Integrable $\mathcal{N} = 2$ Landau-Ginzburg Theories from Quotients of Fusion Rings†

Eli J. Mlawer, Harold A. Riggs, and Howard J. Schnitzer

Department of Physics
Brandeis University
Waltham, MA 02254

Abstract

The discovery of integrable $\mathcal{N} = 2$ supersymmetric Landau-Ginzburg theories whose chiral rings are fusion rings suggests a close connection between fusion rings, the related Landau-Ginzburg superpotentials, and $\mathcal{N} = 2$ quantum integrability. We examine this connection by finding the natural $\mathfrak{so}(N)_K$ analogue of the construction that produced the superpotentials with $\mathfrak{sp}(N)_K$ and $\mathfrak{su}(N)_K$ fusion rings as chiral rings. The chiral rings of the new superpotentials are not directly the fusion rings of any conformal field theory, although they are natural quotients of the tensor subring of the $\mathfrak{so}(N)_K$ fusion ring.

The new superpotentials yield solvable (twisted $\mathcal{N} = 2$) topological field theories. We obtain the integer-valued correlation functions as sums of $\mathfrak{so}(N)_K$ Verlinde dimensions by expressing the correlators as fusion residues. The $\mathfrak{so}(2n+1)_{2k+1}$ and $\mathfrak{so}(2k+1)_{2n+1}$ related topological Landau-Ginzburg theories are isomorphic, despite being defined via quite different superpotentials.

September 1993

†Supported in part by the DOE under grant DE-FG02-92ER40706
1. Introduction

The quantum field theories on worldsurfaces that possess $\mathcal{N} = 2$ supersymmetry continue to yield new surprises and find new areas of application,\textsuperscript{1–5} even after several years of intense scrutiny.\textsuperscript{6} In addition to playing crucial roles in attempts to extract testable physics from string theory,\textsuperscript{7} in the discovery of mirror symmetry in algebraic geometry,\textsuperscript{8} and in the understanding of how matter couples to quantum gravity in two dimensions,\textsuperscript{9–13} such theories have recently found realizations in the critical behaviour of exactly solvable lattice models\textsuperscript{14} and have even led to the solution of certain long-standing problems in (experimentally realizable) two-dimensional polymer physics.\textsuperscript{3}

Those $\mathcal{N} = 2$ field theories that are characterized by a superpotential, the $\mathcal{N} = 2$ Landau-Ginzburg theories,\textsuperscript{15, 16} are important not least because many of their properties are explicitly calculable. Special one-parameter families of superpotentials have been found that smoothly deform $\mathcal{N} = 2$ superconformal theories into massive $\mathcal{N} = 2$ supersymmetric—and quantum integrable\textsuperscript{17–19}—field theories. In many such cases the chiral ring (the ring of the chiral, $\mathcal{N} = 2$ primary fields) of the superconformal theory\textsuperscript{16} deforms to a ring isomorphic to the fusion ring of a rational conformal field theory. For example, the superpotentials whose chiral rings are Grassmannian cohomology rings\textsuperscript{16} can be perturbed in different directions to obtain massive (apparently integrable) supersymmetric theories with chiral rings that are isomorphic to the $\mathfrak{su}(N)_K$ and $\mathfrak{sp}(N)_K$ current algebra fusion rings.\textsuperscript{20, 21} These particular fusion rings have properties (such as automorphisms generated by simple currents) that an arbitrary fusion ring\textsuperscript{22} does not possess. This leaves open the possibility that polynomial potentials which yield integrable theories only arise from fusion rings with such extra structure. Since any finite set of distinct points in an $n$-dimensional complex vector space can be represented as the critical points of some polynomial in $n$ variables,\textsuperscript{23, 24} the existence itself of a polynomial whose critical points reproduce the solutions of a fusion ring cannot be a crucial factor for integrability.\textsuperscript{†} The direct construction of covering-space potentials\textsuperscript{25} faces the problem of translation to physical variables. The interesting question concerning such integrable deformations remains: What is the crucial principle behind the magic of these special polynomial potentials? A promising direction is the connection between the structure of certain graph rings\textsuperscript{23} and integrability.

In this paper we construct a new family of massive superpotentials. They are natural $\mathfrak{so}(N)_K$ parallels of the $\mathfrak{su}(N)_K$ and $\mathfrak{sp}(N)_K$ related potentials and have, we expect, a good chance of leading to integrable theories. However, the associated chiral rings are not the fusion rings of any conformal field theory. While

\textsuperscript{†}It remains unclear in which cases potentials exist that flow smoothly (i.e., without changing the number of superconformal chiral primary fields) to a rational fusion ring.
these chiral rings are related to the $\text{SO}(N)_K$ fusion rings, they reproduce only a quotient of an $\text{SO}(N)_K$ fusion subring. Further, unlike the $\text{SU}(N)_K$ and $\text{Sp}(N)_K$ fusion rings, they do not possess a simple-current automorphism. This suggests that simple-current automorphisms are not prerequisites for natural (possibly integrable) Landau-Ginzburg constructions. We will only consider the case $\text{SO}(2n+1)_{2k+1}$ in this paper. A similar but more involved construction is possible for $\text{SO}(N)_K$ for any $N$ and $K$ with similar results, the details of which will be presented elsewhere.

In section two we use the idea of Young tableau transposition symmetry to define quotient rings of the tensor subring of the $\text{SO}(2n+1)$ representation ring and establish a connection with the $\text{SO}(2n+1)_{2k+1}$ fusion ring. In section three we show that although these rings are fusion rings, they cannot, in general, be the fusion rings of any conformal field theory. In section four we construct potentials whose local rings exactly reproduce these quotient rings. In section five, we use these potentials as the superpotentials of massive $\mathcal{N} = 2$ supersymmetric field theories, and find that the superpotential related to $\text{SO}(2n+1)_{2k+1}$ is a new deformation of the superpotential whose chiral ring is the homology intersection ring of the Grassmannian

$$\frac{U(n+k)}{U(n) \otimes U(k)}.$$  

This same Grassmannian superpotential can be perturbed in other directions to obtain Landau-Ginzburg models with chiral rings isomorphic to the $\text{SU}(n+1)_k$ and $\text{Sp}(n)_k$ current algebra fusion rings. As an immediate application, we consider the twisted $\mathcal{N} = 2$ topological Landau-Ginzburg models\textsuperscript{27,9–12} that can be obtained from these $\text{SO}(2n+1)_{2k+1}$ related superpotentials, and write the correlation functions on any genus in terms of the Verlinde numbers of the $\text{SO}(2n+1)_{2k+1}$ fusion ring. We also show that the twisted versions of the $\text{SO}(2n+1)_{2k+1}$ and $\text{SO}(2k+1)_{2n+1}$ based topological Landau-Ginzburg theories are identical (on surfaces of any genus). In the concluding section we comment on the pattern formed by the fusion-related potentials discovered to date.

2. Transposition Ideals and Cominimal Quotients

First we will exhibit $n$ independent generating relations for a sequence of ideals in the ring of polynomials in the $n$ fundamental tensor characters $\chi_i$ of $\text{SO}(2n+1)$, $\mathbb{Z}[\chi_1, \ldots, \chi_n]$. Then we write the values of the $\chi_i$ that satisfy the generating relations of the $k^{th}$ ideal as ratios of $\text{SO}(2n+1)_{2k+1}$ modular transformation matrix elements. This is possible due to the close connection of the related quotient ring,

$$\mathcal{R}_{n,k} = \frac{\mathbb{Z}[\chi_1, \ldots, \chi_n]}{\mathcal{I}_k},$$

to the $\text{SO}(2n+1)_{2k+1}$ fusion algebra.
2.1 The Transposition Ideals

The irreducible representations of the complex Lie algebras \( \text{so}(2n+1) \) (i.e., \( B_n \) for \( n \geq 3 \), \( C_2 \) for \( n = 2 \), and \( A_1 \) for \( n = 1 \)) are naturally classified by the Young tableaux with at most \( n \) rows. The row lengths of the tableau for a given irreducible representation are related to the Dynkin indices of its highest weight by

\[
\ell_j = \sum_{i=j}^{n-1} a_i + \frac{1}{2} a_n \quad 1 \leq j \leq n - 1 \quad \ell_n = \frac{1}{2} a_n .
\]

(2.1)

We will only consider tensor representations (for \( n = 1 \) and \( n = 2 \) only real representations) in the following so that the \( \ell_i \) will always be integral; the Dynkin index of the last root, \( a_n \), is then even. For \( n = 2 \), \( a_2 \) labels, contrary to custom, the short root.† The \( \text{so}(2n+1) \) tensor language allows an elegant and uniform description of all topics dealt with in this paper for all \( n \).

Let \( K_i \) denote the character of the representation associated with the single-row tableau of length \( \ell_1 = i \). Then the character of an arbitrary representation, specified by a tableau \( a \) with \( m_1 \) non-zero row lengths \( \ell_i \), is given by the \( m_1 \times m_1 \) determinant

\[
\text{char}_a = \frac{1}{2} \det | K_{\ell_i-i+j} + K_{\ell_i-i+2-j} |
\]

(2.2)

where \( i, j = 1, \ldots, m_1 \), and \( K_j = 0 \) for \( j < 0 \). This determinant is well-defined for any tableau, so that we can always refer to the character of a given tableau, even if that tableau does not correspond to a representation of \( \text{so}(2n+1) \). In particular, let \( \chi_i \) denote the single column tableau of height \( m_1 = i \) for \( i = 0, \ldots, \infty \). If \( 1 \leq i \leq n \) then \( \chi_i \) denotes the character of the \( i^{\text{th}} \) fundamental tensor representation. It is important to realize that the \( \chi_i \) with \( i > n \) do not all vanish, but instead satisfy

\[
\chi_{n+j} - \chi_{n-j+1} = 0 \quad 1 \leq j \leq n + 1
\]

\[
\chi_{n+j} = 0 \quad j > n + 1 .
\]

(2.3)

This follows from the explicit expression for the \( K_i \) as traces in the relevant representations as well as the definition of the \( \chi_i \) in (2.2). These identities allow one to write the character of any tableau with more than \( n \) rows in terms of a character with at most \( n \) rows, as follows.

With \( \chi_0 = 1 \) (the character of the identity representation) and \( \chi_j = 0 \) for \( j < 0 \), the character of any tableau \( a \) with \( \ell_1 \) non-zero column lengths \( m_i \) is given by the \( \ell_1 \times \ell_1 \) determinant

\[
\text{char}_a = \frac{1}{2} \det | \chi_{m_i-i+j} + \chi_{m_i-i+2-j} |
\]

(2.4)

†Note that the \( \text{SO}(5) \) tableau (2.1) for a given representation of \( C_2 \) differs significantly from the standard \( \text{Sp}(2) \) tableau for the same representation. Similarly, any \( \text{SO}(3) \) tableau has half the cells of the \( \text{SU}(2) \) tableau for the same \( A_1 \) representation.
with \( i,j = 1, \ldots, \ell_1 \). Note that formula (2.4) transforms into formula (2.2) under the interchange of rows and columns, as long as the identities (2.3) are disregarded. Upon using these latter identities in the determinant (2.4) one obtains the character associated with any tensor representation (or with any tableau) as a polynomial in the fundamental characters \( \chi_i \) with \( i = 0, 1, \ldots, n \). It is remarkable that, as a consequence of (2.3) and (2.4), the character of a tableau with more than \( n \) rows can be transformed (up to sign) into the character of a single tableau with at most \( n \) rows, according to a simple rule. This rank modification rule, which implements certain products of Weyl group reflections via the removal of strips of cells on the tableau boundary, embodies the implications of (2.3) for the characters of arbitrary tableaux.

