Rotating Black Holes in Cubic Gravity

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Using on-shell amplitude methods, we derive a rotating black hole solution in a generic theory of Einstein gravity with additional terms cubic in the Riemann tensor. We give an explicit expression for the metric in Einsteinian Cubic Gravity (ECG) and low energy effective string theory, which correctly reproduces the previously discovered solutions in the zero angular-momentum limit. We show that at first order in the coupling, the classical potential can be written to all orders in spin as a differential operator acting on the non-rotating potential, and we comment on the relation to the Janis-Newman algorithm. Furthermore, we derive the classical impulse and scattering angle for such a black hole and comment on the phenomenological interest of such quantities.

INTRODUCTION

We have recently entered the era of gravitational wave astronomy, where one of the key subjects of observation will be black holes [1, 2]. Analyses of the initial black hole data suggests further confirmation of General Relativity (GR) as the correct low-energy description of gravity. Having said this, there still remain pressing open issues in cosmology, such as the dark sector of the universe and more recently, the $H_0$–problem. From a theoretical viewpoint, it is reasonable to expect that GR is not the final say on gravity, and therefore it is prudent to study gravitational theories that extend beyond GR. This can be considered either in an effective field theory (EFT) sense, in which one considers the higher-order curvature terms expected from e.g. low-energy string theory, or simply as modified theories of gravity. In this paper, we shall study the effects of adding cubic curvature contributions to the Einstein-Hilbert action, specifically considering rotating black hole solutions to the field equations. Such analyses are important in order to test the validity of GR in the strong gravity regime of black holes, and further, to identify if such modifications to GR are necessary.

Since angular momentum is conserved, we would expect that nearly all astrophysical black holes in the universe will be spinning, irrespective of the specific theory of gravity under consideration. In a recent paper, two of the present authors used modern amplitude methods to derive non-rotating black hole solutions in cubic theories of gravity [3]. In this work, we extend this analysis to include rotating solutions by considering scattering amplitudes involving particles with arbitrary spin. In general, finding solutions to the field equations in covariant theories of gravity is fraught with difficulty, due to the non-linearity of gravity, the choice of coordinate systems and the sheer number of terms one often has to consider, not to mention the unwieldy complications of rotating solutions. Modern amplitude techniques have been extremely useful in this regard [4–21], partly due to their manifest gauge invariance and partly to the technology that has been developed over the last decade to greatly simplify computations. One particularly efficient method to compute the classical (perturbative) solutions that arise from amplitudes is the so-called Leading Singularity [8], the highest codimension singularity of a given amplitude that allows one to efficiently compute the classical contributions by taking appropriate limits.

In theories of gravity, spin-effects can be found in post-Newtonian or post-Minkowskian multipole expansions [7, 22–29]. At first post-Minkowskian order (1PM), classical effective matching gives rise to an all order in spin expansion [27] in the two-body problem in general relativity. A similar analysis can be obtained directly from the amplitudes, as we will do here, by matching the results obtained from the amplitude calculations with an effective action [11, 12, 16, 17].

We will consider a light scalar particle of mass $m_A$ and momentum $p_1$ probing the spacetime generated by a heavy spinning particle of mass $m_B$, momentum $p_3$ and spin $s$.

![Gravitational probe of spinning particles in cubic gravity](image_url)
In order to compute the classical contributions from such an interaction, we will compute the Leading singularities in the Holomorphic classical limit (HCL) [30]. We will consider a generic six-derivative theory in four dimensions described by the action

\[ S = \int \mathcal{d}^4x \sqrt{g} \left( \frac{2}{\kappa^2} R + \lambda \mathcal{P} \right), \tag{1} \]

where the coupling has mass dimension \( |\lambda| = -2 \) and

\[ \mathcal{P} = \beta_1 R\alpha\beta R^{\alpha\beta\gamma\delta} R_{\gamma\delta\mu\nu} + \beta_2 R_{\mu\nu} \alpha\beta R^{\alpha\beta\lambda\sigma} R_{\lambda\sigma\mu\nu} + \beta_3 R_{\mu\nu\alpha\beta} R^{\mu\alpha\beta} + \beta_4 R_{\mu\nu} \beta_5 R^{\alpha\beta}, \tag{2} \]

where the \( \beta_i \) are generic coefficients which we will leave undefined for the time being.

