THE GEOMETRY OF $C^{1,\alpha}$ FLAT ISOMETRIC IMMERSIONS

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Abstract. We show that any isometric immersion of a flat plane domain into $\mathbb{R}^3$ is developable provided it enjoys the little Hölder regularity $c^{1,2/3}$. In particular, isometric immersions of local $C^{1,\alpha}$ regularity with $\alpha > 2/3$ belong to this class. The proof is based on the existence of a weak notion of second fundamental form for such immersions, the analysis of the Gauss-Codazzi-Mainardi equations in this weak setting, and a parallel result on the very weak solutions to the degenerate Monge-Ampère equation analyzed in [43].

1. Introduction

1.1. Background. It often happens, in differential geometry, as in other contexts where nonlinear PDEs appear, that the solutions to a given nonlinear system of equations satisfy also systems of higher order equations carrying significant geometric or analytic information. A very basic example is the Gauss-Codazzi-Mainardi equations satisfied by any smooth isometric immersion of a 2 dimensional Riemannian manifold into $\mathbb{R}^3$. Note that, in their turn, these equations can be used to prove certain rigidity statements for the immersion, e.g. to show that any isometric immersion of a flat domain is developable, i.e. roughly speaking, at any given point $p$ its image contains a segment of $\mathbb{R}^3$ passing through $p$.

The subtlety which is often glossed over in the context of classical differential geometry, where all the mappings are assumed smooth, or at least twice or thrice continuously differentiable, is that, due to nonlinearity, the passage from say a given set of first order equations, e.g. the system of isometric immersion equations in our example, to the higher order equations requires a minimum of regularity. There is no guarantee that in the absence of this regularity the geometric information hidden in the higher order equations would be accessible. The above mentioned subtlety could be a source of confusion. Coming back to our example, it is often assumed that all isometric images of flat domains must be ruled, which is untrue. Already, at the dawn of modern differential geometry, no less figures than Lebesgue and Picard intuited the existence of a surface which could be flattened onto the plane with no distortion of relative distances, but nowhere contained a straight segment. To quote Picard:

According to general practice, we suppose in the preceding analysis, as in all infinitesimal geometry of curves and surfaces, the existence of derivatives which we need in the calculus. It may seem premature to entertain a theory of surfaces in which one does not make such hypotheses. However, a curious result has been pointed out by Mr Lebesgue (Comptes Rendus, 1899 and thesis) according to which one may, by the aid of continuous functions, obtain surfaces corresponding to a plane, of such sort that every rectifiable line of the plane has a corresponding rectifiable line of the same length of the surface, nevertheless the surfaces obtained are no

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[53, Page 555], translated and quoted in [7].

[24]. See also [8] Pages 164-165 and the discussion therein of Lebesgue’s heuristic construction.
longer ruled. If one takes a sheet of paper, and crumples it by hand, one obtains a surface applicable to the plane and made up of a finite number of pieces of developable surfaces, joined two and two by lines, along which they form a certain angle. If one imagines that the pieces become infinitely small, the crumpling being pushed everywhere to the limit, one may arrive at the conception of surfaces applicable to the plane and yet not developable [in the sense that there is no envelope of a family of planes of one parameter and not ruled].

The suggested constructions by Lebesgue and then Picard fall short of being totally satisfactory as they point to continuous but yet not differentiable mappings. However, they were perhaps the heralds of a celebrated result due to Kuiper [41], who based on the work by John Nash [49, 50] proved that any Riemannian manifold can be $C^1$ isometrically embedded into a Riemannian manifold of one dimension higher. (Actually, by Nash and Kuiper’s construction, $C^1$ isometric embeddings can approximate any short immersion). As a consequence, the flat disk can be $C^1$-isometrically embedded into any given small ball of $\mathbb{R}^3$ and hence the image cannot be ruled. To make the matters worse, these embeddings can be made through recurrent self-similar corrugations [3, 4], whose images seem, without a proof at hand, not to include any straight segment of $\mathbb{R}^3$ in any scale. In contrast, and returning to our original developability statement in a higher regularity setting where the Gauss-Codazzi equations hold true, Hartman and Nirenberg [29] proved, incidentally almost at the same time Nash and Kuiper showcased their results, that a $C^2$-regular isometric immersion of a flat domain must have a locally ruled surface as its image. We have come full circle, and this discussed dichotomy leaves us with a fundamental question any analyst would like to ask: At what regularity threshold or thresholds between $C^1$ and $C^2$ the unruled isometries transition into the ruled regime?

1.2. Recent developments. To put the problem described above in a broader context, we note that the above dichotomy -known in the recent literature as flexibility vs. rigidity- is not specific to the case of isometric immersions. The Nash and Kuiper scheme for creating highly oscillatory anomalous solutions of typically lower regularity through iterations could be studied under the broader topic of convex integration [26, 56] for the differential inclusions (or PDEs re-cast in this framework), and the involved density or flexibility related results are usually referred to as h-principle. The existence of such h-principle is usually accompanied by a parallel rigidity property which indicates that the construction cannot be carried out in high regularity scales. A notable example is the recent discovery by the first author and Székelyhidi [18]: They showed that the system of Euler equations in fluid dynamics is given to the convex integration method and that Nash and Kuiper’s iterations can be adapted in this case in order to create non-conservative compactly supported continuous flows approximating an open set of subsolutions (see [57, 19] for a thorough discussion of the connection between the two problems of turbulence and isometric immersions). Their results, and the subsequent work in improving the regularity of the anomalous solutions stood in contrast with the rigidity result reflected in the conservation of energy for solutions passing a certain regularity threshold ($\alpha > 1/3$ for $C^{1,\alpha}$-Hölder continuous solutions) in [13, 21], as conjectured by Onsager [51]. Their approach finally lead to the complete resolution of this conjecture by Isett in [33]; see [20] for a review of the history of the problem and the intermediate results.

Coming back to the question of isometric immersions, a parallel question is whether the images of isometric immersions into $\mathbb{R}^3$ of closed convex surfaces (i.e. 2-manifolds with no boundary and positive Gaussian curvature) are rigid motions. This result follows from the convexity of the image of such surfaces. This convexity fails for $C^{1,\alpha}$ isometric immersions of convex surfaces if $\alpha < 1/5$, in which regime the above mentioned h-principle for isometric immersions holds true,
as shown in [17, 10], improving on the results by [6, 14]. On the other hand, Conti, De Lellis and Székelyhidi proved in [14, 15] that when $\alpha > \frac{2}{3}$, the isometric image of a closed convex surface is convex, (from which it follows that the immersion is a rigid motion), and that more generally the h-principle cannot hold true for isometric immersions of any elliptic 2-manifold with or without boundary. Mischa Gromov conjectures in [27, Section 3.5.5.C, Open Problem 34-36] that the transition threshold is $\alpha = \frac{1}{2}$ in this case. The best evidence so far for this conjecture is provided in [16], where the authors show that $C^{1,1/2}$ regularity could be the borderline regularity for the identity of intrinsic covariant derivative of the 2-manifold and its geometric interpretation.

This article concerns the fundamental question of the sub-$C^2$ regularity scales for which, given an isometric immersion of a flat domain, one is able to make sense of fundamentally geometric objects such as the second fundamental form and the Gauss-Codazzi equations. The final goal is to show that the developability property of such isometric immersions (as defined below in Definition 1 and proven for $C^2$ isometries in [29]) survives for $C^{1,\alpha}$ isometric immersions if $\alpha > 2/3$. As we shall see, parts of the project, e.g. the definition of the second fundamental form, can be carried over for $\alpha > 1/2$, but our analysis falls short of proving a weak variant of the Codazzi-Mainardi equations for $1/2 < \alpha < 2/3$. In this manner, the question of the optimal threshold remains open in the flat case as it is for the elliptic case.

Before proceeding, it should be mentioned that the developability of flat isometric immersions have been discussed in the literature in other regularity regimes. Pogorelev [54, Theorem 1, p. 603] proves this developability under the rather weak assumptions of $C^1$ regularity and the existence and vanishing of the total extrinsic curvature (defined as a measure). This result lies at the background of our conclusion through the analysis made in [43] and [39, 40]. On the other hand parallel results have been shown by the second author for $W^{2,2}$ isometric immersions [52], where a slight $C^1$ regularity gain in this case is also proven, and by Jerrard in [36] for the class of Monge-Ampère functions [25]. The former result was based on observations made by Kirchheim on solutions to the degenerate Monge-Ampère equation [38, Chapter 2], and lead to a subsequent statement regarding the smooth density of isometric immersions. These statements were generalized to higher dimensional domains for the co-dimension 1 case in [45]. The latter approach by Jerrard [35, 36], anchored in geometric measure theory, opened the path for proving $C^1$ regularity and a full range of developability results for isometric immersions of any dimension and co-dimension of sub-critical Sobolev regularity [37].

1.3. The main result. We first begin by a definition to clarify the notion of developability as we understand it.

**Definition 1.** Let $\Omega \subset \mathbb{R}^2$ be a domain and $u : \Omega \to \mathbb{R}^k$ be $C^1$-smooth. We say $u$ is developable if for all $x \in \Omega$, either $u$ is affine in a neighborhood of $x$, or there exists a line $L_x$ containing $x$ such that the Jacobian derivative $\nabla u$ is constant on $L_x$, the connected component of $x$ in $L_x \cap \Omega$.

**Remark 1.1.** For bounded domains, $L_x$ is always a segment whose both ends lie on $\partial \Omega$. We allow for unbounded domains, and hence $L_x$ could be a complete line, a ray, or an open segment. With some abuse of notation, we will refer to all of them as “segments” throughout the paper.

There are other equivalent formulations for the developability whose equivalence we will show further on in Proposition 2.1. In particular, local and global developability, as laid out in Corollary 2.2, are equivalent. Also, from Corollary 2.8 and Lemma 2.9 it follows that when $u$ is developable, the segments $L_x$ are uniquely determined outside the maximal affine region; they do neither intersect each other nor pass through the local constancy regions of the Jacobian derivative.

In this article we prefer to work with the little Hölder spaces. For our choice of notations and precise definitions of these spaces see Section 3. Working with the little Hölder scale pinpoints the
needed control -for the effectiveness of our arguments- on the oscillations of the mapping gradient. The corresponding results in the classical Hölder regime is then obtained as a corollary.

The main result of this article is the following theorem:

**Theorem 1.** Let $\Omega \subset \mathbb{R}^2$ be a domain and $2/3 \leq \alpha < 1$. If $u \in c^{1,\alpha}(\Omega,\mathbb{R}^3)$ is an isometric immersion, then $u$ is developable. In particular, if $\alpha > 2/3$, we have $C^{1,\alpha}(\Omega) \subset c^{1,2/3}(\Omega)$ and therefore all isometric immersions $u : \Omega \to \mathbb{R}^3$ of local $C^{1,\alpha}$ regularity are developable.

**Remark 1.2.** Based on the observations made in this article, the proofs of [14, 15] for the rigidity statements are also valid under little Hölder $c^{1,2/3}$ regularity assumption. A more difficult task would be to consider other types of regularity regimes, e.g. the fractional Sobolev one. For this direction, See [44] and the discussions therein.

Per [54, Theorem 1, p. 603], what we need to show is that the image of the Gauss map $\vec{n}$ is of measure zero. Following [14, Corollary 5] it is possible to prove that the Brouwer degree $\deg(\vec{n},\Omega)$ of $\vec{n}$ of the immersion vanishes for all open sets $V \subset \Omega$ and $y \in \mathbb{R}^2$, when defined. However this is not sufficient to conclude with the needed statement $|\vec{n}(\Omega)| = 0$ for developability: Through a similar approach as in [46, Section 5], for each $\alpha \in (0,1)$, one can construct a map in $C^{0,\alpha}(\Omega,\mathbb{R}^2)$ whose local degree vanishes everywhere, but whose image is onto the unit square. Hence the needed degree Gauss map could still have an image of positive measure and this obstacle necessitates another strategy. Indeed, our argument uses a slight improvement on the parallel result proved for the degenerate *very weak Monge-Ampère equation* in [43], which uses the degree formula for the gradient of the solution, and its affine perturbations of positive degree, as follows:

The very weak Hessian determinant [34, 22] of a given function $v \in W^{1,2}_{\text{loc}}(\Omega)$ is defined to be

$$\text{Det} D^2v := -\frac{1}{2}\left(\partial_{11}(\partial_2 v)^2 + \partial_{22}(\partial_1 v)^2 - 2\partial_{12}(\partial_1 v \partial_2 v)\right) = \frac{1}{2} \text{curl}^T \text{curl} (\nabla v \otimes \nabla v)$$

in the sense of distributions. The operator $\text{Det} D^2$ coincides with the usual Monge-Ampère operator $\det \nabla^2 v$ when $v \in C^2$ and $\nabla^2 v$ stands for the Hessian matrix field of $v$. (A $C^3$ regularity is needed for a straightforward calculation). The following statement regarding the degenerate Monge-Ampère equation is a crucial ingredient of our analysis (Compare with [43, Theorem 1.3]).

**Theorem 2.** Let $\Omega \subset \mathbb{R}^2$ be an open domain and let $v \in c^{1,2/3}(\Omega)$ be such that

$$\text{Det} D^2v = 0,$$

in $\Omega$. Then $v$ is developable.

**Remark 1.3.** Note that per Corollary [22] the assumption $v \in C^{1,\alpha}(\overline{\Omega})$ in [43, Theorem 1.3] can too be easily relaxed to the local regularity $v \in C^{1,\alpha}(\Omega)$ in any bounded or unbounded domain.

Theorem [2] can be proved following the exact footsteps of Lewicka and the second author in proving [43, Theorem 1.3], taking into account Lemma [3,1] and Proposition [3,4] below in showing the following degree formulas for any open set $U \subset \Omega$:

\begin{align}
&(\forall y \in \mathbb{R}^2 \setminus \nabla v(\partial U)) \quad \deg(\nabla v, U, y) = 0, \\
&(\forall 0 < \delta \ll 1) \quad (\forall y \in F_\delta(U) \setminus F_\delta(\partial U)) \quad \deg(F_\delta, U, y) \geq 1,
\end{align}

where $F_\delta(x_1, x_2) := \nabla v(x_1, x_2) + \delta(-x_2, x_1)$. See [43, Equations (7.6) and (7.9)]. The rest of the proof remains unchanged. We will leave the verification of details to the reader.

