Coefficient Estimates for Initial Taylor-Maclaurin Coefficients for a Subclass of Analytic and Bi-Univalent Functions Defined by Al-Oboudi Differential Operator

Serap Bulut

Civil Aviation College, Kocaeli University, Arslanbey Campus, 41285 İzmit-Kocaeli, Turkey

Correspondence should be addressed to Serap Bulut; bulutserap@yahoo.com

Received 5 August 2013; Accepted 7 October 2013

Academic Editors: H. Bulut and J. Park

1. Introduction

Let $\mathbb{R} = (-\infty, \infty)$ be the set of real numbers, $\mathbb{C}$ the set of complex numbers, and
\[ \mathbb{N} := \{1, 2, 3, \ldots\} = \mathbb{N}_0 \setminus \{0\} \tag{1} \]
the set of positive integers.

Let $A$ denote the class of all functions of the form
\[ f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{2} \]
which are analytic in the open unit disk
\[ \mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}. \tag{3} \]

We also denote by $B$ the class of all functions in the normalized analytic function class $A$ which are univalent in $\mathbb{U}$.

For two functions $f$ and $g$, analytic in $\mathbb{U}$, we say that the function $f$ is subordinate to $g$ in $\mathbb{U}$ and write
\[ f(z) < g(z) \quad (z \in \mathbb{U}), \tag{4} \]
if there exists a Schwarz function $\omega$, which is analytic in $\mathbb{U}$ with
\[ \omega(0) = 0, \quad |\omega(z)| < 1, \quad (z \in \mathbb{U}) \tag{5} \]
such that
\[ f(z) = g(\omega(z)), \quad (z \in \mathbb{U}). \tag{6} \]

Indeed, it is known that
\[ f(z) < g(z), \quad (z \in \mathbb{U}) \implies f(0) = g(0), \tag{7} \]
\[ f(\mathbb{U}) \subset g(\mathbb{U}). \]

Furthermore, if the function $g$ is univalent in $\mathbb{U}$, then we have the following equivalence:
\[ f(z) < g(z), \quad (z \in \mathbb{U}) \iff f(0) = g(0), \tag{8} \]
\[ f(\mathbb{U}) \subset g(\mathbb{U}). \]

For $f \in A$, Al-Oboudi [1] introduced the following operator:
\[ D_0^\delta f(z) = f(z), \tag{9} \]
\[ D_\delta^1 f(z) = (1 - \delta) f(z) + \delta z f'(z) =: D_\delta f(z), \quad (\delta \geq 0), \tag{10} \]
\[ D_\delta^n f(z) = D_\delta \left( D_\delta^{n-1} f(z) \right), \quad (n \in \mathbb{N}). \tag{11} \]
If \( f \) is given by (2), then from (10) and (11) we see that
\[
D^n_\delta f(z) = z + \sum_{k=1}^{\infty} [1 + (k - 1)\delta]^{n} a_k z^k, \quad (n \in \mathbb{N}_0),
\]
with \( D^n_\delta f(0) = 0 \). When \( \delta = 1 \), we get Sălăgean's differential operator \( D^n_1 f = D^n \), [2].

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk \( U \). In fact, the Koebe one-quarter theorem [3] ensures that the image of \( U \) under every univalent function \( f \in \mathcal{S} \) contains a disk of radius \( 1/4 \). Thus every function \( f \in \mathcal{S} \) has an inverse \( f^{-1} \), which is defined by
\[
 f^{-1}(f(z)) = z \quad (z \in \mathbb{U}),
\]
\[
f^{-1}(w) = w - a_1 w^2 + (2a_2^2 - a_1) w^3 - \left(5a_3^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots. \tag{14}
\]

A function \( f \in \mathcal{S} \) is said to be bi-univalent in \( U \) if both \( f \) and \( f^{-1} \) are univalent in \( U \). Let \( \Sigma \) denote the class of bi-univalent functions in \( U \), given by (2). For a brief history and interesting examples of functions in the class \( \Sigma \), see [4] (see also [5, 6]). In fact, the aforementioned work of Srivastava et al. [4] essentially revived the investigation of various subclasses of the bi-univalent function class \( \Sigma \) in recent years; it was followed by such works as those by Frasin and Aouf [7], Porwal and Darus [8], and others (see, e.g., [9–17]).

