Exponential stability of linear systems under a class of Desch–Schappacher perturbations

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In this paper, we investigate the uniform exponential stability of the system
\[
\frac{dx(t)}{dt} = Ax(t) - \rho Bx(t), \quad (\rho > 0),
\]
where the unbounded operator \(A\) is the infinitesimal generator of a linear \(C_0\)-semigroup of contractions \(S(t)\) in a Hilbert space \(X\) and \(B\) is a Desch–Schappacher operator. Then we give sufficient conditions for exponential stability of the above system. The obtained stability result is then applied to prove the uniform exponential stabilization of some bilinear partial differential equations.

KEYWORDS
bilinear control, exponential stabilization, linear system, unbounded control operator

MSC CLASSIFICATION
37L15, 93D15, 47D06, 93C05

1 INTRODUCTION

Let \(X, Z\) be Banach spaces such that \(Z \subset X\) with continuous and dense embedding and define a linear operator \(L : Z \to X\) and a boundary operator \(G : Z \to X\) and consider the following Boundary control system:

\[
\begin{align*}
   \dot{x}(t) &= Lx(t), \quad t > 0, \\
   Gx(t) &= u(t), \quad t \geq 0, \\
   x(0) &= x_0.
\end{align*}
\]

Choosing \(u(.) = M(x(.))\), where \(M : X \to X\) is a linear operator, system (1) can be rewritten as the following Cauchy problem:

\[
\begin{align*}
   \dot{x}(t) &= Ax(t), \quad t > 0, \\
   x(0) &= x_0,
\end{align*}
\]

where \(A := L, \quad D(A) := \{x \in Z : Gx = Mx\}\). Moreover, in order to reformulate system (1) as a distributed linear system, additional conditions on \(L\) and \(G\) should be satisfied. In the literature, there are natural conditions on \(G\) and \(L\) such as \(G\) is onto and the restricted operator \(A \subset L\) with domain \(D(A) = \ker G\) generates a \(C_0\)-semigroup on \(X\) (see Greiner [1]). It is well-known that these conditions imply that the input equation associated to (1) can be reformulate as follows:

\[
\begin{align*}
   \dot{x}(t) &= Ax(t) + Bu(t), \quad t \geq 0, \\
   x(0) &= x_0,
\end{align*}
\]

for some unbounded control operator \(B : X \to X_{-1}\), where \(X_{-1}\) is an extension of the state space (see Section 2 for the definition). We shall recall that the perturbation theory of boundary problems was mainly developed in earlier studies.
[1–4] and the references therein. Greiner [1] showed that (2) is well-posed using feedback theory. Recently, the case of unbounded boundary perturbations has been considered in earlier studies [5, 6]. In this paper, we lead to consider the problem of exponential stability of (3) in the case of Hilbert space $X$ by taking $u(t) = -px(t)$. This leads to the following abstract system:

$$\begin{align*}
\dot{x}(t) &= Ax(t) - pBx(t), \quad t > 0, \\
x(0) &= x_0,
\end{align*}$$

where the unbounded operator $(A, D(A))$ generates a $C_0$–semigroup $S(t)$ on $X$. Here, $B$ is an unbounded linear operator of $X$ in the sense that it is bounded from $X$ to some extrapolating space of $X$. In the case of various real problems, the modeling may lead to mathematical models of the form (4) with an operator $B$ of type Desch-Schappacher. Such a perturbation $B$ appears, for instance, in case of control actions exercised through the boundary of the geometrical domain of partial differential equations and also in many other situations of internal control. It is worth noting that the solution of (4) does not exist, in general, with values in $X$. Thus, to confront this difficulty the concept of admissibility is needed, which requires the introduction of extrapolating space of the state space $X$.

Our goal in this paper is to investigate the uniform exponential stability of the system (4). This consists on looking for a set of parameters $\rho$ for which there exists a global $X$–valued mild solution $x(t)$ of (4) which is such that $\|x(t)\| \leq Ke^{-\sigma t}\|x_0\|$, $\forall t \geq 0$ for some constants $K, \sigma > 0$. As an application, one can consider the stabilization of bilinear systems by means of switching controllers, which leads to a closed-loop system like (4). This problem has been considered in Liu et al. [7] for a bounded operator $B$, and the case of a Miyadera–Voigt type operator has been investigated in Ouzahra et al. [8]. Moreover, in Ammari et al. [9], the case of $1$–admissibility in Banach space has been considered. However, the $1$–admissibility assumption prevents us to consider the case of Hilbert state space as in this case, the operator $B$ will be necessary bounded (see Weiss [10]). In other words, the $1$–admissibility condition excludes several applications that are also available in Hilbert space. Moreover, in Ammari et al. [9], it was assumed that $D((A_{-1} - \rho B)(X) = D(A_{-1}) \cap D(B)(X)$, which played an essential role in the proofs of the stabilization results (in a technical point of view). Unfortunately, there are several examples in which this domain condition is not fulfilled (see, e.g., Examples 1&2 in Section 3). In this paper, we will rather use the $p$–admissibility property with $p > 1$. Then we introduce new sufficient conditions for uniform exponential stability of system (4), which are easily checkable. In the sequel, we proceed as follows: The main results of this paper are contained in Section 2. In Section 3, we provide applications to feedback stabilization of bilinear heat and transport equations. The conclusion is given in Section 4.

## 2 | EXPONENTIAL STABILITY

In this section, we state and prove our two main stabilization results. We start by introducing the necessary tools regarding the notion of admissibility in connection with the generation results, and then provide some a priori estimates of the solution of (4). In the sequel, we consider system (4) in Hilbert space $X$.

### 2.1 | Preliminary results

As pointed out in the introduction, the unbounded aspect of the operator $B$ does not guarantee the existence of an $X$–valued solution $x(t)$ of (4). However, one may extend the system at hand in a larger (extrapolating) space $X_{-1}$ of the state space $X$ in which the existence of the solution $x(t)$ is ensured and then give the required admissibility conditions of $B$ so that the solution $x(t)$ lies in $X$. Classically, the space $X_{-1}$ can be viewed as the completion of $X$ with respect to the norm $\|x\|_{-1} := \|(\lambda I - A)^{-1}x\|_X, \ x \in X,$ for some $\lambda$ in the resolvent set of $A$. This space is independent of the choice of $\lambda$ and we have the following continuous and dense embedding: $X \hookrightarrow X_{-1}$. Moreover, $X_{-1}$ is the dual of $D(A^*)$ with respect to the pivot space $X$. That way the unbounded operator $B$ becomes bounded from $X$ to the extrapolating space $X_{-1}$, that is, $B \in \mathcal{L}(X, X_{-1})$. Thus, in order to give a meaning to solutions of (4), we have to use the fact that the semigroup $S(t)$ can be extended to a $C_0$–semigroup $S_{-1}(t)$ on $X_{-1}$, whose generator $A_{-1}$ has $D(A_{-1}) = X$ as domain and is such that $A_{-1}x = Ax,$ for any $x \in D(A)$. Recall that for any given initial state $x_0 \in X$, a mild solution of (4) is an $X$-valued continuous function $x$ on $[0, T]$ satisfying the following variation of parameter formula:

$$x(t) = S(t)x_0 - \rho \int_0^t S_{-1}(t-s)Bx(s)ds, \ \forall t \geq 0.$$
which always makes sense in \(X\), and system (4) can be rewritten in the large space \(X\) in the following abstract form:

\[
\begin{cases}
\dot{x}(t) = A_1 x(t) - \rho Bx(t), & t > 0, \\
x(0) = x_0,
\end{cases}
\]

(5)

which is well-posed in \(X\) whenever \(A - \rho B\) is the generator of a \(C_0\)-semigroup on \(X\) (see Engel & Nagel [11], Section II.6). The well-posedness of systems like (4) has been studied in many works using different approach earlier studies (see, e.g., previous studies, [5, 10, 12–15]).

