About a question of Gateva-Ivanova and Cameron on square-free set-theoretic solutions of the Yang-Baxter equation

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1. Introduction

The Yang-Baxter equation is one of the basic equations in mathematical physics. Finding all the solutions of this equation is a difficult task, for this reason Drinfeld [8] posed the question of finding a particular subclass of these solutions, the so-called set-theoretic solutions. A set-theoretic solution is a pair \((X, r)\), where \(X\) is a non-empty set and \(r : X \times X \rightarrow X \times X\) a bijective map, satisfying

\[ r_1 r_2 r_1 = r_2 r_1 r_2, \]

where \(r_1 := r \times \text{id}_X\) and \(r_2 := \text{id}_X \times r\). Recall that, if \(\lambda_x : X \rightarrow X\) and \(\rho_y : X \rightarrow X\) are maps such that

\[ r(x, y) = (\lambda_x(y), \rho_y(x)) \]

for all \(x, y \in X\), the set-theoretic solution of the Yang-Baxter equation \((X, r)\) is said to be a left [right] non-degenerate if \(\lambda_x \in \text{Sym}(X)\) and \(\rho_x \in \text{Sym}(X)\) for every \(x \in X\) and non-degenerate if it is left and right non-degenerate. Moreover, a solution is called involutive if \(r^2 = \text{id}_{X \times X}\) and square-free if \(r(x, x) = (x, x)\) for every \(x \in X\). Recall that, if \((X, r)\) is an involutive non-degenerate solution, it is possible to consider an equivalence relation \(\sim\) on \(X\) which induces in a classical way an involutive solution \(\text{Ret}(X, r) := (X/\sim, \bar{r})\), the so-called retraction of \((X, r)\) (for more details see [9]). An involutive non-degenerate solution \((X, r)\) is said to be retractable if the relation \(\sim\) coincides with the trivial equivalence relation on \(X\), i.e., if \(x \sim y\) if and only if \(x\) is equal to \(y\); otherwise \((X, r)\) is called retractable. Moreover, an involutive solution is called multipermutational of level \(m\) and we write \(\text{mpl}(X, r) = m\) if \(|\text{Ret}^{m-1}(X, r)| > 1\) and \(|\text{Ret}^m(X, r)| = 1\), where \(\text{Ret}^k(X, r)\) is defined inductively as \(\text{Ret}^k(X, r) := \text{Ret}(\text{Ret}^{k-1}(X, r))\), for every natural number \(k\) greater than 1. In particular, because of their links with other algebraic structures,
multipermutational square-free involutive solutions have been considered in several papers [6, 7, 11, 18]. In that regard, several methods to construct multipermutational solutions were developed: for example, Cedó, Jespers, and Okniński [7] constructed the first family of square-free involutive solutions $X_m$ of level $m$ and abelian associated permutation group. In 2011, Gateva-Ivanova and Cameron [11] posed the following question:

**Question.** [11, Open Question 6.13] For each positive integer $m$ denote by $N_m$ the minimal integer so that there exists a square-free involutive multipermutational solution $(X_m, r_m)$ of order $|X_m| = N_m$, and with $mpl(X_m, r_m) = m$. How does $N_m$ depend on $m$?

They showed that $N_m \leq 2^{m-1} + 1$ and asked if the equality holds for every $m \in \mathbb{N}$, after observing that it holds for $m \in \{1, 2, 3\}$. Vendramin [18, Example 3.2] answered in negative sense constructing an involutive square-free solution of cardinality 6 and multipermutational level 4. The next year Lebed and Vendramin [15] inspected the involutive finite solutions of small size and they showed that $N_4 = 6$ and $N_5 = 8$. Moreover, they considered the relation between two consecutive terms of the succession $N_m$ and they showed that $N_{m+1} \leq 2N_m$ : in this way, since $N_5 = 8$ they indirectly obtained that $N_m \leq 2^{m-2}$, for every $m > 4$.

The goal of this article is to give a new estimation of $N_m$, by the introduction of a new sequence $\bar{N}_k$. Let $X$ be the minimal square-free involutive solution $X$ of multipermutational level $m$, having an automorphism $\sigma$ such that $\sigma_{[k-1]}(x) \neq \sigma_{[k-1]}(\sigma(x))$ for some $x \in X$, where $\sigma_{[n]}$ is the epimorphism from $X$ to $\text{Ref}^n(X)$ defined inductively as $\sigma_{[0]}(x) := x$ and $\sigma_{[n]}(x) := \sigma_{\sigma_{[n-1]}(x)}$. Defining $\bar{N}_m$ as the cardinality of $X$, we prove that $N_m \leq \bar{N}_m + 1$. Unfortunately, we are still unable to give an estimation of $\bar{N}_m$ but, despite this, we are able to improve the upper bound of $N_m$ through the estimation of some values of $\bar{N}_m$. In fact, in the main theorem of this article, we prove a stronger relation between $N_m$ and the new sequence $\bar{N}_m$, i.e.,

$$N_m \leq \bar{N}_k \cdot 2^{m-k-1} + 1 \quad (1)$$

for every $k < m$. In particular, $N_m \leq \bar{N}_{m-1} + 1$. For this reason, any value of $\bar{N}_k$ with $k < m$ or even just an estimation of $\bar{N}_k$, could improve the estimation of $N_m$. At this purpose, we prove that $\bar{N}_5 \leq 10$ and, with this result, we obtain a significant improvement of the estimations obtained in [11] and [15].

The main tool of this article is the algebraic structure of *left cycle sets*, introduced by Rump [16] and also considered in several papers (see for example [2–4, 12, 15, 17, 18]). Recall that a non-empty set $X$ with a binary operation $\cdot$ is a *left cycle set* if each left multiplication $\sigma_x : X \rightarrow X, y \mapsto x \cdot y$ is invertible and

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z)$$

for all $x, y, z \in X$. Moreover, a left cycle set $(X, \cdot)$ is *non-degenerate* if the squaring map $\alpha : X \rightarrow X, x \mapsto x \cdot x$ is bijective. Rump [16] proved that if $(X, \cdot)$ is a non-degenerate left cycle set, the map $r : X \times X \rightarrow X \times X$, defined by $r(x, y) = (\lambda_x(y), \rho_y(x))$, where $\lambda_x(y) := \sigma_x^{-1}(y)$ and $\rho_y(x) := \lambda_x(y) \cdot x$, is a non-degenerate involutive solution of the Yang-Baxter equation. Conversely, if $(X, r)$ is a non-degenerate involutive solution and $\cdot$ the binary operation given by $x \cdot y := \lambda_{x}^{-1}(y)$ for all $x, y \in X$, then $(X, \cdot)$ is a non-degenerate left cycle set. The existence of this bijective correspondence allows to move the study of involutive non-degenerate solutions to non-degenerate left cycle sets. In this context, we prove the inequality (1) by a mixture of two well-known extension-tools of left cycle sets: the one-sided extension of left cycle sets, developed in terms of set-theoretic solutions [9] by Etingof, Schedler, and Soloviev, and the dynamical extension of left cycle sets developed by Vendramin [18].

