RELATIVISTIC QUANTUM THEORY
WITH FRACTIONAL SPIN AND STATISTICS

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1. WHY RELATIVISTIC ANYONS?

Quantum mechanical particles can have only integer or half-integer spin, and their statistics must be accordingly Bose or Fermi — this was the standard wisdom until a few years ago. Now we know that the truth of this statement depends on the number of space-time dimensions: it fails if the space dimension is less than three. Both spin and statistics are physically not very interesting in one dimension. On the other hand, many realistic systems have effectively only two dimensions, and the possibility of generalizing spin and statistics in the planar case seems intriguing and attractive. However, despite the excitement it has generated, no really solid application to realistic physical systems of this discovery has been found yet. Applications to the formulation of a theory of high-$T_c$ superconductivity are still at best tentative. Even in the case of the fractional Quantum Hall effect, where the best available theory seems to display this phenomenon, fractional spin is hardly more than an a posteriori feature of the theory, rather than providing a physical explanation of the observed effect.

However, quantum mechanics with generic spin and statistics offers the possibility of rediscovering some of the basic ideas of quantum mechanics in a new light, whatever its phenomenological success in condensed matter physics. Perhaps, new, purely quantum mechanical effects may be found. Certainly, a more general framework to understand the quantum mechanics of spin appears.

But then, the dynamics of spin is nontrivial only in the relativistic case, as we can see by looking at the simplest physical example, that of a free spinning particle. Whereas in non-relativistic quantum mechanics the wave function is just the tensor product of a spatial wave function $\psi(\vec{x})$ times a spin wave function $u$, in the relativistic case the spin and translational degrees of freedom are coupled dynamically by the Dirac equation $(\partial + m)\psi(x) = 0$ which must be satisfied by the spinor wave function $\psi(x)$.

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1 It has become customary to call generalized spin and statistics fractional statistics. This is somewhat of a misnomer, since in fact all real values of the spin and statistics are possible in two spatial dimensions, and not only rational ones as “fractional” would seem to suggest. We shall however stick to this now conventional nomenclature. Excitations with generic spin are often also called anyons (as opposed to bosons and fermions). Fractional statistics should not be confused with parastatistics. Particles with parastatistics have ordinary (i.e., integer or half-integer) spin, but satisfy a modified exclusion principle. Particles with fractional statistics satisfy the ordinary exclusion principle (i.e., no two particles can occupy the same quantum state), but have generic spin.
Example: the Hausdorff dimension of paths for a free particle

The dynamical coupling of spin and translational degrees of freedom has observable, physical effects even for a single free particle. This can be seen by computing the Hausdorff dimension of the paths traversed by a quantum mechanical free particle, i.e., the scaling law which gives the typical length $L$ of a path traversed by a quantum particle in terms of the distance $R$ between the initial and final points of the path. Indeed, it can be shown that the paths that contribute to the quantum propagation amplitude, i.e. to the path integral for the particle propagation satisfy the scaling law

$$R = L^{1/d_H}, \quad (1.1)$$

where the Hausdorff dimension $d_H$ takes a well-defined value. This value turns out to be different for spinning and spinless particles.

We can understand that via an argument due to Polyakov,\[1\] by considering the quantum propagator $K(x', x) = \langle x' | x \rangle$ that connects two points $x$, $x'$ such that $|x' - x| = R$. This is just the Fourier transform of the usual momentum-space propagator $K(p)$:

$$\langle x' | x \rangle = \int d\vec{p} e^{ip(x' - x)} K(p) \quad (1.2)$$

where, for free bosons and fermions, respectively

\begin{align*}
\textit{Bosons: } K(p) &= K^B(p) = \frac{1}{p^2 + m^2} \quad \text{Fermions: } K(p) &= K^F(p) = \frac{1}{\not{p} + m}, \quad (1.3)
\end{align*}

i.e., the propagator is respectively equal to the Klein-Gordon and the Dirac one. The propagators (1.2) may be rewritten as

\begin{align*}
K^B(x', x) &= \int_0^\infty dL e^{-m^2L} \int d\vec{p} e^{ip(x' - x) - L\not{p}^2} \\
K^F(x', x) &= \int_0^\infty dL e^{-mL} \int d\vec{p} e^{ip(x' - x) - L\not{p}}. \quad (1.4)
\end{align*}

The integration over $L$ in Eq.(1.4) can be actually viewed as the integration over the lengths of the paths which contribute to the propagator. That is, if we express $K(x', x)$ as a Feynman path integral, and we define $K(L; x', x)$ by the relation

$$K(x', x) = \int dL e^{-Lm^{d_H}} K(L; x', x), \quad (1.5)$$
where \(d_H = 2\) for bosons and \(d_H = 1\) for fermions, then \(K(L; x', x)\) is the contribution to the propagator \(K(x', x)\) which is obtained restricting the Feynman sum over paths to paths of length \(L\). A rigorous proof of this goes beyond our point; the simplest way of seeing that this is the case is to notice that the propagator \(K(x', x)\) provides a solution to the Klein-Gordon and Dirac equations in the cases of bosons and fermions respectively. All the solution of these equations can be obtained from the solutions to the equation

\[
\frac{\partial}{\partial s} \phi = \hat{O} \phi
\]

where \(\hat{O} = \partial_\mu \partial^\mu\) and \(\hat{O} = \hat{\rho}\) in the respective cases, by projecting out the solutions \(\phi\) that satisfy \(\frac{\partial}{\partial s} \phi = m^{d_H} \phi\). But if we identify \(s\) with proper time, then the solutions to Eq. (1.6) are given by the Feynman sum over paths with fixed length \(L\), while \(K(x', x)\) is given by Eq. (1.5) with \(L\) identified with the path length.

Inspection of Eq. (1.4) then reveals immediately that the bulk of the contribution to the propagator \(K(x', x)\) comes from paths which satisfy

\[
L \sim \frac{1}{p^{d_H}},
\]

which gives immediately Eq. (1.1) because \(L \sim \frac{1}{R}\). We see thus that indeed the scaling (Hausdorff) dimension \(d_H\) equals two for bosonic paths, and one for fermionic ones. In other words the value of spin modifies the (fractal) properties of the paths traversed by a free quantum particle: spin makes its presence felt even in the absence of any explicit spin dependent potential or coupling to other particles.

But if it is only in a relativistic theory that the spin dynamics is nontrivial, we are naturally lead to ask, does the dynamics of particles and fields with fractional spin and statistics admit a relativistic formulation? And if yes, how will it look like? It is the purpose of the present lectures to answer these questions.

In order to make our treatment as self-contained as possible, we shall start by reviewing in Sect.II the nonrelativistic theory, concentrating on the path-integral approach, which is particularly suited to a relativistic generalization, and we shall reformulate it in a relativistically covariant way. Then we shall discuss the group-theoretical underpinnings of relativistic fractional spin; this will lead us to discuss the possibility of quantizing spin.
without using anticommuting variables or spinors. We shall thus introduce, in Sect.III, the path integral quantization of spin, which will allow us to derive the fermion propagator without introducing spinors. In Sect.IV this will be used to turn the nonrelativistic path integral of Sect.II into one which describes an arbitrary number of (first-quantized) relativistic point particles with generic spin and statistics. In Sect.V we will finally discuss some of the ideas and problems in the formulation of a consistent relativistic (second-quantized) field theory with fractional spin and statistics.

Our approach will be throughout of a rather explicit nature. We will not aim at mathematical rigour, but rather at introducing basic ideas and techniques. Also, we will not try to give a comprehensive or fair review of the subject, which is rapidly growing; rather, we will present a subjective perspective. The reader is referred to the numerous good review papers in this subject both for a survey of different approaches, and exhaustive references to the original literature[2]. The lectures are scattered with a few exercises, which the reader interested in getting a working knowledge of the field is invited to work out.
2. FROM THE NONRELATIVISTIC TO THE RELATIVISTIC THEORY

In two space dimensions angular momentum and statistics are not quantized\footnote{3}. This is an immediate consequence of the symmetry structure of the wave function of a two-dimensional system. The wave function carries a representation (reducible, in general) of the rotation group, which in \(d\) dimensions is the group \(O(d)\). If \(d > 2\), then \(O(d)\) is doubly connected (\(i.e., \pi_1[O(d)] = \mathbb{Z}_2\)); its universal cover is the group \(\text{Spin}(d)\), which, in the usual \(d = 3\) case, is isomorphic to \(\text{SU}(2)\). But when \(d = 2\), the rotation group is \(O(2)\), which, being isomorphic to the circle \(S^1\) is infinitely connected (\(\pi_1[O(2)] = \mathbb{Z}\)); its universal cover is the real line \(\mathbb{R}\). It follows that whereas in more than two dimensions the wave function can carry either a simple-valued or a double valued representation of the rotation group, in two dimensions it may carry an arbitrarily multivalued one. This means that upon rotation by \(2\pi\) the wave function may acquire an arbitrary phase; the multivaluedness of the representation is classified by the value of this phase:

\[
R^{2\pi} \psi = e^{2\pi i \ell} \psi,
\]

\(i.e.,\) by the parameter \(\ell\). Because rotations are generated by the orbital angular momentum operator, the allowed values of \(\ell\) provide the spectrum of allowed values of the angular momentum. It follows that in more than two dimensions the angular momentum can only be integer or half-integer, whereas in two dimensions it can be generic. We are intentionally speaking of (orbital) angular momentum rather than spin here; the distinction shall become clear in a relativistic framework, in Sect.IV.

Also, the wave function for a system of \(n\) particles carries a one-dimensional unitary representation of \(\pi_1(C)\), the fundamental group of the (quantum-mechanical) configuration space \(C\). This representation is characterized by the phase which the wave function acquires upon the interchange of any two particles, which is universal if the particles are identical if we denote by \(q_i\) the set of quantum numbers carried by the \(i\)-th particle, then

\[
\psi(q_1, \ldots, q_i, \ldots, q_j, \ldots, q_n) = e^{2\pi i \sigma} \psi(q_1, \ldots, q_j, \ldots, q_i, \ldots, q_n).
\]

The parameter \(\sigma\) in Eq.(2.2) is called the statistics of the particles.

For \(n\) identical particles in \(d\) dimensions \(C\) is obtained by identifying all sets of \(n\) \(d\)-component vectors (which can be thought of as the particles’ positions) after having
removed its subset $D$ of all configurations where two or more particles coincide:

$$C = \frac{\mathbb{R}^{dn} - D}{S_n},$$

(2.3)

where $S_n$ is the set of permutations of $n$ vectors. If $d > 2$, then $\pi_1(C) = S_n$, which admits only two one-dimensional representations, the trivial one, and the alternating one; these correspond respectively to integer or half-integer values of the statistics $\sigma$ in Eq. (2.2). If $d = 2$, then $\pi_1(C) = B_n$, the braid group, which admits an infinity of representations, and the statistics can be arbitrary.

Because the interchange in Eq. (2.2) can be performed by means of a rotation by $\pi$ of the two particles, a rotation by $\pi$ generated by the total orbital angular momentum exchanges all couples of particles. This establishes a relation between the value of $\sigma$ and the allowed spectrum of values of $\ell$:

$$\ell = k + \sigma (n - 1); \quad k \in \mathbb{Z}.$$  

(2.4)

Notice that this is not (yet) a spin-statistics connection, because $\ell$ is the orbital angular momentum, being the eigenvalue of the generator $\hat{l}$ of space rotations (as distinct from spin, which generates rotations in an internal space).

### 2.1. The path integral approach

In the non-relativistic case a system of particles with generic spin and statistics can be easily constructed starting from a theory of particles with ordinary (say, bosonic) spin and statistics. In general, the quantum dynamics is entirely described by $S$-matrix elements, i.e., the transition amplitudes from an initial state $|i\rangle$ to a final state $|f\rangle$:

$$S_{fi} \equiv \langle \psi_f | \psi_i \rangle,$$  

(2.5)

The exclusion of points where two particles coincide can be performed without loss of generality because such points are measure zero in the configuration space. More precisely, it can be shown that under reasonable assumptions of regularity of the wave function the contribution to the path integral from intersecting paths vanishes. Eq. (2.2) shows however that if $\sigma \neq 0$ then the wave function is either vanishing or singular when two particles coincide, which may be viewed as the manifestation of the exclusion principle for generic statistics. It is interesting to observe that if $\sigma$ is not zero but not half-integer the exclusion principle is somewhat weaker: for example, point particles can have contact (i.e., delta-like) interactions in such case, whereas fermions cannot.

In this case the value of the phase depends of course on whether the interchange is performed by a clockwise or counterclockwise rotation of the two particles around their center of mass. Conventionally the statistics is defined assuming that the interchange in Eq. (2.2) is counterclockwise.
which can be expressed in terms of the quantum propagator \( K(q', q) \equiv \langle q', t'|q, t \rangle \), where \( q \) denotes a point in configuration space (such as, e.g., the set of positions of all particles):

\[
S_{fi} = \langle \psi_f | q' t' \rangle \langle q' t' | q t \rangle \langle q t | \psi_i \rangle = \int dq dq' \psi_f^*(q') K(q', q) \psi_i(q). \tag{2.6}
\]

The propagator is in turn given in terms of the Lagrangian \( L \) of the system by the Feynman path integral

\[
K(q', t'; q, t) = \int_{q(t) = q; q(t') = q'} Dq(t_0) e^{i \int_{t'}^{t} dt_0 L[q(t_0)]}. \tag{2.7}
\]

Now, given a theory with bosonic spin and statistics, described by a Lagrangian \( L_0 \), a theory with generic statistics is obtained by adding to \( L_0 \) an interaction term \( L_t \):

\[
L = L_0 + L_t \tag{2.8}
\]

where the interaction \( L_t \) is given in terms of the position vectors \( \vec{x}_i \) of the \( n \) particles by\footnote{In the sequel our notational conventions shall be as follows: latin indices take the values 1,2, while greek indices run from 0 to 2; \( x^1, x^2 \) are space coordinates and \( x^0 \equiv t \) is the time coordinate; the three-dimensional Minkowski metric is \((+,−,−)\); the vector notation always denotes the (two) spatial components of vectors; \( \rho, \phi \) are polar coordinates on the space plane; repeated indices are summed over; \( \epsilon^{ab} \) and \( \epsilon^{\mu \nu \rho} \) are, respectively, the two- and three-dimensional completely antisymmetric tensors, with the convention \( \epsilon^{12} = \epsilon^{012} = 1 \); the exterior product of two vectors is defined as \( \vec{v} \times \vec{w} = \epsilon^{ab} v^a w^b \) (notice that it is a scalar).}

\[
L_t = -s \sum_{i \neq j} \frac{d}{dt} \Theta(\vec{x}_i - \vec{x}_j) \tag{2.9}
\]

\[
\Theta[\vec{x}] = \tan^{-1} \left( \frac{x^2}{x^1} \right).
\]

The function \( \Theta(\vec{x}) \) is just the polar angle of the vector \( \vec{x} \); it is defined as a multivalued function on the punctured plane \( \mathbb{R}^2_p \equiv \mathbb{R}^2 - \{0\} \); it is single-valued on its universal cover \( \tilde{\mathbb{R}}^2_p \), which is the Riemann surface of the complex logarithm. The choice of branch can be fixed defining

\[
\Theta(q) = \int_{q_0}^{q} dq' \frac{d}{dq'} \Theta(q'), \tag{2.10}
\]

where \( q \in \mathbb{R}^2_p \) is a point in the punctured plane spanned by \( \vec{x}_i - \vec{x}_j \) for all \( i, j \), and the integration runs along a path which joins a fiducial reference point \( q_0 \in \mathbb{R}^2_p \) to the point
Fig. 1: Linking number of particles trajectories (solid lines). The linking number is defined by joining the endpoints to infinity along a fixed direction (dashed lines). a) \( l = -1 \); b) \( l = 0 \); c) \( l = 1 \).

\( q \) at which \( \Theta \) is evaluated. It is essential that the multivalued definition of the angle be taken (i.e., if \( \vec{x} \) is rotated by \( 2\pi \) then \( \Theta \) also changes by \( 2\pi \)) even though the choice of branch (i.e., the choice of \( q_0 \)) is immaterial.

\( L_t \) is called a topological Lagrangian because, being a total derivative, it leads to a contribution to the action (i.e. to the path integral) which does not depend on the details of the paths. Rather, it depends on the endpoints, and, because of the multivaluedness of the function \( \Theta \), on the topology of the paths. Indeed, \( L_t \) is a sum over all particle pairs of terms of the form

\[
l = \frac{1}{2\pi} \int dt \frac{d}{dt} \Theta (\vec{x}_i - \vec{x}_j),
\]

up to a coefficient of \(-2\pi s\), where \( l \) Eq. (2.11) is an expression for the linking number of the curves \( \vec{x}_i, \vec{x}_j \), i.e., it is equal to the number of times the two paths link (see Fig.1).

It is easy to check that indeed the theory with Lagrangian \( L \) (2.8) describes generic spin and statistics: the propagator of this theory is

\[
K(q', t'; q, t) = \int_{q(t) = q; q(t') = q'} Dq(t_0) e^{i \int_{t}^{t'} dt_0 \left( L[q(t_0)] - s \sum_{i \neq j} \frac{d}{dt_0} \Theta [\vec{x}_i(t_0) - \vec{x}_j(t_0)] \right) - i n_{ij}} e^{-is (\sum_{i \neq j} \dot{\Theta}_{ij}(t') + 2\pi n_{ij})} K_0^{(n)}(q', t'; q, t) e^{is \sum_{i \neq j} \dot{\Theta}_{ij}(t)},
\]

(2.12)

\(^5\) For open paths this may be defined by joining the endpoints to a point at infinity along a fixed direction and in a fixed order.
where for short $\Theta_{ij} = \Theta(\vec{x}_i - \vec{x}_j)$ and $\hat{\Theta} \equiv [\Theta \mod \mathbb{Z}]$, the sums over $n_{ij}$ correspond to contributions to the path integral from paths that wind $n_{ij}$ times on the configuration space, and $K^{(n)}$ is the path integral (2.7), computed from the Lagrangian $L_0$, but restricting the sum over paths in such a way that for each set of values of $n_{ij}$ only paths with the corresponding winding numbers are included.

Now, all the effects of the topological interaction can be absorbed in a redefinition of the wave function: if we define a new wave function

$$\psi_0(q,t) = e^{is\sum_{i\neq j} \Theta_{ij}(t)} \psi(q,t) \quad (2.13)$$

then it is an obvious consequence of Eq.(2.6) that the same $S$-matrix elements can be equivalently obtained propagating the wave function $\psi$ with the propagator $K$, Eq.(2.12), or propagating the new wave function (2.13) with the usual propagator $K_0$. In other words, what the topological interaction does is to lift the wave function from the configuration space to its universal cover: the wave function $\psi_0$ (2.13) at point $q$ carries a path joining $q_0$ to $q$; roughly speaking, this allows it to “remember” along its evolution the sheet of the Riemann surface on which it should be evaluated. Hence, we constructed a theory which differs from the starting one only by the boundary conditions satisfied by the wave function: if we rotate by an angle $\alpha$ the wave function Eq.(2.13) we get

$$R^\alpha \psi_0 = e^{is\alpha(n-1)} R^\alpha \psi. \quad (2.14)$$

In particular, if $\alpha = 2\pi$ comparing this with Eq.(2.1) shows that if $\psi$ is left unchanged (i.e. if we started with a theory of bosons) then $\psi_0$ has angular momentum $j = sn(n-1)$. Thus, the topological interaction has induced arbitrary angular momentum, and, due to the relation Eq.(2.4), fractional statistics as well.

Of course, the formulation with topological interaction and “conventional” wave functions $\psi$ is completely equivalent to that without interaction and “twisted” wave functions $\psi_0$, thus all the physical observables are the same in the two cases. Consider in particular

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6 More rigorously, the set of paths from $q_0$ to each point $q$ in configuration space defines a homotopy mesh, i.e., it provides a unique prescription to close an open path, by joining its endpoints to $q_0$. Then, an open path can be uniquely assigned to a homotopy class, determined by its linking number (compare Fig.1). This, in turn, determines the motion of the point on the Riemann surface, because the linking number of the path is equal to the number of sheets the point has travelled in the course of its motion.
the angular momentum. In the approach where there is no topological interaction, but the wave function is $\psi_0$, which obeys the boundary condition (2.14), the angular momentum operator has the form $\hat{l}_0$ that it would have in a free theory, but the phase $\Theta_{ij}$ provides an extra contribution to its spectrum. In the approach where the wave function satisfies the usual boundary condition, but there is a topological interaction, the angular momentum operator $\hat{l}$ receives a contribution from the topological term, and it is related to the angular momentum operator of a free theory by

$$\hat{l} = \hat{l}_0 + sn(n - 1), \quad (2.15)$$

where the spectrum of $\hat{l}_0$ is the usual one (i.e. the integers). In both cases, the spectrum of angular momentum is given by Eq.(2.4) with $\sigma = s$.

