Kählerian K3 surfaces and Niemeier lattices. I

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Abstract. Using the results obtained in [1], Remark 1.14.7, we clarify the relation between Kählerian K3 surfaces and Niemeier lattices. We emphasize that all 24 Niemeier lattices are important in the description of K3 surfaces, not only the one related to the Mathieu group.

Keywords: K3 surface, Kählerian surface, automorphism group, integer quadratic form.

Dedicated to I. R. Shafarevich on his 90th birthday

§ 1. Introduction

The study of finite symplectic automorphism groups of Kählerian K3 surfaces began in our papers [2], [3] of 1976–1979. We developed some general theory and completely classified the Abelian ones (14 non-trivial Abelian groups). In [1], Remark 1.14.7, we further showed, in particular, that all the finite symplectic automorphism groups of K3 surfaces can be obtained from the automorphism groups of negative-definite even unimodular lattices using primitive embeddings of negative-definite even lattices in unimodular ones (these results are reviewed in § 3). By the results in [1] on the existence of primitive embeddings of even lattices in even unimodular lattices, in the case of K3 surfaces it suffices to use negative-definite even unimodular lattices of rank 24. These are the Niemeier lattices.

In 1988 all finite symplectic automorphism groups of Kählerian K3 surfaces were classified as abstract groups by Mukai [4] (see also [5]). In 1998 Kondō [6] showed that this classification can be obtained using primitive embeddings of lattices in Niemeier lattices (see also Mukai’s important Appendix in [6]). This is similar to our considerations in [1], Remark 1.14.7. Hashimoto [7] has recently used Niemeier lattices in a similar way to classify the finite symplectic automorphism groups of Kählerian K3 surfaces in the same way as in our classification of the Abelian ones in [3].

Thus it becomes clear that the use of negative-definite even unimodular lattices and Niemeier lattices is a very powerful method.

In this paper we use the ideas and results in [1], Remark 1.14.7, to show that all 24 Niemeier lattices play an important role in the description of Kählerian K3 surfaces, their geometry and their finite symplectic automorphism groups.

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(It has been common practice to use only the Niemeier lattice with root system $24A_1$, which is related to the Mathieu group $M_{24}$.) From our point of view, all 24 Niemeier lattices are important for K3 surfaces.

We define and use a *marking* of a Kählerian K3 surface $X$ by a Niemeier lattice. In particular, one can use this marking to study the finite symplectic automorphism groups and non-singular rational curves on $X$. We demonstrated in [3] that it is important in the study of finite symplectic automorphism groups of K3 surfaces to work not only with algebraic K3 surfaces, but also with arbitrary Kählerian K3 surfaces. A general Kählerian K3 surface $X$ has negative-definite Picard lattice $S_X$ of rank $\text{rk} S_X \leq 19$. By the results of [1], there is a primitive embedding of $S_X$ in one of the 24 Niemeier lattices. This primitive embedding can be used to study the arithmetic and geometric properties of $S_X$ and $X$. It is called the marking of the K3 surface $X$ by the Niemeier lattice. All 24 Niemeier lattices are important in this process. We also use some modifications of these markings for those Kählerian K3 surfaces $X$ whose Picard lattice $S_X$ is semi-negative definite or hyperbolic.

In §2 we recall the results in [1] on primitive embeddings of lattices in even unimodular lattices. In §3 we recall the classification of Niemeier lattices and the results in [1], Remark 1.14.7, about their use (and the use of any other even negative-definite unimodular lattices) in the study of automorphism groups that act trivially on the discriminant group of a lattice. This study proceeds by embedding the given lattice in an even negative-definite unimodular lattice.

In §4 we recall definitions and basic results concerning Kählerian K3 surfaces $X$. These results are used to define the marking $S \subset N_i$ of $X$ by a Niemeier lattice $N_i$.

In §5 we apply markings of K3 surfaces $X$ by Niemeier lattices to the study of the finite symplectic automorphism groups and non-singular rational curves on $X$.

In §6 we consider examples of markings of Kählerian K3 surfaces by concrete Niemeier lattices $N_i$, $1 \leq i \leq 23$, and their applications. In particular, we show that for every Niemeier lattice $N_i$, $i = 1, 2, 3, 5, \ldots, 9, 11, \ldots, 23$, there is a Kählerian K3 surface $X$ that can be marked only by $N_i$ (see Theorem 6.1). We believe that similar results hold for the remaining Niemeier lattices. All Niemeier lattices are important for the marking of Kählerian K3 surfaces.

We hope to consider other examples and applications in further publications. A preliminary version of this paper was published as a preprint [8]. It also contains some further results.

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**§2. The existence of primitive embeddings of even lattices in even unimodular lattices, following [1]**

In this paper we use the notation, definitions and results of [1] concerning lattices, that is, non-degenerate integral (over $\mathbb{Z}$) symmetric bilinear forms. In particular, $\oplus$ stands for the orthogonal sum of lattices or quadratic forms. Given a prime $p$, we denote the ring of $p$-adic integers (resp. its group of invertible elements) by $\mathbb{Z}_p$ (resp. $\mathbb{Z}_p^*$).

Let $S$ be a lattice, $A_S = S^*/S$ its discriminant group, and $q_S$ the discriminant quadratic form on $A_S$. Here $S$ is assumed to be even, that is, $x^2$ is even for any $x \in S$. The minimal number of generators of the finite Abelian group $A_S$ is denoted
by \( l(A_S) \), and the order of \( A_S \) by \(|A_S|\). Given a prime \( p \), we write \( q_{S,p} = q_S \otimes \mathbb{Z}_p \) for the \( p \)-component of \( q_S \) (equivalently, for the discriminant quadratic form of the \( p \)-adic lattice \( S \otimes \mathbb{Z}_p \)). A quadratic form on a group of order 2 is denoted by \( q^2_\theta(2) \). A \( p \)-adic lattice of rank \( l(A_{S,p}) \) with discriminant quadratic form \( q_{S,p} \) is denoted by \( K(q_{S,p}) \). It is unique (up to an isomorphism) for \( p \neq 2 \), and for \( p = 2 \) if \( q_{S,2} \not\equiv q^2_\theta(2) \oplus q' \). We have the following result, where an embedding \( S \subset L \) of lattices is said to be primitive if \( L/S \) has no torsion.

**Theorem 2.1** ([1], Theorem 1.12.2). Let \( S \) be an even lattice of signature \((t_+, t_-)\), and let \( l_+, l_- \) be integers. Then a primitive embedding of \( S \) in an even unimodular lattice of signature \((l_+, l_-)\) exists if and only if the following conditions hold.

1) \( l_+ - l_- \equiv 0 \mod 8 \);

2) \( l_+ - t_+ \geq 0, l_- - t_- \geq 0, l_+ + l_- - t_+ - t_- \geq l(A_S) \);

3) \((-1)^{l_+ - t_+}|A_S| \equiv \det K(q_{S,p}) \mod (\mathbb{Z}_p^2)^2 \) for every odd prime \( p \) such that \( l_+ + l_- - t_+ - t_- = l(A_S) \); \( q_{S,2} \not\equiv q^2_\theta(2) \oplus q' \).

We note that if the last inequality in condition 2) of Theorem 2.1 is strict, then one does not need conditions 3) and 4): they hold automatically. If \( q_{S,2} \cong q^2_\theta(2) \oplus q' \), then one does not need condition 4).

§ 3. Niemeier lattices and their primitive sublattices

Negative-definite even unimodular lattices \( N \) of rank 24 are called **Niemeier lattices**. They were classified by Niemeier [9] (see also [10], Chs. 16, 18). All elements with square \((-2)\) in such lattices determine root systems \( \Delta(N) \) and generate root sublattices \([\Delta(N)] = N^{(2)} \subset N\), which are orthogonal sums of the irreducible root lattices \( A_n, n \geq 1, D_m, m \geq 4, E_k, k = 6, 7, 8, \) with the corresponding root systems \( \mathbb{A}_n, \mathbb{D}_m, \mathbb{E}_k \). The Dynkin diagrams of these root lattices are also denoted by \( \mathbb{A}_n, \mathbb{D}_m, \mathbb{E}_k \). We fix the standard bases of simple roots for these root systems and lattices as in [11] (see Fig. 1). If there are several irreducible components, we introduce an additional second index to enumerate them.

By [9], there are 24 Niemeier lattices \( N \) up to isomorphism, and they are characterized by their root sublattices. Here is a list:

\[
N^{(2)} = [\Delta(N)] = (1) D_{24}, (2) D_{16} \oplus E_8, (3) 3E_8, (4) A_{24}, (5) 2D_{12}, (6) A_{17} \oplus E_7, (7) D_{10} \oplus 2E_7, (8) A_{15} \oplus D_9, (9) 3D_8, (10) 2A_{12}, (11) A_{11} \oplus D_7 \oplus E_6, (12) 4E_6, (13) 2A_9 \oplus D_6, (14) 4D_6, (15) 3A_8, (16) 2A_7 \oplus D_5, (17) 4A_6, (18) 4A_5 \oplus D_4, (19) 6D_4, (20) 6A_4, (21) 8A_3, (22) 12A_2, (23) 24A_1.
\]

This list contains 23 Niemeier lattices \( N_i \), where \( i \) is shown in the brackets. The last one is the **Leech lattice** (24) with \( N^{(2)} = \{0\} \); it has no roots. We also denote the Niemeier lattice with root system \( R \) by \( N(R) \).
We recall that a simple-root basis $P(N)$ of the root lattice $[\Delta(N)]$ is determined up to the reflection group $W(N)$, which is generated by the reflections $s_\delta : x \to x + (x \cdot \delta)\delta$, $x \in N$, in the roots $\delta \in \Delta(N)$. Its Dynkin diagram is denoted by $\Gamma(P(N))$. Let $A(N) \subset O(N)$ be the group of symmetries of a fixed basis $P(N)$. In other words, $g \in A(N)$ if and only if $g(P(N)) = P(N)$. Then we have a semi-direct product decomposition

$$O(N) = A(N) \rtimes W(N)$$

of the automorphism group of $N$. Thus, choosing a basis $P(N)$ is equivalent to choosing a fundamental chamber for $W(N)$, and $A(N)$ is the group of symmetries of that fundamental chamber.

Let $N$ be the disjoint union of all 24 Niemeier lattices $N_i$, $i = 1, 2, \ldots, 24$, with their zeros identified. Naturally, $N_i = N_i \subset N$ stands for the corresponding Niemeier sublattice of this set and is called a Niemeier component of $N$. If $K$ is a lattice, then an embedding $K \subset N$ means an embedding of $K$ as a sublattice in one of the lattices $N_i \subset N$. If $K \neq \{0\}$, then $N_i$ is uniquely determined and is called the component of the embedding $K \subset N$. The embedding is said to be primitive if $N_i/K$ has no torsion.

We now fix bases $P(N_i)$ of the sets $\Delta(N_i)$ for all 24 components $N_i$ of the set $N$. The direct product

$$A(N) = \prod_{i=1}^{24} A(N_i)$$

acts naturally on $N$. By definition, $A(N_i)$ acts as $A(N_i)$ on the component $N_i = N_i$ and as the identity on all other components $N_j$, $j \neq i$.

Since Niemeier lattices have signature $(0,24)$, Theorem 2.1 yields the following assertion.

**Theorem 3.1** (corollary of Theorem 1.12.2 in [1]). An even lattice $S$ has a primitive embedding in $N$ (equivalently, in one of the Niemeier lattices) if and only if the following conditions hold.

1. $S$ is negative definite and $\text{rk} S + l(A_S) \leq 24$;
2. $|A_S| \equiv \det(q_{S_p}) \mod (\mathbb{Z}_p^*)^2$ for every odd prime $p$ such that $\text{rk} S + l(A_{S_p}) = 24$;
3. $|A_S| \equiv \pm \det(q_{S_2}) \mod (\mathbb{Z}_2^*)^2$ if $\text{rk} S + l(A_{S_2}) = 24$ and $q_{S_2} \not\equiv q_0^{(2)}(2) \oplus q'$.

Theorem 3.1 gives simple conditions on $S$ under which it may be regarded as a primitive sublattice of one of the Niemeier lattices (equivalently, of $N$). We can use this to obtain important information about $S$ and all primitive sublattices of Niemeier lattices. Later, these results will be applied to Picard lattices of Kählerian K3 surfaces.

Let $S$ be a negative-definite lattice. We write $H(S)$ for the kernel of the natural homomorphism

$$\pi : O(S) \to O(q_S).$$

Equivalently, an automorphism $\varphi \in O(S)$ of $S$ belongs to $H(S)$ if and only if $\varphi$ is equal to the identity on the discriminant group $S^*/S$. 
In [1], Remark 1.14.7, it was observed that $\pi$ has a non-trivial kernel if and only if $S$ contains some exceptional sublattices. These exceptional sublattices can be found from the negative-definite even unimodular lattices and their automorphism groups. We remark that $H(L) = O(L)$ for unimodular lattices $L$. It was suggested in [1], Remark 1.14.7, that for negative-definite even lattices $S$, $H(S)$ might be studied using primitive embeddings of $S$ in negative-definite even unimodular lattices. For example, if the hypotheses of Theorem 3.1 hold for $S$, then one can use primitive embeddings in Niemeier lattices (or, equivalently, in $\mathcal{N}$). We now summarize the results in [1], Remark 1.14.7.

Let $S$ be an even negative-definite lattice and $\Delta(S)$ the set of roots $\delta \in S$ with $\delta^2 = -2$. It is easy to see that the reflection $s_\delta$ belongs to $H(S)$. Thus, $[\delta] = \langle-2\rangle$ is the first example of an exceptional sublattice. The Weyl group $W(S)$ generated by the reflections in all elements of $\Delta(S)$ is a normal subgroup of $H(S)$. We choose a basis $P(S)$ of $\Delta(S)$ (or, equivalently, a fundamental chamber for the Weyl group $W(S)$). Put

$$A(S) = \{ \varphi \in H(S) \mid \varphi(P(S)) = P(S) \}$$

(equivalently, $A(S) \subset H(S)$ is the group of symmetries of the fundamental chamber). Then $H(S) = A(S) \rtimes W(S)$ is a semi-direct product. Let

$$\mathcal{L}(S) = S_{A(S)} = (S^{A(S)})_S\perp$$

be the orthogonal complement in $S$ of the fixed part $S^{A(S)}$ of the action of $A(S)$. Clearly, the sublattice $\mathcal{L}(S) = S_{A(S)} \subset S$ is uniquely determined up to the action of $W(S)$. It is called the co-invariant sublattice of $S$ for $A(S)$.

The following properties of the primitive sublattice $\mathcal{L}(S)$ were established in [1], Remark 1.14.7, Proposition 1.14.8: $\mathcal{L}(S)^{A(S)} = \{ 0 \}$ (this is obvious), $\mathcal{L}(S)$ has no roots with square $\langle -2 \rangle$, $\mathcal{L}(L(S)) = \mathcal{L}(S)$ and $A(\mathcal{L}(S)) = A(S)\mathcal{L}(S)$. Thus $H(\mathcal{L}(S)) = A(\mathcal{L}(S))$, and $A(\mathcal{L}(S))$ determines $A(S)$: it suffices to extend $A(\mathcal{L}(S))$ by the identity to the orthogonal complement of $\mathcal{L}(S)$ in $S$.

The sublattice $\mathcal{L}(S) = S_{A(S)}$ was called the Leech-type sublattice of $S$ in [1], Remark 1.14.7. The group $H(S)$ and the natural subgroups $W(S)$ and $A(S)$ of $H(S)$ are completely determined by the basis $P(S)$ and the Leech-type sublattice $\mathcal{L}(S)$. Indeed, $W(S)$ is generated by reflections in $P(S)$, and we have $A(S) = A(\mathcal{L}(S))$ after extending $A(\mathcal{L}(S))$ by the identity to the orthogonal complement $\mathcal{L}(S)\perp$.

Now let $S \subset N_i$ be a primitive sublattice of a Niemeier lattice (or any other negative-definite even unimodular lattice). Replacing $S$ by $w(S)$, where $w \in W(N_i)$, we can regard $P(S)$ as a subset of $P(N_i)$. We shall always assume that the primitive embedding $S \subset N_i$ satisfies the following condition:

$$P(S) = S \cap P(N_i) \text{ is a basis of } \Delta(S). \quad (3.1)$$

Then, extending $A(S)$ by the identity to $S_{N_i}^\perp$ (this is possible since $A(S)$ acts as the identity on $S^*/S$), we get a subgroup of $A(N_i)$. It follows that

$$A(S) = \{ \varphi|S \mid \varphi \in A(N_i) \text{ and } \varphi|S_{N_i}^\perp \text{ is the identity} \}. \quad (3.2)$$
Moreover, \( A(S) \subset A(N_i) \) and \( P(S) \subset P(N_i) \) satisfy an obvious important condition:

\[
P(S) \text{ is invariant with respect to } A(S).
\]

Since the bases \( P(N_i) \), their graphs \( \Gamma(P(N_i)) \), and the groups \( A(N_i) \) are known for all Niemeier lattices \( N_i \), we obtain an effective tool for constructing primitive sublattices \( S \subset N_i \) with various graphs \( \Gamma(P(S)) \) and groups \( A(S) \).

**Example 3.1.** The Niemeier lattices \( N = N_1 = N(D_{24}) \), \( N_2 = N(D_{16} \oplus E_8) \) have trivial groups \( A(N) \) (see [10], Ch. 16). It follows that all the primitive sublattices \( S \subset N \) have trivial groups \( A(S) \) and trivial Leech-type sublattices \( \mathcal{L}(S) \).

Suppose that \( \Gamma(S^{(2)}) \) has a subgraph \( \mathbb{D}_n \), where \( n \geq 13 \). Then, by the classification of Niemeier lattices, \( S \) can only have primitive embeddings in \( N(D_{24}) \) and \( N(D_{16} \oplus E_8) \), and the group \( A(S) \) is also trivial if \( S \) satisfies Theorem 3.1.

Further examples will be given in § 5.

§ 4. Additional markings of Kählerian K3 surfaces by Niemeier lattices

We consider Kählerian K3 surfaces, that is, compact simply connected complex surfaces \( X \) admitting a holomorphic differential 2-form \( \omega_X \in H^{2,0}(X) \) without zeros. One can regard \( \omega_X \) as a complex volume form (see [12], Ch. 9 and [13] for Kählerian K3 surfaces).

It is known that the cohomology lattice \( H^2(X, \mathbb{Z}) \) with the intersection pairing is an even unimodular lattice of signature \((3, 19)\) without torsion. All even unimodular lattices of signature \((3, 19)\) are isomorphic (the same holds for all indefinite even unimodular lattices of equal signature). We have the Hodge decomposition

\[
H^2(X, \mathbb{C}) = H^2(X, \mathbb{Z}) \otimes \mathbb{C} = H^{2,0}(X) + H^{1,1}(X) + H^{0,2}(X),
\]

where \( H^{2,0}(X) = \mathbb{C} \omega_X \) is one-dimensional, \( H^{0,2}(X) = H^{2,0}(\overline{X}) \), and \( H^{1,1}(X) = H^{1,1}(\overline{X}) \) is 20-dimensional. It follows that the Picard lattice

\[
S_X = H^2(X, \mathbb{Z}) \cap H^{1,1}(X)
\]

of \( X \) is a sublattice in the hyperbolic real subspace \( H^{1,1}_R(X) \subset H^{1,1}(X) \) of signature \((1, 19)\). Therefore the Picard lattice \( S_X \) is a primitive sublattice of \( H^2(X, \mathbb{Z}) \) belonging to one of the following classes:

(a) negative definite with \( 0 \leq \text{rk} S_X \leq 19 \),

(b) semi-negative definite with one-dimensional kernel and \( 1 \leq \text{rk} S_X \leq 19 \),

(c) hyperbolic (that is, of signature \((1, \text{rk} S_X - 1)\)) with \( 1 \leq \text{rk} S_X \leq 20 \) (this holds for all algebraic \( X \)).

We denote the Picard number of \( X \) by \( \rho(X) = \text{rk} S_X \).

Let \( L_{K3} \) be an abstract even unimodular lattice of signature \((3, 19)\). Then the Picard lattice \( S_X \) of any Kählerian K3 surface \( X \) has a primitive embedding \( S_X \subset L_{K3} \) and satisfies one of the conditions (a), (b) or (c). Since the period map for K3 surfaces is epimorphic (14–16), the Picard lattices \( S_X \) of K3 surfaces are characterized by these properties. An even lattice which is either negative definite, semi-negative definite with one-dimensional kernel, or hyperbolic is isomorphic.
to the Picard lattice of a Kählerian K3 surface if and only if it has a primitive embedding in the lattice $L_{K3}$.

Theorem 2.1 yields a complete description of Picard lattices of Kählerian K3 surface.

**Theorem 4.1** (corollary of Theorem 1.12.2 in [1]). An even negative-definite lattice $M$ is isomorphic to the Picard lattice $S_X$ of a Kählerian K3 surface $X$ (equivalently, $M$ has a primitive embedding in $L_{K3}$) if and only if

1. $\text{rk} M \leq 19$ and $\text{rk} M + l(A_M) \leq 22$;
2. $-|A_M| \equiv \det K(q_{M,p}) \mod (\mathbb{Z}_p)^2$ for every odd prime $p$ such that $\text{rk} M + l(A_{M,p}) = 22$;
3. $|A_M| \equiv \pm \det K(q_{M_2}) \mod (\mathbb{Z}_2)^2$ if $\text{rk} M + l(A_{M_2}) = 22$ and $q_{M_2} \not\equiv q_{2}^{(2)}(2) \oplus q'$.

**Theorem 4.2** (corollary of Theorem 1.12.2 in [1]). An even semi-negative definite lattice $M$ with one-dimensional kernel $\text{Ker} M$ is isomorphic to the Picard lattice $S_X$ of a Kählerian K3 surface $X$ (equivalently, $M$ has a primitive embedding in $L_{K3}$) if and only if $\tilde{M} = M/\text{Ker} M$ satisfies

1. $\text{rk} \tilde{M} \leq 18$ and $\text{rk} \tilde{M} + l(A_{\tilde{M}}) \leq 20$;
2. $|A_{\tilde{M}}| \equiv \det K(q_{\tilde{M},p}) \mod (\mathbb{Z}_p)^2$ for every odd prime $p$ such that $\text{rk} \tilde{M} + l(A_{\tilde{M},p}) = 20$;
3. $|A_{\tilde{M}}| \equiv \pm \det K(q_{\tilde{M}_2}) \mod (\mathbb{Z}_2)^2$ if $\text{rk} \tilde{M} + l(A_{\tilde{M}_2}) = 20$ and $q_{\tilde{M}_2} \not\equiv q_{2}^{(2)}(2) \oplus q'$.

**Theorem 4.3** (corollary of Theorem 1.12.2 in [1]). An even hyperbolic lattice $M$ (that is, $M$ has signature $(1, \text{rk} M - 1)$) is isomorphic to the Picard lattice $S_X$ of an algebraic K3 surface $X$ over $\mathbb{C}$ (equivalently, $M$ has a primitive embedding in $L_{K3}$) if and only if

1. $\text{rk} M \leq 20$ and $\text{rk} M + l(A_M) \leq 22$;
2. $|A_M| \equiv \det K(q_{M,p}) \mod (\mathbb{Z}_p)^2$ for every odd prime $p$ such that $\text{rk} M + l(A_{M,p}) = 22$;
3. $|A_M| \equiv \pm \det K(q_{M_2}) \mod (\mathbb{Z}_2)^2$ if $\text{rk} M + l(A_{M_2}) = 22$ and $q_{M_2} \not\equiv q_{2}^{(2)}(2) \oplus q'$.

We recall the definition of the period domain $\tilde{\Omega}$ for Kählerian K3 surfaces (see [13]). Fix an even unimodular lattice $L_{K3}$ of signature $(3, 19)$. The domain $\tilde{\Omega}$ consists of all triples

$$(H^{2,0}, V^+, P).$$

Here $H^{2,0} \subset L_{K3} \otimes \mathbb{C}$ is a one-dimensional complex vector subspace such that

$$\omega \cdot \omega = 0, \quad \omega \cdot \varnothing > 0$$

for all $\omega \in H^{2,0}$, $\omega \neq 0$ (we extend the symmetric bilinear form on $L_{K3}$ to a $\mathbb{C}$-bilinear form on $L_{K3} \otimes \mathbb{C}$). Such subspaces $H^{2,0} \subset L_{K3} \otimes \mathbb{C}$ form a 20-dimensional complex homogeneous manifold, which is denoted by $\Omega$.

For a fixed $H^{2,0} \subset L_{K3} \otimes \mathbb{C}$ in $\Omega$, we denote the orthogonal complement of $H^{2,0}$ in $L_{K3} \otimes \mathbb{R}$ by $H^{1,1}_R$. This is a hyperbolic form of signature $(1, 19)$. It contains the light cone

$$V = \{ x \in H^{1,1}_R \mid x^2 > 0 \}.$$
Let $V^+$ be one of the two halves (connected components) of $V$. It determines a hyperbolic space $\mathcal{H}^+ = V^+ / \mathbb{R}^+$, which is the projectivization of $V^+$.

