Large equilateral sets in subspaces of $\ell_n^\infty$ of small codimension

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Abstract

For fixed $k$ we prove exponential lower bounds on the equilateral number of subspaces of $\ell_n^\infty$ of codimension $k$. In particular, we show that if the unit ball of a normed space of dimension $n$ is a centrally symmetric polytope with at most $\frac{4n}{k} - o(n)$ pairs of facets, then it has an equilateral set of cardinality at least $n + 1$. These include subspaces of codimension 2 of $\ell_n^{4+2}$ for $n \geq 9$ and of codimension 3 of $\ell_n^{5+3}$ for $n \geq 15$.

1 Introduction

Let $(X, \| \cdot \|)$ be a normed space. A set $S \subseteq X$ is called $c$-equilateral if $\| x - y \| = c$ for all distinct $x, y \in S$. $S$ is called equilateral if it is $c$-equilateral for some $c > 0$. The equilateral number $e(X)$ of $X$ is the cardinality of the largest equilateral set of $X$. Petty [Pet71] made the following conjecture regarding lower bounds on $e(X)$.

Conjecture 1 (Petty [Pet71]). For all normed spaces $X$ of dimension $n$, $e(X) \geq n + 1$.

Petty [Pet71] proved Conjecture 1 for $n = 3$, and Makeev [Mak05] for $n = 4$. For $n \geq 5$ the conjecture is still open, except for some special classes of norms. The best general lower bound is $e(X) \geq \exp(\Omega(\sqrt{\log n}))$, proved by Swanepoel and Villa [SV08]. Regarding upper bounds on the equilateral number, a classical result of Petty [Pet71] and Soltan [Sol75] shows that $e(X) \leq 2^n$ for any $X$ of dimension $n$, with equality if and only if the unit ball of $X$ is an affine image of the $n$-dimensional cube. For more background on the equilateral number see Section 3 of the survey [Swa18].

Kobos studied subspaces of $\ell_n^\infty$ of codimension 1, and proved the lower bound $e(X) \geq 2^{\frac{n}{2}}$, which in particular implies Conjecture 1 for these spaces for $n \geq 6$.

In the same paper he proposed as a problem to prove Petty’s conjecture for subspaces of $\ell_n^\infty$ of codimension 2. In Theorem 1 we prove exponential lower bounds on the equilateral number of subspaces of $\ell_n^\infty$ of codimension $k$. This, in particular, solves Kobos’ problem if $n \geq 9$.

Theorem 1. Let $X$ be a $(n - k)$-dimensional subspace of $\ell_n^\infty$. Then

$$e(X) \geq \frac{2^{n-k}}{(n-k)^k},$$

(1)

$$e(X) \geq 1 + \frac{1}{2^{k-1}} \sum_{r=1}^{\ell} \binom{n - kr}{r} \text{ for every } 1 \leq \ell \leq n/(k+1), \text{ and}$$

(2)

$$e(X) \geq 1 + \sum_{r=1}^{\ell} \binom{n - 2rk}{r} \text{ for every } 1 \leq \ell \leq n/(2k+1).$$

(3)

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Note that none of the three bounds follows from the other two in Theorem 1, hence none of them is redundant. Comparing (1) and (3), for fixed $k$ we have $\max_i \sum_{1 \leq i \leq \ell} \left(\frac{n-2k}{r}\right)^i = O(2^k n)$ for some $0 < c_k < 1$, while $\frac{2^n}{(n-k)^r} = 2^{n-k-\log(n-k)} = 2^{n-o(n)}$. On the other hand, when we let $k$ vary, it can be as large as $\Omega(n)$ in (3) to still give a non-trivial estimate, while $k$ can only be chosen up to $O(n/\log n)$ for (1) to be non-trivial. Finally, (2) is beaten by (1) and (3) in most cases, however for $k = 2, 3$, and for small values of $n$ (2) gives the best bound.

For any two $n$-dimensional normed spaces $X, Y$ we denote by $d_{BM}(X,Y) = \inf T\{\|T\|\|T^{-1}\|\}$ their Banach-Mazur distance, where the infimum is over all linear isomorphisms $T : X \to Y$. The metric space of isometry classes of normed spaces endowed with the logarithm of the Banach-Mazur distance is the Banach-Mazur compactum. It is not hard to see that $e(X)$ is upper semi-continuous on the Banach-Mazur compactum. This, together with the fact that any convex polytope can be obtained as a section of a cube of sufficiently large dimension (see for example Page 72 of Grünbaum’s book [Gru03]) implies that it would be sufficient to prove Conjecture 1 for $k$-codimensional subspaces of $c_{\infty}$ for all $1 \leq k \leq n - 4$ and $n \geq 5$. (This was also pointed out in [Kob14].) Unfortunately, our bounds are only non-trivial if $n$ is sufficiently large compared to $k$. However, we deduce an interesting corollary.

