NON LINEAR INTEGRAL EQUATION AND EXCITED–STATES SCALING FUNCTIONS IN THE SINE-GORDON MODEL

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Abstract

The NLIE (the non-linear integral equation equivalent to the Bethe Ansatz equations for finite size) is generalized to excited states, that is states with holes and complex roots over the antiferromagnetic ground state. We consider the sine-Gordon/massive Thirring model (sG/mT) in a periodic box of length $L$ using the light-cone approach, in which the sG/mT model is obtained as the continuum limit of an inhomogeneous six vertex model. This NLIE is an useful starting point to compute the spectrum of excited states both analytically in the large $L$ (perturbative) and small $L$ (conformal) regimes as well as numerically.

We derive the conformal weights of the Bethe states with holes and non-string complex roots (close and wide roots) in the UV limit. These weights agree with the Coulomb gas description, yielding a UV conformal spectrum related by duality to the IR conformal spectrum of the six vertex model.

I. INTRODUCTION

The NLIE proposed in ref. [1] allows to treat in an unified way the thermodynamics of magnetic chains and the finite size corrections to vertex models solvable by Bethe Ansatz. Moreover, in ref. [1] we derived (using the light-cone approach) the NLIE that describes the ground state of the sine–Gordon/massive–Thirring (sG/mT) field theory on a finite

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spatial volume (with periodic boundary conditions). The NLIE has been successfully used in different situations [7,8,11].

In the present paper we generalize the NLIE to excited states. That is, we derive the nonlinear integral equation equivalent to the Bethe Ansatz equations for states with holes and complex BA roots around the antiferromagnetic ground state in a periodic box of size $L$ (the case with only holes in certain special configurations has been treated also in [13]). In our framework the complex roots do not generally appear in the form of Bethe or Takahashi strings [4]. That is, their imaginary parts take continuous values (even for infinite volume) which are determined by the BAE themselves.

One can derive in an analogous way the NLIE for the finite–size effect on excited states in the six vertex model, as done in [14] and the so–called excited–states thermodynamics of the XXZ chain at temperature $T = 1/L$.

NLIE closely related to ours are obtained in ref. [8] along different lines which starts from the Perturbed Conformal Field Theory in the continuum. This method, however, is not yet directly applicable to the sG/mT model.

As is known [2,9], the sine–Gordon model with coupling $\beta$ admits an integrable U(1)–invariant light–cone lattice regularization based on the $R$–matrix of the six–vertex model with anisotropy $\gamma = \pi - \beta^2/8$. The energy–momentum spectrum is then calculated exactly by means of the Algebraic Bethe Ansatz, or Quantum Inverse Scattering Method: a Bethe Ansatz state is identified by an unordered set of distinct, generally complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_M$ which satisfy the famous Bethe Ansatz equations

$$\left[ s_\gamma(\lambda_j + \Theta + i\pi/2) \right]^N \left[ s_\gamma(\lambda_j + \Theta - i\pi/2) \right] = - \prod_{n=1}^M \frac{s_\gamma(\lambda_j - \lambda_n + i\pi)}{s_\gamma(\lambda_j - \lambda_n - i\pi)} \quad (1.1)$$

where $s_\gamma(x) = \sinh(\gamma x/\pi)$, $N$ stands for the number of sites, $L = N\delta$ is the physical size of the system (with periodic boundary conditions) and $\delta$ is the lattice spacing (that is the inverse of the UV cutoff). The energy $E$ and momentum $P$ of this BA state can be extracted from the relation

$$e^{-i(E \pm P)\delta/2} = \prod_{j=1}^M \frac{s_\gamma(i\pi/2 + \Theta \pm \lambda_j)}{s_\gamma(i\pi/2 - \Theta \mp \lambda_j)} \quad (1.2)$$

The real parameter $\Theta$ plays the role of rapidity cutoff and will diverge in the continuum limit $\delta \to 0$ in such a way to keep the the physical mass scale $m$ fixed.

To be precise, the continuum relativistic QFT defined on the infinite Minkowski plane follows by first taking the IR limit $L \to \infty$ (that is $N \to \infty$ at fixed $\delta$) and then the UV limit $\delta \to 0$ near the critical point $\Theta = \infty$, holding $m \sim \delta^{-1} \exp(-\Theta)$ fixed. On the other hand, by taking the continuum limit ($\delta \to 0$) at fixed $L$, we get instead the same QFT on a ring of length $L$. It is this second procedure that we wish to study here.

Let us also recall that the sG model has two distinct regimes, one repulsive, for $0 < \gamma < \pi/2$, and one attractive, for $\pi/2 < \gamma < \pi$. In the repulsive regime the spectrum contains only solitons and antisolitons, with U(1) charge $S = +1/2$ and $-1/2$ respectively (this charge is properly quantized w.r.t. the nonlocal hidden SU(2)$_q$ symmetry of the model [10]), while in the attractive regime there are also neutral bound state of these, the so–called breathers. In the BA solution the soliton/antisoliton states appears as holes in the ground
state distribution of BA roots; the breather states appear instead as special configurations of complex roots (see below).

We treat in an unified way both the repulsive and the attractive regimes.

The central object in the NLIE is the counting function $Z(\lambda)$. Its name follows from the fact that $\frac{1}{2\pi} Z(\lambda_j)$ is an integer for odd $S$ and a half-odd integer for even $S$ at the roots $\lambda_j$ of the BAE. In addition, $Z(\lambda)$ is monotonically increasing in the bulk. $Z(\lambda) + \frac{1}{4}(1 + (-1)^S)$ can take values which are integer multiple of $2\pi$ also for real $\lambda$ which are not roots of the BAE. These are the so called holes and together with the complex roots describe the excited states.

We find that $Z(\lambda)$ may be decreasing at some roots and holes. We call such points special roots/holes. They appear in the borders of the bulk where the root density becomes sparse. The presence of special real roots/holes turns to be a crucial feature in the analysis of the excited states.

The NLIE takes the following form in the sG model for arbitrary excited states with U(1) charge $S$ (notice that the adopted periodic boundary conditions force $S$ to be an integer)

$$Z(\lambda) = mL \sinh \lambda + g(\lambda) + \int_{-\infty}^{+\infty} dx \ G(\lambda - x) \ Q(x) \ ,$$  \hspace{1cm} (1.3)

where the unknown $Z(\lambda)$ is the counting function, $Q(x)$ is the nonlinearity

$$Q(x) = -i \log \frac{1 + (-)^S e^{iZ(x+i\epsilon)}}{(-)^S + e^{-iZ(x-i\epsilon)}}$$

and the function $g(\lambda)$ contains the information about the excited state considered

$$g(\lambda) = z_H(\lambda) + z_S(\lambda) + z_C(\lambda) \ ,$$  \hspace{1cm} (1.4)

where $z_H(\lambda)$, $z_S(\lambda)$ and $z_C(\lambda)$ stand for the contribution of the holes, the special holes and the complex roots.

We have,

$$z_H(\lambda) = \sum_{j=1}^{N_H} \chi(\lambda - h_j)$$

$$z_S(\lambda) = -2 \sum_{j=1}^{N_S} \chi(\lambda - y_j)$$  \hspace{1cm} (1.5)

where the $h_j$ stand for the positions of the holes and $y_j$ for those of the special root/holes. The form of $z_C(\lambda)$ depends whether $\gamma < \pi/2$ (repulsive regime) or $\gamma > \pi/2$ (attractive regime). It is given in eqs. (4.11) - (4.12).

The kernel $G(\lambda - x)$, explicitly written in eqs. (4.8), is just $(2\pi)^{-1}$ times the logarithmic derivative of the soliton–soliton scattering amplitude.

For the ground state, the NLIE (1.3) reduces to the form presented in [1,11].

A new way to write the NLIE follows by explicitly performing the limit $\epsilon \to 0$ in eq. (5.3):

$$Q(x) = \{Z(x) + \frac{1}{2}[1 - (-1)^S]\pi\} \ mod \ 2\pi$$  \hspace{1cm} (1.6)
where the mod $2\pi$ restriction may be written as
\[
X \mod 2\pi = X - 2\pi \text{sign}(X) \left\lfloor \frac{|X|}{2\pi} + \frac{1}{2} \right\rfloor \quad (1.7)
\]
for any real number $X$ ($|x|$ stands for the integer part of $x$).

Using eqs. (1.6) and (1.7) reduces the nonlinearity in eq. (1.3) to an integer part calculation. We use this new form of the NLIE to solve it for $mL \to 0$ and $\Theta = \pm\infty$ in sec. VII. Namely, in the plateau regions where the counting function is flat the NLIE reduces to a simple algebraic equation. For instance, as $\lambda \to \infty$ with finite $\Theta$ we get [see eq. (1.20)],
\[
X = b + \frac{\chi_\infty}{\pi} (X \mod 2\pi) \quad (1.8)
\]
Where $X = Z_N(+\infty) + \delta_S \pi$, $b$ is a known constant and $\chi_\infty = \frac{\pi^2/2 - \gamma}{1-\gamma/\pi}$. Since $X \mod 2\pi = X - 2\pi n$ for a suitable integer $n$, eq.(1.8) is solved immediately by
\[
X = 2(1 - \gamma/\pi)b - 2n(\pi - 2\gamma)
\]
provided
\[
|b - 2\pi n| \leq \frac{\pi}{2(1 - \gamma/\pi)}.
\]

The energy–momentum for an arbitrary excited state can be expressed in terms of the counting function as follows,
\[
E \pm P = E_V + m \sum_{j=1}^{N_H} e^{\pm h_j} - 2m \sum_{j=1}^{N_S} e^{\pm y_j} + E^\pm_C = m \int_{-\infty}^{+\infty} \frac{dx}{2\pi} e^{\pm x} Q(x) \quad (1.9)
\]
where $E_V$ is the ground state bulk energy,
\[
E_V = N\delta^{-1} \left[ -2\pi + \int_{-\infty}^{+\infty} d\lambda \frac{\phi_{1/2}(\pi\lambda/\gamma + 2\Theta)}{\pi \cosh \lambda} \right]
\]
$E^\pm_H$ and $E^\pm_S$ stand for the contributions from holes and special holes, respectively. $E^\pm_C$ represents instead the contributions of the complex roots. Its form depends whether $\gamma < \pi/2$ (repulsive regime) or $\gamma > \pi/2$ (attractive regime). $E^\pm_C$ is given in eqs.(6.10) and (6.11), respectively.

We provide in the present paper a thorough analysis of the excited states in the six vertex/sine-Gordon model as a basic step to derive the NLIE. We analyze the distribution of real roots for generic excited states in secs. II and III for large but finite $\Theta$ and $N$. There are two bulk seas of real roots centered around $\lambda = \pm\Theta$ and possibly a few isolated roots at the two extremities and in between the two seas. In such regions the counting function exhibits a, possibly non–monotonic behaviour characterized by one or two plateaus.

We derive a general relation for the spin in terms of the number of holes and complex roots [eq.(2.13)]. In the continuum limit (sG model), it takes the form
\[
2S = N_H - 2N_S - M_{close} - 2M_{wide} \theta(\pi - 2\gamma)
\]
where $N_H$ is the number of holes, $N_S$ is the number of special holes, $M_{\text{close}}$ the number of close roots and $M_{\text{wide}}$ the number of wide roots. Notice that $N_H - 2N_S$ acts as the effective hole number.

Concerning the complex roots, our analysis fully uses the non-string complex roots \[4\]. Bethe strings do not appear as generic solutions in our approach. These non-string complex roots do not contribute to the energy–momentum in the repulsive regime. They have different properties depending on their distance to the real axis (close or wide roots). They are just internal quantum numbers describing the collective $U(1)$ spin state of the excitation. In addition, wide roots in the attractive regime appear as independent excitations not carrying any spin. In the attractive regime, regular arrays of complex roots that do contribute to the energy–momentum appear in the infinite volume limit. Such arrays only contain wide roots and describe hole bound states (breathers).

The NLIE \([1.3]\) is therefore an useful starting point to find the spectrum of excited states both numerically or analytically in the large $mL$ (perturbative) and small $mL$ (conformal) regimes.

In section VIII we derive the scaling limit of the NLIE describing the conformal limit (UV regime) of the sG theory. This NLIE is simpler than the full sG-NLIE but it cannot be solved at present in closed form. However, we succeed to compute the eigenvalues exactly in the conformal regime for all excited states with the help of the Lemma \([8.10]\). This Lemma yields the integral involved in the eigenvalue calculation in close form without knowing the solution of the NLIE. We then show that the sG model exhibits in the UV limit, the conformal spectrum of a Coulomb gas with unit central charge and compactification radius

$$R = \sqrt{2(1 - \gamma/\pi)} = \frac{\beta}{\sqrt{4\pi}}$$

The Bethe states provide the primary as well as the secondary conformal states. We find that the (UV) sG conformal spectrum is related with the (IR) six-vertex conformal spectrum (which coincides with the IR conformal spectrum for the XXZ chain) by a duality transformation $R \leftrightarrow R^{-1}$.

Some simple excited states are discussed in detail in sec. VIII.

II. GENERALITIES

The BA roots are either real or come in complex conjugated pairs $(\xi, \bar{\xi})$, with the exception of single roots at $\text{Im} \lambda = \frac{\pi^2}{2\gamma}$ or $\text{Im} \lambda = -\frac{\pi^2}{2\gamma}$, which are actually self–conjugated due to $i\pi^2/\gamma$ periodicity [see eqs. \((1.1)-(1.2)\)]. It is well known that the ground state, or vacuum in the QFT language, corresponds to the unique BA solution with $N$ real roots for the entire range $0 < \gamma < \pi$. Then in any physically relevant transfer matrix eigenstate there are $M_R$ real roots $r_1, r_2, \ldots, r_{M_R}$ and $M_C = M - M_R$ complex roots $\xi_1, \xi_2, \ldots, \xi_{M_C}$ with $M_R$ of order $N$ and $M_C$ of order one. Moreover, we recall that $S = N - M$ is the eigenvalue of conserved $U(1)$ charge (the third component of the total spin in the six–vertex language).
Taking the logarithm of eq. (1.1) one finds,

\[ N[\phi_{1/2}(\lambda_k + \Theta) + \phi_{1/2}(\lambda_k - \Theta)] - \sum_{j=1}^{M} \phi_1(\lambda_k - \lambda_j) = 2\pi I_k \]  

(2.1)

where the \( I_k \) are half-odd-integers for \( M \) even and integers for \( M \) odd and

\[ \phi_{\nu}(\lambda) \equiv i \log \frac{s_\gamma(i\nu\pi + \lambda)}{s_\gamma(i\nu\pi - \lambda)} = i \log \frac{\sinh[\gamma(i\nu + \lambda/\pi)]}{\sinh[\gamma(i\nu - \lambda/\pi)]} \]  

(2.2)

We choose the logarithmic cuts to run parallel to the real axis and such that \( \phi_{\nu}(\lambda) \) is an odd function of complex \( \lambda \) (see figs. 1 and 2). This removes any \( 2\pi \) ambiguity on the “quantum numbers” \( I_k \). Notice that \( \phi_{\nu}(\lambda) \) is monotonic function. Moreover, since \( N \) may be chosen to be even (eventually \( N \to \infty \)), the parity of \( M \) is the same of \( S + N - M \), so that the \( I_k \) are half-odd-integers for \( S \) even and integers for \( S \) odd. Therefore we may write in general

\[ I_k = \text{integer} + \frac{1}{2}(1 - \delta_S), \quad \delta_S \equiv \frac{1}{2}[1 - (-1)^S] \]

• The counting function

Given a solution \( \lambda_1, \lambda_2, \ldots, \lambda_M \) of the BAE, we define the corresponding counting function as

\[ Z_N(\lambda) = N[\phi_{1/2}(\lambda + \Theta) + \phi_{1/2}(\lambda - \Theta)] - \sum_{k=1}^{M} \phi_1(\lambda - \lambda_k) \]  

(2.3)

Comparing eq. (2.3) to eq. (1.1) we have by definition

\[ Z_N(\lambda_k) = 2\pi I_k, \quad k = 1, 2, \ldots, M \]  

(2.4)

As stated above, the BA root \( \lambda_1, \lambda_2, \ldots, \lambda_M \) must all be distinct (the corresponding BA state would otherwise vanish). Once specialized to the real roots, eq. (2.4) reads

\[ Z_N(r_j) = 2\pi I_{R_j}, \quad j = 1, 2, \ldots, M_R \]  

(2.5)

We naturally assume the ordering \( r_1 < r_2 < \ldots < r_{M_R} \). Furthermore, we anticipate that the size of the distribution of real roots is of order \( 2 \log N \) for large \( N \), that is \( r_1 \sim -\log N \), \( r_{M_R} \sim +\log N \).

