A plat form for surface-links

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Abstract

It is known that every surface-link, which is a closed surface embedded in 4-space, can be described as the closure of a 2-dimensional braid providing it is orientable. In this paper, we introduce a new method of describing a surface-link using a braided surface, which we call a plat form. We prove that every surface-link, not necessarily orientable, can be described in a plat form. The plat index for a surface-link is defined. In classical knot theory, the plat index of a link coincides with the bridge index. The plat index of a surface-link we introduce here is an analogy of them.

1 Introduction

In classical knot theory, there are two methods of presenting a link using braids: One is the closure of a braid and the other is a plat form. A plat form is the closure of a braid as in Figure 2 and we call such a closure the plat closure.

Every link is presented by braids in each method. The plat index of a link is the half of the minimum degree of braids whose plat closures are equivalent to the link. However, it is equal to the bridge index of the link.

A surface-link is a closed surface embedded in $\mathbb{R}^4$, and a 2-knot is a 2-sphere embedded in $\mathbb{R}^4$. The closure of a 2-dimensional braid was introduced by Viro and Kamada so that every orientable surface-link can be described by the closure of a 2-dimensional braid $[6,7,8]$.

In this paper, we introduce a new method of presenting a surface-link which we call a plat form. It is an analogy of a plat form of a link.

The following is our main theorem.
Theorem 1.1. Every (not necessarily orientable) surface-link is equivalent to a surface-link in a plat form.

We emphasize that our method enables us to present any surface-link, while the closure of a 2-dimensional braid presents an orientable surface-link.

A genuine plat form is a special case of a plat form. Some surface-links can be presented in a genuine plat form. For an orientable surface-link, we have the following.

Theorem 1.2. Every orientable surface-link is equivalent to a surface-link in a genuine plat form.

These theorems yield new surface-link invariants, called the plat index and the genuine plat index. (The latter is not defined for every surface-link.) Both of the invariants are analogies of the plat index and the bridge index of a link. We compute the (genuine) plat index for trivial surface-links and a non-trivial 2-knot.

This paper is organized as follows. In Section 2 we recall the notions of braids, surface-links, and braided surfaces. In Section 3 we define a plat form for surface-links. In Section 4 we prove Theorems 1.1 and 1.2. In Section 5 we define the plat index of a surface-link, show a relationship between the plat index and the braid index, and give some examples.

We work in the PL category, and surfaces embedded in 4-space are considered to be embedded locally flatly.

2 Preliminaries

2.1 A plat form for links and wickets

We recall the notions of a plat form for classical links and wickets.

In this paper, we assume that $m$ and $n$ are positive integers, $I = [0, 1]$ is the interval, $D$ is the square $I^2$ or a 2-dimensional disk, and $Q_n = \{q_1, \ldots, q_n\}$ is a fixed set of $n$ points $D$ such that $q_k = \left(\frac{1}{2}, \frac{k}{m^n}\right) \in I^2 = D$ for $k = 1, 2, \ldots, n$.

When we consider a (geometric) $n$-braid in $D \times I$, we assume that the boundary of the braid is $Q_n \times \partial I$. We regard the $n$-braid group $B_n$ as the fundamental group of the configuration space of $n$ points of Int $D$ based on $Q_n$ and also the group of equivalence classes of geometric $n$-braids. We denote by $\sigma_1, \ldots, \sigma_{n-1}$ the standard generators of $B_n$ due to Artin (1).

A wicket in $D \times I$ is a semicircle in $D \times I$ that meets $D \times \{0\}$ orthogonally at its endpoints in Int $D$ (3). An $m$-wicket system means a disjoint union of $m$ wickets. Note that the boundary of an $m$-wicket system consists of $2m$ points of Int $D$, and conversely, if a set of $2m$ points of Int $D$ equipped with a partition into $m$ pairs bounds an $m$-wicket system, then such a system is unique.

The $Q_{2m}$ equipped with the partition $\{(q_1, q_2), \ldots, (q_{2m-1}, q_{2m})\}$ bounds an $m$-wicket system, which we call the standard $m$-wicket system.

Let $\beta$ be a (geometric) $2m$-braid in $D \times I$. Attach the standard $m$-wicket system to the top of the braid and (the mirror image of) another standard $m$-wicket system under the bottom, and assume that it is in the 3-space $\mathbb{R}^3$, then we have a link. It is called the plat closure of $\beta$ and denoted by $\tilde{\beta}$. See Figure 3.
A link is said to be in a plat form when it is the plat closure of a braid. Every link is equivalent to a link in a plat form.

We may define the plat index of a link \( L \), denoted by \( \text{plat}(L) \), by the half of the minimum degree of braids whose plat closures are equivalent to \( L \). It is nothing more than the bridge index of a link; nevertheless, we adopt \( \text{plat}(L) \) for comparison with the case of surface-links.

A plat presentation of a link is not unique. For example, the plat closure of a (geometric) \( 2m \)-braid \( \beta \) is equivalent as a link to the plat closure of a \( 2n \)-braid \( \beta' \) obtained by the transformation depicted in Figure 4 for any \( n > m \), which we call stabilization.

