A NUMERICAL TREATMENT OF ANISOTROPIC RADIATION FIELDS COUPLED WITH RELATIVISTIC RESISTIVE MAGNETOFLOWS

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ABSTRACT

We develop a numerical scheme for solving fully special relativistic, resistive radiation magnetohydrodynamics. Our code guarantees conservation of total mass, momentum, and energy. The radiation energy density and the radiation flux are consistently updated using the M–1 closure method, which can resolve an anisotropic radiation field, in contrast to the Eddington approximation, as well as the flux-limited diffusion approximation. For the resistive part, we adopt a simple form of Ohm’s law. The advection terms are explicitly solved with an approximate Riemann solver, mainly the Harten–Lax–van Leer scheme; the HLLC and HLLD schemes are also solved for some tests. The source terms, which describe the gas–radiation interaction and the magnetic energy dissipation, are implicitly integrated, relaxing the Courant–Friedrichs–Lewy condition even in an optically thick regime or a large magnetic Reynolds number regime. Although we need to invert $4 \times 4$ matrices (for the gas–radiation interaction) and $3 \times 3$ matrices (for the magnetic energy dissipation) at each grid point for implicit integration, they are obtained analytically without preventing massive parallel computing. We show that our code gives reasonable outcomes in numerical tests for ideal magnetohydrodynamics, propagating radiation, and radiation hydrodynamics. We also applied our resistive code to the relativistic Petschek-type magnetic reconnection, revealing the reduction of the reconnection rate via radiation drag.

Key words: hydrodynamics – magnetohydrodynamics (MHD) – radiative transfer – relativistic processes

1. INTRODUCTION

Radiation and/or magnetic fields, relativity, and resistivity play crucial roles in a number of high-energy astrophysical phenomena such as black-hole accretion disks, jets, disk winds, pulsar winds, magnetar flares, core-collapse supernovae, and gamma-ray bursts. For example, a geometrically thick disk is supported by radiation pressure, which dominates the total pressure in the case of a near- or super-critical accretion rate. The radiation force is thought to accelerate the matter, producing jets (Lynden-Bell 1978; Icke 1980, 1989; Tajima & Fukue 1996). In contrast, the radiation drag reduces the velocity of the relativistic outflow. The magnetic field lines enhanced in the inner part of the accretion disks launch jets/outflows (Blandford & Payne 1982; Uchida & Shibata 1985; Kudoh & Shibata 1997). The magnetorotational instability (MRI) is thought to be the origin of the disk viscosity, by which the angular momentum is transported outward (Velikhov 1959; Chandrasekhar 1960; Balbus & Hawley 1991). The resistivity would cause magnetic energy to be converted to the energy of the matter through magnetic reconnection. Also, the resistivity might influence the evolution and/or saturation of the MRI in the disks (Lesur & Longaretti 2007; Fromang et al. 2007, 2013; Simon & Hawley 2009).

The global structure of accretion disks and outflows is investigated by radiation hydrodynamics (RHD) simulations (Eggum et al. 1987, 1988; Okuda & Fujita 2000; Ohhsuga et al. 2005; Ohhsuga 2006), magnetohydrodynamics (MHD) simulations (Machida et al. 2006; Kato et al. 2008; McKinney & Blandford 2009; Tchekhovskoy et al. 2010), and radiation-MHD (RMHD) simulations (Ohhsuga et al. 2009; Takeuchi et al. 2010; Ohhsuga & Mineshige 2011). Especially, Takeuchi et al. (2010) showed that high-velocity jets, which are magnetically collimated, are powered by the radiation force. Also, RMHD simulations of the local patch of the disk are performed (Hirose et al. 2009; Jiang et al. 2013). Although such works were highly successful, they should extend to relativistic simulations.

Many approximate methods have been proposed to solve the radiation transfer, since the computational cost for a rigorous method is too expensive to perform. In the flux-limited diffusion (FLD) approximation, a zeroth moment equation of the radiation transfer equation is solved to update the radiation energy density. The radiation flux as well as the radiation stress tensor is given based on the gradient of the radiation energy density. The FLD is quite a useful technique and gives appropriate radiation fields within the optically thick regime, but it does not always give precise radiation fields in the regime where the optical depth is around unity or less (see Ohhsuga & Mineshige 2011). In contrast to the FLD approximation, both zeroth and first moment equations are solved in the Eddington approximation. However, this method is somewhat problematic for anisotropic radiation fields, since the Eddington tensor is evaluated by assuming the isotropic radiation fields. Additionally, the speed of light is effectively reduced in this method.

Although the variable Eddington tensor method proposed by Stone et al. (1992) is known to give better results, it is very complex and expensive. One of the reasonable methods is the so-called M–1 closure, in which the Eddington tensor is obtained as a function of the radiation energy density and radiation flux (Minerbo 1978; Levermore 1984). The anisotropy of radiation fields is approximately taken into consideration, and the radiation propagates with the speed of light in an optically thin medium. The M–1 closure is adopted for the non-relativistic RHD code (González et al. 2007) and recently for the general relativistic (GR) code (Sadowski et al. 2013). Another truncated
Relativistic RMHD or RHD simulations were recently initiated. Farris et al. (2008) first proposed a numerical scheme of GR-RMHD, in which the Eddington approximation is employed. Zanotti et al. (2011) adopted a GR RMHD code to the Bondi–Hoyle accretion onto black holes. However, in their works, the explicit integration method is employed even for the gas–radiation interaction. In relativistic phenomena, the dynamical timescale as well as the timescale that the characteristic wave passes through the system could be comparable to the light crossing time. Thus, although the numerical time step becomes slightly shorter via the explicit treatment of the propagating radiation, the computational cost does not increase as much. In contrast, if the absorption/scattering opacity is very large, the timescale of the gas–radiation interaction could be much shorter than the other timescales, making the computation time consuming. In the non-relativistic RHD/RMHD simulations, such a difficulty is avoided by implicitly solving the gas–radiation interaction terms. We should employ such an implicit treatment in the relativistic code (Roedig et al. 2012; Sgdowski et al. 2013; Takahashi et al. 2013).

For resistive simulations, the magnetic energy dissipation should be implicitly solved to relax the Courant–Friedrichs–Lewy (CFL) condition in the regime of a large magnetic Reynolds number. Here, note that including the resistivity is a lot more complicated in the relativistic MHD than in the non-relativistic MHD, since we have to solve four additional equations for calculating the time evolution of the electric fields and charge density. The numerical treatment of relativistic MHD simulations with resistivity was developed by Komissarov (2007), Watanabe & Yokoyama (2006), Palenzuela et al. (2009), and Takamoto & Inoue (2011). The relativistic resistive RMHD simulations are a challenging task.

In the present paper, we propose an explicit–implicit scheme for solving special relativistic RMHD (SR-RMHD) and special relativistic resistive RMHD (SR-R2MHD) equations. Here, the radiation fields in the observer’s frame are used and we solve zeroth and first moment equations with the M-1 method. Since the M-1 closure is constructed such that the radiation energy momentum tensor is covariant, the Lorentz transformation for the radiation fields is unnecessary in our procedure. Our scheme ensures a conservation of total energy and momentum (matter, magnetic field, and radiation). An advection of magnetofluids and the radiation is explicitly solved, and the gas–radiation interaction as well as the magnetic energy dissipation via the resistivity is implicitly treated. Note that although we propose SR code in the present study, the extension to the GR version would be straightforward except for the M-1 closure. The procedure for the M-1 closure in the GR code is shown in Shibata et al. (2011).

This paper is organized as follows: in Section 2, we introduce argument equations for SR-RMHD and SR-R2MHD. The numerical scheme is explained in Section 3, and we show the results in Section 4. Finally, Section 5 is devoted to a summary.

2. BASIC EQUATIONS

In the following, we take the light speed as unity and assume the Minkowski flat spacetime. The metric is described by $g^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. Greek indices range over 0, 1, 2, 3 and Latin indices range over 1, 2, 3, where 0 indicates the time component and 1, 2, 3 indicate space components.

A set of equations for the fully special relativistic radiation electro-MHD consists of conservation of mass,

$$\partial_t (\rho u^\nu) = 0, \quad (1)$$

conservation of energy,

$$\partial_t \left[\rho \gamma^2 - \rho g + \frac{E^2 + B^2}{2}\right] + \nabla \cdot [\rho g u + E \times B] = G^0, \quad (2)$$

conservation of momentum,

$$\partial_t \left[\rho g u + (E \times B)\right] + \nabla \cdot \left[\rho u u + p g \delta + \left( -EE - BB + \delta |E|^2 + |B|^2 \right) \right] = G, \quad (3)$$

and the Maxwell equations

$$\partial_t B + \nabla \times E = 0, \quad (4)$$

$$\partial_t E - \nabla \times B = -j, \quad (5)$$

where $\delta$ is the Kronecker delta, $\rho$, $p_g$, and $h$ are the proper mass density, gas pressure, and gas specific enthalpy, respectively. The exchange of energy and momentum between the gas and the radiation, $G^0$ and $G$, are shown in Equations (12) and (13).

The bulk four velocity $u^\mu$ is related to the three velocity $v$ by

$$u^\mu = \gamma(1, v), \quad (6)$$

where $\gamma = \sqrt{1 + |v|^2}$ is the Lorentz factor.

