CONVERGENCE OF PASSIVE SCALAR FIELDS IN
ORNSTEIN-UHLENBECK FLOWS TO KRAICHNAN’S MODEL

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ABSTRACT. We prove that the passive scalar field in the Ornstein-Uhlenbeck velocity field with
wave-number dependent correlation times converges, in the white-noise limit, to that of Kraichnan’s
model with higher spatial regularity.

1. Introduction

A passive scalar field \( T(t, x) \) in a given fluid velocity \( u(t, x) \) satisfies the advection-diffusion
equation

\[
\frac{\partial T}{\partial t} = u \cdot \nabla T + \frac{\kappa}{2} \Delta T, \quad T(0, x) = T_0(x)
\]

where \( \kappa \geq 0 \) is the molecular diffusivity. Kraichnan’s model for passive scalar has been widely
studied to understand turbulent transport in the inertial range because of its tractability (see, e.g.,
[16], [14], [8] and the references therein). The model and its variant postulate a white-noise-in-time,
compressible or incompressible velocity field \( u \) which can be described as the time derivative of a
zero mean, isotropic Brownian field \( B_t \) with the two-time structure function

\[
\mathbb{E}[B_t(x) - B_t(y)] \otimes [B_s(x) - B_s(y)] = \min(t, s) \int 2[\exp(ik \cdot (x - y)) - 1]\alpha^{-1}\mathcal{E}(\eta + 1, k)|k|^{-d}dk, \quad \alpha > 0
\]

Here \( 2\alpha^{-1}\mathcal{E}(\eta + 1, k) \) is the spatial power spectrum with

\[
\mathcal{E}(\eta + 1, k) = E_0(k)|k|^{-2\eta - 1} \quad \text{for} \quad \ell_0^{-1} \ll |k| \ll \ell_1^{-1}, \quad \eta \in (0, 1)
\]

where \( E_0(k) \) is a positive-definite matrix whose entries are homogeneous functions of degree zero,
\( \ell_0 \) and \( \ell_1 \) are the integral and the viscous scales respectively and they determine the so-called
inertial range. Below the viscous scale \( \ell_1 \) the velocity field is smooth. The spatial Hurst exponent
\( \eta \) characterizes the roughness of the velocity field in the inertial range and equals 1/3 in the case of

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Kolmogorov’s theory of turbulence. The tractability of this model lies in the Gaussian and white-noise nature of the velocity field. To fix the idea, we interpret eq. (1) in the sense of Stratonovich’s integral

\[ dT = \nabla T \circ dB_t + \frac{\kappa_0}{2} \Delta T \, dt, \quad \kappa_0 \geq 0, \quad T(0, x) = T_0(x). \]

To study the effect of a more realistic temporal structure, one naturally considers the Ornstein-Uhlenbeck (OU) velocity field

\[ u(t, x) = \frac{1}{\varepsilon} V(\frac{t}{\varepsilon^2}, x) \]

with a similar spatial structure but a wave-number-\( k \) dependent correlation time \( a^{-1}|k|^{-2\beta} \), \( a > 0, \beta > 0 \), where \( \varepsilon > 0 \) is the scaling parameter. The two-time structure function has the spectral representation

\[ \mathbb{E}[V(t, x) - V(t, y)] \otimes [V(s, x) - V(s, y)] = \int_{\mathbb{R}^d} \left[ \exp (ik \cdot (x - y)) - 1 \right] \exp (-a|k|^{2\beta}|t - s|) \mathcal{E}(\alpha, k)|k|^{-d} \, dk \]

where \( \mathcal{E} \) is the power spectrum given by (6) with \( 1 < \alpha < 2 \) (see [6], [7]). The spatial Hurst exponent of the velocity equals \( \alpha - 1 \) in the inertial range. The parameters \( \alpha, \beta \) have the value \( 4/3, 1/3 \), respectively, in the case of Kolmogorov’s theory of turbulence.

In this paper we study the relation between these two model. For simplicity of the presentation we set

\[ \mathcal{E}(\alpha, k) = \begin{cases} E_0(k)|k|^{-2\alpha}, & \text{for } |k| \in (\ell_0^{-1}, \ell_1^{-1}) \\ 0, & \text{for } |k| \not\in (\ell_0^{-1}, \ell_1^{-1}). \end{cases} \]

with \( \ell_0 < \infty, \ell_1 > 0 \). We defer the discussion of the meaning of solutions of (1) and (3) until Section 2.

First we consider the situation of a non-vanishing ultraviolet cutoff \( \ell_1 > 0 \). We have the following correspondence principle.

**Theorem 1.** Let \( \ell_0 < \infty, \ell_1 > 0 \) be fixed. Let \( \kappa = \kappa(\varepsilon) \geq 0 \) and \( \lim_{\varepsilon \to 0} \kappa = \kappa_0 < \infty \). Let \( T_0 \in L^\infty(\mathbb{R}^d) \).

Then the solution \( T_\varepsilon^t \) of (1) with the drift (3) converges in distribution, as \( \varepsilon \to 0 \), in the space \( D([0, t_0); L^\infty_w(\mathbb{R}^d)), \forall t_0 < \infty \) to the unique solution \( T_t \) of the martingale problem (cf. (20)) corresponding to eq. (3), where the Brownian velocity field has the spatial covariance with the power spectrum \( 2a^{-1}\mathcal{E}(\alpha + \beta, k) \). Here \( D([0, t_0); L^\infty_w(\mathbb{R}^d)) \) is the space of \( L^\infty(\mathbb{R}^d) \)-valued right continuous
processes with left limits endowed with the Skorohod metric \( \mathbb{B} \) and \( L_w^\infty(\mathbb{R}^d) \) is the standard space \( L^\infty(\mathbb{R}^d) \) endowed with the weak* topology.

This result suggests that in the limit of rapid temporal decorrelation the OU flow resembles Kraichnan’s model with a higher spatial regularity \( \eta = \alpha + \beta - 1 \). In particular, the strict Kolmogorov’s theory \( \alpha = 4/3, \beta = 1/3 \) now corresponds to \( \eta = 2/3 \) in Kraichnan’s model.

In the next theorem we let \( \ell_1 \) vanish along with the scaling factor \( \epsilon \). In such a limit Theorem 1 is not expected to hold for *compressible flows* in the entire range of \( \alpha, \beta \) for the Stratonovich correction term in the limiting Kraichnan model is well-defined only if \( \alpha + \beta > 3/2 \). Moreover, for \( \alpha + \beta < 2 \) and \( \ell_1 = 0 \), the Kraichnan model with compressible velocity field may not have a unique solution for a given initial condition due to the spatial non-Lipschitzness of the velocity field (cf. [10], [14]).

