A Counterexample to a Result on Lotka–Volterra Systems

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Abstract In the article of Dancsó et al. (Acta Appl. Math. 23:103–127, 1991) the authors claim the existence of a Hopf bifurcation which in general does not exist.

Keywords Lotka–Volterra system · Hopf bifurcation

Mathematics Subject Classification 37K10 · 37C27 · 37K05

1 Introduction and the counterexample

In [1] the generalized Lotka–Volterra model of the form

\[ \begin{align*}
\dot{x} &= x^{\hat{p}(\mu)} - x^{p(\mu)} y^{q(\mu)}, \\
\dot{y} &= C(\mu) \left( x^{p(\mu)} y^{q(\mu)} - y^{\hat{q}(\mu)} \right),
\end{align*} \tag{1} \]

with \( C(\mu) > 0 \) is analyzed. The authors claim to have the following result.

Theorem 1 Assume that the functions \( \hat{p}(\mu), p(\mu), \hat{q}(\mu), q(\mu) \) and \( C(\mu) \) are continuously differentiable and for admissible values of \( \mu \) these functions are positive and satisfy

\[ \hat{p}q + p\hat{q} - \hat{p}\hat{q} > 0. \tag{2} \]

If, for some \( \mu_0 \),

\[ \hat{p}(\mu_0) - p(\mu_0) = C(\mu_0) (\hat{q}(\mu_0) - q(\mu_0)) \tag{3} \]

and

\[ \frac{d}{d\mu} \left( \hat{p} - p - C(\hat{q} - q) \right) \bigg|_{\mu = \mu_0} \neq 0, \tag{4} \]

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then the system undergoes an Andronov–Hopf bifurcation at $\mu_0$. Moreover, the bifurcation is supercritical, resp. subcritical according to

$$p(\hat{p} - p)(\hat{q} - 1)(\hat{q} + q) - q(\hat{q} - q)(\hat{p} - 1)(\hat{p} + p)|_{\mu = \mu_0} < 0,$$

resp. $> 0$.

Unfortunately there is a problem in the proof of Theorem 1 due to the following counterexample.

**Theorem 2** Consider the polynomial differential system

$$\begin{align*}
\dot{x} &= x - x^2 y^2 = P(x, y), \\
\dot{y} &= (1 + \mu)(x^2 y^2 - y) = Q(x, y),
\end{align*}$$

with $1 + \mu > 0$. System (5) satisfies all the assumptions of Theorem 1 with $\mu = \mu_0 = 0$ but it does not exhibit an Andronov–Hopf bifurcation.

**Proof** Comparing system (1) with system (5) we have

$$\hat{p}(\mu) = \hat{q}(\mu) = 1, \quad p(\mu) = q(\mu) = 2, \quad C(\mu) = 1 + \mu.$$

Then we have

$$\hat{pq} + p\hat{q} - \hat{p}\hat{q} = 3 > 0.$$

So, condition (2) holds.

Take $\mu_0 = 0$. Then, condition (3) is immediately satisfied, and for condition (4) we obtain

$$\left. \frac{d}{d\mu}(\hat{p} - p - C(\hat{q} - q)) \right|_{\mu = 0} = 1 \neq 0.$$

In short, all the conditions of Theorem 1 are satisfied by system (5).

The unique equilibria of system (5) are the $(0, 0)$ and $(1, 1)$. Around the equilibrium point $(0, 0)$ cannot exist periodic orbits because the straight lines $x = 0$ and $y = 0$ are invariant by the flow of system (5), i.e. they are formed by orbits of system (5). Therefore, if there are periodic orbits these must surround the equilibrium point $(1, 1)$. We recall that in the bounded region limited by a periodic orbit of a differential system in the plane it must be an equilibrium point, see for instance Theorem 1.31 of [2].

We claim that the unique periodic orbits of systems (5) for all $\mu$ exist when $\mu = 0$, and they are the periodic orbits surrounding the center $(1, 1)$ of system (5) with $\mu = 0$. Now we shall prove the claim. Clearly once the claim is proved it follows that system (5) cannot exhibit an Andronov–Hopf bifurcation.

System (5) with $\mu = 0$ has the first integral $H = x + y + 1/(xy)$, because the derivative of $H$ on the orbits of system (5) with $\mu = 0$ satisfies that

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}(x - x^2 y^2) + \frac{\partial H}{\partial y}(x^2 y^2 - y) = 0.$$

Since the eigenvalues of the linear differential system (5) with $\mu = 0$ at the equilibrium $(1, 1)$ are $\pm \sqrt{3}i$, this equilibrium either is a focus or a center, but it cannot be a focus.
because the first integral $H$ is defined at $(1, 1)$. Hence, we have proved that the equilibrium $(1, 1)$ for system (5) with $\mu = 0$ is a center. Now we shall see that the periodic orbits of this center filled all the positive quadrant $Q = \{(x, y) : x > 0 \text{ and } y > 0\}$. Assume that they do not filled all that quadrant. Since in that quadrant the unique equilibrium is the $(1, 1)$, the external boundary of the continuum set of periodic orbit surrounding the center $(1, 1)$ must be a periodic orbit $\gamma$, and after that orbit the nearby orbits must spiral. Consider the Poincaré map defined in an analytic transversal arc $\Sigma$ to this periodic orbit $\gamma$. Since the flow of the polynomial differential system (5) with $\mu = 0$ is analytic, such a Poincaré map is analytic, but it is not possible that an analytic map of one variable be the identity on the piece of the arc $\Sigma$ contained in the bounded region limited by $\gamma$, and different to the identity on the piece of the arc $\Sigma$ outside the bounded region limited by $\gamma$. So such a last periodic orbit $\gamma$ does not exist and the periodic orbits surrounding the center $(1, 1)$ filled all the positive quadrant $Q$. See a picture of the phase portrait of system (5) with $\mu = 0$ on the Poincaré disc in Fig. 1, for more details on the Poincaré disc see Chap. 5 of [2].

For completing the proof of the claim we must show that system (5) with $\mu \neq 0$ has no periodic solutions surrounding the equilibrium $(1, 1)$. We shall use the Dulac criterium: Let $P$ and $Q$ be the polynomials defined in (5). If there exists a $C^1$ function $B(x, y)$ in a simply connected region $R$ such that $\partial(BP)/\partial x + \partial(BQ)/\partial y$ has constant sign and is not identically zero in any subregion of $R$, then system (5) does not have a periodic orbit lying entirely in $R$. For a proof of this criterium see for instance Theorem 7.12 of [2].

Consider the function $B = 1/(x^2y^2)$ defined in the positive quadrant $Q$. Then

$$\frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial x} = \frac{\mu}{x^2y^2} \neq 0 \quad \text{in } Q \text{ if and only if } \mu \neq 0.$$ 

Therefore, by the Dulac criterium, system (5) with $\mu \neq 0$ has no periodic solutions surrounding the equilibrium $(1, 1)$, and this prevents the existence of a Hopf bifurcation. The proof of the claim and of Theorem 2 is completed.

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References

1. Dancsó, A., Farkas, H., Farkas, M., Szabó, G.: Investigations into a class of generalized two-dimensional Lotka–Volterra schemes. Acta Appl. Math. 23, 103–127 (1991)
2. Dumortier, F., Llibre, J., Artés, J.C.: Qualitative Theory of Planar Differential Systems. Springer, New York (2006)