Abstract We illustrate how nuclear polarization corrections in muonic atoms can be formally connected to inelastic response functions of a nucleus. We first discuss the point-nucleon approximation and then include finite-nucleon-size corrections. As an example, we compare our ab-initio calculation of the third Zemach moment in $\mu^4$He$^+$ to previous phenomenological results.

Keywords Nuclear polarization · Lamb shift · Muonic atoms · Zemach moment

1 Introduction

Recent measurements of the $\mu$H Lamb shift $\Delta E$ (2S-2P transition) at PSI \cite{1,2} allowed for an unprecedented precise determination of the proton charge radius, which deviates, however, by 7σ from the CODATA value based on $e$H measurements \cite{3}. This discrepancy challenges our understanding of experimental errors and theoretical calculations. To investigate this discrepancy, spectroscopy measurements in other muonic atoms, e.g., $\mu$D, $\mu^3$He$^+$ and $\mu^4$He$^+$, are planned at PSI \cite{4}. The extraction of the nuclear charge radius $\langle R_c^2 \rangle^{1/2}$ from a $\Delta E$ measurement, based on the expression

$$\Delta E \equiv \delta_{QED} + \delta_{pol} + \delta_{Zem} + m^3(\alpha)^4 \langle R_c^2 \rangle / 12.$$  \hspace{1cm} (1)

must be accompanied by theoretical estimates of the QED corrections $\delta_{QED}$, as well as the Zemach moment $\delta_{Zem}$ and the nuclear polarization $\delta_{pol}$, which are the elastic and inelastic nuclear structure...
corrections, respectively. The Zemach contribution is defined via the nuclear charge density \( \tilde{\rho}_0(R) \) as
\[
\delta_{Zem} = -\frac{m^4}{24}(Z\alpha)^5 \int dR dR' |R - R'|^3 \tilde{\rho}_0(R)\tilde{\rho}_0(R').
\] (2)

In the non-relativistic limit, the general Hamiltonian for a muonic atom can be expressed by
\[
H = H_{\text{nuc}} + H_\mu - \Delta H,
\] (3)
where \( H_{\text{nuc}} \) is the nuclear Hamiltonian, \( H_\mu = \frac{p^2}{2m_r} - Z\alpha/r \) the muonic Hamiltonian, \( m_r \) the reduced mass and \( \Delta H \) the difference between the muonic Coulomb interaction with the point-nucleus and the sum of the interactions with each of the \( Z \) protons, located at a distance \( R_a \) from the center of mass:
\[
\Delta H = \sum_a^Z \Delta V(r, R_a) \equiv \sum_a^Z \alpha \left( \frac{1}{|r - R_a|} - \frac{1}{r} \right).
\] (4)

Nuclear structure corrections are calculated using second-order perturbation theory on \( \Delta H \). Since a muon interacts more closely with the nucleus than does an electron, muonic atoms are very sensitive to nuclear corrections \([3, 4]\). Calculations in \( \mu D \) and \( \mu^4 \text{He}^+ \) have demonstrated the importance of using state-of-the-art methods and nuclear Hamiltonians to predict such corrections at a percentage accuracy \([5, 8, 9, 10]\). Here, we bridge the nuclear effects to inelastic response functions, and to charge and transition densities. Then, we present results on the Zemach moment in \( \mu^4 \text{He}^+ \) (see also \([3]\)).

