Quantum Key Recycling with optimal key recycling rate based on a noise level

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Abstract. Quantum Key Recycling (QKR) protocol can send a secret classical message via a quantum channel and recycle the per-shared keys. An important issue of QKR protocols is how to increase the tolerance of noise in the quantum channel. The studies [1-4] have answered this question, but we find out that there is a security loophole in their solutions. To avoid this loophole, this study proposes a new QKR protocol which recycles the used keys according to the noise level and optimizes the key recycling rate of the pre-shared keys. Further, the proposed protocol also eliminates the need for classical authentication channels by delaying recycling some of the keys. The security proofs and analyses are given to prove the security of the recycled keys is universal composable.

Keywords: Quantum Cryptography, Quantum Key Recycling, Key Recycling Rate, Authenticated Quantum Protocol, Universal Composable security, Quantum Key Distribution, Key Rate.

1 Introduction

1.1 Background

Quantum cryptography uses the properties of quantum mechanics to achieve things that non-quantum cryptography cannot. For example, the most well-known QKD protocol BB84 [5] can share an unconditionally secure key between two participants using a quantum channel and a classical authentication channel. QKD uses the measurement uncertainty of the quantum state to detect the existence of the adversary Eve and prove that the shared key is secure. If Eve wants to get the information of the shared key, she will disturb the qubits and be exposed. The participants can use the classical authentication channel to communicate and check the existence of Eve. Before BB84 was proposed, another idea called Quantum Key Recycling (QKR) was proposed in 1982 [6]. The QKR protocols are used to send a message (and can also be used to share a new key). The concept of QKR is recycling the keys that are used to
encrypt messages. Though QKD and QKR do not work in the same way, their main concept is the same. They both use the uncertainty of measurement in the quantum state to achieve their goals. The QKR protocols encode the cipher into the quantum state. If Eve wants to get the information of the cipher, she will disturb the qubits and has a chance to be exposed. In QKR protocols, the participants check the existence of Eve using Message Authentication Code (MAC) instead of a public discussion via a classical authentication channel. If Eve does not exist, then we can know for sure the cipher is secure and not taken or copied. Afterward, we can recycle all the keys that were used to encrypt the message, including those that were used for One-Time Pad (OTP).

Ever since the first QKR protocol was proposed in 1982, there had been no research on this topic until 2005. In 2005, a QKR protocol with a strong security proof [7] was proposed, but its proof depended on a technique called \( W_nC_m \)-cipher and needed a quantum computer to implement. In 2017, a new version of QKR that has a security proof and uses the single BB84 qubits [4] was proposed. This research also raised the question about how to make QKR protocols noise-tolerable (error-tolerable). This question is very important because the assumption that there is a noiseless channel is not realistic. The regular QKR protocol can recycle some of the keys even when the protocol is under attack and fails, but it will not be able to transmit any messages. The study [4] tried to answer this question but it could only tolerate a few errors. This research was followed by many other studies [1-3] that focused on the question about how to tolerate more errors and keep the security.

1.2 Proposed Security Loophole

The most straightforward way to increase error tolerance is using an Error Correction Code (ECC) such as a hamming code [8]. The concept of ECC is using redundant information to detect and correct errors. This redundant information also needs to be protected. If Eve gets this redundant information, she can get part of the message. The previous studies [1-4] focused on how to protect the redundant information added by ECC. To reach the goal, [4] used a special technique introduced by Dodis and Smith [9], and the other studies [1-3] used OTP.

However, none of the exiting QKR protocols [1-4] consider an important issue which Eve’s attacking traces may be erased by ECC; this is, a security loophole occurs in this situation\(^1\). More specifically speaking, if Eve wants to get information that is in the quantum state, she will disturb the quantum state and cause errors. These studies [1-4] only focus on the noise tolerance (i.e., correcting noise errors), but they ignored that these errors may be created by Eve’s attacks. Therefore, this allows Eve to steal information without being detected.

