Quantum $f$-divergences and error correction

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Abstract

Quantum $f$-divergences are a quantum generalization of the classical notion of $f$-divergences, and are a special case of Petz’ quasi-entropies. Many well-known distinguishability measures of quantum states are given by, or derived from, $f$-divergences; special examples include the quantum relative entropy, the Rényi relative entropies, and the Chernoff and Hoeffding measures. Here we show that the quantum $f$-divergences are monotonic under substochastic maps whenever the defining function is operator convex. This extends and unifies all previously known monotonicity results for this class of distinguishability measures. We also analyze the case where the monotonicity inequality holds with equality, and extend Petz’ reversibility theorem for a large class of $f$-divergences and other distinguishability measures. We apply our findings to the problem of quantum error correction, and show that if a stochastic map preserves the pairwise distinguishability on a set of states, as measured by a suitable $f$-divergence, then its action can be reversed on that set by another stochastic map that can be constructed from the original one in a canonical way. We also provide an integral representation for operator convex functions on the positive half-line, which is the main ingredient in extending previously known results on the monotonicity inequality and the case of equality. We also consider some special cases where the convexity of $f$ is sufficient for the monotonicity, and obtain the inverse Hölder inequality for operators as an application. The presentation is completely self-contained and requires only standard knowledge of matrix analysis.

Keywords: relative entropy, quasi-entropy, $f$-divergences, Rényi relative entropies, Schwarz maps, stochastic maps, substochastic maps, operator convex functions, Chernoff distance, Hoeffding distances

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1 Introduction

In the stochastic modeling of systems, the probabilities of the different outcomes of possible measurements performed on the system are given by a state, which is a probability distribution in the case of classical systems and a density operator on the Hilbert space of the system in the quantum case. In applications, it is important to have a measure of how different two states are from each other and, as it turns out, such measures arise naturally in statistical problems like state discrimination. Probably the most important statistically motivated distance measure is the relative entropy, given as

\[ S(\rho \parallel \sigma) := \begin{cases} 
\text{Tr} \rho (\log \rho - \log \sigma), & \text{supp} \rho \leq \text{supp} \sigma, \\
+\infty, & \text{otherwise},
\end{cases} \]

for two density operators \( \rho, \sigma \) on a finite-dimensional Hilbert space. Its operational interpretation is given as the optimal exponential decay rate of an error probability in the state discrimination problem of Stein’s lemma \[7, 21, 38, 45\], and it is the mother quantity for many other relevant notions in information theory, like the entropy, the conditional entropy, the mutual information and the channel capacity \[7, 45\].

Undisputably the most relevant mathematical property of the relative entropy is its monotonicity under stochastic maps, i.e.,

\[ S(\Phi(\rho) \parallel \Phi(\sigma)) \leq S(\rho \parallel \sigma) \tag{1.1} \]

for any two states \( \rho, \sigma \) and quantum stochastic map \( \Phi \) \[45\]. Heuristically, \(1.1\) means that the distinguishability of two states cannot increase under further randomization. The monotonicity inequality yields immediately that if the action of \( \Phi \) can be reversed on the set \( \{\rho, \sigma\} \), i.e., there exists another stochastic map \( \Psi \) such that \( \Psi(\Phi(\rho)) = \rho \) and \( \Psi(\Phi(\sigma)) = \sigma \), then \( \Phi \) preserves the relative entropy of \( \rho \) and \( \sigma \), i.e., inequality \(1.1\) holds with equality. A highly non-trivial observation, made by Petz in \[43, 44\], is that the converse is also true: If \( \Phi \) preserves the relative entropy of \( \rho \) and \( \sigma \) then it is reversible on \( \{\rho, \sigma\} \) and, moreover, the reverse map can be given in terms of \( \Phi \) and \( \sigma \) in a canonical way. This fact has found applications in the theory of quantum error correction \[25, 26, 39\], the characterization of quantum Markov chains \[18\] and the description of states with zero quantum discord \[10, 14\], among many others.

Relative entropy has various generalizations, most notably Rényi’s \( \alpha \)-relative entropies \[47\] that share similar monotonicity and convexity properties with the relative entropy and are also related to error exponents in binary state discrimination problems \[9, 35\]. A general approach to quantum relative entropies was developed by Petz in 1985 \[41\], who introduced the concept of quasi-entropies (see also \[42\] and Chapter 7 in \[40\]). Let \( \mathcal{A} := B(\mathbb{C}^n) \) denote the algebra of linear operators on the finite-dimensional Hilbert space \( \mathbb{C}^n \) (which is essentially the algebra of \( n \times n \) matrices with complex entries, and hence we also use the term matrix algebra). For a positive \( A \in \mathcal{A} \) and a strictly positive \( B \in \mathcal{A} \), a general \( K \in \mathcal{A} \) and a real-valued continuous function \( f \) on \([0, +\infty)\), the quasi-entropy is defined as

\[ S^K_f (A \parallel B) := \langle KB^{1/2}, f(\Delta(A/B))(KB^{1/2}) \rangle_{\text{HS}} = \text{Tr} B^{1/2} K^* f(\Delta(A/B))(KB^{1/2}), \]

where \( \langle X, Y \rangle_{\text{HS}} := \text{Tr} X^* Y, X, Y \in \mathcal{A}, \) is the Hilbert-Schmidt inner product, and \( \Delta(A/B) : \mathcal{A} \to \mathcal{A} \) is the so-called relative modular operator acting on \( \mathcal{A} \) as \( \Delta(A/B) X := AXB^{-1}, X \in \mathcal{A} \).
A. The relative entropy can be obtained as a special case, corresponding to the function
\( f(x) := x \log x \) and \( K := I \), and Rényi’s \( \alpha \)-relative entropies are related to the quasi-entropies corresponding to \( f(x) := x^\alpha \).

The two most important properties of the quasi-entropy are its monotonicity and joint convexity. Let \( \Phi : A_1 \to A_2 \) be a linear map between two matrix algebras \( A_1 \) and \( A_2 \), and let \( \Phi^* : A_2 \to A_1 \) denote its dual with respect to the Hilbert-Schmidt inner products. A trace-preserving map \( \Phi : A_1 \to A_2 \) is called a stochastic map if \( \Phi^* \) satisfies the **Schwarz inequality** \( \Phi^*(Y^*)\Phi^*(Y) \leq \Phi^*(Y^*Y) \), \( Y \in A_2 \). The following monotonicity property of the quasi-entropies was shown in [41, 42]: Assume that \( f \) is an operator monotone decreasing function on \( [0, +\infty) \) with \( f(0) \leq 0 \) and \( \Phi : A_1 \to A_2 \) is a stochastic map. Then

\[
S^K_f(\Phi(A)||\Phi(B)) \leq S^K_f^{|(K)}(A||B)
\]

(1.2)

holds for any \( K \in A_2 \) and invertible positive operators \( A, B \in A_1 \). If \( f \) is an operator convex function on \( [0, +\infty) \), then \( S^K_f(A, B) \) is jointly convex in the variables \( A \) and \( B \) [40, 41, 42], i.e.,

\[
S^K_f\left(\sum_i p_i A_i \left\| \sum_i p_i B_i\right\|\right) \leq \sum_i p_i S^K_f(A_i||B_i)
\]

for any finite set of positive invertible operators \( A_i, B_i \in A \) and probability weights \( \{p_i\} \).

Quasi-entropy is a quantum generalization of the \( f \)-divergence of classical probability distributions, introduced independently by Csiszár [31] and Ali and Silvey [11], which is a widely used concept in classical information theory and statistics [31, 32]. This motivates the terminology “quantum \( f \)-divergence”, which we will use in this paper for the quasi-entropies with \( K = I \). Actually, our notion of \( f \)-divergence is also a slight generalization of the quasi-entropy in the sense that we extend it to cases where the second operator is not invertible. This extension is the same as in the classical setting, and was already considered in the quantum setting, e.g., in [51]. We give the precise definition of the quantum \( f \)-divergences in Section 2 where we also give some of their basic properties, and prove that they are continuous in their second variable; the latter seems to be a new result. In Section 8 we collect various technical statements on positive maps, which are necessary for the succeeding sections. In particular, we introduce a generalized notion of Schwarz maps, and investigate the properties of this class of positive maps.

The monotonicity \( S_f(\Phi(A)||\Phi(B)) \leq S_f(A||B) \) of the \( f \)-divergences was proved in [42] for the case where \( f \) is operator monotone decreasing and \( \Phi \) is a stochastic map, and where \( f \) is operator convex and \( \Phi \) is the restriction onto a subalgebra; in both cases \( B \) was assumed to be invertible. This was extended in [30] to the case where \( f \) is operator convex, \( \Phi \) is stochastic and both \( A \) and \( B \) are invertible, using an integral representation of operator convex functions on \( (0, +\infty) \), and in [51] to the case where \( f \) is operator convex and \( \Phi \) is a completely positive trace-preserving map, without assuming the invertibility of \( A \) or \( B \), using the monotonicity under restriction onto a subalgebra and Lindblad’s representation of completely positive maps. In Section 4 we give a common generalization of these results by proving the monotonicity relation for the case where \( f \) is operator convex, \( \Phi \) is a substochastic map which preserves the trace of \( B \), and both \( A \) and \( B \) are arbitrary positive semidefinite operators. This is based on the continuity result proved in Section 2 and an integral representation of operator convex functions on \( [0, +\infty) \) that we provide in Section 8. To the best of our knowledge, this representation is new, and might be interesting in itself.

It has been known [25, 26, 43] for the relative entropy and some Rényi relative entropies that the monotonicity inequality for two operators and a 2-positive trace-preserving map
holds with equality if and only if the action of the map can be reversed on the given operators. We extend this result to a large class of $f$-divergences in Section 5, where we show that if a stochastic map $\Phi$ preserves the $f$-divergence of two operators $A$ and $B$ corresponding to an operator convex function which is not a polynomial then it preserves a certain set of "primitive" $f$-divergences, corresponding to the functions $\varphi_t(x) := -x/(x + t)$ for a set $T$ of $t$'s. Moreover, if this set has large enough cardinality (depending on $A$, $B$ and $\Phi$) and $\Phi$ is 2-positive then there exists another stochastic map $\Psi$ reversing the action of $\Phi$ on $\{A, B\}$, i.e., such that $\Psi(\Phi(A)) = A$ and $\Psi(\Phi(B)) = B$. In Section 6 we formulate equivalent conditions for reversibility in terms of the preservation of measures relevant to state discrimination, namely the Chernoff distance and the Hoeffding distances, and we also show that these measures cannot be represented as $f$-divergences. In Section 7 we apply the above results on reversibility to the problem of quantum error correction, and give equivalent conditions for the reversibility of a quantum operation on a set of states in terms of the preservation of pairwise $f$-divergences, Chernoff and Hoeffding distances, and many-copy trace-norm distances. Related to the latter, we also analyze the connection with the recent results of [6], where reversibility was obtained from the preservation of single-copy trace-norm distances under some extra technical conditions, and show that the approach of [6] is unlikely to be recovered from our analysis of the preservation of $f$-divergences, as the quantum trace-norm distances cannot be represented as $f$-divergences. This is in contrast with the classical case, and is another manifestation of the significantly more complicated structure of quantum states and their distinguishability measures, as compared to their classical counterparts.

In our analysis of the monotonicity inequality $S_f(\Phi(A)\|\Phi(B)) \leq S_f(A\|B)$ and the case of the equality, it is essential that $f$ is operator convex; it is an open question though whether this is actually necessary. In Appendix A we consider some situations where convexity of $f$ is sufficient; this includes the case of commuting operators, which is essentially a reformulation of the classical case, and the monotonicity under the pinching operation defined by the reference operator $B$, which was first proved in [13] for the Rényi relative entropies. Although both of these cases are very special and their proofs are considerably simpler than the general case, they are important for applications. As an illustration, we derive from these results the exponential version of the operator Hölder inequality and the inverse Hölder inequality, and analyse the case when they hold with equality.

## 2 Quantum $f$-divergences: definition and basic properties

Let $\mathcal{A}$ be a finite-dimensional $C^*$-algebra. Unless otherwise stated, we will always assume that $\mathcal{A}$ is a $C^*$-subalgebra of $\mathcal{B}(\mathcal{H})$ for some finite-dimensional Hilbert space $\mathcal{H}$, i.e., $\mathcal{A}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$ that is closed under taking the adjoint of operators. For simplicity, we also assume that the unit of $\mathcal{A}$ coincides with identity operator $I$ on $\mathcal{H}$; if this is not the case, we can simply consider a smaller Hilbert space. The Hilbert-Schmidt inner product on $\mathcal{A}$ is defined as

$$\langle A, B \rangle_{\text{HS}} := \text{Tr} A^* B, \quad A, B \in \mathcal{A},$$

with induced norm $\|A\|_{\text{HS}} := \sqrt{\text{Tr} A^* A}$, $A \in \mathcal{A}$.

We will follow the convention that powers of a positive semidefinite operator are only taken on its support; in particular, if $0 \leq X \in \mathcal{A}$ then $X^{-1}$ denotes the generalized inverse of $X$ and $X^0$ is the projection onto the support of $X$. For a real $t \in \mathbb{R}$, $X^t$ is a unitary on supp $X$ but
not on the whole Hilbert space unless \(X^0 = I\). We denote by \(\log^*\) the extension of \(\log\) to the domain \([0, +\infty)\), defined to be 0 at 0. With these conventions, we have \(\frac{d}{dz}X^z\big|_{z=0} = \log^* X\). We also set

\[
0 \cdot \pm \infty := 0, \quad \log 0 := -\infty, \quad \text{and} \quad \log +\infty := +\infty.
\]

For a linear operator \(A \in \mathcal{A}\), let \(L_A, R_A \in \mathcal{B}(\mathcal{A})\) denote the left and the right multiplications by \(A\), respectively, defined as

\[
L_A : X \mapsto AX, \quad R_A : X \mapsto XA, \quad X \in \mathcal{A}.
\]

Left and right multiplications commute with each other, i.e., \(L_AR_B = R.BL_A\), \(A,B \in \mathcal{A}\). If \(A,B\) are positive elements in \(\mathcal{A}\) with spectral decompositions \(A = \sum_{a \in \text{spec}(A)} aP_a\) and \(B = \sum_{b \in \text{spec}(B)} bQ_b\) (where \(\text{spec}(X)\) denotes the spectrum of \(X \in \mathcal{A}\)) then the spectral decomposition of \(L_AR^{-1}\) is given by \(L_AR^{-1} = \sum_{a \in \text{spec}(A)} \sum_{b \in \text{spec}(B)} ab^{-1}LP_aRQ_b\), and for any function \(f\) on \(\{ab^{-1} : a \in \text{spec}(A), b \in \text{spec}(B)\}\), we have

\[
f(L_AR^{-1}) = \sum_{a \in \text{spec}(A)} \sum_{b \in \text{spec}(B)} f(ab^{-1})LP_aRQ_b.
\]

(Note that we have \(0^{-1} = 0\) in the above formulas due to our convention.)

**2.1 Definition.** Let \(A\) and \(B\) be positive semidefinite operators on \(\mathcal{H}\) and let \(f : [0, +\infty) \to \mathbb{R}\) be a real-valued function on \([0, +\infty)\) such that \(f\) is continuous on \((0, +\infty)\) and the limit

\[
\omega(f) := \lim_{x \to +\infty} \frac{f(x)}{x}
\]

exists in \([-\infty, +\infty]\). The \(f\)-divergence of \(A\) with respect to \(B\) is defined as

\[
S_f(A\|B) := \langle B^{1/2}, f(L_AR_{B^{-1}})B^{1/2}\rangle_{\text{HS}}
\]

when \(\text{supp}\, A \leq \text{supp}\, B\). In the general case, we define

\[
S_f(A\|B) := \lim_{\varepsilon \searrow 0} S_f(A\|B + \varepsilon I).
\]

**2.2 Proposition.** The limit in (2.2) exists, and

\[
\lim_{\varepsilon \searrow 0} S_f(A\|B + \varepsilon I) = \langle B^{1/2}, f(L_AR_{B^{-1}})B^{1/2}\rangle_{\text{HS}} + \omega(f) \text{Tr} (A - B^0).
\]

In particular, Definition [2.1] is consistent in the sense that if \(\text{supp}\, A \leq \text{supp}\, B\) then

\[
\lim_{\varepsilon \searrow 0} S_f(A\|B + \varepsilon I) = \langle B^{1/2}, f(L_AR_{B^{-1}})B^{1/2}\rangle_{\text{HS}}.
\]

**Proof.** By (2.1), we have \(S_f(A\|B + \varepsilon I) = \sum_{a \in \text{spec}(A)} \sum_{b \in \text{spec}(B)} (b + \varepsilon)f(a/(b + \varepsilon)) \text{Tr} P_{a}Q_{b}\), and the assertion follows by a straightforward computation using that for any \(a,b \geq 0\),

\[
\lim_{0 < b \to b} \tilde{b}f(a/\tilde{b}) = \begin{cases} b\omega(f), & b > 0, \\ a\omega(f), & b = 0. \end{cases}
\]

\(\square\)
2.3 Corollary. For $A, B$ and $f$ as in Definition 2.1

$$S_f(A\|B) = \langle B^{1/2}, f (L_A R_{B^{-1}}) B^{1/2}\rangle_{HS} + \omega(f) \Tr A(I - B^0)$$

$$= f(0) \Tr B + \langle B^{1/2}, (f - f(0)) (L_A R_{B^{-1}}) B^{1/2}\rangle_{HS} + \omega(f) \Tr A(I - B^0)$$

$$= \sum_{a \in \text{spec}(A)} \left( \sum_{b \in \text{spec}(B) \setminus \{0\}} b f(a/b) \Tr P_a Q_b + a \omega(f) \Tr P_a Q_0 \right).$$

and $S_f(A\|B) = \langle B^{1/2}, f (L_A R_{B^{-1}}) B^{1/2}\rangle_{HS}$ if and only if $\text{supp} A \leq \text{supp} B$ or $\lim_{x \to +\infty} \frac{f(x)}{x} = 0$.

2.4 Remark. Note that $L_A R_{B^{-1}} = \Delta(A/B)$, given in the Introduction, and hence the $f$-divergence is a special case of the quasi-entropy (with $K = I$) when $\text{supp} A \leq \text{supp} B$ or $\lim_{x \to +\infty} \frac{f(x)}{x} = 0$

2.5 Corollary. Let $A, A_1, A_2, B, B_1, B_2$ and $f$ be as in Definition 2.1. We have the following:

(i) For every $\lambda \in [0, +\infty)$,

$$S_f(\lambda A\|\lambda B) = \lambda S_f(A\|B).$$

(ii) If $A_1^0 \perp B_1^0 \perp A_2^0 \perp B_2^0$ then

$$S_f(A_1 + A_2\|B_1 + B_2) = S_f(A_1\|B_1) + S_f(A_2\|B_2).$$

(iii) If $V : \mathcal{H} \to \mathcal{K}$ is a linear or anti-linear isometry then

$$S_f(V A V^*\|V B V^*) = S_f(A\|B).$$

(iv) If $x$ is a unit vector in some Hilbert space $\mathcal{K}$ then

$$S_f(A \otimes |x\rangle\langle x|\|B \otimes |x\rangle\langle x|) = S_f(A\|B).$$

Proof. Immediate from (2.6). □

2.6 Remark. Note that if $V$ is an anti-linear isometry then there exists a linear isometry $\tilde{V}$ and a basis $\mathcal{B}$ such that $V A V^* = \tilde{V} A^T \tilde{V}^*$, $A \in \mathcal{A}_+$, where the transposition is in the basis $\mathcal{B}$. Hence, (iii) of Corollary 2.5 is equivalent to the $f$-divergences being invariant under conjugation by an isometry and transposition in an arbitrary basis.

2.7 Example. Let $f_\alpha(x) := x^\alpha$ for $\alpha > 0$, $x \geq 0$. For $\alpha = 0$, we define $f_0(x) := 1$, $x > 0$, $f_0(0) := 0$. A straightforward computation yields that

$$S_{f_\alpha}(A\|B) = \Tr A^\alpha B^{1-\alpha} + \left( \lim_{x \to +\infty} x^{\alpha-1} \right) \Tr A(I - B^0)$$

for any $A, B \in \mathcal{A}_+$, and hence, if $0 \leq \alpha < 1$ then

$$S_{f_\alpha}(A\|B) = \Tr A^\alpha B^{1-\alpha},$$

whereas for $\alpha > 1$ we have

$$S_{f_\alpha}(A\|B) = \begin{cases} \Tr A^\alpha B^{1-\alpha}, & \text{supp } A \leq \text{supp } B, \\ +\infty, & \text{otherwise.} \end{cases}$$
The Rényi relative entropy of $A$ and $B$ with parameter $\alpha \in [0, +\infty) \setminus \{1\}$ is defined as

$$S_\alpha(A\parallel B) := \frac{1}{\alpha - 1} \log S_{f_\alpha}(A\parallel B) = \begin{cases} \frac{1}{\alpha - 1} \log \text{Tr} A^\alpha B^{1-\alpha}, & \text{supp } A \leq \text{supp } B \text{ or } \alpha < 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

The choice $f(x) := x \log x$ yields the relative entropy of $A$ and $B$,

$$S_f(A\parallel B) = \begin{cases} \text{Tr } A (\log^* A - \log^* B), & \text{supp } A \leq \text{supp } B, \\ +\infty, & \text{otherwise,} \end{cases}$$

where the second case follows from $\lim_{x \to +\infty} \frac{x \log x}{x} = +\infty$.

The following shows that the representing function for an $f$-divergence is unique:

**2.8 Proposition.** Assume that a function $D : A_+ \times A_+ \rightarrow \mathbb{R}$ can be represented as an $f$-divergence. Then the representing function $f$ is uniquely determined by the restriction of $D$ onto the trivial subalgebra as

$$f(x) = D(xI\parallel I)/\dim \mathcal{H}, \quad x \in [0, +\infty). \quad (2.8)$$

In particular, for every $D : A_+ \times A_+ \rightarrow \mathbb{R}$ there is at most one function $f$ such that $D = S_f$ holds.

**Proof.** Formula (2.8) is obvious from (2.6), and the rest follows immediately. \[\square\]

In most of the applications, $f$-divergences are used to compare probability distributions in the classical, and density operators in the quantum case, and one might wonder whether there is more freedom in representing a measure as an $f$-divergence if we are only interested in density operators instead of general positive semidefinite operators. The following simple argument shows that if a measure can be represented as an $f$-divergence on quantum states then its values are uniquely determined by its values on classical probability distributions.

Given density operators $\rho$ and $\sigma$ with spectral decomposition $\rho = \sum_{a \in \text{spec } (\rho)} a P_a$ and $\sigma = \sum_{b \in \text{spec } (\sigma)} b Q_b$, we can define classical probability density functions $(\rho : \sigma)_1$ and $(\rho : \sigma)_2$ on $\text{spec } (\rho) \times \text{spec } (\sigma)$ as

$$(\rho : \sigma)_1 (a, b) := a \text{Tr } P_a Q_b, \quad (\rho : \sigma)_2 (a, b) := b \text{Tr } P_a Q_b.$$ 

This kind of mapping from pairs of quantum states to pairs of classical states was introduced in [37], and is one of the main ingredients in the proofs of the quantum Chernoff and Hoeffding bound theorems.

**2.9 Lemma.** For any two density operators $\rho, \sigma$ and any function $f$ as in Definition 2.1,

$$S_f(\rho\parallel \sigma) = S_f((\rho : \sigma)_1 \parallel (\rho : \sigma)_2).$$

**Proof.** It is immediate from (2.6). \[\square\]

**2.10 Corollary.** Let $f$ and $g$ be functions as in Definition 2.1. If $S_f$ and $S_g$ coincide on classical probability distributions then they coincide on quantum states as well.

**Proof.** Obvious from Lemma 2.9 \[\square\]
2.11 Example. For two density operators \( \rho, \sigma \), their quantum fidelity is given by \( F(\rho, \sigma) := \text{Tr} \sqrt{\rho^{1/2}\sigma\rho^{1/2}} \). For classical probability distributions, the fidelity coincides with \( S_{f_{1/2}} \), where \( f_{1/2}(x) = x^{1/2} \). If the fidelity could be represented as an \( f \)-divergence for quantum states then the representing function should be \( f_{1/2} \), due to Corollary 2.10. However, the corresponding quantum \( f \)-divergence is \( S_{f_{1/2}}(\rho||\sigma) = \text{Tr} \rho^{1/2}\sigma^{1/2} \), which is not equal to \( F(\rho, \sigma) \) in general. This shows that the fidelity of quantum states cannot be represented as an \( f \)-divergence.

In Sections 6 and 7 we give similar non-representability results for measures related to state discrimination on the state spaces of individual algebras.

Our last proposition in this section says that when \( \omega(f) \) is finite, the \( f \)-divergence is continuous in the second variable.