In other words, for any \( n > m \), stabilization changes a (geometric) \( 2m \)-braid \( \beta \) to a (geometric) \( 2n \)-braid \( \beta' \) with

\[
\beta' = \beta \sigma_{2m} \sigma_{2(m+1)} \sigma_{2(m+2)} \cdots \sigma_{2(n-1)}.
\]

Hilden’s subgroup \( K_{2m} \) is the subgroup of \( B_{2m} \) generated by \( \sigma_1, \sigma_2 \sigma_1^2 \sigma_2, \sigma_2 \sigma_{2i-1} \sigma_{2i+1} \sigma_2 \) for each \( i = 1, 2, \ldots, m-1 \) ([4], cf. [2]).

**Proposition 2.1** (Birman [2]). Let \( m_i \) (\( i = 1, 2 \)) be positive integers and \( \beta_i \) be (geometric) \( m_i \)-braids. The plat closure \( \tilde{\beta}_1 \) is equivalent as a link to \( \tilde{\beta}_2 \) if and only if there exists an integer \( t \geq \max\{m_1, m_2\} \) such that for each \( m \geq t \) the equivalence classes of the \( 2m \)-braids \( \beta_i' \) (\( i = 1, 2 \)) obtained from \( \beta_i \) by stabilization belong to the same double coset of \( B_{2m} \) modulo \( K_{2m} \).

Let \( W_m \) be the space consists of all configuration of \( m \)-wicket systems in \( D \times I \), and \( W_m = \pi_1(W_m) \) be the fundamental group of \( W_m \) based on the standard wicket system.

**Proposition 2.2** (Brendle-Hatcher [3]). For each \( m \geq 0 \), Hilden’s group \( K_{2m} \) and \( W_m \) are isomorphic.
The isomorphism from $W_m$ to $K_{2m}$ introduced in [3] is given as follows: Let $f : I \to W_m$ be a loop in $W_m$ based on the standard wicket system. Consider a geometric $2m$-braid $\beta_f$ defined by

$$\beta_f = \{ \partial f(t) \times \{t\} \subset D \times I | t \in I \}.$$ 

The map sending $[f] \in W_m$ to $[\beta_f] \in B_{2m}$ induces the isomorphism.

In this paper, we call a geometric $2m$-braid $\beta$ adequate if there exists a loop $f : I \to W_m$ based on the standard $m$-wicket system such that $\beta = \beta_f$. Remark that the Hilden’s subgroup $K_{2m}$ consists of elements represented by some adequate $2m$-braids.

### 2.2 Surface-links

A surface-link is a closed surface embedded in $\mathbb{R}^4$, and a surface-knot is a connected surface-link. An embedded 2-sphere in $\mathbb{R}^4$ is called 2-knot. Two surface-links $F$ and $F'$ are equivalence if they are ambient isotopic in $\mathbb{R}^4$, and we denote it $F \approx F'$. A banded link in $\mathbb{R}^3$ means a pair $(L, B)$ of a link $L$ and a family of mutually disjoint bands $B = \{b_1, \ldots, b_n\}$ attaching to $L$. We write $L_B$ for a link obtained from $L$ by surgery along the bands belonging to $B$. A banded link is admissible if two links $L$ and $L_B$ are trivial.

Let $(L, B)$ be an admissible banded link in $\mathbb{R}^3$. Let $d$ (or $D$, resp.) be a union of mutually disjoint 2-disks embedded in $\mathbb{R}^3$ bounded by $L$ (or $L_B$, resp.). Consider a surface-link $F$ in $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$ such that

$$p(F \cap \mathbb{R}^3 \times \{t\}) = \begin{cases} 
D & (t = 1), \\
L_B & (0 < t < 1), \\
L \cup |B| & (t = 0), \\
L & (-1 < t < 0), \\
d & (t = -1), \text{ and} \\
\emptyset & \text{otherwise,}
\end{cases}$$

(1)

where $p : \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ is the first factor projection and $|B|$ is the union of the bands belonging to $B$. We denote this surface by $F(L, B)$ and call it a closed realizing surface of $(L, B)$. It depends on a choice of $d$ and $D$. However, the equivalence class as a surface-link does not depend on them (cf. [9][11]).

**Lemma 2.3** (Kawauchi-Shibuya-Suzuki [11]). Let $(L, B)$ and $(L', B')$ be admissible banded links, and let $F$ and $F'$ be their closed realizing surfaces. If $(L, B)$ is ambient isotopic to $(L', B')$, i.e., there is an ambient isotopy carrying $L$ to $L'$ and the bands of $B$ to those of $B'$, then a closed realizing surface $F(L, B)$ is equivalent to $F(L', B')$.

**Lemma 2.4** (cf. [11]). For any surface-link $F$, there exists an admissible banded link $(L, B)$ such that $F$ is equivalent to a closed realizing surface $F(L, B)$.
2.3 Braided surfaces and 2-dimensional braids

A braided surface was introduced by Rudolph [13] and a 2-dimensional braid was introduced by Viro (cf. Kamada [6, 7, 8]). Let \( D_1 \) and \( D_2 \) be 2-disks (or squares) and let \( \text{pr}_i : D_1 \times D_2 \to D_i \) (\( i = 1, 2 \)) be the \( i \)-th factor projection. Let \( y_0 \in D_2 \) be a fixed base point of \( D_2 \) with \( y_0 \in \partial D_2 \).