The main strategy of our proof of Theorem [1] is hence to relate the given isometric immersion $u \in c^{1,2/3}$ to a scalar function $v$ which satisfies the assumptions of Theorem [2]. In a first step, we show that a notion of the second fundamental form $A$ in the weak sense exists for immersions of
$C^{1,\alpha}$-regularity for $\alpha > 1/2$. Next, the Codazzi-Mainardi equations is used to prove that if $u$ is isometric, $A$ is curl free, implying that it must be the Hessian matrix of a scalar function, namely the sought after function $v$. We then show the required regularity for $v$, and proceed to prove using the Gauss equation that $v$ satisfies the very weak degenerate Monge-Ampère equation as required by the assumptions of Theorem 2. Finally, we need to prove that developability of $v$, as derived from Theorem 2 implies the developability of the isometric immersion $u$. Apart from Theorem 2 the $2/3$-Hölder exponent regularity is only required for proving that a weak version of Codazzi-Mainardi equations holds for isometric $u \in C^{1,2/3}$, i.e. when we need to show that $A$ is curl free.

The article is organized as follows: In Section 2 we will present and prove a few statements regarding the developability properties of $C^1$ mappings. In Section 3 we will gather a few analytical tools which deal with properties of Hölder continuous functions and with quadratic differential expressions in terms of functions of low regularity. In particular, our basic proposition 3.6 will allow us to define a second fundamental form for the immersions of Hölder type regularity. The subsequent section is dedicated to the definition and properties of this second fundamental form. In Section 5, building on the previous sections, we complete the proof of Theorem 1. Appendix I is dedicated to a side result (Proposition 1.1) on the developability of each component of the immersion, which can be shown independently with a shorter proof. Finally, in Appendix II the proof of some standard facts regarding little Hölder spaces are presented.

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2. Preliminaries: Developable mappings

We have gathered in this section a few statements we will need, and their proofs, regarding the developable mappings in two dimensions. Most of the material in this section are well known and can be found in one form or another in the literature on the topic [29, 54, 12, 38, 58, 52, 48, 39, 40, 30, 31, 36, 45, 37, 43], but the authors do not know of any instance where the following statements are explicitly formulated in concert. As already observed in [30], much of the developability properties of a mapping concern the level sets of its Jacobian derivative, so we formulate our statements having the mappings $f = \nabla u$ in mind, where $u$ is a developable mapping.

2.1. Global vs. local developability. Developability can be defined by local or global formulations, which turn out to be equivalent. There is a risk of confusion, which must be avoided, due to slight differences between the possible formulations (see e.g. the footnote on p. 875 in [40]). Also, it would be useful to have a set of equivalent statements at hand to streamline the arguments. The following proposition states three equivalent conditions which the Jacobian derivative of a $C^1$ mapping can satisfy to be developable:

**Proposition 2.1.** Let $\Omega \subset \mathbb{R}^2$ be any domain and let $f : \Omega \to \mathbb{R}^m$. If $f$ is continuous, then the following three conditions are equivalent.

(a) For all $x \in \Omega$, either $f$ is constant in a neighborhood of $x$, or there exists a line $L$ containing $x$ such that $f$ is constant on the connected component of $x$ in $L \cap \Omega$. 


(b) For all \( x \in \Omega \), there exists an open disk \( B_x \subset \Omega \) centered at \( x \) with the following property: To all \( p \in B_x \), we can associate a line \( L_p \) containing \( p \) such that \( f \) is constant on the segment \( \tilde{l}_p = L_p \cap B \). Moreover, for any \( y, z \in B_x \), \( \tilde{l}_y \cap \tilde{l}_z = \emptyset \), or \( \tilde{l}_y = \tilde{l}_z \).

(c) For all \( x \in \Omega \), there exists an open disk \( B_x \) centered at \( x \) in \( \Omega \) such that for all \( p \in B_x \), there exists a line \( L \subset \mathbb{R}^2 \) containing \( p \) so that \( f \) is constant on the segment \( L \cap B_x \).

**Corollary 2.2.** \( u \in C^1(\Omega, \mathbb{R}^k) \) is developable if and only if for any point in \( x \in \Omega \), there exists a neighborhood \( V \) of \( x \) on which the restriction \( u \vert_V \) is developable.

**Remark 2.3.** Local regularity assumptions are sufficient in the assumptions of Theorems 1 and 2 because of the equivalence of global and local developability as formulated above.

**Remark 2.4.** The continuity of \( f \) is necessary for the implications (a) \( \Rightarrow \) (b), (c) in Proposition 2.1 as the following example demonstrates. The function

\[
f : \mathbb{R}^2 \to \mathbb{R}, \quad f(s, t) := \begin{cases} 
0 & \text{if } st \neq 0 \\
1 & \text{if } st = 0,
\end{cases}
\]

satisfies condition (a) but not (b) or (c). Also, the geometric configuration of the level sets of \( f \) in [52, Figure 1] shows the necessity of the distinction between the points in the local constancy regions from other points in condition (a). In other words, a global formulation of condition (b) is not a statement equivalent to (b): A continuous function on a convex domain \( \Omega \) can satisfy conditions (a) and (b) while there are points \( x \in \Omega \) such that for no line \( L \) containing \( x \) the function is constant on the intersection \( L \cap \Omega \).

**Remark 2.5.** In the recent literature on similar problems, global (and hence inequivalent) versions of condition (b) are formulated as “developability of \( f \)” in [38, Definition 2.29] and “\( \Omega \)-developability of \( f \) on \( \Omega \)” in [30, Section 3.1]. On the other hand condition (a) appears in the statement of [52, Theorem II], and then in [48] as the “condition (L)”. It is also equivalent to \( f \) being “countably developable” according to [30, Definition 1] and [31]. (Note that [30, Assumption (24)] is redundant for continuous \( f \); see Lemma 2.9). Finally, it is condition (c) that is stated as the property of the gradients (with empty-interior images) of \( C^1 \) mappings in the main result of [39, 40]. (This local result was an ingredient of the proof of Theorem 2 in [43]). Generalizations of (a) to weaker regularity or higher dimensional settings can be found in [38, 36, 37].

**Remark 2.6.** The developability of a \( C^1 \) mapping on \( \Omega \), as formulated in Definition 1 translates therefore to its Jacobian derivative satisfying condition (a). To clarify some discrepancy in the literature, we emphasize that we prefer to reserve the term developable for those mappings whose Jacobian derivatives are constant along lower dimensional flat strata in one form or another [45, 37], rather than directly using it for their derivative mappings as done in [38, Definition 2.29] or [30, 31]. This is for historical reasons, since this term is conventionally used to refer to (ruled) smooth surfaces of zero Gaussian curvature [11].

**Proof.**

(a) \( \Rightarrow \) (b):

This is implicitly proved in [45, Lemma 3.7]. We pursue a slightly different strategy. For any \( f : \Omega \to \mathbb{R} \), let \( C_f \) be the set on which \( f \) is locally constant:

\[
C_f := \{ x \in \Omega; \text{ } f \text{ is constant in a neighborhood of } x \}.
\]

Note that \( f \) is constant on the connected components of the open set \( C_f \). If \( f \) satisfies the condition (a) of the proposition, for each \( x \in \Omega \setminus C_f \) we choose a line \( L_x \) so that \( f \) is constant on the connected component \( x \in L_x \cap \Omega \), denoted by \( \tilde{l}_x \).
We begin by the following simple but useful lemma.

**Lemma 2.7.** Let $f : \Omega \to \mathbb{R}^m$ be continuous satisfy condition (a) in Proposition 2.1. Let $\Delta \subset \Omega$ be a closed triangular domain. If $f$ is constant on two edges of $\Delta$, then $f$ is constant on $\Delta$.

**Proof.** Let $x, y, z$ be the vertices of $\Delta$ and assume that $f$ is constant on $[x, y]$ and $[x, z]$. For any point $w \in \Delta \setminus C_f$, $l_w$ must cross one of the two segments $[x, y]$ or $[x, z]$. This implies $f(w) = f(x)$. If $w \in \Delta \cap C_f$, and if the segment $[w, x] \subset C_f$, then once again $f(w) = f(x)$. Finally, if $[w, x] \notin C_f$, let $\tilde{w}$ be the closest element to $w$ on the segment $[w, x]$ not in $C_f$. We already proved that $f(\tilde{w}) = f(x)$. But since $\tilde{w}$ is in the closure of the connected component of $w$ in $C_f$, and $f$ is continuous, we have also $f(w) = f(\tilde{w}) = f(x)$. We conclude that $f|_\Delta \equiv f(x)$. \qed

We now make a crucial observation on the constancy directions $L_x$.

**Corollary 2.8.** Let $f$ be continuous and satisfy condition (a) in Proposition 2.1. If $x \in \Omega \setminus C_f$, there exists only one line $L$ for which $f$ is constant on the connected component $l$ of $x$ in $L \cap \Omega$.

**Proof.** Let $x \in \Omega \setminus C_f$ and assume that for two lines $L \neq \Lambda$, $f$ is constant on the connected components $l$ and $\lambda$ of $x$ in respectively $L \cap \Omega$ and $\Lambda \cap \Omega$. Choose a disk $B(x, \rho) \subset B(x, 2\rho) \subset \Omega$ and note that $\partial B(x, \rho)$ intersects $l$ and $\lambda$ on 4 points $x_1, x_2 \in l$ and $\chi_1, \chi_2 \in \lambda$. Now the assumptions of Lemma 2.7 is satisfied for the four closed triangular domains with vertices $x, x_i, \chi_j$, $i, j \in \{1, 2\}$. This implies that $x \in C_f$, which contradicts the first assumption on $x$. \qed

As a consequence, when $f$ is continuous, for each $x \in \Omega \setminus C_f$, there is only one choice for $L_x$ and there is no ambiguity in the notation. We further observe:

**Lemma 2.9.** Let the continuous mapping $f : \Omega \to \mathbb{R}^m$ satisfy condition (a) in Proposition 2.1. Then for all $x \in \Omega \setminus C_f$, $l_x \subset \Omega \setminus C_f$. Moreover, for all $y, z \in \Omega \setminus C_f$, $l_y \cap l_z = \emptyset$, or $l_y = l_z$.

**Proof.** For showing the first conclusion, we argue by contradiction: Let $y \in C_f \cap l_x$ and let $V$ be a open neighborhood of $y$ in $\Omega$ on which $f$ is constant. We observe that for $\delta > 0$ small enough $B(x, \delta) \setminus l_x$ cannot entirely lie in $C_f$, since otherwise, $B(x, \delta) \setminus l_x$ having only two connected components on both sides of $l_x$, and $f$ being continuous, the value of $f$ would be constant on $B(x, \delta)$, contradicting $x \notin C_f$. Therefore, there is a sequence $x_k \in \Omega \setminus C_f$ converging to $x$ such that $f(x_k) \neq f(x)$. Since the value of $f$ on $x_k$ and $x$ differ, $l_{x_k} \cap l_x = \emptyset$ for all $k$. As $x_k \to x$, we deduce that $l_{x_k}$ must locally uniformly converge to $l_x$. This implies that $l_{x_k} \cap \Omega$ is non-empty for $k$ large enough and hence $f(x_k) = f(y) = f(x)$, which is a contradiction.

To prove the second statement, also assume by contradiction that $l_y \neq l_x$, and that $x \in l_y \cap l_x = \emptyset$. Choose a disk $B(x, \rho) \subset B(x, 2\rho) \subset \Omega$ and note that $\partial B(x, \rho)$ intersects $l_y$ and $l_z$ on 4 points $y_1, y_2 \in l_y$ and $z_1, z_2 \in l_z$. Now the assumptions of Lemma 2.7 is satisfied for the four closed triangular domains with vertices $x, y_i, z_j$, $i, j \in \{1, 2\}$. This implies that $x \in C_f$, which contradicts the first statement, as $y \notin C_f$ but $x \in l_y \cap C_f$. \qed

We are now ready to prove (b) assuming (a). We first note that the condition (b) is obvious if $x \in C_f$. If $x \notin C_f$, without loss of generality, and through rotations, dilations and translations, we can assume that $x = (0, 0)$ and that $l_x$ is parallel to the horizontal axis. We show that we can find $\delta^-, \delta^+ > 0$ such that the condition (b) holds true for $B^+(x, \delta^+)$ and $B^-(x, \delta^-)$ in the upper and lower planes, where the open half disks are defined by

$$B^\pm(x, \delta) := B(x, \delta) \cap \{(s, t) \in \mathbb{R}^2; \pm t > 0\}.$$ 
Then we can choose $B = B(x, \min\{\delta^+, \delta^-\})$ and the proof is complete.

Without loss of generality we concentrate on the upper half plane. If there exists $\delta^+ > 0$ such that $B(x, \delta^+) \subset C_f$, the conclusion is obvious: We choose $L_y$ always parallel to the horizontal
axis. Otherwise there exists a sequence \( x_k \notin C_f \) in the upper open half plane converging to \( x \). Note that by Lemma 2.9, any two (possibly unbounded) segments \( l_y, l_x \) do not intersect within \( \Omega \) unless they are the same. This implies that \( l_{x_k} \cap l_x = \emptyset \) and that the \( l_{x_k} \) must converge locally uniformly to \( l_x \). We choose \( \rho > 0 \) such that the closed rectangular box
\[
S^+_\rho := \{(s,t) \in \mathbb{R}^2; \ |s| \leq \rho, 0 \leq t \leq \rho\}
\]
is a subset of \( \Omega \). We let \( x_0^\pm := (\pm \rho, 0), \ x_0 = (\pm \rho, \rho) \) and choose \( k \) large enough such that for \( y = x_k, \ l_y = l_{x_k} \) intersects the two segments \( [x_0^+, x_0^-] \) in their respective interior points \( y^- \) and \( y^+ \). The closure of the convex open quadrilateral \( P \) created by the four vertices \( x_0^+, x_0^-, y^+, y^- \) lies in \( S^+_\rho \) and we have that \( [y^-, y^+] \subset l_y \) and \( [x_0^-, x_0^+] \subset l_x \).

