Stochastics theory of log-periodic patterns

Enrique Canessa

The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy

Abstract

We introduce an analytical model based on birth-death clustering processes to help understanding the empirical log-periodic corrections to power-law scaling and the finite-time singularity as reported in several domains including rupture, earthquakes, world population and financial systems. In our stochastics theory log-periodicities are a consequence of transient clusters induced by an entropy-like term that may reflect the amount of cooperative information carried by the state of a large system of different species. The clustering completion rates for the system are assumed to be given by a simple linear death process. The singularity at $t_o$ is derived in terms of birth-death clustering coefficients.

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1E-mail: canessae@ictp.trieste.it
1 Overview

Increasing evidence of accelerated patterns having an overall power law behaviour with superimposed log-periodic oscillations has been found in a variety of applied domains. These observations have been reported in a series of experiments on rupture in heterogeneous media [1, 2] and from the historical data analysis of earthquakes [3, 4, 5, 6], world population [7] and financial stock markets [8, 9, 10, 11, 12, 13], (see examples in Fig.1). It has been also argued that log-periodic corrections to scaling should be present in a wider class of out-of-equilibrium dynamical systems (see, e.g., [14, 15]). The logarithmic modulations are periodic in $t - t_o$ and not on $t$ and are precursors to a spontaneous finite-time singularity $t_o$ at which they accumulate.

The interest in log-periodic corrections to power-law scaling is twofold. On one hand they enhance the fit quality to observed data with better precision than simple power laws by adjusting the (frequency, local minima and maxima of) oscillations. On the other hand their real-time monitoring could, in principle, allow for an enhancement of predictions in different contexts [15, 16].

At the theoretical level, log-periodic oscillatory structures has been associated to the existence of complex fractal dimensions [9] and critical exponents [15, 16, 17, 18]. However, as pointed out in [19], predictions of stock market crashes using complex critical exponents should be taken with some concern not only because of the many fitting parameters required but also because the time period used to perform the fit is rather long [20]. This does not mean that the apparent acceleration and the log-periodic modulation do not actually exist -the whole subject deserves further investigations.

Most recently, log-periodic patterns associated to financial crashes have also been shown to stem from models for stock markets inspired by percolation phenomena [21, 22, 23]. Furthermore, logarithmic oscillations have been found in an off-lattice bead-spring model of a polymer chain in a quenched porous medium under the influence of
an external field \([2]\) and in a uniform spin model on a fractal \([2]\). However, besides these simulation studies, there is no a convincing microscopic theoretical model which substantiates the idea that the singularity at finite \(t_o\) (e.g., a financial crash) is a critical point. Also, there is as yet no a fundamental theory that substantiates the claim for the precursory, universal log-periodic oscillations on large time scales.

In this work we made an attempt to provide a new scenario within which to elaborate an analytical microscopic theory and contribute towards an understanding of the underlying physics of log-periodic patterns. We introduce an stochastics model based on birth-death clustering processes to sustain the claim of log-periodic corrections to scaling and of a finite-time singularity. In our theory transient clusters are formed following an entropy-like formula that may reflect the amount of cooperative information (or disorder) carried by the state of a large system of different species. The clustering completion rates for the system are assumed to be exponentially distributed according to a simple linear death process. The singularity at \(t_o\) is derived in terms of birth-death clustering coefficients.

\section{Stochastics theory}

We write the governing equation for power law behaviour decorated by large scale log-periodic oscillations as a superposition of two terms

\[ G(t) = G_0(t) + G_\infty(t) , \]

where \(G_0\) takes the form of a pure power law and \(G_\infty\) represents the (universal periodic) corrections.

Similarly to \([4, 5]\), the first term is taken to be

\[ \frac{dG_0(t)}{dt} = \kappa(t_o - t)^{\alpha-1} , \]
with \( t_o \) a finite time at which a singularity appears and the exponent \( \alpha \) satisfies \( \alpha \neq 1 \).

