Quantum Fluctuations and Particle Production of Coherently Oscillating Inflaton

Jung Kon Kim and Sang Pyo Kim*

Department of Physics
Kunsan National University
Kunsan 573-701, Korea

Abstract

We study a massive inflaton minimally coupled to the FRW Universe using the semiclassical gravity derived from canonical quantum gravity. It is found that the semiclassical quantum gravity leads to the power-law expansion $t^{2/3}$ in an oscillatory phase of the inflaton. The particle production of the inflaton in the expanding Universe restricts the duration of stable coherent oscillations and parametric resonance. The semiclassical gravity shows a significant difference that the Hubble constant does not oscillate in contrast with the oscillatory behavior in classical gravity.

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*Electronic address: sangkim@knusun1.kunsan.ac.kr
I. INTRODUCTION

In the quantum field theory in a curved spacetime the spacetime is treated as a classical background geometry, while matter fields are quantized. There are two typical approaches to it, one from conventional field theoretical approach \[1\] and the other from canonical quantum gravity \[2\]. Since the full consideration of quantum fluctuating geometry and matter field are not still at hand, it would be rather meaningful to consider the semiclassical gravity obeying the semiclassical Einstein equation

\[
G_{\mu \nu} = \frac{8\pi}{m_P^2} \langle \hat{T}_{\mu \nu} \rangle, \tag{1}
\]

where the quantum field represented by a scalar field $\phi$, is governed by the time-dependent Schrödinger equation \[2\]

\[
i \frac{\partial}{\partial t} \Phi(\phi, t) = \hat{H}_m(\phi, t)\Phi(\phi, t). \tag{2}
\]

To study a quantum inflaton from the point of view of the semiclassical gravity will be of particular importance and interest in cosmology.

To be more specific, we consider a massive inflaton minimally coupled to a spatially flat Friedmann-Robertson-Walker (FRW) Universe with the metric

\[
ds^2 = -N^2(t)dt^2 + a^2(t)d\Omega^2, \tag{3}
\]

where $N(t)$ is the lapse function and $a(t)$ is the scale factor representing the size of the Universe. The time-time component classical equation is the Friedmann equation

\[
\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{8\pi}{3m_P^2} T_{00} a^3(t), \tag{4}
\]

where

\[
T_{00} = a^3(t) \left(\frac{\dot{\phi}^2(t)}{2} + m^2 \frac{\phi^2(t)}{2}\right), \tag{5}
\]

\[1\] We use the units such that $\hbar = c = k_B = 1$ and $G = \frac{1}{m_P^2}$. 

2
is the energy density of the inflaton. The classical equation of motion for the inflaton is

\[ \ddot{\phi}(t) + 3\frac{\dot{a}(t)}{a(t)} \dot{\phi}(t) + m^2 \phi(t) = 0. \] (6)

This is the first scheme of our applying semiclassical gravity to cosmology. The main purpose of this paper is to study the quantum fluctuations and particle production of a coherently oscillating inflaton, a homogeneous massive scalar field minimally coupled to the spatially flat FRW Universe, using the recently introduced semiclassical gravity in which the quantum back reaction of matter field included \[3\]. We find exactly the back-reaction originated from the quantum inflaton which is complex valued and recognize that it looks quite similar as the classical energy density as a result. We resort to the WKB-like solution of the quantum inflaton to study analytically the semiclassical Einstein gravity equation. It is shown that the quantum fluctuations in an oscillatory phase leads to a period of power-law expansion of matter dominated era until it decays into light bosons and that the coherently oscillating inflaton suffers from particle production whose amount is proportional to the duration of coherent oscillations. The particle production can restrict the duration of stable coherent oscillations of the inflaton and affect in a certain way the abundant (catastrophic) particle production due to the parametric resonance of bosonic fields coupled to this coherently oscillating inflaton.

The organization of this paper is as follows. In Section II, we find the quantum states of inflaton and study analytically the semiclassical equation in the oscillatory phase of inflaton. In Section III, we discuss the particle creation due to the quantum fluctuation of the inflaton. And in Section IV, we compare the semiclassical gravity with the classical gravity.

