A Semi-analytic Stress Solution for Elastic/Plastic FGM Discs Subject to External Pressure

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Abstract. A new semi-analytic plane stress solution for the elastic/plastic distribution of stress in a thin annular disc subject to pressure over its outer radius is presented. The yield stress varies along the radius of the disc and is a monotonically decreasing function of the radius. It is shown that the general solution consists of several stages and the general structure of the solution depends on material and geometric parameters. An example illustrating the general solution is provided.

1. Introduction

The expansion/contraction of a disc is one of the classical problems of plasticity. Therefore, there is a vast amount of literature on this topic. However, most studies for elastic/plastic models for stationary and rotating discs have only focused on deformation theories of plasticity, unless Tresca’s yield criterion is adopted [1-6]. An overview of solutions for the flow theory of plasticity based on several smooth yield criteria has been provided in [7]. All these solutions are for discs of uniform properties. Purely elastic solutions for functionally graded discs are given, for example, in [8, 9]. Modelling of creep in functionally graded discs is presented in [10]. The objective of the present paper is to extend the method for finding stress and strain distributions in thin discs developed in [7] to functionally graded discs subject to external pressure. The solution for a chosen distribution of the yield stress along the radius is semi-analytic. A numerical technique is only necessary to solve transcendental equations.

2. Statement of the problem

It is assumed that a thin hollow disc of Poisson’s ratio $\nu$, Young’s modulus $E$, outer radius $b_0$, and inner radius $a_0$ is subject to uniform pressure $p_0$ over its outer radius. It is assumed that both Poisson’s ratio and Young’s modulus are constant. The disc has no stress at the initial instant. Strains are supposed to be infinitesimal. It is convenient to choose a cylindrical coordinate system $(r, \theta, z)$ in which the solution is independent of $\theta$. The stress components referred to this coordinate system are
denoted as $\sigma_r$, $\sigma_\theta$ and $\sigma_z$. These stresses are the principal stresses. Moreover, $\sigma_z = 0$. The circumferential displacement vanishes everywhere. The stress boundary conditions are

$$\sigma_r = 0$$

(1)

for $r = a_0$ and

$$\sigma_r = -p_0$$

(2)

for $r = b_0$. It is assumed that the magnitude of $p_0$ is large enough to initiate plastic yielding. Therefore, there are elastic and plastic regions in the disc. The constitutive equations in the elastic region are

$$e^e_r = \frac{\sigma_r - v\sigma_\theta}{E}, \quad e^e_\theta = \frac{\sigma_\theta - v\sigma_r}{E}, \quad e^e_z = -v(\sigma_r + \sigma_\theta).$$

The superscript $e$ denotes the elastic part of the strain. In the elastic region the whole strain is elastic. Equation (3) is also valid in plastic regions. However, this equation is not required in such regions for finding stress solutions. The von Mises plane stress yield criterion for functionally graded materials reads

$$\sigma_r^2 + \sigma_\theta^2 - \sigma_r\sigma_\theta = \sigma_0^2 \Phi(r).$$

Here $\sigma_0$ is a reference stress and $\Phi(r)$ is some given function of $r$. In the present paper, it is assumed that

$$d\Phi(r)/dr \leq 0.$$ 

(5)

The flow rule associated with the von Mises yield criterion is

$$\dot{e}^p_r = \dot{\lambda}(\sigma_r - \sigma), \quad \dot{e}^p_\theta = \dot{\lambda}(\sigma_\theta - \sigma), \quad \dot{e}^p_z = \dot{\lambda}(\sigma_z - \sigma)$$

(6)

where $\dot{e}^p_r$, $\dot{e}^p_\theta$ and $\dot{e}^p_z$ are the plastic strain rates, $\dot{\lambda}$ is a non-negative multiplier and $\sigma = (\sigma_r + \sigma_\theta + \sigma_z)/3$. Since $\sigma_z = 0$ in the case under consideration, equation (6) becomes

$$\dot{e}^p_r = \frac{\dot{\lambda}}{3}(2\sigma_r - \sigma_\theta), \quad \dot{e}^p_\theta = \frac{\dot{\lambda}}{3}(2\sigma_\theta - \sigma_r), \quad \dot{e}^p_z = -\frac{\dot{\lambda}}{3}(\sigma_r + \sigma_\theta).$$

(7)

Since the boundary value problem is statically determinate, the flow rule is not required for the stress solution. The only nontrivial equilibrium equation reduces to

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0.$$ 

(8)

It is convenient to introduce the following dimensionless quantities

$$a = \frac{a_0}{b_0}, \quad \rho = \frac{r}{b_0}, \quad p = \frac{p_0}{\sigma_0}.$$ 

