On a Weighted Singular Integral Operator with Shifts and Slowly Oscillating Data

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Abstract. Let $\alpha, \beta$ be orientation-preserving diffeomorphism (shifts) of $\mathbb{R}_+ = (0, \infty)$ onto itself with the only fixed points 0 and $\infty$ and $U_\alpha, U_\beta$ be the isometric shift operators on $L^p(\mathbb{R}_+)$ given by $U_\alpha f = (\alpha')^{1/p}(f \circ \alpha)$, $U_\beta f = (\beta')^{1/p}(f \circ \beta)$, and $P^\pm_2 = (I \pm S_2)/2$ where

$$(S_2 f)(t) := \frac{1}{\pi i} \int_0^\infty \left( \frac{t}{\tau} \right)^{1/2 - 1/p} \frac{f(\tau)}{\tau - t} \, d\tau, \quad t \in \mathbb{R}_+,$$

is the weighted Cauchy singular integral operator. We prove that if $\alpha', \beta'$ and $c, d$ are continuous on $\mathbb{R}_+$ and slowly oscillating at 0 and $\infty$, and

$$\limsup_{t \to s} |c(t)| < 1, \quad \limsup_{t \to s} |d(t)| < 1, \quad s \in \{0, \infty\},$$

then the operator $(I - cU_\alpha)P^+_2 + (I - dU_\beta)P^-_2$ is Fredholm on $L^p(\mathbb{R}_+)$ and its index is equal to zero. Moreover, its regularizers are described.

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1. Introduction

Let $\mathcal{B}(X)$ be the Banach algebra of all bounded linear operators acting on a Banach space $X$, and let $\mathcal{K}(X)$ be the ideal of all compact operators in

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An operator \( A \in \mathcal{B}(X) \) is called Fredholm if its image is closed and the spaces \( \ker A \) and \( \ker A^* \) are finite-dimensional. In that case the number

\[
\text{Ind } A := \dim \ker A - \dim \ker A^*
\]

is referred to as the index of \( A \) (see, e.g., [2, Sections 1.11–1.12], [4, Chap. 4]).

For bounded linear operators \( A \) and \( B \), we will write \( A \simeq B \) if \( A - B \in \mathcal{K}(X) \).

Recall that an operator \( B_r \in \mathcal{B}(X) \) (resp. \( B_l \in \mathcal{B}(X) \)) is said to be a right (resp. left) regularizer for \( A \) if

\[
AB_r \simeq I \quad (\text{resp. } B_l A \simeq I).
\]

It is well known that the operator \( A \) is Fredholm on \( X \) if and only if it admits simultaneously a right and a left regularizer. Moreover, each right regularizer differs from each left regularizer by a compact operator (see, e.g., [4, Chap. 4, Section 7]). Therefore we may speak of a regularizer \( B = B_r = B_l \) of \( A \) and two different regularizers of \( A \) differ from each other by a compact operator.

A bounded continuous function \( f \) on \( \mathbb{R}_+ = (0, \infty) \) is called slowly oscillating (at 0 and \( \infty \)) if for each (equivalently, for some) \( \lambda \in (0, 1) \),

\[
\lim_{r \to s} \sup_{t, \tau \in [\lambda r, r]} |f(t) - f(\tau)| = 0, \quad s \in \{0, \infty\}.
\]

The set \( \mathcal{SO}(\mathbb{R}_+) \) of all slowly oscillating functions forms a \( C^* \)-algebra. This algebra properly contains \( C(\mathbb{R}_+) \), the \( C^* \)-algebra of all continuous functions on \( \mathbb{R}_+ := [0, +\infty] \). Suppose \( \alpha \) is an orientation-preserving diffeomorphism of \( \mathbb{R}_+ \) onto itself, which has only two fixed points 0 and \( \infty \). We say that \( \alpha \) is a slowly oscillating shift if \( \log \alpha' \) is bounded and \( \alpha' \in \mathcal{SO}(\mathbb{R}_+) \). The set of all slowly oscillating shifts is denoted by \( \mathcal{SOS}(\mathbb{R}_+) \).

Throughout the paper we suppose that \( 1 < p < \infty \). It is easily seen that if \( \alpha \in \mathcal{SOS}(\mathbb{R}_+) \), then the shift operator \( W_\alpha \) defined by \( W_\alpha f = f \circ \alpha \) is bounded and invertible on all spaces \( L^p(\mathbb{R}_+) \) and its inverse is given by \( W_\alpha^{-1} = W_{\alpha^{-1}} \), where \( \alpha^{-1} \) is the inverse function to \( \alpha \). Along with \( W_\alpha \) we consider the weighted shift operator

\[
U_\alpha := (\alpha')^{1/p} W_\alpha
\]

being an isometric isomorphism of the Lebesgue space \( L^p(\mathbb{R}_+) \) onto itself. Let \( S \) be the Cauchy singular integral operator given by

\[
(Sf)(t) := \frac{1}{\pi i} \int_0^\infty \frac{f(\tau)}{\tau - t} \, d\tau, \quad t \in \mathbb{R}_+.
\]

where the integral is understood in the principal value sense. It is well known that \( S \) is bounded on \( L^p(\mathbb{R}_+) \) for every \( p \in (1, \infty) \). Let \( \mathcal{A} \) be the smallest closed subalgebra of \( \mathcal{B}(L^p(\mathbb{R}_+)) \) containing the identity operator \( I \) and the operator \( S \). It is known (see, e.g., [3], [5 Section 2.1.2], [18] Sections 4.2.2–4.2.3], and [19]) that \( \mathcal{A} \) is commutative and for every \( y \in (1, \infty) \) it contains...
the weighted singular integral operator
\[
(S_y f)(t) := \frac{1}{\pi i} \int_0^\infty \left( \frac{t}{\tau} \right)^{1/y-1/p} \frac{f(\tau)}{\tau - t} \, d\tau, \quad t \in \mathbb{R}_+,
\]
and the operator with fixed singularities
\[
(R_y f)(t) := \frac{1}{\pi i} \int_0^\infty \left( \frac{t}{\tau} \right)^{1/y-1/p} \frac{f(\tau)}{\tau + t} \, d\tau, \quad t \in \mathbb{R}_+,
\]
which are understood in the principal value sense. For \( y \in (1, \infty) \), put
\[
P_{\pm}^y := \frac{1}{2} \left( I \pm S_y \right).
\]
This paper is in some sense a continuation of our papers [7, 8, 9], where singular integral operators with shifts were studied under the mild assumptions that the coefficients belong to \( SO(\mathbb{R}_+) \) and the shifts belong to \( SOS(\mathbb{R}_+) \). In [7, 8] we found a Fredholm criterion for the singular integral operator
\[
N = (aI - bW_\alpha)P^+_p + (cI - dW_\alpha)P^-_p
\]
with coefficients \( a, b, c, d \in SO(\mathbb{R}_+) \) and a shift \( \alpha \in SOS(\mathbb{R}_+) \). However, a formula for the calculation of the index of the operator \( N \) is still missing. Further, in [9] we proved that the operators
\[
A_{ij} = U^i_\alpha P^+_p + U^j_\beta P^-_p, \quad i, j \in \mathbb{Z},
\]
with \( \alpha, \beta \in SOS(\mathbb{R}_+) \) are all Fredholm and their indices are equal to zero. This result was the first step in the calculation of the index of \( N \). Here we make the next step towards the calculation of the index of the operator \( N \).

For \( a \in SO(\mathbb{R}) \), we will write \( 1 \gg a \) if
\[
\limsup_{t \to s} |a(t)| < 1, \quad s \in \{0, \infty\}.
\]

**Theorem 1.1 (Main result).** Let \( 1 < p < \infty \) and \( \alpha, \beta \in SOS(\mathbb{R}_+) \). Suppose \( c, d \in SO(\mathbb{R}_+) \) are such that \( 1 \gg c \) and \( 1 \gg d \). Then the operator
\[
V := (I - cU_\alpha)P^+_p + (I - dU_\beta)P^-_p,
\]
is Fredholm on the space \( L^p(\mathbb{R}_+) \) and \( \text{Ind} V = 0 \).

The paper is organized as follows. In Section 2 we collect necessary facts about slowly oscillating functions and slowly oscillating shifts, as well as about the invertibility of binomial functional operators \( I - cU_\alpha \) with \( c \in SO(\mathbb{R}_+) \) and \( \alpha \in SOS(\mathbb{R}_+) \) under the assumption that \( 1 \gg c \). Further we prove that the operators in the algebra \( A \) commute modulo compact operators with the operators in the algebra \( F\mathcal{O}_{\alpha, \beta} \) of functional operators with shifts and slowly oscillating data. Finally, we show that the ranges of two important continuous functions on \( \mathbb{R} \) do not contain the origin. In Section 3 we recall that the operators \( P^+_y \) and \( R_y \), belonging to the algebra \( A \) for every \( y \in (1, \infty) \), can be viewed as Mellin convolution operators and formulate two relations between \( P^+_y \), \( P^-_y \), and \( R_y \). Section 4 contains results
on the boundedness, compactness of semi-commutators, and the Fredholm-
ness of Mellin pseudodifferential operators with slowly oscillating symbols of limited smoothness (symbols in the algebra \(\tilde{E}(\mathbb{R}^+, V(\mathbb{R}))\)). Results of this section are reformulations/modifications of corresponding results on Fourier pseudodifferential operators obtained by the second author in [12] (see also [13, 14, 15]). Notice that those results are further generalizations of earlier results by Rabinovich (see [17, Chap. 4] and the references therein) obtained for Mellin pseudodifferential operators with \(C^\infty\) slowly oscillating symbols.

In [9, Lemma 4.4] we proved that the operator \(U_\gamma R_y\) with \(\gamma \in SOS(\mathbb{R}^+)\) and \(y \in (1, \infty)\) can be viewed as a Mellin pseudodifferential operator with a symbol in the algebra \(\tilde{E}(\mathbb{R}^+, V(\mathbb{R}))\). In Section 5 we generalize that result and prove that \((I - \nu U_\gamma) R_y\) and \((I - \nu U_\gamma)^{-1} R_y\) with \(y \in (1, \infty)\), \(\gamma \in SOS(\mathbb{R}^+)\), and \(\nu \in SO(\mathbb{R}^+)\) satisfying \(1 \gg \nu\), can be viewed as Mellin pseudodifferential operators with symbols in the algebra \(\tilde{E}(\mathbb{R}^+, V(\mathbb{R}))\). This is a key result in our analysis.

Section 6 is devoted to the proof of Theorem 1.1. Here we follow the idea, which was already used in a simpler situation of the operators \(A_{ij}\) in [9]. With the aid of results of Section 2 and Section 5, we will show that for every \(\mu \in [0, 1]\) and \(y \in (1, \infty)\), the operators

\[
[I - \mu c U_\alpha] P_y^+ + (I - \mu d U_\beta) P_y^- \cdot [(I - \mu c U_\alpha)^{-1} P_y^+ + (I - \mu d U_\beta)^{-1} P_y^-] \\
[I - \mu c U_\alpha)^{-1} P_y^+ + (I - \mu d U_\beta)^{-1} P_y^-] \cdot [(I - \mu c U_\alpha) P_y^+ + (I - \mu d U_\beta) P_y^-]
\]

are equal up to compact summands to the same operator similar to a Mellin pseudodifferential operator with a symbol in the algebra \(\tilde{E}(\mathbb{R}^+, V(\mathbb{R}))\). Moreover, the latter pseudodifferential operator is Fredholm whenever \(y = 2\) in view of results of Section 4. This will show that each operator

\[V_{\mu, 2} = (I - \mu c U_\alpha) P_2^+ + (I - \mu d U_\beta) P_2^-\]

is Fredholm on \(L^p(\mathbb{R}^+)\). Considering the homotopy \(\mu \mapsto V_{\mu, 2}\) for \(\mu \in [0, 1]\), we see that the operator \(V\) is homotopic to the identity operator. Therefore, the index of \(V\) is equal to zero. This will complete the proof of Theorem 1.1.

