Random Walk in Dynamic Markovian Random Environment

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Abstract
We consider a model, introduced by Boldrighini, Minlos and Pellegrinotti [4, 5], of random walks in dynamical random environments on the integer lattice $\mathbb{Z}^d$ with $d \geq 1$. In this model, the environment changes over time in a Markovian manner, independently across sites, while the walker uses the environment at its current location in order to make the next transition. In contrast with the cluster expansions approach of [5], we follow a probabilistic argument based on regeneration times. We prove an annealed SLLN and invariance principle for any dimension, and provide a quenched invariance principle for dimension $d > 7$, providing for $d > 7$ an alternative to the analytical approach of [5], with the added benefit that it is valid under weaker assumptions. The quenched results use, in addition to the regeneration times already mentioned, a technique introduced by Bolthausen and Sznitman [7].

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1 Introduction

In recent years there has been a great deal of study on random walks in random environments (RWRE) on $\mathbb{Z}^d$, $d \geq 1$, where first the environment is chosen at random and kept fixed throughout the time evolution, and then a walker moves randomly in such a way that given the environment, its position forms a time homogeneous Markov chain whose transition probabilities depend on the environment. Even though a lot is known about this model, there are many challenging problems left open, see [13] and [12] for surveys.

In the current paper, we consider random walks in dynamical random environments, where along with the walker the environment changes over time. Such model was studied in an abstract setting by Kifer [9]. More relevant to us, the model we consider was first introduced by Boldrighini, Minlos and Pellegринotti [4], where they consider the case where the environment changes over time in an i.i.d. fashion. In [4] and in [6], they proved under certain assumptions (omitted here) that for almost every realization of the dynamic environment, the position of the random walk, properly centered and scaled, satisfies a central limit theorem, with covariance that does not depend on the particular realization of environment. Further, in [5], they obtain similar results for $d \geq 3$ for certain environment evolving over time as a Markov chain, independently at each site. The proofs in these papers are based on cluster expansion, and involve a heavy analytic machinery. In the i.i.d. case, a somewhat simpler proof can be found in Stannat [11].

Our goal in this paper is to describe a probabilistic treatment of this model, which is arguably simpler than those that have appeared in the literature. We recover most of the results of [5], at least when $d > 7$, under weaker and more natural hypotheses. Our approach is based on the introduction of appropriate “regeneration times”, borrowing this concept, if not the details of the construction, from the RWRE literature. Our approach to quenched results is based on a technique introduced in [7].

The paper is divided as follows. The next section describes precisely the model and states our main results: a strong law of large numbers, an annealed CLT (in any dimension), and a quenched CLT (for $d > 7$). Section 3 constructs the regeneration times alluded to above, and derives their basic properties. Section 4 provides the proofs of our main results. Finally, we present in Section 5 several remarks and open problems.

2 Description of the model and main results

In what follows we will consider $\mathbb{Z}^d$, for a fixed $d \geq 1$ with nearest neighbor graph structure as our basic underlying graph. Let $N_0 := \{\pm e_i\}_{i=1}^d \cup \{0\}$ be the neighbors of the origin $0$ (including itself), where $e_i$ is the unit vector in the $i^{th}$ coordinate direction. We use $N_x := x + N_0$ to denote the collection of neighbors of an $x \in \mathbb{Z}^d$.

Let $\mathcal{S} := \mathcal{S}_0$ be a collection of probability measures on the $(2d + 1)$ elements
of $N_0$. To simplify the presentation and avoid various measurability issues, we assume that $S$ is a Polish space (including the possibilities that $S$ is finite or countably infinite). For each $x \in \mathbb{Z}^d$, $S_x$ denotes a copy of $S$, with all elements of $S$ shifted to have support on $N_x$. Formally, an element $\omega(x, \cdot)$ of $S_x$, is a probability measure satisfying

$$\omega(x, y) \geq 0 \quad \forall \ x, y \in \mathbb{Z}^d \quad \text{and} \quad \sum_{y \in N_x} \omega(x, y) = 1$$

Let $B_S$ denote the Borel $\sigma$-field on $S$. We let $K : S \times B_S \to [0, 1]$ denote a Markov transition probability on $S$ with a unique stationary distribution $\pi$. Let $P^\pi$ be the probability distribution on the standard product space $S^{\mathbb{N}_0}$, where $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$, giving the $\pi$-stationary Markov $K$-chain on $S$. For each $x \in \mathbb{Z}^d$, let $K_x, \pi_x$ and $P_x^{\pi_x}$ be just the copies of respective quantities when we replace $S$ by $S_x$. Let

$$\Omega := \prod_{x \in \mathbb{Z}^d} S_x^{\mathbb{N}_0}, \quad P^\pi := \bigotimes_{x \in \mathbb{Z}^d} P_x^{\pi_x},$$

where the measure is defined on $\mathcal{F}$, the standard product $\sigma$-algebra on $\Omega$. An element $\omega \in \Omega$ will be written as $\{ (\omega_t(x, \cdot))_{t \geq 0} \big| x \in \mathbb{Z}^d \}$. We note that under $P^\pi$, the canonical variable $\omega \in \Omega$ follows a distribution such that

- For each $x \in \mathbb{Z}^d$, $(\omega_t(x, \cdot))_{t \geq 0}$ is a stationary Markov chain with transition kernel $K$.

- The chains $(\omega_t(x, \cdot))_{t \geq 0}$ are i.i.d. as $x$ varies over $\mathbb{Z}^d$.

We now turn to define a random walk $(X_t)_{t \geq 0}$. Given an environment $\omega \in \Omega$, $(X_t)_{t \geq 0}$ is a time inhomogeneous Markov chain taking values in $\mathbb{Z}^d$ with transition probabilities

$$P_\omega \left( X_{t+1} = y \big| X_t = x \right) = \omega_t(x, y).$$

For each $\omega \in \Omega$, we denote by $P^\omega_x$ the law induced by $(X_t)_{t \geq 0}$ on $\left( (\mathbb{Z}^d)^{\mathbb{N}_0}, \mathcal{G} \right)$, where $\mathcal{G}$ is the $\sigma$-algebra generated by the cylinder sets, such that

$$P^\omega_x (X_0 = x) = 1.$$  \(3\)

Naturally $P^\omega_x$ is called the *quenched law* of the random walk $\{X_t\}_{t \geq 0}$, starting at $x$.

We note that for every $G \in \mathcal{G}$, the function

$$\omega \mapsto P^\omega_x (G)$$

\(\text{By this we mean that } (\omega_t(x, \cdot))_{t \geq 0} \text{ is a stationary Markov chain on } S_x \text{ with transition probabilities } K_x \text{ and stationary law } \pi_x. \text{ To alleviate notation, we identify in the sequel } S_x \text{ with } S, \pi_x \text{ with } \pi, \text{ and } K_x \text{ with } K \text{ whenever no confusion occurs.}$$
is $\mathcal{F}$-measurable. Hence, we may define the measure $\mathbb{P}^x$ on $\left(\Omega \times \left(\mathbb{Z}^d\right)^{\mathbb{N}_0}, \mathcal{F} \otimes \mathcal{G}\right)$ from the relation

$$
\mathbb{P}^x(F \times G) = \int_F \mathbb{P}_\omega^x(G) \mathbb{P}^\pi(d\omega), \quad \forall \, F \in \mathcal{F}, \, G \in \mathcal{G}.
$$

With a slight abuse of notation, we also denote the marginal of $\mathbb{P}^x$ on $(\mathbb{Z}^d)^{\mathbb{N}_0}$ by $\mathbb{P}^x$, whenever no confusion occurs. This probability distribution is called the \textit{annealed law} of the random walk $\{X_t\}_{t \geq 0}$, starting at $x$. Note that under $\mathbb{P}^x$, the random walk $\{X_t\}_{t \geq 0}$ is not, in general, a Markov chain. However, when $K(s, \cdot) = K(s_0, \cdot)$ for some $s_0 \in \mathcal{S}$, i.e. the environment is i.i.d. in time, then of course $\{X_t\}_{t \geq 0}$ is actually a random walk on $\mathbb{Z}^d$ under $\mathbb{P}^x$, with deterministic increment distribution given by the mean of $\pi$.

