On a variety related to the commuting variety of a reductive Lie algebra.
Jean-Yves Charbonnel

To cite this version:
Jean-Yves Charbonnel. On a variety related to the commuting variety of a reductive Lie algebra.. 2016. hal-01178429v2

HAL Id: hal-01178429
https://hal.science/hal-01178429v2
Preprint submitted on 5 Aug 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
ON A VARIETY RELATED TO THE COMMUTING VARIETY OF A REDUCTIVE LIE ALGEBRA.

JEAN-YVES CHARBONNEL

Abstract. For a reductive Lie algebra over an algebraically closed field of characteristic zero, we consider a Borel subgroup $B$ of its adjoint group, a Cartan subalgebra contained in the Lie algebra of $B$ and the closure $X$ of its orbit under $B$ in the Grassmannian. The variety $X$ plays an important role in the study of the commuting variety. In this note, we prove that $X$ is Gorenstein with rational singularities.

CONTENTS

1. Introduction 1
2. On solvable algebras 3
3. Solvable algebras and main varieties 17
4. Normality for solvable Lie algebras 22
5. Rational singularities for solvable Lie algebras 30
Appendix A. Rational Singularities 37
Appendix B. About singularities 39
References 39

1. Introduction

In this note, the base field $k$ is algebraically closed of characteristic 0, $\mathfrak{g}$ is a reductive Lie algebra of finite dimension, $\ell$ is its rank, $\dim \mathfrak{g} = \ell + 2n$ and $G$ is its adjoint group. As usual, $\mathfrak{b}$ denotes a Borel subalgebra of $\mathfrak{g}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, contained in $\mathfrak{b}$, and $B$ the normalizer of $\mathfrak{b}$ in $G$.

1.1. Main results. Let $X$ be the closure in $\text{Gr}_\ell(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under the action of $B$. By a well known result, $G.X$ is the closure in $\text{Gr}_\ell(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under the action of $G$. By [Ri79], the commuting variety of $\mathfrak{g}$ is the image by the canonical projection of the restriction to $G.X$ of the canonical vector bundle of rank $2\ell$ over $\text{Gr}_\ell(\mathfrak{g})$. So $X$ and $G.X$ play an important role in the study of the commuting variety. As it is explained in [CZ16], $X$ and $G.X$ play the same role for the so called generalized commuting varieties and the so called generalized isospectral commuting varieties. The main result of this note is the following theorem:

**Theorem 1.1.** The variety $X$ is Gorenstein with rational singularities.

An induction is used to prove this theorem. So we introduce the categories $\mathcal{C}'_t$ and $\mathcal{C}_t$ with $t$ a commutative Lie algebra of finite dimension. Their objects are nilpotent Lie algebras of finite dimension, normalized by $t$ with additional conditions analogous to those of the action of $\mathfrak{h}$ in $\mathfrak{u}$. In particular the minimal dimension

\textit{Date:} August 5, 2016.

\textit{1991 Mathematics Subject Classification.} 14A10, 14L17, 22E20, 22E46 .

\textit{Key words and phrases.} polynomial algebra, complex, commuting variety, desingularization, Gorenstein, Cohen-Macaulay, rational singularities, cohomology.
of the objects in \( \mathcal{C}_t \) is the dimension of \( t \) and an object of dimension \( \dim t \) is a commutative algebra. The category \( \mathcal{C}_t \) is a full subcategory of \( \mathcal{C}'_t \). For \( a \) in \( \mathcal{C}'_t \), we consider the solvable Lie algebra \( r := t + a \) and \( R \) the adjoint group of \( r \). Denoting by \( X_R \) the closure in \( \text{Gr}_{\dim t}(r) \) of the orbit of \( t \) under \( R \), we prove by induction on \( \dim a \) the following theorem:

**Theorem 1.2.** The variety \( X_R \) is normal and Cohen-Macaulay.

The result for the category \( \mathcal{C}'_t \) is easily deduced from the result for the category \( \mathcal{C}_t \) by Corollary 2.2. One of the key argument in the proof is the consideration of the fixed points under the action of \( R \) in \( X_R \). As a matter of fact, since the closure of all orbit under \( R \) in \( X_R \) contains a fixed point, \( X_R \) is Cohen-Macaulay if so are the fixed points by openness of the set of Cohen-Macaulay points. Then, by Serre’s normality criterion, it suffices to prove that \( X_R \) is smooth in codimension 1. For that purpose the consideration of the restriction to \( X_R \) of the tautological vector bundle of rank \( \dim t \) over \( \text{Gr}_{\dim t}(r) \) is very useful.

For the study of the fixed points, we introduce Property \((P)\) and Property \((P_1)\) for the objects of \( \mathcal{C}'_t \):

- Property \((P)\) for \( a \) in \( \mathcal{C}'_t \) says that for \( V \) in \( X_R \), contained in the centralizer \( r^s \) of an element \( s \) of \( t \), \( V \) is in the closure of the orbit of \( t \) under the centralizer \( r^s \) of \( s \) in \( R \).
- Property \((P_1)\) for \( a \) in \( \mathcal{C}'_t \) says that for \( V \) in \( X_R \) normalized by \( t \) and such that \( V \cap t \) is the center of \( r \), then the non zero weights of \( t \) in \( V \) are linearly independent.

Property \((P_1)\) for \( a \) results from Property \((P)\) for \( a \) and Property \((P)\) for \( a \) results from Property \((P_1)\) for \( a \) and Property \((P)\) for the objects of \( \mathcal{C}'_t \) of dimension smaller than \( \dim a \). So, the main result for the objects of \( \mathcal{C}'_t \) is the following proposition:

**Proposition 1.3.** The objects of \( \mathcal{C}'_t \) have Property \((P)\).

From this proposition, we deduce some structure property for the points of \( X_R \).

The second part of Theorem 1.1, that is Gorensteinness property and Rational singularities, is obtained by considering a subcategory \( \mathcal{C}_{t,s} \) of \( \mathcal{C}_t \). This category is defined by an additional condition on the objects. The main point for \( a \) in \( \mathcal{C}_{t,s} \) is the following result:

**Proposition 1.4.** Let \( k \geq 2 \) be an integer. Denote by \( \mathcal{E}^{(k)} \) the \( R \)-equivariant vector subbundle of \( X_R \times r^k \) whose fiber at \( t \) is \( t^k \). Then there exists on the smooth locus of \( \mathcal{E}^{(k)} \) a regular differential form of top degree without zero.

From Proposition 1.4 and Theorem 1.2, we deduce that \( \mathcal{E}^{(k)} \) and \( X_R \) are Gorenstein with rational singularities.

This note is organized as follows. In Section 2, categories \( \mathcal{C}'_t \) and \( \mathcal{C}_t \) are introduced for some space \( t \). In particular, \( a \) is an object of \( \mathcal{C}_b \). In Subsection 2.3, we define Property \((P)\) for the objects of \( \mathcal{C}'_t \) and we deduce some result on the structure of points of \( X_R \). In Subsection 2.4, we define Property \((P_1)\) for the objects of \( \mathcal{C}'_t \) and we prove that Property \((P_1)\) is a consequence of Property \((P)\). In Subsection 2.5, we give some geometric constructions to prove Property \((P)\) by induction on the dimension of \( a \). At last, in Subsection 2.6, we prove Proposition 1.3. In particular, the proof of [CZ16, Lemma 4.4,(i)] is completed. In Section 3, we are interested in the singular locus of \( X_R \). In Subsection 3.3, regularity in codimension 1 is proved with some additional properties analogous to those of [CZ16, Section 3]. Moreover, the constructions of Subsection 2.5 are used to prove the results by induction on the dimension of \( a \). In Section 4, Cohen-Macaulayness property is proved by induction. In Section 5, the category \( \mathcal{C}_{t,s} \) is introduced and Proposition 1.4 is proved. Then with some results given in the appendix, we finish the proof of Theorem 1.1.
1.2. Notations. • An algebraic variety is a reduced scheme over \( k \) of finite type. For \( X \) an algebraic variety, its smooth locus is denoted by \( X_{\text{sm}} \).

• Set \( k^* := k \setminus \{0\} \). For \( V \) a vector space, its dual is denoted by \( V^* \).

• All topological terms refer to the Zariski topology. If \( Y \) is a subset of a topological space \( X \), denote by \( \overline{Y} \) the closure of \( Y \) in \( X \). For \( Y \) an open subset of the algebraic variety \( X \), \( Y \) is called a big open subset if the codimension of \( X \setminus Y \) in \( X \) is at least 2. For \( Y \) a closed subset of an algebraic variety \( X \), its dimension is the biggest dimension of its irreducible components and its codimension in \( X \) is the smallest codimension in \( X \) of its irreducible components. For \( X \) an algebraic variety, \( \mathcal{O}_X \) is its structural sheaf, \( k[X] \) is the algebra of regular functions on \( X \), \( k(X) \) is the field of rational functions on \( X \) when \( X \) is irreducible and \( \Omega_X \) is the sheaf of regular differential forms of top degree on \( X \) when \( X \) is smooth and irreducible.

• If \( E \) is a subset of a vector space \( V \), denote by \( \operatorname{span}(E) \) the vector subspace of \( V \) generated by \( E \). The grassmannian of all \( d \)-dimensional subspaces of \( V \) is denoted by \( \operatorname{Gr}_d(V) \).

• For a Lie algebra, \( V \) a subspace of \( \mathfrak{a} \) and \( x \) in \( \mathfrak{a} \), \( V^x \) denotes the centralizer of \( x \) in \( V \). For \( A \) a subgroup of the group of automorphisms of \( \mathfrak{a} \), \( A^x \) denotes the centralizer of \( x \) in \( A \). An element \( x \) of \( \mathfrak{a} \) is regular if \( \mathfrak{a}^x \) has dimension \( \ell \) and the set of regular elements of \( \mathfrak{g} \) is denoted by \( \mathfrak{g}_{\text{reg}} \).

• The Lie algebra of an algebraic torus is also called a torus. In this note, a torus denoted by a gothic letter means the Lie algebra of an algebraic torus.

• For a Lie algebra, the Lie algebra of derivations of \( \mathfrak{a} \) is denoted by \( \text{Der}(\mathfrak{a}) \). By definition \( \text{Der}(\mathfrak{a}) \) is the Lie algebra of the group \( \text{Aut}(\mathfrak{a}) \) of the automorphisms of \( \mathfrak{a} \).

• Let \( b \) be a Borel subalgebra of \( \mathfrak{g} \), \( \mathfrak{h} \) a Cartan subalgebra of \( \mathfrak{g} \) contained in \( b \) and \( \mathfrak{n} \) the nilpotent radical of \( b \).

2. On solvable algebras

Let \( t \) be a vector space of positive dimension \( d \). Denote by \( \tilde{C}_t \) the subcategory of the category of finite dimensional Lie algebras whose objects are finite dimensional nilpotent Lie algebras \( \mathfrak{a} \) such that there exists a morphism

\[
\begin{array}{ccc}
\mathfrak{a} & \xrightarrow{\varphi} & \text{Der}(\mathfrak{a}) \\
\end{array}
\]

whose image is the Lie algebra of a subtorus of \( \text{Aut}(\mathfrak{a}) \). For \( \mathfrak{a} \) and \( \mathfrak{a}' \) in \( \tilde{C}_t \), a morphism \( \psi \) from \( \mathfrak{a} \) to \( \mathfrak{a}' \) is a morphism of Lie algebras such that \( \psi \circ \varphi_a(t) = \varphi_{\mathfrak{a}'}(t) \circ \psi \) for all \( t \) in \( \mathfrak{t} \). For \( x \) in \( \mathfrak{t} \), \( x \) is a semisimple derivation of \( \mathfrak{a} \). Denote by \( \mathcal{R}_{\mathfrak{t}, \alpha} \) the set of weights of \( \mathfrak{t} \) in \( \mathfrak{a} \). Let \( \tilde{C}'_t \) be the full subcategory of objects \( \mathfrak{a} \) of \( \tilde{C}_t \) verifying the following conditions:

1. \( 0 \) is not in \( \mathcal{R}_{\mathfrak{t}, \alpha} \),
2. for \( \alpha \) in \( \mathcal{R}_{\mathfrak{t}, \alpha} \), the weight space of weight \( \alpha \) has dimension 1,
3. for \( \alpha \) in \( \mathcal{R}_{\mathfrak{t}, \alpha} \), \( k\alpha \cap (\mathcal{R}_{\mathfrak{t}, \alpha} \setminus \{\alpha\}) \) is empty.

For \( \mathfrak{a} \) in \( \tilde{C}'_t \) and \( \mathfrak{a}' \) a subalgebra of \( \mathfrak{a} \), invariant under the adjoint action of \( \mathfrak{t} \), \( \mathfrak{a}' \) is in \( \tilde{C}'_t \). Denote by \( \mathcal{C}_t \) the full subcategory of objects \( \mathfrak{a} \) of \( \tilde{C}'_t \) such that \( \varphi_a \) is an embedding. For example \( \mathfrak{n} \) is in \( \mathcal{C}_t \).

For \( \mathfrak{a} \) in \( \tilde{C}_t \), denote by \( \mathfrak{r}_{\mathfrak{t}, \alpha} \) the solvable algebra \( t + \alpha \). \( \pi_{\mathfrak{t}, \alpha} \) the quotient morphism from \( \mathfrak{r}_{\mathfrak{t}, \alpha} \) to \( \mathfrak{t} \). \( R_{\mathfrak{t}, \alpha} \) the adjoint group of \( \mathfrak{r}_{\mathfrak{t}, \alpha} \), \( A_{\mathfrak{t}, \alpha} \) the connected closed subgroup of \( R_{\mathfrak{t}, \alpha} \) whose Lie algebra is \( \text{ad} \mathfrak{a} \), \( X_{R_{\mathfrak{t}, \alpha}} \) the closure in \( \text{Gr}_d(\mathfrak{r}_{\mathfrak{t}, \alpha}) \) of the orbit of \( t \) under \( R_{\mathfrak{t}, \alpha} \) and \( E_{\mathfrak{t}, \alpha} \) the restriction to \( X_{R_{\mathfrak{t}, \alpha}} \) of the tautological vector bundle over \( \text{Gr}_d(\mathfrak{r}_{\mathfrak{t}, \alpha}) \). The variety \( X_{R_{\mathfrak{t}, \alpha}} \) is called the main variety related to \( \mathfrak{r}_{\mathfrak{t}, \alpha} \). For \( \alpha \) in \( \mathcal{R}_{\mathfrak{t}, \alpha} \), let \( \mathfrak{a}'' \) be the weight space of weight \( \alpha \) under the action of \( \mathfrak{t} \) in \( \mathfrak{a} \).
In the following subsections, a vector space \( t \) of positive dimension \( d \) and an object \( a \) of \( \mathcal{C}_1 \) are fixed. We set:

\[
\mathcal{R} := \mathcal{R}_{t,a}, \quad r := r_{t,a} \quad \pi := \pi_{t,a}, \quad R := R_{t,a}, \quad A := A_{t,a}, \quad n := \dim a.
\]

Let \( \mathfrak{z} \) be the orthogonal complement of \( \mathcal{R} \) in \( t \) and \( d^\# \) its codimension in \( t \). Then \( n \geq d^\# \).

2.1. **General remarks on \( \mathcal{C}_1 \).** For \( x \) in \( t \), we say that \( x \) is semisimple if so is \( \text{ad} x \) and \( x \) is nilpotent if so is \( \text{ad} x \). For \( s \) a commutative subalgebra of \( t \), we say that \( s \) is a torus if \( \text{ad} s \) is the Lie algebra of a subtorus of \( \text{GL}(t) \).

**Lemma 2.1.** Let \( x \) be in \( t \) and \( s \) a commutative subalgebra of \( t \).

(i) The center of \( t \) is equal to \( \mathfrak{z} \).

(ii) The element \( x \) is semisimple if and only if \( R.x \cap t \) is not empty.

(iii) The element \( x \) is nilpotent if and only if \( x \) is in \( \mathfrak{z} + a \).

(iv) The algebra \( a \) is in \( \mathcal{C}_1 \) if and only if \( \mathfrak{z} = \{0\} \). In this case, \( x \) has a unique decomposition \( x = x_s + x_n \) with \( [x_s, x_n] = 0 \). \( x_s \) semisimple and \( x_n \) nilpotent.

(v) The algebra \( s \) is a torus if and only if \( s \cap a = \{0\} \) and \( \pi(s) \) is a subtorus of \( t \). In this case, \( s \) and \( \pi(s) \) are conjugate under \( R \).

**Proof.** By definition \( \text{ad} r_{t,a} \) is an algebraic solvable subalgebra of \( \text{gl}(r_{t,a}) \) and \( \text{ad} t \) is a maximal subtorus of \( \text{ad} r_{t,a} \).

(i) Let \( \mathfrak{z}' \) be the center of \( t \). As \( \{t, \mathfrak{z}'\} = \{0\} \),

\[
\mathfrak{z}' = \mathfrak{z}' \cap t \oplus \bigoplus_{a \in \mathcal{R}} \mathfrak{z}' \cap a^\alpha.
\]

So, by Condition (1), \( \mathfrak{z}' \) is contained in \( t \). For \( t \) in \( t \), \( t \) is in \( \mathfrak{z}' \) if and only if \( \alpha(t) = 0 \) for all \( \alpha \) in \( \mathcal{R}_{t,a} \), whence \( \mathfrak{z}' = \mathfrak{z} \).

(ii) As the elements of \( t \) are semisimple by definition, the condition is sufficient since the set of semisimple elements of \( t \) is invariant under the adjoint action of \( R \). Suppose that \( x \) is semisimple. By [Hu95, Ch. VII], for some \( g \) in \( R \), \( \text{Ad} g(x) \) is in \( \text{ad} t \), whence \( g(x) \) is in \( t \) by (i).

(iii) As \( \text{ad} a \) is the set of nilpotent elements of \( \text{ad} r \), \( x \) is in \( \mathfrak{z} + a \) if and only if it is nilpotent by (i).

(iv) By definition, \( \mathfrak{z} \) is the kernel of \( \varphi_a \). Hence \( \mathfrak{z} = \{0\} \) if and only if \( a \) is in \( \mathcal{C}_1 \). As \( \text{ad} r \) is an algebraic subalgebra of \( \text{gl}(t) \), it contains the components of the Jordan decomposition of \( \text{ad} x \). As a result, when \( a \) is in \( \mathcal{C}_1 \), \( x \) has a unique decomposition \( x = x_s + x_n \) with \( [x_s, x_n] = 0 \). \( x_s \) semisimple and \( x_n \) nilpotent.

(v) Suppose that \( s \) is a torus. By (i), \( s \cap a = \{0\} \) and by [Hu95, Ch. VII], for some \( g \) in \( R \), \( \text{ad} g(s) \) is contained in \( \text{ad} s \) since \( \text{ad} s \) is a maximal torus of \( \text{ad} t \). Then, by (i), \( g(s) \) is a subtorus of \( t \). Moreover, \( g(s) = \pi(s) \) since \( g(y) - y \) is in \( a \) for all \( y \) in \( t \). Conversely, if \( s \cap a = \{0\} \) and \( \pi(s) \) is a subtorus of \( t \), \( \text{ad} s \) is conjugate to the subtorus \( \text{ad} \pi(s) \) of \( \text{ad} t \) by [Hu95, Ch. VII] so that \( s \) and \( \pi(s) \) are conjugate under \( R \).

Denoting by \( t^\# \) a complement to \( \mathfrak{z} \) in \( t \), \( a \) is an object of \( \mathcal{C}_{t^\#} \) since \( \varphi_a(t) = \varphi_{a^\#}(t^\#) \) and the restriction of \( \varphi_a \) to \( t^\# \) is injective. Set \( t^\# := t^\# + a \) and denote by \( R^\# \) the adjoint group of \( t^\# \). Let \( X_{R^\#} \) be the closure in \( \text{Gr}_{a^\#}(t^\#) \) of the orbit of \( t^\# \) under \( R^\# \).

**Corollary 2.2.** All element of \( X_R \) is a commutative algebra containing \( \mathfrak{z} \). Moreover, the map

\[
X_{R^\#} \longrightarrow X_R, \quad V \longmapsto V \oplus \mathfrak{z}
\]

is an isomorphism.
Proof. As the set of commutative subalgebras of dimension \( d \) of \( t \) is a closed subset of \( \text{Gr}_d(t) \) containing \( t \) and invariant under \( R \), all element of \( X_R \) is a commutative algebra. According to Lemma 2.1(i), all element of \( R.t \) contains \( \mathfrak{z} \) and so does all element of \( X_R \). For \( g \in R \), denote by \( \overline{g} \) the image of \( g \) in \( R^\# \) by the restriction morphism. Then

\[
g(t) = \overline{g}(t^\#) + \mathfrak{z} \quad \text{and} \quad \overline{g}(t^\#) = g(t) \cap t^\#.
\]

Hence the map

\[
X_{R^\#} \longrightarrow X_R, \quad V \mapsto V \oplus \mathfrak{z}
\]

is an isomorphism whose inverse is the map \( V \mapsto V \cap t^\# \). \( \square \)

For \( a \) of dimension \( d^\# \), \( R := \{\beta_1, \ldots, \beta_{d^\#}\} \), and for \( I \) subset of \( \{1, \ldots, d^\#\} \), denote \( X_{R,I} \) the image of \( \mathfrak{k}^I \) by the map

\[
\mathfrak{k}^I \longrightarrow X_R, \quad (z_i, \ i \in I) \mapsto \mathfrak{z} \oplus \text{span}([t_i + z_i x_i, \ i \in I]) \oplus \bigoplus_{i \notin I} \mathfrak{a}^{\beta_i}
\]

with \( x_i \in \mathfrak{a}^{\beta_i} \) for \( i = 1, \ldots, d^\# \) and \( t_1, \ldots, t_{d^\#} \) in \( t \) such that \( \beta_i(t_j) = \delta_{i,j} \) for \( 1 \leq i, j \leq d^\# \), with \( \delta_{i,j} \) the Kronecker symbol.

**Lemma 2.3.** Suppose that \( a \) has dimension \( d^\# \). Denote by \( \beta_1, \ldots, \beta_{d^\#} \) the elements of \( R \).

(i) The algebra \( a \) is commutative.

(ii) The set \( X_R \) is the union of \( X_{R,I} \), \( I \subset \{1, \ldots, d^\#\} \).

**Proof.** (i) As \( \mathfrak{z} \) has codimension \( d^\# \) in \( t, \beta_1, \ldots, \beta_{d^\#} \) are linearly independent. Hence for \( i \neq j, \beta_i + \beta_j \) is not in \( \mathfrak{z} \). As a result, \( a \) is commutative.

(ii) According to Corollary 2.2, we can suppose \( d = d^\# \) so that \( t_1, \ldots, t_d \) is the dual basis of \( \beta_1, \ldots, \beta_d \). For \( I \) subset of \( \{1, \ldots, d\} \), denote by \( I' \) the complement to \( I \) in \( \{1, \ldots, d\} \) and \( \mathfrak{z}_{I'} \) the orthogonal complement to \( \beta_i, \ i \in I' \) in \( t \) and set:

\[
V_I := \mathfrak{z}_{I'} \oplus \bigoplus_{i \in I'} \mathfrak{a}^{\beta_i}.
\]

By (i), for \( i \) in \( I \),

\[
\exp(z_i \text{ad} x_i + \cdots + z_d \text{ad} x_d)(t_i) = t_i - z_i x_i.
\]

Hence \( X_{R,I} \) is the orbit of \( V_I \) under \( A \) and its closure in \( X_R \) is the union of \( X_{R,J} \), \( J \subset I \). As a result, \( X_R \) is the union of \( X_{R,I} \), \( I \subset \{1, \ldots, d\} \) since \( X_{R,\{1, \ldots, d\}} \) is the orbit of \( t \) under \( A \). \( \square \)

### 2.2. On some subsets of \( \mathcal{R} \).

For \( \alpha \) in \( \mathcal{R} \), let \( x_\alpha \) be in \( a' \setminus \{0\} \). For \( \Lambda \) subset of \( \mathcal{R} \), denote by \( t_\Lambda \) the intersection of the kernels of its elements and set:

\[
a_\Lambda := \bigoplus_{\alpha \in \Lambda} a^\alpha \quad \text{and} \quad t_\Lambda := t \oplus a_\Lambda.
\]

When \( \Lambda \) has only one element \( \alpha \), set \( t_\alpha := t_\Lambda \).

**Definition 2.4.** Let \( \Lambda \) be a subset of \( \mathcal{R} \). We say that \( \Lambda \) is a complete subset of \( \mathcal{R} \) if it contains all element of \( \mathcal{R} \) whose kernel contains \( t_\Lambda \).

For \( \Lambda \) complete subset of \( \mathcal{R} \), \( a_\Lambda \) is a subalgebra of \( a \) and \( t_\Lambda \) is a subalgebra of \( t \). In particular, \( a_\Lambda \) is in \( C_t' \). In this case, denote by \( R_{\Lambda} \) the connected closed subgroup of \( R \) whose Lie algebra is \( \text{ad} t_\Lambda \).

**Lemma 2.5.** Let \( \Lambda \) be a complete subset of \( \mathcal{R} \), strictly contained in \( \mathcal{R} \). Then \( a_\Lambda \) is contained in an ideal \( a' \) of \( t \) of dimension \( \dim a - 1 \) and contained in \( a \).
Proof. As $\Lambda$ is complete and strictly contained in $R$, $a_{a_{\Lambda}}$ is a subalgebra of $r$, strictly contained in $a$. Then, by Lie’s Theorem, there is a sequence

$$a_{\Lambda} = a_0 \subset \cdots \subset a_m = a$$

of subalgebras of $r$ such that $a_i$ is an ideal of codimension 1 of $a_{i+1}$ for $i = 0, \ldots, m - 1$, whence the lemma.

For $s$ in $t$, denote by $\Lambda_s$ the subset of elements of $R$ whose kernel contains $s$.

Lemma 2.6. Let $s$ be in $t$.

