Gravitational radiation and isotropic change of the spatial geometry

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Abstract

To simplify a number of considerations in the weak field approximation, including the determination of the response of interferometric gravitational wave detectors, the “transverse traceless” (TT) gauge is often used. While the identification of the corresponding gauge invariant part of the metric perturbations in the pure vacuum case is obvious, it is not widely known that the determination and the use of the TT part is much more complicated and, in turn, much less rewarding when sources are involved. It is shown here that likewise the transverse part of the electric current in the Coulomb gauge within Maxwell’s theory the sources of the TT gauge part of the metric perturbations become non-local. This, in practice, invokes the necessity of the use of more adequate projection operators then the ones applied, e.g, in the weak field limit, and in many post-Newtonian considerations. It is also pointed out that, whenever nonlinear effects are taken into account, some of the conclusions concerning the response of interferometric gravitational wave detectors may be influenced. In particular, attention is called on the possibility that gravitational radiation may produce an isotropic change of the spatial geometry.

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1 Introduction

General relativity is a metric theory of gravity which can also be formulated as being a gauge theory. This paper—besides pointing out various analogies between the use of the Coulomb gauge in Maxwell’s theory and the TT gauge in the linearised Einstein’s theory—is to point out the necessity of careful reinvestigation of the standard arguments applied in determining the response of interferometric gravitational wave (GW) detectors. The main motivation for the present work is rooted in the fact that sensitivity of the ground based GW detectors such as LIGO and Virgo has been improved significantly [4, 5, 6, 7], and it is widely held that, if not earlier, then once the advanced detectors will be operating, detection of gravitational waves will become an everyday routine. Therefore, it is getting of obvious interest to have the best possible estimates for the astrophysical parameters of the associated GW sources. In this respect it is worth to be recalled that the optical observations of the change of the parameters of the orbital motion of potential GW sources agree—see, e.g., the reports on the Hulse-Taylor pulsar and similar type of binary systems [1, 2, 3] (see also section 6.2.3 of [12])—, up to a very high precision, with the predictions of Einstein’s theory. More precisely, the energy loss, which is signified by the observed change of the orbital parameters and which is assumed to be yielded by gravitational radiation, is in accordance with the predictions of Einstein’s theory. To ensure the same order of precision in determining the astrophysical parameters based on the independent GW observations it is of crucial importance to be sure that not only the generation but the propagation of GW signals from the sources to our detectors is properly determined in the applied models.

In this paper we intend to provide a simple enough discussion indicating some of the potential sources of imprecision in the current determination of the detector response for the arrival of GWs. In singling out an appropriate framework it turned out to be really useful to have the comprehensive paper by Flanagan and Hughes [8] at hand. Most of the arguments below are going to refer to the results formulated in the first part of this work. There is, however, one point to be mentioned here. As opposed to the ambitious plans manifested by the first part of [8] for certain reasons Flanagan and Hughes did not follow the path chosen there. For instance, after providing a very useful critical summary of the former conventional discussions in section 2.3 they returned to the orthodox arguments in spite of the fact that these are apparently inconsistent with the conclusion of the first part of their paper. In this respect, the main points of the arguments presented in this paper may be considered as natural continuation and completion of the work initiated by Flanagan and Hughes.

As it was indicated above the use of the Coulomb gauge in Maxwell’s theory shares several essential properties with that of the TT gauge in the weak field approximation—in particular, since the argument justifying the non-locality of the pertinent sources
in these gauges are completely parallel—in the rest of this section some of the most important related facts of Maxwell’s theory are recalled. The gauge invariant Maxwell tensor $F_{\alpha\beta}$ is given in terms of a vector potential $A_\alpha$ as $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, and two vector potentials $A_\alpha$ and $A'_\alpha$ are known to be physically equivalent, i.e., they yield the same Maxwell tensor, if there exists a real function $\chi$ such that

$$A'_\alpha = A_\alpha + \partial_\alpha \chi. \quad (1)$$

The field equations, whenever the gauge dependent vector potential $A_\alpha$ satisfies the Lorentz gauge condition $\partial^\alpha A_\alpha = 0$, read as (see, e.g., [9, 10])

$$\Box A_\alpha = -4\pi J_\alpha \quad (2)$$

where $\Box = -\partial_t^2 + \nabla^2$ and $J_\alpha$ stands for the electric four current vector. It is known that by choosing the real function $\chi$ appropriately the Lorentz gauge condition can always be guaranteed to hold. Recall also that we still have the freedom of applying a restricted gauge transformation of the form (1) provided that the generator $\chi$ is subject to the equation $\Box \chi = 0$ since then the Lorentz gauge condition remains intact.