### 2 Polarization in the point-nucleon approximation

From second order perturbation theory, the nuclear polarization \( \delta_{pol}^A \) becomes\([4]\)
\[
\delta_{pol}^A = - \sum_{N\neq N_0} \int dR dR' \rho_N^A(R) P(R, R', \omega_N) \rho_N(R'),
\] (5)
where \( \rho_N(R) = \langle N|\hat{\rho}(R)|N_0 \rangle \) is the matrix element of the point-nucleon charge density operator \( \hat{\rho}(R) \equiv \sum_a^Z \delta(R - R_a), \omega_N = E_N - E_{N_0}, \) and \( E_{N_0}, E_N, |N_0 \rangle \) and \( |N \rangle \) are the nuclear ground- and excited-state energies and wave-functions, respectively. The symbol \( \sum \) indicates a sum over discrete and an integration over continuum states. The nucleus is excited into all possible states but \( E_{N_0} \), thus \( \delta_{pol}^A \) is an inelastic contribution. The muonic matrix element \( P(R, R', \omega_N) \equiv P \) is given by
\[
P = -Z^2 \int dr dr' \Delta V(r, R)(|\mu_o| r)|r| \frac{1}{H_\mu + \omega_N - \epsilon_{\mu o}} |r'| \langle \mu_o | \Delta V(r', R'),
\] (6)
where \( \epsilon_{\mu o} \) and \( |\mu_o \rangle \) are the unperturbed muon atomic energy and wave-function in either the 2S or 2P states. The leading contribution to \( \delta_{pol}^A \) is obtained by neglecting in Eq. (6) the Coulomb-potential part of \( H_\mu \). Considering only contributions to the 2S state we get
\[
P = -Z^2 \delta^2(0) \int \frac{dq}{(2\pi)^3} \left( \frac{4\pi\alpha}{q^2} \right)^2 \left( 1 - e^{iqR} \right) \frac{1}{q^2/2m_r + \omega_N} \left( 1 - e^{-iqR'} \right),
\] (7)
where \( \delta^2(0) = (m_r^2\alpha)^3/8\pi \) is the \( \mu \) wave function at the origin. Since \( q \) can be large in the integral, we should not expand the plane wave in multipole. Instead, we first integrate over \( q \) in Eq. (7), obtaining
\[
P = -\frac{2\pi^2 Z^2 \delta^2(0)}{m_r \omega_N^2} \frac{1}{|R - R'|} \left[ e^{-\sqrt{2m_r \omega_N}|R - R'|} - 1 + \sqrt{2m_r \omega_N} |R - R'| - m_r \omega_N |R - R'|^2 \right].
\] (8)

The quantity \( |R - R'| \) indicates the “virtual” distance a proton travels during the two-photon exchange. Due to uncertainty principle it becomes \( |R - R'| \sim 1/\sqrt{2M_A \omega_N} \). Therefore, \( |R - R'| \sqrt{2M_\alpha \omega_N} \)

\(^1\) The upper index \( A \) indicates that the intrinsic nucleon polarization is not included, see \([3]\) for more details.
\[ \omega \approx \sqrt{m_r/M_A} \] can be used as a systematic expansion parameter. Up to 4\textsuperscript{th} order this expansion yields
\[ P(|R - R'|, \omega_N) \approx \frac{m^3(2\alpha)^5}{12} \sqrt{\frac{2m_r}{\omega_N}} \left| |R - R'|^2 - \frac{\sqrt{2m_r\omega_N}}{4} |R - R'|^3 + \frac{m_r\omega_N}{10} |R - R'|^4 \right|. \] (9)