II. QUANTUM STATES OF INFLATON

The massive inflaton in the flat FRW cosmological model is described as a time-dependent harmonic oscillator

\[ H_m = \frac{1}{2a^3} \pi_\phi^2 + \frac{m^2a^3}{2} \phi^2. \] (7)
The Fock space of the Hamiltonian (7) was constructed in [3]

\[ \hat{A}^\dagger(t)\hat{A}(t) |n, \phi, t\rangle = n |n, \phi, t\rangle, \quad (8) \]

where

\[ \hat{A}(t) = \phi^*(t)\hat{\pi}_\phi - a^3(t)\dot{\phi}^*(t)\dot{\phi}. \quad (9) \]

Furthermore, the expectation value of the Hamiltonian was given by a simple form in Ref. [4]. By taking the expectation value with respect to the number state \( |n, \phi, t\rangle \), which is a quantum state of Eq. (2), we obtain the semiclassical Einstein equation

\[ \left( \frac{\dot{a}(t)}{a(t)} \right)^2 = \frac{8\pi}{3m_P^2} \frac{1}{a^3} \langle n, \phi, t | \hat{H}_m | n, \phi, t \rangle \]

\[ = \frac{8\pi}{3m_P^2} \left( n + \frac{1}{2} \right) \left( \dot{\phi}^*(t)\dot{\phi}(t) + m^2\phi^*(t)\phi(t) \right). \quad (10) \]

In the above the \( \phi \) and \( \phi^* \) satisfy Eq. (6) and the boundary condition

\[ a^3(t) \left( \dot{\phi}^*(t)(\dot{\phi}(t) - \phi^*(t)\dot{\phi}(t)) \right) = i. \quad (11) \]

The boundary condition, a Wronskian, fixes the normalization constants of two independent solutions. The boundary condition is a necessary requirement for the exact quantum back-reaction at least for a massive inflaton. It should be remarked that the semiclassical Einstein equation (10) has almost the same form as the Friedmann equation (4) except for its complex valued behavior and that it has the boundary condition entirely determined by Eq. (11) in strong contrast with the classical theory, in which the boundary condition for the scalar field is \( \phi(t_0) \) and \( \dot{\phi}(t_0) \) at an initial time \( t_0 \).

We now solve analytically the self-consistent semiclassical Einstein equation (10). As is well known it is very difficult to find the classical solutions of the scalar field in an analytic form, we rely on the WKB method. Let the solution have the form

\[ \phi(t) = \frac{1}{a^{3/2}(t)}\varphi(t), \quad (12) \]

then
\[ \ddot{\varphi}(t) + \left( m^2 - \frac{3}{4} \left( \frac{\dot{a}(t)}{a(t)} \right)^2 - \frac{3}{2} \frac{\ddot{a}(t)}{a(t)} \right) \varphi(t) = 0. \]  

We focus on the oscillatory phase of inflaton after inflation. In the parameter region of

\[ m^2 > \frac{3\dot{a}^2}{4a^2} + \frac{3\ddot{a}}{2a}, \]  

the inflaton has an oscillatory solution of the form

\[ \varphi(t) = \frac{1}{\sqrt{2\omega(t)}} \exp(-i \int \omega(t) dt), \]  

where

\[ \omega^2(t) = m^2 - \frac{3}{4} \left( \frac{\dot{a}(t)}{a(t)} \right)^2 - \frac{3}{2} \frac{\ddot{a}(t)}{a(t)} + \frac{3}{4} \left( \frac{\dot{\omega}(t)}{\omega(t)} \right)^2 - \frac{1}{2} \frac{\ddot{\omega}(t)}{\omega(t)}. \]

The normalization constant has already been fixed to satisfy the boundary condition \((11)\).

It should be remarked that the vacuum state in \((8)\) is nothing but the adiabatic vacuum \([4]\), since the WKB-type solutions are used.