It is also well-known that gauge independent quantities can be built up from the vector potential. The pertinent gauge is frequently referred as “Coulomb”, “radiation” or “transverse” gauge and it can be introduced as follows. Start by picking out an inertial reference system, $(t, \mathbf{x})$, of the underlying Minkowski spacetime. Then a vector potential $A_\alpha$ may be decomposed as $A_\alpha = (-\phi, A_i)$. The spatial part $A_i$ of $A_\alpha$ can be split up into ‘transversal’ and ‘longitudinal’ part as $A_i = A^T_i + \partial_i \phi$, where $A^T_i$ is such that $\partial^a A^T_a = 0$. This decomposition is unique if in addition the potential $\phi$ is guaranteed to tend to zero while $r \to \infty$ as then the elliptic equation $\nabla^2 \phi = \partial^a A_a$ possesses a unique solution. It is straightforward to see that once a gauge transformation (1) is applied the variables $\phi$ and $\phi$ will be changed. It is well-known, however, that their combination $\Phi = \phi + \partial_t \phi$, along with $A^T_i$, is gauge invariant [9]. The field equation (2), pertinent for $\Phi$ and $A^T_i$, reads then as

$$\nabla^2 \Phi = -4\pi \rho \quad (3)$$

$$\Box A^T_i = -4\pi J^T_i \quad (4)$$

where the decomposition $J_\alpha = (-\rho, J_i)$ has been used. Notice that the source for the transverse part of the vector potential $A^T_i$, $J^T_i = J_i - \frac{1}{4\pi} \partial_i (\partial_t \Phi)$
extends over all space even if the spatial part $J_i$ of $J_\alpha$ is localised \[9\]. This is a direct consequence of the fact that $\Phi$ is subject to the Poisson equation \[3\], i.e., $\Phi$ is non-local. Notice that $\Phi$ is time dependent even though it does not time evolve as a wave.

An immediate consequence of the non-locality of the transverse part of spatial vector fields is that, once sources are involved, one has to be careful in determining, e.g., the transverse part $A^T_i$ of the spatial part of vector potential $A_i$, by making use of a projection operator. A projection operator\[2\] $P^i_j$ of this type is formally defined referring to the inverse Laplace operator, \[\nabla^2\], as $P^i_j = \delta^i_j - \partial_i \nabla^2 \partial^j$. However, the precise form of this projection operator—see, e.g., the discussion on page 242 of Jackson’s book \[9\]—, making the non-locality of the involved fields completely transparent, assigns to a spatial vector $V_i$ its transverse part $V^T_i = P^i_j V_j$, as

$$P^i_j V_j = \delta^i_j V_j + \frac{1}{4\pi} \frac{\partial^i}{|x-x'|} \int \frac{\partial^j V_j(x')}{|x-x'|} d^3x', \quad (6)$$

where the relation $\nabla^2 \frac{1}{|x-x'|} = -4\pi \delta(x-x')$ has implicitly been used.

It is straightforward to verify that whenever the spatial vector $V_i$ possesses the form of a plane wave solution, i.e., $V_i = V^0_i \cdot \cos(kx - \omega t + \psi_0)$, with constant amplitude $V^0_i$ and phase $\psi_0$, and with $\omega = |k|$, then $P^i_j$ can be given as

$$P^i_j = \delta^i_j - n_i n^j, \quad (7)$$

where the spatial unit vector $n_i$ is given as $n_i = k_i/\omega$. It is important to emphasise that the projection operator $P^i_j$, given by \[6\] for the generic case, reduces to the form of \[7\] if and only if $V_i$ is given as a linear superposition of plane wave solutions such that all the spatial wavenumber vectors are parallel. Accordingly, the application of the projection operator $P^i_j$ \[7\] does not yield the TT-part of $V_i$ besides this exceptional case. It is worth to be mentioned that it does not even do the job for slightly more general solutions to the sourceless wave equation, $\Box V_i = 0$. All these observations imply then that whenever sources are involved the only adequate projection operator must possess the form of \[6\].