We claim that the condition \((b)\) is valid for \( P \) (standing for \( B_x \)), which completes the proof since there exists some \( \delta^+ > 0 \), such that \( B^+(x, \delta^+) \subset P \). For any \( z \in P \), if \( z \notin C_f \), then we can choose \( \tilde{l}_z = l_z \cap P \), which lies between \( \tilde{l}_{z_0} \) and \( \tilde{l}_y \). If, on the other hand, \( z \in C_f \), let the points \( z_0 \in [y^-, y^+] \) and \( z_0 \in [x_0^-, x_0^+] \) be those points on \( \partial P \) which lie vertically above and below \( z \). Since \( z_0, z_1 \notin C_f \), we can choose the two closest elements of \( P \setminus C_f \) to \( z \) on the segment \([z_0, z_1] \), which we name respectively by \( \tilde{z}_0, \tilde{z}_1 \). We observe that \( l_{\tilde{z}_0} \) and \( l_{\tilde{z}_1} \) cannot intersect \( l_x \) and \( l_y \), which contain the upper and lower boundaries of \( P \), and hence will intersect the two vertical boundaries of \( P \) between \( x_0^- \) and \( y^+ \) in \( \tilde{z}_0^+ \) and \( \tilde{z}_1^- \), \( i = 0, 1 \). We claim that \( f \) is constant in the region \( X \) enclosed between \( l_{\tilde{z}_0} \) (which is above \( \tilde{l}_x \) and below \( l_{\tilde{z}_1} \)) and \( l_{\tilde{z}_1} \) (which is above \( \tilde{l}_{\tilde{z}_0} \), and below \( \tilde{l}_y \)) in \( P \). Note that \( X \) is the convex quadrilateral created by \( \tilde{z}_0^- , \tilde{z}_0^+, \tilde{z}_1^+, \tilde{z}_1^- \) and contains \( z \) in its interior.

The latter claim about the constancy of \( f \) on \( X \), if proven, completes the proof of the former claim regarding \( P \), since we are free to choose our non-intersecting constancy segments in this region \( X \) (in particular for \( z \)) in a manner that no such two segments intersect within \( P \): If the directions \( l_{\tilde{z}_0} \) and \( l_{\tilde{z}_1} \) are parallel, choose the line \( L_p \) for all points \( p \in X \) to be the line parallel to them passing through \( p \). If, on the other hand, the extensions of \( l_{\tilde{z}_0} \) and \( l_{\tilde{z}_1} \) meet at a point \( q \) (outside of \( \Omega \)), for all \( p \in X \) we choose the line \( L_p \) to be the one passing through \( p \) and \( q \). If \( \tilde{l}_p \), for any \( p \in X \), were to intersect any other \( \tilde{l}_q \) for \( \tilde{p} \in P \setminus X \), it would have to first intersect one of the two \( \tilde{l}_{\tilde{z}_i} \), \( i = 0, 1 \), which does not happen by construction. Note that \( X \) is the connected component of \( C_f \cap P \) containing \( z \). We can therefore foliate \( P \) in constancy segments by exhausting all the connected components of \( C_f \cap P \).

To prove our final claim, that is the constancy of \( f \) on the region \( X \), we note that \( f \) is constant on the segment \( \tilde{l}_{\tilde{z}_1} = [\tilde{z}_1^- , \tilde{z}_1^+] \), and on the segment \([z_0, \tilde{z}_1] \), whose interior lies in \( C_f \). Since \( \tilde{z}_1 \in [z_0^- , z_0^+] \), by Lemma 2.7, \( f \) must be constant in the triangular domain created by \( \tilde{z}_0, \tilde{z}_0^- , z_0^+ \). But then this last assertion implies that \( f \) is constant on the segments \([z_0^- , z_0^+] \) and \([\tilde{z}_0, \tilde{z}_0^+] \), which combined with Lemma 2.7 again, using the constancy of \( f \) on segments \([z_0, \tilde{z}_0^-] \) and \([\tilde{z}_0, \tilde{z}_0^+] \subset l_{\tilde{z}_0} \), proves the claim that \( f \) is constant on \( X \). The proof of \((a) \Rightarrow (b) \) is complete.

\((b) \Rightarrow (c) \):
It is logically obvious.

\((c) \Rightarrow (a) \):
The proof can be found in the last page of [43]. We reproduce it here for completeness. Let \( x \in \Omega \) and consider a Ball \( B_x \) over which the condition \((c) \) is satisfied. In particular, \( f \) is constant on \( L_x \cap B_x \) for some line \( L_x \) passing through \( x \). Now, consider the (possibly infinite) segment
\[c_x := \text{The connected component of } x \text{ in } \{y \in L_x \cap \Omega; f(y) = f(x)\} \]
We claim that \( c_x \), which contains the segment \( L_x \cap B_x \), is either equal to \( l_x \), i.e. the connected component of \( x \) in \( L_x \cap \Omega \), or \( x \in C_f \). This proves that \( f \) is constant either on \( l_x \), or in a neighborhood of \( x \), and so \((a) \) holds true.
We assume that \( c_x \neq l_x \), and prove \( x \in C_f \). In this case \( c_x \) must admit at least one endpoint within \( \Omega \). Let \( y \in \Omega \) be that endpoint. Consider the open disk \( B_y \) centered at \( y \) in which all points admit constancy segments within \( B_y \). Let the line \( L_y \) passing through \( y \) be such that \( f \) is constant on the segment \( L_y \cap B_y \) and let \( z, w \) be the endpoints of this segment. \( L_y \) and \( L_x \) cannot be parallel, since then \( c_x \) can be extended along \( L_x = L_y \) to include either \( z \) or \( w \), which contradicts the fact that it must be a maximal connected constancy subset of \( l_x \). Therefore, we can choose an element \( \tilde{y} \in B_y \cap c_x \) to form an open triangle \( \Delta \) with vertices \( \tilde{y}, z, w \). \( f \) is constant on \([y, z]\) and on \([\tilde{y}, y]\), where \( y \in (z, w) \). Since no segment \( L_p \cap B_y \) departing from a point \( p \in \Delta \) can reach \( \partial B_y \) on its both ends without crossing \([w, z]\) or \([\tilde{y}, y]\), we deduce that \( f|_\Delta = f(y) = f(x) \).

We observe that \( C_f \) contains \( \Delta \), and hence also the open segment \((\tilde{y}, y)\). We have thus found a non-empty interval in \( c_x \) from which we can propagate our local constancy property and so reach to the desired conclusion of \( x \in C_f \). We argue by contradiction: If \( x \notin C_f \), let \( \tilde{x} \) be the closest point to \( y \) on \([x, y]\) which is not in \( C_f \). Certainly \( C_f \supset (\tilde{x}, y) \supset (\tilde{y}, y) \) and \((x, y) \supset (\tilde{x}, y)\). Consider once more the open disk \( B_{\tilde{x}} \) according to the condition (c). Since \( C_f \) is open, for some \( \tilde{y} \in (\tilde{x}, y) \) close enough to \( \tilde{x} \), \( f \) is constant on a segment \([\tilde{z}, \tilde{w}] \subset B_{\tilde{x}} \cap C_f \) orthogonal to \( c_x \) at \( \tilde{y} \), with \( \tilde{y} \in (\tilde{z}, \tilde{w}) \). Also note that for \( \tilde{x} \in [x, \tilde{y}] \), \( f \) is constant on the portion \([\tilde{x}, \tilde{y}] = [x, \tilde{y}] \cap B_{\tilde{x}} \supset [\tilde{x}, \tilde{y}] \) of the segment \([x, \tilde{y}] \subset c_x \). Since \( f \) takes the value \( f(x) \) on both of \([\tilde{x}, \tilde{y}] \) and \([\tilde{z}, \tilde{w}] \), both within the set \( B_{\tilde{x}} \), a similar argument as above shows that \( f \) is constant on the open triangle \( \Delta \) with vertices \( \tilde{x}, \tilde{z}, \tilde{w} \), and so \( \Delta \subset C_f \). But then \( \tilde{x} \in (\tilde{x}, \tilde{y}) \subset \Delta \subset C_f \), which contradicts its choice.

In view of Proposition 2.11 and as already observed in [35, Proposition 2.30], we have:

**Corollary 2.10.** Let \( \Omega \subset \mathbb{R}^2 \) be a domain and let the continuous mapping \( f : \Omega \to \mathbb{R}^m \) satisfy any of the equivalent conditions in Proposition 2.7. Then for all \( x \in \Omega \), there exists \( L > 0 \), an open disk \( B \) centered at \( x \), and a unit Lipschitz vector field \( \eta : B \to \mathbb{R}^2 \) with \( \text{Lip} \eta \leq L \), for which

\[
\forall y \in B \quad \eta(y) = \eta(y + s\eta(y)) \quad \text{and} \quad f(y) = f(y + s\eta(y)) \quad \text{for all } s \text{ for which } y + s\eta(y) \in B.
\]

**Proof.** We choose a disk \( B_x = B(x, 2\delta) \subset \Omega \) according to the condition (b) and we let \( B = B(x, \delta) \).

For any \( p \in B \), there exists a line \( L_p \) such that \( f \) is constant on \( \tilde{l}_p = L_p \cap B_x \), and no two such lines meet within \( B_x \). We define a the mapping \( \Lambda : B \to \mathbb{RP}^1 \) which associates to each point \( p \in B \) the element of the real projective line \( \mathbb{RP}^1 \) determined by the direction of \( L_p \), and we note that it is constant along the segments \( L_p \cap B \) and continuous. Since \( B \) is simply connected, \( \Lambda \) can be lifted to a continuous mapping \( \tilde{\eta} : B \to S^1 \). By constancy of \( \Lambda \) along the segments \( L_p \cap B \), \( \tilde{\eta} \) can only take two distinct values along them, and so its continuity implies that \( \tilde{\eta} \) is constant along these directions, which are now determined by \( \tilde{\eta} \) itself. Since two distinct lines \( L_p \) and \( L_q \), for \( p, q \in B \), do not meet except possibly at an at least a \( \delta \)-distance from \( \partial B \), we conclude that \( \text{Lip} \tilde{\eta} \leq 1/\delta \).

### 2.2. Developability through test functions.

The following lemma will allow us to translate the developability properties of \( v \in C^1 \) into a property for the distribution \( \nabla^2 v \):

**Lemma 2.11.** Let \( B \subset \mathbb{R}^2 \) be an open disk and \( f : B \to \mathbb{R}^m \) be continuous. Also, let \( \tilde{\eta} : B \to \mathbb{R}^2 \) be a unit Lipschitz vector field such that the following property holds true:

\[
\forall x \in B \quad \tilde{\eta}(x) = \tilde{\eta}(x + s\tilde{\eta}(x)) \quad \text{for all } s \text{ for which } x + s\tilde{\eta}(x) \in B.
\]

Then the following two properties are equivalent:

(a) \( \forall x \in B \quad f(x) = f(x + s\tilde{\eta}(x)) \quad \text{for all } s \text{ for which } x + s\tilde{\eta}(x) \in B \),

(b) \( \forall \psi \in C^\infty_c(B) \quad \int_B f \text{div}(\psi\tilde{\eta}) = 0 \).
This yields $\gamma$ that we have.

Note that $t$ which gives for $\tilde{t}$ (2.1)

Passing to the limit we obtain $s$ such that for all $k, s, s', t, t' \in [-t_0, t_0]$, and all $s, s' \in \mathbb{R}$,

$$
\gamma(t) + s\tilde{\eta}(\gamma(t)), \gamma(t') + s'\tilde{\eta}(\gamma(t')) \in B \implies \gamma(t) + s\tilde{\eta}(\gamma(t)) \neq \gamma(t') + s'\tilde{\eta}(\gamma(t')),
$$

unless $t = t'$ and $s = s'$. Indeed, assume by contradiction that there exist sequences $t_k, t'_k \to 0$, $s_k, s'_k$, such that for all $k, (t_k, s_k) \neq (t'_k, s'_k)$, and

$$
\gamma(t_k) + s_k\tilde{\eta}(\gamma(t_k)) = \gamma(t'_k) + s'_k\tilde{\eta}(\gamma(t'_k)) \in B.
$$

Note that $t_k = t'_k$ implies $s_k = s'_k$, and hence we must have $t_k \neq t'_k$ for all $k$, implying on its turn that $\gamma(t_k) \neq \gamma(t'_k)$ for all $k$. On the other hand, by the main property of $\tilde{\eta}$ we obtain

$$
\tilde{\eta}(\gamma(t_k)) = \tilde{\eta}(\gamma(t_k) + s_k\tilde{\eta}(\gamma(t_k))) = \tilde{\eta}(\gamma(t'_k) + s'_k\tilde{\eta}(\gamma(t'_k))) = \tilde{\eta}(\gamma(t'_k)).
$$

This yields

$$
\gamma(t_k) - \gamma(t'_k) = (s'_k - s_k)\tilde{\eta}(\gamma(t_k)),
$$

implying that

$$
\frac{\gamma(t_k) - \gamma(t'_k)}{t_k - t'_k} \cdot \tilde{\xi}(\gamma(t_k)) = 0.
$$

Passing to the limit we obtain $|\tilde{\xi}(x)|^2 = \gamma'(0) \cdot \tilde{\xi}(\gamma(0)) = 0$, which is a contradiction.

Since $\gamma([-t_0, t_0])$ is compact in $B$, we choose $0 < s_0 < \frac{1}{2L}$ such that the image of $[-t_0, t_0] \times [-s_0, s_0]$ under

$$
\Phi(t, s) := \gamma(t) + s\tilde{\eta}(\gamma(t)), \quad t \in [-t_0, t_0], s \in [-s_0, s_0],
$$

lies compactly in $B$. We note that $\Phi$ is one-to-one and Lipschitz on $U := (-t_0, t_0) \times (-s_0, s_0)$ and that we have

$$
\det \nabla \Phi(t, s) = 1 + sk(t) \geq \frac{1}{2} \text{ a.e. in } U.
$$

We conclude that $\Phi : U \to \Phi(U)$ is a bilipschitz change of variable.