Electric $E$ and magnetic $B$ fields are redefined to absorb a factor of $1/\sqrt{4\pi}$. We should specify Ohm’s law to relate the charge density $j$ and $E$. When we assume an ideal MHD, the closure relation is given by

$$E = -v \times B. \quad (7)$$

Then, electric fields are determined without solving Equation (5).

For a resistive MHD, we adopt a simple form of Ohm’s law:

$$j = \rho_e v + \eta^{-1} \gamma [E + v \times B - (v \cdot E)v], \quad (8)$$

where $\eta$ is electric resistivity (Blackman & Field 1993) and $\rho_e$ is the charge density, obtained by solving the charge conservation equation,

$$\partial_t \rho_e + \nabla \cdot j = 0, \quad (9)$$

(Komissarov 2007). Since $E$ should evolve according to Equation (5), we have to solve four additional equations in relativistic resistive MHD. In our numerical code, we can switch the resistivity on/off.

Equations (4) and (5) satisfy the divergence conditions $\nabla \cdot B = 0$ and $\nabla \cdot E = \rho_e$, if they are satisfied at the initial state. But these conditions are violated due to numerical errors. We adopted a generalized Lagrange multiplier (GLM) method (Dedner et al. 2002; Komissarov 2007) to overcome these problems. We do not describe the details of this scheme, but it appears in Dedner et al. (2002) and Komissarov (2007).

The radiation field obeys the following conservation equation,

$$T_{\text{rad}, \nu}^{\mu\nu} = -G^\nu. \quad (10)$$
where the energy momentum tensor of radiation $T_{\alpha \beta}^{\text{rad}}$ is given by

$$T_{\alpha \beta}^{\text{rad}} = \left( \begin{array}{c} E_r \\ F_r \\ P_r \end{array} \right),$$

where $E_r$, $F_r$, and $P_r$ are the radiation energy density, flux, and stress measured in the laboratory frame, respectively.

The radiation exchanges its energy and momentum with fluids by absorption/emission and scattering processes through the radiation four force $G^\alpha$:

$$G^\alpha = -\rho \kappa \left( 4\pi B y - \gamma E_r + u_j F^j_r \right)$$

$$- \rho \sigma_r \left[ \gamma |u|^2 E_r + \gamma u_j u_k P^j_k - \left( \gamma^2 + |u|^2 \right) u_j F^j_r \right],$$

and

$$G^j = -4\pi \rho \kappa B u^j + \rho \left( \kappa + \sigma_r \right) \left( \gamma F^j_r - u_k P^k_{j} \right)$$

$$- \rho \sigma_r u^j \left( \gamma^2 E_r - 2\gamma u_k F^k_{j} + u_k u_l P^k_{l} \right),$$

where $\kappa$ and $\sigma_r$ are the absorption and scattering coefficients measured in the comoving frame, respectively (e.g., Kato et al. 2008).

The blackbody intensity $B$ is described by gas temperature $T_g$ as

$$B = \frac{a_R T_g^4}{4\pi},$$

where $a_R$ is the radiation constant. The gas temperature is determined by the Boyle–Charles’s law:

$$p_g = \frac{\rho k_B T_g}{\mu m_p},$$

where $k_B$ and $m_p$ are the Boltzmann constant and proton mass, respectively, and $\mu$ is the mean molecular weight.

Finally, closure relations should be provided by specifying the equation of state for the matter and radiation fields. For the fluids, we assume a constant $\Gamma$ law, relating the specific enthalpy with the gas pressure by

$$h = 1 + \frac{\Gamma}{\Gamma - 1} \frac{p_g}{\rho},$$

where $\Gamma$ is the specific heat ratio.

For the radiation field, $P^j_r$ is assumed to be related to $E_r$ and $F^j_r$ through the Eddington tensor $P^j_r = D^j_{ik} E_r$. In this paper, we assume an M-1 closure given by Levermore (1984), which is explicitly described as

$$D^j_{ik} = \frac{1 - \chi}{2} \delta^j_{ik} + \frac{3\chi - 1}{2} n^i n^k,$$

where

$$\chi = \frac{3 + 4 f^2}{5 + 2\sqrt{4 - 3|f|^2}},$$

$$f^j = \frac{F^j_r}{E_r},$$

$$n^j = \frac{F^j_r}{|F_r|}.$$

We have to note that the Eddington tensor of the M-1 model is a function of $E_r$ and $F^j_r$, which can be evaluated in the laboratory frame. For the Eddington approximation, which is another class of the closure relation, the Eddington tensor $D^j_{ik}$ should be evaluated in the comoving frame. Then we need to transform the Lorentz transformation to obtain $P^j_r$ from $D^j_{ik}$, $E_r$, and $F_r$ (Takahashi et al. 2013). On the other hand, the M-1 closure is constructed such that the radiation energy momentum tensor is covariant. Thus, we can directly obtain $P^j_r$ from $E_r$ and $F_r$ without the Lorentz transformation.

Here we note that the M-1 closure given by Levermore (1984) is useful in the non-relativistic or special relativistic cases. The extension to general relativity is proposed by Shibata et al. (2011).

Now, we have 12 hyperbolic equations for SR-RMHD and 16 hyperbolic equations for SR-R2MHD. When the GLM method is adopted to preserve divergence-free conditions, 13 and 18 equations should be numerically solved for SR-RMHD and SR-R2MHD, respectively.

3. NUMERICAL SCHEME

In this section, we show how to solve the SR-RMHD and SR-R2MHD equations. First, we show a numerical scheme to solve the SR-RMHD equations in Section 3.1. Next, we show how to extend the SR-RMHD code to SR-R2MHD by taking into account an electric resistivity in Section 3.2.

3.1. SR-RMHD

Summarizing, an argument system of SR-RMHD is

$$\partial_t D + \nabla \cdot (D \mathbf{v}) = 0,$$

$$\partial_t \rho e + \nabla \cdot \mathbf{m} = G^0,$$

$$\partial_t \mathbf{m} + \nabla \cdot \mathbf{P} = \mathbf{G},$$

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0,$$

$$\partial_t \mathbf{E} + \nabla \cdot \mathbf{F} = -G^0,$$

$$\partial_t \mathbf{F} + \nabla \cdot \mathbf{P} = -\mathbf{G},$$

where

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B},$$

$$D = \rho \gamma,$$

$$e = \rho \gamma \gamma^2 - \frac{|\mathbf{E}|^2 + |\mathbf{B}|^2}{2},$$

$$m = \rho \gamma \mathbf{u} + \mathbf{E} \times \mathbf{B},$$

$$\mathbf{P} = \rho \mathbf{u} u - \delta p_g - \mathbf{EE} - \mathbf{BB} + \frac{\delta}{2}(|\mathbf{E}|^2 + |\mathbf{B}|^2).$$

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In Cartesian coordinates, the system can be described by a simple phase equation,
\[
\frac{\partial \mathcal{U}(\mathcal{P})}{\partial t} + \frac{\partial \mathcal{F}^k(\mathcal{P})}{\partial x} = S(\mathcal{P}),
\]
where \( \mathcal{P} \), \( \mathcal{U} \), \( \mathcal{F} \), and \( S \) are primitive variables, conserved variables, fluxes, and source terms, respectively:
\[
\mathcal{P} = \begin{pmatrix} \rho \\ \rho u^1 \\ B^1 \\ E_r \\ P_r^1 \end{pmatrix}, \quad \mathcal{U} = \begin{pmatrix} D \\ m^1 \\ e \\ B^1 \\ E_r \\ F_r^1 \end{pmatrix}, \quad \mathcal{F}^k = \begin{pmatrix} D^k \\ \Pi^{lk} \\ m^k \\ \varepsilon^{jkl} E_l \\ F^k \\ P_r^k \end{pmatrix},
\]
and
\[
S \equiv \begin{pmatrix} 0 \\ -S_E^i \\ -S_F^i \\ 0 \\ S_E^i \\ S_F^i \end{pmatrix} = \begin{pmatrix} 0 \\ G^0 \\ G^i \\ 0 \\ -G^0 \\ -G^i \end{pmatrix},
\]
where \( \varepsilon^{jkl} \) is the Levi–Civita antisymmetric tensor. In the following, we consider one-dimensional problems along the \( x \) direction without a loss of generality:
\[
\frac{\partial \mathcal{U}}{\partial t} + \frac{\partial \mathcal{F}^x}{\partial x} = S. \tag{35}
\]
Extensions to multi-dimensional problems and to curved space are straightforward.

We solve Equation (35) using an operator-splitting method as
\[
\frac{\partial \mathcal{U}}{\partial t} + \frac{\partial \mathcal{F}^x}{\partial x} = 0, \tag{36}
\]
\[
\frac{\partial \mathcal{U}}{\partial t} = S. \tag{37}
\]
The conservative discretization of the one-dimensional Equations (36) and (37) over a time step \( \Delta t \) from \( t = n\Delta t \) is
\[
\mathcal{U}^n_{i+1/2} = \mathcal{U}^n_i - \frac{\Delta t}{\Delta x} \left( f^+_{i+1/2} - f^-_{i-1/2} \right), \tag{38}
\]
\[
\mathcal{U}^{n+1} = \mathcal{U}^n + S^{n+1} \Delta t, \tag{39}
\]
where \( \Delta x \) is the grid spacing and \( i \) denotes the grid point, \( x = i \Delta x \). \( f \) is the numerical flux described below.

