ALMOST HOMOGENEOUS CURVES OVER
AN ARBITRARY FIELD

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Abstract. We classify the pairs \((C, G)\) where \(C\) is a seminormal curve over an arbitrary field \(k\) and \(G\) is a smooth connected algebraic group acting faithfully on \(C\) with a dense orbit, and we determine the equivariant Picard group of \(C\). We also give a partial classification when \(C\) is no longer assumed to be seminormal.

1. Introduction

A variety which is homogeneous under the action of an algebraic group is a very symmetric object. The study of equivariant compactifications of homogeneous varieties leads to the notion of almost homogeneous varieties. These are the varieties with a dense orbit. For example, toric varieties are the normal almost homogeneous varieties under the action of a torus.

In case of curves, the situation is rather nice for several reasons. First, a curve is almost homogeneous under the action of a smooth connected algebraic group if and only if the action is non-trivial. Moreover the complement of the dense orbit consists only of finitely many fixed points.

Second, we have natural compactifications. More precisely, if \(C\) is a regular curve over a field \(k\) then there exists a regular projective curve \(\hat{C}\) and an open immersion \(C \hookrightarrow \hat{C}\). Such a curve satisfies a universal property: for any proper scheme \(Y\) over \(k\), every morphism \(C \rightarrow Y\) extends uniquely to a morphism \(\hat{C} \rightarrow Y\). In particular \(\hat{C}\) is unique up to unique isomorphism. We call it the regular completion of \(C\) and the points of \(\hat{C}\setminus C\) are called the points at infinity.

Finally, if \(C\) is a projective curve then the functor which associates with a \(k\)-scheme \(S\) the abstract group \(\text{Aut}_S(C \times S)\) of \(S\)-automorphisms of \(C \times S\) is representable by an algebraic group denoted by \(\text{Aut}_C\) (see [Br17, Ex. 7.1.2]).

The study of automorphisms of curves has a long history. A first step was a theorem of Adolf Hurwitz stating that a compact Riemann surface of genus at least 2 has finitely many automorphisms. This result was generalized several times, and in particular Maxwell Rosenlicht gave in [Ro55, Thm., p. 4] a version for smooth
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projective curves over an arbitrary field, in the language of algebraic function fields in one variable. He also classified in [Ro55, Thm., p. 10] the algebraic function fields which he called “exceptional”, that is, those of genus at least 1 having infinitely many automorphisms (fixing a given place if the genus equals 1). The link with regular curves is standard: there is an anti-equivalence between the category of regular curves over $k$ and the category of algebraic function fields in one variable over $k$, which associates with a curve $C$ its function field $k(C)$, and the closed points of $C$ correspond to the places of $k(C)$. Some years after, Peter Russell gave in [Ru, Thm. 4.2] an interpretation of Rosenlicht’s classification in geometric terms, which we state for simplicity in case $k$ is separably closed: a regular projective curve $C$ of genus at least 1 has infinitely many automorphisms (fixing a given closed point if the genus equals 1) if and only if $C$ is the regular completion of a torsor under a non-trivial form of the additive group $\mathbf{G}_{a,k}$.

From the geometric point of view, a natural approach to classify almost homogeneous curves is to understand the regular completions of homogeneous curves, and the behavior of the points at infinity. Vladimir Popov thus obtained in [Po, Chap. 7] a full classification of almost homogeneous curves over an algebraically closed field of characteristic zero, and determined their abstract automorphism group. Our objective is to extend this result by classifying the pairs $(C, G)$ where $C$ is a curve over an arbitrary field and $G$ is a smooth connected algebraic group acting faithfully on $C$ with a dense orbit.

The complexity of the classification depends on the class of singularity of the curves. After regular curves, the next class to look at is the class of seminormal curves. Roughly speaking, these are the curves whose branches intersect as transversally as possible. For example, the nodal curve $(y^2 - x^3 - xy = 0)$ is seminormal but the cusp $(y^2 - x^3 = 0)$ is not seminormal. Seminormal curves can be easily described in the language of pinchings developed by Daniel Ferrand in [Fe] (see Section 3.1 and Lemma 3.8). We obtain the following theorem.

**Theorem 1.1.** Let $C$ be a seminormal curve, $G$ a smooth connected algebraic group and $\alpha : G \times C \to C$ an action. The action is faithful and $C$ is almost homogeneous if and only if one of the following cases holds:

1. (homogeneous curves)
   1. $C$ is a smooth projective conic and $G \simeq \text{Aut}_C$;
   2. $C \simeq \mathbf{A}^1_k$ and $G \simeq \mathbf{G}_{a,k} \times \mathbf{G}_{m,k}$ (acting by affine transformations);
   3. $G$ is a form of $\mathbf{G}_{a,k}$ and $C$ is a $G$-torsor;
   4. $G$ is a form of $\mathbf{G}_{m,k}$ and $C$ is a $G$-torsor;
   5. $C$ is a smooth projective curve of genus 1 and $G \simeq \text{Aut}_C^*;

2. (regular, non-homogeneous curves)
   1. $C \simeq \mathbf{P}^1_k$ and $G \simeq \mathbf{G}_{a,k} \times \mathbf{G}_{m,k};$
   2. $G$ is a form of $\mathbf{G}_{a,k}$ and $C$ is the regular completion of a $G$-torsor;
   3. $C \simeq \mathbf{A}^1_k$ or $\mathbf{P}^1_k$ and $G \simeq \mathbf{G}_{m,k};$
   4. $C$ is a smooth projective conic and $G$ is the centralizer of a separable point of degree 2;

3. (seminormal, singular, non-homogeneous curves)
1. \( G \) is a non-trivial form of \( \mathbb{G}_{a,k} \) and \( C \) is obtained by pinching the point at infinity \( \tilde{P} \) of the regular completion of a \( G \)-torsor on a point \( P \) whose residue field \( \kappa(P) \) is a strict subextension of \( \kappa(\tilde{P})/k \);

2. \( C \) is obtained by pinching two \( k \)-rational points of \( \mathbb{P}^1_k \) on a \( k \)-rational point and \( G \simeq \mathbb{G}_{m,k} \);

3. \( C \) is obtained by pinching a separable point \( \tilde{P} \) of degree 2 of a smooth projective conic \( \tilde{C} \) on a \( k \)-rational point, and \( G \) is the centralizer of \( \tilde{P} \) in \( \text{Aut}_{\tilde{C}} \).

Since in the case 3.1 the curves are parameterized by subextensions of \( \kappa(\tilde{P})/k \), these curves can form an uncountable family (see Remark 3.13).

When \( C \) is not necessarily seminormal, the situation is more complex and requires studying representations of algebraic groups. We give a classification in Sections 4.1 and 4.2, except for almost homogeneous curves under the action of \( \mathbb{G}_{a,k} \times \mathbb{G}_{m,k} \) or a form of \( \mathbb{G}_{a,k} \) when the field \( k \) has positive characteristic, because in this case the representations of the forms of \( \mathbb{G}_{a,k} \) (and even \( \mathbb{G}_{a,k} \) itself) are not so well-understood. The full classification remains an open problem.

Another open problem is to determine the automorphism group scheme of almost homogeneous projective curves, especially for the curves obtained by pinching the regular completion of a torsor under a non-trivial form of \( \mathbb{G}_{a,k} \).

For projective almost homogeneous curves under the action of an algebraic group \( G \), one may want to try to embed them in the projectivization of a \( G \)-module. The standard tool is the notion of \( G \)-linearized line bundles. We describe in Proposition 5.10 the \( G \)-linearized line bundles over a variety obtained by a pinching. We use this description to determine in Theorem 5.13 the equivariant Picard groups of the curves appearing in Theorem 1.1.

**Notations and conventions.** We fix a base field \( k \) and an algebraic closure \( \overline{k} \), and denote by \( k_s \) the separable closure of \( k \) in \( \overline{k} \). For all \( k \)-schemes \( X \) and \( S \), the base change \( X \times_k S \) shall be denoted by \( X_S \); in case \( S = \text{Spec} \, K \) for some field extension \( K/k \), we simply write \( X_K \).

All morphisms between \( k \)-schemes are morphisms over \( k \). A variety over \( k \) is a separated scheme of finite type over Spec \( k \) which is geometrically integral. A curve is a variety of dimension 1. An algebraic group over \( k \) is a group scheme of finite type over Spec \( k \). A subgroup of an algebraic group is a (closed) subgroup scheme. We may consider non-smooth groups, but all algebraic groups acting on a variety shall be assumed smooth.

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2. Regular almost homogeneous curves

2.1. Scheme-theoretic actions

In this section we recall some elementary facts about the actions of algebraic groups, in the setting of schemes. We use [DG] as a general reference.
Let $X$ be a variety, $G$ a smooth algebraic group with neutral element $e$ and an action
\[
\alpha : \begin{cases} 
G \times X & \to X, \\
(g, x) & \mapsto gx
\end{cases}
\]
For any $k$-rational point $x \in X(k)$ the orbit morphism
\[
\alpha_x : \begin{cases} 
G & \to X, \\
g & \mapsto gx
\end{cases}
\]
factorizes as $G \to Gx \to X$ where $Gx$ is a smooth scheme, the first morphism is faithfully flat and the second one is an immersion. Thus $Gx$ is a $G$-stable (locally closed) subscheme of $X$, called the orbit of $x$ (see [DG, II 5.3.1]).

For any closed subscheme $Y$ of $X$, the scheme-theoretic normalizer $N_G(Y)$ and centralizer $C_G(Y)$ are representable by subgroups of $G$ (see [DG, II 1.3.6]). In case $Y = X$, the centralizer is called the kernel of the action and is a normal subgroup of $G$. In case $Y$ is a $k$-rational point $x$, the centralizer is denoted by $G_x$ and is called the isotropy subgroup of $x$; it is also the fiber at $x$ of $\alpha_x$, and $\alpha_x$ induces an isomorphism $G/G_x \simeq Gx$.

Similarly, the functor of fixed points is representable by a closed subscheme $X^G$ of $X$. Moreover $X^G(\overline{k})$ is the set of the elements of $X(\overline{k})$ which are fixed under the action of $G(\overline{k})$.

One would like to define $X$ to be homogeneous (resp. almost homogeneous) if there exists a unique orbit (respectively a dense orbit). Since $X$ may not have any $k$-rational point, a more convenient definition can be given as follows.

**Definition 2.1.** The variety $X$ is said to be homogeneous (resp. almost homogeneous) under the action of $G$ if the morphism
\[
\gamma = (\alpha, \text{pr}_2) : \begin{cases} 
G \times X & \to X \times X, \\
(g, x) & \mapsto (gx, x)
\end{cases}
\]
is surjective (resp. dominant).

However, one recovers the natural definition in terms of orbits after a suitable field extension, as shown by the following lemmas.

**Lemma 2.2.** Let $K/k$ be a field extension. The following assertions are equivalent:

i) The variety $X$ is homogeneous under the action of $G$.

ii) The variety $X_K$ is homogeneous under the action of $G_K$.

iii) The action of the abstract group $G(\overline{k})$ on the set $X(\overline{k})$ is transitive.

iv) If there exists a $K$-rational point $x \in X(K)$ then the orbit $G_K x$ equals $X_K$.

**Proof.** The assertions i), ii) and iii) are equivalent because a morphism $f : Y \to Y'$ between schemes of finite type over $k$ is surjective if and only if $f_K$ is surjective, and also if and only if the induced map of sets $f : Y(\overline{k}) \to Y'(\overline{k})$ is surjective (see [SGA1, Exp. XII, Prop. 3.2]). In particular this holds for $\gamma$. Moreover for any $K$-rational point $x$ of $X_K$ the equality of schemes $G_K x = X_K$ is equivalent to the surjectivity of the orbit morphism $G_K \to X_K$, to the surjectivity of the induced map $G(\overline{k}) \to X(\overline{k})$ and to the transitivity of the action of $G(\overline{k})$ on $X(\overline{k})$. \qed
Lemma 2.3. The following assertions are equivalent:

i) The variety $X$ is almost homogeneous under the action of $G$.

ii) For every field extension $K/k$, the variety $X_K$ is almost homogeneous under the action of $G_K$.

iii) There exists a field extension $K/k$ such that the variety $X_K$ is almost homogeneous under the action of $G_K$.

iv) There exists a $G$-stable and homogeneous open subscheme $U$ of $X$.

v) There exists a field extension $K/k$ and a $K$-rational point $x \in X(K)$ such that the orbit $G_K x$ is an open subscheme of $X_K$.

In v) the extension $K/k$ can be chosen to be finite and separable. Furthermore, the open subscheme $U$ is unique, and we call it the open orbit.

Proof. Let $\pi : X_K \times X_K \to X \times X$ be the projection, which is an open morphism. The set-theoretic image of the morphism $\gamma_K : G_K \times X_K \to X_K \times X_K$ is $\pi^{-1}(\gamma(G \times X))$. So $\gamma_K(G_K \times X_K)$ contains a nonempty open subset of $X_K \times X_K$ if and only if $\gamma(G \times X)$ contains a nonempty open subset of $X \times X$. Thus by Chevalley’s theorem, $\gamma$ is dominant if and only if $\gamma_K$ is dominant (see [DG, I 3.3.8]).

Assume that $X$ is almost homogeneous. By [Dem, Sect. 5, p. 519], there exists a finite separable extension $K/k$ and a point $x \in X(K)$ such that the orbit $G_K x$ is open in $X_K$. Any element $\sigma$ of $\text{Gal}(K/k)$ induces an automorphism of $X_K$. Then the open subscheme $\sigma(G_K x)$ is the orbit of $\sigma(x)$. The two open orbits $G_K x$ and $\sigma(G_K x)$ must have a nonempty intersection, so they are equal. In other words, $G_K x$ is $\text{Gal}(K/k)$-stable. Thus, by Galois descent, there exists an open subscheme $U$ of $X$ such that $U_K = G_K x$, and this subscheme is $G$-stable and homogeneous. Moreover, if $U'$ is a $G$-stable and homogeneous open subscheme of $X$ then $U_K$ and $U'_K$ are two open orbits in $X_K$ so again we have $U_K = U'_K$ and $U = U'$.

If $X$ contains a $G$-stable and homogeneous open subscheme $U$ then the image of $\gamma$ contains $U \times U$ so $\gamma$ is dominant.

If v is true then the orbit $G_K x$ is a $G_K$-stable and homogeneous open subscheme of $X_K$ so $X_K$ is almost homogeneous.

Lemma 2.4. Let $G$ be a smooth connected algebraic group and $Z$ a 0-dimensional reduced $k$-scheme of finite type. The only action $\alpha : G \times Z \to Z$ is the trivial one.

Proof. We can assume that $k$ is separably closed. The scheme $Z$ is noetherian and 0-dimensional so it is affine and its underlying topological space is a finite set endowed with the discrete topology. The components of $Z$ are stable open subschemes so we can assume that $Z$ is integral. We write $Z = \text{Spec} K$ for some field extension $K/k$. Since $Z$ is of finite type over $k$, $K/k$ is a finite extension, and hence it is purely inseparable. The automorphism of $Z$ induced by an element $g \in G(k)$ corresponds to a $k$-automorphism of $K$. But $K/k$ is purely inseparable, so the unique $k$-automorphism of $K$ is the identity. Since $G(k)$ is dense in $G$ ($G$ is geometrically reduced), the action is trivial.

Remark 2.5. This result is not true in general if $G$ is not smooth. Indeed, let $k$ be an imperfect field of characteristic $p$, $a \in k$ which is not a $p$th power, $K = k((1/p)) = k[X]/(X^p - a)$ and $Z = \text{Spec} K$. Then $Z$ is a 0-dimensional reduced
Lemma 2.6. Let $G$ be a smooth connected algebraic group, $C$ a curve and $\alpha : G \times C \to C$ an action. The curve $C$ is almost homogeneous if and only if the action $\alpha$ is non-trivial. In this case, the open orbit is the complement of the closed subscheme $Z = C^G$ of fixed points.

Proof. If $C$ is not almost homogeneous then for $x \in C(\overline{k})$, the closure of the orbit $G_{\overline{k}}x$ in $C_{\overline{k}}$ is irreducible and cannot have dimension 1, so the orbit is trivial.

Assume that $C$ is almost homogeneous. Let $U = C \setminus Z$. Then $U_{\overline{k}}$ is the open orbit in $C_{\overline{k}}$. Indeed, let $x \in C(\overline{k})$ such that the orbit $G_{\overline{k}}x$ is open. By Lemma 2.4, the complement of this orbit is the set of fixed points of $C(\overline{k})$. So we have $U_{\overline{k}} = G_{\overline{k}} \setminus Z_{\overline{k}} = G_{\overline{k}} \setminus Z(\overline{k}) = G_{\overline{k}}x$. □

We now show how to restrict to faithful actions. First recall that if $H$ is a subgroup of a smooth algebraic group $G$ then by [SGA3, Exp. VIA, 3.2] the quotient $G/H$ exists, it is a smooth scheme of finite type and we have a $G$-equivariant morphism $\pi : G \to G/H$ which is a $H$-torsor. If $H$ is a normal subgroup then $G/H$ is an algebraic group and for every $k$-scheme $S$, we have an exact sequence of abstract groups $1 \to H(S) \to G(S) \to (G/H)(S)$. The last morphism need not be surjective, but it is surjective if $S = \text{Spec} \overline{k}$. More generally, for any $k$-scheme $S$ and $\overline{g} \in (G/H)(S)$ there exists a morphism $S' \to S$ which is faithfully flat of finite presentation and an element $g \in G(S')$ such that $\overline{g} = \pi(g)$ in $(G/H)(S')$.

Lemma 2.7. Let $H$ be the kernel of the action $\alpha$. Then there exists a unique action $\beta : G/H \times X \to X$ making the diagram

\[
\begin{array}{ccc}
G \times X & \xrightarrow{\alpha} & X \\
\pi \times \text{id}_X & \downarrow & \beta \\
G/H \times X & & 
\end{array}
\]

commute. The action $\beta$ is faithful, and $X$ is homogeneous (resp. almost homogeneous) under the action of $G$ if and only if it is so under the action of $G/H$.

Proof. The morphism $\pi \times \text{id}_X : G \times X \to G/H \times X$ is a $H$-torsor so it is a categorical quotient. Since $\alpha : G \times X \to X$ is $H$-invariant, there exists a unique morphism $\beta$ such that the diagram commutes.

Let us show that $\beta$ is an action. Let $S$ be a $k$-scheme, $x \in X(S)$ and $\overline{g}_1 \in (G/H)(S)$ and $\overline{g}_2 \in (G/H)(S)$. We have $\pi(e)x = x$. Let $S' \to S$ be a morphism which is faithfully flat and $g_1 \in G(S')$, $g_2 \in G(S')$ such that $\overline{g}_1 = \pi(g_1)$ and $\overline{g}_2 = \pi(g_2)$ in $(G/H)(S')$. Then in $X(S')$ we have $\overline{g}_1(\overline{g}_2x) = \overline{g}_1(g_2x) = (g_1g_2)x = (\overline{g}_1\overline{g}_2)x$. Since the map $X(S') \to X(S'')$ is injective (because $S' \to S$ is an epimorphism), the equality $\overline{g}_1(\overline{g}_2x) = (\overline{g}_1\overline{g}_2)x$ already holds in $X(S)$.

