Second Order Stochastic Optimization in Linear Time

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Abstract

Stochastic optimization and, in particular, first-order stochastic methods are a cornerstone of modern machine learning due to their extremely efficient per-iteration computational cost. Second-order methods, while able to provide faster per-iteration convergence, have been much less explored due to the high cost of computing the second-order information. In this paper we develop a second-order stochastic method for optimization problems arising in machine learning based on novel matrix randomization techniques that match the per-iteration cost of gradient descent, yet enjoy the linear-convergence properties of second-order optimization. We also consider the special case of self-concordant functions where we show that a first order method can achieve linear convergence with guarantees independent of the condition number. We demonstrate significant speedups for training linear classifiers over several convex benchmarks.

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1 Introduction

In recent literature stochastic first-order optimization has taken the stage as the primary workhorse for training learning models, due in large part to its affordable computational costs, requiring linear time (in the data representation) per iteration. The main research effort in improving first-order methods is devoted to improving their convergence rates, which are in general sublinear and depend polynomially on the condition number of the objective. Such efforts have introduced elegant ideas and algorithms in recent years, including adaptive regularization [DHS11], variance reduction [JZ13], duality coordinate descent [SSZ13], and many more.

In contrast, second-order methods based on Newton’s method have been much less explored. The main hinderance in applying second-order optimization to machine learning is the prohibitive computational cost per iteration which requires computation of the Hessian in addition to a matrix inversion. These operations are infeasible for large scale problems in high dimensions. While some recent work does try to bring stochastic ideas to second-order optimization for machine learning, these attempts either still require inversion of matrices that is prohibitive or introduce dimension reduction techniques [EM15] that can potentially distort the spectral information.

In this paper we propose a novel second-order algorithm, LiSSA (Linear time Stochastic Second-Order Algorithm) for convex optimization that attains the best of both worlds: it inherits the fast second-order convergence rates known from offline optimization and their mild dependence on the condition number, while also allowing for an implementation with linear time per-iteration cost, matching the running time of gradient-based methods.

Formally, the main optimization problem we are concerned with in this paper is:

$$\min_{\theta \in \mathbb{R}^d} f(\theta) = \min_{\theta \in \mathbb{R}^d} \left\{ \frac{1}{m} \sum_{k=1}^{m} f_k(\theta) + R(\theta) \right\}$$

where each $f_k(\theta)$ is a convex function and $R(\theta)$ is a convex regularizer. Most commonly used objectives in machine learning can be written in the form given above. Examples include Logistic Regression, SVMs, etc.

In this paper we concern ourselves with second-order optimization methods, the following Newton update being the standard example:

$$\theta_{t+1} = \theta_t - \eta \nabla^2 f(\theta_t) \nabla f(\theta_t)$$

The Hessian provides curvature information to the update step, and indeed it can be shown that the update rule eventually reaches quadratic convergence [Nes04] under reasonable assumptions on the condition number of the function. Although Newton’s method comes with good theoretical guarantees, the complexity per step grows as $\Omega(md^2 + d^3)$ (the former term for computing the Hessian and the latter for inversion) making it prohibitive in practice. Recent methods such as NewSamp [EM15] try to relax these restrictions by sub-sampling the Hessian and taking an inverse. Formally, letting $i_t$ be a random number between $\{1\ldots m\}$, the estimator of the inverse is defined as $\hat{X} = \left( \sum_t \nabla^2 f_{i_t} \right)^{-1}$. They then perform the Newton step with respect to $\hat{X}$ in place of the true Hessian inverse. The above approach has two major shortcomings. First, it can be seen that $\mathbb{E}[\hat{X}] \neq \nabla^2 f$, i.e. the estimator is not unbiased. In addition, the procedure above still costs $\Omega(d^3)$ time per iteration. Even in the case of low rank projections as in [EM15], the method still requires $\Omega(d^2)$ time per step at the cost of losing spectral information.
Our algorithm LiSSA remedies both the problems described above by creating a novel unbiased estimator of the Hessian inverse. The key idea underlying our construction is the following well known fact about the Taylor series expansion of the matrix inverse.

**Fact 1.1.** For a matrix $A \in \mathbb{R}^{d \times d}$ such that $A \succeq 0$ and $\|A\| \leq 1$, we have that

$$A^{-1} = \sum_{i=0}^{\infty} (I - A)^i$$

We propose two unbiased estimators based on the above series. These estimators are formally defined in Section 3.1. The key idea underlying both the estimators is to create an unbiased estimator for the polynomial term $(I - A)^j$. This can be achieved via independent and unbiased samples $\{\tilde{A}_1 \ldots \tilde{A}_j\}$ of the matrix $A$ by considering $\prod (I - \tilde{A}_t)$ which is readily seen to be an unbiased estimator of the polynomial term above. To define the second estimator we consider a recursive reformulation of the above series which lets us capture the entire Taylor series at once without increasing the sample complexity.

Our second key contribution comes from the following observation. In the case of Generalized Linear Models and other popular objectives in machine learning, each of the functions $f_k$ typically is a function of $y_i \langle \theta, x_i \rangle$ where $x_i$ is the $i$th sample and $y_i$ is the corresponding label. The typical regularization function used in these settings is $\|\theta\|_2$. In such cases it can be seen that a sub-sample of the Hessian has the form $X_{ij} = \alpha I + \beta x_i x_j^\top$, so a typical update using our estimator has the following form

$$\theta_{t+1} = \theta_t - \prod_{j=1}^{i} (I - (\alpha I + \beta x_j x_i^\top)) \nabla f(\theta_t)$$

Note that the above product can be performed in time $O(d)$ because it essentially reduces to vector-vector multiplication. 1 This gives us a **linear (in the dimension) time update step**. We state our Algorithm (1) and then provide the formal guarantee with respect to step size and convergence (Theorem 3.3) in Section 3.2. Further in section 4 we show that LiSSA can be modified to run in time linear in sparsity of the input data. We also provide an alternative interpretation of our estimator which allows us to combine any first order method([JZ13, SSZ16]) with LiSSA to achieve better dependence of our algorithm on the condition number. The following theorem formally described in Section 4.2 shows a typical result we prove in the paper.

**Theorem** (Informal). After a sufficient number of initial gradient descent steps, Algorithm 2 run with Acc-SDCA returns a point $x_t$ such that with probability at least $1 - \delta$

$$f(x_t) \leq \min_{x^*} f(x^*) + \varepsilon$$

in total time $\tilde{O}(m + \min \{\sqrt{\kappa l m, \kappa l}\} d \log \left(\frac{1}{\delta}\right) \log \log \left(\frac{1}{\varepsilon}\right))$.

The convergence guarantees for our algorithm depends on the local condition number $\kappa_l$, which is a bound on the condition number of the Hessian at any point in the set. The convergence rate of most first order methods on the other hand depend on a global notion of condition number.

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1We note that using the techniques described in [Pea94] a Hessian vector performed in linear time in much more general cases than the rank one restriction. Since that is the case with the most common Machine Learning objectives we stick with these cases.
We define these notions formally in Section 2. Furthermore, in Section 6 we provide the results of our empirical evaluation of LiSSA as compared to other standard algorithms on various real-world datasets.

In the second half of the paper we turn our attention to the special case when the function $f$ is self-concordant. Self-concordant functions are a sub-class of convex functions which have been extensively studied in convex optimization literature in the context of Interior Point Methods [Nem04]. For self-concordant functions we show that the basic Newton’s method takes a constant number of iterations independent of the condition number to get into a region where one can find an ellipsoid in which both the function is well conditioned and the true minimum is guaranteed to be contained in the ellipsoid. We then show that a standard first order method can be used to achieve linear convergence to the optimum. Owing to the fact that the function is well conditioned on the ellipsoid, we show that SVRG converges linearly with a complexity independent of the condition number. We state our Algorithm (3) and prove the convergence guarantees formally in Section 5.

1.1 Related Work

Due to their time complexity advantage, first order methods have dominated the space of optimization algorithms. Especially for machine learning applications, stochastic first order methods have proven quite successful, starting from Stochastic Gradient Descent (SGD) [RM51] to more recent work, including SDCA [SSZ13], SAGA [RSB12, DBLJ14] and variance reduction methods such as SVRG [JZ13, ZMJ13].

Second-order methods and their variants have been the cornerstone for the development of Interior Point Methods [Nem04]. However, these techniques typically deal with full Hessian information, which is not suitable for machine learning applications. Cheaper alternatives such as Quasi-Newton methods [Bro70, Fle70, Gol70, Sha70, Nes04] have been proposed and proven to be effective. The line of work which is closest to ours employs subsampling to create estimates of the true Hessian [Mar10, BCNN11]. The related work of [EM15] is the first such result to provide convergence guarantees for such algorithms. Other methods include taking a hybrid approach that combine the above two methods to gain improvement [SYG07, BHNS14].

