Higher-Order Soft Corrections to Lepton Pair and Higgs Boson Production

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Abstract

Utilizing recent three-loop results on the quark and gluon splitting functions and form factors, we derive the complete threshold-enhanced third-order (N\textsuperscript{3}LO) QCD corrections to the total cross sections for the production of lepton pairs and the Higgs boson in hadron collisions. These results, for the latter case obtained in the heavy top-quark limit, are employed to extend the threshold resummation for these processes to the fourth logarithmic order. We investigate the numerical impact of the higher-order corrections for Higgs boson production at the TEVATRON and the LHC. Our results, suitably treated in Mellin $N$-space, provide a sufficiently accurate approximation to the full N\textsuperscript{3}LO contributions. Corrections of about 5\% at the LHC and 10\% at the TEVATRON are found for typical Higgs masses. The N\textsuperscript{3}LO predictions exhibit a considerably reduced dependence on the renormalization scale with, for the first time, stationary points close to the Higgs mass.
The production of lepton pairs and especially the Higgs boson $H$, see Refs. [1] for detailed reviews, are among the most important processes in high-energy proton collisions. The corresponding cross sections receive sizable higher-order QCD corrections, necessitating calculations beyond the standard next-to-leading order (NLO) of perturbative QCD. After the early calculation of the next-to-next-to-leading order (NNLO) corrections to the Drell-Yan process in Refs. [2, 3], considerable progress has been achieved in this field during the last five years. The NNLO corrections are now completely known also for Higgs production in the heavy top-quark limit [4–10], and the all-order resummation of the threshold logarithms [11, 12] has been extended to the next-to-next-to-leading logarithmic (NNLL) accuracy for both processes [13, 14]. Still, at colliders energies especially for Higgs production, corrections of yet higher order are not entirely negligible.

In this letter we derive the complete logarithmically enhanced soft-emission corrections to both lepton pair and Higgs boson production at the third order (N$^3$LO) in the strong coupling constant $\alpha_s$, and extend the corresponding threshold resummations to the N$^3$LL contributions. The first step is achieved by analysing the general mass-factorization structure of the third-order coefficient functions $c_3$ in terms of recent results on the three-loop splitting functions [15, 16] and the quark and gluon form factors [17, 18], analogous to the second-order procedure of Ref. [19]. For the second step we just have to determine one resummation coefficient, usually denoted by $D_3$, from these results, as the structure of the resummation exponent has already been derived in Ref. [20] to the required accuracy. As discussed below our results on $c_3$ for Higgs production, suitably treated in Mellin $N$-space, provide a very good approximation to the complete N$^3$LO corrections, thus facilitating improved predictions for the cross sections at the TEVATRON and the LHC.

In the soft limit, i.e., retaining only contributions of the forms

$$ D_k = \left[ \ln^{k}(1-x) \right]_+, \delta(1-x) $$

(1)

to the coefficient functions, only the respective subprocesses $q\bar{q} \rightarrow \gamma^* \rightarrow l^+l^-$ and $gg \rightarrow H$ contribute to the Drell-Yan process and Higgs boson production. In the latter case, the $Hgg$ vertex is an effective interaction in the limit of a heavy top quark,

$$ \mathcal{L}_{\text{eff}} = -\frac{1}{4} C_H H G_{\mu\nu}^{a} G^{a,\mu\nu}, $$

(2)

where $G_{\mu\nu}^{a}$ denotes the gluon field strength tensor, and the prefactor $C_H$ includes all QCD corrections to the top-quark loop. This coefficient is of order $\alpha_s$ and known up to N$^3$LO ($\alpha_s^3$) [21]. The analysis of the higher-order corrections in the heavy-top limit is justified by the agreement between this approximation and the full calculations at NLO [22–24].

