Part of the D - dimensional Spiked harmonic oscillator spectra

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Abstract

The pseudoperturbative shifted - $l$ expansion technique PSLET [5,20] is generalized for states with arbitrary number of nodal zeros. Interdimensional degeneracies, emerging from the isomorphism between angular momentum and dimensionality of the central force Schrödinger equation, are used to construct part of the $D$ - dimensional spiked harmonic oscillator bound - state spectra. PSLET results are found to compare excellently with those from direct numerical integration and generalized variational methods [1,2].
1 Introduction

The simplest model of realistic interaction potentials in atomic, molecular, and nuclear physics is provided by the spiked harmonic oscillator

\[ V(q) = c_1 q^2 + c_2 q^{-b}, \quad c_1, c_2, b > 0, \quad q \in (0, \infty). \]  

(1)

The construction of its bound - states has attracted attention over the last few years [1-9]. It is an interesting model not only because of being a singular potential representing a repulsive core in realistic interactions, but also because of its intrinsic properties in view of mathematical physics [10-16]. However, most of the studies on this model potential (1) were devoted to one spatial dimension (1D, the hyperquantum limit in view of Herschbach [17,18]). It was just very recently, to the best of our knowledge, that Hall and Saad have generalized their variational analysis [1a] and smooth transformation [2] methods, VAM and STM, respectively, to the D - dimensional case and studied its bound - states. They have also used direct numerical integration (DNI) for comparison purposes. It is therefore interesting to carry out systematic studies of the bound - state spectra generated by this interesting class of singular potentials (1).

On the other hand, results from exactly solvable potentials (an interesting field of mathematical physics in itself) are essential ingredients for the description of realistic physical problems [1-5,19]. The solutions of these can be used in perturbation and pseudoperturbation theories, or they can be combined with numerical calculations. Nevertheless, in the simplest case, analytical calculations can aid numerical studies in areas where numerical techniques might not be safely controlled. For example, when bound - state wave functions with arbitrary nodal zeros are required for certain singular potentials (a next level of complexity), analytical solutions can supply a
basis for numerical calculations. Moreover, in many problems the Hamiltonian does not contain any physical parameter suitable for a perturbation expansion treatment. More often, the Hamiltonian contains physical parameters, but, typically, zeroth-order solutions for special values of these are not tractable or good starting approximations. One would therefore resort to variational calculations [1], pseudoperturbation expansions (artificial in nature) [5,18-26], etc.

Recently, we have introduced a pseudoperturbative shifted -l (l is the angular momentum quantum number) expansion technique (PSLET) to solve for nodeless states of Schrödinger equation. It simply consists of using 1/l as a pseudoperturbation parameter, where l = l − β and β is a suitable shift. The shift β is vital for it removes the poles that would emerge, at lowest orbital states with l=0, in our proposed expansions below. Our analytical, or often semianalytical, methodical proposal PSLET has been successfully applied to quasi-relativistic harmonic oscillator [20], spiked harmonic oscillator [5], anharmonic oscillators [21], and to the two-dimensional (flatland, in view of Godson and López-Cabrera in [17]) hydrogenic atom in an arbitrary magnetic field [22].

Encouraged by its satisfactory performance in handling nodeless states, we generalize PSLET recipe (in section 2) for states with arbitrary number of nodal zeros, k ≥ 0. Moreover, in the underlying "radical" time-independent radial Schrödinger equation, in ℏ = m = 1 units,

\[ \left[ -\frac{1}{2}\frac{d^2}{dq^2} + \frac{l(l + 1)}{2q^2} + V(q) \right] \Psi_{k,l}(q) = E_{k,l}\Psi_{k,l}(q), \]  

the isomorphism between orbital angular momentum l and dimensionality D invites interdimensional degeneracies to obtain [17]. Which, in effect, allows us to generate the ladder of excited states for any given k and nonzero l from
the \( l=0 \) result, with that \( k \), by the transcription \( D \rightarrow D + 2l \). That is, if \( E_{k,l}(D) \) is the eigenvalue in \( D \)-dimensions, then

\[
E_{k,l}(2) \equiv E_{k,l-1}(4) \equiv \ldots \equiv E_{k,1}(2l) \equiv E_{k,0}(2l + 2)
\]

for even \( D \), and

\[
E_{k,l}(3) \equiv E_{k,l-1}(5) \equiv \ldots \equiv E_{k,1}(2l + 1) \equiv E_{k,0}(2l + 3)
\]

for odd \( D \). For more details the reader may refer to ref.s \([17,18,27]\). We therefore calculate, in section 3, the energies for \( D = 2 \) and \( D = 3 \) spiked harmonic oscillators, for a given number of nodes \( k \) and different values of \( l \), and construct part of its \( D \)-dimensional bound-state spectra. We compare our results with those reported by Hall and Saad via generalized variational analysis VAM, and direct numerical integration DNI methods \([1,2]\). Section 4 is devoted for concluding remarks.