**Corollary 1.** Let $P$ be an origin-symmetric convex polytope in $\mathbb{R}^d$ with at most $\frac{4d}{3} - \frac{1 + \sqrt{8d+9}}{6} = \frac{4d}{3} - o(d)$ opposite pairs of facets. If $X$ is a $d$-dimensional normed space with $P$ as a unit ball, then $e(X) \geq d + 1$.

There have been some extensions of lower bounds obtained on the equilateral number of certain normed spaces to other norms that are close to them according to the Banach-Mazur distance. These results are based on using the Brouwer Fixed-Point Theorem, first applied in this context by Brass [Bra99] and Dekster [Dek00]. We prove the following.

**Theorem 2.** Let $X$ be an $(n-k)$-dimensional subspace of $c_{\infty}$, and $Y$ be an $(n-k)$-dimensional normed space such that $d_{BM}(X,Y) \leq 1 + \frac{\ell}{2(n-2k-\ell k-1)}$ for some integer $1 \leq \ell \leq \frac{n-2k}{k}$. Then $e(Y) \geq n - k(2 + \ell)$.

## 2 Norms with polytopal unit ball and small codimension

We recall the following well known fact to prove Corollary 1. (For a proof, see for example [Bal97].)

**Lemma 1.** Any centrally symmetric convex $d$-polytope with $f \geq d$ opposite pairs of facets is a $d$-dimensional section of the $f$-dimensional cube.

**Proof of Corollary 1** By Lemma 1 $P$ can be obtained as an $d$-dimensional section of the $(\frac{4d}{3} - \frac{1 + \sqrt{8d+9}}{6})$-dimensional cube. Choose $n = \frac{4d}{3} - \frac{1 + \sqrt{8d+9}}{6}$, $\ell = 2$ and $k = \frac{d}{3} - \frac{d + \sqrt{8d+9}}{6}$, and apply inequality (3) from Theorem 1. This yields $e(X) \geq d + 1$. □

To confirm Petty’s conjecture for subspaces of $c_{\infty}$ of codimension 2 and 3 when $n \geq 9$ and respectively $n \geq 15$, apply inequality (2) from Theorem 1 with $\ell = 2$.

## 3 Large equilateral sets

**Notation**

We denote vectors by bold lowercase letters, and the $i$-th coordinate of a vector $a \in \mathbb{R}^n$ by $a^i$. We treat vectors by default as column vectors. By subspace we mean linear subspace. We
write \( \text{span}(\mathbf{a}_1, \ldots, \mathbf{a}_k) \) for the subspace spanned by \( \mathbf{a}_1, \ldots, \mathbf{a}_k \in \mathbb{R}^n \). For a subspace \( X \subseteq \mathbb{R}^n \) we denote by \( X^\perp \) the orthogonal complement of \( X \). We denote by \([n]\) the set \( \{1, \ldots, n\} \), by \( 2^n \) the set of all subsets of \([n]\), by \( \binom{[n]}{m} \) the set of all subsets of \( S \) of cardinality \( m \), and by \( \binom{S}{m} \) the set of all non-empty subsets of \( S \) of cardinality at most \( m \). Further, for \( j \in \mathbb{R} \) and \( S \subseteq \mathbb{R} \) let \( j + S = \{ j + s : s \in S \} \). \( \mathbf{0} \) denotes the vector \((0, \ldots, 0) \in \mathbb{R}^n \). For two vectors \( \mathbf{a} \) and \( \mathbf{b} \), let \( \mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^n a_i b_i \) be their scalar product.

**Idea of the constructions**

For two vectors \( \mathbf{x}, \mathbf{y} \in X \) we have \( \| \mathbf{x} - \mathbf{y} \|_\infty = c \) if and only if the following hold.

There is an \( 1 \leq i \leq n \) such that \( |x^i - y^i| = c \), and

\[
|x^i - y^i| \leq c \text{ for all } 1 \leq i \leq n.
\] 

In our constructions of \( c \)-equilateral sets \( S \subseteq X \), we split the index set \([n]\) of the coordinates into two parts \([n] = N_1 \cup N_2 \). In the first part \( N_1 \), we choose all the coordinates from the set \( \{0, 1, -1\} \), so that for each pair from \( S \) there will be an index in \( N_1 \) for which \((5)\) holds, and \((5)\) is not violated by any index in \( N_1 \). We use \( N_2 \) to ensure that all of the points we choose are indeed in the subspace \( X \). For each vector, this will lead to a system of linear equations. The main difficulty will be to choose the values of the coordinates in \( N_1 \) so that the coordinates in \( N_2 \), obtained as a solution to those systems of linear equations, do not violate \((5)\).

**Proof of Theorem 1**

For vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^k \) let \( B(\mathbf{v}_1, \ldots, \mathbf{v}_k) \in \mathbb{R}^{k \times k} \) be the matrix whose \( i \)-th column is \( \mathbf{v}_i \). For a matrix \( B \in \mathbb{R}^{k \times k} \), a vector \( \mathbf{v} \in \mathbb{R}^k \) and an index \( i \in [k] \), we denote by \( B(i, \mathbf{v}) \) the matrix obtained from \( B \) by replacing its \( i \)-th column by \( \mathbf{v} \).

