REPRESENTATIONS AND THE FOUNDATIONS OF MATHEMATICS

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Abstract. The representation of mathematical objects in terms of (more) basic ones is part and parcel of (the foundations of) mathematics. In the usual foundations of mathematics, i.e. ZFC set theory, all mathematical objects are represented by sets, while ordinary, i.e. non-set theoretic, mathematics is represented in the more parsimonious language of second-order arithmetic. This paper deals with the latter representation for the rather basic case of continuous functions on the reals and Baire space. We show that the logical strength of basic theorems named after Tietze, Heine, and Weierstrass changes significantly upon the replacement of ‘second-order representations’ by ‘third-order functions’. We discuss the implications and connections to the Reverse Mathematics program and its foundational claims regarding predicativist mathematics and Hilbert’s program for the foundations of mathematics. Finally, we identify the problem caused by representations of continuous functions and formulate a criterion to avoid problematic codings within the bigger picture of representations.

1. Introduction

Lest we be misunderstood, let our first order of business be to formulate the following blanket caveat:

any formalisation of mathematics generally involves some kind of representation (aka coding) of mathematical objects in terms of others.

Now, the goal of this paper is to critically examine the role of representations based on the language of second-order arithmetic; such an examination perhaps unsurprisingly involves the comparison of theorems based on second-order representations versus theorems formulated in third-order arithmetic. To be absolutely clear, we do not claim that the latter represent the ultimate mathematical truth, nor do we (wish to) downplay the role of representations in third-order arithmetic.

As to content, we briefly introduce representations based on second-order arithmetic in Section 1.1, while our main results are sketched in Section 1.2.

1.1. Representations and second-order arithmetic. The representation of mathematical objects in terms of (more) basic ones is part and parcel of (the foundations of) mathematics. For instance, in the usual foundation of mathematics, ZFC set theory, the natural numbers \( \mathbb{N} = \{0, 1, 2, \ldots\} \) can be represented via sets by the \textit{von Neumann ordinals} as follows: 0 corresponds to the empty set \( \emptyset \), and \( n + 1 \) corresponds to \( n \cup \{n\} \). The previous is more than just a simple technical
device: critique of this sort of representation has given rise to various structuralist and nominalist foundational philosophies [5].

A more conceptual example is the vastly more parsimonious (compared to set theory) language $L_2$ of \textit{second-order arithmetic}, essentially consisting of natural numbers and sets thereof. Nonetheless, this language is claimed to allow for the formalisation of ordinary or core mathematics by e.g. Simpson as follows.

The language $L_2$ comes to mind because it is just adequate to define the majority of ordinary mathematical concepts and to express the bulk of ordinary mathematical reasoning. [64, I.12]

We focus on the language of second-order arithmetic, because that language is the weakest one that is rich enough to express and develop the bulk of core mathematics. [64, Preface]

Simpson’s claims can be substantiated via a number of representations of uncountable objects as second-order objects, including (continuous) functions on $\mathbb{R}$ [64, II.6], topologies [64, I.8.2], metric spaces [64, X.1], et cetera. One usually refers to these second-order representations as ‘codes’ and the associated practice of using codes as ‘coding’. The aim of this paper is a critical investigation of this coding practice.

To be absolutely clear, we do not judge the correctness of Simpson’s above grand claims in this paper, nor will we make any general judgement as to whether coding is good or bad \textit{per se}. What we shall show is much more specific, namely that introducing codes for continuous functions in certain theorems of core or ordinary mathematics \textit{changes the minimal axioms required to prove these theorems}. The knowledgable reader of course recognises the final italicised sentence as follows.

We therefore formulate our \textit{Main Question} as follows: \textit{Which set existence axioms are needed to prove the theorems of ordinary, non-set-theoretic mathematics?} [64, I.1]; emphasis original.

Indeed, the previous question is the central motivation of the \textit{Reverse Mathematics} program (RM hereafter), founded by Friedman in [22,23]. An overview of RM may be found in [63,64], while an introduction for the ‘scientist in the street’ is [66]. We provide a brief overview of Reverse Mathematics, including Kohlenbach’s higher-order variety, in Section A.

Thus, if one is interested in the Main Question of RM, it seems \textit{imperative} that the aforementioned coding practice does not change the minimal axioms needed to prove a given theorem. This becomes even more pertinent in light of Simpson’s strong statements regarding the use of ‘extra data’ in constructive mathematics.

The typical constructivist response to a nonconstructive mathematical theorem is to modify the theorem by adding hypotheses or “extra data”. In contrast, our approach in this book is to analyze the provability of mathematical theorems as they stand, passing to stronger subsystems of $\mathbb{Z}_2$ if necessary. [64, I.8]

This situation has prompted some authors, for example Bishop/Bridges [20, page 38], to build a modulus of uniform continuity into their

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1Simpson describes \textit{ordinary mathematics} in [64, I.1] as \textit{that body of mathematics that is prior to or independent of the introduction of abstract set theoretic concepts}.
definitions of continuous function. Such a procedure may be appropriate for Bishop since his goal is to replace ordinary mathematical theorems by their “constructive” counterparts. However, as explained in chapter I, our goal is quite different [from Bishop’s Constructive Analysis]. Namely, we seek to draw out the set existence assumptions which are implicit in the ordinary mathematical theorems as they stand. [64, IV.2.8]; emphasis original.

Hence, if one takes seriously the claim that RM studies theorems of mathematics ‘as they stand’, it is of the utmost important that the latter standing is not changed by the coding practise of RM. Indeed, Kohlenbach has shown in [38, §4] that the existence of a code for a continuous function is the same as the latter having a modulus of continuity, a slight constructive enrichment going against the above.

Moreover, coding is not a tangential topic in RM in light of the following: the two pioneers of RM, Friedman and Simpson, actually devote a section titled ‘The Coding Issue’ to the coding practise in RM in their ‘issues and problems in RM’ paper [25], from which we consider the following.

Most mathematics naturally lies within the realm of complete separable metric spaces and continuous functions between them defined on open, closed, compact or $G_δ$ subsets. This is the central coding issue.

PROBLEM. Continuation of the previous problem: Show that Simpson’s neighborhood condition coding of partial continuous functions between complete separable metric spaces is ”optimal”. (It amounts to a coding of continuous functions on a $G_δ$.)

We emphasize our view that the handling of the critical coding in $\text{RCA}_0$ in [Simpson’s monograph [64]] is canonical, but we are asking for theorems supporting this view. [25, p. 135]

Moreover, in his ‘open problems in RM’ paper, Montalbán discusses Friedman’s strict RM program [24] as follows.

Friedman has proposed the development what he calls strict reverse mathematics (SRM). The objective of SRM is to eliminate the following two possible criticisms of reverse mathematics: that we need to code objects in cumbersome ways (something that is not part of classical mathematics), [...].

As for the coding issue, Friedman says that, for each area $X$ of mathematics, there will be a SRM for $X$. The basic concepts of $X$ will be taken as primitives, avoiding the need for coding. This would also allow consideration of uncountable structures, thereby getting around this limitation of reverse mathematics. [44, p. 449]

The previous quotes suggest that on one hand the coding practise of RM is definitely on the radar in RM, while on the other hand there is an optimistic belief that coding does not cause any major problems in RM. This brings us to the next section in which we discuss the main results of this paper.

1.2. Main results. The conclusion from the previous section was that on one hand the coding practise of RM is definitely on the radar in RM, while on the other hand there is an optimistic belief that coding does not cause any major problems in RM.
Our main goal is to dispel this belief in that we identify basic theorems about continuous functions, including basic ones named after Heine, Tietze, and Weierstrass, where the (only) change from ‘continuous third-order function’ to ‘second-order code for continuous function’ changes the strength of the theorem at hand, namely as in Figure 1 below. Since we want to consider objects from third-order arithmetic, we shall work in Kohlenbach’s higher-order RM introduced in [39] and discussed in Section A.1. The associated base theory is $\text{RCA}_0$, which proves the same $L_2$-sentence as $\text{RCA}_0$, up to insignificant logical details, by Remark A.4.

Figure 1 below provides an overview of our results to be proved in Section 2-4. The first column lists a theorem, while the second (resp. third) column lists the theorem’s logical strength for the formulation involving second-order codes (resp. third order functions). The reader should (only) view these two formulations as two possible formalisations of the same theorem, based on respectively second-order and third-order arithmetic. In particular, we discuss the question whether third-order objects are a more natural form of representation than second-order objects in Section 5. We wish to stress that it would perhaps not come as a surprise that the coding of topologies or measurable sets in second-order arithmetic has its problems [34, 50], but the theorems in Figure 1 are quite elementary in comparison.

Note that we use the usual RM-definition of (separably) closed sets from [64, II.4] and [7, 8] everywhere, i.e. the only difference between the second and third column in Figure 1 is the change from ‘(continuous) third-order function’ to ‘second-order code’. We also note that the final rows are formulated over Cantor and Baire space, i.e. the coding of the reals is not relevant here, and we could obtain versions of the other theorems for the latter spaces. Finally, the first instance of Ekeland’s variational principle involves honest codes.

| Theorem                  | RM-codes ($L_2$) | Third-order functions |
|-------------------------|------------------|-----------------------|
| Heine’s theorem for     | equivalent       | provable in $\text{RCA}_0^{\omega} + \text{WKL}_0$ |
| separably closed sets   | to $\text{ACA}_0$|                       |
| Weierstrass’ theorem    | equivalent       | provable in $\text{RCA}_0^{\omega} + \text{WKL}_0$ |
| for separably closed    | to $\text{ACA}_0$|                       |
| sets                    |                  |                       |
| Tietze’s theorem        | equivalent       | provable in $\text{RCA}_0^{\omega}$ |
| for separably closed    | to $\text{ACA}_0$|                       |
| sets                    |                  |                       |
| One-point-extension    | equivalent       | provable in $\text{RCA}_0^{\omega}$ |
| theorem (Theorems 2.9   | to $\text{ACA}_0$|                       |
| and 2.10)               |                  |                       |
| Ekeland’s variational   | equivalent       | provable in a        |
| principle on $2^\mathbb{N}$ | to $\text{ACA}_0$ | conservative extension |
| Ekeland’s variational   | equivalent       | provable in a        |
| principle on $\mathbb{N}$ | to $\text{ACA}_0$ | conservative extension |

**Figure 1. Summary of our results**

We caution the reader not to over-interpret the above: for instance, we do not claim that the third column of Figure 1 provides the ultimate (RM) analysis of the theorems in the first column. What Figure 1 does establish is that coding can

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2Honest codes are a (rather) technical device from [20, Def. 5.1].
change the logical strength of the most basic theorems. Hence, coding as done in
RM is neither good nor bad in general, but in the specific case of the Main Question
of RM, coding seems problematic lest the wrong minimal axioms be identified.