We let $V_x = \Phi(U)$, which is an open neighborhood of $x$ in $B$. We calculate for any $\psi \in C^\infty_c(V_x)$:

$$
div(\psi \tilde{\eta}) = \partial_{\tilde{\xi}}(\psi \tilde{\eta}) \cdot \tilde{\eta} + \partial_{\tilde{\xi}}(\psi \tilde{\eta}) \cdot \tilde{\xi} = \nabla \psi \cdot \tilde{\eta} + \psi \partial_{\tilde{\xi}} \tilde{\eta} \cdot \tilde{\xi},
$$

which gives for $\tilde{\psi} = \psi \circ \Phi$:

$$
div(\psi \tilde{\eta})(\Phi(t, s)) = \partial_s \tilde{\psi} + \tilde{\psi} \frac{k(t)}{1 + sk(t)}.
$$

Therefore, letting $\tilde{f}(t, s) = f \circ \Phi(s, t) = f(\gamma(t) + s\tilde{\eta}(\gamma(t))$ we obtain

$$
\int_{V_x} f \text{div}(\psi \tilde{\eta}) = \int_U f \text{div}(\psi \tilde{\eta}) \circ \Phi(s, t) (1 + sk(t)) \, ds \, dt
$$

(2.1)
If (a) is satisfied, then \( \tilde{f}(t, s) = \tilde{f}(t) \) for all values of \( s \) for which the function is defined, and hence, since \( \tilde{\psi} \) is compactly supported in \( U \), we obtain:

\[
(2.2) \quad \forall \psi \in C^\infty_c(V_x) \quad \int_{V_x} f \text{div}(\psi \eta) = 0.
\]

Now for \( \psi \in C^\infty_c(B) \), we let \( K := \text{supp}(\psi) \subset B \). \( K \) is compact and so it admits a finite covering of open sets of the form \( V_{x,i} \), for \( i = 1, \ldots, n \). We consider a partition of unity associated with this covering \( \theta_i \in C^\infty_c(V_{x,i}), \sum_{i=1}^{n} \theta_i = 1 \) on \( K \), and we conclude with (b) by (2.2):

\[
\int_B f \text{div}(\psi \eta) = \sum_{i=1}^{n} \int_{B} f \text{div}(\theta_i \psi \eta) = \sum_{i=1}^{n} \int_{V_{x,i}} f \text{div}(\theta_i \psi \eta) = 0.
\]

To prove the converse, assume (b) and let \( \varphi \in L^1(U) \), be such that

\[
(2.3) \quad \int_{-s_0}^{s_0} \varphi(t, s) \, ds = 0,
\]

and define

\[
\phi(t, s') := \frac{1}{1 + s'k(t)} \int_{-s_0}^{s'} \varphi(t, s) \, ds.
\]

We can construct a sequence \( \psi_k \in C^0_{c,1}(V_x) \) such that

\[
\psi_k \circ \Phi \to \phi \quad \text{in} \quad L^1(U) \quad \text{and} \quad \partial_s(\psi_k \circ \Phi) \to \partial_s \phi \quad \text{in} \quad L^1(U).
\]

Therefore by (b)

\[
\int_{V_x} f \text{div}(\psi \eta) = 0 \quad \forall \psi \in C^\infty_c(V_x) \quad \implies \quad \int_{V_x} f \text{div}(\psi_k \eta) = 0,
\]

which in view of (2.1) implies

\[
\int_U \tilde{f} \varphi \, dsdt = \int_U \tilde{f}(t, s) \partial_s((1 + sk(t)\phi) \, dsdt = \lim_{k \to \infty} \int_U \tilde{f}(t, s) \partial_s((1 + sk(t)\psi_k \circ \Phi) \, dsdt = 0.
\]

This fact being true for all \( \varphi \in L^1(U) \) satisfying (2.3) implies that

\[
\forall t \in (-t_0, t_0) \quad \forall s \in (-s_0, s_0) \quad \tilde{f}(t, s) = \int_{-s_0}^{s_0} \tilde{f}(t, s) \, ds.
\]

This means that for \( |s| < s_0 \):

\[
f(x + s\eta(x)) = \tilde{f}(0, s) = \tilde{f}(0, 0) = f(x).
\]

The global property (a) is a direct consequence of this local property around each \( x \in B \), and the fact that \( \eta \) itself is constant on \( x + s\eta(x) \in B \) for all \( s \).
3. A Hölder continuity toolbox

3.1. Notations. For any open set $U \subset \mathbb{R}^n$, sufficiently differentiable function $f : U \to \mathbb{R}$, and nonnegative integer $j$ we will denote by $[\cdot]_{j;U}$ the supremum norm of its $j$th (multi-index) derivatives over $U$ and by

$$\|f\|_{k;U} := \sum_{j=0}^{k} [f]_{j;U}$$

its $C^k$ norm. For $0 < \alpha \leq 1$, the corresponding Hölder seminorms and norms are identified by the standard conventions

$$[f]_{k,\alpha;U} := \sup_{x,y \in U, x \neq y} \frac{|D^k f(x) - D^k f(y)|}{|x - y|^{\alpha}}.$$ 

The modulus of continuity of $f : U \to \mathbb{R}$ is defined by

$$\omega_{f,U}(r) := \sup_{x,y \in U, 0 < |x - y| \leq r} |f(x) - f(y)|.$$ 

We also introduce the $\alpha$-Hölder modulus of $f$ for $0 < \alpha < 1$:

$$[f]_{0,\alpha;U|r} := \sup_{x,y \in U, 0 < |x - y| \leq r} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}}.$$ 

We will drop the subscript $U$ when denoting these quantities for the specific case of an open set named $U$, hence $\| \cdot \| = \| \cdot \|_{0;U}$, etc.

For a fixed mollifying kernel $\phi \in C^\infty_c(B(0,1))$ with $\int \phi = 1$, and $x \in \mathbb{R}^n$, we denote the standard convolution of a given mapping $f$ with $\phi$ over the length scale $\varepsilon$ by

$$f_\varepsilon(x) := f \ast \phi_\varepsilon(x) = \int_{\mathbb{R}^n} f(x - y)\phi_\varepsilon(y)dy,$$

where for all $x$

$$\phi_\varepsilon(x) := \varepsilon^{-n} \phi\left(\frac{x}{\varepsilon}\right).$$

Naturally $f_\varepsilon(x)$ is defined for $\varepsilon$ small enough provided $f$ is integrable in some neighborhood of $x$.

Throughout this article, the universally bounded constant $c = O(1)$ might change but is independent of all the data, unless specified by an argument. We will also use the usual little-o convention, i.e. for any two functions $b, \tilde{b} : [0, \varepsilon_0) \to \mathbb{R}^+$, we have:

$$\tilde{b}(\varepsilon) \text{ is } o(b(\varepsilon)) \iff \forall \varepsilon \quad \tilde{b}(\varepsilon) \leq cb(\varepsilon) \quad \text{and} \quad \lim_{\varepsilon \to 0^+} \frac{\tilde{b}(\varepsilon)}{b(\varepsilon)} = 0.$$

We will repeatedly refer to the following convolution estimates in our arguments:

**Lemma 3.1.** Let $V \subset U \subset \mathbb{R}^n$ be open sets such that $\text{dist}(\overline{V}, \partial U) > 0$. Assume that $f, h : U \to \mathbb{R}$ are locally integrable. Then for any $0 < \alpha < 1$ and $\varepsilon < \text{dist}(\overline{V}, \partial U)$:

(i) $\|f_\varepsilon - f\|_{0;V} \leq c[f]_{0,\alpha;\varepsilon}^{\varepsilon\alpha},$

(ii) $\|\nabla f_\varepsilon\|_{0;V} \leq c[f]_{0,\alpha;\varepsilon}^{\varepsilon^{\alpha-1}},$

(iii) $\|f_\varepsilon h_\varepsilon - (fh)_\varepsilon\|_{1;V} \leq c[f]_{0,\alpha;\varepsilon} [h]_{0,\alpha;\varepsilon^{2\alpha-1}}.$
Note the basic estimate

\[ \forall x \in V \quad \forall y \in B(0, \varepsilon) \quad |f(x) - f(x - y)| \leq \omega_{f, U}(\varepsilon) \leq |f|_{0, \alpha; \varepsilon^\alpha}. \]

Based on this estimate, the proof follows the same lines as in [14] Lemma 1 and is left to the reader. The commutator estimate Lemma 3.1-(iii) will be crucial for our analysis. It will be used to prove the quadratic estimate (1.5) ([14] Proposition 1) for the pull-back metric of mollified isometric immersions. It is also a necessary ingredient of the proof of Theorem 2 through establishing the degree formulas (1.1) for \( v \in c^{1/2\alpha} \), following [43].

### 3.2. Little Hölder spaces

Our goal here is to define and characterize the elements of the little Hölder spaces \( c^{1,\alpha} \). We denote by \( C^k(U) \) the functions which are continuously differentiable up to the \( k \)th order in \( U \), and by \( C^k(U) \) the functions whose derivatives up to order \( k \) in \( U \) exist and admit a continuous extension to \( U \). For any open bounded weakly Lipschitz domain \( U \subset \mathbb{R}^n \), we recall the definition of the Hölder space

\[ C^{k,\alpha}(U) := \{ f \in C^k(U); \| f \|_{k,\alpha; U} < \infty \}. \]

By uniform continuity of derivatives, all the derivatives of \( f \) up to the order \( k \) have continuous extensions to \( U \) and so \( C^{k,\alpha}(U) \subset C^k(U) \). For an arbitrary open set \( \Omega \subset \mathbb{R}^n \) we say \( f \in C^{k,\alpha}(\Omega) \) if and only if for any point \( x \in \Omega \), \( f \in C^{k,\alpha}(U) \) for some open neighborhood \( U \subset \Omega \) of \( x \).

**Remark 3.2.** We have restricted our attention to weakly Lipschitz bounded domains \( U \) in order to insure the inclusion \( C^1(U) \subset C^{0,\alpha}(U) \) and to avoid unnecessary technical complications in what follows.

**Definition 2.** Let \( 0 < \alpha < 1 \) and \( U \subset \mathbb{R}^n \) be an open bounded weakly Lipschitz domain. The little Hölder space \( c^{0,\alpha}(U) \) is the closure of \( C^1(U) \) with respect to \( \| \cdot \|_{0,\alpha; U} \) in \( C^{0,\alpha}(U) \). For any open set \( \Omega \subset \mathbb{R}^n \), we say \( f \in c^{0,\alpha}(\Omega) \) if and only if for any point \( x \in \Omega \), \( f \in c^{0,\alpha}(U) \) for an open neighborhood \( U \subset \Omega \) of \( x \). For \( k \in \mathbb{N} \), the space \( c^{k,\alpha}(U) \) (resp. \( C^{k,\alpha}(\Omega) \)) is defined to be the set of all functions \( f \in C^k(U) \) (resp. \( C^k(\Omega) \)) for which the components of \( D^k f \) belong to \( c^{0,\alpha}(U) \) (resp. \( c^{0,\alpha}(\Omega) \)).

**Remark 3.3.** Under the above regularity assumptions on \( U \), it can be shown that \( c^{k,\alpha}(U) \) is the closure of \( C^\infty(U) \) in \( C^{k,\alpha}(U) \). We will not use this fact in our arguments.

It is useful to identify the elements of the little Hölder spaces in terms of their modulus of continuity. In this regard we state the following well known equivalence, whose proof is presented for the sake of completeness (see also Appendix [II]).

**Proposition 3.4.** Let \( U \) be a bounded weakly Lipschitz domain, \( 0 < \alpha < 1 \). Then the following statements are equivalent.

1. \( f \in c^{0,\alpha}(U) \).
2. \( f : U \to \mathbb{R} \) admits an extension \( \tilde{f} \in c^{0,\alpha}(\mathbb{R}^n) \) and \( \tilde{f}\varepsilon|_U \to f \) in \( C^{0,\alpha}(U) \).
3. \( \omega_{f, U}(r) = o(r^\alpha) \).
4. \( \lim_{r \to 0} \| f \|_{0,\alpha; U} |r| = 0 \).

**Proof.** (i) \( \Rightarrow \) (iv): Assuming \( f \in c^{0,\alpha}(U) \), let \( f_k \in C^1(U) \) be a sequence of functions converging to \( f \) in \( C^{0,\alpha}(U) \) as \( k \to \infty \). We estimate:

\[ |f|_{0,\alpha; U} \leq |f - f_k|_{0,\alpha; U} + |f_k|_{0,\alpha; U} \leq |f - f_k|_{0,\alpha} + c(U)\| \nabla f_k \|_r^{1-\alpha}, \]

and a straightforward argument by the auxiliary argument \( k \) implies the required convergence.
(iv) ⇒ (iii): This is straightforward since for all \( r > 0 \):
\[
\frac{1}{r^\alpha} \omega_{f;U}(r) \leq [f]_{0,\alpha;U|r}.
\]

(iii) ⇒ (ii): By the assumption \( f \) is uniformly continuous in \( U \). We extend \( f \) to \( \tilde{f} : \mathbb{R}^n \to \mathbb{R} \) by applying Proposition II.2 obtaining that \( \omega_{\tilde{f};\mathbb{R}^n}(r) \leq C \omega_{f;U}(r) \) is \( o(r^\alpha) \). We conclude that
\[
[f]_{0,\alpha;\mathbb{R}^n|r} = \sup_{0 \leq s \leq r} \frac{1}{s^\alpha} \omega_{\tilde{f};\mathbb{R}^n}(s)
\]
is \( o(1) \) as a function of \( r \). Consider any bounded weakly Lipschitz domain \( V \subset \mathbb{R}^n \) and note that \( \tilde{f}|_V \in C^{0,\alpha}(\overline{V}) \). Applying Lemma II.1 to \( V \) implies that \( f_\varepsilon|_V \to \tilde{f}|_V \) in \( C^{0,\alpha}(\overline{V}) \) as \( \varepsilon \to 0 \) and thus \( \tilde{f}|_V \in c^{0,\alpha}(\overline{V}) \). Both conclusions in (ii) follow.

(ii) ⇒ (i): This statement trivially follows from Definition 2.

Corollary 3.5. Let \( 0 < \alpha < \beta < 1 \) and let \( U \subset \mathbb{R}^n \) be a bounded weakly Lipschitz domain. Then
\[ c^{1,\beta}(\overline{U}) \subset c^{1,\beta}(\overline{U}) \subset c^{1,\alpha}(\overline{U}). \]

Proof. The first inclusion is trivial. If \( f \in C^{1,\beta}(\overline{U}) \), then \( \omega_{f;U}(r) \) is \( O(r^\beta) \) and so the second inclusion follows by Proposition 3.4.

3.3. A distributional product and a criteria for Hölder continuity. The following two propositions are practically known by the community at large. Similar or more general statements in the same spirit have appeared in various contexts, e.g. in the discussion of the Young integral [53] Chapter 4. For examples, see [2] Theorem 2.52 and [24] Theorem 13.16, or compare with [17] Theorem 22. Here, we have formulated and proved rather accessible straightforward versions for the little Hölder spaces which are more adapted to our needs.

