Explaining Constraint Programming

Krzysztof R. Apt\textsuperscript{1,2,3}

\textsuperscript{1} School of Computing, National University of Singapore
\textsuperscript{2} CWI, Amsterdam
\textsuperscript{3} University of Amsterdam, the Netherlands

Abstract. We discuss here constraint programming (CP) by using a proof-theoretic perspective. To this end we identify three levels of abstraction. Each level sheds light on the essence of CP.

In particular, the highest level allows us to bring CP closer to the computation as deduction paradigm. At the middle level we can explain various constraint propagation algorithms. Finally, at the lowest level we can address the issue of automatic generation and optimization of the constraint propagation algorithms.

1 Introduction

Constraint programming is an alternative approach to programming which consists of modelling the problem as a set of requirements (constraints) that are subsequently solved by means of general and domain specific methods.

Historically, constraint programming is an outcome of a long process that has started in the seventies, when the seminal works of Waltz and others on computer vision (see, e.g., [30]) led to identification of constraint satisfaction problems as an area of Artificial Intelligence. In this area several fundamental techniques, including constraint propagation and enhanced forms of search have been developed.

In the eighties, starting with the seminal works of Colmerauer (see, e.g., [16]) and Jaffar and Lassez (see [21]) the area constraint logic programming was founded. In the nineties a number of alternative approaches to constraint programming were realized, in particular in ILOG solver, see e.g., [20], that is based on modeling the constraint satisfaction problems in C++ using classes. Another, recent, example is the Koalog Constraint Solver, see [23], realized as a Java library.

This way constraint programming eventually emerged as a distinctive approach to programming. In this paper we try to clarify this programming style and to assess it using a proof-theoretic perspective considered at various levels of abstraction. We believe that this presentation of constraint programming allows us to more easily compare it with other programming styles and to isolate its salient features.

2 Preliminaries

Let us start by introducing the already mentioned concept of a constraint satisfaction problem. Consider a sequence $X = x_1, \ldots, x_m$ of variables with respective domains $D_1, \ldots, D_n$. By a constraint on $X$ we mean a subset of $D_1 \times \ldots \times D_m$. A constraint
satisfaction problem (CSP) consists of a finite sequence of variables $x_1, \ldots, x_n$ with respective domains $D_1, \ldots, D_n$ and a finite set $C$ of constraints, each on a subsequence of $X$. We write such a CSP as

$$\{ C \mid x_1 \in D_1, \ldots, x_n \in D_n \}.$$

A solution to a CSP is an assignment of values to its variables from their domains that satisfies all constraints. We say that a CSP is consistent if it has a solution, solved if each assignment is a solution, and failed if either a variable domain is empty or a constraint is empty. Intuitively, a failed CSP is one that obviously does not have any solution. In contrast, it is not obvious at all to verify whether a CSP is solved. So we introduce an imprecise concept of a ‘manifestly solved’ CSP which means that it is computationally straightforward to verify that the CSP is solved. So this notion depends on what we assume as ‘computationally straightforward’.

In practice the constraints are written in a first-order language. They are then atomic formulas or simple combinations of atomic formulas. One identifies then a constraint with its syntactic description. In what follows we study CSPs with finite domains.

3 High Level

At the highest level of abstraction constraint programming can be seen as a task of formulating specifications as a CSP and of solving it. The most common approach to solving a CSP is based on a top-down search combined with constraint propagation.

The top-down search is determined by a splitting strategy that controls the splitting of a given CSP into two or more CSPs, the ‘union’ of which (defined in the natural sense) is equivalent to (i.e., has the same solutions as) the initial CSP. In the most common form of splitting a variable is selected and its domain is partitioned into two or more parts. The splitting strategy then determines which variable is to be selected and how its domain is to be split.

In turn, constraint propagation transforms a given CSP into one that is equivalent but simpler, i.e., easier to solve. Each form of constraint propagation determines a notion of local consistency that in a loose sense approximates the notion of consistency and is computationally efficient to achieve. This process leads to a search tree in which constraint propagation is alternated with splitting, see Figure 1.

