Collective Spin-Hall Effect for Electron-Hole Gratings

Ka Shen and G. Vignale

Department of Physics and Astronomy, University of Missouri, Columbia, Missouri 65211, USA

(Dated: May 7, 2014)

We show that an electric field parallel to the wavefronts of an electron-hole grating in a GaAs quantum well generates, via the electronic spin Hall effect, a spin grating of the same wave vector and with an amplitude that can exceed 1% of the amplitude of the initial density grating. We refer to this phenomenon as "collective spin Hall effect". A detailed study of the coupled-spin charge dynamics for quantum wells grown in different directions reveals rich features in the time evolution of the induced spin density, including the possibility of generating a helical spin grating.

The spin Hall effect (SHE), i.e., the generation of a transverse spin current from a charge current and vice versa, has attracted much attention in the past decade [1–10], and has now become one of the standard tools for the generation and detection of spin currents in magnetoelectronic devices [11–13]. Theoretically, both intrinsic and extrinsic mechanisms have been shown to contribute to the SHE in semiconductors. While the intrinsic mechanism originates from the spin-orbit coupling (SOC) in the band structure [1, 2], the extrinsic one results from the SOC with impurities [1, 3]. Experimentally, the first evidence of SHE in semiconductors was the observation of a spin accumulation at the edges of n-doped GaAs [3]. This is clearly a single-particle effect taking place in a macroscopically homogeneous sample. Recently, Anderson et al. [16] have proposed an interesting collective manifestation of the SHE in a periodically modulated electron gas. They suggested that an optically induced spin density wave (transient spin grating [17–19]) in a two-dimensional electron gas could be partially converted into a density wave when an electric field perpendicular to the grating wave vector is applied.

There are some difficulties with the implementation of this idea. First of all, the electric field due to the induced charge density, when properly taken into account, effectively prevents the accumulation of charge. Second, the SOC considered in that work comes solely from band structure (i.e., it is purely "intrinsic") and, for this reason, the spin-charge coupling is found to be of third order in the, presumably small, strength of the SOC.

In the present work, we re-examine the coupled spin-density transport in a periodically modulated electron gas in a novel set-up which is free of the above-mentioned difficulties. Differing from Ref. [16], we start from an electrically neutral electron-hole grating (uniform spin density) in an n-type semiconductor quantum well and show that an electric field parallel to the wavefronts of the grating generates, via SHE, a periodic spin modulation of the same wave vector as the initial electron-hole grating (see Fig. 1). Since any local charge imbalance is screened quickly by the background electrons, we can safely assume that the system remains charge-neutral throughout its evolution and, in particular, no additional electric field is created. Furthermore, going beyond the treatment of

\[ H_0 = \frac{k^2}{2m_e} + \frac{1}{m} k \cdot A, \]  

where \( A = m(\lambda_1 \sigma_y + \gamma_y \sigma_z, \lambda_2 \sigma_x - \gamma_x \sigma_z) \) is the spin-dependent vector potential that describes SOC. Specifically, if \( \alpha \) and \( \beta \) denote the Rashba [26, 27] and Dresselhaus coefficients [28], we have \( \lambda_1 = \beta + \alpha, \lambda_2 = \beta - \alpha \).
and $\gamma_i = \frac{\lambda_i^2}{\hbar}eE_i$ in a (001) QW (the $x$ and $y$ axes are in the $[110]$ and $[\bar{1}0\bar{0}]$ directions, forming 45° angles with the cubic axes), while $\lambda_1 = \alpha$, $\lambda_2 = -\alpha$, $\gamma_x = \frac{\lambda_x^2}{\hbar}eE_x$ and $\gamma_y = \frac{\lambda_y^2}{\hbar}eE_y - \beta$ in a (110) QW. The terms containing the effective Compton wavelength $\lambda_c$ (~4.6 Å in GaAs) describe the SOC from the applied electric field. The Hamiltonian for the (heavy) holes has a similar form with a different effective mass $m_h$. In this case, however, we assume that the spin polarization is quenched, due to strong spin-orbit interaction in the valence band, on a time scale that is shorter than that of the diffusion process. For this reason, no spin-dependent terms are included for the holes.

