BCJ duality and the double copy in the soft limit

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Abstract

We examine the structure of infrared singularities in QCD and quantum General Relativity, from the point of view of the recently conjectured double copy property which relates scattering amplitudes in non-Abelian gauge theories with gravitational counterparts. We show that IR divergences in both theories are consistent with the double copy procedure, to all orders in perturbation theory, thus providing all loop-level evidence for the conjecture. We further comment on the relevance, or otherwise, to the so-called dipole formula, a conjecture for the complete structure of IR singularities in QCD.

1 Introduction

Scattering amplitudes in quantum field theories continue to generate a large amount of interest, due to the many phenomenological and theoretical applications. Much of this work (motivated by the relevance of QCD to hadron colliders) focuses on non-Abelian gauge theories, including supersymmetric counterparts (e.g. $\mathcal{N} = 4$ Super-Yang Mills theory) whose simpler structure provides a useful testing ground for theoretical techniques. Amplitudes have also been studied in gravitational field theories, namely General Relativity and its supersymmetric counterparts such as $\mathcal{N} = 8$ supergravity, which may provide an ultraviolet finite field theory of gravity [1]. Although gravitational theories look superficially very different from non-Abelian gauge theories, a recent programme of work has suggested that gauge and gravity amplitudes may be related in an intriguing way [2–4].

Firstly, gauge theory amplitudes may be written in a special form which manifests an explicit duality between colour and kinematics, the so-called BCJ duality of [5]. Secondly, an $m$-point, $L$-loop order gauge theory amplitude in BCJ dual form can be translated into an equivalent gravity amplitude by the double copy procedure, in which colour factors are replaced by kinematic factors. That gauge theory amplitudes admit BCJ duality is known to be the case at tree level [3, 6–9], where the double copy property is equivalent to the known KLT relations [10] relating amplitudes in gauge and gravity theories. At loop level there is no formal proof of the double copy conjecture, although it is known that this should hold pending the existence of BCJ duality to all orders in the gauge theory, for the cases of pure gravity and $\mathcal{N} = 8$ super-gravity [3]. Loop-level checks of the duality have been carried out up to four loop order in $\mathcal{N} = 4$ SYM theory [2, 11–13] for amplitudes.
involving up to five external particles, and two loop order (for four point scattering with maximal helicity violation) in QCD [2].

As is perhaps clear from the above comments, much work on BCJ duality and the double copy property has focused on \( \mathcal{N} = 4 \) SYM theory, in which a restricted set of loop diagrams contribute [4]. A significant motivation for studying \( \mathcal{N} = 4 \) SYM is that the double copy relates this to \( \mathcal{N} = 8 \) supergravity, so that one may investigate the issue of whether the latter theory is ultraviolet finite [1, 14, 15]. To date, less work has been invested in examining the non-supersymmetric context of pure Yang-Mills theory, where the double copy relates this to General Relativity. Examples include [16, 17] (see also [18]), in which amplitudes in \( 4 \leq \mathcal{N} \leq 8 \) supergravity are considered by combining BCJ dual \( \mathcal{N} = 4 \) SYM amplitudes with those in a less supersymmetric theory, exploiting the fact that the two sets of kinematic factors in the double copy procedure need not both satisfy BCJ duality. The double copy has also been investigated in the context of \( \mathcal{N} = 1 \) and \( \mathcal{N} = 2 \) SYM theory [66]. Recently, an extension of the duality was considered, to Yang-Mills theory deformed by higher dimensional operators [19].

The aim of this paper is to point out that the infrared limit of QCD² provides a clean environment in which to examine the double copy property in a non-supersymmetric context. We will argue that the known structure of infrared singularities in both QCD and gravity is consistent with the double copy, via BCJ duality, to all loop orders. The infrared behaviour of gauge theory amplitudes has been intensively studied over many years (see e.g. [20–29]). In QCD, the state of current knowledge regarding massless scattering amplitudes can be summarised in the so-called dipole formula [30–34], a conjecture which states that soft and collinear singularities exponentiate such that the exponent contains colour correlations between at most pairs of external legs. The formula is known to be exact up to two loops, and the structure of possible corrections at three loop order and beyond has been further studied in [35–39]. It is known that the dipole formula fails already at two loop level for massive external particles [40–48], a fact which is not immediately relevant here due to the fact that the framework of BCJ duality and the double copy procedure is set up only for massless particles.

The study of infrared singularities in General Relativity was first examined in [49], and there has recently been a revival of interest, particularly in the work of [50–52] which aims to describe soft graviton physics in the same language used in a gauge theory context, and which we will review in what follows. It is now well established that the soft limit of gravity is one loop exact. That is, infrared singularities exponentiate such that the exponent contains only one loop graphs. Note that this implies that all IR singularities in gravity are dipoles, in the sense that at one loop in the exponent only pairs of particles can be correlated.

The above comments, and the relationship between gauge theory and gravity offered by the double copy procedure, beg the question: could the QCD dipole formula have an essentially gravitational origin? This thought motivated the present study, and we will see that in fact the answer appears to be no. Nevertheless, we will find that the structure of IR singularities in QCD match up at all orders with the known structure of singularities in General Relativity, providing all-loop-level

²We consider pure gluodynamics (QCD without quarks) throughout the paper. For convenience, we refer to this as QCD throughout.
evidence for the double copy conjecture. The argument is insensitive to possible corrections to the dipole formula - we will see explicitly that many singularities disappear when the double copy is taken. This includes collinear singularities, which are already known to vanish in gravity after summing over diagrams \[49,52\].

The structure of the paper is as follows. In section 2 we review necessary background material in more detail, namely concepts relating to BCJ duality, the double copy, and the structure of infrared singularities in QCD and gravity. In section 3 we examine BCJ duality and the double copy in the soft limit at one loop order, before extending this analysis to two loop order in section 4. In section 5 we generalise our remarks to all loop orders, before summarising our results and concluding in section 6.

2 Review of necessary concepts

2.1 BCJ duality and the double copy

In this section we briefly summarise BCJ duality \[5\] and the associated double copy conjecture \[2,3\], which together provide a map from gauge theory to gravity amplitudes, potentially at the multiloop level \[3\].

A general massless \(m\)-point \(L\)-loop gauge theory amplitude \(A_{(m)}^{(L)}\) in \(D\) space-time dimensions can be written as

\[
A_{(m)}^{(L)} = i^L g^{m-2+2L} \sum_{i \in \Gamma} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_i \prod_{\alpha} p_{\alpha}^2} \prod_{i} n_i C_i \prod_{\alpha} p_{\alpha}^2,
\]

where \(g\) is the coupling constant. Here the sum is over the complete set of graphs involving triple gluon vertices, consistent with the given loop order and number of external particles, and \(S_i\) a symmetry factor for each graph \(i\) (the dimension of its automorphism group). The denominator contains all relevant propagator momenta, and \(n_i\) is the kinematic numerator associated with each graph. Finally, \(C_i\) is the colour factor of each graph, obtained by dressing each three gluon vertex with a factor (adopting the same conventions as \[4\])

\[
\tilde{f}^{abc} = i\sqrt{2} f^{abc},
\]

where \(f^{abc}\) are the usual SU(3) structure constants. Noteworthy points regarding this formula are:

- The restriction to graphs with cubic vertices only does not mean that graphs with four gluon vertices have been excluded. Rather, one can always choose to rewrite the latter in terms of cubic graphs, by introducing extra momenta in the denominator (which are compensated by additional factors in the numerator). This relies upon the fact that the colour factor for a four gluon vertex is a product of two structure constants, consistent with a pair of three gluon vertices.

- The numerators in eq. (1) may or may not come from individual Feynman diagrams. They may also have been obtained from e.g. a generalised unitary-based approach (see e.g. \[53\] and references therein), in which case each \(n_i\) collects all kinematic information associated with a particular scalar integral (fixed by the denominator).

\[\text{See also section II(A) of \[4\] for a recent and pedagogical review of this material.}\]
From any graph at a given loop order, one may construct two other graphs related by taking BCJ transformations involving crossing of s-channel like subgraphs into t and u-channel subgraphs. This is best illustrated pictorially, as in figure 1. Such transformations exist for any internal line (propagator) in a given cubic graph, leading to a set of inter-related graphs, which can be generated from a single diagram. The colour factors associated with the three graphs in figure 1 satisfy the identity
\[ c_s = c_t + c_u, \]  
(3)

where \( c_i \) is the colour factor associated with the diagram containing an \( i \)-channel-like subgraph. This follows from the Jacobi identity for the structure constants, after factoring out the part of each colour factor which is the same for each graph. Note there is an ambiguity in how one defines signs in this identity (which can be compensated by a corresponding choice for the numerators).

