Bell inequality and complementarity loophole

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A simple classical, deterministic, local situation violating the Bell inequality is described. The detectors used in the experiment are ideal and the observers who decide which pair of measuring devices to choose for a given pair of particles have free will. The construction uses random variables which are not jointly measurable in a single run of an experiment and the hidden variables have a nonsymmetric probability density. Such random variables are complementary but still fully classical. An assumption that classical random variables cannot satisfy any form of complementarity principle is false, and this is the loophole used in this example. A relationship to the detector inefficiency loophole is discussed.

I. INTRODUCTION

Consider some system which is characterized by the following properties. It consists of a classical source which emits pairs of classical particles which are classically correlated. There are two observers, Alice and Bob, who measure binary random variables, say $A$ and $A'$ for Alice and $B$ and $B'$ for Bob. Alice and Bob use perfect detectors and all pairs emitted by the source are detected in coincidence. Neither Alice and Bob nor the particles themselves communicate and the observers have a possibility of deciding at the very last moment which configuration of $A$ and $A'$, $B$ and $B'$ to measure. Therefore not only Alice (Bob) does not know what are the settings of $B$ and $B'$ ($A$ and $A'$) for a given pair, but also the particle that propagates towards Bob (Alice) cannot predict which detector will be finally chosen for the measurement of the binary random variable. Alice and Bob have free will which means they either decide on their own how to configure the measuring devices, or leave this to a random generator. Once this choice is made the result is completely determined by the classical state of the particles.

I believe that after such a characterization of what is going to happen most of the readers will conclude that the Bell inequality cannot be violated in this experiment. However, in what follows I will give an example of a simple classical system (pairs of billiard balls) which satisfy all those requirements and yet violate the Bell inequality. There is of course no magic in this result. I simply take advantage of another loophole which exists in the proof of the Bell theorem: Although the particles and the observables are classical, the experiment is devised in such a way that on a given pair one can measure either $A$ and $B$, or $A$ and $B'$, or $A'$ and $B$, or $A'$ and $B'$. Once we have a result, say $A = +1$, it makes no sense to compare it to the result of $A'$, because there is no such result for this pair! We have a kind of classical complementarity here and it turns out to be sufficient for the violation. Before I explain how the model works let me therefore begin with a few comments on complementarity principle in a context of hidden variables.

II. COMPLEMENTARITY OF RANDOM VARIABLES

There exists a prejudice that the very idea of hidden variables contradicts the notion of *complementarity*. It is argued that once we know a hidden variable state $\lambda$ we know — by definition — also the values of all random variables $A(\lambda)$, $A'(\lambda)$, etc., even if their corresponding quantum counterparts (observables) $\hat{A}$, $\hat{A}'$, are not simultaneously measurable. In particular, it is assumed that at a hidden variables level one can safely consider expressions such as $A(\lambda) + A'(\lambda)$ even though the two random variables represent non-orthogonal linear polarizations of a single photon. The Bell inequality

$$|\mathcal{B}_{A,A',B,B'}| \leq 2$$

is obtained if one averages the “Bell random variable”

$$\mathcal{B}_{A,A',B,B'}(\lambda) = [A(\lambda) + A'(\lambda)]B(\lambda) + [A(\lambda) - A'(\lambda)]B'(\lambda).$$

That something may be fundamentally wrong with random variables of the form $A + A'$ is illustrated by the simple classical Aerts model discussed in [1]. The system he considered consisted of a mass $m$ placed on a circle and a measuring device composed of two masses $m_1$, $m_2$, satisfying $m_1 + m_2 = 1$ and placed antipodally on the circle. A measurement gives “+1” if the Newton force between $m$ and $m_1$ is greater than this between $m$ and $m_2$. In such a case the mass $m$ falls on the location point of $m_1$. To make another measurement one leaves $m$ at the point of its arrival, removes the masses $m_1$ and $m_2$ and puts another, similar pair $m_1'$, $m_2'$ of masses tilted by some $\theta$ with respect to the previous one. The conditional probability of “first +1 and second +1” for two such measurements made *one after another* is $\cos^2(\theta/2)$ which is typical of spin-1/2 particles. Since the probability is of a Malusian type there exists a classical Bell-type inequality which can be derived from the classical Bayes rule but which is not satisfied by the system of masses on
discusses another such case. It will be shown that a local complementarity may be also sufficient for a violation of the Bell inequality. This is what I call the complementarity loophole. The model I describe is macroscopic, local, fully deterministic, involves 100% efficient detectors, can be made by any kid at home, and yet maximally violates the Bell inequality. It is similar to the situation I once discussed in ③.