Put $H^{1,1}_Z = H^{1,1}_R \cap L_{K3}$. Then $H^{1,1}_Z$ is a lattice which is either negative definite, semi-negative definite with one-dimensional kernel, or hyperbolic. Let $\Delta(H^{1,1}_Z)$ be the set of all elements in this lattice with square $(-2)$. The group $W(H^{1,1}_Z)$ generated by reflections in all elements of $\Delta(H^{1,1}_Z)$ is a discrete reflection group in $V^+$ and $\mathcal{H}^+$. We write $P$ for the set of all vectors in $\Delta(H^{1,1}_Z)$ that are perpendicular to one of the fundamental chambers $\mathcal{M}$ of this reflection group and are directed outward from this chamber. Thus,

$$\mathcal{M} = \{ x \in V^+ \mid x \cdot P \geq 0 \},$$

(4.4)

every codimension-one face of the polytope $\mathcal{M}$ is perpendicular to precisely one $\delta \in P$ and, vice versa, every $\delta \in P$ is perpendicular to a codimension-one face of $\mathcal{M}$. Briefly, $P = P(\mathcal{M})$.

The set of all triples (4.1) is denoted by $\tilde{\Omega}$. It is a non-Hausdorff 20-dimensional complex manifold, which gives an étale covering of $\Omega$.

We recall that for every Kählerian K3 surface $X$ there is a canonical triple

$$\alpha(K2,0(X), V^+(X), P(X))$$

(4.5)

(see [13] for details), where $H^{2,0}(X) \subset H^2(X, \mathbb{Z}) \otimes \mathbb{C}$ has already been defined and considered above, $V^+(X)$ is the half of the light cone

$$V(X) = \{ x \in H^{1,1}_R(X) \mid x^2 > 0 \}$$

(4.6)

containing the Kähler class of $X$, and

$$P(X) \subset S_X$$

(4.7)

is the set of classes of all non-singular rational curves on $X$. All these classes have square $(-2)$, and any exceptional curve on $X$ (that is, an irreducible complex curve $C \subset X$ with $C^2 < 0$) is one of them. They determine the nef cone

$$\text{NEF}(X) = \{ x \in \overline{V^+(X)} \mid x \cdot P(X) \geq 0 \},$$

which is a fundamental chamber for the reflection group $W(S_X)$, and $P(X)$ is the set of vectors perpendicular to the codimension-one faces of $\text{NEF}(X)$.

We recall that a marking of a Kählerian K3 surface $X$ is an isomorphism of lattices

$$\alpha : H^2(X, \mathbb{Z}) \cong L_{K3},$$

(4.8)

The pair $(X, \alpha)$ is called a marked Kählerian K3 surface.

Taking

$$\alpha(K2,0(X), V^+(X), P(X))$$

$$= ((\alpha \otimes \mathbb{C})(H^{2,0}(X)), (\alpha \otimes \mathbb{R})(V^+(X)), \alpha(P(X))) \in \tilde{\Omega},$$

(4.9)

we get the period map $\alpha$ from the moduli space of marked Kählerian K3 surfaces to the period domain $\Omega$ of marked Kählerian K3 surfaces.
By the global Torelli theorem for Kählerian K3 surfaces (see [17] for the algebraic case and [13]) and since the period map for Kählerian K3 surfaces is epimorphic (see [14] for the algebraic case and [15], [16]), the period map \( \alpha \) is an isomorphism of complex spaces.

Wishing to introduce an additional marking of Kählerian K3 surfaces by Niemeier lattices \( N_i \) (or \( N \)), we define the corresponding period domain \( \tilde{\Omega}_N \) as the set of all quadruples

\[
(H^{2,0}, V^+, P, \tau: S \subset N_i),
\]

where the triples \((H^{2,0}, V^+, P) \in \tilde{\Omega}\) are as above and \( \tau: S \subset N_i \) will now be described. First, \( S \subset H_{Z}^{1,1} \) is a maximal negative-definite sublattice in \( H_{Z}^{1,1} \). Thus we have \( S = H_{Z}^{1,1} \) if \( H_{Z}^{1,1} \) is negative definite, and \( H_{Z}^{1,1} = S \oplus \mathbb{Z}c \) if \( H_{Z}^{1,1} \) has a one-dimensional kernel \( \mathbb{Z}c \). If \( H_{Z}^{1,1} \) is hyperbolic, then \( S \subset H_{Z}^{1,1} \) is a negative-definite sublattice of \( H_{Z}^{1,1} \) with \((S)_{H_{Z}^{1,1}}^\perp = \mathbb{Z}h \), where \( h^2 > 0 \). In the parabolic case, when \( S \) has a one-dimensional kernel \( \mathbb{Z}c \), we also require that \( P \cap S = P(S) \). In the hyperbolic case we also require \( h \) to be nef, that is, \( h \in V^+ \) and \( h \cdot P \geq 0 \).

Equivalently, \( h \) belongs to the fundamental chamber for \( W(H_{R}^{1,1}) \) determined by \( V^+ \) and \( P \). Second, \( \tau: S \subset N_i \) is a primitive embedding of \( S \) in one of the Niemeier lattices \( N_i \) (or, equivalently, in \( N \)) such that \( \tau(P \cap S) \subset P(N_i) \). Thus \( \tau \) may be regarded as an additional marking by Niemeier lattices. Since \( S \) is negative definite and has a primitive embedding in \( L_{K3} \), it satisfies Theorem 4.1. In particular, \( \text{rk} \ S \leq 19 \) and \( \text{rk} \ S + l(A_S) \leq 22 \). Then \( \text{rk} \ S + l(A_S) < 24 \). Thus \( S \) satisfies Theorem 3.1 and has a primitive embedding in one of the 24 Niemeier lattices \( N_i \). The same argument shows that \( S \oplus A_1 \) satisfies Theorem 3.1. It follows that \( S \) has a primitive embedding in one of the 23 Niemeier lattices with non-empty set of roots. This is an important trick of Kondô [6], which enables us to avoid the difficult Leech lattice. It follows that

\[
\tau: \tilde{\Omega}_N \to \tilde{\Omega}
\]

is a natural surjective ‘covering’ of \( \tilde{\Omega} \). Unfortunately, we cannot claim that \( \tilde{\Omega}_N \) is a manifold. We can only claim that it is a topological space whose open subsets are pre-images of open subsets of \( \tilde{\Omega} \), and \( \tau \) is one-to-one over the generic points of \( \tilde{\Omega} \) (where \( H_{Z}^{1,1} = \{0\} \)) since we identify in \( N \) the zeros of all Niemeier lattices \( N_i \), \( i = 1, \ldots, 24 \). These points of \( \tilde{\Omega} \) form the complement of an infinite family of divisors in \( \tilde{\Omega} \). Thus we can only claim that \( \tilde{\Omega}_N \) is similar to a manifold over these points of \( \tilde{\Omega} \). Moreover, \( \tau \) has finite fibres over those points of \( \tilde{\Omega} \) where the lattice \( H_{Z}^{1,1} \) is negative definite. Thus \( \tilde{\Omega}_N \) is very similar to a manifold over these points. The fibres of \( \tau \) over the remaining points of \( \tilde{\Omega} \) are generally countable.

We similarly introduce an additional marking for Kählerian K3 surfaces \( X \). It is given by an embedding

\[
\tau: S \subset N.
\]

Here \( S \subset S_X \) is a maximal negative-definite sublattice. Hence \( S = S_X \) if \( S_X \) is negative definite. If \( S_X \) is semi-negative definite with one-dimensional kernel generated by the class \( c \) of an elliptic curve \( C \) on \( X \), then \( S_X = S \oplus \mathbb{Z}c \) and \( S \cap P(X) = P(S) \). If \( X \) is algebraic, then \( S = h_{S_X}^\perp \), where \( h \in S_X \) is primitive,
$h^2 > 0$, and $h$ is nef (that is, $h \cdot D \geq 0$ for every effective divisor $D$ on $X$). Moreover, $	au: S \subset \mathcal{N}$ is a primitive embedding of $S$ in one of the 24 Niemeier lattices $N_i$ such that $P(X) \cap S = P(S) \subset P(N_i)$.

The standard marking $\alpha: H^2(X, \mathbb{Z}) \cong L_{K3}$ of a Kählerian K3 surface $X$ endowed with an additional marking $\tau: S \subset \mathcal{N}$ by Niemeier lattices determines the periods $\alpha(X, \tau) \in \tilde{\Omega}_\mathcal{N}$. They are

$$\alpha(H^{2,0}(X), V^+(X), P(X), \tau: S \subset \mathcal{N})$$

$$= (\langle \alpha \otimes \mathbb{C} \rangle(H^{2,0}(X)), \langle \alpha \otimes \mathbb{R} \rangle(V^+(X)), \alpha(P(X)), \tau \alpha^{-1}: \alpha(S) \subset \mathcal{N}) \in \tilde{\Omega}_\mathcal{N}. \quad (4.12)$$

By the global Torelli theorem and since the period map for Kählerian K3 surfaces is epimorphic (see the references above), we obtain an isomorphism between the moduli space of Kählerian K3 surfaces $X$ with an additional marking by Niemeier lattices and their period space $\tilde{\Omega}_\mathcal{N}$.

**Remark 4.1.** Our definitions and considerations yield the following procedure for constructing a Kählerian K3 surface $X$ with marking $\tau: S \subset N_i$ by a Niemeier lattice $N_i$.

(a) Check that $S$ satisfies Theorem 4.1 (equivalently, that there is a primitive embedding $S \subset L_{K3}$).

(b) Choose a primitive embedding $S \subset L_{K3}$.

(c1) To construct $X$ with negative-definite $S_X$, choose a sufficiently general $H^{2,0} \subset (S)_{L_{K3}}^\perp \otimes \mathbb{C}$ such that $H^1_{Z,1} = S$. Choose $V^+ \subset H^1_{\mathbb{R}}$ and put $P = P(S) = P(N_i) \cap S$. Then $(H^{2,0}, V^+, P, \tau: S \subset N_i)$ are the periods of some marked Kählerian K3 surface $(X, \alpha)$ with $S_X = \alpha^{-1}(S)$ and with marking $\tau \alpha: S_X = \alpha^{-1}(S) \subset N_i$ by the Niemeier lattice $N_i$. We have $P(X) = \alpha^{-1}(P(N_i) \cap S)$.

(c2) To construct $X$ with semi-negative-definite $S_X$, check that $S \subset L_{K3}$ satisfies Theorem 4.2. Then there is a primitive non-zero element $c \in (S)_{L_{K3}}^\perp$ such that $S \oplus \mathbb{Z}c \subset L_{K3}$ is a primitive sublattice with kernel $\mathbb{Z}c$. Choose a sufficiently general $H^{2,0} \subset (S \oplus \mathbb{Z}c)_{L_{K3}}^\perp \otimes \mathbb{C}$ such that $H^1_{Z,1} = S \oplus \mathbb{Z}c$. Choose $V^+ \subset H^1_{\mathbb{R}}$ in such a way that $c \in V^\perp$. Put $P = P(S) = P(N_i) \cap S$. For every connected component $P_i$, $i = 1, 2, \ldots, k$, of the Dynkin diagram of $P$ take a maximal root $\delta_i$ and put $p_i = c - \delta_i$. Then

$$\tilde{P} = P \cup \{p_1, p_2, \ldots, p_k\} = P(S \oplus \mathbb{Z}c)$$

and $(H^{2,0}, V^+, \tilde{P}, \tau: S \subset N_i)$ are the periods of a marked Kählerian K3 surface $(X, \alpha)$ with $S_X = \alpha^{-1}(S \oplus \mathbb{Z}c)$ and with marking $\tau \alpha: \alpha^{-1}(S) \subset N_i$ by the Niemeier lattice $N_i$. We have $P(X) = \alpha^{-1}(\tilde{P})$.

(c3) To construct $X$ with hyperbolic $S_X$ (thus $X$ is algebraic), take a primitive element $h \in (S)_{L_{K3}}^\perp$ with $h^2 > 0$ (such elements obviously exist). Let $\tilde{S} = [S \oplus \mathbb{Z}h]_{pr}$ be the primitive sublattice of $L_{K3}$ generated by $S$ and $h$. Then $\tilde{S}$ is hyperbolic. Choose a sufficiently general $H^{2,0} \subset (\tilde{S})_{L_{K3}}^\perp \otimes \mathbb{C}$ such that $H^1_{Z,1} = \tilde{S}$, and choose $V^+ \subset H^1_{\mathbb{R}}$ in such a way that $h \subset V^+$. Put $P = P(S) = P(N_i) \cap S$ and take a fundamental chamber $\mathcal{M}$ for $W(\tilde{S})$ such that $h \in \mathcal{M}$ and $P \subset P(\mathcal{M})$, where $P(\mathcal{M})$ is the set of vectors with square $(-2)$ that are perpendicular to codimension-one faces of $\mathcal{M}$ and directed outward from $\mathcal{M}$. Then $(H^{2,0}, V^+, P(\mathcal{M}), \tau: S \subset N_i)$
are the periods of some marked algebraic K3 surface \((X, \alpha)\) with \(S_X = \alpha^{-1}(\tilde{S})\), nef element \(\alpha^{-1}(h)\), \(P(X) = \alpha^{-1}(P(M))\), and marking \(\tau \alpha: \alpha^{-1}(S) \subset N_i\) by the Niemeier lattice \(N_i\). We have \(P(X) = \alpha^{-1}(P(M))\), and the linear system \(|nh|\), \(n > 0\), contracts non-singular rational curves with classes in \(\alpha^{-1}(S \cap P(N_i))\) and only these curves.

§ 5. Applications of markings by Niemeier lattices

Let \(X\) be a Kählerian K3 surface with marking \(\tau: S \subset N_i\) by a Niemeier lattice \(N_i\). Using the conditions imposed on \(\tau\) and our considerations in § 3 (which follow from [1], Remark 1.14.7), we obtain the following result.

**Theorem 5.1.** Let \(X\) be a Kählerian K3 surface with marking \(\tau: S \subset N_i\) by a Niemeier lattice \(N_i\). Then \(P(X) \cap S = P(S)\) is a basis for the root system \(\Delta(S)\), and \(P(S) = P(X) \cap S\) is a subset of \(P(N_i)\). Thus the Dynkin diagram \(\Gamma(P(X) \cap S)\) is the Dynkin diagram \(\Gamma(P(S))\) and a subdiagram of the Dynkin diagram \(\Gamma(P(N_i))\).

In particular, if \(S_X\) is negative definite, then \(S = S_X\) and \(P(X) = P(N_i) \cap S\) is the set of classes of all non-singular rational curves on \(X\). If \(X\) is algebraic, then \(P(X) \cap S = P(S) = P(N_i) \cap S\) is the set of classes of all non-singular rational curves on \(X\) which are contracted by the linear system \(|nh|\) for \(n > 0\), where \(h\) is the primitive nef element of \(S_X\) that generates the orthogonal complement of \(S\) in \(S_X\).

It follows that for Kählerian K3 surfaces \(X\), the markings by Niemeier lattices describe the sets of non-singular rational curves on \(X\) whose classes are contained in \(S \subset S_X\).

We recall that an automorphism \(\varphi\) of a Kählerian K3 surface \(X\) is said to be symplectic if \(\varphi\) preserves the holomorphic form \(\omega_X\), that is, \(\varphi^*(\omega_X) = \omega_X\). This is equivalent (see [3]) to requiring that \(\varphi\) acts as the identity on the transcendental part

\[
H^2(X, \mathbb{Z})/S_X. \tag{5.1}
\]

It is known (see [17] and [13]) and follows formally from the global Torelli theorem for K3 surfaces that the kernel of the action of \(\text{Aut} X\) on \(H^2(X, \mathbb{Z})\) is trivial. A subgroup of \(\text{Aut} X\) is said to be symplectic if all its elements are symplectic. We denote the group of all symplectic automorphisms of \(X\) by \((\text{Aut} X)_0\).

Given a marking \(\tau: S \subset N_i\) of \(X\) by a Niemeier lattice \(N_i\), we consider the group of automorphisms

\[
\text{Aut}(X, S)_0 = \{ f \in (\text{Aut} X)_0 \mid f(S) = S \text{ and } f|S_{H^2(X, \mathbb{Z})}^{\perp} = \text{the identity}\}. \tag{5.2}
\]

If \(S_X\) is negative definite or hyperbolic, then the requirement for \(f|S_{H^2(X, \mathbb{Z})}^{\perp}\) to be the identity follows from the other conditions in (5.2). In any case, the definition of \(\text{Aut}(X, S)_0\) can be rewritten as

\[
\text{Aut}(X, S)_0 = \{ f \in \text{Aut} X \mid f(S) = S \text{ and } f|S_{H^2(X, \mathbb{Z})}^{\perp} = \text{the identity}\}. \tag{5.3}
\]

By [1], Proposition 1.5.1, an automorphism \(\varphi \in O(S)\) can be extended to an automorphism in \(O(H^2(X, \mathbb{Z}))\) by the identity on \(S_{H^2(X, \mathbb{Z})}^{\perp}\) if and only if \(\varphi\) is equal
to the identity on $A_S = S^*/S$. By the global Torelli theorem for K3 surfaces (see [17], [13]), we then have $\varphi = f^*$ for some $f \in \text{Aut} X$. It follows from (5.1) that $f \in (\text{Aut} X)_0$. We shall identify the group $\text{Aut}(X, S)_0$ with its action on $S$.

Using our considerations in § 3 (which follow from [1], Remark 1.14.7), we obtain the following result.

**Theorem 5.2.** Let $X$ be a Kählerian K3 surface with marking $S \subset N_i$ by a Niemeier lattice $N_i$, $i = 1, \ldots, 24$. Let $\text{Aut}(X, S)_0$ be the symplectic automorphism group of $X$ consisting of all automorphisms of $X$ that are equal to the identity on $S^\perp_{H^2(X, \mathbb{Z})}$ (they are all symplectic). Then the action of $\text{Aut}(X, S)_0$ on $S$ identifies it with the subgroup

$$A(S) = \{ \varphi \in A(N_i) \mid \varphi|_{S^\perp_{N_i}} \text{is the identity} \}$$

of $A(N_i)$. We have

$$\text{Aut}(X, S)_0 = A(S)|S.$$

The Leech-type sublattice (or the coinvariant sublattice) $\mathcal{L}(S) \subset S$ is given by

$$\mathcal{L}(S) = S_{A(S)} = (S^{A(S)})^\perp_S = ((N_i)^{A(S)})^\perp_{N_i},$$

(5.5)

Let $P(X) \cap S$ be the set of all classes of non-singular rational curves on $X$ that lie in $S$. Then $P(X) \cap S$ is invariant with respect to $A(S)$.

**Remark 5.1.** Theorems 5.1, 5.2 yield the following procedure for constructing Kählerian K3 surfaces with all possible markings $S \subset N_i$ (in what follows we omit $\tau$) and all possible $P(X) \cap S$ and $\text{Aut}(X, S)_0$.

(a) Given a Niemeier lattice $N_i$, choose a subgroup $A \subset A(N_i)$ such that $\mathcal{L} = (N_i)_A = ((N_i)^A)^\perp_{N_i}$ satisfies Theorem 4.1 (equivalently, there is a primitive embedding $\mathcal{L} \subset L_{K3}$). In what follows such subgroups are referred to as KahK3-subgroups (that is, Kählerian K3 surface subgroups). Let $\text{Clos}(A) \subset A(N_i)$ (we use the notation of [7]) be the maximal subgroup of $A(N_i)$ with the same coinvariant sublattice $\mathcal{L} = (N_i)_A = ((N_i)^A)^\perp_{N_i}$. These and only these subgroups Clos($A$) correspond to the full symplectic automorphism groups $\text{Aut}(X, S)_0$ of Kählerian K3 surfaces which can be marked by $N_i$. The subgroup $A$ can only be a proper subgroup, $A \subset \text{Clos}(A) = \text{Aut}(X, S)_0$.

We note that the list of all possible abstract groups $A$ for all Niemeier lattices together is known (see [3] for Abelian groups $A$ and [4]–[6] for arbitrary $A$). The Leech-type lattices $\mathcal{L}$ corresponding to $A$ (for all Niemeier lattices) are also known (see [3] for Abelian groups $A$ and [7] for arbitrary $A$). The Leech-type lattices $\mathcal{L}$ along with the action of $A$ on $\mathcal{L}$ are uniquely determined up to isomorphism for almost all abstract groups $A$.

(b) Choose an $A$-invariant subset $P \subset P(N_i)$ such that the primitive sublattice $S_0 = [\mathcal{L}, P]_{\text{pr}} \subset N_i$ generated by $\mathcal{L}$ and $P$ satisfies Theorem 4.1 (equivalently, there is a primitive embedding $S_0 = [\mathcal{L}, P]_{\text{pr}} \subset L_{K3}$). Then $P \subset P(S_0) = S_0 \cap P(N_i)$ and $A \subset \text{Aut}(S_0)$.

(c) Extend $S_0$ to a primitive sublattice $S$, $S_0 \subset S \subset N_i$, with the following properties: $S \cap P(N_i) = P$,

$$\{ \varphi \in A(N_i) \mid \varphi(S) = S \text{ and } \varphi|_{S^\perp_{N_i}} \text{is the identity} \} = \text{Clos}(A)$$
and $S$ satisfies Theorem 4.1. Then $A(S) = \text{Clos}(A)$, $\mathcal{L}(S) = \mathcal{L}$, $P(S) = P$ and there is a primitive embedding $S \subset L_{K3}$.

(d) Follow Remark 4.1 to construct a Kählerian K3 surface $X$ with marking $S \subset N_i$ by the Niemeier lattice $N_i$. Then $\text{Aut}(X, S)_0 = \text{Clos}(A)$ and $P(X) \cap S = P(S) = P$.

§ 6. Markings of K3 surfaces by concrete Niemeier lattices $N_1, \ldots, N_{23}$

In what follows we endow the root lattices $A_n$, $D_n$ and $E_k$, $k = 6, 7, 8$, with the bases shown in Fig. 1.

![Figure 1. Bases of the Dynkin diagrams $A_n$, $D_n$, $E_k$.](image)

For $A_n$, $n \geq 1$, we put $\varepsilon_1 = (\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n)/(n + 1)$. This is a generator of the discriminant group $A_n^*/A_n \cong \mathbb{Z}/(n + 1)\mathbb{Z}$.

For $D_n$, $n \geq 4$ and $n \equiv 0 \mod 2$, we put $\varepsilon_1 = (\alpha_1 + \alpha_3 + \cdots + \alpha_{n-3} + \alpha_{n-1})/2$, $\varepsilon_2 = (\alpha_{n-1} + \alpha_n)/2$, $\varepsilon_3 = (\alpha_1 + \alpha_3 + \cdots + \alpha_{n-3} + \alpha_n)/2$. These are all non-zero elements of the discriminant group $D_n^*/D_n \cong (\mathbb{Z}/2\mathbb{Z})^2$.

For $D_n$, $n \geq 4$ and $n \equiv 1 \mod 2$, we put $\varepsilon_1 = (\alpha_1 + \alpha_3 + \cdots + \alpha_{n-2})/2 + \alpha_{n-1}/4 - \alpha_n/4$, $\varepsilon_2 = (\alpha_{n-1} + \alpha_n)/2$, $\varepsilon_3 = (\alpha_1 + \alpha_3 + \cdots + \alpha_{n-2})/2 - \alpha_{n-1}/4 + \alpha_n/4$. These are all non-zero elements of $D_n^*/D_n \cong \mathbb{Z}/4\mathbb{Z}$.

For $E_6$ we put $\varepsilon_1 = (\alpha_1 - \alpha_3 + \alpha_5 - \alpha_6)/3$, $\varepsilon_2 = (-\alpha_1 + \alpha_3 - \alpha_5 + \alpha_6)/3$. These are all non-zero elements of $E_6^*/E_6 \cong \mathbb{Z}/3\mathbb{Z}$.

For $E_7$ we put $\varepsilon_1 = (\alpha_2 + \alpha_5 + \alpha_7)/2$. This is the non-zero element of $E_7^*/E_7 \cong \mathbb{Z}/2\mathbb{Z}$.

If the Dynkin diagram of a root lattice has several connected components, then the second index of the basis elements labels the connected component.

We shall apply the results obtained above to concrete Niemeier lattices $N_i$. Let $X$ be a Kählerian K3 surface.