Here, Equation (36) is integrated explicitly, while Equation (37) is solved implicitly (Roedig et al. 2012; Sadowski et al. 2013; Takahashi et al. 2013). Although absorption or scattering timescales, \( \sim 1/(\rho c) \) or \( \sim 1/(\rho a) \), can be much shorter than the dynamical time scale in an optically thick medium, the implicit integration of Equation (37) allows us to use a time step, \( \Delta t \), larger than the absorption/scattering timescales. Since Equation (36) has a hyperbolic form, \( \Delta t \) in our code is determined using the maximum wave velocities for the radiation field \( c/\tau \) and magnetofluids \( \lambda_f \) as \( \Delta t = C_{\text{ad}} \Delta x / \max(\{c/\tau, \lambda_f\}) \), where \( C_{\text{ad}} < 1 \) is a CFL number and \( c/\tau \) and \( \lambda_f \) are obtained by computing maximum values of eigenvalues for radiation fields and magnetofluids (discussed later).

For the first step, we compute the surface values of the primitive variables \( \mathcal{P}_{i \pm 1/2, s} \) from the cell-centered variables \( \mathcal{P}_i \) as
\[
\mathcal{P}_{i \pm 1/2, s} = \mathcal{P}_i \pm \frac{\delta r \mathcal{P}}{2}, \tag{40}
\]
where \( s = L(R) \) denotes left (right) state variables. The spatial accuracy of the numerical codes depends on the choice of slope \( \delta r \). Many types of slope limiters that preserve monotonicity are proposed. In this paper, we utilize a harmonic mean proposed by van Leer (1977), which has a second-order accuracy in space:
\[
\delta r \mathcal{P} = \frac{2\max(0, \Delta \mathcal{P}_+, \Delta \mathcal{P}_-)}{\Delta \mathcal{P}_+ + \Delta \mathcal{P}_-}, \tag{41}
\]

where
\[
\Delta \mathcal{P}_\pm = \pm (\mathcal{P}_{i+1} - \mathcal{P}_i). \tag{42}
\]
Extensions to higher order schemes are straightforward (e.g., Colella & Woodward 1984; Martí & Müller 1996; Komissarov 1999; De Zanna & Bucciantini 2002).

For the second step, the numerical fluxes \( f_{i \pm 1/2} \) are computed from the reconstructed primitive variables \( \mathcal{P}_{i \pm 1/2, s} \). We adopt an approximate Riemann solver to evaluate the numerical fluxes. We utilize the Harten–Lax–van Leer (HLL; Harten et al. 1983) scheme to evaluate \( f_{i \pm 1/2} \) given by
\[
f_{i \pm 1/2} = \frac{\lambda^+ F_L - \lambda^- F_R + \lambda^+ \lambda^- (U_R - U_L)}{\lambda^+ - \lambda^-}, \tag{43}
\]
where \( \lambda^+ \) and \( \lambda^- \) are the maximum and minimum wave velocity, respectively. The wave velocity is obtained by computing eigenvalues of the Jacobian matrix \( \partial F / \partial \mathcal{U} \). We note that the wave speed of the radiation fields is independent of the fluid velocities and radiation independently. For the radiation field, the wave velocities, \( \lambda^\pm \), are numerically computed from the Jacobian matrix and are tabulated before time integration in our scheme, since the computation is time consuming (González et al. 2007). Here, we note that such a wave velocity is overestimated when the system is highly optically thick. In this limit, the radiation energy should be slowly diffused out with the diffusion velocity, \( c/\tau \), in the comoving frame, where \( \tau \) is the optical thickness. However, the eigenvalue computed from the Jacobian matrix has a large value (\( c/\sqrt{3} \)), causing a large numerical diffusion. Thus, following Sadowski et al. (2013), we modify the wave velocities as
\[
\lambda^+ \rightarrow \min \left( \lambda^+; \frac{4}{3 \tau^i} \right), \tag{44}
\]
\[
\lambda^- \rightarrow \max \left( \lambda^-; -\frac{4}{3 \tau^i} \right), \tag{45}
\]
where \( \lambda^\pm \) are the right- and left-going wave velocities in the comoving frame and \( \tau^i \) is the total optical depth in a cell. This modification drastically reduces numerical diffusion in the optically thick case.
For the MHD, wave speeds are computed by solving a quartic equation,
\[
\rho h (1 - c_s^2) a^4 = (1 - \lambda^2) \left[ \left( b_i^2 \right)^2 + \rho h c_s^2 \right] a^2 - c_s^2 B^2, \tag{46}
\]
with \( \lambda = \gamma (\gamma - 1) \) and \( B = b_i^2 - \lambda B_0 \) (Mignone & Bodo 2006). Here, \( c_s \) is the sound speed and \( b_i = \gamma (B \cdot v, B / \gamma^2 + v \cdot v) \) is the covariant form of the magnetic fields. The fast magnetosonic wave velocities \( \lambda, \gamma \) are obtained by taking the maximum and minimum values of the roots \( \lambda \). Numerical fluxes for the magnetofluids are computed from Equation (43) using \( \lambda, \gamma \).

We note that higher order approximate Riemann solvers such as the HLLC (Mignone & Bodo 2006; Honkila & Janhunen 2007) and the HLLD (Mignone et al. 2009) can be adopted to compute numerical fluxes for magnetofluids. For most of the cases, we utilize the HLL scheme, but we show one-dimensional numerical tests with the HLL, HLLC, and HLLD schemes in Section 4.1.

For the third step, we solve Equation (38) using the numerical fluxes \( f_{i \pm 1/2} \) and obtain auxiliary conserved variables, \( \mathbf{u}^n_i = (D^*, m^*, e^*, B^*, E^*_r, F^*_r) \), where the superscript asterisks indicate that the quantity is computed at the third step. With the procedures so far, the advection terms are already solved, and only the gas–radiation interaction (Equation (39)) remains. Hence, we obtain two of the conserved variables at the next time step, \( D^{n+1} = D^* \) and \( B^{n+1} = B^* \). In addition, although the gas–radiation interaction changes the energy and momentum of the radiation magnetofluids in a local grid are conserved. It implies that the total energy and the total momentum at the next time step, \( e^{n+1} = e^n + \Delta E^*_r \), and \( \mathbf{m}^{n+1} \), are obtained as
\[
e^{n+1} = e^n + \Delta E^*_r, \tag{47}
\]
\[
\mathbf{m}^{n+1} = m^n + \mathbf{F}^*_r. \tag{48}
\]

For the fourth step, we calculate \( m^{n+1}, e^{n+1}, E^{n+1}_r, \) and \( F^{n+1}_r \). In particular, we calculate \( E^{n+1}_r \) and \( F^{n+1}_r \) by solving the gas–radiation interaction (Equation (39)), and \( m^{n+1} \) and \( e^{n+1} \) are evaluated by \( m^{n+1} = m^n + \mathbf{F}^{n+1}_r \) and \( e^{n+1} = e^n = E^{n+1}_r \). The primitive variables at the next time step, \( \mathbf{p}^{n+1} \), are simultaneously computed.

In this step, we iteratively solve Equation (39) for the radiation energy density and the radiation flux. The source terms include the primitive variables of fluids, \( S = S(E_r, F_r, D^{ij}, \mathbf{p}_h, \mathbf{B}) \), where \( \mathbf{p}^{(m)}_h \) represents the primitive variables of fluids (i.e., \( \rho, u, \) and \( p_h \)). We evaluate \( E_r \) and \( F_r \) at the \( (m+1) \) step in an implicit manner using \( \mathbf{p}^{(m)} \) and \( \mathbf{p}^{(m+1)} \) as
\[
\mathbf{U}^{(m+1)} = \mathbf{U}^* + \Delta t S (E_r^{(m+1)}, F_r^{(m+1)}, D_r^{ij,(m)}, \mathbf{p}^{(m)}_h, \mathbf{B}^{(m+1)}). \tag{49}
\]
The explicit form of this equation is shown later. After solving Equation (49), we calculate
\[
\mathbf{e}^{(m+1)} = e^{n+1} - E_r^{(m+1)}, \tag{50}
\]
\[
\mathbf{m}^{(m+1)} = m^{n+1} - F_r^{(m+1)}. \tag{51}
\]
Since all the conserved variables at the \( (m+1) \) step are obtained, we recover the primitive variables \( \mathbf{p}^{(m+1)} \) from \( \mathbf{U}^{(m+1)} \) (the recovery method is discussed later). Then, we again solve Equation (49) using the updated primitive variables \( \mathbf{p}_h^{(m+1)} \) and \( D^{jk,(m)} = D^{jk}_r (E_r^{(m+1)}_r, F_r^{(m+1)}) \) (a similar method is found in relativistic resistive MHD by Palenzuela et al. 2009). By setting \( \mathbf{p}_h^{(0)} \) and \( D^{jk,(0)}_r \) to \( D^{jk}_r (E_r^{(0)}_r, F_r^{(0)}) \), we continue the iteration until successive variables \( \Delta \mathbf{p}_h^{(m+1)} \equiv E_r^{(m+1)}_r - E_r^{(m)}_r, \Delta F_r^{(m+1)} \equiv F_r^{(m+1)} - F_r^{(m)} \), and \( \delta D^{(m+1)}_r \equiv D^{(m+1)}_r - D^{(m)}_r \) fall below a specified tolerance. When the solutions converge, we apply them to the solutions at the \( n+1 \) time step (\( \mathbf{p}^{(n+1)}_h = \mathbf{p}^{(m+1)}_h \)).

