Research Article

Marc Joye*, Oleksandra Lapiha, Ky Nguyen, and David Naccache

The Eleventh Power Residue Symbol

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Abstract: This paper presents an efficient algorithm for computing $11^{th}$-power residue symbols in the cyclotomic field $\mathbb{Q}(\zeta_{11})$, where $\zeta_{11}$ is a primitive $11^{th}$ root of unity. It extends an earlier algorithm due to Caranay and Scheidler (Int. J. Number Theory, 2010) for the $7^{th}$-power residue symbol. The new algorithm finds applications in the implementation of certain cryptographic schemes.

Keywords: Power residue symbol, Cyclotomic field, Reciprocity law, Cryptography

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1 Introduction

Quadratic and higher-order residuosity is a useful tool that finds applications in several cryptographic constructions. Examples include [6, 13, 14, 19] for encryption schemes and [1, 2, 12] for authentication schemes and digital signatures. A central operation therein is the evaluation of a residue symbol of the form $\left[\frac{a}{\lambda}\right]$ without factoring the modulus $\lambda$ in the cyclotomic field $\mathbb{Q}(\zeta_{p})$, where $\zeta_{p}$ is a primitive $p^{th}$ root of unity.

For the case $p = 2$, it is well known that the Jacobi symbol can be computed by combining Euclid’s algorithm with quadratic reciprocity and the complementary laws for $-1$ and $2$; see e.g. [10, Chapter 1]. This eliminates the necessity to factor the modulus. In a nutshell, the computation of the Jacobi symbol $\left(\frac{a}{n}\right)$ proceeds by repeatedly performing 3 steps: (i) reduce $a$ modulo $n$ so that the result (in absolute value) is smaller than $n/2$, (ii) extract the sign and the powers of $2$ for which the symbol is calculated explicitly with the complementary laws, and (iii) apply the reciprocity law resulting in the ‘numerator’ and ‘denominator’ of the symbol being flipped. Eventually, the numerator of the symbol becomes $\pm 1$ and the algorithm terminates with the value of $\left(\frac{a}{n}\right)$. Under certain conditions, this methodology naturally extends to higher values for $p$.

The case $p = 3$ is discussed in [4, 14, 19], the case $p = 4$ in [4, 18], the case $p = 5$ in [14], the case $p = 7$ in [3], and the case $p = 8$ in [10, Chapter 9]. Caranay and Scheidler describe a generic algorithm in [3, Section 7] for computing the $p^{th}$-power residue symbol for any prime $p \leq 11$, building on Lenstra’s norm-Euclidean algorithm. They also provide a detailed implementation for the case $p = 7$. The case $p = 11$ is difficult. We quote from [3]:

“Even for the case $p = 11$, for which Euclidean division remains straightforward, the other details of the method get increasingly complicated. Finding explicit conditions for a cyclotomic integer to be primary becomes more and more technical, as does an algorithm to find a primary associate. The cyclotomic field generated by an $11^{th}$ primitive root of unity has four fundamental units, so complementary laws need to be found for three of them as well as for the ramified prime lying above 11 (or for 11 itself).”

The general case is addressed in a recent algorithm by de Boer and Pagano [5].

*Corresponding Author: Marc Joye: Zama, Paris, France; Email: marc.joye@zama.ai
Oleksandra Lapiha: École normale supérieure, Paris, France
Ky Nguyen: École normale supérieure, Paris, France
David Naccache: École normale supérieure, Paris, France

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Our contributions
This paper takes up the challenge put forward in [3] and presents the first implementation of the Caranay–Scheidler algorithm for the 11th-power residue symbol. The contributions of this paper are three-fold: We provide explicit conditions for primary algebraic integers in \( \mathbb{Z}[\zeta_{11}] \); we devise an efficient algorithm for finding a primary associate; and we give explicit complementary laws for a set of four fundamental units and for the special prime 1 − \( \zeta_{11} \).

Organization
The rest of this paper is organized as follows. In Section 2, we review some basic definitions and known results on cyclotomic fields. Section 3 particularizes to the 11th cyclotomic field. We establish and prove an efficient criterion for primary cyclotomic integers. We also define a set of four fundamental units and give explicit formulas to find their index. Section 4 is the core of the paper. We present the ingredients and develop the companion algorithms for the computation of the eleventh power residue symbol.

2 Higher-Order Power Residue Symbols
Throughout this section, \( p \leq 13 \) denotes an odd rational prime.

2.1 Basic definitions and notation
Fix \( \zeta = \zeta_p = e^{2\pi i/p} \) a primitive \( p \)th root of unity and let \( \omega = 1 - \zeta \). The number field \( \mathbb{Q}(\zeta) \) defines the \( p \)th cyclotomic field. The ring of integers of \( \mathbb{Q}(\zeta) \) is \( \mathbb{Z}[\zeta] \) and is norm-Euclidean [9, 11] (in particular, it is a unique factorization domain). Since \( \zeta, \zeta^2, \ldots, \zeta^{p-1} \) form an integral basis for \( \mathbb{Q}(\zeta) \), any element \( a \in \mathbb{Z}[\zeta] \) can be expressed as \( \sum_{j=1}^{p-1} a_j \zeta^j \) with \( a_j \in \mathbb{Z} \). The norm and trace of an element \( a \in \mathbb{Z}[\zeta] \) are the rational integers respectively given by

\[
N(a) = \prod_{k=1}^{p-1} \sigma_k(a) \quad \text{and} \quad T(a) = \sum_{k=1}^{p-1} \sigma_k(a),
\]

where \( \sigma_k : \zeta \mapsto \zeta^k \). The group of units of \( \mathbb{Z}[\zeta] \) is the direct product of \( \langle \zeta \rangle \) and a free abelian group \( \mathcal{E} \) of rank \( r = (p - 3)/2 \). The generators of \( \mathcal{E} \) are called fundamental units and will be denoted by \( \eta_1, \ldots, \eta_r \). Two elements \( a \) and \( \beta \) are called associates if they differ only by a unit factor. We write \( a \sim \beta \).

We follow the approach of Kummer. A central notion is that of primary elements (see [7, p. 158]) in \( \mathbb{Z}[\zeta] \).

**Definition 2.1.** An element \( a \in \mathbb{Z}[\zeta] \) is said to be primary whenever it satisfies

\[
a \not\equiv 0 \pmod{\omega}, \quad a \equiv B \pmod{\omega^2}, \quad a\overline{a} \equiv B^2 \pmod{p}
\]

for some \( B \in \mathbb{Z} \).

**Lemma 2.2 ([3, Lemma 2.6]).** Every element \( a \in \mathbb{Z}[\zeta] \) with \( a \not\equiv 0 \pmod{\omega} \) has a primary associate \( a^* \) of the form

\[
a^* = \pm \zeta^{e_0} \eta_1^{e_1} \cdots \eta_r^{e_r} a \quad \text{where} \quad 0 \leq e_0, e_1, \ldots, e_r \leq p - 1.
\]

Moreover, \( a^* \) is unique up to its sign.

2.2 Kummer’s reciprocity law
Let \( a, \pi \in \mathbb{Z}[\zeta] \) with \( \pi \) prime, \( \pi \not\equiv \omega \), and \( \pi \nmid a \). The \( p \)th-power residue symbol \( \left[ \frac{a}{\pi} \right]_p \) is then defined to be the \( p \)th-root of unity \( \zeta^i \) such that

\[
a^{\left(N(a)-1\right)/p} \equiv \zeta^i \pmod{\pi}.
\]
This exponent $i$ (with $0 \leq i \leq p - 1$) is called the index of $a$ w.r.t. $\pi$ and is noted $\text{ind}_\pi(a)$. If $\pi$ divides $a$ then $\left[ \frac{a}{\pi} \right]_{\pi} = 0$.