As a by-product of the proof of the main result, in Section 7 we describe all regularizers of a slightly more general operator

\[W = (I - c U_\alpha^{\varepsilon_1}) P_2^+ + (I - d U_\beta^{\varepsilon_2}) P_2^-\]

where \(\varepsilon_1, \varepsilon_2 \in \{-1, 1\}\) and show that

\[G_y W \simeq R_y\]

for every \(y \in (1, \infty)\), where \(G_y\) is an operator similar to a Mellin pseudodifferential operator with a symbol in \(\tilde{E}(\mathbb{R}^+, V(\mathbb{R}))\) with some additional properties. The latter relation for \(y = 2\) will play an important role in the proof of an index formula for the operator \(N\) in our forthcoming work [11].
2. Preliminaries

2.1. Fundamental Property of Slowly Oscillating Functions

For a unital commutative Banach algebra \( \mathfrak{A} \), let \( M(\mathfrak{A}) \) denote its maximal ideal space. Identifying the points \( t \in \mathbb{R}_+ \) with the evaluation functionals \( t(f) = f(t) \) for \( f \in C(\mathbb{R}_+) \), we get \( M(C(\mathbb{R}_+)) = \mathbb{R}_+ \). Consider the fibers

\[
M_s(SO(\mathbb{R}_+)) := \{ \xi \in M(SO(\mathbb{R}_+)) : \xi|_{C(\mathbb{R}_+)} = s \}
\]

of the maximal ideal space \( M(SO(\mathbb{R}_+)) \) over the points \( s \in \{0, \infty\} \). By [14, Proposition 2.1], the set

\[
\Delta := M_0(SO(\mathbb{R}_+)) \cup M_\infty(SO(\mathbb{R}_+))
\]

coincides with \((\text{clos}_{SO^*} \mathbb{R}_+) \setminus \mathbb{R}_+ \) where \( \text{clos}_{SO^*} \mathbb{R}_+ \) is the weak-star closure of \( \mathbb{R}_+ \) in the dual space of \( SO(\mathbb{R}_+) \). Then \( M(SO(\mathbb{R}_+)) = \Delta \cup \mathbb{R}_+ \). By [8, Lemma 2.2], the fibers \( M_s(SO(\mathbb{R}_+)) \) for \( s \in \{0, \infty\} \) are connected compact Hausdorff spaces. In what follows we write

\[
a(\xi) := \xi(a)
\]

for every \( a \in SO(\mathbb{R}_+) \) and every \( \xi \in \Delta \).

Lemma 2.1 ([14, Proposition 2.2]). Let \( \{a_k\}_{k=1}^\infty \) be a countable subset of \( SO(\mathbb{R}_+) \) and \( s \in \{0, \infty\} \). For each \( \xi \in M_s(SO(\mathbb{R}_+)) \) there exists a sequence \( \{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+ \) such that \( t_n \to s \) as \( n \to \infty \) and

\[
a_k(\xi) = \lim_{n \to \infty} a_k(t_n) \quad \text{for all} \quad k \in \mathbb{N}.
\]

(2.1)

Conversely, if \( \{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+ \) is a sequence such that \( t_n \to s \) as \( n \to \infty \), then there exists a functional \( \xi \in M_s(SO(\mathbb{R}_+)) \) such that (2.1) holds.

2.2. Slowly Oscillating Functions and Shifts

Repeating literally the proof of [6, Proposition 3.3], we obtain the following statement.

Lemma 2.2. Suppose \( \varphi \in C^1(\mathbb{R}_+) \) and put \( \psi(t) := t\varphi'(t) \) for \( t \in \mathbb{R}_+ \). If \( \varphi, \psi \in SO(\mathbb{R}_+) \), then

\[
\lim_{t \to s} \psi(t) = 0 \quad \text{for} \quad s \in \{0, \infty\}.
\]

Lemma 2.3 ([7, Lemma 2.2]). An orientation-preserving shift \( \alpha : \mathbb{R}_+ \to \mathbb{R}_+ \) belongs to \( SOS(\mathbb{R}_+) \) if and only if

\[
\alpha(t) = te^{\omega(t)}, \quad t \in \mathbb{R}_+,
\]

for some real-valued function \( \omega \in SO(\mathbb{R}_+) \cap C^1(\mathbb{R}_+) \) such that the function \( t \mapsto t\omega'(t) \) also belongs to \( SO(\mathbb{R}_+) \) and \( \inf_{t \in \mathbb{R}_+} (1 + t\omega'(t)) > 0 \).

Lemma 2.4 ([7, Lemma 2.3]). If \( c \in SO(\mathbb{R}_+) \) and \( \alpha \in SOS(\mathbb{R}_+) \), then \( c \circ \alpha \) belongs to \( SO(\mathbb{R}_+) \) and

\[
\lim_{t \to s} (c(t) - c[\alpha(t)]) = 0 \quad \text{for} \quad s \in \{0, \infty\}.
\]
Lemma 2.5 ([9] Corollary 2.5). If $\alpha, \beta \in \text{SOS}(\mathbb{R}_+)$, then $\alpha \circ \beta \in \text{SOS}(\mathbb{R}_+)$ for all $i, j \in \mathbb{Z}$.

Lemma 2.6. If $\alpha \in \text{SOS}(\mathbb{R}_+)$, then
\[
\omega(t) := \log[\alpha(t)/t], \quad \tilde{\omega}(t) := \log[\alpha(t)/t], \quad t \in \mathbb{R}_+,
\]
are slowly oscillating functions such that $\omega(\xi) = -\tilde{\omega}(\xi)$ for all $\xi \in \Delta$.

Proof. From Lemma 2.5 with $i = -1$ and $j = 0$ it follows that $\alpha_{-1}$ belongs to $\text{SOS}(\mathbb{R}_+)$. Then, by Lemma 2.5 $\omega$, $\tilde{\omega} \in \text{SOS}(\mathbb{R}_+)$. It is easy to see that
\[
\tilde{\omega}(t) = \frac{\alpha_{-1}(t)}{t} = -\log \frac{t}{\alpha_{-1}(t)} = -\log \frac{T(t)}{\alpha_{-1}(t)} = -\omega(\alpha_{-1}(t))
\]
for all $t \in \mathbb{R}_+$. Hence, from Lemma 2.4 it follows that $\omega \circ \alpha_{-1} \in \text{SO}(\mathbb{R}_+)$ and
\[
\lim_{t \to s}(\omega(t) + \tilde{\omega}(t)) = \lim_{t \to s}(\omega(t) - \omega(\alpha_{-1}(t))) = 0, \quad s \in \{0, \infty\}. \quad (2.2)
\]
Fix $s \in \{0, \infty\}$ and $\xi \in M_s(\text{SO}(\mathbb{R}_+))$. By Lemma 2.1 there is a sequence $\{t_j\}_{j \in \mathbb{N}} \subseteq \mathbb{R}_+$ such that $t_j \to s$ and
\[
\omega(\xi) = \lim_{j \to \infty} \omega(t_j), \quad \tilde{\omega}(\xi) = \lim_{j \to \infty} \tilde{\omega}(t_j).
\]
From (2.2)–(2.3) we obtain
\[
\omega(\xi) = \lim_{j \to \infty} \omega(t_j) - \lim_{j \to \infty} (\omega(t_j) + \tilde{\omega}(t_j)) = -\lim_{j \to \infty} \tilde{\omega}(t_j) = -\tilde{\omega}(\xi),
\]
which completes the proof. \qed

2.3. Invertibility of Binomial Functional Operators

From [7] Theorem 1.1] we immediately get the following.

Lemma 2.7. Suppose $c \in \text{SO}(\mathbb{R}_+)$ and $\alpha \in \text{SOS}(\mathbb{R}_+)$. If $1 \gg c$, then the functional operator $I - cU_\alpha$ is invertible on the space $L^p(\mathbb{R}_+)$ and
\[
(I - cU_\alpha)^{-1} = \sum_{n=0}^\infty (cU_\alpha)^n.
\]

2.4. Compactness of Commutators of SIO’s and FO’s

Let $\mathcal{B}$ be a Banach algebra and $\mathcal{G}$ be a subset of $\mathcal{B}$. We denote by $\text{alg}_B \mathcal{G}$ the smallest closed subalgebra of $\mathcal{B}$ containing $\mathcal{G}$. Then
\[
\mathcal{A} = \text{alg}_{\mathcal{B}(L^p(\mathbb{R}_+))}\{I, S\}
\]
is the algebra of singular integral operators (SIO’s). Fix $\alpha, \beta \in \text{SOS}(\mathbb{R}_+)$ and consider the Banach algebra of functional operators (FO’s) with shifts and slowly oscillating data defined by
\[
\text{FO}_{\alpha, \beta} := \text{alg}_{\mathcal{B}(L^p(\mathbb{R}_+))}\{U_{\alpha}, U_{\alpha}^{-1}, U_{\beta}, U_{\beta}^{-1}, aI : a \in \text{SO}(\mathbb{R}_+)\}.
\]

Lemma 2.8. Let $\alpha, \beta \in \text{SOS}(\mathbb{R}_+)$. If $A \in \text{FO}_{\alpha, \beta}$ and $B \in \mathcal{A}$, then
\[
AB - BA \in \mathcal{K}(L^p(\mathbb{R}_+)).
\]
2.5. Ranges of Two Continuous Functions on $\mathbb{R}$

Given $a \in \mathbb{C}$ and $r > 0$, let $D(a, r) := \{ z \in \mathbb{C} : |z - a| \leq r \}$. For $x \in \mathbb{R}$, put

$$p_2^+(x) := \frac{e^{2\pi x}}{e^{2\pi x} + 1}, \quad p_2^-(x) := \frac{1}{e^{2\pi x} + 1}. \quad (2.4)$$

**Lemma 2.9.** Let $\psi, \zeta \in \mathbb{R}$ and $v, w \in \mathbb{C}$. If

$$f(x) := (1 - ve^{i\psi x})p_2^+(x) + (1 - we^{i\zeta x})p_2^-(x), \quad (2.5)$$

then $f(\mathbb{R}) \subset D(1, r)$, where $r := \max(\{|v|, |w|\})$.

**Proof.** From (2.4) and (2.5) we see that for every $x \in \mathbb{R}$ the point $f(x)$ lies on the line segment connecting the points $1 - ve^{i\psi x}$ and $1 - we^{i\zeta x}$. In turn, these points lie on the concentric circles

$$\{z \in \mathbb{C} : |z - 1| = |v|\}, \quad \{z \in \mathbb{C} : |z - 1| = |w|\}, \quad (2.6)$$

respectively. Thus, each line segment mentioned above is contained in the disk $D(1, r) = \{ z \in \mathbb{C} : |z - 1| \leq \max(\{|v|, |w|\}) \}$.