2 Preliminaries

2.1 Definitions and Notations

We adopt the convention of denoting vectors and scalars as small alphabets, matrices as large alphabets, and vectors in boldface. Let $x, y \in \mathbb{R}^d$ and let $A \in \mathbb{R}^{d \times d}$ be such that $A \succeq 0$. We denote the matrix norm of $x$ with respect to $A$ as $\|x\|_A = \sqrt{x^T A x}$. We will use $\|\cdot\|$ without a subscript to denote the $\ell_2$ norm for vectors and the spectral norm for matrices.

Throughout the paper we denote $x^* \triangleq \arg\min_{x \in K} f(x)$. For a convex set $K$ and a positive semi-definite matrix $A \succeq 0$, define $\Pi^A_K(x)$ to be projection of $x$ onto $K$ in the norm $\|\cdot\|_A$. In the case of standard Euclidean projection we will omit the norm and refer to it as $\Pi_K(x)$. In the cases that we are interested in, the set $K$ will be the unit ball in some norm $\|\cdot\|_A$ around a point $x_0$ and we would take the projection in the norm given by the matrix $A$ itself. In such a case it can be seen that $\Pi^A_K(x) = x_0 + \frac{(x-x_0)}{\|x-x_0\|_A}$. A convex function $f$ over a set $K \subseteq \mathbb{R}^d$ is defined to be $\alpha$-strongly convex with respect to a
norm $\| \cdot \|$ if, for all $x, y \in K$,
\[
f(y) \geq \nabla f(x)^\top (y - x) + \frac{\alpha}{2} \| y - x \|^2
\]

Similarly, $f$ is defined to $\beta$-smooth if
\[
f(y) \leq \nabla f(x)^\top (y - x) + \frac{\beta}{2} \| y - x \|^2
\]

2.2 Different Notions of Condition Number

We now define several measures for the condition number of a function $f$. The differences between these notions are subtle and we use them to precisely characterize the running time for our algorithms. For an $\alpha$-strongly convex and $\beta$-smooth function $f$, we will denote the condition number of the function as $\kappa(f) \triangleq \frac{\beta}{\alpha}$ or $\kappa$ when the function is clear from the context. If $f$ is of the form $f(x) = \frac{1}{m} \sum_{k=1}^{m} f_k(x)$, let $\beta_{\max}(x) \triangleq \max_{k} \lambda_{\max}(\nabla^2 f_k(x))$ and $\alpha_{\min}(x) \triangleq \min_{k} \lambda_{\min}(\nabla^2 f_k(x))$. Then, we define
\[
\hat{\kappa} \triangleq \frac{\max_{x} \beta_{\max}(x)}{\alpha}, \quad \hat{\kappa}_{\max} \triangleq \frac{\max_{x} \beta_{\max}(x)}{\min_{x} \alpha_{\min}(x)}
\]

The running time for our algorithm depends on a more local definition of condition number which we denote as $\kappa_l$ and is defined as
\[
\kappa_l \triangleq \max_{x} \frac{\lambda_{\max}(\nabla^2 f(x))}{\lambda_{\min}(\nabla^2 f(x))}
\]

It can be seen that $\kappa_l \leq \kappa$.

Similarly, there is a notion of local condition number for $\hat{\kappa}$, namely
\[
\hat{\kappa}_l \triangleq \max_{x} \frac{\beta_{\max}(x)}{\lambda_{\min}(\nabla^2 f(x))}
\]

It is also necessary to describe formally the notion of a “worst case” local condition number, as it plays a crucial role in the analysis of Theorem 3.3. Namely, we define
\[
\hat{\kappa}_{l,\max} \triangleq \max_{x} \frac{\beta_{\max}(x)}{\alpha_{\min}(x)}
\]

In the next subsection we collect key concepts and pre-existing results that we use for our analysis in the rest of the paper.

2.3 Matrix Concentration

The following lemma is a standard concentration of measure result for sums of independent matrices. \footnote{The theorem in the reference states the inequality for more nuanced bounded variance case. We only state the simpler bounded spectral norm case which suffices for our purposes.} An excellent reference for this material is [Tro12].
Theorem 2.1 (Matrix Bernstein [Tro12]). Consider a finite sequence \( \{X_k\} \) of independent, random, Hermitian matrices with dimension \( d \). Assume that
\[
E[X_k] = 0 \text{ and } \|X_k\| \leq R
\]
Define \( Y = \sum_k X_k \). Then we have for all \( t \geq 0 \)
\[
Pr(\|Y\| \geq t) \leq d \exp \left( -\frac{t^2}{4R^2} \right)
\]

2.4 Self-Concordant Functions

In this section we define and collect some of the well known properties of self-concordant functions. An excellent reference for this material is the lecture notes on this subject by Nemirovski [Nem04]. We begin by defining self-concordant functions.

Definition 2.2 (Self-Concordant Functions). Let \( K \subseteq \mathbb{R}^n \) be a non-empty open convex set, and let \( f : K \mapsto \mathbb{R} \) be a \( C^3 \) convex function. Then, \( f \) is said to be self-concordant if
\[
|\nabla^3 f(x)[h, h, h]| \leq 2\left( h^\top \nabla^2 f(x)h \right)^{3/2},
\]
where we have
\[
\nabla^k f(x)[h_1, \ldots, h_k] \triangleq \frac{\partial^k}{\partial t_1 \cdots \partial t_k} |_{t_1=\ldots=t_k} f(x + t_1 h_1 + \cdots + t_k h_k).
\]

Another key object in the analysis of self concordant functions is the notion of a Dikin Ellipsoid, which is the unit ball around a point in the norm given by the Hessian \( \| \cdot \|_{\nabla^2 f} \) at the point. We will refer to this norm as the local norm around a point and denote it as \( \| \cdot \|_x \). Formally,

Definition 2.3 (Dikin ellipsoid). The Dikin ellipsoid of radius \( r \) centered at a point \( x \) is defined as
\[
W_r(x) \triangleq \{ y \mid \|y - x\|_{\nabla^2 f(x)} \leq r \}
\]

One of the key properties of self-concordant functions that we use is that inside the Dikin ellipsoid, the function is well conditioned with respect to the local norm at the center. The next lemma makes this formal. The proof of this lemma can be found in [Nem04].

Lemma 2.4 (See [Nem04]). For all \( h \) such that \( \|h\|_x < 1 \) we have that
\[
(1 - \|h\|_x)^2 \nabla^2 f(x) \preceq \nabla^2 f(x + h) \preceq \frac{1}{(1 - \|h\|_x)^2} \nabla^2 f(x)
\]

The next lemma establishes that the function is smooth in the Dikin ellipsoid. The proof of this lemma can be found in [Nem04]

Lemma 2.5 (See [Nem04]). For all \( h \) such that \( \|h\|_x < 1 \) we have that
\[
f(x + h) \leq f(x) + \langle \nabla f(x), h \rangle + \rho(\|h\|_x)
\]
where \( \rho(x) \triangleq -\ln(1-x) - x \)
Another key quantity which is used both as a potential function as well as a dampening for the step size in the analysis of Newton’s method in general is the Newton Decrement which is defined as $\lambda(x) \triangleq \| \nabla f(x) \|_2 = \sqrt{\nabla f(x)^T \nabla^{-2} f(x) \nabla f(x)}$. The following lemma quantifies how $\lambda(x)$ behaves as a potential by showing that once it drops below 1, it ensures that the minimum of the function lies in the current Dikin ellipsoid. This is the property which we use crucially in our analysis. The proof can be found in [Nem04].

**Lemma 2.6** (See [Nem04]). If $\lambda(x) < 1$ then
\[
\| x - x^* \|_x \leq \frac{\lambda(x)}{1 - \lambda(x)}
\]

### 3 LiSSA: Linear (time) Stochastic Second-Order Algorithm

#### 3.1 Estimators for the Hessian Inverse

We propose two unbiased estimators based on the above series. To define the first estimator pick a probability distribution over non-negative integers $\{ p_i \}$ and sample $\hat{i}$ from the above distribution. Let $X_1, ... X_i$ be independent samples of the Hessian $\nabla^2 f$ and define the estimator as

**Definition 3.1** (Estimator 1).
\[
\tilde{\nabla}^{-2} f = \frac{1}{p_i} \prod_{j=1}^{\hat{i}} (I - X_j)
\]

Observe that our estimator of the Hessian inverse is unbiased, i.e. $E[\tilde{X}] = \nabla^{-2} f$ at any point. We describe a version of LiSSA based on Estimator 1 in the Appendix A.1 and theoretically analyze its convergence. Estimator 1 has the disadvantage that in a single sample it incorporates only one term of the Taylor series. Prima facie in order to estimate all the terms to a certain level one needs a quadratically growing number of samples which worsens the sample complexity. We address this issue by proposing a second estimator which is based on the observation that the above series has the following succinct recursive definition.