So the nodes in the tree are CSPs with the root (level 0) being the original CSP. At the even levels the constraint propagation is applied to the current CSP. This yields exactly one direct descendant. At the odd levels splitting is applied to the current CSP. This yields more than one descendant. The leaves of the tree are CSPs that are either failed or manifestly solved. So from the leaves of the trees it is straightforward to collect all the solutions to the original CSP.

The process of tree generation can be expressed by means of proof rules that are used to express transformations of CSPs. In general we have two types of rules. The deterministic rules transform a given CSP into another one. We write such a rule as:

$$\phi \quad \psi$$
Fig. 1. A search tree for a CSP

where \( \phi \) and \( \psi \) are CSPs.

In turn, the **splitting** rules transform a given CSP into a sequence of CSPs. We write such a rule as:

\[
\frac{\phi}{\psi_1 \mid \ldots \mid \psi_n}
\]

where \( \phi \) and \( \psi_1, \ldots, \psi_n \) are CSPs.

It is now easy to define the notion of an **application of a proof rule** to a CSP. In the case of a deterministic rule we just replace (after an appropriate renaming) the part that matches the premise of the rule by the conclusion. In the case of a splitting rule we replace (again after an appropriate renaming) the part that matches the premise of the rule by one of the CSPs \( \psi_i \) from the rule conclusion.

We now say that a deterministic rule

\[
\frac{\phi}{\psi}
\]

is **equivalence preserving** if \( \phi \) and \( \psi \) are equivalent and that a splitting rule

\[
\frac{\phi}{\psi_1 \mid \ldots \mid \psi_n}
\]

is **equivalence preserving** if the union of \( \psi_i \)'s is equivalent to \( \phi \).

In what follows all considered rules will be equivalence preserving. In general, the deterministic rules are more ‘fine grained’ than the constraint propagation step that is modeled as a single ‘step’ in the search tree. In fact, our intention is to model constraint propagation as a repeated application of deterministic rules. In the next section we shall discuss how to schedule these rule applications efficiently.
The search for solutions can now be described by means of derivations, just like in logic programming. In logic programming we have in general two types of finite derivations: successful and failed. In the case of proof rules as defined above a new type of derivations naturally arises.

**Definition 1.** Assume a finite set of proof rules.

- By a **derivation** we mean a sequence of CSPs such that each of them is obtained from the previous one by an application of a proof rule.
- A finite derivation is called
  - **successful** if its last element is a first manifestly solved CSP in this derivation,
  - **failed** if its last element is a first failed CSP in this derivation,
  - **stabilizing** if its last element is a first CSP in this derivation that is closed under the applications of the considered proof rules.

The search for a solution to a CSP can now be described as a search for a successful derivation, much like in the case of logic programming. A new element is the presence of stabilizing derivations.

One of the main problems constraint programming needs to deal with is how to limit the size of a search tree. At the high level of abstraction this matter can be addressed by focusing on the derivations in which the applications of splitting rules are postponed as long as possible. This bring us to a consideration of stabilizing derivations that involve only deterministic rules. In practice such derivations are used to model the process of constraint propagation. They do not lead to a manifestly solved CSP but only to a CSP that is closed under the considered deterministic rules. So solving the resulting CSP requires first an application of a splitting rule. (The resulting CSP can be solved but to determine it may be computationally expensive.)

This discussion shows that at a high level of abstraction constraint programming can be viewed as a realization of the computation as deduction paradigm according to which the computation process is identified with a constructive proof of a formula from a set of axioms. In the case of constraint programming such a constructive proof is a successful derivation. Each such derivation yields at least one solution to the initial CSP.

Because so far no specific rules are considered not much more can be said at this level. However, this high level of abstraction allows us to set the stage for more specific considerations that belong to the middle level.