Let $N$ be the kernel of the action $\beta$. Let $\overline{g} \in N(S)$. Let $S' \to S$ and $g \in G(S')$ as previously. Then for every $S'' \to S'$ and $x \in X(S'')$ we have $gx = \overline{g}x = x$. So
$g \in H(S')$. Then $\overline{g} = \pi(g) = e$ in $(G/H)(S')$ so, as before, $\overline{g} = e$ in $(G/H)(S')$. Hence $N(S)$ is the trivial group.

Finally, since $\pi \times \text{id}_X$ is surjective, the morphism $\gamma : G \times X \to X \times X$ is surjective (resp. dominant) if and only if so is the analogous morphism $G/H \times X \to X \times X$. 
\end{proof}

We can also restrict to the action of the neutral component $G^\circ$ of $G$.

\begin{lemma}
The variety $X$ is homogeneous (resp. almost homogeneous) under the action of $G$ if and only if it is so under the action of $G^\circ$.
\end{lemma}

\begin{proof}
We can assume that $k$ is algebraically closed. Assume first that $X$ is almost homogeneous under the action of $G$. By hypothesis, the morphism $\gamma : G \times X \to X \times X$ defined in Definition 2.1 is dominant. Let us show that its restriction $\gamma_0$ to $G^\circ \times X$ is also dominant. For every irreducible component $G_i$ of $G$, the right multiplication by some $g_i \in G_i(k)$ induces isomorphisms $G^\circ \simeq G_i$ and $\overline{\gamma}(G^\circ \times X) \simeq \overline{\gamma}(G_i \times X)$. Moreover we have $\bigcup_i \overline{\gamma}(G_i \times X) = \bigcup_i \overline{\gamma}(G_i \times X) = \overline{\gamma}(G \times X) = X \times X$ (the first equality holds because there are finitely many irreducible components). Hence $\dim \overline{\gamma}(G^\circ \times X) = \dim X \times X$. But $\overline{\gamma}(G^\circ \times X)$ and $X \times X$ are irreducible, so they are equal and $\gamma_0$ is dominant.

Assume now that $X$ is homogeneous under the action of $G$. Then $X$ is almost homogeneous under the action of $G^\circ$. Pick $x \in X(k)$ such that the orbit $U = G^\circ x$ is open. The cosets $G^\circ g$ for $g \in G(k)$ form a covering of $G$ so, since $Gx = X$, we have $X(k) = \bigcup_{g \in G(k)} G^\circ(k)gx$. For $g \in G(k)$, the orbit $G^\circ gx$ of $gx$ is equal to $gU$. Since $X$ is irreducible, the open subsets $U$ and $gU$ must have a common $k$-rational point. But $G^\circ(k)x = U(k)$ and $G^\circ(k)gx = gU(k)$ are the orbits of $x$ and $gx$ under the action of the abstract group $G^\circ(k)$, hence they are equal. Therefore the abstract group $G^\circ(k)$ acts transitively on $X(k)$. 
\end{proof}

\section{Forms of the multiplicative and additive groups}

Since $\text{Aut}_{k_s,\text{gp}}(G_{m,k_s}) \simeq \mathbb{Z}/2\mathbb{Z}$ (with trivial action of $\Gamma = \text{Gal}(k_s/k)$), the forms of $G_{m,k}$ are classified by the Galois cohomology set $H^1(\Gamma, \mathbb{Z}/2\mathbb{Z})$. This gives the following well-known result.

\begin{lemma}
Let $G$ be a form of $G_{m,k}$. There exists a unique étale quadratic $k$-algebra $K$ such that $G \simeq R^1_{K/k}(G_{m,K})$ where $R_{K/k}(G_{m,K})$ is the Weil restriction and $R^1_{K/k}(G_{m,K})$ is the kernel of the norm $N_{K/k} : R_{K/k}(G_{m,K}) \to G_{m,k}$. The trivial form corresponds to $K = k \times k$. Moreover, if $G$ is a non-trivial form (that is, if $K/k$ is a quadratic Galois field extension) then $G_K \simeq G_{m,K}$.
\end{lemma}

There is also a geometric description of the forms of $G_{m,k}$ and their torsors in terms of conics. We first need a lemma about the group structures of $A^1_k \setminus \{0\}$.

\begin{lemma}
Up to the choice of the neutral element, the only algebraic group structure on $A^1_k \setminus \{0\}$ is $G_{m,k}$.
\end{lemma}

\begin{proof}
Let $G$ be an algebraic group structure on $A^1_k \setminus \{0\}$. After a suitable translation, we may assume that the neutral element is 1. We can also assume
that $k$ is algebraically closed. The left multiplication by an element $x \in G(k)$ is an automorphism of $A^1_k \setminus \{0\}$, which corresponds to an algebra automorphism of $k[T, T^{-1}]$. It is given by $T \mapsto aT^\varepsilon$ for some $a \in k^\times$ and $\varepsilon \in \{-1, 1\}$. The left multiplication by $x$ maps 1 to $x$, so $a = x$. If $\varepsilon = -1$ and $x \neq 1$ then there are fixed points, which is impossible. Thus $\varepsilon = 1$ and the law is given by $(x, y) \mapsto xy$. \hfill $\Box$

**Lemma 2.11.** Let $G$ be an algebraic group and $C$ a curve. Then $G$ is a form of $G_{m,k}$ and $C$ is a $G$-torsor if and only if there exists a smooth projective conic $\hat{C}$ and a closed subscheme $Z$ of $\hat{C}$ such that $Z = \text{Spec} \ K$ for some étale quadratic $k$-algebra $K$, $C = \hat{C} \setminus Z$ and $G$ is the centralizer of $Z$ in $\text{Aut} \hat{C}$.

**Proof.** Let $G$ be a form of $G_{m,k}$, $C$ a $G$-torsor, $\hat{C}$ the regular completion of $C$ and $Z = \hat{C} \setminus C$ endowed with its structure of a reduced closed subscheme. Let $K$ be the unique étale quadratic $k$-algebra such that $G \simeq R^1_{K/k}(G_{m,K})$. If $G$ is the trivial form then $K = k \times k$. By Galois cohomology and Hilbert’s theorem 90, every $G_{m,k}$-torsor is trivial. So $C \simeq A^1_k \setminus \{0\}$ and then $\hat{C} \simeq P^1_k$, $Z = \{0, \infty\} = \text{Spec} \ K$ and $G$ is the centralizer of $Z$ in $\text{Aut} \hat{C} \simeq \text{PGL}_{2,k}$.

Assume now that $G$ is a non-trivial form, so that $K/k$ is a quadratic Galois field extension. We have $G_K \simeq G_{m,K}$ so, as above, $C_K \simeq A^1_K \setminus \{0\}$. Since the extension $K/k$ is separable, the curve $(\hat{C})_K$ is the regular completion of $C_K$. Thus $(\hat{C})_K \simeq P^1_k$ and $Z_K = \{0, \infty\} = \text{Spec}(K \times K)$. Therefore $\hat{C}$ is a smooth projective conic. By Galois descent, we have $Z = \text{Spec}(k \times k)$ or $Z = \text{Spec} \ K$.

In the first case, we have $\hat{C} \simeq P^1_k$ and the underlying space of $G$ is $A^1_k \setminus \{0\}$ so $G \simeq G_{m,k}$, which contradicts the hypothesis. Hence $Z = \text{Spec} \ K$. Moreover $G$ is the centralizer of $Z$ in $\text{Aut} \hat{C}$ since $G_{m,K}$ is the centralizer of $\{0, \infty\}$ in $\text{Aut} P^1_k$.

Conversely, assume that $K/k$ is a quadratic Galois field extension and $G$ is the centralizer of $Z$ in $\text{Aut} \hat{C}$. Then $G$ acts on the complement $C$ of $Z$, which is a $G$-torsor because it becomes so after extension to $K$. As before we have $G_K \simeq G_{m,K}$.

Russell classified the forms of $G_{a,k}$ in [Ru] and obtained the following result.

**Proposition 2.12.** If $C$ (resp. $G$) is a form of $A^1_k$ (resp. $G_{a,k}$) then there exists a unique field extension $K/k$ which is minimal under inclusion and such that $C_K \simeq A^1_k$ (resp. $G_K \simeq G_{a,K}$), and this extension is finite and purely inseparable. If $k$ has characteristic $p > 0$ then the forms of $G_{a,k}$ are the algebraic groups isomorphic to a subgroup of $G_{2,k}^2$ given by an equation of the form $y^p = x + a_1 x^p + \cdots + a_m x^{p^n}$ for some integers $m \geq 0$ and $n \geq 0$ and some coefficients $a_i$ in $k$. If $G$ is given by such an equation then the minimal field extension $K/k$ such that $G_K \simeq G_{a,K}$ is $K = k(a_1^{p^n}, \ldots, a_m^{p^n})$. The $G$-torsors are the schemes isomorphic to a closed subscheme $C$ of $A^2_k$ given by an equation of the form $y^p = b + x + a_1 x^p + \cdots + a_m x^{p^n}$ for some coefficient $b$ in $k$. The complement of $C$ in its regular completion $\hat{C}$ is a non-smooth point $P$ whose residue field is a subextension of $K/k$.

### 2.3. Classification of regular almost homogeneous curves

The key result for the classification is the well-known following proposition.
Proposition 2.13 ([Br17, Exs. 7.1.2, 7.1.3]). Let $C$ be a smooth projective curve of genus $g$. If $g > 1$ (resp. $g = 1$), then $\text{Aut}_C$ (resp. the centralizer $\text{Aut}_{C,Q}$ of a $k$-rational point $Q$ endowed with its structure of a reduced closed subscheme of $C$) is a finite étale group.

The following lemma is a special case of a result of André Weil on birational actions. For the sake of completeness, we give a proof based on ideas of Rosenlicht (see [Ro56, Thm. 15]). Note that Michel Demazure gave in [Dem, Prop. 5] a similar statement for smooth curves.

Lemma 2.14. Let $C$ be a regular curve, $\hat{C}$ its regular completion, $G$ a smooth connected algebraic group and $\alpha : G \times C \to C$ an action. The action $\alpha$ extends uniquely to an action on $\hat{C}$. Moreover, the complement $Z = \hat{C} \setminus C$, endowed with its structure of a reduced closed subscheme of $\hat{C}$, is $G$-stable (and the action of $G$ on it is trivial).

Proof. Let $\hat{\alpha} : G \times \hat{C} \to \hat{C}$ be the rational action defined by $\alpha$. Let $U \subseteq G \times \hat{C}$ be its domain of definition, which contains $G \times C$. It is enough to show that for any $(g,x) \in G(\overline{k}) \times \hat{C}(\overline{k})$ the element $gx$ is defined, that is, $(g,x) \in U(\overline{k})$. Since $G_{ks}$ is the regular completion of $C_{ks}$, we can assume $k = k_s$.

The scheme $G \times \hat{C}$ is normal (since $G$ is smooth and $\hat{C}$ is normal) and $\hat{C}$ is proper over $k$ so $(G \times \hat{C}) \setminus U$ has codimension at least 2 in $G \times \hat{C}$. The subscheme $Z$ of $\hat{C}$ consists of finitely many closed points $P_1, \ldots, P_r$ whose residue fields are purely inseparable extensions of $k$. The irreducible components $Z_i = G \times \text{Spec} \kappa(P_i)$ of $G \times Z$ have codimension 1 in $G \times \hat{C}$. So $U \cap Z_i$ is a nonempty open subset of $Z_i$ (otherwise $Z_i \subseteq (G \times \hat{C}) \setminus U$ and $Z_i$ would have codimension at least 2). By [EGA, IV2, Prop. 2.4.5] the first projection $\pi_i : Z_i \to G$ is a homeomorphism, so $V = \bigcap_{i=1}^r \pi_i(U \cap Z_i)$ is a nonempty open subset of $G$ such that $V \times Z \subseteq U$. For $g \in V(\overline{k})$ and $x \in Z(\overline{k})$ we have $gx \in Z(\overline{k})$. Indeed if $gx \in \hat{C}(\overline{k})$ then $g^{-1}(gx)$ is defined, so we have $g^{-1}(gx) = (g^{-1}g)x = ex = x$, which is impossible because $g^{-1}(gx) \in C(\overline{k})$. Finally $G(\overline{k}) = V(\overline{k})V(\overline{k})$, hence for any $g \in G(\overline{k})$ and $x \in Z(\overline{k})$, the element $gx$ is defined.

Let $N_G(Z)$ be the normalizer of $Z$ in $G$. Its formation commutes with field extensions so we still can assume $k = k_s$. Any element $g \in G(k)$ yields an automorphism of $\hat{C}$, again denoted by $g$, such that the subset $Z$ is stable. Since $Z$ is a reduced subscheme, $g$ restricts to an automorphism of $Z$. Thus we have $N_G(Z)(k) = G(k)$. But $G(k)$ is dense in $G$ so $N_G(Z) = G$, that is, the action on $\hat{C}$ restricts to an action on $Z$. By Lemma 2.4, this action is trivial. \[\square\]

Remark 2.15. In particular, the action of $G$ on $\hat{C}$ gives a morphism $G \to \text{Aut}_{\hat{C}}$ which factorizes as $G \to \text{Aut}_{\hat{C},Z}$, where $\text{Aut}_{\hat{C},Z}$ is the centralizer of $Z$ in $\text{Aut}_{\hat{C}}$.

From the above results we easily deduce a list of possible homogeneous curves (which can be found in [Po, Prop. 7.1.2] in case $k$ is algebraically closed).

Lemma 2.16. Let $C$ be a smooth curve. If $C$ is homogeneous under the action of a smooth algebraic group then $C$ is a form of $\mathbb{P}^1_k$, $\mathbb{A}^1_k$ or $A^1_k \setminus \{0\}$, or $C$ is a projective curve of genus 1.
The converse is false. More specifically, some forms of $A^1_k$ are not homogeneous, as shown by a counterexample mentioned without proof by Russell in [Ru].

**Lemma 2.17.** We assume that $k$ is imperfect. Let $P$ be a point on $P^1_k$ such that the extension $\kappa(P)/k$ is purely inseparable of degree at least 3. We endow $P$ with its structure of a reduced closed subscheme of $P^1_k$. Then $C = P^1_k \setminus \{ P \}$ is a form of $A^1_k$ such that the centralizer $\text{Aut}_{P^1_k, P}$ of $P$ in $\text{Aut}_{P^1_k}$ is infinitesimal.

**Proof.** Since $\kappa(P)/k$ is purely inseparable, there exists a unique point $P_{k_s}$ of $P^1_{k_s}$ above $P$ and we have $[\kappa(P_{k_s}) : k_s] = [\kappa(P) : k]$. Thus we can assume $k = k_s$. Similarly, there exists a unique point $P_k$ of $P^1_k$ above $P$ and it is $\overline{k}$-rational. So $C_k = P^1_k \setminus \{ P_k \} \simeq A^1_k$, thus $C$ is a form of $A^1_k$.

Let $a \in k$ such that $P$ has equation $t^{p^m} - a = 0$, where $[t : u]$ are the usual coordinates on $P^1_k$ and $\kappa(P)/k$ has degree $p^m$. An element of $\text{Aut}_{P^1_k, P}(\overline{k})$ is a homography inducing the identity on the closed subscheme $P_k$ of $P^1_k$, which has equation $(t - a^{p^m})t^{p^m} = 0$. So after conjugating by a homography sending $[a^{p^m} : 1]$ to $[0 : 1]$, and since $p^m \geq 3$, it suffices to show that if a homography $f = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ induces the identity on the subscheme of equation $t^3 = 0$ then $f = \text{id}$.

We have $f(0) = 0$ so $\beta = 0$ and we can set $\delta = 1$. We still denote by $t$ the image of $t$ in $k[t]/(t^3)$. Then $f([t : 1]) = [\alpha t : 1 + \gamma t]$ so $f$ acts on $k[t]/(t^3)$ by $f \cdot t = \alpha t/(1 + \gamma t) = \alpha t - \alpha \gamma t^2$. So $\alpha = 1$ and $\gamma = 0$. Hence $\text{Aut}_{P^1_k, P}(\overline{k})$ is the trivial group. \qed

We can now prove the case 1 of Theorem 1.1.

**Theorem 2.18.** Let $C$ be a curve and $G$ a smooth connected algebraic group acting faithfully on $C$. The curve $C$ is homogeneous under the action of $G$ if and only if one of the following cases holds:

(a) $C$ is a smooth projective conic and $G \simeq \text{Aut}_C$;
(b) $C \simeq A^1_k$ and $G \simeq G_{a,k} \times G_{m,k}$ (acting by affine transformations);
(c) $G$ is a form of $G_{a,k}$ and $C$ is a $G$-torsor;
(d) $G$ is a form of $G_{m,k}$ and $C$ is a $G$-torsor;
(e) $C$ is a smooth projective curve of genus 1 and $G \simeq \text{Aut}^0_C$.

**Proof.** If $C$ is homogeneous then $C_k$ is smooth and we can consider its regular completion $\widehat{C}_k$. By Lemmas 2.14 and 2.6, the action of $G_k$ on $C_k$ extends to an action on $\widehat{C}_k$, and $C_k$ is the complement of the subscheme of fixed points $Z = (\widehat{C}_k)^{G_k}$.

By Lemma 2.16, either $\widehat{C}_k$ has genus 1 or $\widehat{C}_k$ is isomorphic to $P^1_k$. In case $\widehat{C}_k$ has genus 1, it is a torsor under $\text{Aut}_{\widehat{C}_k}^0$ ($\widehat{C}_k$ is an elliptic curve $E$ and we have $\text{Aut}_{\widehat{C}_k}^0 = E$ where $E$ acts on itself by translation). Since $G_k$ is a subgroup of $\text{Aut}^0_{\widehat{C}_k}$ and $\dim \text{Aut}^0_{\widehat{C}_k} = 1$, we have $G_k = \text{Aut}^0_{\widehat{C}_k}$ and $C_k = \widehat{C}_k$.

We now assume $\widehat{C}_k = P^1_k$. Then $G_k$ is a subgroup of $\text{Aut}_{P^1_k} = \text{PGL}_{2,k}$ and $Z$ consists of 0, 1 or 2 points. If $G_k$ is a strict subgroup of $\text{PGL}_{2,k}$ then, by [Bor,
Cor. 11.6], \( G \) is solvable so (up to conjugation) it is a subgroup of the standard Borel subgroup \( G_{a,k} \rtimes G_{m,k} \). Thus if \( Z \) is empty then \( C \simeq \mathbb{P}^1_k \) and \( G \simeq \text{PGL}_{2,k} \).

If \( Z \) consists of 2 points then \( C \simeq \mathbb{A}^1_k \) and \( G \simeq G_{a,k} \) or \( G \simeq G_{a,k} \rtimes G_{m,k} \). Thus if \( Z \) is empty then \( C \simeq \mathbb{A}^1_k \) and \( G \simeq \text{PGL}_{2,k} \).

