Electroosmotic flow in Hele-Shaw configurations with non-uniform surface charge

Evgeniy Boyko*, Shimon Rubin*, Amir Gat, and Moran Bercovici

*Equal contribution

Faculty of Mechanical Engineering,
Technion - Israel Institute of Technology, Haifa, Israel

Abstract

We present an analytical study, validated by numerical simulations, of electroosmotic flow in a Hele-Shaw cell with non-uniform surface charge patterning. Applying the lubrication approximation and assuming thin electric double layer, we obtain a pair of uncoupled Poisson equations which relate the pressure and the stream function, respectively, to gradients in the zeta potential distribution parallel and perpendicular to the applied electric field. We solve the governing equations for the fundamental case of a disk with uniform zeta potential and show that the flow-field in the outer region takes the form of a pure dipole. We illustrate the ability to generate complex flow-fields around smooth convex regions by superposition of such disks with uniform zeta potential and a uniform pressure driven flow. This method may be useful for future on-chip devices, allowing flow control without the need for mechanical components.
INTRODUCTION

Electroosmotic flow (EOF) is the motion of a liquid due to interaction of an externally applied electric field with the net surplus of charged mobile ions in the diffuse part of an electrical double layer (EDL). For solid surfaces with a uniform charge distribution and channel geometry, EOF is characterized by a uniform plug-like velocity profile, making it a useful tool for dispersion-free transport [1]. However, in practice, most surfaces are non-homogeneously charged to some extent, either due to manufacturing limitations or by intended design.

EOF with inhomogeneous zeta potential distribution has been studied thoroughly in cylindrical capillaries. Anderson and Keith Idol [2] found an exact solution to EOF through a capillary with axially varying zeta potential, assuming negligible inertia and a thin Debye layer. Herr et al. [3] investigated analytically and experimentally a particular case of an EOF through a cylindrical capillary with an axial step change in surface charge. They measured experimentally the flow profile and showed good agreement with the theoretical predictions.

Ajdari [4, 5] was the first to analytically study and provide a closed form solution to the two dimensional problem of EOF between an undulating plate and a flat plate with a periodic surface charge distribution. Ajdari focused his analysis on the cross-section of the flow cell and demonstrated net flow generation between the plates, even though the plates are on average electro-neutral. Focusing on parallel flow cell, Long et al. [6] provided an analytical solution for the three dimensional EOF field due to arbitrary distribution of zeta-potential. In particular, by taking the limit of a small gap between parallel plates, Long et al. [6] obtained a solution associated with a Hele-Shaw case, deducing that a localized defect in surface charge distribution induces long-range flow perturbations. Ajdari [7] considered a three dimensional geometry, consisting of two plates with a spatially varying gap and periodic zeta potential in one direction. By applying the lubrication theory, Ajdari formulated the Onsager matrix, relating the applied pressure gradient and electric field to flow rate and electric current. Ghosal [8] also applied a lubrication approximation to investigate EOF in an infinitely long channel with slowly varying cross-section shape and zeta-potential in the axial direction. Ghosal showed that the flow rate through any section can be related to a uniform cylindrical capillary with an equivalent radius and zeta potential, which are determined solely by the geometry and surface charge distribution in the channel. Stroock
et al. [9] performed an experimental study of EOF in flat shallow micro-channels having non-uniform zeta potential and compared the results with those of Ajdari [4, 5] and Long et al. [6]. Stroock et al. [9] considered alternate positive and negative zeta potential patterns and demonstrated that various flow types such as multi-directional or circular flow can be generated, depending on whether the applied field is parallel or perpendicular to gradient of zeta potential.

In this work we present an analytical study, validated by numerical simulations, showing that non-uniform surface potential can be leveraged to obtain complex flow patterns in a chamber composed of two parallel plates separated by a small gap (Hele Shaw cell). We apply a lubrication approximation and derive a pair of Poisson governing equations relating the pressure and the stream function to the surface charge distribution within the cell. Importantly, we obtain a solution of the equation for the fundamental case of a disk with non-zero zeta-potential. We show that for the case of vanishing pressure far from the disk, the flow field outside the disk corresponds to that of a dipole, and demonstrate the superposition of such dipoles for creating complex desired flow fields.

