EXISTENCE AND PHASE SEPARATION OF ENTIRE SOLUTIONS TO A PURE CRITICAL COMPETITIVE ELLIPTIC SYSTEM

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Abstract. We establish the existence of a positive fully nontrivial solution \((u, v)\) to the weakly coupled elliptic system

\[
\begin{align*}
-\Delta u &= \mu_1 |u|^{2^* - 2} u + \lambda |u|^\alpha |v|^\beta u, \\
-\Delta v &= \mu_2 |v|^{2^* - 2} v + \lambda |u|^\alpha |v|^\beta v,
\end{align*}
\]

\(u, v \in D^{1,2}(\mathbb{R}^N)\),

where \(N \geq 4, 2^* := \frac{2N}{N-2}\) is the critical Sobolev exponent, \(\alpha, \beta \in (1, 2], \alpha + \beta = 2^*, \mu_1, \mu_2 > 0\), and \(\lambda < 0\). We show that these solutions exhibit phase separation as \(\lambda \to -\infty\), and we give a precise description of their limit domains.

If \(\mu_1 = \mu_2\) and \(\alpha = \beta\), we prove that the system has infinitely many fully nontrivial solutions, which are not conformally equivalent.

Key words: Competitive elliptic system; critical nonlinearity; entire solution; phase separation.

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1. Introduction

We study the weakly coupled elliptic system

\[
\begin{align*}
-\Delta u &= \mu_1 |u|^{2^* - 2} u + \lambda |u|^\alpha |v|^\beta u, \\
-\Delta v &= \mu_2 |v|^{2^* - 2} v + \lambda |u|^\alpha |v|^\beta v,
\end{align*}
\]

(1.1)

where \(N \geq 4, 2^* := \frac{2N}{N-2}\) is the critical Sobolev exponent, \(\alpha, \beta \in (1, 2], \alpha + \beta = 2^*, \mu_1, \mu_2 > 0\), and \(\lambda \in \mathbb{R}\).

The solutions to this system are solitary waves for a system of coupled Gross-Pitaevskii equations. This type of systems arises, e.g., in the Hartree-Fock theory for double condensates, that is, Bose-Einstein condensates of two different hyperfine states which overlap in space; see [9]. The sign of \(\mu_i\) reflects the interaction of the particles within each single state. If \(\mu_i\) is positive, this interaction is attractive. The sign of \(\lambda\), on the other hand, reflects the interaction of particles in different states. This interaction is attractive if \(\lambda > 0\) and it is repulsive if \(\lambda < 0\). If the condensates...
repel, they separate spatially. This phenomenon is called phase separation and has been described in [28].

Motivated by their physical applications, weakly coupled elliptic systems have received much attention in recent years, and there are many results for the cubic case - where \( \alpha = \beta = 2 \) and \( 2^* \) is replaced by \( 4 \) - in low dimensions \( N \leq 3 \); see, e.g., [1, 2, 3, 7, 15, 16, 18, 19, 23, 24, 25, 29]. In this case, the nonlinear terms are subcritical.

In contrast, there are only few results for the critical case. For a Brezis-Nirenberg type system in a bounded domain of dimension \( N \geq 4 \) existence results were recently obtained by Chen and Zhou in [5, 6]. They also exhibited phase separation for \( N \geq 6 \). An unbounded sequence of sign-changing solutions for \( N \geq 7 \) and \( \alpha = \beta \) was obtained in [17], and spiked solutions were constructed in [22] for \( N = 4 \). Some existence and multiplicity results for a Coron type system in a bounded domain with one or multiple small holes were recently obtained in [20, 21].

We are interested in solutions to the system (1.1) in the whole space \( \mathbb{R}^N \). When \( \lambda = 0 \) this system reduces to the single equation

\[
- \Delta w = |w|^{2^*-2}w, \quad w \in D^{1,2}(\mathbb{R}^N).
\]

It is well known that the problem (1.2) has a positive solution and infinitely many sign-changing solutions. Note that, if \( w \) solves (1.2), then \( u = \mu_1^{\frac{2-N}{2}} w, v = 0 \), and \( u = 0, v = \mu_2^{\frac{2-N}{2}} w \), solve (1.1). So the system has infinitely many solutions with one trivial component. We are interested in solutions where both components, \( u \) and \( v \), are nontrivial. They are called fully nontrivial solutions. A solution is said to be positive if \( u \geq 0 \) and \( v \geq 0 \), and it is said to be synchronized if it is of the form \( (su, tu) \) with \( s, t \in \mathbb{R} \).

In the cooperative case, i.e., when \( \lambda > 0 \), Chen and Zou established the existence of a positive least energy fully nontrivial solution to the system (1.1) with \( \alpha = \beta = \frac{2-N}{2} \) for all \( \lambda > 0 \) if \( N \geq 5 \) and for a wide range of \( \lambda > 0 \) if \( N = 4 \); see [5, 6]. Peng, Peng and Wang [20] studied the system for \( \mu_1 = \mu_2 = 1, \lambda = \frac{1}{2} \) and different values of \( \alpha \) and \( \beta \), and they obtained uniqueness and nondegeneracy results for positive synchronized solutions. Guo, Li and Wei studied the critical system (1.1) in dimension \( N = 3 \) for \( \lambda < 0 \) and they established the existence of positive solutions with \( k \) peaks for \( k \) sufficiently large in [12]. In [10, 11] Gladiali, Grossi and Troestler obtained radial and nonradial solutions to some critical systems using bifurcation methods.

Here we focus our attention to the competitive case, i.e., to \( \lambda < 0 \). In this case, the system (1.1) does not have a least energy fully nontrivial solution; see Proposition 2.2 below. This behavior showcases the lack of compactness of the variational functional, which comes from the fact that system is invariant under
translations and dilations, that allow functions to travel to infinity and to blow up without changing their energy value.

But the conformal invariance of the system (1.1) can also be used to our advantage. There are groups of conformal transformations of \( \mathbb{R}^N \) which have the property that all of their orbits have positive dimension. So, as blow-up can only occur at points, looking for solutions which are invariant under such group actions will restore compactness. W. Ding used this fact in [8] to establish the existence of infinitely many sign-changing solutions for the single equation (1.2). Note that linear isometries of \( \mathbb{R}^N \), on the other hand, do not serve this purpose, because the origin is always a fixed point.

Let \( O(N+1) \) be the group of linear isometries of \( \mathbb{R}^{N+1} \) and let \( \Gamma \) be a closed subgroup of \( O(N+1) \). We shall look for solutions to the system (1.1) which are invariant under the conformal action of \( \Gamma \) on \( \mathbb{R}^N \) induced by the stereographic projection \( \sigma : S^N \to \mathbb{R}^N \cup \{ \infty \} \). Namely, for each \( \gamma \in \Gamma \), we consider the map \( \tilde{\gamma} : \mathbb{R}^N \to \mathbb{R}^N \) given by \( \tilde{\gamma}x := (\sigma \circ \gamma^{-1} \circ \sigma^{-1})(x) \), which is well defined except at a single point. The reason for considering this action is that \( O(N+1) \) contains subgroups \( \Gamma \), which do not act transitively on \( S^N \) (i.e., \( \Gamma p \neq S^N \) for every \( p \in S^N \)), with the property that the \( \Gamma \)-orbit \( \Gamma p \) of every point \( p \in S^N \) has positive dimension. We may take, for example, \( \Gamma := O(m) \times O(n) \) with \( m + n = N + 1 \), \( m, n \geq 2 \). These were the groups considered by W. Ding in [8]; see Examples 3.4 below.

A function \( u \) will be said to be \( \Gamma \)-invariant if

\[
|\det \tilde{\gamma}'(x)|^{1/2} u(\tilde{\gamma}x) = u(x) \quad \text{for all } \gamma \in \Gamma, \ x \in \mathbb{R}^N,
\]

and a pair of functions \( (u, v) \) will be said to be \( \Gamma \)-invariant if each of them is \( \Gamma \)-invariant. We will prove the following results.

**Theorem 1.1.** Let \( \Gamma \) be a closed subgroup of \( O(N+1) \) such that \( \Gamma \) does not act transitively on \( S^N \) and the \( \Gamma \)-orbit of every point \( p \in S^N \) has positive dimension. Then, the following statements hold true:

(a) The system (1.1) has a positive fully nontrivial \( \Gamma \)-invariant solution for each \( \lambda < 0 \).

(b) If \( \mu_1 = \mu_2 =: \mu \) and \( \alpha = \beta \), then, for each \( \lambda \leq -\frac{\mu}{\alpha} \), the system (1.1) has infinitely many fully nontrivial \( \Gamma \)-invariant solutions, which are not conformally equivalent.

(c) There exists a \( \lambda_* < 0 \), which depends on \( \mu_1, \mu_2, \alpha, \beta \), such that the system (1.1) does not have a fully nontrivial synchronized solution if \( \lambda < \lambda_* \).

The next result says that there is phase separation for the positive solutions.
**Theorem 1.2.** Assume that \( \Gamma \) does not act transitively on \( S \) and that the \( \Gamma \)-orbit of every point \( p \in S \) has positive dimension. For \( \lambda < 0 \) with \( \lambda \to -\infty \) let \((u_k, v_k)\) be the positive fully nontrivial \( \Gamma \)-invariant solution for the system (1.1) with \( \lambda = \lambda_k \) given by Theorem 1.1(a). Then, after passing to a subsequence, we have that \( u_k \to u_\infty \) and \( v_k \to v_\infty \) strongly in \( D^{1,2}(\mathbb{R}^N) \), the functions \( u_\infty \) and \( v_\infty \) are continuous, \( u_\infty \geq 0, v_\infty \geq 0 \), \( u_\infty v_\infty \equiv 0 \), \( u_\infty \) solves the problem
\[
-\Delta u = \mu_1 |u|^{2^*-2}u, \quad u \in D^{1,2}_0(\Omega_1),
\]
and \( v_\infty \) solves the problem
\[
-\Delta v = \mu_2 |v|^{2^*-2}v, \quad v \in D^{1,2}_0(\Omega_2),
\]
where \( \Omega_1 := \{ x \in \mathbb{R}^N : u_\infty(x) > 0 \} \) and \( \Omega_2 := \{ x \in \mathbb{R}^N : v_\infty(x) > 0 \} \). Moreover, \( \Omega_1 \) and \( \Omega_2 \) are \( \Gamma \)-invariant and connected, \( \Omega_1 \cap \Omega_2 = \emptyset \) and \( \overline{\Omega_1} \cup \overline{\Omega_2} = \mathbb{R}^N \).

