Ground state solutions for Kirchhoff-type equations with general nonlinearity in low dimension

Jing Chen and Yiqing Li

Abstract
This paper is dedicated to studying the following Kirchhoff-type problem:

\[
\begin{cases}
-m(\|\nabla u\|^2_{L^2(\mathbb{R}^N)}) \Delta u + V(x)u = f(u), & x \in \mathbb{R}^N; \\
u \in H^1(\mathbb{R}^N),
\end{cases}
\]

where \( N = 1, 2 \), \( m : [0, \infty) \to (0, \infty) \) is a continuous function, \( V : \mathbb{R}^N \to \mathbb{R} \) is differentiable, and \( f \in C(\mathbb{R}, \mathbb{R}) \). We obtain the existence of a ground state solution of Nehari–Pohozaev type and the least energy solution under some assumptions on \( V \), \( m \), and \( f \). Especially, the existence of nonlocal term \( m(\|\nabla u\|^2_{L^2(\mathbb{R}^N)}) \) and the lack of Hardy’s inequality and Sobolev’s inequality in low dimension make the problem more complicated. To overcome the above-mentioned difficulties, some new energy inequalities and subtle analyses are introduced.

MSC: 35J20; 35J60

Keywords: Kirchhoff-type problem; Nehari–Pohozaev manifold; Least energy solution

1 Introduction
In this paper, we consider the following Kirchhoff-type equation:

\[
\begin{cases}
-m(\|\nabla u\|^2_{L^2(\mathbb{R}^N)}) \Delta u + V(x)u = f(u), & x \in \mathbb{R}^N; \\
u \in H^1(\mathbb{R}^N),
\end{cases}
\]

where \( N = 1, 2 \), \( m : [0, \infty) \to (0, \infty) \), \( V : \mathbb{R}^N \to \mathbb{R} \) and \( f : \mathbb{R} \to \mathbb{R} \) are continuous functions.

Problem (1.1) has a profound physical meaning for it is related to the stationary analogue of the Kirchhoff equation, which arises in nonlinear vibrations, see Alves, Corrêa, and Figueiredo [1] for more details. The following equation is a special case of (1.1) when \( \mathbb{R}^N \) is replaced by a bounded domain \( \Omega \) and \( f(s) = V(x)s \) by \( f(x, s) \):

\[-m(\|\nabla u\|^2_{L^2(\Omega)}) \Delta u = f(x, u) \quad \text{in } \Omega. \]
Equation (1.2) appears when we search for a stationary solution to
\[ u_{tt} - m \left( \int_{\Omega} |\nabla u|^2 \, dx \right) \Delta u = f(x, t) \quad \text{in } \Omega \times (0, T). \]  
(1.3)

Problem (1.3) was proposed by Kirchhoff [20] when \( m(t) = a + bt \) and \( N = 1 \). We refer the readers to [2–4, 9, 10] for more mathematical and physical background on Kirchhoff-type problems.

Lions [25] proposed an abstract functional analysis framework to the Kirchhoff equation
\[ u_{tt} - \left( a + b \int_{\Omega} |\nabla u|^2 \, dx \right) \Delta u = f(x, t) \quad \text{in } \Omega \times (0, T), \]  
(1.4)

after which extensive attention to (1.4) was aroused. In recent years, the following Kirchhoff-type problem
\[ \begin{cases} 
- (a + b \int_{\mathbb{R}^N} |\nabla u|^2 \, dx) \Delta u + V(x) u = |u|^{q-1} u, & x \in \mathbb{R}^N; \\
 u \in H^1(\mathbb{R}^N) 
\end{cases} \]  
(1.5)

has been studied intensively by many researchers, where \( V \in C(\mathbb{R}^N, \mathbb{R}) \) and \( f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}) \), \( a, b > 0 \) are constants. By variational methods, a number of important results of the existence and multiplicity of solutions for problem (1.5) have been established with \( f \) satisfying various conditions. In the meantime, \( V \) is usually assumed as a constant, or periodic, or radial, or coercive, see for example [1, 5, 7, 8, 11, 14–16, 19, 21, 23, 24, 26–28, 30, 31, 34, 36–38] and the references therein.

Recently, Ikoma [17] investigated the following Kirchhoff-type equations with power type nonlinearity:
\[ \begin{cases} 
- (a \int_{\mathbb{R}^N} |\nabla u|^2 \, dx) \Delta u + V(x) u = |u|^{q-1} u, & x \in \mathbb{R}^N; \\
 u \in H^1(\mathbb{R}^N) 
\end{cases} \]  
(1.6)

Here, \( q \in (1, \infty) \) if \( N = 1, 2 \) and \( q \in (1, \frac{N+2}{N-2}) \) if \( N \geq 3 \). Problem (1.6) in the general dimensions was considered, and the author proved that problem (1.6) has a ground state solution. Precisely, they assumed the following conditions on \( m \) and \( V \) when \( N = 1, 2, \) or \( N \geq 3 \):

\begin{enumerate}
\item [(M1)] \( m \in C([0, \infty)) \) and there exists \( m_0 > 0 \) such that \( 0 < m_0 \leq m(s) \) for all \( s \in [0, \infty) \); \n\item [(M2)] There exist \( q_0 > 0 \) and \( \epsilon_0 > 0 \) such that \( M(s) - \frac{1}{q_0-1} m(s) \geq \epsilon_0 s \) in \( (0, \infty) \); \n\item [(M3)] The function \( s^{-1} M(s) \) is nondecreasing in \( (0, \infty) \); \n\item [(V1)] \( V \in C^1(\mathbb{R}^N, \mathbb{R}) \) such that \( \lim_{|x| \to \infty} V(x) = \sup_{x \in \mathbb{R}^N} V(x) =: V_\infty < +\infty \); \n\item [(V2)] \( 0 < \inf_{x \in \mathbb{R}^N} V(x) =: V_0 \); \n\item [(V3)] Let \( q_0 \) be the constant that appeared in (M2). When \( 1 < q < 2q_0 + 1 \), there exist \( \alpha, \beta > 0 \) such that
\[ |x \cdot \nabla V(x)| \leq C (1 + |x|)^\alpha \quad \text{for all } x \in \mathbb{R}^N, \]
\[ x \cdot \nabla V(x) \leq \beta V(x), \quad \beta \in \left( 0, \frac{(q-1)2 - (N-2)q_0}{2q_0 + 1 - q} \right) \quad \text{for all } x \in \mathbb{R}^N. \]
\end{enumerate}
Through refined topological analysis on energy functional of “limit problem”, Ikoma compared the mountain pass level $c_{\nu, \lambda}$ with the one of the limit equation $c_{\infty, \lambda}$. Specially, in Ikoma’s analysis, the situation of $N = 1$ is different from the one when $N \geq 2$. To get a ground state solution of (1.6), the author used the standard arguments which combined the monotonicity trick to get a bounded (PS)$_{c_{\nu, \lambda}}$ sequence and the concentration–compactness lemma to prove that the sequence has a convergent subsequence. It is worth noting that, in Ikoma’s argument, he had to face the difficulties about obtaining the boundedness of the (PS) sequence and the strongly convergent subsequences when $1 < q < 2q_0 + 1$.

Tang and Chen [32] dealt with the Schrödinger–Kirchhoff type equation

$$\begin{cases}
-(a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx) \Delta u + V(x)u = f(u), & x \in \mathbb{R}^3; \\
u \in H^1(\mathbb{R}^3),
\end{cases} \tag{1.7}$$

when the nonlinearity $f$ is more general. They proved that (1.7) possesses a ground state solution of Nehari–Pohozaev type and the least energy solution with the following assumptions about $f$, here $N = 3$:

(F1) $f \in C(\mathbb{R}, \mathbb{R})$ and there exist constants $C_0 > 0$ and $p_0 \in (2, 2^*)$ such that

$$|f(t)| \leq C_0 (1 + |t|^{p_0 - 1}), \quad \forall t \in \mathbb{R};$$

(F2) $f(t) = o(t)$ as $t \to 0$;

(F3) $\lim_{t \to \infty} \frac{F(t)}{t^p} = +\infty$, where $F(t) := \int_0^t f(s) \, ds$;

(F4) The function $\frac{(t|t|^p + NF(t))^{p}}{|p-p'|^{p}}$ is nondecreasing on $t \in (-\infty, 0) \cup (0, \infty)$, where $p > 2$.

Precisely, they proved that there exists the least energy solution following the standard approach as Ikoma’s result. The difference is that they developed a new trick and compared the level $c_\lambda$ with the energy $I^\infty_{\chi\nu}$ of the minimizer $u^\infty_{\chi\nu}$ more directly. Here, $c_\lambda$ is the limited energy of a bounded (PS) sequence $\{u_n(\lambda)\}$ for almost every $\lambda \in [1/2, 1)$. By using their original highlight inequalities to obtain the comparison, they got a minimizer $u^\infty_{\chi\nu}$ on the Nehari–Pohozaev manifold which is also used in Li [22] and Ruiz [29] for $\lambda \in [1/2, 1]$. Then, by using the global compactness lemma obtained in [22], they got a nontrivial critical point $u_\lambda$ which possesses energy $c_\lambda$. Their approach is applicable to the problems of other types, such as Schrödinger–Poisson problems [32], Choquard equations [33], and so on.

