MINIMAL MASS BLOW-UP SOLUTIONS FOR DOUBLE POWER NONLINEAR SCHRÖDINGER EQUATIONS WITH AN INVERSE POWER POTENTIAL

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Abstract. We consider the following nonlinear Schrödinger equation with double power nonlinearities and an inverse power potential:

\[ \frac{\partial u}{\partial t} + \Delta u + |u|^p u + C_1|u|^{p-1}u + \frac{C_2}{|x|^{2\sigma}}u = 0 \]

in \( \mathbb{R}^N \). From the classical argument, the solution with subcritical mass (\( \|u_0\|_2 < \|Q\|_2 \)) is global and bounded in \( H^1(\mathbb{R}^N) \), where \( Q \) is the ground state of the mass-critical problem. Previous results show the existence of a minimal-mass blow-up solution for the equation with \( C_1 > 0 \) and \( C_2 = 0 \) or \( C_1 = 0 \) and \( C_2 > 0 \) and investigate the behaviour of the solution near the blow-up time. Moreover, they have suggested that a subcritical power nonlinearity and an inverse power potential behave in a similar way with respect to blow-up. On the other hand, the previous results also show the nonexistence of a minimal-mass blow-up solution for the equation with \( C_1 < 0 \) and \( C_2 = 0 \) or \( C_1 = 0 \) and \( C_2 < 0 \). In this paper, we investigate the existence and behaviour of a minimal-mass blow-up solution for the equation with \( C_1 > 0 \) or \( C_1 < 0 \) and \( C_2 > 0 \), that is the subcritical power nonlinearity and the inverse power potential cancel each other’s effects. Furthermore, we give a lower estimate of the arbitrary finite-time blow-up solution with critical mass and show that the energies of critical-mass blow-up solutions are positive when \( (C_1, C_2, p, \sigma) \) is under certain conditions.

1. Introduction

We consider the following nonlinear Schrödinger equation with double power nonlinearities and an inverse power potential:

\[ \frac{\partial u}{\partial t} + \Delta u + |u|^p u + C_1|u|^{p-1}u + \frac{C_2}{|x|^{2\sigma}}u = 0 \]

in \( \mathbb{R}^N \), where \( N \in \mathbb{N} \), \( p \) and \( \sigma \) are positive, \( C_1 \) and \( C_2 \) is real, and double-sign do not correspond. It is well known that if

\[ 1 < p < 1 + \frac{4}{N} \quad \text{and} \quad 0 < \sigma < \min\left\{ \frac{N}{2}, 1 \right\}, \]

then \( (1) \) is locally well-posed in \( H^1(\mathbb{R}^N) \) from [2] Proposition 3.2.2, Proposition 3.2.5, Theorem 3.3.9, and Proposition 4.2.3]. This means that for any initial value \( u_0 \in H^1(\mathbb{R}^N) \), there exists a unique maximal solution \( u \in C((T^*, T^*), H^1(\mathbb{R}^N)) \cap C^1((T^*, T^*), H^{-1}(\mathbb{R}^N)) \) for \( (1) \) with \( u(0) = u_0 \). Moreover, the mass (i.e., \( L^2 \)-norm) and energy \( E \) of the solution \( u \) are conserved by the flow, where

\[ E(u) := \frac{1}{2} \|\nabla u\|_2^2 - \frac{1}{2 + \frac{4}{N}} \|u\|_2^{2 + \frac{4}{N}} - \frac{C_1}{p + 1} \|u\|_{p+1}^{p+1} - \frac{C_2}{2} \|\nabla |x|^{-\sigma} u\|_2^2. \]

Furthermore, the blow-up alternative holds:

\[ T^* < \infty \quad \text{implies} \quad \lim_{t \downarrow T^*} \|\nabla u(t)\|_2 = \infty. \]

We define \( \Sigma^k \) by

\[ \Sigma^k := \left\{ u \in H^k(\mathbb{R}^N) \mid \|x|^ku \in L^2(\mathbb{R}^N) \right\}, \quad \|u\|_{\Sigma^k}^2 := \|u\|_{H^k}^2 + \|\nabla |x|^k u\|_2^2. \]

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Particularly, $\Sigma^1$ is called the virial space. If $u_0 \in \Sigma^1$, then the solution $u$ for (1) with $u(0) = u_0$ belongs to $C((T_*, T^*), \Sigma^1)$ from [2, Lemma 6.5.2].

Moreover, we consider the case

$$1 < p < 1 + \frac{4}{N} \quad \text{and} \quad 0 < \sigma < \min \left\{ \frac{N}{4}, 1 \right\}.$$  

Under this condition, if $u_0 \in H^2(\mathbb{R}^N)$, then the solution $u$ for (1) with $u(0) = u_0$ belongs to $C((T_*, T^*), H^2(\mathbb{R}^N)) \cap C^1((T_*, T^*), L^2(\mathbb{R}^N))$ and $|x|\nabla u \in C((T_*, T^*), L^2(\mathbb{R}^N))$ from [2, Theorem 5.3.1]. Furthermore, if $u_0 \in \Sigma^2$, then the solution $u$ for (1) with $u(0) = u_0$ belongs to $C((T_*, T^*), \Sigma^2) \cap C^1((T_*, T^*), L^2(\mathbb{R}^N))$ and $|x|\nabla u \in C((T_*, T^*), L^2(\mathbb{R}^N))$ from the same proof as in [2, Lemma 6.5.2].

1.1. Critical problem. Firstly, we describe the results regarding the mass-critical problem:

$$i\frac{\partial u}{\partial t} + \Delta u + |u|^4 u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N.$$  

In particular, (1) with $\sigma = 0$ and $p = 1$ is reduced to (4).

It is well known ([1], [12], [13]) that there exists a unique classical solution $Q$ for

$$-\Delta Q + Q - |Q|^4 Q = 0, \quad Q \in H^1(\mathbb{R}^N), \quad Q > 0, \quad Q \text{ is radial},$$

which is called the ground state. If $||u||_2 = ||Q||_2 (||u||_2 < ||Q||_2, ||u||_2 > ||Q||_2)$, we say that $u$ has the critical mass (subcritical mass, supercritical mass, respectively).

We note that $E_{\text{crit}}(Q) = 0$, where $E_{\text{crit}}$ is the energy with respect to (4). Moreover, the ground state $Q$ attains the best constant in the Gagliardo-Nirenberg inequality

$$||v||_{2+\frac{4}{N}}^{2+\frac{4}{N}} \leq \left( 1 + \frac{2}{N} \right) \left( \frac{||v||_2}{||Q||_2} \right)^{\frac{4}{N}} ||\nabla v||_{2}^{2} \quad \text{for} \quad v \in H^1(\mathbb{R}^N).$$

Therefore, for all $v \in H^1(\mathbb{R}^N)$,

$$E_{\text{crit}}(v) \geq \frac{1}{2} ||\nabla v||_{2}^{2} \left( 1 - \left( \frac{||v||_2}{||Q||_2} \right)^{\frac{4}{N}} \right)$$

holds. This inequality and the mass and energy conservations imply that any subcritical-mass solution for (4) is global and bounded in $H^1(\mathbb{R}^N)$.

Regarding the critical mass case, we apply the pseudo-conformal transformation

$$u(t, x) \mapsto \frac{1}{|t|^\frac{N}{2}} u \left( -\frac{1}{t}, \frac{x}{t} \right) e^{\frac{|x|^2}{4t}}$$

to the solitary wave solution $u(t, x) := Q(x)e^{it}$. Then we obtain

$$S(t, x) := \frac{1}{|t|^\frac{N}{2}} Q \left( \frac{x}{t} \right) e^{-\frac{i}{4} |x|^2},$$

which is also a solution for (4) and satisfies

$$||S(t)||_2 = ||Q||_2, \quad ||\nabla S(t)||_2 \sim \frac{1}{|t|} \quad (t \nearrow 0).$$

Namely, $S$ is a minimal-mass blow-up solution for (4). Moreover, $S$ is the only finite time blow-up solution for (4) with critical mass, up to the symmetries of the flow (see [7]).