A (pointed) braided surface of degree \( n \) is a surface \( S \) properly embedded in \( D_1 \times D_2 \) such that the following conditions are satisfied [13]:

1. \( \pi = \text{pr}_2|_S : S \to D_2 \) is a simple branched covering map of degree \( n \).
2. \( \partial S \) is a closed \( n \)-braid in \( D_1 \times \partial D_2 \).
3. \( \text{pr}_1(\pi^{-1}(y_0)) = Q_n \).

(A branched covering map \( \pi \) of degree \( n \) is called simple if the preimage of each branch locus consists of \( n - 1 \) points.)

A 2-dimensional braid of degree \( n \) is a braided surface \( S \) of degree \( n \) such that \( \partial S \) is the trivial closed braid, i.e., \( \text{pr}_1(\pi^{-1}(y)) = Q_n \) for \( y \in \partial D_2 \). The degree of \( S \) is denoted by \( \deg S \). We say that two braided surfaces of the same degree are equivalent if they are ambient isotopic by an isotopy \( \{h_t\}_{t \in I} \) of \( D_1 \times D_2 \) such that for each \( t \in I \), \( h_t \) is fiber-preserving when we regard \( D_1 \times D_2 \) as the trivial \( D_1 \)-bundle over \( D_2 \), and the restriction of \( h_t \) to \( \text{pr}_2^{-1}(y_0) \) is the identity map. A braided surface is trivial if it is equivalent to \( Q_n \times D_2 \).

Lemma 2.5 ([8], Lemma 16.9). A braided surface \( S \) is trivial if and only if \( S \) has no branch points.

Suppose that \( D_1 \) and \( D_2 \) are in \( \mathbb{R}^2 \), and \( D_1 \times D_2 \) is in \( \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4 \). Let \( S \) be a 2-dimensional braid of degree \( n \), and let \( S_t \) be the section \( S \cap D_1 \times (I \times \{t\}) \) for each \( t \in I \). The closure of a 2-dimensional braid \( S \), denoted by \( \overline{S} \), is an orientable surface-link in \( \mathbb{R}^4 \) obtained from \( S \) by attaching 2-disks trivially outside of \( D_1 \times D_2 \) in \( \mathbb{R}^4 \) along the components of \( \partial S \). It is described by the motion picture as in Figure 5. As an analogue of Alexander’s theorem, the following is proved in [7].

Proposition 2.6 (Kamada [7]). Every orientable surface-link is equivalent to the closure of a 2-dimensional braid.

For an orientable surface-link \( F \), the braid index of \( F \), denoted by \( \text{Braid}(F) \), is the minimal degree of 2-dimensional braids whose closures are equivalent to \( F \).

3 A plat form for surface-links

In this section, we introduce a plat form for surface-links. We assume \( D_2 \subset \mathbb{R}^2 \). Let \( N \) be a regular neighborhood of \( \partial D_2 \) in \( \mathbb{R}^2 \setminus \text{Int} D_2 \), which is parameterized with \( (t, x) \in I \times S^1 \) such that \( \partial D_2 = \{0\} \times S^1 \) and \( y_0 = (0, 0) \in I \times S^1 \), where \( S^1 = \mathbb{R}/\mathbb{Z} \).

Definition 3.1. A surface \( A \) in \( D_1 \times N \) is of m-wicket type (or simply of wicket type) if it is a properly embedded surface in \( D_1 \times N \) satisfying the following conditions.
(1) \( A \cap (D_1 \times (I \times \{0\})) \) is the standard \( m \)-wicket system when we identify \( D_1 \times (I \times \{0\}) \) with \( D \times [0, 1] \).

(2) For each \( \theta \in S^1 \), \( A \cap (D_1 \times (I \times \{\theta\})) \) is an \( m \)-wicket system.

**Definition 3.2.** A braided surface \( S \) in \( D_1 \times D_2 \) is *adequate* if there exists a surface of \( m \)-wicket type, \( A \), in \( D_1 \times N \) such that the boundaries of \( S \) and \( A \) coincide: \( \partial S = \partial A \).

It is trivial that the degree of an adequate braided surface is even. Note that for each \( \theta \in S^1 \), the section \( A \cap (D_1 \times (I \times \{\theta\})) \) is determined from the boundary of \( A \). Hence \( A \) is determined by \( \partial S \). Therefore, for an adequate braided surface \( S \), such a surface of wicket type is uniquely determined.

**Definition 3.3.** Let \( S \) be an adequate braided surface, and \( A \) be a surface of wicket type in \( D_1 \times N \) with \( \partial S = \partial A \). The *plat closure* of \( S \), denoted by \( \tilde{S} \), is the union of \( S \) and \( A \) in \( \mathbb{R}^4 \).