2 The weak field approximation of GR

The weak field approximation of general relativity is believed to be adequate in describing weak gravitational effects. In such a case the metric $g_{\alpha\beta}$ of the spacetime

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\[2\] All the index raising and lowering are meant to be done by either of the fixed background metrics $\eta_{\alpha\beta}$ or $\delta_{ij}$ of the Minkowski spacetime or the Euclidean space, respectively. Moreover, the Einstein’s summation convention is used only for identical upper and lower indices.
is supposed to be close to the flat metric $\eta_{\alpha\beta}$ of the Minkowski spacetime. More precisely, it is assumed that Minkowski-type coordinate systems exist such that

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

and that $|h_{\alpha\beta}| \ll 1$.

As a direct consequence of the generic diffeomorphism invariance of Einstein’s theory two linear perturbations $h_{\alpha\beta}$ and $h'_{\alpha\beta}$ of the flat Minkowski spacetime are considered equivalent, whenever they are related as

$$h'_{\alpha\beta} = h_{\alpha\beta} + \partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha,$$

where $\xi^\alpha$ denotes some infinitesimal vector field determining the coordinate transformation

$$x^\alpha \rightarrow x'^\alpha = x^\alpha - \xi^\alpha.$$

Notice that in (8) $h_{\alpha\beta}$ and $\xi^\alpha$ play the same role as the vector potential $A_\alpha$ and the function $\chi$ do in the Maxwell case.

The linearised Einstein equations can then be shown to take—in terms of the trace reversed,

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} h^{\gamma\gamma},$$

of $h_{\alpha\beta}$—the simple form

$$\Box \bar{h}_{\alpha\beta} = -16\pi T_{\alpha\beta}$$

provided that $\bar{h}_{\alpha\beta}$ satisfies the Lorentz gauge condition

$$\partial^\alpha \bar{h}_{\alpha\beta} = 0.$$  

It is well-known but worth to be mentioned that there always exist coordinate transformations of the form $x'^\alpha = x^\alpha - \xi^\alpha$ such that (13) holds in the new gauge. Moreover, the pertinent gauge is not unique since further restricted coordinate transformations with $\xi^\alpha$ subject to the wave equation

$$\Box \xi^\alpha = 0$$

may still be applied as they leave the Lorentz gauge condition (13) intact.

The solution to the inhomogeneous equation (12), given in terms of the retarded Green function, read as

$$\bar{h}_{\alpha\beta}(t, x) = 4 \int T_{\alpha\beta}(t - |x - x'|, x') d^3 x'.$$

In virtue of (12) all the components of $\bar{h}_{\alpha\beta}$ possess radiative degrees of freedom which by many authors (see, e.g., [8, 11]) is considered to be an “unfortunate consequence” of the applied gauge. A more adequate objection could be that the components of $\bar{h}_{\alpha\beta}$ are not gauge invariant thus they cannot be directly applied in determining the response of our GW detectors.
3 The “radiation” or TT gauge

In virtue of the criticism recalled above, more importantly, because of the obvious need for a correct derivation of the response of our ground based laser interferometric detectors like LIGO and Virgo to the arrival of a GW signal, it is important to know whether the true radiative physical degrees of freedom can always be separated in the weak field approximation.

It has been known for long that the gauge independent expressions can be built up from the components of $h_{\alpha\beta}$. In the following short review of the pertinent results we shall follow the discussion of [8] unless otherwise indicated.

Consider first a “1 + 3” decomposition

$$h_{\alpha\beta} = \left( \begin{array}{cc} h_{tt} & h_{t\xi} \\ h_{\xi t} & h_{\xi\xi} \end{array} \right)$$

of $h_{\alpha\beta}$ based on the use of a Minkowski type coordinate system, $(t, x)$, where time-time, time-space and space-space parts are given in terms of the variables $\phi$, $\beta$, $\gamma$, $\varepsilon$, $h_{ij}^{TT}$ and $\lambda$ as

$$h_{tt} = 2\phi$$
$$h_{ti} = \beta_i + \partial_i \gamma$$
$$h_{ij} = h_{ij}^{TT} + \frac{1}{3} H \delta_{ij} + \partial_i \varepsilon_j + \left( \partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2 \right) \lambda,$$

where $H \equiv \delta^{ij} h_{ij}$ denotes the three-dimensional trace which is related to $h = h^{\alpha\alpha}$ as $h = H - 2\phi$. The variables $\gamma, \varepsilon, \lambda$—and, in turn, $\beta_i$ and $h_{ij}^{TT}$, as well—can be seen to be uniquely determined once the relations

$$\partial_i \beta_i = 0, \quad \partial_i \varepsilon_i = 0, \quad \partial_i h_{ij}^{TT} = 0,$$

along with the boundary, or fall off, conditions

$$\gamma \to 0, \quad \varepsilon_i \to 0, \quad \lambda \to 0, \quad \nabla^2 \lambda \to 0 \quad \text{while} \quad r \to \infty,$$

are imposed. Note that, in virtue of (19) and (20) the contraction $\delta^{ij} h_{ij}^{TT}$ vanishes, which along with the last relation of (20), implies that $h_{ij}^{TT}$ is TT.