**Theorem 2.** Suppose that the OU velocity field \( V \) is divergence-free, \( \nabla \cdot V = 0 \). Let \( \ell_0 < \infty \) be fixed and \( \ell_1 = \ell_1(\epsilon) > 0 \) such that \( \lim_{\epsilon \to 0} \ell_1 = 0 \). Let \( \kappa = \kappa(\epsilon) \geq 0, \lim_{\epsilon \to 0} \kappa = \kappa_0 < \infty \). Let \( T_0 \in L^\infty \cap L^2(\mathbb{R}^d) \). If, additionally, any one of the following conditions is satisfied:

1. \( \alpha + 2\beta > 4 \);
2. \( \alpha + 2\beta = 4, \lim_{\epsilon \to 0} \kappa \epsilon^2 \sqrt{\log(1/\epsilon)} = 0; \)
3. \( 3 < \alpha + 2\beta < 4, \lim_{\epsilon \to 0} \kappa \epsilon^2 \ell_1^{2\beta - 4} = 0; \)
4. \( \alpha + 2\beta = 3, \lim_{\epsilon \to 0} \epsilon \sqrt{\log(1/\epsilon)} = \lim_{\epsilon \to 0} \kappa \epsilon^2 \ell_1^{-1} = 0; \)
5. \( 2 < \alpha + 2\beta < 3, \lim_{\epsilon \to 0} \epsilon \ell_1^{\alpha+2\beta-3} = \lim_{\epsilon \to 0} \kappa \epsilon^2 \ell_1^{\alpha+2\beta-4} = 0; \)
6. \( \alpha + 2\beta \leq 2, \lim_{\epsilon \to 0} \epsilon \ell_1^{\alpha+2\beta-3} = 0, \)

then the convergence holds as in Theorem 1 but in the space \( D([0,t_0]; L^\infty_w \cap L^2_w(\mathbb{R}^d)), \forall t_0 < \infty \) where \( L^2_w(\mathbb{R}^d) \) is the usual \( L^2 \)-function space endowed with the weak topology. The Brownian flow of the limiting Kraichnan’s model has the the spatial power spectrum \( 2\alpha^{-1} \bar{E}(\alpha + \beta, k) \) where

\[
\bar{E}(\alpha + \beta, k) = \lim_{\ell_1 \to 0} E(\alpha + \beta, k).
\]

**Remark 1.** The assumption of \( L^2(\mathbb{R}^d) \)-initial condition in Theorem 2 is to ensure uniqueness of the limiting Kraichnan model with \( \ell_1 = 0 \) (see Section 2). The limiting velocity field is only spatially Hölder continuous (for \( \alpha + \beta < 2 \)) with exponent \( \alpha + \beta - 1 \).

**Remark 2.** In Theorem 2, when \( \kappa_0 > 0 \) and \( 2 < \alpha + 2\beta < 3, \lim_{\epsilon \to 0} \kappa \epsilon^2 \ell_1^{2\beta - 4} = 0 \) implies \( \lim_{\epsilon \to 0} \epsilon \ell_1^{\alpha+2\beta-3} = 0. \)

**Remark 3.** In the special case of \( \kappa_0 = 0 \), the limiting Kraichnan model preserves the \( L^2 \)-norm of the initial condition. On the other hand, the energy identity for the pre-limiting model (see Section 2).
\[ \int |T_\varepsilon^t(x)|^2 \, dx + \kappa \int_0^t \int |\nabla T_\varepsilon^s(x)|^2 \, dx \, ds = \int |T_0(x)|^2 \, dx \]

implies that \( \| T_\varepsilon^t \|_2 < \| T_0 \|_2 \). Consequently, the convergence in the sense of the weak-L^2 topology in Theorem 2 implies that \( \lim_{\varepsilon \to 0} \| T_\varepsilon^t \|_2 = \| T_t \|_2 \) and that the convergence is indeed in the strong L^2 sense.

Finally we note that the Gaussianity of the velocity field is not essential to the results. It has been used in the proofs to control the first 4 moments of the velocity fields and to have a mild decay in the tail distributions of the velocity fields (cf. (35)). The comparable result in [11] requires a faster-than-Gaussian decay in the tail distributions and does not apply here. It also requires spatial regularity in the velocity fields.

2. Formulation of solutions

From the general theory of parabolic partial differential equations [9], for any fixed \( \kappa > 0, \varepsilon > 0 \), the solution \( T_\varepsilon^t(x) \) is a \( C^{2+\eta} \)-function, with any \( 0 < \eta < \alpha - 1 \). But the solutions \( T_\varepsilon^t \) may lose all the regularity as \( \kappa \to 0, \varepsilon \to 0 \). So we consider the weak formulation of the equation:

\[ \langle T_\varepsilon^t, \theta \rangle - \langle T_0, \theta \rangle = \frac{\kappa}{2} \int_0^t \langle T_\varepsilon^s, \Delta \theta \rangle \, ds - \frac{1}{\varepsilon} \int_0^t \langle T_\varepsilon^s, \nabla \cdot (\theta V \left( \frac{s}{\varepsilon^2}, \cdot \right) \rangle \, ds \]

for any test function \( \theta \in C_\infty^\infty(\mathbb{R}^d) \), the space of smooth functions with compact supports. We view \( T_\varepsilon^t \) as distribution-valued processes. The solutions \( T_\varepsilon^t \) can be represented as

\[ T_\varepsilon^t(x) = \mathbb{M}[T_0(\Phi_{s}^t, \varepsilon)] \]

where \( \Phi_{s}^t, \varepsilon \) is the unique stochastic flow of the SDE

\[ d\Phi_{s}^t, \varepsilon(x) = -\frac{1}{\varepsilon} V(\Phi_{s}^t, \varepsilon, \frac{s}{\varepsilon^2}) \, ds + \kappa^{1/2} \, dw(t), \quad 0 \leq s \leq t \]

\[ \Phi_{s}^t, \varepsilon(x) = x. \]

In view of the averaging in the representation (9) we have

**Proposition 1.**

\[ \| T_\varepsilon^t \|_\infty \leq \| T_0 \|_\infty \quad a.s. \]

One also has that

\[ \mathbb{E}\{ \| T_\varepsilon^t \|_p \} \leq \| T_0 \|_p, \quad \forall p \geq 1. \]
Indeed, by the spatial homogeneity of the field $V$, the distribution of $\Phi^{t,\varepsilon}_{s}(x)$ is the same as the distribution of $\Phi^{t,\varepsilon}_{s}(0) + x$ for each fixed $x$. Hence we have

$$
\mathbb{E}[\|T^{\varepsilon}_{t}\|_{p}^{p}] \leq \int \mathbb{M}[T_{0}^{p}(\Phi^{t,\varepsilon}_{0}(x))dx = \mathbb{M}[\int T_{0}^{p}(\Phi^{t,\varepsilon}_{0}(0) + x)dx] = \|T_{0}\|_{p}^{p}.
$$

Proposition 1 (resp. (12)) says that, for $T_{0} \in L^{\infty}$ (resp. $L^{p}$), $T^{\varepsilon}_{t}$ is almost surely a $L^{\infty}$ (resp. $L^{p}$)-function for every $t \geq 0$.