The product of characters defined by the product of the polynomials (2.4) without imposition of the identities (2.3) is given by

\[
\text{char}_a \text{ char}_b = \sum_c T_{ab}^c \text{ char}_c \equiv \sum_d \text{ char}\{(a/d) \cdot (b/d)\} .
\]  

(2.5)

The raised dot indicates the product of tableaux that is given by the Littlewood-Richardson rule, and

\[
(a/d) = \sum_e L_{de}^a e
\]

denotes the formal sum of all tableaux \( e \) (with Littlewood-Richardson multiplicity \( L_{de}^a \)) such that \( a \in d \cdot e \). Equation (2.5) represents the tableau product in the ring of polynomials in the infinite set of variables \( \chi_i, \ i = 0, 1, \ldots, \infty \). With \( \tilde{a} \) denoting the transpose of the tableau \( a \) (i.e., \( \ell_i(\tilde{a}) = m_i(a) \)), the transposition symmetry of the Littlewood-Richardson multiplicities

\[
L_{\tilde{a} \tilde{b}}^c = L_{ab}^c
\]

and the aforementioned transposition symmetry of the determinant formulae guarantee that the transposition symmetry continues to hold for the tableau multiplicities in (2.5), so that

\[
T_{\tilde{a} \tilde{b}}^c = T_{ab}^c .
\]  

(2.6)

Since the characters of tableaux with more than \( n \) rows often appear in the product (2.5), the rank modification rules that implement the identities (2.3) must be imposed in order to obtain the representation ring of the tensor representations of \( \text{so}(2n + 1) \) from (2.5). Once this is done, the free ring of polynomials in the fundamental characters \( \chi_0, \chi_1, \ldots, \chi_n \) generates the tensor ring of \( \text{so}(2n + 1) \), with the translation between polynomials in the \( \chi_i \) and tableaux given by the determinant formula in (2.4). However, the transposition symmetry just mentioned is lost.

It is interesting that the generators of the local rings of the \( \mathfrak{su}(N)_K \) and \( \mathfrak{sp}(N)_K \) fusion potentials are transposes (upon interchange of \( N \) and \( K \)) of the generators
of the respective rank modification rules for these Lie algebras. Therefore, it is natural to consider the ideals \( I_k \) of the \( \text{so}(2n + 1) \) tensor representation ring that are generated by the relations

\[
K_{k+j} - K_{k-j+1} = 0 \quad 1 \leq j \leq n .
\] (2.7)

These generators are (upon interchange of \( n \) and \( k \)) exact tableau transposes of (a subset of) the identities (2.3) satisfied by the \( x_i \). We will call these ideals the \textit{transposition ideals}. Since (2.7) does not hold for all \( j > n \) (see equation A.4 in the appendix and the comment that follows there), it is nontrivial that the quotient rings

\[
\mathcal{R}_{n,k} = \mathbb{Z}[x_1, \ldots, x_n] / I_k
\] (2.8)

exhibit a restoration of the transposition symmetry (2.6) that is lost when the rank modification rules are imposed.

\[2.2\] The Cominimal Quotients

We will now describe certain quotients of the polynomial ring \( \mathbb{Z}[x_1, \ldots, x_n] \) that are directly related to the \( \text{so}(2n + 1)_{2k+1} \) fusion rings and their simple currents.

The standard basis elements \( \phi_a \) for the fusion ring associated with conformal-scalar fields carrying representations of the level \( K \) untwisted affine Lie algebra \( \text{so}(2n + 1)_K \) (i.e., \( A_1^{(1)} \) at level \( 2K \) for \( n = 1 \), \( C_2^{(1)} \) at level \( K \) for \( n = 2 \), and \( B_n^{(1)} \) at level \( K \) for \( n \geq 3 \)) are labelled by the irreducible and integrable highest weight representations of \( \text{so}(2n + 1)_K \). These representations are classified by the tableaux with at most \( n \) rows whose (possibly half-integral) row lengths also satisfy \( \ell_1 + \ell_2 \leq K \). We will specify the tensor subring of the \( \text{so}(2n + 1)_{2k+1} \) fusion ring by \( T_{n,k} \) and will call \( k \) the \textit{reduced} level.

The non-negative integers \( N_{ab}^c \) that define the fusion ring product

\[
\phi_a \ast \phi_b = \sum_c N_{ab}^c \phi_c ,
\] (2.9)

are related to the modular transformation matrix elements \( S_{ab} \), which are all real for \( \text{so}(2n + 1)_K \), by Verlinde’s sum\(^{31}\)

\[
N_{ab}^c = \sum_r S_{ar} S_{br} S_{cr} S_{0r} \] (2.10)

over all level \( K \) integrable representations. This formula implies that the fusion coefficients are completely determined by an extension of the Speiser algorithm for calculating ordinary Kronecker products.\(^{32}\) The extended algorithm exhibits a WZW fusion ring as a quotient of the relevant representation ring by a set of relations.
between characters related to certain products of affine Weyl group reflections. We will refer to these character relations as the fusion modification rules\(^\text{33}\) and the ideal they generate, \(F_k\), as the fusion ideal. This means that

\[ \mathcal{T}_{n,k} = \frac{\mathbb{Z}[\chi_1, \ldots, \chi_n]}{F_k}. \]  

The SO\((2n + 1)_{2k+1}\) fusion ring has a nontrivial automorphism associated with the \(\mathbb{Z}_2\) automorphism of the extended SO\((2n + 1)\) Dynkin diagram.\(^\text{34}\) This diagram automorphism induces a map \(\sigma\) between the integrable highest weight representations that label the basis elements of the fusion ring. In terms of the tableau row lengths that label these representations, this map leaves unchanged all row lengths except the first, which transforms according to

\[ \ell_1(\sigma(a)) = K - \ell_1(a). \]  

We will call the quotient of \(\mathcal{T}_{n,k}\) by the ideal \(C_k\) composed of the entire set of identities

\[ \phi_{\sigma(t)} - \phi_t = 0 \]  

where \(t\) is any tableau with \(\ell_1 + \ell_2 \leq K\), the cominimal\(^\dagger\) quotient of \(\mathcal{T}_{n,k}\). The multiplicities that define the tableau product in this quotient ring are given by

\[ M_{ab}^c = N_{ab}^c + N_{ab}^{\sigma(c)}. \]  

Since the representations of SO\((2n + 1)\) are self-conjugate, and since cominimally equivalent representations have just been equated in (2.13), the cominimal quotient of \(\mathcal{T}_{n,k}\) has no apparent non-trivial automorphisms. However, if \(n = k\), there is a hidden automorphism given by tableau transposition, as will become apparent.

We will label the basis elements of this cominimal quotient ring by the \(n\) row tableaux with \(\ell_1 \leq k\). Then we may obtain the quotient-ring product by using the identities (2.13) to eliminate any tableau with first row length \(\ell_1\) greater than the reduced level \(k\) from fusion rule (2.9). The two ideals commute so that we can write

\[ \frac{\mathcal{T}_{n,k}}{C_k} = \frac{\mathbb{Z}[\chi_1, \ldots, \chi_n]}{F_k \cdot C_k}. \]  

If \(a\) denotes any basis tableau of \(\mathcal{T}_{n,k}/C_k\), then (since \(\ell_1(\bar{a}) \leq n\) and \(m_1(\bar{a}) \leq k\)) \(\bar{a}\) is always a basis tableau of \(\mathcal{T}_{k,n}/C_n\). With tableau transposition considered as a map between \(\mathcal{T}_{n,k}/C_k\) and \(\mathcal{T}_{k,n}/C_n\) (as well as an operation on tableaux), the fact that \(\bar{a} = a\), means that transposition gives a one-to-one correspondence between elements

\(^\dagger\)The term arises from the relation between the WZW fields that generate simple-currents and cominimal highest weights.\(^\text{34}\)
of $\mathcal{T}_{n,k}/C_k$ and $\mathcal{T}_{k,n}/C_n$. This correspondence is also an isomorphism: It was shown some time ago\textsuperscript{35} that the $\text{SO}(2n+1)_{2k+1}$ and $\text{SO}(2k+1)_{2n+1}$ fusion multiplicities are related by

$$
\left( N_{ab}^c \right)_{\text{SO}(2n+1)_{2k+1}} = \left( N_{ab}^{-\Delta(c)} \right)_{\text{SO}(2k+1)_{2n+1}}
$$

(2.16)

where $\Delta = r(a) + r(b) - r(c)$, and $r(\tau)$ denotes the number of cells in the tableau $\tau$. Using this in (2.14) gives

$$
\left( M_{ab}^c \right)_{\mathcal{R}_{n,k}} = \left( M_{ab}^{-\bar{c}} \right)_{\mathcal{R}_{k,n}} .
$$

(2.17)

The equalities $\overline{\sigma(c)} = \bar{c}$, $\Delta_{\sigma(c)} = \Delta_c + 1 \mod 2$, and $\sigma^2 = 1$ have been used here. Identity (2.17) is just the statement that $\mathcal{T}_{n,k}/C_k$ and $\mathcal{T}_{k,n}/C_n$ are isomorphic as rings under tableau transposition.

2.3 The Transposition Quotient is the Cominimal Quotient

Now we shall show that the rings $\mathcal{R}_{n,k}$ (2.8) are identical to the cominimal quotients of the $\mathcal{T}_{n,k}$ just defined, i.e. that

$$
\mathcal{R}_{n,k} = \frac{\mathcal{T}_{n,k}}{C_k}
$$

(2.18)

under the correspondence between ring elements given by the natural correspondence of tableau labels:

$$
\phi_a \leftrightarrow \text{char}_a .
$$

(2.19)

Since $\phi_i \leftrightarrow \chi_i$, comparison of (2.8) and (2.15) shows that we must demonstrate the equivalence

$$
\mathcal{I}_k \equiv F_k \cdot C_k
$$

(2.20)

under the correspondence (2.19).

The relations that generate the ideals $\mathcal{I}_k$ (2.7) are transparently special cases of the cominimal equivalence relations (2.13) if $n \leq k + 1$. In general they also include examples of the fusion identities implied by the extended Speiser algorithm (this is shown in part three of the appendix). In all cases the relations that generate $F_k \cdot C_k$ imply the relations that generate the transposition ideals $\mathcal{I}_k$. Therefore $\mathcal{I}_k$ is a subideal of $F_k \cdot C_k$.

In the first two parts of the appendix we use the determinant formula (2.4), and the extended Speiser algorithm to show the converse, namely, that the generators (2.7) imply the entire set of cominimal equivalence relations (2.13) and the entire set of fusion relations (2.9). Therefore $F_k \cdot C_k$ is a subideal of $\mathcal{I}_k$, and the equivalence (2.20) follows.
From (2.17) it follows that the rings related by interchange of rank and reduced level are isomorphic:

\[ R_{n,k} \equiv R_{k,n} , \]  

with the isomorphism given by tableau transposition.