Regarding the supercritical mass case, there exists a solution $u$ for (4) such that

$$||\nabla u(t)||_2 \sim \sqrt{\frac{\log |\log |T^*-t||}{T^*-t}} \quad (t \nearrow T^*).$$

(see [9], [10]).
1.2. Previous results. We describe previous results [4] regarding (1) with $C_2 = 0$:

\begin{equation}
 i \frac{\partial u}{\partial t} + \Delta u + |u|^2 u \pm |u|^{p-1}u = 0, \quad 1 < p < 1 + \frac{4}{N}.
\end{equation}

**Theorem 1.1** ([4], see also [5]). For any energy level $E_0 \in \mathbb{R}$, there exist $t_0 < 0$ and a radially symmetric initial value $u(t_0) \in \Sigma^1$ with

$$
\|u_0\|_2 = \|Q\|_2, \quad E(u_0) = E_0
$$

such that the corresponding solution $u$ for (3) with $\pm = +$ blows up at $T^* = 0$. Moreover,

$$
\left\| u(t) - \frac{1}{\lambda(t)} P \left( t, \frac{x}{\lambda(t)} \right) e^{-i \frac{\lambda(t)}{\lambda(t)} \left| \frac{x}{\lambda(t)} \right|^2 + i \gamma(t)} \right\| \rightarrow 0 \quad (t \nearrow 0)
$$

holds for some blow-up profile $P$, positive constants $C_1(p)$ and $C_2(p)$, positive-valued $C^1$ function $\lambda$, and real-valued $C^1$ functions $b$ and $\gamma$ such that

$$
P(t) \rightarrow Q \text{ in } H^1(\mathbb{R}^N), \quad \lambda(t) = C_1(p) t^{\frac{4-N(p-1)}{2}} (1 + o(1)),
$$

$$
b(t) = C_2(p) t^{\frac{4-N(p-1)}{2}} (1 + o(1)), \quad \gamma(t)^{-1} = O \left( \|t\|^\frac{4-N(p-1)}{2} \right)
$$

as $t \nearrow 0$.

**Theorem 1.2** ([4]). For any critical-mass initial value $u(t_0) \in H^1(\mathbb{R}^N)$, the corresponding solution for (3) with $\pm = -$ is global and bounded in $H^1(\mathbb{R}^N)$.

Similarly to the critical problem, by using the Gagliardo-Nirenberg inequality, we can show that the subcritical mass solution for (3) is global and bounded in $H^1(\mathbb{R}^N)$. Therefore, if there is a minimal-mass blow-up solution, it has a mass greater than or equal to critical mass. In Theorem 1.1 a critical-mass blow-up solution with a blow-up rate of $|t|^{\frac{4-N(p-1)}{2}}$ has been constructed. This blow-up rate is different from the blow-up rate $t^{-1}$ of the critical problem. On the other hand, Theorem 1.2 shows that there is no blow-up solution with critical mass. For any supercritical-mass, there exists a blow-up solution for (5) with $\pm = -$ with that mass [4] Lemma 1.2]. Therefore, Theorem 1.1 states that there is no minimal-mass blow-up solution. Consequently, we see that the perturbation term $|u|^{p-1}u$ affects the existence of the minimal-mass blow solution and its behaviour.

Next, we describe previous result [5] regarding (1) with $C_1 = 0$:

\begin{equation}
 i \frac{\partial u}{\partial t} + \Delta u + |u|^2 u \pm \frac{1}{|x|^{2\sigma}} u = 0.
\end{equation}

**Theorem 1.3** ([5]). Assume $0 < \sigma < \min \left\{ \frac{N}{2}, 1 \right\}$. Then for any energy level $E_0 \in \mathbb{R}$, there exist $t_0 < 0$ and a radially symmetric initial value $u_0 \in \Sigma^1$ with

$$
\|u_0\|_2 = \|Q\|_2, \quad E(u_0) = E_0
$$

such that the corresponding solution $u$ for (6) with $\pm = +$ and $u(t_0) = u_0$ blows up at $T^* = 0$. Moreover,

$$
\left\| u(t) - \frac{1}{\lambda(t)} P \left( t, \frac{x}{\lambda(t)} \right) e^{-i \frac{\lambda(t)}{\lambda(t)} \left| \frac{x}{\lambda(t)} \right|^2 + i \gamma(t)} \right\| \rightarrow 0 \quad (t \nearrow 0)
$$

holds for some blow-up profile $P$ and $C^1$ functions $\lambda : (t_0, 0) \rightarrow (0, \infty)$ and $b, \gamma : (t_0, 0) \rightarrow \mathbb{R}$ such that

$$
P(t) \rightarrow Q \text{ in } H^1(\mathbb{R}^N), \quad \lambda(t) = C_1(\sigma) t^{\frac{1+\sigma}{1+\sigma}} (1 + o(1)),
$$

$$
b(t) = C_2(\sigma) t^{\frac{1+\sigma}{1+\sigma}} (1 + o(1)), \quad \gamma(t)^{-1} = O \left( \|t\|^\frac{1+\sigma}{1+\sigma} \right)
$$

as $t \nearrow 0$.

**Theorem 1.4** ([5]). Assume $N \geq 2$ and $0 < \sigma < \min \left\{ \frac{N}{2}, 1 \right\}$. For any critical-mass initial value $u(t_0) \in H^1_{\text{rad}}(\mathbb{R}^N)$, the corresponding solution for (6) with $\pm = -$ is global and bounded in $H^1(\mathbb{R}^N)$. 
For Theorems 1.3 and 1.4 as in Theorems 1.1 and 1.2 we see that the perturbation term $|x|^{-2\sigma}u$ affects the existence of the minimal-mass blow-up solution and its behaviour.

Let $\alpha_p$ and $\alpha_\sigma$ be defined by

$$\alpha_p := 2 - \frac{N}{2}(p - 1), \quad \alpha_\sigma := 2 - 2\sigma.$$ 

Then the blow-up rates of Theorem 1.1 and 1.3 are represented

$$|t|^{-\frac{\alpha_\sigma}{\alpha_p}}, \quad |t|^{-\frac{\alpha_p}{\alpha_\sigma}},$$

respectively. Therefore, if $\alpha_p = \alpha_\sigma$, then we expect a power nonlinearity $|u|^{p-1}u$ and an inverse power potential $|x|^{-2\sigma}$ to behave in a similar way. Assuming $\alpha_p = \alpha_\sigma$, the subcritical nonlinearity and the inverse power potential may influence each other to reach different conclusions from Theorems 1.1 and 1.3. Therefore, in the following, we consider

$$i\frac{\partial u}{\partial t} + \Delta u + |u|^{p-1}u \pm \frac{1}{|x|^{2\sigma}}u = 0 \quad \text{(double-sign corresponds)},$$

where $C_0 > 0$ and $\alpha_p = \alpha_\sigma$.

Let $\alpha$ and $\omega$ be defined by

$$\alpha := \alpha_p = \alpha_\sigma, \quad \omega := \frac{p + 1}{2} \left| \cdot \right|^{-\sigma}Q_2^\frac{\alpha}{p+1}.$$ 

1.3. **Main results.** Firstly, we show that for (7), a minimal-mass solution, which blows up at a finite time is constructed when the attractive term is not inferior to the repulsive term (Theorems 1.5, 1.6, and 1.7).

**Theorem 1.5** (Existence of a minimal-mass blow-up solution 1). Assume $\omega, C_0 > \omega$, and $\alpha_p = \alpha_\sigma$. Then for any energy level $E_0 \in \mathbb{R}$, there exist $t_0 < 0$ and a radially symmetric initial value $u_0 \in H^1(\mathbb{R}^N)$ with

$$\|u_0\|_2 = \|Q\|_2, \quad E(u_0) = E_0$$

such that the corresponding solution $u$ for (7) with $(\pm, \mp) = (+, -)$ and $u(t_0) = u_0$ blows up at $T^* = 0$. Moreover,

$$\left\| u(t) - \frac{1}{\lambda(t)} \frac{\partial}{\partial t} P \left( t, \frac{x}{\lambda(t)} \right) e^{-i\frac{\lambda\gamma(t)}{\lambda(0)^2} + i\gamma(t)} \right\|_{\Sigma^1} \rightarrow 0 \quad (t \nearrow 0)$$

holds for some blow-up profile $P$ and $C^1$ functions $\lambda : (t_0, 0) \rightarrow (0, \infty)$ and $b, \gamma : (t_0, 0) \rightarrow \mathbb{R}$ such that

$$P(t) \rightarrow Q \quad \text{in } H^1(\mathbb{R}^N), \quad \lambda(t) = C_1(\alpha)|t|^{\frac{\alpha_p}{\alpha_\sigma}} (1 + o(1)),$$

$$b(t) = C_2(\alpha)|t|^{\frac{\alpha_p}{\alpha_\sigma} - 1} (1 + o(1)), \quad \gamma(t)^{-1} = O \left( |t|^{\frac{\alpha_p}{\alpha_\sigma}} \right)$$

as $t \nearrow 0$.