**Definition 3.4.** A surface-link is said to be *in a plat form* if it is the plat closure \( \tilde{S} \) of an adequate braided surface \( S \). Moreover, it is said to be *in a genuine plat form* if \( S \) is a 2-dimensional braid.

We consider a condition for a braided surface to admit the plat closure. For a braided surface \( S \) of degree \( n \), let \( \beta_S \) be a geometric \( n \)-braid obtained by cutting the closed braid \( \partial S \) along \( \pi^{-1}(y_0) \). It is easy to see that \( [\beta_S] = [\beta_S'] \) in the braid group \( B_n \) if two braided surfaces \( S \) and \( S' \) are equivalent.

**Theorem 3.5.** A braided surface \( S \) is equivalent to an adequate one if and only if \( \deg S = 2m \) for some \( m \in \mathbb{N} \) and the braid \( [\beta_S] \) belongs to \( K_{2m} \).

**Proof.** We show the only if part. Let \( S \) be a braided surface equivalent to an adequate one \( S_0 \), and let \( A_0 \) be a surface of \( m \)-wicket type such that \( \partial S_0 = \partial A_0 \) for some \( m \in \mathbb{N} \).
Thus $\deg S = \deg S_0 = 2m$, and $[\beta_S] = [\beta_{S_0}] \in B_{2m}$. Let $f : [0, 1] \to W_m$ be a map defined by $f(t) = A_0 \cap D_1 \times (I \times \{t\})$. Then $f$ is a loop in $W_m$ such that $f(0) = f(1)$ is the standard $m$-wicket system. By the isomorphism in Proposition 2.2 the element $[f] \in W_m$ corresponds to the braid $[\beta_f] \in K_{2m}$. Since $\partial S_0 = \partial A_0$, we have $[\beta_{S_0}] = [\beta_f]$. Therefore, $[\beta_S] \in K_{2m}$.

The if part can be shown by considering the above argument in the opposite direction. □

4 Proofs of Theorems 1.1 and 1.2

In this section, we prove Theorems 1.1 and 1.2.

For subsets $A \subset \mathbb{R}^2$ and $B \subset \mathbb{R}$, the product $A \times B$ is considered as a subset of $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$.

Lemma 4.1. By an ambient isotopy, any banded link $(L, B)$ in $\mathbb{R}^3$ is deformed to a banded link $(L_0, B_0)$ such that the following conditions are satisfied:

1. There exists a disk $D$ in $\mathbb{R}^2$ and a $2m_0$-braid $\beta_0$ in $D \times [-3, 3]$ for some $m_0 \in \mathbb{N}$ such that $\beta_0 = L_0 \cap D \times [-3, 3]$ and $\beta_0 = L_0$.

2. There exist mutually disjoint $2$-disks $D_i$ in $\mathbb{R}^2$ ($i = 1, 2, \ldots, n$), where $n$ is the number of bands belonging to $B_0$, such that for each $i$, the intersection of $L_0$ and $D_i \times [-1, 1]$ is a pair of vertical line segment, and the band $b_i \in B_0$ is in the cylinder $D_i \times [-1, 1] \subset D \times I$ as in Figure 6.

![Figure 6: A band $b_i$ in the cylinder $D_i \times [-1, 1]$](image)

Proof. By an ambient isotopy of $\mathbb{R}^3$, deform $(L, B)$ so that $L$ and the bands belonging to $B$ are in $\mathbb{R}^2 \times [-2, 2]$ and the bands satisfy the condition (2). Let $B_0$ be the set of bands obtained from $B$ by this deformation. Move every maximal (or minimal, resp.) point of $L$ into $\mathbb{R}^2 \times \{3\}$ (or $\mathbb{R}^2 \times \{-3\}$, resp.) by an ambient isotopy of $\mathbb{R}^3$ keeping $D_i \times [-1, 1]$ ($i = 1, \ldots, n$) fixed pointwise. Finally, by an ambient isotopy of $\mathbb{R}^3$ keeping $\mathbb{R}^2 \times [-1, 1]$ fixed pointwise, we deform the link $L$ into a link $L_0$ satisfying the condition (1). □

Let $r$ be a real number, and let $h : \mathbb{R}^2 \times (-\infty, r] \to (-\infty, r]$ be the second-factor projection which we regard as a height function of $\mathbb{R}^2 \times (-\infty, r]$. 

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Lemma 4.2 ([9], Theorem 3.3.2). Let $F$ and $F'$ be compact surfaces properly embedded in $\mathbb{R}^3 \times (-\infty, r]$ such that all critical points of $F$ and $F'$ are minimal points with respect to $h$ and the boundary of $F$ and that of $F'$ in $\mathbb{R}^3 \times \{r\}$ are the same trivial link. Then, $F$ and $F'$ are ambient isotopic in $\mathbb{R}^3 \times (-\infty, r]$ rel $\mathbb{R}^3 \times \{r\}$.

First, we prove Theorem 1.1.

