Steady state for the subcritical contact branching random walk on the lattice with the arbitrary number of offspring and with immigration*

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Abstract
We consider the subcritical contact branching random walk on $\mathbb{Z}^d$ in continuous time with the arbitrary number of offspring and with immigration. We prove the existence of the steady state (statistical equilibrium).

1 Introduction
This paper is the continuation of our previous publication [2]. As in the majority of the publications in the area of population dynamics, we considered binary splitting in [2]. During time interval $[t, t + dt]$, each particle in our population

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either dies with probability \( \mu dt \) or produces with probability \( \beta dt \) one offspring which jumps from the parental particle at the size \( x \in \mathbb{Z}^d \) to the random point \( x + z \) with probability \( b(z) \). We assume \( b(z) = b(-z) \) and \( \sum_{z \in \mathbb{Z}^d} b(z) = 1 \). In other terms, the infinitesimal generating function describing the branching has a form \( \phi(z) = \beta z^2 - (\beta + \mu)z + \mu \). \([1, 5, 6, 7, 12]\) are also based on the binary splitting.

But in many applications, especially in the model of forest introduced in \([12]\), where the particles (i.e., trees) do not move at all but produce the seeds which are randomly distributed around the parental tree (we introduce this option in our more general model \([5]\)), the assumption of the binary splitting is highly artificial. In contrast, the natural assumption here is that typical number of seeds is large (of order hundreds and thousands), i.e., The infinitesimal generating function has now a form \( \phi(z) = \sum_{l=2}^{\infty} \beta_l \psi_l z^l - (\sum_{l=2}^{\infty} \beta_l + \mu) \psi_z + \mu \). It is well known that for heavy tailed distribution \( \beta_l \), the branching process can explode. Since we will use low moment analysis and the Carleman type conditions for the uniqueness of the solution of the moments problem, we will assume that \( \{\beta_l, l \geq 2\} \) have geometrically decay, i.e., \( \phi(z) \) is analytic in the circle \(|z| \leq 1 + \delta, \delta > 0\).

Let us note that for the arbitrary number of offspring, corresponding moments (mean numbers of offspring, variance of this number etc.) can be essentially different. It leads to important phenomenon of the intermittency in the model of the forest.

In this paper, we study the steady state for the subcritical branching random walk on the lattice with the arbitrary number of offspring. It is not only the natural continuation of the publication \([2]\), which consider the binary splitting. It is also a natural continuation of the publication \([3]\), which study the convergence of the population to the statistical equilibrium for critical contact process on the lattice \( \mathbb{Z}^d \). We consider the general model of the subcritical branching random walk on the lattice \( \mathbb{Z}^d \). The structure of this paper is as follows: In section 2, we introduce our model, containing the random walk with generator \( L_a \), mortality rate \( \mu \), splitting rate with arbitrary number of offspring and their distribution around parental particle with some law and immigration rate \( \gamma \). We provide in this section several technical lemmas. In section 3, we prove a Carleman type estimate for the cumulants of subcritical populations and prove the existence of a steady state. The last section contains the summary.

2 Description of the model

Let \( N(t, y) \) to be the particles field on the lattice \( \mathbb{Z}^d \) with continuous time \( t \geq 0 \), i.e., \( N(t, y) \) is the numbers of population at site \( y \in \mathbb{Z}^d \) at the moment \( t \). The evolution of this particle field consists of several elements:

- Each particle independently on others performs (until the transformation:
Each particle in the site \( \tau \) and subpopulation \( n \) at site \( l \) \( n \) \( y \) \( \in \mathbb{Z}^d \). It means that \( \forall z \in \mathbb{Z}^d \), there is some integer \( k \geq 1 \), there are some vectors \( z_1, \cdots, z_k \) and there are some positive integers \( n_1, \cdots, n_k \) such that \( y = \sum_{i=1}^{k} n_i z_i \) and \( a(z_i) > 0 \) for \( i = 1, \ldots, k \).