## 4 Middle Level

The middle level is concerned with the form of derivations that involve only deterministic rules. It allows us to explain the **constraint propagation algorithms** which are used to enforce constraint propagation. In our framework these algorithms are simply efficient schedulers of appropriate deterministic rules. To clarify this point we now introduce examples of specific classes of deterministic rules. In each case we discuss a scheduler that can be used to schedule the considered rules.
**Example 1: Domain Reduction Rules**

These are rules of the following form:

\[
\langle C; x_1 \in D_1, \ldots, x_n \in D_n \rangle \\
\langle C'; x_1 \in D'_1, \ldots, x_n \in D'_n \rangle
\]

where \(D'_i \subseteq D_i\) for all \(i \in [1..n]\) and \(C'\) is the result of restricting each constraint in \(C\) to \(D'_1, \ldots, D'_n\).

We say that such a rule is **monotonic** if, when viewed as a function \(f\) from the original domains \(D_1, \ldots, D_n\) to the reduced domains \(D'_1, \ldots, D'_n\), i.e.,

\[f(D_1, \ldots, D_n) := (D'_1, \ldots, D'_n),\]

it is monotonic:

\[D_i \subseteq E_i\] for all \(i \in [1..n]\) implies \(f(D_1, \ldots, D_n) \subseteq f(E_1, \ldots, E_n)\).

That is, smaller variable domains yield smaller reduced domains.

Now, the following useful result shows that a large number of domain reduction rules are monotonic.

**Theorem 1.** (10) Suppose each \(D'_i\) is obtained from \(D_i\) using a combination of

- union and intersection operations,
- transposition and composition operations applied to binary relations,
- join operation \(\Join\),
- projection functions, and
- removal of an element.

Then the domain reduction rule is monotonic.

This repertoire of operations is sufficient to describe typical domain reduction rules considered in various constraint solvers used in constraint programming systems, including solvers for Boolean constraints, linear constraints over integers, and arithmetic constraints over reals, see, e.g., [10].

Monotonic domain reduction rules are useful for two reasons. First, we have the following observation.

**Note 1.** Assume a finite set of monotonic domain reduction rules and an initial CSP \(P\). Every stabilizing derivation starting in \(P\) yields the same outcome.

Second, monotonic domain reduction rules can be scheduled more efficiently than by means of a naive round-robin strategy. This is achieved by using a **generic iteration algorithm** which in its most general form computes the least common fixpoint of a set of functions \(F\) in an appropriate partial ordering. This has been observed in varying forms of generality in the works of [12], [28], [17] and [7]. This algorithm has the following form. We assume here a finite set of functions \(F\), each operating on a given partial ordering with the least element \(\bot\).
GENERIC ITERATION algorithm

\[ d := \bot; \]
\[ G := F; \]
\[ \text{WHILE } G \neq \emptyset \text{ DO} \]
\[ \text{choose } g \in G; \]
\[ \text{IF } d \neq g(d) \text{ THEN} \]
\[ G := G \cup \text{update}(G, g, d); \]
\[ d := g(d) \]
\[ \text{ELSE} \]
\[ G := G \setminus \{g\} \]
\[ \text{END} \]
\[ \text{END} \]

where for all \( G, g, d \)

\[ \mathbf{A} \{ f \in F - G \mid f(d) = d \land f(g(d)) \neq g(d) \} \subseteq \text{update}(G, g, d). \]

The intuition behind the assumption \( \mathbf{A} \) is that \( \text{update}(G, g, d) \) contains at least all the functions from \( F - G \) for which \( d \) is a fixpoint but \( g(d) \) is not. So at each loop iteration if \( d \neq g(d) \), such functions are added to the set \( G \). Otherwise the function \( g \) is removed from \( G \).

An obvious way to satisfy assumption \( \mathbf{A} \) is by using the following \textit{update} function:

\[ \text{update}(G, g, d) := \{ f \in F - G \mid f(d) = d \land f(g(d)) \neq g(d) \}. \]

The problem with this choice of \textit{update} is that it is expensive to compute because for each function \( f \) in \( F - G \) we would have to compute the values \( f(g(d)) \) and \( f(d) \). So in practice, we are interested in some approximations from above of this \textit{update} function that are easy to compute. We shall return to this matter in a moment.