If \( Z \) consists of 2 points then \( C \simeq \mathbb{A}^1_k \) and \( G \simeq G_{a,k} \) or \( G \simeq G_{a,k} \rtimes G_{m,k} \). If \( Z \) consists of 1 point then \( C \simeq \mathbb{A}^1_k \) and \( G \simeq G_{a,k} \) or \( G \simeq G_{a,k} \rtimes G_{m,k} \).

The classification over \( k \) follows immediately, except in case b. We assume \( C \simeq \mathbb{A}^1_k \) and \( G \simeq G_{a,k} \rtimes G_{m,k} \). Let us prove \( C \simeq \mathbb{A}^1_k \). By Proposition 2.12, we can assume that \( k \) is separably closed. Let \( L/k \) be a (purely inseparable) extension such that \( C \simeq \mathbb{A}^1_L \). As above, we have \( G_L \simeq G_{a,L} \rtimes G_{m,L} \). Let \( T \) be a maximal torus of \( G \). Then \( T_L \) is a maximal torus of \( G_L \) so \( T_L \simeq G_{m,L} \). Since the extension \( L/k \) is purely inseparable, we already have \( T \simeq G_{m,k} \). The curve \( C \) is geometrically reduced so the subset \( C(k) \) is dense in \( C \). If for every \( y \in C(k) \) the orbit morphism \( T \rightarrow C \) of \( y \) were constant then we would have \( C_T \subseteq C \), contradicting the faithfulness of the action.

Let \( y \in C(k) \) be such that the orbit morphism \( T \rightarrow C \) is non-constant. Let \( \tilde{C} \) be the regular completion of \( C \). Then the orbit morphism extends to a non-constant morphism \( \mathbb{P}^1_k \rightarrow \tilde{C} \). Thus \( \tilde{C} \) is a regular projective curve of genus 0 having \( k \)-rational points, so \( \tilde{C} \simeq \mathbb{P}^1_k \). Hence \( C \) is a strict open subscheme of \( \mathbb{P}^1_k \), containing strictly \( T \simeq G_{m,k} \). So \( C \simeq \mathbb{A}^1_k \), and thus \( G \simeq G_{a,k} \rtimes G_{m,k} \).

Remark 2.19. We use the term “conic” as a synonym of “projective curve of arithmetic genus 0”. Indeed, such a curve \( C \) is smooth, and the anticanonical bundle \( \omega_C^{-1} \) is very ample and yields a closed immersion \( C \hookrightarrow \mathbb{P}^2_k \) such that \( C \) is given by a homogeneous equation of degree 2. Let \( q \) be the corresponding quadratic form. Then we have \( \text{Aut}_C \simeq \text{PO}(q) \) and it is a form of \( \text{PGL}_{2,k} \).

The case 2 of Theorem 1.1 is a direct consequence.

Corollary 2.20. Let \( C \) a regular curve, \( G \) is a smooth connected algebraic group and \( \alpha : G \times C \rightarrow C \) an action. The action is faithful and \( C \) is almost homogeneous, but not homogeneous if and only if one of the following cases holds:

(a) \( C \simeq \mathbb{P}^1_k \) and \( G \simeq G_{a,k} \rtimes G_{m,k} \);

(b) \( G \) is a form of \( G_{a,k} \) and \( C \) is the regular completion of a \( G \)-torsor;

(c) \( C \simeq \mathbb{A}^1_k \) or \( \mathbb{P}^1_k \) and \( G \simeq G_{m,k} \);

(d) \( C \) is a smooth projective conic and \( G \) is the centralizer of a separable point of degree 2.

Proof. By Lemma 2.3, if \( C \) is almost homogeneous then it contains a \( G \)-stable open subscheme \( U \) which is a homogeneous curve. Moreover \( C \) is contained in the regular completion of \( U \). For the case d, we use Lemma 2.11.

3. Seminormal almost homogeneous curves

3.1. Normalization and pinching

We shall deduce the classification of seminormal almost homogeneous curves from the case of regular curves. In order to do so, we must link the action of a group \( G \) on a curve with the action on the normalized curve.

We first need to recover a variety \( X \) from its normalization \( \tilde{X} \). In case the field \( k \) is algebraically closed and the singular locus of \( X \) is a finite set, Jean-Pierre
Serre gave in [Se, Chap. IV] an explicit description by constructing the underlying space of $X$ and its structure sheaf. This can also be done in the more general language of “pinchings”.

The conductor $\mathcal{C}$ of the normalization $\nu : \tilde{X} \to X$ is the coherent sheaf of ideals defined as the annihilator

$$\mathcal{C} = \text{Ann}_{\mathcal{O}_X} \nu_* \mathcal{O}_{\tilde{X}} / \mathcal{O}_X.$$ 

For $x \in X$, the stalk $(\nu_* \mathcal{O}_{\tilde{X}})_x$ is the integral closure $\overline{\mathcal{O}}_{X,x}$ of $\mathcal{O}_{X,x}$ and we have

$$\mathcal{C}_x = \text{Ann}_{\mathcal{O}_{X,x}} (\nu_* \mathcal{O}_{\tilde{X}})_x / \mathcal{O}_{X,x} = \{ a \in \mathcal{O}_{X,x} \mid a (\nu_* \mathcal{O}_{\tilde{X}})_x \subseteq \mathcal{O}_{X,x} \}.$$ 

In other words, $\mathcal{C}_x$ is the largest ideal of $\mathcal{O}_{X,x}$ which is an ideal of $\overline{\mathcal{O}}_{X,x}$, that is to say, the conductor ideal in the classical sense.

Let $i : Z \to X$ the closed immersion defined by $\mathcal{C}$ and $\tilde{Z} = \nu^{-1}(Z)$. We have the cartesian square

$$
\begin{array}{ccc}
\tilde{Z} & \xrightarrow{j} & \tilde{X} \\
\lambda \downarrow & & \downarrow \nu \\
Z & \xrightarrow{i} & X
\end{array}
$$

called the conductor square of $X$.

**Lemma 3.1.**

1. The underlying space of $Z$ is the set of non-normal points of $X$.
2. The morphism $\nu$ induces an isomorphism from $\tilde{X} \setminus \tilde{Z}$ to $X \setminus Z$.
3. The morphism $\lambda : \tilde{Z} \to Z$ is finite and schematically dominant.
4. The square is cocartesian in the category of locally ringed spaces.
5. For every separable extension $K/k$,

$$
\begin{array}{ccc}
\tilde{Z}_K & \xrightarrow{j_K} & \tilde{X}_K \\
\lambda_K \downarrow & & \downarrow \nu_K \\
Z_K & \xrightarrow{i_K} & X_K
\end{array}
$$

is the conductor square of $X_K$.

**Proof.** The underlying space of $Z$ is $\{ x \in X \mid 1 \notin \mathcal{C}_x \} = \{ x \in X \mid \mathcal{O}_{X,x} \subseteq \overline{\mathcal{O}}_{X,x} \}$. Then $X \setminus Z$ is the normal locus of $X$ and $\nu$ induces an isomorphism from $\nu^{-1}(X \setminus Z) = \tilde{X} \setminus \tilde{Z}$ to $X \setminus Z$. Moreover $\lambda$ is finite and schematically dominant because so is $\nu$ and the square is cartesian.
The square is cocartesian if and only if for every open subscheme \( U \) of \( X \), the square
\[
\begin{array}{ccc}
\lambda^{-1}i^{-1}(U) & \xrightarrow{j} & \nu^{-1}(U) \\
\downarrow & & \downarrow
\lambda \\
i^{-1}(U) & \xrightarrow{i} & U
\end{array}
\]
is cocartesian. The morphism \( \nu : \nu^{-1}(U) \to U \) is the normalization and \( i^{-1}(U) \) is the closed subscheme of \( U \) defined by the conductor of \( \nu : \nu^{-1}(U) \to U \). Thus we can assume that \( X \) is affine. Let us write \( X = \text{Spec} \, A \). We have \( \tilde{X} = \text{Spec} \, \tilde{A} \) where \( \tilde{A} \) is the integral closure of \( A \). Let \( \mathfrak{c} = \{ a \in A \mid a\tilde{A} \subseteq \tilde{A} \} \) be the conductor ideal. For any prime ideal \( \mathfrak{p} \in \text{Spec} \, A \), the integral closure of the localization \( A_{\mathfrak{p}} \) is \( (\tilde{A})_{\mathfrak{p}} \) and, since \( \tilde{A} \) is a \( A \)-module of finite type, \( \mathfrak{c}_{\mathfrak{p}} \) is the conductor ideal of \( (\tilde{A})_{\mathfrak{p}} \) in \( A_{\mathfrak{p}} \). Thus \( \mathcal{C} \) is the coherent sheaf of ideals associated with \( \mathfrak{c} \) and we have \( Z = \text{Spec} \, A/\mathfrak{c} \) and \( \tilde{Z} = \text{Spec} \, \tilde{A}/\mathfrak{c} \). Consequently, by [Fe, Lem. 1.3 and Thm. 5.1] the square is cocartesian.

The variety \( \tilde{X}_K \) is normal and \( \nu_K \) is the normalization. In order to prove that the diagram obtained after the field extension to \( K \) is the conductor square, we can assume that \( X \) is affine. Then the result is a direct consequence of [Bou, I 2.10, Cor. 2, p. 40]. \( \square \)

Conversely Ferrand showed that under a mild assumption the variety \( X \) is obtained by “pinching” \( \tilde{Z} \) onto \( Z \). As a special case of [Fe, Thm. 5.4 and Prop. 5.6] we have the following result.

**Proposition 3.2.** Let \( \tilde{X} \) be a \( k \)-scheme, \( j : \tilde{Z} \to \tilde{X} \) a closed immersion and \( \lambda : \tilde{Z} \to Z \) a finite and schematically dominant morphism. If every finite set of points of \( \tilde{Z} \) is contained in an affine open subset of \( \tilde{X} \) then there exists a unique \( k \)-scheme \( X \) such that we have a square
\[
\begin{array}{ccc}
\tilde{Z} & \xrightarrow{j} & \tilde{X} \\
\downarrow & & \downarrow \nu \\
Z & \xrightarrow{i} & X
\end{array}
\]
which is cocartesian in the category of locally ringed spaces. Furthermore this square is cartesian, \( \nu \) is finite and schematically dominant, \( i \) is a closed immersion and \( \nu \) induces an isomorphism from \( \tilde{X} \setminus \tilde{Z} \) to \( X \setminus Z \). We call such a square a “pinching diagram”.

If \( \tilde{X} \) is a variety (resp. a proper variety), then so is \( X \). Finally, if \( \tilde{X} \) is normal then \( \nu \) is the normalization.

**Remark 3.3.**

i) A justification of the last part can be found in Sean Howe’s master’s thesis [Ho].
ii) If $\tilde{X}$ is a curve then the condition that every finite set of points of $\tilde{Z}$ is contained in an affine open subset of $\tilde{X}$ is automatically satisfied. In this case, $\tilde{Z}$ and $Z$ are finite schemes and $Z$ is determined by the injective morphism $\lambda: \mathcal{O}(Z) \to \mathcal{O}(\tilde{Z})$. This corresponds to the intuitive picture: a curve has finitely many singular points and to normalize this curve one “separates” the branches; conversely, the curve can be recovered from its normalization by gluing points together.

An action on the variety can be lifted to an action on the normalized variety.

**Lemma 3.4.** Let $G$ be a smooth algebraic group, $X$ a variety, $\alpha: G \times X \to X$ an action and

$$
\begin{array}{ccc}
\tilde{Z} & \xrightarrow{j} & \tilde{X} \\
\lambda & & \nu \\
Z & \xrightarrow{i} & X
\end{array}
$$

the conductor square. There exists a unique action $\tilde{\alpha}: G \times \tilde{X} \to \tilde{X}$ such that $\nu$ is equivariant. Moreover, the square

$$
\begin{array}{ccc}
G \times \tilde{X} & \xrightarrow{\tilde{\alpha}} & \tilde{X} \\
\text{id} \times \nu & & \nu \\
G \times X & \xrightarrow{\alpha} & X
\end{array}
$$

is cartesian and the actions $\alpha$ and $\tilde{\alpha}$ have the same kernel. Finally, the closed subschemes $Z$ and $\tilde{Z}$ are $G$-stable.

**Proof.** The existence of the action $\tilde{\alpha}$ is given in [Br17, Prop. 2.5.1]. The square is cartesian because

$$
\begin{align*}
\alpha: \begin{cases}
G \times X & \to G \times X, \\
(g, x) & \mapsto (g, gx)
\end{cases}
\end{align*}
$$

is an isomorphism making the diagram

$$
\begin{array}{ccc}
G \times X & \xrightarrow{u} & G \times X \\
\alpha & & \text{pr}_2 \\
& & X
\end{array}
$$

commute (and similarly for $\tilde{\alpha}$).

The smooth locus $U$ of $X$ and $\nu^{-1}(U)$ are schematically dense open subschemes of $X$ and $\tilde{X}$. Thus for any $k$-scheme $S$, the open subschemes $U_S$ and $\nu^{-1}(U)_S = (\nu \times \text{id}_S)^{-1}(U_S)$ are schematically dense in $X_S$ and $\tilde{X}_S$. They are $G_S$-stable and the morphism $\nu \times \text{id}_S: \nu^{-1}(U)_S \to U_S$ is a $G_S$-equivariant isomorphism. Hence an
element $g \in G(S)$ induces the identity on $X_S$ if and only if it induces the identity on $U_S$, if and only if it induces the identity on $\nu^{-1}(U)_S$, if and only if it induces the identity on $\tilde{X}_S$. Consequently $\alpha$ and $\tilde{\alpha}$ have the same kernel.

Let $\mathcal{C}$ be the conductor sheaf of $\nu$. By [EGA, IV2, Prop. 6.8.5], the scheme $G \times \tilde{X}$ is normal. Then $\text{id} \times \nu : G \times \tilde{X} \to G \times X$ is the normalization. It follows from [Bou, I.2.10, Cor. 2, p. 40] that $\alpha^* \mathcal{C}$ and $\text{pr}_2^* \mathcal{C}$ are both equal to the conductor sheaf of $\text{id} \times \nu$. In particular they are equal as subsheaves of $\mathcal{O}_{G \times X}$ (using the canonical isomorphisms $\alpha^* \mathcal{O}_X \simeq \mathcal{O}_{G \times X} \simeq \text{pr}_2^* \mathcal{O}_X$). Therefore we have $\alpha^{-1}(Z) = G \times Z$ as subschemes of $G \times X$, thus $Z$ is $G$-stable. Since the morphism $\nu$ is equivariant, $\tilde{Z}$ is $G$-stable too. \qed

**Lemma 3.5.** Let

$$
\begin{array}{c}
\tilde{Z} \xrightarrow{j} \tilde{X} \\
\downarrow \lambda \\
Z \xrightarrow{i} X
\end{array}
$$

be a pinching diagram as in Proposition 3.2. Let $G$ be an algebraic group and $\tilde{\alpha} : G \times \tilde{X} \to \tilde{X}$ and $\beta : G \times Z \to Z$ two actions. If the closed subscheme $\tilde{Z}$ is $G$-stable and $\lambda$ is equivariant then there exists a unique action $\alpha : G \times X \to X$ such that $Z$ is $G$-stable under this action, the induced action on $Z$ is $\beta$ and $\nu$ is equivariant.

**Proof.** It follows from [Ho, Thm. 3.11] that the square

$$
\begin{array}{c}
G \times \tilde{Z} \xrightarrow{id \times j} G \times \tilde{X} \\
\downarrow \text{id} \times \lambda \\
G \times Z \xrightarrow{id \times i} G \times X
\end{array}
$$

is cocartesian in the category of locally ringed spaces. Let $\tilde{\beta} : G \times \tilde{Z} \to \tilde{Z}$ be the action induced by $\tilde{\alpha}$. By assumption the diagram

$$
\begin{array}{c}
G \times \tilde{Z} \xrightarrow{id \times j} G \times \tilde{X} \\
\downarrow \text{id} \times \lambda \\
G \times Z \xrightarrow{id \times i} G \times X
\end{array}
\xrightarrow{\nu}
\begin{array}{c}
\tilde{X} \\
\text{id} \times \nu \\
\downarrow \alpha \\
X
\end{array}
$$

commutes. Then there exists a unique morphism $\alpha : G \times X \to X$ which completes it. It remains to prove that $\alpha$ is an action, that is to say the two composite
morphisms \( G \times G \times X \xrightarrow{\text{id} \times \alpha} G \times X \xrightarrow{\alpha} X \) and \( G \times G \times X \xrightarrow{\mu \times \text{id}} G \times X \xrightarrow{\alpha} X \) are equal, as well as \( \text{Spec} \ k \times X \xrightarrow{c \times \text{id}} G \times X \xrightarrow{\alpha} X \) and the second projection. This can be done in both cases by using the fact that the square

\[
\begin{array}{ccc}
G \times G \times \tilde{Z} & \longrightarrow & G \times G \times \tilde{X} \\
\downarrow && \downarrow \\
G \times G \times Z & \longrightarrow & G \times G \times X
\end{array}
\]

is cocartesian and showing that the two morphisms complete the same diagram. The details are left to the reader. \( \square \)

The normalization behaves well with respect to almost-homogeneity.

**Lemma 3.6.** Let \( G \) be a smooth connected algebraic group, \( X \) a variety with conductor square

\[
\begin{array}{ccc}
\tilde{Z} & \xrightarrow{j} & \tilde{X} \\
\downarrow_{\lambda} & & \downarrow_{\nu} \\
Z & \xrightarrow{i} & X
\end{array}
\]

\( \alpha : G \times X \to X \) an action and \( \tilde{\alpha} : G \times \tilde{X} \to \tilde{X} \) the action given by Lemma 3.4. The variety \( X \) is almost homogeneous if and only if \( \tilde{X} \) is almost homogeneous. In this case, if in addition \( X \) is a curve \( C \) then \( Z \) and \( \tilde{Z} \) are contained in the subschemes \( C^G \) and \( \tilde{C}^G \) of fixed points.

**Proof.** The morphisms \( \gamma : G \times X \to X \times X \) and \( \tilde{\gamma} : G \times \tilde{X} \to \tilde{X} \times \tilde{X} \) of Definition 2.1 make the diagram

\[
\begin{array}{ccc}
G \times \tilde{X} & \xrightarrow{\tilde{\gamma}} & \tilde{X} \times \tilde{X} \\
\downarrow_{\text{id} \times \nu} & & \downarrow_{\nu \times \nu} \\
G \times X & \xrightarrow{\gamma} & X \times X
\end{array}
\]

commute. The morphism \( \nu \) is surjective and closed, so \( \text{id} \times \nu \) is surjective and \( \nu \times \nu \) is surjective and closed. If \( \tilde{\gamma} \) is dominant then \( \gamma \) is dominant too.

Conversely, assume that \( X \) is almost homogeneous. Let \( U \) be the open orbit in \( X \), given by Lemma 2.3. Then \( U \) is smooth so \( \nu^{-1}(U) \simeq U \). Hence \( \nu^{-1}(U) \) is a \( G \)-stable and homogeneous open subscheme of \( \tilde{X} \), so \( \tilde{X} \) is almost homogeneous.

