Convergence and the Length Spectrum

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Abstract: The author defines and analyzes the $1/k$ length spectra, $L_{1/k}(M)$, whose union, over all $k \in \mathbb{N}$ is the classical length spectrum. These new length spectra are shown to converge in the sense that $\lim_{i \to \infty} L_{1/k}(M_i) \subset \{0\} \cup L_{1/k}(M)$ as $M_i \to M$ in the Gromov-Hausdorff sense. Energy methods are introduced to estimate the shortest element of $L_{1/k}$, as well as a concept called the minimizing index which may be used to estimate the length of the shortest closed geodesic of a simply connected manifold in any dimension. A number of gap theorems are proven, including one for manifolds, $M^n$, with $\text{Ricci} \geq (n-1)$ and volume close to $\text{Vol}(S^n)$. Many results in this paper hold on compact length spaces in addition to Riemannian manifolds.

1 Introduction

Recall that a compact length space is a metric space such that every pair of points is joined by a length minimizing rectifiable curve whose length is the distance between the two points. The simplest example of such a space is a Riemannian manifold. A “geodesic” in such a space is a locally length minimizing curve.

A closed geodesic is a map $\gamma : S^1 \to M$ which is locally minimizing around every point in $S^1$. This extends the concept of a smoothly closed geodesic in a manifold. (c.f. [BBI]SoWei) We shall assume throughout that all geodesics are parametrized proportional to arclength with speed $L/(2\pi)$. The length spectrum, $L(M)$, of a length space, $M$, is the set of lengths of closed geodesics. These definitions are just extensions of the classical definitions on Riemannian manifolds.

The length spectrum is not continuous with respect to deformations of the manifold. When a sequence of spaces, $M_i$, converges in the Gromov-Hausdorff sense [Defn 2.2] to a space, $M$, it may have closed geodesics $\gamma_i$ converging to a curve which is not a closed geodesic. That is, there could be a “disappearing length”: $\exists L_0 = \lim_{i \to \infty} L_i \in L(M_i)$ such that $L_0 \notin L(M)$.

In particular, we have this situation in Figure 1. Here the sequence of surfaces, $M_i$, with in-

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creasingly small pairs of handles converges in the Gromov-Hausdorff sense to a standard sphere, $Y$. Notice how the closed geodesics which pass through both handles converge to a geodesic segment but not to a closed geodesic. The lengths, $L_i$, of these closed geodesics converge to $L_0 = \pi/3$ which is not in the length spectrum of the sphere. In fact the shortest closed geodesic in $S^2$ has length $2\pi$. For details see Example 2.2.

In Example 2.1 we will examine the length spectrum of a flat torus created by taking the isometric product of a circle with a small circle. As the smaller circle’s diameter approaches 0, we say the sequence of tori “collapses” in the Gromov-Hausdorff sense to a circle, $S^1$. The length spectrum of the limit space, $S^1$, is just \( \{n\pi : n \in \mathbb{N}\} \), yet the length spectra of the collapsing tori is becoming an increasingly dense collection of points in \([0, \infty)\). Thus we have quite a large collection of disappearing lengths!

Both of these examples will be described in full detail in the first section [Example 2.1] and [Example 2.2], after we have given the rigorous definition of Gromov-Hausdorff convergence.

It is also possible that there is a “suddenly appearing length”: $L_0 \in L(M)$ such that no sequence $L_i \in L(M_i)$ converges to $L$. This occurs even when $M_i$ converges to $M$ in the $C^4$ sense as can be seen in Figure 2.

![Figure 2: The geodesic in $Y$ is suddenly appearing as a limit of the $M_i$ but not as a limit of the $N_i$.](image)

In this paper, we define a new collection length spectra, $L_{1/k}(M)$, [Defn 3.2] such that

$$
\bigcup_{k \in \mathbb{N}} L_{1/k}(M) = L(M) \quad \text{[Theorem 3.1].}
$$

While any collection of length spectra satisfying (1.1) would have to incorporate the sudden appearances observed in Figure 2, we do prove in Theorem 7.1 that

$$
\lim_{k \to \infty} L_{1/k}(M_i) \subset \{0\} \cup L(M),
$$

when $M_i$ converge to $M$ in the Gromov-Hausdorff sense.

Throughout the paper we survey past results and techniques used to study the length spectrum, we relate them to the new $1/k$ length spectra and we suggest new directions of research. As many of the proposed problems in this paper are at a level a graduate student should be able to handle, we have presented this paper in a manner that should easily be read by a student.

In Section 2 we give the necessary background on Gromov-Hausdorff convergence, completely describing how the spheres with tiny handles and the collapsing tori converge [Examples 2.2 and 2.1]. We also present ellipsoids which converge to a singular doubled disk Example 2.3. Readers who are just interested in the $1/k$ length spectra and not their convergence properties may skip Section 2 and easily read everything except Sections 7.1 and 8.
In Section 3, we introduce $1/k$ geodesics, which are geodesics that minimize on any subsegment whose length is $1/k$ of the total length [Defn 3.1]. The set of lengths of such geodesics is $L_{1/k}$ [Defn 3.2] and we prove Theorem 3.1. We also relate $L_{1/k}(M)$ to the diameter and injectivity radius of the space [Lemma 3.2 and Lemma 3.3]. Using these results we describe the $1/k$ length spectra of a sphere and collapsing tori [Examples 3.1 and 3.2].

In Section 3, we also complete a study of closed geodesics. We define the minimizing index of a geodesic as the smallest $k$ which can be used to classify it as a $1/k$ geodesic [Defn 3.3]. Then we define the injectivity radius of a geodesic [Defn 3.4] and relate it to the minimizing index [Lemma 3.4]. We discuss iterated geodesics [Lemma 3.5] and a particularly illustrative example of the equator of an ellipsoid close to a doubled disk [Example 3.6].

In Section 4, we prove that the covering spectrum defined in [SoWei] is a subset of $L_{1/2}$ [Theorem 4.1]. Recall that in [SoWei], the $\text{CovSpec}(M) \cup \{0\}$ was proven to be continuous with respect to Gromov-Hausdorff convergence of the manifold. In other words, there is no sudden appearance of elements in the Covering Spectrum as described in Figure 2. We could not expect to prove such a strong theorem for $L_{1/2}(M)$ because we include all lengths of $L(M)$ in one of the them [Theorem 4.1]. In fact the suddenly appearing geodesic in Figure 2 is a $1/2$ geodesic and an element of $L_{1/2}$. This justifies the lack of an equality in Theorem 7.1.

In Section 5, we turn to a study of the systole of a manifold. Since the systole is the length of the shortest noncontractible curve it is an element of $L_{1/2}$ [Lemma 5.1]. We survey past estimates relating the systole to the volume and diameter of a manifold and extend them to estimates on $\min L_{1/2}$. It should be noted that some of these estimates are only achieved on singular manifolds, so the extension of all concepts to compact length spaces in this paper is further justified.

In Section 5, we estimate the length of the shortest closed geodesic in a compact length space, $\min L(M)$. First we define the minimizing index, $\text{minind}(M)$, of a space and then prove that $\min L(M) \leq \text{minind}(M)diam(M)$ [Defn 5.1 and Theorem 5.1]. We also provide a lower bound on $\min L(M)$ [Theorem 6.2]. We close with an application of an old result of Klingenberg to show that the minimizing index of a manifold without conjugate points is $1/2$ [Corollary 6.3].

In Section 6, we finally prove the convergence theorem mentioned above. We conclude that if there is a sequence of spaces with a disappearing length in the limit, as in Figure 1, then the geodesics that disappear must have minimizing index diverging to infinity [Corollary 7.2]. Theorem 7.1 also immediately implies that $L_{1/k}(M) \cup \{0\}$ is compact [Corollary 7.3]. We discuss the convergence of $L_{1/k}(M_i)$ for the collapsing tori, the flattening ellipsoids and a new example converging to a hexagonal region [Examples 7.1, 7.2 and 7.3]. We remark on Bangert’s Theorem [Remark 7.4].

In Section 6, we rephrase Theorem 7.1 as a gap theorem [Theorem 8.1] and then prove a number of gap theorems. For example, we use Colding’s sphere stability theorem to prove:

**Theorem 1.1** There exists a function $\Psi : \mathbb{R}^+ \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}^+$ such that $\lim_{\delta \to 0} \Psi(\delta, k, n) = 0$ such that if $N^n$ is a compact Riemannian manifold with

$$\text{Vol}(N^n) \geq \text{Vol}(S^n) - \delta$$

and $\text{Ricci}(N^n) \geq (n - 1)$ then

$$L_{1/(2k)}(M^n) \subset \{0, \epsilon_k\} \cup (2\pi - \epsilon_k, 2\pi + \epsilon_k) \cup \cdots (2k\pi - \epsilon_k, 2k\pi + \epsilon_k)$$

for $\epsilon_k = \Psi(\delta, k, n)$.

In light of Example 8.1, we propose that the length spectra on these manifolds do not converge [Problem 11.22]. We obtain similar results for Riemannian manifolds with $\text{Ricci} \geq (n - 1)$ and $\text{rad}(M^n)$ close to $\text{rad}(S^n)$ and for $M^n$ with first Betti number equal to $n - 1$ and Ricci curvature that is almost nonnegative [Theorems 8.3 and 8.4]. Similar gap theorems also exist for $M^n$ which are almost isotropic and have a uniform lower bound on their Ricci curvature [Remark 8.5].
In Section 9, we introduce openly $1/k$ geodesics which are shown to be uniquely defined on Riemannian manifolds by any collection of evenly spaced points [Defn 9.1 and Lemma 9.1]. We then define the openly $1/k$ length spectra and extend most of the results and definitions of the previous sections to this setting. There are no openly $1/2$ geodesics [Lemma 9.2], so the results in Sections 4 and 5 do not apply. Theorem 7.1 doesn’t extend well either due to the open nature of the Defn 9.1 [Theorem 9.2 and Example 9.3]. Otherwise the results carry over. We close with a discussion of the borderline case of a $1/k$ geodesic which is not an openly $1/k$ geodesic on manifolds.

Section 10 extends the theory of geodesics on as critical points of the energy function on the loop space to openly $1/k$ geodesics. First we review the piecewise geodesic version of the theory and then prove Theorem 10.2 which identifies openly $1/k$ geodesics on a convex compact Riemannian manifold with boundary with “rotating” smooth critical points of a uniform energy function on $k$-fold product, $(M)^k$, of the space. We explicitly demonstrate a few examples and then discuss why this theory does not extend well to $1/k$ geodesics and nonsmooth spaces. Nevertheless it can be used to estimate the minimizing index of a Riemannian manifold and to determine whether a convex Riemannian manifold with boundary has any closed geodesics.

Section 11 concludes the paper with a list of open problems most of which should be on the level of a graduate student.

Additional gap theorems related to sectional curvature will appear in future papers along with a survey of sectional curvature results on the length spectrum.

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2 Background

Here we provide the necessary background on Gromov-Hausdorff convergence. Readers who are only interested in studying the $1/k$ length spectrum on a fixed Riemannian manifold may skip to Section 3. Essentially all the material here appeared in [G3] and can also be studied in [BBI].

For those readers who have studied $C^k$ convergence of manifolds, keep in mind that Gromov has proven that if a sequence of compact manifolds $M_i$ converges to $M$ in the $C^k$ sense then they also converge in the Gromov-Hausdorff sense. While $C^k$ convergence requires that the manifolds be diffeomorphic, $GH$ convergence doesn’t even require that they have the same dimension. In fact the spaces need only be compact metric spaces.

We begin with an older concept, the Hausdorff distance between sets.

**Definition 2.1** Given two compact subsets $A, B$ in a metric space $Z$, we can define the Hausdorff distance as follows:

$$d_H^Z(A, B) = \inf\{r : A \subset T_r(B) \text{ and } B \subset T_r(A)\},$$

where $T_r(X)$ is the tubular neighborhood around $X$ of radius $r$:

$$T_r(X) = \{z \in Z : \exists x_z \in X \text{ s.t. } d(x_z, z) < r\}.$$  

The surfaces $M_i$ in Figure 1 would converge in the Hausdorff sense to the standard sphere, $Y$, as subsets of $Z = \mathbb{E}^3$, if they were superimposed. One need only take the radius of the tubular neighborhood large enough to capture the tiny handles. In this respect the Hausdorff distance is blind to the topology of the sets it compares.

