Notes on the Chern-Character

Maakestad H*
Department of Mathematics, NTNU, Trondheim, Norway

Abstract

Notes for some talks given at the seminar on characteristic classes at NTNU in autumn 2006. In the note a proof of the existence of a Chern-character from complex K-theory to any cohomology Lie theory with values in graded Q-algebras equipped with a theory of characteristic classes is given. It respects the Adams and Steenrod operations.

Keywords: Chern-character; Chern-classes; Euler classes; Singular cohomology; De Rham-cohomology; Complex K-theory; Adams operations; Steenrod operations

Introduction

The aim of this note is to give an axiomatic and elementary treatment of Chern-characters of vectorbundles with values in a class of cohomology-theories arising in topology and algebra. Given a theory of Chern-classes for complex vectorbundles with values in singular cohomology one gets in a natural way a Chern-character from complex K-theory to singular cohomology using the projective bundle theorem and the Newton polynomials. The Chern-classes of a complex vectorbundle may be defined using the notion of an Euler class [1] and one may prove that a theory of Chern-classes with values in singular cohomology is unique. In this note it is shown one may relax the conditions on the theory for Chern-classes and still get a Chern-character. Hence the Chern-character depends on some choices.

Many cohomology theories which associate to a space a graded commutative Q-algebra $H^*$ satisfy the projective bundle property for complex vectorbundles. This is true for De Rham-cohomology of a real compact manifold, singular cohomology of a compact topological space and complex K-theory. The main aim of this note is to give a self contained and elementary proof of the fact that any such cohomology theory will recieve a Chern-character from complex K-theory respecting the Adams and Steenrod operations.

Complex K-theory for a topological space $B$ is considered, and characteristic classes in K-theory and operations on K-theory such as the Adams operations are constructed explicitly, following [2].

The main result of the note is the following (Theorem 4.9):

$H^* : Top \rightarrow Q$–algebras

from the category of topological spaces to the category of graded commutative Q-algebras with respect to continuous maps of topological spaces. We say the theory satisfy the projective bundle property if the following axioms are satisfied: For any rank $n$ complex continuous vectorbundle $E$ over a compact space $B$ there is an Euler class.

$$u_E \in H^*(P(E))$$ (1)

Where $P(E)\rightarrow B$ is the projective bundle associated to $E$. This assignment satisfy the following properties: The Euler class is natural, i.e for any map of topological spaces $f:B' \rightarrow B$ it follows:

$$f^*u_{E'} = u_{f^*E}$$ (2)

For $E = \bigoplus_{i=1}^n L_i$ where $L_i$ are linebundles there is an equation:

$$\prod_{i=1}^n (u_{E_i} - u_{L_i}^c) = 0 \text{in } H^{2\chi}(P(E))$$ (3)

The map $\pi^*$ induce an injection $\pi^* : H^*(B) \rightarrow H^*(P(E))$ and there is an equality,

$$H^*(P(E)) = H^*(B)[1, u_E, u_{E_2}^{-1}, \ldots, u_{E_n}^{-1}]$$

Assume $H^*$ satisfy the projective bundle property. There is by definition an equation,

$$u_{E_i}^c - c_i(E)u_{E_i}^{-1} + \cdots + (-1)^i c_i(E) = 0$$

in $H^*(P(E))$.

Definition 2.1: The class $c_i(E) \in H^i(B)$ is the $i$’th characteristic class of $E$.

Example 2.2: If $P(E)\rightarrow B$ is the projective bundle of a complex vectorbundle and $u_E = c_1(\lambda(E)) \in H^1(P(E),\mathbb{Z})$ is the Euler classe of the tautological linebundle $(E)$ on $P(E)$ in singular cohomology as defined in Section 14 [1], one verifies the properties above are satisfied [4]. One gets the Chern-classes $c_i(E) \in H^i(B,\mathbb{Z})$ in singular cohomology.

*Corresponding author: Maakestad H, Dept. of Mathematics, NTNU, Trondheim, Norway, Tel: +47 73 59 35 20; E-mail: Helge.Maakestad@math.ntnu.no

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Definition 2.3: A theory of characteristic classes with values in a cohomology theory $H^*$ is an assignment.

$$E\to c(E)\in H^n(B)$$

for every complex finite rank vector bundle $E$ on $B$ satisfying the following axioms:

$$f^*c(E)=c(f^*E)$$

(4)

If $E\cong F$ it follows $c(E)=c(F)$

(5)

$$c_k(E\oplus F)=\sum_{i+j=k}c_i(E)c_j(F).$$

(6)

Note: if $\phi:H^*\to H^*$ is a functorial endomorphism of $H^*$ which is a ring-homomorphism and $c$ is a theory of characteristic classes, it follows the assignment $E\to \psi(c(E)) = \phi(c(E))$ is a theory of characteristic classes.