The general structure of the expansion coefficients $W_n^b$ of the bare partonic cross section,

$$ W^b = \sum_{n=0}^{\infty} \left( a_s^b \right)^n W_n^b; \quad a_s \equiv \frac{\alpha_s}{4\pi}, $$

(3)

is given by
\[ W_0^b = \delta(1-x) \]
\[ W_1^b = 2 \text{Re} \mathcal{F}_1 \delta(1-x) + S_1 \]
\[ W_2^b = (2 \text{Re} \mathcal{F}_2 + |\mathcal{F}_1|^2) \delta(1-x) + 2 \text{Re} \mathcal{F}_1 S_1 + S_2 \]
\[ W_3^b = (2 \text{Re} \mathcal{F}_3 + 2 |\mathcal{F}_1 \mathcal{F}_2|) \delta(1-x) + (2 \text{Re} \mathcal{F}_2 + |\mathcal{F}_1|^2) S_1 + 2 \text{Re} \mathcal{F}_1 S_2 + S_3 . \]

Here \( \mathcal{F}_n \) denotes the bare \( n \)-loop time-like quark or gluon form factor, calculated in dimensional regularization with \( D = 4 - 2\varepsilon \) and, as all quantities, expanded in terms of \( a_s = \alpha_s/(4\pi) \). The dependence of the pure real-emission contributions \( S_n \) on the scaling variable \( x = M^2/s \) is given by the \( D \)-dimensional + distributions \( f_{2n,\varepsilon} \) defined by

\[ f_{k,\varepsilon}(x) = \varepsilon^{-(k\varepsilon)} (1-x)^{-1-k\varepsilon} = -\frac{1}{k} \delta(1-x) + \sum_{i=0}^{+\infty} \frac{(-k\varepsilon)^i}{i!} \varepsilon^i D_i \]  

with \( D_i \) of Eq. (1). As appropriate for a parallel treatment of the two processes, the coefficient function for Higgs production is defined as in Ref. [8], i.e., \( C_H \) in Eq. (2) is kept as a prefactor.

The expansion coefficients \( W_k \) obtained from Eq. (4) after renormalizing the coupling constant,

\[ a_s^b = a_s - \frac{\beta_0}{\varepsilon} a_s^2 + \left( \frac{\beta_0^2}{\varepsilon^2} - \frac{1}{2} \frac{\beta_1}{\varepsilon} \right) a_s^3 + \ldots , \]  

and multiplying, for Higgs production, with the square of the renormalization constant [25, 26]

\[ Z_{G^2} = \left[ 1 - \beta(\alpha_s)/(a_s\varepsilon) \right]^{-1} , \quad \beta(\alpha_s) = -\beta_0 a_s^2 - \beta_1 a_s^3 - \beta_2 a_s^4 - \ldots , \]  

of the operator \( G_{\mu\nu}^a G^{a,\mu\nu} \) in Eq. (2) obey the Kinoshita-Lee-Nauenberg theorem [27, 28] and the mass-factorization relations (putting \( \mu^2 = M_{f,\mu}^2 \))

\[ W_1 = \frac{2}{\varepsilon} \gamma_0 + c_1 + \varepsilon a_1 + \varepsilon^2 b_1 , \]
\[ W_2 = \frac{1}{\varepsilon^2} \left\{ (2 \gamma_0 - \beta_0) \gamma_0 \right\} + \frac{1}{\varepsilon} \left\{ 2 \gamma_1 + 2c_1 \gamma_0 \right\} + c_2 + 2a_1 \gamma_0 + \varepsilon \left\{ a_2 + 2b_1 \gamma_0 \right\} , \]
\[ W_3 = \frac{1}{3\varepsilon^3} \left\{ 2(\gamma_0 - \beta_0)(2 \gamma_0 - \beta_0) \gamma_0 \right\} + \frac{1}{3\varepsilon^2} \left\{ 6 \gamma_1 \gamma_0 - 2\beta_0 \gamma_1 - 2\beta_1 \gamma_0 \right\} + 3c_1 (2 \gamma_0 - \beta_0) \gamma_0 \] \[ + \frac{1}{3\varepsilon} \left\{ 2 \gamma_2 + 3c_1 \gamma_1 + 6c_2 \gamma_0 + 3a_1 (2 \gamma_0 - \beta_0) \gamma_0 \right\} + c_3 + a_1 \gamma_1 + 2a_2 \gamma_0 + b_1 (2 \gamma_0 - \beta_0) \gamma_0 . \]