Definition 3. Let \( U \subset \mathbb{R}^n \) be a bounded weakly Lipschitz domain. We say that a distribution \( T \in \mathcal{D}'(U,\mathbb{R}^n) \) satisfies the \( \alpha \)-interpolation property on \( W^{1,1}_0(U) \) whenever for \( C > 0 \)
\[ \forall \psi \in C_c^\infty(U) \quad \forall j \in \{1, \cdots, n\} \quad |T_j[\psi]| \leq C \| \psi \|^\alpha_{L^1} \| \partial_j \psi \|_{L^1}^{1-\alpha}. \]
We say that \( T \) satisfies the fine \( \alpha \)-interpolation property on \( W^{1,1}_0(U) \) whenever for \( C > 0 \)
\[ \forall \sigma > 0 \quad \exists \delta = \delta(\sigma) > 0 \quad \forall \psi \in C_c^\infty(U) \quad \forall j \in \{1, \cdots, n\} \quad \| \psi \|_{L^1} \leq \delta \| \partial_j \psi \|_{L^1} \quad \Rightarrow \quad |T_j[\psi]| \leq C \| \psi \|^\alpha_{L^1} \| \partial_j \psi \|_{L^1}^{1-\alpha}, \]
with \( \delta(1) = +\infty \)

Proposition 3.6. Let \( U \subset \mathbb{R}^n \) be a bounded weakly Lipschitz domain and \( V \Subset U \) be an open set compactly contained in \( U \). Let \( 0 < \alpha < 1 \), \( f \in C^{0,\alpha}(\overline{U}) \) and \( h \in c^{0,\alpha}(\overline{U}) \). If \( \alpha > 1/2 \), then as \( \varepsilon \to 0 \), for all \( j = 1, \cdots, n \) the sequence \( f_\varepsilon \partial_j h_\varepsilon \) converges in the sense of distributions to a distribution on \( V \) which we denote by \( f \partial_j h \). More precisely, the convergence is in the dual of \( W^{1,1}_0(V) \), i.e. if \( \psi \in W^{1,1}_0(V) \)
\[ f \partial_j h[\psi] := \lim_{\varepsilon \to 0} \int_V f_\varepsilon \partial_j h_\varepsilon \psi(x) dx \]
exists. Moreover for all \( \psi \in C_c^\infty(V) \)
\[ \left| \int_V f_\varepsilon \partial_j h_\varepsilon \psi(x) dx - f \partial_j h[\psi] \right| \leq \| f \|_{0,\alpha} \| h \|_{0,\alpha} \left( o(\varepsilon^{2\alpha-1}) \| \psi \|_{L^1} + o(\varepsilon^\alpha) \| \partial_j \psi \|_{L^1} \right), \]
(3.2)
and \( T = (f \partial_1 h, \ldots, f \partial_n h) \) satisfies the fine \( \alpha \)-interpolation property \((3.1)\) on \( W_0^{1,1}(U) \) with \( C = c(U,V) \| f \|_{0,\alpha}[h]_{0,\alpha} \).

**Remark 3.7.** If \( h \) is merely in \( C^{0,\alpha}(\overline{U}) \), a similar result holds true with the following estimates for all \( \psi \in W_0^{1,1}(U) \):

\[
| \int_V f \varepsilon \partial_j h_x \psi(x) \, dx - f \partial_j h[\psi] | \leq c \| f \|_{0,\alpha}[h]_{0,\alpha} \left( \varepsilon^{2\alpha-1} \| \psi \|_{L^1} + \varepsilon^\alpha \| \partial_j \psi \|_{L^1} \right)
\]

and

\[
| f \partial_j h[\psi] | \leq C \| \psi \|_{L^1} \| \partial_j \psi \|_{L^1}^{1-\alpha},
\]

i.e. \( T = (f \partial_1 h, \ldots, f \partial_n h) \) satisfies the \( \alpha \)-interpolation property on \( W_0^{1,1}(U) \).

**Proof.** Let \( \psi \in W_0^{1,1}(V) \), and for \( \varepsilon < \text{dist}(\overline{V}, \partial U) \) we define:

\[
a(\varepsilon) := \int_V f \varepsilon \partial_j h_x \psi(x) \, dx.
\]

We first prove that \( a(\varepsilon) \) is a Cauchy sequence. For any \( 0 < \varepsilon' < \varepsilon \) we have

\[
| a(\varepsilon) - a(\varepsilon') | = \left| \int_V (f \varepsilon \partial_j h_x - f \varepsilon' \partial_j h_x \varepsilon) \psi \right|
\leq \left| \int_V f \varepsilon (\partial_j h_x - \partial_j h_x \varepsilon) \psi \right| + \left| \int_V (f \varepsilon - f \varepsilon') \partial_j h_x \psi \right|
\leq \left| \int_V \partial_j f \varepsilon (h_x - h_x \varepsilon) \psi \right| + \left| \int_V f \varepsilon (h_x - h_x \varepsilon) \partial_j \psi \right| + \left| \int_V (f \varepsilon - f \varepsilon') \partial_j h_x \psi \right|
= I_1(\varepsilon, \varepsilon') + I_2(\varepsilon, \varepsilon') + I_3(\varepsilon, \varepsilon')
\]

We estimate, using \( \varepsilon' < \varepsilon \):

\[
I_1(\varepsilon, \varepsilon') = \int_V \psi(x) \left( \int_{\mathbb{R}^n} (f(x - y) \partial_j \phi_x(y)) dy \right) \left( \int_{\mathbb{R}^n} (h(x - z))(\phi_x(z) - \phi_x'(z)) \, dz \right) \, dx
\leq \| \psi \|_{L^1} \left( c[f]_{0,\alpha} \varepsilon^{\alpha-1}([h]_{0,\alpha} \varepsilon^\alpha) \right) \leq \| \psi \|_{L^1} \left( c[f]_{0,\alpha} \varepsilon^{\alpha-1}([h]_{0,\alpha} \varepsilon^\alpha) \right)
\]

where we used Lemma \(3.1\) (i) and Proposition \(3.3\) in the third line. For \( I_3(\varepsilon, \varepsilon') \), we note that \( \varepsilon \leq \varepsilon' \) and hence the same estimate as for \( I_1 \) is not obtained, but through the same calculations as for \( I_1 \), we obtain

\[
I_3(\varepsilon, \varepsilon') \leq \| \psi \|_{L^1} \left( [h]_{0,\alpha} [f]_{0,\alpha} \varepsilon^{\alpha-1} \varepsilon^\alpha \right).
\]
In the same manner, we can also obtain an estimate for $I_2$:

$$I_2(\varepsilon, \varepsilon') = \int_V f(\varepsilon, \varepsilon') \partial_j \psi dx \leq \|\partial_j \psi\|_{L^1} \|f\|_0 \sup_{x \in V} \int_{\mathbb{R}^n} h(x-y)(\phi(\varepsilon(y)-\phi(\varepsilon'(y)))dy)$$

$$= \|\partial_j \psi\|_{L^1} \|f\|_0 \sup_{x \in V} \int_{\mathbb{R}^n} (h(x-y)-h(x))(\phi(\varepsilon(y)-\phi(\varepsilon'(y)))dy$$

$$\leq \|\partial_j \psi\|_{L^1} \|f\|_0 [h]_{0,\alpha} \alpha(\varepsilon(\varepsilon)).$$

Putting the three estimates together in view of (3.3), we obtain a first crude estimate:

$$(3.4) \quad |a(\varepsilon) - a(\varepsilon')| \leq [h]_{0,\alpha} \left( o(\varepsilon^{2\alpha-1}) + \varepsilon^{2\alpha-1} o(\varepsilon) \right) [f]_{0,\alpha} \psi_{L^1} \|f\|_0 \|\partial_j \psi\|_{L^1}. $$

This estimate is not enough to establish the Cauchy property of the sequence $a(\varepsilon)$. Therefore we proceed as follows. If $\varepsilon'/\varepsilon \geq 1/2$, we set $\rho = \varepsilon'/\varepsilon$, $m = 1$. Otherwise there exists $m \in \mathbb{N}$, depending on $\varepsilon'/\varepsilon < 1$, such that $1/4 \leq \rho = (\varepsilon'/\varepsilon)^{1/m} < 1/2$. We now write

$$|a(\varepsilon) - a(\varepsilon')| \leq \sum_{k=1}^{m} |a(\rho^{k-1},\varepsilon) - a(\rho^k,\varepsilon)|.$$

We apply (3.4) successively to $0 < \rho^k \varepsilon < \rho^{k-1} \varepsilon$ (as the new $\varepsilon'$ and $\varepsilon$) to obtain

$$|a(\varepsilon) - a(\varepsilon')| \leq [h]_{0,\alpha} \left( o(\varepsilon^{2\alpha-1}) [f]_{0,\alpha} \psi_{L^1} \left( 1 + \rho^{\alpha-1} \right) \sum_{k=1}^{m} \rho^{(k-1)(2\alpha-1)} ight)$$

$$+ o(\varepsilon) \|f\|_0 \|\partial_j \psi\|_{L^1} \sum_{k=1}^{m} \rho^{(k-1)\alpha}$$

$$\leq (\rho^{\alpha-1} + 1) \frac{1 - \rho^{(2\alpha-1)m}}{1 - \rho^{2\alpha-1}} o(\varepsilon^{2\alpha-1}) [h]_{0,\alpha} [f]_{0,\alpha} \psi_{L^1}$$

$$+ \frac{1 - \rho^\alpha}{1 - \rho^{\alpha}} o(\varepsilon) [h]_{0,\alpha} \|f\|_0 \|\partial_j \psi\|_{L^1}$$

$$\leq o(\varepsilon^{2\alpha-1}) [h]_{0,\alpha} [f]_{0,\alpha} \psi_{L^1} + o(\varepsilon) [h]_{0,\alpha} \|f\|_0 \|\partial_j \psi\|_{L^1},$$

since by our choice either $1/2 \leq \rho < 1$ and $m = 1$, or $1/4 \leq \rho < 1/2$, independent of $\varepsilon', \varepsilon$. This implies that $a(\varepsilon)$ is Cauchy sequence. Hence the limit

$$f \partial_j h(\psi) := a = \lim_{\varepsilon \to 0} a(\varepsilon),$$

exists.

Now, we estimate for a fixed $\varepsilon > 0$, as $\varepsilon' \to 0$:

$$\int_V f(\varepsilon, \varepsilon') \partial_j \psi dx - f \partial_j h(\psi) = |a(\varepsilon) - a| \leq \|\psi\|_{L^1} |[f]_{0,\alpha} [h]_{0,\alpha} o(\varepsilon^{2\alpha-1}) + \|\partial_j \psi\|_{L^1} |f|_0 [h]_{0,\alpha} o(\varepsilon),$$

which establishes (3.2). To obtain (3.1), applying Lemma 3.1(ii) and Proposition 3.4 to $\partial_j h_{\varepsilon}$ we observe that:

$$|f \partial_j h(\psi)| \leq |a(\varepsilon)| + |a(\varepsilon)| - |a| \leq |a(\varepsilon)| + |[f]_{0,\alpha} [h]_{0,\alpha} \|\psi\|_{L^1} o(\varepsilon^{2\alpha-1}) + \|f\|_0 [h]_{0,\alpha} \|\partial_j \psi\|_{L^1} o(\varepsilon)$$

$$\leq \|f\|_0 [h]_{0,\alpha} \|\psi\|_{L^1} o(\varepsilon^{2\alpha-1}) + |[f]_{0,\alpha} [h]_{0,\alpha} \|\psi\|_{L^1} o(\varepsilon^{2\alpha-1}) + \|f\|_0 [h]_{0,\alpha} \|\partial_j \psi\|_{L^1} o(\varepsilon)$$

$$\leq \|f\|_0 [h]_{0,\alpha} \|\psi\|_{L^1} o(\varepsilon^{2\alpha-1}) + \|\partial_j \psi\|_{L^1} o(\varepsilon).$$
In other words,

\begin{equation}
\varepsilon \leq \min\{\delta_0, \text{dist}(V, \partial U)\} \implies |f \partial_j h[\psi]| \leq c\|f\|_{0,\alpha}[h]_0,\alpha \left(\|\psi\|_{L^1} \varepsilon^{\alpha-1} + \|\partial_j \psi\|_{L^1} \varepsilon^\alpha\right) \sigma,
\end{equation}

with \(\delta_0(1) = +\infty\).

We conclude the proof of (3.1): Given, \(\sigma > 0\), we let \(\delta = \left(1 + \frac{\text{diam}(V)}{\text{dist}(V, \partial U)}\right) \delta_0\).

Assume \(\|\psi\|_{L^1} \leq \delta \|\partial_j \psi\|_{L^1}\) for a given nonzero \(\psi\), then letting

\[\varepsilon = \left(1 + \frac{\text{diam}(V)}{\text{dist}(V, \partial U)}\right)^{-1} \|\psi\|_{L^1}^{\frac{1}{\alpha}},\]

we obtain \(\varepsilon < \text{dist}(V, \partial U)\) by the Poincaré inequality \(\|\psi\|_{L^1} \leq \text{diam}(V) \|\partial_j \psi\|_{L^1}\) on \(V\). On the other hand we also have \(\varepsilon \leq \delta_0\) and therefore applying (3.5) we obtain as required:

\[|f \partial_j h[\psi]| \leq c \left(1 + \frac{\text{diam}(V)}{\text{dist}(V, \partial U)}\right)^{1-\alpha} \|f\|_{0,\alpha}[h]_0,\alpha \|\psi\|_{L^1}^{\alpha} \|\partial_j \psi\|_{L^1}^{1-\alpha} \sigma.\]

**Definition 4.** We say \(U \subset \mathbb{R}^n\) is an open coordinate rectangular box when it is a rectangular cuboid with edges parallel to the coordinate axes, i.e.

\[U = \prod_{k=1}^n I_k\]

for \(I_k\) an open interval in \(\mathbb{R}\).

**Proposition 3.8.** Assume that \(U \subset \mathbb{R}^n\) is an open coordinate rectangular box and that for \(f \in L^1(U)\), \(T = \nabla f\) satisfies the fine \(\alpha\)-interpolation property on \(W^{1,1}_0(U)\) according to Definition 4 with \(C = 1\). Then

\[\|f\|_{0,\alpha,U}[\delta(\sigma)] \leq 2^{n+1} \sigma.\]

In particular, by Proposition 3.4 \(f \in C^{0,\alpha}(\overline{U})\) and \([f]_{0,\alpha,U} \leq 2^{n+1}\).

**Proof.** The proof of the proposition follows classical ideas relating the decay of mean oscillations to pointwise behavior of functions, as pioneered by Morrey and Campanato, see for instance [8]. We prove the statement by induction over the dimension.