Our analysis is based on the quantum kinetic equation for the density matrix $\rho_k(r)$ of electrons

$$\partial_t \rho_k + \frac{1}{2} \{ \nabla_k H_0, \nabla_r \rho_k \} + i[H_0, \rho_k] = \partial_t \rho_k|_{\text{scat}}, \quad (2)$$

where $\nabla_r = \nabla_r + eE \partial_r$. In the relaxation time approximation, the scattering term on the right-hand side is

$$\partial_t \rho_k|_{\text{scat}} = -\frac{\rho_k}{\tau} + \frac{\rho_k}{2m}\lambda^2_{\text{scat}}(k \cdot A, \partial_t \rho_k)$$

$$-\frac{1}{2} \alpha_{ss} \sum_{|k'|=|k|} \{ k \times k' \cdot \sigma, \rho_{k'} \}, \quad (3)$$

where $\rho_k = \rho_k$ is the momentum-space angular average of the density matrix. The last term on the right-hand side of Eq. (3) is the skew scattering term with the coefficient $\alpha_{ss} = \frac{\hbar}{8\pi m^2 n_i} \lambda^2_{\text{scat}} \left( \frac{m \alpha_{ss}}{\hbar} \right)^3$, where $n_i$ and $u_i$ are the density and the scattering potential of impurities, respectively. The third term on the right-hand side of Eq. (3), which effectively amounts to shifting the argument of $\rho_k$ from $k$ to $k + A$, is critically important to ensure the vanishing of the spin-charge coupling to linear order in SOC. From Eq. (2) we derive coupled equations of motion for the inhomogeneous density and spin density of electrons and the density of holes. The spin density of the holes is assumed to be zero.

(110) quantum well – For orientation, let us begin with the simplest case, namely a symmetric (110) GaAs QW. What makes this system most interesting from our perspective is that the Dresselhaus effective magnetic field is along the $z$-direction, and therefore preserves the $z$-component of the electron spin, $S_z$. The intrinsic SHE is completely absent. The extrinsic SHE, embodied in the skew-scattering term, is present and clearly conserves $S_z$. Therefore, we can write down separate kinetic equations for spin-up and spin-down electrons:

$$\partial_t n_{\sigma k} + \frac{1}{m} (k \cdot \nabla_r) n_{\sigma k} + eE \cdot \nabla_k n_{\sigma k}$$

$$= -\frac{n_{\sigma k} - n_{\bar{\sigma} k}}{\tau} - \sigma \alpha_{ss} \sum_{|k'|=|k|} (k \times k' \cdot \hat{z}) n_{\sigma k'}, \quad (4)$$

with $\sigma = +, -$ representing spin up and down with respect to $z$-direction, respectively. Following the standard procedure we substitute the “first-order solution”

$$n_{\sigma k} \approx \bar{n}_{\sigma k} - \sigma \alpha_{ss} \sum_{|k'|=|k|} (k \times k' \cdot \hat{z}) \tilde{n}_{\sigma k'}, \quad (5)$$

where $\tilde{n}_{\sigma k} \equiv (1 - \frac{m}{2\tau} \nabla_r - e \tau \text{E} \cdot \nabla_k) n_{\sigma k}$, into Eq. (4), and sum over $k$ to obtain the diffusion equation

$$\partial_t n_{\sigma} - D \nabla^2 n_{\sigma} + v_d \cdot \nabla_r n_{\sigma} - \sigma v_{ss} \cdot \nabla n_{\sigma} = 0, \quad (6)$$

where $n_{\sigma} = \sum_k n_{\sigma k}$ is the total density of electron with spin $\sigma$ and $D = \frac{\hbar^2}{2m \tau} \sigma$ is the diffusion constant. The drift velocity and spin-Hall drift velocity are given by $v_d = \frac{eE}{m}$ and $v_{ss} = 2\sigma \alpha_{ss} \tau D m (E \times \hat{z})$, respectively. We then combine the two equations of different spins and get coupled kinetic equations for the total density ($N = n_+ + n_-$) and the total spin polarization ($S_z = n_+ - n_-$):

$$\partial_t - D \nabla^2 n + v_d \cdot \nabla_r N - v_{ss} \cdot \nabla S_z = 0, \quad (7)$$

$$\partial_t - D \nabla^2 v_d \cdot \nabla_r S_z - v_{ss} \cdot \nabla N = 0. \quad (8)$$

Notice the appearance of a spin-density coupling, which occurs only in a non-uniform system and is proportional to the skew-scattering drift velocity – a quantity of first order in the SOC strength. The equation for the hole density is similar to Eq. (7), with $D$ and $v_d$ replaced by...
the corresponding quantities for the holes, but without the last term, because the spin polarization of the holes is neglected. In fact, the last term can also be neglected on the left-hand side of Eq. (4) for the electrons, since it leads to minute corrections to the evolution of the density. By imposing the local neutrality condition, that is, assuming that the electron density is always equal to the hole density, we combine the diffusion equations for electrons and holes into ambipolar diffusion and spin-density transport equations

\[
\begin{align*}
\left( \partial_t - D_e \nabla^2 + \Gamma \right) N &= 0, \\
\left( \partial_t - D_s \nabla^2 \right) S_z - \mathbf{v}_{ss} \cdot \nabla N &= 0,
\end{align*}
\]