First let us clarify the status of the above algorithm. Recall that a function \( f \) on a partial ordering \((D, \sqsubseteq)\) is called \textit{monotonic} if \( x \sqsubseteq y \) implies \( f(x) \sqsubseteq f(y) \) for all \( x, y \) and \textit{inflationary} if \( x \subseteq f(x) \) for all \( x \).

\textbf{Theorem 2.} (\cite{7}) Suppose that \((D, \sqsubseteq)\) is a finite partial ordering with the least element \( \bot \). Let \( F \) be a finite set of monotonic and inflationary functions on \( D \). Then every execution of the GENERIC ITERATION algorithm terminates and computes in \( d \) the least common fixpoint of the functions from \( F \).

In the applications we study the iterations carried out on a partial ordering that is a Cartesian product of the component partial orderings. More precisely, given \( n \) partial orderings \((D_i, \sqsubseteq_i)\), each with the least element \( \bot_i \), we assume that each considered function \( g \) is defined on a ‘partial’ Cartesian product \( D_{i_1} \times \ldots \times D_{i_l} \). Here \( i_1, \ldots, i_l \) is a subsequence of \( 1, \ldots, n \) that we call the \textit{scheme} of \( g \). Given \( d \in D_1 \times \cdots \times D_n \), where \( d := d_{i_1}, \ldots, d_{i_l} \) and a scheme \( s := i_1, \ldots, i_l \) we denote by \( d[s] \) the sequence \( d_{i_1}, \ldots, d_{i_l} \).

The corresponding instance of the above GENERIC ITERATION algorithm then takes the following form.
GENERIC ITERATION FOR COMPOUND DOMAINS algorithm

\[
d := (\bot_1, \ldots, \bot_n);
\]
\[
d' := d;
\]
\[
G := F;
\]
WHILE \( G \neq \emptyset \) DO
choose \( g \in G \);
\[
d'[s] := g(d[s]), \text{ where } s \text{ is the scheme of } g;
\]
IF \( d'[s] \neq d[s] \) THEN
\[
G := G \cup \{ f \in F \mid \text{scheme of } f \text{ includes } i \text{ such that } d[i] \neq d'[i] \};
\]
ELSE
\[
d[s] := d'[s]
\]
END
END

So this algorithm uses an update function that is straightforward to compute. It simply checks which components of \( d \) are modified and selects the functions that depend on these components. It is a standard scheduling algorithm used in most constraint programming systems.

Example 2: Arc Consistency

Arc consistency, introduced in \[24\], is the most popular notion of local consistency considered in constraint programming. Let us recall the definition.

Definition 2.

- Consider a binary constraint \( C \) on the variables \( x, y \) with the domains \( D_x \) and \( D_y \), that is \( C \subseteq D_x \times D_y \). We call \( C \) arc consistent if
  - \( \forall a \in D_x \exists b \in D_y (a, b) \in C \),
  - \( \forall b \in D_y \exists a \in D_x (a, b) \in C \).
- We call a CSP arc consistent if all its binary constraints are arc consistent.

So a binary constraint is arc consistent if every value in each domain has a support in the other domain, where we call \( b \) a support for \( a \) if the pair \((a, b)\) (or, depending on the ordering of the variables, \((b, a)\)) belongs to the constraint.

In the literature several arc consistency algorithms have been proposed. Their purpose is to transform a given CSP into one that is arc consistent without losing any solution. We shall now illustrate how the most popular arc consistency algorithm, AC-3, due to \[24\], can be explained as a specific scheduling of the appropriate domain reduction rules. First, let us define the notion of arc consistency in terms of such rules.