Motivated by the abovementioned works, we define the following subclass of function class \( \Sigma \).

**Definition 1.** Let \( h: \mathbb{U} \rightarrow \mathbb{C} \) be a convex univalent function such that
\[
h(0) = 1, \quad h(\bar{z}) = h(z) \quad (z \in \mathbb{U}; \mathcal{R}(h(z)) > 0). \tag{15}
\]
A function \( f \), defined by (2), is said to be in the class \( \mathcal{NP}^2_{\Delta}(n, \beta; h) \) if the following conditions are satisfied:
\[
f \in \Sigma, \quad e^{i\beta} \left( 1 - \lambda \right) \frac{D^n_\delta f(z)}{z} + \lambda (D^n_\delta f(z))' < h(z) \cos \beta + i \sin \beta
\]
\[
\quad (z \in \mathbb{U}),
\]
\[
e^{i\beta} \left( 1 - \lambda \right) \frac{D^n_\delta g(w)}{w} + \lambda (D^n_\delta g(w))' < h(w) \cos \beta + i \sin \beta \quad (w \in \mathbb{U}), \tag{16}
\]
where \( \beta \in (-\pi/2, \pi/2) \), \( \lambda \geq 1 \), the function \( g \) is given by
\[
g(w) = w - a_1 w^2 + (2a_2^2 - a_1) w^3 - \left(5a_3^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots, \tag{17}
\]
and \( D^n_\delta \) is the Al-Oboudi differential operator.

**Remark 2.** If we set
\[
h(z) = \frac{1 + A z}{1 + B z} \quad (-1 \leq B < A \leq 1) \tag{18}
\]
in Definition 1, then the class \( \mathcal{NP}^2_{\Delta}(n, \beta; h) \) reduces to the class denoted by \( \mathcal{NP}^2(\Delta, n, \beta, A, B) \) which is the subclass of the functions \( f \in \Sigma \) satisfying
\[
e^{i\beta} \left( 1 - \lambda \right) \frac{D^n_\delta f(z)}{z} + \lambda (D^n_\delta f(z))' \leq \frac{1 + A z}{1 + B z} \cos \beta + i \sin \beta \quad (z \in \mathbb{U}),
\]
\[
e^{i\beta} \left( 1 - \lambda \right) \frac{D^n_\delta g(w)}{w} + \lambda (D^n_\delta g(w))' \leq \frac{1 + A w}{1 + B w} \cos \beta + i \sin \beta \quad (w \in \mathbb{U}), \tag{19}
\]
where \( \beta \in (-\pi/2, \pi/2) \), \( \lambda \geq 1 \), the function \( g \) is defined by (17), and \( D^n_\delta \) is the Al-Oboudi differential operator.

**Remark 3.** If we set
\[
h(z) = \frac{1 + (1 - 2\alpha) z}{1 - z} \quad (0 \leq \alpha < 1) \tag{20}
\]
in Definition 1, then the class \( \mathcal{NP}^2(\Delta, n, \beta; h) \) reduces to the class denoted by \( \mathcal{NP}(\Delta, n, \beta, h) \) which is the subclass of the functions \( f \in \Sigma \) satisfying
\[
\mathfrak{R} \left\{ e^{i\beta} \left( 1 - \lambda \right) \frac{D^n_\delta f(z)}{z} + \lambda (D^n_\delta f(z))' \right\} > \alpha \cos \beta
\]
\[
\quad (z \in \mathbb{U}),
\]
\[
\mathfrak{R} \left\{ e^{i\beta} \left( 1 - \lambda \right) \frac{D^n_\delta g(w)}{w} + \lambda (D^n_\delta g(w))' \right\} > \alpha \cos \beta \quad (w \in \mathbb{U}), \tag{21}
\]
where \( \beta \in (-\pi/2, \pi/2) \), \( \lambda \geq 1 \), the function \( g \) is defined by (17), and \( D^n_\delta \) is the Al-Oboudi differential operator.

