Holography and CFT on Generic Manifolds

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ABSTRACT

In this paper it is shown how the AdS/CFT correspondence extends to a more general situation in which the first theory is defined on $(d+1)$-dimensional manifold $\tilde{M}$ defined as the filling in of a compact $d$-dimensional manifold $M$. The stability of the spectral correspondence mass/conformal-weight under such geometry changes is also proven.

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1 Introduction

During last year there has been a big effort in understanding AdS/CFT correspondence \[3–6\]. Among others, the question arose of how to obtain the CFT on the boundary of the AdS space for various field theories (see also \[8,9,13,14\]). In these papers a method was proposed to demonstrate also a spectral correspondence between the mass spectrum of the theory on the AdS space and the conformal weight spectrum of the CFT on the boundary $\partial \text{AdS}$. This correspondence is of course crucial in establishing the full String/SYM mapping and is at the basis of this realization of the t’Hooft-Susskind holography \[1,2\].

A natural interesting question concerns possible generalizations of this mapping for string vacua of the form $\text{AdS} \times Q$ with $Q$ not a sphere, in order to obtain on the other side CFTs with less supersymmetry \[7,11,12\].

In this letter we address the problem of proving an analogous statement for geometries which are not AdS, but more general. We will consider the geometry of a space $\tilde{M}$, with dimension $d + 1$, obtained by filling up a generic compact manifold $M$, with dimension $d$; we will define it with an appropriate metric such that the conformal transformations on $M$ are included as isometries on $\tilde{M}$. In this construction $M$ plays the role of a boundary compactified space-time.

The general correspondence between the two theories can be expressed as

$$
\int_{\tilde{\phi}|_{\partial \tilde{M} = \tilde{M}}} D[\tilde{\phi}] e^{iS_{\tilde{M}}(\tilde{g},\tilde{\phi})} = \int D[\phi] e^{iS_M(g,\phi)} + i \int_M \sqrt{g} \mathcal{O}_\delta[\phi] \phi_0
$$

(1.1)

(in a regularized sense), where $S_M$ and $S_{\tilde{M}}$ are the actions of the two related theories on $M$ and $\tilde{M}$, $g$ and $\tilde{g}$ are the respective metrics, $\phi$ and $\tilde{\phi}$ represent fields on the manifolds and $\mathcal{O}_\delta[\phi]$ a generic composite field of some conformal dimension $\delta$. Notice that, in general, for interacting bulk theories the problem arises of obtaining sharp conditions on it so that on $\partial \tilde{M} = M$ one is left possibly with a local field theory.

The above program is viable by generalizing the AdS approach \[3,4\]. In the next section we will study the case of the massive interacting scalar boson and the free massive spinor to explain the general method. In the rest of the paper the correspondence QFT on $\tilde{M}$ versus CFT on $M = \partial \tilde{M}$ is explained together with the proof of stability of the spectral map.

Given the above situation, one could obtain potentially interesting string or M-theory vacua of the form $\tilde{M} \times Q$ and speculate on an extension of the string/SYM correspondence. The study of the properties of this class of vacua is not performed in this letter.
2 Some explicit calculations

Let \((M, g)\) be a compact riemannian manifold of dimension \(d\) and let \((\tilde{M}, \tilde{g})\) the riemannian manifold \(\tilde{M} = \mathbb{R}_+ \times M\) equipped with the metric

\[
ds^2 = \frac{d\xi^2 + ds^2}{\xi^2}
\]

where \(ds^2 = g_{ab}dx^a dx^b\) is the metric on \(M\) and \(\xi\) is a coordinate on \(\mathbb{R}_+\).

Notice that each isometry of \((M, g)\) is also an isometry of \((\tilde{M}, \tilde{g})\). More interesting is the fact that each dilatation of \((M, g), ds^2 \to r^2 ds^2\), is also an isometry of \((\tilde{M}, \tilde{g})\), once \(\xi \to r\xi\).

In the following we will exploit a technique to discharge to the harmonic analysis on the manifold \(M\) all its peculiar geometrical data. This is obtained via transferring all the amplitudes calculations in a Fourier transformed form.

2.1 The free massive scalar boson

In this section we analyze the free massive scalar boson theory.