- Each particle in the site \( x \) during the time interval \( [t, t+dt] \) (independently on others and past time) can annihilate (die) with probability \( \mu \) or splits onto \( l \) particles with probabilities \( \beta_l \) \( dt \) where \( l \geq 2 \). In such splitting, one offspring (it can be considered the parental particle) remains at \( x \) and the other \( l - 1 \) particles jump independently from \( x \) to \( x+v \) with probability distribution \( b(v) \), where \( b(v) = b(-v) \) and \( \sum_{v \in \mathbb{Z}^d \setminus \{0\}} b(v) = 1 \). We assume that

\[
\Delta = \mu - \sum_{l=2}^{\infty} (l-1) \beta_l > 0. \tag{2}
\]

- We also assume that for any site \( x \), the new particles (immigrants) appear at the moments \( 0 < \tau_1(x) < \tau_2(x) \cdots \) and \( \tau_{i+1}(x) - \tau_i(x) \sim \text{Exp} (\gamma) \). In different terms, moments \( \tau_i(x) \), \( i \geq 1 \) form a Poissonian point field on \( \{x\} \times [0, \infty) \) with parameter \( \gamma \). Meanwhile, we assume the independence of such point fields for different \( x \in \mathbb{Z}^d \).

Let \( n(t-\tau_i(x), x, y) \) denote the subpopulation, i.e., the number of particles, at site \( y \in \mathbb{Z}^d \) at time \( t \) descended from a particle that appeared at \( x \) (immigrated) at time \( \tau_i(x) < t \). Without loss of generality, we can assume that \( N(0,y) \equiv 0 \), since all subpopulation starting at the moment \( t = 0 \) will vanish to the large moment \( t \) with probability at least \( e^{-\Delta t} \). As a result, we have the following important representation

\[
N(t,y) = \sum_{x \in \mathbb{Z}^d} \sum_{\tau_i(x) \leq t} n(t-\tau_i(x), x, y), \tag{3}
\]

where subpopulation \( n(t-\tau_i(x), x, y) \) are independent for different \( x \in \mathbb{Z}^d \) and \( \tau_i \leq t \).

\[
N(t,y) \overset{\text{Law}}{=} \sum_{x \in \mathbb{Z}^d} \sum_{\tau_i(x) \leq t} n(t-\tau_i(x), x, y),
\]

\[
= \sum_{x \in \mathbb{Z}^d} \sum_{\xi_1+\cdots+\xi_k \leq t} n(\xi_1 + \cdots + \xi_k, x, y),
\]
where $\xi_i \sim Exp(\gamma)$.

Let us consider the subpopulation $n(t, x, y)$. We introduce the generating function for an individual subpopulation

$$u_z(t, x; y) = E_n^{n(t, x, y)}.$$  (4)

We hereafter consider this as a function of the variables $t$ and $x$. For every fixed $y \in \mathbb{Z}^d$, $u_z(t, x; y)$ satisfies the backward Kolmogorov equation (where we omit the arguments $(t, x; y)$):

$$\frac{\partial u_z}{\partial t} = \kappa L_a u_z - \left( \mu + \sum_{l=2}^{\infty} \beta_l \right) u_z + \mu u_z \sum_{l=2}^{\infty} \beta_l (u_z * b)^{l-1}$$  (5)

with initial condition $u_z(0, x; y) = \delta(x - y)$ if $x = y$ and $u_z(0, x; y) = 1$ otherwise.

Here, we use the following designation for the convolution of two functions:

$$u_z * b = (u_z * b)(t, x; y) = \sum_{v \in \mathbb{Z}^d} u_z(t, x - v; y) b(v).$$  (6)

From (5) we can derive equations for the factorial moments

$$m_k(t, x; y) = E[n(t, x, y)(n(t, x, y) - 1) \cdots (n(t, x, y) - k + 1)] = \left. \frac{\partial^k u_z}{\partial z^k} \right|_{z=1} (t, x; y),$$  (7)

where $k = 1, 2, \ldots$. In particular, by differentiating Eq. (5) we obtain an equation for the first moment:

$$\frac{\partial m_1}{\partial t} = \left( \kappa L_a + \sum_{l=2}^{\infty} (l-1)\beta_l L_b \right) m_1 + \Delta m_1,$$  (8)

$$m_1(0, x; y) = \delta(x - y).$$

Here, $L_b$ is defined as (similarly to Eq. (1)):

$$L_b f = (L_b f)(x) = \sum_{v \neq 0} b(v) [f(x + v) - f(x)].$$  (9)