As the components of $h_{\alpha\beta}$ themselves are not gauge invariant the variables $\phi, \gamma, \lambda, H, \beta_i$ and $\varepsilon_i$ are not gauge invariant either. However, the combinations

$$\Phi \equiv -\phi + \partial_t \gamma - \frac{1}{2} \partial_t^2 \lambda$$
$$\Theta \equiv \frac{1}{3} (H - \nabla^2 \lambda)$$
$$\Xi_i \equiv \beta_i - \frac{1}{2} \partial_i \varepsilon_i,$$

3For more details the reader may look up the pertinent part of the argument applied for the analogous decomposition of the energy-momentum tensor below.
along with the $3 \times 3$ matrix $h^{TT}_{ij}$, can be shown to be gauge invariant.

3.1 The decomposition of the energy-momentum tensor

In order to be able to determine the evolution equations for the above introduced gauge invariant expressions we shall need an analogous decomposition of the energy-momentum tensor.

Before providing this decomposition recall first that whenever matter fields are involved the Einstein’s and matter field equations have to be solved simultaneously. Now, partly to simplify our argument, and also to avoid the associated considerable technical difficulties, without choosing any concrete field equations we shall assume that the field values, along with the components of the energy-momentum tensor, $T_{\alpha\beta}$, are determined by some unspecified field equations, governing the time evolution of the sources of the gravitational waves. Once we have the energy-momentum tensor, $T_{\alpha\beta}$, a decomposition, completely analogous to the one applied above for $h_{\alpha\beta}$, can be provided as follows.

Start by a “$1 + 3$” splitting of $T_{\alpha\beta}$

$$T_{\alpha\beta} = \begin{pmatrix} T_{tt} & T_{ti} \\ T_{it} & T_{ij} \end{pmatrix},$$

and by defining the variables $\rho$, $S_i$, $\sigma_{ij}$, $\sigma_i$ and $\sigma$ via the relations

$$T_{tt} = \rho,$$  \hspace{1cm} (26)

$$T_{ti} = S_i + \partial_i S,$$  \hspace{1cm} (27)

$$T_{ij} = \sigma_{ij} + \left( \partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2 \right) \sigma,$$  \hspace{1cm} (28)

where $P = \frac{1}{3} \delta^{ij} T_{ij}$ . As above $S$, $\sigma$ and $\sigma_i$ get to be uniquely determined once the relations

$$\partial^i S_i = 0, \quad \partial^i \sigma_i = 0, \quad \partial^i \sigma_{ij} = 0,$$  \hspace{1cm} (29)

along with the boundary, or fall off, conditions

$$S \to 0, \quad \sigma_i \to 0, \quad \sigma \to 0, \quad \nabla^2 \sigma \to 0 \quad \text{while} \quad r \to \infty,$$  \hspace{1cm} (30)

are imposed. Note that as above, in virtue of (28) and (29) the contraction $\delta^{ij} \sigma_{ij}$ vanishes, which along with the last relation of (29), implies that $\sigma_{ij}$ is TT.

The uniqueness of the above decomposition can be seen as follows. First, the $\partial^i$-divergence of (27) yields $\nabla^2 S = \partial^i T_{ti}$, which has a unique solution by the above boundary condition. Once $S$ is known $S_i$ gets to be uniquely determined by the relation $S_i = T_{ti} - \partial_i S$. Concerning the uniqueness of $\sigma$, take now the $\partial^i \partial^j$-“divergence”
of (28) which yields the Poisson equation \( \nabla^2 \nabla^2 \sigma = \frac{3}{2} [\partial^i \partial^j T_{ij} - \nabla^2 P] \), and which has a unique solution for \( \nabla^2 \sigma \), and, in turn, in virtue of (30), \( \sigma \) becomes uniquely determined, as well. Once \( \nabla^2 \sigma \) is known \( \sigma \) gets also to be uniquely determined by the Poisson equation \( \nabla^2 \sigma = 2 [\partial^j T_{ij} - \partial_i P] - \frac{2}{3} \partial_i \nabla^2 \sigma \), which is yielded by the \( \partial^i \)-divergence of (28), along with the fall off condition \( \sigma_i \to 0 \) while \( r \to \infty \).

