RELATIVE POSITIONS OF MATROID ALGEBRAS

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Abstract. A classification is given for regular positions $D \oplus D \subseteq D$ of Jones index 4 where $D = \text{alg lim} M_n(C)$ is an even matroid algebra and where the individual summands have index 2. A similar classification is obtained for positions of direct sums of 2-symmetric algebras and, in the odd case, for the positions of sums of 2-symmetric $C^*$-algebras in matroid $C^*$-algebras. The approach relies on an analysis of intermediate non-self-adjoint operator algebras and the classifications are given in terms of $K_0$ invariants, partial isometry homology and scales in the composite invariant $K_0(-) \oplus H_1(-)$.

1. Introduction

A matroid algebra is a complex algebra with involution which is a union of a chain of subalgebras each of which is a full matrix algebra. Together with their $C^*$-algebra counterparts, the matroid $C^*$-algebras, these fundamental algebras are classified in terms of the range of a normalised trace on projections, or, equivalently, by $K_0$ as a scaled ordered dimension group. See Dixmier [Dix] and Elliott [Ell]. Although a direct sum of matroid algebras is similarly determined we show that the position of this direct sum as a subalgebra of a matroid superalgebra depends on a number of much finer invariants. Here we say that subalgebras are in the same position if they are conjugate by an automorphism of the superalgebra. The terminology follows the von Neumann algebra usage, as in Ocneanu [Ocn]. An analysis is given of the relative position of the summands of a direct sum subalgebra of a matroid algebra and in particular a complete classification is obtained for what may be viewed as the first nontrivial case, namely that of regular positions

$$D \oplus D \subseteq \tilde{D}$$

where $D$ and $\tilde{D}$ are even matroid algebras and the individual summand inclusions

$$D \oplus 0 \subseteq p_1 \tilde{D} p_1, \quad 0 \oplus D \subseteq p_2 \tilde{D} p_2$$

for the central projections $p_1, p_2$, are standard inclusions of Jones index 2. Thus $D \oplus D$ has index 4 in $\tilde{D}$. See [GDJ] and [Wat]. The regularity of the inclusion requires that the subalgebra and the superalgebra share a standard regular AF masa.

Classifications are given in terms of

(a) the scaled ordered group $(K_0(-), \Sigma_0(-))$ of the subalgebra,

(b) a scaled partial isometry homology group $(H_1(-), \Sigma_1(-))$, which may be realised as a subgroup of $\mathbb{Q}$ together with an interval symmetric about the origin,

(c) a joint scale $\Sigma(-)$ in the composite invariant $K_0(-) \oplus H_1(-)$.

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As the index 2 positions are unique the invariants which augment $K_0$ should be viewed as invariants for the relative position of the matroid algebra summands.

The joint scale in the composite invariant, which need not coincide with the product scale, accounts for a number of obstructions to lifting $K_0 \oplus H_1$ isomorphisms to algebra isomorphisms. The simplest of these is an $H_0H_1$ coupling obstruction which manifests itself as a coset in $\mathbb{R}/\mathbb{Q}$. If this vanishes then so too do the other obstructions and the partial isometry homology group $H_1(-)$ is the remaining invariant for relative position. Thus there are uncountably many positions in this case and there is a unique position with trivial $H_1$ invariant.

Similar classifications are obtained for inclusions of direct sums of the so called 2-symmetric algebras in matroid algebras. In the odd case, which is somewhat more accessible, we obtain an approximate version of the key arguments of section 3 and this enables the classification of inclusions of sums of 2-symmetric C*-algebras.

If the partial isometry homology invariant $H_1(\pi)$ is nonzero for the ordered inclusion $\pi : D \oplus D \rightarrow D$ then we calculate the outer automorphism group $Out_{D \oplus D}(D)$ of subalgebra-respecting automorphisms modulo automorphisms with approximately inner restrictions. In the unital matroid case this group is equal to $\mathbb{Z}_2$ if the $H_0H_1$ coupling obstruction is nontrivial and coincides with $Aut(H_1(\pi))$ otherwise. The generator for $\mathbb{Z}_2$ derives from a homology inverting automorphism. On the other hand, in the 2-symmetric case we show that there can be intriguing obstructions to homology inversion. This is the case, for example, for positions determined by the so called homologically asymmetric 4-cycle algebra systems identified in Donsig and Power [DoP2]. The determination of $Out_{D \oplus D}(D)$ when $H_1(\pi)$ is trivial is left open.

If a partly self-adjoint algebra $A$ is a generating subalgebra of a self-adjoint algebra $B$ then the position $A \cap A^* \subseteq B$ is an invariant for the star extendible isomorphism class of $A$. Dually, the isomorphism type of the algebras in the intermediate subalgebra lattice of an inclusion provide invariants for that inclusion. This natural link between self-adjoint and non-self-adjoint algebra is essential to our approach and it is by considering direct limits of non-self-adjoint finite-dimensional algebras that we obtain classifying invariants and diverse inclusions.

For a regular direct system of general digraph algebras (finite-dimensional incidence algebras) such as

$$A_1 \overset{\alpha_1}{\rightarrow} A_2 \overset{\alpha_2}{\rightarrow} A_3 \rightarrow \ldots,$$

the natural limit homology groups

$$H_n(\{A_k, \alpha_k\}) = \lim_{\rightarrow} (H_n(A_k), H_n(\alpha_k))$$

were introduced in Davidson and Power [DP] and given intrinsic formulations, as partial isometry homology, in [Po4], [Po5]. It is necessary here to consider regular morphisms, that is, homomorphisms that are direct sums of multiplicity one embeddings ([Po2]), and in this case the limit homology groups are well-defined invariants for the regular isomorphism of regular direct systems. With this invariant and the introduction of new scales for $K_0 \oplus H_1$ we completed the $K_0H_1$ classification of certain direct systems of 4-cycle algebras up to regular isomorphism. See Donsig and Power [DoP2]. On the other hand in [DoP1] it was shown that such direct systems can be irregularly isomorphic, in a nontrivial manner, and so the classification of even the algebraic direct limits of the systems was left open. In the
present paper we overcome this obstacle by establishing the well-definedness of the partial isometry homology group invariants. This also leads to the correct homology invariants for the discrimination and classification of subalgebra positions.

The paper is organised as follows. In Section 2 we define the standard index 2 inclusions of matroid algebras, the generic index 4 inclusions of direct sums of matroid algebras, inclusions of 2-symmetric algebras, and generic limits of 4-cycle algebras. The intrinsic formulation of $H_n\left(\{A_k, \alpha_k\}\right)$ as the homology of a chain complex is also indicated. In Section 3 we obtain the key fact that irregular isomorphisms of regular systems of 4-cycle algebras are only possible if the systems are of matroid type and have vanishing homology. In sections 4 and 5 we classify generic limits of 4-cycle algebras, relative positions of direct sums of matroid algebras and 2-symmetric algebras, and we determine $Out_{D_1 \oplus D_2}(D)$. We also indicate obstructions to homology inverting automorphisms. These sections are largely algebraic and concern only algebraic direct limits.

In section 6 the key arguments of sections 3 and 4 are generalised to the case of approximately regular embeddings and the classification of operator algebra limits of generic 4-cycle algebra systems is obtained, in the odd case, in terms of

$$(K_0(-) \oplus H_1(-), \Sigma_0(-), \Sigma(-)).$$

The stability of $K_0H_1$ invariants obtained here leads also to the fact that close limit algebras are isomorphic, in analogy with situation for AF $C^*$-algebras [Chr]; [PR].

The reader may notice a distinct parallel between the $K_0H_1$ classification of cycle algebras and the modern $K_0K_1$ determination of $C^*$-algebras, such as is indicated in Elliott [Ell2] and Riordam [R]. Thus homology inverting automorphisms bear an analogy with $K_1$ inverting automorphisms, such as the flip on the Bunce-Deddens algebras. The results below suggest extending the well-definedness of partial isometry homology further in order to approach the $K_*H_*$ determination of partly self-adjoint subalgebras of such $C^*$-algebras.

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2. Generic Inclusions

Let $F_q$ be an (unclosed) unital matroid algebra associated with the generalised integer $q = q_1q_2\ldots$, with $q_i \geq 2$ for all $i$. Suppose that $q = 2^\infty p$ with $p = p_1p_2\ldots$, and $p_i \geq 2$ for all $i$ and consider the realisation of $F_q$ as a direct limit

$$F_q = \text{alg lim}_{\longrightarrow} (M_{2^k p_1 \ldots p_k} \oplus M_{2^{k+1} p_1 \ldots p_k}, \alpha_k)$$

where

$$K_0 \alpha_k = \begin{bmatrix} p_{k+1} & p_{k+1} \\ p_{k+1} & p_{k+1} \end{bmatrix}$$

for all $k \geq 1$. View the maps $\alpha_k$ as the restrictions to the block diagonal subalgebras of the map

$$\alpha_k : M_{2^{k+1} p_1 \ldots p_k} \rightarrow M_{2^{k+2} p_1 \ldots p_{k+1}}$$

for which
\[
\alpha_k : \begin{bmatrix} a & b \\ d & c \end{bmatrix} \rightarrow \begin{bmatrix} \sigma_k(a) & 0 & 0 & \sigma_k(b) \\ 0 & \sigma_k(c) & \sigma_k(d) & 0 \\ 0 & \sigma_k(b) & \sigma_k(a) & 0 \\ \sigma_k(d) & 0 & 0 & \sigma_k(c) \end{bmatrix}
\]

where \( \sigma_k(a) = a \oplus \cdots \oplus a \) \((p_k+1\) times). In this way obtain an inclusion \( F_q \rightarrow F_{2q} \) \((= F_q)\) with \( F_{2q} = F_q + JF_q \) where \( J \) is the self-adjoint unitary determined by the symmetry \[
\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}
\]
in \( M_{4q^2} \). For convenience we refer to this inclusion \( F_q \rightarrow F_{2q} \) as the standard regular inclusion of index 2 and we remark that it also arises from the crossed product \( F_{2q} = F_q \rtimes \alpha \mathbb{Z}_2 \) where \( \alpha \) is a product type symmetry. (See [FaM]. Note that the abelian diagonal subalgebra

\[
C = \operatorname{alg lim}(C_k; \alpha_k)
\]
in \( F_{2q} \) is a regular diagonal masa for both \( F_{2q} \) and \( F_q \). That is, each of the algebras \( F_{2q} \) and \( F_q \) is generated by the partial isometries that normalise \( C \).

If \( F_q \oplus F_q \rightarrow F_{2q} \oplus F_{2q} \) is the associated index 2 position then it may seem curious, given the uniqueness of the trivial index 2 position \( F_{2q} \oplus F_{2q} \rightarrow F_{2q} \otimes M_2 \), that there are uncountably many positions. Nevertheless this is a consequence of Theorem 4.5.

We now consider embeddings of non-self-adjoint algebras which lead to an apparent variety of relative positions for matroid algebras and 2-symmetric algebras.

Let \( A \subseteq M_n \) be a complex algebra which contains a maximal abelian self-adjoint subalgebra (masa). Then there is a matrix unit system \( \{ e_{ij} : 1 \leq i, j \leq n \} \) for \( M_n \) such that the masa is spanned by the diagonal matrix units and \( A \) itself is spanned by matrix units. We refer to \( A \) as a digraph algebra whose digraph is the directed graph on \( n \) vertices with edges \((i,j)\) directed from \( j \) to \( i \), for each \( e_{ij} \) in \( A \). Let \( \phi : A_1 \rightarrow A_2 \) be an embedding of digraph algebras with \( \phi(C_1) \subseteq C_2 \) for some masas \( C_i \subseteq A_i \), \( i = 1, 2 \), and assume that \( \phi \) is star extendible in the sense that there is a (necessarily unique) \( C^* \)-algebra extension \( \phi : C^*(A_1) \rightarrow C^*(A_2) \). Then the following statements are equivalent.