(9) \hspace{1cm} (10)

where \( D_e \) and \( D_s \) represent the ambipolar and spin diffusion constants, respectively. Here, we have introduced the rate \( \Gamma \) of electron-hole recombination. All these terms can be of comparable magnitude in real systems. (ii) At variance with Ref. 16, our diffusion matrix is non-symmetric: \( D_{ij} \neq D_{ji} \). This lack of symmetry comes from a careful consideration of the operatorial character of \( \mathbf{q} \), whereby \( \epsilon_k \neq \epsilon_k \mathbf{q} \), as explained in the supplemental material 34. Eqs. (14-15) are our main theoretical result: they combine extrinsic and intrinsic contributions to the SHE as well as spin precession, and reduce to the results of the previous sec-

at \( t = (D_0 q^2 + \Gamma - D_s q^2)^{-1} \ln \left( \frac{D_0 q^2 + \Gamma}{D_0 q^2} \right) \). Noting that the quantity within the round brackets is of order 1, we see that the amplitude ratio is roughly the fraction of the grating wavelength covered by an electron that travels at the skew-scattering drift velocity \( (v_{ss}) \) during the diffusion lifetime of the grating \( (1/D_0 q^2) \). Not surprisingly, this ratio shows a non-monotonic dependence on \( q \), reaching a maximum \( A_{\text{max}}^\text{max} \approx 1.4 \times 10^{-2} \) at the optimal wave vector \( q^{\text{opt}} \approx 0.2 \mu \text{m}^{-1} \), with the material parameters listed in the caption of Fig. 2.

(001) quantum well – In a (001) QW, the presence of the in-plane effective magnetic field due to band SOC and the non-conservation of \( S_z \) lead to more complex scenarios. To begin with, the coupling of longitudinal and transverse spin fluctuations leads to a set of drift-diffusion equations of the form

\[
\partial_t (\Delta N, S_x, S_y, S_z)^T = -\mathbf{D}(\mathbf{q}) (\Delta N, S_x, S_y, S_z)^T,
\]

(14)

where \( \mathbf{D}(\mathbf{q}) \) is the 4x4 drift-diffusion matrix acting on the column vector of the Fourier amplitudes of the density at wave vector \( \mathbf{q} \). Here \( \mathbf{q} := \mathbf{q} - ie\mathbf{E} \partial_k \) is a momentum-space operator, which takes into account drift under the action of the electric field \( \mathbf{E} \). Without going into technical details we only summarize the salient results (for details, see Ref. 34). Taking \( \mathbf{q} = q \hat{x} \) and \( \mathbf{E} = E \hat{y} \) and assuming \( \frac{k_F q}{m} \ll 1 \) and \( |\alpha \pm \beta|k_F \ll E_F \) (conditions that define the diffusive regime) we find

\[
\mathbf{D}(\mathbf{q}) = \begin{pmatrix}
Dq^2 & -\frac{1}{2}\tau \lambda_2 q^2 \mathbf{v}_d \\
-\frac{1}{2}\tau \lambda_2 q^2 \mathbf{v}_d + 4\tau \lambda_2 (\gamma_0 + \gamma_y) \mathbf{v}_d & Dq^2 + \frac{1}{\tau_{xx}} \\
-4i\tau D\lambda_1 \lambda_2 q + i\tau \lambda_2 \gamma_y q \mathbf{v}_d & -iq_{ss} \partial \mathbf{\lambda}_2 \mathbf{v}_d \\
-4i\tau D\lambda_1 \lambda_2 q + i\tau \lambda_2 \gamma_y q \mathbf{v}_d - iq_{ss} \partial \mathbf{\lambda}_2 \mathbf{v}_d & 4iD\gamma_y q \\
-4i\tau D\lambda_1 \lambda_2 q - i\tau \lambda_1 \lambda_2 q \mathbf{v}_d & -4i\partial \mathbf{\lambda}_1 \gamma_y - 2\lambda_2 \mathbf{v}_d \\
-4i\tau D\lambda_1 \lambda_2 q - i\tau \lambda_1 \lambda_2 q \mathbf{v}_d - iq_{ss} \partial \mathbf{\lambda}_2 \mathbf{v}_d & Dq^2 + \frac{1}{\tau_{xx}}
\end{pmatrix}
\]

(15)

where \( \mathbf{v}_d = \frac{e\mathbf{E}}{m} \). Here, \( \mathbf{\lambda}_1 = m\lambda_1, \gamma_i = m\gamma_i, \frac{1}{\tau_{xx}} = 4D(\lambda_1^2 + \gamma_0^2), \frac{1}{\tau_y} = 4D(\lambda_2^2 + \gamma_y^2) \), and \( \frac{1}{\tau_z} = 4D(\lambda_1^2 + \lambda_2^2) \).