Assume a binary constraint \( C \) on the variables \( x, y \). We introduce the following two rules.
ARC CONSISTENCY 1

\[
\begin{align*}
(C : x \in D_x, y \in D_y) \\
(C' : x \in D'_x, y \in D'_y)
\end{align*}
\]

where \( D'_x := \{ a \in D_x | \exists b \in D_y (a, b) \in C \} \).

ARC CONSISTENCY 2

\[
\begin{align*}
(C : x \in D_x, y \in D_y) \\
(C : x \in D_x, y \in D'_y)
\end{align*}
\]

where \( D'_y := \{ b \in D_y | \exists a \in D_x (a, b) \in C \} \).

So in each rule a selected variable domain is reduced by retaining only the supported values. The following observation characterizes the notion of arc consistency in terms of the above two rules.

Note 2 (Arc Consistency). A CSP is arc consistent iff it is closed under the applications of the ARC CONSISTENCY rules 1 and 2.

So to transform a given CSP into an equivalent one that is arc consistent it suffices to repeatedly apply the above two rules for all present binary constraints. Since these rules are monotonic, we can schedule them using the GENERIC ITERATION FOR COMPOUND DOMAINS algorithm. However, in the case of the above rules an improved generic iteration algorithm can be employed that takes into account commutativity and idempotence of the considered functions, see \[8\].

Recall that given two functions \( f \) and \( g \) on a partial ordering we say that \( f \) is idempotent if \( f(f(x)) = f(x) \) for all \( x \) and say that \( f \) and \( g \) commute if \( f(g(x)) = g(f(x)) \) for all \( x \). The relevant observation concerning these two properties is the following.

Note 3. Suppose that all functions in \( F \) are idempotent and that for each function \( g \) we have a set of functions \( \text{Comm}(g) \) from \( F \) such that each element of \( \text{Comm}(g) \) commutes with \( g \). If \( \text{update}(G, g, d) \) satisfies the assumption \( \mathbf{A} \), then so does the function \( \text{update}(G, g, d) - \text{Comm}(g) \).

In practice it means that in each iteration of the generic iteration algorithm less functions need to be added to the set \( G \). This yields a more efficient algorithm.

In the case of arc consistency for each binary constraint \( C \) the functions corresponding to the ARC CONSISTENCY rules 1 and 2 referring to \( C \) commute. Also, given two binary constraints that share the first (resp. second) variable, the corresponding ARC CONSISTENCY rules 1 (resp. 2) for these two constraints commute, as well. Further, all such functions are idempotent. So, thanks to the above Note, we can use an appropriately ‘tighter’ \( \text{update} \) function. The resulting algorithm is equivalent to the AC–3 algorithm.
Example 3: Constructive Disjunction

One of the main reasons for combinatorial explosion in search for solutions to a CSP are **disjunctive constraints**. A typical example is the following constraint used in scheduling problems:

\[
\begin{align*}
\text{Start}[\text{task}_1] + \text{Duration}[\text{task}_1] & \leq \text{Start}[\text{task}_2] \lor \\
\text{Start}[\text{task}_2] + \text{Duration}[\text{task}_2] & \leq \text{Start}[\text{task}_1]
\end{align*}
\]

stating that either \text{task}_1 is scheduled before \text{task}_2 or vice versa. To deal with a disjunctive constraint we can apply the following splitting rule (we omit here the information about the variable domains):

\[
\frac{C_1 \lor C_2}{C_1 \mid C_2}
\]

which amounts to a case analysis.

