Automorphic representations and harmonic cochains for $GL_{n+1}$

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Abstract. Let $K$ be a global field of positive characteristic. Let $\infty$ be a fixed place of $K$. This paper gives an explicit isomorphism between the space of automorphic forms (resp. cusp forms) for $GL_{n+1}(K)$ that transform like the special representations and certain spaces of harmonic cochains (resp. those with a finite support) defined on the Bruhat-Tits building of $GL_{n+1}(K_\infty)$.

Keywords. Bruhat-Tits buildings, arithmetic groups, automorphic representations, harmonic cochains, special representations.

Introduction

Let $K$ be a global field of characteristic $p > 0$, that is a function field of a smooth geometrically irreducible projective curve $C$ over a finite field of characteristic $p$. We identify the places of $K$ with the closed points $|C|$ of $C$. Let $\infty \in |C|$ be a fixed place of $K$ and $K_\infty$ be the completion of $K$ at this place. Denote by $\mathbb{A}$ and $\mathbb{A}_f$ respectively the ring of adèles and that of finite adèles of $K$.

Denote by $G$ the reductive group scheme $GL_{n+1}$. Let $M$ be a commutative ring (with unit) and $L$ be an integral $M$-algebra of characteristic zero. $M$ and $L$ are assumed to be endowed with the trivial action of $G(K_\infty)$. For an open compact subgroup $\mathfrak{K}_f$ of $G(\mathbb{A}_f)$ denote by $\mathfrak{Aut}^{\mathfrak{K}_f}(L)$ the space of $L$-valued automorphic forms that are invariant under $\mathfrak{K}_f$. These are functions defined on

$$G(K) \backslash G(\mathbb{A})/\mathfrak{K}_f \times \mathfrak{K}_\infty Z_G(K_\infty)$$
for some open compact subgroup $\mathfrak{K}_\infty$ of $G(K_\infty)$. Denote by $\mathfrak{Aut}_\infty^k(L)$ the subspace of such functions that moreover are cuspidal.

Let $X$ be a set of representatives of the double cosets $G(K)\backslash G(\mathfrak{A}_f)/\mathfrak{K}_f$. It is a finite set. For any $z \in X$ the intersection $\Gamma_z := G(K) \cap z \mathfrak{K}_f z^{-1}$ in $G(\mathfrak{A}_f)$ is an arithmetic subgroup of $G(K)$. For an integer $k$, $0 \leq k \leq n$, denote by $\mathfrak{Harm}_k^z(M,L)^{\Gamma_z}$ and respectively $\mathfrak{Harm}_0^z(M,L)^{\Gamma_z}$ the space of harmonic cochains on the Bruhat-Tits building of $G(K_\infty)$ invariant under the action of $\Gamma_z$ and respectively of those with finite supports modulo $\Gamma_z$.

In this paper, using a result we established in [1] and that gives in any degree $k$, $0 \leq k \leq n$, an explicit isomorphism between the space of $k$-harmonic cochains and the dual of the $k$-special representation $\text{Sp}_k^z(M)$ of $G(K_\infty)$, also following the ideas of Drinfeld that did the work for $G = GL_2$, see [6] or also [10], we prove, Th. 4.1, that for every $k$, $0 \leq k \leq n$, we have $M$-isomorphisms:

$$\bigoplus_{z \in X} \mathfrak{Harm}_k^z(M,L)^{\Gamma_z} \cong \text{Hom}_{M[G(K_\infty)]}(\text{Sp}_k^z(M), \mathfrak{Aut}_\infty^k(L))$$

and

$$\bigoplus_{z \in X} \mathfrak{Harm}_0^z(M,L)^{\Gamma_z} \cong \text{Hom}_{M[G(K_\infty)]}(\text{Sp}_k^z(M), \mathfrak{Aut}_0^k(L)).$$

These isomorphisms are given explicitly.

We organise this paper as follows. In the first section we give a brief description of the Bruhat-Tits building associated to $G(K_\infty)$. In the second section, we recall the notions of harmonic cochains and special representations and give the link between the two. In the third section, we recall the notions of automorphic forms and cusp forms, we look at them as functions defined on the adele groups and we give another interpretation of them as functions defined on the component at infinity, then we give other properties in particular when looking at them through the special representations. Finally, in section 4, we give an explicit link between the automorphic and cusp forms that transform like special representations and the harmonic cochains, cf. the isomorphisms above.

1 The Bruhat-Tits building of $GL_{n+1}(K_\infty)$

For general properties of buildings, see [1] and [7]. An introduction to the Bruhat-Tits building of $G(K_\infty)$ with pointed cells is given in [5].

The Bruhat-Tits building (pointed cells). Let $V$ be the standard vector space $K^{n+1}_\infty$. A lattice in $V$ is a free $O_\infty$-submodule $\Lambda$ of $V$ of rank $n + 1$. The Bruhat-Tits building of $G(K_\infty)$ may be described as a simplicial complex $\mathfrak{I}$ whose vertices are the dilation classes of lattices. More precisely, two lattices $\Lambda$ and $\Lambda'$ are in the same class if $\Lambda' = \lambda \Lambda$ for some $\lambda \in K^\times$. The class of $\Lambda$ is a vertex $v$ and is denoted $v = [\Lambda]$. For $k$, $0 \leq k \leq n$, a $k$-cell $\sigma$ in $\mathfrak{I}$ is a set of $k + 1$ vertices $\{[\Lambda_0], [\Lambda_1], \ldots, [\Lambda_k]\}$ such that:

$$\cdots \supseteq \Lambda_0 \supseteq \Lambda_1 \supseteq \cdots \supseteq \Lambda_k \supseteq \pi_\infty \Lambda_0 \supseteq \cdots$$

(1)
Notice that there is an obvious cyclic ordering (mod. \((k + 1)\)) on the vertices of \(\sigma\).

A pointed \(k\)-cell of \(\mathcal{J}\) is a pair \((\sigma, v)\) consisting of a \(k\)-cell \(\sigma\) together with a distinguished vertex \(v\) of \(\sigma\). Notice, therefore, that in the case of a pointed cell \((\sigma, v)\) there is a precise ordering on the vertices. If \(v = [\Lambda_0]\) we write :

\[(\sigma, v) = (\Lambda_0 \supseteq \Lambda_1 \supseteq \cdots \supseteq \Lambda_k \supseteq \pi_0 \Lambda_0).\]  

(2)

For each \(k\), \(0 \leq k \leq n\), let \(\mathcal{J}^k\) be the set of pointed \(k\)-cells of \(\mathcal{J}\).

**The action of** \(G(K_\infty)\). For a fixed basis of the vector space \(V\), the action of \(G(K_\infty)\) on \(V\) is given by the matrix product \(ug^{-1}\) where \(u \in V\) is considered as a line matrix with respect to the basis of \(V\). This action induces an action of \(G(K_\infty)\) on the vertex set of the building \(\mathcal{J}\) by

\[g.v = [\Lambda g^{-1}].\]

Thus, \(G(K_\infty)\) acts on the cells by acting on their vertices.

**The type of a pointed cell.** (Cf. \([5, \S 1.1]\).) Denote by \(\kappa_\infty = O_\infty/\pi_\infty O_\infty\) the residue field of \(K_\infty\). Let \(\sigma = (\Lambda_0 \supseteq \Lambda_1 \supseteq \cdots \supseteq \Lambda_k \supseteq \pi_\infty \Lambda_0) \in \mathcal{J}^k\) be a pointed \(k\)-cell. The type of \(\sigma\) is defined as follows :

\[t(\sigma) = (d_1, \ldots, d_{k+1})\]

where \(d_i = \dim_{\kappa_\infty} \Lambda_{i-1}/\Lambda_i\) for each \(i = 1, \ldots, k+1\) (here, we suppose \(\Lambda_{k+1} = \pi_\infty \Lambda_0\)).

Since clearly the action of \(G(K_\infty)\) preserves the dimension of the \(\kappa_\infty\)-vector spaces \(\Lambda_{i-1}/\Lambda_i\), it preserves the type of the pointed \(k\)-cells as well.