Throughout this paper we will assume the followings hold,

\textbf{Assumption 1} (A1) There exists $0 < \kappa \leq 1$ such that

$$
K(w, A) \geq \kappa \pi(A), \quad \forall \, w \in \mathcal{S}, \, A \in \mathcal{B} \mathcal{S}.
$$

\textbf{(A2)} There exist $0 < \varepsilon < 1$ and a fixed translation invariant Markov kernel with only nearest neighbor transition $q: \mathbb{Z}^d \times \mathbb{Z}^d \to [0, 1]$ with the property that $q(x, y) = q(y - x)$ and

$$
\left| \sum_{y \in \mathbb{Z}^d: |y| = 1} q(0, y) \, e^{i \ell \cdot y} \right| < 1, \quad \forall \, \ell \in \mathbb{Z}^d \setminus \{0\},
$$

such that

$$
\mathbb{P}^\pi(\omega_t(x, y) \geq \varepsilon \, q(x, y)) = 1, \quad \forall \, x, y \in \mathbb{Z}^d, \, t \geq 0. \quad \text{(6)}
$$

\textbf{(A3)} $\kappa + \varepsilon^2 > 1$.

\textbf{Remark 2} Some comments regarding Assumption (1) are in order.

1. Condition (A1) provides a uniform “fast mixing” rate for the environment chains. If, as in [2], $\mathcal{S}$ is finite and $K$ is irreducible and aperiodic, then while (A1) may fail, it does hold if $K$ is replaced by $K^r$ for some fixed $r \geq 1$. A slight modification of our arguments applies to that case, too.

2. Condition (A2) is the same as that made in [4, 5] and later on in [11]. This condition essentially means that the random environment has a “deterministic” part $q$, which is non-degenerate. We thus refer to condition (A2) as an ellipticity condition.

We note that the assumption that $q$ is translation invariant does not present a loss of generality. This is because under $\mathbb{P}^\pi$ the environment chains $(\omega_t(x, \cdot))_{t \geq 0}$ are assumed i.i.d. as $x$ varies.
Finally note that we could allow $\varepsilon = 1$ in (6). In that case, the environment is given by the deterministic kernel $q$. This makes the walk a classical Markov chain which is well studied. This is the reason why we work with the restriction $\varepsilon < 1$.

3. Condition (A3) is technical but absolutely crucial for our argument. It implies that there is a trade off between the environment Markov chain being fast mixing ($\kappa$ close to 1) and the fluctuation in the environment being “small” ($\varepsilon$ close to 1). The later is a condition assumed in [4, 5] also.

We now formulate our main results as follows.

**Theorem 1 (Annealed SLLN)** Let Assumption 4 hold. Then there exists a deterministic $v \in \mathbb{R}^d$ such that

$$\frac{X_n}{n} \xrightarrow{n \to \infty} v \quad \text{a.s.} \quad [\mathbb{P}^0]$$  \hspace{1cm} (7)

**Remark 3** A (non-explicit) formula for the limit velocity $v$ is given below, see (38). A consequence of that formula is that whenever the transition probabilities $K(s, \cdot)$ are invariant under lattice isometries (i.e., $\int_S K(s, ds') s'(e) = \int_S K(s, ds') s'(T e)$ for any lattice isometry $T$ and any $s \in S$), then $v = 0$.

We also prove

**Theorem 2 (Annealed Invariance Principle)** Let Assumption 4 hold. Then, there exists a $(d \times d)$ strictly positive definite matrix $\Sigma$, such that under $\mathbb{P}^0$,

$$\left( \frac{X_{\lfloor nt \rfloor} - nt v}{\sqrt{n}} \right)_{t \geq 0} \xrightarrow{n \to \infty} \text{BM}_d(\Sigma),$$  \hspace{1cm} (8)

where $\text{BM}_d(\Sigma)$ denotes a $d$-dimensional Brownian motion with covariance matrix $\Sigma$, $\xrightarrow{d}$ denotes weak convergence of the laws, and $\lfloor x \rfloor$ denotes the integer part of $x$.

**Remark 4** An implicit formula for the covariance matrix $\Sigma$ is given in (46).

In order to get a quenched invariance principle, we need some further restrictions. Set

$$\gamma = \log(1 - \kappa)/\log \varepsilon.$$  \hspace{1cm} (9)

**Theorem 3 (Quenched Invariance Principle)** Let Assumption 4 hold. Assume further that $\gamma > 6$ and also

$$d > 1 + \frac{4 + \frac{2\gamma(d-1)}{\gamma(d-3)} - 2(d-1)}{1 - 6/\gamma} > 7.$$  \hspace{1cm} (10)

Then, with the notation of Theorem 2 one has that (8) holds true under $\mathbb{P}^0_\omega$, for $\mathbb{P}^\pi$ almost every $\omega$.  

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Remark 5 Condition (11) is clearly not optimal, as the case $\kappa = 1$ demonstrates (recall that when $\kappa = 1$, the environment is i.i.d. in time, and the quenched CLT statement holds true in any dimension as soon as (A2) holds true, see [11]). We do not know however whether the quenched CLT holds for Markov environments in low dimension.

3 Construction of a “regeneration time”

Since for $\kappa = 1$ the annealed law on the random walk is Markovian with deterministic transition, we may and will in the sequel consider only the case $\kappa < 1$. Our approach is based on the construction of a sequence of a.s. finite (stopping) times $\{\tau_n\}_{n \geq 1}$ (on an enlarged probability space), such that, between two successors, say $\tau_n$ and $\tau_{n+1}$, environment chains at each location visited by the walk go through a time of “regeneration”, in the sense that they have started afresh from new states selected according to the stationary distribution $\pi$. This in turn provides a renewal structure for $\{\tau_n, X_{\tau_n}\}_{n \geq 1}$. We note that regeneration times have been extensively used in the RWRE context, see [12, 13]. What makes the situation considered here particularly simple is that the regeneration times constructed here are actually stopping times.

To introduce the regeneration times, we begin by constructing an extension of our probability space, which is similar to what is done in [13].

Let $W = \{0, 1\}$ and let $W$ denote the $\sigma$-algebra on $W^N$ generated by cylinder sets. For $\varepsilon > 0$ as in Condition (A2), let $Q_\varepsilon$ be the product measure on $(W^N, W)$ such that the coordinate variables, say $(\varepsilon_t)_{t \geq 1}$, are i.i.d. Bernoulli variables with $Q_\varepsilon(\varepsilon_1 = 1) = \varepsilon$.

For any $(\omega, \varepsilon) \in \Omega \times W^N$, we define a probability measure $P^0_{\omega, \varepsilon}$ on $(\mathbb{Z}^d)^{N_0}$ such that $\{X_t\}_{t \geq 0}$ is a Markov chain with state space $\mathbb{Z}^d$ and transition law

$$P^0_{\omega, \varepsilon}(X_{t+1} = y \mid X_t = x) = 1_{[\varepsilon_{t+1} = 1]} q(x, y) + \frac{1_{[\varepsilon_{t+1} = 0]}}{1 - \varepsilon} [\omega_t(x, y) - \varepsilon q(x, y)],$$

where $y \in \{x\} \cup \{x \pm e_i\}_{i = 1}^d$, and also

$$P^0_{\omega, \varepsilon}(X_0 = 0) = 1.$$

Finally we define a measure $P^0$ on $(\Omega \times W^N \times (\mathbb{Z}^d)^{N_0}, \mathcal{F} \otimes W \otimes \mathcal{G})$, such that the coordinate variables has the following distribution

$$(\omega, \varepsilon) \sim P^\pi \otimes Q_\varepsilon \quad \text{and} \quad \{X_t\}_{t \geq 0} \sim P^0_{\omega, \varepsilon} \text{ given } (\omega, \varepsilon).$$

It is immediate to check that under $P^0$, the marginal distribution of the chain $\{X_t\}_{t \geq 0}$ is $P^0$ and also the conditional distribution of $\{X_t\}_{t \geq 0}$ given $\omega$ is $P^0_{\omega}$. We
will call the $\varepsilon$-variables the $\varepsilon$-coin tosses. Heuristically, under $P^0$, the evolution of the random walk at time $t$ is done by first tossing the $\varepsilon$-coin $\varepsilon_t$, and depending on the outcome, taking a step either according to how the environment dictates it (if $\varepsilon_t = 0$) or taking a step according to the fixed transition kernel $q$ (if $\varepsilon_t = 1$).