An explicit form of Equation (49) for the radiation field is given by
\[
\mathbf{C}^{(m)} (E_r^{(m+1)}, F_r^{(m+1)}) = \left[ E_r^{(m+1)} + (4 \pi \rho \gamma \kappa B^2)^{\delta} (F_r^{(m+1)} + (4 \pi \rho \gamma \kappa B^2)^{\delta}) - \Delta t \right], \tag{52}
\]
where
\[
\mathbf{C} = 1 - \Delta t X, \tag{53}
\]
and
\[
X = \left( \begin{array}{c}
\rho \gamma \left[ - \kappa + \kappa (u_2^2 + u_\rho u_\gamma D) \right], \\
\rho \mu \kappa (\gamma - 2) (u_2^2), \\
\rho \left[ \kappa u_\rho D \right] + \kappa (u_2^2 + u_\rho u_\gamma D R), \\
- \rho \gamma \left[ (\kappa + \kappa) u_2^2 + u_\rho u_\gamma D R \right],
\end{array} \right) \tag{54}
\]
By inverting the \( 4 \times 4 \) matrix \( \mathbf{C} \) directly, we obtain the conserved variables \( E_r^{(m+1)} \) and \( F_r^{(m+1)} \).

Here, we mention the recovery method for converting from the conserved variables to the primitive variables. By the third step, \( D^{n+1} \) and \( B^{n+1} \) are obtained as we have already mentioned. In the fourth step, we have \( m^{(m+1)} \) and \( e^{(m+1)} \) by solving Equations (49)–(51). Then, three unknown variables, \( \rho^{(m+1)}, u^{(m+1)}, \) and \( p^{(m+1)}_h \), are computed by solving a single nonlinear equation \( g(W) = 0 \) on \( W = \rho \gamma^2 \) using the Newton-Raphson method,
\[
g(W) = W - \rho_s + \left( 1 - \frac{1}{2} \frac{\gamma^2}{W} \right) \left| B \right|^2 - \frac{S^2}{W^2}, \tag{55}
\]
\[
\gamma = \left[ 1 - \frac{S^2}{W^2} + \frac{W^2}{(W + |B|^2)^2} \right], \tag{56}
\]
\[
p_s = \frac{1}{\gamma} - \frac{S}{\gamma W}, \tag{57}
\]
\[
\frac{d\gamma}{dW} = \frac{\gamma^3}{W^3 (W + |B|^2)} \times \left[ |B|^2 - 2 S W^2 + 3 S^2 |B|^2 W + S |B|^2 \right], \tag{58}
\]
\[
\frac{dp_s}{dW} = \frac{1}{\gamma^3} \frac{dW}{dW} - \frac{2}{\gamma^3} \frac{dW}{dW} - \frac{1}{\gamma^3} \frac{dW}{dW} + \frac{S}{\gamma^3} \frac{dW}{dW} + \frac{S}{\gamma^3} \frac{dW}{dW}, \tag{60}
\]
where \( \Gamma_1 = \gamma (\Gamma - 1) \) and \( S = m \cdot B \) (Mignone & Bodo 2006; Mignone & McKinney 2007). This recovery method in SR-RMHD is the same as that in relativistic pure MHD.
We noted that our scheme does not guarantee the physical constraint $|\mathbf{F}_r| \leq E_r$. If a truncation error leads to $|\mathbf{F}_r| > E_r$, unphysical solutions appear. To avoid this problem, we artificially reduce the radiation flux without changing the direction of the radiation flux, if the condition is violated, as

$$F_r \rightarrow F_r \min \left(1, \frac{E_r}{|\mathbf{F}_r|}\right).$$

We confirmed that $|\mathbf{F}_r|$ rarely exceeds $E_r$ and the above procedure is applied in the test problems described in Section 4.

### 3.2. SR-R2MHD

In SR-R2MHD, we solve Equations (5), (8)–(9), and (21)–(26) so that we have 16 hyperbolic equations. Note that Equation (5) becomes stiff for the ideal limit ($\eta \rightarrow 0$). Thus, we solve the SR-R2MHD equations using operator splitting, as in SR-RMHD. The primitive variables ($P$), conserved variables ($\mathcal{U}$), fluxes ($\mathcal{F}$), and source terms ($\mathcal{S}$) for SR-R2MHD are given by

$$P = \begin{pmatrix} \rho \\ u^j \\ p_g \\ B^j \\ E_j \\ \rho_c \\ E_r \\ F^j_r \end{pmatrix}, \quad \mathcal{U} = \begin{pmatrix} D \\ m^j \\ e \\ B^j \\ E_j \\ \rho_c \\ E_r \\ F^j_r \end{pmatrix}, \quad \mathcal{F} = \begin{pmatrix} D \delta^{jk} \\ \Pi^{jk} \\ m^k \\ e^{ijkl} E_i \\ e^{ijkl} B_l \\ F^j \\ F^j_f \\ P^j_r \end{pmatrix},$$

and

$$\mathcal{S} = \mathcal{S}_a + \mathcal{S}_b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -qv^j \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -S_E \\ E_f \\ 0 \\ 0 \\ 0 \end{pmatrix} \left[ E_j + e^{ijkl} B_m - v^j v_l E_l \right].$$

Here we decompose $\mathcal{S}$ as $\mathcal{S} = \mathcal{S}_a + \mathcal{S}_b$. Note that $\mathcal{S}_a$ makes Equation (39) stiff for the ideal limit ($\eta \rightarrow 0$) or when the cooling/scattering timescale is shorter than the dynamical timescale. On the other hand, $\mathcal{S}_b$ is independent of $\eta$, $\kappa$, and $\sigma_r$ so that we can integrate this term explicitly (Komissarov 2007; Palenzuela et al. 2009). Then, the one-dimensional discretization of Equation (35) is given by

$$\mathcal{U}_{i}^{\eta} - \frac{\Delta t}{\Delta x} \left[ f^i_{+1/2} - f^i_{-1/2} \right] = S_{a,i} \Delta t,$$

$$\mathcal{U}_{i}^{\eta+1} - \mathcal{U}_{i}^{\eta} = \mathcal{S}^{\eta+1} \Delta t.$$ (65)

For the first step, we compute the surface values of the primitive variables. The procedure is the same as that for SR-RMHD given in Equations (40)–(42).

For the second step, we compute the numerical fluxes $f_{i \pm 1/2}$ using the HLL scheme. Similar to SR-RMHD, the eigenvalues and eigenvectors can be computed independently for the electromagnetofluids and the radiation. For a fluid, a fastest wave speed is light speed since we solve a full set of Maxwell’s equations. Thus, we take $\lambda^\pm = \pm 1$ in Equation (43) so that the HLL scheme reduces to the Lax–Friedrich scheme (Komissarov 2007). For the radiation field, we can compute the numerical fluxes using the HLL scheme as described in the previous section.

For the third step, we solve Equation (64) using numerical fluxes. At this step, $S_a$ is integrated explicitly. The conserved variables at the auxiliary step $U^{\eta}$ are then obtained. As discussed in Section 3.1, we can compute total energy ($e_r = e + E_r$) and momentum ($m^j = m^j + F^j_r$) at the $n + 1$-time step from Equations (47) and (48) after the third step, i.e., before solving Equation (65).

For the fourth step, we integrate the source terms $S_b$, which consist of two equations, the radiation moment equations and Ampere’s law. Since $E_r$ and $F_r$ do not appear in $j$, we can integrate Equation (65) for the radiation fields and the electric fields independently.

For the radiation moment equations, we can integrate $\mathcal{S}_b$ using the implicit scheme described in the previous subsection. For Ampere’s law, we adopt an implicit scheme proposed by Palenzuela et al. (2009). In their scheme, the electric field is evaluated at the $(m+1)$th iteration step.

Then, we recover $\mathcal{U}^{(m+1)}$ from $\mathcal{U}^{(m)}$. The gas energy density $\epsilon_h$ and the momentum $m_h$ for fluids are computed from Equations (47) and (48) by

$$\epsilon_{h}^{(m+1)} = \rho_h y^2 - p_g = \epsilon_{h}^{(m+1)} - \frac{|B^{a+1}|^2 + |E^{(m+1)}|^2}{2},$$

$$m_{h}^{(m+1)} = \rho_h y u = m_{h}^{(m+1)} - F_{r}^{(m+1)} - E^{(m+1)} \times B^{a+1}.$$ (68)

Note that we have $E$ at the $(m + 1)$th iteration step since it is both a primitive and a conserved variable in SR-R2MHD. Thus, the electromagnetic energy density and the Poynting flux at the $(m + 1)$-step are already determined. We compute $\rho$, $u$, and $p_g$ from $D$, $m_h$, and $\epsilon_h$ while they are in turn computed from $D$, $m$, and $\epsilon$ in SR-RMHD.