**The standard cells.** Let \(\{u_1, \ldots, u_{n+1}\}\) be the standard basis of \(V = K_\infty^{n+1}\). Consider, for each \(i = 0 \ldots n\), the vertex \(v_i^o = [\Lambda_i^o]\) represented by the lattice :

\[\Lambda_i^o = \pi_\infty O_\infty u_1 \oplus \cdots \oplus \pi_\infty O_\infty u_i \oplus O_\infty u_{i+1} \oplus \cdots \oplus O_\infty u_{n+1}.\]

Since the \(\Lambda_i^o\), \(0 \leq i \leq n\), satisfy (I), we have an \(n\)-cell \(\sigma_0 = \{v_0^o, v_1^o, \ldots, v_n^o\}\) called the fundamental chamber of \(\mathcal{J}\).

Now, once and for all, fix \(\Delta = \{1, \ldots, n\}\). For each \(I \subseteq \Delta\) such that \(\Delta - I = \{i_1 < \cdots < i_k\}\), we have a \(k\)-cell

\[\sigma_I = \{v_0^o, v_{i_1}^o, \ldots, v_{i_k}^o\}.\]  

(3)

The \(\sigma_I\), \(I \subseteq \Delta\), are called the standard cells of the Bruhat-Tits building \(\mathcal{J}\). These cells are the faces of the fundamental chamber \(\sigma_0\) having \(v_0^o\) as vertex, called the fundamental vertex of \(\mathcal{J}\).

We denote by \(T\) the standard maximal torus of \(G(K_\infty)\) of diagonal matrices and by \(N\) its normalizer in \(G(K_\infty)\). Since the Weyl group \(W = N/T\) of \(G(K_\infty)\) with respect to \(T\) is isomorphic to the permutation group \(S_{n+1}\), \(W\) is generated by the set \(S = \{s_i, i \in \Delta\}\) of the reflexions \(s_i\) which correspond to the transpositions \((i, i + 1) \in S_{n+1}\). We have the following lemma :
Lemma 1.0.1. Let $y_i, 0 \leq i \leq n$, be the diagonal matrix $y_i = \text{diag}(1, \ldots, 1, \pi_\infty, \ldots, \pi_\infty)$ and let $w_i = (s_is_{i+1} \cdots s_n)(s_{i-1}s_i \cdots s_{n-1}) \cdots (s_2s_{n-i+1}) \in W$. We have :

$$(\sigma_\emptyset, v_i^0) = y_i w_i (\sigma_\emptyset, v_0^0).$$

If $(\sigma, v_{i_j}^0) = (v_{i_j}^0, \ldots, v_{i_k}^0, v_{i_0}^1, v_{i_1}^1, \ldots, v_{i_{i-1}}^0)$ is a face of the pointed chamber $(\sigma_\emptyset, v_i^0)$, where $0 \leq i_0 < i_1 < \cdots < i_k \leq n$ and $0 \leq j \leq k$, then

$$(\sigma, v_{i_j}^0) = y_{i_j} w_{i_j} (\sigma_{\widehat{I}_{i_j}}, v_0^0)$$

where $\Delta - \widehat{I}_{i_j} = \{ i_{j+1} - i_j < \cdots < i_k - i_j < n + 1 + i_0 - i_j < \cdots < n + 1 + i_{j-1} - i_j \}.$

Proof. The vertices of the fundamental chamber are $v_0^0 = [\Lambda_0^o]$. We can easily check that the representants $\Lambda_l^o$ of these vertices satisfy :

$$\Lambda_l^o y_i w_i = \begin{cases} 
\Lambda_{n+1-l-i}^o & \text{if } 0 \leq l \leq i-1 \\
\Lambda_{l-i}^o \pi_\infty & \text{if } i \leq l \leq n.
\end{cases}$$

Therefore, by taking into account the way in which $G(K_\infty)$ acts on the vertices of $\mathfrak{I}$, it follows that :

$$w_{i_j}^{-1} y_{i_j}^{-1} v_{i_j}^0 = \begin{cases} 
v_{n+1-l-i}^0 & \text{if } 0 \leq l \leq i-1 \\
v_{l-i}^0 & \text{if } i \leq l \leq n,
\end{cases}$$

hence $w_{i_j}^{-1} y_{i_j}^{-1} (\sigma_\emptyset, v_0^0) = (\sigma_\emptyset, v_0^0)$ and, if $(\sigma, v_{i_j}^0)$ and $\widehat{I}_{i_j}$ are as in the lemma, we have $w_{i_j}^{-1} y_{i_j}^{-1} (\sigma, v_{i_j}^0) = (\sigma_{\widehat{I}_{i_j}}, v_0^0).$

Since the action of $G(K_\infty)$ is transitive on the chambers of $\mathfrak{I}$, the lemma above shows that $G(K_\infty)$ acts transitively on the pointed $k$-cells of a given type. So, if we denote by $t_I$ the type of the pointed standard $k$-cell $(\sigma_I, v_0^0)$, by $\widehat{\mathfrak{I}}^{k,t_I}$ the set of all pointed $k$-cells of type $t_I$, and by $B_I$ the pointwise stabilizer in $G(K_\infty)$ of the standard cell $\sigma_I$, we have a one-to-one correspondence

$$G(K_\infty)/B_I \xrightarrow{\sim} \widehat{\mathfrak{I}}^{k,t_I}.$$

(4)

2 Harmonic cochains and special representations

Through all this section, we fix a commutative ring $M$ and an $M$-module $L$. Assume that $G(K_\infty)$ acts trivially on $M$ and that $L$ is endowed with an $M$-linear $G(K_\infty)$-action.
2.1 Harmonic cochains

From now on, we sometimes denote by \( \sigma \) a pointed cell \((\sigma, v)\) when it is clear that it is pointed and which vertex is distinguished.

Let us recall the definition of harmonic cochains given by E. de Shalit ([5, def. 3.1]).

**Definition 2.1.** Let \( k \) be an integer such that \( 0 \leq k \leq n \). A \( k \)-harmonic cochain with values in the \( M \)-module \( L \) is a homomorphism \( h \in \text{Hom}_M(M[\hat{I}^k], L) \) which satisfies the following conditions:

1. **(HC1)** If \( \sigma = (v_0, v_1, \ldots, v_k) \in \hat{I}^k \) is a \( k \)-pointed cell and if \( \sigma' = (v_1, \ldots, v_k, v_0) \) is the same cell but pointed at \( v_1 \), then
   \[ h(\sigma) = (-1)^k h(\sigma'). \]

2. **(HC2)** Fix a pointed \((k - 1)\)-cell \( \eta \in \hat{I}^{k-1} \), fix a type \( t \) of pointed \( k \)-cells, and consider the set \( B(\eta, t) = \{ \sigma \in \hat{I}^k; \eta < \sigma \text{ and } t(\sigma) = t \} \). Then
   \[ \sum_{\sigma \in B(\eta, t)} h(\sigma) = 0. \]

3. **(HC3)** Let \( k \geq 1 \). Fix \( \sigma = (\Lambda_0 \supseteq \Lambda_1 \supseteq \cdots \supseteq \Lambda_k \supseteq \pi_\infty \Lambda_0) \in \hat{I}^k \) and fix an index \( 0 \leq j \leq k \). Let \( C(\sigma, j) \) be the collection of all \( \sigma' = (\Lambda_0' \supseteq \Lambda_1' \supseteq \cdots \supseteq \Lambda'_k \supseteq \pi_\infty \Lambda'_0) \in \hat{I}^k \) for which \( \Lambda'_i = \Lambda_i \text{ if } i \neq j, \Lambda_j \supseteq \Lambda'_j \supseteq \Lambda_{j+1} \text{ and }\dim_k \Lambda'_j / \Lambda_{j+1} = 1 \). Then
   \[ h(\sigma) = \sum_{\sigma' \in C(\sigma, j)} h(\sigma'). \]

4. **(HC4)** Let \( \sigma = (v_0, v_1, \ldots, v_{k+1}) \in \hat{I}^{k+1} \). Let \( \sigma_j = (v_0, \ldots, \hat{v}_j, \ldots, v_{k+1}) \in \hat{I}^k \). Then
   \[ \sum_{j=0}^{k} (-1)^j h(\sigma_j) = 0. \]

For any \( k, \ 0 \leq k \leq n \), we denote by \( \mathcal{H}_{arm}^k(M, L) \) the space of \( k \)-harmonic cochains with values in the \( M \)-module \( L \).