\(^1\) A QKR-like protocol called uncloneable encryption [10] doesn’t have this security loophole. It always abandons some keys to erase the information leaked to Eve. This makes the uncloneable encryption protocol unable to recycle all the keys even when there is no error.
We take [1] as an example to explain this security loophole. Here, assuming the protocol mentioned in [1] can tolerate \( x\% \) of the errors of the message and Eve can use the known-plaintext attack. When the sender, Alice, sends the quantum ciphertext to the receiver, Bob, Eve intercepts all qubits, attacks \( x\% \) qubits (here, she can adopt any attack strategy to obtain the classical information about ciphertext), and then sends all qubits (including attacked and non-attacked qubits) to Bob by using the noiseless quantum channel. Because Eve attacks \( x\% \) qubits, some errors will be caused. However, because these errors will be less than or equal to \( x\% \), Bob will correct the errors (i.e., Eve's attack traces have been eliminated), and thus Bob cannot detect Eve's attack. This situation will make that all the keys used to encrypt the message in this round will still be recycled and used in the next round. Meanwhile, Eve can obtain some information about these recycled keys by using the part of classical cipher and the known-plaintext attack. After a finite number of rounds, Eve will be able to get enough information about the recycled keys and steal all secure messages that Alice sends to Bob. According to the above-mentioned attack strategy, the QKR protocols adopting the same method to tolerate the noise of the quantum channel also have this security loophole.

1.3 Contributions

In this paper, we propose a new QKR protocol for the noisy quantum channels without the above-mentioned security loophole. The core idea of our research is taking the number of errors that are corrected by ECC into account when recycling keys. Our QKR protocol only uses single qubits with conjugate coding and doesn't need any classical communication. We eliminate the classical channel by delaying recycling some of the keys (these keys are recycled in the next round of the protocol). We also give a security analysis of the recycled keys and optimize the key recycling rate (the length of the recycled keys / the length of all the keys used to run the protocol in a round). Our QKR protocol can recycle all the keys when no errors exist and can recycle keys safely and efficiently if the error rate is under a specific level.

We analyze the security of the recycled keys based on information theory. The security of the recycled keys in our QKR protocol is universal composable [11]. The distance between the recycled keys and the ideal keys is limited by a specific security parameter \( \varepsilon \). This means that an adversary only has an \( \varepsilon \) chance to distinguish the difference between our QKR protocol and an ideal one.

2 Preliminaries

Most of the notions will be explained when they first appear in this paper. But we describe some notions here for convenience: the notion \( |X| \) denote the number of possible values of a variable \( X \) and we define \( |X|_l = \log_2 |X| \).
2.1 Information entropy

The concept of information entropy is introduced by Shannon in 1948 [12]. It gives a way to measure the randomness of a message. We denote the information entropy of a random variable $X$ in bits by

$$H(X) = -\sum_x p(x) \log_2 p(x),$$

where $p(x)$ is the probability of possible values $x$. $H(X)$ achieves the maximum when all possible values of $X$ have the uniform probability and will be 0 when the value of $X$ is definite. This entropy is also called the "Shannon entropy". The meaning of the Shannon entropy of a random variable $X$ is the uncertainty of the variable $X$.

We denote the conditional entropy of a random variable $X$ when given another variable $Y$ by

$$H(X|Y) = -\sum_{x,y} p(x,y) \log_2 \frac{p(x,y)}{p(y)}.$$

$H(X|Y) = 0$ if and only if the variable $X$ is completely determined by $Y$. Conversely, $H(X|Y) = H(X)$ if and only if $X$ is completely independent of $Y$.

Using the definition above, we can denote the mutual information of two variables $X$ and $Y$ by

$$I(X;Y) = H(X) - H(X|Y) = H(Y) - H(Y|X).$$

The meaning of mutual information is the uncertainty of a variable when the other variable is known. It is the opposite of conditional entropy. When two variables are completely independent, the mutual information will be 0. And the mutual information will achieve maximum when one variable can completely determine the other one.

2.2 The density operator and the von Neumann entropy

We denote the density operator of a quantum state $Q = |\psi\rangle$ by $\rho_Q = |\psi\rangle\langle\psi|$. We can use the density operator of the quantum state $Q$ to calculate the von Neumann entropy of it. The von Neumann entropy is very similar to the Shannon entropy. They both used to measure uncertainty. But the von Neumann entropy is used to measure the uncertainty of a quantum state instead of a variable. We denote the von Neumann entropy of a quantum state $Q$ by

$$S(Q) = -\text{tr}(\rho_Q \log_2 \rho_Q),$$

where $\text{tr}$ denotes trace. And if we write $\rho_Q$ in terms of its eigenvectors $|1\rangle, \ldots, |j\rangle$ as $\rho_Q = \sum_j \eta_j |j\rangle \langle j|$, then the von Neumann entropy is [13]

$$S(Q) = -\sum_j \eta_j \log_2 \eta_j.$$

It is worth mentioning that the von Neumann entropy is not used to measure the
measurement uncertainty of a quantum state but is used to measure the uncertainty of
a mixed state. The von Neumann entropy is not affected by the measurement opera-
tors (or called measurement basis) used to measure the state. For example: If we use
the z-basis to measure the quantum state $|+\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)$, the measurement
result is random. But the von Neumann entropy of $|+\rangle$ is 0 because it is a pure state.

