Wave energy in quantum mechanics

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Abstract Setting out from accepted physical principles along a purely mathematical line of thought, new ideas about energy are introduced having profound theoretical and practical consequences. Einstein’s 1905 mass energy equivalence equation is shown to have an expanded phasor form that reveals additional subtle aspects of the relationships between energy, mass and momentum. These insights lead to compelling new ideas concerning the nature of quantum mechanical phenomena.

1. The momentum four-vector
The real and imaginary components of the four-vector momentum $p^\mu$ respectively correspond to the linear momentum $p$ and energy $E$ of a particle. Employing natural units ($c = 1$), these two vector components are of identical length for any massless particle (e.g., a photon).

$$|E| = |p|$$  \hspace{1cm} (1)

Figure 1. The four-vector momentum of a photon and its two distinct components

Conventional treatment of the relationship between energy and momentum for massless particles appearing in all physics textbooks expresses it as the following immediate equality.

$$E = pc$$  \hspace{1cm} (2)

However, because the momentum $p$ is real while the energy $E$ is imaginary per Fig. 1, although $pc$ and $E$ are numerically equivalent in magnitude, establishing the mathematical relationship between them in the context of Fig. 1 unequivocally requires an imaginary coefficient.

$$E = ipc$$  \hspace{1cm} (3)
Yet, in common algebraic expressions, total energy $E$ conventionally represents a measurable quantity, which must be real-valued. Therefore, the mathematical expression for the real-valued total energy corresponding to momentum that reflects the subtleties of special relativity with complete and detailed precision is the following complex modulus.

$$E = |ipc| = \sqrt{(0 + ipc)(0 - ipc)} = pc$$

(4)

Far from being trivial, this mathematical subtlety is of the utmost importance for developing a physically accurate conceptual model of energy in microphysics. The relationship commonly expressed by the simple form of Eq. (2) is deceptive because it implicitly disregards the fact that the four-vector components $p$ and $E$ of radiation energy (i.e., wave energy) lie along different dimensions in space-time and so are not equivalent although they are equal in magnitude. Critical physical insights associated with this geometric relationship have been obscured by the conventional algebra. Fig. 1 makes it clear that mathematical equivalence of energy and momentum requires the momentum vector to be rotated in the complex plane so that it is collinear with the energy vector. The explicit appearance of the imaginary coefficient in Eq. (4) based on the fundamental mathematical relationship expressed by Eq. (3) is essential.

In contrast with the preceding case, consider now the rest frame of a material particle, which has a none-zero rest mass. By definition, the momentum is zero for the rest frame; therefore the particle’s four-vector momentum is collinear with the imaginary axis.

$$E = m_0c^2$$

(5)

Accordingly, an elementary mathematical distinction exists between the real-valued intrinsic rest energy of a material particle (5), which is invariant, and the imaginary-valued energy of momentum (3), which is dependent on the reference frame of the observer. The subtle fact that the mathematical descriptions of these two energy manifestations do not coexist on the same number line, which is obscured by the conventional algebraic equations, is made readily apparent by the foregoing intuitive graphical analysis using Fig. 1 and Fig. 2. Clearly, we are not entitled to arbitrarily sum two such energy values together without considering their underlying geometric relationship in the complex plane. It follows that in the general case of a moving material particle having both rest energy and momentum, the total energy of the particle is described most fundamentally by a complex number.

$$E = m_0c^2 + ipc$$

(6)

Again, since total energy $E$ is a measurable quantity, its necessarily real-valued magnitude is the complex norm (i.e., the modulus) of what can only be correctly described by a complex number.
Applying a few elementary algebraic steps to Einstein’s famous 1905 energy equation (8) yields the same well-known relationship between energy, momentum and mass (10), however it is not obvious or even discernable from this prior derivation that the roots of $E^2$ are complex conjugates.