**PROBLEM DEFINITION**

We hereafter denote dimensional variables by tildes and normalized variables without tildes. Figure 1 presents a schematic illustration of the problem’s setup. We consider creeping flow in a narrow gap between two parallel plates. We employ a Cartesian coordinate system $(\tilde{x}, \tilde{y}, \tilde{z})$ whose $\tilde{x}$ and $\tilde{y}$ axes lie at the lower plane and $\tilde{z}$ is perpendicular thereto. The characteristic length of the plates in the $\tilde{x} - \tilde{y}$ plane is $\tilde{l}$, and the gap between the plates is $\tilde{h}$. The lower and upper plates have arbitrary zeta potential distributions, defined as $\tilde{\zeta}^L(\tilde{x}, \tilde{y})$ and $\tilde{\zeta}^U(\tilde{x}, \tilde{y})$, respectively. Hereafter, we adopt the $\parallel$ and $\perp$ subscript to denote parallel and perpendicular directions to the plates, respectively. A uniform two-dimensional external electric field $\tilde{E}_\parallel = (\tilde{E}_x, \tilde{E}_y)$ is applied parallel to the plates. The continuity equation for incompressible fluid is

$$\tilde{\nabla} \cdot \tilde{\u} = 0. \quad (1)$$
FIG. 1. Schematic illustration of the problem. (a) We consider a configuration consisting of two parallel plates of characteristic length $l$, separated by a thin gap $h$. Each of the plates is functionalized with arbitrary surface charge, resulting in surface potentials of $\zeta^L, \zeta^U$ on the lower and upper plates, respectively. The fluid is subject to a uniform electric field $E \parallel$. (b) Averaging over the depth of the cell yields a 2D problem, in which $\langle \zeta \rangle$ denotes the average potential of the two plates, $\langle \zeta \rangle = (\zeta^U + \zeta^L)/2$.

Assuming a thin electric double layer regime and neglecting body forces, the momentum equation is given by

$$\tilde{\rho} \left( \frac{\partial \tilde{u}}{\partial t} + \tilde{u} \cdot \nabla \tilde{u} \right) = -\nabla \tilde{p} + \tilde{\eta} \nabla^2 \tilde{u},$$

where $\tilde{u} = (\tilde{u}, \tilde{v}, \tilde{w})$ is the velocity vector, $\tilde{t}$ is time and $\tilde{\rho}, \tilde{\eta}$ and $\tilde{p}$ denote the fluid density, dynamic viscosity and pressure, respectively. We account for the body forces acting on the double layer by using the Helmholtz-Smoluchowski slip boundary conditions given by

$$\tilde{u}_\parallel \bigg|_{\tilde{z}=0} = -\tilde{\varepsilon} \tilde{\zeta}^L E_\parallel / \tilde{\eta}; \quad \tilde{u}_\parallel \bigg|_{\tilde{z}=h} = -\tilde{\varepsilon} \tilde{\zeta}^U E_\parallel / \tilde{\eta}; \quad \tilde{u}_\perp \bigg|_{\tilde{z}=0,\tilde{h}} = 0,$$

where $\tilde{u}_\parallel = (\tilde{u}, \tilde{v}), \tilde{u}_\perp = \tilde{w} \hat{z}$, $\tilde{\varepsilon}$ is the fluid permittivity while $\tilde{\zeta}^L, \tilde{\zeta}^U$ are position dependent zeta-potentials on each one of the plates.