As the energy-momentum tensor is supposed to be known it is straightforward to see that having \( \sigma \) and \( \sigma_i \) determined, \( \sigma_{ij} \) gets also fixed as

\[
\sigma_{ij} = T_{ij} - P \delta_{ij} - \partial_i \sigma_j - \left( \partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2 \right) \sigma .
\] (31)

In [8] the authors claim that \( \sigma_{ij} \), along with some of the other variables, can be chosen freely. It should be noted that this can be done only if the energy-momentum tensor is not specified.

There is an even more important additional point to be mentioned here. As the variables \( \sigma \) and \( \sigma_i \) satisfy Poisson type equations they are non-local. In consequence of this fact and the above relation (31) the TT-part, \( \sigma_{ij} \), of \( T_{ij} \) cannot be local either regardless whether the energy-momentum tensor, \( T_{\alpha \beta} \), of the matter sources is of compact support or not.

Since this non-locality is in certain extent inconvenient it could be tempting to argue that although the variables \( S, \sigma \) and \( \sigma_i \) were shown to be subject to some very complicated elliptic equations they might be completely negligible. In this respect it is useful to take into account the conservation law \( \partial^\alpha T_{\alpha \beta} = 0 \) which reads as [8]

\[
\nabla^2 S = \partial_t \rho \quad (32)
\]

\[
\nabla^2 \sigma = -\frac{3}{2} P + \frac{3}{2} \partial_t S \quad (33)
\]

\[
\nabla^2 \sigma_i = 2 \partial_t S_i .
\] (34)

These relations—which had a completely different role in the discussion of [8]—provide the following alternative characterisation of the variables \( S, \sigma \) and \( \sigma_i \). They make it immediately transparent that the time derivative of the energy density and that of the impulse—both of these quantities are supposed to provide significant contribution to the gravitational wave production—are the sources for \( S \) and \( \sigma_i \), respectively. Thereby, the more intensive is the considered GW source the more significant the quantities \( S, \sigma \) and \( \sigma_i \) become.

4 The linearised Einstein’s equations

Now we are prepared to present the explicite form of Einstein’s equations relevant for the above introduced gauge invariant quantities in the investigated weak field
approximation. These equations can be given as [8]

\[ \nabla^2 \Theta = -8\pi \rho \]  
\[ \nabla^2 \Phi = 4\pi (\rho + 3P - 3 \partial_t S) \]  
\[ \nabla^2 \Xi_i = -16\pi S_i \]  
\[ \Box h^{TT}_{ij} = -16\pi \sigma_{ij} . \]

The above equations justify the conventional assertion that only the TT part of the metric perturbation satisfies wave equation while all the other gauge invariant expressions, although they are time dependent, do not time evolve as waves since they are subject to Poisson equations. Accordingly, it is frequently said that only the “non-radiative” physical degrees of freedom are tied to the matter sources. What is even more surprising is that conclusions of the following type are drawn based on the above set of equations: Since the sources are at enormous distance from the Earth, in virtue of (38), GW signals can basically be considered as being sourceless and possessing the same type of properties as if they were GW signals in the pure vacuum case. As opposed to this, we would like to emphasise that according to the conclusion of the previous section the source term \( \sigma_{ij} \) in (38) is non-zero, and it is non-local either even though \( T_{\alpha\beta} \) is completely localised.

As an immediate consequence of this non-locality let us make a comment regarding the “conventional” way of determining the TT part \( h^{TT}_{ij} \) of a solution \( h_{\alpha\beta} \) to the evolution equation (12). It is usually assumed in the weak field approximation (see, e.g., Section 4.1 of [8] or section 3.1 of [12]) and, for some surprise, also in the post-Newtonian framework (see, e.g., the sentence involving Eq. (2.2) of [13], section 5.3.4 of [12] or the first paragraph on page 19 in [14]), that the TT part of \( h_{ij} \) may be determined in terms of the projection tensor, \( \Lambda_{ij}^{kl} \) as

\[ h^{TT}_{ij} = \Lambda_{ij}^{kl} h_{kl} . \]  