**Remark 4.** If we set
\[
h(z) = \frac{1 + (1 - 2\alpha) z}{1 - z} \quad (0 \leq \alpha < 1) \tag{22}
\]
in Definition 1, then the class \( \mathcal{NP}(\Delta, n, \beta; h) \) reduces to the class denoted by \( \mathcal{NP}(\Delta, n, \beta, h) \) which is the subclass of the functions \( f \in \Sigma \) satisfying
\[
\mathfrak{R} \left\{ e^{i\beta} \left( 1 - \lambda \right) \frac{D^n_\delta f(z)}{z} + \lambda (D^n_\delta f(z))' \right\} > \alpha \cos \beta
\]
\[
\quad (z \in \mathbb{U}),
\]
\[
\mathfrak{R} \left\{ e^{i\beta} \left( 1 - \lambda \right) \frac{D^n_\delta g(w)}{w} + \lambda (D^n_\delta g(w))' \right\} > \alpha \cos \beta \quad (w \in \mathbb{U}), \tag{23}
\]
where \( \beta \in (-\pi/2, \pi/2) \), \( \lambda \geq 1 \), the function \( g \) is defined by (17), and \( D^n_\delta \) is the Sălăgean differential operator.
Remark 5. If we set
\[ n = 0, \quad h(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} \quad (0 \leq \alpha < 1) \quad (24) \]
in Definition 1, then the class \( N_{\Sigma}^{\alpha, \beta}(n; h) \) reduces to the class denoted by \( N_{\Sigma}^{\alpha}(\beta, \alpha) \) which is the subclass of the functions \( f \in \Sigma \) satisfying
\[ \Re \left\{ e^{i\beta} \left( (1 - \lambda) \frac{f(z)}{z} + \lambda f'(z) \right) \right\} > \alpha \cos \beta \quad (z \in \mathbb{U}), \]
\[ \Re \left\{ e^{i\beta} \left( (1 - \lambda) \frac{g(w)}{w} + \lambda g'(w) \right) \right\} > \alpha \cos \beta \quad (w \in \mathbb{U}), \]
where \( \beta \in (-\pi/2, \pi/2) \), \( \lambda \geq 1 \), and the function \( g \) is defined by (17).

Remark 6. If we set
\[ n = 0, \quad \lambda = 1, \quad h(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} \quad (0 \leq \alpha < 1) \quad (26) \]
in Definition 1, then the class \( N_{\Sigma}^{\alpha, \beta}(n; \beta; h) \) reduces to the class denoted by \( N_{\Sigma}^{\alpha}(\beta, \alpha) \) which is the subclass of the functions \( f \in \Sigma \) satisfying
\[ \Re \left\{ e^{i\beta} f'(z) \right\} > \alpha \cos \beta \quad (z \in \mathbb{U}), \]
\[ \Re \left\{ e^{i\beta} g'(w) \right\} > \alpha \cos \beta \quad (w \in \mathbb{U}), \]
where \( \beta \in (-\pi/2, \pi/2) \) and the function \( g \) is defined by (17).

We note that
\[ N_{\Sigma}^{\alpha}(n, 0, \alpha) = \mathcal{H}_{\Sigma}(n, \alpha, \lambda) \quad \text{(see [8])}, \]
\[ N_{\Sigma}^{\alpha}(0, \alpha) = \mathcal{B}_{\Sigma}(\alpha, \lambda) \quad \text{(see [7])}, \]
\[ N_{\Sigma}^{\alpha}(0, 0) = \mathcal{H}_{\Sigma}(\alpha) \quad \text{(see [4])}. \]

Firstly, in order to derive our main results, we need the following lemma.

Lemma 7 (see [18]). Let the function \( h(z) \) given by
\[ h(z) = \sum_{n=1}^{\infty} b_n z^n \]
be convex in \( \mathbb{U} \). Suppose also that the function \( \varphi(z) \) given by
\[ \varphi(z) = \sum_{n=1}^{\infty} c_n z^n \]
is holomorphic in \( \mathbb{U} \). If \( \varphi(z) < h(z) \) (\( z \in \mathbb{U} \)), then
\[ |c_n| \leq |b_1| \quad (n \in \mathbb{N}). \quad (31) \]

The object of the present paper is to find estimates on the Taylor-Maclaurin coefficients \( |a_2| \) and \( |a_3| \) for functions in this new subclass \( N_{\Sigma}^{\alpha, \beta}(n; \beta; h) \) of the function class \( \Sigma \).