It is conjectured that one can always choose to redefine the numerators \( n_i \) in the amplitude of eq. (1), such that they satisfy a similar identity to eq. (3). That is, the numerators of the \( s \), \( t \) and \( u \)-channel-like graphs in figure 1 can be chosen to obey
\[ n_s = n_t + n_u. \]  
(4)

Furthermore, if a given colour factor picks up a minus sign under reordering of a vertex, then
\[ c_i \rightarrow -c_i \quad \Rightarrow \quad n_i \rightarrow -n_i. \]  
(5)

These properties are collectively known as BCJ duality, after [5]. It has been proven at tree-level, but remains a conjecture at loop level. Transformations required to write the numerators in a suitable form are known as generalised gauge transformations, and an algorithmic procedure exists in principle to establish, for a given loop-level amplitude, what a possible set of numerators actually is [4]. It is not clear, however, whether this algorithm is fully general, particularly in non-supersymmetric theories where more diagrams enter than in supersymmetric cases. In general, one may represent the effect of a generalised gauge transformation on a given numerator \( n_i \) as
\[ n_i \rightarrow n_i + \Delta_i, \]  
(6)

where the \( \{\Delta_i\} \) must satisfy
\[ \sum_{i \in \Gamma} \frac{\Delta_i c_i}{\prod_{\alpha_i} p_{\alpha_i}^2} = 0. \]  
(7)

Again the sum is over all cubic diagrams with colour factors \( c_i \) and propagator denominators \( p_{\alpha_i}^2 \). Indeed, eq. (7) is obtained by substituting the transformation of eq. (6) into eq. (1) and requiring
that the amplitude is invariant. One may further decompose the generalised gauge parameters \( \Delta_i \) (assuming these to be local) as \[^3\]

\[
\Delta_i = \sum_{\alpha_i} \Delta_{i,\alpha_i} p_{\alpha_i}^2.
\]

(8)

That is, the \( \Delta_i \) associated with diagram \( i \) can be expanded in terms of the inverse propagators of this diagram. The \( \Delta_i \) factors then move contributions between diagrams by cancelling propagators, so that contributions from one diagram can be absorbed in another.

Once a form of the amplitude satisfying BCJ duality has been found, a remarkable conjecture states that one can turn it into a gravity amplitude, in a straightforward manner. By stripping off the colour factors in eq. \( (1) \) and replacing them with another set of numerators \( \{ \tilde{n}_i \} \), the double copy conjecture states that \[^2,3\]

\[
\mathcal{M}_m^{(L)} = i^{L+1} \left( \frac{\kappa}{\sqrt{2}} \right)^{m-2+2L} \sum_{i \in \Gamma} \int \frac{d^D p_i}{(2\pi)^D S_i} \prod_{l=1}^L \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} p_{\alpha_i}^2}
\]

(9)

is an \( m \)-point, \( L \)-loop gravity amplitude. Note that the numerators \( \{ n_i \} \) and \( \{ \tilde{n}_i \} \) need not come from the same theory. In this paper, we will be concerned with both sets of numerators coming from QCD, in which case the gravity amplitude corresponds to general relativity (coupled to an antisymmetric tensor and a dilaton). If the two gauge theories are \( \mathcal{N} = N \) and \( \mathcal{N} = M \) Super-Yang-Mills theory, then one obtains an amplitude in \( \mathcal{N} = (N + M) \) supergravity.

This is all we need for what follows. We now turn to the study of infrared singularities in QCD and gravity.

### 2.2 Infrared singularities in non-Abelian gauge theory

Infrared singularities in quantum field theory have been studied over many decades, with a vast accompanying literature. Here we briefly summarise only those facts which are of direct relevance to what follows. For a more detailed pedagogical review, see e.g. \[^34\]. Whilst our statements can be interpreted in the context of a general non-Abelian gauge theory, we explicitly refer to QCD throughout.

It is by now well-known that infrared singularities factorise, such that the general structure of an \( m \)-point scattering amplitude in QCD is, schematically,

\[
\mathcal{A}_m = \mathcal{H}_m \cdot S \cdot \prod_{i=1}^m \frac{J_i}{\mathcal{F}_i}.
\]

(10)

Here \( \mathcal{H}_m \) is an infrared finite hard interaction, which is dressed by a universal soft function \( S \) that collects all infrared singularities. The jet function \( J_i \) collects collinear singularities associated with external leg \( i \), and the eikonal jet function \( \mathcal{F}_i \) removes the double counting of divergences which are both soft and collinear. The soft function is given by a vacuum expectation value of Wilson line

\[^4\]Note that in this paper we use \( \kappa = \sqrt{16\pi G_N} \) rather than \( \kappa = \sqrt{32\pi G_N} \) (see section \(^2,3\)). This modifies the coupling factors in eq. \(^9\) relative to those in \(^2,3\).
operators along the space-time trajectories of the outgoing hard particles. Equivalently, one may calculate the soft part of a given hard interaction by dressing all external lines with all possible soft gluon emissions, according to the *eikonal Feynman rule*

\[ g_s T_i \frac{p^\mu}{p \cdot k}, \]

where \( g_s \) is the strong coupling constant, and \( p \) (\( k \)) the momentum of the hard external line (soft gluon) respectively. Furthermore, \( T_i \) is a colour generator in the representation of external line \( i \), where we have used the notation of [54][55]. Where soft gluons meet off the external lines, they couple according to the usual three and four gluon vertices of QCD.

All singularities appearing in eq. (10) can be shown to exponentiate, so that instead of eq. (10) we may write

\[ A_m = H_m \cdot Z, \]

where

\[ Z = \exp \left[ \sum_{n=1}^{\infty} c_n (\{ p_i \}, \epsilon, \mu) \alpha_S^n \right] \]

contains all soft and collinear singularities, and depends upon the momenta \( \{ p_i \} \), as well as the dimensional regularisation parameter \( \epsilon \) and scale \( \mu \). The structure of the exponent (i.e. the form of the coefficients \( \{ c_n \} \) in eq. (13)) is known explicitly up to two loop order for both massless and massive particles [40–48,56,57]. For massless particles, it has the remarkable property of involving both colour and kinematic correlations between at most pairs of particles, despite the fact that one would naïvely expect correlations between more than two particles to appear at two loop order and beyond. This property motivated the conjecture of the so-called *dipole formula* in QCD [30–34], which gives the all-order structure of eq. (13) as

\[ Z = \exp \left\{ \int_0^{\lambda_2^2/\chi^2} \frac{d\lambda}{\lambda^2} \left[ \frac{1}{8} \gamma_K (\alpha_S \lambda^2, \epsilon) \sum_{(i,j)} \ln \left( \frac{2p_i \cdot p_j e^{i\pi \lambda_{ij}}}{\lambda^2} \right) T_i \cdot T_j - \frac{1}{2} \sum_{i=1}^{m} \gamma_J_i (\alpha_S \lambda^2, \epsilon) \right] \right\}. \]

Here \( \gamma_K \) is the cusp anomalous dimension (itself a perturbative expansion in \( \alpha_S \) with constant coefficients), and \( \gamma_J_i \) a further anomalous dimension associated with jet \( i \), and which collects hard collinear contributions. The double sum in the first term is over all pairs of particles \( (i, j) \), and following [34] we have used the notation \( \lambda_{ij} = 1 \) if \( i \) and \( j \) are both in the initial or both in the final state (otherwise \( \lambda_{ij} = 0 \)). Equation (14) indeed contains correlations between dipoles only, and is known to break down already at two loop order for massive external legs. For the massless case, corrections may occur at three loop order and beyond, either through explicit kinematic dependence of the relevant Feynman integrals, or through higher order Casimir invariants appearing in the cusp anomalous dimension. The form of possible corrections has been investigated in [35–39]. Further progress may be made using recently developed techniques for classifying the structure of the exponent [28,29,58,59], or by considering alternative gauges [60].

### 2.3 Infrared singularities in gravity

Complementary to the above mentioned studies in gauge theory, IR singularities have also been investigated in gravity, commencing with the classic work of [49]. Recently, there has been a revival
of interest, which has focused in particular on writing the structure of gravitational IR divergences using the same language as is used in modern QCD studies. Reference [50] suggested the use of the following gravitational generalisation of eq. (10):

$$M_m = H_m \cdot S^{\text{grav}},$$

(15)

where $M_m$ is an $m$-point gravity amplitude. Here $H_m$ is again a hard interaction which is infrared finite, and $S^{\text{grav}}$ is a universal gravitational soft function, which is given by a vacuum expectation value of suitable Wilson line operators. There are no jet functions, due to the fact that collinear singularities cancel in gravity after summing over all diagrams and using momentum conservation. The latter property was first established in the soft limit [49], but has recently been fully extended to encompass hard collinear singularities [52, 61]. Furthermore, ref. [51] examined the form of eq. (15) in more detail using the path integral approach of [62], also classifying what happens beyond the eikonal approximation. A similar structure of next-to-eikonal corrections was found as in the case of gauge theory, as explored in [62, 63]. Gravitational Wilson lines in a soft-graviton context were further explored in [64] using the radial coordinate space picture of [60], and a continuous deformation was found between the cusp anomalous dimensions of QED and gravity at one loop.

As in the gauge theory case, the soft function in gravity exponentiates. However, a drastic simplification over gauge theory occurs in that the exponent contains only one-loop diagrams, with no higher order corrections. This property is known as one-loop exactness, and has been firmly established by the studies of [49, 50, 52]. It implies that all infrared singularities (i.e. to all orders in perturbation theory) ultimately stem from the exponentiation of the one-loop corrections, in marked contrast to the gauge theory case of eq. (13): even if the dipole formula of eq. (14) occurs to be true, there is still a further perturbation expansion to be carried out in the exponent, whose soft part requires the cusp anomalous dimension to all orders. It is amusing to note, as has already been mentioned in the introduction, that one-loop exactness in gravity implies that all infrared singularities are associated with pairs of particles (the most that can be correlated with a single graviton exchange). This is reminiscent of the QCD dipole formula in some sense (i.e. that higher multipole correlations vanish), and allows us to speculate as to whether the dipole formula may have a gravitational origin. The results of this paper would appear to suggest that this is not the case, due to the disappearance of many singularities upon taking the double copy of a gauge theory. However, one-loop exactness has another important role: it tells us that we know the all-order structure of IR singularities in gravity completely. We can then ask whether the known IR divergence structures in QCD and gravity are consistent with each other, if we apply the double copy procedure (via BCJ duality). If this is so, this provides all-order evidence (at least in a particular limit) for the double copy conjecture. This will be the aim of the rest of the paper.

