A model of jam formation in congested traffic

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Abstract.

We study a model of irreversible jam formation in congested vehicular traffic on an open segment of a single-lane road. The vehicles obey a stochastic discrete-time dynamics which is a limiting case of the generalized Totally Asymmetric Simple Exclusion Process. Its characteristic features are: (a) the existing clusters of jammed cars cannot break into parts; (b) when the leading vehicle of a cluster hops to the right, the whole cluster follows it deterministically, and (c) any two clusters of vehicles, occupying consecutive positions on the chain, may become nearest-neighbors and merge irreversibly into a single cluster. The above dynamics was used in a one-dimensional model of irreversible aggregation by Bunzarova and Pesheva [Phys. Rev. E 95, 052105 (2017)]. The model has three stationary non-equilibrium phases, depending on the probabilities of injection ($\alpha$), ejection ($\beta$), and hopping ($p$) of particles: a many-particle one, MP, a completely jammed phase CF, and a mixed MP+CF phase. An exact expression for the stationary probability $P(1)$ of a completely jammed configuration in the mixed MP+CF phase is obtained. The gap distribution between neighboring clusters of jammed cars at large lengths $L$ of the road is studied. Three regimes of evolution of the width of a single gap are found: (i) growing gaps with length of the order $O(L)$ when $\beta > p$; (ii) shrinking gaps with length of the order $O(1)$ when $\beta < p$; and (iii) critical gaps at $\beta = p$, of the order $O(L^{1/2})$. These results are supported by extensive Monte Carlo calculations.

1. Introduction

We study a discrete-time discrete-space model of congested traffic on finite segments of a single-lane road, which allows a detailed description of the growth of clusters of jammed vehicles under different boundary conditions. The stochastic dynamics is a particular case of the generalized Totally Asymmetric Simple Exclusion Process (gTASEP), which has been used in a one-dimensional model of irreversible aggregation [1].

Various generalizations of the TASEP have been developed and used to model traffic flow, with a special focus on the formation and spreading of traffic jams, see, e.g., [2, 3]. The problem is of particular interest also for various transport processes taking place in the living cells. From the viewpoint of physics, molecular motors are proteins or macromolecular complexes that utilize chemical energy to move collectively on a single track in a way resembling vehicular traffic [4].

In Sec. II we formulate the model, present the phase diagram and some basic properties which were obtained in [1]. In Sec. III an exact (in the thermodynamic limit) analytic expression for the stationary probability $P(1)$ of finding completely jammed configurations is derived. In
Sec. IV we consider the evolution of the gaps between consecutive clusters under different boundary conditions: three regimes of asymptotic behavior of the gaps lifetimes are found and the corresponding asymptotic form of $P(1)$ is evaluated close to the threshold of complete jamming. Perspectives for further developments are given in the concluding Sec. V.

2. The Model

We consider an open chain of $L$ sites, labeled consecutively by the index $i = 1, 2, \ldots, L$. Each site of the lattice can be empty or occupied by just one particle. The bulk dynamics of the generalized totally asymmetric simple exclusion process (gTASEP) corresponds to a discrete-time backward-ordered update with two hopping probabilities $p$ and $\tilde{p}$. The update of the whole configuration starts with the last site: if site $L$ is occupied, the particle is removed from it with probability $\tilde{\beta}$. In the bulk, consecutive updates of all pairs of nearest-neighbor sites are performed in the backward order $((L-1, L), \ldots, (i, i+1), \ldots, (1, 2))$. Single particles, as well as the leading particle of a cluster of close packed particles, hop one site to the right with probability $p$, when that site is vacant. If the leading particle of a cluster has hopped, each of the remaining particles of that cluster hop to the empty site on the right with probability $\tilde{p}$. Finally, the first site of the chain is updated: a particle is injected with probability $\beta > 0$, if the site was vacant at the beginning of the configuration update, and with probability $\tilde{\alpha} = \min\{\alpha\tilde{p}/p, 1\}$, if it was initially occupied by a particle, but became vacant after the update of the bond $(1, 2)$. Note that when $\tilde{p} = p$ one has the TASEP with backward-sequential update, and when $\tilde{p} = 0$ - the TASEP with parallel update. In the general case of $0 < \tilde{p} < 1$, the model was studied under periodic boundary conditions in [5, 6].