**HCL Parametrisation**

We will make heavy use of the HCL parametrisation, which we will review here along with our conventions and notation. The incoming and outgoing particles have momentum

\[ p_2^2 = p_2^2 = m_A^2, \quad p_3^2 = p_4^2 = m_B^2, \tag{3} \]

and the usual Mandelstam relations are given by

\[ s := (p_1 + p_3)^2, \quad t := (p_1 - p_2)^2, \quad u := (p_1 - p_4)^2. \tag{4} \]

We define the exchanged momentum as

\[ K := p_1 - p_2 = |\lambda| \langle \lambda \rangle = (0, q), \quad K^2 = t = -|q|^2, \tag{5} \]

and work in a parametrisation that makes the classical pieces explicit, e.g.

\[ p_3 = |\eta| |\lambda| + |\lambda| |\eta|, \quad p_4 = \beta |\eta| |\lambda| + \frac{1}{\beta} |\lambda| |\eta| + |\lambda| |\lambda|, \tag{6} \]

\[ t = \frac{m_B^2}{m_B^2} = \frac{(\beta - 1)^2}{\beta}, \quad \langle \lambda \eta \rangle = |\lambda \eta| = m_B. \]

In addition, we parametrise the loop momentum \( L \) as

\[ L = z \ell + \omega K, \quad |\ell| = |\eta| + B |\lambda|, \quad \langle \ell \rangle = |\eta| + A |\lambda|. \tag{7} \]

Demanding the on-shell cut conditions \( k_{3,4}^2 = L^2 - m_B^2 \) (and imposing \( t^2 = 0 \)) fixes \( \omega = -\frac{1}{2} \) with \( A = -B = \frac{1 - 2\beta}{1 + \beta} \). Given this parametrisation, the LS is given by

\[ I = \frac{1}{16\sqrt{-tm_B}} \int_{L^2} \frac{dy}{y^5} \mathcal{M}_+^+ \mathcal{M}_+^- \mathcal{M}_-^+ \mathcal{M}_-^-, \tag{8} \]

where we have taken the \( \beta \rightarrow 1 \) limit.

The chosen parameters also induces a convenient parametrisation for \( k_{3,4} \)

\[ |k_3| = \frac{1}{\beta + 1} \left( |\eta| (\beta^2 - 1) y + |\lambda| (1 + \beta y) \right), \]

\[ |k_4| = \frac{1}{\beta + 1} \left( |\eta| (\beta^2 - 1) - \frac{1}{y} |\lambda| (1 + \beta y) \right), \]

\[ |k_4| = \frac{1}{\beta + 1} \left( -\beta |\eta| (\beta^2 - 1)y + |\lambda| (1 - \beta y) \right), \]

\[ |k_4| = \frac{1}{\beta + 1} \left( 1 + \frac{1}{y} |\lambda| (1 - y) \right). \tag{9} \]

We also define the parameters

\[ u := |\lambda| |p_1| \eta, \quad v := |\eta| |p_1| \lambda, \tag{10} \]

with \( u + v = 2p_1 \cdot p_3 \) in the HCL. These are related to the mandelstam variables in the HCL via

\[ u = m_A m_B (\rho + \sqrt{\rho^2 - 1}), \quad v = m_A m_B (\rho - \sqrt{\rho^2 - 1}), \tag{11} \]

where

\[ m_A m_B \sqrt{\rho^2 - 1} = \sqrt{(s - (m_A + m_B)^2)(s - (m_A - m_B)^2)} \]

and the non-dispersive limit is given by \( \rho \rightarrow 1 \).