2.12 Proposition. Assume that \( \omega(f) \) is finite. Let \( A, B, B_k \in A \) with \( A, B, B_k \geq 0 \) for all \( k \in \mathbb{N} \), and assume that \( \lim_{k \to \infty} B_k = B \). Then

\[
\lim_{k \to \infty} S_f(A\|B_k) = S_f(A\|B).
\]

Proof. First, by the assumption on \( \omega(f) \) and Corollary 2.3, note that \( S(A\|B_k) \) is finite for any \( k \). Then by the definition (2.22), we can choose a sequence \( \varepsilon_k > 0 \), \( k \in \mathbb{N} \), such that \( \lim_{k \to \infty} \varepsilon_k = 0 \), and for all \( k \in \mathbb{N} \),

\[
S_f(A\|B_k + \varepsilon_k I) - \frac{1}{k} < S_f(A\|B_k) < S_f(A\|B_k + \varepsilon_k I) + \frac{1}{k}.
\]

Let \( \tilde{B}_k := B_k + \varepsilon_k I \), which is strictly positive for any \( k \in \mathbb{N} \). Obviously, \( \lim_{k \to \infty} \tilde{B}_k = B \), and the assertion will follow if we can show that

\[
\lim_{k \to \infty} S_f(A\|\tilde{B}_k) = S_f(A\|B).
\]  

Let \( A = \sum_{a \in \text{spec}(A)} aP_a \), \( B = \sum_{b \in \text{spec}(B)} bQ_b \) and \( \tilde{B}_k = \sum_{c \in \text{spec}(\tilde{B}_k)} cQ_c^{(k)} \) be the spectral decompositions of the respective operators. Then

\[
S_f(A\|\tilde{B}_k) = \sum_{a \in \text{spec}(A)} \sum_{c \in \text{spec}(\tilde{B}_k)} f(a/c)c\text{Tr} P_aQ_c^{(k)}.
\]

From the continuity of the eigenvalues and the spectral projections when \( \tilde{B}_k \to B \), we see that, for every \( \delta > 0 \) with \( \delta < \frac{1}{2}\min\{|b - b'| : b, b' \in \text{spec}(B), b \neq b'\} \), if \( k \) is sufficiently large, then we have

\[
\text{spec}(\tilde{B}_k) \subset \bigcup_{b \in \text{spec}(B)} (b - \delta, b + \delta) \quad (\text{disjoint union})
\]

and moreover,

\[
Q_b^{(k)} := \sum_{c \in \text{spec}(\tilde{B}_k) \cap (b - \delta, b + \delta)} Q_c^{(k)} \longrightarrow Q_b \quad \text{as } k \to +\infty, \text{ for all } b \in \text{spec}(B).
\]

Due to (2.3), for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) as above such that, for \( a \in \text{spec}(A) \), \( b \in \text{spec}(B) \) and \( c \in \text{spec}(\tilde{B}_k) \),

\[
|cf(a/c) - bf(a/b)| \leq \varepsilon \quad \text{if } b > 0 \text{ and } c \in (b - \delta, b + \delta),
\]

\[
|cf(a/c) - a\omega(f)| \leq \varepsilon \quad \text{if } c \in (0, \delta).
\]
Hence, if \( k \) is sufficiently large, then we have by (2.6)

\[
|S_f(A\|B_k) - S_f(A\|B)|
\]

\[
\leq \sum_{a \in \text{spec}(A)} \sum_{b \in \text{spec}(B) \setminus \{0\}} \left| \sum_{c \in \text{spec}(B_k) \setminus \{0\}} c f(a/c) \text{Tr} P_a Q_c^{(k)} - b f(a/b) \text{Tr} P_a Q_b \right|
\]

\[
+ \sum_{a \in \text{spec}(A)} \left| \sum_{c \in \text{spec}(B_k) \setminus \{0\}} c f(a/c) \text{Tr} P_a Q_c^{(k)} - a \omega(f) \text{Tr} P_a Q_0 \right|
\]

\[
\leq \sum_{a \in \text{spec}(A)} \sum_{b \in \text{spec}(B) \setminus \{0\}} \left\{ \sum_{c \in \text{spec}(B_k) \setminus \{0\}} |c f(a/c) - b f(a/b)| \text{Tr} P_a Q_c^{(k)} + |b f(a/b) \text{Tr} P_a (Q_b^{(k)} - Q_b)| \right\}
\]

\[
+ \sum_{a \in \text{spec}(A)} \left\{ \sum_{c \in \text{spec}(B_k) \setminus \{0\}} |c f(a/c) - a \omega(f)| \text{Tr} P_a Q_c^{(k)} + |a \omega(f) \text{Tr} P_a (Q_0^{(k)} - Q_0)| \right\}
\]

\[
\leq \varepsilon \text{Tr} I + \sum_{a \in \text{spec}(A)} \sum_{b \in \text{spec}(B) \setminus \{0\}} |b f(a/b)| \left\| Q_b^{(k)} - Q_0 \right\|_1 + \sum_{a \in \text{spec}(A)} |a \omega(f)| \left\| Q_0^{(k)} - Q_0 \right\|_1.
\]

This implies that

\[
\limsup_{k \to \infty} |S_f(A\|B_k) - S_f(A\|B)| \leq \varepsilon \text{Tr} I
\]

for every \( \varepsilon > 0 \), and so (2.9) follows. \( \square \)

2.13 Remark. The finiteness assumption on \( \omega(f) \) is essential in the above proposition. Indeed, take \( f \) such that \( \omega(f) = +\infty \) or \( -\infty \). Let \( A = B = |x\rangle \langle x| \) be a rank 1 projection, and \( B_k = |x_k\rangle \langle x_k| \) where \( \|x_k - x\| \to 0 \) and \( x_k \) is not proportional to \( x \) for any \( k \). Then \( S_f(A\|B) = f(1) \) while \( S_f(A\|B_k) = +\infty \) or \( -\infty \), respectively. Note also that \( S_f(A\|B) \) is not continuous in the first variable even when \( \omega(f) \) is finite, unless \( f \) is assumed to be continuous at 0.

3 Preliminaries on positive maps

Let \( A_i \subset \mathcal{B}(\mathcal{H}_i) \) be finite-dimensional \( C^* \)-algebras with unit \( I_i \) for \( i = 1, 2 \). For a subset \( B \subset A_i \), we will denote the set of positive elements in \( B \) by \( B^+ \); in particular, \( A_i^+ \) denotes the set of positive elements in \( A_i \). For a linear map \( \Phi : A_1 \to A_2 \), we denote its adjoint with respect to the Hilbert-Schmidt inner products by \( \Phi^* \). Note that \( \Phi \) and \( \Phi^* \) uniquely determine each other and, moreover, \( \Phi \) is positive/n-positive/completely positive if and only if \( \Phi^* \) is positive/n-positive/completely positive, and \( \Phi \) is trace-preserving/trace non-increasing if and only if \( \Phi^* \) is unital/sub-unital.

For given \( B \in A_{1,+} \) and \( \Phi : A_1 \to A_2 \), we define \( \Phi_B : A_1 \to A_2 \) and \( \Phi_B^* : A_2 \to A_1 \) as

\[
\Phi_B(X) := \Phi(B)^{-1/2} \Phi(B^{1/2} X B^{1/2}) \Phi(B)^{-1/2}, \quad X \in A_1,
\]

\[
\Phi_B^*(Y) := B^{1/2} \Phi^* \left( \Phi(B)^{-1/2} Y \Phi(B)^{-1/2} \right) B^{1/2}, \quad Y \in A_2.
\]
With these notations, we have $(\Phi_B)^* = \Phi_B^*$ and $(\Phi_B^*)^* = \Phi_B$.

For a normal operator $X \in \mathcal{A}_1$, let $P_{(1)}(X)$ denote the spectral projection of $X$ onto its fixed-point set. Note that if $B \in \mathcal{A}_{1,+}$ then $B^0$ is a projection in $\mathcal{A}_1$ and hence $B^0 \mathcal{A}_1 B^0$ is a $C^*$-algebra with unit $B^0$.

**3.1 Lemma.** If $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ is a positive map and $A, B$ are positive elements in $\mathcal{A}_1$ such that $A^0 = B^0$ then $\Phi(A)^0 = \Phi(B)^0$. In particular, $\Phi(B)^0 = (\Phi(B^0))^0$ for any positive $B \in \mathcal{A}_1$.

**Proof.** The assumption $A^0 = B^0$ is equivalent to the existence of strictly positive numbers $\alpha, \beta$ such that $\alpha A \leq B \leq \beta A$, which yields $\alpha \Phi(A) \leq \Phi(B) \leq \beta \Phi(A)$ and hence $\Phi(A)^0 = \Phi(B)^0$.

**3.2 Lemma.** Let $B \in \mathcal{A}_{1,+}$ and let $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a positive map such that $\Phi^*(\Phi(B)^0) \leq I_1$ (in particular, this is the case if $\Phi$ is trace non-increasing). Then

$$\text{Tr} \Phi(B) \leq \text{Tr} B,$$

and the following are equivalent:

(i) $\text{Tr} \Phi(B) = \text{Tr} B$.

(ii) For any function $f$ on $\text{spec}(B)$ such that $f(0) = 0$ if $0 \in \text{spec}(B)$, we have

$$f(B)\Phi^*(\Phi(B)^0) = \Phi^*(\Phi(B)^0)f(B) = f(B).$$

(iii) $B^0 \leq P_{(1)}(\Phi^*(\Phi(B)^0))$.

(iv) $\Phi$ is trace-preserving on $B^0 \mathcal{A}_1 B^0$. (In particular, if $A \in \mathcal{A}_{1,+}$ is such that $A^0 \leq B^0$ then $\text{Tr} \Phi(A) = \text{Tr} A$.)

(v) For the map $\Phi_B^*$ given in (3.2), we have

$$\Phi_B^*(\Phi(B)) = B.$$

**Proof.** By assumption, $\Phi^*(\Phi(B)^0) \leq I_1$ and hence,

$$0 \leq \text{Tr}(I_1 - \Phi^*(\Phi(B)^0))B = \text{Tr} B - \text{Tr} \Phi^*(\Phi(B)^0)B = \text{Tr} B - \text{Tr} \Phi(B)^0\Phi(B) = \text{Tr} B - \text{Tr} \Phi(B).$$

If $\text{Tr} \Phi(B) = \text{Tr} B$ then $(I_1 - \Phi^*(\Phi(B)^0))B = 0$, i.e., $B = \Phi^*(\Phi(B)^0)B$, so we get $B^n = \Phi^*(\Phi(B)^0)B^n$, $n \in \mathbb{N}$, which yields (iii). Hence, the implication (ii) $\Rightarrow$ (iii) holds. If (iii) holds then we have $B^0 = \Phi^*(\Phi(B)^0)B^0$ and hence, for any $x \in \mathcal{H}$ such that $B^0x = x$, we have $x = B^0x = \Phi^*(\Phi(B)^0)B^0x = \Phi^*(\Phi(B)^0)x$, or equivalently, $x \in \text{ran} P_{(1)}(\Phi^*(\Phi(B)^0))$. This yields (iii), and the converse direction (iii) $\Rightarrow$ (ii) is obvious. Assume now that (ii) holds. If $X \in B^0 \mathcal{A}_1 B^0$, then $XB^0 = B^0X = X$, and

$$\text{Tr} \Phi(X) = \text{Tr} \Phi(X)\Phi(B)^0 = \text{Tr} X\Phi^*(\Phi(B)^0) = \text{Tr} XB^0\Phi^*(\Phi(B)^0) = \text{Tr} XB^0 = \text{Tr} X,$$

showing (iv). The implication (iv) $\Rightarrow$ (i) is obvious.

Assume that (ii) holds. Then $\Phi_B^*(\Phi(B)) = B^{1/2}\Phi^*(\Phi(B)^0)B^{1/2} = B$, showing (v). On the other hand, if (v) holds then $B^{1/2}\Phi^*(\Phi(B)^0)B^{1/2} = B$, and hence $0 = B^{1/2}(I_1 - \Phi^*(\Phi(B)^0))B^{1/2}$. Since $I_1 - \Phi^*(\Phi(B)^0) \geq 0$, we obtain $B^{1/2}(I_1 - \Phi^*(\Phi(B)^0))^{1/2} = 0$, which in turn yields $B = B\Phi^*(\Phi(B)^0)$. From this (ii) follows as above.

\[\square\]
3.3 Corollary. Let $A, B \in \mathcal{A}_{1,+}$, and let $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a trace non-increasing positive map. Then $\Phi$ is trace-preserving on $(A + B)\mathcal{A}_1(A + B)^0$ if and only if
\[
\text{Tr } \Phi(A) = \text{Tr } A \quad \text{and} \quad \text{Tr } \Phi(B) = \text{Tr } B.
\]

Proof. Obvious from Lemma 3.2 \hfill \Box

3.4 Corollary. Let $A, B \in \mathcal{A}_{1,+}$ and let $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a trace non-increasing positive map such that $\text{Tr } \Phi(A) = \text{Tr } A$. Then
\[
\text{Tr } \Phi(B)\Phi(A)^0 \geq \text{Tr } BA^0 \quad \text{and} \quad \text{Tr } \Phi(B)(I_2 - \Phi(A)^0) \leq \text{Tr } B(I_1 - A^0).
\]

Note that the first inequality means the monotonicity of the Rényi 0-relative entropy $S_0(A|B) \geq S_0(\Phi(A)||\Phi(B))$ under the given conditions.

Proof. Due to Lemma 3.2, the assumptions yield that $A^0 \leq P_{[1]}(\Phi^*(\Phi(A)^0)) \leq \Phi^*(\Phi(A)^0)$, and hence $0 \leq \text{Tr } B(\Phi^*(\Phi(A)^0) - A^0) = \text{Tr } \Phi(B)\Phi(A)^0 - \text{Tr } BA^0$. The second inequality follows by taking into account that $\text{Tr } \Phi(B) \leq \text{Tr } B$. \hfill \Box

The following lemma yields the monotonicity of the Rényi 2-relative entropies, and is needed to prove the monotonicity of general $f$-divergences. The statement and its proof can be obtained by following the proofs of Theorem 1.3.3, Theorem 2.3.2 (Kadison’s inequality) and Proposition 2.7.3 in [5] using the weaker conditions given here. For readers’ convenience, we include a self-contained proof here.

3.5 Lemma. Let $A, B \in \mathcal{A}_{1,+}$ and $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a positive map. Then
\[
\Phi(B^0AB^0)\Phi(B)^{-1}\Phi(B^0AB^0) \leq \Phi(B^0AB^{-1}AB^0).
\]

In particular, if $A^0 \leq B^0$ then
\[
\Phi(A)\Phi(B)^{-1}\Phi(A) \leq \Phi(AB^{-1}A).
\]

If, moreover, $\Phi$ is also trace non-increasing then
\[
S_{f_2}(\Phi(A)||\Phi(B)) = \text{Tr } \Phi(A)^2\Phi(B)^{-1} \leq \text{Tr } A^2B^{-1} = S_{f_2}(A||B).
\]

Proof. Define $\Psi : \mathcal{A}_1 \to \mathcal{A}_2$ as $\Psi(X) := \Phi(B^{1/2}X\Phi(B)^{1/2})$, $X \in \mathcal{A}_1$. Let $X := B^{-1/2}AB^{-1/2}$ and let $X = \sum_{x \in \sigma(X)} xP_x$ be its spectral decomposition. Then
\[
\hat{X} := \begin{bmatrix} \Psi(X^2) & \Psi(X) \\ \Psi(X) & \Psi(I_1) \end{bmatrix} = \sum_{x \in \sigma(X)} \begin{bmatrix} x^2 & x \\ x & 1 \end{bmatrix} \otimes (P_x) \geq 0,
\]
and hence we have
\[
0 \leq \hat{Y}X\hat{Y}^* = \begin{bmatrix} \Psi(X^2) - \Psi(X)\Psi(I_1)^{-1}\Psi(X) & \Psi(X)(I_2 - \Psi(I))^0 \\ (I_2 - \Psi(I)^0)\Psi(X) & \Psi(I_1) \end{bmatrix},
\]
where
\[
\hat{Y} := \begin{bmatrix} I_2 & -\Psi(X)\Psi(I_1)^{-1} \\ 0 & I_2 \end{bmatrix}.
\]
Hence $\Psi(X^2) \geq \Psi(X)\Psi(I_1)^{-1}\Psi(X)$, which is exactly (3.3). The inequalities in (3.4) and (3.5) follow immediately. \hfill \Box
We say that a map \( \Phi : A_1 \to A_2 \) is a Schwarz map if
\[
\|\Phi\|_S := \inf\{c \in [0, +\infty) : \Phi(X)^*\Phi(X) \leq c\Phi(X^*X), \; X \in A\} < +\infty.
\]

Obviously, if \( \Phi \) is a Schwarz map then \( \Phi \) is positive, and we have \( \|\Phi\| = \|\Phi(I_1)\| \leq \|\Phi\|_S \).
(Note that \( \|\Phi\| = \|\Phi(I_1)\| \) is true for any positive map \( \Phi \) [52 Corollary 2.3.8]). We say that \( \Phi \) is a Schwarz contraction if it is a Schwarz map with \( \|\Phi\|_S \leq 1 \). A Schwarz contraction \( \Phi \) is also a contraction, due to \( \|\Phi\| \leq \|\Phi\|_S \). Note that a positive map \( \Phi \) is a contraction if and only if it is subunital, which is equivalent to \( \Phi^* \) being trace non-increasing. We say that a map \( \Phi \) between two finite-dimensional \( C^* \)-algebras is a substochastic map if its Hilbert-Schmidt adjoint \( \Phi^* \) is a Schwarz contraction, and \( \Phi \) is stochastic if it is a trace-preserving substochastic map. Note that in the commutative finite-dimensional case substochastic/stochastic maps are exactly the ones that can be represented by substochastic/stochastic matrices.

It is known that if \( \Phi \) is 2-positive then it is a Schwarz map with \( \|\Phi\|_S = \|\Phi\| \). In general, however, we might have \( \|\Phi\| < \|\Phi\|_S < +\infty \), as the following example shows. In particular, not every Schwarz map is 2-positive.

**3.6 Example.** Let \( H \) be a finite-dimensional Hilbert space, and for every \( \varepsilon \in \mathbb{R} \), let \( \Phi_\varepsilon : B(H) \to B(H) \) be the map
\[
\Phi_\varepsilon(X) := (1 - \varepsilon)X^T + \varepsilon(\text{Tr} \; X)I/d, \quad X \in B(H),
\]
where \( d := \dim H > 1 \) and \( X^T \) denotes the transpose of \( X \) in some fixed basis \( \{e_1, \ldots, e_d\} \) of \( H \). It was shown in [52] that \( \Phi_\varepsilon \) is positive if and only if \( 0 \leq \varepsilon \leq 1 + 1/(d - 1) \), for \( k \geq 2 \) it is \( k \)-positive if and only if \( 1 - 1/(d + 1) \leq \varepsilon \leq 1 + 1/(d - 1) \), and it is a Schwarz contraction if and only if \( 1 - 1/\left(1 + \sqrt{d + 1/4}\right) \leq \varepsilon \leq 1 + 1/(d - 1) \). This already shows that there are parameter values \( \varepsilon \) for which \( \Phi_\varepsilon \) is a Schwarz contraction but not 2-positive. Moreover, if \( \varepsilon \in [0, 1) \) then for every \( c \in [0, +\infty) \) we have
\[
c\Phi_\varepsilon(X^*X) - \Phi_\varepsilon(X^*)\Phi_\varepsilon(X)
= c(1 - \varepsilon)(X^*X)^T + c\varepsilon(\text{Tr} \; X^*X)I/d - (1 - \varepsilon)^2(X^*)^TX^T
- \varepsilon(1 - \varepsilon)(\text{Tr} \; X)(X^*)^T/d - \varepsilon(1 - \varepsilon)(\text{Tr} \; X^*X)^T/d - \varepsilon^2\text{Tr} \; X^2I/d^2
\geq (\text{Tr} \; X^*X)I/d \left[ (c - d(1 - \varepsilon)^2 - 2\varepsilon(1 - \varepsilon)\sqrt{d} - \varepsilon^2) \right],
\]
where we used that \( |\text{Tr} \; X|^2 \leq (\text{Tr} \; I)(\text{Tr} \; X^*X) \) and \( X^*X \leq \|X\|^2I \leq (\text{Tr} \; X^*X)I \). This shows that \( \Phi_\varepsilon \) is a Schwarz map for every \( \varepsilon \in (0, 1) \) and \( \|\Phi_\varepsilon\|_S \leq (1/\varepsilon)(d(1 - \varepsilon)^2 + 2\varepsilon(1 - \varepsilon)\sqrt{d} + \varepsilon^2) \).

Note that for \( X := |e_1\rangle\langle e_2| \) we have
\[
0 \leq \langle e_1| (\|\Phi_\varepsilon\|_S\Phi_\varepsilon(X^*X) - \Phi_\varepsilon(X^*)\Phi_\varepsilon(X)) e_1 \rangle = \|\Phi_\varepsilon\|_S \varepsilon/d - (1 - \varepsilon)^2,
\]
which yields that \( \|\Phi_\varepsilon\|_S \geq d(1 - \varepsilon)^2/\varepsilon \). In particular, \( \lim_{\varepsilon \searrow 0} \|\Phi_\varepsilon\|_S = +\infty \). Since \( \Phi_\varepsilon \) is a positive unital map for every \( \varepsilon \in [0, 1 + 1/(d - 1)] \), we have \( \|\Phi_\varepsilon\| = 1 \) for every \( \varepsilon \in [0, 1 + 1/(d - 1)] \), while \( \|\Phi_\varepsilon\|_S > 1 \) and hence \( \|\Phi\| < \|\Phi_\varepsilon\|_S \) whenever \( (1 - \varepsilon)^2/\varepsilon > d \).

Similarly, it was shown in [52] that the map
\[
\Psi_\varepsilon(X) := (1 - \varepsilon)X + \varepsilon(\text{Tr} \; X)I/d, \quad X \in B(H),
\]
is completely positive if and only if \( 0 \leq \varepsilon \leq 1 + 1/(d^2 - 1) \), for \( 1 \leq k \leq d - 1 \) it is \( k \)-positive if and only if \( 0 \leq \varepsilon \leq 1 + 1/(dk - 1) \), and it is a Schwarz contraction if and only if
0 ≤ ε ≤ 1 + 1/d. A similar computation as above shows that Ψε is a Schwarz map if and only if 0 ≤ ε < 1 + 1/(d − 1), and \( \lim_{d \to 1/(d-1)} \| \Psi_\epsilon \|_S = +\infty. \)

Finally, the map
\[
\Lambda_\epsilon(X) := (1 - \epsilon)X^T + \epsilon X, \quad X \in \mathcal{B}(\mathcal{H}),
\]
positive if and only if 0 ≤ ε ≤ 1, for each k ≥ 2 it is k-positive if and only if ε = 1, and it is a Schwarz contraction if and only if ε = 1 [52]. Moreover, for \( X := |e_1\rangle \langle e_2| \) and every c ∈ \( \mathbb{R} \) we have \( \langle e_1, (c \Lambda_\epsilon(X^*X) - \Lambda_\epsilon(X^*)\Lambda_\epsilon(X))e_1 \rangle = -(1 - \epsilon)^2 \), and hence \( \Lambda_\epsilon \) is a Schwarz map if and only if ε = 1.

3.7 Lemma. Let \( \Phi : \mathcal{A}_1 \to \mathcal{A}_2 \) be a substochastic map, and assume that there exists a \( B \in \mathcal{A}_{1,+} \setminus \{0\} \) such that \( \operatorname{Tr}(\Phi(B)) = \operatorname{Tr}B \). Then \( \|\Phi^*\|_S = \|\Phi\| = 1 \).

Proof. Let \( \tilde{\mathcal{A}}_1 := B^0\mathcal{A}_1B^0, \tilde{\mathcal{A}}_2 := \Phi(B)^0\mathcal{A}_2\Phi(B)^0 \), and define \( \tilde{\Phi} : \tilde{\mathcal{A}}_1 \to \tilde{\mathcal{A}}_2 \) as \( \tilde{\Phi}(X) := \Phi(B^0XB^0) = \Phi(X), X \in \tilde{\mathcal{A}}_1 \). Then \( \Phi^*(Y) = B^0\Phi^*(Y)B^0, Y \in \tilde{\mathcal{A}}_2 \), and Lemma 3.2 yields that \( \tilde{\Phi}^*(\Phi(B)^0) = B^0 \), i.e., \( \tilde{\Phi}^* \) is unital. Hence, 1 = \( \|\Phi^*\| \leq \|\Phi\| \leq \|\Phi^*\|_S \leq 1 \), from which the assertion follows.