Integration of this equation yields

\[
G_0(t) = G_0(t_o) - \frac{\kappa}{\alpha} (t_o - t)^\alpha; \quad (3)
\]

In the following we seek for an approximate form to the correction term \( G_\infty \).

### 2.1 Galerkin finite elements method for \( G_N(s, t) \)

The starting point of our theory is to assume that the two-dimensional (e.g., energy and time dependent) \( G_N \) function of a discretized system of \( N \) nodes is the solution of some non-linear differential equation (e.g., a diffusion equation with particular boundary conditions) which we do not know but we shall figure out the answer.

Using the standard Galerkin finite elements method described in the Appendix (see also, e.g., [26]), a general trial solution to this unknown differential equation can be approximated as

\[
G_N(s, t) \equiv \sum_{j=0}^{N} g_j(s) \tilde{P}_j(t) , \quad (4)
\]

where \( g_j \) are basic (interpolation) functions, and \( \tilde{P}_j \) are the so-called test functions. The terms \( g_j(s) \) are often referred to as trial functions and Eq.(4) as the trial solution at nodal points.

Without loss of generality, for all \( t \) and different \( j \)-states we can rewrite \( \tilde{P} \) in a more suitable form as the sum of even and odd parts

\[
\tilde{P}_j(t) = p_{2j}(t) + p_{2j+1}(t) \quad (j = 0, 1, 2, \cdots, N) \quad (5)
\]

This means that \( \tilde{P}_0 = p_0 + p_1, \tilde{P}_1 = p_2 + p_3, \tilde{P}_2 = p_4 + p_5, \cdots \)

Hence, in the limit \( N \rightarrow \infty \), we then get

\[
\sum_{j=0}^{\infty} p_j(t) = \sum_{j=0}^{\infty} \tilde{P}_j(t) \equiv 1 , \quad (6)
\]
a result that will prove useful later when we associate the test functions $\tilde{P}_j$ with the equilibrium state probabilities to be characterized by birth-death processes.

### 2.1.1 Birth-death model for $p_j(t)$ - Effect of disorder

The test functions $\tilde{P}_j(t)$ are usually determined by solving a system of differential equations (in time) generated by some governing equation, and if $N$ is made arbitrarily large the error introduced becomes small (see Appendix). In order to gain insight into the dynamics leading to log-periodic structures, we take next a different approach and relate the test functions to a large number of processes forming clusters or aggregates that change as a function of time (e.g., cell populations, customers queueing, interactive multi-agent ensembles, investors groups) acting collectively to pass on information or to introduce system disorder.

We assume that stochastics ”births” and ”deaths” clustering processes occur according to a simple one-dimensional birth-and-death model (see, e.g., \cite{27}). The state probabilities $p_j(t)$ in this case are obtained recursively from

$$\tilde{\lambda}_j(t)p_j(t) = \tilde{\mu}_{j+1}(t)p_{j+1}(t) \quad (j = 0, 1, 2, \ldots) \quad (7)$$

By a choice of the birth coefficients $\hat{\lambda}_j > 0$ and of the death coefficients $\hat{\mu}_j > 0$, various stochastics models can be constructed (e.g., queueing models in which customers corresponds to the ”population”, arrivals are ”births” and departures are ”deaths”). In other words, the quantity $\hat{\lambda}$ is interpreted as the birth rate and $\hat{\mu}$ the death rate when the population is at the state $j$.

Eq.\,(7) together with the normalization condition of Eq.\,(6) can easily be solved to yield the following statistical-equilibrium state distribution (as seen from an arbitrary outside observer)

$$p_j(t) = \frac{\hat{\lambda}_0(t)\hat{\lambda}_1(t)\cdots \hat{\lambda}_{j-1}(t)}{\hat{\mu}_1(t)\hat{\mu}_2(t)\cdots \hat{\mu}_j(t)}p_0(t) \quad . \quad (8)$$
This means that for each time \( t > 0 \) the state probabilities can, in principle, be determined subject to specification of the initial conditions \( p_0(t) \) (i.e., the so-called absorbing state) and the product of birth-death ratios at all previous states.