The next result provides sufficient conditions on a Desch–Schappacher perturbation \(B\) to guarantee the existence and uniqueness of the mild solution of (4) (see Adler et al. [5] and Engel & Nagel [11], p. 182).

**Theorem 1.** Let \(A\) be the generator of a \(C_0\)-semigroup \(S(t)\) on \(X\) and let \(B \in \mathcal{L}(X, X)\) be \(p\)-admissible for some \(1 < p < \infty\), that is, there is a \(T > 0\) such that

\[
\int_0^T S_1(t-s)Bu(s)ds \in X, \quad \forall u \in L^p(0, T; X).
\]

(6)

Then for any \(\rho\), the operator \((A_1 - \rho B)\rvert_X\) defined on the domain \(D((A_1 - \rho B)\rvert_X) := \{x \in X : (A_1 - \rho B)x \in X\}\) by

\[
(A_1 - \rho B)\rvert_X := A_1 x - \rho Bx, \quad \forall x \in D((A_1 - \rho B)\rvert_X)
\]

(7)

is the generator of a \(C_0\)-semigroup \((T(t))_{t \geq 0}\) on \(X\), which verifies the variation of parameters formula (see earlier works [3, 4])

\[
T(t)x = S(t)x - \rho \int_0^t S_1(t-s)BT(s)xds, \quad \forall t \geq 0, \quad \forall x \in X.
\]

An operator \(B \in \mathcal{L}(X, X)\) satisfying condition (6) is called a Desch–Schappacher operator or perturbation. Moreover, the operator defined by (7) is the part \((A_1 - \rho B)\rvert_X\) of \((A_1 - \rho B)\) on \(X\) (see Pazy [16], p. 39 and Engel & Nagel [11], p. 147).

**Remark 1.** [10] Note that if the operator \(B \in \mathcal{L}(X, X)\) is \(p\)-admissible that uses condition (6) that holds true, then there exists \(M > 0\) such that for all \(0 < t \leq T\), we have

\[
\left\| \int_0^t S_1(t-s)Bu(s)ds \right\|_X \leq M\|u\|_{L^p(0, T; X)}, \quad \forall u \in W^{1, p}(0, T; X),
\]

(8)

where \(\|u\|_{L^p(0, T; X)} = \left( \int_0^T \|u(t)\|_X^p dt \right)^{\frac{1}{p}}\).

Let us now show the following lemma that will be needed in the sequel.

**Lemma 1.** Let \(A\) be the generator of a \(C_0\)-semigroup of contractions \(S(t)\) on \(X\) and let \(B \in \mathcal{L}(X, X)\) such that assumption (6) holds. Then for any \(0 < \rho < \frac{1}{T^*M}\), the mild solution \(x(t)\) of system (4) satisfies the following estimate:

\[
\left\| x(\cdot) \right\|_{L^p(0, T; X)} \leq \frac{T^*}{1 - \rho T^*M} \|x_0\|_X, \quad \forall x_0 \in X
\]

(9)

and

\[
\left\| \int_0^t S_1(t-s)Bx(s)ds \right\|_X \leq M_\rho \|x_0\|_X, \quad \forall t \in [T, 2T], \quad \forall x_0 \in X,
\]
where $M$ is given in Remark 1 and $M_{\rho} := \frac{MT^{\frac{1}{\gamma}}}{1 - \rho MT^{\frac{1}{\gamma}}}(2 + \rho \frac{MT^{\frac{1}{\gamma}}}{1 - \rho MT^{\frac{1}{\gamma}}})$.

**Proof.** Let $x_0 \in X$. From Theorem 1, we know that system (4) admits a unique mild solution $x(t)$ which is given by

$$x(t) = S(t)x_0 - \rho \int_0^t S_{-1}(t-s)Bx(s)ds, \ \forall t \geq 0.$$  

(10)

Let us estimate $\|x(\cdot)\|_{L^p(0,T;X)}$. From (10), we get via Minkowski's inequality

$$\|x(\cdot)\|_{L^p(0,T;X)} \leq \left(\int_0^T \|S(t)x_0\|_X^p dt\right)^{\frac{1}{p}} + \rho \left(\int_0^T \left\|\int_0^t S_{-1}(t-s)Bx(s)ds\right\|_X^p ds\right)^{\frac{1}{p}} dt.$$  

Then from Remark 1, we derive

$$\|x(\cdot)\|_{L^p(0,T;X)} \leq T^{\frac{1}{\gamma}}\|x_0\|_X + \rho MT^{\frac{1}{\gamma}}\|x(\cdot)\|_{L^p(0,T;X)},$$  

which gives the estimate (9) for any $0 < \rho < \frac{1}{MT^{\frac{1}{\gamma}}}$.

Let $x_0 \in X$, and let us write for any $t \in [T, 2T],$

$$\int_0^t S_{-1}(t-s)Bx(s)ds = \int_0^T S_{-1}(t-s)Bx(s)ds + \int_T^t S_{-1}(t-s)Bx(s)ds.$$  

It follows that

$$\left\|\int_0^t S_{-1}(t-s)Bx(s)ds\right\|_X \leq S_{-1}(T-t) \int_0^T S_{-1}(T-s)Bx(s)ds + \int_0^{t-T} S_{-1}(t-T-\tau)Bx(T)d\tau.$$  

Then, the admissibility of $B$ together with the contraction property of $S_{-1}(t)$ yields

$$\left\|\int_0^t S_{-1}(t-s)Bx(s)ds\right\|_X \leq M\|x(\cdot)\|_{L^p(0,T;X)} + M\|x(\cdot + T)\|_{L^p(0,T;X)}.$$  

Using inequality (9), we derive to

$$\left\|\int_0^t S_{-1}(t-s)Bx(s)ds\right\|_X \leq \frac{MT^{\frac{1}{\gamma}}}{1 - \rho MT^{\frac{1}{\gamma}}}\|x_0\|_X + \frac{MT^{\frac{1}{\gamma}}}{1 - \rho MT^{\frac{1}{\gamma}}}\|x(T)\|_X.$$  

Based on the V.C.F (10), it follows directly from (9) that for all $t \geq 0$, we have

$$\|x(t)\|_X \leq \left(1 + \frac{\rho MT^{\frac{1}{\gamma}}}{1 - \rho MT^{\frac{1}{\gamma}}}\right)\|x_0\|_X, \ \forall x_0 \in X$$  

(11)
for any $0 < \rho < \frac{1}{MT^p}$. Using this last inequality together with (9), we derive the desired estimate as follows:

$$\left\| \int_0^t S_{-1}(t-s)Bx(s)ds \right\|_X \leq M_\rho \|x_0\|_X, \ \forall t \in [T, 2T].$$

with $M_\rho := \frac{MT^p}{1-\rho MT^p} \left(2 + \frac{\rho MT^p}{1-\rho MT^p}\right)$.

\[ \square \]

### 2.2 A direct approach

Let us now state our main result.

