AN ELLIPTIC SYSTEM WITH DEGENERATE COERCIVITY

LUCIO BOCCARDO, GISSELLA CROCE, CHIARA TANTERI

1. Introduction

1.1. Setting. In this paper we study the existence of solutions of the degenerate elliptic system

\[
\begin{aligned}
  -\text{div} \left( \frac{a(x)\nabla u}{(b(x) + |z|)^2} \right) + u &= f(x), \\
  -\text{div} \left( \frac{A(x)\nabla z}{(B(x) + |u|)^2} \right) + z &= F(x),
\end{aligned}
\]

where \( \Omega \) is a bounded, open subset of \( \mathbb{R}^N \), with \( N > 2 \), \( a(x) \) and \( A(x) \) are measurable matrices such that, for \( \alpha, \beta \in \mathbb{R}^+ \),

\[
\alpha|\xi|^2 \leq a(x)\xi\xi, \quad \alpha|\xi|^2 \leq A(x)\xi\xi; \quad |a(x)| \leq \beta, \quad |A(x)| \leq \beta.
\]

Moreover we assume

\[
0 < \lambda \leq b(x), \quad B(x) \leq \gamma,
\]

for some \( \lambda, \gamma \in \mathbb{R}^+ \) and

\[
f(x), \quad F(x) \in L^2(\Omega).
\]

Theorem 1.1. Under the assumptions \( (1.2), (1.3), (1.4) \), there exist \( u \in W_0^{1,1}(\Omega) \) and \( z \in W_0^{1,1}(\Omega) \), distributional solutions of the system \( (1.1) \).

1.2. Comments. First of all, we note that existence of solutions belonging to the nonreflexive space \( W_0^{1,1}(\Omega) \) is not so usual in the study of elliptic problems. Recently the existence of solutions in \( W_0^{1,1}(\Omega) \) was proved in \( [3], [4], [5] \), for elliptic scalar problems with degenerate coercivity (so that this paper is an extension to the systems of some of those results) and in some borderline cases of the Calderon-Zygmund theory of nonlinear Dirichlet problems in \( [9] \).

The main difficulty of the problem is that even if the differential operator is well defined between \( W_0^{1,2}(\Omega) \) and its dual, it is not coercive on \( W_0^{1,2}(\Omega) \): degenerate coercivity means that when \( |v| \) is “large”, \( \frac{1}{(a(x) + |v|)^2} \) goes to zero: for an explicit example see \( [18] \).

\footnote{\( \text{\small (see [14], [15], [9], [7], [13], [10], [17])} \)}
The study of problems involving degenerate equations begins with the paper [8] and it is developed in [1], [10], [11], [12], [3], [4], [5] (see also [2]).

2. Existence

2.1. A priori estimates. The first existence result is concerned with the case of a bounded data.

We recall the following definitions.

\[ T_k(s) = \begin{cases} 
    s, & \text{if } |s| \leq k; \\
    k \frac{s}{|s|}, & \text{if } |s| > k; 
\end{cases} \]

\[ G_k(s) = s - T_k(s). \]

Proposition 2.1. Let \( \rho > 0 \), \( \sigma > 0 \) and \( g, G \in L^\infty(\Omega) \). Then there exist weak solutions \( w, W \) belonging to \( W^{1,2}_0(\Omega) \) of the system

\[
\begin{align*}
    w &\in W^{1,2}_0(\Omega) \cap L^\infty(\Omega) : -\text{div} \left( \frac{a(x) \nabla w}{(b(x) + |T_\rho(W)|^2} \right) + w = g(x), \\
    W &\in W^{1,2}_0(\Omega) \cap L^\infty(\Omega) : -\text{div} \left( \frac{A(x) \nabla W}{(B(x) + |T_\sigma(w)|^2} \right) + W = G(x).
\end{align*}
\]