However, as already explained in Section 3 it is in general preferable to postpone an application of a splitting rule and try to reduce the domains first. **Constructive disjunction**, see [29], is a technique that occasionally allows us to do this. It can be expressed in our rule-based framework as a domain reduction rule that uses some auxiliary derivations as side conditions:

\[
\text{CONSTRUCTIVE DISJUNCTION}
\]

\[
\frac{\langle C_1 \lor C_2 : x_1 \in D_1, \ldots, x_n \in D_n \rangle}{\langle C'_1 \lor C'_2 : x_1 \in D'_1 \cup D''_1, \ldots, x_n \in D'_n \cup D''_n \rangle}
\]

where \( \text{der}_1, \text{der}_2 \)

with

\[
\text{der}_1 := \langle C_1 : x_1 \in D_1, \ldots, x_n \in D_n \rangle \vdash \langle C'_1 : x_1 \in D'_1, \ldots, x_n \in D'_n \rangle,
\]

\[
\text{der}_2 := \langle C_2 : x_1 \in D_1, \ldots, x_n \in D_n \rangle \vdash \langle C'_2 : x_1 \in D'_1, \ldots, x_n \in D'_n \rangle,
\]

and where \( C'_1 \) is the result of restricting the constraint in \( C_1 \) to \( D'_1, \ldots, D'_n \) and similarly for \( C'_2 \).

In words: assuming we reduced the domains of each disjunct separately, we can reduce the domains of the disjunctive constraint to the respective unions of the reduced domains. As an example consider the constraint

\[
\langle |x - y| = 1 : x \in [4..10], y \in [2..7] \rangle.
\]

We can view \( |x - y| = 1 \) as the disjunctive constraint \( (x - y = 1) \lor (y - x = 1) \). In the presence of the **ARC CONSISTENCY** rules 1 and 2 rules we have then

\[
\langle x - y = 1 : x \in [4..10], y \in [2..7] \rangle \vdash \langle x - y = 1 : x \in [4..8], y \in [3..7] \rangle
\]

\[
\langle x - y = 1 : x \in [4..10], y \in [2..7] \rangle \vdash \langle x - y = 1 : x \in [4..8], y \in [3..7] \rangle
\]
and

\[ \langle y - x = 1 ; x \in [4..10], y \in [2..7] \rangle \vdash \langle y - x = 1 ; x \in [4..6], y \in [5..7] \rangle. \]

So using the **CONSTRUCTIVE DISJUNCTION** rule we obtain

\[ \langle |x - y| = 1 ; x \in [4..8], y \in [3..7] \rangle. \]

If each disjunct of a disjunctive constraint is a conjunction of constraints, the auxiliary derivations in the side conditions can be longer than just one step. Once the rules used in these derivations are of an appropriate format, their applications can be scheduled using one of the discussed generic iteration algorithms. Then the single application of the **CONSTRUCTIVE DISJUNCTION** rule consists in fact of two applications of the appropriate iteration algorithm.

It is straightforward to check that if the auxiliary derivations involve only monotonic domain reduction rules, then the **CONSTRUCTIVE DISJUNCTION** rule is itself monotonic. So the **GENERIC ITERATION FOR COMPOUND DOMAINS** algorithm can be applied both within the side conditions of this rule and for scheduling this rule together with other monotonic domain reduction rules that are used to deal with other, non-disjunctive, constraints.

In this framework it is straightforward to formulate some strengthenings of the constructive disjunction that lead to other modification of the constraints \( C_1 \) and \( C_2 \) than \( C'_1 \) and \( C'_2 \).

**Example 4: Propagation Rules**

These are rules that allow us to add new constraints. Assuming a given set \( A \) of ‘allowed’ constraints we write such rules as

\[
\begin{array}{c}
B \\ \hline
C
\end{array}
\]

where \( B, C \subseteq A \).

This rule states that in presence of all constraints in \( B \) the constraints in \( C \) can be added, and is a shorthand for a deterministic rule of the following form:

\[
\langle B ; x_1 \in D_1, \ldots, x_n \in D_n \rangle \\
\langle B, C ; x_1 \in D_1, \ldots, x_n \in D_n \rangle
\]

An example of such a rule is the transitivity rule:

\[
\frac{x < y, y < z}{x < z}
\]
that refers to a linear ordering < on the underlying domain (for example natural numbers).