Indeed, the foundational claims made in relation to RM are (of course) based on
the correct identification of the minimal axioms need to prove a given theorem. We
are thinking of the development of Russell-Weyl-Feferman predicative mathematics
or Simpson's partial realisation of Hilbert's program for the foundations of math-
ematics. In Section 6.1 we introduce the aforementioned programs and critically
investigate the implications of Figure 1 for these claims and programs

Now, the above is only a stepping stone: we wish to understand why the theorems
in Figure 1 behave as they do. Moreover, this understanding should lead to a
distinction between ‘good’ and ‘bad’ codes, as e.g. the coding of real numbers does
not seem to lead to results as in Figure 1. We discuss these matters in some detail in
Section 6.2. In a nutshell, the cause of the results in Figure 1 is that there are partial
codes that do not correspond to a third-order object in a weak system. Digging
more deeply, we shall observe that second- and third-order objects are intimately
connected, also in non-classical extensions of the base theory, but the same is not
true for (second-order) representations of third-order objects. The absence of this
connection is what makes a code ‘bad’ in the sense of yielding results as in Figure 1.

Finally, to establish the results in Figure 1 we require the ‘excluded middle trick’
introduced in [51]. Since all systems are based on classical logic, one can always
invoke the following disjunction.

Either there exists a discontinuous function on \( \mathbb{R} \) (in which case we
have higher-order \( \text{ACA}_0 \)) or all function on \( \mathbb{R} \) are continuous.

This is the crucial step for the below proofs, as will become clear.

In conclusion, the aim of this paper is to establish the results in Figure 1 and
discuss the associated foundational implications. At the risk of repeating ourselves,
Figure 1 is meant to showcase the difference between the second-order and third-
order formalisations of the theorems in the first column. To be absolutely clear, the
third-order formalisation can also be said to involve a kind of representation/coding,
i.e. the former does not constitute the theorem \( \text{an sich} \). We should not have to point
out that the latter point of view is captured by the centred statement in Section 1.

2. Tietze extension theorem

We establish the results regarding the Tietze extension theorem as in Figure 1.
Hereafter, we assume familiarity with the basic notions of RM, Kohlenbach's base
theory \( \text{RCA}_0 \) in particular. For the reader's convenience, an introduction to RM,
and the definition of \( \text{RCA}_0 \), can be found in Section A. We like to point out that
our results do not call into question the Big Five phenomenon, as will become clear.

2.1. Introduction. We discuss some relevant facts for Tietze's theorem.

First of all, the Tietze extension theorem has been studied in RM in e.g. [7, 27, 62, 64].
In the latter references, it is shown that there are versions of Tietze's extension theorem provable in \( \text{RCA}_0 \), equivalent to \( \text{WKL}_0 \), and equivalent to \( \text{ACA}_0 \).
These different versions make use of different notions of continuity and closed set.
The main result of this section is that a slight change to the Tietze's extension
theorem, namely replacing second-order codes by third-order functions, makes the version at the level of ACA₀ provable in RCA₀^ω.

Secondly, the aforementioned ‘slight change’ definitely has historical antecedent, in that it can be found in Tietze’s paper [70], and is actually the dominating formalism therein. To see this, let us discuss the exact formulation of Tietze’s extension theorem, which expresses that for certain spaces X, if a function f is continuous on a closed C ⊂ X, there is a function g, continuous on all of X, such that f = g on C. There are at least two ways of formulating the antecedent of the previous theorem, say for R → R-functions, namely as follows.

(A) The function f is defined and continuous on C; undefined on R \ C.

(B) The function f is defined everywhere on R and continuous on C.

Tietze proves three theorems in [70], called Satz 1, 2, and 3. The first two are formulated using (B), while the third one is formulated as a corollary to the first and second theorem, and is formulated using (A). Tietze explicitly mentions that f can be discontinuous outside of C in [70, p. 10]. When treating Tietze’s extension theorem, Carathéodory uses both (A) and (B) in [11, §§541-543], while Hausdorff states that (B) is used in [30].

It goes without saying that item (A) corresponds rather well to the RM-definition of ‘continuous function on a closed set’, while (B) is essentially the higher-order definition since all objects are total (= everywhere defined) in Kohlenbach’s RM. Moreover, we show below that the ‘actual’ strength of the second-order Tietze extension theorem comes from the fact that it can extend certain (partial) codes into (total) higher-order functions (working in RCA₀^ω).

In conclusion, given the previous observation regarding items (A) and (B), it seems reasonable to study a version of Tietze’s extension theorem based on (B) in higher-order RM, which can be found in Section 2.2. Hereinafter, ‘continuous’ refers to the usual ‘ε-δ’ definition of third-order functions, while ‘RM-continuous’ refers to the coding used in RM [64, II.6.1].

2.2. Separably closed sets. We establish the results sketched in Figure 1, i.e. we study Tietze’s extension theorem for separably closed sets.

First of all, the aforementioned theorem can be be found in [7, 27] and we note that ‘separably closed sets’ are closed sets represented by a countable dense sub-set (see e.g. [7,8] for details).

Definition 2.1. [RCA₀; separably closed set] A sequence S = (x_n)_{n ∈ N} is a (code for a) separably closed set S in a complete separable metric space A. We say that x ∈ A belongs to S, denoted ‘x ∈ S’ if (∀k ∈ N)(∃n ∈ N)(d(x, x_n) < 1/k).

The second-order version of Tietze’s theorem is then as follows.

Principle 2.2 (TIE^2). For f : C → R RM-continuous on the separably closed C ⊆ [0, 1], there is RM-continuous g : [0, 1] → R such that f(x) = g(x) for x ∈ C.

As suggested above, TIE^2 ↔ ACA₀ over RCA₀ by [27, Theorem 6.9]. The exact choice of domain does not seem to matter much by [7, Theorem 1.35].

Secondly, the higher-order version version of Tietze’s theorem based on item (B) from Section 2.1 is as follows. The reader will agree that the only change from TIE^2 to TIE^ω is that we replaced ‘second-order code’ by ‘third-order function’.
Principle 2.3 (TIE\(^\omega\)). For \( f: [0, 1] \to \mathbb{R} \) continuous on the separably closed \( C \subseteq [0, 1] \), there is continuous \( g: [0, 1] \to \mathbb{R} \) such that \( f(x) =_R g(x) \) for \( x \in C \).

To be absolutely clear, recall we use ‘continuous’ in the sense of the usual ‘\( \epsilon-\delta \)’ definition, while ‘RM-continuous’ refers to the coding used in RM [64, II.6.1]. In contrast to TIE\(^2\) requiring arithmetical comprehension, the higher-order version TIE\(^\omega\) is much weaker.

Theorem 2.4. The system RCA\(_0\)\(^\omega\) proves TIE\(^\omega\).

Proof. We use the law of excluded middle as in \((\exists^2) \lor \neg (\exists^2)\). In case \(\neg (\exists^2)\), all functions on \(\mathbb{R}\) are continuous by [39] §3, and we may use \( g = f \) to obtain TIE\(^\omega\).

In case \(\exists^2\), we use [7] Theorem 1.10 to guarantee that any separably closed set \( C \subseteq \mathbb{R} \) is also closed in the sense of RM, i.e. given by a \( \Pi^0_1 \)-formula in \( L_2 \). Note that \( \exists^2 \) can decide such formulas. By [38] §4, we can obtain an RM-code for \( f \) on \( C \) using \( \exists^2 \). Applying TIE\(^2\) to the RM-code of \( f \), there is code for a continuous function \( g \) such that \( f =_R g \) on \( C \). Apply QF-AC\(^1,0\) to the totality of \( g \) on \([0, 1]\) to obtain the required continuous function. \(\square\)

The above results can be neatly summarised as in (2.1), working over RCA\(_0\)\(^\omega\):

\[
\text{TIE}^2 \to \text{ACA}_0 \to \text{WKL}_0 \to \text{TIE}^\omega \tag{2.1}
\]

Thirdly, we study a ‘reverse coding principle’ that allows one to obtain a (higher-order) function from certain codes.

Principle 2.5 (RCP). For a RM-code \( f \) defined on a separably closed set \( C \subseteq [0, 1] \), there is \( h: [0, 1] \to \mathbb{R} \) with \( (\forall x \in C)(h(x) =_R f(x)) \).

Note that a total RM-code trivially yields a higher-order function with the same values, in contrast to the following theorem.

Theorem 2.6. The system RCA\(_0\)\(^\omega\) proves RCP \(\to\) ACA\(_0\).

Proof. Let \( f \) be a RM-code defined on a separably closed \( C \subset [0, 1] \). Then RCP yields \( h: [0, 1] \to \mathbb{R} \) which is continuous on \( C \). Applying TIE\(^\omega\) to \( h \) (see Theorem 2.4) then yields a (higher-order) version of TIE\(^2\) for \( f \) given by RM-codes, but where \( g: [0, 1] \to \mathbb{R} \) is not given by a RM-code. However, the proof of TIE\(^2\) \(\to\) ACA\(_0\) in [27] Theorem 6.9 still goes through when \( g: [0, 1] \to \mathbb{R} \) is not given by a code but is only continuous on \([0, 1]\). This proof uses \( \Pi^0_1 \)-separation and we note that RCA\(_0\)\(^\omega\) proves separation for \( \Pi^0_1 \)-formulas with type two parameters in the same way as RCA\(_0\) proves \( \Pi^0_1 \)-separation for \( L_2 \)-formulas. The latter result is in [64] IV.4.8. \(\square\)

The previous proof becomes much easier if we work in RCA\(_0\)\(^\omega\) + WKL, as it is established in [38] §4 that the latter system shows that a continuous \( \mathbb{Y}^2 \) has a code on Cantor space; the same goes through for \([0, 1]\) and hence \( g \) from the proof. We shall repeatedly use this ‘coding principle’ in the remainder of this paper. We now consider the following somewhat strange corollary.

