Real hereditarily indecomposable Banach spaces and uniqueness of complex structure

Valentin Ferenczi

Abstract

There exists a real hereditarily indecomposable Banach space \( X = X(\mathbb{C}) \) (resp. \( X = X(\mathbb{H}) \)) such that the algebra \( \mathcal{L}(X)/S(X) \) is isomorphic to \( \mathbb{C} \) (resp. to the quaternionic division algebra \( \mathbb{H} \)).

Up to isomorphism, \( X(\mathbb{C}) \) has exactly two complex structures, which are conjugate, totally incomparable, and both hereditarily indecomposable. So there exist two Banach spaces which are isometric as real spaces but totally incomparable as complex spaces. This extends results of J. Bourgain and S. Szarek [6, 19, 20].

The quaternionic example \( X(\mathbb{H}) \), on the other hand, has unique complex structure up to isomorphism; there also exists a space with an unconditional basis, non isomorphic to \( l_2 \), with unique complex structure. This answers a question of S. Szarek in [20].

1 Introduction

It is well-known that any two real Banach spaces which are isometric must be linearly isometric: this was proved in 1932 by S. Mazur and S. Ulam [18]. Recently, G. Godefroy and N.J. Kalton also proved that if a separable real Banach space embeds isometrically into a Banach space, then it embeds linearly isometrically into it [10].

In 1986, J. Bourgain and S. Szarek proved that Mazur-Ulam’s result is completely false in the complex case: there exist two Banach spaces which are linearly isometric as real spaces but non isomorphic as complex spaces [6, 19, 20]. One of the main results of this paper is the following extension of their result.

\(^1\text{MSC numbers: 46B03, 46B04, 47B99.}\)
Theorem 1 There exist two Banach spaces which are isometric as real spaces, but totally incomparable as complex spaces.

We recall that two spaces are said to be totally incomparable if no subspace of one if is isomorphic to a subspace of the other. We shall also show that this result is optimal, in the sense that there does not exist a family of more than two Banach spaces which are mutually isomorphic as real spaces but totally incomparable as complex spaces.

Our examples are natural modifications of the hereditarily indecomposable Banach space of W.T. Gowers and B. Maurey [13]. Hereditarily indecomposable (or H.I.) Banach spaces were discovered in 1991 by these two authors: a space $X$ is H.I. if no (closed) subspace of $X$ is decomposable (i.e. can be written as a direct sum of infinite dimensional subspaces). Equivalently, for any two subspaces $Y, Z$ of $X$ and $\epsilon > 0$, there exist $y \in Y$, $z \in Z$ such that $\|y\| = \|z\| = 1$ and $\|y - z\| < \epsilon$. Gowers and Maurey gave the first known example $X_{GM}$ of a H.I. space, both in the real and the complex case. After this result, many other examples of H.I. spaces with various additional properties were given. They are too numerous to be all cited here. We refer to [2] for a list of these examples. Let us mention however the remarkable result of S. Argyros and A. Tolias [5]: for any separable Banach space $X$ not containing $\ell_1$, there is a separable H.I. space with a quotient isomorphic to $X$.

A Banach space $X$ is said to have a Schauder basis $(e_i)_{i \in \mathbb{N}}$ if any vector of $X$ may be written uniquely as an infinite sum $\sum_{i \in \mathbb{N}} \lambda_i e_i$. The basis $(e_i)$ is unconditional if any permutation of $(e_i)$ is again a basis. This is equivalent to saying that there exists $C < +\infty$ such that for any vector written $\sum_{i \in \mathbb{N}} \lambda_i e_i$ in $X$, and any scalar sequence $(\mu_i)_{i \in \mathbb{N}}$ such that $\forall i \in \mathbb{N}, |\mu_i| \leq |\lambda_i|$, the inequality $\|\sum_{i \in \mathbb{N}} \mu_i e_i\| \leq C \|\sum_{i \in \mathbb{N}} \lambda_i e_i\|$ holds.

Classical spaces, such as $c_0$, $l_p$ for $1 \leq p < +\infty$, $L_p$ for $1 < p < +\infty$, and Tsirelson’s space $T$ have unconditional bases; or contain subspaces with an unconditional basis in the case of $C([0, 1])$ or $L_1$. The H.I. spaces of Gowers and Maurey were the first known examples of spaces not containing any unconditional basic sequence, thus answering an old open question by the negative. The importance of H.I. spaces also stems from the famous Gowers’ dichotomy theorem [12]: any Banach space contains either a subspace with an unconditional basis or a H.I. subspace. In some sense, H.I. spaces capture even more of the general structure of all separable Banach spaces than classical spaces. By [3], any Banach space containing copies of all separable (reflexive) H.I. spaces must be universal for the
class of separable Banach spaces (i.e. must contain an isomorphic copy of any separable Banach space). On the other hand, all spaces with an unconditional basis may be embedded into the unconditional universal space $U$ of Pełczyński (see i.e [16] about this). The space $U$ has an unconditional basis and thus is certainly not universal (for example, it does not contain $L_1$ nor a H.I. subspace).

H.I. spaces have interesting properties linked to the space of operators defined on them. An operator $s \in \mathcal{L}(Y, Z)$ is strictly singular if no restriction of $s$ to an infinite dimensional subspace of $Y$ is an isomorphism into $Z$. Equivalently, for any $\epsilon > 0$, any subspace $Y'$ of $Y$, there exists $y \in Y'$ such that $\|s(y)\| < \epsilon \|y\|$. The ideal of strictly singular operators is denoted $S(Y, Z)$. It is a perturbation ideal for Fredholm operators, we refer to [16] about this. Gowers and Maurey proved that any complex H.I. space $X$ has what we shall call the $\lambda Id + S$-property, i.e. any operator on $X$ is of the form $\lambda Id + S$, where $\lambda$ is scalar and $S$ strictly singular. Note however that spaces with the $\lambda Id + S$-property which are far from being H.I. were also constructed [4]. It follows from this property that any operator on $X$ is either strictly singular or Fredholm with index 0, and so $X$ cannot be isomorphic to a proper subspace (thus the existence of $X_{GM}$ answers the old hyperplane’s problem, which had been solved previously by Gowers [11]). Then in [7] the author extended the result: if $X$ is complex H.I. and $Y$ is a subspace of $X$, then every operator from $Y$ into $X$ is of the form $\lambda i_{Y,X} + s$, where $\lambda$ is scalar, $i_{Y,X}$ the canonical injection of $Y$ into $X$, and $s \in S(Y, X)$. This property of operators characterizes complex H.I. spaces.

When $X$ is real the situation is more involved. From now on, $X_{GM}$ denotes the real version of the H.I. space of Gowers and Maurey, as opposed to the complex version $X_{GM}^\mathbb{C}$. The real space $X_{GM}$ has the property that for any $Y \subset X_{GM}$, any operator from $Y$ to $X_{GM}$ is of the form $\lambda i_{Y,X_{GM}} + s$, where $s$ is strictly singular [13] (note that this property of operators implies the H.I. property). In general, a real H.I. space $X$ must satisfy that for all $Y \subset X$, $\dim \mathcal{L}(Y, X)/S(Y, X) \leq 4$ [8]. The converse is false: the space $X = X_{GM} \oplus X_{GM}$ is not H.I. but for any $Y \subset X$, $\dim \mathcal{L}(Y, X)/S(Y, X) \leq 4$ (see the proof in Remark [7]). Also when $X$ is real H.I., the algebra $\mathcal{L}(X)/S(X)$ is a division algebra isomorphic to $\mathbb{R}$, $\mathbb{C}$ or the quaternionic division algebra $\mathbb{H}$ [8]. This implies easily, by continuity of the Fredholm index, that any operator on $X$ is either strictly singular or Fredholm with index 0 (this was already proved in [13]), and so $X$ is not isomorphic to a proper subspace.
We will show that each of the values 2 and 4 for \( \dim \mathcal{L}(X)/\mathcal{S}(X) \) is indeed possible. We shall build versions of \( X_{GM} \) for which the algebra \( \mathcal{L}(X)/\mathcal{S}(X) \) is isomorphic to \( \mathbb{C} \) or \( \mathbb{H} \). Furthermore, the complex and the quaternionic examples satisfy \( \dim \mathcal{L}(Y, X)/\mathcal{S}(Y, X) = 2 \) and \( 4 \) respectively, for any subspace \( Y \) of \( X \):

**Theorem 2** There exists a real H.I. Banach space \( X(\mathbb{C}) \) such that for any subspace \( Y \) of \( X(\mathbb{C}) \), \( \dim \mathcal{L}(Y, X(\mathbb{C}))/\mathcal{S}(Y, X(\mathbb{C})) = 2 \), and such that the algebra \( \mathcal{L}(X(\mathbb{C}))/\mathcal{S}(X(\mathbb{C})) \) is isomorphic to the complex field \( \mathbb{C} \).

There exists a real H.I. Banach space \( X(\mathbb{H}) \) such that for any subspace \( Y \) of \( X(\mathbb{H}) \), \( \dim \mathcal{L}(Y, X(\mathbb{H}))/\mathcal{S}(Y, X(\mathbb{H})) = 4 \), and the algebra \( \mathcal{L}(X(\mathbb{H}))/\mathcal{S}(X(\mathbb{H})) \) is isomorphic to \( \mathbb{H} \), the algebra of quaternions.

The initial idea of this construction was given to the author by his Ph. D. advisor B. Maurey in 1996, but the examples were never checked. Recently the question was asked the author by D. Kutzarova and S. Argyros and the interest in this subject was revived by the survey paper of Argyros [2], see also [5]. The author therefore decided to write down the proof. The complex structure properties of our examples turned out to be quite interesting.

We shall write the construction of the real H.I. space denoted \( X(\mathbb{H}) \) in the quaternionic case, assuming familiarity with Gowers-Maurey type constructions as in [13] and mainly [14]. The reader will adapt the construction for the example with complex algebra of operators, denoted \( X(\mathbb{C}) \). We shall then proceed to give the proofs of the operators properties in each case.

In the following, spaces and subspaces are supposed infinite-dimensional and closed, unless specified otherwise. If \( X \) is a complex Banach space, with scalar multiplication denoted \((\lambda + i\mu)x\) for \( \lambda, \mu \in \mathbb{R} \) and \( x \in X \), its conjugate \( X^\prime \) is defined as \( X \) equipped with the scalar multiplication \((\lambda + i\mu).x := (\lambda - i\mu)x\). Note that \( X \) and \( X^\prime \) are isometric as real spaces.

When \( X \) is a real Banach space, a complex structure \( X^I \) on \( X \) is \( X \) seen as a complex space with the law

\[
\forall \lambda, \mu \in \mathbb{R}, (\lambda + i\mu).x = (\lambda Id + \mu I)(x),
\]

where \( I \) is some operator on \( X \) such that \( I^2 = -Id \), and renormed with the equivalent norm

\[
|||x||| = \sup_{\theta \in \mathbb{R}} \|\cos \theta x + \sin \theta Ix\|.
\]
Note that the conjugate of $X^I$ is the complex structure $X^{-I}$.

It is known that complex structures do not always exist on a Banach space, even on a uniformly convex one [20]. Gowers also constructed a space with an unconditional basis on which every operator is a strictly singular perturbation of a diagonal operator [11, 14], and which therefore does not admit complex structure (this answers Pb 7.1. in [20]).

Concerning uniqueness, J. Bourgain [6], S. Szarek [19, 20], used local random techniques to give an example of a complex Banach space not isomorphic to its conjugate; the space is a "gluing" of finite dimensional spaces which are "far" from their conjugates. Therefore there exist spaces which are isometric as real spaces but not isomorphic as complex spaces. Later on, N. J. Kalton [15] constructed a simple example defined as a twisted Hilbert space. Recently, R. Anisca [1] constructed a subspace of $L_p$, $1 \leq p < 2$ which has the same property, and moreover admits continuum many non-isomorphic complex structures. Note that these examples fail to have an unconditional basis, since when a complex Banach space $X$ has an unconditional basis $(e_n)$, the map $c$ defined by $c(\sum_{n \in \mathbb{N}} \lambda_n e_n) = \sum_{n \in \mathbb{N}} \lambda_n e_n$ is a $\mathbb{C}$-linear isomorphism from $X$ onto $\overline{X}$.