We wish to stress that Theorem 1.2 gives very precise information on the domains \( \Omega_1 \) and \( \Omega_2 \), as the following result shows.

**Proposition 1.3.** Let \( \Gamma := O(m) \times O(n) \) with \( m + n = N + 1 \), \( m, n \geq 2 \). Then, after adding a point at infinity and up to relabeling, the domains \( \Omega_1 \) and \( \Omega_2 \) given by Theorem 1.2 have the following shape: \( \Omega_1 \) is diffeomorphic to \( \mathbb{S}^{m-1} \times \mathbb{R}^n \), \( \Omega_2 \) is diffeomorphic to \( \mathbb{R}^m \times \mathbb{S}^{n-1} \), and their common boundary is diffeomorphic to \( \mathbb{S}^{m-1} \times \mathbb{S}^{n-1} \), where \( \mathbb{S}^k \) and \( \mathbb{S}^{k-1} \) denote the open unit ball and the unit sphere in \( \mathbb{R}^k \) respectively.

This paper is organized as follows. In Section 2 we discuss the variational setting and we prove part (c) of Theorem 1.1. Part (a) is proved in Section 3 and part (b) in Section 4. Section 5 is devoted to the proof of Theorem 1.2 and Proposition 1.3.
The fully nontrivial solutions to (1.1) lie on the set
\[ \mathcal{N} := \{(u, v) \in \mathbf{D} : u \neq 0, v \neq 0, f(u, v) = 0, h(u, v) = 0\}, \]
which is called the Nehari manifold and has the following properties.

**Proposition 2.1.** (a) For every \((u, v) \in \mathcal{N}\), one has that
\[ \mu_1^{-\frac{(N-2)}{2}} S^{N/2} \leq \int_{\mathbb{R}^N} |\nabla u|^2, \quad \mu_2^{-\frac{(N-2)}{2}} S^{N/2} \leq \int_{\mathbb{R}^N} |\nabla v|^2, \]
where \(S\) is the best constant for the embedding \(D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)\).

(b) \(\mathcal{N}\) is a closed \(C^1\)-submanifold of codimension 2 of the Hilbert space \(\mathbf{D}\), and the tangent space to \(\mathcal{N}\) at the point \((u, v)\) is the orthogonal complement in \(\mathbf{D}\) of the linear subspace generated by \(\nabla f(u, v)\) and \(\nabla h(u, v)\).

(c) \(\mathcal{N}\) is a natural constraint for the functional \(E\), i.e., a critical point of the restriction of \(E\) to \(\mathcal{N}\) is a critical point of \(E\).

(d) If \((u, v) \in \mathcal{N}\), then \(E(u, v) = \max \{E(su, tv) : s > 0, t > 0\}\).

**Proof.** (a) Let \((u, v) \in \mathcal{N}\). Then, as \(f(u, v) = 0, h(u, v) = 0\) and \(\lambda < 0\), we have that
\[ \int_{\mathbb{R}^N} |\nabla u|^2 \leq \mu_1 \int_{\mathbb{R}^N} |u|^{2^*} \quad \text{and} \quad \int_{\mathbb{R}^N} |\nabla v|^2 \leq \mu_2 \int_{\mathbb{R}^N} |v|^{2^*}. \]
Since \(u \neq 0\) and \(v \neq 0\), using the Sobolev inequality we get that
\[ 0 < S \leq \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{\left(\int_{\mathbb{R}^N} |u|^{2^*}\right)^{2/2^*}} \leq \mu_1^{2/2^*} \left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^{(2^* - 2)/2^*}, \]
and
\[ 0 < S \leq \frac{\int_{\mathbb{R}^N} |\nabla v|^2}{\left(\int_{\mathbb{R}^N} |v|^{2^*}\right)^{2/2^*}} \leq \mu_2^{2/2^*} \left(\int_{\mathbb{R}^N} |\nabla v|^2\right)^{(2^* - 2)/2^*}. \]
This proves (a).

(b) Statement (a) implies that \(\mathcal{N}\) is closed in \(\mathbf{D}\). Next we show that \(\nabla f(u, v)\) and \(\nabla h(u, v)\) are linearly independent for every \((u, v) \in \mathcal{N}\). If \(s \nabla f(u, v) + t \nabla h(u, v) = 0\) for some \((u, v) \in \mathcal{N}, s, t \in \mathbb{R}\), then
\[ 0 = s \langle \nabla f(u, v), (u, 0) \rangle + t \langle \nabla h(u, v), (u, 0) \rangle \]
\[ = s \left(2 \int_{\mathbb{R}^N} |\nabla u|^2 - 2^* \mu_1 \int_{\mathbb{R}^N} |u|^{2^*} - \lambda \alpha^2 \int_{\mathbb{R}^N} |u|^{\alpha} |v|^{\beta}\right) + t \left(-\lambda \alpha^2 \int_{\mathbb{R}^N} |u|^{\alpha} |v|^{\beta}\right) \]
\[ = s \left((2 - 2^*) \mu_1 \int_{\mathbb{R}^N} |u|^2 + \lambda \alpha (2 - \alpha) \int_{\mathbb{R}^N} |u|^{\alpha} |v|^{\beta}\right) + t \left(-\lambda \alpha^2 \int_{\mathbb{R}^N} |u|^{\alpha} |v|^{\beta}\right) \]
\[ =: sa_{11} + ta_{12}, \]
and

\[0 = s \langle \nabla f(u, v), (0, v) \rangle + t \langle \nabla h(u, v), (0, v) \rangle\]

\[= s \left( -\lambda_\alpha \int_{\mathbb{R}^N} |u|^{\alpha} |v|^\beta \right) + t \left( 2 \int_{\mathbb{R}^N} |\nabla v|^2 - 2^* \mu_2 \int_{\mathbb{R}^N} |v|^{2^*} - \lambda \beta^2 \int_{\mathbb{R}^N} |u|^{\alpha} |v|^\beta \right)\]

\[= s \left( -\lambda_\alpha \int_{\mathbb{R}^N} |u|^{\alpha} |v|^\beta \right) + t \left( (2 - 2^*) \mu_2 \int_{\mathbb{R}^N} |v|^{2^*} + \lambda \beta(2 - \beta) \int_{\mathbb{R}^N} |u|^{\alpha} |v|^\beta \right)\]

\[= \det(a_{ij}) \geq (2 - 2^*)^2 c_0^2 > 0,\]

where \(c_0 := \min\{\mu_1^{-(N-2)/2}, \mu_2^{-(N-2)/2}\} S^{N/2}.\) If \(\int_{\mathbb{R}^N} |u|^{\alpha} |v|^\beta \neq 0\) then, as \(\alpha, \beta \in (1, 2)\) and \(\lambda < 0,\) we have that

\[A := -\lambda \int_{\mathbb{R}^N} |u|^{\alpha} |v|^\beta \geq -\lambda \int_{\mathbb{R}^N} |u|^{\alpha} |v|^\beta + \alpha > \left( -\lambda \int_{\mathbb{R}^N} |u|^{\alpha} |v|^\beta + 1 \right) \alpha =: C\alpha,\]

\[B := -\lambda \int_{\mathbb{R}^N} |u|^{\alpha} |v|^\beta \geq -\lambda \int_{\mathbb{R}^N} |u|^{\alpha} |v|^\beta + \beta > \left( -\lambda \int_{\mathbb{R}^N} |u|^{\alpha} |v|^\beta + 1 \right) \beta =: C\beta.\]

We use these inequalities, and the fact that \(\alpha, \beta \in (1, 2)\) and \(\alpha + \beta = 2^*,\) to estimate the determinant

\[
\begin{vmatrix}
(2 - 2^*) \frac{\mu_1 \int_{\mathbb{R}^N} |u|^{2^*}}{\lambda \int_{\mathbb{R}^N} |u|^{\alpha} |v|^\beta} - \alpha(2 - \alpha) & \alpha \\
\alpha \beta & (2 - 2^*) \frac{\mu_2 \int_{\mathbb{R}^N} |v|^{2^*}}{\lambda \int_{\mathbb{R}^N} |u|^{\alpha} |v|^\beta} - \beta(2 - \beta)
\end{vmatrix}
\]

\[= (2 - 2^*)^2 AB - (2 - 2^*) (\beta(2 - \beta)A + \alpha(2 - \alpha)B) + \alpha \beta (2 - \alpha)(2 - \beta) - (\alpha \beta)^2\]

\[\geq C\alpha \beta \left[ (2 - 2^*)^2 (2 - 2^*) (4 - 2^*) \right] + \alpha \beta \left[ (2 - \alpha)(2 - \beta) - \alpha \beta \right]\]

\[= \frac{(2^* - 2) c_0 \alpha \beta}{-\lambda \int_{\mathbb{R}^N} |u|^{\alpha} |v|^\beta}.\]

It follows that

\[(2.1) \quad \det(a_{ij}) \geq (2^* - 2) c_0 \alpha \beta (-\lambda) \int_{\mathbb{R}^N} |u|^{\alpha} |v|^\beta > 0.\]

Thus, in both cases, \(s = t = 0.\) This proves that \(\nabla f(u, v)\) and \(\nabla h(u, v)\) are linearly independent for every \((u, v) \in \mathcal{N}.)\) Therefore, \(\mathcal{N}\) is a \(C^1\)-submanifold of \(D\) and the tangent space to \(\mathcal{N}\) at the point \((u, v)\) is the orthogonal complement in \(D\) of the linear subspace generated by \(\nabla f(u, v)\) and \(\nabla h(u, v)\).

(c) If \((u, v) \in \mathcal{N}\) is a critical point of the restriction of \(E\) to \(\mathcal{N}\), then \(\nabla E(u, v) = s \nabla f(u, v) + t \nabla h(u, v)\) for some \(s, t \in \mathbb{R}.\) Taking the scalar product with \((u, 0)\) and \((0, v)\) we get that

\[s \langle \nabla f(u, v), (u, 0) \rangle + t \langle \nabla h(u, v), (u, 0) \rangle = \langle \nabla E(u, v), (u, 0) \rangle = f(u, v) = 0,\]

\[s \langle \nabla f(u, v), (0, v) \rangle + t \langle \nabla h(u, v), (0, v) \rangle = \langle \nabla E(u, v), (0, v) \rangle = h(u, v) = 0.\]
But we have already shown that this implies that \( s = t = 0 \). Hence, \( \nabla E(u,v) = 0 \), i.e., \((u,v)\) is a critical point of \( E \).