Here the anomalous dimensions \( \gamma_k \) are related by a conventional sign to the diagonal Altarelli-Parisi splitting functions \( P_k \). In \( x \)-space, where these quantities (in the \( \overline{\text{MS}} \) scheme adopted throughout this article) have soft limits of the form [29]

\[ P_{k-1} = A_k \mathcal{D}_0 + P_k \delta(1-x) , \]
the products in Eq. (8) – (10) have to be read as Mellin convolutions. To the required accuracy these convolutions can be readily carried out using, for example, the appendix of Ref. [30].

Eqs. (8) – (10) can be used to derive all $D_k$ contributions to the $\overline{\text{MS}}$ coefficient function $c_n$, once the coefficients $A_{n}$, $P_n^\delta$ are known together with all $1/\varepsilon$ pole terms of the $n$-loop form factor and suitable lower-order information. The salient point for this extraction is the structure (5) of the soft emissions linking the coefficients of $\varepsilon^{-1} \delta(1-x)$ to those of $D_0$; thus a mass-factorization constraint on the former term fixes the latter coefficient. With the results of Refs. [2, 8, 9] and [15–18] the above conditions are fulfilled at $n = 3$ for both lepton pair and Higgs boson production.

Thus, treating Higgs boson production first, we insert the $\varepsilon$-expansion

$$S_n = f_{2n,\varepsilon} \sum_{l=-2n}^\infty 2n s_{n,l} \varepsilon^l$$

into Eqs. (8) – (10) and recursively determine the coefficients $s_{n,l}$. The first-order result is known to all orders in $\varepsilon$. For later convenience, we here present its expansion up to order $\varepsilon^3$,

$$S_1 = 2 f_{2,\varepsilon} C_A \left\{ -\frac{4}{\varepsilon^2} + 6 \zeta_2 + \frac{28}{3} \zeta_3 \varepsilon + \frac{3}{2} \zeta_2^2 \varepsilon^2 + \varepsilon^3 \left[ -14 \zeta_2 \zeta_3 + \frac{124}{5} \zeta_5 \right] + \ldots \right\}. \quad (13)$$

The corresponding second-order coefficients read

$$s_{2,-4} = -8 C_A^2$$

$$s_{2,-3} = -\frac{11}{3} C_A^2 + \frac{2}{3} C_A n_f$$

$$s_{2,-2} = C_A^2 \left[ -\frac{67}{9} + 58 \zeta_2 \right] + \frac{10}{9} C_A n_f$$

$$s_{2,-1} = C_A^2 \left[ -\frac{404}{27} + \frac{77}{3} \zeta_2 + \frac{538}{3} \zeta_3 \right] + C_A n_f \left[ \frac{56}{27} - \frac{14}{3} \zeta_2 \right]$$

$$s_{2,0} = C_A^2 \left[ -\frac{2428}{81} + \frac{469}{9} \zeta_2 + \frac{682}{9} \zeta_3 + \frac{16}{5} \zeta_2^2 \right] + C_A n_f \left[ \frac{328}{81} - \frac{70}{9} \zeta_2 - \frac{124}{9} \zeta_3 \right]. \quad (14)$$

Here $n_f$ denotes the number of effectively massless quark flavours, $C_F$ and $C_A$ are the usual SU(N) colour factors, with $C_F = 4/3$ and $C_A = 3$ for QCD, and $\zeta_n$ represents Riemann’s $\zeta$-function.