We adopt a recovery method developed by Zenitani et al. (2009). In their method, a single quartic equation on $u = \sqrt{|\mathbf{u}|^2}$ is numerically solved:

$$\Gamma_1^2 (\epsilon_h^2 - |m_h|^2) u^2 - 2 \Gamma_1 |m_h|Du^3 + [\Gamma_1^2 \epsilon_h^2 - D^2 - 2 \Gamma_1 (\Gamma_1 - 1) |m_h|^2] u^2 - 2 (\Gamma_1 - 1) D |m_h| u - (\Gamma_1 - 1)^2 |m_h|^2 = 0,$$ (69)

where $\Gamma_1 = \Gamma/(\Gamma - 1)$. $p_g$, $\rho$, and $u$ are obtained by

$$p_g = \frac{1}{\Gamma_1 - 1} \left( \frac{|m_h|}{\gamma \sqrt{\gamma^2 - 1} - D} - \frac{D}{\gamma - 1} \right),$$

$$\rho = \frac{D}{\gamma},$$

$$u = \frac{m_h}{\rho_h \gamma^2}.$$ (72)

Here, we omit a superscript $(m)$ for simplicity.
Now, we obtain all of the primitive variables at the \((m + 1)\)th step. Similar to SR-RMHD, \(P_t\) are evaluated at the \((m)\)th step when Equation (65) is solved. Thus, we again solve Equation (65) using updated \(P_t\). This iteration is continued until successive variables fall below a specified tolerance.

### 4. NUMERICAL TESTS

In this section, we show the results of some numerical tests for one- and two-dimensional problems. Results for one-dimensional problems of relativistic pure MHD are shown in Section 4.1. We present results of one- and two-dimensional problems of propagating radiation energy in Section 4.2 and one-dimensional shock tube problems of relativistic RHD in Section 4.3. In Section 4.4, we attempt to solve the relativistic magnetic reconnection problem using our SR-R2MHD code.

#### 4.1. Relativistic Ideal Magnetohydrodynamics

We perform four numerical tests of one-dimensional shock tube problems without radiation and resistivity. An initial discontinuity is situated at \(x = 0.5\) in a computational domain of \(x = [0, 1]\). Initial states of left \((x < 0.5)\) and right \((x > 0.5)\) regions for each problem are listed in Table 1.

In the following subsection, the relativistic MHD equations are solved using a scheme that is accurate to the first order in space. Numerical fluxes are computed by the HLL (Harten et al. 1983), HLLC (Mignone & Bodo 2006), and HLLD (Mignone et al. 2009) schemes. We note that the results from the alternative HLLC scheme (Honkkila & Janhunen 2007) are consistent with those from the scheme of Mignone & Bodo (2006).

The accuracy of our numerical code is verified by calculating the \(L - 1\) norm:

\[
L_1(g) = \sum_{i=1}^{N_x} \left| g_i^\text{ref} - g_i \right| \Delta x_i, \tag{73}
\]

where \(N_x\) and \(\Delta x_i\) are the number of grid points and the grid spacing, respectively. \(g_i\) is a numerical solution of some physical quantities, while \(g_i^\text{ref}\) is a reference solution. We use numerical results of a second-order HLLD scheme with \(N_x = 6400\) for the reference solutions, which are consistent with those obtained with more grids. In this subsection, the CFL number is fixed to be 0.8.

### 4.1.1. Isolated Contact and Rotational Discontinuities

For tests of stationary isolated contact and rotational discontinuities, which were proposed by Mignone et al. (2009), we employ \(N_x = 40\) and \(\Gamma = 5/3\). At the initial state there is a density jump, while the other quantities are continuous for the isolated contact wave front (see contact wave in Table 1). The velocity and magnetic field vectors are discontinuous, while \(\rho\) and \(P_t\) are invariant for the rotational discontinuity (see rotational wave in Table 1).

In Figure 1, we plot the density, \(\rho\), for the isolated contact discontinuity (left panel) and the \(y\) component of the magnetic fields, \(B_y\), for the isolated rotational discontinuity (right panel) at \(t = 1.0\). Plus signs, crosses, and open circles denote results with the HLL, HLLC, and HLLD solvers, respectively.

We find in the left panel that the HLLC and HLLD schemes, which intrinsically capture an entropy wave, can reproduce the contact surface, while a density profile becomes smoothed out in the case of the HLL scheme. The right panel clearly shows that the HLLD scheme, which can intrinsically capture the Alfven wave, recovers a surface of the rotational discontinuity, in contrast to the HLLC and HLL schemes. Here, we note that the profile of \(B_y\) is slightly steeper for the HLLC scheme than for the HLL scheme at \(x \sim 0.5\). This is because the numerical viscosity is smaller in the HLLC scheme than in the HLL scheme. We recognize that our numerical code can correctly capture the entropy wave by the HLLC and HLLD schemes and the Alfven waves by the HLLD scheme.

### 4.1.2. MHD Shock Tube 1

A relativistic extension of the shock tube problem originally formulated by Brio & Wu (1988) has been proposed by many authors (Balsara 2001; Del Zanna et al. 2003; Mignone & Bodo 2006; Mignone et al. 2009). In this problem (model MHDST1), an initial discontinuity is broken up into a fast rarefaction wave, a compound wave, a contact discontinuity, a slow shock, and a fast rarefaction wave, from left to right in the left and middle panels of Figure 2.

Figure 2 shows the numerical results at \(t = 0.4\). Here, we employ \(\Gamma = 2\). The left and middle panels show the \(\rho\) and \(B_y\) profiles with \(N_x = 400\). Solid, dashed, and dotted curves denote the results of the first-order HLLD, HLLC, and HLL schemes, respectively, while the reference solutions are plotted as thick solid curves. Although the profile of the rarefaction wave front

### Table 1

| Model     | \(\kappa\) | \(\Gamma\) | State | \(\rho\) | \(P_t\) | \(u^i\) | \(u^i\) | \(B^i\) | \(B^i\) | \(E^i\) |
|-----------|-----------|-----------|-------|---------|--------|--------|--------|--------|--------|--------|
| Contact wave | 0, 5/3   | L         | 10    | 1       | 0      | 1.02   | 0.292  | 5      | 1      | 0.5    |
| Rotational wave | 0, 5/3   | L         | 1     | 1       | 0      | 1.02   | 0.292  | 5      | 1      | 0.5    |
| MHDST1     | 0, 2     | L         | 1     | 1       | 0      | 0.066  | -0.424 | 0.707  | 2.4    | -1.7   |
| MHDST2     | 0, 3/4   | L         | 1     | 0.1     | 22.3   | 0      | 0      | 7      | 0      | 0.5    |
| RHDST1     | 0.4, 5/3 | L         | 1     | 1.0     | 3.0 \times 10^{-5} | 0.015   | 0      | 0      | 0      | 0      | 1.0 \times 10^{-8} |
| RHDST2     | 0.3, 2   | L         | 1.0   | 60      | 10     | 0      | 0      | 0      | 0      | 0      | 2.0 \times 10^{-7} |
| RHDST3     | 0.08, 5/3| L         | 1.0   | 6.0 \times 10^{-3} | 0.69   | 0      | 0      | 0      | 0      | 0      | 1.13 \times 10^{3} |

**Note.** Parameter sets of numerical tests. The scattering coefficient \(\sigma_s\) is taken to be zero in all models.
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\[(HLLC)\]

\[
\xrightarrow{L}
\]

\[
\xrightarrow{t}
\]

\[
\xrightarrow{\Delta x}
\]

\[
\xrightarrow{\rho}
\]

\[
\xrightarrow{B_y}
\]

\[
\xrightarrow{\gamma}
\]

\[
\xrightarrow{\gamma}
\]

\[
\xrightarrow{HLLD, HLLC, and HLL schemes, respectively.}
\]

\[
\xrightarrow{0.1–0.3}
\]

\[
\xrightarrow{0.5}
\]

\[
\xrightarrow{0.6}
\]

\[
\xrightarrow{0.65}
\]

\[
\xrightarrow{HLLD}
\]

\[
\xrightarrow{HLLC}
\]

\[
\xrightarrow{HLL}
\]

\[
\xrightarrow{\text{reference}}
\]

\[
\xrightarrow{L_1}\norm{\rho}\text{ norm compared with the reference solutions. Open circles, crosses, and plus signs correspond to the results of the HLLD, HLLC, and HLL schemes, respectively.}
\]

\[
\xrightarrow{1\text{ norm for }\rho\text{ at }t=0.4.}
\]

\[
\xrightarrow{\text{We can see that the error linearly decreases with decreasing grid size in all schemes. We note that the HLLD scheme drastically reduces the numerical errors compared to the other solvers. We find }L_1^{HLL} : L_1^{HLLC} : L_1^{HLLD} = 1.0 : 0.91 : 0.55,\text{ where the superscript }L_1\text{ indicate the scheme. Here, note that the HLLD scheme takes a longer computational time. We find }t_{HLL} : t_{HLLC} : t_{HLLD} = 1 : 1.27 : 1.61,\text{ where }t_{HLL}, t_{HLLC},\text{ and }t_{HLLD}\text{ are the normalized computational times of the HLL, HLLC, and HLLD schemes, respectively.}
\]

\[
\xrightarrow{4.1.3. MHD Shock Tube 2}
\]

\[
\xrightarrow{\text{We perform a test calculation for a collision of oppositely directing relativistic flows (model MHDST2; Balsara 2001; Del Zanna et al. 2003; Mignone & Bodo 2006; Mignone et al. 2009). Here, the bulk Lorentz factor is }\gamma \approx 22.4,\text{ and we set the specific heat ratio, }\Gamma,\text{ to be }5/3.
\]

