Mass, center of mass and isoperimetry in asymptotically flat 3-manifolds

Sérgio Almaraz\(^1\) · Levi Lopes de Lima\(^2\)

Received: 29 October 2021 / Accepted: 26 May 2023 / Published online: 12 July 2023
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2023

Abstract
We revisit the interplay between the mass, the center of mass and the large scale behavior of certain isoperimetric quotients in the setting of asymptotically flat 3-manifolds (both without and with a non-compact boundary). In the boundaryless case, we first check that the isoperimetric deficits involving the total mean curvature recover the ADM mass in the asymptotic limit, thus extending a classical result due to G. Huisken. Next, under a Schwarzschild asymptotics and assuming that the mass is positive we indicate how the implicit function method pioneered by R. Ye and refined by L.-H. Huang may be adapted to establish the existence of a foliation of a neighborhood of infinity satisfying the corresponding curvature conditions. Recovering the mass as the asymptotic limit of the corresponding relative isoperimetric deficit also holds true in the presence of a non-compact boundary, where we additionally obtain, again under a Schwarzschild asymptotics, a foliation at infinity by free boundary constant mean curvature hemispheres, which are shown to be the unique relative isoperimetric surfaces for all sufficiently large enclosed volume, thus extending to this setting a celebrated result by M. Eichmair and J. Metzger. Also, in each case treated here we relate the geometric center of the foliation to the center of mass of the manifold as defined by Hamiltonian methods.

Mathematics Subject Classification 53C21 · 53A10 · 83C05
1 Introduction

Among the large scale invariants that can be attached by means of Hamiltonian methods to an asymptotically flat Riemannian 3-manifold, viewed as the (time-symmetric) initial data set of a solution of Einstein field equations, the ADM mass and the center of mass stand out as the most relevant ones. Besides their undisputed physical prominence, the study of these invariants also reveals deep connections with several areas of Geometric Analysis, including the Yamabe problem [1–3], the inverse mean curvature flow [4], the construction of canonical foliations at infinity [5–10] and the existence of isoperimetric surfaces for sufficiently large enclosed volumes [11]. We recall that in order to have the center of mass well defined, we must supplement the standard ADM decay assumptions with the so-called Regge–Teitelboim conditions [12, 13]. Even though some of the results discussed below do not require the fulfillment of these extra conditions, throughout this Introduction we assume that this is always the case in order to simplify the presentation.

Motivated by questions related to the Yamabe problem on manifold with boundary [14, 15], a version of the positive mass theorem has been established for asymptotically flat manifolds carrying a non-compact boundary [16, 17]. Moreover, a notion of center of mass for this class of manifolds has been introduced in [18], again under a suitable Regge–Teitelboim-type condition. The main purpose of this paper is to confirm that, similarly to what happens in the boundaryless case, these asymptotic invariants also play a central role in the investigation of large scale isoperimetric properties in the presence of a non-compact boundary.

With this goal in mind, we start by slightly broadening our perspective and checking that already in the boundaryless case the ADM mass is recovered as the asymptotic limit of certain isoperimetric deficits involving the enclosed volume, the area and the total mean curvature in various combinations (Theorem 4). This provides interesting extensions of a well-known result due to Huisken [19], where the area/volume case is dealt with. Moreover, it clearly suggests the existence of stable foliations in a neighborhood of infinity satisfying the corresponding curvature conditions, which besides the mean curvature should additionally involve (a modified version of) the Gauss–Kronecker curvature (the product of the principal curvatures). Under a Schwarzschild-type asymptotics, and assuming as usual that the ADM mass is positive, we prove the existence of the foliations by adapting the well-known implicit function method pioneered by Ye [6] and refined by Huang [8] (Theorem 6). Their approach is made feasible here by means of a simple calculation expressing the Gauss–Kronecker curvature of large coordinate spheres in terms of the corresponding mean curvature up to a remainder decaying fast enough (Proposition 13). This is a quite direct consequence of the existence of an almost conformal vector field at infinity and allows us to rely on the computations for the mean curvature in [8, 20]. As a by-product of this procedure we are able to check in each case that the geometric center of the foliation and the center of mass of the manifold always remain at a finite distance from each other and even coincide for certain values of the parameters defining the asymptotics. Besides their intrinsic interest, the existence of the foliations suggest that their leafs may be realized as isoperimetric surfaces for the corresponding isoperimetric problems, which involve minimizing the total mean curvature for large prescribed values of the volume or area, at least if some convexity assumption on the competing surfaces is imposed. Notice that this is in alignment with the well-known fact that round spheres in $\mathbb{R}^3$ constitute global minimizers to these problems among convex surfaces [21–23]; see also Remark 6 for more on this point.

Our next goal is to confirm that some of the classical results referred to above may be suitably extended to the class of asymptotically flat manifolds introduced in [16]. We first prove
that in the presence of a non-compact boundary the relative isoperimetric deficit involving the
area of coordinate hemispheres and the volume they enclose jointly with the boundary also
recovers the mass in the asymptotic limit (Theorem 7). As before, this preliminary remark is
accompanied by a result ensuring, again under a Schwarzschild asymptotics and assuming
that the mass is positive, that a neighborhood of infinity is foliated by stable free boundary
hemispheres of constant mean curvature (CMC) and moreover that the geometric center of
mass of this foliation coincides with the center of mass as defined by Hamiltonian methods
(Theorem 9). We may view these results as the large scale analogues of the main theorems
in [24, 25], even though the technical details are quite distinct in nature. A key step here is
the establishment of a certain identity relating the center of mass to the integral over large
coordinate hemispheres of the higher order terms in the expansion of the corresponding mean
curvature against the asymptotic coordinates along the non-compact boundary (Proposition
19). In the boundaryless case, this kind of identity was first proved in [8, 20] by means of a
quite delicate density argument. Subsequently, an elementary proof appeared in [26] and we
succeed in checking that this latter reasoning adapts well in the presence of a non-compact
boundary. As in these works, the identity is used here to remove the obstruction to the invert-
ibility of the relevant linearized operator coming from the invariance of the mean curvature
under translational isometries preserving the asymptotic boundary, besides playing a crucial
role in checking that the geometric and Hamiltonian centers of mass coincide. Moreover, it
provides an alternate expression for the center of mass as an asymptotic integral involving
the mean curvature of large coordinate hemispheres (Corollary 29). Finally, we complete
our analysis of the large scale geometry of this class of manifolds by checking that these
free boundary CMC hemispheres constitute the unique relative isoperimetric surfaces for
sufficiently large values of the enclosed volume (Theorem 10). This extends to our setting a
previous result by Eichmair–Metzger in the boundaryless case [11]; see also [27].

In order to keep to a minimum the technical features of the exposition, we work here under
a Schwarzschild asymptotics whenever needed. In particular, we are usually quite generous
when imposing orders of decay rates for the asymptotics of geometric quantities. Also, we
have chosen to avoid the consideration of inner minimal horizons (black holes). Moreover,
we restrict ourselves to the time-symmetric case, thus bypassing the complications coming
from the extrinsic geometry of the given initial data set. As a consequence, we are unable
to consider here the rather appealing question of determining how the asymptotic quantities
vary as the initial data set evolves in time under the field equations. Nevertheless, we believe
that the results established here under these mildly restrictive assumptions may be suitably
extended to larger classes of asymptotically flat manifolds, in the line of [7, 9–11, 28–31]
for instance. Another topic we refrain from discussing here in detail is the uniqueness of
the leafs as solutions of the corresponding variational problems. However, we remark that
in the specific case of the free boundary CMC hemispheres in Theorem 9, the appropriate
uniqueness is easily obtained by adapting an argument in [5, Section 4]; see Appendix E.
We note that this uniqueness is crucial when identifying those hemispheres to the relative
isoperimetric hemispheres in Theorem 10 and it eventually guarantees that they are unique
in the class of relative isoperimetric surfaces enclosing large volumes. Finally, it would be
interesting to investigate how our constructions fit into the quasi-local approach to conserved
quantities summarized in [32]. We hope to address some of these questions elsewhere.

This paper is organized as follows. In Sect. 2, after a brief motivation intended to illustrate
the local interplay between isoperimetric quotients and the scalar curvature, we provide
precise statements of all the results mentioned above. The arguments leading to the existence
of the stable foliations are presented in Sects. 3 and 4. The proof of Theorem 10, which
provides the relative isoperimetric regions in the presence of a boundary, is explained in
Sect. 5. In order not to interrupt the exposition in the bulk of the paper, we defer to the appendices the proofs of a few technical results, including a discussion of the isoperimetric variational theories involved, specially in regard to the corresponding stability criteria. These appendices are also used to introduce much of the notation used in the paper. Finally, we also intersperse along the text a few interesting problems in this area of research.

2 Preliminaries and statements of the results

For the sake of motivation, we start by considering an arbitrary Riemannian 3-manifold \((M, g)\). We fix \(q \in M\) and introduce normal coordinates \(z = (z_1, z_2, z_3)\) around \(q\) in the usual way. If \(r = |z|\) is the geodesic distance to \(q\), let \(B_r^M(q)\) be the geodesic ball of radius \(r\) centered at \(q\) and \(S_r^M(q) = \partial B_r^M(q)\) the corresponding geodesic sphere. If \(N\) is the outward unit normal to \(S_r^M(q)\), let \(W_r = \nabla N\) be the shape operator of \(S_r(q)\), where \(\nabla\) is the Levi-Civita connection. For each \(k = 1, 2\) we may consider the curvature integral

\[
Q_{r}^{M,k}(q) = \int_{S_r^M(q)} \sigma_{2-k}(W_r)dS_r^M(q),
\]

where \(\sigma_i(W_r)\) is the elementary symmetric function of degree \(i\) in the eigenvalues of \(W_r\) (the principal curvatures \(\kappa_1, \kappa_2\)). Thus, \(\sigma_0 = 1, \sigma_1 = \kappa_1 + \kappa_2\), the mean curvature, also denoted here by \(H\), and \(\sigma_2 = \kappa_1 \kappa_2\), the Gauss–Kronecker curvature, also denoted here by \(K\). Finally, we set \(Q^{M:3}(q) = \text{vol}_g(B_r^M(q))\) by convention.

For \(1 \leq k < h \leq 3\) we consider the isoperimetric quotient

\[
I_{r}^{M:h,k}(q) = \frac{Q_{r}^{M,k}(q)^{\frac{h}{k}}}{Q_{r}^{M:h}(q)}.\tag{2}
\]

Notice that for \((M, g) = (\mathbb{R}^3, \delta)\), the Euclidean space with the standard flat metric, these quotients do not depend on the pair \((q, r)\) so we denote them simply by \(I^{h,k}\).

**Proposition 1** As \(r \to 0\), there exists \(c_{k, h} > 0\) such that

\[
1 - \frac{I_{r}^{M:h,k}(q)}{I^{h,k}} = c_{k, h}R_g(q)r^2 + O(r^4),
\]

where \(R_g\) is the scalar curvature of \(g\). In particular, if \(R_g(q) \geq 0\) then

\[
I_{r}^{M:h,k}(q) \leq I^{h,k},
\]

for all \(r > 0\) small enough.

**Proof** This folklore result may be checked as follows. The expansion for the volume, namely,

\[
\frac{Q^{M:3}(q)}{Q^{\mathbb{R}^3:3}(0)} = 1 - \frac{R_g(q)}{30}r^2 + O(r^4),
\]

may be found in [33, Section 9.2]. By means of the well-known variational formulae

\[
Q_{r}^{M:2}(q) = \frac{d}{dr}Q_{\rho}^{M:3}(q), \quad Q_{r}^{M:1}(q) = \frac{d}{dr}Q_{\rho}^{M:2}(q),
\]

we see that

\[
\frac{Q_{r}^{M:2}(q)}{Q^{\mathbb{R}^3:2}(0)} = 1 - \frac{R_g(q)}{18}r^2 + O(r^4),
\]

\(\Box\) Springer
and
\[
\frac{Q^M_{\tau;1}(q)}{Q^{\mathbb{R}^3;1}(0)} = 1 - \frac{R_g(q)}{9} r^2 + O(r^4).
\]
From these, the expansions in (3) follow easily. \(\square\)

This result says that the local behavior of the relative isoperimetric quotients \(I^{M,h,k}_r(q)/I^{h,k}_r\) is completely determined by the sign of \(R_g(q)\). We note however that in general this isoperimetric comparison result only holds true at small scales since the argument depends on the corresponding asymptotic expansions as \(r \to 0\). Our first remark here is that large scale versions of this principle hold true in the setting of asymptotically flat manifolds with non-negative scalar curvature and, as we shall see, this is closely related to the positive mass theorem in General Relativity. In the case \((h, k) = (3, 2)\), this was first observed by Huisken [19], as we now pass to describe.

**Definition 1** A manifold \((M, g)\) is asymptotically flat if there exists a compact subset \(U \subset M\) and a diffeomorphism \(M \setminus U \cong \mathbb{R}^3 \setminus B_1(0)\) such that in the corresponding asymptotic coordinates \(x = (x_1, x_2, x_3)\) there holds
\[
e := g - \delta = O_2(r^{-\tau}), \quad \tau > \frac{1}{2}
\]
and
\[
R_g = O(r^{-3-\sigma}), \quad \sigma > 0,
\]
as \(r = |x|_{\delta} \to +\infty\).

Note that (6) implies \(R_g \in L^1(M)\). Also, for a tensor \(f = f(x)\) in the asymptotic region, we say that \(f = O_k(r^{-\tau})\) if \(|(\partial_{\alpha}f)(x)| = O(r^{-\tau-|\alpha|})\) for any multi-index \(\alpha\) with \(0 \leq |\alpha| \leq k\).

Any manifold as in Definition 1 may be viewed as the time-symmetric initial data set of a solution to the Einstein field equations whose geometry at spatial infinity is essentially Minkowskian. In particular, the appropriate Hamiltonian version of Noether’s theorem may be employed to attach to the Riemannian manifold \((M, g)\) certain asymptotic invariants capturing common physical quantities associated with the isolated gravitational system modeled by the solution [12, 13, 34–37]. The most prominent of these invariants is the so-called ADM mass, which is given by
\[
m_{ADM} = \lim_{r \to +\infty} \frac{1}{16\pi} \int_{S^2_r} \mathbb{U}(1, e) \left(\frac{x}{r}\right) dS^2_r.
\]
Here,
\[
\mathbb{U}(f, e) = f(\text{div}_e - d\text{tr}_e - \text{iv}_e f e + \text{tr}_e f d f,
\]
f : \(\mathbb{R}^3 \to \mathbb{R}\) is a smooth function, \(1\) is the function identically equal to 1, and \(S^2_r\) is the coordinate sphere of radius \(r\) in the asymptotic region centered at the origin.

Let us denote by \(A(r)\) (respectively \(V(r)\)) the area of \(S^2_r\) (respectively the volume of the compact region enclosed by \(S^2_r\)). We now recall a classical definition due to Huisken.

**Definition 2** [19] Under the conditions above, we set
\[
J^{M:3,2}_r = \frac{2}{A(r)} \left(V(r) - \frac{1}{6\pi^{1/2}} A(r)^{\frac{3}{2}}\right).
\]
Remark 1 This may be expressed as

\[ J_r^{M;3,2} = 2 \frac{V(r)}{A(r)} \left( 1 - \frac{\bar{I}_r^{M;3,2}}{I^{3,2}} \right), \tag{9} \]

where

\[ \bar{I}_r^{M;3,2} = \frac{A(r)^{3/2}}{V(r)} \]

is the large scale analogue of the isoperimetric quotient in (2) with \((h, k) = (3, 2)\) and \(I^{3,2} = 6\pi^{1/2}\) is the corresponding quotient evaluated on round spheres in \((\mathbb{R}^3, \delta)\). Thus, \(J_r^{M;3,2}\) should be thought of as the asymptotic analogue of the isoperimetric deficit in (3) with \((h, k) = (3, 2)\).

As remarked above, the isoperimetric deficit \(J_r^{M;3,2}\) should somehow be controlled in the case the standard dominant energy condition \(R_g \geq 0\) holds; here we view \((M, g)\) as a time-symmetric initial data set propagating in time to a solution of Einstein field equations, so this energy condition is justified on physical grounds. This link between scalar curvature and isoperimetry holds true indeed and the key result goes as follows.

Theorem 2 [19] One has

\[ \lim_{r \to +\infty} J_r^{M;3,2} = m_{ADM}. \tag{10} \]

Notice that this justifies the appearance of the extra factor \(V(r)/A(r)\) in the right-hand side of (1) as the ADM mass has the dimension of length.

Combining this with the positive mass theorem [38] we obtain the following remarkable result.

Corollary 3 If \((M, g)\) is asymptotically flat with scalar curvature \(R_g \geq 0\) then for \(r > 0\) large enough,

\[ \bar{I}_r^{M;3,2} \leq I^{3,2}, \]

with the strict inequality holding unless \((M, g) = (\mathbb{R}^3, \delta)\) isometrically.

Proof We know that \(m_{ADM} \geq 0\) by the positive mass theorem. If \(m_{ADM} = 0\) then \((M, g) = (\mathbb{R}^3, \delta)\) isometrically and the equality holds. Otherwise, \(m_{ADM} > 0\) and the strict inequality holds. \(\square\)

Remark 2 The discussion above leads naturally to a conjectured \(C^0\) version of the positive mass theorem. More precisely, the subisoperimetry condition in (4) with \((h, k) = (3, 2)\) may be interpreted as the statement that \(R_g(q) \geq 0\) for a metric \(g\) which is only assumed to be \(C^0\). This led Huisken to conjecture that the validity of this subisoperimetry condition at any \(q \in M\) implies that the mass of an asymptotically flat manifold \((M, g)\) is nonnegative, where \(g\) is of class \(C^0\), with the equality holding if and only if \((M, g) = (\mathbb{R}^3, \delta)\) isometrically. Of course, here the mass of \((M, g)\) is defined by the limit in the left-hand side of (10), whenever it exists. To our knowledge, this conjecture, which constitutes a far-reaching generalization of the standard positive mass theorem in [38], is still wide open. In any case, we observe that a similar question may be formulated in the presence of a non-compact boundary; see Remark 7 below.
A proof of Theorem 2 appears in [39] and for our purposes a key observation is that a simple variation of their computations, which we reproduce in Appendix A below, proves that this remarkable connection between isoperimetry and mass also holds true for the other isoperimetric quotients involving the total mean curvature

\[ M(r) := \int_{S^2_r} H \, dS \]

of the coordinate sphere \( S^2_r \). They correspond to the cases \((h, k) = (3, 1)\) and \((h, k) = (2, 1)\) in (3) and are defined as follows.