**GOVERNING EQUATIONS FOR NON UNIFORM EOF IN A HELE-SHAW CELL**

Hereafter we denote characteristic scales of the system using an asterisk superscript. The typical magnitude of in-plane flow velocity, $\tilde{u}^*$, is determined by the Helmholtz - Smoluchowski slip condition as $\tilde{u}^* = -\tilde{\varepsilon} \tilde{\zeta}^* \tilde{E}^* / \tilde{\eta}$, where $\tilde{\zeta}^*$ is characteristic value of zeta potential and $\tilde{E}^*$ is characteristic externally applied electric field. We choose the characteristic time scale as $t^* = \tilde{l} / \tilde{u}^*$, whereas the characteristic velocity in the $\tilde{z}$ direction, $\tilde{w}^*$, and the characteristic pressure, $\tilde{p}^*$, remain to be determined from scaling arguments.
We define the normalized coordinates, \((x, y, z)\), and time, \(t\): \(x = \tilde{x}/\tilde{l}, y = \tilde{y}/\tilde{l}, z = \tilde{z}/\tilde{h}, t = \tilde{t}/\tilde{t}^*\); normalized velocity components, \((u, v, w)\), and pressure, \(p\): \(u_\parallel = \tilde{u}_\parallel/\tilde{u}^*, w = \tilde{w}/\tilde{w}^*, p = \tilde{p}/\tilde{p}^*\); and normalized zeta potential, \(\zeta\), and electric field, \(E_\parallel\): \(\zeta = \tilde{\zeta}/\tilde{\zeta}^*, E_\parallel = \tilde{E}_\parallel/\tilde{E}^*\). We focus our analysis on a shallow flow chamber, \(\epsilon = \tilde{h}/\tilde{l} \ll 1\), \((4)\) and negligible inertia regime. The latter is characterized by negligible reduced Reynolds number, \(\epsilon Re\), defined as
\[
\epsilon Re = \epsilon \frac{\tilde{\rho} \tilde{u}^* \tilde{h}}{\tilde{\eta}} \ll 1.
\](5)
In the following, we will consider the leading order in \(\epsilon, \epsilon Re\) of \(2\). From order of magnitude analysis of (1) we obtain the characteristic velocity in the \(\hat{z}\) direction, \(\tilde{w} = \epsilon \tilde{u}^*\). Order of magnitude analysis of (2) yields the typical pressure scale, \(\tilde{p}^* = \tilde{\eta} \tilde{u}^*/\epsilon^2 \tilde{l}\). We note that as the flow is driven by the wall boundary conditions, the characteristic pressure is independent of the viscosity, \(\tilde{p}^* = -\tilde{\varepsilon} \tilde{\zeta}^* \tilde{E}^*/\epsilon^2 \tilde{l}\). Integrating the leading order of the in-plane components in (2) twice with respect to \(z\) while making use of the boundary conditions (3), we obtain an expression for the in-plane velocity field
\[
\mathbf{u}_\parallel = \frac{1}{2} z (z - 1) \nabla_\parallel p + z (\zeta^U - \zeta^L) \mathbf{E}_\parallel + \zeta^L \mathbf{E}_\parallel,
\](6)
where \(\nabla_\parallel = (\partial/\partial x, \partial/\partial y)\) is the two dimensional gradient. Defining the mean in-plane velocity, as \(\langle \mathbf{u}_\parallel \rangle = \int_{z=0}^{z=1} \mathbf{u}_\parallel dz\), and depth averaging over \((1)\) and \((6)\) yields
\[
\nabla_\parallel \langle \mathbf{u}_\parallel \rangle = 0,
\](7)
and
\[
\langle \mathbf{u}_\parallel \rangle = -\frac{1}{12} \nabla_\parallel p + \langle \zeta \rangle \mathbf{E}_\parallel,
\](8)
where \(\langle \zeta \rangle\) is an arithmetic mean value of the potential on the walls, \(\langle \zeta \rangle = (\zeta^U + \zeta^L)/2\). Assuming a uniform electric field, applying the two dimensional divergence to (8), and using (7), we obtain an equation in terms of the pressure only,
\[
-\frac{1}{12} \nabla_\parallel^2 p + \nabla_\parallel \langle \zeta \rangle \cdot \mathbf{E}_\parallel = 0.
\](9)
Similarly, applying the normal component of the curl operator to (8), leads to an equation for the stream function,
\[
\nabla_\parallel^2 \psi + \nabla_\parallel \langle \zeta \rangle \times \mathbf{E}_\parallel = 0.
\](10)
Eqs. (9) and (10) represent an uncoupled set of Poisson equations for the pressure and the stream function, and extend the Hele-Shaw equation [11] to include non-uniform EOF. Notably, (9) includes a source term that depends on gradients of zeta potential which are parallel to the applied electric field, while (10) relates the vorticity, \( \omega = -\nabla_\parallel^2 \psi \), to changes of zeta potential in a direction normal to the applied electric field. In particular, when \( \nabla_\parallel \langle \zeta \rangle \times \vec{E} \parallel = 0 \) holds in the entire domain, the resulting flow field would be irrotational and a velocity potential function could be defined.