Let us point out that some arguments and tricks used in [32] fail to adapt directly to one- and two-dimensional cases for Hardy’s and Sobolev’s inequalities and do not work at this point. In this sense, it is more complicated than a three-dimensional case. To the best of our knowledge, there are few results concerning (1.1) in one- and two-dimensional case. Based on [17, 32], the main purpose of this paper is to extend and complement the corresponding existence results on (1.7) in a three-dimensional case and above to the lower dimensional situation. However, the general term $m(t)$ is more difficult to deal with than the special form $a + bt$, and the appearance of $m(\|\nabla u\|_{L^2(\mathbb{R}^N)})$ makes problem (1.1) more intricate. In addition, it is noticed that there is no need to distinguish $N = 1$ and $N = 2$ in our approach.

Now we state our hypotheses. Let $N = 1, 2$. For $V$, we suppose (V1), (V2) as above, and we use a mild hypothesis (V3) in place of (V3). Furthermore, (V4) makes some subtle
inequalities hold, which helps us to prove the existence of ground states. For \( f \), we suppose (F1)–(F4). For \( m \), we assume (M1) above, (M2) instead of (\( \tilde{M} \)), (M3), and (M4).

(V3) \( (p - N)\cdot V(x) - \nabla V(x) \cdot x \geq 0 \), where \( p > 2 \);

(V4) The map \( t \mapsto \frac{V(tx)}{p^2 - 1}(\nabla \cdot \nabla V(tx)) \) is nonincreasing on \( (0, +\infty) \) for all \( x \in \mathbb{R}^N \setminus \{0\} \), where \( \cdot \) denotes the inner product in \( \mathbb{R}^N \), \( p > 2 \) holds here and after;

(M2) There exists \( \varepsilon_0 > 0 \) such that \( M(s) - \frac{N}{2N}m(s) \geq \varepsilon_0 s \) for \( s \in (0, \infty) \), where \( M(s) := \int_0^s m(t) \, dt \);

(M3) The map \( t \mapsto -m\cdot m(t)^{p-1} \) is nonincreasing in \( (0, \infty) \) for all \( s \in (0, \infty) \);

(M4) The function \( m \) is nondecreasing in \( [0, \infty) \).

A simple example of \( m \) satisfying all conditions (M1)–(M4) is the following:

\[
m(t) = a + bt^\frac{1}{p},
\]

where \( a > 0, b > 0, p > 2 \). We can easily verify that \( m(t) \) fits all the above hypotheses.

To state our results, we define the norm in \( H^1(\mathbb{R}^N) \)

\[
\|u\| = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) \, dx \right)^{1/2},
\]

(1.8)

\[
\|u\|_s = \left(\int_{\mathbb{R}^N} |u|^s \, dx \right)^{1/s}, \quad 1 \leq s < +\infty,
\]

(1.9)

and the energy functional

\[
E(u) = \frac{1}{2} M(\|\nabla u\|_2^2) + \frac{1}{2} \int_{\mathbb{R}^N} V(x)u^2 \, dx - \int_{\mathbb{R}^N} F(u) \, dx.
\]

(1.10)

Under assumptions (V1), (F1), and (F2), weak solutions to (1.1) correspond to critical points of \( E \) and \( E \in C^1(\mathbb{R}^N, \mathbb{R}) \). For any \( \varepsilon > 0 \), it follows from (F1) and (F2) that there exists \( C_\varepsilon > 0 \) such that

\[
|f(t)| \leq \varepsilon |t| + C_\varepsilon |t|^{p-1}, \quad \forall t \in \mathbb{R}.
\]

(1.11)

Let us define the Pohozaev functional for (1.1) by

\[
\mathcal{P}(u) := \frac{N-2}{2} m(\|\nabla u\|_2^2)\|\nabla u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^N} \left[ NV(x) + (\nabla V(x), x) \right] u^2 \, dx
\]

\[
- N \int_{\mathbb{R}^N} F(u) \, dx,
\]

(1.12)

that is, \( \mathcal{P}(u) = 0 \) if \( u \) is a critical point of \( E \), and define the following constraint set:

\[
\mathcal{M} := \{ u \in H^1(\mathbb{R}^N) \setminus \{0\} : J(u) := \langle E'(u), u \rangle + \mathcal{P}(u) = 0 \},
\]

(1.13)

clearly, \( u \in \mathcal{M} \) if \( u \) is a critical point of \( E \).

Our main results are as follows.

**Theorem 1.1** Suppose that \( m, V, \) and \( f \) satisfy (M1)–(M3), (V1)–(V4), and (F1)–(F4). Then Problem (1.1) has a solution \( \bar{u} \in H^1(\mathbb{R}^N) \) such that \( E(\bar{u}) = \inf_{\mathcal{M}} E > 0 \).
Theorem 1.2 Suppose that \( m, V, \) and \( f \) satisfy (M1)–(M4), (V1)–(V3), and (F1)–(F4). Then Problem (1.1) has a least energy solution \( \bar{u} \in H^1(\mathbb{R}^N) \setminus \{0\} \).

The paper is organized as follows. In Sect. 2, we give some preliminaries and the proof of Theorem 1.1. Section 3 is devoted to proving Theorem 1.2. In this paper, \( C_1, C_2, \ldots \) denote positive constants possibly different in different places.

2 Preliminaries

To obtain the ground state solution of (1.1), we establish the key energy inequality related with \( E \) and \( J \). To this end, we first prove the following two inequalities.

Lemma 2.1 Suppose that (F1) and (F4) hold. Then

\[
    t^N F(tu) - F(u) + \frac{1 - t^{2+p}}{2+p} \left[ f(u)u + NF(u) \right] \geq 0, \quad \forall t \geq 0, u \in \mathbb{R}. \tag{2.1}
\]

Proof It is evident that (2.1) holds for \( u = 0 \). For \( u \neq 0 \), let

\[
    g(t) = t^N F(tu) - F(u) + \frac{1 - t^{2+p}}{2+p} \left[ f(u)u + NF(u) \right], \quad t \geq 0. \tag{2.2}
\]

Then, from (F4), one has

\[
    g'(t) = Nt^{N-1} F(tu) + t^N f(tu)u - t^{1+p} \left[ NF(u) + f(u)u \right] = t^{1+p} \left| u \right|^{2+p-N} \left[ \frac{f(tu)u + NF(tu)}{|tu|^{p-N}} - \frac{f(u)u + NF(u)}{|u|^{p-N}} \right]
\]

\[
    \begin{align*}
        &\geq 0, \quad t \geq 1, \\
        &\leq 0, \quad 0 < t < 1.
    \end{align*}
\]

It follows that \( g(t) \geq g(1) = 0 \) for \( t \geq 0 \). This implies that (2.1) holds. \( \square \)

Lemma 2.2 Suppose that (V1) and (V4) hold. Then

\[
    V(x) - t^{N+2} V(tx) - \frac{1 - t^{2+p}}{2+p} \left[ (N+2) V(x) + \nabla V(x) \cdot x \right] \geq 0. \tag{2.3}
\]

Proof Let

\[
    h(t) := V(x) - t^{N+2} V(tx) - \frac{1 - t^{2+p}}{2+p} \left[ (N+2) V(x) + \nabla V(x) \cdot x \right].
\]

By (V4), one has

\[
    h'(t) = -(N+2)t^{N+1} V(tx) - t^{N+1} \nabla V(tx) \cdot (tx) + t^{1+p} \left[ (N+2) V(x) + \nabla V(x) \cdot x \right] \frac{1}{p-N}
\]

\[
    = t^{1+p} \left[ (N+2) V(x) + \nabla V(x) \cdot x - \frac{(N+2) V(tx) + \nabla V(tx) \cdot (tx)}{p-N} \right].
\]
\[ \begin{cases} \geq 0, & t \geq 1, \\ < 0, & 0 < t < 1. \end{cases} \tag{2.4} \]

It follows that \( h(t) \geq h(1) = 0 \) holds for \( t \geq 0 \). \hfill \square

**Lemma 2.3** Suppose that (F1), (F2), (F4), (V1), (V4), and (M3) hold. Then, for all \( u \in H^1(\mathbb{R}^N) \) and \( t > 0 \), the following key inequality holds:

\[ E(u) \geq E(tu_t) + \frac{1 - t^{2+p}}{2 + p} J(u), \tag{2.5} \]

where \( u_t(x) := u(t^{-1}x) \) are fixed.