The solution of (8) is:

$$m_1(t, x, y) = e^{-\Delta t} p(t, x, y),$$  (10)

where $p(t, x, y)$ (fundamental solution) is the transition probability of the event that a particle that starts at $x \in \mathbb{Z}^d$ arrives at $y \in \mathbb{Z}^d$ during time $t > 0$ for the random walk which is defined by the symmetric isotropic generator $\kappa L_a + \sum_{l=2}^{\infty} (l-1)\beta_l L_b$, i.e. $p(t, x, y)$ satisfies the following equation

$$\frac{\partial p(t, x, y)}{\partial t} = \left( \kappa L_a + \sum_{l=2}^{\infty} (l-1)\beta_l L_b \right) p(t, x, y)$$  (11)

where $p(0, x, y) = \delta_x(y)$. 4
Denote \( \hat{p}(t, k, 0) = \sum_x e^{ikx} p(t, x, 0) \). Applying Fourier transform on both side of the Eq. 11 we have

\[
\frac{\partial \hat{p}(t, k, 0)}{\partial t} = \left( \kappa \hat{L}_a(k) + \sum_{l=2}^{\infty} (l-1) \beta_l \hat{L}_b(k) \right) \hat{p}(t, k, 0)
\]

where \( \hat{L}_a(k) = 1 - \hat{a}(k) \), \( \hat{L}_b(k) = 1 - \hat{b}(k) \), \( \hat{a}(k) = \sum_z \cos(k, z)a(z) \), \( \hat{b}(k) = \sum_z \cos(k, z)b(z) \). As a result,

\[
\hat{p}(t, k, 0) = e^{t(\kappa \hat{L}_a(k) + \sum_{l=2}^{\infty} (l-1) \beta_l \hat{L}_b(k))}.
\]

Therefore, the transition probability of the underlying random walk has the form

\[
p(t, x, y) = \frac{1}{(2\pi)^d} \int_{T^d} e^{t(\kappa \hat{L}_a(k) + \sum_{l=2}^{\infty} (l-1) \beta_l \hat{L}_b(k))} e^{-ik(x-y)} dk,
\]

and \( T^d = [-\pi, \pi]^d \).

Note that

\[
\sum_{y \in \mathbb{Z}^d} p(t, x, y) = 1, \quad (12)
\]

\[
p(t, x, y) \leq p(t, x, x) = p(t, 0, 0) \quad (13)
\]

For each \( x \in \mathbb{Z}^d \), \( \nu_x(t) = \sum_{y \in \mathbb{Z}^d} n(t, x, y) \) is a Galton-Watson process, see [13]. We have the well known equation for the generating function of this process \( \psi_z(t) := E_z^{\nu_x(t)} \):

\[
\frac{\partial \psi_z}{\partial t} = \sum_{l=2}^{\infty} \beta_l \psi_z^{l-1} - \left( \sum_{l=2}^{\infty} \beta_l + \mu \right) \psi_z + \mu = (\psi_z - 1) \left( \sum_{l=2}^{\infty} \beta_l (\psi_z^{l-1} + \psi_z^{l-2} + \cdots + \psi_z) - \mu \right),
\]

\[
\psi_z(0) = z.
\]

Please refer to [13] for more details of discussion for \( \psi_z(t) \).

### 3 Main result

The central goal of this paper is to prove the convergence of the particle field \( N(t, y), y \in \mathbb{Z}^d \) to a steady state (statistical equilibrium).

**Theorem 1.** Let \( N(t, y), y \in \mathbb{Z}^d \) be a random field as described above, assume that for all \( l \geq 2 \),

\[
\beta_l \leq \beta \delta^l \text{ for some } \beta > 0, \delta \in (0, 1).
\]
Then, for all $y \in \mathbb{Z}^d$

$$N(t, y) \xrightarrow{\text{Law}} N(\infty, y).$$

**Remark:**

- The condition that $\beta_l$ decreases geometrically implies that the generating function of this sequence $\sum_{l=2}^{\infty} \beta_l z^l$ is analytic in the disk $|z| < \delta$ for some suitable $\delta > 0$.