For a matrix $A$ define
\[
A^{-1}_j = \sum_{i=0}^{j} (I - A)^i
\]

i.e. the first $j$ terms of the above Taylor expansion. It is easy to see that the following recursion holds for $A^{-1}_j$
\[
A^{-1}_j = I + (I - A)A^{-1}_{j-1}
\]

Using the above recursive formulation, we now describe an unbiased estimator of $\nabla^{-2} f$ by deriving an unbiased estimator $\tilde{\nabla}^{-2} f_j$ for $\nabla^{-2} f_j$.

**Definition 3.2** (Estimator 2). Given $j$ independent and unbiased samples $\{X_1, ... X_j\}$ of the hessian $\nabla^2 f$. Define $\{\tilde{\nabla}^{-2} f_0, ... \tilde{\nabla}^{-2} f_j\}$ recursively as follows
\[
\tilde{\nabla}^{-2} f_0 = I
\]
\[
\tilde{\nabla}^{-2} f_t = I + (I - X_j)\tilde{\nabla}^{-2} f_{t-1}
\]
Algorithm 1 LiSSA: Linear (time) Stochastic Second-Order Algorithm

Input: \( T, f(x) = \sum_{k=1}^{m} f_k(x) \), \( S_1, S_2, T_1, \eta_1 \)

\[ x_1 = \text{GradientDescent}(f(x), T_1, \eta_1) \]

for \( t = 1 \) to \( T \) do
  for \( i = 1 \) to \( S_1 \) do
    \[ X_{[i,0]} = \nabla f(x_t) \]
    for \( j = 1 \) to \( S_2 \) do
      Sample \( \tilde{\nabla}^2 f_{[i,j]}(x_t) \) uniformly from \( \{\nabla^2 f_k(x_t) \mid k \in [m]\} \)
      \[ X_{[i,j]} = \nabla f(x_t) + (I - \tilde{\nabla}^2 f_{[i,j]}(x_t))X_{[i,j-1]} \]
    end for
    \[ X_{[i]} = X_{[i,S_2]} \]
  end for
  \[ X_t = 1/S_1 \left( \sum_{i=1}^{S_1} X_{[i,S_2]} \right) \]
  \[ x_{t+1} = x_t - X_t \]
end for
return \( x_{T+1} \)

It can be readily seen that \( \mathbb{E}[\tilde{\nabla}^{-2} f_j] = \nabla^{-2} f_j \) and therefore \( \mathbb{E}[\tilde{\nabla}^{-2} f_j] \rightarrow \nabla^{-2} f \) as \( j \rightarrow \infty \) giving us an unbiased estimator in the limit.

Although theoretically we have similar guarantees for both our estimators, empirically we have observed that Estimator 1 has a worse performance as compared to Estimator 2 which is what we use in our algorithm.

3.2 Algorithm

In this section we describe our main algorithm LiSSA (1). Our algorithm runs in two phases: in the first phase it runs vanilla gradient descent for \( T_1 \) steps and step size \( \eta_1 \) to shrink the function value to the regime where we can then show linear convergence for our algorithm. It further uses Estimator 2 (Def. 3.2) in place of the Hessian inverse to take Newton steps. We use two parameters \( S_1, S_2 \) to define the Newton step. \( S_2 \) represents the depth to which we capture the Taylor expansion. \( S_1 \) represents the number of unbiased estimators of the Hessian inverse we average to get better concentration for our estimator. In the algorithm, we compute the average Newton step directly instead of estimating the Hessian inverse. As mentioned earlier the purpose is that the average step can be computed in linear time.

3.3 Main Theorem

In this section we present our main theorem which analyzes the convergence properties of LiSSA and provide a proof for it. Before stating our main theoretical result we note that we make the following assumptions on our function \( f \) with respect to the different parameters related to the function. We explicitly state the dependence on these parameters in our running times.

- \( f \) is \( \alpha \)-strongly convex and \( \beta \)-smooth. \( \hat{\kappa}_l \) is the associated local condition number. \( \nabla^2 f \) has a Lipschitz constant bounded by \( M \).
• $f$ is of the form $f = \frac{1}{m} (\sum f_k)$.
• Each $f_k$ is scaled such that, $\frac{1}{\hat{\kappa}_l^{\max}} I \preceq f_k \preceq I$.  

We are now ready to state our main result.

**Theorem 3.3.** Consider Algorithm 1. Set parameters as follows

• $T_1 = 2\frac{\beta}{\alpha} \ln(8(\hat{\kappa}_l^{\max})^2 M)$, $\eta_1 = \frac{1}{\beta}$
• $S_1 = O((\hat{\kappa}_l^{\max})^2 \ln(\frac{d}{\delta}))$, $S_2 \geq 2\hat{\kappa}_l \ln(4\hat{\kappa}_l)$

The following guarantee holds for every $t \geq T_1$ with probability $1 - \delta$,

$$\|x_{t+1} - x^*\| \leq \frac{\|x_t - x^*\|}{2}$$

Moreover, we have that each step of the algorithm takes at most $\tilde{O}(md + (\hat{\kappa}_l^{\max})^2 \hat{\kappa}_l^2 d^2)$ time. Additionally we know that if all $\nabla_f x_i$ are rank one, then each step of the algorithm can be implemented in time $O(md + (\hat{\kappa}_l^{\max})^2 \hat{\kappa}_l d)$.

As an immediate corollary we get that:

**Corollary 3.4.** For $t > T_1$ Algorithm 1 returns a point $x_t$ such that with probability at least $1 - \delta$

$$f(x_t) \leq \min_{x^*} f(x^*) + \varepsilon$$

in total time $\tilde{O}(m + (\hat{\kappa}_l^{\max})^2 \hat{\kappa}_l d \ln \left(\frac{1}{\varepsilon}\right))$.

In the above theorems $\tilde{O}$ hides log factors of $\alpha, \beta, d, \frac{1}{\delta}$.

**Remark 3.5.** We note that the dependency on $\hat{\kappa}_l$ in the above theorem can be improved in two ways. The $(\hat{\kappa}_l^{\max})^2$ term is a pessimistic bound on the variance and can possibly be improved to an average case number. Secondly the dependence on $\hat{\kappa}_l$ can potentially be made $\sqrt{\hat{\kappa}_l}$ by using a lower degree approximation to the inverse by using Chebyshev polynomials instead of the Taylor series.

**Proof of Theorem 3.3.** First note that we have the following standard lemma about Gradient Descent

**Lemma 3.6.** [Folklore] For $\alpha$-strongly convex and $\beta$-smooth functions, the following guarantee holds for gradient descent algorithm with step size $\frac{1}{\beta}$

$$f(x_t) - f(x^*) \leq e^{-\frac{t\alpha}{\beta}}$$

Note that due to the scaling the function $f$ is $\frac{1}{\hat{\kappa}_l^{\max}}$-strongly convex. Therefore we also have that

$$f(x_t) \geq f(x^*) + \frac{1}{2\hat{\kappa}_l^{\max}} \|x_t - x^*\|^2$$

The scaling is without loss of generality even when looking at additive errors since due to the linear convergence this gets picked up in the log-term.
Putting the above facts together we get that
\[ \|x_t - x^*\| \leq 2\hat{k}_1^{\text{max}} \sqrt{e^{\frac{-t}{16}}} \]
Setting \( t = T_1 \) and substituting for the value, it follows that after \( T_1 \) steps of gradient descent we have
\[ \|x_{T_1} - x^*\| \leq (4\hat{k}_1^{\text{max}} M)^{-1} \]  
(1)
As can be easily seen from Definition 3.2 a single step of our algorithm is equivalent to \( x_{t+1} = x_t - \hat{\nabla}^{-2} f(x_t) \nabla f(x_t) \) where \( \hat{\nabla}^{-2} f(x_t) \) is the average of \( S_1 \) independent estimators \( \nabla^{-2} f(x_t) \).
We now make use of the following lemma

**Lemma 3.7.** Let \( x_{t+1} = x_t - \hat{\nabla}^{-2} f(x_t) \nabla f(x_t) \), as per a single iteration of Algorithm 1, and \( S_1, S_2 \) are as defined in Algorithm 1. Then if we choose \( S_2 \geq 2\hat{k}_1 \ln(2\hat{k}_1) \) we get the following guarantee on the convergence rate for every step with probability \( 1 - \delta \)
\[ \|x_{t+1} - x^*\| \leq \gamma \|x_t - x^*\| + M \|\nabla^{-2} f(x_t)\| \|x_t - x^*\|^2 \]
where \( \gamma = 16\hat{k}_1^{\text{max}} \sqrt{\frac{\ln(d\delta^{-1})}{S_1}} + \frac{1}{16}. \)
Substituting the values of \( S_1 \) and \( S_2 \), combining Equation (1) and Lemma 3.7, and noting that \( \|\nabla^{-2} f(x_t)\| \leq \hat{k}_1^{\text{max}} \), we have that at the start of the Newton phase the following inequality holds
\[ \|x_{t+1} - x^*\| \leq \frac{\|x_t - x^*\|^2}{4} + M\hat{k}_1^{\text{max}} \|x_t - x^*\|^2 \leq \frac{\|x_t - x^*\|^2}{2} \]
It can be shown via induction that the above property holds for all \( t \geq T_1 \) which concludes the proof.