Example 2.4: Let $k\in \mathbb{Z}$ and let $\psi^k_B$ be the ring-endomorphism of $H_{even}^*$ defined by $\psi^k_B(x) = k^x$ where $x\in H(B)$. Given a theory $c(E)$ satisfying Definition 2.3 it follows $\bar{c}(E) = \psi^k_B(c(E))$ is a theory satisfying Definition 2.3.

Note furthermore: Assume $y_1$ is the tautological line bundle on $P^1$. Since we do not assume $c(y_1) = Z$ where $Z$ is the canonical generator of $H(P^1(Z))$ it does not follow that an assignment $E\to \bar{c}(E)$ is uniquely determined by the axioms 4-46. We shall see later that the axioms 4-46 is enough to define a Chern-character [5].

Theorem 2.5: Assume the theory $H^*$ satisfy the projective bundle property. It follows $H^*$ has a theory of characteristic classes.

Proof: We verify the axioms for a theory of characteristic classes.

Axiom 4: Assume we have a map of rank $n$ bundles $f:F\to E$ over a map of topological spaces $g:B\to B$. We pull back the equation,

$$u_E^n = c_0(E)u_F^{n-1} + \cdots + (-1)^n c_n(E) = 0$$

in $H^n(P(E))$ to get an equation,

$$u_F^n = f^*c(E)u_F^{n-1} + \cdots + (-1)^n f^*c_n(E) = 0$$

and by unicity we get $\bar{c}(F) = c(F)$. It follows $c(E) = c(F)$ for isomorphic bundles $E$ and $F$, hence Axiom 5 is ok. Axiom 6: Assume $E \cong \bigoplus_{i=1}^n L_i$ is a decomposition into line bundles. There is an equation $\prod_{i}(a_E-u_{L_i})$ hence we get a polynomial relation.

$$u_E^n - a_i(u_{L_i})u_E^{n-1} + \cdots + (-1)^n s_n(u_{L_i}) = 0$$

in $H^n(P(E))$. Since $c(L_i) = -u_{L_i}$ it follows,

$$\prod c(L_i) = \prod (1+c(L_i)) = c(E)$$

and this is ok.

Given a compact topological space $B$. We may consider the Grothendieck-ring $K^*_C(B)$ of complex finite-dimensional vector bundles. It is defined as the free abelian group on isomorphism-classes $[E]$ where $E$ is a vector bundle, modulo the subgroup generated by elements of the type $[E\oplus F] - [E] - [F]$. It has direct sum as additive operation and tensor product as multiplication. Assume $E$ is a complex vector bundle of rank $n$ and let:

$$\pi:P(E)\to B$$

be the associated projective bundle. We have a projective bundle theorem for complex K-theory:

Theorem 2.6: The group $K^*(P(E))$ is a free $K^*(B)$ module of finite rank with generator $u$ - the euler class of the tautological line-bundle. The elements $\{1,u,u^2,\ldots,u^{n-1}\}$ is a free basis.

Proof: See Theorem IV.2.16 in [2].

As in the case of singular cohomology, we may define characteristic classes for complex bundles with values in complex K-theory using the projective bundle theorem: The element $u^a$ satisfies an equation,

$$u^a - c_i(E)u^{a-1} + \cdots + (-1)^{a-1}c_{a-1}(E)u + (-1)^a c_a(E) = 0$$

in $K^*(P(B))$. One verifies the axioms defined above are satisfied, hence one gets characteristic classes $c_i(E)\in K^*_C(B)$ for all $i=0,\ldots,n$.

Theorem 2.7: The characteristic classes $c_i(E)$ satisfy the following properties:

$$f^*c_i(E)=c_i(f^*E)$$

(7)

$$c_k(E\oplus F)=\sum_{i+j=k}c_i(E)c_j(F)$$

(8)

$$c_1(L)=1-Lc_1(L)=0, i>1$$

(9)

where $E$ is any vector bundle, and $L$ is a line bundle [6].

Proof: See Theorem IV.2.17 in [2].