We adopt the above stochastic dynamics with $\tilde{p} = 1$, which implies the following properties: (a) Existing clusters of jammed vehicles do not break into parts; (b) Whenever the leading vehicle of a cluster hops with probability $p$ to an empty site on the right, the whole cluster follows it as a whole entity. When a cluster reaches the last site of the chain, it starts hopping forward with probability $\beta$, rather than $p$. (c) Any two clusters of vehicles, occupying consecutive positions on the chain, may become nearest-neighbors and merge irreversibly into a single cluster.

As shown in [1], this model has three non-equilibrium stationary phases: a many-particle one, MP, a phase with completely filled configuration, CF, and a mixed MP+CF phase, see Fig. 1.

![Figure 1](image)

**Figure 1.** Phase diagram in the plane of injection ($\alpha$) - ejection ($\beta$) probabilities. The small circle with the letter 'T' beside denotes the triple point.

The many-particle phase MP occupies two regions, MP I and MP II; it contains a macroscopic number of clusters of jammed cars of size $O(1)$ as $L \to \infty$; MP I and MP II differ by the shape of the local density profile and the type of evolution of the rightmost gaps. The phase CF consists of completely jammed configurations carrying the current $J = \beta$. The phase MP+CF is mixed: the completely filled configurations are perturbed by short living gaps entering the chain from the first site.
3. Probability of complete jamming

An expression for the probability $P(1)$ of a completely jammed configuration in the mixed MP+CF phase can be derived from the global stationarity condition $J_{in} = J_{out}$, where $J_{in}$ ($J_{out}$) is the input (output) current. If the system is in a completely jammed configuration, which happens with probability $P(1)$, the first site becomes empty with probability $\beta$, as a result a particle ejection from the last site of the chain and a deterministic shift of the whole cluster one site to the right; then a particle may enter the system with probability $\tilde{\alpha} = \alpha/p$. The total probability of this event is $P(1)(\alpha/p)\beta$. When the system is not completely jammed, which takes place with probability $1-P(1)$, a particle enters at the first site with probability $\alpha$, as shown in [1]. By equating the input current $J_{in} = P(1)\alpha\beta/p + [1 - P(1)]\alpha$ to the output current $J_{out} = \beta$ (since $\rho_1^{\text{MP+CF}} = 1$), one solves for $P(1)$ and obtains: $P(1) = p(\alpha - \beta)/[\alpha(p - \beta)]$, $\beta \leq \alpha < p$. This result is in excellent agreement with the Monte Carlo simulation data.

4. Evolution of the system configurations

In particle hopping discrete-space traffic models, one often uses the dual representation of configurations in terms of empty sites positions, instead of particle coordinates. In the case of our model, such a representation leads to a very peculiar dynamics of the inter-cluster gaps.

First, because existing clusters cannot break up, gaps may appear only at the first site of the chain. Second, gaps disappear when two consecutive clusters merge or when the rightmost cluster leaves the system and the next one reaches the last site of the chain. Third, and most interesting feature of the gap dynamics is that, as long as two consecutive gaps exist, the distance between them remains constant due to the conservation of particles in the cluster between them. Thus, in the space-time picture, the neighboring edges of two coexisting gaps are forced to move in parallel and gaps may not cross or merge, see a computer simulation in Fig. 2. The evolution of a mixed configuration in MP+CF, close to the boundary with MP II, is shown in Fig. 3. When $(1 - \tilde{\alpha}) \ll 1$, a vacancy at the first site of a completely filled configuration appears, on the average, $N = [(1 - \tilde{\alpha})\beta]^{-1} \gg 1$ updates (discrete time steps). The width of each gap performs a random walk, which begins with an initial state containing one or several neighboring empty sites, and ends up when the random walk reaches the origin. If the average lifetime $\bar{n}$ of the gap is $\bar{n} < \bar{N}$, then an estimate of the probability $P(1)$ is given by $P(1) \approx 1 - \bar{n}/\bar{N}$.