Hausdorff distance is also blind to the dimensions of the sets: it can easily be seen that $A_0 = [0, 1] \times \{0\} \subset \mathbb{E}^2$ and $A_r = [0, 1] \times [-r, r] \subset \mathbb{E}^2$ satisfy $d_H(A_0, A_r) \leq r$. On the other hand, for small
one can see how it makes sense that $A_r$ could be thought of as close to $A_0$. To quote Cheeger, they look very similar “to the naked eye”.

Gromov extended this concept to compact metric spaces, providing us with a metric between spaces that is also blind to dimension and topology, but captures the idea that the spaces are close in some blurry sense. [G3]

Before we define the Gromov-Hausdorff distance between metric spaces, recall that $f : X \to Z$ is an isometric embedding if it is one-to-one and $d_X(x_1, x_2) = d_Z(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$.

The sphere sitting inside Euclidean space is not isometrically embedded because the distances on the sphere are measure intrinsically (the poles are a distance $\pi$ apart in $S^2$). A plane is isometrically embedded in Euclidean space because it is totally geodesic.

**Definition 2.2 (Gromov)** The Gromov-Hausdorff distance between two compact metric spaces $X$ and $Y$ is defined as follows:

$$d_{GH}(X, Y) = \inf \{d_Z(f(X), g(Y)) : Z, f : X \to Z, g : Y \to Z\}$$

where the set runs through any metric space, $Z$, and any isometric embeddings $f : X \to Z$ and $g : Y \to Z$.

It is an easy exercise to prove the following two lemmas:

**Lemma 2.1 (Gromov)**

$$d_{GH}(X, Y) \leq \text{diam}(X) + \text{diam}(Y) \quad (2.4)$$

**Lemma 2.2 (Gromov)** If $X_i$ converge to $X$ in the Gromov-Hausdorff sense, $d_{GH}(X_i, Y) \to 0$ then $\text{diam}(X_i) \to \text{diam}(X)$.

Gromov proved that both the space of compact metric spaces and the space of compact length spaces are complete with respect to $d_{GH}$. In particular, he proved the difficult theorem that the Gromov-Hausdorff limit of a compact length space is a compact length space. [G3]

We can now give the details of the sequence of tori collapsing to a circle with disappearing lengths that was mentioned in the introduction.

**Example 2.1** Let $M_j = S^{1}_\pi \times S^{1}_{\pi/j}$ be a flat torus formed by taking the isometric product of a circle of diameter $\pi$ with a circle of diameter $1/j$. Note that as $j$ diverges to infinity, $M_j$ converges in the Gromov-Hausdorff sense to $S^{1}_\pi$. This can be seen by taking $Z = M_j$ itself and isometrically embedding $S^{1}_\pi$ as $S^{1}_\pi \times \{0\} \subset M_j$, so

$$d_{GH}(M_j, S^{1}_\pi) \leq d_H^{M_j}(M_j, S^{1}_\pi \times \{0\}) \leq \pi/j. \quad (2.5)$$

It is well known that the length spectrum of $M_j$ is the collection of distances between lattice points $(2\pi a, 2\pi b/j)$ where $a, b \in \mathbb{Z}$. Thus

$$L(M_j) = \{\sqrt{(2\pi a)^2 + (2\pi b/j)^2} : a, b \in \mathbb{N}\} \cup \{\pi, \pi/k\}. \quad (2.6)$$

Notice how this length spectrum becomes increasingly dense as $j$ goes to infinity, so that for any $N$ we get

$$L(M_j) \cap [0, N] \to [0, N]$$

in the Hausdorff sense. (2.7)

In particular, there are lengths $l_j \in L(M_j)$ such that $l_j \to \pi$ even though $\pi$ is not in the length spectrum of $S^1$. 

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The sequence of surfaces in Figure 1 are trickier to deal with as they are not easily isometrically embedded into a common space and even the choice of a metric space $Z$ for each $M_j$ is not obvious.

We first recall Gromov’s concept of an $r$ net on a metric space. A set $N \subset X$ is an $r$ net if $X \subset T_r(N)$. It is clear that $d_H(N, X) \leq r$. When $N$ is a finite collection of points then it is a finite net. Let $N_X(r)$ be the minimum cardinality of all $r$ nets in $X$.

Gromov’s famous compactness theorem states that a class of compact metric spaces, $\{X\}$, with a uniform bound on $N_X(r) \leq N(r)$ is precompact with respect to the Gromov-Hausdorff metric. In particular, the class of complete manifolds with $\text{Ricci} \geq -K$, $\dim = n$ and $\text{diam} \leq D$ is precompact. The limits of the sequences are compact length spaces.

If one considers an $r$ net, $N \subset X$, and endows it with the restricted metric from $X$, then it may not be a length space. However, it is a metric space such that $d_{GH}(X, N) \leq d^N_H(X, N) \leq r$. Using the triangle inequality, one than sees that $d_{GH}(X, Y) \leq 2r$ if both spaces have isometric $r$ nets. We now use this technique to prove the convergence of the surfaces in Figure 1.

**Example 2.2** The surfaces in Figure 1 converge to the standard sphere in the Gromov-Hausdorff sense. Let us suppose that $M_j$ with its handles removed is isometric to a a standard sphere with two disks of radius $1/j$ removed and that the diameter of the handles is $< 4/j$. Now lets form a finite $100/j$ net on $M_j$ such that for any pair of points in the net, the minimizing geodesic between them does not hit either handle. Since the points in the net aren’t on the handles, they correspond isometrically to specific points on $Y = S^2$. That is we have a metric space $N_j$, the net, such that $N_j$ isometrically embeds into both $Y$ and $M_j$ and such that

$$d^M_{GH}(N_j, M_j) \leq 100/j \text{ and } d^Y_{GH}(N_j, Y) \leq 100/j.$$  

Thus

$$d_{GH}(M_j, Y) \leq d_{GH}(M_j, N_j) + d_{GH}(N_j, Y) \leq 200/j$$

and we see that $M_j$ converges to $Y$ is the Gromov-Hausdorff sense.

Note that it is not necessary to find an isometric pair of $r$ nets in two compact metric spaces $X_1$ and $X_2$ to prove they are close in the Gromov-Hausdorff sense. It suffices to find a pair of “almost isometric” nets $N_1$ and $N_2$. Gromov has proven that if both nets have the same cardinality and one can set up a bijection between them: $f : N_1 \to N_2$ such that

$$\sup \{d_{N_1}(x_1, x_2) - d_{N_2}(f(x_1), f(x_2)) : x_1, x_2 \in N_1 \} < \epsilon$$

then one can show $d_{GH}(N_1, N_2) < 2\epsilon$ [Gr], cf [BBI. Cor 7.3.28]. So in that case $d_{GH}(X_1, X_2) < 2r + \epsilon$.

Using (2.10) we see that when $X_i \to X$ in the $C^k$ sense then they also converge in the Gromov-Hausdorff sense. For example, Figure 2 has a smoothly converging sequence of compact Riemannian manifolds. They are all diffeomorphic to the sphere with metrics $g$ that converge smoothly, $C^k$, to the limit space. The diffeomorphisms are almost isomorphisms in the sense described in (2.10) without even requiring the use of finite nets.

In fact one need only find $f_i : X_i \to X$ which are $\epsilon_i$ almost distance preserving,

$$|d_X(f_i(a), f_i(b)) - d_X(a, b)| < \epsilon_i$$

and $\epsilon_i$ almost onto, $T_{\epsilon_i}(f_i(X_i)) \supset X$, to prove that $X_i$ converge to $X$ in the Gromov Hausdorff sense.

**Example 2.3** Let $M_c$ be an ellipsoid

$$(x)^2 + (y)^2 + (z/c)^2 = 1.$$  

If we take $c_j \to \infty$, then $M_j = M_{c_j}$ converges to the doubled disk, $Y$, in the Gromov Hausdorff sense.
More precisely, $Y$, is two flat disks of radius 1 glued together along the circle, so that the distance between points on a common disk is the usual Euclidean distance and the distance between points $x$ and $y$ on different disks is:

$$d_{M_\infty}(x, y) = \inf_{z \in S^1}(|x - z| + |y - z|).$$

(2.13)

Gluing is significantly more complicated when the shapes aren’t convex (c.f. [BBI]).

To prove that $M_j$ converge to $Y$ in the Gromov-Hausdorff sense, we just employ the maps $f_i : M_i \to M_i$ defined to be $f_i(x, y, z) = (x, y, \text{sgn}(z))$, where $\text{sgn}(z)$ is just used to indicate whether we are on the upper or lower disk. Naturally the edge where $z = 0$ doesn’t need a sign.

Example 2.1 is said to be “collapsing” because the dimension of the limit is less than the dimension of the sequence. Example 2.2 is considered to be “noncollapsing” because the dimension of the manifolds in the sequence is the same as the dimension of the limit space. On the other hand, the injectivity radius is decreasing to 0 in this example.

The following definition is a simple extension of a Riemannian injectivity radius.

**Definition 2.3** The injectivity radius of a compact length space, $M$, is

$$\text{injrad}_x = \sup\{t : \text{any geodesic segment of length } t \text{ is minimal}\}$$

(2.14)

**Example 2.4** The Hawaiian Earring, a compact length space consisting of a collection of circles of radius $1/j$ for each $j \in \mathbb{N}$ all joined at a common point, has an injectivity radius equal to 0. It is also known to have no universal cover (c.f. [Sp]). That is, there is no covering space which covers all the other covering spaces.

For completeness of exposition, we now review the fact that there are no disappearing lengths when the sequence of compact length spaces has a uniform lower bound on injectivity radius.

**Proposition 2.3** Suppose $M_j$ are compact length spaces with a common positive lower bound, $i_0$, on their injectivity radius, and $M_j$ converge to $Y$ in the Gromov-Hausdorff metric, then for any $R > 0$ we have the following Hausdorff limit:

$$L(M_j) \cap [0, R] \to L_\infty \subset L(Y) \cap [0, R].$$

(2.15)

That is, for all $\epsilon > 0$, $\exists N_{\epsilon, R} \in \mathbb{N}$ such that

$$L(M_j) \cap [0, R] \subset T_\epsilon(L(Y)).$$

(2.16)

Note that when manifolds converge smoothly, they do have a common lower bound on their injectivity radius. So Figure 2 demonstrates that one still might have suddenly appearing geodesics in this case.

One can see that $N_{\epsilon, R}$ depends on $R$, just by examining a sequence of circles of radius $r_j \to \pi$ converging to the standard circle. The errors accumulate as you wrap repeatedly around the same geodesic. So one needs a common upper bound, $R$, on the length of the geodesic to get a common rate of convergence.

**Proof:** Let $\gamma_i$ be the geodesics of length $L_i \to L_\infty$. For $i$ sufficiently large, $L_i \in [2i_0, 2L_\infty]$. Thus the $\gamma_i : S^1 \to M_i$ are equicontinuous. By the Grove-Petersen Arzela-Ascoli Theorem, a subsequence of the $\gamma_i$ converge to $\gamma_\infty : S^1 \to Y$, of length $L_\infty$. [GrPet]

One may consider Proposition 2.3 as a kind of semicontinuity of the length spectrum with respect to Gromov-Hausdorff convergence. One of the goals of this paper is to prove a similar convergence theorem for spaces without a common lower bound on injectivity radius.
3 1/k Geodesics

We now introduce a new length spectrum, $L_{1/k}$, which we will later prove has a strong relationship with Gromov-Hausdorff convergence. Here we focus on the properties of this new concept on a fixed compact length space or Riemannian manifold and its relationship with the traditional length spectrum.

**Definition 3.1** A $1/k$ geodesic is a closed geodesic, $\gamma : S^1 \to M$, which is minimizing on all subsegments of length $L/k$ where $L = \text{Length}(\gamma)$:

$$d(\gamma(t), \gamma(t + 2\pi/k)) = L_{\gamma}([t, t + 2\pi/k]) = L/k \quad \forall t \in S^1.$$ (3.1)

**Definition 3.2** Let the $1/k$ Length Spectrum, $L_{1/k}(M) \subset L(M)$, be the set of lengths of $1/k$ geodesics.

The following lemma is immediate:

**Lemma 3.1** $L_{1/k}(M) \subset L_{1/(k+1)}(M)$.

**Definition 3.3** The minimizing index, $\text{minind}(\gamma)$, of a geodesic, $\gamma$, is the smallest $k \in \mathbb{N}$ such that $\gamma$ is a $1/k$ geodesic.