The principles of special relativity require the underlying mathematical representation of total energy to employ the complex numbers, yet apparently Einstein never understood this. In 1908, simultaneously with the introduction of the “world-postulate” in which space and time were unified into an independent reality, it was Hermann Minkowski who also introduced the concept of the momentum four-vector. This vector in the complex plane is defined by

$$p^\mu \equiv n |^\eta^\mu$$

where $|\eta^\mu$ is the velocity four-vector, i.e.,

$$p^\mu = \begin{bmatrix} \gamma m_0 c \\
\gamma m_0 v^1 \\
\gamma m_0 v^2 \\
\gamma m_0 v^3 \end{bmatrix}$$

Employing Eq. (8) and the definition of relativistic linear momentum, $p^\mu$ is succinctly expressed

$$p^\mu = \begin{bmatrix} E \\
c/E \\
p^1 \\
p^2 \\
p^3 \end{bmatrix}$$

The Minkowski norm of $p^\mu$ (i.e., the rest energy) is Lorentz invariant.

$$p^\mu p_\mu = \frac{E^2}{c^2} - p^2 = m_0^2 c^2$$

Again, but more formally, this is the identical relationship between energy, momentum and rest mass as previously derived algebraically (8–10). Similarly, the equations presented in this approach do not readily reveal that the roots of $E^2$ are fundamentally complex conjugates (6, 7). The natural doctrine that energy as a measurable must be real-valued contributed to obscuring the fact that total energy in the context of the special theory of relativity cannot be accurately described by the real numbers, alone.

Define the phase angle $\alpha$ as follows.
\[ \alpha \equiv \sin^{-1} \beta \quad \beta \equiv \frac{v}{c} \]  

(15)

One may then conveniently define a phasor associated with the relativistic description of energy.

\[ e^{i\alpha} = \cos \alpha + i \sin \alpha = \sqrt{1 - \beta^2} + i\beta \]  

(16)

\[ \therefore mc^2 e^{i\alpha} = mc^2 \gamma^{-1} + imvc = m_0c^2 + ipc \]  

(17)

Eq. (7) is then more elegantly and succinctly written

\[ E = \left| mc^2 e^{i\alpha} \right| \]  

(18)

Figure 3. Energy represented as a phasor

In addition to demonstrating mass-energy equivalence, this new energy formula (18), of which Einstein’s 1905 equation (8) is a naïve form, explicitly reveals the non-trivial fact that the roots of \( E^2 \) are complex conjugates (i.e., that mass energy and momentum energy are mathematically and also physically distinct). This necessitates profound changes to conventional interpretations of quantum mechanical phenomena, in particular the wave-particle duality.

2. The energy of matter waves

Some two decades after the photoelectric effect had demonstrated the particle-like behavior of light, Louis de Broglie initially put forward the proposition that material particles must exhibit wave-like behavior. In a 1923 published paper that immediately preceded his celebrated 1924 PhD thesis, de Broglie stated the following:

…the atom of light should be considered as a moving object of a very small mass (\(<10^{-50} \text{ g}\)) that moves with a speed very nearly equal to \( c \) (although slightly less). We come therefore to the following conclusion: The atom of light, which is equivalent by reason of its total energy to a radiation of frequency, is the seat of an internal periodic phenomenon that, seen by the fixed observer, has at each point of space the same phase as a wave of frequency propagating in the same direction with a speed very nearly equal (although very slightly greater) to the constant called the speed of light.²

Although this line of thinking concerning the nature of light is erroneous and even nonsensical, it apparently lead de Broglie to correctly predict that the relationship between momentum and wavelength that holds for massless particles (19) also holds for all particles.

\[ \lambda = \frac{h}{p} \]  

(19)
One may judge from de Broglie’s statement, quoted above, that although he was successful in making an important quantitative empirical prediction, the genesis of the wave theory of matter was associated with a confused and incorrect understanding of physical principles. Furthermore, these ideas drove the conceptual model of the wave manifestation of material particles as a wave packet moving with group velocity \( v \) (the relative velocity of the particle). This model rests on two fundamental a priori mathematical assumptions. The first is that the energy of the wave packet \( hf \) corresponds to the total energy of the material particle, where \( f \) represents frequency rather than the Greek \( \nu \) (\( \nu \)), which is difficult to distinguish from the italic \( v \) (\( v \)) of the English alphabet used herein to represent velocity.

