Non-integrability criterion for homogeneous Hamiltonian systems via blowing-up technique of singularities

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Abstract

It is a big problem to distinguish between integrable and non-integrable Hamiltonian systems. We provide a new approach to prove the non-integrability of homogeneous Hamiltonian systems with two degrees of freedom. The homogeneous degree can be chosen from real values (not necessarily integer). The proof is based on the blowing-up theory which McGehee established in the collinear three-body problem. We also compare our result with Molares-Ramis theory which is the strongest theory in this field.

1 INTRODUCTION

Let $H : \mathcal{D} \to \mathbb{R}$ be a smooth function where $\mathcal{D}$ is an open set in $\mathbb{R}^{2k}$. The Hamiltonian system is defined by the ordinary differential equations

$$
\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}(p, q), \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j}(p, q) \quad (j = 1, \ldots, k)
$$

(1)

where $(p, q) = (p_1, \ldots, p_k, q_1, \ldots, q_k) \in \mathcal{D}$. The function $H$ is called the Hamiltonian and $k$ is called the degrees of freedom.

A function $F : \mathcal{D} \to \mathbb{R}$ is called the first integral of (1) if $F$ is conserved along each solution of (1). For two functions $F, G : \mathcal{D} \to \mathbb{R}$, the Poisson bracket is the function defined by

$$
\{F, G\} = \sum_{j=1}^{k} \frac{\partial F}{\partial q_j} \frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j} \frac{\partial G}{\partial q_j}.
$$

A function $F : \mathcal{D} \to \mathbb{R}$ is a first integral of (1) if and only if $\{F, H\}$ is identically zero. Hamiltonian system (1) is called integrable if there are $k$ first integrals $F_1(= H), F_2, \ldots, F_k$ such that $dF_1, \ldots, dF_k$ are linearly independent in an open dense set of $\mathcal{D}$ and that $\{F_i, F_j\} = 0$ for any $i, j = 1, \ldots, k$. 

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The dynamics of the integrable systems are well understood because of the Liouville-Arnold theorem (see [1], Chapter 10) while the dynamics of the non-integrable Hamiltonian systems may be chaotic. Therefore it is important to distinguish between integrable and non-integrable Hamiltonian systems.

This problem have been studied for quite long time. Bruns [2] proved that in the 3-body problem there is no algebraic first integral which is independent from the known ones. After that, Poincaré [4] proved that the perturbed Hamiltonian systems there is no analytic first integral depending analytically on a parameter. Then by applying it to the restricted 3-body problem, he proved the non-existence of an analytic first integral depending analytically on a mass parameter.

Another theory in this field was originated by Kovalevskaya [3]. By studying the property of singularities she discovered a new integrable case in the rigid body model. As a development of her approach, Ziglin [5, 6] established the theory of singularity for proving the non-integrability. By applying the Ziglin analysis, Yoshida [7] provided a criterion for the non-integrability of the homogeneous Hamiltonian systems. Morales-Ruiz & Ramis [8, 9] extended the Ziglin analysis by applying the Differential Galois theory (Picard-Vessiot theory). The Morales-Ramis theory is the strongest in this field now.

Our purpose is to prove the non-integrability of Hamiltonian systems from a new approach. We consider a Hamiltonian system of 2 degrees of freedom with a homogeneous potential of degree $\beta \in \mathbb{R}$. Its Hamiltonian is represented by

$$H(p, q) = \frac{1}{2} \|p\|^2 + U(q) \quad ((p, q) \in \mathbb{R}^2 \times (\mathbb{R}^2 \setminus \{0\})).$$ (2)

Here $U$ is a real-meromorphic function on $\mathbb{R}^2 \setminus \{0\}$ and satisfies the homogeneous property:

$$U(\lambda q) = \lambda^\beta U(q) \quad (q \in \mathbb{R}^2 \setminus \{0\}, \lambda > 0).$$

Let $V(\theta) = U(\cos \theta, \sin \theta)$.