Let \(\Delta\) and \(\tilde{\Delta}\) be the scalar Laplacians on \(M\) and \(\tilde{M}\) respectively. By definition, the following relation holds

\[
\tilde{\Delta} = \xi^2 \Delta - \xi^{d+1} \partial_\xi \xi^{1-d} \partial_\xi .
\] (2.2)

Let now \(m\) be the mass of the scalar \(\tilde{\phi}\): its equation of motion is

\[
\left(\tilde{\Delta} + m^2\right) \tilde{\phi} = 0 .
\] (2.3)

We write its solution as an integral in terms of the would-be boundary value \(\phi_0\)

\[
\tilde{\phi}(\xi, x) = \int_M d^d y \sqrt{g(y)\phi_0(y)} \Gamma(\xi| x, y) .
\]

The Green function is to be expanded as

\[
\Gamma(\xi| x, y) = \sum_{\lambda} \gamma(\xi| \lambda) \psi_\lambda(x) \psi_\lambda(y) ,
\]

where we denoted by \(\{\psi_\lambda\}\) a complete orthonormal set of solutions of \(\Delta \psi_\lambda = \lambda \psi_\lambda\) on \(M\).

Using (2.2) we obtain from (2.3) the following equation for \(\gamma(\xi| \lambda)\)

\[
\left(\xi^2 \lambda - \xi^{d+1} \partial_\xi \xi^{1-d} \partial_\xi\right) \gamma + m^2 \gamma = 0
\]
which can be shown to be reducible to the modified Bessel equation of order \( \nu = \sqrt{(d/2)^2 + m^2} \).

The solution which is regular at \( \xi \sim \infty \) and nicely matches at \( \lambda \sim 0 \) is then

\[
\gamma(\xi|\lambda) = c' \left( \sqrt{\frac{\lambda}{2}} \right)^\nu \xi^{\frac{\nu}{2}} K_\nu \left( \xi \sqrt{\lambda} \right),
\]

where \( K_\nu \) is the relative modified Bessel function of order \( \nu \) and \( c' \) is a normalization.

We can now perform the calculation of the regularized classical action for the field configuration \( \tilde{x} \) on \( \tilde{M} \)

\[
\tilde{I}_r (\tilde{\phi}) = \frac{1}{2} \int_{\tilde{M}} d^{d+1} \tilde{x} \sqrt{\tilde{g}} \left( \tilde{g}^{AB} \partial_A \tilde{\phi} \partial_B \tilde{\phi} + m^2 \tilde{\phi}^2 \right)
= \frac{1}{2} \int_{\tilde{M}} d^{d+1} \tilde{x} \partial_A \left( \sqrt{\tilde{g}} \tilde{g}^{AB} \tilde{\phi} \partial_B \tilde{\phi} \right).
\]

This reduces, using Stoke’s theorem, to

\[
\frac{1}{2} \int_M d^d x \left( \tilde{\phi}_{|\xi=\epsilon} e^{1-d} \sqrt{g} \partial_\xi \tilde{\phi}_{|\xi=\epsilon} \right) = \frac{1}{2} \sum_\lambda \hat{\phi}_0(\lambda)^2 e^{1-d} \left[ \gamma \partial_\xi \gamma \right]_{|\xi=\epsilon},
\]

where \( \hat{\phi}_0(\lambda) = \int_M d^d x \sqrt{g} \psi_\lambda \phi_0 \) and where we used completeness and orthonormality of the armonics \( \{\psi_\lambda\} \).

Extracting the finite part of \( e^{1-d} \gamma \partial_\xi \gamma \) as \( \epsilon \to 0 \)

\[
\text{fin} \left[ e^{1-d} \gamma \partial_\xi \gamma \right] = c \left( \frac{d}{2} + \nu \right) \lambda^\nu,
\]

we get

\[
\tilde{I}_r \text{fin} \left( \tilde{\phi} \right) = c \left( \frac{d}{2} + \nu \right) \frac{1}{2} \sum_\lambda \hat{\phi}_0(\lambda)^2 \cdot \lambda^\nu
= c \left( \frac{d}{2} + \nu \right) \frac{1}{2} \int_M d^d x d^d y \sqrt{g}(x) \sqrt{g}(y) \phi_0(x) \phi_0(y) \Omega_\nu(x,y)
\]

where \( \Omega_\nu(x,y) = \sum_\lambda \psi_\lambda(x) \psi_\lambda(y) \cdot \lambda^\nu \). This is exactly the generating functional for 2-point functions of a CFT for a field of conformal dimension \( \delta = \frac{d}{2} + \nu \). In fact, under a scale transform \( g_{ab} \to r^2 g_{ab} \), we get \( \lambda \to r^{-2} \lambda \) and \( \psi_\lambda \to r^{-d/2} \psi_\lambda \) and then \( \Omega_\nu \to r^{-d-2\nu} \Omega_\nu \).

In the case \( M = \mathbb{R}^d \cup \{\infty\} \) everything above reduces to the usual AdS calculation and \( \Omega_\nu(x,y) \sim \frac{1}{|x-y|^{d+2\nu}} \).