### 3.4 Proof of Lemma 3.7

In this section we provide the proof of Lemma 3.7.

**Proof of Lemma 3.7.** Define \( \chi(x_t) = \int_0^1 \nabla^2 f(x^* + \tau(x_t - x^*))d\tau \). Note that \( \nabla f(x_t) = \chi(x_t)(x_t - x^*) \).
Following an analysis similar to that of Nesterov [Nes04], we have that
\[ \|x_{t+1} - x^*\| = \|x_t - x^* - \hat{\nabla}^{-2} f(x_t) \nabla f(x_t)\| \]
\[ = \|x_t - x^* - \hat{\nabla}^{-2} f(x_t) \chi(x_t)(x_t - x^*)d\tau\| \]
\[ \leq \|I - \hat{\nabla}^{-2} f(x_t) \chi(x_t)\| \|x_t - x^*\| \]
Following from the previous equations we have that
\[ \frac{\|x_{t+1} - x^*\|}{\|x_t - x^*\|} \leq \|I - \hat{\nabla}^{-2} f(x_t) \chi(x_t)\| = \| \left( I - \hat{\nabla}^{-2} f(x_t) \chi(x_t) \right) - \frac{\hat{\nabla}^{-2} f(x_t) - \nabla^{-2} f(x_t)}{a} \chi(x_t) \| \]
We now analyze the above two terms \( a, b \) separately.
\[ \|a\| = \|I - \nabla^{-2} f(x_t) \chi(x_t)\| \]
\[ \leq \|\nabla^{-2} f(x_t)\| \int_0^1 \nabla^2 f(x_t) - \nabla^2 f(x^* + \tau(x_t - x^*))d\tau\| \]
\[ \leq M \|\nabla^{-2} f(x_t)\| \|x_t - x^*\| \]
The second inequality follows from the Lipschitz bound on the Hessian. The second term can be bounded as follows

\[ \|b\| = \left( \| \hat{\nabla}^{-2} f(x_t) - \nabla^{-2} f(x) \| \| \chi(x_t) \| \right) \leq \gamma \]

The previous claim follows from Lemma 3.8 which shows a concentration bound on the sampled estimator and by noting that due to our assumption on the function we have that \( \forall x \| \nabla^2 f(x) \| \leq 1 \) and hence \( \| \chi(x) \| \leq 1 \).

Putting the above two bounds together and using triangle inequality we get that

\[ \frac{\|x_{t+1} - x^*\|}{\|x_t - x^*\|} \leq M \|\nabla^{-2} f(x_t)\| \|x_t - x^*\| + \gamma \]

which concludes the proof. \( \square \)

**Lemma 3.8.** Let \( \hat{\nabla}^{-2} f(x_t) \) be the average of \( S_1 \) independent samples of \( \nabla^{-2} f_i(x_t)S_2 \) defined in 3.2 and used in the per step update of Algorithm 1 and let \( \nabla^2 f(x_t) \) be the true Hessian. If we set \( S_2 \geq 2\hat{\kappa} l \ln(\hat{\kappa} l S_1) \), then we have that,

\[ \Pr \left( \|\hat{\nabla}^{-2} f(x_t) - \nabla^{-2} f(x_t)\| > 16\hat{\kappa} \max \sqrt{\frac{\ln(d)}{S_1}} + 1/16 \right) \leq \delta \]

**Proof of Lemma 3.8.** First note the following statement which is a straightforward implication of our construction of the estimator

\[ \mathbb{E}[\hat{\nabla}^{-2} f_i(x_t)] = \mathbb{E}[\hat{\nabla}^{-2} f_i(x_t)S_2] = \sum_{i=1}^{S_2} (I - \nabla^2 f(x_t))^i \]

We also know from Fact 1.1 that for matrices \( X \) such that \( \|X\| \leq 1 \)

\[ X^{-1} = \sum_{i=0}^{\infty} (I - X)^i \]

Since we have scaled the function such that \( \|\nabla^2 f_k\| \leq 1 \), it follows that

\[ \nabla^{-2} f(x_t) = \mathbb{E} \left[ \hat{\nabla}^{-2} f_i(x_t) \right] + \sum_{i=S_2}^{\infty} (I - \nabla^2 f(x_t))^i \]

Also note that we have \( \nabla^2 f(x_t) \geq \frac{L}{\hat{\kappa} l} \) which implies that \( \|I - \nabla^2 f(x_t)\| \leq 1 - \frac{1}{\hat{\kappa} l} \). Observing the second term in the above equation,

\[ \| \sum_{i=S_2}^{\infty} (I - \nabla^2 f(x_t))^i \| \leq \|(I - \nabla^2 f(x_t))\|^{S_2} \left( \sum_{i=0}^{\infty} \|I - \nabla^2 f(x_t)^i \| \right) \]

\[ \leq \left( 1 - \frac{1}{\hat{\kappa} l} \right)^{S_2} \left( \sum_{i=0}^{\infty} \left( 1 - \frac{1}{\hat{\kappa} l} \right)^i \right) \]

\[ \leq \left( 1 - \frac{1}{\hat{\kappa} l} \right)^{S_2} \hat{\kappa} l \]

\[ \leq \exp \left( -\frac{S_2}{\hat{\kappa} l} \right) \hat{\kappa} l \]
Since we have chosen $S_2 \geq 2\tilde{k}_i \ln(4\tilde{k}_i)$ we get that the above term is bounded by $1/16$. We will now show using Matrix Bernstein inequality 2.1 that the estimate $\nabla^2 f$ is concentrated around its expectation. To apply the inequality we first need to bound the spectral norm of each random variable. Towards that end we note that $\nabla^2 f_{S_2}$ has maximum spectral norm bounded by

$$
\|\nabla^2 f_{S_2}\| \leq \sum_{i=0}^{S_2} (1 - 1/\tilde{k}_i^{\text{max}})^i \leq \tilde{k}_i^{\text{max}}
$$

We can now apply the Matrix Bernstein inequality 2.1 which gives the following

$$
\Pr\left(\|\nabla^2 f - \mathbb{E}[\nabla^2 f]\| > \varepsilon\right) \leq 2d \exp\left(\frac{-\varepsilon^2 S_1}{64(\tilde{k}_i^{\text{max}})^2}\right)
$$

Setting $\varepsilon = 16\tilde{k}_i^{\text{max}} \sqrt{\frac{\ln(\frac{2}{S_1})}{S_1}}$ gives us that the probability above is bounded by $\delta$. Now putting together the bounds and Equation (5) we get the required result.

4 Further Extensions

4.1 Leveraging Sparsity in the Input Data

A key property of real-world datasets is that although the input is a high dimensional vector, the number of non-zero entries is typically very low. In this section we show how LiSSA can be implemented in a way to leverage the underlying sparsity of the data. Our key observation is that for cases of interest, i.e. the loss function takes the form $f_k(\theta) = \ell(y_k, \theta^T x_k)$, the rank one Hessian-vector product can be performed in $O(s)$ time where $s$ is the sparsity of the input $x_k$.

**Theorem 4.1.** Consider Algorithm 1, let $f$ be of the form described above, and let $s$ be such that the number of non zero entries in $x_i$ is bounded by $s$. Then each step of the algorithm can be implemented in time $O(ms + (\tilde{k}_i^{\text{max}})^2\kappa_i s)$.

We also have as a corollary of the above theorem:

**Corollary 4.2.** For $t > T_1$ Algorithm 1 returns a point $x_t$ such that with probability at least $1 - \delta$

$$
f(x_t) \leq \min_{x^*} f(x^*) + \varepsilon
$$

in total time $\tilde{O}(ms + (\tilde{k}_i^{\text{max}})^2\kappa_i s) \ln\left(\frac{1}{\delta}\right)$.