One more extension of the probability space is needed before we can define the "regeneration time" and it is for the Markov evolution of the environment at each sites. Notice under Condition (A1), a step in the $K$-chain on $S$ can be taken as follows. First, toss a coin independently with probability $\kappa$ of turning head, if it turns out head, then select a state according to the stationary distribution $\pi$ independently; otherwise if the coin lands in tail, then take a step according to the Markov transition kernel $\tilde{K}(w, A) = K(w, A) - \kappa \pi(A) 1 - \kappa$, $w \in S$, $A \in B_S$.

A rigorous construction is as follows. Extend the measurable space $(\Omega, F)$ to accommodate i.i.d. Bernoulli ($\kappa$) variables $\{ (\alpha_t(x))_{t \geq 1} \mid x \in \mathbb{Z}^d \}$, such that under $P^\pi$ the Markov evolution of the environment $\omega_t(x, \cdot)$ at a site $x \in \mathbb{Z}^d$ is obtained by the above description using the $\kappa$-coin tosses $(\alpha_t(x))_{t \geq 1}$. We note that in this construction $\{ (\alpha_t(x))_{t \geq s+1} \mid x \in \mathbb{Z}^d \}$ is independent of $\{ (\omega_t(x, \cdot))_{0 \leq t \leq s} ; (X_t)_{0 \leq t \leq s} \}$, for every $s \geq 0$.

For $t \geq 0$ and $x \in \mathbb{Z}^d$, define

$$I_t(x) := \sum_{s=0}^{t} 1 \{ X_s = x, \varepsilon_s = 0 \},$$

which is the number of "proper" visits to the site $x$ by the chain up to time $t$ ("proper" visits are those visits in which the walker "learns" about the environment, that is those in which the next move depends on the random environment rather than on the auxiliary $\varepsilon$-coin). If $I_t(x) > 0$ let

$$\gamma_t(x) := \sup \left\{ s \leq t \mid X_s = x, \varepsilon_s = 0 \right\},$$

be the time of last "proper" visit to $x$ before time $t$, and

$$\eta_t(x) := \inf \left\{ s \geq 0 \mid \alpha_{\gamma_t(x)+s}(x) = 1 \right\},$$

be the first time after $\gamma_t(x)$ when the environment chain at site $x$ takes a step according to the stationary measure $\pi$. For completeness we will take $\gamma_t(x) = \eta_t(x) = 0$ if $I_t(x) = 0$. Finally we define

$$\tau_1 := \inf \left\{ t > 0 \mid \gamma_t(x) + \eta_t(x) < t \ \forall \ x \in \mathbb{Z}^d \right\}.$$

$\tau_1$ will be called a "regeneration time", because from time $\tau_1$ the environment chains at all sites look like they have started afresh from stationary distribution.
Proposition 4  
Let Assumption 1 hold. Then, \( \mathbb{E} \left( \tau_1^2 \right) < \infty \).

Proof: Due to the strict inequality in Condition (A3), we can find \( 0 < \delta < 1 \) such that \( \varepsilon^2 > (1 - \kappa)\delta \). Define

\[
L(t) = \left\lfloor \frac{-\delta \log t}{\log \varepsilon} \right\rfloor,
\]

which is an increasing sequence of integers going to \( \infty \) with \( L(t) < t \) for large \( t \).

For fixed \( t \geq 1 \), let \( \beta_t \) be the first time there is a run of length \( L(t) \) of non-zero \( \epsilon \)-coin tosses ending at it, that is

\[
\beta_t := \inf \left\{ s \geq L(t) \mid \epsilon_s = 1, \epsilon_{s-1} = 1, \ldots, \epsilon_{s-L(t)+1} = 1 \right\}.
\]

From the definition of \( \tau_1 \) we get

\[
P_0(\tau_1 > t) \leq P_0(\beta_t > t) + P_0(\beta_t \leq t, \exists x \in \mathbb{Z}^d \text{ s.t. } \eta_{\beta_t}(x) \geq \beta_t - \gamma_{\beta_t}(x))
\]

We will consider the first and second terms in the right hand side of (20) separately, as follows.

Consider first the second term, noting that for each \( x \in \mathbb{Z}^d \) and for any \( t \geq 1 \) the time \( \eta_t(x) \) is nothing but a Geometric (\( \kappa \)) random variable. Thus,

\[
P_0(\beta_t \leq t, \exists x \in \mathbb{Z}^d \text{ s.t. } \eta_{\beta_t}(x) \geq \beta_t - \gamma_{\beta_t}(x)) \leq C_0 \sum_{r=0}^{\infty} r^{d-1} \exp \left( -\lambda \left( L(t) \vee r \right) \right)
\]

where \( C_0 = C_0(d) > 0 \) is such that an \( L_1 \) ball in \( \mathbb{Z}^d \) of radius \( r \) contains less than \( C_0 r^{d-1} \) points, and \( \lambda = -\log (1 - \kappa) \). Indeed, the first inequality in (21) follows from the observation that

\[
\beta_t - \gamma_{\beta_t}(x) \geq L(t) \vee |x - X_{\beta_t-L(t)}|,
\]

with \( | \cdot | \) denoting the \( L_1 \)-norm on \( \mathbb{Z}^d \), and then computing the probability by conditioning on \( X_{\beta_t-L(t)} \).

Concerning the first term in (20), note that

\[
P_0(\beta_t > t) \leq P_0 \left( \frac{t-L(t)}{L(t)} \sum_{j=0}^{t-L(t)} \left[ \epsilon_{jL(t)+1} = 1, \epsilon_{jL(t)+2} = 1, \cdots, \epsilon_{(j+1)L(t)+1} = 1 \right] \right)
\]

\[
\leq \left( 1 - \varepsilon L(t) \right) \frac{t}{L(t)^2}.
\]
Using the choice of $L(t)$ and equations (20), (21) and (22) one concludes that
\[ P^0 (\tau_1 > t) = \frac{C_1}{t^{2+\zeta}}, \]
where $C_1, \zeta > 0$ are some constants (depending on $\delta, d$). This completes the proof.

**Remark 6** An inspection of the proof reveals that in fact,
\[ \forall \gamma' < \frac{\log(1 - \kappa)}{\log \varepsilon} =: \gamma, \quad P^0 [\tau_1^{\gamma'}] < \infty. \]
This will be useful when deriving the quenched invariance principle.

Define now a $\sigma$-algebra,
\[ \mathcal{H}_1 := \sigma \left( \tau_1; \{X_t\}_{0 \leq t \leq \tau_1}; \{(\alpha_t (\cdot))_{t \leq \tau_1} \} \right). \]
The following is the most crucial lemma.