**Definition 3** Under the conditions above, set

\[ J_{M;3,1}^r = \frac{4}{3rM(r)} \left( V(r) - \frac{1}{3 \cdot 2^7 \cdot \pi^2} M(r)^3 \right) = \frac{4}{3} \frac{V(r)}{rM(r)} \left( 1 - \frac{\widehat{I}_{M;3,1}^r}{I^{3,1}} \right), \]

and

\[ J_{M;2,1}^r = \frac{1}{M(r)} \left( A(r) - \frac{1}{16\pi} M(r)^2 \right) = \frac{A(r)}{M(r)} \left( 1 - \frac{\widehat{I}_{M;2,1}^r}{I^{2,1}} \right), \]

where

\[ \widehat{I}_{M;3,1}^r = \frac{M(r)^3}{V(r)}, \quad \widehat{I}_{M;2,1}^r = \frac{M(r)^2}{A(r)}, \quad I^{3,1} = 3 \cdot 2^7 \cdot \pi^2, \quad I^{2,1} = 16\pi. \]

**Theorem 4** If \((M, g)\) is asymptotically flat then

\[ \lim_{r \to +\infty} J_{M;3,1}^r = m_{ADM}, \quad \lim_{r \to +\infty} J_{M;2,1}^r = m_{ADM}. \quad (11) \]

Again, if combined with the positive mass theorem, this result has the following nice consequence.

**Corollary 5** If \((M, g)\) is asymptotically flat with scalar curvature \( R_g \geq 0 \) then for all \( r > 0 \) large enough, we have the inequalities

\[ \widehat{I}_{r;3,1} \leq I^{3,1}, \quad \widehat{I}_{r;2,1} \leq I^{2,1}, \]

with the strict inequality holding in each case unless \((M, g) = (\mathbb{R}^3, \delta)\) isometrically.

We present the proof of Theorem 4 in Appendix A.

**Remark 3** The subisoperimetry condition in (4) with either \((h, k) = (3, 1)\) or \((h, k) = (2, 1)\) may be interpreted as the statement that \( R_g(q) \geq 0 \) for a metric \( g \) which is only assumed to be \( C^1 \). Thus, similarly to the discussion in Remark 2, we may ask whether the validity of any of these subisoperimetry conditions everywhere on an asymptotically flat manifold endowed with a \( C^1 \) metric implies that the associated mass in nonnegative, with the rigidity statement holding as well. Here, the mass should be defined by the corresponding limit in the left-hand sides of (11), whenever it exists. It would be interesting to examine this question in light of recent developments on the positive mass theorem for metrics with low regularity; see [40] and the references therein.

We now recall the connection between isoperimetry and another basic invariant of asymptotically flat 3-manifolds, namely, the center of mass [12, 13]. In the following, given a tensor \( f = f(x) \) on the asymptotic region, we set \( f^{(1)}(x) = (f(x) + f(-x))/2 \) and
In fact, convergence takes place if and only if each \( x_i \) expansion of the metric in respectively, the first order term and the odd part of the second order term in the asymptotic \( m \) where \( \partial_{\alpha} f \)\(^{(a)}(1+|\alpha|)\) (respectively, \( \partial_{\alpha} f \)\(^{(a)}(1+|\alpha|)\)) for any multi-index \( \alpha \) with \( 0 \leq |\alpha| \leq k \).

**Definition 4** We say that an asymptotically flat 3-manifold as in Definition 1 satisfies the Regge–Teitelboim (RT) conditions if there holds

\[
g^{(-1)}(x) = \hat{O}(r^{-1-\tau}), \quad \tau > \frac{1}{2},
\]

and

\[
R^{(-1)}(x) = \hat{O}(r^{-4-\sigma}), \quad \sigma > 0.
\]

Note that (13) guarantees that each \( x_i R^{(-1)}_g \in L^1(M) \) for \( i = 1, 2, 3 \). Also, it is clear that \( x_i R^{(1)}_g \) has the property that its integral over the region enclosed by two coordinate spheres vanishes. So, we may use the method in [41], with \( x_i \) as a static potential, to ensure that to any manifold as in Definition 4 with \( m_{ADM} \neq 0 \) we may attach a (Hamiltonian) center of mass defined by

\[
\mathcal{C}_i = \lim_{r \to +\infty} \frac{1}{16\pi m_{ADM}} \int_{S^2_r} U(x_i, e) \left( \frac{x}{r} \right) dS^2_r, \quad i = 1, 2, 3.
\]

**Example 1** Recall that the Schwarzschild metric on \( \mathbb{R}^3 \setminus \{c\}, c \in \mathbb{R}^3 \), is given by

\[
g_{m,c}(x) = \left( 1 + \frac{m}{2|x-c|} \right)^4 \delta,
\]

where \( m \in \mathbb{R} \). We denote \( g_m = g_{m,0} \) for simplicity. Clearly, \( g_{m,c} \) is asymptotically flat and since \( R_{g_{m,c}} = 0 \), it also satisfies the RT conditions. The associated asymptotic invariants may be easily determined if we expand (15) as \( r = |x|_{\delta} \to +\infty \):

\[
g_{m,c}(x) = \left( 1 + \frac{2m}{r} + \frac{2mc \cdot x}{r^3} + \frac{3m^2}{2r^2} + O(r^{-3}) \right) \delta,
\]

where the dot is the Euclidean inner product. A direct computation then gives \( m_{ADM} = m \) and \( C = c \).

This example indicates that the mass and the center of mass are somehow captured by, respectively, the first order term and the odd part of the second order term in the asymptotic expansion of the metric in \( r^{-1} \). This motivates the consideration of the following important class of examples.

**Definition 5** We say that an asymptotically flat 3-manifold \( (M, g) \) (with \( \tau = 1 \)) is asymptotically Schwarzschild \((\text{aS})\) if in the asymptotic region there holds

\[
g \to \left( 1 + \frac{2m}{r} \right) \delta = O_3(r^{-2}).
\]

In general, an \text{aS} manifold may not satisfy the RT conditions as (13) might be violated. In this case, we would still have \( m_{ADM} = m \) but if \( m \neq 0 \) the limit in (14) might not exist. In fact, convergence takes place if and only if each \( x_i R_g \) is integrable [42]. Examples where the limit defining the center of mass diverges may be found in [43].

**Definition 6** If an \text{aS} manifold satisfies the RT conditions we say that it is an \text{aSRT} manifold.
Motivated by (16), we also consider a more flexible kind of asymptotics.

**Definition 7** We say that an asymptotically flat 3-manifold \((M, g)\) (with \(\tau = 1\)) is \(\epsilon\)-asymptotically Schwarzschild (\(\epsilon\)-aS) if in the asymptotic region there holds

\[
g - f_{m,c}^{\gamma_1, \gamma_2} \delta = O_3(r^{-2-\epsilon}), \quad f_{m,c}^{\gamma_1, \gamma_2} = 1 + \frac{2m}{r} + \gamma_1 \frac{c \cdot x}{r^3} + \gamma_2 \frac{1}{r^2}
\]

with \(\epsilon \geq 0, m \neq 0, \gamma_1, \gamma_2 \in \mathbb{R}\) and \(c \in \mathbb{R}^3\).

**Remark 4** Clearly, an \(\epsilon\)-aS manifold is aS. Also, the case \(\epsilon = 0\) does not impose any further conditions on an aS 3-manifold and \(\gamma_1, \gamma_2\) and \(c\) are irrelevant in this case. If \(\epsilon > 0\) then \((M, g)\) is aSRT with center of mass

\[
C = \frac{\gamma_1}{2m} c.
\]

The most relevant choice in this case is \(\gamma_1 = 2m\) and \(\gamma_2 = 3m^2/2\), when satisfying (18) is equivalent to

\[
g - g_{m,c} = O_3(r^{-2-\epsilon}).
\]

With the appropriate definitions at hand, we now come back to the isoperimetry discussion motivated by Theorems 2 and 4. These results suggest that for aSRT manifolds with \(m > 0\), large coordinate spheres might be perturbed to yield global solutions to the corresponding isoperimetric problems (here we should assume that the competing surfaces are convex in a suitable sense so as to make sure that \(M(r) > 0\)). In the classical case \((h, k) = (3, 2)\), this has been established in [11]. Very likely, a similar result should hold in the remaining cases treated in Theorem 4 and we hope to address this issue elsewhere.

In any case, a first check toward this goal is to show that under these same asymptotic conditions a neighborhood of infinity can be foliated by surfaces satisfying the corresponding curvature conditions and moreover that the geometric center of the foliations relate to the center of mass of \((M, g)\) defined by Hamiltonian methods as in (14). In the case \((h, k) = (3, 2)\), the existence of a canonical foliation by stable CMC spheres and the identification of the geometric center of this foliation with the Hamiltonian center of mass has been first established by Huisken and Yau [5] and then investigated further by many authors under varying asymptotic assumptions; see for instance [6–10, 26, 28, 29, 31, 44, 45]. Here we rely on the implicit function approach developed in [6, 8, 9, 20] to treat the cases \((h, k) = (3, 1)\) and \((h, k) = (2, 1)\) as follows.

**Theorem 6** Let \((M, g)\) be an aSRT 3-manifold with positive ADM mass. Then there exists a neighborhood of infinity which is foliated by strictly stable spheres satisfying any of the curvature conditions

\[
\tilde{K} := K - \frac{1}{2} \text{Ric}_g (v, v) = \text{const.},
\]

or

\[
\tilde{K} = \gamma H.
\]

Here, \(K\) is the Gauss–Kronecker curvature of the leaf, \(v\) is its outward unit normal and \(\gamma\) is a suitably chosen Lagrangian multiplier (varying with the leaf). Also, the geometric center \(C_{\mathcal{F}}\) of any of these foliations remains at a finite distance from the Hamiltonian center of mass \(C\) of \((M, g)\). Finally, if we further assume that the manifold is \(\epsilon\)-aS with \(\epsilon > 0\) then \(C_{\mathcal{F}} = C\).
The curvature conditions (19) and (20) express the fact that the surfaces are critical configurations for the corresponding isoperimetric quotients, i.e., they extremize the total mean curvature under a volume or area constraint, respectively. The stability statement should be understood in this variational sense; see Appendix B for a discussion on this issue. The proof of Theorem 6 is presented in Sect. 3.

Remark 5 It is not clear whether the rather unsatisfactory assumption $\epsilon > 0$ in the last statement of Theorem 6 is an artifact of our method of proof or an intrinsic feature of the invariants involved.

Remark 6 It is well-known that in Euclidean space $(\mathbb{R}^3, \delta)$ the isoperimetric quotients in (3) with $(h, k) = (3, 1)$ or $(h, k) = (2, 1)$ are minimized at round spheres if some convexity assumption is imposed on the competing surfaces. In the convex case ($K \geq 0$) this follows from classical Brunn-Minkowski theory [21]. The result holds more generally if the competing surfaces are assumed to be mean convex ($H \geq 0$) and star-shaped [22] and it is conjectured that star-shapedness may be dispensed with; see [23] for a survey of recent contributions in this direction. The problem of extending these isoperimetric inequalities (and their analogues involving quermassintegrals of higher order) to other geometries, again under suitable convexity requirements, is quite challenging from a technical viewpoint. The class of aSRT 3-manifolds seems to be a natural choice for starting this research program, as it is reasonable to conjecture that the leaves of the foliations in Theorem 6 constitute global solutions to the corresponding isoperimetric problems for large values of the area or volume. See [46] for a possible approach to a version of this problem in space forms using methods from Integral Geometry.

We now turn our attention to asymptotically flat 3-manifolds carrying a non-compact boundary as in [16]. We set $\mathbb{R}^3_+ = \{x \in \mathbb{R}^3; x_3 \geq 0\}$, the Euclidean half-space with the standard flat metric $\delta^+ = \delta|_{\mathbb{R}^3_+}$.

Definition 8 ([16]) A 3-manifold $(M, g)$ is asymptotically flat with a non-compact boundary $\Sigma$ if there exists a compact subset $V \subset M$ and a diffeomorphism $M \setminus V \sim \mathbb{R}^3_+ \setminus B_1(\vec{0})$ such that in the corresponding asymptotic coordinates $x = (x_1, x_2, x_3)$ there holds

$$e^+ := g - \delta_+ = O_3(r^{-\tau}), \quad \tau > \frac{1}{2},$$

and

$$R_g = O(r^{-3-\sigma}), \quad H_\Sigma = O(r^{-2-\sigma}), \quad \sigma > 0,$$

as $r = |x|_\delta \to +\infty$. Here, $H_\Sigma$ is the mean curvature of $\Sigma$.

Note that (22) implies that $R_g \in L^1(M)$ and $H_\Sigma \in L^1(\Sigma)$.

In this setting, the asymptotic invariant corresponding to (7) has been defined in [16] and is given by

$$m = \lim_{r \to +\infty} \frac{1}{16\pi} \left( \int_{S^2_{r,+}} \mathbb{U}(1, e^+) \left( \frac{x}{r} \right) dS^2_{r,+} - \int_{S^1_{r}} e^+ \left( \frac{x}{r}, \vartheta \right) dS^1_{r} \right),$$

where $S^2_{r,+}$ is the coordinate hemisphere of radius $r$, centered at the origin of $\mathbb{R}^3$, in the asymptotic region, $S^1_{r} = \partial S^2_{r,+} \subset \Sigma$ and $\vartheta$ is the outward unit co-normal vector field to $S^2_{r,+}$ along $S^1_{r}$ (with respect to the flat metric $\delta^+$). In order to introduce the corresponding relative
isoperimetric deficit we denote by $A(r)$ (respectively $V(r)$) the area of $S_{r,+}^2$ (respectively, the volume of the compact region enclosed by $S_{r,+}^2$ and $\Sigma$). Similarly to (8) we may consider

$$J_r^{M;3,2} = \frac{1}{A(r)} \left( V(r) - \frac{1}{3 \cdot 2^{1/2} \pi^{1/2}} A(r)^{3/2} \right) = \frac{V(r)}{A(r)} \left( 1 - \frac{I_r^{M;3,2}}{I_r^{3,2}} \right),$$

where

$$I_r^{M;3,2} = \frac{A(r)^{3/2}}{V(r)},$$

and $I_r^{3,2} = 3 \cdot 2^{1/2} \pi^{1/2}$ is the relative isoperimetric quotient of a hemisphere centered at a point in $\mathbb{R}^2 = \partial \mathbb{R}^3$. Similarly to Theorem 2 we now have

**Theorem 7** Under the conditions above,

$$\lim_{r \to +\infty} J_r^{M;3,2} = m. \tag{24}$$

This result, whose proof is postponed to Appendix A, has the following notable consequence.

**Theorem 8** Let $(M, g)$ be asymptotically flat with a non-compact boundary $\Sigma$ as above. Assume further that $R_g \geq 0$ everywhere and $H_\Sigma \geq 0$ along the boundary. Then for all $r > 0$ large enough,

$$I_r^{M;3,2} \leq I_r^{3,2},$$

with the strict inequality holding unless $(M, g) = (\mathbb{R}^3, \delta^+)$ isometrically.

**Proof** Apply the positive mass theorem in [16]. $\Box$

**Remark 7** Pick any smooth Riemannian 3-manifold $(M, g)$ with boundary $\partial M$ and fix $q \in \partial M$. After introducing Fermi coordinates around $q$, we may consider the coordinate hemisphere $S_{r,+}^M(q)$ of small radius $r > 0$ centered at $q$. By using the calculations in [24, 25] we may easily check that, as $r \to 0$,

$$1 - \frac{I_r^{M,\partial M;3,2}(q)}{I_r^{3,2}} = c^+ H_\partial M(q)r + O(r^2).$$

Here, $c^+ > 0$ is a universal constant and

$$I_r^{M,\partial M;3,2}(q) = \frac{A(r)^{3/2}}{V(r)},$$

where $A(r)$ and $V(r)$ are respectively the area of $S_{r,+}^{M,\partial M}(q)$ and the volume it encloses jointly with $\partial M$, and $H_\partial M(q)$ is the mean curvature of $\partial M$ at $q$. Thus, if the metric $g$ is assumed to be merely $C^0$, we may interpret the (boundary) subisoperimetry condition

$$I_r^{M,\partial M;3,2}(q) \leq I_r^{3,2}, \tag{25}$$

for all $r > 0$ small enough, as the statement that $H_\partial M(q) \geq 0$ in this weak sense. Thus, in the spirit of Remark 2 above, we are led to conjecture that for an asymptotically flat 3-manifold with a non-compact boundary endowed with a $C^0$ metric, the fulfillments of the subisoperimetry conditions (4) for $q$ in the interior and (25) for $q$ on the boundary imply that the mass of the manifold is nonnegative, with the equality taking place if only if it is
isometric to \((\mathbb{R}^3_+, \delta^+)\). Of course, here the mass should be defined by the limit in the left-hand side of (24), whenever it exists. Presumably, after a doubling across the boundary, this turns out to be equivalent to the original conjecture in Remark 2. In any case, this would lead to a far-reaching generalization of the main result in [16].