**Exercise:** Prove that the topological Lagrangian $L_t$ Eq.(2.9) contributes to the canonical Noether angular momentum $\hat{l} = \frac{dL}{dq} \delta R_q$, where $\delta R$ is the variation of the Lagrangian $L$ upon infinitesimal rotation. Show that the contribution shifts the angular momentum according to Eq.(2.15).

2.2. **Covariant formulation and Chern-Simons theory**

The path-integral approach discussed in the previous section is non-covariant, in that it relies crucially on the non-covariant parametrization of paths with time. It is also non-relativistic, in that spin, i.e. intrinsic angular momentum for one-particle states is missing, even in the case of particles which obey fermionic statistics. However, it may be derived from a fully covariant formalism. This is accomplished in two subsequent steps.[5]

*From the Chern-Simons theory to the Hopf interaction*

We start with a theory of point particles with (say) bosonic statistics, defined by the Lagrangian $L_0$. The point particle excitations are carried by a current $j^\mu$ which may be written as a sum of Dirac deltas:

$$j^\mu = \sum_{i=1}^{n} \left(1, \frac{d\vec{x}_i}{dt}\right) \delta^{(2)}(\vec{x} - \vec{x}_i)$$

$$= \sum_{i=1}^{n} \int ds \delta^{(3)}(x - x_i) \frac{dx^\mu}{ds}.$$  \quad (2.16)
Let us now construct a new theory, whose Lagrangian is obtained by adding to the particle Lagrangian $L_0$ a coupling $L_c$ to an abelian gauge field whose dynamics is provided by the Lagrangian $L_f$:

$$L = L_0 + L_c + L_f$$

$$L_c = e \sum_i (\dot{x}_i \cdot \vec{A} - A^0)$$

$$L_f = -\frac{1}{2s} \int d^2y \left( \vec{A}(\vec{y}) \times \dot{\vec{A}}(\vec{y}) + 2A^0(\vec{y})B(\vec{y}) \right).$$

The action $I$ associated to the Lagrangian (2.17) may be written in covariant notation as

$$I = I_0 + I_c + I_f$$

$$I_c = \int d^3x j^\mu(x)A_\mu(x)$$

$$I_f = -\frac{1}{2s} \int d^3x \epsilon^{\mu\nu\rho}A_\mu(x)\partial_\nu A_\rho(x).$$

The field action $I_f$ Eq.(2.18) is the Abelian version of the Chern-Simons action. Its peculiar properties are due to the fact that the field is coupled through the $\epsilon^{\mu\nu\rho}$ tensor, which is a generally covariant object. It is often referred to as a topological action because of its sensitivity to the global features of the gauge potential $A$. For our purposes, however, it is enough to observe that the action $I_f$ is quadratic in the field $A^\mu$. We can therefore compute the path integral over the $A^\mu$ field exactly, i.e., we may determine the effective action

$$I_{\text{eff}}[j] \equiv -i \ln \int \mathcal{D}A^\mu \, e^{i(I_c + I_f)}.$$  

The result is equal to the so-called Hopf action (the reason of the name will be clarified in Sect.IV.1):

$$I_{\text{eff}} = I_H = \pi s \int d^3x d^3y j^\mu(x)K_{\mu\nu}(x,y)j^\nu(y),$$

where the bilocal kernel

$$K_{\mu\nu}(x,y) = -\frac{1}{2\pi} \epsilon_{\mu\rho\nu} \frac{(x-y)^\rho}{|x-y|^3}$$

is the inverse of the operator $\epsilon^{\mu\nu\rho} \partial_\nu$ when acting on the current $j^\nu$, i.e., it satisfies

$$\epsilon_{\mu\rho\sigma} \partial_\nu K^{\rho\sigma}(x,y) = \delta_\mu^\sigma \delta^{(3)}(x-y).$$
Exercise: a) Construct the Green function of the 2+1 dimensional Laplacian \( G(x - y) \), which satisfies \( \partial_\mu \partial^\mu G(x - y) = \delta^{(3)}(x - y) \).

b) Prove Eq. (2.22).

From the Hopf interaction to the topological action

Eq. (2.20) shows that the effect of coupling the point-particle current to the Chern-Simons Lagrangian is to induce the current-current self-interaction \( I_H \). We show now that this, in the non-relativistic limit, leads back to the topological interaction \( L_t \), Eq. (2.9).

First, we notice that because the current (2.16) is a sum of deltas, the Hopf interaction reduces to a sum over all pairs of particles:

\[
I_H = s \sum_{i,j} I_{ij}
\]

\[
I_{ij} = -\frac{1}{2} \int dx_i^\mu dx_j^\nu \epsilon_{\mu\rho\nu} (x_i - x_j)^\rho |x_i - x_j|^3.
\]

Then, we study the generic term \( I_{ij} \) in the sum. This can be simplified greatly by noticing that one can write

\[
\frac{x_i^\mu}{|x|^3} = \epsilon^{\mu\alpha\beta} \partial_\alpha \tilde{A}_\beta(x).
\]

The function \( \tilde{A}_\mu \) must be singular, because the l.h.s. of Eq. (2.24) may be written as a divergence. As a matter of fact, the l.h.s. of Eq. (2.42) is the field of a Dirac magnetic monopole and Eq. (2.24) defines \( \tilde{A} \) as its potential, which notoriously has a string of singularities (that can be put anywhere by a choice of gauge). The geometrical reason for the appearance here of the Dirac monopole will be clarified in Sect.IV. Anyway, for our purposes it is enough to pick a particular form of \( \tilde{A} \) that satisfies (2.24); a convenient one is

\[
\tilde{A}_0(t, \vec{x}) = 0; \quad \tilde{A}_a(t, \vec{x}) = -\frac{\epsilon_{ab}x^b}{r(t - r)},
\]

where \( r^2 = |x|^2 = t^2 - x_1^2 - x_2^2 \).

Notice that at this step we are already singling out time as special in that the string of singularities is put along the time axis. We also parametrize paths with time, and then, using the expression (2.24), (2.25) for the interaction kernel in the action \( I_{ij} \), Eq. (2.23), we get

\[
I_{ij} = -\frac{1}{2} \int_0^T dt \int_0^T dt' \frac{dx_i^\mu(t)}{dt} \frac{dx_j^\nu(t')}{dt'} (\partial_\mu \tilde{A}_\nu(x_i - x_j) - \partial_\nu \tilde{A}_\mu(x_i - x_j)) \frac{dx_j^\nu(t')}{dt'}
\]

\[
= \int_0^T dt \epsilon^{ab} \left( \frac{dx_i^a(t)}{dt} - \frac{dx_j^a(t)}{dt} \right) \frac{(x_i(t) - x_j(t))^b}{|x_i(t) - x_j(t)|^2} + I_g,
\]

13
where
\[ I_g = -\frac{1}{2} \int_0^T dt \left( \tilde{A}_\mu(x_i(t) - x_j(T)) \frac{dx^\mu_i}{dt} - \tilde{A}_\mu(x_i(0) - x_j(t)) \frac{dx^\mu_i}{dt} \right) + x_i \leftrightarrow x_j, \quad (2.27) \]

**Exercise:** a) Provide the intermediate steps in Eq.(2.26)
b) Prove that
\[ \partial_a \Theta(\vec{x}) = -\epsilon^{ab} \frac{x^b}{|x|^2}. \quad (2.28) \]

Now we should distinguish two cases, either \( i = j \) or \( i \neq j \). If \( i = j \) the bilocal kernel in Eq.(2.23) looks singular when \( t = t' \). However the last step in Eq.(2.26) shows that in fact when \( i = j \) the entire integral vanishes. This result can be arrived at in a more rigorous way by regulating the divergence in the kernel. We will discuss this in Sect.IV, where we shall see that the vanishing of the self-intercation, even though true in the nonrelativistic limit, cannot hold in a relativistic treatment. If instead \( i \neq j \) we may use Eq.(2.28) to get
\[ I_{ij} = -\int dt \frac{d}{dt} \Theta(\vec{x}_i - \vec{x}_j) + I_g, \quad (2.29) \]
which, up to the \( I_g \) term, coincides with the topological action, i.e., with the linking number Eq.(2.11). Notice that the assumption that \( \Theta(\vec{x}) \) is a multivalued function is implicitly made when using Eq.(2.28), which is correct (as we will see in more detail in Sect.V.2) only if a multivalued determination of \( \Theta \) is used.

The terms \( I_g \) vanish for closed paths; for open paths they are associated to a contribution to the Lagrangian which does not modify angular momentum and statistics (as it can be explicitly verified by checking that it is rotationally invariant) and need not concern us here. We have thus succeeded in reproducing the topological interaction \( L_t \) by coupling the point-particle current to itself through a Chern-Simons field. At this point we have gone as far as possible in making the nonrelativistic theory of particles with fractional statistics and angular momentum look covariant. In order to use this knowledge to construct relativistic quantum mechanics with fractional spin we need a deeper understanding of the relevant symmetry structure. Before we even try to construct such a theory, we must ask whether a relativistic wave function may carry a multivalued representation of rotations. But, just like a nonrelativistic wave function carries a representation of the rotation group, a relativistic one carries a representation of the Lorentz group. Hence we need to understand the structure of the Lorentz group in 2+1 dimensions, just like we did in the beginning of this section for the spatial rotation group.
Mathematical digression: The Lorentz group in 2+1 dimensions

First we list some basic facts about the structure of the group. The generators in the fundamental representation are the $3 \times 3$ matrices

$$L^{(\mu\nu)}_{\alpha\beta} = -i \left( g^{\mu\alpha} g^{\nu\beta} - g^{\nu\alpha} g^{\mu\beta} \right).$$

The operator $\frac{1}{2}(L^{(12)} - L^{(21)}) \equiv R$ generates the compact rotation subgroup, while the operators $\frac{1}{2}(L^{(0a)} - L^{(a0)}) \equiv B^a$ generate the non-compact boosts. The Lie algebra is

$$[B^a, R] = -i\epsilon^{ab} B^b, \quad [B^a, B^b] = i\epsilon^{ab} R,$$

or, in covariant notation,

$$[L^{(\mu\nu)}, L^{(\rho\sigma)}] = i \left( g^{\mu\sigma} L^{(\nu\rho)} + g^{\nu\rho} L^{(\mu\sigma)} - g^{\mu\rho} L^{(\nu\sigma)} - g^{\nu\sigma} L^{(\mu\rho)} \right).$$

This is the same as the Lie algebra of SL(2, $\mathbb{R}$): $[X^0, X^+] = X^-; \quad [X^+, X^-] = -X^0; \quad [X^-, X^0] = -X^+$,

hence the two groups admit the same universal cover. SL(2, $\mathbb{R}$) is the group of matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with real elements and such that $\det(A) = 1$.

**Exercise:**
a) Show that the condition $\det(A) = 1$ may be rewritten as the equation of a three-dimensional $(2, 1)$ one-sheeted hyperboloid. This is the group manifold of the universal cover of SO(2,1).

b) Work out the correspondence between generators of SL(2, $\mathbb{R}$) and SO(2,1).

c) Show that the elements of a rotation subgroup of SO(2,1) correspond to points on a “neck” of the hyperboloid.

Because the group manifold of SL(2, $\mathbb{R}$) is a one-sheeted hyperboloid, it follows that the Lorentz group in 2+1 dimensions is infinitely connected: $\pi_1[SO(2)] = \pi_1[SL(2, \mathbb{R})] = \mathbb{Z}$. Also, non-contractible paths on the group manifold correspond to non-contractible paths in its rotation subgroup. Hence, multivalued representations of SO(2,1) correspond to multivalued representations of rotations. Notice that if instead we considered theories
defined in Euclidean space-time the Lorentz group would be SO(3). The group manifold
of SO(3) is notoriously doubly connected [the universal cover is SU(2)], \( \pi_1(SO(3)) = \mathbb{Z}_2 \).
This implies that representations of SO(3) can only be either single-valued or double-valued, that is, that spin may be only either integer or half-integer: the Minkowski signature of the metric is essential if we wish to consider fractional spin.

We can now look at the irreducible representations (irreps) of SO(2,1). These are
classified by the eigenvalue of the Casimir operator \( Q = (B^1)^2 + (B^1)^2 - R^2 \), and obtained
diagonalizing the rotation generator \( R \).

**Exercise**: Show that the most general solution of the commutation
relations (2.31) has the form

\[
\begin{align*}
R\xi_m &= m\xi_m \\
B^+\xi_m &= \sqrt{(d + m)(-d + m + 1)}\xi_{m+1} \\
B^-\xi_m &= \sqrt{(-d + m)(d + m + 1)}\xi_{m-1},
\end{align*}
\]

(2.35)

where the raising and lowering operators are defined in terms of the
boost generators as \( B^\pm = B^1 \pm iB^2 \), and the eigenvalue of \( Q \) associated
to a given irrep is \( d(d-1) \).

All irreps are ladders of states of the form (2.35). These fall into three classes:

a) If \( d \) is integer or half-integer then there is a \( 2d + 1 \)-dimensional irrep, spanned by
\( \xi_m, m = -|d|, -|d| + 1, \ldots, |d| \). These are the analogue of the usual irreps of the rotation
group in three dimensions.

b) If \( d \) is not integer or half integer there are two semi-infinite irreps, bounded either
from below or from above, and spanned respectively by \( m = d, d + 1, d + 2, \ldots \) and \( m =
-d, -d - 1, \ldots, -d - 2, \ldots \).

c) For every \( 0 \leq d < 1 \) and every \( d \neq j \pmod{\mathbb{Z}} \) and \( d \neq -j \pmod{\mathbb{Z}} \) there exists
a doubly infinite irrep spanned by \( m = \ldots, j - 2, j - 1, j, j + 1, j + 2, \ldots \).

Unitary irreps are obtained requiring \( B^\pm \dagger = B^\mp \) and \( R^\dagger = R \). The latter condition is
satisfied only if

\[
(\xi_{m_1}, \xi_{m_2}) = \alpha_{m_1} \delta_{m_1, m_2}
\]

(2.36)

where \( \alpha_{m_1} \) is a real positive constant. It then follows from Eq.(2.35) that

\[
\begin{align*}
(\xi_{m+1}, B^+\xi_m) &= \alpha_{m+1} \sqrt{(d + m)(-d + m + 1)} \\
(B^-\xi_{m+1}, \xi_m) &= \alpha_m \left( \sqrt{(d + m)(-d + m + 1)} \right)^*.
\end{align*}
\]

(2.37)
Thus unitary irreps are obtained when the parameter $C_d = \sqrt{(d + m)(-d + m + 1)}$ is real.

In the three above cases:

a) $C_d$ is purely imaginary, hence no representation is unitary (except the trivial one $d = j = 0$).

b) $C_d$ is real if and only if $d > 0$; these irreps are unitary.

c) $C_d$ is real either if $d = \frac{1}{2} + i\alpha$ (principal series of representations) or if $d$ is a real number such that $\frac{1}{2} - |j - \frac{1}{2}| < d < \frac{1}{2} + |j - \frac{1}{2}|$ (supplementary series).

**Exercise**: prove the conditions for unitarity a-c.

From this classifications it follows that there exist no finite-dimensional unitary irreps. Furthermore, even if we are willing to give up unitarity (after all, the usual spinor representation of the Lorentz group is not unitary) finite-dimensional representations are at most double-valued. Hence if we try to generalize to arbitrary statistics the usual route used in constructing theories of fermions, namely, go to a wave function which carries a true representation of the universal cover of the rotation and Lorentz groups, we must introduce an infinite-dimensional wave function. This is physically unpleasant, because if an infinite-component wave function is to describe a finite number of degrees of freedom, then it must be subject to an infinite number of constraints.

However, this rather unpalatable option can be avoided by taking advantage of an alternative formulation of the spin dynamics which dispenses us from considering wave functions defined on the universal cover of the group. To understand this, we must step back to the familiar case of fermions. We will see that their dynamics can be formulated without ever introducing wave functions which carry representations of the universal cover of the rotation group, *i.e.*, without using spinors. Rather, a formulation purely in terms of bosonic variables is possible.
3. SPIN WITHOUT SPINORS

The classical and quantum dynamics of a spin-$1/2$ object is traditionally formulated in terms of functions defined on the double cover of the rotation group, i.e. the group Spin($d$) in $d$ dimensions (in three dimensions this is SU(2)), i.e. in terms of spinors. However, it also possible to formulate the dynamics of spin in terms of phase-space variables. This, upon quantization, leads to a formulation of a spin path-integral in terms of bosonic (as opposed to anticommuting) variables, and to a wave function defined in phase-space, rather than on the group. When this formalism is used to quantize spinning particles, one obtains phase-space wave functions which carry the representations of the Poincaré group associated to particle states without having to introduce wave functions defined on the group. We will first present the phase-space approach to spin in the simple and familiar case of a spin degree of freedom in three spatial dimensions; then we will discuss the group theory which underlies the construction of the relativistic quantum mechanics of point-particle states, i.e., the theory of the Poincaré group in 2+1 dimensions; and finally we will provide a formulation of the dynamics of 2+1 dimensional fermions without using spinors, and show that it is equivalent to the usual approach.

3.1. Path integrals for spin

The formulation of spin in terms of phase-space variables is accomplished in a Lagrangian framework, through the formulation of a spin action\[7\]. This leads naturally to quantization in the path-integral approach. We study a single spin degree of freedom in three dimensions. Classically, this is defined as a system whose only degrees of freedom are the components of the angular momentum, whose modulus is fixed. The configuration space is the sphere $S^2$, which is convenient to view as the coset SO(3)/SO(2), and to parametrize with spherical angles $\theta$, $\phi$ as\[8\]

$$
\vec{e} = \begin{pmatrix}
\sin \theta \cos \phi \\
\sin \theta \sin \phi \\
\cos \theta
\end{pmatrix}.
$$

(3.1)

The spin vector is then $\vec{J} = s\vec{e}$. Eventually, we shall be interested in the quantization of system which have as a configuration space SO(2,1)/U(1), which, roughly speaking, is the

\[7\] In this section, the vector notation denotes three-dimensional Euclidean vectors, latin indices run from 1 to 3.
“Wick rotation” to Minkowski space of this, and is the one-sheeted \((1,1)\) two-dimensional hyperboloid spanned by

\[
\bar{e} = \begin{pmatrix}
\cosh \theta \\
\sinh \theta \cos \phi \\
\sinh \theta \sin \phi
\end{pmatrix}.
\] (3.2)

To this purpose, we will use throughout a covariant notation, such that all results carry over to the Minkowski case by replacing sines and cosines of \(\theta\) with their hyperbolic counterparts, and the coordinate \(x^3\) with a Minkowski coordinate \(x^0\).

An Ansatz for the spin action

We first propose an Ansatz for the spin action and explain its meaning, then we verify that it works both classically and quantum-mechanically.

The action

\[
I_s = \int dt \mathcal{L}(\theta, \phi) = s \int dt \cos \theta \dot{\phi}
\] (3.3)

describes the spin dynamics both at the classical and quantum level. At the classical level, this means that the action Eq.(3.3) leads to the classical spin canonical structure.