• Holes

Distinct real numbers \( h_j, j = 1, 2, \ldots, N_H \), that are also distinct from the real BA roots \( r_1, r_2, \ldots, r_{M_R} \) but satisfy the same quantization rule (2.3), that is

\[ Z_N(h_j) = 2\pi I_{H_j} \]  

(2.6)

with the \( I_{H,j} \) integers or half-integers, are called holes. Again we assume the ordering \( h_1 < h_2 < \ldots < h_{N_H} \). Together with the real BA roots the holes form the complete set of real zeroes of the function

\[ 1 + (-1)^S e^{iZ_N(\lambda)}. \]

We denote this set as \( \{x_j; j = 1, 2, \ldots, M_R + N_H\} \) and assume it ordered.
• Special root/holes

In the counting function (2.3) the term \(N[\phi(\lambda + \Theta, \gamma/2) + \phi(\lambda - \Theta, \gamma/2)]\), which is monotonically increasing over the real line, acts as source of possible quantum numbers for real roots and holes (a sort of phase space). Each summand in the sum over roots either subtricts or add phase space depending on the sign of \(\pi - 2\gamma\) and the imaginary part of the root. In general, in any zero–temperature physical state, the global effect is to produce a \(Z_N(x)\) which is monotonically increasing on the real line, but there could be exceptions, as we shall now discuss.

Let us call “bulk regions” the portions of the real line where \(Z_N'(x)\) is positive and of order \(N\) for large \(N\). In such regions the spacing between consecutive real roots is of order \(1/N\). Therefore, the number of roots within the bulk is of order \(N\). The regions around the two extremities of the real distribution are by definition not in the bulk and there \(Z_N'(x)\) may indeed change sign due to one or more isolated roots. Another place where \(Z_N'(x)\) might change sign is the middle of the real distribution, which is peaked around \(+\Theta\) and \(-\Theta\).

The limit \(N \to \infty\) forces \(Z_N(x)\) to be monotonic increasing for any fixed \(x\) and \(\Theta\). But for any arbitrarily large \(N\) there could be specific configurations of quantum numbers and large enough values of \(|x|\) and/or \(\Theta\), such that \(Z_N\) is decreasing at the two extremities and/or in the middle of the real distribution. This possibilities are easily verified numerically.

If \(Z_N(x)\) decreases, it may decrease enough to cross downward a quantization point already crossed upwards. Therefore the following classification become necessary. We shall define as non-degenerate those configurations of quantum numbers such that the union

\[I_R \cup I_H = \{I_{R,j}; j = 1, 2, \ldots M_R\} \cup \{I_{H,j}; j = 1, 2, \ldots N_H\}\]

contains only distinct integers or half-odd-integers. This is automatically the case if \(Z_N'(x_j) > 0\) at all zeroes of

\[1 + (-1)^Se^{iZ_N(\lambda)}\]

so that the ordering of the \(x_j\)’s implies the ordering of the corresponding quantum numbers \(I_j\). The degenerate configurations are those where one or more quantum numbers appear more than once in \(I_R \cup I_H\). Evidently to such quantum numbers are associated at the same time real zeroes of

\[1 + (-1)^Se^{iZ_N(\lambda)}\]

with \(Z_N'(x_j) > 0\) and real zeroes with \(Z_N'(x_j) < 0\). We call the latter special real roots or holes, root/holes for short, as opposed to the normal ones with \(Z_N'(x_j) > 0\). Notice that two (or more) roots cannot be associated to the same integer. If this would be the case, by continuity in \(\gamma\) and \(\Theta\), one could cause these two roots to “merge” and thus obtain a double root of the BAE which is to be discarded. Hence to a degenerate quantum number is associated at most one real root, while the other are holes. The same continuity argument in the two free parameters \(\Theta\) and \(\gamma\) serves to deal with the exceptional cases \(Z_N'(x_j) = 0\). We may regard such cases as mergings of a real roots and a hole or of two holes. In either case they do not require a special treatment.

Extensive numerical studies have shown that degenerate configurations are restricted to few types. For large \(N\) at fixed \(\Theta\) special root/holes may appear only at either one of the two tails of the real distribution, due to a single real or self–conjugated root or to a cluster...
of complex roots that become isolated from the bulk when $\gamma$ gets close to one of a special set of rational multiples of $\pi$. If $\Theta$ is large enough at fixed $N$, then the distribution of real roots separates into two distinct distributions peaked around $+\Theta$ and $-\Theta$, respectively. In this case there could be special root/holes whenever one or more roots remain at a distance of order 1, rather than $\Theta$, from the origin and therefore become isolated.

In this section we consider large $N$ at fixed $\Theta$ and shall deal with the opposite regime of large $\Theta$ at fixed $N$ in section [11].

Let us consider here an explicit example of root/holes in the outside tails. We consider the BA states obtained by removing $S$ real roots from the ground state distribution. If $2\gamma S < \pi$, then one finds that there are $N + S$ allowed values for the quantum numbers. This follows from the limiting values

$$\frac{1}{2\pi} Z_N(\pm \infty) = \pm \frac{1}{2\pi} [N\pi + (\pi - 2\gamma)S] = \pm \left[\frac{N - S + 2S}{2} - \frac{\gamma S}{\pi}\right].$$

This is just right to accommodate $N - S$ real roots (as required) and $2S$ holes. We are thus dealing with a non-degenerate configuration and this is confirmed numerically (it can be done, for $N$ in the thousands and to high precision, on any modern personal computer). One finds also the following: if the smallest and largest of the $N + S$ real zeroes of $Z_N(x)$ (say $x_1$ and $x_{N+S}$) are holes, then $Z_N(x)$ is globally monotonic; if instead $x_1$ is a real root, then $Z_N(x)$ tends to $Z_N(-\infty)$ from below and $Z_N'(x)$ changes sign just before $x_1$; if $x_{N+S}$ is a real root, then $Z_N(x)$ tends to $Z_N(+\infty)$ from above and $Z_N'(x)$ changes sign just after $x_{N+S}$. Suppose now we let $2\gamma S \to \pi^-$ while keeping all quantum numbers fixed. One finds that when $x_{N+S}$ is a hole, it simply goes to $+\infty$ and then “jumps” to the line $\text{Im} \lambda = -\pi^2/(2\gamma)$ (so that it ceases to be a real zero) when $2\gamma S$ exceeds $\pi$; in the meantime $Z_N(x)$ remains monotonic in the right tail. The same scenario applies to the left tail if $x_1$ is a hole when $2\gamma S < \pi$, with the only difference that $x_1$ jumps to the line $\text{Im} \lambda = +\pi^2/(2\gamma)$. [The choice of sign in the two jumps is a consequence of our logarithmic branch conventions for the function $\phi_\nu(\lambda)$ and the request that all quantum numbers stay fixed]. On the other hand, if $x_{N+S}$ is a root, then $Z_N(+\infty)$ tends to $2\pi I_{R_{N+S}}$ as $2\gamma S \to \pi^-$, so that we may say that a special hole appears at $+\infty$. When $2\gamma S$ exceeds $\pi$ this special hole moves in to a finite values $h$ until it collides from the right with the root $x_{N+S}$ for a certain, configuration–dependent value of $\gamma$. Then the root and the hole split again, but now with $h < x_{N+S}$ and $Z_N(h) < 0$, while $x_{N+S}$ is now a special root.

Finally, as $\gamma$ approaches the universal value $\pi/(S + 1)$, $x_{N+S}$ tends to $+\infty$ and then jumps to $\text{Im} \lambda = -\pi^2/(2\gamma)$ when $\gamma$ exceeds $\pi/(S + 1)$. Notice that the interval $\pi/(2S) \leq \gamma \leq \pi/(S + 1)$ of non–monotonicity shrinks to the single free–field point $\pi/2$ for $S = 1$. One can also check that, after the jump, $h$ is still the largest real zero of $Z_N(\lambda)$ when $S \geq 2$, while a new special hole appears when $S = 1$. A specular description applies in the left tail.

It should also be clear that the mechanisms just described may repeat for larger values of $\Theta$, provided we start from suitable configurations. Likewise, it is possible that clusters of $n > 1$ complex roots become isolated and are then “pushed to infinity” for certain configurations and special values of $\gamma$. Looking at the original BAE one sees that these complex roots must tend asymptotically to form $q$–strings with the same real parts and spacing $\pi^2/(q\gamma)$ in the imaginary direction [3].
• Complex roots

Besides the classification of real roots and holes into normal and special types, it proves convenient to classify the complex roots \( \xi_j, j = 1, 2, \ldots , M_C \) into close roots

\[
\{c_j; j = 1, 2, \ldots , M_{\text{close}}\}, \quad |\text{Im } c_j| < \min(\pi, \pi(\frac{\pi}{\gamma} - 1))
\]

and wide roots [4]

\[
\{w_j; j = 1, 2, \ldots , M_{\text{wide}}\}, \quad \min(\pi, \pi(\frac{\pi}{\gamma} - 1)) < |\text{Im } w_j| \leq \frac{\pi^2}{2\gamma}
\]

The self–conjugate roots need to be further divided into two distinct classes. The first class is formed by the real roots that have jumped to \( \text{Im } \lambda = \pm \frac{\pi}{2\gamma} \) by passing through \( \text{Re } \lambda = \mp \infty \) upon suitably varying \( \gamma \) at fixed quantum numbers, as we have described in the example above. The second class is formed by all other self–conjugated roots. That is, the first class are those self–conjugated roots that become real roots if we make \( \gamma \) small enough keeping the state fixed.

When \( \gamma < \pi/2 \) the total number, locations and quantum numbers of the self–conjugated roots of the second class are connected to those of the holes. When \( \gamma > \pi/2 \) they are independent variables (see eq. (2.11) below) and describe the lightest breather states. In either case the presence of such roots properly characterizes the BA state as an excited state and cannot be eliminated just by varying \( \gamma \).

• Relation among the various numbers of roots and holes

It is fairly easy to establish a relation among the U(1) charge and the number of holes, special root/holes and complex roots. From the definition itself of the counting function, eq.(2.3) and the asymptotic values of the function \( \phi_\nu(\lambda) \) we read

\[
Z_N(+\infty) = +N\pi + (\pi - 2\gamma)S + 2\pi\text{sign}(\pi - 2\gamma)M_{\text{wide},\downarrow}
\]

\[
Z_N(-\infty) = -N\pi - (\pi - 2\gamma)S - 2\pi\text{sign}(\pi - 2\gamma)M_{\text{wide},\uparrow}
\]

where \( M_{\text{wide},\uparrow} (M_{\text{wide},\downarrow}) \) is the number of wide roots above (below) the real line. On the other hand we have

\[
Z_N(+\infty) = 2\pi(I_{\text{max}} + \frac{1}{2}) + \zeta_+
\]

\[
Z_N(-\infty) = 2\pi(I_{\text{min}} - \frac{1}{2}) - \zeta_-
\]

where \( I_{\text{max}} (I_{\text{min}}) \) is the quantum number corresponding to the largest (smallest) real root or hole and \( \zeta_\pm \) is the mod \( 2\pi \) residue (obviously \( |\zeta_\pm| < \pi \))

\[
\zeta_\pm = [\pm Z_N(\pm\infty) + \pi\delta_S] \text{ mod } 2\pi = -2\gamma S + 2\pi \left[\frac{1}{2} + \frac{\pi}{\gamma} S\right]
\]

so that

\[
I_{\text{max}} + \frac{1}{2} = +\frac{1}{2}(N + S) + \text{sign}(\pi - 2\gamma)M_{\text{wide},\downarrow} - \left[\frac{1}{2} + \frac{\pi}{\gamma} S\right]
\]

\[
I_{\text{min}} - \frac{1}{2} = -\frac{1}{2}(N + S) - \text{sign}(\pi - 2\gamma)M_{\text{wide},\uparrow} + \left[\frac{1}{2} + \frac{\pi}{\gamma} S\right]
\]
Evidently the total number of real zeroes of $1 + (-1)^S e^{i N(\lambda)}$ is

$$M_R + N_H = I_{\text{max}} - I_{\text{min}} + 1 + 2 N_S$$

$$= N + S - 2 \left[ \frac{1}{2} + \frac{2}{\pi} S \right] + \text{sign}(\pi - 2\gamma) M_{\text{wide}} + 2 N_S$$

where $N_S$ stands for the total number of special roots/holes.

On the other hand we have by definition

$$M_R = N - S - M_{\text{close}} - M_{\text{wide}}$$

so that we obtain the following general constraint between the total number of holes and of complex roots and the U(1) charge $S = N - M$ [$\theta(x)$ is the step function]:

$$N_H - 2 N_S = 2 \left( S - \left[ \frac{1}{2} + \frac{2}{\pi} S \right] \right) + M_{\text{close}} + 2 M_{\text{wide}} \theta(\pi - 2\gamma)$$

(2.11)

The important fact about eq.(2.11) is that it involves only quantities which are finite as $N \to \infty$. Notice also that $N_H$ turns out to be always even. We may write eq.(2.11) in a different way by introducing $M'_{s-c}$, the number of self–conjugated roots of the first class and the “effective hole number”

$$N_{H,\text{eff}} \equiv N_H - 2 N_S - 2 \theta(\pi - 2\gamma) M'_{s-c} + 2 \left[ \frac{1}{2} + \frac{2}{\pi} S \right]$$

(2.12)

Notice that $N_{H,\text{eff}}$ indeed stays constant throughout the processes described in the examples above. We now have the relation

$$N_{H,\text{eff}} = 2 S + M_{\text{close}} + 2 \theta(\pi - 2\gamma)(M_{\text{wide}} - M'_{s-c})$$

(2.13)

where evidently $M_{\text{wide}} - M'_{s-c}$ is the number of wide roots which are not self–conjugated roots of the first class.

We may now develop the following interpretation for the complex roots. First of all we observe that, for any given configuration of quantum numbers, there exists sufficiently small values of $\gamma$ such that $\left[ \frac{1}{2} + \frac{2}{\pi} S \right] = 0$ and $N_S = M'_{s-c} = 0$. Then $N_{H,\text{eff}} = N_H$ from eq.(2.12) and we see [from eq.(2.13)] that the holes carry a U(1) charge $S = N_H/2$ if no complex roots are present, while this charge is lowered by 1 for each close root and by 2 for each wide root. Next, as $\gamma$ is raised while keeping all quantum numbers fixed, $\left[ \frac{1}{2} + \frac{2}{\pi} S \right]$, $N_S$ and $M'_{s-c}$ might become nonzero but $N_{H,\text{eff}}$ stays fixed. In particular, if the $N_H$ holes present for small enough $\gamma$ are all well within the real distribution, in the sense that their quantum numbers are of order 1 rather than $N$, then these holes stay normal for any value of $\gamma$ and are exactly those counted by $N_{H,\text{eff}}$. We anticipate that that these configurations are exactly those relevant in the continuum limit (see section [VII]).