Given these definitions and kinematics, we find the following relations

\[ \langle \lambda | p_1 | \lambda \rangle = -\frac{(\beta - 1)^2}{\beta} m_B^2 + (1 - \beta) (v - \frac{u}{\beta}), \]

\[ \langle \eta | p_1 | \eta \rangle = \frac{uv - m_A^2 m_B^2}{(u - v)(\beta - 1)} + \mathcal{O}(\beta - 1)^0. \tag{12} \]

Spin effects are given in terms of the Pauli-Lubanski spin-vector, which we will use in its mass-rescaled form, given by

\[ a^\mu = -\frac{1}{m_B^2} (P^\nu \tilde{\sigma}_\mu). \tag{13} \]

The spin dependence is characterised by identifying [4]

\[ \epsilon_{\mu\nu\rho\sigma} p_1^\nu p_3^\rho K^\sigma a^\sigma = (E_A + E_B)(a \cdot p \times q), \tag{14} \]

where \( \mathbf{p} \) is the relative momentum. We will work in the anti-chiral HCL basis, defining the relative momentum-dependence of the amplitudes in terms of the rapidity \( w \). By working in the anti-chiral basis, we lose any information that might have been contained in the polarisation tensors of the external spinning particles. In order to restore these, we will turn to the Generalised Expectation Value (GEV) introduced in Ref. [11], which in our case amounts to normalising the amplitude via

\[ \langle \mathcal{M} \rangle = e^{-K \cdot a} \mathcal{M}. \tag{15} \]

Armed with the HCL parametrisation and this normalisation, we can compute the amplitudes relevant to derive the classical potential.
At tree level, there is no $O(\lambda)$ contribution, and we need only to consider the diagram

\[ \text{FIG. 2. Tree Level LS} \]

The minimally coupled three-particle amplitude with one graviton and two spin-$s$ particles is given by \[31\]

\[
\mathcal{M}_3[1, 2, K^{+2}] = \frac{\kappa}{2} (m x_{12})^2 \left( \frac{12}{m^{2s}} \right)^2,
\]

\[ \mathcal{M}_3[1, 2, K^{-2}] = \frac{\kappa}{2} \left( \frac{m}{x_{12}} \right)^2 \left( \frac{12}{m^{2s}} \right)^2, \quad (16) \]

where $x_{ij}$ is defined via

\[ x_{ij} \lambda_i^a \frac{\lambda_j^a}{m} \quad \text{and} \quad x_{ij} = \frac{p_j^a \lambda_i^a}{m}. \quad (17) \]

The tree-level leading singularity is simply the residue in the $t$ channel, in this case given by the product of the above amplitudes. Considering the HCL parametrisation and working in the chiral representation, this is given by

\[
\mathcal{M}_4^t = - \left( \frac{\kappa}{2} \right)^2 \frac{m_A m_B^2}{t} \left( \frac{x_{34}}{x_{12}} \left( 1 + \frac{K \cdot a}{s} \right)^{2s} + \frac{x_{12}}{x_{34}} \right),
\]

\[
= - \left( \frac{\kappa}{2} \right)^2 \frac{1}{t} \left( u^2 \left( 1 + \frac{K \cdot a}{s} \right)^{2s} + v^2 \right), \quad (18) \]

where we have formally defined the spin vector $a = 2s\hat{a}$ and noted that

\[ u = m_A m_B \frac{x_{34}}{x_{12}}, \quad v = m_A m_B \frac{x_{12}}{x_{34}}. \quad (19) \]

For spin zero, in the non-relativistic limit, we find then the amplitude

\[
\mathcal{M}_4^0 = - \left( \frac{\kappa^2}{2} \right) \frac{m_A m_B^2}{t} \quad . \quad (20) \]

**ONE LOOP LS**

In order to derive a black hole solution at order $O(\lambda)$ for cubic gravity, we only need to consider one loop diagram.

**TREE LEVEL LS**

We now work out the tree-level components that go into the loop, expressed in terms of the parameters above. For a particle of generic spin $s$ emitting a graviton, the three-particle amplitude is given by

\[
\mathcal{M}_3[1^s, 2^s, K^{+2}] = - \left( \frac{\kappa}{2} \right)^2 m_{12}^2 \left( 1 + \frac{K \cdot a}{s} \right)^{2s}, \quad (21) \]

\[
\mathcal{M}_3[1^s, 2^s, K^{-2}] = - \left( \frac{\kappa}{2} \right) \left( \frac{m_{12}^2}{x_{12}} \right), \quad (22) \]