(a) \( \phi \) is a direct sum of multiplicity one embeddings.

(b) \( \phi \) maps the (partial isometry) normaliser of \( C_1 \) into the normaliser of \( C_2 \).

In this event we say that \( \phi \) is a regular star extendible embedding.

A 4-cycle matrix algebra is a complex subalgebra of \( M_r \) consisting of partitioned matrices of the form

\[
\begin{bmatrix}
  a_{11} & 0 & a_{13} & a_{14} \\
  0 & a_{22} & a_{23} & a_{24} \\
  0 & 0 & a_{33} & 0 \\
  0 & 0 & 0 & a_{44}
\end{bmatrix}
\]

where \( a_{ij} \) is a matrix in \( M_{n_i \times n_j} \), the space of \( n_i \) by \( n_j \) matrices, and \( r = n_1 + n_2 + n_3 + n_4 \). The simplest of these algebras are the 4-cycle algebra \( A_1 \) in \( M_4 \), which is also denoted \( A(D_4) \), and the equidimensional 4-cycle algebras, which are the algebras \( A_1 \otimes M_n \). Along with
their $2n$-cycle counterparts, these algebras have featured prominently in the consideration of Hochschild cohomology for non-self-adjoint operator algebras. In fact

$$\text{Hoch}^1(A_1 \otimes M_n; A_1 \otimes M_n) = \mathbb{C}$$

for all $n$. See [GiS], [DP], for example.

Taking a more geometric homological perspective, the algebra $A_1 \otimes M_n$ determines, through a matrix unit system, a simplicial complex $\Delta(A_1 \otimes M_n)$ and therefore one may associate the integral simplicial homology group, $H_1(\Delta(A_1 \otimes M_n)) = \mathbb{Z}$. This simplicial complex is the direct product of the complete complex on $n$ vertices with the 4 cycle complex $\Delta(A_1)$ indicated by the diagram

```
      3
     / \  /
    1   2
     \  /  \n     4
```

More generally, this homology group association is available for any digraph algebra; edges of the digraph $G$ of $A$ contribute 1-simplices to $\Delta(A)$ and complete (undirected) subgraphs of $G$ on $t+1$ vertices determine the $t-$simplices. Note that regular homomorphisms induce homology group homomorphisms.

An alternative, homologically equivalent association is to take $\Delta(A)$ to be the complex determined by the reduced digraph of the digraph algebra $A$. (Then $\Delta(A_1 \otimes M_n) = \Delta(A_1)$.)

Consider now the 4-cycle digraph algebra $A_1 = A(D_4)$. Its digraph $D_4$ has edges $(1,3)$, $(1,4)$, $(2,4)$, $(2,3)$, $(1,1)$, $(2,2)$, $(3,3)$, $(4,4)$. A multiplicity one star-extendible embedding

$$\phi : A(D_4) \otimes M_n \rightarrow A(D_4) \otimes M_m$$

is said to be rigid if the images of the rank one partial isometries $e_{13} \otimes p$, $e_{14} \otimes p$, $e_{24} \otimes p$, $e_{23} \otimes p$, with $p$ a rank one projection of $M_n$, are inequivalent rank one partial isometries in the block off-diagonal part of the range algebra. Since the initial and final projections of these images are rank one projections in $A(D_4) \otimes M_m$ it follows that the images belong, in some order, to the distinct block subspaces

$$e_{13} \otimes M_m, e_{14} \otimes M_m, e_{24} \otimes M_m, e_{23} \otimes M_m.$$ 

There are precisely four inner equivalence classes of such embeddings and these classes correspond to the four symmetries of $D_4$.

**Definition 2.1.** An embedding $\alpha$ between 4-cycle algebras is said to be generic if it is conjugate (inner unitarily equivalent) to a direct sum of rigid rank one embeddings, with at least one summand of each of the four types.

A general direct sum of multiplicity one rigid embeddings, without the constraint of the definition, is also called rigid. Generic and rigid embeddings between general (nonequidimensional) 4-cycle algebras are defined similarly.
Definition 2.2. (i) A 4-cycle algebra direct system $\mathcal{A} = (A_k, \alpha_k)$ is said to be a generic if infinitely many of the embeddings $\alpha_k$ are generic. In this case the associated limit algebra $A = \text{alg lim}(A_k, \alpha_k)$ is said to be a generic.

(ii) An inclusion $D_1 \oplus D_2 \to B$ of matroid algebras is said to be a generic index 4 regular inclusion if it is conjugate, by a star automorphism of $B$, to an inclusion $A \cap (A)^* \to B$ where $A$ is a generic 4-cycle limit algebra as in (i), and $B$ is the self-adjoint algebra generated by $A$. In this case the system $\{A_k, \alpha_k\}$ is said to be of matroid type.

If $\mathcal{A}$ is a generic system then it has a subsystem in which all the embeddings are generic. Henceforth we shall assume this normalisation throughout the paper without further comment.

For an explicit distinguished example consider the stationary system $\{A_k, \alpha_k\}$ where $A_k = A(D_4) \otimes M_{4k}$ and $\alpha_k$ is the embedding

$$
\begin{bmatrix}
  a & 0 & 0 & 0 \\
  0 & a & 0 & 0 \\
  0 & 0 & b & 0 \\
  0 & 0 & 0 & b
\end{bmatrix}
\to
\begin{bmatrix}
  a & 0 & 0 & 0 & y & 0 & 0 & x & 0 & 0 \\
  0 & a & 0 & 0 & x & 0 & 0 & y & 0 & 0 \\
  0 & 0 & b & 0 & w & 0 & 0 & 0 & 0 & z \\
  0 & 0 & 0 & b & 0 & 0 & 0 & z & w & 0 & 0 \\
  c & 0 & 0 & 0 \\
  0 & c & 0 & 0 \\
  0 & 0 & d & 0 \\
  0 & 0 & 0 & d
\end{bmatrix}
$$

One can check that a rigid system is not generic if and only if for all large $k$ each of the embeddings $\alpha_k$ is a sum of multiplicity one rigid embeddings corresponding to at most two classes.

Not every inclusion $A \cap A^* \to B$ arising from a generic limit algebra $A$ as in (i) is a direct sum of matroid algebras. In general $A \cap A^*$ is a direct sum of two 2-symmetric algebras and we also refer to these inclusions as generic inclusions (of unspecified index).

Definition 2.3. A 2-symmetric algebra is the algebraic direct limit algebra of a system of algebras each of which is a direct sum of two matrix algebras and where the partial embeddings have multiplicities indicated by a symmetric $2 \times 2$ integral matrix of the form $\begin{bmatrix} x & y \\ y & x \end{bmatrix}$.

The explicit $K_0$ classification of these algebras and their C*-algebra closures, the 2-symmetric C*-algebras, is given in Fack and Marechal [FaM] for the unital equidimensional case, and in Donsig and Power [DoP2] for the general case.
The most conspicuous invariant for the subalgebra position \( D_1 \oplus D_2 \subseteq B \) is the lattice of intermediate algebras partially ordered by inclusion. That there are thirteen such algebras follows readily from the fact that the intermediate algebras are generated by matrix units. (See also the discussion of inductivity in Chapter 4 of [Po3].) The intermediate subalgebra lattice is indicated in the diagram below in the case of a generic unital matroid algebra inclusion with \( D_1 = D_2 \) where \( D \subseteq \tilde{D} \) indicates the standard index 2 position. The properly nonselfadjoint intermediate algebras are the algebras \( A, A^* \) together with
\[
\langle A, \tilde{D} \oplus D \rangle, \langle A^*, \tilde{D} \oplus D \rangle, \langle A, D \oplus \tilde{D} \rangle, \langle A^*, D \oplus \tilde{D} \rangle, \langle A, \tilde{D} \oplus \tilde{D} \rangle, \langle A^*, \tilde{D} \oplus \tilde{D} \rangle.
\]

To identify finer invariants for the inclusion we may focus on the pair \( \{A, A^*\} \).

For a rigid embedding \( \alpha \) between 4-cycle algebras define the multiplicity signature of \( \alpha \) as the ordered 4-tuple \( \{r_1, r_2, r_3, r_4\} \) where \( r_1, \ldots, r_4 \) are the number of summands of each of the four types. If \( r_2 \) and \( r_4 \) correspond to the number of rank one rigid embeddings associated with the two reflection symmetries of \( D_4 \) then the induced \( H_1 \) group homomorphism \( H_1(\alpha) \) can be identified with the map \( \mathbb{Z} \to \mathbb{Z} \) which is multiplication by \( r_1 - r_2 + r_3 - r_4 \). This group homomorphism is determined up to sign, this sign being fixed only after an identification of the reduced graphs of \( A_1 \) and \( A_2 \) and realisations of \( H_1(A_1) \) and \( H_1(A_2) \).

It is an elementary but significant fact that the inner equivalence class of \( \alpha \) is determined by the pair \( K_0(\alpha), H_1(\alpha) \). See [Po3], [DoP2].

More generally one may define the induced homology map \( H_n(\alpha) \) for any regular embedding between digraph algebras, and for a regular star extendible direct system \( \{A_k, \alpha_k\} \) one has the abelian group
\[
H_n(\{A_k, \alpha_k\}) = \lim_{\longrightarrow}(H_n(A_k), H(\alpha_k)).
\]
This is evidently a well-defined invariant for the regular isomorphism of systems. For the stationary example above it is the zero group for all \( n \geq 1 \).

One may define \( H_n(\{A_k, \alpha_k\}) \) more intrinsically in the following manner. For related forms and variations see Power \([Po1], [Po4], [Po5]\).

Let \( \text{Pisom}_{reg}(\{A_k, \alpha_k\}) \) be the set of partial isometries \( v \) in \( A_k \), for some \( k \), which are regular. By this we mean that \( v \) is a sum of rank one partial isometries \( w \) for which both \( w^*w \) and \( wv^* \) belong to \( A_k \). The unitary equivalence classes of such partial isometries in a fixed algebra \( A_k \) are in bijective correspondence with the edges of the digraph of \( A_k \). Define the 1-chain group of \( \{A_k, \alpha_k\} \) to be the free group generated by all these unitary equivalence classes \( [v] \), for all \( k \), subject to the relations arising from the regular embeddings \( \alpha_k \). Thus \( [v] = [v_1] + [v_2] + \cdots + [v_m] \) if \( v \) in \( A_k \) is equal to \( v_1 + v_2 + \cdots + v_m \) in \( A_l \) for \( l > k \). Whereas partial isometries generate 1-simplices, triangles of partial isometries in \( \{A_k, \alpha_k\} \) (with \( v_3 = v_1v_2 \) and \( v_1^*v_1 = v_2^*v_2^* \)) determine 2-simplices and a corresponding 2-chain group (again, modulo inclusion relations). There are obvious boundary maps and \( H_1(\{A_k, \alpha_k\}) \) is defined to be the appropriate homology group.