**Theorem 1.6** (Existence of a minimal-mass blow-up solution 2). Assume $\omega, 0 < C_0 < \omega$, and $\alpha_p = \alpha_\sigma$. Then for any energy level $E_0 \in \mathbb{R}$, there exist $t_0 < 0$ and a radially symmetric initial value $u_0 \in H^1(\mathbb{R}^N)$ with

$$\|u_0\|_2 = \|Q\|_2, \quad E(u_0) = E_0$$

such that the corresponding solution $u$ for (7) with $(\pm, \mp) = (-, +)$ and $u(t_0) = u_0$ blows up at $T^* = 0$. Moreover,

$$\left\| u(t) - \frac{1}{\lambda(t)} \frac{\partial}{\partial t} P \left( t, \frac{x}{\lambda(t)} \right) e^{-i\frac{\lambda\gamma(t)}{\lambda(0)^2} + i\gamma(t)} \right\|_{\Sigma^1} \rightarrow 0 \quad (t \nearrow 0)$$

holds for some blow-up profile $P$ and $C^1$ functions $\lambda : (t_0, 0) \rightarrow (0, \infty)$ and $b, \gamma : (t_0, 0) \rightarrow \mathbb{R}$ such that

$$P(t) \rightarrow Q \quad \text{in } H^1(\mathbb{R}^N), \quad \lambda(t) = C_1(\alpha)|t|^{\frac{\alpha_p}{\alpha_\sigma}} (1 + o(1)),$$

$$b(t) = C_2(\alpha)|t|^{\frac{\alpha_p}{\alpha_\sigma} - 1} (1 + o(1)), \quad \gamma(t)^{-1} = O \left( |t|^{\frac{\alpha_p}{\alpha_\sigma}} \right)$$

as $t \nearrow 0$.

In particular, when the attractive term is balanced by the repulsive term, there is a minimal-mass blow-up solution with a blow-up rate $t^{-1}$ like in the critical problem:
Then for any energy level $E_0 > 0$, there exist $t_0 < 0$ and a radially symmetric initial value $u_0 \in H^1(\mathbb{R}^N)$ with

$$\|u_0\|_2 = \|Q\|_2, \quad E(u_0) = E_0$$

such that the corresponding solution $u$ for (7) with $u(t_0) = u_0$ blows up at $T^* = 0$. Moreover,

$$\left\| u(t) - \frac{1}{\lambda(t)} S \left( t, \frac{x}{\lambda(t)} \right) e^{-i \frac{\mu(t)}{\lambda(t)^2} t + \gamma(t)} \right\|_{\Sigma^1} \to 0 \quad (t \nearrow 0)$$

holds for $C^1$ functions $\lambda : (t_0, 0) \to (0, \infty)$ and $b, \gamma : (t_0, 0) \to \mathbb{R}$ such that

$$\lambda(t) = \sqrt{\frac{8E_0}{\| yQ \|_2^2} |t| (1 + o(1)), \quad b(t) = \frac{8E_0}{\| yQ \|_2^2} |t| (1 + o(1)), \quad \gamma(t)^{-1} = O(|t|)$$

as $t \nearrow 0$.

On the other hand, minimal-mass blow-up solutions do not exist when the attractive term is inferior to the repulsive term (Theorems 1.8 and 1.9).

**Theorem 1.8** (Nonexistence of a minimal-mass blow-up solution 1). Assume (8), $C_0 > \omega$, and $\alpha_p = \alpha_{\sigma}$. Then for any critical-mass initial value $u(t_0) \in H^1(\mathbb{R}^N)$, the corresponding solution for (7) with $(\pm, \mp) = (-, +)$ is global and bounded in $H^1(\mathbb{R}^N)$.

**Theorem 1.9** (Nonexistence of a minimal-mass blow-up solution 2). Assume $N \geq 2$, (9), $0 < C_0 < \omega$, and $\alpha_p = \alpha_{\sigma}$. Then for any critical-mass initial value $u(t_0) \in H^1_{\text{rad}}(\mathbb{R}^N)$, the corresponding solution for (7) with $(\pm, \mp) = (+, -)$ is global and bounded in $H^1(\mathbb{R}^N)$.

Furthermore, optimal lower estimates are given for blow-up rates of critical-mass solution that blows up at a finite time in general (Theorems 1.10 and 1.11). In particular, optimal lower estimations of blow-up rates are almost unknown except for classical results and cases where the uniqueness of the blow-up solution is known.

**Theorem 1.10** (Behaviour of a minimal-mass blow-up solution 1). Assume $N \geq 2$ and that $u$ is a radial $H^1$-solution for (7) with $C_0 \in (0, \omega) \cup (\omega, \infty)$, (9) with $\pm = +$, or (10) with $\pm = +$ which has critical mass and blows up at a finite time $T$. Then

$$\| \nabla u(t) \|_2 \gtrsim \frac{1}{|T - t|^{\frac{1}{2}}} \quad (t \to T)$$

holds.

**Theorem 1.11** (Behaviour of a minimal-mass blow-up solution 2). Assume $N \geq 2$ and that $u$ is a radial $H^1$-solution for (7) with $C_0 = \omega$ and $\| u(0) \|_2 = \| Q \|_2$. Then

$$E(u) > 0$$

holds. Moreover, if $u$ blows up at a finite time $T$, then the blow-up rate is estimated by

$$\| \nabla u(t) \|_2 \gtrsim \frac{1}{|T - t|} \quad (t \to T).$$

Theorems 1.8, 1.9 are shown in the same way as in the proofs in [4, 5]. Therefore, their proofs are only leave in Remark 2.2. Henceforth, we only prove Theorems 1.7, 1.11 and 1.10.

1.4. **Comments regarding the main results.** Theorem 1.7 implies that the power nonlinearity and the inverse power potential behave in the same way with respect to blow-up. In this paper, we construct blow-up profile $P$ as an approximate solution for

$$i \frac{\partial P}{\partial s} + \Delta P - P + |P|^p P \mp C_0 \lambda_{\alpha|P|}^{-1} P \mp \lambda^{\alpha \sigma} \frac{1}{|y|^{2\beta}} P = 0$$

and treat $\lambda_{\alpha|P|}^{-1} P$ and $\lambda^{\alpha \sigma} \frac{1}{|y|^{2\beta}} P$ as perturbation terms. If $\alpha_p \neq \alpha_{\sigma}$, then we expect that the blow-up is dominated by the perturbation term for which $\alpha$ is smaller since $\lambda(s) \to 0$ as $s \to \infty$. Namely, it is expected that similar conclusion to the result in [4] or [5] will hold. Likewise, if $\alpha_p = \alpha_{\sigma}$, then we expect that the blow-up for (1) is dominated by the perturbation term for which $C_j$ is larger. Theorems 1.5, 1.6, 1.8 and 1.9 imply that the
threshold of these results is \( C_1 = \omega C_2 \). Moreover, Theorem 1.7 states that (11) with \( C_1 \pm \omega \) and \( C_2 = \mp 1 \) (i.e., (7) with \( C_0 = \omega \)) has a minimal-mass blow-up solution with the same blow-up rate \( t^{-1} \) as for the mass-critical problem.

In Theorem 1.7, unlike Theorems 1.5 and 1.6, there is a condition that the energy level \( E_0 \) is positive. In [4, 5], \( \lambda_1 \) and \( b_1 \) are defined by

\[
s_1 := \int_{\lambda_1}^{\lambda_0} \frac{1}{\mu^{2+\frac{1}{2}+1}} \sqrt{\frac{2^N}{\mu - \frac{2^N}{2}} + \frac{4\beta_1}{\|y_1\|^2}} \, d\mu, \quad E(P_{\lambda_1, b_1, 0}) = E_0
\]

for sufficiently large \( s_1 \). Since \( \beta = 0 \) holds under the assumptions of Theorem 1.7, a formal application of this definition requires \( E_0 > 0 \). Thus, in the proof of Theorem 1.7, the condition is merely a technical requirement. However, Theorem 1.11 states that the critical-mass blow-up solution for (7) (not necessarily a finite time blow-up) has always positive energy. Therefore, the assumption of positive energy in Theorem 1.7 is inevitable.

The lower estimation in Theorem 1.11 or 1.10 could be shown by modifying Lemma 2.4 and separating Lemma 3.1 from Lemma 3.2. Firstly, assuming that the solution blows up at \( T \), it can be decomposed by using Lemma 2.4. The parameters \( \lambda \) and \( \tilde{\varepsilon} \) of the decomposition are expected to converge to 0 as \( t \to T \). We have corrected the "There exist \( C > 0 \)" in the old decomposition lemma (e.g., [5, Lemma 4.1]) to "For any \( \epsilon > 0 \)" in Lemma 2.4. Thus, easily showing that the parameters \( \lambda \) and \( \tilde{\varepsilon} \) converge to 0 as \( t \to T \). If we want to show the uniqueness of blow-up rates, we need to estimate these parameters \( \lambda \) and \( \tilde{\varepsilon} \). For this purpose, one can consider using a bootstrap, as in the case of the construction of a minimal-mass blow-up solution. However, while this bootstrap assumes \( \tilde{\varepsilon}(t_1) = 0 \), it is not clear whether this holds for \( \tilde{\varepsilon} \) of general critical-mass blow-up solutions. Therefore, we avoid this approach and partially solve the problem by using Lemma 3.1. In particular, the lower estimations in Theorems 1.11 and 1.10 are optimal estimations, because the solutions with the blow-up rates are actually constructed in Theorems 1.5, 1.6 and 1.7.