where we have chosen to work in the purely anti-chiral basis and expose the spin dependence of positive helicity amplitudes. We can therefore work out the product of three-particle amplitudes that go into the LS

\[
\mathcal{M}_3[3^s, -L^s, k_{34}^2] \mathcal{M}_3[4^s, L^s, k_{34}^2] = \left( \frac{\kappa}{2} \right)^2 m_{12}^4 y^4 \times \left( 1 + \frac{1 + y^2}{4y} K \cdot a \right)^{2s} \left( 1 - \frac{(1 - y^2 K \cdot a}{4y} s \right)^{2s}, \quad (23) \]

where we have used the fact that $x_{3L} = x_{4L} = -y$ in this parametrisation.

We can also determine the four-point that goes into the LS in terms of these parameters. Noting that we can express

\[
k_{3,1} = k_3 \cdot p_1 + O(\beta - 1)^2 = (\beta - 1) \frac{(1 - y^2)(v - u)}{8y}, \quad (24) \]

we find
\[ M_4[p_1, p_2, k_3^2, k_4^2] = \frac{3}{16} \frac{k^4 \lambda (k_3 k_4)^4}{(p_1 - p_2)^2} \left[ \beta_1 \left( (k_3 \cdot p_1 + k_3 \cdot p_2)^2 - m_A^2 k_3 \cdot k_4 \right) + 8 \beta_2 (k_3 \cdot p_1)(k_3 \cdot p_2) \right] \]

\[ = - \frac{3}{64} \frac{k^4 \lambda m_B^2 (\beta - 1)^6}{y^4} \left[ 2\beta_1 m_A^2 m_B^2 - (\beta_1 + 2\beta_2) \left( 1 - \frac{y^2}{4} \right)^2 (v - u)^2 \right] \]

Putting this all together, we find the LS to be evaluated is

\[ \mathcal{T} = - \frac{3}{4096} \frac{k^6 \lambda m_B(-t)^{3/2}}{\left( 1 + \frac{(1 + y)^2}{4y} K \cdot a \right)^2 s} \left( 1 - \frac{(1 - y)^2}{4y} K \cdot a \right)^2 s \times \left[ 2\beta_1 m_A^2 m_B^2 - (\beta_1 + 2\beta_2)(v - u)^2(1 + y)^2 \right]. \] (26)

In order to check that this is the correct expression, we first choose \( s = 0 \) to ensure we are able to recover the results obtained in [3]. Taking the first term in eq. (26), and summing together with \( m_A \leftrightarrow m_B \), we find

\[ \mathcal{T} = - \frac{3}{1024} \beta_1 k^6 \lambda m_A^2 m_B^2 (m_A + m_B) q^3. \] (27)

This precisely matches the small \( t \) expansion of the pure \( \beta_1 \) term of eq. (5.2) in [3]. Moving on to the second term, we find that the LS is

\[ \mathcal{T} = - \frac{3}{1024} (\beta_1 + 2\beta_2) k^6 \lambda (m_A + m_B) (v - u)^2 q^3 \]

\[ = - \frac{3}{1024} (\beta_1 + 2\beta_2) k^6 \lambda (m_A + m_B)^3 p^2 q^3, \] (28)

which agrees with the pure \( (\beta_1 + 2\beta_2) \) term of eq. (5.2) in [3]. This leaves us confident that eq. (26) is indeed the correct expression, and thus we shall move on to consider cases in which \( s \neq 0 \).

**ALL ORDER IN SPIN CLASSICAL POTENTIAL**

In momentum space, the classical potential is a related to the amplitude \( \mathcal{M} \) by

\[ V(q, p) = \frac{\langle \mathcal{M} \rangle}{4E_A E_B}. \] (29)

where \( E_i = \sqrt{m_i^2 + p_i^2 + q_i^2} \).