(i) The centralizer $r^s$ of $s$ in $r$ is the direct sum of $t$ and $a_{\Lambda_s}$.  
(ii) The center of $r^s$ is equal to $t_{\Lambda_s}$.

Proof. By definition, $\Lambda_s$ is a complete subset of $R$. Let $x$ be in $r$. Then $x$ has a unique decomposition

$$x = x_0 + \sum_{\alpha \in R} c_{\alpha} x_{\alpha}$$

with $x_0$ in $t$ and $c_{\alpha}, \alpha \in R$ in $k$.

(i) Since $s$ is in $t$, $x$ is in $r^s$ if and only if $c_{\alpha} = 0$ for $\alpha \in R \setminus \Lambda_s$, whence the assertion.

(ii) The algebra $a_{\Lambda_s}$ is in $C_r$ and $t_{\Lambda_s}$ is the orthogonal complement to $\Lambda_s$ in $t$. So, by (i) and Lemma 2.1(i), $t_{\Lambda_s}$ is the center of $r^s$.

2.3. Property (P) for objects of $C_r$. Let $T$ be the connected closed subgroup of $R$ whose Lie algebra is $\text{ad} t$. For $s$ in $t$, denote by $X^s_R$ the subset of elements of $X_R$ contained in $r^s$ and $\overline{R_0 \cdot t}$ the closure in $\text{Gr}_d(r)$ of the orbit of $t$ under $R^s$. Then $\overline{R_0 \cdot t}$ is contained in $X^s_R$.

Definition 2.7. Say that $a$ has Property (P) if $X^s_R$ is equal to $\overline{R_0 \cdot t}$ for all $s$ in $t$.

By Corollary 2.2, a has Property (P) if and only if the object $a$ of $C_r$ has Property (P).

Lemma 2.8. If $a$ has dimension $d^\#$, then $a$ has Property (P).

Proof. According to Corollary 2.2, we can suppose $d = d^\#$. Denote by $\beta_1, \ldots, \beta_d$ the elements of $R$. Then $\beta_1, \ldots, \beta_d$ is a basis of $t^\ast$. Let $t_1, \ldots, t_d$ be the dual basis, $s$ in $t$ and $V$ in $X^s_R$. By Lemma 2.3(ii), for some subset $I$ of $\{1, \ldots, d\}$, $V$ is in $X^{\beta_I}_R$. Then for some $(z_i, i \in I)$,

$$V = \text{span}((t_i + z_i x_i i \in I)) \oplus \bigoplus_{i \in I'} a^{\beta_i}$$

with $I'$ the complement to $I$ in $\{1, \ldots, d\}$ and $x_i$ in $a^{\beta_i}$ for $i = 1, \ldots, d$. Setting

$$I' := I' \cup \{i \in I \mid z_i \neq 0\},$$

for $i$ in $\{1, \ldots, d\}$, $i$ is in $I''$ if and only if $\beta_i(s) = 0$. So, by Lemma 2.5(i),

$$r^s = t \oplus \bigoplus_{i \in I''} a^{\beta_i}.$$

Then by Lemma 2.3(ii), $V$ is in $\overline{R_0 \cdot t}$.

By definition, an algebraic subalgebra $t$ of $r$ is the semi-direct product of a torus $s$ contained in $t$ and $t \cap a$.

Lemma 2.9. Suppose that $a$ has Property (P). Let $V$ be in $X_R$, $x$ in $V$ and $y$ in $r$ such that $\text{ad} y$ is the semisimple component of $\text{ad} x$. Then the center of $r^s$ is contained in $V$.  

6
Suppose that \( R \) orthogonal complement of \( W \) we say that \( \Lambda \) in \( V \). In this case, \( r \) fixed points since \( V \) in \( X \). So, by Property (P), \( \exists \gamma \) contained in \( g(V) \) since \( \exists \gamma \) is in \( k(t) \) for all \( k \) in \( R \). Hence, \( \exists \gamma \) whence the lemma.  

**Corollary 2.10.** Suppose that \( \alpha \) has Property (P). Let \( V \) be in \( X \). Then \( V \) is a commutative algebraic subalgebra of \( \tau \) and for some subset \( \Lambda \) of \( R \), the biggest torus contained in \( V \) is conjugate to \( t_{\Lambda} \) under \( R \).

**Proof.** According to Corollary 2.2, \( V \) is a commutative subalgebra of \( \tau \) and we can suppose \( d = \alpha_{\#} \). Let \( s \) be the set of semisimple elements of \( V \). Then \( s \) is a subspace of \( V \). By Lemma 2.9, \( V \) contains the semisimple components of its elements so that \( V \) is the direct sum of \( s \) and \( V \cap a \). Let \( s \) be in \( s \) such that the center of \( \tau^s \) has maximal dimension. After conjugation by an element of \( R \), we can suppose that \( s \) is in \( t \). By Lemma 2.6(ii), \( t_{\Lambda} \) is the center of \( \tau^s \). Hence, by Lemma 2.9, \( t_{\Lambda} \) is contained in \( s \). Suppose that the inclusion is strict. A contradiction is expected. Let \( s' \) be in \( s \), \( t_{\Lambda} \). Since \( V \) is contained in \( \tau^s \), for some \( g \) in \( \tau^s \), \( g(s') \) is in \( t_{\Lambda} \). Moreover, \( g(s) \) is the set of semisimple elements of \( g(V) \) and \( t_{\Lambda} \) is contained in \( g(s) \). Denoting by \( \Lambda' \) the set of elements of \( \Lambda_{\#} \) whose kernel contains \( g(s') \), for some \( z \) in \( k^s \), \( \Lambda' \) is the set of elements of \( R \) such that \( \alpha(s + zg(s')) = 0 \). By Lemma 2.9, \( t_{\Lambda'} \) is contained in \( g(V) \). So, by minimality of \( |\Lambda_{\#}| \), \( \Lambda' = \Lambda_{\#} \) and \( g(s') \) is in \( t_{\Lambda} \), whence the contradiction since \( g(s') \) is in \( g(s) \). As a result, \( t_{\Lambda} = s \) and \( V = t_{\Lambda} + V \cap a \), whence the corollary.  

### 2.4. Fixed points in \( X \) under \( T \) and \( R \)

For \( V \) subspace of dimension \( d \) of \( \tau \), denote by \( R \) the set of elements \( \beta \) of \( R \) such that \( \alpha_{\#} \) is contained in \( V \), \( r \) the rank of \( R \) and \( \beta_{\#} \) its orthogonal complement in \( t \) so that \( \dim \beta_{\#} = d - r \). As \( Gr_d(\tau) \) and \( X \) are projective varieties, the actions of \( T \) and \( R \) in these varieties have fixed points since \( T \) and \( R \) are connected and solvable.

**Definition 2.11.** We say that \( a \) has Property (P) if for \( V \) fixed point under \( T \) in \( X \) such that \( V \cap t = \beta_{\#} \), \( r_V = |R_{\#}| \).

**Lemma 2.12.** Suppose that \( a \) has Property (P). Let \( V \) be in \( Gr_d(\tau) \).

(i) The element \( V \) is a fixed point under \( T \) in \( X \) if and only if \( V \) is a commutative subalgebra of \( \tau \) and

\[
V = \beta_{\#} \oplus \bigoplus_{\beta \in R_{\#}} \beta_{\#}.
\]

In this case, \( r_V = |R_{\#}| \).

(ii) The element \( V \) is a fixed point under \( R \) in \( X \) if and only if \( V \) is a commutative ideal of \( \tau \) and \( \beta_{\#} \) is the orthogonal complement of \( R_{\#} \) in \( t \). In this case, \( r_V = |R_{\#}| = d_{\#} \).

**Proof.** If \( V \) is a fixed point under \( T \),

\[
V = V \cap t \oplus \bigoplus_{\beta \in R_{\#}} \beta_{\#}.
\]

(i) Suppose that \( V \) is a fixed point under \( T \) in \( X \). Then \( R_{\#} \) is not empty. Let \( s \) be an element of \( \beta_{\#} \) such that \( \beta(s) \neq 0 \) if \( \beta \) is not a linear combination of elements of \( R_{\#} \). Then \( V \) is contained in \( \tau^s \). So, by Property (P), \( V \) is in \( \tau^s \). By Lemma 2.6(i), \( \beta_{\#} \) is the center of \( \tau^s \). Hence \( \beta_{\#} \) is contained in \( V \) and \( \beta_{\#} = V \cap t \) since \( V \cap t \) is contained in \( \beta_{\#} \). As a result, \( \beta_{\#} \) has dimension \( d - |R_{\#}| \) and \( r_V = |R_{\#}| \).

Conversely, suppose that \( V \) is a commutative algebra and

\[
V = \beta_{\#} \oplus \bigoplus_{\beta \in R_{\#}} \beta_{\#}.
\]
Set:
\[ a_V := \bigoplus_{\beta \in \mathcal{V}_V} a^\beta, \quad r_V := t \oplus a_V. \]

Then \( a_V \) is a commutative Lie algebra and \( a_V \) is in \( \mathcal{C}'_t \). Moreover, \( \mathfrak{z}_V \) is the center of \( r_V \) by Lemma 2.1(i). By Lemma 2.3(ii), \( V \) is in the closure of the orbit of \( t \) under the action of the adjoint group of \( r_V \) in \( \text{Gr}_d(\mathfrak{r}_V) \), whence the assertion.

(ii) The element \( V \) of \( \text{Gr}_d(\mathfrak{r}) \) is a fixed point under \( R \) if and only if \( V \) is an ideal of \( r \). So, by (i), the condition is sufficient. Suppose that \( V \) is a fixed point under the action of \( R \) in \( X_R \). By (i),
\[ V = \mathfrak{z}_V \oplus \bigoplus_{\beta \in \mathcal{R}_V} a^\beta. \]

As \( V \) is an ideal of \( r \), \( \mathfrak{z}_V \) is contained in the kernel of all elements of \( \mathcal{R} \) so that \( \mathfrak{z}_V = \mathfrak{z} \). In particular, \( |\mathcal{R}_V| = d^# \) and the elements of \( \mathcal{R}_V \) are linearly independent. \( \square \)

2.5. **On some varieties related to** \( X_R \). Let \( a' \) be an ideal of \( r \) of dimension \( \dim a - 1 \) and contained in \( a \). As a subalgebra of a normalized by \( t \), \( a' \) is in \( \mathcal{C}'_t \). Denote by \( t' \) the subalgebra \( t + a' \) of \( r \), \( A' \) and \( R' \) the connected closed subgroups of \( R \) whose Lie algebras are \( \mathfrak{ad} a' \) and \( \mathfrak{ad} t' \) respectively. Let \( X_{R'} \) be the closure in \( \text{Gr}_d(\mathfrak{r}) \) of the orbit of \( t \) under \( R' \) and \( \alpha \) the element of \( \mathcal{R} \) such that
\[ a = a' \oplus a'^\alpha. \]

For \( \delta \) in \( \mathcal{R} \) denote again by \( \delta \) the character of \( T \) whose differential at the identity is \( \mathfrak{ad} x \mapsto \delta(x) \).

Setting:

\[ \mathfrak{6}_{d-1,d,d+1} := \text{Gr}_{d-1}(\mathfrak{r}) \times \text{Gr}_{d}(\mathfrak{r}) \times \text{Gr}_{d}(\mathfrak{r}) \times \text{Gr}_{d+1}(\mathfrak{r}) \quad \text{and} \quad \mathfrak{6}_{d-1,d,d+1} := \text{Gr}_{d-1}(\mathfrak{r}) \times \text{Gr}_{d}(\mathfrak{r}) \times \text{Gr}_{d+1}(\mathfrak{r}), \]

denote by \( \theta_\alpha \) and \( \theta'_{\alpha} \) the maps
\[ \mathbb{k} \times A' \xrightarrow{\theta_\alpha} \mathfrak{6}_{d-1,d,d+1}, \quad (z, g) \mapsto (g.t_{\alpha}, g.t, g \exp(zd \alpha).t, g.(t + a'^\alpha)), \]
\[ A' \xrightarrow{\theta'_{\alpha}} \mathfrak{6}_{d-1,d,d+1}, \quad g \mapsto (g.t_{\alpha}, g.t, g.(t + a'^\alpha)). \]

Let \( I_\alpha \) and \( S_\alpha \) be the closures in \( \text{Gr}_{d-1}(\mathfrak{r}) \) and \( \text{Gr}_{d+1}(\mathfrak{r}) \) of the orbits of \( t_{\alpha} \) and \( t + a'^\alpha \) under \( A' \) respectively.

**Lemma 2.13.** Let \( \Gamma \) and \( \Gamma' \) be the closures in \( \mathfrak{6}_{d-1,d,d+1} \) and \( \mathfrak{6}_{d-1,d,d+1} \) of the images of \( \theta_\alpha \) and \( \theta'_{\alpha} \).

(i) The varieties \( \Gamma \) and \( \Gamma' \) have dimension \( n \) and \( n - 1 \) respectively. Moreover, they are invariant under the diagonal actions of \( \mathcal{R}' \) in \( \mathfrak{6}_{d-1,d,d+1} \) and \( \mathfrak{6}_{d-1,d,d+1} \).

(ii) The image of \( \Gamma \) by the first, second, third and fourth projections are equal to \( I_\alpha \), \( X_{R'} \), \( X_R \), \( S_\alpha \) respectively.

(iii) The set \( \Gamma' \) is the image of \( \Gamma \) by the projection
\[ \mathfrak{6}_{d-1,d,d+1} \xrightarrow{\varpi} \mathfrak{6}_{d-1,d,d+1}, \quad (V_1, V', V, W) \mapsto (V_1, V', W). \]

(iv) For all \( (V_1, V', V, W) \) in \( \Gamma \), \( V_1 \) is contained in \( V' \cap V \) and \( V' + V \) is contained in \( W \).

(v) Let \( (V_1, V', V, W) \) be in \( \Gamma \) such that \( V' \) is contained in \( t_{\alpha} + a' \). Then \( W \) is contained in \( t_{\alpha} + a \).

(vi) Let \( (V_1, V', V, W) \) be in \( \Gamma \) such that \( V' \) is not contained in \( t_{\alpha} + a \). Then \( W \) is not commutative.

**Proof.** (i) The maps \( \theta_\alpha \) and \( \theta'_{\alpha} \) are injective since \( t \) is the normalizer of \( t \) by Condition (1), whence \( \dim \Gamma = n \) and \( \dim \Gamma' = n - 1 \). For \( (z, g, k) \) in \( \mathbb{k} \times A' \times A' \), \( \theta_\alpha(z, k) = k.\theta_\alpha(z, g) \) and \( \theta'_{\alpha}(k) = k.\theta'_{\alpha}(g) \). Hence
\( \Gamma \) and \( \Gamma' \) are invariant under the diagonal action of \( A' \) in \( \mathfrak{g}_{d-1,d,d+1} \) and \( \mathfrak{g}_{d-1,d,d+1} \). Let \( k \) be in \( T \). For all \((z, g) \) in \( \mathfrak{k} \times A' \),

\[
kg.t_a = kgk^{-1}(t_a), \quad kg.t = kgk^{-1}(t),
\]

\[
kg.(t + a^\alpha) = kgk^{-1}.(t + a^\alpha), \quad kg \exp(zad.x_a).t = kgk^{-1} \exp(z\alpha(\text{ad} x_a)).t
\]

so that the images of \( \theta \) and \( \theta' \) are invariant under \( T \), whence the assertion.

(ii) Since \( \text{Gr}_d(\tau), \text{Gr}_{d-1}(\tau), \text{Gr}_{d+1}(\tau) \) are projective varieties, the images of \( \Gamma \) by the first, second, third and fourth projections are closed subsets of their target varieties. Since the image of \( \theta_{a} \) is contained in the closed subset \( I_{\alpha} \times X_{\alpha} \times X_{R} \times S_{\alpha} \) of \( \mathfrak{g}_{d-1,d,d+1} \), they are contained in \( I_{\alpha} \times X_{R}, X_{R} \) and \( S_{\alpha} \) respectively. By definition, \( R', t_{\alpha}, R'.t \) and \( R'.(t + a^\alpha) \) are contained in the images of \( \Gamma \) by the first, second and fourth projections and \( R.t \) is contained in the image of \( \Gamma \) by the third projection since \( A \) is the image of \( \mathfrak{k} \times A' \) by the map \((z, g) \) ↦ \( g \exp(zad.x_a) \), whence the assertion.

(iii) As \( \text{Gr}_d(\tau) \) is a projective variety, \( \pi_\alpha(\Gamma) \) is a closed subset of \( \mathfrak{g}_{d-1,d,d+1} \) containing the image of \( \theta_{a} \), since \( \pi_\alpha \theta_{a}(z, g) = \theta_{a}'(g) \) for all \((z, g) \in \mathfrak{k} \times A' \). Moreover, \( \Gamma \) is contained in \( \pi_\alpha^{-1}(\Gamma') \), whence \( \Gamma' = \pi_\alpha(\Gamma) \).

(iv) The subset \( \Gamma \) of elements \((V_1, V', V, W) \) of \( \mathfrak{g}_{d-1,d,d+1} \) such that \( V_1 \) is contained in \( V' \) and \( V \) and such that \( V' \) and \( V \) are contained in \( W \), is contained in \( \mathfrak{k} \times A' \). For all \((z, g) \) in \( \mathfrak{k} \times A' \),

\[
g \exp(zad.x_a).(t + a^\alpha) = g.(t + a^\alpha)
\]

Hence the image of \( \theta_{a} \) and \( \Gamma \) are contained in \( \Gamma \) so that \( V_1 \) and \( V + V' \) are contained in \( V' \cap V \) and \( W \) respectively for all \((V_1, V', V, W) \) in \( \Gamma \).

(v) Denote by \( \Gamma_\alpha \) the closure in \( \text{Gr}_d(\tau) \times \text{Gr}_{d+1}(\tau) \) of the image of the map

\[
A' \xrightarrow{\theta_{a,x} \times \theta_{a}} \text{Gr}_d(\tau) \times \text{Gr}_{d+1}(\tau), \quad g \mapsto (g(t), g(t + a^\alpha)).
\]

For all \((T_1, T', T, T_2) \) in the image of \( \theta_{a,x} \), \((T', T_2) \) is in the image of \( \theta_{a,x} \). Then \( \Gamma_\alpha \) is the image of \( \Gamma \) by the projection

\[
\mathfrak{g}_{d-1,d,d+1} \xrightarrow{\theta_{a,x}} \text{Gr}_d(\tau) \times \text{Gr}_{d+1}(\tau), \quad (T_1, T', T, T_2) \mapsto (T', T_2).
\]

Denote by \( \tau \) the quotient morphism

\[
x \xrightarrow{\tau} x/a' = t + a^\alpha.
\]

For \( g \) in \( A' \) and \( x \) in \( \tau \), \( \tau\circ g(x) = \tau(x) \). Set:

\[
X := \{(g, t, z, z', v, w) \in A' \times t_{\alpha} \times \mathbb{K}^2 \times T \times \tau; \mid v = g(zs + t), \ w = g(zs + t + z'x_a)\}
\]

and denote by \( Y \) the closure in \( T \times \tau \) of the image of \( X \) by the canonical projection

\[
A' \times t_{\alpha} \times \mathbb{K}^2 \times T \times \tau \longrightarrow T \times \tau.
\]

As for all \((g, t, z, z', v, w) \) in \( X \),

\[
\tau(v) = zs + t \quad \text{and} \quad \tau(w) = zs + t + z'x_a,
\]

\[
\alpha \circ \pi \circ \tau(v) = \alpha \circ \pi \circ \tau(w)
\]

for all \((v, w) \) in \( Y \). By definition, for all \((T, T') \) in \( \Gamma_\alpha \), \( T \times T' \) is contained in \( Y \). By hypothesis, \( V' \) is contained in the kernel of \( \alpha \circ \pi \) and \( (V', W) \) is in \( \Gamma_\alpha \). Hence \( W \) is contained in the kernel of \( \alpha \circ \pi \).

(vi) Denote by \( \Gamma_\alpha' \) the closure in \( \mathfrak{g}_{d-1,d,d+1} \times \text{Gr}_1(\tau) \) of the image of the map

\[
\mathbb{K} \times A' \xrightarrow{\theta_{a,x} \times \theta_{a}} \mathfrak{g}_{d-1,d,d+1} \times \text{Gr}_1(\tau), \quad (z, g) \mapsto (\theta_{a}(z, g), g(a^\alpha))
\]
and \( \Gamma'_{s} \) the closure in \( \text{Gr}_{d}(t') \times \text{Gr}_{1}(r) \) of the image of the map
\[
A' \twoheadrightarrow \text{Gr}_{d}(t') \times \text{Gr}_{1}(r), \quad g \mapsto (g(t), g(a^0)).
\]
For all \((T_1, T', T, T_2, T_2')\) in the image of \(0_{s+a}, T'+T_2'\) is contained in \(T_2\). Then so is it for all \((T_1, T', T, T_2, T_2')\) in \(\Gamma'\). As \(6_{d-1,d,d+1} \times \text{Gr}_{1}(r)\) and \(\text{Gr}_{1}(r)\) are projective varieties, \(\Gamma\) and \(\Gamma'_{s} \) are the images of \(\Gamma'\) by the projections
\[
6_{d-1,d,d+1} \times \text{Gr}_{1}(r) \twoheadrightarrow \text{Gr}_{d}(t') \times \text{Gr}_{1}(r), \quad (T_1, T', T, T_2, T_2') \mapsto (T_1, T', T, T_2),
\]
respectively.

Set:
\[
X' := \{(g, t, z, v, w) \in A' \times t \times \k \times t' \times r \mid v = g(t), \ w = g(zx_a)\}
\]
and denote by \(Y'\) the closure in \(t' \times r\) of the image of \(X'\) by the canonical projection
\[
A' \times t \times \k \times t' \times r \twoheadrightarrow t' \times r.
\]
As for all \((g, t, z, v, w)\) in \(X'\),
\[
[v, w] = g([t, zx_a]) = \alpha(t)g(zx_a) = \alpha \circ \pi(v)w,
\]
\([v, w] = \alpha \circ \pi(v)w\) for all \((v, w)\) in \(Y'\). By definition, for all \((T, T')\) in \(\Gamma'_{s}\), \(T \times T'\) is contained in \(Y'\). For some \(W'\) in \(\text{Gr}_{1}(r), (V_1, V', V, W, W')\) is in \(\Gamma'_{s}\). By hypothesis, \(V'\) is not contained in the kernel of \(\alpha \circ \pi\). Hence, for some \(v\) in \(V'\) and \(w\) in \(W \setminus \{0\}\), \(\alpha \circ \pi(v) \neq 0\) and \([v, w] = \alpha \circ \pi(v)w\). \(\square\)

**Corollary 2.14.** Suppose that \(a'\) has Property \((P)\). Let \(s\) be in \(t\) such that \(r^s\) is contained in \(a'\) and \((V_1, V', V, W)\) be in \(\Gamma\) such that \(V\) is contained in \(r^s\) and \([s, V']\) is contained in \(V'\).

(i) If \(W\) is not commutative then \(V' = V\) and \(V\) is in \(R^s.t\).

(ii) Suppose that for some \(v\) in \(a, s + v\) is in \(V\). Then \(V' = V\) and \(V\) is in \(R^s.t\).

**Proof.** By Lemma 2.13(ii), \(V\) and \(V'\) are in \(X_{R}\) and \(X_{R'}\) respectively.

(i) If \(V' = V, V\) is in \(R^s.t\) by Property \((P)\) for \(a'\). Suppose \(V' \neq V\). A contradiction is expected. Then, by Lemma 2.13(iv), for some \(x\) and \(y\) in \(W\),
\[
V = V_1 \oplus \k x, \quad V' = V_1 \oplus \k y, \quad W = V_1 \oplus \k x \oplus \k y.
\]
Moreover, as \(V\) is contained in \(r^s\) and \([s, V'] \subset V'\), \(W\) is contained in \(r^s\) and we can choose \(y\) so that \([s, y] \in \k y\). Since \(V\) and \(V'\) are commutative subalgebras of \(r, [x, y] \neq 0\). We have two cases to consider:

(a,1) \(V'\) is contained in \(r^s\),

(a,2) \(V'\) is not contained in \(r^s\).

(a,1) By Property \((P)\) for \(a', s\) is in \(V'\). Hence \(s = ty + v\) for some \(t, v\) in \(\k \times V_1\). As \(V\) is a commutative subalgebra of \(r^s\), containing \(V_1\) and \(x, \ 0 = [x, s] = t[x, y] \).

Then \(s = v\) is in \(V_1, \) whence a contradiction since \(\alpha(s) \neq 0\) and \(V_1\) is contained in \(t_a + a'\) by Lemma 2.13(ii).

(a,2) For some \(a\) in \(\k^+, [s, y] = ay\). Then \(y\) is in \(a'\) and \(V'\) is contained in \(t_a + a'\) since so is \(V_1\). As a result, by Lemma 2.13(v), \(V\) and \(W\) are contained in \(t_a + a'\) since \(V\) is contained in \(a'\). As \([s, [x, y]] = a[x, y], [x, y] = by\) for some \(b\) in \(\k^+\) since the eigenspace of eigenvalue \(a\) of the restriction of \(ad_s\) to \(V'\) is generated by \(y\). Then \(ad_x\) is not nilpotent. Let \(x_0\) be in \(r^s\) such that \(ad_{x_0}\) is the semisimple component of \(ad\). Then \(x_0\) is in \(t_a + a'\), \([s, x_0] = 0\) and \([x_0, V_1] = 0\) since \([s, x] = 0\) and \([x, V_1] = 0\). Moreover, \([ax_0 - bx_0, y] = 0\). Then \(ax_0 - bx_0\) is a semisimple element of \(r^s\) such that \([ax_0 - bx_0, V'] = 0\). As it is conjugate under \(R'\) to an
element of $t$ by Lemma 2.1(ii), by Property (P) for $a', ax_b - bs$ is in $V'$, whence a contradiction since $V'$ is contained in $t_a + a'$ and $ax_b - bs$ is not in $t_a + a'$.