The real spaces $X(\mathbb{C})$ and $X(\mathbb{H})$ possess an operator $I$ such that $I^2 = -Id$; the associated complex structures are H.I.. In fact the space $X(\mathbb{C})$ will look a lot like the complex version of Gowers-Maurey’s space $X_{GM}^C$ seen with its $\mathbb{R}$-linear structure. However it doesn’t seem at all obvious that the properties of $X(\mathbb{C})$ are shared by $X_{GM}^C$ seen as a real space. Technically, the difference comes from the fact that the functionals used in the definition of the norm in our construction of $X(\mathbb{C})$ are $\mathbb{R}$-linear but cannot be allowed to be $\mathbb{C}$-linear when viewed in the complex setting (see Lemma 13 which prevents this).

We show that $X(\mathbb{C})$, with the complex structure $X^J(\mathbb{C})$ associated to some canonical operator $J$, is totally incomparable with its conjugate. Therefore we have examples of Banach spaces which are isometric as real spaces but totally incomparable as complex spaces. Furthermore, it turns out that $X^J(\mathbb{C})$ and its conjugate are the only complex structures on $X(\mathbb{C})$ up to isomorphism.

The space $X(\mathbb{H})$, on the other hand, admits a unique complex structure up to isomorphism. This answers a question of Szarek from [19]: he asked whether the Hilbert space was the only space with unique complex structure.

**Theorem 3** There exists a real H.I. Banach space which admits exactly two complex structures up to isomorphism. Moreover, these two complex structures are
conjugate and totally incomparable.

There also exists a real H.I. Banach space with unique complex structure up to isomorphism.

We shall see that the space $X(\mathbb{C})$ is in some sense the only possible example of space with totally incomparable complex structures: if a space $X$ admits two totally incomparable complex structures, then these structures must be conjugate up to isomorphism and both saturated with H.I. subspaces. It follows that there cannot be more than two mutually totally incomparable structures on a Banach space. We shall also note that for any $n \in \mathbb{N}$, the space $X(\mathbb{C})^n$ admits exactly $n + 1$ complex structures up to isomorphism.

Our constructions are different from those of the previous authors, although as noted by Maurey in [17], there are some similarities between the "few operators" properties of Gowers-Maurey’s spaces and the "few operators" properties of the finite-dimensional spaces glued together in Bourgain-Szarek’s example. The peculiar complex structure properties of the spaces $X(\mathbb{C})$ and $X(\mathbb{H})$ follow directly from their few operators properties. For example, $J$ and $-J$ are, up to strictly singular operators, the only operators on $X(\mathbb{C})$ whose square is $-Id$.

We also prove that whenever \{$X_i, 1 \leq i \leq N$\} is a family of pairwise totally incomparable real Banach spaces with the $\lambda Id + S$-property, and $n_i, 1 \leq i \leq N$, are integers, the direct sum $\sum_{1 \leq i \leq N} \oplus X_i^{2n_i}$ has a unique complex structure up to isomorphism. This provides additional examples of spaces not isomorphic to a Hilbert space, which have a unique complex structure. We also provide an unconditional version of the space $X(\mathbb{C})$:

**Theorem 4** There exists a real Banach space with an unconditional basis, not isomorphic to $l_2$, with unique complex structure up to isomorphism.

Finally, note that a complex Banach space, which is H.I. as a real space, is always complex H.I.. Indeed if it contained two $\mathbb{C}$-linear subspaces in a direct sum, these would in particular form a direct sum of $\mathbb{R}$-linear subspaces. We shall show that the converse is false (the complexification of the real $X_{GM}$ will do).

## 2 Some properties of H.I. spaces

The following was proved in [7], [8].
Theorem 5 [7, 8] Let $X$ be a real H.I. space. Then there exists a division algebra $E$ which is isomorphic to either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, and, for $Y \subset X$, linear embeddings $i_Y$ of $E_Y = \mathcal{L}(Y, X)/\mathcal{S}(Y, X)$ into $E$. Let $\preceq$ be the relation between subspaces of $X$ defined by $Z \preceq Y$ iff $Z$ embeds into $Y$ by an isomorphism of the form $i_Z + s$, where $s \in \mathcal{S}(Z, X)$. For any $Z \preceq Y$, the canonical restriction map modulo strictly singular operators $p_{YZ} : E_Y \to E_Z$ embeds $E_Y$ into $E_Z$ and satisfies $i_Y = i_Z p_{YZ}$. The algebra $E$ is actually the inductive limit of the system $(E_Y, p_{YZ})$ under $\preceq$, which is a filter relation. Furthermore the map $i_X$ embeds $\mathcal{L}(X)/\mathcal{S}(X)$ as a subalgebra of $E$.

A technical lemma (Lemma 2 in [8]) will be very useful. Given a Banach space $X$, a subspace $Y$ of $X$ is was defined in [8] to be quasi-maximal in $X$ if $Y + Z$ for $Z$ infinite dimensional in $X$ is never a direct sum. Equivalently the quotient map from $X$ onto $X/Y$ is strictly singular. An obvious remark is that a space $X$ is H.I. if and only if any subspace of $X$ is quasi-maximal in $X$.

Lemma 6 [7] let $X$ be a Banach space, let $T$ be an operator from $X$ into some Banach space and let $Y$ be quasi-maximal in $X$. Then $T$ is strictly singular if and only if $T|_Y$ is strictly singular.

In particular, if $X$ is H.I. and $Y$ a subspace of $X$, then $T$ is strictly singular if and only if $T|_Y$ is strictly singular.

The "filter property" will denote the fact that if $X$ is H.I. and $Y, Z$ are subspaces of $X$, then there exist a subspace $W$ such that $W \preceq Y$ and $W \preceq Z$ - in particular $W$ embeds into $Y$ and $Z$ (Lemma 1 in [7]).

We recall that a space $X$ is said to be $HD_n$ if $X$ contains at most and exactly $n$ infinite dimensional subspaces in a direct sum [8]. For example, a space is $HD_1$ if and only if it is H.I.

We finally recall that $X_{GM}$ denotes the real H.I. space of Gowers and Maurey. The following remark shows that real H.I. spaces are not characterized by the property that $\dim \mathcal{L}(Y, X)/\mathcal{S}(Y, X) \leq 4$ for all subspaces $Y$ of $X$.

Remark 7 Let $X = X_{GM} \oplus X_{GM}$. For any $Y \subset X$, $\dim \mathcal{L}(Y, X)/\mathcal{S}(Y, X) \leq 4$.

Proof : By [8] Corollary 2, $X$ is $HD_2$ as a direct sum of two H.I. spaces. Let $Y$ be an arbitrary subspace of $X$, then $Y$ is either H.I. or $HD_2$. Let $d$ be the dimension of $\mathcal{L}(Y, X)/\mathcal{S}(Y, X)$. Assume $Y$ is $HD_2$. Then $Y$ contains a direct sum of H.I. spaces $Z_1 \oplus Z_2$. By [8] Corollary 3, passing to subspaces and by
the filter property, we may assume that $Z_1$ and $Z_2$ are isomorphic and embed into $X_{GM}$. Since $\dim \mathcal{L}(Z_1, X_{GM})/\mathcal{S}(Z_1, X_{GM}) = 1$, we deduce that $\dim \mathcal{L}(Z_1 \oplus Z_2, X)/\mathcal{S}(Z_1 \oplus Z_2, X) \leq 4$.

Since $Z_1 \oplus Z_2$ is $HD_2$ ([8] Corollary 2), it is quasimaximal in $Y$, so the restriction map $r$ defined from $\mathcal{L}(Y, X)/\mathcal{S}(Y, X)$ into $\mathcal{L}(Z_1 \oplus Z_2, X)/\mathcal{S}(Z_1 \oplus Z_2, X)$ by $r(\widetilde{T}) = \widetilde{T}|_{Z_1 \oplus Z_2}$ is well defined and injective (Lemma 6) - here $\widetilde{T}$ denotes the class of $T$ modulo strictly singular operators. It follows that

$$d \leq \dim \mathcal{L}(Z_1 \oplus Z_2, X)/\mathcal{S}(Z_1 \oplus Z_2, X) = 4.$$  

If $Y$ is H.I., we do a similar proof, passing to a subspace $Z$ of $Y$ which embeds into $X_{GM}$, and obtain by the same H.I. properties that

$$d \leq \dim \mathcal{L}(Z, X)/\mathcal{S}(Z, X) = 2.$$

$\square$

3 Construction of real H.I. spaces $X(\mathbb{H})$ and $X(\mathbb{C})$

Let $c_{00}$ be the vector space of all real sequences which are eventually 0. Let $(e_n)_{n \in \mathbb{N}}$ be the standard basis of $c_{00}$. Given a family of vectors \{x_i, i \in I\}, $[x_i, i \in I]$ denotes the closed linear span of \{x_i, i \in I\}. For $k \in \mathbb{N}$, let $F_k = [e_{4k-3}, e_{4k-2}, e_{4k-1}, e_{4k}]$. The sequence $F_k$ will provide a 4-dimensional decomposition of $X(\mathbb{H})$.

We proceed to definitions which are adaptations of the usual Gowers-Maurey definitions to the 4-dimensional decomposition context.

If $E \subset \mathbb{N}$, then we shall also use the letter $E$ for the projection from $c_{00}$ to $c_{00}$ defined by $E(\sum_{i \in \mathbb{N}} x_i) = \sum_{i \in E} x_i$, where $x_i \in F_i, \forall i \in \mathbb{N}$. If $E, F \subset \mathbb{N}$, then we write $E \prec F$ to mean that $\max E < \min F$, and if $k \in \mathbb{N}$ and $E \subset \mathbb{N}$, then we write $k \prec E$ to mean $k < \min E$. The support of a vector $x = \sum_i x_i \in c_{00}, x_i \in F_i$, is the set of $i \in \mathbb{N}$ such that $x_i \neq 0$. An interval of integers is the intersection of an interval of $\mathbb{R}$ with $\mathbb{N}$. The range of a vector, written $\text{ran}(x)$, is the smallest interval containing its support. We shall write $x \prec y$ to mean $\text{ran}(x) \prec \text{ran}(y)$. If $x_1 < \cdots < x_n$, we shall say that $x_1, \ldots, x_n$ are successive.

The class of functions $\mathcal{F}$ is defined as in [13], and the function $f \in \mathcal{F}$ is defined on $[1, +\infty)$ by $f(x) = \log_2(1 + x)$. Let $\mathcal{X}_4$ be the set of normed spaces of the form $X = (c_{00}, \|\cdot\|)$ such that $(F_i)_{i=1}^\infty$ is a monotone Schauder decomposition.
of $X$ where each $e_k$ is normalized. If $f \in \mathcal{F}$, $X \in \mathcal{X}_4$ and every $x \in X$ satisfies the inequality

$$
\|x\| \geq \sup \{ f(N)^{-1} \sum_{i=1}^{N} \|E_i x\| : N \in \mathbb{N}, E_1 < \cdots < E_n \},
$$

where the $E_i$'s are intervals, then we shall say that $X$ satisfies a lower $f$-estimate (with respect to $(F_i)_{i=1}^{\infty}$).

Special vectors are considered in [14]. We give their definitions in our context, as well as some lemmas without proof: indeed their proof is essentially the proof from [14] word by word, with the difference that "successive" and "lower $f$-estimate" mean with respect to $(e_i)$ in their case and to $(F_k)$ in ours. As our definitions are also based on the decomposition $(F_k)$ instead of $(e_i)$, it is easy to check that the proofs are indeed valid. We shall only point out the parts of the proofs which require a non-trivial modification. Alternatively, in the case of the real H.I. space with complex algebra of operators, these lemmas correspond exactly to the lemmas of [14] for the complex space $X^c_{GM}$, with our 2-dimensional real decomposition interpreted as a Schauder basis on $\mathbb{C}$.

For $X \in \mathcal{X}_4$, $x \in X$, and every integer $N \geq 1$, we consider the equivalent norm on $X$ defined by

$$
\|x\|_{(N)} = \sup \sum_{i=1}^{N} \|E_i x\|
$$

where the supremum is over all sequences $E_1, E_2, \ldots, E_N$ of successive intervals.