Corollary 2.7. The following are equivalent over RCA\(_0\)\(^\omega\) to ACA\(_0\):

(a) TIE\(^2\)
(b) RCP

3Just apply QF-AC\(^1,0\) to the formula ‘\( \alpha(x) \) is defined for all \( x \in \mathbb{R} \)’ for the RM-code \( \alpha \).
(c) For a RM-code $f$ defined on a separably closed set $C \subseteq [0,1]$, there is continuous $g : [0,1] \to \mathbb{R}$ such that $(\forall x \in C)(g(x) = f(x))$.

Proof. Since $\text{TIE}^2 \leftrightarrow \text{ACA}_0$ by [27, Theorem 6.9], we only need to prove (1) $\Rightarrow$ (3) in light of the theorem. In case $\neg (3)$, all functions on $\mathbb{R} \to \mathbb{R}$ are continuous by [39, §3]. In case (3), we also have $\text{ACA}_0$ and applying $\text{TIE}^2$ yields a code for a continuous $g : \mathbb{R} \to \mathbb{R}$ as in item (2). This code in turn yields the function required for item (3) by applying $\text{QF-AC}^{1,0}$ to the totality of this code. The law of excluded middle now finishes the proof. □

We note that the results in Corollary 2.7 are extremely robust: we can endow $g$ in RCP with any property weaker than continuity and the equivalence would still go through. Moreover, we observe that RCP is the weakest set existence axiom equivalent to $\text{TIE}^2$ over $\text{RCA}_0^\omega$, while the former constitutes the non-constructive part of the Tietze extension theorem as in $\text{TIE}^2$. Indeed, working over $\text{RCA}_0^\omega$, once the original function as in $\text{TIE}^2$ is made total via RCP, there is no additional strength required to obtain a continuous extension function, as $\text{TIE}^2$ is provable in $\text{RCA}_0^\omega$.

Finally, we show that item (A) from Section 2.1 is rather strong when the associated notions are interpreted in second-order RM. For simplicity, we work over Cantor space $2^\mathbb{N}$.

**Theorem 2.8 (RCA$^\omega_0$).** Let $D \subset 2^\mathbb{N}$ be a RM-closed set and let $f : D \to \mathbb{N}$ be a RM-code for a continuous function. Then $(\forall x \in 2^\mathbb{N})(x \in D \leftrightarrow f(x) \text{ is defined})$ implies that ‘$x \in D$’ is decidable, i.e. there is $\Phi^2$ such that $(\forall x \in 2^\mathbb{N})(x \in D \leftrightarrow \Phi(x) = 0)$.

Proof. Note that ‘$x \in D$’ is $\Pi^1_1$ while ‘$f(x)$ is defined’ is $\Sigma^0_1$. Applying $\text{QF-AC}^{1,0}$ to the forward implication yields the required $\Phi^2$. □

We briefly discuss the foundational implications of the above, while a more detailed investigation may be found in Section 5.1. Simpson states in [54, IX.3.18] that the partial realisation of Hilbert’s program for the foundations of mathematics is a ‘very important direction of research’. In a nutshell, the equivalence $\text{TIE}^2 \leftrightarrow \text{ACA}_0$ suggests that the Tietze extension theorem for separably closed sets is out of reach for this partial realisation, while Theorem 2.3 implies *that this is not the case*.

Finally, we study a ‘one-point-extension theorem’ from [76, §3], which was suggested to us by K. Yokoyama. The reversal in Theorem 2.2 is attributed to K. Tanaka in [70].

**Theorem 2.9.** The following assertions are pairwise equivalent over $\text{RCA}_0$.

(1) $\text{ACA}_0$.

(2) If $f$ is a RM-continuous function from $(0,1)$ to $\mathbb{R}$ such that $\lim_{x \to 0^+} f(x) = 0$, then there exists a RM-continuous function $\tilde{f}$ from $[0,1)$ to $\mathbb{R}$ such that $\tilde{f}(x) = f(x)$ for $x \in (0,1)$ and $\tilde{f}(0) = 0$.

This theorem is interesting, as the ‘extension’ of $f$ consist only of one (obvious) point, while separably closed sets are not used. In contrast to $\text{ACA}_0$ in the previous theorem, we have the following theorem.

**Theorem 2.10 (RCA$^\omega_0$).** If $f : \mathbb{R} \to \mathbb{R}$ is continuous on $(0,1)$ and such that $\lim_{x \to 0^+} f(x) = 0$, then there exists a function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ such that $\tilde{f}(x) = f(x)$ for $x \in (0,1)$ and $\tilde{f}(0) = 0$, i.e. $\tilde{f}$ is continuous on $[0,1)$.
Proof. We make use of \((\exists^2) \lor \neg(\exists^2)\). In the latter case, all functions on \(\mathbb{R}\) are continuous by \([39, \S 3]\) and \(f = \tilde{f}\). In the former case, define \(\tilde{f}(x) = 0\) if \(x = 0\) and \(f(x)\) otherwise, using \(\exists^2\). Clearly, \(\tilde{f}\) is continuous on \([0, 1)\). □

In conclusion, we have established the results from Figure 1 pertaining to the Tietze extension theorem. In doing so, we have identified \(\text{RCP}\) as the source of ‘non-constructivity’ in \(\text{TIE}_2\). Our results were proved using the ‘excluded middle trick’ described in Section 1.2. This kind of proof suggests that third-order functions \(f\) as in \(\text{TIE}_\omega\) come with a certain kind of constructive enrichment: the extension \(g\) is implicit in \(f\) via a simple (classical) case distinction. In particular, the weakness of \(\text{TIE}_\omega\) relative to \(\text{TIE}_2\) is due to this additional (implicit) information. The reader may interpret this as \(\text{TIE}_2\) and \(\text{TIE}_\omega\) being variations of the same theorem with different coding or constructive enrichment. We discuss the latter view in more detail in Section 5, as well as the question whether second-order or third-order representations are more natural.

3. Theorems by Heine and Weierstrass

We establish the results sketched in Figure 1 for Heine’s continuity theorem and Weierstrass’ approximation theorem. These theorems are studied in RM in [64, IV.2]. We shall study these theorems for separably closed sets; our results are similar to those for Tietze’s theorem and we shall therefore be brief.

First of all, the following two principles are the \(L_2\)-versions of the Heine and Weierstrass theorems for separably closed sets.

**Principle 3.1 (HEI\(_2^2\)).** Let \(C \subseteq [0, 1]\) be separably closed. Any RM-continuous \(f : C \to \mathbb{R}\) is also uniformly continuous with a modulus on \(C\).

**Principle 3.2 (WEI\(_2^2\)).** Let \(C \subseteq [0, 1]\) be separably closed. For RM-continuous \(f : C \to \mathbb{R}\), there is a sequence of polynomials \((p_n)_{n \in \mathbb{N}}\) with \((\forall x \in C)((|f(x) - p_n(x)| < \frac{1}{2^n}))\).

We have the following expected second-order RM result.

**Theorem 3.3.** The system \(\text{RCA}_0\) proves \(\text{ACA}_0 \leftrightarrow \text{HEI}^2 \leftrightarrow \text{WEI}^2\).

Proof. We have \(\text{TIE}_2^2 \leftrightarrow \text{ACA}_0\) by [27] Theorem 6.9], where the former was introduced in Section 2.2. To prove \(\text{ACA}_0 \to \text{HEI}_2^2\) or \(\text{ACA}_0 \to \text{WEI}_2^2\), let \(f\) be RM-continuous on a separably closed set \(C \subseteq [0, 1]\). Use \(\text{TIE}_2^2\) to obtain the ‘extension’ \(g\) of \(f\) and use \(\text{WKL}\) to obtain the uniform continuity and approximation results for \(g\), which all follow from [64, IV.2.3-5]. These properties are then inherited from \(g\) to \(f\) on \(C\), and the forward implications are done. Alternatively, \(\text{ACA}_0\) is equivalent to the Heine-Borel theorem for countable covers of separably closed sets [33], which readily yields \(\text{HEI}_2^2\) and \(\text{WEI}_2^2\).

For the implication \(\text{HEI}_2^2 \to \text{ACA}_0\), note that the antecedent for \(C = [0, 1]\) yields \(\text{WKL}_0\) by [64, IV.2.3]. In turn, \(\text{WKL}_0\) yields the strong Tietze extension theorem for RM-continuous functions with a modulus of uniform continuity by [27] Theorem 6.14]. Putting together the above, we obtain \(\text{HEI}_2^2 \to \text{TIE}_2^2\) and \(\text{ACA}_0\) follows.

For the implication \(\text{WEI}_2^2 \to \text{ACA}_0\), note that \(\text{WKL}_0\) follows from the antecedent in the same way. Now, \(\text{WKL}_0\) proves that an RM-continuous function has a modulus of uniform continuity, and the same holds for sequences of RM-continuous functions.
This provides a modulus of uniform continuity for $f$ as in $\text{WEI}^2$, and the rest of the proof is now the same as in the previous paragraph.

An alternative to the previous two paragraphs is provided by the second part of the proof of [27, Theorem 6.9] of $\text{TIE}^2 \rightarrow \text{ACA}_0$. Indeed, this proof is based on uniform continuity and $\text{HEI}^2$ (more) directly applies. The same holds for $\text{WEI}^2$ by noting that polynomials are uniformly continuous on $[0, 1]$. Another alternative proof is based on the aforementioned result concerning the Heine-Borel theorem for countable covers of separably closed sets. □

Thirdly, we consider the higher-order versions of $\text{HEI}^2$ and $\text{WEI}^2$, where the only change is that we replaced ‘second-order code’ by ‘third-order function’.

**Principle 3.4 ($\text{HEI}^\omega$).** Let $C \subseteq [0, 1]$ be separably closed. Any $f : \mathbb{R} \rightarrow \mathbb{R}$ which is continuous on $C$ is also uniformly continuous with a modulus on $C$.