From $s_{2,-4} \ldots s_{2,-1}$ and the first-order results one recovers the $D_k$ terms [4, 5] of the NNLO coefficient function at the renormalization and factorization scales $\mu_r^2 = \mu_f^2 = M_H^2$,

$$c_2(x) = 128 C_A^2 D_3 - \left\{ \frac{176}{3} C_A^2 - \frac{32}{3} C_A n_f \right\} D_2 + \left\{ C_A^2 \left[ \frac{1072}{9} - 160 \zeta_2 \right] - \frac{160}{9} C_A n_f \right\} D_1$$

$$+ \left\{ C_A^2 \left[ -\frac{1616}{27} + \frac{176}{3} \zeta_2 + 312 \zeta_3 \right] + C_A n_f \left[ \frac{224}{27} - \frac{32}{3} \zeta_2 \right] \right\} D_0$$

$$+ \left\{ C_A^2 \left[ 93 + \frac{536}{9} \zeta_2 - \frac{220}{3} \zeta_3 - \frac{4}{5} \zeta_2^2 \right] - C_A n_f \left[ \frac{80}{3} + \frac{80}{9} \zeta_2 + \frac{8}{3} \zeta_3 \right] \right\} \delta(1-x). \quad (15)$$
The SU(N) result (14) for the coefficient $s_{2,0}$, on the other hand, is fixed by the corresponding $\delta(1-x)$ contribution to Eq. (15) derived in Ref. [9]. Actually this colour-factor decomposition is checked (and could have been predicted from the N = 3 QCD results of Refs. [4,5]) by the absence of a $C_F n_f$ term in $S_2$ obvious from the colour structure of the contributing Feynman diagrams.

The quantity $a_2$ in Eq. (9) has not been computed so far, hence the coefficient $s_{2,1}$ is unknown at this point. This suggests a problem, as this coefficient enters the $e^0$ part of Eq. (10). However, its contribution to the $D_0$ term (but not the $\delta(1-x)$ piece) of $c_3$ is found to cancel in the end. Keeping $s_{2,1}$ as an unknown in the intermediate relations, the third-order coefficients in Eq. (13) are

$$s_{3,-6} = -\frac{32}{3} C_A^3$$

$$s_{3,-5} = -\frac{44}{3} C_A^3 + \frac{8}{3} C_A^2 n_f$$

$$s_{3,-4} = C_A^3 \left[ -\frac{2896}{81} + 184 \zeta_2 + 536 \frac{C_A^2 n_f}{81} - 16 \frac{C_A n_f^2}{81} \right]$$

$$s_{3,-3} = C_A^3 \left[ -\frac{21052}{243} + 6710 \frac{\zeta_2}{27} + 2440 \frac{\zeta_3}{3} + 536 \frac{C_A^2 n_f}{243} - 16 \frac{C_A n_f^2}{243} \right]$$

$$s_{3,-2} = C_A^3 \left[ -\frac{51322}{243} + 48856 \frac{\zeta_2}{81} + 29876 \frac{\zeta_3}{27} - 7592 \frac{\zeta_2}{45} \right] + C_A C_F n_f \left[ -\frac{1104}{27} + \frac{32}{9} \zeta_3 \right]$$

$$s_{3,-1} = C_A^3 \left[ -\frac{617525}{2187} + \frac{251942}{243} \frac{\zeta_2}{27} + \frac{56032}{27} \zeta_3 - 1661 \frac{\zeta_2}{10} \right] + C_A C_F n_f \left[ -\frac{1711}{81} - \frac{22}{3} \zeta_2 - \frac{304}{27} \zeta_3 - \frac{32}{15} \zeta_2 \right]$$

$$+ C_A^2 n_f \left[ -\frac{164194}{2187} - \frac{55154}{243} \zeta_2 - \frac{31520}{81} \zeta_3 + \frac{97}{3} \zeta_2 \right] + 4 C_A s_{2,1}. \quad (16)$$

Analogous to $s_{2,0}$ discussed above, the coefficient $s_{3,0}$ cannot be derived by mass-factorization arguments, but requires a third-order calculation like in the case of deep-inelastic scattering [31].