Case 0. Consider markings of $X$ by negative-definite even unimodular lattices $K$ of rank 16. There are two such lattices: $K_1 = \Gamma_{16}$ with root system $D_{16}$ and $K_2 = 2E_8$ with root system $2E_8$. All the primitive sublattices $S \subset K_i$, $i = 1, 2$, satisfy Theorem 4.1 because $\text{rk} S + \text{rk}(A_S) \leq 16 < 22$ by Theorem 2.1. This gives
a marking of some $X$ if $P(S) \subset P(K_1)$. Therefore this case is easy (it was actually considered in [1], Remark 1.14.7).

The group $A(K_1)$ is trivial, and $A(K_2) = C_2$ is the group of order 2 permuting the two components $E_8$.

Let $X$ be marked by a primitive sublattice $S \subset K_1 = \Gamma_{16}$. Then $\text{Aut}(X,S)_0 = A(K_1)$ is trivial, $P(X) \cap S = P(S)$ and $\Gamma(P(S)) \subset \Gamma(P(K_1)) = \mathbb{D}_{16}$. Any such subgraph is possible.

Let $A = A(K_2)$ be the group of order 2. Then the coinvariant sublattice (or Leech-type sublattice) is

$$L_A = (K_2)_A = E_8(2) = \{[\alpha_{i1} - \alpha_{i2} \mid 1 \leq i \leq 8]\} \subset K_2 = 2E_8.$$ 

It is isomorphic to $E_8$ with the form multiplied by 2.

Let $X$ be marked by $S \subset K_2$. Then $P(X) \cap S = P(S)$. If $E_8(2) \not\subset S$, then the symplectic group $A(X,S)_0$ is trivial and $\Gamma(P(X) \cap S) \subset 2\mathbb{E}_8$. If $E_8(2) \subset S$, then $A(X,S)_0 = A(K_2)$ is the group of order 2. The graph $\Gamma(P(X) \cap S)$ is a subgraph of $2\mathbb{E}_8$ and is invariant under permutation of the two components of $2\mathbb{E}_8$. All such subgraphs are possible.

We now consider Niemeier lattices $N_i$ such that $A(N_i)$ has no non-trivial Kählerian K3 surface subgroups (KahK3-subgroups). If $X$ has a marking $S \subset N_i$ by such a lattice $N_i$, then the group $	ext{Aut}(X,S)_0$ is trivial.

**Case 1.** For the Niemeier lattice $N_1 = N(D_{24}) = [D_{24},\varepsilon_1]$, the group $A(N_1)$ is trivial (see [10], Ch. 16).

Let $X$ be marked by a primitive sublattice $S \subset N_1$. Then $S$ must satisfy Theorem 4.1 and $\Gamma(P(S)) \subset \Gamma(P(N_1)) = \mathbb{D}_{24}$. Every such $S$ gives a marking of some $X$, and $P(X) \cap S = P(S)$. The group $\text{Aut}(X,S)_0 = A(N_1)$ is trivial.

Suppose that $S = D_n$, $17 \leq n \leq 19$. This lattice satisfies Theorem 4.1. Hence there is a surface $X$ marked by $S = D_n \subset N_1$. By the classification of Niemeier lattices, only $N_1 = N(D_{24})$ is possible for marking $X$ by such a sublattice $S \subset S_X$.

**Case 2.** For the Niemeier lattice $N_2 = N(D_{16} \oplus E_8) = [D_{16},\varepsilon_{11}] \oplus E_8$, the group $A(N_2)$ is trivial (see [10], Ch. 16).

Let $X$ be marked by a primitive sublattice $S \subset N_2$. Then $S$ must satisfy Theorem 4.1 and $\Gamma(P(S)) \subset \Gamma(P(N_2)) = \mathbb{D}_{16}\mathbb{E}_8$. Any such $S$ gives a marking of some $X$, and $P(X) \cap S = P(S)$. The group $\text{Aut}(X,S)_0 = A(N_2)$ is trivial.

Suppose that $S = D_n \oplus E_8$, $8 \leq n \leq 11$. This sublattice satisfies Theorem 4.1. Hence there is a surface $X$ marked by $S \subset N_2$. By the classification of Niemeier lattices, only $N_2 = N(D_{16} \oplus E_8)$ is possible for marking $X$ by such a sublattice $S \subset S_X$.

**Case 4.** For the Niemeier lattice $N = N_4 = N(A_{24}) = [A_{24},5\varepsilon_1]$, the group $A(N)$ has order 2 (see [10], Ch. 16). It gives a non-trivial involution of the diagram $P(N_4) = A_{24}$. We easily see that $L = N_{A(N)} = (N^{A(N)})_{\overline{N}}$ does not satisfy Theorem 4.1. Hence there are no non-trivial KahK3-subgroups $A \subset A(N)$. (This also follows from the results of [3], which say that $\text{rk} L = 8$ for any KahK3-subgroup of order 2.)

Let $X$ be marked by a primitive sublattice $S \subset N_4$. Then $S$ must satisfy Theorem 4.1 and $\Gamma(P(S)) \subset \Gamma(P(N_4)) = A_{24}$. Any such $S$ gives a marking of some $X$, and $P(X) \cap S = P(S)$. The group $\text{Aut}(X,S)_0$ is trivial.
Suppose that $S = A_n$, $18 \leq n \leq 19$. This lattice satisfies Theorem 4.1. Hence there is a surface $X$ marked by $S \subset N_4 = N(A_{24})$. By the classification of Niemeier lattices, only $N_1 = N(D_{24})$ and $N_4 = N(A_{24})$ are possible for marking $X$ by such a sublattice $S \subset S_X$.

Case 5. For the Niemeier lattice $N = N_5 = N(2D_{12}) = [2D_{12}, \varepsilon_{11} + \varepsilon_{22}, \varepsilon_{21} + \varepsilon_{12}]$, the group $A(N)$ has order 2 and permutes the two components $D_{12}$. We easily see that $L = N_A(N) = (N_A(N))^{\frac{1}{N}}$ does not satisfy Theorem 4.1. Hence there are no non-trivial KahK3-subgroups $A \subset A(N)$.

Let $X$ be marked by a primitive sublattice $S \subset N_5$. Then $S$ must satisfy Theorem 4.1 and $\Gamma(P(S)) \subset \Gamma(P(N_5)) = 2D_{12}$. Any such $S$ gives a marking of some $X$, and $P(X) \cap S = P(S)$. The group $\text{Aut}(X, S)_0$ is trivial.

Suppose that $S = D_{10} \oplus D_9$. This sublattice satisfies Theorem 4.1. Hence there is a surface $X$ marked by $S \subset N_5$. By the classification of Niemeier lattices, only $N_5 = N(2D_{12})$ is possible for marking $X$ by such a sublattice $S \subset S_X$.

Case 10. For the Niemeier lattice $N = N_{10} = N(2A_{12}) = [2A_{12}, \varepsilon_{11} + 5\varepsilon_{12}]$, the group $A(N)$ is cyclic of order 4. For both non-trivial subgroups $A \subset A(N)$ we easily see that $L = N_A(N) = (N_A(N))^{\frac{1}{4}}$ does not satisfy Theorem 4.1. It suffices to check this for the subgroup of order 2 that gives a non-trivial involution of the graph $A_{12}$ for each of the two components $A_{12}$. Thus there are no non-trivial KahK3-subgroups $A \subset A(N)$.

Let $X$ be marked by a primitive sublattice $S \subset N_{10}$. Then $S$ must satisfy Theorem 4.1 and $\Gamma(P(S)) \subset \Gamma(P(N_{10})) = 2A_{12}$. Any such $S$ gives a marking of some $X$, and $P(X) \cap S = P(S)$. The group $\text{Aut}(X, S)_0$ is trivial.

Suppose that $S = A_{10} \oplus A_9$. This lattice satisfies Theorem 4.1. Hence there is a K3 surface $X$ marked by $S \subset N_{10}$. By the classification of Niemeier lattices, only $N_1 = N(D_{24})$, $N_4 = N(A_{24})$, $N_5 = N(2D_{12})$ and $N_{10} = N(2A_{12})$ are possible for marking $X$ by such a sublattice $S \subset S_X$.

In the following cases, the orders of all Kählerian K3 surface subgroups (KahK3-subgroups) $A$ of the group $A(N_i)$ are equal to 1 or 2. If $X$ has a marking $S \subset N_i$ by such a lattice $N_i$, then the group $\text{Aut}(X, S)_0$ is either trivial or of order 2.

Case 3. For the Niemeier lattice $N = N_3 = N(3E_8)$, the group $A(N)$ is $\mathfrak{S}_3$ acting by permutations of the three components $E_8$. We denote these components by $(E_8)_j$, $j = 1, 2, 3$. Simple calculations show that if $A \subset A(N) = \mathfrak{S}_3$ is either $\mathfrak{S}_3$ or the alternating group $\mathfrak{A}_3$, then the lattice $L = N_A = ((N^A)^{\frac{1}{N}})$ does not satisfy Theorem 4.1. It satisfies Theorem 4.1 only when $A$ is either trivial or generated by a transposition. These are all KahK3-subgroups.

If $A = [(kl)] \subset A(N)$ is generated by a transposition $(kl)$, $1 \leq k < l \leq 3$, then the Leech-type sublattice (or the coinvariant sublattice) is given by

$$E_8(2)_{kl} = N_A = \{[\alpha_{ik} - \alpha_{il} \mid 1 \leq i \leq 8]\}.$$

Let $X$ be marked by a primitive sublattice $S \subset N_3 = N(3E_8)$. Then $S$ must satisfy Theorem 4.1 and $\Gamma(P(S)) \subset \Gamma(P(N_3)) = 3E_8$. Any such $S$ gives a marking of some $X$, and $P(X) \cap S = P(S)$. If $S$ contains no sublattices $E_8(2)_{kl}$, $1 \leq k < l \leq 3$, then the group $\text{Aut}(X, S)_0$ is trivial. If $E_8(2)_{kl} \subset S$ for some $k, l$, $1 \leq k < l \leq 3$, then $\text{Aut}(X, S)_0 = [(kl)]$ is the group of order 2.
Suppose that $S = 2E_8$. Then $S$ satisfies Theorem 4.1 and gives a marking $S \subset N_3 = N(3E_8)$ of some surface $X$. By the classification of Niemeier lattices, only $N_3 = N(3E_8)$ is possible for marking $X$ by such a sublattice $S \subset S_X$.

Case 6. For the Niemeier lattice $N = N_6 = N(A_{17} \oplus E_7) = [A_{17} \oplus E_7, 3\epsilon_{11} + \epsilon_{12}]$, the group $A(N_6)$ has order 2 (see [10], Ch. 16). Its generator is trivial on $E_7$ and induces a non-trivial involution of the diagram $A_{17}$.

The coinvariant sublattice for $A(N)$ is given by

$$E_8(2) = N_{A(N)} = \left\{ \alpha_{i_1} - \alpha_{(18-i_1)} \mid 1 \leq i \leq 8 \right\}, \quad \frac{1}{3} \sum_{i=1}^{8} i(\alpha_{i_1} - \alpha_{(18-i_1)}) \subset N_6.$$ 

Let $X$ be marked by a primitive sublattice $S \subset N_6 = N(A_{17} \oplus E_7)$. Then $S$ must satisfy Theorem 4.1 and $\Gamma(P(S)) \subset \Gamma(P(N_6)) = \mathbb{A}_{17}E_7$. Any such $S$ gives a marking of some $X$, and $P(X) \cap S = P(S)$. If $E_8(2) \not\subset S$, then the group $\text{Aut}(X, S)_0$ is trivial. If $E_8(2) \subset S$, then $\text{Aut}(X, S)_0 = A(N_6)$ has order 2.

In particular, $S = [A_{17}, 6\epsilon_{11}] \subset N_6$ satisfies Theorem 4.1 and gives a marking of some surface $X$. By the classification of Niemeier lattices, $X$ can be marked only by $N_6 = N(A_{17} \oplus E_7)$ for such a sublattice $S \subset S_X$.

Case 7. For the Niemeier lattice $N = N_7 = N(D_{10} \oplus 2E_7) = [D_{10} \oplus 2E_7, \epsilon_{11} + \epsilon_{12}, \epsilon_{31} + \epsilon_{13}]$, the group $A(N_7)$ has order 2 (see [10], Ch. 16). Its generator permutes the two diagrams $E_7$ and induces a non-trivial involution of the diagram $D_{10}$.

The coinvariant sublattice for $A(N)$ is given by

$$E_8(2) = N_{A(N)} = \left\{ \alpha_{91} - \alpha_{10,1}, \{\alpha_{i_2} - \alpha_{i_3} \mid 1 \leq i \leq 7\}, \frac{1}{2}(\alpha_{91} - \alpha_{10,1} + \alpha_{22} - \alpha_{23} + \alpha_{52} - \alpha_{53} + \alpha_{72} - \alpha_{73}) \right\} \subset N_7.$$ 

Let $X$ be marked by a primitive sublattice $S \subset N_7 = N(D_{10} \oplus 2E_7)$. Then $S$ must satisfy Theorem 4.1 and $\Gamma(P(S)) \subset \Gamma(P(N_7)) = \mathbb{D}_{10}2E_7$. Any such $S$ gives a marking of some $X$, and $P(X) \cap S = P(S)$. If $E_8(2) \not\subset S$, then the group $\text{Aut}(X, S)_0$ is trivial. If $E_8(2) \subset S$, then $\text{Aut}(X, S)_0 = A(N_7)$ has order 2.

In particular, $S = [\alpha_{91}, \alpha_{10,1}, 2E_7]_{pr} \subset N_7$ satisfies Theorem 4.1 and gives a marking of some surface $X$. Moreover, $\text{Aut}(X, S)_0$ has order 2. By the classification of Niemeier lattices and our calculations above, $X$ can be marked only by $N_7 = N(D_{10} \oplus 2E_7)$ for such a sublattice $S \subset S_X$.

Case 8. For the Niemeier lattice $N = N_8 = N(A_{15} \oplus D_9) = [A_{15} \oplus D_9, 2\epsilon_{11} + \epsilon_{12}]$, the group $A(N_8)$ has order 2 (see [10], Ch. 16). Its generator induces non-trivial involutions of the diagrams $A_{15}$ and $D_9$.

The coinvariant sublattice for $A(N)$ is given by

$$E_8(2) = N_A = \left\{ \alpha_{i_1} - \alpha_{(16-i_1)} \mid 1 \leq i \leq 7\right\}, \alpha_{82} - \alpha_{92}, \quad \frac{1}{4} \sum_{i=1}^{7} (\alpha_{i_1} - \alpha_{(16-i_1)}) + \frac{1}{2}(\alpha_{82} - \alpha_{92}) \subset N_8.$$
Let $X$ be marked by a primitive sublattice $S \subset N_8 = N(A_{15} \oplus D_9)$. Then $S$ must satisfy Theorem 4.1 and $\Gamma(P(S)) \subset \Gamma(P(N_8)) = A_{15}D_9$. Any such $S$ gives a marking of some $X$, and $P(X) \cap S = P(S)$. If $E_8(2) \not\subset S$, then the group Aut$(X,S)_0$ is trivial. If $E_8(2) \subset S$, then Aut$(X,S)_0 = A(N_8)$ has order 2.

In particular, $S = [A_{15}, \alpha_{92}]_{pr} \subset N_8$ satisfies Theorem 4.1 and gives a marking of some surface $X$. Moreover, Aut$(X,S)_0$ has order 2. By the classification of Niemeier lattices and our calculations above, $X$ can be marked only by $N_8 = N(A_{15} \oplus D_9)$ for such a sublattice $S \subset S_X$.

Case 9. For the Niemeier lattice $N = N_9 = N(3D_8) = [3D_8, \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{23}, \varepsilon_{21} + \varepsilon_{12} + \varepsilon_{23}, \varepsilon_{21} + \varepsilon_{22} + \varepsilon_{13}]$, the group $A(N)$ is $G_3$ acting by permutations of the three components $D_8$ and permutations of $\alpha_{71}, \alpha_{72}$ and $\alpha_{73}$. Simple calculations show that if $A \subset A(N) = G_3$ is equal to $G_3$ or $A_3$, then the sublattice $L = N_A = ((N^A)^{1/2})$ does not satisfy Theorem 4.1. It satisfies Theorem 4.1 only when $A$ is either trivial or generated by a transposition. These are all KahK3-subgroups.

Let $A = [(k,l)]$, $1 \leq k < l \leq 3$, be a transposition on $(D_8)_j$, $j = 1, 2, 3$. Then the coinvariant sublattice is given by

$$N_A = E_8(2)_{kl} = \left\{ \alpha_{ik} - \alpha_{il} \mid 1 \leq i \leq 8 \right\} \frac{1}{2}(\alpha_{1k} - \alpha_{1l} + \alpha_{3k} - \alpha_{3l} + \alpha_{5k} - \alpha_{5l} + \alpha_{8k} - \alpha_{8l})$$

Let $X$ be marked by a primitive sublattice $S \subset N_9 = N(3D_8)$. Then $S$ must satisfy Theorem 4.1 and $\Gamma(P(S)) \subset \Gamma(P(N_9)) = 3D_8$. Any such $S$ gives a marking of some $X$, and $P(S) = P(X) \cap S$. If $S$ contains no sublattices $E_8(2)_{kl}$, $1 \leq k < l \leq 3$, then Aut$(X,S)_0$ is trivial. If $E_8(2)_{kl} \subset S$ for some $k, l$, $1 \leq k < l \leq 3$, then Aut$(X,S)_0 = [(kl)]$ is the group of order 2.

Suppose that $S = [2D_8]_{pr} \subset N_9$. Then $S$ satisfies Theorem 4.1 and gives a marking of some surface $X$. Moreover, Aut$(X,S)_0$ has order 2. By the classification of Niemeier lattices and our calculations above, $X$ can be marked only by $N_9 = N(3D_8)$ for such a sublattice $S \subset S_X$.

Case 11. For the Niemeier lattice $N = N_{11} = N(A_{11} \oplus D_7 \oplus E_6) = [A_{11} \oplus D_7 \oplus E_6, \varepsilon_{11} + \varepsilon_{12} + \varepsilon_{13}]$, the group $A(N_{11})$ has order 2 (see [10], Ch.16). The generator of $A(N_{11})$ induces non-trivial involutions of the subdiagrams $A_{11}, D_7$ and $E_6$. The coinvariant sublattice for the group $A = A(N_{11})$ is given by

$$E_8(2) = N_A = \left\{ \alpha_{i1} - \alpha_{(12-i)1} \mid 1 \leq i \leq 5 \right\}, \alpha_{62} - \alpha_{72}, \alpha_{13} - \alpha_{63}, \alpha_{33} - \alpha_{53},$$

$$\frac{1}{6} \sum_{i=1}^{5} i(\alpha_{i1} - \alpha_{(12-i)1}) + \frac{1}{2}(\alpha_{62} - \alpha_{72}) + \frac{1}{3}(-\alpha_{13} + \alpha_{63} + \alpha_{33} - \alpha_{53}) \subset N_{11}.$$

Let $X$ be marked by a primitive sublattice $S \subset N_{11} = N(A_{11} \oplus D_7 \oplus E_6)$. Then $S$ must satisfy Theorem 4.1 and $\Gamma(P(S)) \subset \Gamma(P(N_{11})) = A_{11}D_7E_6$. Any such $S$ gives a marking of some $X$, and $P(S) = P(X) \cap S$. If $S$ does not contain $E_8(2)$, then the group Aut$(X,S)_0$ is trivial. If $E_8(2) \subset S$, then Aut$(X,S)_0 = A(N_{11})$ has order 2.
In particular, \( S = [A_{11} \oplus E_6, \alpha_{62}, \alpha_{72}]_{pr} \subset N_{11} \) satisfies Theorem 4.1 and gives a marking of some surface \( X \). The group \( \text{Aut}(X, S)_0 \) has order 2. By the classification of Niemeier lattices and our calculations above, \( X \) can be marked only by \( N_{11} = N(A_{11} \oplus D_7 \oplus E_6) \) for such a sublattice \( S \subset S_X \).

Case 15. For the Niemeier lattice

\[ N = N_{15} = N(3A_8) = [3A_8, 4\varepsilon_{11} + \varepsilon_{12} + \varepsilon_{13}, 4\varepsilon_{12} + \varepsilon_{13}, 4\varepsilon_{11} + \varepsilon_{12} + 4\varepsilon_{13}] \]

the group \( A(N) \) has order 12 and is the direct product of the group of order 2 inducing non-trivial involutions on all three components \( A_8 \) and the group \( S_3 \) acting by permutations of the three components \( A_8 \) and of the elements \( \alpha_{11}, \alpha_{12} \) and \( \alpha_{13} \) (see [10], Ch. 16).

Simple calculations show that the coinvariant sublattice \( L = N_A = ((N^A)^N_3) \) of a subgroup \( A \subset A(N) \) satisfies Theorem 4.1 only when \( A \) is trivial or \( A = [(kl)] \), \( 1 \leq k < l \leq 3 \), is generated by the transposition \( (kl) \) of the components \( (A_8)_k, (A_8)_l \) and elements \( \alpha_{1k}, \alpha_{1l} \) and acts as the identity on the remaining component \( (A_8)_j \). These are all KahK3-subgroups. The coinvariant sublattice for \( A = [(kl)] \) is given by

\[ E_8(2)_{kl} = N_A = \left\{ \alpha_{ik} - \alpha_{il} \mid 1 \leq i \leq 8 \right\}, \frac{1}{3} \sum_{i=1}^{8} i(\alpha_{ik} - \alpha_{il}) \right\} \subset N_{15}. \]

Let \( X \) be marked by a primitive sublattice \( S \subset N_{15} = N(3A_8) \). Then \( S \) must satisfy Theorem 4.1 and \( \Gamma(P(S)) \subset \Gamma(P(N_{15})) = 3A_8 \). Any such \( S \) gives a marking of some \( X \), and \( P(X) \cap S = P(S) \). If \( S \) contains no sublattices \( E_8(2)_{kl}, 1 \leq k < l \leq 3 \), then the group \( \text{Aut}(X, S)_0 \) is trivial. If \( E_8(2)_{kl} \subset S \) for \( 1 \leq k < l \leq 3 \), then \( \text{Aut}(X, S)_0 = [(kl)] \) has order 2.

Suppose that \( S = [2A_8]_{pr} \subset N_{15} \). Then \( S \) satisfies Theorem 4.1 and gives a marking of some surface \( X \). Moreover, \( \text{Aut}(X, S)_0 \) has order 2. By the classification of Niemeier lattices and our calculations above, \( X \) can be marked only by \( N_{15} = N(3A_8) \) for such a sublattice \( S \subset S_X \).

The cases studied below are more complicated. We shall use the following simple general assertions along with the computer programs listed in the Appendix, § 7 (of course, they could also have been used in all the cases above, where we omitted the simple calculations).

**Proposition 6.1.** Let \( N \) be a Niemeier lattice (or any other even negative-definite unimodular lattice), \( P(N) \) a basis of the root system of \( N \), and

\[ A(N) = \{ \varphi \in O(N) \mid \varphi(P(N)) = P(N) \}. \]

If \( A \subset A(N) \) is a KahK3-subgroup with Leech-type sublattice (or coinvariant sublattice) \( N_A = (N^A)^N_N \) (equivalently, there is a primitive embedding \( N_A \subset L_{K3} \)), then the conjugate subgroup \( A^g = gAg^{-1}, g \in A(N) \), is also a KahK3-subgroup with Leech-type sublattice (coinvariant sublattice) \( N_{Ag} = g(N_A) \). These groups determine all conjugacy classes of KahK3-subgroups of \( A(N) \). Thus, to describe the KahK3-subgroups \( A \subset A(N) \) and their coinvariant sublattices, it suffices to describe their representatives in all KahK3-conjugacy classes in \( A(N) \) and their coinvariant sublattices.

Proof. \( N_{A^g} \) is isomorphic to \( N_A \). If \( N_A \) has a primitive embedding \( N_A \subset L_{K3} \), then \( N_{A^g} = g(N_A) \subset N \) also has a primitive embedding \( N_{A^g} \subset L_{K3} \).

The following proposition can be used to calculate the coinvariant sublattices.

**Proposition 6.2.** Let \( N \) be a Niemeier lattice (or any other even negative-definite unimodular lattice), \( P(N) \) a basis of the root system of \( N \), and

\[
A(N) = \{ \varphi \in O(N) \mid \varphi(P(N)) = P(N) \}.
\]

Let \( A_1 \subset A(N) \) and \( A_2 \subset A(N) \) be subgroups. Suppose that \( A = \langle A_1, A_2 \rangle \subset A(N) \) is generated by \( A_1 \) and \( A_2 \). Then the coinvariant sublattice \( N_A \) is the primitive sublattice \( N_A = [N_{A_1}, N_{A_2}]_{pr} \subset N \) of \( N \) generated by the coinvariant sublattices \( N_{A_1} \), and \( N_{A_2} \) of the subgroups \( A_1 \) and \( A_2 \). Therefore \( A = \langle A_1, A_2 \rangle \) is a KahK3-subgroup.