**Proof** Note that

\[ E(tu_t) = \frac{1}{2} M(t^N \| \nabla u \|^2_2) + \frac{t^{N+2}}{2} \int_{\mathbb{R}^N} V(tx)u^2 \, dx - t^N \int_{\mathbb{R}^N} F(tu) \, dx \tag{2.6} \]

and

\[ J(u) = \langle E'(u), u \rangle + \mathcal{P}(u) \]

\[ = \frac{N}{2} m(\| \nabla u \|^2_2) \| \nabla u \|^2_2 + \frac{1}{2} \int_{\mathbb{R}^N} \left[ (2 + N)V(x) + \nabla V(x) \cdot x \right] u^2 \, dx \]

\[ - \int_{\mathbb{R}^N} [f(u)u + NF(u)] \, dx. \tag{2.7} \]

Thus, by (1.10), (2.1), (2.3), and (2.6), one has

\[ E(u) - E(tu_t) \]

\[ = \frac{1}{2} M(\| \nabla u \|^2_2) - \frac{1}{2} M(t^N \| \nabla u \|^2_2) + \frac{1}{2} \int_{\mathbb{R}^N} \left[ V(x) - t^{N+2}V(tx) \right] u^2 \, dx \]

\[ + \int_{\mathbb{R}^N} \left[ t^N F(tu_t) - F(u) \right] \, dx \]

\[ = \frac{1 - t^{2+p}}{2 + p} J(u) + \frac{1}{2} M(\| \nabla u \|^2_2) - \frac{1}{2} M(t^N \| \nabla u \|^2_2) - \frac{1 - t^{2+p}}{2 + p} \cdot \frac{N}{2} m(\| \nabla u \|^2_2) \| \nabla u \|^2_2 \]

\[ + \frac{1}{2} \int_{\mathbb{R}^N} \left[ V(x) - t^{N+2}V(tx) \right] \frac{1 - t^{2+p}}{2 + p} \left[ (N + 2)V(x) + \nabla V(x) \cdot x \right] u^2 \, dx \]

\[ + \int_{\mathbb{R}^N} \left[ t^N F(tu_t) - F(u) + \frac{1 - t^{2+p}}{2 + p} \left[ f(u)u + NF(u) \right] \right] \, dx \]

\[ \geq \frac{1 - t^{2+p}}{2 + p} J(u) + \frac{1}{2} M(\| \nabla u \|^2_2) - \frac{1}{2} M(t^N \| \nabla u \|^2_2) - \frac{1 - t^{2+p}}{2 + p} \cdot \frac{N}{2} m(\| \nabla u \|^2_2) \| \nabla u \|^2_2. \]

In fact, the following assertion holds:

\[ L(t) := \frac{1}{2} M(\| \nabla u \|^2_2) - \frac{1}{2} M(t^N \| \nabla u \|^2_2) - \frac{1 - t^{2+p}}{2 + p} \cdot \frac{N}{2} m(\| \nabla u \|^2_2) \| \nabla u \|^2_2 \geq 0. \tag{2.8} \]
From (M3), we have

\[ L'(t) = \frac{N}{2} t^{1+p} m(\|\nabla u\|_2^2) \|\nabla u\|_2^2 - \frac{N}{2} t^{N-1} m(\|\nabla u\|_2^2) \|\nabla u\|_2^2 \]

\[ = \frac{N}{2} t^{1+p} \left[m(\|\nabla u\|_2^2) \|\nabla u\|_2^2 - m(t^N \|\nabla u\|_2^2) t^N \|\nabla u\|_2^2 \right] \]

\[ \begin{cases} 
\geq 0, & t \geq 1, \\
< 0, & 0 < t < 1. 
\end{cases} \quad (2.9) \]

Then \( L(t) \geq L(1) = 0 \). That implies (2.5) holds.

From Lemma 2.3, we have the following corollary.

**Corollary 2.4** Suppose that (F1), (F2), (F4), (V1), and (V4) hold. Then, for \( u \in M \),

\[ E(u) = \max_{t>0} E(tu). \quad (2.10) \]

**Lemma 2.5** Suppose that (F1)–(F3) and (M2) hold. Then, for any \( u \in H^1(\mathbb{R}^N) \setminus \{0\} \), there exists unique \( t_u > 0 \) such that \( tu u_t \in M \).

**Proof** Let \( u \in H^1(\mathbb{R}^N) \setminus \{0\} \) be fixed. Clearly, for \( E(tu) \) defined as (2.6) on \((0, \infty)\), we have

\[ \frac{dE(tu)}{dt} = 0 \Leftrightarrow \frac{N}{2} m(t^N \|\nabla u\|_2^2) t^{N-1} \|\nabla u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^N} (N + 2) t^{N+1} V'(tx) \]

\[ \quad + t^{N+1} \nabla V(tx) \cdot (tx) \] \[ \quad \int_{\mathbb{R}^N} \left[ NF(tu) + f(tu)tu \right] dx = 0 \]

\[ \Leftrightarrow J(tu) = 0 \Leftrightarrow tu \in M. \quad (2.11) \]

By (F1) and (F2), it is easy to verify that \( E(0) = 0 \) when \( t = 0, E(tu) > 0 \) for \( t > 0 \) small. By (M2), there exists \( C_1 > 0 \) such that \( M(s) \leq M(1) \frac{2^{M1}}{2\pi} + C_1 s \) for any \( s \geq 0 \), which yields

\[ E(tu) \leq \frac{1}{2} \left\{ M(1)(t^N \|\nabla u\|_2^2) \frac{2^{M1}}{2\pi} + C_1 t^N \|\nabla u\|_2^2 \right\} + \frac{t^{N+2}}{2} \int_{\mathbb{R}^N} V(tx) u^2 dx \]

\[ - t^N \int_{\mathbb{R}^N} F(tu) dx, \quad \forall tu \in M, \quad (2.12) \]

then \( E(tu) < 0 \) for \( t \) large follows from (F3). Therefore \( \max_{t \in (0, \infty)} E(tu) \) is achieved at \( tu > 0 \) so that \( \frac{dE(tu)}{dt}|_{tu} = 0 \) and \( tu u_t \in M \).

Next we claim that \( tu \) is unique for any \( u \in H^1(\mathbb{R}^N) \setminus \{0\} \). In fact, for any given \( u \in H^1(\mathbb{R}^N) \setminus \{0\} \), let \( t_1, t_2 > 0 \) such that \( t_1 u t_1, t_2 u t_2 \in M \). Then \( f(t_1 u t_1) = f(t_2 u t_2) = 0 \). Jointly with (2.5), we have

\[ E(t_1 u t_1) \geq E(t_2 u t_2) + \frac{t_1^{2+p} - t_2^{2+p}}{(2 + p)t_1^{2+p}} f(t_1 u t_1) = E(t_2 u t_2). \quad (2.13) \]

Also,

\[ E(t_2 u t_2) \geq E(t_1 u t_1) - \frac{t_2^{2+p} - t_1^{2+p}}{(2 + p)t_2^{2+p}} f(t_2 u t_2) = E(t_1 u t_1). \quad (2.14) \]

(2.13) and (2.14) imply \( t_1 = t_2 \). Therefore, \( tu > 0 \) is unique for any \( u \in H^1(\mathbb{R}^N) \setminus \{0\} \). \( \square \)
Lemma 2.6 Suppose that (F1)–(F4) hold. Then
\[
\inf_{u_0 \in \mathcal{M}} E(u) = I = \inf_{u \in H^1(\mathbb{R}^N), \|u\|_\infty = 1} \max_{t \neq 0} E(tu).
\]

Proof Both Corollary 2.4 and Lemma 2.5 imply the above lemma. \qed

Lemma 2.7 Suppose that (V1), (V4), and (M1) hold. Then there exists \( \omega_1 > 0 \) such that
\[
\omega_1 \|u\|^2 \leq Nm(\|\nabla u\|_2^2)\|\nabla u\|_2^2 + \int_{\mathbb{R}^N} [(N + 2)V(x) + \nabla V(x) \cdot x]u^2 \, dx, \quad \forall u \in H^1(\mathbb{R}^N).
\] (2.15)

Proof From (2.3), one has
\[
(N + 2)V(x) + \nabla V(x) \cdot x \geq (2 + p) p^{N-p} V(tx) - (2 + p) t^{-(2 + p)} V(x), \quad \forall t > 0, x \in \mathbb{R}^N.
\] (2.16)
Taking \( t \to \infty \) in (2.16), we deduce that
\[
(N + 2)V(x) + \nabla V(x) \cdot x \geq 0, \quad \forall x \in \mathbb{R}^N.
\] (2.17)
Arguing by contradiction, suppose that there exists a sequence \( \{u_n\} \subset H^1(\mathbb{R}^N) \) such that \( \|u_n\| = 1 \) and
\[
Nm(\|\nabla u_n\|_2^2)\|\nabla u_n\|_2^2 + \int_{\mathbb{R}^N} [(N + 2)V(x) + \nabla V(x) \cdot x]u_n^2 \, dx = o(1). \quad (2.18)
\]
Thus there exists \( \bar{\omega} \in H^1(\mathbb{R}^N) \) such that \( u_n \rightharpoonup \bar{\omega} \). Then \( u_n \to \bar{\omega} \) in \( L^s_{\text{loc}}(\mathbb{R}^N) \) for \( 2 \leq s < \infty \) and \( u_n \to \bar{\omega} \) a.e. in \( \mathbb{R}^N \). By (2.17), (2.18), the weak semicontinuity of norm, and Fatou’s lemma, we have
\[
0 = \lim_{n \to \infty} \left\{ Nm(\|\nabla u_n\|_2^2)\|\nabla u_n\|_2^2 + \int_{\mathbb{R}^N} [(N + 2)V(x) + \nabla V(x) \cdot x]u_n^2 \, dx \right\}
\geq Nm_0 \|\nabla \bar{\omega}\|_2^2 + \int_{\mathbb{R}^N} [(N + 2)V(x) + \nabla V(x) \cdot x]\bar{\omega}^2 \, dx,
\] (2.19)
which implies \( \bar{\omega} = 0 \). Thus, from (V1) and (V4), one has
\[
\int_{\mathbb{R}^N} [(N + 2)(V(x) - V_\infty) + \nabla V(x) \cdot x]u_n^2 \, dx = o(1), \quad n \to \infty.
\] (2.20)
Both (2.18) and (2.20) imply
\[
o(1) = Nm(\|\nabla u_n\|_2^2)\|\nabla u_n\|_2^2 + \int_{\mathbb{R}^N} [(N + 2)V(x) + \nabla V(x) \cdot x]u_n^2 \, dx
\geq Nm_0 (N + 2)V_\infty \|u_n\|_2^2 + o(1)
\geq \min \{Nm_0, (N + 2)V_\infty\} \|u_n\|^2 + o(1),
\] (2.21)
which is a contradiction, and it shows that there exists \( \omega_1 > 0 \) such that (2.15) holds. \qed
Lemma 2.8 \textit{Suppose that (F1)--(F3), (M2), and (V2) hold. Then}

(i) there exists $\rho_0 > 0$ such that $\|u\| \geq \rho_0$, $\forall u \in \mathcal{M}$;

(ii) $I = \inf_{u \in \mathcal{M}} E(u) > 0$.