The above results, after combination with the gluon splitting function [16] and form factor [18] according to Eqs. (3) and (4), lead to the following soft-emission contribution to the third-order (N^3LO) coefficient function for Higgs boson production at $\mu_f^2 = \mu_r^2 = M_H^2$:

$$c_3 \bigg|_{d_5} = 512 C_A^3, \quad (17)$$

$$c_3 \bigg|_{d_4} = -\frac{7040}{9} C_A^3 + \frac{1280}{9} C_A^2 n_f, \quad (18)$$

$$c_3 \bigg|_{d_3} = C_A^3 \left[ -\frac{59200}{27} - 3584 \zeta_2 \right] - \frac{10496}{27} C_A^2 n_f + \frac{256}{27} C_A n_f^2, \quad (19)$$
Eq. (22) represents a new result of the present study. Eqs. (17) – (21) agree with the results derived from the NNLL threshold resummation in Ref. [14]. Accordingly the coefficients of $s_n, l$ for the Drell-Yan case are related to Eqs. (17) and (18), respectively, by factors

$$c_3 \bigg|_{D_3} = \frac{7744}{27} C_A C_F + C_A C_F^2 \left[ \frac{17152}{9} - 512 \zeta_2 \right] - C_F^3 \left[ 2048 + 3072 \zeta_2 \right] - \frac{2816}{27} C_A C_F n_f - \frac{2560}{9} C_F n_f + \frac{256}{27} C_F n_f^2,$$

$$c_3 \bigg|_{D_2} = C_A C_F \left[ \frac{28480}{27} + \frac{704}{3} \zeta_2 \right] - C_A C_F^2 \left[ \frac{4480}{9} - \frac{11264}{3} \zeta_2 - 1344 \zeta_3 \right] + 10240 \zeta_3 C_F^3 + C_A C_F n_f \left[ \frac{9248}{27} - \frac{128}{3} \zeta_2 \right] + C_F n_f^3 \left[ \frac{544}{9} - \frac{2048}{3} \zeta_2 \right] - \frac{640}{27} C_F n_f^2,$$

$$c_3 \bigg|_{D_1} = C_A C_F \left[ \frac{124024}{81} - \frac{12032}{9} \zeta_2 - 704 \zeta_3 + \frac{704}{5} \zeta_2^2 \right] - C_A C_F^2 \left[ \frac{35572}{9} + \frac{11648}{9} \zeta_2 + 5184 \zeta_3 - \frac{3648}{5} \zeta_2^2 \right] + C_F^3 \left[ 2044 + 2976 \zeta_2 - 960 \zeta_3 - \frac{14208}{5} \zeta_2^2 \right] - C_A C_F n_f \left[ \frac{32816}{81} - 384 \zeta_2 \right] - \frac{16}{3} \zeta_2.$$
Higgs boson production these coefficients read

\[ + C_F^2 n_f \left[ \frac{4288}{9} + \frac{2048}{9} \zeta_2 + 1280 \zeta_3 \right] + C_F n_f^2 \left[ \frac{1600}{81} - \frac{256}{9} \zeta_2 \right], \]

(25)

\[ c_3 \bigg|_{D_b} = C_A^2 C_F \bigg[ \frac{594058}{729} + \frac{98224}{81} \zeta_2 + \frac{40144}{27} \zeta_3 - \frac{2992}{15} \zeta_2^2 - \frac{352}{3} \zeta_2 \zeta_3 - 384 \zeta_5 \bigg] + C_A C_F \bigg[ \frac{25856}{27} - \frac{12416}{27} \zeta_2 + \frac{26240}{9} \zeta_3 + \frac{1408}{3} \zeta_2^2 - 1472 \zeta_2 \zeta_3 \bigg] - C_F^3 \bigg[ 4096 \zeta_3 + 6144 \zeta_2 \zeta_3 - 12288 \zeta_5 \bigg] - C_F n_f^2 \left[ \frac{3712}{729} - \frac{640}{27} \zeta_2 - \frac{320}{27} \zeta_3 \right] + C_A C_F n_f \left[ \frac{125252}{729} - \frac{29392}{81} \zeta_2 - \frac{2480}{9} \zeta_3 + \frac{736}{15} \zeta_2^3 \right] - C_F^2 n_f \left[ 6 - \frac{1952}{27} \zeta_2 + \frac{5728}{9} \zeta_3 + \frac{1472}{15} \zeta_2^3 \right]. \]

(26)

Also Eqs. (23) – (25) agree with the results derived from the NNLL threshold resummation [13], while Eq. (26) is a new result of this study. The additional coefficients for \( \mu_R^2 \neq M_Y^2 \) and \( \mu_F^2 \neq \mu_F^2 \) can be derived analogously to Eqs. (2.16) – (2.18) of Ref. [32] or using the N3LL threshold resummation expression [20], but will be skipped here for brevity.

