On the mystery of the missing pie in graphene

October 1, 2009

Paola Giacconi
Istituto Nazionale di Fisica Nucleare,
Sezione di Bologna, 40126 Bologna, Italy

Roberto Soldati
Dipartimento di Fisica, Università di Bologna,
Istituto Nazionale di Fisica Nucleare,
Sezione di Bologna, 40126 Bologna, Italy

Abstract
We investigate in some detail the structure of the electromagnetic current density for the pseudo-relativistic massless spinor effective model for graphene. It is shown that the pseudo-relativistic massless Dirac field theory in $2+1$ space-time dimensions and in the presence of a constant homogeneous electric field actually leads to the measured current density and to the minimum quantum conductivity.

PACS numbers: 11.10.Wx, 02.30.Sa, 73.43.-f

1 Introduction

One among the many far intriguing features of graphene is that its conductivity at zero magnetic field doesn’t vanish close to the charge carriers neutrality point of concentration $n \leq 10^{-11} \text{ cm}^{-2}$ . Actually, on the one hand, the measured values for the lowest conductivity $\sigma_{\text{min}}$ near the neutrality point are very close to the conductivity quantum $e^{2}/h$ per carrier type. This zero-field quantum conductivity does not apparently depend on the
chemical potential and keeps its single carrier value down to liquid helium temperatures [1]. On the other hand, it turns out that the insofar known theoretical predictions fail [3] in reproducing $\sigma_{\text{min}}$, a disagreement known as the mystery of the missing pie, but for a very recent calculation [4]. There, however, the deduction for the actual value of $\sigma_{\text{min}}$ is certainly elegant and correct but admittedly rather formal. In particular, the direction in the graphene’s sample plane of the minimum quantum current density is not at all clear from the above mentioned theoretical derivation.

As a further important ambipolar electric field effect, which was not explained in [4], it turns out that the single layer graphene resistivity $\rho$ decreases rapidly to zero with adding charge carriers, showing their very high-mobility [1]. It is the aim of the present short note to fulfill this gap, by developing a new derivation for the canonical quantum current density, within the well established effective model [8, 9] of a massless Dirac field theory in 2+1 space-time dimensions and in the presence of constant and homogeneous electromagnetic classical background fields. In fact, the latter field theoretic model appears to be exactly solvable, in such a manner that all the gauge invariant physical observables, like e.g. the minimum quantum conductivity, can be explicitly calculated and eventually compared with experimental data.

In ref. [4] it has been shown that the Euclidean effective action, or the grand potential in statistical mechanics language, in the presence of a background uniform electrostatic field turns out to be independent from the (complex) chemical potential and from the temperature. Conversely, in the presence of a uniform magnetic field, the dependence of the effective action upon the chemical potential is crucial in order to reproduce the measured values for the quantum Hall conductivity [5]. It follows thereby that in the present note we can restrict ourselves, without any loss of generality, to zero temperature and zero chemical potential planar quantum electrodynamics in a constant homogeneous electric field. Hence, according to the leading order expansion around the Dirac points at the corner of the Brillouin zone in graphene [6], the classical pseudo-relativistic lagrangian in the Nikishov [10] temporal gauge $A^\mu = (0, -Ect, 0)$ reads

$$\mathcal{L} = \bar{\psi}(x) \left( i\hbar \partial_\mu + \frac{e}{c} A_\mu \right) \tilde{\gamma}^\mu \psi(x) \quad \bar{\psi} = \psi^\dagger \sigma_1$$

Here $(-e)$ is the electron charge, $c$ the light velocity in vacuum, $v_F \approx 1/300$ is the Fermi velocity of the massless Dirac quasiparticles in graphene, $\tilde{\gamma}^\mu$ are
the rescaled Dirac matrices, i.e.,
\[ \tilde{\gamma}^0 = c \gamma^0 \quad \tilde{\gamma}^\ell = v_F \gamma^\ell \quad (\ell = 1, 2) \]
while \( F_{01} = E_x = E > 0 \) denotes the constant and homogeneous electric field pointing towards the positive \( OX \) axis. We remark that in the planar QED on the \( 2+1 \) dimensional Minkowski space-time the spinor field has canonical dimensions of the inverse of a length, i.e., of a wave number.