For $0 < \epsilon \leq 1$ and $f \in \mathcal{F}$, we say that a sequence $x_1, \ldots, x_N$ of successive vectors satisfies the RIS($\epsilon$) condition (for $f$) if there exists a sequence $n_1 < \ldots < n_N$ of integers such that $\|x_i\|_{(n_i)} \leq 1$ for each $i = 1, \ldots, N$, $n_1 > (2N/f'(1))f^{-1}(N^2/\epsilon^2)$, and $\epsilon \sqrt{f(n_i)} > |\text{ran}(\sum_{j=1}^{i-1} x_j)|$, for $i = 2, \ldots, N$. Observe that when $x_1, \ldots, x_N$ satisfies the RIS($\epsilon$) condition, then $E x_1, \ldots, E x_N$ also does for any interval $E$.

Given $g \in \mathcal{F}$, $M \in \mathbb{N}$ and $X \in \mathcal{X}_4$, an $(M, g)$-form on $X$ is a functional of norm at most 1 which can be written $\sum_{j=1}^{M} x_j^*$ for a sequence $x_1^* < \ldots < x_M^*$ of successive functionals of norm at most $g(M)^{-1}$. Observe that if $x^*$ is an $(M, g)$-form then $|x^*(x)| \leq g(M)^{-1} \|x\|_{(M)}$ for any $x$.

**Lemma 8** Let $X \in \mathcal{X}_4$. Let $f, g \in \mathcal{F}$ be such that $\sqrt{f} \leq g$. Assume that $x_1, \ldots, x_N \in X$ satisfies the RIS($\epsilon$)-condition for $f$. If $x^*$ is a $(k, g)$-form for
some integer \( k \geq 2 \) then

\[
|x^*(\sum_{i=1}^{N} x_i)| \leq \epsilon + 1 + \frac{N}{\sqrt{f(k)}}.
\]

In particular, \( |x^*(x_1 + \ldots + x_N)| < 1 + 2\epsilon \) when \( k > f^{-1}(N^2/\epsilon^2) \).

**Proof:** Reproduce the proof of [14] Lemma 1, noting that, for \( x \in c_{00} \), \( \|x\|_{c_0} = \max_{i \in \mathbb{N}} \|x_i\| \) if \( x = \sum_{i \in \mathbb{N}} x_i \) with \( x_i \in F_i, \forall i \in \mathbb{N} \).

**Lemma 9** Let \( X \in \mathcal{X}_4 \). Let \( f, g \in \mathcal{F} \), \( \sqrt{f} \leq g \), and let \( x_1, \ldots, x_N \in X \) satisfy the RIS(\( \epsilon \)) condition for \( f \). Let \( x = \sum_{i=1}^{N} x_i \), and suppose that

\[
\|Ex\| \leq 1 \vee \sup \{|x^*(Ex)| : x^*(k, g) - form, k \geq 2\},
\]

for every interval \( E \). Then \( \|x\| \leq (1 + 2\epsilon)N g(N)^{-1} \).

**Proof:** Reproduce the proof of [14] Lemma 3, using [14] Lemma 4 in its 4-dimensional decomposition version.

**Lemma 10** Let \( X \in \mathcal{X}_4 \) satisfy a lower \( f \)-estimate. Then for every \( n \in \mathbb{N} \) and \( \epsilon > 0 \), every subspace of \( X \) generated by a sequence of successive vectors contains a vector \( x \) of finite support such that \( \|x\| = 1 \) and \( \|x\|_{(n)} \leq 1 + \epsilon \). Hence, for every \( N \in \mathbb{N} \), every subspace generated by a sequence of successive vectors contains a sequence \( x_1, \ldots, x_N \) satisfying the RIS(\( \epsilon \)) condition with \( \|x_i\| \geq (1 + \epsilon)^{-1} \).

**Proof:** Given a sequence \((x_n)_{n \in \mathbb{N}}\) of successive vectors in \( X \), \((x_n)\) is basic bi-monotone, and for vectors in \([x_n, n \in \mathbb{N}]\), the notions of lower \( f \)-estimate, successive vectors, etc... with respect to \((F_k)_{k \in \mathbb{N}}\) correspond to the usual notions of lower \( f \)-estimate, successive vectors, etc... with respect to the basis \((x_n)\). Therefore the conclusion holds by Lemma 4 in [14].

We now pass to the definition of \( X = X(\mathbb{H}) \). Let \( Q \subset c_{00} \) be the set of sequences with rational coordinates and modulus at most 1. Let \( J \subset \mathbb{N} \) be a set such that if \( m < n \) and \( m, n \in J \), then \( \log \log \log n \geq 2m \). We write \( J \) in
increasing order as \( \{j_1, j_2, \ldots \} \). We shall also need \( f(j) > 256 \) where \( f(x) \) is still the function \( \log_2(x + 1) \). Let \( K, L \subset J \) be the sets \( \{j_1, j_3, \ldots \} \) and \( \{j_2, j_4, \ldots \} \).

Let \( \sigma \) be an injection from the collection of finite sequences of successive elements of \( \mathbb{Q} \) to \( L \). Given \( X \in X_4 \) and \( f \in \mathcal{F} \) such that \( X \) satisfies a lower \( f \)-estimate (with respect to \( (F_k) \)), and given an integer \( m \in \mathbb{N} \), let \( A_m^*(X) \) be the set of functionals of the form \( f(m)^{-1} \sum_{i=1}^m x_i^* \) such that \( x_i^* < \ldots < x_m^* \) and \( \|x_i^*\| \leq 1 \).

If \( k \in \mathbb{N} \), let \( \Gamma_k^X \) be the set of sequences \( y_1^* < \ldots < y_k^* \) such that \( y_i^* \in \mathbb{Q} \) for each \( i \), \( y_i^* \in A_{j_2}^*(X) \) and \( y_{i+1}^* \in A_{s(y_i^*, \ldots, y_j^*)}^*(X) \) for each \( 1 \leq i \leq k - 1 \). These sequences are called special sequences. If \( (g_i)_{i=1}^k \), \( k \in K \), is a special sequence, then the functional \( f(k)^{-1/2} \sum_{j=1}^k g_j \) is a special functional on \( X \) of size \( k \). The set of such functionals is denoted \( B_k^*(X) \). If \( f \in \mathcal{F} \) and \( g(k) = f(k)^{1/2} \), then a special functional of size \( k \) is also a \((k, g)\)-form.

The quaternionic division algebra may be represented as an algebra of operators on \( \mathbb{R}^4 \). It is then generated by a family \( \{Id_{\mathbb{R}^4}, u, v, w\} \), where \( u, v, w \) satisfy the relations \( u^2 = v^2 = w^2 = -Id_{\mathbb{R}^4} \), \( uv = -vu = w, vw = -wv = u \), and \( wu = -uw = v \). We may identify \( u, v, \) and \( w \) with operators \( u_k, v_k, \) and \( w_k \) on each \( F_k \) using the identification to \( \mathbb{R}^4 \) via the canonical basis \( e_{4k-3}, \ldots, e_{4k} \) of \( F_k \). We then define linear operators \( U, V \) and \( W \) on \( c_{00} \) by, for all \( k \in \mathbb{N} \), \( U_{|F_k} = u_k \) (resp. \( V_{|F_k} = v_k, W_{|F_k} = w_k \)).

In particular it is clear that \( U^2 = V^2 = W^2 = -Id \) and that \( UV = -VU = W, VW = -WV = U, WU = -UW = V \), so that \( Id, U, V \) and \( W \) generate an algebra which is isomorphic to \( \mathbb{H} \).

Our space \( X = X(\mathbb{H}) \) is then defined inductively as the completion of \( c_{00} \) in the smallest norm satisfying the following equation:

\[
\|x\| = \|x\|_{c_0} \vee \sup\{f(n)^{-1} \sum_{i=1}^n \|E_ix\| : 2 \leq n, E_1 < \cdots < E_n\}
\]

\[
\vee \sup\{ \|x^*(Ex)\| : k \in K, x^* \in B_k^*(X), E \subset \mathbb{N}\} \vee \|Ux\| \vee \|Vx\| \vee \|Wx\|,
\]

where \( E, \) and \( E_1, \ldots, E_n \) are intervals of integers.

We may immediately observe that \( U, V \) and \( W \) extend to norm 1 operators on \( X \), and even isometries on \( X \), by the quaternionic relations between them. Note also that \( U, V \) and \( W \) commute with any interval projection, and that whenever \( x < y \) and \( T \in \{U, V, W\} \), we have \( Tx < Ty \). It follows that \( \|Tx\|_{(N)} = \|x\|_{(N)} \).
whenever \( N \geq 1 \) and \( T \in \{U,V,W\} \); when a sequence \( x_1, \ldots, x_N \) satisfies the RIS(\( \epsilon \)) condition, then so does \( T x_1, \ldots, T x_N \). The adjoints \( Id_{X^*}, W^*, V^*, U^* \), in this order, also satisfy the quaternionic commutation relations, the commutation with interval projections, and the relation with successive vectors. It follows that when \( x^* \in A^*_m \) for some \( m \in \mathbb{N} \) and \( T \in \{U,V,W\} \), we have \( T^* x^* \in A^*_m \). However, and this is fundamental, the sequence \( T^* x^*_1, \ldots, T^* x^*_N \) is not in general special when \( x^*_1, \ldots, x^*_N \) is a special sequence.

The next lemma is taken directly from [14].

**Lemma 11** For every \( K_0 \subset K \), there is a function \( g \in F \) such that \( f \geq g \geq f^{1/2} \), \( g(k) = f(k)^{1/2} \) whenever \( k \in K_0 \), and \( g(x) = f(x) \) whenever \( N \in J \setminus K_0 \) and \( x \) in the interval \([\log N, \exp N] \).

**Lemma 12** Let \( 0 < \epsilon \leq 1, 0 \leq \delta < 1 \), \( M \in L \) and let \( n \) and \( N \) be integers such that \( N/n \in [\log M, \exp M] \) and \( f(N) \leq (1+\delta)f(N/n) \). Assume that \( x_1, \ldots, x_N \) satisfies the RIS(\( \epsilon \)) condition and let \( x = x_1 + \ldots + x_N \). Then \( \| (f(N)/N)x \|_{(n)} \leq (1+\delta)(1+3\epsilon) \). In particular, if \( n = 1 \), we have \( \| (f(N)/N)x \| \leq 1 + 3\epsilon \).

**Proof:** We may reproduce the proof of Lemma 7 from [14], provided we show that if a vector \( Ex \) is such that \( \| Ex \| > 1 \), then it is normed by a \((k,g)\)-form, where \( g \) is given by Lemma 11 in the case \( K_0 = K \setminus \{k\} \). To see this, note that the only new case with respect to the classical Gowers-Maurey’s proof is if \( Ex \) was normed by some \( T_1^* E_1^* \ldots T_m^* E_m^* x^* \), with \( T_i \in \{U,V,W\} \) and \( E_i \) an interval projection, for all \( 1 \leq i \leq m \), and \( x^* \) a \((k,g)\)-form. By the commutation properties of \( U,V,W \), we may assume \( Ex \) is normed by \( T^* x^* \) with \( T \in \{U,V,W\} \) and \( x^* \) a \((k,g)\)-form. But in this case, \( T^* x^* \) is also a \((k,g)\)-form, since \( T^* \) is an isometry which respects successive vectors.

To reproduce the proof of Gowers and Maurey, after having added the isometries \( U, V \) and \( W \) in the definition of the norm, we shall need to distinguish the action of a functional \( x^* \) from the actions of \( U^* x^* \), \( V^* x^* \) and \( W^* x^* \). This is expressed by the next lemma.