In the last section, we will see that the same approach is useful to construct further interesting examples of left cycle sets that are new counterexamples to the Gateva-Ivanova’s Conjecture. In 2004, Gateva-Ivanova [10, Question 2.28] conjectured that every finite square-free involutive non-degenerate solution $(X, r)$ is multipermutational. Cedó, Jespers, and Okniński [6] proved that the
2. Some preliminary results

A non-empty set $X$ with a binary operation $\cdot$ is a left cycle set if the left multiplication $\sigma_x : X \to X, y \mapsto x \cdot y$ is invertible and
\begin{equation}
(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z)
\end{equation}
for all $x, y, z \in X$. A left cycle set $(X, \cdot)$ is non-degenerate if the squaring map $q : X \to X, x \mapsto x \cdot x$ is bijective. Rump [16] proved that if $(X, \cdot)$ is a non-degenerate left cycle set, the map $r : X \times X \to X \times X$ given by $r(x, y) = (\lambda_x(y), \rho_y(x))$, where $\lambda_x(y) := \sigma_x^{-1}(y)$ and $\rho_y(x) := \lambda_x(y) \cdot x$ is a non-degenerate involutive solution of the Yang-Baxter equation. Conversely, if $(X, r)$ is a non-degenerate involutive solution, the binary operation $\cdot$ defined by $x \cdot y := \lambda_x^{-1}(y)$ for all $x, y \in X$ makes $X$ into a non-degenerate left cycle set. The existence of this correspondence allows to move the study of involutive non-degenerate solutions to left cycle sets, as recently done in [2–5, 14, 15, 17, 18], and clearly to translate in terms of left cycle set the classical concepts related to the non-degenerate involutive set-theoretic solutions.

Therefore, a left cycle set is said to be square-free if the squaring map $q$ is the identity on $X$. The image $\sigma(X)$ of the map $\sigma : X \to \text{Sym}(X), x \mapsto \sigma_x$ can be endowed with an induced binary operation
\begin{equation}
\sigma_x \cdot \sigma_y := \sigma_{x \cdot y}
\end{equation}
which satisfies (2). Rump [16] showed that $(\sigma(X), \cdot)$ is a non-degenerate left cycle set if and only if $(X, \cdot)$ is non-degenerate. The left cycle set $\sigma(X)$ is called the retraction of $(X, \cdot)$.

The left cycle set $(X, \cdot)$ is said to be irretractable if $(\sigma(X), \cdot)$ is isomorphic to $(X, \cdot)$, otherwise it is called retractable.

A non-degenerate left cycle set $(X, \cdot)$ is said to be multipermutational of level $m$, if $m$ is the minimal non-negative integer such that $\sigma^n(X)$ has cardinality one, where
\begin{equation}
\sigma^0(X) := X \quad \text{and} \quad \sigma^n(X) := \sigma(\sigma^{n-1}(X)), \quad \text{for} \quad n \geq 1.
\end{equation}
In this case, we write $mpl(X) = m$. Obviously, a multipermutational left cycle set is retractable but the converse is not necessarily true.

From now on, by a left cycle set we mean a non-degenerate left cycle set. The permutation group $G(X)$ of $X$ is the subgroup of $\text{Sym}(X)$ generated by the image $\sigma(X)$ of $\sigma$.

In order to construct new examples of left cycle sets, Vendramin [18] introduced the concept of dynamical cocycle. If $I$ is a left cycle set and $S$ a non-empty set, then $\varepsilon : I \times I \times S \to \text{Sym}(S), (i, j, s) \mapsto \varepsilon_{i,j}(s, -)$ is a dynamical cocycle if
\begin{equation}
\varepsilon_{e,i,j,k}(\varepsilon_{i,j}(r, s), \varepsilon_{i,k}(r, t)) = \varepsilon_{j,i,k}(\varepsilon_{i,j}(s, r), \varepsilon_{j,k}(s, t))
\end{equation}
for all $i, j, k \in I, r, s, t \in S$. Moreover, if $\varepsilon$ is a dynamical cocycle, then the left cycle set $S \times \varepsilon I := (S \times I, \cdot)$, where
\begin{equation}
(s, i) \cdot (t, j) := (\varepsilon_{i,j}(s, t), i \cdot j)
\end{equation}
for all \(i, j \in I, s, t \in S\), is called dynamical extension of \(I\) by \(\alpha\).

A dynamical cocycle \(\alpha : I \times I \times S \to \text{Sym}(S)\) is said to be constant [18] if \(\alpha(i, j)(r, -) = \alpha(i, j)(s, -)\) for all \(i, j \in I, r, s \in S\). An example of a constant dynamical cocycle, extensively used in [15] because of its simplicity to compute, is the following:

**Example 1.** Let \(S\) be a finite abelian group, \(X\) a left cycle set and \(f\) a function from \(X \times X\) to \(S\) such that

\[
    f(i, k) + f(i \cdot j, i \cdot k) = f(j, k) + f(j \cdot i, j \cdot k)
\]

for all \(i, j, k \in X\). Then \(\alpha : X \times X \times X \to \text{Sym}(S)\) given by

\[
    \alpha(i, j)(s, t) := t + f(i, j),
\]

for all \(i, j \in X\) and \(s, t \in S\), is a constant dynamical cocycle.