In particular, if \( A = \langle g_1, \ldots, g_n \rangle \) is generated by \( g_1, \ldots, g_n \in A \), then \( N_A = [N(g_1), \ldots, N(g_n)]_{pr} \subset N \). Moreover, \( A \) is a KahK3-sublattice if and only if the sublattice \( [N(g_1), \ldots, N(g_n)]_{pr} \subset N \) has a primitive embedding in \( L_{K3} \).

Proof. Indeed, since \( A_1, A_2 \subset A \), we have \( [N_{A_1}, N_{A_2}]_{pr} \subset N_A \). Put \( M = [N_{A_1}, N_{A_2}]_{pr} \subset N \). Since \( N_{A_1} \subset M \) and \( N_{A_2} \subset M \), we have \( A_1 \subset A(M) \), \( A_2 \subset A(M) \) and \( A = \langle A_1, A_2 \rangle \subset A(M) \). It follows that \( A = \langle A_1, A_2 \rangle \) is equal to the identity on \( M_{\frac{1}{N}} \) and \( M \subset N_{\langle A_1, A_2 \rangle} \). Therefore \( M = N_A \).

The following simple but important assertion will also be used to calculate the coinvariant sublattices \( N_A \) of subgroups \( A \subset A(N) \).

**Proposition 6.3.** Let \( N \) be a Niemeier lattice (or any other even negative-definite unimodular lattice), \( P(N) \) a basis of the root system of \( N \), and

\[
A(N) = \{ \varphi \in O(N) \mid \varphi(P(N)) = P(N) \}.
\]

Suppose that \( P(N) \) generates \( N \) over \( \mathbb{Q} \) (otherwise use an \( A \)-invariant basis in \( N \otimes \mathbb{Q} \) instead of \( P(N) \)). Write \( P(N) = \{ e_1, e_2, \ldots, e_n \} \) and let \( \{ e_1^*, e_2^*, \ldots, e_n^* \} \subset N \otimes \mathbb{Q} \) be the dual elements, that is, \( e_i^* \cdot e_j = \delta_{ij} \), where \( \delta_{ij} \) is the Kronecker delta. Let \( A \subset A(N) \) be a subgroup, \( N_A \subset N \) its coinvariant sublattice \( N_A = (N^A)^{1/2}_N \), \( \{ e_{i_1}, \ldots, e_{i_{2t}} \}, \ldots, \{ e_{i_{2t+1}}, \ldots, e_{i_{2t+t}} \} \) all the orbits of \( A \) in \( P(N) \), and \( t \) the number of orbits. Then the elements

\[
\{ e_{i_{1t+1}}^* - e_{i_{2t+1}}^* , \ldots , e_{i_1(j_{1t-1})}^* - e_{i_{t_{1t}}}^* \}, \ldots , \{ e_{i_{2t+1}}^* - e_{i_{2t+1}}^* , \ldots , e_{i_1(j_{1t-1})}^* - e_{i_{t_{1t}}}^* \}
\]

form a basis in \( N_A \otimes \mathbb{Q} \). In particular, \( \text{rk} N_A = \text{rk} N - t \).

A primitive sublattice \( K_{pr} \subset N \) is determined by its rational basis in the vector space \( K \otimes \mathbb{Q} \). Indeed, \( K_{pr} = K \otimes \mathbb{Q} \cap N \subset N \otimes \mathbb{Q} \). Therefore Proposition 6.3 yields an effective way of calculating the sublattice \( N_A \) and its invariants (for Theorem 4.1) which are used to decide whether \( N_A \) has a primitive embedding in \( L_{K3} \) and whether \( A \) is a KahK3-subgroup.
The Appendix (§7) contains Program 0 which uses Programs 1–4 to calculate a normal basis (or Smith basis) of the primitive sublattice $K_{pr} \subset N$ for any sublattice $K$ of $N$ given by generators in $K \otimes \mathbb{Q}$. We use the root basis $P(N)$ of a Niemeier lattice $N$ (corresponding to the basis columns $(0,\ldots,0,1,0,\ldots,0)^t$ of length 24), the integer matrix $r$ of the lattice $N$ in this basis (which is equivalent to the Dynkin diagram) and additional coding data of $N$. The sublattice $K$ is given by the rational columns in this basis (the matrix SUBL of size $(24 \times \cdot)$) in Program 0. This program calculates the normal basis (also called the basis of elementary divisors or Smith basis) of the sublattice $K_{pr}$ (denoted by the matrix SUBLP in Program 0) for the embedding $K_{pr} \subset K_{pr}^*$. It also calculates the elementary divisors (or Smith invariants) of this embedding (denoted by DSUBLpr in Program 0). In particular, it calculates $\text{rk} K_{pr}$ and the discriminant group $A_{K_{pr}} = K_{pr}^*/K_{pr}$, along with a minimal number $l(A_{K_{pr}})$ of generators. Moreover, the normal basis can be used to calculate the Jordan form of the lattice $K_{pr} \otimes \mathbb{Z}_p$ over the ring of $p$-adic integers and the discriminant form $q_{K_{pr}}$. The last vectors of this basis give the unimodular part of the lattice $K_{pr}$ over $\mathbb{Z}_p$. Thus Program 0 (see §7) gives all the necessary invariants for Theorem 4.1 to decide whether $K_{pr}$ has a primitive embedding in $L_{K3}$. This program is very fast in all the cases considered below.

Case 13. For the Niemeier lattice

$$N = N_{13} = N(2A_9 \oplus D_6) = [2A_9 \oplus D_6, 2\varepsilon_{11} + 4\varepsilon_{12}, 5\varepsilon_{11} + \varepsilon_{13}, 5\varepsilon_{12} + \varepsilon_{33}],$$

the group $A(N)$ is the cyclic group $C_4$ of order 4 (see [10], Ch. 16). Its elements are determined by permutations of the ends of the Dynkin diagrams $A_9$ and $D_6$. The generator $\varphi$ corresponds to the permutation

$$\varphi = \left(\begin{array}{ccccccc}
\alpha_{11} & \alpha_{91} & \alpha_{12} & \alpha_{92} & \alpha_{53} & \alpha_{63} \\
\alpha_{12} & \alpha_{92} & \alpha_{91} & \alpha_{11} & \alpha_{63} & \alpha_{53}
\end{array}\right).$$

By Proposition 6.3, the coinvariant sublattice $N_{[\varphi]}$ is

$$N_{[\varphi]} = [\alpha_{11}^* - \alpha_{12}^*, \alpha_{12}^* - \alpha_{91}^*, \alpha_{91}^* - \alpha_{92}^*, \alpha_{21}^* - \alpha_{22}^*, \alpha_{22}^* - \alpha_{81}^*, \alpha_{81}^* - \alpha_{82}^*,
\alpha_{31}^* - \alpha_{32}^*, \alpha_{32}^* - \alpha_{71}^*, \alpha_{71}^* - \alpha_{72}^*, \alpha_{41}^* - \alpha_{42}^*, \alpha_{42}^* - \alpha_{61}^*, \alpha_{61}^* - \alpha_{62}^*,
\alpha_{51}^* - \alpha_{52}^*, \alpha_{52}^* - \alpha_{63}^*]_{pr} \subset N.$$  

Using Program 0 (see §7), we obtain that $\text{rk} N_{[\varphi]} = 14$, $N_{[\varphi]}^*/N_{[\varphi]} \cong (\mathbb{Z}/4\mathbb{Z})^4 \times (\mathbb{Z}/2\mathbb{Z})^2$ and $l(N_{[\varphi]}^*/N_{[\varphi]}) = 6$. By Theorem 4.1, the lattice $N_{[\varphi]}$ has a primitive embedding in $L_{K3}$, and $[\varphi] = A(N)$ is a KahK3-subgroup.

All the subgroups of $A(N)$ are also KahK3-subgroups. The only non-trivial subgroup of $A(N)$ is $[\varphi^2]$ of order 2. Similar calculations show that $N_{[\varphi^2]} \cong E_8(2)$.

Let $X$ be marked by a primitive sublattice $S \subset N_{13} = N(2A_9 \oplus D_6)$. Then $S$ must satisfy Theorem 4.1 and $\Gamma(P(S)) \subset \Gamma(P(N_{13})) = 2A_9D_6$. Any such $S$ gives a marking of some $X$, and $P(X) \cap S = P(S)$.

If $N_{[\varphi]} \subset S$, then $\text{Aut}(X,S)_{0} = [\varphi] \cong C_4$ is the cyclic group of order 4. If this is not the case but $N_{[\varphi^2]} \subset S$, then $\text{Aut}(X,S)_{0} = [\varphi^2] \cong C_2$ has order 2. If $N_{[\varphi^2]} \not\subset S$, then $\text{Aut}(X,S)_{0}$ is trivial.

Suppose that $S = [2A_9, N_{[\varphi]}]_{pr} = \left[\alpha_{11}, \alpha_{21}, \ldots, \alpha_{81}, N_{[\varphi]}\right]_{pr} \subset N_{13}$. Then (using Program 0 in §7) we see that $\text{rk} S = 19$ and $S^*/S \cong \mathbb{Z}/4\mathbb{Z}$. Thus $S$ satisfies
Theorem 4.1 and gives a marking of some $X$. We have $\text{Aut}(X, S)_0 \cong C_4$ and $2A_9 \subset \Gamma(P(X) \cap S) = P(S)$. By the classification of Niemeier lattices and our calculations above, $X$ can be marked only by $N_{13} = N(2A_9 \oplus D_6)$ for such a sublattice $S \subset S_X$. Case 16. For the Niemeier lattice $N = N_{16} = N(2A_7 \oplus 2D_5) = [2A_7 \oplus 2D_5, \varepsilon_{11} + \varepsilon_{12} + \varepsilon_{13} + \varepsilon_{24}, \varepsilon_{11} + 7\varepsilon_{12} + \varepsilon_{23} + \varepsilon_{14}]$, the group $A(N)$ is the dihedral group of order 8 (see [10], Ch. 16). We identify it with the group of symmetries of a square. Its elements are determined by permutations of the terminals of the Dynkin diagrams $A_7$ and $D_5$. The central symmetry $\varphi_0$ corresponds to the involution

$$\varphi_0 = \left( \begin{array}{cccccccc} \alpha_{11} & \alpha_{71} & \alpha_{12} & \alpha_{72} & \alpha_{43} & \alpha_{53} & \alpha_{44} & \alpha_{54} \\ \alpha_{71} & \alpha_{11} & \alpha_{72} & \alpha_{12} & \alpha_{43} & \alpha_{53} & \alpha_{44} & \alpha_{54} \end{array} \right).$$

The generating symmetries $\varphi_1$ and $\varphi_2$ correspond to the involutions

$$\varphi_1 = \left( \begin{array}{cccccccc} \alpha_{11} & \alpha_{71} & \alpha_{12} & \alpha_{72} & \alpha_{43} & \alpha_{53} & \alpha_{44} & \alpha_{54} \\ \alpha_{12} & \alpha_{72} & \alpha_{11} & \alpha_{71} & \alpha_{43} & \alpha_{53} & \alpha_{44} & \alpha_{54} \end{array} \right),$$

$$\varphi_2 = \left( \begin{array}{cccccccc} \alpha_{11} & \alpha_{71} & \alpha_{12} & \alpha_{72} & \alpha_{43} & \alpha_{53} & \alpha_{44} & \alpha_{54} \\ \alpha_{11} & \alpha_{71} & \alpha_{72} & \alpha_{12} & \alpha_{43} & \alpha_{53} & \alpha_{44} & \alpha_{53} \end{array} \right),$$

where $\varphi_2 \varphi_1$ is the rotation through $90^\circ$ and has order 4.

The coinvariant sublattice $N_H$ of the cyclic subgroup $H = [\varphi_2 \varphi_1]$ of order 4 has $\text{rk} N_H = 16$ and $\text{l}(N_H^*/N_H) = 8$ (we use Proposition 6.3 and Program 0). Hence $N_H$ has no primitive embeddings in $L_{K3}$ and, by Theorem 4.1, $H$ is not a KahK3-subgroup. It follows that $A(N)$ is not a KahK3-subgroup.

The subgroups $H = [\varphi_0, \varphi_1]$ and $H = [\varphi_0, \varphi_2]$ are conjugate in $A(N)$ and isomorphic to $C_2 \times C_2$. For them, using Proposition 6.3 and Program 0, we get $\text{rk} N_H = 12$ and $N_H^*/N_H \cong (\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^6$. Hence $N_H$ has a primitive embedding in $L_{K3}$ by Theorem 4.1 and these subgroups $H \cong C_2 \times C_2$ are KahK3-subgroups.\(^1\)

All the other non-trivial subgroups of $A(N)$ are cyclic of order 2: $[\varphi_0]$ and the conjugate subgroups $[\varphi_1]$, $[\varphi_1 \varphi_0]$, $[\varphi_2]$, $[\varphi_2 \varphi_0]$. Being subgroups of the groups $H$ considered above, they are also KahK3-subgroups. The coinvariant sublattices $N_{[\varphi_0]}$, $N_{[\varphi_1]}$, $N_{[\varphi_1 \varphi_0]}$, $N_{[\varphi_2]}$, $N_{[\varphi_2 \varphi_0]}$ are isomorphic to $E_8(2)$.

Let $X$ be marked by a primitive sublattice $S \subset N_{16} = N(2A_7 \oplus 2D_5)$. Then $S$ must satisfy Theorem 4.1 and $\Gamma(P(S)) \subset \Gamma(P(N_{16})) = 2A_7 2D_5$. Any such $S$ gives a marking of some $X$, and $P(X) \cap S = P(S)$.

If $N_{[\varphi_0, \varphi_1]} \subset S$, then $\text{Aut}(X, S)_0 = [\varphi_0, \varphi_1] \cong C_2 \times C_2$. If $N_{[\varphi_0, \varphi_2]} \subset S$, then $\text{Aut}(X, S)_0 = [\varphi_0, \varphi_2] \cong C_2 \times C_2$. If neither of these inclusions holds but we have $N_{[\varphi]} \subset S$ for some $\varphi \in \{\varphi_0, \varphi_1, \varphi_1 \varphi_0, \varphi_2, \varphi_2 \varphi_0\}$, then $\text{Aut}(X, S)_0 = [\varphi] \cong C_2$. If $N_{[\varphi]} \not\subset S$ for such a $\varphi$, then $\text{Aut}(X, S)_0$ is trivial.

Suppose that $S = [2A_7, N_{[\varphi_0, \varphi_1]}]_{\text{pr}} \subset N_{16}$. Then, using Program 0 in § 7, we obtain that $\text{rk} S = 16$ and $S^*/S \cong (\mathbb{Z}/2\mathbb{Z})^4$. Hence $S$ satisfies Theorem 4.1 and gives

\(^1\)These calculations show that, for a symplectic group $G = C_2 \times C_2$ on a Kählerian K3 surface, we have $S^*_G/S_G = S^*_{(2,2)}/S_{(2,2)} \cong (\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^6$. We must therefore correct our calculation of the last group in [3], Proposition 10.1.
a marking of some $X$. We have $\text{Aut}(X,S)_{0} \cong C_{2} \times C_{2}$ and $2A_{7} \subset \Gamma(P(X) \cap S) = P(S)$. By the classification of Niemeier lattices and our calculations above, $X$ can be marked only by $N_{16} = N(2A_{7} \oplus 2D_{5})$ for such a sublattice $S \subset S_{X}$.

In our further calculations we do not mention the use of Proposition 6.3 and Program 0 in § 7. We use them all the time.

Case 17. For the Niemeier lattice $N = N_{17} = N(4A_{6})$

\[ = [4A_{6}, \varepsilon_{11} + 2\varepsilon_{12} + \varepsilon_{13} + 6\varepsilon_{14}, \varepsilon_{11} + 6\varepsilon_{12} + 2\varepsilon_{13} + \varepsilon_{14}, \varepsilon_{11} + \varepsilon_{12} + 6\varepsilon_{13} + 2\varepsilon_{14}], \]

the group $A(N)$ has centre $[\varphi_{0}]$ of order 2 which preserves the components $4A_{6}$, and $A(N)/[\varphi_{0}] = A_{4}$ is the alternating group on the components $4A_{6}$ (see [10], Ch. 16). The elements of this group are determined by permutations of the ends of the Dynkin diagrams $A_{6}$. The group itself is generated by the involution

\[ \varphi_{0} = (\alpha_{11}\alpha_{61})(\alpha_{12}\alpha_{62})(\alpha_{13}\alpha_{63})(\alpha_{14}\alpha_{64}) \]

and the following elements $\varphi_{1}, \varphi_{4}$ of order 3:

\[ \varphi_{1} = (\alpha_{12}\alpha_{13}\alpha_{14})(\alpha_{62}\alpha_{63}\alpha_{64}), \]

\[ \varphi_{4} = (\alpha_{11}\alpha_{12}\alpha_{63})(\alpha_{61}\alpha_{62}\alpha_{13}). \]

Here $\varphi_{4}\varphi_{1}$ has order 4 and $(\varphi_{4}\varphi_{1})^{2} = \varphi_{0}$. All the elements of $A(N)$ are conjugate to $\varphi_{0}, \varphi_{1}, \varphi_{0}\varphi_{1}$ (of order 6) or $\varphi_{4}\varphi_{1}$.

The coinvariant sublattice $N_{[\varphi_{0}]}$ has rk $N_{[\varphi_{0}]} = 12$ and $N_{[\varphi_{0}]/[\varphi_{0}]} \cong (\mathbb{Z}/2\mathbb{Z})^{12}$. By Theorem 4.1, $N_{[\varphi_{0}]}$ has no primitive embeddings in $L_{K3}$ and $[\varphi_{0}]$ is not a KahK3-subgroup.

For the cyclic group $[\varphi_{1}] \cong C_{3}$ we have rk $N_{[\varphi_{1}]} = 12$ and $N_{[\varphi_{1}]/[\varphi_{1}]} \cong (\mathbb{Z}/3\mathbb{Z})^{6}$. By Theorem 4.1, $N_{[\varphi_{1}]}$ has a primitive embedding in $L_{K3}$ and $[\varphi_{1}]$ is a KahK3-subgroup. All the cyclic subgroups of order 3 in $A(N)$ are conjugate to it. They are $[\varphi_{1}], [\varphi_{4}]$ and $[\varphi_{2}], [\varphi_{3}]$, where $\varphi_{2} = \varphi_{1}\varphi_{4}\varphi_{1}^{-1}, \varphi_{3} = \varphi_{1}^{2}\varphi_{4}\varphi_{1}^{-2}$. It follows from the structure of $A(N)$ that the conjugate subgroups

\[ H = [\varphi_{i}] \cong C_{3}, \quad i = 1, \ldots, 4, \]

are all non-trivial KahK3-subgroups of $A(N_{17})$. We have rk $N_{H} = 12$, $N_{H}^{*}/N_{H} \cong (\mathbb{Z}/3\mathbb{Z})^{6}$.

Let $X$ be marked by a primitive sublattice $S \subset N_{17} = N(4A_{6})$. Then $S$ must satisfy Theorem 4.1 and $\Gamma(P(S)) \subset \Gamma(P(N_{17})) = 4A_{6}$. Any such $S$ gives a marking of some $X$, and $P(X) \cap S = P(S)$.

If $N_{[\varphi_{i}]} \subset S$ for some $i = 1, 2, 3, 4$, then $\text{Aut}(X,S)_{0} = [\varphi_{i}] \cong C_{3}$. If $N_{[\varphi_{i}]} \not\subset S$ for all $i = 1, 2, 3, 4$, then $\text{Aut}(X,S)_{0}$ is trivial.

Suppose that $S = [(A_{6})_{2} = [\alpha_{12}, \ldots, \alpha_{62}], N_{[\varphi_{i}]}]_{\text{pr}} \subset N_{17}$. We have rk $S = 18, S^{*}/S \cong \mathbb{Z}/7\mathbb{Z}$. Hence $S$ satisfies Theorem 4.1 and gives a marking of some surface $X$. We have $\text{Aut}(X,S)_{0} \cong C_{3}$ and $3A_{6} \subset \Gamma(P(X) \cap S) = P(S)$. By the classification of Niemeier lattices and our calculations above, $X$ can be marked only by $N_{17} = N(4A_{6})$ for such a sublattice $S \subset S_{X}$. 


Case 14. For the Niemeier lattice

\[ N = N_{14} = N(4D_6) = [4D_6, \text{even permutations of } 0_1 + \varepsilon_{12} + \varepsilon_{23} + \varepsilon_{34}] \]
\[ = [4D_6, \varepsilon_{12} + \varepsilon_{23} + \varepsilon_{34}, \varepsilon_{32} + \varepsilon_{13} + \varepsilon_{24}, \varepsilon_{22} + \varepsilon_{33} + \varepsilon_{14}; \varepsilon_{11} + \varepsilon_{33} + \varepsilon_{24}, \varepsilon_{21} + \varepsilon_{13} + \varepsilon_{34}; \varepsilon_{31} + \varepsilon_{23} + \varepsilon_{14}; \varepsilon_{21} + \varepsilon_{32} + \varepsilon_{14}; \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{34}, \varepsilon_{31} + \varepsilon_{12} + \varepsilon_{24}; \varepsilon_{31} + \varepsilon_{22} + \varepsilon_{13}, \varepsilon_{11} + \varepsilon_{32} + \varepsilon_{23}, \varepsilon_{21} + \varepsilon_{12} + \varepsilon_{33}] \]

the group \( A(N) \) can be identified with \( \mathfrak{S}_4 \) acting on the components \( 4D_6 \) (see [10], Ch. 16). Its elements are determined by permutations of the ends of the connected components of \( 4D_6 \). We enumerate these components by the numbers 1, \ldots, 4. Even permutations give rise to the corresponding permutations of \( (\alpha_{51}, \alpha_{52}, \alpha_{53}, \alpha_{54}) \) and \( (\alpha_{61}, \alpha_{62}, \alpha_{63}, \alpha_{64}) \). Odd permutations interchange \( \alpha_{5i} \) and \( \alpha_{6i} \). For example, the transposition (12) determines the permutation

\[ (12) = (\alpha_{51}\alpha_{62})(\alpha_{61}\alpha_{52})(\alpha_{53}\alpha_{63})(\alpha_{54}\alpha_{64}). \]

For the element \( (12)(34) \in \mathfrak{S}_4 \) of order 2 we have \( \text{rk} N_{[(12)(34)]} = 12 \) and \( N_{[(12)(34)]}/N_{[(12)(34)]} \cong (\mathbb{Z}/2\mathbb{Z})^2 \). By Theorem 4.1, \( N_{[(12)(34)]} \) has no primitive embeddings in \( L_{K3} \). Hence \( [(12)(34)] \) and its conjugates are not KahK3-subgroups.

Let \( (\mathfrak{D}_6)_4 = \mathfrak{S}_3(1,2,3) \) be the group of all elements of \( \mathfrak{S}_4 \) that preserve 4. We denote the conjugates of \( (\mathfrak{D}_6)_4 \) by \( (\mathfrak{D}_6)_k \), \( k = 1, \ldots, 4 \). They are isomorphic to the dihedral group \( \mathfrak{D}_6 \) of order 6. We have

\[ \text{rk} N_{(\mathfrak{D}_6)_4} = 14, \quad N^*_{(\mathfrak{D}_6)_4}/N_{(\mathfrak{D}_6)_4} \cong (\mathbb{Z}/6\mathbb{Z})^2 \times (\mathbb{Z}/3\mathbb{Z})^3. \]

By Theorem 4.1, \( N_{(\mathfrak{D}_6)_4} \) has a primitive embedding in \( L_{K3} \) and \( (\mathfrak{D}_6)_4 \) is a KahK3-subgroup.

The subgroups of \( (\mathfrak{D}_6)_4 \) are also KahK3-subgroups. They are either trivial, or conjugate to \( [(12)] \cong C_2 \) or \( (C_3)_4 = [(123)] \cong C_3 \). We have \( N_{[(12)]} \cong E_8(2) \) and

\[ \text{rk} N_{[(123)]} = 12, \quad N^*_{[(123)]}/N_{[(123)]} \cong (\mathbb{Z}/3\mathbb{Z})^6. \]

It follows from the structure of \( \mathfrak{S}_4 \) that all the non-trivial KahK3-subgroups of \( A(N_{14}) \) are the conjugate subgroups \( (\mathfrak{D}_6)_k \), \( k = 1, \ldots, 4 \) (which are isomorphic to \( \mathfrak{D}_6 \)), the conjugate subgroups \( [(ij)] \), \( 1 \leq i < j \leq 4 \) (which are isomorphic to \( C_2 \)), and the conjugate subgroups \( (C_3)_k \subset (\mathfrak{D}_6)_k \), \( k = 1, \ldots, 4 \) (which are isomorphic to \( C_3 \)).