Assume that \( X \) is an almost homogeneous curve \( C \). By Lemma 2.6, \( U \) and \( \nu^{-1}(U) \) are the complements of the subschemes of fixed points. Since \( Z \) is the singular locus of \( C \) and \( U \) is smooth, \( Z \) is contained in \( C \setminus U \). Then \( \tilde{Z} = \nu^{-1}(Z) \) is contained in \( \tilde{C} \setminus \nu^{-1}(U) \). \( \square \)
3.2. Classification of seminormal almost homogeneous curves

A reminder on seminormality can be found in Appendix A. We have the following characterization of seminormal curves, which is a consequence of the well-known characterization [GT, Cor. 2.7] of seminormal Japanese rings satisfying Serre’s condition $S_2$ (a reduced noetherian ring of Krull dimension 1 satisfies this condition).

**Lemma 3.7.** Let $C$ be a curve and

$$
\begin{array}{ccc}
\tilde{Z} & \xrightarrow{j} & \tilde{C} \\
\downarrow{\lambda} & & \downarrow{\nu} \\
Z & \xrightarrow{i} & C
\end{array}
$$

the conductor square. The curve $C$ is seminormal if and only if $\tilde{Z}$ is reduced.

**Proof.** Let $U$ be an affine open subscheme of $C$. Then

$$
\begin{array}{ccc}
\lambda^{-1}i^{-1}(U) & \xrightarrow{j'} & \nu^{-1}(U) \\
\downarrow{\lambda} & & \downarrow{\nu} \\
i^{-1}(U) & \xrightarrow{i} & U
\end{array}
$$

is the conductor square. So by Corollary A.6, we can assume than $C$ is affine. Then the result follows from [GT, Cor. 2.7]. $\square$

In this case $Z$ is reduced too. Therefore if a smooth connected algebraic group $G$ acts on $C$ then by Lemmas 2.4 and 3.4, $G$ acts trivially on $Z$ and $\tilde{Z}$. The following lemma shows that the converse is true, even when we consider a pinching diagram which is not the conductor square.

**Lemma 3.8.** Let $C$ be a curve obtained by a pinching diagram

$$
\begin{array}{ccc}
\tilde{Z} & \xrightarrow{j} & \tilde{C} \\
\downarrow{\lambda} & & \downarrow{\nu} \\
Z & \xrightarrow{i} & C
\end{array}
$$

where $\tilde{C}$ is a regular curve and $\tilde{Z}$ is a reduced finite scheme. Then $C$ is seminormal. Moreover, if a smooth connected algebraic group $G$ acts on $\tilde{C}$ so that $\tilde{Z}$ is $G$-stable then the action on $\tilde{C}$ descends to an action on $C$ such that $Z$ is $G$-stable.
Proof. Let us show that \( C \) is seminormal. The morphism \( \nu \) induces an isomorphism from \( \tilde{C} \setminus \tilde{Z} \) to \( C \setminus Z \) so it suffices to show that for every point \( P \in Z \), the ring \( \mathcal{O}_{C,P} \) is seminormal. Since \( \lambda \) is schematically dominant and \( \tilde{Z} \) is reduced, \( Z \) is a reduced finite scheme. Let \( U \) be an affine open subscheme of \( C \) containing \( P \) but no other point of \( Z \). The square

\[
\begin{array}{ccc}
\lambda^{-1}(\text{Spec } \kappa(P)) & \longrightarrow & \nu^{-1}(U) \\
\downarrow \quad \lambda & & \quad \downarrow \nu \\
\text{Spec } \kappa(P) & \longrightarrow & U
\end{array}
\]

is still a pinching diagram. Thus we can assume that \( C \) is affine and \( Z = \text{Spec } \kappa(P) \).

We write \( C = \text{Spec } A \) and \( \tilde{C} = \text{Spec } \overline{A} \) where \( \overline{A} \) is the integral closure of \( A \). The point \( P \) corresponds to a maximal ideal \( m \) of \( A \) and the points of \( \tilde{Z} \) to the maximal ideals \( \overline{m}_1, \ldots, \overline{m}_r \) of \( \overline{A} \) above \( m \). Then \( Z = \text{Spec } A/m \) and \( \tilde{Z} = \bigsqcup_{i=1}^r \text{Spec } \overline{A}/\overline{m}_i = \text{Spec } \overline{A}/(\overline{m}_1 \cdots \overline{m}_r) \). By [Fe, Lem. 1.3], we have \( \overline{m}_1 \cdots \overline{m}_r = m \). In other words, the maximal ideal \( mA_m \) of \( \mathcal{O}_{C,P} \) is the Jacobson radical of its integral closure \( \overline{\mathcal{O}}_{C,P} \). Since the only prime ideals of \( \mathcal{O}_{C,P} \) are \( (0) \) and \( mA_m \) (because \( \mathcal{O}_{C,P} \) is an integral local ring of Krull dimension 1), it follows immediately from the definition of \( \mathcal{O}_{C,P}^+ \) that \( \mathcal{O}_{C,P} \) is seminormal.

By Lemma 2.4, \( G \) acts trivially on \( Z \) and \( \tilde{Z} \). Hence by Lemma 3.5, the action of \( G \) on \( \tilde{C} \) descends to an action on \( C \). \( \square \)

A consequence of Lemma 3.8 is that the property of being a seminormal almost homogeneous curve descends by separable field extensions (we do not use this result in the sequel, but it has its own interest). To see this, we first need the following particular case of [CGP, Lem. C.4.1, p. 649].

Lemma 3.9. Let \( G \) be an algebraic group. There exists a largest smooth subgroup \( G^{\text{sm}} \) of \( G \). Moreover, for every separable field extension \( K/k \), we have \( G^{\text{sm}}(K) = G(K) \) and \( (G_K)^{\text{sm}} = (G^{\text{sm}})_K \).

Proposition 3.10. Let \( C \) be a seminormal curve and \( K/k \) a separable field extension. If \( C_K \) is almost homogeneous under the action of a smooth connected algebraic group then so is \( C \).

Proof. Let \( G \) be a smooth connected algebraic group over \( K \) acting on \( C_K \) such that \( C_K \) is almost homogeneous. We can assume that the action is faithful. Let

\[
\begin{array}{ccc}
\tilde{Z} & \longrightarrow & \tilde{C} \\
\lambda \downarrow & & \nu \downarrow \\
Z & \longrightarrow & C
\end{array}
\]

be the conductor square, \( \tilde{C} \) the regular completion of \( C \) and \( Z' = (\tilde{C} \setminus \tilde{C}) \cup \tilde{Z} \) endowed with its structure of a reduced closed subscheme of \( \tilde{C} \). Since the extension
$K/k$ is separable, by Lemma 3.1 the diagram

\[
\begin{array}{ccc}
\tilde{Z}_K & \longrightarrow & \tilde{C}_K \\
\downarrow & & \downarrow \\
Z_K & \longrightarrow & C_K
\end{array}
\]

is still the conductor square. Moreover $\tilde{C}_K$ is the regular completion of $\tilde{C}_K$ and $Z'_K = (\tilde{C}_K \setminus \tilde{C}_K) \sqcup \tilde{Z}_K$ is reduced. By Lemmas 2.14 and 3.4, the action of $G$ on $C_K$ lifts to an action on $\tilde{C}_K$ such that $Z'_K$ is fixed. In other words, up to isomorphism, $G$ is a subgroup of the centralizer $\text{Aut}_{\tilde{C}_K,Z'_K}$ of $Z'_K$ in $\text{Aut}_{\tilde{C}_K}$. In addition $G$ is smooth, so by Lemma 3.9, it is a subgroup of $(\text{Aut}_{\tilde{C}_K,Z'_K})^\text{sm}$. Then $\tilde{C}_K$ is almost homogeneous under the action of $(\text{Aut}_{\tilde{C}_K,Z'_K})^\text{sm}$. We have $(\text{Aut}_{\tilde{C}_K,Z'_K})^\text{sm} = ((\text{Aut}_{\tilde{C},Z'})^\text{sm})_K$ so $\tilde{C}$ is almost homogeneous under the action of $(\text{Aut}_{\tilde{C},Z'})^\text{sm}$. By definition, this group acts trivially on $\tilde{Z}$ so, by Lemma 3.5, the action on $\tilde{C}$ descends to an action on $C$ and $C$ is almost homogeneous. Finally, by Lemma 2.8, $C$ is almost homogeneous under the action of $((\text{Aut}_{\tilde{C},Z'})^\text{sm})^\circ$.  

\[\square\]

Remark 3.11.

i) Proposition 3.10 does not extend to inseparable extensions, even for smooth curves. Indeed by Lemma 2.17 there exists a curve $C$ and a purely inseparable extension $K/k$ such that $C_K \simeq \mathbb{A}^1_K$ and every action of a smooth connected algebraic group on $C$ is trivial.

ii) With the notations of the proof, the largest smooth connected group acting faithfully on $C$ is the neutral component $((\text{Aut}_{\tilde{C},Z'})^\text{sm})^\circ$.

We deduce the classification of seminormal almost homogeneous curves, which is the case 3 of Theorem 1.1.

**Theorem 3.12.** Let $C$ be a singular seminormal curve, $G$ a smooth connected algebraic group and $\alpha : G \times C \to C$ an action. The action is faithful and $C$ is almost homogeneous if and only if one of the following cases holds:

(a) $G$ is a non-trivial form of $G_{a,k}$ and $C$ is obtained by pinching the point at infinity $\tilde{P}$ of the regular completion of a $G$-torsor on a point $P$ whose residue field $k(P)$ is a strict subextension of $k(\tilde{P})/k$;

(b) $C$ is obtained by pinching two $k$-rational points of $\mathbb{P}^1_k$ on a $k$-rational point and $G \simeq G_{m,k}$;

(c) $C$ is obtained by pinching a separable point $\tilde{P}$ of degree 2 of a smooth projective conic $\tilde{C}$ on a $k$-rational point, and $G$ is the centralizer of $\tilde{P}$ in $\text{Aut}_{\tilde{C}}$.

**Proof.** Let $\nu : \tilde{C} \to C$ be the normalization. By Lemmas 3.6 and 3.8, $C$ is almost homogeneous if and only if $\tilde{C}$ is almost homogeneous, and $C$ is obtained by pinching a set $\tilde{Z}$ of fixed points of $\tilde{C}$ endowed with its structure of a reduced
closed subscheme, on a reduced scheme \( Z \). The different possible pairs \( (\tilde{C}, G) \) are given by Corollary 2.20.

We cannot have \( \tilde{C} = \mathbb{P}^1_k \) and \( G = G_{a,k} \times G_{m,k} \) or \( G_{a,k} \) since we would have to pinch the \( k \)-rational point \( \infty \) on a \( k \)-rational point, so \( \nu \) would be an isomorphism and \( C \) would be regular. Similarly, we cannot have \( \tilde{C} = \mathbb{A}^1_k \) and \( G = G_{m,k} \).

If \( G \) is a non-trivial form of \( G_{a,k} \) and \( \tilde{C} \) is the regular completion of a \( G \)-torsor \( U \) with point at infinity \( \tilde{P} \) then \( \tilde{C} \setminus U = \{ \tilde{P} \} \). So we have to pinch \( \tilde{P} \) on a point \( P \). If \( \kappa(P) = \kappa(\tilde{P}) \) then \( \nu \) would be an isomorphism.

Similarly, if \( \tilde{C} \) is a smooth projective conic and \( G \) is the centralizer of a separable point \( \tilde{P} \) of degree 2 then we have to pinch \( \tilde{P} \) on a point \( P \), and \( \kappa(P) \not\subseteq \kappa(\tilde{P}) \).

If \( \tilde{C} = \mathbb{P}^1_k \) and \( G = G_{m,k} \) then we have to pinch 0 or \( \infty \) or \( \tilde{Z} = \{ 0, \infty \} = \text{Spec} \ k \sqcup \text{Spec} \ k \). As above, we cannot pinch only 0 or \( \infty \). The reduced scheme \( Z \) consists of one or two points. It cannot consist of two points, because those would have to be \( k \)-rational and \( \nu \) would again be an isomorphism. Therefore \( Z \) consists of one point, which must be \( k \)-rational. \( \square \)

Remark 3.13.

i) Let \( G \) be a non-trivial form of \( G_{a,k} \), \( \tilde{C} \) the regular completion of a \( G \)-torsor, \( \tilde{P} \) the point at infinity, \( K_1 \) and \( K_2 \) strict subextensions of \( \kappa(\tilde{P})/k \), and \( C_1 \) and \( C_2 \) the curves obtained by the pinching diagrams

\[
\begin{align*}
\text{Spec} \kappa(\tilde{P}) & \xrightarrow{j} \tilde{C} \\
\lambda_1 & \downarrow \quad \nu_1 \\
\text{Spec} K_1 & \xrightarrow{i_1} C_1
\end{align*}
\quad \text{and} \quad
\begin{align*}
\text{Spec} \kappa(\tilde{P}) & \xrightarrow{j} \tilde{C} \\
\lambda_2 & \downarrow \quad \nu_2 \\
\text{Spec} K_2 & \xrightarrow{i_2} C_2
\end{align*}
\]

These diagrams are the conductor squares. If there exists an isomorphism \( \psi : C_1 \to C_2 \) then, by the universal property of normalization, \( \psi \) can be lifted to an automorphism \( \tilde{\psi} \) of \( \tilde{C} \). Then \( \tilde{\psi} \) induces an automorphism of \( \kappa(\tilde{P}) \) mapping \( K_1 \) to \( K_2 \). But the extension \( \kappa(\tilde{P})/k \) is purely inseparable, so the unique automorphism of \( \kappa(\tilde{P}) \) is the identity and we must have \( K_1 = K_2 \). Therefore, distinct subextensions yield distinct curves.

ii) The family of seminormal curves which are almost homogeneous under the action of a non-trivial form of \( G_{a,k} \) can be very large (see the following example).

Example 3.14. Set \( p = 2 \), \( k = \mathbb{F}_2(a,b) \) and consider the form \( G \) of \( G_{a,k} \) given by the equation \( y^4 = x + ax^2 + b^2x^4 \) (see Proposition 2.12). We claim that the residue field of the point at infinity of \( G \) is \( \mathbb{F}_2(a^{1/2},b^{1/2}) \). For \( c \in k \), \( k(a^{1/2} + cb^{1/2}) \) is a subextension of \( \mathbb{F}_2(a^{1/2},b^{1/2})/k \), and different values of \( c \) yield pairwise non-isomorphic subextensions. As a consequence, we have a family parameterized by \( k \) of almost homogeneous curves under the action of \( G \) with the same normalization.

We now prove our claim on the residue field. The scheme-theoretic closure \( \overline{G} \) of \( G \) in \( \mathbb{P}_k^2 \) has the homogeneous equation \( y^4 = xz^3 + ax^2z^2 + b^2x^4 \) and the regular completion of \( G \) is the normalization of \( \overline{G} \). In the chart \( (x = 1) \), the equation reads \( y^4 = z^3 + az^2 + b^2 \). Set \( A = k[y,z]/(y^4 - z^3 - az^2 - b^2) \) and
$B = k[y,w]/(y^2 - w^3 - aw - b)$. Let us show that the normalization in this chart is given by the ring morphism $A \to B$ sending $z$ to $w^2 - a$. First this morphism in injective and finite. The ring $B$ is an integral domain and has the same fraction field as $A$ because $w = (y^2 - b)/z$. By the Jacobian criterion, every point of Spec $B$ is smooth except the point given by the ideal $m = (w^2 - a)$ (which, besides, corresponds to the point at infinity of $G$). This point is nonetheless regular because $m$ is principal. Hence $B$ is integrally closed and is the integral closure of $A$. Thus the residue field of the point at infinity is

$$k[y,w]/(y^2 - w^3 - aw - b)/\mathfrak{m} \simeq F_2(a^{1/2}, b^{1/2}).$$

4. Arbitrary curves

Popov gave a classification of almost homogeneous curves over an algebraically closed field in [Po, Chap. 7] and used the language of Serre ([Se, Chap. IV]) to describe them. We extend the result to an arbitrary field (except for almost homogeneous curves under the action of $G_{a,k}$ or a form of $G_{a,k}$) by using the language of pinchings.

4.1. Almost homogeneous curves under the action of $G_{a,k}$ in characteristic zero

We consider the projective coordinates $[t : u]$ on $P^1_k$. For $n \geq 0$, we denote by $P^1_{k,n}$ the curve defined by the pinching diagram

$$\begin{array}{ccc}
\text{Spec } k[u]/(u^{n+1}) & \overset{j}{\longrightarrow} & P^1_k \\
\downarrow \lambda & & \downarrow \nu \\
\text{Spec } k & \overset{i}{\longrightarrow} & P^1_{k,n}
\end{array}$$

where Spec $(k[u]/(u^{n+1}))$ is the $n$th infinitesimal neighborhood of the point $\infty$. By Lemma 3.5 the actions of $G_{a,k}$ and $G_{a,k} \ltimes G_{m,k}$ on $P^1_k$ descend to faithful actions on $P^1_{k,n}$.

**Theorem 4.1.** We assume that $k$ has characteristic zero. Let $C$ be a curve and $\alpha : G_{a,k} \times C \to C$ an action. The action is faithful (and $C$ is almost homogeneous) if and only if $C \simeq A^1_k$ or if there exists $n \geq 0$ such that $C \simeq P^1_{k,n}$, and the action is the natural one.

**Proof.** We adapt Popov’s proof of [Po, Thm. 1.1, p. 171]. Assume that the action
is faithful (so $C$ is almost homogeneous). Let

\[
\begin{array}{ccc}
\tilde{Z} & \xrightarrow{j} & \tilde{C} \\
\lambda & \downarrow & \nu \\
Z & \xrightarrow{i} & C
\end{array}
\]

be the conductor square. Every $G_{a,k}$-torsor is trivial so, by Theorem 1.1, we have $\tilde{C} = \mathbb{A}^1_k$ or $\tilde{C} = \mathbb{P}^1_k$ (with the natural action of $G_{a,k}$). In the first case $\tilde{C}$ is homogeneous so $C$ is homogeneous too and $C = \mathbb{A}^1_k$. We assume henceforth that $\tilde{C} = \mathbb{P}^1_k$. By Lemma 3.6, $C$ is obtained by pinching a closed subscheme of $\mathbb{P}^1_k$ supported by the point $\infty$. Thus we write $\tilde{Z} = \text{Spec} \ k[u]/(u^{N+1})$ for some $N \geq 0$. Then $A = \mathcal{O}(Z)$ is a subalgebra of $k[u]/(u^{N+1})$ which is $G_{a,k}$-invariant.