## 2 The generalization of PSLET

With the shifted angular momentum, equation (2) reads

\[
\left\{-\frac{1}{2} \frac{d^2}{d q^2} + \frac{\bar{\ell}^2 + (2\beta + 1)\bar{\ell} + \beta(\beta + 1)}{2q^2} + \frac{\bar{\ell}^2}{Q} V(q)\right\} \Psi_{k,l}(q) = E_{k,l} \Psi_{k,l}(q),
\]

where \( Q \) is a constant that scales the potential \( V(q) \) at large-\( l_D \) limit (the pseudoclassical limit \([17]\)) and is set, for any specific choice of \( l_D \) and \( k \), equal to \( \bar{\ell}^2 \) at the end of the calculations. Here \( l_D = l + (D - 3)/2 \), to incorporate the interdimensional degeneracies associated with the isomorphism between
angular momentum and dimensionality $D$. Hence, $\bar{l} \rightarrow \bar{l} = l_D - \beta$ throughout this paper. Next, we shift the origin of the coordinate system through $x = \bar{l}^{1/2}(q - q_o)/q_o$, where $q_o$ is currently an arbitrary point to be determined below. Expansions about this point (see Appendix for more details), $x = 0$ (i.e. $q = q_o$), obviously localize the problem at an arbitrary point $q_o$ and the derivatives, in effect, contain information not only at $q_o$ but also at any point on $q$-axis, in accordance with Taylor’s theorem. It is then convenient to expand $E_{k,l}$ as

$$E_{k,l} = \sum_{n=-2}^{\infty} E^{(n)}_{k,l} \bar{l}^{-n}.$$  

(6)

Equation (5) thus becomes

$$\left[ -\frac{1}{2} \frac{d^2}{dx^2} + \sum_{n=0}^{\infty} v^{(n)} \bar{l}^{-n/2} \right] \Psi_{k,l}(x) = \left[ \sum_{n=1}^{\infty} q_o^2 E^{(n-1)}_{k,l} \bar{l}^{-n} \right] \Psi_{k,l}(x).$$  

(7)

Up to this point, one would conclude that the above procedure is nothing but an imitation of the eminent shifted large-N expansion (SLNT) [25,26,28-30]. However, because of the limited capability of SLNT in handling large-order corrections via the standard Rayleigh-Schrödinger perturbation theory, only low-order corrections have been reported, sacrificing in effect its preciseness. Therefore, one should seek for an alternative and proceed by setting the wave functions with any number of nodes as

$$\Psi_{k,l}(x(q)) = F_{k,l}(x) \exp(U_{k,l}(x)).$$  

(8)

In turn, equation (7) readily transforms into the following Riccati equation:
\[
\begin{align*}
F_{k,l}(x) & \left[ -\frac{1}{2} \left( U^{''}_{k,l}(x) + U^{'}_{k,l}(x)U^{'}_{k,l}(x) \right) + \sum_{n=0}^{\infty} v^{(n)}(x) \tilde{l}^{-n/2} \\
- \sum_{n=1}^{\infty} q^{2}_{o} E^{(n-1)}_{k,l} \tilde{l}^{-n} \right] - F^{'}_{k,l}(x)U^{'}_{k,l}(x) - \frac{1}{2} F^{''}_{k,l}(x) = 0,
\end{align*}
\]

where \( F^{'}_{k,l}(x) \) denotes derivatives with respect to \( x \). It is evident that this equation admits solution of the form

\[
U^{'}_{k,l}(x) = \sum_{n=0}^{\infty} U^{(n)}_{k}(x) \tilde{l}^{-n/2} + \sum_{n=0}^{\infty} G^{(n)}_{k}(x) \tilde{l}^{-(n+1)/2},
\]

\[
F_{k,l}(x) = x^{k} + \sum_{n=0}^{k-1} \sum_{m=0}^{\infty} a_{p,k}^{(n)} x^{p} \tilde{l}^{-n/2},
\]

where

\[
U^{(n)}_{k}(x) = \sum_{m=0}^{n+1} D_{m,n,k} x^{2m-1} ; \quad D_{0,n,k} = 0,
\]

\[
G^{(n)}_{k}(x) = \sum_{m=0}^{n+1} C_{m,n,k} x^{2m}.
\]