Let \( X \) be marked by a primitive sublattice \( S \subset N_{14} = N(4D_6) \). Then \( S \) must satisfy Theorem 4.1 and \( \Gamma(P(S)) \subset \Gamma(P(N_{14})) = 4D_6 \). Any such \( S \) gives a marking of some \( X \), and \( P(X) \cap S = P(S) \).

If \( N_{(\mathfrak{D}_6)_k} \subset S \) for some \( k \) in the range 1, \ldots, 4, then \( \text{Aut}(X, S)_0 = (\mathfrak{D}_6)_k \cong \mathfrak{D}_6 \). If this is not the case but \( N_{(C_3)_k} \subset S \) for some \( k \) in the range 1, \ldots, 4, then \( \text{Aut}(X, S)_0 = (C_3)_k \cong C_3 \). If \( N_{(C_3)_k} \not\subset S \) but \( N_{[(ij)]} \subset S \) for some \( i, j, 1 \leq i < j \leq 4 \), then \( \text{Aut}(X, S)_0 = [(ij)] \cong C_2 \). If \( N_{[(ij)]} \not\subset S \), then the group \( \text{Aut}(X, S)_0 \) is trivial.

Suppose that \( S = [(D_6)_1 = [\alpha_{11}, \ldots, \alpha_{61}], N_{(\mathfrak{D}_6)_4}]_{pr} \subset N_{14} \). We have \( \text{rk} S = 19, S^*/S \cong \mathbb{Z}/4\mathbb{Z} \). Hence \( S \) satisfies Theorem 4.1 and gives a marking of some surface \( X \). We have \( \text{Aut}(X, S)_0 \cong \mathfrak{D}_6 \) and \( 3\mathfrak{D}_6 \subset \Gamma(P(X) \cap S) = P(S) \). By the
classification of Niemeier lattices and our calculations above, \(X\) can be marked only by \(N_{14} = N(4D_6)\) for such a sublattice \(S \subset S_X\).

**Case 12.** For the Niemeier lattice

\[ N = N_{12} = N(4E_6) = \left[ 4E_6, \varepsilon_{11} + \varepsilon_{13} + \varepsilon_{24}, \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{14}, \varepsilon_{11} + \varepsilon_{12} + \varepsilon_{23} \right], \]

the group \(A(N)\) has centre \([\varphi_0]\) of order 2 which preserves the components \(4E_6\), and \(A(N)/[\varphi_0] = \mathfrak{S}_4\) is the symmetric group on the components \(4E_6\) (see [10], Ch. 16). The elements of \(A(N)\) are determined by permutations of the ends of the Dynkin diagrams \(E_6\). This group is generated by the involutions

\[ \varphi_0 = (\alpha_{11}\alpha_{61})(\alpha_{12}\alpha_{62})(\alpha_{13}\alpha_{63})(\alpha_{14}\alpha_{64}), \]

\[ \widetilde{(12)} = (\alpha_{11}\alpha_{12})(\alpha_{61}\alpha_{62})(\alpha_{14}\alpha_{64}), \]

\[ \widetilde{(23)} = (\alpha_{11}\alpha_{61})(\alpha_{12}\alpha_{13})(\alpha_{62}\alpha_{63}), \]

\[ \widetilde{(34)} = (\alpha_{11}\alpha_{61})(\alpha_{13}\alpha_{14})(\alpha_{63}\alpha_{64}). \]

We similarly define involutions \(\widetilde{ij}\), \(1 \leq i < j \leq 4\), which act as transpositions of the components \((E_6)_i\), \((E_6)_j\) and the elements \(\alpha_{11}, \alpha_{1j}\).

The element \(\widetilde{(34)(12)}\) has order 4, and \((\widetilde{(34)(12)})^2 = \varphi_0\). The coinvariant sublattice \(N_{[\widetilde{(34)(12)}]}\) has \(\text{rk } N_{[\widetilde{(34)(12)}]} = 16\) and \(N_{[\widetilde{(34)(12)}]}^*/N_{[\widetilde{(34)(12)}]} = (\mathbb{Z}/4\mathbb{Z})^4 \times (\mathbb{Z}/2\mathbb{Z})^4\).

By Theorem 4.1, \(N_{[\widetilde{(34)(12)}]}\) has no primitive embeddings in \(L_{K3}\) and \([\widetilde{(34)(12)}]\) is not a KahK3-subgroup.

For \(k = 1, \ldots, 4\) let \((\mathfrak{D}_{12})_k\) be the set of all elements of \(A(N_{12})\) that preserve the component \((E_6)_k\). The subgroups \((\mathfrak{D}_{12})_k\) are conjugate to each other and are isomorphic to the dihedral group \(D_{12}\) of order 12. We have

\[ \text{rk } N_{(\mathfrak{D}_{12})_k} = 16, \quad N^*_{(\mathfrak{D}_{12})_k}/N_{(\mathfrak{D}_{12})_k} \cong (\mathbb{Z}/6\mathbb{Z})^4. \]

By Theorem 4.1, \(N_{(\mathfrak{D}_{12})_k}\) has a primitive embedding in \(L_{K3}\) and \((\mathfrak{D}_{12})_k\) is a KahK3-subgroup.

It follows from the structure of \(A(N)\) that the conjugate subgroups

\[ (\mathfrak{D}_{12})_k \cong \mathfrak{D}_{12}, \quad k = 1, \ldots, 4, \]

and all their subgroups are precisely all the KahK3-subgroups of \(A(N_{12})\).

Here are all non-trivial such subgroups (where the subscript \(k\) means the subgroups of \((\mathfrak{D}_{12})_k\): the subgroups

\[ (C_6)_k, \quad k = 1, \ldots, 4, \]

which are isomorphic to \(C_6\) and conjugate to \((C_6)_4 = [(\widetilde{(12)})(\widetilde{(23)})]\) with the sublattice \(N_{(C_6)_k} = N_{(\mathfrak{D}_{12})_k}\) (therefore \(\text{Clos}(C_6)_k = (\mathfrak{D}_{12})_k\)); the subgroups

\[ (\mathfrak{D}_6)_k, \quad k = 1, \ldots, 4, \quad i = 1, 2, \]

which are isomorphic to \(\mathfrak{D}_6\) and have \(\text{rk } N_{(\mathfrak{D}_6)_k} = 14\) and \(N^*_{(\mathfrak{D}_6)_k}/N_{(\mathfrak{D}_6)_k} \cong (\mathbb{Z}/6\mathbb{Z})^2 \times (\mathbb{Z}/3\mathbb{Z})^3\); the conjugate subgroups

\[ (C_3)_k, \quad k = 1, \ldots, 4, \]
which are isomorphic to $C_3$ and have $\text{rk} N_{(C_3)_k} = 12$ and $N^{*}_{(C_3)_k}/N_{(C_3)_k} \cong (\mathbb{Z}/3\mathbb{Z})^6$; the conjugate subgroups

$$[\widetilde{(ij)}, \varphi_0], \quad 1 \leq i < j \leq 4,$$

which are isomorphic to $C_2 \times C_2$ and have $\text{rk} N_{[\widetilde{(ij)}, \varphi_0]} = 12$ and $N^{*}_{[\widetilde{(ij)}, \varphi_0]}/N_{[\widetilde{(ij)}, \varphi_0]} \cong (\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^6$; the subgroups

$$[\varphi_0], \quad [(\widetilde{ij})], \quad [(\widetilde{ij})\varphi_0], \quad 1 \leq i < j \leq 4,$$

which are isomorphic to $C_2$ and have coinvariant sublattice isomorphic to $E_8(2)$.

Let $X$ be marked by a primitive sublattice $S \subset N_{12} = N(4E_6)$. Then $S$ must satisfy Theorem 4.1 and $\Gamma(P(S)) \subset \Gamma(P(N_{17})) = 4E_6$. Any such $S$ gives a marking of some $X$, and $P(X) \cap S = P(S)$.

If $N_{(D_{12})_k} \subset S$ for some $k$ in the range $1, \ldots, 4$, then $\text{Aut}(X, S)_0 = [\mathcal{D}_{12}]_k \cong \mathcal{D}_{12}$. If this is not the case but $N_{(D_{0})_{k}} \subset S$ for some $k$ in the range $1, \ldots, 4$, then $\text{Aut}(X, S)_0 = (C_3)_k \cong C_3$. If this is not the case, but $N_{[\tilde{(ij)}, \varphi_0]} \subset S$ for some $i, j$, $1 \leq i < j \leq 4$, then $\text{Aut}(X, S)_0 = [\tilde{(ij)}, \varphi_0] \cong C_2 \times C_2$. If we only have $N_{H} \subset S$ for some $H \in \{[\varphi_0] \cup \{[(\tilde{ij})], \{[(\tilde{ij})\varphi_0] \mid 1 \leq i < j \leq 4\}$, then $\text{Aut}(X, S)_0 = H \cong C_2$. Otherwise the group $\text{Aut}(X, S)_0$ is trivial.

Suppose that $S = [(E_6)_{1}] = [\alpha_{11}, \ldots, \alpha_{61}], N_{(C3)_{4}}]_{\text{pr}} \subset N_{12}$. We have $\text{rk} S = 18$, $S^{\ast}/S \cong \mathbb{Z}/3\mathbb{Z}$. Therefore $S$ satisfies Theorem 4.1 and gives a marking of some surface $X$. We have $\text{Aut}(X, S)_0 \cong C_3$ and $3E_6 \subset \Gamma(P(X) \cap S) = P(S)$. By the classification of Niemeier lattices and our calculations above, the K3 surface $X$ can be marked only by $N_{12} = N(4E_6)$ for such a sublattice $S \subset S_X$.

Case 18. For the Niemeier lattice

$$N = N_{18} = N(4A_5 \oplus D_4) = [4A_5 \oplus D_4, 2\varepsilon_{11} + 2\varepsilon_{13} + 4\varepsilon_{14}, 2\varepsilon_{11} + 4\varepsilon_{12} + 2\varepsilon_{14},$$

$$2\varepsilon_{11} + 2\varepsilon_{12} + 4\varepsilon_{13}, 3\varepsilon_{11} + 3\varepsilon_{12} + \varepsilon_{15}, 3\varepsilon_{11} + 3\varepsilon_{13} + \varepsilon_{25}, 3\varepsilon_{11} + 3\varepsilon_{14} + \varepsilon_{35}],$$

the group $A = A(N)$ has centre $[\varphi_0]$ of order 2 which preserves the components $4A_5$ and $D_4$. The group $A(N)/[\varphi_0] = \mathcal{G}_4$ is the symmetric group on the components $4A_5$ (see [10], Ch. 16). The elements of $A$ are determined by their action on the ends of the Dynkin diagram $4A_5 \mathcal{D}_4$. The group $A$ is generated by the involutions

$$\varphi_0 = (\alpha_{11} \alpha_{51})(\alpha_{12} \alpha_{52})(\alpha_{13} \alpha_{53})(\alpha_{14} \alpha_{54}),$$

$$[\tilde{12}] = (\alpha_{11} \alpha_{12})(\alpha_{51} \alpha_{52})(\alpha_{14} \alpha_{54})(\alpha_{15} \alpha_{35}),$$

$$[\tilde{23}] = (\alpha_{11} \alpha_{51})(\alpha_{12} \alpha_{13})(\alpha_{52} \alpha_{53})(\alpha_{15} \alpha_{45}),$$

$$[\tilde{34}] = (\alpha_{11} \alpha_{51})(\alpha_{13} \alpha_{14})(\alpha_{53} \alpha_{54})(\alpha_{15} \alpha_{35}).$$

We similarly define involutions $[(\tilde{ij})], 1 \leq i < j \leq 4$, which act as transpositions of the components $(\tilde{A}_5)_i, (\tilde{A}_5)_j$ and the elements $\alpha_{ii}, \alpha_{jj}$.

The group $A = A(N)$ is isomorphic to $T_{48} = Q_8 \times \mathcal{G}_3$ (this notation is standard, see [4]). The coinvariant lattice $N_A$ has $\text{rk} N_A = 19$ and $N_A^{\ast}/N_A \cong \mathbb{Z}/24\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$
By Theorem 4.1, $N_A$ has a primitive embedding in $L_{K3}$ and $A = A(N)$ is a KahK3-subgroup. All of its subgroups are also KahK3-subgroups. These facts seem to have been first observed in [7].

Here are all (up to conjugation) the non-trivial subgroups $H \subset A$ of the group $A = A(N)$ and the invariants of their coinvariant sublattices $N_H$.

1. A subgroup $H$ isomorphic to $T_{24}$ (the binary tetrahedral group) and consisting of the elements of $A$ that induce even permutations of the components $4A_5$. The coinvariant sublattice $N_H$ is equal to $N_A$. Therefore $\operatorname{Clos}(H) = A$ is isomorphic to $T_{48}$ and we have $\operatorname{rk} N_H = 19$, $N_H^*/N_H \cong \mathbb{Z}/24\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. 

2. Subgroups $H$ isomorphic to $SD_{16}$ (2-Sylow subgroups of $A$) and conjugate to $[34](23)(12), (13), (24)]$ with $\operatorname{rk} N_H = 18$ and $N_H^*/N_H \cong (\mathbb{Z}/8\mathbb{Z})^2 \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. 

3. Subgroups $H$ isomorphic to $C_8$ and conjugate to $[34](23)(12)]$ while the subgroup $\operatorname{Clos}([34](23)(12)]) = ([34](23)(12), (13), (24)]$ is isomorphic to $SD_{16}$. Thus $N_{[34](23)(12)]} = N_{[34](23)(12), (13), (24)]}$ and $\operatorname{rk} N_H = 18$, $N_H^*/N_H \cong (\mathbb{Z}/8\mathbb{Z})^2 \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. 

4. A subgroup $H$ isomorphic to $Q_8$ and given by $H = [34](12), (24)(13)]$ with $\operatorname{rk} N_H$ equal to 17 and $N_H^*/N_H \cong (\mathbb{Z}/8\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^3$. 

5. Subgroups $H$ isomorphic to $\mathcal{O}_{12}$ and conjugate to $[12], (23)]$ with $\operatorname{rk} N_H = 16$ and $N_H^*/N_H \cong (\mathbb{Z}/6\mathbb{Z})^4$. 

6. Subgroups $H$ isomorphic to $C_6$ and conjugate to $[23](12)]$ while the subgroup $\operatorname{Clos}([23](12)]) = ([12], (23)]$ is isomorphic to $\mathcal{O}_{12}$. Thus $N_{[23](12)]} = N_{[12], (23)]}$ and $\operatorname{rk} N_H = 16$, $N_H^*/N_H \cong (\mathbb{Z}/6\mathbb{Z})^4$. 

7. Subgroups $H$ isomorphic to $\mathcal{O}_8$ and conjugate to $[12], (34)]$ with $\operatorname{rk} N_H = 15$ and $N_H^*/N_H \cong (\mathbb{Z}/4\mathbb{Z})^5$. 

8. Subgroups $H$ isomorphic to $\mathcal{O}_6$ and conjugate to $[\varphi_0(23), (12)]$ or $[\varphi_0(13), (23)]$ with $\operatorname{rk} N_H = 14$ and $N_H^*/N_H \cong (\mathbb{Z}/6\mathbb{Z})^2 \times (\mathbb{Z}/3\mathbb{Z})^3$. 

9. Subgroups $H$ isomorphic to $C_4$ and conjugate to $[34](12)]$ with $\operatorname{rk} N_H = 14$ and $N_H^*/N_H \cong (\mathbb{Z}/4\mathbb{Z})^4 \times (\mathbb{Z}/2\mathbb{Z})^2$. 

10. Subgroups $H$ isomorphic to $C_2 \times C_2$ and given by $[(i,j), \varphi_0]$, $1 \leq i < j \leq 4$, with $\operatorname{rk} N_H = 12$ and $N_H^*/N_H \cong (\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^6$. 

11. Subgroups $H$ isomorphic to $C_3$ and conjugate to $[\varphi_0(23)(12)]$ with $\operatorname{rk} N_H = 12$ and $N_H^*/N_H \cong (\mathbb{Z}/3\mathbb{Z})^6$. 

12. Subgroups $H$ isomorphic to $C_2$ and given by $[\varphi_0]$, $[(i,j)]$ and $[(i,j)\varphi_0]$, $1 \leq i < j \leq 4$. The lattice $N_H$ is isomorphic to $E_8(2)$.

Let $X$ be marked by a primitive sublattice $S \subset N = N_{18} = N(4A_5 \oplus D_4)$. Then $S$ must satisfy Theorem 4.1 and $\Gamma(P(S)) \subset \Gamma(P(N_{18})) = 4A_5 \oplus D_4$. Any such $S$ gives a marking of some $X$, and $P(X) \cap S = P(S)$.

If $S = N_A$, then $\operatorname{Aut}(X, S)_0 = A \cong T_{48}$. If this is not the case but $N_H \subset S$ for $H \cong SD_{16}$ or $H \cong \mathcal{O}_{12}$, then $\operatorname{Aut}(X, S)_0 = H$. If this is not the case but $N_H \subset S$ for $H \cong Q_8$, $\mathcal{O}_8$ or $\mathcal{O}_6$, then $\operatorname{Aut}(X, S)_0 = H$. If this is not the case but $N_H \subset S$ for $H \cong C_4$ or $H \cong C_3$, then $\operatorname{Aut}(X, S)_0 = H$. If this is not the case but $N_H \subset S$ for $H \cong C_2$, then $H = \operatorname{Aut}(X, S)_0 = H \cong C_2$. Otherwise the group $\operatorname{Aut}(X, S)_0$ is trivial.
Suppose that \( H = [(12), (34)] \cong D_8 \) and \( S = [(A_5)_1 = [\alpha_{11}, \ldots, \alpha_{51}], N_H]_{pr} \subset N_{18} \). We have \( \text{rk} \ S = 18 \), \( S^*/S \cong (\mathbb{Z}/4\mathbb{Z})^4 \times (\mathbb{Z}/2\mathbb{Z})^2 \) and \( q_{S_2} = q_0^{(2)}(2) \oplus q' \). Hence \( S \) satisfies Theorem 4.1 and gives a marking of some surface \( X \). We have \( H \subset \text{Aut}(X, S)_0 \), \( H \cong D_8 \), and \( 2A_5 \subset \Gamma(P(X) \cap S) = P(S) \). By the classification of Niemeier lattices and our calculations above, \( X \) can be marked only by \( N_{18} = N(4A_5 \oplus D_4) \) for such a sublattice \( S \subset S_X \).

**Case 19.** For the Niemeier lattice

\[
N = N_{19} = N(6D_4)
\]

\[
= [6D_4, \varepsilon_{13} + \varepsilon_{14} + \varepsilon_{15} + \varepsilon_{16}, \varepsilon_{12} + \varepsilon_{14} + \varepsilon_{25} + \varepsilon_{36}, \varepsilon_{11} + \varepsilon_{14} + \varepsilon_{35} + \varepsilon_{26},
\varepsilon_{23} + \varepsilon_{24} + \varepsilon_{25} + \varepsilon_{26}, \varepsilon_{22} + \varepsilon_{24} + \varepsilon_{35} + \varepsilon_{16}, \varepsilon_{21} + \varepsilon_{24} + \varepsilon_{15} + \varepsilon_{36}],
\]

the group \( A = A(N) \) contains a cyclic group \([\varphi] \) of order 3 which preserves the connected components of \( 6D_4 \), and \( A/\langle \varphi \rangle = S_6 \) is the symmetric group on the 6 components of \( 6D_4 \) (see [10], Chs. 16, 18). The elements of \( A \) are determined by their action on the ends of the components of \( 6D_4 \). This group is generated by the element

\[
\varphi = (\alpha_{11}\alpha_{31}\alpha_{41})(\alpha_{12}\alpha_{32}\alpha_{42})(\alpha_{13}\alpha_{33}\alpha_{43})(\alpha_{14}\alpha_{34}\alpha_{44})(\alpha_{15}\alpha_{35}\alpha_{45})(\alpha_{16}\alpha_{36}\alpha_{46})
\]

and the involutions

\[
\tilde{(12)} = (\alpha_{41}\alpha_{42})(\alpha_{11}\alpha_{32})(\alpha_{31}\alpha_{12})(\alpha_{13}\alpha_{33})(\alpha_{14}\alpha_{34})(\alpha_{15}\alpha_{35})(\alpha_{16}\alpha_{36}),
\tilde{(23)} = (\alpha_{11}\alpha_{31})(\alpha_{12}\alpha_{33})(\alpha_{32}\alpha_{13})(\alpha_{42}\alpha_{43})(\alpha_{14}\alpha_{34})(\alpha_{15}\alpha_{45})(\alpha_{36}\alpha_{46}),
\tilde{(34)} = (\alpha_{11}\alpha_{31})(\alpha_{12}\alpha_{32})(\alpha_{43}\alpha_{44})(\alpha_{13}\alpha_{34})(\alpha_{33}\alpha_{14})(\alpha_{15}\alpha_{35})(\alpha_{16}\alpha_{36}),
\tilde{(45)} = (\alpha_{11}\alpha_{41})(\alpha_{32}\alpha_{42})(\alpha_{13}\alpha_{33})(\alpha_{44}\alpha_{45})(\alpha_{14}\alpha_{35})(\alpha_{34}\alpha_{15})(\alpha_{16}\alpha_{36}),
\tilde{(56)} = (\alpha_{11}\alpha_{31})(\alpha_{12}\alpha_{32})(\alpha_{13}\alpha_{33})(\alpha_{14}\alpha_{34})(\alpha_{15}\alpha_{35})(\alpha_{35}\alpha_{16})(\alpha_{45}\alpha_{46}).
\]

Here and in what follows, \( \tilde{x} \) stands for an element of \( \pi^{-1}(x) \), where \( x \in S_6 \) and \( \pi: A \rightarrow S_6 \) is the canonical homomorphism. We similarly use the ‘tilde’ for subgroups \( G \subset S_6 \). Hence \( \tilde{G} \) is a subgroup of \( A \) such that \( \pi \) is an isomorphism between \( \tilde{G} \) and \( G \). We write \( \tilde{G} \) for \( \pi^{-1}(G) \), where \( G \subset S_6 \).

We now describe the KahK3-subgroups \( H \) of \( A \). They can be of two types.

**Type I.** The KahK3-subgroups \( H \subset A = A(N_{19}) \) that contain the subgroup \([\varphi] \). Then \( H = \tilde{G} = \pi^{-1}(G) \) for some \( G \subset S_6 \). It suffices to describe all possible conjugacy classes of such subgroups \( G \) in \( S_6 \).

Since \( \text{rk} \ N_H \leq 19 \) for KahK3-subgroups \( H \), the group \( G \) must have more than two orbits in \( \{1, \ldots, 6\} \). Therefore either \( G \subset S_{1,1,4} \), \( G \subset S_{1,2,3} \), or \( G \subset S_{2,2,2} \). Here we use the standard notation of [4]. For example, \( S_{1,2,3} \) consists of all permutations in \( S_6 \) that preserve the subsets \( \{1\} \), \( \{2, 3\} \) and \( \{4, 5, 6\} \). The alternating subgroup of \( S_{1,2,3} \) is denoted by \( \mathfrak{A}_{1,2,3} \).

If \( H = \tilde{G} = \tilde{S}_{1,1,4} = [\varphi, (34), (45), (56)] \cong \mathfrak{A}_{3,4} \), then \( \text{rk} \ N_H = 18 \) and \( (N_H)^*/N_H \cong (\mathbb{Z}/12\mathbb{Z})^2 \times \mathbb{Z}/3\mathbb{Z} \). By Theorem 4.1, \( N_H \) has a primitive embedding in \( L_{K3} \) and \( H \) is a KahK3-subgroup.
If \( H = \tilde{G}_{1,2,3} = [\varphi, (23), (45), (56)] \simeq G_{3,3} \), then \( \text{rk} N_H = 18 \) and \((N_H)^*/N_H \simeq \mathbb{Z}/18\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^2\). Let \( q_{N_H} \) be the 3-component of the discriminant form \( q_{N_H} \) of the lattice \( N_H \). Then

\[
\det K(q_{N_H}) = \det \begin{pmatrix}
-6948 & -5022 & 108 & 144 \\
-5022 & -3684 & 78 & 108 \\
108 & 78 & -18 & 0 \\
144 & 108 & 0 & -12
\end{pmatrix}
\]

over \( \mathbb{Z}_3 \) and \( \det K(q_{N_H}) \equiv 2^5 \cdot 3^5 \cdot 61 \cdot 109 \equiv -3^5 \) mod \( (\mathbb{Z}_3)^2 \). By Theorem 4.1, \( N_H \) has a primitive embedding in \( L_{K3} \) and \( H \) is a KahK3-subgroup.