We note that our diffusion matrix differs from the one reported in Ref. 16 in two ways: (i) in addition to the “standard” terms linear in \( \mathbf{q} \) and cubic in the SOC strength, we include terms of second order in both \( \mathbf{q} \) and the SOC strength as well as terms of third order in \( \mathbf{q} \) and first order in SOC. All these terms can be of comparable magnitude in real systems. (ii) At variance with Ref. 16, our diffusion matrix is non-symmetric: \( D_{1i} \neq D_{1i} \). This lack of symmetry comes from a careful consideration of the operatorial character of \( \mathbf{q} \), whereby \( \epsilon_k \neq \epsilon_k \mathbf{q} \), as explained in the supplemental material 34. Eqs. (14-15) are our main theoretical result: they combine extrinsic and intrinsic contributions to the SHE as well as spin precession, and reduce to the results of the previous sec-

In Fig. 2, we plot the time evolution of the induced-spin gratings as well as the density grating. One can see that the amplitude of the spin grating initially increases and then begins to decrease after a maximum around 1% the amplitude of the initial density grating. The induced spin gratings show a \( \frac{\pi}{2} \) phase shift from the density grating. From Eq. (12), we see that, for a given \( q \), \( S_z \) reaches the maximal value

\[
A_{\text{max}}^\text{max}(q) = \frac{v_{ss} q}{D_0 q^2 + \Gamma} \left( \frac{D_0 q^2 + \Gamma}{D_0 q^2} \right) \frac{D_0 q^2}{v_{ss}^2 + \Gamma - D_0 q^2},
\]

(13)
tion if the intrinsic SOCs appropriate for (110) QW are used.

(001) quantum well with balanced SOC – The case of a (001) QWs with identical Dresselhaus and Rashba coefficients, $\alpha = \beta$ (corresponding to the condition $\lambda_2 = 0$) with $\mathbf{q}$ oriented along the [110] direction gives us the opportunity to demonstrate a particularly interesting application of Eqs. (14-15). Just as in a symmetric (110) QW, only the skew scattering contributes to the collective SHE, but now $S_z$ is not conserved. Since $\gamma_q$ is negligibly small (two orders smaller than the band SOC), the $S_y$ component decouples from the $S_x$ and $S_z$ components and the diffusion matrix reduces to

$$D(q) = \begin{pmatrix} Dq^2 & 0 & -iqv_{ss} \\ q_0v_{ss} & D(q^2 + q_0^2) & 0 \\ 0 & 0 & 2iDq_0 \end{pmatrix},$$

(16)

with $q_0 = \frac{4n\beta}{\hbar^2} \approx 3.5 \mu m^{-1}$. After imposing the charge-neutrality condition, the diffusion equations for the density and the two helical components of the spin density $S_\pm = \frac{1}{\sqrt{2}}(S_x \pm iS_z)$ are found to be

$$\partial_t \Delta N = -D_q\alpha^2\Delta N - \frac{1}{\sqrt{2}}qv_{ss}S_- + \frac{1}{\sqrt{2}}qv_{ss}S_+,$$

(17)

$$\partial_t S_- = -\frac{1}{\sqrt{2}}\left(q + q_0\right)v_{ss}N - D_s\left(q + q_0\right)^2S_-,$$

(18)

$$\partial_t S_+ = -\frac{1}{\sqrt{2}}\left(q - q_0\right)v_{ss}N - D_s\left(q - q_0\right)^2S_+.$$

(19)

As in the previous calculations, we neglect the last two terms on the right-hand side of Eq. (17). Then the solution for the density reduces to a simple diffusion process, and the solution for the two helical modes is given by

$$S_\pm = \frac{1}{\sqrt{2}}A_0e^{iqv_{ss}t}\left[e^{-\left(D_s\eta_0^2 - D_sq_0^2\right)t} - 1\right]e^{-D_sq_0^2t},$$

(20)

which yields the spin-polarization

$$S_x = \pm \sum_{\pm} \frac{q_0v_{ss}}{D_s\eta_0^2 - D_sq_0^2}\left[e^{-D_s\eta_0^2t} - e^{-D_sq_0^2t}\right],$$

(21)

$$S_z = -\sum_{\pm} \frac{q_0v_{ss}}{D_s\eta_0^2 - D_sq_0^2}\left[e^{-D_s\eta_0^2t} - e^{-D_sq_0^2t}\right].$$

(22)

These amplitudes show a strong dependence on the wave vector. One can see that the contributions from the $S_-$ mode is proportional to $q_+ = q + q_0$ while the contribution from the $S_+$ mode is proportional to $q_- = q - q_0$ (the correspondence is reversed if we switch the sign of $q$). Further, the $S_+$ mode is long-lived, due to the slowly decaying term $e^{-D_sq_0^2t}$, while the $S_-$ mode is short-lived $\frac{1}{q_0^2}$. The long-time behavior of $S_z$ being dominated by the $S_\pm$ component, is positive for $q > q_0$ and negative for $q < q_0$. In the special case $q = q_0$ – a practically realizable case – $S_z$ vanishes identically, and the amplitude of $S_z$ decays to zero most rapidly. In this case, the skew scattering converts the initial density grating into a helical wave of wave vector $q_0$. Further interpretation of this intriguing effect, based on an

SU(2) gauge transformation that eliminates the intrinsic SOC [22, 23], is given in Ref. [54].