Let $F$ be a surface-link. By Lemma 2.4 there exists an admissible banded link $(L, B)$ such that the closed realizing surface is equivalent to $F$. Applying Lemma 4.1 to $(L, B)$, we have an admissible banded link $(L_0, B_0)$ and $2m_0$-braid $\beta_0$ satisfying the two conditions of the lemma. By Lemma 2.3 the closed realizing surface of $(L_0, B_0)$ is equivalent to $F$.

Let $c$ be the number of components of the link $L_0$ and $c_B$ be that of $(L_0)(B_0)$, and let $1_n$ denote the trivial geometric $n$-braid $Q_n \times I$. Since $L$ (or $L_B$, resp.) is a $c$-component (or $c_B$-component, resp.) trivial link, $L$ (or $L_B$, resp.) is equivalent to the plat closures of $\beta_0$ and $1_n$ (or $(\beta_0)_B$ and $1_{2c_B}$, resp.). By Proposition 2.1 we see that there exist adequate geometric $2m$-braids $\gamma$, $\gamma'$, $\delta$, and $\delta'$ for some $m \in \mathbb{N}$, such that

$$[\beta] = [\gamma][\alpha'][\gamma']$$

and

$$[\beta_B] = [\delta][\alpha_B][\delta']$$

in $B_{2m}$,

where $\beta$ (or $\beta_B$, $\alpha$, or $\alpha_B$, resp.) is the geometric $2m$-braid obtained from the braid $\beta_0$ (or $(\beta_0)_B$, $1_{2c_B}$, or $1_{2c_B}$, resp.) by stabilization.

By definition of stabilization, the geometric $2m$-braid $\beta$ (in $D \times I$) contains the geometric $2m_0$-braid $\beta_0$ in a subcylinder of $D \times I$. Thus, we define $B_1$ as the set of bands attaching to $\beta$ obtained from $B_0$ by identifying the cylinder $D \times I$ of $\beta_0$ with the subcylinder above. Hence, the geometric braid $\beta_B$ is equal to $\beta_{B_1}$ (which is obtained from $\beta$ by surgery along the bands belonging to $B_1$).

Next, we construct a surface $S_0$ properly embedded in $D_1 \times D_2$ using these $2m$-braids. Let $0 = t_0 < t_1 < \cdots < t_5 < t_6 = 1$ be a partition of $I = [0, 1]$. We divide $D_2 = I \times I$ into seven pieces $E_0, \ldots, E_6$ with $E_i = I \times [t_i, t_{i+1}]$ $(i = 0, \ldots, 6)$.

![Figure 7: A partition of $D_2$](attachment:image_url)

Let $p_1 : D_1 \times I \rightarrow D_1$ and $p_2 : D_1 \times I \rightarrow I$ be the first and second factor projections of $D_1 \times I$, and let $p_i : D_1 \times D_2 \rightarrow D_1$ be the $i$-th factor projections of $D_1 \times D_2$. For a geometric braid $b \subset D_1 \times I$ and $t \in I$, we denote by $b_t$ the subset $p_1(b \cap p_2^{-1}(t))$ of $D_1$.  

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Now, we define a surface $S_0$ over $E_i (i = 0, \ldots, 6)$ as follows:

(0) We define $S_0$ over $E_0$ as $Q_{2m} \times E_0$ in $D_1 \times E_0$.

(1) Write a point of $E_1$ by $(s, t) \in E_1 = I \times [t_1, t_2]$. Then, we define $S_0$ over $E_1$ as follows:

$$
\text{pr}_1(S_0 \cap \text{pr}_2^{-1}(s, t)) = \begin{cases} 
Q_{2m} & ((s, t) \in I \times [t_1, (t_1 + t_2)/2]), \\
\alpha_{(1-i)} & ((s, t) \in I \times ((t_1 + t_2)/2, t_2]), 
\end{cases}
$$

and $S_0 \cap D_1 \times (I \times [(t_1 + t_2)/2])$ is a trivial braid $1_{2m}$ with bands attaching to $1_{2m}$ as in Figure 8 such that the surgery result of $1_{2m}$ is $\alpha$. We denote by $B_1$ the set of these bands.

(2) Write a point of $E_2$ by $(s, t) \in E_2 = I \times [t_2, t_3]$. First, we define $S_0$ over $\partial E_2$ ($= I \times \{t_2, t_3\} \cup \{0, 1\} \times [t_2, t_3]$) by

$$
\text{pr}_1(S_0 \cap \text{pr}_2^{-1}(s, t)) = \begin{cases} 
\beta_{(1-i)} & ((s, t) \in I \times [t_3]), \\
\gamma_{(t-t_3)/(t_2-t_3)} & ((s, t) \in I \times [t_2, t_3]), \\
\alpha_{(1-i)} & ((s, t) \in I \times [t_2]), \text{ and} \\
\gamma'_{(t-t_3)/(t_1-t_2)} & ((s, t) \in [0] \times [t_2, t_3]).
\end{cases}
$$

See Figure 8. Since $[\beta] = [\gamma][\alpha][\gamma']$, the $2m$-braid $[\gamma][\alpha][\gamma'][\beta]^{-1}$ is the identity element of $B_{2m}$. We define $S_0$ over $E_2$ as a braided surface of degree $2m$ without branch points, which is uniquely determined up to equivalence as a braided surface by Lemma 2.5.
(3) We construct $S_0$ over $E_3$ by a similar way of the case (1). We define $S_0$ over $E_3$ as follows:

$$\text{pr}_1(S_0 \cap \text{pr}_2^{-1}(s, t)) = \begin{cases} 
\beta_{(1-s)} & ((s, t) \in I \times [(t_3 + t_4)/2, t_4]), \\
(\beta_B)_{(1-s)} & ((s, t) \in I \times ((t_3 + t_4)/2, t_4)),
\end{cases}$$

and $S_0 \cap D_1 \times (I \times [(t_3 + t_4)/2])$ is a geometric $2m$-braid $\beta$ with bands belonging to $B_1$.