III. THE MODEL

We consider two systems, say billiard balls, which are initially somehow connected and form a “molecule”. Such a “molecule” is placed at random in a room which has eight square holes cut in its floor. The holes form a chessboard-like pattern shown in Fig. 1. We assume that each “molecule” will finally fall out through one of the holes, and once it reaches the next floor it breaks and separates into two disconnected billiard balls which start to roll towards two perpendicular walls of the room. Let us call the walls the “Alice wall” and the “Bob wall”. We assume that the room and the balls are devised in such a way that if one of the balls moves towards the Alice wall the other one moves towards the Bob one. Now, once they reach the walls they fall into one of two drawers, say A and A′ for the Alice wall, and B and B′ for the Bob wall. Each of the drawers contains two boxes: the right one and the left one. Once Alice or Bob hear a sound of a ball falling into her or his drawer they open it and give a result “+1” if they find the ball in the right box and “−1” if they find it in the left one. Each drawer defines a binary random variable: A, A′ for the Alice wall and B, B′ for the Bob one.

Mathematically we may model the experiment as follows. The chessboard is the [0, 4] × [0, 4] square. The falling “molecule” falls at a point with coordinates (a, b). If a ∈ [0, 2] then Alice finds the ball in A′ and obtains “−1” if a ∈ [0, 1] and “+1” if a ∈ (1, 2]. For a ∈ (2, 4] Alice finds the ball in A and obtains “−1” if a ∈ (2, 3] and “+1” if a ∈ (3, 4]. We similarly describe the measurements made by Bob.

We finally add one more element. We assume that both Alice and Bob can freely exchange the locations of the drawers. So, for example, Alice can cross the possible paths of the balls which move towards her drawers according to the scheme shown at Fig. 2. She does it without knowing whether a given ball moves towards A or A′. She makes the decision either according to her free will or leaves it to a random generator. In either case the probability that the drawers will be exchanged is pA for Alice and pB for Bob. So neither Alice nor Bob can know for sure which experiment is made by her or his partner.

Let us note that the arrangement is in a sense similar to the experiment of Aspect [10] where the devices remain in fixed positions but photons from the singlet state pair choose the devices at random. In the Aspect-type
experiment the choice depends on whether the photon is reflected or transmitted at an optical switch and this, in principle, depends on both the (hidden variable) state of the photon and this of the switch.

A. Locality

From the description given above it follows that Alice and Bob do not communicate. The results they obtain depend on the initial correlation (state) of the “molecule”, that is the coordinates of the point on the floor where the “molecule” splits into two balls, and the local actions they undertake. Therefore the situation is local.

B. Determinism

The system is deterministic. The determinism does not necessarily apply to the “switch” since a free will of Alice and Bob can change its state. A “particle” does not “know” which “detector” will be chosen since the “switch” can be changed to another state at any time.

C. Detection efficiency

To determine the detector efficiency we compare the number of detections in coincidence to those where only one of the detectors “clicks”. Here whenever Alice’s A or A’ “clicks” then either B or B’ does the same, and vice versa. We conclude that the detectors are perfect.

D. Averages

Averages of the “one arm” random variables vanish: \( \langle A \rangle = \langle A' \rangle = \langle B \rangle = \langle B' \rangle = 0 \) but the Bell average depends on the type of actions undertaken by Alice and Bob:

\[
\langle AB \rangle + \langle AB' \rangle + \langle A'B \rangle - \langle A'B' \rangle = 4p_{AB} \geq 0.5. \tag{1}
\]

The Bell inequality is violated for \( p_{AB} > 0.5 \).

E. Perfect correlations

The averages occurring in (1) are \( \pm 1 \) if \( p_A = p_B = 1 \) so that the inequality can be violated even for perfect correlations. This makes our “state” in a sense similar to the Greenberger-Horn-Zeilinger (GHZ) one [9].

F. Complementarity

Similarly to real Bell-type experiments Alice and Bob measure, on a single pair, only either A and B, or A and B’, or A’ and B, or A’ and B’. Alice cannot measure simultaneously A and A’. This is simply logically impossible because the hidden variable \( a \) (the coordinate of the “molecule”) cannot simultaneously satisfy \( a \leq 1 \) and \( a > 1 \).

G. Symmetry

A transition between A and A’, and B and B’ is obtained by a shift of the domains of the random variables by 2 to the left or to the right. The initial statistical state of the pair (the chessboard) is not invariant under this operation. This element is technically responsible for the violation (one cannot change the variables under integrals and do not change the probability density).

H. Symmetry vs. complementarity

The above discussion shows that alternative measurements A and A’ may be called complementary if a transformation \( A \rightarrow A' \) applied to random variables is not a symmetry of the probability density of the hidden variables. So an additional care is needed if one tries to describe in hidden variables terms states which do not possess symmetries. This remark applies in particular to the theorems of Home and Selleri [8], and Gisin and Peres [9], which use general entangled states, and to the GHZ states which are not rotationally invariant.

I. Free will of Alice and Bob

Alice and Bob are free to manipulate with the detectors. The degree of their freedom is measured by \( p_A \) and \( p_B \). Their different actions result in different averages they obtain. In fact, the more active they are the more violation is obtained. This clearly distinguishes our example from the situation discussed by Brans in [10] where the choice of detectors is pre-determined since his universe is totally deterministic.