Proof. The existence is a consequence of the Leray-Lions theorem (or Schauder theorem) since the principal part is not degenerate, thanks to the presence of \( T_\rho \) and \( T_\sigma \). Moreover, if we take \( G_h(w) \) as test function in the first equation and \( G_k(W) \) as test function in the second equation, we have, dropping two positive terms,

\[
\begin{align*}
    \int_\Omega [||w| - |g(x)||] |G_h(w)| &\leq 0, \\
    \int_\Omega [||W| - |G(x)||] |G_k(w)| &\leq 0.
\end{align*}
\]

Then the choice \( h = \|g\|_{L^\infty(\Omega)} \), \( k = \|G\|_{L^\infty(\Omega)} \) implies

\[
\begin{align*}
    \|w\|_{L^\infty(\Omega)}, \\
    \|W\|_{L^\infty(\Omega)}.
\end{align*}
\]

Thus, if we set \( \rho = \|g\|_{L^\infty(\Omega)} \) and \( \sigma = \|G\|_{L^\infty(\Omega)} \), we can say that \( w \) and \( W \) are bounded weak solutions of the system

\[
\begin{align*}
    w &\in W^{1,2}_0(\Omega) \cap L^\infty(\Omega) : -\text{div} \left( \frac{a(x) \nabla w}{(b(x) + |W|^2} \right) + w = g(x), \\
    W &\in W^{1,2}_0(\Omega) \cap L^\infty(\Omega) : -\text{div} \left( \frac{A(x) \nabla W}{(B(x) + |w|^2} \right) + W = G(x).
\end{align*}
\]
Now we define
\[ f_n = \frac{f}{1 + \frac{1}{n}|f|}, \quad F_n = \frac{F}{1 + \frac{1}{n}|F|}, \]
so that
\[ \|f_n - f\|_{L^2(\Omega)} \to 0, \quad \|F_n - F\|_{L^2(\Omega)} \to 0. \]

Thanks to the Proposition 2.1, there exists a solution \((u_n, z_n)\) of the system
\[ \begin{align*}
    u_n \in W^{1,2}_0(\Omega) : & \quad -\text{div}\left( \frac{a(x)\nabla u_n}{(b(x) + |z_n|)^2} \right) + u_n = f_n(x), \\
    z_n \in W^{1,2}_0(\Omega) : & \quad -\text{div}\left( \frac{A(x)\nabla z_n}{(B(x) + |u_n|)^2} \right) + z_n = F_n(x),
\end{align*} \]

Now we prove our first estimates.

**Lemma 2.2.** The sequences \(\{u_n\}\) and \(\{z_n\}\) are bounded in \(L^2(\Omega)\).

**Proof.** We take \(G_k(u_n)\) as a test function in the first equation and we have
\[ \alpha \int_{\Omega} \left\| \nabla G_k(u_n) \right\|^2 + \int_{\Omega} |G_k(u_n)|^2 \leq \int_{\Omega} |f| |G_k(u_n)| \]
If we drop the first positive term and we use the H"older inequality, then we have
\[ \left[ \int_{\Omega} |G_k(u_n)|^2 \right]^{\frac{1}{2}} \leq \left[ \int_{\{k \leq |u_n|\}} |f|^2 \right]^{\frac{1}{2}}. \]

Similar estimates hold true for \(z_n\). In particular, taking \(k = 0\), we have the boundedness of the sequences \(\{u_n\}\) and \(\{z_n\}\) in \(L^2(\Omega)\). So we have that there exist \(u, z\) such that, up to subsequences,
\[ u_n \rightharpoonup u, \quad z_n \rightharpoonup z \quad \text{weakly in } L^2(\Omega). \]

Then if we drop the second term in (2.3), we have
\[ \alpha \int_{\Omega} \left\| \nabla G_k(u_n) \right\|^2 \leq \int_{\{k \leq |u_n|\}} |f|^2. \]
A similar estimate for \(z_n\) comes from the second equation. \(\square\)