**Principle 3.5 ($\text{WEI}^\omega$).** Let $C \subseteq [0, 1]$ be separably closed. For $f : \mathbb{R} \rightarrow \mathbb{R}$ which is continuous on $C$, there is a sequence of polynomials $(p_n)_{n \in \mathbb{N}}$ with $(\forall x \in C)((|f(x) - p_n(x)| < \frac{1}{2^n}))$.

We now prove the following results from Figure 1.

**Theorem 3.6.** The system $\text{RCA}_0^\omega + \text{WKL}$ proves $\text{HEI}^\omega$ and $\text{WEI}^\omega$.

**Proof.** We use $(\exists^2) \lor \neg(\exists^2)$ as in the proof of Theorem 2.4. In case $\neg(\exists^2)$, all functions on $\mathbb{R}$ are continuous by [39, §3]. The usual proof using $\text{WKL}_0$ of the Heine and Weierstrass theorems now goes through, in light of the coding results in [38, §4]. Indeed, it is established in the latter that continuous functions on $2^\mathbb{N}$ have RM-codes, and the same holds for $[0, 1]$.

In case $(\exists^2)$, we use [7, Theorem 1.10] to guarantee that any seperably closed set $C \subseteq \mathbb{R}$ is also closed in the sense of RM, i.e. given by a $\Pi^0_1$-formula in $L_2$. Note that $\exists^2$ can decide such formulas. By [38, §4], we can obtain an RM-code for $f$ on $C$, and the theorem now follows from Theorem 3.3. □

The following equivalence is immediate, but conceptually important. Indeed, while the change from second-order codes to third-order functions fundamentally changes the logical strength of the Heine and Weierstrass theorems for separably closed sets, the higher-order versions are still equivalent to one of the Big Five.

**Corollary 3.7.** The system $\text{RCA}_0^\omega$ proves $\text{HEI}^\omega \leftrightarrow \text{WKL} \leftrightarrow \text{WEI}^\omega$.

**Proof.** For $C = [0, 1]$, the theorem reduces to the well-known results from [64, IV.2]. A RM-code defined on $[0, 1]$ gives rise to a continuous third-order functional by applying $\text{QF-AC}^1_{\omega,0}$ to the totality of the RM-code. □

Next, an interesting observation regarding RM is made by Montalbán as follows.

To study the big five phenomenon, one distinction that I think is worth making is the one between robust systems and non-robust systems. A system is robust if it is equivalent to small perturbations of itself. [44, p. 432]

As it happens, the above versions of the Tietze, Heine, and Weierstrass theorems are robust too: we can restrict to e.g. bounded functions and the same equivalences as in Theorem 3.3 and Corollary 3.7 go through. Similarly, we could study Brown's
version of the Tietze extension theorem from [7] equivalent to ACA₀, and the results would be the same as for the above versions.

Finally, note that Tietze’s extension theorem was the first theorem for which we established the results as in Figure 1. Following [51], one could say that the Heine and Weierstrass theorems also ‘suffer from the Tietze syndrome’, if the latter name were not taken already.

4. **Ekeland’s variational principle**

We establish the results pertaining to Ekeland’s variational principle as sketched in Figure 1. In a nutshell, we show that the equivalences for fragments of this principle involving ACA₀ and Π₁¹-CA₀ disappear when we formulate this principle in higher-order arithmetic. We shall not study the exact details of the latter formulation, only what happens if this is done.

4.1. **At the level of arithmetical comprehension.** We show that the equivalence between ACA₀ and a fragment of Ekeland’s variational principle disappears when we formulate this principle in higher-order arithmetic.

First of all, the RM-properties of Ekeland’s principle are studied in [20] inside the framework of second-order arithmetic; Ekeland’s weak variational principle from [16] is called ‘FVP’ in [20], where the following results are proved.

**Theorem 4.1.**

(a) Over RCA₀, ACA₀ is equivalent to FVP restricted to total honestly coded potentials on [0, 1] (or 2ᴺ).

(b) Over RCA₀, WKL₀ is equivalent to FVP restricted to total continuous potentials on [0, 1] (or 2ᴺ).

Now let FVP⁰ₜot and FVP⁰ₜot,cont be some higher-order versions of FVP for total (and continuous in the second case) potentials on [0, 1], i.e. the versions of FVP from respectively items (a) and (b), but not involving codes. To be absolutely clear, FVP⁰ₜot has the form (∀f : [0, 1] → ℝ)A(f), while FVP⁰ₜot,cont has the form (∀f ∈ C([0,1]))A(f), which is readily expressed in the language of higher-order arithmetic. The exact formulation does not matter, as long as the aforementioned syntactical form is available. Moreover, since we could also use 2ᴺ instead of [0, 1], none of what follows has anything to do with the coding of real numbers.

As it happens, the higher-order principles FVP⁰ₜot,cont and FVP⁰ₜot cannot satisfy the same properties as in items (a) and (b) above, by the following theorem.

**Theorem 4.2.** Assume the following proofs are given.

(a’) The system ACA₀ proves FVP⁰ₜot.

(b’) The system RCA₀ + WKL proves FVP⁰ₜot,cont.

Then FVP⁰ₜot is already provable in RCA₀ + WKL.

**Proof.** We use the law of excluded middle (∃²) ∨ ¬(∃²). The proof in the former case follows thanks to item (a). In case ¬(∃²), all functions on ℝ are continuous by [38] §4. Hence, FVP⁰ₜot, which has the form (∀f : [0, 1] → ℝ)A(f), reduces to FVP⁰ₜot,cont, which has the form (∀f ∈ C([0,1]))A(f). The latter has a proof thanks to item (b’), and we are done. □
To be absolutely clear, we do not claim that anything is wrong with the RM of FVP as in items (a) and (b) above. We do claim that no higher-order version of FVP can satisfy the same properties. Indeed, items (a') and (b') are enough to make $\text{FVP}_{\text{tot}}^\omega$ provable in $\text{RCA}_0^\omega + \text{WKL}$, and hence $\text{FVP}_{\text{tot}}^\omega$ cannot imply $\text{ACA}_0$. The discrepancy between the second- and higher-order RM is (presumably) caused by the use of codes in second-order arithmetic, in particular the concept of ‘honest code’ from item (a). Note that every continuous function on $\mathbb{N}$ has an RM-code on $2^\mathbb{N}$ given $\text{WKL}$ [48 §4], i.e. item (b') seems rather natural.

The observation from this section is not an isolated incident, as we will see next.

4.2. At the level of hyperarithmetical comprehension. We show that the equivalence between $\Pi^1_1$-CA$_0$ and a fragment of Ekeland’s variational principle disappears when we formulate this principle in higher-order arithmetic.

The following results regarding FVP are from [20].

**Theorem 4.3.** The system $\text{RCA}_0$ proves the following equivalences.

(c) $\Pi^1_1$-CA$_0$ is equivalent to FVP restricted to total potentials on $\mathbb{N}$.

(d) $\text{ACA}_0$ is equivalent to FVP restricted to total continuous potentials on $\mathbb{N}$.

Now let $\text{FVP}_{\text{tot}}^\omega(\mathbb{N})$ and $\text{FVP}_{\text{tot,cont}}^\omega(\mathbb{N})$ be some higher-order versions of FVP for total (and continuous in the second case) potentials on $\mathbb{N}$, i.e. the versions of FVP from respectively items (c) and (d), but not involving codes. To be absolutely clear, $\text{FVP}_{\text{tot}}^\omega(\mathbb{N})$ has the form $(\forall f : \mathbb{N} \to \mathbb{N}) A(f)$, while $\text{FVP}_{\text{tot,cont}}^\omega$ has the form $(\forall f \in C(\mathbb{N})) A(f)$, which is readily expressed in the language of higher-order arithmetic.

As it happens, the above higher-order principles cannot satisfy the same properties as in items (c) and (d) above: the following two items already imply that $\text{FVP}_{\text{tot}}^\omega(\mathbb{N})$ is provable in a conservative extension of $\text{ACA}_0$. Note that $\text{RCA}_0^\omega + \text{ACA}_0 + (\kappa_3^0)$ is conservative over $\text{ACA}_0$ by [39 Prop. 3.12].

**Theorem 4.4.** Assume the following proofs are given.

(c') The system $Z^\Omega_2$ proves $\text{FVP}_{\text{tot}}^\omega(\mathbb{N})$.

(d') The system $\text{RCA}_0^\omega + \text{ACA}_0$ proves $\text{FVP}_{\text{tot,cont}}^\omega(\mathbb{N})$.

Then $\text{RCA}_0^\omega + \text{ACA}_0$ and $(\kappa_3^0)$ already proves $\text{FVP}_{\text{tot}}^\omega(\mathbb{N})$.

**Proof.** As mentioned in [18 §6] or [59], $\text{RCA}_0^\omega$ proves $(\exists^2) \leftrightarrow [(\exists^2) + (\kappa_3^0)]$, which was first proved by Kohlenbach in a private communication. Now use $(\exists^2) \lor \neg(\exists^2)$ as in the previous proof. \qed

To be absolutely clear, we do not claim that anything is wrong with the RM of FVP as in items (c) and (d) above. We do claim that no higher-order version of FVP can satisfy the same properties. Indeed, items (c') and (d') are enough to make $\text{FVP}_{\text{tot}}^\omega(\mathbb{N})$ provable in a conservative extension of $\text{ACA}_0$ by Theorem 1.1 and hence $\text{FVP}_{\text{tot}}^\omega(\mathbb{N})$ cannot imply $\Pi^1_1$-CA$_0$. The discrepancy between the second- and higher-order RM is (presumably) caused by the use of codes in RM. Note that we use $\text{RCA}_0^\omega + \text{ACA}_0$ in the theorem and item (d'), rather than $\text{ACA}_0^\omega$. This does not really constitute a weakening as the former system already proves that every continuous functional on Baire space has an RM-code by [38 §4].

4The system $\text{RCA}_0^\omega + \text{MUC}$ from [39 Prop. 3.12] readily proves $(\kappa_3^0)$, and hence the associated conservation result for the latter follows.
The results in Theorem 4.4 are also foundationally significant as follows: item (d) above could lead one to claim that the associated version of Ekeland’s variational principle is not available in predicativist mathematics (see [64, I.11.9] or [19]). Such a claim is debatable as Theorem 4.4 provides a way of ‘pushing down’ theorems to predicatively reducible mathematics. We discuss this observation in more detail in Section 6.1.