Let \( u \) be blow-up solution for

\[
i \frac{\partial u}{\partial t} + \Delta u + \mu |u|^{p-1}u = 0, \quad 1 + \frac{4}{N} \leq p < \frac{4}{N-2} \quad \left(1 + \frac{4}{N} \leq p < \infty \text{ if } N = 1\right),
\]

where \( \mu > 0 \). It is known that

\[
\|\nabla u(t)\|_2 \gtrsim \frac{1}{|T - t|^{\frac{1}{2} - \frac{2}{\alpha} + 2}} \quad \text{as } t \to T
\]

if \( u \) blows up at a finite time \( T \) (see [2, Theorem 6.5.13]). In particular, when \( p = \frac{4}{N} + 1 \), the lower estimation is as follows:

\[
\|\nabla u(t)\|_2 \gtrsim \frac{1}{|T - t|^{\frac{1}{2}}} \quad \text{as } t \to T.
\]

Compared to this lower estimate, the lower estimate in Theorem 1.11 is better, although there is a requirement that the solution has a critical mass.

In Theorem 1.7 \( \alpha > 1 \) is required. Assuming \( \alpha < 1 \), there may be a minimal-mass blow-up solution with a blow-up rate that is not \( t^{-1} \). For \( P_{0,0}^+ \), which constitutes the blow-up profile \( P \) (see (10)), if \( \langle L_+ P_{0,0}^+, P_{0,0}^- \rangle \neq 0 \), then we obtain

\[
\langle L_+ P_{0,0}^+, P_{0,0}^- \rangle > 0
\]

since \( (P_{0,0}^+, Q)_p = 0 \). Therefore, \( \beta_{0,1} > 0 \) and

\[
\lambda_{\text{app}}(s) = \left(\alpha \sqrt{\frac{\beta_{0,1}}{1 - \alpha}}\right)^{-\frac{1}{\alpha}} s^{-\frac{2}{\alpha}}, \quad b_{\text{app}}(s) = \frac{1}{\alpha s}
\]

are solutions for

\[
\frac{\partial b}{\partial s} + b^2 - \beta_{0,1} \lambda^{2\alpha} = 0, \quad \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b = 0
\]

in \( s > 0 \). Accordingly, we expect the existence of a minimal-mass blow-up solution with a blow-up rate \(|t|^{-\frac{1}{\alpha}}\).
1.5. Notations. In this section, we introduce the notation used in this paper.

Let 
\[ N := \mathbb{Z}_{\geq 1}, \quad N_0 := \mathbb{Z}_{\geq 0}. \]

We define 
\[ (u, v)_2 := \Re \int_{\mathbb{R}^N} u(x)\overline{\nu}(x)dx, \quad \|u\|_p := \left( \int_{\mathbb{R}^N} |u(x)|^p dx \right)^{\frac{1}{p}}, \]
\[ f(z) := |z|^\frac{p}{2} z, \quad F(z) := \frac{1}{2 + \frac{k}{N}} |z|^{2 + \frac{k}{N}} \]
\[ g(z) := |z|^{p-1} z, \quad G(z) := \frac{1}{p+1} |z|^{p+1} \quad \text{for } z \in \mathbb{C}. \]

By identifying \( \mathbb{C} \) with \( \mathbb{R}^2 \), we denote the differentials of \( f, g, F, \) and \( G \) by \( df, dg, dF, \) and \( dG \), respectively. We define 
\[ \Lambda := \frac{N}{2} + x \cdot \nabla, \quad L_+ := -\Delta + 1 - \left( 1 + \frac{4}{N} \right) Q^\frac{4}{N}, \quad L_- := -\Delta + 1 - Q^\frac{4}{N}. \]

Namely, \( \Lambda \) is the generator of \( L^2 \)-scaling, and \( L_+ \) and \( L_- \) come from the linearised Schrödinger operator to around \( Q \). Then 
\[ L_- Q = 0, \quad L_+ \Lambda Q = -2Q, \quad L_- |x|^2 Q = -4\Lambda Q, \quad L_+ \rho = |x|^2 Q \]
hold, where \( \rho \in S(\mathbb{R}^N) \) is the unique radial solution for \( L_+ \rho = |x|^2 \rho \). Note that there exist \( C_\alpha, \kappa_\alpha > 0 \) such that 
\[ \left| \left( \frac{\partial}{\partial x} \right)^{\alpha} Q(x) \right| \leq C_\alpha Q(x), \quad \left| \left( \frac{\partial}{\partial x} \right)^{\alpha} \rho(x) \right| \leq C_\alpha (1 + |x|)^{\kappa_\alpha} Q(x), \]
for any multi-index \( \alpha \). Furthermore, there exists \( \mu > 0 \) such that for all \( u \in H^1(\mathbb{R}^N) \), 
\[ (L_+ \Re u, \Re u) + (L_- \Im u, \Im u) \geq \mu \|u\|_{H^1}^2 - \frac{1}{\mu} \left( (\Re u, Q)^2 + (\Re u, |x|^2 Q)^2 + (\Im u, \rho)^2 \right) \]
(e.g., see \[8 \] \[9 \] \[11 \] \[12 \]). We denote by \( \mathcal{Y} \) the set of functions \( g \in C^\infty(\mathbb{R}^N \setminus \{0\}) \cap C(\mathbb{R}^N) \cap H^1_{\text{rad}}(\mathbb{R}^N) \) such that 
\[ \exists C_\alpha, \kappa_\alpha > 0, \quad |x| \geq 1 \Rightarrow \left| \left( \frac{\partial}{\partial x} \right)^{\alpha} g(x) \right| \leq C_\alpha (1 + |x|)^{\kappa_\alpha} Q(x) \]
for any multi-index \( \alpha \). Moreover, we defined by \( \mathcal{Y}' \) the set of functions \( g \in \mathcal{Y} \) such that 
\[ \Lambda g \in H^1(\mathbb{R}^N) \cap C(\mathbb{R}^N). \]

Finally, we use the notation \( \lesssim \) and \( \gtrsim \) when the inequalities hold up to a positive constant. We also use the notation \( \approx \) when \( \lesssim \) and \( \gtrsim \) hold. Moreover, positive constants \( C \) and \( \epsilon \) are sufficiently large and small, respectively.

2. Construction of a blow-up profile

For \( K \in N_0 \), let 
\[ \Sigma_K = \{ (j, k) \in N_0^2 \mid j + k \leq K \}. \]

**Proposition 2.1** (Existence of a blow-up profile). Let \( K, K' \in N_0 \) be sufficiently large. Let \( \lambda(s) > 0 \) and \( b(s) \in \mathbb{R} \) be \( C^1 \) functions of \( s \) such that \( \lambda(s) + |b(s)| \ll 1 \). Then for any \( (j, k) \in \Sigma_K + K' \), there exist \( P_{j, k}^+, P_{j, k}^- \in \mathcal{Y}' \), \( \beta_{j, k} \in \mathbb{R} \), and \( \Psi : (\lambda, b) \mapsto \Psi(\lambda, b) \in H^1(\mathbb{R}^N) \) such that \( P \) satisfies 
\[ i \frac{\partial P}{\partial s} + \Delta P - P + f(P) \mp C_\alpha \lambda^\alpha g(P) \mp \lambda^\alpha \frac{1}{|y|^{2\alpha}} P \mp \frac{\theta |y|^2}{4} P = \Psi, \]
where \( P \) and \( \theta \) defined by 
\[ P(s, y) = Q(y) + \sum_{(j, k) \in \Sigma_K + K'} \left( b(s)^{2j} \lambda(s)^{(k+1)\alpha} P_{j, k}^+(y) + i b(s)^{2j+1} \lambda(s)^{(k+1)\alpha} P_{j, k}^-(y) \right) \]
\[ \theta(s) = \sum_{(j, k) \in \Sigma_K + K'} b(s)^{2j} \lambda(s)^{(k+1)\alpha} \beta_{j, k} \]
In particular, 
\[ \beta_{0,0} = 0 \]
holds.

Moreover, for some sufficiently small \( \epsilon' > 0 \),
\begin{equation}
\left\| e^{\epsilon'|\Psi|} \right\|_{H^1} \lesssim \lambda^\alpha \left( \left| b + \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} \right| + \left| \frac{\partial b}{\partial s} + b^2 - \theta \right| + (b^2 + \lambda^\alpha)^{K+2} \right)
\end{equation}
holds.