(d) Fix \((u,v) \in \mathcal{N}\) and let \((\hat{s},\hat{t})\) be a critical point of the function \( e(s,t) := E(su,tv) \) in \((0,\infty) \times (0,\infty)\). Then, as \( s \frac{\partial E}{\partial s}(s,t) = f(su,tv) \) and \( t \frac{\partial E}{\partial t}(s,t) = h(su,tv) \), we have that \((\hat{s}u,\hat{t}v) \in \mathcal{N}\). Moreover,

\[
\hat{s}^2 \frac{\partial^2 e}{\partial s^2}(\hat{s},\hat{t}) = (2 - 2^*) \mu_1 \int_{\mathbb{R}^N} |\hat{s}u|^2 + \lambda \alpha (2 - \alpha) \int_{\mathbb{R}^N} |\hat{s}u|^\alpha |\hat{t}v|^\beta,
\]
\[
\hat{t}^2 \frac{\partial^2 e}{\partial t^2}(\hat{s},\hat{t}) = (2 - 2^*) \mu_2 \int_{\mathbb{R}^N} |\hat{t}v|^2 + \lambda \beta (2 - \beta) \int_{\mathbb{R}^N} |\hat{s}u|^\alpha |\hat{t}v|^\beta,
\]
\[
\hat{s}\hat{t} \frac{\partial^2 e}{\partial s \partial t}(\hat{s},\hat{t}) = -\lambda \alpha \beta \int_{\mathbb{R}^N} |\hat{s}u|^\alpha |\hat{t}v|^\beta.
\]

Hence, \( \frac{\partial^2 e}{\partial s^2}(\hat{s},\hat{t}) < 0, \frac{\partial^2 e}{\partial t^2}(\hat{s},\hat{t}) < 0 \) and, as shown in part (b),

\[
(\hat{s}\hat{t})^2 \left( \frac{\partial^2 e}{\partial s^2}(\hat{s},\hat{t}) \frac{\partial^2 e}{\partial t^2}(\hat{s},\hat{t}) - \left( \frac{\partial^2 e}{\partial s \partial t}(\hat{s},\hat{t}) \right)^2 \right) > 0.
\]

Therefore, \((\hat{s},\hat{t})\) is a strict local maximum of \( e \). This implies that \((1,1)\) is the only critical point of \( e \) in \((0,\infty) \times (0,\infty)\) and it is a global maximum; see Lemma \( \text{A.2} \) in the appendix.

\[\square\]

The following statement was proved in \([\text{5}]\) for \( \alpha = \beta = \frac{2^*}{2} \). We give a simpler proof which applies to all \( \alpha, \beta \).

**Proposition 2.2.** \( \inf_{(u,v) \in \mathcal{N}} E(u,v) = \frac{1}{N}(\mu_1^{-(N-2)/2} + \mu_2^{-(N-2)/2})S^{N/2} \) and this value is not attained by \( E \) on \( \mathcal{N} \).

**Proof.** If \((u,v) \in \mathcal{N}\), then Proposition 2.1(a) yields

\[
E(u,v) = E(u,v) - \frac{1}{2 \alpha} E'(u,v) [(u,v)]
= \frac{1}{N} \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) \geq \frac{1}{N} \left( \mu_1^{-(N-2)/2} + \mu_2^{-(N-2)/2} \right) S^{N/2}.
\]

To prove the opposite inequality, we choose a sequence of functions \( w_k \in C_c^\infty (B_1(0)) \) in the unit ball \( B_1(0) := \{ x \in \mathbb{R}^N : |x| < 1 \} \) which satisfies

\[
\int_{B_1(0)} |\nabla w_k|^2 = \int_{B_1(0)} |w_k|^2 \quad \text{and} \quad \int_{B_1(0)} |\nabla w_k|^2 \to S^{N/2}.
\]

Such a sequence exists because

\[
S = \inf_{w \in D_0^{2,\infty} (\Omega)} \frac{\int_{\Omega} |\nabla w|^2}{\left( \int_{\Omega} |w|^2 \right)^{2/2}}
\]

for every domain \( \Omega \in \mathbb{R}^N \); see, e.g., \([20]\). Fix \( \xi \in \mathbb{R}^N \) with \( |\xi| = 1 \) and define \( u_k(x) := \mu_1^{-(N-2)/4} w_k(x - \xi) \) and \( v_k(x) := \mu_2^{-(N-2)/4} w_k(x + \xi) \). As \( u_k \) and \( v_k \) have
disjoint supports, we have that \( \int_{\mathbb{R}^N} |u_k|^\alpha |v_k|^\beta = 0 \). Hence,

\[
 f(u_k, v_k) = \mu_1^{(2-N)/2} \left( \int_{B_1(0)} |\nabla w_k|^2 - \int_{B_1(0)} |w_k|^2 \right) = 0,
\]

\[
 h(u_k, v_k) = \mu_2^{(2-N)/2} \left( \int_{B_1(0)} |\nabla w_k|^2 - \int_{B_1(0)} |w_k|^2 \right) = 0,
\]

i.e., \((u_k, v_k) \in \mathcal{N}\), and

\[
 E(u_k, v_k) = \frac{1}{N} \left( \frac{1}{\mu_1} \int_{B_1(0)} |\nabla w_k|^2 + \frac{1}{\mu_2} \int_{B_1(0)} |\nabla w_k|^2 \right)
\]

\[
 \rightarrow \frac{1}{N} (\mu_1^{-(N-2)/2} + \mu_2^{-(N-2)/2}) S^{N/2}.
\]

This proves that \( \inf_{(u,v) \in \mathcal{N}} E(u,v) = \frac{1}{N} (\mu_1^{-(N-2)/2} + \mu_2^{-(N-2)/2}) S^{N/2} \).

To show that this value is not attained, we argue by contradiction. Assume that \((u_0, v_0) \in \mathcal{N}\) is a minimum of \( E \) on \( \mathcal{N}\). As \( (|u_0|, |v_0|) \in \mathcal{N}\) and \( E(|u_0|, |v_0|) = E(u_0, v_0) \), the pair \((|u_0|, |v_0|)\) is also a minimum of \( E \). So, we may assume that \( u_0 \geq 0 \) and \( v_0 \geq 0 \). We consider two cases. If \( \int_{\mathbb{R}^N} u_0^\alpha v_0^\beta = 0 \), then \( u_0^\alpha v_0^\beta = 0 \) a.e. in \( \mathbb{R}^N \) and \( u_0 \) solves the equation \(-\Delta u = \mu_1 |u|^{2^*-2} u\). As \( v_0 \) is nontrivial, we have that \( u_0 = 0 \) in a set of positive measure. This is a contradiction. If, on the other hand, \( \int_{\mathbb{R}^N} u_0^\alpha v_0^\beta > 0 \), then

\[
 \int_{\mathbb{R}^N} |\nabla u_0|^2 < \mu_1 \int_{\mathbb{R}^N} |u_0|^{2^*} \quad \text{and} \quad \int_{\mathbb{R}^N} |\nabla v_0|^2 < \mu_2 \int_{\mathbb{R}^N} |v_0|^{2^*},
\]

and from the Sobolev inequality we derive

\[
 \mu_1^{-(N-2)/2} S^{N/2} < \int_{\mathbb{R}^N} |\nabla u_0|^2 \quad \text{and} \quad \mu_2^{-(N-2)/2} S^{N/2} < \int_{\mathbb{R}^N} |\nabla v_0|^2.
\]

This implies that \( E(u_0, v_0) > \frac{1}{N} (\mu_1^{(N-2)/2} + \mu_2^{(N-2)/2}) S^{N/2} \), which is, again, a contradiction.

\[\square\]

**Proposition 2.3.** There exists a \( \lambda_* < 0 \), depending on \( \mu_1, \mu_2, \alpha, \beta \), such that

\[
 \mathcal{N} \cap \{(su, tu) : s, t \in \mathbb{R}, u \in D^{1,2}(\mathbb{R}^N)\} = \emptyset \quad \text{if} \quad \lambda < \lambda_*.
\]

**Proof.** To highlight the role of \( \lambda \), we write \( \mathcal{N}_\lambda, \ f_\lambda \) and \( h_\lambda \), instead of \( \mathcal{N}, \ f \) and \( h \).

Arguing by contradiction, assume there exists a sequence \((\lambda_k)\) with \( \lambda_k \to -\infty \) for which there are \( s_k, t_k \in \mathbb{R} \) and \( u_k \in D^{1,2}(\mathbb{R}^N) \) such that \((s_k u_k, t_k u_k) \in \mathcal{N}_\lambda_k\). Then \( s_k \neq 0, t_k \neq 0 \) and \( u_k \neq 0 \). So, after replacing \( u_k \) with \( r_k u_k \) for some suitable \( r_k > 0 \), we may assume that \( \int_{\mathbb{R}^N} |\nabla u_k|^2 = \int_{\mathbb{R}^N} |u_k|^{2^*} \). We may also assume that \( s_k > 0, t_k > 0 \). Then, dividing the equations \( f_\lambda (s_k u_k, t_k u_k) = 0 \) and \( h_\lambda (s_k u_k, t_k u_k) = 0 \) by \( \int_{\mathbb{R}^N} |u_k|^{2^*} \), we obtain that \((s_k, t_k)\) solves the system

\[
 \begin{align*}
 1 &= \mu_1 s_k^{2^*-2} + \lambda_k \alpha \gamma s_k^{-2} t_k^\beta, \\
 1 &= \mu_2 t_k^{2^*-2} + \lambda_k \beta s_k^{\gamma-2} t_k^\beta.
\end{align*}
\]
Recall that $\alpha + \beta = 2^*$. Dividing the first equation by $s_k^{\alpha - 2} t_k^\beta$ and the second one by $s_k^{\alpha} t_k^{\beta - 2}$ we get that
\[
\mu_1 \left( \frac{s_k}{t_k} \right) ^\beta = \frac{1}{s_k^{\alpha - 2} t_k^\beta} - \lambda_k \alpha \geq -\lambda_k \alpha,
\]
\[
\mu_2 \left( \frac{t_k}{s_k} \right) ^\alpha = \frac{1}{s_k^{\alpha} t_k^{\beta - 2}} - \lambda_k \beta \geq -\lambda_k \beta.
\]
It follows that both sequences $(\frac{s_k}{t_k})$ and $(\frac{t_k}{s_k})$ are unbounded. This is a contradiction. □

3. Symmetries and compactness

Let $(\mathbb{S}^N, g)$ be the standard sphere and $q \in \mathbb{S}^N$ be the north pole. The stereographic projection $\sigma: \mathbb{S}^N \setminus \{q\} \rightarrow \mathbb{R}^N$ is a conformal diffeomorphism. The coordinates of the standard metric $g$ in the chart given by $\sigma^{-1}: \mathbb{R}^N \rightarrow \mathbb{S}^N \setminus \{q\}$ are $g_{ij} = \psi^{2^* - 2} \delta_{ij}$, where
\[
\psi(x) := \left( \frac{2}{1 + |x|^2} \right)^{(N - 2)/2}, \quad x \in \mathbb{R}^N.
\]
For $u \in C^\infty(\mathbb{S}^N)$, we set $u := \psi \circ \sigma^{-1}$ and we write $\nabla_g u$ for its gradient.