**Lemma 13** Let \( x \) be a finitely supported vector in \( X \). Then there exists a functional \( x^* \) of norm at most 1 such that \( x^*(x) \geq 1/2 \| x \| \) and such that \( x^*(Ux) = x^*(Vx) = x^*(Wx) = 0 \).
Proof: We observe that for any reals $\alpha, \beta, \gamma$, the inverse of the operator $Id - \alpha U - \beta V - \gamma W$ is equal to $(1 + \alpha^2 + \beta^2 + \gamma^2)^{-1}(Id + \alpha U + \beta V + \gamma W)$. It follows that

$$
\| (Id - \alpha U - \beta V - \gamma W)^{-1} \| 
\leq \frac{1 + |\alpha| + |\beta| + |\gamma|}{1 + \alpha^2 + \beta^2 + \gamma^2} \leq \frac{3}{2}
$$

by elementary calculus. So for any $x \in c_{00}$,

$$
d(x, [Ux, Vx, Wx]) \geq \frac{2}{3} \| x \| .
$$

We conclude using the Hahn-Banach theorem. \(\square\)

Given $N \in L$, and $\delta > 0$, define a $\delta$-norming $N$-pair to be a pair $(x, x^*)$ defined as follows. Let $y_1, \ldots, y_N$ be a sequence satisfying the RIS(1) condition. Let $x = N^{-1}f(N)(y_1 + \ldots + y_N)$. Let, for $1 \leq i \leq N$, $y_i^*$ be a functional of norm at most 1, such that $\text{ran}(y_i^*) \subset \text{ran}(y_i)$ and $y_i^*(y_i) = \delta$. Let $x^* = f(N)^{-1}(y_1^* + \ldots + y_N^*)$. Note that if $(x, x^*)$ is a $\delta$-norming $N$-pair, then $x^*(x) = \delta$ and Lemma 12 implies that $\| x \|_{(\sqrt{N})} \leq 8$. By Lemma 10 and Hahn-Banach Theorem, $\delta$-norming $N$-pairs $(x, x^*)$ with arbitrary constant $\delta \leq 1/2$ exist with $x$ in an arbitrary block-subspace and $N$ arbitrary.

Proposition 14 The space $X(\mathbb{H})$ is hereditarily indecomposable.

Proof: Write $X = X(\mathbb{H})$. Let $Y$ and $Z$ be subspaces of $X$ and $\epsilon > 0$. We may assume that $Y$ and $Z$ are generated by successive vectors in $X$. Let $k \in K$ be such that $(\epsilon/72)\sqrt{k}/k^{1/2} > 1$. We construct sequences $x_1, \ldots, x_k$ and $x_1^*, \ldots, x_k^*$ as follows. Let $N_1 = j_{2k}$ and by Lemma 10 let $(x_1, x_1^*)$ be a 1/3-norming $N_1$-pair such that $x_1 \in Y$, with $\| x_1^*(Tx_1) \| < k^{-1}$ if $T = U, V$ or $W$: this is possible by Lemma 12 applied to each of the $N_1$ vectors forming $x_1$. Since we allow an error term $k^{-1}$, $x_1$ and the functional $x_1^*$ may be perturbed so that $x_1^*$ is in Q and $\sigma(x_1^*) < f^{-1}(4)$. In general, after the first $i - 1$ pairs were constructed, let $(x_i, x_i^*)$ be a 1/3-norming $N_i$-pair such that $x_i$ and $x_i^*$ are supported after $x_{i-1}$ and $x_{i-1}^*$, with $x_i \in Y$ if $i$ is odd and $x_i \in Z$ if $i$ is even, such that $\| x_i^*(Tx_i) \| < k^{-1}$ if $T = U, V$ or $W$, having perturbed $x_i^*$ in such a way that $N_{i+1} = \sigma(x_i^*, \ldots, x_k^*)$ satisfies $f(N_{i+1}) > 2^{i+1}$ and $\sqrt{f(N_{i+1})} > 2\| \text{ran}(x_i^*, \ldots, x_k^*) \|$. Now let $y = x_1 + x_3 + \ldots + x_{k-1}, z = x_2 + x_4 + \ldots + x_k$. Let also $x^* = f(k)^{-1/2}(x_1^* + \ldots + x_k^*)$. Our construction guarantees that $x^*$ is a special functional, and therefore of norm at most 1. It is also clear that $y \in Y, z \in Z$, and that

$$
\|y + z\| \geq x^*(y + z) \geq 1/3k f(k)^{-1/2}.
$$
Our aim is now to obtain an upper bound for $\|y - z\|$. Let $x = y - z$. Let $g$ be the function given by Lemma \[\text{III}\] in the case $K_0 = K \setminus \{k\}$. By the definition of the norm, all vectors $Ex$ are either normed by $(M, g)$-forms, by special functionals of length $k$, by images of such functionals by $U$, $V$ or $W$, or they have norm at most 1. In order to apply Lemma \[\text{VIII}\] it is enough to show that $|T^*z^*(Ex)| = |z^*(TEx)| \leq 1$ for any special functional $z^*$ of length $k$ in $K$, $E$ an interval, $T$ in the set $\{Id, U, V, W\}$. Let $z^* = f(k)^{-1/2}(z^*_{\ell} + \ldots + z^*_k)$ be such a functional with $z^*_i \in A^*_m$ for $1 \leq i \leq k$.

We evaluate $|z^*_i(ETx_i)|$ for $1 \leq i \leq k$ and $1 \leq i \leq k$. Recall that $T$ and $E$ commute.

Let $t$ be the largest integer such that $m_t = N_t$. Then $z^*_i = x^*_i$ for all $i < t$. There are at most two values of $i < t$ such that $x_i \neq Ex_i \neq 0$ or $z^*_i \neq Ez^*_i \neq 0$, and for them $|z^*_i(ETx_i)| \leq 1$. The values of $i < t$ for which $x_i = Ex_i$ and $z^*_i = Ez^*_i$ form an interval $e$ and satisfy $z^*_i(Tx_i) = x^*_i(x_i) = 1/3$ if $T = Id$, or $|z^*_i(Tx_i)| = |x^*_i(Tx_i)| \leq k^{-1}$, when $T = U$, $V$ or $W$. Therefore $|\sum_{i \in e} z^*_i(ET(-1)^ix_i)| \leq 1$.

When $i = l = t$, we obtain $|z^*_i(TEx_i)| \leq 1$.

If $l > t$ or $l \neq t$ then $z^*_i(Tx_i) = (T^*z^*_i)(x_i)$ and we have $T^*z^*_i \in A^*_{m_l}$ for some $m_l$. Moreover, because $\sigma$ is injective and by definition of $t$, $m_t \neq N_t$. If $m_t < N_t$, then by the remark after the definition of $N$-pairs, $\|x_i\|\sqrt{N_t} \leq 8$, so the lower bound of $j_{2k}$ for $m_1$ tells us that $|T^*z^*_i(x_i)| \leq k^{-2}$. If $m_t > N_t$ the same conclusion follows from Lemma \[\text{VIII}\] There are also at most two pairs $(i, l)$ for which $0 \neq z^*_i(ETx_i) \neq z^*_l(Tx_i)$, in which case $|z^*_i(ETx_i)| \leq 1$.

Putting these estimates together we obtain that

$$|z^*(TEx)| = |z^*(TE\sum_{i=1}^k (-1)^ix_i)| \leq f(k)^{-1/2}(2 + 1 + 1 + 2 + k^2k^{-2}) \leq 1.$$ 

We also know that $(1/8)(x_1, \ldots, x_k)$ satisfies the RIS(1) condition. Hence, by Lemma \[\text{VIII}\] $\|x\| \leq 24k f(k)^{-1} = 24k f(k)^{-1}$. It follows that $\|y - z\| \leq 72 f(k)^{-1/2} \|y + z\| \leq \epsilon \|y + z\|$. It follows that $Y$ and $Z$ do not form a direct sum and so $X$ is H.I..

We may also construct a complex version $X(\mathbb{C})$ of our space, with a 2-dimensional decomposition, and a canonical isometry $J$ satisfying $J^2 = -Id$ corresponding to a representation of the complex numbers as operators on each 2-dimensional summand. We leave the reader adapt our definitions and proofs to that case. Alternatively one could use the previous 4-dimensional decomposition.
setting and put only the operator $U$, instead of $U, V$ and $W$, in the definitions and the proofs. We therefore obtain:

**Proposition 15** The space $X(C)$ is hereditarily indecomposable.

### 4 Properties of operators on $X(H)$ and on $X(C)$

We now turn to the operator properties of our spaces $X(C)$ and $X(H)$. The quaternionic case is immediate from Theorem 5.

**Proposition 16** Let $X = X(H)$. Then the algebra $L(X)/S(X)$ is isomorphic to $H$. Furthermore, for any $Y \subset X$, $\dim L(Y, X)/S(Y, X) = 4$, i.e. every operator from $Y$ into $X$ is of the form $ai_{Y,X} + bU|Y + cV|Y + dW|Y + s$, where $a, b, c, d$ are reals and $s$ is strictly singular.

*Proof:* The operators $Id, U, V$, and $W$ generate a quaternionic division algebra, so $L(X)/S(X)$ is of dimension at least $4$. By Theorem 5 it is isomorphic to $H$, and furthermore, since $L(X)/S(X)$ embeds into $E_Y = L(Y, X)/S(Y, X)$, and $E_Y$ is of dimension at most $4$ for any $Y \subset X$ by [8], we deduce that $\dim E_Y = 4$ for any $Y \subset X$.

The complex case requires the following lemma, which is inspired by Lemma 4.14 from [2].

**Lemma 17** Let $X$ be a real H.I. space and $J$ be an operator on $X$ such that $J^2 = -Id$. Let $Y$ be a subspace of $X$. Let $T \in L(Y, X)$ be an operator which is not of the form $\lambda i_{Y,X} + \mu J|Y + s$, with $\lambda, \mu$ scalars and $s$ strictly singular. Then there exists a finite codimensional subspace $Z$ of $Y$ and some $\alpha > 0$ such that:

$$\forall z \in Z, d(Tz, [z, Jz]) \geq \alpha \|z\|.$$

*Proof:* Otherwise we may construct a normalized basic sequence $(y_n) \in Y$, and scalars sequences $(\lambda_n), (\mu_n)$ with for all $n \in \mathbb{N},$

$$\|T y_n - \lambda_n y_n - \mu_n J y_n\| \leq 2^{-n}.$$

It follows that for $C = 1 + \|T\|$, for all $n \in \mathbb{N},$

$$\|\lambda_n y_n + \mu_n J y_n\| \leq C.$$
We may assume for convenience that $J$ is isometric. We note that

$$(\lambda_n Id + \mu_n J)^{-1} = \frac{\lambda_n Id - \mu_n J}{\lambda_n^2 + \mu_n^2},$$

from which it follows that

$$1 = \|y_n\| \leq C \frac{\|\lambda_n Id - \mu_n J\|}{\lambda_n^2 + \mu_n^2},$$

so

$$\lambda_n^2 + \mu_n^2 \leq C(|\lambda_n| + |\mu_n|).$$

It follows immediately that $\max(|\lambda_n|, |\mu_n|) \leq 2C$. Thus we may assume that the sequences $(\lambda_n)$ and $(\mu_n)$ converge, and, passing to a subsequence, deduce that for some $\lambda, \mu$,

$$\|T y_n - \lambda y_n - \mu J y_n\| \leq 3.2^{-n}.$$  

From this it follows that the restriction of $T - \lambda i_{Y,X} - \mu J_{|Y}$ to the space generated by the basic sequence $(y_n)$ is compact, therefore strictly singular. By Lemma 6, we deduce that $T - \lambda i_{Y,X} - \mu J_{|Y}$ is strictly singular on $Y$, a contradiction. □

**Proposition 18** Let $X = X(\mathbb{C})$. Then the algebra $\mathcal{L}(X)/\mathcal{S}(X)$ is isomorphic to $\mathbb{C}$. Furthermore, for any $Y \subset X$, $\dim \mathcal{L}(Y, X)/\mathcal{S}(Y, X) = 2$, i.e. any operator from $Y$ into $X$ is of the form $\lambda i_{Y,X} + \mu J_{|Y} + s$, where $\lambda, \mu$ are reals and $s$ is strictly singular.

**Proof**: Let $Y$ be a subspace of $X$. The operator $J_{|Y}$ is not of the form $\lambda i_{Y,X} + s$, $s$ strictly singular, otherwise by the H.I. property, Lemma 6 $J - \lambda Id$ would be strictly singular, which would contradict the fact that $J^2 = -Id$. It follows that $\dim \mathcal{L}(Y, X)/\mathcal{S}(Y, X) \geq 2$.