**Theorem 3.1** Any closed geodesic is a $1/k$ geodesic for a sufficiently large number $k$. So

$$\bigcup_{k=1}^{\infty} L_{1/k}(M) = L(M).$$ (3.2)

**Proof:** If $\gamma : S^1 \to M$ is a closed geodesic then for all $t$, there exists $\epsilon_t > 0$ such that $\gamma$ is minimizing from $t - \epsilon_t$ to $t + \epsilon_t$. The intervals $(t - \epsilon_t, t + \epsilon_t)$ form an open cover of $S^1$ and since $S^1$ is compact, there is a finite subcover and a Lebesgue number, $\rho > 0$, for this cover. The $\gamma$ is a $1/k$ geodesic for any $k > 2\pi/\rho$. □

**Lemma 3.2** If $\text{diam}(M) \leq D$ then $\text{minind}(\gamma) \geq L(\gamma)/D$ and $L_{1/k}(M) \subset (0, Dk]$.

**Proof:** If $\text{minind}(\gamma) = k$, then $\gamma$, must be minimizing on segments of length $L(\gamma)/k$. So $L(\gamma)/k \leq D$. □

Recall Definition 2.3 of the injectivity radius. It is easy to see that $L(M) \subset [2\text{injrad}(M), \infty)$. The injectivity radius also provides the following useful relationship between $L(M)$ and $L_{1/k}(M)$.

**Lemma 3.3** If $M$ has $\text{injrad}(M) \geq i_0 > 0$ then for all $L_0 > 0$,

$$L(M) \cap [0, L_0] = L_{1/k}(M) \cap [0, L_0] \text{ where } k \geq L_0/i_0.$$ (3.3)

**Proof:** Let $\gamma : S^1 \to M$ be any closed geodesic with $L(\gamma) \leq L_0$. Then any segment from $\gamma(t)$ to $\gamma(t + L/(2\pi k))$ has length $\leq L_0/k \leq i_0$. Thus it is minimising on this interval and $\gamma$ is a $1/k$ geodesic. □

This estimate is only sharp when the injectivity radius of the manifold is achieved by a pair of points on the geodesic. There is no reason that a distant pair of cut points or a cut point perpendicular to the geodesic should affect its minimising index. In Example 3.2 below we will see that in a thin torus, $S^1 \times S^1$, there is a $1/2$ geodesic of length $2\pi$ no matter how small the injectivity radius, $\delta$, of the torus is. For this reason we make the following new definition.

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Definition 3.4 The injectivity radius, \( injrad(\gamma) \) of a closed geodesic \( \gamma \), is

\[
\text{injrad}(\gamma) = \inf \{ h_t : t \in S^1 \}
\]

where

\[
h_t = \sup \{ h : \gamma \text{ is minimizing on } [t, t+h] \}. \quad (3.5)
\]

The following lemma is easily follows from Definitions 3.1 and 3.4.

Lemma 3.4 A closed geodesic \( \gamma \) of length \( L \) satisfies

\[
\frac{L}{\text{minind}(\gamma)} \leq \text{injrad}(\gamma) < \frac{L}{(\text{minind}(\gamma) - 1)}. \quad (3.6)
\]

Recall that a prime geodesic is a closed geodesic whose period is \( 2\pi \). All closed geodesics are either prime geodesics or iterated geodesics of the form \( \gamma_n(t) = \gamma_1(nt) \) where \( \gamma_1 \) is a prime geodesic, and \( n \in \mathbb{N} \).

Lemma 3.5 If \( \gamma_1 : S^1 \to M \) is a \( 1/k \) closed geodesic and \( \gamma_n : S^1 \to M \) is defined by \( \gamma_n(t) = \gamma_1(nt) \), then \( \gamma_n \) is an \( 1/(kn) \) geodesic. In fact

\[
\text{minind}(\gamma_n) \in [n(\text{minind}(\gamma_1) - 1), n \text{ minind}(\gamma_1)] \cap [2n, \infty). \quad (3.7)
\]

Proof: Let \( L \) be the length of \( \gamma_1 \) and \( k = \text{minind}(\gamma_1) \). Then \( nL = L(\gamma_n) \) and

\[
d(\gamma_n(t), \gamma_n(t + (2\pi/(kn))) = d(\gamma_1(nt), \gamma_1(nt + 2\pi/(k))) = L/k = L(\gamma_n)/(nk), \quad (3.8)
\]

which implies that \( \text{minind}(\gamma_n) \leq (nk) \). On the other hand, suppose \( j = \text{minind}(\gamma_n) \), then

\[
nL/j = d(\gamma_n(t), \gamma_n(t + (2\pi/j)) = d(\gamma_1(nt), \gamma_1(nt + 2\pi n/j)) \quad (3.9)
\]

So \( \text{injrad}(\gamma_1) \geq nL(\gamma_1)/j \) and applying Lemma 3.4 we get

\[
L(\gamma_1)/(\text{minind}(\gamma_1) - 1) \geq \text{injrad}(\gamma_1) \geq nL(\gamma_1)/j \quad (3.10)
\]

which implies \( \text{minind}(\gamma_1) - 1 \leq (j/n) \leq \text{minind}(\gamma_n)/n \).

Our final consideration is that \( \text{injrad}(\gamma_1) \leq L(\gamma_1)/2 \), so \( \text{minind}(\gamma_n) \geq 2n \). \( \square \)

We can now apply these lemmas to examine some examples.

Example 3.1 Suppose \( S^2 \) is the standard sphere. It is well known that all its prime closed geodesics have length \( 2\pi \). These geodesics can easily be seen to be \( 1/2 \) geodesics. By Lemma 3.4, we then have

\[
2k\pi \in L_{1/2k}(S^2) \quad (3.11)
\]

and by Lemma 3.4

\[
\{2\pi, 4\pi, ..., 2k\pi \} \subset L_{1/2k}(S^2) \subset L_{1/(2k+1)}(S^2). \quad (3.12)
\]

This also follows directly from Lemma 3.3. On the other hand, by Lemma 3.4

\[
L_{1/j}(S^2) \subset L(S^2) \cap (0, j\pi] \quad (3.13)
\]

which gives

\[
L_{1/2k}(S^2) = L_{1/(2k+1)}(S^2) = \{2\pi, 4\pi, ..., 2k\pi \}. \quad (3.14)
\]
Example 3.2 Let $M_j = S^1_\pi \times S^1_{\pi/j}$ be a flat torus from Example 2.1. The closed geodesics of the torus are of the form
\[
\gamma(t) = ((at + x_0) \mod 2\pi, (bt/j + y_0) \mod 2\pi/j),
\]
where $a, b \in \mathbb{Z}$ and $x_0, ky_0 \in [0, 2\pi]$. It is minimizing until $|a|t = \pi$ or $|b|t/j = \pi/j$, that is until $t = \min\{\pi/|a|, \pi/|b|\}$. So it’s minimizing index is
\[
\min\text{ind}(\gamma) = \max\{2|a|, 2|b|\}. \tag{3.16}
\]

Note that $\gamma$ is a prime geodesic whenever $a$ and $b$ are relatively prime or $ab = 0$ and one of them has absolute value 1. In particular the geodesic with $b = a + 1$ is a prime geodesic. So for any natural number, $k$, there is a prime geodesic in the torus with minimizing index $= 2k$.

The length of our arbitrary geodesic, $\gamma$, is
\[
L(\gamma) = \sqrt{(2\pi a)^2 + (2\pi b/j)^2} \tag{3.17}
\]
So, skipping the trivial geodesic, we have
\[
L_{1/(2k)}(M_j) = \{ \sqrt{(2\pi a)^2 + (2\pi b/j)^2} : a, b = 0, 1, 2, \ldots k \} \setminus \{0\}. \tag{3.18}
\]

and $L_{1/(2k-1)}(M_j) = L_{1/(2k)}(M_j)$.

Recall Lemma 3.5 which states that the nth iterate of $1/k$ geodesic is a $1/(nk)$ geodesic. This does not mean its minimizing index is $nk$. In fact in Example 3.3 we will demonstrate that the minimizing index of an iterated geodesic may take on any natural number allowed in the Lemma.

Example 3.3 We claim that for any $k \in \mathbb{N}$ there exists $c_k \in (0, 1]$ such that the the ellipsoid, $M(c_k)$:
\[
(x)^2 + (y)^2 + (z/c_k)^2 = 1. \tag{3.19}
\]
has a prime geodesic $\gamma_{c_k}(t) = (\cos(t), \sin(t), 0)$ whose minimizing index is $k + 1$. Furthermore, for any $n \in \mathbb{N}$, there exists $c_{n,k} \in (0, 1]$ such that $\gamma_{c_k}(nt)$ has minimizing index equal to $k + n$.

The brute force proof of the claim is to use the recent work of Itoh and Kiyohara to explicitly determine the cut locus of the points on this geodesic [IK]. One sees that $inj\text{rad}(\gamma_{c_k})$ varies continuously with $c$ taking on all values in $(0, \pi)$. The claim then follows by applying Lemma 3.4

Lemma 3.2 implies that the minimizing index of a closed geodesic, $\gamma : S^1 \to M$ satisfies
\[
\min\text{ind}(\gamma) \geq L(\gamma)/\text{Diam}(M). \tag{3.20}
\]
Thus any sequence of indefinitely increasingly long geodesics, like the prime geodesics found by Gromoll-Myer [GlMy], have minimizing index approaching infinity. This is also known to be true of the Morse Index which will be discussed later in Section 10.

4 1/2 Geodesics and the Covering Spectrum

In this section we produce a wealth of 1/2 geodesics in length spaces which are not simply connected.

Lemma 4.1 A closed geodesic which is the shortest among all noncontractible closed geodesics is a 1/2 geodesic.

This lemma is a consequence of the following one applied to the universal cover.

Lemma 4.2 If $\tilde{M}$ is a covering space of $M$ and $c$ is the shortest curve which lifts nontrivially to $\tilde{M}$, then $c$ is a 1/2 geodesic.
Proof: If \( c : S^1 \to M \) is not a 1/2 geodesic, then \( \exists t_0 > 0 \) such that \( d(c(t_0), c(t_0 + \pi)) < L/2 \). So we can join \( c(t_0) \) to \( c(t_0 + \pi/L) \) by a geodesic segment \( \eta \) of length \( < L/2 \). Let \( c_1 \) to be the curve created by taking \( c \) restricted to \( [t_0, t_0 + \pi] \) followed by \( \eta^{-1} \) and \( c_2 \) to be \( \eta \) followed by \( c \) from \( t_0 + \pi \) wrapping around to \( t_0 \). Since both \( c_i \) are shorter than \( c \), they must lift trivially to \( \tilde{M} \). This forces \( c \) to lift trivially as well, contradicting the hypothesis. \( \square \)

Note that closed geodesics which are minimizers in their homotopy classes are not necessarily 1/2 geodesics. This can be seen by looking at the geodesics in Figure 1 or by considering iterated geodesics in a torus. The covering spaces which unwrap these geodesics, also unwrap shorter geodesics, thus these geodesics do not satisfy the hypothesis of Lemma 4.2.

Geodesics which do satisfy the hypothesis of Lemma 4.2 were studied in SoWei. Their lengths correspond to the elements of the covering spectrum, which is defined using a special selection of covering spaces [Defn 3.1, Theorem 4.12 SoWei]. We can now improve this theorem.

**Theorem 4.1** If \( X \) is a compact space with a simply connected universal cover, then

\[
2CovSpec(X) \subset L_{1/2}(X). \tag{4.1}
\]

**Proof:** Theorem 4.12 of SoWei, stated that \( 2CovSpec(X) \subset L(X) \). A key step in the proof is Lemma 4.9 of SoWei, where one takes any element \( \delta \in CovSpec(X) \) and produces a a corresponding curve \( c \) of length \( 2\delta \) which satisfies the hypothesis of Lemma 4.2 above. \( \square \)

## 5 Systoles and 1/2 Geodesics

Recall that the systole of a manifold, \( sys(M) \), is the length of the shortest noncontractible closed geodesic (c.f. CrKt). This definition easily extends to any compact length space with a universal cover or a positive injectivity radius. Without a positive injectivity radius the systole may be 0 (see Example 2.4).