In order to have a well defined center of mass in the setting of Definition 8, we need the analogue of the RT conditions (12) and (13). Here, given a tensor \(f = f(x)\) in the asymptotic region, we set \(f^{(1)}(x) = (f(x', x_3) + f(-x', x_3))/2\), with \(x' = (x_1, x_2)\), and \(f^{(-1)} = f - f^{(1)}\). Since, in this region, \(x_3\) is an even function of \(x'\) which vanishes on \(\Sigma\), namely, \(x_3 = \sqrt{r^2 - |x'|^2}\), we may think of \(f^{(1)}\) and \(f^{(-1)}\) as the even and odd parts of \(f\). In particular, we can mimic the discussion preceding Definition 4 so as to make sense of any of the conditions \(f^{(1)}, f^{(-1)} \in \hat{O}_k(r^{-\tau})\).

**Definition 9** We say that an asymptotically flat 3-manifold with a non-compact boundary as in Definition 8 satisfies the Regge–Teitelboim (RT) conditions if there holds

\[
g^{(-1)}(x) = \hat{O}(r^{-1-\tau}), \quad \tau > 1/2,
\]

and

\[
R^{(-1)}_g(x) = \hat{O}(r^{-4-\sigma}), \quad H^{(-1)}(x) = \hat{O}(r^{-3-\sigma}), \quad \sigma > 0.
\]

Note that (27) implies that each \(x_a R^{(-1)} g \in L^1(M)\) and each \(x_a H^{(-1)}(x) \in L^1(\Sigma)\), \(a = 1, 2\). Also, \(x_a R^{(-1)} g\) has the property that its integral over the region enclosed by two large coordinate hemispheres vanishes, and similarly for the integral of \(x_a H^{(1)}(x)\) over the region in the boundary enclosed by two coordinate circles. Thus, we may use the method in [41], as adapted in [17] and with \(x_a\) as a static potential, to ensure that to any manifold as in Definition 9 with \(m \neq 0\) we may attach a (Hamiltonian) center of mass by

\[
C^+_\alpha = \lim_{r \to \infty} \frac{1}{16\pi m} \left( \int_{S^2} \mathbb{U}(x_a, e^+) \left( \frac{x}{r} \right) dS^2_{r,+} - \int_{S^1} x_a e^+ \left( \frac{x}{r}, \vartheta \right) dS^1_r \right),
\]

where \(\alpha = 1, 2\). This invariant has been introduced in [18].

As in the boundaryless case, we may also consider the *half-Schwarzschild metric* \(g^+_m = g_m |_{\mathbb{R}_+^3 \setminus \{0\}}\).

**Definition 10** An asymptotically flat 3-manifold with a non-compact boundary \((M, g, \Sigma)\) is *asymptotically half-Schwarzschild (ahS)* if a neighborhood of infinity is diffeomorphic to the complement of a hemisphere in \(\mathbb{R}_+^3\) so that

\[
g = \left( 1 + \frac{2m}{r} \right) \delta^+ + p^+, \quad p^+ = O_3(r^{-2}).
\]

As usual, if we assume further that each \(x_a R_g\) and \(x_a H_\Sigma\) are integrable then the RT\(^+\) conditions are satisfied and the center of mass \(C^+\) is well defined for ahS manifolds.

Theorem 8 suggests that for an ahS manifold with \(m > 0\), large coordinate hemispheres may be perturbed to yield global solutions of the corresponding relative isoperimetric problem, where each competing surface \(S\) satisfies \(\partial S \subset \Sigma\) and \(\text{int } S \cap \Sigma = \emptyset\), with the relevant constrained volume being the one enclosed by \(S\) and \(\Sigma\). As a first step towards this goal we establish here the following result:
Theorem 9 Assume that \((M, g, \Sigma)\) is an ahS 3-manifold with a non-compact boundary \(\Sigma\) and satisfying the RT^+ conditions. If \(m = m/2 > 0\) then there exists a neighborhood of infinity which is foliated by strictly stable free boundary CMC hemispheres. Moreover, the geometric center of this foliation coincides with the Hamiltonian center of mass \(C^+\) of \((M, g, \Sigma)\).

The proof of this result is presented in Sect. 4. Again, the stability statement above should be interpreted in the sense of Appendix B.

Our next result solves the relative isoperimetric problem referred to above by extending a celebrated result due to Eichmair–Metzger [11] to our setting. To state it, recall that the relative isoperimetric profile \(I_g : [0, +\infty) \to [0, +\infty)\) of \((M, g, \Sigma)\) is given by

\[
I_g(V) = \inf_{\Omega} \mathcal{P}(\partial^*\Omega, M^o),
\]

where \(M^o = M \setminus \Sigma\) is the interior of \(M\) and \(\partial^*\Omega\) is the relative reduced boundary of a Borel set \(\Omega \subset M\) satisfying:

(i) \(\text{vol}(\Omega) = V\); (ii) the relative perimeter \(\mathcal{P}(\partial^*\Omega, M^o)\) is finite.

We stress that in the computation of \(\mathcal{P}(\partial^*\Omega, M^o)\) only that part of the boundary area inside \(M^o\) counts, hence the qualification “relative”. A solution to the relative isoperimetric problem is a Borel subset \(\Omega\) such that \(\text{vol}(\Omega) = V\) and \(\mathcal{P}(\partial^*\Omega, M^o) = I_g(V)\) for some \(V > 0\). We then say that \(\Omega\) is a (relative) isoperimetric region and its boundary \(\partial\Omega \cap M^o\) is a (relative) isoperimetric surface.

Theorem 10 Let \((M, g, \Sigma)\) be as in Theorem 9. Then there exists \(V_0 > 0\) such that for any \(V \geq V_0\) a bounded isoperimetric region with volume \(V\) exists. That region has a connected and smooth relative boundary which remains close to a centered coordinate hemisphere. Moreover, it sweeps out the whole manifold as \(V \to +\infty\). In particular, the corresponding isoperimetric surfaces coincide with the leafs of the foliation in Theorem 9, thus being unique for each value of the enclosed volume.

This result, whose proof is presented in Sect. 5, provides a very precise description of the relative isoperimetric profile of \((M, g, \Sigma)\) as above for all sufficiently large volume values.

3 The foliations in the boundaryless case: the proof of Theorem 6

The purpose of this section is to explain how the well-known implicit function method presented in [6, 8] may be adapted to prove Theorem 6. We consider, in an aSRT manifold as in that theorem, the coordinate sphere \(S^2_\rho(a)\) of radius \(\rho > 0\) centered at some \(a \in \mathbb{R}^3\) to be chosen later. A justifiable assumption here is that \(a\) varies in a bounded region and we will take it for granted. In particular, we will sometimes omit the dependence on \(a\) in the estimates below.

The following two results play a central role in our argument.

Proposition 11 [8, Equation (5.1)] In an aS manifold, the mean curvature of \(S^2_\rho(a)\) at a point \(x\) is

\[
H_{\rho,a}(x) = \frac{2}{\rho} - \frac{4m}{\rho^2} + \frac{9m^2}{\rho^3} + \frac{6m(x - a) \cdot a}{\rho^4} + G_{\rho,a}(x) + O(\rho^{-4}), \quad (30)
\]

where

\[
G_{\rho,a}(x) = \frac{1}{2} p_{ij,k}(x) r_i r_j r_k + 2 \frac{p_{ij}(x)}{\rho} r_i r_j - p_{ij,i}(x) r_j
\]
\[-\frac{p_{ii}(x)}{\rho} + \frac{1}{2} p_{ij}(x) \tau_j, \quad (31)\]

with \( p = g - \left( 1 + \frac{2m}{r} \right) \delta \) and \( \tau = (x-a)/\rho \).

**Proposition 12** [8, Lemma 5.1] The center of mass \( C \) of an aSRT manifold satisfies

\[
\int_{S^2_\rho(a)} (x_i - a_i) G_{\rho,a}(x) dS^2_\rho(a) = -8\pi m C_i + O(\rho^{-1}), \quad i = 1, 2, 3, \quad (32)
\]

where \( dS^2_\rho(a) \) is the area element induced by the flat metric.

Let \((M, g)\) be an aSRT 3-manifold as in Theorem 6 and assume that it is also \( \epsilon \)-aS with \( \epsilon \geq 0 \) and some fixed \( c \in \mathbb{R}^3 \) (recall that \( c \) is irrelevant for the case \( \epsilon = 0 \)). The key observation now is that we can express the Gauss–Kronecker curvature \( K_{\rho,a} \) of \( S^2_\rho(a) \) in terms of the mean curvature \( H_{\rho,a} \) up to terms decaying fast enough. This will allow us to make use of Propositions 11 and 12.

**Proposition 13** There holds

\[
2\rho K_{\rho,a} = H_{\rho,a} \left( 1 - \frac{2m}{\rho} + \frac{9m^2 - 3\gamma_2}{2\rho^2} + \frac{3m}{\rho^3} x \cdot a - \frac{3\gamma_1}{2\rho^3} x \cdot c + O(\rho^{-2-\epsilon}) \right) + O(\rho^{-5}). \quad (33)
\]

The proof of Proposition 13, which relies on the fact that the Schwarzschild space carries a radial conformal vector field, is deferred to Appendix C.

**Corollary 14** There holds

\[
2\rho \tilde{K}_{\rho,a} = \frac{2}{\rho} - \frac{6m}{\rho^2} + \frac{26m^2 - 3c\gamma_2}{\rho^3} + \frac{12m}{\rho^4} x \cdot a - \frac{3\gamma_1}{\rho^4} x \cdot c + G_{\rho,a}(x) + O(\rho^{-3-\epsilon}). \quad (34)
\]

**Proof** Combine (33), (30) and the fact that

\[
\text{Ric}_g(v, v) = -\frac{2m}{\rho^3} + O(\rho^{-4-\epsilon}). \quad (35)
\]

To proceed we consider for a function \( \phi \in C^2(a)(S^2_\rho(a)) \) the corresponding normal graphical surface over \( S^2_\rho(a) \):

\[
S^2_\rho(a, \phi) = \left\{ x + \rho^{-\theta} \phi(x) v(x); x \in S^2_\rho(a) \right\}, \quad \theta \in (0, 1). \quad (36)
\]

By the Taylor’s formula, the modified Gauss–Kronecker curvature \( \tilde{K}_{\rho,a}(a, \rho^{-\theta} \phi) \) of \( S^2_\rho(a, \phi) \) expands as

\[
\tilde{K}_{\rho,a}(a, \rho^{-\theta} \phi) = \tilde{K}_{\rho,a}(0, 0) + d\tilde{K}_{\rho,a}(a, 0)(\rho^{-\theta} \phi) + \int_0^1 (1-s)d^2\tilde{K}_{\rho,a}(a, s\rho^{-\theta} \phi)(\rho^{-\theta} \phi, \rho^{-\theta} \phi)ds. \quad (37)
\]

We now observe that \( \tilde{K}_{\rho,a}(0, 0) = \bar{K}_{\rho,a} \) and \( d\tilde{K}_{\rho,a}(a, 0) = L_{S^2_\rho(a)} \) is the Jacobi operator appearing in (B14), whose asymptotic behavior we need to determine. With this goal in mind
we introduce local coordinates \( \{ y_1, y_2 \} \) on \( S^2_\rho(a) \) and let \( \delta_\rho \) be the induced Euclidean metric, so that \( h = g|S^2_\rho(a) \) is given by

\[
h = \left( 1 + \frac{2m}{\rho} \right) \delta_\rho + O(\rho^{-2}).
\]

**Proposition 15** One has

\[
L_{S^2_\rho(a)} = -\frac{1}{\rho} \left( \Delta_\rho + \frac{2}{\rho^2} \right) + O(\rho^{-4}),
\]

where \( \Delta_\rho \) is the Laplacian with respect to \( \delta_\rho \).

**Proof** With the notation of Appendix B we have

\[
\Lambda_{S^2_\rho(a)} = \frac{1}{\sqrt{\det h}} \partial_B \left( \sqrt{\det h} h^{AC} \Pi^B_A \partial_C \right), \quad \partial_A = \partial/\partial y_A.
\]

Since

\[
\sqrt{\det h} = \left( 1 + \frac{2m}{\rho} \right) \sqrt{\det \delta_\rho} + O(\rho^{-2}), \quad h^{AC} = \left( 1 + \frac{2m}{\rho} \right)^{-1} \delta^{AC} + O(\rho^{-2}),
\]

and

\[
\Pi^B_A = \rho^{-1} \delta^B_A + O(\rho^{-2}), \quad (38)
\]

we compute that

\[
\Lambda_{S^2_\rho(a)} = \frac{1}{\sqrt{\det h}} \partial_B \left( \rho^{-1} \sqrt{\det \delta_\rho} \delta^BC \partial_C + O(\rho^{-1}) \right)
\]

\[
= \frac{1}{\rho} \Delta_\rho + O(\rho^{-4}).
\]

On the other hand, from (30) and (34),

\[
H_{\rho,a} K_{\rho,a} = \frac{2}{\rho^3} + O(\rho^{-4}).
\]

Also, from (35), (38) and the fact that

\[
\text{Riem}^\nu_g = \frac{1}{2} \text{Ric}_g(v, v)h + O(\rho^{-4}), \quad (39)
\]

we see that

\[
\frac{1}{2}(\nabla_v \text{Ric}_g)(v, v) - \text{tr}_h(\Pi\text{Riem}^\nu_g) = O(\rho^{-4}). \quad (40)
\]

The result follows. \( \square \)

We now have at our disposal the ingredients needed to prove the existence of a foliation satisfying (19) in Theorem 6. Indeed, from (34) and (37) we see that finding \( \phi \) so that

\[
\tilde{K}_\rho(a, \rho^{-\theta} \phi) = \frac{1}{\rho^2} - \frac{3m}{\rho^3} \quad (41)
\]
is equivalent to solving
\[ 2\rho^{-\theta} \Delta_{(\rho)} \phi(x) = \frac{26m^2 - 3\gamma_2}{\rho^4} + \frac{12m}{\rho^4} x \cdot a - \frac{3\gamma_1}{\rho^4} x \cdot c + G_{\rho,a}(x) + E_{\rho,\phi}(x), \] (42)
where \( \Delta_{(\rho)} = \Delta_\rho + 2\rho^{-2} \) and the remainder term \( E_{\rho,\phi} \) is controlled as
\[ E_{\rho,\phi}(x) = O \left( \rho^{-3-\theta} |\phi| + \rho^{-3-2\theta} |\phi|^2 + \rho^{-1-2\theta} |\phi|^3 |\delta^2 \phi| \right) + O(\rho^{-3-\epsilon}). \] (43)

We next pull back this equation under the map \( F : S^2_1(\tilde{0}) \to S^2_\rho(a) \), \( F(\tau) = a + \rho \tau \), so as to obtain an equation for \( \psi = F^* \phi \) on \( S^2_1(\tilde{0}) \):
\[ 2\Delta_{(1)} \psi(\tau) = \mathcal{F}(\tau, a, \psi) \] (44)
where \( \Delta_{(1)} = \Delta_1 + 2 \) and
\[ \mathcal{F}(\tau, a, \psi) = \frac{26m^2 - 3\gamma_2}{\rho^{1-\theta}} + \frac{12m}{\rho^{1-\theta}} \tau \cdot a - \frac{3\gamma_1}{\rho^{1-\theta}} \tau \cdot c + \rho^{2+\theta} F^* G_{\rho,a}(\tau) \]
\[ + \rho^{2+\theta} F^* E_{\rho,\phi}(\tau) + 12m \rho^{-2+\theta} |a|^2 - 3\gamma_1 \rho^{-2+\theta} a \cdot c. \]

We note that the primary obstruction to solving (44) is the fact that the operator
\[ \Delta_{(1)} : C^{2,\alpha}(S^2_1(\tilde{0})) \to C^\alpha(S^2_1(\tilde{0})) \]
has a nontrivial cokernel generated by the functions \( \tau_i, i = 1, 2, 3 \). Thus, in order to remove this obstruction, we shall calculate
\[ \int_{S^2_1(\tilde{0})} \tau_i \mathcal{F}(\tau, a, \psi) \, dS^2_1(\tilde{0}). \]
Note that by symmetry
\[ \int_{S^1(\tilde{0})} \tau_i \left( \frac{26m^2 - 3\gamma_2}{\rho^{1-\theta}} + 12m \rho^{-2+\theta} |a|^2 - 3\gamma_1 \rho^{-2+\theta} a \cdot c \right) \, dS^2_1(\tilde{0}) = 0. \]
Using (32) we easily see that
\[ \int_{S^2_1(\tilde{0})} \tau_i \left( \frac{12m}{\rho^{1-\theta}} \tau \cdot a - \frac{3\gamma_1}{\rho^{1-\theta}} \tau \cdot c + \rho^{2+\theta} F^* G_{\rho,a}(\tau) \right) \, dS^2_1(\tilde{0}) \]
\[ = \frac{16\pi m}{\rho^{1-\theta}} \left( a_i - \frac{\gamma_1}{4m} c_i - \frac{C_i}{2} \right) + O(\rho^{\theta-2}) \]
\[ = \frac{16\pi m}{\rho^{1-\theta}} \left( a_i - \frac{\gamma_1}{2m} c_i \right) + O(\rho^{\theta-2}), \]
where in the last step we used that \( C = \frac{\gamma_1}{24m} c \), according to Remark 4. The remaining integral
\[ \int_{S^1(\tilde{0})} \tau_i \rho^{2+\theta} F^* E_{\rho,\phi}(\tau) \, dS^2_1(\tilde{0}) \]
may be estimated so as to yield
\[ \int_{S^1(\tilde{0})} \tau_i \mathcal{F}(\tau, a, \psi) dS^2_1(\tilde{0}) = \frac{16\pi m}{\rho^{1-\theta}} \left( a_i - \frac{\gamma_1}{2m} c_i \right) + \frac{1}{\rho^{1-\theta}} \widehat{E}_i, \] (45)
where
\[ \widehat{E}_i = O((\rho^{-1} + \rho^{-\theta}) ||\psi||_{C^2}) + O(\rho^{-\epsilon}). \] (46)
Thus, for each $\rho$ large enough we may choose $a_\rho \in \mathbb{R}^3$ such that
\[
(a_\rho)_i = \frac{\gamma_i}{2m} c_i - \frac{1}{16\pi m} \hat{E}_i,
\] (47)
so as to have
\[
\int_{S_1^2(0)} \tau_i \tilde{\mathcal{F}}(\tau, a_\rho, \psi) \, dS_1^{2,\delta}(\tilde{0}) = 0, \quad i = 1, 2, 3,
\] (48)
for any $\psi$ with $\|\psi\|_{C^2}$ bounded.