## 2 Uniform electric field

Consider now a graphene sample in the presence of a constant electrostatic field pointing towards the positive \( OX \) axis, i.e.,
\[ F_{01} = F_{10} = E_x = E \quad (E > 0) \]
In this section, we will choose a gauge which leads to non-stationary sets of solutions. In particular, to solve the Dirac equation in the present \( 2+1 \) dimensional massless case, it is convenient to employ the representation for the gamma matrices
\[ \gamma^0 = \sigma_1 \quad \gamma^1 = i \sigma_2 \quad \gamma^2 = i \sigma_3 \quad (2) \]
After setting \((x^0, x^1, x^2) = (ct, x, y)\) we get the massless Dirac operator in the Nikishov temporal gauge \( A_\mu = (0, Ect, 0) \), i.e.,
\[ i\slashed{\partial} = \frac{i\hbar}{v_F} \partial_t \sigma_1 + (i\hbar \partial_x + eEt) i\sigma_2 - \hbar \partial_y \sigma_3 \quad (3) \]
It follows that in the Nikishov temporal gauge the 1-particle hamiltonian
\[ H_t = \hbar v_F \left[ \sigma_3 \left( i \partial_x - eEt/\hbar \right) - i \partial_y \sigma_2 \right] \quad (4) \]
is explicitly time dependent, which means that the energy does not lead to a good quantum number for the specification of the Fock space states of the quantized massless Dirac spinor field. Thus, at variance with the magnetic field case, in the present context the Dirac hamiltonian does not play here any key role. Notice that The classical Action for the massless Dirac field in
the 2+1 dimensional Minkowski space-time and in the presence of a uniform background electric field takes the form

\[ \mathcal{A} = c \int_{-\infty}^{\infty} dt \int d\mathbf{r} \ \bar{\Psi}(t, \mathbf{r}) \left[ \frac{\mathit{i\hbar}}{v_F} \partial_t \sigma_1 + (\mathit{i\hbar} \partial_x + eEt) i \sigma_2 - \hbar \partial_y \sigma_3 \right] \Psi(t, \mathbf{r}) \]

whence it is apparent that the canonical physical dimensions of the spinor field are equal to the inverse of a length. In order to obtain the solutions in this gauge, it is convenient to introduce the spatial Fourier transforms

\[ \Psi(t, x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dp \ e^{ipx + iky} \tilde{\Psi}(t, p, k) \]

with \( \mathbf{p} = (p_x, p_y) \equiv (p, k) \) as well as the quantum electric length

\[ \ell_E \equiv \sqrt{\frac{\hbar v_F}{eE}} \]

and the dimensionless quantities

\[ \xi \equiv \left( p - \frac{eEt}{\hbar} \right) \ell_E \quad \lambda \equiv k^2 \ell_E^2 \]

so that the Dirac operator can be recast in the suitable form

\[ i\not{D} = \frac{\hbar}{\ell_E} \begin{pmatrix} -ik\ell_E & -\mathit{id}_\xi - \xi \\ -\mathit{id}_\xi + \xi & ik\ell_E \end{pmatrix} \]

If we set

\[ \Psi(t, x, y) = \begin{pmatrix} \varphi(t, x, y) \\ \chi(t, x, y) \end{pmatrix} \quad \tilde{\Psi}(t, p, k) = \begin{pmatrix} \tilde{\varphi}(t, p, k) \\ \tilde{\chi}(t, p, k) \end{pmatrix} \]

we obtain the coupled differential equations

\[ \begin{cases} ik\ell_E \tilde{\varphi} + (\mathit{id}_\xi + \xi) \tilde{\chi} = 0 \\ (-\mathit{id}_\xi + \xi) \tilde{\varphi} + ik\ell_E \tilde{\chi} = 0 \end{cases} \]