**Lemma 2.10.** Let $\psi, \zeta \in \mathbb{R}$ and $v, w \in \mathbb{C}$ with $|v| < 1$, $|w| < 1$. If

$$g(x) := (1 - ve^{i\psi x})^{-1}p_2^+(x) + (1 - we^{i\zeta x})^{-1}p_2^-(x), \quad x \in \mathbb{R}, \quad (2.7)$$

then $g(\mathbb{R}) \subset D((1 - r^2)^{-1}, (1 - r^2)^{-1}r)$, where $r = \max(\{|v|, |w|\}) < 1$.

**Proof.** From (2.4) and (2.7) we see that for every $x \in \mathbb{R}$ the point $g(x)$ lies on the line segment connecting the points $1 - ve^{i\psi x}$ and $1 - we^{i\zeta x}$.

In turn, these points lie on the images of the circles given by (2.6) under the inversion mapping $z \mapsto 1/z$. The image of the first circle in (2.6) is the circle $T_v := \{ z \in \mathbb{C} : |z - b| = \rho \}$ with center and radius given by

$$b = [(1 - |v|)^{-1} + (1 + |v|)^{-1}] / 2 = (1 - |v|^2)^{-1},$$

$$\rho = [(1 - |v|)^{-1} - (1 + |v|)^{-1}] / 2 = (1 - |v|^2)^{-1} |v|.$$  

Analogously, the image of the second circle in (2.6) is the circle

$$T_w := \{ z \in \mathbb{C} : |z - (1 - |w|^2)^{-1}| = (1 - |w|^2)^{-1} |w| \}.$$  

Let $D_v$ and $D_w$ be the closed disks whose boundaries are $T_v$ and $T_w$, respectively. Obviously, one of these disks is contained in another one, namely, $D_v \subset D_w$ if $|v| \leq |w|$ and $D_w \subset D_v$ otherwise. Then each point $g(x)$, lying on the segment connecting the points $(1 - ve^{i\psi x})^{-1} \in T_v$ and $(1 - we^{i\zeta x})^{-1} \in T_w$, belongs to the biggest disk in $\{D_v, D_w\}$, that is, to the disk with center $(1 - r^2)^{-1}$ and radius $(1 - r^2)^{-1}r$, where $r = \max(\{|v|, |w|\}) < 1$.

From Lemmas 2.9 and 2.10 it follows that the ranges $f(\mathbb{R})$ and $g(\mathbb{R})$ do not contain the origin if $|v| < 1$ and $|w| < 1$. 

Proof. In view of [7 Corollary 6.4], we have $aB - BaI \in \mathcal{K}(L^p(\mathbb{R}_+))$ for all $a \in SO(\mathbb{R}_+)$ and all $B \in \mathcal{A}$. On the other hand, from [9 Lemma 2.7] it follows that $U_B \in \mathcal{K}(L^p(\mathbb{R}_+))$ for all $\gamma \in \{ \alpha, \beta \}$ and $B \in \mathcal{A}$. Hence, $AB - BA \in \mathcal{K}(L^p(\mathbb{R}_+))$ for each generator $A$ of $\mathcal{F}\mathcal{O}_{\alpha, \beta}$ and each $B \in \mathcal{A}$. Thus, the same is true for all $A \in \mathcal{F}\mathcal{O}_{\alpha, \beta}$ by a standard argument. □
3. Weighted Singular Integral Operators Are Similar to Mellin Convolution Operators

3.1. Mellin Convolution Operators

Let $F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denote the Fourier transform,

$$(Ff)(x) := \int_{\mathbb{R}} f(y)e^{-ixy}dy, \quad x \in \mathbb{R},$$

and let $F^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the inverse of $F$. A function $a \in L^\infty(\mathbb{R})$ is called a Fourier multiplier on $L^p(\mathbb{R})$ if the mapping $f \mapsto F^{-1}aFf$ maps $L^2(\mathbb{R}) \cap L^p(\mathbb{R})$ onto itself and extends to a bounded operator on $L^p(\mathbb{R})$. The latter operator is then denoted by $W^0(a)$. We let $M_p(\mathbb{R})$ stand for the set of all Fourier multipliers on $L^p(\mathbb{R})$. One can show that $M_p(\mathbb{R})$ is a Banach algebra under the norm

$$\|a\|_{M_p(\mathbb{R})} := \|W^0(a)\|_{\mathcal{B}(L^p(\mathbb{R}))}.$$

Let $d\mu(t) = dt/t$ be the (normalized) invariant measure on $\mathbb{R}_+$. Consider the Fourier transform on $L^2(\mathbb{R}_+,d\mu)$, which is usually referred to as the Mellin transform and is defined by

$$(Mf)(x) := \int_{\mathbb{R}_+} f(t)t^{-ix}dt, \quad x \in \mathbb{R}.$$}

It is an invertible operator, with inverse given by

$$(M^{-1}g)(t) = \frac{1}{2\pi} \int_{\mathbb{R}} g(x)t^{ix}dx.$$}

Let $E$ be the isometric isomorphism

$$E : L^p(\mathbb{R}_+,d\mu) \to L^p(\mathbb{R}_+), \quad (Ef)(x) := f(e^x), \quad x \in \mathbb{R}. \quad (3.1)$$

Then the map $A \mapsto E^{-1}AE$ transforms the Fourier convolution operator $W^0(a) = F^{-1}aF$ to the Mellin convolution operator

$$\text{Co}(a) := M^{-1}aM$$

with the same symbol $a$. Hence the class of Fourier multipliers on $L^p(\mathbb{R})$ coincides with the class of Mellin multipliers on $L^p(\mathbb{R}_+,d\mu)$.

3.2. Algebra $\mathcal{A}$ of Singular Integral Operators

Consider the isometric isomorphism

$$\Phi : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+,d\mu), \quad (\Phi f)(t) := t^{1/p}f(t), \quad t \in \mathbb{R}_+, \quad (3.2)$$

The following statement is well known (see, e.g., [3], [5, Section 2.1.2], and [18, Sections 4.2.2–4.2.3]).

**Lemma 3.1.** For every $y \in (1, \infty)$, the functions $s_y$ and $r_y$ given by

$$s_y(x) := \coth[\pi(x + iy)], \quad r_y(x) := 1/\sinh[\pi(x + iy)], \quad x \in \mathbb{R},$$
belong to \( M_p(\mathbb{R}) \), the operators \( S_y \) and \( R_y \) belong to the algebra \( \mathcal{A} \), and
\[
S_y = \Phi^{-1} \text{Co}(s_y)\Phi, \quad R_y = \Phi^{-1} \text{Co}(r_y)\Phi.
\]

For \( y \in (1, \infty) \) and \( x \in \mathbb{R} \), put
\[
p_y^\pm(x) := \frac{(1 \pm s_y(x))}{2}.
\]
This definition is consistent with (2.4) because \( s^2_y(x) = \tanh(\pi x) \) for \( x \in \mathbb{R} \).

In view of Lemma 3.1 we have
\[
P_y^\pm = (I \pm S_y)/2 = \Phi^{-1} \text{Co}(p_y^\pm)\Phi.
\]

**Lemma 3.2.**

(a) For \( y \in (1, \infty) \) and \( x \in \mathbb{R} \), we have
\[
p_y^+p_y^-(x) = -\frac{(r_y(x))^2}{4}, \quad (p_y^\pm(x))^2 = p_y^\pm(x) + \frac{(r_y(x))^2}{4}.
\]
(b) For every \( y \in \mathbb{R}_+ \), we have
\[
P_y^+P_y^- = P_y^-P_y^+ = -\frac{R_y^2}{4}, \quad (P_y^\pm)^2 = P_y^\pm + \frac{R_y^2}{4}.
\]

**Proof.** Part (a) follows straightforwardly from the identity \( s^2_y(x) - r^2_y(x) = 1 \).
Part (b) follows from part (a) and Lemma 3.1. \( \square \)

### 4. Mellin Pseudodifferential Operators and Their Symbols

#### 4.1. Boundedness of Mellin Pseudodifferential Operators

In 1991 Rabinovich [16] proposed to use Mellin pseudodifferential operators with \( C^\infty \) slowly oscillating symbols to study singular integral operators with slowly oscillating coefficients on \( L^p \) spaces. This idea was exploited in a series of papers by Rabinovich and coauthors. A detailed history and a complete bibliography up to 2004 can be found in [17, Sections 4.6–4.7]. Further, the second author developed in [12] a handy for our purposes theory of Fourier pseudodifferential operators with slowly oscillating symbols of limited smoothness (much less restrictive than in the works mentioned in [17]). In this section we translate necessary results from [12] to the Mellin setting with the aid of the transformation
\[
A \mapsto E^{-1}AE,
\]
where \( A \in \mathcal{B}(L^p(\mathbb{R})) \) and the isometric isomorphism \( E : L^p(\mathbb{R}_+, d\mu) \to L^p(\mathbb{R}) \) is defined by (3.1).

Let \( a \) be an absolutely continuous function of finite total variation
\[
V(a) := \int_\mathbb{R} |a'(x)|dx
\]
on \( \mathbb{R} \). The set \( V(\mathbb{R}) \) of all absolutely continuous functions of finite total variation on \( \mathbb{R} \) becomes a Banach algebra equipped with the norm
\[
\|a\|_{V} := \|a\|_{L^\infty(\mathbb{R})} + V(a). \tag{4.1}
\]
Following [12][13], let $C_b(\mathbb{R}_+, V(\mathbb{R}))$ denote the Banach algebra of all bounded continuous $V(\mathbb{R})$-valued functions on $\mathbb{R}_+$ with the norm
\[
\|a(\cdot, \cdot)\|_{C_b(\mathbb{R}_+, V(\mathbb{R}))} = \sup_{t \in \mathbb{R}_+} \|a(t, \cdot)\|_V.
\]
As usual, let $C_0^\infty(\mathbb{R}_+)$ be the set of all infinitely differentiable functions of compact support on $\mathbb{R}_+$.

The following boundedness result for Mellin pseudodifferential operators follows from [13 Theorem 6.1] (see also [12 Theorem 3.1]).

**Theorem 4.1.** If $a \in C_b(\mathbb{R}_+, V(\mathbb{R}))$, then the Mellin pseudodifferential operator $\mathrm{Op}(a)$, defined for functions $f \in C_0^\infty(\mathbb{R}_+)$ by the iterated integral
\[
[\mathrm{Op}(a)f](t) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}_+} a(t, x) \left( \frac{t}{\tau} \right)^{ix} f(\tau) \frac{d\tau}{\tau} \quad \text{for} \quad t \in \mathbb{R}_+,
\]
extends to a bounded linear operator on the space $L^p(\mathbb{R}_+, d\mu)$ and there is a number $C_p \in (0, \infty)$ depending only on $p$ such that
\[
\|\mathrm{Op}(a)\|_{B(L^p(\mathbb{R}_+, d\mu))} \leq C_p \|a\|_{C_b(\mathbb{R}_+, V(\mathbb{R}))}.
\]

Obviously, if $a(t, x) = a(x)$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, then the Mellin pseudodifferential operator $\mathrm{Op}(a)$ becomes the Mellin convolution operator
\[
\mathrm{Op}(a) = \mathrm{Co}(a).
\]

**4.2. Algebra $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$**

Let $SO(\mathbb{R}_+, V(\mathbb{R}))$ denote the Banach subalgebra of $C_b(\mathbb{R}_+, V(\mathbb{R}))$ consisting of all $V(\mathbb{R})$-valued functions $a$ on $\mathbb{R}_+$ that slowly oscillate at 0 and $\infty$, that is,
\[
\lim_{r \to 0} \text{cm}_r^C(a) = \lim_{r \to \infty} \text{cm}_r^C(a) = 0,
\]
where
\[
\text{cm}_r^C(a) := \max \left\{ \|a(t, \cdot) - a(\tau, \cdot)\|_{L^\infty(\mathbb{R})} : t, \tau \in [r, 2r] \right\}.
\]

Let $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ be the Banach algebra of all $V(\mathbb{R})$-valued functions $a \in SO(\mathbb{R}_+, V(\mathbb{R}))$ such that
\[
\lim_{|h| \to 0} \sup_{t \in \mathbb{R}_+} \|a(t, \cdot) - a^h(t, \cdot)\|_V = 0
\]
where $a^h(t, x) := a(t, x + h)$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$.