A consistent classical limit is only reached by taking \( s \to \infty \) as \( \hbar \to 0 \) keeping \( sh \) fixed, since the intrinsic angular momentum of a spin \( s \) particle is proportional to \( sh \) [32]. We now make a further identification for the variables \( u \) and \( v \) as being

\[ u = m_A m_B e^w, \quad v = m_A m_B e^{-w}, \] (30)

where \( w \) is the rapidity. Plugging this into the tree-level four point and taking the infinite spin limit gives

\[ \mathcal{M}_4^\infty = - \frac{k^2 m_A^2 m_B^2}{4} (e^{2w} e^{2K \cdot a} + e^{-2w} e^{-2K \cdot a}). \] (31)

After normalisation with the GEV, we then find

\[ \langle \mathcal{M}_4^\infty \rangle = - \frac{k^2 m_A^2 m_B^2}{4t} \left( e^{2w} e^{K \cdot a} + e^{-2w} e^{-K \cdot a} \right). \] (32)

Doing the same for the loop amplitude yields

\[ \mathcal{I}^\infty = - \frac{3k^6 \lambda}{2048} (-t)^{3/2} m_A^2 m_B^3 \beta(w)(1 + e^{2K \cdot a}), \] (33)

where we have defined \( \beta(w) := (\beta_1 + 2(\beta_1 + 2\beta_2) \sinh^2 w \). Again normalising this using the GEV gives

\[ \langle \mathcal{I}^\infty \rangle = - \frac{3k^6 \lambda}{1024} m_A^2 m_B^3 \beta(w) q^3 \cosh q \cdot a. \] (34)

The momentum space potential is therefore given by

\[ V = \frac{k^2 m_A m_B}{16} \left( e^{2w} e^{aq} + e^{-2w} e^{-aq} \right) - \frac{3k^6 \lambda m_A m_B^2}{4096} \beta(w) q^3 \cosh q \cdot a. \] (35)

Restricting ourselves to the non-dispersive terms (i.e. \( w = 0 \)) and taking the Fourier transform, we find then that the all-order in spin potential is given by

\[ V(r) = - \frac{k^2}{32\pi} m_A m_B \cos (a \cdot \mathbf{\nabla}) \left( \frac{1}{r} - \frac{9k^4 \lambda \beta_1 m_A + m_B}{32\pi} \right), \]

\[ = - G m_A m_B \cos (a \cdot \mathbf{\nabla}) \left( \frac{1}{r} - \frac{228\pi G^2 \lambda \beta_1 m_A + m_B}{r^6} \right). \] (36)

We note that this form of the potential allows us to interpret the attachment of spin-factors to the
amplitudes generating the spacetime as the on-shell avatar of the Newman-Janis algorithm for higher derivative gravity, in precisely the same way as it does in the Kerr and Kerr-Newman cases [19, 20]. This is because attachment of such factors gives rise to a differential operator which performs a complex deformation of the coordinates \( r \rightarrow r + ia \) (a simple extension of the translation operator), i.e.

\[
\cos(a \cdot \nabla)f(r) = 2\Re(f(r + ia)),
\]

\[
\sin(a \cdot \nabla)f(r) = 2\Im(f(r + ia)),
\]

for any holomorphic function \( f(r) \).

**METRIC CONSTRUCTION**

In momentum space, the classical potential for a gravitomagnetic system is of the form

\[
V(q) = m\Phi(q) + p \cdot w, \tag{39}
\]

where \( \Phi \) is the gravito-electric field, \( w \) is gravito-magnetic field and \( p \) is kinetic momentum. The perturbative metric can be decomposed in terms of these fields as

\[
h_{00} = 2\Phi, \quad h_{0i} = -w_i, \quad h_{ij} = 2\Phi\delta_{ij}. \tag{40}
\]

We can then identify the relevant components of the metric

\[
\Phi = \lim_{m_A \rightarrow 0} \frac{1}{m_A} V, \quad w_i = \lim_{m_A \rightarrow 0} 2\frac{\partial V}{\partial p_i},
\]

where we note that we will often write \( p_i = m_A u_i \) in the limit we are interested in.