\textit{Proof} (i). For all $u \in \mathcal{M}$, by (F1), (1.11), (2.7), (2.15), and the Sobolev embedding theorem, one has

$$
\frac{\omega_1}{2} \|u\|^2 \leq \frac{N}{2} m(\|\nabla u\|^2_2) \|\nabla u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^N} [(2 + N)V(x) + \nabla V(x) \cdot x]u^2 \, dx \\
= \int_{\mathbb{R}^N} [f(u)u + NF(u)] \, dx \\
\leq \frac{\omega_1}{4} \|u\|^2 + C_2 \|u\|^{p_0}.
$$

\textit{(2.22)}

This implies

$$
\|u\| \geq \rho_0 := \left( \frac{\omega_1}{4C_2} \right)^{1/(p_0-2)}, \quad \forall u \in \mathcal{M}.
$$

\textit{(2.23)}

(ii). Together a direct calculation with (V2), it follows from (2.3) that

$$(p - N)V(x) - \nabla V(x) \cdot x$$

$$\geq (2 + p)t_0^{N+2}V_0 - t_0^{2+p}[(N + 2)\|V\|_\infty + \|\nabla V(x) \cdot x\|_\infty]$$

\textit{(2.24)}

for all $t \in \mathbb{R}$, $x \in \mathbb{R}^N$. There exists $t_0 > 0$ small enough such that

$$(2 + p)t_0^{N+2}V_0 - t_0^{2+p}[(N + 2)\|V\|_\infty + \|\nabla V(x) \cdot x\|_\infty] \geq \frac{V_0}{4},$$

\textit{(2.25)}

then

$$
(p - N)V(x) - \nabla V(x) \cdot x \geq (2 + p)t_0^{N+2}V_0 - t_0^{2+p}[(N + 2)\|V\|_\infty + \|\nabla V(x) \cdot x\|_\infty] \\
\geq \frac{V_0}{4} > 0.
$$

\textit{(2.26)}

Let $t \to 0$ in (2.1), then

$$
f(u)u - (2 + p - N)F(u) \geq 0, \quad \forall u \in \mathbb{R}.
$$

\textit{(2.27)}

From (M2), (2.26), and (2.27), for all $u \in \mathcal{M}$, one has

$$
E(u) = E(u) - \frac{1}{2 + p} I(u) \\
= \frac{1}{2} M(\|\nabla u\|^2_2) + \frac{1}{2(2 + p)} \int_{\mathbb{R}^N} [(p - N)V(x) - \nabla V(x) \cdot x]u^2 \, dx \\
- \frac{1}{2 + p} \frac{N}{2} m(\|\nabla u\|^2_2) \|\nabla u\|_2^2 + \frac{1}{2 + p} \int_{\mathbb{R}^N} [f(u)u - (2 + p - N)F(u)] \, dx
$$
\[
\geq \frac{1}{2} M(\|\nabla u\|_2^2) - \frac{N}{2(2 + N)} m(\|\nabla u\|_2^2) \|\nabla u\|_2^2 + \frac{V_0}{8(2 + p)} \|u\|_2^2 \\
\geq \frac{\varepsilon_0}{2} \|\nabla u\|_2^2 + \frac{V_0}{8(2 + p)} \|u\|_2^2 \\
\geq \min\left\{\frac{\varepsilon_0}{2}, \frac{V_0}{8(2 + p)}\right\} \|u\|_2^2 := C_3 \|u\|_2^2 \geq C_3 \rho_0^2 > 0. \tag{2.28}
\]

This shows that \( I = \inf_{u \in \mathcal{M}} E(u) > 0. \)

The following lemma has been proved in [12] and [32].

**Lemma 2.9** Assume that (F1) and (F2) hold. If \( u_n \rightharpoonup u \) in \( H^1(\mathbb{R}^N) \), then along a subsequence of \( \{u_n\} \),

\[
\lim_{n \to \infty} \sup_{\varphi \in H^1(\mathbb{R}^N), \|\varphi\| \leq 1} \left| \int_{\mathbb{R}^N} \left[ f(u_n) - f(u_n - u) - f(u) \right] \varphi \, dx \right| = 0. \tag{2.29}
\]

Similar to [31, Lemma 2.10], by using Lemma 2.9, we have the following lemma.

**Lemma 2.10** Assume that (F1)–(F4) hold. If \( u_n \rightharpoonup u \) in \( H^1(\mathbb{R}^N) \), then

\[
E(u_n) = E(u) + E(u_n - u) + o(1) \tag{2.30}
\]

and

\[
J(u_n) = J(u) + J(u_n - u) + o(1). \tag{2.31}
\]

Similar to [31, Lemma 2.13], by using the key inequality (2.5), the deformation lemma, and the intermediate value theorem of continuous function, the following lemma is given. We omit the proof here.

**Lemma 2.11** Suppose that (M1)–(M3), (V1)–(V4), and (F1)–(F4) hold. If \( \bar{u} \in \mathcal{M} \) and \( E(\bar{u}) = I \), then \( \bar{u} \) is a critical point of \( E \).

To overcome the lack of the compactness of Sobolev embedding, we define its limit problem related to (1.1) by

\[
\begin{align*}
-m(\|\nabla u\|_2^2) \Delta u + V_\infty u &= f(u), \quad x \in \mathbb{R}^N; \\
u &\in H^1(\mathbb{R}^N).
\end{align*} \tag{2.32}
\]

Under assumptions (F1) and (F2), weak solutions to (2.32) correspond to critical points of the energy functional defined in \( H^1(\mathbb{R}^N) \) by

\[
E^\infty(u) = \frac{1}{2} M(\|\nabla u\|_2^2) + \frac{1}{2} \int_{\mathbb{R}^N} V_\infty u^2 \, dx - \int_{\mathbb{R}^N} F(u) \, dx. \tag{2.33}
\]

Similar to (1.13) and (2.7), we define the functional

\[
J^\infty(u) := \frac{N}{2} m(\|\nabla u\|_2^2) \|\nabla u\|_2^2 + \frac{2 + N}{2} \int_{\mathbb{R}^N} V_\infty |u|^2 - \int_{\mathbb{R}^N} \left[ f(u)u + NF(u) \right] \, dx, \tag{2.34}
\]
the constraint set
\[ M^\infty := \{ u \in H^1(\mathbb{R}^N) \setminus \{0\} : J^\infty(u) = 0 \}, \] (2.35)
and the minimizer
\[ I^\infty := \inf_{u \in M^\infty} E(u). \] (2.36)

Since \( V(x) \equiv V_\infty \), which is well covered by (V1), (V2), and (V4), then all the above conclusions on \( E \) are true for \( E^\infty \). Through discussing the corresponding limit equation (2.32), we will get the proof of Theorem 1.1.

**Lemma 2.12** Assume that (M1)–(M3) and (F1)–(F4) hold. Then \( I^\infty := \inf_{u \in M^\infty} E^\infty(u) \) is achieved.

**Proof** For any \( u \in M^\infty \), we have \( E^\infty(u) \geq I^\infty \). Let \( \{u_n\} \subset M^\infty \) such that \( E^\infty(u_n) \to I^\infty \) as \( n \to \infty \). Since \( J^\infty(u_n) = 0 \), then it follows from (2.27) that
\[
I^\infty + o(1) = E^\infty(u_n) - \frac{1}{2 + p} J^\infty(u_n) \\
= \frac{1}{2} M(\|\nabla u_n\|^2_2) - \frac{N}{2(2 + p)} m(\|\nabla u_n\|^2_2)\|\nabla u_n\|^2_2 \\
+ \frac{p - N}{2(2 + p)} V_\infty \|u_n\|^2_2 + \frac{1}{2 + p} \int_{\mathbb{R}^N} [f(u_n)u_n - (p + 2 - N)F(u_n)] \, dx \\
\geq \frac{1}{2} M(\|\nabla u_n\|^2_2) - \frac{N}{2(2 + p)} m(\|\nabla u_n\|^2_2)\|\nabla u_n\|^2_2 + \frac{p - N}{2(2 + p)} V_\infty \|u_n\|^2_2. \] (2.37)

Similar to (2.28), this shows that \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^N) \).