**Example 2.** Let \(X\) be a left cycle set, \(k\) a natural number and \(S := \mathbb{Z}/k\mathbb{Z}\). Let \(\alpha : X \times X \times S \to \text{Sym}(S)\) be the function given by

\[
    \alpha(i, j)(s, t) := \begin{cases} 
    t & \text{if } i = j \\
    t + 1 & \text{if } i \neq j.
    \end{cases}
\]

Then, \(\alpha\) is a constant dynamical cocycle and so \(S \times_k X\) is a left cycle set.

An important family of dynamical extensions was obtained by Bachiller, Cedó, Jespers, and Okniński.

**Proposition 1** ([1], Section 2). Let \(A\) and \(B\) be non-trivial abelian groups and let \(I\) be a set with \(|I| > 1\). Let \(\varphi_1 : A \to B\) be a function such that \(\varphi_1(a)\) for every \(a \in A\) and let \(\varphi_2 : B \to A\) be a homomorphism. On \(X(A, B, I) := A \times B \times I\) we define the following operation

\[
    (a, b, i) \cdot (c, d, j) := \begin{cases} 
    (c, d - \varphi_1(a \cdot c), i), & \text{if } i = j \\
    (c, \varphi_2(b), d), & \text{if } i \neq j,
    \end{cases}
\]

for all \(a, c \in A, b, d \in B\) and \(i, j \in I\). Then \((X(A, B, I), \cdot)\) is a non-degenerate left cycle set.

Recently, we constructed a large family of dynamical extensions that includes the one obtained by Bachiller et al.

**Proposition 2** ([3], Theorem 2). Let \(A, B\) be a non-empty sets, \(I\) a non-degenerate left cycle set and \(\beta : A \times A \times I \to \text{Sym}(B), \; \gamma : B \to \text{Sym}(A)\).

Put \(\beta(a, c, i) := \beta(a, c, i), \gamma_b := \gamma(b)\) for \(a, c \in A\) and \(b \in B\). Assume that

1. \(\gamma_{a'}d = \gamma_{a}d'\gamma_{b}\),
2. \(\beta(a, c, i) = \beta(\gamma_{a}(a), \gamma_{c}(c), i)\),
3. \(\gamma_{\beta(a, c, i)}(d) \gamma_{b} = \gamma_{\gamma_{a}(a)}(b) \gamma_{d}\),
4. \(\beta(a, c, i) \beta(a', c, i) = \beta(a', c, i) \beta(a, c, i)\)

hold for all \(a, a', c \in A, b, d \in B\) and \(i, j \in I, i \neq j\).

Let \(\cdot\) be the operation on \(A \times B \times I\) defined by

\[
    (a, b, i) \cdot (c, d, j) := \begin{cases} 
    (c, \beta(a, c, i)(d), i \cdot j), & \text{if } i = j \\
    (\gamma_{b}(c), d, i \cdot j), & \text{if } i \neq j.
    \end{cases}
\]

Then \(X(A, B, I, \beta, \gamma) := (A \times B \times I, \cdot)\) is a non-degenerate left cycle set.

### 3. Left cycle sets and automorphisms

Before being able to prove our main result in the next section, some preliminary results are requested. If \(X\) is a left cycle set, an element \(\alpha \in \text{Sym}(X)\) is an automorphism of \(X\) if \(\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)\) for all \(x, y \in X\).
In this section, we want to show the importance of the automorphisms group in the study of the left cycle sets. At this purpose we will see that the automorphisms of a left cycle set are useful to construct further examples of left cycle sets and to understand the structure of particular families of left cycle sets. Moreover it is natural ask which is the automorphism group $\text{Aut}(X)$ of a given left cycle set $X$. For example, if $X$ is the left cycle set given by $x \cdot y := y$ for all $x, y \in X$, then $\text{Aut}(X) = \text{Sym}(X)$. At this stage of studies, facing this problem in the general case seems to be very hard. However, the following two propositions are useful to find some automorphisms of particular left cycle sets.

**Proposition 3.** Let $X$ be a left cycle set and $\alpha \in \text{Aut}(X)$. Consider the retraction $\sigma(X)$ and let $\tilde{\alpha} : \sigma(X) \to \sigma(X)$ be the function given by $\tilde{\alpha}(\sigma_x) := \sigma_{\alpha(x)}$ for every $x \in X$. Then $\tilde{\alpha} \in \text{Aut}(\sigma(X))$.

**Proof.** If $x, y \in X$, then
\[
\sigma_x = \sigma_y \iff \forall z \in X \quad x \cdot z = y \cdot z
\]
\[
\iff \forall z \in X \quad \alpha(x) \cdot \alpha(z) = \alpha(y) \cdot \alpha(z)
\]
\[
\iff \sigma_{\alpha(x)} = \sigma_{\alpha(y)}
\]
hence $\tilde{\alpha}$ is well-defined and injective. With a straightforward calculation, it is possible to show that $\tilde{\alpha}$ is an epimorphism. \hfill $\square$

**Proposition 4.** Let $X$ be a left cycle set, $f \in \text{Aut}(X)$ and set inductively $X_0 := X$ and $X_m$ the left cycle set on $X_{m-1} \times \mathbb{Z}/2\mathbb{Z}$ given by
\[
(x, s) \cdot (y, t) := \begin{cases} (x \cdot y, t) & \text{if } x = y \\ (x \cdot y, t + 1) & \text{if } x \neq y \end{cases}
\]
for all $m \in \mathbb{N}$. Then, the function $f_m : X_m \to X_m$ given by $f_m(x, s_1, ..., s_m) := (f(x), s_1, ..., s_m)$ for every $(x, s_1, ..., s_m) \in X_m$ is an automorphism of $X_m$ for every $m \in \mathbb{N}$.