3.8 Lemma. The set of Schwarz maps is closed under composition, taking the adjoint, and positive linear combinations. Moreover, for \( \alpha \geq 0 \) and \( \Phi, \Phi_1, \Phi_2 : \mathcal{A}_1 \to \mathcal{A}_2 \),

\[
\|\alpha\Phi\|_S = \alpha\|\Phi\|_S, \quad \|\Phi_1 + \Phi_2\|_S \leq \|\Phi_1\|_S + \|\Phi_2\|_S. \tag{3.6}
\]

Proof. The assertion about the composition is obvious. To prove closedness under the adjoint, assume that \( \Phi : \mathcal{A}_1 \to \mathcal{A}_2 \) is a Schwarz map. Our goal is to prove that \( \Phi^* \) is a Schwarz map, too. Let \( \iota_k \) be the trivial embedding of \( \mathcal{A}_k \) into \( \mathcal{B}(\mathcal{H}_k) \) for \( k = 1, 2 \). The adjoint \( \pi_k := \iota_k^* \) of \( \iota_k \) is the trace-preserving conditional expectation (or equivalently, the Hilbert-Schmidt orthogonal projection) from \( \mathcal{B}(\mathcal{H}_k) \) onto \( \mathcal{A}_k \). Since \( \iota_k \) is completely positive, so is \( \pi_k \), and since \( \pi_k \) is unital, it is also a Schwarz contraction. Let \( \tilde{\Phi} := \iota_2 \circ \Phi \circ \pi_1 \), the adjoint of which is \( \tilde{\Phi}^* = \iota_1 \circ \Phi^* \circ \pi_2 \). Note that \( \tilde{\Phi} \) is a Schwarz map, too, with \( \|\tilde{\Phi}\|_S = \|\Phi\|_S \), since for any \( X \in \mathcal{B}(\mathcal{H}_1) \),

\[
\tilde{\Phi}(X^*)\tilde{\Phi}(X) = \iota_2(\Phi(\pi_1(X^*))\Phi(\pi_1(X))) \leq \|\Phi\|_S \iota_2\Phi(\pi_1(X^*)\pi_1(X)) \leq \|\Phi\|_S \tilde{\Phi}(X^*X).
\]

Hence, for any vector \( v \in \mathcal{H}_1 \) and any orthonormal basis \( \{e_i\}_{i=1}^{d_1} \) in \( \mathcal{H}_1 \), we have

\[
\|\Phi\|_S \tilde{\Phi}(|v\rangle\langle v|) \geq \tilde{\Phi}(|v\rangle\langle e_i|\tilde{\Phi}(|e_i\rangle\langle v|), \quad i = 1, \ldots, d_1,
\]

where \( d_1 := \dim \mathcal{H}_1 \). Let \( Y \in \mathcal{A}_2 \) be arbitrary. Multiplying the above inequality with \( Y \) from the left and \( Y^* \) from the right, and taking the trace, we obtain

\[
\|\Phi\|_S \langle v, \tilde{\Phi}^*(Y^*Y) \rangle v = \|\Phi\|_S \operatorname{Tr} Y \tilde{\Phi}(|v\rangle\langle v|)Y^* \geq \operatorname{Tr} Y \tilde{\Phi}(|e_i\rangle\langle e_i|)\tilde{\Phi}(|e_i\rangle\langle v|)Y^*.
\]

Note that \( \operatorname{Tr} : \mathcal{A}_2 \to \mathbb{C} \) is completely positive, and hence it is a Schwarz map with \( \|\operatorname{Tr}\|_S = \|\operatorname{Tr}(I_2)\| = d_2 := \dim \mathcal{H}_2 \). Hence, the above inequality can be continued as

\[
d_2 \|\Phi\|_S \langle v, \tilde{\Phi}^*(Y^*Y) \rangle v \geq \operatorname{Tr} Y \tilde{\Phi}(|e_i\rangle\langle e_i|) \operatorname{Tr} \tilde{\Phi}(|e_i\rangle\langle v|)Y^* = \langle v, \tilde{\Phi}^*(Y^*)e_i\rangle\langle e_i, \tilde{\Phi}^*(Y)v \rangle,
\]

and summing over \( i \) yields

\[
d_1d_2 \|\Phi\|_S \langle v, \tilde{\Phi}^*(Y^*Y) \rangle v \geq \langle v, \tilde{\Phi}^*(Y^*)\tilde{\Phi}^*(Y)v \rangle.
\]
Since the above inequality is true for any \( v \in \mathcal{H}_1 \), and \( \Phi^*(Y) = \Phi^*(Y) \) for any \( Y \in \mathcal{A}_2 \), the assertion follows.

The assertion on positive linear combinations follows from \((3.6)\), and the first identity in \((3.6)\) is obvious. To see the second identity, assume first that \( \Phi \) and \( \Phi_1 \) are Schwarz contractions. Then, for any \( \varepsilon \in [0,1] \) and any \( X \in \mathcal{A}_1 \) we have

\[
((1 - \varepsilon)\Phi + \varepsilon\Phi_2)(X^*X) - ((1 - \varepsilon)\Phi + \varepsilon\Phi_2)(X)^*((1 - \varepsilon)\Phi + \varepsilon\Phi_2)(X) \\
= (1 - \varepsilon)[\Phi(X^*X) - \Phi_1(X^*)\Phi_1(X)] + \varepsilon[\Phi_2(X^*X) - \Phi_2(X^*)\Phi_2(X)] \\
+ \varepsilon(1 - \varepsilon)[(\Phi_1(X) - \Phi_2(X))^*(\Phi_1(X) - \Phi_2(X))] \geq 0,
\]

and hence \((1 - \varepsilon)\Phi + \varepsilon\Phi_2\) is a Schwarz contraction for any \( \varepsilon \in [0,1] \). Finally, let \( \Phi_1, \Phi_2 : \mathcal{A}_1 \to \mathcal{A}_2 \) be non-zero Schwarz maps. Then \( \tilde{\Phi}_k := \Phi_k/\|\Phi_k\|_S \) is a Schwarz contraction for \( k = 1, 2 \), and choosing \( \varepsilon := \|\Phi_2\|_S/\|\Phi_1\|_S + \|\Phi_2\|_S \), we get

\[
\|\Phi_1 + \Phi_2\|_S = (\|\Phi_1\|_S + \|\Phi_2\|_S)\|(1 - \varepsilon)\Phi_1 + \varepsilon\Phi_2\|_S \leq \|\Phi_1\|_S + \|\Phi_2\|_S. \quad \square
\]

Lemma \((3.9)\) and Corollary \((3.10)\) below are well-known when \( \Phi \) and \( \gamma \) are unital 2-positive maps. Their proofs are essentially the same for Schwarz contractions, which we provide here for the readers’ convenience.

### 3.9 Lemma
Let \( \Phi : \mathcal{A}_1 \to \mathcal{A}_2 \) be a Schwarz map, and let

\[
\mathcal{M}_{\Phi} := \{X \in \mathcal{A}_1 : \Phi(X)\Phi(X^*) = \|\Phi\|_S \Phi(XX^*)\}.
\]

Then

\[
X \in \mathcal{M}_{\Phi} \quad \text{if and only if} \quad \Phi(X)\Phi(Z) = \|\Phi\|_S \Phi(XZ), \quad Z \in \mathcal{A}_1. \quad (3.7)
\]

Moreover, the set \( \mathcal{M}_{\Phi} \) is a vector space that is closed under multiplication.

**Proof.** We may assume that \( \|\Phi\|_S > 0 \), since otherwise \( \Phi = 0 \) and the assertions become trivial. Define \( \gamma(X_1, X_2) := \|\Phi\|_S^{-1} \Phi(X_1X_2) - \Phi(X_1)\Phi(X_2)^* \), \( X_1, X_2 \in \mathcal{A}_1 \). Let \( X \in \mathcal{M}_{\Phi}, \quad Z \in \mathcal{A}_1 \) and \( t \in \mathbb{R} \). Then

\[
0 \leq \gamma(tX + Z, tX + Z) = t^2\gamma(X, X) + t[\gamma(X, Z) + \gamma(Z, X)] + \gamma(Z, Z) \\
= t[\gamma(X, Z) + \gamma(Z, X)] + \gamma(Z, Z).
\]

Since this is true for any \( t \in \mathbb{R} \), we get \( \gamma(X, Z) + \gamma(Z, X) = 0 \), and repeating the same argument with \( iZ \) in place of \( Z \), we get \( \gamma(X, Z) - \gamma(Z, X) = 0 \). Hence, \( \Phi(X)\Phi(Z) = \|\Phi\|_S \Phi(XZ) \). The implication in the other direction is obvious. The assertion about the algebraic structure of \( \mathcal{M}_{\Phi} \) follows immediately from \((3.7)\). \( \square \)

For a map \( \gamma \) from a \( C^* \)-algebra into itself, we denote by \( \ker (\text{id} - \gamma) \) the set of fixed points of \( \gamma \).

### 3.10 Corollary
Let \( \gamma : \mathcal{A} \to \mathcal{A} \) be a Schwarz contraction, and assume that there exists a strictly positive linear functional \( \alpha \) on \( \mathcal{A} \) such that \( \alpha \circ \gamma = \alpha \). Then \( \|\gamma\|_S = \|\gamma\| = 1 \), \( \ker (\text{id} - \gamma) \) is a non-zero \( C^* \)-algebra, \( \gamma \) is a \( C^* \)-algebra morphism on \( \ker (\text{id} - \gamma) \), and \( \gamma_\infty := \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \gamma^k \) is an \( \alpha \)-preserving conditional expectation onto \( \ker (\text{id} - \gamma) \).
Proof. The assumption $\alpha \circ \gamma = \alpha$ is equivalent to $\gamma^*(A) = A$, where $\alpha(X) = \text{Tr} AX$, $X \in \mathcal{A}$, and $A$ is strictly positive definite. Thus 1 is an eigenvalue of $\gamma^*$ and therefore also of $\gamma$. Hence, the fixed-point set of $\gamma$ is non-empty, and it is obviously a linear subspace in $\mathcal{A}$, which is also self-adjoint due to the positivity of $\gamma$. If $X \in \ker(\text{id} - \gamma)$ then $0 \leq \alpha (\gamma(X^*X) - (\gamma(X))X(X)) = \alpha (\gamma(X^*X)) - \alpha(X^*X) = 0$, and hence $\gamma(X^*X) = \gamma(X^*)\gamma(X) = X^*X$, i.e., $X^*X \in \ker(\text{id} - \gamma)$. The polarization identity then yields that ker $(\text{id} - \gamma)$ is closed also under multiplication, so it is a $C^*$-subalgebra of $\mathcal{A}$. Let $\tilde{I}$ be the unit of ker $(\text{id} - \gamma)$; then $1 = \|\tilde{I}\| = \|\gamma(\tilde{I})\| \leq \|\gamma\| \leq \|\gamma\|_S \leq 1$, so $\|\gamma\|_S = 1$. Repeating the above argument with $X^*$ yields that ker $(\text{id} - \gamma) \subset \mathcal{M}_\gamma \cap \mathcal{M}_\gamma^*$, where $\mathcal{M}_\gamma$ is defined as in Lemma 3.9. Moreover, by Lemma 3.9 $\gamma$ is a $C^*$-algebra morphism on $\mathcal{M}_\gamma \cap \mathcal{M}_\gamma^*$, and hence also on ker $(\text{id} - \gamma)$. Note that $\langle X, Y \rangle := \alpha(X^*Y)$ defines an inner product on $\mathcal{A}$ with respect to which $\gamma$ is a contraction, and hence $\gamma(\infty)$ exists and is the orthogonal projection onto ker $(\text{id} - \gamma)$, due to von Neumann’s mean ergodic theorem. By Lemma 3.9 we have $\gamma(\langle X, Y \rangle) = \gamma(X)\gamma(Y) = \gamma(Y)$ for any $X \in \ker(\text{id} - \gamma)$ and $Y \in \mathcal{A}$, which yields that $\gamma(\infty)$ is a conditional expectation.

3.11 Lemma. Let $B_1 := B \in \mathcal{A}_{1,+}$ be non-zero, and let $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a trace non-increasing $2$-positive map such that $\text{Tr} \Phi(B) = \text{Tr} B$. Let $B_2 := \Phi(B)$. Then exist decompositions $\text{supp} B_m = \bigoplus_{k=1}^r \mathcal{H}_{m,k,L} \otimes \mathcal{H}_{m,k,R}$, $m = 1, 2$, invertible density operators $\omega_{B,k}$ on $\mathcal{H}_{1,k,R}$ and $\tilde{\omega}_{B,k}$ on $\mathcal{H}_{2,k,R}$, and unitaries $U_k : \mathcal{H}_{1,k,L} \to \mathcal{H}_{2,k,L}$ such that

$$\ker(\text{id} - \Phi^*_B \circ \Phi)_+ = \bigoplus_{k=1}^r \mathcal{B}(\mathcal{H}_{1,k,L})_+ \otimes \omega_{B,k},$$

$$\Phi(A_{1,k,L} \otimes \omega_{B,k}) = U_k A_{1,k,L} U_k^* \otimes \tilde{\omega}_{B,k}, \quad A_{1,k,L} \in \mathcal{B}(\mathcal{H}_{1,k,L}).$$

(3.8)

Proof. Let $\tilde{A}_1 := B^0 A_1 B^0$, $\tilde{A}_2 := \Phi(B) A_2 \Phi(B)^0$, and define $\tilde{\Phi} : \tilde{A}_1 \to \tilde{A}_2$ as $\tilde{\Phi}(X) := \Phi(B^0 X B^0) = \Phi(X)$, $X \in \tilde{A}_1$. Then $\tilde{\Phi}(Y) = B^0 \Phi^*(Y) B^0$, $Y \in \tilde{A}_2$, and a straightforward computation verifies that $\tilde{\Phi}(B)(X) := \Phi(B)^{-1/2} \Phi(B^{1/2} X B^{1/2}) \Phi(B)^{-1/2} = \Phi_B(X)$, $X \in \tilde{A}_1$, and $\tilde{\Phi}_B(Y) := B^{1/2} \Phi^*(B^{-1/2} Y B^{-1/2}) B^{1/2} = \Phi_B(Y)$, $Y \in \tilde{A}_2$. Let $\gamma_1 := \Phi^* \circ \Phi_B$ and $\gamma_2 := \Phi_B \circ \Phi^*$. Obviously, $\gamma_1$ and $\gamma_2$ are again $2$-positive and, since

$$\gamma_1(B^0) = \Phi^*(\Phi(B)^0) = B^0 \Phi^*(\Phi(B)^0) B^0 = B^0,$$

$$\gamma_2(B^0) = \Phi(B)^{-1/2} \Phi(B^{1/2} \Phi^*(\Phi(B)^0) B^{1/2}) \Phi(B)^{-1/2} = \Phi(B)^0$$

due to Lemma 3.2 they are also unital. Hence, $\|\gamma_i\| = \|\gamma_i\|_S = 1$, $i = 1, 2$. Note that if $A_1 := A \in \ker(\text{id} - \Phi^*_B \circ \Phi)_+$ then $A^0 \leq B^0$ and hence $A \in \tilde{A}_1$, and

$$\gamma_1^*(A + B) = \Phi_B^*(\Phi(A + B)) = A + B, \quad \gamma_2^*(\Phi(A + B)) = \Phi_B^*(\Phi(A + B)) = \Phi(A + B).$$

Let $A_2 := \Phi(A_1)$. By the above, $\gamma_m$ leaves the faithful state $\alpha_m$ with density $(A_m + B_m) / \text{Tr}(A_m + B_m)$ invariant, and hence, by Corollary 3.10 $\ker(\text{id} - \gamma_m)$ is a $C^*$-algebra of the form ker $(\text{id} - \gamma_m) = \bigoplus_{k=1}^r \mathcal{B}(\mathcal{H}_{m,k,L}) \otimes \mathcal{H}_{m,k,R}$, where $\bigoplus_{k=1}^r \mathcal{H}_{m,k,L} \otimes \mathcal{H}_{m,k,R}$ is a decomposition of supp $B_m$. Moreover, $\lim_{m \to \infty} \frac{1}{n} \sum_{k=1}^n \gamma_m^k$ gives an $\alpha_m$-preserving conditional expectation onto ker $(\text{id} - \gamma_m)$, for $m = 1, 2$. Hence, by Takesaki’s theorem 50, $(A_m + B_m)^{it}$ ker $(\text{id} - \gamma_m) (A_m + B_m)^{-it} = \ker(\text{id} - \gamma_m)$. Now the argument of Section 3 in 34 yields the existence of invertible density operators $\omega_{A,B,k}$ on $\mathcal{H}_{1,k,R}$ and positive definite operators $X_{1,k,L,A,B}$ on $\mathcal{H}_{1,k,L}$ such that $A + B = \bigoplus_{k=1}^r X_{1,k,L,A,B} \otimes \omega_{A,B,k}$. By Theorem 9.11 in 40, we have $(A + B)^{it} B^{-it} \in \ker(\text{id} - \gamma_1)$ for every $t \in \mathbb{R}$, which yields that $\omega_{A,B,k}$ is independent of $A$, and hence that every $A \in \ker(\text{id} - \Phi_B^* \circ \Phi)_+$ can be written in the form $A = \bigoplus_{k=1}^r A_{1,k,L} \otimes \omega_{B,k}$.
with $\omega_{B,k} := \omega_{A,B,k}$ and some positive semidefinite operators $A_{1,k,L}$ on $\mathcal{H}_{1,k,L}$. This shows that $\ker (\text{id} - \Phi_B) \subset \bigoplus_{k=1}^r B(\mathcal{H}_{1,k,L})_+ \otimes \omega_{B,k}$. For the proof of (3.8), we refer to Theorem 4.2.1 in [33]. Finally, the decomposition $B = \bigoplus_{k=1}^r B_{1,k,L} \otimes \omega_{B,k}$ together with (3.8) shows that $\ker (\text{id} - \Phi_B) \subset \bigoplus_{k=1}^r B(\mathcal{H}_{1,k,L})_+ \otimes \omega_{B,k}$. \hfill $\Box$

4 Monotonicity

Now we turn to the proof of the monotonicity of the $f$-divergences under substochastic maps. Let $\mathcal{A}_i \subset B(\mathcal{H}_i)$ be finite-dimensional $C^*$-algebras for $i = 1, 2$. Recall that we call a map $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ substochastic if $\Phi^*$ satisfies the Schwarz inequality

$$\Phi^*(Y^*)\Phi^*(Y) \leq \Phi^*(Y^*Y), \quad Y \in \mathcal{A}_2,$$

and $\Phi$ is called stochastic if it is a trace-preserving substochastic map.

For a $B \in \mathcal{A}_{1,+}$ and a substochastic map $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$, we define the map $V : \mathcal{A}_2 \to \mathcal{A}_1$ as

$$V(X) := \Phi^*(X\Phi(B)^{-1/2})B^{1/2}, \quad X \in \mathcal{A}_2. \tag{4.1}$$

Note that $V = R_{B^{1/2}} \circ \Phi^* \circ R_{\Phi(B)^{-1/2}}$ and hence $V^* = R_{\Phi(B)^{-1/2}} \circ \Phi \circ R_{B^{1/2}}$, which yields

$$V^*(B^{1/2}) = \Phi(B)^{1/2}. \tag{4.2}$$

4.1 Lemma. We have the following equivalence:

$$V(\Phi(B)^{1/2}) = B^{1/2} \quad \text{if and only if} \quad \text{Tr} \, \Phi(B) = \text{Tr} \, B.$$

Proof. By definition,

$$V(\Phi(B)^{1/2}) = \Phi^*(\Phi(B)^{1/2}\Phi(B)^{-1/2})B^{1/2} = \Phi^*(\Phi(B)^0)B^{1/2}.$$

Hence, if $\text{Tr} \, \Phi(B) = \text{Tr} \, B$ then $V(\Phi(B)^{1/2}) = B^{1/2}$ due to Lemma 3.2. On the other hand, $B^{1/2} = V(\Phi(B)^{1/2}) = \Phi^*(\Phi(B)^0)B^{1/2}$ yields $\Phi^*(\Phi(B)^0)B^n = B^n$, $n \in \mathbb{N}$, and hence also (ii) of Lemma 3.2 which in turn yields $\text{Tr} \, \Phi(B) = \text{Tr} \, B$. \hfill $\Box$

4.2 Lemma. The map $V$ is a contraction and

$$V^* \left( L_AR_{B^{-1}} \right) V \leq L_{\Phi(A)}R_{\Phi(B)^{-1}}. \tag{4.3}$$

Moreover, when $\Phi^*$ is a $C^*$-algebra morphism, $V$ is an isometry if $\Phi(B)$ is invertible, and $V$ holds with equality if $B$ is invertible.

Proof. Let $X \in \mathcal{A}_2$. Then,

$$\|VX\|^2_{\text{HS}} = \text{Tr}(VX)^*(VX) = \text{Tr} B^{1/2}\Phi^*(\Phi(B)^{-1/2}X^*)\Phi^*(X\Phi(B)^{-1/2})B^{1/2} \leq \|\Phi^*\|_S \text{Tr} B^{1/2}\Phi^*(\Phi(B)^{-1/2}X^*)\Phi^*(X\Phi(B)^{-1/2})B^{1/2} \leq \|\Phi^*\|_S \text{Tr} \Phi(B)^{-1/2}X^*\Phi(B)^{-1/2} = \|\Phi^*\|_S \text{Tr} \Phi(B)^0XX^* \leq \|\Phi^*\|_S \text{Tr} XX^* = \|\Phi^*\|_S \|X\|^2_{\text{HS}} \leq \|X\|^2_{\text{HS}}. \tag{4.4}$$

(4.5)
If $\Phi^*$ is a $C^*$-algebra morphism then $\|\Phi^*\|_S = 1$ and the inequality in (4.1) holds with equality, and if $\Phi(B)$ is invertible then the inequality in (4.5) holds with equality. Similarly,

$$
\langle X, V^* (L_AR_{B^{-1}}) VX \rangle_{HS} = \text{Tr}(VX)^* A(VX)B^{-1}
$$

$$
= \text{Tr} B^{1/2}\Phi^* (\Phi(B)^{-1/2}X^*) A\Phi^*(X\Phi(B)^{-1/2})B^{1/2}B^{-1}
$$

$$
= \text{Tr} A\Phi^*(X\Phi(B)^{-1/2})B^0\Phi^*(\Phi(B)^{-1/2}X^*)
$$

$$
\leq \text{Tr} A\Phi^*(X\Phi(B)^{-1/2})\Phi^*(\Phi(B)^{-1/2}X^*)
$$

$$
\leq \|\Phi^*\|_S \text{Tr} A\Phi^*(X\Phi(B)^{-1/2})\Phi^*(\Phi(B)^{-1/2}X^*)
$$

$$
= \|\Phi^*\|_S \text{Tr} \Phi(A)X\Phi(B)^{-1}X^* = \|\Phi^*\|_S \langle X, L_{\Phi(A)}R_{\Phi(B)^{-1}}X \rangle_{HS}
$$

$$
\leq \langle X, L_{\Phi(A)}R_{\Phi(B)^{-1}}X \rangle_{HS}.
$$

If $\Phi^*$ is a $C^*$-algebra morphism then $\|\Phi^*\|_S = 1$ and the inequalities in (4.7) and (4.8) hold with equality, and if $B$ is invertible then (4.6) holds with equality.

Recall that a real-valued function $f$ on $[0, +\infty)$ is operator convex if $f(tA + (1-t)B) \leq tf(A) + (1-t)f(B)$, $t \in [0, 1]$, for any positive semi-definite operators $A, B$ on any finite-dimensional Hilbert space (or equivalently, on some infinite-dimensional Hilbert space). For a continuous real-valued function $f$ on $[0, +\infty)$, the following are equivalent (see [13, Theorem 2.1]): (i) $f$ is operator convex on $[0, +\infty)$ and $f(0) \leq 0$; (ii) $f(V^*AV) \leq V^*f(A)V$ for any contraction $V$ and any positive semi-definite operator $A$. The function $f$ is operator monotone decreasing if $f(A) \geq f(B)$ whenever $A$ and $B$ are such that $0 \leq A \leq B$. If $f$ is operator monotone decreasing on $[0, +\infty)$ then it is also operator convex (see the proof of [13, Theorem 2.5] or [4, Theorem V.2.5]). A function $f$ is operator concave (resp., operator monotone increasing) if $-f$ is operator convex (resp., operator monotone decreasing). An operator convex function on $[0, +\infty)$ is automatically continuous on $(0, +\infty)$, but might be discontinuous at 0. For instance, a straightforward computation shows that the characteristic function $1_{[0]}$ of the set $\{0\}$ is operator convex on $[0, +\infty)$. It is easy to verify that the functions

$$
\varphi_t(x) := -\frac{x}{x+t} = -1 + \frac{t}{x+t}
$$

are operator monotone decreasing and hence operator convex on $[0, +\infty)$ for every $t \in (0, +\infty)$.

**4.3 Theorem.** Let $A, B \in \mathcal{A}_{1, +}$, let $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a substochastic map such that $\text{Tr} \Phi(B) = \text{Tr} B$, and let $f$ be an operator convex function on $[0, +\infty)$. Assume that

$$
\text{Tr} \Phi(A) = \text{Tr} A \quad \text{or} \quad 0 \leq \omega(f).
$$

Then,

$$
S_f(\Phi(A)\|\Phi(B)) \leq S_f(A\|B).
$$

**Proof.** First we prove the theorem when $f$ is continuous at 0. Due to Theorem 8.1 we have the representation

$$
f(x) = f(0) + ax + bx^2 + \int_{(0, \infty)} \left( \frac{x}{1+t} + \varphi_t(x) \right) \, d\mu(t), \quad x \in [0, +\infty),
$$

where $b \geq 0$ and $\varphi_t(x)$ is given in (4.9). Define

$$
\Delta := L_AR_{B^{-1}} \quad \text{and} \quad \tilde{\Delta} := L_{\Phi(A)}R_{\Phi(B)^{-1}}.
$$
Then
\[
S_f(A\|B) = f(0) \operatorname{Tr} B + a \operatorname{Tr} AB^0 + b \operatorname{Tr} A^2 B^{-1} + \int_{(0, +\infty)} \left( \frac{\operatorname{Tr} AB^0}{1 + t} + S_{\varphi_t}(A\|B) \right) d\mu(t) + \omega(f) \operatorname{Tr} A(I - B^0).
\]

(4.12)

Note that \( \operatorname{Tr} B = \operatorname{Tr} \Phi(B) \) by assumption and, since \( b \geq 0 \), we have \( b \operatorname{Tr} A^2 B^{-1} \geq b \operatorname{Tr} \Phi(A)^2 \Phi(B)^{-1} \) due to Lemma 3.3. Since \( \varphi_t \) is operator convex, operator monotonic decreasing and \( \varphi_t(0) = 0 \), we have \( V^* \varphi_t(\Delta)V \geq \varphi_t(V^*V) \geq \varphi_t(\Delta) \) (4.13) for the contraction \( V \) defined in 4.11, due to (4.3) and [13, Theorem 2.1] as mentioned above. Hence, by Lemma 4.1,
\[
S_{\varphi_t}(A\|B) = \langle B^{1/2}, \varphi_t(\Delta) B^{1/2} \rangle_{HS} = \langle V \Phi(B)^{1/2}, \varphi_t(\Delta) V \Phi(B)^{1/2} \rangle_{HS} \geq \langle \Phi(B)^{1/2}, \varphi_t(\Delta) \Phi(B)^{1/2} \rangle_{HS} = S_{\varphi_t}(\Phi(A)\|\Phi(B)).
\]

(4.14)

Therefore, in order to prove the monotonicity inequality 4.11, it suffices to prove the monotonicity of the remaining terms in (4.12).