At the quantum level, this means that if the action \(I_s\) (3.3) is used as a weight in the Feynman path integral, it leads to spin quantization in the sense that it leads to the same \(S\)-matrix element that one would obtain using the usual spin Hamiltonian. That is, the \(S\)-matrix element are

\[
\langle f | i \rangle = \langle \phi_f | e^{i \int H(t) dt} | \phi_i \rangle,
\] (3.4)

where

\[
| \phi \rangle = | m \rangle \langle m | \phi \rangle; \quad \langle m | \phi \rangle = \frac{e^{-im\phi}}{\sqrt{2\pi}}
\] (3.5)

and \(H(t)\) is a (generally time dependent) interaction Hamiltonian such as, for example, the coupling to an external magnetic field \(H = s \vec{J} \cdot \vec{B}\); they can be computed as

\[
\langle f | i \rangle = \int_{\vec{e}(t_f) = \vec{e}(\phi_f); \vec{e}(t_i) = \vec{e}(\phi_i)} D\vec{e} \ e^{i \int dt \mathcal{L}_s - V(\vec{J})}
\] (3.6)

where the boundary conditions can be imposed only on the value of \(\phi\) (or of \(\theta\)), because quantum-mechanically \(\phi\) and \(\theta\) do not commute (they determine different components of the angular momentum operator), and, because the Lagrangian \(\mathcal{L}_s\) is first-order in time derivatives, the potential \(V\) coincides with the Hamiltonian. More in general, path-integration with this weight gives the matrix elements of functions of spin operators:

\[
\langle f | F(\vec{J}) | i \rangle = \int_{\vec{e}(t_f) = \vec{e}(\phi_f); \vec{e}(t_i) = \vec{e}(\phi_i)} D\vec{e} \ e^{i \int dt \mathcal{L}_s - V(\vec{J})} F(\vec{J}),
\] (3.7)
The rationale for this Ansatz is clear if one considers the case of a closed time evolution, i.e., one where the initial and final states coincide. In such case, the spin action \( I_s \), evaluated along a closed path \( C \) (on the sphere \( S^2 \)), equals

\[
I_s = s \int_C \cos \theta \dot{\phi} dt = s \int_C \cos \theta d\phi
= s \int_S d \cos \theta d\phi = s \int_S dS \cdot \vec{e} = s \int_S \left( \frac{\partial \vec{e}}{\partial s} \times \frac{\partial \vec{e}}{\partial t} \right) \cdot \vec{e},
\]

where \( S \) is a surface on the sphere bound by \( C \), we have used Stokes’ theorem, and in the last step we have introduced a parametrization of the surface \( S \) in terms of two parameters \( s, t \). The last expression is immediately recognized as the expression of the solid angle subtended by the curve \( C \). Thus the action \( I_s \) Eq. (3.3) is the analogue for a spin degree of freedom of the action for a free massive particle: just like the latter, it is given by the simplest geometric invariant of the trajectory. This is the arc-length for a particle, and the solid angle for a spin.

The classical spin action and the Faddeev-Jackiw canonical formalism

We wish to check that the spin action \( I_s \) leads to the canonical structure of a classical spin degree of freedom, i.e., to the Poisson bracket

\[
\{J^i, J^j\} = \epsilon^{ijk} J^k.
\]

This can be done in a simple and elegant way through the Faddeev-Jackiw formalism for the determination of the canonical structure (and quantization) of systems with Lagrangians which are first-order in time derivatives[8]. The formalism applies whenever the Lagrangian can be written in the form

\[
L = f_i(x) \frac{dx_i}{dt} - V(x)
\]

where \( x \) are phase-space variables and \( f_i(x) \) are arbitrary functions.

The Euler-Lagrange equations for such a system are

\[
\frac{\partial V}{\partial x_i} = f_{ij} \frac{dx_j}{dt}
\]

where

\[
f_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}.
\]
The Euler-Lagrange equations (3.11) can be rewritten in canonical form as
\[
\frac{dx^i}{dt} = \{x^j, x^i\} \frac{\partial V}{\partial x^j} = \{V, x^i\}
\] (3.13)
provided the Poisson bracket is defined as
\[
\{x^i, x^j\} = (f^{-1})^{ji}
\] (3.14)
where \(f^{-1}\) is the inverse of the matrix \(f\) defined in Eq.(3.12). Hence, in this formalism the variables \(x\) are viewed as phase-space variables (coordinates and momenta) with Poisson bracket given by Eq.(3.14); in a conventional treatment \(x\) would be coordinates, whose conjugate momenta are fixed by a constraint equation because the action is first-order. Of course, after resolution of the constraints this would lead to the same results.

Let us now apply this to the case of the spin action (3.3), which can be further rewritten by defining
\[
\vec{e} = \vec{\nabla} \times \vec{\tilde{A}}[\vec{e}].
\] (3.15)
Here the potential \(\vec{\tilde{A}}\) as a function of \(\vec{e}\) is the same as the Dirac monopole (2.24),(2.25) discussed in the previous section. We won’t however need its explicit form. Using Eq.s (3.15),(3.8) the spin action becomes
\[
I_s = s \int_S d\vec{S} \cdot \vec{e} = s \int_S d\vec{S} \cdot \vec{\nabla} \times \vec{\tilde{A}}[\vec{e}] = s \int_C dt \frac{d\vec{e}}{dt} \cdot \vec{\tilde{A}}[\vec{e}].
\] (3.16)
This has the form of the Eq.(3.11), with
\[
 f_i = s \tilde{A}_i[\vec{e}]; \quad f_{ij} = s \left( \partial_i \tilde{A}_j - \partial_j \tilde{A}_i \right) = se^{ijk} \epsilon^k \quad f_{ij}^{-1} = \frac{1}{s^2} f_{ij},
\] (3.17)
It follows immediately from Eq.(3.14) that
\[
\{e^i, e^j\} = \frac{1}{s} e^{ijk} e^k
\] (3.18)
which, identifying \(\vec{J} = se\), coincides with the canonical angular momentum Poisson bracket Eq.(3.9), which is what we set out to prove.

**Geometrical formulation of the spin action**
In order to proceed to the quantization of the spin action, it is convenient to introduce a little more formalism, and rewrite the action in a geometrically more transparent way, as an action defined on the space of orbits upon group transformations. This has the advantage of leading directly to geometric quantization. To this purpose, we write the vector $\vec{e}(t)$ as the result of acting on an arbitrary reference vector $\vec{e}_0$ with an SO(3) matrix $\Lambda(t)$:

$$\vec{e}(t) = \Lambda(t)\vec{e}_0. \quad (3.19)$$

This fixes two out of the three Euler angles which parametrize $\Lambda$, while leaving the angle corresponding to rotations around the $\vec{e}_0$ axis undetermined. Eq. (3.19) also implies

$$\dot{\vec{e}}(t) = \dot{\Lambda}(t)\vec{e}_0 \quad (3.20)$$

where the dot denotes total differentiation with respect to $t$. The path traversed by the vector $\vec{e}(t)$ on the sphere as a function of $t$ has thus been mapped to a path traversed by the matrix $\Lambda(t)$ on manifold of the coset $\text{SO}(3)/\text{SO}(2)$ — the space of orbits of $\vec{e}_0$ upon action of $\Lambda$.

We can lift this to a path on the group manifold of SO(3) by fixing the third Euler angle which determines $\Lambda$. Because $\dot{\vec{e}} \cdot \vec{e} = 0$, we may do it by requiring that

$$\vec{n}(t) = \Lambda(t)\vec{n}_0 \quad (3.21)$$

where

$$\vec{n}(t) = \frac{\dot{\vec{e}}(t)}{|\dot{\vec{e}}(t)|}. \quad (3.22)$$

The vectors $\vec{e}(t)$ and $\vec{n}(t)$, together with

$$\vec{b}(t) = \vec{e}(t) \times \vec{n}(t) \quad (3.23)$$

define an orthonormal frame, whose motion in space is generated by the action of the matrix $\Lambda$. We may exploit this to rewrite the spin action as an element of the Lie algebra of SO(3), by defining further

$$\vec{v}^{(3)} = \vec{e}_0$$

$$\vec{v}^{(1)} = \vec{n}_0$$

$$\vec{v}^{(2)} = \vec{b}_0; \quad \vec{b}_0 = \vec{e}_0 \times \vec{n}_0, \quad (3.24)$$
and choosing the values

\[ v_i^{(a)} = \delta_i^a, \] (3.25)

for the three vectors which form the reference frame.

**Exercise:** Prove that

a) \[
\left( \Lambda^{-1} \dot{\Lambda} \right)_{ij} = \vec{v}^{(i)} \cdot \vec{v}^{(j)}; \] (3.26)

b) \( \Lambda^{-1} \dot{\Lambda} \) is an element of the Lie algebra (called the Maurer-Cartan form), i.e. it can be written as a linear combination of the generators

\[
\left( \Lambda^{-1} \dot{\Lambda} \right)_{ij} = \sum_{ab} C_{ab} (M^{ab})_{ij}, \] (3.27)

where \((M^{ab})_{ij} = (\delta_i^a \delta_j^b - \delta_j^a \delta_i^b)\) are the SO(3) generators in the fundamental representation and \(C_{ab}\) are three independent real constants;

c) the constants \(C_{ab}\) are given by

\[
C_{ij} = \frac{1}{4} \text{tr} \left( M_{ij} \Lambda^{-1} \dot{\Lambda} \right) = \frac{1}{2} \vec{v}^{(i)} \cdot \vec{v}^{(j)}. \] (3.28)

d) the spin action equals

\[
I_s = s \left( \text{tr} \int dt \frac{1}{2} \left( \Lambda^{-1} \dot{\Lambda} M_{12} \right) + \text{integers} \right). \] (3.29)

**Hint** to point d): prove first that \( \int_S \left( \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right) \cdot \vec{e} = \int dt \vec{b} \cdot \vec{n} + \text{integers}. \)

The meaning of the integers will be discussed in Sect.IV.1.

Eq.(3.29) expresses the kinetic term in the spin action in terms of the matrix \( \Lambda \). A potential term, written in terms of \( \vec{J} = s \vec{e} \), can be expressed in terms of \( \Lambda \) as well using the identity

\[
e^i = \frac{1}{2} \epsilon^{ijk} \left( \Lambda^{-1} \frac{M_{12}}{2} \Lambda \right)_{jk}. \] (3.30)

The advantage of this formulation is that, once the phase space variables \( \vec{e} \) are expressed in terms of a dynamical group variable (i.e., the SO(3) matrix \( \Lambda \)) the dynamics depends only on the algebra of the group, hence, it does not depend on the choice of a representation.

---

8 Notice that the results below hold true also in Minkowski space with the obvious replacement of \( \delta_{ij} \) with the Minkowski metric.
Even though we can recover formulations in terms of any group representation we please by choosing an explicit form of the generators, we are not forced to do so, which is ultimately what we are trying to accomplish.

As a simple exercise, let us see how the formulation in terms of spinors can be recovered. To this purpose, choose the spinor representation

\[ M_{ij} = -i \epsilon^{ijk} \sigma_k \]  \hspace{1cm} (3.31)

where \( \sigma_i \) are the usual Pauli matrices. Then

\[ \text{tr} \left( \frac{1}{2} \left( \Lambda^{-1} \dot{\Lambda} M_{12} \right) \right) = \text{tr} \left( \Lambda^{-1} \dot{\Lambda} \frac{\sigma_3}{2i} \right) = \left( \Lambda^{-1} \dot{\Lambda} \left( \frac{I + \sigma_3}{2i} \right) \right). \]  \hspace{1cm} (3.32)

But if we define two-component spinors

\[ \psi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \psi(t) = \Lambda(t) \psi_0, \]  \hspace{1cm} (3.33)

then the spin action (with \( s = \frac{1}{2} \)) reduces to

\[ I_s = \frac{1}{2} \int \frac{dt}{i} \psi^*(t) \frac{d}{dt} \psi(t), \]  \hspace{1cm} (3.34)

while the generic spin vector can be represented as

\[ \vec{e} = \psi^*(t) \vec{\sigma} \psi(t). \]  \hspace{1cm} (3.35)

**Quantization of the spin action**

A detailed proof that indeed using the spin action \( I_s \) in a Feynman path integral leads to spin quantization according to Eq.s (3.6),(3.7) can be accomplished by explicit computation of the path integral\[7]. Rather than going through this rather elaborate procedure, we notice that, once expressed in terms of the \( \Lambda \) matrices, Eq.(3.19), the quantization of a spin degree of freedom is a particular case of the more general problem of quantization of a system whose classical configuration space is the set of orbits of a group \( G \), in our case the rotation group. It can be shown that for all such systems the axioms of quantum mechanics fix uniquely both the structure of the Hilbert space, and the form of the action which upon path integration yields quantization of the system\[9].

Namely, the Hilbert space of the quantized system is the representation space of the universal cover \( \tilde{G} \) of the given group \( G \), so that if \( T(\tilde{G}) \) is a unitary representation of \( \tilde{G} \)
then the Hilbert space is spanned by the vectors $|\phi\rangle = T(g)|\phi_0\rangle$ where $g$ are all elements $g \in \tilde{G}$. Thus, all quantum evolutions of the system can be viewed as trajectories traversed by $g$ in the representation space, and the path-integral has the general form

$$\langle f|i \rangle = \int Dg \ e^{i I_w[g]}$$

(3.36)

with the given boundary conditions. Furthermore, the axioms of quantum mechanics, and in particular, the principle of superposition of quantum amplitudes $\langle f|i \rangle = \langle f|f'\rangle\langle f'|i \rangle$ fixes uniquely the kinetic term in the action in the path integral (3.36):

$$I_w[g] = \int dt \langle \phi_0| \left[ T(g^{-1}(t)) \frac{d}{dt} T(g(t)) - H(g(t)) \right] |\phi_0\rangle,$$

(3.37)

where $H(g)$ is a Hamiltonian. Identifying $T(g)$ with the matrix $\Lambda$, this is recognized to coincide with the form (3.29),(3.34) of the spin action.

We may understand this prescription in a rough and ready way in the spin-$\frac{1}{2}$ case, where it is easy to work out the path integral directly from the Hamiltonian form, using the spinor formalism of Eq.(3.33)-(3.35): a generic $S$-matrix element has the form

$$\langle f|i \rangle = \langle \psi_f| e^{i \int H dt} |\psi_i \rangle = \prod_{j=1}^{N} \int d\Lambda_j \langle \psi_{j+1}| e^{i \Delta t H(t_j)} |\psi_j \rangle,$$

(3.38)

where the time interval has been sliced according to $\Delta t = \frac{t_f - t_i}{N}$, where $N$ eventually tends to infinity, $t_j = t_i + (j - 1)\Delta t$, and $\psi_j$ is given by Eq.(3.33) with $\Lambda = \Lambda_j$. In the limit of large $N$

$$\langle \psi_{j+1}| e^{i \Delta t H(t_i)} |\psi_j \rangle \approx \langle \psi_{j+1}| (1 + i \Delta t H(t_j)) |\psi_j \rangle = 1 - \frac{1}{2} \Delta t \frac{d}{dt} \langle \psi_j| e^{i \Delta t H(t_j)} |\psi_j \rangle$$

(3.39)

which gives the path integral (3.6) with the form (3.37),(3.34) of the action. This shows explicitly that the first-order action is obtained directly from the time evolution of the state vectors, according to the prescription Eq. (3.37); because the action is already written in terms of phase-space variables (coordinates and momenta) no integration over momenta is required.

Finally, it is interesting to observe that the form Eq.(3.24) of the action is that which leads to the so-called quantization of the coadjoint orbits [10] of the group of which $\Lambda$ is an element. In this formalism, quantization is enforced by imposing the commutation
relations which follow from the Poisson brackets (3.18); this leads to wave functions which are characters of the given group. In the present case, these are the Wigner functions, i.e., the spin wave functions (3.4)-(3.5).

The formalism for the quantization of a three-dimensional spin degree of freedom discussed so far reduces the problem to that of the quantization of the orbits of a normalized vector upon SO(3) action. This suggests that by simply Wick rotating SO(3) to SO(2,1) we may obtain quantization of the Lorentz group in 2+1 dimension; furthermore, the quantization based on the spin action Eq.(3.29) allows to abstract from the choice of a specific representation and seems therefore to lend itself naturally to be generalized to the case of arbitrary spin. However, if we wish to quantize spinning particles, rather than a fixed spin degree of freedom, spin must be coupled to the translational degrees of freedom. The way this is done is fixed by the representation theory of the Poincaré group, since, according to Wigner[11], one-particle states are in one-to-one correspondence with Poincaré irreps. We must therefore study this group and its representation theory.

Mathematical digression: The Poincaré group and point particles in 2+1 dimensions

The Poincaré group is the semidirect product of the Lorentz group and the translation group; in 2 + 1 dimensions it is the group ISO(2, 1) = $T^3 \otimes \text{SO}(2, 1)$[12]. Its Lie algebra is generated by the three generators $L^{(\mu \nu)}$ of the Lorentz group, and the three generators $P^\mu$ of translations (physically interpreted as momentum operators), and it is given by extending the Lorentz algebra (5.2) by the further relations:

\[ [P^\mu, P^\nu] = 0, \quad [L^{(\mu \nu)}, P^\rho] = i \left( P^\mu g^{\nu \rho} - P^\nu g^{\mu \rho} \right). \quad (3.40) \]

The Casimir operators are

\[ P^2 = P_\mu P^\mu \text{u}(p) \quad W = \epsilon_{\mu \nu \rho} P^\mu M^{\nu \rho}, \quad (3.41) \]

i.e. the total momentum, and the Pauli-Lubanski scalar $W$ which generates rotations around the momentum axis (and is a vector in the familiar 3+1 dimensional case). The group is infinitely connected, its universal cover $\widetilde{\text{ISO}}(2, 1)$ is obtained by taking the

\footnote{We revert henceforth to the notational conventions of Footnote 3.}
semidirect product of translations with the universal cover $\widetilde{SO}(2,1)$ of the Lorentz group: $\text{ISO}(2,1) = T^3 \otimes \widetilde{SO}(2,1)$.

The unitary irreps of $\widetilde{\text{ISO}}(2,1)$ may be easily classified and constructed through Wigner’s method of induced representations. According to this method, all unitary irreps of the Poincaré group (or its universal cover) are induced by unitary irreps of the stability group of an orbit under Lorentz action of a point in the space $\tilde{N}$ dual to a carrier space $N$ of an irrep of the translation group. This means that all Poincaré irreps are constructed through the following procedure:

a) List all unitary irreps of $T^3$. These have all the form of momentum eigenfunctions, i.e., plane waves $e^{i v \cdot p}$ where $v \in T^3$. They are classified by the values of the momentum eigenvalues, i.e., all vectors $p \in \tilde{T}$ which span the dual space $\tilde{T}$.

b) List all the distinct orbits in $\tilde{T}$. These are all the distinct sets of momentum values which can be obtained by acting on a reference momentum vector with a generic Lorentz transformation. They are classified by all the distinct eigenvalues $m^2$ of the total momentum $P^\mu P_\mu$.

c) Construct all the distinct irreps of the stability subgroup of the momentum vector, i.e., the subgroup of $\text{SO}(2,1)$ which leaves that vector invariant. These are generated the operator $W$, Eq. (3.41), hence classified by all its distinct eigenvalues $sm$.

Physically, $m$ and $s$ are interpreted as the mass and spin of the particle, respectively, and each distinct irrep provides the wave function $u(p)$ of a one-particle state with fixed mass and spin, which is thus an eigenfunction of the two Casimir operators:

$$P_\mu P^\mu u(p) = m^2 u(p)$$

$$\epsilon_{\mu\nu\rho} P^\mu M^{\nu\rho} u(p) = ms u(p).$$

The transformation properties of these wave functions are easy to construct explicitly following the above procedure. The transformation of $u(p)$ upon translation along $a^\mu$ ($a^\mu a_\mu = 1$) is

$$e^{iP^\cdot a} u(p) = e^{i p^\cdot a} u(p).$$

**Exercise:** a) Prove that if when $p_0 = \begin{pmatrix} m \\ 0 \\ 0 \end{pmatrix}$, the action of a rotation by $\theta$ on $u(p)$ is given by a certain representation function $D_s[\theta]$ according to $e^{i \theta R} u(p_0) = D_s[\theta] u(p_0)$ then the action of a generic Lorentz transformation $U(\Lambda)$ on a generic state $u(p)$ is

$$U(\Lambda) u(p) = D_s[\Gamma^{-1}(p)\Lambda \Gamma(\Lambda^{-1} p)] U(\Lambda^{-1} p)$$

(3.44)
where $\Gamma(p)$ is the Lorentz transformation which takes $p_0$ to $p$: $\Gamma(p)p_0 = p$.

b) Prove that for an infinitesimal rotation $R(\epsilon)$ and an infinitesimal boost $B(\epsilon\vec{\theta})$ along $\vec{\theta}$ ($|\vec{\theta}| = 1$)

$$
\Gamma^{-1}(p)R(\epsilon)\Gamma(R^{-1}(\epsilon)p) = R(\epsilon),
$$

$$
\Gamma^{-1}(p)B(\epsilon)\Gamma(B^{-1}(\epsilon\vec{\theta})p) = R\left(e^{\frac{e^{ab}\theta_a p_b}{E + m}}\right).
$$

The transformations upon infinitesimal rotations and boosts are determined by Wigner’s procedure, according to Eq.s (3.44),(3.45) in terms of the representation functions $D_s$ of the stability subgroup. Because this is just the abelian rotation group $U(1)$, $D_s(\theta) = e^{is\theta}$, and the transformations upon infinitesimal rotations and boosts are

$$
e^{ieR}\ u(p) = e^{ies}\ u(e^{-ieR}p),
$$

$$
e^{ie\theta_a B^a}\ u(p) = e^{ies\left(e^{ab}\theta_a p_b\right)\left(e^{E + m}\right)}\ u(e^{-ie\theta_a B^a}p).
$$

where $E$ denotes the time component of $p_\mu$ (the energy).