If \( H = \tilde{G}_{2,2,2} = [\varphi, (12), (34), (56)] \), where \( G_{2,2,2} \simeq (C_2)^3 \), then \( \text{rk} N_H = 18 \) and \((N_H)^*/N_H \simeq (\mathbb{Z}/6\mathbb{Z})^3 \times (\mathbb{Z}/2\mathbb{Z})^3\). By Theorem 4.1, \( N_H \) has no primitive embeddings in \( L_{K3} \) and \( H \) is not a KahK3-subgroup. The same coinvariant lattice \( N_H \) appears for those subgroups \( H \subset \tilde{G}_{2,2,2} \) that are conjugate to \([\varphi, (12)(34), (56)], [\varphi, (12)(34), (34)(36)] \) or \([\varphi, (12)(34)(56)] \).

It follows that the KahK3-subgroups of type I are precisely those subgroups of type I which are conjugate to subgroups of \( \tilde{G}_{1,1,4} = [\varphi, (34), (45), (56)] \simeq \mathfrak{A}_{3,4} \) and \( \tilde{G}_{1,2,3} = [\varphi, (23), (45), (56)] \simeq \mathfrak{S}_{3,3} \). It is easy to list them all.

The subgroups conjugate to subgroups of type I in \( \tilde{G}_{1,1,4} = [\varphi, (34), (45), (56)] \simeq \mathfrak{A}_{3,4} \) are conjugate to one of the following subgroups.

(1.1) \( H = [\varphi, (34), (45), (56)] = \tilde{G}_{1,1,4} \simeq \mathfrak{A}_{3,4} \) with \( \text{rk} N_H = 18 \) and \((N_H)^*/N_H \simeq (\mathbb{Z}/12\mathbb{Z})^2 \times \mathbb{Z}/3\mathbb{Z}\).

(1.2) \( H = [\varphi, (34)(45), (34)(56)] = \tilde{G}_{1,1,4} \simeq C_3 \times \mathfrak{A}_4 \) with \( \text{Clos}(H) = [\varphi, (34), (45), (56)] = \tilde{G}_{1,1,4} \simeq \mathfrak{A}_{3,4} \) as above and \( \text{rk} N_H = 18 \), \((N_H)^*/N_H \simeq (\mathbb{Z}/12\mathbb{Z})^2 \times \mathbb{Z}/3\mathbb{Z}\).

(1.3) \( H = [\varphi, (34)(45)(56), (35)] = \tilde{D}_8 \simeq D_8 \times C_3 \) with \( \text{Clos}(H) = [\varphi, (34), (45), (56)] = \tilde{G}_{1,1,4} \simeq \mathfrak{A}_{3,4} \) as above and \( \text{rk} N_H = 18 \), \((N_H)^*/N_H \simeq (\mathbb{Z}/12\mathbb{Z})^2 \times \mathbb{Z}/3\mathbb{Z}\).

(1.4) \( H = [\varphi, (34)(56), (34)(45)(34)(56)] = \tilde{K}_4 \simeq C_2 \times C_6 \) with \( \text{Clos}(H) = [\varphi, (34), (45), (56)] = \tilde{G}_{1,1,4} \simeq \mathfrak{A}_{3,4} \) as above and \( \text{rk} N_H = 18 \), \((N_H)^*/N_H \simeq (\mathbb{Z}/12\mathbb{Z})^2 \times \mathbb{Z}/3\mathbb{Z}\). Here \( K_4 \simeq C_2 \times C_2 \) is the Klein normal subgroup\(^2\) of the group \( \tilde{G}_{1,1,4} \simeq \mathfrak{G}_4 \).

(1.5) \( H = [\varphi, (34)(45)(56)] = \tilde{C}_4 \simeq Q_{12} \) with \( \text{Clos}(H) = [\varphi, (34), (45), (56)] = \tilde{G}_{1,1,4} \simeq \mathfrak{A}_{3,4} \) as above and \( \text{rk} N_H = 18 \), \((N_H)^*/N_H \simeq (\mathbb{Z}/12\mathbb{Z})^2 \times \mathbb{Z}/3\mathbb{Z}\). Here \( C_4 \subset \tilde{G}_{1,1,4} \simeq \mathfrak{G}_4 \) is the cyclic subgroup of order 4 and \( Q_{12} \) is the binary dihedral group of order 12.

(1.6) \( H = [\varphi, (45), (56)] \) \( \simeq \mathfrak{A}_{3,3} \) with \( \text{rk} N_H = 16 \) and \((N_H)^*/N_H \simeq \mathbb{Z}/9\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^4\).

\(^2\)These calculations show that, for a symplectic group \( G = C_2 \times C_6 \) on a Kählerian K3 surface, we have \( S_{G_2}(1)/S_{G_2}(G) = S_{G_2(2,6)}/S_{G_2(2,6)} \simeq (\mathbb{Z}/12\mathbb{Z})^2 \times \mathbb{Z}/3\mathbb{Z}\). We must therefore correct our calculation of the last group in [3], Proposition 10.1.
(I.7) \( H = [\varphi, (45)\bar{(56)}] = [(45)(56)] \cong (C_3)^2 \) with \( \text{Clos}(H) = [\varphi, (45), \bar{(56)}] = [(45), (56)] \cong \mathfrak{A}_{3,3} \) as above and \( \text{rk} N_H = 16 \). The group \( (N_H)^*/N_H \) is isomorphic\(^3\) to \( \mathbb{Z}/9\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^4 \).

(I.8) \( H = [\varphi, (34), \bar{(56)}] = [(34)(56)] \cong \mathfrak{O}_{12}, \) \( \text{rk} N_H = 16, (N_H)^*/N_H \cong (\mathbb{Z}/6\mathbb{Z})^4 \).

(I.9) \( H = [\varphi, (34)\bar{(56)}] = [(34)(56)] \cong C_6 \) with \( \text{Clos}(H) = [\varphi, (34), \bar{(56)}] = [(34), (56)] \cong \mathfrak{O}_{12} \) as above and \( \text{rk} N_H = 16, (N_H)^*/N_H \cong (\mathbb{Z}/6\mathbb{Z})^4 \).

(I.10) \( H = [\varphi, (56)] = [(56)] \cong \mathfrak{O}_6, \) \( \text{rk} N_H = 14, (N_H)^*/N_H \cong (\mathbb{Z}/6\mathbb{Z})^2 \times (\mathbb{Z}/3\mathbb{Z})^3 \).

(I.11) \( H = [\varphi] = \{e\} \cong C_3 \) with \( \text{rk} N_H = 12, (N_H)^*/N_H \cong (\mathbb{Z}/3\mathbb{Z})^6 \).

The subgroups conjugate to subgroups of type I in \( \tilde{\mathfrak{S}}_{1,2,3} = [\varphi, (23), (45), \bar{(56)}] \cong \mathfrak{S}_{3,3} \) and different from the subgroups (I.1)–(I.11) above are conjugate to one of the following subgroups.

(I’.1) \( H = [\varphi, (23), (45), \bar{(56)}] = \tilde{\mathfrak{S}}_{1,2,3} \cong \mathfrak{S}_{3,3} \) and \( \text{rk} N_H = 18, (N_H)^*/N_H \cong \mathbb{Z}/18\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^2 \).

(I’.2) \( H = [\varphi, (23)(45)\bar{(56)}] = [(23)(45)(56)] \cong C_3 \times \mathfrak{O}_6 \) with \( \text{Clos}(H) = [\varphi, (23), \bar{(23)}, (45), \bar{(56)}] = \tilde{\mathfrak{S}}_{1,2,3} \cong \mathfrak{S}_{3,3} \) and \( \text{rk} N_H = 18, (N_H)^*/N_H \cong \mathbb{Z}/18\mathbb{Z} \times (\mathbb{Z}/6\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})^2 \).

Type II. The KahK3-subgroups \( H \subset A = A(N_{19}) \) that do not contain the subgroup \([\varphi]\). Then \( \pi \) establishes an isomorphism \( \pi : H \to \pi(H) = G \subset \mathfrak{S}_6 \). It suffices to describe all possible conjugacy classes of \( G \) in \( \mathfrak{S}_6 \) and their lifts to \( H = \tilde{G} \subset A \).

Since \( \text{rk} N_H \leq 19 \), the subgroup \( G \) must have at least two orbits on \( \{1, \ldots, 6\} \). Therefore \( H = \tilde{G} \), where either \( G \subset \mathfrak{S}_{1,5}, G \subset \mathfrak{S}_{2,4}, \) or \( G \subset \mathfrak{S}_{3,3} \) (up to conjugacy).

Consider the cases when \( G \subset \mathfrak{S}_{1,5} \). For \( G = \mathfrak{S}_{1,5} \) we easily obtain the following result (up to conjugacy in \( A \)).

(II.1) \( H = [(23), (34), (45)\varphi, \bar{(56)}] = \tilde{\mathfrak{S}}_{1,5} \cong \mathfrak{S}_5 \) with \( \text{rk} N_H = 19 \) and \( (N_H)^*/N_H \cong \mathbb{Z}/60\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \). By Theorem 4.1, \( N_H \) has a primitive embedding in \( L_{K3} \) and \( H \) is a KahK3-subgroup.

In particular, for any \( G \subset \mathfrak{S}_{1,5} \) we obtain a KahK3-subgroup

\[ \tilde{G} = \pi^{-1}(G) \cap [(23), (34), (45)\varphi, \bar{(56)}] \cong G. \]

One can also verify that this is the only lift of the subgroup \( G \subset \mathfrak{S}_{1,5} \) to a subgroup \( \tilde{G} \subset A \) up to conjugacy in \( A \) provided that \( G \) is different from \( \mathfrak{S}_4 \) and \( \mathfrak{A}_4 \). Each of the subgroups \( \mathfrak{S}_4, \mathfrak{A}_4 \) has two lifts.

We now list the subgroups \( H = \tilde{G} \subset [(23), (34), (45)\varphi, \bar{(56)}] \) for each of the 17 non-trivial conjugacy classes of subgroups \( G \subset \mathfrak{S}_5 \) and the additional conjugacy classes for \( G \cong \tilde{\mathfrak{S}}_4, \mathfrak{A}_4 \).

---

\(^3\)These calculations show that, for a symplectic group \( G = (C_3)^2 \) on a Kählerian K3 surface, we have \( S_{(G)}/S_{(G)} = S_{(3,3)}^T/S_{(3,3)} \cong \mathbb{Z}/9\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^4 \). We must therefore correct our calculation of the last group in [3], Proposition 10.1.
(II.2) $H = \{(23)(34)(45)\varphi(56), (23)(34)\} \cong [(23456), (234)] \cong \mathfrak{A}_5$ with $\text{rk}N_H = 18$ and $(N_H^+)/N_H \cong \mathbb{Z}/30 \mathbb{Z} \times \mathbb{Z}/10 \mathbb{Z}$.

(II.3) $H = \{(23)(34)(45)\varphi(56), (34)(56)\varphi(45)\} = [(23456), (3465)] \cong \text{Hol}(C_5) = C_5 \times \text{Aut}(C_5)$ with $\text{rk}N_H = 18$ and $(N_H^+)/N_H \cong (\mathbb{Z}/10 \mathbb{Z})^2 \times \mathbb{Z}/5 \mathbb{Z}$.

(II.4) $H = \{(23)(34)(45)\varphi(56), (56)(34)(45)\varphi(34)(56)(45)\varphi\} = [(23456), (36)(45)] \cong \mathfrak{D}_{10}$ with $\text{rk}N_H = 16$ and $(N_H^+)/N_H \cong (\mathbb{Z}/5 \mathbb{Z})^4$.

(II.5) $H = \{(23)(34)(45)\varphi(56)\} = [(23)(34)(56)] \cong C_5$ with $\text{Clos}(H) = [(23)(34)(45)\varphi(56), (56)(34)(45)\varphi(34)(56)(45)\varphi\} = [(23456), (36)(45)] \cong \mathfrak{D}_{10}$ as above and $\text{rk}N_H = 16$, $(N_H^+)/N_H \cong (\mathbb{Z}/5 \mathbb{Z})^4$.

(II.6) $H = \{(34), (45)\varphi, (56)\} = [(34), (45), (56)] \cong \mathfrak{G}_4$ with $\text{rk}N_H = 17$ and $(N_H^+)/N_H \cong (\mathbb{Z}/12 \mathbb{Z})^2 \times \mathbb{Z}/4 \mathbb{Z}$.

(II.6') $H = \{(34), (45), (56)\} = [(34), (45), (56)] \cong \mathfrak{G}_4$ with $\text{rk}N_H = 17$ and $(N_H^+)/N_H \cong (\mathbb{Z}/12 \mathbb{Z})^2 \times \mathbb{Z}/4 \mathbb{Z}$.

(II.7) $H = [(34)(56), (45)\varphi(56)] = [(34)(45)(56)] \cong \mathfrak{A}_4$ with $\text{rk}N_H = 16$ and $(N_H^+)/N_H \cong (\mathbb{Z}/12 \mathbb{Z})^2 \times (\mathbb{Z}/2 \mathbb{Z})^2$.

(II.8) $H = [(23), (45)\varphi, (56)] = [(23), (45), (56)] \cong \mathfrak{D}_{12}$ with $\text{rk}N_H = 16$ and $(N_H^+)/N_H \cong (\mathbb{Z}/6 \mathbb{Z})^4$.

(II.9) $H = [(23)(45)\varphi(56)] = [(23)(45)(56)] \cong C_6$ with $\text{Clos}(H) = [(23), (45)\varphi, (56)] \cong \mathfrak{D}_{12}$ as above and $\text{rk}N_H = 16$, $(N_H^+)/N_H \cong (\mathbb{Z}/6 \mathbb{Z})^4$.

(II.10) $H = [(23)(45)\varphi, (23)(56)] = [(23)(45), (23)(56)] \cong \mathfrak{D}_6$ with $\text{rk}N_H = 14$ and $(N_H^+)/N_H \cong (\mathbb{Z}/6 \mathbb{Z})^2 \times (\mathbb{Z}/3 \mathbb{Z})^3$.

(II.11) $H = [(45)\varphi, (56)] = [(45), (56)] \cong \mathfrak{D}_6$ with $\text{rk}N_H = 14$ and $(N_H^+)/N_H \cong (\mathbb{Z}/6 \mathbb{Z})^2 \times (\mathbb{Z}/3 \mathbb{Z})^3$.

(II.12) $H = [(34)(45)\varphi(56), (34)(45)\varphi(34)] = [(34)(45)(56), (35)] \cong \mathfrak{D}_8$ with $\text{rk}N_H = 15$ and $(N_H^+)/N_H \cong (\mathbb{Z}/4 \mathbb{Z})^5$.

(II.13) $H = [(34)(45)\varphi(56)] = [(34)(45)(56)] \cong C_4$ with $\text{rk}N_H = 14$ and $(N_H^+)/N_H \cong (\mathbb{Z}/4 \mathbb{Z})^4 \times (\mathbb{Z}/2 \mathbb{Z})^2$.

(II.14) $H = [(45)\varphi(56)] = [(45)(56)] \cong C_3$ with $\text{rk}N_H = 12$ and $(N_H^+)/N_H \cong (\mathbb{Z}/3 \mathbb{Z})^6$.

(II.15) $H = [(34)(56), (45)\varphi(34)(56)(45)\varphi] = [(34)(56), (35)(46)] \cong (C_2)^2$ with $\text{rk}N_H = 12$ and $(N_H^+)/N_H \cong (\mathbb{Z}/4 \mathbb{Z})^2 \times (\mathbb{Z}/2 \mathbb{Z})^6$.

(II.16) $H = [(34), (56)] = [(34), (56)] \cong (C_2)^2$ with $\text{rk}N_H = 12$ and $(N_H^+)/N_H \cong (\mathbb{Z}/4 \mathbb{Z})^2 \times (\mathbb{Z}/2 \mathbb{Z})^6$.

(II.17) $H = [(34)(56)] = [(34)(56)] \cong C_2$ with $\text{rk}N_H = 8$ and $(N_H^+)/N_H \cong (\mathbb{Z}/2 \mathbb{Z})^8$.

(II.18) $H = [(56)] = [(56)] \cong C_2$ with $\text{rk}N_H = 8$ and $(N_H^+)/N_H \cong (\mathbb{Z}/2 \mathbb{Z})^8$.  

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If $G \subset \mathfrak{S}_{3,3}$ and $G$ has no one-element orbits (equivalently, $G$ is not conjugate to a subgroup of $\mathfrak{S}_{1,5}$), we easily see that $G$ contains an element of cycle type $[3,3]$. Then $H = \bar{G}$ contains one of the subgroups $H_1 = [(12)(23)(45)(56)], [(12)(23)(45)(56)\varphi], [(12)(23)(45)(56)\varphi^2]$ (up to conjugacy) with $\text{rk} N_{H_1} = 16$ and $N_{H_1}/N_{H_1} \cong (\mathbb{Z}/2\mathbb{Z})^6$. By Theorem 4.1, $N_{H_1}$ has no primitive embeddings in $L_{K3}$ and none of these subgroups $H_1$ is a KahK3-subgroup. Hence $H = \bar{G}$ is not a KahK3-subgroup.

If $G \subset \mathfrak{S}_{2,4}$ and $G$ has no one-element orbits, then we similarly see that $G$ either contains an element of cycle type $[2,2,2]$ or $[2,4]$, or is conjugate to the subgroup $[(12)(34), (12)(56)]$, which is isomorphic to $(C_2)^2$. In the first case, the subgroup $H = \bar{G}$ contains a subgroup $H_1 = [(12)(34)]$ (up to conjugacy) with $\text{rk} N_{H_1} = 12$ and $N_{H_1}/N_{H_1} \cong (\mathbb{Z}/2\mathbb{Z})^{12}$. In the second case, the subgroup $H = \bar{G}$ contains a subgroup $H_1 = [(12)(34)(45)(56)]$ (up to conjugacy) with $\text{rk} N_{H_1} = 16$ and $N_{H_1}/N_{H_1} \cong (\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})^4$. By Theorem 4.1, neither of these lattices $N_{H_1}$ has a primitive embedding in $L_{K3}$, and neither of these subgroups $H_1$ is a KahK3-subgroup. Hence $H = \bar{G}$ is not a KahK3-subgroup. The last case gives the following subgroup.

(II'.1) $H = [(12)(34), (12)(56)] \cong (C_2)^2$ (up to conjugacy) with $\text{rk} N_{H} = 12$ and $(N_{H})^*/N_{H} \cong (\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})^6$. By Theorem 4.1, $N_{H}$ has a primitive embedding in $L_{K3}$ and $H$ is a KahK3-subgroup.

This case is interesting because the invariant lattice $N_{H}$ has no roots (with square $(-2)$).

Thus we finally obtain a list of all KahK3-conjugacy classes of subgroups in $A = A(N_{19})$. It is formed by the subgroups $(I.1)$–$(I.11)$, $(I'.1)$, $(I'.2)$, $(II.1)$–$(II.18)$, $(II.6')$, $(II.7')$, $(II'.1)$.

Let $X$ be marked by a primitive lattice $S \subset N = N_{19} = N(6D_4)$. Then $S$ must satisfy Theorem 4.1 and $\Gamma(P(S)) \subset \Gamma(P(N_{19})) = 6\mathbb{D}_4$. Any such $S$ gives a marking of some $X$, and $P(X) \cap S = P(S)$.

If $N_{H} \subset S$ for some $H$ of type $(I.1)$ ($H \cong \mathfrak{A}_{3,4}$), $(I'.1)$ ($H \cong \mathfrak{S}_{3,3}$), $(II.1)$ ($H \cong \mathfrak{S}_{4}$) or $(II'.1)$ ($H \cong (C_2)^2$), then $\text{Aut}(X, S)_0 = H$. If this is not the case but $N_{H} \subset S$ for some $H$ of type $(I.6)$ ($H \cong \mathfrak{A}_{3,3}$), $(I.8)$ ($H \cong \mathfrak{D}_{12}$), $(II.2)$ ($H \cong \mathfrak{A}_{5}$), $(II.3)$ ($H \cong \text{Hol}(C_5)$), $(II.6)$ ($H \cong \mathfrak{S}_{1}$), $(II.8)$ ($H \cong \mathfrak{D}_{12}$) or $(II.7')$ ($H \cong \mathfrak{A}_{4}$), then $\text{Aut}(X, S)_0 = H$. If this is not the case but $N_{H} \subset S$ for some $H$ of type $(I.10)$ ($H \cong \mathfrak{D}_{6}$), $(II.4)$ ($H \cong \mathfrak{D}_{10}$), $(II.7)$ ($H \cong \mathfrak{A}_{4}$), $(II.10)$ ($H \cong \mathfrak{D}_{6}$), $(II.11)$ ($H \cong \mathfrak{D}_{6}$) or $(II.12)$ ($H \cong \mathfrak{D}_{8}$), then $\text{Aut}(X, S)_0 \cong H$. If this is not the case but $N_{H} \subset S$ for some $H$ of type $(II.13)$ ($H \cong C_4$), $(II.14)$ ($H \cong C_3$), $(II.15)$ ($H \cong (C_2)^2$) or $(II.16)$ ($H \cong (C_2)^2$), then $\text{Aut}(X, S)_0 = H$. If this is not the case but $N_{H} \subset S$ for some $H$ of type $(II.17)$ ($H \cong C_2$) or $(II.18)$ ($H \cong C_2$), then $\text{Aut}(X, S)_0 \cong H$. Otherwise the group $\text{Aut}(X, S)_0$ is trivial.

Suppose that $H = [(34), (56)] = [(34), (56)] \cong \mathfrak{S}_{4}$ is of type (II.6) and put $S = [(D_4)_6 = [\alpha_{16}, \ldots, \alpha_{46}], N_{H}]_{pr} \subset N_{19}$. We have $\text{rk} S = 19$ and $S^*/S \cong \mathbb{Z}/12\mathbb{Z}$. Hence $S$ satisfies Theorem 4.1 and gives a marking of some surface $X$. We have $H \subset \text{Aut}(X, S)_0$, where $H \cong \mathfrak{S}_{4}$ and $4\mathbb{D}_4 \subset \Gamma(P(X) \cap S) = P(S)$. By the classification of Niemeier lattices and our calculations above, $X$ can be marked only by $N_{19} = N(6D_4)$ for such a sublattice $S \subset S_X$. 
Case 20. For the Niemeier lattice

\[ N = N_{20} = N(6A_4) = [6A_4, [1(01441)]] = [6A_4, \varepsilon_{11} + \varepsilon_{13} - \varepsilon_{14} - \varepsilon_{15} + \varepsilon_{16}, \varepsilon_{11} + \varepsilon_{12} + \varepsilon_{14} - \varepsilon_{15} - \varepsilon_{16}, \varepsilon_{11} - \varepsilon_{12} - \varepsilon_{13} + \varepsilon_{14} + \varepsilon_{16}, \varepsilon_{11} + \varepsilon_{12} - \varepsilon_{13} - \varepsilon_{14} + \varepsilon_{15}] \]

the group \( A = A(N_{20}) \) contains a cyclic group \([\varphi]\) of order 2 that preserves the connected components of \(6A_4\), and \(A/[\varphi] = \mathfrak{S}_5\) acts faithfully by permutations on the 6 connected components of \(6A_4\) (see [10], Ch. 16, 18). The group \( \mathfrak{S}_5 \) acts on the following five triples of classes of elements in \( N_{20}/6A_4 \):

\[ \tilde{1} = \{2\varepsilon_{11} - 2\varepsilon_{12} + \varepsilon_{14} + \varepsilon_{15}, \varepsilon_{13} + 2\varepsilon_{14} - 2\varepsilon_{15} - \varepsilon_{16}, 2\varepsilon_{11} + 2\varepsilon_{12} - \varepsilon_{13} - \varepsilon_{16} \}, \]

\[ \tilde{2} = \{2\varepsilon_{11} - \varepsilon_{12} + 2\varepsilon_{13} - \varepsilon_{14}, \varepsilon_{12} - \varepsilon_{14} - 2\varepsilon_{15} + 2\varepsilon_{16}, 2\varepsilon_{11} - 2\varepsilon_{13} + \varepsilon_{15} + \varepsilon_{16} \}, \]

\[ \tilde{3} = \{2\varepsilon_{11} - \varepsilon_{13} + 2\varepsilon_{14} - \varepsilon_{15}, \varepsilon_{12} - 2\varepsilon_{13} + 2\varepsilon_{15} - \varepsilon_{16}, \varepsilon_{11} - \varepsilon_{12} + 2\varepsilon_{14} - \varepsilon_{16} \}, \]

\[ \tilde{4} = \{-2\varepsilon_{11} - \varepsilon_{12} - \varepsilon_{13} + 2\varepsilon_{15}, \varepsilon_{12} - \varepsilon_{13} + 2\varepsilon_{14} - 2\varepsilon_{15}, 2\varepsilon_{11} - \varepsilon_{14} + 2\varepsilon_{15} - \varepsilon_{16} \}, \]

\[ \tilde{5} = \{2\varepsilon_{11} + \varepsilon_{13} + \varepsilon_{14} - 2\varepsilon_{15}, \varepsilon_{12} + 2\varepsilon_{13} - 2\varepsilon_{14} - \varepsilon_{15}, \varepsilon_{11} - 2\varepsilon_{12} - 2\varepsilon_{15} - \varepsilon_{16} \}. \]

It is easy to give their exact definition.