In Fig. 3, we plot the amplitude of the $z$-component of the electron spin, $S_z$, as function of time at a distance $x = -0.25L$ from a peak of the density grating. Notice the reversal of sign of the long-time behavior and the quick decay of the signal at $q = q_0$, due to the vanishing of the $S_+$ mode.

In summary, we have studied the collective spin Hall effect in a periodically modulated electron gas in the presence of an in-plane electric field perpendicular to the wave vector of the initial density modulation. In the symmetric (110) quantum well the amplitude of the induced spin density is controlled solely by skew scattering and can be as large as 1% of that of the initial density modulation. This should be observable in state-of-the-art experiments [17-19]. Similarly, the collective spin Hall effect in (001) QWs with identical Rashba and Dresselhaus SOC strengths is also entirely controlled by skew scattering. In this case, the skew scattering generates a spiral spin density wave when the wave vector of the initial grating matches the wave vector of the spin-orbit coupling.

We gratefully acknowledge support from NSF Grant No. DMR-1104788.
[7] S. Murakami, Adv. Solid State Phys. 45, 197 (2005).

[8] J. Wunderlich, B. Kaestner, J. Sinova, and T. Jungwirth, Phys. Rev. Lett. 94, 047204 (2005).

[9] H.-A. Engel, E. I. Rashba, and B. I. Halperin, in *Handbook of Magnetism and Advanced Magnetic Materials*, edited by H. Kronmüller and S. Parkin (Wiley, Chichester, UK, 2007), vol. V, pp. 2858-2877.

[10] E. M. Hankiewicz and G. Vignale, J. Phys.: Condens. Matter 21, 253202 (2009).

[11] I. Zutic, J. Fabian, and S. DasSarma, Rev. Mod. Phys. 76, 323 (2004).

[12] J. Fabian, A. Matos Abague, C. Ertler, P. Stano, and I. Zutic, Acta Physica Slovaca 57, 565 (2007).

[13] D. Awschalom and E. M. Flatté, Nature Physics 3, 153 (2007).

[14] M. W. Wu, J. H. Jiang, and M. Q. Weng, Phys. Rep. 493, 61 (2010).

[15] L. Liu, O. J. Lee, T. J. Gudmundsen, D. C. Ralph, and R. A. Buhrman, Phys. Rev. Lett 109, 096602 (2012).

[16] B. Anderson, T. D. Stanescu, and V. Galitski, Phys. Rev. B 81, 121304R (2010).

[17] A. R. Cameron, P. Riblet, and A. Miller, Phys. Rev. Lett. 76, 4793 (1996).

[18] C. P. Weber, N. Gedik, J. E. Moore, J. Orenstein, J. Stephens, and D. D. Awschalom, Nature 437, 1330 (2005).

[19] L. Yang, J. D. Koralek, J. Orenstein, D. R. Tibbetts, J. L. Reno, and M. P. Lilly, Nature Phys. 8, 153 (2012).

[20] That is, if only the spin-density coupling that is linear in the wave vector $q$ of the grating is considered.

[21] A. A. Burkov, A. S. Nunez and A. H. MacDonald, Phys. Rev. B 70, 155308 (2004).

[22] E. G. Mishchenko, A. V. Shytov, and B. I. Halperin, Phys. Rev. Lett. 93, 226602 (2004).

[23] B. A. Bernevig, J. Orenstein and S. C. Zhang, Phys. Lett. Phys. 97, 236601 (2006).

[24] T. D. Stanescu and V. Galitski, Phys. Rev. B 75, 125307 (2007).

[25] J. Smit, Physica 24, 39 (1958).

[26] E. I. Rashba, Sov. Phys. Solid State 2, 1109 (1960).

[27] Y. A. Bychkov and E. I. Rashba, J. Phys. C: Solid State Phys. 17, 6039 (1984).

[28] G. Dresselhaus, Phys. Rev. 100, 580 (1955).

[29] J. L. Cheng and M. W. Wu, J. Phys.: Condens. Matter 20, 085209 (2008).

[30] R. Raimondi, P. Schwab, C. Gorini, and G. Vignale, Ann. Phys. 524, 153 (2012).