Analogously to the Legendre symbol, the $p$th-power residue symbol generalizes: For any $a$, $\lambda \in \mathbb{Z}[\zeta]$ with $\lambda$ non-unit and $\gcd(\lambda, \omega) \sim 1$, writing $\lambda = \prod_j \pi_j^{\ell_j}$ for primes $\pi_j$ in $\mathbb{Z}[\zeta]$, the generalized $p$th-power residue symbol $\left[ \frac{a}{\pi} \right]_{\pi}$ is defined as $\left[ \frac{a}{\pi} \right]_{\pi} = \prod_j \left[ \frac{\lambda}{\pi_j} \right]_{\pi_j}$.

Kummer [7] stated the reciprocity law in 1850 (see also [15, Art. 54]). It is restricted to so-called “regular” primes, which include odd primes $p \leq 13$. Although initially formulated for primary primes in $\mathbb{Z}[\zeta]$, the reciprocity law readily extends to all primary elements; see [3, Corollary 3.6].

**Theorem 2.3** (Kummer’s Reciprocity Law). Let $a$ and $\lambda$ be two primary elements in $\mathbb{Z}[\zeta]$. Then $\left[ \frac{a}{\lambda} \right]_{\lambda} = \left[ \frac{\lambda}{a} \right]_{a}$.

### 2.3 Complementary laws

The special prime $\omega$ and its conjugates are excluded from Kummer’s reciprocity law. Moreover, it does not apply to units other than $\pm 1$ as they are not primary. For these elements, the $p$th-power residue symbol is determined through complementary laws, also stated by Kummer [7, 8] (see also [15, Art. 55]). The complementary laws rely on the logarithmic differential quotients given by

$$\Delta_n(a) = \left. \frac{d^n \ln(F(e^n))}{d e^n} \right|_{e=0}$$

for any $a = \sum_{j=1}^{p-1} a_j \zeta^j \in \mathbb{Z}[\zeta]$ with $T(a) \neq 0$ and where $F(X) = \sum_{j=0}^{p-2} b_j X^j \in \mathbb{Z}[X]$ whose coefficients are $b_0 = -a_{p-1}$ and $b_j = a_j - a_{p-1}$ for $1 \leq j \leq p - 2$. Notice that $a = F(\zeta)$.

**Theorem 2.4** (Complementary Laws). Let $\pi$ be a primary prime in $\mathbb{Z}[\zeta]$. Then,

1. $\text{ind}_\pi(p) \equiv \frac{\Delta_0(\pi)}{p} \pmod{p}$;
2. for any unit $e \in \mathbb{Z}[\zeta]^*$,

$$\text{ind}_\pi(e) \equiv \Delta_1(e) \frac{\text{N}(\pi)-1}{p} + \sum_{i=1}^{r} \Delta_2(e) \Delta_{p-2i}(\pi) \pmod{p}$$

where $r = (p - 3)/2$.

For completeness, we give the complementary laws for $\pm 1$ and $\zeta$. Alternatively, they can be obtained directly from the definition of the $p$th-power residue symbol. The next corollary is a straightforward extension to composite moduli.

**Corollary 2.5** ([3, Corollary 3.6]). Let $\lambda \in \mathbb{Z}[\zeta]$ such that $\omega \nmid \lambda$. Then $\left[ \frac{\zeta}{\lambda} \right]_{\lambda} = 1$ and $\left[ \frac{\zeta}{\lambda} \right]_{\lambda} = \zeta^{\frac{\text{N}(\pi)-1}{p}} \pmod{p}$.

### 3 The Case $p = 11$

This section presents results for the special case $p = 11$. We henceforth assume that $\zeta := \zeta_{11}$ is a primitive $11$th root of unity.

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1 An odd prime $p$ is said to be regular if it does not divide the class number of $\mathbb{Q}(\zeta_p)$. 
3.1 Primary elements

Definition 2.1 explicitly characterizes primary elements. The next proposition specializes it for prime \( p = 11 \) in order to have a simple criterion involving only rational integers.

It is useful to introduce some notation. For \( \alpha = \sum_{j=1}^{10} a_j \zeta^j \in \mathbb{Z}[\zeta] \), we define the rational integers \( A_k(\alpha) = \sum_{j=0}^{k-1} a_j \zeta^j, \) for \( 0 \leq k \leq 9 \). Also, when \( A_0(\alpha) \equiv 0 \pmod{11} \), we define \( a_j(\alpha) = \frac{A_0(\alpha)}{A_k(\alpha)} \pmod{11} \), for \( 1 \leq j \leq 9 \). Notice that \( A_0(\alpha) = -T(\alpha) \).

**Proposition 3.1.** Let \( \alpha = \sum_{j=1}^{10} a_j \zeta^j \in \mathbb{Z}[\zeta] \) and \( A_k := A_k(\alpha) \). Then \( \alpha \) is primary if and only if the following conditions hold:

1. \( A_0 \not\equiv 0 \pmod{11} \);
2. \( A_1 \equiv 0 \pmod{11} \);
3. \( A_2 \equiv A_4 \equiv 0 \pmod{11} \);
4. \( A_0 A_6 + A_3^2 \equiv 0 \pmod{11} \);
5. \( A_3 A_5 - A_0 A_8 \equiv 0 \pmod{11} \).

**Proof.** Let \( \omega = 1 - \zeta \). From Definition 2.1, \( \alpha \) is primary if and only if \( \alpha \equiv (\omega \alpha) \equiv B \pmod{11} \) for some rational integer \( B \). We have \( \sum_{j=1}^{10} a_j \omega^j = \sum_{j=1}^{10} a_j (1 - \omega)^j = \sum_{j=1}^{10} a_j (1 - j \omega) \equiv A_0 - A_1 \omega \pmod{11} \). As a result, the condition \( \alpha \equiv 0 \pmod{11} \) is equivalent to \( A_0 \equiv 0 \pmod{11} \), and the condition \( \alpha \equiv B \pmod{11} \) with \( B = A_0 \in \mathbb{Z} \) is equivalent to \( A_1 \omega \equiv 0 \pmod{11} \). We observe that rational integers are congruent modulo \( \omega \) if and only if they are congruent modulo \( p \) in this case 11. We therefore have \( A_0 \equiv 0 \pmod{11} \iff A_0 \equiv 0 \pmod{11} \) and \( A_1 \equiv 0 \pmod{11} \iff A_1 \equiv 0 \pmod{11} \).