Adams Operations and Newton Polynomials

We introduce some cohomology operations in complex K-theory and Newton-polynomials and prove elementary properties following the book [2].

Let $\Phi(B)$ be the abelian monoid of elements of the type $\sum n[E]$ with $n\geq 0$. Consider the bundle $\lambda(E)\times E$ and the association.

$$\lambda(E) = \sum_{i\geq 0}\lambda_i(E)E^i$$

giving a map,

$$\lambda: \Phi(X) \to 1 + i\lambda K^*_C(B)[[t]]$$

One checks,

$$\lambda([E\oplus F]) = \lambda([E])\lambda([F])$$

to the set of powerseries with constant term equal to one [7]. Explicitly the map is as follows:

$$\lambda([nE]) = \lambda([E])\lambda([F])^{-n}.$$

When $n$ denotes the trivial bundle of rank $n$ we get the explicit formula.

$$\lambda([E]) = \lambda([E]) (1+t)^{-n}.$$

Let $u=t/1-t$. We may define the new powerseries, $\gamma(E) = \lambda_u(E) = \sum_{k=0}^{\infty} \lambda_k(E)u^k$. It follows.

$$\gamma([E\oplus F]) = \lambda_u([E\oplus F]) = \lambda_u([E])\gamma_u([F]).$$

We may write formally,

$$\gamma_u([E]) = \sum_{k=0}^{\infty} \gamma_k(E)u^k \in K^*_C(B)[[t]]$$.
Hence it follows that,
\[ \gamma^k(E) = \sum_{i+j=k} \gamma^i(E)\gamma^j(E). \]

We get operations,
\[ \gamma^i : K^*_C(B) \to K^*_C(B) \]
for all \( i \geq 1 \). We next define Newton polynomials using the elementary symmetric functions. Let \( u_1, u_2, u_3, \ldots \) be independent variables over the integers \( \mathbb{Z} \), and let \( Q_k = u_1^k + u_2^k + \cdots + u_k^k \) for \( k \geq 1 \). It follows \( Q_k \) is invariant under permutations of the variables \( u_i \) for any \( \sigma \in \mathfrak{S}_k \), we have \( \sigma Q_k = Q_k \) hence we may express \( Q_k \) as a polynomial in the elementary symmetric functions \( \sigma_i \):
\[ Q_k = Q_1(\sigma_1, \sigma_2, \ldots, \sigma_k). \]

We define,
\[ \Delta_i(\sigma) = \Delta_i(\sigma_1, \sigma_2, \ldots, \sigma_i) \]
to be the \( k \)th Newton polynomial in the variables \( \sigma_1, \sigma_2, \ldots, \sigma_i \) where \( \sigma_i \) is the \( i \)th elementary symmetric function. One checks the following:
\[ \Delta_i(\tau_1, \tau_2, \ldots, \tau_{i-1}) = \sigma_i^2 - 2 \sigma_i, \]
and \( \Delta_2(\sigma_1, \sigma_2, \sigma_3) = \sigma_1^3 - 3 \sigma_1 \sigma_2 + 3 \sigma_3 \)
and so on.

Let \( n \geq 1 \) and consider the polynomial.
\[ p(t) = (1 + tu_1)(1 + tu_2)\ldots(1 + tu_n)^{t^n - t^{n-1} - \cdots - t + 1} \]
where,
\[ \sigma_i = \sum_{u_1, u_2, \ldots, u_i} \]
is the \( i \)th elementary symmetric polynomial in the variables \( u_1, u_2, \ldots, u_n \).

**Lemma 3.1:** There is an equality.
\[ Q_k(\sigma_1(u_{11}, u_{12}, \ldots, u_{1n}), \sigma_2(u_{11}, u_{12}, \ldots, u_{1n}), \ldots, \sigma_k(u_{11}, u_{12}, \ldots, u_{1n})) = u_1^k + u_2^k + \cdots + u_n^k. \]

**Proof:** Trivial.

Assume we have virtual elements \( x = E - n = \bigoplus (L_i - 1) \) and \( y = \bigoplus_p (R_j - 1) \) in complex K-theory \( K^*_C(B) \). We seek to define a cohomology-operation \( c \) on complex K-theory using a formal powerseries.
\[ f(u) = a_0 + a_1u + a_2u^2 + \cdots + \in \mathbb{Z}[u]. \]

We define the element.
\[ c(x) = a_0 Q_0(y(x)) + a_1 Q_0(y(x), y_2(x)) + a_2 Q_0(y(x), y(x), y(x)) + \cdots. \]

**Proposition 3.2:** Let \( L \) be a linebundle. Then \( y(L - 1) = 1 + t(L - 1) = c(L/t) \) hence \( y(L - 1) = L - 1 \) and \( (L - 1) = 0 \) for \( i > 1 \).

