Recovery of defects from the information at detectors

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Abstract
A discrete wave equation in a multidimensional uniform space with local defects and sources is considered. The characterization of all possible defect configurations corresponding to given amplitudes of waves at the receivers (detectors) is provided.

Keywords: lattice with defects, propagating waves, localized waves, inverse analysis, cloaking

(Some figures may appear in colour only in the online journal)

1. Introduction
The recovery of defects from the available information about the amplitudes of waves at the detectors is very important in non-destructive testing in general and appears in areas such as structural geology inversion, medical imaging, and modeling of cloaking devices, see the discussion in [1]. The popular methods of solving this problem are based on stochastic inversion. In this paper we describe analytically the set of all possible defects corresponding to given amplitudes of waves at the detectors.

The current research is inspired by [2, 3], where the authors considered the problem of recovering a smooth and compactly supported potential $q$ in the continuous equation of the Schrödinger type $U_t - \Delta U + qU = F$ from its far backscattering data. Some interesting observations devoted to inverse scattering problems on discrete periodic graphs are given in [4, 5]. Note also that for continuous media with sparse distributed point scatters there are stable and efficient methods of recovery of scattering properties (see [6, 7]), especially if the multiple scattering is negligible. In this paper, we consider the discrete wave equation

$$S_n \frac{\partial^2 U_n}{\partial t^2} - \Delta_{\text{discr}} U_n = F_n, \quad n \in \mathbb{Z}^d$$

(1)
(\(d\) is the dimension) and try to recover the slownesses \(S_n\) from the information about the amplitudes of waves observed at some nearby points \(R \subset \mathbb{Z}^d\), see examples in figure 1. The discrete Laplacian in (1) is

\[
\Delta_{\text{disc}} U_n = \sum_{n' \sim n} (U_{n'} - U_n),
\]

where \(\sim\) means neighboring points (we have \(2d\) neighboring points). We assume that the slowness \(S_n^2 = s^2\) is uniform at each point of the lattice \(\mathbb{Z}^d\) except some defect points \(\mathcal{N}\), where

\[
S_n^2 = s^2 + s_n^2, \quad n \in \mathcal{N}.
\]

The term \(s_n\) is a so-called defect perturbation of the constant slowness \(s\). We assume also that the sources have the form

\[
\begin{align*}
F_n &= \sum_{j=1}^{M} e^{-i\omega_j t} \sum_{m \in \mathcal{F}_j} F_{mn} \delta_{mn},
\end{align*}
\]

where \(\delta\) is the Kronecker delta. The set \(\mathcal{F}_j \subset \mathbb{Z}^d\) consists of locations of sources of the same frequency \(\omega_j\), and the constant amplitudes \(F_{mn}\) are all non-zero. It is natural to assume that all frequencies are different \(\omega_i \neq \omega_j, i \neq j\). Note that it is possible to have many frequencies at one point since \(\mathcal{F}_i \cap \mathcal{F}_j\) can be non-empty for \(i \neq j\). The number of defects \(N = \#\mathcal{N}\), the receivers \(R = \#\mathcal{R}\), and the different frequencies \(M\) are finite numbers. For simplicity, we assume that all \(\omega_j^2\) do not belong to the spectrum \([0, 4d]\). As is shown in [1] the solution of the equation (1) has the form

\[
U_n = \sum_{j=1}^{M} U_n^j e^{-i\omega_j t}, \quad n \in \mathbb{Z}^d,
\]

where the constant amplitudes \(U_n^j\) can be explicitly expressed as rational functions of defect perturbations; see (15), (17), and details in [1]. We will focus on the inverse problem. Suppose that we record amplitudes at the receivers \(\mathcal{R}\). Thus, we know the vectors

\[
u_j = (U_n^j)_{n=1}^{R} \quad \text{for} \quad j = 1, \ldots, M,
\]

Figure 1. Wave fields in the uniform medium with defects and sources (computed by (17)), where the red defect corresponds to a slow material and the green defect is fast. The inverse problem consists of the recovery of defect properties from the information about the amplitudes of the waves observed at the detectors (blue points).
where \( \mathcal{R} = \{ r_j \}_{j=1}^N \). Suppose that we know the approximation location of the defects. For convenience we can assume that the set \( \mathcal{N} \) is known, and it is a sufficiently large set which covers all defect points. If some point \( n \) of this large set \( \mathcal{N} \) is a non-defect point then \( s_n = 0 \).