- In order to prove Theorem (11), we first will estimate all factorial moments of a subpopulation, i.e. $m_k(t, x; y), k \geq 1$, see Eq. (7). From this and the relationship between moments and cumulants, we can estimate cumulants for the total population $N(t, y)$ uniformly in $t$. Using the monotonicity in $t$ and boundedness of these cumulants, we can conclude that their limit exists at $t \to \infty$. Then we will use the Carleman conditions to establish a unique limiting distribution.

Under the model assumption, it is trivial that $m_0(t, x; y) \equiv u_z(t, x; y) |_{z=1} = 1$. For all $k \geq 2$, differentiate Eq. (5) $k$-times differentiation, we can derive equation for the $k$-th factorial moments:

$$\frac{\partial m_k}{\partial t} = \left( \kappa \mathcal{L}_a + \sum_{l=2}^{\infty} (l - 1) \beta_l \mathcal{L}_b \right) m_k - \Delta m_k +$$

$$\sum_{l=2}^{\infty} \beta_l \sum_{n=1}^{k-1} \frac{n!}{l!} \sum_{j_1 + \ldots + j_l = k - n} \sum_{j_s \geq 0} k! \cdot \ldots \cdot (m_{j_1} \ast b) \cdot \ldots \cdot (m_{j_l} \ast b) +$$

$$\sum_{l=2}^{\infty} \beta_l \sum_{\sum_{j_s \geq 0} j_s = k} \frac{k!}{j_1! \cdot \ldots \cdot j_l!} (m_{j_1} \ast b) \cdot \ldots \cdot (m_{j_l} \ast b)$$

with the initial condition $m_k(0, x; y) = 0$ when $k \geq 2$. Without loss of generality, we assume that $y = 0$.

Let us first recall Duhamel’s principal.

**Lemma 2.** *(Duhamel’s principal)* if $f(t, x), t \geq 0, x \in \mathbb{Z}^d$ is the fundamental solution of the homogeneous equation:

$$\frac{\partial f}{\partial t}(t, x) = \mathcal{L} f(t, x)$$

with the initial condition $f(0, x) = \delta(x)$, then the equation:

$$\frac{\partial F}{\partial t}(t, x) = \mathcal{L} F(t, x) + h(t, x)$$

with the initial condition $F(0, x) = 0$ has the solution:

$$F(t, x) = \int_0^t ds \sum_{z \in \mathbb{Z}^d} f(t - s, x - z) h(s, z).$$
In order to estimate all factorial moments of a subpopulation, the proof of the next lemma will be similar to the proofs in [3].

Lemma 3. Under the conditions of Theorem 1, for all \( k \geq 1 \)

\[
m_k(t, x; 0) \leq k! B^{k-1} D_k e^{-\Delta t} p(t, x, 0),
\]

where

\[
B = \max \left\{ 1, \beta \int_0^{\infty} e^{-\Delta s} p(s, 0, 0) ds \right\} < \infty
\]

and the sequence \( D_k \) is recursively defined as: \( D_1 = 1 \) and, for \( k \geq 2 \)

\[
D_k = \sum_{l=2}^{\infty} \beta_l \sum_{n=1}^{k-1} \left( \begin{array}{c} l - 1 \\ i \end{array} \right) \sum_{j_x = 1 \atop j_x \geq 1}^{\sum_{i=1}^{l-1} j_x = k-n} D_{j_1} \cdots D_{j_l} + \sum_{l=2}^{\infty} \beta_l \sum_{n=1}^{k-1} \left( \begin{array}{c} l - 1 \\ i \end{array} \right) \sum_{j_x = 1 \atop j_x \geq 1}^{\sum_{i=1}^{l-1} j_x = k} D_{j_1} \cdots D_{j_l}.
\]