First let \(n = 1\) and \(U \subset \mathbb{R}\) and be an open interval and let \(I \subset U\) be any open subinterval. Note that for any \(\phi \in C^\infty_c(I)\), we can construct a sequence \(\tilde{\phi}_k \in C^\infty_c(I)\) such that \(\|\tilde{\phi}_k\|_0 \leq 1 + 2\|\phi\|_0\), and \(\tilde{\phi}_k\) converges strongly in \(L^1\) to \(\phi - \int_I \phi\). Now fix \(\varphi \in C^\infty_c(I)\) with \(\int_I \varphi = 1\) and let

\[\phi_k := \tilde{\phi}_k - \left(\int_I \tilde{\phi}_k\right) \varphi \xrightarrow{L^1} \phi - \int_I \phi.\]

By the dominated convergence theorem we have

\[\lim_{k \to \infty} \int_I f \phi_k = \int_I f(\phi - \int_I \phi).\]
Let $\psi_k(t) = \int_{-\infty}^{t} \phi_k(s) \, ds$ and note that $\psi_k \in C_c^\infty(I)$. We have therefore for all $\phi \in C_c^\infty(I)$ and $\sigma > 0$, provided $|I| \leq \delta(\sigma)$:

$$
\left| \int_I (f - \overline{f}) \phi \right| = \left| \int_I f(\phi - \phi) \right| = \lim_{k \to \infty} |\int_I f(\phi_k - \phi)| \leq \lim_{k \to \infty} |\int_I f(\phi_k)| = \lim_{k \to \infty} |T[\psi_k]|
$$

$$
\leq \liminf_{k \to \infty} \|\psi_k\|_{L^1} \|\phi_k\|_{L^1} \leq \liminf_{k \to \infty} \|\psi_k\|_{L^1} |I|^{\sigma}
$$

$$
= \liminf_{k \to \infty} \|\phi_k\|_{L^1} |I|^{\sigma} = \|\phi - \overline{\phi}\|_{L^1} |I|^{\sigma} \leq 2\|\phi\|_{L^1} |I|^{\sigma},
$$

where we used the Poincaré inequality on $I$. Since $(L^1)' = L^\infty$, and $C_c^\infty(I)$ is dense in $L^1(I)$, we conclude that

$$
\forall \sigma > 0 \quad |I| \leq \delta(\sigma) \implies \|f - \overline{f}\|_{L^\infty(I)} \leq 2|I|^{\sigma}.
$$

Now, let $y \in U$, and let $d = \text{dist}(y, \partial U)$, and for $z \in (y - d/2, y + d/2)$ and $r < d/2$ define

$$
h_r(z) := \int_{z-r}^{z+r} f(x) dx.
$$

$h_r$ is continuous in $z$ and converges a.e. to $f$ on $(y - d/2, y + d/2)$ as $r \to 0$. On the other hand, for all $r' < r < d/2$:

$$
\left| h_r(z) - h_{r'}(z) \right| \leq \int_{(z-r,z+r) \times (z-r',z+r')} |f(x) - f(x')| dx dx' \leq 4|z - r, z + r|^\alpha = 2^{\alpha + 2} r^\alpha,
$$

where we applied the bound (3.6) to $I = (z - r, z + r)$, $\sigma = 1$, $\delta(1) = +\infty$. As a consequence, $h_r$ is a Cauchy sequence in the uniform norm and $f$ is continuous as the uniform limit of the $h_r$ for $r \to 0$. Now, applying once again (3.6) to $I = (x, y) \subset U$ we obtain that

$$
\forall \sigma > 0 \quad \forall x, y \in U \quad |x - y| \leq \delta(\sigma) \implies |f(x) - f(y)| \leq 4|x - y|^{\sigma},
$$

which implies $|\int_{0,\alpha U} \delta(\sigma)| \leq 4\sigma$, as required.

Now, assume that $n > 1$ and that the statement is true for $n - 1$. Let $Q \subset U$ be any coordinate rectangular box, i.e.

$$
Q := \prod_{k=1}^{n} I_k \subset U,
$$

with the open intervals $I_k \subset \mathbb{R}$. Let $\hat{x} := (x_1, \cdots, x_{n-1})$ and note that for all $h \in L^1(Q)$,

$$
\text{for a.e. } \hat{x} \in Q^{n-1} := \prod_{j=1}^{n-1} I_k, \quad \hat{h}(\hat{x}) := \int_{I_n} h(x) dx
$$

is well defined and belongs to $L^1(Q)$. We claim that:

$$
(3.7) \quad \forall \sigma > 0 \quad |I_n| \leq \delta(\sigma) \implies \text{ess sup}_{x \in Q} \left| f(x) - \hat{f}(\hat{x}) \right| \leq 2|I_n|^{\sigma}.
$$

Let us first show that the conclusion holds assuming the claim (3.7) is true for all coordinate boxes $Q \subset U$. For $a, b$ the extremities of $I_n$, choose $0 \leq \theta_k \leq 1$ in $C_c^\infty(I_n)$ such that $\theta_k \equiv 1$ on $(a + 1/k, b - 1/k)$. Given $\sigma > 0$, for $1 \leq j \leq n - 1$, and $\psi \in C_c^\infty(Q^{n-1})$, assuming that
\[ \|\psi\|_{L^1(Q^{n-1})} \leq \delta(\sigma)\|\partial_j \psi\|_{L^1(Q^{n-1})}, \]  
we have by our main assumption on \( f \):

\[
\left| \int_{Q^{n-1}} \hat{f} \partial_j \psi \right| = \frac{1}{|I_n|} \int_Q f(x) \partial_j \psi(\hat{x}) \, dx = \lim_{k \to \infty} \frac{1}{|I_n|} \int_Q f(x) \theta_k(x) \partial_j \psi(\hat{x}) \, dx 
\]

\[
= \lim_{k \to \infty} \frac{1}{|I_n|} \int_Q f(\theta_k(\hat{x})) \leq \lim_{k \to \infty} \frac{1}{|I_n|} \|\theta_k \psi\|_{L^1(Q)} \|\partial_j(\theta_k \psi)\|_{L^1(Q)}^{1-\alpha} = \frac{1}{|I_n|} \|\psi\|_{L^1(Q)} \|\partial_j \psi\|_{L^1(Q)}^{1-\alpha}.
\]

since \( \theta_k \in C_c^\infty(Q) \) and

\[
\|\theta_k \psi\|_{L^1(Q)} = \left( \int_a^b \theta_k \right) \|\psi\|_{L^1(Q^{n-1})} \leq \delta(\sigma) \left( \int_a^b \theta_k \right) \|\partial_j \psi\|_{L^1(Q^{n-1})} = \delta(\sigma) \|\partial_j \psi\|_{L^1(Q^{n-1})}.
\]

Applying the induction assumption, we deduce that

\[ (3.8) \quad [\hat{f}]_{\theta, \alpha, Q^{n-1}, \delta(\sigma)} \leq 2^n. \]

To prove that \( f \) is continuous, fix \( y \in U \) and consider a box \( Q_d = Q^{n-1} \times (y_n - d, y_n + d) \subset U \) containing \( y \). For all \( z \in Q^{n-1} \times (y_n - d/2, y_n + d/2) \), and \( r < d/2 \) we define

\[ h_r(z) := \int_{z_n-r}^{z_n+r} f(\tilde{z}, s) \, ds \]

Note that, if \( r \) is fixed, the vertical averages of \( f \) are continuous in \( \tilde{z} \) as established in \( (3.8) \), and so \( h_r \) is continuous in \( z \). Applying \( (3.7) \) with \( \sigma = 1 \) to \( Q = Q^{n-1} \times (z_n - r, z_n + r) \) we have for \( r' < r < d/2 \):

\[ \left| h_r(z) - h_r(z') \right| \leq \int_{(z-r,z+r) \times (z'-r',z'+r')} |f(\tilde{z}, s) - f(\tilde{z}, s')| \, ds \, ds' \leq 4(z - r, z + r)^\alpha = 2^{n+2} r^\alpha, \]

which implies that \( h_r \) locally uniformly converge to their limit, which happens to be \( f \). Hence \( f \) is continuous in \( U \).

To obtain a Hölder estimate, let \( x, y \in U \). First, assume \( x_j = y_j \) for some \( 1 \leq j \leq n \). We re-arrange the coordinates so that \( j = n \) and note that for any sequence of coordinate rectangular boxes \( Q_k \) of height \( 1/k \), containing \( x \) and \( y \), such that \( x_n = x_n \) is the midpoint of \( I_{n,k} \), we have by \( (3.8) \)

\[ \forall \sigma > 0 \quad |x - y| / \delta(\sigma) \Rightarrow |f(x) - f(y)| = \lim_{k \to \infty} |\hat{f}Q_k(\tilde{x}) - \hat{f}Q_k(\tilde{y})| \leq 2^n |\tilde{x} - \tilde{y}| \alpha \sigma \leq 2^n |x - y| \alpha \sigma. \]

If, on the other hand, \( x_j \neq y_j \) for all \( j \), we can choose a coordinate rectangular box \( Q \subset U \) which has \( x \) and \( y \) as its opposite vertices on the largest diameter. We obtain by applying \( (3.7) \) and \( (3.8) \) to the now continuous \( f \), provided \( |x - y| \leq \delta(\sigma) \):

\[ |f(x) - f(y)| \leq |f(x) - \hat{f}(\tilde{x})| + |\hat{f}(\tilde{x}) - \hat{f}(\tilde{y})| + |f(y) - \hat{f}(\tilde{y})| \leq 4|x_n - y_n| \alpha \sigma + 2^n |\tilde{x} - \tilde{y}| \alpha \sigma \leq (4 + 2^n) |x - y| \alpha \sigma \]

\[ \leq 2^{n+1} |x - y| \alpha \sigma. \]

In both cases we can conclude with the desired bound, i.e. \([\hat{f}]_{\theta, \alpha, U, \delta(\sigma)} \leq 2^{n+1} \alpha \sigma \). Note that the sequence \( Q_k \) in the first case and the box \( Q \) in the second case exist since \( U \) is assumed to be a coordinate box itself.
It remains to prove the claim (3.7). Given \( \sigma > 0 \) assume that \(|I_n| \leq \delta(\sigma)\) as required in (3.7). For any \( \phi \in C_c^\infty(\Omega) \), consider a sequence \( \tilde{\phi}_k \in C_c^\infty(\Omega) \), \( \|\tilde{\phi}_k\|_0 \leq 1 + 2\|\phi\|_0 \), converging strongly in \( L^1 \) to \( \phi - \tilde{\phi}(\bar{x}) \). As a consequence,
\[
\lim_{k \to \infty} \int_{I_n} \tilde{\phi}_k(x)dx_n = 0 \quad \text{in} \quad L^1(Q^{n-1}).
\]
Choose a \( \varphi \in C_c^\infty(I_n) \) such that \( f_{I_n} \varphi = 1 \) and define
\[
\phi_k(x) = \tilde{\phi}_k(x) - \left( \int_{I_n} \tilde{\phi}_k(\bar{x}, s)ds \right) \varphi(x_n).
\]
Then \( \phi_k \in C_c^\infty(Q) \) converges in \( L^1 \) to \( \phi - \tilde{\phi}(\bar{x}) \) and \( \hat{\phi}_k(\bar{x}) = \int_{I_n} \phi_k(x)dx_n = 0 \). Letting \( \psi_k(\bar{x}, t) = \int_{-\infty}^t \phi_k(\bar{x}, s)ds \) we obtain for all \( \phi \in C_c^\infty(Q) \), \( \psi_k \in C_c^\infty(Q) \). We hence obtain
\[
\left| \int_Q (f - \hat{f}(\bar{x})) \phi \right| = \left| \int_Q f(\phi - \hat{\phi}(\bar{x})) \right| = \lim_{k \to \infty} \int_Q f \phi_k \leq \lim_{k \to \infty} \int_Q f \partial_n \psi_k = \lim_{k \to \infty} T_n[\psi_k]
\]
\[
\leq \liminf_{k \to \infty} \|\psi_k\|_L^1 \|\partial_n \psi_k\|_L^{1-\alpha} \leq \liminf_{k \to \infty} \|I_n\|^\alpha \|\partial_n \psi_k\|_L^1 \|\partial_n \psi_k\|_L^{1-\alpha} \sigma
\]
\[
= \lim_{k \to \infty} \|I_n\|^\alpha \|\phi_k - \int_{I_n} \phi_k(x)dx_n\|_L^1 \sigma = \|I_n\|^\alpha \|\phi - \int_{I_n} \phi(x)dx_n\|_L^1(\Omega) \sigma
\]
\[
\leq 2\|I_n\|^\alpha \|\phi\|_L^1(\Omega) \sigma,
\]
where we used the Poincaré inequality
\[
\|\psi_k\|_L^1(\Omega) \leq \|I_n\| \|\partial_n \psi_k\|_L^1(\Omega) \leq \delta(\sigma) \|\partial_n \psi_k\|_L^1(\Omega)
\]
on \( I_n \). Once again, using the fact that \( L^\infty \) is the dual of \( L^1 \), and the density of \( C_c^\infty(Q) \) in \( L^1(Q) \), we conclude with (3.7).

4. A SECOND FUNDAMENTAL FORM FOR \( c^{1,\alpha} \) ISOMETRIES

Let \( u \in c^{1,\alpha}(\Omega, \mathbb{R}^3) \) be an immersion with \( 2/3 \leq \alpha < 1 \) and let \( V \Subset \Omega \) be any smooth simply-connected domain compactly contained in \( \Omega \), e.g. an open disk. We intend to show the following three statements, which we will formulate more precisely later on. We shall prove:

1. A weak notion of the second fundamental form \( A_{ij} = -\partial_i u \cdot \partial_j n \) makes sense for the immersion \( u \) on \( V \). \( A = [A_{ij}] \) is symmetric. This part can be carried out for \( \alpha > 1/2 \).

2. If \( u \) is isometric, then \( A \) is curl free, and is equal to \( \nabla^2 v \) for a scalar function \( v \in c^{1,\alpha}(V) \).

3. \( \text{Det}(D^2v) = -\frac{1}{2} \text{curl}^T \text{curl} (\nabla v \otimes \nabla v) = 0 \) in \( \mathcal{D}'(V) \).