Just like the Shannon entropy, we can denote the conditional entropy and mutual
information of the von Neumann entropy in a similar way. The meanings of them are
also similar except the von Neumann entropy is used to measure the uncertainty of a
quantum state.

2.3 Universal composable security

The universal composable security of a component (a system, a protocol or a key) is
defined as a real component close to an ideal (secure) one. We can say, for example, a
key $S$ is $\epsilon$-secure with respect to an ideal key $K$, which is independent and uniformly
distributed, if the distance between $S$ and $K$ is less than or equal to $\epsilon$, where $\epsilon$ is a
chosen secure parameter that $\epsilon \geq 0$.

We denote the trace distance between density operators of two quantum states $\rho$
and $\sigma$ by

$$
\delta(\rho,\sigma) = \frac{1}{2} \text{tr}(|\rho - \sigma|) = \frac{1}{2} \text{tr}\left[\sqrt{(\rho - \sigma)^\dagger(\rho - \sigma)}\right].
$$

We can say $\rho$ is $\epsilon$-close to $\sigma$ if $\delta(\rho,\sigma) \leq \epsilon$. The distance between classical variables
can be seen as a special case of the trace distance. Let $X$ and $Y$ be two random vari-
ables. The variational distance between the probability distribution of $X$ and $Y$ is de-
noted by

$$
\delta(P_X, P_Y) = \frac{1}{2} \sum_i |P_X(i) - P_Y(i)|,
$$

where $i$ is the possible value of $X$ and $Y$. The variational distance is equal to the trace
distance $\delta([|X|], [|Y|])$, where $[|X|]$ and $[|Y|]$ are the state representations of $X$ and $Y$, i.e.,

$$
\delta(P_X, P_Y) = \delta([|X|], [|Y|]).
$$

The universal composable security provides many good properties. First, a key that
has universal composable security can maintain its security in any context. If a secret
key $S$ has universal composable security, the remains bits in $S$ are secure even some
bits of $S$ are given to an adversary. Second, the security of a complex system can be
calculated more easily by simply summing the secure parameters $\epsilon$ of its components.
We can do this because the trace distance is subadditive with respect to the tensor
product. Third, there are no quantum operators that can increase the trace distance of
two quantum states by applying the same operator to them.

2.4 Privacy amplification

Let’s say that there is a string held by two participants called Alice and Bob, and they
want to use it as their secret key. But some information about this string is held by an adversary called Eve, and Alice and Bob only know how many bits have been leaked, they don’t know which ones are leaked. Then, is it possible to use part of this string as a secret key? The answer is affirmative. We can use the privacy amplification to extract a shorter secret key from an insecure string by erasing the information held by Eve. The extracted secret key can have universal composable security and be $\varepsilon$-secure with respect to an ideal key, where $\varepsilon$ is the secure parameter chosen by Alice and Bob.

Let $S$ denote the insecure string and $\mathcal{F}$ denote a family of hash functions. The function $\mathcal{F}$ from domain $X$ to range $Y$ is said to be two-universal \cite{14,15} if $\Pr_{f \leftarrow \mathcal{F}}[f(x) = f(x')] \leq \frac{1}{|Y|}$, for any $x, x' \in X$. The research \cite{11} of the properties of the two-universal hash function shows we can use it to do privacy amplification. The length of the secret key that can be extracted from the insecure string is limited by the information about the string held by Eve. We can bound the key extracted rate (the length of secret key extracted from an insecure string divide by the length of the string) by \cite{16,17}

$$\text{key extracted rate} \geq S(A|E) - H(A|B)$$

$$= I(A;B) - I(A;E),$$

where $E$ is the quantum information of the string held by Eve and $A$ and $B$ are the string held by Alice and Bob.

In this paper, we use privacy amplification to extract a secret key from a used key. We then call the key extracted rate key recycling rate.

### 3 The protocol

Before Alice sends a message $\text{Meg} \in \{0,1\}^n$ to Bob. Alice and Bob need to share a secret key pool $K$ which they can pick up secret keys from it synchronously. They also share an authentication function $Au()$, an error correction function $\text{ECC}_d()$ and a key updating function $\text{Upd}()$ defined below.