A possible criticism of Theorem 4.4 is that the axiom ($\kappa_3^0$) is somewhat ad hoc and not that natural. One can replace this axiom by more natural theorems as follows: the Lindelöf lemma for Baire space together with ($\exists^2$) proves $\Pi^1_1$-CA$^0_0$, while the Heine-Borel theorem for uncountable covers of Cantor space yields ATR$_0$ when combined with ($\exists^2$) (see [49, 51] for these results). Thus, one readily modifies Theorem 4.4 and its proof to work with the aforementioned theorems rather than ($\kappa_3^0$). Note that in the case of the Lindelöf lemma, the resulting theorem still applies to item (c) and $\Pi^1_1$-CA$^0_0$ in particular.

5. Codes of various order and naturalness

5.1. Introduction. The above results raise the obvious question whether third-order objects are a more natural form of representation than second-order objects. We discuss various possible (partial) answers in this section, as follows.

First of all, we discuss some relevant history of mathematics in Section 5.2, namely pertaining to the modern concept of function. We establish that the latter predates set theory and therefore should be studied in RM. A more definitive answer shall be obtained in Section 6.2.2.

Secondly, we discuss the coding of open sets in Section 5.3, motivated by the close connection between the coding of open sets and continuous functions in second-order RM. We argue that the higher-order representation of open sets is closer to the mathematical mainstream, based in part on Section 5.2.

Thirdly, we discuss related results in Section 5.4, namely pertaining to ‘totality versus partiality’ and closely related results connected to coding from [59].

5.2. The history of functions. As discussed throughout the literature, RM studies ordinary mathematics, which is generally qualified as:

that body of mathematics that is prior to or independent of the introduction of abstract set theoretic concepts. ([64, I.1])

Now, the study of arbitrary, in particular discontinuous, functions clearly predates set theory. Indeed, the modern concept of ‘arbitrary’ or ‘general’ function is generally credited to Lobachevsky ([32]) and Dirichlet ([13]) in 1834-1837. A detailed history may be found in [77, §13], where Euler is credited (much earlier) for the general definition. Regardless of priority, Fourier’s famous work ([21]) on Fourier series gave great impetus to the (study of the) general definition, as follows.

This is the first of its [=Fourier series] services which I wish to emphasize, the development and complete clarification of the concept of a function. ([73, p. 116], emphasis in original)

Moreover, discontinuous functions had already enjoyed a rich history by the time set theory came to the fore. Indeed, Euler’s use of pulse functions, which are zero except at one point, is discussed in [77, p. 71], while Dirichlet famously mentions the characteristic function of $\mathbb{Q}$ in [15] around 1829; Riemann studied discontinuous
functions in his 1854 Habilitationsschrift, which propelled them in the mathematical mainstream, according to Kleiner [37, p. 115]. The 1870 dissertation of Hankel, a student of Riemann, has ‘discontinuous functions’ in its title (29).

The previous is not meant to settle priority disputes, but rather (only) establish that arbitrary -in particular discontinuous- functions predate set theory ‘by a large margin’. From this point of view, the study of such functions squarely belong to ordinary mathematics.

In our opinion, the aforementioned study is best undertaken in a language that has third-order objects as first-class citizens, like higher-order arithmetic. In Section 6.2.2 we provide additional arguments in favour of this view, based on the fundamental connections between second- and third-order objects; such connections do not really exist for second-order representations. In this way, the study of $\mathbb{I}E^{\omega}$ etc. is not an afterthought, but the very bread and butter of RM.

5.3. Coding open sets. The coding of continuous functions in second-order RM is intrinsically linked to the coding of open sets in light of [64, II.7.1]. The latter essentially establishes that these two concepts are (effectively) interchangeable in the base theory $\text{RCA}_0$. Hence, the question whether third-order functions are a more natural form of coding than second-order (codes for) functions directly translates to open sets. We discuss the coding of the latter in this section.

First of all, open sets are represented in second-order RM as $\Sigma^0_1$-formulas with an additional extensionality requirement (see [64, II.5.6-7]). Intuitively, such formulas represent countable unions of basic open intervals. In turn, open sets in separable spaces can be represented via such unions, explaining the coding. It goes without saying that the notion of $\Sigma^0_1$-formula is a construct from mathematical logic The aforementioned [64, II.7.1] shows that open sets in second-order RM are (equivalently) represented by (second-order codes for) continuous characteristic functions.

Secondly, we have previous studied open sets represented via third-order characteristic functions in [50,52,53]. Characteristic (aka indicator) functions are used to represent sets in mathematical logic in e.g. [34,40], but also in mainstream mathematics: most textbooks develop the Lebesgue integral using (arbitrary) characteristic functions of sets (see [68, p. xi] for an example), while Dirichlet already considered the characteristic function of $\mathbb{Q}$ in [15] in 1829. We also mention Thomae’s function, similar to Dirichlet’s function and introduced in [69, p. 14] around 1875.

In light of the previous, it is not an exaggeration to claim that in the case of open sets, the third-order representation of open sets via (arbitrary) characteristic functions is ‘more mainstream mathematics’ than the representation in second-order RM via $\Sigma^0_1$-formulas or (second-order codes for) continuous characteristic functions. In particular, Section 5.2 dictates the study of arbitrary characteristic functions in RM.

5.4. Related results. We discuss results related to the representations of objects in second- and third-order arithmetic. In light of the above results on RCP, an important role is played by the partial versus total distinction, as discussed in Section 5.4.1. We discuss closely related RM-results connected to coding in Section 5.4.2; these results were first published in [59].
5.4.1. Partial versus total objects. We discuss the use of partial and total objects in mathematics motivated by RCP and Corollary 2.7.

First of all, as explained nicely in [65, p. 10], the study of computability based on Turing’s framework ([72]) is fundamentally based on partial computable functions for the simple reason that those can be listed in a computable way. By contrast, the total computable functions are susceptible to diagonalisation by their very nature. Since second-order RM makes heavy use of results in (Turing) computability theory, it is not a surprise that partial functions (like codes in \( \text{TIE}^2 \)) play an important role.

Secondly, Kleene’s notion of computability based on S1-S9 ([36]) is similarly based on total objects, while the system RCA\(_0\) is officially a type theory in which all functionals are total. In general, Martin-Löf’s intuitionistic type theory (see e.g. [43]) is similarly based on total objects, and the same for the associated proof assistants. One can accommodate partial functions in type theory using certain advanced techniques (see e.g. [6]).

Thus, mathematics and computer science boast frameworks based on partial objects and frameworks based on total objects. The choice of framework may therefore depend on personal goals and preconceptions. As discussed in Section 5.4.1, Tietze discusses both total and partial functions, namely as in items (A) and (B).

In conclusion, the partial versus total distinction does not (directly) provide arguments for or against a particular kind of coding. However, as discussed next, the choice of framework may be the cause of certain observed phenomena.

5.4.2. Results related to coding. As discussed in Section 5.4.1, mathematics and computer science boast frameworks based on partial objects and frameworks based on total objects. The choice of framework may therefore depend on personal goals and preconceptions. In particular, one is free to chose between the development of RM in second- or higher-order arithmetic.

However, one should keep in mind that certain observed phenomena are (only) an artifact of this choice. As an example, we discuss splittings and disjunctions in RM, as studied in [59].

As to splittings, there are some examples in RM of theorems \( A, B, C \) such that \( A \iff (B \land C) \), i.e. \( A \) can be split into two independent (fairly natural) parts \( B \) and \( C \). As to disjunctions, there are (very few) examples in RM of theorems \( D, E, F \) such that \( D \iff (E \lor F) \), i.e. \( D \) can be written as the disjunction of two independent (fairly natural) parts \( E \) and \( F \). By contrast, there is a plethora of (natural) splittings and disjunctions in higher-order RM, as shown in [59] and witnessed by Figure 2 below.

| MUC ↔ [WKL + (\( k_3 \)) + \( \lnot(3^2) \)] | (\( \exists \)) ↔ [(\( \exists \)) + (\( \exists \))] | (\( \exists \)) ↔ [(\( \exists \)) + FF] |
| MUC ↔ [WKL + (\( k_3 \)) + \( \lnot(5^2) \)] | (\( \exists \)) ↔ [(\( \exists \)) + (\( \exists \))] | (\( \exists \)) + WKL ↔ [(\( \exists \)) \lor MUC] |
| MUC ↔ [WKL + (\( k_3 \)) + \( \lnot(3^3) \)] | (\( \exists \)) ↔ [FF + (\( \exists \)) + \( \lnot MUC \)] | FF ↔ [(\( \exists \)) \lor MUC] |
| MUC ↔ [FF + \( \lnot(\exists) \)] | (\( \exists \)) ↔ [FF + MUC] | FF ↔ [(\( \exists \)) \lor (\( \exists \))] |
| MUC ↔ [FF + (\( Z^3 \)) + \( \lnot(5^2) \)] | (\( \exists \)) ↔ [FF + MUC] | (\( \exists \)) ↔ [(\( \exists \)) \lor (\( \exists \))] |
| MUC ↔ [FF + (\( Z^3 \)) + \( \lnot(3^3) \)] | WKL ↔ [(\( \exists \)) \lor HBU] | (\( \exists \)) ↔ [(\( \exists \)) \lor \( \lnot FF \) \lor MUC] |
| T\(_2\) ↔ [T\(_6\) \lor \( \Sigma_2^0 \)-IND] | WWKL ↔ [(\( \exists \)) \lor WHBU] | LIN ↔ [HBU \lor \( \lnot WKL \)] |

Figure 2. Summary of (some of) the results in [59]
We refer to [59, §2] for the relevant definitions not found in Appendix A.2. Some results in Figure 2 are proved in extensions of RCA_0\omega.

In a nutshell, splittings and disjunctions are rare in second-order RM, but rather common in higher-order arithmetic. As discussed at length in [59, §6] and suggested by Figure 2, the splittings and disjunctions in Figure 2 are all directly based on the axiom (\exists^2) and/or the associated ‘excluded middle trick’.