[31] We have temporarily reinstated $\hbar$ to point out that $\alpha_{ss}$ has the dimensions of a diffusion constant.

[32] We point out that the correction due to SOC with the in-plane electric field accounts for only one half of the so-called side-jump contribution to the extrinsic spin Hall effect.

[33] D. Culcer, E. M. Hankiewicz, G. Vignale, and R. Winkler, Phys. Rev. B 81, 125332 (2010).

[34] See the Supplemental Material File for a discussion of technical details relevant to the main text.

[35] C. P. Weber, J. Orenstein, B. A. Bernevig, S. C. Zhang, J. Stephens, and D. D. Awschalom, Phys. Rev. Lett. 98, 076604 (2007).

[36] M. Q. Weng, M. W. Wu, and H. L. Cui, J. Appl. Phys. 103, 063714 (2008).

[37] V. A. Slipko, I. Savran, and Y. V. Pershin, Phys. Rev. B 83, 193302 (2011).

[38] M. P. Walser, C. Reichl, W. Wegscheider, and G. Salis, Nature Phys. 8, 757 (2012).

[39] I. V. Tokatly and E. Ya. Sherman, arXiv:1302.2121
1. DERIVATION OF DRIFT-DIFFUSION EQUATION IN (001) GAAS QUANTUM WELLS

By defining the $x$ and $y$ axes in the [110] and [ ¯110] direction separately, we express the Hamiltonian of electron gas in (001) GaAs QWs as

$$H_0 = \frac{k^2}{2m_e} + k_x(\lambda_1\sigma_y + \gamma_y\sigma_z) + k_y(\lambda_2\sigma_x - \gamma_x\sigma_z),$$

where $\lambda_1 = \beta + \alpha$ and $\lambda_2 = \beta - \alpha$ with $\alpha$ and $\beta$ being the Rashba and Dresselhaus coefficients. $\gamma_i = \frac{\hbar e}{m_e}E_i$ describes the SOC due to the in-plane electric field with $\lambda_i$ corresponding to the effective Compton wavelength.

In the relaxation time approximation, the kinetic equation for the local electron density matrix $\rho_k(r)$ is given by $[3, 2]$

$$\frac{\partial}{\partial t}\rho_k + \frac{1}{2}(\nabla_k H_0, \nabla_r \rho_k) + i[H_0, \rho_k] = -\rho_k \frac{\partial}{\partial t} + \frac{1}{2m_e} \{k \cdot A, \partial_{\mu} \rho_k\} - \frac{1}{2} \alpha_{ss} \sum_{|k'|=|k|} \{k \times k' \cdot \sigma, \rho_{k'}\},$$

where $\nabla_r = \nabla_r + eE\partial_{\mu}$. The last term on the right-hand side is the skew scattering term from the second-order Born approximation $[3, 4]$. With Fourier transform with respect to $t$ and $r$, we rewrite Eq. (2) as $[3]$

$$(I + \mathcal{K}_k) \begin{pmatrix} g_k^0 \\ g_k^f \\ g_k^g \\ g_k^s \end{pmatrix} = (I + \mathcal{T}_k) \begin{pmatrix} g_k^0 \\ g_k^f \\ g_k^g \\ g_k^s \end{pmatrix} + \mathcal{M}_{k,k'} \begin{pmatrix} g_{k'}^0 \\ g_{k'}^f \\ g_{k'}^g \\ g_{k'}^s \end{pmatrix},$$

with $\rho_k = \sum_i g_k^i \sigma_i$ and $\rho_k = \sum_i g_k^i \sigma_i$. Here $\mathcal{I}$ represents the $4 \times 4$ identical matrix, and $\mathcal{T}$ and $\mathcal{K}$ are given by

$$\mathcal{T}_k = \begin{pmatrix} 0 & -B_x \partial_{\mu} & -B_y \partial_{\mu} & -B_z \partial_{\mu} \\ -B_x \partial_{\nu} & 0 & 0 & 0 \\ -B_y \partial_{\nu} & 0 & 0 & 0 \\ -B_z \partial_{\nu} & 0 & 0 & 0 \end{pmatrix},$$