The group \( H_0(\{A_k, \alpha_k\}) \) can also be defined as the 0-chain group of \( \{A_k, \alpha_k\} \) which is similarly defined in terms of projection classes \([p] \) and it is routine to verify that

\[
H_0(\{A_k, \alpha_k\}) = \lim_{\longrightarrow}(K_0(C^*_{\infty}(A_k)), K_0(\alpha_k)),
\]

where \( \alpha_k \) is the star extension of \( \alpha_k \).

We have introduced the generic index 4 inclusions of matroid algebras by means of systems of 4-cycle algebras with generic embeddings and this presentation emphasises the strong connection with non-self-adjoint algebras which will be important in the sequel. The following proposition provides an alternative direct definition.

First we note that there is a distinguished index 4 inclusion arising from a rigid non-generic system. In the unital equidimensional case this is the natural inclusion of \( F_q \otimes \mathbb{C}^2 \) in \( (F_q + JF_q) \otimes M_2 \) and we refer to this as the homologically extreme inclusion.

**Proposition 2.4.** Let \( D_1 \oplus D_2 \) be a unital subalgebra of \( D_3 \) where \( D_1, D_2 \) and \( D_3 \) are isomorphic to the unital even matroid algebra \( F_q \). Then the position \( D_1 \oplus D_2 \subseteq D_3 \) is a generic index 4 inclusion (or the homologically extreme inclusion) if and only if there is a regular presentation

\[
D_3 = \text{alg lim}(M_{2^k s_k} \otimes M_4, \alpha_k)
\]

\[
D_1 \oplus D_2 = \text{alg lim}(M_{2^k s_k} \otimes \mathbb{C}^4, \beta_k)
\]

for which

(i) \( \beta_k \) is the restriction of \( \alpha_k \) to the block diagonal and has the form

\[
\beta_k(a \oplus b \oplus c \oplus d) = (\sigma_k(a) \oplus \sigma_k(b)) \oplus (\sigma_k(a) \oplus \sigma_k(b)) \oplus (\sigma_k(c) \oplus \sigma_k(d)) \oplus (\sigma_k(c) \oplus \sigma_k(d))
\]

where \( \sigma_k \) is the standard embedding of multiplicity \( q_k + 1 \),

(ii) if \( p_1 \) and \( p_2 \) are the orthogonal central projections of \( D_1 \oplus D_2 \) then the inclusions

\( D_i \subseteq p_i D_3 p_i, i = 1, 2 \) are standard index 2 inclusions.
3. Regular and Irregular Factorisations

The next lemma is crucial and will be used with Lemma 3.2 to obtain the well-definedness of partial isometry homology for algebraic direct limits.

The following definitions will be convenient. Let \( \phi : A_1 \to A_2 \) be a star extendible embedding of 4-cycle algebras. Then \( \phi \) is said to be \textit{locally regular} if the image of each rank one partial isometry in \( A_1 \) is a regular partial isometry in \( A_2 \), that is, each such image is an orthogonal sum of rank one partial isometries in \( A_2 \) whose initial and final projections lie in \( A_2 \). Also \( \phi \) is said to be a \textit{proper} embedding if, firstly, \( \phi(A^r_1) \subseteq A^r_2 \), where \( A^r_i \) denotes the block upper triangular part of \( A_i \), and secondly, for each rank one partial isometry \( v \) in \( A_1 \) the partial isometry \( \phi(v) \) has support in each of the four block subspaces of \( A^r_2 \). Equivalently put, all the matrices for \( \phi \) given in Table 1 are nonzero. The generic embeddings defined earlier are therefore the same as the proper rigid embeddings.

**Lemma 3.1. Factorisation dichotomy.** Let \( \phi : A_1 \to A_2, \psi : A_2 \to A_3, \eta : A_3 \to A_4 \), be star extendible proper embeddings between 4-cycle algebras. If the compositions \( \psi \circ \phi \), \( \eta \circ \psi \), are rigid embeddings then either \( \eta \) is locally regular or \( H_1(\psi \circ \phi) = 0 \).

Before beginning the proof we establish some notation for the triple \( \phi, \psi, \psi \circ \phi \).

Assume that \( A_1 = A(D_4) \), that \( A_2 \) and \( A_3 \) are equidimensional and that the maps are all unital. This is the essential case to consider.

Choose matrix units for \( A_2 \cap A^*_2 \) so that \( \phi|(A_1 \cap A^*_1) \) has the form

\[
\phi : \lambda_1 \oplus \lambda_2 \oplus \lambda_3 \oplus \lambda_4 \to (\lambda_1 P' + \lambda_2 R') \oplus (\lambda_1 Q' + \lambda_2 S') \oplus (\lambda_3 P + \lambda_4 R) \oplus (\lambda_3 Q + \lambda_4 S)
\]

where \( P', R', Q', S', P, R, Q, S \) are orthogonal projections which are sums of diagonal matrix units. Set \( p' = \text{rank } P', r' = \text{rank } R', \ldots, s = \text{rank } S \). Since \( \phi \) is star extendible we have

\[
p' + q' = r' + s = p + q = r + s = \rho,
\]

where \( \rho \) is the multiplicity of \( \phi \). Furthermore

\[
\phi : e_{13} \to \begin{bmatrix}
0 & 0 & \alpha_1 & 0 & \beta_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \delta_1 & 0 & \gamma_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

where \( \begin{bmatrix} \alpha_1 & \beta_1 \\ \delta_1 & \gamma_1 \end{bmatrix} \) is a unitary matrix in \( M_\rho \). Similarly \( \phi(e_{14}), \phi(e_{24}), \phi(e_{23}) \) have associated unitary matrices.
\[
\begin{bmatrix}
\alpha_2 & \beta_2 \\
\delta_2 & \gamma_2
\end{bmatrix},
\begin{bmatrix}
\alpha_3 & \beta_3 \\
\delta_3 & \gamma_3
\end{bmatrix},
\begin{bmatrix}
\alpha_4 & \beta_4 \\
\delta_4 & \gamma_4
\end{bmatrix}.
\]

The dimensions of the matrix entries of these matrices are indicated in the following table

|   | p | r | q | s |
|---|---|---|---|---|
| p' | α_1 | α_2 | β_1 | β_2 |
| r' | α_4 | α_3 | β_4 | β_3 |
| q' | δ_1 | δ_2 | γ_1 | γ_2 |
| s' | δ_4 | δ_3 | γ_4 | γ_4 |

Table 1

Similarly matrix units for \( A_3 \cap A_3^* \) may be chosen to standardise the map \( \psi : A_2 \cap A_2^* \rightarrow A_3 \cap A_3^* \), and there is a \( \sigma \times \sigma \) matrix \( \begin{bmatrix} a_1 & a_2 \\ a_4 & a_3 \end{bmatrix} \) so that

\[
\psi : e_{13} \otimes x \rightarrow \begin{bmatrix}
0 & 0 & a_1 \otimes x & 0 & a_2 \otimes x & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_4 \otimes x & 0 & a_3 \otimes x & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

Here \( A_2 \) is identified with \( A(D_4) \otimes M_k \), with \( k = \rho \), and \( x \in M_k \). The map \( \psi \) is determined by four \( \sigma \times \sigma \) unitary matrices each with a \( 2 \times 2 \) block decomposition, whose entries and dimensions are indicated in Table 2.

|   | u | w | v | x |
|---|---|---|---|---|
| u' | a_1 | b_1 | a_2 | b_2 |
| w' | d_1 | c_1 | d_2 | c_2 |
| v' | a_4 | b_4 | a_3 | b_3 |
| x' | d_4 | c_4 | d_3 | c_3 |

Table 2
Thus
\[ u + v = w + x = u' + v' = w' + x' = \sigma. \]

We have
\[
\psi \circ \phi(e_{13}) = \begin{bmatrix}
0 & 0 & v_1 & 0 & v_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & v_4 & 0 & v_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

where
\[
\begin{bmatrix}
v_1 & v_2 \\
v_4 & v_3 \\
\end{bmatrix} = \begin{bmatrix}
\alpha_1 \otimes a_1 & \beta_1 \otimes b_1 & \alpha_1 \otimes a_2 & \beta_1 \otimes b_2 \\
\delta_1 \otimes d_1 & \gamma_1 \otimes c_1 & \delta_1 \otimes d_2 & \gamma_1 \otimes c_2 \\
\alpha_1 \otimes a_4 & \beta_1 \otimes b_4 & \alpha_1 \otimes a_3 & \beta_1 \otimes b_3 \\
\delta_1 \otimes d_4 & \gamma_1 \otimes c_4 & \delta_1 \otimes d_3 & \gamma_1 \otimes c_3 \\
\end{bmatrix}
\]

with similar expressions for the $\rho \sigma \times \rho \sigma$ unitary matrices
\[
\begin{bmatrix}
w_1 & w_2 \\
w_4 & w_3 \\
\end{bmatrix}, \quad \begin{bmatrix}
x_1 & x_2 \\
x_4 & x_3 \\
\end{bmatrix}, \quad \begin{bmatrix}
y_1 & y_2 \\
y_4 & y_3 \\
\end{bmatrix}
\]

arising from $\psi \circ \phi(e_{14}), \psi \circ \phi(e_{24}), \psi \circ \phi(e_{23})$ respectively. Thus
\[
\psi \circ \phi(e_{14}) = \begin{bmatrix}
0 & 0 & 0 & w_1 & 0 & w_2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & w_4 & 0 & w_3 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

where
\[
\begin{bmatrix}
w_1 & w_2 \\
w_4 & w_3 \\
\end{bmatrix} = \begin{bmatrix}
\alpha_2 \otimes a_1 & \beta_2 \otimes b_1 & \alpha_2 \otimes a_2 & \beta_2 \otimes b_2 \\
\delta_2 \otimes d_1 & \gamma_2 \otimes c_1 & \delta_2 \otimes d_2 & \gamma_2 \otimes c_2 \\
\alpha_2 \otimes a_4 & \beta_2 \otimes b_4 & \alpha_2 \otimes a_3 & \beta_2 \otimes b_3 \\
\delta_2 \otimes d_4 & \gamma_2 \otimes c_4 & \delta_2 \otimes d_3 & \gamma_2 \otimes c_3 \\
\end{bmatrix}
\].

*Proof of Lemma 3.1.* Assume first that $\psi$ is not locally regular. (This is a consequence of the non local regularity of $\eta$.) Without loss of generality we may assume that $a_1$ is not a partial isometry. As the map $\psi \circ \phi$ is regular it follows that the matrix
\[
v_1 = \begin{bmatrix}
\alpha_1 \otimes a_1 & \beta_1 \otimes b_1 \\
\delta_1 \otimes d_1 & \gamma_1 \otimes c_1 \\
\end{bmatrix}
\]
is a partial isometry and from this it follows that \( \alpha_1 \) is a strict contraction. Indeed if this is not the case then the matrix unit system for \( A_2 \) may be chosen so that, in addition, \( \alpha_1 \) has the form

\[
\begin{bmatrix}
1 & 0 \\
0 & *
\end{bmatrix}.
\]

Thus the unitary matrix

\[
\begin{bmatrix}
\alpha_1 & \beta_1 \\
\delta_1 & \gamma_1
\end{bmatrix}
\]

has an associated form

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & * & * \\
0 & * & *
\end{bmatrix}.
\]

This implies that the matrix \( a_1 \) appears as an orthogonal part of \( v_1 \), contrary to the fact that \( v_1 \) is a partial isometry and \( a_1 \) is not.