In conclusion, as discussed at the end of Section 2.2, the use of third-order functionals in TIE_\omega constitutes an implicit constructive enrichment, based on (\exists^2) and the excluded middle trick. However, the latter are also responsible for Figure 2, and more generally most results in [59].

6. Foundational implications

We discuss the foundational implications of the above results. In Section 6.1, we investigate how our results affect the foundational claims made in RM, especially pertaining to predicativist mathematics and Hilbert’s program for the foundations of mathematics. In Section 6.2, we speculate on why the theorems in Figure 1 behave as they do. We also formulate a general criterion for judging coding.

6.1. Lines in the sand. Certain results in RM are viewed as contributions to foundational programs by Hilbert and Russell-Weyl. We critically examine such claims in this section. In a nutshell, these programs draw a line in the sand in that they identify a certain threshold of logical strength above which the associated mathematics is somehow no longer meaningful or acceptable. Our above results are very relevant as we have shown how certain theorems can drop in strength, even below the aforementioned thresholds as it turns out.

6.1.1. Hilbert’s program for the foundations of mathematics. We discuss Hilbert’s program for the foundations of mathematics, described in [32], an outgrowth of Hilbert’s second problem from his famous list of 23 open problems [31]. The aim of this program was to establish the consistency of mathematics using only so-called finitistic methods. Tait has argued that Hilbert’s finitistic mathematics is captured by the formal system PRA in [67].

Now, Gödel’s incompleteness theorems establish that any logical system that can accommodate arithmetic, cannot even prove its own consistency [28]. Thus, it is generally believed that Gödel’s results show that Hilbert’s program is impossible. According to Simpson [64, IX.3.18], a partial realisation of Hilbert’s program is however possible as follows: a certain theorem T is called finitistically reducible if T is provable in a \Pi^0_2-conservative extension of PRA. The most prominent of the latter systems is the Big Five system WKL_0. The intuitive idea is that while T may deal with infinitary objects (and is therefore not finitistic), T does not yield any (new) \Pi^0_2-sentences that PRA cannot prove. We note that Burgess criticises the above in [10], but we will go along with Simpson.

In light of the previous, the ‘line in the sand’ is as follows: the \Pi^0_2-sentences provable in PRA are finitistic, and any theorem T that does not prove any new \Pi^0_2-sentences may be called ‘reducible to finitism’. All other theorems are ‘not finitistically reducible’. Our results pose a problem as follows: the Tietze, Heine,

\footnote{By contrast, Detlefsen [12,13] and Artemov [1] argue at length why Gödel’s results do not show that Hilbert’s program is impossible.}
and Weierstrass theorems for separably closed are not finitistically reducible when formulated with second-order codes (Theorem 3.3), while these theorems drop to finitistically reducible when formulated in third-order arithmetic (Theorems 2.4 and 3.6). In our opinion, the foundational status of a theorem, i.e. finitistically reducible or not, should not depend on technical details like coding.

6.1.2. Predicativist mathematics. We discuss the development of predicativist mathematics due to Russell, Weyl, and Ferferman [17, 55, 75]. In a nutshell, motivated by contradictions in naive set theory, predicativist mathematics also draws a line in the sand, namely at the level of ATR₀. Our results from Section 4.2 therefore yield a ‘drop’ similar to the previous section, namely from the level of Π₁¹-CA₀ (above ATR₀) to level of ACA₀ (below ATR₀).

First of all, Russell’s paradox shows that naive set theory is inconsistent, the problematic entity being the ‘set of all sets’. The root cause of this paradox, according to Russell, is implicit definition: the ‘set of all sets’ is defined as the collection of all sets including itself [55]. Thus, to define the ‘set of all sets’, one implicitly assumes that it exists already, a kind of vicious circle.

According to Russell, one should in general avoid such implicit definitions and vicious circles, which are also called impredicative, or non-predicative in [55], leading to the term predicativist or predicative for that mathematics that makes no use of impredicative notions. Weyl’s Das Kontinuum is a milestone in the development of mathematics on predicative grounds [75], while the main protagonist of the modern development is Feferman [18].

The upper limit of predicativist mathematics was identified independently by Feferman and Schütte as being the ordinal Γ₀, which is also the proof-theoretic ordinal of the Big Five system ATR₀ [19, 64]. Similar to the previous section, systems like ACA₀ may involve impredicative notions, but since its ordinal is (well) below Γ₀, the former system is called predicatively reducible.

Having found the line drawn in the sand by predicativist mathematics, we note that our results pose a problem: the Ekeland variational principle for ℕ¹ is not predicatively reducible when formulated with second-order codes (Theorem 4.3), while this theorems drop to predicatively reducible when formulated in third-order arithmetic (Theorem 4.4). We repeat that, in our opinion, the foundational status of a theorem, i.e. predicatively reducible or not, should not depend on technical details like coding.

6.2. Cause and effect. In this section, we speculate on why the theorems in Figure 1 behave as they do. We first discuss Kohlenbach’s result on coding in Section 6.2.1, which yields a satisfactory answer to the aforementioned question ‘by contrast’. As suggested above, this understanding should ideally lead to a distinction between ‘good’ and ‘bad’ codes, as e.g. the coding of real numbers does not seem to lead to results as in Figure 1. We discuss the ‘good’ versus ‘bad’ distinction for codes in Section 6.2.2.

As in the previous section, some people disagree and claim that predicativist mathematics can extend beyond ATR₀ [74].
6.2.1. **Coding comes to a head.** We sketch the ‘coding’ results from [38] §4 and identify the cause of the behaviour of the theorems in Figure 1 in contrast to Kohlenbach’s results.

Kohlenbach proves in [38] §4 that for continuous type two functionals, the existence of a RM-code is equivalent to the existence of a continuous modulus of continuity. Thus, RM-codes constitute a constructive enrichment as follows: when interpreted in higher-order arithmetic, a second-order theorem about RM-continuous functions only seems to apply to higher-order functionals that also come with an RM-code (or the aforementioned modulus). In particular, it is not clear whether such a theorem applies to all continuous functionals: ‘out of the box’ it only seems to apply to the sub-class ‘continuous functionals with a continuous modulus’. To remove (part of) this uncertainty, note that WKL suffices to show that continuous functionals have an RM-code on $2^\mathbb{N}$ [38] §4, i.e. the RM of WKL does not change if we replace any leading universal quantifier over RM-codes by a quantifier over higher-order functionals that are continuous in the relevant theorems.

The observation in the previous paragraph gives rise to the concept of coding overhead of a given theorem $T$, which is the minimal axioms(s) needed to prove (over RCA$_0^\omega$) that $T$ formulated with second-order codes implies $T$ formulated in third-order arithmetic, whenever the former seems less general than the latter. However, coding overhead is not all there is, much to our own surprise, as follows.

The problem with the theorems in Figure 1 turns out to be the opposite of the direction indicated by the concept of coding overhead: our results for e.g. the Tietze extension theorem imply that, working again in Kohlenbach’s base theory, the class of ‘second-order codes defined on a separably closed set $C$’ is much bigger than ‘third-order functions continuous on $C$’. In this way, statements like RCP connecting the aforementioned classes are at the level of ACA$_0$, while the associated version of Tietze’s extension theorem $\text{TIE}^\omega$ is provable in RCA$_0^\omega$. In particular, Theorem 2.6 suggests we also define the notion of coding underhead of a theorem $T$, which is the minimal axioms(s) needed to prove (over RCA$_0^\omega$) that $T$ formulated in third-order arithmetic implies $T$ formulated with second-order codes, and this whenever the former seems less general than the latter, like in the case of separably closed sets.

In conclusion, past results on coding have focused on the constructive enrichment provided by codes, as this causes the class of represented objects to be (potentially) smaller than the class of actual objects. By contrast, the second-order Tietze, Weierstrass, and Heine theorems in Figure 1 pertain to a class of codes that is larger than the class of third-order objects (working in RCA$_0^\omega$). This ‘opposite’ difference in size is the cause of the behaviour observed in Figure 1.

6.2.2. **Beyond good and bad coding.** Having identified the cause of the behaviour of the theorems in Figure 1 we now expand our point of view and attempt to provide a criterion by which coding can be judged as ‘good’ or ‘bad’, in light of the Main Question of RM. We reiterate that we do not judge the coding practise of RM as good or bad. We only wish to establish a criterion that avoids the ‘bad’ behaviour of coding as summarised in Figure 1 assuming one judges the latter as such.

Firstly, we consider the following -in our opinion- uncontroversial statement. Codes for a certain class of objects are meant to capture/represent/reflect, as well as possible, the original class of objects.
Simpson’s quotes on ‘theorems as they stand’ from Section 1 support this statement, while the following quote expresses the same idea.

There is a key criterion for choice of coding that is implicitly used: that $\text{RCA}_0$ should prove as much as possible. E.g., if real numbers are Cauchy sequences of rationals, we can’t prove in $\text{RCA}_0$ that every real number is the limit of a sequence of rationals with arbitrarily fast convergence. [25, p. 134]

Indeed, this criterion is used to choose the particular coding of real numbers used in RM, namely fast-converging Cauchy sequences as in [64, II.4]. By contrast, other possible representations of real numbers (like Dedekind cuts) are rejected because $\text{RCA}_0$ cannot prove basic properties about the representations.

The problem with the previous is that one does not fully take the role of the base theory into account. Indeed, in our opinion, the following addition is crucial.

Codes for a certain class of objects are meant to capture/represent/reflect, as well as possible, the original class of objects also in non-classical extensions of the base theory.

Our rationale is that the base theories $\text{RCA}_0$ and $\text{RCA}^\omega_0$ are weak enough to be consistent with non-classical axioms like e.g. the axiom CT which expresses that every $\mathbb{N} \rightarrow \mathbb{N}$-function is computable (in the sense of Turing). The axiom CT is called *Church’s thesis* in constructive mathematics [3]. Also found in the latter is ‘Brouwer’s principle’ BP, which expresses that all functions on $\mathbb{N}^\mathbb{N}$ are continuous; BP yields a conservative extension of $\text{RCA}^\omega_0$, while the associated *intuitionistic fan functional* MUC from [59, §3] yields a conservative extension of WKL$_0$. Ishihara connects BP to $\neg(\exists^2)$, where the latter is formulated with $\Pi^1_1$-formulas in [35].