**GENERAL SOLUTION FOR AN AXIALLY SYMMETRIC CASE**

**Disk with constant \( \zeta \)-potential**

Consider the case where the \( \zeta \)-potential distribution acquires a non-zero value, \( \zeta_0 \), in the inner region \( (r_0 < r) \) and vanishes in the outer region, \( (r_0 > r) \). In the following, we adopt the superscripts \( \text{in} \) and \( \text{out} \) to distinguish between physical quantities in each one of the two regions. The corresponding distribution is given by

\[
\langle \zeta(r) \rangle = \zeta_0 H(r_0 - r), \tag{11}
\]

where \( r \) is the distance as measured from the center of our coordinate system which coincides with the center of the disk, and \( H \) denotes the Heaviside step function. Yariv [12], Khair and Squires [13] showed that discontinuity in \( \zeta \)-potential may affect the electric field in the bulk, near the discontinuity. It has been shown [13] that the characteristic length scale of this effect is associated with the ratio of surface to bulk conductivities and is termed healing-length, \( \tilde{L}_H \). Importantly, in our work we assume that the corresponding healing-length is much smaller than all other length scales, and as a result the electric field is uniform everywhere. Furthermore, we assume the radius of the disk, \( r_0 \), is much larger than the height of the channel, \( \epsilon \). These assumptions can be summarized as \( \tilde{L}_H / \tilde{l} \ll \epsilon \ll r_0 \ll 1 \).

Under these assumptions, for an electric field directed along the \( x \) axis, (9) takes the following form in polar coordinates

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 p}{\partial \theta^2} - 12E \cos(\theta) \frac{d \langle \zeta \rangle}{dr} = 0, \tag{12}
\]

where \( \theta \) is the azimuthal angle in the plane and \( x = r \cos(\theta) \). The last term in (12) suggests
FIG. 2. The pressure distribution (color-map) and streamlines (white lines) due to a uniform electric field in the $x$ direction, applied to a Hele-Shaw configuration having non-zero zeta-potential in a disk of radius $r_0$. (a) The case of no incident flow. The depth averaged flow field described by (16), results in uniform flow in the inner region and dipole flow in the outer region. Note that the points of extremum pressure and vorticity correspond to locations where the source terms in (9) and (10) attain their minimal or maximal values. (b) The case where an external pressure gradient (counter-flow) is applied such that the velocity inside the disk vanishes. The result coincides with that of potential flow around a cylinder and is characterized by open streamlines which do not enter the disk region. Both solutions are normalized such that the pressure varies between $\pm 1$.