For tightness as well as identification of the limit, the following infinitesimal operator $A^{\varepsilon}$ will play an important role. Let $V^{\varepsilon}_{t} \equiv V(t/\varepsilon^{2}, \cdot)$. Let $F^{\varepsilon}_{t}$ be the $\sigma$-algebras generated by $\{V^{\varepsilon}_{s}, s \leq t\}$ and $E^{\varepsilon}_{t}$ the corresponding conditional expectation w.r.t. $F^{\varepsilon}_{t}$. Let $M^{\varepsilon}$ be the space of measurable function adapted to $\{F^{\varepsilon}_{t}, \forall t\}$ such that $\sup_{t < t_{0}}\mathbb{E}|f(t)| < \infty$. We say $f(\cdot) \in D(A^{\varepsilon})$, the domain of $A^{\varepsilon}$, and $A^{\varepsilon}f = g$ if $f, g \in M^{\varepsilon}$ and for $f^{\delta}(t) \equiv \delta^{-1}[\mathbb{E}^{\varepsilon}_{t}f(t + \delta) - f(t)]$ we have

$$
\sup_{t, \delta} \mathbb{E}|f^{\delta}(t)| < \infty,
$$

$$
\lim_{\delta \to 0} \mathbb{E}|f^{\delta}(t) - g(t)| = 0, \ \forall t.
$$

For $f(t) = \phi(\langle T^{\varepsilon}_{t}, \theta \rangle), f'(t) = \phi'(\langle T^{\varepsilon}_{t}, \theta \rangle), \forall \phi \in C^{3}_{c}(\mathbb{R})$ (i.e. $C^{3}$-function with a compact support) we have the following expression from (8) and the chain rule

$$
A^{\varepsilon}f(t) = \frac{\kappa}{2}f'(t) \langle T^{\varepsilon}_{t}, \Delta \theta \rangle - \frac{1}{\varepsilon}f'(t) \langle T^{\varepsilon}_{t}, \nabla^{\varepsilon}_{t}(\theta) \rangle
$$

where

$$
\nabla^{\varepsilon}_{t}(\theta) \equiv \nabla \cdot [\theta V^{\varepsilon}_{t}].
$$

A main property of $A^{\varepsilon}$ is that

$$
f(t) - \int_{0}^{t} A^{\varepsilon}f(s)ds \text{ is a } F^{\varepsilon}_{t}\text{-martingale, } f \in D(A^{\varepsilon}).
$$

Also,

$$
\mathbb{E}^{\varepsilon}_{t}f(s) - f(t) = \int_{t}^{s} \mathbb{E}^{\varepsilon}_{t} A^{\varepsilon}f(\tau)d\tau \ \forall s > t \ \text{a.s.}
$$

(see [12]). We can view $T^{\varepsilon}_{t}$ as the distribution-valued stochastic solutions to the martingale problem (15).

Likewise we formulate the solutions for the Kraichnan’s model (9) as the solutions to the corresponding martingale problem. We will first describe the limiting martingale problem for Theorem 1 and then discuss the changes due to $\ell_{1} \to 0$ in Theorem 2. We rewrite (9) as an Itô’s SDE

$$
dT_{t} = \left(\frac{\kappa_{0}}{2} \Delta + \frac{1}{a} B\right) T_{t} dt + \sqrt{2a^{-1/2}} \nabla T_{t} \cdot dW_{t}^{(1)}
$$
where \( W_t^{(1)}(x) \) is the Brownian vector field with the spatial covariance
\[
\Gamma^{(1)}(x - y) = \int \exp(ik \cdot (x - y))E(\alpha + \beta, k)|k|^{1-d}dk
\]
and the operator \( \mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2 \) is given by
\[
\mathcal{B}_1 \phi = \sum_i \left[ \frac{\partial}{\partial x_i} \Gamma^{(1)}_{ij}(0) \right] \frac{\partial \phi}{\partial x_j} \tag{18}
\]
\[
\mathcal{B}_2 \phi = \sum_{i,j} \Gamma^{(1)}_{ij}(0) \frac{\partial^2 \phi}{\partial x_i \partial x_j}. \tag{19}
\]

Eq. (17) can be formulated as the martingale problem: Find a measure \( \mathbb{P} \) (of \( T_t \)) on the subspace of \( D([0, t_0]; L^\infty_w(\mathbb{R}^d)) \) whose elements have a given initial data in \( L^\infty_w(\mathbb{R}^d) \) such that
\[
f(\langle T_t, \theta \rangle) - \int_0^t \left\{ f'(\langle T_s, \theta \rangle) \left[ \frac{\kappa_0}{2} (T_s, \Delta \theta) + \frac{1}{a} (T_s, \mathcal{B}^* \theta) \right] + \frac{1}{2} f''(\langle T_t, \theta \rangle) \langle \theta, \mathcal{K}^{(1)}_{T_t} \theta \rangle \right\} ds
\]
is a martingale w.r.t. the filtration of a cylindrical Wiener process, for each \( f \in C^3_c(\mathbb{R}) \)

where \( \mathcal{B}^* \) is the adjoint of \( \mathcal{B} \) and \( \mathcal{K}^{(1)}_{T_t} \) is a positive-definite operator given formally as
\[
\mathcal{K}^{(1)}_{T_t} \theta = \int \theta(y) \nabla T_t(x) \cdot (x - y) \nabla T_t(y) dy \tag{20}
\]
such that
\[
\left< \theta_1, \mathcal{K}^{(1)}_t \theta_2 \right> = \int \int \phi(x) \phi(y) G^{(1)}_{\theta_1, \theta_2}(x, y) dx dy
\]

When \( \ell_1 \rightarrow 0 \) (Theorem 2) \( \Gamma^{(1)} \) in the preceding discussion should be replaced by
\[
\tilde{\Gamma}^{(1)}(x - y) = \lim_{\ell_1 \rightarrow 0} \Gamma^{(1)}(x - y) \tag{21}
\]
and all objects (such as \( \mathcal{B}, G_{\theta_1, \theta_2}, \mathcal{K}^{(1)}_{T_t} \)) related to \( \Gamma^{(1)} \) should be replaced accordingly (by \( \tilde{\mathcal{B}}, \tilde{G}_{\theta_1, \theta_2}, \tilde{\mathcal{K}}^{(1)}_{T_t} \)).

In particular, \( \mathcal{B}_1 \) is well-defined only for \( \alpha + \beta > 3/2 \) in general for compressible flows. Namely, the martingale problem (20) is not well defined in the compressible case unless the limiting Brownian velocity has a spatial Hurst exponent which is bigger than 1/2.

In the case of divergence-free vector fields, \( \tilde{\mathcal{B}}_1 = 0 \) and
\[
\tilde{\mathcal{B}} \phi = \sum_{i,j} \tilde{\Gamma}^{(1)}_{ij}(0) \frac{\partial^2 \phi}{\partial x_i \partial x_j}. \tag{25}
\]

Also,
\[
G^{(1)}_{\theta_1, \theta_2} \equiv \sum_{i,j} \tilde{\Gamma}^{(1)}_{ij}(x - y) \frac{\partial \theta_1(x)}{\partial x_i} \frac{\partial \theta_2(y)}{\partial y_j}. \tag{23}
\]
2.1. **Uniqueness of the limiting Kraichnan model.** When \( \lim_{\varepsilon \to 0} \ell_1 > 0 \) the limit Brownian velocity field is spatially smooth and generates a unique flow of diffeomorphisms on \( \mathbb{R}^d \) \([11, 2]\) from which it follows the uniqueness of the martingale solution.