3.2. The relativistic spinning particle

One-particle states carry, according to Wigner, irreps of the Poincaré group, or generally its universal cover. Poincaré representation theory tells us that such irreps correspond to eigenstates of the two Casimir operators $P^\mu P_\mu$ and $W$; the eigenvalue of the former is interpreted as the square mass whereas the eigenvalue of the latter, which is the generator of rotations around the axis defined by the particle’s momentum, is the product of the particle’s mass and spin. This suggests that quantization of a spinning particle can be accomplished by supplementing the quantization of the translational degrees of freedom of a relativistic (massive) spinless particle with the further quantization of the spin degree of freedom, which is just that corresponding to rotations around the momentum axis. It is quite conceivable that this, in turn, should be accomplished by the procedure which we described in Sect.III.2.

An Ansatz for the spin action

We posit an Ansatz for the action of a relativistic spinning particle, based on Poincaré representation theory\cite{13}. We then verify then that it leads to the correct classical and quantum theory in the spin-$\frac{1}{2}$ case.
The free massive spinning particle action is written as the sum of the free spinless particle action \( I_0 = I_0[x, p] \) and the spin action \( I_s \) Eq.(3.3). The spin vector is attached to the particle by requiring that the constraint Eq.(3.42) be satisfied. Defining a generalized spin operator

\[
S^\mu = \epsilon^{\mu\nu\rho} M_{\nu\rho},
\]

for a momentum eigenstate \( u(p) \) with momentum \( p^\mu \) the constraint Eq.(3.42) takes the form

\[
S^\mu e_\mu u(p) = su(p),
\]

where we have defined a unit momentum vector

\[
e^\mu \equiv \frac{p^\mu}{m}.
\]

In other words, the constraint Eq.(3.42) just means that the spin and momentum vectors are parallel, and the action takes the form

\[
I = I_0[x, p] + I_s[e] - V[x, e],
\]

where \( V[x, e] \) is a potential, and \( I_s \) is given by Eq.(3.3), with \( e \) expressed in terms of the momentum according to Eq.(3.49). The action (3.50) can be written compactly in terms of the Lagrangian

\[
L = p^\mu \frac{dx^\mu}{dt} + \text{str} \left( \Lambda^{-1} \dot{\Lambda} M_{12} \right)
\]

with the constraints

\[
p^\mu p_\mu = m^2
\]

\[
\Lambda^\mu_\nu p_\nu = p^\mu; \quad p_0^\mu \equiv \begin{pmatrix} m \\ 0 \\ 0 \end{pmatrix}.
\]

Notice that using this constraint the Lagrangian may be expressed as \( L = L[x, \Lambda] \). The action (3.51) is written in first-order form in order to emphasize the analogy of the case of a spinning particle to that of a spin degree of freedom discussed in Sect.III.1.

**Exercise:** prove that in the spinless case \( s = 0 \) the Lagrangian (3.51) with Eq.(3.52) is the first-order form of the standard Lagrangian for a massive spinless particle

\[
L_0 = m\sqrt{\dot{x}^2};
\]
i.e., prove that after resolution of the constraints these two Lagrangians coincide.

In the particular case of spin-$\frac{1}{2}$ the spin action may be written in according to Eq. (3.34) in terms of a spinor $\psi$ (3.33), while the constraint Eq. (3.48) takes the form of a Dirac equation:

$$\sigma^\mu p_\mu \psi = m\psi,$$ (3.54)

where $\sigma^\mu$ are the Wick-rotated Pauli matrices, defined according to Eq. (3.31) in terms of the Lorentz generators in the spinor representation, and satisfy a 2+1 dimensional Clifford algebra, i.e., they are 2+1 dimensional Dirac matrices.

**Classical theory**

It is very easy to verify that the Balachandran Lagrangian Eq. (3.51)-(3.52) defines the classical dynamics of a spinning particle in the free case. The classical equations of motion for such a system are just the conservation laws for linear and generalized angular momentum. The former

$$\dot{p}^\mu = 0$$ (3.55)

follows trivially by varying the Lagrangian with respect to a generic variation of $x^\mu$. Let us check the latter.

The most general variation of $\Lambda$ is the infinitesimal Lorentz transformation $\delta \Lambda = i\omega^{\mu\nu} M_{\mu\nu} \Lambda$, where $\omega^{\mu\nu}$ is an antisymmetric infinitesimal parameter matrix. The variation of the Lagrangian is thus

$$\delta L = -i \text{tr} (\omega^{\mu\nu} M_{\mu\nu} K) + \frac{i}{2} \text{tr} \left( s \frac{d}{dt} \omega^{\mu\nu} M_{\mu\nu} \right)$$

$$K_{\mu\nu} = (\dot{x}_\mu p_\nu - x_\nu \dot{p}_\mu)$$

$$S_{\mu\nu} = s (\Lambda^{-1} M_{12} \Lambda)_{\mu\nu},$$ (3.56)

where the trace refers to the matrix indices. Hence the Euler-Lagrange equation is

$$\frac{d}{dt} (x^\mu p^\nu - x^\nu p^\mu + S^{\mu\nu}) = 0.$$ (3.57)

This is just the conservation of the total (orbital and spin) angular momentum. Eqs. (3.55) and (3.57) show that indeed the classical equations of motion of a free spinning particle follow from the Lagrangian (3.51)-(3.52). It may be further verified that by introducing minimal coupling to an electromagnetic field according to the replacement $p^\mu \to p^\mu - eA^\mu$
one obtains the correct coupling of a charged particle to an electromagnetic field, hence the Michel-Bargmann-Telegdi equations of motion follow, and so forth. We shall not pursue this further, and turn to the quantum theory.

Quantization

The Lagrangian \((3.51), (3.52)\), which is written in first order form, can be quantized along the lines discussed in the case of a spin degree of freedom in Sect.III.1 [9]. Also, it may be shown [1] that the path integral obtained from the action \((3.50)\) coincides with that of the so-called Brink-di Vecchia-Howe superparticle, which is equivalent to a spin-\(\frac{1}{2}\) particle. Finally, it may be verified [14] that such path integral is equal to the scaling limit of a sum over directed random walks (i.e., random walks with Hausdorff dimension \(d_H = 1\)), which is known to reproduce the Dirac propagator. Rather than following any of these paths, we shall show explicitly (following Ref.[1]) that this path integral leads (in the free spin-\(\frac{1}{2}\) case) to the known form of the Dirac propagator.

First, however, we need to rewrite the path integral for a free massive spinless particle in a more convenient way. We start with the expression for the Euclidean space propagator

\[
\langle x' | x \rangle = \int_{x(0) = x, x(1) = x'} Dx(s) \ e^{-m \int_0^1 ds \sqrt{\dot{x}^2 - g}},
\]

obtained from the Lagrangian \((3.53)\), where \(s\) is a covariant parametrization of the paths \(x(s)\), chosen so that the paths from \(x\) to \(x'\) are traversed as \(s\) varies \(0 \leq s \leq 1\). The path integral Eq.(3.58) can be rewritten by introducing a function \(g(s) \equiv \dot{x}^2\) which may be viewed as an induced metric along the curve with respect to the parameter \(s\), because it satisfies by construction

\[
dx^2 = g(s) \, ds^2.\]

We get

\[
\langle x' | x \rangle = \int_{x(0) = x, x(1) = x'} Dx(s) Dg(s) \ \delta^{(\infty)}(\dot{x}^2 - g) \ e^{-m \int_0^1 ds \sqrt{g}},
\]

where the the constraint Eq.(3.59) along the path is enforced for all \(s\) by means of a functional Dirac delta which we have denoted by \(\delta^{(\infty)}\).

Once written in terms of \(g(s)\), the path integral \((3.60)\) is manifestly invariant upon reparametrization of the paths: if we let \(s \rightarrow f(s)\), then, because of Eq.(3.59) \(g(s) \rightarrow g(f(s))[\dot{f}(s)]^2\). We may exploit this invariance to choose a parametrization such that \(g(s) = \text{const.} \equiv L^2\). The parameter \(L\) is just the length of the path, because

\[
\int_0^1 ds \sqrt{\dot{x}^2} = \int_0^1 ds \sqrt{g(s)} = L.
\]
This may be viewed as a choice of gauge, which we can enforce by introducing one more functional delta:

\[
\langle x' | x \rangle = \int_0^\infty dL \int_{x(0)=x; x(1)=x'} Dx(s) Dg(s) \, \delta^{(\infty)}(\dot{x}^2 - g) \, \delta^{(\infty)}(g - L^2) \, e^{-mL},
\]

at the expense of an extra (ordinary) integration over the path length. The functional integration over \( g \) has become trivial:

\[
\langle x' | x \rangle = \int_0^\infty dL \int_{x(0)=x; x(1)=x'} Dx(s) \, \delta^{(\infty)}(\dot{x}^2 - L^2) \, e^{-mL}.
\]

We may trade the path integration over \( x \) for a path integration over the unit tangent vectors \( e \) to the path

\[
e^\mu = \frac{\dot{x}^\mu}{|\dot{x}|} = \frac{\dot{x}^\mu}{L}.
\]

However, the constraint that the endpoints of the path be at \( x \) and \( x' \) is nonlocal when expressed in terms of the tangent vectors \( e \), and must be enforced by an (ordinary three-dimensional) Dirac delta. We get thus

\[
\langle x' | x \rangle = \int d\vec{p} e^{-i\vec{p} \cdot (x' - x)} \int dL e^{iL} \int D\epsilon(s) e^{-mL} \delta^{(\infty)}(\epsilon^2 - 1) \, e^{i L \int_0^L ds \epsilon^\mu(s)} \delta^{(\infty)}(\epsilon^2 - 1),
\]

which is the sought-for form of the spinless particle propagator.

We may now use the form Eq. (3.65) of the path integral to construct that for spinning particles using the Lagrangian (3.51). Because the path integral is already written in terms of the unit momentum vector \( e \), it is enough to add the spin action to the weight in the sum over paths, while the constraints Eq. (3.52) are automatically satisfied if the spin vector is identified with \( e \):

\[
\langle x' | x \rangle = \int d\vec{p} e^{i\vec{p} \cdot (x' - x)} \int dL e^{-mL} \int D\epsilon(s) e^{-i\vec{p} \cdot \int_0^L ds \epsilon(s)} \, e^{i I_s[e]} \delta^{(\infty)}(\epsilon^2 - 1),
\]

where \( I_s[e] \) is the spin action Eq. (3.3), written in terms of the parametrization Eq. (3.2) of the vector \( e \). Now, for a closed path

\[
\int D\bar{\epsilon}(s) e^{i I_s[e]} [\epsilon_1 e_2 \ldots e_n] = \text{tr} (\sigma_1^{\mu_1} \sigma_2^{\mu_2} \ldots \sigma_n^{\mu_n}).
\]
Exercise: Prove Eq.(3.67). Hint: Prove first the cases $n = 1$ and $n = 2$
using the commutator Eq.(3.18), then proceed by induction.

In general, for an open path, we may use Eq.(3.7), which, being true for all matrix elements, implies the equation at the operator level

$$
\int D\bar{e}(s) e^{iL^a[e]} F(e^\mu) = F(\sigma^\mu)
$$

(3.68)
in the spin-$\frac{1}{2}$ case. Thus, using this result in Eq.(3.66) obtains

$$
\langle x'|x \rangle = \int d\vec{p} e^{ip\cdot(x'-x)} \int dL e^{-mL} e^{-ip\cdot\sigma}
= \int d\vec{p} e^{ip\cdot(x'-x)} \frac{1}{p+m},
$$

(3.69)

which is the Dirac propagator.

We have thus obtained the Dirac propagator starting with a formulation where the spin degrees of freedom are expressed in terms of phase-space variables, identified with the unit tangent vectors to the particle paths, and weighted with the spin action discussed in the previous section. Even though in the particular case of spin-$\frac{1}{2}$ this leads back to the usual formulation in terms of spinors and Dirac matrices, our starting point, namely the action (3.50) seems to be valid for any value of the spin parameter $s$, and does not necessarily require the use of variables, like spinors, defined on the cover of the gauge group. This is the formulation which we shall try to generalize to the case of generic spin and statistics.
4. POINT PARTICLES WITH GENERIC SPIN

In Sect.II we have described an approach to the path integral for particles with fractional spin and statistics which seems to lend itself to a covariant formulation, in that it may be derived from the covariant Chern-Simons action, Eq.(2.18). In Sect.III we have seen that spin-$\frac{1}{2}$ path integrals may be formulated in a way which seems to be amenable to generalization to the case of generic spin. We would now like to merge these two approaches. What is missing from the treatment of Sect.III is the discussion of the arbitrarily multivalued representations of the Poincaré group which are associated to particles with generic spin, as well as the particle-particle interaction which is expected to lead to the generic values of the statistics (and therefore, according to Eq.(2.4), of the orbital angular momentum) when several particles with generic spin are present. On the other hand, in the treatment of Sect.II whereas the statistics interaction follows from a covariant coupling, what seems to be entirely missing is some interaction which generates fractional spin in the case of a single particle: indeed, in the nonrelativistic limit considered there, the Chern-Simons coupling only leads to a coupling of each particle with each other particle, and has no effect whatsoever for a single-particle system, while we know from experience with the $s = \frac{1}{2}$ case, that the dynamical effects of spin are present even for a single particle. Therefore, we shall first study the case of a one-particle system, and show that the spin action can be obtained from a covariant Chern-Simons coupling. After pausing to describe the mathematical underpinnings of our construction, we shall prove that indeed this lead to wave function which carries the Poincaré irreps associated to generic values of spin and statistics.

4.1. The Hopf action and the spinning particle

Let us consider a system of a single particle, with action given by Eq.(2.18). In such case, the Chern-Simons coupling produces only one interaction term of the form (2.23), with $i = j$, i.e.

$$I_{ii} = -\frac{1}{2} \int ds \, dt \, \epsilon_{\mu\nu\rho} \frac{dx^\mu(s)}{ds}(x(s) - x(t))^\nu \frac{dx^\rho(t)}{dt},$$  \hspace{1cm} (4.1)

where the two integrations run along the same curve $x(s)$ traversed by the particle, and $s, t$ are invariant parameters along the curve, for example the arc-length $ds^2 = dx^\mu dx^\nu g_{\mu\nu}$. The bilocal kernel (2.21) in Eq.(1.1) is singular as $s \rightarrow t$. Nevertheless, expanding $x(s)$ in Taylor series in the vicinity of $s = t$ we get

$$\epsilon_{\mu\nu\rho} \dot{x}^\mu(s) \dot{x}^\nu(t) \frac{(x(s) - x(t))^\nu}{|x(s) - x(t)|^3} = -\frac{1}{6} |s - t| \epsilon_{\mu\nu\rho} \frac{\dot{x}^\mu(s) \dot{x}^\nu(s) \ddot{x}^\rho(s)}{|x(s)|^3} + O(|s - t|^2),$$  \hspace{1cm} (4.2)
where the dot denotes differentiation with respect to $s$. This expression is $O(|s - t|)$ as $s \to t$, implying that the integrand in Eq.(4.1) is regular, and actually vanishing when $s \to t$.

The particle self-coupling induced through the coupling to the Chern-Simons term is therefore perfectly well-defined, despite the singularity of the kernel (2.21). Whereas in the nonrelativistic limit the self-coupling contribution was just set to zero, we shall now show that in the relativistic case it automatically produces the spin action discussed in the previous section[15]. To this purpose, we must compute the integral Eq.(4.1) for a generic space-time curve[16].

The writhing number of a space-time curve

Let us first consider, for simplicity, the case of a closed curve $x(s)$. In order to treat the singularity of the kernel Eq.(2.21) it is convenient to introduce a “framing” of the curve $x(s)$, i.e., define a new curve

$$x_\epsilon^\mu(s) = x^\mu(s) + \epsilon n^\mu(s), \quad (4.3)$$

where $n^\mu$ satisfies $n \cdot n = 1$ and $n \cdot \frac{dx}{ds} = 0$ and $\epsilon \to 0$ (see Fig.2). Let us further define

$$I_\epsilon = -\frac{1}{2} \int ds \int dt \epsilon_{\mu\nu\rho} \frac{dx_\epsilon^\mu(s)}{ds} \frac{(x_\epsilon(s) - x(t))}{|x_\epsilon(s) - x(t)|^3} \frac{dx^\rho(t)}{dt}. \quad (4.4)$$

This is just the integral $I_{ij}$ Eq.(2.23), computed for the two curves $x$ and $x_\epsilon$, which, according to Eq.(2.29), is proportional to their linking number $l$ Eq.(2.11):

$$I_\epsilon = -2\pi l(\epsilon), \quad (4.5)$$

where $l(\epsilon)$ is an integer for closed paths (recall Fig. 1). For $\epsilon$ sufficiently small, $l$ does not depend on $\epsilon$ (Fig. 2), and we can define

$$l = -\frac{1}{2\pi} \lim_{\epsilon \to 0} I_\epsilon. \quad (4.6)$$
But \( l \) Eq.(4.6) depends on the choice of framing, \( i.e., \) on the choice of \( n \) Eq.(4.3). Clearly, this cannot be equal to \( I_{ii} \) Eq.(4.4), \( i.e., \) \( \lim_{\epsilon \to 0} I_{\epsilon} \neq I_{ii} \), because \( l \) Eq.(4.6) is manifestly framing-dependent, while \( I_{ii} \) is a well-defined integral with no reference to framing, hence it must be framing-independent. This entails that the integral and the limit in Eq.(4.4) do not commute:

\[
I_{\delta} \equiv -\frac{1}{2} \lim_{\epsilon \to 0} \int ds \ dt - \int ds \ dt \lim_{\epsilon \to 0} \epsilon_{\mu \nu \rho} \frac{dx_{\mu}(s)}{ds} \frac{(x_{\epsilon}(s) - x(t))^\nu}{|x_{\epsilon}(s) - x(t)|^3} \frac{dx^\rho(t)}{dt} = -2\pi l - I_{ii} \neq 0.
\]  

We may determine \( I_{\delta} \) by considering the integral which defines \( I_{\epsilon} \) according to Eq.(4.4), and separating a small neighborhood of the point \( s = t \) from its region of integration:

\[
I_{\epsilon} = -\frac{1}{2} \int_{0}^{T} \int_{s-\delta}^{s+\delta} dt \int_{s-\delta}^{s+\delta} \epsilon_{\mu \nu \rho} \frac{dx_{\mu}(s)}{ds} \frac{(x_{\epsilon}(s) - x(t))^\nu}{|x_{\epsilon}(s) - x(t)|^3} \frac{dx^\rho(t)}{dt},
\]  

where eventually we shall let \( \delta \to 0 \). When \( s \neq t \) the bilocal interaction kernel is always regular, hence the \( \epsilon \to 0 \) limit commutes with the integration in the region with the point \( s = t \) excluded. It follows that

\[
-\frac{1}{2} \lim_{\epsilon \to 0} \int_{0}^{T} \int_{s-\delta}^{s+\delta} dt \int_{s-\delta}^{s+\delta} \epsilon_{\mu \nu \rho} \frac{dx_{\mu}(s)}{ds} \frac{(x_{\epsilon}(s) - x(t))^\nu}{|x_{\epsilon}(s) - x(t)|^3} \frac{dx^\rho(t)}{dt} =
\]

\[
= -\frac{1}{2} \int_{0}^{T} \int_{s-\delta}^{s+\delta} dt \int_{s-\delta}^{s+\delta} \epsilon_{\mu \nu \rho} \frac{dx_{\mu}(s)}{ds} \frac{(x_{\epsilon}(s) - x(t))^\nu}{|x_{\epsilon}(s) - x(t)|^3} \frac{dx^\rho(t)}{dt} = I_{ii} + O(\delta),
\]

where the last step follows from the vanishing of the integrand at \( s = t \), Eq.(4.4). Comparing this with Eq.(4.7) shows that the noncommutativity, \( i.e., \) the difference between \( I_{ii} \) and the linking number \( l \) comes entirely from the \( s \approx t \) region, \( i.e. \)

\[
I_{\delta} = -\frac{1}{2} \lim_{\delta \to 0} \lim_{\epsilon \to 0} \int_{0}^{T} \int_{s-\delta}^{s+\delta} dt \int_{s-\delta}^{s+\delta} \epsilon_{\mu \nu \rho} \frac{dx_{\mu}(s)}{ds} \frac{(x_{\epsilon}(s) - x(t))^\nu}{|x_{\epsilon}(s) - x(t)|^3} \frac{dx^\rho(t)}{dt}.
\]  

**Exercise:** Prove that

\[
I_{\delta} = \int \frac{ds}{2\pi} \epsilon_{\mu \nu \rho} e^\mu n^\nu \frac{dn^\rho}{ds} \equiv 2\pi \tau,
\]  

\[36\]
by expanding the integrand in Taylor series around the point \( s = t \)
(compare Eq.(4.12)).