**Proof.** By Example 2 and by induction on $m$, it is easy to see that $X_m$ is a left cycle set. By induction on $m$, one can also see that $(x, s_1, s_2, ..., s_m) \cdot (y, t_1, t_2, ..., t_m)$ is equal to
\[
(x \cdot y, t_1 + 1 - \delta_{x,y}, t_2 + 1 - \delta(x, s_1), ..., t_m + 1 - \delta(x, s_1, ..., s_{m-1})), (y, t_1, ..., t_m-1)),
\]
for all $m \geq 1$, where $\delta_{x, \beta} = 1$ if $x = \beta$ and 0 otherwise. Let $m$ be a positive integer. We have
\[
\begin{align*}
\text{f}_m((x, s_1, s_2, ..., s_m) \cdot (y, t_1, t_2, ..., t_m)) &= f_m(x \cdot y, t_1 + 1 - \delta_{x,y}, t_2 + 1 - \delta(x, s_1), (y, t_1), ..., t_m + 1 - \delta(x, s_1, ..., s_{m-1}), (y, t_1, ..., t_m-1)) \\
&= (f(x) \cdot f(y), t_1 + 1 - \delta_{x,y}, t_2 + 1 - \delta(x, s_1), (y, t_1), ..., t_m + 1 - \delta(x, s_1, ..., s_{m-1}), (y, t_1, ..., t_m-1)) \\
&= (f(x) \cdot f(y), t_1 + 1 - \delta(f(x), f(y)), t_2 + 1 - \delta(x, s_1), (f(y), t_1), ..., t_m + 1 - \delta(f(x), s_1, ..., s_{m-1}), (f(y), t_1, ..., t_m-1)) \\
&= (f(x), s_1, s_2, ..., s_m) \cdot (f(y), t_1, t_2, ..., t_m) \\
&= f_m(x, s_1, s_2, ..., s_m) \cdot f_m(y, t_1, t_2, ..., t_m).
\end{align*}
\]
Hence $f_m \in \text{Aut}(X_m)$ and the result follows. \hfill $\square$

Gateva-Ivanova and Cameron [11] and Etingof et al. [9] showed that automorphisms of left cycle sets allow to construct other examples of left cycle sets. We prove that, if $X$ is a left cycle set, $\alpha \in \text{Aut}(X)$ and $z \not\in X$, then, under suitable hypothesis, the retraction $\sigma(X \cup \{z\})$ is isomorphic to the left cycle set having the disjoint union $\sigma(X) \cup \{z\}$ as underlying set.

**Proposition 5.** Let $X$ be a left cycle set, $\alpha \in \text{Aut}(X), z \not\in X$ and $(X \cup \{z\}, \circ)$ the algebraic structure given by
Then the pair \((X \cup \{z\}, \circ)\) is a left cycle set.
Moreover, suppose that \(x \neq \sigma_x\) for all \(x \in X\). Then the retraction \(\sigma(X \cup \{z\})\) is isomorphic to the left cycle set \((\sigma(X) \cup \{\sigma_z\}, \circ)\) given by
\[
\sigma_x \circ \sigma_y := \begin{cases} 
\sigma_x \cdot \sigma_y & \text{if } \sigma_x, \sigma_y \in X \\
\sigma_y & \text{if } y = z \\
\sigma_{\sigma(y)} & \text{if } y \in X, x = z.
\end{cases}
\]

**Proof.** By [9, Section 2], \((X \cup \{z\}, \circ)\) is a left cycle set. Now, since \(x \neq \sigma_x\), for every \(x \in X\), there is a natural bijection \(\phi\) from \(\sigma(X) \cup \{\sigma_z\}\) to the disjoint union \(\sigma(X) \cup \{\sigma_z\}\) given by \(\phi(\sigma_x) := \sigma'_x\), for every \(\sigma_x \in \sigma(X) \cup \{\sigma_z\}\), where \(\sigma'_x\) and \(\sigma_x\) are the left multiplications in \(\sigma(X) \cup \{\sigma_z\}\) and \(\sigma(X) \cup \{\sigma_z\}\) respectively. It is easy to see that \(\phi\) is a homomorphism, so the thesis follows. \(\square\)

### 4. A new estimation of \(N_m\)

The goal of this section is to provide an estimation of \(N_m\) depending on another sequence, which we denote by \(N_k\), that will allow us to improve the estimation obtained by Lebed and Vendramin [15].

Following Gateva-Ivanova and Cameron [11], if \(X\) is a left cycle set and \(n\) a natural number, we indicate by \(\sigma_{[n]}\) the epimorphism from \(X\) to \(\sigma^n(X)\) defined inductively by
\[
\sigma_{[0]}(x) := x \quad \sigma_{[n]}(x) := \sigma_{\sigma_{[n-1]}(x)}
\]
for all \(n \in \mathbb{N}\) and \(x \in X\). The following Lemma, due to Gateva-Ivanova and Cameron, involves the function \(\sigma_{[n]}\) and it will be useful in our work.

**Lemma 6** ([11], Proposition 7.8(3)). Let \(X\) be a finite square-free left cycle set of multipermutation level \(k\). Then, the sets \(\sigma_{[k-1]}(x)\) are \(G(X)\)-invariant.

We indicate by \(N_k\) the cardinality of the minimal square-free left cycle set \(X\) of level \(k\) having an automorphism \(\alpha\) such that there exists \(x \in X\) with \(\sigma_{[k-1]}(x) \neq \sigma_{[k-1]}(\alpha(x))\). For example, let \(X := \{a, b\}\) be the left cycle of level 1 given by \(\sigma_a = \sigma_b := id_X\) and put \(\alpha := (ab)\). Then, \(\alpha \in Aut(X)\) and \(\sigma_{[0]}(a) \neq \sigma_{[0]}(\alpha(a))\), therefore \(N_1 = 2\).

In order to prove the main result of this article, we need some preliminary results.

**Lemma 7.** Let \(X\) be a square-free left cycle set of level \(k\), \(\{z\}\) a set with a single element such that \(z \not\in X\) and \(\alpha \in Aut(X)\) such that there exists \(x \in X\) with \(\sigma_{[k-1]}(x) \neq \sigma_{[k-1]}(\alpha(x))\). Then, the left cycle set \((X \cup \{z\}, \circ)\) given by
\[
x \circ y := \begin{cases} 
x \cdot y & \text{if } x, y \in X \\
y & \text{if } y = z \\
\alpha(y) & \text{if } y \in X, x = z
\end{cases}
\]
has level \(k + 1\).