**Theorem 2.** Let $B \in L(X, X_1)$ and let $A$ be the infinitesimal generator of a linear $C_0$-semigroup of contractions $S(t)$ on $X$, and assume that for some $T > 0$, we have

(i) there exists $1 < p < \infty$ such that for all $x \in L^p(0, T; X)$, we have

$$\int_0^T S_{-1}(T-s)Bu(s)ds \in X,$$

(ii) for some $\delta > 0$, we have

$$\int_0^T \Re \langle S(t)x, B^*S(t)x \rangle_x dt \geq \delta \|S(T)x\|_X^2, \ \forall x \in X. \quad (12)$$

Then there is a $1 > \rho_1 > 0$ such that for all $\rho \in (0, \rho_1)$, system (4) is exponentially stable on $X$.

**Proof.** For any $\rho > 0$, we set $A_{\rho B} := (A_{-1} - \rho B)|_X$. According to assumption (i), we deduce from Theorem 1 that system (4) admits a unique mild solution which is given, for $x_0 \in X$, by the variation of parameters formula (see Desch & Schappacher [13]):

$$x(t) = S(t)x_0 - \rho \int_0^t S_{-1}(t-s)Bx(s)ds, \ \forall t \geq 0. \quad (13)$$

For $\lambda \in \rho(A)$, $(\rho(A)$ is the resolvent set of $A)$; we consider system (4) with $B_\lambda := \lambda R(\lambda, A_{-1})B$ instead of $B$. Observing that the operator $B_\lambda$ is bounded, we deduce that the corresponding system admits a unique mild solution denoted by $x_\lambda$, which satisfies the following formula

$$x_\lambda(t) = S(t)x_0 - \rho \int_0^t S_{-1}(t-s)B_\lambda x_\lambda(s)ds, \ \forall t \geq 0, x_0 \in X. \quad (14)$$

We claim that $x_\lambda(t)$ converges to $x(t)$ as $\lambda \to +\infty$. Indeed, for all $t > 0$, we have

$$x(t) - x_\lambda(t) = \rho \int_0^t S_{-1}(t-s)B_\lambda x_\lambda(s)ds - \rho \int_0^t S_{-1}(t-s)Bx(s)ds$$

$$= \rho \int_0^t S_{-1}(t-s)B_\lambda(x_\lambda(s) - x(s))ds + \rho \int_0^t S_{-1}(t-s)B_\lambda x(s)ds - \rho \int_0^t S_{-1}(t-s)Bx(s)ds.$$
Then, using (8), this yields for all $t \in [0, T]$

$$
\|x_\lambda(t) - x(t)\|_X \leq \rho M \|x_\lambda(\cdot) - x(\cdot)\|_{L^p(0,T; X)} + \rho \left\| \int_0^t S_{-1}(t-s)B_\lambda x(s)ds - \int_0^t S_{-1}(t-s)Bx(s)ds \right\|_X,
$$

which by integrating and using Minkowski inequality gives for all $t \in [0, T]$

$$
\|x_\lambda(\cdot) - x(\cdot)\|_{L^p(0,T; X)} \leq T^\frac{1}{p} \rho M \|x_\lambda(\cdot) - x(\cdot)\|_{L^p(0,T; X)} + \rho \left( \int_0^T \left\| \int_0^t S_{-1}(t-s)B_\lambda x(s)ds - \int_0^t S_{-1}(t-s)Bx(s)ds \right\|_X dt \right)^\frac{1}{p}.
$$

It follows that

$$
\|x_\lambda(\cdot) - x(\cdot)\|_{L^p(0,T; X)}^p \leq \alpha_\rho \int_0^T \left\| \int_0^t S_{-1}(t-s)B_\lambda x(s)ds - \int_0^t S_{-1}(t-s)Bx(s)ds \right\|_X dt,
$$

with $\alpha_\rho := \left( \frac{\rho}{1 - \rho M T^\frac{1}{p}} \right)^p$.

Now, observe that

$$
\int_0^t S_{-1}(t-s)B_\lambda x(s)ds = \lambda R(\lambda, A_\lambda) \int_0^t S_{-1}(t-s)Bx(s)ds,
$$

and taking into account the admissibility assumption, we get $\int_0^t S_{-1}(t-s)B_\lambda x(s)ds = \lambda R(\lambda, A) \int_0^t S_{-1}(t-s)Bx(s)ds$. Then, we have $\lim_{\lambda \to \infty} \int_0^t S_{-1}(t-s)B_\lambda x(s)ds = \int_0^t S_{-1}(t-s)Bx(s)ds$ in $X$. Moreover, by using the fact that $\|\lambda R(\lambda, A)\|_{L^p(X)} \leq 1$ and inequality (8), we get

$$
\int_0^t S_{-1}(t-s)B_\lambda x(s)ds - \int_0^t S_{-1}(t-s)Bx(s)ds \leq 2M \|x(\cdot)\|_{L^p(0,T; X)}, \forall \lambda \in (A), \forall t \in [0, T].
$$

Then, according to the dominated convergence theorem we have

$$
\lim_{\lambda \to \infty} \|x_\lambda(\cdot) - x(\cdot)\|_{L^p(0,T; X)}^p \leq \alpha_\rho \lim_{\lambda \to \infty} \int_0^T \left\| \int_0^t S_{-1}(t-s)B_\lambda x(s)ds - \int_0^t S_{-1}(t-s)Bx(s)ds \right\|_X dt
$$

$$
= \alpha_\rho \int_0^T \lim_{\lambda \to \infty} \left\| \int_0^t S_{-1}(t-s)B_\lambda x(s)ds - \int_0^t S_{-1}(t-s)Bx(s)ds \right\|_X dt
$$

$$
= 0.
$$

Let $x_0 \in D(A_{\rho \beta})$ be fixed. For all $t > 0$ we have $x_\lambda(t) \in X$. Moreover, $x_\lambda(t)$ has a weak derivative $\frac{d}{dt}x_\lambda(t) = T_\lambda(t)A_{\rho \beta}x_0$, which is weakly continuous and we have

$$
\left\| \frac{d}{dt}(x_\lambda(t), f) \right\| \leq L \|A_{\rho \beta}x_0\|_X \|f\|. \forall f \in X^*(L > 0 \text{ is a constant}).
$$
Thus, the weak derivative of $x_i(t)$ is bounded in any bounded time-interval and the function $t \to \|x_i(t)\|_X$ is Lipschitz continuous, so it follows that $\|x_i(t)\|_X$ is differentiable almost everywhere and for a.e. $t > 0$, we have (see [17])
\[
\frac{d}{dt}\|x_i(t)\|^2_X = 2\Re\langle A_{R^i}x_i(t), x_i(t) \rangle_X, \quad \forall t > 0.
\]
The dissipativeness of $A$ implies
\[
\frac{d}{dt}\|x_i(t)\|^2_X \leq -2\rho\Re\langle B_{R^i}x_i(t), x_i(t) \rangle_X, \quad \forall t > 0. \tag{15}
\]