Using Eq.s (4.7),(4.10),(4.11) the particle self-interaction is found to be

\[
I_{ii} = 2\pi (\tau - l). \tag{4.12}
\]

If we frame the curve using in Eq.(4.3) the principal normal, defined in terms of the tangent \( e \) as

\[
n^\mu_p = \frac{\dot{e}^\mu}{|\dot{e}|}; \quad e^\mu = \frac{\dot{x}}{|\dot{x}|}, \tag{4.13}
\]
then \( l \) is called the self-linking number of the curve, while

\[
\tau = \int \frac{ds}{2\pi b \cdot \dot{n}}; \quad b^\mu = e^{\mu\rho} \epsilon_{\nu\rho} n^\nu
\]

is the geometric torsion of the curve, while \( b \) is the binormal vector. Then \( I_{ii} \) Eq.(4.12) is called the writhing number of the given curve.\(^{10}\)

Eq.(4.12), with the expressions (4.9) and (4.11) for the quantities \( l \) and \( \tau \) provides the desired expression of the interaction induced by coupling a one-particle current to the Chern-Simons term. Several remarks on this result are in order:

a) Even though the interaction Eq.(4.12) has been obtained from a bilocal coupling of each point to each other point along the particle trajectory, according to Eq.(4.12) it may be expressed as the integral along the curve of a local function of the curve and its derivative.
b) The result Eq.(4.12) is independent of the choice of framing Eq.(4.3) which we have introduced in order to arrive at it. However its decomposition into the two terms (4.6) and (4.11) is framing-dependent; the framing dependence of these two contributions cancel against each other.
c) Whereas \( l \) Eq.(4.6) is a topological quantity (i.e., it is invariant upon small deformations of the curve), \( \tau \) Eq.(4.11) is a metric quantity (i.e., it varies continuously upon small variations of the curve), thus so is also the interaction \( I_{ii} \) Eq.(4.12).
d) The self-linking number (i.e. \( l \) [Eq.(4.6)] computed when the curve is framed with the

\(^{10}\) Strictly speaking this terminology applies to the case of an Euclidean metric, i.e., to the case of curves in three-dimensional space, rather than 2+1-dimensional space-time. There is, however, no obstacle to defining linking, self-linking, etc. for a Minkowski metric, either by performing the computations in Euclidean space and Wick-rotating the result (supplementing appropriate factors of \( i \)), or by performing the computation in Minkowski space directly.
principal normal) has a geometric interpretation[17] as the number of intersection of the curve with the envelope of its normals. It is a measure of the number of coils which the curve forms.

If the curve $x(s)$ is open, rather than closed, Eq.(4.12) is still true, with $\tau$ given by Eq.(4.11), while $l$ receives a correction, as in the nonrelativistic computation Eq.(2.29), which is present because for an open path there is a certain ambiguity in the definition of the self-linking number, just as there is one in the definition of the linking number. It may, however, be set to zero by a choice of phase of the wave function and will be neglected henceforth.

The writhing number and the spin action

We may now proceed to our final step, and show that the particle self-interaction, Eq.(4.12) reproduces indeed the spin action[13]. To this purpose, we must go back to the formalism introduced in Sect.III.1: we define a frame of three vectors $e^\mu(s)$, $n^\mu(s)$ and $b^\mu(s)$, and construct a matrix $\Lambda(s)$ which generates the time evolution of this frame when acting on a reference frame, which we may take as the configuration of the given frame at initial time $t = 0$ (as in Eq.s (3.19),(3.21)). The vector $e^\mu$ is now the unit tangent to the curve, while $n^\mu$ is the framing vector introduced in Eq.(4.3). In the particular case in which the curve is framed with the principal normal this is the so-called Frenet frame of the curve. Of course, vectors are normalized with respect to the Minkowski metric, thus $\Lambda$ is an SO(2,1) matrix. For convenience, we also introduce the labelling Eq.(3.24) of the three vectors of the frame as $u^\mu(\nu)$, where both indices are raised and lowered with the Minkowski metric, and we make the choice Eq.(3.25) for the reference frame. Using the Minkowski version of Eq.s (3.26)-(3.28) it then follows immediately that $\tau$ Eq.(4.14) is given by

$$\tau = \int dt \frac{1}{2\pi} \text{tr} \left( \Lambda^{-1} \dot{\Lambda} R \right),$$

(4.15)

where $R$ is the generator of the rotation subgroup of SO(2,1) defined as in Eq.(2.31). With this expression for $\tau$, the self-interaction is very close to the form Eq.(3.29) of the spin action.

In order to show the complete equivalence, we introduce an explicit parametrization of the matrix $\Lambda$ with Euler angles:

$$\Lambda(s) = e^{i\phi(s)R} e^{i\theta(s)B_2} e^{i\psi(s)R},$$

(4.16)
where \( B_2 \) is the generator of boosts along the \( y \) axis (Eq. (2.31)). With this parametrization, the angles \( \theta(s) \) and \( \phi(s) \) parameterize the tangent vector \( e \) according to Eq. (3.2), while \( \psi \) determines the direction of the vector \( n \) in the plane orthogonal to \( e \). It is easy to work out the form of \( \tau \), \( l \) and \( I_{ii} \):

\[
\tau = \int \frac{ds}{2\pi} \left( \dot{\phi} \cosh \theta + \dot{\psi} \right)
\]

\[
l = \int \frac{ds}{2\pi} \dot{\psi}
\]

\[
I_{ii} = \int ds \dot{\phi} \cosh \theta.
\]

(4.17)

Eq. (4.17) shows manifestly the framing-independence of the coupling \( I_{ii} \); it also shows that \( I_{ii} \) coincides with the Minkowski form of the spin action Eq. (3.3). Thus coupling the current of a theory of bosonic point particles to the Chern-Simons term in the one-particle sector of the theory induces an interaction term in the Lagrangian which is identical to that which, if the coefficient of the Chern-Simons coupling in Eq. (2.17) is fixed with \( s = \frac{1}{2} \) coincides with that which leads to quantization of spin-\( \frac{1}{2} \) particles. It is interesting to observe that Eq. (4.17) also clarifies the relationship between the form Eq. (3.3) (in terms of the angles \( \theta, \phi \)) and the alternate form Eq. (3.29) (in terms of the matrix \( \Lambda \)) of the spin action: the former coincides with \( I_{ii} \), while the latter is written as the sum of \( \tau \) and a framing correction. The framing correction is equal to \(-l\) and is what was alluded to as “integers” in Eq. (3.29).

Because in the present approach there is nothing special about the value \( s = \frac{1}{2} \), and since we know that in the non-relativistic, many particle case the Chern-Simons coupling produces automatically nontrivial statistics, this suggests that this approach will lead to physical states which carry generic spin and statistics automatically, by just taking an arbitrary number of particles and an arbitrary value for \( s \). Before we show that this is indeed the case we pause to study some of the mathematics which underlies the peculiar features of the ubiquitous spin action.
Mathematical digression: The spin action, the Hopf map, and the Dirac monopole

Whereas the expressions Eq.(4.17) of $\tau$, $l$ and $I_{ii}$ are true with any choice of framing, they take forms which have a particularly simple geometrical interpretation when specific choices of framing are made. One such choice we have already discussed, and corresponds to taking for $n$ the canonical (Frenet) normal, so that $\tau$ is just the (Minkowski analytic continuation of) the torsion, and $l$ the self-linking number. It is important to notice that in such case the third Euler angle $\psi$ takes values $-\infty \leq \psi \leq \infty$, in keeping with the interpretation of $l$ as a (self)linking number: when $n$ goes one full loop around $e$, then $l$ increases by one unit; and analogously the total torsion is an increasing function along the curve. Recalling the discussion of the Lorentz group in Sect.II.2, this means that the matrix $\Lambda$ Eq.(4.16) is actually an element of the universal cover of SO(2,1), since $\psi$ parametrizes the compact rotation subgroup.

An alternative simple possibility is to choose $n$ as the vector such that

$$\psi = -\phi. \quad (4.18)$$

With this choice,

$$\tau = \int \frac{ds}{2\pi} \dot{\phi} (\cosh \theta - 1)$$
$$l = -\int \frac{dt}{2\pi} \dot{\phi} \quad (4.19)$$

(while $I_{ii}$ is of course unchanged). Then both $\tau$ and $l$ have simple interpretations in the case of a closed curve: $\tau$ Eq.(4.19) is an expression for the solid angle subtended by the path traversed by $e$ (or rather, its Euclidean counterpart)\[11\]. The value of $l$ instead gives the homotopy class of the path traversed by $e$ after its manifold of definition has been punctured to remove the point $\theta = 0$. In the Euclidean case, for instance, this is just the linking number of a path on the sphere from which the north pole has been removed, i.e., the number of times the path loops around the north pole.

All these mathematical structures have a simple interpretation in terms of the so-called Hopf fibration\[18\]. This consists of viewing the sphere $S^3$ as a fiber bundle with

\[11\] This definition of solid angle and that of Eq.(3.8) correspond to the two possibilities of defining the solid angle of a curve on a sphere as that of the surface bound by that curve and containing the north pole or the south pole, respectively. Otherwise stated, with the definition Eq.(3.8) the solid angle of a small closed circle around the north pole is close to $2\pi$, whereas with the definition Eq.(4.19) it is close to 0.
base space $S^2$ and fibre $S^1$. In the Euler angle parametrization $S^2$ is spanned by $e_3$, i.e., by the Euler angles $\theta, \phi$, whereas the fiber is spanned by the third Euler angle $\psi$. A choice of framing is just a section of this bundle: in particular Eq.(4.18) is the so-called natural section of the bundle.

The expression Eq.(4.17) for $\tau$ is recognized as the parallel transport (holonomy) with respect to the induced U(1) connection along the fibre: the connection is given by the rotation component of the Maurer-Cartan form (recall Eqs (3.26),(3.27)), i.e.

$$\hat{A}_\mu = \text{tr} \frac{1}{2i} \left[ \Lambda^{-1} \partial_\mu \Lambda(t) R \right], \quad (4.20)$$

and the holonomy is

$$2\pi \tau = \int dx^\mu \hat{A}_\mu, \quad (4.21)$$

where the integration runs along a path traversed by $\Lambda$ on the bundle space. This can be decomposed in the motion along the instantaneous fibre, given by $2\pi l$, and the induced motion (holonomy) due to the motion on the base space, given by $I_{ii}$. The latter physically is the Thomas precession due to the motion of the frame (3.24) in space.

The connection $\hat{A}_\mu$ is well-known in physics as the Dirac monopole potential; the possibility of expressing it with different sections of the bundle corresponds to the possibility of choosing gauge inequivalent potentials, because in that application the fibre degree of freedom is viewed as a gauge degree of freedom. As is well-known, this potential has in general a singularity, which has a different location in different gauges. This is seen explicitly in the various expressions for $\hat{A}_\mu$ which we have given so far: for example Eq.(4.19) corresponds to choosing in Eq.(4.21)

$$\hat{A}_\mu[e] = \left(0, -\frac{\epsilon_{ab}e^b}{(e^0 - 1)}\right), \quad (4.22)$$

which is singular at the north pole, and is the same form of the monopole potential which was used in Eq.(2.25) (notice however that in Eq.(2.25) the monopole was in the space of positions, here it is in the space spanned by tangent vectors). The choice of framing $\psi = 0$, which gives $I_{ii} = \tau$, hence

$$I_{ii} = \int dx^\mu \hat{A}'_\mu \quad (4.23)$$

12 As usual we refer to the Euclidean case which is geometrically simple, although all results can be formally extended to the Minkowski case.
corresponds to the choice
\[
\tilde{A}_\mu'[e] = \left(0, -\frac{\epsilon_{abc}e^0e^b}{(e^0)^2 - 1}\right),
\] (4.24)
which has singularities both at the north and south pole, and so forth.

The need for a singularity in the potential may be understood by observing that the integral of the connection (4.20) along a closed loop C which bounds a surface S is
\[
\oint dt \frac{de}{dt} \cdot \tilde{A}[e] = \int_S dS^\mu \epsilon_{\mu}^{\nu\rho} \partial_\nu \tilde{A}_\rho'[e] = \int_S dS^\mu \Omega_\mu,
\] (4.25)
where \(\Omega_\mu\) is the first Chern class of the Hopf bundle (the field of the monopole). Because the Hopf bundle is nontrivial, the Chern class is closed but not exact, hence the potential (connection) is not globally well-defined. The two expressions Eq.(4.17) and Eq. (4.19) of the holonomy of the connection (4.20) correspond to two different options to avoid this singularity. In Eq.(4.17) the potential is formulated on the full bundle space, rather than on the base space only. In Eq.(4.19) the potential is expressed on the base space only, but the bundle is trivialized globally by puncturing: by removing a point from the base space (the north pole of the sphere) the bundle is globally trivial.

In the Minkowski case the full “bundle” space is the group manifold of SO(2,1) discussed in Sect.II.1, i.e. a (2,1) one-sheeted hyperboloid; the base space, spanned by \(e\), is a (1,1) two-sheeted hyperboloid; and the fibre, parametrized by \(\psi\), is still a circle \(S^1\). Whereas in the Euclidean version, the base is simply connected, the bundle is doubly connected, and the fibre is infinitely connected, in the Minkowski case, the bundle and the fibre are infinitely connected, while the base is simply connected. In the latter case, the holonomy Eq.(4.21), evaluated along a path \(P\) on the bundle space, is (if \(P\) goes through the identity, i.e., the point \(\theta = \phi = \psi = 0\)) an expression for the winding number \(w\) of \(P\) over that space (which is infinitely connected), because it is equal to the total projected motion along the circle which is the non-contractible neck of the space:
\[
w = \int_P \frac{dt}{2\pi} w(t); \quad w(t) = \frac{1}{2i} \text{tr} \left(\Lambda^{-1} \dot{\Lambda} R\right).
\] (4.26)

Equipped with this geometric knowledge we can now proceed to study the path integral for particles interacting through the induced Hopf interaction Eq.(2.23) in the case of generic values of the parameter \(s\).
4.2. Path integral and multivalued relativistic wave functions

We can proceed in a relativistic setting as we did in Sect.II.2 for a nonrelativistic theory: we start with a (now relativistic) theory of bosonic particles, and we couple the covariant, conserved point particle current Eq.(2.16) to a Chern-Simons term according to Eq.(2.17), so that effectively a bilocal current-current Hopf interaction of the form \((2.20),(2.21)\) is generated. This, once the explicit form of the current as a sum of \(n\) Dirac deltas at the particles’ locations is used, is seen \([Eq.(2.23)\)] to separate into \(n\) particle self-interaction terms \(I_{ii}\), and \(n(n-1)\) interactions of each particle with each other particle.

We shall first concentrate on the self-interaction, by considering the case of a one-particle system, and show that for generic values of the coupling parameter \(s\) it indeed leads to fractional spin. Then we study an \(n\) particle system, see how generic statistics is also induced, and finally discuss the spin-statistics relation.

The path-integral in the one-particle case

For a one-particle system the induced Hopf interaction reduces to the single term \(I_{ii}\) which, as we have shown in the previous section Eq.(4.17), coincides with the spin action Eq.(3.3). Hence, the full action coincides with the relativistic particle action Eq.(3.50) discussed in Sect.III.2, but now with generic values of the spin parameter \(s\). The path integral is thus

\[
K(x',t';x,t) = \int_{x(t)=x;\ x(t')=x'} \sum_{n=-\infty}^{\infty} Dx^{(n)}(t_0) e^{-is(\hat{\psi}(t') + 2\pi n)} \left[ e^{i \int_{x}^{x'} dt_0 \{ L_0(x(t_0)) + 2\pi s \theta [\epsilon] \}} \right] e^{is\hat{\psi}(t)},
\]

(4.27)

where \(x\) is a point in 2+1 dimensional Minkowski space-time, \(t\) is an invariant parameter along the curve (such as the proper time), and \(\hat{\psi} \equiv \psi \mod \mathbb{Z}\). If we use the canonical framing Eq.(4.13),(4.14), the sum runs over “self-linking classes”, i.e., paths are classified according to their self-linking number, and for fixed \(n\) only paths with self-linking equal to \(n\) are included in the path-integration.\(^\text{13}\)

The measure of integration over paths of the \(n\)-th self-linking class \(Dx^{(n)}\) is in practice rather complicated. For practical purposes, it is more convenient to use the framing

\(^\text{13}\) Notice that the integration runs over all possible paths from \(x\) to \(x'\), including those which go backwards in time. This means that the tangent vector \(e\) may be space-like, hence \(\theta\) might be imaginary.
Eq. (4.18); then, the spin action is entirely expressed according to Eq. (4.19) in terms of the tangent vector $e$, now defined on a punctured hyperboloid, as discussed in the previous Sect. It is thus convenient to write the path-integral with a separate integration over the vector $e(t)$ along the path, and a functional $\delta^{(\infty)}$ to enforce the constraint that $e$ be parallel to the tangent to the path at every point, as we did in the spin-$\frac{1}{2}$ case in Sect.III.2:

$$K(x', t'; x, t) = \int_{x(t)=x; \; x(t')=x'} Dx(t_0) e^{i \int_t^{t'} dt_0 L_0[x(t_0)]}$$

$$\times \int \sum_{n=-\infty}^{\infty} De^{(n)}(t_1) \delta^{(\infty)} \left( \frac{\dot{x}(t_1)}{|\dot{x}(t_1)|} - e(t_1) \right)$$

$$\times e^{is(\phi[e(t')]-\phi[t]) + 2\pi n} \left[ e^{i2\pi s \int_t^{t'} dt_1 \tau[e(t_1)]} \right] e^{-is\phi[e(t)]},$$

where (using for simplicity the Euclidean nomenclature) $e(t)$ traverses a curve on the unit sphere in the course of its evolution, and $0 \leq \phi \leq 2\pi$ is an azimuthal angle on this sphere. The $e$-integration in Eq. (4.28) is extended to all paths on this sphere. Whereas $\tau$ is (for closed paths) the solid angle subtended by each path, the phase factor of

$$w = \frac{1}{2\pi} [\phi(t') - \phi(t)] + n$$

counts the total winding of each path about the axis through the poles, and $n$ is just the homotopy class of the path on the sphere punctured at the poles, which has fundamental group $\pi_1(S^2 - \{\text{poles}\}) = \mathbb{Z}$.