A solution of the form

$$p(r, \theta) = \alpha(r) \cos(\theta),$$

(13)

where $\alpha(r)$ is yet to be determined. Combining (11)-(13) yields the following ODE for $\alpha(r)$

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{d\alpha(r)}{dr} \right) - \frac{\alpha(r)}{r^2} + 12E\zeta_0 \delta(r_0 - r) = 0$$

(14)

where $\delta$ is the Dirac delta function. Solving (14) separately in the inner and outer regions, and requiring regularity yields the following expression for the pressure

$$p(r, \theta) = \begin{cases} 
\frac{a^{\text{out}}}{r} + b^{\text{out}} r & \text{if } r > r_0 \\
0 & \text{if } r < r_0 
\end{cases} \cos(\theta)$$

(15)

where the dipole strength $a^{\text{out}}$, $b^{\text{in}}$, and $b^{\text{out}}$ are coefficients to be determined from boundary conditions. Utilizing (9), (11) and (15) provides the associated closed-form expression for
the flow field

\[
\langle u_\parallel \rangle = \begin{cases} 
\frac{1}{12} \left[ \left( \frac{a^{\text{out}}}{r^2} - b^{\text{out}} \right) \cos(\theta) \hat{r} + \left( \frac{a^{\text{out}}}{r^2} + b^{\text{out}} \right) \sin(\theta) \hat{\theta} \right] & r > r_0 \\
\frac{1}{12} \left[ (-b^{\text{in}} + 12E\zeta_0) \cos(\theta) \hat{r} + (b^{\text{in}} - 12E\zeta_0) \sin(\theta) \hat{\theta} \right] & r < r_0.
\end{cases}
\]

(16)

Remarkably, in the inner region, the pressure changes linearly in the direction of the electric field, which results in a uniform velocity vector field, explicitly given by

\[
\langle u_\parallel \rangle^{\text{in}} = \left( -\frac{1}{12} \frac{\partial p}{\partial x} + E\zeta_0 \right) \hat{x} = \left( -\frac{1}{12} b^{\text{in}} + E\zeta_0 \right) \hat{x}.
\]

(17)

The coefficients are determined by demanding continuity of the pressure and radial velocity component at \( r = r_0 \), leading to

\[
a^{\text{out}} = 6E\zeta_0 r_0^2; \quad b^{\text{in}} - b^{\text{out}} = 6E\zeta_0.
\]

(18)

The coefficient \( a^{\text{out}} \) represents a contribution to the pressure as a result of the local heterogeneity in \( \zeta \)-potential, while the constant \( b^{\text{out}} \) represents the magnitude of a uniform flow field in the \( x \) direction which stems from an externally applied pressure gradient in that direction, \( b^{\text{out}} = \Delta p/\Delta x \) (e.g. for the case of constant pressure at \( r \to \infty \), \( b^{\text{out}} = 0 \)). \( b^{\text{in}} \) is then readily obtained from (18).

The difference between \( b^{\text{in}} \) and \( b^{\text{out}} \) represents discontinuity in the tangential component of the depth averaged velocity on the surface \( r = r_0 \), induced by the vorticity source specified by (10). From (16) we find that the corresponding value of tangential velocity jump (i.e. difference between outer to inner values) is \( -E\zeta_0 \). This result can be also obtained by using the relation \( \langle u_\theta \rangle = -\partial \psi/\partial r \) and radially integrating (10) along an infinitesimal region around \( r_0 \).

Several interesting observations can be made regarding the dipole solution (15) in the context of potential theory. First, the velocity in the outer region is identical in form to that obtained from the superposition of a dipole and uniform flow in potential flow theory. The pressure (15) in the outer region, can thus be interpreted (up to a factor of \(-1/12\)) as the velocity potential. Consistently, the corresponding solution of (10) for the stream function is also a dipole of the form (15), rotated by \( \pi/2 \) in the counterclockwise direction.

Figure 2(a) presents the streamlines and pressure distribution map of a single electroosmotic dipole for the case of vanishing pressure far from the disk (\( b^{\text{out}} = 0 \)). As expected, the boundary conditions in the inner region dictate flow from low to high pressure, while
the flow in the outer region is from high to low pressure. Interestingly, taking advantage of
the uniform flow field in the inner region, we can superpose it with other depth averaged
flow of equal magnitude and opposite direction, to achieve zero net flow in the inner region,
i.e. \( \langle u_{||} \rangle^{in} = 0 \). This can be achieved by applying a mean pressure gradient \( \Delta p/\Delta x = b^{out} = 6E\zeta_0 \), or by adding a bias value of \(-\zeta_0/2\) to the zeta potential everywhere in the
domain. Figure 2(b) presents the streamlines and pressure distribution for this case. The
corresponding flow field around the disk coincides then with the well-known solution of
potential flow past an infinitely long cylinder.