To obtain the above transient solution for \( p_j(t) \) (i.e., for finite \( t \)) in closed form, it is necessary to postulate basic expressions for \( \hat{\lambda}_j \) and \( \hat{\mu}_j \). Here we assume that -for the case of a finite probability distribution, i.e., \( p_j(t) > 0 \) (with \( j = 1, 2, \cdots, N \))- clusters form via an entropy-like formula

\[
\hat{\lambda}_j(t) = - \sum_{k=1}^{M} \lambda_k(t) \ln \lambda_k(t) > 0 ; \quad (j = 0, 1, 2, \cdots, N - 1)
\]  

with \( \lambda > 0 \) for all \( k \). According to information theory \cite{28}, the shape of our birth coefficients may reflect the measure of cooperative information carried by the outcomes \( \lambda_1, \cdots, \lambda_M \) (or the amount of disorder in the discrete observable \( \hat{\lambda}_j \)) in a system of \( M \) different species or types (e.g., human gender, financial traders).

On the other hand, in analogy to Erlang loss systems \cite{27}, we assume the clustering completion rate for the system in state \( j \) to be exponentially distributed (with rate \( \mu > 0 \)), hence we deal with simple linear death processes

\[
\hat{\mu}_j(t) = j\mu(t) ; \quad (j = 1, 2, \cdots, N) .
\]  

Then, the equilibrium state probabilities given by Eq.(8) become

\[
p_j(t) = \frac{(-1)^j}{j!} \left[ \sum_{k=1}^{M} a_k \ln \lambda_k(t) \right]^{j-1} p_0(t) , \quad (j = 0, 1, 2, \cdots, N)
\]  

where the per-capita ratio \( a_k \equiv \lambda_k(t)/\mu(t) > 0 \) is, for simplicity, assumed to be time independent. This means that \( \lambda_k \) and \( \mu \) should both scale as power laws of the form \( \sim \Delta t^{\pm n} \).

By using the normalization condition of Eq.(8) plus Eq.(11) and the Taylor series
expansions for the exponential function, we also obtain
\[ p_0(t) ≡ \left[ \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \Gamma_j^M(t) \right]^{-1} = e^{\Gamma_M(t)} , \] (12)
where
\[ \Gamma_M(t) ≡ \sum_{k=1}^{M} a_k \ln \lambda_k(t) = \ln \prod_{k=1}^{M} \lambda^a_k(t) = \ln p_0(t) . \] (13)

Now that we have a precise formulation for \( \tilde{P}_j(t) \), we can set trial functions for the basic interpolation functions \( g_j \) to solve for \( G_N(s,t) \) given in Eq.(4).

2.1.2 Trial \( g_j(s) \) functions

The efficiency of the Galerkin formulation is very dependent on making the correct choice of the approximating test and trial functions (see Appendix). Of the many nodal unknowns that could be candidates, here we consider the polynomial expansion
\[ g_j(s) ≡ \frac{\gamma}{(s-1)^j} = \gamma \left\{ (-1)^j + \binom{j}{1} (-1)^{j-1} s + \binom{j}{2} (-1)^{j-2} s^2 + \ldots \right\} , \] (14)
with \( s \) a dimensionless variable.

Substitution of Eq.(14) into (5) and using Eq.(13) plus these trial functions, it gives the approximate trial solution
\[ G_N(s, t) = \gamma p_0(t) \sum_{j=0}^{N} \frac{1}{(s-1)^j} \left\{ \frac{(-1)^{2j}}{(2j)!} \Gamma_{2j}^M(t) + \frac{(-1)^{2j+1}}{(2j+1)!} \Gamma_{2j+1}^M(t) \right\} , \] (15)
as is easily verified.

We shall see next that our \( g_j \) functions make a judicious choice.
2.2 Onset of log-periodicity

Let us consider large systems in statistical equilibrium and adopt the following notation for the required correction term of Eq. (1) near \( t_0 \):

\[
G_\infty(t) \equiv \lim_{N \to \infty} G_N(s \approx 0, t).
\]  

(16)

For the sake of simplicity we have set \( s \approx 0 \) in order to gain insight into the genesis of log-periodicities.