**proof.** The proof is the same as for [4][5]. We prove only \( \beta_{0,0} = 0 \). From the proofs in [4][5], \( P^+_0,0 \) satisfies
\[ L + P^+_0,0 - \beta_{0,0} \frac{|y|^2}{4} Q \mp C_0 g(Q) + \frac{1}{|y|^{2s}} Q = 0, \quad (P^+_0,0, Q)_2 = 0. \]
Therefore, since \( C_0 = \omega \),
\[ 0 = (P^+_0,0, Q)_2 = \frac{1}{2} \left< L + P^+_0,0, \Lambda Q \right> = \frac{1}{2} \left< \beta_{0,0} \frac{|y|^2}{4} Q \mp C_0 g(Q) + \frac{1}{|y|^{2s}} Q, \Lambda Q \right> \]
\[ = \frac{1}{2} \left( \frac{\beta_{0,0}}{4} \| | \cdot | Q \|_2^2 + C_0 \frac{N(p-1)}{2(p+1)} || Q ||_{p+1}^p + \sigma \| | -^\sigma Q \|_2^2 \right). \]
Accordingly, \( \beta_{0,0} = 0 \).

**Remark 2.2.** Assume \((\pm, \mp) = (+, -)\) and \( C_0 > \omega \). Then we obtain
\[ 0 = (P^+_0,0, Q)_2 = \frac{1}{2} \left( \frac{\beta_{0,0}}{4} \| | \cdot | Q \|_2^2 - (C_0 - \omega) \frac{N(p-1)}{2(p+1)} || Q ||_{p+1}^p \right). \]
Therefore, \( \beta_{0,0} > 0 \). Likewise, if \((\pm, \mp) = (-, +)\) and \( C_0 < \omega \), we obtain \( \beta_{0,0} > 0 \). Thus, Theorems [4][5] and [1][0] can be shown as in [4][5].

For the blow-up profile \( P \), the following properties are obtained by direct calculation:

**Proposition 2.3.** Let define
\[ P_{\lambda, b, \gamma}(s, x) := \frac{1}{\lambda(s)^{\frac{3}{2}}} P \left( s, \frac{x}{\lambda(s)} \right) e^{-\frac{|x|^2}{2\lambda(s)}} e^{i\gamma(s)}. \]
Then,
\[ \left| \frac{d}{ds} \| P_{\lambda, b, \gamma} \|_2^2 \right| \lesssim \lambda^\alpha \left( \left| b + \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} \right| + \left| \frac{\partial b}{\partial s} + b^2 - \theta \right| + (b^2 + \lambda^\alpha)^{K+2} \right), \]
\[ \left| \frac{d}{ds} E(P_{\lambda, b, \gamma}) \right| \lesssim \frac{1}{\lambda^2} \left( \left| b + \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} \right| + \left| \frac{\partial b}{\partial s} + b^2 - \theta \right| + (b^2 + \lambda^\alpha)^{K+2} \right) \]
hold. Moreover,
\begin{equation}
\left| 8 E(P_{\lambda, b, \gamma}) - \| | Q \|_2^2 b^2 \right| \lesssim \frac{\lambda^\alpha (b^2 + \lambda^\alpha)}{\lambda^2} \end{equation}
holds.

This section closes with a decomposition lemma, which is key to this paper. See [4][5] for the proof.

**Lemma 2.4** (Decomposition). For any \( \epsilon_0 > 0 \), there exist \( \delta > 0 \) such that the following statement. Let \( I \) be an interval. We assume that \( u \in C(I, H^1(\mathbb{R}^N)) \cap C^4(I, H^{-1}(\mathbb{R}^N)) \) satisfies
\[ \forall t \in I, \left\| \lambda(t)^{\frac{3}{2}} u(t, \lambda(t) y) e^{i\gamma(t)} \right\|_{H^1} < \delta \]
for some functions $\lambda : I \to (0, \bar{t})$ and $\gamma : I \to \mathbb{R}$. Then there exist unique functions $\tilde{\lambda} : I \to (0, \infty)$, $\tilde{b} : I \to \mathbb{R}$, and $\tilde{\gamma} : I \to \mathbb{R}/2\pi\mathbb{Z}$ such that

\begin{equation}
 u(t, x) = \frac{1}{\lambda(t)} \left( P + \tilde{\varepsilon} \right) \left( t, \frac{x}{\lambda(t)} \right) e^{-i\int_0^t \frac{b(s)^2}{\lambda(s)} + \tilde{\gamma}(t)} ,
\end{equation}

hold, where $|\cdot|_{\mathbb{R}/2\pi\mathbb{Z}}$ is defined by

$$|c|_{\mathbb{R}/2\pi\mathbb{Z}} := \inf_{m \in \mathbb{Z}} |c + 2\pi m|,$$

and that $\tilde{\varepsilon}$ satisfies the orthogonal conditions

\begin{equation}
 (\tilde{\varepsilon}, iAP)_2 = (\tilde{\varepsilon}, |y|^2P)_2 = (\tilde{\varepsilon}, \rho)_2 = 0
\end{equation}
on $I$. In particular, $\tilde{\lambda}$, $\tilde{b}$, and $\tilde{\gamma}$ are $C^1$ functions and independent of $\lambda$ and $\gamma$.

**Remark.** This lemma is slightly different from \[5, Lemma 4.1\], with “There exist $\mathcal{C} > 0$” changed to “For any $\epsilon_0$”.

Not essential for the construction of a minimal-mass blow-up solution, but required in the proof of Theorems 1.10 and 1.11 (see (22)).

This modification follows easily from the implicit function theorem. Indeed, for a $C^1$-function $f : U \to \mathbb{R}^{n+m}$ with $f(x_0, y_0) = 0$ and $\det \frac{\partial f}{\partial y}(x_0, y_0) \neq 0$, from the implicit function theorem, there exist a neighbourhood $V$ of $x_0$, a neighbourhood $W$ of $y_0$, and a $C^1$-function $S : V \to W$ such that $f(x, S(x)) = 0$ for any $x \in V$. In particular, from the continuity and uniqueness of $S$, for any $\varepsilon > 0$, there exists some $\delta > 0$ and $S : B(x_0, \delta) \to B(y_0, \varepsilon)$ also $f(x, S(x)) = 0$ for any $x \in B(x_0, \delta)$. This $\varepsilon$ corresponds to $\epsilon_0$ in Lemma 2.4 and $\delta$ corresponds to $\delta$ and $\tilde{t}$ in Lemma 2.4.

### 3. Uniformity estimates for decomposition parameters

For $s_1 > 0$, let $\lambda_1$ and $b_1$ be defined by

$$\lambda_1 := \sqrt{\frac{\|yQ\|^2}{8E_0} s_1^{-1}}, \quad E(P_{\lambda_1}, b_1, 0) = E_0 .$$

Note that such $b_1 > 0$ exists from (12). Moreover, let functions $\lambda_{app}$ and $b_{app}$ be defined by

$$\lambda_{app}(s) := \sqrt{\frac{\|yQ\|^2}{8E_0} s^{-1}}, \quad b_{app}(s) := s^{-1} .$$

Let $u(t)$ be the solution for (7) with an initial value

\begin{equation}
 u(t, x) := \frac{1}{\lambda_1} \left( P \right) \left( \frac{x}{\lambda_1} \right) e^{-i\frac{b_1}{\lambda_1} |x|^2} .
\end{equation}

Then since $u$ satisfies the assumption of Lemma 2.4 in a neighbourhood of $t_1$, there exists a decomposition $(\tilde{\lambda}_1, \tilde{b}_1, \tilde{\gamma}_1, \tilde{\varepsilon}_1)$ such that (13) in a neighbourhood $I$ of $t_1$. The rescaled time $s_{t_1}$ is defined by

$$s_{t_1} := s_1 - \int_1^{t_1} \frac{1}{\tilde{\lambda}_1(\tau)} d\tau .$$

Moreover, let $I_{t_1}$ be the maximal interval such that a decomposition as (13) is obtained and we define

$$J_{s_1} := s_{s_1}(I_{t_1}) .$$

Then, since $s_{t_1} : I_{t_1} \to J_{s_1}$ is strictly monotonically increasing, we can define inverse function $s_{t_1}^{-1} : J_{s_1} \to I_{t_1}$. Furthermore, we define

$$t_{t_1} := -\frac{\|yQ\|^2}{8E_0} s_{t_1}^{-1} , \quad \lambda_{t_1}(s) := \tilde{\lambda}(t_{t_1}(s)) , \quad b_{t_1}(s) := \tilde{b}(t_{t_1}(s)) ,$$

$$\gamma_{t_1}(s) := \tilde{\gamma}(t_{t_1}(s)) , \quad \varepsilon_{t_1}(s, y) := \tilde{\varepsilon}(t_{t_1}(s), y) .$$
For the sake of clarity in notation, we often omit the subscript $t_1$. In particular, it should be noted that $u \in C((T_*, T^*), \mathbb{S}^2(\mathbb{R}^N))$ and $|x| \nabla u \in C((T_*, T^*), L^2(\mathbb{R}^N))$. Additionally, let $s_0$ be sufficiently large, $s_1 \geq s_0$, and $s^* := \max \{s_0, \inf J_{s_1}\}$.