### 2.2 The interacting massive scalar boson

In this subsection we analyze perturbatively the interacting scalar boson which we denote by \( \tilde{\Phi} \). The perturbative analysis will be performed in the classical limit for the cubic potential \( \frac{\hbar}{3!} \tilde{\Phi}^3 \) in the first order in \( \hbar \). Higher orders in \( \hbar \) can be analyzed in the same way. In the
following formulas we will omit sometimes the specific indication of higher order corrections \( o(h) \).

The e.o.m. of the field is

\[
(\bar{\Delta} + m^2) \Phi + \frac{h}{2} \Phi^2 = 0. \tag{2.4}
\]

We write its solution as an integral in terms of the would-be boundary value \( \Phi_0 = \phi_0 + h\varphi_0 \)

\[
\Phi(\xi, x) = \int_M d^d y \sqrt{g(y)} \sum_{\lambda} \psi_\lambda(x) \psi_\lambda(y) \gamma(\xi|\lambda) [\phi_0(y) + hg(\xi|\lambda)\varphi_0(y)] = \tilde{\phi}_0 + h\tilde{\varphi}_0
\]

where \( g \) is normalized as \( g(\xi|\lambda) = 1 + O(\xi) \) near the boundary. Eq. (2.4) splits in

\[
(\bar{\Delta} + m^2) \tilde{\phi}_0 = 0; \quad (\bar{\Delta} + m^2) \tilde{\varphi}_0 + \frac{1}{2} \tilde{\phi}_0^2 = 0.
\]

and we obtain for \( g(\xi|\lambda) \) the equation

\[
\partial_\xi \left( \xi^{-d+1} \gamma(\xi|\lambda)^2 \partial_\xi g(\xi|\lambda) \right) \tilde{\phi}_0(\lambda) = -\xi^{-(d+1)} \gamma(\xi|\lambda) \sum_{\lambda' \lambda''} \gamma(\xi|\lambda') \gamma(\xi|\lambda'') c_{\lambda\lambda'\lambda''} \tilde{\phi}_0(\lambda') \tilde{\phi}_0(\lambda'')
\]

where \( c_{\lambda\lambda'\lambda''} \equiv \int_M d^d x \sqrt{g(x)} \psi_\lambda(x) \psi_{\lambda'}(x) \psi_{\lambda''}(x) \).

The classical action is evaluated as follows

\[
\bar{I}^h_\epsilon \left( \Phi \right) = \frac{1}{2} \int_{M^d} d^{d+1} \tilde{x} \sqrt{\tilde{g}} \left( g^{AB} \partial_A \Phi \partial_B \Phi + m^2 \Phi^2 + \frac{h}{3} \Phi^3 \right) = \frac{1}{2} \int_{M^d} d^{d+1} \tilde{x} \left[ \partial_A \left( \sqrt{\tilde{g}} g^{AB} \partial_B \Phi \right) + \sqrt{\tilde{g}} \left( \bar{\Delta} + m^2 \right) \Phi + \frac{h}{3} \Phi^3 \right] = \bar{I}_\epsilon (\phi) + h \int_{M^d} d^{d+1} \tilde{x} \left[ \partial_A \left( \frac{1}{2} \sqrt{\tilde{g}} g^{AB} \partial_B (\tilde{\phi}) \right) - \frac{1}{12} \sqrt{\tilde{g}} \tilde{\phi}^3 \right]
\]

and, using the Stokes theorem and (2.3), we get its finite part

\[
\text{fin } [\bar{I}^h_\epsilon \left( \Phi \right)] = \frac{1}{2} \int_M d^d x \sqrt{g(x)} \int_M d^d y \sqrt{g(y)} \Phi_0(x) \Omega_\nu(x, y) \Phi_0(y) + \frac{h}{3!} \int_M d^d x \sqrt{g(x)} \int_M d^d y \sqrt{g(y)} \int_M d^d z \sqrt{g(z)} \Phi_0(x) \Phi_0(y) \Phi_0(z) \Omega_\nu(x, y, z), \tag{2.6}
\]

where

\[
\Omega_\nu(x, y) = c \left( \frac{d}{2} + \nu \right) \Omega_\nu(x, y)
\]

\[
\Omega_\nu(x, y, z) = b \sum_{\lambda\lambda'\lambda''} \psi_{\lambda}(x) \psi_{\lambda'}(y) \psi_{\lambda''}(z) c_{\lambda\lambda'\lambda''} (\lambda\lambda'\lambda'')^{\nu/2} H (\lambda, \lambda', \lambda''),
\]