Let \( \{ \mathbf{a}_i : 1 \leq i \leq k \} \) be a set of \( k \) linearly independent vectors in \( \mathbb{R}^n \) spanning \( X^\perp \). That is, \( \mathbf{x} \in X \) if and only if \( \mathbf{a}_i \cdot \mathbf{x} = 0 \) for all \( 1 \leq i \leq k \). Further, let \( A \in \mathbb{R}^{k \times n} \) be the matrix whose \( i \)-th row is \( \mathbf{a}_i^T \), and let \( \mathbf{b}_j = (\mathbf{a}_{i_1}^T, \ldots, \mathbf{a}_{i_k}^T) \) be the \( j \)-th column of \( A \). For \( I \subseteq [n] \) and for \( \sigma \in \{\pm 1\}^n \) let \( \mathbf{b}_I = \sum_{i \in I} \mathbf{b}_i \) and \( \mathbf{b}_{I, \sigma} = \sum_{i \in I} \sigma_i \mathbf{b}_i \).

**Proof of (1).** We construct a 2-equilateral set of size \( \frac{\varphi(n-k)}{(n-k)} \). Let \( B = B(\mathbf{b}_{n-k+1}, \mathbf{b}_{n-k+2}, \ldots, \mathbf{b}_n) \). We may assume without loss of generality that \( |\det B| \geq |\det B(\mathbf{b}_{i_1}, \ldots, \mathbf{b}_{i_k})| \) for all possible choices of \( i_1, \ldots, i_k \in [n] \). The vectors \( \{ \mathbf{a}_i : i \in [k] \} \) are linearly independent, hence \( \det B \neq 0 \). The first part of the indices \( (N_1) \) now will be \([n - k]\), and for these indices we choose coordinates from the set \( \{0, 1, -1\} \). For \( J \subseteq [n - k] \) we define the first \( n - k \) coordinates of the vector \( \mathbf{w}(J) \in \mathbb{R}^n \) as

\[
\mathbf{w}(J)^i = \begin{cases} 
1 & \text{if } i \in J \\
-1 & \text{if } i \in [n - k] \setminus J.
\end{cases}
\]

To ensure that \( \mathbf{w}(J) \in X \) we must have \( A \mathbf{w}(J) = \mathbf{0} \). This means \( (\mathbf{w}(J)^{n-k+1}, \ldots, \mathbf{w}(J)^n) \) is a solution of

\[
B \mathbf{x} = \mathbf{b}_{[n-k] \setminus J} - \mathbf{b}_J.
\] 

By Cramer’s rule \( \mathbf{x} = (x^1, \ldots, x^k) \) with

\[
x^i = \frac{\det B(i, \mathbf{b}_{[n-k] \setminus J} - \mathbf{b}_J)}{\det B}
\]
is a solution of \([\mathfrak{R}]\). Thus we obtain that \(w(J)\), defined by

\[
w(J)^i = \begin{cases} 
1 & \text{if } i \in J \\
-1 & \text{if } i \in [n-k] \setminus J \\
\frac{\det B(i-n+k,b_{[n-k]\setminus J}+b_J)}{\det B} & \text{if } i \in [n] \setminus [n-k],
\end{cases}
\]

is in \(X\). By the multilinearity of the determinant we have

\[
\det B(i-n+k, b_{[n-k]\setminus J} - b_J) = \sum_{j \in [n-k]\setminus J} \det B(i-n+k, b_j) - \sum_{j \in J} \det B(i-n+k, b_j).
\]

Thus by the maximality of \(|\det B|\) and by the triangle inequality:

\[
|\det B(i-n+k, b_{[n-k]\setminus J} - b_J)| \leq (n-k)|\det B|.
\]

This implies that for each \(J\) and \(i \in [n] \setminus [n-k]\) we have \(-(n-k) \leq w(J)^i \leq n-k\).

Consider the set \(W = \{w(J) : J \in 2^{[n-k]}\}\). \(W\) is not 2-equilateral, because for \(J_1, J_2 \in 2^{[n-k]}\) and for \(i \in [n] \setminus [n-k]\) we only have that \(|w(J_1)^i - w(J_2)^i| \leq 2(n-k)\). However we can find a 2-equilateral subset of \(W\) that has large cardinality, as follows.

First we split \(W\) into \(n-k\) parts such that if \(w(J_1)\) and \(w(J_2)\) are in the same part, then \(|w(J_1)^{n-k+1} - w(J_2)^{n-k+1}| \leq 2\), and keep the largest part. Then we split the part we kept into two parts again similarly, based on \(w(J)^{n-k+2}\), and keep the largest part. We continue in the same manner for \(w(J)^{n-k+3}, \ldots, w(J)^{n}\).