Substituting equations (10) - (13) into equation (9) implies

\[
\begin{align*}
F_{k,l}(x) & \left[ -\frac{1}{2} \sum_{n=0}^{\infty} \left( U^{(n)}_{k} \tilde{l}^{-n/2} + G^{(n)}_{k} \tilde{l}^{-(n+1)/2} \right) \\
- \frac{1}{2} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left( U^{(m)}_{k} U^{(n-m)} \tilde{l}^{-n/2} + G^{(m)}_{k} G^{(n-m)} \tilde{l}^{-(n+2)/2} \right) \right]
\end{align*}
\]
The above procedure obviously reduces to the one described by Mustafa and Odeh [5,20-22], for \( k = 0 \). Moreover, the solution of equation (14) follows from the uniqueness of power series representation. Therefore, for a given \( k \) we equate the coefficients of the same powers of \( \bar{l} \) and \( x \), respectively. For example, when \( k = 1 \) one obtains

\[
D_{1,0,1} = -w, \quad U_1^{(0)}(x) = -w x, 
\]

\[
C_{1,0,1} = -\frac{B_3}{w}, \quad a_{0,1}^{(1)} = -\frac{C_{0,0,1}}{w}, 
\]

\[
C_{0,0,1} = \frac{1}{w} (2C_{1,0,1} + 2\beta + 1), 
\]

\[
D_{2,2,1} = \frac{1}{w} \left( \frac{C_{1,0,1}^2}{2} - B_4 \right), 
\]

\[
D_{1,2,1} = \frac{1}{w} \left( \frac{5}{2} D_{2,2,1} + C_{0,0,1} C_{1,0,1} - \frac{3}{2} (2\beta + 1) \right), 
\]

\[
E_{1,\ell}^{(0)} = \frac{1}{q_o^2} \left( \frac{\beta(\beta + 1)}{2} + a_{0,1}^{(1)} C_{1,0,1} - \frac{3}{2} D_{1,2,1} - \frac{C_{0,0,1}^2}{2} \right), 
\]
etc. Here, we reported the nonzero coefficients only and give the definitions of the related parameters in the Appendix. One can then calculate the energy eigenvalues and eigenfunctions from the knowledge of $C_{m,n,k}$, $D_{m,n,k}$, and $a_{p,k}^{(n)}$ in a hierarchical manner. Nevertheless, the procedure just described is suitable for a software package such as MAPLE to determine the energy eigenvalue and eigenfunction corrections up to any order of the pseudoperturbation series (6).

Although the energy series, equation (6), could appear divergent, or, at best, asymptotic for small $\bar{l}$, one can still calculate the eigenenergies to a very good accuracy by forming the sophisticated $[N,M]$ Padé approximation [24]

$$P^M_N(1/\bar{l}) = (P_0 + P_1/\bar{l} + \cdots + P_M/\bar{l}^M)/(1 + q_1/\bar{l} + \cdots + q_N/\bar{l}^N)$$

to the energy series (6). The energy series (6) is calculated up to $E^{(8)}_{k,l}/\bar{l}^8$ by

$$E_{k,l} = \bar{l}^2 E^{(-2)}_{k,l} + E^{(0)}_{k,l} + \cdots + E^{(8)}_{k,l}/\bar{l}^8 + O(1/\bar{l}^9), \quad (21)$$

and with the $P_4^4(1/\bar{l})$ Padé approximant it becomes

$$E_{k,l}[4,4] = \bar{l}^2 E^{(-2)}_{k,l} + P^4_4(1/\bar{l}). \quad (22)$$

Our recipe is therefore well prescribed.

3 D - spiked harmonic oscillator spectra

In this section we consider the spiked harmonic oscillator potential (1) and illustrate the above mentioned procedure. The substitution of equation (1)
in (45), for $k \geq 0$, implies

$$w = \sqrt{\frac{8c_1 q_o + bc_2 (b - 2) q_o^{- (b+1)}}{2c_1 q_o - bc_2 q_o^{- (b+1)}}}, \quad \beta = -\frac{1}{2} (1 + [2k + 1]w). \quad (23)$$

Equation (44), in turn, reads

$$l_D + \frac{1}{2} \left( 1 + [2k + 1] \sqrt{\frac{8c_1 q_o + bc_2 (b - 2) q_o^{- (b+1)}}{2c_1 q_o - bc_2 q_o^{- (b+1)}}} \right) = q_o^2 \sqrt{c_1 - \frac{bc_2}{2} q_o^{- (b+2)}},$$

which is explicit in $q_o$. However, in the absence of a closed-form solution for $q_o$, which is often the case (hence the notion that PSLET is often semi-analytical), numerical solutions of (24) could resolve this issue. Once $q_o$ is determined the coefficients $C_{m,n,k}$, $D_{m,n,k}$, and $a_{p,k}^{(n)}$ are determined in a sequential manner. Hence, the eigenvalues, equation (21), and eigenfunctions, equations (10)-(13), are calculated in the same batch for each value of $k$, $D$, $l$, $c_1$, $c_2$, and $b$.