2. A Set of General Coefficient Estimates

In this section, we state and prove our general results involving the bi-univalent function class \( N_{\Sigma}^{\alpha, \beta}(n; \beta; h) \) given by Definition 1.

Theorem 8. Let the function \( f(z) \) given by the Taylor-Maclaurin series expansion (2) be in the function class
\[ N_{\Sigma}^{\alpha, \beta}(n; \beta; h) \quad (\beta \in (-\pi/2, \pi/2), \lambda \geq 1, \delta \geq 0, n \in \mathbb{N}_0) \quad (32) \]
with
\[ h(z) = 1 + B_1 z + B_2 z^2 + \cdots. \quad (33) \]

Then
\[ \left| a_2 \right| \leq \min \left\{ \left| B_1 \right| \cos \beta \left( 1 + \delta \right)^n \left( 1 + \lambda \right)^n \left( 1 + 2\delta \right)^n \left( 1 + 2\lambda \right)^n \right\}, \quad (34) \]
\[ \left| a_3 \right| \leq \frac{\left| B_1 \right| \cos \beta}{\left( 1 + 2\delta \right)^n \left( 1 + 2\lambda \right)^n}. \quad (35) \]

Proof. It follows from (16) that
\[ e^{i\beta} \left( (1 - \lambda) \frac{D_n f(z)}{z} + \lambda(D_n f(z))' \right) \]
\[ = p(z) \cos \beta + i \sin \beta \quad (z \in \mathbb{U}), \]
\[ e^{i\beta} \left( (1 - \lambda) \frac{D_n g(w)}{w} + \lambda(D_n g(w))' \right) \]
\[ = q(w) \cos \beta + i \sin \beta \quad (w \in \mathbb{U}), \]
where \( p(z) < h(z) \) and \( q(w) < h(w) \) have the following Taylor-Maclaurin series expansions:
\[ p(z) = 1 + p_1 z + p_2 z^2 + \cdots, \]
\[ q(w) = 1 + q_1 w + q_2 w^2 + \cdots, \]
respectively. Now, upon equating the coefficients in (36) and (37), we get
\[ e^{i\beta} (1 + \delta)^n (1 + \lambda) a_2 = p_1 \cos \beta, \quad (40) \]
\[ e^{i\beta} (1 + 2\delta)^n (1 + 2\lambda) a_3 = p_2 \cos \beta, \quad (41) \]
\[ -e^{i\beta} (1 + \delta)^n (1 + \lambda) a_2 = q_1 \cos \beta, \quad (42) \]
\[ e^{i\beta} \left[ -(1 + 2\delta)^n (1 + 2\lambda) a_3 + 2(1 + 2\delta)^n (1 + 2\lambda) a_2^2 \right] = q_2 \cos \beta. \quad (43) \]

From (40) and (42), we obtain
\[ p_1 = -q_1, \quad (44) \]
\[ 2e^{2i\beta} (1 + \delta)^n (1 + \lambda)^2 a_2^2 = \left( p_1^2 + q_1^2 \right) \cos^2 \beta. \quad (45) \]
Also, from (41) and (43), we find that
\[ a_2^2 = \frac{e^{-i\beta} (p_2 + q_2) \cos \beta}{2(1 + 2\delta)^n (1 + 2\lambda)}. \] (46)

Since \( p, q \in h(\mathbb{U}) \), according to Lemma 7, we immediately have
\[ |p_k| = \left| \frac{p^{(k)}}{k!} \right| \leq |B_1| \quad (k \in \mathbb{N}), \]
\[ |q_k| = \left| \frac{q^{(k)}}{k!} \right| \leq |B_1| \quad (k \in \mathbb{N}). \] (47)

Applying (47) and Lemma 7 for the coefficients \( p_1, p_2, q_1, \) and \( q_2 \), from the equalities (45) and (46), we obtain
\[ |a_2|^2 \leq \frac{|B_1|^2 \cos^2 \beta}{(1 + \delta)^{2n}(1 + \lambda)^2}, \]
\[ |a_2|^2 \leq \frac{|B_1| \cos \beta}{(1 + 2\delta)^n (1 + 2\lambda)} \] (48)

respectively. So we get the desired estimate on the coefficient \( |a_2| \) as asserted in (34).