We may thus draw the conclusion that in the repulsive regime $\gamma < \pi/2$ all complex roots (other than self–conjugated roots of the first class) act as parameters characterizing the various U(1) polarization states of $N_{H,\text{eff}}$ normal holes well within the distribution of real roots, which have U(1) charge 1/2. The allowed values for the quantum numbers of these complex roots must then be of order $N_H$ rather than $N$, as can be verified by carefully analyzing the counting function of BA solutions found numerically.
In the attractive regime $\gamma > \pi/2$ this interpretation holds for the close roots only. From eq.\,(2.13) we see that adding wide roots does not change the U(1) charge of the $N_{H,\text{eff}}$ holes since $\theta(\pi - 2\gamma) = 0$. Thus, when $\gamma > \pi/2$, the wide roots appear as independent excitations not carrying any U(1) charge and with a phase space of order $N$. These picture is made more precise but fully confirmed by appropriate calculations (including those of the energy–momentum spectrum) in the $N \to \infty$ limit at fixed $\Theta$ (see section VII).

III. LARGE $\Theta$ LIMIT AND RELATION TO THE HOMOGENEOUS SIX-VERTEX MODEL AND THE XXZ CHAIN

When $\Theta$ vanishes the BAE (1.1) reduce to those of the homogeneous six-vertex model and the XXZ spin 1/2 chain with $2N$ sites. This is the obvious correspondence. Less obvious and more important for our purposes is the relation which follows in the limit $\Theta \to \infty$ at fixed $N$. In this limit the BAE split into two separate sets of twisted BAE each relative to an XXZ chain with $N$ sites. The two sets remain coupled by the twists, which depend on global properties of the original BA state as well as on those of the final pair. To see all this we may work directly with the counting function $Z_N(\lambda, \Theta)$, in which we write out for clarity the dependence on $\Theta$. Notice that this dependence is both explicit, in the source term proportional to $N$, and implicit, through the BA roots themselves.

As $\Theta \to \infty$ at fixed $N$, almost all the roots separate into two sets, the left–moving and right–moving seas, having “center of mass” in $+\Theta$ and $-\Theta$, respectively, and spreading of order $\log N$ for large $N$. Few roots may stay within finite regions for special values of $\gamma$ or sufficiently symmetric root configurations, as will become clear below. Let us split the $M$ BA roots into the two sets $\{\pm\Theta + \lambda_{j^\pm}; j = 1, 2, \ldots, M\}$ of the roots diverging as $\pm\Theta$, which we shall call right– and left–moving, respectively, plus the $M_0$ roots with a finite limit for $\Theta \to \infty$. Each $\lambda_{j^\pm}$ measures the distance of the diverging root from its center of mass and tends to a finite limit. Then we find for $Z_N(\lambda, \Theta)$:

$$
\lim_{\Theta \to +\infty} Z_N(\lambda \pm \Theta, \Theta) \equiv Z_N^\pm(\lambda) = N \phi_{1/2}(\lambda) - \sum_{k=1}^{M^\pm} \phi_1(\lambda - \lambda_{k^\pm}^\pm) \\
\pm(\pi - 2\gamma)(S - S^\pm) \pm 2\pi \left[ \frac{1}{4} N + n_{\text{wide}}^\pm \right]
$$

where $S^\pm = N/2 - M^\pm$ is the U(1) charges of the left and right seas, respectively, and $n_{\text{wide}}^\pm$ are integer which depend on the wide roots. Evidently the $M^0$ roots that do not diverge contribute $S^0 = -M^0$ to the total charge, so that $S = S^- + S^0 + S^+$. We anticipate that, while the global charge $S$ is nonnegative, in certain cases one of the two partial charges $S^\pm$ may take also the value $-1$. For completeness, we also observe that, due to our choice of logarithmic branches for the function $\phi_1(\lambda)$, the integers $n_{\text{wide}}^\pm$ are given by

$$
n_{\text{wide}}^+ = \text{sign}(\pi - 2\gamma)[M_{\text{wide},\downarrow} - M_{\text{wide},\uparrow}^+], \\
n_{\text{wide}}^- = \text{sign}(\pi - 2\gamma)[M_{\text{wide},\uparrow} - M_{\text{wide},\downarrow}^-]. \tag{3.1}
$$

The quantization rules

$$
Z_N^\pm(\lambda_{j^\pm}^\pm) = 2\pi I_j^\pm
$$
now reproduce, upon exponentiation, the twisted BAE of two XXZ chains.

By direct inspection one finds
\[\Omega_\pm \equiv Z_N^{\pm}(\mp \infty) = \pm(\pi - 2\gamma)(S - 2S^\pm) \pm 2\pi \ell_\text{wide}^\pm \]

(3.2)

where
\[\ell_\text{wide}^+ = \text{sign}(\pi - 2\gamma)[M_{\text{wide},\downarrow} - M_{\text{wide},\uparrow}^+],
\]
\[\ell_\text{wide}^- = \text{sign}(\pi - 2\gamma)[M_{\text{wide},\uparrow} - M_{\text{wide},\downarrow}^-],
\]

and \(M_{\text{wide},\uparrow}(M_{\text{wide},\downarrow})\) is the number of wide roots above (below) the real line. We may also write
\[\Omega_\pm = \pm[(\pi - 2\gamma)S^0 + 2\pi \ell_0^\pm]
\]

(3.3)

where
\[\ell_0^\pm = \frac{1}{2}(\ell_\text{wide}^+ - \frac{1}{2}\ell_\text{wide}^-),\]
\[\ell_\text{wide} = \frac{1}{2}(\ell_\text{wide}^+ + \frac{1}{2}\ell_\text{wide}^-)
\]

Evidently \(\Omega\) is the common value of the right asymptote of \(Z_N^-(\lambda)\) and the left asymptote of \(Z_N^+(\lambda)\) whenever \(S^0 = 0\) (which implies \(\ell_0^0 = 0\) too). In this case we may approximate \(Z_N(\lambda, \Theta)\) for \(\Theta\) very large but finite, as
\[Z_N(\lambda, \Theta) \simeq Z_N^-(\lambda - \Theta) + Z_N^+(\lambda + \Theta) - \Omega
\]

Thus \(Z_N(\lambda, \Theta)\) has a large almost flat plateau which tends to \(\Omega\) as \(\Theta \to \infty\). This observation applies also when \(N\) is very large, provided \(\Theta \gg \log N\), since the size of each sea is of order \(\log N\). In case of non-degenerate configurations \(\Omega\), the height of the plateau, fixes the way the quantum numbers \(I_j\) relative to real roots and holes are subdivided into those belonging to the two seas, that is the \(I_j^\pm\)'s associated to real roots and holes. In fact, just by definition we must have:
\[2\pi I^-_{\text{max}} < \Omega < 2\pi I^+_{\text{min}} = 2\pi(I^-_{\text{max}} + 1)
\]

(3.4)

On the other hand, if \(S^0\) is nonzero, then \(Z_N(\lambda, \Theta)\) keeps a nontrivial structure in the neighborhood of the origin; this structure interpolates between two large plateaus with heights \(\Omega_\pm\). For instance, if \(S^0 = -1\) due to a single real root at \(\lambda = x_0\), then \(Z_N(\lambda, \Theta)\) can be approximated as
\[Z_N(\lambda, \Theta) \simeq Z_N^-(\lambda - \Theta) + Z_N^+(\lambda + \Theta) - \Omega - \phi_1(\lambda - x_0)
\]

But since \(Z_N(x_0, \Theta) = 2\pi I_0\), where \(I_0\) is some integer plus \(\frac{1}{2}\delta S\), to leading order in \(\Theta\) we get \(\Omega = 2\pi I_0\). Comparing with eq. (3.3) this shows that the real root may stay at a finite location \(x_0\) only for certain special values of \(\gamma\), unless separately \(I_0 = 0\) and \(\Omega = 0\).
As a matter of fact the situations characterized by a special value of $\gamma$ bear a strict correspondence with those discussed in the example of section I. There we found that for certain special values of $\gamma$ one extremal real root was pushed to infinity with respect to the bulk of the roots, which stayed localized in a finite region. Here we find the same with respect to a bulk that moves to $\pm \infty$. The same argument may be repeated also for the threshold values of $\gamma$ beyond which special root/holes appear; it suffices to translate the discussion of section I to the inner tails of the real distributions, that is the right tail of the left sea and the left tail of the right sea. There is an important difference, however: in the discussion of section I all special values of $\gamma$ are functions of the total spin $S$, which is the sum $S^+ + S^-$ for generic situations; here they are functions of the difference $\Delta S = S^+ - S^-$. 

So, let us recall that in the example of section I we dealt with the states obtained by removing $S$ real roots from the ground state distribution. As in section I we begin with the range $0 < 2S\gamma < \pi$ for which the configuration is non-degenerate and then follow its evolution as $\gamma$ grows.

The simplest case is when $S$ is even and the $2S$ holes are equally divided into $N_H^+ = S$ right–moving and $N_H^- = S$ left–moving ones (so that $\Omega = 0$, see eq.(3.3)) while $I_{\text{max}} = -1/2$ and $I_{\text{min}}^+ = 1/2$ are occupied by holes. In this case nothing special happens in the central part of the root distribution when $\Theta$ is very large and $\gamma$ raises beyond $\pi/(2S)$. This is because the relevant phase space of each sea (the right tail of the left sea and the left tail of the right sea) receives from the twist caused by other sea exactly the same contribution that it looses due to the growth of $\gamma$. On the other hand the boundaries of the distribution (the left tail of the left sea and the right tail for the right sea) may develop special root/holes as discussed in section I. Actually this scenario applies to the entire range of $\gamma$ and therefore also applies to the passage through those points, such as $\gamma = \pi/(S + 1)$, when extremal real roots jumps to the lines $\text{Im} \lambda = \pm \pi^2/(2\gamma)$. This particular class of BA states is the one treated in ref. [5].

Let us now allow $N_H^+$ to be different from $N_H^-$, but still impose that the $I_{\text{max}}$ and $I_{\text{min}}^+$ are occupied by holes. Then we find that, as long $2|\Delta S|\gamma < \pi$, so that $-\pi(\Delta S + 1) < \Omega < -\pi(\Delta S - 1)$, we have $N_H^+ = 2S^+$ and $N_H^- = 2S^-$ [recall eq. (3.4)]. As $\gamma$ exceeds $\pi/(2|\Delta S|)$, one hole passes from the sea with higher partial U(1) charge to the other, since the actual $I_{\text{max}}^-$ and $I_{\text{min}}^+$ increase by one for positive $\Omega$ or decreases by one for negative $\Omega$. When $\gamma$ is exactly equal to $\pi/(2|\Delta S|)$ there is a hole with a finite limit as $\Theta \to \infty$, since $2\pi I_{\text{max}}^+ = \Omega > 0$ or $2\pi I_{\text{min}}^- = \Omega < 0$. This example shows that even if $N_H = 2S$, $N_H^+ - N_H^-$ may differ from $2(S^+ - S^-)$. In particular in the exchange of one hole from one sea to the other $\Delta S$ stays constant as $N_H^+ - N_H^-$ changes by two. Notice also that a hole may stay trapped into a finite region only if $\gamma$ assumes exactly one of a discrete set of values. Hence we can regard this case as exceptional and always assume $N_H^+ + N_H^- = N_H$ by continuity.

When either one, or both, of the two quantum numbers closest to $\Omega/(2\pi)$ are occupied by real roots the situation gets more involved due to the appearance of special root/holes. For instance, if $\Delta S$ is a negative even and $I_{\text{max}}^- = (|\Delta S| - 1)/2$ is occupied by a real root, then, as $\gamma$ exceeds $\pi/(2|\Delta S|)$, the plateau $\Omega$ gets below $2\pi I_{\text{max}}^-$ which therefore becomes triply degenerate: two new holes, one left–moving and special, the other right–moving and normal, appear to the right of the root. As $\gamma$ grows further the root and the left–moving hole “collide” and then exchange their positions, the hole now being normal and the root special; when $\gamma$ reaches $\pi/(|\Delta S| + 1)$ the root comes back from large negative values of order $-\Theta$ to
finite values, so that $S^-$ increases by one while $S^0$ changes from 0 to $-1$ and two plateaus are formed, with the left one $2(\pi - 2\gamma) = 2\pi(|\Delta S| - 1)/(|\Delta S| + 1)$ higher than the other; then the root passes to the right–moving sea when $\gamma$ exceeds $\pi/(|\Delta S| + 1)$ and we have once more a unique plateau with height $\Omega$ larger than the original one by $2(\pi - 2\gamma)$, in accordance with the fact that $\Delta S$ has decreased by two units; next the root “collides” with the right–moving hole and exchanges place with it; at this stage we have a normal left–moving hole, a special right–moving hole and a real root associated to the degenerate quantum number $I_{\text{max}}$ and the two holes disappears as soon as the new $\Omega$ gets smaller than $2\pi I_{\text{max}}$, that is when $\gamma$ exceeds $3\pi/(2|\Delta S|)$. In figs. 3 and 4 the changing shape of the central portion of $Z_N(\lambda)$ is depicted for a special case of the type just discussed. It is obtained by numerically solving the BAE.

Another interesting observation concern that case when initially $S^+ = 0$, so that the right sea is a twisted ground state. At the end we find a state with $S^+ = -1$, that is with $M^+ = N/2 + 1$. This is possible exclusively thanks to the twist induced by the other sea.