The path-integral over $e$ in Eq. (4.28) is akin to that for the propagation of a charged particle in the field of a Dirac monopole, with two important differences: first, the monopole is in the space of tangent vectors, rather than position space, and then, the topology of the space is different. Whereas in the monopole case the singularity of the action Eq. (4.19) is treated by expressing it in terms of $\hat{A}$ according to Eq. (4.21), and then exploiting gauge invariance to choose a form of the potential $\hat{A}$ which is always free of singularities, here the singularity is treated by puncturing the sphere. In other words, in the monopole case there is no sum over $n$ in the path-integral Eq. (4.28), and a single-valued determination of $\tau$ is chosen by choice of gauge. These two options correspond to different choices of space of quantization, even though the local canonical structure of the theory is the same. A more careful treatment of the spin-$\frac{1}{2}$ case reveals that the correct transition amplitudes, both for the Euclidean spin degree of freedom [7] studied in Sect.III.1 [Eq. (3.7)] and for the spinning particle [14] of Sect.III.2 [Eq. (3.66)] are reproduced only if the multivalued prescription of...
Eq. (4.28) is used, rather than the monopole prescription¹⁴. In the Chern-Simons approach which we are pursuing this is an automatic consequence of the computation. We shall see shortly that this prescription is also mandatory if we wish to obtain the multivalued Lorentz and Poincaré representations associated to fractional spin.

We can now proceed to prove that in the case of generic \( s \) the propagator (4.28) defines the dynamics of a particle with fractional spin. We do this by proceeding in analogy to what we did in Sect.II.1 in the nonrelativistic case: we eliminate the “spin” interaction from the propagator by a suitable redefinition of the wave function, and we show that the redefined wave function carries the multivalued Poincaré irreps associated to generic spin.

**Elimination of the interaction**

It is clear that the nonrelativistic construction cannot be reproduced literally, since the effect of the Hopf term is not merely to endow the path integral with the integral of a total derivative, as it should necessarily be the case if the action Eq. (4.17) were purely topological. Rather, the action Eq. (4.17) may be viewed as a Wess-Zumino term i.e., as a total derivative in one dimension more, by rewriting \( I_{ii} \) according to Eq.s (4.23), (4.25).

Then, the interaction can be eliminated, but at the expense of introducing a wave function defined on a path, rather than on a point, i.e., defined in one dimension more. In particular, consider a wave function \( \psi(x,t) \) propagated by the path integral Eq. (4.27) according to Eq. (2.6), where now the configuration space \( C \) is that for a relativistic particle, i.e., \( q \) is a point in 2+1 dimensional Minkowski space-time, and \( t \) is a covariant parameter along the curve as in Eq. (4.27). Then, we define a new wave function \( \psi_0 \) which depends not only on the point \( x \in C \), but also on a path \( P_0 \) that joins a reference point \( x_0 \) to \( x \):

\[
\psi_0(x) = e^{-is\Theta_{P_0}(x)}\psi(x); \quad \Theta_{P_0}(x) = \int_{x_0}^{x} dx' \frac{de}{dx'} \cdot \hat{A}'[e], \quad (4.30)
\]

where \( \hat{A}' \) is the Dirac monopole potential (in \( e \)-space) as given by Eq. (4.24), \( x_0 \) is a reference point in space-time, and \( P_0 \) is a path that joins \( x_0 \) to \( x \).

Because causality dictates that boundary conditions be imposed on a space-like surface, the wave function must have support in one such surface, hence the path \( P_0 \) must be contained in a space-like surface, too.

¹⁴ The need for a multivalued phase may be understood as the consequence of the fact that the coherence effects that yield the desired quantization rules are effective only if one path-integrates over a noncompact phase space.
**Exercise**: Prove that if the path $P_0$ is planar, the phase $\Theta_{P_0}$ Eq.(4.30) is invariant upon deformations of $P_0$.

Without loss of generality, we may take the path $P_0$ to be a straight line joining $x$ to spatial infinity along a space-like plane. The set of paths $P_0[x(t)]$ provides us with a mesh over space time thereby allowing to reduce the computation of the spin terms for an open path $P$ to the determination of the writhing number of the closed path $P_C$ which is obtained by joining the endpoints $x_i$, $x_f$ of $P$ to $x_0$ through $P_0(x_i)$ and $P_0(x_f)$, respectively (recall footnote 6).

The $S$ matrix elements computed for the wave functions $\psi_0$ and $\psi$ are related in a simple way: the former contains an extra weight in the sum over paths, due to the transport of the phase $\Theta_{P_0}$ Eq.(4.30), which equals

$$I_{\Theta} = -i \int dt \left( \langle \psi_0 | \frac{d}{dt} | \psi_0 \rangle - \langle \psi | \frac{d}{dt} | \psi \rangle \right),$$

(4.31)

where the integral runs along the given path. Explicitly

$$I_{\Theta} = s \int_P dt \int_{P_0(t)} \cosh \theta d\phi = s \int_S d\cosh \theta d\phi,$$

(4.32)

where $S$ is surface swept by the path $P_0(t)$ when $t$ runs along the path $P$. But this is of course the same surface $S$ that appears in Eq.(4.30), whereas the integrand is equal to that in Eq.(4.17). Hence, the $S$ matrix elements computed using in Eq.(2.6) the wave function $\psi$ and the propagator Eq.(4.27) is identically equal to that computed by replacing $\psi$ with $\psi_0$ and $K$ with $K_0$.

The Hopf interaction has thus been shown to amount to a phase redefinition of the wave function. Because the Hopf interaction is not purely topological, in that it contains the metric term $\tau$ Eq.(4.17), the wave function has to be lifted to a function defined on a path, rather than on a point [19]. We should now like to check that this is enough to endow the one-particle states of the theory with the multivalued Lorentz and Poincaré representations associated to fractional spin. Before we do that, we would like to understand how this is possible: in Sect.III we have shown how a path integral for spinning particles can be constructed using phase-space variables, rather than variables on the covering of the Lorentz group (i.e. spinors); now, we would like to extend this from path integrals to wave functions.

*Cocycles and Poincaré irreps*
The multivalued wave function Eq. (2.13) introduced in order to construct the non-relativistic theory of particles with fractional statistics is a particular example of a more general case [20].

Quite in general, consider a wave function \( \psi(q) \) defined on a certain configuration space \( \mathcal{C} \), with a symmetry group \( G \), and such that \( \psi \) is single-valued when the group \( G \) acts on the configuration space (just like the rotation groups acts on the nonrelativistic configuration space). If the group \( G \) is multiply connected, with universal cover \( \tilde{G} \), we may construct a multivalued wave-function \( \psi_0 \) as

\[
\psi_0(q) = e^{is\alpha_0(q)}\psi(q),
\]

(4.33)

where \( \alpha_0(q) \) is multivalued upon action of the group \( G \) on the configuration space. In particular, if \( g^n_0 \in \tilde{G} \) is in the \( n \)-th Riemann sheet of the group manifold of \( \tilde{G} \), but projects down to the identity of \( G \), we require that

\[
\alpha_0(q^{g^n_0}) - \alpha_0(q) = n,
\]

(4.34)

where \( q^g \) denotes the transform of point \( q \) upon action of the element \( g \) of the group \( G \).

If the action \( U(g) \) of the group element \( g \) on the Hilbert space spanned by wave functions is given by

\[
U(g)\psi_0(q) = \psi_0(q^g),
\]

(4.35)

then the wave function \( \psi_0 \) Eq. (4.33) upon group action transforms with an extra phase prefactor (cocycle) \( \omega_1(q; g) \), according to

\[
U(g)\psi_0(q) = e^{i\omega_1(q; g)}\psi_0(q^g).
\]

(4.36)

The cocycle is given by

\[
\omega_1(q; g) = s(\alpha_0(q^g) - \alpha_0(q)) = s\Delta^g\alpha_0.
\]

(4.37)

The phase Eq. (4.37) is a 1-cocycle over the group, in that if we require that the transformation law Eq. (4.36) preserves associativity of the group, then \( \omega_1(q; g) \) must satisfy the 1-cocycle condition

\[
\omega_1(q; g_2g_1) = \omega_1(q; g_1) + \omega_1(q^{g_1}; g_2).
\]

(4.38)
Because of Eq. (4.37) the condition Eq. (4.38) is automatically satisfied. A cocycle which may be expressed according to Eq. (4.37) is said to be trivial; nevertheless, due to the multivaluedness Eq. (4.34) of $\alpha_0$ the triviality is only local, i.e., it is not possible to eliminate the cocycle by a global phase redefinition of the wave function.

An explicit expression of $\omega_1(q; g)$ can be given in terms of the winding number density over the group G. The winding number density is a function $w[g(t)]$ which, integrated along a non-contractible path over the group manifold (which exists since by assumption $G$ is multiply connected) gives the homotopy class of the path, i.e., an integer which identifies the class of equivalence to which the path belongs. Explicitly, if $P$ is a path over the group manifold of the $p$-th homotopy class, which we may express as a one-parameter smooth family of elements of the group $g(t)$ parameterized by $t$, then

$$\oint_P \frac{dt}{2\pi} w[g(t)] = p. \quad (4.39)$$

The cocycle is then constructed by choosing a reference point $q_0$ in configuration space, and it is given by integrating the winding number density along a path from $q_0$ to the given point $q$:

$$\omega_1(q; g) = s \int_{t_0}^{t_1} w[g(t)]$$

$$q_0^{g(t_0)} \equiv \Lambda(t_0)q_0 = q$$

$$q_0^{g(t_1)} \equiv \Lambda(t_1)q_0 = q^g. \quad (4.40)$$

Because of Eq. (4.34), the wave function Eq. (4.33) carries a multivalued representation of $G$. The multivaluedness is fixed by the value of the parameter $s$. Hence, if we can find an expression for a function $\alpha_0(q)$ such that the cocycle Eq. (4.37) computed from it is equal to the desired expression Eq. (4.40), then $\psi_0$ provides us with a wave function defined on configuration space, but which (thanks to the phase prefactor) carries a representation of the universal cover of the group. In other words, the cocycle lifts the representation carried by the wave function from the group to its cover. Thus there is no need to define the wave function as a function on the universal cover of the group, and the desired multivaluedness is produced by the cocycle.

Hence, if the wave function $\psi_0$ upon Lorentz transformation acquires a cocycle related according to Eq. (4.40) to the winding number over the Lorentz group (explicitly given by Eq. (4.26)), then that wave function carries a multivalued Lorentz representation, and generic spin. The analogy with the nonrelativistic treatment, and the way both can be
understood within the framework which we just discussed, is summarized by the following table.

| General nonrelativistic theory | Relativistic theory |
|--------------------------------|---------------------|
| $G$                            | SO(2)               |
| $w(t)$                         | $\Theta(\vec{x}_i - \vec{x}_j)$               |
| $\alpha_0(q)$                  | $\Theta(q) = \int_{q_0}^{q} dq' \frac{d}{dq} \Theta(q')$ |
|                                | $\Theta_{P_0}(x) = \int_{x_0}^{x_{P_0}} dx' \frac{d}{dx} \cdot \hat{A}[e]$ |

Tab. 1  The cocycle construction

**Exercise:** a) Prove that

$$
\Theta_{P_0}(\Lambda(g)x) = \Theta_{P_0}(x) + \int_x^{\Lambda(g)x} dx' \frac{d}{dx'} \cdot \hat{A}'[e],
$$

(4.41)

where on the r.h.s. the integration runs over a path of tangent vectors obtained by acting on $e(x)$ with a path of matrices $\Lambda(t)$ that joins the unit of the group to the given element $\Lambda(g)$ of SO(2,1).

b) Prove that for a closed path in the space of $e$ vectors

$$
\int_x^{\Lambda(g)x} dx' \frac{d}{dx'} \cdot \hat{A}'[e] = -\int_{t_0}^{t_1} \frac{1}{2t} \left(\Lambda^{-1}\dot{\Lambda}R\right),
$$

(4.42)

where on the r.h.s. of Eq. (4.42) the integration runs on a path on the group manifold which joins the identity to $\Lambda(g)$ (notice the minus sign on the r.h.s.). **Hint:** consider boosts and rotations separately and use Eq. (3.45).

It immediately follows from Eqs. (4.41) and (4.42) that the wave function Eq. (4.30) transforms with the Lorentz cocycle Eq. (4.40), (4.26), as per the above table. Notice that the cocycle is a function $\omega_1(e,g)$, i.e., it depends on the group transformation $g$, and on the configuration-space point $q$ which is a tangent vector $e$, because the phase $\Theta_{P_0}$, the monopole potential Eq. (4.20), and the SO(2,1) winding number Eq. (1.21) are defined as functions of $e$. This entails that the wave function $\psi$ in Eq. (1.30) ought to be defined as a
momentum eigenstate, with $e = \frac{p}{m}$. In general, it should be Fourier decomposed in terms of momentum eigenstates, each of which will carry a different phase $\Theta_{P_0}$.

The Lorentz cocycle, when evaluated on a momentum eigenstate, reproduces automatically the transformation law associated to irreducible Poincaré representations according to Eq.(3.46).

**Exercise:** Prove that

$$
\omega_1(e, ) + eR) = s e
\omega_1(e, ) + e\theta^a B^a = ise \left( \frac{\epsilon_{ab} \theta^a e^b}{1 + e^0} \right),
$$

where $\omega_1$ is the cocycle given by Eq.(4.40) in terms of the SO(2,1) winding number Eq.(4.21). **Hint:** use the explicit form Eq.(4.22) of the monopole potential.

In sum, the wave function $\psi_0$ Eq.(4.30) carries the Poincaré irreps associated to generic spin.

Because the formulation of the theory in terms of the wave function $\psi_0$ and the propagator $K_0$ (without Chern-Simons-Hopf interaction) is completely equivalent to that in terms of the propagator Eq.(4.27) and the conventional wave function $\psi$, the dynamics is seen to admit a dual formulation, just as in the nonrelativistic case discussed in Sect.II.1.

**Exercise:** a) Prove that the Hopf interaction Eq.(2.23) with $I_{ii}$ given by Eq.(4.12) provides contributions to the canonical conserved Noether charges for angular momentum $R$ and boosts $B^a$ which have the form

$$
R_H = s
B_H^a = s \frac{\epsilon^{ab} e^b}{e_0 + 1}.
$$

b) Prove that if the operators $P_{0\mu}, B_{0a}, R_0$ satisfy the Poincaré algebra Eq.s (3.40),(2.31), then also the operators

$$
P^\mu = P_{0\mu}
R = R_0 + s
B^a = B_{0a} + s \frac{\epsilon^{ab} P^b}{P_0 + m}
$$

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satisfy the same algebra.\cite{21}

In the formulation in terms of $\psi_0$, the spectrum of the angular momentum operator $J$ is shifted because of the phase prefactor $\Theta_{P_0}$ in the wave function. In the formulation in terms of $\psi$, it is the canonical angular momentum operator which is shifted by the topological interaction according to

$$R = R_0 + R_H,$$

where $R_0$ is the operator in the absence of Hopf interaction, and $R_H$ is given by Eq.\cite{44}. In the relativistic theory, a shift of the angular momentum is consistent with the Poincaré algebra only if the boost generators are shifted as well. Indeed, the spectrum of boosts is also shifted according to

$$B = B_0^\alpha + B_H^\alpha,$$

where again $B_H^\alpha$ is given by Eq.\cite{44}, and is due either to the Hopf interaction which affects the operator, or to the phase prefactor $\Theta_{P_0}$ which affects its spectrum. Because for momentum eigenstates $e^\mu = \frac{p^\mu}{m}$ in terms of the momentum eigenvalue $p^\mu$, the shift \cite{44,47} can be viewed as a redefinition of the Poincaré generators, which has the form Eq.\cite{45}. Thus, the possibility of introducing generic spin through a topological interaction is related to the possibility of a redefinition of the Poincaré generators which preserves the algebra, but shifts the angular momentum spectrum.

**Multiparticle states, spin and statistics**

We may proceed to study the general case of an $n$-particle system. In this case, on top of $n$ copies of the spin action Eq.\cite{17} the action contains $n(n-1)$ particle-particle interaction terms. These are actually the same which were discussed in the nonrelativistic case, Eq.\cite{23}: indeed, the linking number of two space-time curves is covariant, it is only the parametrization of paths with time introduced in Eq.\cite{29} which is not. Since, however, Eq.\cite{29} is clearly invariant with respect to reparametrizations of the path (as it is manifest from its form Eq.\cite{10}) it is enough to replace $t$ with any invariant parameter to obtain a covariant result.

In general, the explicit invariance will be broken by the choice of boundary conditions; however, Lorentz covariance is preserved: if, for example, we impose on the path integral boundary conditions at fixed time by requiring the initial and final states to be $\langle \vec{x}; t | \psi_{i,f} \rangle = \psi_{i,f}(\vec{x}; t)$, then, upon Lorentz transformation by $\Lambda$, the initial and final states become
\[ \langle \Lambda \vec{x}; \Lambda t | \psi_{i,f} \rangle = \psi_{i,f}(\Lambda \vec{x}; \Lambda t) \]. The only effect of the linking-number terms is to endow the path integral with multivalued phases \( \Theta_{ij} \) Eq.(2.13) which depend on the endpoints of the path. In a relativistic treatment, these phases are defined as polar angles on the arbitrary space-like plane on which boundary conditions at initial and final times are imposed. Without further ado, we can give to all the results derived in the nonrelativistic case in Sect.II.1 a Lorentz covariant interpretation. In particular, the propagator from \( \psi_i(\vec{x}_1, \ldots, \vec{x}_n; t) \) to \( \psi_f(\vec{x}_1', \ldots, \vec{x}_n'; t') \) is

\[
K(\vec{x}_1', \ldots, \vec{x}_n'; t'; \vec{x}_1, \ldots, \vec{x}_n; t) =
\sum_{n, \tau} e^{-i\sigma (\sum_{i<j} \hat{\Theta}_{ij}(t') + 2\pi n_{ij})}
\times \hat{K}(\vec{x}_1', \ldots, \vec{x}_n'; t'; \vec{x}_1, \ldots, \vec{x}_n; t) e^{i\sigma \sum_{i<j} \hat{\Theta}_{ij}(t)}.
\]

where \( \hat{\psi} \) and \( \tau \) are as in the one-particle propagator Eq.(4.27), \( \hat{\Theta}_{ij} \) is as in Eq.(2.12), and the parameter \( \sigma = s \) has been introduced in order to ease the discussion of spin and statistics. In the non-relativistic limit the writhing number is ill-defined, since the unit tangent to the curves in three-space is always \( e = (1,0,0) \), so that \( \phi \) in Eq.(4.17) is ill-defined. We can then set \( \phi = 0 \) conventionally, in which case the nonrelativistic limit of the propagator (4.48) reproduces Eq.(2.12).

The extra multivaluedness introduced by the phases \( \Theta_{ij} \) due to the terms particle-particle interaction is the same as that which is present in the nonrelativistic treatment and can be handled in the same way. All the terms induced by the Hopf interaction in the propagator Eq.(4.48) may be absorbed in a redefinition of the wave function which has the form Eq.(4.30) for the self-interaction, and of Eq.(2.13) case for the particle-particle interaction. The former phase is defined for momentum eigenstates, whereas the latter, being a functional of the particle’s positions is sharp for position eigenstates. The
redefinition of the wave function for, say, a position eigens state, takes thus the form

$$\psi_0(\vec{x}_1, \ldots, \vec{x}_n; t) = e^{i\sigma} \sum_{i=1}^{n} \sum_{j=1}^{n} \Theta_{ij}(t) \int \frac{m}{E_1} d^2 k_1 \ldots \frac{m}{E_n} d^2 k_n \times e^{-is} \sum_{i=1}^{n} \Theta_{P_0}(k_i) \langle k_1, \ldots, k_n | \psi(\vec{x}_1, \ldots, \vec{x}_n; t) \rangle.$$  \hspace{1cm} (4.49)

Hence, both the multivalued phase which leads to fractional spin $\Theta_{P_0}$ and that which leads to fractional statistics (and orbital angular momentum) $\Theta_{ij}$ are generated by the same interaction, expressed in terms of a Dirac monopole potential (compare Eq.s (4.30) and (2.24), respectively). The latter, however, is a function in the space of relatives positions of the particle, while the former is a function in the space of tangent vectors to the particle trajectories (identified with momenta).