(9)

3. **Purely elastic solution**

The general elastic solution is well known (see, for example, [11]). In our nomenclature, this solution can be written as

$$\frac{\sigma_r}{\sigma_0} = \frac{A}{\rho^2} + B, \quad \frac{\sigma_\theta}{\sigma_0} = \frac{-A}{\rho^2} + B.$$ 

(10)
Here $A$ and $B$ are constants of integration. These constants are found from the boundary conditions (1) and (2). In particular, using (9) it is possible to get $A = A_1 = p a^2 \left(1 - a^2 \right)^{-1}$ and $B = B_1 = -p \left(1 - a^2 \right)^{-1}$. Let $p_e$ be the value of $p$ at which the initiation of plastic yielding occurs. Replacing $A$ and $B$ in (10) with $A_1$ and $B_1$, respectively, and substituting the resulting expressions into (4) shows that if the initiation of plastic yielding occurs at $\rho = a$ then the corresponding value of $p$ is $p = p_{e1} = \Phi(a_1) \left(1 - a^2 \right)^{1/2}$. Analogously, if the initiation of plastic yielding occurs at $\rho = 1$ then the corresponding value of $p$ is $p = p_{e2} = \Phi(b_2) \left(1 - a^2 \right)^{1/\sqrt{3a^4 + 1}}$. Thus $p_e = \min \{p_{e1}, p_{e2}\}$. It follows from this equation that there are three different general structures of the solution. In particular, (i) plastic yielding initiates at $\rho = a$ if $\Phi(a_0) \sqrt{3a^4 + 1} < 2\Phi(b_0)$, (ii) at $\rho = 1$ if $\Phi(a_0) \sqrt{3a^4 + 1} > 2\Phi(b_0)$, and (iii) at $\rho = a$ and $\rho = 1$ simultaneously if $\Phi(a_0) \sqrt{3a^4 + 1} = 2\Phi(b_0)$.

In what follows, it is assumed that $\Phi(a_0) \sqrt{3a^4 + 1} < 2\Phi(b_0)$.

4. Elastic/plastic solution with one plastic region

The following solution is for the function $\Phi(\rho)$ in the form

$$
\Phi(r) = (r/b_0)^n = \rho^n.
$$

The condition (5) requires that $n \leq 0$ and the condition (11) that $a^2 \sqrt{3a^4 + 1} < 2$. The yield criterion (4) is valid in the plastic region. This criterion is satisfied by the following substitution:

$$
\frac{\sigma_0}{\sigma_0} = -2\rho^n \sin \psi, \quad \frac{\sigma_0}{\sigma_0} = -\rho^n \left(\frac{\sin \psi}{\sqrt{3}} + \cos \psi\right).
$$

Here equation (12) has been taken into account. Substituting (13) into (8) yields

$$
\frac{2\cos \psi d\psi}{\sqrt{3} \cos \psi - (1 + 2n) \sin \psi} = \frac{d\rho}{\rho}.
$$

It follows from (1), (9) and (13) that $\psi = 0$ for $\rho = a$. The solution of equation (14) satisfying this boundary condition is

$$
\ln \frac{\rho}{a} = \frac{\sqrt{3} \psi}{2(1 + n + n^2)} - \frac{(1 + 2n)}{2(1 + n + n^2)} \ln \left[\cos \psi - \frac{(1 + 2n)}{\sqrt{3}} \sin \psi\right].
$$

The distribution of the stresses in the plastic region is determined from (13) and (15) in parametric form. The distribution of the stresses in the elastic region is given by (10) but $A \neq A_1$ and $B \neq B_1$. This distribution must satisfy the boundary condition (2). Then, using (9) and (10) it is possible to find that

$$
A + B = -p.
$$

and

$$
\frac{\sigma}{\sigma_0} = -\frac{p}{\rho^2} + B \left(1 - \frac{1}{\rho^2}\right), \quad \frac{\sigma}{\sigma_0} = \frac{p}{\rho^2} + B \left(1 + \frac{1}{\rho^2}\right)
$$

in the elastic region. Let $\rho_1$ be the dimensionless radius of the elastic/plastic boundary and $\psi$ be the corresponding value of $\psi$. It follows from (15) that
\[ \ln \frac{\rho}{a} = \frac{\sqrt{3}\psi_e}{2(1+n+n^2)} - \frac{(1+2n)}{2(1+n+n^2)} \ln \left[ \cos \psi_e - \frac{(1+2n)}{\sqrt{3}} \sin \psi_e \right]. \] (18)