Assume first that \( \operatorname{supp} A \leq \operatorname{supp} B \), and hence also \( \operatorname{Tr} \Phi(A) = \operatorname{Tr} A \) (see Lemma 3.2). Then \( \operatorname{Tr} AB^0 = \operatorname{Tr} A = \operatorname{Tr} \Phi(A) = \operatorname{Tr} \Phi(A) \Phi(B)^0 \), which also yields \( \operatorname{Tr} A(I_1 - B^0) = \operatorname{Tr} \Phi(A)(I_2 - \Phi(B)^0) \). Hence, all the terms in (4.12) are monotonic non-increasing under \( \Phi \), and therefore we have the inequality (4.11).

If \( \omega(f) = +\infty \), then either \( \operatorname{supp} A \nsubseteq \operatorname{supp} B \), in which case
\[
S_f(A\|B) = +\infty \geq S_f(\Phi(A)\|\Phi(B)),
\]
or we have \( \operatorname{supp} A \leq \operatorname{supp} B \), and hence (4.11) follows by the previous argument.

Next, assume that \( \operatorname{Tr} \Phi(A) = \operatorname{Tr} A \), and define \( B_\varepsilon := B + \varepsilon A, \varepsilon > 0 \). Then \( \operatorname{Tr} \Phi(B_\varepsilon) = \operatorname{Tr} \Phi(B) + \varepsilon \operatorname{Tr} \Phi(A) = \operatorname{Tr} B + \varepsilon \operatorname{Tr} A = \operatorname{Tr} B_\varepsilon \), and \( \operatorname{supp} A \leq \operatorname{supp} B_\varepsilon \). Hence, by the previous argument,
\[
S_f(\Phi(A)\|\Phi(B_\varepsilon)) \leq S_f(A\|B_\varepsilon).
\]

(4.15)

By the previous paragraph, it is sufficient to consider the case where \( \omega(f) \) is finite, and therefore Proposition 2.12 can be used to obtain (4.11) by taking the limit \( \varepsilon \searrow 0 \) in (4.15).

Finally, assume that \( 0 \leq \omega(f) < +\infty \). By Proposition 8.4, this yields the representation
\[
f(x) = f(0) + \omega(f)x + \int_{(0, \infty)} \varphi_t(x) d\mu(t),
\]
and hence
\[
S_f(A\|B) = f(0) \operatorname{Tr} B + \omega(f) \operatorname{Tr} AB^0 + \int_{(0, +\infty)} S_{\varphi_t}(A\|B) d\mu(t) + \omega(f) \operatorname{Tr} A(I - B^0).
\]

(4.16)

Since \( \operatorname{Tr} \Phi(A) \leq \operatorname{Tr} A \), inequality (4.11) follows.
So far, we have proved the theorem for the case where \( f \) is continuous at \( 0 \). Consider the functions \( \bar{f}_\alpha(x) := -x^\alpha, \ x \geq 0, \ 0 < \alpha < 1 \). Then \( \bar{f}_\alpha \) is operator convex, continuous at \( 0 \) and \( \omega(\bar{f}_\alpha) = 0 \) for all \( \alpha \in (0, 1) \). Hence, by the above, we have

\[
- \text{Tr} \Phi(A)^\alpha \Phi(B)^{1-\alpha} = S_{f_\alpha}(\Phi(A)\|\Phi(B)) \leq S_{f_\alpha}(A\|B) = -\text{Tr} A^\alpha B^{1-\alpha}, \quad \alpha \in (0, 1). \tag{4.16}
\]

Taking the limit \( \alpha \searrow 0 \), we obtain

\[
\text{Tr} \Phi(A)^0 \Phi(B) = \text{Tr} A^0 B,
\tag{4.17}
\]

which in turn yields

\[
S_{1_{(0)}}(\Phi(A)\|\Phi(B)) = \text{Tr} \Phi(B) - \text{Tr} \Phi(A)^0 \Phi(B) \leq \text{Tr} B - \text{Tr} A^0 B = S_{1_{(0)}}(A\|B). \tag{4.18}
\]

Assume now that \( f \) is an operator convex function on \([0, +\infty)\), that is not necessarily continuous at \( 0 \). Convexity of \( f \) yields that \( f(0^+) := \lim_{x \searrow 0} f(x) \) is finite, and \( \alpha := f(0) - f(0^+) \geq 0 \). Note that \( \tilde{f} := f - \alpha 1_{(0)} \) is operator convex and continuous at \( 0 \), \( \omega(\tilde{f}) = \omega(f) \), and \( S_{\tilde{f}}(A\|B) = S_{\tilde{f}}(A\|B) + \alpha S_{1_{(0)}}(A\|B) \) for any \( A, B \in \mathcal{A}_{1,+} \). Applying the previous argument to \( \tilde{f} \) and using (4.18), we see that

\[
S_{\tilde{f}}(\Phi(A)\|\Phi(B)) = S_{\tilde{f}}(\Phi(A)\|\Phi(B)) + \alpha S_{1_{(0)}}(\Phi(A)\|\Phi(B))
\leq S_{\tilde{f}}(A\|B) + \alpha S_{1_{(0)}}(A\|B) = S_{\tilde{f}}(A\|B)
\]

if any of the conditions in (4.10) holds, completing the proof of the theorem. \( \square \)

4.4 Remark. Note that \( \text{supp} A \leq \text{supp} B \) is also sufficient for (4.11) to hold, due to Lemma 3.2

4.5 Example. Let \( A, B \in \mathcal{A}_{1,+} \) and \( \Phi : \mathcal{A}_1 \to \mathcal{A}_2 \) be a substochastic map such that \( \text{Tr} \Phi(B) = \text{Tr} B \). Let \( \text{sgn} x := x/|x|, \ x \neq 0 \), and define \( \bar{f}_\alpha := \text{sgn}(\alpha - 1)f_\alpha \), \( 0 < \alpha \neq 1 \), where \( f_\alpha \) is given in Example 2.7. Since \( f_\alpha \) is operator convex, and \( \omega(f_\alpha) \geq 0 \) for all \( \alpha \in [0, 2] \setminus \{1\} \), Theorem 4.3 yields that

\[
\text{sgn}(\alpha - 1) \text{Tr} \Phi(A)^\alpha \Phi(B)^{1-\alpha} = S_{f_\alpha}(\Phi(A)\|\Phi(B)) \leq S_{f_\alpha}(A\|B) = \text{sgn}(\alpha - 1) \text{Tr} A^\alpha B^{1-\alpha} \tag{4.19}
\]

when \( \alpha \in (1, 2] \) and \( \text{supp} A \leq \text{supp} B \). (Note that \( S_{f_\alpha}(\Phi(A)\|\Phi(B)) \leq S_{f_\alpha}(A\|B) = +\infty \) is trivial when \( \alpha \in (1, 2] \) and \( \text{supp} A \not\subseteq \text{supp} B \).) The same inequality has been shown in the proof of Theorem 4.3 for \( \alpha \in [0, 1) \); see (4.16) and (4.17). This yields the monotonicity of the Rényi relative entropies,

\[
S_{\alpha}(\Phi(A)\|\Phi(B)) = \frac{1}{\alpha - 1} \log S_{f_\alpha}(\Phi(A)\|\Phi(B)) \leq \frac{1}{\alpha - 1} \log S_{f_\alpha}(A\|B) = S_{\alpha}(A\|B) \tag{4.20}
\]

for \( \alpha \in [0, 2] \setminus \{1\} \).

Since \( \omega(f) \geq 0 \) for \( f(x) := x \log x \), Theorem 4.3 also yields the monotonicity of the relative entropy,

\[
S(\Phi(A)\|\Phi(B)) \leq S(A\|B).
\]
4.6 Remark. In the proof of Theorem 4.3 it was essential that \( f \) is operator convex, but it is not known if it is actually necessary. See Appendix A for some special cases where convexity of \( f \) is sufficient.

Theorem 4.3 yields the joint convexity of the \( f \)-divergences:

4.7 Corollary. Let \( A_i, B_i \in \mathcal{A}_+ \) and \( p_i \geq 0 \) for \( i = 1, \ldots, r \), and let \( f \) be an operator convex function on \([0, +\infty)\). Then

\[
S_f \left( \sum_i p_i A_i \| \sum_i p_i B_i \right) \leq \sum_i p_i S_f (A_i \| B_i).
\]

Proof. Let \( \delta_1, \ldots, \delta_r \) be a set of orthogonal rank-one projections on \( \mathbb{C}^r \), and define \( A := \sum_{i=1}^r p_i A_i \otimes \delta_i, B := \sum_{i=1}^r p_i B_i \otimes \delta_i \). The map \( \Phi : \mathcal{A} \otimes \mathcal{B}(\mathbb{C}^r) \to \mathcal{A} \), given by \( \Phi(X \otimes Y) := X \operatorname{Tr} Y, X \in \mathcal{A}, Y \in \mathcal{B}(\mathbb{C}^r) \), is completely positive and trace-preserving and hence, by Theorem 4.3

\[
S_f \left( \sum_i p_i A_i \| \sum_i p_i B_i \right) = S_f (\Phi(A) \| \Phi(B)) \leq S_f (A \| B) = \sum_i p_i S_f (A_i \| B_i),
\]

where the last identity is due to Corollary 2.5.

4.8 Remark. For an operator convex function \( f \) on \([0, +\infty)\) let \( \mathcal{M}_f(\mathcal{A}_1, \mathcal{A}_2) \) denote the set of positive linear maps \( \Phi : \mathcal{A}_1 \to \mathcal{A}_2 \) such that the monotonicity \( S_f(\Phi(A) \| \Phi(B)) \leq S_f(A \| B) \) holds for all \( A, B \in \mathcal{A}_1 \). The joint convexity of the \( f \)-divergences shows that \( \mathcal{M}_f(\mathcal{A}_1, \mathcal{A}_2) \) is convex. Indeed, if \( \Phi_1, \Phi_2 \in \mathcal{M}_f(\mathcal{A}_1, \mathcal{A}_2) \) then Corollary 4.7 yields

\[
S_f((1 - \lambda)\Phi_1(A) + \lambda \Phi_2(A) \| (1 - \lambda)\Phi_1(B) + \lambda \Phi_2(B)) \\
\leq (1 - \lambda)S_f(\Phi_1(A) \| \Phi_1(B)) + \lambda S_f(\Phi_2(A) \| \Phi_2(B)) \\
\leq (1 - \lambda)S_f(A \| B) + \lambda S_f(A \| B) = S_f(A \| B)
\]

for any \( \lambda \in [0, 1] \) and \( A, B \in \mathcal{A}_1 \). Note also that if \( \Phi_1 \in \mathcal{M}_f(\mathcal{A}_1, \mathcal{A}_2) \) and \( \Phi_2 \in \mathcal{M}_f(\mathcal{A}_2, \mathcal{A}_3) \) then \( \Phi_2 \circ \Phi_1 \in \mathcal{M}_f(\mathcal{A}_1, \mathcal{A}_3) \).

We say that a linear map \( \Phi : \mathcal{A}_1 \to \mathcal{A}_2 \) is a co-Schwarz map if there is a \( c \in [0, \infty) \) such that

\[
\Phi(X^\ast)\Phi(X) \leq c\Phi(X X^\ast), \quad X \in \mathcal{A}_1,
\]

and it is a co-Schwarz contraction if the above inequality holds with \( c = 1 \). It is easy to see that a linear map \( \Phi : \mathcal{A}_1 \to \mathcal{A}_2 \) is a co-Schwarz map (resp., a co-Schwarz contraction) if and only if there is a Schwarz map (resp., a Schwarz contraction) \( \hat{\Phi} : \mathcal{A}_1^\ast \to \mathcal{A}_2 \) such that

\[
\Phi = \hat{\Phi} \circ T, \quad T(X) := X^\ast \text{ denotes the transpose of } X \in \mathcal{A}_1 \text{ with respect to a fixed orthonormal basis of } \mathcal{H}_1, \quad \mathcal{A}_1^\ast := \{X^\ast : X \in \mathcal{A}_1\} \subset B(\mathcal{H}_1).
\]

Furthermore, we say that \( \Phi \) is co-substochastic (resp., co-stochastic) if \( \Phi^\ast \) is a co-Schwarz contraction (resp., a unital co-Schwarz contraction). Theorem 4.3 holds also when \( \Phi : \mathcal{A}_1 \to \mathcal{A}_2 \) is a co-substochastic map. This follows immediately from Theorem 4.3 and the fact that transpositions leave every \( f \)-divergences invariant (see (iii) of Corollary 2.5). Alternatively, this can be proved by replacing the operator \( V \) defined in (4.1) with the conjugate-linear map

\[
\hat{V}(X) := \Phi^\ast(\Phi(B)^{-1/2} X^\ast) B^{1/2}, \quad X \in \mathcal{A}_2,
\]

and following the proofs of Lemma 4.2 and Theorem 4.3 with \( \hat{V} \) in place of \( V \).
Recall that a positive map is called decomposable if it can be written as the sum of a completely positive map and a completely positive map composed with a transposition. By the above, a similar notion of decomposability is sufficient for the monotonicity of the $f$-divergences. Namely, if a trace-preserving positive map $\Phi : A_1 \to A_2$ is decomposable in the sense that it can be written as a convex combination of a stochastic and a co-stochastic map then $\Phi \in \mathcal{M}_f(A_1, A_2)$ for any operator convex function $f$ on $[0, +\infty)$. Example 3.6 provides simple examples of trace-preserving positive maps that are decomposable in this sense but which are neither stochastic nor co-stochastic.

## 5 Equality in the monotonicity

In this section we analyze the situation where the monotonicity inequality

$$S_f(\Phi(A)\|\Phi(B)) \leq S_f(A\|B)$$

holds with equality, based on the integral representation of operator convex functions that we give in Section 8.

By Theorem 8.1, every operator convex function $f$ on $[0, +\infty)$ admits a decomposition

$$f(x) = \alpha 1_{\{0\}}(x) + f(0^+) + ax + \int_{(0,\infty)} \left( \frac{x}{1+t} + \varphi_t(x) \right) d\mu_f(t), \quad x \in [0, +\infty), \quad (5.1)$$

where $\alpha, b \geq 0$, $f(0^+) := \lim_{x \downarrow 0} f(x)$, $1_{\{0\}}$ is the characteristic function of the singleton $\{0\}$, $f_2(x) := x^2$, $\varphi_t(x)$ is given in (4.9), and $\mu_f$ is a positive measure on $(0, +\infty)$.

Recall that $\text{spec}(X)$ denotes the spectrum of an operator $X$. We will use the notation $|H|$ to denote the cardinality of a set $H$. Given $B \in A_{1,+}$ and a positive map $\Phi : A_1 \to A_2$, let $\Phi_B : A_1 \to A_2$ and $\Phi^*_B : A_2 \to A_1$ be the maps defined in (3.1) and (3.2).

### 5.1 Theorem

Let $A, B \in A_{1,+}$ be such that $\text{supp} A \leq \text{supp} B$, let $\Phi : A_1 \to A_2$ be a substochastic map such that $\text{Tr } \Phi(B) = \text{Tr } B$, and define

$$\Delta := L_A R_{B^{-1}} \quad \text{and} \quad \tilde{\Delta} := L_{\Phi(A)} R_{\Phi(B)^{-1}}.$$

Then, for the following conditions [(i)] [(x)] we have

\begin{align*}
(i) \implies (ii) \implies (iii) \implies (iv) \iff (v) \iff (vi) \iff (vii) \iff (viii) \iff (ix) \iff (x)
\end{align*}

and if $\Phi$ is 2-positive then [(x)] $\implies (i)$ holds as well.

(i) There exists a stochastic map $\Psi : A_2 \to A_1$ such that

$$\Psi(\Phi(A)) = A, \quad \Psi(\Phi(B)) = B. \quad (5.2)$$

(ii) There exists a substochastic map $\Psi : A_2 \to A_1$ such that (5.2) holds.

(iii) For every operator convex function $f$ on $[0, +\infty)$,

$$S_f(\Phi(A)\|\Phi(B)) = S_f(A\|B). \quad (5.3)$$

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(iv) The equality in (5.3) holds for some operator convex function $f$ on $[0, +\infty)$ such that
\[ |\text{supp } \mu_f| \geq |\text{spec}(\Delta) \cup \text{spec}(\tilde{\Delta})|. \tag{5.4} \]

(v) There exists a $T \subset (0, +\infty)$ such that $|T| \geq |\text{spec}(\Delta) \cup \text{spec}(\tilde{\Delta})|$ and
\[ S_{\varphi_t}(\Phi(A)\|\Phi(B)) = S_{\varphi_t}(A\|B), \quad t \in T. \]

(vi) $B^0\Phi^*(\Phi(B)^{-2}\Phi(A)^z) = B^{-2}A^z$ for all $z \in \mathbb{C}$.

(vii) $B^0\Phi^*(\Phi(B)^{-\alpha}\Phi(A)^{\alpha}) = B^{-\alpha}A^{\alpha}$ for some $\alpha \in (0, 2) \setminus \{1\}$.

(viii) $B^0\Phi^*(\Phi(B)^{-it}\Phi(A)^{it}) = B^{-it}A^{it}$ for all $t \in \mathbb{R}$.

(ix) $B^0\Phi^*(\log^* \Phi(A) - (\log^* \Phi(B))\Phi(A)^0) = \log^* A - (\log^* B)A^0$.

(x) $\Phi_{B^0}(\Phi(A)) = A$.

Moreover, (ii) $\implies$ (iii) holds without assuming that $\text{supp } A \leq \text{supp } B$. If $\Phi$ is $n$-positive/completely positive then $\Psi$ in (i) can also be assumed to be $n$-positive/completely positive.

**Proof.** The implication (i)$\implies$(ii) is obvious. Assume that (ii) holds, and let $\hat{A} := \Phi(A)$, $\hat{B} := \Phi(B)$. Then $\text{Tr } A = \text{Tr } \hat{A} \leq \text{Tr } \hat{\Phi}(A) \leq \text{Tr } \hat{A}$ and similarly for $B$ and $\hat{B}$, which yields $\text{Tr } \Phi(A) = \text{Tr } \hat{A}$, $\text{Tr } \Phi(B) = \text{Tr } \hat{B}$ and $\text{Tr } \Phi(A) = \text{Tr } A$, $\text{Tr } \Phi(B) = \text{Tr } B$ (note that this latter is automatic here, and not necessary to assume from the beginning). Applying Theorem 4.3 twice, we get that $S_f(A\|B) = S_f(\Phi(A)\|\Phi(B)) \leq S_f(\hat{A}\|\hat{B}) = S_f(\Phi(A)\|\Phi(B)) \leq S_f(A\|B)$ for any operator convex function $f$ on $[0, +\infty)$, proving (iii). The implication (iii)$\implies$(iv) is again obvious.

Note that if $A = 0$ then $S_f(A\|B) = f(0)\text{Tr } B$ for any function $f$, and (i) (x) hold true automatically. Hence, for the rest we will assume that $A \neq 0$ and hence also $B \neq 0$.

Assume that (iv) holds, i.e., $S_f(\Phi(A)\|\Phi(B)) = S_f(A\|B)$ for an operator convex function $f$ on $[0, +\infty)$ satisfying (5.4). By (5.1), we have
\[ S_f(A\|B) = \alpha S_{1_{(0)}}(A\|B) + f(0^+)\text{Tr } A + a \text{Tr } A + bS_{f_2}(A\|B) + \int_{(0, +\infty)} \left( \frac{\text{Tr } A}{1 + t} + S_{\varphi_t}(A\|B) \right) d\mu(t) \]
(cf. (1.2)). Note that $\text{Tr } \Phi(B) = \text{Tr } B$ by assumption and $\text{Tr } \Phi(A) = \text{Tr } A$ follows due to Lemma 3.2. Thus,
\[ 0 = S_f(A\|B) - S_f(\Phi(A)\|\Phi(B)) = \alpha \left( S_{1_{(0)}}(A\|B) - S_{1_{(0)}}(\Phi(A)\|\Phi(B)) \right) + b \left( S_{f_2}(A\|B) - S_{f_2}(\Phi(A)\|\Phi(B)) \right) + \int_{(0, +\infty)} (S_{\varphi_t}(A\|B) - S_{\varphi_t}(\Phi(A)\|\Phi(B))) d\mu_f(t). \]

By Theorem 4.3 the $f$-divergences corresponding to $1_{(0)}$, $f_2$ and $\varphi_t$ are monotonic non-increasing under $\Phi$, and hence the above equality yields that
\[ S_{\varphi_t}(\Phi(A)\|\Phi(B)) = S_{\varphi_t}(A\|B) \]
for all $t \in \operatorname{supp} \mu_f$. This gives (v) with $T := \operatorname{supp} \mu_f$.

Assume now that (v) holds. This means that for every $t \in T$,

$$0 = S_{\varphi_t}(A\|B) - S_{\varphi_t}(\Phi(A)\|\Phi(B)) = (\Phi(B)^{1/2}, (V^*\varphi_t(\Delta)V - \varphi_t(\tilde{\Delta}))\Phi(B)^{1/2})_{\text{HS}},$$

where we used that $V\Phi(B)^{1/2} = B^{1/2}$ due to Lemma 4.1 (note that $\omega(\varphi_t) = 0$, $t > 0$). By (4.13) this is equivalent to

$$V^*\varphi_t(\Delta)V\Phi(B)^{1/2} = \varphi_t(\tilde{\Delta})\Phi(B)^{1/2}, \quad t \in T,$$

or equivalently,

$$V^*[-I_1 + t(\Delta + tI_1)^{-1}] B^{1/2} = [-I_2 + t(\tilde{\Delta} + tI_2)^{-1}] \Phi(B)^{1/2}, \quad t \in T.$$

By (1.2) we get

$$V^*(\Delta + tI_1)^{-1} B^{1/2} = (\tilde{\Delta} + tI_2)^{-1} \Phi(B)^{1/2}, \quad t \in T.$$

Using Lemma 5.2 below and the assumption that $|T| \geq |\text{spec}(\Delta) \cup \text{spec}(\tilde{\Delta})|$, we obtain

$$V^*h(\Delta) B^{1/2} = h(\tilde{\Delta}) \Phi(B)^{1/2} \quad (5.5)$$

for any function $h$ on $\text{spec}(\Delta) \cup \text{spec}(\tilde{\Delta})$. In particular,

$$V^*(\Delta + tI_1)^{-\gamma} B^{1/2} = (\tilde{\Delta} + tI_2)^{-\gamma} \Phi(B)^{1/2}, \quad \gamma, t > 0. \quad (5.6)$$

Using (5.6) with $\gamma = 1$ and $\gamma = 2$, we obtain

$$\|V^*(\Delta + tI_1)^{-1} B^{1/2}\|^2_{\text{HS}} = \langle (\tilde{\Delta} + tI_2)^{-1} \Phi(B)^{1/2}, (\tilde{\Delta} + tI_2)^{-1} \Phi(B)^{1/2} \rangle_{\text{HS}}$$

$$= \langle (\tilde{\Delta} + tI_2)^{-2} \Phi(B)^{1/2}, \Phi(B)^{1/2} \rangle_{\text{HS}}$$

$$= \langle V^*(\Delta + tI_1)^{-2} B^{1/2}, \Phi(B)^{1/2} \rangle_{\text{HS}}$$

$$= \langle (\Delta + tI_1)^{-2} B^{1/2}, B^{1/2} \rangle_{\text{HS}}$$

$$= \| (\Delta + tI_1)^{-1} B^{1/2} \|^2_{\text{HS}}.$$

Therefore, we have $\|V^*x\|^2_{\text{HS}} = \|x\|^2_{\text{HS}}$ for $x := (\Delta + tI_1)^{-1} B^{1/2}$, and since $V$ is a contraction, we get $0 \leq \|VV^*x - x\|^2_{\text{HS}} = \|VV^*x\|^2_{\text{HS}} - 2 \|V^*x\|^2_{\text{HS}} + \|x\|^2_{\text{HS}} = \|VV^*x\|^2_{\text{HS}} - \|x\|^2_{\text{HS}} \leq 0$, by which $VV^*(\Delta + tI_1)^{-1} B^{1/2} = (\Delta + tI_1)^{-1} B^{1/2}$. Substituting (5.6) with $\gamma = 1$, we finally obtain

$$V(\tilde{\Delta} + tI_2)^{-1} \Phi(B)^{1/2} = (\Delta + tI_1)^{-1} B^{1/2}, \quad t > 0, \quad (5.7)$$

and using again Lemma 5.2 we get

$$Vh(\tilde{\Delta}) \Phi(B)^{1/2} = h(\Delta) B^{1/2}$$

for any function $h$ on $\text{spec}(\Delta) \cup \text{spec}(\tilde{\Delta})$. By the definition (1.1) of $V$, this means that

$$\Phi^* \left( \left( h(\tilde{\Delta}) \Phi(B)^{1/2} \right) \Phi(B)^{-1/2} \right) B^{1/2} = h(\Delta) B^{1/2}.$$

In particular, the choice $h(x) := x^z$, $x > 0$, $h(0) := 0$, yields

$$\Phi^* \left( \left( \Phi(A)^z \Phi(B)^{-z} \right) B^{1/2} \right) B^{1/2} = A^z B^{1/2 - z}, \quad z \in \mathbb{C}. \quad (5.8)$$
Multiplying from the right with $B^{-1/2}$ and taking the adjoint, we obtain $[\text{vi}]$