**Definition 1** \cite{18}. $Au()$ is an $\varepsilon$-almost XOR universal$_2$ ($\varepsilon$-AXU$_2$) hash function. The $\varepsilon$-AXU$_2$ hash function is a family of hash functions $Au(u, \text{Meg}) = MAC$ for $MAC \in \{0,1\}^t$ that if $u$ is a random key and for all $m_1, m_2 \in \text{Meg}$ that $m_1 \neq m_2$ and all $t \in MAC$,

$$\Pr_u[Au(u, m_1) \oplus Au(u, m_2) = t] \leq \varepsilon,$$

where $\varepsilon$ is a secure parameter.

**Definition 2.** The error correction function $\text{ECC}_d(\text{Meg}) = C$ for $C \in \{0,1\}^{n+s}$ can encode a message $\text{Meg}$ into a binary linear code $C$ that has minimum Hamming distance $d$. $\text{ECC}_d()$ has a corresponding decoding function that can decode $C$ back to $\text{Meg}$.
When decoding $C$, we can know the errors’ number and positions if the error number is not beyond $\frac{d-1}{2}$ and then we can correct $\lfloor\frac{d-1}{2}\rfloor$ errors [19]. If there are too many errors, the decoding of $C$ will fail and output a wrong message.

**Definition 3.** Rec($k$) = $k'$, where $k$ is the old key, $k'$ is the recycled key and $0 \leq |k'|_1 \leq |k|_1$, is a two-universal hash function. We can say $k'$ is $\varepsilon$-secure if for $k_{\text{ideal}} \in \{0,1\}^{|k|_1}$ is an ideal key,

$$\delta(k_{\text{ideal}}, k') \leq \varepsilon,$$

where $\delta(\cdot, \cdot)$ denotes the trace distance and $\varepsilon$ is a secure parameter.

Rec($k$) is used to do the privacy amplification to make sure the security of the recycled key. We can then define a key updating function $Upd(k)$ based on Rec($k$).

**Definition 4.** A key updating function $Upd(k) = k'||k^\text{new} \in \{0,1\}^{|k|_1}$, where $k^\text{new}$ is a fresh key picked up from $K$ used to extend the length of the output, is used to maintain the security of a key $k$ without shorting its length. The key recycling rate of $Upd(k)$ is defined as $r = \frac{|k|_1}{|k|_1}$.

Now we are ready to run our QKR protocol. The steps of our protocol are described below:

1. Alice’s classical part: Alice picks up a key: $u \in \{0,1\}^u$ from $K$. Then she computes the message authentication code $MAC = Au(u,\text{Meg}) \in \{0,1\}^t$. And encodes $\text{Meg}||MAC$ into $C = ECC_d(\text{Meg}||MAC) \in \{0,1\}^{u+t+s}$. Alice picks up another key: $k_v \in \{0,1\}^{n+t+s}$ from $K$ and computes the cipher $M = C \oplus k_v$.

2. Alice’s quantum part: Alice picks up a key: $k_b \in \{0,1\}^{n+t+s}$ from $K$. And encodes $M$ into qubits sequence $Q$ in the $z$-basis $|0\rangle, |1\rangle$ or the $x$-basis $|+\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle), |\rangle = 1/\sqrt{2}(|0\rangle - |1\rangle)$ according to $k_b$. Then she sends $Q$ to Bob. After this, Alice recycles $k_b$ and $u$.

3. Bob’s quantum part: When Bob receives $Q'$, he picks up keys $u, k_v, k_b$ from $K$. He measures $Q'$ according to $k_b$ and gets the cipher $M' = C' \oplus k_v$.

4. Bob’s classical part: Bob computes the plaintext $C' = M' \oplus k_v$. Then decodes $C'$ to correct potential errors and gets $\text{Meg}'||MAC'$, the errors’ number $q$ is recorded. Then, Bob computes $MAC = Au(u,\text{Meg}')$ and checks $MAC = MAC'$. Bob updates $k_v$ to $k_v' = Upd(k_v)$ according to $q$ and recycles $k_b$ and $u$.

5. In the next time Bob sends a message to Alice, he will tell Alice $q$ to let Alice update $k_v$ to $k_v'$.
4 Security Analysis

In this section, we will analyze the security of the message $M$ and the recycled keys $A$, $k'_v$, and $k_b$. These analyses assume the adversary Eve, who wants to get $M$ or keys, has full control of the quantum channel that Alice and Bob used. Eve also has unlimited computation power and any other sources expect she needs to follow the laws of physics. We will analyze the security under the collective attacks [20] but assume the implementation of the protocol is ideal so Eve cannot use the hacking attacks [21] to get information. When we analyze the security of the recycled keys, we assume that Eve can do the known-plaintext attack. Although Eve is unable to get the full message in real scenarios because she does not know about $A$ and thus does not know about $M_AE_E$ either. We still assume that Eve can get the whole plaintext.