**The action of \( G(K_\infty) \).** The action of \( G(K_\infty) \) on \( \mathcal{H}_{arm}^k(M, L) \) is induced from its natural action on \( \text{Hom}_M(M[\hat{I}^k], L) \), namely
\[ (g \cdot h)(\sigma) = g \cdot h(g^{-1} \sigma) \]
for any \( h \in \mathcal{H}_{arm}^k(M, L) \), any \( g \in G(K_\infty) \), and any \( \sigma \in \hat{I}^k \).
2.2 Special representations

Let \( P \) be the upper triangular Borel subgroup of \( G(K_\infty) \). A parabolic subgroup of \( G(K_\infty) \) is a closed subgroup which contains a Borel subgroup. The subgroups which contain \( P \) are said to be special; these subgroups are completely determined by the subsets \( I \) of \( \Delta \).

Indeed, if for each \( I \subseteq \Delta \), let \( W_I \) be the subgroup of \( W \) generated by the \( s_i, i \in I \), it has been shown that the subset \( P_I = PW_I P = \tilde{P} N_I \tilde{P} \) where \( N_I \subseteq N \) is such that \( N_I/T = W_I \) is a subgroup of \( G(K_\infty) \) containing \( P \), and that every subgroup of \( G(K_\infty) \) containing \( P \) is a certain \( P_I \) for \( I \subseteq \Delta \). Note that \( P = P_\emptyset \).

The group \( G(K_\infty) \) is a locally compact topological group, its topology being induced from that of the non-archimedean field \( K_\infty \). We know, that for any \( I \subseteq \Delta \), the homogeneous space \( G(K_\infty)/P_I \) is compact with respect to the quotient topology.

Denote by \( C^\infty(G(K_\infty)/P_I, M) \) the set of locally constant functions on \( G(K_\infty)/P_I \) with values in \( M \).

The action of \( G(K_\infty) \). The action on \( C^\infty(G(K_\infty)/P_I, M) \) is induced by the action by left translations on \( G(K_\infty)/P_I \).

For any \( I_1 \subseteq I_2 \subseteq \Delta \), we have natural commutative diagrams of \( M[G(K_\infty)] \)-monomorphisms

\[
C^\infty(G(K_\infty)/P_I, M) \to C^\infty(G(K_\infty)/P_{I_2}, M) \to C^\infty(G(K_\infty)/P_{I_1}, M),
\]

and the special representations of \( G(K_\infty) \) are defined as follows:

**Definition 2.2.** Let \( k \) be an integer with \( 0 \leq k \leq n \) and let \( J_k \) be the subset \([1, n-k]\) of \( \Delta \). A \( k \)-special representation of \( G(K_\infty) \) is the \( M[G(K_\infty)] \)-module:

\[
\text{Sp}^k(M) = \frac{C^\infty(G(K_\infty)/P_{J_k}, M)}{\sum_{j=n-k+1}^{n} C^\infty(G(K_\infty)/P_{J_{k}\cup\{j\}}, M)}.
\]

In case \( k = n \), this is the ordinary Steinberg representation.

One thing important to know about the special representations is that they are cyclic. Indeed:

For each \( I \subseteq \Delta \), we denote by \( B^*_I \) the open compact subgroup of \( G(O_\infty) \) which is the inverse image of the standard parabolic subgroup \( P_I(K_\infty) \) of \( G(K_\infty) \) by the map “reduction mod. \( \pi_\infty \)”: \( G(O_\infty) \to G(K_\infty) \). The parahoric subgroups of \( G(K_\infty) \) are the conjugates in \( G(K_\infty) \) of the \( B^*_I \), \( I \subseteq \Delta \). Note that we have

\[
B_I = B^*_I K_\infty,
\]

(5)
and that then the $B_I$ is compact open modulo the center of $G(K_\infty)$.

Let $I \subseteq \Delta$. For any subset $H$ of $G(K_\infty)$, we denote by $\chi_{HP_I} \in C^\infty(G(K_\infty)/P_I, M)$ the characteristic function of $HP_I/P_I$.

**Proposition 2.1.** (P. Schneider and U. Stuhler) The $M[G(K_\infty)]$-module $C^\infty(G(K_\infty)/P_I, M)$ is generated by the characteristic function $\chi_{B_I P_I}$.

*Proof.* See [11, §4, prop. 8’ and cor. 9’]. \qed

### 2.3 Harmonic cochains and special representations

Let $k$ be an integer, $0 \leq k \leq n$. In [1] we proved that the space of harmonic cochains of degree $k$ is isomorphic to the dual of the $k$-special representation. For this purpose, inspired by the definition of the harmonic cochains and using parahoric subgroups, we defined a certain type of sets. Here we recall these essential points.

Let $I \subseteq \Delta$ such that $\Delta - I = \{i_1 < i_2 < \cdots < i_k\}$. It is clear that for any $m = 1, \ldots, k$, we have $i_m \leq n - k + m$. Set

$$C^*_I = B^*_{[i_k+1,n]}B^*_{[i_1+1,n-k+1]}B^*_{[1,n-k]} \quad \text{and} \quad C_I = B_{[i_k+1,n]}B_{[i_1+1,n-k+1]}B_{[1,n-k]} \quad (6)$$

The set $C^*_I$ is compact open in $G(K_\infty)$ and we have $C_I = C^*_I K^*_\infty$, see [2.2]. Hence, the set $C_I P_k/P_J = C^*_I P_k/P_J$ is compact open in $G(K_\infty)/P_J$.

**Theorem 2.1.** For any $k$, $0 \leq k \leq n$, there is an $M[G(K_\infty)]$-isomorphism

$$\mathcal{H}_{arm}^k(M, L) \cong \text{Hom}_M(\text{Sp}^k(M), L)$$

*Proof.* The proof here involves hard computations of combinatorial nature. The interested reader could find it in [1]. For later use we just need to describe the isomorphism. The isomorphism is given by the map:

$$\Phi : \mathcal{H}_{arm}^k(M, L) \longrightarrow \text{Hom}_M(\text{Sp}^k(M), L)$$

which to a harmonic cochain $h$ associates the $M$-linear map $\varphi_h$ defined by

$$\varphi_h(g\chi_{B_J P_J}) = h(g(\sigma_J v_0)) \quad (7)$$

for any $g \in G(K_\infty)$. The inverse map

$$H : \text{Hom}_M(\text{Sp}^k(M), L) \longrightarrow \mathcal{H}_{arm}^k(M, L)$$

sends $\varphi$ to $h_\varphi$ given by

$$h_\varphi(g(\sigma_J v_0)) = \varphi(g\chi_{C_I P_J}) \quad (8)$$

for any $g \in G(K_\infty)$. \qed
3 Automorphic forms

3.1 Automorphic forms

Let $\mathbb{A} = \prod_{\nu \in |C|} K_{\nu}$ be the ring of adèles of $K$, i.e. the restricted product of the family $(K_{\nu})_{\nu \in |X|}$ with respect to the family of the compact open subrings $(O_{\nu})_{\nu \in |C|}$:

$$\mathbb{A} = \{(a_{\nu}) \in \prod_{\nu \in |C|} K_{\nu} | a_{\nu} \in O_{\nu} \text{ for almost all } \nu \in |C|\}$$

We denote by $O = \prod_{\nu \in |C|} O_{\nu}$, it is an open compact subset of $\mathbb{A}$. We can write $\mathbb{A} = \mathbb{A}_f \times K_\infty$ et $\mathcal{O} = \mathcal{O}_f \times O_\infty$, where we have set $\mathbb{A}_f = \prod_{\nu \in |C| - \{\infty\}} K_{\nu}$ the ring of finite adèles of $K$ and $\mathcal{O}_f = \prod_{\nu \in |C| - \{\infty\}} O_{\nu}$.