More formally, for each vector \(s \in \{-n-k, -n-k, \ldots, n-k - 2\}^k = T^k\) let \(W(s)\) be the set of those vectors \(w(J)\) for which

\[
w(J)^{n-k+i} \in [s^i, s^i+2] \text{ for every } i \in k.
\]

We have \(W \subseteq \bigcup_{s \in T^k} W(s)\), and hence there is an \(s\) for which \(|W(s)| \geq 2^{n-k}/(n-k)^m\).

It is not hard to check that \(W(s)\) is 2-equilateral. Indeed, for every \(J_1, J_2 \in W(s)\), we have \(|w(J_1)^i - w(J_2)^i| \leq 2\) for \(i \in [n] \setminus [n-k]\) by the definition of \(W(s)\), and for \(i \in [n-k]\) by the definition of \(w(J)\). Further, by the definition of \(w(J)\) there is an index \(j \in [n-k]\) for which \(\{w(J_1)^i, w(J_2)^i\} = \{1, -1\}\) (assuming \(J_1 \neq J_2\)).

**Proof of (2).** Fix some \(1 \leq \ell \leq n/(k+1)\). We will construct a 1-equilateral set of cardinality \(\frac{1}{2^{\ell}} \sum_{i \leq \ell} \binom{n-k}{r} + 1\). Let \(I_1, \ldots, I_k \subseteq \binom{[n]}{\ell}\) and \(\sigma \in \{\pm 1\}^n\) be such that the determinant of \(B = B(b_{I_1,\sigma}, \ldots, b_{I_k,\sigma})\) is maximal among all possible choices of \(k\) disjoint \(I_1, \ldots, I_k(\binom{[n]}{\ell})\) and \(\sigma \in \{\pm 1\}^n\). Note that \(\det B > 0\) since the vectors \(a_1, \ldots, a_k\) are linearly independent. Let \(I = \bigcup_{i \in [k]} I_i\) and \(|I| = m\). By re-ordering the coordinates, we may assume that \(I = [n] \setminus [n-m]\).

The first part of the indices now will be \([n-m]\), and for these indices we choose all the coordinates from the set \([-1, 0, 1]\). For a set \(J \in \binom{[n-m]}{\leq \ell}\) we define the first \(n-m\) coordinates of the vector \(w(J) \in \mathbb{R}^n\) as

\[
w(J)^i = \begin{cases} 
-\sigma^i & \text{if } i \in J \\
0 & \text{if } i \in [n-m] \setminus J.
\end{cases}
\]

To ensure that \(w(J) \in X\) we must have \(Aw(J) = 0\). This means that \((w(J)^{n-m+1}, \ldots, w(J)^n)\) has to be a solution of

\[
B(b_{n-m+1}, b_{n-m+2}, \ldots, b_n)x = b_{J,\sigma}.
\]
We will find a solution of (17) of a specific form, where for each \( j \in [k] \), if \( i_1, i_2 \in I_j \), then \( \sigma^{i_1}x^{i_1} = \sigma^{i_2}x^{i_2} \). For this, let \( y = (y^1, y^2, \ldots, y^k) \) be a solution of

\[ By = b_{J, \sigma}, \]

and for each \( j \in [k] \) and \( i \in I_j \) let \( x^i = \sigma^i y^i \). Then \( (x^{n-m+1}, \ldots, x^n) \) is a solution of (17), and by Cramer’s rule we have \( y^i = \frac{\det B(i, b_{J, \sigma})}{\det B} \). Thus we obtained that \( w(J) \), defined as

\[ w(J)^i = \begin{cases} -\sigma^i & \text{if } i \in J \\ 0 & \text{if } i \in [n-m] \setminus J \\ \frac{\sigma^i \det B(i, b_{J, \sigma})}{\det B} & \text{if } i \in I_j \text{ for some } j \in [k], \end{cases} \]

is in \( X \). Note that \( B(j, b_{J, \sigma}) = B(b_{J_1, \sigma}, \ldots, b_{J_k, \sigma}) \) for some disjoint sets \( J_1, \ldots, J_k \), hence by the maximality of \( \det B \) we have

\[ |w(J)^i| \leq 1 \text{ for each } 1 \leq i \leq n. \tag{8} \]

Consider the set \( W = \{ w(J) : J \in \binom{[n-m]}{\leq \ell} \} \). \( W \) is not a 1-equilateral set, because for \( J_1, J_2 \in \binom{[n-m]}{\leq \ell} \) and for some \( i_1 \in I_1 \cup \cdots \cup I_{k-1} \) we only know that \( w(J_1)^i, w(J_2)^i \in [-1, 1] \), and thus \( |w(J_1)^i - w(J_2)^i| \leq 2 \). However we can find a 1-equilateral subset of \( W \) that has large cardinality.