We are now in a position to produce general formulae analogous to eqs. (2.11) and (2.13). Our purpose is to suitably relate the plateau heights $\Omega^\pm$ to the hole and complex root content of the BA state. We proceed as in section II and define $\omega^\pm$, with $|\omega^\pm| < \pi$, by the relations

$$\Omega^- = 2\pi (I_{\text{max}}^- + \frac{1}{2}) + \omega^-, \quad \Omega^+ = 2\pi (I_{\text{min}}^+ - \frac{1}{2}) - \omega^+$$

We may also write

$$\omega^\pm = (\mp \Omega^\pm + \pi \delta_S) \mod 2\pi$$

But $\mp \Omega^\pm + \delta_S \pi$ is just $2\gamma(S - 2S^\pm)$ plus some integer multiple of $2\pi$ (see eqs. (3.2) and (3.3)), so that we obtain

$$\omega^\pm = 2\gamma(S - 2S^\pm) + 4\pi(S^\pm - \hat{S}^\pm)$$

where

$$\hat{S}^\pm = S^\pm + \frac{1}{2}\text{sign}(S - 2S^\pm) \left[ \frac{1}{2} + \frac{\pi}{2} |S - 2S^\pm| \right]$$

It is convenient to relate $\hat{S}^\pm$ to the content of the two seas in terms of holes, special root/holes and complex roots. To this end we write

$$\omega^- = \Omega^- - 2\pi (I_{\text{max}}^- + \frac{1}{2})$$

$$= \Omega^- - 2\pi (I_{\text{min}} - M^- + N^- - 2N^- S^- - \frac{1}{2})$$

and similarly

$$\omega^+ = -\Omega^+ + 2\pi (I_{\text{min}}^+ - \frac{1}{2})$$

$$= -\Omega^+ + 2\pi (I_{\text{max}} - M^+ - N^+ + 2N^+ S^+ + \frac{1}{2})$$

Then using eqs. (2.10) and the obvious relation

$$M^\pm = \frac{1}{2}N - S^\pm - M^\pm_{\text{close}} - M^\pm_{\text{wide}}$$
yields

\[ \hat{S}^\pm = \frac{1}{2} [ N_{H,\text{eff}}^\pm - M_{\text{close}}^\pm - 2 \theta(\pi - 2\gamma)(M_{\text{wide}}^\pm - M_{\text{s-c}}^\pm)] \] (3.9)

where the effective hole number of each sea reads (compare with eq. (2.12))

\[ N_{H,\text{eff}}^\pm = N_H^\pm - 2 N_S^\pm - 2 \theta(\pi - 2\gamma) M_{\text{s-c}}^\pm + \left\lfloor \frac{1}{2} + \frac{\gamma}{\pi} S \right\rfloor \] (3.10)

We see that \( \hat{S}^+ \) and \( \hat{S}^- \) may be regarded as the partial U(1) charges induced by the holes, with the complex roots of each sea fixing the global polarization state of the holes (recall the interpretation in section II). \( \hat{S}^\pm \) does not coincide in general with \( S^\pm \) because the splitting of holes in right– and left–moving ones is only partially correlated to that of the roots (recall that \( S^\pm \) is defined by \( N/2 - M^\pm \)), due to the coupling of the two seas caused by the twists and/or the possibilities that \( N_H^+ + N_H^- < N_H \) or \( S^0 \neq 0 \). It should also be remarked that, unlike \( S^\pm \), \( \hat{S}^\pm \) can be half-odd-integers.

The expression (3.9) in terms of holes and complex root will prove itself very useful in section VIII.

**IV. THE FUNDAMENTAL NONLINEAR INTEGRAL EQUATION**

In this section we derive a NLIE (nonlinear integral equation) which is fully equivalent to the original BAE (1.1) for any excited state. The crucial property of such NLIE is that it depends analytically on the number of sites \( N \) and does not contain explicitly the real roots (which number is of order \( N \)), but only holes and complex roots (which number is of order 1).

In the definition (2.3) of the counting function, let us rewrite the sum over all roots as

\[
\sum_{j=1}^{M} \phi_1(\lambda - \lambda_k) = \sum_{j=1}^{M + N_H} \phi_1(\lambda - x_j) - \sum_{j=1}^{N_H} \phi_1(\lambda - h_j) + \sum_{j=1}^{M_C} \phi_1(\lambda - \xi_j) \] (4.1)

where we recall that the \( x_j \)'s are all the the points on the real lines where \( e^{iZ_N(\lambda)} = (-1)^S \).

That is, real roots and holes. Next we convert the sum over the \( x_j \)'s into a contour integral

\[
\sum_{k=1}^{M + N_H} \phi(\lambda - \lambda_k, \gamma) = \oint_{\Gamma} \frac{d\mu}{2\pi i} \phi_1(\lambda - \mu) \frac{d}{d\mu} \log[1 + (-1)^Se^{iZ_N(\mu)}] \] (4.2)

where \( \Gamma \) is a closed curve that lies in the analyticity domain of the integrand and encircles anti-clockwise all the \( x_j \)'s once. At this initial stage we prefer to work with derivatives to avoid worrying about boundary integration constants. We also consider \( \lambda \) in the neighborhood of the real axis. Thus we can write the derivative of eq. (2.3) in the form

\[
Z_N^\prime(\lambda) = N z_0^\prime(\lambda) - \oint_{\Gamma} \frac{d\mu}{2\pi i} \phi_1(\lambda - \mu) \frac{d}{d\mu} \log[1 + (-1)^Se^{iZ_N(\mu)}] \]

where the source term reads

\[
z_0(\lambda) = \phi_{1/2}(\lambda + \Theta) + \phi_{1/2}(\lambda - \Theta) + \frac{1}{N} \sum_{j=1}^{N_H} \phi_1(\lambda - h_j) - \frac{1}{N} \sum_{j=1}^{M_C} \phi_1(\lambda - \xi_j) \] (4.3)
We next choose $\Gamma$ to be the boundary of the infinite rectangle centered in the origin with horizontal sides extending from $-\infty$ to $+\infty$ and vertical sides of length $2\eta$, with $0 < \eta < \frac{1}{2}\min(\pi, \pi(\frac{\pi}{2} - 1), \sigma)$, $\sigma$ being the absolute value of the imaginary part of the complex root of $e^{i\mathcal{N}(\lambda)} = (-1)^S$ closest to the real line. We denote with $\Gamma_+$ and $\Gamma_-$ the upper and lower halves of $\Gamma$, respectively. They are both oriented from left to right, so that

$$\oint_{\Gamma} d\mu \ldots = \int_{\Gamma_-} d\mu \ldots - \int_{\Gamma_+} d\mu \ldots$$

By construction, the counting function enjoys the property

$$\mathcal{N}(\bar{\mu}) - \mathcal{N}(\mu) = 0 \mod 2\pi \quad (4.4)$$

Moreover, by analyticity we have $\mathcal{N}(x + iy) = iy\mathcal{N}'(x) + \ldots$ so that $\text{Im} \mathcal{N}(\mu)$ has the same sign of $\mathcal{N}'(\text{Re} \mu)$ for $\text{Im} \mu$ positive and small enough. Assuming for the time being that we are in a non-degenerate case, this suggests to extract $\mathcal{N}(\mu)$ from the lower half of the integration rectangle by writing $1 + (-1)^S e^{i\mathcal{N}} = e^{i\mathcal{N}}[(-1)^S + e^{-i\mathcal{N}}]$, so that the remaining logarithms have no cut ambiguities for $\eta$ small enough. The necessary corrections for degenerate cases will be introduced at the end.

By extracting $\mathcal{N}(\mu)$ we obtain a convolution of $\phi'_{1}(\lambda - \mu)$ with $\mathcal{N}'(\mu)$ which can be moved to the real axis thanks to analyticity, yielding

$$[(1 + K) * \mathcal{N}](\lambda) = N\mathcal{N}(\lambda) - i\int_{\Gamma_+} \frac{d\mu}{2\pi} \phi'_{1}(\lambda - \mu) \frac{d}{d\mu} \log \left[1 + (-1)^S e^{i\mathcal{N}(\mu)}\right]$$

$$+ i\int_{\Gamma_-} \frac{d\mu}{2\pi} \phi'_{1}(\lambda - \mu) \frac{d}{d\mu} \log \left[(-1)^S + e^{-i\mathcal{N}(\mu)}\right] \quad (4.5)$$

where $K * f$ stands for the convolution

$$(K * f)(\lambda) = \int_{-\infty}^{+\infty} \frac{dx}{2\pi} \phi'_{1}(\lambda - x)f(x)$$

Applying $(1 + K)^{-1}$ to both sides of eq. (4.5) and integrating by parts leads to the following nonlinear integral equation (NLIE)

$$\mathcal{N}(\lambda) = N\mathcal{N}(\lambda) - i\int_{\Gamma_+} \frac{d\mu}{2\pi} \mathcal{G}(\lambda - \mu) \log \left[1 + (-1)^S e^{i\mathcal{N}(\mu)}\right]$$

$$+ i\int_{\Gamma_-} \frac{d\mu}{2\pi} \mathcal{G}(\lambda - \mu) \log \left[(-1)^S + e^{-i\mathcal{N}(\mu)}\right] \quad (4.6)$$

where the ‘dressed’ source term $z(\lambda)$ is by construction the sum of bulk, hole and complex root contributions (compare with eq. (4.3)), plus an integration constant $C$ to be determined later

$$z(\lambda) = ([1 + K]^{-1}s)(\lambda) = z_{V}(\lambda) + \frac{1}{N} [z_{H}(\lambda) + z_{C}(\lambda)] + C \quad (4.7)$$
and $G(\lambda)$ stands for the kernel of the convolution operator $G = [1 + K]^{-1} \ast K$. Through Fourier transforms one obtains the following explicit expressions

$$
G(\lambda) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ik\lambda/2} \left( \frac{\sinh(\pi/2 - \gamma)k}{2\sinh(\pi - \gamma)k/2 \cosh(\gamma k/2)} \right) - 1
$$

(4.8)

$$
z_V(\lambda) = \text{gd}(\Theta + \lambda) - \text{gd}(\Theta - \lambda)
$$

(4.9)

$$
z_H(\lambda) = \sum_{j=1}^{N_H} \chi(\lambda - h_j)
$$

(4.10)

where $\text{gd}(x) = \arctan[\sinh(\lambda)]$ is the so-called hyperbolic amplitude (or Gudermannian) and $\chi(\lambda)$ is the odd primitive of $2\pi G(\lambda)$. Notice that $\chi(\lambda)$ coincides with the soliton–soliton two-body scattering phase shift $^{13}$. On the other hand the complex root contribution is different in the two regimes: when $\gamma < \pi/2$ we have

$$
z_C(\lambda) = -\sum_{j=1}^{M_{\text{close}}} \chi(\lambda - c_j) - \sum_{j=1}^{M_{\text{wide}}} \phi_{\alpha/2}(\alpha(\lambda - w_j'))
$$

(4.11)

where $\phi_{\nu}(\lambda)$ was defined in eq.(2.2), $\alpha = (1 - \gamma/\pi)^{-1}$ and

$$
w_j' = w_j - \text{sign}(|\text{Im}w_j|) \frac{\pi}{2}
$$

while when $\gamma > \pi/2$ we have

$$
z_C(\lambda) = -\sum_{j=1}^{M_{\text{close}}} \chi(\lambda - c_j) - \sum_{j=1}^{M_{\text{wide}}} \left[ \text{gd}(\lambda - w'' + i\frac{\pi^2}{2\gamma}) + \text{gd}(\lambda - w'' - i\frac{\pi^2}{2\gamma}) \right]
$$

(4.12)

where now

$$
w_j'' = w_j - \text{sign}(|\text{Im}w_j|) \frac{\pi}{2\gamma}
$$

A rather more compact form of eq. (4.6) is obtained by choosing $\eta$ to be infinitesimal:

$$
Z_N(\lambda) = N z(\lambda) + (G \ast Q_N)(\lambda) + C
$$

(4.13)

where (recall that we assumed $Z_N'(x) > 0$, so that the logarithms are always in their principal determination)

$$
Q_N(x) = -i \log \frac{1 + (-)^S e^{iZ_N(x+i\epsilon)}}{(-)^S + e^{-iZ_N(x-i\epsilon)}} \xrightarrow{\epsilon \to 0} \{Z_N(x) + \delta_S \pi\} \mod 2\pi
$$

(4.14)

Hence $Q_N(x)$ jumps downward by $2\pi$ each time $Z_N(x)$ crosses $2\pi$ times one of the quantum numbers $I_k$, that is when $x$ passes through a root or a hole, as required. Notice that the assumption that $Z_N'(x_j) > 0$ for any real root or hole $x_j$ could actually be dropped after the limit $\epsilon \to 0$ in (4.14). In practice, when $\epsilon$ is strictly zero we identify $Q_N(x)$ as the unique real function with the following three properties:

$$
e^{iQ_N(x)} = (-1)^S e^{iZ_N(x)}, \quad |Q_N(x)| \leq \pi, \quad Z_N \to -Z_N \implies Q_N \to -Q_N
$$
Thus \( Q_N(x) \) jumps by \( 2\pi \) upwards at a special root/hole where \( Z_N'(x_j) < 0 \). Of course one may generalize in the same way the definition of \( Q_N(x) \) in terms of logarithms, provided we suitably change determination whenever \( Z_N'(x_j) < 0 \).

We may clarify this matter by performing a simple exercise which backtracks the derivation of the NLIE, eq. (4.13). By definition we have, provided \( Z_N'(x_j) > 0 \)

\[
Q'_N(x) = Z'_N(x) - 2\pi \sum_{j=1}^{M_R+N_H} \delta(\lambda - x_j)
\]

which inserted into the derivative of eq. (4.13) leads to a complete cancelation of the hole contribution, so that we obtain

\[
(1 - G) * Z'_N = Nz'_V + z'_C
\]

But \( 1 - G = (1 + K)^{-1} \) and therefore the last equation is just \( (1 + K)^{-1} \) applied to the derivative of the original definition of the counting function, eq. (2.3).

It should by now be clear how to modify the NLIE in eq. (4.13) in the degenerate cases. In general we have

\[
Q'_N(x) = Z'_N(x) - 2\pi \sum_{j=1}^{M_R+N_H} \text{sign}(Z_N'(x_j)) \delta(\lambda - x_j)
\]

Let us now denote with \( y_k, k = 1, \ldots, N_S \), the locations of the special root/holes. Then we also have

\[
Q'_N(x) = Z'_N(x) - 2\pi \sum_{j=1}^{M_R+N_H} \delta(\lambda - x_j) + 4\pi \sum_{k=1}^{N_S} \delta(\lambda - y_k)
\]

Inserting this into eq. (4.13) would not reproduce \( (1 + K)^{-1} \) applied to eq. (2.3) just because of the last sum over special root/holes. Hence the convolution of \( G \) with this sum must be subtracted from the source term \( Nz'(\lambda) \) in eq. (4.13) yielding the modification

\[
z_H(\lambda) \rightarrow z_H(\lambda) + z_S(\lambda), \quad z_S(\lambda) = -2 \sum_{j=1}^{N_S} \chi(\lambda - y_j)
\]

on the hole source. With this simple but crucial change, eq. (4.13) holds true in general. Of course it could be recast into the alternative form (4.6) by analytic continuation and contour deformation.

Let us now take care of the integration constant \( C \). One can easily establish that \( C \) must vanish. In fact \( Z_N(x) \) is by definition an asymptotically odd function up to \( 2\pi \) times an integer, that is \( [Z_N(x) + Z_N(-x)] \to 0 \mod 2\pi \) as \( x \to \infty \). Thus \( (G * Q_N)(x) \) is asymptotically odd and we need only verify that \( z(+\infty) - z(-\infty) \) has the required extra \( 2\pi \) times an integer.

Let us also remark that in the limit \( N \to \infty \) at fixed \( \Theta \) we have to leading order \( Z_N(\lambda) \simeq Nz_V(\lambda) \). Hence to leading order the distribution of real roots is exponentially peaked around \(+\Theta\) and \(-\Theta\), with a spreading of order \( \log N \). This confirm the anticipation made in section II about the size of the real distribution.
We now observe that the NLIE in eqs. (4.6) and (4.13) are manifestly analytic in $\lambda$, so that they may be used to define $Z_N(\lambda)$ away from the real axis. This definition may actually differ, for $|\text{Im} \lambda|$ large enough, from the original definition in eq. (2.3) due to a different cut structure. We shall adopt the new definition implied by the NLIE: since in the NLIE there is only one term of order $N$ and is explicitly known, we have a better control on the possible values of the quantum numbers relative to the complex roots. This redefinition implies that these quantum numbers may be shifted by integers w.r.t. their original definition.