**Proof of Theorem 4.1.** Proof by induction. Let $c_0 = 1$, $v_0 = 0$, and consider the update rules $c_{j+1} = 1 + (1 - \lambda)c_j$ and $v_{j+1} = (1 - \lambda)v_j - \nabla^2 f_{[i,j+1]}(x)(c_j \nabla f(x) + v_j)$, where $\lambda$ is the regularization parameter. For the base case, note that $X_{[i,0]} = c_0 \nabla f(x) + v_0 = \nabla f(x)$, as is the case in Algorithm 1. Furthermore, suppose $X_{[i,j]} = c_j \nabla f(x) + v_j$. Then we see that

$$
X_{[i,j+1]} = \nabla f(x) + (I - \lambda I - \nabla^2 f_{[i,j+1]}(x))X_{[i,j]}
= \nabla f(x) + ((1 - \lambda)I - \nabla^2 f_{[i,j+1]}(x))(c_j \nabla f(x) + v_j)
= (1 + (1 - \lambda)c_j)\nabla f(x) + (1 - \lambda)(v_j) - \nabla^2 f_{[i,j+1]}(x)(c_j \nabla f(x) + v_j)
= c_{j+1} \nabla f(x) + v_{j+1}
$$

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Note that updating $c_{j+1}$ takes constant time, and $\tilde{\nabla}^2 f_{[i,j+1]}(x)(c_j \nabla f(x))$ and $\tilde{\nabla}^2 f_{[i,j+1]}(x)v_j$ can each be calculated in $O(s)$ time. It can also be seen that each product gives an $s$-sparse vector, so subtracting them from $(1 - \lambda)(v_j)$ takes $O(s)$ time. Since $\nabla f(x)$ can be calculated in $O(ms)$ time, and since $v_0$ is 0-sparse which implies the number of non-zero entries of $v_j$ is at most $js$, it follows that the total time to calculate $X_t$ is $O(ms + (\kappa_t^{\max})^2 \kappa_t s)$.

4.2 Improvements in Running Time Using General First Order Algorithms

In this section we first provide an alternative interpretation of our estimator for $\nabla^{-2}$. For this purpose consider the following quadratic approximation of the function at a point $x_t$ where we are looking to perform the Newton step.

$$Q_t(x) = f(x_t) + \langle \nabla f(x_t), x - x_t \rangle + \frac{1}{2}(x - x_t)^T \nabla^2 f(x_t)(x - x_t)$$

It can be seen that the Newton step i.e. $\nabla^{-2} f(x_t) \nabla f(x_t)$ corresponds exactly to the minimizer of the above quadratic, which we denote by $x_t^*$. Let's consider running Gradient Descent on the above quadratic $Q_t$ and let $y_t^i$ be the $i$th step in this process. By definition we have that

$$y_t^{i+1} = y_t^i - \nabla Q_t(y_t^i) = (I - \nabla^2 f(x_t))y_t^i - \nabla f(x_t)$$

The above expression corresponds exactly to the steps taken in LiSSA, the difference being that we use a sample of the Hessian instead of the true Hessian. This corresponds exactly to doing Stochastic Gradient Descent on the quadratic. It is important to note that the SGD performed in LiSSA is a special form of SGD where we use the true gradient of the function and an estimate for the Hessian. This is required for the linear convergence guarantees we show for LiSSA.

The above interpretation allows for the possibility of any first order linearly convergent scheme to be used for approximating the minimizer of the quadratic. In particular consider an algorithm $A$ which given the quadratic function $Q_t$ and an error value $\varepsilon$ produces a point $y$ such that

$$Q_t(y) - Q_t(x_t^*) \leq \varepsilon$$

with probability at least $1 - \delta_A$. Let the total time taken by the algorithm $A$ to produce the point be $T_A(\varepsilon)$. $T_A$ usually depends on the parameters, $m, d, \kappa$, of the quadratic $Q_t$. For the applications we are interested in $A$ will be a linearly convergent, i.e. $T_A \sim \log(\frac{1}{\varepsilon})$.

Given such an algorithm $A$, LiSSA2($A$) described in Algorithm 2 implements the above idea by modifying LiSSA by replacing the inner loop with $A$.

We show the following meta theorem about LiSSA2($A$).

**Theorem 4.3.** Let $\{x_t\}$ be defined as in Algorithm 2. Suppose the algorithm $A$ satisfies the condition 3 with probability $1 - \delta_A$ under the appropriate setting of parameters $A_{params}$. Assume that the given function is $\alpha$-strongly convex, $\beta$-smooth, its Hessian is $M$-Lipschitz, and we have a bound $B$ on the diameter of the function. Set the parameters in the algorithm as follows:

- $T_{GD} = 10\frac{B}{\alpha} \ln(\frac{M}{\alpha})$, $\eta_{GD} = 1/\beta$
- $T = \log \log(1/\varepsilon)$
Algorithm 2 LiSSA2(A)

Input: \( T, f(x) = \sum_{k=1}^{m} f_k(x) \), \( S_1, A, A_{params}, T_{GD}, \eta_{GD}, \varepsilon \)

\[ x_0 = \text{GradientDescent}(f(x), T_{GD}, \eta_{GD}) \]

for \( t = 1 \) to \( T \) do
  for \( i = 1 \) to \( S_1 \) do
    \[ y_i^t = A(Q_t, \varepsilon, A_{params}) \]
  end for
  \[ x_t = \arg\min_{i} Q_t(y_i^t) \]
end for

return \( x_{T+1} \)

- \( S_1 = \log_{1/\delta\alpha}(T/\delta) \)

where \( \varepsilon \) is the final error guarantee one wishes to achieve. Then we have that after \( T \) steps with probability at least \( 1 - \delta \)

\[
\min_{t=1\ldots T} \| x_t - x^* \| \leq \varepsilon.
\]

Using the above characterization we get the following immediate corollaries.\(^4\)

**Corollary 4.4.** If \( A \) is replaced by SVRG [JZ13] then under suitable setting of parameters we get that LiSSA2(SVRG) produces a point \( x \) such that

\[
f(x) - f(x^*) \leq \varepsilon
\]

with probability at least \( 1 - \delta \), in total time \( \tilde{O}(m + \kappa l) d \log(1/\varepsilon) \log \log(1/\varepsilon) \).

**Corollary 4.5.** If \( A \) is replaced by Acc-SDCA [SSZ16] then under suitable setting of parameters we get that LiSSA2(Acc-SDCA) produces a point \( x \) such that

\[
f(x) - f(x^*) \leq \varepsilon
\]

with probability at least \( 1 - \delta \), in total time \( \tilde{O}(m + \min \{ \sqrt{\kappa_1 m}, \kappa_1 \}) d \log(1/\varepsilon) \log \log(1/\varepsilon) \).

Here the \( \tilde{O} \) above hides logarithmic factors in \( \kappa, \alpha, \beta, d, \delta \), but not in \( \varepsilon \). We provide the proof of the above theorem in the Appendix. We note that the idea of applying SVRG as a sub-routine to achieve better guarantees have also recently been used in developing faster algorithms for PCA [GH15, Sha15]. We now provide a proof of Theorem 4.3.

**Proof of Theorem 4.3.** We run the algorithm \( A \) to achieve accuracy \( \frac{\varepsilon^2 \alpha}{2} \) on each of the intermediate quadratics. Since the function is \( \alpha \)-strongly convex we have that the quadratics \( Q_t \) are \( \alpha \)-strongly convex too which implies via a union bound that for all \( t \leq T \)

\[
\| x_{t+1} - x_t^* \| \leq \sqrt{\frac{2(Q_t(x_{t+1}) - Q(x_t^*))}{\alpha}} \leq \varepsilon^2
\]

\(^4\)The proof for the corollaries follow directly from the corresponding bounds on the algorithms. The bound for SVRG as stated in the paper holds in Expectation but can easily be seen to hold with high probability with a simple application of Markov’s inequality.
with probability at least $1 - \delta$.

Assume that $\forall t < T$, $\|x_t - x^*\| \geq \varepsilon$ (otherwise the theorem is trivially true). Using the Newton’s method analysis as before we get that $\forall t \leq T$

$$
\|x_{t+1} - x^*\| \leq \|x_t - x^*\| + \|x_{t+1} - x_t\| \\
\leq \|x_t - \nabla^{-2}f(x_t)\nabla f(x_t) - x^*\| + \|x_{t+1} - x_t\| \\
\leq \frac{M}{4\alpha}\|x_t - x^*\|^2 + \varepsilon^2 \\
\leq \left(\frac{M}{4\alpha} + 1\right)\|x_t - x^*\|^2
$$

where the second inequality follows from the analysis in the proof of Theorem 3.3 and Equation 4. From the analysis of gradient descent detailed in the proof of Theorem 3.3 we have that $\|x_0 - x_t\| \leq \sqrt{\frac{T}{M}}$. Applying the above inductively and using the value of $T$ prescribed by the theorem statement we get that $\|x_T - x^*\| \leq \varepsilon$.

5 Condition Number Independent Algorithms

In this section we change gears and describe an efficient linearly convergent Algorithm (3) for optimization of self-concordant functions for which the running time is independent of the condition number. We have not tried to optimize the complexity of algorithms in terms of $d$ in this section as our main focus is to make them condition number independent.

The key idea here is the ellipsoidal view of Newton’s method whereby we show that after making a constant number of full Newton steps, one can identify an ellipsoid and a norm such that the function is well conditioned with respect to the norm in the ellipsoid. This is depicted in Figure 1.