We assume also that the information about the sources is available. All of these data will be used to determine the unknown defect perturbations \( s_n, n \in \mathcal{N} \). Let \( k = (k_j) \in [-\pi, \pi]^d \) and introduce

\[
A_j = 2d - (\omega_j)^2 - 2 \sum_{i=1}^{d} \cos k_i, \quad \cdots = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \cdots dk,
\]

where \( \mathcal{N} = \{ n_j \}_{j=1}^N \) and \( \text{diag}(\text{vector}) \) denotes a diagonal matrix with components of the vector on the main diagonal. Introduce

\[
\mathbf{C}_j = \left\langle \mathbf{c}^A_{m} \right\rangle, \quad \mathbf{a}_m = \left\langle \frac{\mathbf{a}^A_{m}}{A_j} \right\rangle, \quad \mathbf{s} = (s_{n_j})_{j=1}^N, \quad \mathbf{S} = \text{diag}(\mathbf{s}),
\]

where \( ^* \) denotes the Hermitian conjugation. Let us introduce the sets of so-called \textit{admissible} slownesses \( \mathbf{s} \):

\[
\mathcal{G}_{\text{adm}}^j = \bigcap_{j=1}^M \mathcal{G}_{\text{adm}}^j, \quad \text{where} \quad \mathcal{G}_{\text{adm}}^j = \{ \mathbf{s} : \det \mathbf{G}_j = 0 \}, \quad \mathbf{G}_j = \mathbf{I} - \omega_j^2 \mathbf{A}_j \mathbf{S}.
\]

For \textit{admissible} defects the direct problem of determining the amplitudes \( \mathbf{U}_n \), which satisfy (1), has a unique solution. Almost all defects encountered in nature are admissible. Only the structures with very unusual properties can contain \textit{non-admissible} defects. So the consideration of \textit{admissible} defects is probably not a significant limitation. The next theorem describes analytically the set of all the possible defects \( \mathbf{s} \) which correspond to given amplitudes \( \mathbf{u}_j \) of the waves measured at the receivers.

**Theorem 1.1.** If the amplitudes \( \mathbf{u}_j, j = 1, \ldots, M \) at the receivers correspond to the defect \( \mathbf{s} \) then there are \( \mathbf{x}_j \in \mathbb{C}^N \) such that

\[
\mathbf{C}_j \mathbf{x}_j = \mathbf{u}_j - \sum_{m \in \mathcal{F}_j} \mathbf{c}^A_{m} \mathbf{F}^A_{m}, \quad j = 1, \ldots, M
\]

and \( \mathbf{s} \in \mathcal{S} = \bigcap_{j=1}^M \mathcal{S}_j \), where

\[
\mathcal{S}_j = \left\{ \frac{\omega_j^{-2}(\mathbf{x} + \mathbf{x}_j)}{A_j, \mathbf{x} + A_j \mathbf{x}_j + \sum_{m \in \mathcal{F}_j} \mathbf{a}^A_{m} \mathbf{F}^A_{m}} : \mathbf{x} \in \ker \mathbf{C}_j \right\}
\]

(ratio/means the component-wise ratio of two vectors). Moreover, for the defects \( \mathbf{s} \in \mathcal{S} \cap \mathcal{G}_{\text{adm}} \) the amplitudes at the receivers are \( \mathbf{u}_j, j = 1, \ldots, M \).