We assume $\dim \mathcal{L}(Y, X)/\mathcal{S}(Y, X) > 2$ and look for a contradiction. Let $T \in \mathcal{L}(Y, X)$ which is not of the form $\lambda i_{Y,X} + \mu J_{|Y} + s$ and assume without loss of generality that $\|T\| \leq 1$. Then by Lemma 17 we may find some $\alpha > 0$ and some subspace $Z$ of $Y$ such that for all $z \in Z$,

$$d(Tz, [z, Jz]) \geq \alpha \|z\|.$$  

We may assume that $Z$ is generated by successive vectors with respect to the 2-dimensional decomposition of $X$. 16
We fix a sequence \((y_n)\) in \(Z\) such that for all \(n, y_{n+1}\) and \(T y_{n+1}\) are supported after \(y_n\) and \(T y_n\), \(\|y_n\|_{(n)} \leq 1\) while \(\|y_n\| \geq 1/2\). Let \(k \in K\) and construct sequences \(x_1, \ldots, x_k\) and \(x_1^*, \ldots, x_k^*\) as follows. Let \(N_1 = j_{2k}\). Let \(x_1 = N_1^{-1} f(N_1(y_{n_1} + \ldots + y_{n_{N_1}}))\) where \(y_{n_i}\) is a subsequence satisfying the RIS(1) condition. Then \(T x_1 = N_1^{-1} f(N_1(T y_{n_1} + \ldots + T y_{n_{N_1}}))\), where the sequence \((T y_{n_i})\) satisfies the RIS(1) condition as well. We let \(N_2 = j_{2k}\). Let \(x_1^* = N_2^{-1} f(N_2(x_1) + \ldots + x_{k})\) be associated to \(T x_1\) so that \((x_1^*, T x_1)\) is an \(\alpha/2\)-norming \(N_1\)-pair, and such that \(|x_1^*(x_1)| < k^{-1}\) and \(|x_1^*(J x_1)| < k^{-1}\) (to find \(y_{n_i}\)'s realizing this, apply the inequality from Lemma 17 to each \(y_{n_i}\), with Hahn-Banach theorem). Lemma 12 implies that \(\|T x_1\|_{(\sqrt{N_1})} \leq 8\). Repeating this, and up to perturbations, we build, for \(1 \leq i \leq k\), \(x_i \in Z\) and \(x_i^*\) so that \((x_i^*, T x_i)\) is an \(\alpha/2\)-norming \(N_i\)-pair, and such that \(x_1^*, \ldots, x_k^*\) is a special sequence. We let \(x = x_1 + \ldots + x_k\) and \(x^* = f(k)^{-1/2}(x_1^* + \ldots + x_k^*)\). We therefore have

\[
\|x\| \geq \|T x\| \geq x^*(T x) \geq (\alpha/2) k f(k)^{-1/2}.
\]

We now use Lemma 9 to obtain an upper estimate for \(\|x\|\). Let \(g\) be the function given by Lemma 11 in the case \(K_0 = K \setminus \{k\}\). By the definition of the norm, all vectors \(Ex\) are either normed by \((M, g)\)-forms, by special functionals of length \(k\), by images of such functionals by \(J\), or they have norm at most 1. In order to apply Lemma 9, it is enough to show that \(|z^*(Ex)| \leq 1\) and \(|J^* z^*(Ex)| = |z^*(JEx)| \leq 1\) for any special functional \(z^*\) of length \(k\) in \(K\) and any interval \(E\). Let \(z^* = f(k)^{-1/2}(z_1^* + \ldots + z_k^*)\) be such a functional with \(z_i^* \in A_{m_i}\).

We evaluate \(|z_i^*(Ex_i)|\) and \(|z_i^*(EJx_i)|\) for \(l \leq k\) and \(i \leq l\).

Let \(t\) be the largest integer such that \(m_t = N_t\). Then \(z_i^* = x_i^*\) for all \(i < t\). There are at most two values of \(i < t\) such that \(x_i \neq Ex_i \neq 0\) or \(z_i^* \neq Ez_i^* \neq 0\), and for them \(|z_i^*(Ex_i)| \leq 1\) and \(|z_i^*(JEx_i)| \leq 1\). The values of \(i < t\) for which \(x_i = Ex_i\) and \(z_i^* = Ez_i^*\) give \(|z_i^*(Ex_i)| = |x_i^*(x_i)| < k^{-1}\) and \(|z_i^*(EJx_i)| = |x_i^*(Jx_i)| < k^{-1}\).

When \(i = l = t\), we obtain \(|z_t^*(Ex_t)| \leq 1\) and \(|z_t^*(EJx_t)| \leq 1\).

If \(i = t \neq l\) then \(z_t^*(Jx_i) = (J^* z_t^*)(x_i)\) and we have as before \(J^* z_t^* \in A_{m_t}\) for some \(m_t \neq N_t\). If \(m_t < N_t\), then as we remarked above, \(\|x_i\|_{(\sqrt{N_t})} \leq 8\), so the lower bound of \(j_{2k}\) for \(m_t\) tells us that \(|J^* z_t^*(x_i)| \leq k^{-2}\). If \(m_t > N_t\) the same conclusion follows from Lemma 8. There are also at most two pairs \((i, l)\) for which \(0 \neq z_i^*(EJx_i) \neq z_i^*(Jx_i)\), in which case \(|z_i^*(EJx_i)| \leq 1\). The same proof holds for \(|z_i^*(Ex_i)|\).

Putting these estimates together we obtain that

\[
|z^*(Ex)| \vee |z^*(EJx)| \leq f(k)^{-1/2}(2 + k.k^{-1} + 1 + 2 + k^2.k^{-2}) \leq 1.
\]
We also know that \((1/8)(x_1, \ldots, x_k)\) satisfies the RIS(1) condition. Hence, by Lemma\(^8\) \(\|x\| \leq 24kg(k)^{-1} = 24kf(k)^{-1}\).

Finally we deduce that \(\alpha\sqrt{f(k)} \leq 48\), a contradiction, since \(k\) was arbitrary in \(K\). We conclude that \(\dim L(Y, X)/S(Y, X) = 2\) for any \(Y \subset X\), and that \(L(X)/S(X)\) is isomorphic to \(\mathbb{C}\). \(\square\)

5 Complex structures on \(X(\mathbb{C})\) and \(X(\mathbb{H})\)

Given \(X\) a real Banach space, let \(\mathcal{GL}(X)\) denote the group of invertible operators on \(X\), and let \(\mathcal{I}(X)\) denote the subset of operators \(I\) on \(X\) such that \(I^2 = -Id\).

Lemma 19 Let \(X\) be a real Banach space. Let \(I \in \mathcal{I}(X)\), and let \(S \in S(X)\) be such that \(I + S \in \mathcal{I}(X)\). Then the complex structures \(X^I\) and \(X^{I+S}\) associated to \(I\) and \(I+S\) respectively are isomorphic.

Proof: Write \(T = I + S\). From \(T^2 = -Id\), we deduce \(S^2 = -IS - SI\). Now let \(\alpha = Id - (SI)/2\). This is an \(\mathbb{R}\)-linear map on \(X\). Moreover, it is easy to check, using the relation satisfied by \(S^2\), that

\[
\alpha I = I + (S/2) = (I + S)\alpha = T\alpha.
\]

This relation ensures that \(\alpha\) may be seen as a \(\mathbb{C}\)-linear operator from \(X^I\) into \(X^T\).

Furthermore, by properties of strictly singular operators (see e.g. [16]), the \(\mathbb{R}\)-linear operator \(\alpha\) is Fredholm of index 0 on \(X\) as a strictly singular perturbation of \(Id\). This means that \(\alpha(X)\) is closed, and that

\[
\dim_{\mathbb{R}} Ker\alpha = \dim_{\mathbb{R}}(X/\alpha(X)) < +\infty.
\]

This is also true when \(\alpha\) is seen as \(\mathbb{C}\)-linear (note in particular that \(Ker\alpha\) is \(I\)-stable and \(\alpha(X)\) is \(T\)-stable). That is, \(\alpha(X^I)\) is closed in \(X^T\), and

\[
\dim_{\mathbb{C}} Ker\alpha = \dim_{\mathbb{C}}(X^T/\alpha(X^I)) < +\infty.
\]

Therefore \(\alpha\) is \(\mathbb{C}\)-Fredholm with index 0, i.e. there exist \(\mathbb{C}\)-linear decompositions \(X^I = X_0 \oplus F_0\) and \(X^T = Y_0 \oplus G_0\), with \(\dim_{\mathbb{C}} F_0 = \dim_{\mathbb{C}} G_0 < +\infty\), such that the restriction of \(\alpha\) to \(X_0\) is a \(\mathbb{C}\)-linear isomorphism onto \(Y_0\). Since \(F_0\) and \(G_0\) are isomorphic, we deduce that there exists a \(\mathbb{C}\)-linear isomorphism from \(X^I\) onto \(X^T\). \(\square\)
Let $\pi$ denote the quotient map from $\mathcal{L}(X)$ onto $\mathcal{L}(X)/S(X)$. We also let $(\mathcal{L}(X)/S(X))_0$ denote the group $\pi(\mathcal{GL}(X))$, and $\tilde{I}(X)$ denote the set of elements of $(\mathcal{L}(X)/S(X))_0$ whose square is equal to $-\pi(Id)$. In the following we shall identify a complex structure on $X$ with the associated operator $I \in \tilde{I}(X)$.

**Proposition 20** Let $X$ be a real Banach space. Then the quotient map $\pi$ induces an injective map $\tilde{\pi}$ from the set of isomorphism classes of complex structures on $X$ into the set of conjugation classes of elements of $\tilde{I}(X)$ for the group $(\mathcal{L}(X)/S(X))_0$. The image of $\tilde{\pi}$ is the set of conjugation classes of elements of $\tilde{I}(X)$ which may be lifted to an element of $I(X)$. If $S(X)$ admits a supplement in $\mathcal{L}(X)$ which is a subalgebra of $\mathcal{L}(X)$, then $\tilde{\pi}$ is bijective.

**Proof:** For any operator $T$ on $X$, we write $\tilde{T} = \pi(T)$. Let $I$ and $T$ be operators in $\tilde{I}(X)$. If $\alpha$ is a $\mathbb{C}$-linear isomorphism from $X'$ onto $X$, then the $\mathbb{C}$-linearity means that $\alpha I = T \alpha$. Therefore $\tilde{\alpha} \tilde{I} = \tilde{T} \tilde{\alpha}$, and $\tilde{I}$ and $\tilde{T}$ satisfy a conjugation relation. Conversely, if $\tilde{I} = \tilde{\alpha}^{-1} \tilde{T} \tilde{\alpha}$ for some $\alpha \in \mathcal{GL}(X)$, then $\tilde{\alpha}^{-1} T \alpha = I + S$, where $S$ is strictly singular. Note that $(I + S)^2 = -Id$, and since $T \alpha = \alpha(I + S)$, $\alpha$ is a $\mathbb{C}$-linear isomorphism from $X^{I+S}$ onto $X$. By Lemma[19] it follows that $X'$ and $X$ are isomorphic. This proves that $\tilde{\pi}$ is well-defined and injective.

If $\mathcal{H}(X)$ is a subalgebra of $\mathcal{L}(X)$ which supplements $S(X)$, then let $T \in \mathcal{L}(X)$ be such that $\tilde{T}^2 = -\tilde{I}d$; we may assume that $T$ (and therefore $T^2$) belongs to $\mathcal{H}(X)$. Then since $T^2 = -Id + S$, $S$ strictly singular, $T^2$ must be equal to $-Id$.

Any class $\tilde{T} \in \tilde{I}(X)$ may therefore be lifted to an element of $I(X)$. \qed

We recall that $X^J(\mathbb{C})$ denotes the complex structure on $X(\mathbb{C})$ associated to the operator $J$, and that two Banach spaces $X$ and $X'$ are said to be totally incomparable if no subspace of $X$ is isomorphic to a subspace of $X'$.

**Proposition 21** The space $X^J(\mathbb{C})$ and its conjugate $X^{-J}(\mathbb{C})$ are complex H.I. and totally incomparable. Moreover, any complex structure on $X(\mathbb{C})$ is isomorphic either to $X^J(\mathbb{C})$ or to $X^{-J}(\mathbb{C})$.

**Proof:** Any complex structure on $X(\mathbb{C})$ is H.I., since $X(\mathbb{C})$ is real H.I.. We have $\mathcal{L}(X(\mathbb{C})) = [Id, J] \oplus S(X(\mathbb{C}))$, and

$$(\mathcal{L}(X(\mathbb{C}))/S(X(\mathbb{C})))_0 \simeq \mathbb{C}^*.$$
The only two elements of \( \mathbb{C} \) of square \(-1\) are \( i \) and \(-i\). By Proposition 20, it follows that \( X^J(\mathbb{C}) \) and \( X^{-J}(\mathbb{C}) \) are the only two complex structures on \( X(\mathbb{C}) \) up to isomorphism.