**Arbitrary axially symmetric distribution**

Employing the linearity of the governing equations enables to superpose the basic dipole
solution \( b^{out} = 0 \), [15], in order to to construct more complex, axially symmetric solutions.
Specifically, we consider the case where \( \zeta_0 \) in (11) is now some function, \( \zeta_0(r) \), of the radial
coordinate. Using superposition we obtain the corresponding pressure distribution, \( p_{tot} \),
\[
 p_{tot}(r, \theta) = \begin{cases} 
 \frac{12E}{r} \cos(\theta) \left( \int_0^{r_0} \zeta_0(r_0') r_0' dr_0' \right) & r > r_0 \\
 12E \cos(\theta) \left[ \frac{1}{r} \left( \int_0^r \zeta_0(r_0') r_0' dr_0' \right) + \frac{r}{2} \left( \int_r^{r_0} \zeta_0(r_0') r_0' dr_0' \right) \right] & r < r_0.
\end{cases}
\]  
(19)

Figure 3 presents an analytical solution for superposition of two concentric disks of radii \( r_0/2 \) and \( r_0 \) with zeta potentials \( \zeta_0 \) and \(-\zeta_0/4\) respectively. For this special case, the pressure
vanishes outside the larger disk, and results in flow that is confined to the boundaries of the larger disk, whereas a uniform velocity field is obtained in the smaller disk, as expected.

**Flow around an Airfoil in a Hele-Shaw cell**

We here present an example, which demonstrates the ability to engineer desired EOF, in a Hele-Shaw cell with a uniform electric field. Specifically, we apply the panel method, commonly used in aerodynamics [14], to design EOF around a symmetric NACA 0015 airfoil profile. For concreteness, we choose the direction of the electric field in our example along the $x$ axis, and specify a known pressure gradient, $\Delta p/\Delta x$, along this direction. We place a set of 24 disks, each having a uniform (but potentially different) zeta potential, also along this axis. We then utilize the closed form dipole solution generated by each disk, and linearity of governing equations, to calculate the velocity vector at 24 points along the airfoil curve. Demanding no-penetration at each of the points, results in a set of linear algebraic equations for the associated dipole strength values. Figure 4(a) shows the resulting dipole strength of each of the disks as well as the corresponding location along the $x$ axis. Figure 4(b) presents the streamlines (solid lines) obtained from the analytical solution. To validate the results, we performed a three dimensional direct numerical simulation using Comsol Multiphysics, in which the Stokes flow equations are solved, coupled to Helmholtz - Smoluchowski boundary conditions. We used equally sized disks having zeta potential values as determined by the analytical solution. The velocity field is depth-averaged, and streamlines of this numerical solution are also presented in the figure (dashed), showing good agreement with the analytic result.

**EFFECT OF PATTERNING GEOMETRY AND ORIENTATION ON GENERATION OF PRESSURE GRADIENTS AND VORTICITY**

Eqs. (9) and (10) are both Poisson-type equations for the pressure and vorticity, in which gradients in zeta potential serve as source terms. The respective dot-product and curl operators in these source terms suggest that the solutions would fundamentally depend on the geometry of surface patterning. To further highlight the roles of these source terms, let us consider two simple cases of EOF arising from a thin finite line with uniform zeta
FIG. 4. Superposition of EOF dipole elements enables creation of complex flow patterns, satisfying no-penetration on desired convex surfaces. (a) 24 circular disks are distributed along the x axis. The non-dimensional dipole strength of each of the disks is set such that a no-penetration boundary condition is obtained over a NACA 0015 airfoil. (b) Comparison of analytical (solid lines) and numerical results (dashed lines) of the flow field obtained from the distribution. Analytical results were obtained from 2D Hele-Shaw model, whereas the numerical results are depth-averaged from a direct 3D numerical solution using Comsol Multiphysics.

