Aspects of duality in gravitational theories.

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Chapter 1

Introduction

One of the biggest challenges of modern physics is the quantization of gravity. The theory introduced by Einstein in 1916 is still the best we have nearly 100 years later. Unfortunately, it is a classical theory and we know that physics is fundamentally quantum. In some sense it is even worse, Einstein’s gravity predicts its own downfall: it predicts black-holes, objects so massive that spacetime breaks down and singularities appear.

Multiple attempts have been made to build a consistent quantum theory of gravity. String theory is maybe the most successful but we think it is fair to say that there is no complete solution yet. New ideas are needed.

A different way to solve the problem is to study the properties of General Relativity. The hope is that a better understanding of the classical theory will impose constraints on the form of the quantum version or even hint at a solution. Easier said than done. The equations of motion of General relativity are non-linear partial differential equations and those are prone to very peculiar and complicated solutions.

The usual tool to deal with those kind of problems or at least to obtain some control is the use of symmetries. In themselves, they simplify the analysis but due to the famous Noether theorem, they also give rise to conserved quantities. In the most favorable scenario, this can even lead to a complete analytic solution of the problem.

The concept of symmetry is central in modern physics and you will encounter it throughout this work but it is not the focus point of our analysis. As the title says, we are interested in another kind of symmetries: the concept of dualities. The symmetries described above are internal symmetries, dualities can be considered as “external” symmetries: they are symmetries between different theories. Two theories are said to be dual if they describe the same problem through different means, the duality providing the dictionary to go from one description to the other.

Dualities can be very powerful. For instance, some questions can be very difficult to handle in one description and trivial in the dual one. Those dualities can first be studied
on the classical level but one can also hope for a dual version of gravity that would be quantizable.

In this work, we considered two dualities. The first one is known as the electromagnetic duality. Discovered in Maxwell’s theory, this duality exchanges the role of electric and magnetic fields. Recently, it has been extended to gravity at least at the linearized level. The second one is the gauge/gravity correspondance. It links a gravity theory in $n$ dimensions with a gauge theory in $n - 1$ dimensions. In the past 10 years, it received a lot of interest and it constitutes a very active subject of research.

1.1 Electromagnetic duality

Without sources, Maxwell’s equations in 4 dimensions are invariant under the exchange of electric and magnetic fields. If we add the usual electric sources, this symmetry is broken. To restore it, Dirac introduced the notion of a magnetic monopole: a source for the magnetic field playing the dual role of the electric sources. Even if nobody has observed them experimentally, the theoretical study of these magnetic monopoles has brought a lot of interesting results.

One of the most surprising comes from Dirac himself. He proved that the existence of magnetic monopoles could be an explanation of the quantization of electric charge. He showed the following quantization condition: All electric and magnetic charge, $e_i$ and $g_j$, must satisfy:

$$e_i g_j = \frac{1}{2} n_{ij} \quad \text{where } n_{ij} \in \mathbb{Z}. \quad (1.1.1)$$

In other terms, the presence of one monopole of charge $g$ anywhere in the universe will force every electric charge to be of the form:

$$\frac{n}{2g} \quad \text{where } n \in \mathbb{Z}. \quad (1.1.2)$$

For a long time after the introduction by Dirac of magnetic monopoles in 1931, these monopoles were a mere theoretical curiosity. We were able to build theories with monopoles but we didn’t have to use them to describe nature. Everything changed in 1974 when ’t Hooft and Polyakov showed that some quantum field theories relevant in particle physics inevitably contain magnetic monopoles. In fact, all reasonable grand unified theories necessarily contain them [1].

How is this? A usual quantum field theory will contain a bunch of fundamental fields: some electrically charged fermions coupled to gauge fields like the electromagnetic field. Because we want to describe all interactions at the same time (so the name: grand unified theories), the gauge fields will be more complicated but in the limit of low energy one should recover an electromagnetic field. The gauge fields describe fundamental particles,
and the only charged particles are electrically charged. In these kinds of theories, the magnetic monopoles cannot arise as point particles. In fact, ’t Hooft and Polyakov showed the existence of extended objects (solitons) that behave exactly as magnetic monopoles. If you are near one of them, you would see a complex internal structure composed of more elementary particles but if you look at it from a distance you would only see a magnetic monopole.

Three years later, Montonen and Olive conjectured that the theory considered by ’t Hooft and Polyakov contained electric magnetic duality in the following sense. The duality would not be a symmetry of the theory, it would be the link between two theories: on one hand we would find the ‘electric’ theory described above and on the other hand we would find another theory (‘magnetic’ theory) where the role of the components are exchanged. The magnetic monopoles would be the elementary particles whereas the electric particles would become extended objects. One could do computations in any of the two theories, the results would be the same provided one uses an appropriate dictionary between the two theories.

This duality conjectured by Montonen-Olive would be one example of what is called a strong-weak coupling duality. For the ‘electric’ theory, the electric charge $e$ is the coupling constant of the theory, the magnetic one $g$ being some constant characterizing the soliton. For the ‘magnetic’ theory, the situation is reversed, the coupling constant is given by the magnetic charge of our monopoles. Following Dirac’s argument, we know that the product of electric and magnetic charges is quantized. In particular, we have:

$$eg = \frac{1}{2},$$

where we chose $n = 1$ for simplicity. If we have a weakly coupled ‘electric’ theory, $e$ being very small, the coupling constant $g$ of the dual theory is very large: the theory is strongly coupled.

Those kinds of dualities are very interesting because, usually, strongly coupled theories are not tractable. If one is able to find a dual theory with a small coupling constant, one would be able to use perturbation theory to compute physical quantities and, after computations have been done, to use the dictionary to get back the quantities of the original theory.

There are some hints that the same duality exits for General Relativity in 4 dimensions. In particular, the Taub-NUT solution seems to be the electromagnetic dual to the Schwarzschild black-hole[2]. Unfortunately, any precise definition of the duality is difficult due to the non-linear form of the equations of motion. On the other hand, the linearized Einstein’s equations describing a free spin 2 particle are invariant under the duality and, recently, it has been shown that it is a symmetry of the associated reduced phase space action[3].
The goal of our work is a generalization to spin 2 of the so-called double potential formalism for spin 1 fields \([4, 5]\), which has been extended so as to include couplings to dynamical dyons by using Dirac strings \([6, 7]\). In the original double potential formalism, Gauss’s constraint is solved in terms of new transverse vector potentials for the electric field so that electromagnetism is effectively formulated on a reduced phase space with all gauge invariance eliminated. Alternatively, one may choose \([8]\) to double the gauge redundancy of standard electromagnetism by using a description with independent vector and longitudinal potentials for the magnetic and electric fields and 2 scalar potentials that appear as Lagrange multipliers for the electric and magnetic Gauss constraints. In this framework, the string-singularity of the solution describing a static dyon is resolved into a Coulomb-like solution. Furthermore, magnetic charge no longer appears as a topological conservation law but as a surface charge on a par with electric charge.

The aim of the present work is to apply the same strategy to the spin 2 case. Doubling the gauge invariance by keeping all degrees of freedom of symmetric tensors now leads to a second copy of linearized lapse and shifts as Lagrange multipliers for the new magnetic constraints. As a consequence, the string singularity of the gravitational dyon, the linearized Taub-NUT solution is resolved and becomes Coulomb-like exactly as the purely electric linearized Schwarzschild solution. Furthermore, as required by manifest duality, magnetic mass, momentum and Lorentz charges also appear as surface integrals.

Our work thus presents a manifestly duality invariant alternative to \([9]\) where the coupling of spin 2 fields to conserved electric and magnetic sources has been achieved in a manifestly Poincaré invariant way through the introduction of Dirac strings.

We start in chapter \([2]\) with a summary of the results obtained in the spin 1 case. In chapter \([8]\) we then apply the same analysis to the spin 2 case. The idea is that both problems are closely related and the reader can consider the electromagnetic case as a physical toy model for linearized gravity.

As a result of this work, a really interesting link between electromagnetic duality and soliton theory appeared. We show in chapter \([4]\) that massless higher spin gauge fields are bi-Hamiltonian and consequently integrable systems. Even if for now this only holds at the free level, this insight relates two very different domains of physics. Our hope is that it can allow the use of the techniques of soliton theory in the study of fundamental fields.

### 1.2 Gauge/Gravity conjecture

The idea of a duality between gauge theories and gravitational theories was first proposed by Maldacena in his famous paper \([10]\). At some level, this duality relates a supergravity theory in anti-de Sitter background to a gauge theory defined on its boundary. Since then,
people believe that the duality extend to any background.

One example where this idea was most successful is the 3-dimensional case. In 1985, Brown and Henneaux showed that the infinitesimal symmetries of asymptotically $AdS_3$ spacetimes \cite{11,12,13} provide a representation of the algebra of conformal Killing vectors of the flat boundary metric. In this case, the boundary being 2-dimensional, this algebra is infinite dimensional. This was surprising as people were expecting this symmetry group to be just $SO(2,2)$ the exact symmetry group of $AdS_3$. Using Hamiltonian methods, they also showed that the algebra of surface charges form a centrally extended representation of the 2-dimensional algebra. Those results imply that any consistent quantum theory of gravity in 3 dimension should be a conformal theory.

The presence of this infinite-dimensional algebra allows the use of the powerful techniques of 2-dimensional conformal field theory. Maybe the most interesting result was then obtained by Strominger \cite{14}. Using the Cardy formula for the growth of states in a 2D conformal theory, he was able to reproduce the value of the Bekenstein-Hawking area formula for the entropy of BTZ black-holes.

Historically, the first example where the asymptotic symmetry group is enhanced with respect to the isometry group of the background metric and becomes infinite-dimensional is the one of asymptotically flat 4-dimensional spacetimes at null infinity \cite{15,16,17}. The asymptotic symmetry group of non singular transformations is the well-known Bondi-Metzner-Sachs group. It consists of the semi-direct product of the Lorentz group, times the infinite-dimensional abelian normal subgroup of so-called supertranslations.

The starting point of our work is the following observation. If one focuses on infinitesimal transformations and does not require the associated finite transformations to be globally well-defined, then there is a further enhancement. The symmetry algebra is then the semi-direct sum of the infinitesimal local conformal transformations of the 2-sphere with the abelian ideal of supertranslations, and now both factors are infinite-dimensional.

The aim of the present work is to reconsider from the point of view of local conformal transformations the 4-dimensional case which is, in some sense at least, of direct physical relevance. In particular, we provide a detailed derivation of the natural generalization of the $bms_4$ algebra discussed above. No modification of well studied boundary conditions is needed and the transformations are carefully distinguished from conformal rescalings.

A major motivation for our investigation comes from Strominger’s derivation described above. More recently, a similar analysis has been used to derive the Bekenstein-Hawking entropy of an extreme 4-dimensional Kerr black hole \cite{18}. One of our hopes is to make progress along these lines in the non extreme case, either directly from an analysis at null infinity or by making a similar analysis at the horizon, as discussed previously for instance in \cite{19,20,21,22,23,24,25,26,27,28,29,30,31}.

Related work includes \cite{32,33} on asymptotic quantization where for instance the im-
Applications of supertranslations for the gravitational $S$-matrix have been discussed. Asymptotically flat spacetimes at null infinity in higher spacetime dimensions have been investigated for instance in [34, 35, 36, 37], while various aspects of holography in 4 dimensions have been studied in some details in [38, 39, 40, 41, 42, 43, 44]. In particular, a symmetry algebra of the kind that we derive and study here has been conjectured in [45].

We will start in chapter 5 with an analysis of the $AdS_3$ case. We basically rederive the results of Brown-Henneaux using a different setup. In chapter 6 we will do the same analysis for spacetimes that are asymptotically flat at null infinity in 3 dimensions. Finally, in chapter 7 we present our results on $bms_4$. The reader should view the first two cases as intermediate problems because the $bms_4$ case has features in common with both the $AdS_3$ and the $bms_3$ cases.

1.3 Conventions

We will work with natural units: $c = 1$, $\epsilon_0 = 1$ and $\hbar = 1$. In the following, we will use index notation for vectors. Summation over repeated indices will be understood. The 1-forms $dx^\mu$ are treated as fermionic variables and the skew-symmetric epsilon tensor in $n$-dimension is defined as $\epsilon_0 \ldots (n-1) = 1$.

1.4 Summary of original results

- Introduction of an extended double potential formalism for spin 2 and definition of surface charges in a duality invariant way.

- Description of massless integer spin gauge fields as bi-Hamiltonian systems

- Exact representation by bulk vector fields of the symmetry algebra of asymptotically $AdS_3$ spacetimes through the introduction of a modified Lie bracket.

- Definition of an extended $bms_4$ algebra with both supertranslation and superrotations: non trivial extension of the Poincaré algebra containing Virasoro algebra due to the presence of gravity.

- Derivation of the $bms_4$ charge algebra containing a field dependent central extension.

The publications containing those results are

1. G. Barnich and C. Troessaert, “Manifest spin 2 duality with electric and magnetic sources,” *JHEP* 01 (2009) 030, [0812.0552].
2. G. Barnich and C. Troessaert, “Duality and integrability: Electromagnetism, linearized gravity and massless higher spin gauge fields as bi-Hamiltonian systems,” *J. Math. Phys.* **50** (2009) 042301, [0812.4668](https://arxiv.org/abs/0812.4668).

3. G. Barnich and C. Troessaert, “Symmetries of asymptotically flat 4 dimensional spacetimes at null infinity revisited,” *Phys. Rev. Lett.* **105** (2010) 111103, [0909.2617](https://arxiv.org/abs/0909.2617).

4. G. Barnich and C. Troessaert, “Aspects of the BMS/CFT correspondence,” *JHEP* **05** (2010) 062, [1001.1541](https://arxiv.org/abs/1001.1541).

5. G. Barnich and C. Troessaert, “Supertranslations call for superrotations,” [1102.4632](https://arxiv.org/abs/1102.4632).

6. G. Barnich and C. Troessaert, “BMS charge algebra,” to appear soon.
Chapter 2

Double-potential formalism for spin 1

In this chapter, we will present the double potential formalism for electromagnetism. As said in the introduction, the reader should view this as a physical toy model for the spin 2. Indeed, the ideas are the same in the two cases; everything is just more complicated for linearized gravity.

We start by a quick introduction to electromagnetic duality for spin 1. Then, we describe the double potential formalism used to write a duality invariant action. Finally, we introduce the extended double potential formalism developed in [8] in order to couple to both electric and magnetic sources in a duality invariant way.

2.1 Electro-magnetic duality

Classically, electromagnetism is described by two fundamental vector fields: the electric field and the magnetic field. The dynamics of these fields and their interaction with charged particles is governed by the well known Maxwell equations (see [46],[47]). In empty space, they take the following form:

\[
\begin{align*}
\nabla \cdot \vec{E} &= 0, \\
\nabla \cdot \vec{B} &= 0, \\
\n\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0, \\
\n\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} &= 0.
\end{align*}
\]

These equations possess an unusual symmetry consisting of the exchange of the electric and magnetic field. More precisely, the following map leaves the above equations invariant:

\[ \vec{E}' = -\vec{B}, \quad \vec{B}' = \vec{E}. \]
In addition to leaving the equations invariant, this map also doesn’t change physical quantities like the total energy or the total momentum of the electromagnetic field:

\[ H = \int_V d^3x \frac{1}{2} \left( |\vec{E}|^2 + |\vec{B}|^2 \right), \]  

(2.1.6)

\[ \vec{P} = \int_V d^3x \vec{E} \times \vec{B}, \]  

(2.1.7)

where both integrals are evaluated over the entire space \( V \). This symmetry is called electromagnetic duality.

Nature seems to break this symmetry. Indeed, the only charged matter we are aware of is electrically charged: it only sources the electric field. In the presence of these sources, the Maxwell equations become

\[ \vec{\nabla} \cdot \vec{E} = \rho_e, \]  

(2.1.8)

\[ \vec{\nabla} \cdot \vec{B} = 0, \]  

(2.1.9)

\[ \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0, \]  

(2.1.10)

\[ \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j}_e, \]  

(2.1.11)

where \( \rho_e \) and \( \vec{j}_e \) are respectively the charge and the current densities of electric particles. 

This consistency of the above equations implies the conservation of the electric charge: \( \partial_0 \rho_e + \partial_i j^i_e = 0 \). The duality map given above does not leave these equations invariant any longer. To restore the symmetry, Dirac postulated the existence of magnetic monopoles: particles that source the magnetic field. If these magnetic monopoles are present, the electromagnetic equations will take the form:

\[ \vec{\nabla} \cdot \vec{E} = \rho_e, \]  

(2.1.12)

\[ \vec{\nabla} \cdot \vec{B} = \rho_m, \]  

(2.1.13)

\[ \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = -\vec{j}_m, \]  

(2.1.14)

\[ \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j}_e, \]  

(2.1.15)

with \( \rho_m \) and \( \vec{j}_m \) respectively the charge and the current densities of magnetic particles. In this setting, one can see that, provided we also exchange the sources, the equations are again invariant under the duality:

\[ \vec{E}' = -\vec{B}, \quad \vec{B}' = \vec{E}, \]  

(2.1.16)

\[ \vec{j}_e' = -\vec{j}_m, \quad \vec{j}_m' = \vec{j}_e, \]  

(2.1.17)

\[ \rho'_e = -\rho_m, \quad \rho'_m = \rho_e. \]  

(2.1.18)
The simplest solution to these equations is called a dyon: a point-particle carrying both electric and magnetic charges, respectively $e$ and $g$. The electromagnetic field induced by this source is given by twice the well known Coulomb solution:

$$\vec{E} = \frac{e \vec{r}}{4\pi r^3}, \quad (2.1.19)$$

$$\vec{B} = \frac{g \vec{r}}{4\pi r^3}. \quad (2.1.20)$$

The above results can easily be written in term of the usual 4 dimensional quantities. The electric and magnetic tensors are put together in a 2-form $F = F_{\mu\nu} dx^\mu dx^\nu$:

$$F_{i0} = E_i, \quad F_{ij} = \epsilon_{ijk} B^k. \quad (2.1.21)$$

Using this notation, one can easily write the Maxwell equations in a manifestly Lorentz-invariant way:

$$\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = j^\mu_e, \quad (2.1.22)$$

$$\partial_\mu F^{\mu\nu} = j^\nu_m. \quad (2.1.23)$$

where we have defined the electric and the magnetic currents respectively as $j^\mu_e = (\rho_e, j^i_e)$ and $j^\mu_m = (\rho_m, j^i_m)$. Remark that for consistency of the equations of motion (2.1.22) and (2.1.23), both currents must be conserved. The duality transformation on the electromagnetic fields takes the following form:

$$F'_\mu\nu = \ast F = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}. \quad (2.1.24)$$

At the level of the classical equations of motion, the duality is well behaved and the magnetic monopoles can be added quite easily. The next questions are: How can one describe this in term of an action? and Is it possible to write an action for electromagnetism in presence of both electric and magnetic sources? This was done by Dirac in his famous paper [48] by introducing string-like singularities known as Dirac strings.

If the magnetic sources are absent, Maxwell’s equations become

$$dF = 0 \quad (2.1.25)$$

$$\partial_\nu F^{\mu\nu} = j^\mu_e \quad (2.1.26)$$

in term of the external differential $d = dx^\mu \partial_\mu$. Using the Poincaré Lemma in $\mathbb{R}^4$ this identity implies the existence of a 1-form $A = A_\mu dx^\mu$ such that $F = dA$ or $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Note that there is a redundancy in the description, the transformation $\delta A_\mu = \partial_\mu \epsilon(x^\nu)$ leaves the tensor $F_{\mu\nu}$ invariant for any function $\epsilon$. Such a transformation
is called a gauge transformation. The usual action for electromagnetism is build from this vector potential $A_\mu$:

$$S[A_\mu] = \int d^4x \ \left( -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + A_\mu j_\mu \right).$$  \hfill (2.1.27)

When using the description of electromagnetism in terms of $A_\mu$, we have chosen a side; all Maxwell’s equations are not on the same footing. Half of them (2.1.25) are consistency equations known as the bianchi identities, they are coming from the identity $d^2 = 0$ and the definition $F = dA$. The other half (2.1.26) are dynamical equations coming from the variation of the action (2.1.27).

Adding magnetic sources is problematic. The above parametrisation is based on the identity $dF = 0$ but in presence of magnetic sources, it is no longer valid. The solution to this problem was found by Dirac in his famous paper [48]. The idea is to introduce a parametrisation that is valid everywhere except on some points where it must be corrected. The electromagnetic field is now defined as

$$F = dA + G.$$  \hfill (2.1.28)

The 2-form $G$ is a function of the magnetic source, zero nearly everywhere and such that

$$\star dG = j_m.$$  \hfill (2.1.29)

If $j_m$ is produced by a set of point-like monopoles, $G_{\mu\nu}$ can be chosen to be non-zero on a set of strings (called Dirac’s strings), each one of them attached to one of the monopoles and going to infinity. Notice that the potential $A_i$ must be singular along the strings to compensate the infinity of $G$ and to produce a $F$ regular outside the source.

In the case of one monopole sitting at the origin, the potential 4-vector and the associated string term are given by:

$$A_0 = 0,$$  \hfill (2.1.30)

$$A_i = \frac{g}{4\pi} \left( \frac{y}{r(r-z)}, -\frac{x}{r(r-z)}, 0 \right),$$  \hfill (2.1.31)

$$G_{ij} = \epsilon_{ijz} g \delta(x) \delta(y) \Theta(z),$$  \hfill (2.1.32)

$$G_{0i} = 0,$$  \hfill (2.1.33)

where $\delta(x)$ and $\Theta(z)$ are the Dirac and the Heaviside functions. The magnetic field produced by the potential (2.1.31) alone would be the following:

$$\tilde{B}^i = \epsilon^{ijk} \partial_j A_k = \frac{g}{4\pi} \frac{x^i}{r^3} - g\delta(x) \delta(y) \Theta(z) \delta^{i z}.$$  \hfill (2.1.34)

Without the string term, this magnetic field $\tilde{B}^i$ is the one created by a semi-infinite solenoid which is infinitely thin; it describes a string of infinitely concentrated magnetic flux that spreads out from the origin to infinity and creates the magnetic flux of the monopole.
The string term is then added to remove this semi-infinite solenoid and get the magnetic monopole alone.

\[ A_\mu dx^\mu = \frac{g}{4\pi} (-1 - \cos \theta) d\phi. \]  \hspace{1cm} (2.1.35)

In that case, the singularity is hidden in the shortcomings of the parametrisation. Indeed, the spherical parametrisation of \( \mathbb{R}^3 \) is not well defined at the poles \( \theta = 0, \pi \) corresponding to the \( z \) axis.

Using this new definition for \( F_{\mu\nu} \), one can use the same action as before to describe electromagnetism:

\[ S[A_\mu] = \int_V d^4x \left( -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + A_\mu j^\mu \right). \]  \hspace{1cm} (2.1.36)

The two Maxwell equations concerning magnetic sources will follow from the definition of \( F_{\mu\nu} \) and \( G_{\mu\nu} \), the other two will be the equations of motion coming from the action. One can see the difference in the treatment of the magnetic and the electric part. The ‘magnetic’ equations are imposed by construction, the ‘electric’ equations are imposed dynamically. This description of electromagnetism is not duality invariant.
The above action generates the correct Lorenz force for charged particles provided Dirac’s veto holds. This veto, stating that two strings cannot cross each other, also implies the quantization of electric charge described in the introduction.

2.2 Double potential formalism

At the outset, the electromagnetic duality is a symmetry of the equations of motion but not of the action. As we saw, the formalism used to describe the theory breaks the duality. More directly, one can write the action without sources in term of the electric and magnetic fields:

\[ S = \int d^4x \frac{1}{2} \left( E^2 - B^2 \right). \tag{2.2.1} \]

This is clearly not invariant under the exchange \( E' = -B \) and \( B' = E \).

An important question is whether one can construct an action which is invariant under the duality. We will now describe the construction of such an action. This construction is based on the Hamiltonian formalism and, as such, is not manifestly invariant under the Lorentz group.

2.2.1 Canonical formulation of electromagnetism

The usual Hamiltonian action without sources is given by

\[ S[E^i, A_i, A_0] = \int d^4x \left( -E^i \dot{A}_i - \mathcal{H}_{EM} - A_0 G \right), \tag{2.2.2} \]

\[ \mathcal{H}_{EM} = \frac{1}{2} \left( E^2 + B^2 \right), \tag{2.2.3} \]

\[ G = \partial_i E^i, \tag{2.2.4} \]

where we have used the notation \( B^i = \epsilon^{ijk} \partial_j A_k \). The function \( \mathcal{H}_{EM} \) is the Hamiltonian density, \( G \) is the Gauss constraint, the generator of the gauge transformation, and associated to it we have \( A_0 \) which is the Lagrange multiplier. The associated Poisson bracket can be read easily from the kinetic term:

\[ \{F, G\} = \int d^3x \left( \frac{\delta F}{\delta E^i(x)} \frac{\delta G}{\delta A_i(x)} - F \leftrightarrow G \right). \tag{2.2.5} \]

One can check easily that this action generates the Maxwell equations. Furthermore, it reduces to the usual Lagrangian action when the auxiliary field \( E^r \) is solved for.

2.2.2 Vector decomposition

In the rest of the first part, we will assume that we have boundary conditions on all our fields such that the Laplacian operator \( \Delta = \partial_i \partial^i \) is invertible; in other words, the Laplace
equation \( \Delta \phi = \psi \) has one and only one solution for each allowed value of the field \( \psi \) (where \( \phi \) and \( \psi \) stands for any field under consideration). Using this property, any vector \( V^i \) can be divided in two parts: a longitudinal and a transverse part as follows:

\[
V^i = V^{Ti} + V^{Li},
\]

\[
V^{Li} = \partial^i \Delta^{-1} \partial_j V^j,
\]

with the transverse part \( V^T \) defined as the reminder \( V^T = V - V^L \). We have \( \partial_i V^{Ti} = 0 \).

We can apply the Poincaré lemma to introduce a potential for the transverse part of \( V \):

\[
V^{TI} = \mathcal{O} W^i = \epsilon^{ijk} \partial_j W_k.
\]

Remark that the curl operator \( \mathcal{O} \) kills the longitudinal part of \( W^i \). In that sense, all the interesting information is stored in the transverse part of \( W \) which is unique and given by \( W^{Ti} = -\mathcal{O} \Delta^{-1} \mathcal{O} V^{Ti} \). From the three components of a general vector in 3D, two are stored in the transverse part and one is in the longitudinal part.

The elements of the decomposition are orthogonal under integration if boundary terms can be neglected,

\[
\int d^3 x V^i W_i = \int d^3 x (V^{Ti} W^T_i + V^{Li} W^L_i).
\]

and the operator \( \mathcal{O} \) is self-adjoint, e.g.,

\[
\int d^3 x (\mathcal{O} V)^j W_i = \int d^3 x V^i (\mathcal{O} W)_i.
\]

### 2.2.3 Reduced phase space and duality invariant action

The Gauss constraint implies that \( E^i \) must be transverse. If we solve the constraint by putting \( E^L \) to zero, we must also fix the gauge freedom associated to it. This gauge freedom is \( \delta A_i = \partial_i \epsilon \) and it can be used to fix the longitudinal part of \( A \) to zero. The resulting action is

\[
S[ E^{Ti}, A^T_i ] = \int d^4 x \left( -E^{Ti} \dot{A}^{T_i} - \mathcal{H} \right),
\]

\[
\mathcal{H} = \frac{1}{2} ( E^{T2} + B^2 ) .
\]

This is the reduced phase-space formulation of electromagnetism: all the constraints and gauge freedom have been removed. The only degrees of freedom left are the 2 physical degrees of freedom describing the photon. They are stored in the canonical pairs \((A^T_i, -E^{Ti})\).
As we saw above, we can then introduce a new potential \( Z^T_k \) to describe the transverse part of the electric field:

\[
E^i = \mathcal{O} Z^T_i = \epsilon^{ijk} \partial_j Z^T_k.
\] (2.2.13)

Putting this in the action gives

\[
S[A^T_i, Z^T_i] = \int d^4x \left\{ \mathcal{O} Z^T_i \dot{A}^T_i - \frac{1}{2} \left( (\mathcal{O} Z^T)^2 + (\mathcal{O} A^T)^2 \right) \right\}.
\] (2.2.14)

This action is already duality invariant \([4]\) but the symmetry can be made more manifest if we rename our fields as \( A^T a_i = (A^T_i, Z^T_i) \) and \( B^ia_i = (B^i, E^i) \) with \( a = 1, 2 \) \([5]\). After some integrations by part, the action becomes

\[
S[A^T a_i] = \int d^4x \left\{ \epsilon_{ab} \frac{1}{2} \mathcal{O} A^T_{ai} \dot{A}^T_{bi} - \mathcal{H} \right\},
\] (2.2.15)

\[
\mathcal{H} = \frac{1}{2} B_{ai} B_{ai},
\] (2.2.16)

\[
B_{ai} = \mathcal{O} A^T_{ai},
\] (2.2.17)

where \( \epsilon_{ab} \) is antisymmetric with \( \epsilon_{12} = 1 \) \((a, b, ... \) are moved up and down using the Kronecker symbol \( \delta_{ab} \)). The electromagnetic duality is the \( SO(2) \) rotation acting on the duality index:

\[
\delta_D A^T a_i = \epsilon^{ab} A^T_{bi}.
\] (2.2.18)

The fact that the action is invariant is obvious because every term is a scalar under this rotation. The duality generator is

\[
D = -\frac{1}{2} \int d^3x A^T_{ai} \mathcal{O} A^T_{ai}
\] (2.2.19)

which is simply an \( SO(2) \) Chern-Simons action \([49]\).

### 2.2.4 Poincaré transformations

To obtain a duality invariant action, we had to use the Hamiltonian formalism and thus loose the manifest Lorentz invariant of the action. To be precise, we still have manifest invariance under the spatial part of the group but not for the boosts. This is a common fact in the Hamiltonian formalism: the 3+1 split breaks manifest Lorentz invariance because time has become a special coordinate. On the other hand, going to the Hamiltonian formulation doesn’t change the symmetries of the action (see e.g. \([50]\)). The boosts should be hidden somewhere. Tracing them back from the Lagrangian formulation, the Poincaré generator associated to the usual electromagnetic action \((2.2.2)\) are given by

\[
Q_G(\omega, a) = \frac{1}{2} \omega_{\mu\nu} J^{\mu\nu} - a_\mu P^\mu
\] (2.2.20)
where \( P^0 \) is the time translation given by the Hamiltonian, \( P^\mu \) are the spatial translations, \( J^{\mu\nu} \) the spatial rotations and the boosts. They can be constructed from the symmetric energy-momentum tensor as follows:

\[
T^{00} = H_{EM}, \quad T^{i0} = \epsilon^{ijk} E_j B_k, \quad (2.2.21)
\]

\[
P^\mu = \int d^3 x \, T^{\mu 0}, \quad J^{\mu\nu} = - \int d^3 x \left( x^\mu T^{\nu 0} - x^\nu T^{\mu 0} \right), \quad (2.2.22)
\]

From these generators, one can deduce the generators of the reduced phase space theory by evaluating them on the constraint surface. The generators of the double potential formalism are then obtained by introducing the new potential. The final result is given by (2.2.20) with

\[
T^{00} = H, \quad T^{i0} = -\frac{1}{2} \epsilon^{ijk} \epsilon_{ab} B^a_j B^b_k = \epsilon^{ijk} E_j B_k, \quad (2.2.23)
\]

\[
P^\mu = \int d^3 x \, T^{\mu 0}, \quad J^{\mu\nu} = - \int d^3 x \left( x^\mu T^{\nu 0} - x^\nu T^{\mu 0} \right), \quad (2.2.24)
\]

and the associated transformations

\[
\delta_\xi A^T_a = -\epsilon^{ab} B_i \xi^0 - \epsilon_{ijk} \xi^j B^a_k, \quad (2.2.25)
\]

\[
\xi^\mu = a^\mu + \omega^\mu_{\nu} x^\nu \quad \text{with} \quad \omega^\mu_{\nu} = -\omega_{\nu\mu}. \quad (2.2.26)
\]

One can also show by direct computation that these generators form a representation of the Poincaré algebra under the Poisson bracket associated with the action:

\[
\{ F, G \} = \int d^3 x \left( \frac{1}{2} \frac{\delta F}{\delta A^a_i(x)} \epsilon_{ijkl} \partial^j \Delta^{-1} \frac{\delta G}{\Delta^a_k(x)} \right). \quad (2.2.27)
\]

### 2.3 Double potential formalism with external sources

The action presented in the previous section concerns only the reduced phase space in absence of sources, only the dynamical degrees of freedom of electromagnetism. It obviously doesn’t say anything about the sources. The next logical step would be to add them. To build an action that is invariant under the duality in presence of sources, one has two possibilities to describe the interactions. The first one is to treat both electric and magnetic sources as topological information by introducing strings for both sides [7]. What we will present here is the other natural possibility: treat both sides as dynamical information. This was done in [8] and we will mainly review the part of their results that will be interesting in the spin 2 case.

Let’s first have a look at the Hamiltonian formulation in presence of electric source
\[ j^\mu_c \text{ (this source must be conserved } \partial_\mu j^\mu_c = 0) \text{. We have} \]
\[
S[A_i, E^i, A_0] = \int d^4x \left( -E^i \dot{A}_i - \mathcal{H} - A_0(\partial_i E^i - j^0_i) \right), \tag{2.3.1}
\]
\[
\mathcal{H} = \frac{1}{2} (E^2 + B^2 - A_i j^i_0). \tag{2.3.2}
\]

The source term modifies the Gauss constraint. The longitudinal part of \( E \) is no longer zero and part of the information about the sources is stored in the gauge sector. To add magnetic sources, we need an equivalent mechanism for the dual part.

The solution proposed by the authors of [8] is to double the gauge freedom. To do so, they introduce the following parametrisation for the magnetic and electric fields \( B^{ai} = (B^i, E^i) \):
\[
B^{ai} = \epsilon^{ijk} \partial_j A^a_k + \partial^i C^a. \tag{2.3.3}
\]
The fields \( A^a_i \) and \( C^a \) are respectively potentials for the transverse part and the longitudinal parts of \( B^a \). The electromagnetic action they propose is the following:
\[
S[A^a_i, C^a, A^a_0] = S_{EM}[A^a_i, C^a, A^a_0] + S_I[A^a_i, A^a_0; j^{a\mu}] \tag{2.3.4}
\]
where
\[
S_{EM}[A^a_i, C^a, A^a_0] = \int d^4x \left( \frac{1}{2} \epsilon_{ab}(B^{ai} + \partial^i C^a) \dot{A}^b_i - A^a_0 \mathcal{G}_a - \mathcal{H} \right), \tag{2.3.5}
\]
\[
\mathcal{H} = \frac{1}{2} B^{ai} B_{ai}, \tag{2.3.6}
\]
\[
\mathcal{G}_a = \epsilon_{ab} \partial^b B^{bi}. \tag{2.3.7}
\]
is the substitute for the usual Maxwell action and
\[
S_I[A^a_i, A^a_0; j^{a\mu}] = \int d^4x \epsilon_{ab} j^{a\mu}_b \tag{2.3.8}
\]
is the “interaction” action. The external magnetic and electric currents are labeled as \( j^{a\mu} = (j^{\mu}_m, j^{\mu}_e) \) and are conserved, \( \partial_\mu j^{a\mu} = 0 \). This action is manifestly invariant under duality rotations that acts on the \( a \) indices
\[
\delta_D A^a_\mu = \epsilon^{ab} A^b_\mu, \quad \delta_D C^a = \epsilon^{ab} C^b, \quad \delta_D j^{a\mu} = \epsilon^{ab} j^{b\mu}. \tag{2.3.9}
\]

We will start by studying the electromagnetic core of this action, namely \( S_{EM} \).

### 2.3.1 Canonical structure

After some integration by parts, one can write the kinetic term as
\[
\int d^4x \frac{1}{2} \epsilon_{ab}(B^{ai} + \partial^i C^a) \dot{A}^b_i = \int d^4x \left( -\mathcal{O} A^{2T_i} \dot{A}^{T^T_1} + \partial^i C^{1L_i} \dot{A}^{L^L_1} - \partial^2 C^2 \dot{A}^{L_1} \right) \tag{2.3.10}
\]
The canonically conjugate pairs are identified as

\[ (A^T_1(x), -\mathcal{O}A^{2Ti}(y)), (A^{L2}_i(x), \partial^iC^1(y)), (A^{L1}_i(x), -\partial^iC^2(y)) \]. \tag{2.3.11} \]

The usual canonical pairs of electromagnetism can be chosen in term of the new variables as

\[ (A^T_1(x), -\mathcal{O}A^{2Ti}(y)), (A^{L2}_i(x), \partial^iC^1(y)) \]. \tag{2.3.12} \]

The new canonical pair

\[ (A^{L1}_i(x), -\partial^iC^2(y)) \] \tag{2.3.13} \]

is just the magnetic dual of the pure gauge pair \((A^{L2}_i(x), \partial^iC^1(y))\) of the usual formulation.

Those results imply

\[ \{A^{ai}(x), B^{bj}(y)\} = -\epsilon^{ab}\delta^{ij}\delta^3(x, y), \quad \{B^{ai}(x), B^{bj}(y)\} = -\epsilon^{ab}\epsilon^{ijk}\partial^k\delta^3(x, y). \] \tag{2.3.14} \]

\subsection*{2.3.2 Gauge structure and degree of freedom count}

In this action, there are 2 first-class constraints:

\[ \mathcal{G}_a = \epsilon_{ab}\partial_iB^{bi} = \epsilon_{ab}\Delta C^b \approx 0. \] \tag{2.3.15} \]

For \(a = 1\) we have the usual Gauss constraint \(\partial_iE^i = 0\). On the other hand, \(a = 2\) gives its dual: the magnetic Gauss constraint \(\partial_iB^i = 0\). In the usual formulation, this Maxwell equation is imposed by the formalism. We see that in this case, it has become a dynamical equation.

The gauge transformations generated by \(\int d^3x \lambda^a \mathcal{G}_a\) can be easily computed:

\[ \delta_\lambda A^a_\mu = \partial_\mu \lambda^a, \quad \delta_\lambda C^a = 0. \] \tag{2.3.16} \]

Exactly as expected, we have the usual gauge transformation parametrized by \(\lambda^1\) and a new dual one parametrized by \(\lambda^2\). The gauge sector of the theory has been doubled in a duality invariant way.

The new constraint \(\mathcal{G}_2\) can be gauge fixed through the condition \(A^{L2}_i = 0\). This partially gauge fixed theory corresponds to the usual electromagnetic theory in Hamiltonian form as described in section 2.2.1. More precisely, the observables of a Hamiltonian field theory with constraints are defined as equivalence classes of functionals that have weakly vanishing Dirac brackets with the constraints and where two functionals are considered as equivalent if they agree on the surface defined by the constraints (see e.g. \[50\]). The new
constraint together with the gauge fixing condition form second class constraints. The Dirac bracket algebra of observables of this (partially) gauge fixed formulation is isomorphic to the Poisson bracket algebra of observables of the extended formulation on the one hand, and to the Poisson bracket algebra of observables of usual electromagnetism on the other hand.

We start with 4 degrees of freedom per spacetime point. The 2 constraints, being first class, remove 2 of those degrees of freedom. In the end, the fully gauge fixed theory contains 2 physical degrees of freedom per spacetime point described by the transverse vector potential $A^T_i = A^{T1}_i$ and its canonically conjugate variable $-E^T_i = -O A^{2T_i}$, as it should.

### 2.3.3 Duality generator

The duality generator is the $SO(2)$ Chern-Simons term suitably extended to the longitudinal potentials:

$$D = -\frac{1}{2} \int d^3 x \left( B^{ai} + \partial^i C^a \right) A_{ai}. \quad (2.3.17)$$

This generator commutes with the Hamiltonian and the other Poincaré generators given below but it is only weakly gauge invariant:

$$\{ G_a, D \} = \epsilon_{ab} G^b. \quad (2.3.18)$$

### 2.3.4 Poincaré transformations

Consider now a symmetry generator of the usual Hamiltonian action of electromagnetism (2.2.2). It is defined by an observable $K[A, E] = K[A^1, B^2]$ whose representative is weakly conserved in time,

$$\frac{\partial}{\partial t} K + \{ K, H_{EM} \} \approx 0. \quad (2.3.19)$$

Since the new Hamiltonian differs from the usual one by terms proportional to the new constraint,

$$H = H_{EM} + \int d^3 x \ G_2 k, \quad k = \frac{1}{2} C^1, \quad (2.3.20)$$

and since $K$, when expressed in terms of the new variables, does not depend on $A^{L2}$, so that $\{ K, \int d^3 x \ G_2 k \} \approx 0$ in the extended theory, it follows that $K$ is also weakly conserved and thus a symmetry generator of the extended theory,

$$\frac{\partial}{\partial t} K + \{ K, H \} \approx 0. \quad (2.3.21)$$
Consider then the Poincaré generators \( Q_G(\omega, a) \) of electromagnetism (2.2.20). When expressed in terms of the new variables, they are representatives for the Poincaré generators of the extended theory. Indeed, we just have shown that they are symmetry generators, while we have argued in Section 2.3.2 that their Poisson algebra is isomorphic when restricted to their respective constraint surfaces.

As expected, the extended theory is invariant under Poincaré transformations but the generators obtained above are not invariant under the duality. Generators being defined up to terms proportional to the constraints, one can try to change the representative of the Poincaré generators to make them invariant under the duality. It is possible: one solution is to keep expressions (2.2.20-2.2.24) but with the new definition of \( B^a_i \). The associated transformations are now:

\[
\begin{align*}
\delta_\xi C^a &= 0, \\
\delta_\xi A^a_i &= \partial_i \lambda^a - \epsilon^{ab} B^0_b \xi^a - \epsilon_{ijk} \xi^i B^a_k, \\
\delta_\xi A^a_0 &= \partial_0 \lambda^a + \epsilon^{ab} B^i_b \xi^a.
\end{align*}
\]

where

\[
\lambda^a = -\epsilon^{ab} \Delta^{-1} \partial_i \left( B^i_b \xi^b \right) + \Delta^{-1} \partial_i \left( \epsilon^{ijk} B^a_j \xi^k \right).
\]

### 2.3.5 Equations of motion and dyon solution

The equations of motion can be easily computed and shown to be equivalent to the usual Maxwell equations. The variation of \( A^a_0 \) gives the two Gauss laws:

\[
\partial_i B^a_i = \Delta C^a = j^a. 
\]

The variation of \( C^a \) gives

\[
\epsilon_{ab} \Delta A^b_0 = \epsilon_{ab} \partial_i A^b_i - \Delta C^a.
\]

We are working with boundary conditions such that \( \Delta \) is invertible. With this assumption, both \( A^a_0 \) and \( C^a \) are auxiliary fields in the sense that their equations of motion can be solved to express them in terms of the other fields without the need for initial conditions.

The variation of \( A^a_i \) gives the remaining Maxwell’s equations

\[
- \epsilon_{ab} \dot{B}^{b} + \epsilon^{ijk} \partial_j B_{ak} = \epsilon_{ab} j^{b}. 
\]

The simplest non-trivial source is a dyon sitting at the origin with charges \( Q^a = (P, Q) \):

\[
\dot{\mathbf{j}}^a = 4\pi Q^a \delta^0 \delta^3 (x).
\]

The Maxwell’s equation in the above form are now solved by

\[
A^a_\mu = -\frac{\epsilon^{ab} Q_b}{r} \delta^0_\mu, \quad C^a = -\frac{Q^a}{r}.
\]

In this set-up, the string-like singularity has been completely removed.
2.3.6 Surface charges

The purpose of this extended double potential formalism is to have both sectors of the theory on the same footing and to be able to compute both the electric and the magnetic charge as dynamical conserved quantities. Doubling of the gauge freedom gives exactly that: two gauge freedoms imply two surface charges. In the original paper, the authors used the techniques developed in [51, 52, 53] to compute those charges and by coupling this theory to gravitation and were able to derive the first law for R-N black holes charged with both electric and magnetic sources. Unfortunately, in the spin 2 case, the theory will not be local as a Hamiltonian gauge theory and those techniques will not be available. We thus revert to the original Hamiltonian method of [54, 55]. We refer the reader to appendix A.1 for a quick summary adapted to the problem at hand where there is no need to discuss fall-off conditions.