Now the semiclassical Einstein equation can be rewritten as

\[ a(t) = \left[ \frac{4\pi}{3m_p^2} \left( n + \frac{1}{2} \right) \frac{1}{\omega(t)} \left( \frac{\dot{a}(t)}{a(t)} \right)^2 \left( m^2 + \omega^2(t) + \frac{1}{4} \left( \frac{\dot{\omega}(t)}{\omega(t)} + 3 \frac{\dot{a}(t)}{a(t)} \right)^2 \right) \right]^{1/3}. \]

We solve the Eq. \((17)\) perturbatively. Starting from an approximation ansatz \(\omega_0(t) = m, \ a_0(t) = a_0 t^{2/3}\), we obtain the next order perturbative solution

\[ a_1(t) = \left[ \frac{6\pi}{m_p^2} \left( n + \frac{1}{2} \right) m t^2 \left( 1 + \frac{1}{2m^2 t^2} \right) \right]^{1/3}. \]

and

\[ \omega_1(t) = m \left[ 1 + \frac{1}{2m^4 t^4 \left( 1 + \frac{1}{2m^4 t^2} \right)^2} \right]^{1/2}. \]

The perturbative solutions agree with those in Appendix obtained from the fixed background \(a(t) = a_0 t^{2/3}\). During the later stage \((mt > 1)\) of evolution after inflation, the expansion-law is that of the matter dominated era \(a_1(t) \sim t^{2/3} \) \([5]\). The power-law expansion of the Universe continues until the inflaton decays into light bosons \(\chi\) through interactions of the form \(\lambda_1 \phi \chi^2, \lambda_2 \phi^2 \chi^2\), and etc.
III. QUANTUM FLUCTUATION AND PARTICLE PRODUCTION

The quantum fluctuations of the inflaton can be found from the dispersion relations

\[
(\Delta \phi)^2_n = \langle n, \phi, t | (\dot{\phi} - <\dot{\phi}>_n)^2 | n, \phi, t \rangle = \phi_c^*(t)\phi_c(t)(2n + 1),
\]

(20)

and

\[
(\Delta \pi^\phi)^2_n = a^6(t)\dot{\phi}_c^*(t)\dot{\phi}_c(t)(2n + 1).
\]

(21)

In the limit \(mt >> 1\), the dispersions become

\[
(\Delta \phi)_n = \sqrt{\frac{1}{a^3_m t^2}} \left( n + \frac{1}{2} \right)
\]

\[
(\Delta \pi^\phi)_n = \sqrt{a^3_m t^2 \left( 1 + \frac{1}{m^2 t^2} \right)} \left( n + \frac{1}{2} \right).
\]

(22)

The dispersion of the field decreases inversely proportionally to \(t\). This means that the quantum field \(\dot{\phi}\) becomes strongly peaked along the trajectory, but it does not describe the coherent state along the classical trajectory, instead \(<\dot{\phi}> = 0\). Since the dispersion of momentum increases proportionally to \(t\), the uncertainty of the quantum state keeps an asymptotical constant value

\[
(\Delta \phi)_n(\Delta \pi^\phi)_n = \sqrt{1 + \frac{1}{m^2 t^2}} \left( n + \frac{1}{2} \right).
\]

(23)

The uncertainty criterion on the emergence of classical fields [6] contrasts with the common view that the inflaton behaves as a classical background field after inflation, i.e. in the oscillatory phase. It has a physical significance to keep the quantum properties of the oscillating inflaton.

The particle production can be explained correctly by semiclassical gravity. The Fock space used in this paper has a one-parameter dependence on the cosmological time \(t\). The vacuum defined at the initial time \(t_0\) contains particles at a later time \(t\).
\[ N_0(t, t_0) = \langle 0, \phi, t_0 | \hat{N}(t) | 0, \phi, t_0 \rangle = a^6 |\phi(t)\dot{\phi}(t_0) - \dot{\phi}(t)\phi(t_0)|^2, \] 

(24)

and the number of particle produced in the \( n \)th quantum state is

\[ \langle n, \phi, t_0 | \hat{N}(t) | n, \phi, t_0 \rangle = n + (2n + 1)N_0(t, t_0). \] 

(25)