potential, which is either aligned with the electric field or perpendicular to it. In both cases, we model the lines using the linearity of the governing equations, and sum up the pressure contributions of dipoles having a strength per unit length $a_{out}$

$$dp(x, y) = \frac{a_{out}(x - x_0(l))}{(x - x_0(l))^2 + (y - y_0(l))^2} dl,$$

where the dipole position is parametrized by $l$, and $dl$ is an infinitesimal interval along the curve $l$. We then use (8) to derive the fluid velocity profile outside the dipole distribution. Without loss of generality, we choose an electric field directed along the x axis.

For the case of a line of dipoles is directed along the electric field the pressure distribution and the mean velocity are given by

$$p(x, y) = a_{out} \ln \left( \frac{r_+}{r_-} \right),$$

$$\langle u_\parallel \rangle = -\frac{a_{out}}{12} \left( \frac{\hat{r}_+}{r_+} - \frac{\hat{r}_-}{r_-} \right),$$

while for a case where the line is perpendicular to electric field these are given by

$$p(x, y) = a_{out} \left( \tan^{-1} \left( \frac{y - y_-}{x - x_0} \right) - \tan^{-1} \left( \frac{y - y_+}{x - x_0} \right) \right),$$

$$\langle u_\parallel \rangle = \frac{a_{out}}{12} \left( \frac{\hat{\theta}_+}{r_+} - \frac{\hat{\theta}_-}{r_-} \right).$$
FIG. 5. Analytical solution showing the flow field due to a line of uniform zeta potential aligned (a) along the direction of the electric field and (b) perpendicular to the electric field. Each solution is obtained from superposition of dipoles, according to [20]. Color-maps describe the normalized pressure distribution and the white lines show streamlines. In configuration (a) gradients in zeta potential are all in the directions of the electric field, resulting in source/sink terms at the edges of the line for the pressure equation (9). The streamline equation (10), on the other hand, remains homogenous and no vorticity is formed. In (b), the line’s edges form a zeta potential gradient perpendicular to the electric field, resulting in vorticity and localized circulation around each for the edges. In addition, a pressure difference forms across the line, reflecting the fact the zeta potential changed in a direction parallel to the direction of the electric field.

Here, \( \hat{\theta}_\pm \) are unit vectors which perpendicular to \( \hat{r}_\pm \); \( r_- \), \( r_+ \) are corresponding distances from the point \((x, y)\) to the edge points \( r_- = (x_-, y_0) \), \( r_+ = (x_+, y_0) \), respectively.

The solution (21) can be interpreted as a sink and a source, located at the edges of the uniform zeta potential line. Figure 5(a) presents the resulting pressure and velocity field for this case. Clearly, the apparent violation of continuity, \( \nabla \parallel \cdot \langle u_\parallel \rangle = 0 \), condition at the two points is an artifact of the fact that (21a) and (21b) represent the flow outside of the array of dipoles, and additional flow that connects \( r_- \) and \( r_+ \) exists within this array. Note that in this case all zeta potential gradients are in the direction of the electric field and thus, consistent with the streamlines equation (10), the flow field, (21b), is free of vorticity. The solution (22) is effectively equivalent to a pair of equal strength and opposite sign vortices, centered at the edges of the uniform \( \zeta \) line. Figure 5(b) presents the exact analytical solution for the pressure and velocity, as given by (22) and (22). Consistent with the equation for the pressure, (9), high pressure gradients are formed between the two sides.
of the line, where the zeta potential changes in a direction parallel to the electric field. At
the line’s edges the pressure difference results in circulatory flow, which is again consistent
with (10) which predicts vorticity where the zeta potential changes in a direction normal to
electric field (in this case, the uniform zeta line terminating into the zeta-free surface). The
circulation, defined as the line integral, $\Gamma_\pm = \oint \langle \mathbf{u} \rangle \cdot d\mathbf{l}$, along a closed path around each
edge point coincides (up to a factor of $\pi/6$ ) with individual dipole strength $\Gamma_\pm = \pm \pi a^{\text{out}}/6$. Clearly, $\Gamma_+ + \Gamma_- = 0$, i.e. the total circulation remains zero. More generally, from (10) it
follows that the total vorticity generated by non-homogeneity of an arbitrary shape which
hosts a constant zeta potential must vanish: the component of the zeta potential gradient
perpendicular to the electric field is $\langle \zeta \rangle \sin(\theta)$, and thus its line integral along the closed
curve must vanish.