The analysis of appendix A.1 is not directly applicable to our case since we do not have Darboux coordinates and the Poisson brackets of the fundamental variables are non local. In the spin 1 case, everything is under control and well behaved in the end. But, because it will not be the case for the spin 2, we will spend some time here to present the method used in the following chapter to deal with the surface charges of linearized gravity in the double potential formalism.

The non-locality brings two problems: \( \delta \epsilon^a_z z^A = 0 \) may not have non trivial solutions (see spin 2), and \( \Delta^{-1} \) applied to localized sources will spread them throughout space. In view of this, the idea is to redo the analysis of appendix A.1 while keeping the sources explicitly throughout the argument.

In the presence of the sources, the constraints are

\[
\mathcal{G}_a = \epsilon_{ab} \partial_i B^{bi} - \epsilon_{ab} j^{b0}. \tag{2.3.31}
\]

Instead of (A.1.11), we can write

\[
\lambda^a \mathcal{G}_a = -\partial_i \lambda^a \epsilon_{ab} B^{bi} - \lambda^a \epsilon_{ab} j^{b0} - \partial_i \tilde{k}^i_{A} \lambda^a [z^A] \tag{2.3.32}
\]

with

\[
\tilde{k}^i_{A} [z^A] = -\lambda^a \epsilon_{ab} B^{bi}. \tag{2.3.33}
\]

Consider now gauge parameters \( \lambda^a (x) \) satisfying

\[
\partial_i \lambda^a_s = 0 \quad \Leftrightarrow \quad \lambda^a_s = \text{cst.} \tag{2.3.34}
\]

It follows that the surface charges

\[
Q_{\lambda s} = \int_S d^3 x \tilde{k}^i_{A} [z_s], \tag{2.3.35}
\]
only depend on the homology class of $S$ outside of the sources. The equation of motion (2.3.28) implies

$$\partial_0 k^i_{\lambda s} [z_s] = \lambda^a \epsilon_{abj} x^b - \epsilon^{ijk} \partial_j (\lambda^a B_{ak}) .$$

(2.3.36)

The surface charges (2.3.35) are thus time-independent outside of the sources. Breaking them into their two components, we obtain both the electric and the magnetic charges:

$$Q = \oint_S d^3 x_i E^i, \quad \mathcal{P} = \oint_S d^3 x_i B^i .$$

(2.3.37)

The result is what we expected and is what was obtained in [8] using the covariant approach to surface charges.
Chapter 3

Double potential formalism for spin 2

In this chapter, we will introduce a double potential formalism for linearized gravity in presence of both type of gravitational sources: the “electric” source which is the mass or the energy and the “magnetic” source. The strategy is the same as the one presented in the previous chapter for electromagnetism. The two questions are closely related but the spin 2 case is more complicated.

We will start by a small review on gravitational electromagnetic duality. After this introduction, the results of Henneaux-Teitelboim [3] for the invariance of the action in absence of sources will be presented. Finally, the last section will be devoted to our construction of an extended double potential formalism for linearized gravity [56].

3.1 Electromagnetic duality for gravity

The first hint at the existence of the electromagnetic duality in gravitation came with the discovery of the Taub-Nut solution to the Einstein equations in 4 dimensions. The Taub-Nut metric [57, 9] is given by

\[
ds^2 = -V(r) [dt + 2N(1 - \cos \theta)d\phi]^2 + V(r)^{-1}dr^2 + (r^2 + N^2)(d\theta^2 + \sin^2 \theta d\phi^2),
\]

with

\[
V(r) = 1 - \frac{2(N^2 + Mr)}{(r^2 + N^2)} = \frac{r^2 - 2Mr - N^2}{R^2 + N^2},
\]

where \(M\) is the usual mass and \(N\) a second parameter. The study of this solution showed that this \(N\) plays the role of an electromagnetic dual to \(M\). A closer look at the \(g_{t\phi}\) component of the metric shows a string-like singularity quite similar to the one appearing in the vector potential for the magnetic monopole in spherical coordinates (2.1.35). This string-like singularity is in this case known as the Misner string. Unfortunately, the attempt to interpret it as a non-physical object, as an artefact coming from the metric description
of gravity, led to problems [58]. The non-linear nature of gravity implies that for this singularity to be spurious, we need the time coordinate to be periodic. In that case, the Taub-NUT acquires some pathological behavior like time-like closed curves.

A lot of work has been done in an attempt to better understand the duality in the full non-linear theory but it is still a difficult question. On the other hand, the linear version, describing a spin 2 particle in a Minkowski background, is nicer. The theory is described by the Pauli-Fiertz action:

\[ S_{PF}[h_{\mu\nu}] = -\frac{1}{4} \int d^4x \left( \partial^\rho h^{\mu\nu} \partial_\rho h_{\mu\nu} - 2 \partial_\mu h^{\mu\nu} \partial_\rho h_\nu^\rho + 2 \partial^\mu h_\rho^\nu \partial^\rho h_{\mu\nu} - \partial^\mu h_\rho^\sigma \partial_\mu h^{\rho\sigma} \right) \]  
\[ (3.1.3) \]

where \( h_{\mu\nu} \) is symmetric. This action can be obtained from the full Einstein action by linearizing around flat space, \( h_{\mu\nu} \) being the deviation of the full metric from the Minkowski metric. The associated equations of motions can be written as

\[ R_{\mu\nu} = R^\rho_{\mu\rho\nu} = 0 \]  
\[ (3.1.4) \]

where \( R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho} \) is the linearized Riemann tensor defined as

\[ R_{\mu\nu\rho\sigma} = \partial_{[\mu} h_{\nu\rho\sigma]} \]  
\[ (3.1.5) \]

By construction, this tensor satisfies to the following identities

\[ R_{\mu[\nu\rho\sigma]} = 0 \]  
\[ (3.1.6) \]
\[ R_{\mu\nu[\rho\sigma,\alpha]} = 0. \]  
\[ (3.1.7) \]

It follows that \( R_{\mu\nu\rho\sigma} \) is symmetric for the exchange of the pairs \( (\mu\nu) \) and \( (\rho\sigma) \):

\[ R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}, \]  
\[ (3.1.8) \]

and, using (3.1.7) and (3.1.4), one can easily prove

\[ \partial^\mu R_{\mu\nu\rho\sigma} = 0. \]  
\[ (3.1.9) \]

It turns out that equations (3.1.6) and (3.1.7) are the conditions for the existence of a tensor potential \( h_{\mu\nu} \) for a general tensor \( R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho} \).

Equations (3.1.4), (3.1.6)-(3.1.7) and (3.1.9) are the equivalent of Maxwell’s equation for linearized gravity. In term of the dual tensor \( S_{\mu\nu\rho\sigma} = -S_{\nu\mu\rho\sigma} = -S_{\mu\nu\sigma\rho} \) defined as

\[ S_{\mu\nu\rho\sigma} = -\frac{1}{2} \epsilon_{\mu\nu\alpha\beta} R^\alpha_{\rho\sigma} \]  
\[ (3.1.10) \]

they will keep the same form. Indeed, the structural equations (3.1.6) and (3.1.7) become

\[ S_{\mu\nu} = S^\rho_{\mu\rho\nu} = 0 \]  
\[ (3.1.11) \]
\[ \partial^\mu S_{\mu\nu\rho\sigma} = 0. \]  
\[ (3.1.12) \]
On the other hand, the equations of motion (3.1.4) and (3.1.9) become
\[ S_{\mu[\nu\rho]} = 0, \quad S_{\mu\nu[\rho\sigma,\alpha]} = 0. \] (3.1.13)
(3.1.14)

The electromagnetic duality for the spin 2 is defined as the $SO(2)$ rotation between $R_{\mu\nu\rho\sigma}$ and its dual $S_{\mu\nu\rho\sigma}$:
\[ \delta_D R_{\mu\nu\rho\sigma} = S_{\mu\nu\rho\sigma}, \quad \delta_D S_{\mu\nu\rho\sigma} = -R_{\mu\nu\rho\sigma}. \] (3.1.15)

It obviously leaves the set of equations invariant.

A linearized Riemann tensor satisfying to (3.1.6) and (3.1.4) can be completely parametrized by two 3 dimensional symmetric tensors $E_{ij}$ and $B_{ij}$ defined as
\[ E_{ij} = R_{0i0j}, \quad B_{ij} = -\frac{1}{2} \epsilon_{kl} R_{0j}^{kl}. \] (3.1.16)

The are called the electric and magnetic part of the Weyl tensor which in this case is equal to the Riemann tensor. The duality transformations are equivalent to
\[ \delta_D E_{ij} = B_{ij}, \quad \delta_D B_{ij} = -E_{ij}. \] (3.1.17)

The case we described until here is a free spin 2 field without any external source. Adding the usual “electric” source can be done easily by adding an interacting term to the Pauli-Fierz action:
\[ S[h_{\mu\nu}; T^{\mu\nu}] = \frac{1}{16\pi G} S_{PF} + \frac{1}{2} \int d^4 x h_{\mu\nu} T^{\mu\nu}. \] (3.1.18)

This source is an energy-momentum tensor, it is symmetric and conserved:
\[ T^{\mu\nu} = T^{\nu\mu}, \quad \partial_\mu T^{\mu\nu} = 0. \] (3.1.19)

The new equations of motion are
\[ G^{\mu\nu} = 8\pi G T^{\mu\nu}, \quad G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} T^{\mu\nu} R^\rho_\rho \] (3.1.20)
where $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} T^{\mu\nu} R^\rho_\rho$ is the linearized Einstein tensor. This also implies a modification of the identity (3.1.9):
\[ \partial_\mu R^{\mu\nu\rho\delta} = 8\pi G \left( \partial^\nu \bar{T}^{\rho\delta} - \partial^\delta \bar{T}^{\nu\rho} \right) \] (3.1.21)
where for a tensor $K^{\mu\nu}$, we have defined $\bar{K}^{\mu\nu} = K^{\mu\nu} - \frac{1}{2} T^{\mu\nu} K^\rho_\rho$. The structural equations (3.1.6) and (3.1.7) don’t change in presence of electric sources. The equations are no longer invariant under the duality: for instance, the duality will send equation (3.1.20) to
\[ S^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} S^\rho_\rho = 8\pi G T^{\mu\nu} \quad \iff \quad R_{\mu[\nu|\rho\sigma]} = -\frac{8}{3} \pi G \epsilon_{\nu|\rho\sigma\delta} \bar{T}^\delta_\mu. \] (3.1.22)
The idea to restore the duality is exactly the same in the spin 2 case as what was done by Dirac for electromagnetism: adding dual sources. The “magnetic” sources are represented by a dual, magnetic, energy-momentum tensor $\Theta^{\mu\nu}$. As for $T^{\mu\nu}$, it must be conserved: $\partial_\mu \Theta^{\mu\nu} = 0$. The new equations are [9]:

$$G^{\mu\nu} = 8\pi G T^{\mu\nu},$$

$$R_{\mu[\nu\rho\sigma]} = -\frac{8}{3} \pi G \epsilon_{\nu\rho\sigma\delta} \Theta^\delta_{\mu},$$

$$R_{\mu\nu[\rho\sigma,\alpha]} = \frac{8}{3} \pi G \epsilon_{\rho\sigma\alpha\beta} \left( \partial_\nu \Theta^\beta_{\mu} - \partial_\mu \Theta^\beta_{\nu} \right),$$

$$\partial_\mu R^{\mu\rho\gamma\delta} = 8\pi G \left( \partial_\gamma \bar{T}^{\rho\delta} - \partial_\delta \bar{T}^{\rho\gamma} \right).$$

The equations are symmetric under the generalized duality:

$$\delta_D R^{\mu\nu\rho\sigma} = S_{\mu\nu\rho\sigma},$$

$$\delta_D S_{\mu\nu\rho\sigma} = -R_{\mu\nu\rho\sigma},$$

$$\delta_D T^{\mu\nu} = \Theta^{\mu\nu},$$

$$\delta_D \Theta^{\mu\nu} = -T^{\mu\nu}.$$

Again, the same problem appears in this case. One can write duality invariant equations but writing an action is more difficult. As for electromagnetism, the necessary conditions to introduce the potential $h_{\mu\nu}$, namely equations (3.1.6) and (3.1.7), are no longer valid in presence of magnetic sources. In [9], the authors proposed a solution by introducing string-like terms. Following the ideas of Dirac, they managed to write an action invariant under Poincaré containing both electric and magnetic external sources. They also derived a quantification condition for the spin 2 conserved charges $P^\gamma$ and $Q^\gamma$ (the electric and magnetic 4-momentum of the spin 2 field):

$$\frac{4GP_\gamma Q^\gamma}{\hbar} \in \mathbb{Z}.$$

### 3.2 Duality symmetric action without sources

The strategy in this case is the same as the one used for electromagnetism: write the Hamiltonian action, completely fix the gauge to go to the reduced phase space and introduce new potentials. Doing this, the authors of [3] managed to write a duality invariant action for linearized gravity. In the following, we will present their results in a different notation.
3.2.1 Canonical formulation of Pauli-Fierz theory

The Hamiltonian formulation of general relativity linearized around flat spacetime is

\[
S_{PF}[h_{mn}, \pi^{mn}, n_m, n_n] = \int dt \left[ \int d^3x \left( \pi^{mn} \dot{h}_{mn} - n^m \mathcal{H}_m - n^H \right) - H_{PF} \right],
\]  
(3.2.1)

with

\[
H_{PF}[h_{mn}, \pi^{mn}] = \int d^3x \left( \pi^{mn} \pi_{mn} - \frac{1}{2} \pi^2 + \frac{1}{4} \partial^r h^{mn} \partial_r h_{mn} - \frac{1}{2} \partial_m h^{mn} \partial_r h_{rn} + \frac{1}{2} \partial^m h \partial^m h_{mn} - \frac{1}{4} \partial^m h \partial_m h \right),
\]  
(3.2.2)

and

\[
\mathcal{H}_m = -2 \partial^m \pi_{mn}, \quad \mathcal{H}_\perp = \Delta h - \partial^m \partial^n h_{mn}.
\]  
(3.2.3)

Again, indices are lowered and raised with the flat space metric \( \delta_{mn} \) and its inverse, \( h = h^m_m, \pi = \pi^m_m \) and \( \Delta = \partial_m \partial^n \) is the Laplacian in flat space. The linearized 4 metric is reconstructed using \( h_{00} = -2n \) and \( h_{0i} = n_i \).

3.2.2 Decomposition of symmetric rank two tensors

Symmetric rank two tensors \( \phi_{mn} \) decompose as \([59, 60]\)

\[
\phi_{mn} = \phi_{mn}^{TT} + \phi_{mn}^T + \phi_{mn}^L,
\]  
(3.2.4)

\[
\phi_{mn}^L = \partial_m \psi_n + \partial_n \psi_m,
\]  
(3.2.5)

\[
\phi_{mn}^T = \frac{1}{2} (\delta_{mn} \Delta - \partial_m \partial_n) \psi^T.
\]  
(3.2.6)

Here \( \phi_{mn}^{TT} \) is the transverse-traceless part, containing two independent components. The tensor \( \phi_{mn}^T \) contains the trace of the transverse part of \( \phi_{mn} \) and only one independent component. The last three components are the longitudinal part contained in \( \phi_{mn}^L \). In terms of the original tensor \( \phi_{mn} \) the potentials for the longitudinal part and the trace are given by

\[
\psi_m = \Delta^{-1} \left( \partial^n \phi_{mn} - \frac{1}{2} \Delta^{-1} \partial_m \partial^k \phi_{kl} \right),
\]  
(3.2.7)

\[
\psi^T = \Delta^{-1} \left( \phi - \Delta^{-1} \partial^m \partial^n \phi_{mn} \right),
\]  
(3.2.8)

while the transverse traceless part is then defined as the remainder,

\[
\phi_{mn}^{TT} = \phi_{mn} - \phi_{mn}^T - \phi_{mn}^L.
\]  
(3.2.9)
This implies
\[
\Delta^2 \phi_{mn}^{TT} = \Delta^2 \phi_{mn} - \Delta \partial_m \partial^k \phi_{kn} - \Delta \partial_n \partial^k \phi_{km} - \frac{1}{2} \Delta (\delta_{mn} \Delta - \partial_m \partial_n) \phi + \frac{1}{2} (\delta_{mn} \Delta + \partial_m \partial_n) \partial^k \partial^l \phi_{kl},
\]
(3.2.10)
\[
\int d^3 x \phi_{mn}^{TT} \Delta^2 \phi_{mn}^{TT} = \int d^3 x \left( \Delta \phi_{mn}^{TT} \Delta \phi_{mn} + 2 \partial_m \phi_{mn} \Delta \partial^k \phi_{kn} - \frac{1}{2} (\Delta \phi)^2 + \partial_m \partial_n \phi_{mn} \Delta \phi + \frac{1}{2} \partial_m \partial_n \phi_{mn} \partial^k \partial^l \phi_{kl} \right).
\]
(3.2.11)

Alternatively, one can introduce the local operator \( P_{TT} \)
\[
(P_{TT} \phi)_{mn} = \frac{1}{2} \left[ \epsilon_{mpq} \partial^p (\Delta \phi^q_m - \partial_n \partial_r \phi^{qr}) + \epsilon_{npq} \partial^p (\Delta \phi^q_m - \partial_n \partial_r \phi^{qr}) \right],
\]
(3.2.12)
which projects out the longitudinal and trace parts and onto a transverse-traceless tensor,
\[
(P_{TT} \phi)_{mn} = P_{TT} (\phi_{TT})_{mn} = (P_{TT} \phi_{TT})_{mn}.
\]
(3.2.13)

In addition,
\[
(P_{TT} (P_{TT} \phi))_{mn} = -\Delta^3 \phi_{mn}^{TT}.
\]
(3.2.14)

As a consequence, the transverse-traceless tensor \( \phi_{mn}^{TT} \) can be written as \( P_{TT} \) acting on a suitable potential \( \psi_{TT}^{TT} \),
\[
\phi_{mn}^{TT} = (P_{TT} \psi_{TT}^{TT})_{mn}; \quad \psi_{mn}^{TT} = -\Delta^{3} (P_{TT} \phi)_{mn}.
\]
(3.2.15)

When acting on a transverse-traceless tensor, the last two terms of \( P_{TT} \) can be dropped. In this case, it is related to the generalized curl [61, 62],
\[
(O \phi)_{mn} = \frac{1}{2} \left( \epsilon_{mpq} \partial^p \phi_{mn}^q + \epsilon_{npq} \partial^p \phi_{mn}^q \right),
\]
(3.2.16)
\[
(P_{TT} \phi_{TT}^{TT})_{mn} = \Delta (O \phi_{TT}^{TT})_{mn}.
\]
(3.2.17)

Remark that the generalized curl acting on a transverse-traceless tensor will produce a transverse-traceless tensor. A second operator that projects out the longitudinal and trace parts and onto a transverse-traceless tensor is \( Q_{TT} \),
\[
(Q_{TT} \phi)_{mn} = \epsilon_{mpq} \epsilon_{nrs} \partial^p \partial^r \Delta \phi^{qs} - \frac{1}{2} (\delta_{mn} \Delta - \partial_m \partial_n) (\Delta \phi - \partial^p \partial^r \phi_{pr}).
\]
(3.2.18)

In this case,
\[
(Q_{TT} (Q_{TT} \phi))_{mn} = \Delta^4 \phi_{mn}^{TT},
\]
(3.2.19)
so that the transverse-traceless tensor \( \phi_{mn}^{TT} \) can be written as \( Q_{TT} \) acting on another potential \( \chi_{mn}^{TT} \),
\[
\phi_{mn}^{TT} = (Q_{TT} \chi_{TT}^{TT})_{mn} ; \quad \chi_{mn}^{TT} = \Delta^{-4} (Q_{TT} \phi)_{mn}.
\]
(3.2.20)
In turn this operator is related to the way the constraints $\mathcal{H}_m = 0$ are solved by expressing the momenta $\pi^{mn}$ in terms of superpotentials in [3]. When acting on a transverse-traceless tensor, the last term can again be dropped and it is related to the square of the generalized curl,

$$\left( Q^{TT} \phi^{TT} \right)_{mn} = \Delta \left( \mathcal{O}(\mathcal{O} \phi^{TT}) \right)_{mn} = -\Delta^2 \phi_{mn}^{TT}. \quad (3.2.21)$$

The elements of the decomposition are orthogonal under integration if boundary terms can be neglected,

$$\int d^3x \, \phi^{mn} \varphi_{mn} = \int d^3x \left( \phi^{TTmn} \varphi_{mn}^{TT} + \phi^{Lmn} \varphi_{mn}^L + \phi^{Tmn} \varphi_{mn}^T \right). \quad (3.2.22)$$

and the operators $\mathcal{P}^{TT}, \mathcal{Q}^{TT}, \mathcal{O}$ are self-adjoint, e.g.,

$$\int d^3x \left( \mathcal{P}^{TT} \phi \right)^{mn} \varphi_{mn} = \int d^3x \varphi_{mn} \left( \mathcal{P}^{TT} \phi \right)^{mn}. \quad (3.2.23)$$

### 3.2.3 Duality invariant action without sources

Because of the orthogonality of the decomposition, the canonically conjugate pairs can be directly read off from the kinetic term and are given by

$$\left( h_{mn}^{TT} (\vec{x}), \pi_{mn}^{TT} (\vec{y}) \right), \quad \left( h_{mn}^L (\vec{x}), \pi_{mn}^L (\vec{y}) \right), \quad \left( h_{mn}^T (\vec{x}), \pi_{mn}^T (\vec{y}) \right). \quad (3.2.24)$$

The first class constraints $\mathcal{H}_m = 0 = \mathcal{H}$ are equivalent to $\pi_{mn}^{kl} = 0 = h_{mn}^T$. They can be gauge fixed through the conditions $h_{mn}^L = 0 = \pi_{mn}^L$. The reduced theory only depends on 2 degrees of freedom (per spacetime point), the transverse-traceless components $(h_{mn}^{TT}(\vec{x}), \pi_{mn}^{TT}(\vec{y}))$ and the reduced Hamiltonian simplifies to

$$H^R = \int d^3x \left( \pi_{TT}^{mn} \pi_{TT} + \frac{1}{4} \partial_r h_{mn}^{TT} \partial^r h_{mn}^{TT} \right). \quad (3.2.25)$$

Using the same strategy than in the electromagnetic case, the next step is to introduce new potentials. In this case, the authors of [3] introduced two new potentials:

$$\pi_{TT}^{mn} = -\Delta H_{TT}^{Dmn} \quad \text{and} \quad h_{mn}^{TT} = 2 \left( \mathcal{O} H_{TT}^{TT} \right)_{mn}. \quad (3.2.26)$$

Plugging this into the reduced action and doing some integrations by parts brings it to

$$S[H_{TT}, H_{TT}^D] = \int dt \left[ \int d^3x \left[ -2\Delta \left( \mathcal{O} H_{TT}^D \right)_{mn} \dot{H}_{mn}^{TT} \right] - H \right], \quad (3.2.27)$$

$$H = \int d^3x \left( \Delta^2 H_{TT}^{mn} H_{mn}^{TT} + \Delta^2 H_{TT}^{mn} H_{mn}^{TT} \right). \quad (3.2.28)$$
As before, one can introduce a notation better suited to the duality $H^a_{TT} = (H_{TT}, H^D_{TT})$. Using this, the action takes a duality invariant form:

$$S[H^a_{TT}] = \int dt \left[ \int d^3x \epsilon_{ab} \Delta (O H^a_{TT})^{mn} \dot{H}^{TT}_{mn} - H \right], \quad (3.2.29)$$

$$H = \int d^3x \Delta^2 H^m_{mn} H^a_{TT} \dot{H}^{TT}_{mn}. \quad (3.2.30)$$

The duality is the $SO(2)$ rotation generated by

$$\delta_D H^T_{mn} = \epsilon^{ab} H^T_{bn}. \quad (3.2.31)$$

This transformation obviously leaves the action (3.2.29) invariant. The associated duality generator is

$$D = -\int d^3x \mathcal{P}_{TT} (H^a)_{mn} H^m_{mn}. \quad (3.2.32)$$

In [3], it was cast in the form of a Chern-Simons term.

The fact that this transformation is indeed the electromagnetic duality is less clear for the spin 2. In the electromagnetism case, the two potentials were easily related to the electric and the magnetic fields. It was then easy to interpret the $SO(2)$ rotation on the $a$ index as the duality introduced at the level of the equations of motion. Following [3], we see that the electric and magnetic part of the Weyl tensor are given in terms of the new potentials as:

$$E_{mn} = 2\epsilon_{mpq} \partial^p \Delta H^q_{n}, \quad (3.2.33)$$

$$B_{mn} = 2\epsilon_{mpq} \partial^p \Delta H^D_{n}. \quad (3.2.34)$$

We see that the duality defined in (3.2.31) induce the duality (3.1.17) at the level of the Riemann tensor.

### 3.2.4 Poincaré generators

As for electromagnetism, the invariance of the action under the Poincaré transformations is no longer manifest but is of course still present. The strategy here is the same: introduce the new potentials in the reduced phase space generators which are given by the generators of Pauli-Fierz evaluated on the constraint surface.

To build the Poincaré generators of Pauli-Fierz, we will use the fact that it is the linearization of general relativity. In this section, we assume that the canonical variables vanish sufficiently fast at the boundary so that integrations by parts can be used even if the gauge parameters do not vanish at the boundary.

In the Hamiltonian formulation of general relativity [60], the canonically conjugate variables are the spatial 3 metric $g_{ij}$ and the extrinsic curvature $\pi^{ij}$. The constraints are
The associated generators of gauge transformation $H[\xi] = \int d^3x \left( \mathcal{H}_\perp \xi^\perp + \mathcal{H}_i \xi^i \right)$ satisfy the so-called surface deformation algebra [63, 64],

\[
\{H[\xi], H[\eta]\} = H[[\xi, \eta]_{SD}],
\]

\[
[\xi, \eta]_{SD}^\perp = \xi^i \partial_i \eta^\perp - \eta^i \partial_i \xi^\perp,
\]

\[
[\xi, \eta]_{SD}^i = g^{ij} \left( \xi^k \partial_k \eta^i - \eta^k \partial_k \xi^i \right) + \xi^j \partial_j \eta^i - \eta^j \partial_j \xi^i.
\]

When the parameters $f, g$ of gauge transformations depend on the canonical variables, (3.2.36) is replaced by [12]

\[
\{H[f], H[g]\} = H[k],
\]

\[
k = [f, g]_{SD} + \delta_y f - \delta_y g - m,
\]

\[
m^\perp = \int d^3x' \left[ \left\{ f^\perp, g^\perp(x') \right\} \mathcal{H}_\perp(x') + \left\{ f^\perp, g^i(x') \right\} \mathcal{H}_i(x') \right],
\]

\[
m^i = \int d^3x' \left[ \left\{ f^i, g^\perp(x') \right\} \mathcal{H}_\perp(x') + \left\{ f^i, g^j(x') \right\} \mathcal{H}_j(x') \right],
\]

where

\[
\delta_\xi g_{ij} = \nabla_i \xi_j + \nabla_j \xi_i + 2D_{ijkl} \pi^{kl} \xi^\perp,
\]

\[
D_{ijkl} = \frac{1}{2\sqrt{g}} (g_{ik}g_{jl} + g_{jk}g_{il} - g_{ij}g_{kl}),
\]

\[
\delta_\xi \pi^{ij} = -\xi^i \sqrt{g} (R^{ij} - \frac{1}{2} g^{ij} R) + \frac{\xi^\perp}{2\sqrt{g}} g^{ij} (\pi^{kl} \pi_{kl} - \frac{1}{2} \pi^2)
\]

\[
- 2\frac{\xi^\perp}{\sqrt{g}} (\pi^{im} \pi^m - \frac{1}{2} \pi^m \pi) + \sqrt{g} (\nabla_j \nabla^i \xi^\perp - g^{ij} \nabla_m \nabla^m \xi^\perp)
\]

\[
+ \nabla^m (\pi_{ij} \pi^m) - \nabla_m \pi^{ij} - \nabla_m \pi^{mj} - \nabla_m \xi^i \pi^{mj}.
\]

Let $g_{ij} = \delta_{ij} + h_{ij}$ and consider the canonical change of variables from $g_{ij}, \pi^{kl}$ to $z^A = (h_{ij}, \pi^{kl})$. We will expand in terms of the homogeneity in the new variables and use the flat metric $\delta_{ij}$ to raise and lower indices in the remainder of this appendix. Furthermore, Greek indices take values from 0 to 3 with $\mu = (\perp, i)$. Indices are lowered and raised with $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ and its inverse. Let $\tilde{\omega}_{\mu\nu} = -\tilde{\omega}_{\nu\mu}$.

To lowest order, i.e., when $g_{ij} = \delta_{ij}$, the vector fields

\[
\xi_P(\tilde{\omega}, \tilde{\alpha})^\mu = -\tilde{\omega}^\mu \xi^i + \tilde{\alpha}^\mu,
\]

with bracket the surface deformation bracket form a representation of the Poincaré algebra [54],

\[
[\xi_P(\tilde{\omega}_1, \tilde{\alpha}_1), \xi_P(\tilde{\omega}_2, \tilde{\alpha}_2)]_{SD}^{(0)} = \xi_P([\tilde{\omega}_1, \tilde{\omega}_2], \tilde{\omega}_1 \tilde{\alpha}_2 - \tilde{\omega}_2 \tilde{\alpha}_1).
\]
For the gauge generators, we find \( H[\xi] = H^{(1)}[\xi] + H^{(2)}[\xi] + H^{(3)}[\xi] + \cdots \), where

\[
H^{(1)}[\xi] = \int d^3 x \left( -2 \partial^i \pi_{ij}^i \xi^i + (\partial^i \partial^j h_{ij} - \Delta h) \xi^j \right) \tag{3.2.48}
\]

\[
= \int d^3 x \left( H_m^{(1)} \xi^m + H_\perp^{(1)} \xi^\perp \right) \tag{3.2.49}
\]

are the gauge generators associated to the constraints (3.2.3) of the Pauli-Fierz theory. Because

\[
H[[\xi,\eta]_{SD}] = H^{(1)}[[\xi,\eta]_{SD}] + H^{(2)}[[\xi,\eta]_{SD}] + H^{(1)}[[\xi,\eta]_{SD}] + O(z^3), \tag{3.2.50}
\]

we have to lowest non trivial order

\[
\{ H^{(1)}[\xi], H^{(2)}[\eta] \} = H^{(1)}[[\xi,\eta]_{SD}]. \tag{3.2.51}
\]

This means that \( H^{(2)}[\eta] \) are observables, i.e., weakly gauge invariant functionals.

One can use integrations by parts to show that \( H^{(1)}[\xi_P] = 0 \). It then follows that

\[
\{ H[\xi_P], H[\eta_P] \} = \{ H^{(2)}[\xi_P], H^{(2)}[\eta_P] \} + O(z^3). \tag{3.2.52}
\]

For vectors \( \xi_P(\tilde{\omega}, \tilde{a}), \eta_P(\tilde{\theta}, \tilde{b}) \) of the form (3.2.46), the first term on the RHS of (3.2.50) vanishes on account of (3.2.47). To lowest non trivial order, (3.2.36) then implies

\[
\{ H^{(2)}[\xi_P], H^{(2)}[\eta_P] \} = H^{(2)}[[\xi_P,\eta_P]_{SD}] + H^{(1)}[[\xi_P,\eta_P]_{SD}]. \tag{3.2.53}
\]

The generators \( H^{(2)}[\xi_P] \) equipped with the Poisson bracket thus form a representation of the Poincaré algebra when the constraints of the Pauli-Fierz theory are satisfied. Explicitly, the term proportional to the constraints is

\[
H^{(1)}[[\xi,\eta]_{SD}] = -2 \int d^3 x \partial^i \pi_{ji} h^{ik} (\xi^i_P \theta^k - \eta^i_P \omega^k), \tag{3.2.54}
\]

while

\[
H_i^{(2)} = -2 \partial_j \left( \pi^{jk} h_{ik} \right) + \pi^{jk} \partial_i h_{jk} \tag{3.2.55}
\]

\[
H_\perp^{(2)} = \pi^{ij} \pi_{ij} - \frac{1}{2} \pi^2 + \frac{1}{4} \partial_k h_{ij} \partial^k h^{ij} - \frac{1}{2} \partial_k h^{kj} \partial^j h_{ij} + \frac{1}{2} \partial_i h \partial_j h^{ij} - \frac{1}{4} \partial_i h \partial^j h
\]

\[
+ \partial_i \left( \frac{1}{2} h \partial^j h - h^{ij} \partial^j h_{ij} - \frac{1}{2} h \partial_j h^{il} - h^{il} \partial_j h + \frac{3}{2} h^{ij} \partial^j h_{ij} + \frac{1}{2} h_{ij} \partial^j h^l \right). \tag{3.2.56}
\]
Isolating terms proportional to the constraints, we find

\[ H^{(2)}[\xi] = \int d^3x \left( \mathcal{H}_m h^{mi} \xi_i + \frac{1}{2} \mathcal{H} h \xi_\perp \right) + \bar{H}^{(2)}[\xi], \]  
(3.2.57)

\[ \hat{\mathcal{H}}_i^{(2)} = -\pi^{jk}(\partial_j h_{ki} + \partial_k h_{ji} - \partial_i h_{jk}), \]  
(3.2.58)

\[ \bar{\mathcal{H}}_\perp^{(2)} = \pi^{ij} \pi_{ij} - \frac{1}{2} \pi^{2} \]  
\[ + \frac{1}{4} \partial_k h_{ij} \partial^k h_{ij} - \frac{1}{2} \partial_k h^{ki} \partial^j h_{ij} + \frac{1}{4} \partial_i h \partial^i h \]  
\[ + \partial_l \left( -h_{ij} \partial^j h_{ij} - h_{il} \partial_i h + \frac{3}{2} h_{lj} \partial^j h_{ij} + \frac{1}{2} h_{ij} \partial^i h_{jl} \right), \]  
(3.2.59)

with \( \bar{H}^{(2)}[\xi] = \int d^3x \left( \bar{\mathcal{H}}_i^{(2)} \xi^i + \bar{\mathcal{H}}_\perp^{(2)} \xi_\perp \right) \). On account of (3.2.51) and the analog of (3.2.39) for \( \mathcal{H}^{(1)}[f] \), it follows that

\[ \{ \bar{H}^{(2)}[\xi_P], \bar{H}^{(2)}[\eta_P] \} \approx \bar{H}^{(2)}[\{\xi_P, \eta_P\}_SD], \]  
(3.2.60)

where \( \approx \) means an equality up to terms proportional to the constraints \( \mathcal{H}_m, \mathcal{H}_\perp \) of Pauli-Fierz theory. Note that the functionals \( H^{(2)}[\xi_P] \) and \( \bar{H}^{(2)}[\xi_P] \) generate transformations of the canonical variables that are equivalent because they differ at most by a gauge transformations of the Pauli-Fierz theory when restricted to the constraint surface.

The generators for global Poincaré transformations of Pauli-Fierz theory can then be identified as

\[ Q_G(\omega, a) = \frac{1}{2} \omega_{\mu\nu} P^\mu_G - a_\mu P^\mu_G = \bar{H}^{(2)}[\xi_P, \bar{a}_G] \]  
(3.2.61)

Indeed, differentiating (3.2.53) with respect to \( b_\perp \) gives

\[ \{ H, Q_G(\omega, a) \} = \frac{\partial}{\partial t} Q_G(\omega, a) + 2 \int d^3x \partial^i \pi_{ij} h^{ik} \omega_{lk}. \]  
(3.2.62)

When combined with (3.2.53) and (3.2.61), this shows that, on the constraint surface, the generators \( Q_G(\omega, a) \) are conserved and satisfy the Poincaré algebra.

Finally, we can further simplify the explicit expression for \( \bar{H}^{(2)}[\xi_P] \) by using linearity of \( \xi_P \) in \( x^i \) and integrations by parts to show that

\[ \int d^3x \bar{\mathcal{H}}_\perp^{(2)} \xi_\perp = \int d^3x \left[ \pi^{ij} \pi_{ij} - \frac{1}{2} \pi^2 + \frac{1}{4} \partial_k h_{ij} \partial^k h_{ij} - \frac{1}{2} \partial_k h^{ki} \partial^j h_{ij} \right. \]  
\[ + \frac{1}{4} \partial_i h \partial^i h + \partial_l \left( h \partial_l h^{il} + h^{lj} \partial^j h_{ij} \right) \xi_\perp. \]  
(3.2.63)
The expansion of the gauge transformations \((3.2.43), (3.2.45)\) gives to first order:

\[
\delta^{(0)}_{\xi} h_{ij} = \partial_i \xi_j + \partial_j \xi_i, \quad \delta^{(0)}_{\xi} \pi_{ij} = (\partial^i \partial^j - \delta^{ij} \Delta) \xi^l, \quad (3.2.64)
\]

\[
\delta^{(1)}_{\xi} h_{ij} = \frac{1}{2} h (\partial^i \partial^j - \delta^{ij} \Delta) \xi^l - h^{im} \partial_m \partial^j \xi^l - h^{jm} \partial_m \partial^i \xi^l + h^{ij} \Delta \xi^l + \partial^{ij} h^{mn} \partial_m \partial^l \xi^l + \partial_m (\pi^{ij} \xi^m) - \pi^{mj} \partial_m \xi^i - \pi^{mi} \partial_m \xi^j + \frac{1}{2} \partial_k \xi^l \left[ - \partial^j h^{kl} - \partial^i h^{kj} + \partial^k h^{ij} + \delta^{ij} (\Delta h - \partial_k \partial_l h^{kl}) \right]. \quad (3.2.65)
\]

\[
\delta^{(1)}_{\xi} \pi_{ij} = \frac{1}{2} \omega_{\mu \nu} \eta^\mu_{\nu} G^\mu = \int d^3 x \left( \overline{\mathcal{H}}^{(2)}_i \xi^i + \overline{\mathcal{H}}^{(2)}_{\perp} \right) \quad (3.2.67)
\]

\[
\overline{\mathcal{H}}^{(2)}_i = -\pi^{TT} \partial_j h^{TT}_{ki} + \partial_k h^{TT}_{ji} - \partial_j h^{TT}_{kj}, \quad (3.2.68)
\]

\[
\overline{\mathcal{H}}^{(2)}_{\perp} = \pi^{TT} \partial^{TT} \partial_{\perp} + \frac{1}{4} \partial_k \partial^{TT} h^{TT} \partial^k h^{TT} \partial^{TT}_{ij}. \quad (3.2.69)
\]

Finally, the Poincaré generators for the double potential formalism are obtained by introducing the new potentials in the above expression for \(Q_G\).

### 3.3 Duality symmetric action with external sources

Having briefly recalled the double formalism potential for gravity, we will now introduce our work on the extension of this formalism to include external sources, as was done in the spin 1 case. The main difference is that the spin 2 action was built piece by piece due to its complexity.

The analysis starts with a degree of freedom count that shows that the phase space of duality invariant spin 2 fields with doubled gauge invariance can be taken to consist of 2 symmetric tensors, 2 vectors and 2 scalars in 3 dimensions. We then define the metric, extrinsic curvature and their duals in terms of the phase space variables and propose our duality invariant action principle with enhanced gauge invariance. We proceed by identifying the canonically conjugate pairs and discuss the gauge structure, Hamiltonian and duality generators of the theory. In the absence of sources, we then show how the generators for global Poincaré transformations can be extended to the duality invariant theory.

The next step is the introduction of the external sources. The equations of motion are then solved in the simplest case of a point-particle dyon sitting at the origin. They are
Coulomb-like without string singularities. By identifying the Riemann tensor in terms of the canonical variables and computing it for this case, we then show that this solution indeed describes the linearized Taub-NUT solution.

Finally, we discuss the surface charges of the theory and show that they include electric and magnetic energy-momentum and angular momentum. Because of the non-locality of the Poisson structure, we proceed indirectly and show that the expressions obtained by generalizing the surface charges of Pauli-Fierz theory in a duality invariant way fulfill the standard properties. Finally, we investigate how the surface charges transform under a global Poincaré transformation of the sources.

### 3.3.1 Degree of freedom count

In order to be able to couple to sources of both electric and magnetic type in a duality invariant way, we want to keep all components and double the gauge invariance of the theory. With 2 degrees of freedom,

\[
\# \text{dof} = 2
\]

and 8 first class constraints,

\[
\# \text{fcc} = 8
\]

we thus need

\[
10\text{ canonical pairs},
\]

\[
\# \text{cp} = 10
\]

according to the degree of freedom count \[65\]

\[
2 \times (\# \text{cp}) = 2 \times (\# \text{dof}) + 2 \times (\# \text{fcc}).
\]

This can be done by taking 2 symmetric tensors, 2 vectors and 2 scalars as fundamental canonical variables,

\[
z^A = (\mathcal{H}^a_{mn}, A^a_m, C^a).
\]

### 3.3.2 Change of variables and duality rotations

For \(a = 1, 2\), consider \(h^a_{mn} = (h_{mn}, h_D^a)\) and \(\pi^a_{mn} = (\pi^a_D, \pi_{mn})\) and the definitions

\[
h^a_{mn} = \epsilon_{mpq}\partial^p\mathcal{H}^{aq}_n + \epsilon_{mpq}\partial^p\mathcal{H}^{aq}_m + \partial_m A^a_n + \partial_n A^a_m + \frac{1}{2}(\delta_{mn}\Delta - \partial_m \partial_n)C^a
\]

\[
= 2\Delta^{-1} (\mathcal{D}^{TT} H^a)_{mn} + \partial_m (\Delta^{-1} \epsilon_{mpq}\partial^p\partial^q H^{aq} + A^a_n) + \frac{1}{2}(\delta_{mn}\Delta - \partial_m \partial_n)C^a,
\]

\[
\pi^a_{mn} = \epsilon_{mpq}\epsilon_{nrs}\partial^p\partial^q H^{aq}_{rs} - \partial_m \partial^r H^a_{rn} - \partial_n \partial^r H^a_{rm} - (\delta_{mn}\Delta - \partial_m \partial_n)H^a + \delta_{mn}\partial^p\partial^q H^{aq}_{kl}
\]

\[
= \Delta^{-1} (\mathcal{Q}^{TT} H^a)_{mn} - \partial_m \partial^r H^a_{rn} - \partial_n \partial^r H^a_{rm} - \frac{1}{2}(\delta_{mn}\Delta - \partial_m \partial_n)H^a + \frac{1}{2}(\delta_{mn}\Delta - \partial_m \partial_n)\partial^p\partial^r H^a_{pr}
\]

\[
= -\Delta H^a_{mn}.
\]

The relations for \(h_{mn}[H^1, A^1, C^1]\) and \(\pi_{mn}[H^2]\) are the local change of coordinates from the standard canonical variables of linearized gravity to the new variables. They are invertible and, as usual, the inverse is not local. The relations for \(h^2_{mn} = h^D_{mn}, \pi^1_{mn} = \)
serve to denote convenient combinations of the new variables in terms of which expressions below will simplify. As indicated by the notation, the infinitesimal duality rotations among the fundamental variables are

$$\delta_D H^a_{mn} = \epsilon^{ab} H_{bmn}, \quad \delta_D A^a_m = \epsilon^{ab} A_{bm}, \quad \delta_D C^a = \epsilon^{ab} C_b. \quad (3.3.5)$$

Since $h^a_{mn}, \pi^a_{mn}$ are linear combinations of the fundamental variables, they are rotated in exactly the same way. We can thus consider $h^2_{mn} = h^D_{mn}, \pi^1_{mn} = \pi^D_{mn}$ as the dual spatial metric and the dual extrinsic curvature in the linearized theory.