For any $a \in K$, we have $\nu(a) = 0$, for almost all $\nu \in |C|$, which means that $K$ can be seen as a subfield of $\mathbb{A}$ imbedded diagonally.

In what follows we work with $G(K), G(\mathbb{A})$ et $G(\mathcal{O})$ ... that have the same properties than $K, \mathbb{A}, \mathcal{O} ...$ recalled above. For example, $G(K)$ is embedded diagonally in $G(\mathbb{A}), G(\mathcal{O})$ is a compact open subgroup of $G(\mathbb{A})$ ...

Let $M$ be a commutative ring with a unit and $L$ be an integral $M$-algebra of characteristic zero. Assume these are endowed with the trivial action of $G(\mathbb{A})$.

**Definition 3.1.** (G. Harder, [8]) An automorphic form with values in $L$, with respect to an open compact subgroup $\mathfrak{K}$ of $G(\mathcal{O})$, is a function $f : G(\mathbb{A}) \to L$, such that $f(\gamma g k) = f(g)$ for $\gamma \in G(K), g \in G(\mathbb{A})$ and $k \in \mathfrak{K}^{ZG}(K_\infty)$.

An automorphic form is a cusp form if moreover :

$$\int_{U_I(K) \setminus U_I(\mathbb{A})} f(ug) du_I = 0$$

for every $I \subseteq \Delta$, ($du_I$ a Haar measure which is normalized with respect to the compact $U_I(K) \setminus U_I(\mathbb{A})$).

In fact this integral is a finite sum, so it has a sense for any choice of $L$ of characteristic zero.

Denote by $\mathfrak{Aut}(\mathfrak{K}, L)$ (resp. $\mathfrak{Aut}_0(\mathfrak{K}, L)$) the set of $L$-valued automorphic forms with respect to an open compact subgroup $\mathfrak{K}$ of $G(\mathcal{O})$ (resp. the set of those that are cusp forms).

From now on, once and for all we fix an open compact subgroup $\mathfrak{K}_f$ of $G(\mathcal{O}_f)$, $X = X_{\mathfrak{K}_f}$ a set of representatives of the double cosets $G(K) \setminus G(\mathbb{A}_f) / \mathfrak{K}_f$. For every $x \in X$, we let $\Gamma_x$ the arithmetic group $x \mathfrak{K}_f x^{-1} \cap G(K)$. Put 

$$\mathfrak{Aut}^\mathfrak{K}_f(L) = \bigcup_{\mathfrak{K}_\infty} \mathfrak{Aut}(\mathfrak{K}_f \times \mathfrak{K}_\infty, L) \quad \text{and} \quad \mathfrak{Aut}_0^\mathfrak{K}_f(L) = \bigcup_{\mathfrak{K}_\infty} \mathfrak{Aut}_0(\mathfrak{K}_f \times \mathfrak{K}_\infty, L)$$

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where $\mathfrak{K}_\infty$ runs through the open compact subgroups of $G(O_\infty)$.

- The action $G(K_\infty) : G(K_\infty)$ acts on $\mathfrak{Aut}^{K_f}(L)$ by the formula
  \[(g.f)(g) = f(gg).\] (9)

Indeed, it is not difficult to see that if $f$ is in $\mathfrak{Aut}(K_f \times \mathfrak{K}_\infty, L)$ then $g.f$ belongs to $\mathfrak{Aut}(K_f \times g\mathfrak{K}_\infty g^{-1} \cap G(O_\infty), L) \subseteq \mathfrak{Aut}^{K_f}(L)$.

We end this paragraph by stating the following important result about cusp forms:

**Theorem 3.1.** (G. Harder) Let $\mathfrak{K}$ be an open compact subgroup of $G(\mathbb{A})$. Then there exists an open subset $\mathcal{U}_{\mathfrak{K}}$ of $G(\mathbb{A})$ such that $G(K)\mathcal{U}_{\mathfrak{K}}G(K_\infty) = \mathcal{U}_{\mathfrak{K}}$, the quotient $G(K)\mathcal{U}_{\mathfrak{K}}/G(K_\infty)$ is finite, and we have:

$$\text{supp}(f) \subseteq \mathcal{U}_{\mathfrak{K}}, \quad \text{pour tout } f \in \mathfrak{Aut}_0(\mathfrak{K}, L).$$

**Proof.** See [8, cor. 1.2.3].

### 3.2 Automorphic forms as functions on $G(K_\infty)$

**Definition 3.2.** A subgroup $\Gamma$ of $G(\mathfrak{K})$ is said to be arithmetic if it is commensurable with $G(A)$, i.e. if $\Gamma \cap G(A)$ is a subgroup of finite index in both $\Gamma$ and $G(A)$.

For any subgroup $\mathfrak{H}_f$ of $G(\mathfrak{A}_f)$, choose a set $X_{\mathfrak{H}_f} \subseteq G(\mathfrak{A}_f)$ of representatives of the double cosets $G(K)\backslash G(\mathfrak{A}_f)/\mathfrak{H}_f$. For every $\underline{x} \in X_{\mathfrak{H}_f}$, let

$$\Gamma_{\underline{x}} = G(K) \cap \underline{x}\mathfrak{H}_f\underline{x}^{-1}$$

be the intersection in $G(\mathfrak{A}_f)$. It is a discrete subgroup of $G(K_\infty)$.

The following is a well known result, see [8].

**Proposition 3.1.** If a subgroup $\mathfrak{H}_f$ of $G(\mathfrak{A}_f)$ contains an open compact subgroup of $G(\mathfrak{A}_f)$, then $X_{\mathfrak{H}_f}$ is a finite set. If, moreover, $\mathfrak{H}_f$ is open compact in $G(\mathfrak{A}_f)$, the $\Gamma_{\underline{x}}$ ($\underline{x} \in X_{\mathfrak{H}_f}$) are arithmetic.

We have:

**Lemma 3.2.1.** Let $H_\infty$ be a subgroup of $G(K_\infty)$ and $\mathfrak{H}_f$ be a subgroup of $G(\mathfrak{A}_f)$. For any $\underline{x} \in X_{\mathfrak{H}_f}$, let $\Gamma_{\underline{x}} = \underline{x}\mathfrak{H}_f\underline{x}^{-1} \cap G(K)$. We have a bijective correspondence:

$$G(K)\backslash G(\mathbb{A})/(\mathfrak{H}_f \times H_\infty) \cong \coprod_{\underline{x} \in X_{\mathfrak{H}_f}} \Gamma_{\underline{x}}\backslash G(K_\infty)/H_\infty$$

given by the map which sends $G(K)\underline{g}(\mathfrak{H}_f \times H_\infty)$ to the double class $\Gamma_{\underline{x}}^{-1}g_\infty H_\infty$, for any $\underline{g} = (\tau\underline{x}_h, g_\infty) \in G(\mathbb{A})$. 

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Proof.
The map is well defined. Indeed, let \( g = (g_f, g_\infty) \in G(\mathbb{A}) \) and let \( \tau, \tau' \in G(K) \) and \( h_f, h_f' \in \mathfrak{H}_f \) be such that \( g_f = \tau x h_f = \tau' x h_f' \). This implies that \( \tau^{-1} \tau' = x h_f h_f'^{-1} x^{-1} \in \mathfrak{H}_f \mathfrak{H}_f^{-1} \cap G(K) = \Gamma_\infty \), then:

\[
\tau^{-1} g_\infty = (\tau^{-1} \tau') x^{-1} g_\infty \in \Gamma_\infty \mathfrak{H}_f^{-1} g_\infty.
\]

For the infinite component, The right invariance of the map \( H_\infty \) is clear.

The map is surjective. For any \( \xi \in X_{\mathfrak{H}_f} \) and any \( g_\infty \in G(K_\infty) \), the double coset \( \Gamma_\infty g_\infty H_\infty \) is the image of \( G(K)(\xi, g_\infty)(\mathfrak{H}_f \times H_\infty) \).