**Proof.** We prove the thesis by induction on \(k\). If \(k = 1\), then \(x \cdot y = y\) for all \(x, y \in X\) and there exist \(x, y \in X\) such that \(x \neq y\) and \(\alpha(x) = y\). This implies that for the left cycle set \((X \cup \{z\}, \circ)\) we have that \(\sigma_x = id_{X \cup \{z\}}\) for every \(x \in X\) and \(\sigma_z = \alpha \neq id_{X \cup \{z\}}\), hence \(|\sigma(X \cup \{z\})| = 2\) and \(mpl(X \cup \{z\}) = 2\).
Now, if \( X \) has level \( k \), then \( \text{mpl}(X \cup \{z\}, \circ) = 1 + \text{mpl}(\sigma(X \cup \{z\})) \) and, by Lemma 6, \( \sigma_z \neq \sigma_x \) for every \( x \in X \). Hence, by Proposition 5, \( \sigma(X \cup \{z\}) \) is isomorphic to the left cycle set \( (\sigma(X) \cup \{\sigma_z\}, \circ) \) given by

\[
\sigma_x \circ \sigma_y := \begin{cases} 
\sigma_x \cdot \sigma_y & \text{if } \sigma_x, \sigma_y \in X \\
\sigma_y & \text{if } y = z \\
\sigma_{x(y)} & \text{if } y \in X, x = z.
\end{cases}
\]

Moreover, by Proposition 3, we have that \( \sigma_x \) is an element of \( \text{Aut}(\sigma(X)) \). If \( x \) and \( y \) are elements of \( X \) such that \( \sigma_{[k-1]}(x) \neq \sigma_{[k-1]}(y) \) and \( \sigma(x) = y \), it follows that

\[
\sigma_x(\sigma(x)) = \sigma_{z\cdot x} = \sigma_{x(y)} = \sigma_y
\]

hence we can apply the inductive hypothesis. Therefore,

\[
\text{mpl}(X \cup \{z\}, \circ) = 1 + \text{mpl}(\sigma(X) \cup \{\sigma_z\}) = 1 + (k - 1 + 1) = k + 1
\]

and the thesis follows. \( \square \)

**Lemma 8.** Let \( X \) be a square-free left cycle set of level \( k \), \( \{z\} \) a set with a single element \( z \notin X \) and \( \sigma \in \text{Aut}(X) \) such that there exists \( x \in X \) with \( \sigma_{[k-1]}(x) \neq \sigma_{[k-1]}(\sigma(x)) \). Moreover, set inductively \( X_0 = Z_0 := X, X_m \) the left cycle set on \( X_{m-1} \times \mathbb{Z}/2\mathbb{Z} \) given by

\[
(x, s) \cdot (y, t) := \begin{cases} 
(x \cdot y, t) & \text{if } x = y \\
(x \cdot y, t + 1) & \text{if } x \neq y
\end{cases}
\]

for all \( x, y \in X_{m-1} \) and \( s, t \in \mathbb{Z}/2\mathbb{Z} \) and \( (Z_m, \circ) \) the algebraic structure given by \( Z_m := X_{m-1} \cup \{z\} \) and

\[
x \circ y := \begin{cases} 
x \cdot y & \text{if } x, y \in X_{m-1} \\
y & \text{if } y = z \\
\sigma_{x(y)}, \ldots, y_{m-1} & \text{if } y = (y_0, \ldots, y_{m-1}) \in X_{m-1}, x = z
\end{cases}
\]

for every \( m \in \mathbb{N} \). Then \( Z_m \) is a square-free left cycle set of level \( k + m \).

**Proof.** By Propositions 4 and 5, the pair \( (Z_m, \circ) \) is a square-free left cycle set. By induction on \( m \) we prove that \( (Z_m, \circ) \) has level \( k + m \). If \( m = 1 \), the thesis follows by the previous lemma. Now, let us suppose the thesis true for a natural number \( m \). Since

\[
\text{mpl}(Z_{m+1}) = 1 + \text{mpl}(\sigma(Z_{m+1})) = 1 + \text{mpl}(\sigma(X_m \cup \{z\}))
\]

and, by Propositions 4 and 5, \( \sigma(X_m \cup \{z\}) \) is isomorphic to the left cycle set \( (\sigma(X_m) \cup \{\sigma_z\}, \circ) \) given by

\[
\sigma_x \circ \sigma_y := \begin{cases} 
\sigma_x \cdot \sigma_y & \text{if } \sigma_x, \sigma_y \in X_m \\
\sigma_y & \text{if } y = z \\
\sigma_{x(y)} & \text{if } y \in X_m, x = z
\end{cases}
\]

it follows that \( \text{mpl}(Z_{m+1}) = 1 + \text{mpl}(\sigma(X_m) \cup \{\sigma_z\}) \). Finally, by [15, Theorem 10.6 and Corollary 10.7], \( \sigma(X_m) \) is isomorphic to \( X_{m-1} \), and so we obtain that \( \sigma(X_m) \cup \{\sigma_z\} \) is isomorphic to \( Z_m \). By the inductive hypothesis, we have that

\[
\text{mpl}(Z_{m+1}) = 1 + \text{mpl}(Z_m) = 1 + k + m,
\]

hence the thesis. \( \square \)
Theorem 9. Let $N_m$ be the cardinality of the minimal square-free left cycle set of level $m$ and $N_k$ the cardinality of the minimal square-free left cycle set $X$ of level $k$ such that there exists an automorphism $\alpha$ and an element $x \in X$ with $\sigma_{[k-1]}(x) \neq \sigma_{[k-1]}(\alpha(x))$. Then, the inequality
\[ N_m \leq \tilde{N}_k \cdot 2^{m-k-1} + 1 \] holds for every $k < m$. In particular, $N_m \leq \tilde{N}_{m-1} + 1$.

Proof. If $r$ is a natural number and $X$ is a left cycle set of level $k$ and cardinality $\tilde{N}_k$ having an automorphism $\alpha$ such that there exists $x \in X$ with $\sigma_{[k-1]}(x) \neq \sigma_{[k-1]}(\alpha(x))$, then the left cycle set $Z_r$, constructed as in the previous Lemma, is a square-free left cycle set of level $k + r$ and cardinality $\tilde{N}_k \cdot 2^r + 1$, hence $N_{r+k} \leq \tilde{N}_k \cdot 2^{r-1} + 1$. Setting $m := r + k$, we obtain
\[ N_m \leq \tilde{N}_k \cdot 2^{m-k-1} + 1 \] hence the thesis. \(\square\)

Even if we still do not know an estimation of the sequence $\tilde{N}_m$, the previous result allows us to improve the upper bounds of $N_m$ obtained in [11] and [15] using a suitable estimation of some values of the sequence $\tilde{N}_m$.