Then, we can write

\[ \begin{cases} \tilde{\chi} = (\mathit{i/k\ell_E})(-\mathit{id}_\xi + \xi) \tilde{\varphi} \\ (d_\xi^2 + \xi^2 + \lambda + i) \tilde{\varphi} = 0 \quad (k \neq 0) \end{cases} \]
\[
\begin{cases}
(id_\xi + \xi) \tilde{\chi} = 0 \\
(-id_\xi + \xi) \tilde{\varphi} = 0
\end{cases} \quad (k = 0)
\]

Thus, in the presence of a homogeneous electric static field, one actually finds the following complete and orthonormal sets of time dependent solutions of the Dirac equation: namely, for \( k \neq 0 \) we have either the \textit{outgoing normal modes}

\[
\begin{aligned}
\left. u_{p^+}(t, \mathbf{r}) = \frac{1}{2\pi \ell_E} \exp \left\{ i \mathbf{p} \cdot \mathbf{r} - \frac{1}{8} \pi \lambda \right\} \times \left\{ \frac{1}{2} (1 + i) \sqrt{\lambda} D_{i\lambda/2}(-z_-) \right\} 
\right. \\
& \left. \times \left\{ D_{i\lambda/2}(-z_-) \right\} \right)
\end{aligned}
\]

which correspond to the normal modes of an outgoing quasiparticle (massless electron or negatively charged neutrino) with momentum \( \mathbf{p} \) and charge \(-e\), while

\[
\begin{aligned}
\left. v_{p^+}(t, \mathbf{r}) = \frac{1}{2\pi \ell_E} \exp \left\{ i \mathbf{p} \cdot \mathbf{r} - \frac{1}{8} \pi \lambda \right\} \times \left\{ \frac{1}{2} (1 - i) \sqrt{\lambda} D_{-i\lambda/2}(-z_-) \right\} \right.
\end{aligned}
\]

(13)

do correspond to the normal modes of an outgoing antiquasiparticle (hole) with momentum \( \mathbf{p} \) and charge \( e \), where we have set

\[
z_\pm \equiv (1 \pm i) \xi
\]

Furthermore the \textit{incoming normal modes} will be given by

\[
\begin{aligned}
\left. u_{p^-}(t, \mathbf{r}) = \frac{1}{2\pi \ell_E} \exp \left\{ i \mathbf{p} \cdot \mathbf{r} - \frac{1}{8} \pi \lambda \right\} \times \left\{ -\frac{1}{2} (1 + i) \sqrt{\lambda} D_{i\lambda/2}(-z_-) \right\} \right.
\end{aligned}
\]

(14)

which correspond to the normal modes of an incoming quasiparticle (massless electron or negatively charged neutrino) with momentum \( \mathbf{p} \) and charge \(-e\), while

\[
\begin{aligned}
\left. v_{p^-}(t, \mathbf{r}) = \frac{1}{2\pi \ell_E} \exp \left\{ i \mathbf{p} \cdot \mathbf{r} - \frac{1}{8} \pi \lambda \right\} \times \left\{ \frac{1}{2} (1 - i) \sqrt{\lambda} D_{-i\lambda/2}(-z_-) \right\} \right.
\end{aligned}
\]

(15)

\footnote{The complete and orthonormal sets of solutions listed below do coincide with those ones obtained in \cite{4} up to the overall dimensional factor \( \ell_E^{-1} \).}
do correspond to the normal modes of an incoming antiquasiparticle (hole) with momentum $p$ and charge $e$.