For all $t > 0$, we have the following equality:
\[
\langle B_{R^i}S(t)x_0, S(t)x_0 \rangle_X = \langle B_{R^i}S(t)x_0, S(t)x_0 - x_i(t) \rangle_X + \langle B_{R^i}S(t)x_0 - B_{R^i}x_i(t), x_i(t) \rangle_X + \langle B_{R^i}x_i(t), x_i(t) \rangle_X,
\]
which gives
\[
\langle S(t)x_0, B^* \lambda R^*(\lambda, A)S(t)x_0 \rangle_X = \langle S(t)x_0, B^* \lambda R^*(\lambda, A)(S(t)x_0 - x_i(t)) \rangle_X + \langle S(t)x_0 - x_i(t), B^* \lambda R^*(\lambda, A)x_i(t) \rangle_X + \langle B_{R^i}x_i(t), x_i(t) \rangle_X.
\]

For every $t > 0$, we have, for some positive constant $C$,
\[
\Re\langle S(t)x_0, B^* \lambda R^*(\lambda, A)S(t)x_0 \rangle_X \leq C\langle S(t)x_0, B^* \lambda R^*(\lambda, A)(S(t)x_0 - x_i(t)) \rangle_X + C\|B^*\|_{L(D(A^*),X)}\|\lambda R^*(\lambda, A)\|_{L(X)}\|S(t)x_0 - x_i(t)\|_X + \Re\langle B_{R^i}x_i(t), x_i(t) \rangle_X.
\]

where we have taken into consideration the identification $X_{-1} \cong D(A^*)^*$. Using the fact that $S(t)$ is a contraction, it comes
\[
\Re\langle S(t)x_0, B^* \lambda R^*(\lambda, A)S(t)x_0 \rangle_X \leq C\|B^*\|_{L(D(A^*),X)}\|x_0\|_X\|S(t)x_0 - x_i(t)\|_X + C\|B^*\|_{L(D(A^*),X)}\|\lambda R^*(\lambda, A)\|_{L(X)}\|x_i(t)\|_X\|S(t)x_0 - x_i(t)\|_X + \Re\langle B_{R^i}x_i(t), x_i(t) \rangle_X.
\]

Let us estimate each term of this last expression.

From (i), we deduce according to Remark 1 that for some constant $M > 0$ and for all $u \in L^p(0, T; X)$, we have
\[
\left\| \int_0^T S_{-1}(T-s)Bu(s)ds \right\|_X \leq M\|u\|_{L^p(0, T; X)}. \tag{16}
\]

Formula (14) combined with the estimate (8) gives
\[
\|S(t)x_0 - x_i(t)\|_X = \rho \left\| \int_0^i S_{-1}(t-s)Bx_i(s)ds \right\|_X \\
\leq \rho M\|x_i(\cdot)\|_{L^p(0, T; X)} \\
\leq \rho M\|x(\cdot)\|_{L^p(0, T; X)}, \quad \forall t \in [0, T].
\]

Then, according to Lemma 1, we conclude that
\[
\|S(t)x_0 - x_i(t)\|_X \leq \frac{\rho MT^3}{1 - \rho MT^2}\|x_0\|_X, \quad \forall t \in [0, T]. \tag{17}
\]
Using (11) and (17), we deduce that for all \( t \in (0, T] \), we have
\[
\Re \langle S(t)x_0, B^* R^*(\lambda, A)S(t)x_0 \rangle_X \leq \frac{\rho \text{CLMT}^\frac{1}{2}}{1 - \rho T\tau M} ||x_0||_X^2 
+ \frac{\rho \text{CLMT}^\frac{1}{2}}{1 - \rho T\tau M} \left( 1 + \frac{\rho MT^\frac{1}{2}}{1 - T\tau \rho M} \right) ||x_0||_X^2 + \Re \langle B_1 x_1(t), x_1(t) \rangle_X,
\]
with \( L := \|B^*\|_{C([A^*], X)} \). Then, integrating the last inequality and using (15), we get for all \( t \in [0, T] \)
\[
2\rho \int_0^T \Re \langle S(t)x_0, B^* S(t)x_0 \rangle_X dt \leq 2\rho^2 k_\rho ||x_0||_X^2 + ||x_0||_X^2 - ||x(T)||_X^2.
\]
Thus, letting \( \lambda \to +\infty \), we derive
\[
\frac{\rho^2 \text{CLMT}^{\frac{1}{2}+1}}{1 - \rho T\tau M} \left( 2 + \frac{\rho MT^\frac{1}{2}}{1 - T\tau \rho M} \right).\]
Applying inequality (12), it follows that
\[
2\rho \delta ||S(T)x_0||_X^2 - 2\rho^2 k_\rho ||x_0||_X^2 \leq ||x_0||_X^2 - ||x(T)||_X^2. \tag{18}
\]
Now, using Lemma 1, we deduce via the variation of constants formula (13) that for all \( t \in [T, 2T] \), we have
\[
||x(t)||_X \leq ||S(t)x_0||_X + \rho \int_0^t S_1(t - s)Bx(s)ds ||_X \\
\leq ||S(T)x_0||_X + \rho M_p ||x_0||_X.
\]
By reiterating the processes for \( t \in [(k + 1)T, (k + 2)T], k \geq 1 \), we deduce that
\[
||x(t)||_X \leq ||S(T)x(kT)||_X + \rho M_p ||x(kT)||_X.
\]
Then for all \( k \geq 1 \), we have
\[
||x((k + 1)T)||_X^2 \leq 2||S(T)x(kT)||_X^2 + 2\rho^2 M_p^2 ||x(kT)||_X^2. \tag{19}
\]
Then (18) reads as follows:
\[
2\rho \delta ||S(T)x(kT)||_X^2 - 2\rho^2 k_\rho ||x(kT)||_X^2 \leq ||x(kT)||_X^2 - ||x((k + 1)T)||_X^2. \tag{20}
\]
This together with (19) implies
\[
\rho \delta \left( ||x((k + 1)T)||_X^2 - 2\rho^2 M_p^2 ||x(kT)||_X^2 \right) - 2k_\rho \rho^2 ||x(kT)||_X^2 \leq ||x(kT)||_X^2 - ||x((k + 1)T)||_X^2.
\]
Hence,
\[
(1 + \rho \delta)||x((k + 1)T)||_X^2 \leq \left( 2\delta \rho^2 M_p^2 + 2k_\rho \rho^2 + 1 \right) ||x(kT)||_X^2, \quad k \geq 0.
\]
This implies
\[ \|x((k+1)T)\|^2_X \leq K_\rho \|x(kT)\|^2_X, \]
where \( K_\rho = \frac{2\rho^2 (\rho M^2 + k_\rho) + 1}{1 + \rho^2} \).