In our subsequent calculations we will use the following conventions (a number of different choices exist in the literature - see e.g. [65] for a convenient reference). The graviton field $h_{\mu \nu}$ is defined in terms of the metric tensor $g_{\mu \nu}$ via

$$g_{\mu \nu} = \eta_{\mu \nu} + \kappa h_{\mu \nu},$$

(16)

where $\eta_{\mu \nu}$ is the Minkowski space metric, $\kappa = \sqrt{16\pi G_N}$ and $G_N$ is Newton’s constant. Emission of soft gravitons with momentum $k$ from a hard external line of momentum $p$ is then given by the
eikonal Feynman rule (see [64] for a recent derivation using the above conventions)

\[
\frac{\kappa}{2} \frac{p_\mu p_\nu}{p \cdot k},
\]

which may be compared with the gauge theory eikonal Feynman rule of eq. (11). Finally, we will need the graviton propagator, for which we use the result in the de Donder gauge:

\[
D_{\mu\nu,\alpha\beta}(k) = -i P_{\mu\nu,\alpha\beta} \frac{k^2}{k^2 - i\epsilon}, \quad P_{\mu\nu,\alpha\beta} = \eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\nu\alpha} - \frac{2}{D-2} \eta_{\mu\nu} \eta_{\alpha\beta}.
\]

Having reviewed the necessary material for what follows, we begin our investigation of the double copy procedure in the soft limit in the following section.

3 One loop analysis

In the previous section, we reviewed various theoretical ideas concerning BCJ duality, the double copy, and the structure of infrared singularities in both QCD and gravity. This motivated the central question of our study: do the known IR singularity structures in QCD and gravity match up with each other upon taking the double copy of QCD? The aim of this paper is to argue that this is indeed the case, and we will begin at one-loop order with the following strategy:

1. We classify all possible BCJ relations of the form of eq. (4), between sets of three diagrams containing cubic vertices, in the case of pure Yang-Mills theory.

2. Next, we write down all possible diagrams which give infrared singularities at this order, and show that they can be matched up in sets of three, each of which offers an explicit solution of a BCJ relation in the soft limit. We will also see that additional BCJ relations cannot be explicitly satisfied (or otherwise) in the Feynman gauge in the soft limit, but that this will be irrelevant for the double copy.

3. We then take the double copy of all infrared divergent diagrams, and verify that this reproduces the one-loop divergences in GR.

The first step is to obtain the BCJ relations, which we do in the following subsection.

3.1 BCJ relations

To start with, we must write down all possible one-loop graphs, and for a concrete example we consider the case of 4-point scattering, labelled as in figure 2. We take all momenta outgoing, and define Mandelstam invariants according to

\[
s = (p_1 + p_2)^2, \quad t = (p_1 + p_4)^2, \quad u = (p_1 + p_3)^2,
\]

for ease of comparison with e.g. [5]. For each graph, we specify the orientation of momentum on each line, together with a set of basis momenta for the corresponding numerator function (three external, and one internal). All other momenta in each graph can then be fixed from these momenta. The box, triangle and bubble graphs are shown in figures 3, 4 and 5 respectively.
Figure 2: Four point scattering, where all momenta are taken to be outgoing.

Figure 3: Set of all box graphs at one-loop order, where we depict the basis momenta entering the numerator for each graph.

Figure 4: Set of all triangle graphs at one-loop order, where we depict the basis momenta entering the numerator for each graph.
Figure 5: Set of all bubble graphs at one-loop order, where we show the choice of internal basis momentum for each graph. All external momenta are labelled as in graph (b9).
As explained in section 2, the diagrams in figures 3-5 form sets of three, related by subjecting internal lines to a BCJ transformation (replacement of t-channel-like exchange by s- and u-channel exchanges). The colour factors of such a set satisfy the Jacobi identity, and BCJ duality, if satisfied, then requires a corresponding set of functional equations to hold for the kinematic numerators associated with each graph. The BCJ relations in the present case can be divided into four classes. Firstly, there are relations which relate two box topologies to a triangle. These are (using the labels and basis momenta from figures 3 and 4)

\begin{equation}
nt_1(p_1, p_2, p_3, k) = n_{B1}(p_1, p_2, p_3, k) - n_{B2}(p_1, p_2, p_3, -p_1 - p_3 + k);
nnt_2(p_1, p_2, p_3, k) = n_{B1}(p_1, p_2, p_3, -p_2 - p_3 - k) - n_{B2}(p_1, p_2, p_3, k);
nnt_3(p_1, p_2, p_3, k) = n_{B1}(p_1, p_2, p_3, -k - p_2) - n_{B3}(p_1, p_2, p_3, -p_3 + k);
nnt_4(p_1, p_2, p_3, k) = n_{B1}(p_1, p_2, p_3, k + p_1) - n_{B3}(p_1, p_2, p_3, k + p_1);
nnt_5(p_1, p_2, p_3, k) = n_{B3}(p_1, p_2, p_3, k - p_2 - p_3) - n_{B2}(p_1, p_2, p_3, k - p_1 - p_3);
nnt_6(p_1, p_2, p_3, k) = n_{B3}(p_1, p_2, p_3, -k) - n_{B2}(p_1, p_2, p_3, k - p_3).
\end{equation}

We have chosen to write these in terms of the triangles evaluated with loop momentum k as the fourth argument, for reasons that will become clear. Next, there are BCJ relations that relate each triangle topology to the same topology (evaluated with a different loop momentum argument) and a bubble. There are two subclasses - firstly those relations which involve an external self-energy:

\begin{equation}
nnt_1(p_1, p_2, p_3, k) = -n_{T1}(p_1, p_2, p_3, -p_2 - k) + n_{B1}(p_1, p_2, p_3, k);
nnt_2(p_1, p_2, p_3, k) = -n_{T1}(p_1, p_2, p_3, p_1 - k) + n_{B2}(p_1, p_2, p_3, k);
nnt_2(p_1, p_2, p_3, k) = -n_{T2}(p_1, p_2, p_3, -p_3 - k) + n_{B3}(p_1, p_2, p_3, k);
nnt_3(p_1, p_2, p_3, k) = -n_{T3}(p_1, p_2, p_3, -p_2 - k) + n_{B4}(p_1, p_2, p_3, k);
nnt_3(p_1, p_2, p_3, k) = -n_{T3}(p_1, p_2, p_3, -p_1 - p_2 - p_3 + k) + n_{B6}(p_1, p_2, p_3, k - p_3);
nnt_4(p_1, p_2, p_3, k) = -n_{T4}(p_1, p_2, p_3, p_1 + p_2 + p_3 - k) + n_{B8}(p_1, p_2, p_3, k - p_3);
nnt_4(p_1, p_2, p_3, k) = -n_{T4}(p_1, p_2, p_3, -p_1 - p_2 - p_3 + k) + n_{B9}(p_1, p_2, p_3, -p_1 - p_2 - p_3 - k);
nnt_5(p_1, p_2, p_3, k) = -n_{T5}(p_1, p_2, p_3, -p_1 - p_2 - p_3 + k) + n_{B11}(p_1, p_2, p_3, k - p_2);
nnt_5(p_1, p_2, p_3, k) = -n_{T5}(p_1, p_2, p_3, p_1 + p_2 + p_3 - k) + n_{B13}(p_1, p_2, p_3, -k);
nnt_6(p_1, p_2, p_3, k) = -n_{T6}(p_1, p_2, p_3, -p_1 - k) + n_{B12}(p_1, p_2, p_3, p_1 + k);
nnt_6(p_1, p_2, p_3, k) = -n_{T6}(p_1, p_2, p_3, p_3 - k) + n_{B14}(p_1, p_2, p_3, -k).
\end{equation}

Secondly, there are those which involve an internal self-energy:

\begin{equation}
nnt_1(p_1, p_2, p_3, k) = -n_{T1}(p_1, p_2, p_3, p_1 - p_2 - k) + n_{B5}(p_1, p_2, p_3, -p_2 - k);
nnt_2(p_1, p_2, p_3, k) = -n_{T2}(p_1, p_2, p_3, -p_1 - p_2 - 2p_3 - k) + n_{B5}(p_1, p_2, p_3, p_3 + k);
nnt_3(p_1, p_2, p_3, k) = -n_{T3}(p_1, p_2, p_3, -p_2 - k) + n_{B10}(p_1, p_2, p_3, -p_2 - k);
nnt_4(p_1, p_2, p_3, k) = -n_{T4}(p_1, p_2, p_3, -2p_1 - p_2 - p_3 - k) + n_{B10}(p_1, p_2, p_3, k + p_1);
nnt_5(p_1, p_2, p_3, k) = -n_{T5}(p_1, p_2, p_3, p_1 + 2p_2 + p_3 - k) + n_{B15}(p_1, p_2, p_3, -k + p_1 + p_2 + p_3);
nnt_6(p_1, p_2, p_3, k) = -n_{T6}(p_1, p_2, p_3, -p_1 + p_3 - k) + n_{B15}(p_1, p_2, p_3, p_1 + k).
\end{equation}
Next, there are relations which relate the numerators of three bubble graphs:

\[
\begin{align*}
    n_{b1}(p_1, p_2, p_3, k) &= n_{b6}(p_1, p_2, p_3, -p_2 - k) + n_{b11}(p_1, p_2, p_3, -p_2 - k) \\
    n_{b2}(p_1, p_2, p_3, k) &= n_{b7}(p_1, p_2, p_3, k) + n_{b12}(p_1, p_2, p_3, k) \\
    n_{b3}(p_1, p_2, p_3, k) &= n_{b8}(p_1, p_2, p_3, k) + n_{b14}(p_1, p_2, p_3, k) \\
    n_{b4}(p_1, p_2, p_3, k) &= n_{b9}(p_1, p_2, p_3, k) + n_{b13}(p_1, p_2, p_3, -p_1 - p_2 - p_3 - k).
\end{align*}
\]  

(23)

Each of these has the form of a complete set of s-, t- or u-channel tree-level graphs, dressed by a common external self-energy. Finally, there are relations which relate two bubble graphs with a tadpole. Here we assume that tadpole graphs can be set to zero, in which case the remaining relations impose constraints on the forms of the bubble numerators as follows:

\[
n_{b6}(p_1, p_2, p_3, k) = n_{b6}(p_1, p_2, p_3, -k) \quad \forall \quad 1 \leq i \leq 15.
\]  

(24)

A similar set of BCJ relations has been presented recently in [66], in the context of \( \mathcal{N} = 1 \) and \( \mathcal{N} = 2 \) SYM theory. However, in that case it was found that the bubble numerators could be set to zero.