**Lemma 2.3.** The sequences \(\{u_n\}\) and \(\{z_n\}\) are bounded in \(W^{1,1}_0(\Omega)\).

**Proof.** A consequence of (2.6) and of the H"older inequality is
\[ \int_{\Omega} |\nabla G_k(u_n)| = \int_{\Omega} \frac{|\nabla G_k(u_n)|}{(b(x) + |z_n|)} (b(x) + |z_n|) \leq \left[ \int_{\{k \leq |u_n|\}} |f|^2 \right]^{\frac{1}{2}} \left( \|b\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \right). \]
Similar estimates hold true for $z_n$. In particular, with $k = 0$, we have

$$\int_{\Omega} |\nabla u_n| \leq \frac{\|f\|_{L^2(\Omega)} \left( \|b\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \right)}{\alpha^{\frac{1}{2}}},$$

$$\int_{\Omega} |\nabla z_n| \leq \frac{\|F\|_{L^2(\Omega)} \left( \|b\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \right)}{\alpha^{\frac{1}{2}}}.$$ \hfill (2.7)

Now we improve the convergence (2.5).

**Lemma 2.4.** The sequences $\{u_n\}$ and $\{z_n\}$ are compact in $L^2(\Omega)$.

**Proof.** The estimates (2.7) imply, thanks to the Rellich embedding Theorem, the $L^1$ compactness and then the a.e. convergences

$$u_n(x) \to u(x), \quad z_n(x) \to z(x).$$ \hfill (2.8)

Now we use the Vitali Theorem: since we have the a.e. convergences (2.8), the compactness is achieved if we prove the equiintegrability. Let $E$ be a measurable subset of $\Omega$. Since $u_n = T_k(u_n) + G_k(u_n)$, we have (we use (2.4))

$$\int_E |u_n|^2 \leq 2 \int_E |T_k(u_n)|^2 + 2 \int_E |G_k(u_n)|^2 \leq 2 k^2 |E| + 2 \int_{\Omega} |G_k(u_n)|^2 \leq 2 k^2 |E| + 2 \int_{\{k \leq |u_n|\}} |f|^2,$$

where $|E|$ denotes the measure of $E$. Now we recall that a consequence of Lemma 2.3 is that the sequence $\{u_n\}$ is bounded in $L^1(\Omega)$, so that if we fix $\epsilon > 0$, there exists $k_\epsilon$ such that (uniformly with respect to $n$)

$$\int_{\{k \leq |u_n|\}} |f|^2 \leq \epsilon, \quad k \geq k_\epsilon.$$

Then

$$\int_{E} |u_n|^2 \leq 2 k^2 |E| + 2 \epsilon$$

implies

$$\lim_{|E| \to 0} \int_{E} |u_n|^2 \leq 2 \epsilon, \quad \text{uniformly with respect to } n.$$

Similar inequality holds true for $z_n$. \hfill \square

**Lemma 2.5.** The sequences $\{u_n\}$ and $\{z_n\}$ are weakly compact in $W^{1,1}_0(\Omega)$.
AN ELLIPTIC SYSTEM WITH DEGENERATE COERCIVITY

Proof. Here we follow [4, 5]. Let again $E$ be a measurable subset of $\Omega$, and let $i$ be in $\{1, \ldots, N\}$. Then

$$
\int_E |\partial_i u_n| \leq \int_E |\nabla u_n| = \int_E \frac{|\nabla u_n|}{b(x) + |z_n|} (b(x) + |z_n|)
\leq \left[ \int_{\Omega} \frac{|\nabla u_n|^2}{(b(x) + |z_n|)^2} \right]^{\frac{1}{2}} \left[ \int_E (b(x) + |z_n|)^2 \right]^{\frac{1}{2}}
\leq \left[ \frac{1}{\alpha} \int_{\Omega} |f|^2 \right]^{\frac{1}{2}} \left\{ \left[ \int_E b(x) \right]^{\frac{1}{2}} + \left[ \int_E |z_n|^2 \right]^{\frac{1}{2}} \right\},
$$