When \( \lim_{\varepsilon \to 0} \ell_1 = 0 \) the limiting velocity field is only spatially Hölder continuous and we establish the uniqueness of the martingale solution by proving the uniqueness of the \( n \)-point correlation function

\[
F_n(x_1, x_2, x_3, \ldots, x_n) = \mathbb{E}_{T_0}[T_t(x_1)T_t(x_2)\cdots T_t(x_n)].
\]

The evolution of the \( n \)-point correlation function is given by a weakly continuous (hence strongly continuous) sub-Markovian semigroup on \( L^p(\mathbb{R}^n) \), \( \forall p \in (1, \infty) \) whose generator can be deduced by taking the test function \( f(r) = r^n \) in the martingale formulation:

\[
(26) \quad \mathcal{L}_n \Phi(x_1, \cdots, x_n) = \frac{\kappa_0}{2} \sum_{j=1}^n \Delta_{x_j} \Phi + \frac{1}{a} \sum_{i,j=1}^n \bar{\Gamma}^{(1)}(x_i - x_j) : \nabla_{x_i} \nabla_{x_j} \Phi, \quad \Phi \in C_c^\infty(\mathbb{R}^n), \quad \kappa_0 \geq 0.
\]

Note that the symmetric operator \( \mathcal{L}_n \) (26) is an essentially self-adjoint positive operator on \( C_c^\infty(\mathbb{R}^n) \), which then induces a **unique** symmetric Markov semigroup of contractions on \( L^2(\mathbb{R}^n) \). The essential self-adjointness is due to the sub-Lipschitz growth of the square-root of \( \bar{\Gamma}^{(1)}(x_1 - x_2) \) at large \( |x_1|, |x_2| \) (hence no escape to infinity) \([5]\).

In the sequel we will adopt the following notation

\[
f(t) \equiv f(T_t^\varepsilon, \theta), \quad f'(t) \equiv f'(T_t^\varepsilon, \theta), \quad f''(t) \equiv f''(T_t^\varepsilon, \theta), \quad f'''(t) \equiv f'''(T_t^\varepsilon, \theta) \quad \forall f \in C^3_c(\mathbb{R}).
\]

Namely, the prime stands for the differentiation w.r.t. the original argument \( (T_t^\varepsilon, \theta) \) not \( t \) of \( f, f' \) etc.

### 3. Proof of Theorem 1

The proofs are a refinement of that of \([4]\) to deal with the wave-number dependence of the correlation time and the lack of spatial regularity in the velocity fields. For the reader’s convenience, we will repeat some of the calculations in \([4]\) and refer the reader to \([13]\) for the full exposition of the perturbed test function method used here. The perturbed test function method is initiated in \([17]\).

#### 3.1. **Tightness.** A family of distribution-valued right-continuous with left limits processes \( \{T^\varepsilon, 0 < \varepsilon < 1\} \) is tight if and only if the family of real-valued, right-continuous with left limits processes \( \{(T^\varepsilon, \theta), 0 < \varepsilon < 1\} \) is tight for all \( \theta \in C_c^\infty(\mathbb{R}^d) \). We use the tightness criterion of \([13]\) (Chap. 3,
Theorem 4), namely, we will prove: Firstly,

\[ \lim_{N \to \infty} \limsup_{\varepsilon \to 0} \mathbb{P}\left\{ \sup_{t \leq t_0} \vert \langle T_\varepsilon, \theta \rangle \vert \geq N \right\} = 0, \quad \forall t_0 < \infty. \tag{27} \]

Secondly, for each \( f \in C_c^3(\mathbb{R}) \) there is a sequence \( f^\varepsilon(t) \in D(\mathcal{A}^\varepsilon) \) such that for each \( t_0 < \infty \)
\{\( A^\varepsilon f^\varepsilon(t), 0 < \varepsilon < 1, 0 < t < t_0 \)\} is uniformly integrable and

\[ \lim_{\varepsilon \to 0} \mathbb{P}\left\{ \sup_{t \leq t_0} \left| f^\varepsilon(t) - f(\langle T_\varepsilon, \theta \rangle) \right| \geq \delta \right\} = 0, \quad \forall \delta > 0. \tag{28} \]

Then it follows that the laws of \{\( \langle T_\varepsilon, \theta \rangle, 0 < \varepsilon < 1 \)\} are tight in the space \( L^\infty_{w^*}(\mathbb{R}^d) \).

Condition (27) is satisfied as a result of Proposition 1. Let
\[
\hat{f}_1^\varepsilon(t) = \frac{1}{\varepsilon} \int_t^\infty \mathbb{E}_t \hat{f}'(t) \langle T_\varepsilon^\varepsilon, \hat{V}_\varepsilon^\varepsilon(\theta) \rangle \, ds
\]
be the 1-st perturbation of \( f(t) \). Using the spectral representation

\[ \mathbb{E}_t \hat{V}_s^\varepsilon = \int e^{ix \cdot k} e^{-a |k|^{2\beta} |s-t|^\varepsilon^{-2}} \hat{V}_t^\varepsilon(dk), \quad \forall s \geq t, \tag{29} \]

we obtain

\[ f_1^\varepsilon(t) = \frac{\varepsilon}{a} \hat{f}'(t) \langle T_\varepsilon^\varepsilon, \hat{V}_\varepsilon^\varepsilon(\theta) \rangle \tag{30} \]

with

\[ \hat{V}_\varepsilon^\varepsilon(\theta) = \nabla \cdot [\theta \hat{V}_\varepsilon^\varepsilon] \]

\[ \hat{V}_\varepsilon^\varepsilon(\theta) = \hat{V} \left( \frac{t}{\varepsilon^2} \right) \equiv \varepsilon^{-2} \int_t^\infty \mathbb{E}_t \hat{V}_s^\varepsilon \, ds \tag{31} \]

where \( \hat{V} \) has the power spectrum \( E(\alpha + 2\beta, k) \).

**Proposition 2.**

\[ \lim_{\varepsilon \to 0} \sup_{t < t_0} \mathbb{E} \left| f_1^\varepsilon(t) \right| = 0, \quad \lim_{\varepsilon \to 0} \sup_{t < t_0} \left| f_1^\varepsilon(t) \right| = 0 \quad \text{in probability} \]