**Theorem 1.** Assume the following 6 properties:

1. the homogeneous degree $\beta$ is a real number excluding $-2$ and $0$:

$$\beta \in \mathbb{R}\setminus\{-2, 0\};$$

2. there are three critical points $\theta_l$ of $V$:

$$\frac{\partial V}{\partial \theta}(\theta_l) = 0, \quad \theta_{-1} < \theta_0 < \theta_1, \quad (l = -1, 0, 1);$$

3. the function $V$ is negative between $\theta_{-1}$ and $\theta_1$:

$$V(\theta) < 0 \quad (\theta \in [\theta_{-1}, \theta_1]);$$

4. the derivative of $V$ does not vanish between these critical points:

$$\frac{\partial V}{\partial \theta}(\theta) \neq 0 \quad (\theta \in (\theta_{-1}, \theta_0) \cup (\theta_0, \theta_1));$$
5. the second derivative of $V$ is negative at critical points $\theta_{\pm 1}$:

$$\frac{\partial^2 V}{\partial \theta^2}(\theta_{\pm 1}) < 0;$$

6. at critical point $\theta_0$, the following inequality satisfies:

$$-\frac{1}{8}(\beta + 2)^2V(\theta_0) < \frac{\partial^2 V}{\partial \theta^2}(\theta_0).$$

Then the Hamiltonian system of (2) has no real-meromorphic first integral independent from $H$.

![Figure 1: Function $V(\theta)$](image)

Above we used the word “real-meromorphic”. We call a real function $f(p, q)$ real-meromorphic if and only if $f(p, q)$ is analytic in all but possibly a discrete subset of $\mathbb{R}^2 \times (\mathbb{R}^2 \setminus \{0\})$ and these exceptional points must be poles.

**Remark 1.** The case of $\theta_1 = \theta_{-1} + 2\pi$ is allowed in assumption 2. These two critical points are essentially identical. In this case, just two critical points of $V$ are necessary.

**Remark 2.** In the case of $\beta = -2$, the Hamiltonian system is integrable. Because a function

$$G(p, q) = (q \cdot p)^2 - 2\|q\|^2H(p, q)$$

is a first integral. Hence this case does not need to be studied.

**Remark 3.** In the case of $V(\theta) > 0$ on $[\theta_{-1}, \theta_1]$, if $V$ is analytic in the complex domain $\mathbb{C}^2 \setminus \{(0, 0)\}$, $V$ can be replaced by changing coordinates with $(P, Q) = (\sqrt{-1}p, \sqrt{-1}q)$, and then the new equations satisfy the assumption 2 of this theorem.

If $V$ is a constant, the system is integrable. Hence we need to consider the non-constant functions. Generically there are several critical points of $V$ and the graph is convex at some of them. The assumption 1-5 of this theorem is not strong, and only assumption 6 is a little strong.
This paper is organized as follows. In Section 2 we introduce the McGehee’s blowing-up technique for the homogeneous Hamiltonian systems. We prove our theorem in Section 3 by using the McGehee’s technique. We present two applications of the theorem in Section 4. In the final section we compare our theorem with the Morales-Ramis theorem.

2 MCGEHEE’S BLOWING UP TECHNIQUE

McGehee [10] established a blowing-up technique for the triple collision singularity in the collinear three-body problem. We can easily extend the technique for the general homogeneous Hamiltonian systems [2].

We first consider the case of $\beta < 0$. The McGehee coordinates $(r, \theta, v, w)$ are defined by

$$q = r(\cos \theta, \sin \theta),\quad p = r^{\beta/2}(v(\cos \theta, \sin \theta) + w(-\sin \theta, \cos \theta))$$

and the time variable $t$ is changed into $\tau$ according to $dt = r^{1-\beta/2}d\tau$. The map $(r, \theta, v, w) \mapsto (p, q)$ are analytic in \{(r, \theta, v, w) \mid r > 0, \theta \in \mathbb{R}/2\pi\mathbb{Z}, v, w \in \mathbb{R}\}. Then the equations become

$$\frac{dr}{d\tau} = rv$$

$$\frac{d\theta}{d\tau} = w$$

$$\frac{dv}{d\tau} = -\frac{\beta}{2}v^2 + w^2 - \beta V(\theta)$$

$$\frac{dw}{d\tau} = -\left(\frac{\beta}{2} + 1\right)vw - \frac{\partial V}{\partial \theta}(\theta).$$

In these coordinates the total energy is

$$h = r^\beta \left(\frac{v^2 + w^2}{2} + V(\theta)\right).$$

Fix the energy constant at any non-zero value ($h \neq 0$).