At this point one can run any first order algorithm one wishes. We in particular choose SVRG and prove its fast convergence. Algorithm 4 states the modified SVRG routine for general norms used in Algorithm 3.

We now state and prove the following theorem regarding the convergence of Algorithm 3.

**Theorem 5.1.** Let $0 < r < 1$, let $\gamma \geq 1$, and let $\nu$ be a constant depending on $\gamma, r$. Set $\eta = 10(1 - r)^2$, $S_1 = c_r = \frac{50}{(1-r)^2}$, where $c_r$ is a constant depending on $r$, and $T = \frac{f(x_1) - f(x^*)}{\nu}$. Then,

**Algorithm 3 EllipsoidCover**

1: Input : Self-concordant $f(x) = \sum_{i=1}^{m} f_i(x)$, $T$, $r \in \{0, 1\}$, initial point $x_1 \in K$, $\gamma > 1$, $S_1$, $\eta$, $T$

2: Initialize: $x_{curr} = x_1$

3: while $\lambda(x_{curr}) > \frac{r}{1+r}$ do

4: step = $\gamma(1 + \lambda(x_{curr}))$

5: $x_{curr} = x_{curr} - \frac{1}{\text{step}} \nabla^{-2}f(x_{curr})\nabla f(x_{curr})$

6: end while

7: $x_{T+1} = N - \text{SVRG}(W_r(x), f(x), \nabla^2 f(x_{curr}), S_1, \eta, T)$

8: return $x_{T+1}$

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Figure 1: Visualization of the steps taken in Algorithm 3.

after \( t > T \) the following linear convergence guarantee holds between two full gradient steps for Algorithm 3

\[
E[f(w_s) - f(w^*]) \leq \frac{1}{2} E[f(w_{s-1}) - f(w^*)]
\]

Moreover, the time complexity of each full gradient step is \( O(md + c_r d^2) \) where \( c_r \) is a constant depending on \( r \) (independent of the condition number).

Proof. It follows from Lemma 5.3 that at \( t = \frac{f(x_t) - f(x^*)}{\nu} \) the minimizer is contained in the Dikin Ellipsoid of radius \( r \) where \( \nu \) is a constant depending on \( \gamma, r \). This coupled with Lemma 5.2 shows that the function satisfies the following property with respect to \( W_{x_t} \),

\[
\forall x \in W_{x_t}, (1 - r)^2 \nabla^2 f(x_T) \preceq \nabla^2 f(x) \preceq (1 - r)^{-2} \nabla^2 f(x_T)
\]

Using the above fact and Lemma 5.4 and substituting for the parameters concludes the proof. \( \square \)

We collect the proofs of the following lemmas in the Appendix.

**Lemma 5.2.** Let \( f \) be a self-concordant function over \( K \), let \( 0 < r < 1 \), and consider \( x \in K \). Let \( W_r(x) \) be the Dikin ellipsoid of radius \( r \) and let \( \nabla f(x) \). Then, for all \( h \) s.t. \( x + h \in W_r(x) \),

\[
\alpha \nabla^2 f(x) \preceq \nabla^2 f(x + h) \preceq \beta \nabla^2 f(x)
\]

**Lemma 5.3.** Let \( f \) be a self-concordant function over \( K \), let \( 0 < r < 1 \), let \( \gamma \geq 1 \), and consider following the damped Newton step as described in Algorithm 3 with initial point \( x_1 \). Then, the
Algorithm 4 N-SVRG

**Input:** Convex set $\mathcal{K}$, $f(x) = \sum_{i=1}^{m} f_i(x)$, Norm $A \succeq 0$, update frequency $S_1$, and learning rate $\eta$

Initialize $\tilde{x}_0$

for $s = 1, 2 \ldots$ do

$\tilde{x} = \tilde{x}_{s-1}$

$\tilde{\mu} = \frac{1}{n} \sum_{k=1}^{m} \nabla f_k(\tilde{x})$

$x_0 = \tilde{x}$

for $t = 1$ to $S_1$ do

Randomly choose $i_t \in \{1 \ldots m\}$

$g = \nabla \psi_{i_t}(x_{t-1}) - \nabla \psi_{i_t}(\tilde{x}) + \tilde{\mu}$

$x_t = \Pi_{\mathcal{K}}^{\tilde{x}} (x_{t-1} - \eta A^{-1} g)$

end for

$\tilde{x}_s = x_t$ for randomly chosen $t \in \{1, \ldots, m\}$

end for

number of steps $t$ of the algorithm before the minimizer of $f$ is contained in the Dikin ellipsoid of radius $r$ of the current iterate, i.e. $x^* \in W_r(x_t)$, is at most

$$t = \frac{f(x_1) - f(x^*)}{\nu},$$

where $\nu$ is a constant depending on $\gamma, r$.

**Lemma 5.4.** Let $f$ be a convex function. Suppose there exists a convex set $\mathcal{K}$ and a positive semi-definite matrix $A$ such that $\forall x \in \mathcal{K} \ aA \leq \nabla^2 f(x) \leq \beta A$. Then the following holds between two full gradient steps of Algorithm 4

$$\mathbb{E}[f(x_s) - f(x^*)] \leq \left( \frac{1}{\alpha \eta (1 - 2\eta \beta)} n + \frac{2\eta \beta}{(1 - 2\eta \beta)} \right) \mathbb{E}[f(x_{s-1}) - f(x^*)]$$

6 Experiments

In this section we present experimental evaluation for our theoretical results. We perform the experiments for a classification task over two labels. We use the following popular and standard regularized objective function Logistic Regression, defined as

$$\lambda \|\theta\|^2 + \sum_{k} \log(1 + exp(-y_i(\theta, x_i))).$$

For all of the classification tasks we choose two values of $\lambda : \frac{1}{m}$ and $\frac{10}{m}$, where $m$ is the number of training examples. These values are generally treated as the standard value of the regularization that is needed. We perform the above classification tasks over three data sets: MNIST, CoverType, Mushrooms. The details of the datasets are provided in Table 1. To make sure our functions are scaled (such that the norm of the Hessian is bounded) we scale the above dataset points to unit norm.
Table 1: A description of datasets used in the experiment

| Data set       | m   | d   | Reference |
|----------------|-----|-----|-----------|
| MNIST4-9       | 11791 | 784 | [LC10]    |
| Mushrooms      | 8124 | 112 | [Lic13]   |
| CoverType      | 100000 | 54  | [BD99]    |

Figure 2: Differing convergence rates for LiSSA based on different choices of the $S_2$ parameter.

6.1 Comparison with Standard Algorithms

In Figure 3 we present a comparison between the running times of our algorithm with the running times of five standard and popular algorithms: AdaGrad [DHS11], BFGS [Bro70, Fle70, Gol70, Sha70], Gradient Descent, SGD, and SVRG [JZ13]. The figure displays a graphical comparison between the algorithms for every combination of the three datasets and both choices of $\lambda$. Note that the graphs plot $\log(\text{Error}) = \log(\text{Current Value} - \text{Optimum Value})$ on the Y-axis. We obtained the optimum value for each case by running our algorithm for a long enough time until it converged to the point of machine precision. The $\log(\text{Error})$ is plotted as a function of the time elapsed from the start of the run for each algorithm.

We next describe our choice of parameters for the algorithms. For AdaGrad we used the faster diagonal scaling version proposed by [DHS11]. We implemented the basic version of BFGS with backtracking line search. For Gradient Descent in each experiment we find a reasonable step size using parameter line tuning. For Stochastic Gradient Descent we use the variable step size $\eta_t = \gamma/\sqrt{t}$ which is usually the prescribed step size, and we hand tune the parameter $\gamma$. For SVRG we set the number of inner iterations to $2m$ as recommended in the original paper for convex functions, and parameter tuned the step size $\eta$.

To pick the parameters for our algorithm, we observe that it exhibits smooth behavior even in the case of $S_1 = 1$, so this is used for the experiment. However, we observe that increasing $S_2$ has a positive effect on the convergence of the algorithm up to a certain point, as a higher $S_2$ leads to a
Figure 3: Performance of LiSSA as compared to a variety of related optimization methods for different datasets and choices of regularization parameter $\lambda$. $S_1 = 1$, $S_2 \sim \kappa \ln(\kappa)$

larger per-iteration cost. This behavior is consistent with the theoretical analysis. We summarize the comparison between the per-iteration convergence and the value of $S_2$ in Figure 2. As the theory predicts $S_2$ to be of the order $\kappa \ln(\kappa)$, for our experiments we determine an estimate for $\kappa$ and set $S_2$ to around $\kappa \ln(\kappa)$. This value is typically equal to $m$ in our experiments. We observe that setting $S_2$ in this way resulted in the experimental results displayed in Figure 3.