Next, in order to find the bound on the coefficient \( |a_3| \), we subtract (43) from (41). We thus get
\[ 2(1 + 2\delta)^n (1 + 2\lambda) a_3 - 2(1 + 2\delta)^n (1 + 2\lambda) a_2^2 = e^{-i\beta} (p_2 - q_2) \cos \beta. \] (50)

Upon substituting the value of \( a_2^2 \) from (45) into (50), it follows that
\[ a_3 = \frac{e^{-i\beta} (p_2 + q_2) \cos \beta}{2(1 + 2\delta)^n (1 + 2\lambda)} + \frac{e^{-i\beta} (p_2 - q_2) \cos \beta}{2(1 + 2\delta)^n (1 + 2\lambda)}. \] (51)

So we get
\[ |a_3| \leq \frac{|B_1|^2 \cos^2 \beta}{(1 + \delta)^{2n}(1 + \lambda)^2} + \frac{|B_1| \cos \beta}{(1 + 2\delta)^n (1 + 2\lambda)}. \] (52)

On the other hand, upon substituting the value of \( a_2^2 \) from (46) into (50), it follows that
\[ a_3 = \frac{e^{-i\beta} (p_2 + q_2) \cos \beta}{2(1 + 2\delta)^n (1 + 2\lambda)} + \frac{e^{-i\beta} (p_2 - q_2) \cos \beta}{2(1 + 2\delta)^n (1 + 2\lambda)}. \] (53)

And we get
\[ |a_3| \leq \frac{|B_1| \cos \beta}{(1 + 2\delta)^n (1 + 2\lambda)}. \] (54)

Comparing the inequalities in (52) and (54) completes the proof of Theorem 8.

3. Corollaries and Consequences

By setting
\[ h(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1) \] (55)
in Theorem 8, we have the following corollary.

**Corollary 9.** Let the function \( f(z) \) given by the Taylor-Maclaurin series expansion (2) be in the function class
\[ \mathcal{N}^\alpha_{\delta} \lambda (n, \beta; A, B) \]
\( (\beta \in (-\pi/2, \pi/2), \lambda \geq 1, \delta \geq 0, -1 \leq B < A \leq 1, n \in \mathbb{N}_0) \). (56)

Then
\[ |a_2| \leq \min \left\{ \frac{(A - B) \cos \beta}{(1 + \delta)^{2n}(1 + \lambda)}, \sqrt{\frac{(A - B) \cos \beta}{(1 + 2\delta)^n (1 + 2\lambda)}} \right\}, \]
\[ |a_3| \leq \frac{(A - B) \cos \beta}{(1 + 2\delta)^n (1 + 2\lambda)}. \] (57)

By setting
\[ h(z) = \frac{1 + (1 - 2\alpha) z}{1 - z} \quad (0 \leq \alpha < 1) \] (58)
in Theorem 8, we have the following corollary.

**Corollary 10.** Let the function \( f(z) \) given by the Taylor-Maclaurin series expansion (2) be in the function class
\[ \mathcal{N}^\alpha_{\delta} \lambda (n, \beta, \alpha) \]
\( (\beta \in (-\pi/2, \pi/2), \lambda \geq 1, \delta \geq 0, 0 \leq \alpha < 1, n \in \mathbb{N}_0) \). (59)

Then
\[ |a_3| \leq \min \left\{ \frac{2(1 - \alpha) \cos \beta}{(1 + \delta)^{2n}(1 + \lambda)}, \sqrt{\frac{2(1 - \alpha) \cos \beta}{(1 + 2\delta)^n (1 + 2\lambda)}} \right\}, \]
\[ |a_3| \leq \frac{2(1 - \alpha) \cos \beta}{(1 + 2\delta)^n (1 + 2\lambda)}. \] (60)

By setting
\[ \delta = 1, \quad h(z) = \frac{1 + (1 - 2\alpha) z}{1 - z} \quad (0 \leq \alpha < 1) \] (61)
in Theorem 8, we have the following corollary.