\[
hf = mc^2 \tag{20}
\]

The second is that the phase velocity \( w \) is superluminal and unbounded (i.e., “unphysical”) according to the idea that the wave packet’s group velocity is that of the material particle (\( v \)).

\[
w = \frac{c^2}{v} \tag{21}
\]

The de Broglie wavelength then follows from the definition of the phase velocity \( (w = \lambda f) \).

\[
\lambda = \frac{c^2}{v} \cdot \frac{h}{mc^2} = \frac{h}{mv} \tag{22}
\]

Because a mass term explicitly appears in this equation, it fails to be meaningful for massless particles and so does not represent a generalized solution. Also, although experiment confirms the empirical validity of this relationship for particles of non-zero rest mass, the derivation and associated conceptual model behind the equation leads to a conspicuous logical inconsistency between Eqs. (22) and (20). For the rest frame of a particle, the velocity is zero, which implies an infinite wavelength. An infinite wavelength suggests the model of a string of arbitrary length in which no displacement can be found (i.e., a null wave). It is therefore the case that the string contains no wave energy whatsoever (i.e., the frequency is necessarily zero). However, because de Broglie’s model for wavelike behavior of material particles assumes no distinction between a particle and its wave manifestation (i.e., that the energy of the wave \( hf \) is equal to the total energy of the particle \( mc^2 \)) the frequency of the wave associated with the zero momentum condition corresponds to the rest energy. While it is unequivocally true that the measurable manifested wavelength of any particle is inversely proportional to its momentum according to Eq. (19), it is also true that the frequency of these “matter waves”, which has not been subject to empirical measurement, cannot possibly correspond to Eq. (20). Something is clearly amiss in the conventional conceptual foundations of quantum mechanics that must be addressed.

Unlike Eq. (8), Eq. (18) makes a vivid mathematical distinction between the rest energy of a particle and the energy contribution due to its motion (i.e., momentum). In consideration of a massless particle such as a photon, for which energy manifests exclusively in the form of a wave characterized by a particular frequency, rest mass plays no part in the wave manifestation associated with the momentum carried by the particle. The wave manifestation of energy associated with a frequency is then associated with the imaginary (i.e., time) part of the total energy exclusive of the real (i.e., space) part.

\[
hf = \text{Im} \left[ mc^2 e^{i\alpha} \right] = pc \tag{23}
\]

The logical inconsistency between Eq. (22) and Eq. (20) arising from de Broglie’s initial assumption of the latter relationship (20) implies that Eq. (23) is a general physical principle that applies to all particles; the energy of the momentum-driven wave manifestation of any particle is
an independent contribution to the total energy that is not inclusive of any rest energy. — As this was not previously known, the resulting inaccurate understanding of fundamental phenomena underlying quantum mechanics will have caused numerous puzzling theoretical impasses in physics. This correction is certain to yield a number of important breakthroughs.

3. The manifestation of matter waves
Combining Eq. (23) with Eq. (19), the phase velocity of matter waves is determined to be fixed and equal to the speed of light, rather than superluminal and unbounded.

\[ w = \frac{\hbar f}{p c} = c \]  \hspace{1cm} (24)

Then the wave manifestation of a material particle on the microphysics scale is a periodic field that surrounds the source particle with some similarity to the distinction between a macroscopic material body and its associated gravitational field. The particle produces its momentum-induced wave manifestation, but the particle and the wave it produces are not one and the same thing. With the exception of the rest frame and the massless particle, neither of the two forms of energy (wave/radiation or particle/mass) can be identified with total energy; only the geometric vector sum of each manifestation’s distinct energy component in the complex plane yields total energy. Given that Eq. (23) correctly describes the energy of matter waves, we must abandon the long-held conventional idea that these waves manifest as a wave packet with group velocity \( v \) and are formed by the superposition of many waves with different wavelengths. Rather, the newly emergent model required by Eq. (18) is that of a particle surrounded by a spherical standing wave. This wave is characterized by a single well-defined wavelength associated with momentum (19) that has a radially decreasing amplitude, a group velocity of zero, a phase velocity of \( c \) (24) and an energy \( pc \) (23) that is not inclusive of rest energy. Like the gravitational field, matter waves are clearly higher-dimensional geometric objects because wave amplitude is a fourth coordinate for points in three-dimensional space. The standing wave model requires both an outbound and an inbound matter wave. While the former is produced by the particle, the latter can be envisioned as a response of spacetime to the presence of mass-energy.