### 3.3.3 Action principle and locality

The duality invariant local action principle that we propose is of the form

$$S_G[z^A, u^a] = \int d^4x \left( a_A[z] \dot{z}^A - u^a \gamma_a[z] \right) - \int dt H[z], \quad (3.3.6)$$

where $u^a$ denote the 8 Lagrange multiplies and $\gamma_a$ the constraints.

Let us stress here that we use the assumption that the flat space Laplacian $\Delta$ is invertible in order to show equivalence with the usual Hamiltonian or covariant formulation of Pauli-Fierz theory and also to disentangle the canonical structure. The action principle (3.3.6) itself and the associated equations of motion will be local both in space and in time independently of this assumption. The theory itself is not local as a Hamiltonian gauge theory (see e.g. [50], chapter 12) because the Poisson brackets among the canonical variables will not be local.

### 3.3.4 Canonical structure

The explicit expression for the kinetic term is

$$a_A \dot{z}^A = \epsilon_{ab} H^{amn} \left( (\mathcal{P}^{TT} \dot{H}^b)_{mn} + \partial_m \Delta \dot{A}^b_n + \partial_n \Delta \dot{A}^b_m + \frac{1}{2} (\delta^{mn} \Delta - \partial^m \partial^n) \Delta \dot{C}^b \right). \quad (3.3.7)$$

The canonically conjugate pairs are identified by writing the integrated kinetic term as

$$\int d^4x a_A \dot{z}^A = \int d^4x \left( -2 \Delta (\mathcal{O} H^2_{TT})^{mn} \dot{H}^{1TT}_{mn} + 2 \Delta \partial_m H^{2mn} \dot{A}_1^m 
- 2 \Delta \partial_m \dot{H}^{1mn}_L A^2_n \frac{1}{2} \Delta (\Delta H^2_T - \partial_p \partial_q H^2_{pq}) \dot{C}^1 + \frac{1}{2} \Delta (\Delta H^1_T - \partial_p \partial_q H^1_{pq}) \dot{C}^2 \right). \quad (3.3.8)$$

This means that the usual canonical pairs of linearized gravity can be chosen in terms of the new variables as

$$\left( H^{1TT}_{mn}(x), -2 \Delta (\mathcal{O} H^2_{TT})^{kl}(y) \right), \left( C^1(x), -\frac{1}{2} \Delta (\Delta H^2_T - \partial_p \partial_q H^2_{pq}) (y) \right), \left( A^1_m(x), 2 \Delta \partial_v H^{2mn}_L (y) \right). \quad (3.3.9)$$
The 4 additional canonical pairs are
\[
\left( A_m^2(x), -2 \Delta \partial_r H_L^{1m}(y) \right), \left( C^2(x), \frac{1}{2} \Delta (\Delta H_T^1 - \partial_r \partial_q H_T^{1pq})(y) \right).
\] (3.3.10)

In particular, it follows that
\[
\{ h_{mn}^a(x), \pi_{bkl}^a(y) \} = \epsilon_{ab} \frac{1}{2} (\delta_m^k \delta_n^l + \delta_m^l \delta_n^k) \delta^3(x, y).
\] (3.3.11)

### 3.3.5 Gauge structure

The constraints \( \gamma_\alpha \equiv (H_{am}, H_{a\perp}) \) are chosen as
\[
H_{am} = -2 \epsilon_{ab} \partial^a \pi_{mn}^b = 2 \epsilon_{ab} \Delta \partial^a H_{mn}^b, \\
H_{a\perp} = \Delta h_a - \partial_m \partial_n h_{mn}^a = \Delta^2 C_a.
\] (3.3.12) (3.3.13)

The constraints \( H_{1m}, H_{1\perp} \) are those of the standard Hamiltonian formulation of Pauli-Fierz theory expressed in terms of the new variables. The constraints are first class and abelian
\[
\{ \gamma_\alpha, \gamma_\beta \} = 0.
\] (3.3.14)

The new constraints \( \gamma^N = 0 \) are \( H_{2m} = 0 = H_{2\perp} \). They are equivalent to \( \partial^r H_T^1_{rm} = 0 = C^2 \) and are gauge fixed through the conditions \( A_m^2 = 0 = H_T^{1T}_{mn} \). This does not affect \( \pi^{bkl} \), while \( h_{mn}^1 \) is changed by a gauge transformation. The partially gauge fixed theory corresponds to the usual Pauli-Fierz theory in Hamiltonian form as described in section 3.2.1.

In the same way, the original constraints \( H_{1m} = 0 = H_{1\perp} \) are equivalent to \( \partial^r H_T^2_{rm} = 0 = C^1 \) and are gauge fixed through \( A_m^2 = 0 = H_T^{2T}_{mn} \), leading to the completely reduced theory in terms of the 2 transverse-traceless physical degrees of freedom.

If \( \xi^a = (\xi^{am}, \xi^{a\perp}) \) collectively denote the gauge parameters, the gauge symmetries are canonically generated by the smeared constraints,
\[
\delta_x z^A = \{ z^A, \Gamma[\xi] \}, \quad \Gamma[\xi] = \int d^3 x \, \gamma_\alpha \epsilon^a, \quad (3.3.15)
\]
so that
\[
\delta_x H_{mn}^a = -\Delta^{-1} \epsilon_{ab} (\delta_{mn} \Delta - \partial_m \partial_n) \xi^b, \quad \delta_x A_m^a = \xi_m^a, \quad \delta_x C^a = 0, \quad (3.3.16)
\]
which implies in particular
\[
\delta_x h_{mn}^a = \partial_m \xi_n^a + \partial_n \xi_m^a, \quad \delta_x \pi_{mn}^a = \epsilon_{ab} (\delta_{mn} \Delta - \partial_m \partial_n) \xi_b^a. \quad (3.3.17)
\]

Note that a way to get local gauge transformations for the fundamental variables is to multiply the constraints by \( \Delta \), which is allowed when the flat space Laplacian is invertible. This amounts to introducing suitable potentials for the gauge parameters and Lagrange multipliers.
3.3.6 Duality generator

The canonical generator for the infinitesimal duality rotations (3.3.5) is

\[
D = \int d^3x \left( - (\mathcal{P} TT H^a)_{mn} H^{mn}_a + 2\Delta \partial_r H^{rn}_a A^a_m \\
- \frac{1}{2} \Delta (\Delta H^a - \partial^m \partial^n H^{mn}_a) C^a \right). \tag{3.3.18}
\]

On the constraint surface, it reduces to

\[
D \approx - \int d^3x \mathcal{P} TT (H^a)_{mn} H^{mn}_a, \tag{3.3.19}
\]

which is also the duality generator of the non-extended double potential formalism (see section 3.2.3).

This generator is only weakly gauge invariant,

\[
\{ \mathcal{H}_a, D \} = \epsilon_{ab} \mathcal{H}_b^a, \quad \{ \mathcal{H}_a, D \} = \epsilon_{ab} \mathcal{H}_b^a. \tag{3.3.20}
\]

3.3.7 Hamiltonian

In terms of the new variables (3.3.3)-(3.3.4), the Pauli-Fierz Hamiltonian reads

\[
H_{PF} = \int d^3x \left( H^{amn} \Delta^2 H_{amn}^{TT} - 2\Delta \partial_r H^{rn}_a \partial_s H^{sm}_a - \\
\partial^r \partial^s H^{rs}_a \Delta H^2 - \frac{1}{2} (\partial^r \partial^s H^{rs}_a)^2 + \frac{1}{8} \Delta C^1 \Delta^2 C^1 \right), \tag{3.3.21}
\]

where one can use (3.2.10) to expand the first term as a local functional of \( H^a_{mn} \).

The local Hamiltonian \( H = \int d^3x h \) of the manifestly duality invariant action principle (3.3.6) is

\[
H = \int d^3x \left( H^{amn} \Delta^2 H_{amn}^{TT} - 2\Delta \partial_r H^{rn}_a \partial_s H^{sm}_a - \\
- \partial^r \partial^s H^{rs}_a \Delta H_a - \frac{1}{2} \partial^r \partial^s H^{rs}_a \partial_k \partial_l H^{kl}_a + \frac{1}{8} \Delta C^a \Delta^2 C^a \right), \tag{3.3.22}
\]

which simplifies to

\[
H = \int d^3x \left( \Delta H^a_{mn} \Delta H^{mn}_a - \frac{1}{2} \Delta H^a \Delta H_a + \frac{1}{8} \Delta C^a \Delta^2 C^a \right). \tag{3.3.23}
\]

It is equivalent to the Pauli-Fierz Hamiltonian since it reduces to the latter when the additional constraints \( \partial^r H^1_{rn} = 0 = C^2 \) hold. Note that the terms proportional to \( \partial^r H^{rn}_a \) and \( C^a \) may be dropped since they vanish on the constraint surface, \( H \approx \int d^3x H^{amn} \Delta^2 H_{amn}^{TT} \).
The Hamiltonian is gauge invariant on the constraint surface,

\[ \{H, \Gamma[\xi]\} = \int d^3 x \, \mathcal{H}_m^a \partial^m \xi_a^\perp. \] (3.3.24)

In order for the action (3.3.6) to be gauge invariant, it follows from (3.3.24) that the Lagrange multipliers \( u^\alpha \) need to transform as

\[ \delta \xi u^{am} = \dot{\xi}^{am} - \partial^m \xi^a, \quad \delta \xi u^a = \dot{\xi}^a. \] (3.3.25)

### 3.3.8 Poincaré generators

The same argumentation we used in section (2.3.4) to show that the symmetry generators of the usual Hamiltonian action of electromagnetism are also symmetry generators of the extended double potential formalism is still valid. The symmetry generators of Pauli-Fierz are symmetry generators of the extended theory. It follows that the Poincaré generators \( Q_G(\omega, a) \) of Pauli-Fierz theory as described in section, when expressed in terms of the new variables, are representatives for the Poincaré generators of the extended theory.

As before, the generators obtained that way are not invariant under the duality. We now want to show that one can find representatives for the Poincaré generators that are duality invariant,

\[ \{Q_D^G(\omega, a), D\} = 0, \] (3.3.26)

by adding terms proportional to the new constraints.

The first step in the proof consists in showing that the reduced phase space generators, i.e., the generators \( Q_G(\omega, a) \) for which all variables except for the physical \( H^{TT} \) have been set to zero, are duality invariant. All other contributions to \( Q_G(\omega, a) \) are then shown to be proportional to the constraints of Pauli-Fierz theory. Both these steps follow from straightforward but slightly tedious computations. For the generators of rotations and boosts for instance the computation is more involved because the explicit \( x^i \) dependence has to be taken into account when performing integrations by parts.

In terms of the new variables, the terms proportional to the constraints are bilinear in \( (h^1, A^2), (\pi^2, A^2), (h^1, C^1) \) and \( (\pi^2, C^1) \). The duality invariant generators \( Q_D^G(\omega, a) \) are then obtained by adding the same terms with the substitution \( h^1 \rightarrow h^2, A^2 \rightarrow -A^1, \pi^2 \rightarrow -\pi^1 \) and \( C^1 \rightarrow C^2 \), while keeping unchanged the terms involving only the physical variables \( H^{TT} \).

As a consequence, the duality invariant Poincaré transformations of \( h^1, \pi^2 \) are unchanged on the extended constraint surface. They are given by (3.2.65)-(3.2.66) where \( \xi^\perp = -\omega^0 x^\nu + a^0 \) and \( \xi^i = -\omega^i x^\nu + a^i \). Because of (3.3.26), those for of \( h^2, -\pi^1 \) are obtained, on the constraint surface, by applying a duality rotation to the right hand-sides of (3.2.65)-(3.2.66).
3.3.9 Interacting variational principle

We define

\[ h_a^{m} = r_a^{m} = h_{0m} = -2n^a, \]  
(3.3.27)

and consider the action

\[ S_T[z^A, \alpha^A; T^{\mu\nu}] = \frac{1}{16\pi G} S_G + S^J, \]  
(3.3.28)

with \( S_G \) given in (3.3.6) and the gauge invariant interaction term

\[ S^J = \int d^4x \frac{1}{2} h_a^{\mu\nu} T_a^{\mu\nu}, \quad \partial_\mu T_a^{\mu\nu} = 0, \]  
(3.3.29)

where \( T_a^{\mu\nu} \equiv (T^{\mu\nu}, \Theta^{\mu\nu}) \) are external, conserved electric and magnetic energy-momentum tensors.

3.3.10 Equations of motion

Our goal in this section is to show that our interacting variational principle (3.3.28) generates the duality invariant equations of motion (3.1.23)-(3.1.24) introduced in section 3.1. To do so, we need to give the expression of the full Riemann tensor and its dual in terms of our fields \( (H_a^m, \alpha_a^m, C^a) \). We will start by introducing a new decomposition of the Riemann tensor and its dual in electric and magnetic part. This decomposition is a generalization of the decomposition of a Weyl tensor in an electric and magnetic part. After that, we will derive the expression of this new parametrisation in term of our canonical variables.

We will use a duality invariant notation: \( R_a^{\mu\nu\rho\sigma} = (R^{\mu\nu\rho\sigma}, S^{\mu\nu\rho\sigma}) \). The Ricci tensors and Einstein tensors are defined as

\[ R_a^{\mu\nu} = R^{\alpha\mu\nu\rho} \quad G_a^{\mu\nu} = G^{\alpha\mu\nu\rho} = R^{\alpha}_{\mu\nu}\rho - \frac{1}{2} \eta_{\mu\nu} R^a. \]  
(3.3.30)

A general Riemann tensor \( R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho} \) has 36 independent components.

The equations of motion (3.1.23) and (3.1.24) becomes

\[ G_a^{\mu\nu} = 8\pi G T_a^{\mu\nu} \iff R_a^{\mu\nu\rho\sigma} + R_a^{\mu\nu\sigma\rho} + R_a^{\mu\rho\nu\sigma} = 8\pi G e^{ab} \epsilon_{\delta\nu\rho\sigma} T_b^{\delta\mu}. \]  
(3.3.31)

They imply in particular that, on-shell, the tensors \( R_a^{\mu\nu}, G_a^{\mu\nu} \) are symmetric [9]. Furthermore, the Bianchi “identities” (3.1.25) and (3.1.26) read

\[ \partial_\lambda R_a^{\gamma\delta\sigma\rho} + \partial_\gamma R_a^{\lambda\delta\sigma\rho} + \partial_\delta R_a^{\gamma\lambda\sigma\rho} = 8\pi G e^{ab} \epsilon_{\epsilon\alpha\beta\rho}(\partial_\gamma T_b^{\epsilon\delta\lambda} - \partial_\lambda T_b^{\epsilon\delta\gamma}) \]

\[ \iff \partial_\mu R_a^{\gamma\delta\sigma\mu} = 8\pi G \left( \partial^\sigma T_a^{\gamma\delta} - \partial^\gamma T_a^{\delta\sigma} \right), \]  
(3.3.32)

while the contracted Bianchi identities are

\[ \partial_\nu G_a^{\mu\nu} = 0. \]  
(3.3.33)
Let
\[ K^\lambda_{\mu\nu\rho\sigma} \left[ R^a_{\lambda\tau} \right] = \frac{1}{2} \left[ \eta_{\mu\rho} R^a_{\sigma\rho} + \eta_{\mu\sigma} R^a_{\rho\rho} - \eta_{\mu\rho} R^a_{\sigma\rho} - \eta_{\mu\sigma} R^a_{\rho\rho} \right] - \frac{R^a}{6} \left[ \eta_{\mu\rho} \eta_{\nu\sigma} - \eta_{\mu\sigma} \eta_{\nu\rho} \right]. \] (3.3.34)

Defining
\[ \tilde{R}^a_{\mu\nu\rho\sigma} = R^a_{\mu\nu\rho\sigma} - \frac{1}{2} \epsilon^{ab} \epsilon_{\rho\alpha\beta} K^\lambda_{\mu\nu} \alpha^\beta \left[ R^a_{b\lambda\tau} \right], \] (3.3.35)
the tensor \( \tilde{R}^a_{\mu\nu\rho\sigma} \) is skew in the first and last pairs of indices, satisfies the cyclic identity because \( \epsilon^{\gamma\nu\rho\sigma} R^a_{\mu\nu\rho\sigma} = \epsilon^{\gamma\nu\rho\sigma} \frac{1}{2} \epsilon^{ab} \epsilon_{\rho\alpha\beta} K^\lambda_{\mu\nu} \alpha^\beta \left[ R^a_{b\lambda\tau} \right] \) and, as a consequence, is also symmetric in the exchange of the first and last pair of indices, \( \tilde{R}^a_{\mu\nu\rho\sigma} = \tilde{R}^a_{\rho\mu\sigma\nu} \). The associated Ricci tensors \( \tilde{R}^a_{\nu\sigma} = \tilde{R}^a_{\nu\sigma} \) and \( \tilde{R}^a_{\nu\sigma} = \frac{1}{2} \epsilon^{ab} \epsilon_{\nu\sigma\mu\alpha} R^b_{\mu\alpha} \). The Weyl tensors are then defined as usual in terms of \( \tilde{R}^a_{\mu\nu\rho\sigma} \),
\[ C^a_{\mu\nu\rho\sigma} = \tilde{R}^a_{\mu\nu\rho\sigma} - K^\lambda_{\mu\nu\rho\sigma} \left[ R^a_{\lambda\tau} \right], \] (3.3.36)
and satisfy all standard symmetry properties: skew-symmetry in the first and last pairs of indices, tracelessness (because \( \tilde{R}^a_{\nu\sigma} = K^\lambda_{\mu\nu\rho\sigma} \left[ R^a_{\lambda\tau} \right] \)), the cyclic identity (because \( \epsilon^{\gamma\nu\rho\sigma} K^\lambda_{\mu\nu\rho\sigma} \left[ R^a_{\lambda\tau} \right] = 0 \)), which implies also symmetry in the exchange of the first and last pair of indices,
\[ C^a_{\mu\nu\rho\sigma} = -C^a_{\mu\nu\rho\sigma}, \] (3.3.37)
\[ C^a_{\mu\nu\rho\sigma} = 0, \quad \epsilon^{\gamma\nu\rho\sigma} C^a_{\mu\nu\rho\sigma} = 0, \quad C^a_{\mu\nu\rho\sigma} = C^a_{\rho\mu\nu\sigma}. \] (3.3.38)
As before, the 10 independent components of the Weyl tensor can be parametrized by the electric and magnetic components \( E^a_{mn} = (E_{mn}, B_{mn}) \), symmetric and traceless tensors defined by
\[ E^a_{mn} = C^a_{0m}\delta_{0n} = \frac{1}{2} \epsilon_{njk} \epsilon^{ab} C^a_{b0}\delta_{0m}. \] (3.3.39)

Putting all definitions together, the relation between the Riemann and Weyl tensors is
\[ R^a_{\mu\nu\rho\sigma} = C^a_{\mu\nu\rho\sigma} + K^\lambda_{\mu\nu\rho\sigma} \left[ R^a_{\lambda\tau} \right] + \frac{1}{2} \epsilon^{ab} \epsilon_{\rho\alpha\beta} K^\lambda_{\mu\nu} \alpha^\beta \left[ R^a_{b\lambda\tau} \right], \] (3.3.40)
\[ = C^a_{\mu\nu\rho\sigma} + K^\lambda_{\mu\nu\rho\sigma} \left[ R^a_{\lambda\tau} \right] + \frac{1}{2} \epsilon^{ab} \epsilon_{\rho\alpha\beta} K^\lambda_{\mu\nu} \alpha^\beta \left[ R^a_{b\lambda\tau} \right]. \] (3.3.41)

In particular, it follows that the 36 independent components of the Riemann tensor \( R^1_{\mu\nu\rho\sigma} \) can be parameterized by the 10 independent components of the Weyl tensor \( C^1_{\mu\nu\rho\sigma} \), the 16 components of the Ricci tensor \( R^1_{\lambda\tau} \), and the 10 components of \( R^2_{\lambda\tau} \).

If we define
\[ E^a_{mn} = R^a_{0(m)\delta_{(0n)}}, \quad F^a_{mn} = \frac{1}{2} \epsilon^{mjk} R^a_{0j(0k)}, \quad R^a_{mn} = R^a_{(mn)} + E^a_{mn} \] (3.3.42)
the parameterization consisting in choosing the symmetric tensors \( E^a_{mn}, R^a_{mn} \) (24 components), \( F^a_{mn} \) (6 components), and \( R^1_{\mu\nu} (= R^2_{\mu\nu}) \) (6 components) is more useful for our purpose. That all tensors can be reconstructed from these variables follows from the fact that
\[ R^a_{0m} = -2 \epsilon^{ab} F^a_{0m}, \quad R^a_{0\delta_{(0n)}} = E^a, \quad R^a_{(mn)} = R^a_{mn} - E^a_{mn}. \] (3.3.43)

This means that the symmetric part of the Ricci tensors can be reconstructed from the variables. Since the antisymmetric parts belong to the variables, so can the complete Ricci tensors \( R^a_{\mu\nu} \). Using now (3.3.40) and definitions (3.3.39), (3.3.42), (3.3.44), we find
\[ E^a_{mn} = \frac{1}{2} \left( E^a_{mn} + R^a_{mn} \right) - \frac{\delta_{mn}}{6} \left( E^a + R^a \right). \] (3.3.44)
It follows that the Weyl tensors and then, using again (3.3.40), the Riemann tensors can be reconstructed.

In terms of the new parameterization, the equations of motion (3.3.31) read \( R^a_{(\mu i)} = 0 \) and

\[
-2\epsilon^{ab} F_{bm} = 8\pi GT_{0m},
\]

\[
\frac{1}{2} R^a = 8\pi GT_{00}^a,
\]

\[
R^a_{mn} - \mathcal{E}^a_{mn} + \delta_{mn}(\mathcal{E}^a - \frac{1}{2} R^a) = 8\pi GT_{mn}^a.
\]

Using these equations of motion, the Bianchi identities (3.3.32) are equivalent to

\[
\partial^b(\epsilon_{ikm} F^{am} + \mathcal{R}^a_{ik}) = \frac{1}{2} \partial_i R^a,
\]

\[
2\epsilon^{ab} \partial_i F_{bn} = \partial^a (\epsilon^{a}_{mn} + \epsilon_{mnk} F^{ak}) - \partial_m \mathcal{E}^a,
\]

\[
\partial_0 \mathcal{R}^a_{ik} = \frac{1}{2} \epsilon^{ab} [\epsilon_{ijkl} \partial^i \mathcal{E}^{jk} + \epsilon_{ijkl} \partial^j \mathcal{E}^{ik}] - 2\delta_{ik} \partial_j F^b - \partial_i F_{bk} - \partial_k F_{bi} \iff
\]

\[
\epsilon^{ab} \partial_0 (\mathcal{R}^a_{ik} - \frac{1}{2} \delta^{ik} R_b) = -\frac{1}{2} [\epsilon_{klm} \partial_j \mathcal{E}^{ai} + \epsilon_{lm} \partial \mathcal{E}^{a_k} + 2\epsilon_{ik} \partial^j F^a - \partial^i F^{ak} - \partial^k F^{aj}].
\]

We will now express the Riemann tensor in terms of the canonical variables in such a way that the covariant equations (3.3.45)-(3.3.50) coincide with the Hamiltonian equations deriving from (3.3.28).

From the constraints with sources, we find

\[
\mathcal{R}^a = \partial^m \partial^a h_m^a - \Delta h^a = -\Delta^2 C^a,
\]

\[
F^a_m = \frac{1}{2} \Delta \partial^m H_m^a.
\]

Assuming \( \Delta \) to be invertible, which we do in the rest of this section, \( \mathcal{R}^a \) and \( C^a \), respectively \( F^a_m \) and \( \partial^m H_m^a \) determine each other. By taking the divergence, the Bianchi identity (3.3.48) implies that

\[
\partial^m \partial^a \mathcal{R}^a_{mn} = -\frac{1}{2} \Delta^3 C^a.
\]

Similarly, the Bianchi identity (3.3.49) implies in particular that \( \Delta \mathcal{E}^a - \partial^m \partial^a \mathcal{E}^a_{mn} = \epsilon^{ab} \partial_0 \Delta \partial^m \partial^a H_{bmn} \). When combined with (3.3.47), the equations of motion following from variation with respect to \( C^a \) read

\[
\frac{1}{2} \Delta^3 C^a + \epsilon^{ab} \Delta \partial_0 (\Delta H_b - \partial^m \partial^a H_{bmn}) + 2\Delta^2 n^a = \Delta \mathcal{E}^a - \partial^m \partial^a (\mathcal{R}^a_{mn} - \mathcal{E}^a_{mn}).
\]

When combined with the previous relations, they imply that

\[
\mathcal{E}^a = -\frac{1}{2} \epsilon^{ab} \partial_0 \Delta H_b + \Delta n^a,
\]

\[
\partial^m \partial^a \mathcal{E}^a_{mn} = -\frac{1}{2} \epsilon^{ab} \partial_0 \Delta (\Delta H_b - 2\partial^m \partial^a H_{bmn}) + \Delta^2 n^a.
\]

The rest of the Bianchi identities (3.3.48), (3.3.49) are taken into account by applying a curl. This gives \( \epsilon^{st} \partial_r \partial^c R^a_{ik} = \frac{1}{2} \Delta (\Delta \partial^k H_{a}^{ar} - \partial^r \partial^a \partial^k H_{a}^{mr}) \) and \( \epsilon^{st} \partial_r \partial^c \mathcal{E}^a_{ik} = \epsilon^{st} \epsilon^{ab} \partial_0 \partial_r \mathcal{R}^a_{ki} - \partial^r \partial^k F^a_k + \Delta F^{ar} \).

Yet another curl gives \( \partial_0 \partial^m \partial^a R_m^a - \Delta \partial^a \mathcal{R}^a_{kn} = \frac{1}{2} \epsilon_{klr} \partial^d \Delta \partial^a H_{a}^{nr} \) and \( \partial_0 \partial^m \partial^a \mathcal{E}^a_{kn} - \Delta \partial^a \mathcal{E}^a_{kn} = 2\epsilon^{ab} \partial_0 (\partial_0 \partial^m F_{bn} - \Delta F_{bn}) + \epsilon_{klr} \partial^d \Delta F^{ar} \). Using the previous relations we then get

\[
\partial^m \mathcal{R}^a_{kn} = -\frac{1}{2} \partial_0 \Delta^2 C^a - \frac{1}{2} \epsilon_{klr} \partial^d \Delta \partial^a H_{a}^{nr},
\]

\[
\partial^a \mathcal{E}^a_{kn} = \epsilon^{ab} \partial_0 \Delta (-\frac{1}{2} \partial_0 H_b + \partial^a H_{bkn}) + \partial_0 \Delta n^a - \frac{1}{2} \epsilon_{klr} \partial^d \Delta \partial^a H_{a}^{nr}.
\]
The equations of motion following from variation with respect to $A_m^a$ are then identically satisfied.

Defining $D_{mn}^a = \mathcal{R}_{mn}^a - \mathcal{E}_{mn}^a$ and using definition (3.2.12) of $\mathcal{P}_TT$ combined with (3.3.47), the equations of motion following from variation with respect to $H_{mn}^a$ read

$$
\epsilon_{ab}\partial_0\left[2(\mathcal{P}_TT H^b)_{mn} + \partial_m \Delta A_n^a + \partial_n \Delta A_m^b + \frac{1}{2}(\delta_{mn} \Delta - \partial_m \partial_n)C^b\right] - \epsilon_{ab}\Delta(\partial_m n_n^b + \partial_m n_n^b) -
-2 \Delta^2 H_{mn}^a + \delta_{mn} \Delta^2 H^a = -\epsilon_{mpq}\partial^p D_{am}^q - \epsilon_{mpq} \partial^p D_{qm}^a. \tag{3.3.53}
$$

Taking into account definition (3.2.12) and previous relations, we can extract

$$
- \Delta^{-1}(\mathcal{P}_TT D_{mn}^a)_{mn} = \frac{1}{2} \epsilon_{ab}\partial_0\left[2(\mathcal{P}_TT H^b)_{mn} - \epsilon_{mpq}\partial_n \partial^p \partial^q H_{r}^{bq} + \partial_m \Delta A_n^b + \partial_n \Delta A_m^b + \frac{1}{2}(\delta_{mn} \Delta - \partial_m \partial_n)C^b\right] - \epsilon_{ab}\Delta(\partial_m n_n^b + \partial_m n_n^b) -
-2 \Delta^2 H_{mn}^a + \delta_{mn} \Delta^2 H^a. \tag{3.3.54}
$$

In order to extract the remaining information from (3.3.53), we first apply $\delta_{mn} \Delta - \partial_m \partial^n$ to get

$$
\epsilon_{ab}\partial_0 \Delta C^b + 2 \partial^m \partial^a H_{amn} = 0, \tag{3.3.55}
$$

and then a divergence $\partial^m$ giving

$$
\epsilon_{ab}\partial_0(\Delta A_n^b - \epsilon_{mpq}\partial^p \partial^q H_{r}^{bq}) = \epsilon_{ab}\Delta n_n^b + 2 \Delta \partial^k H_{an}^k - \frac{1}{2} \partial_0 \Delta H^a - \partial_0 \partial^k \partial^l H_{kl}. \tag{3.3.56}
$$

We can now inject the latter relations into (3.3.53) and use (3.2.13), (3.2.6) to get

$$
D_{mn}^{TT} = -\epsilon_{ab}\partial_0 \Delta H_{mn}^{TT} - (\mathcal{P}_TT H^a)_{mn}, \tag{3.3.57}
$$

$$
D_{mn}^a = -\epsilon_{ab}\partial_0 \Delta \left[H_{amn} - \frac{1}{2} \delta_{mn} H^a\right] - (\mathcal{P}_TT H^a)_{mn} = -\epsilon_{ab}\Delta(\partial_m n_n^b + \partial_m n_n^b) -
-\frac{1}{4} \delta_{mn} \Delta + \partial_m \partial_n) \Delta C^a. \tag{3.3.58}
$$

Injecting into the second form of the last Bianchi identity (3.3.50) and using previous relations gives

$$
\epsilon_{ab}\partial_0 \mathcal{R}_{ij}^b = - (\mathcal{O} \mathcal{R}_a)_{ij} + \Delta^2 H_{aij} + \frac{1}{4} \Delta \partial_i \partial^k H_{akj} + \frac{1}{4} \Delta \partial_j \partial^k H_{aki} - \frac{1}{2} \partial_i \partial_j \partial^k \partial^l H_{akl}
- \frac{1}{2} \epsilon_{ab}\partial_0\left[\epsilon_{iqn} \partial^q \Delta H_{ij}^{ln} + \epsilon_{jqn} \partial^q \Delta H_{ij}^{ln} + \frac{1}{2} (\delta_{ij} \Delta + \partial_i \partial_j) \Delta C^b\right]. \tag{3.3.59}
$$

Identifying the terms with time derivatives gives

$$
\mathcal{R}_{ij}^a = -\frac{1}{2} \left[\epsilon_{iqn} \partial^q \Delta H_{ij}^{an} + \epsilon_{jqn} \partial^q \Delta H_{ij}^{an} + \frac{1}{2} (\delta_{ij} \Delta + \partial_i \partial_j) \Delta C^a\right]
= \frac{1}{2} \left[\partial_i \partial^k h_{kj}^i + \partial_j \partial^k h_{ki}^j - \partial_i \partial_j h^a - \Delta h_{ij}^a - \epsilon_{ijkl} \partial^k \partial^p \partial^q \partial^r H_{ij}^{kl} - \epsilon_{ijkl} \partial^k \partial^p \partial^q \partial^r H_{ij}^{kl}\right]. \tag{3.3.60}
$$

The terms without time derivatives in (3.3.59) then cancel identically. Together with (3.3.58) this then finally gives

$$
\mathcal{E}_{ti}^a = \epsilon_{ab}\partial_0\Delta \left[H_{tij}^a - \frac{1}{2} \delta_{ij} H^a\right] + \partial^i \partial^j n_a - \frac{1}{2} \epsilon_{ijkl} \partial_k \partial^j \partial^p H_{alp} - \frac{1}{2} \epsilon_{ijkl} \partial_k \partial^j \partial^p H_{alp}
= -\epsilon_{ab}\partial_0 \left[\partial_i \partial^j h_{ij}^a + \partial^i \partial^j n_a - \frac{1}{2} \epsilon_{ijkl} \partial_k \partial^j \partial^p H_{alp} - \frac{1}{2} \epsilon_{ijkl} \partial_k \partial^j \partial^p H_{alp}\right]. \tag{3.3.61}
$$
3.3.11 Linearized Taub-NUT solution

We start by considering the sources corresponding to a point-particle gravitational dyon with electric mass $M$ and magnetic mass $N$ at rest at the origin of the coordinate system, for which

$$ T_\mu^\nu(x) = \delta_0^\mu \delta_0^\nu M_a \delta^{(3)}(x), \quad M_a = (M, N). \quad (3.3.62) $$

In this case, only the constraints (3.3.13) are affected by the interaction and become

$$ \mathcal{H}_{a\perp} = -16\pi G M_a \delta^{(3)}(x). \quad (3.3.63) $$

They are solved by

$$ \Delta C^a = GM^a \left( \frac{4}{r} \right), \quad (3.3.64) $$

where $r = \sqrt{x^i x_i}$. It is then straightforward to check that all equations of motions are solved by

$$ C^a = GM^a (2r), \quad n^a = GM^a \left( -\frac{1}{r} \right), \quad A_m^a = n^{am} = H_{mn}^a = 0, $$

$$ h_m^{an} = GM^a \left( \delta_m^{an} + \frac{x_m^a x_n}{r^3} \right), \quad \pi_m^{an} = 0. \quad (3.3.65) $$

The usual Schwarzschild form is obtained by adding a pure gauge solution with parameter $\xi^{am} = GM^a \left( -\frac{1}{2} \frac{x_m^a}{r} \right), \xi^{a\perp} = 0$. The solution then reads

$$ C^a = GM^a (2r), \quad n^a = GM^a \left( -\frac{1}{r} \right), \quad A_m^a = GM^a \left( -\frac{1}{2} \frac{x_m}{r} \right), \quad n^{am} = H_{mn}^a = 0, $$

$$ h_m^{an} = GM^a \left( \frac{2x_m x_n}{r^3} \right), \quad \pi_m^{an} = 0. \quad (3.3.66) $$

To show that this solution describes the linearized Taub-NUT solution, we need to compute its Riemann tensor using the relations given in the previous section. Following for instance [66] section A.1,2 and using a regularization in Fourier space, we find

$$ R_i^{aj} = GM^a \left[ \frac{16\pi}{3} \delta_i^j \delta^3(x) + \frac{\eta(r)}{r^3} \left( \delta_i^j - \frac{3x_i x_j}{r^2} \right) \right], \quad (3.3.67) $$

$$ E_i^{aj} = GM^a \left[ \frac{4\pi}{3} \delta_i^j \delta^3(x) + \frac{\eta(r)}{r^3} \left( \delta_i^j - \frac{3x_i x_j}{r^2} \right) \right], \quad (3.3.68) $$

where $\eta(r)$ is a regularizing function that suppresses the divergence at the origin and is 1 away from the origin. We then find

$$ R_{00}^a = GM^a 4\pi \delta^3(x), \quad R_i^{aj} = GM^a 4\pi \delta_i^j \delta^3(x), \quad (3.3.69) $$

$$ E_i^{aj} = GM^a \frac{\eta(x)}{r^3} \left( \delta_i^j - \frac{3x_i x_j}{r^2} \right), \quad (3.3.70) $$
and all other components of $R^a_{\mu\nu}$ vanishing. For the Riemann tensor, this implies

$$R^a_{0i0j} = GM^a\left[\frac{4\pi}{3}\delta_{ij}\delta^3(x) + \frac{\eta(x)}{r^3}(\delta_{ij} - \frac{3x_i x_j}{r^2})\right],$$

$$R^a_{0ijk} = -\epsilon^{ab}\epsilon_{jk}GM^a\left[\frac{4\pi}{3}\delta_{il}\delta^3(x) + \frac{\eta(x)}{r^3}(\delta_{il} - \frac{3x_i x_l}{r^2})\right],$$

with all other components obtained through the on-shell symmetries of the Riemann tensor. This is the usual Riemann tensor for the linearized Taub-NUT solution.

As in the electromagnetism case, this formalism resolves the string singularity of the linearized Taub-NUT solution present in the standard Pauli-Fierz formulation. In spherical coordinates, the latter can for instance be described by

$$h_{rr} = \frac{2GM}{r} = h_{00}, \quad h_{0\phi} = -2N(1 - \cos \theta),$$

and all other components vanishing, with a string-singularity along the negative $z$-axis.

### 3.3.12 Electric and magnetic energy-momentum and angular momentum surface charges

As for the spin 1 case, the analysis of appendix is not directly applicable since we do not have Darboux coordinates and the Poisson brackets of the fundamental variables are non-local. Another problem is that the gauge transformations (3.3.16) do not allow for non-trivial solutions to $\delta_\varepsilon A^A = 0$. As before, we will use the idea of the appendix to derive expressions for the surface charges. We still have to keep the sources explicitly throughout the argument because of the presence of $\Delta^{-1}$.

In the presence of the sources, the constraints $\gamma^{a}_{\alpha} = (H^J_{am}, H^J_{a\perp})$ are determined

$$H^J_{am} = H_{am} - (16\pi G)T^0_{am}, \quad H^J_{a\perp} = H_{a\perp} - (16\pi G)T^0_{a\perp},$$

Instead of (A.1.11), we can write

$$\gamma^{a}_{\alpha} \varepsilon^a = \left(\partial^m \xi^{am} + \partial^n \xi^{an}\right)\epsilon_{ab}\pi_{mn} + \left(\delta^{mn}\Delta - \delta^{m\Delta}\delta^{n}\right)\xi^{a\perp}h_{amn} - \partial_i \tilde{h}^j_{\varepsilon}[z]$$

$$- (16\pi G)\left(T^0_{am}\xi^{am} + T^0_{a\perp}\xi^{a\perp}\right),$$

where

$$\tilde{h}^j_{\varepsilon}[z] = 2\xi^{a\perp}(x)\epsilon_{ab}\pi^{bmi} - \xi^{a\perp}(\delta^{mn}\partial^j - \delta^{mj}\partial^n)h_{amn} + h_{amn}(\delta^{mn}\partial^j - \delta^{nj}\partial^m)\xi^{a\perp}.$$

Consider now gauge parameters $\epsilon^{a}_{\alpha}(x)$ satisfying the conditions

$$\left\{ \begin{array}{l}
\partial^m \xi^{am} + \partial^n \xi^{an} = 0 = \partial^m \xi^{a\perp}, \\
(\delta^{mn}\Delta - \delta^{m\Delta}\delta^{n})\xi^{a\perp} = 0 = \partial^m \xi^{a\perp},
\end{array} \right.$$
The general solution to conditions (3.3.77) can be written as

$$\xi^a_{\mu s} = -\omega^{a}_{\mu [\nu} x^{\nu} + a_{\mu}^a,$$  

(3.3.78)

for some constants $a_{\mu}^a, \omega^{a}_{\mu [\nu]} = -\omega^{a}_{[\mu \nu]}$. It follows in particular that the surface charges

$$Q_{[s]}[z] = \frac{1}{16\pi G} \oint_S d^3 x_i \tilde{k}_{[s]}^i [z],$$  

(3.3.79)

do not depend on the homology class of $S$ outside of sources.

Assuming $\Delta$ invertible, the equations of motion associated to $\mathcal{L}_T = \frac{1}{16\pi G} \mathcal{L}_H + \mathcal{L}^J$ imply in particular that

$$\partial_0 h_{mn}^a = \partial_m n_n^a + \partial_n n_m^a - 2\epsilon^{ab} \Delta H_{bmn} + \epsilon^{ab} \delta_{mn} \Delta H_b + (16\pi G)\epsilon^{ab} (\Delta^{-1} (\mathcal{O}T)_b)_{mn} + \frac{1}{2} \Delta^{-2} \partial_m \epsilon_{npq} \partial^p \partial_k T_b^{kq} + \frac{1}{2} \Delta^{-2} \partial_n \epsilon_{mpq} \partial^p \partial_k T_b^{kq}),$$

(3.3.80)

$$\epsilon_{ab} \partial_0 \pi_{mn}^b = (\mathcal{P}^{TT} H_a)_{mn} + (8\pi G) T_{anm} - \frac{1}{2} (\delta^{mn} \Delta - \partial^m \partial^n)(2n_a + \frac{1}{2} \Delta C_a).$$  

(3.3.81)

By direct computation using the equations of motion, one then finds

$$\partial_0 \tilde{k}_{[s]}^i [z] = (16\pi G)(\xi^a_{\mu s} T_{a}^i) - \partial_j k_{[s]}^{[ij]} [z, u_s],$$  

(3.3.82)

with

$$k_{[s]}^{[ij]} [z, u] = \left(2n_a \partial_i \xi^a_{s \perp} + \xi^a_{s \perp} \partial^i n_s^a + \xi^a_{s} \partial^j (2n_a + \frac{1}{2} \Delta C_a) + \epsilon^{amn} \epsilon_{mpq} \partial_p \partial^j H_{aq} + \omega^{aj} \partial^k H_{ak} + \omega^{aj} \partial^i H_{a} + 2\omega^{ak} \partial^j H_{ak} + 16\pi G \epsilon^{ab} \epsilon_{mpq} \partial^m \partial_2 T_b^{pq} \partial_n \xi^a_{s \perp} + 8\pi G \epsilon^{ab} \epsilon_{mpq} \partial_m \partial_n \xi_{as}^a - (i \leftrightarrow j) \right)$$

$$+ \epsilon^{ijk} \omega^a_{k}(2n_a + \frac{1}{2} \Delta C_a) - \epsilon^{amn} (\Delta H_{amk} - \partial_m \partial^r H_{ark}) - 16\pi G \epsilon^{ab} \Delta^{-2} \partial^r T_{brk} \xi^a_{as} + 8\pi G \epsilon^{ab} (\Delta^{-1} T_{bk}^{am} + \Delta^{-2} \partial^m \partial^r T_{brk}) \partial_m \xi^a_{as} \right),$$  

(3.3.83)

where $\omega^{a}_{mn} = \omega^{ak} \epsilon_{kmn}$. The surfaces charges (3.3.79) are thus also time-independent outside of sources.

Finally, the surface charges are gauge invariant,

$$\tilde{k}_{[s]}^i [\delta_{\eta} z] = \partial_j r_{[s, \eta]}^{[ij]},$$  

(3.3.84)

$$r_{[s, \eta]}^{[ij]} = \left(2\epsilon^{aij} \partial^i \eta^a_{s \perp} + \eta^a_{s} \partial^j \xi^a_{s \perp} + \xi^a_{s} \partial^i \eta^a_{s} - (i \leftrightarrow j) \right) - 2\epsilon^{ijk} \omega^a_{k} \eta^a_{s \perp}.$$  

(3.3.85)

Defining

$$Q_{[s]}[z] = \frac{1}{2} \omega^{a}_{[\mu \nu} J^a_{\nu} - a^{a}_{\mu} P^a_{\mu},$$  

(3.3.86)
we get for the individual generators

\begin{align}
(16\pi G) P_a^\perp &= - \oint_{S^\infty} d^3 x_m \partial^m \Delta C_a = \oint_{S^\infty} d^3 x_m (\partial_t h_a^{mn} - \partial^m h_a) , \\
(16\pi G) P_a^m &= 2 \oint_{S^\infty} d^3 x_m \epsilon_{ab} \Delta H^{bmn} = -2 \oint_{S^\infty} d^3 x_m \epsilon_{ab} \pi^{bmn} , \\
(16\pi G) J_a^{kl} &= 2 \oint_{S^\infty} d^3 x_m \epsilon_{ab} (\Delta H^{bmk} x^l - \Delta H^{bml} x^k) \\
&= -2 \oint_{S^\infty} d^3 x_m \epsilon_{ab} (\pi^{bmk} x^l - \pi^{bml} x^k) , \\
(16\pi G) J_a^{\perp k} &= \oint_{S^\infty} d^3 x_m (\Delta C_a \delta^{mk} - \partial^m \Delta C_a x^k) \\
&= \oint_{S^\infty} d^3 x_m \left[ (\partial_t h_a^{mn} - \partial^m h_a) x^k - h_a^{mk} + h_a \delta^{mk} \right].
\end{align}

The only non-vanishing surface charges of the dyon sitting at the origin are

\[ P_a^\perp = M_a. \]

As expected, they measure the electric and magnetic mass of the dyon.