**Proof.** By Proposition 1 we have

\[ \mathbb{E} \left[ \left| f_1^\varepsilon(t) \right| \right] \leq \frac{\varepsilon}{a} \| f' \|_\infty \| T_0 \|_\infty \left[ \| \theta \|_\infty \int_{|x| \leq M} \mathbb{E} \left| \hat{V}_t^\varepsilon(x) \right| \, dx + \| \nabla \theta \|_\infty \int_{|x| \leq M} \mathbb{E} \left| \nabla \cdot \hat{V}_t^\varepsilon \right| \, dx \right] \tag{33} \]

and

\[ \sup_{t < t_0} \left| f_1^\varepsilon(t) \right| \leq \frac{\varepsilon}{a} \| f' \|_\infty \| T_0 \|_\infty \left[ \| \theta \|_\infty \sup_{t < t_0} \int_{|x| \leq M} \left| \hat{V}_t^\varepsilon(x) \right| \, dx + \| \nabla \theta \|_\infty \sup_{t < t_0} \int_{|x| \leq M} \left| \nabla \cdot \hat{V}_t^\varepsilon \right| \, dx \right]. \tag{34} \]
By the temporal stationarity of $\tilde{V}^\varepsilon(t)$ we can replace $E[\tilde{V}^\varepsilon(x)], E[\nabla \cdot \tilde{V}^\varepsilon(x)]$ in (33) by $E[\tilde{V}(0, x)], E[\nabla \cdot \tilde{V}(0, x)]$. By the Gaussianity, temporal stationarity and spatial homogeneity of $\tilde{V}$, we can replace $\sup_{t < t_0} \int_{|x| \leq M} |\tilde{V}^\varepsilon(x)| \, dx$ in (34) by

$$M^d \sup_{|x| \leq M} \left| \tilde{V} \left( \frac{t}{\varepsilon}, x \right) \right| \leq C \log \left[ \frac{M^d t_0}{\varepsilon^2} \right] = o \left( \frac{1}{\varepsilon} \right)$$

with a random constant $C$ possessing a distribution with a finite moment (Indeed, a Gaussian-like tail by Chernoff’s bound). A similar inequality holds for $\nabla \cdot \tilde{V}$.

Proposition 2 now follows from (33), (34) and (35).

Set $f^\varepsilon(t) = f(t) - f_1(t)$. A straightforward calculation yields

$$\mathcal{A}^\varepsilon f_1^\varepsilon = -\frac{k \varepsilon}{2a} f''(t) \left( T^\varepsilon_t, \Delta \theta \right) \left( T^\varepsilon_t, \tilde{V}^\varepsilon_t(\theta) \right) + \frac{k \varepsilon}{2a} f'(t) \left( T^\varepsilon_t, \Delta \tilde{V}^\varepsilon_t(\theta) \right)$$

$$+ \frac{1}{a} f''(t) \left( T^\varepsilon_t, V^\varepsilon_t(\theta) \right) \left( T^\varepsilon_t, \tilde{V}^\varepsilon_t(\theta) \right) - \frac{1}{a} f'(t) \left( T^\varepsilon_t, V^\varepsilon_t(\theta) \right)$$

and, hence

$$\mathcal{A}^\varepsilon f^\varepsilon(t) = \frac{K}{2} f'(t) \left( T^\varepsilon_t, \Delta \theta \right) - \frac{1}{a} f'(t) \left( T^\varepsilon_t, V^\varepsilon_t(\theta) \right) - \frac{1}{a} f''(t) \left( T^\varepsilon_t, V^\varepsilon_t(\theta) \right)$$

$$+ \frac{k \varepsilon}{2a} \left[ f''(t) \left( T^\varepsilon_t, \Delta \theta \right) \left( T^\varepsilon_t, V^\varepsilon_t(\theta) \right) - f'(t) \left( T^\varepsilon_t, \Delta \tilde{V}^\varepsilon_t(\theta) \right) \right]$$

$$= A^\varepsilon_1(t) + A^\varepsilon_2(t) + A^\varepsilon_3(t) + A^\varepsilon_4(t)$$

where $A^\varepsilon_2(t)$ and $A^\varepsilon_3(t)$ are the $O(1)$ statistical coupling terms.

For the tightness criterion stated in the beginnings of the section, it remains to show

**Proposition 3.** \{\mathcal{A}^\varepsilon f^\varepsilon\} are uniformly integrable and

$$\lim_{\varepsilon \to 0} \sup_{t < t_0} E[|A^\varepsilon_4(t)|] = 0$$

**Proof.** We show that \{\mathcal{A}^\varepsilon f^\varepsilon\}, $i = 1, 2, 3, 4$ are uniformly integrable. To see this, we have the following estimates.

$$|A^\varepsilon_i(t)| = \frac{K}{2} \left| f'(t) \left( T^\varepsilon_t, \Delta \theta \right) \right| \leq \frac{K}{2} \| f' \|_\infty \| T_0 \|_\infty \| \Delta \theta \|_1$$
Thus $A^*_1$ is uniformly integrable since it is uniformly bounded.

$$
|A^*_2(t)| = \frac{1}{a} |f'(t) \langle T^*_t, V^*_t(\theta) \rangle |
\leq \frac{C}{a} ||f'||\infty ||T_0||\infty \left[ \int_{|x|<M} |V^*_t|^2 \, dx + \int_{|x|<M} |\nabla \cdot V^*_t|^2 \, dx \right]^{1/2} \times
\left[ \int_{|x|<M} |\hat{V}^*_t|^2 \, dx + \int_{|x|<M} |\nabla \cdot \hat{V}^*_t|^2 \, dx \right]^{1/2}.
$$

Thus $A^*_2$ is uniformly integrable in view of the uniform boundedness of the 4-th moment of $V^*_t, \hat{V}^*_t$ and their spatial derivatives due to Gaussianity and the ultraviolet cutoff $\ell_1 > 0$.

$$
|A^*_3(t)| = \frac{1}{a} |f''(t) \langle T^*_t, V^*_t(\theta) \rangle \langle T^*_t, \hat{V}^*_t(\theta) \rangle |
\leq \frac{C}{a} ||f'||\infty ||T_0||\infty \left[ \int_{|x|<M} |V^*_t|^2 \, dx + \int_{|x|<M} |\nabla \cdot V^*_t|^2 \, dx + \int_{|x|<M} |\hat{V}^*_t|^2 \, dx + \int_{|x|<M} |\nabla \cdot \hat{V}^*_t|^2 \, dx \right].
$$

Thus $A^*_3$ is uniformly integrable for the similar reason that $A^*_2$ is uniformly integrable.

$$
|A^*_4| = \frac{K_\varepsilon}{2a} |f''(t) \langle T^*_t, \Delta \theta \rangle \langle T^*_t, \hat{V}^*_t(\theta) \rangle - f'(t) \langle T^*_t, \Delta \hat{V}^*_t(\theta) \rangle |
\leq \frac{C K_\varepsilon}{2a} \left[ \int_{|x|<M} |\hat{V}^*_t|^2 \, dx + \int_{|x|<M} |\nabla \cdot \hat{V}^*_t|^2 \, dx \right] \times
\left[ \int_{|x|<M} |\hat{V}^*_t|^2 \, dx + \int_{|x|<M} |\nabla \hat{V}^*_t|^2 \, dx + \int_{|x|<M} |\Delta \hat{V}^*_t|^2 \, dx \right]^{1/2}.
$$