It remains to look at the third condition, \( \alpha \omega \equiv A_0^2 \pmod{11} \). Using matrix notation and defining the Vandermonde matrix

\[
V = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & 2 & 2^2 & \ldots & 2^9 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 10 & 10^2 & \ldots & 10^8
\end{pmatrix} \pmod{11},
\]

we can express \( (A_0, \ldots, A_9) \) as \( (A_0, \ldots, A_9) = (a_1, \ldots, a_{10}) V \). In turn, we can write \( \alpha \equiv \sum_{j=1}^{10} a_j (1 - \omega)^j \equiv (a_1, \ldots, a_{10}) (1 - \omega, \ldots, (1 - \omega)^{10}) \equiv (A_0, \ldots, A_9) V^{-1} ((1 - \omega), \ldots, (1 - \omega)^{10}) \pmod{11} \). Similarly, we can write \( \alpha \omega \equiv \sum_{j=1}^{10} a_j \zeta^{11-j} \equiv \sum_{j=1}^{10} a_j (1 - \omega)^{11-j} \equiv (A_0, \ldots, A_9) V^{-1} ((1 - \omega)^{10}, \ldots, (1 - \omega)) \pmod{11} \). Hence, noting that \( \omega^{10} \equiv 11 \), the product \( \alpha \omega \pmod{11} \) can be put after a little algebra in the form of a degree-9 polynomial in \( \omega \):

\[
\alpha \omega \equiv A_0^2 (A_0 A_2 - A_1^2) \omega^5 + (A_0 A_2 - A_1^2) \omega^3 + (A_0 A_4 - 4 A_1 A_3 + 3 A_2^2) \omega^6
+ (-A_0 A_2 + 2A_0 A_6 + A_2 + 3 A_1 A_3 - 5 A_2^2) \omega^5 + (A_0 A_4 + A_6 - 4 A_1 A_3 - 4 A_2^2 - 4 A_3 A_5 + 2 A_5 A_7)
+ (A_2^2 - 5 A_2 A_6 - 4 A_3^2) \omega^6
+ (4 A_0 A_2 - 2 A_0 A_6 - A_2 A_3 - 4 A_1 A_3 - 3 A_1 A_5 + 2 A_2 A_4 - 2 A_3 A_5 + A_4^2) \omega^7
+ (-5 A_0 A_4 - A_4 A_6 - A_2 A_6 + 2 A_1 A_5 - 3 A_1 A_7
- 4 A_2^2 - 4 A_2 A_6 - A_3^2 + 4 A_3 A_5 + 3 A_4^2) \omega^8
+ (4 A_0 A_2 + 5 A_0 A_6 - 5 A_0 A_8 - 4 A_1 A_3 + 2 A_1 A_5
- 4 A_1 A_7 + 4 A_2^2 - A_2 A_4 + 3 A_2 A_6 - 3 A_3^2 + 5 A_3 A_5 + A_4^2) \omega^9
\pmod{11}.
\]

For conciseness, we write \( C_j \) the rational coefficient of \( \omega^j \) in the right hand side of Eq. (1): \( \alpha \omega \equiv A_0^2 + \sum_{j=2}^9 C_j \omega^j \pmod{11} \). The condition \( \alpha \omega \equiv A_0^2 \pmod{11} \) can be thus rewritten as \( \sum_{j=2}^9 C_j \omega^j \equiv 0 \pmod{11} \). Further, since \( \omega^j \not\equiv 0 \pmod{11} \) for \( 2 \leq j \leq 9 \), we get \( C_2 \omega^2 \equiv 0 \pmod{11} \iff C_2 \equiv 0 \pmod{11} \), \( C_2 \omega^2 + C_3 \omega^3 \equiv C_3 \omega^3 \equiv 0 \pmod{11} \), and so on: \( C_4 \equiv 0 \pmod{11} \), \( C_5 \equiv 0 \pmod{11} \), and so on: \( C_4 \equiv 0 \pmod{11} \), and so on: \( C_4 \equiv 0 \pmod{11} \), and so on: \( C_4 \equiv 0 \pmod{11} \), and so on: \( C_4 \equiv 0 \pmod{11} \).
We now apply Theorem 2.4 to find the index of the fundamental units (mod 11), (mod 11) is so given by

\[ \eta \equiv \zeta^5 + \zeta^{-5}, \quad \eta_2 \equiv \zeta + \zeta^{-1}, \quad \eta_3 \equiv \zeta^{-1}(1 + \zeta + \zeta^2), \quad \eta_4 \equiv \zeta^3 + \zeta^{-3}, \]

and \( \eta_i = \sigma_{-1}(\eta_i) \) for \( 1 \leq i \leq 4 \), where \( \sigma_{-1} : \zeta \mapsto \zeta^{-1} \). We will use this set \( \{\eta_1, \eta_2, \eta_3, \eta_4\} \) of fundamental units in later computations.

\[ \forall a \in \mathbb{Z}[\zeta] \text{ is real (i.e., } a = \overline{a} \text{). Then } 2A_1(a) \equiv A_1(2a) \equiv A_1(a + \overline{a}) \equiv 0 \text{ (mod 11)} \iff A_1(a) \equiv 0 \text{ (mod 11).} \]

We now apply Theorem 2.4 to find the index of the fundamental units \( \eta_i, 1 \leq i \leq 4 \), and of special prime \( \omega \).

**Proposition 3.3.** Let \( \pi \) be a primary prime in \( \mathbb{Z}[\zeta] \). Then

\[ \text{ind}_\pi(\eta_1) \equiv a_3(\pi) + 3a_5(\pi) + 4a_7(\pi) + 4a_6(\pi) \eta_3(\pi) + 3a_9(\pi) \pmod{11} \]

\[ \text{ind}_\pi(\eta_2) \equiv 3a_3(\pi) + 5a_5(\pi) - 2a_7(\pi) + 5a_6(\pi) \eta_3(\pi) + a_9(\pi) \pmod{11} \]

\[ \text{ind}_\pi(\eta_3) \equiv 2a_3(\pi) - 2a_5(\pi) + 3a_7(\pi) - 4a_6(\pi) \eta_3(\pi) - 3a_9(\pi) \pmod{11} \]

\[ \text{ind}_\pi(\eta_4) \equiv 4a_3(\pi) + 4a_5(\pi) + 3a_7(\pi) + a_6(\pi) \eta_3(\pi) - 2a_9(\pi) \pmod{11} \]

and

\[ \text{ind}_\pi(\omega) \equiv 5a_6(\pi) a_3(\pi) - 2a_6(\pi) a_5(\pi) - A_1(\pi) - 5 \frac{N(\pi)}{11} - 5 \pmod{11} \]

where \( A_1(\pi) = \frac{A_1(\omega) \mod{11}}{11} \).

**Proof.** Let \( \pi \in \mathbb{Z}[\zeta] \) be a primary prime. Recalling that \( A_1(\eta_i) \equiv 0 \pmod{11} \) for \( 1 \leq i \leq 4 \), we have \( \Delta(\eta_i) \equiv 0 \pmod{11} \); see Appendix A. Applied to \( \eta_1 \), the general formula for computing the index w.r.t. a primary prime (cf. Theorem 2.4) becomes \( \text{ind}_\pi(\eta_1) \equiv \Delta_2(\eta_1) \Delta_9(\pi) + \Delta_4(\eta_1) \Delta_7(\pi) + \Delta_6(\eta_1) \Delta_5(\pi) + \Delta_8(\eta_1) \Delta_3(\pi) \pmod{11} \). An application of the formulas given in Appendix A yields

| \( \Delta_2(\eta_1) \pmod{11} \) | \( \Delta_4(\eta_1) \pmod{11} \) | \( \Delta_6(\eta_1) \pmod{11} \) | \( \Delta_8(\eta_1) \pmod{11} \) |
|---|---|---|---|
| \( i = 1 \) | 3 | 4 | 3 | 1 |
| \( i = 2 \) | 1 | 9 | 5 | 3 |
| \( i = 3 \) | 8 | 3 | 9 | 2 |
| \( i = 4 \) | 9 | 3 | 4 | 4 |

Further, \( \pi \) being primary and thus \( A_0(\pi) \neq 0 \pmod{11} \) and \( a_1(\pi) \equiv a_2(\pi) \equiv a_6(\pi) \equiv a_6(\pi) + a_3(\pi)^2 \equiv 0 \pmod{11} \), those formulas also yield, modulo 11,

\[ A_3(\pi) \equiv a_3(\pi), \quad A_5(\pi) \equiv a_5(\pi), \quad A_7(\pi) \equiv a_7(\pi), \quad A_9(\pi) \equiv -a_3(\pi)^3 + 4a_6(\pi) a_3(\pi) + a_9(\pi) \equiv 5a_6(\pi) a_3(\pi) + a_9(\pi). \]

Plugging all these quantities in the above expression for \( \text{ind}_\pi(\eta_1) \) gives the desired result.