For later use, we combine \( k_{ij}^{[i] \epsilon} \) into the \( n - 2 \) forms \( k_{ij}^{[i] \epsilon} \) through the following expressions in Cartesian coordinates,

\[ k_{ij}^{[i] \epsilon} = k_{ij}^{[i] \epsilon} = \frac{1}{k!} \epsilon_{\mu_1 \ldots \mu_{k+1} \ldots \mu_{n-k}} dx^{\mu_{k+1}} \ldots dx^{\mu_{n}}. \]

Equations (3.3.82) can then be summarized by

\[ dk_{ij}^{[i] \epsilon} \approx -(16\pi G) T_{\epsilon s}, \quad T_{\epsilon s} = T_{\alpha \beta} A_{\alpha}^\mu d^3 x \mu, \quad dT_{\epsilon s} = 0, \]

where closure of the \( n - 1 \)-forms \( T_{\epsilon s} \) follows from the conservation of the sources, the symmetry of the energy-momentum tensor and (3.3.78).

### 3.3.13 Poincaré transformations of surface charges

Suppose now that \( z_{s}^{A}, u_{s}^{\alpha} \) solve the equations of motions for the conserved sources \( T_{\epsilon s}^{\mu \nu} \). Let \( z_{s}^{A}, u_{s}^{\alpha} \) be the solution associated to new sources \( T_{\epsilon}^{\mu \nu}(x) \) related to \( T_{\epsilon s}^{\mu \nu}(x) \) through a (proper) Poincaré transformation, \( x^\mu = \Lambda^\mu \nu x^\nu + b^\mu \) with \( |\Lambda| = 1 \),

\[ T_{\epsilon}^{\mu \nu}(x') = \Lambda^\mu \alpha \Lambda^\nu \beta T_{\epsilon s}^{\alpha \beta}(x). \]

For instance, starting from the conserved energy-momentum tensors (3.3.62) of a dyon sitting at the origin with world-line \( z^\mu = \delta^\mu_0 s \), one can obtain in this way the conserved
energy-momentum tensors of a dyon moving along a straight line, \( z'^\mu = u^\mu s + a^\mu \) with \( u^\mu, a^\mu \) constant, \( u^\mu u_\mu = -1 \) and \( s \) the proper time,

\[
T'_{\alpha\beta}(x') = M_\alpha u^\alpha \delta(4)(x' - z'(\lambda)) \frac{dz^\mu}{d\lambda} = M_\alpha \frac{u^\mu u^\nu}{u^0} \delta^{(4)}(x'^\mu - z'^\mu(x^0)).
\]

(3.3.98)

when \( \Lambda^\mu_0 = u^\mu \).

Assume then that the \( \xi^\alpha_{a\beta}(x) \) transform like vectors

\[
\xi^\alpha_{a\beta}(x') = \Lambda^\mu_\alpha \xi^\alpha_{a\beta}(x) = -(\Lambda \omega \Lambda^{-1} x')^\nu + (\Lambda \omega \Lambda^{-1} b + \Lambda a)_{\nu},
\]

(3.3.99)

which implies that the \( T'_{\epsilon s} \) are closed Poincaré invariant \( n - 1 \) forms,

\[
T'_{\epsilon s}(x', dx') = T_{\epsilon s}(x, dx).
\]

(3.3.100)

We can then use the following variant of the tube lemma. If at fixed time \( t \), \( T^0_{\alpha \nu}(x) \xi^\alpha_{a\nu}(x) \) has compact support and there exists a tube, i.e., a space-time volume \( \mathcal{W} \) connecting the hypersurfaces \( \Omega : x^0 = t \) and \( \Omega' : t = x'^0 = \Lambda^0_\nu x^\nu + b^0 \) such that \( T_{\epsilon s} \) is entirely contained in \( \mathcal{W} \), it follows from Stokes’ theorem that

\[
\int_\Omega T_{\epsilon s} = \int_{\Omega'} T'_{\epsilon s}.
\]

(3.3.101)

If we now compute the surface charges for a large enough sphere \( S \) at fixed \( t \) containing both \( T^0_{\alpha \nu}(x) \xi^\alpha_{a\nu}(x) \) and \( T^0_{\alpha \nu}(x) \xi^\alpha_{a\nu}(x) \), it finally follows from (3.3.75) that the surface charges evaluated for the new solutions \( z'^A_\epsilon \) are obtained from those of the old solutions \( z^A_\epsilon \) through

\[
\mathcal{Q}_{\epsilon s}[z'_s] = \mathcal{Q}_{\epsilon s}[z_s].
\]

(3.3.102)

### 3.4 Conclusion

In this chapter, we have developed an extended double potential formalism for spin 2. This allowed us to write a manifestly duality invariant action in presence of both electric and magnetic external sources. We derived the expression of the surface charges in term of the fundamental canonical fields obtaining in a duality invariant way both the mass and the NUT charge. Those charges can also be constructed using Lagrangian methods but, as such, are not duality invariant (see e.g. [67, 68, 69]).

In fact we have shown here that the standard expressions for surface charges in Pauli-Fierz theory, when extended in a duality invariant way, have all the expected properties. More interesting would be to develop the theory of surface charges from scratch in theories of the current type where the Poisson brackets among the fundamental variables are not local to see if the ones we have found exhaust all possibilities. From the preceding discussion we see that pseudo-differential operators will play a crucial role for a
discussion of these generalized conservation laws, as they do in the discussion of ordinary conservation laws for evolution equations of the Korteweg-de Vries type for instance.

This association with the soliton theory is genuine. As we will see in the next chapter, there is a close relation between electromagnetic duality and integrable systems.
Chapter 4

Electromagnetic duality and integrability

A cornerstone of soliton theory is the discovery that the evolution equations are Hamiltonian systems \([70, 71]\). In this context, the occurrence of hierarchies of evolution equations sharing the same infinite set of conservation laws can be understood as a consequence of the existence of a second compatible Hamiltonian structure giving rise to the same evolution equations \([72, 73]\).

The equations of motion associated to the theories for the known fundamental forces of nature, electromagnetism, Yang-Mills theories and gravitation, are variational and thus Hamiltonian. This is no coincidence, since these theories are fundamentally quantum, at least the first three of them, and only for variational theories quantization is sufficiently well understood.

In order to have Poincaré invariance, respectively diffeomorphism invariance, manifestly realized, most modern investigations of these equations are carried out in the Lagrangian framework. This could be the reason why the bi-Hamiltonian structure underlying these equations and discussed below has hitherto remained unnoticed.

At the heart of our analysis is an important exception to this paradigm, namely the question whether the duality invariance of the four dimensional Maxwell or linearized gravity equations admits a canonical generator. This question has been answered to the affirmative in the reduced phase space of these theories and generalized to massless higher spin gauge fields \([4, 3, 61]\).

We will show in this chapter that the reduced phase space formulation of massless higher spin gauge fields is bi-Hamiltonian. We will start by a quick review of the theory of bi-Hamiltonian systems. After that, we will first study the electromagnetic case and then go to linearized gravity and massless gauge fields of spin higher than 2 where the analysis of spin 1 can be carried over easily by taking care of additional spatial indices.
4.1 Bi-Hamiltonian systems

We will now present the basic results of the bi-Hamiltonian theory using the famous Korteweg-de Vries equation as an exemple. We refer the reader to the book of Olver [74] for a more complete presentation.

A bi-Hamiltonian system is remarkable in the sense that its evolution equations can be written in Hamiltonian form in not just one but two different ways. We are then interested in systems of the form

$$\frac{\partial z^A}{\partial t} = K^A_1[z] = \{ z^A, \mathcal{H}_1 \}_1 = \{ z^A, \mathcal{H}_0 \}_0$$

(4.1.1)

where \( \{.,\}_1 \) and \( \{.,\}_0 \) are two different Poisson brackets associated to two different Hamiltonians \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \). For instance, the KdV equation is given by

$$\frac{\partial u(x)}{\partial t} = \partial^3_x u + u \partial_x u.$$  

(4.1.2)

It can be written as a Hamiltonian equation in two different ways:

$$\{ F, G \}_1 = \int dx \frac{\delta F}{\delta u(x)} \frac{\delta G}{\delta u(x)}$$

$$\mathcal{H}_1 = \int dx \left( -\frac{1}{4} (\partial_x u)^2 + \frac{1}{6} u^3 \right),$$

(4.1.3)

and

$$\{ F, G \}_0 = \int dx \frac{\delta F}{\delta u(x)} \left( \partial_x^2 + \frac{2}{3} u \partial_x u + \frac{1}{3} \partial_x^3 u \right) \frac{\delta G}{\delta u(x)}$$

$$\mathcal{H}_0 = \int dx \frac{1}{2} u^2.$$  

(4.1.4)

(4.1.5)

Being Hamiltonians, both \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are conserved quantities and, as such, generate symmetries through both Poisson brackets. The following three transformations are then symmetries of the equations of motion (4.1.1):

$$\delta_0 z^A = K^A_0[z] = \{ z^A, \mathcal{H}_0 \}_1,$$  

(4.1.7)

$$\delta_1 z^A = K^A_1[z] = \{ z^A, \mathcal{H}_1 \}_1 = \{ z^A, \mathcal{H}_0 \}_0,$$  

(4.1.8)

$$\delta_2 z^A = K^A_2[z] = \{ z^A, \mathcal{H}_1 \}_0.$$  

(4.1.9)

Let’s assume that \( \delta_2 z^A \) is a Hamiltonian vector field for \( \{.,\}_1 \), i.e. there exists \( \mathcal{H}_2 \) such that

$$\delta_2 z^A = \{ z^A, \mathcal{H}_2 \}_1.$$  

(4.1.10)

In that case, \( \mathcal{H}_2 \) is a new conserved quantity and it generates a new symmetry through the other canonical structure:

$$\delta_3 z^A = K^A_3[z] = \{ z^A, \mathcal{H}_2 \}_0.$$  

(4.1.11)
If this continues, one can build an infinite tower of conserved quantities \( \mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \ldots \) associated to an infinite tower of symmetries whose characteristics are \( K_0, K_1, K_2, \ldots \).

A key point in this argument is the assumption that \( \delta z^A \) is a Hamiltonian vector field for \( \{ \cdot \}, \) \( 1 \). In order to control this property, we need to introduce the notion of compatible Hamiltonian structures:

**Definition 4.1.1.**

- Two Hamiltonian structures \( \{ \cdot \}, 1 \) and \( \{ \cdot \}, 0 \) are said to be compatible if for all \( a, b \in \mathbb{R} \), \( a \{ \cdot \}, 0 + b \{ \cdot \}, 1 \) is a Hamiltonian structure. They form a Hamiltonian pair.

- A system of evolution equations is a bi-Hamiltonian system if it can be written in the form (4.1.1) where \( \{ \cdot \}, 1 \) and \( \{ \cdot \}, 0 \) form a Hamiltonian pair.

To any Poisson bracket \( \{ \cdot \}, \) we can associate a skew-adjoint linear differential operator \( J^{AB} \) such that:

\[
\{ F, G \} = \int d^nx \frac{\delta F}{\delta z^A(x)} J^{AB} \left( \frac{\delta G}{\delta z^B(x)} \right) = -\int d^nx J^{BA} \left( \frac{\delta F}{\delta z^A(x)} \right) \frac{\delta G}{\delta z^B(x)}.
\]  
(4.1.12)

This operator is called a Hamiltonian operator. We will use the notation \( D^{AB} \) for the Hamiltonian operator associated to \( \{ \cdot \}, 1 \) and \( E^{AB} \) the one associated to \( \{ \cdot \}, 0 \). The two Hamiltonian operators associated to the KdV equation are given by

\[
D = \partial_x, \quad E = \partial_x^3 + \frac{2}{3} u \partial_x + \frac{1}{3} \partial_x u.
\]  
(4.1.13)

With those structures, we can state the main theorem on bi-Hamiltonian systems.

**Theorem 4.1.2.** Let

\[
\frac{\partial z^A}{\partial t} = K_1^A[z] = \{ z^A, \mathcal{H}_1 \}, 1 = D^{AB} \frac{\delta \mathcal{H}_1}{\delta z^B} = \{ z^A, \mathcal{H}_0 \}, 0 = E^{AB} \frac{\delta \mathcal{H}_0}{\delta z^B}
\]  
(4.1.14)

be a bi-Hamiltonian system of evolution equations. Assume that the operator \( D^{AB} \) is non-degenerate and define \( R_B^A = E^{AC} D^{-1}_{CB} \). Let \( K_0^A[z] = \{ z^A, \mathcal{H}_1 \}, 0 \) and assume that for all \( n \in \mathbb{N} \) we can recursively define

\[
K_n^A[z] = R_B^A K_{n-1}^B[z]
\]  
(4.1.15)

meaning that for each \( n \), \( K_{n-1} \) lies in the image of \( D \). Then, there exists a sequence of functionals \( \mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \ldots \) such that
• for all \( n \geq 1 \), the evolution equation
\[
\frac{\partial z^A}{\partial t} = K_n^A[z] = \{ z^A, \mathcal{H}_n \}_1 = \{ z^A, \mathcal{H}_{n-1} \}_0 \tag{4.1.16}
\]
is a bi-Hamiltonian system;

• the Hamiltonian functionals \( \mathcal{H}_n \) are all in involution with respect to either Poisson bracket:
\[
\{ \mathcal{H}_m, \mathcal{H}_n \}_1 = 0 = \{ \mathcal{H}_m, \mathcal{H}_n \}_0 \quad n, m \geq 0 \tag{4.1.17}
\]
and hence provide an infinite collection of conservation laws for each of the bi-Hamiltonian systems (4.1.16).

In the KdV case, the operator \( R \) is given by
\[
R = \partial_x^2 + \frac{2}{3} u + \frac{1}{3} \partial_x u \partial_x^{-1}, \tag{4.1.18}
\]
where \( \partial_x^{-1} \) is a formal operator acting only on functions that are total derivative (if \( Q = \partial_x P \), then we set \( P = \partial_x^{-1} Q \). We can remove the ambiguity of the additive constant by normalizing \( P|_{u=0,x=0} = 0 \)). The hierarchy of evolution equations generated by the theorem is the usual KdV hierarchy:
\[
\begin{align*}
K_0 &= \partial_x u, \tag{4.1.19} \\
K_1 &= \partial_x^3 u + u \partial_x u, \tag{4.1.20} \\
K_2 &= \partial_x^5 u + \frac{5}{3} u \partial_x^3 u + \frac{10}{3} \partial_x u \partial_x^2 u + \frac{5}{6} u^2 \partial_x u, \tag{4.1.21} \\
K_3 &= \ldots \tag{4.1.22}
\end{align*}
\]
associated to an infinite amount of constants of motion:
\[
\begin{align*}
\mathcal{H}_0 &= \int dx \left( \frac{1}{2} u^2 \right), \tag{4.1.23} \\
\mathcal{H}_1 &= \int dx \left( -\frac{1}{2} (\partial_x u)^2 + \frac{1}{6} u^3 \right), \tag{4.1.24} \\
\mathcal{H}_2 &= \int dx \left( \frac{1}{2} (\partial_x^2 u)^2 + \frac{5}{72} u^4 + \frac{5}{16} u^2 \partial_x^2 u \right), \tag{4.1.25} \\
\mathcal{H}_3 &= \ldots \tag{4.1.26}
\end{align*}
\]

The operator \( R^A_B \) defined in the theorem is a recursion operator for the system in the sense that if \( \delta_Q z^A = Q^A[z] \) is a symmetry of the evolution equation (4.1.14) and \( Q^A \) is in the image of \( \mathcal{D} \), then
\[
\delta_{RQ} z^A = R^A_B Q^B[z] \tag{4.1.27}
\]
is also a symmetry of the system. A repeated use of this operator allows the creation of an infinite tower of symmetries from any given symmetry. Let us point out that they are not necessarily Hamiltonian even if the starting one is. In the case of Korteweg-de Vries, this operator is known as the Lenard recursion operator.
4.2 Electromagnetism

As we saw in chapter 2, the reduced phase space action of electromagnetism can be written as

\[ S_R[A_{ai}^T] = \int dt \left[ \int d^3x \frac{1}{2} \epsilon_{ab} (\mathcal{O} A_{ai}^T)^b \partial_0 A_{ai}^T - H_1 \right], \]  

(4.2.1)

\[ H_1 = \frac{1}{2} \int d^3x \left( \mathcal{O} A_{ai}^T \right)_i \left( \mathcal{O} A_{ai}^T \right)^i = -\frac{1}{2} \int d^3x A_{ai}^T \Delta A_{ai}^T, \]  

(4.2.2)

where in the second expression for the Hamiltonian, we have used that \( \mathcal{O} \) is “self-adjoint”.

The standard Poisson bracket determined by the kinetic term is

\[ \{ A_{ai}^T(x), A_{bj}^T(y) \}_1 = \epsilon_{ab} \Delta^{-1} \epsilon^{ijkl} \delta^{T(3)}_{ij}(x - y) = \epsilon_{ab} \Delta^{-1} \left( \mathcal{O} \delta_{ij}^{T(3)}(x - y) \right)^j, \]  

(4.2.3)

where \( \delta_{ij}^{T(3)}(x - y) \) is the transverse delta function, see e.g. [66] section A.1.2. In vacuum, Maxwell’s equations for the physical degrees of freedom read

\[ \partial_0 A_{ai}^T(x) = \{ A_{ai}^T(x), H_1 \}_1 = -\epsilon_{ab} \left( \mathcal{O} A_{ai}^T \right)^{(x)}, \]  

(4.2.4)

while the generator for duality rotations is

\[ H_0 = -\frac{1}{2} \int d^3x A_{ai}^T \left( \mathcal{O} A_{ai}^T \right)_i, \quad \{ H_0, H_1 \}_1 = 0. \]  

(4.2.5)

When presented in this way, the second Hamiltonian structure is obvious and a lot simpler than the one induced from the covariant action principle. Indeed, a natural Poisson bracket on reduced phase space is simply

\[ \{ A_{ai}^T(x), A_{bj}^T(y) \}_0 = \epsilon_{ab} \delta_{ij}^{T(3)}(x - y), \]  

(4.2.6)

in terms of which the duality generator is the Hamiltonian for Maxwell’s equations,

\[ \{ A_{ai}^T(x), H_1 \}_1 = \{ A_{ai}^T(x), H_0 \}_0. \]  

(4.2.7)

This is the main result of this chapter.

At this stage, one can pause and ask whether electromagnetism and its quantization should not be based on this new Hamiltonian structure. A good reason to favor the old, more complicated structure is that, by construction, the Poincaré and conformal symmetries admit canonical generators for the old structure, while not all of them do for the new one. We plan to return to this question in detail elsewhere.

The rest of the analysis is standard. Following the previous section, the associated recursion operator is defined by

\[ R_{ai}^{bj} = -\delta_i^u (\mathcal{O})^j_u. \]  

(4.2.8)
Consider, for $p \geq 1$, $K_p^{Tai} = (-)^p \epsilon^{ab} (\mathcal{O}^p A^T_b)_i$, or equivalently,

$$K_{2n+1}^{Tai} (x) = (-)^{n+1} \epsilon^{ab} \Delta^n (\mathcal{O} A^T_i)_b, \quad K_{2n+2}^{Tai} (x) = (-)^{n+1} \epsilon^{ab} \Delta^{n+1} A^T_i(x), \quad (4.2.9)$$

for $n \geq 0$. The evolution equations of the hierarchy

$$\partial_0 A^{Tai}(x) = K_p^{Tai}(x), \quad \forall p \geq 1, \quad (4.2.10)$$

are also bi-Hamiltonian,

$$K_p^{Tai}(x) = \{ A^{Tai}(x), H_p \}_1 = \{ A^{Tai}(x), H_{p-1} \}_0, \quad (4.2.11)$$

where $H_{p-1} = \frac{(-)^p}{2} \int d^3 x A^T_i (\mathcal{O}^p A^T)_i$,

$$H_{2n} = \frac{(-)^{n+1}}{2} \int d^3 x A^{Tai} \Delta^n (\mathcal{O} A^T_i)_b, \quad H_{2n+1} = \frac{(-)^{n+1}}{2} \int d^3 x A^T_i \Delta^{n+1} A^T_i. \quad (4.2.12)$$

with Hamiltonians that are in involution,

$$\{ H_n, H_m \}_1 = 0 = \{ H_n, H_m \}_0, \quad \forall n, m \geq 0. \quad (4.2.13)$$

## 4.3 Linearized gravity

Following the result presented in chapter 3, the reduced phase space action of linearized gravity can be written as

$$S^R[H^{TT}] = \int dt \left[ \int d^3 x \epsilon_{ab} \Delta (\mathcal{O} H^{TT})^{mn} \partial_0 H^{TTb} - H_1 \right], \quad (4.3.1)$$

$$H_1 = \int d^3 x \left( H^{TTann} \Delta^2 H^{TT}_{ann} \right). \quad (4.3.2)$$

The standard Poisson bracket determined by the kinetic term is

$$\{ H^{TTa}_{mn}(x), H^{TTbkl}(y) \}_1 = \frac{1}{2} \epsilon^{ab} \Delta^{-2} (\mathcal{O}^2 \delta^{(3)TT}_{mn} (x - y))^{kl}, \quad (4.3.3)$$

where $\delta^{(3)TT}_{mn}(x - y)$ denotes the projector on the transverse-traceless part of a symmetric rank two tensor. The duality generator is

$$D = - \int d^3 x H^{TTann} \Delta (\mathcal{O} H^{TT}_a)_{mn}, \quad \{ D, H_1 \}_1 = 0. \quad (4.3.4)$$

The analogy with the spin 1 case can be made perfect by the change of variables,

$$H^{1TT}_{mn} = \frac{1}{\sqrt{2}} \Delta^{-1} (\mathcal{O} A^{2TT})_{mn}, \quad H^{2TT}_{mn} = \frac{1}{\sqrt{2}} \Delta^{-1} (\mathcal{O} A^{1TT})_{mn}. \quad (4.3.5)$$
in terms of which
\[ H_1 = \frac{1}{2} \int d^3 x \left( (O A^{T a})^{mn} (O A^{T T}_a)^{mn} \right) = -\frac{1}{2} \int d^3 x \left( A^{TT a mn} \Delta A^{T T a}_{mn} \right), \tag{4.3.6} \]
\[ \{ A^{TT a}_{mn}(x), A^{TT bkl}(y) \}_1 = \epsilon^{ab} \Delta^{-1} \left( O^y \delta^{(3)T T}_{mn}(x-y) \right)^{kl}, \tag{4.3.7} \]
\[ H_0 = -D = -\frac{1}{2} \int d^3 x \ A^{TT a mn} (O A^{T T}_a)^{mn}. \tag{4.3.8} \]

All formulae of section 4.2 below equation (4.2.5) now generalize in a straightforward way to massless spin 2 fields by replacing $T$ (transverse) by $TT$ (transverse-traceless) and contracting over the additional spatial index.

### 4.4 Massless higher spin gauge fields

The extension of these results to massless higher spin gauge fields [75] (see also [76]) follows directly from the observation that the Hamiltonian reduced phase space formulation of these theories merely involves additional spatial indices [61], so that all above results generalize in a straightforward way.

This can be seen for instance by starting from the approach inspired from string field theory, where the Lagrangian action for massless higher spin gauge fields is written as the mean value of the BRST charge for a suitable first quantized particle model [77, 78, 79] (see also [80, 81] for further developments). In this framework the reduction of the action to the light-cone gauge corresponds to the elimination of BRST quartets composed of ghost and light-cone oscillators (see e.g. [82, 83]). In exactly the same way, the ghost, temporal and longitudinal oscillators form quartets that can be eliminated to yield the Lagrangian gauge fixed action for a massless field of spin $s \geq 1$,

\[ S_L[\phi^{TT}_{i_1 \ldots i_s}] = -\frac{1}{2} \int d^4 x \ \partial_{\mu} \phi^{TT}_{i_1 \ldots i_s} \partial^{\mu} \phi^{TT i_1 \ldots i_s} \tag{4.4.1} \]

where the field $\phi^{TT}_{i_1 \ldots i_s}$ is real, completely symmetric, traceless and transverse,

\[ \phi^{TT}_{j_{i_1 \ldots i_s}} = \phi^{TT}_{(i_1 \ldots i_s)}, \ \phi^{TT}_{i_{i_2 \ldots i_s}} = 0, \ \partial^{i} \phi^{TT}_{i_{i_2 \ldots i_s}} = 0. \tag{4.4.2} \]

The Hamiltonian formulation is direct, the momenta being $\pi^{TT}_{i_1 \ldots i_s} = \partial_{0} \phi^{TT}_{i_1 \ldots i_s}$.

Consider then the Fock space defined by $\{ a^i, a_i^\dagger \} = \delta^i_j$, $a_i |0 \rangle = 0$, the number operator $N = a_i^\dagger a^i$, and the “string field” $\phi^{TT}_s (x) = \frac{1}{\sqrt{s!}} a_{i_1}^\dagger \ldots a_{i_s}^\dagger |0 \rangle \phi^{TT}_{i_1 \ldots i_s} (x)$ and the inner product

\[ \langle \phi^{TT}_s, \psi^{TT}_s \rangle = \int d^4 x \langle \phi^{TT}_s, \psi^{TT}_s \rangle_F = \int d^4 x \phi^{TT}_{i_1 \ldots i_s} \psi^{TT i_1 \ldots i_s}. \tag{4.4.3} \]

With this inner product, the generalized curl [61]

\[ O = \frac{1}{N} \epsilon^{ijk} a_i^\dagger \partial_j a_k \tag{4.4.4} \]
is again self-adjoint. Furthermore, it squares to $-\Delta$ inside the inner product involving transverse-traceless fields,

$$O^2 = \frac{1}{N^2} \left[ -\Delta N^2 + (\partial \cdot a^\dagger)(\partial \cdot a) + (a^\dagger \cdot a^\dagger)\Delta (a \cdot a) + 2(\partial \cdot a^\dagger)N(\partial \cdot a) - (\partial \cdot a^\dagger)^2(a \cdot a) - (a^\dagger \cdot a^\dagger)(\partial \cdot a) \right], \quad \implies \langle \phi_s^{TT}, \mathcal{O}^2 \psi_s^{TT} \rangle = -\langle \phi_s^{TT}, \Delta \psi_s^{TT} \rangle. \quad (4.4.5)$$

The change of variables making duality invariance transparent is

$$\phi^{TT}_{i_1...i_s} = A^{TT1}_{i_1...i_s}, \quad \pi^{TT}_{i_1...i_s} = \left( \mathcal{O} A^{TT2} \right)_{i_1...i_s}. \quad (4.4.6)$$

The first order reduced phase space variational principle becomes

$$S^R[A^{Ta}] = \int dt \left[ \int d^3 x \left\{ \frac{1}{2} \varepsilon_{ab} \left( \mathcal{O} A^{Ta} \right)_{i_1...i_s} \partial_0 A^{Tb}_{i_1...i_s} - H_1 \right\} \right], \quad (4.4.7)$$

$$H_1 = -\frac{1}{2} \int d^3 x A^{Ta}_{i_1...i_s} \Delta A^{TTi_1...i_s}. \quad (4.4.8)$$

Again, all formulae of section 4.2 below equation (4.2.4), including the one for the duality generator, suitably generalize by contracting over the additional spatial indices.

### 4.5 Conclusion

We have shown in this chapter that the reduced phase space formulation of massless higher spin gauge fields is bi-Hamiltonian. The second Poisson bracket on reduced phase space turned out to be more natural than the one induced from the covariant variational principle, while the generator for duality rotations played the role of the second Hamiltonian. This result trivially generalizes to Yang-Mills theory with an invariant, non degenerate metric, linearized around a zero potential by decorating the expressions obtained in the electromagnetic case with an additional Lie algebra index.

Several generalizations and extensions are suggested by this result. A first exercise consists in studying the consequences for symmetries and conservation laws of both the Maxwell and the higher spin equations and compare them to known results (see e.g. [84, 85, 86] and references therein). Another obvious question is to investigate more general backgrounds. For instance, the generalization to massless spins propagating on (anti-) de Sitter spaces instead of Minkowski spacetime and the inclusion of fermionic gauge fields should be straightforward. In Yang-Mills theories (anti) self-dual backgrounds could be promising in view of their close connection to integrable systems.

The most important problem is however the inclusion of interactions. When comparing to the Korteweg-de Vries equation for instance, the present work corresponds to the bi-Hamiltonian structure for the linearized equation. The question is then to find interactions that preserve this structure.
Chapter 5

Gravitational features of the AdS$_3$/CFT$_2$ correspondence

In this chapter, we will work with asymptotically $AdS_3$ space-times in Fefferman-Graham form. In that case, the gauge is completely fixed as all subleading orders of the asymptotic Killing vectors are uniquely determined. Equipped with a new modified Dirac-type Lie bracket taking into account their metric dependence due to the gauge fixing, those asymptotic Killing vectors form a representation of the local conformal algebra to all orders in $r$. We give a short description of the possible central extensions of the 2 dimensions conformal algebra.

We then solve the Einstein’s equations for metrics in the Fefferman-Graham form and compute how the local conformal algebra in 2 dimensions is realized on solution space. The last step is the covariant computation of the surface charge algebra and the value of the central extension as reviewed in appendix A.2.

This chapter is based on results originally derived in [12] and developed further in [87, 88, 89, 90, 91, 92, 93, 94, 95, 96]. The only original result is the introduction of the modified Lie Bracket in section 5.2.

5.1 Asymptotically AdS$_3$ spacetimes in Fefferman-Graham form

The Fefferman-Graham form for the line element of a 3 dimensional asymptotically anti-de Sitter spacetime is

$$ds^2 = \frac{l^2}{r^2} dr^2 + g_{AB}(r, x^C) \, dx^A dx^B,$$

(5.1.1)
with \( g_{AB} = r^2 \hat{\gamma}_{AB}(x^C) + O(1) \), where \( \hat{\gamma}_{AB} \) is a conformally flat 2-dimensional metric. For explicit computations we will sometimes choose the parametrization \( \hat{\gamma}_{AB} = e^{2\varphi} \eta_{AB} \) with \( \varphi(x^C) \) and \( \eta_{AB} \) the flat metric on the cylinder, \( \eta_{AB} dx^A dx^B = -d\tau^2 + d\varphi^2 \), \( \tau = \frac{1}{t} \).

### 5.2 Asymptotic symmetries

The transformations leaving this form of the metric invariant are generated by vector fields satisfying

\[
\mathcal{L}_\xi g_{rr} = 0 = \mathcal{L}_\xi g_{rA}, \quad \mathcal{L}_\xi g_{AB} = O(1),
\]

which implies

\[
\begin{align*}
\xi^r &= -\frac{1}{2} \psi r, \\
\xi^A &= Y^A + I^A, \\
I^A &= -\frac{r^2}{2} \partial_B \psi \int_r^\infty \frac{d\eta}{\eta} g^{AB} = -\frac{r^2}{4} \hat{\gamma}^{AB} \partial_B \psi + O(r^{-4}),
\end{align*}
\]

where \( Y^A \) is a conformal Killing vector of \( \hat{\gamma}_{AB} \), and thus of \( \eta_{AB} \), while \( \psi = \bar{D}_A Y^A \) is the conformal factor.

Indeed, the inverse to metric (5.1.1) is

\[
g^{\mu\nu} = \begin{pmatrix}
\frac{r^2}{\tau^2} & 0 \\
0 & g^{AB}
\end{pmatrix}
\]

where \( g^{AB} g_{BC} = \delta^A_C \). From \( \mathcal{L}_\xi g_{rr} = 0 \), we find \( \xi^r = Ar \) for some \( A(x^C) \). From \( \mathcal{L}_\xi g_{rA} = 0 \) we find \( \partial_r \xi^A = -g^{AB} \frac{\partial_B A}{r} \) so that \( \xi^A = Y^A + I^A \) for some \( Y^A(x^C) \) and where \( I^A = \int_r^\infty d\eta' g^{AB} r^B \). Finally, \( \mathcal{L}_\xi g_{AB} = O(1) \) requires \( Y^A \) to be a conformal Killing vector of \( \hat{\gamma}_{AB} \) and \( A = -\frac{1}{2} \psi \).

Let \( \hat{Y}^A = [Y_1, Y_2]^A, \hat{\psi} = \bar{D}_A \hat{Y}^A, \) denote by \( \delta^\mu_1 \xi_2^\mu \) the change induced in \( \xi_2^\mu(g) \) due to the variation \( \delta^\mu_1 g_{\mu\nu} = \mathcal{L}_{\xi_1} g_{\mu\nu} \) and define

\[
[\xi_1, \xi_2]^\mu_M = [\xi_1, \xi_2]^\mu - \delta^\mu_1 \xi_2^\mu + \delta^\mu_2 \xi_1^\mu.
\]

For vectors \( \xi_1, \xi_2 \) given in (5.2.2), we have

\[
[\xi_1, \xi_2]^r_M = -\frac{1}{2} \hat{\psi} r, \quad [\xi_1, \xi_2]^A_M = \hat{Y}^A + \hat{I}^A,
\]

where \( \hat{I}^A \) denotes \( I^A \) with \( \psi \) replaced by \( \hat{\psi} \).

Indeed, for the \( r \) component, we have \( \delta^r_1 \xi_2^r = 0 \) and the result follows by direct computation of the Lie bracket. Similarly, \( \lim_{r \to \infty} [\xi_1, \xi_2]^A_M = \hat{Y}^A. \) Finally, using \( \partial_r \xi^r = \frac{1}{r} \xi^r \) and \( \partial_r \xi^A = -\frac{r^2}{2} \partial_B \xi^r g^{BA} \) a straightforward computation shows that \( \partial_r ([\xi_1, \xi_2]^A_M) = \partial_B ([\xi_1, \xi_2]^A_M) g^{BA} \), which gives the result. It thus follows that on an asymptotically anti-de Sitter spacetime in the sense of Fefferman-Graham (solving or not Einstein’s equations with cosmological constant):
The spacetime vectors (5.2.2) equipped with the bracket $[\cdot, \cdot]_M$ form a faithful representation of the conformal algebra.

By conformal algebra, we mean here the direct sum of 2 copies of the Witt algebra. Furthermore, since $\delta^g_{\xi_1, \xi_2} = 0$, $\delta^g_{\xi_1, \xi_2} = O(r^{-4})$, it follows that these vectors form a representation of the conformal algebra only up to terms of order $O(r^{-4})$ when equipped with the standard Lie bracket.

More generally, one can also consider the transformations that leave the Fefferman-Graham form of the metric invariant up to a Weyl rescaling of the boundary metric $\bar{\gamma}_{AB}$. They are generated by spacetime vectors such that

$$L[\xi, g_{rr}] = 0 = L[\xi, g_{rA}],$$
$$L[\xi, g_{AB}] = 2\omega g_{AB} + O(1).$$

(5.2.4)

It is then straightforward to see that the general solution is given by the vectors (5.2.2), where $\psi$ is replaced by $\tilde{\psi} = \psi - 2\omega$. When equipped with the modified Lie bracket $[\cdot, \cdot]_M$ these vectors now form a faithful representation of the extension of the two dimensional conformal algebra defined by elements $(Y, \omega)$ and the Lie bracket $(\tilde{Y}, \tilde{\omega}) = [(Y_1, \omega_1), (Y_2, \omega_2)]$

$$\tilde{Y}^A = Y_1^B \partial_B Y_2^A - Y_2^B \partial_B Y_1^A, \quad \tilde{\omega} = 0.$$

(5.2.5)

with $\omega(x^C)$ arbitrary and $Y^A$ conformal Killing vectors of $\bar{\gamma}_{AB}$ and thus also of $\eta_{AB}$. The asymptotic symmetry algebra is then the direct sum of the abelian ideal of elements of the form $(0, \omega)$ and of 2 copies of the Witt algebra.

Indeed, we have $\lim_{r \to \infty} (\frac{1}{r}[\xi_1, \xi_2]_M) = -\frac{1}{2} Y_1^A \partial_A \tilde{\psi}_2 + \partial_C \omega_1 Y_2^C + (1 \leftrightarrow 2) = -\frac{1}{2} \tilde{\psi}$ and $\partial_r (\frac{1}{r}[\xi_1, \xi_2]_M) = 0$, while the proof for the $A$-component is unchanged.

5.3 Conformal algebra and central extension

In terms of light-cone coordinates, $x^\pm = \tau \pm \phi$, $2\partial_\pm = \frac{\partial}{\partial \tau} \pm \frac{\partial}{\partial \phi}$, we have $\bar{\gamma}_{AB}dx^A dx^B = -e^{2\phi} dx^+ dx^-$, and if,

$$Y^\pm(x^\pm)\partial_{\pm} = \sum_{n \in \mathbb{Z}} c_n l_n^\pm, \quad l_n^\pm = e^{inx^\pm} \partial_{\pm},$$

(5.3.1)

the algebra in terms of the basis vectors $l_n^\pm$ reads

$$i[l_m^+, l_n^+] = (m-n)l_m^+, \quad i[l_m^-, l_n^-] = 0.$$

(5.3.2)

The definition of the basic vectors is different from [97]. This new definition is better suited to the periodicity of $\phi$ and will allow us to do the flat limit ($l \to \infty$) in the next chapter.
Up to equivalence, the most general central extension of the conformal algebra in 2 dimensions is given by
\[
\begin{cases}
  il_{\pm}^{m, n} = (m - n)l_{\pm}^{m} + \frac{c^{\pm}}{12} m(m + 1)(m - 1)\delta_{m+n}^{0}, \\
  il_{\pm}^{m, n} = 0.
\end{cases}
\] (5.3.3)

The proof follows from doing twice the one for the Witt algebra \(w\), see e.g \([98, 99, 100]\).

### 5.4 Solution space

Let us now start with an arbitrary metric of the form (5.1.1), without any assumptions on the behavior in \(r\) and let \(k_{B}^{A} = \frac{1}{2}g^{AC}g_{CB,r}\). One can then define \(K_{B}^{A}\) through the relation \(k_{B}^{A} = \frac{1}{r}\delta_{B}^{A} + \frac{1}{r^3}K_{B}^{A}\). We have
\[
\Gamma_{rr}^{r} = -\frac{1}{r}, \quad \Gamma_{rA}^{rA} = 0, \quad \Gamma_{rr}^{A} = 0,
\]
\[
\Gamma_{AB}^{r} = -\frac{r^2}{l^2}k_{B}^{A}, \quad \Gamma_{rB}^{A} = k_{B}^{A}, \quad \Gamma_{BC}^{A} = (2)\Gamma_{BC}^{A},
\]
where \((2)\Gamma_{BC}^{A}\) denotes the Christoffel symbol associated to the 2-dimensional metric \(g_{AB}\), which is used to lower indices on \(k_{B}^{A}\). If \(K_{AB}^{T}\) denotes the traceless part of \(K_{B}^{A}\), the equations of motion are organized as follows
\[
g^{AB}G_{AB} - \frac{2}{l^2} = 0 \iff \partial_{r} \Gamma = -r^{-3}(\frac{1}{2}k^{2} + K_{B}^{A}K_{B}^{T^{A}}), \quad (5.4.1)
\]
\[
G_{AB} - \frac{1}{2}g_{AB}g^{CD}G_{CD} = 0 \iff \partial_{r} K_{B}^{A} = -r^{-3}KK_{B}^{T^{A}}, \quad (5.4.2)
\]
\[
G_{rA} \equiv r^{-3}(2D_{B}K_{B}^{A} - \partial_{A}K) = 0, \quad (5.4.3)
\]
\[
G_{rr} - \frac{1}{l^2}g_{rr} \equiv \frac{1}{l^2} [r^{-6}(\frac{1}{2}K^{2} - K_{B}^{A}K_{B}^{T^{A}}) + 2r^{-4}K - \frac{l^2}{r^2}(2)R] = 0. \quad (5.4.4)
\]

Combining the Bianchi identities \(2(\sqrt{-g}G_{\alpha}^{\beta, \beta} + \sqrt{-g}G_{\beta, \gamma}g^{\beta\gamma, \alpha} \equiv 0\) with the covariant constancy of the metric, we get the identities
\[
2(\frac{r}{l}\sqrt{|(2)g|G_{rA}}),_{r} + 2(\frac{l}{r}\sqrt{|(2)g|}g^{BC}[G_{CA} - \frac{1}{l^2}g_{CA}]),_{B} +
\]
\[
+ (\frac{l}{r}\sqrt{|(2)g|})(G_{BC} - \frac{1}{l^2}g_{BC})g^{BC},_{A} \equiv 0, \quad (5.4.5)
\]
\[
(\frac{r}{l}\sqrt{|(2)g|}[G_{rr} - \frac{1}{r^2}]),_{r} + (\frac{l}{r}\sqrt{|(2)g|}g^{BA}G_{Ar}),_{B} +
\]
\[
+ (\frac{l}{r}\sqrt{|(2)g|})(G_{AB} - \frac{1}{l^2}g_{AB})k^{AB} \equiv 0. \quad (5.4.6)
\]
To solve the equations of motion, we first contract (5.4.2) with $K^T_A$, which gives
\[ \partial_r (K^T_B K^B_A) = -2r^{-3}K K^T_B K^B_A. \]
If we assume $K^T_B K^B_A = \frac{1}{2}K^2$, we can take the sum and difference with (5.4.1) to get
\[ \partial_r (K + K) = -\frac{1}{2}r^{-3}(K + K)^2, \quad \partial_r (K - K) = -\frac{1}{2}r^{-3}(K - K)^2, \]
which can be solved in terms of 2 integration “constants” $C(x^B), D(x^B)$
\[ K = -\frac{1}{C + \frac{1}{2}r^{-2}} - \frac{1}{D + \frac{1}{2}r^{-2}}, \quad K^T_B K^B_A = \frac{(D - C)^2}{2(C + \frac{1}{2}r^{-2})^2(D + \frac{1}{2}r^{-2})^2}. \]
When used in (5.4.2), we find
\[ K^T_B = A^T_B \left( \frac{1}{C + \frac{1}{2}r^{-2}} - \frac{1}{D + \frac{1}{2}r^{-2}} \right), \quad A^T_B A^T_A = \frac{1}{2}, \]
and can now reconstruct the metric from the equation $\partial_r g_{AB} = 2g_{AC}K^C_B$. Defining $\Theta = \frac{1}{D} + \frac{1}{C}, \Omega = \frac{1}{D} - \frac{1}{C}$, we get
\[ g_{AB} = r^2\tilde{\gamma}_{AB} \left[ 1 + \frac{1}{2r^2} \Theta + \frac{1}{16r^4} (\Theta^2 + \Omega^2) \right] + A^T_{AB} \left[ \Omega + \frac{1}{4r^2} \Theta \Omega \right], \quad (5.4.7) \]
where $\tilde{\gamma}_{AB}$ are additional integration constants, restricted by the condition that $\tilde{\gamma}_{AB}$ is symmetric, of signature $-1$. The index on $A^T_B$ is lowered with $\gamma_{AB}$, with $A^T_{AB}$ requested to be symmetric. It follows that $A^T_B$ contains only 1 additional independent integration constant. Writing $g_{AB} = r^2\tilde{\gamma}_{AB} + \gamma_{AB}$, with $\gamma_{AB} = \tilde{\gamma}_{AB} + o(r^0)$, we have $K^A_B = -\tilde{\gamma}^A_B + o(r^0)$ where the index on $\tilde{\gamma}^A_B$ has been lifted with $\tilde{\gamma}^{AB}$, the inverse of $\tilde{\gamma}_{AB}$.