Table 1 shows PSLET results for the ground-state energies, covering a wide range of the coupling $c_2$ when $b = 2.5$, along with those reported by Hall and Saad [1a], via a generalized variational analysis and direct numerical integration methods. Using the interdimensional degeneracies, equations (3) and (4), or directly the dimensionality $D$ in $l_D$, we display the energies for $V(q) = (q^2 + 10/q^{1.9})/2$ in table 2. Clearly, our results compare excellently with those from direct numerical integrations. However, it should be noted that in [5] we have calculated the energy series up to $E_{0, l}^{(4)}/l^4$ correction. Therefore, slight discrepancies obtain between the present results in table 1 and those reported in table 2 of [5].

Adhering to the implicated wisdom in equations (3) and (4), that the
two- and three-dimensional (2D and 3D, respectively) cases are the basic ingredients of the energy ladder at larger dimensions, we report (in table 3) the 2D- and 3D-nodal bound-state energies when the coupling $c_2 = 1000$ and $b = 0.5, 1, \ldots, 2.5, 3$. The stability of the last three approximants of the Padé sequence indicate that the results are exact. For more details on this issue the reader may refer to refs [20,24]. Nevertheless, our results $E_{0,0}$ for the 3D-spiked harmonic oscillator are in exact accord with those from direct numerical integrations [2]. Following the same strategy, we display in table 4 the $k=1$ and 2 nodal bound-state energies for $V(q) = (q^2 + 1000/q^{3/2})/2$. Eventually, the leading term of PSLET, $\bar{l}^2 E^{(-2)}_{k,l'}$, turns out to be a good starting approximation. Tables 1, 2 and 4 bear this out.

Moreover, for the spiked harmonic oscillator, with $b = 2$, one would rewrite the effective potential term $(l(l+1)+c_2)/2q^2 + q^2/2$ as $l'(l'+1)/2q^2 + q^2/2$ with $l' = -1/2 + \sqrt{(l+1/2)^2 + c_2}$. For this particular case, PSLET procedure yields, respectively, $w = 2$, $\beta = -(2k + 3/2)$, $\bar{l} = 2k + l' + 3/2$, $q_o^2 = \bar{l}$, $\bar{l}^2 E^{(-2)}_{k,l'} = 2k + l' + 3/2$ (the exact well known energies),

$$E^{(0)}_{k,l'} = E^{(1)}_{k,l'} = \cdots = E^{(8)}_{k,l'} = \cdots = E^{(n)}_{k,l'} = 0,$$

and when $k = 0$, for example,

$$U_{0,l'}(x) = -\frac{1}{2} \left( y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5} - \frac{y^6}{6} + \frac{y^7}{7} - \frac{y^8}{8} + \cdots \right) + i \left( y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5} - \frac{y^6}{6} + \frac{y^7}{7} - \frac{y^8}{8} + \cdots \right) - \bar{l} \frac{y^2}{2} - \bar{l}y,$$

where $y = x\bar{l}^{-1/2}$. Obviously, the terms in brackets in equation (26) are the
infinite geometric series expansions for \( \ln(1+y) \). Equation (26) thus becomes

\[
U_{0,l'}(x) = \ln(1+y)^{-1/2} + \ln(1+y)\tilde{t} - \tilde{t}y - \frac{y^2}{2}.
\] (27)

Hence Eq.(8) (with \( F_{0,l'}(x) = 1 \) from (11)) reads

\[
\Psi_{0,l'}(q) = N_{0,l'} q^{l'+1} e^{-q^2/2},
\] (28)

the exact well known solutions [31], where \( N_{0,l'} \) are the normalization constants. Proceeding exactly as above, one could obtain the well known solutions with \( k \geq 1 \). However, this already lies far beyond the scope of our present methodical proposal.