(4) We construct $S_0$ over $E_4$ by a similar way of the case (2). First, we define $S_0$ over the boundary of $E_4$ by

$$\text{pr}_1(S_0 \cap \text{pr}_2^{-1}(s, t)) = \begin{cases} 
\beta_B(1-s) & ((s, t) \in I \times \{t_4\}), \\
\delta_{(t-t_4)/(t_4-t_5)} & ((s, t) \in \{1\} \times [t_4, t_5]), \\
(\alpha_B)_{(1-s)} & ((s, t) \in I \times \{t_5\}), \text{and} \\
\delta'_{(t-t_5)/(t_4-t_5)} & ((s, t) \in \{0\} \times [t_4, t_5]).
\end{cases}$$

Since $[\beta_B] = [\delta][\alpha_B][\delta']$, we can define $S_0$ over $E_4$ as a braided surface of degree $2m$ without branch points.

(5) We construct $S_0$ over $E_5$ by a similar way of the case (1). We define $S_0$ over $E_5$ as follows:

$$\text{pr}_1(S_0 \cap \text{pr}_2^{-1}(s, t)) = \begin{cases} 
(\alpha_B)_{(1-s)} & ((s, t) \in I \times [t_5, (t_5 + t_6)/2]), \\
Q_{2m} & ((s, t) \in I \times [(t_1 + t_2)/2, t_6]),
\end{cases}$$

and $S_0 \cap D_1 \times I \times \{(t_5 + t_6)/2\}$ is the trivial braid $1_{2m}$ with bands attaching to $1_{2m}$ as in the opposite direction of Figure 8 such that the surgery result of $1_{2m}$ is $\alpha_B$.

We denote by $B_1^+$ the set of these bands.

(6) Finally, we define $S_0$ over $E_6$ as $Q_{2m} \times E_6$.

![Figure 9: Braids over $\partial E_i$ ($i = 0, \ldots, 6$)](image-url)
As a result, we have the properly embedded surface $S_0$ in $D_1 \times D_2$. It is a braided surface of degree $2m$ except in neighborhoods of the bands appearing in the cases (1), (3), and (5).

However, by an ambient isotopy of the neighborhood of each band, we can deform the band to a branch point as shown in Figure 10. Then we have a braided surface of degree $2m$, denoted by $S$.

![Figure 10: Two motion pictures of band surgery and a simple branch point ($i = 1, 3, 5$)](image)

The braided surface $S$ is adequate because the geometric $2m$-braid $\beta_S$ (obtained by cutting $\partial S$ along $\pi^{-1}(y_0)$) is a composition of adequate ones.

Finally, we show that the surface-link $F$ is equivalent to the plat closure $\tilde{S}$ of $S$.

Let $p : \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ be the first-factor projection, $h : \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ be the second-factor projection which we regard as a height function of $\mathbb{R}^4$, and $A$ be a surface of wicket type such that $\partial A = \partial S$. Note that $\partial \tilde{S} = \partial S_0$. Let $F_0$ be the union $S_0 \cup A$, which is a surface-link equivalent to $\tilde{S} = S \cup A$.

By an ambient isotopy of $\mathbb{R}^4$, deform $F_0$ to a surface-link $F_1$ such that

\[ p(F_1 \cap \mathbb{R}^3 \times \{t\}) = \begin{cases} D_1 & (t = t_1), \\ 1_{2m} & ((t_5 + t_6)/2 < t < t_7), \\ 1_{2m} \cup |B_1^-| & (t = (t_5 + t_6)/2), \\ p(F_0 \cap \mathbb{R}^3 \times \{t\}) & ((t_1 + t_2)/2 < t < (t_5 + t_6)/2), \\ 1_{2m} \cup |B_1^-| & (t = (t_1 + t_2)/2), \\ 1_{2m} & (t_0 < t < (t_1 + t_2)/2), \\ d_1 & (t = t_0), \text{ and } \\ \emptyset & \text{otherwise}, \end{cases} \tag{2} \]

where $d_1$ (or $D_1$, resp.) are mutually disjoint $m$ 2-disks in $\mathbb{R}^3$ bounded by $1_{2m}$ which are disjoint from $|B_1^-|$ (or $|B_1^+|$, resp.) as in the left of Figure 11.