and

$$\mathcal{K}_k = \begin{pmatrix} \Omega & i\lambda_2 \tau \hat{q}_y & i\lambda_1 \tau \hat{q}_x & i\tau(\gamma_y \hat{q}_x - \gamma_x \hat{q}_y) \\ i\lambda_2 \tau \hat{q}_y & \Omega & -2B_z \tau & 2B_y \tau \\ i\lambda_1 \tau \hat{q}_x & 2B_z \tau & \Omega & -2B_x \tau \\ i\tau(\gamma_y \hat{q}_x - \gamma_x \hat{q}_y) & -2B_y \tau & 2B_x \tau & \Omega \end{pmatrix}.$$
\[ D(q) = \begin{pmatrix}
Dq^2 & -\frac{1}{2}\tau\lambda_2 q^2 v_d & -i\frac{\tau}{\tau_{ss}}\lambda_2 q^2 v_d & -4i\tau D\lambda_1\tilde{\lambda}_2^2 q + i\tau\tilde{\lambda}_2\gamma y q v_d & -4i\tau D\tilde{\lambda}_2^2\gamma y q - i\tau\lambda_1\tilde{\lambda}_2 q v_d \\
-\frac{1}{2}\tau\lambda_2 q^2 v_d & Dq^2 + \frac{1}{\tau_{ss}} & 4iD\gamma y q & -iv_{ss} D\lambda_2 / v_d & -iqv_{ss}
\end{pmatrix} \]

with \( v_d = \frac{\tau E}{m} \) and \( v_{ss} = 2\alpha_{ss}\tau eDmE \). Here, \( \tilde{\lambda}_i = m\lambda_i \), \( \tilde{\gamma}_i = m\gamma_i \), \( \frac{1}{\tau_{xx}} = 4D(\tilde{\lambda}_1^2 + \tilde{\gamma}_y^2) \), \( \frac{1}{\tau_{xy}} = 4D(\tilde{\lambda}_1^2 + \tilde{\gamma}_y^2) \) and \( \frac{1}{\tau_{ss}} = 4D(\tilde{\lambda}_1^2 + \tilde{\lambda}_2^2) \).

2. ASYMMETRY OF THE DIFFUSION MATRIX

The above diffusion matrix, Eq. (10) [i.e., Eq. (15) in main text], clearly shows that the spin-charge coupling is non-symmetric, i.e., \( D_{ij} \neq D_{ji} \). This comes from our careful consideration of the order of the quantity \( \epsilon_k \) and the operator \( \partial_{x_k} \) in \( \tilde{q}_y \), e.g.,

\[
\langle \tilde{q}_y \epsilon_k \rangle = -i\frac{1}{N} eE \sum_k \partial_{x_k} (\epsilon_k \rho_k) = 0, \quad (11)
\]

\[
\langle \epsilon_k \tilde{q}_y \rangle = -i\frac{1}{N} eE \sum_k \epsilon_k \partial_{x_k} \rho_k = i\frac{m}{\tau} v_d. \quad (12)
\]

\[
\partial_t \begin{pmatrix}
N \\
S_x \\
S_y \\
S_z
\end{pmatrix} = - \begin{pmatrix}
4\tau\lambda_2(\tilde{\lambda}_1^2 + \tilde{\gamma}_y^2) v_d + 2\tilde{\lambda}_1 v_{ss} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\tau_{xx}} & -4D\tilde{\lambda}_1\gamma y + 2\tilde{\lambda}_2 v_d & 0 & 0
\end{pmatrix}
\begin{pmatrix}
N \\
S_x \\
S_y \\
S_z
\end{pmatrix}. \quad (13)
\]

Obviously, the complete vanishing of the \( D_{11} \) guarantees the conservation of the particle number (i.e., the \( q = 0 \) component of the particle density) – an exact physical constraint. However, a non-zero \( D_{12} \) arising from the incorrect treatment of the ordering of the operators would violate this constraint when a uniform spin polarization is present. On the other hand, the non-zero matrix element \( D_{21} \) in Eq. (13) predicts the generation of a uniform in-plane spin polarization by a steady electric current, in the presence of SOC. This is a well-known effect – the so-called Edelstein effect \([7,8]\) – and has been observed in experiments \([9]\). For example, for a pure Rashba SOC (\( \lambda_1 = -\lambda_2 = m\alpha \)) in the steady state we obtain, to linear order in the electric field,

\[
P = \frac{S_x}{N} = \frac{\alpha \tau}{D} v_d - \frac{v_{ss}}{2D\alpha m}.
\]

The first term is solely due to Rashba SOC \([7,8]\), while the second term describes the correction due to the skew scattering \([3,4]\). We should point out that in our derivation so far the spin relaxation has been assumed to be dominated by the D’yakonov-Perel’ (DP) mechanism: therefore, \( \alpha \) should be finite. In the limit \( \alpha \to 0 \), different spin relaxation mechanisms, e.g., the Elliott-Yafet (EY) mechanism, become dominant. By replacing \( \frac{1}{\tau_{yy}} \) by \( \frac{1}{\tau_{yy}} + \frac{1}{\tau_{xx}} \) in our theory, we see that the spin polarization vanishes in the \( \alpha \to 0 \) limit as physically expected, and in agreement with previous work \([4]\).
3. ANALYSIS OF THE BALANCED CASE: \( \alpha = \beta \)