With (2.8), the rest of the proof is similar to [31, Lemma 2.12], so we omit it. \( \square \)

The same as the case of dimension three in [31], we get the following relation between \( I \) and \( I^\infty \).

**Lemma 2.13** Suppose that (M1)–(M3), (V1)–(V4), and (F1)–(F4) hold. Then \( I < I^\infty \).

**Lemma 2.14** Suppose that (M1)–(M3), (V1)–(V4), and (F1)–(F4) hold. Then \( I \) is achieved.

**Proof** Step 1. Choosing a minimizing sequence \( \{u_n\} \subset M \) of \( I \) and showing that the sequence is bounded in \( H^1(\mathbb{R}^N) \). This part of argument is the same as Lemma 2.12. So we omit it here.

Step 2. Showing that \( \{u_n\} \) is convergent in \( H^1(\mathbb{R}^N) \). By Lion’s concentration compactness principle [35, Lemma 1.21], similar to [31, Lemma 3.2], one can easily obtain that result, so we also omit it here. \( \square \)

**Proof of Theorem 1.1** Under Lemmas 2.6, 2.11, and 2.14, there exists \( \tilde{u} \in M \) such that
\[
E(\tilde{u}) = I = \inf_{u \in H^1(\mathbb{R}^N) \setminus \{0\}} \max_{t > 0} E(tu_t), \quad E'(\tilde{u}) = 0. \] (2.38)

This shows that \( \tilde{u} \) is a ground state solution of Nehari–Pohozaev type for (1.1). \( \square \)
3 The least energy solutions for (1.1)

In this section, we give the proof of Theorem 1.2.

Proposition 3.1 ([18]) Let $X$ be a Banach space and $\Lambda \subset \mathbb{R}^+$ be an interval. We consider a family $\{\Phi_{\lambda}\}_{\lambda \in \Lambda}$ of $C^1$-functionals on $X$ of the form

$$\Phi_{\lambda}(u) = A(u) - \lambda B(u), \quad \forall \lambda \in \Lambda,$$

where $B(u) \geq 0$, $\forall u \in X$, and such that either $A(u) \to +\infty$ or $B(u) \to +\infty$, as $\|u\| \to \infty$. We assume that there are two points $v_1, v_2$ in $X$ such that

$$c_\lambda := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi_{\lambda}(\gamma(t)) > \max \{\Phi_{\lambda}(v_1), \Phi_{\lambda}(v_2)\}, \quad (3.1)$$

where

$$\Gamma = \{\gamma \in C([0,1],X) : \gamma(0) = v_1, \gamma(1) = v_2\}.$$

Then, for almost every $\lambda \in \Lambda$, there is a bounded $(PS)_{c_\lambda}$ sequence for $\Phi_{\lambda}$, that is, there exists a sequence such that

(i). $\{u_n(\lambda)\}$ is bounded in $X$;

(ii). $\Phi_{\lambda}(u_n(\lambda)) \to c_\lambda$;

(iii). $\Phi_{\lambda}'(u_n(\lambda)) \to 0$ in $X^*$, where $X^*$ is the dual of $X$.

To apply Proposition 3.1, we introduce two families of functionals defined by

$$E_{\lambda}(u) = \frac{1}{2} M(|\nabla u|^2) + \frac{1}{2} \int_{\mathbb{R}^N} V(x)u^2 \, dx - \lambda \int_{\mathbb{R}^N} F(u) \, dx \quad (3.2)$$

and

$$E_{\lambda}^\infty(u) = \frac{1}{2} M(|\nabla u|^2) + \frac{1}{2} \int_{\mathbb{R}^N} V_{\infty}u^2 \, dx - \lambda \int_{\mathbb{R}^N} F(u) \, dx \quad (3.3)$$

for $\lambda \in [1/2, 1]$.

Lemma 3.2 ([13]) Suppose that $(V1)$, $(V2)$, $(F1)$, and $(F2)$ hold. Let $u$ be a critical point of $E_{\lambda}$ in $H^1(\mathbb{R}^N)$, then we have the following Pohozaev-type identity:

$$\mathcal{P}_{\lambda}(u) := \frac{N - 2}{2} m(\|u\|_2^2)\|u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^N} \left[NV(x) + (\nabla V(x), x)\right]u^2 \, dx$$

$$- N\lambda \int_{\mathbb{R}^N} F(u) \, dx = 0. \quad (3.4)$$

We set $J_{\lambda}(u) := \langle E'_{\lambda}(u), u \rangle + \mathcal{P}_{\lambda}(u)$, then

$$J_{\lambda}(u) = \frac{N}{2} m(\|u\|_2^2)\|u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^N} \left[(N + 2)V(x) + (\nabla V(x), x)\right]u^2 \, dx$$

$$- \lambda \int_{\mathbb{R}^N} [f(u)u + NF(u)] \, dx \quad (3.5)$$
for $\lambda \in [1/2, 1]$. Correspondingly, we also let

$$J_\lambda^\infty(u) = \frac{N}{2} m(\|\nabla u\|_2^2) \|\nabla u\|_2^2 + \frac{N + 2}{2} V_\infty \|u\|_2^2 - \lambda \int_{\mathbb{R}^N} \left[ f(u)u + NF(u) \right] \, dx$$

(3.6)

for $\lambda \in [1/2, 1]$. Set

$$\mathcal{M}_\lambda^\infty := \{ u \in H^1(\mathbb{R}^N) \setminus \{ 0 \} : J_\lambda^\infty(u) = 0 \}, \quad I_\lambda^\infty := \inf_{u \in \mathcal{M}_\lambda^\infty} E_\lambda^\infty(u).$$

(3.7)

By Lemma 2.3, we have the following lemma.

**Lemma 3.3** Suppose that (M1)–(M3), (V1), (F1), (F2), and (F4) hold. Then

$$E_\lambda^\infty(u) \geq E_\lambda^\infty(tu) + \frac{1 - \lambda^2}{2 + 2\lambda} J_\lambda^\infty(u), \quad \forall u \in H^1(\mathbb{R}^N), t > 0, \lambda \geq 0.$$  

(3.8)

In view of Theorem 1.1, $E_\lambda^\infty$ has a minimizer $u_1^\infty$ on $\mathcal{M}_1^\infty$, i.e.,

$$u_1^\infty \in \mathcal{M}_1^\infty, \quad (E_1^\infty)'(u_1^\infty) = 0, \quad \text{and} \quad I_1^\infty = E_1^\infty(u_1^\infty).$$

(3.9)

**Lemma 3.4** Suppose that (M1)–(M3), (V1), (V2), and (F1)–(F3) hold. Then

(i). there exists $T > 0$ independent of $\lambda$ such that $E_\lambda(T(u_1^\infty)_T) < 0$ for all $\lambda \in [1/2, 1]$;

(ii). there exists $\kappa_0 > 0$ independent of $\lambda$ such that, for all $\lambda \in [1/2, 1]$,

$$c_\lambda := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} E_\lambda(\gamma(t)) \geq \kappa_0 > \max \{ E_\lambda(0), E_\lambda(T(u_1^\infty)_T) \},$$

(3.10)

where

$$\Gamma = \left\{ \gamma \in C([0, 1], H^1(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) = T(u_1^\infty)_T \right\};$$

(iii). $c_\lambda$ and $I_\lambda^\infty$ are nonincreasing on $\lambda \in [1/2, 1]$.

The proof of Lemma 3.4 is standard, the reader can refer to [6, Lemma 4.4].

We use the ingenious assumptions on $V$ borrowed from [31], that is, for $V \in C(\mathbb{R}^N, \mathbb{R})$ and $V(x) \leq V_\infty$ but $V(x) \not= V_\infty$, there exist $\bar{x} \in \mathbb{R}^N$ and $\bar{r} > 0$ such that

$$V_\infty > V(x) \quad \text{and} \quad \left| (u_1^\infty)(x) \right| > 0, \quad \text{a.e.} \ |x - \bar{x}| \leq \bar{r}. \quad (3.11)$$

**Lemma 3.5** Suppose that (M1)–(M3), (V1), (V2), and (F1)–(F4) hold. Then there exists $\bar{\lambda} \in [1/2, 1]$ such that $c_\lambda < I_\lambda^\infty$ for $\lambda \in [\bar{\lambda}, 1]$.

**Proof** It is easy to see that $E_\lambda(t(u_1^\infty)_T)$ is continuous on $t \in (0, \infty)$. Hence, for any $\lambda \in [1/2, 1]$, we can choose $t_\lambda \in (0, T)$ such that $E_\lambda(t_\lambda(u_1^\infty)_T) = \max_{t \in [0, T]} E_\lambda(t(u_1^\infty)_T)$. Define

$$\gamma_0(t) = \begin{cases} tT(u_1^\infty)_T, & \text{for } t > 0, \\ 0, & \text{for } t = 0. \end{cases}$$

Then $\gamma_0$ is a solution of (M3), (V1), (V2), and (F1)–(F4) with $\lambda = \bar{\lambda}$, and

$$E_\lambda(t(u_1^\infty)_T) \geq E_\lambda(\gamma_0(t)) \geq \kappa_0 > \max \{ E_\lambda(0), E_\lambda(T(u_1^\infty)_T) \},$$

for $t \in (0, T)$. Therefore, we have $c_\lambda < I_\lambda^\infty$ for $\lambda \in [\bar{\lambda}, 1]$. 