Combining Lemma 4.2 applied to the universal cover of \( M \) we immediately obtain the following lemma:

**Lemma 5.1** If \( M \) is a compact length space with a universal cover, then the shortest 1/2 geodesic has length

\[
sys(M) \in L_{1/2}(M) \subset (0, 2diam(M)] \tag{5.1}
\]

thus \( \min L_{1/2}(M) \leq sys(M) \).

Note that it is quite possible that this is a strict inequality as the shortest 1/2 geodesic could be contractible and wrapped around some small “knob” (c.f. CoHng).

There is a significant body of research providing upper bounds on the systole of various surfaces, and thus also \( \min L_{1/2} \). See for example, Croke and Katz’s recent survey article CrKt. Croke and Katz combine an inequality of Gromov G2 with a theorem by Pu Pu, to obtain the following proposition which we rephrase using Corollary 5.4.

**Proposition 5.2 (Pu)** If \( M^2 \) is not diffeomorphic to a sphere, then

\[
(min L_{1/2}(M))^2 \leq sys(M)^2 \leq \pi Vol(M)/2 \tag{5.2}
\]

and when equality holds, \( M^2 \) is the standard \( \mathbb{R}P^2 \) with constant sectional curvature.

We may also rephrase Loewner’s result (c.f. CrKt).
Proposition 5.3 (Loewner) If $M^2$ is diffeomorphic to a torus, then

\[(\min L_{1/2}(M))^2 \leq \text{sys}(M)^2 \leq 2\text{Vol}(M)/\sqrt{3}\]  

(5.3)

and when equality holds, $M$ is a skewed flat torus with a 120 degree angle.

In the following example the equality is only achieved on a singular manifold, so it is necessary to use our compact length definition of a $1/2$ geodesic. \[Bav\][Sak]

Proposition 5.4 (Bavard, Sakai) If $M^2$ is diffeomorphic to $\mathbb{R}P^2 \# \mathbb{R}P^2$, then

\[(\min L_{1/2}(M))^2 \leq \text{sys}(M)^2 \leq \pi \text{Vol}(M)/2^{3/2}.\]  

(5.4)

In this case equality doesn’t hold on a manifold, but rather on a singular space formed by gluing together two moebius strips of constant sectional curvature 1, with width $\pi/2$ and central curve of length $\pi$ along a singular circle. The singular circle is a geodesic in the metric space sense and achieves the minimal length, $\sqrt{2}\pi = \pi \text{Vol}(M)/2^{3/2}$.

In the case of manifolds diffeomorphic to $S^2$, all geodesics are contractible and the situation is much more complicated. Calabi and Croke have conjectured that on a surface diffeomorphic to a sphere,

\[\min L \leq \sqrt{12} \sqrt{\text{Vol}(M)},\]  

(5.5)

and an example achieving this inequality appears in \[Cr\].

The strongest result in this direction is by Rotman \[Ro\].

Proposition 5.5 (Rotman) If $M^2$ is diffeomorphic to a sphere then

\[\min L \leq 4\sqrt{2} \sqrt{\text{Vol}(M)}.\]  

(5.6)

The best estimate based on diameter is in \[NaRo\] and independantly \[Sab\]:

Proposition 5.6 (Nabutovsky-Rotman, Sabourou) If $M^2$ is diffeomorphic to a sphere then

\[\min L \leq 4 \text{Diam}(M).\]  

(5.7)

Note that neither Proposition 5.5 nor 5.6 provide bounds on $\min L_{1/2} \leq \min L$. It would be interesting to examine their proofs and see whether their techniques would provide upper bounds on the various $\min L_{1/k}$ [Problem 11.6].

Remark 5.7 Note that if we were to try to extend these volume estimates to compact length spaces then we would need a measure and a dimension for the spaces. One might study compact length spaces with finite second Hausdorff measure. See Problem 11.7.

There are many beautiful results estimating $\min L(M)$ for manifolds with curvature bounds, but discussion of such results and their relation to the $L_{1/k}$ must be postponed to future papers.

6 Estimating the Length of the Shortest Closed Geodesic

In this section we discuss how $1/k$ geodesics may be used to estimate the length of the shortest closed geodesic in a Riemannian manifold or compact length space.

Gromov \[G2\] has conjectured that for a compact Riemannian manifold $M$,

\[\min L(M) \leq c(n)\text{Vol}(M^n)^{1/n}\]  

(6.1)
and another well-known conjecture is that
\[ \min L(M) \leq c(n) \text{Diam}(M^n). \] (6.2)

In fact Rotman suggests that
\[ \min L(M) \leq 2 \text{Diam}(M^n) \] (6.3)

and there are no known counterexamples. Note that (6.3) is trivially true when the manifold is not simply connected (c.f. covspec lemma and Remark 6.1).

**Remark 6.1** It follows from Lemma 3.2, that if \( L_{1/2}(M) \) is nonempty, then
\[ \min L(M) \leq \min L_{1/2}(M) \leq 2 \text{diam}(M). \] (6.4)

However, the author has recently been informed that Wing Kai Ho has produced examples of smooth manifolds diffeomorphic to \( S^2 \) which have no \( 1/2 \) geodesics \([H]\). In Problem 11.8 we ask what properties can be imposed on a manifold that would guarantee the existence of a \( 1/2 \) geodesic.

To provide an estimate on \( \min L \) it is necessary to define the following quantity:

**Definition 6.1** The minimizing index, \( \min \text{ind}(M) \), of a compact length space, \( M \), is the smallest \( k \) such that there is a geodesic of minimizing index \( k \).

**Theorem 6.1** If \( M \) is a compact length space then
\[ \min L(M) \leq \min \text{ind}(M) \text{diam}(M). \] (6.5)

**Proof:** Setting \( k = \min \text{ind}(M) \), we know We know \( \min L(M) \leq \min L_{1/k}(M) \leq k \text{diam}(M) \) by Lemma 3.2.

The following theorem might help one find counterexamples to overly sharp conjectures regarding \( \min L(M) \). See Problems 11.11 and 11.12.

**Theorem 6.2** If \( M \) is a compact length space and \( k = \min \text{ind}(M) \) then
\[ \min L(M) \geq \min \{k \text{injrad}(M), \min L_{1/k}(M)\}. \] (6.6)

**Proof:** By Lemma 3.2 if \( M \) has \( \text{injrad}(M) \geq i_0 > 0 \) and minimality index \( k \), then taking \( L_0 = i_0k \) we have
\[ L(M) \cap [0, L_0] = L_{1/k}(M) \cap [0, L_0]. \] (6.7)

Thus \( \min L(M) \) either = \( \min L_{1/k}(M) \) or it is > \( i_0k \).

When \( M \) is a manifold we can use Klingenberg’s old lemma to estimate its minimizing index as follows \([K]\) (c.f. \([KCC]\)):

**Lemma 6.2 (Klingenberg)** Let \( M \) be a compact Riemannian manifold. If \( x \) is the closest cut point to \( y \), then either \( x \) is a conjugate point of \( y \), or there are exactly two geodesics from \( y \) to \( x \) and they meet at 180°. If \( x \) and \( y \) are cut points such that \( d_M(x, y) = \text{injrad}(M) \), then either they are conjugate points or there are exactly two geodesics from \( x \) to \( y \) and together they form a closed geodesic.

**Corollary 6.3** If \( M \) is a compact Riemannian manifold with no conjugate points then the shortest closed geodesic is a \( 1/2 \) geodesic of length \( 2 \text{injrad}(M) \). So \( \min \text{ind}(M) = 2 \) and \( \min L = 2 \text{injrad}(M) \).
In Problem 11.5 we ask for an appropriate extension of the definition of conjugate point to compact length spaces which might allow one to extend Corollary 6.3.

Note that Klingenberg applied his lemma to manifolds with negative sectional curvature as these spaces have no conjugate points. We suggest in Problem 11.4 that Corollary 6.3 might extend to CAT(0) spaces.

In general, however, the author leaves the discussion of the rich literature concerning the length spectrum and sectional curvature bounds out of this paper. We will discuss Ricci curvature bounds as such bounds lead to applications relating to Gromov-Hausdorff convergence.

7 Convergence without Sudden Disappearances

In this section we prove our main convergence theorem and present some simple illustrative examples.

Theorem 7.1 If $M_i \rightarrow M$ in the GH sense then $L_{1/k}(M_i) \converges to a subset of L_{1/k}(M) \cup \{0\}$ in the Hausdorff sense. That is, for all $\epsilon, R > 0$, there exists $N \in \mathbb{N}$ such that

$$L_{1/k}(M_i) \cap [0, R] \subset T_r(L_{1/k}(M) \cup \{0\}) \quad \forall i \geq N.$$  \hspace{1cm} (7.1)

Recall that in Figure 2 we gave an example with suddenly appearing $1/2$ geodesics, which are not the limits of such a sequence of $L_i$ or even $L_i$ selected from $L(M_i)$.

**Proof:** Suppose, on the contrary, there exists $\epsilon$ such that for a subsequence of the $i$ there are $L_i \in L_{1/k}(M_i) \setminus T_r(L_{1/k}(M) \cup \{0\})$. \hspace{1cm} (7.2)

By Lemma 3.2 and the fact that Gromov-Hausdorff convergence implies that $\text{diam}(M_i) \rightarrow D = \text{diam}(M), L_i \in [0, k\text{diam}(M_i)] \subset [0, 2kD]$. So a further subsequence must converge, $L_i \rightarrow L_0$, where

$$L_0 \in (0, 2kD) \setminus L_{1/k}(M).$$ \hspace{1cm} (7.3)

Thus there exists corresponding $1/k$ geodesics $\gamma_i : S^1 \rightarrow M_i$ of speed $L_i/(2\pi)$. By Grove Petersen’s Arzela-Ascoli Theorem [GrPet], we know that a subsequence of the $\gamma_i$ converges to a curve $c : S^1 \rightarrow M$ of length $L_0$. Furthermore

$$d_M(c(t - \pi/k, c(t + \pi/k)) = \lim_{i \rightarrow \infty} d_{M_i}(\gamma_i(t - \pi/k), \gamma_i(t + \pi/k)) = \lim_{i \rightarrow \infty} L_i/k = L_0/k,$$ \hspace{1cm} (7.4)

so $c$ is either a $1/k$ geodesic or it is trivial. This contradicts (7.3). \hfill $\square$

**Remark 7.1** The same proof could be used to show that if $\gamma_i$ all have $\text{injrad}(\gamma_i) \geq r_0$, then their limit does as well.

The following two corollaries are immediate:

**Corollary 7.2** Suppose $L_i \in L(M_i)$ and $L_i \rightarrow L_\infty \notin L(M)$ where $M$ is the Gromov-Hausdorff limit of the $M_i$. Then the geodesics $\gamma_i : S^1 \rightarrow M_i$ of length $L(\gamma_i) = L_i$ have $\text{minind}(\gamma_i)$ diverging to infinity.

**Corollary 7.3** Given any compact length space $M$, $L_{1/k}(M) \cup \{0\}$ is compact.
Example 7.1 Recall $M_j = S^1_\pi \times S^1_{\pi/j}$, the flat tori of Example 2.1 that converged to a circle whose length spectra had disappearing lengths. In Example 3.2 we showed $L_{1/(2k)}(M_j)$ is the union:
\[
\{ \sqrt{(2\pi a)^2 + (2\pi b/j)^2} : a, b = 1, 2, ... k \} \cup \{ 2\pi b/j : b = 1, 2, ... k \} \cup \{ 2\pi a : a = 1, 2, ... k \}
\]
(7.6)
and $L_{1/(2k-1)}(M_j) = L_{1/(2k)}(M_j)$. As $j$ diverges to infinity, $L_{1/(2k)}(M_j)$, converges in the Hausdorff sense to the union
\[
\{ \sqrt{(2\pi a)^2 + (0b)^2} : a, b = 1, 2, ... k \} \cup \{(0b : b = 1, 2, ... k) \} \cup \{ 2\pi a : a = 1, 2, ... k \}
\]
which is $\{ 2\pi a : a = 0, 1, 2, ... k \} = L_{1/(2k)}(S^1)$.