The terms in Eq. (9) contribute to \( \delta_{pol}^4 \) at different orders in the \( \sqrt{m_r/M_A} \) expansion, and are evaluated using nuclear response functions defined as
\[ S_0(\omega) \equiv \frac{1}{2J_0 + 1} \sum_{N \neq N_0, J} |\langle N_0 J_0 | \hat{O} | N J \rangle|^2 \delta(\omega - \omega_N), \] (10)
where \( \hat{O} \) is a general operator and \( J_0 (J) \) is the angular momentum of the ground (excited) state. The leading \( |R - R'|^2 \) term in Eq. (9) is related to the electric-dipole excitation and yields
\[ \delta_{D1}^{(0)} = -\frac{2\pi m^3}{9} (2\alpha)^5 \int_{\omega_{th}}^{\infty} d\omega \sqrt{\frac{2m_r}{\omega}} S_{D1}(\omega), \] (11)
where \( \omega_{th} \) is the lowest excitation energy of the nucleus. The sub-leading \( |R - R'|^3 \) term is independent of \( \omega \), which allows substituting \( \sum_{N \neq N_0} |N \rangle \langle N| \) with \( 1 - |N_0 \rangle \langle N_0| \), and relating this inelastic term to two elastic ones
\[ \delta^{(1)} = \delta_{R\alpha pp}^{(1)} + \delta_{Z\delta 3}^{(1)} = -\frac{m^4}{24} (2\alpha)^5 \int dRdR'|R - R'|^3 |N_0 \rangle \delta(\hat{R}) \hat{\rho}(\hat{R}^\prime) \langle N_0| \] (12)
\[ + \frac{m^4}{24} (2\alpha)^5 \int dRdR'|R - R'|^3 \rho_0(\hat{R}) \rho_0(\hat{R}^\prime) \]
where \( \rho_0(\hat{R}) \equiv \rho_{N_0}(\hat{R}) = \langle N_0 | \hat{\rho}(\hat{R}) | N_0 \rangle \) is the nuclear charge density distribution in the point-nucleon approximation. The first term in Eq. (12), \( \delta_{R\alpha pp}^{(1)} \), is the 3\textsuperscript{rd} moment of the proton charge correlation function. The term \( \delta_{Z\delta 3}^{(1)} \) is the 3\textsuperscript{rd} Zemach moment \( 12 \), which cancels the elastic contribution \( \delta_{zem} \) in Eq. (2), when \( \hat{\rho}_0(\hat{R}) \) is defined in the point-nucleon limit as \( \rho_0(\hat{R}) \). Finally, contributions from the sub-sub-leading \( |R - R'|^4 \) term are
\[ \delta^{(2)} = \frac{m^5}{18} (2\alpha)^5 \int_{\omega_{th}}^{\infty} d\omega \sqrt{\frac{2m_r}{\omega}} \left[ S_{R2}(\omega) + \frac{16\pi}{25} S_Q(\omega) + \frac{16\pi}{5} S_{D1D3}(\omega) \right], \] (13)
where \( S_{R2} \) and \( S_Q \) are, respectively, the monopole \( \hat{R}^2 = \frac{1}{2} \sum_a Z_a R_a^2 \) and quadrupole \( \hat{Q} = \frac{1}{2} \sum_a Z_a R_a^2 Y_2(\hat{R}_a) \) structure functions. \( S_{D1D3} \) indicates the interference between \( \hat{D}_1 \) and \( \hat{D}_3 = \frac{1}{2} \sum_a R_a^2 Y_1(\hat{R}_a) \).

3 Corrections due to the finite nucleon charge distributions

Considering the finite charge distributions of the nucleons, the proton position in Eq. (1) should be replaced by a convolution over the proton charge density, and similarly for the neutron. Therefore, the point-proton muonic matrix element \( P \) in Eq. (7) is replaced by three finite-size terms: proton-proton \( P_{pp} \), neutron-proton \( P_{np} \), and neutron-neutron \( P_{nn} \). The muonic matrix elements \( P_{pp} \) and \( P_{np} \) are
\[ P_{pp} = -Z^2 \phi^2(0) \int \frac{dq}{(2\pi)^3} \left( \frac{4\pi \alpha}{q^2} \right)^2 \left[ 1 - G_p^E(q^2) e^{iq \cdot R} \right] \frac{1}{q^2/2m_r + \omega} \left[ 1 - G_n^E(q^2) e^{-iq \cdot R} \right], \] (14)
\[ P_{np} = 2NZ \phi^2(0) \int \frac{dq}{(2\pi)^3} \left( \frac{4\pi \alpha}{q^2} \right)^2 \left[ 1 - G_p^E(q^2) e^{iq \cdot R} \right] \frac{1}{q^2/2m_r + \omega} G_n^E(q^2) e^{-iq \cdot R}, \] (15)
and we neglect the neutron-neutron term. \( G_p^E \) and \( G_n^E \) are respectively the proton and neutron charge form factors. Here, we adopt the form factors defined in Ref. 13: \( G_p^E(q^2) = (1 + q^2/\beta^2)^{-2} \) and
\( G^E_{\nu}(q^2) = \lambda_0 q^2 (1 + q^2/\beta^2)^{-3} \), where \( \beta \) and \( \lambda \) are fixed by the proton and neutron charge radii via \( \beta = \sqrt{\langle 2 \rangle / \langle r_p^2 \rangle^{1/2}} \) and \( \lambda = -\langle r_n^2 \rangle / 6 \). Similar to solving Eq. (7), we integrate over \( q \) in Eq. (14) and (15), and extract the dominant nucleon-size corrections. For \(^4\)He the calculation of the nucleon-size corrections is further simplified, since \( N = Z \) and \( \rho_0^{(N)}(R) = \rho_0^{(Z)}(R) = \rho_0(R) \) assuming isospin symmetry in the ground-state. Because corrections to \( \delta^{(i)} \) vanish, the leading nucleon-size (NS) correction is