Having presented the one-loop relations in this section, we proceed to consider the soft limit. Our aim is to show that the BCJ relations are satisfied in this limit, up to corrections which are subleading in soft momentum, and which are irrelevant for the double copy to gravity. This is the subject of the following section.

3.2 Soft limit

In this section, we consider all possible infrared singular diagrams at one loop order, and show that their numerators are consistent with the BCJ relations obtained in the previous section. A full solution of all relations requires the inclusion of higher order terms in the soft loop momentum. However, these corrections will be seen to be irrelevant in taking the double copy to gravity, if one is focusing solely on infrared divergences.

IR singularities at one loop are generated by dressing tree level diagrams with soft gluon emissions. Considering once again the concrete example of 4-point scattering, we may write the full tree level amplitude as

\[
\mathcal{A}^{(0)} = \sum_{x \in \{s, t, u\}} \mathcal{A}_x, \quad \mathcal{A}_x = \frac{c_x n_x(p_1, p_2, p_3)}{x},
\]  

(25)

where \( \{n_x\} \) is a suitable set of numerators (with colour factors \( \{c_x\} \) ) obeying the tree-level BCJ relations

\[
n_s(p_1, p_2, p_3) - n_t(p_1, p_2, p_3) - n_u(p_1, p_2, p_3) = 0, \quad c_s - c_t - c_u = 0.
\]  

(26)

Note that each tree-level numerator requires three basis momenta, and we label these according to figure 6 (n.b. the choice of external momenta is similar to that used in figures 3-5). Note that we

\footnote{Note that we here adopt the phase conventions of eq. (1). Associating a factor \( i \) and \( -i \) with each propagator and vertex respectively, one may absorb a further factor of \( 1/i \) in the numerators \( n_x \) to obtain an overall power of \( i^L \) at L-loop order. Performing the double copy involves an extra numerator, and thus an additional explicit factor of \( i \) in the gravity amplitude.}
Figure 6: Labelling of basis momenta for the tree-level numerators $n_x(p_1, p_2, p_3)$.

have absorbed coupling factors into the tree level numerators, so that all powers of the coupling in subsequent equations correspond to higher order corrections. Also, implicit in eq. (25) is the fact that the four-gluon vertex graph has been rewritten in terms of cubic graphs. The effect of dressing a tree-level amplitude $A_x$ with a soft gluon exchanged between legs $i$ and $j$ is given by

$$g_s^2 T_i \cdot T_j I_{ij} A_x,$$

where the eikonal integral factor

$$I_{ij} = i \int \frac{d^Dk}{(2\pi)^D} \frac{p_i \cdot p_j}{k^2 p_i \cdot k p_j \cdot k},$$

as results from connecting two eikonal Feynman rules (eq. (11)) with a gluon propagator (note we use the Feynman gauge). The full infrared singular part of the one-loop amplitude can then be written

$$A^{(1)}_{IR} = ig_s^2 \sum_x \sum_{i<j} T_i \cdot T_j I_{ij} \frac{A_x}{x},$$

where the first sum is over the three tree-level cubic topologies, and the second sum is over all pairs of external legs, where each pair is counted only once. There are no contributions from soft emissions which begin and end on the same external line, due to the fact that all outgoing hard particles are massless ($p_i^2 = 0$). Furthermore, there are no contributions from internal self-energy diagrams: such graphs contain additional hard propagators that remove the infrared singularity.

We can now interpret eq. (28) from the viewpoint of BCJ duality. First, let us examine the relations of eq. (20), relating pairs of box graphs with a triangle. These relations may be satisfied by grouping the terms appearing in eq. (28) into sets of three. Interchanging the orders of summation in eq. (28), one may pick out a particular pair $(i, j)$, to give

$$\sum_x g_s^2 T_i \cdot T_j \int \frac{d^Dk}{(2\pi)^D} \frac{p_i \cdot p_j}{k^2 p_i \cdot k p_j \cdot k} \frac{c_x n_x(p_1, p_2, p_3)}{x},$$

where we have substituted in the explicit forms for $I_{ij}$ and $A_x$ from eqs. (25) and (27). This is three separate terms, each of which can be interpreted as a soft limit of one of the graphs appearing in figures 3-4. Taking the pair (3,4) as an example, we may interpret eq. (29) as shown in figure 7.
Figure 7: Diagrams involving a soft gluon exchange between legs 3 and 4, where solid (curly) lines denote hard (soft) gluons respectively, and we have suppressed the orientations on all lines for brevity.

From eq. (29), we may associate with each term a collected numerator and colour factor.

\[ \hat{n}_{x,ij} = (p_i \cdot p_j)n_x(p_1, p_2, p_3), \quad c_{x,ij} = T_i \cdot T_j c_x, \quad (30) \]

To make this mapping more precise, we may identify the soft limits of the numerator functions appearing in eq. (20) as follows. For the triangles, one has

\[ n_{T1}(p_1, p_2, p_3, k) = \hat{n}_{s,12} + O(k); \]
\[ n_{T2}(p_1, p_2, p_3, k) = \hat{n}_{s,34} + O(k); \]
\[ n_{T3}(p_1, p_2, p_3, k) = \hat{n}_{t,23} + O(k); \]
\[ n_{T4}(p_1, p_2, p_3, k) = \hat{n}_{t,14} + O(k); \]
\[ n_{T5}(p_1, p_2, p_3, k) = \hat{n}_{u,24} + O(k); \]
\[ n_{T6}(p_1, p_2, p_3, k) = \hat{n}_{u,13} + O(k). \quad (31) \]

For the boxes, there is more than one soft limit for each graph, corresponding to the fact that there is an infrared singular region when any of the internal lines has vanishing momentum. We may express these limits as

\[ n_{B1}(p_1, p_2, p_3, k) = \hat{n}_{t,12} + O(k); \]
\[ n_{B1}(p_1, p_2, p_3, -p_2 - p_3 - k) = \hat{n}_{t,34} + O(k); \]
\[ n_{B1}(p_1, p_2, p_3, -k - p_2) = \hat{n}_{s,23} + O(k); \]
\[ n_{B1}(p_1, p_2, p_3, k + p_1) = \hat{n}_{s,14} + O(k); \]
\[ n_{B2}(p_1, p_2, p_3, k) = -\hat{n}_{u,34} + O(k); \]
\[ n_{B2}(p_1, p_2, p_3, -p_1 - p_3 + k) = -\hat{n}_{u,12} + O(k); \]
\[ n_{B2}(p_1, p_2, p_3, k - p_1 - p_2 - p_3) = -\hat{n}_{s,24} + O(k); \]
\[ n_{B2}(p_1, p_2, p_3, k - p_3) = -\hat{n}_{s,13} + O(k); \]
\[ n_{B3}(p_1, p_2, p_3, k) = -\hat{n}_{t,13} + O(k); \]

---

6 Strictly speaking the eikonal integral of eq. (27) is zero, being a scaleless integral in dimensional regularisation. However, this is due to the cancellation of the IR divergence with a spurious UV pole which results from the replacement of quadratic propagators by linear ones, such that one may recover the IR divergence using an additional regularisation procedure.

7 Note that the colour generators $T_i$ are understood with indices ordered in the direction of momentum flow on each external line.

8 Note that care is needed regarding minus signs in eq. (32). The sign of each soft limit is fixed by the collected colour factor in eq. (30), and its comparison with figures 3 and 6.
\[ n_{B3}(p_1, p_2, p_3, -p_3 + k) = \hat{n}_{u,23} + \mathcal{O}(k); \]
\[ n_{B3}(p_1, p_2, p_3, k + p_1) = \hat{n}_{u,14} + \mathcal{O}(k); \]
\[ n_{B3}(p_1, p_2, p_3, k - p_2 - p_3) = -\hat{n}_{t,24} + \mathcal{O}(k). \]

(32)

One may then verify the relations

\[ \hat{n}_{s,ij} - \hat{n}_{u,ij} - \hat{n}_{t,ij} = 0, \quad c_{s,ij} - c_{u,ij} - c_{t,ij} = 0, \]

(33)

which follow from the fact that the tree level colour factors and numerators satisfy eq. (26), and that these are multiplied by a common factor in eq. (30). Equation (33) contains six different relations (the number of ways of choosing two external legs out of four), which correspond precisely to soft limits of the BCJ relations in eq. (20). Take, for example, the case shown in figure (7). These diagrams consist of the \( k \to 0 \) limits of \( n_{B1}(p_1, p_2, p_3, -p_2 - p_3 - k) \), \( n_{B2}(p_1, p_2, p_3, k) \) and \( n_{T3}(p_1, p_2, p_3, k) \), and the relevant BCJ relation from eq. (33) thus corresponds to the second relation in eq. (20).