\[
\xrightarrow{\text{Figure 3 shows results at }t=0.4.\text{ The left and middle panels show the profiles of }p_s\text{ and }B_y,\text{ respectively. The solid, dashed, and dotted curves represent the results of the first-order HLLD, HLLC, and HLL schemes with }N_x = 400,\text{ respectively, while a reference solution (second-order HLLD scheme with }N_x = 6400)\text{ is shown by the thick solid curve.}
\]

\[
\xrightarrow{\text{We find in Figure 3 that two slow mode shocks }x \approx 0.45, 0.55\text{ are sandwiched between two fast mode shocks }x \approx 0.2, 0.8\text{ and that all of the approximate Riemann solvers we adopted can capture the fastest mode (fast magnetosonic wave). Note that although a slow mode is not taken into account in the HLLD scheme, less numerical viscosity leads to an optimal solution.}
\]

\[
\xrightarrow{\text{The }L_1\text{ norm for }B_y\text{ is shown in the right panel of Figure 3. Filled circles, crosses, and plus signs indicate results with the HLLD, HLLC, and HLL schemes, respectively. The error linearly decreases with the grid spacing for all numerical schemes and the HLLD scheme is approved as the best numerical scheme compared with the other approximate Riemann solvers. This panel also shows that the accuracy of the HLLC scheme is better than that of the HLL scheme (see also Figure 7 in Mignone et al. 2009). We find }L_1^{HLL} : L_1^{HLLC} : L_1^{HLLD} = 1.0 : 0.74 : 0.51\text{ and }t_{HLL} : t_{HLLC} : t_{HLLD} = 1 : 1.11 : 1.57.
\]

\[
\xrightarrow{\text{In addition to the problems mentioned above (contact wave, rotational wave, MHDST1, MHDST2), we performed several conventional one-dimensional relativistic shock tube problems proposed by authors and demonstrated the relativistic self-similar expansion of magnetic loops in two dimensions (Takahashi et al. 2011a). We confirmed that our numerical code can solve these problems.}
\]

\[
\xrightarrow{4.2. Tests for Propagating Radiation Energy}
\]

\[
\xrightarrow{\text{We show the results of numerical tests for propagating radiation energy. We recover a light speed }c\text{ in this subsection.}}
\]
Although we solve a full set of SR-RMHD equations throughout this subsection, the RHD are not proved. The propagation of radiation energy in the static fluid is virtually tested, since the radiation force as well as the gas pressure force is negligible and the velocity of the matter is almost kept null.

4.2.1. Point Explosion

We show an expansion of the radiation field from a point source. We performed two-dimensional simulations in the $x$-$y$ plane with a volume bounded by $x = [-L, L]$ and $y = [-L, L]$, where $L = 1$ cm. We use uniformly spaced grids of 200 × 200 cells. We assume a static ($v = 0$) and constant density profile with $\rho = 1$ g cm$^{-3}$. The absorption coefficient is given by $\kappa = 0.1$ cm$^{-2}$ g$^{-1}$, and the scattering coefficient is set to be null. Thus, a computational domain is optically thin, $\tau = \rho \kappa L = 0.1$. The radiation energy and flux are initially given by

\[ E_r = 10^{10} E_{LTE} \exp \left( \frac{x^2 + y^2}{0.01} \right), \]
\[ F_x = c E_r \frac{x}{\sqrt{x^2 + y^2}}, \]
\[ F_y = c E_r \frac{y}{\sqrt{x^2 + y^2}}, \]

where $E_{LTE} = a_R T_r^4 = 10^{10}$ erg cm$^{-3}$ is the radiation energy density at the local thermodynamic equilibrium (LTE; $T_r = T_e$). Here, $T_r$ is the radiation temperature. We solve the SR-RMHD equations with first-order accuracy in space and time.

The left panel of Figure 4 shows a bird’s eye view of the radiation energy density, $E_r$, at $t = 0.75L/c$. Also, one-dimensional profiles of $E_r$ on the $x$ axis (solid curve), $y$ axis (open circles), and $y = x$ (open triangles) are plotted in the right panel. Following initial enhancements in the radiation energy, the radiation energy propagates in a circle at light speed. Since most of the radiation energy is transported without being absorbed by matter, the radiation energy density decreases with distance from the center ($r \equiv (x^2 + y^2)^{1/2}$) as approximately $E_r \propto r^{-1}$.

Such a caldera structure also appears even if we employ the Eddington approximation (Takahashi et al. 2013). However, in this method, the wave front propagates with a speed of $c/\sqrt{3}$. Such a reduction in speed is induced by the assumption in the Eddington approximation that the radiation field is isotropic ($D_j = 1/3$). On the other hand, the Eddington tensor is given by taking into account the non-isotropic radiation fields in the M-1 closure (see Equations (17)–(20)). Turner & Stone (2001) attempted a similar test problem using the FLD approximation. They also succeeded in reproducing the radiation energy propagating at the speed of light. However, since the radiation flux is basically given by the gradient of the radiation energy density in the FLD approximation, a caldera structure is not formed and the top-hat-shaped distribution of the radiation energy density appears. The M-1 closure has an advantage when radiation transport in an optically thin medium is considered.

The right panel of Figure 4 shows the one-dimensional profiles of $E_r$ at $t = 0.75L/c$. The solid curve denotes the profile at $y = 0$, while the open circles and open triangles denote $E_r$ at $x = 0$ and $y = x$, respectively. As discussed above, the radiation energy is transported at the speed of light and a caldera structure is formed. We can see that these three profiles are consistent, indicating that the space symmetry is assured in our numerical code.

In this test problem, non-zero radiation fluxes are initially given by $|F_x| = c E_r$. Then, the radiation energy propagates radially at the speed of light. When we take $F_x = 0$ initially, the radiation energy slowly expands compared to previous results. This can be confirmed in the right panel of Figure 4. The dashed curve shows $E_r(x, y = 0)$ at $t = 0.75L/c$ for a model where $F_x$ is initially zero. We find that the wave front is slightly delayed for $|F_x(t = 0)| = 0$ in comparison with that for $|F_x(t = 0)| = c E_r$. This is because the Eddington tensor becomes 1/3 when $|F_x| = 0$, leading to a wave velocity of $c/\sqrt{3}$. Since the expansion speed approaches the speed of light as time goes on, the gap of the wave fronts does not widen any further. Note that the characteristic wave velocity remains $c/\sqrt{3}$ around the origin via the nearly isotropic radiation fields. It makes profiles more diffusive. Hence, the radiation energy density for the case of $|F_x(t = 0)| = 0$ is not null around the origin.

4.2.2. Beam

We show radiation transport with a certain angle to a grid (Richling et al. 2001; González et al. 2007). We used 400 × 400 grid points that cover the computational domain $x = [0, L]$
and $y = [0, L]$, where $L = 1$ cm. We assume a constant density profile of $\rho = 1$ g cm$^{-3}$. We assume $\kappa = 1$ g cm$^{-3}$ and $\sigma_s = 0$, leading to an optical depth of $\tau = \rho \kappa L = 1$. The radiation field is in LTE with matter, whose energy is $E_r = E_{\text{LTE}} = 10^{10}$ erg cm$^{-3}$. We inject radiation from the boundary $x = 0$ and $y = [0.1, 0.2]$. The injected radiation energy is $E_{\text{inj}} = 10^5 E_{\text{LTE}}$ and the radiation flux is given by $F_x = F_y = c E_r / \sqrt{2}$. We adopted a free-boundary condition at the other boundaries. The SR-RMHD equations are solved with second-order accuracy in space and time.

Figure 5 shows a snapshot of $E_r$ at $t = 2.0L/c$. We can see that a beam profile is sharply captured in our numerical scheme thanks to its second-order accuracy. Since the radiation energy is absorbed by the matter and the emission of the matter is negligible, the radiation energy density decreases with increasing $l$. Then, the profile of $E_r$ is analytically expressed as

$$E_r(l) = E_{\text{inj}} \exp(-\rho \kappa l),$$

(77)

within the wave front of $l = ct$ (dashed curve). We can see that our numerical results excellently recover the analytical solution.

4.2.3. Shadow

We show the light propagation around a piece of dense matter. This test was proposed by Hayes & Norman (2003)
and González et al. (2007) adopted the M-1 formulation to the problem. We perform simulations with the M-1 closure and the Eddington approximation.

We utilize a simulation box bounded by \( x = [-5, 10] \) km and \( y = [0, 5] \) km with 300 \( \times \) 100 grid points. By setting \( C_{\text{crit}} = 0.5 \), the time step is \( \Delta t = C_{\text{crit}} \Delta x/c = 8.3 \times 10^{-8} \) s. We set \( \sigma_t = 0 \) in the whole range of the domain. We consider a dense clump embedded in less dense matter. The clump is located at the origin. The radius and the mass density of the clump are \( r_0 = 2 \) km and \( \rho_1 = 30 \) g cm\(^{-3} \). The density of the surrounding rarefied matter is set to \( \rho_0 = 10^{-6} \) g cm\(^{-3} \). Since we suppose here that \( \kappa = 10^{-2} \) cm\(^2\) g\(^{-1}\), the optical thickness of the clump is \( \rho_1 \kappa r_0 = 6 \times 10^4 \), and the less dense region is optically thin, \( \rho_1 \kappa \sim 15 \) km. The radiation energy density is set to be constant, \( E_r = E_{\text{LTE}} = 10^5 \) erg cm\(^{-3} \), and the rarefied matter is in LTE, initially. We assume a uniform gas temperature with \( T_g = T_L \) at the initial state.