Upon Lorentz transformation the wave function Eq.(4.49) acquires $n$ copies of the cocycle Eq.(4.40),(4.26), and $n(n-1)$ phases due to the transformation of $\Theta_{ij}$. These can be straightforwardly shown to give again the same cocycle. Thus, upon Lorentz transformation the wave function Eq.(4.49) acquires $n + n(n-1) = n^2$ copies of the cocycle Eq.(4.40),(4.26). Upon spatial rotation, in particular, the wave function acquires a phase which shifts the total angular momentum spectrum $j_0$ of a theory without Hopf interaction to

$$j = j_0 + [n + n(n-1)]s = j_0 + n^2s.$$  \hspace{1cm} (4.50)

We can now finally discuss the spin-statistics relation. It should be noticed that there are two, distinct relations between statistics and angular momentum. The first one is expressed by the fact that the operator $L_{x_i x_j}$ that generates rotations of the $i$-th and $j$-th particle about each other is identified with their relative angular momentum operator. This implies the relation Eq.(2.4) between the statistics and the spectrum of eigenvalues of the orbital angular momentum $\ell$. This is a purely kinematical relation which is always true regardless of the dynamics, and follows from the definitions of statistics and angular momentum. The second relation is a relation between the spin angular momentum (which may be measured, modulo integer, as the total angular momentum modulo integer of a one-particle wave function), and the statistics of an $n$-particle wave function. Otherwise stated, this is a relation between the values of the coefficients $\sigma$ and $s$ in the path integral and wave function Eq. (4.48),(4.49).

The spin-statistics theorem states that for Boson and Fermion fields $\sigma = s \mod (\mathbb{Z})$. In the theory under investigation $\sigma = s$ and the spin-statistics theorem is automatically
satisfied. This is presumably related to the fact that the theory can be derived as a particular limit of the local field theory defined by the Chern-Simons Lagrangian Eq.(2.17). The orbital angular momentum $\ell$, total spin $S$ and total angular momentum $j$ in our theory are thus in general given by

$$
\ell = n(n - 1)\sigma + k; \quad k \in \mathbb{Z}
$$

$$
S = ns
$$

$$
j = \ell + S,
$$

which, setting $\sigma = s$ leads back to Eq.(4.50).

This concludes our discussion of relativistic particles with fractional spin and statistics. We have shown that the Hopf interaction induces both a particle self-coupling, and a particle-particle coupling, which may be eliminated by a phase-redefinition of the wave function, provided the latter is localized on a path. The redefined wave function carries a multivalued representation of the Lorentz group. The multivaluedness is partly due to the fact that the wave function carries multivalued representations of the Poincaré group due to phases depending on each particle’s momentum. It is also partly due to the fact that on the space-like plane on which boundary conditions to the quantum evolution are defined the wave function carries nontrivial representations of the braid group due to phases depending on the relative position of each couple of particles. The former leads to fractional spin, the latter to fractional statistics and angular momentum. Because these phases are generated by the same interaction, a spin-statistics relation holds. The wave function is defined in phase space, and the multivaluedness associated to fractional spin and statistics is generated by an interaction-induced phase cocycle which takes values on the universal cover of configuration space, obtained acting on the configuration space with the universal cover of the gauge group. This allows to obtain multivalued representations of the Lorentz group without introducing a wave function which carries a representation of the universal cover of the group (as it is usually done for fermions), and which would necessarily be infinite-dimensional.

\[\text{\footnote{A formulation in terms of infinite-component wave functions with an infinite number of constraints is however also possible\cite{21}. Realistic dynamical calculations will presumably have to use a mix of the two approaches.}}\]
5. RELATIVISTIC FIELD THEORY

Fractional spin and statistics in field theory may be introduced by considering theories (like the O(3) model) which support localized topological solitons in 2+1 dimensions. Then, in the limit in which the separation of the solitons is much larger than their size, fractional spin and statistics may be introduced by approximating the solitons with point particles and then proceeding as in the previous section. It is difficult to go beyond this first simple step because of the lack of renormalizability of these theories.

A perhaps more fundamental problem, however, is the construction of a field theory whose elementary (point-like) excitations carry generic spin and statistics. On the one hand, it is clear that most of the machinery which has been introduced in the study of quantum mechanics with fractional statistics will fail to work in field theory, because it relies heavily on the concepts of particle trajectory, particle location, etc., which have no field-theoretic analogue. On the other hand, the construction of quantum mechanics with fractional statistics described so far is based on the coupling of a conserved current to itself by coupling it to a Chern-Simons term [Eq.(2.17)] and integrating the Chern-Simons field out. This is a field-theoretic construction, and one may hope that proceeding along the same lines when the conserved current is a smooth field current, rather than a point-particle one, will anyway lead to physical states with generic spin. A naive analysis suggests that this is indeed the case, but encounters several difficulties.

Field theory with fractional spin and statistics is still very much of an open subject, and there exists no comprehensive approach, even though attempts have been made in several directions. Here we shall only sketch some of the main problems, and briefly described one possible avenue to solving them, in order to give a feeling of the issues which are involved. We will first describe the simplest, canonical approach, display the problems it runs into, and try to understand their origin. Then we will discuss how these problems are resolved in a path-integral approach.

5.1. The Klein-Gordon-Chern-Simons theory

We consider the simplest field-theoretic generalization of the theory of particles coupled to Chern-Simons of Eq.(2.17): namely, a charged (complex) scalar field coupled to Chern-Simons.\[16\] We consider its canonical quantization and seek for the effects of the Chern-Simons coupling.\[22\]

\[16\] The field must be complex if we want to allow for generic statistics. Because particles and antiparticles, which are generated by complex-conjugate operators, have equal and opposite spin and statistics, it follows that a real field is necessarily bosonic or fermionic.
We start with the Lagrangian
\[ \mathcal{L} = (\partial_{\mu} + i A_{\mu}) \phi^* (\partial_{\mu} - i A_{\mu}) \phi - m^2 \phi^* \phi + \frac{1}{4\pi s} \epsilon^{\alpha \beta \gamma} A_{\alpha} \partial_{\beta} A_{\gamma}. \] (5.1)

The action associated to this Lagrangian can be written in the form Eq.(2.18), as the coupling to a Chern-Simons term of the field having action
\[ I_0 = \int d^3 x \left[ \partial_{\mu} \phi^* \partial^\mu \phi - m^2 \phi^* \phi \right] \] (5.2)
through the conserved current
\[ j^\mu(x) = i \left( \phi(x) \pi^\mu(x) - \phi^*(x) \pi_\mu^*(x) \right); \quad \pi_\mu^*(x) = (\partial_{\mu} - i A_{\mu}) \phi(x). \] (5.3)

**Canonical quantization**

We want to quantize the theory canonically in the \( A_0 = 0 \) gauge. To this purpose we must first discuss the constraints of the theory. The equation of motion for the \( A_0 \) field is a constraint (primary, first class), because there is no time derivative of \( A_0 \) in the Lagrangian:
\[ \pi(x) = 0, \] (5.4)
where \( \pi(x) \) is the momentum canonically conjugate to \( \phi(x) \), \( i.e. \), in terms of \( \pi_\mu(x) \) Eq.(5.3), \( \pi(x) = \pi_0(x) \). Requiring this constraint to be preserved by the time evolution, \( i.e. \) requiring \( \{ H, \pi(x) \} = 0 \), we get the secondary constraint
\[ j^0 = \frac{1}{2\pi s} \epsilon^{ij} \partial_i A_j, \] (5.5)
which is the Gauss law.

**Exercise**: Prove that Eq.(5.3) is solved by
\[ A^i(\vec{x}, t) = -s \int d^2 y \epsilon^{ij} \frac{(x - y)^j}{|\vec{x} - \vec{y}|^2} j^0(\vec{y}, t). \] (5.6)

**Hint**: Prove first that
\[ \partial_i \partial_i \frac{1}{2\pi} \ln |\vec{x}| = \delta^{(2)}(\vec{x}). \] (5.7)
The constraint Eq.(5.5) determines the nonvanishing components of the Chern-Simons field $A^i(\vec{x}, t)$. Using Eq.(5.6) and the property Eq.(2.28) of the angle function $\Theta(\vec{x})$ which we have repeatedly used in the previous sections, we see that $A^i$ is given by

$$A^i(\vec{x}, t) = s \int d^2y \frac{\partial}{\partial x^i} \Theta(\vec{x} - \vec{y}) j^0(\vec{y}, t) = \partial_i s S(\vec{x})$$

$$S(\vec{x}) = \int d^2y \Theta(\vec{x} - \vec{y}) j^0(\vec{y}),$$

Thus, if the interchange of derivative and integral in the last step of Eq.(5.8) is allowed, then $A^i$ Eq.(5.8) is a pure gauge, and may be removed by a gauge transformation. This is however a highly nontrivial assumption, because of the singular nature of the function $\Theta(\vec{x})$, as we shall discuss below.

Anyway, if we proceed naively, we conclude that the interaction with the gauge field can be completely eliminated by defining the gauge-transformed fields

$$\phi_0(\vec{x}) = e^{2isS(\vec{x})} \phi(\vec{x})$$
$$\pi_0(\vec{x}) = e^{2isS(\vec{x})} \pi(\vec{x})$$

where $S(\vec{x})$ is as in Eq.(5.8).

**Graded commutators**

Now we can impose canonical commutation relations

$$[\phi(\vec{x}), \pi(\vec{y})] = \delta^{(2)}(\vec{x} - \vec{y})$$
$$[\phi(\vec{x}), \phi(\vec{y})] = [\pi(\vec{x}), \pi(\vec{y})] = 0,$$

which imply in particular

$$[j^0(\vec{y}), \phi^\dagger(\vec{x})] = \delta^{(2)}(\vec{x} - \vec{y}) \phi^\dagger(\vec{x}),$$

*i.e.* the field operator $\phi^*$ acts as a creation operator (it creates one unit of charge). But then it follows that

$$\left[ S(\vec{x}), \phi^\dagger(\vec{y}) \right] = \Theta(\vec{x} - \vec{y}) \phi^\dagger(\vec{y}),$$

hence the commutation relations satisfied by the gauge-transformed fields $\phi_0, \pi_0$ Eq.(5.9) differ from Eq.(5.10), because of the extra non-commutativity of the gauge function $S(\vec{x})$.

**Exercise:** Prove that

$$e^{-2isS(\vec{x})} \phi_0^\dagger(\vec{y}) e^{2isS(\vec{x})} = e^{-2is\Theta(\vec{x} - \vec{y})} \phi_0^\dagger(\vec{y}).$$

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Indeed, using Eq. (5.13) and the canonical commutator Eq. (5.10) it follows immediately that the commutation relation satisfied by the $\phi_0$ fields is

$$\phi_0^\dagger(\vec{x})\phi_0(\vec{y}) - e^{\pm i\pi s} \phi_0^\dagger(\vec{y})\phi_0(\vec{x}) = 0,$$

(5.14)

where the minus (plus) sign applies if the fields are commuted by clockwise (counterclockwise) rotation. The other commutators are modified in an analogous way. Thus, choosing the value of the parameter $s$, the fields $\phi_0, \pi_0$ can be made to satisfy anticommutators, or generalized commutators (usually referred to as graded commutators). It would thus appear that after eliminating the Chern-Simons coupling, the field operators have acquired generic statistics

**Problems and paradoxes**

Despite its simplicity, the construction of field operators which satisfy generalized commutation relations and carry generalized statistics which we have just presented runs into several problems and paradoxes. In particular, a closer look to the derivation which has led to Eq. (5.14) reveals a physical problem, and a mathematical paradox. Let us discuss them in turn.

Physically, if the operators $\phi_0$ are to be interpreted as field operators with fractional statistics, we would expect them also to create excitations with fractional spin. This must be true in the specific case of fermions: if we choose the value of $s$ so that Eq. (5.14) gives an anticommutator, then if the field $\phi_0^\dagger$ is a *bona fide* fermi field operator, when acting on the vacuum it must create a state which carries half-integer spin. This implies that if we rotate such a state by $2\pi$, we must get the same state multiplied by the phase $-1$.

Now, the states $\phi_0$ do have peculiar rotational properties because the function $S(\vec{x})$  Eq. (5.8) is not rotationally invariant (if we take $\Theta$ to be a multivalued function):

$$R^{2\pi} S(\vec{x}) = S(\vec{x}) + 2\pi \int d^2x j^0(\vec{x}),$$

(5.15)

which implies that

$$R^{2\pi} \phi_0^\dagger(\vec{x})|0\rangle = e^{i2\pi} \int d^2x j^0(\vec{x}) \phi_0^\dagger(\vec{x})|0\rangle = e^{i2\pi s} \phi_0^\dagger(\vec{x})|0\rangle,$$

(5.16)

where in the last step we have used the commutator Eq. (5.11), i.e., the fact that $\phi^\dagger$ creates one unit of charge.
But comparing the graded commutator Eq. (5.14) with the transformation law of the fields Eq. (5.16) it is immediately clear that the correct spin-statistics relation is not compatible with the interpretation of the redefined field operators $\phi_0$ as creation operators for field with generalized statistics, even in the case of fermions. Indeed, if we take $s = 1$, then Eq. (5.14) gives an anticommutator, but the field $\phi_0$ is invariant upon rotation by $2\pi$. If we take $s = \frac{1}{2}$ then the field acquires a factor $-1$ upon $2\pi$-rotation, as a fermion should, but Eq. (5.14) is no longer an anticommutator. Thus there is no way we can obtain from $\psi_0$ simultaneously the good rotational and commutation relations of a fermion field, let alone those of a field with generic statistics. It seems that the construction of $\phi_0$ as field operators with fractional spin and statistics is physically inconsistent [23].

There is also a mathematical problem in the previous derivation. The derivation was based on the fact that the expression Eq. (5.6) for $A^i$ satisfies Eq. (5.5). Let us check this explicitly. The expression Eq. (5.7) of the Green function of the two-dimensional Laplacian, taken jointly with the property Eq. (2.28) of the function $\Theta(\vec{x})$ implies

$$\epsilon^{ij} \partial_i \partial_j \Theta(\vec{x}) = 2\pi \delta^{(2)}(\vec{x}), \quad (5.17)$$

which even though unusual is not contradictory because $\Theta(\vec{x})$ is clearly ill-defined in the origin, thus derivatives acting on it may not commute in that point. Then, using Eq. (5.17) in the expression Eq. (5.8) of $A^i$ we indeed get

$$\epsilon^{ij} \partial_i A^j(\vec{x}) = s \int d^2 y \epsilon^{ij} \partial_i \partial_j \Theta(\vec{x} - \vec{y}) j^0(\vec{y}, t) = 2\pi sj^0(\vec{x}), \quad (5.18)$$

consistently with the claim that $A^i$ Eq. (5.6) solves the constraint Eq. (5.5).

However, we may compute Eq. (5.18) in an alternate, presumably equivalent way: we change the integration variable $\vec{y} \to \vec{x} - \vec{y}$ in the definition of $S(\vec{x})$, Eq. (5.8). Then we get

$$\epsilon^{ij} \partial_i A^j(\vec{x}) = s \int d^2 y \epsilon^{ij} \partial_i \partial_j [\Theta(\vec{y}) j^0(\vec{x} - \vec{y}, t)] = 0, \quad (5.19)$$

if the charge density is a smooth function. Thus we arrive at a contradiction, unless the charge density vanishes or is singular. It seems thus that the construction is also mathematically inconsistent [23].

These two problems point towards some deeper difficulties in extending the construction of fractional spin and statistics from first-quantized point-particle mechanics to second-quantized field theory:

a) In quantum mechanics we have seen that it is possible to redefine the wave functions in
such a way that their transformation properties upon, say, rotations are modified, while
the spectrum of physical observables, such as spin and statistics, is unchanged. In order to
establish the spectrum of physical observables knowledge of the interaction dynamics, and
not only the state vectors, is required. This suggests that knowledge of the transformation
properties of the field operators is not enough to determine whether the spin and statistics
of the fundamental excitations of the theory. The first problem above suggests indeed that
a naive identification of rescaled field operators such as $\psi_0^\dagger$ Eq.(5.9) with creation operators
is incorrect.

b) In particle mechanics unusual transformation properties upon rotations and boosts are
induced on physical states through coupling to a superficially rotationally invariant action
(the Chern-Simons action) thanks to the divergence of the particle-particle interaction at
the singular points where two particles coincide. In field theory, the current $j^\mu(x)$ which
carries the physical excitations is a smooth function over space-time, hence a covariant inter-
action Lagrangian cannot consistently change the transformation laws of smooth fields.

c) In field theory, the spin carried by a physical state ought to depend on its particle con-
tent. In the approaches pursued so far, spin is induced on physical state by means of
a peculiar interaction. Whereas in the first-quantized theory the form of the interaction
depends on the number of particles, in the second quantized theory the interaction is fixed
and it is unclear how it can affect physical quantum numbers in a way which depends on
the particle content.

Even though a systematic treatment is not yet available, we discuss a framework where
at least these problem find a satisfactory answer[25].

5.2. The operator cocycle approach

We wish to pursue the same logic which has lead to fractional spin and statistics in the
first-quantized theory. To this purpose, we need to generalize suitably the concepts of wave
function and propagator. This may be done in the Schrödinger functional formulation of
field theory. Time is singled out, and the fields $\phi(\vec{x})$ are quantized canonically at fixed $t$.
The state vectors are then functionals of the field configurations, and functions of time,
while the fields play the role of coordinates:

$$\langle q, t | \Psi \rangle = \langle \phi(\vec{x}), t | \Psi \rangle = \Psi[\phi(x); t].$$

(5.20)

The state functionals Eq.(5.20) are propagated by

$$K(\phi' (\vec{x}), t'; \phi(\vec{x}), t) = \int \mathcal{D}\phi(\vec{x}, t_0) \exp \left( i \int_t^{t'} dt_0 \int d\vec{x} \mathcal{L}[\phi(\vec{x}, t_0)] \right),$$

(5.21)
where $\mathcal{L}$ is the Lagrangian of the given field theory, and the boundary conditions are the field configurations $\phi(\vec{x})$ at initial time $t$ and $\phi'(\vec{x})$ at final time $t'$. Even though this procedure is not manifestly relativistically invariant at intermediate stages, physical amplitudes ($S$-matrix elements) are. Rather, the state functionals transform covariantly: upon a Lorentz boost that takes the vector $\hat{t}$ into the time-like vector $\hat{n}$ the physical states are transformed into functionals of the fields quantized on the plane orthogonal to $\hat{n}$, at fixed values of the coordinate along $\hat{n}$. In general, one may choose to quantize the system canonically on a space-like plane $\Sigma$ and take the coordinate orthogonal to $\Sigma$ to parametrize its evolution.

Then, suppose we start with a theory of bosons with Lagrangian $\mathcal{L}_0$, and we add to it a topological Lagrangian $\mathcal{L}_t$, \textit{i.e.}, a Lagrangian which may be expressed as the total divergence of a three-vector density:

\begin{equation}
\mathcal{L}_0 = \mathcal{L} + \mathcal{L}_t
\end{equation}

\begin{equation}
\mathcal{L}_t = \partial_\mu \Omega^\mu(x).
\end{equation}

If we demand that fields fall off at infinity, this leads to nonvanishing contributions at initial and final times only, since there the field configuration is nontrivial because of the boundary conditions:

\begin{equation}
\int d\vec{x} dt_0 \partial_\mu \Omega^\mu[\phi(\vec{x}, t_0)] = H(t') - H(t)
\end{equation}

\begin{equation}
H(t) = \int d\vec{x} \Omega_0(\vec{x}, t).
\end{equation}

But then we can proceed exactly as in particle mechanics: we note that the propagator of the theory with Lagrangian $\mathcal{L}$ Eq.(5.22) is related by

\begin{equation}
K(q', t'; q, t) = e^{iH(t')} K_0(q', t'; q, t)e^{-iH(t)}
\end{equation}

to the propagator $K_0(q', t'; q, t)$ of the theory with Lagrangian $\mathcal{L}_0$. Again, the state functionals may be redefined according to

\begin{equation}
\Psi_0[\phi(\vec{x}), t] = e^{-iH(t)} \Psi[\phi(\vec{x}), t],
\end{equation}

and then it follows that the $S$ matrix elements computed acting on the states $\Psi$ with the propagator Eq.(5.24) is the same as that found acting with the propagator $K_0$ free of topological interaction on the redefined states (5.25).
However, in a quantized field theory the fields are operators. Hence, the phase prefactor which enters the redefinition Eq. (5.25) of the state vectors is itself an operator, whose effect on the physical states will in general depend on their particle content. Furthermore, the quantized field operators will display short-distance divergencies which may take the place of the divergencies which appeared in the first-quantized theory when two particles came to coincide. We shall take advantage of these facts to solve the problems discussed above.