Then \( \gamma_0 \in \Gamma \) defined by Lemma 3.4(ii), i.e., \( \gamma_0(0) = 0, \gamma_0(1) = T(u_1^\infty) \). Moreover,

\[
E_t( u_1^\infty, t, \gamma_0) = \max_{t \in [0,1]} E_t( \gamma_0(t)) \geq c_\lambda. 
\] (3.12)

It follows from (2.27) that the function \( F(t)/|t|^{2p-N} \) is nondecreasing on \( t \in (-\infty,0) \cup (0, +\infty) \). Since \( t_\lambda \in (0,T) \), then we have

\[
F(t_\lambda u_1^\infty) \leq \frac{F(Tu_1^\infty)}{T^{\frac{2p-N}{2}}}. 
\]

Let

\[
\zeta_0 := \min \left\{ \frac{3\bar{r}}{8(1 + |\bar{x}|)}, 1/4 \right\}. 
\] (3.13)

Then it follows from (3.11) and (3.13) that

\[
|x - \bar{x}| \leq \frac{\bar{r}}{2} \quad \text{and} \quad \tau \in [1 - \zeta_0, 1 + \zeta_0] \quad \Rightarrow \quad |\tau x - \bar{x}| \leq \frac{\bar{r}}{2}. 
\] (3.14)

Let

\[
\tilde{\lambda} := \max \left\{ \frac{1}{2}, 1 - \frac{(1 - \zeta_0)^{N/2}}{2T^2} \int_{\mathbb{R}^N} |V_\infty - V(\tau x)| \| u_1^\infty \|^2 \, dx \right\}, 
\] (3.15)

where \( L(t) \) is defined in (2.8). Then it follows from (3.11) and (3.14) that \( 1/2 \leq \tilde{\lambda} < 1 \). We have two cases to distinguish:

Case (i). \( t_\lambda \in [1 - \zeta_0, 1 + \zeta_0] \). From (3.2), (3.3), (3.8)–(3.12), (3.14), (3.15), and Lemma 3.4(iii), we have

\[
I_\lambda^\infty \geq I_1^\infty = E_1^\infty(u_1^\infty) \geq E_1^\infty(t_\lambda u_1^\infty, t_\lambda) 
\]

\[
= E_\lambda(t_\lambda u_1^\infty, t_\lambda) - (1 - \lambda)T^2 \int_{\mathbb{R}^N} F(t_\lambda u_1^\infty) \, dx + \frac{(1 - \zeta_0)^{N/2}}{2} \int_{\mathbb{R}^N} |V_\infty - V(\tau x)| \| u_1^\infty \|^2 \, dx 
\]

\[
> c_\lambda - (1 - \lambda)T^2 \int_{\mathbb{R}^N} F(Tu_1^\infty) \, dx 
\]

\[
+ \frac{(1 - \zeta_0)^{N/2}}{2} \min_{\tau \in [1 - \zeta_0, 1 + \zeta_0]} \int_{\mathbb{R}^N} |V_\infty - V(\tau x)| \| u_1^\infty \|^2 \, dx 
\]

\[
\geq c_\lambda, \quad \forall \lambda \in [\tilde{\lambda}, 1]. 
\]

Case (ii). \( t_\lambda \in (0,1 - \zeta_0) \cup (1 + \zeta_0, T] \). From (2.5), (3.2), (3.3), (3.8), (3.11), (3.12), (3.15), Assertion 1, and Lemma 3.4(iii),

\[
I_\lambda^\infty \geq I_1^\infty = E_1^\infty(u_1^\infty) 
\]

\[
\geq E_1^\infty(t_\lambda u_1^\infty, t_\lambda) + \frac{1}{2} M(\| \nabla u_1^\infty \|^2_2) - \frac{1}{2} M(t_\lambda^N \| \nabla u_1^\infty \|^2_2) 
\]
\[- \frac{1 - \delta_2^{2+p}}{2 + p} N \frac{m}{2} \| \nabla u_1^\infty \|^2 \| \nabla u_1^\infty \|^2 \]

\[ \geq E_\lambda (t_\ast (u_1^\infty)) - (1 - \lambda) L_2^2 \int_{\mathbb{R}^N} F(t_\ast u_1^\infty) \, dx + \frac{E_\lambda^{N+2}}{2} \int_{\mathbb{R}^N} \left[ V_\infty - V(t_\ast x) \right] |u_1^\infty|^2 \, dx \]

\[ + \frac{1}{2} M \left( \| \nabla u_1^\infty \|^2 \right) - \frac{1 - \delta_2^{2+p}}{2 + p} N \frac{m}{2} \| \nabla u_1^\infty \|^2 \| \nabla u_1^\infty \|^2 \]

\[ > c_5 - (1 - \lambda) T^2 \int_{\mathbb{R}^N} F(T u_1^\infty) \, dx + \frac{1}{2} M \left( \| \nabla u_1^\infty \|^2 \right) - \frac{1 - \delta_2^{2+p}}{2 + p} N \frac{m}{2} \| \nabla u_1^\infty \|^2 \| \nabla u_1^\infty \|^2 \]

\[ \geq c_5, \quad \forall \lambda \in (\bar{\lambda}, 1]. \]

Combining both the above cases, we have \( c_5 < l_\lambda^\infty \) for \( \lambda \in (\bar{\lambda}, 1]. \) \( \square \)

**Lemma 3.6** Suppose that (V1), (V2), and (F1)–(F3) hold. Let \( \{u_n\} \) be a bounded \((PS)_{\lambda^*}\) sequence for \( E_\lambda \) with \( \lambda \in [1/2, 1]. \) Then there exist a subsequence of \( \{u_n\} \), still denoted by \( \{u_n\} \), an integer \( l \in \mathbb{N} \), and \( u_0 \in H^1(\mathbb{R}^N) \) such that

(i) \( A^2_l := \lim_{n \to \infty} \| \nabla u_n \|^2_2, u_n \rightharpoonup u_l \) in \( H^1(\mathbb{R}^N) \) and \( E_\lambda (u_l) = 0; \)

(ii) there exist \( w^1, \ldots, w^l \in H^1(\mathbb{R}^N) \setminus \{0\} \) such that \( (E_\lambda^\infty)'(w^k) = 0 \) for \( 1 \leq k \leq l; \)

(iii) \( c + \frac{1}{4} m(A^2_l) \| \nabla u_l \|^2_2 + \sum_{k=1}^l E_\lambda^\infty (w^k); \)

\[ A^2_l = \| \nabla u_l \|^2_2 + \sum_{k=1}^l \| \nabla w^k \|^2_2, \]

where

\[ E_\lambda (u) = \frac{1}{2} m(A^2_l) \| \nabla u \|^2_2 + \frac{1}{2} \int_{\mathbb{R}^N} V(x) u^2 \, dx - \lambda \int_{\mathbb{R}^N} F(u) \, dx \] \hspace{1cm} (3.16)

and

\[ E_\lambda^\infty (u) = \frac{1}{2} m(A^2_l) \| \nabla u \|^2_2 + \frac{V_\infty}{2} \int_{\mathbb{R}^N} u^2 \, dx - \lambda \int_{\mathbb{R}^N} F(u) \, dx. \] \hspace{1cm} (3.17)

We agree that in the case \( l = 0 \) the above holds without \( w^k. \)

Analogous to the proof of Lemma 2.3 in [22], we can prove Lemma 3.6. We omit it here.

**Lemma 3.7** Suppose that (V1), (V2), (V3), (M4), and (F1)–(F3) hold. Then, for almost every \( \lambda \in (\bar{\lambda}, 1], \) there exists \( u_\ast \in H^1(\mathbb{R}^2) \setminus \{0\} \) such that

\[ E_\lambda (u_\ast) = 0, \quad E_\lambda(u_\ast) = c_\lambda. \] \hspace{1cm} (3.18)

**Proof** Under (F1)–(F3), Lemma 3.4 implies that \( E_\lambda(u) \) satisfies the assumptions of Proposition 3.1 with \( X = H^1(\mathbb{R}^N) \) and \( \Phi_\lambda = E_\lambda. \) So, for almost every \( \lambda \in [1/2, 1], \) there exists a
bounded sequence \( \{u_n(\lambda)\} \subset H^1(\mathbb{R}^N) \) (for simplicity, we denote \( \{u_n\} \) instead of \( \{u_n(\lambda)\} \)) such that

\[
E_\lambda(u_n) \to c_\lambda > 0, \quad \|E'_\lambda(u_n)\| \to 0. \tag{3.19}
\]

By Lemma 3.6, there exist a subsequence of \( \{u_n\} \), still denoted by \( \{u_n\} \), and \( u_k \in H^1(\mathbb{R}^N) \) such that \( A^2_k := \lim_{n \to \infty} \|\nabla u_n\|^2 \) exists, \( u_n \to u_k \) in \( H^1(\mathbb{R}^N) \) and \( (E'_\lambda)(u_k) = 0 \).