The group \( A = A(N_{20}) \) is determined by its action on the ends of the components of \(6A_4\). This group is generated by the involution

\[ \varphi = (\alpha_{11}\alpha_{41})(\alpha_{12}\alpha_{42})(\alpha_{13}\alpha_{43})(\alpha_{14}\alpha_{44})(\alpha_{15}\alpha_{45})(\alpha_{16}\alpha_{46}) \]

and the corresponding transpositions of the elements \( \tilde{1}, \ldots, \tilde{5} \):

\[ (\tilde{12}) = (\alpha_{11}\alpha_{16}\alpha_{42}\alpha_{46})(\alpha_{12}\alpha_{42}\alpha_{46})(\alpha_{13}\alpha_{44}\alpha_{43}\alpha_{14}), \]

\[ (\tilde{23}) = (\alpha_{11}\alpha_{16}\alpha_{41}\alpha_{46})(\alpha_{12}\alpha_{13}\alpha_{42}\alpha_{43})(\alpha_{14}\alpha_{45}\alpha_{44}\alpha_{15}), \]

\[ (\tilde{34}) = (\alpha_{11}\alpha_{12}\alpha_{41}\alpha_{42})(\alpha_{13}\alpha_{14}\alpha_{43}\alpha_{44})(\alpha_{15}\alpha_{46}\alpha_{45}\alpha_{16}), \]

\[ (\tilde{45}) = (\alpha_{11}\alpha_{13}\alpha_{41}\alpha_{43})(\alpha_{12}\alpha_{16}\alpha_{42}\alpha_{46})(\alpha_{14}\alpha_{15}\alpha_{44}\alpha_{45}), \]

which have order 4 in \( A \).

For \( H = [\varphi] \) we have \( \text{rk} N_H = 12 \) and \( (N_H)^*/N_H \cong (\mathbb{Z}/2\mathbb{Z})^{12} \). By Theorem 4.1, \( N_H \) has no primitive embeddings in \( L_{K3} \) and \( H = [\varphi] \) is not a KahK3-subgroup. It follows that the canonical projection \( \pi: A \to \mathfrak{S}_5 \) establishes an isomorphism \( \pi|H: H \to \pi(H) = G \subset \mathfrak{S}_5 \) for every KahK3-subgroup \( H \subset A \). Then we write \( H = \tilde{G} \). Consider all possible \( G \) and \( H \) for KahK3-subgroups \( H \subset A \) up to conjugacy (in \( \mathfrak{S}_5 \) and \( A \)).

We have

\[ (\tilde{12})^2 = ((\tilde{12})(\tilde{23}))^3 = ((\tilde{12})(\tilde{23})(\tilde{34})(\tilde{45})\varphi)^5 = ((\tilde{12})(\tilde{34})(\tilde{45}))^6 = \varphi. \]

It follows that

\[ [(\tilde{12})], [(\tilde{12})\varphi], [(\tilde{12})(\tilde{23})] = [(\tilde{12})], \]

\[ [(\tilde{12})(\tilde{23})(\tilde{34})(\tilde{45})\varphi] = [(\tilde{12})(\tilde{34})], [(\tilde{12})(\tilde{34})(\tilde{45})], [(\tilde{12})(\tilde{34})(\tilde{45})\varphi] = [(\tilde{12})(\tilde{34})] \]

are not KahK3-subgroups.
The subgroup $H = [(12)(34)] \cong C_2$ has $\text{rk} N_H = 12$ and $(N_H)^*/N_H \cong (\mathbb{Z}/2\mathbb{Z})^{12}$. By Theorem 4.1, it is not a KahK3-subgroup.

Using these calculations and the known list of conjugacy classes of subgroups of $\mathfrak{S}_5$, we get the following list of all KahK3-subgroups of $A = A(6A_4)$ up to conjugacy.

1) $H = [(12)(23)(34)(45), (45)(23)(34)] \cong \text{Hol}(C_5)$ and $H = [(12)(23)(34)(45), (45)(23)(34)] \cong \text{Hol}(C_5)$ with $\text{rk} N_H = 18$ and $(N_H)^*/N_H \cong (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/5\mathbb{Z}$.

2) $H = [(12)(23)(34)(45), (45)(23)(34)(23)(45)(34)] = [(12345), (25)(34)] \cong \mathfrak{D}_{10}$ with $\text{rk} N_H = 16$ and $(N_H)^*/N_H \cong (\mathbb{Z}/5\mathbb{Z})^4$.

3) $H = [(12)(23)(34)(45)] = [(12345)] \cong C_5$ with $\text{Clos}(H) = [(12)(23)(34)(45), (45)(23)(34)(23)(45)(34)] \cong \mathfrak{D}_{10}$ as above and $\text{rk} N_H = 16$, $(N_H)^*/N_H \cong (\mathbb{Z}/5\mathbb{Z})^4$.

4) $H = [(12)(23)(34)] = [(1234)] \cong C_4$ and $H = [(12)(23)(34)] \varphi = [(1234)] \cong C_4$ with $\text{rk} N_H = 14$ and $(N_H)^*/N_H \cong (\mathbb{Z}/4\mathbb{Z})^4 \times (\mathbb{Z}/2\mathbb{Z})^2$.

5) $H = [(12)(34)] = [(12)(34)] \cong C_2$ with $\text{rk} N_H = 8$ and $(N_H)^*/N_H \cong (\mathbb{Z}/2\mathbb{Z})^8$.

Let $X$ be marked by a primitive sublattice $S \subset N = N_{20} = N(6A_4)$. Then $S$ must satisfy Theorem 4.1 and $\Gamma(P(S)) \subset \Gamma(P(N_{20})) = 6A_4$. Any such $S$ gives a marking of some $X$, and $P(X) \cap S = P(S)$.

If $N_H \subset S$ for some $H$ of type 1) $(H \cong \text{Hol}(C_5))$, 2) $(H \cong \mathfrak{D}_{10})$ or 4) $(H \cong C_4)$, then $\text{Aut}(X, S)_0 = H$. If this is not the case but $N_H \subset S$ for some $H$ of type 5) $(H \cong C_2)$, then $\text{Aut}(X, S)_0 = H \cong C_2$. Otherwise the group $\text{Aut}(X, S)_0$ is trivial.

Suppose that $H = [(12)(23)(34)] \cong C_4$ has type 4) and put $S = [(A_4)_{1} = [\alpha_{11}, \ldots, \alpha_{41}], A_2 = [\alpha_{22}, \alpha_{32}], N_H]_{\text{pr}} \subset N_{20}$. Then $\text{rk} S = 19$ and $S^*/S \cong \mathbb{Z}/10\mathbb{Z}$. Hence $S$ satisfies Theorem 4.1 and gives a marking of some surface $X$. We have $H \subset \text{Aut}(X, S)_0$, where $H \cong C_4$, and $4A_4A_2 \subset \Gamma(P(X) \cap S) = P(S)$, where $H \cong C_4$ acts transitively on the four components of $4A_4$ and permutes the elements $\alpha_{22}, \alpha_{32}$ of the component $A_2$. By the classification of Niemeier lattices and our calculations above, $X$ can be marked only by $N_{20} = N(6A_4)$ for such a sublattice $S \subset S_X$.

Case 21. For the Niemeier lattice $A = A(N_{21})$ contains a cyclic group $[\varphi]$ of order 2 that preserves the connected components of $8A_3$, and $A/[\varphi] = \text{Aff}(3, \mathbb{F}_2)$ is the affine group of the 3-dimensional affine space over $\mathbb{F}_2$, which is given by the eight components $(A_3)_i, 1 \leq i \leq 8$, of the diagram $8A_3$. Here a 4-element subset $\{i, j, k, l\} \subset \{1, \ldots, 8\}$ is a plane in the affine space if and only if $2(\varepsilon_{1i} + \varepsilon_{1j} + \varepsilon_{1k} + \varepsilon_{1l}) \in N/8A_3$. For example, $\{1, 5, 7, 8\}, \{1, 2, 6, 8\}, \{1, 2, 3, 7\}, \{1, 3, 4, 8\}, \{1, 2, 4, 5\}, \{1, 3, 5, 6\}$ and $\{1, 4, 6, 7\}$ are all the planes containing 1. Thus one can take affine coordinates with $1 = (000), 2 = (100), 3 = (010), 4 = (001), 5 = (101), 6 = (111), 7 = (110)$,
Here $T_{ij}(i) = j$ and $\pi(\tilde{T}_{ij}) = T_{ij}$ for the canonical homomorphism $\pi: A \to \text{Aff}(3, \mathbb{F}_2)$.

The subgroup $H = [\varphi] \cong C_2$ has $\text{rk} N_H = 8$ and $(N_H)^*/N_H \cong (\mathbb{Z}/2\mathbb{Z})^8$. Thus $N_H$ satisfies Theorem 4.1 and $H = [\varphi] \cong C_2$ is a KahK3-subgroup.

The subgroup $H = [\tilde{T}_{12}] \cong C_2$ has $\text{rk} N_H = 12$ and $(N_H)^*/N_H \cong (\mathbb{Z}/2\mathbb{Z})^{12}$. Thus $N_H$ does not satisfy Theorem 4.1 and $H = [\tilde{T}_{12}] \cong C_2$ is not a KahK3-subgroup. The same holds for the subgroup $H = [\varphi \tilde{T}_{12}] \cong C_2$.

It follows that all the KahK3-subgroups $H \subset A$ have trivial translation part in $\text{Aff}(3, \mathbb{F}_2)$. This can be stated in the following way.

(*) Let $\pi: A \to \text{Gl}(3, \mathbb{F}_2)$ be the canonical homomorphism. Then the kernel of $\overline{\pi}|H: H \to \text{Gl}(3, \mathbb{F}_2)$ is contained in $[\varphi] \cong C_2$.

We now describe all possible KahK3-subgroups $H \subset A$ and their images $G = \pi(H)$ under the canonical homomorphism $\pi: A \to \text{Aff}(3, \mathbb{F}_2)$.

Suppose that $H \subset A$ is a KahK3-subgroup, that is, $N_H$ satisfies Theorem 4.1. Then $\text{rk} N_H \leq 19$. It follows that the group $G = \pi(H)$ must have at least two orbits on $\{1, \ldots, 8\}$. We claim that $G$ has a one-element orbit.

Indeed, assume that $G$ has a 2-element orbit $\{1, 2\}$ and no one-element orbits. Then $G$ preserves the vector $12 = (100)$, and the linear part $\overline{G}$ is contained in $\mathcal{S}_4$, where

$$\mathcal{S}_4 = \{f \in \text{Aff}(3, \mathbb{F}_2) \mid f(1) = 1, f(2) = 2\}$$

consists of matrices with eigenvalue 1 and eigenvector $\{1, 2\} = (100)$. For the eigenvalue 1 and basis $\{1, 2\} = (100), \{1, 3\} = (010), \{1, 4\} = (001)$, this is the symmetric group $\mathcal{S}_4$ acting by conjugation on the four cyclic subgroups of order 3 in $\mathcal{S}_4$, which are

$$C_{13} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad C_{23} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

$$C_{33} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad C_{43} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Thus, to construct $G$ and then $H$, we must take a subgroup $\overline{G} \subset \mathcal{S}_4$ and replace some of its generators $g$ by $T_{12}g$. The resulting group $G$ must be isomorphic to its linear part $\overline{G}$ and must have no one-element orbits on $\{1, \ldots, 8\}$.
Suppose that $\overline{G} = [F_{13}] = C_{13} \cong C_3$, where $F_{13} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The transformation $T_{12}F_{13}$ gives the permutation $(12)(358746)$, which has order 6 instead of 3. Hence we do not get a KahK3-subgroup by property (*) above.

Suppose that $\overline{G} = [F_{4}] \cong C_4$, where $F_{4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The transformation $T_{12}F_{4}$ gives the permutation $(12)(3574)$, which is of the same order 4, but has one-element orbits $\{6\}$ and $\{8\}$. Hence we do not get a KahK3-subgroup without a one-element orbit.

Suppose that $\overline{G} = [F_{12}] \cong C_2$, where $F_{12} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ is an odd element of $\mathcal{G}_4$. The transformation $T_{12}F_{12}$ gives the permutation $(12)(37)(46)(58)$. It induces translations $T_{12}$ in the plane $\{1, 2, 3, 7\}$ and $T_{46}$ in the parallel plane $\{4, 5, 6, 8\}$ and lifts to an element

$$T_{12}\overline{F}_{12} = (\alpha_{11}\alpha_{12}\alpha_{31}\alpha_{32})(\alpha_{13}\alpha_{17}\alpha_{33}\alpha_{37})(\alpha_{14}\alpha_{36}\alpha_{34}\alpha_{16})(\alpha_{15}\alpha_{18}\alpha_{35}\alpha_{38})$$

of $A$ such that $(T_{12}\overline{F}_{12})^2 = \varphi$. The subgroup $H = [T_{12}\overline{F}_{12}] \subset A$ has $\text{rk} N_H = 16$ and $(N_H)^*/N_H \cong (\mathbb{Z}/4\mathbb{Z})^4 \times (\mathbb{Z}/2\mathbb{Z})^4$. The lattice $N_H$ does not satisfy Theorem 4.1 and $H$ is not a KahK3-subgroup.

Suppose that $\overline{G} = [F_{22}] \cong C_2$, where $F_{22} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is an even element of $\mathcal{G}_4$. The transformation $T_{12}F_{22}$ gives the permutation $(12)(45)$. It induces a translation $T_{12}$ in the plane $\{1, 2, 4, 5\}$ and acts as the identity on the parallel plane $\{3, 6, 7, 8\}$. Thus it has one-element orbits and does not give a KahK3-subgroup $H$ without a one-element orbit.

Suppose that $\overline{G} = [F_{12}, F_{12}'] \subset \mathcal{G}_4$ is a Klein non-normal subgroup, where $F_{12}$ and $F_{12}'$ are commuting odd transpositions. To obtain the group $G$, one must replace one of the generators $F_{12}$, $F_{12}'$ by one of the elements $T_{12}F_{12}$, $T_{12}F_{12}'$. We have seen above that this does not give a KahK3-subgroup.

Suppose that $\overline{G} = [F_{22}, F_{22}'] \subset \mathcal{G}_4$ is the Klein normal subgroup, where $F_{22}$ and $F_{22}'$ are distinct even elements of order 2. To obtain the group $G$, one must replace one of the generators $F_{22}$, $F_{22}'$ by one of the elements $T_{12}F_{22}$, $T_{12}F_{22}'$. But then our considerations above show that $G$ has one-element orbits, and we do not get a KahK3-subgroup without a one-element orbit.

Suppose that $\overline{G} = [F_{4}, F_{12}'] \cong \mathcal{D}_8$, where $F_{4}$ is the matrix above and $F_{12}' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ is an odd element of order 2 in $\mathcal{G}_4$. We have seen that $G$ is $[T_{12}F_{4}, F_{12}']$. Here $T_{12}F_{4}$ (resp. $F_{12}'$) determines the transposition $(12)(3574)$ (resp. $(34)(57)$). The group $G$ has one-element orbits $\{6\}$ and $\{8\}$. Thus we do not get a KahK3-subgroup without a one-element orbit.

Suppose that $\overline{G} = [F_{13}, F_{12}'] \cong \mathcal{S}_3$, where $F_{13}$ is the matrix above and $F_{12}' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ is an odd element of order 2 in $\mathcal{G}_4$ such that $F_{12}'F_{13}F_{12}' = (F_{13})^2$. We have seen that none of the generators $F_{13}$, $F_{12}'$ can be replaced by $T_{12}F_{13}, T_{12}F_{12}'$ respectively because this does not give a KahK3-subgroup. Thus we do not get a KahK3-subgroup without a one-element orbit.

Suppose that $\overline{G} = [F_{13}, F_{13}'] \cong \mathcal{A}_4$, where $F_{13}$ is the matrix above and $F_{13}'$ is another matrix of order 3. We have seen that neither of the generators $F_{13}, F_{13}'$ can be replaced by $T_{12}F_{13}, T_{12}F_{13}'$ respectively because this does not give a KahK3-subgroup. Thus we do not get a KahK3-subgroup without a one-element orbit.
Suppose that $\overline{G} = [F_{12}, F_{12}', F_{12}'] = \mathcal{S}_4$, where $F_{12}, F_{12}', F_{12}'$ are odd elements of order 2. We have seen that none of the generators $F_{12}, F_{12}', F_{12}'$ can be replaced by $T_{12}F_{12}, T_{12}F_{12}', T_{12}F_{12}'$ respectively because this does not give a Kahlerian K3-subgroup. Thus we do not get a Kahlerian K3-subgroup without a one-element orbit.

As a result, we have proved that $G$ must have a one-element orbit whenever it has a 2-element orbit.

We now assume that $G$ has no one-element orbits and no 2-element orbits, but there is a 3-element orbit. These three elements span a 4-element plane. The remaining element of the plane is then a one-element orbit, a contradiction.

We now assume that $G$ has two 4-element orbits.

First, suppose that these orbits are in general position, that is, they are not planes. We can assume that one of the orbits is $\{1, 2, 3, 4\}$. Then $G$ must contain an even transposition $g$ of order 2 on $\{1, 2, 3, 4\}$. We can assume that this is $(13)(24)$. Hence it acts on all elements as $(13)(24)(78)(65)$ and lifts to an element

$$g = (\alpha_{11}\alpha_{13}\alpha_{31}\alpha_{33})(\alpha_{12}\alpha_{14}\alpha_{32}\alpha_{34})(\alpha_{17}\alpha_{38}\alpha_{37}\alpha_{18})(\alpha_{16}\alpha_{15}\alpha_{36}\alpha_{35})$$

of $A$ such that $g^2 = \varphi$. Then $H = [g]$ must be a Kahlerian K3-subgroup. We have $\text{rk } N_H = 16$ and $(N_H)^*/N_H \cong (\mathbb{Z}/4\mathbb{Z})^4 \times (\mathbb{Z}/2\mathbb{Z})^4$. By Theorem 4.1, $H$ cannot be a Kahlerian K3-subgroup and we get a contradiction.

Second, suppose that these 4-element orbits are parallel planes. We can assume that they are $\{1, 3, 5, 6\}$ and $\{2, 4, 7, 8\}$. Then $G$ must contain an even permutation $g$ of order 2 on $\{1, 3, 5, 6\}$. We can assume that $g$ is $(13)(65)$. If $g$ is not the identity on $\{2, 4, 7, 8\}$ and, for example, $g(2) = 4$, then $g = (13)(24)(78)(65)$. As above, we get a contradiction. Thus $g$ must be equal to the identity on $\{2, 4, 7, 8\}$ and, therefore, $g = (13)(56)$. Considering the other plane $\{2, 4, 7, 8\}$, we similarly construct an even permutation $g' \in G$ of order 2 on $\{2, 4, 7, 8\}$, which is the identity on $\{1, 3, 5, 6\}$. For example, we can assume that $g' = (24)(78)$. Then $gg' = (13)(24)(78)(65)$ and we get a contradiction as above.

Furthermore, we define a group

$$\text{Gl}(3, \mathbb{F}_2) = \{ f \in \text{Aff}(3, \mathbb{F}_2), \ f(1) = 1 \}$$

of order $2^3 \cdot 3 \cdot 7 = 168$. Hence we identify zero with $1 = (000)$ and use the basis $2 = (100), 3 = (010)$ and $4 = (001)$. Consider all possible subgroups $G \subset \text{Gl}(3, \mathbb{F}_2)$ and their lifts to Kahlerian K3-subgroups $H \subset A$ up to conjugacy. We have seen that they give all the Kahlerian K3-conjugacy classes of subgroups of $A$.

Given an element $F \in \text{Gl}(3, \mathbb{F}_2)$, we write $\tilde{F}$ for its lift to an element $\tilde{F} \subset A$ such that $\tilde{F}(\alpha_{11}) = \alpha_{11}$. Given a subgroup $G \subset \text{Gl}(3, \mathbb{F}_2)$, we put

$$\tilde{G} = \{ \tilde{F} \mid F \in G \}.$$

Clearly, $\pi|\tilde{G} : \tilde{G} \cong G$ is an isomorphism.

Consider the following elements of $\text{Gl}(3, \mathbb{F}_2)$ and $A$:

$$F_{17} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = (2345678),$$

$$\tilde{F}_{17} = (\alpha_{12}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18})(\alpha_{32}\alpha_{33}\alpha_{34}\alpha_{35}\alpha_{36}\alpha_{37}\alpha_{38})$$

of order 7;
\[ F_{14} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = (3475)(68), \]
\[ \tilde{F}_{14} = (\alpha_{13}\alpha_{14}\alpha_{37}\alpha_{15})(\alpha_{33}\alpha_{34}\alpha_{17}\alpha_{35})(\alpha_{16}\alpha_{18}\alpha_{36}\alpha_{38}), \]
\[ F_{24} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = (2346)(78), \]
\[ \tilde{F}_{24} = (\alpha_{12}\alpha_{13}\alpha_{34}\alpha_{16})(\alpha_{32}\alpha_{33}\alpha_{14}\alpha_{36})(\alpha_{15}\alpha_{35})(\alpha_{17}\alpha_{38}\alpha_{37}\alpha_{18}) \]
of order 4;
\[ F_{13} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = (346)(587), \]
\[ \tilde{F}_{13} = (\alpha_{13}\alpha_{14}\alpha_{16})(\alpha_{33}\alpha_{34}\alpha_{36})(\alpha_{15}\alpha_{18}\alpha_{17})(\alpha_{35}\alpha_{38}\alpha_{37}) \]
such that \( F_{13}F_{17}(F_{13})^{-1} = (F_{17})^2 \),
\[ F_{23} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = (234)(578), \]
\[ \tilde{F}_{23} = (\alpha_{12}\alpha_{33}\alpha_{34})(\alpha_{32}\alpha_{13}\alpha_{14})(\alpha_{15}\alpha_{17}\alpha_{38})(\alpha_{35}\alpha_{37}\alpha_{18}) \]
of order 3;
\[ F_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = (48)(56), \]
\[ \tilde{F}_{12} = (\alpha_{12}\alpha_{32})(\alpha_{14}\alpha_{38})(\alpha_{34}\alpha_{18})(\alpha_{15}\alpha_{16})(\alpha_{35}\alpha_{36})(\alpha_{17}\alpha_{37}) \]
of order 2.

Considering all possible subgroups \( G \subset \text{Gl}(3, \mathbb{F}_2) \) and their lifts to subgroups of \( A \), we get the following KahK3-subgroups of \( A \) up to conjugacy.

(II.1) \( H = [\tilde{F}_{17}, \tilde{F}_{14}] = \tilde{\text{Gl}}(3, \mathbb{F}_2) \cong L_2(7) = \text{PSL}(2, \mathbb{F}_7) \) with \( \text{rk} N_H = 19 \) and \( (N_H)^*/N_H \cong \mathbb{Z}/28\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z} \).

(II.2) \( H = [\tilde{F}_{17}, \tilde{F}_{13}] = [\tilde{F}_{17}] \times [\tilde{F}_{13}] \cong C_7 \times C_3 \) with \( \text{rk} N_H = 18 \) and \( (N_H)^*/N_H \cong (\mathbb{Z}/7\mathbb{Z})^3 \).

(II.2a) \( H = [\tilde{F}_{17}] \cong C_7 \) with \( \text{Clos}(H) = [\tilde{F}_{17}, \tilde{F}_{13}] \cong C_7 \times C_3 \) as above and \( \text{rk} N_H = 18, (N_H)^*/N_H \cong (\mathbb{Z}/7\mathbb{Z})^3 \).

Cases (II.1)–(II.2a) describe all subgroups \( H \) with 7-element orbit \( \{2, 3, 4, 5, 6, 7, 8\} \) (equivalently, the order of \( H \) is divisible by 7).
The following cases describe all subgroups $H$ with one-element orbits \{1\} and \{2\}.

(I.1) \( H = [\varphi, \F_{14}, \F_{13}] = [\varphi] \times [\F_{14}, \F_{13}] \cong C_2 \times \mathfrak{A}_4 \) with \( \text{rk} N_H = 18 \), \((N_H)^*/N_H \cong (\mathbb{Z}/12\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^2 \) and
\[
\det(K((N_H)_2)) \equiv \det\begin{pmatrix}
-75192 & 85308 & 63780 & -36 \\
85308 & -97356 & -72420 & 48 \\
63780 & -72420 & -54112 & 34 \\
-36 & 48 & 34 & -4
\end{pmatrix} = 2^6 \cdot 3^4 \cdot 23^2 \cdot 239 \equiv \pm 12^2 \cdot 2^2 \mod (\mathbb{Z}_2)^2.
\]

Thus \( N_H \) satisfies Theorem 4.1 and \( H \) is a Kah\é\-ler K3-subgroup. The following are its subgroups.