It is well known that there is a solution for basis elements \( \phi_a \) of the \( \text{SO}(2n+1)_{2k+1} \) fusion ring for every integrable highest weight representation \( r \) of \( \text{SO}(2n+1)_{2k+1} \) of the form

\[ \phi_a(r) = \frac{S_{ar}}{S_{0r}} . \]  

(2.22)

Given this fact and isomorphism (2.18) it follows that the solutions of the \( n \) polynomial equations (2.7) for the \( n \) variables \( \chi_i \) are exactly those solutions (2.22) of the fusion rule algebra that in addition satisfy the cominimal equivalence relations (2.13). Since

\[ S_{\sigma(a)}r = \begin{cases} +S_{ar} & \text{for } r \text{ a tensor} \\ -S_{ar} & \text{for } r \text{ a spinor} \end{cases} , \]  

(2.23)

imposition of the cominimal equivalence relations (2.13) excludes precisely the solutions (2.22) for which \( r \) is a spinor. Therefore, the values of the fundamental characters

\[ \chi_i = \frac{S_{it}}{S_{0t}} , \]

where \( t \) is any tensor representation specified by a tableau with \( \ell_1 \leq k \), give a complete set of solutions of (2.7). From this result and determinant formula (2.4), the values of the \( R_{n,k} \) basis character corresponding to an arbitrary \( n \) row tableau \( a \) with \( \ell_1(a) \leq k \) are

\[ \text{char}_a(t) = \frac{S_{at}}{S_{0t}} , \]  

(2.24)

where \( t \) (a tableau with \( \ell_1 \leq k \)) labels the solution.

It follows from (2.23) that the relation between the \( R_{n,k} \) multiplicities (2.14) and the \( \text{SO}(2n+1)_{2k+1} \) modular matrix elements,

\[ M_{ab}^c = 4 \sum_{t} \frac{S_{at}S_{bt}S_{ct}}{S_{0t}} , \]  

(2.25)

involves a sum only over the integrable tensor representations with \( \ell_1 \leq k \). The matrices \( S_{ab} \) that diagonalize the \( R_{n,k} \) tableau basis are related to the \( \text{SO}(2n+1)_{2k+1} \) modular transformation matrices \( S_{ab} \) (note the difference in typeface) by

\[ S_{ab} = 2S_{ab} . \]  

(2.26)

The orthonormality condition

\[ \sum_{t} S_{at}S_{tb} = \delta_{ab} \]  

(2.27)
for the matrices $S_{ab}$ then follows from the orthonormality condition for $S_{ab}$.

3. Fusion Rings and Topological Metrics

After pointing out that the $R_{n,k}$ are fusion rings, but that they are not the fusion rings of any conformal field theory, we identify a special field whose properties ensure that the $R_{n,k}$ will admit an invertible topological metric.

3.1 The Quotients $R_{n,k}$ as Fusion Rings

A rational fusion ring is a finite-dimensional, commutative, associative, $\mathbb{Z}$-ring with identity that also has an involutive conjugation automorphism and a special \textit{multiplicity basis} in which the structure constants $N_{ab}^c$ that define the fusion product (2.9) between the special basis elements are non-negative integers.\textsuperscript{22} The rings $R_{n,k}$ clearly have all these properties (all fields are self-conjugate and the tableau basis is the special multiplicity basis), so that they are rational fusion rings.

We will now show that the $R_{n,k}$ are not, in general, the fusion rings of any conformal field theory. This result is expected since (diagonal) modular invariant partition functions of the $\text{so}(2n+1)_{2k+1}$ fusion ring (of conformal scalars) with the spinors removed are not known. While the cominimal quotient of the entire $\text{so}(2n+1)_{2k+1}$ fusion ring likely is the fusion ring of a conformal field theory, the presence of the spinors makes finding potentials that might lead to integrable $N = 2$ theories more difficult.

The simple case $R_{1,1}$, which is associated with $\text{so}(3)_3$, has two elements $1$, and $\Box$, which obey the fusion rule

$$\Box \times \Box = 1 + 2 \Box. \quad (3.1)$$

It is known that this fusion ring cannot satisfy certain constraints on the conformal weights and modular matrix elements required in a conformal field theory.\textsuperscript{36, 22}

To see how this works out in a more complicated case consider the ring $R_{2,1}$, which is associated with $\text{so}(5)_3$. Its three elements correspond to the tableaux $1$, $\Box$, and $\Box$. If this ring were the fusion ring of a conformal field theory, then the properly normalized matrices $S$ and $T$ that diagonalize the fusion rules would have to satisfy (since $S$ is real and all fields are self-conjugate)

$$(ST)^3 = I. \quad (3.2)$$

In addition, the conformal weights $h_i$ and the symmetric fusion coefficients $N_{ijk}$ must satisfy Vafa’s constraint\textsuperscript{37}

$$\prod_r (\alpha_i \alpha_j \alpha_k \alpha_l) N_{ijr} N_{rkl} = \prod_r \alpha_i N_{ijr} N_{rkl} + N_{ikr} N_{rjl} + N_{irl} N_{rjk}, \quad (3.3)$$
where \( \alpha_i = e^{2\pi i h_i} \).

The nontrivial fusion rules of \( \mathcal{R}_{2,1} \) are

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\text{\( \square \times \square \)}
\end{array}
\end{array}
&= 1 + \square + \square \\
\begin{array}{c}
\begin{array}{c}
\text{\( \square \times \square \)}
\end{array}
\end{array}
&= \square + 2 \square \\
\begin{array}{c}
\begin{array}{c}
\text{\( \square \times \square \)}
\end{array}
\end{array}
&= 1 + 2 \square + 2 \square
\end{align*}
\]

(3.4)

and the diagonalizing matrix \( S_{ab} \) is

\[
S_{ab} = \frac{1}{2\sqrt{3}} \times \begin{pmatrix}
1 & \square & \square \\
\frac{\sqrt{3}-1}{2} & 2 & \frac{\sqrt{3}+1}{2} \\
\frac{\sqrt{3}+1}{2} & 2 & -2 \\
-2 & \frac{\sqrt{3}-1}{2} & \frac{\sqrt{3}+1}{2}
\end{pmatrix}.
\]

(3.5)

The 15 equations (3.3) yield three independent constraints on the \( \alpha_i \), which are, with the indices \((i, j, k, l)\) of (3.3) listed on the right,

\[
\begin{align*}
\alpha_1 &= 1 \quad (1, 1, 1, 1) \\
\alpha^3_\square &= \alpha^3 \quad (\square, \square, \square, \square) \\
\alpha^6_\square &= \alpha^3 \quad (\square, \square, \square, \square)
\end{align*}
\]

(3.6)

It follows that the conformal weight of the identity is integral \((\alpha_1 = 1)\), and that there are nine possible solutions to the two remaining constraints \(\alpha^3 = 1\) and \(\alpha^3 = 1\).

For any Virasoro central charge \( c \),

\[
\mathcal{T}_i = e^{[2\pi i (h_i - c/24)]},
\]

(3.7)

and the identity-identity matrix element of (3.2) becomes

\[
\sum_{i,j} S_{0i} S_{ij} S_{j0} \alpha_i \alpha_j = e^{\pi ic/4}.
\]

(3.8)

By explicit computation, we find that none of the nine possible values for the \( \alpha_i \) satisfy (3.8) for any real central charge \( c \). Therefore, \( \mathcal{R}_{2,1} \) cannot be the fusion ring of any conformal field theory. While it is difficult to adapt this form of argument to the arbitrary case, we expect that the rings \( \mathcal{R}_{n,k} \) do not in general underlie conformal field theories. Although this might be expected (since the spinors have been omitted) it raises the interesting problem of characterizing the conditions under which a fusion quotient of a conformal-field-theory fusion ring also underlies a conformal field theory.
The topological Landau-Ginzburg theories described in section five have chiral rings isomorphic to the fusion rings $R_{n,k}$. In the natural tableau basis for these rings the topological metric $\eta_{ab}$ is given in terms of the $R_{n,k}$ multiplicities (2.14) by

\[
\eta_{ab} = M_{ab}^{\hat{c}},
\]

where $\hat{c}$ is the tableau with $n$ rows each of length $k$. It can be seen that $\eta_{ab}$ is an invertible matrix as follows.

Let $\rho$ denote the operation of tableau complement in an $n \times k$ rectangle. In terms of the row lengths, $\ell_i(a)$, of a tableau $a$, the tableau $\rho(a)$ has row lengths $\ell_i(\rho(a)) = k - \ell_{n+1-i}(a)$. Diagrammatically the operation $\rho$ forms the complement of the tableau $a$ in the $n \times k$ rectangle, and then rotates this complement by 180 degrees to put it in standard position. The complement of the identity, $\rho(0) = \hat{c}$, is the unique tableau with the maximal number of cells $(nk)$.

In order to show that $\eta_{ab}$ is invertible, we will only need to calculate $M_{ab}^{\hat{c}}$ when $r(a) + r(b) \leq nk$. We will perform the calculation by imposing the rank and fusion modification rules on the tableau product (2.5) of $a$ and $b$. All the terms in (2.5) with $d$ not equal to the identity $(0)$ have fewer boxes than $\hat{c}$. This means that only the terms with $d = 0$, which are given by applying the Littlewood-Richardson rule to $a$ and $b$, can directly produce $\hat{c}$. All we need to know about the modification rules is the

Modification Rule Property: Any sequence of modification rules that connect a tableau with cells outside the $n \times k$ rectangle (i.e., one with $\ell_1 > k$ or $m_1 > n$) to one inside this rectangle (i.e., one with $\ell_1 \leq k$ and $m_1 \leq n$) relates the outer tableau to an inner tableau with fewer cells.

(3.10)

If $r(a) + r(b) < nk = r(\hat{c})$ then (3.10) implies that $\hat{c}$ cannot possibly appear in (2.5) so that $M_{ab}^{\hat{c}} = 0$ for all such cases. In the cases with $r(a) + r(b) = nk$, $M_{ab}^{\hat{c}}$ just equals, again due to (3.10), the Littlewood-Richardson multiplicity, $L_{ab}^{\hat{c}}$.

In the context of $\text{su}(n)$ representation theory, the tableau $\hat{c}$ just denotes the $\text{su}(n)$ identity representation, and $\rho(a)$, the complement of $a$ in an $n \times k$ rectangle, denotes the complex conjugate representation, $\overline{\rho}(a)$, of the $\text{su}(n)$ representation $a$. Since $L_{ab}^{\hat{c}}$ equals the multiplicity of the identity in the $\text{su}(n)$ tensor product $a \otimes b$, since $a \otimes b$ does not contain the identity unless $b = \overline{a}$, and since $a \otimes \overline{a}$ contains the identity just once, it is clear that $L_{ab}^{\hat{c}} = \delta_{\rho(a)}$ for $n \geq 2$. (The same result holds for $n = 1$ trivially.) Therefore,

\[
\eta_{ab} = \delta_{\rho(a)} \quad \text{for} \quad r(a) + r(b) \leq nk.
\]

(3.11)
Unlike $L_{ab}^c$, the fusion multiplicities $M_{ab}^c$ do not vanish when $r(a) + r(b) > nk$. Since each row of $\eta_{ab}$ in fact contains several nonzero elements, demonstration of its invertibility requires a little more work.