For \( \omega \), since \( \omega^{10} \sim 11 \), there exists a unit \( \varepsilon \) such that \( \varepsilon \omega^{10} \equiv 11 \). This holds for \( \varepsilon = \prod_{j=1}^{10} \frac{1 - \zeta^j}{1 - \zeta} = -\zeta^6 \eta_1^3 \eta_2^2 \eta_3^2 \eta_4^{-2} \). As a result, owing to the multiplicative nature of the power residue symbol, we have
We need to investigate the multiplicative properties of the \( \sum \) Proposition 4.1. Let \( \text{Computation of the Eleventh Power Residue Symbol} \)

\[ A_k(a\beta) \equiv \sum_{j=0}^{k} \binom{k}{j} A_j(a)A_{k-j}(\beta) \pmod{11}. \]

**Proof.** Using \( \zeta^{11} = 1 \), the product of \( \alpha = \sum_{j=1}^{10} a_j \zeta^j \) and \( \beta = \sum_{j=1}^{10} b_j \zeta^j \) satisfies \( \alpha \beta = \sum_{\ell=2}^{20} c_\ell \zeta^\ell = c_{11} + c_{12} \zeta + \sum_{j=2}^{9} (c_j + c_{j+11}) \zeta^j + c_{10} \zeta^{10} \), where \( c_\ell = \sum_{m+n=\ell \mod 11} a_m b_n \). Hence, we get

\[ A_k(\alpha \beta) = A_k(c_{11}) + A_k(c_{12} \zeta + \sum_{j=2}^{9} (c_j + c_{j+11}) \zeta^j + c_{10} \zeta^{10}) \]
\[ = A_k(c_{11}) + c_{12} + \sum_{j=2}^{9} (c_j + c_{j+11}) j^k + c_{10} 10^k \]
\[ = A_k(c_{11}) + \sum_{\ell \neq 11} 2^{\ell 20} c_\ell \ell^k \pmod{11}. \]

Furthermore, \( A_k(c_{11}) = A_k(c_{11} - \sum_{j=1}^{10} \zeta^j) \equiv -c_{11} (\sum_{j=1}^{10} j^k) \pmod{11} \). So, for \( k = 0 \), we get \( A_0(c_{11}) \equiv c_{11} \pmod{11} \) and thus \( A_0(\alpha \beta) \equiv \sum_{\ell \neq 11} 2^{\ell 20} c_\ell \ell^k \pmod{11} \). For \( k \geq 1 \), we get \( A_k(c_{11}) \equiv 0 \pmod{11} \), which leads to \( A_k(\alpha \beta) \equiv \sum_{\ell \neq 11} 2^{\ell 20} c_\ell \ell^k \pmod{11} \). Therefore, in all cases, we have

\[ A_k(\alpha \beta) \equiv \sum_{\ell \neq 11} 2^{\ell 20} c_\ell \ell^k \pmod{11}, \]

which completes the proof. \( \square \)

**Corollary 4.2.** Let \( \alpha \in \mathbb{Z}[\zeta] \) and let \( n \in \mathbb{N} \). Then

\[ A_0(\alpha^n) \equiv A_0(\alpha)^n \pmod{11} \]

and, provided that \( A_0(\alpha) \neq 0 \pmod{11} \), for \( 1 \leq k \leq 9 \),

\[ a_k(\alpha^n) \equiv \sum_{\ell=1}^{k} \binom{n}{\ell} h_{k,\ell}(\alpha) \pmod{11} \]

where

\[ h_{k,\ell}(\alpha) = \begin{cases} a_k(\alpha) & \text{for } \ell = 1 \\ \sum_{j=1}^{k-\ell+1} \binom{j}{\ell} h_{k-j,\ell-1}(\alpha)a_j(\alpha) & \text{for } 2 \leq \ell \leq k. \end{cases} \]
Proof. This is a direct application of Proposition 4.1. □

Proposition 4.3. Let \( \eta_1 = \zeta^5 + \zeta^{-5}, \eta_2 = \zeta + \zeta^{-1}, \eta_3 = \zeta^{-1}(1 + \zeta + \zeta^2), \) and \( \eta_4 = \zeta^3 + \zeta^{-3}. \) Let also \( \alpha \in \mathbb{Z}[\zeta] \) with \( A_0(\alpha) \neq 0 \) (mod 11) and \( A_1(\alpha) \equiv 0 \) (mod 11). If rational integers \( 0 \leq e_1, e_2, e_3, e_4 \leq 10 \) verify

\[
\begin{align*}
\sum_{i=1}^{4} a_2(\eta_i) e_i &\equiv -a_2(\alpha) \pmod{11} \\
\sum_{i=1}^{4} (a_4(\eta_i) - 3 a_2(\eta_i)^2) e_i &\equiv -a_4(\alpha) + 3 a_2(\alpha)^2 \pmod{11} \\
\sum_{i=1}^{4} (a_6(\eta_i) - 4 a_4(\eta_i) a_2(\eta_i) - 3 a_2(\eta_i)^3) e_i &\equiv -a_6(\alpha) + 4 a_4(\alpha) a_2(\alpha) - a_3(\alpha)^2 + 3 a_2(\alpha)^3 \pmod{11} \\
\sum_{i=1}^{4} (a_8(\eta_i) + 5 a_6(\eta_i) a_2(\eta_i) - 2 a_4(\eta_i)^2 + 2 a_4(\eta_i) a_2(\eta_i)^2 - 3 a_2(\eta_i)^4) e_i &\equiv -a_8(\alpha) - 5 a_6(\alpha) a_2(\alpha) + 2 a_4(\alpha) a_2(\alpha)^2 + 3 a_2(\alpha)^2 \\
\end{align*}
\]

(2)

then \( \alpha' = \eta_1^{e_1} \eta_2^{e_2} \eta_3^{e_3} \eta_4^{e_4} \alpha \) is a primary associate of \( \alpha. \)

Proof. Write \( \varepsilon = e_1 e_2 e_3 e_4 \) and \( e_1 = \eta_1^{e_1}. \) Hence, \( \alpha' = \varepsilon \alpha \) is an associate of \( \alpha. \) It is worth noting that \( \varepsilon \) is real and thus \( A_1(\varepsilon) \equiv A_1(\alpha) \equiv 0 \) (mod 11). Proposition 3.1 lists the conditions for \( \alpha' \) being primary. From Proposition 4.1 and Corollary 4.2 (see also Appendix A), we get \( A_0(\varepsilon) \equiv \prod_{i=1}^{4} a_2(\eta_i) e_i \equiv \prod_{i=1}^{4} a_2(\eta_i) \equiv 1 \) (mod 11). Therefore, since \( A_0(\eta_i) \equiv 0 \) (mod 11), it follows that \( A_0(\alpha') \equiv A_0(\varepsilon) A_0(\alpha) \neq 0 \) (mod 11). Further, since \( A_1(\alpha') \equiv A_1(\varepsilon) A_1(\alpha) \equiv 0 \) (mod 11).