Since \( \|x(t)\|_X \) decreases, we get for \( k = E\left( \frac{t}{T} \right) \) (where \( E(\cdot) \) is the integer part function).
\[ \|x(t)\|^2_X \leq (K_\rho)^k \|x_0\|^2_X, \]
which gives the following exponential decay
\[ \|x(t)\|_X \leq Ne^{-\sigma t} \|x_0\|. \quad \forall t \geq 0 \]
where \( N = (K_\rho)^{-\frac{1}{2}} \) and \( \sigma = \frac{-\ln(K_\rho)}{2T} \). This estimate extends by density to all \( x_0 \in X \). Taking \( 0 < \rho < \rho_1 \), where \( \rho_1 \) is such that \( 0 < \rho_1 < \frac{1}{T^2 M} \). Since \( M \) and \( k_\rho \) are bounded with respect to \( \rho \), then \( \frac{2\rho^2 (\rho M^2 + k_\rho) + 1}{1 + \rho^2} \in (0, 1) \) is equivalent to \( 2\rho(\rho M^2 + k_\rho) < \delta \), which holds for \( \rho \) small enough. Hence, the uniform exponential stability holds. \( \square \)

**Remark 2.** • Note that for \( p \geq 1 \), the \( p \)–admissibility condition (6) always implies the following \( p \)–resolvant condition (see Staffans [3], p. 201):
\[ \|R(\lambda, A)B\|_{L(X)} \leq \frac{C}{Re(\lambda)^{1-\frac{1}{p}}} \quad \forall \lambda \in \mathbb{C}_\alpha, \]  \hspace{1cm} (21)
where \( \mathbb{C}_\alpha := \{ \lambda \in \mathbb{C} | Re(\lambda) > \alpha \} \) with \( \alpha > 0 \). Moreover, there are many results concerning the validity or invalidity of the converse for some classes of \( C_0 \)–semigroups. As far as the analytic semigroups are concerned, it has been shown in previous research [18–20] that the equivalence holds if and only if \((-A)^{\frac{1}{2}}\) is \( p \)-admissible.

• Condition (12) is verified if the following holds:
\[ \int_0^T Re \langle S(t)x, B^*S(t)x \rangle \, dt \geq \delta \|x\|^2_X, \quad \forall x \in X, \]  \hspace{1cm} (22)
in the case of linear systems. Moreover, by taking \( B = LL^* \), where \( L \) is the output operator, condition (22) reads as follows:
\[ \int_0^T \|B^*S(t)x\|^2_X \, dt \geq \delta \|x\|^2_X, \quad \forall x \in X, \]  \hspace{1cm} (23)
which is an observability exact condition. For some discussions and partial results about equivalent conditions to (23), we refer the reader to earlier studies [21, 22].

### 2.3 A range decomposition method

Let \( X \oplus X_{-1} \) be a direct sum in \( X_{-1} \), where \( X = i(X) \) (\( i \) being the canonical injection of \( X \) in \( X_{-1} \)), so we can write \( X = X \). Then for any \( C \in \mathcal{L}(X_{-1}) \) such that \( rg(C) \subset X \oplus X_{-1} \), we set \( X_C := P_X C \), where \( P_X \) is the projection of \( X \) according to \( X \oplus X_{-1} \). Now, given a pair of operators \((K, L) \in \mathcal{L}(X, X_{-1}) \times \mathcal{L}(X, X_{-1}) \), the decomposition \( X \oplus X_{-1} \) is said to be admissible for \((K, L) \) if the three following properties hold:

(a) \( rg(K) \subset X \oplus X_{-1} \) and \( rg(L) \subset X \oplus X_{-1} \),

(b) \( X_K \) is dissipative on \( D((K + L)|_X) := \{ x \in X : Kx + Lx \in X \} \),

(c) \( X_L \in \mathcal{L}(X) \).
For our stabilization problem, we will be interested with admissible decompositions for the pairs \((A_{-1}, -\rho B)\) with \(\rho > 0\) small enough. Note that if the domain of the operator \((A_{-\rho B})_X\) is independent of \(\rho > 0\) (small enough), which is equivalent to \(D((A_{-\rho B})_X) = D(A) \cap D(B|_X)\), then for the sum \(X \oplus X_{-1}\) to be admissible for the pairs \((A_{-1}, -\rho B), \rho > 0\), it suffices to be admissible for the pair \((A_{-1}, B)\).

We are ready to state our second main result.

**Theorem 3.** Let \(A\) be the infinitesimal generator of a linear \(C_0\)-semigroup of contractions \(S(t)\) on \(X\) and let \(B \in \mathcal{L}(X, X_{-1})\). Let \(X \oplus X_{-1}\) be an admissible decomposition for the pair \((A_{-1}, -\rho B)\) for any \(\rho > 0\) small enough, and assume that for some \(T > 0\), the operator \(B\) is \(p\)-admissible for some \(1 < p < \infty\) and satisfies the estimate:

\[
\int_{0}^{T} \Re(\langle XBS(t)x, S(t)x \rangle) x dt \geq \delta \|S(T)x\|_{X}^2, \quad \forall x \in X,
\]

for some \(T, \delta > 0\). Then there is a \(\rho_1 > 0\) such that system (4) is exponentially stable on \(X\) for all \(\rho \in (0, \rho_1)\).

**Proof.** Let \(0 < \rho < \frac{1}{1 + \frac{T}{\rho} M}\), and let \(x(t)\) be the unique mild solution of system (4) given by formula (13).

Estimates (8) and (9) imply the following estimate for \(t \in [0, T]\):

\[
\|x(t) - S(t)x_0\|_X \leq \frac{\rho M T_{\rho}^{\frac{1}{p}}}{1 - \rho M T_{\rho}^{\frac{1}{p}}} \|x_0\|_X.
\]

(25)

Moreover, observing that \(A_{\rho B}x(t) = (A_{\rho B})x(t)\), we can write for \(x_0 \in D((A_{\rho B})_X)\)

\[
\frac{d}{dt}\|x(t)\|_X^2 = 2\Re(\langle X(A_{-1})x(t) - \rho X Bx(t), x(t) \rangle)_X, \quad \forall t > 0.
\]

Integrating this last equality and using the dissipativeness of \(X(A_{-1})\) gives

\[
2\rho \int_{s}^{t} \Re(\langle X Bx(\tau), x(\tau) \rangle)_X d\tau \leq \|x(s)\|_X^2 - \|x(t)\|_X^2, \quad t \geq s \geq 0.
\]

(26)

We have the following equality

\[
\langle XBS(t)x_0, S(t)x_0 \rangle_X = \langle XBS(t)x_0 - X Bx(t), S(t)x_0 \rangle_X + \langle X Bx(t), S(t)x_0 - x(t) \rangle_X + \langle X Bx(t), x(t) \rangle_X.
\]