**Proof:** We have by definition.
\[ \gamma_i(E) = \lambda_n(1-E) = \sum_{k=0}^{\infty} \lambda^k(E)u^k = \sum_{k=0}^{\infty} \lambda^k(E)(t / 1 - t)^k. \]

We have that,
\[ y_i(nE - mF) = \lambda_i(1 - E)^{n-m}. \]
We get,
\[ y_i(L - 1) = \lambda_i(1)^{1}. \]

We have,
\[ \lambda_i(n) = (1 + t)^n \]
Hence,
\[ \gamma_i(n) = \lambda_i(n) = (1 + u)^n = (1 + t)(1 - t)^n = (1 - t)^n. \]
We get:
\[ \gamma_i(L - 1) = \gamma_i((L - 1)^{1} = \lambda_i((L - 1)^{-1}) \]
\[ = (1 + L - 1)^{-1} = (1 + (1/t - 1) - (1 - t)^{-1}) \]
\[ = 1 + (L - 1) - (1 - t) = 1 + t(L - 1) = 1 - c(L/t). \]

And the proposition follows.

**Note:** if \( x = L - 1 \) we get,
\[ c(x) = \sum_{k=0}^{\infty} a_k Q_k(y(x), y^2(x), \ldots, y^k(x)) = \sum_{k=0}^{\infty} a_k y(y(x)) = \sum_{k=1}^{\infty} a_k y^k(L - 1)^k. \]

We state a Theorem:

**Theorem 3.3:** Let \( E \to B \) be a complex vectorbundle on a compact topological space \( B \). There is a map \( \bar{E} \to B \) such that \( \pi^E \) decompose into linebundles, and the map \( \pi \bar{E} : H^*(B) \to H^*(B) \) is injective [8].

**Proof:** See [2] Theorem IV.2.15.

As we are a split exact sequence.
\[ 0 \to K^*_C(B) \to K^*_C(B) \to H^0(B, \mathcal{Z}) \to 0 \]
hence the group \( K^*_C(B) \) is generated by elements of the form \( E - n \) where \( E \) is a rank \( n \) complex vectorbundle.

**Proposition 3.4:** The operation \( c \) is additive, i.e. for any \( x, y \in K^*_C(B) \) we have,
\[ c(x+y) = c(x) + c(y). \]

**Proof:** The proof follows the proof in [2]. Proposition IV.7.11. We may also from Theorem 3.3 assume \( F = \bigoplus R_j \) and \( F = \bigoplus R_j \) where \( L_j, R_j \) are linebundles. We get the following:
\[ \gamma_i(x+y) = \sum_{j=0}^{\infty} \gamma_i((L_j - 1)) \prod_{j=0}^{\infty} (1 + t_{R_j} - 1) \]
\[ = \sum_{j=0}^{\infty} (1 + t_{L_j} - 1) \sum_{j=0}^{\infty} (1 + t_{R_j} - 1)^j \]
\[ = \sum_{j=1}^{\infty} (1 + t_{L_j} - 1) \sum_{j=0}^{\infty} (1 + t_{R_j} - 1)^j \]
Hence,
\[ y(x+y) = c_i(u_1, u_2, v_1, v_2, \ldots). \]

We get:
\[ Q_i(y(x+y)) - y^i(x+y) = Q_i(\sigma_i(u_1, v_1), \ldots, \sigma_i(u_n, v_n)) \]
which by Lemma 3.1 equals,
\[ u_1^i + \cdots + u_n^i + v_1^i + \cdots + v_n^i = Q_1(\sigma_1(u_1), \ldots, \sigma_1(u_n)) + \cdots + Q_n(\sigma_n(v_1), \ldots, \sigma_n(v_n)) = Q_i(y(x)) + Q_i(y(x)). \]
\[ \psi^k : K^*_C(B) \to K^*_C(B) \]

with the properties:

\[
\psi^k(x+y)=\psi^k(x)+\psi^k(y) \quad (10)
\]

\[
\psi^k(L)=L^k \quad (11)
\]

\[
\psi^k(xy)=\psi^k(x)\psi^k(y) \quad (12)
\]

\[
\psi^k(1)=1 \quad (13)
\]

where \( L \) is a line bundle. The operations \( \psi^k \) are the only operations that are ring-homomorphisms - the Adams operations.