Example 7.2 Recall the sequence of ellipsoids, $M_j^2$, from Example 2.3 converging to a doubled disk, $Y$.

Note that the curves $h_j(t) = (\cos(t), \sin(t), 0)$ mapped into $M_j$ are closed geodesics. Their pointwise limit as $c_j \to 0$ is $h_\infty(t) = (\cos(t), \sin(t), 0)$ mapped into $M_\infty$. Note that $h_\infty$ is parametrized by arclength but
\[
d_{M_\infty}(h_\infty(t - \epsilon), h_\infty(t + \epsilon)) = 2\sin(\epsilon) < 2\epsilon.
\]
(7.8)
So $h_\infty$ is not a closed geodesic. Thus there exists no uniform lower bound on the minimizing index of the $h_j : S^1 \to M_j$. This can also be seen using the recent work of [ItKi].

The next example demonstrates why it is necessary to study $1/k$ geodesics rather than just smooth regular polygons that are minimizing between only $k$ specific regularly spaced points instead of any collection of $k$ regularly spaced points.

Example 7.3 In Figure 3 we see $M_\epsilon \subset \mathbb{E}^3$, the boundary of the $\epsilon$ tubular neighborhood around a flat solid regular square $Z \in \mathbb{E}^2 \times \{0\}$. For $\epsilon$ sufficiently small we can see that the geodesic, $\gamma_\epsilon : S^1 \to M_\epsilon$ running around the equator looks almost like a square. If one chooses a specific regularly spaced selection of four points on $\gamma_\epsilon$ each of which is close to the corner of the square, one sees that $\gamma_\epsilon$ is minimizing between these points. However $\gamma_\epsilon$ is not a $1/4$ geodesic.

Figure 3: The black closed geodesic, $\gamma_\epsilon$, is minimizing between the four corners but not their midpoints as indicated by the white geodesic segment.

As in Example 7.2, the limit space as $\epsilon$ approaches 0 is a doubled copy of $Z$ glued to itself along the square boundary. The square boundary is only piecewise minimizing between the corners and is not a closed geodesic. Thus the $\gamma_\epsilon$ do not converge to a closed geodesic in the limit space.

Remark 7.4 If $M_i$ converge to the standard $S^n$ smoothly, then Bangert proved that the prime geodesics in $M_i$ either have lengths converging to $2\pi$ or to $\infty$ [Bng]. Note that the prime geodesics $\gamma_i$ whose lengths diverge to infinity, have minimal indices also diverging to infinity by Lemma 3.2.

Bangert’s Theorem does not extend to $M_i$ converging to $S^n$ in the Gromov-Hausdorff sense. If we take $M_j = S^n \times S^1_{\pi/j}$ and prime geodesics wrapping once around the equator of $S^n$ while wrapping $j$ times around the $S^1_{\pi/j}$. These all have length $4\pi$. Also the geodesics in Example 7 are prime geodesics converging to a length $< \pi$. 

In Problem 11.14 we ask whether Bangert’s Theorem described here holds when one assumes the \( M_i \) converge in the Gromov Hausdorff sense with a uniform positive lower bound on injectivity radius.

8 Gap Theorems and Ricci Curvature

In this section we apply the length spectrum convergence theorem [Theorem 7.1] to force the existence of gaps in the length spectrum of certain manifolds with Ricci curvature bounds. We begin by rephrasing Theorem 7.1 as a gap theorem:

**Theorem 8.1** Fix a compact length space, \( M \), and choose any \( \epsilon > 0 \) and \( b > 0 \), then there exists \( \delta_{\epsilon,b,M} \) such that if

\[
(a, b) \cap L1/k(M) = \emptyset
\]

then

\[
[a + \epsilon, b - \epsilon] \cap L1/k(N) = \emptyset
\]

for all compact length spaces \( N \) such that \( d_{GH}(N, M) < \delta_{\epsilon,b,M} \).

**Proof:** Suppose on the contrary that there exists \( \epsilon > 0 \), \( k \in \mathbb{N} \) and \( N_i \) converging to \( M \) with

\[
L_i \in [a + \epsilon, b - \epsilon] \cap L1/k(N_i).
\]

Then by Theorem 7.1 a subsequence of the \( L_i \) converge to some \( L \in L1/k(M) \). Since \( L_i \in [a + \epsilon, b - \epsilon] \), so is \( L \) which contradicts (8.1).

The next gap theorem refers directly to the length spectrum.

**Theorem 8.2** Fix a compact length space, \( M \), and choose any \( \epsilon > 0 \) and \( b > 0 \), then there exists \( \delta_{\epsilon,b,M} \) such that if

\[
(ai_0, bi_0) \cap L(M) = \emptyset
\]

then

\[
[ai_0 + \epsilon, bi_0 - \epsilon] \cap L(N) = \emptyset
\]

for all compact length spaces \( N \) such that \( d_{GH}(N, M) < \delta_{\epsilon,b,M} \) with \( injrad(N) \geq i_0 \).

**Proof:** First note that (8.3) implies that

\[
(ai_0, bi_0) \cap L1/k(M) = \emptyset \quad \forall k \in \mathbb{N}.
\]

Now we choose \( k \geq b \), and apply Theorem 8.1 for that \( k \), and take \( \delta_{\epsilon,b,M} := \delta_{\epsilon,k,M} \), which implies that

\[
[a + \epsilon, b - \epsilon] \cap L1/k(N) = \emptyset.
\]

Restricting to \( N \) with \( injrad(N) \geq i_0 \) and applying Lemma 3.3 we get (8.5). □

Notice if one happens to take a sequence of \( N_i \to M \) whose injectivity radii converge to 0, we can still apply Theorem 8.2 but the gaps slide over towards 0 and shrink. This is seen to be exact in Example 2.1.

When one has \( C^2 \) convergence of the manifolds, Ehrlich has proven the injectivity radii converge, in which case Theorem 8.2 is significantly stronger and basically already known [EH].

It is crucial to understand that even with smooth convergence we do not get uniform \( \delta \) depending only on \( \epsilon \). They will always depend on the manifold itself. Otherwise we would never have suddenly appearing geodesics. That is, if \( \delta_{M,b,\epsilon} \) did not depend on \( M \), then take \( M_i \) converging
Proposition 8.1 (Colding) If $N^n$ has $\text{Ricci}(N^n) \geq -(n-1)$, then for all $\epsilon > 0$, there exists $\delta_{\epsilon,n} > 0$ such that
\[ \text{Vol}(N^n) \geq \text{Vol}(S^n) - \delta_{\epsilon,n} \] (8.8)
implies $d_{GH}(N^n, S^n) < \epsilon$.

This proposition combined with Theorem 8.1 and the length spectrum of $S^n$ in Example 8.1 implies the following:

Proposition 8.2 For all $\epsilon > 0$, and any $k \in \mathbb{N}$, there exists $\delta_{\epsilon,k,n} > 0$ such that
\[ \text{Vol}(N^n) \geq \text{Vol}(S^n) - \delta_{\epsilon,k,n} \] (8.9)
and $\text{Ricci}(N^n) \geq -(n-1)$ then for all $j \in \{0, 1, 2, \ldots\}$ we have:
\[ [2j\pi + \epsilon, 2(j+1)\pi - \epsilon] \cap L_{1/(2\epsilon)}(N) = \emptyset. \] (8.10)

Combining this proposition with Lemma 8.4 and Lemma 8.5 we obtain Theorem 1.1 which was stated in the introduction.

Remark 8.3 Problem 11.21 asks for precise estimates on the estimating function in Theorem 1.1. Given a precise estimate, one would be able to bound the volume of a manifold $N^n$ with $\text{Ricci} \geq (n-1)$ depending on the length and minimal index of one of its closed geodesics.

Although Colding did later prove convergence in the $C^{1,\alpha}$ topology, there are manifolds, $M^n_i$, satisfying $\text{Ricci}(M^n_i) \geq (n-1)$, $\text{Vol}(M_i) \to \text{Vol}(S^n)$ whose injectivity radii $\text{injrad}(M_i) \to 0$ [Example 8.1]. So we cannot presume to improve the length spectrum’s convergence or obtain an estimate on $\min L(N^n)$ without imposing an additional condition on the injectivity radius.

Example 8.1 We now construct smooth American footballs, $M^2_j$, with $\text{sect} \geq 1$ (and thus $\text{Ricci} \geq 1$) whose volume $\text{Vol}(M_j) \geq a_j^2\text{Vol}(S^2) - \epsilon_j$ with $a_j \to 1$ and $\text{injrad}(M_j) \leq r_j \to 0$.

Start with the standard $S^2$, remove a wedge of angle $(1 - a_j)2\pi < \pi/4$, and glue the edges to themselves to get a singular manifold, $F_j$, of volume $a_j^2\text{Vol}(S^2)$. For small $r_j > 0$, take two points $p_j, p'_j$ both $r_j/2$ away from a singular point and maximally far apart. There are two distinct geodesics running between them of length less than $r_j$. Let $h_j$ be the distance from these geodesics to the singular point. Then $h_j > r_j\cos(a_j\pi/2)$, the height of the Euclidean comparison triangle.

Now if we remove balls of radius $h_j/2$ about the two singular points in $F_j$, we can cap off these regions smoothly with caps whose sect $\geq 1$. This gives our surfaces $M_j$ and the points $p_j$ and $p'_j$ are still cut points in $M_j$ and so $\text{injrad}(M_j) < r_j$ and $\text{Vol}(M_j) \geq \text{Vol}(F_j) - \pi r_j^2$.

It is not clear how the length spectrum of these $M_j$ behave. Are there examples of $M_j$ with $\min L(M_j) \to 0$ or a disappearing length? [Problems 11.21 and 11.22].
Myers Theorem states that any manifold $M^n$ with $\text{Ricci} \geq (n-1)$ has $\text{diam}(M^n) \leq \pi$ because any geodesic of length $\pi$ must have a conjugate point (c.f. [doC], [MV]). Cheng's Sphere Rigidity Theorem states that this inequality is only achieved on a sphere [Chng].

Cheng's Theorem doesn't have a stability theorem like Proposition 8.1, as is demonstrated by Otsu's examples [Ot]. Otsu's five dimensional manifolds satisfy the Ricci bound and their diameter approaches $\pi$ but they converge in the Gromov-Hausdorff sense to a singular manifold not a sphere. This limit space contains only two points which are a distance $\pi$ apart.

**Remark 8.4** Cheeger-Colding have proven that a manifold with $\text{Ricci} \geq (n-1)$ and diameter close to $\pi$ is Gromov-Hausdorff close to a spherical suspension over a subset of the manifold [ChCo]. This is called an “almost rigidity” result rather than a stability result because they do not prove it is close to a particular metric space, but rather than the metric behaves in an almost rigid manner. In Problem [11.23] we question whether one can obtain a gap theorem based on such a result. One of the biggest difficulties there would be turning this spherical suspension into a length space and not just a metric space. Then naturally one would need to know if there are any uniform properties of the length spectrum on spherical suspensions [Problem 11.24].

To avoid the issue arising in Otsu’s examples, Colding instead examined the radius:

**Definition 8.1** The radius of a compact metric space, $M$, is the smallest $r > 0$ such that $M \subset \bar{B}_p(r)$ for some $p$. In fact

$$\text{rad}(M) = \inf_{p \in M} \sup_{q \in M} d(p,q) \leq \text{diam}(M).$$

(8.11)

When a manifold with $\text{Ricci} \geq (n-1)$ has radius close to $\pi$, then every point in the manifold has a point almost maximally distant from it, thus it is approaching the inequality in Myer’s Theorem along every geodesic in the manifold. Colding proved the following stability result [Co2].

**Proposition 8.5** [Colding] Given $n \geq 2$ and $\epsilon > 0$ there exists $\delta(n,\epsilon) > 0$ such that if $N^n$ is a compact Riemannian manifold whose $\text{Ricci}(M) \geq (n-1)$ and $\text{rad}(M) > \pi - \delta$ then $\text{Vol}(M) > \text{Vol}(S^n) - \epsilon$.