If (ii) occurs, i.e., there exist \( l \in \mathbb{N} \) and \( w^1, \ldots, w^l \in H^1(\mathbb{R}^N) \setminus \{0\} \) such that \( (E^\infty_\lambda)(w^k) = 0 \) for \( 1 \leq k \leq l \),

\[
c_\lambda + \frac{1}{4} m(A^2_\lambda)\|\nabla u_k\|^2 = E_\lambda(u_k) + \sum_{k=1}^l E^\infty_\lambda(w^k) \tag{3.20}
\]

and

\[
A^2_k = \|\nabla u_k\|^2 + \sum_{k=1}^l \|\nabla w^k\|^2. \tag{3.21}
\]

Since \( (E'_\lambda)(u_k) = 0 \), then we have the Pohozaev identity of the functional \( E_\lambda \)

\[
\tilde{P}_\lambda(u_k) := \frac{N - 2}{2} m(A^2_\lambda)\|\nabla u_k\|^2 + \frac{1}{2} \int_{\mathbb{R}^N} \left[ (p - N) V(x) - \nabla V(x) \cdot x \right] u_k^2 \, dx - N \lambda \int_{\mathbb{R}^2} F(u_k) \, dx
\]

\[
= 0. \tag{3.22}
\]

It follows from (2.27), (3.16), (3.22), and (V3) that

\[
E_\lambda(u_k) = E_\lambda(u_k) - \frac{1}{2 + p} \left[ (E'_\lambda(u_k), u_k) + \tilde{P}_\lambda(u_k) \right]
\]

\[
= \frac{2 + p - N}{2(2 + p)} m(A^2_\lambda)\|\nabla u_k\|^2 + \frac{1}{2(2 + p)} \int_{\mathbb{R}^N} \left[ (p - N) V(x) - \nabla V(x) \cdot x \right] u_k^2 \, dx
\]

\[
+ \frac{\lambda}{2 + p} \int_{\mathbb{R}^N} \left[ f(u_k) u_k - (2 + p - N) F(u_k) \right] \, dx
\]

\[
\geq \frac{1}{4} m(A^2_\lambda)\|\nabla u_k\|^2. \tag{3.23}
\]

Since \( (E^\infty_\lambda)(w^k) = 0 \), then we have the Pohozaev identity of the functional \( E^\infty_\lambda \)

\[
\tilde{P}^\infty_\lambda(w^k) := \frac{N - 2}{2} m(A^2_\lambda)\|\nabla w^k\|^2 + \frac{N}{2} \int_{\mathbb{R}^N} (w^k)^2 \, dx - N \lambda \int_{\mathbb{R}^2} F(w^k) \, dx = 0. \tag{3.24}
\]

Thus, from (3.6), (3.17), (3.21), and (3.24), we have

\[
0 = \langle (E^\infty_\lambda)(w^k), w^k \rangle + \tilde{P}^\infty_\lambda(w^k)
\]

\[
= \frac{N}{2} m(A^2_\lambda)\|\nabla w^k\|^2 + \frac{N + 2}{2} \int_{\mathbb{R}^N} (w^k)^2 - \lambda \int_{\mathbb{R}^N} \left[ f(w^k) w^k + NF(w^k) \right] \, dx
\]

\[
\geq J^\infty_\lambda(w^k). \tag{3.25}
\]
Since \( w^k \in H^1(\mathbb{R}^N) \setminus \{0\} \), in view of Lemma 2.5, there exists \( t_k > 0 \) such that \( t_k(w^k) \in M^\infty_\lambda \).

From (3.3), (3.6), (3.8), (3.17), and (3.21), one has

\[
\mathcal{E}_\lambda^\infty(w^k) = \mathcal{E}_\lambda^\infty(w^k) - \frac{1}{2 + p} \left[ \left( \mathcal{E}_\lambda^\infty(w^k)'(w^k), w^k \right) + \mathcal{P}_{\lambda}^\infty(w^k) \right] \\
= \frac{2 + p - N}{2(2 + p)} m(A_{\lambda}^2) \| \nabla w^k \|_2^2 + \frac{N}{2(2 + p)} m(\| \nabla w^k \|_2^2) \| \nabla w^k \|_2^2 - \frac{1}{2} M(\| \nabla w^k \|_2^2) \\
+ \mathcal{E}_\lambda^\infty(w^k) - \frac{1}{2 + p} \mathcal{P}_{\lambda}^\infty(w^k) \\
\geq \frac{2 + p - N}{2(2 + p)} m(A_{\lambda}^2) \| \nabla w^k \|_2^2 + \frac{N}{2(2 + p)} m(\| \nabla w^k \|_2^2) \| \nabla w^k \|_2^2 - \frac{1}{2} M(\| \nabla w^k \|_2^2) \\
+ \mathcal{E}_\lambda^\infty(t_k(w^k)) - \frac{t_k^{2 + p}}{2 + p} \mathcal{P}_{\lambda}^\infty(w^k). \tag{3.26}
\]

Let

\[
H(t) = \frac{2 + p - N}{2(2 + p)} m(A_{\lambda}^2) t + \frac{N}{2(2 + p)} m(t) t - \frac{1}{2} M(t), \tag{3.27}
\]

then by (M4) one has

\[
H'(t) = \frac{2 + p - N}{2(2 + p)} m(A_{\lambda}^2) + \frac{N}{2(2 + p)} m'(t) t - \frac{2 + p - N}{2(2 + p)} m(t) \geq 0, \tag{3.28}
\]

\[
H(0) = -\frac{1}{2} M(0) = 0. \tag{3.29}
\]

Then, from (3.21), (3.25), and (3.26), one has

\[
\mathcal{E}_\lambda^\infty(w^k) \geq I_{\lambda}^\infty. \tag{3.30}
\]

It follows from (3.20), (3.21), (3.23), and (3.26) that

\[
c_{\lambda} + \frac{1}{4} m(A_{\lambda}^2) \| \nabla u_\lambda \|_2^2 = \mathcal{E}_\lambda(u_\lambda) + \sum_{k=1}^{l} \mathcal{E}_\lambda^\infty(w^k) \\
\geq I_{\lambda}^\infty + \frac{1}{4} m(A_{\lambda}^2) \| \nabla u_\lambda \|_2^2 \\
\geq I_{\lambda}^\infty + \frac{1}{4} m(A_{\lambda}^2) \| \nabla u_\lambda \|_2^2, \quad \forall \lambda \in [\bar{\lambda}, 1],
\]

which together with Lemma 3.5 implies that \( l = 0 \) and \( \mathcal{E}_\lambda(u_\lambda) = c_{\lambda} + \frac{1}{4} m(A_{\lambda}^2) \| \nabla u_\lambda \|_2^2 \). Thus, it follows from (3.21) that \( A_{\lambda} = \| u_\lambda \|_2, \; u_n \rightharpoonup u_\lambda \) in \( H^1(\mathbb{R}^N) \) and \( \mathcal{E}_\lambda(u_\lambda) = c_{\lambda}. \) \( \square \)

**Proof of Theorem 1.2** In view of Lemma 3.7, there exist two sequences of \( \{ \lambda_n \} \subset [\bar{\lambda}, 1] \) and \( \{ u_{\lambda_n} \} \subset H^1(\mathbb{R}^N) \), denoted by \( \{ u_{\lambda_n} \} \), such that

\[
\lambda_n \to 1, \quad E_{\lambda_n}(u_{\lambda_n}) = 0, \quad E_{\lambda_n}(u_{\lambda_n}) = c_{\lambda_n}. \tag{3.31}
\]
From Lemma 3.4(iii), (2.27), (3.2), (3.5), and (3.31), one has
\[
c_{1/2} \geq c_{\lambda_n} = E_{\lambda_n}(u_n) - \frac{1}{2 + p} f_{\lambda_n}(u_n) \\
= \frac{1}{2} M \left( \| \nabla u_n \|^2 \right) + \frac{1}{2(2 + p)} \int_{\mathbb{R}^N} \left[ (p - N) V(x) - \nabla V(x) \cdot x \right] u_n^2 \, dx \\
- \frac{N}{2(2 + p)} m \left( \| \nabla u_n \|^2 \right) \| \nabla u_n \|^2 + \frac{\lambda_n}{2 + p} \int_{\mathbb{R}^N} \left[ f(u_n) u_n - (2 + p - N) F(u_n) \right] \, dx \\
\geq \frac{\epsilon_0}{2} \| \nabla u_n \|^2 + \frac{V_0}{8(2 + p)} \| u_n \|^2 \\
\geq C_4 \| u_n \|^2. \quad (3.32)
\]
This shows that \( \{ u_n \} \) is bounded in \( H^1(\mathbb{R}^N) \). Since \( c_{\lambda_n} \rightarrow c_1 \), then similar to the proof of Lemma 3.7, there exists \( \tilde{u} \in H^1(\mathbb{R}^N) \setminus \{0\} \) such that
\[
E'(\tilde{u}) = 0, \quad 0 < E(\tilde{u}) = c_1. \quad (3.33)
\]
Let
\[
\Sigma := \left\{ u \in H^1(\mathbb{R}^2) \setminus \{0\} : E'(u) = 0 \right\}, \quad \hat{I} = \inf_{u \in \Sigma} E(u),
\]
it follows from (3.33) that \( \Sigma \neq \emptyset \) and \( \hat{I} \leq c_1 \). For any \( u \in \Sigma \), Lemma 3.2 yields \( P_\lambda(u) = P_1(u) = 0 \). Therefore, it follows from (3.23) that \( E(u) = E_1(u) > 0 \), thus \( \hat{I} \geq 0 \). Set \( \{ u_n \} \subset \Sigma \) such that
\[
E'(u_n) = 0, \quad E(u_n) \rightarrow \hat{I}. \quad (3.34)
\]
By Lemma 3.5, we have \( \hat{I} \leq c_1 < I^{\infty}_1 \). Through a similar argument as in the proof of Lemma 3.7, we can certify that there exists \( \tilde{u} \in H^1(\mathbb{R}^N) \setminus \{0\} \) such that
\[
E'(\tilde{u}) = 0, \quad E(\tilde{u}) = \hat{I}. \quad (3.35)
\]
This shows that \( \tilde{u} \) is a nontrivial least energy solution of (1.1). \( \square \)