**Proof:** We need:

\[
\psi^k(L-1)=\psi^k(L)-\psi^k(1)=L^k-1.
\]

We have in K-theory:

\[
\psi^k(L-1)=\psi^k(L-1)^k-1-\sum_{i=0}^{k-1}k_i(L-1)^{k-1-i}-1 = \left(\begin{array}{c} k \\ 1 \end{array}\right)(L-1)^{k-1}+\left(\begin{array}{c} k \\ 2 \end{array}\right)(L-1)^{k-2}+\cdots+\left(\begin{array}{c} k \\ k \end{array}\right)(L-1)^0.
\]

We get the series,

\[
c=\sum_{i=1}^{k}u_i^k \in \mathbb{Z}[u].
\]

The following operator,

\[
\psi^k = \sum_{i=1}^{k}Q_i(y_1^i,\ldots,y_k^i)
\]

is an explicit construction of the Adams-operator. One may verify the properties in the theorem, and the claim follows.

Assume \( E,F \) are complex vector bundles on \( B \) and consider the Chern-polynomial,

\[
\zeta(E\oplus F) = 1 + \zeta(E)\zeta(F) + \cdots + \zeta(E\oplus F)^{r^k},
\]

where \( N=rk(E)+rk(F) \). Assume there is a decomposition \( E=\oplus L_i \) and \( F=\oplus F_i \) into line bundles. We get a decomposition,

\[
\zeta(E\oplus F)=\prod_{i=1}^{r}c_i(L_i)c_i(R) = (1+a_1t)(1+a_2t)(1+b_1t)(1+b_2t)\cdots(1+b_r t)
\]

where \( a_i=c_i(L_i), b_j=c_j(R). \) We get thus,

\[
\zeta(E\oplus F) = \sigma(a_1^1,\ldots,a_r^1, b_1^1,\ldots,b_r^1).
\]

Let,

\[
Q_k = u_1^k + \cdots + u^k = Q_k(\sigma_1,\ldots,\sigma_k)
\]

where \( \sigma_i \) is the \( i \)th elementary symmetric function in the \( u_i \)’s.

**Proposition 3.6:** The following holds:

\[
Q_k(c(E\oplus F),\ldots,c(E\oplus F)) = Q_k(c(E)) + Q_k(c(F)).
\]

**Proof:** We have,

\[
Q_k(c(E\oplus F)) = Q_k(a_1,b_1) = a_1^k + \cdots a_n^k + b_1^k + \cdots b^k = Q_k(c(E)) + Q_k(c(F))
\]

and the claim follows.

**The Chern-Character and Cohomology Operations**

We construct a Chern-character with values in singular cohomology, using Newton-polynomials and characteristic classes following [2]. The \( k \)th Newton-class \( s_j(E) \) of a complex vector bundle will be defined using characteristic classes of \( E: \zeta(E),\ldots,\zeta(E) \) and the \( k \)th Newton-polynomial \( s_j(\sigma_1,\ldots,\sigma_k) \). We use this construction to define the Chern-character \( Ch(E) \) of the vector bundle \( E \).

We first define Newton polynomials using the elementary symmetric functions. Let \( u, u_1, u_2, \ldots \) be independent variables over the integers \( \mathbb{Z} \), and let \( Q_k = u^k_1 + u^k_2 + \cdots + u^k_B \) for \( k \geq 1 \). It follows \( Q_k \) is invariant under permutations of the variables \( u_i \) for any \( \sigma \in S_k \) we have \( \sigma Q_k = Q_k \) hence we may express \( Q_k \) as a polynomial in the elementary symmetric functions \( \sigma \):

\[
Q_k = Q_k(\sigma_1,\ldots,\sigma_k).
\]

We define,

\[
s_j(\sigma) = Q_j(\sigma_1,\ldots,\sigma_j)
\]

(14)

to be the \( k \)th Newton polynomial in the variables \( (\sigma_1,\ldots,\sigma_j) \) where \( \sigma_i \) is the \( i \)th elementary symmetric function. One checks the following:

\[
s_1(\sigma) = \sigma_1,
\]

\[
s_2(\sigma_1,\sigma_2) = \sigma_1^2 - 2\sigma_2,
\]

and,

\[
s_j(\sigma_1,\sigma_2,\ldots,\sigma_j) = \sigma_1^j - 3\sigma_j \sigma_2 + 3\sigma_3
\]

and so on.