Combining Proposition 8.5 with Theorem 8.1 we obtain:

**Theorem 8.3** There exists a function $\Psi : \mathbb{R}^+ \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}^+$ such that $\lim_{\delta \to 0} \Psi(\delta,k,n) = 0$ such that if $N^n$ is a compact Riemannian manifold with

$$\text{rad}(N^n) \geq \pi - \delta$$

and $\text{Ricci}(N^n) \geq (n-1)$ then

$$L_1(2k)(M^n) \subset [0,\epsilon_k) \cup (2\pi - \epsilon_k, 2\pi + \epsilon_k) \cup \cdots (2k\pi - \epsilon_k, 2k\pi + \epsilon_k)$$

(8.13)

for $\epsilon_k = \Psi(\delta,k,n)$.

**Remark 8.6** Note that Example 8.1 also has $\text{rad}(M_i) \to \text{rad}(S^2)$, so here we also have no lower bound on injectivity radius and cannot directly conclude a stronger convergence of the length spectrum. See Problems [11.25] and [11.26].

Gromov proved that any Riemannian manifold $M^n$ with $\text{Ricci} \geq 0$ and first Betti number $b_1(M) = n$ is isometric to a torus [GT]. The corresponding stability theorem is hidden in Colding’s proof that any $M^n$ with $b_1(M^n) = n$ and $\text{Ricci} \geq -(n-1)\epsilon$ is homeomorphic to $\mathbb{T}^n$ if $\epsilon$ is sufficiently small [Co3].
Proposition 8.7 (Colding) For any $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that if $M^n$ has $b_1(M^n) = n$ and $\text{Ricci} \geq -(n-1)\delta_\epsilon$, then

$$d_{GH}(M^n, T^n) < \epsilon.$$ (8.14)

Combining this with Theorem 8.1 and the fact that

$$L_{1/(2k)}(T^n) = \{ L^k_1, L^k_2, \ldots, L^k_{m(k)} \}$$ (8.15)

we get the following:

Theorem 8.4 For all $\epsilon > 0$, and any $k \in \mathbb{N}$, there exists $\delta_{\epsilon,k} > 0$ such that if $N^n$ is a Riemannian manifold with

$$\text{Ricci}(N^n) \geq -(n-1)\delta_{\epsilon,k}$$ (8.17)

and $b_1(N^n) = n$ then then for all $j \in \{0, 1, 2, \ldots, m(k)\}$ we have:

$$[L^k_j + \epsilon, L^k_{j+1} - \epsilon] \cap L_{1/(2k)}(N) = \emptyset,$$ (8.18)

where $0 = L^k_0 < L^k_1 < L^k_2 < \ldots$ is given in (8.15).

In Problems 11.27 and 11.28 we ask for relevant examples with arbitrarily small min $L$ or disappearing lengths. Examples with injectivity radius approaching 0 have been described by Colding. One begins by removing balls of radius $1/2$ from a flat torus, gluing in very flat cones, and then smoothly capping them off carefully to keep the injectivity radius exactly as in Example 8.1 and finally smoothing the boundaries of the balls which adds the slightly negative curvature.

Remark 8.8 In [So1], the author has proven another stability theorem, that a locally almost isotopic manifold with $\text{Ricci} \geq -(n-1)H$ is Gromov-Hausdorff close to an isotopic manifold. When the manifold is compact it close to a Riemannian manifold homothetic to a sphere. Thus we would also get a gap theorem for such a manifold. Since the definition of locally almost isotopic is complicated, we do not give the complete explanation here. See Problem 11.29.

9 Openly 1/k Geodesics

In this section we define a new collection of geodesics and lengths which behaves a bit better than 1/k geodesics on manifolds.

Definition 9.1 An openly 1/k geodesic $\gamma : S^1 \rightarrow M$ is a 1/k geodesic which has $\text{injrad}(\gamma) > L(\gamma)/k$.

The great advantage of an openly 1/k geodesic is the following lemma.

Lemma 9.1 If $M$ is a compact Riemannian manifold and $\gamma$ is an openly 1/k geodesic, then it is uniquely determined by any collection of $k$ evenly spaced points up to reparametrization by an isometry of $S^1$.

Proof: This follows from the fact that if $\gamma$ is minimizing on $[a, b]$ then it is uniquely determined on $[a, c]$ for any $c \in (a, b)$ (c.f. [doC]).
Lemma 9.1 does not hold on a compact length space:

Example 9.1 Let $Y$ be a graph with four ordered vertices, $\{v_1, v_2, v_3, v_4 = v_0\}$, and two unit edges $e_i^+$ and $e_i^−$ between $v_i$ and $v_{i+1}$ for $i = 0, 1, 2, 3$. Let $γ$ be the geodesic which traverses $e_1^+, e_2^−, e_3^+, e_4^−$. It is in fact a 1/2 geodesic and thus an openly 1/4 geodesic. However, it is not uniquely determined by the 4 evenly spaced points $v_1, v_2, v_3, v_4$ as there is another geodesic sharing those points which traverses $e_1^-$, $e_2^+$, $e_3^-$, and $e_4^+$. 

Lemma 9.1 does not hold if one only assumes $γ$ is a $1/k$ geodesic instead of openly $1/k$ as we see in the next example:

Example 9.2 If we take a Riemannian manifold depicted in Figure 4

$$M_ε = \partial T_ε([-1,1] \times [-1,1] \times \{0\}) \subset \mathbb{R}^3,$$

and choose the points

$$p_j = ((1 + \varepsilon)\cos(j\pi/2), (1 + \varepsilon)\sin(j\pi/2), 0)$$

then we claim the piecewise geodesic, $γ$, which runs minimally with positive $z$ from $p_0$ to $p_1$, negative $z$ from $p_1$ to $p_2$, positive $z$ from $p_2$ to $p_3$ and negative $z$ from $p_3$ to $p_0$, is a 1/4 geodesic.

![Figure 4: Here we have two copies of $M_ε$. The geodesic $γ$ is depicted on the right.](image)

To prove we show that in fact $γ$ is actually minimizing on $[j\pi/2 - s, j\pi/2 + s]$ for any $s < \pi/2$. First note the $z$ components of $γ(j\pi/2 - s)$ is the negative of $γ(j\pi/2 + s)$. So if $σ$ runs minimally between these two points it must have an $s_0$ where its $z$ component is 0. By symmetry $σ(s_0)$ must be located at $p_1$, thus $σ$ must agree with $γ$. So $γ$ is actually minimizing between any $t$ and $t + 2\pi/4$!

Note that $γ$ is also not uniquely determined by the $p_i$ because there is another geodesic which is the reflection of $γ$ through the xy plane running through the same four points also depicted on the left in Figure 4.

We now develop the theory of openly $1/k$ geodesics.

Definition 9.2 Let $L_{1/k}^{open}(M)$ be the collection of lengths of openly $1/k$ geodesics.

The following lemma is an easy exercise:

Lemma 9.2 For any $k > 2$ we have

$$L_{1/(k-1)}(M) \subset L_{1/k}^{open} \subset L_{1/k}(M)$$

and $L_{1/2}^{open}(M) = \emptyset$.

In particular there are no openly $1/2$ geodesics. Lemma 9.2 combined with Theorem 3.1 immediately implies:
**Theorem 9.1**  On any compact length space, $M$,

$$L(M) = \bigcup_{k=3}^{\infty} L_{1/k}^{\text{open}}(M).$$  \hfill (9.4)

**Definition 9.3**  Let the openly minimizing index, $\text{opind}(\gamma)$, of a geodesic, $\gamma$, be the smallest $k$ such that $\gamma$ is an openly $1/k$ geodesic.

Let $\text{opind}(M) = \min\{\text{opind}(\gamma) : \gamma : S^1 \to M\}$.

Lemma 9.2 immediately implies:

**Lemma 9.3**

$$\minind(\gamma) \leq \text{opind}(\gamma) \leq \minind(\gamma) + 1.$$  \hfill (9.5)

Note that in the flat torus and in the sphere all closed geodesics have $\minind(\gamma) < \text{opind}(\gamma)$ [Examples 9.2 and 9.1]. Manifolds with this property are of significant interest because we are able to bound the open index in Theorems 10.2 and 10.3 below. See Problem 11.10.

Theorem 7.1 and Lemma 9.2 together imply:

**Theorem 9.2**  If $M_j \to M$ in the GH sense then

$$\lim_{j \to \infty} L_{1/k}^{\text{open}}(M_j) \subset L_{1/k}(M).$$  \hfill (9.6)

We do cannot improve this to

$$\lim_{j \to \infty} L_{1/k}^{\text{open}}(M_j) \subset L_{1/k}^{\text{open}}(M).$$  \hfill (9.7)

as can be seen in the following example.

**Example 9.3**  In Example 7.2 we demonstrated that the equators, $\gamma_c$, of flattening ellipsoids $(x^2 + y^2 + (z/c)^2 = 1$ had $\text{injrad}(\gamma_c)$ varying continuously with $c$ and converging to 0 as $c \to 0$.

When $c = 1$, $\minind(\gamma) = 1/2$ and $\text{injrad}(\gamma) = \pi$. As $c$ decreases, the injectivity radius decreases continuously, and at some $c_0 > 0$ the injectivity radius hits $2\pi/3$ for the first time. So for all $c > c_0$, $\gamma_c$ is an openly $1/3$ geodesic but $\gamma_{c_0}$ is not.

So if $c_j$ decrease to $c_0$,

$$\min_{j \to \infty} L_{1/3}^{\text{open}}(M_{c_j}) \notin L_{1/3}^{\text{open}}(M_{c_0})$$  \hfill (9.8)

even though $M_{c_j}$ converges to $M_{c_0}$ in the $C^\infty$ and Gromov-Hausdorff sense.

It is of some interest to understand what is special about $1/k$ geodesics which are not openly $1/k$ geodesics. On such a geodesic, $\gamma$, there is a pair of cut points which are a distance $L(\gamma)/k$ apart.

**Proposition 9.4**  If $M$ is a compact Riemannian manifold with no conjugate points and $\gamma$ is a $1/k$ geodesic of length $\text{injrad}(M)$ which is not an openly $1/k$ geodesic then either $k = 2$ or $\gamma$ is the iterate of a $1/2$ geodesic and $k$ is even.

**Proof:**  If $\gamma : S^1 \to M$ is a $1/k$ geodesic which is not an openly $1/k$ geodesic, then it has a pair of cut points on it which are a distance $L(\gamma)/k$ apart. If $L(\gamma)/k = \text{injrad}(M)$ and $M$ has no conjugate points, then by Klingenberg’s Lemma 6.2, these two points are joined by exactly two geodesics which close up smoothly, thus either $k = 2$ or $\gamma$ is an iterated geodesic $\gamma(t) = \gamma_0(kt/2)$ with $k$ even. Furthermore $\gamma_0$ is a $1/2$ geodesic because $d(\gamma_0(t), \gamma_0(t + \pi)) \geq \text{injrad}(M) = L(\gamma_0)/2.$

**Proposition 9.4** is not true on metric spaces.

**Example 9.4**  Let $M$ be the metric space which is a graph with two vertices and three unit length edges each running from one vertex to the other. Then $\text{injrad}(M) = 1$ and any path which runs back and forth between the endpoints with constant speed and never traverses back on the edge it just crossed over is a geodesic. Thus for any $k \in \mathbb{N}$ $M$ has many prime $1/(2k)$ geodesics of length $2\text{injrad}(M)$.  \hfill \Box
10 Energy and Openly \(1/k\) Geodesics

Here we introduce an energy method which may be used to prove the existence of a \(1/k\) geodesic on a given space with certain properties, thus allowing one to estimate \(\text{minind}(M)\) and thus \(\text{min} L(M)\) via Theorem 10.1 and Lemma 9.3.

In this section we limit ourselves to convex compact Riemannian manifolds with boundary so that we can discuss the derivative of a geodesic. The convexity assumption guarantees the geodesics won’t touch the boundary. Background material may be found in [BTZ] and [Mil].