The familiar fundamental interpretation of general relativity, applicable on the macro-scale, is “matter tells spacetime how to warp and warped spacetime tells matter how to move.” It is now suggested that on the microphysics scale, the momentum of a particle tells spacetime how to wave (i.e., warp in a unique way within the immediate vicinity of the particle). It is also true that this locally warped spacetime, typically in the form of interfering waveforms, will constrain the motion of elementary particles, which move through spacetime modified by their own wave manifestation. This quantitative and qualitative re-evaluation of the microphysics scale wave nature of matter provides a new and intuitive interpretation of quantum mechanics, in particular the measurement problem.

Before proceeding, it is worth mentioning as an aside that there is an intriguing conceptual analogy between the unified distinction between the rest and momentum energy and the unified distinction between the electric and magnetic fields. The electric field is produced by non-moving charges and likewise rest energy is associated with the static condition. In contrast, the distinct magnetic field is produced exclusively by moving charges and likewise the distinct momentum energy is associated with the dynamic condition. Perhaps there is some illuminating conceptual and mathematical correlation between the electromagnetic (charge) field and the gravitational (mass) field to be discovered by further pondering this similarity.

4. The wave-particle duality
The wavefunction in quantum mechanics described by the Schrödinger equation describes a linear superposition of different states, but actual measurements are always made of a physical
system in a definite state. For example, in the two-slit particle diffraction experiment, it is the actual impact locations of whole and indivisible particles (e.g., electrons) that are observed. However, the mathematics describes only the “fuzzy” statistical probability of the lateral distribution of electrons on the screen. The critical question that was not answered satisfactorily by any previous interpretation of quantum mechanics is how the mathematically calculated probabilities are converted into a distinctive measured physical outcome. It is clear that this question cannot possibly have been correctly answered without incorporating the idea that nature makes a sharp distinction between the real-valued rest energy (which may be zero) and the imaginary-valued momentum-driven wave energy in accord with Eq. (18). Because de Broglie failed to make a distinction between a particle and its momentum-driven wave manifestation (i.e., he assumed \( hf = mc^2 \) rather than \( hf = |pc| \)), the various interpretations of observed and calculated quantum effects that arose from his conceptual model of the wave-particle duality were manifestly flawed, leading to the general disaffection and vigorous debate concerning these various interpretations to the present day.

An idea is now introduced that transforms our understanding of quantum mechanics in an intuitive way. Indeed, it is arguably the case that this very simple idea is the realization of “Einstein’s dream” of a robust semi-classical picture of quantum mechanical effects. Recall that Einstein’s equivalence principle simply states that there is no distinction between inertial acceleration and gravitational acceleration for a reference frame of limited size such that tidal effects are negligible and that this single idea forms the foundation of general relativity. In like manner we shall state the quantum equivalence principle. This principle states that \( \Psi^*\Psi \) implies an underlying spacetime energy structure correlated to the calculated probability distribution of the particle. In short, matter waves manifest as a metric field through which particles propagate. When we speak of the energy of a matter wave, we must ask, “What is waving?” The answer is clear; spacetime is waving. If spacetime is waving, then this implies a local geometry of spacetime correlated to the amplitude of the linear superposition of matter waves.