Lemma 3.1. For every $u, v \in C^\infty(\mathbb{S}^N)$ we have that
\[
\int_{\mathbb{S}^N} \left( |\nabla_g u|^2 + \frac{N(N - 2)}{4} u^2 \right) dV_g = \int_{\mathbb{R}^N} |\nabla u|^2 \, dx,
\]
\[
\int_{\mathbb{S}^N} |u|^{2^*} \, dV_g = \int_{\mathbb{R}^N} |u|^{2^*} \, dx,
\]
\[
\int_{\mathbb{S}^N} |u|^\alpha |v|^\beta \, dV_g = \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \, dx.
\]

Proof. If $(M, h)$ is a Riemannian manifold of dimension $n \geq 3$, the operator $L_h := -\Delta_h + \frac{n - 2}{4(n - 1)} R_h$, where $\Delta_h := \text{div}_h \nabla_h$ is the Laplace-Beltrami operator (without a sign) and $R_h$ is the scalar curvature with respect to the metric $h$, is called the conformal Laplacian. It has a certain conformal invariance, which in our case is expressed by the identity
\[
-\Delta_g u + \frac{N(N - 2)}{4} u = -\psi^{1 - 2^*} \Delta u;
\]
see, e.g., [13, Proposition 6.1.1]. Note that the Riemannian volume element on $(\mathbb{S}^N, g)$ is $dV_g = \sqrt{\det(g_{ij})} dx = \psi^{2^*} dx$. So, multiplying this identity by $u$ and integrating by parts, we obtain
\[
\int_{\mathbb{S}^N} \left( |\nabla_g u|^2 + \frac{N(N - 2)}{4} u^2 \right) dV_g = \int_{\mathbb{S}^N} \left( -\Delta_g u + \frac{N(N - 2)}{4} u^2 \right) dV_g = \int_{\mathbb{R}^N} (-\Delta u) u \, dx = \int_{\mathbb{R}^N} |\nabla u|^2 \, dx.
This is the first identity in the statement of the lemma. The other two are immediate. \( \square \)

Taking \( \left( \int_{\mathbb{S}^N} (|\nabla_g u|^2 + \frac{N(N-2)}{4} u^2) dV_g \right)^{1/2} \) as the norm in \( H^1_g(\mathbb{S}^N) \), we obtain a linear isometry of Hilbert spaces \( \iota: H^1_g(\mathbb{S}^N) \to D^{1,2}(\mathbb{R}^N) \) given by

\[
(3.1) \quad \iota(u) := \psi(u \circ \sigma^{-1}).
\]

\( \mathbb{S}^N \) is invariant under the action of the group \( O(N+1) \) of linear isometries of \( \mathbb{R}^{N+1} \), so each \( \gamma \in O(N+1) \) induces a linear isometry \( \gamma: H^1_g(\mathbb{S}^N) \to H^1_g(\mathbb{S}^N) \) given by

\[
(\gamma u)(p) := u(\gamma^{-1} p), \quad p \in \mathbb{S}^N, \quad u \in H^1_g(\mathbb{S}^N).
\]

Therefore, the composition \( \iota \circ \gamma \circ \iota^{-1}: D^{1,2}(\mathbb{R}^N) \to D^{1,2}(\mathbb{R}^N) \) is a linear isometry. This gives an action of \( O(N+1) \) on \( D \), defined by \( \gamma(u, v) := (\gamma u, \gamma v) \), where

\[
\gamma u := (\iota \circ \gamma \circ \iota^{-1}) u, \quad \gamma \in O(N+1), \quad u \in D^{1,2}(\mathbb{R}^N).
\]

Set \( \tilde{\gamma} := \sigma \circ \gamma^{-1} \circ \sigma^{-1} \). As \( \sigma^{-1} \) is a conformal map and \( \gamma^{-1} \) is a linear isometry, we have that \( |\det d_x \sigma^{-1}| = \psi^2(x) \) and \( |\det d_p \gamma^{-1}| = 1 \). Therefore,

\[
|\det \tilde{\gamma}'(x)| = |\det d_{(\gamma^{-1} \circ \sigma^{-1})(x)} \sigma||\det d_x \sigma^{-1}| = \frac{|\det d_x \sigma^{-1}|}{|\det d_\tilde{\gamma}(x) \sigma^{-1}|} \quad \left( \frac{\psi(x)}{\psi(\tilde{\gamma}(x))} \right)^{2^*}
\]

and, since \( \iota^{-1}(u) = \frac{1}{\psi \circ \gamma} u \circ \sigma \), we conclude that

\[
(3.2) \quad \gamma u = (\iota \circ \gamma \circ \iota^{-1}) u = \frac{\psi}{\psi \circ \tilde{\gamma}} u \circ \tilde{\gamma} = |\det \tilde{\gamma}'|^{1/2^*} u \circ \tilde{\gamma}.
\]

Using Lemma \( 3.1 \) it is easy to see that the functional \( E \) is invariant under this action, i.e.,

\[
E(\gamma(u, v)) = E(u, v) \quad \text{for every } \gamma \in O(N+1), \ (u, v) \in D,
\]

and so are \( f \) and \( h \). If \( \Gamma \) is a closed subgroup of \( O(N+1) \), we write

\[
D^\Gamma := \{(u, v) \in D : \gamma(u, v) = (u, v) \quad \text{for every } \gamma \in \Gamma\}
\]

for the \( \Gamma \)-fixed point set of \( D \). By \( 3.2 \) we have that \( (u, v) \in D^\Gamma \) iff \( (u, v) \) is \( \Gamma \)-invariant in the sense defined in the introduction. Define

\[
A^\Gamma := \{(u, v) \in D^\Gamma : u \neq 0, \ v \neq 0, \ f(u, v) = 0, \ h(u, v) = 0\}.
\]

Recall that a group \( \Gamma \) is said to act transitively on a set \( X \) if \( X \) has only one \( \Gamma \)-orbit.

**Lemma 3.2.** If \( \Gamma \) does not act transitively on \( \mathbb{S}^N \), then \( N^\Gamma \neq \emptyset \).

**Proof.** Since \( \Gamma \) does not act transitively on \( \mathbb{S}^N \), there are two points in \( \mathbb{S}^N \) whose \( \Gamma \)-orbits are disjoint. Taking two nontrivial \( \Gamma \)-invariant functions in \( C^\infty(\mathbb{S}^N) \) whose supports lie in disjoint neighborhoods of these orbits, and composing them with
the inverse of the stereographic projection, we obtain a pair of nontrivial functions 
\((u, v) \in D^F\) with \(\text{supp}(u) \cap \text{supp}(v) = \emptyset\). Setting \(s, t \in (0, \infty)\) such that 
\[
\int_{\mathbb{R}^N} |\nabla (su)|^2 = \mu_1 \int_{\mathbb{R}^N} |su|^2^\ast \quad \text{and} \quad \int_{\mathbb{R}^N} |\nabla (tv)|^2 = \mu_2 \int_{\mathbb{R}^N} |tv|^2^\ast,
\]
we get that \((su, tv)\) is nontrivial.

We assume from now on that \(\Gamma\) does not act transitively on \(S^N\).

It is easy to see that \(\nabla E(u, v), \nabla f(u, v), \nabla h(u, v) \in D^F\) for every \((u, v) \in D^F\); cf. Theorem 1.28 in [30]. Then, it follows from Proposition 2.1 that \(N^T\) is a closed \(C^1\)-submanifold of \(D^F\) and a natural constraint for \(E\). The tangent space to \(N^T\) at the point \((u, v)\) is the orthogonal complement in \(D^F\) of the linear subspace generated by \(\nabla f(u, v)\) and \(\nabla h(u, v)\).

The following fact plays a crucial role in the proof of Proposition 3.6 below.

**Proposition 3.3.** If the \(\Gamma\)-orbit \(\Gamma p := \{\gamma p : \gamma \in \Gamma\}\) of every point \(p \in S^N\) has positive dimension, then the embedding \(D^F \hookrightarrow L^2^\ast (\mathbb{R}^N) \times L^2^\ast (\mathbb{R}^N)\) is compact.

**Proof.** It is shown in [3] that the embedding \(H^1_2(S^N)^\Gamma \hookrightarrow L^2_g(S^N)\) is compact. The map defined in [3.1] is an isometry between the \(\Gamma\)-fixed point spaces \(\iota : H^1_2(S^N)^\Gamma \to D^{1, 2}(\mathbb{R}^N)^\Gamma\) and the Lebesgue spaces \(\iota : L^2_g(S^N) \to L^2_g(\mathbb{R}^N)\). Therefore, the embedding \(D^{1, 2}(\mathbb{R}^N)^\Gamma \hookrightarrow L^2_g(\mathbb{R}^N)\) is compact, and so is
\[
D^F = D^{1, 2}(\mathbb{R}^N)^\Gamma \times D^{1, 2}(\mathbb{R}^N)^\Gamma \hookrightarrow L^2_g(\mathbb{R}^N) \times L^2_g(\mathbb{R}^N),
\]
as claimed.