Let $s_*$ be defined by
\[
(16) \quad s_* := \inf \{ \sigma \in (s', s_1) \mid (17) \text{ holds on } [\sigma, s_1] \},
\]
where
\[
(17) \quad \| \varepsilon(s) \|^2_{H^1} + b(s)^2 \| y \varepsilon(s) \|^2_L \leq s^{-2K}, \quad \left| \frac{\lambda(s)}{\lambda_{\text{app}}(s)} - 1 \right| + \left| \frac{b(s)}{b_{\text{app}}(s)} - 1 \right| < s^{-M}
\]
with some $0 < M < 2(\alpha - 1)$.

By direct calculation with (7), (13), and (9),
\[
(18) \quad \frac{\partial \varepsilon}{\partial s} + \Delta \varepsilon - \varepsilon + f(P + \varepsilon) - f(P) \pm C_0 \lambda^\alpha (g(P + \varepsilon) - g(P)) \mp \lambda^\alpha \frac{1}{|y|^{2\sigma}} \varepsilon + \theta |y|^2 \varepsilon
\]
holds in $J_{s_1}$.

**Lemma 3.1.** For $s \in J_{s_1}$,
\[
| \text{Mod}(s) | \lesssim |(\varepsilon, P)_2| + \lambda^\alpha \left\| e^{-i\frac{\lambda |y|^2}{4}} \right\|^2_{H^1} + \left\| e^{-i\frac{b |y|^2}{4}} \right\|^2_{H^1} + (\hat{b}^2 + \lambda^\alpha)^{K+2}
\]
holds, where
\[
\text{Mod}(s) := \left( \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b, \frac{\partial b}{\partial s} + b^2, 1 - \frac{\partial \gamma}{\partial s} \right).
\]

**Proof.** See [4, 5] for details of the proof.

According to the orthogonality properties (14), we have
\[
0 = \frac{d}{ds} (i \varepsilon, \Lambda P)_2 = \left( \frac{\partial \varepsilon}{\partial s}, \Lambda P \right)_2 + \left( i \varepsilon, \frac{\partial (\Lambda P)}{\partial s} \right)_2.
\]

By direct calculation and (10), we obtain
\[
\left| \left( i \varepsilon, \frac{\partial (\Lambda P)}{\partial s} \right)_2 \right| \lesssim |\text{Mod}| \| \varepsilon \|_2 + \lambda^\alpha \| \varepsilon \|_2.
\]

From (18), we obtain
\[
\left( \frac{\partial \varepsilon}{\partial s}, \Lambda P \right)_2 = \left( \mathcal{L}_s \text{ Re } \varepsilon + i \mathcal{L}_s \text{ Im } \varepsilon - (f(P + \varepsilon) - f(P) - df(Q)(\varepsilon)) \mp C_0 \lambda^\alpha (g(P + \varepsilon) - g(P)) \pm \lambda^\alpha \frac{1}{|y|^{2\sigma}} \varepsilon - \theta |y|^2 \varepsilon
\]
\[
+ i \left( \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b, \frac{\partial b}{\partial s} + b^2 - \theta \right) \frac{|y|^2}{4} (P + \varepsilon)
\]
\[
+ \left( \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b \right) \frac{|y|^2}{4} (P + \varepsilon) + \frac{\partial \gamma}{\partial s} \right)_2.
\]
Noting that
\[ \nabla \varepsilon = \nabla \left( \varepsilon e^{-ib|\varepsilon|^2} \right) e^{ib|\varepsilon|^2} + ib \frac{y}{2} \varepsilon, \]
we obtain
\[ \langle L_+ \Re \varepsilon, \Lambda P \rangle = -2(\varepsilon, P)_2 + O \left( \lambda^\alpha \left\| \varepsilon e^{-ib|\varepsilon|^2} \right\|_{H^1} \right). \]

Therefore, since \( \|yP\|_2^2 = \|yQ\|_2^2 + O(\lambda^\alpha) \), we obtain
\[
\left( \frac{\partial \varepsilon}{\partial s} \right)_2 \Lambda P = -\frac{1}{4} \|yQ\| \left( \frac{\partial \partial_s + b^2 - \theta}{\partial s} \right) - 2(\varepsilon, P)_2 + O \left( \lambda^\alpha \left\| \varepsilon e^{-ib|\varepsilon|^2} \right\|_{H^1} \right) + O \left( \left\| \varepsilon e^{-ib|\varepsilon|^2} \right\|_{H^1}^2 \right) \\
+ O \left( (b^2 + \lambda^\alpha)^{K+2} + o(\|\text{Mod}\|) \right).
\]

Similar calculations are performed for other orthogonal conditions in \( 14 \) to obtain the estimate of Lemma 3.1. \( \square \)

**Remark 3.2.** The estimate in Lemma 3.1 is independent of the initial values \( \lambda_1, b_1 \), and \( 0 \) of \( \lambda, b, \) and \( \varepsilon \), respectively. Therefore, a similar estimate can be obtained if \( \lambda, b, \) and \( \varepsilon \) is sufficiently small.

From Lemma 3.1, we obtain the following lemma. See \( 4, 5 \) for details of the proof.

**Corollary 3.3.** For \( s \in (s_*, s_1) \),
\[ \| (\varepsilon(s), Q) \| \lesssim s^{-(K+\alpha)}, \quad |\text{Mod}(s)| \lesssim s^{-(K+\alpha)}, \quad \| e'||\Psi \|_{H^1} \lesssim s^{-(K+2\alpha)} \]
hold.

**proof.** Let
\[ s_\ast := \inf \left\{ s \in [s_*, s_1] \left| |(\varepsilon(\tau), P)_2| < \tau^{-(K+\alpha)} \text{ holds on } [s, s_1] \right. \right\}. \]

Then from Lemma 3.1 it is clearly
\[ |\text{Mod}(s)| \lesssim s^{-(K+\alpha)} \]
on \( (s_\ast, s_1) \). Therefore, from \( 11 \), we obtain
\[ \| e'||\Psi \|_{H^1} \lesssim s^{-(K+2\alpha)}. \]

Accordingly, from Proposition 2.3 we obtain
\[ |(\varepsilon(\tau), P)_2| \lesssim s^{-(K+2\alpha)}. \]
Namely, \( s_* = s_* \) for sufficiently large \( s_0 \). Consequently, Corollary 3.3 holds. \( \square \)

Let \( m > 0 \) be sufficiently large and define
\[ S(s, \varepsilon) := \frac{1}{\lambda^m} \left( \frac{1}{2} \| \varepsilon \|_{H^1}^2 + b^2 \| y\varepsilon \|_2^2 - \int_{\mathbb{R}^N} (F(P + \varepsilon) - F(P) - dF(P)(\varepsilon)) \, dy \right) \]
\[ + C_0 \lambda^\alpha \lambda^\alpha \int_{\mathbb{R}^N} (G(P + \varepsilon) - G(P) - dG(P)(\varepsilon)) \, dy \pm \frac{1}{2} \lambda^\alpha \| y^{-\sigma} \varepsilon \|_2^2. \]

**Lemma 3.4 (Estimates of \( S \)).** For \( s \in (s_*, s_1) \),
\[ \frac{1}{\lambda^m} \left( \| \varepsilon \|_{H^1}^2 + b^2 \| y\varepsilon \|_2^2 + O(s^{-(2(K+\alpha))}) \right) \lesssim S(s, \varepsilon) \lesssim \frac{1}{\lambda^m} \left( \| \varepsilon \|_{H^1}^2 + b^2 \| y\varepsilon \|_2^2 + O(s^{-(2K+2\alpha-1)}) \right) \]
hold. Moreover,
\[ \frac{d}{ds} S(s, \varepsilon(s)) \gtrsim \frac{b}{\lambda^m} \left( \| \varepsilon \|_{H^1}^2 + b^2 \| y\varepsilon \|_2^2 + O(s^{-(2K+2\alpha-1)}) \right) \]
holds for \( s \in (s_*, s_1) \).

**proof.** The proof is the same as for \( 4, 5 \). \( \square \)

We use the estimates obtained in Lemma 3.4 and the bootstrap to establish the estimates of the parameters. Namely, we confirm \( 17 \) on \( [s_0, s_1] \).
Lemma 3.5 (Re-estimation). For \( s \in (s_*, s_1] \),
\[
(19) \quad \|\varepsilon(s)\|_{H^2}^2 + b(s)^2 \|y\varepsilon(s)\|_2^2 \lesssim s^{-(2K+\alpha)},
\]
\[
(20) \quad \left| \frac{\lambda(s)}{\lambda_{app}(s)} - 1 \right| + \frac{b(s)}{b_{app}(s)} - 1 \lesssim s^{-2(\alpha - 1)}.
\]

proof. See [4, 5] for details of the proof.