An element $a \in G_{a,k}(k)$ acts on $\mathbb{P}^1_k$ by the matrix

\[
\begin{pmatrix}
1 & a \\
0 & 1
\end{pmatrix}
\]

that is, for a point with projective coordinates $[1 : u]$, we have $a \cdot [1 : u] = [1 + au : u] = [1 : u/(1 + au)]$. We still denote by $u$ the image of $u$ in $k[u]/(u^{N+1})$. In $k[u]/(u^{N+1})$, the element $1 + au$ is invertible and we have $u/(1 + au) = u - au^2 + \cdots + (-1)^{N-1}a^{N-1}u^{N-1}$. Moreover, for $1 \leq i \leq N$, we have $a \cdot u^i = (a \cdot u)^i$. Hence in the basis $(1, u, \ldots, u^N)$, the endomorphism of $k[u]/(u^{N+1})$ induced by $a$ has a triangular matrix of the form

\[
\begin{pmatrix}
1 & a & 0 & \cdots & 0 \\
0 & 1 & a & \cdots & 0 \\
\vdots & 0 & 1 & \ddots & \vdots \\
0 & \cdots & 0 & 1 & -(N-1)a \\
0 & \cdots & 0 & -a & 1
\end{pmatrix}
\]

The subspaces $k$ and $V = \text{Vect}(u, \ldots, u^N)$ of $k[u]/(u^{N+1})$ are $G_{a,k}$-stable. We can write $A = k \oplus (A \cap V)$. Let us notice that $A \cap V$ is the unique maximal ideal $m$ of $A$. Then $A$ is $G_{a,k}$-stable if and only if $m$ is stable. The image of $G_{a,k}$ in $\text{GL}(V)$ is a unipotent group so, by the Lie-Kolchin theorem, every nonzero stable subspace of $V$ contains a stable line. But, since $k$ has characteristic zero, the above matrix shows that the only stable line in $V$ is $\text{Vect}(u^N)$. If $m = 0$ then $A = k$. We now assume that $m$ is stable and nonzero. We extend $u^N$ to a basis $(e_1, \ldots, e_d, u^N)$ of $m$. By putting the coordinates of the $e_i$ in an echelon form, we can assume that there exist integers $n_1, \ldots, n_{d+1}$ and scalars $b_{i,j} \in k$ such that we have

\[1 \leq n_1 < \ldots < n_d < n_{d+1} = N\]

and, for $1 \leq i \leq d$,

\[e_i = u^{n_i} + b_{i,n_i+1}u^{n_i+1} + \cdots + b_{i,N-1}u^{N-1} + b_{i,N}u^N.\]

Then in the basis $(1, u, \ldots, u^N)$ of $k[u]/(u^{N+1})$, the $n_i$ first coordinates of $a \cdot e_i$ equal zero, the $(n_i + 1)$th one equals 1 and the $(n_i + 2)$th one equals $-n_ia + b_{i,n_i+1}$
(which is therefore different from $b_{i,n_i+1}$ if $a \neq 0$). Thus for the images of $e_i$ to be in $m$, we must have $n_{i+1} = n_i + 1$. So $m = \text{Vect}(u^{n_1}, u^{n_1+1}, \ldots, u^N)$ and $A = k[u^{n_1}, u^{n_1+1}, \ldots, u^N, u^{N+1}] / (u^{N+1})$, which conversely is $G_{a,k}$-stable.

The ideal $u^{n_1}$ of $k[u] / (u^{N+1})$ is also an ideal of $k[u^{n_1}, u^{n_1+1}, \ldots, u^N, u^{N+1}] / (u^{N+1})$. Hence by [Fe, Lem. 1.3] the square of rings

$$
\begin{array}{ccc}
\frac{k[u^{n_1}, u^{n_1+1}, \ldots, u^N]}{(u^{N+1})} & \rightarrow & \frac{k[u]}{(u^{N+1})} \\
\downarrow & & \downarrow \\
k \simeq \frac{k[u^{n_1}, u^{n_1+1}, \ldots, u^N]}{(u^{n_1})} & \rightarrow & \frac{k[u]}{(u^{n_1})} \simeq \frac{k[u]}{(u^{n_1})}
\end{array}
$$

is cartesian. Moreover the morphisms in this square are $G_{a,k}$-equivariant. So the square

$$
\begin{array}{ccc}
\text{Spec } k[u] / (u^{n_1}) & \longrightarrow & \tilde{Z} \\
\lambda' \downarrow & & \downarrow \lambda \\
\text{Spec } k & \longrightarrow & Z
\end{array}
$$

is a pinching diagram with $G_{a,k}$-equivariant morphisms. By concatenation, we get a square

$$
\begin{array}{ccc}
\text{Spec } k[u] / (u^{n_1}) & \longrightarrow & \mathbb{P}^1_k \\
\lambda' \downarrow & & \downarrow \nu \\
\text{Spec } k & \longrightarrow & C
\end{array}
$$

which is a pinching diagram with $G_{a,k}$-equivariant morphisms. \ \Box

Remark 4.2.

i) Contrary to what happened in Remark 3.13 with non-trivial forms of $G_{a,k}$, if $k$ has characteristic zero then the family of curves which are almost homogeneous under the action of $G_{a,k}$ is countable.

ii) The proof of Theorem 4.1 strongly uses the fact that $k$ has characteristic zero and does not seem to be immediately adaptable in positive characteristic, since it leads to a problem of representations of $G_{a,k}$ which are not so well understood. Moreover, Remark 3.13 again suggests that for forms of $G_{a,k}$ the classification should be substantially more complicated, even in the seminormal case. Therefore the classification problem remains open for almost homogeneous curves under the action of $G_{a,k} \rtimes G_{m,k}$ or a form of $G_{a,k}$ in positive characteristic.
Corollary 4.3. We assume that \( k \) has characteristic zero. Let \( C \) be a curve and \( \alpha : G_{a,k} \times G_{m,k} \times C \to C \) an action. The action is faithful (and \( C \) is almost homogeneous) if and only if \( C \cong \mathbb{A}^1_k \) or if there exists \( n \geq 0 \) such that \( C \cong \mathbb{P}^1_{k,n} \), and the action is the natural one.

Proof. We use the previous notations. It follows from Theorem 1.1 that if \( G_{a,k} \times G_{m,k} \) acts faithfully on \( C \) then we have \( \tilde{C} \cong \mathbb{A}^1_k \) or \( \mathbb{P}^1_{k} \) (with the natural action).

The subschemes \( \tilde{Z} \) and \( Z \) are stable under the action of \( G_{a,k} \times G_{m,k} \) so they are stable under the action of the subgroup \( G_{a,k} \). Thus \( C \) is one of the curves given in Theorem 4.1. Conversely, \( G_{a,k} \times G_{m,k} \) acts faithfully on these curves.

4.2. Almost homogeneous curves under the action of a form of \( G_{m,k} \)

In this section we first classify the almost homogeneous curves under the action of \( G_{m,k} \). Then we use the link between the forms of \( G_{m,k} \) and the conics to deduce the classification of almost homogeneous curves under the action of a form of \( G_{m,k} \).

We consider again the projective coordinates \([t : u]\) on \( \mathbb{P}^1_k \). If \( \mathcal{J} \) is a subsemigroup of \((\mathbb{N},+)\) containing 0 and all the integers that are large enough then there exist a unique integer \( m \) and a unique tuple \( c = (c_0, \ldots, c_p) \) such that \( 0 = c_0 < \cdots < c_p < m \) and \( \mathcal{J} = \{c_0, \ldots, c_p\} \cup \{r \in \mathbb{N} \mid r \geq m + 1\} \). The subsemigroup \( \mathcal{J} \) is determined by \( m \) and \( c \), and we denote it by \( \mathcal{J}_m(c) \).

For any subsemigroups \( \mathcal{J}_m(c) \) and \( \mathcal{J}_n(d) \) of \((\mathbb{N},+)\), we denote respectively by \( \mathbb{A}^1_{k,m,n}(c,d) \), \( \mathbb{P}^1_{k,m,n}(c,d) \) and \( \mathbb{P}^1_{k,m,n}(c,d)' \) the curves defined by the pinching diagrams

\[
\begin{array}{ccc}
Z_m & \longrightarrow & \mathbb{A}^1_k \\
\downarrow & & \downarrow \\
Z_m(c) & \longrightarrow & \mathbb{A}^1_{k,m}(c) \\
\downarrow & & \downarrow \\
Spec k \sqcup Spec k & \longrightarrow & \mathbb{P}^1_{k,m,n}(c,d) \\
\downarrow & & \downarrow \\
Spec k & \longrightarrow & \mathbb{P}^1_{k,m,n}(c,d)'
\end{array}
\]

where \( Z_m = Spec k[t]/(t^{m+1}) \) and \( Z_n = Spec k[u]/(u^{n+1}) \) are the \( m \)th and \( n \)th infinitesimal neighborhoods of the points 0 and \( \infty \),

\[
Z_m(c) = Spec k[t^{c_0}, \ldots, t^{c_p}, t^{m+1}]/(t^{m+1}),
\]

\[
Z_n(d) = Spec k[u^{d_0}, \ldots, u^{d_q}, u^{n+1}]/(u^{n+1}),
\]

and \( Spec k \sqcup Spec k \) is the reduced subscheme of \( Z_m(c) \sqcup Z_n(d) \).

Remark 4.4. For example, \( \mathbb{A}^1_k = \mathbb{A}^1_{k,m}(c) \) and \( \mathbb{P}^1_k = \mathbb{P}^1_{k,m,n}(c,d) \) for \( m = n = 0 \) and \( c = d = (0) \).

Theorem 4.5. Let \( C \) be a curve. The group \( G_{m,k} \) acts non-trivially on \( C \) if and only if \( C \cong \mathbb{A}^1 \setminus \{0\} \) or if there exist subsemigroups \( \mathcal{J}_m(c) \) and \( \mathcal{J}_n(d) \) of \((\mathbb{N},+)\) such that \( C \) is isomorphic to \( \mathbb{A}^1_{k,m}(c) \), \( \mathbb{P}^1_{k,m,n}(c,d) \) or \( \mathbb{P}^1_{k,m,n}(c,d)' \).
The argument is essentially the same as in the proof of Popov’s theorem \([Po, \text{Thm. 1.2, p. 171}]\) for \(A_{k,m}(\mathcal{C})\) and \(P_{k,m,n}(\mathcal{C},d)\). For \(P_{k,m,n}(\mathcal{C},d)'\) the argument differs from \([Po, \text{Thm. 1.3, p. 175}]\); the classification of the curves in terms of conductor squares is obtained rather quickly and then we just need to simplify the diagrams, as in the proof of Theorem 4.1.

It follows from Lemmas 3.5 and 3.6 that \(G_{m,k}\) acts on \(A_{k,m}(\mathcal{C})\), \(P_{k,m,n}(\mathcal{C},d)\) and \(P_{k,m,n}(\mathcal{C},d)'\). Conversely, assume that \(G_{m,k}\) acts non-trivially on \(C\). Let

\[
\begin{array}{c}
\tilde{Z} \\
\downarrow \\
Z
\end{array} \longrightarrow \begin{array}{c}
\tilde{C} \\
\downarrow \\
C
\end{array}
\]

be the conductor square. By Theorem 1.1 and Lemma 3.6, \(\tilde{C}\) is isomorphic either to \(A_{k}^1\) or to \(P_{k}^1\).

We first treat the case \(\tilde{C} \simeq A_{k}^1\). The closed subscheme \(\tilde{Z}\) of \(A_{k}^1\) is supported by 0 so we can write \(\tilde{Z} = Z_M = \text{Spec} k[t]/(t^{M+1})\) for some integer \(M\). Then \(Z = \text{Spec} A\) for some \(G_{m,k}\)-invariant subalgebra \(A\) of \(k[t]/(t^{M+1})\). We still denote by \(t\) the image of \(t\) in \(k[t]/(t^{M+1})\). For \(a \in G_{m,k}(k_s)\) and \(0 \leq i \leq M\), we have

\[
a \cdot t^i = a^i t^i
\]

so \(t^i\) is an eigenvector of weight \(i\). Thus the \(k\_s\)-vector space \(k_s[t]/(t^{M+1})\) is the direct sum of stable lines for pairwise distinct weights. Hence the subspace \(A \otimes_k k_s\) of \(k_s[t]/(t^{M+1})\) is spanned by some \(t^i\)'s. Since it is a subring, if it contains \(t^i\) and \(t^j\) then it contains \(t^{i+j}\) too. Thus there exists a subsemigroup \(\mathcal{J}_m(\mathcal{C})\) of \((N,+)\) such that \(m \leq M\) and \(A \otimes_k k_s\) is spanned by the \(t^i\)'s for \(i \in \mathcal{J}_m(\mathcal{C})\), that is,

\[
A \otimes_k k_s = k_s[t^{c_0},\ldots,t^{c_p},t^{m+1},\ldots]/(t^{M+1}).
\]

By Galois descent for vector subspaces, we consequently have \(A = k[t^{c_0},\ldots,t^{c_p},t^{m+1},\ldots]/(t^{M+1})\), which conversely is a \(G_{m,k}\)-stable subalgebra of \(k[t]/(t^{M+1})\). Finally, as in the proof of Theorem 4.1, we can replace \(k[t]/(t^{M+1})\) and \(k[t^{c_0},\ldots,t^{c_p},t^{m+1},\ldots]/(t^{M+1})\) with \(k[t]/(t^{m+1})\) and \(k[t^{c_0},\ldots,t^{c_p},t^{m+1},\ldots]/(t^{m+1})\) with \(k[t]/(t^{m+1})\) and \(k[t^{c_0},\ldots,t^{c_p},t^{m+1},\ldots]/(t^{m+1})\). Therefore \(C \simeq A_{m,k}(\mathcal{C})\).

We now assume \(\tilde{C} \simeq P_{k}^1\). Similarly, the closed subscheme \(\tilde{Z}\) of \(P_{k}^1\) is supported by \([0, \infty)\) so we can write \(\tilde{Z} = Z_M \cup Z_N\) for some integers \(M\) and \(N\). Then \(Z\) is supported by one or two \(k\)-rational points.

In the latter case there exist \(G_{m,k}\)-invariant subalgebras \(A\) and \(B\) of \(k[t]/(t^{M+1})\) and \(k[u]/(u^{N+1})\) such that \(Z = \text{Spec} A \cup \text{Spec} B\), and by the same arguments as before we get an isomorphism \(C \simeq P_{k,m,n}(\mathcal{C},d)\).

In the former case there exists a \(G_{m,k}\)-invariant subalgebra \(A\) of \(k[t]/(t^{M+1})\) such that \(Z = \text{Spec} A\). Since \((t^i,0)\) and \((0,u^j)\) are eigenvectors of respective weights \(i\) and \(-j\), the subspace \(A \otimes_k k_s\) of \(k_s[t]/(t^{M+1}) \times k_s[u]/(u^{N+1})\) is spanned by some \((t^i,0)'s\) and \((0,u^j)'s\) with \(i \geq 1\) and \(j \geq 1\), and a subspace of \(k_s \times k_s\) containing \(k_s \cdot (1,1)\) (which is \(k_s \cdot (1,1)\) or the whole \(k_s \times k_s\). As above and by Galois descent for vector subspaces, there exist two subsemigroups \(\mathcal{J}_m(\mathcal{C})\) and \(\mathcal{J}_n(\mathcal{C})\) of \((N,+)\) such that \(A\) is equal to

\[
\frac{\text{Vect}(t^{c_1},\ldots,t^{c_p},t^{m+1},\ldots)}{(t^{M+1})} \times \frac{\text{Vect}(u^{d_1},\ldots,u^{d_q},u^{n+1},\ldots)}{(u^{N+1})} \oplus k \cdot (1,1)
\]
Moreover, $A$ has a unique maximal ideal, so we must have

$$A = \frac{\text{Vect}(t_{c_1}, \ldots, t_{c_p}, t_{m+1}, \ldots)}{(t_{M+1})} \times \frac{\text{Vect}(u_{d_1}, \ldots, u_{d_q}, u^{n+1}, \ldots)}{(u_{N+1})} \oplus k \cdot (1, 1).$$

By the same argument as above again, we can replace $k[t]/(t_{M+1}) \times k[u]/(u_{N+1})$ and $A$ with $k[t]/(t_{m+1}) \times k[u]/(u_{n+1})$ and

$$A' = \frac{\text{Vect}(t_{c_1}, \ldots, t_{c_p}, t_{m+1})}{(t_{m+1})} \times \frac{\text{Vect}(u_{d_1}, \ldots, u_{d_q}, u^{n+1})}{(u_{n+1})} \oplus k \cdot (1, 1).$$

We have a pinching diagram

$$\begin{array}{ccc}
Z_m \sqcup Z_n & \longrightarrow & \mathbb{P}^1_k \\
\downarrow & & \downarrow \\
\text{Spec } A' & \longrightarrow & C
\end{array}$$

Let $C'$ be the curve defined by the pinching diagram

$$\begin{array}{ccc}
Z_m(c) \sqcup Z_n(d) & \longrightarrow & \mathbb{P}^1_{k,m,n}(c, d) \\
\downarrow & & \downarrow \\
\text{Spec } A' & \longrightarrow & C'
\end{array}$$

The diagram

$$\begin{array}{ccc}
Z_m \sqcup Z_n & \longrightarrow & \mathbb{P}^1_k \\
\downarrow & & \downarrow \\
Z_m(c) \sqcup Z_n(d) & \longrightarrow & \mathbb{P}^1_{k,m,n}(c, d) \\
\downarrow & & \downarrow \\
\text{Spec } A' & \longrightarrow & C'
\end{array}$$

is commutative and the two squares are cocartesian, so the entire composed square is cocartesian. By unicity of the scheme obtained by pinching, we have $C' \simeq C$. By [Fe, Lem. 1.3], the square of rings

$$\begin{array}{ccc}
A' & \longrightarrow & \frac{k[t_{c_0}, \ldots, t_{c_p}, t_{m+1}]}{(t_{m+1})} \times \frac{k[u_{d_0}, \ldots, u_{d_q}, u^{n+1}]}{(u^{n+1})} \\
\downarrow & & \downarrow \\
k & \longrightarrow & k \times k
\end{array}$$
is cartesian. Moreover the morphisms in this square are $G_{m,k}$-equivariant. Therefore we have a pinching diagram

$$
\begin{array}{ccc}
\text{Spec } k \sqcup \text{Spec } k & \longrightarrow & \mathbb{P}^1_{k,m,n}(\xi, \delta) \\
\downarrow & & \downarrow \\
\text{Spec } k & \longrightarrow & C
\end{array}
$$

\[\Box\]

Let $G$ be a non-trivial form of $G_{m,k}$. Let $\tilde{C}$ be a smooth projective conic and $\tilde{P}$ a separable point of degree 2 of $\tilde{C}$ such that $G$ is the centralizer of $\tilde{P}$ in $\text{Aut}_{\tilde{C}}$ (see Lemma 2.11). Let $K = \kappa(\tilde{P})$, $\Gamma = \text{Gal}(K/k) \cong \mathbb{Z}/2\mathbb{Z} \cdot 3m(\xi)$ a subsemigroup of $(\mathbb{N}, +)$ and $\tilde{Y}$ the $m$th infinitesimal neighborhood of $\tilde{P}$ (which is a $G$-stable closed subscheme of $\tilde{C}$). There exists an isomorphism $\tilde{C}_K \cong \mathbb{P}^1_K$ such that $\tilde{Y}_K$ is the $m$th infinitesimal neighborhood of $\{0, \infty\}$, that is, $\tilde{Y}_K = (Z_m)_K \sqcup (Z_m)_K$ (with the previous notations). Set $Y' = (Z_m(\xi))_K \sqcup (Z_m(\xi))_K$. The Galois action of $\Gamma$ on $\mathbb{P}^1_K$ exchanges the two components of $\tilde{Y}_K$, as well as the two components of $Y'$, and the natural morphism $\lambda' : (\tilde{Y})_K \to Y'$ is $\Gamma$-equivariant. Thus, by Galois descent, there exists a unique subalgebra $A$ of $\mathcal{O}(\tilde{Y})$ such that the scheme $\tilde{Y}_K = \tilde{Y}$ and that $\lambda'$ is the morphism deduced from the natural morphism $\lambda : \tilde{Y} \to Y$. Let $\tilde{C}_m(\tilde{P}, \xi)$ and $\tilde{C}_m(\tilde{P}, \xi)'$ be the curves defined by the pinching diagrams

$$
\begin{array}{ccc}
\tilde{Y} & \longrightarrow & \tilde{C} \\
\downarrow & & \downarrow \\
Y & \longrightarrow & \tilde{C}_m(\tilde{P}, \xi) \\
\text{Spec } K & \longrightarrow & \tilde{C}_m(\tilde{P}, \xi) \\
\downarrow & & \downarrow \\
\text{Spec } k & \longrightarrow & \tilde{C}_m(\tilde{P}, \xi)'
\end{array}
$$

Since after field extension we still have pinching diagrams, by unicity we have $\tilde{C}_m(\tilde{P}, \xi)_K \cong \mathbb{P}^1_{K,m,m}(\xi, \xi)$ and $\tilde{C}_m(\tilde{P}, \xi)' \cong \mathbb{P}^1_{K,m,m}(\xi, \xi)'$. We have $G_K \cong G_{m,k}$ and the action $G_{m,k} \times Y' \to Y'$ is $\Gamma$-equivariant so, by Galois descent again, we have an action of $G$ on $Y$ such that $\lambda$ is equivariant. Hence $G$ acts faithfully on $\tilde{C}_m(\tilde{P}, \xi)$ and $\tilde{C}_m(\tilde{P}, \xi)'$.