Similarly it follows, on consideration of the partial isometries

\[
w_1 = \begin{bmatrix}
\alpha_2 \otimes a_1 & \beta_2 \otimes b_1 \\
\delta_2 \otimes d_1 & \gamma_2 \otimes c_1
\end{bmatrix}, \quad x_1 = \begin{bmatrix}
\alpha_3 \otimes a_1 & \beta_3 \otimes b_1 \\
\delta_3 \otimes d_1 & \gamma_3 \otimes c_1
\end{bmatrix}, \quad y_1 = \begin{bmatrix}
\alpha_4 \otimes a_1 & \beta_4 \otimes b_1 \\
\delta_4 \otimes d_1 & \gamma_4 \otimes c_1
\end{bmatrix}
\]

that \( \alpha_2, \alpha_3 \) and \( \alpha_4 \) are strict contractions. Also, by the properness of \( \phi \) each of \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) is nonzero. From this it follows that none of the matrices of Table 1 is nonzero and none is a partial isometry.

We now show that the sixteen contractions \( \alpha_i, \beta_i, \gamma_i, \delta_i, 1 \leq i \leq 4 \), must all have the same rank.

Since \( \phi \) is star extendible we have the equations

\[
\alpha_1 \alpha_1^* + \beta_1 \beta_1^* = P', \quad \alpha_1^* \alpha_1 + \delta_1^* \delta_1 = P.
\]

If \( \text{rank} \ (\alpha_1) < p' = \text{rank} \ (P') \) then there is a rank one projection \( E \leq P' \) with \( E\alpha_1 = 0 \). But then \( E\beta_1 \beta_1^* E = E \), contrary to the fact that \( \beta_1 \) is a strict contraction. Thus \( \text{rank} \ (\alpha_1) = p' \). Similarly the other equation ensures that \( \text{rank} \ (\alpha_1) = p = \text{rank} \ (P) \). For otherwise there is a rank one projection \( F \leq P \) with \( \alpha_1 F = 0 \) and it follows that \( \delta_1 \) has an isometric part. We conclude then that \( p = p' \). Similar arguments, or an appeal to the symmetry of 4-cycle algebras, leads to the equidimensionality condition

\[
p = q = r = s = p' = q' = r' = s'
\]

and this implies that \( \rho \) is even and the common value above is \( \rho/2 \).

Assume now that \( \eta \) is not locally regular. Then, by the argument above, \( \psi \) satisfies the equirank condition

\[
u = v = w = x = u' = v' = w' = x' = \sigma/2.
\]

We conclude then that

\[
\text{rank}(v_1) \geq \text{rank} \ (\alpha_1 \otimes a_1) = \text{rank} \ (\alpha_1) \text{rank}(a_1) = \rho\sigma/4.
\]
and similarly
\[ \text{rank}(v_i) \geq \rho \sigma / 4, \quad \text{for } 1 \leq i \leq 4. \]

Since \( \psi \circ \phi \) has multiplicity \( \rho \sigma \) it follows from the fact that each \( v_i \) is a partial isometry that
\[ \text{rank}(v_i) = \rho \sigma / 4, \quad \text{for } 1 \leq i \leq 4. \]
Thus
\[ H_1(\psi \circ \phi) = [ \text{rank}(v_1) - \text{rank}(v_2) + \text{rank}(v_3) - \text{rank}(v_4)] = 0 \]
as desired. \( \square \)

That irregular factorisations of rigid embeddings are possible is shown in [DoP1]. In the key stationary example there \( \phi \) has multiplicity 2, \( \psi = \phi \otimes I_2 \) and \( \phi \) has the matrix table

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & \sqrt{2} & \sqrt{2} & \sqrt{2} \\
1 & \sqrt{2} & -\sqrt{2} & \sqrt{2} \\
1 & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\
1 & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\
1 & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\
\end{array}
\]

**Lemma 3.2.** Let \( \phi : A_1 \to A_2, \psi : A_2 \to A_3 \) be locally regular star extendible embeddings between 4-cycle algebras. If the composition \( \psi \circ \phi \) is a rigid embedding then \( \phi, \psi \) are rigid embeddings.

**Proof.** We may assume that \( A_1 = A(D_4) \) and that the maps are unital. The general case follows readily from this one. By hypothesis the matrices of Tables 1 and 2 are partial isometries. Since \( \psi \circ \phi \) is rigid it follows that \( v_1 \) and \( w_2 \) have the same final projections and hence, in particular, the matrices
\[
[\alpha_1 \otimes a_1 \quad \beta_1 \otimes b_1] \quad \text{and} \quad [\alpha_2 \otimes a_2 \quad \beta_2 \otimes b_2]
\]
have the same final projections. The partial isometries \( a_1 \) and \( a_2 \) have orthogonal final projections since they appear in the unitary matrix \( \begin{bmatrix} a_1 & a_2 \end{bmatrix} \) and so it follows that the pair \( \alpha_1 \otimes a_1 \) and \( \beta_2 \otimes b_2 \) and hence the pair \( a_1, b_2 \), have coincidental final projections.
Similarly \( b_2, c_3 \) have the same initial projection, \( c_3, d_4 \) have the same final projection and \( d_4, a_1 \) have the same initial projection. By the star extendibility of \( \psi \) we have the equality \( d_4^* c_1 b_2 a_1 = a_1^* a_1 \) and it follows that the matrices \( a_1, b_2, c_3, d_4 \) determine a direct summand
of $\psi$ which is a rigid embedding. In fact this summand is (inner) unitarily equivalent to a direct sum of $k$ equivalent rank one rigid embeddings where $k = \text{rank}(a_1)$.

Continuing in this way it follows that $\psi$ is a rigid embedding with $H_1(\psi) = [\delta]$ where

$$\delta = \text{rank}(a_1) - \text{rank}(b_1) + \text{rank}(c_1) - \text{rank}(d_1).$$

Since $\psi$ and $\psi \circ \phi$ are rigid embeddings it follows that $\phi$ is also a rigid embedding.

**Theorem 3.3.** For generic limit algebras of 4-cycle algebras the limit homology group is a well-defined invariant for star extendible isomorphism.

**Proof.** Suppose that $\{A_k, \alpha_k\}, \{A'_k, \alpha'_k\}$ are generic systems with limits $A$ and $A'$ and that $A$ and $A'$ are star extendibly isomorphic. Then there is a star extendible commuting diagram isomorphism between $\{A_k, \alpha_k\}$ and $\{A'_k, \alpha'_k\}$. Composing embeddings we may assume that the crossover maps are proper. This is so because the systems are generic. By Lemmas 3.1 and 3.2 either $\{A_k, \alpha_k\}$ and $\{A'_k, \alpha'_k\}$ are regularly isomorphic or both $H_1(\{A_k, \alpha_k\})$ and $H_1(\{A'_k, \alpha'_k\})$ are zero.

**Remark.** A similar result holds for limits of $2n$-cycle algebras for $n \geq 3$ details of which will appear in [DoP3]. It is curious that the complexity of the embeddings between such higher order cycle algebras is offset by the fact that they can be shown to be automatically locally regular. Thus one can then move to the consideration of a countertop to Lemma 3.3 and bypass entirely the detailed analysis of irregular factorisations of regular embeddings that we have given here and in [DoP1].

4. $H_1$ CLASSIFICATIONS.

If $\alpha$ is a star extendible embedding between connected digraph algebras then its multiplicity is defined to be the multiplicity of its star algebra extension. Alternatively, the multiplicity of $\alpha$ in the case of a regular embedding is $|r|$ where $H_0(\alpha)$ is realised as multiplication by the integer $r$.

A direct system of digraph algebras, with star extendible embeddings, is said to be an odd system if only finitely many of its embeddings have even multiplicity.

**Theorem 4.1.** A star extendible isomorphism between odd generic direct systems of 4-cycle algebras or between systems with nonzero $H_1$ invariant is necessarily a regular isomorphism.

**Proof.** The homology of the maps of an odd generic system are eventually nonzero and so the argument of Theorem 3.3 applies.

Star extendible isomorphisms between the algebraic direct limits are automatically regular isomorphisms and so the algebraic limit algebras possess all the invariants that their systems possess for regular isomorphism.

It is still the case that generic even systems with trivial $H_1$ invariant are regularly isomorphic if they are star extendibly isomorphic, but in the absence of automatic regularity this regular isomorphism must be constructed.
Theorem 4.2. Let $\mathcal{A}, \mathcal{A}'$ be generic systems of 4-cycle algebras with algebraic direct limits $\mathcal{A} = \text{alg lim} \mathcal{A}$, $\mathcal{A}' = \text{alg lim} \mathcal{A}'$ respectively. Then $\mathcal{A}$ and $\mathcal{A}'$ are star extendibly isomorphic if and only if $\mathcal{A}$ and $\mathcal{A}'$ are regularly isomorphic.

Proof. By the analysis of the last section we may assume that $\mathcal{A}, \mathcal{A}'$ have embeddings $\alpha_k, \alpha'_k$ which satisfy the equidimensionality conditions. In particular $\mathcal{A}, \mathcal{A}'$ are of matroid type and have trivial $H_1$ invariant. Thus $\mathcal{A} = \{A_k, \alpha_k\}, \mathcal{A}' = \{A'_k, \alpha'_k\}$ have the form

$$\begin{bmatrix}
  t_k & t_k & 0 & 0 \\
  t_k & t_k & 0 & 0 \\
  0 & 0 & t_k & t_k \\
  0 & 0 & t_k & t_k
\end{bmatrix},
\begin{bmatrix}
  u_k & u_k & 0 & 0 \\
  u_k & u_k & 0 & 0 \\
  0 & 0 & u_k & u_k \\
  0 & 0 & u_k & u_k
\end{bmatrix}. $$

Since $\mathcal{A}$ and $\mathcal{A}'$ are star extendibly isomorphic it follows that there is a scaled group isomorphism between the direct systems $K_0 A, K_0 A'$ in the form of a commuting diagram. Moreover we may assume that he crossover maps have the form

$$\begin{bmatrix}
  v_k & v_k & 0 & 0 \\
  v_k & v_k & 0 & 0 \\
  0 & 0 & v_k & v_k \\
  0 & 0 & v_k & v_k
\end{bmatrix}$$

where $v_k$ is even for all $k$. (If this is not already so then the crossover maps have the form

$$\begin{bmatrix}
  a & b' & 0 & 0 \\
  b & a' & 0 & 0 \\
  0 & 0 & c & d' \\
  0 & 0 & d & c'
\end{bmatrix}$$

where, by star extendibility, $a + b = a' + b' = c + d = c' + d'$. Composing such a map with a given $K_0$ map gives a new crossover map of the desired type.) Thus each crossover map has a lifting to a generic embedding with zero $H_1$ map. Construct liftings of the crossover maps, in order, to rigid embeddings with trivial homology, and use $K_0 H_1$ uniqueness to arrange commuting triangles. \qed

Remarks. 1. In fact the generic requirement can be dropped and Theorem 4.2 holds for general rigid systems of 4-cycle algebras. As Allan Donsig has observed, it is straightforward to obtain the automatic regularity of the factors $\phi, \psi$ of a composition $\psi \circ \phi$ which is of nongeneric type. This follows essentially from the fact that a partial isometry with support in three block subspaces is necessarily regular.