Due to the fixed cutoff $\ell_1 > 0$, the higher derivatives of $\hat{V}^*_t$ do not cause any difficulty and they all have uniformly bounded, say, the 4-th moments. Hence $A^*_3$ is uniformly integrable. Clearly

$$
\lim_{\varepsilon \to 0} \sup_{t<t_0} \mathbb{E}|A^*_4(t)| = 0.
$$

3.2. Identification of the limit. Once the tightness is established we can use another result in [13] (Chapter 3, Theorem 2) to identify the limit. Let $A$ be a diffusion or jump diffusion operator such that there is a unique solution $\omega_t$ in the subspace of $D([0, t_0); L^\infty_w(\mathbb{R}^d)), \forall t_0 < \infty$, whose elements have the given initial data in $L^\infty_w(\mathbb{R}^d)$ such that

$$
f(\omega_t) - \int_0^t A f(\omega_s) \, ds
$$

(38)
is a martingale. We shall show that for each \( f \in C^3_3(\mathbb{R}) \) there exists \( f^\varepsilon \in D(\mathcal{A}^\varepsilon) \) such that

\[
\sup_{t < t_0, \varepsilon} \mathbb{E}|f^\varepsilon(t) - f(\langle T^\varepsilon_t, \theta \rangle)| < \infty
\]

(39)

\[
\lim_{\varepsilon \to 0} \mathbb{E}|f^\varepsilon(t) - f(\langle T^\varepsilon_t, \theta \rangle)| = 0, \quad \forall t < t_0
\]

(40)

\[
\sup_{t < t_0, \varepsilon} \mathbb{E} |\mathcal{A}^\varepsilon f^\varepsilon(t) - \mathcal{A} f(\langle T^\varepsilon_t, \theta \rangle)| < \infty
\]

(41)

\[
\lim_{\varepsilon \to 0} \mathbb{E} |\mathcal{A}^\varepsilon f^\varepsilon(t) - \mathcal{A} f(\langle T^\varepsilon_t, \theta \rangle)| = 0, \quad \forall t < t_0.
\]

(42)

Then the aforementioned theorem implies that any tight processes \( \langle T^\varepsilon_t, \theta \rangle \) converges in law to the unique process generated by \( \mathcal{A} \). As before we adopt the notation \( f(t) = f(\langle T^\varepsilon_t, \theta \rangle) \).

For this purpose, we introduce the next perturbations \( f^\varepsilon_2, f^\varepsilon_3 \). Let

\[
A_2^{(1)}(\phi) \equiv \left\langle \theta, K^{(1)}_\phi \theta \right\rangle
\]

(43)

\[
A_3^{(1)}(\phi) \equiv \left\langle \phi, \mathbb{E} \left[ V^\varepsilon_t(\tilde{V}^\varepsilon_t(\theta)) \right] \right\rangle
\]

(44)

where the positive-definite operator \( K^{(1)}_\phi \) is defined in (21). It is easy to see that

\[
A_2^{(1)}(\phi) = \mathbb{E} \left[ \langle \phi, V^\varepsilon_t(\theta) \rangle \langle \phi, \tilde{V}^\varepsilon_t(\theta) \rangle \right]
\]

(45)

\[
A_3^{(1)}(\phi) = \langle \mathcal{B} \phi, \theta \rangle
\]

(46)

where the operator \( \mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2 \) is given by (18) and (19).

Define

\[
f^\varepsilon_2(t) = \frac{1}{a} f''(t) \int_t^\infty \mathbb{E}^\varepsilon_t \left[ \langle T^\varepsilon_s, V^\varepsilon_s(\theta) \rangle \langle T^\varepsilon_t, \tilde{V}^\varepsilon_t(\theta) \rangle - A_2^{(1)}(T^\varepsilon_t) \right] ds
\]

(47)

\[
f^\varepsilon_3(t) = \frac{1}{a} f'(t) \int_t^\infty \mathbb{E}^\varepsilon_t \left[ \langle T^\varepsilon_s, V^\varepsilon_s(\tilde{V}^\varepsilon_s(\theta)) \rangle - A_3^{(1)}(T^\varepsilon_t) \right] ds.
\]

(48)

Let

\[
G^{(2)}_{\theta_1, \theta_2} = \sum_{i,j} \frac{\partial^2}{\partial x_i \partial y_j} \left[ \theta_1(x) \theta_2(y) \Gamma^{(2)}_{ij}(x - y) \right]
\]

(49)

\[
\left\langle \theta_1, K^{(2)}_\phi \theta_2 \right\rangle = \iint \phi(x) \phi(y) G^{(2)}_{\theta_1, \theta_2}(x, y) \, dx \, dy
\]

where the covariance function \( \Gamma^{(2)}(x - y) = \mathbb{E} \left[ \tilde{V}^\varepsilon_t(x) \otimes \tilde{V}^\varepsilon_t(y) \right] \) has the spectral density \( \mathcal{E}(\alpha + 2\beta, k) \), and let

\[
A_2^{(2)}(\phi) \equiv \left\langle \theta, K^{(2)}_\phi \theta \right\rangle
\]

(50)

\[
A_3^{(2)}(\phi) \equiv \left\langle \phi, \mathbb{E} \left[ \tilde{V}^\varepsilon_t(\tilde{V}^\varepsilon_t(\theta)) \right] \right\rangle.
\]
Noting that
\[
\mathbb{E}_t^\varepsilon [V_s^\varepsilon (x) \otimes \tilde{V}_s^\varepsilon (y)] \\
= \int e^{i (x-y) \cdot k} \hat{V}_t^\varepsilon (dk) \otimes \hat{\tilde{V}}_t^\varepsilon (dk) e^{-2a|k|^2|s-t|\varepsilon^{-2}} \\
+ \int e^{i (x-y) \cdot k} \left[1 - e^{-2a|k|^2|s-t|\varepsilon^{-2}}\right] \mathcal{E}(\alpha + \beta, k) \, dk
\]
we then have
\[
f_2^\varepsilon (t) = \frac{\varepsilon^2}{2a^2} f''(t) \left[ \left\langle T_t^\varepsilon, \tilde{V}_t^\varepsilon (\theta) \right\rangle - A_2^{(2)} (T_t^\varepsilon) \right]
\]
and similarly
\[
f_3^\varepsilon (t) = \frac{\varepsilon^2}{2a^2} f'(t) \left[ \left\langle T_t^\varepsilon, \tilde{V}_t^\varepsilon (\theta) \right\rangle - A_3^{(2)} (T_t^\varepsilon) \right].
\]

In view of the prefactor \( \varepsilon \) in (48) and (49) and the fact that all terms involved are regular and uniformly bounded, we have

**Proposition 4.**
\[
\lim_{\varepsilon \to 0} \sup_{t < t_0} \mathbb{E}|f_2^\varepsilon (t)| = 0, \quad \lim_{\varepsilon \to 0} \sup_{t < t_0} \mathbb{E}|f_3^\varepsilon (t)| = 0.
\]

The proof of Proposition 4 is analogous to that of Proposition 2.