where we have used the inequality (2.6) in the last passage. Since the sequence $\{u_n\}$ is compact in $L^2(\Omega)$, we have that the sequence $\{\partial_i u_n\}$ is equiintegrable. Thus, by Dunford-Pettis theorem, and up to subsequences, there exists $Y_i$ in $L^1(\Omega)$ such that $\partial_i u_n$ weakly converges to $Y_i$ in $L^1(\Omega)$. Since $\partial_i u_n$ is the distributional derivative of $u_n$, we have, for every $n$ in $\mathbb{N}$,

$$
\int_{\Omega} \partial_i u_n \phi = -\int_{\Omega} u_n \partial_i \phi, \quad \forall \phi \in C_0^\infty(\Omega).
$$

We now pass to the limit in the above identities, using that $\partial_i u_n$ weakly converges to $Y_i$ in $L^1(\Omega)$, and that $u_n$ strongly converges to $u$ in $L^2(\Omega)$; we obtain

$$
\int_{\Omega} Y_i \phi = -\int_{\Omega} u \partial_i \phi, \quad \forall \phi \in C_0^\infty(\Omega),
$$

which implies that $Y_i = \partial_i u$, and this result is true for every $i$. Since $Y_i$ belongs to $L^1(\Omega)$ for every $i$, $u$ belongs to $W^{1,1}_0(\Omega)$. A similar result holds true for $z_n$.

Thus, thanks to Lemma 2.4 and Lemma 2.5, we can improve the convergence (2.5):

(2.9)

$$
\begin{cases}
\text{\(u\_n\) converges weakly in \(W^{1,1}_0(\Omega)\) and strongly in \(L^2(\Omega)\) to \(u\),} \\
\text{\(z\_n\) converges weakly in \(W^{1,1}_0(\Omega)\) and strongly in \(L^2(\Omega)\) to \(z\).}
\end{cases}
$$

2.2. Proof of Theorem 1.1 - First of all, we use the equiintegrability proved in Lemma 2.5 fix $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that, for every measurable subset $E$ with $|E| \leq \delta(\varepsilon)$, we have

$$
\int_E |\nabla u_n| \leq \varepsilon.
$$

Taking into account the absolute continuity of the Lebesgue integral, we have, for some $\tilde{\delta}(\varepsilon) > 0$,

$$
\int_E |\nabla u_n| \leq \varepsilon, \quad \int_E |\nabla u| \leq \varepsilon,
$$

for every measurable subset $E$ with $|E| \leq \tilde{\delta}(\varepsilon)$. 