Hall and Saad [1a] have therefore used, indirectly, the transformation of the angular momentum quantum number and cast the Hamiltonian of the spiked harmonic oscillator (1) as

\[
H = -\frac{1}{2} \frac{d^2}{dq^2} + \frac{l_H(l_H + 1)}{2q^2} + \frac{q^2}{2} + \frac{c_2}{2q^b} - \frac{A}{2q^2}.
\] (29)

Where \( l_H = -1/2 + \sqrt{(l+1/2)^2 + A} \), and \( A \) is used as a further variational refinement in their generalized variational analysis method. They found that \( A = c_2 \) is a good general estimate for the value of \( A \). Indeed this optimum value of \( A \), which reduced substantially the number of the basis function needed for a given accuracy in [1a], enhances the convergence and accuracy of approximation methodical recipes. Practically, it minimizes the effect of the perturbation term \( c_2q^{-b} \) over the harmonic oscillator one (with the irrational quantum number \( l' \)), especially for values of \( b \to 2 \). In table 5, the results of PSLET are obtained using such prescription. They compare
excellently with direct numerical integrations and do not contradict with the upper bounds from the generalized variational estimates.

4 Concluding remarks

We have generalized our pseudoperturbative shifted \(-l\) expansion technique PSLET \([5,20-22]\) for states with arbitrary number of nodal zeros, \(k \geq 0\). Starting with the central force problem, represented by the radial Schrödinger equation, and augmenting the orbital angular momentum by \(l \rightarrow l_D = l + (D - 3)/2\), we have incorporated interdimensional degeneracies. To test PSLET performance, we have treated the spiked harmonic oscillator problem in \(D\) - dimensions. and used results from direct numerical integrations and generalized variational analysis methods \([1,2]\) to compare with. The comparison is readily satisfactory.

The salient features of the attendant proposal PSLET are in order.

It avoids troublesome questions as those pertaining to the nature of small parameter expansions, the trend of convergence to the exact numerical values (marked in tables 1-3 and 5), the utility in calculating the eigenvalues and eigenfunctions in one batch to sufficiently higher - orders (documented through the solution (28) of (1), with \(b = 2\)), and the applicability to a wide range of potentials. Provided that the potential \(V(q)\) gives rise to one minimum of \(E_{k,l}^{(-2)}\) and an infinite number of bound - states. Moreover, beyond its promise as being quite handy (on the computational and practical methodical sides), it offers a useful perturbation prescription where the zeroth - order approximation \(l^2 E_{k,l}^{(-2)}\) inherits a substantial amount of the total energy.

The above has been a very limited review and a number of other useful approaches such as those presented by Papp \([9,32]\), Bender and Wu \([33]\), etc, have been left unattended. However, their accomplishments are indeed of
actual novelties.

Finally, the scope of PSLET applicability extends beyond the present $D$-dimensional spiked harmonic oscillator model. It could be applied to angular momentum states of multi-electron atoms [34-36], relativistic and non-relativistic quark-antiquark models [37], etc.
Appendix

Although some of the following expressions have appeared in previous articles [5,20-22], we would like to repeat them to make this article self contained.

Expansions about \( x = 0 \) (i.e. \( q = q_o \)), yield

\[
\frac{1}{q^2} = \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)}{q_o^2} x^n \bar{l}^{-n/2},
\]

(30)

\[
V(x(q)) = \sum_{n=0}^{\infty} \left( \frac{d^n V(q_o)}{dq_o^n} \right) \frac{(q_ox)^n}{n!} \bar{l}^{-n/2}.
\]

(31)

Equation (5) thus becomes

\[
\left[ -\frac{1}{2} \frac{d^2}{dx^2} + \frac{q_o^2}{l} V(x(q)) \right] \Psi_{k,l}(x) = \frac{q_o^2}{l} E_{k,l} \Psi_{k,l}(x),
\]

(32)

with

\[
\begin{align*}
\frac{q_o^2}{l} \bar{V}(x(q)) &= q_o^2 l \left[ \frac{1}{2q_o^2} + \frac{V(q_o)}{Q} \right] + \bar{l}^{1/2} B_1 x + B_2 x^2 + \frac{(2\beta + 1)}{2} \\
&\quad + (2\beta + 1) \sum_{n=1}^{\infty} (-1)^n \frac{(n+1)}{2} x^n \bar{l}^{-n/2} + \sum_{n=3}^{\infty} B_n x^n \bar{l}^{-(n-2)/2} \\
&\quad + \beta(\beta + 1) \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)}{2} x^n \bar{l}^{-(n+2)/2},
\end{align*}
\]

(33)

\[
B_n = (-1)^n \frac{(n+1)}{2} + \left( \frac{d^n V(q_o)}{dq_o^n} \right) \frac{q_o^{n+2}}{n!Q}.
\]