The Hopf interaction in field theory

An obvious guess for a candidate topological Lagrangian Eq. (5.22) is the Hopf current-current interaction Eq. (2.20), which we can write in any theory which admits a conserved current $j^\mu$. Using the same trick as in Sect. II.2, namely rewriting the bilocal kernel in the Hopf interaction Eq. (2.21) in terms of a monopole potential, Eq. (2.24), we may cast the Hopf action in the form

$$I_H = -\frac{s}{2} \int d^3 x \, d^3 y \, j^\mu(x) \left[ \partial_\mu \tilde{A}_\nu(x - y) - \partial_\nu \tilde{A}_\mu(x - y) \right] j^\nu(y) \quad (5.26)$$

where

$$\Omega_\mu(x) = j^\mu(x) \int d^3 y \left( \tilde{A}_\rho(x - y) j^\rho(y) + \tilde{A}_\rho(y - x) j^\rho(y) \right). \quad (5.27)$$

This is not quite topological, because $\Omega_\mu(x)$ Eq. (5.27) is still nonlocal in time (it is defined as an integral over all times). However, the divergent nature of the monopole potential when $x \to y$ allows a local determination of the surface terms according to Eq. (5.23). An almost verbatim rerun of the computation which lead from the Hopf interaction Eq. (2.26) to the winding number Eq. (2.29) leads, in field theory, to

$$I_t = -2s \left[ H(t') - H(t) \right] + I_{cov} \quad (5.28)$$

where the surface term is explicitly given by

$$H(t) = \frac{1}{2} \int d^2 x \, d^2 y \, j^0(\vec{x}, t) \Theta(\vec{x} - \vec{y}) j^0(\vec{y}, t), \quad (5.29)$$

and $I_{cov}$ denotes a contribution which cannot be simply cast as a surface term, but is covariant upon Lorentz transformation, i.e., it is the field theoretic analogue of the terms $I_g$ found in particle mechanics [Eq. (2.29)]. The result Eq. (5.28) is derived under the only
assumption that the current $j^\mu$ be conserved as a symmetry (Noether) current (so that its conservation holds at the quantum level, as expressed by Ward identities).

The function $H(t)$ Eq.(5.29) may seem at first quite ill-defined, since the function $\Theta(\vec{x})$ is ill-defined when $|\vec{x}| \rightarrow 0$. Indeed, at the classical level $I(t)$ Eq.(5.28) is Lorentz invariant, implying that $H(t)$ is a rotationally invariant and Lorentz covariant quantity. However, if we use a point-particle expression (i.e. the usual sum of deltas Eq.(2.16)) for the charge densities in Eq.(5.28) then $H(t)$ reduces to the point-particle result Eq.(2.29), which is manifestly rotationally noninvariant. This is at the root of the mathematical difficulties discussed at the end of Sect.V.2. However, in quantum field theory the propagation kernel Eq.(5.24) is an operator, a functional of the field operators on which the currents $j^\mu$ depend. The phases $e^{2i\bar{s}H(t)}$ Eq.(5.25) induced on the state functionals are thus indeed to be viewed as operator-valued quantities. The fact that the bilocal kernel in $H(t)$ is ill-defined at $x = y$ is then irrelevant because the product of the two charge densities diverges when their arguments coincide as $j^0(x)j^0(y) \sim |x-y|^4$. This point is thereby effectively excluded from the integration domain in Eq.(5.29).

Rather than going through the technically involved checks that this is enough to make $H(t)$ Eq.(5.28) well-defined, let us discuss what is the effect of using the form Eq.(5.29) of the phase induced by the topological interaction in the redefinition of physical states according to Eq.(5.25). Then we shall see how the two problems discussed at the end of Sect.V.1 are resolved.

**Operator cocycle and physical states**

An $n$-particle state

$$\Psi^n[\phi(\vec{x}), t] = \phi^\dagger(\vec{x}_1) \ldots \phi^\dagger(\vec{x}_n)|0\rangle$$

will lead to the redefined state Eq.(5.25)

$$\Psi_0[\phi(\vec{x}), t] = e^{2i\bar{s}H(t)}\phi^\dagger(\vec{x}_1) \ldots \phi^\dagger(\vec{x}_n)|0\rangle.$$  

(5.31)

Now, whenever the fields which are integrated over in the expression of $H(t)$ come close to the points $x_1, \ldots, x_n$ there will be short-distance divergences, which lead to the sought-for singularities. Thus, the phase prefactor $e^{2i\bar{s}H(t)}$ should be viewed as an operator-valued cocycle, namely, as an operator which, acting on physical states, produces as its eigenvalues the cocycles appropriate to the various states. These can be computed by performing an operator-product expansion in order to extract the leading singularities in Eq.(5.31).
Again, rather than following this procedure, we deduce the same result by somewhat formal, even though much simpler manipulations. We make use of the commutation relation

\[ [H(t), \phi^\dagger(\vec{z}, t)] = S(\vec{z}, t)\phi^\dagger(\vec{z}, t). \] (5.32)

This follows from the assumption that there exists a creation operator \( \phi^\dagger \) which satisfies the basic commutator Eq. (5.11).

**Exercise:** Prove that

\[ e^{2isH}\Psi^n_0 = \prod_{i=1}^n \left(e^{2isS(\vec{x}_i)}\phi^\dagger(\vec{x}_i)\right) |0\rangle \]

\[ = e^{-2is\sum_{j=1}^n\sum_{i=1}^{j-1}\Theta(\vec{x}_i-\vec{x}_j)} \left[ e^{2is\sum_{i=1}^n S(\vec{x}_i) \prod_{i=1}^n \phi^\dagger(\vec{x}_i)} |0\rangle \right], \] (5.33)

where \( \Psi^n_0 \) is given by Eq. (5.31) and

\[ |0\rangle = e^{2isH}|0\rangle. \] (5.34)

Then, the redefined one- and two-particle state functionals are

\[ \Psi^1_0 = e^{2isH}\phi^\dagger(\vec{x}; t)|0\rangle = e^{2isS(\vec{x})}\phi^\dagger(\vec{x}; t)|\widetilde{0}\rangle \]

\[ \Psi^2_0 = e^{2isH}\phi^\dagger(\vec{x}; t)\phi^\dagger(\vec{y}; t)|0\rangle = e^{2is[S(\vec{x})+S(\vec{y})] - 2is\Theta(\vec{x}-\vec{y})}\phi^\dagger(\vec{x}; t)\phi^\dagger(\vec{y}; t)|\widetilde{0}\rangle, \] (5.35)

and so forth. Here \( |\widetilde{0}\rangle \) is a redefined vacuum, which is generally different from \( |0\rangle \) because even though \( Q|0\rangle = 0 \), in general \( j^0(\vec{x})|0\rangle \neq 0 \). However, the redefinition does not affect the Poincaré invariance of the vacuum, and amounts to normal ordering. Hence, the operator cocycle provides phase prefactors both on one- and many-particle states, while leaving the vacuum invariant (up to normal ordering).

The many-particle states, however, may still be symmetrized: namely, we are free to choose the symmetry of physical states by symmetrizing the states on which the operator phase \( e^{2isH} \) acts, so that the general two-particle (say) state will be

\[ \Psi^2_0 = e^{2is[S(\vec{x})+S(\vec{y})] + is\Theta(\vec{x}-\vec{y})}\phi^\dagger(\vec{x}; t)\phi^\dagger(\vec{y}; t)|0\rangle, \] (5.36)

where \( \sigma \) is a free parameter. Now, it is clear that the parameters \( s \) and \( \sigma \) control respectively the spin and statistics of \( n \)-particle states. Indeed, the latter coincides with the
statistics as defined in Eq. (2.2), because the $\Theta$ dependent phases in Eq. (5.36) just symmetrize the state with respect to the interchange of the quantum numbers $x_1, x_2$. This also leads to a contribution to angular momentum, due to the (kinematical) relation between statistics and angular momentum discussed in the end of Sect. IV.2, according to Eqs. (2.4). The former is related to spin because of the transformation law of $S(\vec{x})$, Eq. (5.13). Also, that Eq. shows that the variation of the function $S(x)$ upon rotations, i.e. its contribution to spin will depend on the charge of the state. In particular for an $n$ particle state, if there are $n$ phase prefactors:

$$R^{2\pi} e^{2is[S(\vec{x}_1) + \ldots + S(\vec{x}_n)]} \phi^\dagger(\vec{x}_1) \ldots \phi^\dagger(\vec{x}_n)|0\rangle = e^{2isn^2} \phi^\dagger(\vec{x}_1) \ldots \phi^\dagger(\vec{x}_n)|0\rangle,$$

(5.37)

i.e., the dependence of the spin on the number of particles is quadratic.

It may be furthermore shown, through straightforward generalization of the techniques discussed in Sect. III.1, that upon generic Lorentz transformations the phase prefactors $S(x)$ and $\Theta(x)$ in Eq. (5.36) transform with the correct Lorentz and Poincaré cocycles appropriate to their particle content. The spectrum of spin, orbital and total angular momentum is thus found to be equal to

$$\ell = n(n - 1)\sigma + k; \quad k \in \mathbb{Z}$$

$$S = n^2 s'$$

$$j = \ell + S,$$

(5.38)

where $s' = 2s$, to be contrasted with the point-particle results, Eq. (4.51).

Comparison of the point particle and field theoretical angular momentum spectra, Eqs. (4.51) and (5.38) shows that: i) the dependence of the spin and statistics on the coefficient of the topological action is by a factor of 2 larger in the field theory; ii) the statistics is a free parameter in the field theory while it is fixed a priori in the particle theory; iii) the dependence of the statistics on the number of particles is the same while that of the spin isn’t. This means that the second quantization of the theory does not commute with the point particle limit, and can be traced to the different way the repulsive core which gives rise to fractional spin and statistics is treated. Namely, in both case the repulsive interaction which gives rise to a multiply connected configuration space (required for fractional spin) and the exclusion principle (required for fractional statistics) is due to the divergence of the bilocal kernel Eq. (2.21) as its arguments $x, y$ coincide. This divergence however is regulated differently. In particle mechanics it is regulated geometrically, by
evaluating the kernel over particle trajectories, which leads to the regular integrand of Eq.(2.23). In field theory it is regulated by the current-current repulsion due to their short-distance divergencies.

We notice finally that all the results derived here have been obtained without explicitly specifying the Lagrangian of the theory whose conserved current is coupled through the Hopf interaction, and rely only on the assumption that a conserved current and a creation operator satisfying Eq.(5.11) exist. Even though this includes a large class of theories, it is worth pointing out that the Lagrangian Eq.(5.1) discussed in Sect.V.1 is not in this class: indeed, in that theory the conserved current is not a symmetry current of the theory defined without Chern-Simons interaction, but rather a dynamically conserved current which depends on the Chern-Simons field.

Resolution of the paradoxes

Even without getting into the details of the operator cocycle approach we can see how the paradoxes discussed at the end of Sect.V.2 are resolved[26]. The spin-statistics paradox requires us to take a closer look at the spin-statistics relation in the present approach. Eq.(5.38) shows that if we require the field theory to be local and well-defined in the thermodynamic limit then a particular spin-statistics relation is singled out. Indeed, noninteracting in and out states can exist only if the total angular momentum (which is an additive quantum number, because the rotation group is abelian) is linear in the number of particles. Otherwise, either noninteracting states do not exist, in which case the thermodynamic limit is ill-defined, or causality is violated. To see that this is true, perform the following Gedankenexperiment. First, measure the spectrum of $j$ for a localized system of $n$ particles; then, add a particle to the system and measure the spectrum again. If $j$ isn’t linear in $n$ the difference in normalization of the two spectra depends on the total number of particles which are arbitrarily far and causally disconnected from the system, i.e., it depends on the “wave function of the universe”.

The requirement of linearity of $j$ Eq.(5.38) in the number of particles $n$ is satisfied if

$$s' = -\sigma$$

which implies

$$j = k + ns' \quad k \in \mathbb{Z}.$$  \hspace{1cm} (5.40)

This is a genuine spin-statistics theorem; it has the opposite sign as that which one might have been naively guessed, and which is displayed by the point particle theory, Eq.(4.51).
However, if spin is integer or half-integer Eq. (5.38) reduces to the usual relation and there is no difference between the field theory and the point particle case.

In order that the spin-statistics relation Eq. (5.39) be satisfied, a nontrivial symmetry has to be imposed on physical states, i.e., the statistics must differ from that automatically generated by the operator phase, and displayed in Eq. (5.33). This prevents the identification of redefined operators as creation operators for particles with fractional spin, because an extra symmetrization is required after the creation operators and the operator cocycle have been applied. Thus this analysis shows that creation and annihilation operators for generic statistics cannot be identified, yet the spin-statistics theorem is satisfied (and has a nontrivial generalization to the fractional case). This resolves the spin-statistics paradox.

The paradox in the manipulations using the purported expression of the Chern-Simons field as a pure gauge, Eq. (5.8) requires us to specify more carefully the integration domains in the definitions of the functions $H(t)$ Eq. (5.29) and $S(\vec{x})$ Eq. (5.8). Indeed, a careful analysis of the computation which leads from Eq. (5.27) to the expression Eq. (5.29) of the surface terms generated by the Hopf action reveals that the extremes of integration in the definition of $H(t)$ read

$$H(t) = \frac{1}{2} \int_{0}^{\infty} \rho_x d\rho_x \int_{0}^{\alpha+2\pi} d\theta_x \int_{0}^{\infty} \rho d\rho \int_{\theta_x}^{\theta_x+2\pi} d\theta \theta \left[ j^0(\vec{x}, t) j^0(\vec{x} + \vec{r}, t) \right], \quad (5.41)$$

where $(\rho_x, \theta_x)$ and $(\rho, \theta)$ are polar components of the vectors $\vec{x}, \vec{r}$, respectively, and $\alpha$ is an arbitrary (multivalued) reference angle, which may be chosen, as usual, by defining $H(t)$ as the integral of its time derivative from a reference field configuration to the given one.

This means that the precise definition of $S(\vec{x})$ is

$$S(\vec{x}) = \int_{\theta_x}^{\theta_x+2\pi} d\theta \int_{0}^{\infty} dr \theta j^0(\vec{x} + \vec{r}', t), \quad (5.42)$$

where $\vec{y}' = (r \cos \theta, r \sin \theta)$. Hence, the function $\Theta$ in the definition of $S(\vec{x})$ is a multivalued function of the polar component $\theta_x$ of the vector $\vec{x}$ on which $S$ depends, but it is a single-valued function in the integration domain with respect to $\vec{y}$.

**Exercise:** Prove that $S(\vec{x})$ defined by Eq. (5.42) is multivalued upon rotations according to Eq. (5.13), by computing the action of the angular momentum operator $L = -ie^{ab}x^a \partial_b$ on it and showing that

$$LiS(\vec{y}) = \int d^2 y e^{ab} x^a (\epsilon^{bc} \frac{(x - y)^c}{\vert \vec{x} - \vec{y} \vert^2} j^0(\vec{y}) + \int \frac{d^2 y j^0(y)}{|y| > |x|} = \int d^2 y j^0(\vec{y}). \quad (5.43)$$
Assume for simplicity that \( j^0 = \) is rotationally invariant.

The paradox described above then disappears because the crucial relation Eq.(5.17) no longer holds. Indeed, Eq.(5.42) shows that in the definition Eq.(5.8) the integration over \( \vec{y} \) is to be performed by taking a definition of \( \Theta \) which has a branch cut along \( \theta_x \), \( i.e., \theta_x \leq \Theta(\vec{x} - \vec{y}) \leq \theta_x + 2\pi \). But if \( \Theta \) has a branch cut, then along the cut the basic property of \( \Theta \), Eq.(2.28), is modified because of the discontinuity in the value of \( \Theta \) along the cut. If, \( e.g., \) the cut is along the positive \( x \)-axis then Eq.(2.28) is modified to

\[
\frac{\partial}{\partial x^a}\Theta(\vec{x}) = -\epsilon_{ab} \frac{x^a}{|\vec{x}|^2} - 2\pi H(x^1)\delta(x^2) \begin{bmatrix} 0 \\ 1 \end{bmatrix}_a,
\]

(5.44)

where \( H \) is the Heaviside step function.\(^{17}\) But from Eq.(5.44) it follows that

\[
\epsilon^{ij} \partial_i \partial_j \Theta(\vec{x}) = 0,
\]

(5.45)

thus the paradox disappears.

Equivalently, the definition Eq.(5.42) of \( S(\vec{x}) \) may be viewed as the result of taking a multivalued determination of \( \Theta \), and extending the integration over the full Riemann surface, but with a particle density which is nonvanishing only between \( \theta_x \) and \( \theta_x + 2\pi \).

**Exercise**: Assume that in the definition of \( S(\vec{x}) \) Eq.(5.8) the integration is extended on the full Riemann surface of the logarithm, \( i.e., \), \( \infty \leq \Theta(\vec{x} - \vec{y}) \leq \infty \), but \( j^0 \) is nonvanishing only in a range of values \( \theta_0 \leq \Theta(\vec{x} - \vec{y}) < \theta_0 + 2\pi \). Show that under such assumptions the interchange of integral and derivative in Eq.(5.8) fails. Compute the correction and show that (assuming for simplicity \( \theta_0 = 0 \)) it is given by \( ^{27} \)

\[
A^i(\vec{x}, t) - s \partial_i S(\vec{x}) - 2\pi \int_{-\infty}^x dx' j^0(x', y, t).
\]

(5.46)

In this case Eq.(2.28) holds, but because of Eq.(5.46) \( S(\vec{x}) \) cannot be written as a pure gauge; derivative and integral do not commute and the non-commutativity resolves the discrepancy between Eq.(5.18) and Eq.(5.19).

The definition of \( S \) Eq.(5.42) may seem awkward, but it is actually the physically natural generalization of the definition adopted in point particle theory. Recall that in

\(^{17}\) which we refrain from calling as usual \( \Theta \) for obvious reasons

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that case Θ was viewed as a multivalued relative polar angle between all couples of particles. This function takes values on the cover of the two-particle relative space, which is the infinitely-sheeted Riemann surface of the logarithm. In Eq. (5.42) $S$ is also the multivalued relative polar angle, weighted by the particle density. Because, however, the particle density is defined on configuration space, the integration is defined on the configuration space, rather than on the infinitely-sheeted surface, whereas the multivaluedness is contained in the determination of the relative angle. Equivalently, the charge density as viewed by the creation operator $\phi^\dag (x)$ occupies a $2\pi$ range on the infinitely-sheeted Riemann surface, along which it may move in the course of the time evolution of the system (see Fig. 3).

\[18\] Notice that the dependence of $S(\vec{x})$ on $\vec{x}$ is due to the commutation relation Eq. (5.32) with $\phi^\dag (\vec{x})$, i.e. it is induced by the creation operator.
6. OPEN PROBLEMS

In these lectures we have discussed an approach to the quantization of systems with fractional spin and statistics which is amenable to a relativistic treatment, and seems to carry through from quantum mechanics to field theory. The approach is based on the path-integral quantization of spin using phase-space variables. Even though this approach has allowed us to give a complete and consistent description of the kinematics of these systems, dynamical results are conspicuously missing. However elegant the mathematics involved, we have only proven that the particle content of the theories we have studied is consistent with the desired transformation properties under the Lorentz and Poincaré groups. Nevertheless, our analysis has led to at least one nontrivial result, namely that the spin-statistics relation seems to have a different structure in quantum mechanics and field theory.

The first steps we described in the formulation of relativistic quantum mechanics should hopefully open the way to an investigation of dynamical problems, such as bound states problems, scattering, electromagnetic coupling. This should also shed light on the relationship between the cocycle approach, described here, and the alternative approach based on infinite-component wave functions. The construction of a satisfactory field theory with fractional statistics still has a long way to go: we would like, first of all, to understand the structure of the Hilbert space for such a theory. This should allow one to reach a deeper understanding of spin-statistics relations, and eventually of the relativistic statistical mechanics of such systems. Perhaps, the intricacies we found are hinting to new physical structures which are awaiting discovery.

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