(ii) If $V = V'$, $V$ is in $\mathbb{R}^+$ by Property (P) for $a'$. Suppose $V \neq V'$. A contradiction is expected. As $V$ is contained in $r^3$, $[s, v] = 0$. Let $x$ and $y$ be as in (i). As $V_1$ is contained in $t_a + a'$, $s + v$ is not in $V_1$ since $\alpha(s) \neq 0$. So we can choose $s + v = x$. By (i), $W$ is commutative. Then $[s + v, y] = 0$ and $[ad_s, ad_y] = 0$ since $ad_s$ is the semisimple component of $ad(s + v)$. Hence, by Lemma 2.1(i), $[s, y] = 0$ since $[s, y]$ is in $a$. As a result, $V'$ is contained in $r^3$ since so is $V_1$. So, by Property (P) for $a'$, $s$ is in $V'$ and $W$ is not commutative by Lemma 2.13(vi), whence a contradiction.

For $(T_1, T', T_2)$ in $\Gamma'$, denote by $\Gamma_{T_1, T', T_2}$ the subset of elements $(T_1, T', T, T_2)$ of $G_{d-1, d, d, d+1}$ such that $T$ is contained in $T_2$ and contains $T_1$. Then $\Gamma_{T_1, T', T_2}$ is a closed subvariety of $G_{d-1, d, d, d+1}$, isomorphic to $\mathbb{P}^1(\mathbb{R})$. Let $(V_1, V', V, W)$ be a fixed point under $T$ of $\Gamma$.

Lemma 2.15. (i) For some affine open neighborhood $\Omega$ of $(V_1, V', W)$ in $\Gamma'$, $\Omega$ is invariant under $T$.

(ii) For $i = 0, \ldots, n - 2$, there exist $Y_i$ and $O_i$ such that

(a) $Y_i$ is an irreducible closed subset of dimension $n - 1 - i$ of $\Omega$, containing $(V_1, V', W)$ and invariant under $T$,

(b) $O_i$ is a locally closed subvariety of dimension $n - 1 - i$ of $A'$, invariant under $T$ by conjugation,

(c) $O_i'(O_i)$ is contained in $Y_i$ and $(V_1, V', V, W)$ is in the closure of $0_o(\mathbb{R} \times O_i)$ in $\Gamma$.

(iii) There exist a smooth projective curve $C$, an action of $T$ on $C$, $x_1, \ldots, x_m$ in $C$ and two morphisms

$$C \setminus \{x_1, \ldots, x_m\} \xrightarrow{\mu} A', \quad C \xrightarrow{\nu} \Gamma'$$

satisfying the following conditions:

(a) $x_1, \ldots, x_m$ are the fixed points under $T$ in $C$,

(b) for $g$ in $T$ and $x$ in $C \setminus \{x_1, \ldots, x_m\}$, $\mu(g.x) = g \mu(x) g^{-1}$ and $\nu(g.x) = g \nu(x)$,

(c) $\nu(x_1) = (V_1, V', W)$,

(d) $(V_1, V', V, W)$ is in the closure of the image of $\mathbb{R} \times (C \setminus \{x_1, \ldots, x_m\})$ by the map $(z, x) \mapsto 0_o(z, \mu(x))$.

Proof. (i) As $\Gamma'$ is a projective variety with a $T$ action and $(V_1, V', W)$ is a fixed point under $T$, there exists an affine open neighborhood $\Omega$ of $(V_1, V', W)$ in $\Gamma'$, invariant under $T$.

(ii) Prove the assertion by induction on $i$. For $i = 0$, $Y_i = \Omega$ and $O_i$ is the inverse image of $\Omega$ by $0_o'$. Suppose that $Y_i$ and $O_i$ are known. Let $Y'_j$ be the closure in $\Gamma$ of $0_o(\mathbb{R} \times O_i)$. By Condition (c), $Y'_j$ is invariant under $T$ and $0_o(\mathbb{R} \times O_i)$ is a $T$-invariant dense subset of $Y'_j$. So, it contains an $T$-invariant dense open subset $O'_i$ of $Y'_j$. As $0_o'$ is an orbital injective morphism, $0_o'(O_i)$ is a dense open subset of $Y_i$. Set:

$$Z' := Y'_j \setminus O'_i, \quad Z := Y_i \setminus 0_o(O'_i), \quad Z_o := \Omega \cap (\varnothing(Z) \cup Z').$$

Then $Z_o$ is a $T$-invariant closed subset of $Y_i$, containing $(V_1, V', W)$.

Denote by $Z_{ox}$ the union of the ideals of definition in $\mathbb{R}[Y_i]$ of the irreducible components of $Z_{ox}$ and $I$ the union of the ideals of definition in $\mathbb{R}[Y_i]$ of the irreducible components of $Z_{ox}$. Let $p$ be in $\mathbb{R}[Y_i] \setminus I$, semi-invariant under $T$ and such that $p((V_1, V', W)) = 0$. Denote by $Y'_{i+1}$ an irreducible component of the null variety of $p$ in $Y'_j \setminus \varnothing(I)$, containing $(V_1, V', V, W)$ and $Y_{i+1}$ the closure in $\Omega$ of $\varnothing(Y'_{i+1})$. Then $Y_{i+1}$ has dimension $n - i - 1$ and its intersection with $0_o'(O_i)$ is not empty so that $O_{i+1} := 0_o'^{-1}(O_i \cap 0_o'(O_i))$ is a nonempty locally closed subset of dimension $n - i - 1$ of $A'$. Moreover, $Y_{i+1}$ and $O_{i+1}$ are invariant under $T$ since $p$ is semi-invariant under $T$. As $0_o(\mathbb{R} \times O_{i+1})$ is the intersection of $Y'_{i+1}$ and $0_o(\mathbb{R} \times O_i)$, it is dense in $Y'_{i+1}$ so that $(V_1, V', V, W)$ is in the closure of $0_o(\mathbb{R} \times O_{i+1})$ and $(V_1, V', V, W)$ is in $Y_{i+1}$.
Lemma 2.16. Let $\nu$ be the closure of $Y_{n-2}$, $C$ its normalization and $\nu$ the normalization morphism. Then $C$ is a smooth projective curve. As $Y_{n-2}$ is invariant under $T$, so is $Y_{n-2}$ and there is an action of $T$ on $C$ such that $\nu$ is an equivariant morphism. As the restriction of $\theta_0'$ to $O_{n-2}$ is an isomorphism onto a dense open subset of $Y_{n-2}$, the actions of $T$ on $Y_{n-2}$ and $C$ are not trivial since $\theta_0'$ is equivariant under the actions of $T$. As a result, $T$ has an open orbit $O_\nu$ in $Y_{n-2}$ and $Y_{n-2} \setminus O_\nu$ is the set of fixed points under $T$ of $Y_{n-2}$ since $T$ has dimension 1. Hence the restriction of $\nu$ to $\nu^{-1}(O_\nu)$ is an isomorphism, $C \setminus \nu^{-1}(O_\nu)$ is finite, its elements are fixed under $T$ and there exists a $T$-equivariant morphism $\mu$ from $\nu^{-1}(O_\nu)$ to $A'$ such that $\theta_0' \cdot \mu = \nu$. As $(V_1, V', W)$ is a fixed point under $T$, for some $x_1$ in $C \setminus \nu^{-1}(O_\nu)$, $\nu(x_1) = (V_1, V', W)$ since $(V_1, V', W)$ is in $\nu(C)$. Moreover, $(V_1, V', V, W)$ is in the closure of the map

$$\mathbb{A} \times (C \setminus \nu^{-1}(O_\nu)) \to \Gamma, \quad (z, x) \mapsto \theta_0(z, \mu(x))$$

since it is in $\theta_0(\mathbb{A} \times O_{n-2})$. □

Denote by $\eta$ the morphism

$$\mathbb{A} \times (C \setminus \{x_1, \ldots, x_m\}) \to \mathbb{A} \times (C \setminus \{x_1, \ldots, x_m\}) \to \Gamma, \quad (z, x) \mapsto \theta_0(z, \mu(x))$$

and $\Delta$ the closure of the graph of $\eta$ in $\mathbb{P}^1(\mathbb{A}) \times C \times \Gamma$. Let $\nu$ be the restriction to $\Delta$ of the canonical projection

$$\mathbb{P}^1(\mathbb{A}) \times C \times \Gamma \to \mathbb{P}^1(\mathbb{A}) \times C$$

and for $(z, x)$ in $\mathbb{P}^1(\mathbb{A}) \times C$, let $F_{z,x}$ be the subset of $\Gamma$ such that $\{\nu(z, x)\} \times F_{z,x}$ is the fiber of $\nu$ at $(z, x)$. We have an action of $T$ in $\mathbb{P}^1(\mathbb{A})$ given by

$$t.\z := \begin{cases} \alpha(t)\z & \text{if } \z \in \mathbb{A}^* \\ \z & \text{if } \z \in \{0, \infty\} \end{cases}.$$ 

Lemma 2.16. Let $\Delta_\nu$ be the graph of $\nu$.

(i) The set $\Delta_\nu$ is the image of $\Delta$ by the map $(z, x, y) \mapsto (x, \sigma(y))$.

(ii) For $t$ in $T$ and $(z, x, y)$ in $\Delta$, $t.(z, x, y) := (t.\z, t.\x, t.y)$ is in $\Delta$.

(iii) For $(z, x)$ in $\mathbb{P}^1(\mathbb{A}) \times C$, $\eta$ is regular at $(z, x)$ if and only if $F_{z,x}$ has dimension 0. In this case, $|F_{z,x}| = 1$.

(iv) For $(z, x)$ in $\mathbb{P}^1(\mathbb{A}) \times C \setminus \{0, \infty\} \times \{x_1, \ldots, x_m\}$, $\eta$ is regular at $(z, x)$.

(v) For $t = 1, \ldots, m$, there exists a regular map $\eta_t$ from $\mathbb{P}^1(\mathbb{A})$ to $C$ such that $\eta_t(z) = \eta(z, x_t)$ for all $z$ in $\mathbb{A}^*$. Moreover, its image is contained in $\sigma^{-1}(\nu(x_t))(\Gamma)$.

Proof. (i) As $\mathbb{P}^1(\mathbb{A})$ and $\text{Gr}_T(\nu)$ are projective varieties, the image of $\Delta$ by the map $(z, x, y) \mapsto (x, \sigma(y))$ is a closed subset of $C \times \Gamma'$ contained in $\Delta_\nu$ since $\sigma \circ \eta(z, x) = \nu(x)$ for all $(z, x)$ in $\mathbb{A} \times (C \setminus \{x_1, \ldots, x_m\})$, whence the assertion since the inverse image of $\Delta_\nu$ by this map is a closed subset of $\mathbb{P}^1(\mathbb{A}) \times C \times \Gamma$ containing the graph of $\nu$.

(ii) From the equality

$$t \exp(z \zad x_n) r^{-1} = \exp(\alpha(t) \zad x_n)$$

for all $(t, \z)$ in $T \times \mathbb{A}$, we deduce the equality

$$t.\eta(z, x) = t.\theta_0(z, \mu(x)) = 0_\nu(\alpha(t)z, \mu(t.x)) = \eta(t.\z, t.x)$$

for all $(t, z, x)$ in $T \times \mathbb{A} \setminus (C \setminus \{x_1, \ldots, x_m\})$ since $\theta_0$ and $\mu$ are $T$-equivariant, whence the assertion.

(iii) As $\Gamma$ is a projective variety, $\nu$ is a projective morphism. Moreover, it is birational since $\Delta$ is the closure of the graph of $\eta$. So, by Zariski’s Main Theorem [H77, Ch. III, Corollary 11.4], the fibers of $\nu$ are connected of dimension 0 or 1 since $\mathbb{P}^1(\mathbb{A}) \times C$ is normal of dimension 2. Let $(z, x)$ be in $\mathbb{P}^1(\mathbb{A}) \times C$ such that $F_{z,x}$ dimension 0. There exists a neighborhood $O_{z,x}$ of $(z, x)$ in $\mathbb{P}^1(\mathbb{A}) \times C$ such that $F_x$ has dimension 0 for $y$ in $\mathbb{A}^*$. Therefore, by the Hironaka’s resolution theorem, $\nu(x)$ is smooth at $(z, x)$, and

$$\nu^{-1}(O_{z,x}) = \Gamma \setminus \nu^{-1}(O_{z,x})$$

is a normal crossings divisor on $\mathbb{P}^1(\mathbb{A}) \times C$. Since $\mathbb{P}^1(\mathbb{A}) \times C$ is normal, $\nu$ is a birational morphism.
in $O_{z,x}$. In other words, the restriction of $\nu$ to $\nu^{-1}(O_{z,x})$ is a quasi finite morphism. Moreover, it is birational and surjective. So, again by Zariski’s Main Theorem [Mu88, §9], it is an isomorphism. Hence $\eta$ is regular at $(z, x)$. Conversely, if $\eta$ is regular at $(z, x)$, $\eta(z, x)$ is an isolated point in $F_{z,x}$, whence $F_{z,x} = \{\eta(z, x)\}$ since $F_{z,x}$ is connected.

(iv) The variety $\mathbb{k} \times (C \setminus \{x_1, \ldots, x_m\})$ is an open subset of the smooth variety $\mathbb{P}^1(\mathbb{k}) \times C$ and $\Gamma$ is a projective variety. Hence $\eta$ has a regular extension to a big open subset of $\mathbb{P}^1(\mathbb{k}) \times C$ by [Sh94, Ch. 6, Theorem 6.1]. By Condition (a) of Lemma 2.15(iii), $\{0, \infty\} \times \{x_1, \ldots, x_m\}$ is the set of fixed points under $T$ in $\mathbb{P}^1(\mathbb{k}) \times C$ and by (ii), $t\eta(z, x) = \eta(tz, tx)$ for all $(t, z, x)$ in $T \times \mathbb{P}^1(\mathbb{k}) \times (C \setminus \{x_1, \ldots, x_m\})$. Hence $\eta$ is regular on $P^1(\mathbb{k}) \times C \setminus \{0, \infty\} \times \{x_1, \ldots, x_m\}$.

(v) The restriction of $\eta$ to $\mathbb{k}^* \times \{x_i\}$ is a regular map from a dense open subset of the smooth variety $\mathbb{P}^1(\mathbb{k}) \times \{x_i\}$ to the projective variety $\Gamma$. So, again by [Sh94, Ch. 6, Theorem 6.1], this map has regular extension to $\mathbb{P}^1(\mathbb{k}) \times \{x_i\}$, whence the assertion by (i).

\[ \square \]

Let $I$ be the set of indices such that $\nu(x_i) = (V_1, V', W)$. Denote by $S$ the image of $\Delta$ by the canonical projection $\mathbb{P}^1(\mathbb{k}) \times C \times \Gamma \longrightarrow \Gamma$, $S_n$ its normalization, $\sigma$ the normalization morphism, $S^T$ and $S_n^T$ the sets of fixed points under $T$ in $S$ and $S_n$ respectively. Set

$C_s := \mathbb{P}^1(\mathbb{k}) \times C \setminus \{(0, \infty) \times \{x_1, \ldots, x_m\}\}.$

By Lemma 2.15(iv), $\eta$ is a dominant morphism from $C_s$ to $S$, whence a commutative diagram

\[
\begin{array}{ccc}
C_s & \overset{\eta}{\longrightarrow} & S_n \\
\downarrow{\eta} & & \downarrow{\sigma} \\
S & & & \end{array}
\]

since $C_s$ is smooth. Let $\Delta_n$ be the closure in $\mathbb{P}^1(\mathbb{k}) \times C \times S_n$ of the graph of $\eta_n$ and $\nu_2$ the restriction to $\Delta_n$ of the canonical projection

$\mathbb{P}^1(\mathbb{k}) \times C \times S_n \longrightarrow S_n.$

**Lemma 2.17.** Suppose $V' \neq V$ and $V$ and $V'$ contained in $z + a$.

(i) The variety $\Delta$ is the image of $\Delta_n$ by the map $(z, x, y) \mapsto (z, x, \sigma(y))$.

(ii) The morphism $\nu_2$ is projective and birational.

(iii) There exists a $T$-equivariant morphism

$$(S_n \setminus S_n^T) \overset{\varphi}{\longrightarrow} C_s$$

such that $\eta \circ \varphi$ is the restriction of $\sigma$ to $S_n \setminus S_n^T$.

(iv) For some $i \in I$, $\eta_i(1)$ is not invariant under $T$.

**Proof:** (i) As $S$ is a projective variety, so are $S_n$, $\mathbb{P}^1(\mathbb{k}) \times C \times S_n$, $\Delta_n$ and the image of $\Delta_n$ by the map $(z, x, y) \mapsto (z, x, \sigma(y))$, whence the assertion since the image of the graph of $\eta_n$ by this map is the graph of $\eta$.

(ii) As $\Delta_n$ is projective so is $\nu_2$. Since $\theta_n$ is injective, so is the restriction of $\eta$ to $\mathbb{k} \times (C \setminus \{x_1, \ldots, x_m\})$.

Hence $\nu_2$ is birational.

(iii) By (ii) and Zariski’s Main Theorem [H77, Ch. III, Corollary 11.4], the fibers of $\nu_2$ are connected. For $y$ in $S_n \setminus S_n^T$ and $(z, x)$ in $\mathbb{P}^1(\mathbb{k}) \times C$ such that $(z, x, y)$ is in $\Delta_n$, $\sigma \circ \sigma(y) = \nu(x)$ by (i). If $x$ is not in $\{x_1, \ldots, x_m\}$, $\nu^{-1}(\sigma \circ \sigma(y)) = \{x\}$ by Condition (b) of Lemma 2.15(iii) and $z$ is the element of $\mathbb{k}$ such that $\theta_n(z, \mu(x)) = \sigma(y)$. Suppose $x = x_i$ for some $i = 1, \ldots, m$. Let $z$ and $z'$ be in $\mathbb{k}^*$ such that $(z, x_i, y)$ and $(z', x_i, y)$ are in $\Delta_n$. Then $(z, x_i, \sigma(y))$ and $(z', x_i, \sigma(y))$ are in $\Delta$. By Lemma 2.16.(iii) and (iv), $\sigma(y) = \eta(z, x_i) = \eta(z', x_i)$. For some $t$
in $T$, $t' = t_{z}$ so that $t \sigma(y) = \sigma(y)$. As $y$ is not invariant under $T$ so is $\sigma(y)$ since the fibers of $\sigma$ are finite. Hence the stabilizer of $\sigma(y)$ in $T$ is finite and so is the fiber of $\nu_{2}$ at $y$. As a result, the restriction of $\nu_{2}$ to $\Delta_{n} \setminus \mathbb{P}^{1}(\mathbb{k}) \times C \times S_{n}^{T}$ is an injective morphism. So, again by Zariski’s Main Theorem [Mu88, §9], this map is an isomorphism, whence a morphism

$$(S_{n} \setminus S_{n}^{T}) \overset{\varphi}{\longrightarrow} C_{\ast}.$$  

Moreover, $\varphi$ is $T$-equivariant since so is $\nu_{2}$. For $y$ in $S_{n}$ such that $\sigma(y) = \eta(z, x)$ for some $(z, x)$ in $\mathbb{k}^{*} \times (C \setminus \{x_{1}, \ldots, x_{m}\})$, $(z, x, y)$ is the unique element of $\Delta_{n}$ above $y$. Hence $\eta \varphi = \sigma$.

(iv) Suppose that for all $i$ in $I$, $\eta_{i}(1)$ is invariant under $T$. A contradiction is expected. As $V \neq V'$, $V_{1} = V \cap V'$ and $V + V' = W$ by Lemma 2.13(iv). Moreover, since $V$ and $V'$ are contained in $\mathfrak{s} + a$, for some $\beta$ and $\gamma$ in $\mathcal{R}$,

$$V = V_{1} + a^{\beta} \quad \text{and} \quad V' = V_{1} + a^{\gamma}.$$  

Then $\Gamma_{V_{1}, V', W}$ is invariant under $T$. More precisely, $\Gamma_{V_{1}, V', W}$ is a union of one orbit of dimension 1 and the set $\{(V_{1}, V', W), (V_{1}, V', V, W), (V_{1}, V', V, W)\}$ of fixed points. As a result, $\Gamma_{V_{i}, V', W} \cap S$ is equal to $\{(V_{1}, V', W), (V_{1}, V', V, W)\}$ or $\Gamma_{V_{i}, V', W}$ since $S$ is invariant under $T$. By Lemma 2.16.(ii) and (v), for $i$ in $I$, $\eta_{i}(\mathbb{P}^{1}(\mathbb{k}))$ is equal to $(V_{1}, V', V, W)$ since $\nu_{1}(x_{i}) = (V_{1}, V', W)$. Hence for some $\beta$ and $\gamma$ in $\mathcal{R}$,  

$$\lim_{z \to 0} \eta_{i}(0) = (V_{1}, V', V, W) \quad \text{and} \quad \lim_{z \to \infty} \eta_{i}(\infty) = (V_{1}, V', V, W),$$  

whence a contradiction.

Suppose $\Gamma_{V_{1}, V', W} \cap S = \Gamma_{V_{1}, V', W}$. Let $y$ be in $S_{n}$ such that

$$\sigma(y) \in \Gamma_{V_{1}, V', W} \setminus \{(V_{1}, V', V, W), (V_{1}, V', V, W)\}.$$  

By (iii), for some $i$ in $I$ and some $z$ in $\mathbb{k}^{*}$, $\varphi(t_{z}, y) = (t_{z}, x_{i})$ and $t \sigma(y) = t \eta_{i}(z, x_{i}) = t \eta_{i}(z)$ for all $t$ in $T$, whence a contradiction since $(V_{1}, V', V', W)$ and $(V_{1}, V', V', W)$ are in $T \sigma(y)$. \hfill \Box

**Corollary 2.18.** Let $(V_{1}, V', V, W)$ be a fixed point under $T$ of $\Gamma$ such that $V \cap t = V' \cap t = \mathfrak{s}$. Then $V' = V$.

**Proof.** Suppose $V' \neq V$. A contradiction is expected. By Lemma 2.13(iv), $V_{1} = V \cap V'$ and $W = V + V'$. As $V \cap t = V' \cap t = \mathfrak{s}$, $V$ and $V'$ are contained in $\mathfrak{s} + a$. So, for some $\beta$ in $\mathcal{R}$ and $\gamma$ in $\mathcal{R} \setminus \{a\}$,

$$V = V_{1} + a^{\beta} \quad \text{and} \quad V' = V_{1} + a^{\gamma}.$$  

since $(V_{1}, V', V, W)$ is invariant under $T$. By Lemma 2.17(iv), for some $i$ in $I$, $\eta_{i}(1)$ is not fixed under $T$. Then, by Lemma 2.13(iii), $\eta_{i}(\mathbb{P}^{1}(\mathbb{k})) = \Gamma_{V_{1}, V', W}$. Denoting by $\eta_{i}(z)$ the third component of $\eta_{i}(z)$, for all $z$ in $\mathbb{P}^{1}(\mathbb{k})$, $V_{1}$ is contained in $\eta_{i}(z)$ and $\eta_{i}(z)$ is contained in $W$. Hence for some $a$ in $\mathbb{k}^{*}$,

$$\eta_{i}(1)_{3} = V_{1} \oplus \mathbb{k}(x_{2} + ax_{y}) \quad \text{and} \quad \eta_{i}(a(t))_{3} = V_{1} \oplus \mathbb{k}(\beta(t)x_{2} + a(t)ax_{y})$$  

for all $t$ in $T$. For some $t_{1}$ and $t_{2}$ in $T$, for all $\delta$ in $\mathcal{R}$, $\delta(t_{1})$ and $\delta(t_{2})$ are positive rational numbers and

$$\alpha(t_{1}) > 1, \quad \alpha(t_{2}) > 1, \quad \beta(t_{1}) < \gamma(t_{1}), \quad \beta(t_{2}) > \gamma(t_{2}).$$  

Then

$$\lim_{k \to \infty} V_{1} \oplus \mathbb{k}(\beta(t_{1}^{k})x_{2} + a^{k}ax_{y}) = V_{1} \oplus a^{\delta}, \quad \lim_{k \to \infty} V_{1} \oplus \mathbb{k}(\beta(t_{2}^{k})x_{2} + a^{k}ax_{y}) = V_{1} \oplus a^{\delta},$$

$$\lim_{k \to \infty} \eta_{i}(\alpha(t_{1}^{k})) = \lim_{k \to \infty} \eta_{i}(\alpha(t_{2}^{k})) = \eta_{i}(\infty),$$  

whence $V = V'$ and the contradiction. \hfill \Box
2.6. **Property (P) and Property (P\(_1\)).** In this subsection we suppose that all objects of \(C^i\) of dimension smaller than \(n\) has Property (P). For \(V\) a fixed point of \(X_R\) under \(T\), denote by \(\Lambda V\) the orthogonal complement to \(3V\) in \(R\) and set:

\[ r_V := r_{\Lambda V}, \quad R_V := R_{\Lambda V}. \]

**Lemma 2.19.** Let \(V\) be a fixed point under \(T\) in \(X_R\).

(i) The action of \(R_V\) in \(\overline{R_V.V}\) has fixed points. For \(V\) in such a point,

\[ V = V \cap t \oplus \bigoplus_{\beta \in \mathcal{R}_V} d^\beta, \quad |\mathcal{R}_V| = |\mathcal{R}_{V_\infty}|, \quad r_V \geq r_{V_\infty}. \]

(ii) The set \(\mathcal{R}_V\) has rank at least \(|\mathcal{R}_V| - 1\).

(iii) Suppose that \(a\) has Property (P\(_1\)). Then \(\mathcal{R}_V\) has rank \(|\mathcal{R}_V|\).

(iv) If \(a\) has Property (P\(_1\)), for \(s\) in \(t\) such that \(V\) is contained in \(v^s\), \(V\) is in \(R_t.t\).