Order the basis tableaux that label the rows $a$ and columns $b$ of $\eta_{ab}$ in terms of the increasing number of cells $r(a)$ and $r(b)$ in the respective basis tableaux. This means that any tableaux $a$ ($b$) with more cells appears below (to the right of) tableaux with fewer cells. For example, the identity 0 labels the first row and column and $\rho(0)$ labels the last row and column. Similarly, labels the second row and column, and $\rho(\blacksquare)$ labels the next-to-last row and column. Let

$$D = \binom{n+k}{n}$$

denote the dimension of the matrix $\eta_{ab}$. This is just the dimension of the ring $\mathcal{R}_{n,k}$ (i.e., the number of $n$ column tableaux with $\ell_1 \leq k$). From (3.11) we see that any matrix element with row label $a$ (and appearing at the $i$th row), and with column label $b$ with $r(b) < nk - r(a)$ (so that it appears in the $j$th row with $j < D + 1 - i$) vanishes. In the remaining cases, there occur subblocks positioned along the $(i,D+1-i)$ anti-diagonal with each subblock labeled by tableaux with the same number of cells. Result (3.11) implies that in each row of any subblock there is only one nonzero entry (which is unity) so that a rearrangement of columns will, in all cases, put $\eta_{ab}$ in the form of a matrix with all anti-diagonal matrix elements $(i,D+1-i)$ equal to unity, and all upper anti-triangular matrix elements (i.e., all those having coordinates $(i,j)$ with $j < D + 1 - i$) equal to zero. The determinant of such a $(D \times D)$ matrix is $(-1)^{D(D-1)/2}$ so that

$$|\det \eta_{ab}| = 1 \quad (3.12)$$

and the matrix inverse of $\eta_{ab}$ exists. It is clear from the cofactor expression for $\eta_{ab}^{-1}$ and the integrality of the determinant that the inverse of the integral matrix $\eta_{ab}$ is also an integral matrix.

For example, the complete matrix $\eta_{ab}$ for $\mathcal{R}_{2,1}$, with the basis ordering $1$, $\blacksquare$, and with $\hat{c} = \blacksquare$, can be written down directly from the fusion rules (3.4):

$$\eta_{ab} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix} \quad \text{and} \quad \eta_{ab}^{-1} = \begin{pmatrix} 2 & -2 & 1 \\ -2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad (3.13)$$

As claimed, all entries in $\eta^{-1}$ are integers.

The complement operation also figures prominently in the $su(N)_K$ and $Sp(N)_K$ deformations of the Grassmannian superpotentials. In these cases the complement
map is a simple-current fusion-rule automorphism, and the invertibility of the topological metric follows directly from this fact. In the present \( \text{SO}(N)_K \) case, the complement operation retains the property of leading to an invertible topological metric, even though it is not a fusion rule automorphism. In the following subsection we recall that the complement operation is related to Poincaré duality.

### 3.3 The Grassmannian Schubert Calculus

While the results in this section are (presumably well-known but somewhat) implicit in previous work,\(^{38, 20, 39}\) we state them here to make salient certain relevant points.

The integral homology of the Grassmannian

\[
G_{n,k} = \frac{U(n+k)}{U(n) \times U(k)} \tag{3.14}
\]

is freely generated by the Schubert cycles (or subvarieties) \( \sigma_a \) where \( a \) is a tableau with first column length \( m_1(a) \leq n \) and first row length \( \ell_1(a) \leq k \). All cycles with \( m_1 > n \) or \( \ell_1 > k \) are null.\(^{38}\) Since the intersection of two such subvarieties can be written as an integral linear combination of the generating Schubert cycles,

\[
\sigma_a \star \sigma_b = \sum_c G_{ab}^c \sigma_c, \tag{3.15}
\]

the Schubert cycles form a \( \mathbb{Z} \)-ring under the intersection product \( \star \). With \( K_i^\sigma \) for \( i = 1, \ldots, k \) denoting the cycles corresponding to single-row tableaux of length \( \ell_1 = i \), an arbitrary cycle can be written as a polynomial in the \( K_i^\sigma \) via the \( m_1(a) \times m_1(a) \) determinant\(^{38}\)

\[
\sigma_a = \det |K_{\ell_1-i+j}^\sigma| \tag{3.16}
\]

Here, \( K_i^\sigma \) with \( i > k \) or \( i < 0 \) should be set to zero. Since this determinant formula is identical to the expansion of \( U(n) \) characters in terms of row characters of covariant \( U(n) \) tableaux, the tableau product given by polynomial multiplication (\textit{without} setting \( K_i^\sigma \) to zero for \( i > k \)) is just that given by the Littlewood-Richardson rule. To recover the actual intersection product we must impose the

\textit{Grassmannian Modification Rules:} Remove any tableau with first column length greater than \( n \) (the rank modification rule) and any tableau with first row length greater than \( k \) (the fusion modification rule).

Note that these modification rules possess property (3.10). If just the rank modification rule is imposed, the ring we obtain is exactly the ring of \( U(n) \) characters, \textit{i.e.}, the \( U(n) \) representation ring of covariant tensors. This means that the intersection ring of Schubert cycles is a quotient of the \( U(n) \) representation ring of covariant
tensors by the ideal defined by the fusion modification rule. Since the intersection multiplicities $G_{ab}^c$ can be calculated by imposing the Grassmannian rank and fusion modification rules on the Littlewood-Richardson product, the proof of the last section applies directly so that

$$G_{ab}^c = \delta_{b,\rho(a)} .$$  \hspace{1cm} (3.17)

In the context of algebraic geometry this result is demonstrated by an appeal to Poincaré duality (and the fact that analytic cycles, such as the Schubert cycles considered here, intersect positively).\textsuperscript{38}

From the manifest symmetry under interchange of $n$ and $k$ given by row and column interchange, one also has

$$\sigma_a = \det \left| \chi_{\sigma i - i + j}^\sigma \right|$$ \hspace{1cm} (3.18)

where $\chi_i^\sigma = 0$ if $i > n$ or $i < 0$. From this determinant it follows that the conditions

$$\chi_i^\sigma = 0 \quad \text{for} \quad i = n + 1, \ldots$$ \hspace{1cm} (3.19)

generate the ideal defined by the rank modification rules. Then, it follows from (3.16) that the fusion ideal is generated by the polynomial identities

$$K_i^\sigma(\chi_j^\sigma) = 0 \quad \text{for} \quad i = k + 1, \ldots, k + n .$$ \hspace{1cm} (3.20)

These polynomials integrate to a quasi-homogeneous potential. The $N = 2$ superconformal field theories characterized by these superpotentials\textsuperscript{16} have the intersection rings (3.15) as chiral rings. Since the topological metric is given by $\eta_{ab}^0 = G_{ab}^c$, (3.17) yields the well-known result

$$\eta_{ab}^0 = \delta_{b,\rho(a)} ,$$ \hspace{1cm} (3.21)

and the invertibility of the Grassmannian topological metric follows immediately.
4. Potentials with the $\mathcal{R}_{n,k}$ Fusion Rings as Local Rings

Now we are ready to construct potentials that have the rings $\mathcal{R}_{n,k}$ as local rings, i.e. we will find a $V$ such that

$$\mathcal{R}_{n,k} = \mathbb{Z}[\chi_1, \ldots, \chi_n] / dV $$

(4.1)

for each $n$ and $k$. The construction is natural in the context of $\text{SO}(N)$ group theory. Although many potentials that have $\mathcal{R}_{n,k}$ as a local ring can be found, we claim that those constructed here are special, in that they do yield (we expect integrable) $\mathcal{N} = 2$ Landau-Ginzburg theories, and in that the associated (twisted $\mathcal{N} = 2$) topological theories have several special properties, as we shall see. (We have found that the covering-space potentials of ref. [25] do not always project onto potentials in the physical variables, since, at least in several examples, all the critical-point vanishings appear in the Jacobian rather than in the derivatives of the physical potential.)

The characters $\chi_0, \chi_1, \ldots$ can be written as specialized versions of the elementary symmetric functions

$$\chi_j = E_j = E_j(q_1, \ldots, q_n, q_1^{-1}, \ldots, q_n^{-1}, 1)$$

(4.2)

where the $E_j$ are defined in terms of the auxiliary variables $q_i$ ($i = 1, \ldots, n$) via the generating function

$$E(t) = \sum_{j=0}^{\infty} E_j t^j = (1 + t) \prod_{i=1}^{n} (1 + q_i t)(1 + q_i^{-1} t) .$$

(4.3)

It follows that $E_0 = 1$, $E_{2n+1-j} = E_j$ for $j = 0, \ldots, 2n + 1$, and $E_j = 0$ if $j > 2n + 1$, so that the identities (2.3) are indeed satisfied.

It is a standard result that the determinant (2.4) and the identity (4.2) imply that the character $K_i$ of the single row tableau of length $i$ satisfies

$$K_i = H_i - H_{i-2} ,$$

(4.4)

where the $i^{th}$ complete symmetric function $H_i$ of the variables $q_i$ can be obtained from the generating function

$$H(t) = \sum_{j=0}^{\infty} H_j t^j = \frac{1}{1 - t} \prod_{i=1}^{n} \frac{1}{(1 - q_i t)(1 - q_i^{-1} t)} .$$

(4.5)

\(^{\dagger}\) Without imposition of the $\mathcal{I}_k$ generating relations, the $q_i$ are exponentials of certain Cartan subalgebra elements; when the relations (2.7) are imposed they take fixed values (5.20).
Note that \( H(t)E(-t) = 1 \).

The similarity between identities (4.3–4.5) and those used in the construction of the \( \text{Sp}(N)_K \) potentials in ref. [21] suggests that we consider the generating function

\[
V(t) = \log E(t) = \log(1 + t) \prod_{i=1}^{n} (1 + q_i t)(1 + q_i^{-1} t)
\]

\[
= \sum_{m=1}^{\infty} (-1)^{m-1} V_m t^m ,
\]

in which case

\[
V_m = \frac{1}{m} \sum_{i=1}^{n} (q_i^m + q_i^{-m}) + \frac{1}{m} .
\]

Differentiation of (4.7) and the use of (4.6) yields

\[
\frac{\partial V_m}{\partial E_i} = \sum_{0 \leq j \leq 2n+1} (-1)^j H_{m-j} \frac{\partial E_j}{\partial E_i} .
\]

Since \( \chi_j = E_j = E_{2n+1-j} \), we find

\[
\frac{\partial V_m}{\partial \chi_i} = (-1)^i (H_{m-i} - H_{m-2n+1-i}) \quad \text{for} \quad 1 \leq i \leq n .
\]

We now consider, as candidates for the potentials we want, the differences

\[
\mathcal{V}_m = V_m - V_{m-2} ,
\]

which satisfy

\[
\frac{\partial \mathcal{V}_m}{\partial \chi_i} = (-1)^i (K_{m-i} - K_{m-2n+1+i}) \quad \text{for} \quad 1 \leq i \leq n .
\]

Although the potentials \( \mathcal{V}_m \) are defined in terms of the \( q_i \), it is clear that they are also polynomials in the \( \chi_i \). Upon setting \( m = n + k + 1 \), we see that the critical point conditions

\[
\frac{\partial \mathcal{V}_{n+k+1}}{\partial \chi_i} = 0 \quad 1 \leq i \leq n
\]

exactly coincide (by setting \( i = n + 1 - j \) in eq. 4.12) with the generators of the transposition ideal \( \mathcal{I}_k \) (2.7).

Therefore, the local ring of the polynomial \( \mathcal{V}_{n+k+1} \), written in terms of the fundamental variables \( \chi_1, \ldots, \chi_n \), is, in a natural way, exactly \( \mathcal{R}_{n,k} \).

16
While the shortest path to obtaining $V_{n+k+1}$ as a polynomial in the $\chi_i$, for given $n$ and $k$, is to integrate (4.12), it is useful to have in hand a recursion relation for the potentials. Since the coefficient of $q^j$ in the expansion of the polynomial

$$P(q) = \prod_{i=1}^{n} (q - q_i)(q - q_i^{-1})$$

(4.13)

is given by

$$\sum_{p=0}^{j} (-1)^p \chi_p,$$

and since the $q_i^{\pm 1}$ all satisfy $P(q) = 0$, one obtains from this identity the recursion relation

$$\sum_{i=0}^{2n-1} \left( \sum_{j=0}^{i} (-1)^j \chi_j \right) [(l + 2n - i)V_{l+2n-i} - 1] + \left\{ \begin{array}{ll} 2n & l = 0 \\ lV_l - 1 & l > 0 \end{array} \right\} = 0.$$  (4.14)

for the $V_m$.