Consider the following intuitive classical picture. A steady wind implying a vector field blows through the ideal barrier in the form of a symmetric “mountain range” shown in Fig. 4. Although this vector field is smooth and predictable on the large scale, there are also intrinsic chaotic processes at play. If we release a neutrally buoyant balloon (i.e., a test particle), far away from the windward side of this barrier, at what lateral location on a plane coordinate boundary near to the leeward side will the “particle” end up? A particular balloon will end up at some definite lateral coordinate, but if we repeat the process many times, the results will clearly be statistical. The most likely location for any particular balloon to be found will be behind the large central valley designated region \( A \). Another but far less likely location will be behind either one of the narrower and higher peripheral valleys (\( C \)). It is very unlikely that the balloon will be found directly behind one of the peripheral peaks (\( B \)). Assuming appropriately constrained initial conditions, it is intuitively the case that if one releases many such balloons one at a time and then plots their final location on a virtual plane surface just leeward of the barrier, the population distribution of the balloons would essentially be the inverted image of the barrier [Fig. 5].

Figure 4. An ideal symmetric “mountain range” and central valley
With the a priori knowledge that a kind of \textit{geometric} barrier was responsible for the observed lateral distribution of balloons shown in Fig. 5, but being privy only to the data and not to the direct observation of this barrier, one would naturally intuit the essential form of Fig. 4.

The quantum equivalence principle states that what we see in the data of the single-electron double-slit experiment, which is similar to that shown in Fig. 5, is indicative of a kind of geometric spacetime barrier that exists in the region of space between the diffraction grating and the screen, when both slits are open. The cause of this effective barrier is intuitively obvious; it is the interference and linear superposition of the wave manifestation of each individual electron.

Opening the second slit allows the wave energy $\hbar c$ (i.e., the matter wave field) to travel through both slits, clearly preceding and accompanying the particle itself $m_0 c^2$ (i.e., the wave center), which can travel through only one slit or the other. In the case of a photon there is still a wave center, although the rest mass energy associated with it is zero.

Therefore, within the space between the diffraction grating and the target screen, the single indivisible particle travels through a spacetime with a local geometry temporarily modified during its transit by the interference of its own distinct wave manifestation, which emanates from two separate laterally displaced points in space. On the microphysics scale, the situation is similar to the vector field of the balloon analogy. The local microphysics spacetime geometry created by the interference of the two distinct matter wave sources can be envisioned as a peculiar kind of “gravitational field” in which the strongest potential points to the center of the screen and comparatively narrower and weaker symmetric potentials interspersed by barriers are peripherally distributed. In short, the amplitude of the wavefunction (i.e., by how much spacetime is waving) correlates to the strength of the temporary local periodic potential on the microphysics scale.

So, there is nothing mysterious about the observed statistical distribution of individual particle impacts on the screen in the single electron double-slit experiment. The observed intensity of these impacts exhibiting “organized chaos” is essentially identical to the observed and intuitive
lateral distribution of balloons whose path towards their final plotted semi-random destinations is constrained by the geometry of an idealized vector field.

5. Conclusion
In summary, a new fundamental interpretation of quantum mechanics and in particular the empirical results of the single-electron double-slit experiment arises from the realization that a particle and its wave manifestation are distinct energy manifestations according to Eq. (18). The squared amplitude of the wavefunction is interpreted to correspond to a spacetime geometry that correlates to periodic local potentials. When the statistical probability distribution of a particle (e.g., an electron) is determined, whether in the double-slit experiment or for atomic electron orbits, what is indirectly being measured is the underlying local spacetime geometry typically caused by the interference of matter waves.

While the ideas presented herein may be theoretically compelling, they can only be confirmed by empirical evidence. An experiment that can demonstrate a predicted physical differentiation between the new ideas and preceding convention will be essential. One possibility with practical implications is in the field of nuclear physics. It is apparent that the binding force within the nucleus, nuclear dimensions, the nuclear shell structure, and the statistical nature of radioactive decay can be comprehensively modeled by wave mechanics, although this new idea requires considerable development. A wave model of the nucleus allows for the possibility of inducing controlled nuclear fusion for practical energy production by effectively “tuning” individual nucleons from a source to have a high likelihood of fusing with target nuclei. This is achieved by very precisely controlling the speed at which incident nucleons impinge on an appropriate target. The energy cost of the tuning process is an insignificant fraction of the energy produced by the fusion, which is not dependent on containment of high-temperature plasma.

References
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