With the obstruction so removed we may now use a standard fixed point argument to check that (42) has a unique solution $\phi_\rho$ for all such $\rho$. More precisely, from (48) we see that $\tilde{\mathcal{F}}(\tau, a_\rho, \psi)$ lies in $\text{Ran} \Delta(1)$ if $\|\psi\|_{C^2} \leq 1$. Therefore, we may uniquely solve
\[
2\Delta(1) \tilde{\psi} = \tilde{\mathcal{F}}(\tau, a_\rho, \psi),
\] (49)
for $\tilde{\psi} \in C^{2,\alpha}(S_1^2(0)) \cap (\ker \Delta(1))^\perp$ satisfying
\[
\|\tilde{\psi}\|_{C^{2,\alpha}} \leq C \|\tilde{\mathcal{F}}(\tau, a_\rho, \psi)\|_{C^{0,\alpha}} \leq C' \rho^{\max\{\theta, \theta-1\}} = C' \rho^{\max\{\theta, \theta-1\}},
\] (50)
and this is $\leq 1$ if $\rho$ is large enough. Thus, the map $\psi \mapsto \tilde{\psi}$ has a fixed point which yields a solution $\psi_\rho$ of (44) and hence a solution $\phi_\rho$ of (42). In particular, the graphical surface associated to $\phi_\rho$, denoted simply $S^2_\rho(\phi_\rho)$, has constant modified Gauss–Kronecker curvature given by the right-hand side of (41). Moreover, if we choose $\theta = 1/2$ in (50) then this analysis guarantees that $\phi_\rho \in C^{2,\alpha}(S^2_\rho(a_\rho))$ satisfies
\[
\sum_{|I|\leq 2} \rho^{|I|} |\partial_I \phi_\rho| + \sum_{|I|=2} \rho^{2+\alpha} [\partial_I \phi_\rho]_a \leq C'' \rho^{1/2},
\] (51)
so the corresponding graphical surface $S^2_\rho(\phi_\rho)$, which actually involves the function $\rho^{-1/2} \phi_\rho$, remains at a bounded distance of $S^2_\rho(a_\rho)$ while becoming rounder as $\rho \to +\infty$. The geometric center of mass of the foliation is given by
\[
C_\mathcal{F} = \lim_{\rho \to +\infty} \frac{\int_{S^2_\rho(\phi_\rho)} z \, dS^2_\rho(\phi_\rho)}{\int_{S^2_\rho(\phi_\rho)} dS^2_\rho(\phi_\rho)},
\]
where $z$ is the position vector. We easily see from (46), (47) and (51) that either $C_\mathcal{F}$ remains at a finite distance from $C$ if $\epsilon = 0$ or there holds $C_\mathcal{F} = C$ if $\epsilon > 0$ (recall that $C = \frac{\gamma_1}{2m} c$).

Also, each such surface may be viewed as a graph over $S^2_\rho(c)$. For simplicity of notation, we still denote such a surface by $S^2_\rho(\phi_\rho)$. Finally, for further use we note that by (51) we may determine the asymptotic expansions of the geometric invariants of $S^2_\rho(\phi_\rho)$.

The result is
\[
\begin{align*}
K &= \rho^{-2} - 4m \rho^{-3} + O(\rho^{-4}), \\
H &= 2\rho^{-1} - 4m \rho^{-2} + O(\rho^{-3}), \\
|W|^2 &= 2\rho^{-2} - 8m \rho^{-3} + O(\rho^{-4}), \\
\text{Ric}(v, v) &= -2m \rho^{-3} + O(\rho^{-4}), \\
K_G &= \rho^{-2} - 2m \rho^{-3} + O(\rho^{-4}).
\end{align*}
\] (52)

Here, $K_G$ is the Gaussian curvature.

It remains to check that for $\rho_0$ large enough, the family of surfaces $S^2_\rho(\phi_\rho)$, $\rho \geq \rho_0$, defines a foliation whose leaves are strictly stable in the appropriate sense. We first tackle the stability issue.
Theorem 16 If \( m > 0 \) then \( S_{\rho}^2(\phi_{\rho}) \) is strictly stable for all \( \rho \) large enough.

Proof According to Proposition 25, we must estimate from below the quadratic form

\[
V(f) = \int_S \left( \langle \Pi \nabla f, \nabla f \rangle - f^2 \left( HK + \frac{1}{2} \langle \nabla_v \text{Ric}_g(v, v) + \text{tr}_S(\Pi \text{Riem}_g^v) \rangle \right) \right) dS,
\]

where \( f \in \mathcal{G}(S) \) and here we set \( S = S_{\rho}^2(\phi_{\rho}) \) for simplicity. From (52) we have

\[
-HK = -\frac{2}{\rho^3} + \frac{12m}{\rho^4} + O(\rho^{-5}).
\]

Also, from (35), (38) and (39), we may improve (40) to

\[
-\frac{1}{2} \langle \nabla_v \text{Ric}_g(v, v) - \text{tr}_S(\Pi \text{Riem}_g^v) \rangle = -\frac{m}{\rho^4} + O(\rho^{-5}).
\]

Thus,

\[
V(f) \geq \int_S \langle \Pi \nabla f, \nabla f \rangle dS + \left( -\frac{2}{\rho^3} + \frac{11m}{\rho^4} + O(\rho^{-5}) \right) \int_S f^2 dS. 
\] (53)

We now observe that the Newton tensor of \( S = S_{\rho}^2(\phi_{\rho}) \) satisfies

\[
\Pi = \left( \frac{1}{\rho} - \frac{2m}{\rho^2} \right) I + O(\rho^{-3}).
\] (54)

On the other hand, by the well-known Lichnerowicz eigenvalue bound,

\[
\int_S |\nabla_S f|^2 dS \geq 2 \inf K_G \int_S f^2 dS, \quad f \in \mathcal{G}(S).
\]

where by (52),

\[
\inf K_G \geq \frac{1}{\rho^2} - \frac{2m}{\rho^3} - C \rho^{-4}, \quad C > 0.
\]

We thus conclude that

\[
V(f) \geq \left( \frac{3m}{\rho^4} - C \rho^{-5} \right) \int_S f^2 dS,
\]

and the result follows.

\[\Box\]

We now check that the surfaces define a foliation. Since the argument, as explained for instance in [20], is well-known by now and it may be easily adapted to our setting, here we merely sketch the proof.

Proposition 17 Let \( \xi_0 < \xi_1 \) be the first two (unconstrained) eigenvalues of \( L_{S_{\rho}^2(\phi_{\rho})} \). Then

\[
\xi_0 = -\frac{2}{\rho^3} + \frac{9m}{\rho^4} + O(\rho^{-5}), \quad \xi_1 \geq \frac{m}{\rho^4} - C \rho^{-5}.
\]

In particular, \( L_{S_{\rho}^2(\phi_{\rho})} : C^{2,\alpha}(S_{\rho}^2(\phi_{\rho})) \to C^\alpha(S_{\rho}^2(\phi_{\rho})) \) is invertible for all \( \rho \) large enough.

The proof of this statement is basically a refinement of the stability analysis above. In any case, for any such fixed \( \rho_0 \) large enough, it implies the existence of a unique \( f_{\rho_0} \in C^{2,\alpha}(S_{\rho_0}^2(\phi_{\rho_0})) \) such that \( L_{S_{\rho_0}^2(\phi_{\rho_0})} f_{\rho_0} = 1 \).
Proposition 18 The function $f_{\rho_0}$ vanishes nowhere on $S^2_{\rho_0}(\phi_{\rho_0})$.

Proof This follows immediately from the estimate
\[
\sup_{S^2_{\rho_0}(\phi_{\rho_0})} |f_{\rho_0} - \overline{f_{\rho_0}}| \leq C \rho_0^{-1} |\overline{f_{\rho_0}}|,
\]
where $C > 0$ is a constant depending only on $g$ and the overline stands for the average over $S^2_{\rho_0}(\phi_{\rho_0})$. The method of proof, which is explained in [20, Section 5.2], makes use of Nash–Moser iteration and equally applies here due to the fact that $L_{S^2_{\rho_0}(\phi_{\rho_0})}$ is elliptic of divergence type; see Appendix B. \qed

Now, we may organize the graphical surfaces $S^2_{\rho_0}(\phi_{\rho})$ with $\rho$ close to $\rho_0$ in a smooth deformation
\[
F : (\widetilde{K}_{\rho_0} - \varepsilon, \widetilde{K}_{\rho_0} + \varepsilon) \times S^2_{\rho_0}(\phi_{\rho_0}) \to M, \quad \varepsilon > 0,
\]
where $\widetilde{K}_{\rho_0}$ is the modified Gauss–Kronecker curvature of $S^2_{\rho_0}(\phi_{\rho_0})$ and $F(\widetilde{K}, \cdot)$ has constant modified Gauss–Kronecker curvature equal to $\widetilde{K}$. Clearly,
\[
F(\widetilde{K}, x) = \exp_{S^2_{\rho_0}(\phi_{\rho_0})}(\widetilde{f}_{\widetilde{K}} \nu),
\]
for some function $\widetilde{f}_{\rho_0}$ on $S^2_{\rho_0}(\phi_{\rho_0})$ with $\widetilde{f}_{\rho_0} \equiv 0$. Let
\[
\widetilde{f}_0 := -\frac{\partial \widetilde{f}_{\rho_0}}{\partial \widetilde{K}}|_{\widetilde{K}_{\rho_0}} = \left\langle \frac{\partial F}{\partial \widetilde{K}}|_{\widetilde{K}_{\rho_0}}, \nu \right\rangle.
\]
By (B13),
\[
L_{S^2_{\rho_0}(\phi_{\rho_0})} \widetilde{f}_0 = \frac{d}{dk}(\widetilde{K}_{\rho_0} + k)|_{k=0} = 1,
\]
so that $\widetilde{f}_0 = f_{\rho_0}$ by uniqueness. In particular, $\widetilde{f}_0$ never vanishes and we may use the inverse function theorem to conclude that $F$ is a diffeomorphism onto a small neighborhood of $S^2_{\rho_0}(\phi_{\rho_0})$ in $M$. This shows that the surfaces define a foliation and completes the proof of the first part of Theorem 6.

We now sketch the proof of the existence of a stable foliation satisfying the curvature condition in (20), whose leafs correspond to surfaces extremizing the total mean curvature under an area constraint by Appendix B. The Lagrange multiplier $\gamma$ is determined by observing that (33) leads to
\[
\widetilde{K}_{\rho,a} = \frac{1}{2} - \frac{m}{2\rho^2} + O(\rho^{-3}).
\]
With this choice of $\gamma = O(\rho^{-1}) > 0$, it then follows from (54), (B13) and (B9) that the quadratic form associated to the linearization of (20) at $S = S_\rho(\alpha)$ has
\[
\int_S (\nabla S f, \nabla S f) dS = \left(\frac{1}{2\rho^2} - \frac{3m}{2\rho^2} + O(\rho^{-3})\right) \int_S |\nabla S f|^2 dS
\]
as its principal part. Hence, the linearization is self-adjoint and elliptic. In fact, a computation shows that the rescaled linearization on $S^2_{\rho_0}(\phi_{\rho_0})$ is $2(1 - \gamma)\Delta_{(1)}$; compare with the left-hand side of (44). Thus, we are in a position to run the implicit function method above in order to construct a graphical surface over $S_\rho(\alpha)$ satisfying (20). As in the preceding case, this step only uses that $m \neq 0$. If $m > 0$ then a further analysis shows that these graphical
surfaces comprise a foliation of a neighborhood of infinity whose leafs are strictly stable in the appropriate sense. Moreover, the geometric center of this foliation remains at a finite distance from the Hamiltonian center of mass. In this way, the proof of Theorem 6 is completed.

4 The foliation by free boundary CMC hemispheres: the proof of Theorem 9

In this section we present the proof of Theorem 9. In any ahS manifold as in that theorem we consider the coordinate hemisphere $S_{\rho,+}^2(b)$ centered at some $b \in \mathbb{R}^2$ and use the notation of Appendix B. As in Proposition 11, we compute the mean curvature $H_{\rho,+}^b$ of this hemisphere to obtain

$$H_{\rho,+}^b(x) = \frac{2}{\rho} - \frac{4m}{\rho^2} + \frac{9m^2}{\rho^3} + \frac{6m(x - b) \cdot b}{\rho^4} + G_{\rho,b}(x) + O(\rho^{-4}),$$

where $G_{\rho,b}(x)$ is as in (31) with $\tau = (x - b)/\rho$. We also need the analogue of Proposition 12, which goes as follows.

**Proposition 19** If $(M, g, \Sigma)$ is an ahS manifold meeting the RT$^+$ condition then its center of mass $C^+$ satisfies

$$\int_{S_{\rho,+}^2(b)} (x - b_\alpha) G_{\rho,b}(x) dS_{\rho,+}^{2,\delta^+}(b) = -8\pi m \mathcal{C}_{\alpha}^+ + O(\rho^{-1}), \quad \alpha = 1, 2,$$  

where $dS_{\rho,+}^{2,\delta^+}(b)$ is the area element induced by the flat metric.

The proof of this proposition, which adapts an argument first appearing in [26, Appendix F], is presented in Appendix D.

We now proceed to the proof of Theorem 9 via the standard implicit function method [6, 8]. We consider, for a function $\phi \in C^{2,\alpha}(S_{\rho,+}^2(b))$ satisfying Neumann boundary condition along $S_{\rho}^1 = \partial S_{\rho,+}^2(b)$, the corresponding normal graphical surface over $S_{\rho,+}^2(b)$:

$$S_{\rho,+}^2(b, \phi) = \{ x + \rho^{-\theta} \phi(x) v(x); \ x \in S_{\rho,+}^2(b) \}, \quad \theta \in (0, 1).$$

By the Taylor’s formula, the mean curvature $H_{\rho,+}^b(b, \rho^{-\theta} \phi)$ of $S_{\rho,+}^2(b, \phi)$ expands as

$$H_{\rho,+}(b, \rho^{-\theta} \phi) = H_{\rho,+}(b, 0) + dH_{\rho,+}(b, 0)(\rho^{-\theta} \phi) + \int_0^1 (1 - s)d^2H_{\rho,+}(b, s\rho^{-\theta} \phi)(\rho^{-\theta} \phi, \rho^{-\theta} \phi) ds.$$

We now observe that $H_{\rho,+}(b, 0) = H_{\rho,+}^b$ and $dH_{\rho,+}(a, 0) = \mathcal{L}_{S_{\rho,+}^2(b)}$ is the Jacobi operator appearing in (B7). Thus, finding $\phi$ so that

$$H_{\rho,+}(b, \rho^{-\theta} \phi) = \frac{2}{\rho} - \frac{4m}{\rho^2},$$

is equivalent to solving

$$\rho^{-\theta} \Delta(\phi) = \frac{9m^2}{\rho^3} + \frac{6m(x - b) \cdot b}{\rho^4} + G_{\rho,b}(x) + E_{\rho,\phi}(x),$$

where as usual $\Delta(\phi) = \Delta + 2\rho^{-2}$ (recall that $\Delta$ is the Laplacian with respect to $\delta^+_\rho$, the induced round metric on $S_{\rho,+}^2$) and the remainder $E_{\rho,\phi}$ has the same bound as in (43).
We next pull back this equation under the map $F : S_{1,+}^2(0) \to S_{\rho,+}^2(b)$, $F(\tau) = b + \rho \tau$, so as to obtain an equation for $\psi = F^* \phi$ on $S_{1,+}^2(0)$:

$$\Delta_{(1)} \psi = \frac{9m^2}{\rho^{1-\theta}} + \frac{6m \tau \cdot b}{\rho^{1-\theta}} + \rho^{2+\theta} F^* G_{\rho,b}(\tau) + \rho^{2+\theta} F^* E_{\rho,\phi+}, \quad (60)$$

where $\Delta_{(1)} = \Delta_1 + 2$.

We now recall that the operator $\Delta_{(1)} : C^2(S_{1,+}^2(\tilde{0})) \to C^2(S_{1,+}^2(\tilde{0}))$, where the star means that we impose the Neumann boundary condition, has a nontrivial cokernel generated by the functions $r_\alpha$, $\alpha = 1, 2$. Clearly, this poses an obstruction to solving (60). However, using (56) we easily calculate that

$$\int_{S_{1,+}^2(\tilde{0})} \tau_\alpha \left( \frac{9m^2}{\rho^{1-\theta}} + \frac{6m \tau \cdot b}{\rho^{1-\theta}} + \rho^{2+\theta} F^* G_{\rho,b}(\tau) \right) dS_{1,+}^{2,\delta^+} = \frac{8\pi m}{\rho^{1-\theta}} (b_\alpha - C_\alpha^+) + O(\rho^{-1} \|\psi\|_{C^2}),$$

so we end up with

$$\int_{S_{1,+}^2(\tilde{0})} \tau_\alpha G(\tau, b, \psi) dS_{1,+}^{2,\delta^+} = \frac{8\pi m}{\rho^{1-\theta}} (b_\alpha - C_\alpha^+) + \frac{1}{\rho^{1-\theta}} \hat{E}_\alpha,$$

where $G(\tau, b, \psi)$ is the right-hand side of (60) and

$$\hat{E}_\alpha = O((\rho^{-1} + \rho^{-\theta}) \|\psi\|_{C^2}) \quad (61)$$

Thus, for each $\rho$ large enough we may choose $b_\rho$ such that

$$(b_\rho)_\alpha = C_\alpha^+ - \frac{1}{8\pi m} \hat{E}_\alpha, \quad (62)$$

so as to have

$$\int_{S_{1,+}^2(\tilde{0})} \tau_\alpha G(\tau, b_\rho, \psi) dS_{1,+}^{2,\delta^+} = 0, \quad \alpha = 1, 2, \quad (63)$$

for any $\psi$ with $\|\psi\|_{C^2}$ bounded. This eliminates the obstruction mentioned earlier.