We now consider the remaining BCJ relations. It has already been remarked upon above that bubble graphs do not contribute to the infrared singular part of the one-loop scattering amplitude: the eikonal Feynman rules ensure that external self-energies vanish, and internal bubbles vanish due to the fact that a gluon cuts a hard internal propagator. It is equally true that bubble topologies which result from rewriting four-gluon vertex graphs in terms of bubble topologies (snail graphs in the language of e.g. [4]) are not infrared singular. Thus, in the soft limit, we may set

\[ \hat{n}_{bi} = \mathcal{O}(k) \quad \forall \quad i, \]

(34)

where the hat above the numerator implies that we are evaluating this only in the soft limit. Such numerators are defined only up to arbitrary terms containing at least one power of a soft gluon momentum \( k \), which in principle must come from a full solution of the BCJ relations linking bubbles with other topologies. However, such terms are irrelevant for the IR singularities of the amplitude, and certainly once the double copy to gravity is taken. In the copy, the numerator of each topology is squared, whilst the denominator remains unchanged. Thus, bubble graphs will also not contribute IR singularities on the gravity side.

With the bubble numerators set to zero, one does not have to worry about the relations of eqs. (23) and eq. (24). However, one must still address the relations of eq. (21), which link each triangle numerator to the same numerator function (evaluated with different loop momenta), and a bubble. For these, the soft limit alone provides insufficient information as to whether such relations are satisfied. Clearly the full \( k \) dependence (represented by the \( \mathcal{O}(k) \) terms in eqs. (31) and (34)) is necessary in order to verify these relations, and thus we are not able to solve these relations explicitly. However, this is again irrelevant for taking the double copy. The triangle numerators above are \( \mathcal{O}(k^0) \), whereas the bubble numerators are \( \mathcal{O}(k) \). In modifying them by a generalised gauge transformation, they may potentially change by superpositions of denominator factors as given by eq. (8). Given that all relevant denominators are at least \( \mathcal{O}(k) \), this means that denominators which fully satisfy the BCJ relations have the same limits as given above. Corrections \( \sim \mathcal{O}(k) \) will not give rise to additional infrared singularities, and this remains true after taking the double copy.
given that the denominators do not change.

We have now seen that of the BCJ relations derived in the previous section (for the case of full QCD), it is possible to identify the soft limits of the numerator functions, such that these satisfy the relations linking boxes with triangles. For the remaining relations, the bubbles can be set to zero given that they do not give rise to IR singularities. Furthermore, the relations linking triangles with bubbles imply corrections to the leading soft behaviour that are irrelevant as far as infrared singularities are concerned.

Armed with the above knowledge, we are now permitted to take the double copy of eq. (28) as prescribed in [2], to give

\[ \mathcal{M}_{\text{IR}}^{(1)} = - \sum_x \sum_{i<j} \left( \frac{\kappa}{\sqrt{2}} \right)^2 \int \frac{d^Dk}{(2\pi)^D} \frac{(p_i \cdot p_j)^2}{k^2 p_i \cdot k} \frac{n_x n_x}{x} \]

where we write \( n_x \equiv n_x(p_1, p_2, p_3) \). If the double copy procedure is valid, this result should give the infrared singular parts of the 4-point, 1 loop amplitude in GR. That this is indeed the case can be seen as follows. Firstly, the factor

\[ \frac{i n_x n_x}{x} \]

is, by the tree-level double copy procedure, a gravitational tree level graph for the s, t or u topology. This is then dressed by an eikonal integral, which corresponds to the appropriate gravitational generalisation of eq. (27). To check this, one may contract eikonal Feynman rules on legs \( i \) and \( j \) with the propagator of eq. (18) to get

\[ \mathcal{I}_{\text{grav}}^{ij} = \left( \frac{\kappa}{2} \right)^2 \int \frac{d^Dk}{(2\pi)^D} \frac{p_i}{p_i \cdot k} \frac{p_j}{p_j \cdot k} \left( \eta_{\mu\nu} \eta_{\alpha\beta} + \eta_{\mu\beta} \eta_{\nu\alpha} - \frac{2}{D-2} \eta_{\mu\nu} \eta_{\alpha\beta} \right) \]

where the minus sign in the second eikonal Feynman rule results from reversing the sign of the soft gluon momentum. Dressing the tree-level gravitational interaction with equation (36) is in exact agreement with eq. (35).

Note that, as first observed in [49], soft collinear singularities cancel after summing over all diagrams. Taking those which are associated with line \( i \) as an example, these are generated by the sum of eikonal integrals

\[ \sum_{j \neq i} \int \frac{d^Dk}{(2\pi)^D} \lim_{k \to p_i} \frac{4(p_i \cdot p_j)^2}{k^2 p_i \cdot k p_j \cdot k} = \sum_{j \neq i} \int \frac{d^Dk}{(2\pi)^D} \lim_{k \to p_i} \frac{4(p_i \cdot p_j)}{k^2 p_i \cdot k} . \]

\[ \tag{37} \]

\[^9\text{Here we have again adopted the phase conventions of eq. (1) - see the footnote on p. 12.}\]

\[^{10}\text{Recall that we absorbed gauge theory coupling factors into the tree-level numerators. Implicit in eq. (35) is that these have been replaced appropriately in the double copy to gravity.}\]
Applying momentum conservation

\[ p_i = - \sum_{j \neq i} p_j \]  

then gives zero on the right-hand side of eq. (37), up to non-singular terms. Note that the QCD equivalent of eq. (37) (also including colour factors) is

\[ \sum_{j \neq i} T_i \cdot T_j \int \frac{d^Dk}{(2\pi)^D} \lim_{k \to p_i} \left[ \frac{(p_i \cdot p_j)}{k^2 p_i \cdot k p_j \cdot k} \right] = \sum_{j \neq i} T_i \cdot T_j \int \frac{d^Dk}{(2\pi)^D} \lim_{k \to p_i} \left[ \frac{1}{k^2 p_i \cdot k} \right]. \]  

(38)

Colour conservation

\[ T_i = - \sum_{j \neq i} T_j \]  

then gives

\[ \sum_{j \neq i} T_i \cdot T_j \int \frac{d^Dk}{(2\pi)^D} \lim_{k \to p_i} \left[ \frac{(p_i \cdot p_j)}{k^2 p_i \cdot k p_j \cdot k} \right] = C_i \int \frac{d^Dk}{(2\pi)^D} \lim_{k \to p_i} \left[ \frac{1}{k^2 p_i \cdot k} \right], \]  

(40)

where \( C_i \) is the quadratic Casimir associated with external line \( i \) (\( C_i = C_A = N_c \) for pure gluodynamics). This remains singular, and a simple physical way to interpret this is that collinear singularities can only depend on the quantum numbers of a single particle, which include the squared charge of the appropriate external line. In QCD, this is the quadratic Casimir operator in the relevant representation, whereas in gravity this is the squared 4-momentum, which vanishes if collinear singularities are to be present (i.e. if the leg is massless). The double copy procedure has here provided an interesting mechanism for the cancellation of soft-collinear singularities on the gravity side: the colour dependence in QCD gets replaced by additional momentum factors, which generate the necessary squared 4-momentum. Note that this means that there are singularities on the gauge theory side that vanish upon performing the double copy. We will see this happening more generally at two-loop order and beyond.

We have now verified that the soft limit of GR is precisely reproduced upon taking the double copy of the soft limit of QCD at one-loop order. Although we here focused on the particular case of 4-point scattering, the argument is easily generalised to any number of external legs. Having examined the one-loop case, we proceed to two loops in the following section.

4 Two loop analysis

In the previous section, we showed that the known infrared singularities of GR are reproduced by double copying those of QCD at one-loop order. Application of the double copy relied upon the fact that the BCJ relations could be satisfied in the soft limit. The solution of relations linking boxes with triangles was rather straightforward at this order, and relied ultimately on the fact that corresponding relations were satisfied (after removal of a common soft gluon) at tree level. Put another way, the soft BCJ relations of eq. \((33)\) all had the form of a common eikonal factor \((p_i \cdot p_j)\) multiplying a tree-level relation. Further relations involving at least one bubble graph could not be satisfied explicitly by the Feynman gauge results in the soft limit. However, this was irrelevant given that bubble graphs could be consistently set to zero as far as IR singularities were
concerned. Furthermore, corrections to triangle numerators were subleading in the soft limit, and thus irrelevant for IR singularities, on both the gauge theory and gravity sides of the double copy. This simple structure will not quite be the case at two loop order. We will again see that numerators for individual graphs, computed in the Feynman gauge, do not automatically satisfy BCJ relations in the soft limit. This will now affect IR singularities in the gauge theory, but will remain irrelevant for the double copy, and hence for reproducing the known infrared singularities of GR.

Our strategy will be the same as at one-loop order, and begins by writing down the BCJ relations in full QCD. Then one draws all possible soft topologies, and demonstrates that the resulting numerators satisfy BCJ relations in the soft limit. Our experience at one loop tells us that we do not have to worry about most of the bubble diagrams. The numerators for these (whilst non-zero in full QCD) can be set to zero in the soft limit as before: most external and internal bubbles are IR-suppressed. The only exception to this is the presence of self-energies associated with soft gluons, as we will see in what follows. There is a large number of diagrams at two-loop order and we do not collect them all here. Rather, we consider directly relevant soft topologies, and show that these can indeed be matched up with BCJ relations.