Finally, for $p_y = k = 0$ we come to the so called longitudinal normal modes or zero modes

$$u_{p\pm}(t, x) = \frac{1}{\sqrt{2\pi \ell_E}} \exp \left\{ ipx + \frac{i}{2} \xi^2(t) \right\} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv u_p(t, x) \quad (16)$$

$$v_{p\pm}(t, x) = \frac{1}{\sqrt{2\pi \ell_E}} \exp \left\{ ipx - \frac{i}{2} \xi^2(t) \right\} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv v_p(t, x) \quad (17)$$

The above sets of time dependent massless Dirac spinors have canonical physical dimensions of a wave number, as it does, and are normalized in order to satisfy the following orthonormality relations, viz.,

$$\int dr \ u_{q\pm}^\dagger(t, r) \ u_{p\pm}(t, r) = \ell_E^{-2} \delta(p - q) \quad (18)$$

$$\int dr \ v_{q\pm}^\dagger(t, r) \ v_{p\pm}(t, r) = \ell_E^{-2} \delta(p - q) \quad (19)$$

$$\int dr \ u_{q\pm}^\dagger(t, r) \ v_{p\pm}(t, r) = 0 \quad \forall \ p, q \in \mathbb{R}^2 \quad (20)$$

$$\int_{-\infty}^{\infty} dx \ u_q^\dagger(t, x) \ u_p(t, x) = \ell_E^{-2} \delta(p - q) \quad (21)$$

$$\int_{-\infty}^{\infty} dx \ v_q^\dagger(t, x) \ v_p(t, x) = \ell_E^{-2} \delta(p - q) \quad (22)$$

$$\int_{-\infty}^{\infty} dx \ u_q^\dagger(t, x) \ v_p(t, x) = 0 \quad \forall \ p, q \in \mathbb{R} \quad (23)$$

$$\int dr \ u_{q\pm}^\dagger(t, r) \ u_{p\pm}(t, x) = 0 = \int dr \ u_{q\pm}^\dagger(t, r) \ v_{p\pm}(t, x) \quad (24)$$

$$\int dr \ v_{q\pm}^\dagger(t, r) \ v_{p\pm}(t, x) = 0 = \int dr \ v_{q\pm}^\dagger(t, r) \ u_{p\pm}(t, x) \quad (25)$$
The electric current density vector is provided by the invariance under U(1) phase transformations, according to the Noether theorem, and reads

\[ c J_0(t, r) = \varrho(t, r) = q \Psi(t, x, y) \Psi_\ast(t, x, y) \]  
\[ J_x(t, r) = q v_F \bar{\Psi}(t, x, y) \gamma^1 \Psi(t, x, y) = -q v_F \Psi_\ast(t, x, y) \sigma_3 \Psi(t, x, y) \]  
\[ J_y(t, r) = q v_F \bar{\Psi}(t, x, y) \gamma^2 \Psi(t, x, y) = q v_F \Psi_\ast(t, x, y) \sigma_2^\ast \Psi(t, x, y) \]  

(26) (27) (28)

where \( q \) is the charge. For example, an outgoing particle (massless electron or negatively charged neutrino) of momentum \( p = (p, k) \), \( k \neq 0 \), and negative charge \(-e\) carries the spatially homogeneous current density

\[ \varrho = -e \ u_\ast_{p+}(t, r) u_{p+}(t, r) = -\frac{e^2 E}{2\pi \hbar v_F} \]  
\[ J_x(t ; p) = -\frac{e^2 E}{2\pi \hbar} \exp \{ -\frac{1}{4} \pi \lambda \} \]  
\[ \times \left( | D_{i\lambda/2}(-z_-) |^2 - \frac{1}{2} \lambda | D_{i\lambda/2-1}(-z_-) |^2 \right) \]  
\[ J_y(t ; p) = \frac{ie^2 E}{2\pi \hbar} \exp \{ -\frac{1}{4} \pi \lambda \} \]  
\[ \times D_{i\lambda/2}(-z_-) \frac{1}{2} (1 - i) \sqrt{\lambda} \ D_{-i\lambda/2-1}(-z_+) + \text{c.c.} \]  

(29) (30) (31)

Now, since the transverse momentum of a normal mode solution is different from zero by definition, it turns out that in the weak field limit \( \lambda \to \infty \) the current density components of all the normal modes become exponentially small, for any outgoing and/or incoming particle and/or antiparticle normal mode massless Dirac spinor. Conversely, for the remaining set of longitudinal normal modes with \( p_y = k = 0 \) we immediately find for quasiparticles

\[ \varrho = -\frac{e^2 E}{\hbar v_F}, \quad J_x = -\frac{e^2 E}{\hbar}, \quad J_y = 0, \quad \forall p \in \mathbb{R} \]  