As in the proof of Theorem 6 above, we may now use the standard fixed point argument to check that (59) has a unique solution $\phi_\rho$ for all such $\rho$. In particular, the graphical surface corresponding to $\phi_\rho$ as in (57), denoted $S_{\rho,+}(\phi_\rho)$, has constant mean curvature given by the right-hand side of (58). Also, the Neumann condition imposed on $\phi_\rho$ implies that this graphical surface is free boundary. Moreover, an estimate similar to (51) also holds true here, so $S_{\rho,+}(\phi_\rho)$ remains at a bounded distance of $S_{\rho,+}(b_\rho)$ while becoming rounder as $\rho \to +\infty$.

The geometric center of mass of this family of surfaces is given by

$$C_H^+ = \lim_{\rho \to +\infty} \frac{\int_{S_{\rho,+}^2(\phi_\rho)} z dS_{\rho,+}^{2,\delta^+}(\phi_\rho)}{\int_{S_{\rho,+}^2(\phi_\rho)} dS_{\rho,+}^{2,\delta^+}(\phi_\rho)},$$

where $z$ is the position vector. Clearly, $C_H^+ = C^+$. Also, each such surface may be viewed as a graph over $S_{\rho}^2(S^+)$, so the stability issue.

$$\hat{\text{Springer}}$$
Theorem 20  If \( m = m/2 > 0 \) then \( S_{\rho, +}(\phi_\rho) \) is strictly stable for all \( \rho \) large enough.

For the proof we first note that the geometric invariants of \( S_{\rho, +}(\phi_\rho) \) expand as

\[
\begin{align*}
|W|^2 &= 2\rho^{-2} - 8m\rho^{-3} + O(\rho^{-4}), \\
\text{Ric}(v, v) &= -2m\rho^{-3} + O(\rho^{-4}), \\
K_G &= \rho^{-2} - 2m\rho^{-3} + O(\rho^{-4}).
\end{align*}
\]

(64)

We also need the asymptotic expansion of the second fundamental of \( \Sigma \), the non-compact boundary of \( M \).

Lemma 21  The second fundamental form \( \mathcal{B} \) of \( \Sigma \) satisfies

\[
\mathcal{B}_{\alpha\beta} = O(\rho^{-3}), \quad \alpha, \beta = 1, 2.
\]

(65)

Proof  Recall that

\[
g = \left( 1 + \frac{2m}{r} \right) \delta^+ + p^+, \quad p^+ = O(\rho^{-2}).
\]

Outside a compact subset of \( M \), \( \Sigma \) is defined by \( x_3 = 0 \) and its tangent space is generated by \( \{\partial_1, \partial_2\} \). If \( \eta = \eta^i \partial_i \) is the inward unit normal along \( \Sigma \) then

\[
\mathcal{B}_{\alpha\beta} = \langle \eta, \nabla_{\partial\alpha} \partial_\beta \rangle = \Gamma_{\alpha\beta}^i \langle \eta, \partial_i \rangle = \Gamma_{\alpha\beta}^3 g_{3i} \eta^i.
\]

Since

\[
\Gamma_{\alpha\beta}^3 = \frac{1}{2} g^{33} (g_{\alpha3,\beta} + g_{\beta3,\alpha} - g_{\alpha\beta,3}) + O(\rho^{-4})
\]

\[
= -\frac{1}{2} \left( 1 + \frac{2m}{r} \right)^{-1} g_{\alpha\beta,3} + O(\rho^{-3}),
\]

and

\[
g_{\alpha\beta,3} = -2mr^{-2} \frac{\partial r}{\partial x_3} \delta_{\alpha\beta} + O(\rho^{-3})
\]

\[
= -2mr^{-3} x_3 \delta_{\alpha\beta} + O(\rho^{-3})
\]

\[
= O(\rho^{-3}),
\]

the result follows. \( \Box \)

By Proposition 23, the proof of Theorem 20 involves estimating from below the quadratic form

\[
Q(f) = \int_S (|\nabla_S f|^2 - (|W|^2 + \text{Ric}(v, v) f)^2) dS - \int_{\partial S} \kappa f^2 d\partial S, \quad f \in \mathcal{F}(S),
\]

where here we set \( S = S_{\rho, +}(\phi_\rho) \) for simplicity. We may assume that \( \int_S f^2 dS = 1 \), which implies \( f = O(\rho^{-1}) \). Hence, using (64),

\[
Q(f) = \int_S |\nabla_S f|^2 dS - \int_{\partial S} \kappa f^2 d\partial S - \frac{2}{\rho^2} + \frac{10m}{\rho^3} + O(\rho^{-4}).
\]

(66)

Thus, we are left with the task of estimating from below the quadratic form

\[
\hat{Q}(f) = \int_S |\nabla_S f|^2 dS - \int_{\partial S} \kappa f^2 d\partial S, \quad f \in \mathcal{F}(S).
\]
which is equivalent to estimating from below the first eigenvalue \( \hat{\lambda} \) of the eigenvalue problem

\[
\begin{cases}
-\Delta_S f = \lambda f & \text{in } S \\
\frac{\partial f}{\partial \mu} = \kappa f & \text{on } \partial S
\end{cases}
\]  

where \( f \in F(S) \). Notice that a comparison with the first eigenvalue \( 2/\rho^2 \) of the Neumann (\( \kappa = 0 \)) eigenvalue problem on \((S_{\rho,+}(a), \delta_+^\rho) \leftrightarrow (R^3_+, \delta^+)\) already shows that \( \hat{\lambda} > 0 \) and provides the preliminary but useful estimate \( \hat{\lambda} = O(\rho^{-2}) \).

If \( f \) is an eigenfunction of (67) with eigenvalue \( \lambda = \hat{\lambda} \), we see that

\[
\int_S |\nabla_S f|^2 dS = \int_{\partial S} \kappa f^2 d\partial S + \hat{\lambda}.
\]  

From (65), \( \kappa = B_{\alpha\beta} v^\alpha v^\beta = O(\rho^{-3}) \), so that \( \int_{\partial S} \kappa f^2 d\partial S = O(\rho^{-4}) \) and \( \partial f/\partial \mu = O(\rho^{-4}) \), where (67) was used in the latter step. Thus, \( \int_S |\nabla_S f|^2 dS = O(\rho^{-2}) \), so that \( \nabla_S f = O(\rho^{-2}) \) and, moreover, from (67) we get \( \Delta_S f = O(\rho^{-3}) \).

We now apply a well-known integral identity due to Reilly [47]. In our setting (dim \( S = 2 \)) it simplifies to

\[
\int_S \left( (\Delta_S f)^2 - |\nabla_S^2 f|^2 \right) dS = 2 \int_{\partial S} \frac{\partial f}{\partial \mu} \Delta_{\partial S} f d\partial S + \int_S H_{\partial S} \left( \frac{\partial f}{\partial \mu} \right)^2 + |\nabla_{\partial S} f|^2 d\partial S
\]

\[+ \int_S K_G |\nabla_S f|^2 dS, \tag{69}\]

where \( H_{\partial S} \) is the mean (in fact, geodesic) curvature of \( \partial S \) in \( S \). Since \( S \) is free boundary, \( H_{\partial S} = B(T, T) = O(\rho^{-3}) \), where \( T \) is a unit tangent vector along \( \partial S \) and we used (65). Thus, the second integral in the right-hand side equals

\[
\int_{\partial S} H_{\partial S} |\nabla_S f|^2 d\partial S = O(\rho^{-6}).
\]

On the other hand,

\[\Delta_{\partial S} f = \Delta_S f - \frac{\partial^2 f}{\partial \mu^2} = O(\rho^{-3}), \]

so that the first integral in the right-hand side is also \( O(\rho^{-6}) \). Finally, using (64) and (68),

\[
\int_S K_G |\nabla_S f|^2 dS \geq \left( \frac{1}{\rho^2} - \frac{2m}{\rho^3} - C \rho^{-4} \right) \left( \hat{\lambda} + O(\rho^{-4}) \right), \quad C > 0,
\]

and combining this with the fact that \( |\nabla_S^2 f|^2 \geq (\Delta_S f)^2/2 \) and \( \hat{\lambda} = O(\rho^{-2}) \) we get from (69) that

\[\hat{\lambda} \left( \hat{\lambda} - \frac{2}{\rho^2} + \frac{4m}{\rho^3} \right) \geq -C' \rho^{-6}, \quad C' > 0.\]

Since we already know that \( \hat{\lambda}^{-1} = O(\rho^2) \) is positive, this gives

\[\hat{\lambda} \geq \frac{2}{\rho^2} - \frac{4m}{\rho^3} - C'' \rho^{-4}, \quad C'' > 0.\]
Combining this with (66) we finally have
\[ Q(f) \geq \frac{6m}{\rho^3} - C\rho^{-4}, \]
which completes the proof of Theorem 20.

From this point on, the proof that the family of free boundary CMC hemispheres comprises a foliation follows from a simple variation of the standard argument. Indeed, if \( \chi_0 < \chi_1 \) are the first two (unconstrained) eigenvalues of \( L_{S_{\rho,+}(\phi_\rho)} \), a spin-off of the analysis above leads to
\[ \chi_0 = -\frac{2}{\rho^2} + \frac{10m}{\rho^3} + O(\rho^{-4}), \quad \chi_1 \geq \frac{6m}{\rho^3} - C\rho^{-4}. \]
(70)

In other words, \( L_{S_{\rho,+}(\phi_\rho)} : C^{2,\alpha}_\bullet(S_{\rho,+}(\phi_\rho)) \to C^\alpha(S_{\rho,+}(\phi_\rho)) \) is injective, where the bullet indicates that the boundary condition in (67) is imposed. Since it is known that this Jacobi operator is Fredholm of index zero [48, Section 2], we see that it is surjective as well. In particular, there exists \( f_\rho \in C^{2,\alpha}_\bullet(S_{\rho,+}(\phi_\rho)) \) such that \( L_{S_{\rho,+}(\phi_\rho)} f_\rho = 1 \). On the other hand, just like in the discussion after the proof of Proposition 18 above, we may realize \( f_\rho \) as the variational function associated to a deformation of \( S^2_{\rho,+}(\phi_\rho) \) by the graphical free boundary CMC hemispheres, now parameterized by their mean curvature \( H \in (H_\rho - \epsilon, H_\rho + \epsilon) \), \( \epsilon > 0 \).

Since the Nash–Moser scheme may be implemented to make sure that \( f_\rho \) never vanishes, the standard argument using the inverse function theorem shows that this deformation actually provides a diffeomorphism of \( (H_\rho - \epsilon, H_\rho + \epsilon) \times S^2_{\rho,+}(\phi_\rho) \) onto a small neighborhood of \( S^2_{\rho,+}(\phi_\rho) \) in \( M \). This proves the existence of the foliation and completes the proof of Theorem 9.

5 Large relative isoperimetric hemispheres: the proof of Theorem 10

Here we follow [11] closely and present the proof of Theorem 10. We start by briefly reviewing the argument leading to their main result, which is based on three ingredients:

- An effective area comparison result for large volume, off-center regions in Schwarzschild space, which refines Bray’s characterization of isoperimetric regions as being those enclosed by centered spheres (together with the minimal horizon) [49, 50]. This is then transplanted to an effective estimate for large volume, off-center regions in asymptotically Schwarzschild manifolds; see Proposition 3.3 and Theorem 3.4 in [11].
- A precise understanding of the behavior of minimizing sequences of regions attaining the corresponding isoperimetric profile, to the effect that they split as the disjoint union of a (possibly empty) isoperimetric region (for the volume it encloses) that remains at a finite distance of a given point and a coordinate ball of radius \( r \geq 0 \) which slides away toward the asymptotic region. Moreover, if none of these regions degenerate (in particular, \( r > 0 \)) then the boundary of the isoperimetric region left behind has constant mean curvature \( 2/r \); see [11, Proposition 4.2], which relies on [51, Theorem 2.1].
- Existence of a foliation by CMC spheres filling out the asymptotic region as in [5, 6, 8].

The first item above is used to make sure that the boundary of a sufficiently large isoperimetric region remains close to a centered coordinate sphere bounding the same volume in the sense that the scale invariant \( C^2 \)-norm of the function describing such large isoperimetric surface as a normal graph over the centered sphere tends to zero as the enclosed volume goes to infinity. Otherwise, by suitably scaling down the region one is able to check that it
is off-center, hence not isoperimetric by the effective area estimate, a contradiction; see [11, Theorem 5.1]. With this information at hand, one sees from the second item above that for a large enclosed volume the worst case scenario takes place whenever the runaway ball does not degenerate, for in this case the isoperimetric region attaining this volume splits as the disjoint union of two large balls each roughly with the same radius. Since this configuration is far from being isoperimetric, we get a contradiction. Thus, the runaway ball actually disappears and the isoperimetric region starts filling out the whole manifold as the enclosed volume diverges. Moreover, as its boundary remains close to a centered coordinate sphere, it has to coincide with a leaf of the foliation appearing in the third item.

It turns out that all of these ingredients also work fine in our setting. Of course, the existence of the relevant foliation, in our case by free boundary CMC hemispheres, is the content of Theorem 9 above. We now discuss the validity of the remaining ones.

First, it is clear that in half-Schwarzschild space, the region bounded by the minimal horizon \( r = m/2 \) and a coordinate sphere of radius \( r > m/2 \) is the only one attaining the relative isoperimetric profile for the corresponding volume. Otherwise, after reflecting upon the totally geodesic boundary \( x_3 = 0 \) we obtain a region in “boundaryless” Schwarzschild space which is isoperimetric but differs from any of the symmetric regions realizing the corresponding isoperimetric profile as in Bray’s result. Essentially the same argument yields an effective area comparison for off-center regions in ahS manifolds.

**Definition 11** Let \((M, g, \Sigma)\) be as in Theorem 10. Given \((\tau, \eta) \in (1, +\infty) \times (0, 1)\), a bounded Borel set \(\Omega \subset M\) of finite relative perimeter is said to be \((\tau, \eta)\)-off-center if:

1. there exists a large coordinate hemisphere \(S^2_{r, +}, r > 1\), whose enclosed region, say \(M_r\), has the same volume as \(\Omega\);
2. \(\mathcal{H}^2_g(\partial^* \Omega \setminus M_{\tau r}) \geq \eta \mathcal{H}^2_g(S^2_{r, +})\).

Here, \(\mathcal{H}^2_g\) is Hausdorff measure with respect to \(g\).

The next proposition provides the analogue of the first item above to our setting.

**Proposition 22** Let \((M, g, \Sigma)\) be as in Theorem 10. For every \((\tau, \eta) \in (1, +\infty) \times (0, 1)\) there exists \(V_0 > 0\) and \(\Theta > 0\) such that the following holds. Let \(\Omega \subset M\) be a bounded Borel set with finite relative perimeter whose volume is at least \(V_0\) and which is \((\tau, \eta)\)-off-center and further satisfies \(\mathcal{H}^2_g(\partial^* \Omega)^{1/2} \text{vol}_g(\Omega)^{-1/3} \leq \Theta\) and \(\mathcal{H}^2_g(M_{\sigma} \cap \partial^* \Omega) \leq \Theta \sigma^2\) for all \(\sigma \geq 1\). Then,

\[
\mathcal{H}^2_g(S^2_{r, +}) + cr \leq \mathcal{H}^2_g(\partial^* \Omega), \quad c = c(m, \tau, \eta) > 0.
\]

**Proof** First note that an effective bound similar to (71) holds in case \(\Omega\) is a subset of the half-Schwarzschild space. Indeed, upon reflecting this \(\Omega\) across the totally geodesic boundary \(x_3 = 0\) we obtain a region in the exact “boundaryless” Schwarzschild space to which [11, Proposition 3.3] applies. By halving the so obtained estimate, the sought for bound follows. As already emphasized in [11], this bound is robust enough to provide, via a suitable scaling argument, the effective area estimate (71) for a \((\tau, \eta)\)-off-center region in a ahS manifold as in the theorem. The argument is virtually identical to the one appearing in the proof of [11, Theorem 3.4], so it is omitted here.

By a scaling argument as in the proof of [11, Theorem 5.1], we check that isoperimetric surfaces remain close to a centered hemisphere as the volume diverges. We now take a divergent sequence of volumes \(V_i \to +\infty\). Arguing as in [11, Proposition 4.2], we see that
there exists a fixed isoperimetric region $\Omega_i$ and a coordinate half-ball of radius $r_i \geq 0$ which is disjoint from $\Omega_i$ and contributes to the attained isoperimetric profile in the expected manner:
\[
\text{vol}_3(\Omega_i) + \frac{2\pi r_i^3}{3} = V_i, \quad \mathcal{H}_g^2(\partial^* \Omega_i) + 2\pi r_i^2 = I_g(V_i).
\]
This is our analogue of the volume splitting in the second item above. Moreover, if $r_i > 0$ as $i \to +\infty$ then the mean curvature of $\partial^* \Omega_i$ is $2/r_i$, so the relative isoperimetric region associated to $V_i$ encompasses two disjoint half-balls each roughly with the same radius $r_i$. This contradiction shows that $r_i = 0$ for all $i$ large enough. Thus, to each volume greater than some $V_0$ the corresponding isoperimetric region stays at a finite distance from a given point on the manifold. Since we already known that this region centers around a large coordinate hemisphere, it certainly sweeps out the whole manifold as the volume diverges and its boundary necessarily coincides with a free boundary CMC hemisphere described in Theorem 9; see Appendix E in regard to this last point. This completes our sketch of the proof of Theorem 10.