Assume we have a cohomology theory \( H^\ast \) satisfying the projective bundle property. One gets characteristic classes \( \zeta(E) \) for a complex vector bundle \( E \) on \( B \):

\[
\zeta(E) \in H^\ast(B).
\]

Let the class \( S_j(E) = s_j(c_1(E),c_2(E),\ldots,s_j(E)) \) in \( H^\ast(B) \) be the \( k \)th Newton-class of the bundle \( E \). One gets:

\[
s_j(\sigma_1,\ldots,0) = \sigma_j^1
\]

for all \( k \geq 1 \). Assume \( E,F \) line bundles. We see that,

\[
S_j(E\oplus F) = c_j(E\oplus F)^1 - 2s_j(E\oplus F) = (c_j(E) + c_j(F))^1 - 2(c_j(E)c_j(F) + c_j(E)c_j(F)) = c_j(E) + 2s_j(E)c_j(F) + c_j(F) - 2(c_j(E)c_j(F) + c_j(E)c_j(F)) - 2s_j(E)c_j(F) = c_j(E) + 2s_j(E)c_j(F) - 2s_j(E)c_j(F) - S_j(E) + S_j(F).
\]

This holds in general:

**Proposition 4.1:** For any vector bundles \( E,F \) we have the formula,

\[
S_j(E\oplus F) = S_j(E) + S_j(F).
\]

**Proof:** This follows from 3.6.

Let \( k^\ast(\beta) \) be the Grothendieck-group of complex vector bundles on
B, i.e the free abelian group modulo exact sequences $K^*_e(B) = \oplus \mathbb{Z}[E]/U$ where $U$ is the subgroup generated by elements $[E \oplus F]-[E]-[F]$.

**Definition 4.2:** The class, 
$$Ch(E) = \sum_{k \geq 0} \frac{1}{k!} S_k(E) \in H^{even}(B)$$

is the Chern-character of $E$.

**Lemma 4.3:** The Chern-character defines a group-homomorphism, $Ch: K^*_e(B) \to H^{even}(B)$

between the Grothendieck group $K^*_e(B)$ and the even cohomology of $B$ with rational coefficients.

**Proof:** By Proposition 4.1 we get the following: For any $E,F$ we have,

$$Ch(E \oplus F) = \sum_{k \geq 0} \frac{1}{k!} S_k(E \oplus F) = \sum_{k \geq 0} \frac{1}{k!} (S_k(E) + S_k(F)) = \sum_{k \geq 0} \frac{1}{k!} S_k(E) + \sum_{k \geq 0} \frac{1}{k!} S_k(F) = Ch(E) + Ch(F).$$

We get,

$$Ch([E \oplus F]-[E]-[F]) = Ch(E \oplus F) - Ch(E) - Ch(F) = 0$$

and the Lemma follows.

**Example 4.4:** Given a real continuous vectorbundle $F$ on $B$ there exist Stiefel-Whitney classes $w_i(F) \in H^*(B,\mathbb{Z}/2)$ (see [1]) satisfying the necessary conditions, and we may define a "Chern-character"

$$Ch: K^*_e(B) \to H^*(B,\mathbb{Z}/2)$$

by

$$Ch(F) = \sum_{k \geq 0} Q_k(w_1(F),...,w_k(F)).$$

This gives a well-defined homomorphism of abelian groups because of the universal properties of the Newton-polynomials and the fact $H^*(B,\mathbb{Z}/2)$ is commutative. The formal properties of the Stiefel-Whitney classes $w_i$ ensures that for real bundles $E,F$ Proposition 3.6 still holds: We have the formula,

$$Q_k(w_1(F)) = Q_k(w_1(F)) + Q_k(w_2(F)).$$

Since $S_k(\sigma_1,0,...,0) = \sigma_k^2$ we get the following: When $E,F$ are linebundles we have:

$$S_k((E \otimes F),0,...,0) = c_i((E \otimes F)) = (c_i(E) + c_i(F)) = \sum_{i+j+k} \frac{1}{i!j!k!} c_i(E)^j c_j(F)^k.$$\hspace{1cm} (14)

This property holds for general $E,F$.