**Proof.** (i) As \(\overline{R_V.V}\) is a projective variety and \(R_V\) is connected and solvable, \(R_V\) has fixed points in \(\overline{R_V.V}\). Denote by \(V\) in such a point. Since \(V\) is fixed under \(T\),

\[ V = V \cap t \oplus \bigoplus_{\beta \in \mathcal{R}_V} d^\beta. \]

Moreover, \(V \cap t\) is contained in \(3V\) since \(V\) is commutative. By Lemma 2.6(ii), \(3V\) is the center of \(r_V\). Hence \(V \cap t\) is contained in all element of \(R_V.V\). Moreover, all element of \(R_V.V\) is contained in \(V \cap t + a_{\Lambda V}\). Then

\[ V = V \cap t \oplus \bigoplus_{\beta \in \mathcal{R}_{V_\infty}} d^\beta, \]

whence \(|\mathcal{R}_V| = |\mathcal{R}_{V_\infty}|\). Since \(\mathcal{R}_{V_\infty}\) is contained in \(\Lambda V\) and \(r_V = d - \dim 3V\), \(r_V \geq r_{V_\infty}\).

(ii) By (i), we can suppose that \(V\) is invariant under \(R_V\). By Lemma 2.5, \(a_{\Lambda V}\) is contained in an ideal \(\alpha'\) of \(r\) of dimension \(\dim a - 1\) and contained in \(a\). We then use the notations of Lemma 2.13. Set \(\Gamma_V := \sigma_V^{-1}(V)\). By Lemma 2.13(i), \(\Gamma_V\) is a projective variety invariant under \(R_V\) since so is \(V\). Then \(R_V\) has a fixed point in \(\Gamma_V\). Let \((V_1, V', V, W)\) be such a point. As \(\alpha'\) has Property (P), by Lemma 2.12(i),

\[ V' = 3V' \oplus \bigoplus_{\beta \in \mathcal{R}_{V'}} d^\beta. \]

and the elements of \(\mathcal{R}_{V'}\) are linearly independent.

If \(V' = V\) then \(\mathcal{R}_{V'} = \mathcal{R}_V\) so that \(r_V = r_{V'} = |\mathcal{R}_V|\). Suppose \(V' \neq V\). Then, by Lemma 2.13(iv),

\[ V_1 = 3V' \cap V \cap t \oplus \bigoplus_{\beta \in \mathcal{R}_V \cap \mathcal{R}_{V'}} d^\beta. \]

As \(V_1\) has codimension 1 in \(V\) and \(V'\), \(\mathcal{R}_{V'} = \mathcal{R}_V\) or \(3V' = V \cap t\). In the first case, \(r_V = |\mathcal{R}_V|\) and in the second case,

\[ |\mathcal{R}_V \cap \mathcal{R}_{V'}| = |\mathcal{R}_V| - 1 = |\mathcal{R}_{V'}| - 1, \]

whence \(r_V \geq |\mathcal{R}_V| - 1\) since the elements of \(\mathcal{R}_{V'}\) are linearly independent.

(iii) Prove the assertion by induction on \(\dim 3V\). If \(3V = 3\), then \(r_V = |\mathcal{R}_V|\) by Property (P\(_1\)). Suppose \(\dim 3V = \dim 3 + 1\) and \(V \cap t = 3\). Then \(|\mathcal{R}_V| = d\) and \(r_V = d\) - 1. By Property (P\(_1\)), it is impossible. Hence \(V \cap t = 3V\) since \(V \cap t\) is contained in \(3V\). As a result \(r_V = |\mathcal{R}_V|\).

Suppose \(\dim 3V \geq 2 + \dim 3\), the assertion true for the integers smaller than \(\dim 3V\) and \(r_V < |\mathcal{R}_V|\). A contradiction is expected. By (ii), \(V \cap t\) has dimension at least \(\dim 3V - 1\). Then, for some \(a\) in \(\mathcal{R}\), \(V \cap t_a\) is strictly contained in \(V \cap t\). Let \(\Lambda\) be the orthogonal complement to \(3V \cap t_\alpha\) in \(R\). As \(\overline{\Lambda V}\) is a projective variety and \(R_\Lambda\) is connected, \(R_\Lambda\) has a fixed point in \(\overline{\Lambda V}\). Let \(V_\infty\) be such a point. By Lemma 2.6(ii),
\[ z_V \cap t_\alpha \text{ is the center of } t_\alpha. \text{ Hence } V \cap t_\alpha \text{ is contained in all element of } R_A \cdot V. \text{ Moreover, all element of } R_A \cdot V \text{ is contained in } V \cap t + a_A. \text{ As } V_{\infty} \text{ is an ideal of } t_\alpha, V \cap t \text{ is not contained in } V_{\infty} \text{ since it is not contained in the kernel of } \alpha. \text{ Then}
\[
V_{\infty} = V \cap t_\alpha \oplus \bigoplus_{\beta \in R_{V_{\infty}}} a^\beta.
\]
By (ii), \( r_{V_{\infty}} \geq |\mathcal{R}_{V_{\infty}}| - 1 \), whence
\[
dim z_{V_{\infty}} \leq \dim V \cap t_\alpha + 1 = \dim V \cap t < \dim z_V.
\]
So, by induction hypothesis, \( |\mathcal{R}_{V_{\infty}}| = r_{V_{\infty}} \) and \( z_{V_{\infty}} = V \cap t_\alpha \). Since \( z_V \cap t_\alpha \) is the center of \( t_\alpha, z_V \cap t_\alpha \) is contained in \( z_{V_{\infty}}, \) whence
\[
dim z_V - 1 \leq \dim V \cap t_\alpha = \dim V \cap t - 1.
\]
As a result, \( z_V = V \cap t \) since \( V \cap t \) is contained in \( z_V \), whence a contradiction.

(iv) Suppose that \( a \) has Property \((P_1)\). By (iii),
\[
V = z_V \oplus \bigoplus_{\beta \in R_V} a^\beta
\]
and \( r_V = |\mathcal{R}_V| \). As a result, the centralizer of \( V \) in \( t \) is equal to \( z_V \). Set
\[
a'_V = \bigoplus_{\beta \in R_V} a^\beta, \quad t'_V := t + a'_V.
\]
Denote by \( R'_V \) the connected closed subgroup of \( R \) whose Lie algebra is \( \text{ad} a'_V \). The algebra \( a'_V \) is in \( C'_t \) and has dimension \( d - \dim z_V \). Then, by Lemma 2.3(ii), \( V \) is in \( R'_V \cdot t \), whence the assertion since \( t'_V \) is contained in \( r^t \). \( \square \)

**Corollary 2.20.** Suppose that \( a \) has Property \((P_1)\). Then \( a \) has Property \((P)\).

**Proof.** Let \( V \) be in \( X_R \) and \( s \) in \( t \setminus z \) such that \( V \) is contained in \( r^s \). As \( \overline{T \cdot V} \) is a projective variety and \( T \) is a connected commutative group, \( T \) has a fixed point in \( \overline{T \cdot V} \). Let \( V_{\infty} \) be such a point. Since all element of \( T \cdot V \) is contained in \( r^s \), so is \( V_{\infty} \). Then, by Lemma 2.19(iv), \( V_{\infty} \) is in \( R_{\overline{T \cdot V}} \cdot t \). In particular, \( s \) is in \( V_{\infty} \). Let \( E \) a complement to \( V_{\infty} \) in \( t \), invariant under \( T \). The map
\[
\text{Hom}_k(V_{\infty}, E) \xrightarrow{\kappa} \text{Gr}_{d}(r) , \quad \varphi \mapsto \kappa(\varphi) := \text{span}(\{v + \varphi(v) \mid v \in V_{\infty}\})
\]
is an isomorphism onto an open neighborhood \( \Omega_E \) of \( V_{\infty} \) in \( \text{Gr}_{d}(r) \). For \( \varphi \) in \( \text{Hom}_k(V_{\infty}, E) \) such that \( \kappa(\varphi) \) is in \( T \cdot V, \varphi(s) \) is in \( a^s \). Then, for some \( g \) in \( T \) and for some \( v \) in \( a^s \), \( s + v \) is in \( g(V) \) and the semisimple component of \( \text{ad}(s + v) \) is different from \( 0 \) since \( s \) is not in \( z \). Let \( x \) be in \( r^s \) such that \( \text{ad} x \) is the semisimple component of \( \text{ad}(s + v) \). By Lemma 2.1(ii), for some \( k \) in \( r^s, k(x) \) is in \( t \). Then, by Corollary 2.14(ii), \( k(V) \) is in \( R_{k(x)} \cdot t \). As \( k(x) \) is not in \( z, d(k(x)) \) is an object of \( C'_t \) of dimension smaller than \( n \). By hypothesis, \( d(k(x)) \) has Property \((P)\). Moreover, \( k(V) \) is contained in \( r^s \cap t^{k(x)} \). Hence, by Property \((P)\) for \( d(k(x)), k(V) \) is in \( R_{t} \cdot t \), whence \( V \) is in \( R^s \cdot t \) since \( k(V) \) is in \( R^s \). \( \square \)

**Proposition 2.21.** The objects of \( C'_t \) have Property \((P)\).

**Proof.** Prove by induction on \( n \) that \( a \) has Property \((P)\). By Lemma 2.8, it is true for \( n = d^a \). Suppose that it is true for the integers smaller than \( n \). By Corollary 2.20, it remains to prove that \( a \) has Property \((P_1)\).

Suppose that \( a \) has not Property \((P_1)\). A contradiction is expected. For some fixed point \( V \) under \( T \) in \( X_R \) such that \( V \cap t = z, r_V \neq |\mathcal{R}_V| \). By Lemma 2.19(ii), \( r_V = |\mathcal{R}_V| - 1 \). Then the orthogonal complement of \( \mathcal{R}_V \) in \( t \) is generated by \( z \) and an element \( s \) in \( t \setminus z \). In particular, \( V \) is contained in \( r^s \). According to Lemma 2.5,
for some ideal $a'$ of codimension 1 of $a$, normalized by $t$, $a^t$ is contained in $a'$. Denote by $a$ the element of $\mathcal{R}$ such that

$$a = a' \oplus a'$$

and consider $\theta_{a}$ and $\Gamma$ as in Subsection 2.5. Denote by $\Gamma_V$ the set of elements of $\Gamma$ whose image by the projection

$$\Gamma \rightarrow \text{Gr}_d(t), \quad (T_1, T', T, T_2) \mapsto T$$

is equal to $V$. By Lemma 2.13(ii), $\Gamma_V$ is not empty and it is invariant under $T$ by Lemma 2.13(i). As it is a projective variety, it has a fixed point under $T$. Denote by $(V_1, V', V, W)$ such a point. As $a'$ has Property (P), it has Property (P) by Lemma 2.12. Hence $r_{V'} = |R_{V'}|$ and $V' \neq V$ since $r_V \neq R_V$. Then, by Lemma 2.13(iv),

$$V = V' \cap V' \quad \text{and} \quad W = V' + V.$$ 

As a result, $V' \cap t = V \cap t = 3$ since $R_{V'} \neq R_V$ and $V_1$ has codimension 1 in $V$ and $V'$. Then $V' = V$ by Corollary 2.18, whence a contradiction.$\square$

The following corollary results from Proposition 2.21, Corollary 2.10 and Lemma 2.22.

**Corollary 2.22.** Let $V$ be in $X_R$.

(i) The space $V$ is a commutative algebraic subalgebra of $t$ and for some subset $\Lambda$ of $\mathcal{R}$, the biggest torus contained in $V$ is conjugate to $t_{\Lambda}$ under $R$.

(ii) If $V$ is a fixed point under $R$, then $V$ is an ideal of $t$ and the elements of $R_V$ are linearly independent.

3. **Solvable algebras and main varieties**

Let $t$ be a vector space of positive dimension $d$ and $a$ in $C_t$. Set:

$$\mathcal{R} := \mathcal{R}_{t,a}, \quad t := t_{t,a}, \quad \pi := \pi_{t,a}, \quad R := R_{t,a}, \quad A := A_{t,a}, \quad E := E_{t,a}, \quad n := \dim a.$$ 

In this section, we give some results on the singular locus of $X_R$. For $a'$ in $C_t$, denote by $X_{R_{t,a'}}$ the subset of elements of $X_{R_{t,a'}}$ contained in $a'$.

**3.1. Subvarieties of $X_R$.** Denote by $\mathcal{P}_c(\mathcal{R})$ the set of complete subsets of $\mathcal{R}$ and for $\Lambda$ in $\mathcal{P}_c(\mathcal{R})$ denote by $X_{R_{\Lambda}}$, the closure in $\text{Gr}_d(t)$ of the orbit $R_{t_{\Lambda}} t$.

**Proposition 3.1.** Let $Z$ be an irreducible closed subset of $X_R$, invariant under $R$.

(i) For a well defined complete subset $\Lambda$ of $\mathcal{R}$, all element of a dense open subset of $Z$ is conjugate under $R$ to the sum of $t_{\Lambda}$ and a subspace of $a$.

(ii) All element of $Z$ is contained in $t_{\Lambda} \oplus a$.

(iii) For some irreducible closed subset $Z_{\Lambda}$ of $X_{R_{\Lambda}}$, invariant under $R_{\Lambda}$, $R_Z\Lambda$ is dense in $Z$.

**Proof.** (i) For $\Lambda$ in $\mathcal{P}_c(\mathcal{R})$, let $Y_{\Lambda}$ be the subset of elements $V$ of $Z$ such that $\pi(V) = t_{\Lambda}$. Since $Z$ is invariant under $R$, so is $Y_{\Lambda}$. Moreover, by Corollary 2.22(i),

$$Y_{\Lambda} \subset Y_{\Lambda} \cup \bigcup_{\Lambda' \in \mathcal{P}_c(\mathcal{R}) \backslash \Lambda' \neq \Lambda} Y_{\Lambda'}.$$ 

According to Corollary 2.22(i), $Z$ is the union of $Y_{\Lambda}, \Lambda \in \mathcal{P}_c(\mathcal{R})$. As a result, since $\mathcal{R}$ is finite and $Z$ is irreducible, for a well defined complete subset $\Lambda$ of $\mathcal{R}$, $Y_{\Lambda}$ contains a dense open subset of $Z$. By Lemma 2.1(v), all element of $Y_{\Lambda}$ is conjugate under $R$ to the sum of $t_{\Lambda}$ and a subspace of $a$.

(ii) By (i), for all $V$ in a dense subset of $Z$, $V$ is contained in $t_{\Lambda} \oplus a$, whence the assertion.
(iii) Let $Z_r$ be the subset of elements of $Z$, containing $t_A$. Denote by $s$ an element of $t_A$ such that $\alpha(s) \neq 0$ for all $\alpha$ in $\mathcal{R} \setminus \Lambda$. By Lemma 2.6(i),

$$v^s = t \oplus a_A.$$ 

Hence $Z_r$ is contained in $X_{R_A}$ by Proposition 2.21. Moreover, $Z_r$ is invariant under $R_A$ since $Z$ is invariant under $R$. By (i), $R.Z_r$ is dense in $Z$. So, for some irreducible component $Z_A$ of $Z_r$, $R.Z_A$ is dense in $Z$. Moreover, $Z_r$ is invariant under $R_A$ since so is $Z_r$. \hfill \square

For $\Lambda$ in $\mathcal{P}_c(\mathbb{R})$, denote by $t^\#_A$ a complement to $t_A$ in $t$ and set:

$$t^\#_A := t_A + a_A.$$ 

Let $R^\#_A$ be the adjoint group of $t^\#_A$ and $A^\#_A$ the connected closed subgroup of $R^\#_A$ whose Lie algebra is $\text{ad} a_A$.

**Lemma 3.2.** Let $\Lambda$ be in $\mathcal{P}_c(\mathbb{R})$, nonempty and strictly contained in $\mathbb{R}$.

(i) The tori $t_A$ and $t^\#_A$ have positive dimension and $a_A$ is in $C^t_A$. Moreover,

$$\dim a_A - \dim t^\#_A \leq \dim a - \dim t.$$ 

(ii) The map $V \mapsto V \oplus t_A$ is an isomorphism from $X_{R^\#_A}$ onto $X_{R_A}$.

**Proof.** Since $\Lambda$ is a complete subset of $\mathcal{R}$ strictly contained in $\mathbb{R}$, $t_A$ has positive dimension and since $\Lambda$ is not empty, $t_A$ is strictly contained in $t$. By definition, $\Lambda$ is the set of weights of $t$ in $a_A$ so that $a_A$ is in $C^t_A$. Then $a_A$ is in $C^t_A$ and Assertion (ii) results from Corollary 2.2.

By Lemma 2.1, (i) and (iv), $\mathcal{R}$ generates $t^\#$. Hence

$$|\Lambda| + \dim t_A \leq |\mathcal{R}|.$$ 

By Condition (2) of Section 2, $a$ has dimension $|\mathcal{R}|$ and $a_A$ has dimension $|\Lambda|$. As a result,

$$\dim a - \dim t = |\mathcal{R}| - \dim t_A - \dim t^\# \geq \dim a_A - \dim t^\#.$$ 

\hfill \square

### 3.2. Smooth points of $X_R$ and commutators.

Denote by $t_{\text{reg}}$ the complement in $t$ to the union of $t_{\alpha}$, $\alpha \in \mathcal{R}$ and $t_{\text{reg}}$ the set of elements $x$ of $t$ such that $v^x$ has minimal dimension.

**Lemma 3.3.** (i) The set $t_{\text{reg}}$ is a dense open subset of $t$, contained in $t_{\text{reg}}$. Moreover, $R.t_{\text{reg}}$ is a dense open subset of $r$.

(ii) For all $x$ in $t_{\text{reg}}$, $v^x$ is in $X_R$.

(iii) The set $t_{\text{reg}}$ is a big open subset of $t$.

**Proof.** (i) By definition, $t_{\text{reg}}$ is a dense open subset of $t$. According to Lemma 2.6(i), for $x$ in $t_{\text{reg}}$, $v^x = t$. Then $R.x = A.x = x + a$ since $A.x$ is a closed subset of $x + a$ of dimension $\dim a$. As a result, $R.t_{\text{reg}} = t_{\text{reg}} + a$ is a dense open subset of $r$. Hence $R.t_{\text{reg}}$ is contained in $t_{\text{reg}}$ since $v^x$ is conjugate to $t$ for all $x$ in $R.t_{\text{reg}}$ and $t_{\text{reg}}$ is a dense open subset of $\mathcal{R}$.

(ii) By (i), for all $x$ in $t_{\text{reg}}$, $v^x$ has dimension $d$, whence a regular map

$$t_{\text{reg}} \xrightarrow{\theta} \text{Gr}_d(t), \quad x \mapsto v^x.$$ 

As a result, by (i), for all $x$ in $t_{\text{reg}}$, $v^x$ is in $X_R$.

(iii) Suppose that $t_{\text{reg}}$ is not a big open subset of $r$. A contradiction is expected. Let $\Sigma$ be an irreducible component of codimension 1 of $r \setminus t_{\text{reg}}$. Since $\Sigma \cap A.t_{\text{reg}}$ is empty, $\pi(\Sigma)$ is contained in $t_{\alpha}$ for some $\alpha$ in $r$. Then $\Sigma = t_{\alpha} + a$ since $\Sigma$ has codimension 1 in $r$. By Condition (3) of Section 2, for some $s$ in $t_{\alpha}$, $\gamma(s) \neq 0$ for
all $\gamma$ in $\mathcal{R} \setminus \{\alpha\}$. Then $r^{x+\alpha} = t_\alpha + \alpha^\gamma$ so that $s + x_\alpha$ is in $r_{\text{reg}}$ by (i) and Condition (2) of Section 2, whence the contradiction. \hfill $\Box$

Denote by $X'_R$ the image of $\theta$.

**Proposition 3.4.** (i) The complement to $R.t$ in $X_R$ is equidimensional of dimension $\dim \alpha - 1$.

(ii) The set $X'_R$ is a smooth open subset of $X_R$, containing $R.t$.

**Proof.** (i) As $R$ is solvable and $R.t$ is dense in $X_R$, $R.t$ is an affine open subset of $X_R$. So, by [EGAIV, Corollaire 21.12.7], $X_R \setminus R.t$ is equidimensional of dimension $\dim \alpha - 1$ since $X_R$ has dimension $\dim \alpha$.

(ii) By definition, $E$ is the subvariety of elements $(V, x)$ of $X_R \times r$ such that $x$ is in $V$. Let $\Gamma$ be the image of the graph of $\theta$ by the isomorphism

$$\mathbb{A} \ni t \mapsto Gr_d(t), \quad z \mapsto \exp(z(ad)x)(t),$$

Then $\Gamma$ is the intersection of $E$ and $X'_R \times r_{\text{reg}}$. Since $\Gamma$ is isomorphic to $r_{\text{reg}}$, $\Gamma$ is a smooth open subset of $E$ whose image by the bundle projection is $X'_R$. As a result, $X'_R$ is a smooth open subset of $X_R$ by [MA86, Ch. 8, Theorem 23.7]. \hfill $\Box$

For $\alpha$ in $\mathcal{R}$, set $V_\alpha := t_\alpha \oplus \alpha^\gamma$ and denote by $\theta_\alpha$ the map

$$\mathbb{A} \ni t \mapsto Gr_d(t), \quad z \mapsto \exp(z(ad)x)(t),$$

By Condition (2) of Section 2, $V_\alpha$ has dimension $d$.

**Lemma 3.5.** Let $\alpha$ be in $\mathcal{R}$. Set $X_{R,\alpha} := A.V_\alpha$.

(i) The map $\theta_\alpha$ has a regular extension to $\mathbb{P}^1(\mathbb{A})$ such that $\theta_\alpha(\infty) = V_\alpha$.

(ii) The variety $X_{R,\alpha}$ has dimension $\dim \alpha - 1$ and it is an irreducible component of $X_R \setminus R.t$.

(iii) The intersection $X_{R,\alpha} \cap X'_R$ is not empty.

**Proof.** (i) Let $h_\alpha$ be in $t$ such that $\alpha(h_\alpha) = 1$. Since $X_R$ is a projective variety, the map $\theta_\alpha$ has a regular extension to $\mathbb{P}^1(\mathbb{A})$ by [Sh94, Ch. 6, Theorem 6.1]. For $z$ in $\mathbb{A}$,

$$\theta_\alpha(z) = t_\alpha \oplus \mathbb{A} \times h_\alpha - zx_\alpha.$$  

Hence $\theta_\alpha(\infty) = V_\alpha$.

(ii) By (i), $X_{R,\alpha}$ is contained in $X_R$ and its elements are contained in $t_\alpha \oplus \mathbb{A}$ so that $X_{R,\alpha}$ is contained in $X_R \setminus R.t$. By Condition (3) of Section 2, for $\gamma$ in $\mathcal{R} \setminus \{\alpha\}$ and $v$ in $\mathbb{A}^\gamma$, $[t_\alpha, v] = \mathbb{A}v$ so that no element of $\mathbb{A}^\gamma$ normalizes $V_\alpha$. As a result, the normalizer of $V_\alpha$ in $r$ is equal to $t + \alpha^\gamma$ so that $X_{R,\alpha}$ has dimension $\dim \alpha - 1$. Hence $X_{R,\alpha}$ is an irreducible component of $X_R \setminus R.t$.

(iii) According to Condition (3) of Section 2, for some $s$ in $t_\alpha$, $\gamma(s) \neq 0$ for all $\gamma$ in $\mathcal{R} \setminus \{\alpha\}$. Then $V_\alpha = t^{x+\alpha}$ so that $s + x_\alpha$ is in $r_{\text{reg}}$, whence the assertion. \hfill $\Box$

**3.3. On the singular locus of $X_R$.** In this subsection we suppose $\dim \alpha > d$ and we fix an ideal $\alpha'$ of codimension 1 in $\mathcal{A}$, normalized by $t$ and such that $\alpha'$ is in $\mathcal{C}_t$. For example, all ideal of $r$ of dimension $\dim \alpha - 1$, contained in $\mathcal{A}$ and containing a fixed point under $R$ in $X_R$ is in $\mathcal{C}_t$ by Corollary 2.22(ii). Set:

$$t' := v_{t,\alpha'}, \quad \pi' := \pi_{t,\alpha'}, \quad R' := R_{t,\alpha'}, \quad A' := A_{t,\alpha'}, \quad \mathcal{R}' := \mathcal{R}_{t,\alpha'}.$$  

Let $\alpha$ be in $\mathcal{R}$ such that

$$\alpha = \alpha' \oplus \alpha^\gamma$$

and $\Gamma$ as in Subsection 2.5. Denote by $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$ the restrictions to $\Gamma$ of the first, second, third, fourth projections. Let $Z$ be an irreducible component of $X_{R,n}$. According to Lemma 2.13(ii), for some irreducible
component $T$ of $\varpi_3^{-1}(Z)$, $\varpi_3(T) = Z$. Denote by $Z'$ the image of $T$ by $\varpi_2$ and by $T_1$ the image of $T$ by the projection $\varpi_1 \times \varpi_4$. Then $Z'$ and $T_1$ are irreducible closed subsets of $\text{Gr}_d(r)$ and $\text{Gr}_{d-1}(r) \times \text{Gr}_{d+1}(r)$ respectively. Let $T_0$ be the subset of elements $(V_1, V', V, W)$ of $T$ such that $V' = V$. Then $T_0$ is a closed subset of $T$. If $T_0 = T$, $Z' = Z$ and $Z$ is contained in $X_{r,n}$. Otherwise, $O := T \setminus T_0$ is a dense open subset of $T$. According to Lemma 2.13(iv), for all $(V_1, V', V, W)$ in $O$, $V_1 = V' \cap V$ and $V' + V = W$. Denote by $O_1$ an open subset of $T_1$, contained and dense in $\varpi_1 \times \varpi_4(O)$.