The implication $[\text{vi}] \Rightarrow [\text{vii}]$ is obvious. Assume now that $[\text{vii}]$ holds, i.e., $B^{-\alpha}A^\alpha = B^0\Phi^* (\Phi(B)^{-\alpha} \Phi(A)^\alpha)$ for some $\alpha \in (0,2) \setminus \{1\}$. Multiplying by $B$ and taking the trace, we obtain

$$S_{f_\alpha}(A \parallel B) = \Tr A^\alpha B^{1-\alpha} = \Tr B \Phi^* (\Phi(B)^{-\alpha} \Phi(A)^\alpha) = \Tr B \Phi(B) (\Phi(B)^{-\alpha} \Phi(A)^\alpha)$$

$$= S_{f_\alpha}(\Phi(A) \parallel B(\Phi)),$$

where $f_\alpha(x) := x^\alpha$, $x \geq 0$. Since the support of the representing measure $\mu_{f_\alpha}$ is $(0, +\infty)$ (see Example 8.3), we see that $[\text{vii}]$ implies $[\text{iv}]$. The equivalence of $[\text{vi}]$ and $[\text{viii}]$ is obvious from the fact that the functions $z \mapsto B^0 \Phi^* (\Phi(B)^{-z} \Phi(A)^z)$ and $z \mapsto B^{-z} A^z$ are both analytic on the whole complex plane. Differentiating $[\text{viii}]$ at $t = 0$, we obtain $[\text{ix}]$. A straightforward computation shows that $[\text{ix}]$ yields $[\text{iv}]$ for $f(x) := x \log x$, that is, the equality for the standard relative entropy (note that the support of the representing measure for $x \log x$ is $(0, +\infty)$ by Example 8.3). Hence, we have proved that

$[\text{i}] \Rightarrow [\text{ii}] \Rightarrow [\text{iii}] \Rightarrow [\text{iv}] \Leftrightarrow [\text{v}] \Leftrightarrow [\text{vi}] \Leftrightarrow [\text{vii}] \Leftrightarrow [\text{viii}] \Leftrightarrow [\text{ix}]$

Assume now that $[\text{vi}]$ holds. In particular, the choice $z = 0$ yields

$$B^0 \Phi^* (\Phi(A)^0) = A^0$$  \hspace{1cm} (5.9)

(recall that $A^0 \leq B^0$). Since $\Phi$ is substochastic, we have $\Phi^* (Y^* Y) \geq \Phi^* (Y^* Y) B^0 \Phi^* (Y)$, and multiplying from both sides by $B^0$, we obtain that $\Psi(Y) := B^0 \Phi^* (Y)^B$,

$Y \in A_2$, is a Schwarz contraction. For $u_t := \Phi(B)^{-it} \Phi(A)^it$ and $w_t := B^{-it} A^it$, we have

$$u_t u_t^* = B^{-it} A^it B^it, \quad w_t w_t^* = B^{-it} A^it B^it, \quad t \in \mathbb{R}.$$  \hspace{1cm} (5.9)

Note that $[\text{vi}]$ says that $B^0 \Phi^* (u_t) = w_t$, and hence $\Psi(u_t) = w_t B^0 = w_t$. Thus,

$$0 \leq \Tr B^{1/2} (\Psi(u_t u_t^*) - \Psi(u_t) \Psi(u_t^*)) B^{1/2} = \Tr B \Phi^* (u_t u_t^*) - \Tr B w_t w_t^*$$

$$= \Tr \Phi(B) \Phi(B)^{-it} \Phi(A)^0 \Phi(B)^{it} - \Tr B B^{-it} A^it B^it = \Tr \Phi(B) \Phi(A)^0 - \Tr B A^it$$

$$= \Tr B \Phi^* (\Phi(A)^0) - \Tr B A^it = \Tr B A^it - \Tr B A^it = 0,$$

where we used $[\text{viii}]$. Hence, $B^{1/2} \Psi(u_t u_t^*) B^{1/2} = B^{1/2} \Psi(u_t) \Psi(u_t^*) B^{1/2}$, and multiplying from both sides with $B^{-1/2}$, we obtain $\Psi(u_t u_t^*) = \Psi(u_t) \Psi(u_t^*)$. Since $\Psi(u_t) \neq 0$, and $\Psi$ is a Schwarz contraction, this yields that $\| \Psi \|_S = 1$ and $u_t \in M_\Psi$. Hence, by Lemma 3.9, $\Psi(u_t Y) = \Psi(u_t) \Psi(Y) = w_t \Phi^* (Y)^B$ for all $Y \in A_2$ and $t \in \mathbb{R}$, i.e.,

$$B^0 \Phi^* (\Phi(B)^{-it} \Phi(A)^it Y) B^0 = B^{-it} A^it \Phi^* (Y)^B B^0, \quad t \in \mathbb{R}, Y \in A_2.$$  \hspace{1cm} (5.9)

Note that the maps $z \mapsto B^0 \Phi^* (\Phi(B)^{-z} \Phi(A)^z Y) B^0$ and $z \mapsto B^{-z} A^z \Phi^* (Y)^B B^0$ are analytic on the whole complex plane and coincide on $i\mathbb{R}$ and thus they are equal for every $z \in \mathbb{C}$. Choosing $z = 1/2$ and $Y := \Phi(A)^{1/2} \Phi(B)^{-1/2}$, we get

$$B^0 \Phi^* (\Phi(B)^{-1/2} \Phi(A)^{1/2} \Phi(B)^{-1/2}) B^0 = B^{-1/2} A^{1/2} \Phi^* (\Phi(A)^{1/2} \Phi(B)^{-1/2}) B^0$$

$$= B^{-1/2} A^{1/2} A^{1/2} B^{-1/2},$$

where we used the adjoint of $[\text{vi}]$ with $z = 1/2$. Multiplying from both sides by $B^{1/2}$, we obtain $[\text{x}]$.
Finally, assume that \( (x) \) holds, and hence

\[
\Phi^*_B(\Phi(A)) = A, \quad \Phi^*_B(\Phi(B)) = B.
\]

Note that \( \Phi^*_B \) is not necessarily trace-preserving, as \( (\Phi^*_B)^*(I) = \Phi_B(I) = \Phi(B)^0 \), which might be strictly smaller than \( I_2 \). However, if \( \rho \) is a density operator on \( \mathcal{H}_1 \) then the map \( X \mapsto \Phi_B(X) + (\Tr \rho X)(I_2 - \Phi(B)^0) \) is obviously unital and hence its adjoint \( \Psi : \mathcal{A}_2 \to \mathcal{A}_1 \), \( \Psi(Y) = \Phi_B^*(Y) + [\Tr(I_2 - \Phi(B)^0)Y]\rho \) is trace-preserving. Moreover, \( \Psi(\Phi(A)) = \Phi^*_B(\Phi(A)) \) and \( \Psi(\Phi(B)) = \Phi^*_B(\Phi(B)) \), as one can easily verify. Since \( \Psi \) is obtained from \( \Phi^* \) by composing it with completely positive maps and adding a completely positive map, it inherits the positivity of \( \Phi^* \), i.e., if \( \Phi \), and hence \( \Phi^* \), is \( n \)-positive/completely positive then so is \( \Psi \). In particular, if \( \Phi \) is \( 2 \)-positive then \( \Psi^* \) is a unital \( 2 \)-positive map and hence it is also a Schwarz contraction, i.e., \( \Psi \) is stochastic. Thus \( (x) \Rightarrow (i) \) holds in this case. \( \square \)

5.2 Lemma. If \( f \) is a complex-valued function on finitely many points \( \{x_i\}_{i \in I} \subset [0, +\infty) \) then for any pairwise different positive numbers \( \{t_i\}_{i \in I} \), there exist complex numbers \( \{c_i\}_{i \in I} \) such that \( f(x_i) = \sum_{j \in I} c_{x_i, t_j} t_j \), \( i \in I \).

Proof. The matrix \( C \) with entries \( C_{ij} := \frac{1}{x_i - t_j} \), \( i, j \in I \), is a Cauchy matrix which is invertible due to the assumptions that \( x_i \neq x_j \) and \( t_i \neq t_j \) for \( i \neq j \). From this the statement follows. \( \square \)

5.3 Corollary. Assume that \( \text{supp} A_i \leq \text{supp} B_i \), \( i = 1, \ldots, r \), in the setting of Corollary 4.7. Then equality holds in (5.11) if and only if

\[
p_i A_i = p_i B_i^{1/2} \left( \sum_j p_j B_j \right)^{-1/2} \left( \sum_j p_j A_j \right) \left( \sum_j p_j B_j \right)^{-1/2} B_i^{1/2}, \quad i = 1, \ldots, r.
\]

Proof. It is immediate from writing out the equality \( A = \Phi_B^*(\Phi(A)) \) given in (x) in the setting of Corollary 4.7. \( \square \)

5.4 Remark. Note that if \( \text{supp} A \leq \text{supp} B \) and \( \text{Tr} \Phi(B) = \text{Tr} B \) then for a linear function \( f(x) = f(0) + ax \), the preservation of the \( f \)-divergence is automatic, and has no implication on the reversibility of \( \Phi \) on \( \{A, B\} \). Indeed, we have \( \text{Tr} \Phi(A) = \text{Tr} A \) due to Lemma 3.2 and

\[
S_f(\Phi(A)||\Phi(B)) = f(0) \text{Tr} \Phi(B) + a \text{Tr} \Phi(A) = f(0) \text{Tr} B + a \text{Tr} A = S_f(A||B).
\]

The \( f \)-divergence corresponding to the quadratic function \( f_2(x) := x^2 \) is \( S_{f_2}(A||B) = \text{Tr} A^2 B^{-1} \) (when \( \text{supp} A \leq \text{supp} B \)). Preservation of the \( f \)-divergence by a stochastic map is not automatic in this case; however, it is not sufficient for the reversibility of the map, either. Indeed, it was shown in Example 2.2 of [25] that there exists a positive definite operator \( D_{123} \) on a tripartite Hilbert space \( \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \), such that

\[
D_{123}(\tau_1 \otimes D_{23})^{-1} = (D_{12} \otimes \tau_3)(\tau_1 \otimes D_2 \otimes \tau_3)^{-1}, \tag{5.10}
\]

but

\[
D_{123}^{it}(\tau_1 \otimes D_{23})^{-it} \neq (D_{12} \otimes \tau_3)^{it}(\tau_1 \otimes D_2 \otimes \tau_3)^{-it} \quad \text{for some } t \in \mathbb{R}, \tag{5.11}
\]

where \( \tau_i := \frac{1}{\text{Tr} \mathcal{H}_i} I_i \), and \( D_{23} := \text{Tr}_{\mathcal{H}_1} D_{123} \), \( D_{12} := \text{Tr}_{\mathcal{H}_3} D_{123} \), \( D_2 := \text{Tr}_{\mathcal{H}_1 \otimes \mathcal{H}_3} D_{123} \). Define \( \mathcal{H} := \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \), \( A := D_{123} \) and \( B := \tau_1 \otimes D_{23} \). Let \( A_1 := \mathcal{B}(\mathcal{H}_1) \), \( A_2 := \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \otimes I_3 \) and let \( \Phi^* \) be the identical embedding of \( \mathcal{A}_2 \) into \( \mathcal{A}_1 \). Then, (5.10) reads as

\[
AB^{-1} = \Phi(A)\Phi(B)^{-1}.
\]

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Multiplying both sides by $A$ and taking the trace, we obtain
\[
\text{Tr } A^2 B^{-1} = \text{Tr } A \Phi(A) \Phi(B)^{-1}.
\] (5.12)

Note that $\Phi$ is the orthogonal (with respect to the Hilbert-Schmidt inner product) projection from $A_1$ onto $A_2$, i.e., $\Phi$ is the conditional expectation onto $A_2$ with respect to $\text{Tr}$, and $\Phi(A) \Phi(B)^{-1} \in A_2$. Hence, we have $\text{Tr } A \Phi(A) \Phi(B)^{-1} = \text{Tr } \Phi(A)^2 \Phi(B)^{-1}$. Hence, (5.12) can be rewritten as
\[
S_{\Phi}(A\|B) = \text{Tr } A^2 B^{-1} = \text{Tr } \Phi(A)^2 \Phi(B)^{-1} = S_{\Phi}(\Phi(A)\|\Phi(B)).
\]

However, (5.11) tells that
\[
A^t B^{-it} \neq \Phi^* \left( (\Phi(A)^t \Phi(B)^{-it}) \right)
\]
for some $t \in \mathbb{R}$,

and hence (viii) in Theorem 5.1 is not satisfied. Since $\Phi$ is 2-positive (actually, completely positive), it means that none of (i) (x) of Theorem 5.1 are satisfied.

5.5 Remark. It was shown in [8] that, in the classical setting, preservation of an $f$-divergence by $\Phi$ is equivalent to the reversibility condition (x) of Theorem 5.1 whenever $f$ is strictly convex. We reformulate the classical case in our setting in Appendix A and use the condition for equality to give a necessary and sufficient condition for the equality in the operator H"older and inverse H"older inequalities.

5.6 Remark. The classical case suggests that the support condition (5.4) might be too restrictive in general. On the other hand, [24] provides an example where the $f$-divergence corresponding to a function $f$ with $|\text{supp } \mu_f| = 1$ is preserved and yet the reversibility property (x) of Theorem 5.1 fails to hold. This shows that the support condition (5.4) cannot be completely removed in general.

5.7 Remark. Theorem 5.1 holds also if we replace $\Phi$ and $\Psi$ with co-(sub)stochastic maps, and change conditions (vi) (viii) to the following:
\[
\text{(vi)} \quad B^0 \Phi^* (\Phi(A)^z \Phi(B)^{-z}) = B^{-z} A^z \quad \text{for all } z \in \mathbb{C}.
\]
\[
\text{(vii)} \quad B^0 \Phi^* (\Phi(A)^\alpha \Phi(B)^{-\alpha}) = B^{-\alpha} A^\alpha \quad \text{for some } \alpha \in (0, 2) \setminus \{1\}.
\]
\[
\text{(viii)} \quad B^0 \Phi^* (\Phi(A)^{it} \Phi(B)^{-it}) = B^{-it} A^{it} \quad \text{for all } t \in \mathbb{R}.
\]

In the proof of (v) (vi), let $u_t := \Phi(A)^{it} \Phi(B)^{-it}$ and $w_t := B^{-it} A^{it}$; then
\[
u^*_t u_t = \Phi(B)^{-it} \Phi(A)^0 \Phi(B)^{it}, \quad w_t \nu^*_t = B^{-it} A^0 B^{it}, \quad t \in \mathbb{R}.
\]

Using that $\Phi$ is a co-Schwarz contraction, we have $\Phi(u_t^* u_t) = \Phi(u_t) \Phi(u_t^*)$. From the multiplicative domain for a co-Schwarz contraction, we have $\Phi(Y u_t) = \Phi(u_t) \Phi(Y) = w_t \Phi^* (Y) B^0$ for all $Y \in A_2$ and $t \in \mathbb{R}$. The rest of the proof is as before with $Y = \Phi(B)^{-1/2} \Phi(A)$. The implication (x) (i) holds also if we assume $\Phi$ to be 2-copositive.
5.8 Remark. Note that the assumption that $\Phi$ is substochastic guarantees that $(\Phi_B^\ast)^\ast = \Phi_B$ is a Schwarz map, which is also subunital. However, as Example 3.6 shows, there exist subunital Schwarz maps that are not Schwarz contractions. Even more, it was shown in [21] that if $\Phi$ is not 2-positive then there exists a positive invertible $B$ such that $\Phi_B$ is not a Schwarz contraction. To circumvent this problem, we assumed that $\Phi$ is 2-positive in the proof of Theorem 5.1. Note on the other hand that the monotonicity inequality holds not only for substochastic maps but also for Schwarz decomposable maps, i.e., those maps that can be decomposed as a convex combination of a substochastic and a co-substochastic map; see Remark 4.8. Hence, the implication (i) $\implies$ (iii) might still hold even if $\Phi_B^\ast$ is not a substochastic map. It is easy to see that this is the case, for instance, if $\Phi$ is 2-decomposable, i.e., it is the convex combination of two trace non-increasing maps, one being 2-positive and the other a composition of a 2-positive map with a transposition. It is an open question whether the Schwarz decomposability of $\Phi$ implies that $\Phi_B^\ast$ is Schwarz decomposable for every positive semidefinite $B$.

6 Distinguishability measures related to binary state discrimination

Let $\mathcal{A} \subset B(\mathcal{H})$ be a $C^\ast$-algebra, where $\mathcal{H}$ is a finite-dimensional Hilbert space, and let $S(\mathcal{A})$ be the state space of $\mathcal{A}$, i.e., $S(\mathcal{A}) := \{A \in \mathcal{A}_+ : \text{Tr} A = 1\}$ is the set of density operators in $\mathcal{A}$.

6.1 Definition. For $A, B \in \mathcal{A}_+$, the Chernoff distance $C(A\|B)$ of $A$ and $B$ is defined as

$$C(A\|B) := \sup_{0 \leq \alpha < 1} \{(1 - \alpha) S_\alpha(A\|B)\} = -\min_{0 \leq \alpha \leq 1} \psi(\alpha|A\|B),$$

(6.1)

where $S_\alpha(A\|B)$ is the Rényi relative entropy defined in Example 2.7 and

$$\psi(\alpha|A\|B) := \log \text{Tr} A^\alpha B^{1-\alpha}, \quad \alpha \in \mathbb{R}.$$  

(6.2)

For every $r \in \mathbb{R}$, we define the Hoeffding distance $H_r(A\|B)$ of $A$ and $B$ as

$$H_r(A\|B) := \sup_{0 \leq \alpha < 1} S_\alpha(e^r A\|B) = \sup_{0 \leq \alpha < 1} \left\{ -\frac{\alpha r}{1 - \alpha} + S_\alpha(A\|B) \right\} = \sup_{0 \leq \alpha < 1} \frac{-\alpha r - \psi(\alpha|A\|B)}{1 - \alpha}.$$

(6.3)

6.2 Remark. Note that

$$H_r(A\|B) = \sup_{s \geq 0} \{-sr - \tilde{\psi}(s|A\|B)\},$$

(6.4)

where

$$\tilde{\psi}(s|A\|B) := (1 + s)\psi(s/(1 + s)|A\|B), \quad s \in [0, +\infty), \quad \tilde{\psi}(s|A\|B) := +\infty, \quad s < 0.$$

For simplicity, we will use the notation $\psi(\alpha) = \psi(\alpha|A\|B)$ and $\tilde{\psi}(s) := \tilde{\psi}(s|A\|B)$. Let $\psi^\ast(r) := \sup_{s \in \mathbb{R}} \{sr - \tilde{\psi}(s)\}$ be the polar function, or Legendre-Fenchel transform of $\tilde{\psi}$ [12]. By [6,4], $H_r(\rho|\sigma) = \psi^\ast(-r), \quad r \in \mathbb{R}$. It is easy to see (by computing its second derivative) that $\psi$ is convex, and hence so is $\tilde{\psi}$. Furthermore, $\psi'(s) = \psi(s/(1 + s)) + \psi'(s/(1 + s))/(1 + s), \quad s \in [0, +\infty).$
on a fixed number (say $n$), and $\partial^+ \tilde{\psi}(0) = \psi(0) + \psi'(0)$, where $\partial^+ \tilde{\psi}(0)$ is the right derivative of $\tilde{\psi}$ at 0. In particular, $\lim_{s \to +\infty} \psi'(s) = \psi(1)$. Hence,

$$H_r(A\|B) = \tilde{\psi}^*(-r) = \begin{cases} -\tilde{\psi}(0) = -\psi(0), & -r < \psi(0) + \psi'(0), \\ +\infty, & -r > \psi(1). \end{cases}$$

It is easy to see that

$$\psi(0) = -S_0(A\|B), \quad \text{and if } A^0 \geq B^0 \quad \text{then } \psi'(0) = -S(B\|A),$$

$$\psi(1) = -S_0(B\|A), \quad \text{and if } A^0 \leq B^0 \quad \text{then } \psi'(1) = S(A\|B).$$

Being a polar function, $\tilde{\psi}^*$ is convex, and hence so is the function $r \mapsto H_r(\rho\|\sigma)$. Moreover, $\tilde{\psi}$ is lower semicontinuous and thus the bipolar theorem (see, e.g., Proposition 4.1 in [12]) yields that $\tilde{\psi}$ is the polar function of its polar $\tilde{\psi}^*$. Hence, for every $s \in [0, +\infty)$, we have

$$(1 + s)\psi\left(\frac{s}{1 + s}\right) = \tilde{\psi}(s) = \sup_{r \in \mathbb{R}} \{sr - \tilde{\psi}^*(r)\} = \sup_{\psi(0) + \psi'(0) \leq -r \leq \psi(1)} \{-rs - \tilde{\psi}^*(-r)\}.$$

Replacing $s$ with $\alpha/(1 - \alpha)$, we finally get that for every $\alpha \in [0, 1),$

$$-S_{\alpha}(A\|B) = \frac{\psi'(\alpha)}{1 - \alpha} = \sup_{r \in \mathbb{R}} \left\{ -\frac{r\alpha}{1 - \alpha} - H_r(A\|B) \right\} = \sup_{-\psi(1) \leq r \leq -\psi(0) - \psi'(0)} \left\{ -\frac{r\alpha}{1 - \alpha} - H_r(A\|B) \right\}.$$  \hspace{1cm} (6.5)

That is, the Rényi $\alpha$-relative entropies with parameter $\alpha \in [0, 1]$ and the Hoeffding distances mutually determine each other.

If $\text{Tr} A \leq 1$ then $\psi(1) = \log \text{Tr} AB^0 \leq 0$, and hence the optimization is over non-negative values of $r$ in the last formula of (6.5). Thus, $\alpha \mapsto S_{\alpha}(A\|B)$ is monotonic increasing on $[0, 1)$ and hence

$$H_0(A\|B) = \lim_{\alpha \to 1} S_{\alpha}(A\|B) =: S_1(A\|B).$$

Note that $\tilde{\psi}^*$ is lower semicontinuous (see, e.g., Proposition 4.1 and Corollary 4.1 in [12]), and hence $\tilde{\psi}^*(0) \leq \liminf_{r \to 0^+} \tilde{\psi}^*(-r)$. On the other hand, it is obvious from the definition that $r \mapsto H_r(A\|B) = \tilde{\psi}^*(-r)$ is monotonic decreasing on $\mathbb{R}$, and hence we finally obtain

$$\lim_{r \to 0^+} H_r(A\|B) = \lim_{r \to 0^-} \tilde{\psi}^*(-r) = \tilde{\psi}^*(0) = H_0(A\|B) = S_1(A\|B). \hspace{1cm} (6.6)$$

Finally, it is easy to verify that

$$S_1(A\|B) = S(A\|B) \quad \text{if} \quad \text{Tr} A = 1. \hspace{1cm} (6.7)$$

The importance of the above measures comes from the problem of binary state discrimination, that we briefly describe below. Assume that we have several identical copies of a quantum system, and we know that either all of them are in a state described by a density operator $\rho$, or all of them are in a state described by a density operator $\sigma$. We assume that the system’s Hilbert space $\mathcal{H}$ is finite-dimensional. Our goal is to give a good guess on the true state of the system, based on the outcome of a binary POVM measurement $(T, I - T)$ on a fixed number (say $n$) copies, where $T$ is an operator on $\mathcal{H}^\otimes n$ satisfying $0 \leq T \leq I$. If the outcome corresponding to $T$ happens then we conclude that the state of the system is $\rho$, and
an error occurs if the true state is $\sigma$, which has probability $\beta_n(T) := \text{Tr} \sigma^{\otimes n} T$. Similarly, the outcome corresponding to $I - T$ yields the guess $\sigma$ for the true state, and the probability of error in this case is $\alpha_n(T) := \text{Tr} \rho^{\otimes n} (I - T)$. If, moreover, there are prior probabilities $p$ and $1 - p$ assigned to $\rho$ and $\sigma$, then the optimal Bayesian error probability is given by

$$P_{n,p} := \min_{0 \leq T \leq I} \{ p\alpha_n(T) + (1 - p)\beta_n(T) \} = (1 - \|p\rho^{\otimes n} - (1 - p)\sigma^{\otimes n}\|)/2,$$

where the minimum is reached at $T = \{p\rho^{\otimes n} - (1 - p)\sigma^{\otimes n} > 0\}$, the spectral projection corresponding to the positive part of the spectrum of $p\rho^{\otimes n} - (1 - p)\sigma^{\otimes n}$. For every $p \in (0, 1)$, let

$$T_p (\rho^{\otimes n} \| \sigma^{\otimes n}) := \begin{cases} - \log \frac{1}{2p} (1 - \|p\rho^{\otimes n} - (1 - p)\sigma^{\otimes n}\|_1) = - \log \frac{1}{p} P_{n,p}, & 0 < p \leq 1/2, \\ - \log \frac{1}{2(1-p)} (1 - \|p\rho^{\otimes n} - (1 - p)\sigma^{\otimes n}\|_1) = - \log \frac{1}{1-p} P_{n,p}, & 1/2 < p < 1. \end{cases}$$

The theorem for the quantum Chernoff bound \cite{31,37} says that, as the number of copies $n$ tends to infinity, the error probabilities $P_{n,p}$ decay exponentially, and the rate of the decay is given by the Chernoff distance. More formally,

$$- \lim_{n \to \infty} (1/n) \log P_{n,p} = \lim_{n \to \infty} (1/n) T_p (\rho^{\otimes n} \| \sigma^{\otimes n}) = C(\rho \| \sigma), \quad p \in (0, 1).$$

In the asymmetric setting of the quantum Hoeffding bound, the error probabilities $\alpha_n$ are required to be exponentially small, and $\beta_n$ is optimized under this constraint, i.e., one is interested in the quantities

$$\beta_{n,r} := \min \{ \beta_n(T) : \alpha_n(T) \leq e^{-nr}, T \in \mathcal{B}(\mathcal{H}^{\otimes n}), 0 \leq T \leq I\},$$

where $r$ is some fixed positive number. The theorem for the quantum Hoeffding bound \cite{15,36} says that, for every $r > 0$, the error probabilities $\beta_{n,r}$ decay exponentially fast as $n$ goes to infinity, and the decay rate is given by the Hoeffding distance with parameter $r$. Moreover, if $\text{supp } \rho \leq \text{supp } \sigma$, then for every $r > 0$ we have a real number $a_r$ such that $\beta_{n,r}$ is given by

$$- \lim_{n \to \infty} (1/n) \log \beta_{n,r} = \lim_{n \to \infty} (1/n) T_{-nar} (\rho^{\otimes n} \| \sigma^{\otimes n}) = H_r(\rho \| \sigma).$$

Note that for density operators $\rho$ and $\sigma$, $\psi(\alpha | \rho \| \sigma) = \log \text{Tr} \rho^\alpha \sigma^{1-\alpha} \leq 0$ for every $\alpha \in [0, 1]$ due to H"older’s inequality \cite{38}. Hence, $C(\rho \| \sigma) \geq 0$, and $C(\rho \| \sigma) = 0$ if and only if equality holds in H"older’s inequality, which is equivalent to $\rho = \sigma$. Similarly, $H_r(\rho \| \sigma) \geq 0$ for every $r \in \mathbb{R}$, and $H_r(\rho \| \sigma) = 0$ if and only if $\rho = \sigma$, or $\text{supp } \rho \leq \text{supp } \sigma$ and $r \geq S(\sigma \| \rho)$.