**Lemma 7** For any measurable sets $A, B, C$ we have,
\[ P^0 \left( \{X_{\tau_1 + t} - X_{\tau_1}\}_{t \geq 0} \in A, \{(\omega_{\tau_1 + t} (\cdot, \cdot))_{t \geq 0}\} \in B, \{(\alpha_t (\cdot))_{t \geq \tau_1 + 1}\} \in C \right) \]
\[ = P^0 \left( \{X_t\}_{t \geq 0} \in A, \{(\omega_t (\cdot, \cdot))_{t \geq 0}\} \in B, \{(\alpha_t (\cdot))_{t \geq 1}\} \in C \right). \]

**Proof:** Let $h$ be a $\mathcal{H}_1$ measurable function. Write $1_A := 1 \left( \{X_t - X_0\}_{t \geq 0} \in A \right)$,
$1_B := 1 \left( \{(\omega_t (\cdot, \cdot))_{t \geq 0}\} \in B \right)$ and $1_C := 1 \left( \{(\alpha_t (\cdot))_{t \geq 1}\} \in C \right)$. Note that for every $m \in \mathbb{N}$ and $x \in \mathbb{Z}^d$, since $h$ is $\mathcal{H}_1$-measurable, there exists a random variable $h_{x,m}$ which is measurable with respect to $\sigma \left( \{X_t\}_{t \geq 0}; \{(\alpha_t (\cdot))_{t \leq m}\} \right)$, such that $h = h_{x,m}$ on the event $[\tau_1 = m, X_{\tau_1} = x]$. Writing $\theta$ for the time shift,
\[ P^0 [1_A \circ \theta^{\tau_1} \cdot 1_B \circ \theta^{\tau_1} \cdot 1_C \circ \theta^{\tau_1} \cdot h] \]
\[ = \sum_{m=1}^{\infty} \sum_{x_m \in \mathbb{Z}^d} P^0 [1_A \circ \theta^m \cdot 1_B \circ \theta^m \cdot 1_C \circ \theta^m \cdot 1 (\tau_1 = m) \cdot 1 (X_m = x_m) \cdot h_{x,m,m}] \]
\[ = \sum_{m=1}^{\infty} \sum_{x_m \in \mathbb{Z}^d} P^0 [1_B \circ \theta^m \cdot E^0 [1_A \circ \theta^m \cdot 1_C \circ \theta^m \cdot 1 (\tau_1 = m) \cdot 1 (X_m = x_m) \cdot h_{x,m,m}]] \]
\[ = \sum_{m=1}^{\infty} \sum_{x_m \in \mathbb{Z}^d} \sum_{t_1 \leq \tau_1 \leq m} \sum_{t,1 \leq t \leq m} E^0 [1_B \circ \theta^m \cdot E^0 [1_A \circ \theta^m \cdot 1_C \circ \theta^m \cdot 1 [\tau_1 = m] \cdot 1 [X_t = x_t, 1 \leq t \leq m] \cdot 1 [t_1 = \tau_1 = t, 1 \leq t \leq m] \cdot h_{x,m,m}]] \]
Lemma 8

Proof:

Define $\mathcal{H}_n := \sigma \left( \tau_1, \tau_2, \ldots, \tau_n; (X_t)_{0 \leq t \leq \tau_n}; \left\{ (\alpha_t(x))_{t \geq \tau_n} \right\} \right)$, (29)
then an obvious rerun of the proof of Lemma \( \mathbb{L} \) yields that for measurable sets \( A, B \) and \( C \),

\[
\mathbb{P}^0 \left( \{X_{\tau + t} - X_{\tau} \}_{t \geq 0} \in A, \{\omega_{\tau + t} (\cdot)\}_{t \geq 0} \in B, \{(\alpha_{\tau + t} (\cdot))_{t \geq 1}\} \in C \bigg| \mathcal{H}_n \right) = \mathbb{P}^0 \left( \{X_t\}_{t \geq 0} \in A, \{\omega_t (\cdot, \cdot)\}_{t \geq 0} \in B, \{(\alpha_t (\cdot))_{t \geq 1}\} \in C \right). \tag{30}
\]

So first of all we get from Proposition \( \mathbb{P} \) that \( \mathbb{P}^0 (\tau_n < \infty) = 1 \), for all \( n \geq 1 \), and also under \( \mathbb{P}^0 \)

\[
(\tau_1, X_{\tau_1}), (\tau_2 - \tau_1, X_{\tau_1} - X_{\tau_1}), \ldots, (\tau_n - \tau_{n-1}, X_{\tau_n} - X_{\tau_{n-1}}), \ldots
\]

are i.i.d. of random vectors. \( \blacksquare \)

4 Proofs of the main results

4.1 Proof of Theorem \( \mathbb{Q} \)

Fix \( a \in \mathbb{R}^d \). From Proposition \( \mathbb{P} \) we get that \( \mathbb{E}^0 [\tau_1] < \infty \). Since the random walk \( \{X_t\}_{t \geq 0} \) has bounded increments, it follows that \( \mathbb{E}^0 [a \cdot X_{\tau_1}] < \infty \) as well. Let \( Y_n := a \cdot (X_{\tau_n} - X_{\tau_{n-1}}), \) for \( n \geq 1 \), taking \( \tau_0 = 0 \). Using Lemma \( \mathbb{L} \) we conclude

\[
\frac{\tau_n}{n} \mathop{\longrightarrow}_{n \to \infty} \mathbb{E}^0 [\tau_1] \quad \text{a.s.} \quad \mathbb{P}^0 \], \tag{31}
\]

and

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} Y_k \mathop{\longrightarrow}_{n \to \infty} \mathbb{E}^0 [a \cdot X_{\tau_1}] \quad \text{a.s.} \quad \mathbb{P}^0 \]. \tag{32}
\]

We can find a (possibly random) sequence of numbers \( \{k_n\}_{n \geq 1} \) increasing to \( \infty \) such that for all \( n \geq 1 \) we have \( \tau_{k_n} \leq n < \tau_{k_n+1} \). Then from \( \mathbb{Q} \) we get

\[
\frac{n}{k_n} \mathop{\longrightarrow}_{n \to \infty} \mathbb{E}^0 [\tau_1] \quad \text{a.s.} \quad \mathbb{P}^0 \]. \tag{33}
\]

Since the increments of the random walk are bounded, we have for any \( n \geq 1 \) that

\[
\left| a \cdot (X_n - X_{\tau_{k_n}}) \right| \leq \|a\|_2 (n - \tau_{k_n}) \leq \|a\|_2 (\tau_{k_n+1} - \tau_{k_n}). \tag{34}
\]

On the other hand, since by Lemma \( \mathbb{L} \) the random variables \( (\tau_n - \tau_{n-1})_{n \geq 1} \) are identically distributed and of finite mean, one gets for any \( \delta > 0 \) that

\[
\sum_{n=0}^{\infty} \mathbb{P}^0 (\tau_{n+1} - \tau_n > \delta n) = \sum_{n=0}^{\infty} \mathbb{P}^0 (\tau_1 > \delta n) < \infty.
\]

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It follows from an application of the Borel-Cantelli Lemma that
\[
\frac{\tau_{n+1} - \tau_n}{n} \rightarrow 0 \text{ a.s. } P^0. \tag{35}
\]
So using 34 and 36 we get
\[
\frac{|a \cdot (X_n - X_{\tau_n})|}{n} \rightarrow 0 \text{ a.s. } P^0 \tag{36}
\]
This together with 32 and 33 gives
\[
\frac{a \cdot X_n}{n} \rightarrow E^0 [a \cdot X_{\tau_1}] \text{ a.s. } P^0 \tag{37}
\]
Finally taking \(a = e_i\) for \(i = 1, 2, \ldots, d\) we conclude
\[
\frac{X_n}{n} \rightarrow E^0 [X_{\tau_1}] \text{ a.s. } P^0 \tag{38}
\]
which completes the proof.

### 4.2 Proof of Theorem 2

Fix \(a \in \mathbb{R}^d\), let \((Y_n)_{n \geq 1}\) be as defined above. Put \(Y_n := Y_n - (\tau_n - \tau_{n-1}) a \cdot v\), where \(v = E^0 [X_{\tau_1}] / E^0 [\tau_1]\). Let \(S_n := Y_1 + Y_2 + \cdots + Y_n\), for \(n \geq 1\). By Proposition \(4\), \(\tau_1\) has finite second moment and, due to the boundedness of the increments of the random walk \((X_t)_{t \geq 0}\), so does \(X_{\tau_1}\). Further, by definition, \(E^0 [S_n] = 0\) for all \(n \geq 1\).

Thus by Lemma \(4\) and Donsker’s invariance principle, see e.g. [2, Theorem 14.1], we have
\[
\frac{S_{\lfloor nt \rfloor}}{\sigma_a \sqrt{n}} \xrightarrow{d} \text{BM} (1), \tag{39}
\]
where \(\text{BM}(1) := \text{BM}_1(1)\) and \(\sigma_a^2 := E^0 \left[ \frac{Y_1^2}{\tau_1} \right] > 0\), where the last inequality is due to Condition (A2) and 4.