Assume now \( \alpha \) is a \( \mathbb{C} \)-linear map from a \( \mathbb{C} \)-linear subspace \( Y \) of \( X^J(\mathbb{C}) \) into \( X^{-J}(\mathbb{C}) \). This is in particular an \( \mathbb{R} \)-linear map from \( Y \) into \( X(\mathbb{C}) \). So by Proposition 18, \( \alpha = aId_{|Y} + bJ_{|Y} + s \), where \( s \) is strictly singular. The fact that \( \alpha \) is \( \mathbb{C} \)-linear means that \( \alpha J_{|Y} = -J \alpha \). This implies by ideal properties of strictly singular operators that \( aId_{|Y} + bJ_{|Y} \) is strictly singular and therefore, \( a = b = 0 \). It follows that \( \alpha \) is \( \mathbb{R} \)-strictly singular and thus \( \mathbb{C} \)-strictly singular. Therefore \( X^J(\mathbb{C}) \) and \( X^{-J}(\mathbb{C}) \) are totally incomparable. \( \square \)

Following [13], we shall say that an operator \( W \in L(Y, Z) \) is finitely singular if the restriction of \( W \) to some finite-codimensional subspace of \( Y \) is an isomorphism into \( Z \). This means that \( WY \) is closed and that the Fredholm index \( i(W) \) is defined with values in \( \mathbb{Z} \cup \{ -\infty \} \), where

\[
i(W) = \dim(Ker(W)) - \dim(Z/WY).
\]

The next proposition and corollaries show that the example of \( X(\mathbb{C}) \) is essentially the only one to ensure the total incomparability property.

**Proposition 22** Let \( X \) be a real Banach space, \( T, U \in \mathcal{I}(X) \). Then \( X^T \) is isomorphic to \( X^U \), or \( T - U \) induces a \( \mathbb{C} \)-linear isomorphism from a subspace of \( X^T \) onto a subspace of \( X^{-U} \).

**Proof:** We note that \( (T + U)T = U(T + U) \), therefore \( T + U \) is \( \mathbb{C} \)-linear from \( X^T \) into \( X^U \). The similar result holds for \( T - U \) between \( X^T \) and \( X^{-U} \).

Assume \( T - U \) does not induce a \( \mathbb{C} \)-linear isomorphism from a subspace of \( X^T \) onto a subspace of \( X^{-U} \). Then \( T - U \) is strictly singular on \( X^T \), and we intend to deduce that \( T + U \) is "essentially" an isomorphism from \( X^T \) onto \( X^U \).

First we note that \( T + U \) is finitely singular on \( X \). To see this it is enough by definition to prove that \( T + U \) is finitely singular as a \( \mathbb{C} \)-linear operator from \( X^T \) into \( X^U \). If this were false, then by [16] Proposition 2.c.4, we could find a \( \mathbb{C} \)-linear infinite dimensional subspace \( Y \) of \( X^T \) such that \( \|(T + U)y\| < \|T\|^{-1} \). Since \( T - U \) is strictly singular on \( X^T \), we would find a norm 1 vector \( y \) in \( Y \) with \( \|(T - U)y\| < \|T\|^{-1} \). We would then deduce that

\[
\|Ty\| \leq 2^{-1}(\|(T + U)y\| + \|(T - U)y\|) < \|T\|^{-1} \|y\|,
\]
a contradiction.

We now prove that whenever \( \lambda \in [0, 1] \), the operator \( T + \lambda U \in \mathcal{L}(X) \) is finitely singular. Assume on the contrary that \( T + \lambda U \) is not finitely singular for some \( \lambda \in [0, 1] \). Let \( c = (1 - \lambda^2)(2(1 + \|T\| + 2\lambda\|U\|))^{-1} \) and take an arbitrary \( 0 < \epsilon < c \). As before there exists some infinite-dimensional (\( \mathbb{R} \)-linear) subspace \( Y \) of \( X \) such that \( \|(T + \lambda U)_Y\| < \epsilon \). Therefore, for all \( y \in Y \),

\[
(1) \quad \|Ty + \lambda Uy\| \leq \epsilon \|y\| \leq c\|y\|
\]

and by composing by \( U \),

\[
(2) \quad \|UTy - \lambda y\| \leq \epsilon \|U\| \|y\| \leq c\|U\| \|y\|.
\]

We deduce from this that \( Y \) and \( TY \) form a direct sum. Indeed, if there existed norm 1 vectors \( y \) and \( z \) in \( Y \) with \( \|z - Ty\| \leq c \), then we would have, by (1), \( \|z + \lambda Uy\| \leq 2c \), therefore

\[
(3) \quad \|Uz - \lambda y\| \leq 2c\|U\|,
\]

but also \( \|Tz + y\| \leq c\|T\| \), so by (1) again,

\[
(4) \quad \|y - \lambda Uz\| \leq c(1 + \|T\|).
\]

From (3) and (4), we would get

\[ 1 - \lambda^2 = \|(1 - \lambda^2)y\| \leq c(1 + \|T\| + 2\lambda\|U\|), \]

a contradiction by choice of \( c \). Therefore \( Y + TY \) forms a direct sum with projection constants depending only on \( \lambda \), \( \|T\| \) and \( \|U\| \).

Now since \( T - U \) is \( \mathbb{C} \)-strictly singular from \( X^T \) into \( X^{-U} \), and \( Y \oplus TY \) is a \( \mathbb{C} \)-linear subspace of \( X^T \), there exist \( y, z \in Y \) with \( \|y + Tz\| = 1 \) and \( \|(T - U)(y + Tz)\| \leq \epsilon \), which means \( \|-z - UTy - Uy\| \leq \epsilon \). By (1) and (2), and the fact that \( Y \oplus TY \) is direct, we deduce

\[ \|-z - \lambda z + Ty + \lambda^{-1}Ty\| \leq \epsilon + \epsilon \|U\| \|z\| + \epsilon \lambda^{-1} \|y\| \leq C\epsilon, \]

where \( C \) depends only on \( \lambda \), \( \|T\| \) and \( \|U\| \). In the same way,

\[ \|-z - \lambda z + Ty + \lambda^{-1}Ty\| \geq C'((1 + \lambda) \|z\| + (1 + \lambda^{-1}) \|Ty\|) \geq C'', \]

21
where again $C''$ depends only on $\lambda$, $\|T\|$ and $\|U\|$. As $\epsilon$ was arbitrary, we obtain a contradiction.

We have therefore proved that $T + \lambda U$ is finitely singular whenever $\lambda \in ]0, 1]$, and this is obvious for $\lambda = 0$. In other words, the Fredholm index of $T + \lambda U$ is defined for all $\lambda \in [0, 1]$. By continuity of the Fredholm index, [16] Proposition 2.c.9, we deduce that $\text{ind}(T + U) = \text{ind}(T) = 0$, since $T$ is an isomorphism. Therefore $T + U$ is Fredholm with index 0. It is therefore also Fredholm with index 0 as a $\mathbb{C}$-linear operator from $X^T$ into $X^U$, and we deduce that $X^T$ and $X^U$ are isomorphic.

**Corollary 23** Let $X$ be a real Banach space with two totally incomparable complex structures. Then these complex structures are conjugate up to isomorphism and both saturated with H.I. subspaces.

**Proof:** Assume $X^T$ is totally incomparable with $X^U$. By Proposition 22 (applied to $U$ and $-T$), $X^U$ is isomorphic to $X^{-T}$. To show that $X^T$ is HI-saturated, it is enough by Gowers’ dichotomy theorem to prove that $X^T$ does not contain a subspace with an unconditional basis. Indeed, if $Y$ were such a subspace, then by the remark in the introduction, $Y$ would be isomorphic to $\overline{Y}$, $\mathbb{C}$-linear subspace of $X^{-T}$, which would contradict the total incomparability of $X^T$ with $X^{-T}$. □

**Corollary 24** There cannot exist more than two mutually totally incomparable complex structures on a Banach space.

Note that $X^J(\mathbb{C}) \oplus X^{-J}(\mathbb{C})$ is a non H.I. space which is totally incomparable with its conjugate $X^{-J}(\mathbb{C}) \oplus X^J(\mathbb{C})$. Therefore there is no obvious direction in which to improve Corollary 23.

G. Godefroy asked the author whether there also existed a Banach space with exactly three complex structures. The answers turns out to be yes.

**Proposition 25** For any $n \in \mathbb{N}$, the space $X(\mathbb{C})^n$ has exactly $n + 1$ complex structures up to isomorphism.

**Proof:** We have that $\mathcal{L}(X(\mathbb{C})^n) \simeq \mathcal{M}_n(\mathbb{C}) \oplus \mathcal{S}(X(\mathbb{C})^n)$, and

$$(\mathcal{L}(X(\mathbb{C})^n)/\mathcal{S}(X(\mathbb{C})^n))_0 \simeq \mathcal{G}\mathcal{L}_n(\mathbb{C}).$$
Any complex \((n, n)\)-matrix whose square is equal to \(-\text{Id}\) has minimal polynomial \(X^2 + 1\), and is therefore similar to a diagonal matrix with \(i\) or \(-i\)'s down the diagonal; and there are \(n + 1\) similarity classes of such matrices according to the number of \(i\)'s. Therefore by Proposition 20 there are \(n + 1\) complex structures on \(X(\mathbb{C})^n\), up to isomorphism. If we let \(X = X^J(\mathbb{C})\), these structures are isomorphic to the spaces \(X^k \oplus \overline{X}^{n-k}\), \(0 \leq k \leq n\). □

We now turn to the complex structures on \(X(\mathbb{H})\). Note that since \(U^2 = -\text{Id}\), the real space \(X(\mathbb{H})\) may be equipped with a complex structure associated to \(U\). Let \(X = X^U(\mathbb{H})\) denote this complex space. As it is a real H.I. space, it is H.I. as a complex space.

Any \(\mathbb{R}\)-linear operator \(T\) on \(X\) is of the form \(a\text{Id} + bU + cV + dW + S\), where \(S\) is strictly singular. Saying that \(T\) is \(\mathbb{C}\)-linear means that \(T\) commutes with \(U\), which implies that \(c = d = 0\). Therefore we deduce that any \(\mathbb{C}\)-linear operator on \(X\) is of the form \(a\text{Id} + bU + S = (a + ib)\text{Id} + S\), as expected since \(X\) is H.I.!

(here we used that any \(\mathbb{R}\)-strictly singular operator on \(X\) is \(\mathbb{C}\)-strictly singular).

S. Szarek asked whether there exists a Banach space not isomorphic to a Hilbert space, with unique complex structure ([20], Pb 7.2). The following proposition answers this question by the positive.

**Proposition 26** The space \(X(\mathbb{H})\) admits a unique complex structure up to isomorphism.

**Proof:** We may write \(\mathcal{L}(X(\mathbb{H})) = [\text{Id}, U, V, W] \oplus \mathcal{S}(X(\mathbb{H}))\), and

\[
(\mathcal{L}(X(\mathbb{H}))/\mathcal{S}(X(\mathbb{H})))_0 \simeq \mathbb{H}^*.
\]

Write the generators of \(\mathbb{H}\) as \(\{1, i, j, k\}\). The elements of \(\mathbb{H}\) of square \(-1\) are of the form \(r = bi + cj + dk\) with \(b^2 + c^2 + d^2 = 1\). Any element \(r\) of this form is in the conjugation class of \(i\), since \(i(i + r) = (i + r)i\), for \(r \neq -i\), and \(ij = -ji\), for \(r = -i\). Therefore by Proposition 20 all complex structures on \(X(\mathbb{H})\) are isomorphic. □

### 6 Some other spaces with unique complex structure

We show how to construct various other examples of Banach spaces with a unique complex structure. Note that non H.I. examples of real spaces on which operators
are of the form $\lambda I + S$, $S$ strictly singular, may be found e.g. in [4], [9].

**Proposition 27** Let $\{X_i, 1 \leq i \leq N\}$ be a family of pairwise totally incomparable real Banach spaces with the $\lambda I + S$-property. For $1 \leq i \leq N$, let $n_i \in \mathbb{N}$. Then $\sum_{1 \leq i \leq N} \oplus X_i^{2n_i}$ has a unique complex structure up to isomorphism.