**Proposition 4.5:** Let $E,F$ be complex vectorbundles on a compact topological space $B$. Then the following formulas hold:

$$S_k(E \otimes F) = \sum_{i+j+k} \frac{i+j+k}{i!j!k!} S_i(E) S_j(F)/S_k.$$\hspace{1cm} (14)

**Proof:** We prove this using the splitting principle and Proposition 4.1. Assume $E,F$ are complex vectorbundles on $B$ and $f: B \to B$ is a map of topological spaces such that $f^* E = \oplus L, f^* F = \oplus M$ where $L,M$ are linebundles and the pull-back map $f^*: H^*(B) \to H^*(B')$ is injective. We get the following calculation:

$$f^* S_k(E \otimes F) = \sum_{i+j+k} \frac{i+j+k}{i!j!k!} S_i(E) S_j(F)/S_k.$$\hspace{1cm} (14)

hence by Lemma 4.1 we get,

$$\sum_{i,j} \sum_{u+v \neq k} \frac{u+v}{u} S_u(L_i) S_v(M_j) = \sum_{i,j} \sum_{u+v \neq k} \frac{u+v}{u} S_u(L_i) S_v(M_j) = \sum_{i,j} \sum_{u+v \neq k} \frac{u+v}{u} S_u(\oplus L_i) S_v(\oplus M_j) = Ch: K^*_e(B) \to H^{even}(B),$$

and the result follows since $f^*$ is injective.

**Theorem 4.6:** The Chern-character defines a ring-homomorphism.

$$Ch: K^*_e(B) \to H^{even}(B).$$

**Proof:** From Proposition 4.5 we get:

$$Ch(E \otimes F) = \sum_{k \geq 0} \frac{1}{k!} S_k(E \otimes F) = \sum_{k \geq 0} \frac{1}{k!} \sum_{i+j+k} \frac{i+j+k}{i!j!k!} S_i(E) S_j(F) = \sum_{k \geq 0} \frac{1}{k!} S_k(E) S_k(F) = Ch(E)Ch(F)$$

and the Theorem is proved.

**Example 4.7:** For complex K-theory $K^n_e(B)$ we have for any complex vectorbundle $E$ characteristic classes $c_1(E) \in K^n_e(B)$ satisfying the necessary conditions, hence we get a group-homomorphism.

$$Ch_Z: K^*_e(B) \to K^*_e(B)$$

defined by,

$$Ch_Z(E) = \sum_{k \geq 0} Q_k(c_1(E),...,c_k(E)).$$

If we tensor with the rationals, we get a ring-homomorphism.

$$Ch_Q: K^*_e(B) \to K^*_e(B) \otimes \mathbb{Q}$$

defined by,

$$Ch_Q(E) = \sum_{k \geq 0} Q_k(c_1(E),...,c_k(E)).$$

**Theorem 4.8:** Let $B$ be a compact topological space. The Chern-character,

$$Ch^N: K^*_e(B) \otimes \mathbb{Q} \to H^{even}(B,\mathbb{Q})$$

is an isomorphism. Here $H^*(B,\mathbb{Q})$ denotes singular cohomology with rational coefficients.

**Proof:** See [2].

The Chern-character is related to the Adams-operations in the
following sense: There is a ring-homomorphism.

\[ \psi^k_B : H^{even}(B) \rightarrow H^{even}(B) \]

deﬁned by,

\[ \psi^k_B(x) = k^x \]

when \( x \in H^2(B) \). The Chern-character respects these cohomology operations in the following sense:

**Theorem 4.9:** There is for all \( k \geq 1 \) a innovative diagram.

\[ K^*_C(B, \text{Ch})^k H^{even}(B) \xrightarrow{\psi^k_B} K^*_C(B)^\text{Ch} H^{even}(B) \]

where \( \psi^k_B \) is the Adams operation defined in the previous section.

**Proof:** The proof follows Theorem V.3.27 in [2]: We may assume \( L \) is a linebundle and we get the following calculation: \( \psi^k(B) = L^k \) and \( c_1(L^k) = k c_1(L) \) hence,

\[ Ch(\psi^k_B(L)) = \exp(k c_1(L)) = \sum_{i \geq 0} \frac{1}{i!} k^i c_1(L)^i \]

and the claim follows.

Hence the Chern-character is a morphism of cohomology-theories respecting the additional structure given by the Adams and Steenrod-operations.

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