We prove (19) by contradiction. Let \( C_1 > 0 \) be sufficiently large and define
\[
s^*_1 := \inf \left\{ \sigma \in (s_*, s_1] \mid \|\varepsilon(\tau)\|_{H^2}^2 + b(\tau)^2 \|y\varepsilon(\tau)\|_2^2 \leq C_1 \tau^{-(2K+\alpha)} \ (\tau \in [\sigma, s_1]) \right\}.
\]
Then \( s^*_1 < s_1 \) holds. Here, we assume that \( s^*_1 > s_* \) and define
\[
s^+_1 := \sup \left\{ \sigma \in (s_*, s_1] \mid \|\varepsilon(\tau)\|_{H^2}^2 + b(\tau)^2 \|y\varepsilon(\tau)\|_2^2 \geq \tau^{-(2K+\alpha)} \ (\tau \in [s^*_1, \sigma]) \right\}.
\]
Then we obtain \( s^+_1 > s^*_1 \). Since \( s \mapsto S(s, \varepsilon(s)) \) is increasing on \([s^*_1, s^+_1]\), we obtain
\[
C_1(C_1 - 1) \leq 2C_2.
\]
It is a contradiction. Namely, \( s_* = s_1 \).

From Proposition 2.3 and Lemma 3.3, we obtain
\[
|E(P_{\lambda, b, \gamma}) - E_0| \lesssim s^{-(K+\alpha - 1)} \quad \text{and} \quad |b^2\|yQ\|_2^2 - 8\lambda^2 E_0| \lesssim s^{-2\alpha}.
\]
Therefore,
\[
\left| \frac{\partial}{\partial s} \left( \frac{\|yQ\|_2^2}{8E_0} - s \right) \right| \lesssim s^{-(2\alpha - 1)}, \quad \text{i.e.,} \quad \left| \frac{\lambda_{app}(s)}{\lambda(s)} - 1 \right| \lesssim s^{-(2\alpha - 1)}
\]
holds. Consequently, we obtain (20). \( \square \)

Furthermore, from Lemma 3.5 and (16) we obtain the following corollary:

Corollary 3.6. If \( s_0 \) is sufficiently large, then \( s_* = s^* = s_0 \) for any \( s_1 > s_0 \).

Finally, we rewrite the estimates for \( s \) in Lemma 3.5 into estimates for \( t \).

Lemma 3.7. Let \( s_0 \) be sufficiently large. Then there exists \( t_0 < 0 \) such that
\[
[t_0, t_1] \subset s_0^{-1}([s_0, s_1]), \quad \|C s_{t_1}^{-1} - |t|\| \lesssim |t|^{M+1} \ (t \in [t_0, t_1])
\]
hold for all \( t_1 \in (t_0, 0) \), where \( C := \|yQ\|_2^2/8E_0 \).

Consequently, combining Lemma 3.5 and Lemma 3.7, the following lemma. See [6] for the proofs.

Lemma 3.8 (Conversion of estimates). For any \( t_1 \in (t_0, 0) \) and \( t \in [t_0, t_1] \),
\[
\tilde{\lambda}_{t_1}(t) = \sqrt{\frac{8E_0}{\|yQ\|_2^2}} \left| t \right| \left( 1 + \varepsilon_{\tilde{\lambda};t_1}(t) \right), \quad \tilde{b}_{t_1}(t) = \frac{8E_0}{\|yQ\|_2^2} \left| t \right| \left( 1 + \varepsilon_{\tilde{b};t_1}(t) \right),
\]
\[
\|\tilde{\varepsilon}_{t_1}(t)\|_{H^2} \lesssim |t|^{K+\frac{\alpha}{2}}, \quad \|y\tilde{\varepsilon}_{t_1}(t)\|_2 \lesssim |t|^{K+\frac{\alpha}{2} - 1}
\]
holds for some functions \( \varepsilon_{\tilde{\lambda};t_1} \) and \( \varepsilon_{\tilde{b};t_1} \). Furthermore,
\[
\sup_{t_1 \in (0, t_0]} \left| \varepsilon_{\tilde{\lambda};t_1}(t) \right| \lesssim |t|^M, \quad \sup_{t_1 \in [0, t_0]} \left| \varepsilon_{\tilde{b};t_1}(t) \right| \lesssim |t|^M.
\]
4. PROOF OF THEOREM 1.7

In this section, we prove Theorem 1.7. See [4, 6, 5] for more details.

**Proof of Theorem 1.7** Let \((t_n)_{n \in \mathbb{N}} \subset (t_0, 0)\) be a increasing sequence such that \(\lim_{n \to \infty} t_n = 0\). For each \(n \in \mathbb{N}\), let \(u_n\) be the solution for (7) with the initial value
\[
 u_n(t_n, x) := \frac{1}{\lambda_{1,n}} P \left( \frac{x}{\lambda_{1,n}} \right) e^{-i \frac{b_{1,n} |x|^2}{\lambda_{1,n}^2}}
\]
at \(t_n\), where
\[
 s_n := \frac{\|yQ\|_2^2}{8E_0} t_n^{-1}, \quad \lambda_n := \sqrt{\frac{\|yQ\|_2^2}{8E_0} s_n^{-1}}, \quad E(P_{\lambda_n, b_n, 0}) = E_0.
\]

According to Lemma 2.4, there exists the decomposition
\[
u_n(t, x) = \frac{1}{\lambda_n(t)} (P + \tilde{\epsilon}_n) \left( t, \frac{x}{\lambda_n(t)} \right) e^{-i \frac{b_{1,n} |x|^2}{\lambda_n(t)^2} + i \tilde{\gamma}_n(t)}
\]
on \([0, t_n]\). Up to a subsequence, there exists \(u_\infty(t_0) \in \Sigma^1\) such that
\[
u_n(t_0) \rightharpoonup u_\infty(t_0) \quad \text{weakly in} \quad \Sigma^1, \quad u_n(t_0) \to u_\infty(t_0) \quad \text{in} \quad L^2(\mathbb{R}^N) \quad (n \to \infty).
\]
Moreover, since \(u_n : [t_0, 0) \to \Sigma^1\) is locally uniformly bounded,
\[
u_n \to u_\infty \quad \text{in} \quad C([t_0, T'), L^2(\mathbb{R}^N)), \quad u_n(t) \to u_\infty(t) \quad \text{in} \quad \Sigma^1 \quad (n \to \infty)
\]
holds (see [6]). Particularly, we have \(\|u_\infty(t)\|_2 = \|Q\|_2\).

According to weak convergence in \(H^1(\mathbb{R}^N)\) and Lemma 2.4 we decompose \(u_\infty\) to
\[
u_\infty(t, x) = \frac{1}{\lambda_\infty(t)} (Q + \tilde{\epsilon}_\infty) \left( t, \frac{x}{\lambda_\infty(t)} \right) e^{-i \frac{b_\infty(t) |x|^2}{\lambda_\infty(t)^2} + i \tilde{\gamma}_\infty(t)}
\]
on \([0, 0)\). Furthermore, as \(n \to \infty\),
\[
 \tilde{\lambda}_n(t) \to \tilde{\lambda}_\infty(t), \quad \tilde{b}_n(t) \to \tilde{b}_\infty(t), \quad e^{i \tilde{\gamma}_n(t)} \to e^{i \tilde{\gamma}_\infty(t)}, \quad \tilde{\epsilon}_n(t) \to \tilde{\epsilon}_\infty(t) \quad \text{weakly in} \quad \Sigma^1
\]
hold for any \(t \in [t_0, 0]\). Therefore, we obtain
\[
 \tilde{\lambda}_\infty(t) = \sqrt{\frac{8E_0}{\|yQ\|_2^2}} |t| (1 + \epsilon_{\tilde{\lambda}, 0}(t)), \quad \tilde{b}_\infty(t) = \frac{8E_0}{\|yQ\|_2^2} |t| (1 + \epsilon_{\tilde{b}, 0}(t)),
\]
\[
 \|\tilde{\epsilon}_\infty(t)\|_{H^1} \lesssim |t|^{K + \frac{3}{2}}, \quad \|\tilde{\epsilon}_\infty(t)\|_2 \lesssim |t|^{K + \frac{3}{2} - 1}, \quad \|\epsilon_{\tilde{\lambda}, 0}(t)\| \lesssim |t|^M, \quad \|\epsilon_{\tilde{b}, 0}(t)\| \lesssim |t|^M
\]
from the uniform estimates in Lemma 3.8. Consequently, we obtain Theorem 1.7.