We consider tree-level \( m \)-point scattering dressed by soft gluons up to two-loop order. Possible soft topologies can then link two, three or four external lines. Examples are shown in figure 8, where we label each topology, depicted in a schematic way (i.e. for brevity, we do not label momenta and their orientations) We may also write the eikonal integrals that go with each soft topology, as

\[
\mathcal{I}^{(\text{II} k)}_{ijkl} = -\int \frac{d^Dk}{(2\pi)^D} \int \frac{d^Dl}{(2\pi)^D} \frac{p_i \cdot p_j p_k \cdot p_l}{k^2 l^2 p_i \cdot k p_j \cdot k p_k \cdot l p_l \cdot l'}
\]
obtained by picking a particular term in eq. (44), and keeping the sum over tree-level topologies. This is the only scenario that was possible at one-loop order, and the relevant three graphs are 

correspond to the soft limit of a BCJ relation. There are two distinct classes of relation. Firstly, as in the one-loop case, we can match up terms in eq. (44) into sets of three, such that they 

correspond to a Y graph which is reflected with respect to the former. The infrared singular part of the two-loop gauge theory amplitude can now be written as 

\[
A^{(2)}_{\text{IR}} = -g_s^4 \sum_{x \in \{s, t, u\}} \left[ \sum_{(ijkl)} T_i \cdot T_j T_k \cdot T_l T_{ijkl}^{(||)} + \sum_{(ijk)} \left( T_i \cdot T_j T_k \cdot T_l T_{ijkl}^{(||)} + \tilde{f}^{abc} T_i^a T_j^b T_k^c T_{ijkl}^{(Y_3)} \right) + \tilde{f}^{abc} T_i^a T_j^b T_k^c T_{ijkl}^{(X)} + \tilde{f}^{abc} \tilde{f}^{bcd} T_i^a T_j^b T_k^d T_{ijkl}^{(O)} \right] \times \frac{n_x}{x},
\]

where 

\[
V_{\mu \nu \rho}(k, l) \sim O(k, l)
\]

is the three-gluon vertex, which in this case couples together only soft momenta. Our notation for the eikonal integrals specifies which indices are coupled together by soft gluon emissions, and note that the ordering of these indices can be important (e.g. \( T_{ij}^{(Y)} \) is not the same as \( T_{ij}^{(X)} \): the latter corresponds to a Y graph which is reflected with respect to the former). The infrared singular part of the two-loop gauge theory amplitude can now be written as 

\[
A^{(2)}_{\text{IR}} = -g_s^4 \sum_{x \in \{s, t, u\}} \left[ \sum_{(ijkl)} T_i \cdot T_j T_k \cdot T_l T_{ijkl}^{(||)} + \sum_{(ijk)} \left( T_i \cdot T_j T_k \cdot T_l T_{ijkl}^{(||)} + \tilde{f}^{abc} T_i^a T_j^b T_k^c T_{ijkl}^{(Y_3)} \right) + \tilde{f}^{abc} T_i^a T_j^b T_k^c T_{ijkl}^{(X)} + \tilde{f}^{abc} \tilde{f}^{bcd} T_i^a T_j^b T_k^d T_{ijkl}^{(O)} \right] \times \frac{n_x}{x},
\]

where the notation \( (ij \ldots k) \) denotes that one must sum over all distinct multiples, and we have used 

the colour vertex factor of eq. (2). This expression is obtained by dressing the tree-level interaction with all possible eikonal integrals, and is the two-loop generalisation of eq. (28). Furthermore, we have taken a factor of \( t^2 = -1 \) out of the eikonal integrals, so as to make manifest the phase convention of eq. (1).

As in the one-loop case, we can match up terms in eq. (44) into sets of three, such that they correspond to the soft limit of a BCJ relation. There are two distinct classes of relation. Firstly, there are relations in which the same eikonal integral dresses different tree-level interaction graphs. This is the only scenario that was possible at one-loop order, and the relevant three graphs are obtained by picking a particular term in eq. (44), and keeping the sum over tree-level topologies \( x \). As an example, consider the contribution 

\[
g_s^4 \sum_x (T_i \cdot T_j)^2 T_{ij}^{(||)} ,
\]

where
obtained by selecting a particular pair in the first term in the second line of eq. (44). This corresponds to the graphs shown in figure 9(a). These indeed correspond to the soft limit of three full QCD graphs which enter a BCJ relation, namely those shown in figure 9(b). From eq. (45), one may associate a collected numerator and colour factor for each graph, according to

$$c_x^{(||)} = (T_i \cdot T_j)^2 c_x, \quad \hat{n}_x^{(||)} = (p_i \cdot p_j)^2 n_x(p_1, p_2, p_3),$$

where the hat once again reminds us that such numerators are defined in the soft limit, and may have arbitrary terms \( \sim O(k, l) \) added. The numerators thus defined satisfy the relations

$$\hat{n}_s^{(||)} - \hat{n}_u^{(||)} - \hat{n}_t^{(||)} = 0,$$

by virtue of the fact that this is satisfied by the tree-level numerators, and that these are all multiplied by a common factor. Although we have been somewhat schematic in our notation, eq. (47) is indeed the relevant limit of the BCJ relation with all basis momenta properly accounted for, analogously to eq. (33).

It is clear that the above argument generalises to any set of three graphs which have the same soft topology \( Z \). That is, one may form soft numerators

$$\hat{n}_x^{(Z)} = I_{ij...k}^{(Z)} \bigg|_{\text{num.}} n_x(p_1, p_2, p_3),$$

where \( I_{ij...k}^{(Z)} \bigg|_{\text{num.}} \) denotes the numerator of the relevant eikonal integral, such that

$$\hat{n}_s^{(Z)} - \hat{n}_u^{(Z)} - \hat{n}_t^{(Z)} = 0.$$  

This then corresponds to the soft limit of a BCJ relation.

The second class of soft BCJ relations at two loop order is more complicated, and consists of sets of three graphs in which the hard part of the interaction (in this case a single tree-level topology)
Figure 10: Sets of three graphs, each corresponding to the soft limit of a BCJ relation, such that the hard interaction is the same in each term.
explicitly infrared singularities of gravity via the double copy, we do not have to solve the BCJ relations through superpositions of denominator factors. If, however, all we care about is verifying the general form of such a transformation is given in eq. (6), and consists of redefining each numerator

\[ \hat{n}_{x,ijk} + \hat{n}_{x,kji} - \hat{n}_{x,ijk} = 0, \]

\[ -\hat{n}_{x,ij} + \hat{n}_{x,ij} + \hat{n}_{x,ij} = 0, \]

\[ \hat{n}_{x,ij} + \hat{n}_{x,ij} - \hat{n}_{x,ij} = 0, \]

(50)

(where appropriate momentum arguments are understood) corresponding to each of the sets of graphs in figure 10 and where \( \hat{n}_{x,i...k}^{(Z)} \) is the soft numerator associated with a diagram where soft graph \( Z \) dresses tree-level topology \( x \). As is clear from the eikonal integrals of eq. (42), the numerators of the graphs in the Feynman gauge do not automatically satisfy these relations. The numerators are given by

\[ \hat{n}_{x,ijk} = p_i \cdot p_j \cdot p_k n_x, \]

\[ \hat{n}_{x,ijk}^{(Y)} = V_{\mu\nu\rho}(k,l) p_i^\mu p_j^\nu p_k^\rho n_x, \]

\[ \hat{n}_{x,ij} = V_{\mu\nu\rho}(k,l) n_x, \]

\[ \hat{n}_{x,ij} = (p_i \cdot p_j)^2 n_x, \]

\[ \hat{n}_{x,ij}^{(O)} = V_{\mu\nu\rho}(k,l) n_x, \]

(51)

where we have used a tilde to denote the fact that these soft numerators do not respect BCJ duality. In order to find those that do, one must effect a generalised gauge transformation \( \hat{n} \rightarrow \tilde{n} \). The general form of such a transformation is given in eq. (6), and consists of redefining each numerator through superpositions of denominator factors. If, however, all we care about is verifying the infrared singularities of gravity via the double copy, we do not have to solve the BCJ relations explicitly. To see this, note the numerator factors depend on soft momenta as

\[ \hat{n}_{x,ijk}, \hat{n}_{x,ij}, \hat{n}_{x,ij} \sim \mathcal{O}(K^0), \quad \hat{n}_{x,ijk}, \hat{n}_{x,ij} \sim \mathcal{O}(K), \quad \hat{n}_{x,ij} \sim \mathcal{O}(K^2), \]

(52)

where \( K \equiv (k,l) \) represents a soft momentum scale. Furthermore, all relevant denominator factors scale as at least \( \mathcal{O}(K) \). Thus, from eq. (6) one must have

\[ \hat{n}_{x,ijk}^{(Y)} = \hat{n}_{x,ijk} + \mathcal{O}(K), \]

\[ \hat{n}_{x,ij}^{(X)} = \hat{n}_{x,ij} + \mathcal{O}(K), \]

\[ \hat{n}_{x,ij}^{(O)} = \hat{n}_{x,ij} + \mathcal{O}(K), \]

(53)

and

\[ \hat{n}_{x,ijk}^{(Y)}, \hat{n}_{x,ij}^{(Y)}, \hat{n}_{x,ij}^{(O)} \sim \mathcal{O}(K). \]

(54)

We may summarise this more generally as follows. The BCJ dual numerators for soft graphs involving multiple single gluon emissions between pairs of external lines are the same as those in
the Feynman gauge, up to corrections subleading in soft momentum. These corrections will not lead to additional IR singularities, so can be ignored in the soft limit. This is consistent with the first two BCJ relations in eq. (50), which due to the subleading nature of \( \hat{n}_{x,ijk} \) and \( \hat{n}_{x,ij} \) amount to

\[
\hat{n}_{x,ijk} = \hat{n}_{x,kji}, \quad \hat{n}_{x,ij} = \hat{n}_{x,ji},
\]

as is indeed observed already in eq. (51).