Let $(V_1, W)$ be in $O_1$. Denote by $E$ a complement to $V_1$ in $r$ and by $E'$ a complement to $W$ in $r$ contained in $E$. Let $\kappa$ be the map

$$\text{Hom}_k(V_1, W \cap E) \times \text{Hom}_k(W, E') \overset{\kappa}{\longrightarrow} \text{Gr}_{d-1}(r) \times \text{Gr}_{d+1}(r),$$

$$(\varphi, \psi) \longmapsto (\text{span}\{v + \varphi(v) + \psi(v) | v \in V_1\}, \text{span}\{v + \psi(v) | v \in W\}).$$

Then $\kappa$ is an isomorphism from its source to an open neighborhood of $(V_1, W)$ in the subvariety of elements $(W_1, W_2)$ of $\text{Gr}_{d-1}(r) \times \text{Gr}_{d+1}(r)$ such that $W_1$ is contained in $W_2$. Denote by $\Omega$ the inverse image by $\kappa$ of the intersection of $T_1$ and the image of $\kappa$. Let $(e_1, e_2)$ be a basis of $W \cap E$ and let $\kappa_0$ be the map

$$\Omega \times (k^2 \setminus \{(0, 0)\}) \overset{\kappa_0}{\longrightarrow} \text{Gr}_d(r),$$

$$(\varphi, \psi, x_1, x_2) \longmapsto \text{span}\{v + \varphi(v) + \psi(v) + \psi_0 \varphi(v) | v \in V_1\} \cup \{x_1(e_1 + \psi(e_1)) + x_2(e_2 + \psi(e_2))\}).$$

**Lemma 3.6.** Suppose that $O$ is not empty. Denote by $\tilde{\Omega}$ the image of $\kappa_0$ and $\tilde{Z}$ the closure of $\tilde{\Omega}$ in $\text{Gr}_d(r)$.

(i) The intersections $\tilde{\Omega} \cap Z'$ and $\tilde{\Omega} \cap Z$ are dense in $Z'$ and $Z$ respectively. In particular $Z'$ and $Z$ are contained in $\tilde{Z}$.

(ii) For $V$ in $\tilde{\Omega}$, there exists $(V', V'')$ in $Z' \times Z$ such that

$$V' \cap V'' \subset V, \quad V \subset V' + V'', \quad (V' \cap V'', V' + V'') \in \kappa(\Omega).$$

(iii) Let $F'$ be the fiber of $\kappa_0$ at some element $V$ of $\kappa_0(\Omega)$. Denote by $F$ the subset of elements $(\varphi, \psi)$ of $\Omega$ such that $V$ contains the first component of $\kappa(\varphi, \psi)$ and is contained in the second component of $\kappa(\varphi, \psi)$. Then $F' = F \times k^2(x_1, x_2)$ for some $(x_1, x_2)$ in $k^2 \setminus \{(0, 0)\}$.

(iv) The varieties $\tilde{Z}$ and $Z$ have dimension at most $\dim Z' + 1$.

**Proof.** (i) Since $T$ is irreducible so are $T_1$ and $\Omega$. Hence $\tilde{Z}$ is irreducible. For some $(V', V)$ in $Z' \times Z$, $V_1$ is contained in $V'$ and $V$ and $V'$ are contained in $W$. Since $\kappa(\Omega)$ is an open neighbourhood of $(V_1, W)$ in $T_1$, $\varpi_2(\varpi_1 \times \varpi_4^{-1}(\kappa(\Omega)) \cap T)$ and $\varpi_3(\varpi_1 \times \varpi_4^{-1}(\kappa(\Omega)) \cap T)$ are dense subsets of $Z'$ and $Z$ respectively. For all $(\varphi, \psi)$ in $\Omega$, all element of $\varpi_2(\varpi_1 \times \varpi_4^{-1}(\kappa(\varphi, \psi)) \cap T)$ contains the first component of $\kappa(\varphi, \psi)$ and is contained in the second component of $\kappa(\varphi, \psi)$. Hence all element of $\varpi_2(\varpi_1 \times \varpi_4^{-1}(\kappa(\Omega)) \cap T)$ is in the image of $\kappa_0$. As a result, $\tilde{\Omega} \cap Z'$ is dense in $Z'$ and $Z'$ is contained in $\tilde{Z}$. In the same way, $\tilde{\Omega} \cap Z$ is dense in $Z$ and $Z$ is contained in $\tilde{Z}$.

(ii) According to Lemma 2.13(iv), for all $(V_1', V', V, W')$ in $O$, $V_1' = V' \cap V$ and $W' = V' + V$. By definition, $\kappa(\Omega)$ is contained in $\varpi_1 \times \varpi_4(O)$ and for $V$ in $\tilde{\Omega}$, $V_1' \subset V$ and $V \subset W'$ for some $(V_1', W')$ in $\kappa(\Omega)$, whence the assertion.

(iii) For $(\varphi, \psi)$ in $F$ and for $(x_1, x_2)$ in $k^2 \setminus \{(0, 0)\}$ such that $V = \kappa_0(\varphi, \psi, x_1, x_2)$, the subset of elements $(y_1, y_2)$ of $k^2$ such that $(\varphi, \psi, y_1, y_2)$ is in $F'$ is equal to $k^2(x_1, x_2)$. Moreover, for all $(\varphi, \psi, y_1, y_2)$ in $F'$, $(\varphi, \psi)$ is in $F$, whence the assertion.
(iv) In (iii), we can choose \( V \) such that \( F' \) has minimal dimension so that
\[
\dim \tilde{Z} = \dim \Omega + 2 - (\dim F + 1) = \dim \Omega - \dim F + 1.
\]
By (ii), for some \( V' \) in \( Z' \), for all \((\varphi, \psi)\) in \( F \), \( V' \) contains the first component of \( k(\varphi, \psi) \) and is contained in the second component of \( k(\varphi, \psi) \). So, again by (iii) and (ii),
\[
\dim Z' \geq \dim \Omega - \dim F,
\]
whence \( \dim \tilde{Z} \leq \dim Z' + 1 \) and \( \dim Z \leq \dim Z' + 1 \) since \( Z \) is contained in \( \tilde{Z} \) by (i). \( \square \)

**Proposition 3.7.** The variety \( X_{R,n} \) has dimension at most \( n - d \).

**Proof.** Prove this by induction on \( n \). According to Lemma 2.3(ii), it is true for \( n - d = 0 \). Suppose that \( n - d \) is positive and that it is true for all integer smaller than \( n - d \). In particular, \( X_{R',n} \) has dimension at most \( n - d - 1 \). Let \( Z \) be an irreducible component of \( X_{R,n} \). According to Lemma 2.13(ii), for some irreducible component \( T \) of \( \sigma_3^{-1}(Z) \), \( \sigma_3(T) = Z \). Denote by \( Z' \) the image of \( T \) by \( \sigma_2 \). Let \( T_0 \) be the subset of elements \((V_1, V', V, W)\) of \( T \) such that \( V' = V \). Consider the following cases:

(a) \( T_0 = T \),
(b) \( T_0 \neq T \) and \( Z' \) is contained in \( X_{R',n} \),
(c) \( Z' \) is not contained in \( X_{R',n} \).

(a) In this case, \( Z' = Z \) and \( \dim Z \leq n - d - 1 \) by induction hypothesis.
(b) By induction hypothesis, \( \dim Z' \leq n - d - 1 \) and by Lemma 3.6(iv), \( \dim Z \leq \dim Z' + 1 \), whence \( \dim Z \leq n - d \).
(c) In this case, \( T_0 \neq T \), whence \( \dim Z \leq \dim Z' + 1 \) by Lemma 3.6(iv). Since \( Z \) is an irreducible component of \( X_{R,n} \), \( Z \) is invariant under \( R \). By Lemma 2.13(i), \( \sigma_2 \) and \( \sigma_3 \) are equivariant under the action of \( R' \) in \( \Gamma \) so that \( T \) and \( Z' \) are invariant under \( R' \). For all \((V_1, V', V, W)\) in \( T \setminus T_0 \), \( V_1 = V' \cap V \). Hence all element of a dense open subset of \( Z' \) contains a subspace of dimension \( d - 1 \) of \( a' \). Then, by Proposition 3.1, for some complete subset \( \Lambda \) of \( R' \) such that \( t_A \) has dimension 1 and for some closed subset \( Z_\Lambda \) of \( X_{R,\Lambda}, R'Z_\Lambda \) is dense in \( Z' \) so that
\[
\dim Z' \leq \dim Z_\Lambda + \dim a' - \dim a_\Lambda.
\]
If \( \dim a_\Lambda - \dim t + 1 = n - d \), then \( \Lambda = R' \). In this case, since \( a' \) is in \( C_t \), \( \Lambda \) generates \( t^* \). As \( t_A \) has dimension 1, it is impossible. As a result,
\[
\dim Z_\Lambda \leq \dim a_\Lambda - \dim t + 1 \quad \text{and} \quad \dim Z' \leq n - d
\]
by Lemma 3.2 and induction hypothesis for \( a_\Lambda \). Then \( \dim Z \leq n - d + 1 \). According to Lemma 3.6(i) and (iv), \( \tilde{Z} \) is an irreducible variety of dimension at most \( \dim Z' + 1 \), containing \( Z' \) and \( Z \). If \( \dim Z' = n - d \) and \( \dim Z = n - d + 1 \), then \( Z = \tilde{Z} \). In particular, \( Z' \) is contained in \( Z \). It is impossible since all element of \( Z \) is contained in \( a \). As a result, \( \dim Z \leq n - d \), whence the proposition. \( \square \)

**Corollary 3.8.** (i) The irreducible components of \( X_R \setminus R.t \) are the \( X_{R,a}, \alpha \in R \).

(ii) The set \( X'_a \) is a smooth big open subset of \( X_R \), containing \( R.t \).

**Proof.** According to Proposition 3.4(ii) and Lemma 3.5(iii), Assertion (ii) results from Assertion (i). Prove Assertion (i) by induction on \( n = \dim a \). For \( n = 1 \), \( d = 1 \) by Lemma 2.1(i) and (iv) so that \( X_R \) is the union of \( R.t \) and \( a^\ast \), whence Assertion (i) in this case. Suppose \( n \geq 2 \) and the assertion true for the integers smaller than \( n \). By Lemma 2.1(i), Condition (2) and Condition (3) of Section 2, \( d \geq 2 \). According to Lemma 3.5(ii), for all \( \alpha \in R, X_{R,\alpha} \) is an irreducible component of \( X_R \setminus R.t \). Let \( Z \) be an irreducible component of \( X_R \setminus R.t \). By Proposition 3.4(i), \( Z \) has dimension \( n - 1 \). So, by Proposition 3.7, \( Z \) is not contained in \( X_{R,n} \). Moreover,
Z is invariant under $R$. Then, by Proposition 3.1, for some complete subset $\Lambda$ of $\mathcal{R}$, strictly contained in $\mathcal{R}$ and for some irreducible closed subset $Z_\Lambda$ of $X_{R^\mathcal{R}_\Lambda}$, invariant under $R_\Lambda$, $R.Z_\Lambda$ is dense in $Z$. By Lemma 3.2, $a_\Lambda$ is in $\mathcal{C}_a$ and $Z_\Lambda$ is the image of a closed subset $Z'_\Lambda$ of $X_{R^\mathcal{R}_\Lambda}$, invariant by $R^\mathcal{R}_\Lambda$, by the map $V \mapsto V \oplus t_\Lambda$. Since $Z_\Lambda$ is contained in $Z$, $Z'_\Lambda \cap R_{\Lambda}^\mathcal{R},t_\Lambda$ is empty. As $\Lambda$ is strictly contained in $\mathcal{R}$, $\dim a_\Lambda$ is smaller than $n$. So, by induction hypothesis, for some $\alpha$ in $\Lambda$, $Z'_\Lambda$ is contained in $X_{R^\mathcal{R}_\Lambda,\alpha}$. As a result, $Z_\Lambda$ and $Z$ are contained in $X_{R,\alpha}$, whence $Z = X_{R,\alpha}$, since $Z$ is an irreducible component of $X_R \setminus R.t$.

4. Normality for solvable Lie algebras

Let $t$ be a vector space of positive dimension $d$ and $a$ in $\mathcal{C}_d$. Set:

\[ \mathcal{R} := \mathcal{R}_{t,a}, \quad t := v_{t,a} \quad \pi := \pi_{t,a}, \quad R := R_{t,a}, \quad A := A_{t,a}, \quad \mathcal{E} := \mathcal{E}_{t,a}, \quad n := \dim a. \]

The goal of the section is to prove that $X_R$ is normal and Cohen-Macaulay.

4.1. The case $\dim a = \dim t$. By Condition (2) of Section 2, $\mathcal{R}$ has $d$ elements $\beta_1, \ldots, \beta_d$ linearly independent. Denote by $t_1, \ldots, t_d$ the dual basis in $t$. For $i = 1, \ldots, d$, let $v_i$ be a generator of $\mathcal{R}^\mathcal{R}_i$.

**Lemma 4.1.** If $\dim a = \dim t$ then $X_R$ is a smooth variety. Moreover, for all $(z_1, \ldots, z_d)$ in $k^d$, the subspace generated by $v_1 + z_1 t_1, \ldots, v_d + z_d t_d$ is in $X_R$.

**Proof.** According to Lemma 2.3, $a$ is in in $X_R$ and the map

\[ k^d \rightarrow X_R, \quad (z_1, \ldots, z_d) \mapsto \text{span}(v_1 + z_1 t_1, \ldots, v_d + z_d t_d) \]

is an isomorphism onto an open neighborhood of $a$ in $X_R$. Hence $a$ is a smooth point of $X_R$. By Corollary 2.22, $R$ has only one fixed point $\alpha$ in $X_R$. Since for all $V$ in $X_R$, $R$ has a fixed point in $R.V$ and $X_{R_{s,m}}$ is an open subset of $X_R$, invariant under $R$, $X_R = X_{R_{s,m}}$.

4.2. Cohen-Macaulayness property for some algebras. Let $A_s$ be an integral domain and a local commutative $k$-algebra with maximal ideal $m$ and $u_1, \ldots, u_s$ a regular sequence in $A_s$ of elements of $m$. Let $T_1, \ldots, T_s$ be indeterminates. Set $B_s := A_s[T_1, \ldots, T_s]$ and denote by $P_s$ and $P'_s$ the ideals of $B_s$ generated by the sequences $u_j T_k - u_k T_j, 1 \leq j, k \leq s$ and $u_j T_1 - u_1 T_j, 1 \leq j \leq s$ respectively.

**Lemma 4.2.** The ideal $P_s$ is a prime ideal of $B_s$.

**Proof.** For $s = 1$, $P_s = \{0\}$. Suppose $s \geq 2$. Let $\tilde{P}$ be the ideal of $B_s[T_1^{-1}]$ generated by $P_s$. For $1 \leq j, k \leq s$,

\[ T_1(u_j T_k - u_k T_j) = T_k(u_j T_1 - u_1 T_j) + T_j(u_1 T_k - u_k T_1). \]

Hence $\tilde{P}$ is the ideal of $B_s[T_1^{-1}]$ generated by $P'_s$. Setting $S_j := T_j/T_1$ for $j = 2, \ldots, s$, denote by $C$ the polynomial algebra $A_s[S_2, \ldots, S_s]$ over $A_s$ so that $B_s[T_1^{-1}] = C[T_1, T_1^{-1}]$ and $\tilde{P}$ is the ideal of $B_s[T_1^{-1}]$ generated by $u_j - u_1 S_j, j = 2, \ldots, s$.

**Claim 4.3.** Let $Q$ be the ideal of $C$ generated by $u_j - u_1 S_j, j = 2, \ldots, s$. Then $Q$ is prime.

**Proof.** [Proof of Claim 4.3] Let $\tilde{Q}$ be the ideal of $C[u_1^{-1}]$ generated by $Q$. Then $\tilde{Q}$ is prime since it is generated by $u_j u_1^{-1} - S_j, j = 2, \ldots, s$. As a result, for $p$ and $q$ in $C$ such that $pq$ is in $Q$, for some nonnegative integer $m$, $u_1^m p$ or $u_1^n q$ is in $Q$. So it remains to prove that for $p$ in $C$, $p$ is in $Q$ if so is $u_1 p$.

Let $p$ be in $C$ such that $u_1 p$ is in $Q$. For some $q_2, \ldots, q_s$ in $C$,

\[ u_1 p = \sum_{j=2}^s q_j (u_j - u_1 S_j) \quad \text{whence} \quad \sum_{j=1}^s q_j u_j = 0 \quad \text{with} \quad q_1 := -(p + \sum_{j=2}^s q_j S_j). \]
By hypothesis, the sequence \( u_1, \ldots, u_s \) is regular in \( C \). So for some sequence \( q_{j,k}, 1 \leq j, k \leq s \) in \( C \) such that \( q_{j,k} = -q_{k,j} \),

\[
q_j = \sum_{k=1}^{s} q_{j,k}u_k
\]

for \( j = 1, \ldots, s \). As a result,

\[
\begin{align*}
u_1p &= \sum_{j=1}^{s} \sum_{k=1}^{j} q_{j,k}u_k(u_j - u_1S_j) \\
&= \sum_{j=2}^{s} q_{j,1}u_j - \sum_{j=2}^{s} \sum_{k=1}^{j} q_{j,k}u_ku_1S_j \\
&= u_1(\sum_{j=2}^{s} q_{j,1}(u_j - u_1S_j) + \sum_{2 \leq j < k \leq s} q_{j,k}(u_jS_k - u_kS_j)).
\end{align*}
\]

For \( 2 \leq j, k \leq s \),

\[
u_jS_k - u_kS_j = (u_j - u_1S_j)S_k - (u_k - u_1S_k)S_j \in Q,
\]

whence the claim. \( \square \)

By the claim, \( \tilde{P} \) is a prime ideal of \( B_s[T_1^{-1}] \) since it is generated by \( Q \). As a result for \( p \) and \( q \) in \( B_s \) such that \( pq \) is in \( P_s \), for some nonnegative integer \( m \), \( T_1^mp \) or \( T_1^mq \) is in \( P_s' \) since \( T_1P_s \) is contained in \( P_s' \) by the equality

\[
T_1(u_jT_k - u_kT_j) = T_k(u_jT_1 - u_1T_j) + T_j(u_1T_k - u_kT_1)
\]

for \( 1 \leq i, j \leq s \). So it remains to prove that for \( p \) in \( B_s, p \) is in \( P_s \) if \( T_1p \) is in \( P_s' \).

Let \( p \) be in \( B_s \) such that \( T_1p \) is in \( P_s' \). For some \( r_2, \ldots, r_s \) in \( B_s \),

\[
T_1p = \sum_{j=2}^{s} r_j(u_jT_1 - u_1T_j).
\]

For \( j = 2, \ldots, s \), \( r_j \) has an expansion

\[
r_j = r_{j,0} + T_1r_{j,1}
\]

with \( r_{j,0} \) and \( r_{j,1} \) in \( B'_s := A_s[T_2, \ldots, T_s] \) and \( B_s \) respectively. Set:

\[
p' := p - \sum_{j=2}^{s} r_{j,1}(u_jT_1 - u_1T_j).
\]

Then

\[
T_1p' = \sum_{j=2}^{s} r_{j,0}(u_jT_1 - u_1T_j)
\]

so that the element

\[
\sum_{j=2}^{s} r_{j,0}u_1T_j \in B'_s
\]

is divisible by \( T_1 \) in \( B_s \), whence

\[
\sum_{j=2}^{s} r_{j,0}T_j = 0.
\]

As \( T_2, \ldots, T_s \) is a regular sequence in \( B_s \), for some sequence \( r_{j,k,0}, 2 \leq j, k \leq s \) in \( B_s \) such that \( r_{j,k,0} = -r_{k,j,0} \) for all \( (j, k) \),

\[
r_{j,0} = \sum_{k=2}^{s} r_{j,k,0}T_k
\]
for $j = 2, \ldots, s$. Then

$$T_1 p' = \sum_{2 \leq j, k \leq s} r_{j,k,0} T_k (u_j T_1 - u_1 T_j) = T_1 \sum_{2 \leq j, k \leq s} r_{j,k,0} (T_k u_j - T_j u_k).$$

As a result $p'$ and $p$ are in $P_s$, whence the lemma. \hfill \square

Denote by $P_s'$ the ideal of $B_s$ generated by $P_{s-1}$ and $u_1 T_1 - u_1 T_s$. Let $\mathcal{B}$ and $\mathcal{B}_s'$ be the quotients of $B_s$ by $P_s$ and $P_s'$ respectively. The restrictions to $A_s$ of the quotient morphisms $B_s \longrightarrow \mathcal{B}_s'$ and $B_s \longrightarrow \mathcal{B}$ are embeddings. For $j = 1, \ldots, s$, denote again by $T_j$ its images in $\mathcal{B}_s'$ and $\mathcal{B}$ by these morphisms.

Lemma 4.4. Denote by $\overline{P_s}$ the image in $\mathcal{B}_s'$ of $P_s$ by the quotient morphism.

(i) The intersection of $\overline{P_s}$ and $T_1 \mathcal{B}_s'$ is equal to $[0]$.

(ii) The $\mathcal{B}_s'$-modules $T_1 \mathcal{B}_s'$ and $\mathcal{B}$ are isomorphic.

Proof. Let $a$ be in $B_s$ such that $T_1 a$ is in $P_s$. According to Lemma 4.2, $P_s$ is a prime ideal of $B_s$. Hence $a$ is in $P_s$ since $T_1$ is not in $P_s$. Moreover, for $j = 1, \ldots, s$,

$$T_1 (u_j T_k - u_k T_j) = T_s (u_j T_1 - u_1 T_j) + T_j (u_1 T_s - u_s T_1).$$

Hence $T_1 P_s$ is contained in $P_s'$. As a result, $\overline{P_s}$ is the kernel of the endomorphism $a \mapsto T_1 a$ of $\mathcal{B}_s'$ and the intersection of $\overline{P_s}$ and $T_1 \mathcal{B}_s'$ is equal to $[0]$. As $\mathcal{B}$ is the quotient of $\mathcal{B}_s'$ by $\overline{P_s}$, the endomorphism $a \mapsto T_1 a$ defines through the quotient an isomorphism

$$\mathcal{B} \longrightarrow T_1 \mathcal{B}_s$$

of $\mathcal{B}_s'$-modules. \hfill \square

Let $Q_s$ be the ideal of the polynomial algebra $A_s[T_2, \ldots, T_s]$ generated by the sequence $u_i T_k - u_k T_i$, $2 \leq i, k \leq s$ and denote by $\mathcal{B}_s^\#$ the quotient of $A_s[T_2, \ldots, T_s]$ by $Q_s$.

Lemma 4.5. (i) The quotient of the algebra $\mathcal{B}_s/T_1 \mathcal{B}_s$ by the ideal generated by $u_1$ is equal to the quotient of $\mathcal{B}_s^\#$ by the ideal generated by $u_1$.

(ii) The canonical map $A_s \longrightarrow \mathcal{B}_s/T_1 \mathcal{B}_s$ is an embedding.

(iii) The ideal of $\mathcal{B}_s/T_1 \mathcal{B}_s$ generated by $u_1$ is isomorphic to $A_s$.

Proof. Denote by $Q'_s$ the ideal of $B_s$ generated by $P_s$ and $T_1$.

(i) As the ideal of $B_s$ generated by $Q'_s$ and $u_1$ is equal to the ideal generated by $u_1$, $T_1$ and $Q_s$, $\mathcal{B}_s^\#$ is equal to the quotient of $\mathcal{B}_s/T_1 \mathcal{B}_s$ by the ideal generated by $u_1$.

(ii) Since the intersection of $A_s$ and $Q'_s$ is equal to $[0]$, the canonical map $A_s \longrightarrow \mathcal{B}_s/T_1 \mathcal{B}_s$ is an embedding.

(iii) For $k = 2, \ldots, s$, $u_1 T_k$ is in $Q'_s$. Hence $u_1 B_s$ is contained in the sum of $u_1 A_s$ and $Q'_s$. As a result, $u_1 A_s$ is equal to $u_1 \mathcal{B}_s/T_1 \mathcal{B}_s$ by (ii), whence the assertion since $A_s$ is an integral domain. \hfill \square

Proposition 4.6. Suppose that $A_s$ is Cohen-Macaulay.

(i) The algebra $\mathcal{B}_s$ is an integral domain and a Cohen-Macaulay algebra of dimension $\dim A_s + 1$.

(ii) For $a_1, \ldots, a_m$ regular sequence in $A_s$ of elements of $\mathfrak{m}$ an for $v$ prime ideal of $\mathcal{B}_s$ containing it, $a_1, \ldots, a_m$ is a regular sequence in the localization of $\mathcal{B}_s$ at $v$.

Proof. (i) Prove the assertion by induction on $s$. As $\mathcal{B}_1$ is the polynomial algebra $A_s[T_1]$, the assertion is true for $s = 1$ since $A_s$ is an integral domain and a Cohen-Macaulay algebra. Suppose the assertion true for $s - 1$. By induction hypothesis, $\mathcal{B}_{s-1}[T_1]$ is an integral domain and a Cohen-Macaulay algebra as a polynomial algebra over $\mathcal{B}_{s-1}$ and its dimension is equal to $\dim A_s + 2$. As a result, $\mathcal{B}_s'$ is Cohen-Macaulay
of dimension \( \dim A_s + 1 \) as the quotient of the integral domain and a Cohen-Macaulay algebra \( \mathcal{B}_s[T_s] \) by the ideal generated by \( T_s u_1 - T_s u_s \). As \( \mathcal{B}_s \) is the quotient of \( \mathcal{B}_s' \) by \( \mathcal{P}_s \), \( \mathcal{B}_s \) has dimension at most \( \dim A_s + 1 \).

By Lemma 4.2, \( \mathcal{B}_s \) is an integral domain so that \( \mathcal{B}_s/T_1 \mathcal{B}_s \) has dimension at most \( \dim A_s \).