as \( h^{TT}_{ij} = \Lambda_{ij}^{kl} h_{kl} \), where the “elementary projection operator” \( P^j_i \) is supposed to possess the form \( P_i^j = \delta_i^j - n_i n^j \). We would like to emphasise here that, as it follows from the argument outlined at the end of section II, this form of \( P^j_i \) cannot adequately be applied even to the superposition of plane wave solutions to the sourceless wave equation unless all the spatial wave number vectors are parallel. Since the evolution equation (12) for \( h_{\alpha\beta} \) must have sources in astrophysical situations, as actually we wish to observe these sources, the projection operator \( \Lambda_{ij}^{kl} \), when it is expressed in term of the “elementary projection operator” via (39)—as opposed to the generic resolution applied in various calculations (see, e.g., [8] [12] [13] [14])—has to be constructed by making use of (11) instead of applying (7). The corresponding projection operator—the complexity of which is expected to reflect all the technical difficulties related to
the non-locality of $\sigma_{ij}$—will produce the adequate TT-part of $h_{ij}$. In virtue of these observations there is an obvious need for the reinvestigation of the procedures yielding the wave forms in the asymptotic region by applying the correct projection operator $\Lambda_{ij}^{kl}$, e.g., in the post-Newtonian framework.

5 Further implications of non-localities

In proceeding, let us recall now, that in many of the arguments, aiming to determine the response of laser interferometric detectors to the arrival of a GW signal, the calculations end up (see, e.g., Eq. (3.10) of [8], or Eq. (1.93) of [12]) with the variant of the geodesic deviation equation

$$\frac{d^2 L^i(t)}{dt^2} = -R_{ij}^l L^j_l,$$

where $L^i(t) = L^i_0 + \delta L^i(t)$, with $\delta L^i(t) \ll L^i_0$ and $i, j = 1, 2$, is supposed to denote the coordinates of mirrors at the end of the arms in the proper detector frame.

Then, in the linearised theory, assuming that no sources are present, the relation $R_{i;j} = -\frac{1}{2} \partial^2 h_{TT}^{ij}$, along with the assumption that both $\delta L^i(t)$ and $\partial_t (\delta L^i(t))$ vanish at $t = 0$, is applied to derive the familiar “gauge independent” relation

$$\delta L^i(t) = \frac{1}{2} h_{TT}^{ij} L^j_0.$$

However, as it has already been emphasised above, since we do want to make astrophysical observations, the presence of the sources has to be taken into account. In this more realistic situation, as opposed to the pure vacuum case, the gauge invariant “tidal force components” of the Riemann tensor read as (see, e.g., [8])

$$R_{i;j} = -\frac{1}{2} \partial^2 h_{TT}^{ij} + \partial_i \partial_j \Phi + \partial_i \partial_j \Xi - \frac{1}{2} \partial_t \Theta \delta_{ij}.$$

It is well-known that in the linearised theory, if one takes into account the conservation of the stress energy tensor, $\partial^\alpha T_{\alpha \beta} = 0$, the last three terms can be shown to fall off like $1/r^2$ or faster [8] [3]. Thus, the only term with $1/r$ fall off is the first term on the right hand side of (42). Thereby, within the linearised theory regardless whether sources are present relation (41) determine the response of our detectors.

Up to now only the linearised theory has been considered. However, it has been known for long (see, e.g., the discussion in section 4.4. of [10]) that it has serious
limitations, since, even to have a consistent Newtonian limit one must go beyond the linear approximation. This is justified by the fact that in the linear theory, in virtue of the conservation law $\partial^\rho T_{\alpha\beta} = 0$, e.g., the members of a binary system—instead of orbiting around each other—have to follow timelike geodesics, i.e., straight lines, of the Minkowski spacetime. As opposed to this a nearly Newtonian type of orbiting is produced by the binary if the terms higher order in $h_{\alpha\beta}$ are restored in the Einstein tensor $G_{\alpha\beta}$. Therefore, in a physically consistent description of gravitational wave generation processes the back-reaction has to be taken into account. This can be done, while preserving the simplicity of the basic equations of the linear approximation, by putting all the non-linear terms of the Einstein tensor to the energy-momentum tensor side, or, more precisely, by replacing in (12) the energy-momentum tensor $T_{\alpha\beta}$ by the sum $T_{\alpha\beta} + t_{\alpha\beta}$, where $t_{\alpha\beta} = \mp \frac{1}{8\pi} (^{(n)}G)_{\alpha\beta}$, and $^{(n)}G_{\alpha\beta}$ consists of all the higher order terms in the Einstein-tensor. Note that the assumption, $|h_{\alpha\beta}| \ll 1$, requiring the perturbations to be sufficiently small, could also be relaxed then. In particular, once $T_{\alpha\beta}$ is replaced by the sum $T_{\alpha\beta} + t_{\alpha\beta}$ the yielded equations become suitable to describe the evolution of intrinsically strong GW sources which cannot be done properly in the linearised theory. Note also that the conservation law $\partial^\rho T_{\alpha\beta} = 0$, which is responsible for the above mentioned defects, gets to be replaced by the more adequate relation $\partial^\rho (T_{\alpha\beta} + t_{\alpha\beta}) = 0$. Once the replacement $T_{\alpha\beta} \rightarrow T_{\alpha\beta} + t_{\alpha\beta}$, along with a senseful redefinition of the quantities $\rho$, $S_i$, $S$, $P$, $\sigma_{ij}$, $\sigma_i$, has been done then all the previously derived equations, (32)–(38), can be seen to preserve their forms.