4.1 The security of the message $M$

The security of $M$ is obvious. After Alice’s classical part, the code $C$, which contains the information of $M$, is protected by the XOR encryption using $k_v$. As long as $k_v$ is secure, the cipher $M = C \oplus k_v$ has unconditional security and perfect privacy [22].

4.2 The security of the recycled $u$

We first introduce a new hash function called $\varepsilon$-almost strong universal 2 ($\varepsilon$-ASU 2) hash function [23]. It is a family of hash functions $Au(u, Meg) = MAC$ for $MAC \in \{0,1\}^t$ that if $u$ is a random key and for all $m_1, m_2 \in Meg$ that $m_1 \neq m_2$ and all $t_1, t_2 \in MAC$,

$$\Pr_{u,k_v}[Au(u, m_1) = t_1 \text{ and } Au(u, m_2) = t_2] \leq \frac{\varepsilon}{|MAC|},$$

where $|MAC|$ is the number of $MAC$’s possible values and $\varepsilon$ is a secure parameter.

In our protocol, the $MAC$ in the cipher $M$ is protected by $k_v$. Then, $MAC \oplus k_v$ is $\varepsilon$-ASU 2 because when $u$ and $k_v$ are secret keys and for all $m_1, m_2 \in Meg$ that $m_1 \neq m_2$ and all $t_1, t_2 \in MAC$,

$$\Pr_{u,k_v}[Au(u, m_1) \oplus k_v = t_1 \text{ and } Au(u, m_2) \oplus k_v = t_2] \leq \frac{\varepsilon}{|MAC|}.$$ 

Next theorem, derived from the result of earlier research [24], follows:

**Theorem 1.** We can bound the mutual information of the recycled $u$ and Eve’s information $EI$ after $n$ round of the protocol, for $1 \leq n \leq |MAC|$ by

$$I(u; EI) \leq \log |MAC| - \left(1 - \frac{n}{|MAC|}\right) \log(|MAC| - n),$$

where $EI$ is the information including $Meg, MAC,$ and Bob’s respond (accept or reject) from these $n$ rounds.
Theorem 1 shows that even we use $k_v$ to protect MAC, the information of $u$ still leaks. There is an attack that Eve can disturb one of the first $n$ messages and pass the authentication check (e-ASU$_2$ hash function) with probability at least $\frac{n}{|MAC|}$ even when $k_v$ is completely secure. But the leakage is very small and limited by the number of rounds, and the recycled $u$ is still $e$-secure.

4.3 The security of the recycled $k_b$

Because our protocol always recycle $k_b$, Eve can just intercept every qubit sent from Alice and don’t worry to be detected. After her attack, Eve can also get Bob’s response $R$, which telling Eve how many errors Bob detected. The analysis splits into two parts. First, we analyze the information leakage from the intercepted qubits. Second, we analyze the leakage from Bob’s response.

When Eve intercepts all qubits sent from Alice, she cannot know the value of each qubit even she can do the known-plaintext attack because the values are protected by $k_v$. Eve needs to distinguish the qubits in the z-basis from the qubits in the x-basis. The next Lemma shows Eve can’t get any information about $k_b$ from the intercepted qubits.

**Lemma 1.** When randomly giving Eve one qubit in one of the four states: $|0\rangle, |1\rangle, |+\rangle, |-\rangle$ without giving Eve any other information, Eve cannot know the basis of the qubit.

**Proof:** We assume Eve can measure the qubit with any ancilla bits and use any measurement operators. And we denote the qubit with the ancilla bits is $|\varphi_i\rangle$, for $i \in \{0, 1, +, -\}$, and measurement operators are $\{M_m\}$. Base on the postulates of quantum measurement [25], the probability of measurement outcome $m$ of the state $|\varphi_i\rangle$ is $p(m) = \langle \varphi_i | M_m^* M_m | \varphi_i \rangle$, where $M_m^*$ is the conjugate transpose of $M_m$. By direct computation, for all $M_m$ and all corresponding $m$,

$$p(m)_0 + p(m)_1 = p(m)_+ + p(m)_- .$$

When Eve gets the measurement outcome $m$, she cannot get any information about the basis of the qubit. \[\square\]