Let $a \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$. For every $t \in \mathbb{R}_+$, the function $a(t, \cdot)$ belongs to $V(\mathbb{R})$ and, therefore, has finite limits at $\pm \infty$, which will be denoted by $a(t, \pm \infty)$. Now we explain how to extend the function $a$ to $\Delta \times \overline{\mathbb{R}}$. By analogy with [12 Lemma 2.7] with the aid of Lemma 2.1 one can prove the following.

**Lemma 4.2.** Let $s \in \{0, \infty\}$ and $\{a_k\}_{k=1}^\infty$ be a countable subset of the algebra $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$. For each $\xi \in M_s(SO(\mathbb{R}_+))$ there is a sequence $\{t_j\}_{j \in \mathbb{N}} \subset \mathbb{R}_+$ and functions $a_k(\xi, \cdot) \in V(\mathbb{R})$ such that $t_j \to s$ as $j \to \infty$ and
\[
a_k(\xi, x) = \lim_{j \to \infty} a_k(t_j, x)
\]
for every $x \in \overline{\mathbb{R}}$ and every $k \in \mathbb{N}$.
A straightforward application of Lemma 4.2 leads to the following.

**Lemma 4.3.** Let $b \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$, $m, n \in \mathbb{N}$, and $a_{ij} \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. If

$$b(t, x) = \sum_{i=1}^{m} \prod_{j=1}^{n} a_{ij}(t, x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R},$$

then

$$b(\xi, x) = \sum_{i=1}^{m} \prod_{j=1}^{n} a_{ij}(\xi, x), \quad (\xi, x) \in \Delta \times \mathbb{R}. \quad (4.2)$$

**Lemma 4.4 ([10, Lemma 3.2]).** Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of functions in $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ such that the series $\sum_{n=1}^{\infty} a_n$ converges in the norm of the algebra $C_b(\mathbb{R}_+, V(\mathbb{R}))$ to a function $a \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$. Then

$$a(t, \pm\infty) = \sum_{n=1}^{\infty} a_n(t, \pm\infty) \quad \text{for all} \quad t \in \mathbb{R}_+, \quad (4.2)$$

$$a(\xi, x) = \sum_{n=1}^{\infty} a_n(\xi, x) \quad \text{for all} \quad (\xi, x) \in \Delta \times \mathbb{R}. \quad (4.3)$$

### 4.3. Products of Mellin Pseudodifferential Operators

Applying the relation

$$\text{Op}(a) = E^{-1}a(x, D)E \quad (4.4)$$

between the Mellin pseudodifferential operator $\text{Op}(a)$ and the Fourier pseudodifferential operator $a(x, D)$ considered in [12], where

$$a(t, x) = a(\ln t, x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \quad (4.5)$$

and $E$ is given by (3.1), we infer from [12, Theorem 8.3] the following compactness result.

**Theorem 4.5.** If $a, b \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$, then

$$\text{Op}(a) \text{Op}(b) \simeq \text{Op}(ab).$$

From (3.1), (4.4), (4.5), [12, Lemmas 7.1, 7.2], and the proof of [12, Lemma 8.1] we can extract the following.

**Lemma 4.6.** If $a, b, c \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ are such that $a$ depends only on the first variable and $c$ depends only on the second variable, then

$$\text{Op}(a) \text{Op}(b) \text{Op}(c) = \text{Op}(abc).$$

### 4.4. Fredholmness of Mellin Pseudodifferential Operators

To study the Fredholmness of Mellin pseudodifferential operators, we need the Banach algebra $\tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ consisting of all functions $a$ belonging to $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ and such that

$$\lim_{m \to \infty} \sup_{t \in \mathbb{R}_+} \int_{[-m,m]} \left| \frac{\partial a(t, x)}{\partial x} \right| \, dx = 0.$$

Now we are in a position to formulate the main result of this section.
5. Applications of Mellin Pseudodifferential Operators

5.1. Some Important Functions in the Algebra \( \tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R})) \)

Lemma 5.1 ([9, Lemma 4.2]). Let \( g \in SO(\mathbb{R}_+) \). Then for every \( y \in (1, \infty) \) the functions
\[
g(t, x) := g(t), \quad s_y(t, x) := s_y(x), \quad r_y(t, x) := r_y(x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R},
\]
belong to the Banach algebra \( \tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R})) \).

Lemma 5.2 ([9, Lemma 4.3]). Suppose \( \omega \in SO(\mathbb{R}_+) \) is a real-valued function. Then for every \( y \in (1, \infty) \) the function
\[
b(t, x) := e^{i\omega(t)x}r_y(x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R},
\]
belongs to the Banach algebra \( \tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R})) \) and there is a positive constant \( C(y) \) depending only on \( y \) such that
\[
\|b\|_{C_b(\mathbb{R}_+, V(\mathbb{R}))} \leq C(y) \left( 1 + \sup_{t \in \mathbb{R}_+} |\omega(t)| \right).
\]

5.2. Operator \( U_\gamma R_y \)

Lemma 5.3 ([9, Lemma 4.4]). Let \( \gamma \in SOS(\mathbb{R}_+) \) and \( U_\gamma \) be the associated isometric shift operator on \( L^p(\mathbb{R}_+) \). For every \( y \in (1, \infty) \), the operator \( U_\gamma R_y \) can be realized as the Mellin pseudodifferential operator:
\[
U_\gamma R_y = \Phi^{-1} \text{Op}(\partial) \Phi,
\]
where the function \( \partial \), given for \( (t, x) \in \mathbb{R}_+ \times \mathbb{R} \) by
\[
\partial(t, x) := (1 + tv'(t))^{1/p} e^{i\psi(t)x}r_y(x) \quad \text{with} \quad \psi(t) := \log[\gamma(t)/t],
\]
belongs to the algebra \( \tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R})) \).
5.3. Operator $(I - vU_\gamma)R_y$

The previous lemma can be easily generalized to the case of operators containing slowly oscillating coefficients.

**Lemma 5.4.** Let $y \in (1, \infty)$, $v \in SO(\mathbb{R}^+)$, and $\gamma \in SO(\mathbb{R}^+)$.

(a) the operator $(I - vU_\gamma)R_y$ can be realized as the Mellin pseudodifferential operator:

$$(I - vU_\gamma)R_y = \Phi^{-1} \text{Op}(a)\Phi,$$

where the function $a$, given for $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$ by

$$a(t, x) := (1 - v(t)(\Psi(t))^{1/p} e^{i\psi(t)x})r_y(x)$$

with $\psi(t) := \log[\gamma(t)/t]$ and $\Psi(t) := 1 + t\psi'(t)$, belongs to $\tilde{E}(\mathbb{R}^+, V(\mathbb{R}))$;

(b) we have

$$a(\xi, x) = \begin{cases} (1 - v(\xi)e^{i\psi(\xi)x})r_y(x), & \text{if } (\xi, x) \in \Delta \times \mathbb{R}, \\ 0, & \text{if } (\xi, x) \in (\mathbb{R}^+ \cup \Delta) \times \{\pm \infty\}. \end{cases}$$

**Proof.** (a) This statement follows straightforwardly from Lemmas 5.1, 5.3 and 4.6.

(b) If $t \in \mathbb{R}^+$, then obviously

$$a(t, x) = 0 \quad \text{for} \quad x \in \{\pm \infty\}. \quad (5.1)$$

By Lemma 2.3 $\psi \in SO(\mathbb{R}^+)$. Since $v, \psi \in SO(\mathbb{R}^+)$, from Lemma 5.1 it follows that the functions

$$v(t, x) := v(t), \quad \tilde{\psi}(t, x) := \psi(t), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \quad (5.2)$$

belong to $\tilde{E}(\mathbb{R}^+, V(\mathbb{R}))$. Consider the finite family \(\{a, v, \tilde{\psi}\} \in \tilde{E}(\mathbb{R}^+, V(\mathbb{R}))\).

Fix $s \in [0, \infty)$ and $\xi \in M_s(SO(\mathbb{R}^+))$. By Lemma 4.2 and (5.2), there is a sequence \(\{t_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^+\) and a function $a(\xi, \cdot) \in V(\mathbb{R}^+_s)$ such that

$$\lim_{j \to \infty} t_j = s, \quad v(\xi) = \lim_{j \to \infty} v(t_j), \quad \psi(\xi) = \lim_{j \to \infty} \psi(t_j), \quad (5.3)$$

$$a(\xi, x) = \lim_{j \to \infty} a(t_j, x), \quad x \in \mathbb{R}. \quad (5.4)$$

From Lemmas 2.2 and 2.3 we obtain

$$\lim_{j \to \infty} (\Psi(t_j))^{1/p} = 1. \quad (5.5)$$

From (5.1) and (5.4) we get

$$a(\xi, x) = 0 \quad \text{for} \quad (\xi, x) \in (\mathbb{R}^+ \cup \Delta) \times \{\pm \infty\}. \quad (5.6)$$

Finally, from (5.3)–(5.5) we obtain for $(\xi, x) \in \Delta \times \mathbb{R},$

$$a(\xi, x) = \lim_{j \to \infty} a(t_j, x)$$

$$= \left(1 - \left(\lim_{j \to \infty} v(t_j)\right)\left(\lim_{j \to \infty} (\Psi(t_j))^{1/p}\right)\exp\left(ix \lim_{j \to \infty} \psi(t_j)\right)\right)r_y(x)$$

$$= (1 - v(\xi)e^{i\psi(\xi)x})r_y(x),$$

which completes the proof. \(\square\)
5.4. Operator \((I - vU_\gamma)^{-1}R_y\)

The following statement is crucial for our analysis. It says that the operators \((I - vU_\gamma)R_y\) and \((I - vU_\gamma)^{-1}R_y\) have similar nature.