The radiation is injected at the left boundary, \( x = -5 \) km, where the radiation energy density, \( E_{\text{inj}} \), is set to be \( 10^4 E_{\text{LTE}} \), and the radiation flux is assumed to be \( F^x_{\text{inj}} = c E_r \) with the M-1 closure and \( F^x_{\text{inj}} = c E_r / \sqrt{3} \) with the Eddington approximation. A free boundary condition is employed at the upper (\( x = 10 \) km) and right (\( y = 5 \) km) boundaries. At \( y = 0 \), we use a symmetric boundary.

The radiation energy density at \( t = 41.8 \) \( \mu \)s (left) and \( 167 \) \( \mu \)s (right), which correspond to 0.83 and 3.3 light-crossing times along the \( x \) direction, is presented by color contours in Figure 7. The upper and lower panels are the results with the Eddington and M-1 methods, respectively. In this figure, we can see a shadow behind the clump in the M-1 method (\( x > 2.0 \) km). Since the parallel light injected from the left boundary is absorbed by the dense clump and the photons are not scattered (\( \sigma_t = 0 \)), the lower right region of \( x > 2.0 \) km and \( y < 2.0 \) km is darkened by shadow. Here, we note that the HLL scheme is better than the simple Lax–Friedrich scheme in terms of reproducing such a sharp discontinuity (González et al. 2007). In contrast to the M-1 method, the shadow does not appear in the case of the Eddington model. As we have mentioned, since we assume isotropic radiation fields, the radiation comes around behind the clump even without scattering.

Figure 7 also shows that the radiation energy propagates at the speed of light for the M-1 model. The dashed line in the left lower panel indicates a wave front computed from \( x = -5 \) km + \( ct \). The resulting wave front is in good agreement with the dashed line. On the other hand, the wave speed reduces to \( c/\sqrt{3} \) in the Eddington model as we have discussed above. In the upper left panel, we find that the position of a wave front is \( x \sim 2.2 \) km, which is consistent with the estimation of \( x = -5 \) km + \( ct/\sqrt{3} \) with \( t = 41.8 \) \( \mu \)s.

We stress here again the advantage of an implicit treatment for the gas–radiation interaction (source terms). In this problem, the timescale of the gas–radiation interaction, \( (\rho_1 \kappa) \), is around \( 1.1 \times 10^{-10} \) s in the clump, which is much shorter than the time step, \( 8.3 \times 10^{-8} \) s. If we explicitly integrate the gas–radiation interaction terms, a numerical instability results. We can use a longer time step in the implicit treatment.

### 4.3. Tests for Radiation Hydrodynamics

In this subsection, we show the qualitative difference between the M-1 closure scheme and the Eddington approximation by solving the shock tube problems proposed by Farris et al. (2008). Although these authors obtained semi-analytic solutions by assuming the Eddington approximation, there are no analytic solutions with the M-1 closure due to the nonlinearity in the Eddington tensor. Since our numerical code can recover their analytical solutions by adopting the Eddington approximation in place of the M-1 closure (Takahashi et al. 2013), we clearly understand the feature of the M-1 closure scheme and the difference between this scheme and the Eddington approximation.

A simulation box is bounded by \( x = [-L, L] \), where \( L = 20 \) in normalized units. The number of grid points is fixed at \( N_g = 3200 \) in this subsection. Unlike Farris et al. (2008), initially the discontinuity is situated at \( x = 0 \). The gas and radiation are in local thermal equilibrium in both sides (\( x > 0 \) and \( x < 0 \)). The free boundary condition is applied in both boundaries (\( x = -L \) and \( x = L \)). We again take light speed as unity. The Stefan–Boltzmann constant has a fictitious value of \( a_R = E_{\gamma L} \gamma_{\text{LTE}}^4 / R^4 \), which is used to evaluate \( E_{\gamma L} \gamma_{\text{LTE}}^4 \) (Farris et al. 2008; Zanotti et al. 2011), where the prime denotes a quantity defined in the comoving frame and the subscripts \( L \) and \( R \) denote the left (\( x < 0 \)) and right (\( x > 0 \)) states, respectively. A parameter set of initial conditions is summarized in Table 1.
In the comoving frame, and the ratio of temperature, radiation energy density, and radiation flux measured in the comoving frame, and the ratio of temperature, radiation energy density, and radiation flux measured in the comoving frame, respectively. We plot the mass density, gas temperature, radiation energy density, radiation flux, and the component of the Eddington tensor are plotted. These quantities are measured in the comoving frame. The solid and dashed curves show the results with M-1 and Eddington models, respectively.

4.3.1. The Non-relativistic Strong Shock

Figure 8 shows the result of a non-relativistic strong shock problem (model RHDST1). We plot the mass density, gas temperature, radiation energy density, and radiation flux measured in the comoving frame, and the ratio of $E_r'$ to $E_r'$ at $t = 5000$ from top to bottom. Solid curves denote the solutions of the M-1 model, while dashed curves represent the numerical solutions of the Eddington model.

Since the radiation energy density is much less than the gas energy density and the radiation force does not play an important role, the mass density and the gas temperature have a sharp discontinuity at the shock ($x = 0$) like a shock tube problem with pure hydrodynamics. There is little difference in the density and gas temperature profiles between the two models. In contrast, the profiles of $E_r'$ and $F_r'^{xx}$ in the M-1 model are slightly different from those in the Eddington model. The radiation energy is transported from the shock front ($x = 0$) to the pre-shocked region ($x < 0$) in both models. The radiation flux is approximately given as $-E_r'$ for the M-1 model and as $-E_r'/\sqrt{3}$ for the Eddington model (note that the speed of light is set to unity in this subsection), so that the ratio of $F_r'^{xx}$ to $E_r'$ is smaller at $x \leq 0$ for the M-1 model than for the Eddington model (bottom panel). In addition, the radiation field is attenuated at the precursor region via absorption in both models. However, we find that the gradient of the profiles of $E_r'$ and $F_r'^{xx}$ is smoother in the M-1 model than in the Eddington model (see the region of $x < 0$). The radiation field decreases with distance from the shock front as $e^{-\rho_0|x|}$ for the M-1 model but as $e^{-\sqrt{3}\rho_0|x|}$ for the Eddington model. Such a difference is induced by the fact that the propagating speed of the radiation is decreased to $1/\sqrt{3}$ in the Eddington model as we have discussed above.

4.3.2. The Relativistic Shock

A second shock tube problem is a relativistic shock including radiation (model RHDST2). Here, the four velocity in the upstream is taken to be 10. Figure 9 shows profiles of the mass density, the temperatures of the gas (thick curves), the radiation (thin curves), the radiation energy density, flux, and the component of the Eddington tensor at $t = 5000$ from top to bottom. Solid and dashed curves denote solutions with the M-1 closure and the Eddington approximation, respectively. In this test, the shock front is stationary for the Eddington model, but it moves very slowly with a speed of $1.6 \times 10^{-4}$ for the M-1 model. In this figure, the position of the shock front is readjusted so as to be located at the origin in order to compare the solutions between the two models.

We can see that solutions between two models are qualitatively and quantitatively consistent (except for $D_r^{xx}$). This is because the optical depth is large enough, and, then, the Eddington approximation is valid. However, we find that $D_r^{xx}$ slightly deviates from $1/3$ for the M-1 model, although $D_r^{xx}$ is $1/3$ by definition for the Eddington model. The radiation energy is transported from the shock front to the precursor region, leading to a slightly anisotropic radiation field. In our M-1 model, the maximum of $F_r'^{xx}/E_r'$ is $\leq 0.31$ and then we find $D_r^{xx} = 0.38$.

Here, we note that the gas temperature is higher than the radiation temperature in the pre-shocked region, although the gas temperature cannot exceed the radiation temperature, $T_g \leq T_r$. However, the temperature cannot exceed the radiation temperature, $T_g \leq T_r$.
is reduced by absorption and approaches null with decreasing $x$. Although the radiation energy is transported to only $x \sim -7$ for the Eddington model, the leftward radiation penetrates to $x \sim -12$ for the M-1 model. Since the speed of light is effectively reduced to $1/\sqrt{3}$ as we have discussed above, the leftward radiation flux suddenly decreases via the enhanced absorption for the Eddington model. Therefore, in the M-1 model, the profile of the radiation energy density is smooth and the radiation energy density is enhanced even in the interval $x < -7$.

The leftward radiation flux in the upstream region induces a leftward radiation force (bottom panel), which works to decrease the velocity. Hence, the velocity (density) starts to decrease (increase) at $x \sim -12$ for the M-1 model and at $x \sim -7$ for the Eddington model (see the top and second panels). The bottom panel clearly shows that the pressure gradient force is weaker than the radiation force. Here we note that, in contrast to RHDST1, the precursor strongly affects the upstream gas since the radiation energy greatly exceeds the gas energy density in the present test.