**6.3 Proposition.** Let $A, B \in \mathcal{A}_{1,+}$ and let $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a substochastic map such that $\text{Tr} \Phi(B) = \text{Tr} B$. Then

$$C(\Phi(A) \| \Phi(B)) \leq C(A \| B) \quad \text{and} \quad H_r(\Phi(A) \| \Phi(B)) \leq H_r(A \| B), \quad r \in \mathbb{R}. \quad (6.11)$$

If there exists a substochastic map $\Psi : \mathcal{A}_2 \to \mathcal{A}_1$ such that $\Psi(\Phi(A)) = A$ and $\Psi(\Phi(B)) = B$ then the inequalities in (6.11) hold with equality.

**Proof.** By Example 6.5, $S_\alpha(\Phi(A) \| \Phi(B)) \leq S_\alpha(A \| B)$ for every $\alpha \in [0, 1)$, and equality holds for every $\alpha \in [0, 1)$ if there exists a substochastic map $\Psi : \mathcal{A}_2 \to \mathcal{A}_1$ such that $\Psi(\Phi(A)) = A$ and $\Psi(\Phi(B)) = B$, due to Theorem 6.1. The assertion then follows immediately from the definitions (6.1) and (6.3). \hfill $\square$
Our goal now is to give the converse of the above proposition, i.e., to show that equality in the inequalities of (6.11) yields the existence of a substochastic map $Ψ : A_2 → A_1$ such that $Ψ(Φ(A)) = A$ and $Ψ(Φ(B)) = B$. This would be immediate from Theorem 5.1 if the Chernoff and the Hoeffding distances could be represented as $f$-divergences (at least when $Φ$ is also assumed to be 2-positive). However, no such representation is possible, as is shown in the following proposition:

6.4 Proposition. The Chernoff and the Hoeffding distances cannot be represented as $f$-divergences on the state space of any non-trivial finite-dimensional $C^*$-algebra.

Proof. Let $A ⊂ B(H)$ where $\text{dim } H ≥ 2$, and let $e_1, e_2$ be orthonormal vectors in $H$ such that $|e_j⟩⟨e_j| ∈ A, j = 1, 2$. Define $ρ := |e_1⟩⟨e_1|, σ_p := p|e_1⟩⟨e_1| + (1 - p)|e_2⟩⟨e_2|, p ∈ (0, 1)$. One can easily check that $C(ρ∥σ_p) = H_r(ρ∥σ_p) = −log p$ for every $r > 0$, while $S_f(ρ∥σ_p) = pf(1/p) + (1 - p)f(0)$ for any function $f$ on $[0, +∞)$. Hence, if any of the above measures can be represented as an $f$-divergence, then we have $pf(1/p) + (1 - p)f(0) = −log p$ for the representing function $f$, and taking the limit $p \searrow 0$ yields $ω(f) = +∞$. In particular, $S_f(σ_p∥ρ) = +∞$ for every $p ∈ (0, 1)$. On the other hand, $C(σ_p∥ρ) = −log p$ and $H_r(σ_p∥ρ) = 0$ if $r ≥ −log p$. That is, $C(σ_p∥ρ)$ is finite for every $p ∈ (0, 1)$ and for every $r > 0$ there exists a $p ∈ (0, 1)$ such that $H_r(σ_p∥ρ)$ is finite.

Note, however, that for the applications of Theorems 4.3 and 5.1, it is sufficient to have a more general representability. Indeed, let $A$ be a finite-dimensional $C^*$-algebra and $D : S(A) × S(A) → \mathbb{R}$. We say that $D$ is a monotone function of an $f$-divergence on the state space of $A$ if there exists an operator convex function $f : [0, +∞) → \mathbb{R}$ and a strictly monotonic increasing function $g : \{S_f(ρ∥σ) : ρ, σ ∈ S(A)\} → \mathbb{R} \cup \{±∞\}$ such that $D(ρ∥σ) = g(S_f(ρ∥σ)), \quad ρ, σ ∈ S(A)$.

Obviously, if $D$ is a monotone function of an $f$-divergence then it is monotonic non-increasing under stochastic maps due to Theorem 4.3. Moreover, if $D(Φ(ρ)∥Φ(σ)) = D(ρ∥σ)$ for some stochastic map $Φ$ and $ρ, σ ∈ S(A)$ such that $\text{supp } Φ_ρ ≤ \text{supp } σ$, and the representing function $f$ satisfies $|\text{supp } μ_f| ≥ |\text{spec } L_ρ R_{φ-1} ∪ \text{spec } L_Φ R_{Φ(φ)-1}|$ then $Φ_ρ^*(Φ(ρ)) = ρ$, due to (iv) of Theorem 5.1. For instance, the Rényi $α$-relative entropy is a monotone function of the $f_α$-divergence with $g(x) := \frac{1}{1-α} log \text{sgn}(α - 1)x$, for every $α ∈ [0, 2] \setminus \{1\}$. However, the same argument as in Proposition 6.4 yields that none of the Rényi relative entropies with parameter $α ∈ (0, 1)$ can be represented as $f$-divergences.

6.5 Proposition. For any $r ∈ (0, +∞)$ and any non-trivial $C^*$-algebra $A$, the Hoeffding distance $H_r$ cannot be represented on the state space of $A$ as a monotone function of an $f$-divergence with an operator convex function $f$ on $[0, +∞)$ such that $|\text{supp } μ_f| ≥ 6$.

Proof. Let $A ⊂ B(H)$ be a $C^*$-algebra and let $e_1, e_2$ be orthogonal vectors in $H$ such that $|e_1⟩⟨e_1|, |e_2⟩⟨e_2| ∈ A$. Choose $p, q ∈ (0, 1)$ such that $p ≠ q$ and $q log \frac{2}{p} + (1 - q) log \frac{1}{1-p} < r$, and define $ρ := p|e_1⟩⟨e_1| + (1 - p)|e_2⟩⟨e_2| and σ := q|e_1⟩⟨e_1| + (1 - q)|e_2⟩⟨e_2|$. Then $ψ(0|ρ∥σ) = 0$ and $ψ(0|σ∥ρ) = S(σ∥ρ) = q log \frac{2}{p} + (1 - q) log \frac{1}{1-p} < r$, and hence $H_r(ρ∥σ) = −ψ(0|ρ∥σ) = 0$. Define $Φ : A → A, Φ(X) := (Tr X)I/(dim H)$. Then $Φ$ is completely positive and trace-preserving, $Φ(ρ) = Φ(σ)$, and hence $H_r(Φ(ρ)∥Φ(σ)) = 0 = H_r(ρ∥σ)$. Note that $|\text{spec } L_ρ R_{φ-1}| ≤ 5$ and $|\text{spec } L_Φ R_{Φ(φ)-1}| = 1$. If we had $H_r(ρ∥σ) = g(S_f(ρ∥σ))$ and $H_r(Φ(ρ)∥Φ(σ)) = g(S_f(Φ(ρ)∥Φ(σ)))$ for some strictly monotone $g$ and an operator convex function $f$ on $[0, +∞)$ such that $|\text{supp } μ_f| ≥ 6$ then Theorem 5.1 would yield $Φ_ρ^*(Φ(ρ)) = ρ$. However, $Φ(ρ) = Φ(σ)$ and hence $Φ_ρ^*(Φ(ρ)) = Φ_ρ^*(Φ(σ)) = σ ≠ ρ$. ∎
The above proposition also shows that the preservation of a Hoeffding distance of a pair \((\rho, \sigma)\) by a stochastic map for a given parameter \(r\) might not be sufficient for the reversibility of \(\Phi\) on \(\{\rho, \sigma\}\) in the sense of Theorem 5.1, the reason for this in the above proof is that the Hoeffding distance might be equal to zero even for non-equal states. The Chernoff distance, on the other hand, is always strictly positive for unequal states; yet the following example shows that the preservation of the Chernoff distance is not sufficient for reversibility in general, either.

6.6 Example. Let \(\mathcal{H} := \mathbb{C}^3\) and let \(\mathcal{A}\) be the commutative C*-algebra of operators on \(\mathcal{H}\) that are diagonal in some fixed basis \(e_1, e_2, e_3\). Let \(\rho := (2/3)|e_1\rangle\langle e_1| + (1/3)|e_2\rangle\langle e_2|\), \(\sigma := (1/6)|e_1\rangle\langle e_1| + (1/3)|e_2\rangle\langle e_2| + (1/2)|e_3\rangle\langle e_3|\), and define \(\Phi : \mathcal{A} \to \mathcal{A}\) as

\[
\Phi(|e_1\rangle\langle e_1|) := \Phi(|e_2\rangle\langle e_2|) := |e_1\rangle\langle e_1|, \quad \Phi(|e_3\rangle\langle e_3|) := |e_3\rangle\langle e_3|.
\]

Then \(\Phi\) is completely positive and trace-preserving, and we have \(\Phi(\rho) = |e_1\rangle\langle e_1|\), \(\Phi(\sigma) = (1/2)|e_1\rangle\langle e_1| + (1/2)|e_3\rangle\langle e_3|\). For every \(\alpha \in \mathbb{R}\), we have \(\text{Tr} \rho^\alpha \sigma^{1-\alpha} = 2^{1+4\alpha} = 2^{\alpha - 1}\), and hence

\[
C(\Phi(\rho)\|\Phi(\sigma)) = -\log \psi(0|\Phi(\rho)\|\Phi(\sigma)) = S_0(\Phi(\rho)\|\Phi(\sigma)) = \log 2 = S_0(\rho\|\sigma)
\]

on the other hand, it is easy to see that \(\Phi^*\) of Theorem 5.1 does not hold, and hence \(\Phi\) is not reversible on the pair \(\{\rho, \sigma\}\).

6.7 Remark. Note that in the setting of Theorem 5.1, if \(\Phi\) is 2-positive and \(S_\alpha(\Phi(A)\|\Phi(B)) = S_\alpha(A\|B)\) for some \(\alpha \in (0,1)\) then \(\Phi^*_2(\Phi(A)) = A\), i.e., the preservation of a Rényi \(\alpha\)-relative entropy with some \(\alpha \in (0,1)\) is sufficient for the reversibility of \(\Phi\) on \(\{A, B\}\). The above example shows that the same is not true for the 0-relative entropy.

6.8 Corollary. Let \(\mathcal{A}\) be a C*-algebra which contains at least 3 orthogonal non-zero projections. Then the Chernoff distance cannot be represented on its state space as a monotone function of an \(f\)-divergence with an operator convex \(f\) on \([0, +\infty)\) such that \(|\text{supp } \mu_f| \geq 6\).

Proof. Immediate from Example 6.6.

After the above preparation, we are ready to prove the analogue of Theorem 5.1 for the preservation of the Chernoff and the Hoeffding distances. The preservation of the Chernoff distance was already treated in the proof of Theorem 6 in [23] in the case where both operators are invertible density operators and the substochastic map is the trace-preserving conditional expectation onto a subalgebra. We use essentially the same proof to treat the general case below.

6.9 Theorem. Let \(A, B \in \mathcal{A}_{1,+}\) be such that \(\text{supp } A \leq \text{supp } B\), let \(\Phi : \mathcal{A}_1 \to \mathcal{A}_2\) be a substochastic map such that \(\text{Tr} \Phi(B) = \text{Tr} B\), and assume that \((i)\) or \((ii)\) below holds:

(i) \(C(\Phi(A)\|\Phi(B)) \neq S_0(\Phi(A)\|\Phi(B))\), \(C(\Phi(A)\|\Phi(B)) \neq S_0(\Phi(B)\|\Phi(A))\), and

\[
C(\Phi(A)\|\Phi(B)) = C(A\|B).
\]

(ii) For some \(r \in (-\psi(1|\Phi(A)\|\Phi(B)), -\psi(0|\Phi(A)\|\Phi(B)) - \psi'(0|\Phi(A)\|\Phi(B))\),

\[
H_r(\Phi(A)\|\Phi(B)) = H_r(A\|B).
\]

(6.12)
Then \( \Phi_B^*(\Phi(A)) = A \), and if \( \Phi \) is 2-positive then there exists a stochastic map \( \Psi : A_2 \to A_1 \) such that \( \Psi(\Phi(A)) = A \) and \( \Psi(\Phi(B)) = B \).

**Proof.** Assume first that (i) holds. Due to the assumptions \( C(\Phi(A)||\Phi(B)) \neq S_0(\Phi(A)||\Phi(B)) = -\psi(0|\Phi(A)||\Phi(B)), \) \( C(\Phi(A)||\Phi(B)) \neq S_0(\Phi(B)||\Phi(A)) = -\psi(1|\Phi(A)||\Phi(B)) \), and the definition (6.1) of the Chernoff distance, there exists an \( \alpha^* \in (0, 1) \) such that \( C(\Phi(A)||\Phi(B)) = -\psi(\alpha^*|\Phi(A)||\Phi(B)) \). Using the monotonicity relation (4.14), we get

\[
C(\Phi(A)||\Phi(B)) = -\log \text{Tr} \Phi(A)^{\alpha^*} \Phi(B)^{1-\alpha^*} \leq -\log \text{Tr} A^{\alpha^*} B^{1-\alpha^*} \leq C(A||B) = C(\Phi(A)||\Phi(B)).
\]

Hence, \( \text{Tr} \Phi(A)^{\alpha^*} \Phi(B)^{1-\alpha^*} = \text{Tr} A^{\alpha^*} B^{1-\alpha^*} \), which yields \( \Phi_B^*(\Phi(A)) = A \) due to (iv) of Theorem 5.1.

Assume next that (6.12) holds for some \( r \in (-\psi(1|\Phi(A)||\Phi(B)), -\psi(0|\Phi(A)||\Phi(B)) - \psi'(0|\Phi(A)||\Phi(B)) \). Then there exists an \( s^* \in (0, +\infty) \) such that \( H_r(\Phi(A)||\Phi(B)) = -s^*r - \psi(s^*|\Phi(A)||\Phi(B)) \) (see Remark 6.2). Thus, \( H_r(\Phi(A)||\Phi(B)) = -\alpha^* r/(1-\alpha^*) + S_{\alpha^*}(\Phi(A)||\Phi(B)) \), where \( \alpha^* := \frac{r}{1-s^*} \in (0, 1) \). Using the monotonicity (4.21), we obtain

\[
H_r(\Phi(A)||\Phi(B)) = -\alpha^* r/(1-\alpha^*) + S_{\alpha^*}(\Phi(A)||\Phi(B)) \leq -\alpha^* r/(1-\alpha^*) + S_{\alpha^*}(A||B) \leq H_r(A||B) = H_r(\Phi(A)||\Phi(B)).
\]

Hence, \( \text{Tr} \Phi(A)^{\alpha^*} \Phi(B)^{1-\alpha^*} = \text{Tr} A^{\alpha^*} B^{1-\alpha^*} \), which yields \( \Phi_B^*(\Phi(A)) = A \) due to (iv) of Theorem 5.1.

Finally, if \( \Phi \) is 2-positive then \( \Phi_B^*(\Phi(A)) = A \) yields the existence of \( \Psi \) in the last assertion the same way as in the proof of (x) \( \Rightarrow \) (i) in Theorem 5.1.

**6.10 Corollary.** Assume in the setting of Theorem 6.9 that \( \text{supp} A = \text{supp} B \) and \( \text{Tr} A = \text{Tr} B \). If \( C(\Phi(A)||\Phi(B)) = C(A||B) \) then \( \Phi_B^*(\Phi(A)) = A \).

**Proof.** Let \( \psi(\alpha) := \psi(\alpha|\Phi(A)||\Phi(B)) \), \( \alpha \in \mathbb{R} \). By the assumptions, we have \( \text{supp} \Phi(A) = \text{supp} \Phi(B) \) and \( \text{Tr} \Phi(A) = \text{Tr} \Phi(B) \), and hence \( \psi(0) = \psi(1) \). Since \( \psi \) is convex, there are two possibilities: either \( \psi \) is constant, or the minimum of \( \psi \) on \([0, 1]\) is attained at some \( \alpha^* \in (0, 1) \). In the latter case we have \( C(\Phi(A)||\Phi(B)) \neq S_0(\Phi(A)||\Phi(B)), \) \( C(\Phi(A)||\Phi(B)) \neq S_0(\Phi(B)||\Phi(A)) \), and hence the assertion follows due to Theorem 6.9. If \( \psi \) is constant then we have \( \text{Tr} \Phi(A)^{\alpha} \Phi(B)^{1-\alpha} = e^{\psi(\alpha)} = e^{\psi(1)} = \text{Tr} \Phi(A) = (\text{Tr} \Phi(A)^{\alpha} (\text{Tr} \Phi(B))^{1-\alpha}) \) for every \( \alpha \in [0, 1] \), and the equality case in Hölder’s inequality yields that \( \Phi(A) \) is constant multiple of \( \Phi(B) \) (see Corollary 3.3). Since \( \text{Tr} \Phi(A) = \text{Tr} \Phi(B) \), this yields that \( \Phi(A) = \Phi(B) \). Similarly,

\[
- \min_{0 \leq \alpha \leq 1} \psi(\alpha|A||B) = C(A||B) = C(\Phi(A)||\Phi(B)) = -\log \text{Tr} \Phi(A) = -\log \text{Tr} A = -\psi(0|A||B),
\]

and since \( \text{Tr} A = \text{Tr} B \), we also have \( -\log \text{Tr} A = -\log \text{Tr} B = -\psi(1|A||B) \). Hence, \( \alpha \mapsto \psi(\alpha|A||B) \) is constant on \([0, 1]\), and the same argument as above yields that \( A = B \). Therefore, \( \Phi_B^*(\Phi(A)) = \Phi_B^*(\Phi(B)) = B = A \).

**6.11 Remark.** Note that the interval \((-\psi(1|\Phi(A)||\Phi(B)), -\psi(0|\Phi(A)||\Phi(B)) - \psi'(0|\Phi(A)||\Phi(B)) \) in (ii) of Theorem 6.9 might be empty; this happens if and only if \( \alpha \mapsto \psi(\alpha|\Phi(A)||\Phi(B)) \) is constant. A characterization of this situation was given in Lemma 3.2 of [22].
7 Error correction

Noise in quantum mechanics is usually modeled by completely positive trace non-increasing maps. The aim of error correction is, given a noise operation Φ, to identify a subset C of the state space (called the code) and a quantum operation Ψ such that it reverses the action of the noise on the code, i.e., Ψ(Φ(ρ)) = ρ, ρ ∈ C. It was first noticed in [43] that the preservation of certain distinguishability measures of two states by the noise operation is a sufficient condition for correctability of the noise on those two states. This result was later extended to general families of states in [25, 26]. The measures considered in these papers were the Rényi relative entropies and the standard relative entropy. Recently, the same problem was considered in [6] using the measures \( T_p \) given in (6.8), and similar results were found, although only under some extra technical conditions. Below we summarize these results and extend them to a wide class of measures, based on Theorem 5.1.

Let \( A_i \) be a \( C^* \)-algebra on \( H_i \) for \( i = 1, 2 \), and let \( S(A_i) \) denote the set of density operators in \( A_i \). For a non-empty set \( C \subset S(A_1) \), let \( \overline{\text{co}}C \) denote the closed convex hull of \( C \), and let \( \text{supp} \) be the supremum of the supports of all states in \( C \). Note that there exists a state \( σ \in \overline{\text{co}}C \) such that \( \text{supp} σ = \text{supp} C \). We introduce the notation

\[ d_2 := (\dim H_1)^2 + (\dim H_2)^2. \]

Note that if \( X \in A_1 \) and \( Φ : A_1 \rightarrow A_2 \) is a trace non-increasing positive map then

\[ \| Φ(X) \|_1 = \max \{ \text{Tr} Φ(X)S : S ∈ A_2 \text{ self-adjoint}, -I_2 ≤ S ≤ I_2 \} \]
\[ = \max \{ \text{Tr} XΦ^*(S) : S ∈ A_2 \text{ self-adjoint}, -I_2 ≤ S ≤ I_2 \} \]
\[ ≤ \max \{ \text{Tr} XR : R ∈ A_1 \text{ self-adjoint}, -I_1 ≤ R ≤ I_1 \} = \| X \|_1, \]

which in particular yields that the measures \( T_p \) are monotonic non-increasing under substochastic maps.

7.1 Theorem. Let \( Φ : A_1 \rightarrow A_2 \) be a trace-preserving 2-positive map, and let \( C \subset S(A_1) \) be a non-empty set of states. The following are equivalent:

(i) There exists a stochastic map \( Ψ : A_2 \rightarrow A_1 \) such that for every \( ρ ∈ \overline{\text{co}}C \),

\[ Ψ(Φ(ρ)) = ρ. \] (7.1)

(ii) For every operator convex function \( f \) on \([0, +∞)\), and every \( ρ, σ ∈ \overline{\text{co}}C \),

\[ S_f(Φ(ρ)||Φ(σ)) = S_f(ρ||σ). \] (7.2)

(iii) The equality (7.2) holds for every \( ρ ∈ C \) and for some \( σ ∈ S(A_1) \) such that \( \text{supp} σ ≥ \text{supp} C \), and some operator convex function \( f \) on \([0, +∞)\) such that \( |\text{supp} μ_f| ≥ d_2 \).

(iv) \( S_{φ_t}(Φ(ρ)||Φ(σ)) = S_{φ_t}(ρ||σ) \) for every \( ρ ∈ C \) and for some \( σ ∈ S(A_1) \) such that \( \text{supp} σ ≥ \text{supp} C \), and a set \( T \) of \( t \)'s such that \( |T| ≥ d_2 \).

(v) For every \( ρ, σ ∈ \overline{\text{co}}C \) and every \( r ∈ \mathbb{R} \),

\[ H_r(Φ(ρ)||Φ(σ)) = H_r(ρ||σ). \] (7.3)

(vi) The equality in (7.3) holds for every \( ρ ∈ C \) and for some \( σ ∈ S(A_1) \) such that \( \text{supp} σ ≥ \text{supp} C \), and for every \( r ∈ (0, δ) \) for some \( δ > 0 \).
(vii) For every \( \rho \in \mathcal{C} \) and every \( \sigma \in \mathcal{C} \) such that \( \text{supp} \sigma = \text{supp} \rho \),
\[
\Phi_\rho^*(\Phi(\rho)) = \rho. \tag{7.4}
\]

(viii) The equality (7.4) holds for every \( \rho \in \mathcal{C} \) and some \( \sigma \in \mathcal{S}(A_1) \).

(ix) There exist decompositions \( \text{supp} \mathcal{C} = \bigoplus_{k=1}^r \mathcal{H}_{1,k,L} \otimes \mathcal{H}_{1,k,R} \) and \( \text{supp} \Phi(\mathcal{C}) = \bigoplus_{k=1}^r \mathcal{H}_{2,k,L} \otimes \mathcal{H}_{2,k,R} \), invertible density operators \( \omega_k \) on \( \mathcal{H}_{1,k,R} \) and \( \hat{\omega}_k \) on \( \mathcal{H}_{2,k,R} \), and unitaries \( U_k : \mathcal{H}_{1,k,L} \to \mathcal{H}_{2,k,L} \), \( k = 1, \ldots, r \), such that every \( \rho \in \mathcal{C} \) can be written in the form
\[
\rho = \sum_{k=1}^r p_k \rho_{k,L} \otimes \omega_k
\]
with some density operators \( \rho_{k,L} \) on \( \mathcal{H}_{1,k,L} \) and probability distribution \( \{p_k\}_{k=1}^r \), and
\[
\Phi(\mathcal{A} \otimes \omega_k) = U_k A U_k^* \otimes \hat{\omega}_k, \quad A \in \mathcal{B}(\mathcal{H}_{1,k,L}).
\]
Moreover, if \( \Phi \) is n-positive/completely positive then \( \Psi \) in (i) can also be chosen to be n-positive/completely positive. The implications (i)=⇒(ii)=⇒(iii)=⇒(iv)=⇒(viii) hold also if we only assume \( \Phi \) to be substochastic.

Furthermore, criterion (x) below is sufficient for (i)=(viii) to hold, and it is also necessary if \( \Phi \) is completely positive.