Put next \(m_n := n / E^0 [\tau_1]\). Since \(n \mapsto m_n\) is a deterministic scaling, it follows from 39 that
\[
\frac{S_{\lfloor m_n t \rfloor}}{\sigma_a \sqrt{m_n}} \xrightarrow{d} \text{BM} (1). \tag{40}
\]

Let \((k_n)_{n \geq 1}\) be as before. Our next step is to prove the analogue of 30 with \(m_n t\) replaced by \(k_{\lfloor nt \rfloor}\). A consequence of 33 is that for any fixed \(T < \infty\),
\[
\sup_{t \leq T} \left| \frac{k_{\lfloor nt \rfloor}}{n} - \frac{m_n t}{n} \right| \rightarrow 0, \text{ a.s. } P^0 \tag{41}
\]
Thus,
\[
\frac{S_{k_{\lfloor nt \rfloor}}}{\sigma_a \sqrt{m_n}} \xrightarrow{d} \text{BM (1)}.
\] (42)

Note that for an appropriate \( C = C(a, d) \), using (34),
\[
\sup_{t \leq T} \left| \frac{S_{k_{\lfloor nt \rfloor}} - a \cdot X_{\lfloor nt \rfloor} - nt \cdot a \cdot v}{\sqrt{n}} \right| \leq C \max_{0 \leq i \leq T_{k_{\lfloor nt \rfloor}}} \frac{\tau_{i+1} - \tau_i}{\sqrt{n}}.
\] (43)

Since \( k_{\lfloor nt \rfloor} \leq \lfloor nT \rfloor \), we have that for any \( \delta > 0 \),
\[
\mathbb{P}^0 \left( \max_{0 \leq i \leq T_{k_{\lfloor nt \rfloor}}} \frac{\tau_{i+1} - \tau_i}{\sqrt{n}} > \delta \right) \leq \sum_{i=1}^{\lfloor nT \rfloor} \mathbb{P}^0 (\tau_1 > \delta \sqrt{n}).
\] (44)

Note that, since \( \mathbb{E}^0 [\tau_1^2] < \infty \), one has that
\[
\sum_{i=1}^{\infty} \mathbb{P}^0 (\tau_1 > \delta \sqrt{n}) = \sum_{i=1}^{\infty} \mathbb{P}^0 (\tau_1^2 > \delta^2 \frac{i}{T}) < \infty.
\]

Hence, for each \( \delta_1 > 0 \) there is a deterministic constant \( I_{\delta_1} \) depending on \( d, \delta, T, \delta_1 \) such that
\[
\sum_{i=I_{\delta_1}}^{\infty} \mathbb{P}^0 (\tau_1 > \delta \sqrt{n}) < \delta_1.
\]

Therefore,
\[
\limsup_{n \to \infty} \sum_{i=1}^{\lfloor nT \rfloor} \mathbb{P}^0 (\tau_1 > \delta \sqrt{n})
\]
\[
\leq \limsup_{n \to \infty} \left( \sum_{i=1}^{I_{\delta_1}} \mathbb{P}^0 (\tau_1 > \delta \sqrt{n}) + \sum_{i=I_{\delta_1}+1}^{\infty} \mathbb{P}^0 (\tau_1 > \frac{\delta \sqrt{i}}{\sqrt{T}}) \right) \leq \delta_1.
\]

\( \delta_1 \) being arbitrary, one concludes from the last limit and (44) that
\[
\mathbb{P}^0 \left( \max_{0 \leq i \leq T_{k_{\lfloor nt \rfloor}}} \frac{\tau_{i+1} - \tau_i}{\sqrt{n}} > \delta \right) \xrightarrow{n \to \infty} 0.
\]

Together with (42) and (43), this implies that for every \( a \in \mathbb{R}^d \setminus 0 \) we have
\[
\frac{a \cdot (X_{\lfloor nt \rfloor} - nt \cdot v)}{\sigma_a \sqrt{m_n}} \xrightarrow{d} \text{BM (1)}.
\] (45)

Since \( a \) is arbitrary, this completes the proof of the theorem, with
\[
\Sigma := \frac{\operatorname{Var} \mathbb{E}^0 (X_{\tau_1} - \tau_1 \cdot v)}{\mathbb{E}^0 [\tau_1]}.
\] (46)
4.3 Proof of Theorem \([3]\)

Our argument is based on the technique introduced by Bolthausen and Sznitman in \([7]\), as developed in \([8]\). Let \(B_t^n := (X_{\lfloor nt \rfloor} - nt\pi)/\sqrt{n}\), and let \(\mathcal{B}_t^n\) denote the polygonal interpolation of \((k/n) \to B_{kn}^n\). Consider the space \(C_T := C([0, T], \mathbb{R}^d)\) of continuous \(\mathbb{R}^d\)-valued functions on \([0, T]\), endowed with the distance \(d_T(u, u') = \sup_{t \leq T} |u(t) - u'(t)|/\wedge 1\). By [7] Lemma 4.1, Theorem 3 follows from Theorem 2 once we show that for all bounded Lipschitz function \(F\) on \(C_T\) and \(b \in (1, 2]\),

\[
\sum_{m=1}^{\infty} \text{Var}_{P^*} \left( E^0[F(B^{(b^m)}_{t^m})] \right) < \infty.
\]

(47)

In order to prove (47), we now follow the approach of [8]. We construct the environment \(\omega\) using the variables \((\alpha_t(x))_{t \geq 1, x \in \mathbb{Z}^d}\) as described in Section 3. We next construct two independent sequences of i.i.d. Bernoulli(\(\varepsilon\)) random variables, that we denote by \((\varepsilon_t^{(1)})_{t \geq 1}\) and \((\varepsilon_t^{(2)})_{t \geq 1}\). Given these sequences and the environment \(\omega\), we construct two independent copies of the random walk, denoted \((X_t^{(1)})_{t \geq 1}\) and \((X_t^{(2)})_{t \geq 1}\), following the recipe of Section 3 (we of course use the sequence \(\varepsilon^{(j)}\) to construct \(X^{(j)}\), for \(j = 1, 2\), and introduce the respective linear interpolations \((B_{t^{(j)}}^{n(j)})_{t \geq 0}\). It is then clear that (47) is equivalent to

\[
\sum_m \left( \mathbf{P}^\pi \otimes Q_\varepsilon \otimes Q_\varepsilon \left( E_{\omega, \varepsilon^{(1)}}^0 \otimes E_{\omega, \varepsilon^{(2)}}^0 \left[ F(B(b^mj, (1))) F(B(b^m, (2))) \right] \right) - \mathbf{P}^\pi \otimes \mathbf{P}^\pi \otimes Q_\varepsilon \otimes Q_\varepsilon \left( E_{\omega, \varepsilon^{(1)}}^0 \otimes E_{\omega, \varepsilon^{(2)}}^0 \left[ F(B(b^m, (1))) F(B(b^m, (2))) \right] \right) \right) < \infty,
\]

(48)

where for any probability measure \(\mathbf{P}\) and a measurable function \(f\) by \(\mathbf{P}(f)\) we mean \(E_{\omega, \varepsilon^{(1)}}^0 \otimes \mathbf{P}^\pi \otimes \mathbf{P}^\pi \otimes E_{\omega, \varepsilon^{(2)}}^0\). Recalling the constant \(\gamma\) from (39), we next choose constants \(\theta, \theta', \theta'', \mu, \alpha\) satisfying the following conditions:

\[
0 < \theta < 1, \quad 2/\gamma < \theta' < \theta/2, \quad \theta > 2(\theta' + 1/(d - 1)), \quad \theta' < \theta', \quad (\theta' - \theta'')\gamma > 1,
\]

\[
0 < \mu < 1/2, \quad 1/2 > \alpha > (1/\theta' + 1)/\gamma,
\]

\[
\theta'/(d - 1) - 4 - 2\alpha(d - 1) = \theta''(d - 5 - 2\alpha(d - 1)) > 1.
\]

(49)