As an example, the recursion relation for $n = 2$ yields the potentials related to $\text{so}(5)_{2k+1}$. With $x = \chi_1$ and $y = \chi_2$ it reads, in terms of the auxiliary quantity $U_m = mV_m - 1$,

$$U_{l+4} - (x - 1)U_{l+3} + (y - x + 1)U_{l+2} - (x - 1)U_{l+1} + \left\{ \begin{array}{ll} 4 & l = 0 \\ U_l & l > 0 \end{array} \right\} = 0$$  (4.15)

where $U_1 = x - 1, U_2 = x^2 - 2y - 1, \text{and } U_3 = x^3 - 3xy + 3y - 1$ are the proper initial conditions. Then, $V_m = U_m/m - U_{m-2}/(m-2) - 2/(m^2 - 2m)$ gives the actual potential.

5. $\mathcal{N} = 2$ Landau-Ginzburg Theories

Consider the $\mathcal{N} = 2$ supersymmetric theory characterized by the superpotential $W_{n,k} = \lambda^{n+k+1}V_{n+k+1}(\Phi_i)$ for $n$ chiral superfields $\Phi_i$. A finite number of states are topological (in that their correlation functions do not depend on the locations of the fields), and form a closed ring

$$\mathbb{C}[\Phi_1, \ldots, \Phi_n]/dW_{n,k}$$

(5.1)

clearly isomorphic to $\mathcal{R}_{n,k}$. A tableau basis for this chiral ring is given by associating with any tableau $a$ with $\ell_1 \leq k$ the field

$$\psi_a = \frac{\lambda^{r(a)}}{2} \det |\Phi_{m_i-i+j} + \Phi_{m_i-i+2-j}|$$

(5.2)
where $\Phi_i = 0$ for $i < 0$. We take the analog of (2.3)

$$
\begin{align*}
\Phi_{n+j} &= \Phi_{n-j+1} \\
\Phi_{n+j} &= 0
\end{align*}
$$

as a definition of $\Phi_i$ for $i > n$. While this form of the potential and chiral ring basis makes the isomorphism of the chiral ring with $R_{n,k}$ transparent, the field redefinition $\Phi_i \to \lambda^{-i} \Phi_i$ makes the role of the deformation parameter $\lambda$ clear. Under this rescaling substitution, the critical point vanishing conditions become

$$
\frac{\partial W_{n,k}^\lambda(\Phi_j)}{\partial \Phi_i} = (-1)^{i+1} (K_{k+n+1-i}^\lambda(\Phi_j) - \lambda^{2n+1-2i} K_{k-n+i}^\lambda(\Phi_j)) = 0
$$

(5.4)

where we have set $W_{n,k}^\lambda(\Phi_j) = W_{n,k}(\lambda^{-j} \Phi_j)$, and where, from (5.2),

$$
K_j^\lambda(\Phi_i) = \frac{1}{2} \lambda^i \det |\lambda^{-(mp-p+q)} \Phi_{mp-p+q} + \lambda^{-(mp-p+2-q)} \Phi_{mp-p+2-q}|
$$

$$
\frac{1}{2} \det |\Phi_{mp-p+q} + \lambda^{2q-2} \Phi_{mp-p+2-q}|
$$

(5.5)

with $p, q = 1, \ldots, j$. The column lengths are all one ($m_p = 1$), but we have left this implicit for clarity. The limit $\lambda \to 0$ of (5.4) gives

$$
\frac{\partial W_{n,k}^0(\Phi_j)}{\partial \Phi_i} = (-1)^{i+1} \det |\Phi_{mp-p+q}| = 0
$$

(5.6)

which are exactly the vanishing conditions (3.20) for the cohomology ring of the Grassmannian $G_{n,k}$ which integrate to the known quasi-homogeneous, Grassmannian superpotentials. Similarly, under this scaling the tableau basis (5.2) becomes

$$
\psi_a = \frac{1}{2} \det |\Phi_{mp-p+q} + \lambda^{2q-2} \Phi_{mp-p+2-q}|
$$

so that, as $\lambda$ goes to zero, it flows directly to the standard tableau basis of the Grassmannian chiral ring (3.18). In particular, the special field $\psi_c$ flows smoothly to the unique field with maximal charge. The Landau-Ginzburg potentials of these $\text{SO}(2n+1)_{2k+1}$ related fusion rings therefore arise as deformations of the same $\mathcal{N} = 2$ superconformal field theories that can be deformed to obtain the $\text{Sp}(n)_k$ and $\text{SU}(n+1)_k$ fusion potentials. It would be interesting if all three of these deformation directions yield integrable theories of the kinks that interpolate between the critical points.

One can, of course, obtain the same result by scaling the $q_i$ in section four by $\lambda^{-1}$, in which case one sees that the defining form of the potential,

$$
(W_{n,k})_{\text{quasi-hom}} = \frac{1}{n+k+1} \sum_{i=1}^{n} q_i^{n-k+1},
$$

(5.7)
is identical to that expected for the Grassmannian $G_{n,k}$ in terms of the variables $q_i$.

While the tableau basis (5.2) is a natural deformation of the Grassmannian chiral ring, the metric $\eta_{ab}$, although invertible, is not anti-diagonal, and so depends on the parameter $\lambda$. Therefore the tableau basis is not flat and it is unclear how to establish a general connection to the canonical bases that arise in conformal perturbation theory.

Since the critical points of the rings $R_{n,k}$ are completely non-degenerate we are dealing with completely massive theories for non-zero $\lambda$.

In the $n = 1$ case, the one variable superpotential $W_{1,k}(\Phi)$ with local ring $R_{1,k}$ (necessarily) gives a deformation of an $\mathcal{N} = 2$ minimal models. For example, the first two potentials are,

\[
W_{1,2}^\lambda(\Phi) = \frac{1}{4}\Phi^4 - \lambda\Phi^3 + 2\lambda^3\Phi \\
W_{1,3}^\lambda(\Phi) = \frac{1}{5}\Phi^5 - \lambda\Phi^4 + \frac{2}{3}\lambda^2\Phi^3 + 2\lambda^3\Phi^2 - \lambda^4\Phi - \lambda^5
\] (5.8)

In order to compare these with the known integrable deformations\textsuperscript{17} the field $\Phi$ must first be shifted to remove the leading term of the deformation. In general, the deformation in this new basis is a linear combination of the $\Phi^j$ for $j = 1, \ldots, n+k-1$ so that they appear to be different massive theories than those considered previously.

The first interesting multi-variable potential has $R_{2,1}$ as a chiral ring. With $x = \Phi_1$ and $y = \Phi_2$ it is

\[
\lambda^{-4}W_{2,1}(x,y) = \frac{1}{4}x^4 + \frac{1}{2}y^2 - x^2y + xy - \frac{1}{2}x^2 + y - x .
\] (5.9)

For comparison, the two-variables potentials related to $sp(2)_1$ and $su(3)_1$, are

\[
\lambda^{-4}W_{sp(2)_1}(x,y) = \frac{1}{4}x^4 + \frac{1}{2}y^2 - x^2y + y - \frac{1}{2} \\
\lambda^{-4}W_{su(3)_1}(x,y) = \frac{1}{4}x^4 + \frac{1}{2}y^2 - x^2y + x .
\] (5.10)

All three have the same leading terms

\[
\frac{1}{4}x^4 + \frac{1}{2}y^2 - x^2y
\] (5.11)

which is the quasihomogeneous potential whose local ring is the cohomology ring of the Grassmannian $U(3)/U(2) \otimes U(1)$.

While we will leave the question of the quantum integrability of the theories based on these potentials for another work, we will now show that the $\mathcal{N} = 2$ topological theories that can be constructed from these potentials enjoy rather special properties.
5.1 Twisted $\mathcal{N} = 2$ Topological Field Theories

By twisting the energy-momentum tensor with the $U(1)$ generator present in any $\mathcal{N} = 2$ theory, the topological character of the chiral rings $\mathcal{R}_{n,k}$ can be exploited to produce $\mathcal{N} = 2$ topological field theories composed entirely of the chiral primary fields. Due to the close connection with the $\text{so}(2n+1)_{2k+1}$ fusion algebra the topological field theories can be completely solved.

The genus $g$ correlation functions with an insertion of an arbitrary function $f$ of chiral superfields $\Phi_i$, with superpotential $W$, is

$$\langle f(\Phi_i) \rangle_g = \sum_{dW=0} H^{g-1} f(\Phi_i)$$

(5.12)

where the handle operator is (using the normalization of ref. [39])

$$H = (-1)^{n(n-1)/2} \det(\partial_i \partial_j W).$$

(5.13)

The sum is over the critical points at which $dW = 0$, and there is one critical point for each basis tableau of $\mathcal{R}_{n,k}$. Since

$$\det \left( \frac{\partial^2 W}{\partial \chi_j \partial \chi_k} \right) = \det \left( \frac{\partial^2 W}{\partial E_j \partial E_k} \right) = \frac{1}{\Delta^2} \det \left( \frac{\partial^2 W}{\partial q_j \partial q_k} \right),$$

(5.14)

where (using eq. 4.3)

$$\Delta = \prod_{i=1}^{n} q_i^{-1} \prod_{i=1}^{n} (q_i - q_i^{-1}) \prod_{i<j} ((q_i + q_i^{-1}) - (q_j + q_j^{-1})],$$

(5.15)

and since $\Delta$ does not vanish at the critical points, the handle operator at the critical point corresponding to the tableau $a$ is

$$H(a) = (-1)^{r(a)} [2i(n+k)]^n \left\{ \Delta^{-2} \prod_{i=1}^{n} q_i^{-2} (q_i - q_i^{-1}) \right\}_{q_i=q_i(a)}.$$

(5.16)

This expression can be related to the $\text{so}(2n+1)_{2k+1}$ modular transformation matrices $S_{ab}$ as follows. The explicit form for the modular transformation matrices for $B_n^{(1)}$ which is usually given for $n \geq 3$ also holds for $n = 1 (A_1^{(1)})$ and $n = 2 (C_2^{(1)})$. Therefore for all $n$ we have

$$S_{ab} = (-)^{n(n-1)/2} 2^{n-1} (k+n)^{-n/2} \det \text{Mat}(a,b)$$

(5.17)

with

$$\text{Mat}_{ij}(a,b) = \sin \left( \frac{2\pi \theta_i(a) \theta_j(b)}{k+n} \right),$$

(5.18)
for $i, j = 1, \ldots, n$, and

$$\theta_i(a) = \ell_i(a) - i + n + \frac{1}{2}, \quad i = 1, \ldots, n \tag{5.19}$$

where the $\ell_i(a)$ are the row lengths of the tableau $a$ given in (2.1).

Then, since

$$q_j(a) = e^{i \pi \theta_j(a)/(n+k)} \tag{5.20}$$

are the values that the $q_j$ take at the critical point corresponding to the tableau $a$, we find that (with the $a$ dependence of the $q_j$ suppressed)

$$S_{0a} = \frac{(-1)^{n(n-1)/2}}{2[2(n+k)]^{n/2}} \prod_{i=1}^{n} (q_i^{\frac{1}{2}} - q_i^{-\frac{1}{2}}) \prod_{i<j} \left[ (q_i + q_i^{-1}) - (q_j + q_j^{-1}) \right]. \tag{5.21}$$

In addition,

$$\text{char}_c(a) = S_{\hat{c}a} = S_{0a} A_0 \tag{5.22}$$

By combining (5.16) with (5.21) and (5.22) the handle operator can be written

$$H(a) = \frac{1}{(S_{\hat{c}a})^2} \frac{1}{\text{char}_c(a)} \tag{5.23}$$

where $S_{0a} = 2S_{0a}$ (2.26).