Proof: Denote \( \tilde{m}_k(t, x; 0) = \frac{m_k(t, x; 0)}{k!} \), \( M_j = \tilde{m}_k * b = \sum_{v \in \mathbb{Z}^d} b(v) \tilde{m}_j(t, x + v; 0) \), and \( L_{a, b} = \kappa L_a + \sum_{l=2}^{\infty} (l-1) \beta_l L_b \). Then, Eq. (15) has the form

\[
\frac{\partial \tilde{m}_k}{\partial t} = L_{a, b} \tilde{m}_k + \sum_{l=2}^{\infty} \beta_l \sum_{n=1}^{k-1} \tilde{m}_n \sum_{j_x = 1 \atop j_x \geq 0}^{\sum_{l=1}^{l-1} j_x = k-n} M_{j_1} \cdots M_{j_{l-1}} + \sum_{l=2}^{\infty} \beta_l \sum_{j_x = 1 \atop 0 \leq j_x \leq k-1}^{\sum_{l=1}^{l-1} j_x = k} M_{j_1} \cdots M_{j_{l-1}}.
\]

From Duhamel’s formula, we obtain that

\[
\tilde{m}_k(t, x; 0) = \int_0^t ds e^{-\Delta (t-s)} \sum_{z \in \mathbb{Z}^d} p(t-s, x-z, 0) \sum_{l=2}^{\infty} \beta_l \sum_{n=1}^{k-1} \tilde{m}_n \sum_{j_x = 1 \atop j_x \geq 0}^{\sum_{l=1}^{l-1} j_x = k-n} M_{j_1} \cdots M_{j_{l-1}} (s, z; 0) + \int_0^t ds e^{-\Delta (t-s)} \sum_{z \in \mathbb{Z}^d} p(t-s, x-z, 0) \sum_{l=2}^{\infty} \beta_l \sum_{j_x = 1 \atop 0 \leq j_x \leq k-1}^{\sum_{l=1}^{l-1} j_x = k} M_{j_1} \cdots M_{j_{l-1}} (s, z; 0).
\]

If we excluding \( M_0 \equiv 1 \), then the inner sum of the first term in Eq. (23) can be
\[
\sum_{n=1}^{k-1} \tilde{m}_n \sum_{\sum_{s=1}^{l-1} j_s = k-n, j_s \geq 0} M_{j_1} \cdot \ldots \cdot M_{j_{l-1}} = \\
\sum_{n=1}^{k-1} \tilde{m}_n \sum_{i=1}^{l-1} \binom{l-1}{i} \sum_{\sum_{s=1}^{l-1} j_s = k-n, j_s \geq 0} M_{j_1} \cdot \ldots \cdot M_{j_l}.
\]

(24)

and the inner sum of the second term in Eq. (23) can be written as:

\[
\sum_{\sum_{s=1}^{l-1} j_s = k, 0 \leq j_s \leq k-n} M_{j_1} \cdot \ldots \cdot M_{j_l} \leq \\
\sum_{n=1}^{k-1} \tilde{m}_n \sum_{i=1}^{l-1} \binom{l-1}{i} \sum_{\sum_{s=1}^{l-1} j_s = k-n, j_s \geq 0} M_{j_1} \cdot \ldots \cdot M_{j_l}.
\]

(25)

In the following, we will prove the lemma using mathematical induction.

For \( k = 1 \),

\[ \tilde{m}_1(t, x; 0) = p(t, x, 0). \]

and \( p(t, x, 0) \) is the fundamental solution of Eq. (10) and the base of induction is verified.

Let’s assume that Eq. (19) is true for \( k - 1 \). Then, the right-hand side of Eq. (24) is bounded by

\[
B^{k-1} e^{-\Delta s} p(s, z; 0) \sum_{n=1}^{k-1} D_n \sum_{i=1}^{l-1} \binom{l-1}{i} \left( \frac{e^{-\Delta s} (p \ast b)}{B} \right)^i \sum_{\sum_{s=1}^{l-1} j_s = k-n, j_s \geq 1} D_{j_1} \cdot \ldots \cdot D_{j_l} \leq \\
B^{k-1} e^{-\Delta s} p(s, z; 0) \frac{p(s, 0, 0) e^{-\Delta s}}{B} \sum_{n=1}^{k-1} D_n \sum_{i=1}^{l-1} \binom{l-1}{i} \sum_{\sum_{s=1}^{l-1} j_s = k-n, j_s \geq 1} D_{j_1} \cdot \ldots \cdot D_{j_l},
\]

(26)

where we use simple facts that for all \( x, y \in \mathbb{Z}^d \) \( p(t, x, y) \leq p(t, 0, 0) \) from Eq. (23) and \( (p \ast b)(t, x, 0) \leq p(t, x, 0) \).