The BCJ dual numerators for graphs involving three gluon vertices off the eikonal lines are changed with respect to the Feynman gauge numerators, and are first order in soft momentum. Upon taking the double copy to gravity, these numerators will be squared, whereas the denominators are unchanged. Hence, by power counting, such graphs will not contribute infrared singularities in gravity. In other words, the very BCJ relations that one has to invest effort in solving are irrelevant for the double copy! This also tells us that there are infrared singularities in QCD that vanish when one takes the double copy to gravity. This “information loss” will, as we will see, be a feature at higher loop orders.

The above remarks imply that we can take the double copy of eq. (44) by keeping only the graphs with multiple dipole emissions. Their numerators will be given by eq. (53), where we can safely neglect the corrections which are subleading in soft momenta. Performing the double copy procedure gives a 2-loop gravity amplitude \(^{11}\)

\[
\mathcal{M}_{IR}^{(2)} = -i \left( \frac{\kappa}{2} \right)^4 \sum_{x \in \{s,t,u\}} \left[ \sum_{ijkl} T_{ijkl, grav}^{(||)} + \sum_{ijk} T_{ijk, grav}^{(||)} + \sum_{ij} T_{ij, grav}^{(||)} + T_{ij, grav}^{(X)} \right] \frac{n_k n_{x}}{x},
\]

where

\[
\begin{align*}
T_{ijkl, grav}^{(||)} &= \int \frac{d^4 k}{(2\pi)^4} \int \frac{d^4 l}{(2\pi)^4} \frac{4(p_i \cdot p_j)^2}{k^2 l^2 p_i \cdot k p_j \cdot k p_k \cdot l p_i \cdot l}, \\
T_{ijk, grav}^{(||)} &= \int \frac{d^4 k}{(2\pi)^4} \int \frac{d^4 l}{(2\pi)^4} \frac{4(p_i \cdot p_j)^2}{k^2 l^2 p_i \cdot k p_j \cdot k p_j \cdot (k+l) p_k \cdot l}, \\
T_{ij, grav}^{(||)} &= \int \frac{d^4 k}{(2\pi)^4} \int \frac{d^4 l}{(2\pi)^4} \frac{4(p_i \cdot p_j)^4}{k^2 l^2 p_i \cdot k p_i \cdot (k+l) p_j \cdot k p_j \cdot (k+l)}, \\
T_{ij, grav}^{(X)} &= \int \frac{d^4 k}{(2\pi)^4} \int \frac{d^4 l}{(2\pi)^4} \frac{4(p_i \cdot p_j)^4}{k^2 l^2 p_i \cdot k p_i \cdot (k+l) p_j \cdot l p_j \cdot (k+l)}.
\end{align*}
\]

This indeed agrees with an explicit calculation using the known gravitational eikonal factors, obtained using the Feynman rules of eqs. (17) and (18). We have thus shown that, at two loop order, the infrared singularities of GR are consistent with those of QCD via the double copy procedure.

We can write eq. (57) in a more recognisable form as follows. Firstly, one has

\[
T_{ijkl, grav}^{(||)} = T_{ij, grav}^{grav} T_{kl, grav},
\]

\(^{11}\)For the overall phase factor, see the footnote on p. 12.
where the right-hand side contains the product of two gravitational one-loop eikonal factors from eq. (36). By collecting terms in eq. (57), we can write the entire right-hand side in terms of one-loop integrals. In particular, one has

\[ T_{ij}^{(I_3), \text{grav.}} + T_{kji}^{(I_3), \text{grav.}} = \int \frac{d^Dk}{(2\pi)^D} \int \frac{d^Dl}{(2\pi)^D} \frac{4(p_i \cdot p_j)^2 (p_j \cdot p_k)^2}{k^2 l^2 p_i \cdot k p_k \cdot l p_j \cdot (k + l) \left[ \frac{1}{p_j \cdot k} + \frac{1}{p_j \cdot l} \right]} \]

\[ = \int \frac{d^Dk}{(2\pi)^D} \int \frac{d^Dl}{(2\pi)^D} \frac{4(p_i \cdot p_j)^2 (p_j \cdot p_k)^2}{k^2 l^2 p_i \cdot k p_k \cdot l p_j \cdot k p_j \cdot l} \]

(59)

where in the second line we have used the eikonal identity

\[ \frac{1}{p_j \cdot (k + l)} \left[ \frac{1}{p_j \cdot k} + \frac{1}{p_j \cdot l} \right] = \frac{1}{p_j \cdot k p_j \cdot l}. \]

(60)

One thus has

\[ T_{ij}^{(I_3), \text{grav.}} + T_{kji}^{(I_3), \text{grav.}} = T_{ij}^{\text{grav.}} T_{jk}^{\text{grav.}}. \]

(61)

Also, one has

\[ T_{ij}^{(||), \text{grav.}} + T_{ij}^{(X), \text{grav.}} = \int \frac{d^Dk}{(2\pi)^D} \int \frac{d^Dl}{(2\pi)^D} \frac{4(p_i \cdot p_j)^4}{k^2 l^2 p_i \cdot k p_k \cdot (k + l) p_j \cdot (k + l) \left[ \frac{1}{p_j \cdot k} + \frac{1}{p_j \cdot l} \right]} \]

\[ = \int \frac{d^Dk}{(2\pi)^D} \int \frac{d^Dl}{(2\pi)^D} \frac{4(p_i \cdot p_j)^4}{k^2 l^2 p_i \cdot k p_k \cdot (k + l) p_j \cdot k p_j \cdot l} \]

\[ = \frac{1}{2} \int \frac{d^Dk}{(2\pi)^D} \int \frac{d^Dl}{(2\pi)^D} \frac{4(p_i \cdot p_j)^4}{k^2 l^2 p_i \cdot (k + l) p_j \cdot k p_j \cdot l \left[ \frac{1}{p_i \cdot k} + \frac{1}{p_i \cdot l} \right]} \]

\[ = \frac{1}{2} \int \frac{d^Dk}{(2\pi)^D} \int \frac{d^Dl}{(2\pi)^D} \frac{4(p_i \cdot p_j)^4}{k^2 l^2 p_i \cdot k p_k \cdot l p_j \cdot k p_j \cdot l} \]

\[ = \frac{1}{2} \left( T_{ij}^{\text{grav.}} \right)^2, \]

(62)

where we have again used the eikonal identity of eq. (60), as well as relabelling \( k \leftrightarrow l \). Putting things together, we may rewrite eq. (56) using the results of eqs. (58, 61, 62) to give

\[ M_{\text{IR}}^{(2)} = -i \left( \frac{\kappa}{2} \right)^4 \sum_{x \in \{ s,t,u \}} \left[ \sum_{(ijkl)} T_{ijkl}^{\text{grav.}} T_{kl}^{\text{grav.}} + \sum_{(ijk)} T_{ij}^{\text{grav.}} T_{jk}^{\text{grav.}} + \frac{1}{2} \sum_{(ij)} \left( T_{ij}^{\text{grav.}} \right)^2 \right] n_x n_x x, \]

(63)

where the notation \((ijk)\) denotes summing over all triples, such that the ordering of \(i\) and \(k\) is unimportant. The total soft prefactor (contents of the square bracket in eq. (63) is easily checked to be the second order term in the expansion of

\[ \exp \left[ i \left( \frac{\kappa}{2} \right)^2 \sum_{(ij)} T_{ij}^{\text{grav.}} \right], \]

(64)

as expected from the known exponentiation and one-loop exactness of gravitational infrared divergences [49, 50, 52]. This completes our analysis of BCJ duality and the double copy in the soft limit at two-loop order. In the next section, we generalise our remarks to all loop orders.
5 General remarks

In the previous two sections, we have seen that the structure of infrared singularities in QCD matches on to that of GR after applying the double copy procedure. At one loop, we were able to apply the double copy due to the fact that the Feynman gauge numerators for soft graphs satisfied the appropriate BCJ relations in the soft limit, up to irrelevant corrections (suppressed by powers of soft momentum). At two loops, these relations could be separated into two classes: (a) those involving graphs whose soft topology was the same, but whose underlying hard tree-level topology was different; (b) those involving graphs sharing the same hard interaction, but having different soft topologies. BCJ relations of class (a) were automatically satisfied, due to the fact that the underlying tree-level numerators satisfy BCJ duality. Relations of class (b) were more complicated. Numerators for graphs with no three gluon vertices off the external lines (which we refer to as dipole-like graphs in the following) could be taken to be the same as the Feynman gauge results up to subleading corrections in soft momenta, which are irrelevant from the point of view of infrared singularities. Numerators of graphs involving three-gluon vertices off the external lines did not automatically satisfy BCJ relations, and thus would have to be modified by generalised gauge transformations in order to write down a BCJ-dual representation of a QCD amplitude in the soft limit. However, these graphs are at least linear in soft momenta, and thus vanish upon taking the double copy to gravity. This allowed us to verify that the IR singularities of gravity are correctly reproduced by double copying the QCD results at two-loop order, without having to explicitly solve the BCJ relations.

The aim of this section is to argue that this argument generalises to all loop orders. This is possible because we have already seen all of the necessary ingredients at two loop order. Firstly, the fact that the infrared limit BCJ relations fall into the two classes given above is generally true, independent of the loop order\(^\text{12}\). Then BCJ relations of class (a), and involving a given soft topology \((Z)\), take the generic form of eq. (49), with numerators given by eq. (48). As at two loop order, these relations are satisfied by virtue of the fact that the tree-level numerators satisfy the BCJ relation, and are multiplied by a common factor.