**Corollary 11.** Let the function \( f(z) \) given by the Taylor-Maclaurin series expansion (2) be in the function class
\[ \mathcal{N}^\alpha_{\delta} \lambda (n, \beta, \alpha) \]
\( (\beta \in (-\pi/2, \pi/2), \lambda \geq 1, 0 \leq \alpha < 1, n \in \mathbb{N}_0) \). (62)
Then
\[
|a_2| \leq \min \left\{ \frac{2(1-\alpha) \cos \beta}{2^n(1+\lambda)}, \frac{2(1-\alpha) \cos \beta}{3^n(1+2\lambda)} \right\},
\]
\[
|a_3| \leq \frac{2(1-\alpha) \cos \beta}{3^n(1+2\lambda)}.
\]

Remark 12. When $\beta = 0$, Corollary 11 is an improvement of the following estimates obtained by Porwal and Darus [8].

Corollary 13 (see [8]). Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (2) be in the function class
\[
\mathcal{H}_\Sigma (n, \alpha, \lambda) \quad (\lambda \geq 1, \ 0 \leq \alpha < 1, \ n \in \mathbb{N}_0).
\]

Then
\[
|a_2| \leq \sqrt{\frac{2(1-\alpha)}{3^n(1+2\lambda)}},
\]
\[
|a_3| \leq \frac{4(1-\alpha)^2}{2^n(1+\lambda)^2} + \frac{2(1-\alpha)}{3^n(1+2\lambda)}.
\]

By setting
\[
n = 0, \quad h(z) = \frac{1 + (1-2\alpha)z}{1-z} \quad (0 \leq \alpha < 1)
\]
in Theorem 8, we have the following corollary.

Corollary 14. Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (2) be in the function class
\[
\mathcal{N}_\Sigma (\beta, \alpha) \quad (\beta \in (-\pi/2, \pi/2), \ 0 \leq \alpha < 1).
\]

Then
\[
|a_2| \leq \min \left\{ (1-\alpha) \cos \beta, \sqrt{\frac{2(1-\alpha) \cos \beta}{3}} \right\},
\]
\[
|a_3| \leq \frac{2(1-\alpha) \cos \beta}{3}.
\]

Remark 15. When $\beta = 0$, Corollary 14 is an improvement of the following estimates obtained by Frasin and Aouf [7].

Corollary 16 (see [7]). Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (2) be in the function class
\[
\mathcal{B}_\Sigma (\alpha, \lambda) \quad (\lambda \geq 1, \ 0 \leq \alpha < 1).
\]

Then
\[
|a_2| \leq \sqrt{\frac{2(1-\alpha)}{1+2\lambda}},
\]
\[
|a_3| \leq \frac{4(1-\alpha)^2}{(1+\lambda)^2} + \frac{2(1-\alpha)}{1+2\lambda}.
\]

By setting
\[
n = 0, \quad \lambda = 1, \quad h(z) = \frac{1 + (1-2\alpha)z}{1-z} \quad (0 \leq \alpha < 1)
\]
in Theorem 8, we have the following corollary.

Corollary 17. Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (2) be in the function class
\[
\mathcal{N}_\Sigma (\beta, \alpha) \quad (\beta \in (-\pi/2, \pi/2), \ 0 \leq \alpha < 1).
\]

Then
\[
|a_2| \leq \min \left\{ (1-\alpha) \cos \beta, \sqrt{\frac{2(1-\alpha) \cos \beta}{3}} \right\},
\]
\[
|a_3| \leq \frac{2(1-\alpha) \cos \beta}{3}.
\]

Remark 18. When $\beta = 0$, Corollary 17 is an improvement of the following estimates obtained by Srivastava et al. [4].