By induction hypothesis again, \( \mathcal{B}_s' \) is an integral domain and a Cohen-Macaulay algebra of dimension \( \dim A_s + 1 \). Hence \( \mathcal{B}_s'/u_1 \mathcal{B}_s' \) is Cohen-Macaulay of dimension \( \dim A_s \). According to Lemma 4.5, we have a short exact sequence

\[
0 \longrightarrow A_s/T_1 \mathcal{B}_s \longrightarrow \mathcal{B}_s'/u_1 \mathcal{B}_s' \longrightarrow 0.
\]

Hence the algebra \( \mathcal{B}_s/T_1 \mathcal{B}_s \) is Cohen-Macaulay of dimension \( \dim A_s \) since \( A_s \) and \( \mathcal{B}_s'/u_1 \mathcal{B}_s' \) are Cohen-Macaulay of dimension \( \dim A_s \) and \( \mathcal{B}_s/T_1 \mathcal{B}_s \) has dimension at most \( \dim A_s \). As a result, \( \mathcal{B}_s \) has dimension \( \dim A_s + 1 \). As \( \mathcal{B}_s \) is the quotient of \( \mathcal{B}_s' \) by \( \mathcal{P}_s \), we have a short exact sequence

\[
0 \longrightarrow \mathcal{P}_s + T_1 \mathcal{B}_s' \longrightarrow \mathcal{B}_s' \longrightarrow \mathcal{B}_s/T_1 \mathcal{B}_s \longrightarrow 0.
\]

Then, setting \( M := \mathcal{P}_s + T_1 \mathcal{B}_s' \) and denoting by \( M_a \) the localization of \( M \) at a maximal ideal of \( \mathcal{B}_s' \), containing \( T_1 \),

\[
\text{Ext}^j(\mathfrak{k}, M_a) = 0
\]

for \( j \leq \dim A_s \), \( \mathcal{B}_s \) and \( \mathcal{B}_s/T_1 \mathcal{B}_s \) have dimension \( \dim A_s + 1 \) and \( \dim A_s \). By Lemma 4.4(i), \( M \) is the direct sum \( \mathcal{P}_s \) and \( T_1 \mathcal{B}_s' \). So, denoting by \( (T_1 \mathcal{B}_s)' \), the localization of \( T_1 \mathcal{B}_s' \) at a maximal ideal of \( \mathcal{B}_s' \),

\[
\text{Ext}^j(\mathfrak{k}, (T_1 \mathcal{B}_s)') = 0
\]

for \( j \leq \dim A_s \), \( (T_1 \mathcal{B}_s)' \) is the localization of \( \mathcal{B}_s' \) at this maximal ideal when it does not contain \( T_1 \). As a result, by Lemma 4.4(ii), \( \mathcal{B}_s \) is Cohen-Macaulay since it has dimension \( \dim A_s + 1 \).

(ii) Let \( q \) be a minimal prime ideal of \( \mathcal{B}_s \), containing \( a_1, \ldots, a_m \). Since \( A_s \) is embedded in \( \mathcal{B}_s \), \( q \cap A_s \) is a prime ideal of \( A_s \), containing \( a_1, \ldots, a_m \). As \( A_s \) is Cohen-Macaulay and \( a_1, \ldots, a_m \) is a regular sequence in \( A_s \), \( q \cap A_s \) has height at least \( m \) and \( A_s/q \cap A_s \) has dimension at most \( \dim A_s - m \) by [MA86, Ch. 6, Theorem 17.4]. Then \( \mathcal{B}_s/q \) has dimension at most \( \dim A_s + 1 - m \) since the fraction field of \( \mathcal{B}_s/q \) is generated by the fraction field of \( A_s/q \cap A_s \) and the image of \( T_1 \) by the quotient morphism \( B_s \longrightarrow \mathcal{B}_s/q \). As a result, by (i) and [MA86, Ch. 6, Theorem 17.4], \( q \) has height at least \( m \). As a minimal prime ideal of \( \mathcal{B}_s \) containing \( m \) elements, \( q \) has height at most \( m \). Hence all minimal prime ideal of \( \mathcal{B}_s \), containing \( a_1, \ldots, a_m \), has height \( m \). So, by (i) and [MA86, Ch. 6, Theorem 17.4], \( a_1, \ldots, a_m \) is a regular sequence in the localization of \( \mathcal{B}_s \) at \( p \).

\[\square\]

4.3. Normality and Cohen-Macaulayness property for \( X_R \). Let \( V_0 \) be a fixed point under the action of \( R \) in \( X_R \) and \( \beta_1, \ldots, \beta_d \) the elements of \( \mathcal{R}V_0 \). By Corollary 2.22(ii), \( \beta_1, \ldots, \beta_d \) is a basis of \( t' \). Let \( t_1, \ldots, t_d \) be the dual basis. Denote by \( m \) the codimension of \( V_0 \) in \( a \). According to Lie’s Theorem, for \( m > 0 \), the elements \( \gamma_1, \ldots, \gamma_m \) of \( \mathcal{R} \setminus \{ \beta_1, \ldots, \beta_d \} \) can be ordered so that

\[
a_i := V_0 \oplus a_i^{\gamma_1} \oplus \cdots \oplus a_i^{\gamma_m}
\]

is an algebra of codimension \( m - i \) of \( a \) for \( i = 1, \ldots, m \). Set:

\[
\mathcal{R}' := \mathcal{R} \setminus \{ \gamma_m \}, \quad a' := a_{m-1}, \quad t' := t_{1,a'}, \quad \pi' := \pi_{t,a'}, \quad R' := R_{t,a'}, \quad A' := A_{t,a'}, \quad E := \bigoplus_{i=1}^{m} a_i^{\gamma_i}, \quad E' := E \cap a'.
\]

Denote by \( \kappa \) the map

\[
\text{Hom}_E(V_0, E \oplus t) \xrightarrow{\kappa} \text{Gr}_d(t) \ , \quad \varphi \mapsto \text{span}(\{v + \varphi(v) \mid v \in V_0\}).
\]
Then $\kappa$ is an isomorphism from $\text{Hom}_k(V_0, E \oplus t)$ onto an affine open neighbourhood of $V_0$ in $\text{Gr}_d(r)$. Moreover, there is a short exact sequence

$$0 \longrightarrow \text{Hom}_k(V_0, \kappa x_{\gamma_m}) \longrightarrow \text{Hom}_k(V_0, E \oplus t) \overset{p} \longrightarrow \text{Hom}_k(V_0, E' \oplus t) \longrightarrow 0.$$  

Let $\Omega$ and $\Omega'$ be the inverse images by $\kappa$ of the intersections of the image of $\kappa$ with $X_R$ and $X_{R'}$ respectively. For $\varphi$ in $\Omega$ and $i = 1, \ldots, d$

$$\varphi(v_i) = \sum_{i=1}^{d} z_{i,j}(\varphi)t_j + \sum_{j=1}^{m} a_{i,j}(\varphi)x_{y_j}$$

so that the $z_{i,j}$'s, $1 \leq i, j \leq d$ and the $a_{i,j}$'s, $1 \leq i \leq d$ and $1 \leq j \leq m$ are regular functions on $\Omega$.

Let $\psi$ be the map

$$\kappa \times \Omega' \overset{\psi} \longrightarrow X_R, \quad (s, \varphi) \longmapsto \exp(\text{ad}_x y_m)\kappa(\varphi).$$

**Lemma 4.7.** Let $O$ be the subset of elements $(s, \varphi)$ of $\kappa \times \Omega'$ such that $\psi(s, \varphi)$ is in $\kappa(\Omega)$.

(i) The subset $O$ of $\kappa \times \Omega'$ is open and contains $\{0\} \times \Omega'$.

(ii) The map

$$O \overset{\varphi} \longrightarrow \Omega, \quad (s, \varphi) \longmapsto \kappa^{-1}\psi(s, \varphi)$$

is a birational morphism from $O$ to $\Omega$. In particular, the function $(s, \varphi) \mapsto s$ is in $\kappa(\Omega)$.

**Proof.** (i) As $\kappa(\Omega)$ is an open neighborhood of $V_0$ in $X_R$, $O$ is an open subset of $\kappa \times \Omega'$, containing $\{0\} \times \Omega'$ since $\psi$ is a regular map such that $\psi(0, \varphi) = \kappa(\varphi)$ for all $\varphi$ in $\Omega'$.

(ii) Let $\Omega^c$ be the subset of elements $\varphi$ of $\Omega$ such that $\kappa(\varphi)$ is in $A.t$. Then $\Omega^c$ is a dense open subset of $\Omega$. Let $O^c$ be the inverse image of $\Omega^c$ by $\overline{\psi}$. Let $(s, \varphi)$ and $(s', \varphi')$ be in $O^c$ such that $\overline{\psi}(s, \varphi) = \overline{\psi}(s', \varphi')$, that is

$$\exp(\text{ad}_x y_m)\kappa(\varphi) = \exp(\text{ad}_x y_m)\kappa(\varphi')$$

whence

$$\exp((s - s')\text{ad}_x y_m)\kappa(\varphi) = \kappa(\varphi').$$

According to the above notations, for $i = 1, \ldots, d$,

$$\varphi(v_i) = \sum_{j=1}^{d} z_{i,j}(\varphi)t_j + \sum_{j=1}^{m} a_{i,j}(\varphi)x_{y_j}.$$ 

Since $\kappa(\varphi)$ is in $A.t$,

$$\det ([z_{i,j}(\varphi), 1 \leq i, j \leq d]) \neq 0.$$ 

For $i = 1, \ldots, d$,

$$\exp((s - s')\text{ad}_x y_m)\left(\sum_{j=1}^{d} z_{i,j}(\varphi)t_j\right) = \sum_{j=1}^{d} z_{i,j}(\varphi)t_j - (s - s')(\sum_{j=1}^{d} z_{i,j}(\varphi)y_m(t_j))x_{y_m}.$$

For some $j$, $y_m(t_j) \neq 0$, whence $s = s'$ since $\kappa(\varphi')$ is contained in $r'$. As a result, the restriction of $\overline{\psi}$ to $O'$ is injective, whence the assertion since $\overline{\psi}$ is a dominant morphism.

For $i = 1, \ldots, d$ and $y$ in $t'$, denote by $u_{i,y}$ the function on $\Omega$,

$$u_{i,y} := z_{i,1}y(t_1) + \cdots + z_{i,d}y(t_d).$$

Let $\mathfrak{A}$ be the subalgebra of $k[\Omega]$ generated by the functions $z_{i,j}$'s, $1 \leq i, j \leq d$ and $a_{i,j}$'s, $1 \leq i \leq d$ and $1 \leq j \leq m - 1$. 

26
Lemma 4.8. Let $\iota$ be the restriction morphism from $\Omega$ to $\Omega'$.

(i) The restriction of $\iota$ to $\mathfrak{A}$ is an isomorphism onto $\mathbb{k}[\Omega']$.
(ii) For $1 \leq i, j \leq d$, $u_i, \gamma_m a_{j,m} - u_j, \gamma_m a_{i,m}$ is equal to 0.
(iii) For $i = 1, \ldots, d$ and $\gamma$ in $\mathfrak{t}$, if $\gamma(t_i) \neq 0$ then $u_i, \gamma$ is different from 0.

Proof. (i) For $1 \leq i, j \leq d$, denote by $z_{i,j}'$ the restriction of $z_{i,j}$ to $\Omega'$ and for $1 \leq i \leq d$ and $1 \leq j \leq m - 1$ denote by $a_{i,j}'$ the restriction of $a_{i,j}$ to $\Omega'$. Since $\mathbb{k}[\Omega']$ is generated by the functions

$$z_{i,j}', \ 1 \leq i, j \leq d \quad \text{and} \quad a_{i,j}', \ 1 \leq i \leq d, 1 \leq j \leq m - 1,$$

the restriction of $\iota$ to $\mathfrak{A}$ is surjective. Let $p$ be the kernel of the restriction of $\iota$ to $\mathfrak{A}$. It remains to prove $p = \{0\}$.

For $1 \leq i, j \leq d$ and $k = 1, \ldots, m - 1$, denote by $\overline{z}_{i,j}$ and $\overline{a}_{i,k}$ the functions on $\mathbb{k} \times \Omega'$ such that

$$\exp(\text{sad} x_{\gamma_m}(v_i) + \sum_{j=1}^{d} z_{i,j}'(\varphi)t_j + \sum_{k=1}^{m-1} a_{i,k}'(\varphi)x_{\gamma_k}) -$$

$$(\sum_{j=1}^{d} z_{i,j}(s, \varphi)t_j - \sum_{j=1}^{d} s z_{i,j}(\varphi)\gamma_m(t_j)x_{\gamma_m} + \sum_{k=1}^{m-1} a_{i,k}(s, \varphi)x_{\gamma_k}) \in V_0.$$

Then $\overline{z}_{i,j}$ and $\overline{a}_{i,k}$ are regular functions on $\mathbb{k} \times \Omega'$ as restrictions to $\mathbb{k} \times \Omega'$ of regular functions on $\mathbb{k} \times \text{Hom}(V_0, E' \oplus t)$. Let $\mathfrak{g}$ be the subalgebra of $\mathbb{k}[\Omega'][s]$ generated by the functions

$$\overline{z}_{i,j}, i, j = 1, \ldots, d \quad \text{and} \quad \overline{a}_{i,k}, i = 1, \ldots, d, k = 1, \ldots, m - 1.$$

Since $z_{i,j}'(\varphi) = z_{i,j}(0, \varphi)$ and $a_{i,k}'(\varphi) = a_{i,k}(0, \varphi)$ for all $\varphi$ in $\Omega'$, the restriction to $\mathfrak{g}$ of the quotient morphism $\mathbb{k}[\Omega'][s] \longrightarrow \mathbb{k}[\Omega']$ is surjective. As a result, $\mathfrak{g}$ has dimension $n$ or $n - 1$ since $\Omega'$ and $\mathbb{k}[\Omega'][s]$ have dimension $n - 1$ and $n$ respectively. As $\exp(\text{sad} x_{\gamma_m}(v_i))$ is not necessarily equal to $v_i$,

$$p \psi \neq (\overline{z}_{i,j}, \overline{a}_{i,k}, \ 1 \leq i \leq d, 1 \leq j \leq m - 1).$$

Moreover, $\Omega'$ is contained in $p(\Omega)$ by Lemma 4.7(i) but the inclusion may be strict.

Claim 4.9. The algebra $\mathfrak{g}$ has dimension $n - 1$.

Proof. [Proof of Claim 4.9] There are two cases to consider:

(1) for $i = 1, \ldots, m - 1 \in \mathbb{[a^{\gamma_m}, a^{\gamma_i}]}$ is contained in $V_0$,
(2) for some $i \in \{1, \ldots, m - 1\}$, $[a^{\gamma_m}, a^{\gamma_i}]$ is not contained in $V_0$.

In the first case, $\mathfrak{g} = \mathbb{k}[\Omega']$. Otherwise, denote by $j$ the biggest integer such that $[a^{\gamma_m}, a^{\gamma_j}]$ is not contained in $V_0$ and $a_{i,j}' \neq 0$ for some $i = 1, \ldots, d$. Then, for some $j'$ smaller than $j$, $\gamma_m + \gamma_j = \gamma_{j'}$. Furthermore, for $k < j$ such that $[a^{\gamma_m}, a^{\gamma_k}]$ is not contained in $V_0$, $\gamma_m + \gamma_k$ is in $\mathcal{R} \setminus \{\gamma_{j'}, \ldots, \gamma_m\}$. Then for $k \geq j'$ and $i = 1, \ldots, d$, $a_{i,k}' = a_{i,k}$ and for all $(s, \varphi)$ in $\mathbb{k} \times \Omega'$,

$$\overline{a}_{i,j}(s, \varphi) = a_{i,j}'(\varphi) + sa_{i,j}'(\varphi).$$

As a result, by induction on $m - k$, for $i = 1, \ldots, d$,

$$a_{i,k}' - a_{i,k} \in s\mathfrak{g}[s].$$

Hence $\mathbb{k}[\Omega'][s] = \mathfrak{g}[s]$ and there exists a surjective morphism $\mathbb{k}[\Omega'] \longrightarrow \mathfrak{g}$ so that $\mathfrak{g}$ has dimension $n - 1$. \(\square\)
According to Lemma 4.7(ii), the comorphism of $\overline{\psi}$ is an embedding of $\mathbb{k}[\Omega]$ into $\mathbb{k}[\mathfrak{u}]$ and from this embedding results an isomorphism from $\mathbb{k}(\Omega)$ onto $\mathbb{k}(\Omega')(s)$. Moreover, $\mathbb{A}$ is the image of $\mathbb{A}$ by this embedding so that $\mathbb{A}$ has dimension $n - 1$. As a result, $v = 0$ since $i$ is surjective and $\Omega'$ has dimension $n - 1$.

(ii) Let $\phi$ be in $\Omega$. Since $\kappa(\phi)$ is a commutative algebra, for $1 \leq i, j \leq d$,

$$0 = [v_i + \phi(v_i), v_j + \phi(v_j)] = [\phi(v_i), v_j] + [\phi(v_i), \phi(v_j)].$$

The component on $x_{\gamma_m}$ of the right hand side is

$$\sum_{k=1}^{d} (z_{i,k}a_{j,m}(\phi) - z_{j,k}a_{i,m}(\phi))[t_k, x_{\gamma_m}] = (u_{i,\gamma_m}a_{j,m} - u_{j,\gamma_m}a_{i,m})(\phi)x_{\gamma_m},$$

whence the assertion.

(iii) Denote by $R_0$ the adjoint group of $\mathfrak{t}_0 := t + V_0$ and $X_{R_0}$ the closure in $\text{Gr}_d(\mathfrak{t}_0)$ of $R_0 \cdot t$. Let $\Omega_0$ be the inverse image of $X_{R_0}$ by $\kappa$. According to Lemma 4.1, for $i, j = 1, \ldots, d$, the restriction to $\Omega_0$ of $z_{i,j}$ is equal to $0$ if $j \neq i$, otherwise it is different from $0$. As a result, for $i = 1, \ldots, d$ and $\gamma$ in $t^*$, the restriction of $u_{i,\gamma}$ to $\Omega_0$ is equal to $\overline{z_{i,\gamma}}t_\gamma$ with $\overline{z_{i,\gamma}}$ the restriction of $z_{i,\gamma}$ to $\Omega_0$, whence the assertion. \hfill $\Box$

For $\gamma$ in $t^*$, set:

$$I_\gamma := \{j \in \{1, \ldots, d\} \mid \gamma(t_j) \neq 0\}.$$

**Proposition 4.10.** Denote by $\mathbb{k}[[\Omega]]_0$ the localization of $\mathbb{k}[[\Omega]]$ at $0$.

(i) The local algebra $\mathbb{k}[[\Omega]]_0$ is Cohen-Macaulay.

(ii) For $\gamma$ in $t^*$, $u_{i,\gamma}, i \in I_\gamma$ is a regular sequence in $\mathbb{k}[[\Omega]]_0$ of elements of its maximal ideal.

**Proof.** Prove the proposition by induction on $m$. By Lemma 4.1, for $m = 0$, $\mathbb{k}[[\Omega]]$ is a polynomial algebra of dimension $d$, generated by $z_{1,0}, \ldots, z_{d,0}$. Moreover, for $i = 1, \ldots, d$ and $\gamma$ in $t^*$, $u_{i,\gamma} = z_{i,\gamma}t_\gamma$, whence the proposition for $m = 0$. Suppose $m > 0$ and the proposition true for $m - 1$ and use the notations of Lemma 4.8.

According to Lemma 4.8(i) and the induction hypothesis, the localization $\mathbb{A}_s$ of $\mathbb{A}$ at $0$ is Cohen-Macaulay and for $\gamma$ in $t^*$, $u_{i,\gamma}, i \in I_\gamma$ is a regular sequence in $\mathbb{A}_s$ of elements of its maximal ideal. Denote by $\mathcal{B}$ the polynomial algebra $\mathbb{A}[T_i, i \in I_{\gamma_m}]$ and by $P$ the ideal of $\mathcal{B}$ generated by the sequence $u_{i,\gamma_m}T_j - u_{j,\gamma_m}T_i$, $(i, j) \in I_{\gamma_m}^2$. According to Condition (3) of Section 2, $s := |I_{\gamma_m}| \geq 2$. By Lemma 4.8(ii), $\mathbb{k}[[\Omega]]_0$ is a quotient of the localization of $\mathcal{B}/P$ and by Lemma 4.2, $P$ is a prime ideal of $\mathcal{B}$. By Proposition 4.6(i), $\mathcal{B}/P$ is an integral domain and a Cohen-Macaulay algebra of dimension $n$ since $\mathbb{k}[[\Omega']]$ has dimension $n - 1$. Hence $\mathbb{k}[[\Omega]]_0$ is the localization of $\mathcal{B}/P$ at $0$ since $\mathbb{k}[[\Omega]]_0$ is an integral domain of dimension $n$. As a result, $\mathbb{k}[[\Omega]]_0$ is Cohen-Macaulay and by Proposition 4.6(ii), for $\gamma$ in $t^*$, the sequence $u_{i,\gamma}, i \in I_\gamma$ is regular in $\mathbb{k}[[\Omega]]_0$. \hfill $\Box$

**Theorem 4.11.** The variety $X_R$ is normal and Cohen-Macaulay.

**Proof.** By Corollary 3.8, $X_R$ is smooth in codimension 1. So, by Serre’s normality criterion [Bou98, §1, no 10, Théorème 4], it suffices to prove that $X_R$ is Cohen-Macaulay. According to [MA86, Ch. 8, Theorem 24.5], the set of points $x$ of $X_R$ such that $\mathcal{O}_{X_R,x}$ is Cohen-Macaulay, is open. For $x$ in $X_R$, the closure in $X_R$ of $R.x$ contains a fixed point. So it suffices to prove that for $x$ a fixed point under the action of $R$ in $X_R$, $\mathcal{O}_{X_R,x}$ is Cohen-Macaulay. Let $V_0$ and $\Omega$ be as in Lemma 4.7. Then $\Omega$ is an affine open neighborhood of $V_0$ in $X_R$. By Proposition 4.10(i), $\mathcal{O}_{\Omega,0}$ is Cohen-Macaulay, whence the theorem since $\kappa$ is an isomorphism from $\Omega$ onto an open neighborhood of $V_0$ in $X_R$ and $\kappa(0) = V_0$. \hfill $\Box$
4.4. **Nilpotent cone and regular sequence in** $\mathcal{O}_{\mathcal{E}}$. Let $\beta_1, \ldots, \beta_d$ be a basis of $t^*$. For $i = 1, \ldots, d$, denote again by $\beta_i$ the element of $t^*$ extending $\beta_i$ and equal to 0 on $a$. For $\Lambda$ a complete subset of $\mathcal{R}$, denote by $t^*_\Lambda$ a complement to $t_\Lambda$ in $t$ and set

$$R^*_\Lambda := R^*_{t^*_\Lambda, a_\Lambda} \quad \text{and} \quad E_\Lambda := E_{t^*_\Lambda, a_\Lambda}.$$ 

For $Y$ closed subset of $X_{R^*_\Lambda}$, denote by $E_{\Lambda, Y}$ the restriction of $E_\Lambda$ to $Y$. Let $N^*_\Lambda$ be the image of the map

$$E_{\Lambda, X_{R^*_\Lambda}} \xrightarrow{(V, x) \mapsto (V \oplus t_\Lambda, x)} E,$$

and $N_\Lambda$ the closure in $E$ of $R N^*_\Lambda$.

**Lemma 4.12.** For $i = 1, \ldots, d$, let $\tilde{\beta}_i$ be the function on $E$ defined by $\tilde{\beta}_i(V, x) = \beta_i(x)$. Denote by $N$ the nullvariety of $\tilde{\beta}_1, \ldots, \tilde{\beta}_d$ in $E$.

(i) For all complete subset $\Lambda$ of $\mathcal{R}$, $N_\Lambda$ is a subvariety of $N$ of dimension at most $n$.

(ii) The variety $N$ is the union of $N_\Lambda$, $\Lambda \in \mathcal{P}_i(\mathcal{R})$.

(iii) The variety $N$ is equidimensional of dimension $n$.

**Proof.** (i) Since $a$ is the nullvariety of $\beta_1, \ldots, \beta_d$ in $r$, $N$ is the intersection of $E$ and $X_R \times a$. By definition $N^*_\Lambda$ is contained in $X_R \times a$. Hence $N_\Lambda$ is contained in $N$. By Proposition 3.7,

$$\dim N^*_\Lambda = \dim t^*_\Lambda + \dim X_{R^*_\Lambda, a} \leqslant \dim a_\Lambda.$$ 

Since the image of $X_{R^*_\Lambda, a}$ by the map $V \mapsto V \oplus t_\Lambda$ is invariant by $R_\Lambda$,

$$\dim N_\Lambda \leqslant \dim N^*_\Lambda + \dim a_\Lambda \leqslant \dim a.$$ 

(ii) Let $\sigma_1$ be the bundle projection of the vector bundle $E$ over $X_R$ and $\tau_1$ the restriction to $E$ of the projection $X_R \times r \xrightarrow{\alpha} r$. Let $T$ be an irreducible component of $N$. For all $V$ in $\sigma_1(T)$, $\tau_1(\sigma^{-1}_1(V) \cap T)$ is a closed cone of $a$. Hence $\sigma_1(T) \times \{0\}$ is the intersection of $T$ and $X_R \times \{0\}$ so that $\sigma_1(T)$ is a closed subset of $X_R$. Since $N$ is the intersection of $E$ and $X_R \times a$, $N$ and its irreducible components are invariant under $R$. As a result, $\sigma_1(T)$ is invariant under $R$ and by Proposition 3.1, for some complete subset $\Lambda$ of $\mathcal{R}$ and for some closed subset of $Z_\Lambda$ of $X_{R_\Lambda}$, $\sigma_1(T) = \overline{RT_\Lambda}$. Moreover, by Lemma 3.2, for some closed subset $Z^*_\Lambda$ of $X_{R^*_\Lambda, a}$, $Z_\Lambda$ is the image of $Z^*_\Lambda$ by the map $V \mapsto V \oplus t_\Lambda$. As a result,

$$E_{\Lambda, Z^*_\Lambda} \subset E_{\Lambda, X_{R^*_\Lambda}} \quad \text{and} \quad \sigma^{-1}_1(Z_\Lambda) \cap X_R \times a \subset N^*_\Lambda.$$ 

Then $T$ is contained in $N_\Lambda$, whence the assertion by (i).