In order to identify the spin components, we are required to identify the dispersive terms that multiply \( K \cdot a \) via

\[
m_A m_B \sinh w \cdot K \cdot a = i\epsilon_{\mu\nu\rho\sigma} p^\mu_A p^\nu_B K^\rho a^\sigma. \tag{42}
\]

In the centre of mass frame, we can use this on-shell condition to write

\[
K \cdot a = \frac{i\epsilon_{\mu\nu\rho\sigma} p^\mu_A p^\nu_B K^\rho a^\sigma}{m_A m_B \sinh w} = -i\mathbf{u} \cdot (\mathbf{a} \times \mathbf{q}),
\]

where \( \mathbf{u} \) is the relative four velocity.

With this in hand, we can therefore rewrite the potential, keeping only the necessary dispersive terms, as

\[
V = \cos(\mathbf{u} \cdot (\mathbf{a} \times \mathbf{q})) \left[ \frac{\kappa^2 m_A m_B}{8 \mathbf{q}^2} - \frac{3\kappa^6 \lambda m_A m_B}{4096} \beta(w)\mathbf{q}^2 \right]. \tag{44}
\]

We find then, in momentum space,

\[
\Phi = \cosh(q \cdot a) \left( \frac{\kappa^2 m_B}{8 \mathbf{q}^2} - \frac{3\kappa^6 \lambda m_B^2}{4096} \beta(w)\mathbf{q}^2 \right),
\]

\[
w_i = -\sinh(q \cdot a) \times \left[ \frac{\kappa^2 m_B}{4 \mathbf{q}^2} - \frac{3\kappa^6 \lambda m_B^2}{2048} \beta(w)\mathbf{q}^2 \right] (\mathbf{a} \times \mathbf{q})_i, \tag{45}
\]

which in position space is

\[
\Phi = \cos(a \cdot \nabla) \left( \frac{G m_B}{r} - \frac{288G^3 \pi \lambda \beta(w) m_B^2 \mathbf{q}^2}{r^6} \right),
\]

\[
= \cos(a \cdot \nabla) (\Phi_{Kerr} + \Phi_{R}) ,
\]

\[
w_i = -\sin(a \cdot \nabla) \left( \frac{2G m_B}{r} - 3456 \frac{G^3 \pi \lambda \beta(w) m_B^2}{r^6} \right) (\mathbf{a} \times \mathbf{r})_i
\]

\[
= -2\sin(a \cdot \nabla) (\Phi_{Kerr} + 6\Phi_{R}) (\mathbf{a} \times \mathbf{r})_i,
\]

from which the components of the metric can be determined. Specialising to the case of ECG [33, 34], we choose \( \beta_1 = 12, \beta_2 = 1 \) and \( \lambda = \frac{G^3}{M^4} \), we find the metric

\[
g^E_{00} = 1 - \frac{2GM}{r} + \frac{3328G^4 M^2 \lambda}{r^6} \ldots \tag{49}
\]

\[
g^E_{0i} = \left( 1 + \frac{2GM}{r} - \frac{2592G^4 M^2 \lambda}{r^6} \right) (\mathbf{a} \times \mathbf{r})_i \ldots
\]

\[
g^E_{ij} = \left( 1 + \frac{2GM}{r} - \frac{3456G^4 M^2 \lambda}{r^6} \right) \delta_{ij} \ldots
\]

**CLASSICAL IMPULSE AND SCATTERING ANGLE**

We can also derive the classical impulse imparted to scalar probe particle, given in terms of the scattering amplitude by [35]

\[
\Delta p^\mu_i = \frac{1}{4m_A m_B \sinh w} \int d^4q |\Phi_1|^2 \delta^4(\mathbf{q} \cdot \mathbf{u}_1) \delta(\mathbf{q} \cdot \mathbf{u}_3) e^{-i\mathbf{q} \cdot \mathbf{b}} i\mathbf{q}^\mu \langle \mathcal{M}^\infty \rangle.
\]