Then using the fact that the operator \(X B\) is bounded, it comes

\[
\Re(\langle XBS(t)x_0, S(t)x_0 \rangle_X) \leq \|X B\|_{\mathcal{L}(X)} \|x_0\|_X \|S(t)x_0 - x(t)\|_X
\]

\[
+ \|X B\|_{\mathcal{L}(X)} \|x(t)\|_X \|S(t)x_0 - x(t)\|_X + \Re(\langle X Bx(t), x(t) \rangle_X).
\]

Estimate (25) combined with (11) implies

\[
\Re(\langle XBS(t)x_0, S(t)x_0 \rangle_X) \leq \|X B\|_{\mathcal{L}(X)} \frac{\rho M T_{\rho}^{\frac{1}{p}}}{1 - \rho M T_{\rho}^{\frac{1}{p}}} \|x_0\|_X^2
\]

\[
+ \|X B\|_{\mathcal{L}(X)} \frac{\rho M T_{\rho}^{\frac{1}{p}}}{1 - \rho M T_{\rho}^{\frac{1}{p}}} \left(1 + \frac{\rho M T_{\rho}^{\frac{1}{p}}}{1 - \rho M T_{\rho}^{\frac{1}{p}}}\right) \|x_0\|_X^2
\]

\[
+ \Re(\langle X Bx(t), x(t) \rangle_X), \quad \forall t \in [0, T].
\]
Integrating this inequality and using inequality (24), we deduce that

\[ \delta \rho \| S(T)x(kT) \|^2_X - \rho^2 c_\rho \| x(kT) \|^2_X \leq \int_{kT}^{(k+1)T} \text{Re} \langle xBx(s), x(s) \rangle_X \, ds \]

with \( c_\rho = \frac{MT^{1+\frac{\gamma}{2}}}{1 - \rho MT^\gamma} \| xB \|_{L(X)} \left( 2 + \frac{\rho MT^\gamma}{1 - \rho MT^\gamma} \right) \).

By using (19) and (26), we derive

\[ \rho \delta \left( \| x((k+1)T) \|^2_X - 2\rho^2 M^2 \| x(kT) \|^2_X \right) - 2c_\rho \| x(kT) \|^2_X \leq \| x(kT) \|^2_X - \| x((k+1)T) \|^2_X, \]

or equivalently,

\[ \| x((k+1)T) \|^2_X \leq C_\rho \| x(kT) \|^2_X, \]

where \( C_\rho = \frac{2\rho (\rho M^2 + c_\rho) + 1}{1 + \rho \delta} \). Hence, using the decreasing of \( \| x(t) \|_X \), we deduce the following exponential decay

\[ \| x(t) \|_X \leq Ke^{-\sigma t} \| x_0 \|, \forall t \geq 0, \]

where \( K = (C_\rho)^{-\frac{1}{2}} \) and \( \sigma = -\frac{\ln(C_\rho)}{MT^\gamma} \). This estimate extends by density to all \( x_0 \in X \). Thus, taking \( \rho_1 > 0 \) such that \( 0 < \rho_1 < \frac{1}{M^2} \) and we have \( M_\rho \) and \( c_\rho \) are bounded with respect to \( \rho \), then \( C_\rho \in (0, 1) \) is equivalent to \( 2\rho (\rho M^2 + c_\rho) < \delta \), which is holds for \( \rho \) small enough, then we get the result of the theorem.

\[ \Box \]

3 | EXAMPLES

**Example 1.** Let \( \Omega \) be an open and bounded subset of \( \mathbb{R}^d, \ d \geq 1 \), and let us consider the following bilinear equation of diffusion type

\[
\begin{aligned}
    & \frac{\partial}{\partial t} x(\zeta, t) = \Delta x(\zeta, t) + g(x_0, t) + \nu(t)(-\Delta)^{\frac{1}{2}} x(\zeta, t) & \text{in } \Omega \times (0, \infty), \\
    & x(\zeta, t) = 0 & \text{on } \partial \Omega \times (0, \infty), \\
    & x(\zeta, 0) = x_0 & \text{in } \Omega,
\end{aligned}
\]

where \( g \in L^\infty(\Omega) \), \( \nu \) is a real-valued bilinear control and \( x(t) = x(\cdot, t) \in L^2(\Omega) \) is the state. System (27) is an example of fractional equation of diffusion equations and may describe transport processes in complex systems which are slower than the Brownian diffusion. As practical situations display such anomalous behavior, let us mention the charge carrier transport in amorphous semiconductors, the nuclear magnetic resonance diffusometry in percolative and porous media, and so on (see earlier studies [12, 15, 23, 24]). Here, we aim to prove the exponential stabilization of (27). Let us observe that system (27) can be written in the form of (4) if we close it by the switching feedback control \( \nu(t) = -\rho \mathbf{1}_{\{t \geq 0 \land x(t) \neq 0\}}(-\Delta)^{\frac{1}{2}} x(t) = (-\Delta)^{\frac{1}{2}} x(t), \forall t \geq 0 \). Let us take the state space \( X = L^2(\Omega) \) (endowed with its natural scalar product \( \langle \cdot, \cdot \rangle_X \)) and consider the control operator \( B = (-\Delta)^{\frac{1}{2}} \) and the system’s operator \( A = \Delta + g d \) with \( D(A) = H^2(\Omega) \cap H_0^2(\Omega) \). Operator \( A \) generates an analytic semigroup \( S(t) \) on \( X \) (see Engel & Nagel [11], p. 107 and p. 176) which is given by the following variation of constants formula:

\[ S(t)x = S_0(t)x + \int_0^t S_0(t-s)g(\xi)S(s)x \, ds, \quad t \geq 0, \ x \in L^2(\Omega), \]

where \( S_0(t) \) is the semigroup generated by \( A \) with \( g = 0 \).

Let us verify the assumptions of Theorem 2. In order to make the computation easier, we restrict our self to the mono-dimension case, thus we consider \( \Omega = (0, 1) \). In this case the semigroup \( S_0(t) \) is given by

\[ S_0(t)x = \sum_{j=1}^{\infty} e^{-\sigma_j t} \langle x, \phi_j \rangle_X \phi_j, \ \forall x \in L^2(\Omega) \]
where \( \alpha_j = j^2 \pi^2, j \geq 1 \) is the set of eigenvalues of \(-\Delta\) with the corresponding orthonormal basis of \( L^2(\Omega) \): \( \phi_j(x) = \sqrt{2} \sin(j \pi x) \). Moreover, the semigroup \( S(t) \) is a contraction if in addition

\[
\int_\Omega g(\xi)\gamma^2(\xi)d\xi \leq \|y\|^2_{H^1_0(\Omega)}, \forall y \in H^1_0(\Omega).
\]

Thus, in the sequel, we suppose this condition is satisfied. Operator \( B \) can be expressed as

\[
Bx = \sum_{j \geq 1} \alpha_j^2 \langle x, \phi_j \rangle \phi_j, x \in L^2(\Omega).
\]

Here, \( B \) is unbounded on \( L^2(\Omega) \), but it is bounded from \( L^2(\Omega) \) onto the space \( X_{-1} \); defined as the completion of \( L^2(\Omega) \) for the norm \( \|y\| = \left( \sum_{j \geq 1} \frac{1}{\alpha_j} \langle y, \phi_j \rangle^2 \right)^{\frac{1}{2}} \), \( \forall y \in L^2(\Omega) \), which can be also interpreted as the dual space of \( D((-\Delta)^{\frac{1}{2}}) \) with respect to the \( L^2(\Omega) \)-topology (the space \( L^2(\Omega) \) being the pivot space). Note also that the space \( D((-\Delta)^{\frac{1}{2}}) \) can be doted with the norm \( \|x\|_{D((-\Delta)^{\frac{1}{2}})} = \left( \sum_{j \geq 1} \frac{1}{\alpha_j} \langle x, \phi_j \rangle^2 \right)^{\frac{1}{2}} \).