Finally, check the energy. We obtain
\[
 E(u_n) - E \left( P_{\lambda_n, b_n, \tilde{\gamma}_n} \right) = \int_0^1 \left( E' \left( P_{\lambda_n, b_n, \tilde{\gamma}_n} \right) + \tau \tilde{\epsilon}_n \frac{b_n}{\lambda_n} \tilde{\gamma}_n \right) \, d\tau = O \left( \frac{1}{\lambda_n^2} \left( \|\tilde{\epsilon}_n\|_{H^1} + \|\tilde{\gamma}\|_2 \right) \right) = o(1).
\]
Similarly,
\[
 E(u_\infty) - E \left( P_{\lambda_\infty, b_\infty, \tilde{\gamma}_\infty} \right) = o(1).
\]
From continuity of energy,
\[
 \lim_{n \to \infty} E \left( P_{\lambda_n, b_n, \tilde{\gamma}_n} \right) = E \left( P_{\lambda_\infty, b_\infty, \tilde{\gamma}_\infty} \right).
\]
Therefore, we obtain
\[
 E(u_\infty) = E_0 + o_{t \to 0}(1).
\]
From energy conservation, \(E(u_\infty) = E_0\). \(\square\)
5. Proofs of Theorems 1.11 and 1.10

In this section, let $K = 0$ and $K'$ be sufficiently large.
Firstly, let $u$ be a solution for (1) and define $w$ by

$$w(t, x) := \overline{u}(-t, x).$$

Then $w$ is also a solution for (1). Therefore, if $u$ blows up, we may assume that $u$ blows up at a positive time $T \in (0, \infty]$.

We define $\hat{\lambda}$ and $v$ by

$$\hat{\lambda}(t) := \frac{\|\nabla Q\|_2}{\|\nabla u(t)\|_2}, \quad v(t, x) := \frac{\hat{\lambda}(t)}{\hat{\lambda}(t)} Q u(t, \hat{\lambda}(t)x).$$

Then

$$\|v\|_2 = \|Q\|_2, \quad \|\nabla v\|_2 = \|\nabla Q\|_2, \quad \limsup_{t \to T} E_{\text{crit}}(v(t)) = 0$$

hold. Therefore, there exist $\hat{x} : (0, T) \to \mathbb{R}^N$ and $\hat{\gamma} : (0, T) \to \mathbb{R}$ such that

$$\hat{\lambda}(t) \frac{\hat{\lambda}}{} u(t, \hat{\lambda}(t)(x - \hat{x}(t))) e^{i \hat{\gamma}(t)} \to Q \quad \text{in } H^1 \quad (t \to T)$$

(e.g., see [9]). Assuming that $N \geq 2$ and $u$ is radial, we obtain $\hat{x} = 0$.

From Lemma 2.4, for any $\epsilon_0 > 0$ there exists $t_0$ that is sufficiently close to $T$ such that we obtain the decomposition of $u$ on $(t_0, T)$:

$$u(t, x) = \frac{1}{\lambda(t)} (P + \hat{\epsilon}) \left( t, x \right) e^{-\frac{\lambda(x - \hat{x}(t))}{\lambda(t)}}.$$

In particular, from arbitrariness of $\epsilon_0$ and uniqueness of the decomposition in Lemma 2.4

$$\lim_{t \to T} \frac{\lambda(t)}{\lambda(t)} = 1, \quad \lim_{t \to T} \hat{b}(t) = 0, \quad \lim_{t \to T} e^{(\hat{\gamma}(t) + \hat{\gamma}(t))} = 1$$

hold.

Let $\hat{\epsilon}$ be defined by

$$\hat{\epsilon}(t, y) := \hat{\epsilon}(t, y) e^{-\frac{\lambda(x - \hat{x}(t))}{\lambda(t)}}.$$

Then, from (21) and (22),

$$\lim_{t \to T} \|\hat{\epsilon}\|_{H^1} = 0$$

holds. Moreover, from mass conversation,

$$-(Q, \hat{\epsilon}) = \frac{1}{2} \|\hat{\epsilon}\|_2^2 + O \left( \hat{\lambda} \left( \hat{b}^2 + \hat{\lambda}^\alpha \right) \right) + O(\hat{\lambda}^\alpha \|\hat{\epsilon}\|_2).$$

Therefore, from energy conversation, we obtain

$$\hat{\lambda}^2 E(u_0) = \hat{b}^2 + \frac{1}{2} \|\nabla \hat{\epsilon}\|_2^2 + \frac{1}{2} \|\hat{\epsilon}\|_2^2 - \frac{1}{2} \int_{\mathbb{R}^N} d^2 F(Q)(\hat{\epsilon}, \hat{\epsilon}) dy - \frac{C_1 \hat{\lambda}^\alpha}{\hat{\lambda} + 1} \|Q\|_{p+1} - \frac{C_2 \hat{\lambda}^\alpha}{2} \|\cdot \cdot Q\|_2^2$$

$$+ O(\hat{\lambda}^{2\alpha} + |b| \hat{\lambda}^\alpha + \hat{\lambda}^\alpha \|\hat{\epsilon}\|_{H^1}) + o(\hat{\lambda}^{2\alpha} + \|\hat{\epsilon}\|_{H^1}).$$

Accordingly, from (8),

$$\hat{\lambda}^2 E(u_0) + \frac{C_1 \hat{\lambda}^\alpha}{p+1} \|Q\|_{p+1} + \frac{C_2 \hat{\lambda}^\alpha}{2} \|\cdot \cdot Q\|_2^2 \geq \hat{b}^2 + \|\hat{\epsilon}\|_{H^1}^2.$$)

holds. Consequently, we obtain the following lemma:

**Lemma 5.1.** If $(C_1, C_2) = (\pm \omega, \mp 1)$, then

$$\hat{\lambda}^2 E(u_0) \geq \hat{b}^2 + \|\hat{\epsilon}\|^2_{H^1}.$$

If $(C_1, C_2) = (C_0, -1)$ with $C_0 > \omega$ or $(C_1, C_2) = (-C_0, 1)$ with $0 < C_0 < \omega$, then

$$\hat{\lambda}^\alpha \geq \hat{b}^2 + \|\hat{\epsilon}\|^2_{H^1}.$$

**Corollary 5.2.** If $(C_1, C_2) = (\pm \omega, \mp 1)$ and $u$ is a radial blow-up solution for (1) with critical mass, then

$$E(u) > 0.$$
proof. From Lemma \[5.1\] \( E(u) \geq 0 \) is obvious. We assume \( E(u) = 0 \). Then \( b = 0 \) and \( \xi = 0 \). Namely, from \[10\],
\[
\frac{\lambda}{\lambda(t)} Z + \sum_{k=0}^{K'} (k+1)\alpha \lambda^{(k+1)\alpha+1} \frac{\partial \lambda}{\partial t} P_{0,k}^+ - \tilde{\lambda} \frac{\partial \bar{y}}{\partial t} (Q + \tilde{\lambda}^\alpha Z) \\
\div (Q + \tilde{\lambda}^\alpha Z) + |Q + \tilde{\lambda}^\alpha Z|^{p-1} (Q + \tilde{\lambda}^\alpha Z) \\
\leq \lambda \left( \lambda Z + \sum_{k=1}^{K'} (k+1)\alpha \lambda^{(k+1)\alpha+1} P_{0,k}^+ \right).
\]
Since \( \Lambda \neq 0 \) and \( \tilde{\lambda} \rightarrow 0 \) as \( t \rightarrow T \), \( 0 = \frac{\partial \lambda}{\partial t} \). It means that \( \lambda \) is a constant. However, it contradicts \( \tilde{\lambda} > 0 \) and \( \tilde{\lambda} \rightarrow 0 \) as \( t \rightarrow T \). Consequently, \( E(u) \neq 0 \). \( \square \)

Proofs of Theorems \[1.1\] and \[1.10\] From Lemma \[3.1\]
\[
\left| \frac{\lambda}{\lambda(t)} \frac{\partial \lambda}{\partial t} + b \right| \lesssim \| \xi \|^2_{H^1} + \tilde{\lambda}^{2\alpha}
\]
holds on \((t_0, T)\).

We assume \((C_1, C_2) = (C_0, -1)\) with \( C_0 > \omega \) or \((C_1, C_2) = (-C_0, 1)\) with \( 0 < C_0 < \omega \). Then, since \( |b| \lesssim \tilde{\lambda}^{\frac{\alpha}{2}} \) from Lemma \[5.1\] we obtain
\[
\left| \frac{\lambda}{\lambda(t)} \frac{\partial \lambda}{\partial t} \right| \lesssim \tilde{\lambda}^{\frac{\alpha}{2}}.
\]
Therefore,
\[
1 \gtrsim \left| \frac{\partial \lambda}{\partial t} \tilde{\lambda}^{2-\frac{\alpha}{2}} \right|
\]
holds. Integrating on \((t, T)\), we obtain
\[
\tilde{\lambda}(t)^{2-\frac{\alpha}{2}} \lesssim T - t.
\]
Consequently,
\[
\| \nabla u(t) \|_2 \sim \frac{1}{\tilde{\lambda}(t)} \gtrsim \frac{1}{(T-t)^{\frac{\alpha}{2}}}. 
\]
The same can be proved when assuming \((C_1, C_2) = (\pm \omega, \mp 1)\). \( \square \)

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