When (5.4.1) and (5.4.2) are satisfied, the Bianchi identity (5.4.5) implies that $\sqrt{|\tilde{g}|} G_{ra}$ does not depend on $r$. The equation of motion (5.4.3) then reduces to the condition
\[ \tilde{D}_B \tilde{\gamma}^B_A - \partial_A \tilde{\gamma} = 0, \quad (5.4.8) \]
where $\tilde{\gamma} = \tilde{\gamma}^A_A$. When this condition holds in addition to (5.4.1) and (5.4.2), the remaining Bianchi identity (5.4.6) implies that $r^2\sqrt{|\tilde{g}|} (G_{rr} - \frac{1}{r})$ does not depend on $r$. The equation of motion (5.4.4) then reduces to the condition
\[ \tilde{\gamma} = -\frac{l^2}{2} \tilde{R}, \quad (5.4.9) \]
and also from the leading contribution to $K$ that $\Theta = -\frac{l^2}{2} \tilde{R}$. The constraint (5.4.8) then becomes
\[ \tilde{D}_B \tilde{\gamma}^T_B = -\frac{l^2}{4} \partial_A \tilde{R}. \quad (5.4.10) \]
To solve this equation, one uses light-cone coordinates, \( x^\pm = \tau \pm \phi, 2\partial_\pm = \frac{\partial}{\partial \tau} \pm \frac{\partial}{\partial \phi} \) and the explicit parameterization \( \tilde{\gamma}_{AB} dx^A dx^B = -e^{2\varphi} dx^+ dx^- \). This gives
\[
\tilde{\gamma} = -4l^2 e^{-2\varphi} \partial_+ \partial_- \varphi \iff \tilde{\gamma}_{+-} = l^2 \partial_+ \partial_- \varphi, \tag{5.4.11}
\]
while the general solution to (5.4.10) is
\[
\tilde{\gamma}_{\pm\pm} = l^2 [\Xi_{\pm\pm}(x^\pm) + \partial^2_\pm \varphi - (\partial_\pm \varphi)^2], \tag{5.4.12}
\]
with \( \Xi_{\pm\pm}(x^\pm) \) 2 arbitrary functions of their arguments. Using (5.4.7), one then gets
\[
A_T^{\pm} \Omega = \tilde{\gamma}_{\pm\pm}, \quad A_T^{+-} = 0, \quad \Omega^2 = 16 e^{-4\varphi} \tilde{\gamma}_{++} \tilde{\gamma}_{--}. \tag{5.4.13}
\]
In other words, one can choose \( \varphi(x^+, x^-), \Xi_{\pm\pm}(x^\pm) \) as coordinates on solution space and, by expressing (5.4.7) in terms of these coordinates, we have shown that

*The general solution to Einstein’s equations with metrics in Fefferman-Graham form is given by*
\[
g_{AB} dx^A dx^B = \left( -e^{2\varphi} r^2 + 2\tilde{\gamma}_{++} - r^{-2} e^{-2\varphi} (\tilde{\gamma}_{++} + \tilde{\gamma}_{++}) \right) dx^+ dx^- + \tilde{\gamma}_{++} (1 - r^{-2} e^{-2\varphi} \tilde{\gamma}_{++})(dx^+)^2 + \tilde{\gamma}_{--} (1 - r^{-2} e^{-2\varphi} \tilde{\gamma}_{--})(dx^-)^2, \tag{5.4.13}
\]
with \( \tilde{\gamma}_{AB} \) defined in equations (5.4.11) and (5.4.12).

For instance, in these coordinates, the BTZ black hole\[101,102\] is determined by \( \varphi = 0 \) and
\[
\Xi_{\pm\pm} = 2G(M \pm \frac{J}{l}). \tag{5.4.14}
\]

### 5.5 Conformal properties of solution space

By construction, the finite transformations generated by the spacetime vectors (5.2.2) leave the Fefferman-Graham form invariant, and furthermore transform solutions to solutions.

Using light-cone coordinates and the parametrization \( \tilde{\gamma}_{AB} dx^A dx^B = -e^{2\varphi} dx^+ dx^- \), we have
\[
\begin{aligned}
\{ & \xi^r = -\frac{1}{2} \psi r, \quad \psi = \partial_+ Y^+ + \partial_- Y^- + 2\partial_+ \varphi Y^+ + 2\partial_- \varphi Y^-, \\
& \xi^\pm = Y^\pm + \frac{l^2 e^{-2\varphi}}{2r^2} \partial_+ \psi + O(r^{-4}),
\end{aligned}
\]
and get
\[
\begin{aligned}
\mathcal{L}_\xi g_{\pm\pm} &\approx l^2 \left[ Y^\pm \partial_\pm \Xi_{\pm\pm} + 2\partial_\pm Y^\pm \Xi_{\pm\pm} - \frac{1}{2} \partial^2_\pm Y^\pm \right] + O(r^{-2}), \\
\mathcal{L}_\xi g_{+-} &\approx O(r^{-2}). \tag{5.5.1}
\end{aligned}
\]
It follows that the local conformal algebra acts on solution space as

$$-\delta \Xi_{\pm\pm} = Y^\pm \partial_\pm \Xi_{\pm\pm} + 2\partial_\pm Y^\pm \Xi_{\pm\pm} - \frac{1}{2} \partial_\pm^3 Y^\pm,$$  \hspace{1cm} (5.5.2)

and with $\delta \varphi = 0$. Note that the overall minus sign is conventional and chosen so that $\delta \Xi_{\pm\pm} \equiv \delta Y_{\pm\pm} \Xi_{\pm\pm}$ satisfies $[\delta Y_1,\delta Y_2] \Xi_{\pm\pm} = \delta [Y_1,Y_2] \Xi_{\pm\pm}$.

More generally, when considering the extension of the conformal algebra discussed at the end of section 5.2, we find that

$$L_\xi g_{\pm\pm} \approx l^2 [Y^\pm \partial_\pm \Xi_{\pm\pm} + 2\partial_\pm Y^\pm \Xi_{\pm\pm} - \frac{1}{2} \partial_\pm^3 Y^\pm + \partial_\pm^2 \omega - 2\partial_\pm \varphi \partial_\pm \omega] + O(r^{-2}),$$

$$L_\xi g_{+-} \approx 2\omega (-\frac{r^2}{2} e^{2\varphi}) + l^2 \partial_+ \partial_- \omega + O(r^{-2}),$$

and thus, that the extended algebra acts on solution space as in (5.5.2) with in addition $-\delta \varphi = \omega$.

### 5.6 Centrally extended surface charge algebra

Let us take

$$\varphi = 0.$$  \hspace{1cm} (5.6.1)

in this section. In fact, starting from a Fefferman-Graham metric (5.1.1) with $\bar{\gamma}_{AB} = e^{2\varphi} \eta_{AB}$ one can obtain such a metric with vanishing $\varphi(x^C)$ through the finite coordinate transformation generated by $\xi^r = -\varphi r$ and $\xi^A = -l^2 \partial_B \varphi \int_r^\infty \frac{dr'}{r'} g^{AB}(x,r')$ since $L_\xi g_{rr} = 0 = L_\xi g_{rA}$ and $L_\xi g_{AB} = -2\varphi g_{AB}$.

The background metric is then

$$ds^2 = -r^2 d\tau^2 + \frac{l^2}{r^2} dr^2 + r^2 d\phi^2.$$  \hspace{1cm} (5.6.2)

Furthermore,

$$Y^+ = Y^\tau + Y^\phi, \hspace{0.5cm} Y^- = Y^\tau - Y^\phi, \hspace{0.5cm} \Lambda = \hat{\gamma}_{++} + \hat{\gamma}_{--}, \hspace{0.5cm} \Sigma = \hat{\gamma}_{++} - \hat{\gamma}_{--},$$

and $g_{AB}dx^A dx^B = -r^2 d\tau^2 + r^2 d\phi^2 + h_{AB} dx^A dx^B$ with

$$h_{\tau\tau} \approx \Lambda(x) + O(r^{-2}) \approx h_{\phi\phi}, \hspace{0.5cm} h_{\tau\phi} \approx \Sigma(x) + O(r^{-2}),$$

$$\partial_\tau \Lambda = \partial_\phi \Sigma, \hspace{0.5cm} \partial_\tau \Sigma = \partial_\phi \Lambda.$$  \hspace{1cm} (5.6.3)

For the surface charges, we will use the covariant method of [51] whose results are summarized in appendix A. One can prove the linearity of the charges (A.2.4). We will
then just use equations (A.2.5) and (A.2.1) with $n = 3$ and the surface of integration $\partial \Sigma$ is taken to be the circle at infinity. This gives

$$Q_\xi[g - \tilde{g}, \check{g}] = \frac{l}{16\pi G} \lim_{r \to \infty} \int_0^{2\pi} r d\phi \left[\xi^r (\bar{D}^r h - \bar{D}_r h^r) + \int \int \int \frac{1}{2} h (\bar{D}^r \xi^r - \bar{D}_r \xi) - \frac{1}{2} h^r (\bar{D}^r \xi^r - \bar{D}_r \xi^r)\right].$$

(5.6.4)

Using

$$\bar{D}^r h - \bar{D}_r h^r - \bar{D}^r h^r + \bar{D}_r h^r = r^{-2} \bar{g}^{AB} (\bar{D}_A h + \bar{D}_B h) = r^{-4} (\partial_\phi h_{\tau\phi} - \partial_\tau h_{\phi\phi}),$$

$$\bar{D}^r h - \bar{D}_r h^r - \bar{D}^r h^r + \bar{D}_r h^r = \frac{1}{r^2} (\partial_\phi h_{\tau\phi} - \frac{1}{r^2} h_{\tau\tau}),$$

$$\bar{D}^r h^r - \bar{D}_r h^r = \frac{1}{r^2} (\partial_\phi h_{\tau\phi} - \frac{1}{r^2} h_{\tau\phi}),$$

$$\bar{D}^r \xi^r - \bar{D}_r \xi^r = \frac{2r}{r^2} Y^r - \frac{1}{r^2} \partial_\tau \psi + O(r^{-3}),$$

$$\frac{1}{2} h^r (\bar{D}^r \xi^r - \bar{D}_r \xi^r) - \frac{1}{2} h^r (\bar{D}^r \xi^r - \bar{D}_r \xi^r) = \frac{1}{r^2} h_{\tau A} Y^A + \frac{1}{4r^2} h_{\tau A} \partial_\tau \psi + O(r^{-3}),$$

we find explicitly

$$Q_\xi[g - \tilde{g}, \check{g}] = \frac{l}{16\pi G} \lim_{r \to \infty} \int_0^{2\pi} d\phi \left(2 Y^r h_{\phi\phi} + 2 Y^r h_{\tau\phi}\right)$$

$$\approx \frac{1}{8\pi G} \int_0^{2\pi} d\phi \left(Y^r \Lambda + Y^\phi \Sigma\right) = \frac{l}{8\pi G} \int_0^{2\pi} d\phi \left(Y^r \Xi^r + Y^\phi \Xi^\phi\right).$$

(5.6.5)

The appendix [A] suggests that these charges form a representation of the conformal algebra, or more precisely, that

$$Q_{\xi_1}[L_{\xi_2} g, \check{g}] \approx Q_{[\xi_1, \xi_2]} [g - \tilde{g}, \check{g}] + K_{\xi_1, \xi_2},$$

$$K_{\xi_1, \xi_2} = Q_{\xi_1}[L_{\xi_2} \tilde{g}, \check{g}], \quad [\xi_1, \xi_2]_M = [\xi_1, \xi_2] + \delta^\phi_{\xi_1} \xi_2 - \delta^\phi_{\xi_2} \xi_1.$$  

(5.6.6)

(5.6.7)

An asymptotic Killing vector of the form (5.2.2) depends on the metric, $\xi = [x, \tilde{g}]$ and $\delta^\phi_{\xi_1} \xi_2 = [x, L_{\xi_1}\check{g}]$. From $\delta^\phi_{\xi_1} \xi_2 = O(r^{-4})$ and $\delta^\phi_{\xi_1} \xi_2 = O(r^{-4})$, it follows that only the Lie bracket $[\xi_1, \xi_2]$ contributes on the right hand side, $Q_{\xi_1, \xi_2} [g - \tilde{g}, \check{g}] = Q_{\xi_1, \xi_2} [g - \tilde{g}, \check{g}]$. Using (5.5.1), (5.6.3) and integrations by parts in $\partial_\phi$ and the conformal Killing equation for $Y^A, Y^\phi$ to evaluate the left hand side, one indeed finds

$$Q_{\xi_1}[L_{\xi_2} g, \check{g}] \approx Q_{[\xi_1, \xi_2]} [g - \tilde{g}, \check{g}] + K_{\xi_1, \xi_2},$$

$$K_{\xi_1, \xi_2} = \frac{l}{8\pi G} \int_0^{2\pi} d\phi \left(\partial_\phi Y^1 \partial^2 Y^\phi - \partial_\phi Y^2 \partial^2 Y^\phi\right).$$

(5.6.8)
where $K_{\xi_1,\xi_2}$ is a form of the well-known Brown-Henneaux central charge.

In addition, the covariant expression for the surface charges used above coincides on-shell with those of the Hamiltonian formalism \[51\,53\]. In this context, it follows from the analysis of \[103\,12\,54\] that the surface charge is, after the Fefferman-Graham gauge fixation, the canonical generator of the conformal transformations in the Dirac bracket.

We obtain the numerical value of the central charges $c^\pm$ by evaluating $K_{\xi_1,\xi_2}$ on the generators $l^\pm$:

$$K_{l_m,n}^{+} = i\frac{l}{8G}m^3, \quad K_{l_m,n}^{-} = i\frac{l}{8G}m^3, \quad K_{l_m,n}^{0} = 0,$$

(5.6.9)

which gives $c^+ = c^- = \frac{3l}{2G}$. Remark that this extension is not the one we gave in section 5.3 but an equivalent one. To transform it into the form (5.3.3), one has to redefine $Q_{l_0}^{\pm}$ as $Q_{l_0}^{\pm} + \frac{c^\pm}{24}$. This change corresponds to a change of background from the BTZ black hole with $M = J = 0$ to $AdS_3$. 

Chapter 6

BMS$_3$/CFT$_1$ correspondence

In 3 dimensions, the asymptotic symmetry algebra of asymptotically flat spacetimes at null infinity has been derived in [104, 105]. This algebra, known as bms$_3$ is the semi-direct sum of the conformal transformation of the circle with the abelian algebra of the function on the circle. The algebra of the associated charges has been shown to provide a centrally extended representation of bms$_3$, which has been related by a contraction, similar to that from so(2, 2) to iso(2, 1), to the centrally extended Poisson bracket algebra of surface charges of asymptotically anti-de Sitter spacetimes in 3 dimensions [106].

In this chapter, we will apply the same technique we used on asymptotically AdS$_3$ spacetimes to the case of asymptotically flat spacetimes at null infinity. We start with asymptotically flat metrics at null infinity in a form suggested from the analysis of the 4 dimensional case by Sachs. This form is the analog of the Fefferman-Graham form. In particular, the gauge is completely fixed in the sense that all subleading orders of the asymptotic Killing vectors are again completely determined. Equipped with the modified Lie bracket those vectors form a representation of the bms$_3$ algebra to all orders in $r$. We then study the possible central extensions of the bms$_3$.

After that, we solve the flat equations of motion and compute the representation of the bms$_3$ algebra on solution space. The last section contains the derivation of the surface charge algebra using the same techniques we used for the AdS$_3$ case.

6.1 Asymptotically BMS$_3$ spacetimes

We consider metrics of the form

$$ds^2 = e^{2\beta} \frac{V}{r} du^2 - 2 e^{2\beta} dudr + r^2 e^{2\phi}(d\phi - U du)^2,$$

(6.1.1)
or, equivalently,

\[ g_{\mu\nu} = \begin{pmatrix} e^{2\beta} V r^{-1} + r^2 e^{2\varphi} U^2 & -e^{2\beta} & -r^2 e^{2\varphi} U \\ -e^{2\beta} & 0 & 0 \\ -r^2 e^{2\varphi} U & 0 & r^2 e^{2\varphi} \end{pmatrix} \]

with inverse given by

\[ g^{\mu\nu} = \begin{pmatrix} 0 & -e^{-2\beta} & 0 \\ -e^{-2\beta} & \frac{V}{r} e^{-2\beta} - U e^{-2\beta} & 0 \\ 0 & -U e^{-2\beta} & r^{-2} e^{-2\beta} \end{pmatrix}. \]

Here, \( \varphi = \varphi(u, \phi) \). Three dimensional Minkowski space is described by \( \varphi = 0 = \beta = U \) and \( V = -r \). The fall-off conditions are taken as \( \beta = O(r^{-1}), U = O(r^{-2}) \) and \( V = -2r^2 \partial_u \varphi + O(r) \). In particular, \( g_{uu} = -2r \partial_u \varphi + O(1) \).

### 6.2 Asymptotic symmetries

The transformations leaving this form of the metric invariant are generated by vector fields such that

\[ \mathcal{L}_\xi g_{rr} = 0 = \mathcal{L}_\xi g_{r\phi}, \quad \mathcal{L}_\xi g_{\phi\phi} = 0, \quad \mathcal{L}_\xi g_{uu} = 0, \quad \mathcal{L}_\xi g_{u\phi} = O(1), \quad \mathcal{L}_\xi g_{uu} = O(1). \]  

Equations (6.2.1) imply that

\[
\begin{cases}
\xi^u &= f, \\
\xi^\phi &= Y + I, \\
\xi^r &= -r [\partial_\phi \xi^\phi - \partial_\phi f U + \xi^\phi \partial_\phi \varphi + f \partial_\phi \varphi],
\end{cases}
\]

with \( \partial_r f = 0 = \partial_r Y \). The first equation of (6.2.2) then implies that

\[
\partial_u f = f \partial_u \varphi + Y \partial_\phi \varphi + \partial_\phi Y \quad \iff \quad f = e^\varphi [T + \int_0^u du' e^{-\varphi} (\partial_\phi Y + Y \partial_\phi \varphi)],
\]

with \( T = T(\phi) \), while the second requires \( \partial_u Y = 0 \) and thus \( Y = Y(\phi) \), which implies in turn that the last one is identically satisfied.

The Lie algebra \( \mathfrak{bms}_3 \) is determined by two arbitrary functions \( (Y, T) \) on the circle with bracket \([Y_1, T_1], (Y_2, T_2)] = (\widehat{Y}, \widehat{T})\) determined by \( \widehat{Y} = Y_1 \partial_\phi Y_2 - (1 \leftrightarrow 2) \) and \( \widehat{T} = Y_1 \partial_\phi T_2 + T_1 \partial_\phi Y_2 - (1 \leftrightarrow 2) \). Let \( \mathcal{L} = S^1 \times \mathbb{R} \) with coordinates \( u, \phi \) and consider the vector fields \( \xi = f \frac{\partial}{\partial u} + Y \frac{\partial}{\partial \phi} \) with \( f \) as in (6.2.4) and \( Y = Y(\phi) \). By direct computation, it follows that these vector fields equipped with the commutator bracket provide a faithful representation of \( \mathfrak{bms}_3 \). Furthermore:
The spacetime vectors (6.2.3), with \( f \) given in (6.2.4) and \( Y = Y(\phi) \) form a faithful representation of the \( bms_3 \) Lie algebra on an asymptotically flat spacetime of the form (6.1.1) when equipped with the modified bracket \([\cdot, \cdot]\)_M.

Indeed, for the \( u \) component, there is no modification due to the change in the metric and the result follows by direct computation. As a consequence, \( \hat{f} = [\xi_1, \xi_2]^u_M \) corresponds to \( f \) in (6.2.4) with \( T \) replaced by \( \hat{T} \) and \( Y \) by \( \hat{Y} \). By evaluating \( \mathcal{L}_\xi g^{\mu\nu} \), we find

\[
\begin{align*}
\delta_\xi \varphi &= 0, \\
\delta_\xi \beta &= \xi^\alpha \partial_\alpha \beta + \frac{1}{2} [\partial_a f + \partial_r \xi^r + \partial_\phi f U], \\
\delta_\xi U &= \xi^\alpha \partial_\alpha U + U [\partial_a f + \partial_\phi f U - \partial_\phi \xi^\phi] - \partial_u \xi^\phi - \partial_r \xi^r \frac{Y}{r} + \partial_\phi \xi^r \frac{e^{2(\beta - \varphi)}}{r^2}.
\end{align*}
\]

(6.2.5)

It follows that

\[
\begin{align*}
\delta_\xi^0 \xi^r &= -e^{-2\varphi} \partial_\phi f \int_r^\infty \frac{du'}{u'^2} r^3 \partial_\xi \beta, \\
\delta_\xi^0 \xi^r &= -r (\partial_\phi (\delta_\xi^0 \xi^r) + (\delta_\xi^0 \xi^r) \partial_\phi \varphi - \partial_\phi \xi^r \partial_\xi U).
\end{align*}
\]

(6.2.6)

We also have \( \lim_{r \to \infty} [\xi_1, \xi_2]^0_M = \hat{Y} \). Using \( \partial_r \xi^\phi = \frac{e^{2(\beta - \varphi)}}{r} \partial_\phi f \), (6.2.4) and the expression of \( \xi^r \) in (6.2.3), it follows by a straightforward computation that \( \partial_r ([\xi_1, \xi_2]^0_M) = \frac{e^{2(\beta - \varphi)}}{r} \partial_\phi \hat{f} \), which gives the result for the \( \phi \) component. Finally, for the \( r \) component, we need the relation

\[
\partial_r (\frac{\xi^r}{r}) = -\partial_r (\partial_\phi \xi^\phi + \xi^\phi \partial_\phi f \partial_\phi \varphi - \partial_\phi f U).
\]

We then have \( \lim_{r \to \infty} \frac{[\xi_1, \xi_2]^r_M}{r} = -\partial_\phi \hat{Y} - \hat{Y} \partial_\phi \varphi - \hat{f} \partial_u \varphi \), while direct computation shows that \( \partial_r (\frac{[\xi_1, \xi_2]^r_M}{r}) = -\partial_r (\partial_\phi ([\xi_1, \xi_2]^0_M) - \partial_\phi ([\xi_1, \xi_2]^u_M) U + [\xi_1, \xi_2]^0_M \partial_\phi \varphi \), which gives the result for the \( r \) component.

More generally, one can also consider the transformations that leave the form of the metric (6.1.1) invariant up to a rescaling of \( \varphi \) by \( \omega(u, \phi) \). They are generated by spacetime vectors satisfying

\[
\mathcal{L}_\xi g_{rr} = 0 = \mathcal{L}_\xi g_{r\phi}, \quad \mathcal{L}_\xi g_{\phi\phi} = 2\omega g_{\phi\phi}, \quad \mathcal{L}_\xi g_{uu} = O(r^{-1}), \quad \mathcal{L}_\xi g_{u\phi} = O(1), \quad \mathcal{L}_\xi g_{u\phi} = -2r \partial_u \omega + O(1).
\]

(6.2.7)

(6.2.8)

Equations (6.2.7), (6.2.8) then imply that the vectors are given by (6.2.3), (6.2.4) with the replacement \( \partial_\phi Y \to \partial_\phi \hat{Y} = -\omega \). With this replacement, the vector fields \( \bar{\xi} = \frac{1}{f} \frac{\partial}{\partial u} + \frac{\partial Y}{\partial \phi} \) on \( \mathcal{T} = S^1 \times \mathbb{R} \) equipped with the modified bracket provide a faithful representation of the extension of \( bms_3 \) defined by elements \( (Y, T, \omega) \) and bracket \( [(Y_1, T_1, \omega_1), (Y_2, T_2, \omega_2)] = (\bar{Y}, \bar{T}, \bar{\omega}) \), with \( \bar{Y}, \bar{T} \) as before and \( \bar{\omega} = 0 \).

Indeed, the result is obvious for the \( \phi \) component. Furthermore,

\[
\delta_\xi^0 f_2 = \omega_1 f_2 + e^\varphi \int_0^u du' e^{-\varphi} [-\omega_1 (\partial_\phi Y_2 - \omega_2 + Y_2 \partial_\phi \varphi) + Y_2 \partial_\phi \omega_1].
\]
At $u = 0$, we get $[\xi_1, \xi_2]_M^u|_{u=0} = e^\varphi|_{u=0}\hat{T}$, while direct computation shows that $\partial_u([\xi_1, \xi_2]_M^u) = \hat{f}\partial_u\varphi + \hat{Y}\partial_u\varphi + \partial_u\hat{Y}$, as it should.

Following the same reasoning as before, one can then also show that the spacetime vectors $[6.2.3]$ with the replacement discussed above and equipped with the modified Lie bracket provide a faithful representation of the extended $\mathfrak{bms}_3$ algebra.

Indeed, we have $\xi = \xi + I\frac{\partial}{\partial y} + \xi^r\frac{\partial}{\partial r}$. Furthermore, $[\xi_1, \xi_2]_M = [[\xi_1, \xi_2]]_M = \hat{f}$ as it should. In the extended case, the variations of $\beta, U$ are still given by $[7.4.3]$. We then have

$$\lim_{r\to\infty}[\xi_1, \xi_2]_M = \hat{Y},$$

and find, after some computations, $\delta_r([\xi_1, \xi_2]_M) = -\hat{\delta}_r\hat{Y} - \hat{Y}\partial_u\varphi - \hat{f}\partial_u\varphi$, while direct computation shows that $\delta_r([\xi_1, \xi_2]_M) = -\hat{\delta}_r([\xi_1, \xi_2]_M)\phi + \partial_r$$

$$= (\phi^n + 1)\phi^n,$$

giving the result for the $\phi$ component. Finally, we have

$$\lim_{r\to\infty}\frac{\xi_1, \xi_2]_M^\phi}{r} = -\hat{\phi}\hat{Y} - \hat{Y}\partial_u\varphi - \hat{f}\partial_u\varphi,$$

while direct computation shows that $\delta_r([\xi_1, \xi_2]_M^\phi) = -\hat{\delta}_r([\xi_1, \xi_2]_M^\phi)\phi + \partial_r$$

$$= (\phi^n + 1)\phi^n,$$

which gives the result for the $r$ component.

### 6.3 $\mathfrak{bms}_3$ algebra and central extensions

The $\mathfrak{bms}_3$ algebra can also be viewed as the algebra of vector fields on the circle acting on the functions of the circle and has been originally derived in the context of a symmetry reduction of four dimensional gravitational waves [105] [104].

More precisely, let $y = Y\frac{\partial}{\partial y} \in \text{Vect}(S^1)$ be the vector fields on the circle and $T(d\phi)^{-\lambda} \in \mathcal{F}_\lambda(S^1)$ tensor densities of degree $\lambda$, which form a module of the Lie algebra $\text{Vect}(S^1)$ for the action

$$\rho(y)t = (YT - \lambda Y'T)d\phi^{-\lambda}.$$  

(6.3.1)

The algebra $\mathfrak{bms}_3$ is the semi-direct sum of $\text{Vect}(S^1)$ with the abelian ideal $\mathcal{F}_1(S^1)$, the bracket between elements of $\text{Vect}(S^1)$ and elements $t = Td\phi^{-1} \in \mathcal{F}_1(S^1)$ being induced by the module action, $[y, t] = \rho(y)t$.

Consider the associated complexified Lie algebra and let $z = e^{i\phi}, m, n, k... \in \mathbb{Z}$. Expanding into modes, $y = a^n l_n, t = b^m t_n$, where

$$l_n = e^{in\phi}\frac{\partial}{\partial y} = iz^{n+1}\frac{\partial}{\partial z}, \quad t_n = e^{in\phi}(d\phi)^{-1} = iz^{n+1}(dz)^{-1},$$

the commutation relations read explicitly

$$i[l_n, l_m] = (m - n)l_{m+n}, \quad i[l_n, t_m] = (m - n)t_{m+n}, \quad i[t_m, t_n] = 0.$$  

(6.3.2)

The non-vanishing structure constants of $\mathfrak{bms}_3$ are thus entirely determined by the structure constants $[l_n, l_m] = -if_m^k l_k, f_m^k = \delta_m^{m+n}(m - n)$ of the Witt subalgebra $\mathfrak{w}$ defined by the linear span of the $l_n$.

Up to equivalence, the most general central extension of $\mathfrak{bms}_3$ is given by

$$\begin{cases}
\hat{i}[l_n, l_m] = (m - n)l_{m+n} + \frac{4}{12}m(m + 1)(m - 1)\delta_{m+n}^{m+n}, \\
\hat{i}[l_n, t_m] = (m - n)t_{m+n} + \frac{4}{12}m(m + 1)(m - 1)\delta_{m+n}^{m+n}, \\
\hat{i}[t_m, t_n] = 0.
\end{cases}$$  

(6.3.3)
Proof. In order to get rid of the overall $i$ in (6.3.2), we redefine the generators as $l'_m = il_m$. Inequivalent central extensions of $bms_3$ are classified by the cohomology space $H^2(bms_3)$. More explicitly, the Chevalley-Eilenberg differential is given by

$$
\gamma = -\frac{1}{2} C^m C^{k-m} (2m - k) \frac{\partial}{\partial C^k} - C^m \xi^{k-m} (2m - k) \frac{\partial}{\partial \xi^k},
$$

(6.3.4)

in the space $\Lambda(C, \xi)$ of polynomials in the anticommuting “ghost” variables $C^m, \xi^m$. The grading is given by the eigenvalues of the ghost number operator, $N_{C,\xi} = C^m \frac{\partial}{\partial C^m} + \xi^m \frac{\partial}{\partial \xi^m}$, the differential $\gamma$ being homogeneous of degree 1 and $H^2(bms_3) \cong H^2(\gamma, \Lambda(C, \xi))$. Furthermore, when counting only the ghosts $\xi^m$ associated with supertranslations, $N_\xi = \xi^m \frac{\partial}{\partial \xi^m}$, the differential $\gamma$ is homogeneous of degree 0, so that the cohomology decomposes into components of definite $N_\xi$ degree. The cocycle condition then becomes

$$
\gamma(\omega^0_{m,n} C^m C^n) = 0, \quad \gamma(\omega^1_{m,n} C^m \xi^n) = 0, \quad \gamma(\omega^2_{m,n} \xi^m \xi^n) = 0,
$$

(6.3.5)

with $\omega^0_{m,n} = -\omega^0_{n,m}$ and $\omega^2_{m,n} = -\omega^2_{n,m}$. The coboundary condition reads

$$
\omega^0_{m,n} C^m C^n = \gamma(\eta^0_{m} C^m), \quad \omega^1_{m,n} C^m \xi^n = \gamma(\eta^1_{m} \xi^n).
$$

(6.3.6)

We have $\{ \frac{\partial}{\partial C^0}, \gamma \} = N_{C,\xi}$ with $N_{C,\xi} = m(C^m \frac{\partial}{\partial C^m} + \xi^m \frac{\partial}{\partial \xi^m})$. It follows that all cocycles of $N_{C,\xi}$ degree different from 0 are coboundaries, $\gamma_N = 0$, $N_{C,\xi} \omega_N = N \omega_N$, $N \neq 0$ implies that $\omega_N = \gamma(1_N \frac{\partial}{\partial C^0} \omega_N)$. Without loss of generality we can thus assume that $\omega^0_{m,n} C^m C^n = \omega^0_m C^m C^{-m}$ with $\omega^0_m = -\omega^0_{m,0}$ and in particular $\omega^0_0 = 0$; $\omega^1_m C^m \xi^n = \omega^1_m C^m \xi^{-m}$; $\omega^2_m \xi^m \xi^n = \omega^2_m \xi^m \xi^{-m}$ with $\omega^2_m = -\omega^2_{-m}$ and in particular $\omega^2_0 = 0$. By applying $\frac{\partial}{\partial C^0}$ to the coboundary condition $\omega^0_m C^m C^{-m} = \gamma(\eta^0_{m} C^m)$ we find that $0 = m \eta^0_{m} C^m$. The coboundary condition then gives $\omega^0_m C^m C^{-m} = \gamma(\eta^0_{m} C^m) = -m \eta^0_{m} C^m C^{-m}$. By adjusting $\eta^0_{m}$, we can thus assume without loss of generality that $\eta^0_{m} = 0$ and that the coboundary condition has been entirely used. In the same way $\omega^1_m C^m \xi^{-m} = \gamma(\eta^1_{m} \xi^{m})$ implies first that $\eta^1_{m} = 0$ for $m \neq 0$ and then that one can assume that $\omega^1_m = 0$, with no coboundary condition left.

Taking into account the anticommuting nature of the ghosts, the cocycle conditions become explicitly, $\omega^0_m (2n + m) - \omega^0_n (2m + n) + \omega^0_{m+n} (n - m) = 0, \omega^1_m (2n - m) + \omega^1_n (n - 2m) + \omega^1_{m-n} (n + m) = 0, \omega^2_m (2n + m) + \omega^2_{m+n} (n - m) = 0$. Putting $m = 0$ in the last relation gives $\omega^2_0 = 0$, for $m \neq 0$ and thus for all $m$, putting $m = 1 = n$ in the second relation gives $\omega^1_0 = 0$, while $m = 0$ gives $\omega^1_0 n = -\omega^1_{-1} n$ and thus that $\omega^1_{-n} = -\omega^1_{n}$ for all $n$. Changing $m$ to $-m$ and using this symmetry property, the cocycle conditions for $\omega^0_m$ and $\omega^1_m$ give the same constraints. Putting $m = 1$, one finds the recurrence relation $\omega^0_{n+1} = \frac{n+2}{n+1} \omega^0_n$, which gives a unique solution in terms of $\omega^0_{n}$. The result follows by
setting $c_{1,2} = \frac{1}{2}\omega_{2}^{0,1}$ and checking that the constructed solution does indeed satisfy the cocycle condition.

The above result can also be obtained by a “flat” limit ($l \to \infty$) of the $AdS_3$ result of the previous chapter. First, we write the the 2D conformal charge algebra (5.3.3) in term of the new generators $\tilde{l}_m = l_m^+ - l_m^-$, $\tilde{t}_m = \frac{1}{l}(l_m^+ + l_m^-)$:

$$
\begin{aligned}
&i[\tilde{l}_m, \tilde{l}_n] = (m-n)\tilde{l}_{m+n} + \frac{c^+ - c^-}{12} m(m+1)(m-1)\delta_{m+n}^0, \\
&i[\tilde{l}_m, \tilde{t}_n] = (m-n)\tilde{t}_{m+n} + \frac{c^+ + c^-}{12} m(m+1)(m-1)\delta_{m+n}^0, \\
&i[\tilde{t}_m, \tilde{t}_n] = \frac{1}{l^2} ((m-n)\tilde{l}_{m+n} + \frac{c^+ - c^-}{12} m(m^2 - 1)) .
\end{aligned}
$$

(6.3.7)

Second, we take the limit $l \to \infty$: the new generators reduces to the generators of $bms_3$ if $u = l\tau$ and the algebra goes to (6.3.3) with $c_1 = c^+ - c^-$ and $c_2 = \frac{c^+ + c^-}{l}$.

### 6.4 Solution space

Following [2], the equations of motion are organized in terms of the Einstein tensor $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$ as

$$
\begin{align*}
G_{\alpha\beta} &= 0, \\
G_{AB} - \frac{1}{2} g_{AB} g^{CD} G_{CD} &= 0, \\
G_{uu} &= 0 = G_{uA}, \\
g^{CD} G_{CD} &= 0,
\end{align*}
$$

(6.4.1) \quad (6.4.2) \quad (6.4.3)

and the Bianchi identities are written as

$$
0 = 2\sqrt{-g}G_{\alpha\beta}^{\beta} = 2(\sqrt{-g}G_{\alpha}^{\beta})_{,\beta} + \sqrt{-g}G_{\beta\gamma} g^{\beta\gamma} ,\alpha .
$$

(6.4.4)
For a metric of the form (6.1.1), we have

$$\Gamma^\lambda_{rr} = \delta^\lambda_{2\beta, r}, \quad \Gamma^u_{\lambda r} = 0, \quad \Gamma^r_{\phi r} = \beta_{, \phi} + n, \quad \Gamma^\phi_{\phi r} = \frac{1}{r},$$

$$\Gamma^u_{\phi \phi} = e^{-2\beta + 2\varphi} r, \quad \Gamma^\phi_{\phi \phi} = e^{-2\beta + 2\varphi} U r + \partial_\phi \varphi,$$

$$\Gamma^\phi_{ur} = -\frac{1}{r} U + r^2 e^{2\beta - 2\varphi} (\partial_\phi \beta - n), \quad \Gamma^u_{u \phi} = \beta_{, \phi} - n - e^{-2\beta + 2\varphi} r U,$$

$$\Gamma^r_{ur} = -\frac{1}{2} \left( \partial_r + 2\beta, r \right) \frac{V}{r} - (\beta_{, \phi} + n) U, \quad \Gamma^\phi_{u \phi} = \partial_u \varphi + U (\beta_{, \phi} - n) - e^{-2\beta + 2\varphi} U^2,$$

$$\Gamma^u_{uu} = 2\beta_{, u} + \frac{1}{2} \left( \partial_r + 2\beta, r \right) \frac{V}{r} + 2U n + e^{-2\beta + 2\varphi} U^2,$$

$$\Gamma^r_{\phi \phi} = e^{-2\beta + 2\varphi} (r^2 \partial_\phi U + r^2 \partial_\phi \varphi U + r^2 \partial_u \varphi + V),$$

$$\Gamma^u_{u \phi} = -\frac{V_{, \phi}}{2r} - n \frac{V}{r} - e^{-2\beta + 2\varphi} [r^2 \partial_\phi U + r^2 \partial_\phi \varphi U - r^2 \partial_u \varphi + V],$$

$$\Gamma^\phi_{uu} = 2U \beta_{, u} + \frac{1}{2} U (\partial_r + 2\beta, r) \frac{V}{r} + 2U^2 n + r e^{-2\beta + 2\varphi} U^3 - U, u - 2\partial_u \varphi U$$

$$-\frac{1}{2} e^{2\beta - 2\varphi} r^{-2} (\partial_\phi + 2\partial_\phi \beta) \frac{V}{r} - U (\partial_\phi + \partial_\phi \varphi) U,$$

$$\Gamma^r_{uu} = -\frac{1}{2} \left( \partial_u - 2\partial_u \beta \right) \frac{V}{r} + \frac{1}{2} \frac{V}{r} (\partial_r + 2\partial_r \beta) \frac{V}{r} + V re^{2\varphi - 2\beta} U (\partial_r + \frac{1}{r}) U + r^2 e^{-2\beta + 2\varphi} U^2 \partial_u \varphi$$

$$+ \frac{1}{2} U (\partial_\phi + 2\partial_\phi \beta) \frac{V}{r} + \frac{1}{2} r^2 e^{-2\beta + 2\varphi} U (\partial_\phi + 2\partial_\phi \varphi) U,$$

where the notation $$n = \frac{1}{2} r^2 e^{2\varphi - 2\beta} \partial_u \varphi U$$ has been used.

We start with $$G_{rr} = 0$$. From

$$G_{rr} = R_{rr} = \frac{2}{r} \partial_r \beta,$$

we find $$\beta = 0$$ by taking the fall-off conditions into account. From

$$G_{r \phi} = R_{r \phi} = (\partial_r + \frac{1}{r}) n + \frac{1}{r} \partial_\phi \beta,$$

we then obtain, by using the previous result, that $$n = \frac{N}{r}$$ where the integration constant $$N = N(u, \phi)$$. Using the definition of $$n$$, we get $$U = -r^{-2} e^{-2\varphi} N$$. From $$G_{ru} = -g_{\phi \phi} R_{\phi \phi}$$ and

$$R_{\phi \phi} = e^{-2\beta + 2\varphi} \left( \left( \partial_r - \frac{1}{r} \right) V + 2r \partial_u \varphi + 2r (\partial_\phi + \partial_\phi \varphi) U \right)$$

$$- 2\partial_\phi^2 \beta + 2\partial_\phi \beta \partial_\phi \varphi - 2(\partial_\phi \beta - n)^2 - 2\partial_\phi \varphi n + 2\partial_\phi n$$

$$= e^{2\varphi} \left( \left( \partial_r - \frac{1}{r} \right) V + 2r \partial_u \varphi \right) - 2r^{-2} N^2,$$

we get $$\partial_r \left( \frac{V}{r} \right) = -2\partial_u \varphi + 2r^{-3} e^{-2\varphi} N^2$$ and then

$$V = -2r^2 \partial_u \varphi + r M - r^{-1} e^{-2\varphi} N^2,$$
for an additional integration constant \( M = M(u, \phi) \).

When \( G_{rr} = G_{r\phi} = G_{ru} = 0 \), the Bianchi identity (6.4.4) for \( \alpha = r \) implies that \( G_{\phi\phi} = 0 \). This implies in turn that \( R = 0 \). The Bianchi identity for \( \alpha = \phi \) then gives \( \partial_r (r G_{u\phi}) = 0 \). When \( G_{u\phi} = 0 \), the Bianchi identity for \( \alpha = u \) gives \( \partial_r (r G_{uu}) = 0 \). To solve the remaining equations of motion, there thus remain only the constraints

\[
\lim_{r \to \infty} r R_{u\phi} = 0, \quad \lim_{r \to \infty} r R_{uu} = 0.
\]

to be fulfilled. From

\[
R_{u\phi} = \frac{1}{r} \left( - (\partial_u + \partial_u \varphi) N + \frac{1}{2} \partial_\phi M \right) + O(r^{-2}),
\]

we get

\[
N = e^{-\varphi} \Xi(\phi) + e^{-\varphi} \int_{u_0}^u du \, e^\varphi \frac{1}{2} \partial_\phi M.
\]

while

\[
R_{uu} = \frac{1}{r} \left( - \frac{1}{2} (\partial_u + 2 \partial_u \varphi) M + e^{-2\varphi} \partial_u (\partial_\phi^2 \varphi - \frac{1}{2} (\partial_\phi \varphi)^2) \right) + O(r^{-2})
\]

implies

\[
M = e^{-2\varphi} [\Theta(\phi) - (\partial_\phi \varphi)^2 + 2 \partial_\phi^2 \varphi].
\]

We thus have shown:

**For metrics of the form (6.1.1) with \( \lim_{r \to \infty} \beta = 0 \), the general solution to the equations of motions is given by**

\[
ds^2 = s_{uu} du^2 - 2 du dr + 2 s_{u\phi} du d\phi + r^2 e^{2\varphi} d\phi^2,
\]

\[
s_{uu} = e^{-2\varphi} [\Theta - (\partial_\phi \varphi)^2 + 2 \partial_\phi^2 \varphi] - 2r \partial_u \varphi,
\]

\[
s_{u\phi} = e^{-\varphi} \left[ \Xi + \int_{u_0}^u du \, e^{-\varphi} \left[ \frac{1}{2} \partial_\phi \Theta - \partial_\phi \varphi \Theta - (\partial_\phi \varphi)^2 + 3 \partial_\phi^2 \varphi + \partial_\phi^3 \varphi \right] \right],
\]

where \( \Theta = \Theta(\phi) \) and \( \Xi = \Xi(\phi) \) are arbitrary functions.