J. Data collecting procedures

The experiment with the drawers formally resembles the Aspect-type experiment with the optical switch. There exists another class of experiments [11 12] where the detectors remain in fixed positions during a given run. Alice and Bob may insist on collecting only the data from A and B in a given run of an experiment. In such a case there will be a fraction of particles, namely
those that move towards A' and B', that will not be
detected. This resembles the problem of detector ineffi-
ciency loophole [3]. In experiments one usually tries to
collect as many pairs as possible by using waveguides or
mirrors to direct those propagating in the “wrong” direc-
tions into the fixed detectors so that no data will be lost.
An alternative procedure is to surround a source with
detectors in a way guaranteeing that no particles will es-
cape. The latter case corresponds to the procedure used
in our experiment. Obviously, we could use also the other
procedure by directing the balls which move towards the
“wrong” drawers into the “right” ones. It is interesting
that in such a case the Bell inequality would not be vi-
olated. The reason is that in a formal description of the
experiment we would have to extend domains of all ran-
dom variables appearing in (1) onto the whole space of
hidden variables and then the standard trick leading to
the Bell inequality would work. The two procedures are
therefore, in general, physically inequivalent although the
opposite is typically assumed.

K. Alternative choices of detectors

The alternative choices of our “detectors” correspond
physically to a situation where instead of rotating a single
polarizer from A to A' we have two polarizers A and A'
located in different places and we exchange their locations.
In general if the state one considers is nonsymmetric one
may expect a correlation between the direction of propa-
gation and the internal state which is measured by Alice
or Bob. This is what happens here albeit quite trivially.

L. Quantum description

Any classical model allows also for a quantum descrip-
tion. To find it we can introduce, instead of the billiard
balls, pairs of photons or any other quantum particles
which are emitted by a source which is classically fluc-
tuating in the way determined by the chessboard from
Fig. 1. To each drawer one can associate a projector in
position space which corresponds to a result, say, “a pho-
ton is detected in the left box in the drawer A". Having
these projectors we can construct a density matrix which
describes the source. The observables measured by Alice
and Bob can be also constructed in terms of the project-
tors. Now a warning. Of course any projector has eigen-
values 1 and 0. The states corresponding to a detection
in A' are in a space orthogonal to those corresponding
to a detection in A. Therefore one can say that a result
A' = +1 is simultaneously equivalent to A = 0. There-
fore A has three results, ±1 and 0, and the Bell inequality
should be derived for random variables with three results.
In a sense this is true and this ambiguity was discussed
in some detail in [3]. In more recent literature it is called
a “Kolmogorovian censorship” problem [7]. Still notice
that if we rule out on this basis our problem as trivial, we
have to equally treat the celebrated Aspect experiment
with the optical switch. Indeed, a detection in his ana-
lyzer A means no detection in A' and the whole argument
can be repeated. This also shows that an addition of the
value 0 would lead to averages which are not directly re-
lated to the detector counts. The measured probabilities
are, in fact, the conditional ones and the condition is
“provided the particle is detected in A".

IV. CONCLUSIONS

I have presented an example of a classical system
which is fully deterministic, local, uses perfect detectors,
formally resembles the Aspect experiment with optical
switch, and yet violates the Bell inequality. The ex-
ample uses a version of a hidden variables complement-
arity principle. This is one of the possible versions
of such a principle. The other is the Bell-Aerts one
where the complementarity is related to mutually dis-
turbing measurements. The Bell nonlocality is a kind
of a complementarity-at-a-distance. The complementarity I
use is local. The example is rather primitive and perhaps
somewhat artificial, but this is a result of my inability of
inventing a more sophisticated hidden-variables form of
local complementarity. Nevertheless the fact that I could
not find anything more convincing does not mean that
this is impossible in principle. It seems that one of the
conceptual difficulties lies in a sort of a mental block in-
clining us to use the Komogorovian probability calculu-
s which is naturally related to characteristic functions, i.e.
commuting projectors. The commuting projectors are
inherently related to situations where no complementar-
ity occurs and do not correctly work even in the classical
situation found by Aerts. So complementarity is not, in
general, a quantum property and there is no reason to
believe that it should vanish at levels deeper than the
quantum one.

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FIG. 1. The white squares denote the holes cut in the floor of the upper room.

FIG. 2. Alice has free will. A particle that moves towards the drawer $A$ ($A'$) can be shifted with probability $p_A$ to $A'$ ($A$). Bob can do similar operations with the probability $p_B$. Here the shifts are chosen in such a way that the results $\pm 1$ are not changed. This is not essential but makes calculations simpler. One could consider cases when $\pm 1$ for $A$ goes into $\mp 1$ for $A'$ and so on. To violate the Bell inequality the probabilities of undertaking such actions by Alice or Bob must satisfy, for this particular model, $p_{APB} > 0.5$. 
Fig. 1
M. Czachor, “Bell inequality and complementarity loophole”
Fig.2
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