The point $q = 0$ is singularity of the differential equations, but $r = 0$ is not singular in these differential equations [3]-[7]. It is sufficient to consider the three equations [1], [3] and [5], since these equations are independent from $r$ and since $r$ can be obtained from [7].

The set

$$\mathcal{M} = \left\{(\theta, v, w) \in \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R} \times \mathbb{R} \mid \frac{v^2 + w^2}{2} + V(\theta) = 0\right\}$$

is invariant. In the case of the $n$-body problem, $\mathcal{M}$ is called the collision manifold. Orbits converge to $\mathcal{M}$ as $r \to 0$. 
In the case that \( \beta > 0 \), we can discuss similar argument by letting \( R = r^{-1} \).
The equation (3) becomes
\[
\dot{R} = -Rv
\]  
and the total energy is
\[
H = R^{-\beta} \left( \frac{v^2 + w^2}{2} + V(\theta) \right).
\]  
The equations can be extended to \( R = 0 \). Orbits converge to the invariant set \( \mathcal{M} \) as \( R \to 0 \). It is sufficient to consider the three equations (4), (5) and (6).

The flow on \( \mathcal{M} \) is gradient-like if \( \beta \neq -2 \). This means that the \( v \)-component is monotone along each solution excluding equilibrium points since all orbits on \( \mathcal{M} \) satisfy
\[
\frac{dv}{dt} = \left( \frac{\beta}{2} + 1 \right) w^2 \begin{cases} 
\geq 0 & (\beta > -2) \\
\leq 0 & (\beta < -2)
\end{cases}
\]

If \( \theta_c \) is a critical point of \( V \), i.e. \( \frac{\partial V}{\partial \theta}(\theta_c) = 0 \), \((\theta, v, w) = (\theta_c, \pm \sqrt{-2V(\theta_c)}, 0)\) are equilibrium points of (4), (5), (6). The linearized equations of (4), (5), (6) at \((\theta, v, w) = (\theta_c, \pm \sqrt{-2V(\theta_c)}, 0)\) are
\[
\frac{d}{dt} \begin{pmatrix} \delta \theta \\ \delta v \\ \delta w \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mp \beta \sqrt{-2V(\theta_c)} & 0 \\ -\frac{\partial^2 V}{\partial \theta^2}(\theta_c) & 0 & \mp \left( \frac{\beta}{2} + 1 \right) \sqrt{-2V(\theta_c)} \end{pmatrix} \begin{pmatrix} \delta \theta \\ \delta v \\ \delta w \end{pmatrix}.
\]
The eigenvalues of the coefficient matrix are \( \lambda_1 = \mp \beta \sqrt{-2V(\theta_c)} \), \( \lambda_2 \) and \( \lambda_3 \) where \( \lambda_2 \) and \( \lambda_3 \) are the roots of equation
\[
\lambda^2 \pm \left( \frac{\beta}{2} + 1 \right) \sqrt{-2V(\theta_c)} \lambda + \frac{\partial^2 V}{\partial \theta^2}(\theta_c) = 0.
\]
The eigenspace corresponding to \( \lambda_1 \) is perpendicular to \( \mathcal{M} \) at the equilibrium point and the eigenspace corresponding to \( \lambda_2 \) and \( \lambda_3 \) is tangent to \( \mathcal{M} \).

### 3 PROOF OF THEOREM 1

Assume that \( \Phi(p, q) \) is a real-meromorphic first integral of (2). From the homogeneous property if \((p(t), q(t))\) is a solution, so is \((c^\beta p(c^{\beta - 2} t), c^\beta q(c^{\beta - 2} t))\) for any constant \( c > 0 \). Then \( \Phi(c^\beta p, c^\beta q) \) is also an first integral.