### 6.5 Conformal properties of solution space

By computing \( L_\xi \xi_{\mu\nu} \), we find that the asymptotic symmetry algebra \( \mathfrak{bms}_3 \) acts on solution space according to

\[
-\delta \Theta = Y \partial_\phi \Theta + 2 \partial_\phi Y \Theta - 2 \partial_\phi^3 Y,
\]

\[
-\delta \Xi = Y \partial_\phi \Xi + 2 \partial_\phi Y \Xi + \frac{1}{2} T \partial_\phi \Theta + \partial_\phi T \Theta - \partial_\phi^3 T,
\]

\[
-\delta \varphi = 0.
\]

(6.5.1)

For the extended algebra, the first two relations are unchanged, while \( -\delta \varphi = \omega \).
6.6 Centrally extended surface charge algebra

Let us again take $\varphi = 0$ in this section. For the surface charges computed at the circle at infinity $u = \text{cte}, r = \text{cte} \to \infty$, one starts again from (A.2.3). In this case, one can again prove linearity of the charges (A.2.4) and simplify the expression for the charges to (A.2.5). The background line element, which is used to raise and lower indices, is

$$ds^2 = -du^2 - 2udu + r^2d\phi^2,$$

(6.6.1)

This gives

$$Q_\xi[g - \bar{g}, \bar{g}] = \frac{1}{16\pi G} \lim_{r \to \infty} \int_{0}^{2\pi} r d\phi \left[ \xi^*(\bar{D}^u h - D_\sigma h^u\sigma + \bar{D}^r h^u_r - \bar{D}^u h^r_u) 
- \xi^u(\bar{D}^r h - D_\sigma h^r\sigma - \bar{D}^r h^u_u + \bar{D}^u h^r_u) + \xi^\phi(\bar{D}^r h^u_\phi - \bar{D}^u h^r_\phi) + \frac{1}{2} h(\bar{D}^r \xi^u - \bar{D}^u \xi^r) 
+ \frac{1}{2} h^\sigma(\bar{D}^u \xi_\sigma - D_\sigma \xi^u) - \frac{1}{2} h^\sigma(\bar{D}^r \xi_\sigma - D_\sigma \xi^r) \right].$$

(6.6.2)

Using

$$\bar{D}^u h - D_\sigma h^u\sigma + \bar{D}^r h^u_r = -\frac{1}{r} h_{u\sigma},$$
$$D^r h - D_\sigma h^r\sigma - \bar{D}^r h^u_u + \bar{D}^u h^r_u = -\frac{1}{r} h_{u\sigma} + \frac{2}{r^2} h_{u\phi} + \frac{1}{r^2} \partial_\phi h_{u\phi},$$
$$\bar{D}^r h^u_\phi - \bar{D}^u h^r_\phi = (\frac{1}{r} - \partial_r) h_{u\phi},$$
$$\bar{D}^r \xi^u - \bar{D}^u \xi^r = -2\partial_\phi Y + O(1),$$

we get

$$Q_\xi[g - \bar{g}, \bar{g}] = \frac{1}{16\pi G} \lim_{r \to \infty} \int_{0}^{2\pi} d\phi \left[ (r h_{u\sigma} + uh_{u\sigma}) \partial_\phi Y + h_{u\sigma} T + 2h_{u\phi} Y \right]$$

$$\approx \frac{1}{16\pi G} \int_{0}^{2\pi} d\phi \left[ (\Theta + 1)T + 2\Xi Y \right].$$

(6.6.3)

The surface charges thus provide the inner product that turns the linear spaces of solutions and asymptotic symmetries into dual spaces. It follows that the solutions that we have constructed are all non-trivial as different solutions carry different charges.

From $\delta^t_\xi \xi^u = 0$, $\delta_\xi^t \xi^u = O(r^{-2})$ and $\delta_\xi^t \xi^r = O(r^{-1})$, it follows that only the Lie bracket $[\xi_1, \xi_2]$ contributes on the right hand side of (5.6.6)-(5.6.7). $Q_{[\xi_1, \xi_2]}[g - \bar{g}, \bar{g}] = Q_{[\xi_1, \xi_2]}[g - \bar{g}, \bar{g}]$. Using (6.6.3), (6.5.1) and integrations by parts in $\partial_\phi$ to evaluate the left hand side, one indeed finds

$$Q_{\xi_1}[\mathcal{L}_{\xi_2} g, \bar{g}] \approx Q_{[\xi_1, \xi_2]}[g - \bar{g}, \bar{g}] + K_{\xi_1, \xi_2},$$

(6.6.4)

$$K_{\xi_1, \xi_2} = \frac{1}{8\pi G} \int_{0}^{2\pi} d\phi \left[ \partial_\phi Y_1(T_2 + \partial_\phi^2 T_2) - \partial_\phi Y_2(T_1 + \partial_\phi^2 T_1) \right].$$

(6.6.5)
where $K_{\xi_1, \xi_2}$ is the central charge.

The associated numerical value for the classical central charge is obtained by evaluating $K_{\xi_1, \xi_2}$ for the generators $l_n$ and $t_n$. We obtain

$$c_1 = 0, \quad c_2 = \frac{3}{G}.$$ (6.6.6)

which is the result originally derived in [106]. We saw at the end of section 6.3 that the general form of the $\mathfrak{bms}_3$ algebra can be obtained by a “flat” limit coming from the 2D conformal algebra. This flat limit applied to the anti-de-Sitter case studied in the previous chapter ($c_{\pm} = \frac{3l}{2G}$) leads to the same value for the $\mathfrak{bms}_3$ central charges.

An important question is a complete understanding of the physically relevant representations of $\mathfrak{bms}_3$. Note that in the present gravitational context, the Hamiltonian is associated with $t_0$, so that one is especially interested in representations with a lowest eigenvalue of $t_0$. This question should be tractable, given all that is known on both the Poincaré and Virasoro subalgebras of $\mathfrak{bms}_3$.

It turns out that $\mathfrak{bms}_3$ is isomorphic to the Galilean conformal algebra in 2 dimensions $\mathfrak{gca}_2$ [107]. In a different context, a class of non-unitary representations of $\mathfrak{gca}_2$ have been studied in some details [108].
Chapter 7

BMS$_4$/CFT$_2$ correspondence

This last chapter is devoted to the study of asymptotically flat spacetimes at null infinity in 4 dimensions. As we said in the introduction, this is the first example where the asymptotic symmetry group is enhanced with respect to the isometry group of the background metric and becomes infinite-dimensional [15, 16, 17]. In this case, the induced metric is 2-dimensional because the boundary is a null surface. The asymptotic symmetry group of non singular transformations is the well-known Bondi-Metzner-Sachs group. It consists of the semi-direct product of the group of globally defined conformal transformations of the unit 2-sphere, which is isomorphic to the orthochronous homogeneous Lorentz group, times the infinite-dimensional abelian normal subgroup of so-called supertranslations.

We start by an analysis of the symmetry group of asymptotically flat spacetimes at null infinity in the gauge fixed form introduced by Sachs. The asymptotic Killing vectors form a representation of the group mentioned above through the modified Lie bracket introduced in chapter 5. This is when one focus on globally defined transformations. There is a further enhancement when one focuses on infinitesimal transformations and does not require the associated finite transformations to be globally well-defined. The symmetry algebra is then the semi-direct sum of the infinitesimal local conformal transformations of the 2-sphere with the abelian ideal of supertranslations, and now both factors are infinite-dimensional.

The rest of this chapter is an attempt to apply the analysis of the two previous cases (AdS$_3$ and BMS$_3$) to this new BMS$_4$. We first show that there is no possible central extension. We then solve the equations of motion and show how the group is represented on solution space. The surface charges are non-integrable and we use the proposition of Wald and Zoupas to deal with the non-integrable term. Unfortunately, this spoils the usual covariant definition of the algebra.

The last section is a proposition for a new bracket for the charges. It leads to an algebra of charges that form a representation of the asymptotic symmetries up to a general field-
dependent extension.

7.1 Asymptotically flat 4-d spacetimes at null infinity

Let \( x^0 = u, x^1 = r, x^2 = \theta, x^3 = \phi \) and \( A, B, \cdots = 2, 3 \). Following mostly Sachs [17] up to notation, the metric \( g_{\mu\nu} \) of an asymptotically flat spacetime can be written in the form

\[
d s^2 = e^{2\beta} \frac{V}{r} d u^2 - 2 e^{2\beta} d u d r + g_{AB}(d x^A - U^A d u)(d x^B - U^B d u) \tag{7.1.1}
\]

where \( \beta, V, U^A, g_{AB}(\det g_{AB})^{-1/2} \) are 6 functions of the coordinates, with \( \det g_{AB} = r^4 b(u, \theta, \phi) \) for some fixed function \( b(u, \theta, \phi) \). The inverse to the metric

\[
g_{\mu\nu} = \begin{pmatrix} e^{2\beta} \frac{V}{r} + g_{CD} U^C U^D & - e^{2\beta} & - g_{BC} U^C \\ - e^{2\beta} & 0 & 0 \\ - g_{AC} U^C & 0 & g_{AB} \end{pmatrix}
\]

is given by

\[
g^{\mu\nu} = \begin{pmatrix} 0 & - e^{-2\beta} & 0 \\ - e^{-2\beta} & - \frac{V}{r} e^{-2\beta} & - U^B e^{-2\beta} \\ 0 & - U^A e^{-2\beta} & g^{AB} \end{pmatrix}.
\]

The fall-off conditions are

\[
g_{AB} d x^A d x^B = r^2 \tilde{g}_{\nu\rho} d x^\nu d x^\rho + O(r), \tag{7.1.2}
\]

Sachs chooses \( \tilde{g}_{\nu\rho} = 0 \tilde{g}_{\nu\rho} \) to be the metric on the unit 2 sphere, \( 0 \tilde{g}_{\nu\rho} d x^\nu d x^\rho = d \theta^2 + \sin^2 \theta d \phi^2 \) and \( b = \sin^2 \theta \), but the geometrical analysis by Penrose [109] suggests to be somewhat more general and use a metric that is conformal to the latter, for instance, \( \tilde{g}_{\nu\rho} d x^\nu d x^\rho = e^{2\varphi}(d \theta^2 + \sin^2 \theta d \phi^2) \), with \( \varphi = \varphi(u, x^A) \). We will choose the determinant condition more generally to be \( b(u, x^A) = \det \tilde{g}_{\nu\rho} \). In particular, in the above example, on which we now focus, \( b = e^{4\varphi} \sin^2 \theta \).

The rest of the fall-off conditions are given by

\[
\beta = O(r^{-2}), \quad V/r = -2 r \dot{\varphi} - e^{-2\varphi} + \Delta \varphi + O(r^{-1}), \quad U^A = O(r^{-2}). \tag{7.1.3}
\]

Here, a dot denotes the derivative with respect to \( u \), \( \bar{D}_A \) denotes the covariant derivative with respect to \( \tilde{g}_{\nu\rho} \). We denote by \( \Gamma^A_{BC} \) the associated Christoffel symbols and by \( \Delta \) the associated Laplacian. Similarly, \( 0 D_A, 0 \Gamma^A_{BC}, 0 \Delta \) correspond to \( 0 \tilde{g}_{\nu\rho} \). Note that \( g^{AB} g_{BC} = \delta^A_C \) and that the condition on the determinant implies

\[
\begin{align*}
 g^{AB} \partial_u g_{AB} &= 4 r^{-1}, \\
 g^{AB} \partial_u \tilde{g}_{AB} &= \tilde{g}^{AB} \partial_u \tilde{g}_{AB} = 4 \dot{\varphi}, \\
 g^{AB} \partial_C \tilde{g}_{AB} &= \tilde{g}^{AB} \partial_C \tilde{g}_{AB} = 0 \tilde{g}^{AB} \partial_C \tilde{g}_{AB} + 4 \partial_C \varphi,
\end{align*}
\tag{7.1.4}
\]
where $\tilde{\gamma}^{AB}\tilde{\gamma}_{BC} = \delta^A_C = 0\gamma^{AB}0\gamma_{BC}$. In terms of the metric and its inverse, the fall-off conditions read

$$
\begin{align*}
&g_{uu} = -2r\dot{\varphi} - e^{-2\varphi} + \Delta \varphi + O(r^{-1}), \quad g_{ur} = -1 + O(r^{-2}), \quad g_{uA} = O(1), \\
&g_{rr} = 0 = g_{rA}, \quad g_{AB} = r^2\tilde{\gamma}_{AB} + O(r), \\
&g^{ur} = -1 + O(r^{-2}), \quad g^{uu} = 0 = g^{uA}, \\
&g^{rr} = 2r\dot{\varphi} + e^{-2\varphi} - \Delta \varphi + O(r^{-1}), \quad g^{rA} = O(r^{-2}), \quad g^{AB} = r^{-2}\tilde{\gamma}^{AB} + O(r^{-3}).
\end{align*}
$$

### 7.2 Asymptotic symmetries

With the choice $\varphi = 0$, Sachs studies the vector fields that leave invariant this form of the metric with these fall-off conditions. More precisely, he finds the general solution to the equations

$$
\begin{align*}
&\mathcal{L}_\xi g_{rr} = 0, \quad \mathcal{L}_\xi g_{rA} = 0, \quad \mathcal{L}_\xi g_{AB}^{g^{AB}} = 0, \quad \mathcal{L}_\xi g_{ug} = O(r^{-2}), \quad \mathcal{L}_\xi g_{uA} = O(1), \quad \mathcal{L}_\xi g_{AB} = O(r), \quad \mathcal{L}_\xi g_{uu} = O(r^{-1}).
\end{align*}
$$

(7.2.1)

(7.2.2)

For arbitrary $\varphi$, the general solution to (7.2.1) is given by

$$
\begin{align*}
&\xi^u = f, \\
&\xi^A = Y^A + I^A, \quad I^A = -f_{,B}\int_r^\infty d\rho (e^{2\beta}g^{AB}), \\
&\xi^r = -\frac{1}{2}r(\psi + \chi - f_{,B}U^B + 2f\partial_u \varphi),
\end{align*}
$$

(7.2.3)

with $\partial_r f = 0 = \partial_u Y^A$ and where $\psi = D_A Y^A$, $\chi = D_A I^A$. This gives the expansions

$$
\begin{align*}
&\xi^u = f, \quad \xi^A = Y^A - r^{-1}f_{,B}\tilde{\gamma}^{BA} + O(r^{-2}), \\
&\xi^r = -r(f\dot{\varphi} + \frac{1}{2}\psi) + \frac{1}{2}\Delta f + O(r^{-1}).
\end{align*}
$$

(7.2.4)

The first equation of (7.2.2) then implies that

$$
\dot{f} = f\dot{\varphi} + \frac{1}{2}\psi \iff f = e^\varphi \left[ T + \frac{1}{2} \int_0^u du e^{-\varphi} \psi \right],
$$

(7.2.5)

with $T = T(\theta, \phi)$, while the second requires $\partial_u Y^A = 0$ and thus $Y^A = Y^A(\theta, \phi)$. The third one implies that $Y^A$ is a conformal Killing vector of $\tilde{\gamma}_{AB}$ and thus also of $0\gamma_{AB}$. The last equation of (7.2.2) is then satisfied without additional conditions. For the computation, one uses that $\Delta = e^{-2\varphi}0\Delta$ and $\psi = 0\psi + 2Y^A\partial_A \varphi$, with $0\psi = 0D_A Y^A$ and the following properties of conformal Killing vectors $Y^A$ on the unit 2-sphere,

$$
2_0D_{B0}D_{C0}Y_A = 0\gamma_{CA0}D_{B0}\psi + 0\gamma_{AB0}D_{C0}\psi - 0\gamma_{BC0}D_{A0}\psi + 2Y_{C0}\gamma_{BA} - 2Y_{A0}\gamma_{BC},
$$

(7.2.6)

where the indices on $Y^A$ are lowered with $\tilde{\gamma}_{AB}$. This implies in particular $0\Delta Y^A = -Y^A$ and also that $0\Delta_0 \psi = -2_0 \psi$. 


By definition, the algebra \( \mathfrak{bms}_4 \) is the semi-direct sum of the Lie algebra of conformal Killing vectors \( Y^A \frac{\partial}{\partial x^A} \) of the unit 2-sphere with the abelian ideal consisting of functions \( T(x^A) \) on the 2-sphere. The bracket is defined through

\[
(\hat{Y}, \hat{T}) = [(Y_1, T_1), (Y_2, T_2)],
\]

\[
\hat{Y}^A = Y_1^B \partial_B Y_2^A - Y_2^B \partial_B Y_1^A,
\]

\[
\hat{T} = Y_1^A \partial_A T_2 - Y_2^A \partial_A T_1 + \frac{1}{2} (T_1 \psi_2 - T_2 \psi_1).
\]

(7.2.7)

Let \( I = \mathbb{R} \times S^2 \) with coordinates \( u, \theta, \phi \). On \( I \), consider the scalar field \( \varphi \) and the vectors fields \( \xi(\varphi, T, Y) = f \frac{\partial}{\partial u} + Y^A \frac{\partial}{\partial x^A} \), with \( f \) given in (7.2.5) and \( Y^A \) an \( u \)-independent conformal Killing vector of \( S^2 \). It is straightforward to check that these vector fields form a faithful representation of \( \mathfrak{bms}_4 \) for the standard Lie bracket.

Consider then the modified Lie bracket

\[
[\xi_1, \xi_2]_M = [\xi_1, \xi_2] - \delta^{\theta}_{\xi_1} \xi_2 + \delta^{\theta}_{\xi_2} \xi_1,
\]

(7.2.8)

where \( \delta^{\theta}_{\xi_1} \xi_2 \) denotes the variation in \( \xi_2 \) under the variation of the metric induced by \( \xi_1 \),

\[
\delta^{\theta}_{\xi_1} g_{\mu\nu} = \mathcal{L}_{\xi_1} g_{\mu\nu},
\]

Spacetime vectors \( \xi \) of the form (7.2.3), with \( Y^A(x^B) \) a conformal Killing vectors of the 2-sphere and \( f(u, x^B) \) satisfying (7.2.5) provide a faithful representation of \( \mathfrak{bms}_4 \) when equipped with the modified Lie bracket \([\cdot, \cdot]\). Indeed, for the \( u \) component, there is no modification due to the change in the metric and the result follows by direct computation: \( [\xi_1, \xi_2]_M = \hat{f} \), where \( \hat{f} \) corresponds to \( f \) in (7.2.5) with \( T \) replaced by \( \hat{T} \) and \( Y \) by \( \hat{Y} \). By evaluating \( \mathcal{L}_{\xi_1} g_{\mu\nu} \), we find

\[
\begin{cases}
\delta_{\xi_1} \varphi = 0, \\
\delta_{\xi_1} \beta = \xi_2 \partial_u \beta + \frac{1}{2} \left( \partial_u f + \partial_r 
abla_r + \partial_A f U^A \right), \\
\delta_{\xi_1} U^A = \xi_2 \partial_u U^A + U^A (\partial_u f + \partial_B f U^B) - \partial_B \xi_1 U^B \\
\quad - \partial_u \xi_2 - \partial_r \xi_2 \nabla_r - \partial_B \xi_1 U^B.
\end{cases}
\]

(7.2.9)

It follows that

\[
\begin{align*}
\delta^{\theta}_{\xi_1} \xi_2 &= - \partial_B f_2 \int_r^\infty dr' e^{2\beta} (2 \delta^{\theta}_{\xi_1} \beta g^{AB} + \mathcal{L}_{\xi_1} g^{AB}), \\
\delta^{\theta}_{\xi_1} \xi_2' &= - \frac{1}{2} r \left[ D_A (\delta^{\theta}_{\xi_1} \xi_2') - \partial_A f_2 \delta_{\xi_1} U^A \right].
\end{align*}
\]

We have \( \lim_{r \to \infty} [\xi_1, \xi_2]_M^A = \hat{Y}^A \) and, using \( \partial_r \xi_1 = g^{AB} e^{2\beta} \partial_B f, \) (7.2.5) together with the expression of \( \xi_1 \) in (7.2.4), it follows by a straightforward computation that \( \partial_r ([\xi_1, \xi_2]_M^A) = g^{AB} e^{2\beta} \partial_B \hat{f} \), which gives the result for the \( A \) components. Finally, for the \( r \) component, we need

\[
\partial_r \left( \frac{\xi_2'}{r} \right) = -\frac{1}{2} (\partial_r \chi - \partial_B f \partial_r U).
\]
We then find \( \lim_{r \to \infty} \frac{[\xi_1, \xi_2]}{r} = -\frac{1}{2} (\hat{\psi} + 2\hat{f} \partial_\varphi) \), where \( \hat{\psi} \) corresponds to \( \psi \) with \( Y^A \) replaced by \( \hat{Y}^A \), while \( \partial_r (\frac{[\xi_1, \xi_2]}{r}) = -\frac{1}{2} (\partial_r \hat{\chi} - \partial_B \hat{f} \partial_r U^B) \), where \( \hat{\chi} \) corresponds to \( \chi \) with \( f \) replaced by \( \hat{f} \). This gives the result for the \( r \) component and concludes the proof.

More generally, one can also consider the transformations that leave the form of the metric \( (7.1.1) \) invariant up to a conformal rescaling of \( g_{AB} \), i.e., up to a rescaling of \( \varphi \) by \( \omega(u, x^A) \). They are generated by spacetime vectors satisfying

\[
\mathcal{L}_\xi g_{rr} = 0, \quad \mathcal{L}_\xi g_{rA} = 0, \quad \mathcal{L}_\xi g_{AB} = 4\omega, \quad (7.2.10)
\]

\[
\mathcal{L}_\xi g_{ur} = O(r^{-2}), \quad \mathcal{L}_\xi g_{uA} = O(1), \quad \mathcal{L}_\xi g_{AB} = 2\omega g_{AB} + O(r), \quad (7.2.11)
\]

Equations \((7.2.10), (7.2.11)\) then imply that the vectors are given by \((7.2.3)\) and \((7.2.5)\) with the replacement \( \psi \to \psi - 2\omega \).

With this replacement, the vector fields \( \xi = f \frac{\partial}{\partial u} + Y^A \frac{\partial}{\partial x^A} \) on \( \mathcal{I} = \mathbb{R} \times S^2 \) equipped with the modified bracket provide a faithful representation of the extension of \( \mathfrak{bms}_4 \) defined by elements \( (Y, T, \omega) \) and bracket \( [\xi_1, \xi_2] = (\hat{Y}, \hat{T}, \hat{\omega}) \), with \( \hat{Y}, \hat{T} \) as before and \( \hat{\omega} = 0 \).

Indeed, the result is obvious for the \( A \) components. Furthermore,

\[
\delta^A_\xi f_2 = \omega_1 f_2 + \frac{1}{2} e^\varphi \int_0^u du' e^{-\varphi} [-\omega_1 (\psi_2 - 2\omega_2) + 2Y_2^A \partial_A \omega_1].
\]

At \( u = 0 \), we get \( [\xi_1, \xi_2]^u_{/u=0} = e^\varphi |_{u=0} \hat{T} \), while direct computation shows that \( \partial_u ([\xi_1, \xi_2]^u_{/u=0}) = \hat{f} \hat{\varphi} + \frac{1}{2} D_B Y^B \), as it should.

Following the same reasoning as before, one can then also show that the spacetime vectors \((7.2.3)\) with the replacement discussed above and equipped with the modified Lie bracket provide a faithful representation of the extended \( \mathfrak{bms}_4 \) algebra.

Indeed, we have \( \xi = \xi + I^A \frac{\partial}{\partial x^A} + \xi^r \frac{\partial}{\partial y} \). Furthermore, \( [\xi_1, \xi_2]^u_{/u=0} = [\hat{\xi}_1, \hat{\xi}_2]^u_{/u=0} = \hat{f} \) as it should. In the extended case, the variations of \( \beta, U^A \) are still given by \((7.2.9)\). We then have \( \lim_{r \to \infty} [\xi_1, \xi_2]^A_{/r} = \hat{Y}^A \) and find, after some computations, \( \partial_r ([\xi_1, \xi_2]^A_{/r}) = g^{AB} e^{2\beta} \partial_B \hat{f} \), giving the result for the \( A \) components. Finally, for the \( r \) component, we find \( \lim_{r \to \infty} \frac{[\xi_1, \xi_2]}{r} = -\frac{1}{2} (\hat{\psi} + 2\hat{f} \hat{\varphi}) \), while \( \partial_r ([\xi_1, \xi_2]^r_{/r}) = -\frac{1}{2} (\partial_r \hat{\chi} - \partial_B \hat{f} \partial_r U^B) \), which concludes the proof.

### 7.3 Extended \( \mathfrak{bms}_4 \) Lie algebra

As we showed in the previous section, the algebra \( \mathfrak{bms}_4 \) is the semi-direct sum of the Lie algebra of conformal Killing vectors \( Y^A \frac{\partial}{\partial x^A} \) of the unit 2-sphere with the abelian ideal
consisting of functions $T(x^A)$ on the 2-sphere, the bracket being defined as

$$\tilde{Y}, \tilde{T} = [(Y_1, T_1), (Y_2, T_2)],$$
$$\tilde{Y}^A = Y_1^B \partial_B Y_2^A - Y_2^B \partial_B Y_1^A,$$
$$\tilde{T} = Y_1^A \partial_A T_2 - Y_2^A \partial_A T_1 + \frac{1}{2} (T_1 \partial_2 - T_2 \partial_1).$$

(7.3.1)

In terms of the standard complex coordinates $\zeta = e^{i\phi} \cot \frac{\theta}{2}$, the metric on the sphere is conformally flat,

$$d\theta^2 + \sin^2 \theta d\phi^2 = P^{-2} d\zeta d\bar{\zeta}, \quad P(\zeta, \bar{\zeta}) = \frac{1}{2} (1 + \zeta \bar{\zeta}),$$

and, since conformal Killing vectors are invariant under conformal rescalings of the metric, the conformal Killing vectors of the unit sphere are the same as the conformal Killing vectors of the Riemann sphere.

Depending on the space of functions under consideration, there are then basically two options which define what is actually meant by $\mathfrak{bms}_4$.

The first choice consists in restricting oneself to globally well-defined transformations on the unit or, equivalently, the Riemann sphere. This singles out the global conformal transformations, also called projective transformations, and the associated group is isomorphic to $SL(2, \mathbb{C})/\mathbb{Z}_2$, which is itself isomorphic to the proper, orthochronous Lorentz group. Associated with this choice, the functions $T(\theta, \phi)$, which are the generators of the so-called supertranslations, have been expanded into spherical harmonics. This choice has been adopted in the original work by Bondi, van der Burg, Metzner and Sachs and followed ever since, most notably in the work of Penrose and Newman-Penrose [109, 110], where spin-weighted spherical harmonics and the associated “edth” operator have made their appearance. After attempts to cut this group down to the standard Poincaré group, it has been taken seriously as an invariance group of asymptotically flat spacetimes. Its consequences have been investigated, but we believe that it is fair to say that this version of the BMS group has had only a limited amount of success.

The second choice is motivated by exactly the same considerations that are at the origin of the breakthrough in two dimensional conformal quantum field theories [111]. It consists in focusing on local properties and allowing the set of all, not necessarily invertible holomorphic mappings. In this case, the conformal Killing vectors are given by two copies of the Witt algebra, so that besides supertranslations, there now also are superrotations.

The general solution to the conformal Killing equations is $Y^\zeta = Y(\zeta), \bar{Y}^\bar{\zeta} = \bar{Y}(\bar{\zeta})$, with $Y$ and $\bar{Y}$ independent functions of their arguments. The standard basis vectors are chosen as

$$l_n = -\zeta^{n+1} \frac{\partial}{\partial \zeta}, \quad \bar{l}_n = -\bar{\zeta}^{n+1} \frac{\partial}{\partial \bar{\zeta}}, \quad n \in \mathbb{Z}$$

(7.3.3)
At the same time, let us choose to expand the generators of the supertranslations in terms of
\[ T_{m,n} = P^{-1} \zeta^m \bar{\zeta}^n, \quad m, n \in \mathbb{Z}. \] (7.3.4)

In terms of the basis vector \( l_t \equiv (l_t, 0) \) and \( T_{m,n} = (0, T_{m,n}) \), the commutation relations for the complexified \( \mathfrak{bms}_4 \) algebra read
\[
[l_m, l_n] = (m - n)l_{m+n}, \quad [\bar{l}_m, \bar{l}_n] = (m - n)\bar{l}_{m+n},
[l_t, T_{m,n}] = \left(\frac{l+1}{2} - m\right)T_{m+\bar{t},n}, \quad [\bar{l}_t, T_{m,n}] = \left(\frac{l+1}{2} - n\right)T_{m+n+\bar{t}},
[l_m, \bar{l}_n] = 0, \quad [T_{m,n}, T_{o,p}] = 0.
\] (7.3.5)

The \( \mathfrak{bms}_4 \) algebra contains as subalgebra the Poincaré algebra, which we identify with the algebra of exact Killing vectors of the Minkowski metric equipped with the standard Lie bracket.

Indeed, these vectors form the subspace of spacetime vectors (7.2.3) for which (i) \( \beta = 0 = U^A = \varphi \) while \( V = -r \) and \( g_{AB} = 0\gamma_{AB} \) and (ii) the relations in (7.2.2) hold with 0 on the right hand sides. The former implies in particular that \( I^A = -\frac{1}{2}0\gamma^{AB}\partial_B f \), while a first consequence of the latter is that the modified Lie bracket reduces the standard one.

Besides the previous conditions that \( Y^A \) is an \( u \)-independent conformal Killing vector of the 2-sphere, \( \mathcal{L}_Y 0\gamma_{AB} = 0D_CY^C 0\gamma_{AB} \) and \( f = T + \frac{1}{2}u_0\psi \) with \( T_u = 0 = T_r \), we find the additional constraints
\[
0D_A\partial_B 0\psi + 0D_B\partial_A 0\psi = 0\gamma_{AB} 0\Delta 0\psi,
\]
\[
0D_A\partial_B T + 0D_B\partial_A T = 0\gamma_{AB} 0\Delta T, \quad \partial_A T = -\frac{1}{2}\partial_A (0\Delta T).
\] (7.3.6, 7.3.7)

In the coordinates \( \zeta, \bar{\zeta} \), these constraints are equivalent to \( \partial^2 Y = 0 = \bar{\partial}^2 Y \) and \( \partial^2 \bar{T} = 0 = \bar{\partial}^2 \bar{T} \), where \( T = P^{-1}\bar{T} \) and \( \partial = \partial_{\zeta}, \bar{\partial} = \partial_{\bar{\zeta}} \), so that the complexified Poincaré algebra is spanned by the generators
\[
l_{-1}, l_0, l_1, \quad \bar{l}_{-1}, \bar{l}_0, \bar{l}_1, \quad T_{0,0}, T_{1,0}, T_{0,1}, T_{1,1},
\] (7.3.8)

and the non vanishing commutation relations read
\[
[l_{-1}, l_0] = -l_{-1}, \quad [l_{-1}, l_1] = -2l_0, \quad [l_0, l_1] = -l_1,
[l_{-1}, T_{0,0}] = -T_{0,0}, \quad [l_{-1}, T_{1,1}] = -T_{0,1}, \quad [\bar{l}_{-1}, T_{0,0}] = -T_{0,0}, \quad [\bar{l}_{-1}, T_{1,1}] = -T_{0,1},
[l_0, T_{0,0}] = \frac{1}{2}T_{0,0}, \quad [l_0, T_{0,1}] = \frac{1}{2}T_{0,1}, \quad [l_0, T_{1,0}] = -\frac{1}{2}T_{1,0}, \quad [l_0, T_{1,1}] = -\frac{1}{2}T_{1,1},
[l_1, T_{0,0}] = \frac{1}{2}T_{0,0}, \quad [l_1, T_{0,1}] = -\frac{1}{2}T_{0,1}, \quad [l_1, T_{1,0}] = \frac{1}{2}T_{1,0}, \quad [l_1, T_{1,1}] = -\frac{1}{2}T_{1,1},
[l_{-1}, T_{0,0}] = T_{0,0}, \quad [l_{-1}, T_{0,1}] = T_{1,1}, \quad [\bar{l}_{-1}, T_{0,0}] = T_{0,1}, \quad [\bar{l}_{-1}, T_{0,1}] = T_{1,0}.
\] (7.3.9)
In particular for instance, the generators for translations are associated to \( \frac{1}{2}(T_{1,1} + T_{0,0}) = 1 \), \( \frac{1}{2}(T_{1,1} - T_{0,0}) = \cos \theta \), \( \frac{1}{2}(T_{1,0} + T_{0,1}) = \sin \theta \cos \phi \), \( \frac{1}{2}(T_{1,0} - T_{0,1}) = \sin \theta \sin \phi \). Note that in order for the asymptotic symmetry algebra to contain the Poincaré algebra as a subalgebra, it is essential not to restrict the generators of supertranslations to the sum of holomorphic and antiholomorphic functions on the Riemann sphere.

The quotient algebra of \( \mathfrak{bms}_4 \) by the abelian ideal of infinitesimal supertranslations is no longer given by the Lorentz algebra but by two copies of the Witt algebra. It follows that the problem with angular momentum in general relativity \([112]\), at least in its group theoretical formulation, disappears as now the choice of an infinite number of conditions is needed to fix an infinite number of rotations.

The considerations above apply for all choices of \( \varphi \) which is freely at our disposal. In the original work of Bondi, van der Burg, Metzner and Sachs, and in much of the subsequent work, the choice \( \varphi = 0 \) was preferred. From the conformal point of view, the choice

\[
\varphi = \ln \left[ \frac{1}{2} (1 + \zeta \bar{\zeta}) \right] \tag{7.3.10}
\]

is interesting as it turns \( \bar{\gamma}_{AB} \) into the flat metric on the Riemann sphere with vanishing Christoffel symbols,

\[
\bar{\gamma}_{AB} dx^A dx^B = d\zeta d\bar{\zeta}. \tag{7.3.11}
\]

In this case, \( \psi = \partial_A Y^A \),

\[
f = \tilde{T} + \frac{1}{2} u \psi, \tag{7.3.12}
\]

with \( \tilde{T} = PT \). In terms of \( \tilde{T} \), we get instead of the last of (7.3.1)

\[
\tilde{T} = Y_1^A \tilde{T}_2 + \frac{1}{2} \tilde{T}_1 \partial_A Y^A_2 - (1 \leftrightarrow 2). \tag{7.3.13}
\]

In terms of generators, the algebra (7.3.5) is unchanged if one now expands the supertranslations \( \tilde{T} \) directly in terms of \( \tilde{T}_{m,n} = \zeta^m \bar{\zeta}^n \).

### 7.4 Central extensions of \( \mathfrak{bms}_4 \)

The \( \mathfrak{bms}_4 \) algebra can also be viewed as an abstract structure defined on the 2-sphere with the help of tensor densities introduced in section [6.3]

In stereographic coordinates \( \zeta = e^{i\varphi} \cot \frac{\theta}{2} \) and \( \bar{\zeta} \) for the 2 sphere, the algebra may be realized through the vector fields \( y = Y(\zeta) \partial, \bar{y} = \bar{Y}(\bar{\zeta}) \bar{\partial} \), with \( \partial = \frac{\partial}{\partial \zeta}, \bar{\partial} = \frac{\partial}{\partial \bar{\zeta}} \). They act on tensor densities \( \mathcal{F}_{\frac{1}{2}, \frac{1}{2}} \) of degree \( \left( \frac{1}{2}, \frac{1}{2} \right) \), \( t = T(\zeta, \bar{\zeta}) e^{-\varphi_0} (d\zeta)^{-\frac{1}{2}} (d\bar{\zeta})^{-\frac{1}{2}} \), where \( \varphi_0 = \ln \frac{1}{2} (1 + \zeta \bar{\zeta}) \) through

\[
\rho(y)t = (Y \partial T - \frac{1}{2} \partial Y T) e^{-\varphi_0} (d\zeta)^{-\frac{1}{2}} (d\bar{\zeta})^{-\frac{1}{2}}, \tag{7.4.1}
\]

\[
\rho(\bar{y})t = (\bar{Y} \bar{\partial} T - \frac{1}{2} \bar{\partial} \bar{Y} T) e^{-\varphi_0} (d\zeta)^{-\frac{1}{2}} (d\bar{\zeta})^{-\frac{1}{2}}. \tag{7.4.2}
\]
The algebra \( \mathfrak{bms}_4 \) is then the semi-direct sum of the algebra of vector fields \( y, \bar{y} \) with the abelian ideal \( \mathcal{F}_{\frac{1}{2}, \frac{1}{2}} \), the bracket being induced by the module action, \([y, t] = \rho(y)t, [\bar{y}, t] = \rho(\bar{y})t\).

When expanding \( y = a^nl_n, \bar{y} = a^n\bar{l}_n, t = b^{m,n}T_{m,n} \) with
\[
    l_n = -\zeta^{n+1}\partial, \quad \bar{l}_n = -\bar{\zeta}^{n+1}\bar{\partial}, \quad T_{m,n} = \zeta^m\bar{\zeta}^n e^{-\varphi_0}(d\zeta)^{-\frac{1}{2}}(d\bar{\zeta})^{-\frac{1}{2}},
\]
the enhanced symmetry algebra reads
\[
    [l_m, l_n] = (m-n)l_{m+n}, \quad [\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n},
    [l_m, T_{m,n}] = (\frac{l+1}{2} - m)T_{m+l,n}, \quad [\bar{l}_m, T_{m,n}] = (\frac{l+1}{2} - n)T_{m,n+1},
    [l_m, \bar{l}_n] = 0, \quad [T_{m,n}, T_{o,p}] = 0,
\]
where \( m, n \cdots \in \mathbb{Z} \), which is exactly the algebra we saw in the previous section.

The only non trivial central extensions of \( \mathfrak{bms}_4 \) are the usual central extensions of the 2 copies of the Witt algebra, i.e., they appear in the commutators \([l_m, l_{-m}] \) and \([\bar{l}_m, \bar{l}_{-m}] \). Contrary to three dimensions, there are no central extensions involving the generators for supertranslations. In other words, up to equivalence, the most general central extension of \( \mathfrak{bms}_4 \) is
\[
    [l_m, l_n] = (m-n)l_{m+n} + \frac{\xi}{12}m(m-1)(m+1)b^0_{m+n},
    [\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n} + \frac{\bar{\xi}}{12}m(m-1)(m+1)b^0_{m+n},
    [l_m, T_{m,n}] = (\frac{l+1}{2} - m)T_{m+l,n}, \quad [\bar{l}_m, T_{m,n}] = (\frac{l+1}{2} - n)T_{m,n+1},\]
\[
    [l_m, \bar{l}_n] = 0, \quad [T_{m,n}, T_{o,p}] = 0.
\]

**Proof.** For \( \mathfrak{bms}_4 \), the Chevalley-Eilenberg differential is given by
\[
    \gamma = -\frac{1}{2}C^mC^{k-m}(2m-k)\frac{\partial}{\partial C^k} - \frac{1}{2}\bar{C}^m\bar{C}^{k-m}(2m-k)\frac{\partial}{\partial \bar{C}^k} - C^m\xi^{k-m.n}(3m+1-k)\frac{\partial}{\partial \xi^{k,n}} - C^m\xi^{m,k-n}(3n+1-k)\frac{\partial}{\partial \xi^{m,k}},
\]
in the space \( \Lambda(C, \bar{C}, \xi) \) of polynomials in the anticommuting “ghost” variables \( C^m, \bar{C}^m, \xi^{m,n} \).

The grading is given by the eigenvalues of the ghost number operator, \( N_{C, \xi} = C^m \frac{\partial}{\partial C^m} + \bar{C}^m \frac{\partial}{\partial \bar{C}^m} + \xi^{m,n} \frac{\partial}{\partial \xi^{m,n}} \), the differential \( \gamma \) being homogeneous of degree 1 and \( H^2(\mathfrak{bms}_4) \cong H^2(\gamma, \Lambda(C, \bar{C}, \xi)) \). Furthermore, when counting only the ghosts \( \xi^m \) associated with supertranslations, \( N_{\xi} = \xi^{m,n} \frac{\partial}{\partial \xi^{m,n}} \), the differential \( \gamma \) is homogeneous of degree 0, so that the cohomology decomposes into components of definite \( N_{\xi} \) degree. The cocycle condition then becomes
\[
    \gamma(\omega^0_{m,n}C^mC^m + \omega^0_{m,n}\bar{C}^m\bar{C}^m + \omega^{-1}_{m,n}C^m\bar{C}^m) = 0,
    \gamma(\omega^1_{k,mn}C^k\xi^{m,n} + \omega^1_{k, mn}\bar{C}^k\xi^{m,n}) = 0,
    \gamma(\omega^2_{m,kl}\xi^{m,n}\xi^{k,l}) = 0,
\]
with $\omega_{m,n}^0 = -\omega_{n,m}^0, \bar{\omega}_{m,n} = -\bar{\omega}_{n,m}^0$ and $\omega_{m,kl}^2 = -\omega_{kl,mn}^2$. The coboundary condition reads

$$\omega_{m,n}^0 C^m C^n + \bar{\omega}_{m,kl}^1 C^m C^n + \omega_{m,n}^{-1} C^m C^n - \gamma(\eta_{m}^0 C^n + \bar{\eta}_{m}^0 C^m),$$

$$\omega_{k,mn}^1 C^{k,\xi^m \xi^n} + \bar{\omega}_{k,mn}^1 C^{k,\xi^m \xi^n} = \gamma(\eta_{mn}^1 \xi^m \xi^n). \quad (7.4.8)$$

We have $\{\partial_{\partial C^m}, \gamma\} = N_{C,\xi}$ with $N_{C,\xi} = m C^m \partial_{\partial C^m} + (m - \frac{1}{2}) \xi^m \partial_{\partial \xi^m, n}$ and also $\{\partial_{\partial C^m}, \gamma\} = N_{C,\xi}$ with $N_{C,\xi} = n C^n \partial_{\partial C^n} + (n - \frac{1}{2}) \xi^m \partial_{\partial \xi^m, n}$. It follows again that all cocycles of either $N_{C,\xi}$ or $\bar{N}_{C,\xi}$ degree different from 0 are coboundaries. Without loss of generality we can thus assume that $\omega_{m}^0 C^m C^n + \bar{\omega}_{m,kl}^1 C^m C^n + \omega_{m,n}^{-1} C^m C^n = \omega_{m}^0 C^m C^n + \omega_{m,kl}^1 C^m C^n + \omega_{m,n}^{-1} C^m C^n$ with $\omega_{m}^0 = -\omega_{m}^0$, $\bar{\omega}_{m}^0 = -\bar{\omega}_{m}^0$ and in particular $\omega_{m}^0 = 0 = \bar{\omega}_{m}^0$; none of monomials with one $\xi^m, n$ and either one $C^k$ or one $\bar{C}^k$ can be of degree 0, so $\omega_{k,mn}^1 = 0 = \bar{\omega}_{k,mn}^1; \omega_{m,kl}^1 C^{m,n} \xi^k = 0$. Both the cocycle and the coboundary condition for $\omega_{m}^0 C^m C^n + \bar{\omega}_{m,kl}^1 C^m C^n + \omega_{m,n}^{-1} C^m C^n$ split. For $\omega_{m,0}^0 C^0 \bar{C}^0$ there is no coboundary condition, while the cocycle condition implies $\omega_{m,0}^{-1} = 0$. The rest of the analysis proceeds as in the previous subsection, separately for $\omega_{m}^0 C^m C^n$ and $\bar{\omega}_{m,kl}^1 C^m C^n$, with the standard central extension for $[l_m, l_{-m}]$ and $[\bar{l}_m, \bar{l}_{-m}]$.