(I.2) \( H = [\varphi, \F_{13}, \F_{14}\F_{13}\F_{14}] = [\varphi] \times [\F_{13}, \F_{14}\F_{13}\F_{14}] \cong C_2 \times \mathfrak{A}_4 \) with \( \text{Clos}(H) = [\varphi, \F_{14}, \F_{13}] \cong C_2 \times \mathfrak{S}_4 \) as above, \( \text{rk} N_H = 18 \) and \((N_H)^*/N_H \cong (\mathbb{Z}/12\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^2 \).

(I.3) \( H = [\varphi, \F_{12}, \F_{14}\F_{13}\F_{14}] = [\varphi] \times [\F_{12}, \F_{14}\F_{13}\F_{14}] \cong C_2 \times \mathfrak{D}_6 \cong \mathfrak{D}_{12} \) with \( \text{rk} N_H = 16 \) and \((N_H)^*/N_H \cong (\mathbb{Z}/6\mathbb{Z})^4 \).

(I.4) \( H = [\varphi\F_{14}\F_{13}\F_{14}] \cong C_6 \) with \( \text{Clos}(H) = [\varphi, \F_{12}, \F_{14}\F_{13}\F_{14}] \cong C_2 \times \mathfrak{D}_6 \cong \mathfrak{D}_{12} \) as above, \( \text{rk} N_H = 16 \) and \((N_H)^*/N_H \cong (\mathbb{Z}/6\mathbb{Z})^4 \).

(I.5) \( H = [\varphi, \F_{13}\F_{14}(\F_{13})^{-1}, \F_{12}] = [\varphi] \times [\F_{13}\F_{14}(\F_{13})^{-1}, \F_{12}] \cong C_2 \times \mathfrak{S}_8 \) with \( \text{rk} N_H = 14 \) and \((N_H)^*/N_H \) isomorphic \(^4\) to \((\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^2 \).

(I.6) \( H = [\varphi, \F_{13}(\F_{14})^2(\F_{13})^{-1}, \F_{12}] = [\varphi] \times [\F_{13}(\F_{14})^2(\F_{13})^{-1}, \F_{12}] \cong (C_2)^3 \) with \( \text{rk} N_H = 14 \) and \((N_H)^*/N_H \cong (\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^6 \).

(I.7) \( H = [\varphi, \F_{13}\F_{14}(\F_{13})^{-1}, \F_{13}\F_{14}(\F_{13})^{-1}\F_{12}] = [\varphi] \times [\F_{13}(\F_{14})^2(\F_{13})^{-1}, \F_{13}\F_{14}(\F_{13})^{-1}\F_{12}] \cong (C_2)^3 \) with \( \text{rk} N_H = 14 \) and \((N_H)^*/N_H \cong (\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^6 \).

(I.8) \( H = [\varphi, \F_{13}\F_{14}(\F_{13})^{-1}] = [\varphi] \times [\F_{13}\F_{14}(\F_{13})^{-1}] \cong C_2 \times C_4 \) with \( \text{Clos}(H) = [\varphi, \F_{13}\F_{14}(\F_{13})^{-1}, \F_{12}] \cong C_2 \times \mathfrak{S}_8 \) as above, \( \text{rk} N_H = 16 \) and \((N_H)^*/N_H \cong (\mathbb{Z}/4\mathbb{Z})^4 \times (\mathbb{Z}/2\mathbb{Z})^2 \).

(I.9) \( H = [\varphi, \F_{12}] = [\varphi] \times [\F_{12}] \cong (C_2)^2 \) with \( \text{rk} N_H = 12 \) and \((N_H)^*/N_H \cong (\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^6 \).

(I.10) \( H = [\varphi] \cong C_2 \) with \( \text{rk} N_H = 8 \) and \((N_H)^*/N_H \cong (\mathbb{Z}/2\mathbb{Z})^8 \).

(II.3) \( H = [\F_{14}, \F_{13}] \cong \mathfrak{S}_4 \) with \( \text{rk} N_H = 17 \) and \((N_H)^*/N_H \cong (\mathbb{Z}/12\mathbb{Z})^2 \times (\mathbb{Z}/4\mathbb{Z}) \).

(II.3') \( H = [\varphi\F_{14}, \F_{13}] \cong \mathfrak{S}_4 \) with \( \text{rk} N_H = 17 \) and \((N_H)^*/N_H \cong (\mathbb{Z}/12\mathbb{Z})^2 \times (\mathbb{Z}/4\mathbb{Z}) \).

(II.4) \( H = [\F_{13}, \F_{14}\F_{13}\F_{14}] \cong \mathfrak{A}_4 \) with \( \text{rk} N_H = 16 \) and \((N_H)^*/N_H \cong (\mathbb{Z}/12\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^2 \).

(II.5) \( H = [\F_{12}, \F_{14}\F_{13}\F_{14}] \cong \mathfrak{D}_6 \) with \( \text{rk} N_H = 14 \) and \((N_H)^*/N_H \cong (\mathbb{Z}/6\mathbb{Z})^2 \times (\mathbb{Z}/3\mathbb{Z})^3 \).

\(^4\)These calculations show that, for a symplectic group \( G = (C_2)^3 \) on a Kählerian K3 surface, we have \( S^*_{(G)} / S_{(G)} = S^*_{(2,2,2)} / S_{(2,2,2)} \cong (\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^6 \). We must therefore correct our calculation of the last group in \( [3] \), Proposition 10.1.
(II.5') \( H = [\varphi \widetilde{F}_1, 4, \widetilde{F}_4 \widetilde{F}_3 (\widetilde{F}_1)^{-1}] \cong \mathfrak{D}_6 \) with \( \operatorname{rk} N_H = 14 \) and \( (N_H)^*/N_H \cong (\mathbb{Z}/6\mathbb{Z})^2 \times (\mathbb{Z}/3\mathbb{Z})^3 \).

(II.6) \( H = [\widetilde{F}_1, 4, \widetilde{F}_3 (\widetilde{F}_1)^{-1}] \cong C_3 \) with \( \operatorname{rk} N_H = 12 \) and \( (N_H)^*/N_H \cong (\mathbb{Z}/3\mathbb{Z})^6 \).

(II.7) \( H = [\widetilde{F}_1, 3, \widetilde{F}_4 (\widetilde{F}_3)^{-1}, \widetilde{F}_1, 2] \cong \mathfrak{D}_8 \) with \( \operatorname{rk} N_H = 15 \) and \( (N_H)^*/N_H \cong (\mathbb{Z}/4\mathbb{Z})^5 \).

(II.7') \( H = [\varphi \widetilde{F}_1 \widetilde{F}_4 (\widetilde{F}_1)^{-1}, \widetilde{F}_1, 2] \cong \mathfrak{D}_8 \) with \( \operatorname{rk} N_H = 15 \) and \( (N_H)^*/N_H \cong (\mathbb{Z}/4\mathbb{Z})^5 \).

(II.7'') \( H = [\widetilde{F}_1, 3, \widetilde{F}_4 (\widetilde{F}_3)^{-1}, \varphi \widetilde{F}_1] \cong \mathfrak{D}_8 \) with \( \operatorname{rk} N_H = 15 \) and \( (N_H)^*/N_H \cong (\mathbb{Z}/4\mathbb{Z})^5 \).

(II.8) \( H = [\widetilde{F}_1, 3, (\widetilde{F}_1)^2 (\widetilde{F}_3)^{-1}, \widetilde{F}_1, 2] \cong (C_2)^2 \) with \( \operatorname{rk} N_H = 12 \) and \( (N_H)^*/N_H \cong (\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^6 \) \( H \) is a Klein non-normal subgroup of \( \mathfrak{S}_4 \).

(II.8') \( H = [\varphi \widetilde{F}_1, 3, (\widetilde{F}_1)^2 (\widetilde{F}_3)^{-1}, \widetilde{F}_1, 2] \cong (C_2)^2 \) with \( \operatorname{rk} N_H = 12 \) and \( (N_H)^*/N_H \cong (\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^6 \).

(II.8'') \( H = [\varphi \widetilde{F}_1, 3, (\widetilde{F}_1)^2, \varphi \widetilde{F}_1, 2] \cong (C_2)^2 \) with \( \operatorname{rk} N_H = 12 \) and \( (N_H)^*/N_H \cong (\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^6 \).

(II.9) \( H = [\widetilde{F}_1, 3, (\widetilde{F}_1)^2 (\widetilde{F}_3)^{-1}, \varphi \widetilde{F}_1 (\widetilde{F}_3)^{-1}, \widetilde{F}_1, 2] \cong (C_2)^2 \) with \( \operatorname{rk} N_H = 12 \) and \( (N_H)^*/N_H \cong (\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^6 \).

(II.9') \( H = [\varphi \widetilde{F}_1, 3, (\widetilde{F}_1)^2, \varphi \widetilde{F}_1 (\widetilde{F}_3)^{-1}, \varphi \widetilde{F}_1, 2] \cong (C_2)^2 \) with \( \operatorname{rk} N_H = 12 \) and \( (N_H)^*/N_H \cong (\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^6 \).

(II.10) \( H = [\widetilde{F}_1] \cong C_4 \) with \( \operatorname{rk} N_H = 14 \) and \( (N_H)^*/N_H \cong (\mathbb{Z}/4\mathbb{Z})^4 \times (\mathbb{Z}/2\mathbb{Z})^2 \).

(II.10') \( H = [\varphi \widetilde{F}_1] \cong C_4 \) with \( \operatorname{rk} N_H = 14 \) and \( (N_H)^*/N_H \cong (\mathbb{Z}/4\mathbb{Z})^4 \times (\mathbb{Z}/2\mathbb{Z})^2 \).

(II.11) \( H = [\widetilde{F}_1, 2] \cong C_2 \) with \( \operatorname{rk} N_H = 8 \) and \( (N_H)^*/N_H \cong (\mathbb{Z}/2\mathbb{Z})^8 \).

(II.11') \( H = [\varphi \widetilde{F}_1, 2] \cong C_2 \) with \( \operatorname{rk} N_H = 8 \) and \( (N_H)^*/N_H \cong (\mathbb{Z}/2\mathbb{Z})^8 \).

The following cases describe all subgroups \( H \) that have a unique one-element orbit \( \{1\} \) and a 4-element orbit \( \{2, 3, 4, 6\} \) which is a plane.

(III.1) \( H = [\varphi, \widetilde{F}_2, 4, \widetilde{F}_2, 3] = [\varphi] \times [\widetilde{F}_2, 4, \widetilde{F}_2, 3] \cong C_2 \times \mathfrak{S}_4 \) with \( \operatorname{rk} N_H = 18 \), \( (N_H)^*/N_H \cong (\mathbb{Z}/12\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^2 \) and

\[
\det(K((N_H)^2)) \equiv \det \begin{pmatrix}
-14460 & -12168 & -6276 & -48 \\
-12168 & -11724 & -6072 & -36 \\
6276 & -6072 & -3148 & -18 \\
-48 & -36 & -18 & -4
\end{pmatrix}
= 2^6 \cdot 3^2 \cdot 41 \cdot 9767 \equiv \pm 12^2 \cdot 2^2 \mod (\mathbb{Z}/2)^2.
\]

Thus \( N_H \) satisfies Theorem 4.1 and \( H \) is a KahK3-subgroup. The following cases are its subgroups.

(III.2) \( H = [\varphi, (\widetilde{F}_2)^2, \widetilde{F}_2, 3] = [\varphi] \times [(\widetilde{F}_2)^2, \widetilde{F}_2, 3] \cong C_2 \times \mathfrak{A}_4 \) with \( \operatorname{Clos}(H) = [\varphi, \widetilde{F}_2, 4, \widetilde{F}_2, 3] \cong C_2 \times \mathfrak{S}_4 \) as above, \( \operatorname{rk} N_H = 18 \) and \( (N_H)^*/N_H \cong (\mathbb{Z}/12\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^2 \).

(IV.1) \( H = [\widetilde{F}_2, 4, \widetilde{F}_2, 3] \cong \mathfrak{S}_4 \) with \( \operatorname{rk} N_H = 17 \) and \( (N_H)^*/N_H \cong (\mathbb{Z}/12\mathbb{Z})^2 \times \mathbb{Z}/4\mathbb{Z} \).
(IV.1') $H = [\varphi \tilde{F}_2^4, \tilde{F}_2^3] \cong \mathcal{G}_4$ with $\text{rk} N_H = 17$ and $(N_H)^*/N_H \cong (\mathbb{Z}/12\mathbb{Z})^2 \times \mathbb{Z}/4\mathbb{Z}$.

(IV.2) $H = [(\tilde{F}_2^4)^2, \tilde{F}_2^3] \cong \mathfrak{A}_4$ with $\text{rk} N_H = 16$ and $(N_H)^*/N_H \cong (\mathbb{Z}/12\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^2$.

(IV.2') $H = [\varphi (\tilde{F}_2^4)^2, \tilde{F}_2^3] \cong \mathfrak{A}_4$ with $\text{rk} N_H = 16$ and $(N_H)^*/N_H \cong (\mathbb{Z}/12\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^2$.

Let $X$ be marked by a primitive sublattice $S \subset N = N_{21} = N(8A_3)$. Then $S$ must satisfy Theorem 4.1 and $\Gamma(P(S)) \subset \Gamma(P(N_{21})) = 8A_3$. Any such $S$ gives a marking of some $X$, and $\text{rk}(X) \cap S = P(S)$.

If $N_H \subset S$ for some $H$ of type (I.1), (III.1) $(H \cong C_2 \times \mathcal{G}_4)$ or (II.1) $(H \cong L_2(7) \cong \text{GL}(3, \mathbb{F}_2))$, then $\text{Aut}(X,S)_0 = H$. If this is not the case but $N_H \subset S$ for some $H$ of type (I.3) $(H \cong C_2 \times \mathcal{D}_6 \cong \mathcal{D}_{12})$, (I.5) $(H \cong C_2 \times \mathcal{D}_8)$ or (II.2) $(H \cong C_7 \times C_3)$, then $\text{Aut}(X,S)_0 = H$. If this is not the case but $N_H \subset S$ for some $H$ of type (I.6), (I.7) $(H \cong (C_2)^3)$, (II.3), (II.3'), (IV.1) or (IV.1') $(H \cong \mathcal{G}_4)$, then $\text{Aut}(X,S)_0 = H$. If this is not the case but $N_H \subset S$ for some $H$ of type (I.9) $(H \cong (C_2)^2)$, (II.4), (IV.2), (IV.2') $(H \cong \mathfrak{A}_4)$, (II.7), (II.7'), (II.7'') or (II.7''') $(H \cong \mathcal{D}_8)$, then $\text{Aut}(X,S)_0 = H$. If this is not the case but $N_H \subset S$ for some $H$ of type (I.10) $(H \cong C_2)$, (II.5) or (II.5') $(H \cong \mathcal{D}_6)$, then $\text{Aut}(X,S)_0 = H$. If this is not the case but $N_H \subset S$ for some $H$ of type (II.6) $(H \cong C_3)$, (II.8), (II.8'), (II.8''), (II.9), (II.9') $(H \cong (C_2)^2)$, (II.10) or (II.10') $(H \cong C_4)$, then $\text{Aut}(X,S)_0 = H$. If this is not the case but $N_H \subset S$ for some $H$ of type (II.11) or (II.11') $(H \cong C_2)$, then $\text{Aut}(X,S)_0 = H$. Otherwise the group $\text{Aut}(X,S)_0$ is trivial.

Suppose that $H = [\tilde{F}_2^4, \tilde{F}_2^3] \cong \mathcal{G}_4$ has type (IV.1) and put $S = [(A_3)_2 = [\alpha_{12}, \alpha_{22}, \alpha_{32}], N_H]_{pr} \subset N_{21}$. We have $\text{rk} S = 19$ and $S^*/S \cong \mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. Hence $S$ satisfies Theorem 4.1 and gives a marking of some surface $X$. We have $H \subset \text{Aut}(X,S)_0$, where $H \cong \mathcal{G}_4$, and $(A_3)_2(A_3)_3(A_3)_4(A_3)_6 = \Gamma(P(X) \cap S) = P(S)$. By the classification of Niemeier lattices and our calculations above, $X$ can be marked only by $N_{21} = N(8A_3)$ for such a sublattice $S \subset S_X$.

We now consider the cases when $\text{Aut}(X,S)_0$ can be very large.

**Case 22.** Suppose that the K3 surface $X$ has 18 non-singular rational curves forming a graph $9A_2$. Barth [18] described the primitive sublattice $\tilde{T} \subset S_X$ generated by the classes

$$\alpha_{11}, \alpha_{12}, \ldots, \alpha_{19}, \alpha_{29}$$

of these curves. In particular, $\text{rk} \tilde{T} = 18$ and $\tilde{T}^*/\tilde{T} \cong (\mathbb{Z}/3\mathbb{Z})^3$.

Consider a marking $S \subset N_i$ of the surface $X$ by a lattice $S$ containing the primitive sublattice $\tilde{T} \subset S$. It was shown in [18] that

$$\text{rk}(X) \cap S = \{\alpha_{11}, \alpha_{12}, \ldots, \alpha_{19}, \alpha_{29}\}. $$

Hence $9A_2$ must be a subgraph of the Dynkin diagram of $N_i$, and the corresponding sublattice $\tilde{T} \subset N_i$ is primitive. By the classification of Niemeier lattices, this is possible only for $N(12A_2)$. Thus $X$ can be marked only by $N_{21} = N(12A_2)$.

**Case 23.** Suppose that $X$ has 16 disjoint non-singular rational curves. It is known [19] that the classes $\alpha_1, \alpha_2, \ldots, \alpha_{16}$ of these curves generate a primitive sublattice $\Pi \subset S_X$, which was described in [17] and [19], and $X$ is a Kummer surface. In particular, $\text{rk} \Pi = 16$ and $\Pi^*/\Pi \cong (\mathbb{Z}/2\mathbb{Z})^6$. 

Consider a marking $S \subset N_i$ of the surface $X$ by a lattice $S$ containing the primitive sublattice $\Pi \subset S$. It is known \[19\] that

$$P(X) \cap S = \{\alpha_1, \ldots, \alpha_{16}\}.$$ 

Hence $16A_1$ must be a subgraph of the Dynkin diagram of $N_i$, and the corresponding sublattice $\Pi \subset N_i$ is primitive. By the classification of Niemeier lattices, this is possible only for $N(24A_1)$. Thus $X$ can be marked only by $N_{23} = N(24A_1)$. In this case, $\text{Aut}(X, S)_{\theta}$ is a subgroup of

$$\text{Kum} = \{\varphi \in A(N(24A_1)) \mid \varphi(\Pi) = \Pi\}.$$ 

Some of these subgroups were considered in \[20\] among other things.

Our considerations finally give the following result.

**Theorem 6.1.** For every Niemeier lattice $N_i$, $i = 1, 2, 3, 5, \ldots, 9, 11, \ldots, 23$, there is a Kählerian K3 surface $X$ such that $X$ can be marked only by $N_i$.

We believe that the same result holds for the remaining Niemeier lattices $N_4, N_{10}$. By Kondō’s trick \[6\], which was mentioned in § 4, any Kählerian K3 surface marked by the Leech lattice $N_{24}$ can also be marked by one of the Niemeier lattices $N_i$, $i = 1, \ldots, 23$. On the other hand, the marking by $N_{24}$ is natural for K3 surfaces with $P(X) \cap S$ the empty set. All Niemeier lattices are important for markings of Kählerian K3 surfaces.

We hope to consider the remaining cases $N_i$, $22 \leq i \leq 24$, in more detail in future publications.

### § 7. Appendix: programs

This section contains programs that use GP/PARI Calculator, Version 2.2.13.

**Program 0:** niemeier\general1.txt

\texttt{\ \ for Niemeier lattice Niem given by}
\texttt{\ \ the $24 \times 24$ symmetric matrix $r$ in root basis}
\texttt{\ \ represented by basic of the size $24$ vectors (0,0,...,0,1,0,...,0)\}}
\texttt{\ \ and its rational cording matrix}
\texttt{\ \ cord of size ( . \ \times 24)}
\texttt{\ \ and its sublattice SUBL given by}
\texttt{\ \ (24\times \ ) rational matrix SUBL}
\texttt{\ \ it calculates basis of its primitive sublattice}
\texttt{\ \ SUBLpr (SUBL\otimes Q)\cap Niem as matrix SUBLpr}
\texttt{\ \ and calculates invariants DSUBLpr of \ \ SUBLpr\subsetset SUBLpr*}
\texttt{\ \ and calculates the matrix rSUBLpr of SUBLpr in this}
\texttt{\ \ elementary divisors (Smith) basis SUBLpr}
\texttt{a=matrix(24,24+matsize(cord)[1]);}
\texttt{for(i=1,24,a[i,i]=1);for(i=1,matsize(cord)[1],a[i,24+i]=cord[i,]~ );}
\texttt{L=a;N=SUBL;}
\texttt{r niemeier\latt4.txt;}
\texttt{SUBLpr1=Npr;R=r;B=SUBLpr1;}
\texttt{r niemeier\latt2.txt;}
\texttt{SUBLpr=BB;DSUBLpr=D;rSUBLpr=G;
Program 1: niemeier\latt1.txt
\for a non-degenerate lattice
\L given by a symmetric integer matrix l
\in some generators
\calculates the elementary divisors (Smith) basis of L
\as a matrix b and
\calculates the matrix ll=b'^*l*b
\of L in the bases b
\calculates invariants d of L\subset L*
ww=matsnf(l,1);uu=ww[1];vv=ww[2];dd=ww[3];
nn=matsize(l)[1];nnn=nn;for(i=1,nn,if(dd[i,i]==0,nnn=nnn-1));
b=matrix(nn,nnn,X,Y,vv[X,Y+nn-nnn]);
ll=b'^*l*b;
d=vector(nnn,X,dd[X+nn-nnn,X+nn-nnn]);
kill(ww);kill(uu);kill(vv);kill(dd);kill(nn);kill(nnn);

Program 2: niemeier\latt2.txt
\for a non-degenerate lattice L
\and generators B (n x m) matrix with rational coefficients
\calculates invariants D of L\subset L* of this lattice
\the elementary divisors (Smith) basis BB of this lattice, (n x mm) matrix,
\and matrix G=BB'^*R*BB of L in this basis
l=B'^*R*B;
\r niemeier\latt1.txt;
BB=B*b;G=BB'^*R*BB;D=d;

Program 3: niemeier\latt3.txt
\for a module M
\given by rational columns
\of matrix M, it finds its basis
\as a matrix MM
\and finds matrix VV such that MM=M*VV
gg=gcd(M);M1=M/gg;
ww=matsnf(M1,1);uu=ww[1];vv=ww[2];dd=ww[3];
mm=matsize(dd)[1];nn=matsize(dd)[2];
heap=nn;for(i=1,mm,if(dd[i]==0,heap=heap-1));
VV=matrix(nn,heap);
heap=0;for(i=1,mm,if(dd[i]==0,,heap=heap+1;VV[I,heap]=vv[I]));
M2=M1*VV;MM=M2*gg;
kill(gg);kill(M1);kill(ww);kill(uu);kill(vv);kill(dd);kill(mm);
kill(nn);kill(heap);kill(vv);kill(M2);

Program 4: niemeier\latt4.txt
\for a module L given by
\a rational m x n matrix L
\such that L contains all basic columns
\(0, \ldots, 0, 1, 0, \ldots, 0)\)
and its submodule $N$ given by rational matrix $N$

it finds basis of the primitive submodule $N_{pr} = (N \otimes \mathbb{Q}) \cap L$

as the matrix $N_{pr}$

\[
ggg = \gcd(N); N1 = N/ggg;
\]

\[
M = L;
\]

\`
\r niemeier\latt3.txt;
L1 = MM; kill(VV);
\`

\[
N2 = L1^{-1} \cdot N1;
\]

\[
ww = \text{matsnf}(N2,1); uu = ww[1]; vv = ww[2]; dd = ww[3];
\]

\[
N3 = N2 \cdot vv; mm = \text{matsize}(dd)[1]; nn = \text{matsize}(dd)[2];
\]

\[
nnnn = 0;
\]

for $i = 1, nn, \text{if}(dd[i] == 0, nnn = nnn + 1; ddd = \gcd(dd[i]));$

\[
N4 = \text{matrix}(mm, nnn);
\]

\[
nnn = 0;
\]

for $i = 1, nn, \text{if}(dd[i] == 0, nnnn = nnnn + 1; ddd = \gcd(dd[i]));$

\[
N4 = N3[i]/ddd;
\]

\[
Npr = L1 \cdot N4;
\]

\`
kill(ggg); kill(N1); kill(M); kill(L1); kill(MM);
kill(N2); kill(ww); kill(un); kill(vv); kill(dd);
kill(N3); kill(mm); kill(nn); kill(nnn); kill(nnnn);
\`

\`
kill(dd); kill(N4);
\`

This completes the list of programs.

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