Now, by making use of the above introduced nonlinear setup, we intend to provide a plausibility argument suggesting that the response of our detectors is going to be affected by back-reaction. Before presenting our argument we would like to emphasise that, whenever nonlinearities are taken into account but sufficiently far from the sources $|h_{\alpha\beta}| \ll 1$ holds, it seems to be reasonable to assume that the tidal forces can still be given by (42), with the distinction that now $\Theta$, $\Phi$, $\Xi_i$ and $h_{ij}^{TT}$ refer to the redefined quantities. We shall use this assumption below. Note also that in consequence of the hidden nonlinearities no attempt is made to go beyond providing a plausibility argument, i.e., no quantitative estimates are derived.

In proceeding let us revisit the fall off properties of the last three terms on the right hand side of (42). Note first that since the middle two terms in (42) contain spatial derivatives, it is straightforward to verify that they decay faster than the terms on the sides. In particular, since both $\Phi$ and $\Xi_i$ fall off like $1/r$ the relations $\partial_i \partial_j \Phi \sim r^{-3}$ and $\partial_i \partial_i \Xi_j \sim r^{-2}$ can be seen to hold. Therefore the main issue is whether, besides $h_{ij}^{TT}$, the last term on the right hand side of (42) may also have a $1/r$ fall off in the nonlinear case. In this respect the following simple example provides some important clues.

Assume that in our spacetime we have nothing else but a localised GW source which produces a single short lasting GW burst. Consider now an observer that is
asymptotically far from the source in the distance. For simplicity let us represent
the world-sheet of the GW burst, as it is travelling from the source towards infinity,
by a null shell. The observation occurs where the world-line of the observer meets
this null shell. In advance to the observation the $1/r$ part of $\Theta$ (as measured by the
observer) is proportional to the ADM mass. However, if back-reaction is taken into
account, regardless how tiny is the energy carried by the GW on the future side of
the null shell, the coefficient of the $1/r$ part of $\Theta$ will be smaller. This is so because
the mass felt by the observer will be smaller than the ADM mass as the GW, even
after its detection, goes on, carrying the energy released by the source, towards null
infinity. Accordingly, the coefficient of the $1/r$ part of $\Theta$ will vary in time in spite of
the fact that the total ADM mass is conserved.

The above example suggests that nonlinearities manifest themselves in the fol-
lowing simple way. Whenever $T_{\alpha\beta}$ is replaced with $T_{\alpha\beta} + t_{\alpha\beta}$—in consequence of the fact
that $t_{\alpha\beta}$ extends beyond the observer and it is time dependent—the coefficient of the
$1/r$ part of $\Theta$, at the location of the observer, vary with time. If the observer is not
too far from the sources then this time variation might be oscillatory although it is
expected to be monotonic, as in the above example, if the observer is asymptotically
far from the sources.

Thereby, in the nonlinear regime it seems to be reasonable to assume that second
time derivative of both $\Theta$ and $h^{TT}_{ij}$ fall off like $1/r$. Accordingly, in virtue of (42), for
the variation of the coordinates of the end mirrors the relation

$$\delta L_i(t) \approx \frac{1}{2} \left[ h^{TT}_{ij} + \Theta \delta_{ij} \right] L_0^j$$

applies. What is even more remarkable is that the effect of the second term on the
right hand side of (43) on the arms of the laser interferometric detectors is nothing
but a possibly tiny but isotropic change of the arm lengths.