The map is injective. Let \( g = (\tau x h_f, g_\infty) \) and \( g' = (\tau' x h_f', g_\infty') \) be such that \( \Gamma_\infty \tau^{-1} g_\infty H_\infty = \Gamma_\infty \tau'^{-1} g_\infty' H_\infty \). The union over \( X_{\mathfrak{H}_f} \) being a disjoint union, we must have \( \xi = \xi' \). Thus, if \( \gamma \in \Gamma_\infty \) and \( h_\infty \in H_\infty \) are such that \( \tau^{-1} g_\infty = \gamma \tau'^{-1} g_\infty' h_\infty \), we have:

\[
g = (\tau x h_f, g_\infty) = (\tau x h_f, \tau \gamma \tau'^{-1} g_\infty' h_\infty) = \tau \gamma \tau'^{-1} (\tau' x, g_\infty')(h_f, h_\infty).
\]

Now, since \( \gamma^{-1} \in \Gamma_\infty = \mathfrak{H}_f \mathfrak{H}_f^{-1} \cap G(K) \), there exists \( h_f'' \in \mathfrak{H}_f \) such that \( \gamma^{-1} = x h_f'' x^{-1} \). So:

\[
g = \tau \gamma \tau'^{-1} (\tau' x, g_\infty')(h_f'' h_f, h_\infty) = \tau \gamma \tau'^{-1} g'(h_f'' h_f, h_\infty),
\]

which means that \( g \) and \( g' \) are representatives of the same double coset modulo \( G(K) \) on the left and modulo \( \mathfrak{H}_f \times H_\infty \) on the right.

This lemma 3.2.1 gives us a way to interpret the automorphic forms as functions defined on the quotient sets \( \Gamma_\xi \backslash G(K_\infty) \) right invariant under some open compact subgroup \( \mathfrak{R}_\infty \) of \( G(O_\infty) \) and under \( Z_G(K_\infty) \). More precisely, if for each \( \xi \in X \) one defines:

\[
\mathfrak{Aut}_\xi(L) \quad \text{(resp. } \mathfrak{Aut}_{\xi,0}(L)\text{)}
\]

to be the set of functions \( f : G(K_\infty) \rightarrow L \), left invariant under \( \Gamma_\xi \) and right invariant under \( \mathfrak{R}_\infty Z_G(K_\infty) \), where \( \mathfrak{R}_\infty \) is an open compact subgroup of \( G(O_\infty) \) which depends on \( f \) (respectively, the set of such functions that moreover have a finite support in \( \Gamma_\xi \backslash G(K_\infty) / \mathfrak{R}_\infty Z_G(K_\infty) \)). Then we have the following:

**Proposition 3.2.** We have isomorphism of \( M[G(K_\infty)] \)-modules:

\[
\Xi : \bigoplus_{\xi \in X} \mathfrak{Aut}_\xi(L) \xrightarrow{\sim} \mathfrak{Aut}_{\xi,0}(L)
\]

which maps \( (f_\xi)_{\xi} \) to \( f \) defined by the formula \( f(g) = f_\xi(\gamma^{-1} g_\infty) \) for \( g = (\tau x k_f, g_\infty) \in G(\mathbb{A}) \).

Moreover, by this isomorphism when we restrict to the cusp forms we get an isomorphism of \( M[G(K_\infty)] \)-modules:

\[
\bigoplus_{\xi \in X} \mathfrak{Aut}_{\xi,0}(L) \cong \mathfrak{Aut}_{0}(L)
\]
Proof. The first isomorphism is a direct consequence of Lemma 3.2.1. To get the second isomorphism we only need to use Theorem 3.1.

We finish this section with the following remark that motivates our link between automorphic forms and harmonic cochains.

**Remark 3.1.** Let us for a while get back to the Bruhat-Tits building. For any \( I \subseteq \Delta \) such that \( |\Delta - I| = k \), if in the lemma above we put \( H_\infty = B_I = B_I^o Z_G(K_\infty) \), then, see (4) in §7, we get a one-to-one correspondence:

\[
G(K) \backslash G(\mathbb{A}) / (\mathfrak{H}_f \times B_I^o Z_G(K_\infty)) \cong \prod_{x \in X} \Gamma_x \backslash \hat{\Gamma}^{k,t}.
\]

In this section the link between automorphic forms, functions defined on \( G(K) \backslash G(\mathbb{A}) \) and right invariant by an open compact subgroup of \( G(\mathbb{A}) \) and by \( Z_G(K_\infty) \), and harmonic cochains defined on \( \hat{\Gamma}^k \) and which are invariant under the action of the arithmetic groups \( \Gamma_x \), \( x \in X \).

### 3.3 Automorphic forms through special representations.

Recall, see Proposition 2.1, that as an \( M[G(K_\infty)] \)-module the space \( C^\infty(G(K_\infty)/P_J, M) \) is generated by the characteristic function \( \chi_{B_J P_J} \). So every element \( \varphi \in \text{Sp}^k(M) \) is of the form:

\[
\varphi = \sum_{j=1}^m \alpha_j \varphi^J \chi_{B_J P_J}
\]

with \( \alpha_j \in M \) and \( \varphi^J \in G(K_\infty) \). Therefore by the action of \( G(K_\infty) \) this \( \varphi \) is invariant under the open compact subgroup:

\[
\mathfrak{H}_\infty^J = \bigcap_{j=1}^m \varphi^J \cap G(O_\infty)
\]

of \( G(O_\infty) \).

**Proposition 3.3.** Let \( \mathcal{F} \) be the set of all functions \( f : G(\mathbb{A}_f) \to L \) endowed with the action of \( G(K) \) coming from left translation on \( G(\mathbb{A}_f) \). We have an \( M \)-isomorphism

\[
\Psi : \text{Hom}_M(\text{Sp}^k(M), \mathcal{F})^{G(K)} \xrightarrow{\cong} \text{Hom}_{M[G(K_\infty)]}(\text{Sp}^k(M), \text{Aut}^{\mathfrak{H}_f}(L)).
\]

which sends \( \varphi \) to \( \psi_\varphi \) defined as follows. For any \( \varphi \in \text{Sp}^k(M) \), the function \( \psi_\varphi : G(\mathbb{A}) \to L \) is given by the formula

\[
\psi_\varphi(\varphi) = \varphi(g_\infty \varphi)(g_f)
\]

for \( g = (g_f, g_\infty) \in G(\mathbb{A}) \).
Proof.
Ψ is well defined. Indeed, let us show first that for any \( \varrho \in \mathrm{Sp}^k(M) \), we have \( \psi_\varphi(\varrho) \in \mathfrak{Aut}^{B_f}(L) \). Recall that \( G(K) \) is seen as a subgroup of \( G(\mathbb{A}_f) \) and of \( G(\mathbb{A}) \) embedded diagonally, so let \( \tau \in G(K) \) and write \( \tau = (\mathcal{T}_f, \tau_\infty) \) where \( \mathcal{T}_f = \tau \in G(\mathbb{A}_f) \) and \( \tau_\infty = \tau \in G(K_\infty) \). Let \( \mathfrak{K}^o \) be the open compact subgroup of \( G(O_\infty) \) given by (10) and recall that then \( \varrho \) is invariant under \( \mathfrak{K}^o Z_G(K_\infty) \). Let \( \varrho = (\varrho_f, \varrho_\infty) \in G(\mathbb{A}) \) and \( u_\infty \in G(K_\infty) \), we have :

\[
\psi_\varphi(\varrho)(\tau g k) = \psi_\varphi(\varrho)(\mathcal{T}_f g_f k_f, \tau_\infty g_\infty k_\infty)
\]

\[
= [\varphi(\tau_\infty g_\infty k_\infty, \varrho)](\mathcal{T}_f g_f k_f)
\]

\[
= [\varphi(\tau_\infty) g_\infty k_\infty, \varrho)](g_f k_f)
\]

\[
= \varphi(g_\infty \varrho)(g_f)
\]

\[
= \psi_\varphi(\varrho)(g).
\]

The equality before the last comes from the invariance of \( \varphi \) under \( G(K) \), of \( \varrho \) under \( \mathfrak{K}^o Z_G(K_\infty) \) and of \( \mathcal{F} \) on the right under \( \mathfrak{K} \). Therefore \( \psi_\varphi \in \mathfrak{Aut}(\mathfrak{K} \times \mathfrak{K}^o, L) \subseteq \mathfrak{Aut}^{B_f}(L) \).