2. The theorem above implies a simplification of the isomorphism problem for algebraic limits in the following sense.

Choose partial matrix unit systems $\{e_{ij}^k\}$ for $A_k$ for $k = 1, 2, \ldots$ in the usual way so that each $e_{ij}^k$ is a sum of some of the matrix units of the system $\{e_{ij}^{k+1}\}$. Whilst the semigroup $S = \{e_{ij}^k\}$ formed by the totality of all these matrix units depends on the system $\mathcal{A}$, the semigroup ring $R_A = \mathbb{Z}[S]$ is, by the theorem, a well-defined invariant for $A$. Thus the algebra $A$ comes with a canonical ring inclusion $R_A \to A$ for which $A = \mathbb{C} \otimes_{\mathbb{Z}} R_A$. 

3. It is likely that standard techniques lead to the fact that $A$ and $A'$ are star extendibly isomorphic if and only if they are isomorphic as complex algebras. Alternatively, although more indirectly, in view of the classifications below would be enough to formulate the $K_0H_1$ invariants as algebra isomorphism invariants.

4. It is still an open problem whether the conclusion of Theorem 4.2 holds for the limit algebras of arbitrary regular star extendible systems of digraph algebras.

**Definition 4.3.** For a generic index 4 matroid algebra position $\pi : D_1 \oplus D_2 \to B$, or, more generally, for a generic index 4 position of 2-symmetric algebras, the partial isometry homology group $H_1(\pi)$ is defined to be the abelian group $H_1(A)$ where $A$ is the associated 4-cycle limit algebra.

Let $\zeta : F \oplus F \to F$, with $F = \text{alg lim}_{\to} M_{2k}$, be the unital inclusion determined by the stationary system given in Section 2, based on the simplest generic embeddings. It follows from the argument of the last proof that $\zeta$ is the unique position with trivial homology invariant.

**Theorem 4.4.** Up to position there is a unique generic index 4 matroid algebra inclusion $\zeta : F \oplus F \to F$ with $H_1(\zeta) = 0$.

This uniqueness might suggest that $H_1(\pi)$ provides a complete invariant for the generic index 4 inclusions but this is not the case. The most transparent obstacle is given by a coupling class in $\mathbb{R}/\mathbb{Q}$. If $A = \text{alg lim}_{\to}(A_k, \alpha_k)$ then this class is defined to be that which is determined by any of the products

$$\kappa(A) = \prod_{k=1}^{\infty} \frac{|\delta_k|}{p_k},$$

for suitably large $l$, where $p_k$ is the multiplicity of $\alpha_k$ and $H_1(\alpha_k) = [\delta_k]$.

The following partial classification also follows from the more general Theorem 5.3.

**Theorem 4.5.** Let $\pi_1, \pi_2$ be generic index 4 matroid algebra inclusions of $D_1 \oplus D_2$ in $D$ and assume that $\kappa(\pi_1) = \kappa(\pi_2) = 0$. Then $\pi_1$ and $\pi_2$ are conjugate if and only if the abelian groups $H_1(\pi_1)$ and $H_1(\pi_2)$ are isomorphic.

**Proof.** The necessity of the condition for conjugacy follows from Theorem 3.3. Assume then that $H_1(\pi_1)$ and $H_1(\pi_2)$ are isomorphic. As in the proof of Theorem 4.2 there is a commuting diagram isomorphism between the direct systems $K_0A$ and $K_0A'$ associated with $\pi_1$ and $\pi_2$. Lift the first crossover map of this isomorphism to a rigid embedding $\phi : A_1 \to A'_k$ for some $k$. Under the hypotheses it is possible to lift the next crossover map, $Y_1 : K_0A'_k \to K_0A_l$ say, after increasing $l$ if necessary, to a rigid embedding $\psi_1$ so that $\psi_1 \circ \phi_1$ is equal to the given embedding $i : A_1 \to A_l$. This can be seen from the essential part of the proof of Theorem 11.22 in [Po3]. Thus, by $K_0H_1$ uniqueness it is enough to
choose \( l \) large enough and \( \delta \) in the homology range of \( Y_1 \) so that \( \delta H(\phi_1) = H_1(i) \). This is possible since, for large \( l \) the ratios \( |H_1(i)|/|H_0(i)| \) are arbitrarily small.

One can continue in this way to obtain a commuting diagram of rigid embeddings which determines the desired conjugacy.

5. \( K_0 H_1 \) Classifications

The invariants \( H_1(\pi) \) and \( \kappa(\pi) \) take no account of the order of the summands and so, as formulated, they cannot serve as invariants for the conjugacy of automorphisms with summand respecting restrictions. To determine such conjugacy we now consider \( K_0 H_1 \) invariants for the ordered inclusion and, most decisively, the joint scale in \( K_0 H_1 \) introduced in Donsig and Power [DoP2].

Let \( \{ A_k, \alpha_k \} \) be a generic 4-cycle algebra system with algebraic direct limit \( A \). The identification of the scaled \( K_0 \) group invariant \((K_0 A, \Sigma_0 A)\) and the associated classification of the algebras \( A \cap A^* = D_1 \oplus D_2 \) has been considered in detail in [DoP2]. If \( \{ A_k, \alpha_k \} \) is of matroid type then \( D_1 \) and \( D_2 \) are stably isomorphic even matroid algebras and are isomorphic in the unital equidimensional case. Thus \( K_0(A) \) may be realised naturally as a subgroup of \( \mathbb{Q} \oplus \mathbb{Q} \), with the usual product order and with scale determined by a product of (possibly infinite) intervals, \( I_1 \times I_2 \). The algebra \( A \) is unital if and only if both \( I_1 \) and \( I_2 \) are finite closed intervals. See Dixmier [Dix].

More generally \( K_0 \alpha_k \) has the form

\[
\begin{bmatrix}
  a_k & b_k & 0 & 0 \\
  b_k & a_k & 0 & 0 \\
  0 & 0 & c_k & d_k \\
  0 & 0 & d_k & a_k
\end{bmatrix}
\]

where \( a_k + b_k = c_k + d_k \). Let \( p_k = a_k + b_k \) let \( q_k = a_k - b_k \) (and without loss of generality assume that \( q_k \geq 0 \) for all \( k \)) and let \( G_p \) and \( G_q \) be the subgroups of \( \mathbb{Q} \) associated with the generalised integers \( p = p_1 p_2 \ldots, q = q_1 q_2 \ldots \). One can show that \( K_0 D_1 \) can be realised as \( G_p \oplus G_q \) in this case. In the odd case \( K_0 D_4 \) is generated by \( G_p \oplus G_q \) and \((1/2, 1/2)\).

Note that the scale \( \Sigma_0 A \) can be identified as the set of \( K_0 \) classes of projections \( \phi(1) \) associated with the star extendible injections

\[
\phi : C \rightarrow A_k.
\]

Likewise the scale \( \Sigma_1 A \) of \( H_1 A \) can be defined as the set of elements \((H_1 \psi)(g)\) associated with the morphisms

\[
H_1 \psi : H_1(A(D_4)) \rightarrow H_1 A_k
\]

induced by regular embeddings

\[
\psi : A(D_4) \rightarrow A_k
\]

for some \( k \), where \( g \) is a fixed generator for \( H_1(A(D_4)) \).

The joint scale admits a similar definition.

**Definition 5.1.** Let \( A \) be a generic limit of 4-cycle algebras. Then the joint scale \( \Sigma A \) of \( K_0 A \oplus H_1 A \) is defined to be the subset of \( \Sigma_0 A \times \Sigma_1 A \) consisting of elements

\[
(K_0 \phi([e_{11} \oplus e_{33}]), (H_1 \phi)(g))
\]
associated with the rigid embeddings
\[ \phi : A(D_4) \to A_k \]
for some \( k \geq 1 \).

The next theorem, the lifting Theorem 5.2, can also be obtained from Theorem 4.2 above and the main result in Donsig and Power [DoP2]. For completeness we present a proof.

In [DoP2] we have discussed the circumstances under which it is possible to lift a scaled group homomorphism
\[ \gamma_0 \oplus \gamma_1 : K_0 A_1 \oplus H_1 A_1 \to K_0 A_2 \oplus H_1 A_2 \]
to a rigid embedding between the 4-cycle algebras \( A_1, A_2 \). (We assume that \( \gamma_0 \) respects the ordered summand structure of \( K_0 \), so that \( \gamma_0 \) is necessarily block diagonal.) A necessary and sufficient condition for a lifting to exist is that \( \gamma_0 \) be of rigid type, that is, implemented by a matrix of the form
\[
\begin{bmatrix}
  a & b & 0 & 0 \\
  b & a & 0 & 0 \\
  0 & 0 & c & d \\
  0 & 0 & d & c \\
\end{bmatrix}
\]
with \( a + b = c + d \), and that \( \gamma_0 \oplus \gamma_1 \) maps \( \Sigma A_1 \) into \( \Sigma A_2 \). The rigid type nature of \( \gamma_0 \) is also equivalent to symmetry preservation, by which we mean that \( \gamma_0 \circ \theta = \theta \circ \gamma_0 \), for each of the (four) canonical symmetries \( \theta \) of the ordered \( K_0 \) group.

In general symmetry preservation is not an automatic consequence of joint scale preservation. This can be seen for example for the scaled group homomorphism
\[ \gamma_0 \oplus \gamma_1 = \begin{bmatrix}
  6 & 6 & 0 & 0 \\
  2 & 2 & 0 & 0 \\
  0 & 0 & 6 & 6 \\
  0 & 0 & 2 & 2 \\
\end{bmatrix} \oplus 0 \]
viewed as a map from \( K_0(A(D_4)) \) to \( K_0(A(D_4) \otimes M_{12}) \). However, one can readily check that joint scale preservation does ensure that the column sums of \( \gamma_0 \) are constant (and hence that \( \gamma_0 \) extends to a scaled group homomorphism from \( K_0 C^*(A_1) \) to \( K_0 C^*(A_2) \)). Thus, if in fact \( \gamma_0 \) is a map from \( K_0(A(D_4) \otimes M_r) \) to \( K_0(A(D_4) \otimes M_s) \) which preserves the order unit (and so maps \((r, r, r, r)\) to \((s, s, s, s)\)) then the row sums as well as the column sums coincide, and so \( \gamma_0 \) is automatically of rigid type in this case.

Naturally we say that a (summand respecting scaled group) isomorphism \( \gamma_0 : K_0 A \to K_0 A' \) is of rigid type if it pulls back to a commuting diagram of rigid type embeddings. This condition is similarly equivalent to symmetry preservation for the canonical symmetries on \( K_0 A, K_0 A' \). (See also the discussion in [Po3].)

These remarks explain why one can drop the symmetry preservation hypothesis in the next theorem in the case when the algebras are unital and are determined by equidimensional systems.

**Theorem 5.2.** Let \( A, A' \) be (algebraic) limits of 4-cycle algebras with respect to rigid embeddings and let
\[ \gamma_0 \oplus \gamma_1 : K_0 A \oplus H_1 A \to K_0 A' \oplus H_1 A' \]
be an abelian group isomorphism where \( \gamma_0 \) is a scaled group isomorphism, which in the nonunital case is symmetry preserving. Then the following assertions are equivalent.

(i) There is a star extendible isomorphism \( \phi : A \rightarrow A' \) for which \( \gamma_0 \oplus \gamma_1 = K_0\phi \oplus H_1\phi \).
(ii) The isomorphism \( \gamma_0 \oplus \gamma_1 \) effects a bijection between the joint scales.

Proof. If (i) holds then by Theorem 4.2 and the remarks that follow it there is a regular star extendible isomorphism between the systems \( \{A_k, \alpha_k\} \) and \( \{A'_k, \alpha'_k\} \) for \( A, A' \) with the same induced \( K_0 \) map as \( \phi \). Such an isomorphism induces a \( K_0 \oplus H_1 \) map \( \gamma_0 \oplus \gamma_1 \), with \( \gamma_0 \) symmetry preserving, which effects a bijection between the joint scales.