We note that the \( \zeta \)-function terms of highest transcendentality \( n \), i.e., the coefficients of \( \zeta_n \) and \( \zeta_i \zeta_j \) with \( i + j = n \), in the \( e^{-2l+n} \) contributions to the pure real-emission function \( S_l \) agree between Higgs production and the Drell-Yan process for the Super-Yang-Mills case \( C_A = C_F = n_c \). The same holds for the quark and gluon form factors [17,18] and, consequently, also for the \( \zeta \)-function terms of weight \( n \) in the soft logarithms \( D_{2l-1-n} \) of the coefficient functions for Higgs boson and lepton pair production, see Eqs. (17)–(22) and (23)–(26). By construction, generalizing Eq. (4), this feature extends to all orders of perturbation theory.

We now turn to the threshold resummation. For the processes under consideration, the coefficient functions exponentiate after transformation to Mellin \( N \)-space [11,12],

\[ C^N = (1 + a_s g_{01} + a_s^2 g_{02} + \ldots) \cdot \exp \left( G^N \right) + O(N^{-1} \ln^N N). \]

(27)

Here \( g_{0k} \) collects the \( N \)-independent contributions at the \( k \)-th order, and the resummation exponent \( G^N \) takes the form

\[ G^N(Q^2) = \ln N \cdot g_1(\lambda) + g_2(\lambda) + a_s g_3(\lambda) + a_s^2 g_4(\lambda) + \ldots \]

(28)

with \( \lambda = \beta_0 a_s \ln N \). The functions \( g_3 \) and \( g_4 \) have been determined in Refs. [13,14] and [20], respectively. Besides the quantities \( A_k \) in Eq. (11) and lower-order coefficients, the functions \( g_k \) depend on one parameter, usually denoted by \( D_{k-1} \).

Before we present our new results for the coefficient \( D_3 \), we recall, for the convenience of the reader, the \( N \)-independent first- and second-order contributions which enter its determination. For Higgs boson production these coefficients read

\[ g_{01} = C_A (16 \zeta_2 + 8 \gamma_1^2), \]

(29)
we recover the known coefficients (as always referring to the expansion parameter $a_s = \alpha_s/(4\pi)$)

$$g_{02} = C_A^2 \left[ 93 + \frac{1072}{9} \zeta_2 - \frac{308}{9} \zeta_3 + 92 \zeta_2^2 + \frac{1616}{27} \gamma_e - 56 \gamma_e \zeta_3 + \frac{536}{9} \gamma_e^2 + 112 \gamma_e^2 \zeta_2 \\
+ \frac{176}{9} \gamma_e^3 + 32 \gamma_e^4 \right] + C_A n_f \left[ - \frac{80}{3} - 160 \zeta_2 - 88 \zeta_3 - 224 \gamma_e - \frac{80}{9} \gamma_e^2 - \frac{32}{9} \gamma_e^3 \right] \\
+ C_F n_f \left[ - \frac{67}{3} + 16 \zeta_3 \right].$$ (30)

The corresponding results for the Drell-Yan case are given by

$$g_{01} = C_F (-16 + 16 \zeta_2 + 8 \gamma_e^2),$$ (31)

$$g_{02} = C_F^2 \left[ \frac{511}{4} - 198 \zeta_2 - 60 \zeta_3 + \frac{552}{5} \zeta_2^2 - 128 \gamma_e^2 + 128 \gamma_e^2 \zeta_2 + 32 \gamma_e^4 \right] \\
+ C_A C_F \left[ - \frac{1535}{12} + \frac{376}{5} \zeta_2 + \frac{604}{9} \zeta_3 - \frac{92}{5} \zeta_2^2 + \frac{1616}{27} \gamma_e - 56 \gamma_e \zeta_3 + \frac{536}{9} \gamma_e^2 \\
- 16 \gamma_e^2 \zeta_2 + \frac{176}{9} \gamma_e^3 \right] + C_F n_f \left[ \frac{127}{6} - 64 \zeta_2 + \frac{8}{9} \zeta_3 - \frac{224}{27} \gamma_e - \frac{80}{9} \gamma_e^2 - \frac{32}{9} \gamma_e^3 \right].$$ (32)