First note that we may assume that for any \( j \in [n-m] \) we have \( \det B(k, \sigma^j b_j) \geq 0 \). Indeed, we can ensure this by changing the first \( n - m \) coordinates of \( \sigma \) if necessary. This we may do, since in the definition of \( B \) we only used the last \( m \) coordinates of \( \sigma \). Together with the multilinearity of the determinant, this implies that for \( i \in I_k \) we have

\[ \sigma^i w(J)^i = \frac{\det B(k, b_{J, \sigma})}{\det B} = \frac{\det B(k, \sum_{j \in J} \sigma^j b_j)}{\det B} = \sum_{j \in J} \det B(k, \sigma^j b_j) \geq 0. \tag{9} \]

Next we split \( W \) into two parts such that if \( w(J_1) \) and \( w(J_2) \) are in the same part, then for \( i \in I_1 \), \( w(J_1)^i \) and \( w(J_2)^i \) have the same sign, and we keep the largest part. Then we split that part into two parts again similarly, based on \( I_2 \), and keep the largest part. We continue in the same manner for \( I_3, \ldots, I_{k-1} \).

More formally, for each vector \( s \in \{ \pm 1 \}^{k-1} \) let \( W(s) \subseteq W \) be the set of those vectors \( w(J) \in W \) for which

\[ s^j w(J)^i \sigma^i \geq 0 \text{ for each } i \in I_1 \cup \cdots \cup I_{k-1}, \text{ where } j \in [k-1] \text{ is such that } i \in I_j. \]

Then \( \bigcup_{s \in \{ \pm 1 \}^{k-1}} W(s) \) is a partition of \( W \), hence there is an \( s \) for which \( |W(s)| \geq \frac{1}{2^{k-1}} |W| = \frac{1}{2^{k-1}} \sum_{1 \leq r \leq \ell} \binom{n-m}{r} \geq \frac{1}{2^{k-1}} \sum_{1 \leq r \leq \ell} \binom{n-k}{r} \). \( W(s) \) is a 1-equilateral set, because for any two vectors \( w_1, w_2 \in W(s) \), there is an index \( i \in [n-m] \) for which either \( \{w_1^i, w_2^i\} = \{0, -1\} \) or \( \{w_1^i, w_2^i\} = \{0, 1\} \), and for all \( i \in [n] \) we have \( |w_1^i - w_2^i| \leq 1 \) by (8), by the definition of \( W(s) \) and by (9). Finally, it is not hard to see that we can add \( 0 \) to \( W(s) \). Thus \( W(s) \cup \{0\} \) is a 1-equilateral set of the promised cardinality. \( \square \)

\[ \text{This is the only reason why we took the maximum also over } \sigma \text{ at the beginning of the proof.} \]
Assume now that

We will handle the case when this assumption does not hold at the end of the proof.

The second part have absolute value at most

coordinates corresponding to the first part are from the set \( \ell \)

By working from 2\( \ell \) down to 1, we may assume without loss of generality that for 1 \( \leq i \leq 2\ell \)

| det \( B_i \) | \( \geq | \text{det} \( B_i(j,b_r) \) | \) for all \( j \in [k] \) and \( r \leq N + (i - 1)k \).

(10)

Assume now that

| det \( B_i \) | > 0 for all 1 \( \leq i \leq 2\ell \).

(11)

We will handle the case when this assumption does not hold at the end of the proof.

The first part of the indices now will be \([N]\). We will obtain vectors (denoted by \( y(J) \)) whose coordinates corresponding to the first part are from the set \([0, -1]\), and whose coordinates from the second part have absolute value at most \( \frac{1}{2} \). We do not construct them directly, but as the sum of some other vectors \( w(J,i), z(J,i) \in X \), whose coordinates in the first part are from \([0, -\frac{1}{2}]\).

For a set \( J = \{ j_1, \ldots, j_{|J|} \} \in \binom{[N]}{\frac{|J|}{2}} \) with \( j_1 < \cdots < j_{|J|} \), and for 1 \( \leq i \leq |J| \) let us define the first \( N \) coordinates of \( w(J,i) \in \mathbb{R}^n \) and \( z(J,i) \in \mathbb{R}^n \) as

\[
w(J,i)_j = z(J,i)_j = \begin{cases} -\frac{1}{2} & \text{if } j = j_i \\ 0 & \text{if } j \in [N] \setminus \{ j_i \}. \end{cases}
\]

To ensure that \( w(J,i) \) and \( z(J,i) \) are in \( X \), we must have \( A w(J,i) = A z(J,i) = 0 \). Hence both \( (w(J,i)^{N+1}, w(J,i)^{N+2}, \ldots, w(J,i)^n) \) and \( (z(J,i)^{N+1}, z(J,i)^{N+2}, \ldots, z(J,i)^n) \) are solutions of

\[
Bx = \frac{1}{2} b_{j_i},
\]

(12)

where \( B = (b_{N+1}, b_{N+2}, \ldots, b_n) \).