We choose an open smooth domain \( U \) such that \( V \Subset U \Subset \Omega \). Let \( \phi \in C_c^\infty(B(0,1)) \) be a nonnegative function with \( \int \phi = 1 \). For \( \varepsilon < \text{dist}(\bar{V}, \partial U) \) we define
\[
\bar{n}_\varepsilon := \bar{n} * \phi_\varepsilon
\]
to be the mollification of the unit normal
\[
\bar{n} = \frac{\partial_1 u \times \partial_2 u}{|\partial_1 u \times \partial_2 u|}
\]
of immersion \( u \). Note that by assumption \( C_0 := \|\partial_1 u \times \partial_2 u\|_{0;U} > 0 \) and \( \bar{n} \in c^{0,\alpha}(\Omega, \mathbb{R}^3) \). Let \( u_\varepsilon := u * \phi_\varepsilon : V \to \mathbb{R}^3 \) be the standard mollification of \( u \) on \( V \). If \( \varepsilon \) is small enough \( u_\varepsilon \) is an immersion since \( \nabla u_\varepsilon \) converges uniformly to \( \nabla u \) on \( V \), and
\[
|\partial_1 u_\varepsilon \times \partial_2 u_\varepsilon| \geq \frac{1}{2} C_0.
\]
We define
\[ N^\varepsilon := \frac{\partial_1 u_\varepsilon \times \partial_2 u_\varepsilon}{|\partial_1 u_\varepsilon \times \partial_2 u_\varepsilon|} \]
to be the unit normal corresponding to \( u_\varepsilon \). A direct application of Lemma 3.1 yields:
\begin{equation} \label{4.1}
\|\partial_1 u_\varepsilon - \partial_1 u\|_{0;V} \leq \|u\|_{1,\alpha o(\varepsilon^\alpha)}, \quad \|u_\varepsilon\|_{2;V} \leq \|u\|_{1,\alpha o(\varepsilon^{\alpha-1})}; \quad \|\vec{n}_\varepsilon - \vec{n}\|_{0;V} \leq \|u\|_{1,\alpha o(\varepsilon^\alpha)}. \end{equation}
We also note that
\begin{equation} \label{4.2}
\|N^\varepsilon - \vec{n}\|_{0;V} \leq \frac{1}{C_0} \|u\|_{1,\alpha o(\varepsilon^\alpha)}, \quad \|\nabla N^\varepsilon\|_{0;V} \leq \frac{1}{C_0} \|u\|_{1,\alpha o(\varepsilon^{\alpha-1})}.
\end{equation}
To see the first estimate we observe that
\begin{equation}
\|N^\varepsilon - \vec{n}\|_{0;V} \leq \frac{2}{C_0} \left( \|\partial_1 u_\varepsilon \times \partial_2 u_\varepsilon\|_{0;V} - \|\partial_1 u \times \partial_2 u\|_{0;V} \right) \\
\leq \frac{4}{C_0} \left( \|\partial_1 u_\varepsilon \times \partial_2 u_\varepsilon\|_{0;V} - \|\partial_1 u \times \partial_2 u\|_{0;V} \right) \leq \frac{4}{C_0} \|u\|_{1,\alpha o(\varepsilon)}
\end{equation}
where we used (4.1). The second estimate also follows from (4.1) and a direct calculation.

We finally denote by the symmetric matrix field \( A^\varepsilon := [A^\varepsilon_{ij}] \) the second fundamental form associated with the immersion \( u_\varepsilon \) in the local coordinates, i.e.
\begin{equation} \label{4.3}
A^\varepsilon_{ij} := \partial_{ij} u_\varepsilon \cdot N^\varepsilon = -\partial_i u_\varepsilon \cdot \partial_j N^\varepsilon,
\end{equation}
which by (4.1) satisfies the obvious bound
\begin{equation} \label{4.4}
\|A^\varepsilon\|_{0;V} \leq \|u_\varepsilon\|_{2;V} \leq \|u\|_{1,\alpha o(\varepsilon^{\alpha-1})}.
\end{equation}

### 4.1. Definition of the second fundamental form \( A \).

We are ready now to prove our main claims about the existence of a second fundamental form for the immersion \( u \) and its properties. In what follows the constant \( C = c(U, V, \|u\|_{1,\alpha U}, C_0) \) might change but it has a universal upper bound only depending on the stated quantities.

**Proposition 4.1.** If \( \alpha > 1/2 \), then \( A^\varepsilon_{ij} \) converges in the sense of distributions to a distribution \( A_{ij} \in \mathcal{D}'(V) \). More precisely, the convergence is in the dual of \( W^{1,1}_0(V) \), and we have:
\[ \forall \psi \in C_c^\infty(V) \quad |(A^\varepsilon_{ij} - A_{ij})[\psi]| \leq C \left( o(\varepsilon^{2\alpha-1})\|\psi\|_{L^1} + o(\varepsilon^\alpha)\|\partial_j \psi\|_{L^1} \right). \]
Moreover, for each \( i \), the distribution \( T = (A_{ij})_{j=1}^n \) satisfies the fine \( \alpha \)-interpolation property on \( W^{1,1}_0(V) \) according to Definition 3.

**Proof.** Applying Proposition 3.6 to the components \( f := \partial_i u^m, h := \vec{n}^m \), for \( m \in \{1, 2, 3\} \), we deduce that for a distribution \( A_{ij} \in \mathcal{D}'(V) \),
\[ B^\varepsilon_{ij} := -\partial_i u_\varepsilon \cdot \partial_j \vec{n}_\varepsilon \rightarrow A_{ij} := -\partial_i u \cdot \partial_j \vec{n} \quad \text{in} \quad \mathcal{D}'(V), \]
and
\[ |B^\varepsilon_{ij}[\psi] - A_{ij}[\psi]| \leq C \left( o(\varepsilon^{2\alpha-1})\|\psi\|_{L^1} + o(\varepsilon^\alpha)\|\partial_j \psi\|_{L^1} \right). \]
On the other hand, we have for all \( \psi \in W^{1,1}_0(V) \):
\[
|A^\varepsilon_{ij}[\psi] - A_{ij}[\psi]| \leq |B^\varepsilon_{ij}[\psi] - A_{ij}[\psi]| + \left| \int_B \partial_i u^\varepsilon(x) \cdot (\partial_j \bar{n}^\varepsilon(x) - \partial_j N^\varepsilon(x))\psi(x)dx \right|
\]
\[
\leq C\left(o(\varepsilon^{2\alpha-1})\|\psi\|_{L^1} + o(\varepsilon^\alpha)\|\partial_j \psi\|_{L^1}\right)
\]
\[
+ \left| \int_V \partial_{ij} u^\varepsilon \cdot (\bar{n}^\varepsilon - N^\varepsilon)\psi dx \right| + \left| \int_V \partial_i u^\varepsilon \cdot (\bar{n}^\varepsilon - N^\varepsilon)\partial_j \psi \right|
\]
\[
\leq C\left(o(\varepsilon^{2\alpha-1})\|\psi\|_{L^1} + o(\varepsilon^\alpha)\|\partial_j \psi\|_{L^1}\right),
\]
where we used Proposition 3.6, (4.1) and (4.2). The stated fine \( \alpha \)-interpolation property follows as in Proposition 3.6.

**Definition 5.** For \( 1/2 < \alpha < 1 \), we define the weak second fundamental form of an immersion \( u \in C^{1,\alpha}(\Omega, \mathbb{R}^3) \) on \( V \) as the distribution
\[
A : C_c^\infty(V) \to \mathbb{R}^{2\times 2}_{sym}, \quad A[\psi] := [A_{ij}[\psi]]_{2\times 2}.
\]

**Remark 4.2.** The definition of \( A \) can be uniquely extended to a distribution globally defined on \( \Omega \), i.e. belonging to \( \mathcal{D}'(\Omega, \mathbb{R}^{2\times 2}_{sym}) \), but we will not need this fact.

4.2. **A satisfies Gauss-Codazzi-Mainardi equations (in a weak sense).** From now on we assume that the immersion \( u \) is also isometric, i.e. the pull back-metric \( g := (\nabla u)^T(\nabla u) \) is the Euclidean metric \( \mathcal{E}_2 \) on \( \Omega \). We consider \( \mathcal{G}^\varepsilon := (\nabla u^\varepsilon)^T\nabla u^\varepsilon = u^\varepsilon_3\mathcal{E}_3 \), the pull-back metric induced by \( u^\varepsilon \), where \( \mathcal{E}_3 \) is the Euclidean metric of \( \mathbb{R}^3 \). The commutator estimate Lemma 3.1(iii) implies
\[
\|\mathcal{G}^\varepsilon - (g \ast \phi^\varepsilon)\|_{1;V} = \| (\nabla u^\varepsilon)^T\nabla u^\varepsilon - (\nabla u)^T\nabla u\|_{1;V} \leq \|u\|_{1,\alpha}^2 o(\varepsilon^{2\alpha-1}).
\]
So, in view of the fact that \( \mathcal{E}_2^\varepsilon \ast \phi^\varepsilon = \mathcal{E}_2 \) on \( V \) we obtain:
\[
\|\mathcal{G}^\varepsilon - \mathcal{E}_2\|_{1;V} \leq \|u\|_{1,\alpha}^2 o(\varepsilon^{2\alpha-1}).
\]
Compare with [14, Proposition 1], which is stated under more general settings and where the mollifier needs to be symmetric.

**Proposition 4.3.** We have the uniform estimate
\[
\|\text{curl} A^\varepsilon\|_{0;V} \leq C o(\varepsilon^{3\alpha-2}).
\]
In particular, if \( \alpha \geq 2/3 \), then \( \text{curl} A = 0 \) in \( \mathcal{D}'(V) \).

**Proof.** We recall that the Christoffel symbols associated with a metric \( g \) are given by
\[
\Gamma^i_{jk}(g) = \frac{1}{2} g^{im}(\partial_k g_{jm} + \partial_j g_{km} - \partial_m g_{jk}),
\]
where the Einstein summation convention is used. Hence, in view of (4.5), we have
\[
\Gamma^{i,e}_{kj} := \mathcal{G}^\varepsilon_{ij} \to 0
\]
uniformly, if \( \alpha \geq 1/2 \), with the estimate
\[
\|\Gamma^{i,e}_{kj}\|_{0;V} \leq C o(\varepsilon^{2\alpha-1}).
\]
Writing the Codazzi-Mainardi equations [28, Equation (2.1.6)] for the immersion \( u^\varepsilon \) we have
\[
\partial_2 A^\varepsilon_{11} - \partial_1 A^\varepsilon_{12} = A^\varepsilon_{11}(\Gamma^1_{12} - \Gamma^1_{21}) - A^\varepsilon_{12} \Gamma^2_{12},
\]
\[
\partial_2 A^\varepsilon_{12} - \partial_1 A^\varepsilon_{22} = A^\varepsilon_{12}(\Gamma^1_{22} - \Gamma^1_{21}) - A^\varepsilon_{22} \Gamma^2_{22}.
\]
Hence, by (4.4) and (4.7),
\[ \|\partial_2 A^\varepsilon_{11} - \partial_1 A^\varepsilon_{12}\|_{0,V} \leq C_0(\varepsilon^{\alpha-1}) o(\varepsilon^{2\alpha-1}), \]
which completes the proof.

Before proceeding to the next step, we need to establish a further property of the sequence \( A^\varepsilon \). Observe that if \( u \in C^2(\Omega, \mathbb{R}^3) \) were an isometric immersion, then the determinant of its second fundamental form \( A \) would vanish, since, in this particular case, it would be equal to the Gaussian curvature of the Euclidean metric. In the case considered here, \( u \) does not enjoy the sufficient regularity for \( \det A \) to be defined. However, we can show that for \( \alpha \geq 1/2 \), \( A^\varepsilon \) converges in the sense of \( \varepsilon \)-distributions to 0 as \( \varepsilon \to 0 \), that with a rate which will be crucial in our analysis. The following statement is a slight variant of [14, Proposition 7] recast for our purposes:

**Proposition 4.4.** Let \( A^\varepsilon \) be as defined in (4.3). Then for all \( \psi \in C^\infty_c(\Omega) \),
\[ \left| \int_{\Omega} (\det A^\varepsilon) \psi \right| \leq C \|\psi\|_{W^{1,1}\varepsilon(\Omega)}. \]

**Proof.** In view of the formula for the (0, 4)-Riemann curvature tensor [28, Equation (2.1.2)]
\[ R_{ijlk} = g_{lm}(\partial_k \Gamma^m_{ij} - \partial_j \Gamma^m_{ik} + \Gamma^m_{ks} \Gamma^s_{ij} - \Gamma^m_{js} \Gamma^s_{ik}), \]
and by the Gauss equation [28, Equation (2.1.7)], (4.5) and (4.7) we have on \( \Omega \):
\[ \det A^\varepsilon = R_{1212}(\varphi^\varepsilon) = \varphi^\varepsilon_{1m}(\partial_1 \varphi^\varepsilon_{21} - \partial_2 \varphi^\varepsilon_{12}) + \varphi^\varepsilon_{1s}(\partial_1 \varphi^\varepsilon_{s2} - \partial_2 \varphi^\varepsilon_{s1}) + o(\varepsilon^{2\alpha-1}) \]
\[ = \partial_1(\varphi^\varepsilon_{2m} - \varphi^\varepsilon_{1m}) - \partial_2(\varphi^\varepsilon_{m2} - \varphi^\varepsilon_{m1}) + o(\varepsilon^{2\alpha-1}) \]
\[ = -\text{curl} \text{curl} \varphi^\varepsilon + o(\varepsilon^{2\alpha-1}). \]
The conclusion follows by an integration by parts involving the first term and applying (4.5).  

4.3. \( A \) as the Hessian of a scalar function.

**Proposition 4.5.** Let \( \alpha \geq 2/3 \), \( u \in C^{1,\alpha}(\Omega, \mathbb{R}^3) \) be an isometric immersion and \( V \) be a smooth simply-connected domain compactly contained in \( \Omega \). There exists \( v \in C^{1,\alpha}(V) \cap W^{1,2}(V) \) such that \( A \), the 2nd fundamental form of \( u \) on \( V \), equals \( \nabla^2 v \) in the sense of distributions. Moreover
\[ \text{Det} D^2 v = -\frac{1}{2} \text{curl} \text{curl}(\nabla v \otimes \nabla v) = 0 \quad \text{in} \ \mathcal{D}'(V). \]

**Remark 4.6.** In our setting, it can also be shown that \( v \in C^{1,\alpha}(V) \). In view of Remark 4.2, one can find such \( v \in C^{1,\alpha}(\Omega) \) provided \( \Omega \) is simply connected. We will not need these facts.