(x) For every \( \rho \in \mathcal{C} \), every \( p \in (0,1) \), every \( n \in \mathbb{N} \), and for some \( \sigma \in \mathcal{S}(A_1) \) such that \( \text{supp} \sigma \geq \text{supp} \mathcal{C} \),
\[
T_p(\Phi^\otimes n(\rho^\otimes n) \mid \Phi^\otimes n(\sigma^\otimes n)) = T_p(\rho^\otimes n \mid \sigma^\otimes n). \tag{7.5}
\]

**Proof.** The implications (i)=⇒(ii)=⇒(iii)=⇒(iv)=⇒(viii) follow immediately from Theorem 5.1 under the condition that \( \Phi \) is substochastic (note that in the implication (iii)=⇒(iv) \( T \) can be chosen to be \( \mu_f \), and hence it is independent of the pair \( (\rho, \sigma) \)). If (viii) holds then \( \rho = \Phi_\rho^*(\Phi(\rho)) = \sigma^{1/2} \Phi^\ast(\Phi(\sigma))^{-1/2} \Phi(\rho) (\Phi(\sigma))^{-1/2} \sigma^{1/2} \) implies that \( \supp \rho \leq \supp \sigma \) for every \( \rho \in \mathcal{C} \), and hence \( \Phi_\rho^* \) can be completed to a map \( \Psi \) as required in (i) the same way as in the proof of (x)=⇒(i) in Theorem 5.1. This proves (viii)=⇒(i). Assume that (i) holds. Fixing any \( \rho \in \mathcal{C} \) and \( \sigma \in \mathcal{C} \) such that \( \supp \sigma = \supp \mathcal{C} \), we have \( \Phi(\Phi(\rho)) = \rho \) and \( \Phi(\Phi(\sigma)) = \sigma \), and Theorem 5.1 yields (7.4) for this pair \( (\rho, \sigma) \), proving (i)=⇒(viii). The implication (vii)=⇒(viii) is obvious.

The implication (i)=⇒(x) follows by Proposition 5.3 and the implication (v)=⇒(vi) is obvious. Assume now that (vi) holds. Then, by (6.6) and (6.7), we have \( S(\Phi(A)\|\Phi(B)) = S(A\|B) \), i.e., the equality holds for the standard relative entropy, which is the \( f \)-divergence corresponding to \( f(x) = x \log x \). Since the support of the representing measure for \( x \log x \) is \( (0, +\infty) \), this yields (iii). The implication (x)=⇒(vi) follows from (6.10). Assume that \( \Phi \) is completely positive and (i) holds. Then we can assume \( \Psi \) to be completely positive, and hence \( \Phi^\otimes n \) and \( \Psi^\otimes n \) are positive and trace-preserving for every \( n \in \mathbb{N} \). Thus, by the monotonicity of the measures \( T_p, T_p(\rho^\otimes n \mid \sigma^\otimes n) = T_p(\Psi^\otimes n(\Phi^\otimes n(\rho^\otimes n)) \mid \Psi^\otimes n(\Phi^\otimes n(\sigma^\otimes n))) \leq T_p(\Phi^\otimes n(\rho^\otimes n) \mid \Phi^\otimes n(\sigma^\otimes n)) \leq T_p(\rho^\otimes n \mid \sigma^\otimes n) \), and hence (x) holds.

Finally, (vii)=⇒(ix) follows due to Lemma 5.11 and (ix)=⇒(vii) is a matter of straightforward computation. \( \square \)
Briefly, the above theorem tells that if the noise doesn’t decrease some suitable measure of the pairwise distinguishability on a set of states then its action can be reversed on that set with some other quantum operation; moreover, the reversion operation can be constructed by using the noise operation and any state with maximal support. There are apparent differences between the conditions given above; indeed, (iii) tells that the preservation of one single \( f \)-divergence is sufficient, while (iv) requires the preservation of sufficiently (but finitely) many \( f \)-divergences, (v) requires the preservation of a continuum number of measures, and (x) requires even more. The equivalence between (iii) and (iv) is easy to understand; as we have seen in the proof of Theorem 5.1, as far as monotonicity and equality in the monotonicity are considered, any \( f \)-divergence with an operator convex \( f \) which is not a polynomial is equivalent to the collection of \( \varphi_t \)-divergences with \( t \in \text{supp} \mu_f \), and the condition on the cardinality of \( \text{supp} \mu_f \) is imposed so that any function on the joint spectrum of the relative modular operators can be decomposed as a linear combination of \( \varphi_t \)'s, which in turn is used to construct the inversion map \( \Phi^*_{\sigma} \). It is an open question how much the condition on the cardinality of \( \text{supp} f \) can be improved; cf. Remark 5.6.

Note that (iii) tells in particular that the preservation of the pairwise Rényi relative entropies for one single parameter value \( \alpha \in (0, 2) \) is sufficient for reversibility. This is in contrast with (vi), where the preservation of continuum many Hoeffding distances are required, despite the symmetry suggested by (6.3) and (6.5). On the other hand, we have the following:

**7.2 Proposition.** In the setting of Theorem 7.1, assume that there exists a \( C_0 \subset S(A_1) \) such that \( \overline{\text{co}} C_0 = \overline{\text{co}} C \), and a \( \sigma \in S(A_1) \) such that \( \text{supp} \sigma \geq \text{supp} C \), and the following hold:

\[
0 < m := \inf_{\rho \in C_0} \{ -\psi(0|\Phi(\rho)||\Phi(\sigma)) - \psi'(0|\Phi(\rho)||\Phi(\sigma)) \}
\]

and for some \( r \in (0, m) \),

\[
H_r(\Phi(\rho)||\Phi(\sigma)) = H_r(\rho||\sigma), \quad \rho \in C_0.
\]

Then \( \Phi^*_{\sigma}(\Phi(\rho)) = \rho \) for every \( \rho \in \overline{\text{co}} C \).

**Proof.** Immediate from Theorem 6.9. \( \square \)

Finally, if all the states in \( C \) have the same support then some of the conditions in Theorem 7.1 and Proposition 7.2 can be simplified, and we can give a simple condition in terms of preservation of the Chernoff distance:

**7.3 Proposition.** Let \( \Phi : A_1 \to A_2 \) be a trace-preserving 2-positive map and let \( C \subset S(A_1) \) be a non-empty set of states such that \( \text{supp} \rho = \text{supp} C \) for every \( \rho \in C \). Assume that there exists a \( \sigma \in S(A_1) \) such that \( \text{supp} \sigma = \text{supp} C \) and one of the following holds:

(i) There exists a \( \rho \in (0, 1) \) such that

\[
T_p(\Phi^\otimes n(\rho^\otimes n)||\Phi^\otimes n(\sigma^\otimes n)) = T_p(\rho^\otimes n||\sigma^\otimes n), \quad \rho \in C, \quad n \in \mathbb{N}. \tag{7.6}
\]

(ii) For every \( \rho \in C \),

\[
C(\Phi(\rho)||\Phi(\sigma)) = C(\rho||\sigma).
\]

(iii) There exists a \( C_0 \) such that \( \overline{\text{co}} C_0 = \overline{\text{co}} C \) and an \( r \in (0, \inf_{\rho \in C_0} S(\Phi(\sigma)||\Phi(\rho))) \) such that for every \( \rho \in C_0 \),

\[
H_r(\Phi(\rho)||\Phi(\sigma)) = H_r(\rho||\sigma). \tag{7.7}
\]
Then
\[ \Phi_\sigma^*(\Phi(\rho)) = \rho, \quad \rho \in \mathcal{C}. \] (7.8)

**Proof.** The implication \([i] \Rightarrow [ii]\) is immediate from (6.9), and \([ii]\) implies (7.8) due to Corollary 6.10. Assume now that \([iii]\) holds. Since \(\text{supp} \rho = \text{supp} \sigma, \rho \in \mathcal{C}_0\), we have \(\psi(0|\Phi(\rho)||\Phi(\sigma)) = 0\) and \(-\psi'(0|\Phi(\rho)||\Phi(\sigma)) = S(\Phi(\sigma)||\Phi(\rho)), \rho \in \mathcal{C}_0\). Hence, (7.7) yields (7.3) due to Proposition 7.2.

Note that the conditions (7.3) and (7.4) are very different from the others, as they require the preservation of some measure for arbitrary tensor powers. These conditions could be simplified if the trace-norm distance could be represented as an \(f\)-divergence. Note that this is possible in the classical case; indeed, if \(p\) and \(q\) are probability density functions on some finite set \(\mathcal{X}\), and \(f(x) := |x - 1|, x \in \mathbb{R}\), then
\[ S_f(p||q) = \sum_{x \in \mathcal{X}} q(x)|p(x)/q(x) - 1| = \sum_{x \in \mathcal{X}} |p(x) - q(x)| = \|p - q\|_1. \]

Note, however, that the above \(f\) is not operator convex, and hence the proof given in Theorem 5.1 wouldn’t work for it. Even worse, the trace-norm distance cannot be represented as an \(f\)-divergence, as we show below by a simple argument.

**7.4 Corollary.** If the observable algebra of a quantum system is non-commutative then the trace-norm distance on its state space cannot be represented as an \(f\)-divergence.

**Proof.** Assume that \(\mathcal{A} \subset \mathcal{B}(\mathcal{H})\) is non-commutative; then we can find orthonormal vectors \(e_1, e_2 \in \mathcal{H}\) such that \(|e_i\rangle\langle e_j| \in \mathcal{A}, i = 1, 2\). (For simplicity, we neglect possible higher multiplicities; taking them into account would only result in a constant multiplication factor in the formulas below.) Assume that the trace-norm distance can be represented as an \(f\)-divergence. Then, for every \(s \in [0, 1]\) and \(t \in (0, 1)\), when \(\rho := s|e_1\rangle\langle e_1| + (1 - s)|e_2\rangle\langle e_2|\) and \(\sigma := t|e_1\rangle\langle e_1| + (1 - t)|e_2\rangle\langle e_2|\), we have
\[ tf(s/t) + (1 - t)f((1 - s)/(1 - t)) = S_f(p||\sigma) = \|p - \sigma\|_1 = 2|s - t|. \]

Letting \(s = t\) gives \(f(1) = 0\). Letting \(t \searrow 0\) gives \(s\omega(f) + f(1 - s) = 2s\) for all \(s \in (0, 1]\). This implies that \(\omega(f)\) is finite and \(\omega(f) + f(0) = 2\). Now let \(\rho := |e_1\rangle\langle e_1|\) and \(\sigma := |\psi\rangle\langle \psi|,\) where \(\psi := (e_1 + e_2)/\sqrt{2}\). Then \(\|\rho - \sigma\|_1 = \sqrt{2}\), while by (2.6) one can easily compute
\[ S_f(\rho||\sigma) = \frac{1}{2}f(1) + \frac{1}{2}\omega(f) + \frac{1}{2}f(0) = \frac{1}{2}(\omega(f) + f(0)) = 1. \]

**7.5 Remark.** A similar argument as above can be used to show that for any \(p \in (0, 1)\), the measure \(D_p(\rho||\sigma) := 1 - \|p\rho - (1 - p)\sigma\|_1\) cannot be represented as an \(f\)-divergence on the state space of any non-commutative finite-dimensional \(C^*\)-algebra.

**7.6 Remark.** In general, a function on pairs of classical probability distributions might have several different extensions to quantum states. A function that can be represented as an \(f\)-divergence has an extension given by the corresponding quantum \(f\)-divergence. It is not clear whether this extension has any operational significance in the case of \(f(x) := |x - 1|\).

While the impossibility to represent the trace-norm distance as an \(f\)-divergence shows that the approach followed in Theorem 7.1 cannot be used to simplify the condition in [x] of the theorem, other approaches might lead to better results. Indeed, the results of the recent paper [6] can be reformulated in the following way:
7.7 Theorem. Let \( C \subset S(A_1) \) be a convex set of states and let \( \Phi : A_1 \to A_2 \) be a completely positive trace-preserving map such that
\[
T_p(\Phi(\rho) \mid \mid \Phi(\sigma)) = T_p(\rho \mid \mid \sigma), \quad p \in (0, 1).
\]
Then the fixed-point set of \( \Phi_p \circ \Phi \) is a \( C^* \)-subalgebra of \( PA_1P \), where \( P \) is the projection onto \( \text{supp} C \), and the trace-preserving conditional expectation \( \mathcal{P} \) from \( PA_1P \) onto \( \ker (\text{id} - \Phi_p^* \circ \Phi) \) is \( T_p \)-preserving for all \( p \in (0, 1) \). If, moreover, the restriction of \( \mathcal{P} \) onto \( C \) is surjective onto the state space of \( \ker (\text{id} - \Phi_p^* \circ \Phi) \) then (i)–(x) of Theorem 7.1 hold.

Note that the continuum many conditions requiring the preservation of \( T_p \) for all \( p \in (0, 1) \) in Theorem 7.7 can be simplified to a single condition, requiring that \( \Phi \) is trace-norm preserving on the real subspace generated by \( C \). Note also that the surjectivity condition is sufficient but obviously not necessary. It is, however, an open question whether it can be completely removed. In the approach followed in [6], it is important that one starts with a convex set of states. The same problem was studied in [23] in a different setting, and the following has been shown:

7.8 Theorem. Let \( \rho, \sigma \in S(A) \) be invertible density operators and \( \Phi \) be the trace-preserving conditional expectation onto a subalgebra \( A_0 \) of \( A \). Assume that
\[
T_p(\Phi(\rho) \mid \mid \Phi(\sigma)) = T_p(\rho \mid \mid \sigma), \quad p \in (0, 1),
\]
and \( A_0 \) is commutative or \( \rho \) and \( \sigma \) commute. Then \( \Phi^*_\sigma(\Phi(\rho)) = \rho \) and \( \Phi^*_\rho(\Phi(\sigma)) = \sigma \).

7.9 Remark. In [23] the condition \( T_p(\Phi(\rho) \mid \mid \Phi(\sigma)) = T_p(\rho \mid \mid \sigma), \quad p \in (0, 1) \), was called 2-sufficiency, and
\[
T_p(\Phi^{\otimes n}(\rho^{\otimes n}) \mid \mid \Phi^{\otimes n}(\sigma^{\otimes n})) = T_p(\rho^{\otimes n} \mid \mid \sigma^{\otimes n}), \quad p \in (0, 1), \quad n \in \mathbb{N},
\]
was called \((2, n)\)-sufficiency. It was also shown in Theorem 6 of [23] that in the setting of Theorem 7.8, (7.9) is sufficient for the conclusion of Theorem 7.8 to hold.

8 An integral representation for operator convex functions

Operator monotone and operator convex functions play an important role in quantum information theory [45]. Several ways are known to decompose them as integrals of some families of functions of simpler forms [4, 19]. Here we present a representation that is well-suited for our analysis of \( f \)-divergences, and seems to be a new result.

8.1 Theorem. A continuous real-valued function \( f \) on \([0, +\infty)\) is operator convex if and only if there exist a real number \( a \), a non-negative number \( b \), and a non-negative measure \( \mu \) on \((0, +\infty)\), satisfying
\[
\int_{(0, +\infty)} \frac{d\mu(t)}{(1 + t)^2} < +\infty,
\]
such that
\[
f(x) = f(0) + ax + bx^2 + \int_{(0, +\infty)} \left( \frac{x}{1 + t} - \frac{x}{x + t} \right) d\mu(t), \quad x \in [0, +\infty).
\]
Moreover, the numbers \( a, b \), and the measure \( \mu \) are uniquely determined by \( f \), and
\[
b = \lim_{x \to +\infty} \frac{f(x)}{x^2}, \quad a = f(1) - f(0) - b.
\]
Proof. Obviously, if $f$ admits an integral representation as in (8.2) then $f$ is operator convex, and
\[ f(1) = f(0) + a + b, \quad b = \lim_{x \to +\infty} \frac{f(x)}{x^2}, \]
where the latter follows by the Lebesgue dominated convergence theorem, using (8.1) and that, for $x > 1$,
\[ 0 \leq \frac{1}{x^2} \left( \frac{x}{1+t} - \frac{x}{x+t} \right) = \frac{x-1}{x(x+t)(1+t)} \leq \frac{2x}{x(1+t)(1+t)} = \frac{2}{(1+t)^2}. \]
Hence what is left to prove is that any operator convex function admits a representation as in (8.2), and that the measure $\mu$ is uniquely determined by $f$.

Assume now that $f$ is an operator convex function on $[0, +\infty)$. Then, by Kraus’ theorem (see [29] or Corollary 2.7.8 in [19]), the function
\[ g(x) := \frac{f(x) - f(1)}{x - 1}, \quad x \in [0, +\infty) \setminus \{1\}, \quad g(1) := f'(1), \]
is an operator monotone function on $(0, +\infty)$. Therefore, it admits an integral representation
\[ g(x) = a' + bx + \int_{(0, +\infty)} \frac{x(1+t)}{x+t} dm(t), \quad x \in [0, +\infty), \tag{8.3} \]
where $m$ is a positive finite measure on $(0, +\infty)$, and
\[ a' = g(0) = f(1) - f(0), \quad 0 \leq b = \lim_{x \to +\infty} \frac{g(x)}{x} = \lim_{x \to +\infty} \frac{f(x)}{x^2} \]
(see Theorem 2.7.11 in [19] or pp. 144–145 in [3]). Here, the measure $m$, as well as $a', b$, are unique and
\[ m((0, +\infty)) = g(1) - a' - b = f'(1) - f(1) + f(0) - b. \]
Thus, we have
\[ f(x) = f(1) + g(x)(x - 1) \]
\[ = f(1) + (f(1) - f(0))(x - 1) + bx(x - 1) + \int_{(0, +\infty)} \frac{x(x-1)(1+t)}{x+t} dm(t) \]
\[ = f(0) + (f(1) - f(0) - b)x + bx^2 + \int_{(0, +\infty)} \left( \frac{x}{1+t} - \frac{x}{x+t} \right)(1+t)^2 dm(t) \]
\[ = f(0) + ax + bx^2 + \int_{(0, +\infty)} \left( \frac{x}{1+t} - \frac{x}{x+t} \right) dm(t), \]
where we have defined $a := f(1) - f(0) - b$ and $d\mu(t) := (1+t)^2 dm(t)$. Finiteness of $m$ yields that $\mu$ satisfies (8.1).

Finally, to see the uniqueness of the measure $\mu$, assume that $f$ admits an integral representation as in (8.2). Then, $f$ is operator convex, and hence the function $g$ on $[0, +\infty)$, defined as
\[ g(x) := \frac{f(x) - f(1)}{x - 1} = (a + b) + bx + \int_{(0, +\infty)} \frac{x(1+t)}{x+t} d\mu(t) \quad (1+t)^2, \tag{8.4} \]
is operator monotone. Therefore, it admits an integral representation as in (8.3), and the uniqueness of the parameters of that representation yields that $d\mu(t) = (1+t)^2 dm(t)$. Hence, the measure $\mu$ is uniquely determined by $f$. □
8.2 Corollary. Assume that \( f \) is a continuous operator convex function on \([0, +\infty)\) that is not a polynomial. Then it can be written in the form

\[
\frac{f(x)}{x^2} = \frac{f(0)}{x^2} + \frac{f'(0)x - f(x)}{x^2}, \quad x \in [0, +\infty),
\]

where \( b = \lim_{x \to +\infty} \frac{f(x)}{x^2} \geq 0 \), and \( \mu \) is a non-negative measure on \((0, +\infty)\). Moreover, we can choose

\[
\psi(t) := \frac{1}{1+t} \cdot \frac{f(1) - f(0) - b}{f'(1) - f(1) + f(0) - b} \cdot \frac{1}{(1+t)^2},
\]

and if \( b = 0 \) and \( f'(1) \geq 0 \) then \( \psi(t) \geq 0, \ t \in (0, +\infty) \).

**Proof.** Since \( f \) is operator convex, it can be written in the form \((8.2)\) due to Theorem 8.1. Since \( f \) is not a polynomial, we have \( m((0, +\infty)) > 0 \), where \( dm(t) := d\mu(t)/(1+t^2) \). Moreover, by \((8.4)\), \( f'(1) = g(1) = a + 2b + m((0, +\infty)) \), from which \( m((0, +\infty)) = f'(1) - a - 2b \). Using that \( a = f(1) - f(0) - b \), we finally obtain

\[
a = \frac{a}{m((0, +\infty))} \int_{(0, +\infty)} d\mu(t) = \frac{f(1) - f(0) - b}{f'(1) - f(1) + f(0) - b} \int_{(0, +\infty)} \frac{1}{(1+t)^2} d\mu(t).
\]

Substituting it into \((8.2)\), we obtain \((8.5)\) with \( \psi \) as in \((8.6)\). Note that \((1+t)^2 \psi(t) = 1 + t + \frac{a}{m((0, +\infty))} \geq 1 + \frac{a}{m((0, +\infty))} \). Hence, if \( b = 0 \) and \( 0 \leq f'(1) = a + 2b + m((0, +\infty)) = a + m((0, +\infty)) \) then \( \psi(t) \geq 0 \), proving the last assertion.

8.3 Example.

(i) \( f(x) := x \log x \) admits the integral representation

\[
x \log x = \int_{(0, +\infty)} \left( \frac{x}{1+t} - \frac{x}{x+t} \right) dt.
\]

\( f(0) = a = b = 0 \) and \( \mu \) is the Lebesgue measure in \((8.2)\).

(ii) \( f(x) := -x^\alpha \) (\( 0 < \alpha < 1 \)) admits the integral representation (see [4] Exercise V.1.10)

\[
-x^\alpha = \frac{\sin \alpha \pi}{\pi} \int_{(0, +\infty)} \left( - \frac{x}{x+t} \right) t^{\alpha-1} dt.
\]

\( f(0) = b = 0 \), \( d\mu(t) = \frac{\sin \alpha \pi}{\pi} t^{\alpha-1} dt \), and \( \psi \equiv 0 \) in \((8.3)\). Using that \( \frac{\sin \alpha \pi}{\pi} \int_{(0, +\infty)} \frac{x^\alpha}{1+t} dt = x \), we have

\[
-x^\alpha = -x + \frac{\sin \alpha \pi}{\pi} \int_{(0, +\infty)} \left( \frac{x}{1+t} - \frac{x}{x+t} \right) t^{\alpha-1} dt.
\]

\( f(0) = b = 0 \), \( a = -1 \), and \( d\mu(t) = \frac{\sin \alpha \pi}{\pi} t^{\alpha-1} dt \) in \((8.2)\).

(iii) By the previous point, \( f(x) := x^\alpha \) (\( 1 < \alpha < 2 \)) admits the representation

\[
x^\alpha = \frac{\sin(\alpha - 1) \pi}{\pi} \int_{(0, +\infty)} \frac{x^2 t^{\alpha-2}}{x+t} dt = \frac{\sin(\alpha - 1) \pi}{\pi} \int_{(0, +\infty)} \left( \frac{x}{t} - \frac{x}{x+t} \right) t^{\alpha-1} dt
\]

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\( f(0) = b = 0, \psi(t) = 1/t, \) and \( d\mu(t) = \frac{\sin(\alpha - 1)\pi}{\pi} t^{\alpha - 1} dt \) in (8.2). Using that

\[
\frac{\sin(\alpha - 1)\pi}{\pi} \int_{(0, +\infty)} \left( \frac{x}{t} - \frac{x}{1 + t} \right) t^{\alpha - 1} dt = \frac{\sin(\alpha - 1)\pi}{\pi} \int_{(0, +\infty)} \frac{x t^{\alpha - 2}}{1 + t} dt = x,
\]

we also obtain

\[ x^\alpha = x + \frac{\sin(\alpha - 1)\pi}{\pi} \int_{(0, +\infty)} \left( \frac{x}{1 + t} - \frac{x}{x + t} \right) t^{\alpha - 1} dt. \]

\( f(0) = 0, a = 1, b = 0 \) and \( d\mu(t) = \frac{\sin(\alpha - 1)\pi}{\pi} t^{\alpha - 1} dt \) in (8.2).)

Note that the function \( \psi \) in (8.5) is not unique. For instance, if \( \mu \) is finitely supported on a set \( \{t_1, \ldots, t_r\} \) then only the sum \( \sum_{i=1}^r \psi(t_r) \) is determined by \( f \) while the individual values \( \psi(t_1), \ldots, \psi(t_r) \) are not.

Note also that in general, \( \int_{(0, +\infty)} \frac{1}{1 + t} d\mu(t) \) might not be finite and hence the term \( \int_{(0, +\infty)} \frac{x}{1 + t} d\mu(t) \) cannot be merged with \( ax \) in (8.2). Similarly, the integral \( \int_{(0, +\infty)} \psi(t) d\mu(t) \) might be infinite and hence it might not be possible to separate it as a linear term in the representation (8.5) of \( f \). This is clear, for instance, from (i) of Example 8.3. We have the following:

**8.4 Proposition.** For a continuous real-valued function \( f \) on \([0, +\infty)\) the following are equivalent:

(i) \( f \) is operator convex on \([0, +\infty)\) with \( \lim_{x \to +\infty} f(x)/x < +\infty \);

(ii) there exist an \( \alpha \in \mathbb{R} \) and a positive measure \( \mu \) on \((0, +\infty)\), satisfying

\[
\int_{(0, +\infty)} \frac{d\mu(t)}{1 + t} < +\infty,
\]

such that

\[
f(x) = f(0) + ax - \int_{(0, +\infty)} \frac{x}{x + t} d\mu(t), \quad x \in [0, +\infty).
\]

**Proof.** First, note that if \( f \) is convex on \([0, +\infty)\) as a numerical function, then \( \lim_{x \to +\infty} f(x)/x \) exists in \((-\infty, +\infty)\). In fact, by convexity, \( (f(x) - f(1))/(x - 1) \) is non-decreasing for \( x > 1 \), so that

\[
\lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \frac{f(x) - f(1)}{x - 1}
\]

exists in \((-\infty, +\infty)\). Also, note that condition (8.7) is necessary for \( f(1) \) to be defined in (8.8), and also sufficient to define \( f(x) \) by (8.8) for all \( x \in [0, +\infty) \).