As it happens, both CT and BP imply the *higher-order* versions of the Tietze theorem from Figure 1. Similarly, the higher-order Heine and Weierstrass theorems from Figure 1 follow from MUC or WKL + BP. This behaviour is obviously not reflected in the associated *second-order* versions form Figure 1, as e.g. CT and ACA$_0$ are inconsistent. It should be noted that non-classical axioms like $\neg\text{WKL}_0$ have been studied in classical RM in [4].

Another way of looking things based on Corollary 2.7 is as follows. One one hand, $\text{TIE}^\omega$ follows from CT, while on the other hand RCP is inconsistent with CT. This suggests that $\text{TIE}^\omega$ is excluding the genuinely ‘difficult’ instances of $\text{TIE}^2$, which contributes to the latter being a stronger principle.

The main point of the previous observations is now two-fold, as follows.

On one hand, there are (very) different models of weak systems, which is reflected in the syntax by $(\exists^2) \lor \neg(\exists^2)$ being an axiom of $\text{RCA}^\omega_0$. Depending on which case we are working with, the class of (continuous) functions looks very different: in case $\neg(\exists)$, all functions on $\mathbb{R}$ are continuous, similar to Brouwer’s principle BP, while in case $(\exists^2)$, discontinuous functions are directly available. *By contrast*, whether we work in the case $(\exists^2)$ or $\neg(\exists^2)$ has less of an effect (or even none) on the class of codes for (continuous) functions: in either case there is a code for a discontinuous function, namely as a sequence of functions that converges to $\exists^2$ in $\text{RCA}^\omega_0$.

On the other hand, there is a deep connection between second- and third-order objects: CT is a statement about $\mathbb{N} \rightarrow \mathbb{N}$-functions, but nonetheless causes all $\mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}$-functions to be continuous. Similarly, $(\exists^2)$ is equivalent to the existence of a discontinuous function on $\mathbb{R}$, and the former allows us to solve the Halting
problem, which contradicts CT. By contrast, the connection between second-order objects and representations of third-order objects is much weaker. Indeed, in either case of CT ∨ ¬CT there is a code for a discontinuous function.

In light of the previous, we can say that (i) the class of codes for (continuous) functions is ‘more static’ than the class of (continuous) functions itself and (ii) the class of codes for (continuous) functions is ‘more disconnected’ from the class of second-order objects than the class of (continuous) functions itself.

In conclusion, we may call a coding ‘good’ (meaning it avoids results as in Figure 1) if it satisfies the previous centred statement, i.e. if the correspondence between objects and representations is also valid in non-classical extensions of the base theory.

6.3. White horses and horses that are white. We discuss possible future research topics on coding via existing examples.

Firstly, while it is hard to argue with the statement

\[\text{a countable set is a set that is countable},\]

it should be noted that a countable set in \(L_2\) is actually a sequence by [64, V.4.2]. The latter definition can be said to constitute a constructive enrichment compared to the usual definition based on injections or bijections. Of course, the usual definition of countable set (for subsets of Baire space), is fundamentally third-order and cannot even be expressed in \(L_2\). Nonetheless, it is an interesting question what happens to e.g. theorems from RM if we work in higher-order RM and use the higher-order definition of countable set. This question becomes all the more pertinent in light of the observation that textbooks tend to treat countable sets as follows: to prove that a given set is countable, it is generally only shown that there is an injection to \(\mathbb{N}\), while to prove something about countable sets, one additionally assumes the latter are given by a sequence.

Secondly, a partial answer to the previous question has been published in [57, 58]. In particular, an interesting ‘coding’ result is obtained, as follows. Now, the following \(L_2\)-sentence is equivalent to \(WKL_0\) by [64, IV.4.5].

**Principle 6.1 (\(\text{ORD}^2\)).** Any formally real countable field is orderable.

Let \(\text{ORD}^3\) be the previous principle formulated in higher-order arithmetic for countable subsets of Baire space, where countable is interpreted via the usual definition involving injections to \(\mathbb{N}\), as can be found in Kunen’s textbook [41]. Note that this is the only modification to \(\text{ORD}^2\). Then \(\text{ORD}^3\) implies the Heine-Borel theorem for uncountable covers, which is not provable in \(\Pi^1_k-\text{CA}_0\) for any \(k\).

Thus, the definition of ‘countable’ used in RM also involves constructive enrichments in that the usual definition of ‘countable’ gives rise to a wholly different beast. These results are not as refined or fundamental as the ones in Figure 1.

Thirdly, the closed graph theorem is provable in an extension \(\text{RCA}_0^+\) of \(\text{RCA}_0\) [9 §5]. The former theorem expresses that a linear operator with a graph given by a separably closed set must be continuous. Since \(\text{AC}A_0\) proves \(\text{RCA}_0^+\), the usual ‘excluded middle trick’ yields a higher-order version of the closed graph theorem inside \(\text{RCA}_0^+\).

Fourth, Dag Normann and the author have obtained a lot of results about the RM of countable sets which can be found in the preprints [53,60,61]. Perhaps the
most interesting example is as follows. The following two principles are equivalent over $\text{RCA}_0^\omega$, where ‘countable set’ is defined as above, i.e. based on injections to $\mathbb{N}$.

**Principle 6.2** ($\text{cocode}_0$). For any non-empty countable set $A \subseteq [0, 1]$, there is a sequence $(x_n)_{n \in \mathbb{N}}$ in $A$ such that $(\forall x \in \mathbb{R})(x \in A \leftrightarrow (\exists n \in \mathbb{N})(x_n = R x))$.

**Principle 6.3** ($\text{BW}_0^C$). For any countable $A \subset 2^\mathbb{N}$ and $F : 2^\mathbb{N} \to 2^\mathbb{N}$, the supremum $\sup_{f \in A} F(f)$ exists.

Many similar equivalences can be found in [60]. Moreover, combining $\text{BW}_0^C$ with the Suslin functional, i.e. higher-order $\Pi_1^1\text{-CA}_0$, one obtains $\Pi_2^1\text{-CA}_0$. The system $\Pi_2^1\text{-CA}_0$ has been called the current ‘upper limit’ for RM, previously only reachable via topology (see [45–47]). Moreover, according to Rathjen [54, §3], the strength of $\Pi_2^1\text{-CA}_0$ dwarfs that of $\Pi_1^1\text{-CA}_0$.

**Acknowledgement 6.4.** Our research was supported by the John Templeton Foundation via the grant *a new dawn of intuitionism* with ID 60842. We express our gratitude towards this institution. We thank Anil Nerode, Paul Shafer, and Keita Yokoyama for their valuable advice. We also thank the anonymous referee for the many helpful suggestions, especially for suggesting the need for Section 5. Opinions expressed in this paper do not necessarily reflect those of the John Templeton Foundation.

**Appendix A. Reverse Mathematics**

We introduce *Reverse Mathematics* in Section A.1 as well as its generalisation to *higher-order arithmetic*, and the associated base theory $\text{RCA}_0^\omega$ of Kohlenbach’s *higher-order* RM. We introduce some essential axioms in Section A.2.

**A.1. Reverse Mathematics.** Reverse Mathematics is a program in the foundations of mathematics initiated around 1975 by Friedman [22,23] and developed extensively by Simpson [64]. The aim of RM is to identify the minimal axioms needed to prove theorems of ordinary, i.e. non-set theoretical, mathematics. In almost all cases, these minimal axioms are also equivalent to the theorem at hand (over a weak logical system). The derivation of the minimal axioms from the theorem is the ‘reverse’ way of doing mathematics, lending the subject its name.

We refer to [66] for an introduction to RM and to [63,64] for an overview of RM. We expect basic familiarity with RM, but do sketch some aspects of Kohlenbach’s *higher-order* RM [39] ’essential to this paper, including the base theory $\text{RCA}_0^\omega$ (Definition A.1).

First of all, in contrast to ‘classical’ RM based on *second-order arithmetic* $\mathbb{Z}_2$, higher-order RM uses $\mathbb{L}_\omega$, the richer language of *higher-order arithmetic*. Indeed, while $\mathbb{Z}_2$ is restricted to natural numbers and sets of natural numbers, higher-order arithmetic can accommodate sets of sets of natural numbers, sets of sets of sets of natural numbers, et cetera. To formalise this idea, we introduce the collection of *all finite types* $T$, defined by the two clauses:

(i) $0 \in T$ and (ii) if $\sigma, \tau \in T$ then $(\sigma \to \tau) \in T$,

where 0 is the type of natural numbers, and $\sigma \to \tau$ is the type of mappings from objects of type $\sigma$ to objects of type $\tau$. In this way, $1 \equiv 0 \to 0$ is the type of functions
from numbers to numbers, and where \( n + 1 \equiv n \to \). Viewing sets as given by characteristic functions, we note that \( \mathbb{Z}_2 \) only includes objects of type 0 and 1.

Secondly, the language \( L_\omega \) includes variables \( x^\rho, y^\rho, z^\rho, \ldots \) of any finite type \( \rho \in T \). Types may be omitted when they can be inferred from context. The constants of \( L_\omega \) include the type 0 objects 0, 1 and \( <, +, \times, =_0 \) which are intended to have their usual meaning as operations on \( \mathbb{N} \). Equality at higher types is defined in terms of \( '='_0 \) as follows: for any objects \( x^\tau, y^\tau \), we have

\[
[x =_\tau y] \equiv (\forall z_1^\tau \ldots z_k^\tau)([xz_1 \ldots z_k =_0 yz_1 \ldots z_k]), \tag{A.1}
\]

if the type \( \tau \) is composed as \( \tau \equiv (\tau_1 \to \ldots \to \tau_k \to 0) \). Furthermore, \( L_\omega \) also includes the recursor constant \( R_\sigma \) for any \( \sigma \in T \), which allows for iteration on type \( \sigma \)-objects as in the special case \( A.2 \). Formulas and terms are defined as usual. One obtains the sub-language \( L_{n+2} \) by restricting the above type formation rule to produce only type \( n + 1 \) objects (and related types of similar complexity).