In what follows we focus on another example of propagation rules, membership rules. They have the following form:

\[
\frac{y_1 \in S_1, \ldots, y_k \in S_k}{z_1 \neq a_1, \ldots, z_m \neq a_m}
\]

where \(y_i \in S_i\) and \(z_j \neq a_j\) are unary constraints with the obvious meaning.

Below we write such a rule as:

\[
y_1 \in S_1, \ldots, y_k \in S_k \rightarrow z_1 \neq a_1, \ldots, z_m \neq a_m.
\]

The intuitive meaning of this rule is: if for all \(i \in [1..k]\) the domain of each \(y_i\) is a subset of \(S_i\), then for all \(j \in [1..m]\) remove the element \(a_j\) from the domain of \(z_j\).

The membership rules allow us to reason about constraints given explicitly in a form of a table. As an example consider the three valued logic of Kleene. Let us focus on the conjunction constraint \(\text{and3}(x, y, z)\) defined by the following table:

| t | f | u |
|---|---|---|
| t | f | t |
| f | f | f |
| u | f | u |

That is, \(\text{and3}\) consists of 9 triples. Then the membership rule \(y \in \{u, f\} \rightarrow z \neq t\), or more precisely the rule

\[
\frac{\langle \text{and3}(x, y, z), y \in \{u, f\} \rangle; x \in D_x, y \in D_y, z \in D_z}{\langle \text{and3}(x, y, z), y \in \{u, f\}, z \neq t \rangle; x \in D_x, y \in D_y, z \in D_z}
\]

is equivalence preserving. This rule states that if \(y\) is either \(u\) or \(f\), then \(t\) can be removed from the domain of \(z\).

We call a membership rule is minimal if it is equivalence preserving and its conclusions cannot be established by either removing from its premise a variable or by expanding a variable range. For example, the above rule \(y \in \{u, f\} \rightarrow z \neq t\) is minimal, while neither \(x \in \{u\}, y \in \{u, f\} \rightarrow z \neq t\) nor \(y \in \{u\} \rightarrow z \neq t\) is. In the case of the \(\text{and3}\) constraint there are 18 minimal membership rules.

To clarify the nature of the membership rules let us mention that, as shown in [9], in the case of two-valued logic the corresponding set of minimal membership rules entails a form of constraint propagation that is equivalent to the unit propagation, a well-known form of resolution for propositional logic. So the membership rules can be seen as a generalization of the unit propagation to the explicitly given constraints, in particular to the case of many valued logics.

Membership rules can be alternatively viewed as a special class of monotonic domain reductions rules in which the domain of each \(z_i\) variable is modified by removing
\( a \) from it. So we can schedule these rules using the Generic Iteration for Compound Domains algorithm.

However, the propagation rules, so in particular the membership rules, satisfy an important property that allows us to schedule them using a more efficient, fine-tuned scheduler. We call this property stability. It states that in each derivation the rule needs to be applied at most once: if it is applied, then it does not need to be applied again. So during the computation the applied rules that are stable can be permanently removed from the initial rule set. The resulting scheduler for the membership rules and its further optimizations are discussed in [14].

5 Low Level

The low level allows us to focus on matters that go beyond the issue of rule scheduling. At this level we can address matters concerned with further optimization of the constraint propagation algorithms. Various improvements of the \( AC-3 \) algorithm that are concerned with specific choices of the data structures used belong here but cannot be explained by focusing the discussion on the corresponding \( ARC \) CONSISTENCY 1 and 2 rules.

On the other hand some other optimization issues can be explained in proof-theoretic terms. In what follows we focus on the membership rules for which we worked out the details. These rules allow us to implement constraint propagation for explicitly given constraints. We explained above that they can be scheduled using a fine-tuned scheduler. However, even when an explicitly given constraint is small, the number of minimal membership rules can be large and it is not easy to find them all.

So a need arises to generate such rules automatically. This is what we did in [11]. We also proved there that the resulting form of constraint propagation is equivalent to hyper-arc consistency, a natural generalization of arc consistency to \( n \)-ary constraints introduced in [25].