Moreover, both the radial and circumferential stresses should be continuous across the elastic/plastic boundary. Then, it follows from (13) and (17) that

\[ 2\rho^p \sin \psi_e = \frac{p}{\rho^p} - B \left(1 - \frac{1}{\rho^p} \right), \quad -\rho^p \left( \sin \psi_e + \sqrt{3} \cos \psi_e \right) = \frac{p}{\rho^p} + B \left(1 + \frac{1}{\rho^p} \right). \] (19)

Solving these equations for \( p \) and \( B \) yields

\[ 2\sqrt{3}p = \rho^p \left[ 2(1+\rho^p)\sin \psi_e + (1-\rho^p)\left(\sin \psi_e + \sqrt{3}\cos \psi_e \right) \right], \quad B = -\frac{\rho^p}{2} \left( \sqrt{3}\sin \psi_e + \cos \psi_e \right). \] (20)

Eliminating here \( \rho^p \) by means of (18) supplies the dependencies of \( p \) and \( B \) on \( \psi_e \). Using (16) it is now possible to find the dependence of \( A \) on \( \psi_e \). Then, the variation of the stresses with \( \rho \) at a given value of \( \psi_e \) in the elastic region is determined from (10). The solution with one plastic region is valid if the yield criterion is not violated in the elastic region. Using (4), (10) and (12) this condition can be written as

\[ F(\rho) = \frac{3A^2}{\rho^2} + B^2 - \rho^{2n} \leq 0 \] (21)

in the range \( \rho_e \leq \rho \leq 1 \). The only critical number of the function \( F \) in the range \( 0 < \rho < \infty \) is

\[ \rho_0 = \left[ -\frac{6A^2}{n} \right], \quad n_1 = \frac{1}{2(n+2)}. \] (22)

The second derivative of the function \( F \) with respect to \( \rho \) is

\[ \frac{d^2F}{d\rho^2} = \frac{2}{\rho^6} \left[ 10A^2 - n(2n-1)\rho^{2(n+2)} \right]. \] (23)

Since \( n \leq 0 \), it is evident from (23) that \( d^2F/d\rho^2 > 0 \) at \( \rho = \rho_0 \) and that the function \( F \) attains a minimum at this number. Therefore, another plastic region may start to develop from the surface \( \rho = 1 \) if \( \rho_e < \rho_0 < 1 \). The corresponding value of \( \psi_e \) is denoted as \( \psi_{e1} \) and is determined from the equation \( F(1) = 0 \). Using (21) this equation can be rewritten as

\[ 3A^2 + B^2 - 1 = 0. \] (24)

This equation should be solved numerically. The value of \( p \) at which the second plastic region starts to develop is determined from (20) where \( \psi_e \) should be replaced with \( \psi_{e1} \). This value of \( p \) is denoted as \( p_1 \). There are two plastic regions and one elastic region at \( \psi_e > \psi_{e1} \) (or \( p > p_1 \)).

5. Elastic/Plastic Solution with Two Plastic Regions

Let \( \rho_e \) be the dimensionless radius of the second elastic/plastic boundary and \( \psi_{e1} \) be the corresponding value of \( \psi \). In the elastic region, \( \rho_e \leq \rho \leq \rho_{c1} \), the solution (10) is valid. The solution (13) and (15) is valid in the plastic region \( a \leq \rho \leq \rho_e \). Equations (13) and (14) are valid in the plastic region \( \rho_{c1} \leq \rho \leq 1 \). It follows from (2), (9) and (13) that the boundary condition to equation (14) is

\[ \psi = \psi_{e1} = \arcsin(\sqrt{3}p/2) \] (25)
for $\rho = 1$. The solution of equation (14) satisfying the boundary condition (25) is

$$\ln \rho = \frac{\sqrt{3}(\psi - \psi_f)}{2(1+n+n^2)} - \frac{(1+2n)}{2(1+n+n^2)} \ln \left[ \frac{\cos \psi - (1+2n) \sin \psi \sqrt{3}}{\sqrt{1-3 \rho^2/4 - (1+2n) \rho/2}} \right].$$  

(26)

It follows from this equation that

$$\ln \rho_s = \frac{\sqrt{3}(\psi - \psi_f)}{2(1+n+n^2)} - \frac{(1+2n)}{2(1+n+n^2)} \ln \left[ \frac{\cos \psi - (1+2n) \sin \psi \sqrt{3}}{\sqrt{1-3 \rho_s^2/4 - (1+2n) \rho_s/2}} \right].$$  

(27)