Let \( p > 2, T > 0 \) and let \( u \in L^p(0, T; X) \). Taking in mind that \((-\Delta)^{\frac{1}{2}} \in \mathcal{L}(X, X_{-1})\), we can see that the \( X_{-1} - \) valued integral \( \int_0^T S_{-1}(T-s)(-\Delta)^{\frac{1}{2}} u(s)ds \) is well-defined (see Engel & Nagel [11], chap.II, Theorem 5.34). Moreover, since the semigroup \( S(t) \) is analytic, then so is \( S_{-1}(t) \). This implies that (see Engel & Nagel [11], p. 101) \( S_{-1}(T-s)(-\Delta)^{\frac{1}{2}} u(s) \in X, \forall s \in [0, T) \) and there exists \( K_1 \geq 0 \) such that

\[
\left\| S_{-1}(T-s)(-\Delta)^{\frac{1}{2}} u(s) \right\|_X = \left\| (-\Delta)^{\frac{1}{2}} S_{-1}(T-s)u(s) \right\|_X \leq \frac{K_1}{(T-s)^{\frac{1}{2}}} \|u(s)\|_X.
\]

It follows that

\[
\left\| \int_0^T S_{-1}(T-s)(-\Delta)^{\frac{1}{2}} u(s)ds \right\|_X \leq \int_0^T \left\| S_{-1}(T-s)(-\Delta)^{\frac{1}{2}} u(s) \right\|_X ds
\]

\[
\leq K_1 \int_0^T \frac{1}{(T-s)^{\frac{1}{2}}} \|u(s)\|_X ds.
\]

Let us consider the real \( 0 < r < 2 \) defined by \( \frac{1}{r} + \frac{1}{p} = 1 \). Moreover, by Hölder inequality, we get

\[
\left\| \int_0^T S_{-1}(T-s)(-\Delta)^{\frac{1}{2}} u(s)ds \right\|_X \leq K_2 \|u\|_{L^p(0, T; X)}, \quad (K_2 > 0)
\]

which gives the \( p \)--admissibility of \( B \) for \( p > 2 \).

For all \( x \in L^2(\Omega), t \geq 0, \) and \( j \geq 1 \), we have

\[
\left| \int_0^t S_0(t-s)gS(s)dx, \phi_j \right|_X = \left| \int_0^t \langle S_0(t-s)gS(s)x, \phi_j \rangle_X ds \right| = \left| \int_0^t \langle gS(s)x, e^{-\alpha_j(t-s)} \phi_j \rangle_X ds \right|
\]

\[
\leq \|g\|_{L^\infty(\Omega)} \|x\|_X \frac{1 - e^{-\alpha_j t}}{\alpha_j}.
\]
We deduce that
\[ |\langle S(t)x, \phi_j \rangle_x| \leq e^{-\alpha_j t} \|x\|_X + \|g\|_{L^\infty(\Omega)} \frac{1 - e^{-\alpha_j t}}{\alpha_j} \|x\|_X, \quad t \geq 0, \; j \geq 1. \]  
(28)

Now for any \( x \in X \), we have
\[ BS(t)x = \sum_{j \geq 1} \alpha_j^{\frac{1}{2}} \langle S(t)x, \phi_j \rangle_x \phi_j. \]
This combined with (28) implies that \( BS(t)x \in X \) for all \( x \in X \) and for any \( t > 0 \).
Now, using again the series expansion of \( BS(t)x \) for \( x \in X \), we get
\[ \langle BS(t)x, S(t)x \rangle_x = \sum_{j \geq 1} \alpha_j^{\frac{1}{2}} \langle S(t)x, \phi_j \rangle_x^2 \geq \|S(t)x\|_X^2 \geq \|S(T)x\|_X^2, \quad \forall t \in [0, T]. \]

It follows that assumption (12) is fulfilled.

We conclude by Theorem 2 that for \( \rho > 0 \) small enough, the control \( \nu(t) = -\rho 1_{\{t \geq 0, \; x(t) \neq 0\}} \) guarantees the uniform exponential stability of system (27).

**Example 2.** Consider the following system
\[ (S_0) \left\{ \begin{array}{ll}
\frac{d}{dt} \zeta(t) = \frac{d}{dx} x(\zeta(t), t) - ax(\zeta(t), t) + \nu(t) h(\zeta(t)x(\zeta(t), t) & \text{in } (0, 1) \times (0, \infty), \\
x(1, t) = 0 & \text{in } (0, \infty), \\
x(0, t) = x_0 & \text{in } (0, 1),
\end{array} \right. \]
where \( X = L^2(0, 1), a > 0 \) is a constant, and \( \nu \in L^\infty(0, 1) \) is such that \( h \geq c > 0 \), for some positive constant \( c \). Here, we can take \( A = \frac{d}{dx} - a id \) with domain \( D(A) := \{ x \in H^1(0, 1) : x(1) = 0 \} \). The operator \( A \) is the generator of a contraction semigroup \( S(t) \) given by
\[ (S(t)x) (\zeta) = \begin{cases} 
eq e^{-at}x(\zeta + t) & \text{if } \zeta + t \leq 1, \\
0 & \text{else.} \end{cases} \]

The bounded bilinear system \( (S_0) \) is exponentially stabilizable by the switching feedback control \( \nu(t) = -\rho 1_{\{t \geq 0, \; x(t) \neq 0\}} \).
Indeed, here, the semigroup \( S(t) \) is a contraction (so that \( \|S(t)\| \) is decreasing) and the linear operator \( B_1 := h id \) is bounded linear operator as here \( h \in L^\infty(0, 1) \) and satisfies the observation condition (since \( h \geq c > 0 \)). Let us now consider the following system
\[ (S_1) \left\{ \begin{array}{ll}
\frac{d}{dt} x(\zeta(t)) = x_\zeta(\zeta(t), t) - ax(\zeta(t), t) + \nu(t) h(\zeta(t)x(\zeta(t), t) & \text{in } (0, 1) \times (0, \infty), \\
x(1, t) = 0 & \text{in } (0, \infty), \\
x(0, t) = x_0 & \text{in } (0, 1),
\end{array} \right. \]
where \( \psi : X \to \mathbb{R} \) is a nonnull linear functional of \( X \). This may be seen as a perturbed version of \( (S_0) \) on its boundary condition. According to Riesz representation, one can assume that \( \psi(x) = \int_0^1 f(s)x(s)ds \), \( \forall x \in X \) for some \( f \in X^*(0) \).