It is not hard to verify that the assumptions of Theorem 3 imply that such constants can be found (indeed, verify, using that \(d > 7\) and \(d - 1 > 4/(1 - 6/\gamma)\) which implies that \(\gamma > 6(d - 1)/(d - 5)\), that taking \(\theta' = (d - 3)/2(d - 1)\), \(\alpha = 1/\gamma + 1/\gamma'\) and \(\theta'' = \theta'(\gamma(d - 3) - 2(d - 1))/\gamma(d - 3)\) satisfies the constraints except for the last one with equality, and the last one with inequality due to (10), and argue by continuity). Fix then an integer \(m\). We let \(\bar{\beta}_m\) denote the first time there is a run of length \([m^\theta]\) of non-zero \(\varepsilon\)-coin tosses of both types ending at it, that is

\[
\bar{\beta}_m = \inf \{ s \geq [m^\theta] \mid \epsilon_s^{(1)}(1) = \epsilon_s^{(2)} = \epsilon_{s-1}^{(1)} = \epsilon_{s-1}^{(2)} = \ldots = \epsilon_{s-[m^\theta]+1}^{(1)} = \epsilon_{s-[m^\theta]+1}^{(2)} = 1 \}.
\]
For any $x \in \mathbb{Z}^d$, set $D_m^{(j)}(x) = |x - X_{\tilde{\beta}_m}^{(j)}|$. Setting $n = |\tilde{b}^m|$, define next the events

$$G_m = \{\tilde{\beta}_m \leq n^\mu, X_{\beta_m, \infty}^{(1)} \cap X_{\beta_m, \infty}^{(2)} = \emptyset\},$$

and for $j = 1, 2$,

$$R_m^{(j)} = \{\forall x \in \mathbb{Z}^d, \exists t \in [\tilde{\beta}_m - [n^\theta], \tilde{\beta}_m + D_m^{(j)}(x)] \text{ such that } \alpha_t(x) = 1\}.$$

Finally, set $\hat{G}_m = G_m \cap R_m^{(1)} \cap R_m^{(2)}$. The crucial element of the proof of Theorem 3 is contained in the following lemma, whose proof is postponed.

**Lemma 9** Under the assumptions of Theorem 3 the following estimates hold.

$$\sum_{m=1}^{\infty} P^\pi \otimes Q_\varepsilon \otimes Q_\varepsilon \left( P_{\omega,\varepsilon,\epsilon}^{0} \left[\hat{G}_m^c\right]\right) < \infty \quad (50)$$

$$\sum_{m=1}^{\infty} P^\pi \otimes P^\pi \otimes Q_\varepsilon \otimes Q_\varepsilon \left( P_{\omega,\varepsilon,\epsilon}^{0} \left[\hat{G}_m^c\right]\right) < \infty \quad (51)$$

Equipped with Lemma 9 let us complete the proof (48). Indeed, for any integer $m$,

$$\Delta_m :=
\sum_{m=1}^{\infty} P^\pi \otimes Q_\varepsilon \otimes Q_\varepsilon \left( P_{\omega,\varepsilon,\epsilon}^{0} \left(F(B_{n_{1}}^{(1)})F(B_{n}^{(2)})\right) - P_{\omega,\varepsilon,\epsilon}^{0} \left(F(B_{n_{1}}^{(2)})F(B_{n}^{(2)})\right)\right)$$

$$= P_{\omega,\varepsilon,\epsilon}^{0} \left(F(B_{n_{1}}^{(1)})F(B_{n}^{(2)})\right) - P_{\omega,\varepsilon,\epsilon}^{0} \left(F(B_{n_{1}}^{(1)})F(B_{n}^{(2)})\right); \hat{G}_m$$

$$- P_{\omega,\varepsilon,\epsilon}^{0} \left(F(B_{n_{1}}^{(1)})F(B_{n}^{(2)})\right) + d_m \quad (52)$$

where

$$\sum_{m=1}^{\infty} P^\pi \otimes P^\pi \otimes Q_\varepsilon \otimes Q_\varepsilon (|d_m|) < \infty.$$ 

Let $\beta_t^{n,(j)} = \beta_t^{n,(j)} - \beta_t^{n,(j)}$ for $j = 1, 2$. Recall that on $\hat{G}_m$ one has $\beta_m < n^\mu$ and hence, using the Lipschitz property of $F$, it holds a.s. with respect to $P^0$ on $\hat{G}_m$ that

$$|F(B_{n_{1}}^{(j)}) - F(\hat{B}_{n_{1}}^{(j)})| \leq 2n^{\mu - 1/2}.$$ 

Substituting in (52), we get

$$\Delta_m = P_{\omega,\varepsilon,\epsilon}^{0} \left( F(\hat{B}_{n_{1}}^{(1)})F(\hat{B}_{n}^{(2)})\right)$$

$$- P_{\omega,\varepsilon,\epsilon}^{0} \left( F(\hat{B}_{n_{1}}^{(1)})F(\hat{B}_{n}^{(2)})\right) + e_m \quad (53)$$

where

$$\sum_{m=1}^{\infty} P^\pi \otimes P^\pi \otimes Q_\varepsilon \otimes Q_\varepsilon (|e_m|) < \infty \quad (54).$$
Conditioning on $X^{(j)}_{\bar{\beta}_m}$, $j = 1, 2$, one observes that

$$P_{\pi} \otimes P_{\pi} \otimes Q_{\varepsilon} \otimes Q_{\varepsilon} \left[ F(\hat{B}^{n_1(1)}; \hat{\beta}) F(\hat{B}^{n_1(2)}; \hat{\beta}) \right] = 0.$$  \hspace{1cm} (55)

Together with (53) and (54), one concludes that

$$\sum_{m=1}^{\infty} P_{\pi} \otimes P_{\pi} \otimes Q_{\varepsilon} \otimes Q_{\varepsilon}(\Delta_m) < \infty,$$

as claimed.

**Proof of lemma 9** We begin by considering the event $\{\bar{\beta}_m > n^\mu\}$. By the independence of the i.i.d. sequences $\varepsilon^{(j)}$, we get the estimate

$$Q_{\varepsilon} \otimes Q_{\varepsilon}(\bar{\beta}_m > n^\mu) \leq (1 - \varepsilon^2 |m^\theta| n^\mu |m^\theta|^{-2})$$

$$\leq e^{-n^\mu |m^\theta| |m^\theta| + 2 \varepsilon^2 |m^\theta|} \leq c_1 e^{-c_2 m},$$ \hspace{1cm} (56)

for appropriate constants $c_1$, $c_2$, where the last estimate used that $\theta < 1$.

We next consider the event $(R_m^{(j)})^c \cap \{\bar{\beta}_m \leq n^\mu\}$. Decomposing according to the distance from $X^{(j)}_{\bar{\beta}_m}$, as in the proof of Proposition 4 (see (21)), one gets for some deterministic constants $c_3$, $c_4$ (that may depend on the choice of parameters)

$$P_{\pi} \otimes Q_{\varepsilon}((R_m^{(j)})^c \cap \{\bar{\beta}_m \leq n^\mu\}) \leq c_3 \sum_{r=1}^{\infty} r^{d-1} (1 - \kappa)^r m^\theta \leq c_4 (1 - \kappa)^m.$$ \hspace{1cm} (57)

We next turn to the crucial estimate of the probability of non-intersection after $\bar{\beta}_m$. For this we consider the cases $\mathbf{v} = \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$ separately.