Both the radial and circumferential stresses should be continuous across the surfaces $\rho = \rho_c$ and $\rho = \rho_s$. Using these conditions together with equations (10) and (13) gives

$$\frac{A}{\rho_c^2} + B = -\frac{2 \rho_c^\psi \sin \psi_f}{\sqrt{3}}, \quad \frac{A}{\rho_s^2} - B = \rho_s^\psi \left( \frac{\sin \psi_f}{\sqrt{3}} + \cos \psi_f \right),$$  

$$\frac{A}{\rho_c^2} + B = -\frac{2 \rho_s^\psi \sin \psi_f}{\sqrt{3}}, \quad \frac{A}{\rho_s^2} - B = \rho_s^\psi \left( \frac{\sin \psi_f}{\sqrt{3}} + \cos \psi_f \right).$$  

(28)

It is possible to find from these equations that

$$A = \left[ \rho_c^\psi \left( \sin \psi_f + \sqrt{3} \cos \psi_f \right) - \rho_s^\psi \left( \sin \psi_f + \sqrt{3} \cos \psi_f \right) \right] \rho_s^2 \rho_c^2 \left( \rho_c^2 - \rho_s^2 \right),$$

$$B = -\left[ \rho_c^\psi \left( \sin \psi_f + \sqrt{3} \cos \psi_f \right) - \rho_s^\psi \left( \sin \psi_f + \sqrt{3} \cos \psi_f \right) \right] \rho_s^2 \rho_c^2 \left( \rho_c^2 - \rho_s^2 \right) \frac{2 \rho_s^2 \sin \psi_f}{\sqrt{3}}.$$  

(29)

Since $\rho_c$ is a function of $\psi_c$ and $\rho_s$ is a function of $\psi_s$ due to (18) and (27), eliminating $A$ and $B$ in the third and fourth equations in (28) by means of (29) supplies a system of two equations for determining $\psi_c$ and $\psi_s$ as functions of $p$. This system should be solved numerically. This solution allows $\rho_c$, $\rho_s$, $A$ and $B$ to be found from (18), (27) and (29). Then, the distribution of stresses follows from (10) and (13). The disc becomes fully plastic when $\rho_c = \rho_s$.

6. Illustrative example

Equations (24), (28) and (29) have been solved numerically at $a = 0.3$ and $n = -0.5$. It has been found that $\psi_s \approx 0.078$ and $\rho_c \approx 0.898$. The distribution of the radial and circumferential stresses is depicted in Figs. 1 and 2, respectively, at $p = 0.91$, $p = 0.93$, $p = 0.95$, $p = 0.97$, and $p = 0.99$. It worthy of note that the size of the elastic region is very sensitive to the value of $p$ at $p > 0.99$.

7. Conclusions

A new semi-analytic solution for stress has been derived for functionally graded discs subject to external pressure under plane stress conditions. A numerical technique is only necessary to solve transcendental equations such as (24), (28) and (29). It has been shown that there are several stages of the process. The disc is purely elastic if $p \leq p_c$. In the case treated in the present paper, a plastic region starts to develop from the surface $\rho = a$ at $p = p_c$. The solution with one plastic region exists in the range $p_c \leq p \leq p_1$. Another plastic region starts to develop from the surface $\rho = 1$ at $p = p_1$. The solution with two plastic regions exists until the disc becomes fully plastic at $\rho_c = \rho_s$. 

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Figure 1. Variation of the radial stress with $\rho$ at several values of $\rho$.

Figure 2. Variation of the radial stress with $\rho$ at several values of $\rho$.

8. References
[1] Alujevic A, Legat J and Zupec J 1993 ZAMM 73 T283
[2] Mack W and Bengeri M 1994 Int. J. Mech. Sci. 36 699
[3] Parmaksizoglu C and Guven U 1998 Mech. Struct. Mach. 26 9-20
[4] Mack W and Plochl M 2000 Int. J. Eng. Sci. 38 921
[5] Eraslan A N and Akis T 2003 Mech. Based Des. Struct. Mach. 31 529
[6] Arslan E, Mack W and Eraslan A N 2008 Acta Mech. 195 129
[7] Alexandrov S 2015 Elastic/Plastic Discs Under Plane Stress Conditions (Springer)
[8] Calliglu H, Sayer M and Demair E 2011 Indian J. Eng. Mater. Sci. 18 111
[9] Eray A and Ahmet N 2015 J. Multidiscip. Eng. Sci. Technol. 2 2843
[10] Vandana Gupta and Singh S B 2016 Regen. Eng. Transl. Med. 2 126
[11] Hill R 1950 The Mathematical Theory of Plasticity (Oxford: Clarendon Press)