As an example, our approximate trial solution of Eq. (15) thus becomes

\[
G_\infty(t) = \gamma p_0(t) \left\{ \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} \Gamma_M^{2j}(t) - \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} \Gamma_M^{2j+1}(t) \right\}.
\]  

(17)

Using Taylor series expansions for the cosine and sine functions plus the trigonometric identity \( \cos(a + b) = \cos(a) \cos(b) - \sin(a) \sin(b) \), such that \( a = \pi/4 \) and \( b/2\pi \equiv \Gamma_M(t) \) in radians, the above equation then results in

\[
G_\infty(t) = \sqrt{2} \gamma p_0(t) \cos \left( 2\pi \ln p_0(t) + \frac{\pi}{4} \right),
\]  

(18)

with \( p_0 \) satisfying Eq. (13).

This equation characterizes the complexity of the underlying dynamics of the scaling systems under consideration. It can be seen that in our stochastics theory the log-periodic modulation is a consequence of the entropy-like assumption used for the transmission of information within the birth-death clustering processes.

Using our expression for the initial boundary condition \( p_0 \) leading to Eq. (18), we analyse next the presence of a singularity at a finite time where the oscillations accumulate.
2.3 Finite-time singularity

As discussed in the derivation of Eq.(11) \( \mu \) should scale with \( t \), so we set

\[
\mu(t) \sim (t_o - t)^{\pm n},
\]

(19)

where \( n \neq 0 \) is a given exponent and \( t_o \) characterizes the finite-time singularity. Hence, according to our definitions

\[
\lambda_k(t) \sim a_k(t_o - t)^{\pm n},
\]

(20)

which implies that within our stochastics birth-death theory we can consider finite values of time \( t < t_o \) (or \( n \) even only if \( t > t_o \)) since the coefficients \( \lambda_k \) and \( \mu_k \) (and, therefore, \( a_k \)) are all positive.

By substitution of this scaling into Eq.(13), we finally get

\[
p_0(t) = \prod_{k=1}^{M} \lambda_k^{\alpha_k}(t) = \Theta_M \prod_{k=1}^{M} (t_o - t)^{\pm na_k},
\]

(21)

where

\[
\Theta_M \equiv \prod_{k=1}^{M} a_k^{\alpha_k}.
\]

(22)

Thus from this relation and Eq.(18), we are able to derive log-periodic corrections in the form of \( G_\infty \) for \( t < t_o \).

3 Discussion

Having introduced our theory based on stochastics clustering processes to describe log-periodic corrections to scaling and a finite-time singularity at \( t_o \), we use Eqs.(11), (13), (18) and (21) to obtain

\[
G(t) = A + B(t_o - t)^\alpha + C(t_o - t)^\beta \cos \left(2\pi \beta \ln(t_o - t) + \psi \right),
\]

(23)
where $A \equiv G_0(t_o)$, $B \equiv -\kappa/\alpha$, and $\alpha$ are parameters relating the pure power law term of the governing Eq.(1) and

$$C \equiv \sqrt{2}\gamma \Theta_M , \quad \beta \equiv \pm n \sum_{k=1}^{M} a_k , \quad \psi \equiv 2\pi \ln \Theta_M + \frac{\pi}{4} ,$$

(24)

are the parameters of the log-periodic corrections in terms of our stochastics birth-death model parameters. The fit of this equation to historical random data displaying accelerated precursory patterns and an spontaneous singularity, which indicates the sharp transition to a new regime, is presented in Fig.1.