Let us give some examples.

**Examples 3.4.**

1. If \(m + n = N + 1\), the group \(\Gamma := O(m) \times O(n)\) acts on \(R^{N+1} = R^m \times R^n\) in the obvious way. The \(\Gamma\)-orbit of a point \((x_0, y_0) \in R^m \times R^n\) is the set
\[
\Gamma(x_0, y_0) = \{(x, y) \in R^m \times R^n : |x| = |x_0|, \ |y| = |y_0|\},
\]
so the \(\Gamma\)-orbit of every point \(p \in S^N\) has positive dimension iff \(m, n \geq 2\).

2. For \(N\) odd, another example is obtained by taking \(\Gamma := S^1\) to be the group of unit complex numbers acting on \(C^{(N+1)/2} = \mathbb{R}^{N+1}\) by multiplication on each complex coordinate. Then, the \(\Gamma\)-orbit of every point in \(S^N\) is a circle.

We write \(\nabla N E(u, v)\) for the orthogonal projection of \(\nabla E(u, v)\) onto the tangent space of \(N^T\) at \((u, v)\).

**Lemma 3.5.** If \(((u_k, v_k))\) is a sequence in \(N^T\) such that
\[
E(u_k, v_k) \to c \quad \text{and} \quad \nabla N E(u_k, v_k) \to 0,
\]
then \(((u_k, v_k))\) is bounded in \(D\) and \(\nabla E(u_k, v_k) \to 0\).
Proof. If \((u_k, v_k) \in \mathcal{N}, E(u_k, v_k) \to c\) and \(\nabla \mathcal{N} E(u_k, v_k) \to 0\) then, as

\[
E(u_k, v_k) = E(u_k, v_k) - \frac{1}{2} E'(u_k, v_k) [(u_k, v_k)] = \frac{1}{N} \int_{\mathbb{R}^N} (|\nabla u_k|^2 + |\nabla v_k|^2),
\]

we have that \(((u_k, v_k))\) is bounded in \(D\). This easily implies that \((\nabla f(u_k, v_k))\) and \((\nabla h(u_k, v_k))\) are bounded in \(D\). Let \(s_k, t_k \in \mathbb{R}\) be such that

\[
\nabla E(u_k, v_k) = \nabla \mathcal{N} E(u_k, v_k) + s_k \nabla f(u_k, v_k) + t_k \nabla h(u_k, v_k).
\]

As \((u_k, v_k) \in \mathcal{N}\) and \(\nabla \mathcal{N} E(u_k, v_k) \to 0\), taking the scalar product of this identity with \((u_k, 0)\) and \((0, v_k)\), we get that \(s_k\) and \(t_k\) solve the system

\[
(3.4) \quad \left\{
\begin{array}{l}
o(1) = s_k a_{11}^{(k)} + t_k a_{12}^{(k)}, \\
o(1) = s_k a_{12}^{(k)} + t_k a_{22}^{(k)},
\end{array}
\right.
\]

where \(o(1) \to 0\) as \(k \to \infty\),

\[
a_{11}^{(k)} := (2 - 2^*) \mu_1 \int_{\mathbb{R}^N} |u_k|^{2^*} + \lambda \alpha (2 - \alpha) \int_{\mathbb{R}^N} |u_k|^\alpha |v_k|^\beta,
\]

\[
a_{12}^{(k)} := -\lambda \alpha \beta \int_{\mathbb{R}^N} |u_k|^\alpha |v_k|^\beta := a_{21}^{(k)}
\]

\[
a_{22}^{(k)} := (2 - 2^*) \mu_2 \int_{\mathbb{R}^N} |v_k|^{2^*} + \lambda \beta (2 - \beta) \int_{\mathbb{R}^N} |u_k|^\alpha |v_k|^\beta.
\]

After passing to a subsequence, we have that \(\int_{\mathbb{R}^N} |u_k|^\alpha |v_k|^\beta \to b \in [0, \infty)\). If \(b = 0\), the statement (a) of Proposition 2.1 implies that

\[
\det(a_{ij}^{(k)}) \geq \frac{1}{2} (2 - 2^*)^2 c_0^2 > 0 \quad \text{for } k \text{ large enough},
\]

where \(c_0 := \min \{ \mu_1^{-\frac{(N-2)}{2}}, \mu_2^{-\frac{(N-2)}{2}} \} S^{N/2}\). If \(b > 0\), then (2.1) implies that

\[
\det(a_{ij}^{(k)}) \geq (2^* - 2) c_0 \alpha \beta (-\lambda) \int_{\mathbb{R}^N} |u_k|^\alpha |v_k|^\beta
\]

\[
\geq \frac{1}{2} (2^* - 2) c_0 \alpha \beta (-\lambda) d > 0 \quad \text{for } k \text{ large enough}.
\]

Therefore, the system (3.4) has a unique solution \((s_k, t_k)\) for large enough \(k\) and, as \(((u_k, v_k))\) is bounded in \(D\), after passing to a subsequence, we conclude that \(s_k \to 0\) and \(t_k \to 0\). From the identity (3.3), we get that \(\nabla E(u_k, v_k) \to 0\), as claimed. \(\square\)

**Proposition 3.6.** If the \(\Gamma\) orbit of every point \(p \in S^N\) has positive dimension, then every sequence \(((u_k, v_k))\) in \(\mathcal{N}^\Gamma\) such that

\[
E(u_k, v_k) \to c \quad \text{and} \quad \nabla \mathcal{N} E(u_k, v_k) \to 0
\]

contains a convergent subsequence.

**Proof.** By Lemma 3.5 and Proposition 3.3, \(((u_k, v_k))\) is bounded in \(D\) and the embedding \(D^\Gamma \hookrightarrow L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)\) is compact. So, after passing to a subsequence, we have that \((u_k, v_k) \rightharpoonup (u, v)\) weakly in \(D\) and \((u_k, v_k) \to (u, v)\) strongly
Proof. \(N\) which was proved by A. Szulkin in [27]. Therefore, let Theorem 4.1.

Theorem 3.7. Let \(\Gamma\) be a closed \(C^1\)-submanifold of \(D^\Gamma\) and that \(E\) is of class \(C^1\), bounded below and satisfies the Palais-Smale condition on \(\mathcal{N}^\Gamma\); see Propositions 2.1, 22 and 23. Since \(\mathcal{N}^\Gamma \neq \emptyset\), Theorem 3.1 in [27] asserts that \(c_1 = \inf_{(u,v) \in \mathcal{N}^\Gamma} E(u, v)\) is attained. As \(E(|u|, |v|) = E(u, v)\), \(E\) has a positive minimizer on \(\mathcal{N}^\Gamma\).

\[\begin{align*}
\text{Theorem 3.7.} & \quad \text{If } \Gamma \text{ does not act transitively on } S^N \text{ and the } \Gamma\text{-orbit of every point } p \in S^N \text{ has positive dimension, then } E \text{ has a positive minimizer on } \mathcal{N}^\Gamma. \\
\text{Proof.} & \quad \text{We have shown that } \mathcal{N}^\Gamma \text{ is a } C^1\text{-submanifold of } D^\Gamma \text{ and that } E \text{ is of class } C^1, \text{ bounded below and satisfies the Palais-Smale condition on } \mathcal{N}^\Gamma; \text{ see Propositions 2.1, 22 and 23. Since } \mathcal{N}^\Gamma \neq \emptyset, \text{ Theorem 3.1 in [27] asserts that } c_1 = \inf_{(u,v) \in \mathcal{N}^\Gamma} E(u, v) \text{ is attained. As } E(|u|, |v|) = E(u, v), \text{ then } E \text{ has a positive minimizer on } \mathcal{N}^\Gamma. \\
\text{4. Multiplicity for the symmetric system} \\
\text{To obtain multiple solutions, we adapt a } C^1\text{-Ljusternik-Schnirelmann result, which was proved by A. Szulkin in [27].} \\
\text{Let } X \text{ be a real Banach space with an action of the group } \mathbb{Z}_2 := \{1, -1\} \text{ by linear isometries. A point } z \in X \text{ is called a fixed point if } (-1) \cdot z = z. \text{ A } \mathbb{Z}_2\text{-invariant subset of } X \text{ is called fixed point free if it does not contain a fixed point.} \\
\text{Let } \Sigma \text{ be the collection of all closed } \mathbb{Z}_2\text{-invariant subsets of } X \text{ which are fixed point free. If } Z \in \Sigma \text{ is nonempty, the genus of } Z \text{ is the smallest integer } j \geq 1 \text{ such that there exists a continuous function } \phi : Z \to S^{j-1} \text{ which is } \mathbb{Z}_2\text{-equivariant, i.e., } \phi((-1) \cdot z) = -\phi(z) \text{ for all } z \in Z. \text{ We denote this integer by } \text{genus}(Z). \text{ If no such } j \text{ exists we set } \text{genus}(Z) := \infty. \text{ We define } \text{genus}(\emptyset) := 0. \\
\text{Theorem 4.1.} \quad \text{Let } M \text{ be a closed } \mathbb{Z}_2\text{-invariant } C^1\text{-submanifold of } X \text{ which is fixed point free, and let } F \in C^1(M, \mathbb{R}) \text{ be a } \mathbb{Z}_2\text{-invariant function which is bounded below and satisfies the Palais-Smale condition. If the set } \\
\Sigma_j := \{Z \in \Sigma : Z \subset M, \text{ } Z \text{ is compact and } \text{genus}(Z) \geq j\} \]
is nonempty for every $j \geq 1$, then $F$ has infinitely many critical values.

Proof. Let $K_c := \{ z \in M : F(z) = c \text{ and } F'(z) = 0 \}$, and set

$$c_j := \inf_{Z \in \Sigma_j} \max_{z \in Z} F(z).$$

Since $F$ is bounded below, $\Sigma_{j+1} \subset \Sigma_j$ and the sets $Z$ in $\Sigma_j$ are compact, we have that

$$-\infty < c_1 \leq c_2 \leq \cdots \leq c_j \leq \cdots < \infty.$$

The proof of Theorem 3.1 in [27] can be adapted, in a straightforward manner, to show that, if $c_j = \cdots = c_{j+m} =: c$ for some $m \geq 0$, then

$$\text{genus}(K_c) \geq m + 1.$$

One needs only to replace $\text{cat}_M$ by genus, and take care that the sets involved are $Z_2$-invariant and the maps are $Z_2$-equivariant.