Of course, without carrying out further quantitative investigations there is no
way to argue that this effect is important. Thus, it is of obvious interest to know
whether this effect is large enough, and whether it could be detected by the current
arrangements of our ground based laser interferometric detectors. Unfortunately,
the answer to the latter question is no since the LIGO-Virgo type detectors in their
present form are sensitive only to the relative variation of the arm lengths. We hope
that by a suitable modification of the applied detection schemas, e.g., by adopting
some of the ideas proposed in [15], it may be possible to measure the variation of
the arm lengths separately. If this can be done an isotropic change of the spatial
geometry, simultaneous to the arrival of a gravitational wave train, could hopefully
be detected.
6 Final Remarks

In this paper some of the peculiarities of the TT gauge in the weak field approximation were investigated. The results found indicate that there have been several assumptions applied in determining the TT part of the metric perturbations which are in the air. In particular, the results found have the following non-trivial consequences.

First, it is pointed out that, whenever sources are involved, in determining the TT part of the metric perturbations a new approach—significantly different from the currently applied one—, which is taking into account the non-locality of the sources pertinent for the gauge independent TT variables is needed. It is also indicated that the associated improvements will affect the asymptotic wave forms not only in the linearised theory but in the post-Newtonian framework, as well.

Second, the determination of the response of interferometric gravitational wave detectors may be influenced considerably if back-reaction is taken into account. A plausibility argument was provided justifying that the arrival of a GW train yields an additional isotropic change of the arm lengths. We would like to emphasise that there is an immediate consequence of this effect which may affect the estimates concerning the detectability of gravitational wave signals by our current detectors. This is related to the possibility that the energy released by the astrophysical sources may not completely be transferred into the pure radiative degrees of freedom but some part of it could be used to produce the isotropic change of the spatial geometry in the distance. This variation of the volume may decrease the current estimates of GW amplitudes given in terms of $h_{ij}^{TT}$. It is also indicated that the anticipated isotropic change may be observed by suitably modified versions of the currently applied laser interferometric detectors.

It is worth to be mentioned that in most of the investigations of GW productions, whenever there is an attempt to take into account the nonlinear back-reactions, curiously enough, the nonlinearities themselves are left out almost immediately from the discussions. For instance, in determining the quadrupole tensor, which, in the nonlinear case, should read as

$$ q_{ab} = \int_\Sigma \left[ T_{00} + t_{00} \right] x^a x^b d^3x $$

(44)

it is usually assumed (see, e.g., the bottom of p. 87 in [10], or section 4.2 in [8]) that the contribution of $t_{00}$ can be neglected as the relation $t_{00} \ll T_{00}$ holds at the location of the source. However, in accordance with the last remark of Wald on the top of p. 88 of [10], we would like to emphasise that a higher level of clarity and rigour should be involved here. Note that the main results of the present paper support these necessities simply because $t_{00}$ is known to be global, and, far from the sources, where $t_{00}$ may be small but positive, its contribution could also be significantly amplified.
by the factor $x^\alpha x^\beta$ in (44). Therefore, it may happen that whenever $t^{(0)}$ is sufficiently anisotropic its contribution to the quadrupole moment tensor will become significant. This, in turn, may yield a considerable deformation of the emitted wave forms while travelling from the sources to the observers. If this turns out to be the case we cannot avoid a careful revision of the currently applied template banks.

It is also of obvious interest to know what might be the relation between our findings and “Christodoulou’s nonlinear memory effect” [16]. In this respect we would like to mention that, because of the significant differences of the mathematical setup applied in [16] and in this paper, it is not obvious at all to derive a meaningful relation especially because in [16] no explicit expression is given that could represent the displacement of the mirrors. In particular, as a consequence of these differences while in this paper the nonlinear effects were shown to be isotropic no such conclusion was derived in [16]. Nevertheless, it is worth to be mentioned here that according to the rough estimate provided by Christodoulou the nonlinear effects may be of the same order as the linear ones.

It might be tempting to consider further consequences of the indicated isotropic change of the spatial geometry. Indeed, the pertinent implications might be far-reaching if at certain parts of the universe the monotonous increasing of the function $\Theta$ could be verified. Nevertheless, we would like to emphasise again that in this respect the present paper is far from being conclusive as neither reliable estimates concerning the fraction of the energy converted into the change of $\Theta$ has been derived nor its monotonicity has been studied. To do so further analytic and numerical investigations are needed.

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