As an illustrative example, in Fig.1 we plot the daily Log(S&P500) stock index closing values during the years 1982-1988 [20], the estimated 1000-2000 world population by the U.N. Population Division [29] and the sum of seismic activities measured near the Virgin Islands between Apr. 1979 and Feb. 1980 [30]. Our best fits with Eq.(23) to these data sets have been done as follows:

|                      | A   | B   | C   | $\alpha$ | $\beta$ | $t_o$   | $\psi$ | rms |
|----------------------|-----|-----|-----|----------|---------|---------|-------|-----|
| S&P500 index         | 1.51| 4.34| -0.01| -0.1     | 1.42    | 1988.62 | 1.85  | 0.053|
| World population     | 0.25| 1489.74| -25.24| -1.38    | -1.04   | 2054.61 | -6.34 | 0.047|
| Seismic activity     | -1.38| 1.18| -0.04| -0.74    | -1.25   | 1980.29 | 0.78  | 0.414|

Examination of the plotted curves shows that Eq.(23) can model log-periodic corrections to the leading scaling behaviour and a singularity at $t_o$ in different applied domains similarly to the methods inspired by renormalization group theory entailing complex critical exponents. If we set $\alpha = \beta = 1/f$ we can approach the results obtained in [4, 8, 9] where market crashes, population explosion, and culminating large earthquakes are viewed as critical points in a system with an underlying discrete scale invariance. If we set $\alpha \neq \beta$, our results are then comparable to those obtained using the more general anzat of two different exponents as in [10]. The main difference of our stochastics theory with respect to the critical exponent approach is the presence of the exponent $\beta$ also appearing in the argument of the cosine function. Since $n$ and $a_k$ are
positive then from Eq. (24) we have that $\beta \neq 0$. Furthermore, in our stochastics theory the finite time $t_o$ is also determined in function of $\beta$ as discussed below.

So far, the fitting in Fig.1 allows us to argue that the apparent logarithmic periodicities in scaling systems may also be understood within the context of an stochastics analytical model based on birth-death clustering processes which is the distinctive feature of our stochastics theory. We can interpret the "births" and the "deaths" clusterings in different ways. For financial systems the "births" and "deaths" may represent, e.g., the buyers and the sellers, respectively. Whereas newborns and deceases would be in correspondence with the population growth. Some absorbed- and released- energy may relate the "births" and "deaths" in the case of seismic events for finite times $t < t_o$.

The recurrence Eqs.(7) are conservation-of-flow relations. That is, the long-run rate at which the system moves up from state $j$ to $j + 1$ equals the rate at which the system moves down from state $j + 1$ to $j$ (i.e., rate up = rate down). Thus, birth-death processes describe the stochastic evolution in time of a random variable whose values varies (i.e., increases or decreases) by one in a single event (or state) starting from the absorbing state $p_o$.

The spontaneous singularity is here related to the birth-death coefficients which in turn determinate $p_o$, i.e. the initial boundary condition at the state $j = 0$ via Eq.(12). It is also important to note that the state distribution coefficient defined by

$$-\left(\frac{p_{j+1}-p_j}{p_j}\right) = 1 + \frac{\Gamma _M}{j + 1},$$

depends on the logarithm of the absorbing state via Eq.(13).

In the absence of a well-defined non-linear dynamical equation governing log-periodic corrections to power-law scaling, we have adopted the standard Galerkin finite elements method as the starting point to search for a general trial solution to this "unknown" differential equation. The motivation for our basic interpolation functions $g_j(s)$ follows
computational finite-element methods which are characterized by the use of polynomials for the known test functions (obtained from Eqs. (5) to (13)) as well as for the unknown trial functions of Eq. (14) in subdomains called finite elements [26].

As discussed in the Appendix, our trial \( g_j(s) \)'s would allow to solve the matrix equation for the test functions, which have been related to a large number of (birth-death) processes forming clusters, provided the governing equation of the problem would be known. These forms, using \( s \approx 0 \), were taken for convenience to gain insight into the onset of log-periodicities. If we consider instead small \( s << 1 \), we would obtain rather similar conclusions after some algebra.