In particular, $\text{genus}(K_{c_j}) \geq 1$. Hence, $c_j$ is a critical value. Moreover, as $F$ satisfies the Palais-Smale condition, the sets $K_c$ are compact and have, therefore, finite genus. It follows that, for each $j \geq 1$ there exists $m > 0$ such that $c_j \neq c_{j+m}$.

This proves our claim. 

We derive the following result.

**Theorem 4.2.** Assume that $\mu_1 = \mu_2 =: \mu$ and $\alpha = \beta$. If $\Gamma$ does not act transitively on $S^N$ and the $\Gamma$-orbit of every point $p \in S^N$ has positive dimension, then $E : \mathcal{N}^T \to \mathbb{R}$ has infinitely many critical values for each $\lambda \leq -\frac{\mu}{\alpha}$.

Proof. Consider the action of the group $Z_2$ on $D$ given by

$$(-1) \cdot (u, v) := (-v, -u).$$

As $\mu_1 = \mu_2 =: \mu$ and $\alpha = \beta$, we have that $(-v, -u) \in \mathcal{N}^T$ iff $(u, v) \in \mathcal{N}^T$, and $E(-v, -u) = E(u, v)$ for all $(u, v) \in \mathcal{N}^T$. Note also that $(u, -u) \notin \mathcal{N}$ if $\lambda \leq -\frac{\mu}{\alpha}$.

Otherwise,

$$\int_{\mathbb{R}^N} |\nabla u|^2 = (\mu + \lambda \alpha) \int_{\mathbb{R}^N} |u|^2 \leq 0,$$

which is a contradiction. This means that $E$ and $\mathcal{N}^T$ are $Z_2$-invariant, and $\mathcal{N}^T$ is fixed point free. We have already shown that $E$ is bounded below and satisfies the Palais-Smale condition on $\mathcal{N}^T$; see Propositions 2.2 and 3.6. So, all that is left, is to show is that $\mathcal{N}^T$ contains a compact $Z_2$-invariant subset of genus $\geq j$, for each $j \geq 1$.

Fix $j \geq 1$. As $\Gamma$ does not act transitively on $S^N$, we may choose $2j$ pairwise disjoint open $\Gamma$-invariant subsets $U_1, \ldots, U_{2j}$ of $\mathbb{R}^N$ and nontrivial $\Gamma$-invariant functions $u_i \in C_c^\infty(U_i)$, $v_i \in C_c^\infty(U_{j+i})$, for each $i = 1, \ldots, j$. For $w \in D^{1,2}(\mathbb{R}^N)$, $w \neq 0$,
let $t_w$ be the unique positive number such that $\int_{\mathbb{R}^N} |\nabla (t_w w)|^2 = \mu \int_{\mathbb{R}^N} |t_w w|^{2^*}$ and, for $(u, v) \in D^\Gamma$ with $u \neq 0$ and $v \neq 0$, define

$$g(u, v) := (t_u u, t_v v).$$

Note that $g((-1) \cdot (u, v)) = (-1) \cdot g(u, v)$ and that $g(u, v) \in \mathcal{N}^\Gamma$ if $uv = 0$. Let \{e_1, ..., e_j\} be the canonical basis of $\mathbb{R}^j$. The boundary of the convex hull of the set \{±e_1, ..., ±e_j\}, which is given by

$$Q := \left\{ \sum_{i=1}^j \lambda_i \tilde{e}_i : \tilde{e}_i \in \{e_i, -e_i\}, \lambda_i \in [0, 1], \sum_{i=1}^j \lambda_i = 1 \right\},$$

is symmetric with respect to the origin and radially homeomorphic to the unit sphere $S^{j-1}$. Setting $h(e_i) := g(u_i, v_i)$ and $h(-e_i) := g(-v_i, -u_i)$, and extending this map by

$$h\left(\sum_{i=1}^j \lambda_i \tilde{e}_i\right) := g\left(\sum_{i=1}^j \lambda_i h(\tilde{e}_i)\right),$$

we obtain a map $h : Q \to \mathcal{N}^\Gamma$ which is well defined, continuous and $\mathbb{Z}_2$-equivariant. Then, the set $Z := h(Q)$ is compact and $\mathbb{Z}_2$-invariant. If $\phi : Z \to S^{k-1}$ is continuous and $\mathbb{Z}_2$-equivariant, the composition $\phi \circ h$ yields an odd map $S^{j-1} \to S^{k-1}$. The Borsuk-Ulam theorem implies that $k \geq j$. Hence, $\text{genus}(Z) \geq j$ This finishes the proof.

\[\square\]

\textit{Proof of Theorem 1.1.} The statements (a) and (b) follow from Theorems 3.7 and 4.2 respectively. The statement (c) follows from Proposition 2.3. \[\square\]

5. Phase separation

This section is devoted to the proof of Theorem 1.2. We assume throughout that $\Gamma$ is a closed subgroup of $O(N+1)$ which does not act transitively on $S^N$ and that the $\Gamma$-orbit of every point $p \in S^N$ has positive dimension. To highlight the dependence on $\lambda$, we write $E_\lambda, \mathcal{N}^\Gamma_\lambda, f_\lambda, h_\lambda$ instead of $E, \mathcal{N}^\Gamma, f, h$, and we set

$$c^\Gamma_\lambda := \inf_{(u, v) \in \mathcal{N}^\Gamma_\lambda} E_\lambda(u, v).$$

We denote by $J$ and $\mathcal{M}^\Gamma$ the energy functional and the Nehari manifold of the problem

\begin{equation}
-\Delta w = \mu_1 |w^+|^{2^* - 2} w^+ + \mu_2 |w^-|^{2^* - 2} w^-, \quad w \in D^{1,2}(\mathbb{R}^N)^\Gamma,
\end{equation}

where $w^+ := \max\{w, 0\}$ and $w^- := \min\{w, 0\}$, i.e.,

$$J(w) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla w|^2 - \frac{1}{2^*} \int_{\mathbb{R}^N} (\mu_1 |w^+|^{2^*} + \mu_2 |w^-|^{2^*}),$$

...
Proposition 5.1. For \( E \in H \), hence, \( v \in v^* \). If, moreover, \( uv \leq 0 \), then there exist unique numbers \( s, t \in (0, \infty) \) such that \( su \in M^* \) and \(-tv \in M^* \), namely,

\[
(5.2) \quad s^{2^* - 2} = \int_{\mathbb{R}^N} |\nabla u|^2 \quad \text{and} \quad t^{2^* - 2} = \int_{\mathbb{R}^N} |\nabla v|^2.
\]

If, moreover, \( uw = 0 \), then \( su - tv \in E^* \). Arguing as in the proof of Lemma 3.2, we see that there exist \( u \) and \( v \) with these properties. Hence, \( E^* \neq \emptyset \). We define

\[
c^* = \inf_{w \in E^*} J(w) < \infty.
\]

**Proposition 5.1.** For \( \lambda_k \to -\infty \), let \( (u_k, v_k) \in \mathcal{N}_k \) satisfy \( u_k \geq 0 \), \( v_k \geq 0 \) and \( E_{\lambda_k}(u_k, v_k) = c^* \). Then, after passing to a subsequence, we have that \( u_k \to u_\infty \) and \( v_k \to v_\infty \) strongly in \( D^{1,2}(\mathbb{R}^N)^* \), and these functions satisfy

- (a) \( u_\infty, v_\infty \in M^* \), \( u_\infty \geq 0 \), \( v_\infty \geq 0 \), \( u_\infty v_\infty = 0 \). Thus, \( u_\infty - v_\infty \in E^* \).
- (b) \( \lim_{k \to \infty} c^*_{\lambda_k} = J(u_\infty - v_\infty) = c^* \).
- (c) \( u_\infty - v_\infty \) solves the problem (5.1).

**Proof.** If \( w \in E^* \) then, as \( w^+ w^- = 0 \), we have that \( (w^+, w^-) \in \mathcal{N}^* \) and \( J(w) = E_{\lambda_k}(w^+, w^-) \) for every \( \lambda < 0 \). Hence,

\[
c^*_k \leq c^* \quad \text{for every} \quad \lambda < 0.
\]

This implies, in particular, that

\[
\frac{1}{N} \int_{\mathbb{R}^N} \left( |\nabla u_k|^2 + |\nabla v_k|^2 \right) = E_{\lambda_k}(u_k, v_k) \leq c^* \quad \text{for all} \quad k \in \mathbb{N}.
\]

So, after passing to a subsequence, there exist \( u_\infty, v_\infty \in D^{1,2}(\mathbb{R}^N)^* \) such that

- \( u_k \to u_\infty \), \( v_k \to v_\infty \), weakly in \( D^{1,2}(\mathbb{R}^N) \),
- \( u_k \to u_\infty \), \( v_k \to v_\infty \), strongly in \( L^{2^*}(\mathbb{R}^N) \),
- \( u_k \to u_\infty \), \( v_k \to v_\infty \), a.e. in \( \mathbb{R}^N \).