**Lemma 5.5.** Let \(y \in (1, \infty), v \in SO(\mathbb{R}_+),\) and \(\gamma \in SOS(\mathbb{R}_+).\) If \(1 \gg v,\) then

(a) the operator \(A := I - vU_\gamma\) is invertible on \(L^p(\mathbb{R}_+)\);

(b) the operator \(A^{-1}R_y\) can be realized as the Mellin pseudodifferential operator:

\[
A^{-1}R_y = \Phi^{-1} \text{Op}(c) \Phi,
\]

where the function \(c,\) given for \((t, x) \in \mathbb{R}_+ \times \mathbb{R}\) by

\[
c(t, x) := ry(x) + \sum_{n=1}^{\infty} \left( \prod_{k=0}^{n-1} v[\gamma_k(t)](\Psi[\gamma_k(t)]) \right)^{1/p} e^{i\psi[\gamma_k(t)]x} ry(x)
\]  

with \(\psi(t) := \log[\gamma(t)/t]\) and \(\Psi(t) := 1 + t\psi'(t),\) belongs to \(\tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}));\)

(c) we have

\[
c(\xi, x) = \begin{cases} 
(1 - v(\xi)e^{i\psi(\xi)x})^{-1}ry(x), & \text{if } (\xi, x) \in \Delta \times \mathbb{R}, \\
0, & \text{if } (\xi, x) \in (\mathbb{R}_+ \cup \Delta) \times \{\pm\infty\}.
\end{cases}
\]

**Proof.** (a) Since \(1 \gg v,\) from Lemma 2.7 we conclude that \(A\) is invertible on the space \(L^p(\mathbb{R}_+)\) and

\[
A^{-1} = \sum_{n=0}^{\infty} (vU_\gamma)^n.
\]

Part (a) is proved.

(b) By Lemmas 3.1 and 5.1,

\[
R_y = \Phi^{-1} \text{Op}(c_0) \Phi,
\]

where the function \(c_0,\) given by

\[
c_0(t, x) := ry(x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R},
\]

belongs to \(\tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R})).\)

If \(\gamma \in SOS(\mathbb{R}_+),\) then from Lemma 2.3 it follows that \(\gamma_n \in SOS(\mathbb{R}_+)\) for every \(n \in \mathbb{Z}.\) By Lemma 2.3 the functions

\[
\psi_n(t) := \log \frac{\gamma_n(t)}{t}, \quad \Psi_n(t) := 1 + t\psi'_n(t) \quad t \in \mathbb{R}_+, \quad n \in \mathbb{Z},
\]

are real-valued functions in \(SO(\mathbb{R}_+) \cap C^1(\mathbb{R}_+).\) For every \(n \in \mathbb{N},\)

\[
(vU_\gamma)^n R_y = \left( \prod_{k=0}^{n-1} v \circ \gamma_k \right) U_{\gamma_n} R_y.
\]

By Lemma 5.3,

\[
U_{\gamma_n} R_y = \Phi^{-1} \text{Op}(d_n) \Phi,
\]

where the function \(d_n,\) given by

\[
d_n(t, x) := (\Psi_n(t))^{1/p} e^{i\psi_n(t)x} ry(x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R},
\]
belongs to $\tilde{E}(\mathbb{R}_+, V(\mathbb{R}))$. From (5.11) it follows that
\[
\psi_n(t) = \log \frac{\gamma_n-1[\gamma(t)]}{t} = \log \frac{\gamma_n-1[\gamma(t)]}{\gamma(t)} + \log \frac{\gamma(t)}{t} = \psi_{n-1}[\gamma(t)] + \psi(t).
\]
Therefore
\[
\psi_n'(t) = \psi_{n-1}[\gamma(t)]\gamma'(t) + \psi'(t). \tag{5.15}
\]
By using $\gamma(t) = te^{\psi(t)}$ and $\gamma'(t) = \Psi(t)e^{\psi(t)}$, from (5.11) and (5.15) we get
\[
\Psi_n(t) = t\psi_{n-1}[\gamma(t)]\Psi(t)e^{\psi(t)} + (1 + t\psi'(t))
= \Psi(t)(1 + \gamma(t)\psi'_{n-1}[\gamma(t)]) = \Psi(t)\Psi_{n-1}[\gamma(t)].
\]
From this identity by induction we get
\[
\Psi_n(t) = \prod_{k=0}^{n-1} \Psi[\gamma_k(t)], \quad t \in \mathbb{R}_+, \quad n \in \mathbb{N}. \tag{5.16}
\]
From (5.12)–(5.14) and (5.16) we get
\[
(vU_\gamma)^nR_y = \Phi^{-1}\text{Op}(c_n)\Phi, \quad n \in \mathbb{N},
\tag{5.17}
\]
where the function $c_n$ is given for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ by
\[
c_n(t, x) := a_n(t)b_n(t, x) \tag{5.18}
\]
with
\[
a_n(t) := \prod_{k=0}^{n-1} v[\gamma_k(t)](\Psi[\gamma_k(t)])^{1/p}, \quad b_n(t, x) := e^{i\psi_n(t)x}r_y(x). \tag{5.19}
\]
By the hypothesis, $v \in SO(\mathbb{R}_+)$. On the other hand, $\Psi \in SO(\mathbb{R}_+)$ in view of Lemma 2.3. Hence $\Psi^{1/p} \in SO(\mathbb{R}_+)$. Then, due to Lemmas 2.4 and 2.5 $a_n \in SO(\mathbb{R}_+)$ for all $n \in \mathbb{N}$. Therefore, from Lemma 5.1 we obtain that
\[
a_n(t, x) := a_n(t), (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \text{ belongs to } \tilde{E}(\mathbb{R}_+, V(\mathbb{R})).
\]
On the other hand, by Lemma 5.2 $b_n \in \tilde{E}(\mathbb{R}_+, V(\mathbb{R}))$. Thus, $c_n = a_n b_n$ belongs to $\tilde{E}(\mathbb{R}_+, V(\mathbb{R}))$ for every $n \in \mathbb{N}$.

Following the proof of [6, Lemma 2.1] (see also [1, Theorem 2.2]), let us show that
\[
\lim \sup \|a_n\|_{C^1_b(\mathbb{R}_+)}^{1/n} < 1. \tag{5.20}
\]
By Lemmas 2.2 and 2.3
\[
\lim_{t \to s} \Psi(t) = 1 + \lim_{t \to s} t\psi'(t) = 1, \quad s \in \{0, \infty\}. \tag{5.21}
\]
If $1 \gg v$, then
\[
\lim \sup_{t \to s} |v(t)| < 1, \quad s \in \{0, \infty\}. \tag{5.22}
\]
From (5.21)–(5.22) it follows that
\[
L^s(s) := \lim \sup_{t \to s} |v(t)(\Psi(t))^{1/p}| < 1, \quad s \in \{0, \infty\}.
\]
Fix $\varepsilon > 0$ such that $L^*(s) + \varepsilon < 1$ for $s \in \{0, \infty\}$. By the definition of $L^*(s)$, there exist points $t_1, t_2 \in \mathbb{R}_+$ such that

\[
\begin{align*}
|v(t)(\Psi(t))^{1/p}| &< L^*(0) + \varepsilon & \text{for} & t \in (0, t_1), \\
|v(t)(\Psi(t))^{1/p}| &< L^*(\infty) + \varepsilon & \text{for} & t \in (t_2, \infty). \\
\end{align*}
\]

(5.23)

The mapping $\gamma$ has no fixed points other than 0 and $\infty$. Hence, either $\gamma(t) > t$ or $\gamma(t) < t$ for all $t \in \mathbb{R}_+$. For definiteness, suppose that $\gamma(t) > t$ for all $t \in \mathbb{R}_+$. Then there exists a number $k_0 \in \mathbb{N}$ such that $\gamma_{k_0}(t_1) \in (t_2, \infty)$. Put

\[
M_1 := \sup_{t \in \mathbb{R}_+} |v(t)(\Psi(t))^{1/p}|, \quad M_2 := \sup_{t \in \mathbb{R}_+ \setminus [t_1, \gamma_{k_0}(t_1)]} |v(t)(\Psi(t))^{1/p}|.
\]

Since $v\Psi^{1/p} \in SO(\mathbb{R}_+)$, we have $M_1 < \infty$. Moreover, from (5.23) we obtain

\[
M_2 \leq \max(L^*(0), L^*(\infty)) + \varepsilon < 1.
\]

Then, for every $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$,

\[
|a_n(t)| = \prod_{k=0}^{n-1} |v[\gamma_k(t)](\Psi[\gamma_k(t)])^{1/p}| \leq M_1^{k_0} M_2^{n-k_0} \leq M_1^{k_0} (\max(L^*(0), L^*(\infty)) + \varepsilon)^{n-k_0}.
\]

From here we immediately get (5.20).

Now let us show that

\[
\limsup_{n \to \infty} \|b_n\|_{C^0(\mathbb{R}_+,V(\mathbb{R}))}^{1/n} \leq 1.
\]

(5.24)

By Lemma [5.2] there exists a constant $C(y) \in (0, \infty)$ depending only on $y$ such that for all $n \in \mathbb{N}$,

\[
\|b_n\|_{C^0(\mathbb{R}_+,V(\mathbb{R}))} \leq C(y) \left(1 + \sup_{t \in \mathbb{R}_+} |\psi_n(t)| \right).
\]

(5.25)

From (5.11) we obtain

\[
\psi_n(t) = \log \left(\prod_{k=0}^{n-1} \frac{\gamma[\gamma_k(t)]}{\gamma_k(t)}\right) = \sum_{k=0}^{n-1} \log \frac{\gamma[\gamma_k(t)]}{\gamma_k(t)} = \sum_{k=0}^{n-1} \psi[\gamma_k(t)].
\]

(5.26)

Let

\[
M_3 := \sup_{t \in \mathbb{R}_+} |\psi(t)|.
\]

Since $\gamma_k$ is a diffeomorphism of $\mathbb{R}_+$ onto itself for every $k \in \mathbb{Z}$, we have

\[
M_3 = \sup_{t \in \mathbb{R}_+} |\psi(t)| = \sup_{t \in \mathbb{R}_+} |\psi[\gamma(t)]| = \cdots = \sup_{t \in \mathbb{R}_+} |\psi[\gamma_{n-1}(t)]|.
\]

(5.27)

From (5.26)–(5.27) we obtain

\[
\|b_n\|_{C^0(\mathbb{R}_+,V(\mathbb{R}))} \leq C(y)(1 + M_3 n), \quad n \in \mathbb{N},
\]
which implies (5.24). Combining (5.18), (5.20), and (5.24), we arrive at
\[
\limsup_{n \to \infty} \|c_n\|_{C_b(\mathbb{R}^+, V(\mathbb{R}))}^{1/n} \leq \left( \limsup_{n \to \infty} \|a_n\|_{C_b(\mathbb{R}^+)}^{1/n} \right) \times \left( \limsup_{n \to \infty} \|b_n\|_{C_b(\mathbb{R}^+, V(\mathbb{R}))}^{1/n} \right) < 1.
\]

This shows that the series \( \sum_{n=0}^{\infty} c_n \) is absolutely convergent in the norm of \( C_b(\mathbb{R}^+, V(\mathbb{R})) \). From (5.18)–(5.19) and (5.26) we get for \((t, x) \in \mathbb{R}^+ \times \mathbb{R}\) and \(n \in \mathbb{N}\),
\[
c_n(t, x) = \left( \prod_{k=0}^{n-1} v[\gamma_k(t)] \left( \Psi[\gamma_k(t)] \right)^{1/p} e^{i\psi[\gamma_k(t)]x} \right) r_y(x).
\]

We have already shown that \( c_0 \) given by (5.10) and \( c_n, n \in \mathbb{N}, \) given by (5.28) belong to \( \hat{E}(\mathbb{R}^+, V(\mathbb{R})) \). Thus \( c := \sum_{n=0}^{\infty} c_n \) is given by (5.7) and it belongs to \( \hat{E}(\mathbb{R}^+, V(\mathbb{R})) \).