**Theorem 4.6.** Let $G$ be a non-trivial form of $G_{m,k}$, $C$ a curve and $\alpha : G \times C \to C$ an action. The group $G$ acts faithfully on $C$ (and $C$ is almost homogeneous) if and only if, with the above notations, $C \cong \tilde{C} \setminus \{\tilde{P}\}$ or if there exists a subsemigroup $3m(\xi)$ of $(\mathbb{N}, +)$ such that $C$ is isomorphic to $\tilde{C}_m(\tilde{P}, \xi)$ or $\tilde{C}_m(\tilde{P}, \xi)'$, and the action is the natural one.

**Proof.** Assume that $G$ acts faithfully on $C$. It follows from Theorem 1.1 and Lemmas 2.14 and 3.6 that $\tilde{C}$ is the normalization of $C$ (with the natural action of $G$). Let

$$
\begin{array}{ccc}
\tilde{Z} & \longrightarrow & \tilde{C} \\
\downarrow & & \downarrow \\
Z & \longrightarrow & C
\end{array}
$$
be the conductor square. Then $\tilde{Z}$ is supported by $\tilde{P}$ and $Z$ is supported by a point $P$ such that $\kappa(P) = k$ or $K$. By Lemma 3.1, the diagram

\[
\begin{array}{ccc}
\tilde{Z}_K & \longrightarrow & \mathbb{P}_K^1 \\
\downarrow & & \downarrow \\
Z_K & \longrightarrow & C_K
\end{array}
\]

is still the conductor square.

If $\kappa(P) = K$ then $Z_K$ is supported by two $K$-rational points. As in Theorem 4.5, there exist integers $M$ and $N$ such that $\tilde{Z}_K = \text{Spec} \left( K[t]/(t^M) \times K[u]/(u^N) \right)$ and subsemigroups $\mathfrak{z}_m(\ell)$ and $\mathfrak{z}_n(d)$ of $(\mathbb{N}, +)$ such that

\[
Z_K = \text{Spec} \left( \frac{K[t^c_0, \ldots, t^c_p, t^{m+1}]}{(t^{M+1})} \times \frac{K[u^d_0, \ldots, u^d_q, u^{n+1}]}{(u^{N+1})} \right).
\]

Moreover, the group $\Gamma$ exchanges the two components of $\tilde{Z}_K$, as well as the two components of $Z_K$. Thus we must have $M = N$ and $\mathfrak{z}_m(\ell) = \mathfrak{z}_n(d)$. As before, the squares in

\[
\begin{array}{ccc}
\tilde{Y}_K & \longrightarrow & \tilde{Z}_K \\
\downarrow & & \downarrow \\
Y_K & \longrightarrow & Z_K
\end{array}
\]

are cocartesian, so the entire composed diagram is a pinching diagram. The morphisms are $\Gamma$-equivariant so, by Galois descent, we have $C \simeq \tilde{C}_m(\tilde{P}, \ell)$.

If $\kappa(P) = k$ then $Z_K$ is supported by a $K$-rational point. By a similar argument, there exists a subsemigroup $\mathfrak{z}_m(\ell)$ of $(\mathbb{N}, +)$ such that $C_K \simeq \mathbb{P}_K^1 \simeq \mathbb{P}_{K,m,m}(\ell, \ell)'$ and the morphisms in the pinching diagram defining $\mathbb{P}_{K,m,m}(\ell, \ell)'$ are $\Gamma$-equivariant. Therefore, we have $C \simeq \tilde{C}_m(\tilde{P}, \ell)'$. \qed

5. Equivariant embeddings in a projective space

5.1. Linearized line bundles and pinchings

In this section we describe the linearized line bundles on a scheme obtained by a pinching diagram. We first recall the definition and basic properties of linearized line bundles and the equivariant Picard group. We use [MFK] and [SGA3, Exp. I, §6] as general references.

For a smooth connected algebraic group $G$, we denote by $\hat{G}$ the functor which associates with any $k$-scheme $X$ the group $\hat{G}(X) = \text{Hom}_{X-\text{gp}}(G \times X, \mathbb{G}_{m,k} \times X)$. An element of $\hat{G}(X)$ is given by a morphism $\chi : G \times X \rightarrow \mathbb{G}_{m,k}$ such that $\chi(gg', x) = \chi(g, x) \chi(g', x)$ identically. It is well known that if the abstract group $\hat{G}(k_s)$ is trivial and $X$ is geometrically reduced then $\hat{G}(X)$ is trivial. To check this, let $\chi \in \hat{G}(X)$. We can assume $k = k_s$. Then for $x \in X(k)$, the morphism

\[
\begin{cases}
G & \rightarrow \mathbb{G}_{m,k}, \\
g & \mapsto \chi(g, x)
\end{cases}
\]
is a character, so it is trivial. Since $G(k) \times X(k)$ is dense in $G \times X$ (because $G \times X$ is geometrically reduced), it follows that $\chi$ is the constant morphism $1$.

**Definition 5.1.** Let $G$ be an algebraic group, $X$ be a separated scheme of finite type over $k$, $\alpha : G \times X \to X$ an action and $\pi : L \to X$ a line bundle. We say that $L$ is linearizable if there exists an isomorphism of line bundles $\Phi : \alpha^* L \to \pi^* L$ (over $G \times X$) making the following diagram of line bundles over $G \times G \times X$ commute (cocycle condition):

\[
\begin{array}{ccc}
(id_G \times \alpha)^* \alpha^* L & \xrightarrow{\mathrm{can}} & (\mu \times \text{id}_X)^* \alpha^* L \\
& (id_G \times \alpha)^* \Phi & (\mu \times \text{id}_X)^* \Phi \\
(id_G \times \alpha)^* \pi^* L & \xrightarrow{\mathrm{can}} & \pi^* \alpha^* L \\
& \pi^* \alpha^* \Phi & \pi^* \phi \\
\end{array}
\]

We say that $\Phi$ is a linearization and that $(L, \Phi)$ is a linearized line bundle.

A morphism of linearized line bundles $\varphi : (L_1, \Phi_{L_1}) \to (L_2, \Phi_{L_2})$ is a morphism of line bundles making the square

\[
\begin{array}{ccc}
\alpha^* L_1 & \xrightarrow{\Phi_{L_1}} & \pi^* L_1 \\
& \varphi & \\
\alpha^* \varphi & \pi^* \varphi \\
\alpha^* L_2 & \xrightarrow{\Phi_{L_2}} & \pi^* L_2
\end{array}
\]

commute.

For example, if $X$ is a smooth curve then the action of $G$ on differential forms yields a linearization of the canonical bundle $\omega_X$.

If $(L, \Phi)$ and $(L', \Phi')$ are linearized line bundles over $X$ then $\Phi \otimes \Phi' : \alpha^*(L \otimes L') = \alpha^* L \otimes \alpha^* L' \to \pi^* L \otimes \pi^* L' = \pi^*(L \otimes L')$ is an isomorphism satisfying the cocycle condition, so $L \otimes L'$ is linearizable. Similarly $L^{-1}$ is linearizable.

**Definition 5.2.** We denote by $\text{Pic}^G(X)$ the abelian group of isomorphism classes of linearized line bundles over $X$. It is called the equivariant Picard group.

The linearized line bundles give equivariant embeddings in a projective space.

**Lemma 5.3** ([MFK, Prop. 1.7, p. 35]). Let $G$ be an algebraic group, $X$ a quasi-projective scheme over $k$, $L$ a very ample line bundle over $X$ and $\alpha : G \times X \to X$ an action. The line bundle $L$ is linearizable if and only if there exists an action of $G$ on a projective space $\mathbf{P}_k^n$ and an equivariant immersion $i : X \to \mathbf{P}_k^n$ such that $\mathcal{O}_{\mathbf{P}_k^n}(1)$ is linearizable and $L = i^* \mathcal{O}_{\mathbf{P}_k^n}(1)$.

In other words, a linearized line bundle induces an equivariant immersion of $X$ in the projectivization of a finite-dimensional $G$-module.

The following result on normal schemes is well known.
Proposition 5.4 ([Br15, Thm. 2.14]). Let $G$ be a smooth connected linear algebraic group, $X$ a variety and $\alpha : G \times X \to X$ an action. If $X$ is normal then there exists an integer $n \geq 1$ such that for every line bundle $L$ over $X$, $L^{\otimes n}$ is linearizable.

We want an analogous result without the hypothesis of normality in case $G$ is a form of $G_{a,k}$ in prime characteristic. We first need the following proposition.

Proposition 5.5 ([Br15, Prop. 2.10]). Let $G$ be a smooth linear algebraic group, $X$ a separated reduced scheme of finite type over $k$ and $\alpha : G \times X \to X$ an action. We have a sequence of abstract groups

$$1 \to \mathcal{O}(X)^{\times G} \to \mathcal{O}(X)^{\times} \to \hat{G}(X) \to \text{Pic}^G(X) \to \text{Pic}(X) \xrightarrow{\alpha^*} \frac{\text{Pic}(G \times X)}{\text{pr}_2^* \text{Pic}(X)}$$

where $\text{Pic}^G(X) \to \text{Pic}(X)$ is the forgetful morphism which associates with a linearized line bundle $(L, \Phi)$ the line bundle $L$, and $\mathcal{O}(X)^{\times G}$ denotes the subgroup of $\mathcal{O}(X)^{\times}$ consisting of $G$-invariant elements. In particular, if the abstract group $\hat{G}(k_s) = \text{Hom}_{\text{grp}}(G_{a,k}, \mathbb{G}_m,k)$ is trivial and $X$ is geometrically reduced then the forgetful morphism $\text{Pic}^G(X) \to \text{Pic}(X)$ is injective (that is, a line bundle has at most one linearization).

Proposition 5.6. We assume that $k$ has characteristic $p > 0$. Let $G$ be a form of $G_{a,k}$, $X$ a variety and $\alpha : G \times X \to X$ an action. There exists an integer $r \geq 0$ such that for every line bundle $L$ over $X$, $L^{\otimes p^r}$ is linearizable.

Proof. Because of the exact sequence of Proposition 5.5, it suffices to show that the group $\text{Pic}(G \times X)/\text{pr}_2^* \text{Pic}(X)$ is $p^r$-torsion.

Let us show that we can restrict to the case $G = G_{a,k}$. Let $K/k$ be a finite purely inseparable extension such that $G_K \simeq G_{a,K}$ (see Proposition 2.12). The degree $d$ of this extension is a power of $p$. Let us assume that there exists an integer $s \geq 0$ such that the group $\text{Pic}(G_K \times X_K)/\text{pr}_2^* \text{Pic}(X_K)$ is $p^s$-torsion. Let $L$ be a line bundle over $G \times X$. By hypothesis, there exists a line bundle $M$ over $X_K$ such that $(L_K)^{\otimes p^s} \simeq \text{pr}_2^* M$. Taking norms, we get $L^{d_{p^s}}(L_K)^{\otimes p^s} \simeq N_{K/k}(\text{pr}_2^* M) \simeq N_{K/k}(\text{pr}_2^*(N_{K/k}(M)))$ (see [EGA, II, Prop. 6.5.8]). Thus the group $\text{Pic}(G \times X)/\text{pr}_2^* \text{Pic}(X)$ is $d_{p^s}$-torsion.

We now assume $G = G_{a,k}$. Let $\sigma : X^+ \to X$ be the seminormalization. By Lemma A.7, the pullback morphism $\text{pr}_2^* : \text{Pic}(X^+) \to \text{Pic}(G \times X^+)$ is an isomorphism. Thus the following diagram is commutative and exact in rows:

$$
\begin{array}{cccccc}
0 & \longrightarrow & \text{Pic}(X) & \xrightarrow{\text{pr}_2^*} & \text{Pic}(G \times X) & \longrightarrow & \text{Pic}(G \times X)/\text{pr}_2^* \text{Pic}(X) & \longrightarrow & 0 \\
\sigma^* & & (\text{id}_G \times \sigma)^* & & & & \\
0 & \longrightarrow & \text{Pic}(X^+) & \xrightarrow{\text{pr}_2^*} & \text{Pic}(G \times X^+) & \longrightarrow & 0
\end{array}
$$

Hence the snake lemma gives an exact sequence

$$\ker(\text{id}_G \times \sigma)^* \to \text{Pic}(G \times X)/\text{pr}_2^* \text{Pic}(X) \to \text{coker} \sigma^*.$$
Moreover, the morphism $\alpha : G \times X \to X$ is smooth (because $G$ is smooth) so, by Lemma A.8, the scheme $G \times X^+$ is seminormal and $\text{id}_G \times \sigma : G \times X^+ \to G \times X$ is the seminormalization. Thus it follows from [Br15, Lem. 4.11] that there exists an integer $n \geq 0$ such that $\ker(\text{id}_G \times \sigma)^*$ and $\coker \sigma^*$ are $p^n$-torsion. Hence $\text{Pic}(G \times X)/\text{pr}_2^* \text{Pic}(X)$ is $p^{2n}$-torsion. □

Remark 5.7.

i) This result is not true if $k$ has characteristic zero. Indeed, $G_{a,k}$ acts on the cuspidal curve $C$ obtained by pinching the subscheme $\text{Spec } k[u]/(u^2)$ of $\mathbb{P}^1_k$ supported at $\infty$ onto $\text{Spec } k$ (which can be explicitly realized for example as the curve with homogeneous equation $y^3 = x^2z$ in $\mathbb{P}^2_k$). By [Br15, Ex. 2.16], a line bundle over $C$ is linearizable if and only if it has degree 0.

ii) As a particular case of Proposition 5.6, if $G$ is a non-trivial form of $G_{a,k}$ then its regular completion $C$ can be embedded equivariantly in the projectivization of a $G$-module, but $C$ is not smooth. On the other hand, if $k$ has characteristic zero then the closure of any orbit in the projectivization of a $G_{a,k}$-module is smooth (see [BF, Lem. 2.4]).

If

$$\xymatrix{ \tilde{Z} \ar[r]^j \ar[d]^{\lambda} & \tilde{X} \ar[d]^{\nu} \\ Z \ar[r]_{i} & X }$$

is a pinching diagram then Charles Weibel described the line bundles over $X$ in terms of line bundles over $\tilde{X}$ and $Z$.

Proposition 5.8 ([We, Prop. 7.8]). We have an exact sequence of abstract groups

$$1 \to \mathcal{O}(X)^\times \to \mathcal{O}(\tilde{X})^\times \times \mathcal{O}(Z)^\times \to \mathcal{O}(\tilde{Z})^\times \to \text{Pic } X \to \text{Pic } \tilde{X} \times \text{Pic } Z \to \text{Pic } \tilde{Z}$$

induced by pullbacks and a connecting morphism $\mathcal{O}(\tilde{Z})^\times \to \text{Pic } X$. This sequence is called the Units-Pic sequence.

Loosely speaking, this means that, up to isomorphism, line bundles over $X$ correspond to line bundles over $\tilde{X}$ equipped with trivializations along fibers of $\lambda$.

This can be reformulated in categorical terms. For any scheme $Y$, we denote by $\mathcal{D}_Y$ the category of line bundles over $Y$. With the functors $j^*: \mathcal{D}_{\tilde{X}} \to \mathcal{D}_X$ and $\lambda^*: \mathcal{D}_Z \to \mathcal{D}_\tilde{Z}$ we can form the fiber product category $\mathcal{D}_{\tilde{X}} \times_{\mathcal{D}_Z} \mathcal{D}_Z$. Its objects are the triples $(\tilde{L}, M, \sigma)$ where $\tilde{L}$ and $M$ are line bundles over $\tilde{X}$ and $Z$, and $\sigma$ is an isomorphism $j^*\tilde{L} \simeq \lambda^* M$ of line bundles over $\tilde{Z}$. A morphism between two triples $(\tilde{L}_1, M_1, \sigma_1)$ and $(\tilde{L}_2, M_2, \sigma_2)$ is given by two morphisms $\varphi : \tilde{L}_1 \to \tilde{L}_2$ and $\psi : M_1 \to M_2$ such that $\sigma_2 \circ (j^*\varphi) = (\lambda^*\psi) \circ \sigma_1$.

Proposition 5.9 ([Ho, Thm. 3.13]). The functor

$$T : \begin{cases} \mathcal{D}_X & \to \mathcal{D}_{\tilde{X}} \times_{\mathcal{D}_Z} \mathcal{D}_Z, \\ L & \mapsto (\nu^* L, i^* \tilde{L}, \text{can}) \end{cases}$$
is an equivalence of categories (where \(\text{can}\) is the canonical isomorphism \(j^*\nu^*L \simeq \lambda^*i^*L\)).