In general, when $\beta \neq p$, the size of the rightmost gap performs an asymmetric random walk: its right edge belongs to the cluster which hops one site to the right with the ejection probability $\beta$; on the other hand, its left edge moves one site to the right with probability $p$. Therefore, after each update, the gap width increases by one site with probability $p_g = \beta(1 - p)$, decreases by one site with probability $q_g = p(1 - \beta)$, and remains the same with probability $r = 1 - q_g - p_g$. Three regimes of evolution of the gap width follow:

1. Growing gaps. In the region MP I $\beta > p$ and $p_g > q_g$. Hence, the gap width asymptotically grows with the number $n \gg 1$ of time steps as $W(n) \propto (p_g - q_g)n$. The gap will exist until its lower edge reaches the end of the chain, that is for $L/p$ time steps on the average, while its two edges coexist until the time moment $L/\beta$, when the upper edge reaches the end of the chain. According to the random walk theory, the probability that a gap with unit initial width collapses, before the time moment $L/\beta$, tends to $q_g/p_g < 1$, as $L \to \infty$. Therefore, the probability that the growing gap survives until the end of the chain is $P_{\text{surv}} = 1 - q_g/p_g = (\beta - p)/[\beta(1 - p)]$. On the average, such a gap will appear after $N_e$ updates, hence the estimate $P(1) \approx 1 - \bar{n}/\bar{N}_e$ follows.

2. Short-living gaps. When $\beta < p$, one has $p_g < q_g$ and the gap between the newly growing cluster and the cluster which is leaving the system from its right boundary closes in a finite number of time steps. An upper estimate can be given by the result for a random walk on an infinite chain with initial state at site $i = 1$ and one absorbing state at the origin $i = 0$: $\bar{n} = (q_g - p_g)^{-1} = (p - \beta)^{-1}$. 

$\bar{n} < (q_g - p_g)^{-1} = (p - \beta)^{-1}$. 


(3) Critical gaps. On the line segment $\beta = p$, $\alpha \leq p$, the gap width performs a symmetric random walk with $p_g = q_g = p(1-p)$, $r = 1-2p(1-p)$. The calculation of $\bar{n}$ in this case becomes involved, since the average length of a symmetric random walk on the infinite chain, with initial state at $i = 1$ and one absorbing state at $i = 0$, diverges. Therefore, we have to take into account that the lifetime of the random walk on a finite chain is limited, on the average, by $L/p$. We introduce the average lifetime $\bar{n}_M$ of a gap during $M = L/p$ updates, $\bar{n}_M = \sum_{m=1}^{M} m f_{1,0}^{(m)}$, where $f_{1,0}^{(m)}$ is the probability that the initial gap of unit size will vanish at update $m$. By using the generating function of $f_{1,0}^{(m)}$, see Eq. (53) in Ref. [7], and applying a chain of approximations, we obtain for $M \gg 1$: $\bar{n}_M \simeq \sqrt{L}/[\sqrt{2\pi}p^{3/2}(1-p)]$. This implies the finite-size scaling result $P(1) \simeq 1 - b_1 L^{1/2}(p - \alpha)$, verified by our computer simulations.

5. Conclusions

The gap representation seems promising for calculation of the density profile over the whole chain segment. The generalized TASEP with $\tilde{p} \in (0,1)$ can be applied to take into account processes of cluster fragmentation.

Acknowledgment

VBP acknowledges the support of the RFBR, grant 16-02-00252.

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