From (5.9), (5.17) and Theorem 4.1 we get
\[
\left\| \Phi^{-1} \operatorname{Op}(c) \Phi - \sum_{n=0}^{N} (vU_\gamma)^n R_y \right\|_{L^p(\mathbb{R}^+)} \leq \left\| \Phi^{-1} \left( c - \sum_{n=0}^{N} c_n \right) \Phi \right\|_{L^p(\mathbb{R}^+)} \leq C_p \left\| c - \sum_{n=0}^{N} c_n \right\|_{C_b(\mathbb{R}^+, V(\mathbb{R}))} = o(1) \quad \text{as} \quad N \to \infty.
\]

Hence
\[
\sum_{n=0}^{\infty} (vU_\gamma)^n R_y = \Phi^{-1} \operatorname{Op}(c) \Phi.
\]

Combining this identity with (5.8), we arrive at (5.6). Part (b) is proved.

(c) From (5.10) and (5.28) it follows that \( c_n(t, \pm \infty) = 0 \) for \( n \in \mathbb{N} \cup \{0\} \) and \( t \in \mathbb{R}^+ \). Then, in view of Lemma 4.4,
\[
c(t, \pm \infty) = 0, \quad t \in \mathbb{R}^+.
\]

Since \( v, \psi \in SO(\mathbb{R}^+) \), from Lemma 5.1 it follows that the functions
\[
v(t, x) = v(t), \quad \tilde{\psi}(t, x) := \psi(t), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R},
\]
belong to the Banach algebra \( \hat{E}(\mathbb{R}^+, V(\mathbb{R})) \). Consider the countable family \( \{v, \tilde{\psi}, c\} \cup \{c_n\}_{n=0}^{\infty} \) of functions in \( \hat{E}(\mathbb{R}^+, V(\mathbb{R})) \).

Fix \( s \in \{0, \infty\} \) and \( \xi \in M_q(SO(\mathbb{R}^+)) \). By Lemma 4.2 and (5.30), there is a sequence \( \{t_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^+ \) and functions \( c(\xi, \cdot) \in V(\mathbb{R}^+), c_n(\xi, \cdot) \in V(\mathbb{R}^+), n \in \mathbb{N} \cup \{0\}, \) such that
\[
\lim_{j \to \infty} t_j = s, \quad v(\xi) = \lim_{j \to \infty} v(t_j), \quad \psi(\xi) = \lim_{j \to \infty} \psi(t_j),
\]
and for \( n \in \mathbb{N} \cup \{0\} \) and \( x \in \mathbb{R},
\[
c_n(\xi, x) = \lim_{j \to \infty} c_n(t_j, x), \quad c(\xi, x) = \lim_{j \to \infty} c(t_j, x).
\]
From (5.29) and the second limit in (5.32) we get
\[ c(\xi, \pm\infty) = \lim_{j \to \infty} c(t_j, \pm\infty) = 0. \] (5.33)

Trivially,
\[ c_0(\xi, x) = r_y(x), \quad (\xi, x) \in (\Delta \cup \mathbb{R}_+) \times \mathbb{R}. \] (5.34)

From Lemmas 2.2 and 2.3 we obtain
\[ \lim_{t \to s} \left( \Psi(t) \right)^{1/p} = 1, \quad s \in \{0, \infty\}. \] (5.35)

From Lemma 2.5 it follows that for \( k \in \mathbb{N} \),
\[ \lim_{j \to \infty} v(t_j) = \lim_{j \to \infty} v[\gamma_k(t_j)], \quad \lim_{j \to \infty} \psi(t_j) = \lim_{j \to \infty} \psi[\gamma_k(t_j)]. \] (5.36)

Combining (5.28), (5.31), the first limit in (5.32), and (5.35)–(5.36), we get
for \( x \in \mathbb{R} \) and \( n \in \mathbb{N} \),
\[ c_n(\xi, x) = \lim_{j \to \infty} \left( \prod_{k=0}^{n-1} v[\gamma_k(t_j)] \left( \Psi[\gamma_k(t_j)] \right)^{1/p} e^{i\psi[\gamma_k(t_j)] x} \right) r_y(x) \]
\[ = (v(\xi)e^{i\psi(\xi)x})^n r_y(x). \] (5.37)

From (5.34), (5.37), and Lemma 4.4 we obtain
\[ c(\xi, x) = \sum_{n=0}^{\infty} (v(\xi)e^{i\psi(\xi)x})^n r_y(x). \] (5.38)

Since \( 1 \gg v \), we have
\[ \limsup_{t \to s} |v(t)| < 1, \quad s \in \{0, \infty\}, \]
whence, in view of Lemma 2.1 we obtain
\[ |v(\xi)e^{i\psi(\xi)x}| \leq \max_{s \in \{0, \infty\}} \left( \limsup_{t \to s} |v(t)| \right) < 1. \]

Therefore,
\[ \sum_{n=0}^{\infty} (v(\xi)e^{i\psi(\xi)x})^n = (1 - v(\xi)e^{i\psi(\xi)x})^{-1}. \] (5.39)

From (5.38) and (5.39) we get
\[ c(\xi, x) = (1 - v(\xi)e^{i\psi(\xi)x})^{-1} r_y(x), \quad (\xi, x) \in \Delta \times \mathbb{R}. \] (5.40)

Combining (5.29), (5.33), and (5.40), we arrive at the assertion of part (c). \( \square \)
6. Fredholmness and Index of the Operator $V$

6.1. First Step of Regularization

Lemma 6.1. Let $\alpha, \beta \in SOS(\mathbb{R}_+)$ and let $c, d \in SO(\mathbb{R}_+)$ be such that $1 \gg c$ and $1 \gg d$. Then for every $\mu \in [0, 1]$ and $y \in (1, \infty)$ the following statements hold:

(a) the operators $I - \mu c U_\alpha$ and $I - \mu d U_\beta$ are invertible on $L^p(\mathbb{R}_+)$;
(b) for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, the functions
\[
a_{\mu, y}^{c, \alpha}(t, x) := (1 - \mu c(t)(\Omega(t))^{1/p} e^{i \omega(t)x}) r_y(x),
\]
\[
a_{\mu, y}^{d, \beta}(t, x) := (1 - \mu d(t)(H(t))^{1/p} e^{i \eta(t)x}) r_y(x)
\]
and
\[
c_{\mu, y}^{c, \alpha}(t, x) := r_y(x)
\]
\[
+ \sum_{n=1}^\infty \mu^n \left( \prod_{k=0}^{n-1} c[\alpha_k(t)](\Omega[\alpha_k(t)])^{1/p} e^{i \omega[\alpha_k(t)]x} \right) r_y(x),
\]
\[
c_{\mu, y}^{d, \beta}(t, x) := r_y(x)
\]
\[
+ \sum_{n=1}^\infty \mu^n \left( \prod_{k=0}^{n-1} d[\beta_k(t)](H[\beta_k(t)])^{1/p} e^{i \eta[\beta_k(t)]x} \right) r_y(x),
\]
with
\[
\omega(t) = \log[\alpha(t)/t], \quad \Omega(t) = 1 + t \omega'(t),
\]
\[
\eta(t) = \log[\beta(t)/t], \quad H(t) = 1 + t \eta'(t),
\]
belong to the algebra $\tilde{E}(\mathbb{R}_+, V(\mathbb{R}))$;
(c) the operators
\[
V_{\mu, y} := (I - \mu c U_\alpha)P_y^+ + (I - \mu d U_\beta)P_y^-,
\]
\[
L_{\mu, y} := (I - \mu c U_\alpha)^{-1}P_y^+ + (I - \mu d U_\beta)^{-1}P_y^-
\]
are related by
\[
V_{\mu, y}L_{\mu, y} \simeq L_{\mu, y}V_{\mu, y} \simeq H_{\mu, y},
\]
where
\[
H_{\mu, y} := \Phi^{-1} \text{Op}(h_{\mu, y}) \Phi
\]
and the function $h_{\mu, y}$, given for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ by
\[
h_{\mu, y}(t, x) := 1 + \frac{1}{4} \left[2(r_y(x))^2 - a_{\mu, y}^{d, \beta}(t, x)c_{\mu, y}^{c, \alpha}(t, x) - a_{\mu, y}^{c, \alpha}(t, x)c_{\mu, y}^{d, \beta}(t, x) \right],
\]
belongs to the algebra $\tilde{E}(\mathbb{R}_+, V(\mathbb{R}))$.

Proof. (a) From Lemma 2.7 it follows that the operators
\[
I - \mu c U_\alpha, I - \mu d U_\beta \in F\mathcal{O}_{\alpha, \beta}
\]
are invertible and
\[(I - \mu c U_\alpha)^{-1}, (I - \mu d U_\beta)^{-1} \in \mathcal{FO}_{\alpha,\beta}.\]  \hfill (6.13)

This completes the proof of part (a).

(b) From Lemma 3.1 it follows that
\[R_y^2 = \Phi^{-1} \text{Co}(r_y^2)\Phi = \Phi^{-1} \text{Op}(r_y^2)\Phi,\]  \hfill (6.14)

where \(r_y(t, x) = r_y(x)\) for \((t, x) \in \mathbb{R}_+ \times \mathbb{R}\). From Lemma 5.1 we deduce that \(r_y^2 \in \tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))\). By Lemma 5.4(a),
\[(I - \mu c U_\alpha) R_y = \Phi^{-1} \text{Op}(\alpha^{c,\alpha}_{\mu,y})\Phi,\]  \hfill (6.15)
\[(I - \mu d U_\beta) R_y = \Phi^{-1} \text{Op}(\alpha^{d,\beta}_{\mu,y})\Phi,\]  \hfill (6.16)

where the functions \(\alpha^{c,\alpha}_{\mu,y}\) and \(\alpha^{d,\beta}_{\mu,y}\), given by (6.1) and (6.2), respectively, belong to \(\tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))\). In particular, this completes the proof of part (b).