Smoothly closed geodesics are the critical points of the energy function on the loop space of \(M\):

\[ E(c) = \int_0^1 g(c'(t), c'(t)) \, dt \]  
(10.1)

It is easy to see that when we have a critical point of this energy, one gets a smoothly closed geodesic. Furthermore if \(c\) is a smoothly closed geodesic and is known to be minimizing on subintervals \([t_i, t_{i+1}]\) then the energy satisfies:

\[ E(\gamma) = \sum_{i=1}^{N} d(\gamma(t_{i+1}), \gamma(t_i))^2 / (t_{i+1} - t_i). \]  
(10.2)

So if \(\gamma\) is a \(1/k\) geodesic, then

\[ E(\gamma) = \sum_{i=0}^{k-1} d(\gamma((i + 1)/k), \gamma(i/k))^2 / (1/k). \]  
(10.3)

In Morse Theory one uses a uniform lower bound on injectivity radius and makes a finite dimensional approximation of the loop space. That is any smoothly closed geodesic of length \(\leq L\) can be viewed as a critical point in

\[ \Omega_k(M) \subset (M)^k = M \times M \times \cdots \times M \]  
(10.4)

where

\[ \Omega_k(M) = \{ (x_1, \ldots, x_k) : d(x_i, x_{i+1}) \leq i_0 \}. \]  
(10.5)

of the energy function

\[ E(x_1, \ldots, x_k) = \sum_{i=0}^{k} d(x_i, x_{i+1})^2 / (1/k) \]  
(10.6)

where \(k \geq L/i_0\). Once one finds the \(x_i\) which give a critical value, you join them by the unique geodesic segments between them to get a loop and prove that this loop is a smoothly closed geodesic.

In particular one has the following old theorem:

**Theorem 10.1** [c.f. [Mil]] If \(M\) is a manifold, given a set of length segments \(r_i \in \mathbb{R}^+\) we can define

\[ E_{(r_1, r_2, \ldots, r_k)}(x_1, \ldots, x_k) = \sum_{i=1}^{k} d(x_i, x_{i+1})^2 / r_i \]  
(10.7)

where \(x_{k+1} = x_1\). Then \((y_1, \ldots, y_k)\) is a smooth critical point of \(E : (M)^k \to \mathbb{R}\) iff for all \(i \in \{1, 2, \ldots, k\}\) we have:

\[ d(x_i, x_{i+1})/r_i = d(x_{i-1}, x_i)/r_{i-1} \]  
(10.8)

and

\[ \nabla \rho_{x_{i+1}} = -\nabla \rho_{x_{i-1}} \text{ at } x_i, \]  
(10.9)

where \(\rho_x(y) = d(x, y)\).
Note that $\nabla \rho_x$ is not defined at cut points of $x$. Here however, we avoided this issue by explicitly stating that we are at a smooth critical point.

In particular, if we study $E = E_{1/k,1/k,...,1/k}$ on $\Omega_k$, it is a smooth function when it’s values are less than $L$. So all of its critical points below $k^2L^2$, are smooth geodesics which are minimizing between $k$ evenly spaced points.

**Example 10.1** Let $M_\epsilon = \partial T_\epsilon(Y) \subset \mathbb{R}^3$ where $Y$ is a flat solid regular square in $\mathbb{E}^2 \times \{0\}$ as in Example 7.3. For $\epsilon$ sufficiently small we can see that the geodesic running around the equator looks almost like a square and is the critical point of the energy in $E_{1/k,k,...,1/k}$ for $k=4$ when the $x_i$ are near the vertices of the square. However, it is not a minimizing geodesic between the midpoints of the sides, and so it is not a $1/4$ geodesic.

Nevertheless we would like to use Theorem 10.1 to identify the openly $1/k$ geodesics. First, we do not restrict ourselves to $\Omega_k$ using an injectivity radius, nor do we restrict the values of the energy. This allows us to search for long and short openly $1/k$ geodesics.

**Definition 10.1** Let $E = E_{1/k,1/k,...,1/k} : (M)^k \to \mathbb{R}$ be called the uniform energy.

**Corollary 10.1** For any openly $1/k$ geodesic $\gamma : S^1 \to M$ and any $t \in S^1$ the point

$$ (\gamma(t), \gamma(t + 2\pi/k), \gamma(t + 4\pi/k), ..., \gamma(t - \pi/k)) \in (M)^k $$

(10.10)

is a smooth critical point of the uniform energy on $M^k$. As we run through all values of $t$ we get a critical level set, which is itself a closed geodesic in $(M)^k$.

Before we set up the converse, we add a short lemma about geodesics generated by critical points.

**Lemma 10.2** If $\bar{x} = (x_1, ..., x_k) \in (M)^k$ is a smooth critical point of the uniform energy $E : M^k \to \mathbb{R}$, then it defines a unique closed geodesic, $\gamma_{\bar{x}}$, which runs minimally between the cyclic permutations $(x_1, x_2, ..., x_k)$, $(x_2, x_3, ..., x_k, x_1)$, $(x_3, ..., x_k, x_1, x_2)$ and finally back through $(x_k, x_1, ..., x_{k-1})$ to $(x_1, ..., x_k)$.

**Proof:** We know from Theorem 10.1 that if $\bar{x}$ is a critical point we get a unique geodesic $\gamma : S^1 \to M$ running through $x_1, x_2$, and on through $x_k$ and back to $x_1$. So we can just take

$$ \bar{\gamma}(t) = (\gamma(t), \gamma(t + 2\pi/k), ..., \gamma(t + (k-1)\pi/k)). $$

(10.11)

**Definition 10.2** If $\bar{x}$ is a smooth critical point such that every point on $\gamma_{\bar{x}}$ is also a smooth critical point, then we say $\bar{x}$ is a rotating smooth critical point and $\gamma_{\bar{x}}$ is a rotating smooth critical set.

**Theorem 10.2** Openly $1/k$ geodesics in a convex compact Riemannian manifold with boundary, $M$, have a one to one correspondence with rotating smooth critical points of the uniform energy in $(M)^k$ of nonzero value.

**Proof:** This pretty much follows from Theorem 10.1, Corollary 10.1, and Lemma 10.2.

**Corollary 10.3** Given a manifold $M$, it’s open index, $\text{opind}(M)$, is the smallest value $k$ such that uniform energy $E : M^k \to \mathbb{R}$ has a rotating smooth critical point with a nonzero value.
Example 10.2 Suppose we use this approach to study the length spectrum of $S^1$. First we verify that $L_{1/2}^{open}(S^1) = \emptyset$ because

$$E(s, t) = 4(|s - t| \mod 2\pi)^2 = 4(s - t)^2 \mod 16\pi^2$$

(10.12)

has only $(0, 0)$ as a smooth critical point. For $L_{1/3}^{open}(S^1) = \{2\pi\}$ we examine

$$E(s, t, r) = 3(|s - t| \mod 2\pi)^2 + 3(|t - r| \mod 2\pi)^2 + 3(|r - s| \mod 2\pi)^2$$

(10.13)

This energy is smooth as long as $|s - t| \neq 2k\pi$, $|t - r| \neq 2k\pi$ and $|r - s| \neq 2k\pi$. For $(s_0, t_0, r_0)$ in this domain, there are values $k_1, k_2, k_3 \in \mathbb{Z}$ such that for all $(s, t, r)$ near $(s_0, t_0, r_0)$ we have:

$$(s - t) \mod 2\pi = s - t + 2k_1 \pi$$

(10.14)

$$(t - r) \mod 2\pi = t - r + 2k_2 \pi$$

(10.15)

$$(r - s) \mod 2\pi = r - s + 2k_3 \pi$$

(10.16)

so

$$E(s, t, r) = 3(s - t + 2k_1 \pi)^2 + 3(t - r + 2k_2 \pi)^2 + 3(r - s + 2k_3 \pi)^2$$

(10.17)

Thus we can differentiate and get:

$$0 = \partial E/\partial s = 2(s - t + 2k_1 \pi) - 2(r - s + 2k_2 \pi)$$

(10.18)

$$0 = \partial E/\partial t = -2(s - t + 2k_2 \pi) + 2(t - r + 2k_3 \pi)$$

(10.19)

$$0 = \partial E/\partial r = -2(t - r + 2k_3 \pi) + 2(r - s + 2k_1 \pi)$$

(10.20)

which implies that

$$h = (s - t) \mod 2\pi = (t - r) \mod 2\pi = (r - s) \mod 2\pi.$$ 

(10.21)

Since $3h \mod 2\pi = 0$ we know our smooth critical points have the form $(s, s, s)$ or $(s, s + 2\pi/3, s + 4\pi/3)$ or $(s, s + 4\pi/3, s + 2\pi/3)$. This gives us two nonzero rotating critical points whose energy is $9(2\pi/3)^2 = 4\pi^2$, so their length is $2\pi$. Thus $L_{1/3}^{open} = \{2\pi\}$. Thus $openind(S^1) = 3$.

Using a similar analysis of other compact length spaces one should be able to impose lower bounds on their minimizing index [Problem 11.16].

Lusternick and Fet proved the existence of closed geodesics on an arbitrary compact Riemannian manifold by producing critical points of the energy functional. Such critical points are produced using Morse Theory and the topological properties of the product space. It is much more difficult to prove the existence of rotating critical points. [Problem 11.15] In fact, not all compact length spaces have closed geodesics.

Example 10.3 Let $X = [0, 1]$ with the standard metric $d(s, t) = |s - t|$. Then for any $k \in \mathbb{N}$, we study

$$E(s_1, s_2, \ldots s_k) = \sum_{j=1}^{k} k(s_j - s_{j+1})^2 \text{ where } s_{k+1} = s_1.$$ 

(10.22)

This is a smooth function on $(0, 1)^k \subset [0, 1]^k$, and its critical points satisfy

$$s_{j-1} - s_j = s_j - s_{j+1} \text{ for } j = 1, \ldots k.$$ 

(10.23)

Since we are not on a circle, these points cannot wrap around, so (10.22) implies that all the $s_j = 0$. Thus there are no smooth critical points and by Theorem 10.2, $X = [0, 1]$ has no openly $1/k$ geodesics for any $k$ and by Theorem 9.7 it has no closed geodesics at all.

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In a similar manner Theorems 10.2 and 9.1 could be used to prove other compact length spaces have no closed geodesics. [Problem 11.1]

Remark 10.4 Naturally, one would like to extend Theorem 10.2 to obtain some method of detecting a $1/k$ geodesic which is not openly $1/k$. To do so, one might consider selecting nonsmooth critical points using techniques from Grove-Shiohama’s critical points of distance function or Chang’s critical points of Lipschitz functions [GrShio] [Cng].

Using such techniques one would detect the $1/4$ geodesic in Example 9.2 (see Figure 4). That is the point $(p_0, p_1, p_2, p_3) \subset (M_4)$ defined using the $p_i$ in Example 9.2 is such a nonsmooth critical point.

Similarly, if one were to take a tubular neighborhood of a solid pentagon, $Y$, in the $xy$ plane instead of a square as in Figure 4 and look at five evenly spaced points, $x_j$, on the equator near the midpoints of the sides of the pentagon, then one would again get a nonsmooth critical point in the sense of Chang or of Grove-Shiohama.

However, if we let $\gamma$ run minimally with positive $z$ from $x_0$ to $x_1$ and minimally from $x_1$ to $x_2$, one could verify it was running minimally from $\gamma(t)$ to $\gamma(t + d(x_0, x_1))$, just like the squarelike geodesic in Example 9.2. However, if we continue to extend $\gamma$ in this manner alternating above and below, it returns to $\gamma(0)$ from above creating a corner! So there is no geodesic corresponding to this nonsmooth critical point, although it is halfway around a $1/10$ geodesic.

The author proposes in Problem 11.17 to study nonsmooth critical points.

![Figure 5: The points $x_i \in Y$ correspond to a critical point $(x_0, x_1, x_2, x_3, x_4)$ of the uniform energy function on $(Y)^5$. The geodesic $\gamma$ here is approaching $x_0$ from above.](image)

Remark 10.5 An advantage of focusing on smooth critical points is that we can discuss the Hessian of the energy and degeneracy. Naturally each openly $1/k$ geodesic is a degenerate critical point because of the fact that there is an entire critical level $\gamma_x$. However, a closed geodesic is said to be “degenerate” iff the $\det \text{Hess}_x E = 0$ where we focus on the directions perpendicular to this rotational degeneracy. Such geodesics have smoothly closed Jacobi fields perpendicular to $\gamma'$. [BTZ] [GMS]

In fact, there should be a stronger more global statement describing an openly $1/k$ geodesic which corresponds to a nondegenerate critical point of $E : M^k \to \mathbb{R}$ [Problem 11.18].