Assume now that (ii) holds. Without loss of generality assume that \( A_1 = A'_1 = A(D_4) \) and consider a rigid embedding \( \psi : A(D_4) \rightarrow A_1 \) which determines the triple

\[
([p] \oplus [q], \delta)
\]

where \( p = \psi(e_{11}), q = \psi(e_{33}) \) and \( \delta = (H_1\psi)(g) \). By the joint scale preservation there is a generic embedding \( \eta : A(D_4) \rightarrow A_k' \), for some \( k' \), such that

\[
\gamma_0 \oplus \gamma_1(([p] \oplus [q], \delta) = ([\eta(e_{11} \oplus e_{33})], (H_1\eta)(g)).
\]

Since \( \gamma_0 \) is symmetry preserving, \( \gamma_0 \) and \( K_0\eta \) agree and \( \eta \) is a lifting. (See also Lemma 11.4 of [DoP2].)

Consider now a natural copy of \( A(D_4) \) in \( A_k' \) that is, any copy for which the inclusion is a multiplicity one rigid embedding. The associated injection \( \eta : A(D_4) \rightarrow A_k' \) determines an element \( \gamma_0^{-1} \oplus \gamma_1^{-1}([\eta(e_{11} \oplus e_{33})], H_1\eta(g)) \) in the joint scale of \( K_0A \oplus H_1A \) and so, for some generic embedding \( \xi : \eta(A(D_4)) \rightarrow A_l \) we have

\[
\gamma_0^{-1} \oplus \gamma_1^{-1}([\eta(e_{11} \oplus e_{33})], H_1\eta(g)) = ([\xi(e_{11} \oplus e_{33})], H_1\xi(g)).
\]

By symmetry preservation \( K_0\xi \) agrees with the restriction of \( \gamma_0^{-1} \).

We claim that the embedding \( \xi \) has an extension \( \xi : A_k' \rightarrow A_l \), perhaps after increasing \( l \), which is also generic. Indeed, since \( \gamma_0^{-1} \) is a scaled group isomorphism we can first extend the restriction \( \xi|([\eta(A(D_4)) \cap (\eta(A(D_4)))^* \rightarrow \mathbb{C}^*\text{-}\text{algebra injection}

\[
\xi_* : A_k' \cap (A_k')^* \rightarrow A_l \cap (A_l)^*
\]

so that \( K_0\xi_* \) agrees with \( \gamma_0^{-1}|(A_k' \cap (A_k')^*) \). Each matrix unit \( e \in A_k' \) (for some fixed matrix unit choice) admits a unique factorisation \( e = e_1f e_2 \) with \( f \) a matrix unit in \( \eta(A(D_4)) \) and \( e_1, e_2 \) matrix units in \( A_k' \cap (A_k')^* \). Thus \( \xi \) and \( \xi_* \) have a joint extension \( \xi : A_k' \rightarrow A_l \) and this extension is also generic.

Now \( \xi \circ \eta : A_1 \rightarrow A_l \) has the same \( K_0 \oplus H_1 \) map as the given embedding \( \alpha_{l-1} \circ \alpha_{l-2} \circ \cdots \circ \alpha_1 \) and so by \( K_0H_1 \) uniqueness we may replace \( \xi \) by an inner conjugate embedding to get equality. Continue this process to get a regular isomorphism between the systems and hence a star extendible isomorphism between \( A \) and \( A' \). \( \Box \)

Now let \( \pi : D_1 \oplus D_2 \rightarrow D \) be the ordered subalgebra position determined by the algebra \( A \) as in the last theorem. Define \( K_0H_1(\pi) \) to be the group \( K_0A \oplus H_1A \) and let \( \Sigma(\pi) \) be the joint scale \( \Sigma A \) of \( K_0A \oplus H_1A \). Since \( A \) is determined by the ordered inclusion \( \pi \) the pair \( (K_0H_1(\pi), \Sigma(\pi)) \) is an invariant for ordered inclusion. The next theorem follows immediately from Theorem 5.2 and the remarks above and resolves the problem of classifying generic inclusions (of unspecified index).
Theorem 5.3. Let $D_1, D_2$ be 2-symmetric algebras and let $\pi_1, \pi_2$ be generic ordered inclusions of $D_1 \oplus D_2$ in $D$. Then there is a star isomorphism $\alpha : D \to D$ with $\alpha(\pi_1(D_i)) = \pi_2(D_i)$ for $i = 1, 2$ if and only if there is a group isomorphism

$$\gamma_0 \oplus \gamma_1 : K_0H_1(\pi_1) \to K_0H_1(\pi_2),$$

where $\gamma_0$ is a symmetry preserving scaled group isomorphism and

$$\gamma_0 \oplus \gamma_1(\Sigma(\pi_1)) = \Sigma(\pi_2).$$

It is also of interest to determine the symmetries and automorphisms of these positions and for this one must consider realisations of the invariants. We give an indication of this determination in the remainder of this section.

The appropriate outer automorphism group is given in the following definition. The term approximately inner automorphism indicates automorphisms which are pointwise limits of a sequence of inner unitary automorphisms.

Definition 5.4. Let $\pi : D_1 \oplus D_2 \to D$ be a 2-symmetric algebra inclusion associated with a generic 4-cycle algebra system. Then $\text{Out}_{D_1 \oplus D_2}(D)$ is the group of automorphisms of $D$ which leave invariant the subalgebra $D_1 \oplus D_2$ and for which the restriction to $D_1 \oplus D_2$ is an automorphism which is approximately inner.

Theorem 5.5. If $\pi$ is as above and $H_1(\pi) \neq 0$ then $\text{Out}_{D_1 \oplus D_2}(D)$ is isomorphic to the group of symmetry preserving automorphisms $\gamma_0 \oplus \gamma_1$ of $(K_0H_1(\pi), \Sigma(\pi))$.

Proof. In view of Theorem 5.3 it is enough to show that a subalgebra respecting automorphism which induces the identity map on $(K_0H_1(\pi), \Sigma(\pi))$ has a restriction which is approximately inner. However by $K_0H_1$ uniqueness this is routine.

Theorem 5.6. Let $D_1$ and $D_2$ be even unital matroid algebras and let $\pi : D_1 \oplus D_2 \to D$ be a generic index 4 inclusion with $H_1(\pi) \neq 0$. Then $\text{Out}_{D_1 \oplus D_2}(D)$ is equal to $\text{Aut}(H_1(\pi))$ if $\kappa(\pi) = 0$ and is equal to $\mathbb{Z}_2$ otherwise.

Proof. Let $A$ be the usual 4-cycle algebra limit algebra for which $A \cap A^* = D_1 \oplus D_2$. It follows from Theorem 7.2 of [Dop2] that $\Sigma A$ is invariant under the homology inversion $id \oplus -1$ and that $\Sigma A$ splits as a direct sum, say $\Sigma_{00}A \oplus \Sigma_1 A$. If $\kappa(\pi) > 0$ then $\Sigma_1 A$ is a finite interval in $H_1 A$ and if $\kappa(\pi) = 0$ then $\Sigma_1 A = H_1 A$. Now Theorem 5.5 completes the proof, since symmetry preservation is automatic in the unital case.

It is intriguing that in the 2-symmetric case there may exist obstructions to homology inversion as we now show.

Let $\alpha_1 : A_1 \to A_2$ be a rigid embedding between 4-cycle algebras. Then the abstract homology range of $\alpha_1$, denoted $hr(\alpha_1)$ is the set of maps $H_1(\beta)$ for which $\beta : A_1 \to A_2$ is rigid with $K_0(\beta) = K_0(\alpha_1)$. Thus if, for example, $\alpha_k$ has $K_0$ map

$$\begin{bmatrix} n_k & 2 & 0 & 0 \\ 2 & n_k & 0 & 0 \\ 0 & 0 & n_k & 2 \\ 0 & 0 & 2 & n_k \end{bmatrix}$$
then $\alpha_k$ has possible signatures $\{n_k, 0, 2, 0\}, \{n_k - 1, 1, 1\}, \{n_k - 2, 2, 0, 2\}$ with respective homology maps $[n_k + 2], [n_k - 2], [n_k - 6]$. Thus the homology range is $\{n_k + 2, n_k - 2, n_k - 6\}$.

**Definition 5.7.** [DoP2] The generic 4-cycle algebra system $\{A_k, \alpha_k\}$ is hr-symmetric if for each pair $j, i$ the homology range of $\alpha_j \circ \cdots \circ \alpha_i$ lies in $\mathbb{Z}_+$ or $\mathbb{Z}_-$ and does not contain 0.

Such a system may be constructed as follows. Returning to our earlier notation for $\alpha_k$ note that $K_0(\alpha_k)$ is defined by the triple $(p_k, q_k, r_k)$ where $p_k = a_k + b_k, q_k = a_k - b_k, r_k = c_k - d_k$. (Again, we assume that $c_k \geq d_k$ for all $k$.) The triple for the composition $\alpha_{k+1} \circ \cdots \circ \alpha_1$ is the triple

$$(P_k, Q_k, R_k) = (p_k p_{k-1} \cdots p_1, q_k q_{k-1} \cdots q_1, r_k r_{k-1} \cdots r_1)$$

and the homology range of the composition is contained in the interval $[s, t]$ where

$$t = P_k - |Q_k - R_k|, \quad s = Q_k + R_k - P_k.$$ 

In particular if we choose $\alpha_k$ as above with $n_k$ increasing, such that for all $k$

$$\frac{2(n_k - 2)}{n_k + 2} > \prod_{i=1}^{k-1} \frac{n_i + 2}{n_i - 2},$$

then the homology range of $\alpha_{k+1} \circ \cdots \circ \alpha_1$ is positive for all $k$.

**Proposition 5.8.** Let $A$ be the generic limit algebra of an hr-asymmetric system. Then the joint scale $\Sigma A$ in $K_0 \oplus H_1$ is not invariant under the homology inversion $\gamma_0 \oplus \gamma_1 = id \oplus -1$.

**Proof.** Adding a rank one rigid embedding $A(D_4) \to A_1$ we may assume that $A_1 = A(D_4)$. Thus the element $[e_{11} \oplus e_{33}] \oplus g$ in $\Sigma A_1$ determines an element of the joint scale of $A$. Suppose that $[e_{11} \oplus e_{33}] \oplus -g$ is also an element of the joint scale, appearing as an element of $K_0 A_k \oplus H_1 A_k$ for some $k$. Then we conclude that both $g$ and $-g$ belong to the homology range of $\alpha_{k-1} \circ \cdots \circ \alpha_1$ contrary to hypothesis. \qed

**Remark.** Another possible obstacle to homology inversion, in the odd 2-symmetric case, is a mod 4 congruence class which may be implicit in the joint scale. This is considered in detail in [DoP2]. This somewhat subtle obstruction is purely arithmetic and in contrast to hr-asymmetry it is annihilated by tensoring with an even matroid algebra.

6. **Operator Algebras**

We now obtain approximate versions of the lemmas of section 3 and we retain the notation from that section. The following definition will be useful in arguments involving embeddings which are almost locally regular.