Let \(S\) be the quasi-inverse of \(T\). For \((\tilde{L}, M, \sigma)\) in \(D_{\tilde{X}} \times_{D_{\tilde{Z}}} D_Z\), the isomorphism class of \(L = S(\tilde{L}, M, \sigma)\) is determined by the existence of isomorphisms \(\varphi_\nu : \nu^*L \to \tilde{L}\) and \(\varphi_i : i^*L \to M\) making the diagram

\[
\begin{array}{ccc}
  j^*\nu^*L & \xrightarrow{j^*\varphi_\nu} & j^*\tilde{L} \\
  \downarrow\text{can} & & \downarrow\sigma \\
  \lambda^*i^*L & \xrightarrow{\lambda^*\varphi_i} & \lambda^*M
\end{array}
\]

commute.

We can do the same for linearized line bundles. Let \(G\) be a smooth connected algebraic group. For any scheme \(Y\) which is separated and of finite type over \(k\), and any action \(\alpha : G \times Y \to Y\), we denote by \(D_G^Y\) the category of linearized line bundles over \(Y\). Let \(Y'\) be another such scheme with an action \(\alpha' : G \times Y' \to Y'\) and \(f : Y' \to Y\) an equivariant morphism. For any \((L, \Phi_L)\) in \(D_G^Y\), the line bundle \(f^*L\) over \(Y'\) is equipped with the linearization \(\Phi_{f^*L}\) given by the diagram

\[
\begin{array}{ccc}
  \alpha'^*f^*L & \xrightarrow{\Phi_{f^*L}} & \text{pr}_2^*f^*L \\
  \downarrow\text{can} & & \downarrow\text{can} \\
  (\text{id}_G \times f)^*\alpha^*L & \xrightarrow{(\text{id}_G \times f)^*\Phi_L} & (\text{id}_G \times f)^*\text{pr}_2^*L
\end{array}
\]

If \(\varphi : L_1 \to L_2\) is a morphism of linearized line bundles over \(Y\) then \(f^*\varphi : f^*L_1 \to f^*L_2\) is a morphism of linearized line bundles over \(Y'\). This yields a functor \(f^* : D_G^Y \to D_G^{Y'}\).

We assume that \(X\) is separated and of finite type over \(k\) (thus so are \(\tilde{X}, Z\) and \(\tilde{Z}\)). Let \(\alpha : G \times X \to X\) and \(\tilde{\alpha} : G \times \tilde{X} \to \tilde{X}\) be two actions such that \(Z\) and \(\tilde{Z}\) are \(G\)-stable and \(\nu\) is \(G\)-equivariant (so \(\lambda, i\) and \(j\) are equivariant too).

Let \(\beta : G \times Z \to Z\) and \(\tilde{\beta} : G \times \tilde{Z} \to \tilde{Z}\) be the induced actions. With the functors \(j^* : D_G^X \to D_G^{\tilde{X}}\) and \(\lambda^* : D_G^Z \to D_G^{\tilde{Z}}\) we can form the fiber product category \(D_G^X \times_{D_G^Z} D_G^{\tilde{Z}}\). Its objects are the tuples \((\tilde{L}, \Phi_{\tilde{L}}, M, \Phi_M, \sigma)\) where \((\tilde{L}, \Phi_{\tilde{L}})\) and \((M, \Phi_M)\) are linearized line bundles over \(\tilde{X}\) and \(Z\), and \(\sigma\) is an isomorphism \(j^*\tilde{L} \simeq \lambda^*M\) of linearized line bundles over \(\tilde{Z}\).

**Proposition 5.10.** The functor

\[
T^G : \begin{cases}
  D_G^X & \to & D_G^X \times_{D_G^Z} D_G^{\tilde{Z}}, \\
  (L, \Phi_L) & \mapsto & (\nu^*L, \Phi_{\nu^*L}, i^*L, \Phi_{i^*L}, \text{can})
\end{cases}
\]

is an equivalence of categories (where \(\text{can}\) is the canonical isomorphism \(j^*\nu^*L \simeq \lambda^*i^*L\)).
Proof. It is immediate that the functor $T^G$ is well defined on objects (and its definition on morphisms is the obvious one).

By Proposition 5.9, the functor $T^G$ is faithful. Let us show that it is full. Let $(L_1, \Phi_{L_1})$ and $(L_2, \Phi_{L_2})$ be two linearized line bundles over $X$. A morphism $T^G(L_1, \Phi_{L_1}) \to T^G(L_2, \Phi_{L_2})$ is given by two morphisms of linearized line bundles $\tilde{\varphi} : (\nu^*L_1, \Phi_{\nu^*L_1}) \to (\nu^*L_2, \Phi_{\nu^*L_2})$ and $\psi : (i^*L_1, \Phi_{i^*L_1}) \to (i^*L_2, \Phi_{i^*L_2})$ such that $\sigma_2 \circ (j^*\tilde{\varphi}) = (\lambda^*\psi) \circ \sigma_1$ (where $\sigma_1$ and $\sigma_2$ are the canonical isomorphisms $j^*\nu^*L_1 \simeq \lambda^*i^*L_1$ and $j^*\nu^*L_2 \simeq \lambda^*i^*L_2$). By Proposition 5.9, there exists a morphism of line bundles $\varphi : L_1 \to L_2$ such that $\tilde{\varphi} = \nu^*\varphi$ and $\psi = i^*\varphi$. It remains to show that $\varphi$ is a morphism of linearized line bundles. From the definition of the pullback linearizations $\Phi_{\nu^*L_1}$ and $\Phi_{\nu^*L_2}$, the equality $\tilde{\varphi} = \nu^*\varphi$ and the fact that $\tilde{\varphi}$ is a morphism of linearized line bundles and the different canonical isomorphisms, it follows readily that the diagram

\[
\begin{array}{ccc}
(id_G \times \nu)^* \alpha^*L_1 & \xrightarrow{\Phi_{L_1}} & (id_G \times \nu)^* pr_2^*L_1 \\
(id_G \times \nu)^* \varphi & \downarrow & (id_G \times \nu)^* pr_2^* \varphi \\
(id_G \times \nu)^* \alpha^*L_2 & \xrightarrow{\Phi_{L_2}} & (id_G \times \nu)^* pr_2^*L_2 \\
\end{array}
\]

of line bundles over $G \times \tilde{X}$ commutes (and similarly for the pullback by $id_G \times i$). Moreover, the square

\[
\begin{array}{ccc}
G \times \tilde{Z} & \xrightarrow{id \times j} & G \times \tilde{Z} \\
\downarrow id \times \lambda & & \downarrow id \times \nu \\
G \times Z & \xrightarrow{id \times i} & G \times X \\
\end{array}
\]

is a pinching diagram (see [Ho, Thm. 3.11]) so we can apply Proposition 5.9 to it. Thus the corresponding functor

\[
T' : D_{G \times X} \to D_{G \times \tilde{X}} \times_{D_{G \times \tilde{Z}}} D_{G \times Z}
\]

is (fully) faithful. Consequently the diagram

\[
\begin{array}{ccc}
\alpha^*L_1 & \xrightarrow{\Phi_{L_1}} & pr_2^*L_1 \\
\downarrow \alpha^*\varphi & & \downarrow pr_2^* \varphi \\
\alpha^*L_2 & \xrightarrow{\Phi_{L_2}} & pr_2^*L_2 \\
\end{array}
\]

commutes, that is, $\varphi$ is a morphism of linearized line bundles.

Finally, let us show that $T^G$ is essentially surjective. Let $(\tilde{L}, \Phi_{\tilde{L}}, M, \Phi_M, \sigma)$ be an object of $D_{\tilde{X}} \times_{D_Z} D_{\tilde{Z}}$. By Proposition 5.9 again, there exists a line bundle $L$
over $X$ and isomorphisms $\varphi_\nu : \nu^* L \to \tilde{L}$ and $\varphi_i : i^* L \to M$ making the diagram

\[
\begin{array}{ccc}
j^* \nu^* L & \xrightarrow{\text{can}} & j^* \tilde{L} \\
\downarrow \text{can} & & \downarrow \sigma \\
\lambda^* i^* L & \xrightarrow{\text{can}} & \lambda^* M
\end{array}
\]

commute. We have isomorphisms

\[
\begin{align*}
\sigma_1 : (\text{id} \times j)^* \tilde{\alpha}^* \tilde{L} & \xrightarrow{\text{can}} \tilde{\beta}^* j^* \tilde{L} \xrightarrow{\tilde{\beta}^* \sigma} \tilde{\beta}^* \lambda^* M \xrightarrow{\text{can}} (\text{id} \times \lambda)^* \beta^* M, \\
\sigma_2 : (\text{id} \times j)^* \text{pr}_2^* \tilde{L} & \xrightarrow{\text{can}} \text{pr}_2^* j^* \tilde{L} \xrightarrow{\text{pr}_2^* \sigma} \text{pr}_2^* \lambda^* M \xrightarrow{\text{can}} (\text{id} \times \lambda)^* \text{pr}_2^* M
\end{align*}
\]

of line bundles over $G \times \tilde{Z}$. Since $\sigma : j^* \tilde{L} \to \lambda^* M$ is an isomorphism of linearized line bundles, $\Phi_{\tilde{L}}$ and $\Phi_M$ give an isomorphism $(\tilde{\alpha}^* \tilde{L}, \beta^* M, \sigma_1) \to (\text{pr}_2^* \tilde{L}, \text{pr}_2^* M, \sigma_2)$ in the category $\mathcal{D}_{G \times \tilde{X}} \times \mathcal{D}_{G \times \tilde{Z}} \mathcal{D}_{G \times Z}$. As the functor $T'$ defined above is fully faithful, there exists an isomorphism $\Phi_L : \alpha^* L \to \text{pr}_2^* L$ such that the diagram

\[
\begin{array}{ccc}
(\text{id} \times \nu)^* \alpha^* L & \xrightarrow{\text{can}} & \tilde{\alpha}^* \nu^* L \\
\downarrow (\text{id} \times \nu)^* \Phi_L & & \downarrow \Phi_{\tilde{L}} \\
(\text{id} \times \nu)^* \text{pr}_2^* L & \xrightarrow{\text{can}} & \text{pr}_2^* \nu^* L \xrightarrow{\text{pr}_2^* \varphi_\nu} \text{pr}_2^* \tilde{L}
\end{array}
\]

commutes (and similarly for $\Phi_M$). It remains to show that $\Phi_L$ is a linearization, because then $\varphi_\nu$ and $\varphi_i$ give an isomorphism $T^G(L, \Phi_L) \to (\tilde{L}, \Phi_{\tilde{L}}, M, \Phi_M, \sigma)$. We have to show that $\Phi_L$ satisfies the cocycle condition, that is to say that the diagram

\[
\begin{array}{ccc}
(\text{id}_G \times \alpha)^* \alpha^* L & \xrightarrow{\text{can}} & (\mu \times \text{id}_X)^* \alpha^* L \\
\downarrow (\text{id}_G \times \alpha)^* \Phi_L & & \downarrow \Phi_{L} \\
(\text{id}_G \times \alpha)^* \text{pr}_2^* L & \xrightarrow{\text{can}} & \text{pr}_{23}^* \alpha^* L \xrightarrow{\text{pr}_{23}^* \Phi_L} \text{pr}_{23}^* \text{pr}_2^* L
\end{array}
\]

commutes. It is easily checked that the two diagrams obtained by pulling back by $\text{id}_G \times \text{id}_G \times \nu$ and by $\text{id}_G \times \text{id}_G \times i$ are commutative because $\Phi_{\tilde{L}}$ and $\Phi_M$ are linearizations. But, by the same argument as before, the functor

\[
\mathcal{D}_{G \times G \times X} \to \mathcal{D}_{G \times G \times \tilde{X}} \times \mathcal{D}_{G \times G \times \tilde{Z}} \mathcal{D}_{G \times G \times Z}
\]

is faithful. Therefore the diagram for $\Phi_L$ is indeed commutative. \qed
Corollary 5.11. We have an exact sequence of abstract groups

\[
1 \to \mathcal{O}(X)^{\times G} \to \mathcal{O}(\tilde{X})^{\times G} \times \mathcal{O}(Z)^{\times G} \to \mathcal{O}(\tilde{Z})^{\times G}
\]

induced by pullbacks and a connecting morphism \( \delta : \mathcal{O}(\tilde{Z})^{\times G} \to \mathcal{Pic}^{G}(X) \). We call it “the equivariant Units-Pic sequence”.

Proof. We identify \( \mathcal{O}(\tilde{Z})^{\times} \) with the group of automorphisms of the trivial line bundle \( \tilde{Z} \times A_{k}^{1} \). Then the automorphisms of \( (\tilde{Z} \times A_{k}^{1}, \text{triv}) \) (where \( \text{triv} \) is the trivial linearization) correspond to the elements of \( \mathcal{O}(\tilde{Z})^{\times G} \). The morphism \( \delta \) maps \( \sigma \in \mathcal{O}(\tilde{Z})^{\times G} \) to the isomorphism class of linearized line bundles over \( X \) corresponding (in view of Proposition 5.10) to the isomorphism class of the tuple \( (\tilde{X} \times A_{k}^{1}, \text{triv}, Z \times A_{k}^{1}, \text{triv}, \sigma) \). By construction, \( \delta(\sigma) \) is in the kernel of the morphism \( \mathcal{Pic}^{G}(X) \to \mathcal{Pic}^{G}(\tilde{X}) \times \mathcal{Pic}^{G}(Z) \). Conversely, an element \( (L, \Phi_{L}) \) in this kernel corresponds to a tuple \( (\tilde{X} \times A_{k}^{1}, \text{triv}, Z \times A_{k}^{1}, \text{triv}, \sigma) \) and \( \sigma \) is an automorphism of \( (\tilde{Z} \times A_{k}^{1}, \text{triv}) \), so \( \sigma \in \mathcal{O}(\tilde{Z})^{\times G} \) and \( (L, \Phi_{L}) = \delta(\sigma) \).

Let \( \sigma \in \mathcal{O}(\tilde{Z})^{\times} \), viewed as an automorphism of \( (\tilde{Z} \times A_{k}^{1}, \text{triv}) \). Then \( \sigma \) is in \( \ker \delta \) if and only if there exists an isomorphism \( (\tilde{X} \times A_{k}^{1}, \text{triv}, Z \times A_{k}^{1}, \text{triv}, \sigma) \to (\tilde{X} \times A_{k}^{1}, \text{triv}, Z \times A_{k}^{1}, \text{triv}, \text{can}) \). Such an isomorphism is given by two automorphisms \( \varphi \) of \( (\tilde{X} \times A_{k}^{1}, \text{triv}) \) and \( \psi \) of \( (Z \times A_{k}^{1}, \text{triv}) \) such that \( \varphi \circ (j^{*}\varphi) = (\lambda^{*}\psi) \circ \sigma \). Viewing \( \varphi \) and \( \psi \) as elements of \( \mathcal{O}(\tilde{X})^{\times G} \) and \( \mathcal{O}(Z)^{\times G} \), this condition precisely means that \( \sigma \) is the image of \( (\varphi, \psi) \) by the morphism \( \mathcal{O}(\tilde{X})^{\times G} \times \mathcal{O}(Z)^{\times G} \to \mathcal{O}(\tilde{Z})^{\times G} \). So the sequence is exact at \( \mathcal{O}(\tilde{Z})^{\times G} \).

The image of \( \mathcal{O}(X)^{\times G} \to \mathcal{O}(\tilde{X})^{\times G} \times \mathcal{O}(Z)^{\times G} \) is contained in the kernel of \( \mathcal{O}(\tilde{X})^{\times G} \times \mathcal{O}(Z)^{\times G} \to \mathcal{O}(\tilde{Z})^{\times G} \). Conversely, if \( (\varphi, \psi) \) is in the kernel then, by the Units-Pic sequence, it is the image of an element \( \varphi \in \mathcal{O}(X)^{\times} \). The morphism \( \text{id}_{G} \times \nu : G \times \tilde{X} \to G \times X \) is schematically dominant so the map \( (\text{id}_{G} \times \nu)^{*} : \mathcal{O}(G \times X)^{\times} \to \mathcal{O}(G \times \tilde{X})^{\times} \) is injective. The images of \( \alpha^{*}(\varphi) \) and \( \text{pr}_{2}^{*}(\varphi) \) are respectively \( \tilde{\alpha}^{*}(\varphi) \) and \( \text{pr}_{2}^{*}(\tilde{\varphi}) \). Since \( \tilde{\varphi} \) is \( G \)-invariant, we have \( \tilde{\alpha}^{*}(\tilde{\varphi}) = \text{pr}_{2}^{*}(\tilde{\varphi}) \), so \( \alpha^{*}(\varphi) = \text{pr}_{2}^{*}(\varphi) \) and thus \( \varphi \in \mathcal{O}(X)^{\times G} \).

Let \( ((\tilde{L}, \Phi_{L}), (M, \Phi_{M})) \in \mathcal{Pic}^{G}(\tilde{X}) \times \mathcal{Pic}^{G}(Z) \). Its image in \( \mathcal{Pic}^{G}(\tilde{Z}) \) is trivial if and only if we have an isomorphism of linearized line bundles \( (j^{*}\tilde{L})(\lambda^{*}M)^{-1} \simeq \tilde{Z} \times A_{k}^{1} \), (where \( \tilde{Z} \times A_{k}^{1} \) is endowed with the trivial linearization), that is, if and only if we have an isomorphism of linearized line bundles \( \sigma : j^{*}\tilde{L} \to \lambda^{*}M \). Hence, by Proposition 5.10 again, the sequence is exact at \( \mathcal{Pic}^{G}(\tilde{X}) \times \mathcal{Pic}^{G}(Z) \). \( \square \)

Remark 5.12. If \( X \) is a proper variety then, since \( \mathcal{O}(X) = \mathcal{O}(\tilde{X}) = k \), the equivariant Units-Pic sequence can be simplified as

\[
1 \to \mathcal{O}(\tilde{Z})^{\times G} / \mathcal{O}(Z)^{\times G} \to \mathcal{Pic}^{G}(X) \to \mathcal{Pic}^{G}(\tilde{X}) \times \mathcal{Pic}^{G}(Z) \to \mathcal{Pic}^{G}(\tilde{Z}).
\]

5.2. Equivariant Picard groups of curves

We can determine the equivariant Picard group of the curves of Theorem 1.1.
Theorem 5.13. Let $C$ be a seminormal curve and $G$ a smooth connected algebraic group acting faithfully on $C$.

(1) (homogeneous curves)
1. If $C$ is a smooth projective conic and $G \simeq \Aut_C$ then $\Pic^G(C) = \Pic C = \mathbb{Z} \cdot \omega_C$ where $\omega_C$ is the canonical sheaf, so the map $\frac{1}{2} \deg : \Pic^G(C) \to \mathbb{Z}$ is an isomorphism.
2. If $C \simeq \mathbb{A}^1_k$ and $G \simeq G_{a,k} \times G_{m,k}$ (acting by affine transformations), then $\Pic^G(C) \simeq \hat{G}(C) \simeq \mathbb{Z}$.
3. If $G$ is a form of $G_{a,k}$ or $G_{m,k}$ and $C$ is a $G$-torsor then $\Pic^G(C)$ is trivial.
4. If $C$ is a smooth projective curve of genus 1 and $G \simeq \Aut_C^0$ then $\Pic^G(C)$ is trivial.