Example 3. Let $X := \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ be the left cycle set given by
\[
\begin{align*}
\sigma_1 &= \sigma_2 := id_X, \\
\sigma_3 &= (5 6), \\
\sigma_4 &= (12)(5 6), \\
\sigma_5 &= (3 4), \\
\sigma_6 &= (12)(3 4), \\
\sigma_7 &= (9 10), \\
\sigma_8 &= (3 5)(4 6)(9 10), \\
\sigma_9 &= (7 8), \\
\sigma_{10} &= (3 5)(4 6)(7 8).
\end{align*}
\]

Then, $X$ has level 5, $\sigma_{[4]}(7) \neq \sigma_{[4]}(9)$ and $\alpha := (79)(810) \in Aut(X)$, hence $N_5 \leq 10$.

By Theorem 9 and by the previous example that provide us with an estimation of $\tilde{N}_5$, we are able to improve the upper bound $N_m \leq 2^{m-2}$ for every $m \in \mathbb{N}$, obtained by Lebed and Vendramin [15].

Corollary 10. Let $N_m$ be the cardinality of the minimal square-free left cycle set of level $m$, then
\[ N_m \leq 2^{m-2} - 6 \cdot 2^{m-6} + 1 \] for every $m > 5$.

As seen in the previous Corollary, finding new estimations of some values of $\tilde{N}_k$ could be useful to improve the upper bound of $N_m$.

As an example, in the last part of this section we calculate the values of $\tilde{N}_k$ for $k \in \{1, 2, 3, 4\}$.

Examples 1.

(1) If $X := \{1, 2\}$ is the left cycle set given by $x \cdot y := y$ for all $x, y \in X$, then the permutation $\alpha := (1 2)$ is an automorphism of $X$. Moreover, $\sigma_{[0]}(1) = 1 \neq 2 = \sigma_{[0]}(2)$, hence $\tilde{N}_1 = N_1 = 2$.

(2) The unique square-free left cycle set of size 3 and level 2 is given by $\sigma_1 = \sigma_2 := id_X$ and $\sigma_3 := (1 2)$ and the group of automorphism is generated by $\sigma_3$. Since $\sigma_{[1]}(1) = \sigma_{[1]}(2)$, necessarily $\tilde{N}_2 > 3$. Now, let $X := \{1, 2, 3, 4\}$ be the left cycle set given by
\[
\begin{align*}
\sigma_1 &= \sigma_2 := (3 4), \\
\sigma_3 &= \sigma_4 := (1 2).
\end{align*}
\]

Then, $X$ has level 2, $\alpha := (1 3)(2 4)$ is an automorphism of $X$ and $\sigma_{[1]}(1) \neq \sigma_{[1]}(3)$, hence $\tilde{N}_2 = 4$.

(3) By inspection of small cycle sets [13], we know that $N_3 = 5$. Moreover, there are two square-free left cycle sets of level 3 and size 5. The first one is given by $Y :=
\{1, 2, 3, 4, 5\}, \sigma_1 = \sigma_2 := (3, 4), \sigma_3 = \sigma_4 := (12) \text{ and } \sigma_5 := (13 2 4). \text{ So, the fibers of } \sigma_{[2]} \text{ are } \sigma_{[2]}^{-1}(1) = \{1, 2, 3, 4\} \text{ and } \sigma_{[2]}^{-1}(5) = \{5\}, \text{ but an automorphism that maps 5 to an other element of } X \text{ cannot exists because there are not left multiplications of } X \text{ conjugate to } \sigma_5.

Using the same argument for the other left cycle set of level 3 and size 5, we prove that \( N_3 > 5 \). Let \( X := \{1, 2, 3, 4, 5, 6\} \) be the left cycle set given by

\[
\sigma_1 = \sigma_2 := (7, 8) \quad \sigma_3 := (5, 6) \quad \sigma_4 := (17)(28)(56) \\
\sigma_5 := (34) \quad \sigma_6 := (17)(28)(34) \quad \sigma_7 = \sigma_8 := (12).
\]

Then \( X \) has level 4, \( \sigma_{[3]}(3) \neq \sigma_{[3]}(5) \) and \( \alpha := (35)(46) \in \text{Aut}(X) \), hence \( N_3 = 6 \).

(4) If \( N_4 \) were equal to 7, by Lemma 7 there would be a square-free left cycle set \( C \) of size 8, level 5 and with an element \( c \) such that \( b \cdot c = c \) for every \( b \in C \), but inspecting the left cycle sets of size 8 [13], we know that this is not true. Let \( X := \{1, 2, 3, 4, 5, 6, 7, 8\} \) be the left cycle set given by

\[
\sigma_1 = \sigma_2 := (7, 8) \quad \sigma_3 := (5, 6) \quad \sigma_4 := (17)(28)(56) \\
\sigma_5 := (34) \quad \sigma_6 := (17)(28)(34) \quad \sigma_7 = \sigma_8 := (12).
\]

Then \( X \) has level 4, \( \sigma_{[3]}(3) \neq \sigma_{[3]}(5) \) and \( \alpha := (35)(46) \) is an automorphism of \( X \), hence \( N_4 = 8 \).

5. Irretractable square-free left cycle sets

The multipermutational left cycle sets constructed in the previous sections are obtained by a mixture between some dynamical extensions [18] and a particular case of an extension-tool developed by Etingof, Schedler, and Soloviev called one-sided extension (for more details see [9, Section 2]).

Using the same approach, motivated by [19, Problem 19], we produce new irretractable square-free left cycle sets. Several examples of this kind of left cycle sets were constructed in [1, 3, 18]; however, the examples obtained in this section are different from the previous ones because they are simple.

First of all, we recall the constructions of the irretractable left cycle sets obtained by Bachiller, Cedó, Jespers, and Okniński [1] and by the authors [3].

Proposition 11 ([1], Theorem 3.3(a)-(b)). Let \( A, B, I, \phi_1, \phi_2 \) and \( X(A, B, I) \) be as in Proposition 1.

(1) \( X(A, B, I) \) is square-free if and only if \( \phi_1(0) = 0 \);
(2) If \( \phi_1^{-1}(0) = \{0\} \) and \( \phi_2 \) is injective, then \( X(A, B, I) \) is irretractable.