Corollary 19 (see [4]). Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (2) be in the function class
\[
\mathcal{H}_\Sigma (\alpha) \quad (0 \leq \alpha < 1).
\]

Then
\[
|a_2| \leq \sqrt{\frac{2(1-\alpha)}{3}},
\]
\[
|a_3| \leq \frac{1-\alpha)(5-3\alpha)}{3}.
\]

References

[1] F. M. Al-Oboudi, “On univalent functions defined by a generalized Sălăgean operator,” International Journal of Mathematics and Mathematical Sciences, vol. 2004, no. 27, pp. 1429–1436, 2004.

[2] G. S. Sălăgean, “Subclasses of univalent functions,” in Complex Analysis-Fifth Romanian-Finnish Seminar, Part 1 (Bucharest, 1981), vol. 1013 of Lecture Notes in Mathematics, pp. 362–372, Springer, Berlin, Germany, 1983.

[3] P. L. Duren, Univalent Functions, vol. 259 of Grundlehren der Mathematischen Wissenschaften, Springer, New York, NY, USA, 1983.

[4] H. M. Srivastava, A. K. Mishra, and P. Gochhayat, “Certain subclasses of analytic and bi-univalent functions,” Applied Mathematics Letters, vol. 23, no. 10, pp. 1188–1192, 2010.

[5] D. A. Brannan and T. S. Taha, “On some classes of bi-univalent functions,” in Mathematical Analysis and Its Applications, S. M. Mazhar, A. Hamoui, and N. S. Faour, Eds., vol. 3 of KFAS Proceedings Series, pp. 53–60, Pergamon Press, Elsevier Science, Oxford, UK, 1988.

[6] D. A. Brannan and T. S. Taha, “On some classes of bi-univalent functions,” Studia Universitatis Babeş-Bolyai Mathematica, vol. 31, no. 2, pp. 70–77, 1986.
[7] B. A. Frasin and M. K. Aouf, "New subclasses of bi-univalent functions," *Applied Mathematics Letters*, vol. 24, no. 9, pp. 1569–1573, 2011.

[8] S. Porwal and M. Darus, "On a new subclass of bi-univalent functions," *Journal of the Egyptian Mathematical Society*, vol. 21, no. 3, pp. 190–193, 2013.

[9] S. Bulut, "Coefficient estimates for a class of analytic and bi-univalent functions," *Novi Sad Journal of Mathematics*, vol. 43, no. 2, pp. 59–65, 2013.

[10] S. Bulut, "Coefficient estimates for new subclasses of analytic and bi-univalent functions defined by Al-Oboudi differential operator," *Journal of Function Spaces and Applications*, vol. 2013, Article ID 181932, 7 pages, 2013.

[11] M. Çağlar, H. Orhan, and N. Yağmur, "Coefficient bounds for new subclasses of bi-univalent functions," *Filomat*, vol. 27, no. 7, pp. 1165–1171, 2013.

[12] S. P. Goyal and P. Goswami, "Estimate for initial Maclaurin coefficients of bi-univalent functions for a class defined by fractional derivatives," *Journal of the Egyptian Mathematical Society*, vol. 20, no. 3, pp. 179–182, 2012.

[13] T. Hayami and S. Owa, "Coefficient bounds for bi-univalent functions," *Pan-American Mathematical Journal*, vol. 22, no. 4, pp. 15–26, 2012.

[14] H. Orhan, N. Magesh, and V. K. Balaji, "Initial coefficient bounds for a general class of bi-univalent functions," Preprint.

[15] H. M. Srivastava, S. Bulut, M. Çağlar, and N. Yağmur, "Coefficient estimates for a general subclass of analytic and bi-univalent functions," *Filomat*, vol. 27, no. 5, pp. 831–842, 2013.

[16] Q. H. Xu, Y. C. Gui, and H. M. Srivastava, "Coefficient estimates for a certain subclass of analytic and bi-univalent functions," *Applied Mathematics Letters*, vol. 25, no. 6, pp. 990–994, 2012.

[17] Q. H. Xu, H. G. Xiao, and H. M. Srivastava, "A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems," *Applied Mathematics and Computation*, vol. 218, no. 23, pp. 11461–11465, 2012.

[18] W. Rogosinski, "On the coefficients of subordinate functions," *Proceedings of the London Mathematical Society*, vol. 48, no. 2, pp. 48–82, 1943.