In the limit \( mt_0, mt > 1 \), we use the perturbation solution to get the number of created particle

\[ N_0(t, t_0) = \frac{1}{4\omega(t)\omega(t_0)} \left( \frac{a(t)}{a(t_0)} \right)^3 \left[ \frac{1}{4} \left( \frac{3}{a(t)} \frac{\dot{a}(t)}{a(t_0)} - \frac{3}{a(t_0)} \frac{\dot{a}(t_0)}{a(t)} - \frac{\dot{\omega}(t)}{\omega(t)} + \frac{\omega(t)}{\omega(t_0)} \right)^2 + (\omega(t) - \omega(t_0))^2 \right] \]

\[ \approx \frac{(t - t_0)^2}{4m^2t_0^4}. \] 

(26)

The inflaton after inflation cannot execute coherent oscillations for a sufficiently long interval of time since it suffers from the instability due to the particle production. This means that the inflaton decays into these light bosons when the inflaton is coupled to other light bosonic fields. From the time interval for negligible particle production

\[ N_0(t, t_0) \leq q \] 

(27)

where \( q = O(1) \), we find the duration of stable coherent oscillations

\[ \Delta t < \frac{(mt_0)^2 q}{\pi} T, \] 

(28)

where \( T = \frac{2\pi}{m} \) is the period of the inflaton oscillation. The condition (14) for the oscillations immediately after inflation restricts the interval to the lowest limit

\[ \Delta t < \frac{qT}{\sqrt{2\pi}}. \] 

(29)

We discuss effects on the preheating of the particle production of the inflaton due to the expansion of the Universe. The parametric resonance, an abundant particle production mechanism, requires a sufficient time for the periodic motion of scalar fields coupled to the coherently oscillating background inflaton [7]. As explained above the expansion of the
Universe restricts the duration of the stable coherent oscillations of the inflaton and therefore the amount of particle production due to the parametric resonance [8]. The expansion of the Universe also decreases directly the particle production due the parametric resonance [9]. The limit on the duration of stable coherent oscillations, however, does not exclude completely the preheating. We may be able to get a sufficient time for the stable coherent oscillations if the oscillatory phase of the inflaton begins at a later stage $mt_0 >> 1$, which depends on the inflation models. Semiclassical gravity treatment of bosonic fields coupled to the background inflaton will be done quantitatively in a separate publication.

IV. COMPARISON WITH CLASSICAL GRAVITY

There are several big differences between the classical and semiclassical gravity approaches.

First of all, the boundary condition in the semiclassical gravity shows quite a different feature from that of the classical gravity. In the classical gravity one can choose an arbitrary initial value $\phi_r(t_0)$ and $\dot{\phi}_r(t_0)$. In the oscillatory phase of the inflaton, the initial value can be redefined in terms of the amplitude and phase

$$\phi_r(t) = \frac{A_0}{\alpha^{3/2}(t) \sqrt{\omega(t)}} \sin(\int \omega(t) dt + \delta).$$  \hspace{1cm} (30)

In fact, thanks to the freedom in the choice of the initial values of the inflaton, the chaotic inflation models for a broad class of potentials become possible. However, one can not choose arbitrarily the initial values of the inflaton any more in the semiclassical gravity (10). Rather, the amplitude of the inflaton is fixed by the quantum number $(n + \frac{1}{2})$ and the time-dependence enters only through the classical solutions that satisfy the boundary condition (11). We may choose an arbitrary overall phase factor for the complex solution since it does not affect the boundary condition (11) and the semiclassical equation (10).

Second, the semiclassical equation (10) does not show any oscillatory behavior even in the oscillatory phase of the inflaton. The energy density of quantum state is given by
\[ \langle n, \phi, t | \hat{H}_m | n, \phi, t \rangle = \left( n + \frac{1}{2} \right) \frac{\hat{\omega}}{2\omega} \left( m^2 + \omega^2 + \frac{1}{4} \left( \frac{\hat{\omega}}{\omega} + \frac{3}{a} \right)^2 \right) \]  

(31)

On the other hand, the classical energy density

\[ H_m = \frac{A_0^2}{2} \frac{1}{2\omega} \left[ m^2 + \omega^2 + \frac{1}{4} \left( \frac{\hat{\omega}}{\omega} + \frac{3}{a} \right)^2 + \left( \omega - \frac{1}{4} \left( \frac{\hat{\omega}}{\omega} + \frac{3}{a} \right) \cos 2(\int \omega dt + \delta) \right) \right. \]

\[ \left. - \left( \frac{\hat{\omega}}{\omega} + \frac{3}{a} \right)^2 \sin 2(\int \omega dt + \delta) \right] \]  