First suppose $\mathbf{v} \neq \mathbf{0}$. We start by showing that at time $\bar{\beta}_m$, the two walkers are not likely to be too close in a $(d-1)$-dimensional sub-space. More precisely, let $V^\perp$ be the $(d-1)$ dimensional sub-space of $\mathbb{R}^d$ which is orthogonal to the vector $\mathbf{v}$. Let $L^{d-1}$ be the $(d-1)$-dimensional lattice which is the projection of $\mathbb{Z}^d$ into $V^\perp$. Let $P_q^x$ denote the law of a homogeneous Markov chain on $\mathbb{Z}^d$ with transition probabilities determined by $q$ starting at $x \in \mathbb{Z}^d$ and let $(M_t^x)_{t \geq 0}$ denote the projection of the associated walk into the lattice $L^{d-1}$. In particular, $P_q^x(M_0^x = \hat{x}) = 1$. We use $M^\hat{x}, M^y$ to denote independent copies of such walks starting at $\hat{x}, y$, respectively. For $j = 1, 2$, let $X^{(j)}_{\bar{\beta}_m}$ denote the projection of the vector $X^{(j)}_{\bar{\beta}_m}$ into
\(V^\perp\). Then, with \(A_m = \{|\tilde{X}_{\tilde{\beta}_m}^{(1)} - \tilde{X}_{\tilde{\beta}_m}^{(2)}| \geq |m^{\theta'}|\}\), we have,

\[
P^\pi \otimes P^\pi \otimes Q_\varepsilon \otimes Q_\varepsilon \left( \mathbb{P}_{\omega^{(1)}, \omega^{(1)}, \varepsilon^{(1)}, \varepsilon^{(2)}}^0 \left( A_m^c \right) \right)
\]

\[
\leq \sup_{x, y \in \mathbb{Z}^d} P_{\pi q}^x \otimes P_{\pi q}^y \left( |M_{[m^{\theta}]}^x - M_{[m^{\theta}]}^y| < m^{\theta'} \right)
\]

\[
= \sup_{x, y \in \mathbb{Z}^d} P_{\pi q}^x \left( P_{\pi q}^y \left( |M_{[m^{\theta}]}^x - M_{[m^{\theta}]}^y| < m^{\theta'} \right) \right)
\]

\[
\leq C m^{\theta(d-1)} \sup_{\tilde{x}, \tilde{y} \in \mathbb{Z}^{d-1}} P_{\pi q}^\tilde{x} \left( M_{[m^{\theta}]}^\tilde{x} = \tilde{z} \right)
\]

\[
\leq C m^{\theta(d-1)} \frac{1}{m^{\theta(d-1)/2}} = C m^{-\left(\frac{2}{\theta} - \theta'\right)(d-1)}
\]

(58)

where \(C\) is some deterministic constant and the last inequality is due to the local limit theorem for lattice distribution under \(P_{\pi q}^x\), see [1, Theorem 22.1].

Letting

\[B_m = A_m \cap \{\tilde{\beta}_m \leq n^\mu \} \cap R_m^{(1)} \cap R_m^{(2)},\]

and repeating the estimate (58) without change for \(\mathbb{P}_{\omega^{(1)}, \omega^{(1)}, \varepsilon^{(1)}, \varepsilon^{(2)}}^0\) replacing \(\mathbb{P}_{\omega^{(1)}, \omega^{(1)}, \varepsilon^{(1)}, \varepsilon^{(2)}}^0\), and using that \((d-1)(\theta/2 - \theta') > 1\), we conclude that

\[
\max \left( \sum_{m=1}^{\infty} P^\pi \otimes P^\pi \otimes Q_\varepsilon \otimes Q_\varepsilon \left( \mathbb{P}_{\omega^{(1)}, \omega^{(1)}, \varepsilon^{(1)}, \varepsilon^{(2)}}^0 \left( B_m^c \right) \right) \right),
\]

(59)

\[
\sum_{m=1}^{\infty} P^\pi \otimes P^\pi \otimes Q_\varepsilon \otimes Q_\varepsilon \left( \mathbb{P}_{\omega^{(1)}, \omega^{(1)}, \varepsilon^{(1)}, \varepsilon^{(2)}}^0 \left( B_m^c \right) \right) < \infty
\]

Next, for \(j = 1, 2\), we construct, using the recipe in Section 3 regeneration times \(\tau_{i}^{(j)}\) corresponding to the walks \((X_{t}^{(j)})_{t \geq \beta_m}\) and the chain on the environment.

Note that under the measure \(P^\pi \otimes Q_\varepsilon \otimes Q_\varepsilon \otimes \mathbb{P}_{\omega^{(1)}, \omega^{(1)}, \varepsilon^{(1)}, \varepsilon^{(2)}}^0\), the sequences for \(j = 1\) and \(j = 2\) are not independent if the walks intersect. When no confusion occurs, we let \(P^\pi\) denote the law of this sequence, noting that the sequence \((\tau_{i+1}^{(j)} - \tau_{i}^{(j)})_{i \geq 1}\) is i.i.d. under all measures involved in our construction, as well as, equation (24) holds. Fix an integer \(k \geq 1\), for any fixed \(2 \leq \gamma' < \gamma\) we then have

\[
P^\pi \left( \left| \tau_k^{(j)} \right| - k E^\pi [\tau_1^{(j)}] \right) > k/2
\]

\[
= P^\pi \left( \left| \tau_k^{(j)} \right| - k E^\pi [\tau_1^{(j)}] \right)^{\gamma'} > (k/2)^{\gamma'}
\]

\[
\leq 2^{\gamma'} E^\pi \left[ \left| \tau_k^{(j)} \right| - k E^\pi [\tau_1^{(j)}] \right]^{\gamma'}
\]

\[
\leq \frac{2^{\gamma'} B_{\gamma'} E^\pi \left[ \left| \tau_1^{(j)} \right| - E^\pi [\tau_1^{(j)}] \right]^{\gamma'}}{k^{\gamma'/2}}
\]

(60)
where the last inequality follows from and application of Marcynkiewicz-Zygmund inequality \[10, \text{Pg. 469}\] along with Hölder inequality, and \(B_{\gamma'} > 0\) is a universal constant which depends on \(\gamma'\). Recall, \(\gamma''/2 > 1\) and also \(\gamma > 6\), thus we can choose \(2 \leq \gamma' < \gamma\) such that \(\gamma''/2 > 1\). Now choosing \(k = c_5 m^\theta\), for an appropriate \(c_5\), one concludes the existence of a constant \(c_6\) such that

\[
\sum_{m=1}^\infty P^\tau \left( e^{(j)}_{c_5 m^\theta} > m^\theta/4 \right) < \infty. \quad (61)
\]

On the other hand,

\[
P^\tau \left( \exists i > c_6 m^\theta : \tau_{i+1}^{(j)} - \tau_i^{(j)} > i^\alpha \right) \\
\leq \sum_{i = [c_6 m^\theta] + 1}^\infty P^\tau \left( \tau_{i+1}^{(j)} - \tau_i^{(j)} > i^\alpha \right) \\
= \sum_{i = [c_6 m^\theta] + 1}^\infty P^\tau \left( \tau_{i+1}^{(j)} > i^\alpha \right) \\
\leq \sum_{i = [c_6 m^\theta] + 1}^\infty \frac{C}{i^\gamma} 
\]

for some deterministic constant \(C\), and \(\gamma' < \gamma\). Since our choice of constants implies that \(\theta' (\alpha \gamma - 1) > 1\), we can choose \(\gamma' < \gamma\) such that \(\theta' (\alpha \gamma' - 1) > 1\), and so we conclude that

\[
\sum_{m=1}^\infty P^\tau \left( \exists i > c_6 m^\theta : \tau_{i+1}^{(j)} - \tau_i^{(j)} > i^\alpha \right) < \infty \quad (62)
\]

Let \(c_m^{(j)} = \{ e^{(j)}_{c_6 m^\theta} \leq m^\theta/4 \} \cap \{ \forall i > c_6 m^\theta, \, \tau_{i+1}^{(j)} - \tau_i^{(j)} \leq i^\alpha \} \), and note that for all measures \(P^\tau\) involved,

\[
\sum_{m=1}^\infty P^\tau \left( c_m^{(1)} \cap c_m^{(2)} \right) < \infty. \quad (63)
\]

Next, for \(j = 1, 2\), define \(S_{s}^{(j)} = \bar{X}_r^{(j)}\). Recall that \(\bar{X}_r^{(j)}\) is the orthogonal projection of the vector \(X_r^{(j)}\) into the space \(L^{d-1}\). Note that \(X_r^{(1)} \cap X_r^{(2)} \neq \emptyset\) and \(A_m \cap C_{m}^{(1)} \cap C_{m}^{(2)}\)

\[
\subset \{ |S_0^{(1)} - S_0^{(2)}| \geq m^\theta, \exists \ell, k : |S_\ell^{(1)} - S_k^{(2)}| \leq \ell^\alpha + k^\alpha \} 
\]