Likewise, again from Proposition 4.1 and Corollary 4.2, we find after a little algebra \( a_2(e) \equiv \sum_{i=1}^{4} a_2(\eta_i) e_i \equiv 0 \) (mod 11). Consequently, since \( \alpha' = \varepsilon \alpha, \) the condition \( A_2(\alpha') \equiv 0 \) (mod 11) translates into \( A_2(\alpha') \equiv A_2(\varepsilon) A_2(\alpha) + A_2(\alpha) A_0(\varepsilon) \equiv 0 \) (mod 11) \( \iff \) \( a_2(\varepsilon) \equiv A_2(\alpha) \equiv 0 \) (mod 11), that is \( \sum_{i=1}^{4} a_2(\eta_i) e_i \equiv -a_2(\alpha) \) (mod 11).

Therefore, the calculation for \( a_4(e) \) is more involved and technical. An application of Proposition 4.1 and Corollary 4.2 yields \( a_4(e) \equiv \sum_{i=1}^{4} (a_4(\eta_i) - 3 a_2(\eta_i)^2) e_i (e_i - 1) + 6 \sum_{i=1}^{3} (\sum_{i=1}^{4} a_2(\eta_i) a_2(\eta_i) e_i) e_i \) (mod 11). In turn, \( \sum_{i=1}^{4} a_2(\eta_i) e_i \equiv -a_2(\alpha) \) (mod 11), we obtain \( a_4(e) \equiv \sum_{i=1}^{4} (a_4(\eta_i) - 3 a_2(\eta_i)^2) e_i + 3 (\sum_{i=1}^{4} a_2(\eta_i) e_i)^2 \equiv [\sum_{i=1}^{4} a_2(\eta_i) - 3 a_2(\eta_i)^2] e_i + 3 a_2(\alpha)^2 \) (mod 11). The condition \( A_4(\alpha') \equiv 0 \) (mod 11) so leads to \( a_4(\alpha) + 6 a_2(\alpha)^2 ) e_i \equiv -a_4(\alpha) + 6 a_2(\alpha)^2 \) (mod 11), that is \( \sum_{i=1}^{4} (a_4(\eta_i) - 3 a_2(\eta_i)^2) e_i \equiv -a_4(\alpha) + 3 a_2(\alpha)^2 \) (mod 11).

The two remaining relations are proved similarly as a respective consequence of \( A_0(\alpha') A_6(\alpha') + A_3(\alpha')^2 \equiv 0 \) (mod 11) and \( A_3(\alpha') A_5(\alpha') - A_0(\alpha') A_8(\alpha') \equiv 0 \) (mod 11). Notice that \( a_3(\varepsilon) \equiv a_5(\varepsilon) \equiv a_7(\varepsilon) \equiv 0 \) (mod 11).

From Lemma 2.2, applied to the case \( p = 11, \) we know that every \( \alpha \in \mathbb{Z}[\zeta] \) with \( \alpha \neq 0 \) (mod \( \omega \)) has a primary associate of the form

\[
\alpha' = \pm \zeta^{e_0} \eta_1^{e_1} \eta_2^{e_2} \eta_3^{e_3} \eta_4^{e_4} \alpha
\]

where \( 0 \leq e_0, e_1, e_2, e_3, e_4 \leq 10 \) and \( \alpha' \) is unique up to the sign. First, we observe that the sign does not affect the fact of being primary. Indeed, from Proposition 3.1, it is easily seen that if \( \alpha' \) is primary then so is \( -\alpha'. \) Second, as shown in the proof of Proposition 3.1, we observe that the condition \( \alpha \neq 0 \) (mod \( \omega \)) is equivalent to \( A_0(\alpha) \neq 0 \) (mod 11).

Applying Proposition 4.3 demands that \( \alpha \) satisfies \( A_0(\alpha) \neq 0 \) (mod 11) and \( A_1(\alpha) \equiv 0 \) (mod 11). It turns out that if \( A_0(\alpha) \neq 0 \) (mod 11) then

\[
\alpha' := \zeta^{-a_1(\alpha)} \alpha
\]

satisfies \( A_0(\alpha') \equiv A_0(\zeta^{-a_1(\alpha)}) A_0(\alpha) \equiv A_0(\alpha) \neq 0 \) (mod 11) and \( A_1(\alpha') \equiv A_0(\zeta^{-a_1(\alpha)}) A_1(\alpha) + A_1(\zeta^{-a_1(\alpha)}) A_0(\alpha) \equiv A_1(\alpha) - a_1(\alpha) A_0(\alpha) \equiv 0 \) (mod 11). Note that \( A_0(\zeta) = A_1(\zeta) = 1. \)

Putting all together, we therefore obtain an efficient way to compute a primary associate of an element \( \alpha \in \mathbb{Z}[\zeta] \) with \( A_0(\alpha) \neq 0 \) (mod 11). This is depicted in Algorithm 1. With matrix notation, letting \( (w_1, w_2, w_3, w_4) \) denote the right-hand side of (2) (i.e., \( w_1 = -a_2(\alpha), \) etc) and replacing the \( \eta_i(\zeta_i)'s \) by their respective values, the system of equations (2) can be rewritten as

\[
(e_1, e_2, e_3, e_4) M \equiv (w_1, w_2, w_3, w_4) \pmod{11}
\]
and so
\[(e_1, e_2, e_3, e_4) \equiv (w_1, w_2, w_3, w_4) M^{-1} \pmod{11}\]

where \(M^{-1} = \begin{pmatrix} -4 & 4 & 1 & 3 \\ -3 & 5 & -5 & -3 \\ -2 & 3 & 2 & -3 \\ 1 & 4 & 3 & 1 \end{pmatrix} \pmod{11}\).

\begin{algorithm}
\begin{algorithmic}
\end{algorithmic}
\end{algorithm}

\subsection{4.2 Norm-Euclidean division}

The last ingredient for our algorithm is a norm-Euclidean division. For \(p = 11\), the ring \(\mathbb{Z}[^*]\) is known to be norm-Euclidean, i.e., for all \(\alpha, \lambda \in \mathbb{Z}[^*]\), there exists some \(\rho \in \mathbb{Z}[^*]\) such that \(\rho \equiv \alpha \pmod{\lambda}\) and \(N(\rho) < N(\lambda)\). In [11], Lenstra provides an efficient algorithm for approximating an algebraic number \(\chi \in \mathbb{Q}[^*]\) by an algebraic integer \(i \in \mathbb{Z}[^*]\) satisfying \(N(\chi - i) < 1\). See also [14, Algorithm 5.1] or [3, Algorithm 7.1]. Therefore, if we let \(\chi = \frac{\alpha}{\lambda} = \frac{\alpha \prod \omega_1(\lambda)}{\prod \omega_1(\lambda)} \in \mathbb{Q}[^*]\), we obtain \(i \in \mathbb{Z}[^*]\) such that \(N(\chi - i) < 1\), and thus \(\rho := \alpha - i\lambda\) verifies \(\rho \equiv \alpha \pmod{\lambda}\) and \(N(\rho) = N(\lambda) N(\chi - i) < N(\lambda)\). We write \(\rho = \text{euclid\_div}(\alpha, \lambda)\).