We observe in Figure 3 that the performance of our algorithm is significantly better than AdaGrad, GD, SGD, BFGS, and considerably better than SVRG. We would like to point out that we chose $S_1 = 1$ for the above experiments which seems to work well in practice.

6.2 Comparison with Second-Order Methods

In this subsection we present a comparison between LiSSA, NewSamp [EM15] and vanilla Newton’s method. We perform this experiment on the MNIST database and show the convergence properties of all three algorithms over time as well as over iterations. We could not replicate the results of NewSamp on all of our datasets as it sometimes seems to diverge in our experiments. For Logistic Regression on the MNIST dataset we could get it to converge by setting the value of $S_1$ to be
slightly higher. We observe as is predicted by the theory that when compared with respect to number of iterations, NewSamp and LiSSA perform similarly while vanilla Newton performs the best as it shows a quadratic convergence. This can be seen in Figure 4. However when we consider the performance in terms of time for these algorithms we see that LiSSA has a significant advantage.

7 Conclusion

In this paper we present LiSSA, a linear time stochastic second-order algorithm for convex optimization of standard machine learning objectives. The algorithm makes use of a novel unbiased estimator of the Hessian inverse to provide second-order information, and it has the attractive property that the Newton step can implemented in linear time in the input sparsity. We also show that LiSSA can be coupled with state-of-the-art first order methods to achieve improved convergence guarantees. In addition, we discuss a first-order method for self-concordant functions that can achieve linear convergence with guarantees that are condition number independent. We further show the efficacy of our algorithm through experiments over multiple real world datasets. An aspect of our algorithm we would like to highlight is its stability and robustness across data points in terms of the choice of starting points and hyperparameters.

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A Appendix

A.1 Estimator 1 - Algorithm and Analysis

In this section we present a version of LiSSA (5) which uses Estimator 1 defined in 3.1. The underlying distribution \( p_i \) used in our algorithm is \( p_i = 1/S_2 \) for \( i \leq S_2 \) and 0 otherwise where \( S_2 \) is a parameter provided to the algorithm.

Algorithm 5 LiSSA: Linear (time) Stochastic Second-Order Algorithm, Estimator 1

Input: \( T, f(x) = \sum_{i=1}^{m} f_i(x), S_1, S_2, T_1, \eta_1 \)

\[ x_1 = \text{GradientDescent}(f(x), T_1, \eta_1) \]

for \( t = 1 \) to \( T \) do

for \( i = 1 \) to \( S_1 \) do

Choose \( r \in [0, S_2] \) uniformly

for \( j = 1 \) to \( r \) do

Sample \( \tilde{\nabla}^2 f_{[i,j]}(x_t) \) uniformly from \( \{\nabla^2 f_k(x_t) \mid k \in [m]\} \)

end for

Let \( \tilde{\nabla}^{-2} f_{[i]}(x_t) = S_2 \prod_{j=1}^{r} (I - \tilde{\nabla}^2 f_{[i,j]}(x_t)) \)

end for

\( \tilde{\nabla}^{-2} f(x_t) = 1/S_1 \left( \sum_{i=1}^{S_1} \tilde{\nabla}^{-2} f_{[i]}(x_t) \right) \)

\[ x_{t+1} = x_t - \tilde{\nabla}^{-2} f(x_t) \nabla f(x_t) \]

end for

return \( x_{T+1} \)

We now state the theorem that presents the convergence analysis of Algorithm 5 and provide a proof for it.

**Theorem A.1.** Consider Algorithm 5. Set parameters as follows

- \( T_1 = 2\beta \alpha \ln(8(\hat{\kappa}^\max \ell)^2 M), \eta_1 = \frac{1}{\beta} \)

- \( S_1 = O((\hat{\kappa}^\max \ell)^2 \ln(d)), S_2 \geq 2\hat{\kappa}_l \ln(S_1 \hat{\kappa}_l) \)

The following guarantee holds for every \( t \geq T_1 \) with probability \( 1 - \delta \),

\[ \|x_{t+1} - x^*\| \leq \frac{\|x_t - x^*\|}{2} \]

Moreover, we have that each step of the algorithm takes at most \( \tilde{O}(md + (\hat{\kappa}^\max \ell)^2 \hat{\kappa}_l d^2) \) time. Additionally we know that if all \( \nabla^2 f_i \) are rank one, then each step of the algorithm can be implemented in time \( O(md + (\hat{\kappa}^\max \ell)^2 \hat{\kappa}_l d) \).

As an immediate corollary we get that:

**Corollary A.2.** For \( t > T_1 \) Algorithm 5 returns a point \( x_t \) such that with probability at least \( 1 - \delta \)

\[ f(x_t) \leq \min_{x^*} f(x^*) + \varepsilon \]

in total time \( \tilde{O}(m + (\hat{\kappa}^\max \ell)^2 \hat{\kappa}_l d \ln (\frac{1}{\delta})) \).
The proof of Theorem A.1 is identical to the proof of Theorem 3.3 except for the use of Lemma 3.8. We need to use the following Lemma A.3 in place of Lemma 3.8.

**Lemma A.3.** Let $\tilde{\nabla}^{-2}f(x_t)$ be the estimator of the inverse Hessian defined in Algorithm 1 and let $\nabla^2f(x_t)$ be the true Hessian. If we set $S_2 \geq 2\hat{\kappa}_l \ln(\hat{\kappa}_l S_1)$, then we have that,

$$\Pr\left(\|\tilde{\nabla}^{-2}f(x_t) - \nabla^{-2}f(x_t)\| > 16\hat{\kappa}_l^{\max} \sqrt{\frac{\ln(\frac{\delta}{\hat{\kappa}_l}) \ln(S_1)}{S_1} + 1/S_2^2}\right) \leq \delta$$

**Proof of Lemma 3.8.** First note the following statement which is a straightforward implication of our construction of the estimator

$$E[\tilde{\nabla}^{-2}f_i(x_t)] = \sum_{i=1}^{S_2} (I - \nabla^2 f(x_t))^i$$

We also know from Fact 1.1 that for matrices $X$ such that $\|X\| \leq 1$

$$X^{-1} = \sum_{i=0}^{\infty} (I - X)^i$$

Since we have scaled the function such that $\|\nabla^2 f_k\| \leq 1$, it follows that

$$\nabla^{-2}f(x_t) = E[\tilde{\nabla}^{-2}f_i(x_t)] + \sum_{i=S_2}^{\infty} (I - \nabla^2 f(x_t))^i$$

(5)

Also note that we have $\nabla^2 f(x_t) \succeq \frac{I}{\hat{\kappa}_l}$ which implies that $\|I - \nabla^2 f(x_t)\| \leq 1 - \frac{1}{\hat{\kappa}_l}$. Observing the second term in the above equation,

$$\|\sum_{i=S_2}^{\infty} (I - \nabla^2 f(x_t))^i\| \leq \|(I - \nabla^2 f(x_t))\|^{S_2} \left(\sum_{i=0}^{\infty} \|I - \nabla^2 f(x_t)\|^i\right)$$

$$\leq \left(1 - \frac{1}{\hat{\kappa}_l}\right)^{S_2} \left(\sum_{i=0}^{\infty} \left(1 - \frac{1}{\hat{\kappa}_l}\right)^i\right)$$

$$\leq \left(1 - \frac{1}{\hat{\kappa}_l}\right)^{S_2} \hat{\kappa}_l$$

$$\leq \exp\left(-\frac{S_2}{\hat{\kappa}_l}\right) \hat{\kappa}_l$$

Since we have chosen $S_2 \geq 2\hat{\kappa}_l \ln(\hat{\kappa}_l S_1)$ we get that the above term is bounded by $1/S_2^2$. We will now show using Matrix Bernstein inequality 2.1 that the estimate $\tilde{\nabla}^{-2}f$ is concentrated around its expectation. To apply the inequality we first need to bound the spectral norm of each random variable. Towards that end we note that $\tilde{\nabla}^{-2}f_i$ has maximum spectral norm when $r$ is chosen to be 1 and therefore we have that the following holds always

$$\|\tilde{\nabla}^{-2}f_i\| \leq (1 - 1/\hat{\kappa}_l^{\max}) S_2 \leq 4\hat{\kappa}_l \ln(\hat{\kappa}_l S_1)$$

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We can now apply the Matrix Bernstein inequality 2.1 which gives the following
\[
\Pr \left( \| \tilde{\nabla}^{-2} f - E[\tilde{\nabla}^{-2} f] \| > \varepsilon \right) \leq 2d \exp \left( \frac{-\varepsilon^2 S_1}{64\tilde{\kappa}_i^2 \ln(\tilde{\kappa}_i S_1)} \right)
\]
Setting \( \varepsilon = 16\tilde{\kappa}_i \sqrt{\frac{\ln(\frac{d}{\delta}) \ln(S_1)}{S_1}} \) gives us that the probability above is bounded by \( \delta \). Now putting together the bounds and Equation (5) we get the required result.