(c) From (6.12)–(6.13) and Lemmas 2.8 and 3.1 it follows that
\[(I - \mu c U_\alpha)^T T \simeq T (I - \mu c U_\alpha)^T,\]  \hfill (6.19)
\[(I - \mu d U_\beta)^T T \simeq T (I - \mu d U_\beta)^T;\]  \hfill (6.20)

for every \(t \in \{ -1, 1 \}\) and \(T \in \{ P^+_y, P^-_y, R_y \}\). Applying consecutively relations (6.19)–(6.20) with \(T \in \{ P^+_y, P^-_y \}\), Lemma 3.2(b), and relations (6.19)–(6.20) with \(T = R_y\), we get
\[V_{\mu,y} L_{\mu,y} \simeq (P^+_y)^2 + (I - \mu d U_\beta)(I - \mu c U_\alpha)^{-1} P^-_y P^+_y + (P^-_y)^2 + (I - \mu c U_\alpha)(I - \mu d U_\beta)^{-1} P^+_y P^-_y\]
\[= (P^+_y + \frac{R_y^2}{4}) - (I - \mu d U_\beta)(I - \mu c U_\alpha)^{-1} \frac{R_y^2}{4} \]
\[+ (P^-_y + \frac{R_y^2}{4}) - (I - \mu c U_\alpha)(I - \mu d U_\beta)^{-1} \frac{R_y^2}{4} \]
\[\simeq I + \frac{R_y^2}{2} - \frac{1}{4} (I - \mu d U_\beta) R_y (I - \mu c U_\alpha)^{-1} R_y \]
\[\quad - \frac{1}{4} (I - \mu c U_\alpha) R_y (I - \mu d U_\beta)^{-1} R_y.\]  \hfill (6.21)

Applying equalities (6.15)–(6.18) to (6.21), we obtain
\[V_{\mu,y} L_{\mu,y} \simeq I + \frac{1}{2} \Phi^{-1} \text{Op}(r_y^2)\Phi - \frac{1}{4} \Phi^{-1} \text{Op}(\alpha^{d,\beta}_{\mu,y}) \text{Op}(\alpha^{c,\alpha}_{\mu,y})\Phi \]
\[\quad - \frac{1}{4} \Phi^{-1} \text{Op}(\alpha^{c,\alpha}_{\mu,y}) \text{Op}(\alpha^{d,\beta}_{\mu,y})\Phi.\]  \hfill (6.22)
From Theorem 4.5 we get
\[
\begin{align*}
\text{Op}(a_{\mu,y}^{d,\beta}) \text{Op}(c_{\mu,y}^{\alpha,\mu}) & \simeq \text{Op}(a_{\mu,y}^{d,\beta}c_{\mu,y}^{\alpha,\mu}), \\
\text{Op}(a_{\mu,y}^{c,\alpha}) \text{Op}(c_{\mu,y}^{d,\beta}) & \simeq \text{Op}(a_{\mu,y}^{c,\alpha}d_{\mu,y}^{\beta}).
\end{align*}
\]
(6.23) (6.24)

Combining (6.22)–(6.24), we arrive at
\[
V_{\mu,y}L_{\mu,y} \simeq \Phi^{-1} \text{Op}(h_{\mu,y})\Phi,
\]
where the function \( h_{\mu,y} \), given by (6.11), belongs to the algebra \( \tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R})) \) because the functions (6.1)–(6.4) lie in this algebra in view of part (b). Analogously, it can be shown that
\[
L_{\mu,y}V_{\mu,y} \simeq \Phi^{-1} \text{Op}(h_{\mu,y})\Phi,
\]
which completes the proof.

6.2. Fredholmness of the Operator \( H_{\mu,2} \)

In this subsection we will prove that the operators \( H_{\mu,2} \) given by (6.10) are Fredholm for every \( \mu \in [0, 1] \). To this end, we will use Theorem 4.7.

First we represent boundary values of \( h_{\mu,y} \) in a way, which is convenient for further analysis.

**Lemma 6.2.** Let \( \alpha, \beta \in \text{SOS}(\mathbb{R}_+) \) and let \( c, d \in \text{SO}(\mathbb{R}_+) \) be such that \( 1 \gg c \) and \( 1 \gg d \). If \( h_{\mu,y} \) is given by (6.11) and (6.1)–(6.6), then for every \( \mu \in [0, 1] \) and \( y \in (1, \infty) \), we have
\[
h_{\mu,y}(\xi, x) = \begin{cases} 
v_{\mu,y}(\xi, x)\ell_{\mu,y}(\xi, x), & \text{if } (\xi, x) \in \Delta \times \mathbb{R}, \\
1, & \text{if } (\xi, x) \in (\mathbb{R}_+ \cup \Delta) \times \{\pm \infty\},
\end{cases}
\]
where
\[
\begin{align*}
v_{\mu,y}(\xi, x) & := (1 - \mu c(\xi)e^{i\omega(\xi)x})p_{y}^+(x) + (1 - \mu d(\xi)e^{i\eta(\xi)x})p_{y}^-(x), \\
\ell_{\mu,y}(\xi, x) & := (1 - \mu c(\xi)e^{i\omega(\xi)x})^{-1}p_{y}^+(x) + (1 - \mu d(\xi)e^{i\eta(\xi)x})^{-1}p_{y}^-(x)
\end{align*}
\]
(6.25) (6.26)

for \( (\xi, x) \in \Delta \times \mathbb{R} \).

**Proof.** From (6.11), Lemmas 4.3, 5.4(b), and 5.5(c) it follows that
\[
h_{\mu,y}(\xi, x) = 1 \quad \text{for} \quad (\xi, x) \in (\mathbb{R}_+ \cup \Delta) \times \{\pm \infty\}
\]
and
\[
h_{\mu,y}(\xi, x) = 1 + \frac{1}{4}[2(r_y(x))^2 - (1 - \mu d(\xi)e^{i\eta(\xi)x})r_y(x)(1 - \mu c(\xi)e^{i\omega(\xi)x})^{-1}r_y(x) - (1 - \mu c(\xi)e^{i\omega(\xi)x})r_y(x)(1 - \mu d(\xi)e^{i\eta(\xi)x})^{-1}r_y(x)]
\]
for \((\xi, x) \in \Delta \times \mathbb{R}\). By Lemma 3.2(a),

\[
\mathfrak{h}_{\mu, y}(\xi, x) =
\]

\[
= \left( \frac{p_y^+(x) + (r_y(x))^2}{4} \right) - \left( 1 - \mu d(\xi)e^{i\eta(\xi)x} \right) \left( 1 - \mu c(\xi)e^{i\omega(\xi)x} \right) - 1 \frac{(r_y(x))^2}{4} \\
+ \left( \frac{p_y^-(x) + (r_y(x))^2}{4} \right) - \left( 1 - \mu c(\xi)e^{i\omega(\xi)x} \right) \left( 1 - \mu d(\xi)e^{i\eta(\xi)x} \right) - 1 \frac{(r_y(x))^2}{4}
\]

\[
= (p_y^+(x))^2 + (1 - \mu d(\xi)e^{i\eta(\xi)x})(1 - \mu c(\xi)e^{i\omega(\xi)x})^{-1}p_y^-(x)p_y^+(x) \\
+ (p_y^-(x))^2 + (1 - \mu c(\xi)e^{i\omega(\xi)x})(1 - \mu d(\xi)e^{i\eta(\xi)x})^{-1}p_y^+(x)p_y^-(x)
\]

for \((\xi, x) \in \Delta \times \mathbb{R}\), which completes the proof. \(\square\)

We were unable to prove that \(\mathfrak{h}_{\mu, y}\) satisfies the hypotheses of Theorem 4.7 for every \(y \in (1, \infty)\) or at least for \(y = p\). However, the very special form of the ranges of \(v_{\mu,2}\) and \(\ell_{\mu, 2}\) given by \((6.25)\) and \((6.26)\), respectively, allows us to prove that \(v_{\mu,2}\) and \(\ell_{\mu, 2}\) are separated from zero for all \(\mu \in [0, 1]\), and thus \(\mathfrak{h}_{\mu, 2}\) satisfies the assumptions of Theorem 4.7.

**Lemma 6.3.** Let \(\alpha, \beta \in \text{SOS}(\mathbb{R}_+)\) and let \(c, d \in \text{SO}(\mathbb{R}_+)\) be such that \(1 \gg c\) and \(1 \gg d\). Then for every \(\mu \in [0, 1]\) the operator \(H_{\mu,2}\) given by \((6.10)\) is Fredholm on \(L^p(\mathbb{R}_+)\).

**Proof.** By Lemma 6.2 for the function \(\mathfrak{h}_{\mu,2}\) defined by \((6.11)\) and \((6.1) - (6.6)\) we have

\[
\mathfrak{h}_{\mu,2}(\xi, x) = 1 \neq 0 \quad \text{for} \quad (\xi, x) \in (\mathbb{R}_+ \cup \Delta) \times \{\pm \infty\} \quad (6.27)
\]

and

\[
\mathfrak{h}_{\mu,2}(\xi, x) = v_{\mu,2}(\xi, x)\ell_{\mu,2}(\xi, x) \quad \text{for} \quad (\xi, x) \in \Delta \times \mathbb{R},
\]

where \(v_{\mu,2}\) and \(\ell_{\mu,2}\) are defined by \((6.25)\) and \((6.26)\), respectively. From Lemmas 2.9 and 2.10 it follows that for each \(\xi \in \Delta\) the ranges of the continuous functions \(v_{\mu,2}(\xi, \cdot)\) and \(\ell_{\mu,2}(\xi, \cdot)\) defined on \(\mathbb{R}\) lie in the half-plane

\[
\mathcal{H}^{\mu, \xi} := \{ z \in \mathbb{C} : \text{Re } z > 1 - \mu \max(|c(\xi)|, |d(\xi)|) \}.
\]

From Lemma 2.1 we get

\[
C(\Delta) := \sup_{\xi \in \Delta} |c(\xi)| = \max_{\xi \in \Delta} \left( \limsup_{t \to s} |c(\xi)| \right),
\]

\[
D(\Delta) := \sup_{\xi \in \Delta} |d(\xi)| = \max_{\xi \in \Delta} \left( \limsup_{t \to s} |d(\xi)| \right).
\]

Since \(1 \gg c\) and \(1 \gg d\), we see that \(C(\Delta) < 1\) and \(D(\Delta) < 1\). Therefore, for every \(\xi \in \Delta\) and \(\mu \in [0, 1]\), the half-plane \(\mathcal{H}^{\mu, \xi}\) is contained in the half-plane

\[
\{ z \in \mathbb{C} : \text{Re } z > 1 - \max((|C(\Delta)|, |D(\Delta)|)) \}
\]

and the origin does not lie in the latter half-plane. Thus

\[
\mathfrak{h}_{\mu,2}(\xi, x) = v_{\mu,2}(\xi, x)\ell_{\mu,2}(\xi, x) \neq 0 \quad \text{for all} \quad (\xi, x) \in \Delta \times \mathbb{R}. \quad (6.28)
\]
From (6.27)–(6.28) and Theorem 4.7 we obtain that the operator \( H_{\mu,2} \) is Fredholm on \( L^p(\mathbb{R}_+) \).

\[ \square \]

### 6.3. Proof of the Main Result

For \( \mu \in [0,1] \), consider the operators \( V_{\mu,2} \) and \( L_{\mu,2} \) defined by (6.7) and (6.8), respectively. It is obvious that \( V_{0,2} = P_y^+ + P_y^- = I \) and \( V_{1,2} = V \). By Lemma 6.1(c),

\[ V_{\mu,2} L_{\mu,2} \simeq L_{\mu,2} V_{\mu,2} \simeq H_{\mu,2}, \quad \mu \in [0,1], \tag{6.29} \]

where the operator \( H_{\mu,2} \) given by (6.10) is Fredholm in view of Lemma 6.3.

Let \( H^{(-1)}_{\mu,2} \) be a regularizer for \( H_{\mu,2} \). From (6.29) it follows that

\[ V_{\mu,2} (L_{\mu,2} H^{(-1)}_{\mu,2}) \simeq I, \quad (H^{(-1)}_{\mu,2} L_{\mu,2}) V_{\mu,2} \simeq I, \quad \mu \in [0,1]. \tag{6.30} \]

Thus, \( L_{\mu,2} H^{(-1)}_{\mu,2} \) is a right regularizer for \( V_{\mu,2} \) and \( H^{(-1)}_{\mu,2} L_{\mu,2} \) is a left regularizer for \( V_{\mu,2} \). Therefore, \( V_{\mu,2} \) is Fredholm for every \( \mu \in [0,1] \). It is obvious that the operator-valued function \( \mu \mapsto V_{\mu,2} \in \mathcal{B}(L^p(\mathbb{R}_+)) \) is continuous on \([0,1]\). Hence the operators \( V_{\mu,2} \) belong to the same connected component of the set of all Fredholm operators. Therefore all \( V_{\mu,2} \) have the same index (see, e.g., [4, Section 4.10]). Since \( V_{0,2} = I \), we conclude that

\[ \text{Ind } V = \text{Ind } V_{1,2} = \text{Ind } V_{0,2} = \text{Ind } I = 0, \]

which completes the proof of Theorem 1.1. \( \square \)

### 7. Regularization of the Operator \( W \)

#### 7.1. Regularizers of the Operator \( W \)

As a by-product of the proof of Section 6, we can describe all regularizers of a slightly more general operator \( W \).