**Remark 8** As already pointed out in [11], the existence of relative isoperimetric regions for sufficiently large enclosed volumes via the argument above only requires that $g$ is $C^0$-asymptotic to half-Schwarzschild. The higher order asymptotics, the Regge–Teitelboim condition included, are only needed to make sure that a foliation exists as in Theorem 9, so its leafs may be identified to the isoperimetric hemisphere.

**Remark 9** The argument above assumes the well-known fact that relative isoperimetric surfaces are sufficiently regular (indeed smooth) up to the boundary and hence are stable free boundary CMC surfaces as explained in Appendix B. This most desirable property is explicitly stated in [51, Proposition 2.4] and we refer to the discussion there for the pertaining sources.

**Remark 10** Very likely an analogue of Theorem 9 holds true for the class of asymptotically hyperbolic 3-manifolds with a non-compact boundary introduced in [52]. This would extend a series of results in the boundaryless case literature starting with [53–56]; see also [10] and the references therein for a recent account of the status of this line of research. In the same vein, it might also be possible to characterize the corresponding large relative isoperimetric regions in the line of the main result in [57], so as to extend Theorem 10 accordingly.

**Acknowledgements** S. Almaraz has been partially supported by CNPq/Brazil Grant 309007/2016-0 and FAPERJ/Brazil Grant 202.802/2019, and L. de Lima has been partially supported by CNPq/Brazil grant 312485/2018-2. Both authors have been partially supported by FUNCAP/CNPq/PRONEX Grant 00068.01.00/15. Also, the authors thank A. Freitas, E. Lima, J.F. Montenegro and S. Nardulli for conversations at an early stage of this project.

**Appendix A: The large scale isoperimetric deficits and the mass: the proofs of Theorems 4 and 7**

The arguments to prove Theorems 4 and 7 are simple variations on the computation appearing in [39, Section 2], where a proof of Theorem 2 appears. This justifies the inclusion of a somewhat detailed account of their calculation in what follows.

If $(M, g)$ is asymptotically flat as in Definition 1, we first observe that, since $\partial r / \partial x_i = x_i / r$, we have
\[
\nabla r = g^{ij} \frac{x_i}{r} \frac{\partial}{\partial x_j}.
\]
and hence
\[ |\nabla r|^2 = g^{ij} \frac{x_i x_j}{r^2} = 1 - e_{ij} \frac{x_i x_j}{r^2} + O(r^{-2\tau}). \tag{A2} \]

Also, if \( v \) is the outward unit normal vector field to the coordinate 2-sphere \( S^2_r \) then
\[ v = \frac{x}{r} + O(r^{-\tau}). \tag{A3} \]

Let \( dS^2_r = r^2 dS^2_1 \) be the area element of the Euclidean sphere of radius \( r \). It follows that the area element of the corresponding coordinate sphere \( S^2_r \) expands as
\[ dS^2_r = \left( 1 + \frac{1}{2} h^{ij} e_{ij} + O(r^{-2\tau}) \right) dS^2_r, \tag{A4} \]

where
\[ h_{ij} = g_{ij} - v_i v_j = \delta_{ij} - \frac{x_i x_j}{r^2} + O(r^{-\tau}) \tag{A5} \]
is the induced metric (extended to vanish in the radial direction). Thus, the area of \( S^2_r \) is
\[ A(r) = 4\pi r^2 + \frac{1}{2} \int_{S^2_r} h^{ij} e_{ij} dS^2_r + O(r^{2-2\tau}). \tag{A6} \]

From this we obtain
\[ \frac{d}{dr} A(r) = 8\pi r + \frac{1}{2} \int_{S^2_r} h^{ij} \frac{x_k}{r} e_{ij,k} dS^2_r + \frac{1}{r} \int_{S^2_r} h^{ij} e_{ij} dS^2_r + O(r^{1-2\tau}), \]

where the comma means partial differentiation. Using (A5) we get
\[ \frac{d}{dr} A(r) = 8\pi r + \frac{1}{2} \int_{S^2_r} e_{ii,k} \frac{x_k}{r} dS^2_r - \frac{1}{2} \int_{S^2_r} e_{ij,k} \frac{x_i x_j x_k}{r^3} dS^2_r + O(r^{1-2\tau}). \]

We now work out the third term in the right-hand side. We first note that
\[ \frac{\partial}{\partial x_i} \frac{x_j}{r} = \frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^3}. \tag{A7} \]

We then compute:
\[ \int_{S^2_r} \frac{\partial}{\partial x_k} \left( e_{ij} \frac{x_j}{r} \right) \frac{x_i x_k}{r^2} dS^2_r = \int_{S^2_r} e_{ij,k} \frac{x_i x_j x_k}{r^3} dS^2_r + \int_{S^2_r} e_{ij} \left( \frac{\delta_{jk}}{r} - \frac{x_j x_k}{r^3} \right) \frac{x_i x_k}{r^2} dS^2_r = \int_{S^2_r} e_{ij,k} \frac{x_i x_j x_k}{r^3} dS^2_r, \]

so we have
\[ \int_{S^2_r} e_{ij,k} \frac{x_i x_j x_k}{r^3} dS^2_r = \int_{S^2_r} \frac{\partial}{\partial x_k} \left( e_{ij} \frac{x_j}{r} \right) \frac{x_i x_k}{r^2} dS^2_r. \]
\[
\begin{align*}
\int_2^S \frac{\partial}{\partial x_i} \left( e_{ij} \frac{x_j}{r} \right) dS_r^{2,\delta} \\
- \int_2^S \left( \delta_{ik} - \frac{x_i x_k}{r^2} \right) \frac{\partial}{\partial x_k} \left( e_{ij} \frac{x_j}{r} \right) dS_r^{2,\delta}.
\end{align*}
\]

Using (A5) we have

\[
(I) = \int_2^S e_{ij} \frac{x_j}{r} dS_r^{2,\delta} + \frac{1}{r} \int_2^S \frac{h^{ij} e_{ij}}{r} dS_r^{2,\delta} + O(r^{1-2\tau}).
\]

Also, integration by parts together with (A7) gives

\[
(II) = - \int_2^S \frac{\partial}{\partial x_k} \left( \frac{x_i x_j x_k}{r^3} \right) e_{ij} \frac{x_j}{r} dS_r^{2,\delta} = -2 \int_2^S e_{ij} \frac{x_i x_j x_k}{r^3} dS_r^{2,\delta} + \frac{1}{r} \int_2^S h^{ij} e_{ij} dS_r^{2,\delta} + O(r^{1-2\tau}).
\]

Thus,

\[
\begin{align*}
\frac{d}{dr} A(r) &= 8\pi r + \frac{1}{2} \int_2^S (e_{ii,j} - e_{ij,i}) \frac{x_j}{r} dS_r^{2,\delta} \\
&+ \int_2^S e_{ij} \frac{x_i x_j x_k}{r^3} dS_r^{2,\delta} + \frac{1}{2r} \int_2^S h^{ij} e_{ij} dS_r^{2,\delta} + O(r^{1-2\tau}) \\
&= 8\pi r - 8\pi m + \int_2^S e_{ij} \frac{x_i x_j}{r^3} dS_r^{2,\delta} + \frac{1}{2r} \int_2^S h^{ij} e_{ij} dS_r^{2,\delta} + o(1). \quad (A9)
\end{align*}
\]

Combining this with (A6), we get

\[
\frac{d}{dr} A(r) = \frac{A(r)}{r} + 4\pi r - 8\pi m + \int_2^S e_{ij} \frac{x_i x_j}{r^3} dS_r^{2,\delta} + o(1). \quad (A10)
\]

We now look at the volume \( V(r) \) enclosed by \( S_r^{2} \). By the co-area formula, (A2) and (A4),

\[
\begin{align*}
\frac{1}{r} \frac{d}{dr} V(r) &= \frac{1}{r} \int_2^S |\nabla r|^{-1} dS_r^2 \\
&= \frac{A(r)}{r} + \frac{1}{2} \int_2^S e_{ij} \frac{x_j x_i}{r^3} dS_r^{2,\delta} + o(1). \quad (A11)
\end{align*}
\]

We may now eliminate the integral term in (A10) and (A11). The result is

\[
\frac{d}{dr} (r A(r)) = 4\pi r^2 - 8\pi mr + 2 \frac{d}{dr} V(r) + o(r).
\]

Integrating we obtain a formula relating the volume and area, namely,

\[
V(r) = \frac{1}{2} r A(r) - \frac{2\pi}{3} r^3 + 2\pi mr^2 + o(r^2), \quad (A12)
\]
which gives

\[ J_r^{M;3,2} = r + \frac{4\pi r^2}{A(r)} \left( m - \frac{r}{3} \right) - \frac{2r}{3} \left( \frac{A(r)}{4\pi r^2} \right)^{\frac{1}{2}} + o(1). \]

On the other hand, from (A6),

\[ \frac{A(r)}{4\pi r^2} = 1 + \mathcal{I} + O(r^{-2\tau}), \quad \mathcal{I} := \frac{1}{8\pi r^2} \int_{S_r^2} h_{ij} e_{ij} dS_r^{2,\delta} = O(r^{-\tau}) \]

so that

\[ J_r^{M;3,2} = r + (1 - \mathcal{I} + O(r^{-2\tau})) \left( m - \frac{r}{3} \right) - \frac{2r}{3} \left( 1 + \frac{1}{2} \mathcal{I} + O(r^{-2\tau}) \right) + o(1) \]

\[ = m + o(1), \]

which gives the proof of Theorem 2.

So far we have been following [39] closely. We now explain how a little variation yields the proof of Theorem 4. We will make use of the well-known expansion

\[ H = \frac{2}{r} + O(r^{-\tau}). \]  

(A13)

Together with (A4) this gives

\[ \frac{M(r)}{8\pi r} = 1 + \mathcal{I} + O(r^{-2\tau}). \]  

(A14)

Also, by the first variation formula for the area,

\[
\frac{d}{dr} A(r) = \int_{S_r^2} \left( \frac{\partial}{\partial r}, Hv \right) dS^2_r \\
\stackrel{(A2)+(A13)}{=} \int_{S_r^2} |\nabla r|^{-1} H dS^2_r \\
= M(r) + \int_{S_r^2} e_{ij} \frac{x_i x_j}{r^3} dS_r^{2,\delta} + O(r^{-2\tau+1}),
\]

and combining this with (A10) we get

\[ \frac{1}{2} r^2 M(r) = \frac{1}{2} r A(r) + 2\pi r^3 - 4\pi mr^2 + o(r^2). \]  

(A15)

We now use (A12) to eliminate the area term. Solving for the volume we get

\[ V(r) = \frac{1}{2} r^2 M(r) - \frac{8\pi}{3} r^3 + 6\pi mr^2 + o(r^2), \]  

(A16)

so that, using (A14),

\[ J_r^{M;3,1} = \frac{2}{3} r + \frac{8\pi r}{M(r)} \left( m - \frac{4}{9} r \right) - \frac{2}{9} r \left( \frac{M(r)}{8\pi r} \right)^{\frac{1}{2}} + o(1) \]

\[ = \frac{2}{3} r + (1 - \mathcal{I} + O(r^{-2\tau})) \left( m - \frac{4}{9} r \right) - \frac{2}{9} r \left( 1 + 2\mathcal{I} + O(r^{-2\tau}) \right) + o(1) \]

\[ = m + o(1), \]
which finishes the proof of the first equality in (11). As for the second one, note that by (A15) and (A14),
\[
J_M^{r, 1.2} = \frac{4\pi r}{M(r)} (2m - r) - \frac{1}{16\pi} M(r) + o(1)
\]
\[
= r + \frac{1}{2} \left( 1 - I + O(r^{-2\tau}) \right) (2m - r) - \frac{r}{2} \left( 1 + I + O(r^{-2\tau}) \right) + o(1)
\]
\[
= m + o(1),
\]
which completes the proof of Theorem 4.

We now present the proof of Theorem 7. We first observe that instead of (A6) we now have
\[
\mathcal{A}(r) = 2\pi r^2 + \frac{1}{2} \int_{S_r^+} h^{ij} e^+_{ij} dS_r^{2, \delta^+} + O(r^{-2\tau}). \tag{A17}
\]
Also, the integration by parts leading to (A8) now produces an extra term, so that (II) gets replaced by
\[
(II^+) = -2 \int_{S_r^+} e^+_{ij} \frac{x^i x^j}{r^3} dS_r^{2, \delta^+} - \int_{S_r^+} e^+_{ij} \frac{x^j}{r^3} \partial^k dS_r^{1, \delta^+} + O(r^{-2\tau+1}). \tag{A18}
\]
Thus, instead of (A10) we now have
\[
\frac{d}{dr} \mathcal{A}(r) = \frac{\mathcal{A}(r)}{r} + 2\pi r - 8\pi m + \int_{S_r^+} e^+_{ij} \frac{x^i x^j}{r^3} dS_r^{2, \delta^+} + o(1). \tag{A19}
\]
Hence, proceeding exactly as before we now get
\[
\mathcal{V}(r) = \frac{1}{2} r \mathcal{A}(r) - \frac{\pi}{3} r^3 + 2\pi mr^2 + o(r^2), \tag{A20}
\]
which gives
\[
J_M^{r, 3.2} = \frac{r}{2} + \frac{2\pi r^2}{\mathcal{A}(r)} \left( m - \frac{r}{6} \right) - \frac{r}{3} \left( \frac{\mathcal{A}(r)}{2\pi r^2} \right)^{1/2} + o(1)
\]
\[
= \frac{r}{2} + \left( 1 - \hat{I} + O(r^{-2\tau}) \right) \left( m - \frac{r}{6} \right) - \frac{r}{3} \left( \left( 1 + \frac{1}{2} \hat{I} + O(r^{-2\tau}) \right) \right) + o(1)
\]
\[
= m + o(1),
\]
where
\[
\hat{I} = \frac{1}{4\pi r^2} \int_{S_r^+} h^{ij} e^+_{ij} dS_r^{2, \delta^+} = O(r^{-\tau}).
\]
This completes the proof of Theorem 7.

**Appendix B: The variational setup**

Here we address the variational issues needed in the bulk of the paper. Our aim is twofold. First, we review the well-known variational theory of free boundary constant mean curvature surfaces [58, 59] in a way that is convenient for our purposes. Next, we discuss the much less known variational theory of closed surfaces which are critical for the total mean curvature functional under a volume preserving constraint and develop the corresponding stability
We remark that the variational theory associated to curvature integrals involving elementary symmetric functions of the principal curvatures (quermassintegrals) of hypersurfaces in space forms is a well established subject; see [60] and the references therein.

We start by considering a one-parameter family of compact, embedded surfaces $t \in (-\varepsilon, \varepsilon) \mapsto S_t$ in an arbitrary Riemannian manifold $(M^3, g)$ evolving as

$$\frac{\partial x_t}{\partial t} = Y_t, \quad \text{(B1)}$$

where $x_t$ is the smooth map defining the embedding and $Y_t$ is a vector field along $S_t$, a (not necessarily normal) section of $TM$ restricted to $S_t$. As usual, if $v_t$ is the unit normal vector field along $S_t$, let $W = \nabla v_t$ be the shape operator of $S_t$, so the corresponding principal curvatures (the eigenvalues of $W$) are $\kappa_1$ and $\kappa_2$. Thus, the mean curvature is $H = \kappa_1 + \kappa_2$ and the Gauss–Kronecker curvature is $K = \kappa_1\kappa_2$. For later reference, we recall that

$$\tilde{K} = K - \frac{1}{2} \text{Ric}_g(v, v)$$

is the modified Gauss–Kronecker curvature.

A well-known computation gives

$$\frac{d}{dt} A(t)|_{t=0} = \int_S \text{div}_S Y dS, \quad \text{(B2)}$$

where $A(t)$ is the area of $S_t$ and we agree to drop the subscript $t$ upon evaluation at $t = 0$.

Next we decompose $Y_t$ into its normal and tangential components:

$$Y_t = f_t v_t + Y^\top_t, \quad f_t = \langle Y_t, v_t \rangle. \quad \text{(B3)}$$

Thus, if we assume further that $S_t$ carries a boundary $\partial S_t$,

$$\frac{d}{dt} A(t)|_{t=0} = \int_S f H dS + \int_S \text{div}_S {Y^\top} dS$$

$$= \int_S f H dS + \int_{\partial S} \langle Y, \mu \rangle d\partial S, \quad \text{(B4)}$$

where $\mu$ is the outward unit normal vector field along $\partial S$ and we used that $H = \text{div}_S v$.

Let us assume now that $M$ also carries a boundary, say $\Sigma$, with the variation being admissible in the sense that $\partial S_t \subset \Sigma$. It follows that $S = S_0$ is critical for the area under such variations satisfying the volume preserving condition

$$\int_S f dS = 0 \quad \text{(B5)}$$

if and only if the mean curvature is constant and $S$ meets $\Sigma$ orthogonally along $\partial S$. We then say that $S$ is a free boundary constant mean curvature (CMC) surface.

We now recall the corresponding notion of stability. Assuming that $S = S_0$ is a free boundary CMC as above, a well-known computation [59] gives the second variational formula for the area:

$$\frac{d^2 A}{dt^2}|_{t=0} = \int_S f \mathcal{L}_S f dS + \int_{\partial S} f \left( \frac{\partial f}{\partial \mu} - \kappa f \right) d\partial S, \quad \text{(B6)}$$

where

$$\mathcal{L}_S = -\Delta_S - \left( |W|^2 + \text{Ric}_g(v, v) \right), \quad \text{(B7)}$$

\(\copyright\) Springer
\( \kappa = \langle v, W v \rangle \) and \( W = \nabla \eta \) is the shape operator of the embedding \( \Sigma \hookrightarrow M \). Here, \( \eta \) is the outward unit normal vector to \( M \) along \( \Sigma \).