with \( b = 5 \cdot 2^{-3\nu-1} c^3 \) and

\[
H (\lambda, \lambda', \lambda'') = \int_0^{+\infty} d\xi \xi^{d-1} K_\nu \left( \sqrt{\lambda} \xi \right) K_\nu \left( \sqrt{\lambda'} \xi \right) K_\nu \left( \sqrt{\lambda''} \xi \right).
\]
Using the same rescalings as for the free scalar boson case, under the conformal transformation $g_{ab} \rightarrow r^2 g_{ab}$ we get $\Omega_\nu(x, y, z) \rightarrow r^{-3(\frac{d}{2}+\nu)}\Omega_\nu(x, y, z)$ and therefore $\Omega_\nu(x, y, z)$ represents a good three point function for a conformal field of dimension $\delta = \frac{d}{2} + \nu$.

This shows that under the interaction the mass/conformal weight correspondence remains stable also in this larger framework in which the boundary geometry is generic.

Notice that the above results reproduce the interacting scalar boson amplitudes in the $M = \mathbb{R}^d \cup \{\infty\}$ case. In particular, the coefficient $H(\lambda, \lambda', \lambda'')$ reproduces the interaction vertex as calculated with the Witten-diagrams technique [9, 14].

2.3 The free massive spinor

In this subsection we analyze the free spinor field which we denote by $\tilde{\psi}$. We simplify a little possible questions about the harmonic spinor analysis for manifolds with boundary and torsion (see for example [15] for indication of problems).

The action is

$$\tilde{S}_\epsilon (\tilde{\psi}) = \int_{\tilde{M}_\epsilon} d^{d+1} \tilde{x} \sqrt{\tilde{g}(\tilde{x})} \tilde{\psi}(\tilde{x}) \left( \tilde{\mathcal{D}} - m \right) \tilde{\psi}(\tilde{x}) + \mu \int_{\tilde{M}_\epsilon} \mathcal{L}_{\tilde{\phi}} \left[ d^{d+1} \tilde{x} \sqrt{\tilde{g}(\tilde{x})} \tilde{\psi}(\tilde{x}) \tilde{\psi}(\tilde{x}) \right],$$

which is the usual Dirac action augmented by a boundary term which is the natural covariantization of the analogous term proposed in [8] with $\tilde{\phi}(\tilde{x}) = \xi \partial_\xi + v^a(\xi, x) \partial_{x_a}$ a vector field of this given form and $\mathcal{L}_{\tilde{\phi}} = i_{\tilde{\phi}} \cdot \tilde{d} + \tilde{d} \cdot i_{\tilde{\phi}}$ the Lie derivative. From this technical point of view, it seems unnecessary to impose any condition on the components $v^a$ since they do not appear in the final result. The constant $\mu$ is a weight for the boundary term.

The Dirac operator $\tilde{\mathcal{D}}$ can be written in the form

$$\tilde{\mathcal{D}} = \tilde{e}^A_M \Gamma_A \left( \partial^M + \frac{1}{2} \tilde{\omega}^M_{BC} \Sigma^{BC} \right) = = \xi \left[ e^a_m \Gamma_a \left( \partial^m + \frac{1}{2} \omega^m_{bc} \Sigma^{bc} \right) \right] + \left( \xi \partial_\xi - \frac{d}{2} \right) \Gamma_0 =$$

$$= \xi \mathcal{D} + \left( \xi \partial_\xi - \frac{d}{2} \right) \Gamma_0$$

where $\tilde{e}^A_M$ and $\tilde{\omega}^M_{BC}$ are the components of the inverse vielbein and the spin connection on $\tilde{M}$, respectively. Let $\{\chi_q(x)\}$ be the set of complete orthonormal spinors of the operator $\mathcal{D}$ on $M$, that is $\mathcal{D} \chi_q = i \eta \chi_{q-\eta}$ and $\Gamma_0 \chi_q = \eta \chi_q$.

The relevant equation of motion is the Dirac equation

$$\left( \tilde{\mathcal{D}} - m \right) \tilde{\psi}(\tilde{x}) = 0$$

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and we write its solution as

$$\tilde{\psi}(\tilde{x}) = \sum_{q,\eta} \beta_\eta(\xi|q)\chi_{q\eta}(x)\tilde{\psi}_\eta^0(q),$$  \hspace{1cm} (2.10)

where $\tilde{\psi}_\eta^0(q) = \int_M d^d y \sqrt{g(y)} \chi_{q\eta}(y)\psi_\eta^0(y)$ in terms of the would be boundary spinor $\psi_0(y) = \psi^0_{+1}(y) + \psi^0_{-1}(y)$.