(32)

while for antiquasiparticles we obviously find

\[ \varrho = \frac{e^2 E}{\hbar v_F}, \quad J_x = \frac{e^2 E}{\hbar}, \quad J_y = 0, \quad \forall p \in \mathbb{R} \]  

(33)

This actually means that the only observable nonvanishing current density, in the weak field limit, is the longitudinal one, so that the minimum quantum conductivity\(^2\) for a graphene sample in the presence of a uniform electric field

\(^2\)Notice that in physical units we have \( e^2/\hbar \approx 3.5 \times 10^7 \ \text{cm} \ \text{s}^{-1} \approx 4 \times 10^{-5} \ \Omega^{-1} \)
is provided by \( \sigma_{\text{min}} = 4e^2/h \simeq 1.4 \times 10^8 \text{ cm s}^{-1} \simeq 1.5 \times 10^{-4} \Omega^{-1} \) as indeed observed experimentally, which entails \( \rho_{\text{max}} \simeq 6.6 \text{ K}\Omega \), where the factor four is due \([6, 7]\) to the two inequivalent irreducible two-dimensional spinor representations, as well as to the so-called valley degeneracy. We remark that the present value of \( \sigma_{\text{min}} \) is precisely the very same obtained in \([4]\), but from a quite different method based upon the Euclidean effective action.

On the other side, i.e., in the strong field limit \( \lambda \to 0 \), we readily get for graphene quasiparticles

\[
\rho(\lambda = 0) = \begin{cases} 
-2e^2E/\pi hv_F, & \forall p \in \mathbb{R}^2 \vee k \neq 0 \\
-4e^2E/hv_F, & \forall p \in \mathbb{R} \vee k = 0 
\end{cases} \tag{34}
\]

\[
J_x(\lambda = 0) = \begin{cases} 
-2e^2E/\pi h, & \forall p \in \mathbb{R}^2 \vee k \neq 0 \\
-4e^2E/h, & \forall p \in \mathbb{R} \vee k = 0 
\end{cases} \tag{35}
\]

\[
J_y(\lambda = 0) = 0, \quad \forall p \in \mathbb{R}^2 \tag{36}
\]

while for antiquasiparticles we obviously find the opposite signs. This means that all the quantum states give leading constant contributions, in the strong field regime, to the conductivity of a graphene single layer sample, i.e., \( \mp 4e^2/2\pi h \) for the quasiparticle/antiquasiparticle normal modes with \( k \neq 0 \), while \( \mp 4e^2/h \) for the corresponding zero modes. This feature nicely explains the rapid fall down of graphene resistivity with increasing the charge carriers density, viz., their very high mobility. As a matter of fact, if there are \( N \) electrons in the graphene sample, for instance, then the leading resistivity for a sufficiently strong electric field becomes

\[
\rho \sim 0 \frac{-h}{4Ne^2} \times \begin{cases} 
1 & \text{for } k = 0 \\
2\pi & \text{for } k \neq 0 
\end{cases} \tag{37}
\]

which is vanishing for e.g. a typical density \( n = 10^{19} \text{ electrons cm}^{-2} \), as experimentally observed.

3 Discussion and conclusions

In this note we have analysed the exact current density of graphene within the pseudo-relativistic effective field theoretic model of a massless Dirac field in a 2+1 dimensional space-time under the influence of a uniform background electromagnetic field. We have shown that, in the limit of a vanishing electric field, a minimum quantum conductivity \( \sigma_{\text{min}} \) does survive, which is entirely
due to the longitudinal zero modes, i.e. with \( k = 0 \), and the actual value of which is in agreement with the experimental finding. Moreover, it has been proved that in the strong field regime all the charged graphene quantum states provide a leading, dominant, constant contribution to the conductivity, a feature that simply explains the high charge carriers mobility and the rapid fall down of resistivity with increasing number of the charge carriers. This is the main prediction of the massless, pseudo-relativistic planar QED effective model for graphene, in the presence of a constant homogeneous electric field. In turn, the related minimum quantum conductivity perfectly reproduces its measured values, at variance with many previously obtained different results \([3]\) based upon the Kubo formula. The ultimate reason for this might be that our exact solutions are truly non-perturbative, viz., they do not reproduce the free field spinor when \( E \to 0 \), whilst the Kubo approach is a perturbative linear response approximation.