(34)
Equation (32), along with (33) and (34), is evidently the one-dimensional Schrödinger equation for a perturbed harmonic oscillator

\[
-\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} w^2 x^2 + \varepsilon_o + P(x) \left] X_k(x) = \lambda_k X_k(x), \right.
\]  

(35)

where \( w^2 = 2B_2 \),

\[
\varepsilon_o = \bar{l} \left[ \frac{1}{2} + \frac{q_o^2 V(q_o)}{Q} \right] + \frac{2\beta + 1}{2} + \frac{\beta(\beta + 1)}{2\bar{l}},
\]  

(36)

and \( P(x) \) represents the remaining terms in eq.(33) as infinite power series perturbations to the harmonic oscillator. One would then imply that

\[
\lambda_k = \bar{l} \left[ \frac{1}{2} + \frac{q_o^2 V(q_o)}{Q} \right] + \left[ \frac{2\beta + 1}{2} + (k + \frac{1}{2})w \right] \\
+ \frac{1}{\bar{l}} \left[ \frac{\beta(\beta + 1)}{2} + \lambda_k^{(0)} \right] + \sum_{n=2}^{\infty} \lambda_k^{(n-1)} \bar{l}^{-n},
\]  

(37)

and

\[
\lambda_k = q_o^2 \sum_{n=-2}^{\infty} E_{k,l}(n) \bar{l}^{-(n+1)}. \]  

(38)

Hence, equations (37) and (38) yield

\[
E_{k,l}^{(-2)} = \frac{1}{2q_o^2} + \frac{V(q_o)}{Q},
\]  

(39)

\[
E_{k,l}^{(-1)} = \frac{1}{q_o^2} \left[ \frac{2\beta + 1}{2} + (k + \frac{1}{2})w \right] \]  

(40)

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\[ E_{k,l}^{(0)} = \frac{1}{q_o^2} \left[ \frac{\beta(\beta + 1)}{2} + \lambda_k^{(0)} \right] \]  

(41)

\[ E_{k,l}^{(n)} = \frac{\lambda_k^{(n)}}{q_o^2}; \quad n \geq 1. \]  

(42)

Where \( q_o \) is chosen to minimize \( E_{k,l}^{(-2)} \), i.e.

\[ \frac{dE_{k,l}^{(-2)}}{dq_o} = 0 \quad \text{and} \quad \frac{d^2E_{k,l}^{(-2)}}{dq_o^2} > 0. \]  

(43)

Hereby, \( V(q) \) is assumed to be well behaved so that \( E_{k,l}^{(-2)} \) has a minimum \( q_o \) and there are well-defined bound states. Equation (43) in turn gives, with \( \bar{l} = \sqrt{Q} \),

\[ l_D - \beta = \sqrt{q_o^3V'(q_o)}. \]  

(44)

Consequently, the second term in Eq.(33) vanishes and the first term adds a constant to the energy eigenvalues. It should be noted that the energy term \( \bar{l}^2E_{k,l}^{(-2)} \) corresponds roughly to the energy of a classical particle with angular momentum \( L_z = \bar{l} \) executing circular motion of radius \( q_o \) in the potential \( V(q_o) \).

It thus identifies the zeroth-order approximation, to all eigenvalues, as a classical approximation and the higher-order corrections as quantum fluctuations around the minimum \( q_o \), organized in inverse powers of \( \bar{l} \). The next correction to the energy series, \( \bar{l}E_{k,l}^{(-1)} \), consists of a constant term and the exact eigenvalues of the harmonic oscillator \( w^2x^2/2 \). The shifting parameter \( \beta \) is determined by choosing \( \bar{l}E_{k,l}^{(-1)} = 0 \). This choice is physically motivated.
In addition to its vital role in removing the singularity at \( l = 0 \), it also requires the agreements between PSLET eigenvalues and eigenfunctions with the exact well known ones for the harmonic oscillator and Coulomb potentials. Hence

\[
\beta = -\left[ \frac{1}{2} + (k + \frac{1}{2})w \right],
\]

(45)

where \( w = \sqrt{3 + q_o V''(q_o)/V'(q_o)} \), and primes of \( V(q_o) \) denote derivatives with respect to \( q_o \). Then equation (33) reduces to

\[
\frac{q_o^2}{l} \tilde{V}(x(q)) = q_o^2 \tilde{l} \left[ \frac{1}{2q_o^2} + \frac{V(q_o)}{Q} \right] + \sum_{n=0}^{\infty} v^{(n)}(x) \tilde{l}^{-n/2},
\]

(46)

where

\[
v^{(0)}(x) = B_2 x^2 + \frac{2\beta + 1}{2},
\]