Now let us check that \( \psi \) is a homomorphism of \( M[G(K_\infty)] \)-modules. For any \( \varrho \in \mathrm{Sp}^k(M) \),

\[
\varrho = (\varrho_f, \varrho_\infty) \in G(\mathbb{A}) \) and \( u_\infty \in G(K_\infty) \), we have :

\[
\psi_\varphi(u_\infty \cdot \varrho)(g) = \varphi(g_\infty u_\infty, \varrho)(\varrho_f)
\]

\[
= \psi_\varphi(\varrho)(\varrho_f, g_\infty u_\infty)
\]

\[
= \psi_\varphi(\varrho)(g u_\infty)
\]

\[
= [u_\infty, \psi_\varphi(\varrho)](g),
\]

therefore \( \psi_\varphi(u_\infty \cdot \varrho) = u_\infty \psi_\varphi(\varrho) \) for any \( \varrho \in \mathrm{Sp}^k(M) \) and any \( u_\infty \in G(K_\infty) \). Consequently, \( \psi_\varphi \) is a homomorphism of \( M[G(K_\infty)] \)-modules and \( \Psi \) is well defined clearly \( M \)-linear map.

To prove that \( \Psi \) is an isomorphism we give its reciprocal

\[
\Psi' : \text{Hom}_{M[G(K_\infty)]}(\mathrm{Sp}^k(M), \mathfrak{Aut}_f(L)) \quad \rightarrow \quad \text{Hom}_{M}(\mathrm{Sp}^k(M), \mathcal{F})^{G(K)}
\]

\[
\psi \quad \mapsto \quad \varphi_\psi = \text{Sp}^k(M) \rightarrow \mathcal{F}
\]

where for any \( \varrho \in \mathrm{Sp}^k(M) \), the function \( \varphi_\psi : G(\mathbb{A}_f) \rightarrow L \) is given by the formula :

\[
\varphi_\psi(\varrho)(g_f) = \psi(\varrho)((g_f, 1_\infty))
\]

for \( g_f \in G(\mathbb{A}_f) \).

Let us check that indeed \( \varphi_\psi(\varrho) \in \mathcal{F} \) for \( \varrho \in \mathrm{Sp}^k(M) \). For any \( g_f \in G(\mathbb{A}_f) \) and any \( k_f \in \mathfrak{K}_f \), we have :

\[
\varphi_\psi(\varrho)(g_f k_f) = \psi(\varrho)(g_f k_f, 1_\infty) = \psi(\varrho)((g_f, 1_\infty)(k_f, 1_\infty)),
\]

but \( \psi(\varrho) \in \mathfrak{Aut}_f(L) \), there is then an open compact subgroup \( \mathfrak{K}_\infty \) of \( G(O_\infty) \) such that \( \psi(\varrho) \) is right invariant under \( \mathfrak{K}_f \times \mathfrak{K}_\infty \). We have \( (k_f, 1_\infty) \in \mathfrak{K}_f \times \mathfrak{K}_\infty \), hence :

\[
\psi(\varrho)((g_f, 1_\infty)(k_f, 1_\infty)) = \psi(\varrho)(g_f, 1_\infty) = \varphi_\psi(\varrho)(g_f);
\]

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Indeed, let the image of \( \Psi \) is an automorphic form \( \in \text{Aut}^{\#}(L) \), we have:

\[
[(\tau.\varphi_{\Psi})(\sigma)](g_f) = [(\tau_\gamma.\varphi_{\Psi}(\tau_\gamma^{-1} \cdot \sigma))](g_f) = \varphi_{\Psi}(\tau_\gamma^{-1} \cdot \sigma)(g_{f,1}) = \varphi_{\Psi}(\tau_\gamma^{-1} \cdot \sigma)(g_{f,1}) = \varphi_{\Psi}(\sigma)(g_{f,1}) = \varphi_{\Psi}(\sigma)(g_{f,1}) = \varphi_{\Psi}(\sigma)(g_{f,1}) = \varphi_{\Psi}(\sigma)(g_{f,1}).
\]

Therefore, \( \Psi \) is an \( M \)-isomorphism.
Proposition 3.4. For every integer $k$, $0 \leq k \leq n$. For each $\bar{z} \in X$, we have an isomorphism of $M$-modules:

$$\Psi_{\bar{z}} : \text{Hom}_M(\text{Sp}^k(M), L)^{\Gamma_{\bar{z}}} \xrightarrow{\cong} \text{Hom}_{M[G(K_{\infty})]}(\text{Sp}^k(M), \mathfrak{Aut}_{L}(L)).$$

which to $\varphi$ associates $\psi_{\varphi}$ defined as follows. For any $\rho \in \text{Sp}^k(M)$, the function $\psi_{\varphi}(\rho) : G(K_{\infty}) \to L$ is given by the formula:

$$\psi_{\varphi}(\rho) (g_{\infty}) = \varphi(g_{\infty} \cdot \rho).$$

Proof.

First, we need to prove that $\Psi_{\bar{z}}$ is well defined. That is $\psi_{\varphi}(\rho) \in \mathfrak{Aut}_{L}(L)$ and $\psi_{\varphi}$ is an $M[G(K_{\infty})]$-homomorphism. Since $\varphi : \text{Sp}^k(M) \to L$ is an $M$-linear map invariant under $\Gamma_{\bar{z}}$, and $\rho$ is invariant under $R_{x_{\infty}}G(K_{\infty})$ where $R_{x}$ is the compact open subgroup given by $(10)$ above, then for any $\gamma \in \Gamma_{\bar{z}}$, $g_{\infty} \in G(K_{\infty})$ and $k_{\infty} \in R_{x_{\infty}}G(K_{\infty})$, we have

$$\psi_{\varphi}(\rho)(\gamma g_{\infty} k_{\infty}) = \varphi(\gamma g_{\infty} k_{\infty} \cdot \rho) = (\gamma^{-1} \cdot \varphi)(g_{\infty} (k_{\infty} \cdot \rho)) = \varphi(g_{\infty} \cdot \rho) = \psi_{\varphi}(\rho)(g_{\infty}).$$

It is clear that $\psi_{\varphi}$ is $M$-linear, so it remains to check that it is $G(K_{\infty})$-equivariant. Indeed it is, we have:

$$\psi_{\varphi}(u_{\infty} \cdot \rho)(g_{\infty}) = \varphi(g_{\infty} \cdot (u_{\infty} \cdot \rho)) = \varphi((g_{\infty} u_{\infty}) \cdot \rho) = \psi_{\varphi}(\rho)(g_{\infty} u_{\infty}) = [u_{\infty} , \psi_{\varphi}(\rho)](g_{\infty}).$$

for any $g_{\infty} \in G(K_{\infty})$, any $u_{\infty} \in G(K_{\infty})$ and any $\rho \in \text{Sp}^k(M)$.