**Proof:** If $X$ has the $\lambda I + S$-property, and $n \in \mathbb{N}$, let $\mathcal{M}_{2n}(Id_X)$ be the space of $(2n, 2n)$-matrix operators on $X^{2n}$ with homothetic coefficients, which we identify with the space $\mathcal{M}_{2n}$ of real $(2n, 2n)$-matrices. Let $\mathcal{G}\mathcal{L}_{2n}$ denote the group of invertible real $(2n, 2n)$-matrices. We have that $\mathcal{L}(X^{2n}) = \mathcal{M}_{2n}(Id_X) \oplus S(X^{2n})$ and

$$(\mathcal{L}(X^{2n})/S(X^{2n}))_0 \simeq \mathcal{G}\mathcal{L}_{2n}.$$ 

Now any real $(2n, 2n)$-matrix whose square is $-Id_{2n}$ is diagonalizable with $\mathbb{C}$-eigenvalues $i$ and $-i$, each with multiplicity $n$. So any two such matrices are $\mathbb{C}$-similar and therefore $\mathbb{R}$-similar. By Proposition 20 it follows that all complex structures on $X^{2n}$ are isomorphic.

If $X$ is a direct sum $\sum_{1 \leq i \leq N} \oplus X_i^{2n_i}$, where the $X_i$’s are pairwise totally incomparable, then as $\mathcal{L}(X_i, X_j) = S(X_i, X_j)$ whenever $i \neq j$, we have

$$\mathcal{L}(X) \simeq \left( \sum_{1 \leq i \leq N} \oplus \mathcal{M}_{2n_i}(Id_{X_i}) \right) \oplus S(X),$$

and

$$(\mathcal{L}(X)/S(X))_0 \simeq \Pi_{1 \leq i \leq N} G\mathcal{L}_{2n_i}.$$ 

It follows immediately from the case $N = 1$ that there is a unique conjugacy class of elements of square $-1$ for the group $\Pi_{1 \leq i \leq N} G\mathcal{L}_{2n_i}$. So all complex structures on $X$ are isomorphic. \qed

All the examples considered so far fail to have an unconditional basis. Indeed for each of them the quotient algebra $\mathcal{L}(X)/S(X)$ is finite dimensional. We now show how to construct a real Banach space $X(\mathbb{D}_2)$ with an unconditional basis, not isomorphic to $l_2$, and with unique complex structure (here $\mathbb{D}_2$ stands for ”2-block diagonal”). This is exactly the unconditional version of $X(\mathbb{C})$, precisely in the same way as the space of Gowers [11], on which every operator is the sum of a diagonal and a strictly singular operator, is the unconditional version of Gowers-Maurey’s space $X_{GM}$. In other words, it is a space with ”as few” operators as possible to ensure the existence of an unconditional basis and of a complex
structure. It is not difficult to show that the unconditionality of a 2-dimensional decomposition, and the existence of a map $J$ such that $J^2 = -Id$ defined on each 2-dimensional summand, already imply that any 2-block diagonal operator associated to bounded $(2, 2)$-matrices must be bounded. This will motivate the following definition of $X(D_2)$. We thank B. Maurey for a discussion which clarified this example.

The basis $(e_i)$ is as before the natural basis of $c_{00}(\mathbb{R})$. For $k \in \mathbb{N}$, let $F_k = [e_{2k-1}, e_{2k}]$. Notions of successivity are taken with respect to the 2-dimensional decomposition associated to the $F_k$’s. Let $D_2(c_{00})$ denote the space of 2-block diagonal operators on $c_{00}$, i.e. the space of operators $T$ on $c_{00}$ such that $T(F_k) \subset F_k$ for all $k \in \mathbb{N}$. Any operator in $D_2(c_{00})$ corresponds to a sequence $(M_n) \in \mathcal{M}_2^\mathbb{N}$ of real $(2, 2)$-matrices, and will be denoted $D(M_n)$. For $M \in \mathcal{M}_2$, we shall consider the norm $\|M\|$, when $M$ is seen as an operator on $l_\infty^2$, or sometimes $\|M\|_2$ (resp. $\|M\|_1$), the euclidean norm (resp. the $l_1$-norm) on $\mathcal{M}_2$ identified with $\mathbb{R}^4$.

The space $X(D_2)$ is defined inductively as the completion of $c_{00}$ in the smallest norm satisfying the following equation:

$$
\|x\| = \|x\|_c \vee \sup\{f(n)^{-1} \sum_{i=1}^{n} \|E_i x\| : 2 \leq n, E_1 < \cdots < E_n\} \\
\vee \sup\{|x^*(Ex)| : k \in K, x^* \in B_k^*(X), E \subset \mathbb{N}\} \\
\vee \sup\{\|D(M_n)x\| : \forall n \in \mathbb{N}, \|M_n\| \leq 1\},
$$

where $E$, and $E_1, \ldots, E_n$ are intervals of integers, and $(M_n)$ is a sequence of $(2, 2)$-matrices.

From the definition we observe immediately that any 2-block diagonal operator $D(M_n)$, where the sequence $(M_n)$ is bounded, extends to a bounded operator on $X(D_2)$. The space of such operators will be denoted $D_2(X(D_2))$. Furthermore, the norm on each $F_k$ is the $l_\infty$-norm; and whenever $n \in \mathbb{N}$, and $y_k, z_k$ belong to $F_k$, with $\|z_k\| \leq \|y_k\|$ for all $1 \leq k \leq n$, it follows that

$$
\left| \sum_{k=1}^{n} z_k \right| \leq \left| \sum_{k=1}^{n} y_k \right| .
$$
This is a strong unconditionality property of \(X(\mathbb{D}_2)\) from which we deduce immediately that \((e_i)_{i \in \mathbb{N}}\) is an unconditional basis for \(X(\mathbb{D}_2)\).

**Proposition 28** Any operator on \(X(\mathbb{D}_2)\) may be written \(D + S\), where \(D \in \mathcal{D}_2(X(\mathbb{D}_2))\) is 2-block diagonal and \(S \in \mathcal{S}(X(\mathbb{D}_2))\) is strictly singular.

**Proof:** We sketch how to reproduce a proof from [14]. Let \(T\) be an operator on \(X(\mathbb{D}_2)\) with 0’s down the 2-block diagonal. First we show that if \((x_n)\) is a sequence of successive vectors such that \(\|x_n\|_{(n)} \leq 1\), \(A_n = \text{supp}(x_n)\) and for each \(n\), \(B_n \cup C_n\) is a partition of \(A_n\) in two subsets, then \(C_n TB_n x_n\) converges to 0 (see [14] Lemma 27). The proof is based on a construction of a special sequence in the “usual” way, similar to our proofs for \(X(\mathbb{C})\) and \(X(\mathbb{H})\). Arbitrary choices of signs \(-1\) or 1 in the proof of [14] correspond to arbitrary choices of norm 1 \((2,2)\)-matrices in our case. If \(D = D(M_n) \in \mathcal{D}_2(X)\), with \(\|M_n\| \leq 1\) for all \(n\), then \(D\) preserves successive vectors and \(\|D\| \leq 1\). Therefore the estimates of the end of [14] Lemma 27 based on properties of R.I.S. vectors and \((M, g)\)-forms are still valid. The other argument based on disjointness of supports of \(y_n = B_n x_n\) and \(z_n = C_n T B_n x_n\) is immediately seen to be preserved as well. Corollary 28 from [14] may then be reproduced to obtain that \((Tx_n)\) converges to 0.

Finally if \(T\) is an operator on \(X\), and \(\text{diag}(T)\) is its 2-block diagonal part, then \(((T - \text{diag}(T))x_n)\) converges to 0 whenever \((x_n)\) is a successive sequence such that \(\|x_n\|_{(n)} \leq 1\). By Lemma 10 this implies that \(T - \text{diag}(T)\) is strictly singular.

We recall that \((\mathcal{L}(X(\mathbb{D}_2))/\mathcal{S}(X(\mathbb{D}_2)))_0\) is defined as the group of elements of \(\mathcal{L}(X(\mathbb{D}_2))/\mathcal{S}(X(\mathbb{D}_2))\) which may be lifted to an invertible operator. Likewise, \((l_\infty(\mathcal{M}_2)/c_0(\mathcal{M}_2))_0\) is the group of elements of \(l_\infty(\mathcal{M}_2)/c_0(\mathcal{M}_2)\) which may be lifted to an invertible element of \(l_\infty(\mathcal{M}_2)\), that is, to a sequence \((M_n) \in l_\infty(\mathcal{M}_2)\) of real \((2,2)\) matrices such that \(M_n\) is invertible for each \(n\) and the sequence \((M_n^{-1})\) is bounded.

**Lemma 29** The algebras \(\mathcal{L}(X(\mathbb{D}_2))/\mathcal{S}(X(\mathbb{D}_2))\) and \(l_\infty(\mathcal{M}_2)/c_0(\mathcal{M}_2)\) are isomorphic, and the groups \((\mathcal{L}(X(\mathbb{D}_2))/\mathcal{S}(X(\mathbb{D}_2)))_0\) and \((l_\infty(\mathcal{M}_2)/c_0(\mathcal{M}_2))_0\) are isomorphic.

**Proof:** Write \(X = X(\mathbb{D}_2)\). We note that \(\mathcal{D}_2(X) \cap \mathcal{S}(X)\) is equal to the set \(\{D(M_n) : \lim_{n \to +\infty} M_n = 0\}\). Indeed if \((M_n)\) converges to 0, fix \(\epsilon > 0\) and \(N\)
such that $\|M_n\| \leq \epsilon$ for all $n \geq N$. Let $Y = [F_n]_{n \geq N}$. Any $y \in Y$ may be written $y = \sum_{n \geq N} y_n$, $y_n \in F_n$, and therefore

$$
\|D(M_n)(y)\| = \left\| \sum_{n \geq N} M_n y_n \right\| \leq \epsilon \left\| \sum_{n \geq N} y_n \right\| = \epsilon \|y\|,
$$

by the strong unconditionality properties of the basis. This implies that $D(M_n)$ is compact and therefore strictly singular. Conversely, if $\|M_n\|$ does not converge to 0 then for some $\alpha > 0$ and some infinite set $N$, $\|M_n\| \geq \alpha$ for any $n \in N$, and let $x_n \in F_n$ be a norm 1 vector such that $\|M_n x_n\|$ is bounded above. Modifying the first terms of the sequences $(M_n)$ and $(N_n)$ (up to modifying $S$ and $S'$), we may assume that this is true for all $n \in \mathbb{N}$.

Conversely if $M_n$ is invertible for all $n \in \mathbb{N}$, and $(M_n^{-1})$ is bounded, then $D(M_n)$ is an invertible operator with inverse $D(M_n^{-1})$.

It follows that the elements of $(\mathcal{L}(X)/\mathcal{S}(X))_0$ are those that may be lifted to a diagonal operator $D(M_n)$ where for all $n$, $M_n$ is invertible, and $(M_n^{-1})$ is bounded; such an operator $D(M_n)$ corresponds canonically to an invertible element of $l_\infty(M_2)$, and it follows that

$$(\mathcal{L}(X)/\mathcal{S}(X))_0 \simeq (l_\infty(M_2)/c_0(M_2))_0.$$

\[\square\]

**Lemma 30** Let $A \in M_2$. There exists a map $f_A$ on $M_2$ such that whenever $A^2 = -Id + r$ with $\|r\| < 1$, it follows that $(A + f_A(r))^2 = -Id$, and such that if $\|r\| < 1$ then $\|f_A(r)\| \leq \|A\| ((1 - \|r\|)^{-1/2} - 1)$.  

27
Proof: For \( \|r\| \leq 1 \), let \( f_A(r) = A(Id - r)^{-1/2} - A \), where \((Id - r)^{-1/2}\) is defined as an infinite series \( Id + \sum_{n \geq 1} c_n r^n \). Note that since \( r = A^2 + Id \), \((Id - r)^{-1/2}\) commutes with \( A \). It follows by an elementary computation that \((A + f_A(r))^2 = -Id\). Furthermore,

\[
\|f_A(r)\| \leq \|A\| \sum_{n \geq 1} c_n \|r\|^n = \|A\| ((1 - \|r\|)^{-1/2} - 1).
\]

\[\square\]

Lemma 31 Let \( M \in \mathcal{M}_2 \) be such that \( M^2 = -Id \). Then there exists \( P \in \mathcal{GL}_2 \) such that \( \|P\|_2 = \|P^{-1}\|_2 \leq \sqrt{\|M\|_1} \) and \( P \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} P^{-1} = M \).