We thus argue that log-periodicities are a consequence of transient clusters induced by the entropy-like term given in Eq. (9) which may reflect the amount of cooperative information carried by the state of a large system \( (i.e., N \to \infty) \) of different species \( M \). Using the definition of the amount of (discrete) finite information or entropy, it can be shown that the information is additive under concatenation of independent probabilities as the logarithm function is. It has been proved that it is possible to define information without necessarily using the concept of probability \( (see, e.g., [28]) \). We have adopted the latter definition in this work via Eq. (9) for the birth coefficients. The clustering completion rates for the system are given by a simple linear death process.

The state probabilities \( p_j \) are normalized via Eq. (6). We may also consider that the total sum of outcomes \( \lambda_k \) is constant for all time as in the Shannon theory of information [28]. Therefore, for the whole range \( M \) of different species we set

\[
\sum_{k=1}^{M} \lambda_k(t) \equiv \xi_t > 0 .
\]  

We can then estimate the finite time \( t_\alpha \) at which a singularity appears from Eqs. (20) and (24) by considering the initial time \( t = 0 \) to thus obtain the relation

\[
t_\alpha^{\pm n} = \frac{\pm n \xi_\alpha}{\beta} .
\]
From the examples in Fig.1 we found for the Log(S&P500) stock index: \( n = 0.105, \xi_o = 30 \). World population: \( n = -0.209, \xi_o = 1 \). Accumulated seismic activity: \( n = -0.226, \xi_o = 1 \).

The per-capita ratios \( a_k \) plus the exponent \( n \) appearing in the scaling of Eq.(19) are the minimum ingredients required to derive a complete description of log-periodic corrections to scaling and finite-time singularities within the framework of an stochastics theory based on birth-death clustering processes. The positive state distributions \( p_j \) are determined by \( n \) and \( a_k \) which also relate the exponent \( \beta, C \) and \( \psi \) as in Eq.(23) and \( t_o \) as in Eq.(24). This means that such state distributions of the system drive the log-periodic oscillations. We believe this feature of our stochastic model can help to elaborate a general microscopic theory to understand the underlying mechanisms of log-periodic patterns. In the case of financial systems, such microscopic theory should also explain the peculiar statistical features in short time scales such as the highly correlated variance or volatility of price fluctuations \cite{31, 32}, by exploring the state distribution coefficient given by Eq.(24).
References

[1] J.-C. Anifrani, C. Le Floch'h, D. Sornette and B. Souillard, J. Phys. I (France) 5 (1995) 631.

[2] A. Johansen and D. Sornette, Int. J. Mod. Phys. C 9 433 (1998); see also http://arXiv.org/abs/cond-mat/0003478

[3] D. Sornette and C. Sammis, J. Phys. I (France) 5, 607 (1995).

[4] H. Saleur, C.G. Sammis and D. Sornette, J. Geo. Res. 101, 17661 (1996).

[5] A. Johansen, S. Sornette, H. Wakita, U. Tsunogai, W.I. Newman, H. Saleur, J. Phys. I (France) 6, 1391 (1996).

[6] A. Johansen, H. Saleur and D. Sornette, Eur. Phys. J B 15 551 (2000).

[7] A. Johansen and D. Sornette, preprint http://arXiv.org/abs/cond-mat/0002075

[8] A. Johansen and D. Sornette, Physica A 245, 411 (1997); see also Eur. Phys. J B 17, 319 (2000) and http://arXiv.org/abs/cond-mat/9907270

[9] D. Sornette, A. Johansen, A. Arneodo, J.F. Muzy and H. Saleur, Phys. Rev. Lett. 76, 251 (1996).

[10] J.A. Feigenbaum and P.G.O. Freud, Int. J. Mod. Phys. B 10, 3737 (1996); see also Mod. Phys. Lett. B 12, 57 (1998).

[11] S. Gluzman and V.I. Yukalov, Mod. Phys. Lett. B 12, 75 (1998).

[12] N. Vandewalle, Ph. Boveroux, A. Minguet and M. Ausloos, Physica A 255, 201 (1998); see also Eur. J. Phys. B 4, 139 (1998).

[13] S. Drożdż, F. Ruf, J. Speth and M. Wójcik, Eur. Phys. J. B 10, 589 (1999).
[14] M.F. Shlesinger and B. J. West, Phys. Rev. Lett. 67 2106 (1991).