By Cramer’s rule we have that \( x = (x^1, x^2, \ldots, x^{2k\ell}) \) with

\[
x_j = \begin{cases} 0 & \text{if } j \in [2k\ell] \setminus U_{2i} \\ \frac{\text{det} \left( B_{2i}^{-1}(j-(2i-1)k, b_{j_i}) \right)}{\text{det} \left( B_{2i}^{-1} \right)} & \text{if } j \in U_{2i} \end{cases}
\]

is a solution of (12).

We obtain that \( w(J,i) \) defined as

\[
w(J,i)_j = \begin{cases} -\frac{1}{2} & \text{if } j = j_i \\ 0 & \text{if } j \in [n] \setminus (\{ j_i \} \cup (N + U_{2i})) \\ \frac{\text{det} \left( B_{2i}^{-1}(j-N-(2i-1)k, b_{j_i}) \right)}{\text{det} \left( B_{2i}^{-1} \right)} & \text{if } j \in N + U_{2i} \end{cases}
\]

is in \( X \).

Similarly, by Cramer’s rule we have that \( x = (x^1, x^2, \ldots, x^{2k\ell}) \) with

\[
x_j = \begin{cases} 0 & \text{if } j \in [2k\ell] \setminus U_{2i-1} \\ \frac{\text{det} \left( B_{2i-1}^{-1}(j-(2i-2)k, b_{j_i}) \right)}{\text{det} \left( B_{2i-1}^{-1} \right)} & \text{if } j \in U_{2i-1} \end{cases}
\]
is a solution of \( \{12\} \).

We obtain that \( z(J, i) \) defined as

\[
z(J, i)^j = \begin{cases} -\frac{1}{2} & \text{if } j = j_i \\ 0 & \text{if } j \in [n] \setminus \{j_i\} \cup (N + U_{2i-1}) \\ \frac{\det B_{2i-1}(j - N - (2i - 2)k, \frac{1}{2}b_{j_i})}{\det B_{2i-1}} & \text{if } j \in N + U_{2i-1} \end{cases}
\]

is in \( X \).

Therefore \( y(J) = \sum_{1 \leq i \leq |J|} (w(J, i) + z(J, i)) \in X \). Note that by assumption \( \{10\} \) and by the multilinearity of the determinant we have \( |w(J, i)^j|, |z(J, i)^j| \leq \frac{1}{2} \) for all \( 1 \leq j \leq n \). It is not hard to check that by the construction we have

\[
y(J)^i = -1 \quad \text{if } i \in J, \\
y(J)^i = 0 \quad \text{if } i \in [N] \setminus J, \\
|y(J)^i| \leq \frac{1}{2} \quad \text{if } i \in [n] \setminus [N].
\]

Thus, for any two distinct \( J_1, J_2 \in \binom{[N]}{2} \), there is an \( i \in [N] \) for which \( \{y(J_1)^i, y(J_2)^i\} = \{0, -1\} \), and for all \( 1 \leq i \leq n \) we have \( |y(J_1)^i - y(J_2)^i| \leq 1 \). This means \( ||y(J_1) - y(J_2)||_\infty = 1 \), and \( \{y(J) : J \in \binom{[N]}{2} \} \cup \{0\} \) is a \( 1 \)-equilateral set of cardinality \( \sum_{1 \leq i \leq \ell} \binom{N}{i} + 1 \).

To finish the proof it is only left to handle the case when assumption \( \{11\} \) does not hold. For \( S = \{s_1, \ldots, s_r\} \subseteq [n] \) with \( s_1 < \cdots < s_r \) and \( T = \{t_1, \ldots, t_m\} \subseteq [k] \) with \( t_1 < \cdots < t_m \) let

\[
B(S, T) = \begin{pmatrix} b_{s_1}^{t_1} & \cdots & b_{s_r}^{t_1} \\ \vdots & \ddots & \vdots \\
 b_{s_1}^{t_m} & \cdots & b_{s_r}^{t_m} \end{pmatrix} \in \mathbb{R}^{r \times m}.
\]

We recursively define \( m_i \in \mathbb{N}, B_i \in \mathbb{R}^{m_i \times m_i} \) for \( i \in [2\ell] \cup \{0\}, \) and \( M_i \in \mathbb{N} \) for \( i \in [2\ell] \) as follows. Let \( m_1 = k, M_0 = 0, M_1 = m_1 \) and \( B_1 = B([n] \setminus [n - m_1], [k]) \). By changing the order of \( A \), we may assume that

\[ |\det B_1| \geq |\det B(S, [k])| \text{ for all } S \in \binom{[n]}{m_1}, \tag{13} \]

Assume now that we have already defined \( m_{i-1}, M_{i-1} \) and \( B_{i-1} \). If \( m_{i-1} > 0 \), then let \( m_i = \text{rank } B([n - M_{i-1}, [k]]) \), otherwise let \( m_i = 0 \). If \( m_i > 0 \), then let \( S_i \subseteq \binom{[k]}{m_i} \) such that \( \text{rank } B([n - M_{i-1}], S_i) = m_i \), and let \( B_i = B([n - M_{i-1}] \setminus [n - M_i], S_i) \). Further, let \( M_i = M_{i-1} + m_i = \sum_{r \leq i} m_r \). If \( m_i > 0 \), then again, by re-indexing the first \( n - M_{i-1} \) columns of \( A \), we may assume that

\[ |\det B_i| \geq |\det B(S, S_i)| \text{ for all } S \subseteq \binom{n - M_{i-1}}{m_i}, \tag{14} \]

Finally define \( b_j(i) = B(\{j\}, S_i) \in \mathbb{R}^{m_i} \).