\subsection{4.3 Our algorithm}

The main result is the hendecic reciprocity law.

\begin{thm}[Hendecic Reciprocity] Let \(\alpha\) and \(\lambda\) be two primary elements in \(\mathbb{Z}[^*]\). Then
\[
\begin{pmatrix} \alpha \\ \lambda \end{pmatrix}_{j_{11}} = \begin{pmatrix} \lambda \\ \alpha \end{pmatrix}_{j_{11}}
\]

Moreover,
\[
\begin{pmatrix} \alpha \lambda \end{pmatrix}_{j_{11}} = 1,
\]
\[
\begin{pmatrix} \zeta_5 \alpha \lambda \end{pmatrix}_{j_{11}} = \zeta_5^{N(\lambda) - 1},
\]
\[
\begin{pmatrix} \alpha \lambda^5 \end{pmatrix}_{j_{11}} = \zeta_5^{3a_1(\lambda) + 3a_2(\lambda) + 4a_3(\lambda) + 4a_4(\lambda) + 4a_5(\lambda) + 4a_6(\lambda) + 3a_7(\lambda) + 3a_8(\lambda) + 3a_9(\lambda) + 3a_{10}(\lambda) + 3a_{11}(\lambda)},
\]
\[
\begin{pmatrix} \alpha \lambda \zeta_5 \end{pmatrix}_{j_{11}} = \zeta_5^{3a_1(\lambda) + 3a_2(\lambda) + 3a_3(\lambda) + 3a_4(\lambda) + 3a_5(\lambda) + 3a_6(\lambda) + 3a_7(\lambda) + 3a_8(\lambda) + 3a_9(\lambda) + 3a_{10}(\lambda) + 3a_{11}(\lambda)},
\]
\[
\begin{pmatrix} \alpha \lambda \zeta_5^2 \end{pmatrix}_{j_{11}} = \zeta_5^{2a_1(\lambda) + 2a_2(\lambda) + 2a_3(\lambda) + 2a_4(\lambda) + 2a_5(\lambda) + 2a_6(\lambda) + 2a_7(\lambda) + 2a_8(\lambda) + 2a_9(\lambda) + 2a_{10}(\lambda) + 2a_{11}(\lambda)},
\]
\[
\begin{pmatrix} \alpha \lambda \zeta_5^3 \end{pmatrix}_{j_{11}} = \zeta_5^{a_1(\lambda) + a_2(\lambda) + a_3(\lambda) + a_4(\lambda) + a_5(\lambda) + a_6(\lambda) + a_7(\lambda) + a_8(\lambda) + a_9(\lambda) + a_{10}(\lambda) + a_{11}(\lambda)},
\]
\end{thm}
$$\left[ \frac{\zeta^i + \zeta^{-i}}{\lambda} \right]_{11} = \zeta^{4a_{\lambda}(\lambda)+4a_1(\lambda)+3a_{\lambda}(\lambda)+a_0(\lambda)\lambda-2a_0(\lambda)},$$

and, letting $A_1(\lambda) = A_1(\lambda)/A_i(\lambda) \text{ mod } 121$,

$$\left[ \frac{1-\zeta}{\lambda} \right]_{11} = \zeta^{5a_{\lambda}(\lambda)-2a_0(\lambda)a_1(\lambda)-A_1(\lambda)-5 \text{ mod } 11}.$$

**Proof.** The first statement is Theorem 2.3 for $p = 11$. The second statement for units $\pm 1$ and $\zeta$ is Corollary 2.5 for $p = 11$; note that $\omega \not\mid \lambda$ because $\lambda$ is primary.

The last statements are proved in Proposition 3.3 for a primary prime $\lambda$. We need to show that Proposition 3.3 remains valid for any primary element $\lambda \in \mathbb{Z}[\zeta]$. Actually, it is sufficient to consider the case of $\lambda$ being of the form $\lambda = \pi_1\pi_2$ for two primary primes $\pi_1$ and $\pi_2$. If $\pi_1$ and $\pi_2$ are primary then Proposition 3.1 tells that $a_1(\pi_i) \equiv a_2(\pi_i) \equiv a_i(\pi_i) \equiv a_0(\pi_i) + a_3(\pi_i)^2 \equiv 0$ (mod 11), $i \in \{1, 2\}$. Combined with Proposition 4.1, we so obtain for $j \in \{3, 5, 7\}$, $a_j(\pi_1\pi_2) \equiv a_j(\pi_1) + a_j(\pi_2)$ (mod 11). We also obtain $a_6(\pi_1\pi_2)a_3(\pi_1\pi_2) \equiv a_6(\pi_1)a_3(\pi_1) + a_6(\pi_2)a_3(\pi_2) + 3a_6(\pi_1)a_3(\pi_2) + 3a_6(\pi_2)a_3(\pi_1)$ (mod 11) and $a_9(\pi_1\pi_2) \equiv a_9(\pi_1) + a_9(\pi_2) - 4a_6(\pi_1)a_3(\pi_2) - 4a_6(\pi_2)a_3(\pi_1)$ (mod 11). It is now easily checked, for $\eta_1 = \zeta^5 + \zeta^{-5}$, that $a_3(\pi_1\pi_2) + 3a_7(\pi_1\pi_2) + 4a_7(\pi_1\pi_2)a_3(\pi_1\pi_2) + 3a_9(\pi_1\pi_2) \equiv \text{ind}_{\pi_1}(\eta_1) + \text{ind}_{\pi_2}(\eta_1)$ (mod 11) where the values of $\text{ind}_{\pi_1}(\eta_1)$ and $\text{ind}_{\pi_2}(\eta_1)$ are given by Proposition 3.3; and similarly for the other fundamental units $\eta_2 = \zeta + \zeta^{-1}$, $\eta_3 = \zeta^{-1}(1 + \zeta + \zeta^2)$, and $\eta_4 = \zeta + \zeta^{-3}$. The proof for $\omega = 1 - \zeta$ essentially follows the same lines using a refinement of Proposition 4.1 for $A_1(a\beta)$ mod 121.

Theorem 4.4 gives rise to an efficient algorithm for computing the $11^{\text{th}}$-power residue symbol $\left[ \frac{a}{\lambda} \right]_{11}$. It requires $a$ and $\lambda$ to be co-prime and $T(\lambda) \not\equiv 0$ (mod 11) (as otherwise the symbol is not defined). As a reminder, $\omega$ stands for the special prime $1 - \zeta$ and $\eta_{1} = \zeta^{5} + \zeta^{-5}$, $\eta_{2} = \zeta + \zeta^{-1}$, $\eta_{3} = \zeta^{-1}(1 + \zeta + \zeta^2)$, $\eta_{4} = \zeta^{3} + \zeta^{-3}$ are fundamental units.

### Algorithm 2: Computing $\left[ \frac{a}{\lambda} \right]_{11}$

**Input:** $a, \lambda \in \mathbb{Z}[\zeta]$ with $\gcd(a, \lambda) \sim 1$ and $T(\lambda) \not\equiv 0$ (mod 11)

**Output:** $\left[ \frac{a}{\lambda} \right]_{11}$

1. $\lambda \leftarrow \text{primary}(\lambda)$
2. $j \leftarrow 0$

   while $N(\lambda) > 1$

   1. $\rho \leftarrow \text{euclid\_div}(a, \lambda)$
   2. $s \leftarrow 0$

      while $T(\rho) \equiv 0$ (mod 11)

      1. $s \leftarrow s + 1$
      2. $\rho \leftarrow \rho \div \omega$

   end

   $[p^*, (e_0, e_1, e_2, e_3, e_4)] \leftarrow \text{[primary(\rho), repr(\rho)]}$

   // $p^* = \zeta^{\eta_1(\eta_1^{2} \eta_{2}^{2} \eta_{3}^{2}) \eta_{4}^{2} \rho}$

   $j \leftarrow j + s \times \text{ind}_{\rho}(\omega) - e_0 \times \text{ind}_{\rho}(\zeta) - e_1 \times \text{ind}_{\rho}(\eta_1)$
   $- e_2 \times \text{ind}_{\rho}(\eta_2) - e_3 \times \text{ind}_{\rho}(\eta_3) - e_4 \times \text{ind}_{\rho}(\eta_4)$ (mod 11)

   $a \leftarrow \lambda^j; \lambda^j \leftarrow \rho^*$

end

return $\zeta^i$

**Proposition 4.5.** Algorithm 2 is correct.

**Proof.** Clearly, we have $\left[ \frac{a}{\lambda} \right]_{11} = \left[ \frac{a}{\lambda} \right]_{11} = \left[ \text{euclid\_div}(a, \lambda) \right]_{11}$. 