The impulse is given in terms the incoming probe particle momentum \( p_1 = m_A u_1 \) and its colliding partner \( p_3 = m_B u_3 \), and is simply a measure of the total change in momentum of particle 1 as a result of the collision. The impact parameter is a spacelike vector orthogonal to \( u_1 \) and \( u_3 \), and the delta functions ensure that we pick out the plane in which this lives. Plugging in \( \langle \mathcal{M}^\infty \rangle = \langle \mathcal{M}^\infty_+ \rangle + \langle \mathcal{I}^\infty \rangle \) and performing the integral transforms, we find

\[
\Delta p^\mu_i = \frac{2G m_A m_B}{\sinh w} R \left[ \frac{\hat{b}_i^\mu \cos 2w + 2i \cos w \zeta^\mu}{|\hat{b}_i|^2} \right]
\]

\[
+ 135 G^2 \pi^2 m_B \left( \frac{2i \beta_1 \hat{b}_i^\mu + 2i(\beta_1 + 2\beta_2) \sinh w \zeta^\mu}{|\hat{b}_i|^2} \right)
\]

where \( \zeta^\mu := \epsilon^{\mu\nu\rho\sigma} \hat{b}_\nu u_{1\nu} u_{3\sigma} \) and \( \hat{b}_i = b_i - i\mathbf{a} \).

We can also derive the scattering angle, given in terms of the LS by [11, 36, 37]

\[
\theta = 2 \sin \left( \frac{\theta}{2} \right) = -\frac{E}{(2m_A m_B \sinh w)^2} \frac{\partial}{\partial b} \langle \mathcal{M}^\infty(b) \rangle,
\]
where \( \langle \mathcal{M}^\infty(b) \rangle \) is the LS in impact parameter space. The tree-level LS is given by
\[
\langle \mathcal{M}^\infty(b) \rangle = \left( \frac{k^2}{8\pi} \right) m_B^2 m_B^2 \sum_{\pm} e^{\pm 2w \ln |b \pm a|},
\]
and the loop LS (34) by
\[
\langle \mathcal{I}^\infty(b) \rangle = -\frac{27k^6\lambda}{4096\pi^2} m_B^2 m_B^2 \sum_{\pm} \frac{\beta(w)}{(b \pm a)^3}.
\]
Plugging these back into (52) we find the angle
\[
\theta = \frac{-GE}{\sinh^2 w} \sum_{\pm} \left( \frac{e^{\pm 2w}}{b \pm a} + 270\pi^2 G^2 \lambda m_B \frac{\beta(w)}{(b \pm a)^3} \right).
\]
It can also be checked that this angle is reproduced directly from the classical impulse in eq. (51), by considering [27]
\[
\theta = \frac{\Delta p_{\perp}}{|p_{\perp}|},
\]
where \( \Delta p^\mu = \Delta p_{\perp} \hat{b}^\mu \) and \( |p_{\perp}| = m \sinh w \).

**ROTATING STRINGY BLACK HOLES**

Having so far considered a generic cubic theory of gravity given by the action eq. (63), we can naturally extend our analysis to consider the corrections to the potential that arise from the \( \alpha'^2 \) part of low-energy effective action in string theory,
\[
S = -\frac{2\alpha'^2}{\kappa^2} \int d^4 x \sqrt{-g} \left( \frac{1}{48} I_1 + \frac{1}{24} G_3 \right),
\]
where \( I_1 \) is the \( \beta_2 \) term in our case, and
\[
G_3 = R_{\mu \nu} \alpha^\beta R_{\alpha \beta} \lambda^\sigma R_{\lambda \sigma} - 2 R_{\mu \nu} \alpha^\beta R^\lambda R^\sigma \lambda^\rho R_{\lambda \rho \mu \nu},
\]
is the well known cubic Gauss-Bonnet invariant, which arises from setting \( \beta_1 = -2\beta_2 \).

To compute the \( I_1 \) correction, we take \( \lambda = -\frac{2\alpha'^2}{\kappa^2} \) and then \( \beta_1 = 0, \beta_2 = \frac{1}{6} \) which gives
\[
\Phi_{I_1} = \frac{3(G\alpha')^2}{2} \sinh^2 w \cos(a \cdot \nabla) \frac{m_B^2}{r^6},
\]
\[
w_{I_1} = -18(G\alpha')^2 m_B^2 \sinh^2 w \sin(a \cdot \nabla) \frac{(a \times r)_i}{r^8}.
\]
Then to compute the \( G_3 \) contribution we take \( \beta_1 = -2\beta_2 = -\frac{1}{12} \) to find
\[
\Phi_{G_3} = 3(G\alpha')^2 \cos(a \cdot \nabla) \frac{m_B^2}{r^6},
\]
\[
w_{G_3} = -9(G\alpha')^2 m_B^2 \sin(a \cdot \nabla) \frac{(a \times r)_i}{r^8}.
\]
We can also readily derive the scattering angle in this case, which we find to be
\[
\theta_{I_1} = \frac{45\pi}{32} (G\alpha')^2 \sum_{\pm} \frac{1}{(b \pm a)^5}.
\]