Now, in order to prove that $\Psi_{\bar{z}}$ is an isomorphism, we give its reciprocal map. To an $M[G(K_{\infty})]$-homomorphism $\psi : \text{Sp}^k(M) \to \mathfrak{Aut}_{L}(L)$ we associate a map $\varphi_{\psi} : \text{Sp}^k(M) \to L$ given by the formula:

$$\varphi_{\psi}(\rho) = \psi(\rho)(1_{\infty}),$$

for any $\rho \in \text{Sp}^k(M)$ and where $1_{\infty}$ is the identity element of $G(K_{\infty})$. We need only to check that $\varphi_{\psi}$ is invariant under $\Gamma_{\bar{z}}$ to get clearly an $M$-linear map:

$$\Psi'_{\bar{z}} : \text{Hom}_{M[G(K_{\infty})]}(\text{Sp}^k(M), \mathfrak{Aut}_{L}(L)) \longrightarrow \text{Hom}_M(\text{Sp}^k(M), L)^{\Gamma_{\bar{z}}},$$

For any $\rho \in \text{Sp}^k(M)$ and any $\gamma \in \Gamma_{\bar{z}}$, by the action of $\gamma$ on $\varphi_{\psi}$ we have:

$$(\gamma \cdot \varphi_{\psi})(\rho) = \varphi_{\psi}(\gamma^{-1} \cdot \rho) = \psi(\gamma^{-1} \cdot \rho)(1_{\infty}),$$

than, since $\psi$ is a homomorphism of $M[G(K_{\infty})]$-modules, and by the action of $G(K_{\infty})$ on the function $\psi(\rho) \in \mathfrak{Aut}_{L}(L)$ which is right invariant $\Gamma_{\bar{z}}$, we have:

$$\psi(\gamma^{-1} \cdot \rho)(1_{\infty}) = (\gamma^{-1} \cdot \psi(\rho))(1_{\infty}) = \psi(\rho)(1_{\infty} \cdot \gamma^{-1}) = \psi(\rho)(1_{\infty}) = \varphi_{\psi}(\rho),$$

we conclude, combining these equalities to preceding ones, that we have $\gamma \cdot \varphi_{\psi} = \varphi_{\psi}$ for any $\gamma \in \Gamma_{\bar{z}}$. 

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The $M$-linear maps $\Psi_x$ and $\Psi'_x$ are isomorphisms reciprocal to each other. On one hand we have $\Psi_x \circ \Psi'_x = I$. Indeed, let $\psi \in \text{Hom}_{M[G(K_\infty)]}(\text{Sp}^k(M), \mathfrak{Aut}_x(L))$. For any $\varrho \in \text{Sp}^k(M)$ and any $g_\infty \in G(K_\infty)$, we have:

$$\psi(\varrho(g_\infty)) = \varphi(\psi(g_\infty)) = \psi(\varrho(g_\infty)(1)),$$

than, from that fact $\psi$ is a homomorphism of $M[G(K_\infty)]$-modules and from that $G(K_\infty)$ acts on the function $\psi(g_\infty) \in \mathfrak{Aut}_x(L)$, we deduce:

$$\psi(g_\infty)(1) = \varphi(\psi(g_\infty))(1) = \varphi(\psi(g_\infty)(1)).$$

From these equalities and that of (17), we deduce that $\psi$ is a homomorphism of $M[G(K_\infty)]$-modules and from that $G(K_\infty)$ acts on the function $\psi(g_\infty) \in \mathfrak{Aut}_x(L)$, we deduce:

$$\psi(\varrho(g_\infty))(1) = \varphi(\psi(g_\infty))(1) = \varphi(\psi(g_\infty)(1)).$$

On the other hand $\Psi'_x \circ \Psi_x = I$. Indeed, let $\varphi \in \text{Hom}_{M[\text{Sp}^k(M), L]^{\Gamma_x}}$. Put $\psi = \Psi_x(\varphi)$ and $\varphi_{\psi} = \Psi'_x(\varphi)$. For any $\varrho \in \text{Sp}^k(M)$, we have:

$$\varphi_{\psi}(\varrho) = \psi(\varrho)(1) = \varphi(\psi(\varrho)(1)) = \varphi(\varrho).$$

So $\varphi_{\psi} = \varphi.$

\[ \square \]

4 Automorphic forms and harmonic cochains

4.1 A diagram to summarize the isomorphisms seen so far.

In this paragraph we want to summarize all the preceding results in a commutative diagram. This helps in particular to see how to combine them to get the last result stated in this paper, Theorem 4.1. To complete the diagram we need the following lemma:

**Lemma 4.1.1.** We have an isomorphism of $M$-modules:

$$\Theta : \bigoplus_{\underline{x}} \text{Hom}_M(M[\mathfrak{A}^k], L)^{\Gamma_x} \xrightarrow{\sim} \text{Hom}_M(M[\mathfrak{A}^k], \mathcal{F})^{G(K)}$$

which to a family $(h_{\underline{x}})_{\underline{x} \in \mathcal{X}}$ associates $h := \Theta((h_{\underline{x}})_{\underline{x}})$ defined as follows. For any $\sigma \in \mathcal{Y}^k$, the function $h(\sigma) : G(\mathfrak{A}_f) \to L$ is given by:

$$h(\sigma)(g_f) = h_{\underline{x}}(\tau^{-1}\sigma)$$

for $g_f = \tau x_{k_f} \in G(\mathfrak{A}_f)$ with $\tau \in G(K)$ and $k_f \in \mathfrak{A}_f$.

**Proof.**

Let us prove that $\Theta$ is well defined. First, the definition doesn’t depend on the writing of $g_f = \tau x_{k_f} \in G(\mathfrak{A}_f)$. Indeed, let $\tau' \in G(K)$ and $k'_f \in \mathfrak{A}_f$ be such that $g_f = \tau x_{k_f} = \tau' x_{k'_f}$ as
well. From the equality \( \tau x_k = \tau' x_k' \) we deduce \( \tau^{-1} \tau' = x_k k_f^{-1} x_k' \). Consequently, since \( k_f k_f^{-1} \in \mathfrak{g} \), we have \( \gamma = \tau^{-1} \tau' \in \Gamma = G(K) \cap x \mathfrak{g} \). Therefore, since \( h_{\underline{g}} \) is invariant under \( \Gamma \), we have:

\[
h_{\underline{g}}(\tau^{-1} \sigma) = h_{\underline{g}}(\tau^{-1} \tau' \sigma) = h_{\underline{g}}(\gamma \tau' \sigma) = (\gamma^{-1} . h_{\underline{g}}(. \tau' \sigma) = h_{\underline{g}}(\tau^{-1} \sigma).
\]

Let us prove now that \( h \) is invariant under \( G(K) \). Let \( \tau' \in G(K) \), \( \sigma \in \hat{I} \) and \( g_f = \tau x_k \in G(\mathcal{A}_f) \). We have:

\[
(\tau'.h)(\sigma)(g_f) = [\tau'.h(\tau^{-1} \sigma)](g_f) = h(\tau^{-1} \sigma)(\tau' g_f), \tag{18}
\]

and observing that \( \tau^{-1} g_f = \tau' \tau x_k \), we deduce that:

\[
h(\tau^{-1} \sigma)(\tau^{-1} g_f) = h_{\underline{g}}(\tau^{-1} \tau \tau^{-1} \sigma) = h_{\underline{g}}(\tau^{-1} \sigma) = h(\sigma)(g_f).
\]

From these equalities and that of [18], we get \( (\tau'.h)(\sigma)(g_f) = h(\sigma)(g_f) \); consequently \( \tau'.h = h \).

The inverse map sends \( h \in \text{Hom}_M(M[\hat{I}], F)^{G(K)} \) to \( (h_{\underline{g}})_{\underline{x}} \) which is given by

\[
h_{\underline{g}}(\sigma) = h(\sigma)(\underline{x}).
\]

Let \( \underline{x} \in \underline{X} \), let us check that \( h_{\underline{x}} \) is invariant under \( \Gamma_{\underline{x}} \). Let \( \gamma \in \Gamma_{\underline{x}} \), \( \gamma = (\gamma_f, \gamma_\infty) \) with \( \gamma = \gamma_f \in G(\mathcal{A}_f) \) and \( \gamma = \gamma_\infty \in G(K_\infty) \). For any \( \sigma \in \hat{I} \), we have:

\[
(\gamma.h_{\underline{x}})(\sigma) = h_{\underline{x}}(\gamma^{-1} \sigma) = h(\gamma^{-1} \sigma)(\underline{x}) = h(\gamma^{-1} \sigma)(\underline{x}) + (\gamma.h)(\sigma)(\underline{x}) = h(\sigma)(\underline{x}) = h_{\underline{g}}(\sigma).
\]

It is easy to see that this map is the inverse map of \( \Theta \), and so \( \Theta \) is an isomorphism de \( M \)-modules.