Inserting the above results into the explicit formulae for the resummation exponents [13,14,20], we recover the known coefficients (as always referring to the expansion parameter $a_s = \alpha_s/(4\pi)$)

$$D_1 = 0,$$ (33)

$$D_2 = C_I \left[ C_A \left( - \frac{1616}{27} + \frac{176}{3} \zeta_2 + 56 \zeta_3 \right) + n_f \left( 224 \gamma_e - \frac{32}{3} \zeta_2 \right) \right],$$ (34)

and derive the new third-order contribution

$$D_3 = C_I C_A^2 \left[ - \frac{594058}{729} + \frac{98224}{81} \zeta_2 + \frac{40144}{27} \zeta_3 - \frac{2992}{15} \zeta_2^2 - \frac{352}{3} \zeta_2 \zeta_3 - 384 \zeta_5 \right] \\
+ C_I C_A n_f \left[ \frac{125252}{729} - \frac{29392}{81} \zeta_2 - \frac{2480}{9} \zeta_3 + \frac{736}{15} \zeta_2^2 \right] \\
+ C_I C_F n_f \left[ \frac{3422}{27} - 32 \zeta_2 - \frac{608}{9} \zeta_3 - \frac{64}{5} \zeta_2^2 \right] + C_I n_f^2 \left[ \frac{3712}{729} + \frac{640}{27} \zeta_2 + \frac{320}{27} \zeta_3 \right]$$ (35)

with $C_I = C_F$ for the Drell-Yan case, and $C_I = C_A$ for Higgs production. Hence, not unexpectedly, we find that also $D_3$ is maximally non-abelian, with the quark and gluon cases related by an overall factor $C_F/C_A$. This is the same behaviour as shown by the cusp anomalous dimensions $A_n$ in Eq. (11) [29] and by the form-factor resummation coefficients $f_n$ known up to three loops [9,18]. In fact, there is a simple relation between the coefficients $D_n$ and $f_n$ (using the notation of Ref. [18]),

$$D_2 = 2 \beta_0 s_{1,0} - 2 f_2$$

$$D_3 = 2 \beta_1 s_{1,0} - 4 \beta_0^2 s_{1,1} + 4 \beta_0 \left( s_{2,0} - \frac{36}{5} \zeta_2^2 C_F^2 \right) - 2 f_3,$$ (36)

of which the first line of has been derived before in Ref. [33] from an extension of the threshold resummation to $N$-independent contributions. In the present mass-factorization framework, the $s_{n,l}$ terms in Eqs. (36) can be traced back to the coupling-constant renormalization of Eqs. (4).
We now turn to the numerical impact of the $N^3$LO and resummation corrections to the coefficient functions, confining ourselves to Higgs boson production for brevity. All parameters are taken over from Ref. [8], i.e., we use the values $m_t = 173.4$ GeV (very close to the present world average) and $G_F = 4541.7$ pb for the top mass and Fermi constant in the prefactor $C_H$ in Eq. (2), and the parton distributions of Refs. [34, 35] with their respective values of strong coupling constant at LO, NLO and NNLO, $\alpha_s(M_Z) = 0.130, 0.119$ and 0.115. Anticipating a slight further reduction at N$^3$LO, we employ $\alpha_s(M_Z) = 0.114$ at this order. The N$^3$LO corrections to $C_H$ in the heavy-top limit are taken from Ref. [21]. All higher-order contributions are calculated in the heavy top-quark approximation, but normalized to the full lowest-order result.