CONCLUDING REMARKS

In order to satisfy continuity, EOF on surfaces having non-uniform surface potential
results in internal pressure gradients. In planar flows, these pressures establish velocity
components which are perpendicular to the applied electric field. Thus, a unidirectional
applied electric field is able to drive complex planar flows. We emphasize that the flow fields
we obtained in this work are depth-averaged fields. For example, zero velocity (as in the
center of the disk in uniform flow, figure 2(a)) could correspond to either truly quiescent
flow when (locally) uniform EOF is balanced by pressure driven flow, or to a Couette-type
profile with flow in both directions.

It is important to note that the (8), (9) and (10) are conformal invariant. This follows
directly from the fact that these equations can be rewritten so that each term will be contain
equal number of gradient operators [15]. This opens a door to employ conformal mapping
in studying non-uniform EOF in complex planar geometries. We also note that the stream
function Eq. (10), can be used to define an inverse problem to obtain the required zeta
distribution for obtaining a desired flow field.

In practice, surface patterning could be obtained in several ways, including chemical mod-
ification of the surface, or using an array of surface electrodes. The latter adds complexity,
but has the potential advantage of allowing surface potentials which are much higher, as well
as dynamic modification of the zeta distribution. We believe that flow patterning by EOF,
particularly by dynamic modification of the surface potential, may serve a powerful tool for manipulating fluids without mechanical components in a variety of microfluidic applications.

This research was supported by the Israel Science Foundation (grants No. 512/12 and 818/13) and the FP7 Marie Curie Career Integration (grant No. PCIG09-GA-2011-293576).

[1] H. A. Stone, A. D. Stroock, and A. Ajdari, Annu. Rev. Fluid Mech. 36, 381 (2004).
[2] J. L. Anderson and W. Keith Idol, Chem. Engng. Commun. 38, 93 (1985).
[3] A. E. Herr, J. I. Molho, J. G. Santiago, M. G. Mungal, T. W. Kenny, and M. G. Garguilo, Anal. Chem. 72, 1053 (2000).
[4] A. Ajdari, Phys. Rev. Lett. 75, 755 (1995).
[5] A. Ajdari, Phys. Rev. E 53, 4996 (1996).
[6] D. Long, H. A. Stone, and A. Ajdari, J. Colloid Interface Sci 212, 338 (1999).
[7] A. Ajdari, Phys. Rev. E 65, 016301 (2001).
[8] S. Ghosal, J. Fluid Mech. 459, 103 (2002).
[9] A. D. Stroock, M. Weck, D. T. Chiu, W. T. S. Huck, P. J. A. Kenis, R. F. Ismagilov, and G. M. Whitesides, Phys. Rev. Lett. 84, 3314 (2000).
[10] R. J. Hunter, Zeta potential in colloid science: principles and applications (Academic press London, 1981).
[11] H. S. Hele-Shaw, Nature 58, 520 (1898).
[12] E. Yariv, J. Fluid Mech. 521, 181 (2004).
[13] A. S. Khair and T. M. Squires, J. Fluid Mech. 615, 323 (2008).
[14] J. Katz and A. Plotkin, Low-speed aerodynamics, vol. 13 (Cambridge University Press, 2001).
[15] M. Z. Bazant, Proc. R. Soc. Lond. A Math. Phys. Engng. Sci. 460, 1433 (2004).