**Comments on applications.** (1) If \( \ker \mathbf{C}_j = \{ \mathbf{0} \} \) for some \( j \) then we can uniquely recover the defect \( \mathbf{s} \) from the information about the amplitudes of waves. This can happen when the location of the defect \( \mathcal{N} \) is a moderately sparse set. But for most applications, the defect area is large and dense, e.g. it is a square \( \mathcal{N} = [a_i, b_i]^d \) with a sufficiently large \( b_i - a_i \) in order to cover all possible defects. In this case, the matrices \( \mathbf{C}_j \) have non-trivial kernels (see [1]). Also each \( \mathcal{S}_j \subset \mathbb{C}^N \) is a manifold with the dimension \( N - \text{rank} \mathbf{C}_j \). Thus, if the number of the
different frequencies is greater than \( N \) then we can expect that \( S \) consists of a single point, since the intersection of the manifolds is usually a manifold of a smaller dimension. If \( S \) consists of a unique point then the inverse problem is solved uniquely. This is important for practical applications where the goal is a unique solution of inverse problems. In the case where \( S \) consists of more than one point, we can increase the chance of recovering the defect by using some available additional information; for example, if we know that all slownesses of the defective points are positive and bounded \( \leq B \). Then the set \( [0, B]^M \cap S \) is not large and it can provide useful characteristics of the defect because usually \( x_j \not\in \ker C_j \) and \( S \) consists of complex vectors that are located far from the origin \( 0 \).

(2) In this paragraph we assume that the configuration of sources is fixed. We assume also that the location \( \mathcal{N} \) of the defect is known. Suppose we want to construct a defect that cannot be detected at the receivers. For this we only need to take the unperturbed field \( u = \sum_{m \in \mathcal{N}} c^m F^m \) in theorem 1.1 which leads to the sets \( S^0 \) (12) with \( x_j = 0 \). The intersection \( S^0 = \bigcap_{j=1}^M S^0_j \) is precisely the set of all the possible invisible defects corresponding to given receivers. A defect which is invisible to any location of the external receivers is called a cloaking device. Roughly speaking, we cannot detect this ‘global’ defect from the outside, see figure 2. The set of all possible cloaking devices is now the intersection of all of \( S^0 \) corresponding to all configurations of receivers (enough to take a finite ring of receivers around the defect). Following on from the same argument found in the first paragraph, we may see that if the number of frequencies \( M \) is large then this set can be empty. Nevertheless for monochromatic sources \( (M = 1) \), the cloaking device can be constructed explicitly together with an arbitrary wave field near the device, see [1]. In particular, we can assume that the amplitudes of the wave field are zero inside the cloaking device (cloaking insulator). In this case, any object can be hidden inside the region of zero wave amplitudes and is not detectable from the outside. To sum up, taking \( u = 0 \) and/or \( u = \sum_{m \in \mathcal{N}} c^m F^m \) and/or any other amplitudes we can construct cloaking devices with various properties. Some additional information about the applications of cloaking devices is provided in [8–10].

(3) Almost all the matrices introduced above consist of the elements of the form

\[
a^j_n = \langle A^{-1}_j e^{in\mathbf{k}} \rangle.
\]  

These components are symmetric in \( n \), i.e. if we change the sign of any entry \( n_j \) of \( n \) then \( a^j_n \) does not change. They also satisfy the following identity

\[
\sum_{n \in \mathbb{Z}^N} a^j_n e^{in\mathbf{k}} = \delta_j^0.
\]
\[ \sum_{n=0}^{\infty} a_n^j = (2d - \omega^2 \omega^2) a_n^j + \epsilon_n. \]  

This equation explains why \( \ker C_j \) (and hence \( S_j \)) are non-trivial for dense sets of defects, since \( a_n^j \) is completely determined by \( a_n^j \) where \( n \) are neighboring points. For the same reason it will be more effective to use sparse sets of receivers.

(4) We have considered the frequencies \( \omega_j \) that do not belong to the spectrum \([0, 4d]\) (passband). They can be complex frequencies \( \omega_j = \alpha_j - i\beta_j \) (which means that we have a source attenuation factor \( e^{-\beta_j t} \)), or they can be high real frequencies. At the same time there are many problems where \( \omega_j \in [0, 4d] \); for example, the small real frequencies appear in the problems of long wave propagation. For such frequencies \( A_j^{-1} \) are not definite but we can consider the limit \( \omega \to 0 \) with \( \omega_j \in [0, 4d] \) and \( \epsilon \to +0 \). Except for some specific cases of frequencies \( (\omega = 0, \omega = 4d) \), it is not difficult to show that the limit of \( a_n^j \) exists; hence, we can extend our results to the frequencies from the passband. Such examples are considered in [1], where some computational aspects for \( a_n^j \) with \( \epsilon \to +0 \) are also discussed.