Indeed,

\[
(p \ast b)(t, x, 0) = \sum_z b(z) p(t, x - z, 0) \\
= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{p}(t, k, 0) \hat{b}(k) e^{-ikx} \, dk \\
\leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{p}(t, k, 0) e^{-ikx} \, dk \\
= p(t, x, 0).
\]

Here we use the fact that \( \hat{p}(t, k, 0) \) and \( \hat{b}(k) \) are real and not larger than 1.
From the definition of $B$, see Eq. (20), for all $i \geq 1$ we have
\[
\left( \frac{(p * b)(s, z, 0)e^{-\Delta s}}{B} \right)^i \leq \frac{e^{-\Delta s}p(s, z, 0)}{B} \leq \frac{e^{-\Delta s}p(s, 0, 0)}{B}.
\]
Analogously, the right-hand side of Eq. (25) is bounded by
\[
B^{k-1}p(s, z, 0)e^{-\Delta s} e^{-\Delta s} \sum_{j=1}^{l-1} \left( \begin{array}{c} l-1 \\ i \end{array} \right) \sum_{j=1}^{l-1} \sum_{j_s \geq 1} D_{j_1} \cdots D_{j_l}.
\]
(27)

Now we can substitute it into Eq. (23):
\[
\tilde{m}_k(t, x; 0) \leq B^{k-1} e^{-\Delta t} \int_0^t ds \frac{e^{-\Delta s}p(s, 0, 0)}{B} \sum_{z \in \mathbb{Z}^d} p(t - s, x - z, 0)p(s, z, 0) \sum_{n=1}^\infty \beta_l \left( \sum_{i=1}^{l-1} \sum_{j=1}^{l-1} \left( \begin{array}{c} l-1 \\ i \end{array} \right) \sum_{j=1}^{l-1} \sum_{j_s \geq 1} D_{j_1} \cdots D_{j_l} \right)
\]
(28)

Base on the following facts:

- $\sum_{z \in \mathbb{Z}^d} p(t - s, x - z, 0)p(s, z, 0) = \sum_{z \in \mathbb{Z}^d} p(t - s, x, z)p(s, z, 0) = p(t, x, 0)$ from Chapman-Kolmogorov equation;
- $\beta_l \leq \beta l^l$ from assumption of the lemma;
- $\frac{\beta l^l e^{-\Delta s}p(s, 0, 0)ds}{B} \leq 1$ from Eq. (20),

We can state the lemma using the recursive definition of the sequence $D_k$
Eq. (21) □

**Lemma 4.** The sequence $D_k$ that is determined by $D_1 = 1$ and Eq. (21) increases no faster than geometrically.

The geometrically growth of $D_k$ states in Lemma (4) is proved in Lemma 2 in [3]. From Lemma (3) and Lemma (4), we have the following Corollary.
Corollary 5.

\[ m_k(t, x; 0) \leq c^k k! e^{-\Delta t} p(t, x, 0) \]  
for all \( k \geq 1 \) and

\[ \sum_{x \in \mathbb{Z}^d} m_k(t, x; 0) \leq c^k k! e^{-\Delta t}. \]

Let us now introduce the notation for cumulants. For any random variable \( X \), let \( \phi_X(z) = E z^X \), then the \( l \)-th cumulant

\[ \chi_l(X) = \frac{d^l}{dz^l} \ln(\phi_X(z)) \bigg|_{z=1}. \]