Hence, \( u_\infty \geq 0 \) and \( v_\infty \geq 0 \). Since \( f_{\lambda_k}(u_k, v_k) + h_{\lambda_k}(u_k, v_k) = 0 \), we have that

\[
0 \leq 2^* (-\lambda) \int_{\mathbb{R}^N} |u_k|^{\alpha} |v_k|^{\beta} \leq \mu_1 \int_{\mathbb{R}^N} |u_k|^{2^*} + \mu_2 \int_{\mathbb{R}^N} |v_k|^{2^*} \leq C_0
\]

and, using Fatou’s lemma, we obtain

\[
\int_{\mathbb{R}^N} |u_\infty|^{\alpha} |v_\infty|^{\beta} \leq \liminf_{k \to \infty} \int_{\mathbb{R}^N} |u_k|^{\alpha} |v_k|^{\beta} \leq \frac{C_0}{2^*} \lim_{k \to \infty} \frac{1}{(-\lambda_k)} = 0.
\]
Thus, \( u_\infty v_\infty = 0 \). On the other hand, Proposition 2.1(a) yields
\[
0 < c_0 \leq \int_{\Omega} |\nabla u_k|^2 \leq \mu_1 \int_{\Omega} |u_k|^{2^*},
\]
\[
0 < c_0 \leq \int_{\Omega} |\nabla v_k|^2 \leq \mu_2 \int_{\Omega} |v_k|^{2^*},
\]
Therefore, \( u_\infty \neq 0 \) and \( v_\infty \neq 0 \). Then, as in (5.22), there exist \( s, t \in (0, \infty) \) such that \( su_\infty, -tv_\infty \in M^\Gamma \) and \( su_\infty - tv_\infty \in \mathcal{E}^\Gamma \). So, after passing to a subsequence, we obtain
\[
c_{\infty}^\Gamma \leq \frac{1}{2} \int_{\Omega} (|\nabla (su_\infty)|^2 + |\nabla (tv_\infty)|^2) - \frac{1}{2^2} \int_{\Omega} (\mu_1 |su_\infty|^{2^*} + \mu_2 |tv_\infty|^{2^*})
\]
\[
\leq \frac{1}{2} \liminf_{k \to \infty} \int_{\Omega} (|\nabla (su_k)|^2 + |\nabla (tv_k)|^2) - \frac{1}{2^2} \lim_{k \to \infty} \int_{\Omega} (\mu_1 |su_k|^{2^*} + \mu_2 |tv_k|^{2^*})
\]
\[
\leq \frac{1}{2} \liminf_{k \to \infty} \int_{\Omega} (|\nabla (su_k)|^2 + |\nabla (tv_k)|^2) - \frac{1}{2^2} \lim_{k \to \infty} \int_{\Omega} (\mu_1 |su_k|^{2^*} + \mu_2 |tv_k|^{2^*})
\]
\[
+ \lim_{k \to \infty} (-\lambda_k) \int_{\Omega} |su_k|^\alpha |tv_k|^\beta
\]
\[
\leq \liminf_{k \to \infty} E_{\lambda_k} (su_k, tv_k) \leq \liminf_{k \to \infty} E_{\lambda_k} (u_k, v_k) = \limsup_{k \to \infty} c_{\lambda_k} \leq c_{\infty}^\Gamma,
\]
because \( E_{\lambda_k} (su_k, tv_k) \leq E_{\lambda_k} (u_k, v_k) \); see Proposition 2.1(d). It follows that
\[
\lim_{k \to \infty} (-\lambda_k) \int_{\Omega} |u_k|^\alpha |v_k|^\beta = 0
\]
and that
\[
c_{\infty}^\Gamma = \lim_{k \to \infty} c_{\lambda_k} = \lim_{k \to \infty} E_{\lambda_k} (u_k, v_k) = \liminf_{k \to \infty} E_{\lambda_k} (u_k, v_k)
\]
\[
\leq \limsup_{k \to \infty} E_{\lambda_k} (su_k, tv_k) \leq \lim_{k \to \infty} E_{\lambda_k} (u_k, v_k) = c_{\infty}^\Gamma.
\]
Hence,
\[
\lim_{k \to \infty} \int_{\Omega} (|\nabla (su_k)|^2 + |\nabla (tv_k)|^2) = \int_{\Omega} (|\nabla (su_\infty)|^2 + |\nabla (tv_\infty)|^2)
\]
and, as \( su_k \rightharpoonup su_\infty \) and \( tv_k \rightharpoonup tv_\infty \) weakly in \( D^{1,2}(\Omega) \), we conclude that \( u_k \to u_\infty \) and \( v_k \to v_\infty \) strongly in \( D^{1,2}(\Omega) \). Consequently,
\[
c_{\infty}^\Gamma = \lim_{k \to \infty} E_{\lambda_k} (u_k, v_k)
\]
\[
= \frac{1}{2} \int_{\Omega} (|\nabla u_\infty|^2 + |\nabla v_\infty|^2) - \frac{1}{2} \int_{\Omega} (\mu_1 |u_\infty|^{2^*} + \mu_2 |v_\infty|^{2^*}) = J(u_\infty - v_\infty),
\]
and
\[
0 = \lim_{k \to \infty} f_{\lambda_k} (u_k, v_k) = \int_{\Omega} |\nabla u_\infty|^2 - \mu_1 \int_{\Omega} |u_\infty|^{2^*},
\]
\[
0 = \lim_{k \to \infty} h_{\lambda_k} (u_k, v_k) = \int_{\Omega} |\nabla v_\infty|^2 - \mu_2 \int_{\Omega} |v_\infty|^{2^*}.
\]
This shows that \( u_\infty, v_\infty \in M^\Gamma \), and completes the proof of (a) and (b).

Thus, we have shown that \( u_\infty - v_\infty \) is a minimizer for \( J \) on \( \mathcal{E}^\Gamma \). Since the embedding \( D^{1,2}(\Omega)^\Gamma \hookrightarrow L^{2^*}(\Omega) \) is compact, \( J \) satisfies the Palais-Smale condition on
lemma. We write $B$.

Moreover, \( \langle \pi \rangle \) and \( \{ \pi \} \times \{ 0 \} \) are open, connected and \( \Gamma \)-invariant, and \( \{ \pi \} \times \{ 0 \} \) is a critical point of \( \langle \pi \rangle \).

Next, we prove Proposition 1.3. The proof is based on the following geometric lemma. We write \( \mathbb{B}^k \) and \( \mathbb{S}^{k-1} \) for the open ball and the unit sphere in \( \mathbb{R}^k \).

**Lemma 5.2.** Let \( \Gamma = O(m) \times O(n) \) with \( m + n = N + 1 \), \( m, n \geq 2 \), and let \( U_1 \) and \( U_2 \) be nonempty \( \Gamma \)-invariant open connected subsets of \( \mathbb{S}^N \) such that \( \overline{U_1} \cup \overline{U_2} = \mathbb{S}^N \) and \( U_1 \cap U_2 = \emptyset \). Then, up to relabeling,

1. \( \overline{U}_1 := U_1 \cup (\mathbb{S}^{m-1} \setminus \{ 0 \}) \) and \( \overline{U}_2 := U_2 \cup \{ 0 \} \times \mathbb{S}^{n-1} \) are open, connected and \( \Gamma \)-invariant, and \( \overline{U}_1 \cap \overline{U}_2 = \emptyset \).
2. \( \overline{U}_1 \) is \( \Gamma \)-diffeomorphic to \( \mathbb{S}^{m-1} \times \mathbb{B}^n \) and \( \overline{U}_2 \) is \( \Gamma \)-diffeomorphic to \( \mathbb{B}^m \times \mathbb{S}^{n-1} \).

**Proof.** Consider the function \( \pi : \mathbb{S}^N \rightarrow \mathbb{R}^2 \) given by \( \pi(x, y) := (|x|, |y|) \) where \( x \in \mathbb{R}^m \), \( y \in \mathbb{R}^n \). Then \( \pi \) is continuous and \( \Gamma \)-invariant. Its image is the arc

\[ A := \{(s, t) \in \mathbb{R}^2 : s, t \geq 0, s^2 + t^2 = 1\}. \]

Set \( A_i := \pi(U_i) \). Note that \( A_i \) is nonempty, connected and open in \( A \) for \( i = 1, 2 \). Moreover, \( A_1 \cap A_2 = \emptyset \) and \( \overline{A_1} \cup \overline{A_2} = A \). Therefore, \( A_i \) is an arc and, up to relabeling, \( A_1 := A_1 \cup \{ (1, 0) \} \) and \( A_2 := A_2 \cup \{ (0, 1) \} \) are connected and open in \( A \), and \( A \setminus (\overline{A_1} \cup \overline{A_2}) = \{(s_0, t_0)\} \) with \( s_0 > 0 \) and \( t_0 > 0 \), i.e.,

\[ \overline{A}_1 = \{(s, t) \in A : s > s_0\} \quad \text{and} \quad \overline{A}_2 = \{(s, t) \in A : s < s_0\}. \]
Then, \( \pi^{-1}(\tilde{A}_i) = \tilde{U}_i \) and \( \pi^{-1}(s_0, t_0) = \partial\tilde{U}_1 = \partial\tilde{U}_2 \). Therefore, these sets satisfy (a) and (b).

**Proof of Proposition 1.3.** Adding a point at infinity and applying the inverse of the stereographic projection to \( \Omega_1 \) and \( \Omega_2 \), we obtain two nonempty \( \Gamma \)-invariant open connected subsets \( U_1 \) and \( U_1 \otimes \bar{S}^N \) which, thus, satisfy (a) and (b) of Lemma 5.2.

As \( m + n = N + 1 \) and \( m, n \geq 2 \), the codimension of the sets \( \mathbb{S}^{m-1} \times \{0\} \) and \( \{0\} \times \mathbb{S}^{n-1} \) in \( \mathbb{R}^N \) is at least 2. Therefore, \( D_0^{1,2}(\Omega_i) = D_0^{1,2}(\bar{\Omega}_i) \). Hence, \( u_\infty \) solves the problem

\[
-\Delta u = \mu_1 |u|^{2^* - 2} u, \quad u \in D_0^{1,2}(\bar{\Omega}_1),
\]

and, by the maximum principle, \( u_\infty > 0 \) in \( \bar{\Omega}_1 \). As \( \Omega_1 = \{ x \in \mathbb{R}^N : u_\infty(x) > 0 \} \),

we conclude that \( \Omega_1 = \bar{\Omega}_1 \). Similarly, \( \Omega_2 = \bar{\Omega}_2 \), and the claim is proved.