\[
\delta^{(1)}_{NS} = \delta^{(1)}_{R_{rip}} + \delta^{(1)}_{Z_1} = -m_5^d(Z\alpha)^5 \left[ \frac{2}{\beta \lambda} - \frac{2}{2\beta^2} \right] \int dR dR' |R - R'| \langle N_0 | \hat{\rho}^d(R) \hat{\rho}(R') | N_0 \rangle 
\]

Here, similarly to \( \delta^{(1)} \), contributions to \( \delta^{(1)}_{NS} \) are divided into two terms, \( \delta^{(1)}_{R_{rip}} \) and \( \delta^{(1)}_{Z_1} \). The latter one is the first Zemach moment and is the leading finite-nucleon-size correction to the elastic nuclear structure correction. In fact, the combination of \( \delta^{(1)}_{NS} \) and \( \delta^{(1)}_{Z_1} \) cancels out the elastic Zemach term \( \delta \) in Eq. (2) at the order discussed here.

4 Results: The Zemach elastic term

In Ref. [3], \( \delta_{Z_{em}} \) for \( ^4\)He\(^+\) is approximated as \(-1.40(4)\langle R^2 \rangle^{3/2} \) using the continuous charge distribution approximation in Ref. [11], with the error representing the model dependence. Using \( R_c = 1.681 \) fm determined from \( ^4\)He scattering data [13], one obtains \( \delta_{Z_{em}} = -0.65(19) \) meV with this phenomenological model. We calculated \( \delta_{Z_3}^{(1)} \) and \( \delta_{Z_1}^{(1)} \) in Ref. [3] using state-of-the-art nuclear potentials, i.e., AV18/UIX potentials and EFT N\(^3\)LO (NN) / N\(^2\)LO (NNN) potentials. In Ref. [3], \( \beta \) corresponds to \( \langle r_p^2 \rangle^{1/2} = 0.8409 \) fm from \( ^4\)He Lamb shift measurements [3]. To compare with \( \delta_{Z_{em}} \) based on \( ^4\)He scattering data, for consistency we should use the value of \( \beta \) corresponding to \( \langle r_p^2 \rangle^{1/2} \) from e\(^-\)He data. Using \( \langle r_p^2 \rangle^{1/2} = 0.8775 \) fm [3], we calculated \( \delta_{Z_3}^{(1)} + \delta_{Z_1}^{(1)} \) and obtained 6.12 meV for the AV18/UIX potential and 6.53 meV for the EFT potential. Averaging these two numbers, we obtain \( \delta_{Z_3}^{(1)} + \delta_{Z_1}^{(1)} = 6.32(21) \) meV. This is consistent with the phenomenological value of \( \delta_{Z_{em}} \) mentioned above.

5 Conclusions

We illustrate the calculation of the nuclear corrections to the Lamb shift in muonic atoms, which plays an essential role in the accurate extraction of nuclear charge radii from spectroscopy measurements. We show that the non-relativistic nuclear polarization effects can be systematically expanded in powers of \(|R - R'| \sqrt{2m_e \omega} \). We also discuss finite-nucleon-size corrections. Combining the point-nucleon results with the finite-nucleon-size corrections, we provide an \( ab\)-\textit{initio} calculation of the 3\(^{1}\)\(\Sigma\) Zemach moment in \( ^4\)He\(^+\), which agrees with previous phenomenological calculations.

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