(32)

shows manifestly the oscillatory behavior. As emphasized in Ref. [10], the oscillating terms determine in a significant way the evolution of geometric invariants through the higher derivatives of the scale factor \( a(t) \). The time average over several oscillations gives

\[ \langle H_m \rangle_{t-a} = \frac{A_0^2}{2} \frac{1}{2\omega} \left( m^2 + \omega^2 + \frac{1}{4} \left( \frac{\hat{\omega}}{\omega} + \frac{3}{a} \right)^2 \right) \]  

(33)

assuming that the expansion of the Universe can be neglected during the time interval. The expansion of the Universe due to the time averaged energy density is nearly the same as that by pressureless dusts. We may identify the classical amplitude with \( A_0 = \sqrt{2n + 1} \).

**V. CONCLUSION**

In this paper we studied analytically the semiclassical gravity of a massive inflaton minimally coupled to the spatially flat Friedmann-Robertson-Walker Universe. It was found that the back-reaction of the quantum inflaton obeying the time-dependent Schrödinger equation leads to the power-law expansion \( t^{2/3} \) of the Universe in the oscillatory phase. The power-law expansion is the same as that of the matter (dust particles) dominated era, but not \( t^{1/2} \) of relativistic particles. The criterion based on the uncertainty shows that the oscillating inflaton after inflation is a quantum field rather than a classical one.

The expansion of the Universe driven by the inflaton causes in turn the particle production of the inflaton itself. The particle production can restrict in a certain way the time interval of stable coherent oscillations. Thus due to the particle production of the inflaton originating from the expansion of the Universe, the preheating mechanism is subject to the
limited duration of stable coherent oscillations. This, however, does not mean the complete exclusion of the preheating as a mechanism for the abundant particle and entropy production needed for the formation of the present Universe. The number of particles produced from the light bosons coupled to the inflaton executing coherent oscillations for a limited period shows a power-law increase rather than an exponential increase in the case of the non-decaying inflaton oscillating stably for an infinite period in a static universe.

We also discussed the difference of the boundary conditions between semiclassical and classical gravity. One prominent difference is that the semiclassical gravity does not show any oscillatory behavior of the Hubble constant in strong contrast with the oscillatory behavior of classical gravity. Furthermore, the solution of the inflaton in semiclassical gravity is fixed by the quantum number up to an overall phase factor.

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APPENDIX A: QUANTUM INFLATON IN A FIXED BACKGROUND GEOMETRY

We find the exact quantum states of the inflaton assuming that the expansion of the Universe is fixed by the power-law $a(t) = a_0 t^{2/3}$. The classical field equation of the inflaton

$$\ddot{\phi}_b(t) + \frac{2}{t} \dot{\phi}_b(t) + m^2 \phi_b(t) = 0,$$  \hspace{1cm} (A1)

has a complex solution

$$\phi_b(t) = \sqrt{\frac{\pi}{4a_0^3 t}} H^{(2)}_{-1/2}(mt).$$  \hspace{1cm} (A2)

The complex solution satisfies the boundary condition \([\text{III}]\). Following Ref. \([\text{I}]\), we construct the Fock space of the inflaton in terms of the creation and annihilation operator.
\[ \hat{A}^\dagger(t) = \phi_b(t) \dot{\pi}_\phi - a_0^3 t^2 \dot{\phi}_b(t) \dot{\phi}, \]
\[ \hat{A}(t) = \phi_b^*(t) \dot{\pi}_\phi - a_0^3 t^2 \dot{\phi}_b^*(t) \dot{\phi}. \]  

(A3)

For \( m t >> 1 \) we obtain the asymptotic form

\[ \phi_b(t) = \sqrt{\frac{1}{2a_0^3 m t}} e^{-imt}, \]  

(A4)

which has the same form as the WKB solution obtained without fixing the background geometry.
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