**Remark 4.1.** Let \((h_{\underline{x}})_{\underline{g}}\) and \( h \) be as in the lemma above. It is easy to check that for any \( i, 1 \leq i \leq 4 \), the property \( \text{(HCi)} \) in the definition of harmonic cochains, cf. Definition 2.1, is satisfied by \( h \) if and only if it is by \( h_{\underline{g}} \) for any \( \underline{x} \).

Now we can say that we have the following commutative diagram:
\[ \bigoplus_{x \in X} \mathcal{S}_{\text{arm}}^k(M, L)^{\Gamma_x} \xrightarrow{\Theta} \mathcal{S}_{\text{arm}}^k(M, F)^{G(K)} \]

\[ (\Phi_x)_\ast \downarrow \quad \downarrow \Phi \]

\[ \bigoplus_{x \in X} \text{Hom}_M(\text{Sp}^k(M), L)^{\Gamma_x} \quad \text{Hom}_M(\text{Sp}^k(M), F)^{G(K)} \]

\[ (\Psi_x)_\ast \downarrow \quad \downarrow \Psi \]

\[ \bigoplus_{x \in X} \text{Hom}_{M[G(K_\infty)]}(\text{Sp}^k(M), \text{Aut}_x(L)) \xrightarrow{\Xi} \text{Hom}_{M[G(K_\infty)]}(\text{Sp}^k(M), \text{Aut}^{K_f}(L)) \]

The arrows are isomorphisms of \( M \)-modules. Indeed, \( \Theta \) is a isomorphism given by Lemma 4.1.1 and Remarque 4.1 above, \( \Xi \) is an isomorphism by Proposition 3.2. The \((\Phi_x)_\ast \)'s and \( \Psi \) are isomorphisms by Proposition 3.4 and finally the \((\Phi_x)_\ast \)'s and \( \Phi \) are isomorphisms by Theorem 2.1.

### 4.2 Automorphic forms and harmonic cocycles.

For any integer \( k, \ 0 \leq k \leq n \), for any \( x \in X \), we denote by \( \mathcal{S}_{\text{arm}}^k(M, L)^{\Gamma_x} \) the set of harmonic cochains of degree \( k \) and with finite supports modulo \( \Gamma_x \).

**Theorem 4.1.** Let \( M \) be a commutative ring and \( L \) be an integral \( M \)-algebra of characteristic zero; for every \( k, \ 0 \leq k \leq n \), we have \( M \)-isomorphisms :

\[ \bigoplus_{x \in X} \mathcal{S}_{\text{arm}}^k(M, L)^{\Gamma_x} \cong \text{Hom}_{M[G(K_\infty)]}(\text{Sp}^k(M), \text{Aut}^{\mathcal{K}_f}(L)) \]

and

\[ \bigoplus_{x \in X} \mathcal{S}_{\text{arm}}^k(M, L)^{\Gamma_x} \cong \text{Hom}_{M[G(K_\infty)]}(\text{Sp}^k(M), \text{Aut}^{K_{K_f}}(L)). \]

These isomorphisms are functorial on \( \mathcal{K}_f \) (i.e. compatible with the inclusions \( \mathcal{K}_f' \subseteq \mathcal{K}_f \)).

**Proof.**

The first isomorphism is already given by the preceding diagram, that is \( \Psi \circ \Phi \circ \Theta \) or equally \( \Xi \circ (\Psi_x)_\ast \circ (\Phi_x)_\ast \). We use this last formulation to prove the second isomorphism. Indeed, by Proposition 3.2, \( \Xi \) induces an isomorphism :

\[ \bigoplus_{x \in X} \text{Hom}_{M[G(K_\infty)]}(\text{Sp}^k(M), \text{Aut}_x(L)) \cong \text{Hom}_{M[G(K_\infty)]}(\text{Sp}^k(M), \text{Aut}^{\mathcal{K}_f}(L)), \]
so we need only to prove that, for each \( x \in X \), by the isomorphism
\[
\Psi_x \circ \Phi_x : \mathcal{H}arm^k(M, L)^{\Gamma_x} \to \text{Hom}_{M[G(K_\infty)]}(\text{Sp}^k(M), \text{Aut}_x(L))
\]
we have \( h \in \mathcal{H}arm^k(M, L)^{\Gamma_x} \) if and only if its image \( \psi = \Psi_x \circ \Phi_x(h) \) verifies that \( \psi(g) \in \text{Aut}_{x,0}(L) \) for any \( g \in \text{Sp}^k(M) \). This last assertion is equivalent to \( \psi(x_{B_JP_J}) \in \text{Aut}_{x,0}(L) \); indeed, \( \psi \) is a homomorphism of \( M[G(K_\infty)] \)-modules and \( \text{Sp}^k(M) \) as such is generated by \( g = \chi_{B_JP_J} \). First, let us prove the ”only if” part. Let \( h \in \mathcal{H}arm^k(M, L)^{\Gamma_x} \), so \( h \) has a finite support modulo \( \Gamma_x \). Hence, there exist finitely many \( k \)-cells \( \sigma_1, \ldots, \sigma_r \in \hat{\Gamma}_x \) such that \( \text{supp}(h) \subseteq \Gamma_x \sigma_1 \cup \ldots \cup \Gamma_x \sigma_r \). For every \( i, i = 1, \ldots, r \), set :
\[
\mathcal{U}_i = \{ g \in G(K_\infty), g \sigma_I = \sigma_i \}.
\]
For any \( i \) such that \( \mathcal{U}_i \neq \emptyset \), for any \( g_0 \in \mathcal{U}_i \), we have \( \mathcal{U}_i = g_0 B_J \). Hence, see (5) in \( \S \underline{2.2} \) the \( \mathcal{U}_i \)'s are compact modulo \( Z_G(K_\infty) \) and then so is their union \( \mathcal{U} = \bigcup_{i=1}^r \mathcal{U}_i \). Furthermore, by definition, for any \( g_\infty \in G(K_\infty) \) we have :
\[
\psi(\chi_{B_JP_J})(g_\infty) = h(g_\infty \sigma_J),
\]
therefore, we have supp \( (\psi(\chi_{B_JP_J})) \subseteq \Gamma_x \mathcal{U} \).
Now, the ”if” part. Take \( \psi \in \text{Hom}_{M[G(K_\infty)]}(\text{Sp}^k(M), \text{Aut}_{x,0}(L)) \), thus for any \( g \in \text{Sp}^k(M) \), its image \( \psi(g) \) has a finite support. Let us prove that the harmonic cochain \( h \) that corresponds to \( \psi \) by the isomorphism \( \Psi_x \circ \Phi_x \) is finitely supported modulo \( \Gamma_x \). From the proof of Proposition \( \underline{3.2} \), there exists \( \mathcal{U} \subseteq G(K_\infty) \) so that the quotient \( \Gamma_x \mathcal{U} / B \) is finite, and so that \( \text{supp}(\psi(\chi_{C_I P_J})) \subseteq \mathcal{U} \), for any \( I \subseteq \Delta \) such that \( |\Delta - I| = k \) (for the definition of the set \( C_I \), see (6) in \( \S \underline{2.3} \)). Take \( g_1, \ldots, g_r \) in \( G(K_\infty) \) so that we have :
\[
\Gamma_x \mathcal{U} / B = \{ \Gamma_x g_i B, 1 \leq i \leq r \}.
\]
If \( h \) is not finitely supported modulo \( \Gamma_x \), there would be \( g \in G(K_\infty) \) and \( I \subseteq \Delta \) such that :
\[
(1) \quad g \sigma_I \notin \bigcup_{I \subseteq \Delta} \bigcup_{i=1}^r \Gamma_x g_i \sigma_I
\]
and
\[
(2) \quad h(g \sigma_I) \neq 0.
\]
The second assertion above is equivalent to \( \psi(\chi_{C_I P_J})(g) \neq 0 \), thus \( g \in \mathcal{U} \). Contradiction to the first assertion.
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