**Theorem 7.1.** Let \( 1 < p < \infty \), \( \varepsilon_1, \varepsilon_2 \in \{-1,1\} \), and \( \alpha, \beta \in \text{SOS}(\mathbb{R}_+) \). Suppose \( c,d \in \text{SO}(\mathbb{R}_+) \) are such that \( 1 \gg c \) and \( 1 \gg d \). Then the operator \( W \) given by

\[ W := (I - cU^{\varepsilon_1}) P_2^+ + (I - dU^{\varepsilon_2}) P_2^- \]

is Fredholm on the space \( L^p(\mathbb{R}_+) \) and \( \text{Ind } W = 0 \). Moreover, each regularizer \( W^{(-1)} \) of the operator \( W \) is of the form

\[ W^{(-1)} = [\Phi^{-1} \text{Op}(f) \Phi] \cdot [(I - cU^{\varepsilon_1})^{-1} P_2^+ + (I - dU^{\varepsilon_2})^{-1} P_2^-] + K, \tag{7.1} \]

where \( K \in \mathcal{K}(L^p(\mathbb{R}_+)) \) and \( f \in \tilde{E}(\mathbb{R}_+, V(\mathbb{R})) \) is such that

\[ f(\xi, x) = \begin{cases} 
1, & \text{if } (\xi, x) \in \Delta \times \mathbb{R}, \\
\frac{1}{w(\xi, x)\ell(\xi, x)}, & \text{if } (\xi, x) \in (\mathbb{R}_+ \cup \Delta) \times \{\pm \infty\}, 
\end{cases} \tag{7.2} \]
where
\[
\omega(\xi, x) := (1 - c(\xi)e^{i\xi_1\omega(\xi)x})p_2^+(x) + (1 - d(\xi)e^{i\xi_2\eta(\xi)x})p_2^-(x) \neq 0, \quad (7.3)
\]
\[
\ell(\xi, x) := \frac{p_2^+ (x)}{1 - c(\xi)e^{i\xi_1\omega(\xi)x}} + \frac{p_2^- (x)}{1 - d(\xi)e^{i\xi_2\eta(\xi)x}} \neq 0 \quad (7.4)
\]
for \((\xi, x) \in \Delta \times \mathbb{R}\) with \(\omega(t) := \log[\alpha(t)/t]\) and \(\eta(t) := \log[\beta(t)/t]\) for \(t \in \mathbb{R}_+\).

Proof. Since \(\alpha, \beta \in \text{SOS}(\mathbb{R}_+),\) from Lemma 2.5 it follows that \(\alpha_{-1}\) and \(\beta_{-1}\) also belong to \(\text{SOS}(\mathbb{R}_+).\) Taking into account that \(U_{\alpha}^c = U_{\alpha_1}\) and \(U_{\beta}^c = U_{\beta_2},\) from Theorem 1.1 we deduce that the operator \(W\) is Fredholm and \(\text{Ind} W = 0.\) Further, from (6.30) and Lemma 6.3 it follows that each regularizer \(W^{(-1)}\) of \(W\) is of the form
\[
W^{(-1)} = H^{(-1)}L + K_1, \quad (7.5)
\]
where \(K_1 \in \mathcal{K}(L^p(\mathbb{R}_+)),\)
\[
L := (I - cU_{\alpha}^c)^{-1}P_2^+ + (I - dU_{\beta}^c)^{-1}P_2^-, \quad (7.6)
\]
and \(H^{(-1)}\) is a regularizer of the Fredholm operator \(H\) given by
\[
H := \Phi^{-1} \text{Op}(\mathbf{h})\Phi,
\]
where \(\mathbf{h} \in \tilde{E}(\mathbb{R}_+, V(\mathbb{R}))\) is given for \((t, x) \in \mathbb{R}_+ \times \mathbb{R}\) by
\[
\mathbf{h}(t, x) := 1 + \frac{1}{4} \left[2(r_2(x))^2 - a_{1,2}^d, \beta_2(t, x) \epsilon_{c, \alpha_1}^c, \epsilon_{1,2}^\alpha (t, x) - a_{1,2}^d, \alpha_1 (t, x) \epsilon_{1,2}^d, \beta_2 (t, x)\right],
\]
and the functions \(a_{1,2}^c, \epsilon_{1,2}^\alpha\) and \(a_{1,2}^d, \epsilon_{1,2}^d\) are given by (6.1)–(6.2) and (6.3)–(6.4) with \(\alpha\) and \(\beta\) replaced by \(\alpha_\epsilon\) and \(\beta_\epsilon,\) respectively.

Taking into account Lemma 2.6 by analogy with Lemma 6.2 we get
\[
\mathbf{h}(\xi, x) = \left\{ \begin{array}{ll}
w(\xi, x)\ell(\xi, x), & \text{if } (\xi, x) \in \Delta \times \mathbb{R}, \\
1, & \text{if } (\xi, x) \in (\mathbb{R}_+ \cup \Delta) \times \{\pm \infty\}. \end{array} \right. \quad (7.7)
\]
By Theorem 4.7(b), each regularizer \(H^{(-1)}\) of the Fredholm operator \(H\) is of the form
\[
H^{(-1)} = \Phi^{-1} \text{Op}(\mathbf{f})\Phi + K_2, \quad (7.8)
\]
where \(K_2 \in \mathcal{K}(L^p(\mathbb{R}_+))\) and \(\mathbf{f} \in \tilde{E}(\mathbb{R}_+, V(\mathbb{R}))\) is such that
\[
\mathbf{f}(t, \pm \infty) = 1/\mathbf{h}(t, \pm \infty) \quad \text{for all} \quad t \in \mathbb{R}_+, \quad (7.9)
\]
From (7.5)–(7.6) and (7.8) we get (7.1). Combining (7.7) and (7.9), we arrive at (7.2). \(\square\)
7.2. One Useful Consequence of Regularization of $W$

**Theorem 7.2.** Under the assumptions of Theorem 7.1 for every $y \in (1, \infty)$ there exists a function $g_y \in \tilde{E}(\mathbb{R}_+, V(\mathbb{R}))$ such that

$$ (\Phi^{-1} \text{Op}(g_y) \Phi) W \simeq R_y $$

(7.10)

and

$$ g_y(\xi, x) = \begin{cases} \frac{r_y(x)}{w(\xi, x)}, & \text{if } (\xi, x) \in \Delta \times \mathbb{R}, \\ 0, & \text{if } (\xi, x) \in (\mathbb{R}_+ \cup \Delta) \times \{\pm \infty\}, \end{cases} $$

where the function $w$ is defined for $(\xi, x) \in \Delta \times \mathbb{R}$ by (7.3).

**Proof.** From Theorem 7.1 it follows that

$$ (\Phi^{-1} \text{Op}(f) \Phi) LW \simeq R_y, $$

(7.11)

where $L$ is given by (7.6) and $f \in \tilde{E}(\mathbb{R}_+, V(\mathbb{R}))$ satisfies (7.2). From Lemmas 2.8 and 3.1 we get

$$ WR_y \simeq R_y W. $$

(7.12)

Lemmas 3.1 and 5.5(a)–(b) imply that

$$ LR_y = (I - c_U^{\alpha_1})^{-1} R_y P_2^+ + (I - d_U^{\beta_2})^{-1} R_y P_2^- $$

$$ = \Phi^{-1} \text{Op}(c_{1,y}^{\alpha_1}) \text{Co}(p_2^+) \Phi + \Phi^{-1} \text{Op}(d_{1,y}^{\beta_2}) \text{Co}(p_2^-) \Phi, $$

(7.13)

where the functions $c_{1,y}^{\alpha_1}$ and $d_{1,y}^{\beta_2}$, given by (6.3) and (6.4) with $\alpha$ and $\beta$ replaced by $\alpha_{\varepsilon_1}$ and $\beta_{\varepsilon_2}$, respectively, belong to $\tilde{E}(\mathbb{R}_+, V(\mathbb{R}))$. From (7.13) and Lemmas 5.1 and 4.6 we obtain

$$ LR_y = \Phi^{-1} \text{Op}(c_{1,y}^{\alpha_{\varepsilon_1}} p_2^+ + d_{1,y}^{\beta_{\varepsilon_2}} p_2^-) \Phi, $$

(7.14)

where $c_{1,y}^{\alpha_{\varepsilon_1}} p_2^+ + d_{1,y}^{\beta_{\varepsilon_2}} p_2^- \in \tilde{E}(\mathbb{R}_+, V(\mathbb{R}))$. From (7.11), (7.12), (7.14), and Theorem 4.5 we get (7.10) with

$$ g_y := f(c_{1,y}^{\alpha_{\varepsilon_1}} p_2^+ + d_{1,y}^{\beta_{\varepsilon_2}} p_2^-) \in \tilde{E}(\mathbb{R}_+, V(\mathbb{R})). $$

(7.15)

Obviously,

$$ p_2^+ (\pm \infty) = 1, \quad p_2^\pm (\mp \infty) = 0. $$

(7.16)

By Lemmas 2.6 and 5.5(c),

$$ c_{1,y}^{\alpha_{\varepsilon_1}} (\xi, x) = \begin{cases} \frac{r_y(x)}{1 - c(\xi)e^{i\varepsilon_1 \omega(\xi)x}}, & \text{if } (\xi, x) \in \Delta \times \mathbb{R}, \\ 0, & \text{if } (\xi, x) \in (\mathbb{R}_+ \cup \Delta) \times \{\pm \infty\}, \end{cases} $$

(7.17)

$$ d_{1,y}^{\beta_{\varepsilon_2}} (\xi, x) = \begin{cases} \frac{r_y(x)}{1 - d(\xi)e^{i\varepsilon_2 \eta(\xi)x}}, & \text{if } (\xi, x) \in \Delta \times \mathbb{R}, \\ 0, & \text{if } (\xi, x) \in (\mathbb{R}_+ \cup \Delta) \times \{\pm \infty\}. \end{cases} $$

(7.18)

From (7.15), (7.18), (7.2), (7.4), and Lemma 4.3 we get

$$ g_y(\xi, x) = 0 \quad \text{for} \quad (\xi, x) \in (\mathbb{R}_+ \cup \Delta) \times \{\pm \infty\}. $$
and
\[
\mathfrak{g}_y(\xi, x) = f(\xi, x) \left( \frac{r_y(x)p_2^+(x)}{1 - c(\xi)e^{i\varepsilon_1\omega(\xi)x}} + \frac{r_y(x)p_2^-(x)}{1 - d(\xi)e^{i\varepsilon_2\eta(\xi)x}} \right)
\]
\[
= \frac{\ell(\xi, x)r_y(x)}{w(\xi, x)\ell(\xi, x)} - \frac{r_y(x)}{w(\xi, x)}
\]
for \((\xi, x) \in \Delta \times \mathbb{R}.
\]

Relation (7.10) will play an important role in the proof of an index formula for the operator \(N\) in (11).

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