We now redefine \( U_i \) as

\[ U_i = [n - M_{i-1}] \setminus [n - M_i], \]

and redefine \( w(J, i) \) and \( z(J, i) \) as

\[
w(J, i)^j = \begin{cases} -\frac{1}{2} & \text{if } j = j_i \\ 0 & \text{if } j \in [n] \setminus \{j_i\} \cup U_{2i} \\ \frac{\det B_{2i}(j - n + M_{2i}, \frac{1}{2}b_{j_i}(2i))}{\det B_{2i}} & \text{if } j \in U_{2i} \end{cases}
\]
and
\[ z(J, i)^j = \begin{cases} -\frac{1}{2} & \text{if } j = j_i \\ 0 & \text{if } j \in [n] \setminus \{j_i\} \cup U_{2i-1} \\ \frac{\det B_{2i-1}(j-n+M_{2i-1})}{\det B_{2i-1}} & \text{if } j \in U_{2i-1}. \end{cases} \]

Note that if \( m_{2i} = 0 \) (\( m_{2i-1} = 0 \)), then \( w(J, i)^j = 0 \) (\( z(J, i)^j = 0 \)) for every \( j \neq j_i \), since \( U_{2i} = \emptyset \) (\( U_{2i-1} = \emptyset \)). Further, \( m_i = \text{rank } B([n - M_{i-1}], [k]) = \text{rank } B([n - M_{i-1}] \setminus [n - M_i], S_i) \) implies that \( \text{span} \{a_j^1, \ldots, a_j^{n-M_i} : j \in [k]\} = \text{span} \{a_j^1, \ldots, a_j^{n-M_i} : j \in S_i\} \). This means that if \( v \in \mathbb{R}^n \) is a vector for which \( v^j = 0 \) if \( j > n - M_{i-1} \), then \( v \cdot a_j = 0 \) for all \( j \in S_i \) implies \( v \in X \).

Therefore \( w(J, i), z(J, i) \in X \) for all \( i, J \), thus \( \psi(J) = \sum_{1 \leq i \leq |J|} (w(J, i) + z(J, i)) \in X \). By (13) and (14), and by the multilinearity of the determinant we have \( |w(J, i)^j|, |z(J, i)^j| \leq \frac{1}{2} \) for all \( 1 \leq j \leq n \). The argument that was used under assumption (11) now gives that \( \{\psi(J) : J \in ([n-2k])_1 \cup \{0\} \} \) is 1-equilateral of cardinality \( \sum_{1 \leq r \leq \ell} (\frac{n}{r}) + 1 = \sum_{1 \leq r \leq \ell} (\frac{n}{r}) + 1. \)

4 Equilateral sets in normed spaces close to subspaces of \( \ell^n_\infty \)

The construction we use is similar to the one from [SV08]. Let us fix \( 1 \leq \ell \leq \frac{n-2\ell}{k} \), and let \( N = n - k(2 + \ell) \), and \( c = \frac{\ell}{(n - 2k) - \ell} > 0 \). We assume that the linear structure of \( Y \) is identified with the linear structure of \( X \), and the norm \( \|\cdot\|_Y \) of \( Y \) satisfies
\[
\|x\|_Y \leq \|x\|_\infty \leq (1 + c)\|x\|_Y
\]
for each \( x \in X \). Further let \( M = \{(i,j) : 1 \leq i < j \leq N\} \). For every \( \varepsilon = (\varepsilon_{ij})_{(i,j) \in M} \in [0, c]^M \) and \( j \in N \) we will define a vector \( p_j(\varepsilon) \in \mathbb{R}^n \in Y \) such that
\[
p_j(\varepsilon)^i = -1 & \text{ if } j = i, \quad (15) \\
p_j(\varepsilon)^i = \varepsilon_{ij} & \text{ if } i < j, \quad (16) \\
p_j(\varepsilon)^i = 0 & \text{ if } i \in [N] \setminus [j], \quad (17) \\
|p_j(\varepsilon)^i| \leq \frac{1}{2} & \text{ if } i \in [n] \setminus [N]. \quad (18)
\]

Conditions (15) – (18) imply that \( \|p_s(\varepsilon) - p_t(\varepsilon)\|_\infty = 1 + \varepsilon_{st}^i \) for every \( 1 \leq s < t \leq N \).