Proof: If \( M^2 = -Id \) then \( M \) is of the form \( \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \) with \( a^2 = -1 - bc \). If \( c > 0 \) put \( P = c^{-1/2} \begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix} \), then \( P^{-1} = c^{-1/2} \begin{pmatrix} c & -a \\ 0 & 1 \end{pmatrix} \), and \( P \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} P^{-1} = M \).

Furthermore, \( \|P^{-1}\|_2^2 = \|P\|_2^2 = c^{-1}(1 + a^2 + c^2) = c - b \leq \|M\|_1 \). If \( c \leq 0 \) then \( b > 0 \) and a similar proof holds. \[\square\]

Proposition 32 The space \( X(\mathbb{D}_2) \) has unique complex structure up to isomorphism.

Let \( G \) be the group \( (l_\infty(\mathcal{M}_2)/c_0(\mathcal{M}_2))_0 \) and let \( I = \{g \in G : g^2 = -1\} \). By Lemma 29 and Proposition 20, it is enough to prove that all elements of \( I \) are \( G \)-conjugate.

Let \( g \in I \), so \( g \) is the class of some \( (M_n) \) which is invertible in \( l_\infty(\mathcal{M}_2) \), that is \( \|M_n\|_1 \) and \( \|M_n^{-1}\|_1 \) are bounded by some constant \( C \). Since \( g \in I \), the sequence \( r_n = M_n^2 + Id \) converges to 0. Let \( N \in \mathbb{N} \) be such that for all \( n \geq N \), \( \|r_n\| < 1 \).

Use Lemma 30 to define \( N_n = M_n + f_{M_n}(r_n) \) for \( n \geq N \); we have therefore that \( N_n^2 = -Id \) for all \( n \geq N \) and that \( M_n - N_n \) converges to 0. For \( n < N \) we just put \( N_n = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \). It follows that \( g \) is also the class of \( (N_n) \) modulo \( c_0(\mathcal{M}_2) \).

By Lemma 31 there exists \( P_n \in \mathcal{GL}_2 \) such that \( \|P_n\|_2 = \|P_n^{-1}\|_2 \leq \sqrt{C} \), and such that \( N_n = P_n \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} P_n^{-1} \).

28
Therefore \((P_n)_{n \in \mathbb{N}}\) and \((P_n^{-1})_{n \in \mathbb{N}}\) define inverse elements \(p\) and \(p^{-1}\) in \(G\). Let \(j\) be the element of \(I\) associated to the sequence \((j_n) \in l_\infty(\mathcal{M}_2)\) of constant value \[
abla \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.\]
We deduce that
\[
g = pjp^{-1},
\]
and therefore there is a unique \(G\)-conjugacy class of elements of \(I\).

\[\square\]

## 7 Final remarks and questions

The fact that any complex Banach space which is real H.I. is also complex H.I. raises the following question. Does there exist a complex H.I. space which is not H.I. when seen as a real Banach space? The answer turns out to be yes.

To prove this, we consider the canonical complexification \(X_{GM} \oplus \mathbb{C} X_{GM}\) of the real version of Gowers-Maurey’s space, i.e. \(X_{GM} \oplus X_{GM}\) with the complex structure associated to the operator \(I\) defined by \(I(x, y) = (-y, x)\). Note that by Proposition \[27\] any other complex structure on \(X_{GM} \oplus X_{GM}\) would be isomorphic.

**Proposition 33** The complexification of \(X_{GM}\) is complex H.I. but not real H.I.

**Proof:** Seen as a real space, \(X = X_{GM} \oplus \mathbb{C} X_{GM}\) is clearly not H.I. as a direct sum of infinite dimensional spaces. Let now \(Y\) be a \(\mathbb{C}\)-linear subspace of \(X\). We denote by \(p_1\) and \(p_2\) the projections on the first and the second summand of \(X = X_{GM} \oplus X_{GM}\) respectively. Either \(p_{1|Y}\) or \(p_{2|Y}\) is not strictly singular, and without loss of generality this is true of \(p_{1|Y}\); so \(p_{1|Y}\) is an isomorphism for some \(\mathbb{R}\)-linear subspace \(Y_1\) of \(Y\). We may therefore find a subspace \(Z\) of \(X_{GM}\) and a map \(\alpha : Z \to X_{GM}\) such that \(Y_1 = \{(z, \alpha z), z \in Z\} \subset Y\) (take \(Z = p_1(Y_1)\)) and \(\alpha = p_2(p_{1|Y_1}^{-1})\). By \(\mathbb{C}\)-linearity, \(Y_2 := iY_1 = \{(-\alpha z, z), z \in Z\}\) is also an \(\mathbb{R}\)-linear subspace of \(Y\).

By the properties of \(X_{GM}\), \(\alpha\) is of the form \(\lambda i_{Z, X_{GM}} + s\), where \(s\) is strictly singular. For our computation we may assume that the norm \(||.||\) on \(X \oplus X\) is the \(l_1\)-sum norm. Let \(\epsilon > 0\) be such that \(2(1 + |\lambda| + \epsilon)(1 + |\lambda|)\epsilon < 1\). Passing to a subspace, we may assume that \(||s|| \leq \epsilon\). We prove that whenever \(y_1 \in Y_1, y_2 \in Y_2\) are norm 1 vectors, we have \(||y_1 - y_2|| > \epsilon\). Indeed, otherwise let \(y_1 = (z_1, \alpha z_1)\) be of norm 1 with \(z_1 \in Z\), and \(y_2 = (-\alpha z_2, z_2)\) be of norm 1 with \(z_2 \in Z\), with
\[
\epsilon \geq ||(z_1 + \alpha z_2, \alpha z_1 - z_2)||,
\]
therefore

\[ \epsilon \geq \| z_1 + \lambda z_2 + sz_2 \|, \]

and since \( \| s \| \leq \epsilon \),

\[ \| z_1 + \lambda z_2 \| \leq 2\epsilon. \]

Likewise,

\[ \| \lambda z_1 - z_2 \| \leq 2\epsilon. \]

Combining these two inequalities, we obtain

\[ \| z_2 \| \leq \| (1 + \lambda^2)z_2 \| \leq (1 + |\lambda|)2\epsilon, \]

which implies

\[ 1 = \| y_2 \| = \| z_2 \| + \| \alpha z_2 \| \leq (1 + |\lambda| + \epsilon)(1 + |\lambda|)2\epsilon, \]

a contradiction.

From this we deduce that \( Y_1 \) and \( Y_2 \) form a direct sum in \( Y \). Therefore \( Y \) is not \( \mathbb{R} \)-HI. As \( X \) is \( HD_2 \) as a real space (\([3]\), Corollary 2) and \( Y \subset X \), it follows that \( Y \) is \( HD_2 \) as a real space. It follows that \( Y \) is \( \mathbb{R} \)-quasimaximal in \( X \), and therefore \( \mathbb{C} \)-quasimaximal in \( X \) (this follows immediately from the definition of quasimaximality). As every \( \mathbb{C} \)-linear subspace \( Y \) of \( X \) is \( \mathbb{C} \)-quasimaximal in \( X \), it follows that \( X \) is H.I. as a complex space. \( \square \)

This remark and the previous examples illustrate how various the relations can be between real and complex structure in a complex H.I. space.

**Question 34** By Theorem 5 when \( X \) is real H.I., there exists a division algebra \( E \) isomorphic to \( \mathbb{R}, \mathbb{C} \) or \( \mathbb{H} \), such that for any \( Y \subset X \), \( \mathcal{L}(X)/\mathcal{S}(X) \) embeds into \( \mathcal{L}(Y, X)/\mathcal{S}(Y, X) \) which embeds into \( E \). It follows that for any \( Y \subset X \),

\[ \dim \mathcal{L}(X)/\mathcal{S}(X) \leq \mathcal{L}(Y, X)/\mathcal{S}(Y, X) \leq \dim E. \]

If \( X = X_{GM} \) all these dimensions are equal to 1. We provided examples \( X(\mathbb{C}) \) and \( X(\mathbb{H}) \) for which all these dimensions are equal to 2 or to 4 respectively. It remains open whether these dimensions may differ. For example, does there exist a real H.I. Banach space \( X \) such that every operator on \( X \) is of the form \( \lambda Id_X + S \), but such that there exists an operator \( T \) on some subspace \( Y \) of \( X \) which is not of the form \( \lambda I_{Y,X} + S \)? In this case, \( \mathcal{L}(X)/\mathcal{S}(X) \) would be isomorphic to \( \mathbb{R} \), while \( E \) would be complex or quaternionic.
References

[1] R. Anisca, *Subspaces of $L_p$ with more than one complex structure*, Proc. Amer. Math. Soc. 131 (2003), no. 9, 2819–2829.

[2] S. Argyros, *Ramsey methods in analysis (Saturated and conditional structures in Banach spaces)*, Advanced Courses in Mathematics, CRM Barcelona. Birkhauser Verlag, Basel, (2005).

[3] S. Argyros, *A universal property of reflexive hereditarily indecomposable Banach spaces*, Proc. A.M.S., 129 (2001), 3231–3239.

[4] S. Argyros and A. Manoussakis, *An indecomposable and unconditionally saturated Banach space*, Studia Math. 159 (2003), no. 1, 1–32.

[5] S. Argyros and A. Tolias, *Methods in the theory of hereditarily indecomposable Banach spaces*, Mem. Amer. Math. Soc. 170 (2004), 806.

[6] J. Bourgain, *Real isomorphic complex Banach spaces need not be complex isomorphic*, Proc. Amer. Math. Soc. 96 (1986), no. 2, 221–226.

[7] V. Ferenczi, *Operators on subspaces of hereditarily indecomposable Banach spaces*, Bull. London Math. Soc. 29 (1997), 338–344.

[8] V. Ferenczi, *Hereditarily finitely decomposable Banach spaces*, Studia Mathematica 123 (2) (1997), 135–149.

[9] V. Ferenczi, *Quotient hereditarily indecomposable Banach spaces*, Canad. J. Math. 51 (1999), no. 3, 566–584.

[10] G. Godefroy and N. J. Kalton, *Lipschitz-free Banach spaces*, Studia Mathematica 159 (2003), no. 1, 121–141.

[11] W.T. Gowers, *A solution to Banach’s hyperplane problem*, Bull. London Math. Soc. 26 (1994), no. 6, 523–530.

[12] W.T. Gowers, *An infinite Ramsey theorem and some Banach-space dichotomies*, Ann. of Math. (2) 156 (2002), no. 3, 797–833.

[13] W.T. Gowers and B. Maurey, *The unconditional basic sequence problem*, J. Amer. Math. Soc. 6 (1993), 4, 851–874.
[14] W.T. Gowers and B. Maurey, *Banach spaces with small spaces of operators*, Math. Ann. 307 (1997), 543–568.

[15] N. Kalton, *An elementary example of a Banach space not isomorphic to its complex conjugate*, Canad. Math. Bull. 38 (1995), no. 2, 218–222.

[16] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*, Springer-Verlag, New York, Heidelberg, Berlin (1979).

[17] B. Maurey, *Banach spaces with few operators*, Handbook of the geometry of Banach spaces, vol. 2, edited by W.B. Johnson and J. Lindenstrauss, Elsevier, Amsterdam, 2002.

[18] S. Mazur and S. Ulam, *Sur les transformations isométriques d’espaces vectoriels normés*, C. R. Acad. Sci. Paris 194 (1932), 946–948.

[19] S. Szarek, *On the existence and uniqueness of complex structure and spaces with "few" operators*, Trans. Amer. Math. Soc. 293 (1986), no. 1, 339–353.

[20] S. Szarek, *A superreflexive Banach space which does not admit complex structure*, Proc. Amer. Math. Soc. 97 (1986), no. 3, 437–444.

Valentin Ferenczi,
Université Paris 6,
Equipe d’Analyse, Boîte 186,
4, place Jussieu, 75252 Paris Cedex 05,
France.

e-mail: ferenczi@ccr.jussieu.fr