The NLIE can be analytically continued away from the real axis in a straightforward manner as long as $|\text{Im} \lambda| < \min(\pi, \pi(\frac{\pi}{\gamma} - 1))$. For larger values of $\text{Im} \lambda$ one must take into account that the first singularity of the kernel $G(\lambda)$ can no longer be avoided by deforming the contours $\Gamma_{\pm}$. This is because the real line act as natural boundary for such deformations. For example $\Gamma_{+}$ cannot be deformed through the real line because $F(\mu)$ has modulus larger than one below the real line. Alternatively, one may say that the cuts implied by the poles of the kernel $G(\lambda)$ get pinched by the jump discontinuities of the non-analytic function $Q_N(x)$ when $\text{Im} \lambda$ reaches $\pm \min(\pi, \pi(\frac{\pi}{\gamma} - 1))$. Hence the contribution of such singularity has to be explicitly added via the residue theorem, resulting in, for $|\text{Im} \lambda| > \min(\pi, \pi(\frac{\pi}{\gamma} - 1))$

$$Z_N(\lambda) = N\tilde{z}(\lambda)_II + \int_{-\infty}^{+\infty} G(\lambda - x)_II Q_N(x) \tag{4.16}$$

where for any function $f(\lambda)$ we have defined

$$f(\lambda)_II = \begin{cases} f(\lambda) + f(\lambda - i\pi \text{sign}(\text{Im} \lambda)) & 0 < \gamma < \pi/2 \\ f(\lambda) - f(\lambda - i\frac{\pi}{\gamma}(\pi - \gamma) \text{sign}(\text{Im} \lambda)) & \pi/2 < \gamma < \pi \end{cases} \tag{4.17}$$

Notice that the the second determination $\tilde{z}_V(\lambda)_II$ of the ground–state contribution to the source $z(\lambda)$ identically vanishes in the repulsive regime due to the $i\pi$ anti-periodicity of the sinh function. Hence the wide roots do not have a phase space of order $N$ in the repulsive regime, in agreement with the discussion in section II. On the other hand $\tilde{z}_V(\lambda)_II$ keeps a monotonically increasing term of order $N$ in the attractive regime, as required by the interpretation of the wide roots as independent excitations of the attractive regime.

We remark that the expression (4.16) implies for $Z_N(\lambda)$ a cut structure, in the domain $|\text{Im} \lambda| > \min(\pi, \pi(\frac{\pi}{\gamma} - 1))$, that differs from that of the original definition (2.3).

It is also quite interesting to observe that the notion of second determination (4.17) allows one to write $z_C(\lambda)$, the complex root contribution to the source of the NLIE, in a compact form valid for both regimes:

$$z_C(\lambda) = -\sum_{j=1}^{M_{\text{close}}} \chi(\lambda - c_j) - \sum_{j=1}^{M_{\text{wide}}} \chi(\lambda - w_j)_II \tag{4.18}$$

In fact the soliton–soliton two-body scattering phase shift $\chi(\lambda)$ enjoys the fundamental crossing properties:

$$\chi(\lambda) + \chi(\lambda - i\pi) = \phi_{\alpha/2}(\alpha(\lambda - i\pi/2))$$

in the repulsive regime $\gamma < \pi/2$ and

$$\chi(\lambda) - \chi(\lambda - i\frac{\pi}{\gamma}(\pi - \gamma)) = \text{gd}(\lambda - i\pi/2) + \text{gd}(\lambda + i\pi/2 - i\frac{\pi^2}{\gamma})$$
in the attractive regime $\gamma > \pi/2$ and

Holes, special root/holes and complex roots which specify the source term in the NLIE are constrained by the supplementary quantization rules

$$
Z_N(h_j) = 2\pi I_{H,j}, \quad j = 1, 2, \ldots, N_H
$$
$$
Z_N(y_j) = 2\pi I_{S,j}, \quad j = 1, 2, \ldots, N_S
$$
$$
Z_N(\xi_j) = 2\pi I_{C,j}, \quad j = 1, 2, \ldots, M_C
$$

(4.19)

Of course the second of these relations is just a repetition of the first in case of special holes. In case of special real roots the locations are in general different from those of the holes, but the corresponding quantum numbers form by construction a subset of those of the holes (recall in fact that the case of two real roots with the same quantum numbers is not allowed). Together with the NLIE the above quantization rules provide a framework equivalent to the BAE. The great advantage over the standard algebraic form of the BAE is the analytic dependence on $N$, which allows to explicitly perform the continuum limit (recall that by hypothesis the number of holes and of complex roots stays finite in that limit). More subtle is the question concerning the constructive nature of the NLIE plus supplementary quantization rules, namely whether they completely substitute the original definition (2.3) and the BAE (1.1). Our results show that this is indeed so, provided the proper distinction between normal and special root/holes is made: however, this distinction is based on the sign of $Z'_N(x)$ in certain points, while $Z_N(x)$ is itself the unknown in the NLIE. We shall now verify that this is not a loophole, as it might appear at first sight.

In fact, the NLIE in (4.13) or (4.6) does not admit a solution for arbitrary choices of the source term $z(\lambda)$. Suppose that we take a $z(\lambda)$ with an almost flat behavior in a large portion of the real line (in our case these happens for large $|\text{Re}\lambda|$). Since the convolution with the exponentially peaked kernel $G(x)$ acts as a multiple of the identity on constant functions, we see that in the flat regions the NLIE reduces to a simple algebraic equation. In our case we obtain from (4.6), for instance as $x \to \infty$:

$$
X = b + \frac{\chi_\infty}{\pi} (X \mod 2\pi)
$$

(4.20)

Where $X = Z_N(\infty) + \delta_S \pi$, $b = N z(\infty) + \delta_S \pi$ and

$$
\chi_\infty = \pm \chi(\pm \infty) = \pi \int_{-\infty}^{+\infty} dx G(x) = \frac{\pi/2 - \gamma}{1 - \gamma/\pi}
$$

Let us show that eq.(4.20) admits a solution if the constant $b$ falls in some specific intervals fixed by the ratio $\gamma/\pi$. Since

$$
X \mod 2\pi = X - 2\pi n
$$

(4.21)

for a suitable integer $n$, then eq.(4.20) is solved immediately by

$$
X = 2(1 - \gamma/\pi)b - 2n(\pi - 2\gamma)
$$

which is consistent with eq.(4.21) provided

$$
|b - 2\pi n| \leq \frac{\pi}{2(1 - \gamma/\pi)}.
$$

(4.22)
For $\gamma < \pi/2$, 
\[
\frac{\pi}{2} < \frac{\pi}{2(1 - \gamma/\pi)} < \pi ,
\]
and we see that the sequence of intervals generated by (4.22) varying $n$ does not cover the real line; if $b$ lays in one of the uncovered segments then eq. (4.20) has no solution. The special root/holes cure this problem, by preventing $b = Nz(+\infty) + \delta_S$ $\pi$ to enter into the uncovered segments when $\gamma$ and/or $\Theta$ vary at fixed quantum numbers. Viceversa, in the attractive regime $\gamma > \pi/2$, we have 
\[
\pi < \pi < \infty
\]
and there could be several solutions to eq. (4.20) that differ by integer multiples of $2\pi$. This is connected to the fact that in the attractive regime wide roots act as independent excitations that do not affect the phase space for real roots and holes.

Having derived the fundamental NLIE for a generic BA state, we now turn to the problem of expressing the energy and momentum eigenvalue directly in terms of the function $Z_N(\lambda)$. The methods are based as before on contour integrals.

V. ENERGY AND MOMENTUM AS FUNCTIONALS OF $Z_N$

The energy and momentum of a BA state may be written, taking eq. (1.2) into account
\[
E \delta = \sum_{j=1}^{M} \left[ \phi_{1/2}(\Theta - \lambda_j) + \phi_{1/2}(\Theta + \lambda_j) - 2\pi \right]
\]
(5.1)
\[
P \delta = \sum_{j=1}^{M} \left[ \phi_{1/2}(\Theta - \lambda_j) - \phi_{1/2}(\Theta + \lambda_j) \right]
\]
(5.2)
The choice of logarithmic branch in the energy ensures that the contribution of each real root is negative definite. In this way one finds that the BA states with holes located at the boundaries of the real distribution carry a large energy of order $\delta^{-1}$ and decouple in the continuum limit. Hence, as anticipated in sections II and III, only states with holes well within the real distribution will need to be considered in the continuum limit.

As a preliminary step to relate $E$ and $P$ directly to the counting function we shall first study the quantity
\[
W(\lambda) = \sum_{j=1}^{M} \phi'_{1/2}(\lambda - \lambda_j)
\]
[As before, we first consider the derivative to avoid worrying with boundary integration constants]. The sum over roots in this expression is rewritten as in eq. (4.1). Then the sum over real roots and holes may be transformed into a contour integral as done for $Z_N(\lambda)$, and this integral is then manipulated in much the same way to obtain the following result
\[
W(\lambda) = -\sum_{j=1}^{N_H} \phi'_{1/2}(\lambda - h_j) + \sum_{j=1}^{M_C} \phi'_{1/2}(\lambda - \xi_j) + \int_{-\infty}^{+\infty} \phi'_{1/2}(\lambda - x) Z'_N(x) \, dx
\]
(5.3)
\[
+ i \int_{\Gamma_+} \frac{d\mu}{2\pi} G(\lambda - \mu) \log \left[ 1 + e^{iZ_N(\mu)} \right] - i \int_{\Gamma_-} \frac{d\mu}{2\pi} G(\lambda - \mu) \log \left[ (-1)^S + e^{-iZ_N(\mu)} \right]
\]
We then use the NLIE to eliminate the term linear in $Z'_N(\lambda)$ and add together, for holes and roots, the two fundamental types of contributions, the direct one already present in eq. (5.3) and the back–reaction term coming from the NLIE. We obtain, in compact notation and taking correctly into account the eventual special roots/holes

$$W(\lambda) = W_V(\lambda) + W_H(\lambda) + W_C(\lambda) - \int_{-\infty}^{+\infty} \frac{dx}{2\pi} gd'(\lambda - x) Q'_N(x)$$

where, as above, $gd(\lambda) = \arctan[\sinh(\lambda)]$ and

$$W_V(\lambda) = N \int_{-\infty}^{+\infty} dx \, gd'(x) \phi'_{1/2}(\lambda - x)$$

$$W_H(\lambda) = - \sum_{j=1}^{N_H} gd'(\lambda - h_j) + 2 \sum_{j=1}^{N_S} gd'(\lambda - y_j)$$

while, with the sum still to be performed,

$$W_C(\lambda) = \sum_{j=1}^{M_{close}} \phi'_{1/2}(\lambda - \xi_j) + \int_{-\infty}^{+\infty} dx \, z'_C(x) \phi'_{1/2}(\lambda - x)$$

When $\gamma < \pi/2$ (repulsive regime) one finds that the direct and back–reaction terms cancel completely out in the case of wide roots, independently of their location, yielding

$$W_C(\lambda) = \sum_{j=1}^{M_{close}} gd'(\lambda - c_j)$$

Of course the eventual presence of wide roots would anyway keep affecting $W(\lambda)$ through $Z_N(\lambda)$. In the attractive regime $\gamma > \pi/2$ there is instead no such cancelation for generic wide pair positions and the expression for $W_C(\lambda)$ contains the non-vanishing extra terms due to the wide pairs. Their explicit form is quite long and shall not be written out here. We shall come back on this point later, when we discuss the $L \to \infty$ limit in which these extra terms simplify considerably.

By integrating $W(\pm \Theta)$ as written in eq. (5.4) with respect to $\Theta$ one obtains the expressions of the energy and momentum as functionals of $Z_N$. Some care is required for the integration constants, since they could in principle be state-dependent. This step will be performed only after the $N \to \infty$ limit when it is quite simple.

**VI. THE CONTINUUM LIMIT**

It is well known [2,3,9] that the $N_{H,eff}$ holes interspersed in the bulk of the distribution of real roots are to be identified with physical particles which, in the continuum limit $\delta \to 0$ and $\Theta \to \infty$ on the infinite lattice, acquire a relativistic dispersion relation with mass $m \sim \delta^{-1} \exp(-\Theta)$. They are the solitons and antisolitons of the sG model. The locations $h_j$ of the holes plays the role of rapidities: a hole at $h$ has energy–momentum $m(\cosh h, \sinh h)$ in the continuum limit.

In the standard light–cone approach [3] one reaches the continuum limit by keeping only the leading corrections in $1/L$. These are the terms of order one in the energy and
momentum, that is the particle spectrum with zero energy–momentum density, and the order 1/L corrections in the quantization rules for the particle momenta which yield the S–matrix \[12\]. In practice we can say that in this continuum limit one keeps \( m^{-1} \), the characteristic size of the excitations, much smaller than the size \( L \) of the system.

Here we shall instead take the continuum limit keeping all orders in \( 1/L \). This will allow us to study also the case when \( mL \) is very small, which is relevant for the ultraviolet behavior of the sG model. The objects of interest are the so–called scaling functions, that is the continuum limit of the quantities \( (E - E_V)L \), where \( E \) is the energy of generic excited states and \( E_V \) the vacuum energy.

Let us begin with the counting function. In the continuum limit at fixed \( L \) both \( N \) and \( \Theta \) tend to infinity with the asymptotic relation

\[
\Theta \simeq \log \frac{4N}{mL}
\]  

characteristic of a fixed physical mass for the solitons. In this limit and for any fixed value of \( \lambda \) the vacuum contribution \( N_{zV}(\lambda) \) becomes quite simply \( mL \sinh \lambda \).

As natural in the continuum limit, we consider only BA configurations which have the vacuum structure for large rapidities \( \lambda \) at small enough \( \gamma \): there are only real roots and no hole to the far left and right. Therefore, as \( \gamma \) is raised the first mechanism by which roots get isolated in the tails is exactly that illustrated in section \[13\] first one special hole is formed simultaneously at both extremities; then these holes exchange place with the largest and smallest real roots; finally these tend to infinity and then jump to the lines \( \text{Im} \lambda = \pm \pi^2/(2\gamma) \), respectively. At this stage in each extremity there is a normal holes if \( \gamma < \pi/2 \) or one special hole if \( \gamma > \pi/2 \). Then, as \( \gamma \) raises even further, if \( \gamma < \pi/2 \) the normal holes are first pushed to infinity, so that the vacuum structure is reproduced and the mechanism can start over again; if \( \gamma < \pi/2 \) instead, the mechanism starts over from the point when each special hole is about exchange place with the nearest real root.

Thus, for any value of \( \gamma \) we either have the vacuum structure at each extremity or a symmetric situation such that at each tail we find either a special root/hole or a self–conjugated root plus a normal or special hole. All these deformations of the vacuum structure are then removed to infinity by the continuum limit \( N, \Theta \to \infty \) and their contribution to the source \( z(\lambda) \) cancel out by symmetry. Moreover, since the special root/holes are formed exactly when \( \lfloor \frac{1}{2} + \frac{\pi}{\pi} S \rfloor = 0 \) jumps by one, one finds that the continuum version of eq.(2.12) is just

\[
N_{H,\text{eff}} = N_H - 2N_S
\]  

where now the \( N_S \) special root/holes are in the middle of the distribution, that is for \( x \) of order 1. Likewise, the general relation (2.13) among the numbers of holes, special root/holes complex roots takes now the form

\[
N_{H,\text{eff}} = 2S + M_{\text{close}} + 2 M_{\text{wide}} \theta(\pi - 2\gamma)
\]  

We naturally identify \( N_{H,\text{eff}} \) with the number of solitons and antisolitons.