(47)

\[
v^{(1)}(x) = -(2\beta + 1)x + B_3 x^3,
\]

(48)

and for \( n \geq 2 \)

\[
v^{(n)}(x) = B_{n+2} x^{n+2} + (-1)^n (2\beta + 1) \frac{(n+1)}{2} x^n + (-1)^n \frac{\beta(\beta+1)}{2} (n-1) x^{(n-2)}.
\]

(49)
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Table 1: 3D ground-state energies, in $\hbar = m = 1$ units, for $V(q) = (q^2 + c_2/q^{5/2})/2$. Where $E_P$ represents PSLET results, Eq.(41), and $l^2E^{(-2)}$ is its zeroth-order approximation. $E[4,4]$ shows the effect of the $P^4_4(1/l)$ Padé approximant, Eq.(42). $E_{VAM}$ from VAM, and $E_{DNI}$ from DNI [1a].

| $c_2$ | $l^2E^{(-2)}$ | $E_P$ | $E[4,4]$ | $E_{VAM}$ | $E_{DNI}$ |
|-------|---------------|-------|----------|-----------|-----------|
| 1000  | 44.003142     | 44.9554848 | 44.9554848 | 44.955485 | 44.955485 |
| 100   | 16.666664     | 17.541890  | 17.541890  | 17.541890 | 17.541890 |
| 10    | 7.00149       | 7.73515    | 7.73510    | 7.73511   | 7.73511   |
| 1     | 3.84771       | 4.31578    | 4.31413    | 4.32326   | 4.31731   |
| 0.1   | 3.11132       | 3.26984    | 3.26633    | 3.29602   | 3.26687   |
| 0.01  | 3.0116        | 3.0341     | 3.0344     | 3.0392    | 3.0367    |
| 0.001 | 3.0012        | 3.0035     | 3.0040     | 3.0041    | 3.0040    |
Table 2: $D = 2, \ldots, 10$ ground-state energies, in $\hbar = m = 1$ units, for the potential $V(q) = (q^2 + 10/q^{1.9})/2$. Where $E_P$ represents PSLET results, Eq.(41), and $ar{l}^2E^{(-2)}$ is its zeroth-order approximation, $E[4,4]$ shows the effect of the $P_4^4(1/\bar{l})$ Padé approximant, Eq.(42). $E_{VAM}$ from VAM, and $E_{DNI}$ from DNI [1a].

| $D$ | $\bar{l}^2E^{(-2)}$ | $E_P$     | $E[4,4]$  | $E_{VAM}$ | $E_{DNI}$ |
|-----|----------------|----------|-----------|-----------|-----------|
| 2   | 7.581 139     | 8.485 461| 8.485 369 | 8.485 384 | 8.485 378 |
| 3   | 7.919 880     | 8.564 352| 8.564 355 | 8.564 358 | 8.564 356 |
| 4   | 8.339 920     | 8.795 436| 8.795 440 | 8.795 440 | 8.795 440 |
| 5   | 8.840 678     | 9.163 092| 9.163 093 | 9.163 093 | 9.163 093 |
| 6   | 9.416 352     | 9.646 701| 9.646 701 | 9.646 701 | 9.646 701 |
| 7   | 10.058 042    | 10.225 045| 10.225 045| 10.225 045| 10.225 045|
| 8   | 10.755 870    | 10.879 077| 10.879 077| 10.879 077| 10.879 077|
| 9   | 11.500 402    | 11.592 982| 11.592 982| 11.592 982| 11.592 982|
| 10  | 12.283 349    | 12.354 183| 12.354 183| 12.354 183| 12.354 183|
Table 3: 2D - and 3D - nodeless states energies, with \( l = 0, \ldots, 4 \) (in \( \hbar = m = 1 \) units), for the potential \( V(q) = (q^2 + 1000/q^b)/2 \). Where \( E_{0,l} \) represents PSLET results with the \( P^4_4(1/l) \) Padé approximant, Eq.(42).