The point \((p, q) = (0, 0)\) may be an essential singularity of \( \Phi \). Consider the Laurent series at this point:
\[
\Phi(p, q) = \sum_{k_1, k_2, k_3, k_4 \in \mathbb{Z}} a_{k_1 k_2 k_3 k_4} p_1^{k_1} p_2^{k_2} q_1^{k_3} q_2^{k_4}.
\]

Then we get
\[
\Phi(c^\beta p, c^\beta q) = \sum_{k_1, k_2, k_3, k_4 \in \mathbb{Z}} a_{k_1 k_2 k_3 k_4} c^{\beta(k_1 + k_2) + 2(k_3 + k_4)} p_1^{k_1} p_2^{k_2} q_1^{k_3} q_2^{k_4}.
\]
We gather the terms according to the power of $c$

$$\Phi(c^\beta p, c^2 q) = \sum_{\omega \in \Omega} c^\omega f_\omega(p, q) \quad (10)$$

where

$$\Omega = \{ \beta(k_1 + k_2) + 2(k_3 + k_4) \mid k_j \in \mathbb{Z}, a_{k_1k_2k_3k_4} \neq 0 \}$$

and

$$f_\omega(p, q) = \sum_{\beta(k_1 + k_2) + 2(k_3 + k_4) = \omega} a_{k_1k_2k_3k_4} p_1^{k_1} p_2^{k_2} q_1^{k_3} q_2^{k_4}.$$ 

By substituting $bc$ for $c$ of $(10)$, we get

$$\Phi(b^\beta c^\beta p, b^2 c^2 q) = \sum_{\omega \in \Omega} b^\omega c^\omega f_\omega(p, q), \quad (11)$$

and by substituting $c$, $p$ and $q$ for $b$, $c^\beta p$, $c^2 q$ of $(10)$, we get

$$\Phi(b^\beta c^\beta p, b^2 c^2 q) = \sum_{\omega \in \Omega} b^\omega c^\omega f_\omega(p, q) \quad (12).$$

These equations $(11)$ and $(12)$ deduce

$$\sum_{\omega \in \Omega} b^\omega f_\omega(c^\beta p, c^2 q) = \sum_{\omega \in \Omega} b^\omega c^\omega f_\omega(p, q).$$

Therefore we get

$$f_\omega(c^\beta p, c^2 q) = c^\omega f_\omega(p, q).$$

Moreover since

$$\frac{d}{dt} \Phi(c^\beta p, c^2 q) = \sum_{\omega \in \Omega} c^\omega \frac{d}{dt} f_\omega(p, q) = 0$$

for any $c$, each $f_\omega(p, q)$ is a first integral.

Therefore we can assume that the first integral $\Phi$ satisfies

$$\Phi(c^\beta p, c^2 q) = c^\rho \Phi(p, q) \quad (13)$$

for some constant $\rho$.

From here we focus the case of $-2 < \beta < 0$. Let

$$\Psi(r, \theta, v, w) = \Phi(r^{-\beta/2}(v \cos \theta - w \sin \theta), r^{-\beta/2}(v \sin \theta + w \cos \theta), r \cos \theta, r \sin \theta).$$