**Acknowledgements**
The authors would like to thank the referees for their useful suggestions.

**Funding**
The authors are supported financially by Hunan provincial Natural Science Foundation (No. 2019JJS0146) and Scientific Research Fund of Hunan Provincial Education Department (No: 20B243).

**Availability of data and materials**
Not applicable.

**Competing interests**
The authors declare that they have no competing interests.

**Authors’ contributions**
The research was carried out in collaboration. All authors read and approved the final manuscript.
References

1. Alves, C.O., Corrêa, F.J.S.A., Figueiredo, G.M.: On a class of nonlocal elliptic problems with critical growth. Differ. Equ. Appl. 2, 409–417 (2010)
2. Arosio, A., Panizzi, S.: On the well-posedness of the Kirchhoff string. Trans. Am. Math. Soc. 348, 305–330 (1996)
3. Bernstein, S.: Sur une class d’Equations fonctionelles aux d’rivées partielles. Izv. Akad. Nauk SSSR, Ser. Mat. 4, 17–26 (1940)
4. Cavalcanti, M.M., Domingos Cavalcanti, V.N., Soriano, J.A.: Global existence and uniform decay rates for the Kirchhoff–Carrier equation with nonlinear dissipation. Adv. Differ. Equ. 6, 701–730 (2001)
5. Chen, C.Y., Kuo, Y.C., Wu, T.F.: The Nehari manifold for a Kirchhoff type problem involving sign-changing weight functions. J. Differ. Equ. 250, 1876–1908 (2011)
6. Chen, J., Tang, X.H., Chen, S.T.: Existence of ground states for fractional Kirchhoff equations with general potentials via Nehari–Pohozaev manifold. Electron. J. Differ. Equ. 2018, 142 (2018)
7. Chen, J.S., Li, L.: Multiple solutions for the nonhomogeneous Kirchhoff equations on \( \mathbb{R}^N \). Nonlinear Anal., Real World Appl. 14, 1477–1486 (2013)
8. Chen, S.T., Zhang, B.L., Tang, X.H.: Existence and non-existence results for Kirchhoff-type problems with convolution nonlinearity. Adv. Nonlinear Anal. 9(1), 148–167 (2020)
9. Chipot, M.,Lovat, B.: Some remarks on nonlocal elliptic and parabolic problems. Nonlinear Anal. 30(7), 4619–4627 (1997)
10. D’Ancona, P., Spagnolo, S.: Global solvability for the degenerate Kirchhoff equation with real analytic data. Invent. Math. 108, 247–262 (1992)
11. Deng, Y.B., Peng, S.J., Shuai, W.: Existence and asymptotic behavior of nodal solutions for the Kirchhoff-type problems in \( \mathbb{R}^N \). J. Funct. Anal. 269, 3500–3527 (2015)
12. Ding, Y.H.: Variational Methods for Strongly Indefinite Problems. World Scientific, Singapore (2007)
13. Dong, Z.: Ground states for Kirchhoff equations without compact condition. J. Differ. Equ. 259, 2884–2902 (2015)
14. He, X.M., Zou, W.M.: Infinitely many positive solutions for Kirchhoff-type problems. Nonlinear Anal. 70, 1407–1414 (2009)
15. He, X.M., Zou, W.M.: Existence and concentration behavior of positive solutions for a Kirchhoff equation in \( \mathbb{R}^3 \). J. Differ. Equ. 2, 1813–1834 (2012)
16. He, Y., Li, G.B., Peng, S.J.: Concentrating bound states for Kirchhoff type problems in \( \mathbb{R}^3 \) involving critical Sobolev exponents. Adv. Nonlinear Stud. 14, 483–510 (2014)
17. Ikoma, N.: Existence of ground state solutions to the nonlinear Kirchhoff type equations with potentials. Discrete Contin. Dyn. Syst. 35, 943–966 (2015)
18. Jeanjean, L.: On the existence of bounded Palais–Smale sequence and application to a Landesman–Lazer type problem set on \( \mathbb{R}^N \). Proc. R. Soc. Edinb., Sect. A 129, 787–804 (1999)
19. Li, C., Fang, F., Zhang, B.: A multiplicity result for asymptotically linear Kirchhoff equations. Adv. Nonlinear Anal. 8(1), 267–277 (2019)
20. Kirchhoff, G.: Mechanik. Teubner, Leipzig (1883)
21. Lei, C.Y., Liao, J.F., Tang, C.L.: Multiple positive solutions for Kirchhoff type problems with singularity and critical exponents. J. Math. Anal. Appl. 421, 521–538 (2015)
22. Li, G.B., Ye, H.: Existence of positive ground state solutions for the nonlinear Kirchhoff type equations in \( \mathbb{R}^N \). J. Differ. Equ. 257, 566–600 (2014)
23. Li, Y.H., Li, F.Y., Shi, J.P.: Existence of a positive solution to Kirchhoff type problems without compactness conditions. J. Differ. Equ. 252, 2285–2294 (2012)
24. Liang, S.H., Zhang, J.H.: Existence of solutions for Kirchhoff type problems with critical nonlinearity in \( \mathbb{R}^N \). Nonlinear Anal., Real World Appl. 17, 126–136 (2014)
25. Lions, J.L.: On some questions in boundary value problems of mathematical physics. In: Contemporary Developments in Continuum Mechanics and Partial Differential Equations, Proc. Internat. Sympos. Inst. Mat, Univ. Fed. Rio de Janeiro, 1997. North-Holland Math. Stud., vol. 30, pp. 284–346. North-Holland, Amsterdam (1978)
26. Naimen, D.: The critical problem of Kirchhoff type elliptic equations in dimension four. J. Differ. Equ. 257, 1168–1193 (2014)
27. Peereer, K., Zhang, Z.T.: Nontrivial solutions of Kirchhoff-type problems via the Yang index. J. Differ. Equ. 221, 246–255 (2006)
28. Rabinowitz, PH.: On a class of nonlinear Schrödinger equations. Z. Angew. Math. Phys. 43, 270–291 (1992)
29. Ruiz, D.: The Schrödinger–Poison equation under the effect of a nonlinear local term. J. Funct. Anal. 237, 655–674 (2006)
30. Sun, J.J., Tang, C.L.: Existence and multiplicity of solutions for Kirchhoff type equations. Nonlinear Anal. 74, 1212–1222 (2011)
31. Tang, X.H., Chen, S.T.: Ground state solutions of Nehari–Pohozaev type for Kirchhoff type problems with general potentials. Calc. Var. Partial Differ. Equ. 56, 110 (2017)
32. Tang, X.H., Chen, S.T.: Ground state solutions of Nehari–Pohozaev type for Schrödinger–Poison problems with general potentials. Discrete Contin. Dyn. Syst. 37, 4973–5002 (2017)
33. Tang, X.H., Chen, S.T.: Singularly perturbed Choquard equations with nonlinearity satisfying Berestycki–Lions assumptions. Adv. Nonlinear Anal. 9(1), 413–437 (2020)
34. Wang, J., Tian, L.X., Xu, J.X., Zhang, F.B.: Multiplicity and concentration of positive solutions for a Kirchhoff type problem with critical growth. J. Differ. Equ. 253, 2314–2351 (2012)
35. Willem, M.: Minimax Theorems. Birkhäuser, Boston (1996)
36. Wu, X.: Existence of nontrivial solutions and high energy solutions for Schrödinger–Kirchhoff-type equations in \( \mathbb{R}^N \). Nonlinear Anal., Real World Appl. 12, 1278–1287 (2011)
37. Xiang, M., Radulescu, V.D., Zhang, B.L.: Nonlocal Kirchhoff problems with singular exponential nonlinearity. Appl. Math. Optim. https://doi.org/10.1007/s00245-020-09666-3
38. Ye, H.Y.: Positive high energy solution for Kirchhoff equation in $\mathbb{R}^3$ with superlinear nonlinearities via Nehari–Pohozaev manifold. Discrete Contin. Dyn. Syst. 35, 3857–3877 (2015)