Proposition 12 ([3], Proposition 3). Let \( A, B, I, \beta, \gamma \) be as in Proposition 2, \( |A|, |B|, |I| > 1 \). Then if:

(1) \( A \times A \times \{i\} \cap \beta^{-1}(\{\beta_{(a,a,i)}\}) \subseteq \{(k,k,i)\} \text{ for all } i \in I \text{ and } a \in A, \)
(2) \( \gamma \) is injective,

the non-degenerate left cycle set \( X(A, B, I, \beta, \gamma) \) is irretractable.

In a similar way for the left cycle sets considered in Section 3, we can easily obtain some automorphisms of the left cycle sets constructed by Bachiller, Cedó, Jespers, and Okniński.

Lemma 13. Let \( X(A, B, I) \) be the left cycle set of the Proposition 1, \( m \in \text{Sym}(I) \) and let \( \psi_m \) the function given by \( \psi_m(a, b, i) := (a, b, m(i)) \) for every \( (a, b, i) \in X(A, B, I) \). Then, \( \psi_m \) is an automorphism and the set \( G := \{\psi_m \mid m \in \text{Sym}(I)\} \) is a subgroup of \( \text{Aut}(X(A, B, I)) \) isomorphic to \( \text{Sym}(I) \).
Let $X$. Proposition 14.

**Proof.** By [9, Section 2], we have that such that $x$. Finally, suppose that $A$. Moreover, it is simple: indeed, if $X$ and let $(X, A, B, I) \cup Y, o)$ be the algebraic structure given by

$$x \circ y := \begin{cases} x \cdot y & \text{if } x, y \in X(A, B, I) \\ x \cdot y & \text{if } x, y \in Y \\ y & \text{if } x, y \in X(A, B, I) \text{ and } y \in Y \\ y & \text{if } x \in Y \text{ and } y \in X(A, B, I). \end{cases}$$

Then, the pair $(X(A, B, I) \cup Y)^\circ := (X(A, B, I) \cup Y, o)$ is an irretractable left cycle set.

**Proof.** By [9, Section 2], we have that $(X(A, B, I) \cup Y)^\circ$ is a left cycle set, so it is sufficient to show that $\sigma_x \neq \sigma_y$ for every $x \neq y$. If $x, y \in X(A, B, I)$ or $x, y \in Y$ then clearly $\sigma_x \neq \sigma_y$. Now, suppose that $x := (a, b, i) \in X(A, B, I)$ and $y \in Y$. If $x(y) = \psi_m$ and $m \neq id$, since $\sigma_x(A \times B \times \{j\}) = A \times B \times \{j\}$ and $\sigma_y(A \times B \times \{j\}) = A \times B \times \{m(j)\}$ for every $j \in I$, then necessarily $\sigma_x \neq \sigma_y$. Finally, suppose that $\sigma_x = \sigma_y$. In this case, $x(y) = \psi_{id} = id_X(A, B, I)$. Nor $\sigma_x(c, d, j) = (c, d, j)$ for all $c \in A, d \in B$ and $j \in I$. Since $A$ is a non-trivial group, there exists $c \in A \setminus \{a\}$. Hence

$$\sigma_x(c, d, j) = \sigma_{(a, b, i)}(c, d, i) = (c, d - \varphi_1(a - c), i).$$

Therefore $\varphi_1(a - c) = 0$, in contradiction with the condition $\varphi^{-1}(0) = \{0\}$. 

By the previous result, examples of irretractable square-free left cycle sets that are simple (and hence different from those obtained in [1, 3, 18]) occur in abundance, as we can see in the following examples.

**Example 4.** Let $X(A, B, I)$ be the irretractable square-free left cycle set having 8 element, $m := (12), Y := \{m\}$ the left cycle set having 1 element and $x : Y \to Aut(X(A, B, I))$ the function given by $y(m) := \psi_m$. By Proposition 14, $(X(A, B, I) \cup Y)^\circ$ is an irretractable square-free left cycle set. Therefore, $(X(A, B, I) \cup Y)^\circ$ is a counterexample of the Gateva-Ivanova’s Conjecture of cardinality 9. Moreover, it is simple: indeed, if $C$ is a left cycle set having 3 elements and $p : X(A, B, I) \cup Y \to C$ is a covering map, by [18, Theorem 2.12], there exist a dynamical extension $S \times S C$ such that $(X(A, B, I) \cup Y)^\circ$ is isomorphic to $S \times S C$. Now, we have two cases: $C$ is isomorphic to the left cycle set $X := \{1, 2, 3\}$ given by $\sigma_1 = \sigma_2 = \sigma_3 = id_X$ or $C$ is isomorphic to the left cycle set $Z := \{1, 2, 3\}$ given by $\sigma_1 = \sigma_2 = id_Z$ and $\sigma_3 := (12)$. In both cases, we have that there exist $i \in C$ such that $(s, j) \cdot (t, i) \in S \times \{i\}$ for every $t \in S, (s, j) \in S \times C$, hence, if $O$ is an orbit of $S \times S C$ respect to the action of $G(S \times S C)$, necessarily $|O| \leq 6$. Since the left cycle set $(X(A, B, I) \cup Y)^\circ$ has an orbit of size 8, we have a contradiction.
Example 5. Let $X(A,B,I)$ be the irretractable square-free left cycle set having 8 element, $m := \text{id}_I, Y := \{m\}$ the left cycle set having 1 element and $\alpha : Y \to \text{Aut}(X(A,B,I))$ the function given by $\alpha(m) := \psi_m$. By the previous Proposition, $(X(A,B,I) \cup Y)^\circ$ is an irretractable left cycle set; moreover, since $X(A,B,I)$ is square-free, we have that $(X(A,B,I) \cup Y)^\circ$ is square-free. Therefore, $(X(A,B,I) \cup Y)^\circ$ is a counterexample of the Gateva-Ivanova’s Conjecture of cardinality 9. Moreover, it is not isomorphic to the one of the Example 4 because each left multiplication of the previous Example is different from the identity.

Finding automorphism of the irretractable square-free left cycle set having 8 elements different from those obtained in Lemma 13 can be useful to find other counterexamples to the Gateva-Ivanova’s Conjecture of size 9, as we can see in the following example.