**APPENDIX A. THE ENERGY FUNCTIONAL ON A PLANE**

Consider the function

\[
e(s, t) := a_1 s^2 + a_2 t^2 - b_1 s^p - b_2 t^p + ds^\alpha t^\beta
\]

in \( V := (0, \infty) \times (0, \infty) \), where \( a_i, b_i, d > 0, p > 2, \alpha, \beta > 1 \) and \( \alpha + \beta = p \). We assume that (1,1) is a critical point of \( e \). Then, as

\[
\frac{\partial e}{\partial s}(s, t) = 2a_1 s - pb_1 s^{p-1} + d\alpha s^{\alpha-1} t^\beta,
\]

\[
\frac{\partial e}{\partial t}(s, t) = 2a_2 t - pb_2 t^{p-1} + d\beta s^\alpha t^{\beta-1},
\]

we have that

\[
(A.1) \quad 2a_1 - pb_1 + d\alpha = 0 \quad \text{and} \quad 2a_2 - pb_2 + d\beta = 0.
\]

**Lemma A.1.** There exist \( 0 < r < R < \infty \) and \( \delta > 0 \) such that

\[
\delta \leq \frac{\partial e}{\partial s}(r, t) \quad \text{and} \quad \frac{\partial e}{\partial s}(R, t) \leq -1 \quad \text{for all} \ t \in [r, R],
\]

\[
\delta \leq \frac{\partial e}{\partial t}(s, r) \quad \text{and} \quad \frac{\partial e}{\partial t}(s, R) \leq -1 \quad \text{for all} \ s \in [r, R],
\]

every critical point of \( e \) in \( V \) lies in the interior of \( Q := [r, R] \times [r, R] \), and \( \sup_V e = \max_Q e \).

**Proof.** Let \( t = \tau s \) with \( \tau \in [0, 1] \). Then, from (A.1) we get that

\[
\frac{\partial e}{\partial s}(s, t) = 2a_1 s - (pb_1 - d\alpha) s^{p-1}
\]

\[
\leq 2a_1 s - (pb_1 - d\alpha) s^{p-1} \leq -1 \quad \text{for all} \ s \in [R_1, \infty).
\]

Similarly, if \( s = \tau t \) with \( \tau \in [0, 1] \), we have that

\[
\frac{\partial e}{\partial t}(s, t) \leq -1 \quad \text{for all} \ t \in [R_2, \infty).
\]
On the other hand, as
\[
\frac{\partial e}{\partial s}(s,t) \geq 2a_1s - pb_1s^{p-1} \quad \text{and} \quad \frac{\partial e}{\partial t}(s,t) \geq 2a_2t - pb_2t^{p-1},
\]
there exist \( r, \delta > 0 \) such that
\[
\frac{\partial e}{\partial s}(s,t) > 0 \text{ if } s \in (0, r] \quad \text{and} \quad \frac{\partial e}{\partial s}(r,t) \geq \delta \text{ for all } t \in (0, \infty),
\]
\[
\frac{\partial e}{\partial t}(s,t) > 0 \text{ if } t \in (0, r] \quad \text{and} \quad \frac{\partial e}{\partial t}(s,r) \geq \delta \text{ for all } s \in (0, \infty).
\]
Setting \( R := \max\{R_1, R_2\} \), we obtain our claim.

\[\square\]

**Lemma A.2.** If every critical point of \( e \) in \( V \) is a strict local maximum, then \((1, 1)\) is the only critical point of \( e \) in \( V \) and it is a global maximum.

**Proof.** Let \( Q \) be as in Lemma A.1. Then, \( Q \) is strictly positively invariant under the (positive) gradient flow of \( e \), so this flow defines a map
\[
\eta : \mathbb{R} \times Q \to Q.
\]

As the critical points of \( e \) in \( V \) are contained in the interior of \( Q \) and they are isolated, there are finitely many of them, \( \xi_1, ..., \xi_m \), and, since each of them is a strict local maximum, we may choose \( \varepsilon > 0 \) such that \( \overline{B}_\varepsilon(\xi_i) := \{ x \in \mathbb{R}^2 : |x - \xi_i| \leq \varepsilon \} \subset Q, \overline{B}_\varepsilon(\xi_i) \cap \overline{B}_\varepsilon(\xi_j) = \emptyset \) if \( i \neq j \), and \( \overline{B}_\varepsilon(\xi_i) \) is strictly positively invariant under the gradient flow of \( e \). Set \( \Theta := \overline{B}_\varepsilon(\xi_1) \cup \cdots \cup \overline{B}_\varepsilon(\xi_m) \). Then, the entrance time function \( T_\Theta : Q \to \mathbb{R} \), defined by
\[
T_\Theta(x) := \inf \{ \tau \geq 0 : \eta(\tau, x) \in \Theta \},
\]
is continuous. Therefore, the map \( \pi : Q \to \Theta \) given by
\[
\pi(x) := \eta(T_\Theta(x), x)
\]
is also continuous and it is surjective. As \( Q \) is connected, \( \Theta \) cannot have more than one component. Therefore, \( e \) has only one critical point and it is a global maximum.

\[\square\]

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References

[1] Ambrosetti, Antonio; Colorado, Eduardo: Standing waves of some coupled nonlinear Schrödinger equations. J. Lond. Math. Soc. (2) 75 (2007), no. 1, 67–82.

[2] Bartsch, Thomas; Dancer, Norman; Wang, Zhi-Qiang: A Liouville theorem, a-priori bounds, and bifurcating branches of positive solutions for a nonlinear elliptic system. Calc. Var. Partial Differential Equations 37 (2010), no. 3-4, 345–361.

[3] Bartsch, Thomas; Wang, Zhi-Qiang; Wei, Jun-cheng: Bound states for a coupled Schrödinger system. J. Fixed Point Theory Appl. 2 (2007), no. 2, 353–367.

[4] Castro, Alfonso; Cossio, Jorge; Neuberger, John M.: A sign-changing solution for a superlinear Dirichlet problem. Rocky Mountain J. Math. 27 (1997), no. 4, 1041–1053.

[5] Chen, Zhijie; Zou, Wenming: Positive least energy solutions and phase separation for coupled Schrödinger equations with critical exponent. Arch. Ration. Mech. Anal. 205 (2012), no. 2, 515–551.

[6] Chen, Zhijie; Zou, Wenming: Positive least energy solutions and phase separation for coupled Schrödinger equations with critical exponent: higher dimensional case. Calc. Var. Partial Differential Equations 52 (2015), no. 1-2, 423–467.

[7] Dancer, E. N.; Wei, Jun-cheng; Weth, Tobias: A priori bounds versus multiple existence of positive solutions for a nonlinear Schrödinger system. Ann. Inst. H. Poincaré Anal. Non Linéaire 27 (2010), no. 3, 953–969.

[8] Ding, Wei Yue: On a conformally invariant elliptic equation on $\mathbb{R}^n$. Comm. Math. Phys. 107 (1986), no. 2, 331–335.

[9] Esry, B.D.; Greene, Chris H.; Burke Jr., James P.; Bohn, John L.: Hartree-Fock theory for double condensates. Phys. Rev. Lett. 78 (1997), 3594–3597.

[10] Gladiali, F.; Grossi, M.; Troestler, C.: A non-variational system involving the critical Sobolev exponent. The radial case. J. Anal. Math. to appear. arXiv:1603.05641

[11] Gladiali, F.; Grossi, M.; Troestler, C.: Entire radial and nonradial solutions for systems with critical growth. Preprint arXiv:1612.03510.

[12] Guo, Yuxia; Li, Bo; Wei, Jun-cheng: Entire nonradial solutions for non-cooperative coupled elliptic system with critical exponents in $\mathbb{R}^3$. J. Differential Equations 256 (2014), no. 10, 34633495.

[13] Hebey, Emmanuel: Introduction à l’analyse non linéaire sur les variétés. Diderot, Paris, 1997.

[14] Hebey, Emmanuel; Vaugon, Michel: Sobolev spaces in the presence of symmetries. J. Math. Pures Appl. (9) 76 (1997), no. 10, 859–881.

[15] Lin, Tai-Chia; Wei, Jun-cheng: Ground state of $N$ coupled nonlinear Schrödinger equations in $\mathbb{R}^n$, $n \leq 3$. Comm. Math. Phys. 255 (2005), no. 3, 629–653.

[16] Liu, Jia-quan; Liu, Xian-qi; Wang, Zhi-Qiang: Sign-changing solutions for coupled nonlinear Schrödinger equations with critical growth. J. Differential Equations 261 (2016), no. 12, 7194–7236.

[17] Liu, Zhaoli; Wang, Zhi-Qiang: Multiple bound states of nonlinear Schrödinger systems. Comm. Math. Phys. 282 (2008), no. 3, 721–731.

[18] Maia, L. A.; Montefusco, E.; Pellacci, B.: Positive solutions for a weakly coupled nonlinear Schrödinger system. J. Differential Equations 229 (2006), no. 2, 743–767.

[19] Peng, Shuangjie; Peng, Yan-fang; Wang, Zhi-Qiang: On elliptic systems with Sobolev critical growth. Calc. Var. Partial Differential Equations 55 (2016), no. 6, Paper No. 142, 30 pp.

[20] Pistoia, Angela; Soave, Nicola: On Coron’s problem for weakly coupled elliptic systems. arXiv:1610.07762.

[21] Pistoia, Angela; Tavares, Hugo: Spiked solutions for Schrödinger systems with Sobolev critical exponent: the cases of competitive and weakly cooperative interactions. J. Fixed Point Theory Appl. 19 (2017), no. 1, 407–446.

[22] Sirakov, Boyan: Least energy solitary waves for a system of nonlinear Schrödinger equations in $\mathbb{R}^n$. Comm. Math. Phys. 271 (2007), no. 1, 199–221.

[23] Soave, Nicola: On existence and phase separation of solitary waves for nonlinear Schrödinger systems modelling simultaneous cooperation and competition. Calc. Var. Partial Differential Equations 53 (2015), no. 3-4, 689–718.
[25] Soave, Nicola; Tavares, Hugo: New existence and symmetry results for least energy positive solutions of Schrödinger systems with mixed competition and cooperation terms. J. Differential Equations 261 (2016), no. 1, 505–537.

[26] Struwe, Michael Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems. Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete 34. Springer-Verlag, Berlin, 1996.

[27] Szulkin, Andrzej: Ljusternik-Schnirelmann theory on $C^1$-manifolds. Ann. Inst. H. Poincaré Anal. Non Linéaire 5 (1988), no. 2, 119–139.

[28] Timmermans, E.: Phase separation of Bose-Einstein condensates. Phys. Rev. Lett. 81 (1998), 5718–5721.

[29] Wei, Juncheng; Weth, Tobias: Radial solutions and phase separation in a system of two coupled Schrödinger equations. Arch. Ration. Mech. Anal. 190 (2008), no. 1, 83–106.

[30] Willem, Michel: Minimax theorems. Progress in Nonlinear Differential Equations and their Applications, 24. Birkhäuser Boston, Inc., Boston, MA, 1996.

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