**DISCUSSION**

In this paper, we have derived a new black hole solution in cubic gravity using modern amplitude methods. In particular, we present the all-order (in spin) classical potential, to leading-order in the cubic coupling \( \lambda \). Further, we have shown how the form of this potential allows for an interpretation of the on-shell avatar of the Newman-Janis algorithm, extending it to higher-derivative gravity. This is certainly good motivation to try and establish the precise algorithm that deforms coordinates in some particular coordinate system and allows one to derive the rotating solution directly from the static one. As an additional exercise, we broadened our discussion slightly to consider the leading-order corrections to the potential arising from the \( \alpha'^2 \) part of the low-energy effective action in string theory. We also note that taking the \( a \rightarrow 0 \) limit reproduces the results that can be found elsewhere in the literature [3, 14, 33, 34, 38–40], at least for those quantities that have already been computed.

It should be emphasised that deriving such a black hole solution using traditional geometric methods is a difficult endeavour, a fact which highlights the benefits of using modern amplitude techniques to understand gravitational phenomena. Indeed, this is a rare case in which it has been possible to derive a novel result via modern amplitude methods before it has been done so through the geometric approach, in which the presence of cubic-order curvature terms makes the task almost intractable.

In addition to finding the black hole solution, we also present results for pertinent classical observables, namely the impulse and scattering angle. Given some reliable observational data, these quantities could be used to place a bound on the coupling \( \lambda \), whose parameter space has been only partially constrained [41] – in the model presented, \( \lambda \) can assume arbitrarily large or small values.

There are a number of interesting future directions that immediately present themselves as natural follow ups to this work. It is certainly plausible to consider a binary rotating black hole system cubic gravity, which would amount to including spin on all external particles in the scattering amplitudes and keeping the particle masses general. Furthermore, one could consider gravitational radiation in this setup, which would give rise to observational signatures that could, at least in
principle, be detected by LIGO or future gravitational wave experiments, such as a LISA.

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Appendix A: Integral Transforms

Here we quote the relevant integral transforms that we make use of in this paper:

\[
\int \frac{d^3q}{(2\pi)^3} e^{iq \cdot r} \frac{1}{q^2} = \frac{1}{4\pi r} \tag{A.1}
\]

\[
\int \frac{d^3q}{(2\pi)^3} e^{iq \cdot r} \frac{q_i}{q^2} = \frac{ix_i}{4\pi r^3} \tag{A.2}
\]

\[
\int \frac{d^3q}{(2\pi)^3} e^{iq \cdot r} q_i^n = \frac{(n+1)!}{2\pi^2 r^{3+n}} \sin \left( \frac{3\pi n}{2} \right), \tag{A.3}
\]

\[
\int \frac{d^3q}{(2\pi)^3} e^{iq \cdot r} q_i |q|_n^n = \frac{i(n+1)!}{2\pi^2 r^{5+n}} \sin \left( \frac{3\pi n}{2} \right), \tag{A.4}
\]

\[
\int \frac{d^4q}{(2\pi)^4} \delta(q^0) \delta(q^1 - \beta q^3) e^{iq \cdot b} q^\mu |q|^{n+1} = -\frac{i}{2\pi |\beta\gamma|} \mathcal{H}_1[r^{n+1}] \hat{b}^\mu, \tag{A.5}
\]

\[
\mathcal{H}_\nu[r^n] = \int_0^\infty r^{n+1} J_\nu(kr) = \frac{2^{n+1} \Gamma \left( \frac{1}{2}(2 + \nu + n) \right)}{k^{n+2} \Gamma \left( \frac{1}{2}(\nu - n) \right)}. \tag{A.6}
\]