Next, we analyze the information leakage from Bob’s response. After Bob receiving the qubits from Alice (or Eve), he uses ECC to detect and correct the potential errors and sends the result back to Alice in the next round of the protocol. The response $R$ is an integer which $0 \leq R \leq |K_b|$. We assume Eve can know the response $R$ and use it to get $k_b$. For example: Eve can measure a qubit in the $z$-basis, and then wait for the response by Bob. If Bob detects one error, Eve will know that her measurement caused this error and that the basis of the qubit is not in the $z$-basis.
According to the analysis above, the information leakage is only from Bob’s response $R$. By the properties of mutual information [26] that the mutual information $I(k_b; E')$ will not increase more than the exchanging information between the participants and Eve. We can bound the increasing of $I(k_b; E')$ by the Shannon entropy of $R$. Next theorem gives the desired result.

**Theorem 2.** We can bound $I(k_b; E')$, where $E'$ is Eve’s information after this round of the protocol, by

$$I(k_b; E') \leq I(k_b; E) + H(R),$$

where $I(k_b; E)$ is the mutual information of recycled $k_b$ and Eve’s information before this round of protocol.

We can cover the information leakage from $R$ by slightly updating the $K_b$ using $Upd()$. The key recycling rate of $Upd(K_b)$ is bound by equation (2) and is nearly 1 when the length of the message is long enough.

**4.4 The security of the updated $k'_v$**

Because the security of $k'_v$ is easy to achieve, i.e. $Rec(k_v)$ outputs nothing and $k'_v$ is just a new fresh key picked from $K$, we are more interested in the key recycling rate of $Upd(k_v)$, which denotes how many secret keys can be extracted from the old $k_v$.

To calculate the key recycling rate of $Upd(k_v)$, we use the properties of privacy amplification and bound the key recycling rate by equation (1). We describe the formula here again:

$$r \geq S(A|E) - H(A|B),$$

where $r$ is the key recycling rate, $A$ and $B$ are $K_v$ held by Alice and Bob respectively and $E$ is the quantum state held by Eve. The calculation of the key recycling rate is very similar to the calculation of the key rate of the QKD protocol. When our protocol is considered sending $K_v$ instead of the message to Bob, our QKR protocol’s quantum steps and the BB84 protocol’s quantum steps are similar, and there are a few differences in the classical post-processing:

- The basis of qubits is determined by $K_b$ in our protocol but not random. So Bob can measure every qubit in the correct basis and doesn’t need to communicate with Alice.
- Bob already knows $K_v$, so they don’t need to synchronize the key after the protocol.
- In our protocol, we use the ECC and MAC to get the quantum bit error rate (QBER) instead of using the classical authentication channel to do the public discussion. When the authentication check succeeds, Bob has a $1 - \epsilon$ chance to consider the message is from Alice and not be disturbed by definition 1. We can then
consider the errors that correct by the ECC from $K_v$. We can know there are how many errors when the number of errors is not higher than $\leq \frac{d-1}{2}$ by the ECC. When the error number is higher than $\geq \frac{d-1}{2}$, the decoding will fail and output a wrong message, and the authentication check will fail. We will be able to know the certain number of errors that are not higher than $\leq \frac{d-1}{2}$, and when the authentication check fails, we must set the QBER to 50% to cover the worst situation.

The next theorem shows the effect of these differences.

Theorem 3. We can get $H(A|B) = 0$ because Alice and Bob already know $k_v$. The key recycling rate of $Udp(k_v)$ is then only bounded by

$$r \geq S(A|E).$$

Now we can follow the way in the earlier study [16] to calculate the key recycling rate of $Udp(k_v)$. We denote the joint system held by Alice, Bob and Eve after Eve’s attack by

$$|\psi\rangle_{ABE} := \sum_{i=1}^{4} \sqrt{\lambda_i} |\Phi_i\rangle_{AB} \otimes |v_i\rangle_E,$$

where $|\Phi_1\rangle_{AB}, ..., |\Phi_4\rangle_{AB}$ denote four Bell states in Alice and Bob’s joint system and $|v_1\rangle_E, ..., |v_4\rangle_E$ are some mutually orthogonal states held by Eve. Dependent on Alice and Bob’s measurement results (with respect to the z-basis), we can denote Eve’s system by $|\theta^{a,b}\rangle$, where $a, b$ are Alice and Bob’s measurement results,

$$|\theta^{0,0}\rangle = \frac{1}{\sqrt{2}} (\sqrt{\lambda_1} |v_1\rangle_E + \sqrt{\lambda_2} |v_2\rangle_E)$$

$$|\theta^{0,1}\rangle = \frac{1}{\sqrt{2}} (\sqrt{\lambda_1} |v_1\rangle_E - \sqrt{\lambda_2} |v_2\rangle_E)$$

$$|\theta^{1,0}\rangle = \frac{1}{\sqrt{2}} (\sqrt{\lambda_3} |v_3\rangle_E + \sqrt{\lambda_4} |v_4\rangle_E)$$

$$|\theta^{1,1}\rangle = \frac{1}{\sqrt{2}} (\sqrt{\lambda_3} |v_3\rangle_E - \sqrt{\lambda_4} |v_4\rangle_E).$$