### 4.4. Relativistic Petschek-type Magnetic Reconnection with the Radiation Field

Lastly, we perform an SR-R2MHD simulation of relativistic magnetic reconnection. Recently, some authors have studied relativistic magnetic reconnection without radiation by assuming a uniform resistivity model (Takahashi et al. 2011b) and a spatially localized resistivity model (Watanabe & Yokoyama 2006; Zenitani et al. 2010; Zanotti & Dumbser 2011). The importance of the radiative effects on magnetic reconnection has also been studied (Steinolfson & van Hoven 1984; Jaroschek & Hoshino 2009; Uzdensky & McKinney 2011). In this section, we adopt a spatially localized resistivity model for a fast (Petschek type) magnetic reconnection.

We solve the SR-R2MHD equations in a Cartesian coordinate system on the $x$-$y$ plane. A computational domain consists of $x = [0, 17.4]$ km and $y = [0, 5.7]$ km. We use non-uniform grids and the number of grid points is $N_x = 3500$, $N_y = 800$. A minimum grid size is $\Delta x = \Delta y = 100$ cm. Because of the symmetry of the system, we adopt a point symmetric boundary condition. Scalar quantities are symmetric at $x = 0$ and $y = 0$. At $x = 0$, $u^t$, $B^t$, $E^t$, $B^\perp$, and $E^\perp$ are symmetric while the rest of the vector components are anti-symmetric. At $y = 0$, $u^t$, $B^t$, $E^t$, $B^\perp$, and $E^\perp$ are symmetric and the vector components are anti-symmetric. The free boundary conditions are applied at the other boundaries. We assume an isothermal and uniform gas, $\rho_0 = 0.01$ g cm$^{-3}$ and $T_\infty = 10^8$ K, in the whole domain at the initial state. The gas is initially in LTE, $T_e = T_r$. We assume a force-free magnetic field configuration given by

$$B = B_0 \tanh \left( \frac{y}{\lambda} \right) e_x + B_0 \text{sech} \left( \frac{y}{\lambda} \right) e_y,$$

(Low 1973; Komissarov et al. 2007), where $B_0 = 10^{10}$ G is the amplitude of the magnetic field and $e_x$ and $e_y$ are unit vectors in the $x$ and $y$ directions. $\lambda = 10^4$ cm is the thickness of a current sheet. Since we use a mean molecular weight of 0.5, the plasma $\beta$ in the initial state is $4.1 \times 10^{-5}$.

We adopt a spatially localized resistivity model to attain the fast magnetic reconnection:

$$\eta = \eta_u + \frac{\eta_1 - \eta_u}{\cosh[(x^2 + y^2)/\lambda]^2},$$

(79)
where \( \eta_u \) and \( \eta_i \) are constants. We set the corresponding magnetic Reynolds numbers as \( R_{M,u} = 4\pi \lambda_c / \eta_u = 400 \) and \( R_{M,i} = 4\pi \lambda_c / \eta_i = 50 \). For opacity, we assume electron scattering and free–free absorption. The typical optical depth is 40 for scattering and \( 6.4 \times 10^{-6} \) for absorption.

Figure 11 shows results at \( t = 43.7 \mu s \). Color, arrows, and white curves indicate the radiation energy density, flux, and magnetic field lines in the observer’s frame, respectively. The blue line denotes the position at which there is a steep jump in \( j_z \) in the first quadrant.

Due to an enhancement of the electric resistivity at the origin, magnetic field lines start to reconnect and the gas is evacuated as outflows in the \( \pm x \) directions. Since we adopt a spatially localized resistivity model, four slow shocks attached to the diffusion region form (one of them is indicated by the blue curve; Watanabe & Yokoyama 2006; Zenitani et al. 2010; Zanotti & Dumbser 2011). This indicates that a fast Petschek-type magnetic reconnection is realized even though the radiation field is fulfilled.

We can see that the radiation energy density is confined in exhausts of outflows. The photons suffer from numerous scattering, since the system is very optically thick to scattering. Thus, the radiation flows together with matter via advection, and \( F_r \) is almost parallel to the velocity fields. This implies that the reconnection region is very brightly observed for downstream observers (on the \( x \) axis). On the other hand, it would be difficult to detect the reconnection region for observers around the \( y \) axis or the \( z \) axis.

In order to consider radiation effects on the dynamics, we plot in Figure 12 the pressure gradient force including effects of enthalpy variation (blue), electromagnetic force with a non-adiabatic term (red), radiation force (orange), and total force density (black) along a slow shock denoted by the blue line in Figure 11. This figure clearly shows that the electromagnetic force accelerates the gas and that the pressure gradient force is negligible. The matter is decelerated by the radiation force. Such a deceleration is caused by the radiation drag \((\propto (v^i E_r + v_j P^{j,\gamma}_r))\), which becomes non-negligible compared with the radiation flux force \((\propto F_r)\) when \( F_r^z < (v^i E_r + v_j P^{j,\gamma}_r) \). In the present problem, the condition of \( F_r^z < (v^i E_r + v_j P^{j,\gamma}_r) \) is moderately realized since the large optical thickness reduces the radiation flux. The typical value of \( F_r^z / (v^i E_r + v_j P^{j,\gamma}_r) \) is around unity.

In the present problem, the radiation drag is also non-negligible compared with the electromagnetic force. Here, we recover a light speed \( c \) to avoid misunderstanding. The ratio of the radiation drag to the electromagnetic force is \( \rho c \sigma \nu E_l / 4\pi c B^2 \), where we assume a mildly relativistic plasma and estimate the radiation drag and the electromagnetic force as \( \sim \rho c \sigma \nu E_l / c \) and \( \sim 4\pi \nu^2 / l \), respectively, with \( l \) being a typical length of the current sheet for the order estimation. Such a ratio is rewritten as \( 0.5 \tau_{cs} (E_r / E_{mag}) (c / \nu) \), where \( E_{mag} (= B^2 / 8\pi) \) is the magnetic energy density and \( \tau_{cs} (= \rho c / \lambda c) \) is the optical depth of the current sheet. It implies that the radiation drag tends to play an important role in magnetic reconnection in the high-density and high-velocity plasma. In our simulations, we have \( \tau_{cs} (E_r / E_{mag}) (c / \nu) \sim 3.6 \) by assuming \( l = \lambda \) and \( \nu = \nu_A = B / 4\pi \rho \). Due to the radiation drag, the outflow four velocity is about 10% slower with the radiation field than without the radiation field in our parameter set.

In Figure 13, we show the time evolution of the reconnection rate, which is defined here by \( E_r (0, 0) / B_0 \) (the \( x \) and \( y \) components of the electric fields are null by definition). The solid and dashed curves are the results with and without solving the radiation field, respectively. Due to an enhancement of the localized resistivity, magnetic field lines start to reconnect and the amplitude of the electric field rapidly increases by dissipating the magnetic energy. After \( t = 30 \mu s \), a quasi-steady state is realized and the reconnection rate becomes roughly constant. The reconnection rate at the steady state is about 10% smaller with the radiation field than without the radiation field, as discussed below. As we have already mentioned, the radiation drag force...
sloows down the outflow velocity. Then, the inflow velocity in the quasi-steady state (the downward and upward component of the velocity in the regions of $y > 0$ and $y < 0$) is also reduced. Thus, the $z$ component of the electric fields, $|E_z| = \sqrt{(\mathbf{c} \times \mathbf{E_r})_z / c}$, decreases, inducing the reduction of the magnetic reconnection rate.

Although we show the results for the one-parameter set, the reconnection rate as well as the outflow velocity would depend on the initial parameters. A systematic study of magnetic reconnection with radiation fields will be reported in a forthcoming paper.

5. SUMMARY

We developed a special relativistic-radiation-magnetohydrodynamic (SR-RMHD) code, in which the M-1 closure method is employed and a source term for the gas–radiation interaction is implicitly and iteratively integrated. We also extend our SR-RMHD code to a special relativistic resistive radiation-MHD (SR-R2MHD) code, which includes electric resistivity. Our SR-RMHD code successfully solves some of the test problems, i.e., the shock tube problems of MHD/radiation-HD and propagating radiation, and we demonstrate the radiation drag effect in the relativistic Petschek-type magnetic reconnection using the SR-R2MHD code.

Since our code uses radiation fields only in the observer’s frame, we straightforwardly compute $\mathbf{P}_r$ from $E_r$ and $\mathbf{F}_r$ through the M-1 closure method without the Lorentz transformation. In contrast, the Lorentz transformation is inevitable for the Eddington approximation. By virtue of the M-1 closure method, the anisotropic propagation of radiation is solved and the propagation speed of the radiation is $c$ in optically thin media. The Eddington approximation as well as the PLD approximation is problematic for such an anisotropy. In addition, the speed of light is reduced to $c/\sqrt{3}$ for the Eddington approximation.

In our code, all of the advection terms are explicitly integrated by setting the time step to be a fraction of $\Delta x/c$, with $\Delta x$ being the grid spacing. Implicit integration of the source term prevents the time step from shortening when the timescale of the source term (e.g., the gas–radiation interaction) becomes very small. For the implicit treatment, we directly invert a $4 \times 4$ matrix at each grid point in the SR-RMHD code. In addition to the gas–radiation interaction term, the source term appearing in Ampère’s law is also solved implicitly in the SR-R2MHD code. Then, we need to invert $3 \times 3$ and $4 \times 4$ matrices at each grid point. Such matrix inversion is carried out analytically without communication with neighbor grids. Thus, our code could be massively parallelized without difficulty. Our code would be widely utilized for relativistic astrophysical phenomena, even though the dense and less dense regions are mixed.

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