\[\square\]

A.2 Proof of SVRG Lemma

**Proof of Lemma 5.4.** The proof follows the original proof of SVRG [JZ13] with a few modifications to take into account the general norm. For any \( i \), consider
\[
g_i(x) = f_i(x) - f_i(x^*) - \langle \nabla f_i(x^*), x - x^* \rangle
\]
We know that \( g_i(x^*) = \min_x g_i(x) \) since \( \nabla g_i(x) = 0 \). Therefore
\[
0 = g_i(x^*) \leq \min_{\eta} \left( g_i(x - \eta A^{-1} \nabla g_i(x^*)) \right)
\leq \min_{\eta} \left( g_i(x) - \eta \| g_i(x) \|_{A^{-1}}^2 + 0.5 \beta \eta^2 \| g_i(x) \|_{A^{-1}}^2 \right)
= g_i(x) - \frac{1}{2\beta} \| g_i(x) \|_{A^{-1}}^2
\]
The second inequality follows from smoothness of \( g_i \) (Note that \( g_i \) is as smooth as \( f_i \)). Summing the above inequality over \( i \) and setting \( x^* \) to be the minimum of \( f \) and hence \( \nabla f(x^*) = 0 \) we get
\[
\frac{\sum_{i=1}^{m} \| \nabla f_i(x) - \nabla f_i(x^*) \|_{A^{-1}}^2}{2m\beta} \leq f(x) - f(x^*)
\]
Define \( v_t = \nabla f_{i_t}(x_{t-1}) - \nabla f_{i_t}(x) + \tilde{\mu} \). Conditioned on \( x_{t-1} \) we can take the expectation with respect to \( i_t \) and obtain
\[
E \left[ \| v_t \|_{A^{-1}}^2 \right] \leq 2E \left[ \| \nabla f_{i_t}(x_{t-1}) - \nabla f_{i_t}(x^*) \|_{A^{-1}}^2 \right] + 2E \left[ \| \nabla f_{i_t}(x) - \nabla f_{i_t}(x^*) \|_{A^{-1}}^2 \right]
\leq 2E \left[ \| \nabla f_{i_t}(x_{t-1}) - \nabla f_{i_t}(x^*) \|_{A^{-1}}^2 \right] + 2E \left[ \| \nabla f_{i_t}(x) - \nabla f_{i_t}(x^*) \|_{A^{-1}}^2 \right]
\leq 4\beta \left( f(x_{t-1}) - f(x^*) + f(x) - f(x^*) \right)
\]
The above set inequalities follow by noting the following three facts.

- \( \| a + b \|_{A^{-1}}^2 \leq 2\| a \|_{A^{-1}}^2 + 2\| b \|_{A^{-1}}^2 \)
- \( \tilde{\mu} = \nabla f(\tilde{x}) \)
- \( E \left[ \| X - EX \|_{A^{-1}}^2 \right] \leq E \left[ \| X \|_{A^{-1}}^2 \right] \)
Now note that conditioned on $x_{t-1}$ we have that $Ev_t = \nabla f(x_{t-1})$ and therefore

$$E\|x_t - x^*\|_A^2 \leq \|x_{t-1} - \eta A^{-1}v_t - x^*\|_A^2$$

$$= \|x_{t-1} - x^*\|_A^2 - 2\eta(x_{t-1} - x^*, A A^{-1}E v_t) + \eta^2 E\|v_t\|_{A^{-1}}^2$$

$$\leq \|x_{t-1} - x^*\|_A^2 - 2\eta(x_{t-1} - x^*, \nabla f(x_{t-1})) + 4\eta^2 (f(x_{t-1}) - f(x^*) + f(\tilde{x}) - f(x^*))$$

$$\leq \|x_{t-1} - x^*\|_A^2 - 2\eta (f(x_{t-1}) - f(x^*)) + 4\eta^2 (f(x_{t-1}) - f(x^*) + f(\tilde{x}) - f(x^*))$$

$$= \|x_{t-1} - x^*\|_A^2 - 2\eta(1 - 2\eta\beta) (f(x_{t-1}) - f(x^*)) + 4\beta \eta^2 (f(\tilde{x}) - f(x^*))$$

The first inequality uses the Pythagorean inequality for norms, the second inequality uses the derived inequality for $E\|v_t\|_{A^{-1}}^2$ and the second inequality uses the convexity of $f(x)$. Consider a fixed stage $s$, so that $\tilde{x} = \tilde{x}_{s-1}$ and $\tilde{x}_s$ are selected after all the updates have been completed. By summing the previous inequality over $t$ and taking an expectation over the history we get

$$E\|x_n - x^*\|_A^2 + 2\eta(1 - 2\eta\beta)n E[f(x_s) - f(x^*)] \leq E\|x_0 - x^*\|_A^2 + 4\beta m \eta^2 E[f(\tilde{x}) - f(x^*)]$$

$$= \|\tilde{x} - x^*\|_A^2 + 4\beta m \eta^2 E[f(\tilde{x}) - f(x^*)]$$

$$\leq 2\left(\alpha \eta(1 - 2\eta\beta) + 2\beta m \eta^2\right) E[f(x_s) - f(x^*)]$$

The second inequality uses the strong convexity property. Therefore we have that

$$E[f(x_s) - f(x^*)] \leq \left(\frac{1}{\alpha \eta(1 - 2\eta\beta)} + \frac{2\beta \eta}{(1 - 2\eta\beta)}\right) E[f(x_{s-1}) - f(x^*)]$$

\[\square\]

### A.3 Proof of Ellipsoidal Cover Lemma

**Proof of Lemma 5.3.** Let $\lambda(x)$ be the Newton decrement at $x$. By Lemma 2.6 we know that if $\lambda(x) \leq \frac{r}{1 + r} < 1$, then

$$\|x - x^*\|_x \leq \frac{\lambda(x)}{1 - \lambda(x)} = r.$$

Consider a single iteration of the algorithm at time $t$. If $\lambda(x_t) \leq \frac{r}{1 + r}$, then we may conclude that $x^* \in W_r(x_t)$. Therefore, it is only when $\lambda(x_t) > \frac{r}{1 + r}$ that $x^*$ may not be contained within $W_r(x_t)$. Since our update is of the form

$$x_{t+1} = x_t - \frac{1}{\gamma(1 + \lambda(x_t))} \nabla^2 f(x_t) \nabla f(x_t),$$

We have the following

$$f(x_{t+1}) \leq f(x_t) - \frac{1}{\gamma(1 + \lambda(x_t))} \nabla f(x_t) \nabla^2 f(x_t)^{-1} \nabla f(x_t) + \rho \left(\frac{\lambda(x_t)}{\gamma(1 + \lambda(x_t))}\right)$$

$$= f(x_t) - \frac{(\lambda(x_t))^2}{\gamma(1 + \lambda(x_t))} + \rho \left(\frac{\lambda(x_t)}{\gamma(1 + \lambda(x_t))}\right)$$

$$= f(x_t) - \frac{(\lambda(x_t))^2}{\gamma(1 + \lambda(x_t))} - \ln \left(1 - \frac{\lambda(x_t)}{\gamma(1 + \lambda(x_t))}\right) - \frac{\lambda(x_t)}{\gamma(1 + \lambda(x_t))}$$

$$= f(x_t) - \frac{\lambda(x_t)}{\gamma} + \ln \left(1 + \frac{\lambda(x_t)}{\gamma + (\gamma - 1)\lambda(x_t)}\right)$$

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where the first inequality follows from Lemma 2.5. It now follows that
\[
f(x_t) - f(x_{t+1}) \geq \frac{\lambda(x_t)}{\gamma} - \ln \left( 1 + \frac{\lambda(x_t)}{\gamma + (\gamma - 1)\lambda(x_t)} \right) \\
\geq \frac{\lambda(x_t)}{\gamma} - \frac{\lambda(x_t)}{\gamma + (\gamma - 1)\lambda(x_t)} \\
= \lambda(x_t) \left( \frac{1}{\gamma} - \frac{1}{\gamma + (\gamma - 1)\lambda(x_t)} \right) \\
> \frac{r}{1 + r} \left( \frac{1}{\gamma} - \frac{1}{\gamma + (\gamma - 1)r} \right) \\
> 0
\]
where the second inequality comes from the fact that, for all \( x \in \mathbb{R}, \ln(1 + x) \leq 1 + x \). Let \( \nu = \frac{r}{1 + r} \left( \frac{1}{\gamma} - \frac{1}{\gamma + (\gamma - 1)r} \right) \). Then we see that after
\[
f(x_1) - f(x^*)
\]
steps, we can guarantee that we have arrived at \( x_t \) such that \( x^* \in W_r(x_t) \). \( \square \)