We still have to analyze $\gamma(\omega_{m,n}^2 \xi^m \xi^{-m+1, -n+1}) = 0$. This condition gives $\omega_{m,n}^2 (\frac{3l+1}{2} + m) + \omega_{m,n}^2 (\frac{l-1}{2} - m) = 0$ and also $\omega_{m,0}^2 (\frac{3l+1}{2} + n) + \omega_{m,0}^2 (\frac{l+1}{2} - n) = 0$. Putting $m = 0$ in the first relation gives $\omega_{m,0}^2 (\frac{3l+1}{2}) + \omega_{m,0}^2 (\frac{l-1}{2}) = 0$. Putting $l = -1$ then implies $\omega_{m,0}^2 = 0$ and then also $\omega_{l,0}^2 = 0$ for $l = 1$. But $\omega_{m,0}^2 = -\omega_{k,-n+1}^0 = 0$ which shows that $\omega_{m,n}^2 = 0$ for all $m, n$ and concludes the proof. \(\square\)

### 7.5 Solution space

We start by assuming only that we have a metric of the form \((7.1.1)\) and that the determinant condition holds. Following again \([2]\), the equations of motion are organized in terms of the Einstein tensor $G_{a\beta} = R_{a\beta} - \frac{1}{2} g_{a\beta} R$ as

$$G_{ra} = 0, \quad G_{AB} - \frac{1}{2} g_{AB} g^{CD} G_{CD} = 0, \quad (7.5.1)$$

$$G_{uu} = 0 = G_{uA}, \quad (7.5.2)$$

$$g^{CD} G_{CD} = 0. \quad (7.5.3)$$

Due to the form of the metric and the determinant condition, equation \((7.5.3)\) is a consequence of \((7.5.1)\) on account of the Bianchi identities. Indeed, the latter can be written as

$$0 = 2 \sqrt{-g} G_{\alpha\beta} = 2 (\sqrt{-g} G_{\alpha\beta})_{\beta} + \sqrt{-g} g_{\beta\gamma} g^{\beta\gamma, \alpha}, \quad (7.5.4)$$
When (7.5.1) hold and $\alpha = 1$, we get $G_{AB}g^{AB},_r = 0 = \frac{1}{2}g_{AB}g^{AB},_r g^{CD}G_{CD}$. This implies that (7.5.3) holds by using (7.1.4).

The remaining Bianchi identities then reduce to $2(\sqrt{-g}G^B_A)_\beta = 0 = 2(\sqrt{-g}G^\beta_u)_\beta$. The first gives $(r^2G_{uA})_r = 0$. This means that if $r^2G_{uA} = 0$ for some constant $r$, it vanishes for all $r$. When $G_{uA} = 0$, the last Bianchi identity reduces to $(r^2G_{uu})_r = 0$, so that again, $r^2G_{uu} = 0$ everywhere if it vanishes for some fixed $r$.

Let $k_{AB} = \frac{1}{2}g_{AB}r, l_{AB} = \frac{1}{2}g_{AB,ur}, n_A = \frac{1}{2}e^{-2\beta}g_{AB}U_r^B$ with indices on these variables and on $U^A$ lowered and raised with the 2 dimensional metric $g_{AB}$ and its inverse. Define $K^A_B$ through the relation $k^A_B = \frac{1}{r}\delta^A_B + \frac{1}{r^2}K^A_B$. In particular, the determinant condition implies that $k = \frac{2}{r}$ and thus that $K^A_B = 0$. Similarly, if $l^A_B = \frac{1}{2}L_A^B\gamma_{AB,u} + \frac{1}{2}L_B^A$, the determinant condition implies in particular that $L^D_B$ is traceless, $L^D_D = 0$. Note that for a traceless $2 \times 2$ matrix $M^A_B$, we have $M^A_C M^C_B = \frac{1}{2}M^A_D M^D_C \delta^B_A$.

For a metric of the form (7.1.1), we have
\begin{align*}
\Gamma^A_r &= \delta^A_r 2\beta_r, \quad \Gamma^u_r = 0, \quad \Gamma^r_A = \delta^A_A + n_A, \quad \Gamma^A_B = k^A_B,
\Gamma^u_{AB} &= e^{-2\beta}k_{AB}, \quad \Gamma^A_{BC} = e^{-2\beta}U^A_{k_{BC}} + (2)\Gamma^A_{BC},
\Gamma^A_{ur} &= -k^A_BU^B + e^{2\beta}(\partial_r^A \beta - n^A), \quad \Gamma^u_{ur} = \delta^A_A - n^A - e^{-2\beta}k_{AB}U^B,
\Gamma^r_{ur} &= -\frac{1}{2}(\partial_r + 2\beta_r)U^A_r - (\delta^A_A + n^A)U^B_A,
\Gamma^A_{Bu} &= l^A_B + \frac{1}{2}(2)D^A U_B - \frac{1}{2}(2)D_B U^A + U^A_B(n^B - n_B) - e^{2\beta}k_{BC}U^A U^C,
\Gamma^u_{uu} &= 2\beta_u + \frac{1}{2}(\partial_r + 2\beta_r)U^A_r + 2U^A n_A + e^{-2\beta}k_{AB}U^A U^B,
\Gamma^r_{uu} &= -e^{-2\beta}\left(-\frac{1}{2}(2)D_A U_B + \frac{1}{2}(2)D_B U_A + l_{AB} + \frac{1}{r}k_{AB}\right),
\Gamma^A_{uA} &= \delta^A_A - 2\beta_a U^A_r + \frac{1}{2}U^A_a(\partial_r + 2\beta_r)U^A_r + 2U^A n_B U^B + U^A k_{BC} e^{-2\beta} U^B U^C
- U^A_a - 2\beta_a U^B_r - \frac{1}{2}e^{2\beta}U^A_r - \frac{1}{2}e^{2\beta}(\partial^A + 2\beta^A) \frac{U^A_r}{r} - \frac{1}{2}(2)D^A(U^C U^C),
\Gamma^r_{uu} &= \frac{1}{2}(\partial_r - 2\beta_u)U^A_r + \frac{1}{2}U^A_r(\partial_r + 2\beta_r)U^A_r + \frac{1}{2}U^A(\partial_A + 2\beta_A)U^A_r + 2U^A n_A + \frac{V}{r}e^{-2\beta}k_{AB}U^A U^B + e^{2\beta}l_{AB} U^A U^B + e^{2\beta}U^A U^B(2)D_A U_B.
\end{align*}

To write the equations of motion, we use that $|^{(4)}g| = e^{4\beta} |^{(2)}g|$ and
\begin{align*}
R_{\mu\nu} &= [\partial_\alpha + (2\beta + \frac{1}{2} \ln |^{(2)}g|)_{,\alpha}] \Gamma^\alpha_{\mu\nu} - \partial_\mu \partial_\nu (2\beta + \frac{1}{2} \ln |^{(2)}g|) - \Gamma^\alpha_{\nu\beta} \Gamma_{\mu\alpha}.
\end{align*}

The equation $G_{rr} \equiv R_{rr} = 0$ then becomes
\begin{equation}
\partial_r \beta = -\frac{1}{2r} + \frac{r}{4}k_{AB}k^B_A = \frac{1}{4r^2}K^A_B K^B_A \quad \iff \quad \beta = -\int_r^\infty dr' \frac{1}{4r'^2} K^A_B K^B_A. \tag{7.5.5}
\end{equation}
This equation determines $\beta$ uniquely in terms of $g_{AB}$ because the fall-off condition (7.1.3) excludes the arbitrary function of $u, x^A$ allowed by the general solution to this equation.

The equations $G_{rA} \equiv R_{rA} = 0$ read

$$\partial_r(r^2 n_A) = J_A,$$

$$J_A = r^2(\partial_r - \frac{2}{r})\beta_A - (2)D_B K_A^B = \partial_A(-2r\beta + \frac{1}{4r}K_A^C K_B^C) - (2)D_B K_A^B. \tag{7.5.6}$$

In the original approach [15, 16], it was assumed in particular that the metric $g_{AB}$ admits an expansion in terms powers of $r^{-1}$ starting at order $r^2$. We will assume

$$g_{AB} = r^2 \gamma_{AB} + rC_{AB} + D_{AB} + \frac{1}{4} \gamma_{AB} C_D C_D + o(r^{-\epsilon}), \tag{7.5.7}$$

where indices on $C_{AB}, D_{AB}$ are raised with the inverse of $\gamma_{AB}$. In [2], it was then shown explicitly how (7.5.7) is related to the conformal approach [109, 113] and imposed through differentiability conditions at null infinity.

Under the assumption (7.5.7), $C_D^D = 0 = D_C^C$ and

$$K_A^B = -\frac{1}{2} C_A^B - r^{-1} D_A^B + o(r^{-1-\epsilon}),$$

$$\beta = -\frac{1}{32} r^{-2} C_A^B C_B^A - \frac{1}{12} r^{-3} C_B^A D_A^B + o(r^{-3-\epsilon}), \tag{7.5.8}$$

$$J_A = \frac{1}{2} \bar{D}_B C_A^B + r^{-1} \bar{D}_B D_A^B + o(r^{-1-\epsilon}).$$

These equations then imply $n_A = \frac{1}{2} r^{-1} \bar{D}_B C_A^B + r^{-2} (\ln r \bar{D}_B D_A^B + N_A) + o(r^{-2-\epsilon})$ and involve the arbitrary functions $N_A(u, x^B)$ as integration “constants”. Because $U^A$ has to vanish for $r \to \infty$, we get from the definition of $n_A$

$$U^A = -\frac{1}{2} r^{-2} \bar{D}_B C_B^A - \frac{2}{3} r^{-3} \left[ (\ln r + \frac{1}{3}) \bar{D}_B D_B^A - \frac{1}{2} C_B^A \bar{D}_C C_B^C + N_A \right] + o(r^{-3-\epsilon}), \tag{7.5.9}$$

where the index on $N_A$ has been raised with $\gamma^{AB}$.

It is straightforward to verify that if one trades the coordinate $r$ for $s = r^{-1}$, the only non-vanishing components of the “unphysical” Weyl tensor at the boundary are given by

$$\lim_{s \to 0}(s^2 W_{sAB}) = -D_{AB}, \tag{7.5.10}$$

(see e.g. [114] for a detailed discussion). In [16], the condition $D_{AB} = 0$ was imposed in order to avoid a logarithmic $r$-dependence in the solution to the equations of motion and to avoid singularities on the unit sphere. When one dispenses with this latter restriction, absence of a logarithmic $r$-dependence is guaranteed through the requirement $\bar{D}_B D_B^A = 0$. In the coordinates $\zeta, \tilde{\zeta}$ and with the parametrization $\gamma_{AB} dx^A dx^B = e^{2\tilde{\zeta}} d\zeta d\tilde{\zeta}$, this is equivalent to

$$D_{\zeta\zeta} = d(u, \zeta), \quad D_{\tilde{\zeta}\zeta} = \tilde{d}(u, \tilde{\zeta}), \quad D_{\zeta\tilde{\zeta}} = 0. \tag{7.5.11}$$
A more complete analysis of the field equations when allowing for a logarithmic or, more precisely, a “polyhomogeneous” dependence in $r$ can be found in [115].

Starting from

\[
R_{AB} = (\partial_r + 2\beta_r + \frac{2}{r})\Gamma_{AB}^r - k_A^C\Gamma_{BC}^r - k_B^C\Gamma_{AC}^r + (2)R_{AB} - 2(2)D_B\beta_A + (\partial_u + 2\beta_u + l)\Gamma_{AB}^u - \Gamma_{uA}^u\Gamma_{uB}^u - \Gamma_{rA}^r\Gamma_{rB}^r
- \Gamma_{uA}^C\Gamma_{BC}^u - \Gamma_{uB}^C\Gamma_{AC}^u + (2)D_C(e^{-2\beta}U^Ck_{AB}) + e^{-4\beta}U^Ck_B^Dk_{AC} + 2e^{-2\beta}U^Ck_C^Dk_{AB},
\]

we find

\[
g^{DA}R_{AB} = e^{-2\beta}\left[(\partial_r + \frac{2}{r})(l_B^D + k_B^D\frac{V}{r}) + \frac{1}{2}(2)D_BU^D + \frac{1}{2}(2)D^DU_B\right]
+ k_A^D(2)D_BU^A - k_B^D(2)D_AU^D + (\partial_u + l)k_B^D + (2)D_C(U^Ck_B^D)
+ (2)R_B^D - 2(2)D_B\partial^D\beta + \partial^D\beta\partial_B\beta + n^Dn_B.\quad (7.5.12)
\]

When taking into account the previous equations, $G_{ur} \equiv R_{ur} + \frac{1}{2}e^{2\beta}R = 0$ reduces to $g^{AB}R_{AB} = 0$. Explicitly, we find from the trace of (7.5.12)

\[
\partial_rV = J,
\]

\[
J = e^{2\beta}r^2(2)\Delta\beta + \partial^D\beta\partial_B\beta + n^Dn_D - \frac{1}{2}(2)R - 2rl - r^2(\partial_r + \frac{4}{r})(2)D_BU^B
- 2rl + e^{2\beta}r^2\left[(2)\Delta\beta + (n^A - \partial^A\beta)(n_A - \partial_A\beta) - \frac{1}{2}(2)\bar{R}\right] - 2r(2)D_BU^B
= -2rl - \frac{1}{2}\bar{R} + o(r^{-1-\epsilon}),\quad (7.5.13)
\]

where we have used the previous equation to get the second line. This equation implies

\[
\frac{V}{r} = -rl - \frac{1}{2}\bar{R} + r^{-1}M + o(r^{-1-\epsilon}),\quad (7.5.14)
\]

and implies a third arbitrary function of $M(u, x^B)$ as integration constant.

We have $G_{AB} - \frac{1}{2}g_{AB}g^{CD}G_{CD} = R_{AB} - \frac{1}{2}g_{AB}g^{CD}R_{CD}$. Taking into account the previous equations, it thus reduces to the condition that the traceless part of (7.5.12) vanishes. Using that $\partial_rk_B^D = \partial_r(l_B^D - 2(l_A^Dk_A^B - k_A^Dl_A^B))$, we get

\[
(\partial_r + \frac{2}{r})l_B^D - (l_A^Dk_A^B - k_A^Dl_A^B) + \frac{1}{2}k_B^Dl =
- \frac{1}{2}(\partial_r + \frac{2}{r})(k_B^D\frac{V}{r}) + \frac{1}{2}(2)D_BU^D + \frac{1}{2}(2)D^DU_B
+ k_C^D(2)D_BU^C - k_C^D(2)D_CU^D + (2)D_C(U^Ck_B^D)
+ e^{2\beta}\left[n^Dn_B + (2)D_B\partial^D\beta + \partial^D\beta\partial_B\beta - \frac{1}{2}(2)R_B^D\right],
\]
The various definitions then give

\[ \partial_t L_B^D - \frac{1}{r^2}(F_A K_B^A - K_A^D L_B^A) = J_B^D, \quad (7.5.15) \]

where

\[
J_B^D = -\frac{r^2}{2}(\partial_r + \frac{2}{r})(k_B^D V) + \frac{r^2}{2}k_B^D \left[ 2(\delta A^B) + (n^A - \partial A^B)(n_A - \partial_A B) - \frac{1}{2} \frac{r}{r} \right]
+ \frac{k_B^D}{r} \left[ 2D_B U^D + \frac{r}{2} D_B U^D \right]
- \frac{r^2}{2} \left[ 2D_B U^D + \frac{r}{2} D_B U^D \right]
+ \frac{k_B^D}{r} \left[ 2D_B U^D + \frac{r}{2} D_B U^D \right]
\]

The previous equations imply

\[
J_B^D = -\frac{1}{2}(\partial_r k_B^D + \frac{1}{2} k_B^D V) - \frac{r^2}{2}k_B^D \left[ 2(\delta A^B) + (n^A - \partial A^B)(n_A - \partial_A B) - \frac{1}{2} \frac{r}{r} \right]
- \frac{r^2}{2} \left[ 2D_B U^D + \frac{r}{2} D_B U^D \right]
+ \frac{k_B^D}{r} \left[ 2D_B U^D + \frac{r}{2} D_B U^D \right]
- \frac{r}{2} \left[ 2D_B U^D + \frac{r}{2} D_B U^D \right]
\]

Let \( O_{BC}^{DA} = -\frac{1}{r}(K_A^D \delta_B^C - \delta_A^B K_B^C) \) and \( AR \) denote anti-radial ordering. Equation (7.5.15) without right-hand side has the same form as the Schrödinger equation with time dependent Hamiltonian. If we define

\[
U_{BC}^{DA}(r, r') = \mathcal{A} \mathcal{R} \exp [- \int_{r'}^{r}\mathcal{O}_{BC}^{DA}(r')], \quad (7.5.17)
\]

the solution to the inhomogeneous equation (7.5.15) with non-vanishing \( J_B^D \) can then be obtained by variation of constants and reads

\[
L_B^D(r) = U_{BC}^{DA}(r, \infty)[\frac{1}{2} N_A^C + \int dr' U_{AE}^{CE} (\infty, r') J_E^{DF}(r')], \quad (7.5.18)
\]

and involves two more integration constants encoded in \( N_B^D(u, x^B) \).

In other words, the \( r \)-dependence of \( g_{AB, \nu} \) is completely determined up to two integration constants. It follows that the only variables left in the theory whose \( r \)-dependence is undetermined are the two functions contained in \( R_{AB}(u_0, r, x^C) = g_{AB}(u_0, r, x^C) - r^2 \gamma_{AB}(u_0, x^C) - r C_{AB}(u_0, x^C) - D_{AB}(u_0, x^C) - \frac{1}{4} \gamma_{AB} C_B^D C_C^D \) at some initial fixed \( u_0 \).
When expanding into orders in \( r \), one finds in particular

\[
L^D_B = \frac{1}{2} (\tilde{\gamma}^{DA} C_{AB,u} - C^{DA} \tilde{\gamma}_{AB,u}) + \frac{1}{2} r^{-1} \left[ \tilde{\gamma}^{DA} \partial_u (D_{AB} + \frac{1}{4} \tilde{\gamma}_{AB} C^C C^D) \right] - C^{DA} C_{AB,u} - D^{DA} \tilde{\gamma}_{AB,u} + \frac{1}{4} C^E C^F \tilde{\gamma}^{DA} \tilde{\gamma}_{AB,u} + o(r^{-1-\epsilon}),
\]

\[
J^D_B = \frac{1}{2} \delta^D_B - \frac{1}{2} \tilde{\gamma}^{DA} \tilde{\gamma}_{AB,u} + \frac{1}{4} r^{-1} [C^{DA} \tilde{\gamma}_{AB,u} - \tilde{\gamma}^{DC} \tilde{\gamma}_{CA,u} C^A] + \frac{1}{2} r^{-2} [l^D_B + D^{DA} \tilde{\gamma}_{AB,u} - \tilde{\gamma}^{DC} \tilde{\gamma}_{CA,u} D^A_B] + o(r^{-2+\epsilon}).
\]

When injecting into the equation of motion (7.5.15), the leading order requires that

\[
\tilde{\gamma}_{AB,u} = l \tilde{\gamma}_{AB},
\]

(7.5.19)
or, in other words, that the only \( u \) dependence in \( \tilde{\gamma}_{AB} \) is contained in the conformal factor. This agrees with the assumption of section 7.1 where the \( u \)-dependence of \( \tilde{\gamma}_{AB} \) was contained in \( \exp 2\varphi \) and \( l = 2\partial_u \varphi \), and also with the discussion at the end of the previous subsection, where it was contained in \( \exp 2\tilde{\varphi} \) and \( l = 2\partial_u \tilde{\varphi} \). In the following we always assume that (7.5.19) holds. In particular, this implies

\[
L^D_B = \frac{1}{2} (\tilde{\gamma}^{DA} C_{AB,u} - l C^D_B) + \frac{1}{2} r^{-1} [\tilde{\gamma}^{DA} D_{AB,u} - C^{DA} C_{AB,u} - l D^D_B + \frac{1}{2} C^E C^F \partial_u C^E C^F \delta^D_B] + o(r^{-1-\epsilon}),
\]

\[
J^D_B = \frac{1}{2} r^{-2} l^D_B + o(r^{-2+\epsilon}).
\]

When taking into account the next order of (7.5.15) and comparing to the general solution (7.5.18), we get

\[
\partial_u D_{AB} = 0, \quad N_{AB} = \partial_u C_{AB} - C_{AB} l,
\]

(7.5.20)

where the index on \( N^A_B \) has been lowered with \( \tilde{\gamma}_{AC} \). This implies in turn that

\[
l^A_B = \frac{1}{2} l \delta^A_B + \frac{1}{2} r^{-2} N^A_B - \frac{1}{4} r^{-2} [C^C_N B - N^C B^C + 2 l D^A_B] + o(r^{-2-\epsilon}).
\]

At this stage, equations (7.5.1) have been solved, and then (7.5.3) holds automatically on account of the Bianchi identities. Furthermore \( g^{CD} G_{CD} = 0 \) reduces to \( R_{ur} = 0 \) and we also have \( R = 0 \). Under these assumptions, we only need to discuss the \( r \)-independent part of \( r^2 G_{uA} = 0 \) and then of \( r^2 G_{uu} = 0 \), which reduce to \( r^2 R_{uA} = 0 \) and \( r^2 R_{uu} = 0 \), respectively. The \( r \)-independent part fixes the \( u \) dependence of \( N_A \) and \( M \) in terms of the...
other fields. Explicitly, 

\[ R_{uA} = (-\partial_u + l)\beta_A - \partial_A l - (\partial_u + l)n_A + n_B (2) D^B U_A - \beta_B (2) D_A U^B + 2 U^B (\beta_B \beta_A + n_B n_A) + (2) D_B \left[ l_B^A + \frac{1}{2} (2) D^B U_A - \frac{1}{2} (2) D_A U^B + U^B (\beta_A - n_A) \right] + 2 n_B l_A^B \]

\[ - (\partial_r + 2 \beta_x + \frac{2}{r}) (\frac{V_A}{2r}) - \frac{V}{r} (\partial_r + 2 \beta_x n_A + k_B^B (\frac{V_B}{r} + 2 \frac{V}{r} n_B) \]

\[ - e^{-2 \beta} (\partial_r + 2 \beta_x) \left[ U^B (\frac{1}{2} D_A U_B + \frac{1}{2} (2) D_B U_A + l_A + \frac{V}{r} k_{AB} \right] - e^{-2 \beta} U^B \left[ (\partial_u + l) k_{AB} - 2 l^C k_{CB} - 2 l^C l^{CB} - 2 k_A^C k_{CB} \frac{V}{r} \right] + (2) D_C (k_{AB} U^C) - k_{AC} (2) D^C U_B - k_{BC} (2) D^C U_A \]

and the term proportional to \( r^{-2} \) yields

\[ (\partial_u + l) N_A = \partial_A M + \frac{1}{4} C_B^B \partial_B \bar{R} + \frac{1}{16} \partial_A \left[ N_C^B C_B^C \right] - \frac{1}{4} \bar{D}_A C_B^C N_C^B \]

\[ - \frac{1}{4} \bar{D}_B \left[ C_B^C N_A^C - N_C^B C_A^C \right] - \frac{1}{4} \bar{D}_B \left[ \bar{D}_D C_C^C - \bar{D}_D C_B^C \right] \]

\[ - \frac{1}{32} l \partial_A \left[ C_B^C C_B^C \right] + \frac{1}{16} \partial_A l C_B^C C_B^C + \frac{1}{2} \bar{D}_B \left[ l D_A^B \right]. \quad (7.5.21) \]

Similarly,

\[ R_{uu} = (\partial_u + 2 \beta_u + l) \Gamma_{uu}^A + (\partial_r + 2 \beta_x + \frac{2}{r}) \Gamma_{ru}^A + (\partial_r + 2 \beta_x) \Gamma_{ru}^A + (\partial_u + 2 \beta_x) \Gamma_{uu}^A \]

\[ - 2 \beta_{uu} - \partial_u l - (\Gamma_{uu}^u)^2 - 2 \Gamma_{uu}^u \Gamma_{uu}^A - (\Gamma_{ru}^r)^2 - 2 \Gamma_{rr}^r \Gamma_{uu}^A - \Gamma_{uu}^A \Gamma_{uu}^A \]

and the term proportional to \( r^{-2} \) yields

\[ (\partial_u + \frac{3}{2} l) M = - \frac{1}{8} N_B^A N_B^A - \frac{1}{8} l C_B^A N_B^A - \frac{1}{32} l^2 C_B^A C_B^A + \frac{1}{8} \bar{D}_A \bar{D}_C N_C^A \]

\[ + \frac{1}{4} \bar{D}_A \bar{D}_C C_C^A + \frac{1}{8} l \bar{D}_A \bar{D}_C C_C^A + \frac{1}{4} \bar{D}_C \bar{D}_A \bar{D}_C C_C^A. \quad (7.5.22) \]

All these considerations can be summarized as follows:

For a metric of the form (7.1.1) satisfying the determinant condition and with \( g_{AB} \) as in (7.5.7), the general solution to Einstein’s equations is parametrized by the 2 dimensional background metric \( \gamma_{AB}(u, x^C) \) satisfying (7.5.19), by the mass and angular momentum aspects \( M(u, x^A), N_A(u, x^B) \) satisfying (7.5.22), (7.5.21), by the traceless symmetric news tensor \( N_{AB}(u, x^C) \) defined in (7.5.20), and by the traceless symmetric tensors \( D_{AB}(x^C), C_{AB}(u_0, r, x^C), R_{AB}(u_0, r, x^C) \).

For such spacetimes, the only non vanishing components of the unphysical Weyl tensor at the boundary are given by (7.5.10). When logarithmic terms are required to be
absent in the metric, $D_{AB}(x^C)$ has to satisfy $D_B D^B_A = 0$. In the coordinates $\zeta, \bar{\zeta}$ and the parametrization $\bar{\gamma}_{AB}dx^Adx^B = e^{2\bar{\varphi}}d\zeta d\bar{\zeta}$, this leads to $(7.5.11)$ with $d = d(\zeta)$ and $\bar{d} = \bar{d}(\bar{\zeta})$ by also taking $(7.5.20)$ into account.

In particular, let us now use the parametrization $\bar{\gamma}_{AB}dx^Adx^B = e^{2\bar{\varphi}}d\zeta d\bar{\zeta}$. The determinant condition then reads $\text{det} g_{AB} = -e^{4\bar{\varphi}}r^4$. Even though we will not use it explicitly below, let us point out that the determinant condition can be implemented for instance by choosing the Beltrami representation,

$$h = \frac{g_{\zeta\zeta}}{g_{\bar{\zeta}\bar{\zeta}} + f}, \quad \bar{h} = \frac{g_{\bar{\zeta}\bar{\zeta}}}{g_{\zeta\zeta} + f},$$

$$g_{\zeta\zeta} = \frac{2fh}{1 - y}, \quad g_{\bar{\zeta}\bar{\zeta}} = \frac{2f\bar{h}}{1 - y}, \quad g_{\zeta\bar{\zeta}} = \frac{f(1 + y)}{1 - y},$$

$$g^{\zeta\zeta} = -\frac{2\bar{h}}{f(1 - y)}, \quad g^{\bar{\zeta}\bar{\zeta}} = -\frac{2h}{f(1 - y)}, \quad g^{\zeta\bar{\zeta}} = \frac{1 + y}{f(1 - y)},$$

where $f = \sqrt{-g} y = h\bar{h}$, with $f = \frac{r^2}{2} e^{2\bar{\varphi}}$ fixed, while $h = O(r^{-1}) = \bar{h}$. Alternatively, one can choose

$$g_{\zeta\zeta} = fe^{i\alpha} \sinh \rho, \quad g_{\bar{\zeta}\bar{\zeta}} = fe^{-i\alpha} \sinh \rho, \quad g_{\zeta\bar{\zeta}} = f \cosh \rho,$$

$$g^{\zeta\zeta} = -f^{-1}e^{-i\alpha} \sinh \rho, \quad g^{\bar{\zeta}\bar{\zeta}} = -f^{-1}e^{i\alpha} \sinh \rho, \quad g^{\zeta\bar{\zeta}} = f^{-1} \cosh \rho,$$

where $\rho = O(r^{-1})$ and $\alpha = O(r^0)$.

In the parametrization with the conformal factor introduced with respect to the Riemann sphere, we can write

$$C_{\zeta\zeta} = e^{2\bar{\varphi}}c, \quad C_{\bar{\zeta}\bar{\zeta}} = e^{2\bar{\varphi}}\bar{c}, \quad C_{\zeta\bar{\zeta}} = 0,$$

$$D_{\zeta\zeta} = d, \quad D_{\bar{\zeta}\bar{\zeta}} = \bar{d}, \quad D_{\zeta\bar{\zeta}} = 0. \quad (7.5.23)$$

Equations $(7.5.8)$, $(7.5.9)$ and $(7.5.14)$ read

$$\beta = -\frac{1}{4}r^{-2}c\bar{c} - \frac{1}{3}r^{-3}e^{2\bar{\varphi}}(d\bar{c} + \bar{d}c) + o(r^{-3-\epsilon}),$$

$$U^\zeta = -\frac{2}{r^2}e^{4\bar{\varphi}}\partial(e^{2\bar{\varphi}}c) -$$

$$- \frac{2}{3r^3} \left[(\ln r + \frac{1}{3})4e^{-4\bar{\varphi}}\partial\bar{d} - 4e^{-4\bar{\varphi}}\bar{c}\partial(e^{2\bar{\varphi}}c) + \mathcal{N}^\zeta\right] + o(r^{-3-\epsilon}),$$

$$\frac{V}{r} = -2r\partial_u \bar{\varphi} + 4e^{-2\bar{\varphi}}\partial\bar{\varphi} + r^{-1}2M + o(r^{-1-\epsilon}). \quad (7.5.24)$$
and the evolution equations become

\[
\partial_u (e^{2\tilde{\varphi}} M) = \partial_u \left( e^{\tilde{\varphi}} \left[ \partial^2 \bar{c} + \partial^2 c + 2\partial \tilde{\varphi} \partial \bar{c} + 2 \tilde{\varphi} \partial \bar{c} + 2 \partial^2 \tilde{\varphi} \bar{c} \right] \right) \\
- e^{\tilde{\varphi}} \partial_u (e^{\tilde{\varphi}} c) \partial_u (e^{\tilde{\varphi}} c) + 2 e^{\tilde{\varphi}} \left[ \partial_u (\partial \tilde{\varphi})^2 - \partial_u \partial^2 \tilde{\varphi} \right] + \partial_u (\partial \tilde{\varphi})^2 - \partial_u \partial^2 \tilde{\varphi} \right) \\
+ e^{-\tilde{\varphi}} \left( -4 (\partial \bar{\varphi})^2 \bar{\varphi} + 8 \left( (\partial \partial \bar{\varphi})^2 + \partial \bar{\varphi} \partial \partial^2 \bar{\varphi} + \partial \tilde{\varphi} \partial^2 \bar{\varphi} + \partial \tilde{\varphi} \partial \bar{\varphi} ) \right) \\
\partial_u (e^{2\tilde{\varphi}} N\zeta) = e^{2\tilde{\varphi}} \left[ \partial^2 M + \frac{1}{4} \left\{ (\partial \bar{c} + 3 \bar{\varphi} \partial \bar{c}) \partial_u c - (3c \partial \tilde{\varphi} + 7 \tilde{\varphi} \partial c) \partial_u \bar{c} \right\} + 2 \bar{\varphi} (\partial_u c - c \partial_u \bar{c}) \right. \\
- \frac{1}{2} \partial_u \bar{\varphi} (\partial_u c - c \partial_u \bar{c}) + 2 (\partial \tilde{\varphi} \partial \bar{c} - \partial \bar{\varphi} \bar{c}) \bar{c} \\
+ \partial^3 c + 2 \partial^2 \tilde{\varphi} \partial c - 4 \partial \tilde{\varphi} \partial^2 \bar{\varphi} c - 4 (\partial \bar{\varphi})^2 \partial c \\
- \partial^2 \tilde{\varphi} \partial c - 2 (\partial \bar{\varphi} \partial + \partial^2 \bar{\varphi}) \partial c - 2 (\partial \partial \tilde{\varphi} - \partial \bar{\varphi} \bar{\varphi} - 2 \partial \tilde{\varphi} \partial \bar{\varphi}) \partial c \\
\left. - 2 (\partial \partial \tilde{\varphi} - 2 \partial \bar{\varphi} \partial^2 \bar{\varphi} - 4 \partial \tilde{\varphi} \partial \bar{\varphi} \partial c) \partial c \right].
\]

(7.5.25)

Let us now set \( \bar{\varphi} = 0 \). Note that one can re-introduce an arbitrary \( \bar{\varphi} \) through the finite coordinate transformation generated by \( \xi^u = -u \bar{\varphi}, \xi^A = -\xi_{u, B} \int r^\infty dr' (e^{2\beta} g^{AB}) \), \( \xi^r = -\frac{1}{2} r (\partial_A \xi^A - 2\tilde{\varphi} - f_B U^B) \). The above relations then simplify to

\[
\beta = -\frac{1}{4} r^{-2} \bar{c} - \frac{1}{3} r^{-3} (\partial \bar{c} + \partial \bar{c}) + o(r^{-3-\epsilon}), \\
U^\zeta = -2r^{-2} \partial \bar{c} - \frac{2}{3} r^{-3} \left( (\ln r + \frac{1}{3}) 4 \partial \tilde{\varphi} - 4 \partial \bar{\varphi} c + N^\zeta \right) + o(r^{-3-\epsilon}), \\
V = r^{-1} 2 M + o(r^{-1-\epsilon}), \\
\partial_u M = \left[ \partial^2 \tilde{c} + \partial^2 \bar{c} \right] - \partial \bar{c}, \\
\partial_u N^\zeta = \partial^2 M + \frac{1}{4} \left( (\partial \bar{c} + 5 \partial \bar{c}) \partial \bar{c} - (3c \partial \tilde{\varphi} + 7 \partial \bar{c}) \partial \bar{c} \right] + \partial^3 c - \partial^2 \partial c.
\]

(7.5.26)

When defining \( \widetilde{M} = M - \partial^2 c - \partial^2 \bar{c} \) and \( \widetilde{N}^\zeta = -\frac{1}{12} [2 N^\zeta + 7 \tilde{c} \partial \bar{c} + 3 \bar{c} \partial \bar{c}] \), the evolution equations become

\[
\partial_u \widetilde{M} = -\partial \bar{c}, \\
3 \partial_u \widetilde{N}^\zeta = -\partial \widetilde{M} - 2 \partial^3 c - (\partial \bar{c} + 3 \partial \bar{c}) \partial \bar{c}.
\]

(7.5.27)

\section{Realization of bms\textsubscript{4} on solution space}

In order to compute how bms\textsubscript{4} is realized on solution space we need to compute the Lie derivative of the metric on-shell. We will do so for the extended transformations defined by (7.2.10) - (7.2.11) and use \( -\delta \gamma_{AB} = 2 \omega \gamma_{AB} \). Let \( \tilde{\psi} = \psi - 2 \omega \). This gives

\[
- \delta C_{AB} = \left[ f \partial_u + \mathcal{L}_Y - \frac{1}{2} (\tilde{\psi} + f l) \right] C_{AB} - 2 \bar{D}_A \bar{D}_B f + \bar{A} f \bar{\gamma}_{AB},
\]

(7.6.1)
where (7.5.20) should be used to eliminate $\partial_u C_{AB}$ in favor of $N_{AB}$ and

$$-\delta D_{AB} = \mathcal{L}_Y D_{AB}, \quad (7.6.2)$$

where we have used that

$$D_A D_C f C_B^C + D_B D_C f C_A^C - \bar{\gamma}_{AB} D_C \bar{D}_C f C_{CD} - \bar{\Delta} f C_{AB} = 0,$$

$$\bar{D}_A f D_C C_B^C + D_B f D_C C_A^C + D_C f D_A C_B^C + D_C f D_B C_A^C - 2\bar{D} f D_C C_{AB} - 2\bar{\gamma}_{AB} D_C f D_D C_{CD} = 0,$$

which can be explicitly checked in the parametrization $\bar{\gamma}_{AB} dx^A dx^B = e^{2\bar{\zeta}} d\zeta d\bar{\zeta}$ with $C_{AB}$ defined in (7.5.23). By taking the time derivative of (7.6.1) and using (7.5.20), (7.2.5) with $\psi$ replaced by $\bar{\psi}$, one finds the transformation law for the news tensor,

$$-\delta N_{AB} = [ f \partial_u + \mathcal{L}_Y ] N_{AB} - \left( D_A \bar{D}_B \bar{\psi} - \frac{1}{2} \bar{\Delta} \bar{\psi} \bar{\gamma}_{AB} \right) + \frac{1}{4} ( f \bar{l} + f l^2 + \bar{\psi} l - 4 \omega + 2 Y^C \bar{D}_C l ) C_{AB}$$

$$+ ( \bar{D}_A \bar{D}_B f - \frac{1}{2} \bar{\Delta} f \bar{\gamma}_{AB} - f( \bar{D}_A \bar{D}_B f ) ) = \bar{\Delta} f \bar{\gamma}_{AB}, \quad (7.6.3)$$

We have $g_{uA} = \frac{1}{2} \bar{D}_B C_{AB} + \frac{1}{2} r^{-1} [ ( \ln r + \frac{1}{2} ) \bar{D}_B D_A^B + \frac{1}{4} C_B^B \bar{D}_C C_B^C + N_A ] + o(r^{-1-\epsilon})$, and by computing $\mathcal{L}_Y g_{uA}$ on-shell, we find to leading order that 

$$-\delta( \bar{D}_B C_{AB} ) = [ f \partial_u + \mathcal{L}_Y + \frac{1}{2} ( f \bar{l} + \bar{\psi} ) ] \bar{D}_B D_A^B,$$

which is again consistent with (7.6.2), while the $r^{-1}$ terms, when combined with the previous transformations, give

$$-\delta N_A = [ f \partial_u + \mathcal{L}_Y + \bar{\psi}^A + f l ] N_A - \frac{1}{2} ( f \bar{D}_B l + \bar{D}_B \bar{\psi} + ( \bar{\psi} + l f ) \bar{D}_B ) D_A^B$$

$$+ 3 \bar{D}_A f M - \frac{3}{16} \bar{D}_A f N^B C^C_B + \frac{1}{2} \bar{D}_B f N^C C^B_C + \frac{1}{32} ( \bar{D}_A f l - f \bar{D}_A l - \bar{D}_A \bar{\psi} ) ( C_B^B C^C_B )$$

$$+ \frac{1}{4} \bar{D}_B f \bar{\Delta} f C^C_B - \frac{3}{4} \bar{D}_B f ( \bar{D}_B \bar{D}_C C^C_B ) - \bar{\Delta} f \bar{\gamma}_{AB} D_C C_{CB} + \frac{3}{8} \bar{D}_A ( \bar{D}_C \bar{D}_B f C^{CB} ) \quad (7.6.5)$$

Here $\partial_u N_A$ should be eliminated by using (7.5.21). In the same way, from the order $r^{-1}$ of $\mathcal{L}_Y g_{uAB}$, we get

$$-\delta M = [ f \partial_u + Y^A \partial_A + \frac{3}{2} ( \bar{\psi} + f l ) ] M$$

$$+ \frac{1}{4} \partial_u [ \bar{D}_C \bar{D}_B f C^{CB} + 2 \bar{D}_B f \bar{D}_C C^{CB} ] + \frac{1}{4} [ \bar{D}_A f l - f \bar{D}_A l - \bar{D}_A \bar{\psi} ] \bar{D}_B C^{BA}$$

$$+ \frac{1}{4} \partial_A f ( \bar{\partial}^A R - C^{AB} D_B l ) + \frac{1}{4} [ \bar{D}_C \bar{D}_B f C^{CB} + \bar{D}_B f \bar{D}_C C^{CB} ] \quad (7.6.6)$$
where \( \partial_u M \) should be replaced by its expression from (7.5.22).

Let us now discuss these transformations in the parametrization \( \zeta, \bar{\zeta} \) with \( \bar{\varphi} = 0 = \omega \) so that \( \bar{\gamma}_{AB} dx^A dx^B = d\zeta d\bar{\zeta} \). From the leading and subleading orders of \( \mathcal{L}_\xi g_{\zeta\zeta}, \mathcal{L}_\xi g_{\bar{\zeta}\bar{\zeta}} \), we get

\[
- \delta c = f \dot{c} + Y^A \partial_A c + (\frac{3}{2} \partial Y - \frac{1}{2} \bar{\partial} Y) c - 2 \partial^2 f, \\
- \delta d = Y^A \partial_A d + 2 \partial Y d,
\]

(7.6.7)

with \( f \) given in (7.3.12) and the complex conjugate relation holding for \( \bar{c}, \bar{d} \). In particular, for the news function we find

\[
- \delta \dot{c} = f \ddot{c} + Y^A \partial_A \dot{c} + 2 \partial Y \dot{c} - \partial^3 Y,
\]

(7.6.8)

From the subleading term of \( \mathcal{L}_\xi g^{\zeta\zeta} \) and the leading term of \( \mathcal{L}_\xi g_{uu} \) and we get

\[
- \delta \bar{\mathcal{N}}^\zeta = Y^A \partial_A \bar{\mathcal{N}}^\zeta + (\partial Y + 2 \bar{\partial} Y) \bar{\mathcal{N}}^\zeta + \frac{1}{3} \partial^2 Y \bar{d} \\
- \bar{\partial} f(\bar{\mathcal{M}} + 2 \bar{\partial}^2 \bar{c} + \bar{c} \bar{c}) - \frac{f}{3} [\bar{\partial} \bar{\mathcal{M}} + 2 \bar{\partial}^3 \bar{c} + (\bar{\partial} \bar{c} + 3 \bar{c} \bar{\partial}) \bar{c}],
\]

(7.6.9)

\[
- \delta \bar{\mathcal{M}} = -f \ddot{c} + Y^A \partial_A \bar{\mathcal{M}} + \frac{3}{2} \psi \bar{\mathcal{M}} + \bar{c} \partial^3 Y + c \bar{\partial}^3 Y + 4 \partial^2 \partial^2 \bar{T}.
\]

(7.6.10)

As can be understood by comparing with the 3 dimensional anti-de Sitter and flat cases, this computation already contains information on the central extensions in the surface charge algebra through the inhomogeneous part of the transformation laws for the fields. Although we know that \( BMS_4 \) is free of central extensions in general, in our case we could have field dependent extensions. Signs of this kind of extension are present in the variation of \( M \) for instance where we can see a Schwarzian derivative multiplied by the field \( c \).