On the other hand, since |Ω| is finite and the sequence
\[ D_n = \frac{a(x)}{(b(x) + |z_n|)^2} \]
converges almost everywhere (recall (2.9)), the Egorov theorem says that for every \( q > 0 \), there exists a measurable subset \( F \) of \( \Omega \) such that |\( F \)| < \( q \), and \( D_n \) converges to \( D \) uniformly on \( \Omega \setminus F \). We choose \( q = \tilde{\delta} \) so that we have, for every \( \varphi \in \text{Lip}(\Omega) \),
\[
\int_{\Omega} \left| D_n \nabla u_n \nabla \varphi - D \nabla u \nabla \varphi \right| \leq \int_{\Omega} \left| D_n \nabla u_n \nabla \varphi - D \nabla u \nabla \varphi \right| + \int_F \left| D_n \nabla u_n \nabla \varphi - D \nabla u \nabla \varphi \right| \leq \int_{\Omega \setminus F} |D_n \nabla u_n \nabla \varphi - D \nabla u \nabla \varphi| + 2\varepsilon \beta \lambda^2 \left\| \nabla \varphi \right\|_{L^\infty(\Omega)},
\]
which proves that
\[
(2.10) \quad \int_{\Omega} \frac{a(x)}{(b(x) + |z_n|)^2} \nabla u_n \nabla \varphi \rightarrow \int_{\Omega} \frac{a(x)}{(b(x) + |z|)^2} \nabla u \nabla \varphi.
\]
Thus, thanks to the above limit, (2.11) and Lemma 2.4 it is possible to pass to the limit in the weak formulation of (2.2), for every \( \varphi, \psi \in \text{Lip}(\Omega) \),
\[
(2.11) \quad \left\{ \begin{array}{l}
\int_{\Omega} \frac{a(x)}{(b(x) + |z_n|)^2} \nabla u_n \nabla \varphi + \int_{\Omega} u_n \varphi = \int_{\Omega} f_n(x) \varphi, \\
\int_{\Omega} \frac{A(x)}{B(x) + |u_n|} \nabla z_n \nabla \psi + \int_{\Omega} z_n \psi = \int_{\Omega} F_n(x);
\end{array} \right.
\]
and we prove that \( u \) and \( z \) are solutions of our system, in the following distributional sense
\[
(2.12) \quad \left\{ \begin{array}{l}
\int_{\Omega} \frac{a(x)}{(b(x) + |z|)^2} \nabla u \nabla \varphi + \int_{\Omega} u \varphi = \int_{\Omega} f(x) \varphi, \quad \forall \varphi \in \text{Lip}(\Omega); \\
\int_{\Omega} \frac{A(x)}{B(x) + |u|} \nabla z \nabla \psi + \int_{\Omega} z \psi = \int_{\Omega} F(x) \psi, \quad \forall \psi \in \text{Lip}(\Omega).
\end{array} \right.
\]

Now we show that, in the above definition of solution, it is possible to use less regular test functions: it possible to use functions only belonging to \( W^{1,2}_0(\Omega) \).
Proposition 2.6. The above functions $u$ and $z$ are solutions of our system, in the following sense

\begin{equation}
\begin{aligned}
\int_{\Omega} \frac{a(x) \nabla u \nabla v}{(b(x) + |z|^2)^2} + \int_{\Omega} u v = \int_{\Omega} f(x) v, \quad \forall v \in W^{1,2}_0(\Omega); \\
\int_{\Omega} \frac{A(x) \nabla z \nabla w}{(B(x) + |u|^2)^2} + \int_{\Omega} z w = \int_{\Omega} F(x) w, \quad \forall w \in W^{1,2}_0(\Omega).
\end{aligned}
\end{equation}

Proof. In order to avoid technicalities, here we also assume that

\begin{equation}
a(x) \text{ and } A(x) \text{ are scalar functions.}
\end{equation}

We start with the following inequalities (we use (2.6) with $k = 0$)

\begin{equation}
\int_{\Omega} \left| \frac{a(x) \nabla u_n}{(b(x) + |z_n|^2)^2} \right|^2 \leq \frac{\alpha^2}{\lambda^2} \int_{\Omega} \frac{|\nabla u_n|^2}{(b(x) + |z_n|^2)^2} \leq \frac{\alpha^2}{\lambda^2} \int_{\Omega} |f|^2.
\end{equation}

Thus, up to subsequences, we can say that, for some $\Psi \in (L^2(\Omega))^N$,

\begin{equation}
\int_{\Omega} \frac{a(x) \nabla u_n}{(b(x) + |z_n|^2)^2} \phi \to \int_{\Omega} \Psi \phi,
\end{equation}

for every $\phi \in (L^2(\Omega))^N$. Now we compare the limit (2.10) with the limit (2.15), taking $\phi = \nabla \varphi$, and we deduce that

\begin{equation}
\int_{\Omega} \left[ \frac{a(x) \nabla u}{(b(x) + |z|^2)^2} - \Psi \right] \phi = 0.
\end{equation}

Thus we proved that

\begin{equation}
\frac{a(x) \nabla u_n}{(b(x) + |z_n|^2)^2} \text{ weakly converges in } (L^2(\Omega))^N \text{ to } \frac{a(x) \nabla u}{(b(x) + |z|^2)^2},
\end{equation}

which allows us to pass to the limit in (2.11) only assuming $\varphi, \psi \in W^{1,2}_0(\Omega)$.