Example 6. Let $X(A,B,I)$ be the irretractable square-free left cycle set having 8 element and $f : X(A,B,I) \to X(A,B,I)$ the function given by $f(a,b,i) := (a+1,b,i)$ for every $(a,b,i) \in A \times B \times I$. Then, $f \in \text{Aut}(X(A,B,I))$ : indeed $f$ is clearly bijective and

$$f(a,b,i) \cdot f(c,d,j) = (a+1,b,i) \cdot (c+1,d,j) = (c+1 + (1 - \delta_{i,j})\varphi_j(b), d + \delta_{i,j}\varphi_j(a-c),j) = f((a,b,i) \cdot (c,d,j))$$

for all $(a,b,i),(c,d,j) \in A \times B \times I$. Therefore, $(X(A,B,I) \cup Y)^\circ$ is a counterexample of the Gateva-Ivanova’s conjecture of cardinality 9, where $Y := \{w\}$ is a left cycle set of size 1 and $\alpha(w) := f$. Since $\sigma_x \neq \text{id}_{X(A,B,I) \cup Y}$ for every $x \in X(A,B,I) \cup Y$, this left cycle set cannot be isomorphic to the Example 5. Moreover, this left cycle set cannot be isomorphic to the Example 4: indeed, Example 4 has two orbits respect to the associated permutation group, while this left cycle set has three orbits.

By a similar argument used in Example 4, one can show that the left cycle sets of Examples 5 and 6 are simple.

Other simple irretractable square-free left cycle sets can be obtained among those having a prime number of elements.

Example 7. Let $p$ be a prime number greater than 5 such that $p \equiv 1 (mod\ 4)$ and $k \in \mathbb{N}$ such that $p = 4k + 1$. Moreover, let $a = b := \mathbb{Z}/2\mathbb{Z}, I := \{1, k\}, \varphi_1 = \varphi_2 = \text{id}_A$ and consider the irretractable left cycle set $X(A,B,I)$ as in Proposition 11. Now, let $m \in \text{Sym}(I), Y := \{m\}$ the left cycle set of 1 element and $\alpha : Y \to \text{Aut}(X(A,B,I))$ given by $\alpha(m) := \psi_m$. Then, $(X(A,B,I) \cup Y)^\circ$ is an irretractable square-free left cycle set. Since $X(A,B,I) \cup Y$ has a prime number of elements, $(X(A,B,I) \cup Y)^\circ$ is simple.

However, Proposition 14 allow us to obtain many other left cycle sets arbitrarily large, as we can see in the following examples.

Example 8. Let $p$ be an odd prime and $k \in \mathbb{N}$ such that $p^2 = 4k + 1$. Moreover, let $a = b := \mathbb{Z}/2\mathbb{Z}, I := \{1, k\}, \varphi_1 = \varphi_2 = \text{id}_A$ and consider the irretractable left cycle set $X(A,B,I)$ as in Proposition 11. Moreover, let $m \in \text{Sym}(I), Y := \{m\}$ the left cycle set of 1 element and $\alpha : Y \to \text{Aut}(X(A,B,I))$ given by $\alpha(m) := \psi_m$. Then, $(X(A,B,I) \cup Y)^\circ$ is an irretractable square-free left cycle set having $p^2$ elements. In general this left cycle set is not simple, but, since it has a prime-square number of elements, it cannot be obtained by Propositions 1 and 2.

Example 9. Let $p$ be a prime number such that $p \equiv 1 (mod4)$ and $k \in \mathbb{N}$ such that $p = 4k + 1$. Moreover, let $a = b := \mathbb{Z}/2\mathbb{Z}, I := \{1, k\}, \varphi_1 = \varphi_2 = \text{id}_A$ and consider the irretractable left cycle set $X(A,B,I)$ as in Proposition 11. Moreover, let $m_1 := \text{id}_I, m_2 := \text{id}_I$ and $m_3 := (12)$ and
Moreover, suppose that such that and Proposition 14.

If we consider the left cycle set \( X(A, B, I, \beta, \gamma) \) of Proposition 2, we can easily generalize the argument of Proposition 14.

**Lemma 15.** Let \( X(A, B, I, \beta, \gamma) \) be the left cycle set of Proposition 2, \( m \in \text{Aut}(I) \) such that \( \beta_{(a, b, i)} = \beta_{(a, b, m(i))} \) for every \( (a, b, i) \in A \times B \times I \) and let \( \psi_m \) be the function given by \( \psi_m(a, b, i) := (a, b, m(i)) \) for every \( (a, b, i) \in X(A, B, I) \). Then, \( \psi_m \) is an automorphism of \( X(A, B, I, \beta, \gamma) \) and the set \( G := \{ \psi_m | \psi_m \in \text{Aut}(X(A, B, I, \beta, \gamma)) \} \) is a subgroup of \( \text{Aut}(X(A, B, I, \beta, \gamma)) \) isomorphic to a subgroup of \( \text{Aut}(I) \).

**Proposition 16.** Let \( X(A, B, I, \beta, \gamma) \) be the irretractable left cycle set of Proposition 12, i.e. satisfying conditions (1) and (2), \( (Y, \cdot) \) a left cycle set and \( \chi : Y \rightarrow \text{Aut}(X(A, B, I, \beta, \gamma)) \) an injective function such that \( \chi(Y) \subseteq G \), where \( G \) is the subgroup of \( \text{Aut}(X(A, B, I, \beta, \gamma)) \) of the previous Lemma. Moreover, suppose that \( \chi(a \cdot b)\chi(a) = \chi(b \cdot a)\chi(b) \) for all \( a, b \in Y \) and let \( (X(A, B, I, \beta, \gamma) \cup Y, \circ) \) be the algebraic structure given by

\[
x \circ y := \begin{cases} x \cdot y & \text{if } x, y \in X(A, B, I, \beta, \gamma) \\ x \cdot y & \text{if } x, y \in Y \\ y & \text{if } x, y \in X(A, B, I, \beta, \gamma) \text{ and } y \in Y \\ \chi(x)(y) & \text{if } x \in Y \text{ and } y \in X(A, B, I, \beta, \gamma). 
\end{cases}
\]

Then, the pair \( (X(A, B, I, \beta, \gamma) \cup Y, \circ) \) is an irretractable left cycle set.

We leave the proofs of the previous results because they are similar to the ones of Lemma 13 and Proposition 14.

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