(iii) By (i) and (ii), since $\mathcal{R}$ is finite, the irreducible components of $N$ have dimension at most $n$. As the nullvariety of $d$ functions on the irreducible variety $E_{X_R}$, the irreducible components of $N$ have dimension at least $n$, whence the assertion. \hfill \Box

For $x$ in $E$, denote by $I_x$ the subset of elements $i$ of $\{1, \ldots, d\}$ such that $\tilde{\beta}_i(x) = 0$.

**Corollary 4.13.** For all $x$ in $E$, the sequence $\tilde{\beta}_i, i \in I_x$ is regular in $\mathcal{O}_{\mathcal{E}, x}$.

**Proof.** According to Lemma 4.12, for all subset $I$ of $\{1, \ldots, d\}$, the nullvariety of $\tilde{\beta}_i, i \in I$ in $E$ is equidimensional of dimension $n + d - |I|$. By Theorem 4.11 and Lemma B.1(iii), $E$ is Cohen-Macaulay as a vector bundle over a Cohen-Macaulay variety, whence the corollary by [MA86, Ch. 6, Theorem 17.4]. \hfill \Box
5. Rational singularities for solvable Lie algebras

Let $t$ be a vector space of positive dimension $d$. Denote by $\mathcal{C}_{t,+}$ the full subcategory of $\mathcal{C}_t$ whose objects satisfy the following condition:

(4) there exist regular maps $e_1, \ldots, e_d$ from $\mathfrak{v}_{t,a}$ to $\mathfrak{v}_{t,a}$ such that $e_1(x), \ldots, e_d(x)$ is a basis of $\mathfrak{v}_{t,a}^F$ for all $x$ in $\mathfrak{v}_{t,a}$.

According to [Ko63, Theorem 9], $u$ is in $\mathcal{C}_{b,+}$.

Lemma 5.1. Let $a$ be in $\mathcal{C}_{t,+}$ and $a'$ an ideal of $t + a$, contained in $a$ and containing a fixed point under the action of $R_{t,a}$ in $X_{R_{t,a}}$. Then $a'$ is in $\mathcal{C}_{t,+}$.

Proof: Set $r := t + a$ and $r' := t + a'$. According to Corollary 2.22(ii), $a'$ is in $\mathcal{C}_t$ since it is in $\mathcal{C}_{t,+}$. Set $t_{\text{reg}} := t_{\text{reg}} \cap t$. As $\mathcal{R}_{t,a'}$ is contained in $\mathcal{R}_{t,a}$, $t_{\text{reg}}$ is contained in $t_{\text{reg}}'$ by Lemma 3.3(i). Then $t_{\text{reg}}'$ is contained in $t_{\text{reg}}$ and for all $x$ in $A_{t,a'}, t_{\text{reg}}$, $r^x = r'^x$ since $A_{t,a'}, t_{\text{reg}}$ is a dense open subset of $r'$ by Lemma 3.3(i). So, for all regular maps $e$ from $t$ to $t$ such that $[x, e(x)] = 0$ for all $x$ in $t$, $e(x)$ is in $t'$ for all $x$ in $t'$, whence the lemma.

Let $a$ be in $\mathcal{C}_{t,+}$. Set:

$$\mathcal{R} := \mathcal{R}_{t,a}, \quad r := \mathfrak{v}_{t,a}, \quad \pi := \mathfrak{v}_{t,a}, \quad R := R_{t,a}, \quad A := A_{t,a}, \quad \mathcal{E} := \mathcal{E}_{t,a}, \quad n := \dim a.$$ 

The goal of the section is to prove that $X_R$ is Gorenstein with rational singularities.

For $k$ positive integer, set:

$$\mathcal{E}^{(k)} := \{(u, x_1, \ldots, x_k) \in X_R \times r^k \mid u \ni x_1, \ldots, u \ni x_k\}$$

and denote by $\chi_{R,k}$ the image of $\mathcal{E}^{(k)}$ by the projection

$$(u, x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k).$$

Since $X_R$ is a projective variety, $\chi_{R,k}$ is a closed subset of $r^k$, invariant under the diagonal action of $R$ in $r^k$.

5.1. Differential forms on some smooth open subsets of $\chi_{R,k}$. For $j = 1, \ldots, k$, let $V_j^{(k)}$ be the subset of elements of $\chi_{R,k}$ whose $j$-th component is in $t_{\text{reg}}$.

Lemma 5.2. For $j = 1, \ldots, k$, $V_j^{(k)}$ is a smooth open subset of $\chi_{R,k}$. Moreover, $\Omega_{V_j^{(k)}}$ has a global section without zero.

Proof: Denoting by $\sigma_j$ the automorphism of $r_k$ which permutes the first and the $j$-th component, $\chi_{R,k}$ is invariant under $\sigma_j$ and $\sigma_j(V_j^{(k)}) = V_j^{(k)}$ so that we can suppose $j = 1$. Moreover, for $k = 1$, $\chi_{R,k} = r$ so that we can suppose $k \geq 2$. By definition, $V_1^{(k)}$ is the intersection of $t_{\text{reg}} \times r^{k-1}$ and $\chi_{R,k}$. Hence $V_1^{(k)}$ is an open subset of $\chi_{R,k}$ since $t_{\text{reg}}$ is an open subset of $r$.

Let $e_1, \ldots, e_d$ satisfying Condition (4) with respect to $r$. Let $\theta$ be the map

$$t_{\text{reg}} \times M_{k-1,d}(k) \overset{\theta}{\longrightarrow} r^k, \quad (x, a_{i,j}, 2 \leq i \leq k, 1 \leq j \leq d) \mapsto (x, \sum_{j=1}^d a_{i,j}e_j(x)).$$

Since for all $(x, x_2, \ldots, x_k)$ in $V_1^{(k)}$, $x_2, \ldots, x_k$ are in $r^k$, $\theta$ is a bijective map onto $V_1^{(k)}$. The open subset $t_{\text{reg}}$ has a cover by open subsets $V$ such that for some $e_1, \ldots, e_n$ in $r$, $e_1(x), \ldots, e_d(x)$, $e_1, \ldots, e_n$ is a basis of $r$ for all $x$ in $V$. Then there exist regular functions $\varphi_1, \ldots, \varphi_d$ on $V \times r$ such that

$$v - \sum_{j=1}^d \varphi_j(x,v)e_j(x) \in \text{span}\{e_1, \ldots, e_n\}$$

30
for all \((x,v)\) in \(V \times \mathcal{r}\), so that the restriction of \(\theta\) to \(V \times M_{k-1,d}(\mathcal{k})\) is an isomorphism onto \(X_{R,k} \cap V \times t^{k-1}\) whose inverse is
\[
(x_1, \ldots, x_k) \mapsto (x_1, ((\varphi_1(x_1, x_i), \ldots, \varphi_d(x_1, x_i)), i = 2, \ldots, k))
\]
As a result, \(\theta\) is an isomorphism and \(V_1^{(k)}\) is a smooth variety. Since \(r_{\text{reg}}\) is a smooth open subset of the vector space \(\mathcal{r}\), there exists a regular differential form \(\omega\) of top degree on \(r_{\text{reg}} \times M_{k-1,\ell}(\mathcal{k})\), without zero. Then \(\theta_* (\omega)\) is a regular differential form of top degree on \(V_1^{(k)}\), without zero. \(\square\)

For \(k \geq 2\) set:
\[
V^{(k)} := V_1^{(k)} \cup V_2^{(k)} \quad \text{and} \quad V_{1,2}^{(k)} := V_1^{(k)} \cap V_2^{(k)}.
\]
For \(2 \leq k' \leq k\), the projection
\[
t^k \longrightarrow t^{k'}, \quad (x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_{k'})
\]
duces the projection
\[
X_{R,k} \longrightarrow X_{R,k'}, \quad V_j^{(k)} \longrightarrow V_j^{(k')}
\]
for \(j = 1, \ldots, k'\).

**Lemma 5.3.** Suppose \(k \geq 2\). Let \(\omega\) be a regular differential form of top degree on \(V_1^{(k)}\) without zero. Denote by \(\omega'\) its restriction to \(V_{1,2}^{(k)}\).

(i) For \(\varphi\) in \(\mathcal{k}[V_1^{(k)}]\), if \(\varphi\) has no zero then \(\varphi\) is in \(\mathcal{k}^*\).

(ii) For some invertible element \(\psi\) of \(\mathcal{k}[V_{1,2}^{(2)}]\), \(\omega' = \psi \sigma_{2,*} (\omega')\).

(iii) The function \(\psi(\psi \circ \sigma_2)\) on \(V_{1,2}^{(k)}\) is equal to 1.

**Proof.** The existence of \(\omega\) results from Lemma 5.2.

(i) According to Lemma 5.2, there is an isomorphism \(\theta\) from \(r_{\text{reg}} \times M_{k-1,d}(\mathcal{k})\) onto \(V_1^{(k)}\). Since \(\varphi\) is invertible, \(\varphi_* \theta\) is an invertible element of \(\mathcal{k}[r_{\text{reg}}]\). According to Lemma 3.3(iii), \(\mathcal{k}[r_{\text{reg}}] = \mathcal{k}[\mathcal{r}]\). Hence \(\varphi\) is in \(\mathcal{k}^*\).

(ii) The open subset \(V_{1,2}^{(k)}\) is invariant under \(\sigma_2\) so that \(\omega'\) and \(\omega_{2,*}(\omega')\) are regular differential forms of top degree on \(V_{1,2}^{(k)}\), without zero. Then for some invertible element \(\psi\) of \(\mathcal{k}[V_{1,2}^{(2)}]\), \(\omega' = \psi \sigma_{2,*} (\omega')\). Let \(O_2\) be the set of elements \((x, a_i, e_j, 1 \leq i \leq k-1, 1 \leq j \leq d)\) of \(r_{\text{reg}} \times M_{k-1,d}(\mathcal{k})\) such that
\[
a_{1,1} e_1(x) + \cdots + a_{1,\ell} e_\ell(x) \in r_{\text{reg}}.
\]
Then \(O_2\) is the inverse image of \(V_1^{(k)}\) by \(\theta\). As a result, \(\mathcal{k}[V_{1,2}^{(k)}]\) is a polynomial algebra over \(\mathcal{k}[V_{1,2}^{(2)}]\) since for \(k = 2\), \(O_2\) is the inverse image by \(\theta\) of \(V_{1,2}^{(2)}\). Hence \(\psi\) is in \(\mathcal{k}[V_{1,2}^{(2)}]\) since \(\psi\) is invertible.

(iii) Since the restriction of \(\sigma_2\) to \(V_{1,2}^{(k)}\) is an involution,
\[
\sigma_{2,*} (\omega') = (\psi \circ \sigma_2) \omega' = (\psi \circ \sigma_2) \psi \sigma_{2,*} (\omega'),
\]
whence \((\psi \circ \sigma_2) \psi = 1\). \(\square\)

**Corollary 5.4.** The function \(\psi\) is invariant under the action of \(R\) in \(V_{1,2}^{(k)}\) and for some sequence \(m_\alpha, \alpha \in \mathbb{R}\) in \(\mathcal{Z}\),
\[
\psi(x_1, \ldots, x_k) = \pm \prod_{\alpha \in \mathbb{R}} (a(x_1)a(x_2)^{-1})^{m_\alpha},
\]
for all \((x_1, \ldots, x_k)\) in \(r_{\text{reg}}^2 \times t^{2-2}\).
Proof. First of all, since $V^{(k)}_1$ and $V^{(k)}_2$ are invariant under the action of $R$ in $\mathfrak{X}_{R,k}$, so is $V^{(k)}_{1,2}$. Let $g$ be in $R$. As $\omega$ has no zero, $g.\omega = p_g.\omega$ for some invertible element $p_g$ of $\mathbb{k}[V^{(k)}_1]$. By Lemma 5.3(i), $p_g$ is in $\mathbb{k}^*$. Since $\sigma_2$ is a $R$-equivariant isomorphism from $V^{(k)}_1$ onto $V^{(k)}_2$,

$$g.\sigma_2.(\omega) = p_g.\sigma_2.(\omega) \quad \text{and} \quad p_g.\omega' = g.\omega' = (g.\psi)g.\sigma_2.(\omega') = p_g(g.\psi)\sigma_2.(\omega'),$$

whence $g.\psi = \psi$.

The open subset $t^2_\text{reg}$ of $t^2$ is the complement to the nullvariety of the function

$$(x, y) \mapsto \prod_{\alpha \in \mathbb{R}} \alpha(x)\alpha(y).$$

Then, by Lemma 5.3(ii), for some $a$ in $\mathbb{k}^*$ and for some sequences $m_\alpha$, $\alpha \in \mathbb{R}$ and $n_\alpha$, $\alpha \in \mathbb{R}$ in $\mathbb{Z}$,

$$\psi(x_1, \ldots, x_k) = a \prod_{\alpha \in \mathbb{R}} (\alpha(x_1))^{m_\alpha} \alpha(x_2)^{n_\alpha},$$

for all $(x_1, \ldots, x_k)$ in $t^2_\text{reg} \times t^{k-2}$. By Lemma 5.3(iii),

$$a^2 \prod_{\alpha \in \mathbb{R}} (\alpha(x))^{m_\alpha+n_\alpha} \alpha(y)^{m_\alpha+n_\alpha} = 1,$$

for all $(x, y)$ in $t^2_\text{reg}$. Hence $a^2 = 1$ and $m_\alpha + n_\alpha = 0$ for all $\alpha$ in $\mathbb{R}$. \hfill \Box

According to Lemma 3.5(i), for $\alpha$ in $R$, $\theta_\alpha$ is a bijective regular map from $\mathbb{P}^1(\mathbb{k})$ onto the closed subset $Z_\alpha$ of $X_R$ such that $\theta_\alpha(\infty) = V_\alpha$. Recall that $x_\alpha$ is a generator of $a^\alpha$ and $h_\alpha$ is an element of $t$ such that $\alpha(h_\alpha) = 1$. Denote by $t'_\alpha$ the subset of elements $x$ of $t_\alpha$ such that $\gamma(x) \neq 0$ for all $\gamma$ in $R \setminus \{\alpha\}$. According to Condition (3) of Section 2, $t'_\alpha$ is a dense open subset of $t_\alpha$. Let $x_{-\alpha}$ be in $r^*$ orthogonal to $t + a^\alpha$ for all $\gamma$ in $R \setminus \{\alpha\}$ and such that $x_{-\alpha}(x_\alpha) = 1$.

**Lemma 5.5.** Suppose $k \geq 2$. Let $\alpha$ be in $R$, $x_0$ and $y_0$ in $t'_\alpha$. Set:

$$E := \mathbb{k}x_0 \oplus \mathbb{k}h_\alpha \oplus a^\alpha, \quad E_+ := x_0 \oplus \mathbb{k}h_\alpha \oplus a^\alpha, \quad E_{+1} := x_0 \oplus \mathbb{k}h_\alpha \oplus (a^\alpha \setminus \{0\}), \quad E_{+2} := y_0 \oplus \mathbb{k}h_\alpha \oplus (a^\alpha \setminus \{0\}).$$

(i) For $x$ in $E_{+1}$, $r^x$ is contained in $t_\alpha + E$.

(ii) For $V$ subspace of dimension $d$ of $t_\alpha + E$, $V$ is in $X_R$ if and only if it is in $Z_\alpha$.

(iii) The intersection of $E_{+1} \times E_{+2}$ and $\mathfrak{X}_{R,2}$ is the nullvariety of the function

$$(x, y) \mapsto x_{-\alpha}(y)\alpha(x) - x_{-\alpha}(x)\alpha(y)$$

on $E_{+1} \times E_{+2}$.

**Proof.** (i) If $x$ is regular semisimple, its component on $h_\alpha$ is different from 0 so that $r^x = \theta_\alpha(z)$ for some $z$ in $\mathbb{k}$. Suppose that $x$ is not regular semisimple. Then $x$ is in $x_0 + a^\alpha$. Hence $r^x$ is contained in $t_\alpha + E$ since so is $r^{x_0}$.

(ii) All element of $Z_\alpha$ is contained in $t_\alpha + E$. Let $V$ be an element of $X_R$, contained in $t_\alpha + E$. According to Corollary 2.22(i), $V$ is an algebraic commutative subalgebra of dimension $d$ of $r$. By (i), $V = \theta_\alpha(z)$ for some $z$ in $\mathbb{k}$ if $V$ is in $\mathfrak{A}$. Otherwise, $x_\alpha$ is in $V$. Then $V = \theta_\alpha(\infty)$ since $\theta_\alpha(\infty)$ is the centralizer of $x_\alpha$ in $t_\alpha + E$.

(iii) Let $(x, y)$ be in $E_{+1} \times E_{+2} \cap \mathfrak{X}_{R,2}$. By definition, for some $V$ in $X_R$, $x$ and $y$ are in $V$. By (i) and (ii), $V = \theta_\alpha(z)$ for some $z$ in $\mathbb{P}^1(\mathbb{k})$. For $z$ in $\mathbb{k}$,

$$x = x_0 + s(h_\alpha - zx_\alpha) \quad \text{and} \quad y = y_0 + s'(h_\alpha - zx_\alpha)$$

for some $s$, $s'$ in $\mathbb{k}$, whence the equality of the assertion. For $z = \infty$,

$$x = x_0 + sx_\alpha \quad \text{and} \quad y = y_0 + s'x_\alpha.$$
for some \( s, s' \) in \( k \) so that \( \alpha(x) = \alpha(y) = 0 \). Conversely, let \( (x, y) \) be in \( E_{s,1} \times E_{s,2} \) such that
\[
x_{-\alpha}(y)\alpha(x) - x_{-\alpha}(x)\alpha(y) = 0.
\]
If \( \alpha(x) = 0 \) then \( \alpha(y) = 0 \) and \( x \) and \( y \) are in \( V_0 = \theta_\alpha(\infty) \). If \( \alpha(x) \neq 0 \), then \( \alpha(y) \neq 0 \) and
\[
x \in \theta_\alpha\left(-\frac{x_{-\alpha}(x)}{\alpha(x)}\right) \quad \text{and} \quad y \in \theta_\alpha\left(-\frac{x_{-\alpha}(x)}{\alpha(x)}\right),
\]
whence the assertion. \( \square \)

Set \( V^{(1)} := r_{\text{reg}} \).

**Proposition 5.6.** For \( k \) positive integer, there exists on \( V^{(k)} \) a regular differential form of top degree without zero.

**Proof.** For \( k = 1 \), it is true since \( r_{\text{reg}} \) is an open subset of the vector space \( r \). So we can suppose \( k \geq 2 \). According to Corollary 5.4, it suffices to prove \( m_\alpha = 0 \) for all \( \alpha \) in \( R \). Indeed, if so, by Corollary 5.4, \( \psi = \pm 1 \) on the open subset \( R.(r_{\text{reg}} \times k_{k-2}) \) of \( V^{(k)} \), so that \( \psi = \pm 1 \) on \( V^{(k)}_{1,2} \). Then, by Lemma 5.3(ii), \( \omega \) and \( \pm \sigma_{2,\gamma}(\omega) \) have the same restriction to \( V^{(k)}_{1,2} \), so that there exists a regular differential form of top degree \( \check{\omega} \) on \( V^{(k)} \) whose restrictions to \( V^{(k)}_1 \) and \( V^{(k)}_2 \) are \( \omega \) and \( \pm \sigma_{2,\gamma}(\omega) \) respectively. Moreover, \( \check{\omega} \) has no zero since so has \( \omega \).

Since \( \psi \) is in \( k[V^{(2)}_{1,2}] \) by Lemma 5.3(ii), we can suppose \( k = 2 \). Let \( \alpha \) be in \( R, E, E_{s,1}, E_{s,2} \) as in Lemma 5.3. Suppose \( m_\alpha \neq 0 \). A contradiction is expected. The restriction of \( \psi \) to \( E_{s,1} \times E_{s,2} \cap V^{(2)}_{1,2} \) is given by
\[
\psi(x, y) = ax_{-\alpha}(x)^{m_\alpha x_{-\alpha}(y)^n},
\]
with \( a \) in \( k^* \) and \( (m, n) \) in \( \mathbb{Z}^2 \) since \( \psi \) is an invertible element of \( k[V^{(2)}_{1,2}] \). According to Lemma 5.3(iii), \( n = -m \) and \( a = \pm 1 \). Interchanging the role of \( x \) and \( y \), we can suppose \( m \) in \( \mathbb{N} \). For \( (x, y) \) in \( E_{s,1} \times E_{s,2} \cap V^{(2)}_{1,2} \) such that \( \alpha(x) \neq 0, \alpha(y) \neq 0 \) and
\[
\pm \alpha(x)^m \alpha(y)^{-m} = \pm \prod_{y \in \mathcal{R}} \gamma(x)^{m_\gamma} \gamma(y)^{-m_\gamma}.
\]
As a result, by Corollary 5.4, for \( x \) in \( x_0 + k^* h_{\alpha} \) and \( y \) in \( y_0 + k^* h_{\alpha} \),
\[
(1) \quad \pm \alpha(x)^m \alpha(y)^{-m} = \pm \prod_{y \in \mathcal{R}} \gamma(x)^{m_\gamma} \gamma(y)^{-m_\gamma}.
\]
For \( \gamma \) in \( \mathcal{R} \),
\[
\gamma(x) = \gamma(x_0) + \alpha(x)\gamma(h_{\alpha}) \quad \text{and} \quad \gamma(y) = \gamma(y_0) + \alpha(y)\gamma(h_{\alpha}).
\]
Since \( m \) is in \( \mathbb{N} \),
\[
(2) \quad \pm \alpha(x)^m \prod_{y \in \mathcal{R}} (\gamma(y_0) + \alpha(y)\gamma(h_{\alpha}))^{m_\gamma} \prod_{y \in \mathcal{R}} (\gamma(x_0) + \alpha(x)\gamma(h_{\alpha}))^{-m_\gamma} = \pm \alpha(y)^m \prod_{y \in \mathcal{R}} (\gamma(y_0) + \alpha(y)\gamma(h_{\alpha}))^{m_\gamma} \prod_{y \in \mathcal{R}} (\gamma(x_0) + \alpha(y)\gamma(h_{\alpha}))^{-m_\gamma}.
\]
For \( m_\alpha \) positive, the terms of lowest degree in \( (\alpha(x), \alpha(y)) \) of left and right sides are
\[
\pm \alpha(x)^m \alpha(y)^{-m} \prod_{y \in \mathcal{R}(\alpha)} \gamma(y_0)^{m_\gamma} \prod_{y \in \mathcal{R}(\alpha)} \gamma(x_0)^{-m_\gamma} \quad \text{and} \quad \pm \alpha(y)^m \alpha(x)^{-m} \prod_{y \in \mathcal{R}(\alpha)} \gamma(x_0)^{m_\gamma} \prod_{y \in \mathcal{R}(\alpha)} \gamma(y_0)^{-m_\gamma}
\]
respectively and for $m_\alpha$ negative, the terms of lowest degree in $(\alpha(x), \alpha(y))$ of left and right sides are

\[ \pm \alpha(x)^{m + m_\alpha} \prod_{y \in \mathcal{R} \setminus \{a\}, m_y > 0} \gamma(y_0)^{m_y} \prod_{y \in \mathcal{R} \setminus \{a\}, m_y < 0} \gamma(y_0)^{-m_y} \quad \text{and} \quad \pm \alpha(y)^{m + m_\alpha} \prod_{y \in \mathcal{R} \setminus \{a\}, m_y > 0} \gamma(x_0)^{m_y} \prod_{y \in \mathcal{R} \setminus \{a\}, m_y < 0} \gamma(x_0)^{-m_y} \]

respectively. From the equality of these terms, we deduce $m = \pm m_\alpha$ and

\[ \prod_{y \in \mathcal{R} \setminus \{a\}, m_y > 0} \gamma(y_0)^{m_y} \prod_{y \in \mathcal{R} \setminus \{a\}, m_y < 0} \gamma(x_0)^{-m_y} = \pm \prod_{y \in \mathcal{R} \setminus \{a\}, m_y > 0} \gamma(x_0)^{m_y} \prod_{y \in \mathcal{R} \setminus \{a\}, m_y < 0} \gamma(y_0)^{-m_y}. \]

Since the last equality does not depend on the choice of $x_0$ and $y_0$ in $t'_a$, this equality remains true for all $(x_0, y_0)$ in $t' \times t'$. As a result, as the degrees in $\alpha(x)$ of the left and right sides of Equality (2) are the same,

\[ m = \sum_{y \in X, m_y < 0} m_y = \sum_{y \in X, m_y > 0} m_y. \]

Suppose $m = m_\alpha$. By Equality (1),

\[ \prod_{y \in \mathcal{R} \setminus \{a\}} \gamma(x)^{m_y} \gamma(y)^{-m_y} = \pm 1. \]

Since this equality does not depend on the choice of $x_0$ and $y_0$ in $t'_a$, it holds for all $(x, y)$ in $t_{reg} \times t_{reg}$. Hence $m_y = 0$ for all $y$ in $\mathcal{R} \setminus \{a\}$ and $m = 0$ by Equality (3). It is impossible since $m_\alpha \neq 0$. Hence $m = -m_\alpha$. Then, by Equality (1)

\[ \prod_{y \in \mathcal{R} \setminus \{a\}} \gamma(x)^{m_y} \gamma(y)^{-m_y} = \pm \alpha(x)^{2m} \alpha(y)^{-2m}. \]

Since this equality does not depend on the choice of $x_0$ and $y_0$ in $t'_a$, it holds for all $(x, y)$ in $t_{reg} \times t_{reg}$. Then $m = 0$, whence the contradiction.