### 7.7 \( bms_4 \) charges

In chapter 5 and 6 we were able to integrate the expression for the charges and use equation (A.2.5) to compute them. The charges of \( BMS_4 \) are non-integrable; we have to start from expression (A.2.1) with \( h = \delta g \). We mean by \( \delta g \) the variation of the metric \( g \) generated by a small variation of the parameters characterizing it asymptotically:

\[
\mathcal{X}^T \equiv \{ C_{AB}, N_{AB}, D_{AB}, M, N_A \}.
\]

(7.7.1)

In this section, we will consider metrics with \( \partial_u \varphi = 0 \). Our integration two sphere is given by \( u = u_0 \) constant and \( r = \text{cst} \to \infty \). We will also use the notation \( \int d^2 \Omega^\varphi = \int dx^2 dx^3 \sqrt{\gamma} = \int_0^\pi d\theta \int_0^{2\pi} d\phi \sin \theta e^{2\varphi} \).
We start by writing

\[ \delta Q_{\xi[h,g]} = \frac{1}{16\pi G} \lim_{r \to \infty} \int d^2\Omega^r \, r^2 e^{2\beta} \left[ \xi^r (D^u h - D_\sigma h^{u\sigma} + D^r h_r^u) - D^u h^r_r \right] \]

\[
- \xi^u (D^r h - D_\sigma h^{r\sigma} - D^r h_u^u + D^u h^r_r) + \xi^A (D^r h^u_A - D^u h^r_A) + \frac{1}{2} h(D^r \xi^r - D^u \xi^r) \\
+ \frac{1}{2} h^{r\sigma} (D^u \xi_\sigma - D_\sigma \xi^u) \right] \text{.} \quad (7.7.2)
\]

We have

\[ D^u h - D_\sigma h^{u\sigma} + D^r h^u_r - D^u h^r_r = g^{ur} g^{AB} (D_r h_{AB} - D_A h_{rB}) \]

\[ = -e^{-2\beta} \left( g^{AB} \partial_r h_{AB} - k^{AB} h_{AB} + e^{-2\beta} g^{AB} k_{AB} h_{ru} \right) \]

\[ = \frac{1}{4r^3} C^{AB} \delta C_{AB} + o(r^{-3-\epsilon}) \text{,} \quad (7.7.3) \]

\[ - (D^r h - D_\sigma h^{r\sigma} - D^r h^u_r + D^u h^r_r) = D^A h_A^r - D^r h^A_A \]

\[ = g^{ur} g^{AB} (D_A h_{rAB} - D_\sigma h_{AB}) + O(r^{-3}) \]

\[ = g^{ur} \left( g^{AB} \partial_r (D_B h_{uA} - h_{ur} g^{AB} (l_{AB} + k_{AB} \frac{V}{r}) \right) \]

\[ - k h_{uu} - g^{AB} \partial_u h_{AB} + g^{AB} h_{CA} \Gamma^C_{AB} \right) + O(r^{-3}) \]

\[ = \frac{1}{r^2} \left( 4 \delta M - \frac{1}{2} \bar{D}_A \bar{D}_B \delta C_{AB} + \frac{1}{2} \delta \partial_u (C^{AB} C_{AB}) \right) \]

\[ - \frac{1}{2} \partial_u C_{AB} \delta C^{AB} - C^{AB} \partial_u \delta C_{AB} \right) + o(r^{-2-\epsilon}) \text{,} \quad (7.7.4) \]

\[ D^r h^u_A - D^u h^r_A = \left( g^{ur} \right)^2 (\Gamma_C^{\sigma} h_{uA} - \partial_r h_{uA}) + g^{ur} g^{rB} \left( \Gamma^{C}_{rB} h_{AC} - \partial_r h_{AB} \right) + O(r^{-3}) \]

\[ = \frac{1}{2} \bar{D}_B \delta C^B_A + \frac{2}{3r^2} (2 \ln r - \frac{1}{3}) \bar{D}_B \delta D^B_A + o(r^{-2-\epsilon}) \]

\[ + \frac{1}{r^2} \left( \frac{4}{3} \delta N_A + \frac{2}{3} (C_{AB} D_B C^B_C) \right) \quad (7.7.5) \]

\[ \frac{1}{2} h^{r\sigma} (D^u \xi_\sigma - D_\sigma \xi^u) - \frac{1}{2} h^{u\sigma} (D^r \xi^r - D_\sigma \xi^r) = \]

\[ \frac{1}{2} (h^u + h^r_r) (D^u \xi^r - D^r \xi^u) + \frac{1}{2} h^r_r (D^u \xi^A - D^A \xi^u) \text{.} \]

\[ \frac{1}{2} (h - h^u - h^r_r) (D^r \xi^u - D^u \xi^r) = \frac{1}{2} g^{AB} h_{AB} (D^r \xi^u - D^u \xi^r) = 0 \quad (7.7.6) \]

\[ D^u \xi^A - D^A \xi^u = \frac{g^{ur} \partial_r \xi^A}{g^{AB} \partial_B C^u} + (g^{ur} \Gamma^A_{rC} - g^{AB} \Gamma^{BC}_{uA}) \xi^C + o(r^{-3}) \]

\[ = \frac{-2}{r^2} Y^A + \frac{2}{r^2} C^A C Y^C + O(r^{-3}) \quad (7.7.7) \]
\[
\frac{1}{2} h_A^r = -\frac{1}{4} \tilde{D}_B \delta C^B_A - \frac{1}{3} \ln r + \frac{1}{3} \tilde{D}_B \delta D^B_A + \frac{1}{r} \left( -\frac{1}{3} \delta N_A - \frac{1}{12} \delta (C_{AB} \tilde{D}_C C^C_B) + \frac{1}{4} \delta C_{AB} \tilde{D}_C C^C_B \right) + o(r^{-1 - \epsilon}). \quad (7.7.8)
\]

Putting everything together, we get
\[
Q_\xi[h, g] = \frac{1}{16 \pi G} \lim_{r \to \infty} \int d^2 \Omega^\varphi \left[ r \left( Y^A \frac{1}{2} \tilde{D}_B \delta C^B_A + Y^A \frac{1}{2} \tilde{D}_B \delta C^B_A \right) + f \left( 4 \delta M - \frac{1}{2} \tilde{D}_A \tilde{D}_B \delta C^{AB} + \frac{1}{2} \delta \partial_u (C^{AB} C_{AB}) - \frac{1}{2} \partial_u C^{AB} \delta C_{AB} \right) + \frac{1}{2} \tilde{D}_A f \tilde{D}_B \delta C^{AB} - \frac{1}{4} C_{AB} Y^A \tilde{D}_C C^C_B \right] \]  
\[
+ \frac{1}{16 \pi G} \int d^2 \Omega^\varphi \left[ - \frac{\psi}{8} C^{AB} \delta C_{AB} + \frac{1}{2} \tilde{D}_A f \tilde{D}_B \delta C^{AB} - \frac{1}{4} C_{AB} Y^A \tilde{D}_C C^C_B \right] \]  
\[
+ \frac{1}{16 \pi G} \int d^2 \Omega^\varphi \left[ - \frac{\psi}{16} C^{AB} C_{AB} + 2 Y^A N_A + 4 f M \right] + \frac{1}{16 \pi G} \int d^2 \Omega^\varphi \left[ \frac{f}{2} \partial_u C_{AB} \delta C^{AB} \right]. \quad (7.7.9)
\]

This result can be rewritten as
\[
\delta Q_\xi[h, g] = \delta (Q_s[\mathcal{X}]) + \delta \Theta_s[\mathcal{X}, \delta \mathcal{X}], \quad (7.7.12)
\]

where the integrable part of the surface charge one-form is given by
\[
Q_s[\mathcal{X}] = \frac{1}{16 \pi G} \int d^2 \Omega^\varphi \left[ 4 f M + Y^A \left( 2 N_A + \frac{1}{16} \partial_A (C^{CB} C_{CB}) \right) \right], \quad (7.7.13)
\]

and the non-integrable part is due to the news tensor,
\[
\delta \Theta_s[\mathcal{X}, \delta \mathcal{X}] = \frac{1}{16 \pi G} \int d^2 \Omega^\varphi \left[ \frac{f}{2} N_{AB} \delta C^{AB} \right]. \quad (7.7.14)
\]
The separation into an integrable and a non-integrable part in [7.7.12] is not uniquely defined as this equation also holds in terms $Q'_s = Q_s - N_s$, $\Theta'_s = \Theta_s + \delta N_s$ for some $N_s[\mathcal{X}]$.

These charges are very similar and should be compared to those proposed earlier in [116] in the context of a closely related, but slightly different approach to asymptotically flat spacetimes.

### 7.8 Algebra of charges and extension

One of the big advantages of the covariant techniques to compute charges is that they allow us to define an algebra for those charges. Moreover, when using the equivalence of the Hamiltonian and the covariant approaches, one can infer that this algebra coincides with the Dirac bracket $\{Q_s, Q_s\}^\ast$ of the charges. Unfortunately, in presence of this non-integrable term, the usual definition of algebra (A.2.6) fails.

In the non-integrable case, we propose as a definition

$$\{Q_{s_1}, Q_{s_2}\}[\mathcal{X}] = (-\delta_{s_2})Q_{s_1}[\mathcal{X}] + \phi \Theta_{s_2}[\mathcal{X}, -\delta_{s_1}\mathcal{X}].$$  \hspace{1cm} (7.8.1)

**Theorem 7.8.1.** The $BMS_4$ charges (7.7.13) satisfy

$$\{Q_{s_1}, Q_{s_2}\}[\mathcal{X}] = Q_{[s_1,s_2]}[\mathcal{X}] + K_{s_1,s_2}[\mathcal{X}],$$  \hspace{1cm} (7.8.2)

where the field dependent central extension is

$$K_{s_1,s_2}[\mathcal{X}] = \frac{1}{32\pi G} \int d^2\Omega^\varphi \left[ \left( f_1\partial_A f_2 - f_2\partial_A f_1 \right) \partial^A \hat{R} + C^{BC}(f_1\tilde{D}_B\tilde{D}_C\psi_2 - f_2\tilde{D}_B\tilde{D}_C\psi_1) \right],$$  \hspace{1cm} (7.8.3)

**Proof.** We will start by computing the usual factor,

$$-\delta_{s_2}Q_{s_1}[\mathcal{X}] = \frac{1}{16\pi G} \int d^2\Omega^\varphi \left[ \left( f_1\partial_A f_2 - f_2\partial_A f_1 \right) \partial^A \hat{R} + C^{BC}(f_1\tilde{D}_B\tilde{D}_C\psi_2 - f_2\tilde{D}_B\tilde{D}_C\psi_1) \right] \right],$$  \hspace{1cm} (7.8.4)

and organize according to the different types of terms that appear:

- terms containing $M$

  $$-\delta_{s_2}Q_{s_1}[\mathcal{X}]_M = \frac{1}{16\pi G} \int d^2\Omega^\varphi \left[ f_1^2(\partial_A f_2 + \partial_A f_1) - f_2^2(\partial_A f_1 + \partial_A f_2) \right] + f_1(\partial_A f_1 + \partial_A f_2),$$  \hspace{1cm} (7.8.5)
• terms containing \( N_A \)

\[
- \delta_{s_2} Q_{s_1} [\mathcal{X}]_N = \frac{1}{16\pi G} \int d^2\Omega^\varphi \left[ 2Y_1 A (\mathcal{L}_{Y_2} + \psi_2) N_A \right] \\
= \frac{1}{16\pi G} \int d^2\Omega^\varphi \left[ 2Y_1 A (Y_2 B \bar{D}_B + \psi_2) N_A + 2Y_1 A \bar{D}_A Y_2^B N_B \right] \\
= \frac{1}{16\pi G} \int d^2\Omega^\varphi 2N_A \left[ -Y_2^B \bar{D}_B Y_1 A + Y_1 B \bar{D}_B Y_2 A \right] \\
= \frac{1}{16\pi G} \int d^2\Omega^\varphi 2N_A Y_1^A \quad \text{, (7.8.6)}
\]

• terms containing \( D_{AB} \)

\[
- \delta_{s_2} Q_{s_1} [\mathcal{X}]_D = \frac{1}{16\pi G} \int d^2\Omega^\varphi 2Y_1 A \left[ - \frac{1}{2} [\bar{D}_B \psi_2 + \psi_2 \bar{D}_B] D_A^B \right] \\
= \frac{1}{16\pi G} \int d^2\Omega^\varphi 2Y_1 A \left[ - \frac{1}{2} \bar{D}_B (\psi_2 D_A^B) \right] \\
= \frac{1}{16\pi G} \int d^2\Omega^\varphi \bar{D}^B Y_1 A \psi_2 D_{AB} = 0 \quad \text{, (7.8.7)}
\]

• terms containing the news

\[
- \delta_{s_2} Q_{s_1} [\mathcal{X}]_{\text{news}} = \frac{1}{16\pi G} \int d^2\Omega^\varphi \left[ 2Y_1 A \left( \frac{3}{16} \bar{D} A f_2 N_B^B C_C^C + \frac{1}{2} \bar{D}_B f_2 N_B^B C_A^C \right) \\
+ 2Y_1 A f_2 \left( \frac{1}{16} \partial_A \left[ N_B^B C_C^C \right] - \frac{1}{4} \bar{D}_A C_C^D N_B^B - \frac{1}{4} \bar{D}_B \left[ C_B^B C_A^A - N_B^B C_A^C \right] \right) \\
- \psi_1 \frac{1}{8} C_{AB} f_2 N_{AB} + 4f_1 \left( \frac{1}{4} \bar{D}_B \bar{D}_C f_2 N^B C^C + \frac{1}{2} \bar{D}_B f_2 \bar{D}_C N^B C^C \right) \\
+ 4f_1 f_2 \left( -\frac{1}{8} N_A^A N_A^A + \frac{1}{4} \bar{D}_A \bar{D}_C N^A C^A \right) \right] \\
= \frac{1}{16\pi G} \int d^2\Omega^\varphi - \frac{1}{2} N^A_{BC} f_2 \left[ f_1 N_{BC} + \mathcal{L}_{Y_1} C_{BC} - \frac{1}{2} \psi_1 C_{BC} - 2 \bar{D}_B \bar{D}_C f_1 \right] \\
+ \frac{1}{16\pi G} \int d^2\Omega^\varphi \frac{1}{2} N_B^A C_C^A \left[ Y_1^A \bar{D}^B f_2 + Y_1^B \bar{D}^A f_2 - \bar{\gamma}^{AB} Y_1^D \bar{D}^D f_2 \right] \quad \text{(7.8.8)}
\]

The second line is zero. This is coming from the following identity for the symmetrized product of two traceless tensors in 2 dimensions,

\[
\frac{1}{2} (C_B^A K_C^B + K_B^A C_C^B) = \frac{1}{2} \delta_A^C C_B^D K_D^B \quad \text{, (7.8.9)}
\]

and the conformal Killing equation for the \( Y^A \). The first line can be recognized as,

\[
- \delta_{s_2} Q_{s_1} [\mathcal{X}]_{\text{news}} = \frac{1}{16\pi G} \int d^2\Omega^\varphi - \frac{1}{2} N^A_{BC} f_2 \left[ - \delta_{s_1} C_{BC} \right] \\
= -\partial \Theta_{s_2} [- \delta_{s_1} \mathcal{X}, \mathcal{X}] \quad \text{, (7.8.10)}
\]
\[
\begin{align*}
- \delta_s Q_s &= \frac{1}{16\pi G} \int d^2\Omega^\varphi \left[ 2Y^A_1 \left( -\frac{1}{32} D_A \psi_2 C_B C_C^C + f_2 \frac{1}{4} C_A^B \partial_B \bar{R} \right. \right. \\
&\quad - \frac{1}{4} f_2 \bar{D}_B \left( \bar{D}^B \bar{D}_C C_A^C - \bar{D}_A \bar{D} C B C\right) \\
&\quad + \frac{1}{4} (\bar{D}_B f_2 \bar{R} + \bar{D}_B \bar{\Delta} f_2) C_A^C - \frac{3}{4} \bar{D}_B f_2 (\bar{D}^B \bar{D}_C C_A^C - \bar{D}_A \bar{D} C B C) \\
&\quad + \frac{1}{2} (\bar{D}_A \bar{D}_B f_2 - \frac{1}{2} \bar{\Delta} f_2 \bar{\Delta} f_2) \bar{D}_C C B C + \frac{3}{8} \bar{D}_A (\bar{D} \bar{D}_B f_2 C B C) \\
&\quad - \psi_1 \frac{1}{8} C B C \left( [\mathcal{L}_2 - \frac{1}{2} \psi_2] C_B - 2 \bar{D}_C \bar{D}_B f_2 + \bar{\Delta} f_2 \bar{\Delta} f_2 \right) \\
&\quad + 4f_1 \left( f_2 \frac{1}{8} \bar{\Delta} \bar{R} + \frac{1}{4} \partial_A f_2 \partial A \bar{R} + \frac{1}{8} \bar{D}_C \bar{D}_B \psi_2 C B C \right) \left. \right] \\
= \frac{1}{16\pi G} \int d^2\Omega^\varphi \left[ - Y^A_1 \frac{1}{16} \bar{D}_A \psi_2 C_B C_C - \psi_1 \frac{1}{8} C B C \left( [\mathcal{L}_2 - \frac{1}{2} \psi_2] C_B \right) \right. \\
&\quad + \bar{C} B C \left( \frac{1}{2} f_1 f_2 \bar{D}_B \bar{D}_C \psi_2 + \psi_1 \frac{1}{4} \bar{D}_C \bar{D}_B f_2 + \frac{1}{4} f_2 Y_1 B \partial C \bar{R} \\
&\quad - \frac{3}{4} \psi_1 \bar{D}_B \bar{D}_C f_2 - \bar{D}_C (Y_1^A \bar{D}_A \bar{D}_B f_2) + \frac{1}{2} \bar{D}_C (Y_1 \bar{D}_B f_2) \\
&\quad + \frac{1}{2} \bar{Y}_1 C (\bar{D}_B f_2 \bar{R} + \bar{D}_B \bar{\Delta} f_2) + \frac{1}{2} \bar{D}_C \bar{D}_B (Y_1 \bar{D}_B f_2) \\
&\quad - \frac{3}{4} \bar{D}_C \bar{D}_A (Y_1 \bar{D}_B f_2) + \frac{3}{2} \bar{D}_C \bar{D}_A (Y_1 \bar{D}_B f_2) \\
&\quad + \frac{1}{2} \left( f_1 \partial A f_2 - f_2 \partial A f_1 \right) \partial A \bar{R} \right] . \quad (7.8.11)
\end{align*}
\]

Using the commutation rule for covariant derivatives, this gives

\[
\begin{align*}
- \delta_s Q_s &= \frac{1}{16\pi G} \int d^2\Omega^\varphi \left[ - Y^A_1 \frac{1}{16} \bar{D}_A \psi_2 - Y^A_2 \bar{D}_A \psi_1 \right] C_B C_C^C \\
&\quad + \frac{1}{2} \left( f_1 \partial A f_2 - f_2 \partial A f_1 \right) \partial A \bar{R} + \bar{C}_B C \left( \frac{1}{2} \left( f_1 \bar{D}_B \bar{D}_C \psi_2 - f_2 \bar{D}_B \bar{D}_C \psi_1 \right) \right. \\
&\quad + \frac{1}{4} f_2 Y_1 B \partial C \bar{R} + \frac{1}{2} \bar{D}_C \bar{D}_B \bar{\Delta} Y_1 \bar{B} + \frac{1}{4} \bar{D}_C f_2 Y_1 \bar{B} \bar{R} + \frac{1}{2} \bar{D}_C \bar{D}_B \bar{\Delta} Y_1 \bar{B} \right] \\
= \frac{1}{16\pi G} \int d^2\Omega^\varphi \left[ - Y^A_1 \frac{1}{16} \psi_{\{s,s\}} C_B C_C^C + 2 f_1 \partial A f_2 - f_2 \partial A f_1 \partial A \bar{R} \right. \\
&\quad + \bar{C}_B C \left( \frac{1}{2} \left( f_1 \bar{D}_B \bar{D}_C \psi_2 - f_2 \bar{D}_B \bar{D}_C \psi_1 \right) \right. \\
&\quad + \frac{1}{2} \left( f_1 \partial A f_2 - f_2 \partial A f_1 \right) \partial A \bar{R} \right] , \quad (7.8.12)
\end{align*}
\]

where in the last line we have used the identity \( \bar{\Delta} Y^A = -\frac{1}{7} \bar{R} Y^A \) satisfied by conformal Killing vectors.

Summing everything, we obtain

\[
\begin{align*}
- \delta_s Q_s &= \frac{1}{16\pi G} \int d^2\Omega^\varphi \left[ - Y^A_1 \frac{1}{16} \psi_{\{s,s\}} C_B C_C^C + \frac{1}{2} \left( f_1 \partial A f_2 - f_2 \partial A f_1 \right) \partial A \bar{R} \right. \\
&\quad + \bar{C}_B C \left( \frac{1}{2} \left( f_1 \bar{D}_B \bar{D}_C \psi_2 - f_2 \bar{D}_B \bar{D}_C \psi_1 \right) \right. \\
&\quad + \frac{1}{2} \left( f_1 \partial A f_2 - f_2 \partial A f_1 \right) \partial A \bar{R} \right] \\
&\quad - \left. \Theta_2 \left[ - \delta_1 \{s, s\} \right] \right] \\
= \left. Q_{\{s,s\}} - \phi \Theta_2 \left[ - \delta_1 \{s, s\} \right] \right] + K_{s,s} [\{s,s\}] , \quad (7.8.13)
\end{align*}
\]

with \( K_{s,s} [\{s,s\}] \) defined in (7.8.3).

**Theorem 7.8.2.** *The central extension satisfies the cocycle condition*

\[
K_{\{s,s\}, s} + \delta_s K_{s,s} + \text{cyclic } (1, 2, 3) = 0 . \quad (7.8.14)
\]
Proof. Let us treat the two parts of $K_{s_1,s_2}[\lambda]$ separately:

- for the second part $K_{s_1,s_2} = \frac{1}{16\pi G} \int d^2\Omega^\varphi C^{BC} f_1^A D_B \bar{D}_C \psi_2 - f_2 \bar{D}_B D_C \psi_1$ we have

$$A = \int d^2\Omega^\varphi \left[ (-\delta s_3 C^{BC}) (f_1^A \bar{D}_B \bar{D}_C \psi_2 - f_2 \bar{D}_B D_C \psi_1) + \text{cyclic (1, 2, 3)} \right]$$

$$= \int d^2\Omega^\varphi \left[ [(\delta s_3 + \mathcal{L}_{Y_2} - \frac{1}{2} \psi_3) C_{AB} - 2 \bar{D}_A \bar{D}_B \psi_3 + \Delta f_3 \bar{Y}_{AB}] \right.$$

$$\left. (f_1^A \bar{D}_B \bar{D}_C \psi_2 - f_2 \bar{D}_B D_C \psi_1) + \text{cyclic (1, 2, 3)} \right]$$

$$= \int d^2\Omega^\varphi \left[ - C^{BC} \bar{D}_A ((Y_1^A f_2 - Y_2^A f_1)) \bar{D}_B \bar{D}_C \psi_3 \right.\left. + 2 C^{BC} (\bar{D}_A Y_1^B f_2 - \bar{D}_A Y_2^B f_1) \bar{D}_B \bar{D}_C \psi_3 - \frac{1}{2} C^{BC} (\psi_1 f_2 - \psi_2 f_1) \bar{D}_B \bar{D}_C \psi_3 \right.\left. + 2 (\bar{D}_C f_1 f_2 - \bar{D}_C f_2 f_1) \left( \bar{\Delta} D^C \psi_3 - \frac{1}{2} \bar{D}^C \bar{\Delta} \psi_3 \right) + \text{cyclic (1, 2, 3)} \right]. \quad (7.8.15)$$

The second term is given by

$$B = \int d^2\Omega^\varphi \left[ C^{BC} (f_{s_1,s_2} D_B \bar{D}_C \psi_3 - f_3 \bar{D}_B \bar{D}_C \psi_{s_1,s_2}) + \text{cyclic (1, 2, 3)} \right]$$

$$= \int d^2\Omega^\varphi C^{BC} \left[ \bar{D}_A ((Y_1^A f_2 - Y_2^A f_1) \bar{D}_B \bar{D}_C \psi_3) - \frac{3}{2} (\psi_1 f_2 - \psi_2 f_1) \bar{D}_B \bar{D}_C \psi_3 \right.\left. - (Y_1^A f_2 - Y_2^A f_1) \bar{D}_A \bar{D}_B \bar{D}_C \psi_3 - f_3 \bar{D}_B \bar{D}_C (Y_1^A \bar{D}_A \psi_2 - Y_2^A \bar{D}_A \psi_1) + \text{cyclic (1, 2, 3)} \right]$$

$$= \int d^2\Omega^\varphi C^{BC} \left[ \bar{D}_A ((Y_1^A f_2 - Y_2^A f_1) \bar{D}_B \bar{D}_C \psi_3) - \frac{3}{2} (\psi_1 f_2 - \psi_2 f_1) \bar{D}_B \bar{D}_C \psi_3 \right.\left. - 2 (f_1 \bar{D}_B Y_2^A - f_2 \bar{D}_B Y_1^A) \bar{D}_C \bar{D}_A \psi_3 + \text{cyclic (1, 2, 3)} \right]. \quad (7.8.16)$$

Summing the two, we get

$$A + B = \int d^2\Omega^\varphi C^{BC} \left[ - 2 (f_1 \bar{D}_B Y_2^A + \bar{D}^A Y_2^B) - f_2 (\bar{D}_B Y_1^A + \bar{D}^A Y_1 B)) \bar{D}_C \bar{D}_A \psi_3 \right.\left. - 2 (\psi_1 f_2 - \psi_2 f_1) \bar{D}_B \bar{D}_C \psi_3 + 2 (\bar{D}_C f_1 f_2 - \bar{D}_C f_2 f_1) \left( \bar{\Delta} D^C \psi_3 - \frac{1}{2} \bar{D}^C \bar{\Delta} \psi_3 \right) \right.\left. + \text{cyclic (1, 2, 3)} \right]$$

$$= \int d^2\Omega^\varphi \left[ 2 (\bar{D}_C f_1 f_2 - \bar{D}_C f_2 f_1) \left( \bar{\Delta} D^C \psi_3 - \frac{1}{2} \bar{D}^C \bar{\Delta} \psi_3 \right) + \text{cyclic (1, 2, 3)} \right]. \quad (7.8.17)$$

We can then use the following identities $\bar{\Delta} \psi = - \bar{D}_A (\bar{R}^A)$ and $\Delta \bar{\Delta} \psi = \bar{\Delta} \bar{D} \psi$ to simplify the above to

$$A + B = \int d^2\Omega^\varphi \left[ 2 (\bar{D}_C f_1 f_2 - \bar{D}_C f_2 f_1) \left( - \frac{1}{2} \bar{D}^C (Y_3^A \bar{D}_A R) - \psi_3 \bar{2} \bar{D}^C \bar{R} \right) + \text{cyclic (1, 2, 3)} \right]$$

$$= \int d^2\Omega^\varphi \left[ 2 \bar{D}_C f_1 f_2 - \bar{D}_C f_2 f_1 \left( - \frac{1}{2} \mathcal{L}_{Y_3} \bar{D}^C \bar{R} - \psi_3 \bar{2} \bar{D}^C \bar{R} \right) + \text{cyclic (1, 2, 3)} \right]. \quad (7.8.18)$$

- for the first part $K_{s_1,s_2} = \frac{1}{16\pi G} \int d^2\Omega^\varphi \left[ f_1 \bar{D}_A f_2 - f_2 \bar{D}_A f_1 \right] \bar{\Delta} \bar{R}$, the condition (7.8.14) leads to

$$C = \int d^2\Omega^\varphi \left[ f_{s_1,s_2} \bar{D}_A f_3 - f_3 \bar{D}_A f_{s_1,s_2} \right] \bar{\Delta} \bar{R} + \text{cyclic (1, 2, 3)}$$

$$= \int d^2\Omega^\varphi \left[ \mathcal{L}_{Y_1} (f_2 \bar{D}_A f_3 - f_3 \bar{D}_A f_2) - \psi_1 (f_3 \bar{D}_A f_3 - f_3 \bar{D}_A f_2) \right] \bar{\Delta} \bar{R} + \text{cyclic (1, 2, 3)}$$

$$= \int d^2\Omega^\varphi \left[ (f_2 \bar{D}_A f_3 - f_3 \bar{D}_A f_2) (- \mathcal{L}_{Y_1} - 2 \psi_1) \right] \bar{\Delta} \bar{R} + \text{cyclic (1, 2, 3)} \right]. \quad (7.8.19)$$
The different contributions then sum up to zero, \( A + B + C = 0 \).

An open question is if and in what sense the proposed bracket is indeed a Dirac bracket. The point we want to make here is first of all that theorem 7.8.1 assure the skew-symmetricity of our new definition (7.8.1). Furthermore, (7.8.2) and (7.8.14) imply the Jacobi identity for this bracket, provided that the transformation associated with \( Q_{[s_1,s_2]} [\mathcal{X}] + K_{s_1,s_2} [\mathcal{X}] \) is just \( \delta_{[s_1,s_2]} \) or in other words that the field dependent central extension does not generate a transformation.

When defining as before, \( \{ Q'_{s_1}, Q'_{s_2} \} \ [\mathcal{X}] = (-\delta_{s_2})Q'_{s_1} [\mathcal{X}] + \Theta'_{s_2} [-\delta_{s_1} \mathcal{X}, \mathcal{X}] \), one gets \( \{ Q'_{s_1}, Q'_{s_2} \} = Q'_{[s_1,s_2]} + K'_{s_1,s_2} \), where

\[
K'_{s_1,s_2} = K_{s_1,s_2} + \delta_{s_2} N_{s_1} - \delta_{s_1} N_{s_2} + N_{[s_1,s_2]}. \tag{7.8.20}
\]

Note that \( \delta_{s_2} N_{s_1} - \delta_{s_1} N_{s_2} + N_{[s_1,s_2]} \) is a trivial field dependent 2-cocycle in the sense that it automatically satisfies the cocycle condition (7.8.14).

### 7.9 Conclusion and outlook

In this chapter, we have shown that the symmetry algebra of asymptotically flat 4 dimensional spacetimes is \( \text{bms}_4 \), an algebra that contains both the Poincaré algebra and the non centrally extended Virasoro algebra in a completely natural way. As a first non trivial effect, we have computed the detailed transformation properties of the data characterizing solution space.

Using a covariant method, we constructed the associated surface charges. They agree with the usual definitions found in the literature. In order to define an algebra, we introduced a new bracket. With this new definition, the algebra of the charges form a representation of \( \text{BMS}_4 \) up to a general field-dependent extension. More work is still needed to fully justify the proposal for this Dirac bracket.

We believe that our understanding of the symmetry structure and its action on solution space goes some way in getting quantitative control on “structure X” \([117]\), i.e., on a holographic description of gravity with zero cosmological constant.

In the future, it should be interesting to analyze in more details the consequences of our results on local conformal invariance for the non extremal Kerr/CFT correspondence and for the gravitational S-matrix for instance.
Appendix A

Surface charges

A.1 Hamiltonian approach

A.1.1 Regge-Teitelboim revisited

We present here an adaptation of the original Hamiltonian method of [54, 55]. Let
\[ \mathcal{L}_H = a_A \dot{z}^A - h - \gamma_\alpha u^\alpha, \]
with \( h \) a first class Hamiltonian density and \( \gamma_\alpha \) first class constraints and define \( \phi^i = (z^A, u^\alpha) \). Even though it is not so for our theory, let us first run through the arguments in the case where one has Darboux coordinates for the symplectic structure, i.e., when \( \sigma_{AB} = \frac{\partial a_B}{\partial z^A} - \frac{\partial a_A}{\partial z^B} \) is the constant symplectic matrix. The gauge transformation generated by the smeared constraints is given by
\[ \delta_{\epsilon} z^A = \sigma^{AB} \frac{\delta(\epsilon^\alpha \gamma_\alpha)}{\delta z^B} \]  
(A.1.1)
where \( \sigma^{AB} \) is the inverse of \( \sigma_{AB} \). This transformation is extended to the Legendre multipliers in order to leave the action invariant (see e.g. [50]).

We furthermore suppose that we are in a source-free region of spacetime. In this case one can show that
\[ \delta_{\epsilon} z^A \frac{\delta \mathcal{L}_H}{\delta z^A} + \delta_{\epsilon} u^\alpha \frac{\delta \mathcal{L}_H}{\delta u^\alpha} = -\partial_0 \left( \gamma_\alpha \epsilon^\alpha \right) - \partial_i s^i_{\epsilon}, \]  
(A.1.2)
where \( s^i_{\epsilon} = s^i_{\epsilon}[z, u] \) vanishes when the Hamiltonian equations of motion, including constraints, are satisfied, \( s^i_{\epsilon} \approx 0 \). This identity merely expresses the general fact that the Noether current \( s^i_{\epsilon} \) associated to a gauge symmetry can be taken to vanish when the equations of motions hold (see e.g. [50], chapter 3), \( s^0_{\epsilon} \approx 0 \), and that the integrand of the generator is given by (minus) the constraints contracted with the gauge parameters in the Hamiltonian formalism, \( s^0_{\epsilon} = -\gamma_\alpha \epsilon^\alpha \). An explicit expression for \( s^i_{\epsilon} \) in terms of the structure functions can for instance be found in Appendix D of [53]. Using integrations
by parts, one can write the variations of the constraints under a change of the canonical coordinates $z^A$ as an Euler-Lagrange derivative, up to a total derivative,

$$\delta_z (\gamma \alpha^\alpha) = \frac{\delta A^i}{\delta z^A} - \partial_i k^i. \quad (A.1.3)$$

where $k^i = k^i[\delta z, z]$ depends linearly on $\delta z^A$ and its spatial derivatives. Taking the time derivative of (A.1.3) and using a variation $\delta \phi$ of (A.1.2) to eliminate $\partial_0 \delta_z (\gamma \alpha^\alpha)$, one finds

$$\partial_i \left( \partial_0 k^i - \delta \phi s^i \right) = \partial_0 \left( \frac{\delta A^i}{\delta z^A} \right) + \delta \phi \left( \frac{\partial \delta A^i}{\partial \phi^\alpha} \right). \quad (A.1.4)$$

One now takes $\epsilon^\alpha$ to satisfy $\delta \epsilon^\alpha z^A_s = 0 = \delta \epsilon^\alpha u^\alpha_s$. Note that in the case of Darboux coordinates, this also implies that $\frac{\delta (\alpha \epsilon^\alpha)}{\delta z^A} = 0$. This is where we differ from the original analysis. The authors of [54] considered asymptotic symmetries: $\delta \epsilon^\alpha z^A_s = 0 = \delta \epsilon^\alpha u^\alpha_s$ only as $r$ goes to infinity. We are only considering exact reducibility parameters (in the case of gravity, they are the killing vectors). If furthermore $z^A_s, u^\alpha_s$ is a solution of the Hamiltonian equations of motion and the RHS of (A.1.4) vanishes. By using a contracting homotopy with respect to $\delta \phi^i$ and their spatial derivatives, one deduces that

$$\partial_0 k^i [\delta z_s, z_s] = (\delta \phi s^i)[z_s] - \partial_j k^i [\delta z_s, z_s] \quad (A.1.5)$$

where $k^i [\delta z_s, z_s]$ depends linearly on $\delta \phi^i$ and their spatial derivatives. Finally, when $\delta z^A_s, \delta u^\alpha_s$ satisfy the linearized Hamiltonian equations of motion, including constraints, we find from (A.1.3) and (A.1.5) that

$$\partial_i k^i [\delta z_s, z_s] = 0, \quad \partial_0 k^i [\delta z_s, z_s] - \partial_j t^{ij} = 0. \quad (A.1.6)$$

At a fixed time $t = x^0$, consider a closed 2 dimensional surface $S$, $\partial S = 0$, for instance a sphere with radius $r$ and define the surface charge 1-forms by

$$\delta Q_{z_s}[\delta z_s, z_s] = \oint_S d^3 x_i k^i[\delta z_s, z_s], \quad (A.1.7)$$

where $d^3 x_i = \frac{1}{2} \epsilon_{ijk} dx^j \wedge dx^k$. The first relation of (A.1.6) implies that the surface charge 1-form only depends on the homology class of the closed surface $S$,

$$\oint_{S_1} d^3 x_m k^m_{z_s}[\delta z_s, z_s] = \oint_{S_2} d^3 x_m k^m_{z_s}[\delta z_s, z_s]. \quad (A.1.8)$$

Here $S_1 - S_2 = \partial \Sigma$, where $\Sigma$ is a three-dimensional volume at fixed time $t$ containing no sources. For instance, the surface charge 1-form does not depend on $r$. The second relation of (A.1.6) implies that it is conserved in time and so does not depend on $t$ either,

$$\frac{d}{dt} \delta Q_{z_s}[\delta z_s, z_s] = 0. \quad (A.1.9)$$
The objects obtained by this method are 1-forms on solution space. To build the wanted charges, we still need to integrate them:

\[ Q_{\varepsilon}[z_s] = \int_{Z_0}^{Z_s} \delta Q_{\varepsilon}[\delta_{\gamma} z_s, z_s] \]  

(A.1.10)

where the integration is done along a path \( \gamma \) in solution space going from a reference solution \( Z_0 \) to the solution upon consideration \( Z_s \). The variation \( \delta_{\gamma} z_s \) is the variation of the fields along the path. There is an issue with integrability: the integral may be path dependent; the 1-form \( \delta / Q \) may not be closed, see e.g. [53, 118, 52] for a discussion. This explains the notation \( \delta / \).

### A.1.2 Linear theories

In the case of linear theories, the latter problem does not arise and the whole analysis simplifies. One can replace (A.1.3) by

\[ \gamma_{\alpha}^{\varepsilon} = z^A \delta(\gamma_{\alpha}^{\varepsilon}) / \delta z^A - \partial_ik^i_{\varepsilon}[z], \]  

(A.1.11)

where \( \delta / \delta z^A \) are the (spatial) Euler-Lagrange derivatives and \( k^i_{\varepsilon}[z] \) depends linearly both on the phase space variables \( z^A \) and their spatial derivatives and on the gauge parameters. One then uses (A.1.2) directly to eliminate \( \partial_0(\gamma_{\alpha}^{\varepsilon}) \) from the time derivative of (A.1.11), to get

\[ \partial_i \left[ \partial_0 k^i_{\varepsilon} - s^i_{\varepsilon} \right] = \partial_0 \left[ z^A \delta(\gamma_{\alpha}^{\varepsilon}) / \delta z^A \right] + \delta z^A \delta L_H / \delta z^A + \delta z^A \delta u^\alpha / \delta u^\alpha. \]  

(A.1.12)

For gauge parameters \( \varepsilon_{\alpha} \) that satisfy

\[ \delta_{\varepsilon_{\alpha}} z^A = 0 = \delta_{\varepsilon_{\alpha}} u^\alpha, \]  

(A.1.13)

one then arrives at

\[ \partial_i k^i_{\varepsilon_{\alpha}}[z] = -\gamma_{\alpha}^{\varepsilon}, \quad \partial_0 k^i_{\varepsilon_{\alpha}}[z] = s^i_{\varepsilon_{\alpha}}[z, u] - \partial_j k^j_{\varepsilon_{\alpha}}[z]. \]  

(A.1.14)

For a solution \( z^A_{\varepsilon}, u^\alpha_{\varepsilon} \), the surface charges

\[ Q_{\varepsilon_{\alpha}}[z_s] = \int_S d^3x_i k^i_{\varepsilon_{\alpha}}[z_s], \]  

(A.1.15)

are again independent of \( r \) and \( t \).

When this analysis is applied to the Hamiltonian formulation of Pauli-Fierz theory, one finds the standard expressions

\[ k^i_{\varepsilon}[z] = 2\xi_m \pi^{mi} - \xi^\perp (\delta^{mn} \partial^i - \delta^{mi} \partial^n)h_{mn} + h_{mn}(\delta^{mn} \partial^i - \delta^{mi} \partial^n)\xi^\perp, \]  

(A.1.16)

while the only solutions to (A.1.13) are \( \xi_{\mu s} = -\omega_{\mu |\nu|} x^\nu + a_\mu \), for some constants \( a_\mu \), \( \omega_{\mu |\nu|} = -\omega_{|\nu|\mu} \). In this context of flat space, Greek indices take values from 0 to 3 with \( \mu = (\perp, i) \). Indices \( \mu \) are lowered and raised with \( \eta_{\mu\nu} = \text{diag} (-1, 1, 1, 1) \).
A.2 Covariant approach

The rest of this appendix is devoted to a quick review of the covariant approach developed in [51, 52, 53]. This approach can be applied to any gauge theory but we will focus here on its application to gravity, with or without cosmological constant.

The starting point is the analysis of the linearized theory described by \( h_{\mu\nu} \) around a background \( g_{\mu\nu} \). It has been shown in [119] that the conserved surface charges are completely classified by the Killing vectors \( \xi^\mu \) of the background metric \( g_{\mu\nu} \). Moreover, they form a representation of the Lie algebra of those Killing vectors. The explicit expression for the surface charges of the linearized theory depends only on the Einstein equations of motion and is given by

\[
\delta Q_{\xi}[h, g] = \frac{1}{16\pi G} \int_{\partial \Sigma} (d^{n-2}x)_{\mu\nu} \sqrt{-g} \left[ \xi^\nu D^\mu h - \xi^\nu D_\sigma h^{\mu\sigma} + \xi_\sigma D^\nu h^{\mu\sigma} + \frac{1}{2} hD^\nu \xi^\mu + \frac{1}{2} h^{\nu\sigma} (D^\mu \xi_\sigma - D^\sigma \xi^\mu) \right],
\]

where

\[
(d^{n-k}x)_{\nu\mu} = \frac{1}{k!(n-k)!} \epsilon_{\nu\mu\alpha_1...\alpha_{n-k}} dx^{\alpha_1} \wedge ... \wedge dx^{\alpha_{n-k}}, \quad \epsilon_{01...n-1} = 1.
\]

The algebra of these charges is given by

\[
\{ \delta Q_{\xi_1}[h, g], \delta Q_{\xi_2}[h, g] \} = -\delta Q_{\xi_2}[\mathcal{L}_{\xi_1} h, g] \approx \delta Q_{[\xi_1, \xi_2]}[h, g].
\]

The last equality is an equality on the equations of motion.

In view of these universal properties of the surface charges in the linearized theory, the authors of [51] used them to define surface charges in the full theory associated to asymptotic symmetries. For an asymptotic Killing vector \( \xi \), they define a charge \( Q_\xi \) as

\[
Q_\xi[g] = \int_\gamma \delta Q_\xi[\delta_\gamma g, g]
\]

where \( \delta Q_\xi \) is evaluated on an asymptotic surface \( \delta \Sigma \). For instance, in the case of \( AdS_3 \), \( \delta \Sigma \) will be a circle cross-section of the cylinder at \( r \to \infty \). The integration of (A.2.3) is done along a path \( \gamma \) in solution space going from a reference metric \( \bar{g} \) to the metric upon consideration \( g \). The variation \( \delta_\gamma g \) is the infinitesimal variation of the metric along the path. As in (A.1.10), there may be a problem of integrability.

In the simplest cases, the charges are “linear”:

\[
\delta Q_\xi[\delta_\gamma g, g] = \delta Q_\xi[\delta_\gamma g, \bar{g}]
\]

and the integral can be done easily to give

\[
Q_\xi[g] = \int_\gamma \delta Q_\xi[\gamma g, \bar{g}]
\]
This happens in chapters 5 and 6 with space-times in three dimensions that are asymptotically $AdS_3$ or asymptotically flat at null infinity.

If the charges are integrable, the authors of [53] showed that under some technical conditions the charges form a representation of the algebra of asymptotic Killing vectors up to a central extension:

$$\{ Q_{\xi_1}[g], Q_{\xi_2}[g] \} = -\delta_{\xi_1} Q_{\xi_2}[g]$$

$$= -\delta Q_{\xi_2}[\mathcal{L}_{\xi_1} g, g]$$

$$\approx Q_{[\xi_1, \xi_2]}[g] + K_{\xi_1, \xi_2}$$

where the central extension is given by:

$$K_{\xi_1, \xi_2} = \frac{1}{16\pi G} \int_{\partial \Sigma} (d^{n-2}x)_{\nu\mu} \sqrt{-g} \left[ -2 \bar{D}_\mu \xi_2^\rho \bar{D}\nu \xi_1^\mu + 2 \bar{D}_\mu \xi_1^\rho \bar{D}\nu \xi_2^\mu \\
+ 4 \bar{D}_\rho \xi_2^\nu \bar{D}\nu \xi_1^\mu + (\bar{D}_\rho \xi_1^\nu + \bar{D}\nu \xi_1^\rho)(\bar{D}_\sigma \xi_2^\mu + \bar{D}\sigma \xi_2^\rho) \\
- 2 \bar{R}^\mu_{\nu\rho\sigma} \xi_2^\rho \xi_1^\nu \xi_1^\sigma \right].$$

In addition, the covariant expression for the surface charges described above coincides on-shell with those of the Hamiltonian formalism [51][53]. In this context, it follows from the analysis of [103][12][54] that the surface charge is, after gauge fixation, the canonical generator of the associated asymptotic transformations in the Dirac bracket.
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