Worland *Spectral radial basis functions for full sphere computations*, Journal of Computational Physics 227 (2007) 1209–1224.

[23] T. Matsushima, P. S. Marcus *A spectral method for polar coordinates*, Journal of Computational Physics 120 (1995) 365–374.

[24] M. Medvinsky, E. Turkel, U. Hetmaniuk, *Local absorbing boundary conditions for elliptical shaped boundaries*. J. Comput. Phys. 227 (2008), no. 18, 8254–8267.

[25] K. Mohseni, T. Colonius, *Numerical treatment of polar coordinates singularity*, Journal of Computational Physics 157 (2000) 787–795.

[26] A.F. Oskooi, L. Zhang, Y. Avniel, S.G. Johnson, *The failure of perfectly matched layers, and towards their redemption by adiabatic absorbers*. Opt. Express, 16 (15) (2008), pp. 11376–11392.

[27] B. Perthame and L. Vega, *Energy concentration and Sommerfeld condition for Helmholtz equation with variable index at infinity*, GAFA Geom. Funct. Anal., 17 (2008), 1685–1707.

[28] S. A. Sauter and C. Schwab. *Boundary element methods*. Springer Series in Computational Mathematics, 39. Springer-Verlag, Berlin, 2011.

[29] J. Sylvester, *An estimate for the free Helmholtz equation that scales*, Inverse Probl. Imaging, 3 (2009), 333–351.

[30] L.N. Trefethen, *Spectral Methods in MATLAB*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000.

[31] E. Turkel and A. Yefet, *Absorbing PML boundary layers for wave-like equations*, Appl. Numer. Math. 27 (1998), 533–557.

[32] W. T. M. Verkley, *A spectral model for two-dimensional incompressible fluid flow in a circular basin I*, Journal of Computational Physics 136 (1997) 100–114.

[33] W. T. M. Verkley, *A spectral model for two-dimensional incompressible fluid flow in a circular basin II*, Journal of Computational Physics 136 (1997) 115–131.