Having established these simple facts, we may write down the NLIE satisfied by the continuum limit \( Z(\lambda) \) of \( Z_N(\lambda) \), namely

\[
Z(\lambda) = mL \sinh \lambda + g(\lambda) + \int_{-\infty}^{+\infty} dx \, G(\lambda - x) \, Q(x)
\]  

23
where $Q(x)$ is related to $Z(x)$ as $Q_N(x)$ to $Z_N(x)$ in eq. (1.14), that is
\[
Q(x) = -i \log \frac{1 + (-)^S e^{iZ(x+i\epsilon)}}{(-)^S + e^{-iZ(x-i\epsilon)}} \quad (6.5)
\]
and
\[
g(\lambda) = z_H(\lambda) + z_S(\lambda) + z_C(\lambda) \quad (6.6)
\]
is the excitation part of the source. The various contributions of holes, special root/holes and complex roots are given by eq.(4.10), (4.15) and (4.18).

We recall that the continuum NLIE (6.4) is to be supplemented by the the quantization rules (1.13). We recall also that the quantization rules for wide pairs require the second determination of the counting function, which now reads, according to eqs. (4.16) and (4.17),
\[
Z(\lambda) = mL(\sinh \lambda) + g(\lambda) + \int_{-\infty}^{+\infty} G(\lambda - x) Q(x) \quad (6.7)
\]
Let us now perform the continuum limit on the energy–momentum. We recall that we need to integrate the quantities $W(\pm \Theta)$ w.r.t. $\Theta$, where $W(\lambda)$ is given by eq. (5.4). Thus we obtain for the energy and momentum of a generic excited state,
\[
E \pm P = E_V + E_H^\pm + E_S^\pm + E_C^\mp + m \int_{-\infty}^{+\infty} \frac{dx}{2\pi} e^{\pm x} Q(x) \quad (6.8)
\]
where $E_V$ is the ground state bulk energy
\[
E_V = N\delta^{-1} \left[ -2\pi + \int_{-\infty}^{+\infty} d\lambda \frac{\phi_{1/2}(\pi \lambda/\gamma + 2\Theta)}{\pi \cosh \lambda} \right]
\]
$E_H^\pm$ and $E_S^\pm$ stand for the contributions from holes and special holes, respectively
\[
E_H^\pm = m \sum_{j=1}^{N_H} e^{\pm h_j}, \quad E_S^\pm = -2m \sum_{j=1}^{N_S} e^{\pm y_j} \quad (6.9)
\]
$E_C^\pm$ represents instead the contributions of the complex roots. In the repulsive regime $\gamma < \pi/2$ we have quite simply, from eq. (5.6),
\[
E_C^\pm = -m \sum_{j=1}^{M_\text{close}} e^{\pm c_j} \quad (6.10)
\]
In the attractive regime $\gamma > \pi/2$ their expression can be calculated, with some lengthy algebra, from eq.(5.5) and read
\[
E_C^\pm = -m \sum_{j=1}^{M_\text{close}} e^{\pm c_j} + m \sum_{j=1}^{M_\text{wide}} \left[ e^{\pm w_j} + e^{\pm (w_j-i\pi\epsilon_j\pi^2/\gamma)} \right] \quad (6.11)
\]
where $\epsilon_j = \text{sign}(\text{Im} w_j)$. It is important to observe that the contribution of each wide root coincides with the second determination of the exponential function (see eq.(4.17) and notice that the second determination of the exponential vanishes identically in the repulsive regime) so that we can write for both regimes

$$E^\pm_C = -m \sum_{j=1}^{M_{\text{close}}} e^{\pm c_j} + m \sum_{j=1}^{M_{\text{wide}}} (e^{\pm w_j})^II$$

(6.12)

Eqs.(6.4) and (6.8), with the supplementary quantization rules (4.19) entirely determine the excited–states scaling functions of the sG model.

The inputs required are the quantum numbers of the holes, the special root/holes and the complex roots. Notice that the quantum numbers of the complex roots cannot in general be chosen arbitrarily, since they are coupled to those of the holes and by the NLIE itself. On the other hand they stay fixed for all values of $mL$ and may be determined most conveniently in the infrared limit $mL \to \infty$. This shall be discussed in the next section; it will become apparent that all quantum numbers associated to particles (solitons and breathers) may in practice be freely chosen, while the quantum numbers associated to the configurations of complex roots which describe the internal $U(1)$ states of the solitons must be restricted to a limited number of distinct possible values depending on the particle quantum numbers through the higher level BAE.

One should then solve the NLIE (6.4) with the locations of holes, special root/holes and complex roots as free parameters, to be fixed later by the supplementary conditions (4.19). The practical feasibility of this is limited to simple enough states, but the existence of a given procedure for any given state ensures that each excited–state scaling function can be determined independently from all other states.

VII. $ML \to \infty$: MASS SPECTRUM AND S–MATRIX

We consider here the limit where the physical size $L$ of the system diverges: we then expect the interaction among the physical particles to cease affecting the energy–momentum. Hence the quantities $E - E_V$ and $P$ should approach finite limits equal to a free massive spectrum. In fact as $mL \to \infty$ it is easy to show that the nonlinear integral terms in eqs. (6.4) and (6.8) all vanish exponentially fast. Indeed $Q(x)$ is peaked around $x = 0$ as the exponential of an exponential while $G(\lambda)$ dies exponentially for large $|\text{Re} \theta|$.

Thus the leading form of the counting function is

$$Z(\lambda) = mL \sinh \lambda + g(\lambda)$$

(7.1)

In the infinite volume limit $Z(\lambda)$ is monotonically increasing since the term $mL \sinh \lambda$ dominates. Therefore no special root/holes are present. In the repulsive regime we then obtain for the excitation energy

$$E - E_V = \sum_{j=1}^{N_H} m \cosh h_j - \sum_{j=1}^{M_{\text{close}}} m \cosh c_j$$

(7.2)

with an analogous expression for the momentum. Notice that the counting function is certainly monotonic for large $mL$, so that $N_S = 0$ and $N_{H,\text{eff}} = N_H$. Another important
observation now concerns the close roots $c_j$’s. Since \( \text{Im} \sinh \lambda > 0 \) for \( \pi > \text{Im} \theta > 0 \), from eq. (7.1) it is clear that the quantization conditions for the close roots, \( \exp[iZ(c_j)] = -1 \), require that the \( c_j$’s move exponentially fast in \( mL \) to positions where \( g(c_j) \) develops the right logarithmic singularities. It easy to check that in these positions pairs of close roots are separated by \( i\pi \) so that their contribution cancel out in eq. (7.2) due to the anti-periodicity of the \( \cosh \) function. Notice that no such driving exists for wide roots since the second determination \( (\sinh \lambda)_{II} \) vanishes identically when \( \gamma < \pi/2 \) for the same anti-periodicity (see eqs. (6.4) and (1.17)).

The fact that the excitation energy and momentum do not depend at all on the complex roots confirms their interpretation of quantum numbers describing the collective internal \( U(1) \) states of the solitons. One must notice indeed that the hole rapidities \( h_j \) are free parameters at \( L = \infty \), subject only on the restriction of being distinct. It appear natural, by continuity, to regard the \( I_{II}, j \), the quantum number of the holes, as free distinct half–odd–integers when \( mL \) is finite.

The situation is more complex in the attractive regime \( \gamma > \pi/2 \) when wide roots explicitly enter the expressions for \( E_{C}^{\pm} \). The infrared limit however simplifies the problem, because when \( \gamma > \pi/2 \) the second determination \( (\sinh \lambda)_{II} \) does not vanish anymore, forcing all complex roots, including the wide roots, to fall into special configurations.

These are of two main types. Given the positive integer \( n \) such that
\[
\frac{n}{n+1} < \frac{\gamma}{\pi} < \frac{n+1}{n+2}
\] (7.3)
and defining \( \varrho = \pi(\pi - \gamma)/\gamma \), there are arrays of up to \( 4[n/2] + 4 \) roots of the form
\[ (\chi - il\varrho, \bar{\chi} + i\pi^2/\gamma - il\varrho) ; \quad l = 0, 1, \ldots, \lfloor n/2 \rfloor \]
plus complex conjugates (we chose \( \text{Im} \chi > 0 \) here). The array collapses to one with just \( 2[n/2] + 2 \) roots, if \( n \leq 2 \), whenever \( \text{Im} \chi = \pi/2 \) and has, quite trivially, one less root if \( \chi \) is self–conjugated, that is \( \text{Im} \chi = \pi^2/(2\gamma) = (\pi + \varrho)/2 \). Most importantly, these arrays in any case contain two close roots. These are the configurations of the first type.

The configurations of the second type are made entirely of wide roots and always have fixed imaginary parts; they are odd strings of the form
\[ \chi \pm il\varrho ; \quad \text{Im} \chi = (\pi + \varrho)/2 , \quad l = 0, 1, \ldots, s \]
and even strings of the form
\[ \chi \pm il\varrho ; \quad \text{Im} \chi = \pm\pi/2 , \quad l = 1, 2, \ldots, s \]
where \( 1 \leq s \leq \pi/\varrho \).

The fundamental difference between the two types of arrays is in their effect on the counting function and on the total energy–momentum: the configurations of the first type, which contain close roots, make room for holes (recall eq. (6.3)) but do not affect the \( L = \infty \) energy–momentum, since their contribution to \( E_{C}^{\pm} \) vanishes as can be seen from eq. (6.11); on the contrary the configurations of the second type, which are made solely of wide roots, do not provide room for holes but do affect in \( E_{C}^{\pm} \), which now contain the extra terms
\[
N_h \sum_{j=1}^{N_h} m_{s_j} e^{\pm \text{Re} \chi_j}
\]
where

$$m_s = 2m \sin(s\theta/2)$$  \hspace{1cm} (7.4)$$

is the breather mass spectrum and $N_B$ is the total number of configurations of the second type. These configurations corresponds to the breathers, that is the $S = 0$ soliton–antisoliton bound states. On the other hand the configurations of the first type describe the various polarization states of the soliton–antisoliton system. We can now repeat the argument used for the holes in the repulsive regime: in the attractive regime both the hole parameters (soliton rapidities) and the locations of second–type configurations (breather rapidities) are free in the infinite volume. At finite $mL$, their quantum numbers are free.

The physical $S$–matrix describing the scattering of the solitons and their bound states may be calculated directly from eq. (7.1): clearly $-\exp[ig(h_j)]$ is the total phase which a physical particle with rapidity $h_j$ accumulates by going around the circle, since we assumed periodic boundary conditions. For instance, in a two–hole state with $S = 1$ (no complex roots), we would find, dropping the corrections which vanish in the $N \to \infty$ limit,

$$m \sinh h_1 = \frac{2\pi}{L} I_{H1} - \frac{1}{L} \chi(h_1 - h_2)$$

Hence $\chi(h_1 - h_2)$ is the scattering phase–shift between two solitons or two antisolitons. The rest of the factorizable two–body $S$–matrix of the $sG$ model can be reconstructed by considering more general states with two holes and certain complex roots [12].

As is well known [11], hole positions and complex roots for physical states are connected by the higher level BA equations. This is a finite set of BA-type equations where the holes act as source part and the complex roots appear as BA roots. For a given set of holes, they determine all possible states. The emergence of this higher level BA structure can be seen quite clearly from eq.(7.1). In the repulsive regime it suffices to evaluate $Z(\lambda)$ at the position of each complex root and then sum the result over the two partners of each close pair, to cancel out the imaginary parts proportional to $mL$. There is no need to do this for wide roots, since the second determination $\sinh(\lambda)_{II}$ of $\sinh(\lambda)$ vanishes identically. In the attractive regime things are slightly more complicated: by summing $Z(\lambda) = 2\pi I$ over all the members of an array of the second kind (which corresponds to a breather) one finds the quantization rule for the rapidities of the breathers; by summing $Z(\lambda) = 2\pi I$ over half or over all the members of an array of the first kind, depending on its size, one finds the higher level BA relations between the free parameters of the arrays and the rapidities of the holes.

The exponentially small corrections to the counting function and the energy–momentum can be calculated by iteration. The two next–to–leading orders in the case of the ground state were calculated in this way in ref. [11]. For the excited states one has to take into account that the supplementary quantization rules eq. (4.19) have to be satisfied to the appropriate order in $e^{-mL}$ (notice that the special configurations of complex roots are valid only to leading order). We shall not dwell further here on this infrared expansion.

**VIII. ML → 0 : CONFORMAL SPECTRUM**

When the dimensionless parameter $r \equiv mL$ is very small the regions with positive and negative $\lambda$ where $r \sinh \lambda \sim 1$ are very far apart. We then expect that in a generic case
the central portion of $Z(x)$ broadens to a single plateau, or a two–plateau system, which extends to all $|x|$'s smaller than $\log(2/r)$ and then rapidly disappears (not necessarily in a monotonic fashion) in favor of the dominant exponential growth. The two functions

$$Z_\pm(\lambda) = \lim_{r \to 0} Z(\lambda \pm \log \frac{2}{r}) \quad (8.1)$$

describing this crossover determine entirely the leading terms as $r \to 0$ in the energy–momentum. We shall call $Z_\pm(\lambda)$ “kink” functions, although the “kink” terminology applies more precisely to the function $Q(x)$ for strictly positive $\epsilon$ (recall eq. (6.5)): when $Z(x)$ changes exponentially fast $Q(x)$ dies like the negative exponential of an exponential (it would oscillate exponentially fast for zero $\epsilon$). Thus $Q(x)$ has a central plateau (or a two–plateau region) of width $\sim -\log(r)$ and height $\omega$ with two kink–like drops to zero at the sides of the central region.

Applying the scaling relation to the NLIE satisfied by $Z(\lambda)$ yields the two kink equations

$$Z_\pm(\lambda) = \pm e^{\pm \lambda} + g_\pm(\lambda) + \int_{-\infty}^{+\infty} dx \, G(\lambda - x) \, Q_\pm(x) \quad (8.2)$$

where $Q_\pm(x)$ is related to $Z_\pm(x)$ in the usual way (see eq. (4.14)), while $g_\pm(\lambda)$ follows from $g(\lambda)$ through the scaling $\lambda \to \lambda \pm \log \frac{2}{r}$. The source $g_\pm(\lambda)$ depends on the positions of holes, complex roots and eventual special root/holes as $r \to 0$.

For instance, the hole parameters $\{h_j\}$ may be divided in right–moving, left–moving and the rest according to

$$\{h_j\} = \{h_j^\pm \pm \log \frac{2}{r}, h_j^0\}$$

where the $h_j^\pm$ and $h_j^0$ have finite limits as $r \to 0$. The quantization rules for right–moving and left–moving holes are written

$$Z_\pm(h_j^\pm) = 2\pi I_{H, j}^\pm \quad j = 1, 2, \ldots, N_{H, j}^\pm$$

where by definition $I_{H, j}^+ \geq I_{H, j}^+$ and $I_{H, j}^- \leq I_{H, j}^-$. The hole contribution to $g_\pm(\lambda)$ may now be evaluated to be, (see eqs.(4.10) and (6.6))

$$z_H(\lambda \pm \log \frac{2}{r}) \xrightarrow{r \to 0} z_H^\pm(\lambda) \pm (N_H - N_H^\pm)\chi_{\infty}, \quad z_H^\pm(\lambda) = \sum_{j=1}^{N_H^\pm} \chi(\lambda - h_j^\pm) \quad (8.3)$$

where we recall that

$$\chi_{\infty} = \frac{\pi/2 - \gamma}{1 - \gamma/\pi}$$

Analogous arguments and expressions apply to special root/holes and the complex roots, as evident from eqs.(4.15) and (4.18).