From the property $(13)$, $\Psi$ can be written by

$$\Psi(r, \theta, v, w) = r^{\rho/2} \Psi(1, \theta, v, w).$$

The function $\Psi(1, \theta, v, w)$ is real-meromorphic of $(\theta, v, w)$. Note that we do not need analyticity at $r = 0$ because of $r = 1.$
We denote the equilibrium points by
\[ D_l^\pm = (\theta_l, \pm \sqrt{-2V(\theta_l)}, 0) \quad (l = -1, 0, 1). \]
We also use local coordinates \((\theta, w, z)\) near \(D_l^-\) where
\[ z = \frac{v^2 + w^2}{2} + V(\theta). \]
The transformation \(\{(\theta, v, w) \mid \theta \in \mathbb{R}/2\pi\mathbb{Z}, v < 0, w \in \mathbb{R}\} \rightarrow \{(\theta, z, w) \mid \theta \in \mathbb{R}/2\pi\mathbb{Z}, z \geq \frac{v^2}{2} + V(\theta)\}\) is real-analytic. The surface \(\mathcal{M}\) corresponds to the plane \(z = 0\). In these coordinates, the energy is represented by
\[ h = r^\beta z. \]
Define a function \(g\) on a neighborhood by
\[ g(\theta, z, w) = \Psi(1, \theta, -\sqrt{2z - w^2 - 2V(\theta)}, w) \]
which is real-meromorphic where the coordinates work. Because \(\Psi\) is real-meromorphic, we can consider the Laurent series of \(g\) at \(z = 0\) with respect to \(z\):
\[ g = \sum_{k=\nu}^\infty \gamma_k(\theta, w)z^k \]
where \(\nu\) is an integer and \(\gamma_\nu(\theta, w)\) is not identically zero. Hence the first integral is represented by
\[ \Psi\left((\frac{h}{2})^{\frac{\beta}{2}}, \theta, -\sqrt{2z - w^2 - 2V(\theta)}, w\right) = \left(\frac{h}{2}\right)^{\frac{\beta}{2}} \sum_{k=\nu}^\infty \gamma_k(\theta, w)z^k = : \Xi(\theta, w, z). \]
If \(\Phi\) depends only on \(H\), \(\Xi\) is a constant function. From here the proof varies according to \(\nu - \frac{\beta}{23} < 0\).

**The case of** \(\nu - \frac{\beta}{23} < 0\). Take any \(P \in W^s(D_0^-) \setminus \mathcal{M}\) near \(D_0^-\). Let \(a = \Xi(P)\). We take a small neighborhood of \(P\)
\[ B_\varepsilon = \{Q \in \mathbb{R}^3 \mid |P - Q| < \varepsilon\} \quad (0 < \varepsilon \ll 1), \]
such that for any \(Q \in B_\varepsilon\),
\[ a - 1 \leq \Xi(Q) \leq a + 1 \quad (14) \]
is satisfied. Let \(\varphi_\tau(\theta, z, w)\) be the flow of the differential equations. Since the first integral is conserved along each orbit, (14) holds in
\[ N_\varepsilon = \{\varphi_\tau(Q) \mid \tau \geq 0, Q \in B_\varepsilon\}. \]
From the continuity, (14) also holds its closure \(\overline{N_\varepsilon}\). This set \(\overline{N_\varepsilon}\) includes the unstable manifold \(W^u(D_0^-)\) of \(D_0^-\), and \(W^u(D_0^-)\) is an open set of \(\mathcal{M}\). The \(z\)-component converges to zero as \(Q\) goes close to \(\mathcal{M}\). Hence \(\gamma_\nu\) must be zero on \(W^u(D_0^-)\). From the analyticity, \(\gamma_\nu\) is identically zero. This contradicts the assumption for \(\gamma_\nu\).
Figure 2: A solution converging to $D_0^-$

Figure 3: The stable manifold of $D_1^-$
The case of $\nu - \frac{\rho^2}{\beta} > 0$. Consider the case of $V(\theta_1) \leq V(\theta_{-1})$. The other case is essentially same. Take any $Q \in W^s(D^-_1) \backslash \mathcal{M}$. The first integral has a value $c$ along the orbit passing $Q$:

$$\Xi(\varphi_\tau(Q)) = c \quad (\tau \in \mathbb{R}).$$

The $z$-component of $\varphi_\tau(Q)$ converges to 0 as $\tau$ diverges to infinity, then $c$ must be 0. Therefore $\Xi(Q)$ is zero for all $Q \in W^s(D^-_1) \backslash \mathcal{M}$. The closure of $W^s(D^-_1) \backslash \mathcal{M}$ includes $W^s(D^-_1)$. Because of the continuity, $\Xi(Q)$ is zero on $W^s(D^-_1)$. We can write the function $\Xi$ as

$$\Xi(\theta, w, z) = \left(\frac{h}{z}\right)\nu^\nu \sum_{k=0}^{\infty} \gamma_{k+\nu}(\theta, w)z^k.$$ 

Therefore

$$\sum_{k=0}^{\infty} \gamma_{k+\nu}(\theta, w)z^k = \gamma_\nu(\theta, w) + \gamma_{\nu+1}(\theta, w)z + \cdots = 0$$

satisfies on $W^s(D^-_1) \backslash \mathcal{M}$. From the continuity, $\gamma_\nu = 0$ on $W^s(D^-_1) \cap \mathcal{M}$.