We aim to show that under small valuers of \( \epsilon > 0 \), this system is still exponentially stabilizable.
System \( (S_1) \) can be reformulated as follows:
\[ (S_2) \left\{ \begin{array}{ll}
\frac{d}{dt} x(\zeta(t)) = Ax(\zeta(t), t) + \nu(t) h(\zeta(t)x(\zeta(t), t) & \text{in } (0, 1) \times (0, \infty), \\
x(0) = x_0 & \text{in } (0, 1),
\end{array} \right. \]
where \( x(t) := x(., t) \) and \( A : D(A) \subset X \to X \) is defined by the following:
\[ Ax := Ax - \epsilon h x, \quad \forall x \in D(A) := \{ x \in H^1(0, 1), x(1) + \epsilon \psi(x) = 0 \}. \]
We claim that $\mathcal{A}$ is the generator of a strongly continuous semigroup on $X$. In order to verify this assertion, we will consider $\mathcal{A}$ as a perturbation of the generator $A$.

In order to write system $(S_2)$ in the form (4), let us consider the function $\theta(\zeta) = 1(\zeta) \defeq 1$, $\zeta \in X$, which is such that $A_m \theta = 0$ and $\theta(1) = 1$, where the maximal operator $A_m \defeq \frac{d}{d\zeta}$ has the domain $D(A_m) \defeq H^1(0, 1)$.

Let us introduce the following operator $$Bx = hx - \psi(x)A_{-1} \theta, \; \forall x \in X$$ which is one to one since we have $\theta \not\in D(A)$.

In the sequel, we will verify the assumptions of Theorem 3 and then conclude the stabilization of the perturbed system $(S_1)$.

- From the boundary conditions of $(S_1)$, we can see that $$\forall x \in X, \; x \in D(A) \Leftrightarrow x \in H^1(0, 1) \text{ and } x + e\psi(x) \theta \in D(A).$$

This together with the definition of $\theta$ implies that for $x \in D(A)$, we have

$$X \ni Ax = A_m x - chx = A_m (x + e\psi(x) \theta) - chx = A (x + e\psi(x) \theta) - chx = A_{-1} (x + e\psi(x) \theta) - chx = A_{-1} x - cBx = (A_{-1} - cB) |_{X} x.$$ 

Moreover, for all $x \in D((A_{-1} - cB)|_{X})$, we have $A_{-1} (x + e\psi(x) \theta) \in X$, that is, $x + e\psi(x) \theta \in D(A) \subset H^1(0, 1)$ which implies that $x \in H^1(0, 1)$. Then we have $(A_{-1} - cB) |_{X} x = Ax$. In other words,

$$(A, D(A)) = ((A_{-1} - cB)|_{X}, D(A_{-1} - cB)|_{X}).$$

- The operator $(A_{-1} - cB)|_{X}$ is a generator if we can show that

$$\int_{0}^{1} S_{-1}(1 - r)\psi(u(r))A_{-1} \theta dr \in X, \; \forall u \in L^2(0, 1; X)$$

or equivalently,

$$\int_{0}^{1} S_{-1}(1 - r)\mathbf{1}(\cdot)\psi(u(r)) dr \in D(A), \; \forall u \in L^2(0, 1; X).$$

For $u \in L^2(0, 1; X)$, we have

$$\int_{0}^{1} S_{-1}(1 - r)\mathbf{1}(\cdot)\psi(u(r)) dr = \int_{0}^{1} \psi(u(r)) S(1 - r) \mathbf{1}(\cdot) dr = \int_{0}^{1} e^{-\sigma(1 - r)} \psi(u(r)) dr \defeq g(\cdot).$$

Since $\psi \psi u \in L^2(0, 1)$, this implies that $g \in H^1(0, 1)$ and $g(1) = 0$. In other words, $g \in D(A)$. Hence, for $c > 0$ small enough, system $(S_2)$ and so is $(S_1)$ are well-posed.

- Here we can take $\mathcal{X}_{-1} = \text{span}(A_{-1} \theta)$, so we obtain an admissible decomposition for the pair $(A_{-1}, -cB)$. Indeed, it is clear that $hx \in \mathcal{X}_{-1}$, $\forall x \in X$, so $x \partial B$ is a bounded operator from $X$ to $X$. 
Moreover, for all \( x \in D((\mathcal{A}_1 - \epsilon B)|X) \), we have
\[
\mathcal{A}_1 x = \mathcal{A}_1(x + \epsilon \psi(x) \theta) - \epsilon \psi(x) \mathcal{A}_1 \theta = A(x + \epsilon \psi(x) \theta) - \epsilon \psi(x) \mathcal{A}_1 \theta,
\]
from which it comes that
\[
x(\mathcal{A}_1) x = A(x + \epsilon \psi(x) \theta), \quad \forall x \in D((\mathcal{A}_1 - \epsilon B)|X),
\]
where
\[
D((\mathcal{A}_1 - \epsilon B)|X) = \{ x \in L^2(0, 1) / x + \epsilon \psi(x) \theta \in D(A) \}.
\]
Then for \( x \in D((\mathcal{A}_1 - \epsilon B)|X) \), we have \( (\mathcal{A}_1 - \epsilon B)x \in X \) or equivalently \( x + \epsilon \psi(x) \theta \in D(A) \), and
\[
\langle x(\mathcal{A}_1) x, x \rangle = \langle A(x + \epsilon \psi(x) \theta), x \rangle
= \langle A_m(x + \epsilon \psi(x) \theta), x \rangle
= \langle A_m x, x \rangle
= \int_0^1 x'(s)x(s)ds - \alpha \|x\|^2
\leq \left( \frac{\epsilon^2 \|f\|^2}{2} - \alpha \right) \|x\|^2 - \frac{1}{2} x^2(0).
\]
Thus, the operator \( x(\mathcal{A}_1) \) is dissipative in \( D((\mathcal{A}_1 - \epsilon B)|X) \) for every \( 0 < \epsilon \leq \frac{(2\alpha)^{1/2}}{\|f\|} \).

Finally, the observation estimate follows from the fact that \( h \geq c > 0 \) and that for any \( x \in X \), the mapping \( t \mapsto \|S(t)x\| \) is decreasing.

We conclude by Theorem 3 that for \( \epsilon > 0 \) small enough, the control \( \nu(t) = -\epsilon \mathbf{1}_{\{t \geq 0: x(t) \neq 0\}} \) guarantees the exponentially stabilizing of system \((S_1)\).

4 | CONCLUSIONS

In this paper, we have shown that it is possible for a linear system with dissipative dynamic to be exponentially stable under small Desch–Schapacher perturbations of the dynamic. The main assumptions of sufficiency are formulated in terms of admissibility and observability assumptions of unbounded linear operators. An explicit decay rate of the stabilized state is given. The previous research on this problem concerned either bounded or Miyadera’s type perturbations \([7, 8]\). The main stabilization result is further applied to show the uniform exponential stabilization of unbounded bilinear reaction diffusion and transport equations using a bang bang controller.

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CONFLICT OF INTEREST STATEMENT

This work does not have any conflicts of interest.

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