Define \( \varphi : [0, c]^M \to \mathbb{R}^M \) by
\[
\varphi^j(\varepsilon) = 1 + \varepsilon_{ij} - \|p_i(\varepsilon) - p_j(\varepsilon)\|_Y,
\]
for every \( 1 \leq i < j \leq N \). From
\[
0 = 1 + \varepsilon_{ij}^i - \|p_i(\varepsilon) - p_j(\varepsilon)\|_\infty \leq \varphi^j(\varepsilon) = 1 + \varepsilon_{ij} - \|p_i(\varepsilon) - p_j(\varepsilon)\|_Y \leq 1 + \varepsilon_{ij} - (1 + c)^{-1}\|p_i(\varepsilon) - p_j(\varepsilon)\|_\infty \leq c,
\]
it follows that the image of \( \varphi \) is contained in \([0, c]^M\). Since \( \varphi \) is continuous, by Brouwer’s fixed point theorem \( \varphi \) has a fixed point \( \varepsilon_0 \in [0, c]^M \). Then \( \{p_j(\varepsilon_0) : j \in [N]\} \) is a 1-equilateral set in \( Y \) of cardinality \( N = n - k(2 + \ell) \).
To finish the proof, we only have to find vectors \( p_j(\varepsilon) \) that satisfy conditions (15)–(18). We construct them in a similar way as the equilateral sets in the proof of Theorem 1.

For \( 1 \leq i \leq 2 + \ell \) let
\[
U_i = (i - 1)k + [k],
\]
and
\[
B_i = B(b_{n-ik+1}, b_{n-ik+2}, \ldots, b_{n-(i-1)k}).
\]
By working from \( 2 + \ell \) down to 1 we may assume without loss of generality that for \( 1 \leq i \leq 2 + \ell \)
\[
|\det B_i| \geq |\det B(b_{i_1}, \ldots, b_{i_k})| \tag{19}
\]
for all choices of \( 1 \leq i_1 < \cdots < i_k \leq n - (i - 1)k \) for which \( |\{i_1, \ldots, i_k\} \cap ([n] \setminus [n - 2\ell])| \leq 1 \).
Assume now that
\[
|\det B_i| > 0 \text{ for all } 1 \leq i \leq 2 + \ell.
\]
We can handle the case when this assumption does not hold in a similar way as the case in the proof of inequality (3) in Theorem 1 when assumption (11) did not hold. Therefore we omit the details.

We construct \( p_j(\varepsilon) \) as a sum of \( 2 + \ell \) other vectors \( p_j(\varepsilon, 1), p_j(\varepsilon, 2), \ldots, p_j(\varepsilon, 2 + \ell) \), where \( p_j(\varepsilon, 1) \) is defined as follows.

For \( m \in \{1, 2\} \) let
\[
p_j(\varepsilon, m)^i = \begin{cases} \frac{-1}{2} & \text{if } i = j \\ 0 & \text{if } i \in [n] \setminus \{j\} \cup N + U_m \end{cases}
\]
and for \( m \in \{3, \ldots, 2 + \ell\} \) let
\[
p_j(\varepsilon, m)^i = \begin{cases} \frac{\varepsilon^i}{\ell} & \text{if } i < j \\ 0 & \text{if } i \in [n] \setminus ([j - 1] \cup N + U_m) \end{cases}
\]
where \( s(\varepsilon, j) = \sum_{r<j} \frac{\varepsilon^r}{\ell} b_r \). As before, by Cramer’s rule we have \( p_j(\varepsilon)(m) \in Y \) for all \( m \in \{2+\ell\} \), and thus \( p_j(\varepsilon) = \sum_{m \in \{2+\ell\}} p_j(\varepsilon, m) \in Y \). It follows immediately that \( p_j(\varepsilon) \) satisfies conditions (15)–(17).

Further, by the multilinearity of the determinant, (19), and the triangle inequality for \( m \in \{1, 2\} \) we have
\[
|p_j(\varepsilon, m)^i| = \left| \frac{\det B_m (i - N - km_s(\varepsilon, j))}{\det B_m} \right| \leq \frac{1}{2},
\]
and for every \( m \in \{3, \ldots, 2 + \ell\} \) we have
\[
|p_j(\varepsilon, m)^i| = \left| \frac{\det B_m (i - N - km_s(\varepsilon, j))}{\det B_m} \right| \leq \sum_{r<j} \frac{\varepsilon^r}{\ell} \left| \frac{\det B_m (i - N - km_s(\varepsilon, j))}{\det B_m} \right|
\]
\[
\leq \sum_{r<j} \frac{\varepsilon^r}{\ell} \left| \frac{\det B_m (i - N - km_s(\varepsilon, j))}{\det B_m} \right| \leq \sum_{r<j} \frac{\varepsilon^r}{\ell} \leq (N - 1) \frac{\varepsilon}{\ell} = \frac{1}{2}.
\]
This implies that condition (18) holds for \( p_j(\varepsilon) \) as well, finishing the proof.
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