BCJ relations of class (b) are again more complicated, and can be split into two further subclasses: (i) those involving two dipole-like graphs and a graph containing at least one three gluon vertex off the external lines; (ii) those involving three graphs with at least one three gluon vertex off the external lines. By the same power counting arguments that were used in the previous section, the numerators of dipole-like graphs are the same as their Feynman gauge counterparts up to corrections which are subleading in soft momenta (i.e. which do not contribute infrared singularities). Two dipole graphs which enter the same BCJ relation are related by a permutation of two gluon emissions on an external line (e.g. figure 10(a))\(^\text{13}\). Thus, BCJ relations of subclass (i) set the numerators of such graphs to be equal, which is indeed satisfied in the Feynman gauge, which associates with a single soft gluon emission between lines \(i\) and \(j\) a contribution \(2(p_i \cdot p_j)\), independently of any other gluon emissions. BCJ relations of subclass (ii) are not satisfied by the Feynman gauge numerators for the relevant graphs. However, these numerators are all at least \(O(K)\) (where \(K\) is an arbitrary soft momentum), and remain so after performing a generalised gauge transformation

\textsuperscript{12}At one loop, as we have seen, only class (a) occurs.
\textsuperscript{13}In the language of [29,58,59], such diagrams are in the same multiparton web.
in line with eq. (6). Thus, these graphs do not give infrared singularities after performing the double copy. One thus does not need to solve the BCJ relations explicitly in order to generate the IR divergences of the gravity amplitude. From three-loop order in the Feynman gauge, one must also consider graphs involving the four-gluon vertex off the eikonal lines. This also gives rise to numerators which involve non-zero powers of soft momenta, after rewriting such graphs in terms of cubic topologies, according to the usual BCJ procedure.

Note that there are also BCJ relations which mix up hard and soft information. We have already seen an example of this at one-loop order, namely the relations of eq. (21) which each involve the same triangle numerator evaluated with different momenta. Such relations also arise at higher loop orders, and as at one loop order, the soft limit provides insufficient information regarding whether these relations are satisfied. This is again irrelevant, however, for verifying the infrared singularities in gravity: the explicit solution of such relations requires higher order terms in soft momentum, which do not contribute infrared singularities.

The above remarks allow us to generalise eq. (63) to an arbitrary loop order, as

$$M_{IR}^{(n)} = \left( \frac{\kappa}{2} \right)^{2n} \sum_{x \in \{s,t,u\}} \left[ \sum_{m=2}^{2n} \sum_{\{i_1 \ldots i_m\}} \tilde{T}^{\text{grav.}}_{i_1 \ldots i_m} \right] \frac{n_x n_x}{x}, \quad (65)$$

where we again abbreviate $n_x \equiv n_x(p_1, p_2, p_3)$. In this formula, $\tilde{T}^{\text{grav.}}_{i_1 \ldots i_m}$ is the eikonal integral factor due to the sum of all dipole emissions that link lines $i_1, i_2 \ldots i_m$. The second sum in the square brackets in eq. (65) is then over all multipoles $\langle i_1 \ldots i_m \rangle$ that are linked by such dipole emissions. The first sum is over all possible numbers $m$ of external lines. As in the two-loop analysis of the previous section, one may collect terms in eq. (65) into products of one-loop eikonal integrals, via multiple applications of the (higher-order) eikonal identity, and thus rewrite eq. (65) as

$$M_{IR}^{(n)} = \left( \frac{\kappa}{2} \right)^{2n} \sum_{x \in \{s,t,u\}} \left[ \frac{1}{n!} \left( \sum_{\{(i,j)\}} \frac{i}{d^{D}k} \frac{2p_i \cdot p_j}{(2\pi)^D k^2 p_i \cdot k p_j \cdot k} \right)^n \right] \frac{n_x n_x}{x}. \quad (66)$$

The tree-level amplitude dressed by soft gravitons to all orders, and obtained via the double copy, is then given by

$$M_{IR} = \sum_{n=0}^{\infty} M_{IR}^{(n)} = \exp \left[ \sum_{\{(i,j)\}} \frac{i}{d^{D}k} \frac{2p_i \cdot p_j}{(2\pi)^D k^2 p_i \cdot k p_j \cdot k} \right] \frac{n_x n_x}{x}, \quad (67)$$
in agreement with the known all-order structure of IR divergences in GR [49]. Note that, once we had reached eq. (65), the final result was guaranteed: what mattered was that the double copy procedure correctly reproduces the fact that only dipole-like graphs appear in gravity. The sum over all such graphs automatically leads to the exponentiation of the one-loop corrections [15].

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14 Here we have absorbed factors of $i$ into the eikonal integrals, rather than show these explicitly as in eq. (1).

15 This is the same argument that occurs in the exponentiation of one-loop soft corrections in QED [20,49]. In that theory, however, one-loop exactness is broken due to the presence of fermion bubbles.
This completes our argument that the all-order structure of IR divergences in gravity is consistent with IR singularities in QCD, via the application of the double copy procedure. This can therefore be taken as all-order evidence for the double copy conjecture, albeit in a particular limit. Some further comments are in order. Firstly, the general argument at $O(\alpha_S^2)$ exhibits a property already remarked upon at two loops, namely that many singularities cancel upon taking the double copy to gravity. This means that the gravity side of the correspondence cannot be used to constrain singularities on the gauge theory side. For this reason, the question posed earlier in the paper regarding whether the QCD dipole formula has a gravitational origin appears to have a negative answer: the very singularities which could occur as corrections to the dipole formula vanish when we move to gravity.

Secondly, we have here focused on the case of pure Yang-Mills theory and General Relativity. However, the argument of this paper can also in principle be applied in supersymmetric gauge theories / supergravity. Here we remark that reference [16] obtained amplitudes in $\mathcal{N} \geq 4$ supergravity using the double copy procedure applied to gauge theory amplitudes in $\mathcal{N} = 4$ Super-Yang-Mills theory coupled with $0 \leq \mathcal{N} \leq 4$ Super-Yang-Mills theory. One check on this calculation was the demonstration that infrared singularities, after the double copy, were consistent with exponentiation on the gravity side up to two loop order. The results of this paper generalise this to all loop orders, and non-supersymmetric theories. In supersymmetric theories, it may be possible to extend our arguments beyond the pure soft limit, as non-trivial information can in principle be obtained from infrared singularities [67,68].

Throughout the paper, we have seemingly ignored the fact that the double copy relates pure Yang-Mills theory to General Relativity coupled to a dilaton and two-form. The additional particles may be present on the gravity side of the double copy, and could in principle affect the soft behaviour. However, we have here considered only the leading infrared singularity at each order in perturbation theory, obtained by dressing a tree-level hard interaction consisting of graviton scattering. This will not be affected by the presence of additional particles, which decouple in the soft limit due to power counting arguments [69] (see also [70] for a recent discussion of this point). One may obtain subleading infrared singularities in the gravitational amplitude in principle by coupling higher order hard interactions (including those involving additional particles) to the gravitational soft function discussed throughout the present paper. Whether these subleading singularities match up between the gauge and gravity theories is then purely a statement about the hard function (due to one-loop exactness on the gravity side), and amounts to whether the double copy is satisfied in full QCD.

6 Conclusion

In this paper, we have examined the soft limits of QCD (strictly speaking, pure gluodynamics) and GR, and showed that infrared singular contributions to amplitudes in both theories match up with each other upon using the double copy procedure of [2,3]. The structure of IR divergences in QCD scattering amplitudes is still subject to some uncertainty, and our current state of knowledge can be expressed in terms of the dipole formula of [31–34], plus possible corrections [35–39]. By

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The basic argument is that the gravitational Feynman rules are such that the numerator for a graph involving e.g. a scalar loop is at least $O(K)$, where $K$ is a soft momentum. By a similar argument to that presented in section 4, generalised gauge transformations would not change this.
contrast, the structure of IR singularities is known exactly in GR \cite{GR}, where they exponentiate in terms of one-loop graphs only.

Being able to take the double copy relies on the gauge theory amplitudes displaying manifest BCJ duality \cite{BCJ}. At one-loop order, we saw that amplitudes in the soft limit, as calculated in the Feynman gauge, satisfied BCJ duality up to irrelevant corrections. A similar story occurred at two-loop order, where only dipole-like graphs had numerators which were unmodified (at leading order in soft momentum) by generalised gauge transformations. These are the only graphs that survive upon taking the double copy, so that one does not need to solve the BCJ relations explicitly. We could thus use the double copy procedure to “predict” the structure of IR singularities in GR, finding exact agreement with the known results. Our arguments imply that many singularities vanish upon taking the double copy. If this were not the case, one could have used singularities in gravity to constrain possible corrections to the QCD dipole formula, or even to provide an underlying gravitational explanation for this. Nevertheless, the arguments presented in this paper constitute all-loop level evidence for the validity of the double copy conjecture, which may be more significant in supersymmetric contexts.

As mentioned above, we have only considered the case where the hard interaction consists of tree-level scattering, and hence the leading infrared singularity at each order in perturbation theory on the gravity side of the double copy. In principle, this could contain higher loop orders, and include the double copy scalar and two-form. To show that the double copy is satisfied for the resulting subleading IR singularities then requires BCJ-dual numerators for the hard interaction, which is not possible to all orders without a full proof of BCJ duality in QCD.

There are a number of further questions that can be addressed. It may be possible, for example, to reinterpret our results using the manifestly BCJ-dual effective Lagrangian of \cite{BCJ}. It would also be interesting to examine the full implications of the present analysis in supersymmetric contexts. Finally, a thorough investigation of the role of BCJ duality in non-supersymmetric gauge theory away from the soft limit would potentially provide new insights into QCD and / or gravity. An intermediate step in this regard might be to extend the present analysis to next-to-eikonal order, using the technology of \cite{eikonal,Miller1,Miller2}.

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