Let \(Z_i\) denote a sum of (centered) \(t\) i.i.d. random vectors, each distributed according to \(S_1^{(1)}\), thus taking values in \(L^{d-1}\). Let \((Z_t)^{i \geq 1}\) denote an independent copy of \((Z_t)^{i \geq 1}\). Let \(P^x_P^y\) denote the law of the sequences \(((x + Z_i), (y + Z_i))\).
$Z(t)$, where $x, y \in \mathbb{L}^{d-1}$. Using that under all the measures involved in our computation, the paths $X^{(1)}$ and $X^{(2)}$ are independent until they intersect, one concludes that

$$
P^\pi \otimes P^\pi \otimes Q_\omega \otimes Q_\omega \left( \mathbb{P}_{\omega^{(1)}, \omega^{(2)}, \epsilon^{(1)}, \epsilon^{(2)}} \left( \{X^{(1)}_{[\bar{\beta}_m, \infty]} \cap X^{(2)}_{[\bar{\beta}_m, \infty]} \neq \emptyset \} \cap A_m \cap C^{(1)}_m \cap C^{(2)}_m \right) \right)
\leq \max_{|x| \geq m^{\gamma'}} P^0_Z \left( \exists \ell, k : |Z_\ell - Z'_k| \leq \ell^{\alpha} + k^{\alpha} \right),
\text{with exactly the same estimate when } \mathbb{P}_{\omega^{(1)}, \omega^{(2)}, \epsilon^{(1)}, \epsilon^{(2)}} \text{ replaces } \mathbb{P}_{\omega^{(1)}, \omega^{(2)}, \epsilon^{(1)}, \epsilon^{(2)}}.
$$

But, for some constant $C_\alpha$ whose value may change from line to line,

$$
\max_{|x| \geq m^{\gamma'}} P^0_Z \left( \exists \ell, k : |Z_\ell - Z'_k| \leq \ell^{\alpha} + k^{\alpha} \right)
\leq \max_{|x| \geq m^{\gamma'}} P^0_Z \left( \exists \ell, k : |Z_\ell - Z'_k| \leq C_\alpha (\ell + k)^{\alpha} \right)
\leq \max_{|x| \geq m^{\gamma'}} P^0_Z \left( \exists \ell, k : (\ell + k) < m^{2\gamma''}, |Z_\ell - Z'_k| \leq C_\alpha (\ell + k)^{\alpha} \right)
+ \max_{|x| \geq m^{\gamma'}} P^0_Z \left( \exists \ell, k : (\ell + k) \geq m^{2\gamma''}, |Z_\ell - Z'_k| \leq C_\alpha (\ell + k)^{\alpha} \right)
=: I_1(m) + I_2(m).
$$

For $I_1(m)$ in (66), we observe that because $\alpha < 1/2$, for any $\gamma' < \gamma$, we can find a constant $C_{\gamma'}$ such that for large $m$

$$
\max_{|x| \geq m^{\gamma'}} P^0_Z \left( \exists \ell, k : (\ell + k) < m^{2\gamma''}, |Z_\ell - Z'_k| \leq C_\alpha (\ell + k)^{\alpha} \right)
\leq 2 P^0_Z \left( \max_{\ell \leq m^{2\gamma''}} |Z_\ell| \geq \frac{m^{\gamma'}}{3} \right)
\leq \frac{C_{\gamma'}}{m^{(\theta' - \theta'') \gamma}},
$$

where the first inequality follows from the observation that

$$
|Z_\ell - Z'_k| \geq |x| - |Z_\ell| - |Z'_k - x|,
$$

and the last inequality is due again to the Marcinkiewicz-Zygmund inequality coupled with the Martingale maximal inequality of Burkholder and Gundy [10], Pg. 469. Therefore, since $\theta''$ was chosen such that $(\theta' - \theta'') \gamma > 1$, it follows that

$$
\sum_m I_1(m) < \infty.
$$
Turning to the estimate of the term $I_2(m)$ in (122), we observe, with the value of the constants $C_\alpha, C'_\alpha$ possibly changing from line to line, that

$$
\sum_{\ell \geq k} \max_{|x| \geq m^{\theta''}} \mathbb{E}_Z^{0,x} \left( |Z_\ell - Z'_k| \leq C_\alpha (\ell + k)^\alpha \right)
$$

$$
= \sum_{\ell \geq k} \max_{|x| \geq m^{\theta''}} \mathbb{E}_Z^{0,x} \left[ P_Z^{0,x} \left( |Z_\ell - Z'_k| \leq C_\alpha (\ell + k)^\alpha \right) \right]
$$

$$
\leq \sum_{\ell \geq k} \max_{|x| \geq m^{\theta''}} C'_\alpha (\ell + k)^{(d-1)} \mathbb{E}_Z^{0,x} \left[ \max_{|x - Z'_k| \leq C_\alpha (\ell + k)^\alpha} \mathbb{P}_Z^{0,x} (Z_\ell = z) \right]
$$

$$
\leq \sum_{\ell \geq k} C'_\alpha (\ell + k)^{(d-1)} \frac{1}{(\ell + k)^{(d-1)/2}}
$$

where the last but one inequality follows by applying the local limit estimate [11 Theorem 22.1]. By symmetry, the same estimate holds with $k \geq \ell$ replacing the constraint $\ell \geq k$ in the summation. Thus we conclude that

$$
I_2(m) \leq \sum_{\ell,k} \max_{|x| \geq m^{\theta''}} \mathbb{E}_Z^{0,x} \left( |Z_\ell - Z'_k| \leq C_\alpha (\ell + k)^\alpha \right)
$$

$$
\leq 2 \sum_{\ell \geq k} C'_\alpha (\ell + k)^{(d-1)} \frac{1}{(\ell + k)^{(d-1)/2}}
$$

$$
\leq \sum_{r = \lfloor m^{\theta''} \rfloor + 1} \infty \frac{2C'_\alpha}{r^{d-2-\alpha(d-1)-1}} \leq \frac{2C'_\alpha}{m^{\theta''(d-2\alpha d + 2\alpha - 5)}} \tag{69}
$$

In particular, since $\theta''(d - 2\alpha d + 2\alpha - 5) > 1$, one concludes that

$$
\sum m I_2(m) < \infty \tag{70}
$$

Combining (68) and (70), and substituting in (66) and then in (65), the conclusion of the lemma follows.

Finally for the case $\nu = 0$ we can proceed in two ways. We could chose $V^\perp$ as an arbitrary subspace of co-dimension 1, and simply repeat the argument. Alternatively, we could proceed exactly the same manner except that one works directly with the walkers, rather than their projections. In this case we can replace $(d - 1)$ by $d$ in all the estimates above and the conclusions follows by
observing that the inequalities in (49) holds when \((d - 1)\) is replaced by \(d\). The advantage of the second approach is then that when \(v = 0\), one may replace the constraint on \(d\) in (48) by the weaker constraint involving \(d + 1\).

This completes the proof of the lemma.

\[\]

5 Final Remarks and Open Problems

5.1 The “regeneration” time

We point out that our definition of the “regeneration” time \(\tau_1\) using the coin tosses \(\{(\alpha_t(\cdot))_{t\geq 1}\}\), is quite arbitrary, and was tailored to Condition (A1). What we actually need for the argument is for every environment chain \((\omega_t(\cdot, \cdot))_{t\geq 0}\) a sequence of stopping times such that at these times, the chain starts from a stationary distribution and the times have “good” tail property. The assumption (A1) is used to ensure this is possible with our construction.

5.2 The annealed CLT

Our argument gives a trade-off between the strength of the random perturbation and the mixing rate of the environment. We suspect that such a trade-off is not needed, and that for \(d \geq 3\), an annealed CLT holds true as soon as the environment is mixing enough in time. Our technique does not seem to resolve this question.

5.3 The quenched CLT

We have already emphasized that the condition (10) is not expected to be optimal. In particular, we suspect that if a critical dimension for the quenched CLT exists, it will be \(d = 1\) (see also the comments at the end of [5] hinting that such quenched CLT fails even for small perturbations of a fixed Markovian environment, and the numerical simulations in [3] that are inconclusive).

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