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Let $a^{(in)}(\text{resp. } a^{(out)})$ and $\lambda^{(in)}$ denote the values of $a$ and of $\lambda$ when entering (resp. exiting) the outer while-loop. Define $q := \text{euclid\_div}(a^{(in)}, \lambda^{(in)})$. At the end of the inner while-loop, we have $\rho^* = \zeta^e_0 \eta^e_1 \eta^e_2 \eta^e_3 \rho \omega^{-s}$ and so $\left[\frac{q}{\lambda^{(out)}}\right] = \left[\frac{\rho}{\lambda^{(in)}}\right] \cdot \sum e_i \cdot \text{ind}_r(\lambda^{(out)} \in \mathbb{Z})$. Since $\mathbb{Z} \setminus \mathbb{Q}(\zeta)$ is norm-Euclidean, it follows that the norm of $\lambda^{(out)}$ strictly decreases: $N(\lambda^{(out)}) = N(\rho^*) = N(q) N(\omega)^s$ < $N(q) < N(\lambda^{(in)})$. It eventually becomes 1 and the algorithm terminates.

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A Formulary

In this appendix, we list the general formulas for the logarithmic differential quotients $\Delta_j(a)$ and for the quantities $a_j(a^n)$.

We use Kummer’s notation (cf. Section 2.3) and represent an algebraic integer $a = \sum_{j=1}^{10} a_j \zeta^j \in \mathbb{Z}[\zeta]$ with $T(a) \neq 0$ as $a = F(\zeta)$. The following lemma is useful; it relates $F^{(k)}(1)$ to $A_k(a)$.

**Lemma A.1.** Let $a = \sum_{j=1}^{10} a_j \zeta^j \in \mathbb{Z}[\zeta]$. Then, letting $F^{(k)}$ denote the $k$th derivative of $F$, $F^{(k)}(1) \equiv A_k(a) \pmod{11}$ for $0 \leq k \leq 9$.

**Proof.** The proof is immediate. From the definition of $F$, we have $F(e^v) = \sum_{j=0}^{9} b_j e^{vj}$ where $b_0 = -a_{10}$ and $b_j = a_j - a_{10}$ for $1 \leq j \leq 9$. Hence, evaluating $F(e^v)$ at $v = 0$, we get $F(1) = \sum_{j=0}^{9} b_j = -a_{10} + \sum_{j=1}^{9} (a_j - a_{10}) = A_0(a) - 11a_{10} \equiv A_0(a) \pmod{11}$. Also, for $1 \leq k \leq 9$, $F^{(k)}(1) = \sum_{j=1}^{9} j^k b_j = \sum_{j=1}^{9} j^k (a_j - a_{10}) \equiv A_k(a) - a_{10} \sum_{j=1}^{10} j^k \equiv A_k(a) \pmod{11}$.

We assume that $A_0(a) \neq 0 \pmod{11}$. From $\Delta_1(a) = F^{(1)}(1) \iff F'(1) = \Delta_1(a) F(1)$, we get by induction $F^{(k)}(1) = \sum_{j=1}^{k} (j-1)^{k-1} \Delta_j(a) F^{(k-j)}(1)$ and therefore

$$\Delta_k(a) = \frac{F^{(k)}(1)}{F(1)} - \sum_{j=1}^{k-1} \frac{(k-1)}{j-1} \Delta_j(a) F^{(k-j)}(1). \quad (A1)$$

So, using Lemma A.1 and letting $a_j := a_j(a)$, we obtain the successive logarithmic differential quotients modulo 11.

| $\Delta_1(a)$ | $a_1 \pmod{11}$ |
|----------------|------------------|
| $\Delta_2(a)$ | $-a_1^2 + a_2 \pmod{11}$ |
| $\Delta_3(a)$ | $2a_1^3 - 3a_2a_1 + a_3 \pmod{11}$ |
| $\Delta_4(a)$ | $5a_1^4 + a_2a_1^2 - 4a_3a_1 - 3a_2^2 + a_4 \pmod{11}$ |
| $\Delta_5(a)$ | $2a_1^5 - 5a_2a_1^3 - 3a_3a_1^2 - 3a_2^2a_1 - 5a_4a_1 + a_3a_2 + a_5 \pmod{11}$ |
| $\Delta_6(a)$ | $a_1^6 - 3a_2a_1^4 + a_3a_1^3 + 5a_2^2a_1^2 - 3a_4a_1^2 - a_5a_2a_1 + 5a_5a_1 - 3a_3^2 - 4a_4a_2 + a_5^2 + a_6 \pmod{11}$ |
| $\Delta_7(a)$ | $5a_1^7 - a_2a_1^5 + 4a_3a_1^4 + a_2^2a_1^3 - a_4a_1^3 + 5a_5a_2a_1^2 - 2a_5a_1 - 3a_2^3a_1 + a_4a_2a_1 - 3a_4^2a_1 + 4a_6a_1 + a_3a_2^2 + a_5a_2 - 2a_3a_4 + a_7 \pmod{11}$ |
| $\Delta_8(a)$ | $-2a_1^8 - 3a_2a_1^6 + a_3a_1^5 + a_2^2a_1^4 - 3a_4a_1^4 - 2a_5a_2a_1^3 + 5a_5a_1^3 + 4a_2^2a_1^2 - a_4a_2a_1^2 + 3a_3^2a_1^2 + a_6a_1^2 - 2a_3a_2a_1^2 - a_5a_2a_1 - a_3a_4a_1 + a_3a_7 - a_2^2 - 2a_4a_2^2 - a_5^2 - 2a_3a_2a_1 - 5a_6a_2 - 2a_6^2 - a_5a_3 + a_8 \pmod{11}$ |
| $\Delta_9(a)$ | $5a_1^9 + 5a_2a_1^7 + 2a_3a_1^6 - 2a_2^2a_1^5 + 5a_4a_1^5 - 5a_5a_2a_1^4 - a_5a_1^4 - 5a_3a_3a_1^3 + a_4a_2a_1^3 - 3a_4^3a_1^3 + 2a_6a_1^3 + 3a_3a_2^2a_1^2 - 4a_5a_2a_1^2 - 3a_3a_4a_1^2 - 5a_5a_2^2 - 2a_6a_1^2 + a_4a_2^2a_1 + 5a_5^2a_2a_1 - 2a_6a_2a_1 + 3a_4a_1^2 - 4a_5a_3a_1 + 2a_8a_1 - 3a_3a_2^3 - 3a_5a_2^2 + a_3a_4a_2 - 3a_7a_2 - 5a_3a_4 - a_3^3 \pmod{11}$ |
| $\Delta_{10}(a)$ | $5a_1^{10} + 5a_2a_1^8 + 5a_3a_1^7 + 2a_4a_1^6 - 2a_5a_1^6 - 3a_2^3a_1^5 + 4a_6a_1^5 - f_j(a) = F^{(j)}(1) F(1)$ and obtain

$$\Delta_{11}(a) = 11(-5f_2f_1^9 - 2f_2f_1^8 - f_2^2f_1^7 - 5f_5f_1^7 - 4f_3f_2f_1^6 + f_5f_1^6 + 5f_2^3f_1^5 + 4f_4f_2f_1^5 - f_3^2f_1^4 - 6f_4f_1^5 - 2f_2f_2^2f_1^4 + 3f_5f_2f_1^4 + 5f_5f_1^4 - 5f_1^4 - 2f_2^2f_1^3 + 4f_4f_2f_1^3 - 2f_2^2f_1^3 - 3f_5f_3f_1^3 + 5f_5f_1^3 - 2f_4f_1^3)$$