In general, the relationship between moments and cumulants is given by

\[ \chi_l(X) = l! \sum (-1)^{j_1 + \cdots + j_l - 1}(j_1 + \cdots + j_l - 1) \prod_{k=1}^{l} \left( \frac{m_k(X)}{k!} \right)^{j_k} \]

and

\[ m_l(X) = l! \sum \frac{1}{j_1! \cdots j_l!} \prod_{k=1}^{l} \left( \frac{\chi_k(X)}{k!} \right)^{j_k} \]

where the sign \( \sum \) means the sum over all non-negative integers \((j_1, \cdots, j_l)\) satisfying the constraint

\[ 1j_1 + 2j_2 + 3j_3 + \cdots + lj_l = l. \]

One important property of cumulants is additivity: for independent random variables \( X \) and \( Y \), \( \chi_l(X + Y) = \chi_l(X) + \chi_l(Y) \).

Due to previous remark we obtain that

\[ \chi_l(N(t, 0)) = \chi_l \left( \sum_{x \in \mathbb{Z}^d} \sum_{\tau_i(x) \leq t} n(t - \tau_i(x), x, 0) \right) \]

\[ = \sum_{x \in \mathbb{Z}^d} \chi_l \left( \sum_{\tau_i(x) \leq t} n(t - \tau_i(x), x, 0) \right). \]

In order to calculate \( \chi_l \left( \sum_{\tau_i(x) \leq t} n(t - \tau_i(x), x, 0) \right) \), we will prove the following Lemma.

Lemma 6. Let \( \xi \) be a random variable uniformly distributed on \([0, t]\), then

\[ \chi_l \left( \sum_{\tau_i(x) \leq t} n(t - \tau_i(x), x, 0) \right) = (\gamma t) m_l \left( n(\xi, x, 0) \right). \]
Proof. The generating function of $\chi_l\left(\sum_{\tau_i(x) \leq t} n(t - \tau_i(x), x, 0)\right)$ has the simple form:

$$F(z) = E_{z}^{\sum_{\tau_i(x) \leq t} n(t - \tau_i(x), x, 0)}$$
$$= E_{z}^{\sum_{i=1}^{\Pi_x(t)} n(x, 0)}$$
$$= \sum_{k=0}^{\infty} \frac{e^{-\gamma t}(\gamma t)^l}{l!} \left(E_{z}^{n(x, 0)}\right)^l = \exp\left\{\gamma t (1 - E_{z}^{n(x, 0)})\right\},$$

where $\Pi_x(t)$ is a Poissonian process with parameter $\gamma$ and we use the fact that, if $\Pi_x(t) = l$, then the moments of this process has the distribution of the ordered statistics of $l$ uniformly distributed random variables on $[0, t]$.

The log-generating function is

$$\ln F(z) = \gamma t (1 - E_{z}^{n(x, 0)})$$
$$= \gamma t \left(1 - \sum_{l=0}^{\infty} \frac{m_l(n(x, 0))}{l!}(z - 1)^l\right)$$
$$= \sum_{l=1}^{\infty} \frac{m_l(n(x, 0))}{l!}(z - 1)^l. \quad (34)$$

At the same time,

$$\ln F(z) = \sum_{l=1}^{\infty} \frac{\chi_l\left(\sum_{\tau_i(x) \leq t} n(t - \tau_i(x), x, 0)\right)}{l!}(z - 1)^l. \quad (35)$$

From (34) and (35) we obtain the statement of the lemma. \hfill \Box

**Corollary 7.** $\chi_l(N(t, 0))$ is a monotone function of time $t$ and

$$\chi_l(N(t, 0)) = \gamma \int_{0}^{t} \sum_{x \in \mathbb{Z}^d} m_l(s, x, 0) \, ds.$$

From Corollary 5 and Corollary 7 we obtain

**Corollary 8.**

$$\chi_l(N(t, 0)) \leq c^l l! \gamma \int_{0}^{t} e^{-\Delta_s p}(s, x, 0) \, ds \leq c^l l! \frac{\gamma}{\Delta}. \quad (36)$$