A further improvement can be achieved by removing some rules before scheduling them. This idea was pursued in [14]. Given a set of monotonic domain reduction rules \( R \) we say that a rule \( r \) is redundant if for each initial CSP \( \mathcal{P} \) the unique outcome of a stabilizing derivation (guaranteed by Note 1) is the same with \( r \) removed from \( R \). In general, the iterated removal of redundant rules does not yield a unique outcome but in the case of the membership rules some useful heuristics can be used to appropriately schedule the candidate rules for removal.

We can summarize the improvements concerned with the membership rules as follows:

- For explicitly given constraints all minimal membership rules can be automatically generated.
- Subsequently redundant rules can be removed.
- A fine-tuned scheduler can be used to schedule the remaining rules.
- This scheduler allows us to remove permanently some rules which is useful during the top-down search.
To illustrate these matters consider the 11-valued and11 constraint used in the automatic test pattern generation (ATPG) systems. There are in total 4656 minimal membership rules. After removing the redundant rules only 393 remain. This leads to substantial gains in computing. To give an idea of the scale of the improvement here are the computation times in seconds for three schedulers used to find all solutions to a CSP consisting of the and11 constraint and solved using a random variable selection, domain ordering and domain splitting:

| Rules                | Fine-tuned | Generic | CHR |
|----------------------|------------|---------|-----|
| all rules            | 1874       | 3321    | 7615|
| non-redundant rules  | 157        | 316     | 543 |

CHR stands for the standard CHR scheduler normally used to schedule such rules. (CHR is a high-level language extension of logic programming used to write user-defined constraints, for an overview see [18].) So using this approach a 50 fold improvement in computation time was achieved. In general, we noted that the larger the constraint the larger the gain in computing achieved by the above approach.

6 Conclusions

In this paper we assessed the crucial features of constraint programming (CP) by means of a proof-theoretic perspective. To this end we identified three levels of abstraction. At each level proof rules and derivations played a crucial role. At the highest level they allowed us to clarify the relation between CP and the computation as deduction paradigm. At the middle level we discussed efficient schedulers for specific classes of rules. Finally, at the lowest level we explained how specific rules can be automatically generated, optimized and scheduled in a customized way.

This presentation of CP suggests that it has close links with the rule-based programming. And indeed, several realizations of constraint programming through some form of rule-based programming exist. For example, constraint logic programs are sets of rules, so constraint logic programming can be naturally seen as an instance of rule-based programming. Further, the already mentioned CHR language is a rule-based language, though it does not have the full capabilities of constraint programming. In practise, CHR is available as a library of a constraint programming system, for example ECL’PS (see [11]) or SICStus Prolog (see [3]). In turn, ELAN, see [2], is a rule-based programming language that can be naturally used to explain various aspects of constraint programming, see for example [22] and [15].

In our presentation we abstracted from specific constraint programming languages and their realizations and analyzed instead the principles of the corresponding programming style. This allowed us to isolate the essential features of constraint programming by focusing on proof rules, derivations and schedulers. This account of constraint programming draws on our work on the subject carried out in the past seven years. In particular, the high level view was introduced in [6]. In turn, the middle level summa-
rizes our work reported in [7,8]. Both levels are discussed in more detail in [10]. Finally, the account of propagation rules and of low level draws on [11,14].

This work was pursued by others. Here are some representative references. Concerning the middle level, [26] showed that the framework of Section 4 allows us to parallelize constraint propagation algorithms in a simple and uniform way, while [13] showed how to use it to derive constraint propagation algorithms for soft constraints. In turn, [19] explained other arc consistency algorithms by slightly extending this framework.

Concerning the lowest level, [27] considered rules in which parameters (i.e., unspecified constants) are allowed. This led to a decrease in the number of generated rules. In turn, [4] presented an algorithm that generates more general and more expressive rules, for example with variable equalities in the conclusion. Finally, [5] considered the problem of generating the rules for constraints defined intensionally over infinite domains.

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