In order to get $\Delta_{11}(a) \pmod{121}$, we resort on the general relation (A1). We define $f_j := f_j(a) = \frac{F^{(j)}(1)}{F(1)}$ and obtain

$$\Delta_{11}(a) = 11(-5f_2f_1^9 - 2f_2f_1^8 - f_2^2f_1^7 - 5f_5f_1^7 - 4f_3f_2f_1^6 + f_5f_1^6 + 5f_2^3f_1^5 + 4f_4f_2f_1^5 - f_3^2f_1^4 - 6f_4f_1^5 - 2f_2f_2^2f_1^4 + 3f_5f_2f_1^4 + 5f_5f_1^4 - 5f_1^4 - 2f_2^2f_1^3 + 4f_4f_2f_1^3 - 2f_2^2f_1^3 - 3f_5f_3f_1^3 + 5f_5f_1^3 - 2f_4f_1^3)$$
When \( a \) is primary, we have \( A_0(a) \equiv 0 \pmod{11} \) and \( f_1(a) \equiv f_4(a) \equiv f_5(a) \equiv f_6(a) \equiv 0 \pmod{11} \). For \( a = \sum_{j=1}^{10} a_j j^i \) primary, the previous relation then yields

\[
\frac{A_{11}(a)}{11} = \frac{4f_3f_1^2 - 4f_5f_3f_1}{2f_6f_5} + \frac{f_{11} \mod 121}{11} \equiv 2f_6f_5 + \frac{f_{11} \mod 121}{11} \equiv 2a_6a_5 + \frac{A_{11}(a) \mod 121}{11} \quad (A2)
\]

\[
2a_6a_5 + \frac{A_{11}(a) \mod 121}{11} \equiv 2a_6a_5 + A_1 + 5 - 5a_3 - a_5 + a_7 + a_9 \pmod{11}
\]

where \( A_{11}(a) = \sum_{j=1}^{10} a_j j^{11} \) and

\[
A_1 := A_1(a) = \frac{A_{11}(a) \mod 121}{11},
\]

noting that \( A_{11}(a) - A_1(a) \equiv 11(-a_2 - 4a_4 + 2a_5 - 3a_6 + 3a_7 - a_8 - a_{10}) \equiv 11(5A_0(a) - 5A_3(a) - A_5(a) + A_7(a) + A_9(a)) \pmod{121} \).

We now look at the quantities \( a_j(a^n) \) for \( n > 1 \). We write \( \binom{n}{j} \) as a shortcut to \( j! f_j(a) = n(n - 1) \ldots (n - j + 1) \) and again let \( a_j := a_j(a) \). The next formulas expand those given by Corollary 4.2.

1. \( a_1(a^n) \equiv \binom{n}{1} a_1 \pmod{11} \)
2. \( a_2(a^n) \equiv \binom{n}{1} a_2 + \binom{n}{2} a_1^2 \pmod{11} \)
3. \( a_3(a^n) \equiv \binom{n}{1} a_3 + \binom{n}{3} a_1 a_2 + \binom{n}{3} a_1^3 \pmod{11} \)
4. \( a_4(a^n) \equiv \binom{n}{1} a_4 + \binom{n}{4} (4a_1 a_3 + 3a_2^2) - \binom{n}{5} 5a_1^2 a_3 + \binom{n}{6} a_1^6 \pmod{11} \)
5. \( a_5(a^n) \equiv \binom{n}{1} a_5 + \binom{n}{5} (5a_1 a_4 - a_2 a_3) - \binom{n}{6} a_1 (a_1 a_3 - 4a_2^2) - \binom{n}{6} a_1^3 a_2 + \binom{n}{6} a_1^5 \pmod{11} \)
6. \( a_6(a^n) \equiv \binom{n}{1} a_6 - \binom{n}{2} (5a_1 a_5 - 4a_2 a_4 + a_3)^2 + \binom{n}{3} (4a_1^2 a_4 + 5a_1 a_2 a_3 + 4a_2^3)
- \binom{n}{4} a_1^2 (2a_1 a_3 - a_2^2) + \binom{n}{5} 4a_1^4 a_2 + \binom{n}{6} a_1^6 \pmod{11} \)
7. \( a_7(a^n) \equiv \binom{n}{1} a_7 - \binom{n}{2} (4a_1 a_6 + a_2 a_5 - 2a_3 a_4) - \binom{n}{3} (a_1 a_5 + 5a_1 a_2 a_4 - 4a_3 a_3^2 + 5a_2^2 a_3)
+ \binom{n}{4} a_1 (2a_1^2 a_4 + a_1 a_2 a_3 - 5a_2^3) + \binom{n}{5} a_1^3 (2a_1 a_3 - 5a_2^2) - \binom{n}{6} a_1^5 a_2 + \binom{n}{7} a_1^7 \pmod{11} \)
8. \( a_8(a^n) \equiv \binom{n}{1} a_8 - \binom{n}{2} (3a_1 a_7 + 5a_2 a_6 - a_3 a_5 - 2a_2^2)
- \binom{n}{3} (5a_1 a_6 - 3a_1 a_2 a_5 - 5a_1 a_3 a_4 - 2a_2 a_4 - 5a_2 a_3^2)
+ \binom{n}{4} (a_1 a_5 + 2a_1 a_2 a_4 + 5a_1 a_2 a_3^2 + 4a_2 a_2 a_3 - 5a_2^3)
+ \binom{n}{5} a_1^2 (4a_1^2 a_4 - a_1 a_2 a_3 + 2a_2^2) + \binom{n}{6} a_1^4 (a_1 a_3 + a_2^2) - \binom{n}{7} 5a_1^6 a_2 + \binom{n}{8} a_1^8 \pmod{11} \)
9. \( a_9(a^n) \equiv \binom{n}{1} a_9 - \binom{n}{2} (2a_1 a_8 - 3a_2 a_7 + 4a_3 a_6 - 5a_4 a_5)
+ \binom{n}{3} (3a_1^2 a_7 - a_1 a_3 a_5 - 4a_1 a_2 a_4 + 4a_2^2 a_5 - 5a_2 a_3 a_4 + 5a_3^3)
- \binom{n}{4} (4a_1 a_3 a_6 + 3a_1 a_2 a_5 + 5a_2 a_2 a_4 + 2a_1 a_2 a_3 - a_3 a_2 a_2 + 5a_2^3 a_3)
+ \binom{n}{5} a_1 (5a_1^2 a_4 - 5a_1^2 a_2 a_3 + 4a_1^2 a_2^2 a_3 - 4a_1 a_2 a_3 - a_2^2) + \binom{n}{6} a_1^3 (5a_1^2 a_4 - 5a_1 a_2 a_3 - 5a_2^3)
- \binom{n}{7} a_1^5 (4a_1 a_3 - 4a_2^2) + \binom{n}{8} 3a_1^2 a_2 + \binom{n}{9} a_1^9 \pmod{11} \)