(2) (regular, non-homogeneous curves)
1. If $C \simeq \mathbb{P}^1_k$ and $G \simeq G_{a,k} \times G_{m,k}$ or $G \simeq G_{m,k}$ then we have a split exact sequence

$$1 \to \hat{G}(C) \simeq \mathbb{Z} \to \Pic^G(C) \to \Pic C = \mathbb{Z} \cdot [\infty] \simeq \mathbb{Z} \to 0$$

and the morphism $\Pic^G(C) \to \mathbb{Z}$ is identified with the degree map, so $\Pic^G(C) \simeq \mathbb{Z}^2$.
2. If $G$ is a form of $G_{a,k}$, $C$ is the regular completion of a $G$-torsor and $P$ is the point at infinity then we have $\Pic^G(C) = \mathbb{Z} \cdot [P] \simeq \mathbb{Z}$ and this isomorphism is identified with the map $(1/\kappa(P): k])$ deg.
3. If $C \simeq \mathbb{A}^1_k$ and $G \simeq G_{m,k}$ then $\Pic^G(C) \simeq \hat{G}(C) \simeq \mathbb{Z}$.
4. If $C$ is a smooth projective conic and $G$ is the centralizer of a separable point $\tilde{P}$ of degree 2 on a $k$-rational point then we have a split exact sequence

$$1 \to \hat{G}(C) \simeq \mathbb{Z} \to \Pic^G(C) \to \Pic C = \mathbb{Z} \cdot [\tilde{P}] \simeq \mathbb{Z} \to 0$$

and the morphism $\Pic^G(C) \to \mathbb{Z}$ is identified with $1/\kappa(\tilde{P}): k)]$ deg.

(3) (seminormal, singular, non-homogeneous curves)
1. If $G$ is a non-trivial form of $G_{a,k}$ and $C$ is obtained by pinching the point at infinity $\tilde{P}$ of the regular completion $\tilde{C}$ of a $G$-torsor on a point $P$ whose residue field $\kappa(P)$ is a strict subextension of $\kappa(\tilde{P})/k$ then we have a split exact sequence

$$1 \to \kappa(\tilde{P})^\times /\kappa(P)^\times \to \Pic^G(C) \to \mathbb{Z} \cdot [\tilde{P}] \simeq \mathbb{Z} \to 0$$

and the morphism $\Pic^G(C) \to \mathbb{Z}$ is identified with $(1/\kappa(\tilde{P}): k])$ deg.
2. If $C$ is obtained by pinching two $k$-rational points of $\mathbb{P}^1_k$ on a $k$-rational point and $G \simeq G_{m,k}$ then we have a split exact sequence

$$1 \to (k^\times \times k^\times)/k^\times \to \Pic^G(C) \to \hat{G}(\mathbb{P}^1_k) \simeq \mathbb{Z} \to 0.$$

3. If $C$ is obtained by pinching a separable point $\tilde{P}$ of degree 2 of a smooth projective conic $\tilde{C}$ on a $k$-rational point $P$, and $G$ is the centralizer of $\tilde{P}$ in $\Aut_{\tilde{C}}$ then we have $\Pic^G(C) \simeq \kappa(\tilde{P})^\times /\kappa(P)^\times$. 
Proof. Case 1.1: By Proposition 5.5, the forgetful morphism \( \text{Pic}^G(C) \to \text{Pic} C \) is injective. If \( C = \mathbb{P}^1_k \) and \( G = \text{PGL}_{2,k} \) then it is well known that the sheaf \( \mathcal{O}_{\mathbb{P}^1_k}(1) \) is not linearizable (see [MFK, p. 33]), but the canonical sheaf \( \omega_{\mathbb{P}^1_k} = \mathcal{O}_{\mathbb{P}^1_k}(-2) \) is linearizable. Hence \( \text{Pic}^G(C) \) is the subgroup of \( \text{Pic} C \) generated by \( \omega_{\mathbb{P}^1_k} \). More generally, let \( C \) be a smooth conic and \( G = \text{Aut} C \). There exists a Galois extension \( K/k \) such that \( C_K \cong \mathbb{P}^1_K \). Then the pullback morphism \( \text{Pic} C \to \text{Pic} C_K \cong \mathbb{Z} \) is injective (its kernel consists of the forms of the trivial line bundle \( C_K \times \mathbb{A}^1_K \), so by Galois cohomology and Hilbert’s Theorem 90 it is trivial) and is identified with the degree morphism. Thus if \( C \) is not isomorphic to \( \mathbb{P}^1_k \) then there is no line bundle of degree one (otherwise \( C \) would have a \( k \)-rational point) so \( \text{Pic} C \) is generated by the canonical sheaf \( \omega_C \), which is linearizable.

Case 1.2: Every line bundle over \( C \) is isomorphic to the trivial bundle. By Proposition 5.5, the kernel of the forgetful morphism \( \text{Pic}^G(C) \to \text{Pic} C \) is the group \( \hat{G}(C) \). Moreover, \( \hat{G} \) is the constant sheaf \( \mathbb{Z} \).

Cases 1.3 and 1.4: This follows from [MFK, p. 32] since \( C \) is a \( G \)-torsor.

Case 2.1: By Proposition 5.5 again, the sequence \( 1 \to \hat{G}(C) \to \text{Pic}^G(C) \to \text{Pic} C \) is exact. Let \( \alpha : G \times C \to C \) be the standard action and \( P = \text{Spec} k \) the reduced closed subscheme of \( C \) supported by \( \infty \). Denote by \( [\infty] \) the corresponding Weil divisor, so that \( \text{Pic} C = \mathbb{Z} \cdot [\infty] \). We have the exact sequence of sheaves

\[
0 \to \mathcal{O}_C(-P) \to \mathcal{O}_C \to \mathcal{O}_P \to 0.
\]

Since \( \alpha \) and \( \text{pr}_2 \) are flat, and \( P \) is \( G \)-stable, we have a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & \alpha^* \mathcal{O}_C(-P) & \longrightarrow & \alpha^* \mathcal{O}_C & \longrightarrow & \alpha^* \mathcal{O}_P & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{pr}_2^* \mathcal{O}_C(-P) & \longrightarrow & \text{pr}_2^* \mathcal{O}_C & \longrightarrow & \text{pr}_2^* \mathcal{O}_P & \longrightarrow & 0
\end{array}
\]

which is exact in rows and where the vertical arrows are isomorphisms. This yields an isomorphism \( \alpha^* \mathcal{O}_C(-P) \cong \text{pr}_2^* \mathcal{O}_C(-P) \). Since \( C \) is reduced, by [Br15, Lem. 2.9], this implies that the line bundle corresponding to \( -[P] \) is linearizable. Hence the forgetful morphism \( \text{Pic}^G(C) \to \text{Pic} C \) is surjective.

Case 2.2: As above, the forgetful morphism \( \text{Pic}^G(C) \to \text{Pic} C \) is injective. Let \( U \) be the open subscheme of \( C \) which is a \( G \)-torsor, so that \( P \) is the reduced closed subscheme of \( C \) supported by the point \( C \setminus U \). If \( L \) is a linearizable line bundle over \( C \) then its restriction \( L|_U \) is linearizable. But \( U \) is a \( G \)-torsor so \( \text{Pic}^G(U) \) is trivial. So \( L \) corresponds to a Weil divisor in \( \mathbb{Z} \cdot [P] \). Conversely, by the same argument as above, the line bundle corresponding to \( -[P] \) is linearizable.

Case 2.3: The argument is the same as for the case 1.2.

Case 2.4: The canonical divisor on \( C \) is \( -[P] \) so \( \text{Pic} C = \mathbb{Z} \cdot [P] \). Thus the morphism \( \text{Pic}^G(C) \to \text{Pic} C \) is surjective.
Case 3.1: The curve $C$ is obtained by the pinching diagram

$$
\begin{array}{c}
\tilde{Z} \\
\downarrow \\
Z \\
\end{array}
\xrightarrow{j}
\begin{array}{c}
\tilde{C} \\
\downarrow \\
C \\
\end{array}
$$

where $Z = \text{Spec } \kappa(P)$ and $\tilde{Z} = \text{Spec } \kappa(\tilde{P})$. By Remark 5.12, the equivariant Units-Pic sequence can be written as

$$1 \rightarrow \mathcal{O}(\tilde{Z})^\times / \mathcal{O}(Z)^\times \rightarrow \text{Pic}^G(C) \rightarrow \text{Pic}^G(\tilde{C}) \times \text{Pic}^G(Z) \rightarrow \text{Pic}^G(\tilde{Z}).$$

Since the group of characters $\widehat{G}(k_s)$ is trivial, it follows from Proposition 5.5 that the groups $\text{Pic}^G(Z)$ and $\text{Pic}^G(\tilde{Z})$ are trivial. Moreover by the case 2.2, we have $\text{Pic}^G(\tilde{C}) = \mathbb{Z} \cdot [P]$.

Case 3.2: We adapt the argument of [Br15, Ex. 2.15]. The curve $C$ is obtained by the pinching diagram

$$
\begin{array}{c}
\tilde{Z} \\
\downarrow \lambda \\
Z \\
\end{array}
\xrightarrow{j}
\begin{array}{c}
P^1_k \\
\downarrow \nu \\
C \\
\end{array}
$$

where $\tilde{Z} = \text{Spec } k \sqcup \text{Spec } k$ is the reduced subscheme of $P^1_k$ supported by $\{0, \infty\}$ and $Z = \text{Spec } k$. By Remark 5.12 again, the equivariant Units-Pic exact sequence can be written as

$$1 \rightarrow \mathcal{O}(\tilde{Z})^\times / \mathcal{O}(Z)^\times \rightarrow \text{Pic}^G(C) \rightarrow \text{Pic}^G(P^1_k) \times \text{Pic}^G(Z) \rightarrow \text{Pic}^G(\tilde{Z}).$$

By Proposition 5.5 we have isomorphisms $\text{Pic}^G(P^1_k) \simeq \text{Pic}^G(Z \times Z)$, $\text{Pic}^G(Z) \simeq \widehat{G}(Z) \simeq \mathbb{Z}$ and $\text{Pic}^G(\tilde{Z}) \simeq \widehat{G}(\tilde{Z}) \simeq \mathbb{Z} \times \mathbb{Z}$. Since $G$ acts on the fibers over 0 and $\infty$ of a line bundle $\mathcal{O}_{P^1_k}(n)$ with respective weights $n$ and 0, the morphism $\text{Pic}^G(P^1_k) \times \text{Pic}^G(Z) \rightarrow \text{Pic}^G(\tilde{Z})$ corresponds to

$$\left\{ (Z \times Z) \times \mathbb{Z} \mapsto Z \times \mathbb{Z},
(n, m, \ell) \mapsto (n + m, m) - (\ell, \ell) \right\}.$$

Its kernel is $\mathbb{Z} \cdot ((0,1), 1) \simeq \widehat{G}(P^1_k)$.

Case 3.3: Once more, we have the exact sequence

$$1 \rightarrow \kappa(\tilde{P})^\times / \kappa(P)^\times \rightarrow \text{Pic}^G(C) \rightarrow \text{Pic}^G(\tilde{C}) \times \text{Pic}^G(Z) \rightarrow \text{Pic}^G(\tilde{Z})$$

where $Z = \text{Spec } k$ and $\tilde{Z} = \text{Spec } \kappa(\tilde{P})$. Let $(L, \Phi_L) \in \text{Pic}^G(C)$ and let us show that its image is trivial. By Proposition 5.5, we have $\text{Pic}^G(Z) \simeq \widehat{G}(Z) = \text{Hom}_{k-\text{gp}}(G, \mathbb{G}_{m,k}) = \{1\}$. Hence $i^*(L, \Phi_L)$ is trivial, and so $j^*\nu^*(L, \Phi_L)$ is trivial.
too. The line bundle $L_K$ over $C_K$ is linearizable so, by the case 3.2, it has degree 0. So $L$ and $\nu^*L$ have degree 0 too. Since $\text{Pic} \tilde{C} = \mathbb{Z} \cdot \omega_{C}$, the line bundle $\nu^*L$ is trivial. Its linearization is given by an element $\chi \in \tilde{G}(\tilde{C})$ (and the linearization of $j^*\nu^*L$ is given by $j^*(\chi)$). The morphism $\tilde{G}(\tilde{Z}) \to \text{Pic}^G(\tilde{Z})$ is injective so $j^*(\chi)$ is trivial. After the extension of scalars to $K = \kappa(\tilde{P})$, we have $\chi_K \in \tilde{G}_K(C_K)$ whose pullback in $\tilde{G}_K(\tilde{Z}_K)$ is trivial. But $G_K \simeq G_{m,K}$ so $G_K$ is the constant sheaf $\mathbb{Z}$. Hence $\chi_K$ is trivial, so $\chi$ is trivial too. Therefore the image of $(L, \Phi_L)$ in $\text{Pic}^G(\tilde{C}) \times \text{Pic}^G(Z)$ is trivial.

**Remark 5.14.** Since a line bundle on a projective curve is ample if and only if it has positive degree, we have a full description of ample linearized line bundles over seminormal almost homogeneous curves.

**Corollary 5.15.** Let $C$ be a seminormal curve and $G$ a non-trivial smooth connected algebraic group acting faithfully on $C$. Then $C$ can be embedded in the projectivization of a finite-dimensional $G$-module, except in the cases 1.4, 3.2 and 3.3 of Theorem 5.13.

**A. Seminormality**

In this appendix we recall some properties of seminormal schemes. We use the article [Sw] of Richard Swan as a general reference.

**Definition A.1.** Let $f : Y \to X$ be a morphism between reduced noetherian schemes. We say that the morphism $f$ is subintegral if $f$ is integral and bijective and, for all $y \in Y$, the morphism $f^\sharp : \kappa(f(y)) \to \kappa(y)$ between the residue fields is an isomorphism. We say that $X$ is seminormal if every subintegral morphism $f : Y \to X$ is an isomorphism.

**Remark A.2.**

i) By [EGA, I, Prop. 3.5.8] and [EGA, IV2, Prop. 2.4.5], a subintegral morphism is a universal homeomorphism.

ii) Being a subintegral morphism is a local property on the target.

Normal schemes are obtained by glueing normal rings. Similarly, seminormal schemes are obtained from seminormal rings.

**Definition A.3 ([Sw, Lem. 2.2]).** Let $A$ be a reduced noetherian ring and set

$$A^+ = \{ b \in \overline{A} \mid \forall p \in \text{Spec} A, b \in A_p + R(\overline{A}_p) \subseteq \overline{A}_p \}$$

where $\overline{A}$ is the integral closure of $A$ in its total ring of fractions, $A_p$ and $\overline{A}_p$ are the localizations by the multiplicative set $A \setminus p$ and $R(\overline{A}_p)$ is the Jacobson radical of $\overline{A}_p$. The morphism $\text{Spec} A^+ \to \text{Spec} A$ is subintegral and $A^+$ is the largest subextension of $A \subseteq \overline{A}$ with this property. The ring $A$ is said to be seminormal if $A = A^+$.

Being a seminormal ring is a local property.

**Lemma A.4 ([Sw, Cor. 2.10]).** Let $A$ be a reduced noetherian ring. The following assertions are equivalent:
As an analogue of the normalization, there is a seminormalization. Its construction is well known and we recall it briefly.

**Lemma A.5.** Let $X$ be a reduced noetherian scheme and $\nu : \tilde{X} \to X$ the normalization. There exists a seminormal scheme $X^+$, a subintegral morphism $\sigma : X^+ \to X$ and a factorization $\tilde{X} \to X^+ \xrightarrow{\sigma} X$ of $\nu$, satisfying the following universal property: for every seminormal scheme $Y$ and every morphism $f : Y \to X$, there is a unique factorization $Y \to X^+ \xrightarrow{\sigma} X$ of $f$. The morphism $\sigma$ is called the seminormalization.

**Proof.** If $X$ is affine then we write $X = \text{Spec } A$ and $X^+ = \text{Spec } A^+$. The morphism $\sigma : X^+ \to X$ and the factorization $\tilde{X} \to X^+ \xrightarrow{\sigma} X$ of $\nu$ are given by the inclusions $A \subseteq A^+ \subseteq \overline{A}$. More generally, we cover $X$ by affine open subschemes $U_i = \text{Spec } A_i$ and it follows from [Sw, Prop. 2.9] that we can glue the $\text{Spec } A_i^+$ together to get a seminormal scheme $X^+$, a subintegral morphism $\sigma : X^+ \to X$ and a factorization $\tilde{X} \to X^+ \xrightarrow{\sigma} X$ of $\nu$.

The universal property is given by [Sw, Thm. 4.1] in case $X$ and $Y$ are affine. The general case follows readily by taking affine open coverings.

**Corollary A.6.** Let $X$ be a reduced noetherian scheme. The following assertions are equivalent:

i) The scheme $X$ is seminormal.

ii) For every affine open subscheme $U$ of $X$, the ring $\mathcal{O}_X(U)$ is seminormal.

iii) For every $x \in X$, the ring $\mathcal{O}_{X,x}$ is seminormal.

iv) Every open subscheme of $X$ is seminormal.

v) There exists an open covering of $X$ by seminormal schemes.

Swan proved in [Sw, Thm. 1] that if $X$ is an affine noetherian reduced scheme then the pullback morphism $\text{Pic } X \to \text{Pic}(X \times \mathbb{A}^1)$ is an isomorphism if and only if $X$ is seminormal, generalizing a result of Carlo Traverso ([Tr, Thm. 3.6]). For non-affine schemes, we have the following special case of [Br15, Lem. 3.6].

**Lemma A.7.** Let $X$ be a separated seminormal scheme. The pullback morphism $\text{Pic } X \to \text{Pic}(X \times \mathbb{A}^1)$ is an isomorphism.

The seminormalization commutes with smooth base changes.

**Lemma A.8.** Let $f : Y \to X$ be a smooth morphism between noetherian reduced schemes and $\sigma : X^+ \to X$ the seminormalization. Then $Y \times_X X^+$ is seminormal and the projection $Y \times_X X^+ \to Y$ is the seminormalization.

**Proof.** By Corollary A.6, we can assume that $X$ is affine and write $X = \text{Spec } A$. Let $V = \text{Spec } C$ be an affine open subscheme of $Y$ contained in $f^{-1}(U)$. Then the result follows from [GT, Prop. 5.1].

We now state an analogue of Lemma 3.4 (see [Br15, Lem. 3.5]).
Corollary A.9. Let $G$ be a smooth algebraic group, $X$ a variety, $\alpha : G \times X \to X$ an action and $\sigma : X^+ \to X$ the seminormalization. There exists a unique action $\alpha^+ : G \times X^+ \to X^+$ such that $\sigma$ is equivariant. Moreover, the square

\[
\begin{array}{ccc}
G \times X^+ & \xrightarrow{\alpha^+} & X^+ \\
\downarrow{\text{id} \times \sigma} & & \downarrow{\sigma} \\
G \times X & \xrightarrow{\alpha} & X 
\end{array}
\]

is cartesian, and $\alpha$ and $\alpha^+$ have the same kernel.

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