5.2. Rational singularities and Gorensteinness of $X_R$. For $Y$ subvariety of $Gr_d(t)$, denote by $E_Y$ the restriction to $Y$ of the tautological vector bundle of rank $d$ over $Gr_d(t)$. In particular, for $Y$ contained in $X_R$, $E_Y$ is a subvariety of $E$. For $k$ positive integer, denote by $\tau_k$ and $\varpi_k$ the restrictions to $E(k)$ of the canonical projections

\[ X_R \times t^k \xrightarrow{\tau_k} t^k \quad \text{and} \quad X_R \times t^k \xrightarrow{\varpi_k} X_R. \]

Lemma 5.7. (i) The morphism $\tau_k$ is a projective and birational morphism onto $X_{R,k}$.

(ii) The sets $V(k)$ and $\tau_k^{-1}(V(k))$ are smooth open subsets of $X_{R,k}$ and $E(k)$. Moreover, for $k \geq 2$, they are big open subsets of $X_{R,k}$ and $E(k)$.

(iii) The restriction of $\tau_k$ to $\tau_k^{-1}(V(k))$ is an isomorphism onto $V(k)$.

Proof. Since $X_R$ is a projective variety, $\tau_k$ is projective and its image is $X_{R,k}$ by definition. For $(x_1, \ldots, x_k)$ in $V(k)$ and $(u, x_1, \ldots, x_k)$ in $\tau_k^{-1}((x_1, \ldots, x_k))$, $u = t^{x_1}$ if $x_1$ is in $t_{reg}$ and $u = t^{x_2}$ if $x_2$ is in $t_{reg}$. As a result, the restriction of $\tau_k$ to $\tau_k^{-1}(V(k))$ is a birational morphism onto $V(k)$. Hence $\tau_k$ is a birational morphism and by Zariski’s Main Theorem [Mu88, §9], this restriction is an isomorphism since $V(k)$ is a smooth variety by Lemma 5.2. So it remains to prove that for $k \geq 2$, $\tau_k^{-1}(V(k))$ is a big open subset of $E(k)$.

Suppose that $E(k) \setminus \tau_k^{-1}(V(k))$ has an irreducible component $\Sigma$ of dimension $\dim E(k) - 1$. A contradiction is expected. Since $E(k)$ and $\tau_k^{-1}(V(k))$ are invariant under the automorphisms of $X_R \times t^k$,

\[ (u, x_1, \ldots, x_k) \mapsto (u, t x_1, \ldots, t x_k), \quad (t \in \mathbb{C}^*) \]

so is $\Sigma$. Then $\Sigma \cap X_R \times \{0\} = \varpi_k(\Sigma) \times \{0\}$ so that $\varpi_k(\Sigma)$ is a closed subset of $X_R$. Since $\dim \Sigma = \dim E(k) - 1$, $\dim \varpi_k(\Sigma) \geq \dim X_R - 1$. Suppose $\dim \Sigma = \dim X_R - 1$. Then for all $u$ in $\varpi_k(\Sigma)$, $\{u\} \times t^k$ is in $\Sigma$. It is
impossible since for all $u$ in a dense open subset of $\sigma_\delta(\Sigma)$, $u = \tau x$ for some $x$ in $\tau_{\text{reg}}$ by Corollary 3.8. Hence $\sigma_\delta(\Sigma) = X_R$. Then for all $u$ in a dense open subset of $X'_R$, $\{u\} \times u^k \cap \Sigma$ has codimension 1 in $\{u\} \times u^k$. Since the image of $\{u\} \times u^k \cap \Sigma$ by the projection

$$(u, x_1, \ldots, x_k) \mapsto x_1$$

is not dense in $u$, for all $x_1$ in a dense open subset of its image, $\{u\} \times \{x_1\} \times u^{k-1}$ is contained in $\Sigma$, whence the contradiction since $u \cap \tau_{\text{reg}}$ is not empty. \hfill \Box

By definition, $E^{(k)}$ is the inverse image of $X_R$ by the bundle projection of the vector bundle

$$\{u, x_1, \ldots, x_k\} \in \text{Gr}_d(\tau) \times u^k \mid u \ni x_1, \ldots, u \ni x_k$$

over $\text{Gr}_d(\tau)$ so that $E^{(k)}$ is vector bundle of rank $kd$ over $X_R$. In particular, $E^{(1)} = E$. According to [Hir64], there exists a desingulartization $\Gamma$ of $X_R$ with morphism $\rho$ such that the restriction of $\rho$ to $\rho^{-1}(X_{R_{\text{sm}}})$ is an isomorphism onto $X_{R_{\text{sm}}}$. Let $E^{(1)}$ be the following fiber product

$$\begin{array}{ccc}
\tilde{E}^{(1)} & \xrightarrow{\tilde{\rho}} & E^{(1)} \\
\downarrow & & \downarrow \\
\Gamma & \xrightarrow{\rho} & X_R
\end{array}$$

with $\tilde{\rho}$ the restriction map. Then $\tilde{E}^{(1)}$ is a vector bundle of rank $d$ over $\Gamma$. In particular, it is a smooth variety since $\Gamma$ is smooth.

Let $O$ be a trivialization open subset of the vector bundle $E^{(1)}$ and let $\Phi_1$ be a local trivialization over $O$ of $\tilde{E}^{(1)}$, whence the following commutative diagram

$$\begin{array}{ccc}
\tilde{\sigma}_1^{-1}(O) & \xrightarrow{\Phi_1} & O \times \mathbb{A}^d \\
\downarrow \tilde{\sigma}_1 & & \downarrow \text{id} \\
\sigma_1^{-1}(O) & \xrightarrow{\Phi} & O \times \mathbb{A}^d
\end{array}$$

Then $O$ is a trivialization open subset of the vector bundle $E^{(k)}$. The variety $E^{(1)}$ is a closed subbundle of $E^{(k)}$ over $X_R$ and for some local trivialization $\Phi$ over $O$ of $E^{(k)}$, we have the following commutative diagram

$$\begin{array}{ccc}
\tilde{\sigma}_k^{-1}(O) & \xrightarrow{\Phi} & O \times \mathbb{A}^{kd} \\
\downarrow \tilde{\sigma}_k & & \downarrow \text{id} \\
\sigma_k^{-1}(O) & \xrightarrow{\Phi} & O \times \mathbb{A}^{kd}
\end{array}$$

$\Phi_1$ is the restriction of $\Phi$ to $\tilde{\sigma}_1^{-1}(O)$ and $\Phi(\tilde{\sigma}_1^{-1}(O)) = O \times \mathbb{A}^d \times \{0\}$.

**Lemma 5.8.** Suppose $k \geq 2$. Denote by $\mu$ a generator of $\Omega_{\mathbb{A}^{kd}}$ and by $\tilde{\rho}$ the restriction of $\rho \times \text{id}_{\mathbb{A}^{kd}}$ to $\rho^{-1}(O) \times \mathbb{A}^{kd}$.

(i) The sheaf $\Omega_{E^{(k)}_{\text{sm}}}$ has a global section $\omega$ without zero.

(ii) The sheaf $\Omega_{\tilde{E}^{(1)}_{\text{sm}}}$ has a global section $\omega_{\tilde{\rho}}$ without zero.

(iii) For some $p$ in $\mathbb{A}[O \times \mathbb{A}^{kd}] \setminus \{0\}$, $\tilde{\rho}^*(p(\omega_{\tilde{\rho}} \wedge \mu))$ has a regular extension to $\rho^{-1}(O) \times \mathbb{A}^{kd}$.

**Proof.** (i) According to Proposition 5.6 and Lemma 5.7(iii), $\Omega_{E^{(k)}_{\text{sm}}}^{-1}(V^{(i)})$ has a global section without zero. By Lemma 5.7(ii), $\tau_k^{-1}(V^{(k)})$ is a smooth big open subset of $E^{(k)}$. So, by Lemma A.1, $\Omega_{E^{(k)}_{\text{sm}}}$ has a global section without zero.
(ii) Since $\mu$ is a generator of $\Omega_{x^d}$, there exists a unique $\nu$ in $k[x^d] \otimes k \Gamma(O_{\text{sm}}, \Omega_{\text{sm}})$ such that

$$\Phi_* (\omega |_{\sigma^{-1}(O_{\text{sm}})}) = \nu \land \mu.$$ 

Moreover, $\nu$ has no zero since so has $\omega$. Let $V$ be an affine open subset of $O_{\text{sm}}$ such that the restriction of $\Omega_{\text{sm}}$ to $V$ is locally free, generated by the local section $\omega_V$. Then for some $p_V$ in $k[V \times k^d]$,

$$\Phi_* (\omega |_{\sigma^{-1}(V)}) = p_V \omega_V \land \mu.$$ 

Then $p_V$ has no zero since so has $\nu \land \mu$. As a result, $p_V$ is in $k[V]$ and $p_V \omega_V$ is a local section of $\Omega_{\text{sm}}$ without zero. By the unicity of the decomposition (4), for two different affine open subsets $V$ and $V'$ as above, the differential forms $p_V \omega_V$ and $p_V' \omega_V'$ have the same restriction to $V \cap V'$. As a result, since such affine open subsets cover $O_{\text{sm}}$, for some global section $\omega_O$ of $\Omega_{\text{sm}}$,

$$\Phi_* (\omega |_{\sigma^{-1}(O_{\text{sm}})}) = \omega_O \land \mu.$$ 

Moreover, $\omega_O$ is unique and has no zero.

(iii) Let $\omega_1$ be a generator of $\Omega_1$ and let $\mu_1$ be a generator of $\Omega_{x^d}$. By (i), $\omega_1 \land \mu_1$ is a global section of $\Omega_{\text{sm}} \times k^d$, without zero. So for some regular function $p$ on $O_{\text{sm}} \times k^d$,

$$\Phi_* ((\tau_1)^* (\omega_1) |_{\sigma^{-1}(O_{\text{sm}})}) = p \omega_1 \land \mu_1.$$ 

According to Theorem 4.11, $X_R$ is normal. Then so is $O$ and $p$ has a regular extension to $O \times k^d$. Denote again by $p$ this extension. According to Equality (5), the differential form $\tilde{\rho}^* (p \omega_1 \land \mu_1)$ on $\rho^{-1}(O_{\text{sm}}) \times k^d$ has a regular extension to $\rho^{-1}(O) \times k^d$. In fact, denoting by $\Gamma_1$ the local trivialization over $\rho^{-1}(O)$ of $E^{(1)}$ such that the following diagram

\[
\begin{array}{ccc}
(\sigma_1 \rho^{-1})(O) & \xrightarrow{\Gamma_1} & \rho^{-1}(O) \times k^d \\
\tilde{\rho} \downarrow & & \downarrow \tilde{\rho} \\
\sigma_1^{-1}(O) & \xrightarrow{\Phi_1} & O \times k^d
\end{array}
\]

is commutative, it is the restriction to $\rho^{-1}(O_{\text{sm}}) \times k^d$ of

$$\Gamma_1^* ((\tau_1 \tilde{\rho})^* (\omega_1) |_{(\sigma_1 \rho^{-1})^{-1}(O)}).$$

For some generator $\mu'$ of $\Omega_{k^{d-1}d}$, $\mu = \mu_1 \land \mu'$ and $k[O \times k^d]$ is naturally embedded in $k[O \times k^{kd}]$. As a result, $\tilde{\rho}^* (p \omega_1 \land \mu)$ has a regular extension to $\rho^{-1}(O) \times k^{kd}$. 

\[\square\]

**Proposition 5.9.** The variety $X_R$ is Gorenstein with rational singularities.

**Proof.** According to Theorem 4.11, $X_R$ is normal and Cohen-Macaulay. Then by Lemma 5.8,(ii) and (iii) and Corollary A.5, $O \times k^d$ is Gorenstein with rational singularities. Then so is $O$ by Lemma B.1,(i) and (ii). Since there is a cover of $X_R$ by open subsets as $O$, $X_R$ is Gorenstein with rational singularities. 

As already mentioned, $u$ is in $\mathscr{O}_{b,v}$, whence Theorem 1.1 by Proposition 5.9.
Appendix A. Rational Singularities

Let $X$ be an affine irreducible normal variety.

**Lemma A.1.** Let $Y$ be a smooth big open subset of $X$.

(i) All regular differential form of top degree on $Y$ has a unique regular extension to $X_{sm}$.

(ii) Suppose that $\omega$ is a regular differential form of top degree on $Y$, without zero. Then the regular extension of $\omega$ to $X_{sm}$ has no zero.

**Proof.** (i) Since $\Omega_{X_{sm}}$ is a locally free module of rank one, there is an affine open cover $O_1, \ldots, O_k$ of $X_{sm}$ such that the restriction of $\Omega_{X_{sm}}$ to $O_i$ is a free $\mathcal{O}_{O_i}$-module generated by some section $\omega_i$. For $i = 1, \ldots, k$, set $O'_i := O_i \cap Y$. Let $\omega$ be a regular differential form of top degree on $Y$. For $i = 1, \ldots, k$, for some regular function $a_i$ on $O'_i$, $a_i \omega_i$ is the restriction of $\omega$ to $O'_i$. As $Y$ is a big open subset of $X$, $O'_i$ is a big open subset of $O_i$. Hence $a_i$ has a regular extension to $O_i$ since $O_i$ is normal. Denoting again by $a_i$ this extension, for $1 \leq i, j \leq k$, $a_i \omega_i$ and $a_j \omega_j$ have the same restriction to $O'_i \cap O'_j$ and $O_i \cap O_j$ since $\Omega_{X_{sm}}$ is torsion free as a locally free module. Let $\omega'$ be the global section of $\Omega_{X_{sm}}$ extending the $a_i \omega_i$'s. Then $\omega'$ is a regular extension of $\omega$ to $X_{sm}$ and this extension is unique since $Y$ is dense in $X_{sm}$ and $\Omega_{X_{sm}}$ is torsion free.

(ii) Suppose that $\omega$ has no zero. Let $\Sigma$ be the nullvariety of $\omega'$ in $X_{sm}$. If it is not empty, $\Sigma$ has codimension 1 in $X_{sm}$. As $Y$ is a big open subset of $X$, $\Sigma \cap X_{sm}$ is not empty if so is $\Sigma$. As a result, $\Sigma$ is empty.

Denote by $\psi$ the canonical injection from $X_{sm}$ into $X$.

**Lemma A.2.** Suppose that $\Omega_{X_{sm}}$ has a global section $\omega$ without zero. Then the $\mathcal{O}_X$-module $\psi_*(\Omega_{X_{sm}})$ is free of rank 1. More precisely, the morphism $\theta$:

$$
\mathcal{O}_X \xrightarrow{\theta} \psi_*(\Omega_{X_{sm}}), \quad \psi \mapsto \psi \omega
$$

is an isomorphism.

**Proof.** For $\varphi$ a local section of $\psi_*(\Omega_{X_{sm}})$ above the open subset $U$ of $X$, for some regular function $\psi$ on $U \cap X_{sm}$, $\psi \omega$ is the restriction of $\varphi$ to $U \cap X_{sm}$. Since $X$ is normal, so is $U$ and $U \cap X_{sm}$ is a big open subset of $U$. Hence $\psi$ has a regular extension to $U$. As a result, there exists a well defined morphism from $\psi_*(\Omega_{X_{sm}})$ to $\mathcal{O}_X$ whose inverse is $\theta$.

According to [Hir64], $X$ has a desingularization $Z$ with morphism $\tau$ such that the restriction of $\tau$ to $\tau^{-1}(X_{sm})$ is an isomorphism onto $X_{sm}$. For $U$ open subset of $X$, denote by $\tau_U$ the restriction of $\tau$ to $\tau^{-1}(U)$.

**Proposition A.3.** Suppose that $X$ is Cohen-Macaulay and that there exists a morphism $\mu : \mathcal{O}_Z \longrightarrow \Omega_Z$ such that for some $p$ in $\mathcal{O}_X$, $\mu_\ast$ is an isomorphism onto $p \tau_* (\Omega_Z)$. Then $X$ has rational singularities.

The following proof is the weak variation of the proof of [Hi91, Lemma 2.3].

**Proof.** Since $Z$ and $X$ are varieties over $\mathcal{k}$, we have the commutative diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{\tau} & X \\
p & & \downarrow q \\
\text{Spec}(\mathcal{k}) & & \\
\end{array}
$$

According to [H66, V. §10.2], $p'(\mathcal{k})$ and $q'(\mathcal{k})$ are dualizing complexes over $Z$ and $X$ respectively. Furthermore, by [H66, VII, 3.4] or [Hi91, 4.3(ii)], $p'(\mathcal{k})[-\dim Z]$ equals $\Omega_Z$. Set $\mathcal{D} := q'(\mathcal{k})[-\dim Z]$ so that $\tau'_{\ast} (\mathcal{D}) = \Omega_Z$ by [H66, VII, 3.4] or [Hi91, 4.3(iv)] in particular, $\mathcal{D}$ is dualizing over $X$. 37
Since \( \tau \) is a projective morphism, we have the isomorphism
\[
R\tau_*(\mathcal{H}om_Z(\Omega_Z, \Omega_Z)) \to R\mathcal{H}om_X(\tau_*(\Omega_Z), D)
\]
by [H66, VII, 3.4] or [Hi91, 4.3.(iii)]. Since \( H^i(R\mathcal{H}om_Z(\Omega_Z, \Omega_Z)) = 0 \) for \( i = 0 \) and \( 0 \) for \( i > 0 \), the left hand side of (6) can be identified with \( R\tau_*(\Omega_Z) \).

According to Grauert-Riemenschneider Theorem [GR70], \( R\tau_*(\Omega_Z) \) has only cohomology in degree 0. Since \( \tau \) is projective and birational and \( Z \) is normal, \( \tau_*(\Omega_Z) = \mathcal{O}_X \). So by assumption of the proposition,
\[
R\tau_*(\Omega_Z) \approx \frac{1}{p} \mathcal{O}_X,
\]
whence
\[
R\mathcal{H}om_X(R(\tau)_*(\Omega_Z), D) \approx p\mathcal{O}_X \otimes \mathcal{O}_X D
\]
and (6) can be rewritten as
\[
R\tau_*(\Omega_Z) \approx p\mathcal{O}_X \otimes \mathcal{O}_X D.
\]
Since \( X \) is Cohen-Macaulay, \( D \) has cohomology in only one degree. So, by flatness of the \( \mathcal{O}_X \)-module \( p\mathcal{O}_X \), \( p\mathcal{O}_X \otimes \mathcal{O}_X D \) has cohomology in only one degree. As a result, by (7), \( R^i\tau_*(\Omega_Z) = 0 \) for \( i > 0 \), that is \( X \) has rational singularities.

Denote by \( \mathcal{M} \) the cohomology in degree 0 of \( D \).

**Lemma A.4.** Suppose that \( X \) has rational singularities. Then the \( \mathcal{O}_X \)-modules \( \tau_*(\Omega_Z) \) and \( \mathcal{M} \) are isomorphic. In particular, \( \tau_*(\Omega_Z) \) has finite injective dimension.

**Proof:** Since \( X \) has rational singularities, \( R\tau_*(\Omega_Z) = \mathcal{O}_X \) and \( D \) has only cohomology in degree 0. Moreover, by Grauert-Riemenschneider Theorem [GR70], \( R\tau_*(\Omega_Z) \) has only cohomology in degree 0, whence \( R\tau_*(\Omega_Z) = \tau_*(\Omega_Z) \). Then, by (6), we have the isomorphism
\[
\mathcal{O}_X \to \mathcal{H}om_X(\tau_*(\Omega_Z), \mathcal{M}).
\]
As \( D \) is dualizing, we have the isomorphism
\[
R\tau_*(\Omega_Z) \to R\mathcal{H}om_X(R(\tau)_*(\Omega_Z), D), D)
\]
whence the isomorphism \( \tau_*(\Omega_Z) \to \mathcal{M} \) by (6). As a result, \( \tau_*(\Omega_Z) \) has finite injective dimension since so has \( \mathcal{M} \).

**Corollary A.5.** Let \( Y \) be a smooth big open subset of \( X \). Suppose that the following conditions are verified:

1. \( X \) is Cohen-Macaulay,
2. \( \Omega_Y \) has a global section \( \omega \) without zero,
3. for some global section \( \omega_Z \) of \( \Omega_Z \) and for some \( p \) in \( \mathbb{k}[X] \setminus \{0\} \), the restriction of \( \omega_Z \) to \( \tau^{-1}(Y) \) is equal to \( p\tau_Y^*(\omega) \).

Then \( X \) is Gorenstein with rational singularities. Moreover, its canonical module is free of rank 1.

**Proof:** According to Lemma A.1(ii), \( \omega \) has a unique regular extension to \( X_{\text{sm}} \) and this extension has no zero. Denote again by \( \omega \) this extension. Since \( Z \) is irreducible, \( \tau^{-1}(Y) \) is dense in \( \tau^{-1}(X_{\text{sm}}) \) so that the restriction of \( \omega_Z \) to \( \tau^{-1}(X_{\text{sm}}) \) is equal to \( p\tau_{X_{\text{sm}}}^*(\omega) \) since \( \Omega_Z \) has no torsion. Denote by \( \mu \) the morphism
\[
\mathcal{O}_Z \xrightarrow{\mu} \Omega_Z, \quad \varphi \mapsto \varphi \omega_Z.
\]

\[
\mathcal{O}_Z \xrightarrow{\mu} \Omega_Z, \quad \varphi \mapsto \varphi \omega_Z.
\]
Let $U$ be an open subset of $X$ and $y$ a local section of $\tau_* (\Omega_Z)$ above $U$. Since $\omega$ has no zero and $\tau_{U_{\text{sm}}}$ is an isomorphism onto $U_{\text{sm}}$,

$$\nu|_{\tau^{-1}(U_{\text{sm}})} = \tau^*_{U_{\text{sm}}} (\varphi \omega |_{U_{\text{sm}}})$$

for some $\varphi$ in $k[U_{\text{sm}}]$, whence

$$p\nu|_{\tau^{-1}(U_{\text{sm}})} = \varphi^* \tau_{U_{\text{sm}}} (\Omega_Z|_{\tau^{-1}(U_{\text{sm}})})$$

by Condition (3). Since $X$ is normal, so is $U$ and $U_{\text{sm}}$ is a big open subset of $U$. Hence $\varphi$ has a regular extension to $U$. Denoting again by $\varphi$ this extension,

$$p\nu = \varphi^* \tau_U (\omega |_{\tau^{-1}(U)})$$

since $Z$ is irreducible and $\Omega_Z$ has no torsion. As a result the morphism

$$\tau_* \mu : \tau_* (\Omega_Z) \longrightarrow p\tau_* (\Omega_Z)$$

is an isomorphism since it is obviously injective. So, by Proposition A.3, $X$ has rational singularities. In particular, by [KK73, p.50], $\tau_* (\Omega_X) = \iota_* (\Omega_X)$. Then, by Lemma A.2, the canonical module of $X$ is free of rank 1 and by Lemma A.4, $X$ is Gorenstein.

\section*{Appendix B. About singularities}

In this section we recall a well known result. Let $X$ be a variety and $Y$ a vector bundle over $X$. Denote by $\tau$ the bundle projection.

\textbf{Lemma B.1.} (i) If $Y$ is Gorenstein, then $X$ is Gorenstein.

(ii) The variety $X$ has rational singularities if and only if so has $Y$.

(iii) If $X$ is Cohen-Macaulay, then so is $Y$.

\textbf{Proof.} Let $y$ be in $Y$, $x := \tau (y)$. Denote by $\widehat{\mathcal{O}}_{X,x}$ and $\widehat{\mathcal{O}}_{Y,y}$ the completions of the local rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ respectively.

(i) Since $Y$ is a vector bundle over $X$, $\widehat{\mathcal{O}}_{Y,y}$ is a ring of formal series over $\widehat{\mathcal{O}}_{X,x}$. By [Bru, Proposition 3.1.19,(c)], $\widehat{\mathcal{O}}_{Y,y}$ is Gorenstein. So, by [Bru, Proposition 3.1.19,(b)], $\widehat{\mathcal{O}}_{X,x}$ is Gorenstein. Then by [Bru, Proposition 3.1.19,(c)], $\mathcal{O}_{X,x}$ is Gorenstein, whence the assertion.

(ii) Since $Y$ is a vector bundle over $X$, then there exists a cover of $X$ by open subsets $O$, such that $\tau^{-1} (O)$ is isomorphic to $O \times k^m$ with $m = \dim Y - \dim X$. According to [KK73, p.50], $O \times k^m$ has rational singularities if and only if so has $O$, whence the assertion since a variety has rational singularities if and only if it has a cover by open subsets having rational singularities.

(iii) According to [MA86, Ch. 6, Theorem 17.7], a polynomial algebra over a Cohen-Macaulay algebra is Cohen-Macaulay. Hence for $O$ open subset of $X$ as in (ii), $\tau^{-1} (O)$ is Cohen-Macaulay, whence the assertion since there is a cover of $Y$ by open subsets as $\tau^{-1} (O)$. \qed

\section*{References}

[Bol91] A.V. Bolsinov, \textit{Commutative families of functions related to consistent Poisson brackets}, Acta Applicandae Mathematicae, 24 (1991), n’ 1, p. 253–274.

[Bou98] N. Bourbaki, \textit{Algèbre commutative, Chapitre 10, Éléments de mathématiques}, Masson (1998), Paris.

[Boutot87] J-François. Boutot, \textit{Singularités rationnelles et quotients par les groupes réductifs}, Inventiones Mathematicae 88 (1987), p. 65–68.

[Bru] W. Bruns and J. Herzog, \textit{Cohen-Macaulay rings}, Cambridge studies in advanced mathematics n’ 39, Cambridge University Press, Cambridge (1996).

[CZ16] J-Y. Charbonnel and M. Zaiter \textit{On the Commuting variety of a reductive Lie algebra and other related varieties}, Journal of Algebra 458 (2016), pp. 445–497.