As mentioned above, the $\delta(1-x)$ contributions to the N$^3$LO coefficient functions $c_3$ cannot be derived at this point. However, we note that the coefficients $g_{0k}$ in Eqs. (29) – (32) are much larger than their $\delta(1-x)$ counterparts for $c_1$ and especially for $c_2$. We expect the same behaviour for $c_3$. Moreover, a good approximation (to about 10% or less) to the full double convolutions $g \ast g \ast [c_i(x)/x]$ is obtained at NLO and NNLO by transforming to N-space and keeping only the $\ln^k N$ and $N^0$ terms arising from the $+$-distributions (but not the $\delta$-function) in $c_1$ and $c_2$. Consequently Eqs. (17) – (22) facilitate a sufficient approximation to the complete N$^3$LO correction, to which we assign a conservative 20% uncertainty, i.e., twice the offset found at NLO and NNLO.

The corresponding results are added in Fig. 1 to the total cross sections up to NNLO [6–8] at the TEVATRON and the LHC for the standard choice $\mu_r = \mu_f = M_H$ of the renormalization and factorization scales. The dependence of the cross sections on $\mu_r$ is illustrated in Fig. 2 for two representative values of the Higgs mass $M_H$. Also shown in Fig. 1 are the additional contributions of the N$^3$LL threshold resummation (28), see also Ref. [20], of the terms beyond N$^3$LO. In principle this resummation requires a second coefficient, the four-loop cusp anomalous dimension $A_4$, besides $D_3$ of Eq. (35). However, the effect of $A_4$ can safely be expected to be very small, as corroborated by the Padé estimate of Ref. [17] employed in our numerical analysis. The Mellin-inversion of the exponentiated result (which is entirely dominated by the next few orders in $\alpha_s$ in the present case) has been performed using the standard ‘minimal prescription’ contour [36].

The inclusion of our new result for the coefficient function $c_3$ effects an increase of the cross sections by about 10% at the TEVATRON and 5% at the LHC. The estimated uncertainties due to the approximate character of $c_3$ (see above) thus amount to 2% and 1%, respectively. Contributions of yet higher orders, as estimated by the threshold resummation, lead to a further increase by roughly half the N$^3$LO effect. Lacking N$^3$LO and threshold-resummed (see Ref. [37] for a first study) parton distributions, the NNLO gluon distribution of Ref. [35] has been employed for all results beyond the next-to-leading order. Based on the pattern of the available orders, one may expect slightly smaller (by about 2%) gluon-gluon luminosities at N$^3$LO.

The residual uncertainty due to uncalculated contributions of yet higher order is often estimated by varying $\mu_r$ and/or $\mu_f$. At the LHC the representative variation of $\mu_r$ with fixed $\mu_f$, illustrated in Fig. 2 for two Higgs masses, yields uncertainties of less than 4% for the conventional interval $1/2 M_H \leq \mu_r \leq 2 M_H$ at N$^3$LO, an improvement by almost a factor of three over the 9 to 11% stability of the NNLO cross sections. At the TEVATRON the corresponding $\mu_r$ dependence decreases
Figure 1: The perturbative expansion of the total cross section for Higgs boson production at the Tevatron (left) and the LHC (right) for the standard scale choice $\mu_r = \mu_f = M_H$.

Figure 2: The dependence of the fixed-order predictions for the LHC cross section on the renormalization scale $\mu_r$ at $\mu_f = M_H$ for two representative values of the Higgs boson mass $M_H$. 
from about 11% at NNLO to 5% at N$^3$LO for $M_H = 120$ GeV where, as in Fig. 2 the N$^3$LO cross section exhibits a stationary point close to $\mu_r = 1/2M_H$. Considering these and the above results, 5% at the LHC and 7% at the Tevatron appear to represent conservative estimates of the improved cross-section uncertainties due to the truncation of the perturbation series at N$^3$LO.

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Note added: Shortly after the completion of this letter, Ref. [40] appeared, which addresses the threshold resummation especially for lepton-pair production in the approach of Ref. [33]. In particular, our result (35) for the coefficient $D_3$ for the Drell-Yan process is confirmed by this research.

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