**Definition A.1.** The base theory \( \text{RCA}_0^\omega \) consists of the following axioms.

1. **Basic axioms expressing that \( 0, 1, <, +, \times \) form an ordered semi-ring with equality \( =_0 \).**
2. **Basic axioms defining the well-known \( \Pi \) and \( \Sigma \) combinators (aka \( K \) and \( S \) in [2]), which allow for the definition of \( \lambda \)-abstraction.**
3. **The defining axiom of the recursor constant \( R_\sigma \): For \( m^0 \) and \( f^1 \):
   \[
   R_0(f, m, 0) := m \land R_0(f, m, n + 1) := f(n, R_0(f, m, n)). \tag{A.2}
   \]
4. **The axiom of extensionality: for all \( \rho, \tau \in T \), we have:
   \[
   (\forall x^\rho, y^\rho, \phi^{\rho \to \tau})(x =_\rho y \to \phi(x) =_\tau \phi(y)). \tag{E_{\rho, \tau}}
   \]
5. **The induction axiom for quantifier-free formulas of \( L_\omega \).**
6. **\( \text{QF-AC}^{1,0} \): The quantifier-free Axiom of Choice as in Definition A.2.**

**Definition A.2.** The axiom \( \text{QF-AC} \) consists of the following for all \( \sigma, \tau \in T \):

\[
(\forall x^\sigma)(\exists y^\tau)A(x, y) \to (\exists Y^{\sigma \to \tau})(\forall x^\sigma)A(x, Y(x)), \tag{\text{QF-AC}^{\sigma, \tau}}
\]

for any quantifier-free formula \( A \) in the language of \( L_\omega \).

As discussed in [39, §2], \( \text{RCA}_0^\omega \) and \( \text{RCA}_\rho \) prove the same sentences ‘up to language’ as the latter is set-based and the former function-based. Recursion as in \( A.2 \) is called *primitive recursion*; the class of functionals obtained from \( R_\rho \) for all \( \rho \in T \) is called *Gödel’s system T* of all (higher-order) primitive recursive functionals.

We use the usual notations for natural, rational, and real numbers, and the associated functions, as introduced in [39, p. 288-289].

**Definition A.3** (Real numbers and related notions in \( \text{RCA}_0^\omega \)).

1. **Natural numbers correspond to type zero objects, and we use ‘\( n^0 \)’ and ‘\( n \in \mathbb{N} \)’ interchangeably. Rational numbers are defined as signed quotients of natural numbers, and ‘\( q \in \mathbb{Q} \)’ and ‘\( < \mathbb{Q} \)’ have their usual meaning.**
2. **Real numbers are coded by fast-converging Cauchy sequences \( q_{\langle \rangle} : \mathbb{N} \to \mathbb{Q} \), i.e. such that \( \langle \forall n^0, r^0 \rangle(q_n - q_{n+1} < q \downarrow) \). We use Kohlenbach’s ‘hat function’ from [39, p. 289] to guarantee that every \( q^1 \) defines a real number.**

\footnote{To be absolutely clear, variables (of any finite type) are allowed in quantifier-free formulas of the language \( L_\omega \); only quantifiers are banned.}
(c) We write ‘\(x \in \mathbb{R}\)’ to express that \(x^1 := (q^1_\gamma)\) represents a real as in the previous item and write \([x](k) := q_k\) for the \(k\)-th approximation of \(x\).

(d) Two reals \(x, y\) represented by \(q_\gamma\) and \(r_\gamma\) are equal, denoted \(x =_\mathbb{R} y\), if 
\[
(\forall n^0)(|q_n - r_n| \leq 2^{-n+1}).
\]
Inequality ‘\(\prec \)' is defined similarly. We sometimes omit the subscript ‘\(\mathbb{R}\)’ if it is clear from context.

(e) Functions \(F : \mathbb{R} \to \mathbb{R}\) are represented by \(\Phi^{1 \to 1}\) mapping equal reals to equal reals, i.e. satisfying \((\forall x, y \in \mathbb{R})(x =_\mathbb{R} y \to \Phi(x) =_\mathbb{R} \Phi(y))\).

(f) The relation ‘\(x \leq_\mathbb{R} y\)’ is defined as in (A.1) but with ‘\(\leq_0\)’ instead of ‘\(=\)’.

Binary sequences are denoted ‘\(f^1, g^1 \leq_1 1\)’, but also ‘\(f, g \in \mathbb{C}\)’ or ‘\(f, g \in 2^\mathbb{N}\)’.

Elements of Baire space are given by \(f^1, g^1\), but also denoted ‘\(f, g \in \mathbb{N}^\mathbb{N}\)’.

(g) For a binary sequence \(f^1\), the associated real in \([0, 1]\) is \(\tau(f) := \sum_{n=0}^{\infty} \frac{f(n)}{2^n}\).

(h) Sets of type \(\rho\) objects \(X^{\rho \to 0}\) are given by their characteristic functions \(F^{\rho \to 0}_X \leq_\rho 1\), i.e. we write ‘\(x \in X\)’ for \(F_X(x) = 0\).

Next, we mention the highly useful ECF-interpretation.

**Remark A.4 (The ECF-interpretation).** The (rather) technical definition of ECF may be found in [71, p. 138, §2.6]. Intuitively, the ECF-interpretation \([A]_{ECF}\) of a formula \(A \in \mathcal{L}_\omega\) is just \(A\) with all variables of type two and higher replaced by countable representations of continuous functionals. Such representations are also (equivalently) called ‘associates’ or ‘RM-codes’ [53, §4]. The ECF-interpretation connects \(\text{ACA}_\infty^0\) and \(\text{ACA}_0\) in [39, Prop. 3.1] in that if \(\text{RCA}_0\) proves \(A\), then \(\text{RCA}_0\) proves \([A]_{ECF}\), again ‘up to language’, as \(\text{ACA}_0\) is formulated using sets, and \([A]_{ECF}\) is formulated using types, namely only using type zero and one objects.

In light of the widespread use of codes in RM and the common practise of identifying codes with the objects being coded, it is no exaggeration to refer to ECF as the canonical embedding of higher-order into second-order RM.

### A.2. Some axioms of higher-order RM.

We introduce some functionals which constitute the counterparts of some of the Big Five systems, in higher-order RM. We use the formulation from [39, 49]. First of all, \(\text{ACA}_0\) is readily derived from:

\[
(\exists \mu^2)(\forall f^1)[(\exists n)(f(n) = 0) \to ((f(\mu(f))) = 0) \land (\forall i < \mu(f))f(i) \neq 0] \land [(\forall n)(f(n) \neq 0) \to \mu(f) = 0]],
\]

and \(\text{ACA}_0^{\infty} \equiv \text{RCA}_0^{\infty} + (\mu^2)\) proves the same sentences as \(\text{ACA}_0\) by [34, Theorem 2.5]. The (unique) functional \(\mu^2\) in \((\mu^2)\) is also called Feferman’s \(\mu\) [2], and is clearly discontinuous at \(f = 1, 11, \ldots\); in fact, \((\mu^2)\) is equivalent to the existence of \(F : \mathbb{R} \to \mathbb{R}\) such that \(F(x) = 1\) if \(x \geq 0\), and 0 otherwise [39, §3], and to

\[
(\exists \varphi^2 \leq_2 1)(\forall f^1)[(\exists n)(f(n) = 0) \leftrightarrow \varphi(f) = 0].
\]

Secondly, \(\Pi^1_2\text{-CA}_0\) is readily derived from the following sentence:

\[
(\exists S^2 \leq_2 1)(\forall f^1)[(\exists y^1)(\forall n^0)(f(\gamma n) = 0) \leftrightarrow S(f) = 0]],
\]

and \(\Pi^1_2\text{-CA}_0^{\infty} \equiv \text{RCA}_0^{\infty} + (S^2)\) proves the same \(\Pi^1_2\)-sentences as \(\Pi^1_2\text{-CA}_0\) by [56, Theorem 2.2]. The (unique) functional \(S^2\) in \((S^2)\) is also called the Suslin functional [39]. By definition, the Suslin functional \(S^2\) can decide whether a \(\Sigma^1_k\)-formula as in the left-hand side of \((S^2)\) is true or false. We similarly define the functional \(S^2_\infty\) which decides the truth or falsity of \(\Sigma^1_k\)-formulas; we also define the system \(\Pi^1_2\text{-CA}_0^{\infty}\) as \(\text{RCA}_0^{\infty} + (S^2_\infty)\), where \((S^2_\infty)\) expresses that \(S^2_\infty\) exists. Note that we allow formulas
with function parameters, but not functionals here. In fact, Gandy’s Superjump \cite{20} constitutes a way of extending $\Pi^1_2$-$\text{ACA}_0^{\omega}$ to parameters of type two. We identify the functionals $\exists^2$ and $S_2^2$ and the systems $\text{ACA}_0^{\omega}$ and $\Pi^1_2$-$\text{ACA}_0^{\omega}$ for $k = 0$. Thirdly, full second-order arithmetic $\mathbb{Z}_2$ is readily derived from $\cup_k \Pi^1_2$-$\text{ACA}_0^{\omega}$, or from:

$$((\exists E^3 \leq_i 3)(\forall Y^2)[(\exists f^1) Y(f) = 0 \iff E(Y) = 0]),$$

and we therefore define $Z^2_2 = \text{RCA}_0^{\omega} + (3^2)$ and $Z^2_2' = \cup_k \Pi^1_2$-$\text{ACA}_0^{\omega}$, which are conservative over $\mathbb{Z}_2$ by \cite[Cor. 2.6]{34}. Despite this close connection, $Z^2_2$ and $Z^2_2'$ can behave quite differently, as discussed in e.g. \cite[§2.2]{48}. The functional from $(3^3)$ is also called $\exists^3$, and we use the same convention for other functionals. Note that $(3^3) \leftrightarrow [(3^2) + (\kappa_0^3)]$ as shown in \cite{48, 59}, where the latter is comprehension on $2^\mathbb{N}$:

$$(\exists \kappa_0^3 \leq_i 3)(\forall Y^2)[\kappa_0(Y) = 0 \iff (\exists f \in C) Y(f) = 0].$$

Other ‘splittings’ are studied in \cite{59}, including $(\kappa_0^3)$.

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