**Definition 6.1.** A star extendible homomorphism $\phi$ between 4-cycle algebras is $\epsilon$-strict if each matrix entry $\alpha_i, \beta_i, \gamma_i, \delta_i$, for $1 \leq i \leq 4$, is distance at most $\epsilon$ from a partial isotopy.
Let $\phi : A(D_4) \to A_2$ be a general star extendible embedding and let $\phi' = \alpha \circ \phi$ where $\alpha : A_2 \to A_3$ is a generic embedding. Let $\alpha'_i, \beta'_i, \ldots$, be the matrices corresponding to $\phi'$ as in Table 1. Since $\alpha$ is generic it follows that for each $1 \leq i \leq 4$,

$$||\alpha'_i|| = ||\beta'_i|| = ||\gamma'_i|| = ||\delta'_i|| = \max\{||\alpha_i||, ||\beta_i||, ||\gamma_i||, ||\delta_i||\}$$

A 4-cycle algebra embedding with this property is said to be norm-symmetric.

**Lemma 6.2.** Let $\begin{bmatrix} x & \epsilon_1 \\ \epsilon_2 & y \end{bmatrix}$ be a contraction which is $\eta$-close to a partial isometry. If $\delta = \eta + ||\epsilon_1|| + ||\epsilon_2||$, and $\delta < \frac{1}{8}$, then $x$ is $8\delta$-close to a partial isometry.

**Proof.** The matrix

$$\begin{bmatrix} x^*x + \epsilon_2^2 & x^*\epsilon_1 + \epsilon_2^2y \\ \epsilon_1^*x + y^*\epsilon_2 & \epsilon_1^*\epsilon_1 + y^*y \end{bmatrix}$$

is $2\eta$-close to a projection and so $\begin{bmatrix} x^*x & 0 \\ 0 & y^*y \end{bmatrix}$ is $2\delta$-close to a projection. In particular $||(x^*x)^2 - (x^*x)|| \leq 4\delta$ from which it follows, if $\delta \leq \frac{1}{2}$, that $x$ is $8\delta$-close to a partial isometry. \hfill \square

**Lemma 6.3.** Let $\epsilon < \frac{1}{16}$. Let $\phi : A(D_4) \to A_2$, $\psi : A_2 \to A_3$ be proper embeddings of 4-cycle algebras for which the composition $\psi \circ \phi$ is $\epsilon^2$-strict. If $\psi$ is not $\epsilon$-strict and $\phi$ is norm-symmetric then $\phi$ is not $(\epsilon/51)^2$-strict.

**Proof.** Since $\psi$ is not $\epsilon$-strict at least one of the matrices of Table 2 is not $\epsilon$-close to a partial isometry. Without loss of generality we may assume this matrix to be $a_1$. By assumption the matrix

$$v_1 = \begin{bmatrix} \alpha_1 \otimes a_1 & \beta_1 \otimes b_1 \\ \delta_1 \otimes d_1 & \gamma_1 \otimes c_1 \end{bmatrix}$$

is $\epsilon^2$-close to a regular partial isometry. We now deduce from these two facts that $\alpha_1$ has norm no greater than $1 - (\epsilon/50)^2$.

Let $t = ||\alpha_1||$. This norm is attained and there is a block decomposition

$$\alpha_1 = \begin{bmatrix} t & \epsilon_1 \\ \epsilon_3 & * \end{bmatrix}$$

and an associated induced decomposition of $v_1$;

$$v_1 = \begin{bmatrix} t \otimes a_1 & \epsilon_1 \otimes a_2 & \epsilon_2 \otimes b_1 \\ \epsilon_3 \otimes a_1 & * & * \\ \epsilon_4 \otimes d_1 & * & * \end{bmatrix}.$$
is $\epsilon^2$-close to a partial isometry and so, in view of the last lemma, the matrix $ta_1 = t \otimes a_1$ is $8(\epsilon^2 + ||\epsilon_1|| + ||\epsilon_2||)$ close to a partial isometry. We have
\[
8(\epsilon^2 + ||\epsilon_1|| + ||\epsilon_2||) \leq 8(\epsilon^2 + 2(1 - t^2)\frac{1}{2}) \leq 8\epsilon^2 + 16\sqrt{2}(1 - t)^\frac{1}{2}.
\]
Since $a_1$ is $(1 - t)$-close to $ta_1$, and yet not $\epsilon$-close to a partial isometry, it follows that
\[
\epsilon \leq (1 - t) + 8\epsilon^2 + 16\sqrt{2}(1 - t)^\frac{1}{2}.
\]
Thus
\[
\epsilon(1 - 8\epsilon) \leq (1 + 16\sqrt{2})(1 - t)^\frac{1}{2}
\]
and so
\[
t \leq 1 - \epsilon^2(2(1 + 16\sqrt{2}))^{-2} \leq 1 - (\epsilon/50)^2.
\]
Considering $w_1, x_1$ and $y_1$ in similar ways it follows that $||\alpha_i|| \leq 1 - (\epsilon/50)^2$ for $i = 2, 3, 4$. Since $\phi$ is norm-symmetric it follows that $\phi$ cannot be $(\epsilon/51)^2$-strict. Indeed this would imply that $\alpha_1$ and $\beta_1$ are $(\epsilon/51)^2$-close to partial isometries, one of which at least is necessarily nonzero, and thus of norm 1.

The next lemma is a partial generalisation of the factorisation dichotomy of Lemma 3.1.

**Lemma 6.4.** Let $\epsilon < \frac{1}{16}$ and let $\epsilon_* \leq \frac{1}{17}(\epsilon^4/51^6)$. Let $\phi : A_1 \rightarrow A_2, \psi : A_2 \rightarrow A_3, \eta : A_3 \rightarrow A_4$, be proper norm-symmetric embeddings of 4-cycle algebras. If the compositions are $\epsilon_*$-close to generic embeddings then either $\eta$ is $\epsilon$-strict or $\phi$ has even multiplicity.

**Proof.** Suppose that the compositions are $\epsilon_*$-close to generic embeddings and that $\eta$ is not $\epsilon$-strict. The composition $\eta \circ \psi$ is $\epsilon_*$-strict and $\epsilon_* \leq \epsilon^2$ and so by Lemma 6.3 the map $\psi$ is not $\epsilon_1$-strict where $\epsilon_1 = (\epsilon/51)^2$. Since $\psi \circ \phi$ is $\epsilon_*$-strict it follows similarly, since $\epsilon_* \leq \epsilon_1^2$ and $\psi$ is not $\epsilon_1$-strict, that $\phi$ is not $\epsilon_2$-strict where $\epsilon_2 = (\epsilon_1/51)^2$.

We can now continue in a similar fashion to the proof of Lemma 3.1. Adopting the notation there, since $\psi$ is $\epsilon_1$-strict we may assume that $a_1$ is not $\epsilon_1$-close to a partial isometry. On the other hand our hypotheses imply that the matrix
\[
v_1 = \begin{bmatrix}
\alpha_1 \otimes a_1 & \beta_1 \otimes b_1 \\
\delta_1 \otimes d_1 & \gamma_1 \otimes c_1
\end{bmatrix}
\]
is $\epsilon_*$-close to a partial isometry. We now show, as before, that $\alpha_1$ is a strict contraction.

Suppose that this is not the case. Then, as before, the matrix $a_1$ appears as an orthogonal part of $v_1$. That is, there are projections $p, q$ such that
\[
v_1 = qv_1p + (1 - q)v_1(1 - p)
\]
and $qv_1p = a_1$. However, if
\[
\begin{bmatrix}
z_1 & z_2 \\
z_3 & z_4
\end{bmatrix}
\]
is a partial isometry which is $\epsilon_*$-close to
\[
v_1 = \begin{bmatrix}
qv_1p & 0 \\
0 & (1 - q)v_1(1 - p)
\end{bmatrix}
\]
then \( \|z_2\| \leq \epsilon_s \) and \( \|z_3\| \leq \epsilon_s \), and so Lemma 6.8 implies that \( z_1 \) is \( 16\epsilon_s \)-close to a partial isometry. Thus \( qv_1p \) is \( 17\epsilon_s \)-close to a partial isometry, contrary to the fact that \( 17\epsilon_s < \epsilon_1 \).

Similarly, considering \( w_1, x_1, y_1 \), it follows that \( \alpha_2, \alpha_3, \alpha_4 \) are strict contractions. By the properness of \( \phi \) each \( \alpha_i \) is nonzero and so each of the matrices of Table 1 is nonzero and none is a partial isometry. Thus, by the argument of Lemma 3.1, \( \phi \) has even multiplicity. \( \square \)

We also require an approximate version of Lemma 3.2. For the proof of this the following three approximation principles will be convenient.

First, recall that if \( e, f \) are projections in a \( C^* \)-algebra with \( \|e - f\| < 1 \) then there is a partial isometry \( v \) in the algebra with initial projection \( e \), final projection \( f \) and \( \|v - e\| < 2\|e - f\| \). From this it follows that if \( \pi_1 \) and \( \pi_2 \) are star-homomorphisms between finite-dimensional \( C^* \)-algebras which are \( \epsilon \)-close then there is a unitary \( u \), with \( \|1 - u\| < 2 \), such that \( \pi_1(a) = u\pi_2(a)u^* \) for all \( a \).

Second, if a contraction \( v \) is \( \epsilon \)-close to a partial isometry \( z \) then the range projection of \( v, \) \( rp(v) \), is \( (2\epsilon)^{1/2} \)-close to \( zz^* \).

Finally, we need the following elementary lemma.

**Lemma 6.5.** Let \( E_1, \ldots, E_4 \) be projections for which \( \|E_1 + E_2 - E_3 - E_4\| \leq 2\epsilon \) and \( \|E_1 E_4\| \leq \epsilon, \|E_2 E_3\| \leq \epsilon \). Then \( \|E_1 - E_3\| \leq (6\epsilon)^{1/2} \) and \( \|E_2 - E_4\| \leq (6\epsilon)^{1/2} \).

**Proof.** Let \( x \) be a unit vector with \( E_2 x = x \), so that \( \|E_3 x\| \leq \epsilon \). Then

\[
2\epsilon \geq \langle (E_1 + E_2 - E_3 - E_4)x, x \rangle
\]

and so

\[
3\epsilon \geq \langle (E_1 + E_2 - E_4)x, x \rangle \geq \langle (E_2 - E_4)x, x \rangle.
\]

Thus

\[
\langle (E_2 - E_4)z, z \rangle \leq 3\epsilon
\]

for all unit vector \( z \).

Similarly \( \langle (E_2 - E_2E_4)z, z \rangle \leq 3\epsilon \) and so \( \langle (E_2 - E_4)^2z, z \rangle \leq 6\epsilon \). \( \square \)

**Lemma 6.6.** Let \( \phi : A_1 \to A_2, \psi : A_2 \to A_3 \) be proper \( \epsilon \)-strict star extendible embeddings between 4-cycle algebras. If the composition \( \psi \circ \phi \) is \( \epsilon \)-close to a generic star extendible embedding then \( \psi \) is \( g(\epsilon) \)-close to a rigid embedding where \( g(t) \) is a nonnegative continuous function on \( [0,1] \) with \( g(0) = 0 \).