We have
\[
\mathcal{A}^\varepsilon f_2^\varepsilon (t) = \frac{1}{a} f''(t) \left[ \left\langle T_t^\varepsilon, \tilde{V}_t^\varepsilon (\theta) \right\rangle - A_2^{(1)} (T_t^\varepsilon) \right] + R_2^\varepsilon (t)
\]
\[
\mathcal{A}^\varepsilon f_3^\varepsilon (t) = \frac{1}{a} f'(t) \left[ \left\langle T_t^\varepsilon, \tilde{V}_t^\varepsilon (\theta) \right\rangle - A_3^{(1)} (T_t^\varepsilon) \right] + R_3^\varepsilon (t)
\]
with
\[
R_2^\varepsilon (t) = f''(t) \left[ \frac{\varepsilon^2 \kappa}{4a^2} \left\langle T_t^\varepsilon, \Delta \theta \right\rangle - \frac{\varepsilon}{2a^2} \left\langle T_t^\varepsilon, \tilde{V}_t^\varepsilon (\theta) \right\rangle \right] \left\langle T_t^\varepsilon, \tilde{V}_t^\varepsilon (\theta) \right\rangle - A_2^{(2)} (T_t^\varepsilon) \right]
\]
\[
+ f''(t) \left\langle T_t^\varepsilon, \tilde{V}_t^\varepsilon (\theta) \right\rangle \left[ \frac{\kappa \varepsilon^2}{2a^2} \left\langle T_t^\varepsilon, \Delta \tilde{V}_t^\varepsilon (\theta) \right\rangle - \frac{\varepsilon}{a^2} \left\langle T_t^\varepsilon, \tilde{V}_t^\varepsilon (\tilde{V}_t^\varepsilon (\theta)) \right\rangle \right]
\]
\[
- f''(t) \left[ \frac{\kappa \varepsilon^2}{4a^2} \left\langle T_t^\varepsilon, \Delta G^{(2)}_\theta T_t^\varepsilon \right\rangle - \frac{\varepsilon}{a^2} \left\langle T_t^\varepsilon, \tilde{V}_t^\varepsilon (G^{(2)}_\theta T_t^\varepsilon) \right\rangle \right]
\]
and
\[
R_3^\varepsilon (t) = \frac{\kappa \varepsilon^2}{4a^2} \left\langle T_t^\varepsilon, \Delta G^{(2)}_\theta T_t^\varepsilon \right\rangle - \frac{\varepsilon}{a^2} \left\langle T_t^\varepsilon, \tilde{V}_t^\varepsilon (G^{(2)}_\theta T_t^\varepsilon) \right\rangle
\]
where in (50) \( G^{(2)}_\theta \) denotes the operator
\[
G^{(2)}_\theta \phi \equiv \int G^{(2)}_\theta (x, y) \phi (y) \, dy.
\]
and similarly
\[ R_3^ε(t) = f''(t) \left[ \frac{κε^2}{4a^2} \langle T_1^ε, Δθ \rangle - \frac{ε}{2a^2} \langle T_1^ε, V_i^ε(θ) \rangle \right] \left[ \left\langle T_i^ε, \bar{V}_i^ε(θ) \right\rangle - A_3^{(2)}(T_i^ε) \right] + f'(t) \left[ \frac{κε^2}{4a^2} \left\langle T_i^ε, \Delta \bar{V}_i^ε(θ) \right\rangle - \frac{ε}{2a^2} \left\langle T_i^ε, V_i(\bar{V}_i^ε(θ)) \right\rangle \right] - f'(t) \left[ \frac{κε^2}{4a^2} \left\langle T_i^ε, \Delta E[\bar{V}_i^ε(\bar{V}_i^ε(θ))] \right\rangle + \frac{ε}{2a^2} \left\langle T_i^ε, V_i(\mathbb{E}[\bar{V}_i^ε(\bar{V}_i^ε(θ))]) \right\rangle \right]. \]

Now all terms appearing in \( R_2^ε(t) \) and \( R_3^ε(t) \) are regular and uniformly bounded, we easily have

**Proposition 5.**

\[ \limsup_{ε \to 0, t < t_0} \mathbb{E}[|R_2^ε(t)|] = 0, \quad \limsup_{ε \to 0, t < t_0} \mathbb{E}[|R_3^ε(t)|] = 0. \]

Set
\[ R^ε(t) = A_4^ε(t) + R_2^ε(t) + R_3^ε(t). \]

It follows from Propositions 3 and 5 that
\[ \limsup_{ε \to 0, t < t_0} \mathbb{E}[|R^ε(t)|] = 0. \]

Recall that
\[ M_i^ε(θ) = f^ε(t) - \int_0^t A^ε f^ε(s) \, ds = f(t) - f_1^ε(t) + f_2^ε(t) + f_3^ε(t) - \int_0^t \frac{κ}{2} f'(t) \langle T_i^ε, Δθ \rangle \, ds - \int_0^t \frac{1}{a} \left[ f''(t)(\langle T_i^ε, θ \rangle A_2^{(1)}(T_i^ε) + f'(t) A_3^{(1)}(T_i^ε) \right] \, ds - \int_0^t R^ε(s) \, ds \]

is a martingale. Now that \([39], [12]\) are satisfied we can identify the limiting martingale to be

\[ M_t(θ) = f(t) - \int_0^t \left\{ f'(s) \left[ \frac{κ_0}{2} \langle T_0, Δθ \rangle + \frac{1}{a} A_3^{(1)}(T_0) \right] + \frac{1}{a} f''(s) A_2^{(1)}(T_0) \right\} \, ds. \]

Since \( \langle T_i^ε, θ \rangle \) is uniformly bounded
\[ |\langle T_i^ε, θ \rangle| \leq \|T_0\|_∞ \|θ\|_1 \]

we have the convergence of the second moment
\[ \lim_{ε \to 0} \mathbb{E} \left\{ \langle T_i^ε, θ \rangle^2 \right\} = \mathbb{E} \left\{ \langle T_0, θ \rangle^2 \right\}. \]

Use \( f(r) = r \) and \( r^2 \) in \([51]\)

\[ M_i^{(1)}(θ) = \langle T_t, θ \rangle - \int_0^t \left[ \frac{κ_0}{2} \langle T_0, Δθ \rangle + \frac{1}{a} A_3^{(1)}(T_0) \right] \, ds \]
is a martingale with the quadratic variation
\[ [M^{(1)}(\theta), M^{(1)}(\theta)]_t = \frac{2}{a} \int_0^t A^{(1)}_2(T_s) \, ds = \frac{2}{a} \int_0^t \langle \theta, \mathcal{K}^{(1)} T_s \theta \rangle \, ds. \]
Therefore,
\[ M^{(1)}_t = \sqrt{\frac{2}{a} \int_0^t \sqrt{\mathcal{K}^{(1)}_T} \, dW_s} \]
where \( W_s \) is a cylindrical Wiener process (i.e. \( dW_t(x) \) is a space-time white noise field) and \( \sqrt{\mathcal{K}^{(1)}_T} \) is the square-root of the positive-definite operator given in (21). From (43) and (46) we see that the limiting process \( T_t \) is the (assumed unique) distributional solution to the martingale problem (20) of the Itô’s equation
\[ dT_t = \left( \frac{\kappa_0}{2} \Delta + \frac{1}{a} \mathcal{B} \right) T_t \, dt + \sqrt{2a^{-1} \mathcal{K}^{(1)}_T} \, dW_t \]
where the operator \( \mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2 \) is given by (18)-(19) and \( W^{(1)}_t \) is the Brownian vector field with the spatial covariance \( \Gamma^{(1)}(x - y) \).