| \( D \) | \( b \) | \( E_{0,0} \) | \( E_{0,1} \) | \( E_{0,2} \) | \( E_{0,3} \) | \( E_{0,4} \) |
|-------|-------|---------|---------|---------|---------|---------|
| 2     | 0.5   | 415.886751 | 415.898889 | 415.935293 | 415.995938 | 416.080780 |
|       | 1     | 190.719321  | 190.735267  | 190.783089  | 190.862739  | 190.974135  |
|       | 1.5   | 104.404517  | 104.427341  | 104.495769  | 104.609681  | 104.768874  |
|       | 2     | 65.245553   | 65.277168   | 65.371918   | 65.529521   | 65.749510   |
|       | 2.5   | 44.945030   | 44.986838   | 45.112071   | 45.320150   | 45.610129   |
|       | 3     | 33.303511   | 33.356491   | 33.515080   | 33.778229   | 34.144222   |
| 3     | 0.5   | 415.889786  | 415.914059  | 415.962588  | 416.035338  | 416.132258  |
|       | 1     | 190.72331   | 190.755196  | 190.818940  | 190.914475  | 191.041704  |
|       | 1.5   | 104.41022   | 104.455860  | 104.547051  | 104.683633  | 104.865367  |
|       | 2     | 65.253459   | 65.316665   | 65.442888   | 65.631753   | 65.882705   |
|       | 2.5   | 44.95549    | 45.039054   | 45.205805   | 45.454976   | 45.785438   |
|       | 3     | 33.31676    | 33.422634   | 33.633677   | 33.948503   | 34.365078   |
Table 4: 2D and 3D k - state energies, in $\hbar = m = 1$ units, for the potential $V(q) = (q^2 + 1000/q^{3/2})/2$. Where $E_P$ represents PSLET results, Eq.(41), and $\tilde{l}^2E^{(-2)}$ is its zeroth - order approximation, $E[4, 4]$ shows the effect of the $P_4^4(1/\tilde{l})$ Padé approximant, Eq.(42).

| $D$ | $k$ | $l$ | $\tilde{l}^2E^{(-2)}$ | $E_P$ | $E[4, 4]$ |
|-----|-----|-----|-----------------|------|-----------|
| 2   | 1   | 0   | 105.40419       | 108.15083 | 108.15083 |
|     | 1   |     | 105.67466       | 108.17379 | 108.17379 |
|     | 2   |     | 105.96940       | 108.24263 | 108.24263 |
|     | 3   |     | 106.28970       | 108.35721 | 108.35721 |
| 3   | 0   |     | 105.53648       | 108.15657 | 108.15657 |
|     | 1   |     | 105.81892       | 108.20248 | 108.20248 |
|     | 2   |     | 106.12628       | 108.29421 | 108.29421 |
|     | 3   |     | 106.45983       | 108.43160 | 108.43160 |
| 2   | 2   | 0   | 107.3876        | 111.9017  | 111.9017  |
|     | 1   |     | 107.7382        | 111.9248  | 111.9248  |
|     | 2   |     | 108.1127        | 111.9940  | 111.9940  |
| 3   | 0   |     | 107.5600        | 111.9075  | 111.9074  |
|     | 1   |     | 107.9224        | 111.9536  | 111.9536  |
|     | 2   |     | 108.3092        | 112.0459  | 112.0459  |

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Table 5: $k = 2$ and $l = 1, 2$ energies, in $\hbar = m = 1$ units, for the potential $V(q) = (q^2 + 10/q^2)/2$. Where $E_{2,t,P}$ represents PSLET results, Eq.(41), $E_{2,1,V}$ from VAM, and $E_{2,1,ex}$ from DNI [1a]. $E_{2,t}[4, 4]$ shows the effect of the $P_t^1(1/\bar{l})$ Padé approximant, Eq.(42).

| $D$ | $E_{2,1,ex}$ | $E_{2,1,V}$ | $E_{2,1,P}$ | $E_{2,1}[4, 4]$ | $E_{2,2,P}$ | $E_{2,2}[4, 4]$ |
|-----|---------------|---------------|---------------|----------------|---------------|----------------|
| 2   | 16.543629     | 16.543648     | 16.541951     | 16.543627      | 17.380817     | 17.381708      |
| 3   | 16.904445     | 16.904446     | 16.903172     | 16.904444      | 17.954856     | 17.955444      |
| 4   | 17.381708     | 17.381709     | 17.380817     | 17.381708      | 18.606695     | 18.607067      |
| 5   | 17.955444     | 17.955446     | 17.954856     | 17.955444      | 19.320461     | 19.320691      |
| 6   | 18.607067     | 18.607070     | 18.606695     | 18.607067      | 20.083266     | 20.083406      |
| 7   | 19.320691     | 19.320693     | 19.320461     | 19.320691      | 20.884936     | 20.885021      |
| 8   | 20.083406     | 20.083407     | 20.083266     | 20.083406      | 21.717556     | 21.717608      |
| 9   | 20.885021     | 20.885022     | 20.884936     | 20.885021      | 22.574996     | 22.575027      |
| 10  | 21.717608     | 21.717608     | 21.717556     | 21.717608      | 23.452505     | 23.452524      |