**Proof.** First assume that \( A_1 = A(D_4) \) and that the maps are unital. Using the usual notation of Section 3, since the composition \( \psi \circ \phi \) is \( \epsilon \)-close to a rigid embedding it follows that the pair

\[
v = \begin{bmatrix} v_1 & v_2 \\ v_4 & v_3 \end{bmatrix}, \quad w = \begin{bmatrix} w_1 & w_2 \\ w_4 & w_3 \end{bmatrix}
\]

is \( \epsilon \)-close to a pair

\[
\tilde{v} = \begin{bmatrix} \tilde{v}_1 & \tilde{v}_2 \\ \tilde{v}_4 & \tilde{v}_3 \end{bmatrix}, \quad \tilde{w} = \begin{bmatrix} \tilde{w}_1 & \tilde{w}_2 \\ \tilde{w}_4 & \tilde{w}_3 \end{bmatrix}
\]
where $\tilde{v} = \lambda(e_{13}), \tilde{w} = \lambda(e_{14})$ and $\lambda$ is a rigid embedding. Moreover, by the remarks preceding the lemma, at the expense of replacing $\epsilon$ by $4\epsilon$ we may assume that $\lambda$ and $\psi \circ \phi$ agree on $A \cap A^*$. In particular $v, \tilde{v}$ and $w, \tilde{w}$ have the same initial projections and the same final projections.

Since $\lambda$ is rigid the final projection of $\tilde{v}_1$ is equal to the final projection of $\tilde{w}_2$. On the other hand since $v_1$ is $4\epsilon$-close to the partial isometry $\tilde{v}_1$ the range projection $rp(v_1)$ is $K_1\epsilon^{1/2}$-close to the final projection of $\tilde{v}_1$. (Here $K_1 = 3$ will do.) Thus the range projections

$$rp(v_1) = rp\left[\alpha_1 \otimes a_1 \beta_1 \otimes b_1 \overline{\delta_1} \otimes d_1 \gamma_1 \otimes c_1\right],$$

$$rp(w_2) = rp\left[\alpha_2 \otimes a_2 \beta_2 \otimes b_2 \overline{\delta_2} \otimes d_2 \gamma_2 \otimes c_2\right],$$

are $2K_1\epsilon^{1/2}$-close.

On the other hand, the hypotheses imply that all of the matrices of Table 1 and Table 2 are $\epsilon$-close to partial isometries, say $\|\alpha_i - \alpha'_i\| \leq \epsilon, \ldots, \|\delta_i - \delta'_i\| \leq \epsilon, \|a_i - a'_i\| \leq \epsilon, \ldots, \|b_i - b'_i\| \leq \epsilon$. Thus, with the obvious notation, $\|v'_1 - v_1\| \leq 4\epsilon$ and $\|w'_1 - w_1\| \leq 4\epsilon$ and so $\|v'_1 - \tilde{v}_1\| \leq 5\epsilon, \|w'_1 - \tilde{w}_1\| \leq 5\epsilon$ and

$$\|rp(v'_1) - rp(\tilde{v}_1)\| \leq (10\epsilon)^{1/2}, \quad \|rp(w'_1) - rp(\tilde{w}_1)\| \leq (10\epsilon)^{1/2}.$$

Thus

$$rp\left[\alpha'_1 \otimes a'_1 \beta'_1 \otimes b'_1 \overline{\delta'_1} \otimes d'_1 \gamma'_1 \otimes c'_1\right], \quad rp\left[\alpha'_2 \otimes a'_2 \beta'_2 \otimes b'_2 \overline{\delta'_2} \otimes d'_2 \gamma'_2 \otimes c'_2\right]$$

are $2(10\epsilon)^{1/2}$-close. But $V'_1$, which need not be a contraction, is $2\epsilon$-close to a partial isometry and so its range projection must be close to $v'_1v'_1^*$. In fact calculation shows that it is $7\epsilon^{1/2}$-close. It follows then that

$$\|v'_1v'_1^* - w'_2w'_2^*\| \leq 7\epsilon^{1/2} + 2(10\epsilon)^{1/2} + 7\epsilon^{1/2} \leq 22\epsilon^{1/2}.$$

In particular

$$\|(\alpha'_1 \alpha'_1^* \otimes a'_1 a'_1^* + \beta'_1 \beta'_1^* \otimes b'_1 b'_1^*) - (\alpha'_2 \alpha'_2^* \otimes a'_2 a'_2^* + \beta'_2 \beta'_2^* \otimes b'_2 b'_2^*)\| \leq 22\epsilon^{1/2}.$$

However $a'_1, \ldots, a'_4$ are partial isometries which are the entries of a matrix which is $2\epsilon$-close to a partial isometry and so $a_1$ and $a_2$ have almost orthogonal final projections. In fact $\|q_1q_2\| \leq (10\epsilon)^{1/2}$ where $q_i = rp(a'_i)$, $i = 1, 2$. Likewise $\|e_1e_2\| \leq (10\epsilon)^{1/2}$ where $e_i = rp(b'_i)$, $i = 1, 2$. From the lemma above it follows that

$$\|(\alpha'_1 \alpha'_1^* \otimes a'_1 a'_1^* + \beta'_1 \beta'_1^* \otimes b'_1 b'_1^*)\| \leq (3(22\epsilon)^{1/2})^{1/2} \leq K\epsilon^{1/4}.$$

In a similar way it follows that $b'_2, c'_3$ have $K\epsilon^{1/4}$-close initial projections, and $c'_3, d'_4$ have close final projections. We may thus rechoose $b'_2, c'_3$, so that these initial and final projections match. Set $d'_4 = c'_3(b'_2)^*a'_1$ to obtain a 4-cycle of partial isometries $a'_1, b'_2, c'_3, d'_4$ approximating $a_1, b_2, c_3, d_4$. This 4-cycle of partial isometries defines a rigid embedding $\zeta_1$ (of signature $\{\text{rank}(a'_1), 0, 0, 0\}$) which is an approximate summand of $\psi$.

Similarly we may construct an approximating 4-cycle $b'_1, a'_2, d'_3, c'_4$ and an associated embedding $\zeta_2$ and we can arrange, moreover, that $\zeta_2, \zeta_1$ are orthogonal, so that $\zeta_1 \oplus \zeta_2$ is a rigid embedding. Continuing in this way, obtain the approximating rigid embedding $\zeta_1 \oplus \zeta_2 \oplus \zeta_3 \oplus \zeta_4$. At each stage the approximation factors deteriorate by at most a multiplicative factor or by a fourth root and so the lemma is obtained with $g(t) = C\epsilon^{1/n}$ for some constant $C$ and some integer $n$. \qed
Theorem 6.7. Let $A, A'$ be odd generic systems of 4-cycle algebras with direct limits $A = \lim A$, $A' = \lim A'$ respectively. Then $A$ and $A'$ are star extendibly isomorphic if and only if $A$ and $A'$ are regularly isomorphic.

Whilst it is possible to use the lemmas above to obtain a direct proof of the theorem we prefer to show the key fact that star extendibly isomorphic limit algebras have isomorphic $K_0H_1$ groups and scales. In particular these groups and scales are indeed invariants for the algebra. In fact the arguments given show that close limit algebras of this type have isomorphic $K_0H_1$ invariants and so this approach leads to the fact that close odd generic limit algebras are isomorphic.

Let $x$ be an operator on $\mathbb{C}^n$ with $\|x - u\| \leq \frac{1}{3}$ for some partial isometry $u$. Then the rank of $u$ can be recovered from $x$ as the rank of the spectral projection for $|x|$ for the interval $(\frac{1}{2}, \frac{3}{2})$. Refer to this as the upper rank of $x$, denoted $\text{rank}_u(x)$. From this simple observation it follows that if $\phi : A_1 \to A_2$ is a star extendible embedding between 4-cycle algebras which is $\frac{1}{3}$-close to a rigid embedding $\phi'$ then, in the usual notation,

$$H_1(\phi') = [\text{rank}_u(\alpha_1) - \text{rank}_u(\beta_1) + \text{rank}_u(\gamma_1) - \text{rank}_u(\delta_1)].$$

Thus generic systems $A, A'$ which are linked by an approximately commuting diagram, with approximately rigid crossover maps, have isomorphic $H_1$ maps. Since close maps induce the same $K_0$ maps it follows that the jointly scaled $K_0H_1$ invariants for $A, A'$ are isomorphic and so, by Theorem 5.2, $A, A'$ are star extendibly isomorphic. Note that the approximately commuting diagram need not be asymptotically commuting; $\frac{1}{3}$-close approximations are all that are required.

Proof of the Theorem 6.7 It will be enough to obtain an approximately commuting diagram for $A, A'$. More precisely, it will suffice to obtain embeddings $\eta_1, \eta_2, \ldots$ with

$$A_{t_1} \xrightarrow{\eta_1} A'_{s_1} \xrightarrow{\eta_2} A_{t_2} \xrightarrow{\eta_3} A'_{s_2} \to \ldots$$

such that each $\eta_k$ is $\frac{1}{3}$-close to a rigid embedding and each composition $\eta_{k+1} \circ \eta_k$ is $\frac{1}{3}$-close to the given generic embedding.

Let $\theta : A \to A'$ be a star extendible isomorphism. Note that $A$ admits a direct sum decomposition $A = A \cap A^* + A''$ where $A'$ is the largest ideal whose square is trivial. The ideal $A''$ is also the Jacobson radical. Thus $\theta$ maps $A \cap A^*$ to $A' \cap A''$ and $\theta(A'') = A''$. Since the restriction $\theta : A_1 \to A'$ maps $A_1^r$ into $(A')^r$ it is elementary to construct in $(A_k^r)^r$, for some suitably large $k$, partial isometries $X, Y, Z$ which are $\epsilon_1$-close to $\theta(e_{13}), \theta(e_{14}), \theta(e_{24})$. Note that $W = ZY^*X$ also belongs to $(A_k^r)^r$, and so there is a star extendible proper embedding $\phi_1 : A_1 \to A_k^r$ which is $K\epsilon$-close to the restriction of $\theta$ to $A_1$. Composing $\phi_1$ with the given generic embedding $A_k^r \to A'_{k+1}$ we may also assume that $\phi$ is norm-symmetric. Repeating this idea twice more we obtain the double triangle

$$A_1 \xrightarrow{\phi} A'_{n_1} \xrightarrow{\psi} A_{m_1} \xrightarrow{\eta} A'_{n_2}$$

with $\phi, \psi, \eta$ proper, norm-symmetric, star extendible and with the compositions $\psi \circ \phi, \eta \circ \psi$ close to the given generic embeddings.

Let $\epsilon > 0$, with $\epsilon < \frac{1}{10}$. For suitably small $\epsilon_1$ it follows from Lemma 6.4 that $\eta$ is $\epsilon$-strict.
In this way construct $\epsilon$-strict proper star extendible embeddings $\eta_1 : A_{t_1} \rightarrow A'_{s_1}$, $\eta_2 : A'_{s_1} \rightarrow A_{t_2}$, ..., for which the compositions $\eta_{k+1} \circ \eta_k$ are $\epsilon$-close to the given rigid embeddings. By Lemma 6.6 the embeddings $\eta_k$ are $g(\epsilon)$-close to rigid embeddings. For $\epsilon$ sufficiently small the maps $\eta_k$ give the desired approximately commuting diagram. $\blacksquare$.

Combining the above with Theorem 5.2 one obtains a classification of the operator algebras of odd systems. In particular, in the unital case we have

**Theorem 6.8.** Odd generic unital systems of 4-cycle algebras have operator algebra limits which are star extendibly isomorphic if and only if their invariants

$$(K_0(-) \oplus H_1(-), \Sigma_0(-), \Sigma(-))$$

are isomorphic.

A more detailed perturbational analysis should lead to a full generalisation of Lemma 3.1 from which the even case would similarly follow. This in turn will enable the extension of the relative position analysis of sections 4 and 5 to matroid C*-algebras.

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