[15] H. Saleur and D. Sornette, J. Phys. I (France) 6, 327 (1996).

[16] D. Sornette, Phys. Rep. 297, 239 (1998).

[17] D. Sornette, A. Johansen and J.-P. Bouchaud, J. Phys. I (France) 6, 167 (1996)

[18] D. Sornette and A. Johansen, Physica A 245, 411 (1997).

[19] L. Laloux, M. Potters, R. Cont, J.-P. Aguilar and J.-P. Bouchaud, Eurphys. Lett. 45, 1 (1999).

[20] E. Canessa, in proceedings 2nd EPS Conference on "Application of Physics in Financial Analysis, held in Liège (Belgium) July 2000.

[21] D. Stauffer and D. Sornette, Physica A 252, 271 (1998).

[22] D. Stauffer and N. Jan, Physica A 277, 215 (2000).

[23] D. Stauffer, in proceedings 2nd EPS Conference on "Application of Physics in Financial Analysis, held in Liège (Belgium) July 2000.

[24] V. Yamakov, A. Milchev, G.M. Foo, R.B. Pandey and D. Stauffer, Eur. Phys. J. B 9, 659 (1999).

[25] J.C. Lessa and R.F.S. Andrade, Phys. Rev. E 62, 3083 (2000).

[26] C.A.J. Fletcher, in Computational Galerkin Methods, (Springer-Verlag, NY 1984).

[27] R. Goodman, in Introduction to Stochastic Models, (The Benjamin/Cummings Pub. Comp., Inc, California 1988).

[28] R.S. Ingarden, A. Kossakowski and M. Ohya, in Information Dynamics and Open Systems, (Kluwer Acad. Pub., Dordrecht 1997).
[29] Data taken from http://www.popin.org.

[30] D.J. Varnes and C. G. Bufe, Geophys. J. Int. 124, 149 (1996).

[31] J.-P. Bouchaud an R. Cont, Eur. Phys. J. B 6, 543 (1998).

[32] D. Sornette, P. Simonetti and J.V. Andersen, Phys. Rep. 335, 19 (2000).
Appendix: The Galerkin formulation

In this appendix, the key features of the standard Galerkin finite elements method are stated concisely for completeness [26]. If a 2D problem in a domain \( D(x,y) \) is governed by a linear differential equation \( L(u) = 0 \), with boundary conditions \( S(u) = 0 \) on \( \delta D \), i.e., the boundary of \( D \). Then, the Galerkin method assumes that \( u \) can be accurately represented by the approximate trial solution

\[
u(x,y) = u_o(x,y) + \sum_{j=1}^{N} a_j(y)\phi_j(x)
\]

where the \( \phi_j \)'s are known, trial analytical functions, \( u_o \) is chosen to satisfy the boundary conditions, and the \( a_j \)'s are test functions to be determined.

To obtain the unknown \( a_j \)'s, the inner product of the weighted residual \( R \) is set equal to zero:

\[
(R,\phi_k) \equiv \int \int_D R\phi_k \, dx\,dy = 0 \quad , \quad k = 1, \cdots , N , 
\]

where

\[
R(a_0, a_1 \cdots a_N, x, y) \equiv L(u) = L(u_o) + \sum_{j=1}^{N} a_j(y)L(\phi_j) .
\]

Since this example is based on a linear \( L(u) \), then the above can be rewritten as a matrix equation for the \( a_j \)'s as

\[
\sum_{j=1}^{N} a_j(t)L(\phi_j, \phi_k) = -L(u_o, \phi_k) .
\]

Substitution of the \( a_j \)'s resulting from this equation into Eq.(28) gives the required approximate solution \( u(x, y) \).
Figure caption

- **Fig.1**: Illustrative examples of log-periodic patterns. Full line curves are the fit of our birth-death clustering theory using Eq.(23). These fits allow to estimate the total sum of outcomes $\lambda_k$. 