(i) \( \Rightarrow \) (ii). By assumption, \( f \) is an operator convex function on \([0, +\infty)\) such that \( \lim_{x \to +\infty} f(x)/x \) is finite, hence \( \lim_{x \to +\infty} f(x)/x^2 = 0 \). By Theorem 8.1, we have

\[
f(x) = f(0) + ax + \int_{(0, +\infty)} \left( \frac{x}{1 + t} - \frac{x}{x + t} \right) d\mu(t), \quad x \in [0, +\infty),
\]

where \( a \in \mathbb{R} \) and \( \mu \) is a positive measure on \((0, +\infty)\). We write

\[
\frac{f(x)}{x} = \frac{f(0)}{x} + a + \int_{(0, +\infty)} \left( \frac{1}{1 + t} - \frac{1}{x + t} \right) d\mu(t).
\]
Since
\[ 0 < \frac{1}{1 + t} - \frac{1}{x + t} \xrightarrow{t \to +\infty} \frac{1}{1 + t}, \]
as \( x \to +\infty \), the monotone convergence theorem yields that
\[ \lim_{x \to +\infty} \frac{f(x)}{x} = a + \int_{(0, +\infty)} \frac{d\mu(t)}{1 + t}, \]
which implies (8.7) and
\[ f(x) = f(0) + \left( a + \int_{(0, +\infty)} \frac{d\mu(t)}{1 + t} \right) x - \int_{(0, +\infty)} \frac{x}{x + t} d\mu(t). \]
Hence \( f \) admits a representation of the form (8.8).

(ii) \( \Rightarrow \) (i). It is obvious that \( f \) given in (8.8) is operator convex on \([0, +\infty)\). Since \( \frac{1}{x+t} \leq \frac{1}{1+t} \) for all \( x > 1 \) and all \( t \in [0, +\infty) \), the Lebesgue convergence theorem yields that
\[ \lim_{x \to +\infty} \int_{(0, +\infty)} \frac{d\mu(t)}{x + t} = 0 \]
and so
\[ \frac{f(x)}{x} = \frac{f(0)}{x} + \alpha - \int_{(0, +\infty)} \frac{d\mu(t)}{x + t} \to \alpha \quad \text{as} \quad x \to +\infty. \]
Hence (i) follows.

**8.5 Remark.** Note that the condition \( \lim_{x \to +\infty} \frac{f(x)}{x} < +\infty \) puts a strong restriction on an operator convex function \( f \). Important examples for which it is not satisfied include \( f(x) = x \log x \) and \( f(x) = x^\alpha \) for \( \alpha \in (1, 2] \).

**9 Closing remarks**

Quantum \( f \)-divergences are a quantum generalization of classical \( f \)-divergences, which class in the classical case contains most of the distinguishability measures that are relevant to classical statistics. Although our Corollary 7.4 shows that \( f \)-divergences are less universal in the quantum case, they still provide a very efficient tool to obtain monotonicity and convexity properties of several distinguishability measures that are relevant to quantum statistics, including the relative entropy, the Rényi relative entropies, and the Chernoff and Hoeffding distances.

There are also differences between the classical and the quantum cases in the technical conditions needed to prove the monotonicity. For the approach followed here, it is important that the defining function is not only convex but operator convex, and the map is not only positive but it is also decomposable in the sense of Remark 4.8. It is unknown whether the monotonicity can be proved without these assumptions in general, although Corollary 3.4 and Lemma 3.5 show for instance that positivity of \( \Phi \) might be sufficient in some special cases. For measures that have an operational interpretation in state discrimination, like the relative entropy, the Rényi \( \alpha \)-relative entropies with \( \alpha \in (0, 1) \), and the Chernoff and Hoeffding distances, the monotonicity holds for any positive trace-preserving map \( \Phi \) such that \( \Phi \otimes^n \) is positive for every \( n \in \mathbb{N} \) [14, 35]. Note that both the set of maps satisfying this latter
property and the set of maps that are decomposable in the sense of Remark 4.8 contain all the completely positive trace-preserving maps, but we are not aware of any other explicit relation between these two sets. Moreover, the only example we know for a map \( \Phi \) which is not completely positive but \( \Phi^{\otimes n} \) is positive for every \( n \in \mathbb{N} \) is the transposition, which is trivial in the sense that it preserves any \( f \)-divergence (where \( f \) does not even need to be convex; see Corollary 2.5 and Remark 2.6).

Quantum \( f \)-divergences are essentially a special case of Petz’ quasi-entropies with \( K = I \) (see the Introduction) with the minor modification of allowing operators that are not strictly positive definite. While the monotonicity inequality in Theorem 4.3 can be proved for the quasi-entropies with general \( K \) quite similarly to the case \( K = I \), our analysis of the equality case in Theorem 5.1 doesn’t seem to extend to \( K \neq I \). A special case has been treated recently in [27], where a characterization for the equality case in the joint convexity of the quasi-entropies \( S_{\alpha}^{K}(\cdot,\cdot) \) (see Example 2.7 for \( K = I \)) was given for arbitrary \( K \) and \( \alpha \in (0, 2) \). Note that joint convexity is a special case of the monotonicity under partial traces (see [42, Theorem 6] or Corollary 4.7 of this paper), while monotonicity under partial traces can also be proven from the joint convexity for \( K \)’s of special type [30], which in turn implies the monotonicity under completely positive trace-preserving maps by using their Lindblad representation [51].

For a particularly elegant recent proof of the joint convexity for general \( K \)’s, see [11].

Various characterizations of the equality in the case \( K = I \) have been given before for different types of maps and classes of functions, including the equality case for the strong subadditivity of entropy and the joint convexity of the Rényi relative entropies \( S_{\alpha}^{I}(\cdot,\cdot) \) (see Example 2.7 for \( K = I \) and \( \alpha \in (0, 2) \)). Our Theorem 5.1 extends all these results and it seems to be the most general characterization of the equality, at least in finite dimension. The relevant part from the point of view of application to quantum error correction is that the preservation of some suitable distinguishability measure yields the reversibility of the stochastic operation, and the reversal map can be constructed from the original one in a canonical way. There are various technical conditions imposed in Theorems 5.1 and 7.1 that might be possible to remove. For instance, it is not clear whether the support condition in (5.4) is necessary or maybe the preservation of \( S_{\alpha}^{I}(\cdot,\cdot) \) for one single \( t > 0 \) is sufficient for reversibility. It is also an open question whether the surjectivity condition in Theorem 7.7 can be removed.

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A Commuting operators and the operator Hölder inequality

We will need the following two well-known lemmas in this section. The first one is a generalization of the so-called log-sum inequality, while the second one is a generalization of Jensen’s inequality for the expectation values of self-adjoint operators.

A.1 Lemma. Let \( f : [0, +∞) \to \mathbb{R} \) be a convex function. Let \( a_i \geq 0, b_i > 0, i = 1, \ldots, r \), and define \( a := \sum_{i=1}^{r} a_i, b := \sum_{i=1}^{r} b_i \). Then,

\[
bf(a/b) \leq \sum_{i=1}^{r} b_i f(a_i/b_i). \tag{A.1}
\]

Moreover, if \( f \) is strictly convex, then equality holds if and only if \( a_i/b_i \) is independent of \( i \).

Proof. Convexity of \( f \) yields that

\[
f(a/b) = f \left( \sum_{i=1}^{r} \frac{b_i a_i}{b} \right) \leq \sum_{i=1}^{r} \frac{b_i}{b} f \left( \frac{a_i}{b_i} \right),
\]

which yields (A.1), and the characterization of equality is immediate from the strict convexity of \( f \).

A.2 Lemma. Let \( A \) be a self-adjoint operator and \( \rho \) be a density operator on a finite-dimensional Hilbert space \( \mathcal{H} \). If \( f \) is a convex function on the convex hull of \( \text{spec}(A) \) then

\[
f(\text{Tr} A \rho) \leq \text{Tr} f(A) \rho. \tag{A.2}
\]

If \( f \) is strictly convex then equality holds in (A.2) if and only if \( \rho^0 \) is a subprojection of a spectral projection of \( A \).

Proof. Let \( A = \sum_a a P_a \) be the spectral decomposition of \( A \). Since \( \{ \text{Tr} P_a \rho : a \in \text{spec}(A) \} \) is a probability distribution on \( \text{spec}(A) \), Jensen’s inequality yields

\[
f(\text{Tr} A \rho) = f \left( \sum_a a \text{Tr} P_a \rho \right) \leq \sum_a f(a) \text{Tr} P_a \rho,
\]

and it is obvious that equality holds whenever \( \text{Tr} P_a \rho = 0 \) for all but one \( a \in \text{spec}(A) \). On the other hand, if there are more than one \( a \in \text{spec}(A) \) such that \( \text{Tr} P_a \rho > 0 \) then the above inequality is strict whenever \( f \) is strictly convex.

A.3 Proposition. Let \( A, B \in \mathcal{A}_{1,+} \) be such that \( A \) commutes with \( B \) and let \( \Phi : \mathcal{A}_1 \to \mathcal{A}_2 \) be a substochastic map such that \( \Phi(A) \) commutes with \( \Phi(B) \) and \( \text{Tr} \Phi(B) = \text{Tr} B \). For any convex function \( f : [0, +∞) \to \mathbb{R} \),

\[
S_f(\Phi(A)||\Phi(B)) \leq S_f(A||B). \tag{A.3}
\]

If \( \text{supp} A \leq \text{supp} B \) and \( f \) is strictly convex then equality holds in (A.3) if and only if \( \Phi^*_B(\Phi(A)) = A \).

Proof. Let us consider first the inequality (A.3). Note that if \( \omega(f) = +∞ \) and \( \text{supp} A \not\subseteq \text{supp} B \) then the RHS of (A.3) is \( +∞ \), and hence the inequality holds trivially. On the other hand, if \( \omega(f) < +∞ \) then it is enough to prove that

\[
S_f(\Phi(A)||\Phi(B + \varepsilon I)) \leq S_f(A||B + \varepsilon I)
\]
for every $\varepsilon > 0$, as taking the limit $\varepsilon \searrow 0$ then yields (A.3) due to Proposition 2.12. Hence, for the rest we can assume without loss of generality that $\text{supp} \ A \subseteq \text{supp} \ B$.

Since $A$ and $B$ commute, there exists an orthonormal basis $\{e_x\}_{x \in \mathcal{X}}$ in $\text{supp} \ B$ such that $A = \sum_{x \in \mathcal{X}} A(x) |e_x\rangle \langle e_x|$ and $B = \sum_{x \in \mathcal{X}} B(x) |e_x\rangle \langle e_x|$, where $A(x) := \langle e_x, A e_x \rangle$, $B(x) := \langle e_x, B e_x \rangle$, $x \in \mathcal{X}$. Similarly, there exists a basis $\{f_y\}_{y \in \mathcal{Y}}$ in $\text{supp} \Phi(B)$ such that $\Phi(A) = \sum_{y \in \mathcal{Y}} \Phi(A)(y) |f_y\rangle \langle f_y|$ and $\Phi(B) = \sum_{y \in \mathcal{Y}} \Phi(B)(y) |f_y\rangle \langle f_y|$, where $\Phi(A)(y) := \langle f_y, \Phi(A)f_y \rangle$, $\Phi(B)(y) := \langle f_y, \Phi(B)f_y \rangle$. We have

$$S_f(A\|B) = \sum_x B(x)f \left( \frac{A(x)}{B(x)} \right), \quad S_f(\Phi(A)\|\Phi(B)) = \sum_y \Phi(B)(y)f \left( \frac{\Phi(A)(y)}{\Phi(B)(y)} \right).$$

Let $T_{xy} := \langle f_y, \Phi(|e_x\rangle \langle e_x|)f_y \rangle$; then $\Phi(A)(y) = \sum_x T_{xy} A(x)$, $\Phi(B)(y) = \sum_x T_{xy} B(x)$, and Lemma A.1 yields

$$\Phi(B)(y)f \left( \frac{\Phi(A)(y)}{\Phi(B)(y)} \right) \leq \sum_x T_{xy} B(x)f \left( \frac{T_x A(x)}{T_{xy} B(x)} \right). \quad (A.4)$$

Since $\text{supp} \ |e_x\rangle \langle e_x| \subseteq \text{supp} \ B$, Lemma 3.2 yields that $\text{Tr} \ \Phi(|e_x\rangle \langle e_x|) = \text{Tr} \ |e_x\rangle \langle e_x| = 1$, $x \in \mathcal{X}$, and hence $\sum_{y \in \mathcal{Y}} T_{xy} = 1$, $x \in \mathcal{X}$. Summing over $y$ in (A.4) yields (A.3).

Obviously, equality holds in (A.3) if and only if (A.4) holds with equality for every $y \in \mathcal{Y}$. Assuming that $f$ is strictly convex, we obtain, due to Lemma A.1, that for every $y \in \mathcal{Y}$ there exists a positive constant $c(y)$ such that $T_{xy} A(x) = c(y) T_{xy} B(x)$, i.e.,

$$A(x) = c(y) B(x) \quad (A.5)$$

for every $x$ such that $T_{xy} > 0$. Assume that (A.5) holds; then we have $\Phi(A)(y) = \sum_x T_{xy} A(x) = \sum_x T_{xy} c(y) B(x) = c(y) \Phi(B)(y)$ and hence,

$$\Phi_B(\Phi(A))(x) = B(x) \sum_y T_{xy} \frac{\Phi(A)(y)}{\Phi(B)(y)} = B(x) \sum_y T_{xy} \frac{A(x)}{B(x)} = A(x), \quad x \in \mathcal{X}. \quad \square$$

The following Proposition gives an important special case where the monotonicity inequality (A.3) holds even though $A$ and $B$ don’t commute and $f$ is only assumed to be convex.

**A.4 Proposition.** Let $A, B \in A_+$ be such that $B \neq 0$, let $B = \sum_{b \in \text{spec}(B)} b Q_b$ be the spectral decomposition of $B$ and let $\mathcal{E}_B : \mathcal{X} \to \sum_{b \in \text{spec}(B)} Q_b X Q_b$ be the pinching defined by $B$. For every convex function $f : [0, +\infty) \to \mathbb{R}$,

$$S_f(A\|B) \geq S_f(\mathcal{E}_B(A)\|\mathcal{E}_B(B)) = S_f(\mathcal{E}_B(A)\|B) \geq (\text{Tr} \ B) f \left( \frac{\text{Tr} \ A}{\text{Tr} \ B} \right). \quad (A.6)$$

Moreover, if $\text{supp} \ A \subseteq \text{supp} \ B$ and $f$ is strictly convex then the first inequality in (A.6) holds with equality if and only if $A$ commutes with $B$, and the second inequality holds with equality if and only if $\mathcal{E}_B(A)$ is a constant multiple of $B$. In particular, $S_f(A\|B) = (\text{Tr} \ B) f \left( \frac{\text{Tr} \ A}{\text{Tr} \ B} \right)$ if and only if $A$ is a constant multiple of $B$.

**Proof.** All the assertions are obvious when $A = 0$, so for the rest we assume $A \neq 0$. Assume first that $\text{supp} \ A \subseteq \text{supp} \ B$. For every $b \in \text{spec}(B)$ and $\lambda \in \mathbb{R}$, let $P^{(b)}_\lambda$ be the spectral projection of $Q_b A Q_b$ corresponding to the singleton $\{\lambda\}$, and let $\tilde{P}^{(b)}_\lambda := Q_b P^{(b)}_\lambda Q_b$. Note
that $\tilde{P}_A^{(b)} = P_A^{(b)}$ for every $\lambda \neq 0$, and $Q_b = \sum_\lambda \tilde{P}_A^{(b)}$. The spectral projection of $\mathcal{E}_B(A)$ corresponding to the singleton $\{\lambda\}$ is $\sum_{b \in \text{spec}(B)} \tilde{P}_A^{(b)}$. For every $b \in \text{spec}(B) \setminus \{0\}$ and $\lambda \in \mathbb{R}$, let $\rho_{b, \lambda}$ be a density operator such that $\rho_{b, \lambda} = \tilde{P}_A^{(b)} / \text{Tr} \tilde{P}_A^{(b)}$ whenever $\tilde{P}_A^{(b)} \neq 0$. By (A.6), we have

$$S_f(\mathcal{E}_B(A)\|\mathcal{E}_B(B)) = S_f(\mathcal{E}_B(A)\|B) = \sum_{b \in \text{spec}(B) \setminus \{0\}} \sum_\lambda b f(\lambda/b) \text{Tr} \tilde{P}_A^{(b)} = \sum_{b \in \text{spec}(B) \setminus \{0\}} \sum_\lambda b f(\text{Tr}((A/b)\rho_{b, \lambda})) \text{Tr} \tilde{P}_A^{(b)} \leq \sum_{b \in \text{spec}(B) \setminus \{0\}} \sum_\lambda b \text{Tr} f(A/b)\rho_{b, \lambda} \text{Tr} \tilde{P}_A^{(b)} = \sum_{b \in \text{spec}(B) \setminus \{0\}} \sum_\lambda b \text{Tr} f(A/b)\rho_{b, \lambda} \sum_\lambda \tilde{P}_A^{(b)} = S_f(A\|B),$$

(A.7)

where $A = \sum_a a P_a$ is the spectral decomposition of $A$, and the inequality in (A.7) follows due to Lemma A.2. This yields the first inequality in (A.6). If $A$ commutes with $B$ then $\mathcal{E}_B(A) = A$ and hence the first inequality in (A.6) holds with equality. Conversely, assume that the first inequality in (A.6) holds with equality; then the inequality in (A.7) has to hold with equality as well. If $f$ is strictly convex then this implies that for every $b \in \text{spec}(B) \setminus \{0\}$ and $\lambda \in \mathbb{R}$, there exists an $a(b, \lambda)$ such that $\tilde{P}_A^{(b)} \leq P_{a(b, \lambda)}$, due to Lemma A.2. In particular, $\tilde{P}_A^{(b)}$ commutes with $A$, and, since $Q_b = \sum_\lambda \tilde{P}_A^{(b)}$, so does also $Q_b$, which finally implies that $B$ commutes with $A$.

Consider now the stochastic map $\Phi : A \to \mathbb{C}$, $\Phi(X) := \text{Tr} X$, $X \in A$. Since $\mathcal{E}_B(A)$ and $B$, as well as $\Phi(\mathcal{E}_B(A)) = \text{Tr} A$ and $\Phi(B) = \text{Tr} B$, commute, the second inequality in (A.6) follows due to Proposition A.3 which also yields that this inequality holds with equality if and only if $\mathcal{E}_B(A) = \Phi_B(\mathcal{E}_B(A)) = (\text{Tr} A/\text{Tr} B)B$.

Finally, consider the general case where $\text{supp} A \leq \text{supp} B$ does not necessarily hold. For every $\varepsilon > 0$, let $B_\varepsilon := B + \varepsilon I$. Note that $\text{supp} A \leq \text{supp} B_\varepsilon$ and $\mathcal{E}_{B_\varepsilon} = \mathcal{E}_B$ for every $\varepsilon > 0$, and hence by the above, $S_f(A\|B_\varepsilon) \geq S_f(\mathcal{E}_B(A)\|B_\varepsilon) \geq (\text{Tr} B_\varepsilon) f \left( \frac{\text{Tr} A}{\text{Tr} B_\varepsilon} \right)$ for every $\varepsilon > 0$. Taking the limit $\varepsilon \to 0$ then yields (A.6).

The first inequality above was proved for the case $f = f_\alpha$, $\alpha > 1$, in Section 3.7 of [14], and we followed essentially the same proof here. It was also proved in Section 3.7 of [14] that the monotonicity inequality (A.19) extends for the values $\alpha \in (2, +\infty)$ if $\Phi(A)$ and $\Phi(B)$ commute. We conjecture that this holds in more generality, namely that the monotonicity inequality $S_f(\Phi(A)\|\Phi(B)) \leq S_f(A\|B)$ holds for every convex $f$ if $A$ and $B$ or $\Phi(A)$ and $\Phi(B)$ commute. The inequality $S_f(A\|B) \geq (\text{Tr} B) f \left( \frac{\text{Tr} A}{\text{Tr} B} \right)$ was given in Theorem 3 of [42] for the case where $A$ and $B$ are invertible density operators and $f$ is a non-linear operator convex function. Note that the inequality between the first and the last term in (A.6) is a non-commutative generalization of the generalized log-sum inequality (A.1).

**A.5 Corollary.** For any positive semidefinite operators $A, B$ on a finite-dimensional Hilbert space $\mathcal{H}$, we have

$$\text{Tr} A^\alpha B^{1-\alpha} \leq (\text{Tr} A)^\alpha (\text{Tr} B)^{1-\alpha}, \quad \alpha \in [0, 1].$$

(A.8)
If, moreover, supp $A \leq \text{supp } B$ then

$$\text{Tr } A^\alpha B^{1-\alpha} \geq (\text{Tr } A)^\alpha (\text{Tr } B)^{1-\alpha}, \quad \alpha \in [1, +\infty).$$  \hfill (A.9)$$

If supp $A \leq \text{supp } B$ then $\text{Tr } A^\alpha B^{1-\alpha} = (\text{Tr } A)^\alpha (\text{Tr } B)^{1-\alpha}$ for some $\alpha \in (0, +\infty) \setminus \{1\}$ if and only if $A$ is a constant multiple of $B$.

**Proof.** The assertions are trivial when $A$ or $B$ is equal to zero, and hence we assume that both of them are non-zero. The inequality in (A.8) is obvious when $\alpha = 0$ or $\alpha = 1$, and the inequality in (A.9) is obvious when $\alpha = 1$. For $\alpha \in (0, +\infty) \setminus \{1\}$, the inequalities in (A.8) and (A.9) follow immediately by applying Proposition A.4 to the functions $f_\alpha(x) := \text{sgn}(\alpha - 1)x^\alpha$. Since these functions are strictly convex for every $\alpha \in (0, +\infty) \setminus \{1\}$, if equality holds in (A.8) or (A.9), and supp $A \leq \text{supp } B$, then $A$ is a constant multiple of $B$, due to Proposition A.4. Conversely, the inequalities (A.8) and (A.9) obviously hold with equality if $A$ is a constant multiple of $B$. \hfill \Box

Let $\mathcal{H}$ be a finite-dimensional Hilbert space. For every $A \in \mathcal{B}(\mathcal{H})$ and $p \in \mathbb{R} \setminus \{0\}$, let

$$\|A\|_p := \begin{cases} 0, & A = 0, \\ (\text{Tr } |A|^p)^{1/p}, & A \neq 0, \end{cases}$$

where $|A| := \sqrt{A^*A}$. For $p \in [1, +\infty)$, this is the well-known $p$-norm. Note that

$$\|A^*\|_p = \|A\|_p = \||A||_p$$

for every $A \in \mathcal{B}(\mathcal{H})$ and $p \in \mathbb{R} \setminus \{0\}$.

Corollary A.5 yields the following inverse Hölder inequality:

**A.6 Proposition.** Let $p \in (0, 1)$ and $q < 0$ be such that $1/p + 1/q = 1$. Let $A, B \in \mathcal{B}(\mathcal{H})$ for some finite-dimensional Hilbert space $\mathcal{H}$, and assume that $\text{supp } |A| \leq \text{supp } |B^*|$. Then

$$\|AB\|_1 \geq \|A\|_p \|B\|_q$$  \hfill (A.10)$$

Moreover, the equality case occurs in the above inequality if and only if $|A|^p$ and $|B^*|^q$ are proportional, i.e., $|A|^p = \alpha |B^*|^q$ for some $\alpha \geq 0$.

**Proof.** The assertion is obvious if $A$ or $B$ is zero, and hence we assume that both of them are non-zero. Let $A = U|A|$ and $B^* = V|B^*|$ be the polar decompositions with $U, V$ unitaries. Then $AB = U|A| |B^*|V^*$, and hence $\|AB\|_1 = \|\|A||B^*\||_1$. Let $\tilde{A} := |A|^p$, $\tilde{B} := |B^*|^q$ and $\alpha := 1/p$. Then $\alpha > 1$ and supp $\tilde{A} \leq \text{supp } \tilde{B}$ by assumption, and hence

$$\text{Tr } |A||B^*| = \text{Tr } \tilde{A}^{1/\alpha} \tilde{B}^{1-\alpha} \geq (\text{Tr } \tilde{A})^{\alpha} (\text{Tr } \tilde{B})^{1-\alpha} = (\text{Tr } |A|^p)^{1/p}(\text{Tr } |B^*|^q)^{1/q} = \|A\|_p \|B\|_q,$$

where the inequality follows due to Corollary A.5. It is well-known that $|\text{Tr } X| \leq \|X\|_1$ for every $X \in \mathcal{B}(\mathcal{H})$; indeed, if $X = \sum_i s_i |f_i\rangle \langle e_i|$ is a singular-value decomposition then $|\text{Tr } X| = |\sum_i s_i \langle e_i, f_i\rangle| \leq \sum_i s_i = \text{Tr } |X| = \|X\|_1$. Hence, $\text{Tr } |A||B^*| \leq \||A||B^*\||_1 = \|AB\|_1$, which completes the proof of the inequality (A.10). The characterization of the equality case is immediate from Corollary A.5. \hfill \Box

**A.7 Remark.** Our interest in the inverse operator Hölder inequality was motivated by [16]. The inequality was proved in [17] for positive semidefinite operators, using the usual Hölder inequality. An alternative direct proof for the general case and the condition for the equality was obtained in [20], based on majorization theory [14, 19].
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