Due to (3.12) the inverse operator of $\Phi_{\hat{c}}$,

$$\Phi_{\hat{c}}^{-1} = \sum_b \eta_{ab}^{-1} \Phi_b, \tag{5.24}$$

exists as an integral linear combination of basis elements of the chiral ring $R_{n,k}$. The value of the field $\Phi_{\hat{c}}^{-1}$ at the $a$th critical point is given by one over the character at that point

$$\Phi_{\hat{c}}^{-1}(a) = \frac{1}{\text{char}_c(a)} = \frac{S_{0a}}{S_{\hat{c}a}}. \tag{5.25}$$

Therefore the handle operator satisfies

$$H(a) = (S_{0a})^{-2} \Phi_{\hat{c}}^{-1}(a) \tag{5.26}$$

for all critical points $a$. Due to the non-singularity of the matrix $S_{ab}$ (c.f. 2.27) this uniquely specifies the operators and we have

$$H = H_0 \Phi_{\hat{c}}^{-1}, \tag{5.27}$$

where the untwisted handle operator $H_0$ satisfies

$$H_0(a) = (S_{0a})^{-2}. \tag{5.28}$$
The tableau-basis expansion for the inverse operator begins \( \Phi^{-1} = \Phi \hat{c} + \ldots \) (to be compared with \( \Phi^{-1} = \Phi \hat{c} \) for \( \text{Sp}(N)_K \)). The leading behaviour is therefore \( H = H_0 \Phi \hat{c} \), as expected.

Due to (2.27) the correlation functions are properly normalized. For example, the residue formula (5.12) and (5.26) give

\[
\langle \Phi_a \rangle_{g=0} = \sum_{\ell_1(t) \leq k} \Phi_{\ell_1}(t) \Phi_a(t) (S_{0t})^2 = \sum_{\ell_1(t) \leq k} S_{\ell t} S_{ta} = \delta_{a\hat{c}}, \tag{5.28}
\]

where the sum is over all tableaux with \( \ell_1 \leq k \). Note that \( \langle \Phi^{-1} \rangle = \langle \Phi \rangle = 1 \) at genus zero. The topological metric for the theory based on the \( \text{so}(2n+1)_{2k+1} \) superpotential is therefore given by

\[
\eta_{ab} = \langle \Phi_a \Phi_b \rangle_{g=0} = M_{ab} \hat{c}, \tag{5.29}
\]

as expected. This metric differs from the metric expected on the basis of conformal perturbation theory in a flat basis, \( \eta^0_{ab} = \delta_{b\rho(a)} \) (3.21). In the \( \text{su}(N)_K \) and \( \text{sp}(N)_K \) cases the presence of a simple current associated with the complement operation led to the tableau basis being itself flat. Since \( \eta_{ab} \) is not anti-diagonal the three point function

\[
\langle \Phi_a \Phi_b \Phi_c \rangle_{g=0} = \sum_d M_{ab} \eta_{dc} \tag{5.30}
\]

does not collapse in this basis to a single fusion coefficient, as it did in the \( \text{su}(N)_K \) and \( \text{Sp}(N)_K \) cases.

The correlation function of an arbitrary product of \( s \) chiral primary fields \( \Phi_{a_1}, \ldots, \Phi_{a_s} \) on an genus \( g \) Riemann surface

\[
\langle \Phi_{a_1} \ldots \Phi_{a_s} \rangle_g = \sum_{\ell_1(t) \leq k} S_{0t}^{-2(g-1)} \frac{S_{a_1 t}}{S_{0t}} \ldots \frac{S_{a_s t}}{S_{0t}} \left( \frac{S_{\hat{c} t}}{S_{0t}} \right)^{(g-1)} \tag{5.31}
\]

is almost certainly a non-negative integer. Since the \( \text{so}(2n+1)_{2k+1} \) Verlinde numbers for a genus \( g \) surface

\[
N^g_{a \ldots z} = \sum_r S_{0r}^{-2(g-1)} \frac{S_{ar}}{S_{0r}} \ldots \frac{S_{2r}}{S_{0r}} \tag{5.32}
\]

(where the sum is over all integrable irreducible representations of \( \text{so}(2n+1)_{2k+1} \), including spinors), are known to be non-negative integers, we can prove this result directly for \( g = 0 \) and \( g = 1 \). If \( g = 0 \),

\[
\langle \Phi_{a_1} \ldots \Phi_{a_s} \rangle_0 = 4 \sum_{\ell_1(t) \leq k} S_{0t}^{-2(g-1)} \frac{S_{a_1 t}}{S_{0t}} \ldots \frac{S_{a_s t}}{S_{0t}} \frac{S_{\hat{c} t}}{S_{0t}} \tag{5.33}
\]

\[
= N^0_{a_1 \ldots a_s \hat{c}} + N^0_{a_1 \ldots a_s} \sigma(\hat{c})
\]

22
and, of course, if \( g = 1 \)
\[
\langle \Phi_{a_1} \cdots \Phi_{a_s} \rangle_1 = N_{a_1 \cdots a_s}^{1}.
\] (5.34)

If \( g > 1 \), then identity (5.24) allows an arbitrary \( s \)-point correlation function on any genus to be written as a sum of Verlinde numbers:
\[
\langle \Phi_{a_1} \cdots \Phi_{a_s} \rangle_g = \frac{1}{4g} \sum_{b_1} \eta_{0b_1}^{-1} \cdots \sum_{b_{g-1}} \eta_{0b_{g-1}}^{-1} \left( 4 \sum_{\ell_1(t) \leq k} \frac{S_{0t}^{-2(g-1)}S_{a_1 t}}{S_{0t}} \cdots \frac{S_{a_s t}^{g-1}S_{b_g t}}{S_{0t}} \prod_{j=1}^{g-1} \frac{S_{b_j t}}{S_{0t}} \right)
\]
\( \sum_{b_i} \) is over the \( n \) row tableaux with \( \ell_1 \leq k \). While the Verlinde numbers and the matrix elements \( \eta_{0b_i} \) are all integers, it is not immediately clear whether the correlation functions themselves must be integers due to the fractional prefactor. We have verified the non-negative integrality of many correlation functions by explicit computation. For example, since the one-point functions for \( R_{2,1} \):
\[
\langle \Phi_0 \rangle_g = \langle \Phi_{-1} \rangle_g = 3^{g-1}(2^g + (-1)^{g+1})
\]
\( \langle \Phi_{-2} \rangle_g = 3^{g-1}(2^{g+1} + (-1)^g) \),
(5.36)
are non-negative integers for all \( g \), the correlation functions on any genus are non-negative integers in this case. The fact that the correlation functions are (apparently) all non-negative integers suggests that these numbers might have an interesting geometrical interpretation.

We expect the equality \( \langle \Phi_{-1} \rangle = \langle \Phi_{-2} \rangle \), which follows from (5.36) for \( R_{2,1} \), to hold in general (on arbitrary genus).

5.2 \( \mathcal{N} = 2 \) Rank-Level Duality

Several works\(^20,39\) have commented on the manifest duality\(^†\) of the Grassmannian \( G_{n,k} \) under interchange of \( n \) and \( k \). It was shown in ref. [21] that the chiral rings of the \( \text{sp}(N)_K \) and \( \text{sp}(K)_N \) fusion superpotentials are isomorphic under tableau transposition. (Rank-level-interchange identities connecting the chiral rings of the \( \text{su}(N)_K \) and \( \text{su}(K)_N \) superpotentials were also found in ref. [21], but they do not lead directly to isomorphisms.) Just recently, interesting implications of the rank-level identities for \( B_n, D_n, \) and \( C_n \) conformal weights and modular matrix elements proved in ref. [35] have been found for \( \mathcal{N} = 2 \) superconformal theories.\(^43\)

Here we show that the pair of topological field theories built by twisting the \( \mathcal{N} = 2 \) models associated with the \( \text{so}(2n+1)_{2k+1} \) and \( \text{so}(2k+1)_{2n+1} \) Landau-Ginzburg

\(^†\)Another kind of duality, involving an interchange of the level of the \( \mathcal{N} = 2 \) algebra, has been considered in ref. [42].
potentials are also equivalent. This follows from the isomorphism $\mathcal{R}_{n,k} = \mathcal{R}_{k,n}$ given by tableau transposition (2.21). It also follows from the one-to-one correspondence of primary fields of $\mathcal{R}_{n,k}$ and $\mathcal{R}_{k,n}$ given by transposition, and the identity

$$
\left( S_{\bar{a}b} \right)_{\text{SO}(2n+1)_{2k+1}} = \left( S_{\bar{a}b} \right)_{\text{SO}(2k+1)_{2n+1}},
$$

(5.37)

where $\bar{a}$ denotes the transpose tableau of $a$ (i.e., $\ell_i(\bar{a}) = m_i(a)$). This identity was proved in ref. [35]. The tableau language permits a uniform presentation of rank-level-duality identities that relate each pair of the $A_1^{(1)}$, $C_2^{(1)}$, and $B_n^{(1)}$ fusion rings in various ways, as well as relating $C_2^{(1)}$ level two with $C_2^{(1)}$ level two in a different way than the $\text{sp}(N)_K$ with $\text{sp}(K)_N$ duality exploited in ref. [21]. These results and expression (5.31) imply that

$$
\langle \Phi_{a_1} \Phi_{a_2} \cdots \Phi_{a_s} \rangle_{\mathcal{R}_{n,k}} = \langle \Phi_{\bar{a_1}} \Phi_{\bar{a_2}} \cdots \Phi_{\bar{a_s}} \rangle_{\mathcal{R}_{k,n}}
$$

(5.38)

on Riemann surfaces of any genus. Therefore the (twisted $N = 2$) Landau-Ginzburg models based on the cominimally reduced $\text{so}(2n+1)_{2k+1}$ and $\text{so}(2k+1)_{2n+1}$ theories are exactly dual.

Even in the simple case of $\mathcal{R}_{2,1}$ and $\mathcal{R}_{1,2}$ the rank-level dual potentials look quite different:

$$
V_{2,1}(x,y) = \frac{1}{4}x^4 + \frac{1}{2}y^2 - x^2y + xy - \frac{1}{2}x^2 + y - x
$$

$V_{1,2}(x) = \frac{1}{4}x^4 - x^3 + 2x.

(5.39)

Nevertheless, they lead to identical $N = 2$ topological theories.

6. Comment on the Pattern

The known (massive) $N = 2$ Landau-Ginzburg theories with chiral rings related to WZW fusion rings fit a simple pattern: They all arise as deformations that lead from the quotient of a $U(N)$ representation ring (the Grassmannian cohomology ring) that restores Young tableau transposition symmetry to the quotient of a $U(N)$–subgroup representation ring that also maintains this symmetry. In addition, the tensor language of the $U(N)$–subgroup Young tableaux gives a natural description of the deformed chiral ring. This pattern extends to all the classical groups; no irregularity arises from the varying details of the relevant Dynkin diagrams. It remains a (not impossible) challenge to exhibit the exceptional groups as members of this pattern. In the superconformal case, the quotient of the $U(N)$ representation ring that restores the tableau transposition symmetry has a beautiful, and manifestly rank-level symmetric, geometric realization as the intersection ring of Schubert cycles. It would be very interesting if an analogous geometric interpretation could be found for this symmetry in the massive case.
While we have not given direct evidence that the $\text{SO}(2n+1)_{2k+1}$ related potentials yield integrable theories of the kinks that interpolate between the potentials’ critical points, these theories should at least provide an interesting test case for deciding which fusion ring properties are necessary for integrability. Although it might be tempting to suggest that there is a one-to-one correspondence between the fusion rings of rational conformal field theories and integrable deformations, the results presented here suggest that there are likely more integrable deformations than just those that reproduce conformal-field-theory fusion rings.