Recall that \( S = S_0 \) is strictly stable (as a free boundary CMC surface) if the right-hand side of (B6) is positive for any \( f \neq 0 \) satisfying (B5). Accordingly, we define

\[
\mathcal{F}(S) = \left\{ f \in H^1(S); \int_S f \, dS = 0 \right\}.
\]

**Proposition 23** A free boundary CMC surface \( S \) as above is strictly stable if and only if the first eigenvalue \( \lambda \) of the eigenvalue problem

\[
\begin{align*}
\mathcal{L}_S f &= \lambda f \quad \text{in} \ S, \\
\frac{\partial f}{\partial \mu} &= \kappa f \quad \text{on} \ \partial S,
\end{align*}
\]

is positive, where \( f \in \mathcal{F}(S) \). Equivalently, for any \( 0 \neq f \in \mathcal{F}(S) \),

\[
\int_S \left( |\nabla_S f|^2 - (|W|^2 + \text{Ric}(v, v) f^2) \right) dS - \int_{\partial S} \kappa f^2 d\partial S > 0.
\]

We now turn to the variational theory of the total mean curvature functional \( \int_S H dS \).

Here we assume that \( S_t \) is closed (\( \partial S = \emptyset \)) and the variation is normal (\( Y = f v \)). A simple computation shows that the shape operator evolves as

\[
\frac{\partial W}{\partial t} = -\nabla^2_S f - (W^2 + \text{Riem}_g^v) f,
\]

where \( \nabla^2_S \), the Hessian of \( f \), is viewed as a \((1, 1)\)-tensor, \( \text{Riem}_g^v(\cdot) = \text{Riem}_g(\cdot, v)v \) and \( W^2 = W \circ W \).

**Proposition 24** In a Riemannian 3-manifold \((M, g)\) as above, a closed surface extremizes the total mean curvature under volume (respectively, area) preserving variations if and only if \( \tilde{K} = \text{const} \) (respectively, \( \tilde{K} = \gamma H \), where \( \gamma \) is a constant).

**Proof** From \( \frac{\partial dS_t}{\partial t} = f H dS_t \), the fact that the mean curvature evolves as

\[
\frac{\partial H}{\partial t} = \mathcal{L}_S f,
\]

and the algebraic identity \( |W|^2 = H^2 - 2K \), we immediately see that

\[
\frac{\partial}{\partial t} \left|_{S_t} \right. H dS_t|_{t=0} = 2 \int_S \tilde{K} f dS_t,
\]

which proves the first statement. As for the second one, just combine the computation above with (B4) and take into account that \( \partial S = \emptyset \). \( \square \)

In order to discuss the stability of this variational problem, we now compute the variation of \( \tilde{K} \). First, from \( \frac{\partial v}{\partial t} = -\nabla_S f \),

\[
\frac{\partial}{\partial t} \text{Ric}_g(v, v) = f(\nabla_v \text{Ric}_g)(v, v) - 2\text{Ric}_g(\nabla_S f, v).
\]

As for the variation of \( K \), we first recall the well-known formula

\[
\frac{\partial}{\partial t} K = \text{tr}_S \left( \Pi \frac{\partial}{\partial t} W \right),
\]
where \( \Pi = HI - W \) is the Newton tensor [61]. Using (B8) we then get
\[
\frac{\partial}{\partial t} K = -\text{tr}_S(\Pi \nabla^2_S f) - f \text{tr}_S(\Pi W^2) - f \text{tr}_S(\Pi \text{Riem}^\nu_g).
\]
To proceed we choose an orthonormal frame \( e_A, A = 1, 2 \), tangent to \( S \) with \((\nabla_S) e_A e_B = 0\) at the given point. We compute
\[
\text{tr}_S(\Pi \nabla^2_S f) = \Pi^{AB} e_A(\nabla_S f, e_B)
\]
where the semicolon denotes covariant derivation. By Codazzi equations, recalling that \( h = g|_S \),
\[
\Pi^{AB} e_A(\nabla_S f, e_B)
\]
so that
\[
\text{tr}_S(\Pi \nabla^2_S f) = \text{div}_S(\Pi \nabla_S f) + \text{Ric}_g(\nu, \nabla_S f).
\]
Thus, from (B11) and the algebraic identity \( \text{tr}_S(\Pi W^2) = HK \), which holds in dimension 3,
\[
\frac{\partial}{\partial t} K = -\Lambda_S f - \text{Ric}_g(\nu, \nabla_S f) - f HK - f \text{tr}_S(\Pi \text{Riem}^\nu_g),
\]
where
\[
\Lambda_S f = \text{div}_S(\Pi \nabla_S f).
\]
Together with (B10) this finally gives
\[
\frac{\partial}{\partial t} \tilde{K} = L_S f,
\]
where
\[
L_S = -\Lambda_S - HK - \frac{1}{2}(\nabla_v \text{Ric}_g)(\nu, \nu) - \text{tr}_S(\Pi \text{Riem}^\nu_g)
\]
is the corresponding Jacobi operator. We note that
\[
- \int_S f \Lambda_S \tilde{f} dS = \int_S \langle \Pi \nabla_S f, \nabla_S \tilde{f} \rangle dS,
\]
for any functions \( f \) and \( \tilde{f} \). In particular, \( L_S \) is always self-adjoint. Moreover, it is easy to check that this operator is elliptic whenever \( \Pi \) is positive definite.

We now consider a surface \( S \subset M \) satisfying \( \tilde{K} = \text{const.} \) and with the property that \( \Pi \) is positive definite everywhere. We then say that \( S \) is strictly stable if
\[
\frac{d^2}{dt^2} \int_S H dS|_{t=0} > 0,
\]
for any normal variation as in (B1) with \( f \neq 0 \). As before let us set

\[ G(S) = \left\{ f \in H^1(S) ; \int_S f \, dS = 0 \right\}. \]

**Proposition 25** \( S \) is strictly stable if and only if

\[
\int_S \left( \langle \Pi \nabla S f, \nabla S f \rangle - f^2 \left( HK + \frac{1}{2} (\nabla_v \text{Ric}_g)(v,v) + \text{tr}_S (\Pi \text{Riem}_g^v) \right) \right) \, dS > 0,
\]

for any \( 0 \neq f \in G(S) \). Equivalently, the first eigenvalue \( \lambda_{L_S} \) of the eigenvalue problem

\[ L_S f = \lambda f, \quad f \in G(S), \]

is positive.

**Appendix C: The Gauss–Kronecker curvature in terms of the mean curvature**

In this section we prove Proposition 13. Thus we aim to prove the identity (33) which expresses the Gauss–Kronecker curvature in terms of the mean curvature up to terms decaying fast enough at infinity. Our starting point is the fact that the radial vector field

\[ X = (x_i - a_i) \frac{\partial}{\partial x_i} \]

is conformal with respect to the Euclidean metric, i.e., \( \mathcal{L}_X \delta = 2\delta \), where \( \mathcal{L} \) is the Lie derivative. From this we see that \( X \) is also conformal with respect to the metric \( f_{m,c}^{\gamma_1,\gamma_2} \delta \) where

\[ f_{m,c}^{\gamma_1,\gamma_2} = 1 + \frac{2m}{r} + \frac{\gamma_1 c \cdot x}{r^3} + \frac{\gamma_2}{r^2} \]

for some \( \gamma_1, \gamma_2 \in \mathbb{R} \) and \( c \in \mathbb{R}^3 \). Indeed, there holds \( \mathcal{L}_X (f_{m,c}^{\gamma_1,\gamma_2} \delta) = 2\xi f_{m,c}^{\gamma_1,\gamma_2} \delta \), with

\[
\xi(x) = \frac{1}{f_{m,c}^{\gamma_1,\gamma_2}} \left( f_{m,c}^{\gamma_1,\gamma_2} + \frac{1}{2} \partial_k f_{m,c}^{\gamma_1,\gamma_2} (x_k - a_k) \right)
\]

\[
= 1 - \frac{m}{r} + \frac{2m^2 - \gamma_2}{r^2} + \frac{m}{r^3} x \cdot a - \frac{\gamma_1}{r^3} x \cdot c + O(r^{-3})
\]

\[
= 1 - \frac{m}{\rho} + \frac{2m^2 - \gamma_2}{\rho^2} + \frac{2m}{\rho^3} x \cdot a - \frac{\gamma_1}{\rho^3} x \cdot c + O(\rho^{-3}), \tag{C1}
\]

where in the last step we used that

\[ r^k = \rho^k + k \frac{(x - a) \cdot a}{\rho^{2-k}} + O(\rho^{-2+k}), \quad k \in \mathbb{R}. \tag{C2} \]

Let us consider an aS metric of the form \( g = f_{m,c}^{\gamma_1,\gamma_2} \delta + p \), where \( p = O(r^{2-\epsilon}) \), which satisfies (18) with \( \epsilon \geq 0 \).

**Proposition 26** The vector field \( X \) is almost conformal with respect to \( g \) in the sense that

\[ \mathcal{L}_X g = 2\xi g + B, \quad \text{where} \quad B = O(\rho^{-2-\epsilon}). \tag{C3} \]
Proof A direct computation shows that (C3) holds with $B = \mathcal{L}_X p - 2\xi p$. Note however that

\[(\mathcal{L}_X p)_{jk} = X^i \nabla_i p_{jk} + p_{ik} \nabla_j X^i + p_{lj} \nabla_k X^i,\]

and the result follows given that $X = O(r)$. 

We now take $\{e_1, e_2\}$ to be a local orthonormal frame on $S^2_\rho(a)$ and $\nu$ its outward unit normal vector. Recall that $W = \nabla \nu$ is the shape operator of $S^2_\rho(a)$ and $\Pi = H I - W$ denotes its Newton tensor. If $X^\top = X - \langle X, \nu \rangle \nu$ is the tangential component of $X$, then

\[(L_X g)(\Pi e_A, e_A) = (L_X g)(\Pi e_A, e_A) - 2\langle X, \nu \rangle W(\Pi e_A, e_A),\]

and we obtain from (C3) that

\[\langle \nabla_{\Pi e_A} X^\top, e_A \rangle + \langle \nabla_{e_A} X^\top, \Pi e_A \rangle = 2\xi \langle \Pi e_A, e_A \rangle - 2\langle X, \nu \rangle \langle W \Pi e_A, e_A \rangle + B(\Pi e_A, e_A).\]

Since

\[\langle \nabla_{e_A} X^\top, \Pi e_A \rangle = \langle e_A, \nabla_{\Pi e_A} X^\top \rangle,\]

which is easily verified if we take the frame to be principal with respect to the shape operator $W$, this simplifies to

\[\langle \nabla_{e_A} X^\top, \Pi e_A \rangle = \xi \langle \Pi e_A, e_A \rangle - \langle X, \nu \rangle \langle W \Pi e_A, e_A \rangle + \frac{1}{2} B(\Pi e_A, e_A).\]

Thus, summing over $A$ and using that $H^2_{a,\rho} - |W|^2 = 2K_{a,\rho}$, we obtain

\[\sum_A \langle \nabla_{e_A} X^\top, \Pi e_A \rangle = \xi H^2_{a,\rho} - 2\langle X, \nu \rangle K_{a,\rho} + \frac{1}{2} \sum_A B(\Pi e_A, e_A).\] (C4)

In order to make use of this identity, which first appeared in [62], we need to determine the asymptotics of $X^\top$.

Proposition 27 One has $X^\top = O(\rho^{-1-\epsilon})$.

Proof Recalling that $\tau = (x - a) / \rho$, so that $X = \rho \tau_i \partial / \partial x_i$, one computes

\[\nu = \left( f_{m,c}^{\gamma_1,\gamma_2} \right)^{-1/2} \tau_i \frac{\partial}{\partial x_i} + O(\rho^{-2-\epsilon}),\] (C5)

so that

\[\langle X, \nu \rangle = \left( f_{m,c}^{\gamma_1,\gamma_2} \right)^{1/2} \rho + O(\rho^{-1-\epsilon}).\] (C6)

Thus,

\[\langle X, \nu \rangle \nu = \rho \tau_i \frac{\partial}{\partial x_i} + O(\rho^{-1-\epsilon}),\]

and the result follows. 

We now observe that by (30) we may rewrite (54) as

\[\Pi = \frac{1}{2} H_{\rho,a} I + O(\rho^{-3}).\]
so that
\[
\frac{1}{2} \sum_A B(\Pi e_A, e_A) = \frac{1}{4} H_{\rho, a} \text{tr} S_{\rho}^2(\alpha) B + O(\rho^{-5}),
\]
where we used that \( B = O(\rho^{-2}) \). Also, the left-hand side of (C4) may be treated similarly. Indeed, by Proposition 27,
\[
\sum_A (\nabla_A X^T, \Pi e_A) = \frac{1}{2} H_{\rho, a} \text{div} S_{\rho}^2(\alpha) X^T + O(\rho^{-5}).
\]
Putting all the pieces of our computation together and using (C6) we get
\[
2K_{a, \rho} = \left( \frac{\xi}{(X, v)} + \frac{1}{(X, v)} \left( \frac{1}{4} \text{tr} S_{\rho}^2(\alpha) B - \frac{1}{2} \text{div} S_{\rho}^2(\alpha) X^T \right) H_{a, \rho} + O(\rho^{-6}) \right)
\]
\[
= \left( \frac{\xi}{(X, v)} + O(\rho^{-3-\epsilon}) \right) H_{a, \rho} + O(\rho^{-6}).
\]
The proof of Proposition 13 is completed if we note that by (C1) and (C6),
\[
\frac{\rho \xi}{(X, v)} = \xi \left( (f_{m,c})^{-1/2} + O(\rho^{-2-\epsilon}) \right)
\]
\[
\quad = \left( 1 - \frac{m}{\rho} + \frac{2m^2 - \gamma_2}{\rho^2} + \frac{2m}{\rho^3} x \cdot a - \frac{\gamma_1}{\rho^3} x \cdot c + O(\rho^{-3}) \right)
\]
\[
\quad \cdot \left( 1 - \frac{m}{\rho} + \frac{3m^2 - \gamma_2}{2\rho^2} + \frac{m}{\rho^3} x \cdot a - \frac{\gamma_1}{2\rho^3} x \cdot c + O(\rho^{-2-\epsilon}) \right)
\]
\[
\quad = 1 - \frac{2m}{\rho} + \frac{9m^2 - 3\gamma_2}{2\rho^2} + \frac{3m}{\rho^3} x \cdot a - \frac{3\gamma_1}{2\rho^3} x \cdot c + O(\rho^{-2-\epsilon}).
\]

**Appendix D: The proof of Proposition 19**

Here we indicate how the argument in [26, Appendix F] may be used to prove Proposition 19. In fact, this method allows us to approach the problem in the category of manifolds considered in Definition 9.

**Proposition 28** If \((M, g)\) is an asymptotically flat 3-manifold with a non-compact boundary satisfying the RT condition then
\[
\int_{S_{\rho, +, b}^{2}} (x_a - b_a) \left( H_{\rho, +, b} - \frac{2}{\rho} \right) dS_{\rho, +, b}^{2, \delta^+} = 8\pi m (b_a - C_a^\alpha) + O(\rho^{-\tau}), \quad \alpha = 1, 2. \quad (D1)
\]

**Corollary 29** There holds
\[
C_a^\alpha = -\lim_{\rho \to +\infty} \frac{1}{8\pi m} \int_{S_{\rho, +, b}^{2} \setminus \{0\}} x_a H_{\rho, +, b} dS_{\rho, +, b}^{2, \delta^+} (\vec{0}).
\]

The key ingredient in the proof is an integral identity derived from the fact that \( S_{\rho, +, b}^{2} \) is a free boundary CMC surface with mean curvature \( 2/\rho \) with respect to the metric \( \delta^+ \). In the following, for convenience we shall omit the area element of \( S_{\rho, +, b}^{2} \) and the line element of \( S_\rho^1(b) := \partial S_{\rho, +, b}^{2} \) in the respective integrals.
Proposition 30 There holds
\[
\frac{1}{2} \int_{S^2_{\rho^+}(b)} (x_\alpha - b_\alpha) e_{ij,k}^+ \tau_i \tau_j \tau_k = \int_{S^2_{\rho^+}(b)} (x_\alpha - b_\alpha) \left( \frac{1}{2} e_{ij,k}^+ \tau_j - 2 e_{ij}^+ \frac{\tau_i \tau_j}{\rho} \right)
\]
\[\quad + \frac{1}{2} \int_{S^2_{\rho^+}(b)} (e_{ii}^+ \tau_i + e_{i\alpha}^+ \tau_i)
\]
\[\quad - \frac{1}{2} \int_{S^1_{\rho}(b)} (x_\alpha - b_\alpha) e_{i3}^+ \tau_i,
\]
where \(\tau = (x - b)/\rho\).