Next, we define a surface-link $F_2$ in $\mathbb{R}^4$ by
where $d_2 = d_1 \cup |B^-|$ (or $D_2 = D_1 \cup |B^-|$, resp.) are mutually disjoint $c$ (or $c_B$, resp.) 2-disks in $\mathbb{R}^3$ bounded by $\tilde{\alpha}$ (or $\tilde{\alpha}_B$, resp.) as in the right of Figure 11.

Figure 11: Pushing the bands to $t = t_0$ level

Then, $F_1$ is equivalent to $F_2$. This is seen as follws. Let $C$ be the union of the mutually disjoint 3-balls in $\mathbb{R}^4$ defined as $|B^-| \times [t_0, (t_1 + t_2)/2] \cup |B^-| \times [(t_5 + t_6)/2, t_7]$ in $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$. Then $F_2$ is obtained from $F_1$ by cellular moves along the 3-cells of $C$. (We say that a surface-link $F'$ is obtained from a surface-link $F$ by a cellular move along a 3-cell $C$ if $F \cap C$ is a 2-disk embedded in $\partial C$ and $F' = (F \cup \partial C) \setminus \text{Int}(F \cap \partial C)$. It is known that a cellualr move is realized by an ambient isotopy of $\mathbb{R}^4$ whose support is a regular neighborhood of $C$. [12]) Hence $F_1$ and $F_2$ are equivalent.

Therefore, $F_2$ is equivalent to $\tilde{S}$.

Now, the minimal (or maximal, resp.) disks of $F_2$ with respect to the height function $h$ appear in $t = t_0$ (or $t = t_7$) level as $d_2$ (or $D_2$, resp.), and the saddle bands of $F_2$ appear in $t = (t_3 + t_4)/2$ level as $\tilde{\beta} \cup |B^-|$. By Lemma 4.2 we see that $F_2$ is equivalent to a surface-link $F_3$ such that

$$p(F_3 \cap \mathbb{R}^3 \times \{t\}) = \begin{cases} D_3 & (t = t_7), \\
\tilde{\beta}_B & ((t_3 + t_4)/2 < t < t_7), \\
\tilde{\beta} \cup |B^-| & (t = (t_3 + t_4)/2), \\
\tilde{\beta} & (t_0 < t < (t_3 + t_4)/2), \\
d_3 & (t = t_0), \\
\emptyset & \text{otherwise,} \end{cases} \quad (4)$$
where $d_3$ (or $D_3$, resp.) are mutually disjoint $c$ (or $c_B$, resp.) 2-disks in $\mathbb{R}^3$ bounded by $\beta$ (or $\beta_B$, resp.). Hence $F_3$ is a closed realizing surface of the banded link $(\beta, B_1)$. Since the banded link $(L, B)$ is equivalent to $(\beta, B_1)$ and the surface-link $F$ is equivalent to a closed realizing surface of $(L, B)$, we see that $F$ is equivalent to $\tilde{S}$. □

Next, we prove Theorem 1.2.

For a braided surface $S$ in $D_1 \times D_2 = D_1 \times (I \times I)$, let $S_t$ denote the section $S \cap D_1 \times (I \times \{t\})$ for each $t \in I$. We denote by $\Delta_1$ the trivial 2-braid $Q_2 \times I$ and by $\Delta_m$ for $m \geq 2$ the geometric $2m$-braid $\prod_{k=1}^{m-1}(\sigma_2 \sigma_{2k-1} \cdots \sigma_2 \sigma_1)$. Then the closure of an $m$-braid $b$ is equivalent to the plat closure of a $2m$-braid $\Delta_m b \Delta_m^{-1}$. See Figure 12 and [13] where we assume the braid products are taken from right to left.

![Figure 12: A deformation of $\Delta_m^{\pm 1}$ equipped with the standard $m$-wicket system](image)

![Figure 13: The closure of $b$ and the plat closure of $\Delta_m b \Delta_m^{-1}$, $m = 3$](image)

Let $F$ be an orientable surface-link. By Proposition 2.6 there exists a 2-dimensional $m$-braid $S$ such that the closure $\tilde{S}$ is equivalent to $F$. Let $S'$ be the 2-dimensional $2m$-braid obtained from $S$ by adding trivial $m$ sheets.

Now, we construct 2-dimensional $2m$-braid $S_0$ in $D_1 \times D_2' = D_1 \times I \times [-\varepsilon, 1 + \varepsilon]$ by the motion picture of the section $(S_0)_t$, i.e., for each $t \in I = [0, 1]$, we define the section $(S_0)_t$ in $D_1 \times (I \times \{t\})$ by the composition of $\Delta_m$, $(S')_t$, and $\Delta_m^{-1}$ as in Figure 14. For $t \in [-\varepsilon, 0]$, we define the motion picture by an isotopic deformation from $1_{2m}$ to $\Delta_m \Delta_m^{-1}$. For $t \in [1, 1 + \varepsilon]$ we define the motion picture by an isotopic deformation from $\Delta_m \Delta_m^{-1}$ to $1_{2m}$. 

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Figure 14: Products of $2m$-braids and the section, $m = 3$

Then the closure $\bar{S}$ of $S$ is equivalent to the plat closure $\bar{S}_0$ of $S_0$. (See Figure 15.)