Substituting (2.10) in (2.9) and imposing square integrability, we get

$$\beta_\eta(\xi|q) = c''(q)e^{i\pi\eta/4}\sin \left(\eta m - \frac{1}{2}\right) \xi^{d+1} K_{\frac{d}{2} - \eta m}(q\xi)$$  \hspace{1cm} (2.11)

for $q \neq 0$ and $\beta_\eta(\xi|0) \propto \xi^{\eta m + \frac{d}{2}}$. $\beta_\eta(\xi|q)$ is well defined at $q = 0$ only for the value of $\eta$ dictated by the condition $\eta m \leq 0$ that is, for $m \geq 0$, $\eta = -1$. As a consequence it is also determined $c''(q) = q^{\eta m + \frac{d}{2}} c''$. We are then forced to kill half the boundary value of the spinor from the beginning as

$$\psi^0_{+1}(x) = 0$$

Calculating the classical action from (2.7) we get

$$\text{fin} \left[ \tilde{S}_\epsilon \left( \tilde{\psi} \right) \right] = -\mu a \sum_q \tilde{\psi}_{q-1}^q \tilde{\psi}_{q-1}^q q^{2m} =$$

$$= -\int_M d^d x \sqrt{g(x)} \int_M d^d y \sqrt{g(y)} \tilde{\psi}_0^0(x) \Theta(x,y) \psi^0_{-1}(y),$$  \hspace{1cm} (2.12)

where $a = |c''|^2 \pi \cos^2(\pi m)$ and

$$\Theta(x,y) = \mu a \sum_q \chi_{q-1}(x) q^{2m} \chi_{q-1}(y).$$

Notice that (2.12) is the generating functional for the two point function of a spinorial field of conformal weight $\delta = m + \frac{d}{2}$. In fact, under $g_{ab} \rightarrow r^2 g_{ab}$, we have $q \rightarrow r^{-1} q$ and $\chi_{q\eta} \rightarrow r^{-\frac{d}{2}} \chi_{q\eta}$, so that $\Theta(x,y) \rightarrow r^{-2(\frac{d}{2} + m)} \Theta(x,y)$. Let us point out also that the free spinorial indices in the two point function are acted on by the local Spin$(d)$ group.

Again, in the case $M = \mathbb{R}^d \cup \{\infty\}$, our result reduces to the known ones \[.\]

\section{Conclusions}

As it has been shown above, there is a well defined way to generalize the AdS/CFT correspondence to the more general situation in which the AdS space is replaced by a filling in of a generic compact manifold. This is due to the fact that the procedure of reduction
to the boundary is arranged as a short scale phenomenon on the bulk (see also [10]). At short distances from the boundary and locally, the manifold $\tilde{M}$ can be approximated by a negative constant curvature space and therefore the reduction to the boundary theory holds as if one was on an AdS space. More concretely, one can consider the geometry of $\tilde{M}$ near the boundary. Let $(e^a, \omega^b_a)$ [resp. $(\tilde{e}_A^a, \tilde{\omega}^A_B)$] be the vielbein and spin connection on $M$ [resp. on $\tilde{M}$]: the following relations hold (we used them also to calculate (2.8))

$$\tilde{e}_0 = \frac{d\xi}{\xi}, \quad \tilde{e}_a = \frac{1}{\xi} e^a, \quad \tilde{\omega}_0^a = -\frac{1}{\xi} e^a, \quad \tilde{\omega}_b^a = \omega_a^b$$

and the structure equations on $\tilde{M}$ can be written as

$$\tilde{T}_0 = 0, \quad \tilde{R}_a^0 = -e_a \wedge \frac{d\xi}{\xi^2} - \frac{1}{\xi} T_a$$
$$\tilde{T}_a = \frac{1}{\xi} T_a, \quad \tilde{R}_a^b = -\frac{1}{\xi^2} e_a \wedge e^b + R_a^b,$$

where $T_a$ and $R_a^b$ [resp. $\tilde{T}_A$ and $\tilde{R}_A^B$] are the torsion and curvature 2-forms on $M$ [resp. on $\tilde{M}$]. In the limit $\xi \sim 0$ the first term in the curvature $\tilde{R}_A^B$ dominates and

$$\tilde{R}_A^B \sim -\tilde{e}_A \wedge \tilde{e}_B$$

which means that locally near the boundary the $\tilde{M}$ geometry is AdS (with some torsion if $T_a \neq 0$). This explains also the stability of the spectral correspondence with respect to changes in the geometry of the boundary manifold.

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2The precise meaning of “near the boundary” is $\xi << |R|^{-1/2}$, where $R$ is the scalar curvature on $M$. 

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