Since $\frac{\partial^3 V}{\partial \theta^3}(\theta_1) < 0$, the equilibrium point $D^-_1$ is hyperbolic and $\lambda_2 \lambda_3 < 0$. Hence there are stable and unstable manifolds with dimension 1 on $\mathcal{M}$. The dynamics near the equilibrium point $D^-_0$ is stable focus and the flow on $\mathcal{M}$ is gradient-like with respect to the $v$-component. Hence $W^u(D^-_1)$ twins around $D^-_0$ and $\Xi$ is equal to zero on the spiral curve. $\gamma_\nu$ is also zero there. Therefore from analyticity $\gamma_\nu(\theta, w) \equiv 0$. This is a contradiction.

The case of $\nu - \frac{\rho^2}{\beta} = 0$. In this case $\gamma_\nu$ is a first integral for the flow on $\mathcal{M}$. From the similar argument as the previous case, $\gamma_\nu$ is a constant $c$. $\Xi - c$ is also a first integral. If $\Xi - c$ is not identically zero, $\Xi - c$ has zero point of finite degree at $z = 0$. This is reduced to the case of $\nu - \frac{\rho^2}{\beta} > 0$. This completes the proof for $-2 < \beta < 0$.

The proof for the other $\beta$ is essentially same. We survey the cases.

Consider the case of $\beta < -2$.

The case of $\nu - \frac{\rho^2}{\beta} < 0$. $\gamma_\nu$ must be zero $W^u(D^-_1)$. One branch of $W^u(D^-_{\pm 1})$ twins around $D^-_0$.

The case of $\nu - \frac{\rho^2}{\beta} > 0$. $\gamma_\nu$ must be zero $W^s(D^-_1)$. Since $W^s(D^-_0)$ is an open set of $\mathcal{M}$, $\gamma_\nu$ must be a zero function.

The case of $\nu - \frac{\rho^2}{\beta} = 0$. $\gamma_\nu$ must be constant $W^{s/u}(D^-_1)$. If $\Xi$ is not constant function, this case can be reduced to the case of $\nu - \frac{\rho^2}{\beta} > 0$.

Finally consider the case of $\beta > 0$. 

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The case of $\nu + \frac{\rho}{2\alpha} < 0 \gamma_\nu$ must be zero $W^s(D^-_0)$. One branch of $W^s(D_{\pm 1}^-)$ twins around $D^-_0$.

The case of $\nu + \frac{\rho}{2\alpha} > 0 \gamma_\nu$ must be zero $W^u(D^-_0)$. Since $W^u(D^-_0)$ is an open set of $\mathcal{M}$. $\gamma_\nu$ must be a zero function.

The case of $\nu + \frac{\rho}{2\alpha} = 0 \gamma_\nu$ must be constant $W^{s/u}(D^-_0)$. If $\Xi$ is not constant function, this case can be reduced to the case of $\nu + \frac{\rho}{2\alpha} > 0$.

4 APPLICATION

The Isosceles Three-Body Problem In the planar isosceles three-body problem, we can take the centre of gravity as the origin and the symmetric axis as the $y$-axis, and assume that the equal masses are located at

$$(x, y) \quad \text{and} \quad (-x, y)$$

and the other mass $m_3$ is located at

$$(0, -2\alpha^{-1}y)$$

in the inertial coordinate system, where $\alpha = m_3/m$(Figure 4).

![Figure 4: The planar isosceles three-body problem](image)

By rescaling it, the Hamiltonian is represented by

$$H(p, q) = \frac{1}{2}(p_1^2 + p_2^2) - \frac{1}{q_1} - \frac{4\alpha^{3/2}}{\sqrt{\alpha q_1^2 + (\alpha + 2)q_2^2}}.$$

By applying Theorem 1 we obtain:
Theorem 2. If \( \alpha < \frac{55}{4} \), the isosceles three-body problem has no real-meromorphic first integral independent from \( H \).