(5) Denote the ratio of two vectors in (12) as \( \mathbf{s}(\mathbf{x}) \), where \( \mathbf{x} \in \ker C_j \). By (23) we have the element \( \mathbf{y}_j \) from \( \ker C_j \), satisfying (21), and it is uniquely defined. In other words if \( \mathbf{s}(\mathbf{x}_1) = \mathbf{s}(\mathbf{x}_2) \in S \cap G_{adm} \) then \( \mathbf{x}_1 = \mathbf{x}_2 \), and hence the mapping \( \mathbf{s} \) is a parametrization of the manifold.

2. Proof of theorem 1.1

Taking Fourier series (see details in [1])

\[ u_j = \sum_{n \in \mathbb{Z}} U_n^j e^{i \omega k} \]  

we can equivalently rewrite the problem (1) as a set of integral equations

\[ A_j u_j - \omega_j^2 a^S \langle \mathbf{a} u_j \rangle = \sum_{m \in \mathcal{F}_j} F_m^j e^{i \omega k}, \quad j = 1, \ldots, M. \]  

If \( \mathbf{s} \) is admissible then there is a unique solution of (16):

\[ u_j = A_j^{-1} \sum_{m \in \mathcal{F}_j} F_m^j (\omega_j^2 a^S \mathcal{G}_{j}^{-1} a^j_m + e^{i \omega k}). \]  

Multiplying (16) by \( A_j^{-1} \mathbf{c} \) and taking the integral \( \langle \ldots \rangle \) we obtain

\[ \mathbf{u}_j = \omega_j^2 C_j \mathcal{S} \langle \mathbf{a} u_j \rangle = \sum_{m \in \mathcal{F}_j} F_m^j e^{i \omega k}. \]  

which means that \( \mathbf{x}_j = \omega_j^2 \mathcal{S} \langle \mathbf{a} u_j \rangle \) satisfies (11). This also means that

\[ \mathcal{S} \langle \mathbf{a} u_j \rangle = \omega_j^2 (\mathbf{y}_j + \mathbf{x}_j), \]  

where \( \mathbf{y}_j \in \ker C_j \). Multiplying (16) by \( A_j^{-1} \mathbf{a} \), taking the integral \( \langle \ldots \rangle \), and substituting (19) into (16) we obtain

\[ \langle \mathbf{a} u_j \rangle = A_j \mathbf{y}_j + A_j \mathbf{x}_j + \sum_{m \in \mathcal{F}_j} a_m^j r_m^j. \]  

Equations (19) and (20) mean that \( \mathbf{s} \) belongs to \( S \).

Now, suppose that we have some \( \mathbf{x}_j \) satisfying (11). We take some \( \mathbf{s} \in S \cap G_{adm} \). Then there are \( \mathbf{y}_j \in \ker C_j \) so that
\[ s = \omega_j^{-2}(y_j + x_j) \left( A_j y_j + A_j x_j + \sum_{m \in F_j} a_m^{j} F_m^{j} \right) \]  

(21)

or (because \( S = \text{diag}(s) \))

\[ y_j + x_j = \omega_j^2 S \left( A_j y_j + A_j x_j + \sum_{m \in F_j} a_m^{j} F_m^{j} \right) \]  

(22)

which leads to

\[ y_j + x_j = \omega_j^2 S G_j^{-1} \sum_{m \in F_j} a_m^{j} F_m^{j}. \]  

(23)

Note that in the implication \((22) \Rightarrow (23)\) we use the following well known fact: \( \text{Let } U, V \text{ be two arbitrary square matrices of the same size. If } I - UV \text{ is invertible then } I - VU \text{ is also invertible and} \)

\[ (I - UV)^{-1}U = U(I - VU)^{-1}, \]

where \( I \) denotes the identity matrix. Consider \((17)\), which is the unique solution of \((16)\) with \( S = \text{diag}(s) \). Multiplying \((17)\) by \( c \), taking the integral \( \langle \langle \cdot \rangle \rangle \), and using \((23)\) we obtain

\[ \langle c u_j \rangle = C_j \langle y_j + x_j \rangle + \sum_{m \in F_j} c_m^{j} F_m^{j} = u_j, \]  

(24)

where we also use \( y_j \in \ker C_j \) and \((11)\). Equation \((24)\) shows that \( s \) corresponds to the amplitudes \( u_j \) at the receivers, see \((15)\) and \((6)\).

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