We can write the density operator of Eve’s state according to Alice’s measurement results with respect to the basis $\{|v_1\rangle_E, ..., |v_4\rangle_E\}$,

$$\sigma_E^\theta = \begin{bmatrix}
\lambda_1 & \pm\sqrt{\lambda_1\lambda_2} & 0 & 0 \\
\pm\sqrt{\lambda_1\lambda_2} & \lambda_2 & 0 & 0 \\
0 & 0 & \lambda_3 & \pm\sqrt{\lambda_3\lambda_4} \\
0 & 0 & \pm\sqrt{\lambda_3\lambda_4} & \lambda_4
\end{bmatrix},$$
where ± is a plus sign if \( a = 0 \) and a minus sign if \( a = 1 \). Now we are ready to calculate equation (1). Using the fact that \( S(AE) = H(A) + S(E|A) \), we can get

\[
S(A|E) = S(E|A) + H(A) - S(E)
\]

with

\[
S(E|A) = \frac{1}{2} S(\sigma_0^A) + \frac{1}{2} S(\sigma_1^A),
\]

\[
S(E) = S(\sigma_0^E + \sigma_1^E),
\]

and \( H(A) = 1 \). Our protocol uses two encodings: the \( z \)-basis and the \( x \)-basis. Bob can know the QBER for both encodings by the ECC. This gives two conditions on the state held by Alice and Bob. If we use the Bell basis to express these conditions, and if the QBER equal \( Q \) for both encodings, we can get

\[
\lambda_3 + \lambda_4 = Q \quad \text{and} \quad \lambda_2 + \lambda_4 = Q,
\]

where \( \lambda_1, \ldots, \lambda_4 \) are the diagonal entries of Alice and Bob’s states (with respect to the Bell basis). A straightforward calculation shows when \( \lambda_4 = Q^2 \), the key recycling rate takes its minimum. We show the result in fig 1.

![Fig. 1. The x-axis denotes the real QBER and the y-axis denotes the key recycling rate of \( U(pd(K_p)) \).](image)

It is obvious that our QKR protocol is more efficient than a OTP which cannot recycle any used keys. Our QKR protocol can recycle part of \( K_e \) until the QBER achieves 50% and almost all other keys. Even in the worst case that we need to abandon the entire \( K_e \), the amount of keys we spend to send a message is equal to the OTP. When the QBER is 50%, we can consider the whole quantum message is intercepted by Eve and replaced by a random quantum sequence, and this is the worst situation. If the
authentication check fails, we need to set the QBER to 50% to cover the worst case because the ECC cannot output the right message and the QBER.

5 Conclusion and open questions

In this paper, we found a secure loophole of the existing studies of QKR protocol [1-4] for tolerating the noise of quantum channel. We propose a new QKR protocol without this security loophole, and we also optimize the key recycling rate depending on the QBER. By delaying recycling some of the keys, the participants do not need any classical authentication channel in the proposed QKR protocol. Finally, the universal composable of the recycled key is proved by using the information theory.

In this study, our analysis is under the condition that there is an optimal ECC and the length of transmission messages is infinite. However, the analysis with finite resources can help us understand the potential of the QKR protocols in the practical situation. Therefore, how to analyze the security of our QKR protocol with finite resources [27, 28] will be our future work. Additionally, the core idea that uses the error rate to optimize the key recycling rate in our research may also be used to improve the key recycling rate of quantum authentication protocol [29] which can only recycle the used keys when the quantum channel is noiseless. However, the security proof stated in this paper cannot be directly used in the quantum authentication protocol. Proposed the complete and suitable security analyses for key recycling rate optimization in quantum authentication protocol is also our future work.

Appendix A: compare our QKR protocol to BB84

In this section, we give an interesting result of our QKR protocol. It shows that the QKR protocol is not only a study about the potential of recycling keys, but also a useful protocol.