In fact, it is known that the dynamics is complex in the case of \( \alpha < \frac{55}{4} \). For example there are infinitely many heteroclinic orbits\(^{11, 12}\).

Yoshida’s Example  Consider the Hamiltonian

\[
H(p, q) = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{4}(q_1^4 + q_2^4) + \frac{\varepsilon}{2}q_1^2q_2^2,
\]

(15)

which was written on Yoshida’s paper\(^\text{[13]}\). As we stated at Remark\(^\text{3}\) we can consider the Hamiltonian

\[
G(p, q) = \frac{1}{2}(p_1^2 + p_2^2) - \frac{1}{4}(q_1^4 + q_2^4) - \frac{\varepsilon}{2}q_1^2q_2^2
\]

(16)

instead of \( H \). By applying Theorem\(^\text{1}\) we obtain:

Theorem 3. If \( \varepsilon < -\frac{1}{8} \) or \( \varepsilon > \frac{25}{7} \), the Hamiltonian system\(^\text{[16]}\) has no real-meromorphic first integral independent from \( G \).

From Theorem\(^\text{3}\) and Remark\(^\text{3}\) we obtain:

Theorem 4. If \( \varepsilon < -\frac{1}{8} \) or \( \varepsilon > \frac{25}{7} \), the Hamiltonian system\(^\text{[15]}\) has no meromorphic first integral independent from \( H \).

5 COMPARISON WITH THE MORALES-RAMIS THEORY

We call a configuration \( c \in \mathbb{R}^2 \) the Darboux point of \( U \) if \( \nabla U(c) = c \). Consider the Hessian matrix of \( U \) at \( c \) and call its eigenvalues Yoshida coefficients at \( c \). Since \( U \) is homogeneous with degree \( \beta \), we can easily show that one of Yoshida coefficients is \( \beta - 1 \). As computed by Sansaturio et al\(^\text{[14]}\), the other (non-trivial) Yoshida coefficient is represented by

\[
\lambda = \beta^{-1}V(\theta_c)^{-1}\frac{\partial^2 V}{\partial \theta^2}(\theta_c) + 1
\]

in the polar coordinates where \( \frac{\partial V}{\partial \varphi}(\theta_c) = 0 \).

In our theorem the assumption 6 can be written as

\[
-\frac{1}{8}(\beta + 2)^2 > (\lambda - 1)\beta,
\]

by using \( \lambda \). Then, in other words, if an integrable Hamiltonian system satisfies the assumption 1-5, the Yoshida coefficients at each Darboux point satisfy

\[
-\frac{1}{8}(\beta + 2)^2 \leq (\lambda - 1)\beta.
\]

(17)
The Morales-Ramis theorem gave a list of the Yoshida coefficient which integral systems can have. We have compared the inequality (17) and the Morales-Ramis’ list. The integrable list given by Morales-Ramis is included in our region (17) for $\beta \in \mathbb{Z}\setminus\{\pm 2, 0\}$. For example, in the case of $\beta = -1$, from the Moreles-Ramis theorem, the Yoshida coefficient of an integrable system must be in

$$\left\{-\frac{1}{2}p(p-3) \mid p \in \mathbb{Z}\right\} = \{1, 0, -2, -5, -9, \ldots\}.$$

According to our theorem, the Yoshida coefficient of an integrable system must be no more than $9/8$ if the other assumptions 1-5 are satisfied.

In the example of the isosceles three-body problem, the Morales-Ramis theory guarantees the non-existence of meromorphic first integral for any $\alpha$. In the Yoshida’s example, Morales-Ramis theory guarantees the non-existence of meromorphic first integral excluding $\varepsilon = 0, 1, 3$. The same result have been obtained through the Ziglin analysis [13]. It is known that these exceptional three cases are actually integrable.

We compare our theorem with the Morales-Ramis theory in several viewpoints.