**Acknowledgements**

We are grateful to N. Protasov for a fruitful and stimulating correspondence. We warmly thank Carlota Gabriela Beneventano and Eve Mariel Santangelo for a careful reading of the manuscript and enlightening comments. We wish to acknowledge the support of the Istituto Nazionale di Fisica Nucleare, Iniziativa Specifica PI13, that contributed to the successful completion of this project.

**References**

[1] A.K. Geim and K.S. Novoselov *Nature Materials* **6**, 183 (2007).

[2] K.S. Novoselov, A.K. Geim, S.V. Morozov, D. Jiang, M.I. Katsnelson, I.V. Grigorieva, S.V. Dubonos and A.A. Firsov, *Nature* **438**, 197 (2005).

[3] A.W.W. Ludwig, M.P.A. Fisher, R. Shankar, and G. Grinstein, *Phys. Rev. B* **50**, 7526 (1994); K. Ziegler, *Phys. Rev. B* **55**, 10661 (1997); *Phys. Rev. Lett.* **80**, 3113 (1998); M.I. Katsnelson, *Eur. Phys. J. B* **51**, 157 (2006); N.M.R. Peres, F. Guinea, and A.H. Castro Neto, *Phys. Rev. B* **73**, 125411 (2006); J. Tworzydlo, B. Trauzettel, M. Titov, A. Rycerz, and C. Beenakker, *Phys. Rev. Lett.* **96**, 246802 (2006); J. Cserti, *Phys. Rev. B*
75, 033405 (2007); P.M. Ostrovsky, I.V. Gornyi, A.D. Mirlin, Phys. Rev. B 74, 235443 (2006); S. Ryu, C. Mudry, A. Furusaki, A.W.W. Ludwig, e-print ArXiv: cond-mat/0610598; L.A. Falkovsky and A.A. Varlamov, e-print ArXiv: cond-mat/0606800; K. Ziegler, Phys. Rev. Lett. 97, 266802 (2006); V.P. Gusynin and S.G. Sharapov, Phys. Rev. B 66, 045108 (2002); Phys. Rev. Lett. 95, 146801 (2005); Phys. Rev. B 73, 245411 (2006).

[4] C.G. Beneventano, Paola Giacconi, E.M. Santangelo and Roberto Soldati, J. Phys. A: Math. Theor. 42, 275401 (2009); arXiv:0901.0396v3 [hep-th]

[5] C.G. Beneventano, Paola Giacconi, E.M. Santangelo and Roberto Soldati, J. Phys. A: Math. Theor. 40, F435 (2007).

[6] V.P. Gusynin, S.G. Sharapov, J.P. Carbotte, Int. Jour. Mod. Phys. B 21, 4611 (2007).

[7] A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov and A. K. Geim, Rev. Mod. Phys. 81, 109 (2009).

[8] G.W. Semenoff, Phys. Rev. Lett. 53, 2499 (1984).

[9] D.P. DiVincenzo and E.J. Mele, Phys. Rev. B 29, 1685 (1984).

[10] A.I. Nikishov, Soviet Physics JEPT 30, 660 (1970), transl. Zh. Eksp. Teor. Fiz. 57, 1210 (1969); ArXiv: 0207085v2, 0211088v2 [hep-th].

[11] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series, and Products, Fifth Edition, Alan Jeffrey Editor, Academic Press, San Diego (CA) 1996.

[12] M. Abramowitz and I.A. Stegun, Handbook of Mathematical Formulas, Graphs and Mathematical Tables, Dover, New York, 1978.