To better understand the generation of helical spin modes by skew scattering in the case of balanced SOC \((\alpha = \beta)\) in a (001) GaAs quantum well we recall that in this special case the SOC can be completely eliminated by an SU(2) gauge transformation, which is actually a non-uniform rotation in spin space \([2, 10]\). The spin dynamics in the rotated reference frame is simply determined by the competition of the spin-conserving drift-diffusion process and the gauge-transformed skew-scattering process. The gauge-transformed skew-scattering term has the form

\[
\partial_t \tilde{\rho}_{k|x}\big|_{\text{scat}} = -\frac{\alpha_{xx}}{2} \sum_{|k'|=|k|} (k \times k')_z \{\cos(q_0 x)\sigma_z, \tilde{\rho}_{k'}\} + (k \times k')_z \{\sin(q_0 x)\sigma_z, \tilde{\rho}_{k'}\},
\]

which suggests that the spin Hall velocity in the rotated frame has a sinusoidal variation in space, with the wave vector \(q_0 = \frac{4\pi \beta}{\alpha}\). More precisely, the transverse drift velocity for the \(\tilde{S}_z\) component is \(v^z_{ss}(x) = v_{ss} \cos(q_0 x)\), while that of the \(\tilde{S}_x\) component is \(v^x_{ss}(x) = v_{ss} \sin(q_0 x)\). Since the band SOC is completely gauged away, one has the carrier conservation equation

\[
\frac{d\tilde{\rho}_z(x,t)}{dt} = \partial_t \tilde{\rho}_z(x,t) - \sigma \partial_x[\tilde{\rho}_z(x,t)v^z_{ss}(x)] = 0,
\]

where the diffusion process has been neglected for simplicity. By substituting \(\tilde{\rho}_z(x,t) \simeq \frac{1}{2} A_0 \cos(q_0 x)\), we obtain the equations for \(\tilde{S}_z\) and \(\tilde{S}_x\) in the following form:

\[
\partial_t \tilde{S}_z(x) = -\frac{1}{2} A_0 \cos(q_0 x) \left[ q_- \sin(q_{-} x) + q_+ \sin(q_{+} x) \right],
\]

and

\[
\partial_t \tilde{S}_x(x) = -\frac{1}{2} A_0 \cos(q_0 x) \left[ q_- \cos(q_{-} x) - q_+ \cos(q_{+} x) \right].
\]

The component with wave vector \(q_-\) corresponds to the slowly decaying helical mode \(S_{-}\), whose amplitude is proportional to \(q_-\). Similarly, the component with wave vector \(q_+\) corresponds to the rapidly decaying helical mode \(S_{+}\), with amplitude proportional to \(q_+\). By transforming back to the original, unrotated spin space, we obtain

\[
\partial_t S_x = \cos(q_0 x) \partial_t \tilde{S}_x - \sin(q_0 x) \partial_t \tilde{S}_z = \sum_{\pm} \frac{1}{2} q_{\pm} A_0 \cos(q_0 x) v_{ss},
\]

\[
\partial_t S_z = \sin(q_0 x) \partial_t \tilde{S}_x + \cos(q_0 x) \partial_t \tilde{S}_z = -\sum_{\pm} \frac{1}{2} q_{\pm} A_0 \cos(q_0 x) v_{ss},
\]

which are consistent with Eqs. (21-22) in main text. From this analysis one can see that in the SU(2)-rotated frame the skew scattering cannot create a uniform spin polarization, which would correspond to the \(S_{+}\) mode in the original frame. This is the reason for the vanishing amplitude of the \(\tilde{S}_+\) mode at \(q = q_0\).

[1] E. G. Mischenko, A. V. Shytov, and B. I. Halperin, Phys. Rev. Lett. 93, 226602 (2004).
[2] B. A. Bernevig, J. Orenstein and S. C. Zhang, Phys. Lett. Phys. 97, 236601 (2006).
[3] J. L. Cheng and M. W. Wu, J. Phys.: Condens. Matter 20, 085209 (2008).
[4] R. Raimondi, P. Schwab, C. Gorini, and G. Vignale, Ann. Phys. 324, 153 (2012).
[5] X. Liu and J. Sinova, Phys. Rev. B 86, 174301 (2012).
[6] B. Anderson, T. D. Stanescu, and V. Galitski, Phys. Rev. B 81, 121304R (2010).
[7] A. G. Aronov and Y. B. Lyanda-Geller, JETP Lett. 50, 431 (1989).
[8] V. M. Edelstein, Solid State Commun. 73, 233 (1990).
[9] R. H. Silsbee, J. Phys.: Condens. Matter 16, R179 (2004).
[10] I. V. Tokatly and E. Ya. Sherman, arXiv:1302.2121