**Homogeneous degree**  Our theorem can be applied to the case of any real number $\beta$ excluding $-2, 0$ while the result from an application [8] of Morales-Ramis theory can be apply to the case of any integer excluding $\beta = -2, 0, 2$. The case of $\beta = -2$ does not need to be studied since the systems are integrable as we stated at Remark 2. Our theorem alone can be applied to the case of $\beta = 2$. Neither show anything in the case of $\beta = 0$.

**Degrees of freedom**  Our theorem can be applied to two degrees of freedom while Morales-Ramis theory can be applied to any degrees of freedom.

**Yoshida coefficients**  In the case of integer $\beta$ except 0, $\pm 2$, the assumption which is imposed in the Morales-Ramis theory is wider than ours for proving the non-integrability.

**Class of functions**  Our function class of first integrals is bigger. We prove the non-existence of first integral which is *meromorphic as a real function* in $\mathbb{R}^2 \times (\mathbb{R}^2\setminus\{(0,0)\})$, while M-R theory prove the non-existence of first integrals which is *meromorphic as a complex function*. Moreover only our class of functions allows essential singularities at the exceptional points: $q = 0, q = \infty, p = \infty$.

**Proof methods**  Proofs are quite different. Our proof is simpler and based on dynamics (the behavior of stable and unstable manifolds), the proof of Morales-Ramis theory is far from the theory of the dynamics since that is based on the
complex analysis and the differential Galois theory.

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**References**

[1] Arnoł'd, V. I., *Mathematical Methods of Classical Mechanics*, Springer, 2nd ed. 1989.

[2] Bruns, H., Über die Integrale des Vielkörper-Problems, *Acta Math.*, 1887, vol. 11, no. 1-4, pp. 25–96.

[3] Kovalevskaya, S., Sur Le Problème De La Rotation D’Un Corps Solide Autour D’Un Point Fixe, *Acta Math.*, 1889, vol. 12, no. 1, pp. 177–232.

[4] Poincaré, H., *New methods of celestial mechanics*. Vol. 1. American Institute of Physics, 1993.

[5] Ziglin, S. L., Bifurcation of Solutions and the Nonexistence of First Integrals in Hamiltonian Mechanics. I. *Funktsional. Anal. i Prilozhen.*, 1982, vol. 16, no. 3, pp. 30-41, 96.

[6] Ziglin, S. L., Bifurcation of Solutions and the Nonexistence of First Integrals in Hamiltonian Mechanics. II. *Funktsional. Anal. i Prilozhen.*, 1983, vol. 17, no. 1, pp. 8–23.

[7] Yoshida, H., A Criterion for the Non-existence of an Additional Integral in Hamiltonian Systems with a Homogeneous Potential, *Physica*, 1987, vol. 29D, no. 1-2, pp. 128–142.

[8] Morales-Ruiz, J. J., *Differential Galois Theory and Non-Integrability of Hamiltonian Systems*, Birkhaeuser Basel, 1999.

[9] Morales-Ruiz, J. J., and Ramis, J. P., A Note on the Non-Integrability of Some Hamiltonian Systems with a Homogeneous Potential, *Methods Appl. Anal.*, 2001, vol. 8, no. 1, pp. 113–120.

[10] McGehee, R., Triple Collision in the Collinear Three-Body Problem *Invent. Math.*, 1974, vol. 27, pp. 191–227.

[11] Moeckel, R., Heteroclinic phenomena in the isosceles three-body problem, *SIAM J. Math. Anal.*, 1984, vol. 5, no. 5, 857–876.

[12] Shibayama, M., Yagasaki, K., Heteroclinic connections between triple collisions and relative periodic orbits in the isosceles three-body problem. *Nonlinearity*, 2009, vol. 22, no. 10, 2377–2403.
[13] Yoshida, H., Existence of Exponentially Unstable Periodic Solutions and the Non-Integrability of Homogeneous Hamiltonian Systems, *Physica* 1986, vol. 21D, no. 1, pp. 163–170.

[14] Sansaturio, M. E., Vigo-Aguir, I., and Ferrándiz, J. M., Non-Integrability of Some Hamiltonian Systems in Polar Coordinates. *J. Phys. A: Math. Gen.*, 1997, vol. 30, no. 16, pp. 5869–5876.