As a matter of fact, we can rely on the analysis of the limit $\Theta \to \infty$ performed in section [1]. Indeed, since $\Theta \simeq \log(4N/r)$ in the continuum limit, to reach the scaling form $Z_\pm(\lambda)$ of $Z(\lambda)$ we can follow two different, but equivalent limiting procedures, namely

$$Z_N(\lambda \pm \Theta, \Theta) \xrightarrow{N \to \infty} Z(\lambda \pm \log \frac{2}{r}) \xrightarrow{r \to 0} Z_\pm(\lambda)$$
and
\[ Z_N(\lambda \pm \Theta, \Theta) \xrightarrow{r \to 0} Z_N^\pm(\lambda \pm \log(2N)) \xrightarrow{N \to \infty} Z_\pm(\lambda) \]

The intermediate step of the second procedure involves the functions \( Z_N^\pm(\lambda) \) studied at length in section \([III]\). The subsequent scaling as \( N \to \infty \) does not affect the conclusions drawn there, except that it simplifies some formulae due to the removal of extremal special root/holes and self–conjugated roots of the first class, as discussed in section \([V]\).

Hence the partial “hole–induced” \( U(1) \) charges of the right– and left–moving sea take the form (see eq.(3.10) and eq.(3.9))
\[ \hat{S}^\pm = \frac{1}{2} [N_H^\pm - 2N_S^\pm - M_{\text{close}}^\pm - 2\theta(\pi - 2\gamma) M_{\text{wide}}^\pm] \]

We may write also, with obvious notation
\[ \hat{S}^0 = \frac{1}{2} [N_H^0 - 2N_S^0 - M_{\text{close}}^0 - 2\theta(\pi - 2\gamma) M_{\text{wide}}^0] \]

so that we read from eq.(6.3)
\[ \hat{S}^+ + \hat{S}^- + \hat{S}^0 = S \]

In addition we have \( S^+ + S^- + S^0 = S \) by definition. One can then verify that
\[ g_\pm(x) = z_H(x) + z_S(x) + z_C(x) \pm 2(S - \hat{S}^\pm) \chi_\infty \pm 2\pi n_{\text{wide}}^\pm \]

where the integers \( n_{\text{wide}}^\pm \) are given in eq.(3.1).

From section \([III]\) we read other important relations like
\[ Z_\pm(\mp \infty) = \Omega_\pm = \pm(\pi - 2\gamma)(S - 2S^\pm) \pm 2\pi \ell_{\text{wide}}^\pm \]

and
\[ Q_\pm(\mp \infty) = \mp \omega_\pm = 2\gamma(S - 2S^\pm) - 4\pi(\hat{S}^\pm - S^\pm) = 2\pi(S - 2\hat{S}^\pm) - 2(\pi - \gamma)(S - 2S^\pm) \]

Evidently \( \Omega_\pm \) must satisfy the asymptotic form of eq.(8.2), that is the “plateau equation”
\[ \Omega_\pm = g_\pm(\mp \infty) \pm \frac{\chi_\infty}{\pi} \omega_\pm \]

One easily calculates from eqs.(4.15)–(4.12) and (4.15)
\[ g_\pm(\mp \infty) = \pm 2(S - 2\hat{S}^\pm) \chi_\infty \pm 2\pi \ell_{\text{wide}}^\pm \]

One can check that eqs.(8.4) and (8.5) indeed solve this plateau equation for any value of \( \gamma \) only if \( g_\pm(\mp \infty) \) correctly contains, when required, the contribution \( \pm 2\chi_\infty \) of the special root/holes.

It is convenient to introduce also the function
\[ Q_0(x) = -\Omega + \sum_{j=1}^{M^0} \phi_1(x - \lambda_j^0) \]
(recall that $\Omega = \frac{1}{2}(\Omega_+ + \Omega_-)$ and $M^0 = -S^0 = S^+ + S^- - S$). It is understood that the $\lambda^0$ are the limit values of the roots with a finite limit as $r \to 0$, so that $Q_0(x)$ is $r$–independent. In particular we have $Q_0(\pm \infty) = -Q_\mp(\pm \infty)$. We may now write

$$Q(x) = Q_-(x + \log \frac{r}{\Omega}) + Q_+(x - \log \frac{2r}{\Omega}) + Q_0(x) + q(x)$$  \hspace{1cm} (8.7)$$

where $q(\lambda)$ collects all subleading contributions and vanishes (albeit non–analytically) as $r \to 0$ uniformly in $x$ (in other words $q(x) = o(1)$ for any $x$).

Using the decomposition eq. (8.7) we then find for the energy–momentum

$$E \pm P = E_V + E_H^\mp + E_S^\mp + E_C^\mp \mp E_K^\mp \mp \frac{m}{2\pi} \int_{-\infty}^{+\infty} dx \ e^{\pm x} q(x)$$  \hspace{1cm} (8.8)$$

where the hole contribution reads

$$E_H^\pm = \frac{2}{L} \left[ \sum_{j=1}^{N_H^\pm} e^{\pm h_j^\pm} + \frac{r}{2} \sum_{j=1}^{N_H^0} e^{\pm h_j^0} + \frac{r^2}{4} \sum_{j=1}^{N_H^\pm} e^{\pm h_j^\pm} \right]$$

with a similar expressions for the special root/holes and complex root contribution $E_S^\mp$ and $E_C^\mp$, as can be read from eqs. (6.9) and (6.12). The kink contribution reads

$$E_K^\pm = \frac{m}{2\pi} \int_{-\infty}^{+\infty} dx, e^{\pm x} [Q_-(x + \log \frac{2r}{\Omega}) + Q_+(x - \log \frac{2r}{\Omega}) + Q_0(x)]$$

$$= \frac{1}{L} \int_{-\infty}^{+\infty} dx e^{\mp x} Q_\mp(x) + \frac{m}{2\pi} \int_{-\infty}^{+\infty} d\theta e^{\pm x} [Q_0(x) - Q_0(\pm \infty)]$$

$$+ \frac{m^2 L}{4\pi} \int_{-\infty}^{+\infty} d\theta e^{\pm x} [Q_\mp(x) - Q_\mp(\pm \infty)]$$

$$= \frac{1}{L} \int_{-\infty}^{+\infty} dx e^{\pm x} [Q_\mp(x) + r \frac{Q_0'(x)}{2} + r^2 \frac{Q_0''(x)}{4}]$$

The last integral in eq. (8.8) with $q(\lambda)$ contains corrections vanishing non–analytically as $r \to 0$. Their explicit calculation can be done with Wiener–Hopf techniques as in ref. [11].

On the other hand the terms of order $L^{-1}$, which are those relevant from the conformal theory viewpoint, may be found without even solving the kink equations. The basic ingredient is the following lemma:

**LEMMA.** Assume that $f(x)$ satisfies the nonlinear integral equation

$$-i \log f(x) = \varphi(x) + \int_{-\infty}^{+\infty} dy G(x - y) F(y)$$  \hspace{1cm} (8.9)$$

where $F(x) = 2 \text{Im} \log[1 + f(x + i\epsilon)]$, $\varphi(x)$ is real and $G(x) = G(-x)$ is real too, with bounded integral ($L^1$) and peaked around the origin. Eq. (8.9) tells us that $f(x)$ has unit modulus for real $x$. In addition, we assume that when $f(x + i\epsilon)$ is real then $f(x + i\epsilon) > -1$. Then,

$$\int_{-\infty}^{+\infty} dx \varphi'(x) F(x) = -2 \text{Re} \int_{R} \frac{du}{u} \log(1 + u) - \frac{1}{2} \left[ F_+^2 - F_-^2 \right] \int_{-\infty}^{+\infty} dx G(x)$$  \hspace{1cm} (8.10)$$
where \( F_{\pm} = F(\pm \infty) \) and \( \Gamma \) is any contour in the complex \( u \)-plane that goes from \( f_- = f(-\infty) \) to \( f_+ = f(+\infty) \) (avoiding by hypothesis the logarithmic cut from \( -\infty \) to \(-1\)).

This lemma can be proved as follows. Replacing \( \varphi'(x) \) in the l. h. s. through eq. (8.9) yields

\[
\int_{-\infty}^{+\infty} dx \varphi'(x) F(x) = 2 \text{Im} \int_{-\infty}^{+\infty} dx \left[ -i \frac{d}{dx} \log f(x) - \int_{-\infty}^{+\infty} dy G'(x - y) F(y) \right] \times \log[1 + f(x + i\epsilon)]
\]

\[= -2 \text{Re} \int_{\Gamma} \frac{du}{u} \log(1 + u) - \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy F(x) G'(x - y) F(y)\]

One must be careful now with the double integral, which would seem to vanish by symmetry under \( x \equiv y \). In fact one is allowed to interchange the two integrations only if uniform convergence holds. This is however not true if \( \varphi(x) \) and therefore \( F(x) \) do not vanish at infinity. Proceeding with due care one finds, for \( a < b \),

\[
I(a,b) = \int_{-a}^{a} dx \int_{-b}^{b} dy F(x) G'(x - y) F(y) = \int_{-a}^{a} dx F(x) \left[ \int_{a}^{b} dy G'(x - y) F(y) + \int_{-b}^{-a} dy G'(x - y) F(y) \right]
\]

Hence, upon integration by parts and letting \( b > a \to \infty \)

\[
I(a,b) \approx F_- \int_{-a}^{a} dx [G(x + b) - G(x + a)] + F_+ \int_{-a}^{a} dx [G(x - a) - G(x - b)]
\]

\[\approx \frac{1}{2} \left[ F_+^2 - F_-^2 \right] f_{+\infty} \int_{-\infty}^{+\infty} dx G(x)\]

Therefore we obtain the identity (8.10).

We are now in the position to explicitly calculate the conformal dimensions encrypted in the order \( L^{-1} \) term of the energy–momentum. To make the notation lighter, we shall restrict our attention to \( E + P \). The other chirality \( E - P \) follows by applying the appropriate symmetries. We need to evaluate the quantity

\[
A_+ \equiv \lim_{r \to 0^+} \frac{1}{2} L(E^+_H + E^+_S + E^+_C - E^+_K)
\]

\[= \sum_{j=1}^{N_H^+} e^{h^+_j} - 2 \sum_{j=1}^{N_S^+} e^{y^+_j} - \sum_{j=1}^{M^+_{\text{close}}} e^{c^+_j} + \sum_{j=1}^{M^+_{\text{wide}}} (e^{w^+_j})_H - \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx e^x Q_+(\theta) \tag{8.11}\]

We know that \( Z_+(\theta) \) solves the equation

\[
Z_+(\lambda) = \varphi_+(\lambda) + \int_{-\infty}^{+\infty} dx G(\lambda - x) Q_+(x)
\]

with \( \varphi_+(\lambda) = e^\lambda + g_+(\lambda) \) and that the various unknown parameters \( h^+_j, y^+_j, c^+_j, w^+_j \) are quantized according to

\[
Z_+(h^+_j) = 2\pi I^+_H j, \quad Z_+(y^+_j) = 2\pi I^+_S j, \quad Z_+(c^+_j) = 2\pi I^+_{\text{close}} j, \quad Z_+(w^+_j) = 2\pi I^+_{\text{wide}} j
\]

31
Summing $Z_+(h_j^+)$ over $j$ now yields the relation

$$2\pi I_H^+ \equiv 2\pi \sum_{j=1}^{N_H^+} I_{H,j}^+ = \sum_{j=1}^{N_H^+} \left[ e^{h_j^+} + g_+(h_j^+) \right] + \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx \, z'_{H+}(x) \, Q(x)$$  \hspace{1cm} (8.12)

where we used the relation between $z_H$ and $\chi$ (see eq. (4.10)). Next we sum $Z_+(h_j^+)$ and $Z_+(c_j^+)$ over special root/holes and close roots, respectively; we obtain in a closely parallel way

$$4\pi I_S^+ \equiv 4\pi \sum_{j=1}^{N_S^+} I_{S,j}^+ = 2 \sum_{j=1}^{N_S^+} \left[ e^{y_j^+} + g_+(y_j^+) \right] - \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx \, z'_{S+}(x) \, Q_+(x)$$  \hspace{1cm} (8.13)

$$2\pi I_{\text{close}}^+ \equiv 2\pi \sum_{j=1}^{M_{\text{close}}^+} I_{\text{close},j}^+ = \sum_{j=1}^{M_{\text{close}}^+} \left[ e^{c_j^+} + g_+(c_j^+) \right] - \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx \, z'_{\text{close}+}(x) \, Q_+(x)$$  \hspace{1cm} (8.14)

In the case of wide roots we must recall that the second determination has to be used, according to eqs. (6.7) and (4.17)). Thus we have

$$Z_+(w_j^+) = e_{\Pi}^{w_j^+} + g_+(w_j^+)_{\Pi} + \int_{-\infty}^{+\infty} dx \, G(w_j^+ - x)_{\Pi} \, Q_+(x) = 2\pi I_{\text{wide},j}^+$$

and summing $Z_+(w_j^+)$ over $j$ now gives

$$2\pi I_{\text{wide}}^+ \equiv 2\pi \sum_{j=1}^{M_{\text{wide}}^+} I_{\text{wide},j}^+ = \sum_{j=1}^{M_{\text{close}}^+} \left[ e_{\Pi}^{w_j^+} + g_+(w_j^+)_{\Pi} \right] - \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx \, z'_{\text{wide}+}(x) \, Q_+(x)$$  \hspace{1cm} (8.15)

where we have used the relation

$$z'_{\text{wide}+}(x) = -2\pi \sum_{j=1}^{M_{\text{wide}}^+} G(x - w_j)_{\Pi}$$

which follows from eqs. (4.18).