Acknowledgments

This paper contains the unpublished part of the results presented by the first author in a talk at the conference “Calculus of Variations and Differential Equations - Conférence en l’honneur du 60ème anniversaire de Bernard Dacorogna” (Lausanne, 10-12 juin 2013).

References

[1] L. Boccardo, H. Brezis: Some remarks on a class of elliptic equations. Boll. Unione Mat. Ital. 6 (2003), 521–530.
[2] L. Boccardo, G. Croce: Elliptic partial differential equations. Existence and regularity of distributional solutions. De Gruyter Studies in Mathematics, 55. De Gruyter, Berlin, 2014.
[3] L. Boccardo, G. Croce, L. Orsina: $W^{1,1}_0$ minima of non coercive functionals; Atti Accad. Naz. Lincei, 22 (2011), 513–523.
[4] L. Boccardo, G. Croce, L. Orsina: A semilinear problem with a $W^{1,1}_0$ solution; Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 23 (2012), no. 2, 97–103.
[5] L. Boccardo, G. Croce, L. Orsina: Nonlinear degenerate elliptic problems with $W^{1,1}_0$ solutions; Manuscripta Math. 137 (2012), 419–439.
[6] L. Boccardo, B. Dacorogna: A characterization of pseudo-monotone differential operators in divergence form; Comm. P.D.E. 9 (1984), 1107–1117.
[7] L. Boccardo, B. Dacorogna: Monotonicity of certain differential operators in divergence form. Manuscripta Math. 64 (1989), 253–260.
[8] L. Boccardo, A. Dall’Aglio, L. Orsina: Existence and regularity results for some elliptic equations with degenerate coercivity, dedicated to Prof. C. Vinti (Perugia, 1996). Atti Sem. Mat. Fis. Univ. Modena 46 (1998), 51–81.
[9] L. Boccardo, T. Gallouët: $W^{1,1}_0$ solutions in some borderline cases of Calderon-Zygmund theory; J. Differential Equations, 253 (2012), 2698–2714.
[10] G. Croce: The regularizing effects of some lower order terms in an elliptic equation with degenerate coercivity. Rendiconti di Matematica 27 (2007), 299–314.
[11] G. Croce: An elliptic problem with degenerate coercivity and a singular quadratic gradient lower order term; Discrete Contin. Dyn. Syst. Ser. S 5 (2012), 507–530.
[12] G. Croce: An elliptic problem with two singularities; Asymptot. Anal. 78 (2012), 1–10.
[13] G. Croce, B. Dacorogna: On a generalized Wirtinger inequality; Discrete Contin. Dyn. Syst. 9 (2003), 1329–1341.
[14] B. Dacorogna: Direct methods in the calculus of variations. Applied Mathematical Sciences, 78. Springer-Verlag, Berlin, 1989.
[15] B. Dacorogna: Weak continuity and weak lower semicontinuity of nonlinear functionals. Lecture Notes in Mathematics, 922. Springer-Verlag, Berlin-New York, 1982.
[16] B. Dacorogna, C. Tanteri: On the different convex hulls of sets involving singular values; Proc. Roy. Soc. Edinburgh Sect. A 128 (1998), 1261–1280.
[17] B. Dacorogna, C. Tanteri: Implicit partial differential equations and the constraints of nonlinear elasticity; J. Math. Pures Appl. 81 (2002), 311–341.
[18] A. Porretta: Uniqueness and homogenization for a class of noncoercive operators in divergence form, dedicated to Prof. C. Vinti (Perugia, 1996). Atti Sem. Mat. Fis. Univ. Modena 46 (1998), 915–936.

La Sapienza Università di Roma.
E-mail address: boccardo@mat.uniroma1.it

Université du Havre
E-mail address: gisella.croce@univ-lehavre.fr

École polytechnique fédérale de Lausanne
E-mail address: chiara.tanteri@epfl.ch