The last gives an upper bound uniformly in $t$ for the cumulants of total population $N(t, 0)$. Using this and the monotonicity in $t$ of the cumulants of the total population $\chi_l(N(t, \cdot))$, we conclude the existence and boundedness of $\chi_l(N(\infty, \cdot))$:

$$\chi_l(N(\infty, \cdot)) \leq c^l l!. \quad (36)$$
Finally we may conclude that the behaviour in the limit of the cumulants of the total population $\chi_l(N(\infty, y))$ determines uniquely the limit distribution of $N(\infty, y), y \in \mathbb{Z}^d$. In other words, the classic problem of moments [9] does not take place in this situation. The upper boundary in Eq. (36) implies that the log-generating function for $N(\infty, \cdot)$ is analytical in some neighbourhood of $z = 1$, which is why the sequence of $\chi_l(N(\infty, \cdot))$ uniquely determines the probability distribution of $N(\infty, \cdot)$ [9, Chapter VII, S 6]. Traditionally, these conditions on the sequence of moments or cumulant that are sufficient for the existence of a uniquely determined distribution law are called the Carleman conditions.

**Remark:**

- Similar to the discussion in our previous work [2], one can perform similar analysis in the case when $0 < \Delta^- \leq \Delta(x) = \mu(x) - \sum_{l=2}^{\infty} (l-1)\beta_l(x) \leq \Delta^+ < \infty$. The proof of boundedness of cumulants and moments will be similar and we can prove a result analogous to Theorem 1 and there is a limiting distribution in this case as well.

## 4 Conclusion

We considered a subcritical contact branching random walk on the lattice with the arbitrary number of offspring and with immigration. We showed that, if the rate of mortality is larger than the average number of new particles per unit time (subcritical case), and the tail of the distribution of the number of offspring decreases at least geometrically, then the probability distribution of the population converges to a limiting distribution.

## References

[1] D. Han, S. Molchanov and J. Whitmeyer, Population processes with immigration. In: Panov, V. (ed.) Modern Problems of Stochastic Analysis and Statistics—Selected Contributions in Honor of Valentin Konakov, Springer, Heidelberg (2017), in press.

[2] Elena Chernousova, Yaqin Feng, Stanislav Molchanov and Joseph Whitmeyer. Steady states of lattice population models with immigration, submitted.

[3] Elena Chernousova, Stanislav Molchanov. Steady state for the critical contact branching random walk with the general number of offspring: Intermittency phenomenon, 2017.

[4] Yaqin Feng, Stanislav Molchanov and Joseph Whitmeyer, Random walks with heavy tails and limit Theorems for branching processes with migration and immigration, *Stochastic and Dynamics*, 12(2012), 1-23.
[5] S. Molchanov and J. Whitmeyer, Stationary distributions in Kolmogorov-Petrovski-Piskunov-type models with an infinite number of particles Mathematical Population Studies, Routledge, 24(2017), 147-160.

[6] L. Koralov and S. Molchanov, The structure of the population inside the propagating front, Journal of Mathematical Sciences (Problems In Mathematical Analysis) 189, (2013) 637-658.

[7] A. N. Kolmogorov, I. G. Petrovskii and N. S. Piskunov, A study of the diffusion equation with increase in the quantity of matter, and its application to a biological problem. Bull. Moscow Univ. Math. Ser. A 1(1937), 1-25.

[8] A. N. Kolmogorov, Selected Works of A.N. Kolmogorov: Mathematics and Mechanics Volume 1(Mathematics and its Applications) , (Kluwer Academic Publishers, 1991).

[9] W. Feller, An Introduction to Probability Theory and its Applications. Wiley. Volume I, 2nd edition, (1971).

[10] R. A. Fisher, The wave of advance of advantageous genes, Ann Eugenics 7 (1937) 355-369.

[11] S. Molchanov and E. Yarovaya, Large deviations for a symmetric branching random walk on a multidimensional lattice, Proceedings of the Steklov Institute of Mathematics 282, (2013) 186-201.

[12] Y. Kondratiev, O. Kutovyi, S. Pirogov, Correlation functions and invariant measures in continuous contact model, Infn. Dimens. Anal. Quantum Probab. Relat. Top. 11, (2008) 231-258.

[13] B. A. Sevast’yanov, Branching processes, Nauka, Moscow, (1971).