4. PROOF OF THEOREM 2

The argument is the same as before except with
\[ V^\varepsilon_t(\theta) = V_t \cdot \nabla \theta \]
\[ \tilde{V}^\varepsilon_t(\theta) = \tilde{V}_t \cdot \nabla \theta, \]
instead of (14) and (31), because of the incompressibility of the velocity fields. Also, all the terms containing \( \nabla \cdot V^\varepsilon_t \) and \( \nabla \cdot \tilde{V}^\varepsilon_t \) vanish.

The most severe term to occur in the argument for tightness (in the expression for \( A_2^\varepsilon \)) is
\[ \frac{\kappa \varepsilon}{2a} \left| f'(t) \left( T^\varepsilon_t, \Delta \tilde{V}^\varepsilon_t(\theta) \right) \right| \]
whose second moment can be bounded as
\[ \frac{\kappa \varepsilon}{2a} \sqrt{\mathbb{E} \left| f'(t) \left( T^\varepsilon_t, \nabla^2 \tilde{V}^\varepsilon_t(\theta) \right) \right|^2} \]
\[ \leq C_1 \frac{\kappa \varepsilon}{2a} \| f' \|_\infty \| T_0 \|_\infty \left( \int_{|x| < M} \mathbb{E} \left[ |\Delta \tilde{V}^\varepsilon_t|^2 \right] \, dx \right)^{1/2} \]
\[ \leq C_2 \kappa \varepsilon \times \begin{cases} \ell_1^{\alpha + 2\beta - 3}, & \text{for } \alpha + 2\beta < 3 \\ \sqrt{\log (1/\ell_1)}, & \text{for } \alpha + 2\beta = 3 \\ 1, & \text{for } \alpha + 2\beta > 3. \end{cases} \]
Other possibly divergent terms occurring in identifying the limit can be controlled similarly. For instance, the most severe term without the prefactor $\kappa$ occurs in $R_3^\varepsilon(t)$ and can be bounded as

$$
\varepsilon \mathbb{E} \left| \left\langle T_t^\varepsilon, \mathcal{V}_t^\varepsilon (\tilde{V}_t^\varepsilon (\theta)) \right\rangle \right| \leq \varepsilon \|T_0\|_\infty \mathbb{E}^{1/2} |\mathcal{V}_t^\varepsilon (\tilde{V}_t^\varepsilon (\theta))|^2
$$

$$
\leq C_1 \varepsilon \|T_0\|_\infty \left( \int_{|x|<M} \mathbb{E} |\mathcal{V}_t^\varepsilon|^2 \, dx \right)^{1/2} \times 
$$

$$
\left( \int_{|x|<M} \mathbb{E} |\tilde{\mathcal{V}}_t^\varepsilon|^2 \mathbb{E} \left[ |\nabla^2 \tilde{\mathcal{V}}_t^\varepsilon|^2 \right] \, dx + \int_{|x|<M} \mathbb{E} \left[ |\nabla \tilde{\mathcal{V}}_t^\varepsilon|^4 \right] \, dx \right)^{1/2}
$$

(53)

$$
\leq C_2 \varepsilon \left( \int_{|x|<M} \mathbb{E} \left[ |\nabla^2 \tilde{\mathcal{V}}_t^\varepsilon|^2 \right] \, dx \right)^{1/2}
$$

by the Gaussianity of the fields. The right side of (52) and (53) tends to zero if either

$$
\alpha + 2\beta > 3
$$

or

$$
\alpha + 2\beta = 3, \quad \lim_{\varepsilon \to 0} \varepsilon \sqrt{\log \left( \frac{1}{\ell_1} \right)} = 0
$$

or

$$
\alpha + 2\beta < 3, \quad \lim_{\varepsilon \to 0} \varepsilon (\alpha + 2\beta - 3) = 0
$$

is satisfied. The term involving $\varepsilon \left\langle T_t^\varepsilon, \mathcal{V}_t^\varepsilon (G_\theta^{(2)} T_t^\varepsilon) \right\rangle$ can be similarly estimated.

The most severe term involving the prefactor $\kappa$ occurs in $R_3^\varepsilon$ and can be bounded as

$$
\kappa \varepsilon^2 \mathbb{E} \left| \left\langle T_t^\varepsilon, \Delta \tilde{\mathcal{V}}_t^\varepsilon (\tilde{V}_t^\varepsilon (\theta)) \right\rangle \right| \leq C \kappa \varepsilon^2 \|T_0\|_\infty \left( \int_{|x|<M} \mathbb{E} \left[ |\nabla^3 \tilde{\mathcal{V}}_t^\varepsilon|^2 \right] \right)^{1/2}
$$

(56)

$$
\sim \begin{cases} 
\kappa \varepsilon^2, & \text{for } \alpha + 2\beta > 4 \\
\kappa \varepsilon^2 \sqrt{\log \left( \frac{1}{\ell_1} \right)}, & \text{for } \alpha + 2\beta = 4 \\
\kappa \varepsilon^2 \ell_1^{\alpha + 2\beta - 4}, & \text{for } \alpha + 2\beta < 4 
\end{cases}
$$

the right side of which tends to zero if either

$$
\alpha + 2\beta > 4
$$

or

$$
\alpha + 2\beta = 4, \quad \lim_{\varepsilon \to 0} \kappa \varepsilon^2 \sqrt{\log \left( \frac{1}{\ell_1} \right)} = 0
$$

or

$$
3 < \alpha + 2\beta < 4, \quad \lim_{\varepsilon \to 0} \kappa \varepsilon^2 \ell_1^{\alpha + 2\beta - 4} = 0
$$

(57)
or

\[ 2 < \alpha + 2\beta < 3, \quad \lim_{\varepsilon \to 0} \varepsilon^2 \ell_1^{\alpha+2\beta-4} = \lim_{\varepsilon \to 0} \varepsilon^{\alpha+2\beta-3} = 0. \]

Note that for \( \alpha + 2\beta \leq 2 \) the condition (54) or (55) implies that

\[ \lim_{\varepsilon \to 0} \varepsilon^2 \ell_1^{\alpha+2\beta-4} = 0. \]

Finally we note that in the limit \( (\varepsilon, \ell_1 \to 0) \) the limiting martingale is, instead of (51),

\[ M_t(\theta) = f(t) - \int_0^t \left\{ f'(s) \left[ \frac{\kappa_0}{2} (T_s, \Delta \theta) + \frac{1}{a} \bar{A}_3^{(1)}(T_s) \right] + \frac{1}{a} f''(s) \bar{A}_2^{(1)}(T_s) \right\} ds \]

where

\[ \bar{A}_2^{(1)} = \lim_{\ell_1 \to 0} A_2^{(1)}, \quad \bar{A}_3^{(1)} = \lim_{\ell_1 \to 0} A_3^{(1)} \]

and the limiting process \( T_t \) is the (assumed unique) distributional solution to the martingale problem associated with the SDE

\[ dT_t = \left( \frac{\kappa_0}{2} \Delta + \frac{1}{a} \bar{B} \right) T_t dt + \sqrt{2a^{-1/2}} \nabla T_t \cdot d\bar{W}_t^{(1)} \]

where \( \bar{W}_t^{(1)} \) is the Brownian vector field with the spatial covariance \( \Gamma^{(1)}(x - y) \) given in (24) and the operator \( \bar{B} \) is given in (25).

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