Proof Apply the identity that follows from equating the right-hand sides of (B2) and (B4) with \(\mu = -\partial / \partial x_3\) to the vector field \(Y_\alpha = (x_\alpha - b_\alpha) e_{ij}^+ \tau_i \partial / \partial x_j\) by taking into account that
\[
\text{div}_{S^2_{\rho^+}(b)} Y_\alpha = e_{i\alpha}^+ \tau_i + (x_\alpha - b_\alpha) \left( \frac{e_{ii}^+}{\rho} - 2 \frac{e_{ij}^+}{\rho} \tau_i \tau_j + e_{ij,j}^+ \tau_i - e_{ij,k}^+ \tau_j \tau_k \right).
\]

We now recall the expansion
\[
H_{\rho^+,b} - \frac{2}{\rho} = \frac{1}{2} e_{ij,k}^+ \tau_i \tau_j \tau_k + 2 e_{ij}^+ \frac{\tau_i \tau_j}{\rho} - e_{ij,i}^+ \tau_j + \frac{1}{2} e_{ii,j}^+ \tau_j - e_{ii}^+ \frac{\tau_i}{\rho} + E,
\]
where the remainder satisfies \(E = O(\rho^{-1-2r})\) and \(E^{-1'} = O(\rho^{-2-2r})\); see [20, Lemma 2.1]. This reduces to (55) if we take \(e^+ = 2m r^{-1} \delta^+ + O(r^{-2})\), which provides the link between Propositions 28 and 19. It follows that
\[
\int_{S^2_{\rho^+}(b)} (x_\alpha - b_\alpha) \left( H_{\rho^+,b} - \frac{2}{\rho} \right) = -\frac{1}{2} \int_{S^2_{\rho^+}(b)} (x_\alpha - b_\alpha) \left( e_{ij,i}^+ - e_{ii,j}^+ \right) \tau_j
\]
\[\quad + \frac{1}{2} \int_{S^2_{\rho^+}(b)} (e_{i\alpha}^+ \tau_i - e_{ii}^+ \tau_i)
\]
\[\quad - \frac{1}{2} \int_{S^1_{\rho}(b)} (x_\alpha - b_\alpha) e_{i3}^+ \tau_i + O(\rho^{-r}),
\]
where Proposition 30 has been used to make sure that only those terms which are linear in \(\tau\) survive in the right-hand side. We now observe that under the decay assumptions (including Regge–Teitelboim) the integrals
\[
\int_{S^2_{\rho^+}(b)} x_\alpha \left( e_{ij,i}^+ - e_{ii,j}^+ \right) \frac{b_j}{\rho}, \quad \int_{S^2_{\rho^+}(b)} \left( e_{ij,i}^+ - e_{ii,j}^+ \right) \frac{b_j}{\rho},
\]
and
\[
\int_{S^2_{\rho^+}(b)} \left( e_{i\alpha}^+ \frac{b_i}{\rho} - e_{ii}^+ \frac{b_i}{\rho} \right)
\]
are \(O(\rho^{-r})\), the same happening to the boundary integrals
\[
\frac{b_i}{\rho} \int_{S^1_{\rho}(b)} x_\alpha e_{i3}^+, \quad \frac{b_i b_i}{\rho} \int_{S^1_{\rho}(b)} e_{i3}^+.
\]
Thus, we end up with

$$\int_{S_{\rho, +}^2} (x_a - b_a) \left( H_{\rho, +, b} - \frac{2}{\rho} \right) = -\frac{1}{2} \int_{S_{\rho, +}^2} x_a \left( e_{ij}^+ - e_{ii,j}^+ \right) x_j \frac{1}{\rho}$$

$$+ \frac{1}{2} \int_{S_{\rho, +}^2} e_{i\alpha}^+ x_i - e_{ii\alpha}^+ \frac{x_i}{\rho}.$$ 

Comparing the right-hand side of the above with the definitions of $m$ and $C^+$, the proof of Proposition 28, and hence of Proposition 19, follows.

**Remark 11** The upshot of Corollary 29 is another expression for the center of mass $C^+$, besides (28), derived from Hamiltonian methods, and the isoperimetric one appearing in Theorems 9 and 10. Another rendition of this invariant comes from [18, Theorem 2.4], this time in terms of certain asymptotic flux integrals involving the Einstein tensor of the metric in the interior and the Newton tensor along the boundary; see also [63]. It is remarkable indeed that this kind of invariant admits so many distinct manifestations.

**Appendix E: The uniqueness of the free boundary CMC hemispheres**

The very last piece of the argument leading to Theorem 10 uses the appropriate uniqueness of the free boundary CMC hemispheres in Theorem 9. Here we justify this step by following the reasoning in [5, Section 4]. We know from the analysis in Sect. 4 that for each $\rho$ large enough the corresponding hemisphere is a strictly stable free boundary CMC surface graphically described by a function $\phi_\rho$ on $S_{\rho, +}^2 (C^+)$ satisfying the bound

$$\| \rho^{-1/2} \phi_\rho \|_{C^2, \alpha} (\rho) \leq C,$$

where $C > 0$ is an absolute constant and the weighted Hölder norm is defined as in the left-hand side of (51). The uniqueness claim is that, for $\rho$ large enough depending only on $C$, any other free boundary CMC hemisphere with the same mean curvature and which is graphed by a function satisfying this Hölder bound should coincide with (the graph of) $\phi_\rho$. Indeed, assume there exists another such hemisphere, say associated to a function $\phi_1$. As in [5, Proposition 2.1], the asymptotic roundness of the graphs means that we may interpolate between the corresponding embeddings by setting

$$F_t (x) = F_0 (x) + tu (x) v (x), \quad t \in [0, 1],$$

for some function $u (x) = \langle \vec{a}, v (x) \rangle + q (x)$, where $\vec{a} \in \mathbb{R}^2$ is a vector and $q = O(\rho^{-1})$. A crucial remark at this point is that all of these surfaces are free boundary (with a possibly non-constant mean curvature $H_F$ for $0 < t < 1$) and may be graphed by using functions satisfying the same Hölder bound as $\phi_0$. Since $H_{F_0} = H_{F_1}$, the variational vector field
\[ Y = F_1 - F_0 \text{ satisfies} \]
\[ |Y| \leq \|dH F_0\|^{-1} \sup_t \|d^2 H F_1(Y, Y)\| \leq C_1|Y|^2, \]
where we used (70) applied to \(\mathcal{L} F_0\), the Jacobi operator associated to \(F_0\), and the fact that \(\|d^2 H F_1\| = O(\rho^{-3})\) uniformly in \(t\). Thus, there exists an absolute constant \(C_2 > 0\) such that \(|Y| \leq C_2\) implies \(Y = 0\). We next check that \(|Y|\) (equivalently, \(|\vec{a}|\)) may be chosen small enough so as to fulfill this vanishing criterion if \(\rho\) is large. We first note that, again because \(H F_0 = H F_1\),
\[ \|dH F_0 Y\| \leq \sup_t \|(dH F_1 - dH F_0) Y\|. \quad (E1) \]
As in [64, Proposition 16] we compute that
\[ dH F_1 Y = \mathcal{L} F_1 u + Y^\top H F_1, \]
where \(Y^\top\) is the tangential component of \(Y\). Starting with (C5) we obtain \(|Y^\top| = O(\rho^{-3})\) and hence \(|Y^\top H F_1| = O(\rho^{-4})\). Combining this with (64) we see that the right-hand side of (E1) is \(O(\rho^{-4})\). On the other hand, since \((\vec{a}, \psi)\) is an approximate eigenfunction of \(\mathcal{L} F_0\) under Neumann boundary condition with eigenvalue close to \(6m \rho^{-3}\), the left-hand side of (E1) is \(\geq C_3|\vec{a}|\rho^{-3}\). Thus, \(|\vec{a}| \leq C_4\rho^{-1}\) and the uniqueness claim follows provided we take \(\rho \geq C_2^{-1} C_4\).

References
1. Schoen, R.: Conformal deformation of a Riemannian metric to constant scalar curvature. J. Differ. Geom. 20(2), 479–495 (1984)
2. Lee, J.M., Parker, T.: The Yamabe problem. Bull. AMS 17(1), 37–91 (1987)
3. Brendle, S., Marques, F.: Recent progress on the Yamabe problem. Adv. Lect. Math. 20, 29–47 (2011)
4. Huisken, G., Ilmanen, T.: The inverse mean curvature flow and the Riemannian Penrose inequality. J. Differ. Geom. 59(3), 353–437 (2001)
5. Huisken, G., Yau, S.-T.: Definition of center of mass for isolated physical systems and unique foliations by stable spheres with constant mean curvature. Invent. Math. 124(1–3), 281–311 (1996)
6. Ye, R.: Foliation by constant mean curvature spheres on asymptotically flat manifolds. In: Geometric Analysis and the Calculus of Variations (1996)
7. Metzger, J.: Foliations of asymptotically flat 3-manifolds by 2-spheres of prescribed mean curvature. J. Differ. Geom. 77(2), 201–236 (2007)
8. Huang, L.-H.: On the center of mass of isolated systems with general asymptotics. Class. Quantum Gravity 26(1), 015012 (2008)
9. Huang, L.-H.: Foliations by stable spheres with constant mean curvature for isolated systems with general asymptotics. Commun. Math. Phys. 300(2), 331–373 (2010)
10. Nerz, C.: Foliations by spheres with constant expansion for isolated systems without asymptotic symmetry. J. Differ. Geom. 109(2), 257–289 (2018)
11. Eichmair, M., Metzger, J.: Large isoperimetric surfaces in initial data sets. J. Differ. Geom. 94(1), 159–186 (2013)
12. Regge, T., Teitelboim, C.: Role of surface integrals in the Hamiltonian formulation of general relativity. Ann. Phys. 88(1), 286–318 (1974)
13. Beig, R., Ó Murchadha, N.: The Poincaré group as the symmetry group of canonical general relativity. Ann. Phys. 174(2), 463–498 (1987)
14. Almaraz, S.: Convergence of scalar-flat metrics on manifolds with boundary under a Yamabe-type flow. J. Differ. Equ. 259(7), 2626–2694 (2015)
15. Almaraz, S., de Queiroz, O.S., Wang, S.: A compactness theorem for scalar-flat metrics on 3-manifolds with boundary. J. Funct. Anal. 277(7), 2092–2116 (2019)
16. Almaraz, S., Barbosa, E., de Lima, L.L.: A positive mass theorem for asymptotically flat manifolds with a non-compact boundary. Commun. Anal. Geom. 24(4), 673–715 (2016)
17. Almaraz, S., de Lima, L.L., Mari, L.: Spacetime positive mass theorems for initial data sets with noncompact boundary. In: International Mathematics Research Notices (2020)
18. de Lima, L.L., Girão, F.; Montalbán, A.: The mass in terms of Einstein and Newton. Class. Quantum Gravity 36(7), 075017 (2019)
19. Huisken, G.: An isoperimetric concept for mass and quasilocal mass. Oberwurfach Rep 3, 87–88 (2006)
20. Huang, L.-H.: Center of mass and constant mean curvature foliations for isolated systems. MSRI Lecture Notes (2009)
21. Schneider, R.: Convex bodies: the Brunn–Minkowski theory. In: Encyclopedia of Mathematics and its Applications, vol. 151. Cambridge University Press (2014)
22. Guan, P., Li, J.: The quermassintegral inequalities for k-convex starshaped domains. Adv. Math. 221(5), 1725–1732 (2009)
23. Chang, S.-Y.A., Wang, Y.: On Aleksandrov–Fenchel inequalities for k-convex domains. Milan J. Math. 79(1), 13–38 (2011)
24. Fall, M.M.: Area-minimizing regions with small volume in Riemannian manifolds with boundary. Pac. J. Math. 244(2), 235–260 (2009)
25. Montenegro, J.F.: Foliation by free boundary constant mean curvature leaves. arXiv:1904.11867 (2019)
26. Eichmair, M., Metzger, J.: Unique isoperimetric foliations of asymptotically flat manifolds in all dimensions. Invent. Math. 194(3), 591–630 (2013)
27. Munoz Flores, A.E., Nardulli, S.: The isoperimetric problem of a complete Riemannian manifold with a finite number of asymptotically Schwarzschild ends. Commun. Anal. Geom. 28(7), 1577–1601 (2020)
28. Nerz, C.: Foliations by stable spheres with constant mean curvature for isolated systems without asymptotic symmetry. Calc. Var. Partial Differ. Equ. 54(2), 1911–1946 (2015)
29. Chodosh, O., Eichmair, M., Shi, Y., Yu, H.: Isoperimetry, scalar curvature, and mass in asymptotically flat Riemannian 3-manifolds. Commun. Pure Appl. Math. 74(4), 865–905 (2021)
30. Jauregui, J.L., Lee, D.A.: Lower semicontinuity of mass under $C^0$ convergence and Huisken’s isoperimetric mass. Journal für die reine und angewandte Mathematik 2019(756), 227–257 (2019)
31. Cederbaum, C., Sákovich, A.: On center of mass and foliations by constant spacetime mean curvature surfaces for isolated systems in general relativity. Calc. Var. Partial Differ. Equ. 60(6), 1–57 (2021)
32. Chen, P.-N., Wang, M.-T., Yau, S.-T.: Quasilocal angular momentum and center of mass in general relativity. Adv. Theor. Math. Phys. 20, 671–682 (2016)
33. Gray, A.: Tubes. Progress in Mathematics, vol. 221. Birkhäuser, Basel (2012)
34. Arnowitt, R., Deser, S., Misner, C.W.: The dynamics of general relativity. In: Gravitation: An Introduction to Current Research (1962)
35. Christodoulou, D.: Mathematical Problems of General Relativity I, vol. 1. European Mathematical Society, Helsinki (2008)
36. Harlow, D., Wu, J.: Covariant phase space with boundaries. J. High Energy Phys. 2020(10), 1–52 (2020)
37. de Lima, L.L.: Conserved quantities in general relativity: the case of initial data sets with a non-compact boundary. To appear in “Perspectives in Scalar Curvature”, edited by M. Gromov and H.B. Lawson (2022)
38. Schoen, R., Yau, S.-T.: On the proof of the positive mass conjecture in general relativity. Commun. Math. Phys. 65(3), 45–76 (1979)
39. Fan, X.-Q., Shi, Y., Tam, L.-F.: Large-sphere and small-sphere limits of the Brown–York mass. Comm. Anal. Geom. 17(2), 37–72 (2009)
40. Lee, D.A., LeFloch, P.G.: The positive mass theorem for manifolds with distributional curvature. Commun. Math. Phys. 339(1), 99–120 (2015)
41. Michel, B.: Geometric invariance of mass-like asymptotic invariants. J. Math. Phys. 52(5), 052504 (2011)
42. Chan, P.-Y., Tam, L.-F.: A note on center of mass. Commun. Anal. Geom. 24(3), 471–486 (2016)
43. Cederbaum, C., Nerz, C.: Explicit Riemannian manifolds with unexpectedly behaving center of mass. Ann. Henri Poincaré 16(7), 1609–1631 (2015)
44. Qing, J., Tian, G.: On the uniqueness of the foliation of spheres of constant mean curvature in asymptotically flat 3-manifolds. J. Am. Math. Soc. 20(4), 1091–1110 (2007)
45. Corvino, J., Wu, H.: On the center of mass of isolated systems. Class. Quantum Gravity 25(8), 085008 (2008)
46. de Lima, L.L., Lázaro, I.C.: A Cauchy–Crofton formula and monotonicity inequalities for the Barbosa–Colares functionals. Asian J. Math. 7(1), 81–89 (2003)
47. Reilly, R.C.: Applications of the Hessian operator in a Riemannian manifold. Indiana Univ. Math. J. 26(3), 459–472 (1977)
48. Máximo, D., Nunes, I., Smith, G.: Free boundary minimal annuli in convex three-manifolds. J. Differ. Geom. 106(1), 139–186 (2017)
49. Bray, H.L.: The Penrose inequality in general relativity and volume comparison theorems involving scalar curvature. Ph.D. Thesis, Stanford University (1997)
50. Corvino, J., Gerek, A., Greenberg, M., Krummel, B.: On isoperimetric surfaces in general relativity. Pac. J. Math. 231(1), 63–84 (2007)
51. Ritoré, M., Rosales, C.: Existence and characterization of regions minimizing perimeter under a volume constraint inside Euclidean cones. Trans. Am. Math. Soc. 356(11), 4601–4622 (2004)
52. Almaraz, S., de Lima, L.L.: The mass of an asymptotically hyperbolic manifold with a non-compact boundary. Ann. Henri Poincaré 21(11), 3727–3756 (2020)
53. Rigger, R.: The foliation of asymptotically hyperbolic manifolds by surfaces of constant mean curvature (including the evolution equations and estimates). Manuscr. Math. 113(4), 403–421 (2004)
54. Neves, A., Tian, G.: Existence and uniqueness of constant mean curvature foliation of asymptotically hyperbolic 3-manifolds. Geom. Funct. Anal. 19(3), 910 (2009)
55. Neves, A., Tian, G.: Existence and uniqueness of constant mean curvature foliation of asymptotically hyperbolic 3-manifolds II. J. für die reine und angewandte Mathematik 2010(641), 69–93 (2010)
56. Mazzeo, R., Pacard, F.: Constant curvature foliations in asymptotically hyperbolic spaces. Revista Matematica Iberoamericana 27(1), 303–333 (2011)
57. Chodosh, O.: Large isoperimetric regions in asymptotically hyperbolic manifolds. Commun. Math. Phys. 343(2), 393–443 (2016)
58. Ros, A., Vergasta, E.: Stability for hypersurfaces of constant mean curvature with free boundary. Geom. Dedicata. 56(1), 19–33 (1995)
59. Ros, A., Souam, R.: On stability of capillary surfaces in a ball. Pac. J. Math. 178(2), 345–361 (1997)
60. Barbosa, J.L.M., Colares, A.G.: Stability of hypersurfaces with constant $r$-mean curvature. Ann. Glob. Anal. Geom. 15(3), 277–297 (1997)
61. Rosenberg, H.: Hypersurfaces of constant curvature in space forms. Bulletin des Sciences Mathématiques 117(2), 211–239 (1993)
62. Alías, L.J., Brasil, A., Colares, A.G.: Integral formulae for spacelike hypersurfaces in conformally stationary spacetimes and applications. Proc. Edinb. Math. Soc. 46(2), 465–488 (2003)
63. Chai, X.: Two quasi-local masses evaluated on surfaces with boundary. arXiv preprint arXiv:1811.06168 (2018)
64. Ambrozio, L.C.: Rigidity of area-minimizing free boundary surfaces in mean convex three-manifolds. J. Geom. Anal. 25(2), 1001–1017 (2015)