On smooth Riemannian manifolds the Morse index is the index of the Hessian of the energy of a geodesic, $\int |\gamma'(t)|^2 \, dt$. In particular, index of a closed geodesic, denoted $\text{ind}(\gamma)$, is the dimension of the subspace of smooth closed vector fields perpendicular to $\gamma'$, $V_\lambda$, on which $H$ is negative definite, where

$$H(X, Y) = \int_0^{2\pi} \langle \nabla X, \nabla Y \rangle - \langle R(X, \gamma'(t))\gamma'(t), Y \rangle \, dt. \quad (10.24)$$

Morse proved that for geodesic segments, where the vector fields have no assumption on periodicity, the index bounds the number of conjugate points on a segment. Closed Geodesics have been studied by Klingenberg and Ballman-Thorbbergson-Ziller, relating their index to the Poincare Map [K2] [BTZ].

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It is important to emphasize here that the Morse Index is defined using vector fields and a covariant derivative and thus is not naturally extended to compact length spaces. Even when viewed as a Hessian of an energy on the loop space there is a significant difficulty defining an extension of the concept. Finally, the Poincare Map and even the unique extension of a geodesic is not defined on arbitrary compact length space.

**Theorem 10.3** The Morse Index of a geodesic, \( \gamma \), in a compact Riemannian manifold satisfies:

\[
\text{ind}(\gamma) \leq (n - 1)(\text{opind}(\gamma)).
\]  

\[
\text{ind}(\gamma) \leq (n - 1)(\text{minind}(\gamma) + 1).
\]

**Proof:** Since (10.25) and Lemma 9.3 imply (10.26), we can concentrate on an openly \( 1/k \) geodesic, \( \gamma \).

Let \( t_j = 2\pi j/k \) for \( j = 0 \) to \( k - 1 \). Following [BTZ], we have \( V_{\Lambda} \) equal to the direct sum of \( V^1_{\Lambda} \) and \( V^2_{\Lambda} \) where \( V^1_{\Lambda} \) are piecewise Jacobi along this partition and \( V^2_{\Lambda} \) are smooth vector fields that are 0 on the partition. They are orthogonal with respect to \( H \) and \( H \) is positive definite on \( V^2_{\Lambda} \) because the geodesic is minimal between the points on the partition. Note that the crucial point is that we do not use the injectivity radius here. Instead the number of points in the partition depends on the openly minimizing index of \( \gamma \). This immediately proves that the Morse Index of \( \gamma \) satisfies:

\[
\text{ind}(\gamma) \leq \text{dim}V^1_{\Lambda} = (n - 1)k.
\]

**Example 10.4** A \( 1/k \) geodesic may have Morse index 0 no matter how large \( k \) is, as can be seen in spaces with nonpositive sectional curvature, like a torus, which have no conjugate points.

**Remark 10.6** The crucial difference between the Morse Index and the minimizing index of a closed geodesic is that the Morse index is a purely local concept while the minimizing index is a global concept checking for cut as well as conjugate points.

It would be interesting to investigate whether the minimizing index of an openly \( 1/k \) geodesic is related to the Hessian of the uniform energy in Theorem [10.2] [Problem 11.19].

**Remark 10.7** Now if \( M_i \) converge to \( M \) in the \( C^4 \) sense their finite dimensional loop spaces \( M^k_i \) converge in the \( C^4 \) sense. It is not hard to show (c.f. [Conly]) that a suddenly appearing critical point under \( C^4 \) convergence must be degenerate. Thus it is of significant interest to identify these nondegenerate openly \( 1/k \) geodesics.

**Remark 10.8** One might be tempted to prove that the nondegenerate length spectrum is a continuous function of smoothly converging manifolds. However, this can be seen not to be the case in Figure [C].

In this section we state some open problems, many of which were mentioned earlier in the paper. If you wish to work on one of these problems or have solved one, please let the author know.

**11 Open Problems**

In this section we state some open problems, many of which were mentioned earlier in the paper. If you wish to work on one of these problems or have solved one, please let the author know.
Problem 11.1 What compact length spaces have empty length spectra? Lusternik and Fet proved that on any compact Riemannian manifold there exist closed geodesics by proving the existence of critical points of the energy functional on the loop space (c.f. [Ca]). Here we need more than just critical points, so it would be easier to prove some spaces have empty length spectra using Theorem 10.2 in a manner similar to Example 10.3.

Problem 11.2 Are there upper bounds on \( \min L_{1/k}(M) \) which depend on volume rather than diameter? See also Problem 11.7.

Problem 11.3 Find a compact manifold, \( M \), whose shortest closed geodesic has a larger minimizing index than the manifold.

Problem 11.4 Note that Klingenberg’s Lemma implies that the minimizing index of any manifold without conjugate points is 2 [Corollary 6.3]. This includes all manifolds with sectional curvature \( \leq 0 \). What can one say about the minimizing index of a CAT(0) space? Suggestions for Problem 11.16 may help.

Problem 11.5 Is there an appropriate definition for a conjugate point on a compact length space which will give results as strong as Corollary 6.3? One might look at [So1], which has a definition of conjugate point defined for an entirely different situation. Keep Example 9.4 in mind.

Problem 11.6 Can one use the proofs of Rotman and Nabutovsky-Rotman’s results to provide bounds on \( \min L_{1/k}(M) \)? See Proposition 5.5 or 5.6.

Problem 11.7 Try to extend the volume estimates on \( \min L_{1/2} \) given in Propositions 5.4, 5.3, 5.2 and 5.1 to compact length spaces with finite second Hausdorff measure. It would not be expected that the results would follow without some additional conditions. See Remark 5.7.

Problem 11.8 What properties can be placed on a simply connected manifold to guarantee the existence of a 1/2 geodesic? Note that Theorem 10.2 cannot be used to find a 1/2 geodesic but Problem 11.17 might prove helpful.

Problem 11.9 What properties can be placed on a manifold to allow one to estimate its minimizing index? See Problem 11.16 for one possible approach.

Problem 11.10 In Lemma 9.3 we related the open index to the minimizing index of a geodesic. On the standard sphere the difference between these indices is exactly 1 for all geodesics. What other manifolds share this property? (c.f. and Theorem 10.2).

Problem 11.11 Is there a version of Proposition 6.1 which involves the volume rather than the diameter of the manifold?

Problem 11.12 What happens in the equality case for Proposition 6.1?

Problem 11.13 On a manifold with minimizing index, \( \min ind(M) = k \), is there an exact bound on \( \min L(M) \) which depends on \( k \)?

Problem 11.14 If \( M_i \) converge to \( S^2 \) with the standard metric in the Gromov-Hausdorff sense, and they have a common lower bound on their injectivity radius, \( \operatorname{injrad}(M_i) \geq \rho_0 > 0 \), then do all prime geodesics \( \gamma_i : S^1 \to M_i \) satisfy Bangert’s Theorem that \( L(\gamma_i) \) either converge to \( 2\pi \) or diverge to infinity? See Remark 7.7.
Problem 11.15 Many theorems proving the existence of a closed geodesic on a manifold involve
the study of the Morse Theory of the loop space and the existence of critical point on that loop space. To
produce a $1/k$ geodesic, Theorem 10.2 requires that we find a “rotating” critical point of an energy on
a product space. What conditions can be placed on a manifold or metric space to prove the existence
of such a critical point?

Problem 11.16 Estimate the minimizing index of a compact length space or provide a lower bound
on the minimizing index using Theorem 10.2. See Example 10.3 for a simple case. Such an estimate
would then provide an estimate on $\text{minind}(M)$ and thus $\text{minvol}(M)$ via Theorem 6.1 and Lemma 9.6.

Problem 11.17 In Theorem 10.2 we relate openly $1/k$ geodesics on a compact Riemannian mani-
fold $M$ to special smooth critical points of an energy on $M^k = M \times M \times \cdots \times M$. It would be
interesting to study whether some definition for a nonsmooth critical point might be used that relates
to $1/k$ geodesics. See Remark 10.4. This might help solve Problem 11.8.

Problem 11.18 A degenerate closed geodesic is a geodesic whose energy functional is degenerate.
It has been proven to have a smoothly closed Jacobi field in [BTZ][GlMy]. Is there a similar more
global property concerning nearby geodesics for a degenerate openly $1/k$ geodesic where one defines
degenerate using the Hessian of the uniform energy? See Remarks 10.5, 10.7 and 10.8.

Problem 11.19 Does the index of the Hessian of the uniform energy provide an estimate on the
minimizing index? It would be interesting to investigate whether the minimizing index of an openly
$1/k$ geodesics is related to the Hessian of the uniform energy in Theorem 10.2. See Remark 10.6.

Problem 11.20 In Theorem 11.7 we estimate the location of the length spectrum of a Riemannian
manifold, $N^n$, whose volume is close to that of the sphere and whose Ricci curvature is bounded from
above. Can one find an explicit formula for the estimating function, $\Psi$? How strong is its dependance
on $k$? Can one control $L(N^n)$ and not just $L_{1/k}$? Note that Colding’s Volume Theorem [Co1] does
not give a precise estimate on the Gromov-Hausdorff convergence and getting one from his proof
would be very difficult. However, proving this result directly may be possible. See Remark 11.8.

Problem 11.21 Find a sequence of manifolds $M_j^n$ with $\text{Ricci} \geq (n-1)$, $\text{Vol}(M_j^n) \to \text{Vol}(S^n)$ such
that $\text{minvol}(M_j^n) \to 0$ or prove this cannot occur. Note in Example 11.1 we showed there is no uniform
lower bound on injectivity radius implies by the Ricci and volume conditions.

Problem 11.22 Find a sequence of manifolds $M_j^n$ with $\text{Ricci} \geq (n-1)$, $\text{Vol}(M_j^n) \to \text{Vol}(S^n)$ with
$L_j \in \text{L}(M_j)$ such that $L_j \to L_\infty \notin \text{L}(S^2)$ or prove this cannot occur. Note that by Theorem 11.4 we
know the $\gamma_j$ of length $L_j$ have $\text{minind}(\gamma_j) \to \infty$. It is quite possible that the $M_j$ in Example 11.1
have disappearing geodesics, so these surfaces are worth investigation. One might begin by stretching
elastic loops around footballs in a clever way.

Problem 11.23 Is it possible to get a gap theorem for manifolds with $\text{Ricci} \geq (n-1)$ and diameter
close to $\pi$? See Remark 11.4.

Problem 11.24 Given a length space $X$ what can one say about the length spectrum of the spherical
suspension over $X$? See [BB1] for a rigorous definition of a spherical suspension.

Problem 11.25 Find a sequence of manifolds $M_j^n$ with $\text{Ricci} \geq (n-1)$, $\text{rad}(M_j^n) \to \text{rad}(S^n)$ such
that $\text{minvol}(M_j^n) \to 0$ or prove this cannot occur. See Remark 11.6.

Problem 11.26 Find a sequence of manifolds $M_j^n$ with $\text{Ricci} \geq (n-1)$, $\text{rad}(M_j^n) \to \text{rad}(S^n)$ with
$L_j \in \text{L}(M_j)$ such that $L_j \to L_\infty \notin \text{L}(S^2)$ or prove this cannot occur. Note that by Theorem 11.6 we
know the $\gamma_j$ of length $L_j$ have $\text{minind}(\gamma_j) \to \infty$. 

Problem 11.27 Find a sequence of manifolds $M^n_j$ with $\text{Ricci} \geq -\epsilon_j (n-1) \to 0$, and $b_1(M^n) = n$ such that $\min L(M^n_j) \to 0$ or prove this cannot occur. See Theorem 8.4.

Problem 11.28 Find a sequence of manifolds $M^n_j$ with $\text{Ricci} \geq -\epsilon_j (n-1) \to 0$, and $b_1(M^n) = n$ such that $L_j \to L_\infty \notin L(S^2)$ or prove this cannot occur. Note that by Theorem 8.4 we know the $\gamma_j$ of length $L_j$ have $\min \text{ind}(\gamma_j) \to \infty$.

Problem 11.29 Analyze the length spectra of locally almost isotopic manifolds mentioned in Remark 8.8.

Problem 11.30 In Section 8 we explained how some rigidity theorems with extremal diameters, volumes or eigenvalues relative to Ricci curvature bounds have stability statements. Propositions 5.2, 5.3 and 5.4 do not involve Ricci curvature but do have rigidity results when their equalities have been achieved. Do they have related stability or stability theorems? Without the Ricci curvature bounds one wouldn’t expect these theorems to involve Gromov-Hausdorff convergence.

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