We may now use eqs. (8.12)–(8.15) to eliminate the sum over exponentials in eqs. (8.11); at the same time the derivative of the complete source term is reconstructed in the integral with $Q_+(x)$. Thus we obtain

$$A_+ = 2\pi \left( I_H^+ - 2I_S^+ - I_C^+ \right) - \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx \, \varphi'_+(x) \, Q_+(x) + \Sigma_+$$

$$\Sigma_+ = -\sum_{j=1}^{N_H^+} g_+(h_j^+) + 2 \sum_{j=1}^{N_S^+} g_+(y_j^+) + \sum_{j=1}^{M_{\text{close}}^+} g_+(c_j^+) + \sum_{j=1}^{M_{\text{wide}}^+} g_+(w_j^+)_{\Pi}$$  \hspace{1cm} (8.16)

Now, by exploiting the oddness of $\chi(\theta)$ and $\phi_\nu(\lambda)$, one may verify that all terms in $\Sigma_+$ which depend explicitly on the positions of holes, special root/holes and complex roots cancel out completely, leaving behind only constants, namely

$$\Sigma_+ = -4S^+(S - \tilde{S}^+) \chi_\infty + 2\pi q_{\text{wide}}^+$$  \hspace{1cm} (8.17)
where $q_{\text{wide}}^+$ is a rather involved integer or half-odd-integer which vanishes when no wide roots are present. Its explicit form is more conveniently determined case by case. The integral in eq. (8.16) is computed directly from the lemma, upon the identifications

$$f_\pm = \exp[Z_+(\pm \infty + i \epsilon)] , \quad F_\pm = Q_+(\pm \infty) = 2 \text{Im} \log(1 + f_\pm)$$

The $\epsilon$ is important at $+ \infty$, where $Z_+(\theta)$ diverges exponentially; hence one must let $\epsilon \to 0$ after the $\theta \to + \infty$ limit. On the other hand $Z_+(\theta)$ tends to a constant as $\theta \to - \infty$, so that we can set $\epsilon = 0$ for $f_-$. We have therefore $f_+ = F_+ = 0$, $f_- = e^{i \omega}$ and $F_- = \omega$ and so, Recalling that the integral of $G(\theta)$ over the real axis is just $\chi_\infty / \pi$,

$$\int_{-\infty}^{+\infty} dx \varphi'_+(x) Q_+(x) = -2 \text{Re} \int_{\Gamma} \frac{du}{u} \log(1 + u) + \frac{\omega^2 \chi_\infty}{2\pi}$$

We now choose $\Gamma$ to be the union of the arc of circle from $e^{i \omega +}$ to 1 and the straight segment from 1 to 0, so that

$$-2 \text{Re} \int_{\Gamma} \frac{du}{u} \log(1 + u) = 2 \int_0^1 \frac{du}{u} \log(1 + u) - 2 \int_0^{\omega +} d\alpha \text{Im} \log(1 + e^{i \alpha}) = \frac{\pi^2}{6} - \frac{\omega^2}{2}$$

and

$$\frac{1}{2 \pi} \int_{-\infty}^{+\infty} dx \varphi'_+(x) Q_+(x) = \frac{\pi}{12} - \frac{\omega^2}{8 \pi (1 - \gamma / \pi)}$$

Using eq.(8.14) and extending the derivation to the negative chirality in the obvious way, finally yields

$$\frac{1}{2 \pi} A_\pm = -\frac{1}{24} + \frac{S^2}{4(1 - 2 \pi)} + \frac{1}{4} \left(1 - \frac{\gamma}{\pi}\right)(S - S^\pm)^2 - \frac{1}{2} S^2 + (S - 2 \hat{S}^\pm)(S^\pm - \hat{S}^\pm)
\pm (I_H^+ - 2 I_S^+ - I_C^+ + q_{\text{wide}}^+)$$

$$\pm (I_H^- - 2 I_S^- - I_C^- + q_{\text{wide}}^- - \hat{S}^\pm)$$

This result can be rewritten more conveniently as

$$\frac{1}{2} (E - \frac{1}{2} E_V \pm P) \simeq 2 \pi L^{-1} \left[ -\frac{1}{24} + \Delta_{sG}' \pm n_\pm \right]$$

where

$$\Delta_{sG}^\pm = \frac{[S - (1 - \gamma / \pi)(S - 2 S^\pm)]^2}{4(1 - \gamma / \pi)}$$

and

$$n_\pm = \pm (I_H^+ - 2 I_S^+ - I_C^+ + q_{\text{wide}}^+) - \hat{S}^\pm (S + 2 S^\pm - 2 \hat{S}^\pm)$$

Eqs. (8.20)-(8.22) display the spectrum of a conformal field theory with central charge $c = 1$. The excitations spectrum corresponds to a Coulomb gas and represent the conformal dimensions at the ultraviolet fixed point of the operators which interpolate each given state [see section IX]. $\Delta_{sG}^\pm$ are to be identified with the conformal dimensions of primary operators. These operators are labeled by the U(1) charge $S$ and the partial U(1) charge $S^\pm$. In eq. (8.20) we see the integers $n_\pm$ added to $\Delta_{sG}^\pm$ inside the bracket. It can be verified that they are always nonnegative; when positive, they indicate that the corresponding Bethe state is associated to a secondary conformal operator.

We consider now some relevant examples.
• States with no complex roots

We begin with the states without complex roots of any type, as done already in sections [4] and [11]. For $S$ even and $\gamma$ small enough, we have $N_S = 0$, $N_H = 2S$, $S^+ + S^- = S = \hat{S}^+ + \hat{S}^-$, and $S^\pm = S^\pm = N_H^+/2$. Recall in fact eq. (3.4):

$$\hat{S}^\pm = S^\pm + \frac{1}{2} \text{sign}(S - 2S^\pm) \left( \frac{1}{2} + \frac{\gamma}{\pi} |S - 2S^\pm| \right)$$  

(8.23)

Now the smallest value of $\pm I_H^\pm$, for a given value of $N_H^\pm = 2S^\pm$ is attained when the $I_H^\pm$ are all consecutive half–odd–integers starting from $I_{\min}^+ = (1 - \triangle S)/2$ and $-I_{\max}^- = (1 + \triangle S)/2$, respectively. In this way a single sequence of holes without interruptions is formed and we find

$$\pm I_H^\pm = \sum_{j=1}^{N_H^\pm} \left[ \frac{1}{2} (1 \mp \triangle S) + j - 1 \right] = S^\pm S$$

so that we find $n_\pm = 0$, which shows that these states are interpolated by primary operators. We see that the secondary operators in these conformal towers correspond to the states when the holes are arbitrarily distributed and their sequence has gaps. Thus the full conformal tower reproduces the phase space of the $2S$ holes.

At larger values of $\gamma$ we might have $\hat{S}^\pm \neq S^\pm$. In the case of the primary states this happen because $\hat{S}^\pm$ changes while $S^\pm$ stays fixed. Hence $\Delta_{\text{Ch}}^\pm$ does not change while $S$ gets divided in different ways into $\hat{S}^\pm$ for different values of $\gamma$. At the special values of $\gamma$ where $\hat{S}^\pm$ jumps we have $\hat{S}^0 = 1/2$: one hole is passing from the sea with higher U(1) charge to the other (recall the discussion in section [11]). In any case one finds that $n_\pm$ stays constant an equal to 0 for all $\gamma$: in fact, if $\hat{S}^+ = S^+ - 1/2$ and $\hat{S}^0 = 1/2$, for instance, then $I_H^+ \to I_H^+ - I_{\min}^+ = S^+ S - (1 - \triangle S)/2 = \hat{S}^+(S + 2S^+ - 2\hat{S}^+)$, while the negative chirality sea is not modified at all.

It is easy to check that the same conclusions apply when a special root/hole is formed by raising $\gamma$ when $I_{\max}^-$ and/or $I_{\min}^+$ is occupied by a real root (notice that this implies a secondary state): the conformal dimensions of the state do not change. On the other hand, when the real root passes from one sea to the other as $\gamma$ crosses one of a critical set of rational values (see again the section [11]), $S^\pm$ do change and the state switch from a conformal tower to another. Notice that exactly at the critical value $S^0 = -1$ and there are two plateaus in $Z(\lambda)$.

When $S$ is odd (still no complex root) there is the new possibility that $S^0 = -1$ for all $\gamma$ with $S^\pm = (S + 1)/2$. At the quantization value $Z(x) = 0$ are associated a special real root and two normal holes if $\gamma < \pi$ and just a normal real root if $\gamma > \pi$. Indeed we have $N_H^\pm = 2\hat{S}^\pm = 2S^\pm - \lfloor \frac{1}{2} + \frac{\gamma}{\pi} \rfloor$ by eq. (8.23). However, even if the sequence of holes is interrupted by the real root at $Z(x) = 0$, these $\gamma$–generic two–plateau configurations contain a primary state. In fact, when the holes are maximally packed around the origin, since $I_{\min}^- = -I_{\max}^+ = \lfloor \frac{1}{2} + \frac{\gamma}{\pi} \rfloor$ we find

$$I_H^\pm = \sum_{j=I_{\min}^+}^{S} j = \sum_{j=I_{\min}^+}^{S} j = \frac{1}{2} S(S + 1) = \hat{S}^+(S + 2S^\pm - \hat{S}^+)$$

and eq. (8.22) gives $n_\pm = 0$. 

34
- **Zero charge states**

Let us consider now further illuminating examples. Let us begin with the BA states with two holes and \( S = 0 \) that in the large volume limit describe the scattering of a soliton–antisoliton pair. The antisymmetric state contains two holes and one close pair in both regimes. The symmetric state contains instead two holes and one self–conjugated root in the repulsive regime and two holes and one degenerate array of the first kind based on a self–conjugated root (see section VII) in the attractive regime. In any case one finds from eq.(6.3) that \( S = 0 \), as required. The quantum number of each complex roots is entirely fixed by the number of holes through the higher level BA and takes therefore a unique value.

We now let \( r \to 0 \) keeping all quantum numbers fixed and assuming for simplicity that a unique plateau is formed. The two holes and the complex roots are then either all right–moving or all left–moving, so that \( \hat{S}^\pm = S^\pm = 0 \). One finds that the sum \( I^\pm_C \) over all complex roots identically vanishes (taking into account also the integer or half–odd–integer \( q^\pm_{\text{wide}} \) in eq.(8.22) when there are wide roots). Hence we have \( \Delta^\pm_{\text{sG}} = 0 \) with either \( n^+_1 = 0 \) and \( n^-_1 = I^-_H \geq 2 \) or \( n^-_1 = 0 \) and \( n^+_1 = I^+_H \geq 2 \).

Next let us consider the states with only wide roots and no holes in the attractive regime. This are the breather states (notice that the string–like configurations proper of the large volume limit \( r \to \infty \) of section VII are largely deformed in the opposite limit). Also these states have all \( \Delta^\pm_{\text{sG}} \) since all the various U(1) charges identically vanish. One also finds \( n^\pm \geq 0 \).

Therefore it would appears that the states with zero charge all correspond to descendants of the unit operator from the conformal viewpoint. On the other hand we should not forget that eq.(8.20) represent only the leading term in the \( r \to 0 \) limit. In the subleading corrections there should be differences among the various zero charge states that highlight the different ultraviolet properties of the operators that interpolate such states. In particular the states with only one self–conjugated root in the attractive regime are those of the lightest breather, that is the fundamental boson interpolated by the sG field itself. Since the ultraviolet fixed point of the sG model is the free massless boson field theory, the two–point function of the sG field has a logarithmic singularity at short distances. This has to appear as \( \log(2/r) \) correction in the scaling functions of the lightest breather. We remand a detailed analysis of these aspects of the breather states to further studies.

Finite size corrections in lattice models have been computed in ref. [18] using related but somehow different methods. Our derivation of the NLIE and the calculational methods based on it are simpler and apply to a wider set of models. Moreover, we better control the constant pieces that yield the descendant fields states.

**IX. THE COULOMB GAS AND DUALITY SYMMETRY**

The conformal dimensions for a Coulomb gas (central charge \( c = 1 \)) take the form [14]

\[
\Delta_{e,m}(R) = \frac{1}{2R^2} \left( \frac{e}{2} + mR^2 \right)^2
\]

where \( e, m \in \mathbb{Z} \) stand for the “electric” and “magnetic” charges and \( R \) for the compactification radius.
Notice that the spectrum (9.1) is invariant under ‘electromagnetic’ duality \[ e \leftrightarrow 2m, \quad R \leftrightarrow 1/R \] (9.2)

This duality in fact maps conformal dimensions with even \( e \). It is then an invariance for the subset of primary fields \((e, m)\) with even \( e \).

More generally, for any natural number \( K \) we have the following \( K \)-duality invariance,

\[
\Delta_{e,m}(R) = \Delta_{2mK, e/(2K)}(K/R) \tag{9.3}
\]

This is an endomorphism for conformal states with an ‘electric’ charge \( e \) which is a multiple of \( 2K \). For \( K = 1 \) we recover the duality defined by eq.(9.2).

A look to our results for the sine-Gordon model [eq.(8.21)] shows that the compactification radius has the value

\[
R = \sqrt{2(1 - \gamma/\pi)} = \frac{\beta}{\sqrt{4\pi}} \tag{9.4}
\]

That is,

\[
\Delta_{SG}(R) = \frac{1}{2R^2} \left[ S + \frac{1}{2} R^2 \Delta S \right]^2 \tag{9.5}
\]

Hence the “electric” and “magnetic” charges for sine-Gordon are identified as

\[
e = 2S = 2(S^+ + S^-), \quad m = \frac{1}{2} \Delta S = \frac{1}{2}(S^+ - S^-)
\]

in terms of the two partial \( U(1) \) charges.

Notice that \( 0 \leq R \leq \sqrt{2} \) since \( 0 \leq \gamma \leq \pi \). We see that \( K/R \) does not always belong to this interval. For \( K = 1 \),

\[
\frac{1}{\sqrt{2}} \leq \frac{1}{R} \leq \infty
\]

and there is the nontrivial overlap \((\frac{1}{\sqrt{2}}, \sqrt{2})\) between the allowed values for \( R \) and \( 1/R \). In particular, the invariant point of the duality mapping, \( R = 1 \), is within such interval. \( R = 1 \) corresponds to the free field point \( \gamma = \pi/2 \) and \( \beta^2 = 4\pi \).

\( R = \sqrt{2} \) corresponds to the rational limit of the six-vertex model \( (\gamma = 0) \) and to the strong repulsive limit of sine-Gordon \( (\beta^2 = 8\pi) \), where it becomes strictly renormalizable and equivalent to the \( SU(2) \) Thirring model.

\( R = \frac{1}{\sqrt{2}} \) corresponds to \( \gamma = 3\pi/4 \) in the attractive regime. This is the threshold for the third soliton-antisoliton bound state \((s = 3 \text{ in eq.}(7.4))\).

For \( K = 2 \), only the fixed point \( R = \sqrt{2} \) \( (\gamma = 0) \) is mapped inside the allowed interval. This duality corresponds to the weak-strong coupling mapping in the sinh-Gordon and Toda field theories discussed in ref. \[27\]. [Notice that \( \beta \) becomes \( i\beta \) for sinh-Gordon and Toda theories yielding a different duality structure.]

For \( K > 2 \), \( K/R \) is always outside the interval \((0, \sqrt{2})\).
It is interesting to compare the conformal dimensions (8.21) for the ultra-relativistic (ultraviolet) limit of the sine-Gordon theory with those for the low energy limit (infrared) of the six-vertex model \[9,20\]. These can be written

\[
\Delta_{6V} = \frac{1}{4(1 - \gamma/\pi)} [\Delta S + (1 - \gamma/\pi)S]^2 = \frac{1}{2R^2} \left[\Delta S + \frac{1}{2}R^2 S\right]^2
\]  

(9.6)

where \(S^\pm \equiv \frac{1}{2}(S \pm \Delta S)\) now is the contribution to the third component of the spin due to the right and left tail of the BA distribution, respectively. We see that the \(\Delta_{6V}\) and the \(\Delta_{sG}\) are connected by the exchange \(S \leftrightarrow \Delta S\) between U(1) charge and chiral U(1) charge. Such exchange is equivalent to make \(R \leftrightarrow R^{-1}\).
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Fig. 1: Cuts of the function $\phi_{x/g}(\pi\lambda/\gamma)$ for $x < \pi/2$ (left) and $x > \pi/2$ (right).

Fig. 2: The ground state counting function $Z_N(\lambda)$ for $N = 64$, $\gamma = 7\pi/17$ and $\Theta = 16.4$. It is calculated by solving numerically the corresponding BAE.
Fig.3: The central portion of $Z_N(\lambda)$ for $N = 64$, $\Theta = 16.4$, $S^- = 2$ and $S^0 = S^+ = 0$ as $\gamma/\pi$ changes from 1: $\frac{10}{41}$, 2: $\frac{39}{41}$, 3: 0.296 4: $\frac{20}{61}$ and 5: 0.3332. The rightmost quantization value of the left sea is $\pi$ (the dotted line) and is occupied by a real root which gets closer and closer to the origin. For a detailed description see section III.

Fig.4: The central portion of $Z_N(\lambda)$ for $N = 64$, $\Theta = 16.4$, $S^- = 3$, $S^0 = 0$ and $S^+ = -1$ as $\gamma/\pi$ changes from 6: $\frac{20}{59}$, 7: 0.356, 8: 0.369 and 9: $\frac{5}{13}$. The real root at $Z_N(\lambda) = \pi$ has passed to the right sea. For a detailed description see section III.