Acknowledgement  We thank M. Bourdeau for many useful discussions.

Appendix

Parts one and two give the proof that the transposition ideal $I_k$ contains the cominimal ideal $C_k$ and the entire fusion ideal $F_k$. Part three contains the proof that the extended Speiser algorithm implies those generators of $I_k$ that are not instances of cominimal equivalence. Part four compares this argument with the analogous $su(N)_K$ and $sp(N)_K$ arguments.

1. Proof that the transposition ideal contains all cominimal equivalence relations

We will now show that the relations (2.7) that generate the transposition ideals $I_k$, which read

$$K_{k+j} - K_{k+1-j} = 0 \quad \text{for} \quad j = 1, \ldots, n,$$

(A.1)

generate all of the cominimal equivalence relations (2.13).

Evaluation of the determinant formula (2.4) for the single-row character $K_{k+j}$ yields the useful recursion relation

$$K_{k+j} = \sum_{i=1}^{n} \left[ (-1)^{i+1} \chi_i (K_{k+j-i} - K_{k+j-2n+1+i}) \right] + K_{k+j-2n-1} \quad \text{for} \quad k+j > 2.$$  \hspace{1cm} (A.2)

If $j = n + 1$ then the coefficient of each $\chi_i$ vanishes by assumption (A.1) and relation (A.1) also holds for $j = n + 1$.

An induction argument shows that (A.1) also holds for $n + 2 \leq j \leq k + n$, as follows. First, assume that (A.1) holds for all $j < m \leq k + n$. Using this assumption the recursion relation (A.2) for $j = m$ can be written

$$K_{k+m} = \sum_{i=1}^{n} \left[ (-1)^{i+1} \chi_i (K_{k+1+i-m} - K_{k+2-m+2n-i}) \right] + K_{k+2+2n-m}.$$  \hspace{1cm} (A.3)

Replacing $K_{k+2+2n-m}$ by using (A.2) with $j = 2 + 2n - m$, and cancelling terms, yields $K_{k+m} - K_{k+1-m} = 0$. Therefore, by induction, we now know that

$$K_{k+j} - K_{k+1-j} = 0 \quad \text{for} \quad j = 1, \ldots, n + k.$$  \hspace{1cm} (A.3)
While this is all that is needed in this section, we shall need the case \( j = n + k + 1 \) later. If \( n > 1 \), the induction argument just given applies directly so that equation A.3 also holds for \( j = n + k + 1 \). If \( n = 1 \) (i.e., for \( \text{so}(3) \)), there is a slightly different result. From (A.2) we find (for \( n = 1 \), so that \( j = k + 2 \))

\[
K_{2k+2} = \chi_1(K_{2k+1} - K_{2k}) + K_{2k-1}.
\]

Applying (A.3) to the right-hand side yields

\[
K_{2k+2} = \chi_1(K_0 - K_1) + K_2 = -K_0.
\]

To summarize, we have shown that

\[
\begin{align*}
K_{k+j} - K_{k+1-j} &= 0 & j = 1, \ldots, k + n, k + n + 1 & (n > 1) \\
K_{k+j} - K_{k+1-j} &= 0 & j = 1, \ldots, k + 1 & (n = 1) \\
K_{k+j} + K_0 &= 0 & j = k + 2 & (n = 1)
\end{align*}
\]

(A.4)

(where, in fact, \( K_0 = 1 = K_{2k+1} \)). Note that the equality in the third line is not of the form (A.3) and so is not a transpose of any of the generators (2.3). (The same is true if \( n > 1 \) with the first deviation occurring for \( j = k + 2n \).)

The general cominimal equivalence relation (2.13) for any integrable representation \( \tau \),

\[
\text{char}_\sigma(\tau) - \text{char}_\tau = 0,
\]

(A.5)

now follows from the identities (A.4) and the determinant formula: Given an arbitrary integrable representation \( \tau \) (so that \( \ell_1 + \ell_2 \leq 2k + 1 \)) the determinant in expression (2.2) for its character has first-row matrix elements

\[
K_{\ell_1+j-1} + K_{\ell_1-(j-1)} & j = 1, \ldots, m_1
\]

which equal, due to (A.4),

\[
K_{(2k+1-\ell_1)+j-1} + K_{(2k+1-\ell_1)-(j-1)} & j = 1, \ldots, m_1.
\]

The resulting determinant is exactly that of the character of a representation with first-row length \( 2k + 1 - \ell_1 \), but with all other row lengths the same as \( \tau \). This is exactly the definition of the action of \( \sigma \) (2.12), so that the identities (A.4) imply (A.5) for an arbitrary integrable representation \( \tau \). Therefore, the transposition ideal contains all cominimal equivalence relations (for all \( n \geq 1 \)).

2. Proof that the transposition ideal \( \mathcal{I}_k \) contains the fusion ideal \( \mathcal{F}_k \)

We begin by showing that (A.4) implies a set of boundary fusion relations for representations that just fail to be integrable. Then we will show that these boundary relations generate the entire fusion ideal \( \mathcal{F}_k \).
First consider an arbitrary representation with $\ell_1 + \ell_2 = 2k + 2$. For $n = 1$, $\ell_2 = 0$ and the required relation is just the last identity in (A.4). For $n \geq 2$, define $p$ for $k + 2 > p > 0$ by setting $\ell_1 = k + p$ and $\ell_2 = k + 2 - p$. The determinant in (2.2) for a representation $\tau$ with first row length $\ell_1 = k + p$ has first row matrix elements

$$K_{k+p+j-1} + K_{k+p-(j-1)} \quad j = 1, \ldots, m_1.$$ 

Since there are at most $n$ entries in this row ($m_1 \leq n$), the greatest subscript on $K$, in any possible case, is $2k + n$. Therefore, the identities (A.4) imply that the matrix elements of the first row are

$$K_{(k+2-p)-2+j} + K_{(k+2-p)-j} \quad j = 1, \ldots, m_1.$$ 

However, these are identical (in each column) to the matrix elements of the second row of the determinant so that the character must vanish:

$$\text{char}_\tau = 0 \quad \text{for} \quad \ell_1(\tau) + \ell_2(\tau) = 2k + 2 \quad \text{if} \quad n \geq 2. \quad (A.6)$$

Now consider an arbitrary representation with $\ell_1 + \ell_2 = 2k + 3$ (this case will only be needed for $n \geq 2$). Define $p$ for $k + 2 > p > 0$ by writing $\ell_1 = k + 1 + p$ and $\ell_2 = k + 2 - p$. The matrix elements in the first two rows ($i = 1, 2$ below) of the determinant (2.2) for such a representation $\tau$ are

$$K_{k+p+j} + K_{k+p+2-j} \quad i = 1 \quad j = 1, \ldots, m_1.$$ 

Application of (A.4) transforms these matrix elements into

$$K_{k-p+j-1} + K_{k-p-j+1} \quad i = 1 \quad j = 1, \ldots, m_1.$$ 

Interchanging the first two rows, while leaving all others alone, produces exactly the determinant for the character of a representation $\overline{\tau}$ with $\ell_1 = k + p$, $\ell_2 = k + 1 - p$, and all other row lengths equal to the corresponding row lengths in the original tableau. Therefore, the original character ($\tau$) equals the negative of the character ($\overline{\tau}$) obtained by removing one cell from each of the first two rows of the original representation's tableau, i.e.

$$\text{char}_\tau = -\text{char}_{\overline{\tau}} \quad \text{for} \quad \ell_1 + \ell_2 = 2k + 3 \quad \text{and} \quad n \geq 2. \quad (A.7)$$

A special case occurs when the second and third row lengths in the original representation $\tau$ are equal. Then $\overline{\tau}$ is not a valid tableaux and the determinant produced above vanishes, so that char$_\tau$ does too.

For $n = 1$, the last identity in (A.4) and, for $n \geq 2$, the two identities (A.6) and (A.7), completely determine the fusion product of two arbitrary basis elements $\phi_a$
and $\phi_b$ of the fusion ring, as follows. First we expand $\phi_b$ as a polynomial in the $\chi_i$. Then associativity of the fusion product implies that we can perform the product of $\phi_a$ with a single $\chi_i$ (for each term) first. We compute this fusion product by first computing the tableau product. For $n = 1$ this corresponds to the tensor product between an arbitrary tableaux and a single cell tableaux, which can only increase the first row length by one, a case covered by the last equation in (A.4). (Note that, in terms of $su(2)$ tableaux, we are only dealing with even length $su(2)$ tableaux.) For $n \geq 2$ this tableau product between an arbitrary integrable tableaux and a single-column tableaux can at most increase each of the first two row lengths by one. Therefore, the only non-integrable tableau appearing are those with $\ell_1 + \ell_2 = 2k + 2$, or $2k + 3$, and these are exactly the cases for which we can use (A.6) and (A.7) to replace any such tableau with one with $\ell_1 + \ell_2 \leq 2k + 1$. The iteration of this step involving the fusion of a sum of (integrable) tableaux of the previous step with single column tableaux results in the complete expansion of the fusion product of $\phi_a$ and $\phi_b$ into a sum of $\phi_c$ with $c$ an (integrable) tableaux, without having to consider any non-integrable representations with $\ell_1 + \ell_2 > 2k + 3$. Therefore, the fusion ideal $F_k$ for $so(2n + 1)_{2k+1}$ is generated by the relations satisfied by the two boundary cases $\ell_1 + \ell_2 = 2k + 2$ and $\ell_1 + \ell_2 = 2k + 3$.

Now we will verify that the extended Speiser algorithm implies the relations (A.6) and (A.7) for $n \geq 2$ and as well as the third line of (A.4) for $n = 1$. Let $\varphi$ denote half the sum of positive roots for the algebra being discussed. If $n = 1$, then the $A_1^{(1)}$ Dynkin index vector for a highest weight plus $\varphi$ is $[2K - 2\ell_1, 2\ell_1]$, since $K = 2k + 1$ is half the usual $su(N)$ normalization, and since $a_1 = 2\ell_1$. If $\ell_1 = K + 1$, a single Weyl reflection produces, after removal of $\varphi$, $[0, 2K]$, which is just the highest weight of a single-row tableaux of length $K$, as required by the third line of (A.4).

If $n = 2$, then the $C_2^{(1)}$ Dynkin index vector for a highest weight plus $\varphi$ is $[K + 1 - \ell_1 - \ell_2, 2\ell_2 + 1, \ell_1 - \ell_2 + 1]$. For $\ell_1 + \ell_2 = K + 1$ this weight lies on the boundary of the first Weyl chamber, and the character vanishes. For $\ell_1 + \ell_2 = K + 2$, then a single Weyl reflection produces $[1, 2\ell_2 - 1, \ell_1 - \ell_2 + 1]$. If $\ell_2 = 0$ then a further Weyl reflection produces a weight on the boundary of the first Weyl chamber and the character vanishes. If $\ell_2 > 0$ then the vector is positive and removal of $\varphi$ gives $[0, 2\ell_2 - 2, \ell_1 - \ell_2]$, which is just the highest weight of a tableaux with row lengths $\ell'_i$ given by $\ell'_1 = \ell_1 - 1$ and $\ell'_2 = \ell_2 - 1$. These are exactly the results expected for $n = 2$.