A QKR protocol can also be used to share keys (just like a QKD protocol) because a random number that can be used as a key is also a kind of message. Since our QKR protocol can run in a noisy quantum channel, we can calculate the “key rate” of our QKR protocol and thus can compare our QKR protocol to the regular QKD protocols like BB84. We found our QKR protocol can share keys more efficiently (with a higher key rate) than the BB84 protocol in some situations.

According to the Fig.1, the key recycling rate of \( K_v \) is much better than the key rate of the BB84 protocol which the key rate will be 0 when the QBER is merely 11%. But it doesn’t directly imply our QKR protocol can share keys more efficiently than QKD protocols. When the error-tolerable of our protocol increases, the length of the message and thus the keys used to encrypt it also increase and we will consume more key at the same key recycling rate. In addition, we need to abandon lots of keys if the QBER is higher than the error-tolerable.

Our QKR protocol can be seen as a QKD protocol if the message \( M \) is a random number which can be used to be a new key. We call the keys \( u, K_p, K_v \) used to run our QKR protocol the old keys and define the “key rate” \( kr \) of our QKR protocol as
where we omit the length of MAC. The meaning of the key rate here is how many percents of the message (random number) can be used as a new key. Unlike the QKD protocol like BB84 which needs to do privacy amplification to make sure the security of the new key. In our QKR protocol, if the authentication check succeeds, the entire message is secure and can be used as a new key, but we might need to consume some old keys. According to the secure analysis in the last section, most of the consumed keys are $K_α$ and we can omit the consumption of $u$ and $K_b$ if the message is long enough.

To calculate the key rate of our QKR protocol, we need to know the length of binary linear code $C$ which encoding from the message $M$ by the error correction function. The longer the length of $C$ is, the more error-tolerable the code has. But we need to use more $K_α$ to encrypt $C$. For the sake of convenience, the additional bits added by the ECC will be referred to as “the syndrome” in the following passages. Since the syndrome can fix the errors of the received message $M'$ held by Bob, the amount of useful information contained in the syndrome is equal to the uncertainty of Alice’s message $M$ conditioned on Bob’s message $M'$, which is according to the QBER $Q$ and can be bound by the Shannon entropy $H(Q)$. The length of the syndrome is at least equivalent to $H(Q)$ because one bit message can carry at most one bit information. If an optimal ECC is applied, the length of $C$ which Alice sends to Bob via a quantum channel in our QKR protocol can be denoted by

$$|C|_l = \left(1 + \frac{H(Q)}{1-H(Q)}\right) |M|_l$$

because the syndrome is also a part of the message and needs to correct itself.

Now we are ready to calculate the key rate of our QKR protocol. Unlike the QKD protocol that estimates the QBER after the quantum part of the protocol by the public discussion, our QKR protocol needs to predict the QBER before the quantum part of the protocol and decides how many error-tolerable we need. For this reason, the key rate of our QKR protocol is not only according to the real QBER but also our prediction. If the prediction QBER is less than the real one, our QKR protocol will fail and lose lots of keys. On the other hand, if the prediction QBER is more than the real one, we have to waste some keys to protect the useless part of the syndrome and thus decrease the key rate. When the prediction is perfect, we can get the best key rate at that QBER. The result of the calculation of the key rate of our QKR protocol shows in Fig 2.
Fig. 2. The x-axis denotes the real QBER and the y-axis denotes the key rate. The line error-tolerable 0.07 shows the key rate of our QKR protocol with the prediction QBER 0.07 according to the real QBER. When the real QBER is 0.07, this is the best key rate out of all prediction QBERs for our QKR protocol. The line error-tolerable optimal shows the best key rate we can achieve according to the real QBER. And the line BB84 shows the key rate of the BB84 protocol according to the QBER.

In our QKR protocol, the key $K_r$ used to protect the syndrome can partially recycle and thus is more efficient than the BB84 protocol which leaks all information contained in the syndrome. But the length of the syndrome used in our QKR protocol is longer than the syndrome used in the BB84 protocol because the BB84 protocol uses a classical authentication channel to send the syndrome and thus the syndrome doesn’t need to correct itself. The high key rate in our QKR protocol is achieved at the cost of low error-tolerable and high cost of the keys derived from situations where the real QBER is higher than the prediction. It worth mentioning that when the prediction QBER is 0.11, the key rate of our QKR protocol is just equal to the key rate of BB84 and thus has no benefit. This shows that our QKR protocol is a viable option. When the noise of the quantum channel is fixed and lower than 0.11, using our QKR protocol to share keys is more efficient.

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