Figure 15: Equivalence between $\bar{S}$ and $\bar{S}_0, m = 3$

Hence, the surface-link $F$ is equivalent to the genuine plat closure $\bar{S}_0$.

Remark 4.3. In Lemma 4.1, each band belonging to $B_0$ in the subcylinder $D_i \times [-1, 1]$ was chosen as in the left of Figure 16. However, we may choose as in the right. Then the corresponding branch point of a braided surface obtained in the proof of Theorem 1.1 changes the sign. (A branch point of a braided surface is positive (or negative) if the local monodromy is a conjugate of a standard generator (or its inverse), cf. [8, 9])

Figure 16: A band in the cylinder $D_i \times [-1, 1]$

5 The plat index of a surface-link and examples

In this section, we discuss new invariants of a surface-link called the plat index and the genuine plat index.

Definition 5.1. Let $F$ be a surface-link. The plat index of $F$ is the half of the minimum degree of adequate braided surfaces whose plat closures are equivalent to $F$. It is
denoted by Plat($F$):

$$\text{Plat}(F) = \min \{ \deg S/2 \mid S \text{ is a braided surface, } S \approx F \}. $$

**Definition 5.2.** Let $F$ be a surface-link which admits a genuine plat form. The **genuine plat index of $F$** is the half of the minimum degree of 2-dimensional braids whose plat closures are equivalent to $F$. It is denoted by $g\text{.Plat}(F)$:

$$g\text{.Plat}(F) = \min \{ \deg S/2 \mid S \text{ is a 2-dimensional braid, } S \approx F \}. $$

From the proof of Theorem 1.2, we have the following theorem.

**Theorem 5.3.** Let $F$ be an orientable surface-link. The following inequality holds.

$$\text{Plat}(F) \leq g\text{.Plat}(F) \leq \text{Braid}(F).$$

**Remark 5.4.** Let $L$ be a link, and bridge($L$) be the bridge index of $L$. Then, the following inequality holds.

$$\text{bridge}(L) = \text{plat}(L) \leq \text{braid}(L).$$

In the rest of this paper, we discuss some examples of a plat form. To present braided surfaces, we use a chart description of braided surfaces. A **chart description** is an oriented and labeled planar graph in $D_2$, which is defined by a projection of the crossing curves in a diagram of a braided surface (see [8] in detail).

A surface-knot is **trivial** if it is equivalent to a connected sum of standardly embedded 2-spheres, tori, and projective planes. Trivial projective planes in $\mathbb{R}^4$ are illustrated in Figure 17 (see [8] in detail).

![Figure 17: Motion pictures of trivial projective planes in $\mathbb{R}^4$](image)

**Proposition 5.5.** Let $F$ be a surface-link. Then $\text{Plat}(F) = 1$ if and only if $F$ is either a trivial 2-knot or a non-orientable trivial surface-knot. Furthermore, $g\text{.Plat}(F) = 1$ if and only if $F$ is either trivial 2-knot or a non-orientable trivial surface-knot with the normal Euler number $e(F) = 0$.

This proposition follows immediately from the classification of braided surfaces of degree 2: Let $S$ be a braided surface of degree 2 with $p$ positive branch points and $q$ negative branch points. The equivalence class of $S$ is determined from $p$ and $q$. 


Moreover, $S$ is equivalent to a 2-dimensional 2-braid if and only if $p = q$. In chart description, $S$ has a chart description as in Figure 18 whose $a$, $b$, and $c$ are arbitrarily chosen non-negative integers with $p = a + c$ and $q = b + c$.

![Figure 18: A chart description of a braided surface of degree 2](image)

**Example 1.** Let $F$ be an orientable trivial surface-knot of genus $g > 0$. Then, the braid index of $F$ is equal to 2 so that the (genuine) plat index of $F$ is at most 2 by Theorem 5.3. However, by Proposition 5.5, the (genuine) plat index of $F$ is at least 2, hence we have $g.\text{Plat}(F) = \text{Plat}(F) = 2$. Figure 19 gives such a 2-dimensional 4-braid by a chart description.

![Figure 19: A chart description of a 2-dimensional 4-braid](image)

**Example 2.** Let $F$ be the 2-knot called 2_2 in the table of [10]. Then $g.\text{Plat}(F) = \text{Plat}(F) = 2$. 

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Proof. Note that the 2-knot $2_2$ has a ribbon presentation as in the left of Figure 20 which is another presentation for a ribbon surface-link using a banded link (see [8, 9] for details). By an ambient isotopy of $\mathbb{R}^3$, deform the banded link satisfying the conditions in Lemma 4.1 as in the right of Figure 20.

Hence this presentation provide a 2-dimensional 4-braid presented by the chart description depicted in Figure 21. Since the plat index (and genuine one) of $F$ is at least 2, we have $\text{Plat}(F) = g, \text{Plat}(F) = 2$. □

Figure 20: A deformation of a ribbon presentation of the ribbon 2-knot $2_2$

Figure 21: A chart description of a braided surface of degree 4

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