If $n \geq 3$, the first case to consider is $\ell_1 + \ell_2 = 2k + 2$. The Dynkin indices of such a highest weight plus $\varphi$,

$$(2k + 2 - \ell_1 - \ell_2, \ell_1 - \ell_2 + 1, \ldots, 2\ell_n + 1),$$

give a weight which is on the boundary of the first Weyl chamber, so that the character of any representation with $\ell_1 + \ell_2 = 2k + 2$ vanishes.
For the second case ($\ell_1 + \ell_2 = 2k+3$), let $\ell_1 = k + 1 + p$ and $\ell_2 = k + 2 - p$, where $1 \leq p \leq k + 1$. The highest weight Dynkin indices plus half the sum of positive roots are

$$(-1, 2p, k + 3 - p - \ell_3, \ldots)$$

and we must perform the Weyl reflection corresponding to the affine root to bring the weight into the first Weyl chamber. The result

$$(1, 2p - 1, k + 2 - p - \ell_3, \ldots)$$

has nonnegative entries. The only possible zero occurs for $\ell_3 = k + 2 - p$ (i.e., $\ell_3 = \ell_2$), in which case we again get a vanishing result. Otherwise we obtain the Dynkin indices of an integrable representation, $(0, 2p - 2, k + 1 - p - \ell_3, \ldots)$, corresponding to a representation with $\ell_1 = k + p$, $\ell_2 = k + 1 - p$, and with all other row lengths the same as the original representation. Since an odd number of Weyl reflections were used, the character of the new representation equals the negative of the character of the original one. These results exactly reproduce (A.7), including the special case in which the second and third row lengths are equal.

The final conclusion is that the generating relations of the ideals $I_k$ imply the fusion ideal generating relations and all cominimal equivalence relations, as claimed in the text.

3. **Proof that the extended Speiser algorithm and cominimal equivalence implies the generators of the the transposition ideal $I_k$**

For $n = 1$ and $n = 2$ the generators are instances of the cominimal equivalence relations. For $n \geq 3$ the same is true if $k + 1 \geq n$. For the remaining case $n \geq k + 2$ we only need consider the generators (2.7) for $j = k + 2, \ldots, n$, in which case they read (since $k + 1 - j < 0$)

$$K_{k+j} = 0 \quad j = k + 2, \ldots, n.$$ (A.8)

The highest weight vector plus half the sum of positive roots for $K_{k+j}$ is

$$[k + 2 - j, k + j + 1, 1, \ldots, 1].$$

Since the $(j - k - 1)^{th}$ entry vanishes after $j - k - 2$ Weyl reflections for each $j = k + 2, \ldots, n$, all of these weights are on the boundary of a Weyl chamber, as required by (A.8).

4. **Comparison with the $SU(N)_K$ and $Sp(N)_K$ cases.**

It is instructive to compare the above arguments with the analogous arguments for $SU(N)_K$ and $Sp(N)_K$. The tableaux of an integrable $SU(N)_K$ or $Sp(N)_K$ representation satisfy $\ell_1 \leq K$. The appropriate determinant formula is just that given in (3.18) for $SU(N)_K$ (with the $\chi_i^\sigma$ taken to be the characters of single-column tableaux
and a different set of rank modification rules imposed) and just that given in (2.4) for $\text{Sp}(N)_K$. Using these determinant expansions (and the appropriate rank modification rules) for arbitrary basis fields as polynomials in single-column tableaux, the fusion product of two representations only requires computing a series of fusion products of general integrable representations with single-column tableaux. Hence, any non-integrable tableau occurring at any step in this process will have $\ell_1 = K + 1$. We know from the extended Speiser algorithm that all such tableaux have vanishing characters. In order to find a set of generators which imply that every representation with $\ell_1 = K + 1$ has vanishing character, consider the row-character expansions (3.16) for $\text{su}(N)_K$ and (2.2) for $\text{Sp}(N)_K$. Since the rows beyond the first of any such vanishing determinant must allow arbitrary variations (according to the arbitrary values of the lower rows of the initial tableau), the only way every such determinant can vanish is for the top row of the determinant to also vanish. Therefore, these top-row-entry vanishing conditions (in terms of the characters $\mathcal{K}_j$ of the single-row tableau with $j$ boxes)

$$\mathcal{K}_{K+1+j} + \delta_G \mathcal{K}_{K+1-j} = 0 \quad \text{for } j = 0, \ldots, n-1,$$

where $\delta_G$ is zero for $\text{su}(N)_K$ and one for $\text{Sp}(N)_K$ and $n$ denotes the rank of the relevant group, generate the conformal-scalar fusion rules of the relevant current algebra.

**References**

[1] S. Cecotti and C. Vafa, *Nucl. Phys.* **B367** (1991) 359; “Ising Model and $N = 2$ Supersymmetric Theories,” HUTP-92-A044 (hepth•9209085); “On the Classification of $N = 2$ Supersymmetric Theories,” HUTP-92-A064 (hepth•9211097)

[2] S. Cecotti, P. Fendley, K. Intriligator, and C. Vafa, *Nucl. Phys.* **B386** (1992) 405; M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, *Nucl. Phys.* **B405** (1993) 279

[3] H. Saleur, *Nucl. Phys.* **B382** (1992) 486 and 523; P. Fendley and H. Saleur, *Nucl. Phys.* **B388** (1992) 609

[4] E. Witten, “Quantum Background Independence in String Theory,” IASSNS-HEP-93/29 (hepth•9306122)

[5] S. Kachru and E. Witten, “Computing The Complete Massless Spectrum of A Landau-Ginzburg Orbifold,” IASSNS-HEP-93/40 (hepth•9307038)

[6] For a review see N. Warner, “$N = 2$ Supersymmetric Integrable Models and Topological Field Theories,” Trieste Summer School on High Energy Physics and Cosmology (hepth•9301088)
[7] A. Sen, *Nucl. Phys.* B278 (1986) 289; T. Banks, L. Dixon, D. Friedan, and E. Martinec, *Nucl. Phys.* B299 (1988) 613

[8] L. Dixon, in *Superstrings, Unified Theories and Cosmology 1987*, G. Furlan *et. al.* editors, World Scientific (1988); W. Lerche, C. Vafa, and N. Warner, *Nucl. Phys.* B324 (1989) 427; for references and reviews see *Essays on Mirror Manifolds* (ed. S.-T. Yau) International Press, Hong Kong, 1992

[9] E. Witten, *Comm. Math. Phys.* 117 (1988) 353; 118 (1988) 411; *Nucl. Phys.* B340 (1990) 284

[10] R. Dijkgraaf and E. Witten, *Nucl. Phys.* B342 (1990) 486

[11] K. Li, *Nucl. Phys.* B354 (1991) 711, 725

[12] R. Dijkgraaff, H. Verlinde, and E. Verlinde *Nucl. Phys.* B352 (1991) 59

[13] M. Bershadsky, W. Lerche, D. Nemeschansky, and N. Warner, *Nucl. Phys.* B401 (1993) 304; S. Mukhi and C. Vafa, “Two Dimensional Black Hole as a Topological Coset of c = 1 String Theory,” HUTP-93/A002 (hepth•9301083); S. Panda and S. Roy, “On the Twisted $N = 2$ Superconformal Algebra in 2D Gravity Coupled to Matter,” IC/93/81 (hepth•9305039)

[14] Z. Maassarani, D. Nemeschansky, and N. Warner, *Nucl. Phys.* B393 (1993) 523; D. Nemeschansky and N. Warner, “Off-Critical Lattice Analogues of $N = 2$ Supersymmetric Quantum Integrable Models,” USC-93/018 (hepth•9307141)

[15] E. Martinec, *Phys. Lett.* 217B (1989) 431; C. Vafa and N. Warner, *Phys. Lett.* 218B (1989) 51

[16] W. Lerche, C. Vafa, and N. Warner, *Nucl. Phys.* B324 (1989) 427

[17] P. Fendley, S. Mathur, C. Vafa, and N. Warner, *Phys. Lett.* B243 (1990) 257; P. Fendley, W. Lerche, S. Mathur, and N. Warner, *Nucl. Phys.* B348 (1991) 66

[18] W. Lerche and N. Warner, *Nucl. Phys.* B358 (1991) 571

[19] D. Nemeschansky and N. Warner, *Nucl. Phys.* B380 (1992) 241; A. LeClair, D. Nemeschansky, and N.Warner *Nucl. Phys.* B390 (1993) 653

[20] D. Gepner, *Commun. Math. Phys.* 141 (1991) 381-411

[21] M. Bourdeau, E. Mlawer, H. Riggs, and H. Schnitzer, *Mod. Phys. Let.* A7 (1992) 689 (hepth•9111020)
[22] For a review see J. Fuchs, “Fusion Rules in Conformal Field Theory,” NIKHEF-H-93-15 (hepth・9306162)

[23] P. Di Francesco and J.-B. Zuber, “Fusion potentials I,” J. Phys. A26 (1993) 1441; P. Di Francesco, F. Lesage, and J.-B. Zuber, “Graph Rings and Integrable Perturbations of N = 2 Superconformal Theories,” Saclay preprint SPhT 93/057 (hepth・9306018)

[24] O. Aharony, Phys. Lett. B306 (1993) 276

[25] M. Crescimanno, Nucl. Phys. B393 (1993) 361

[26] C. Gómez and G. Sierra, “On the integrability of N = 2 Landau-Ginzburg models: A graph generalization of the Yang-Baxter equation,” CERN-TH 6963/93 (hepth・9309007)

[27] E. Eguchi and S.-K. Yang, Mod. Phys. Lett. A5 (1990) 1693

[28] W. Fulton and J. Harris, Representation Theory (Springer-Verlag, New York, 1990)

[29] R. King, J. Math. Phys. 12 (1971) 1588

[30] D. Littlewood, The Theory of Group Characters, 2nd ed. (Oxford Univ. Press, Oxford, 1950) p. 110

[31] E. Verlinde, Nucl. Phys. B300 (1988) 360

[32] M. Walton Phys. Lett. B241 (1990) 365; Nucl. Phys. B340 (1990) 777; V. Kac, Infinite-dimensional Lie Algebras, third edition (Cambridge Univ. Press, Cambridge 1990)

[33] C. Cummins, J. Phys. A 24 (1991) 391

[34] J. Fuchs and D. Gepner, Nucl. Phys. B294 (1987) 30

[35] E. J. Mlawer, S. G. Naculich, H. A. Riggs, and H. J. Schnitzer, Nucl. Phys. B 352 (1991) 863

[36] P. Christe and F. Ravanini, Phys. Lett. B217 (1989) 252; G. Rivlis, Mod. Phys. Lett. A5 (1990) 2063

[37] C. Vafa, Phys.Lett. B206 (1988) 421

[38] P. Griffith and J. Harris, Principles of Algebraic Geometry, J. Wiley and Sons (1978)
[39] K. Intriligator, *Mod. Phys. Lett.* **A6** (1991) 3543

[40] C. Vafa, *Mod. Phys. Lett.* **A6** (1991) 337

[41] V. Kac and M. Wakimoto, *Proc. Nat. Acad. Sci.* **85** (1988) 4956

[42] Y. Kazama and H. Suzuki, *Nucl. Phys.* **B321** (1989) 232

[43] J. Fuchs and C. Schweigert, “Level-Rank Duality of WZW Theories and Isomorphisms of $N = 2$ Coset Models,” NIKHEF-H-93-16a, revised August 1993 (hepth•9307107)

[44] D. Gepner and A. Schwimmer, *Nucl. Phys.* **B380** (1992) 147 (hepth•9204020)