**Proof.** Let \( A^\varepsilon \) be as defined in (4.3) and let \( F^\varepsilon \) be the solution to the Neumann problem
\[ \begin{cases} \Delta F^\varepsilon = \text{div} A^\varepsilon & \text{in} \ V \\ \partial_{\nu} F^\varepsilon = A^\varepsilon \cdot \nu & \text{on} \ \partial V \end{cases} \]
Since \( \text{div}(\nabla F^\varepsilon - A^\varepsilon) = 0 \), there exists a vector field \( E^\varepsilon \) such that \( A^\varepsilon = \nabla F^\varepsilon + \nabla^\perp E^\varepsilon \). \( E^\varepsilon \) satisfies
\[ \begin{cases} \Delta E^\varepsilon = \text{curl}(A^\varepsilon - \nabla F^\varepsilon) = \text{curl} A^\varepsilon & \text{in} \ V \\ E^\varepsilon = \text{const.} & \text{on} \ \partial V \end{cases} \]
Since \( A^\varepsilon \) is symmetric, we have \( \partial_2(F^\varepsilon_{2} + E^\varepsilon_{2}) = \partial_1(F^\varepsilon_{1} - E^\varepsilon_{1}) \), and thus we derive the existence of a scalar function \( v(\varepsilon) \in C^\infty(V) \) satisfying
\[ \partial_1 v(\varepsilon) = F^\varepsilon_{1} + E^\varepsilon_{2}, \quad \partial_2 v(\varepsilon) = F^\varepsilon_{2} - E^\varepsilon_{1}, \]
\[ \nabla v^{(\varepsilon)} = F^\varepsilon + (E^\varepsilon_1, -E^\varepsilon_2). \] As a consequence
\[ A^\varepsilon = \nabla^2 v^{(\varepsilon)} - \nabla (E^\varepsilon)^\perp + \nabla^\perp E^\varepsilon. \]

Standard elliptic estimates imply that for any \( p < \infty \)
\[ \| \nabla E^\varepsilon \|_{W^{1,p}(V)} \leq c(p, V) \| \nabla A^\varepsilon \|_{0; V}. \]

As a consequence, fixing \( p > 2 \), and in view of (4.6), \( \| A^\varepsilon - \nabla^2 v^{(\varepsilon)} \|_{0; V} \leq C \varepsilon^{3\alpha - 2} \to 0 \) and therefore \( \nabla^2 v^{(\varepsilon)} \) converges to \( A \) in the sense of distributions.

Now, for all \( \varphi \in C_c^\infty(V) \), there exists a vector field \( \Psi \) in \( V \), vanishing on \( \partial V \), such that \( \text{div } \Psi = \varphi - \int_V \varphi \), for which \( \| \Psi \|_{W^{1,2}(V)} \leq c(V) \| \varphi \|_{L^2(V)} \) (see e.g. \#). By adjusting the \( \nabla v^{(\varepsilon)} \) so that it is of average 0 over \( V \), and in view of Proposition 4.11, we obtain that for all \( \varepsilon, \varepsilon' \leq \delta \):
\[
\left| \int_V (\nabla v^{(\varepsilon)} - \nabla v^{(\varepsilon')}) \varphi \right| = \left| \int_V (\nabla v^{(\varepsilon)} - \nabla v^{(\varepsilon')}) (\varphi - \int_V \varphi) \right| = \left| \int_V (\nabla v^{(\varepsilon)} - \nabla v^{(\varepsilon')}) \text{div } \Psi \right|
\leq \int_V (A^\varepsilon - A^\varepsilon') \Psi + \left| \int_V (\nabla^2 v^{(\varepsilon)} - A^\varepsilon) \Psi \right| + \left| \int_V (\nabla^2 v^{(\varepsilon') - A^\varepsilon') \Psi \right|
\leq C \| \Psi \|_{W^{1,6}(V)} \delta^{3\alpha - 2} \leq C \| \varphi \|_{L^2(V)} \delta^{3\alpha - 2}.
\]

By duality, we conclude that \( \nabla v^{(\varepsilon)} \) converges strongly in \( L^2(V) \) to a vector field \( F \) and that \( \nabla F = A \). It is now straightforward to see that, if necessary by adjusting the \( v^{(\varepsilon)} \) by constants, \( v^{(\varepsilon)} \) converges strongly in \( W^{1,2} \) to some \( v \in W^{1,2}(V) \) and that \( A = \nabla^2 v \).

To complete the proof, it is only necessary to show that (a) \( \text{Det } D^2 v = 0 \) and (b) \( v \in C^{1,\alpha}(V) \).

In order to show (a), we first observe that since \( \nabla v^{(\varepsilon)} \to \nabla v \) in \( L^2(V) \), for all \( \psi \in C_c^\infty(V) \) we obtain:
\[
-\frac{1}{2} \text{curl}^T \text{curl}(\nabla v \otimes \nabla v)[\psi] = \lim_{\varepsilon \to 0} -\frac{1}{2} \text{curl}^T \text{curl}(\nabla v^{(\varepsilon)} \otimes \nabla v^{(\varepsilon)})[\psi] = \lim_{\varepsilon \to 0} \text{det } \nabla^2 v^{(\varepsilon)}[\psi]
= \lim_{\varepsilon \to 0} \int_V \text{det } (A^\varepsilon + \nabla (E^\varepsilon)^\perp - \nabla^\perp E^\varepsilon) \psi
= \lim_{\varepsilon \to 0} \int_V (\text{det } A^\varepsilon) \psi + \lim_{\varepsilon \to 0} \sum_{\text{some indices } i,j,k,l} \int_V A_{ij}^\varepsilon (\partial_k E_l^\varepsilon) \psi
+ \lim_{\varepsilon \to 0} \int_V (\nabla (E^\varepsilon)^\perp - \nabla^\perp E^\varepsilon) \psi.
\]

The third term obviously converges to 0, and so does the first term by Proposition 4.3. For the second term, we have, using Proposition 4.1, 4.6 and 4.8:
\[
\left| \int_V A_{ij}^\varepsilon (\partial_k E_l^\varepsilon) \psi \right| \leq \left| (A_{ij}^\varepsilon - A_{ij}) (\partial_k E_l^\varepsilon) \psi \right| + \left| A (\partial_k E_l^\varepsilon) \psi \right|
\leq C \| (\partial_k E_l^\varepsilon) \psi \|_{W^{1,1}} \varepsilon^{2\alpha - 1} + C \| (\partial_k E_l^\varepsilon) \psi \|_{W^{1,1}}
\leq C \| \psi \|_1 \varepsilon^{2\alpha - 1} + C \| \psi \|_1 \varepsilon^{3\alpha - 2} \to 0.\]

This completes the proof of (a).

It remains to prove (b) \( v \in C^{1,\alpha}(V) \). Since \( \nabla^2 v = A \) is symmetric in \( i, j \), we have \( \partial_{ij} v = \partial_{ji} v = A_{ij} \), as distributions. Proposition 4.11 implies that, for each \( i, \nabla (\partial_i v) \) satisfies the fine \( \alpha \)-interpolation inequality on \( W_0^{1,1}(V) \). For any \( x \in V \), we apply Proposition 5.8 to a coordinate
rectangular box containing $x$ and compactly included in $V$ to conclude that $\partial_i v \in C^{0,\alpha}(V)$ for $i = 1, 2$, which yields $v \in C^{1,\alpha}(V)$.

5. Developability: A proof of Theorem 1

Let $u \in C^{1,\alpha}(\Omega, \mathbb{R}^3)$ be an isometric immersion for $2/3 \leq \alpha < 1$. In order to prove our main Theorem, and in view of Proposition 4.4, it is sufficient to show that $\nabla u$ satisfies condition (c). We first fix an open disk $V$ containing $x$ and compactly contained in $\Omega$ and note the existence of the function $v \in C^{1,\alpha}(V) \subset C^{1,2/3}(V)$ as defined in Proposition 4.3. We apply the key developability result Theorem 2 to $v$ to obtain that $\nabla v$ satisfies any of the equivalent conditions of Proposition 2.1 in $V$. The developability of $u$ is a consequence of Corollary 2.2 and Proposition 5.1 below. The second conclusion follows from Corollary 3.5.

**Proposition 5.1.** Let $1/2 < \alpha < 1$ and $u \in C^{1,\alpha}(\Omega, \mathbb{R}^3)$ be an immersion, with the second fundamental form $A$ defined as in Proposition 4.1 on $V \subset \Omega$. Assume moreover that $A = \nabla^2 v$, where $v \in C^1(V)$. Then $u|_V$ is developable if and only if $v$ is developable.

**Proof.** We consider the following identity [28, Equation (2.1.3)] (also known as the Gauss equation in the literature), which is valid for the mollified sequence of smooth immersions $u_\varepsilon$:

$$\partial_{ij} u_\varepsilon = \Gamma^{k,\varepsilon}_{ij} \partial_k u_\varepsilon + A^{\varepsilon}_{ij} N^\varepsilon,$$

where the standard Einstein summation convention is used.

Let $B \subset V$ be a given disk, $\psi \in C^\infty(B)$ and Lipschitz unit vector field $\hat{\eta} : B \to \mathbb{R}^2$. We denote the coefficients of $u$ and $N^\varepsilon$ respectively by $u^m$ and $N^\varepsilon,m$ and calculate for $m = 1, 2, 3$,

$$\int_B \text{div}(\psi \hat{\eta})(\nabla u^m) = \lim_{\varepsilon \to 0} \int_B \text{div}(\psi \hat{\eta})(\nabla u^m_\varepsilon) = \lim_{\varepsilon \to 0} -\int_B \hat{\eta} \cdot \nabla^2 u^m_\varepsilon$$

$$= \lim_{\varepsilon \to 0} -\int_B \hat{\eta} \cdot (\Gamma^{k,\varepsilon}_{ij} \partial_k u^m_\varepsilon + A^{\varepsilon}_{ij} N^\varepsilon,m) = \lim_{\varepsilon \to 0} -\int_B \hat{\eta} \cdot (A^{\varepsilon} N^\varepsilon,m),$$

where we used (4.7). We have

$$\int_B \hat{\eta} \cdot (A^{\varepsilon} N^\varepsilon,m) = A^{\varepsilon}[N^\varepsilon,m \hat{\eta}] = (A^{\varepsilon} - A)[N^\varepsilon,m \hat{\eta}] + A[N^\varepsilon,m \hat{\eta}].$$

By Proposition 4.1 we have:

$$\left| (A^{\varepsilon} - A)[N^\varepsilon,m \hat{\eta}] \right| \leq C \left( \|N^\varepsilon,m \hat{\eta}\|_{L^1} o(\varepsilon^{2\alpha - 1}) + \|\nabla (N^\varepsilon,m \hat{\eta})\|_{L^1} o(\varepsilon^{\alpha}) \right)$$

$$\leq C \|\hat{\eta}\|_{L^1} o(\varepsilon^{2\alpha - 1}) + \|\nabla (\hat{\eta})\|_{L^1} o(\varepsilon^{\alpha}) \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

where we used (4.2). Note also that in view of $A = \nabla^2 v$, we have for all $\varepsilon$, in view of the fact that $N^\varepsilon,m \psi \in C^\infty(B)$:

$$A[N^\varepsilon,m \hat{\eta}] = -\int_B \text{div}(N^\varepsilon,m \hat{\eta}) \nabla v.$$

Therefore, (5.2) implies

$$\int_B \text{div}(\psi \hat{\eta}) \nabla u^m = \lim_{\varepsilon \to 0} \int_B \text{div}(N^\varepsilon,m \hat{\eta}) \nabla v.$$

If $v$ is developable, we apply Corollary 2.10 to $\nabla v$, and choose accordingly the open disk $B$ around $x \in V$ and the Lipschitz unit vector field $\hat{\eta} : B \to \mathbb{R}^2$. Therefore

$$\forall y \in B \quad \hat{\eta}(y) = \hat{\eta}(y + s \hat{\eta}(y)) \quad \text{and} \quad \nabla v(y) = \nabla v(y + s \hat{\eta}(y)) \quad \text{for all} \quad s \quad \text{for which} \quad y + s \hat{\eta}(y) \in B.$$
We apply Lemma 2.11 to \( f = \nabla v \) and \( B = B_x \) to obtain for any \( \psi \in C^\infty_c(B) \):

\[
\int_B \text{div}(N^\varepsilon,m \psi \vec{\eta}) \nabla v = 0, \quad m = 1, 2, 3,
\]

where we used the fact that \( N^\varepsilon,m \psi \in C^\infty_c(B) \). By (5.3) we conclude that

\[
\hat{B} \text{div}(\psi \vec{\eta}) \nabla u = 0.
\]

Once again applying Lemma 2.11 and (5.4) implies that the Jacobian derivative \( \nabla u \) of the isometric immersion \( u \) is constant along the segments generated by the vector field \( \vec{\eta} \) in \( B \). Therefore \( \nabla u \) satisfies condition (c) of Proposition 2.1, which implies the developability of \( u \mid V \). As already mentioned, this part of the proof concludes the proof of Theorem 1 for \( u \in C^{1,\alpha} \), \( 2/3 \leq \alpha < 1 \).

To finish the proof of Proposition 5.1 we need to prove the converse statement. If, on the other hand, \( u \mid V \) is assumed to be developable, we proceed in a similar manner. According to Corollary 2.10 and Lemma 2.11, for all \( x \in V \) there exist a disk \( B \subset V \) centered at \( x \) and a unit Lipschitz vector field \( \vec{\eta} : B \to \mathbb{R}^2 \) such that for all \( \psi \in C^\infty_c(B) \):

\[
\forall y \in B \quad \vec{\eta}(y) = \vec{\eta}(y + s\vec{\eta}(y)) \quad \text{and} \quad \int_B \text{div}(N^\varepsilon,m \psi \vec{\eta}) \nabla u^m = 0, \quad m = 1, 2, 3.
\]

Hence, using the fact that \( A_{ij}^\varepsilon = \partial_{ij} u_\varepsilon \cdot N^\varepsilon \) we deduce for all \( \psi \in C^\infty_c(B) \):

\[
\int_B \text{div}(\psi \vec{\eta})(\nabla v) = -A[\psi \vec{\eta}] = \lim_{\varepsilon \to 0} A^\varepsilon[\psi \vec{\eta}] = \lim_{\varepsilon \to 0} \int_B \sum_{m=1}^3 (N^\varepsilon,m \nabla^2 u_\varepsilon^m) \psi \vec{\eta}
\]

\[
= \lim_{\varepsilon \to 0} -\int_B \sum_{m=1}^3 \text{div}(N^\varepsilon,m \psi \vec{\eta}) \nabla u_\varepsilon^m
\]

\[
= \lim_{\varepsilon \to 0} -\int_B \sum_{m=1}^3 \text{div}(N^\varepsilon,m \psi \vec{\eta})(\nabla u_\varepsilon^m - \nabla u^m) = 0,
\]

where we used (4.1) and (4.2) to obtain a convergence of order \( o(\varepsilon^{2\alpha-1}) \) in the last line. Following the same line of argument as before \( v \) is hence developable in view of Lemma 2.11.

**Appendix I. Developability of each component can be shown independently**

**Proposition I.1.** Assume \( \Omega \subset \mathbb{R}^2 \) is an arbitrary domain, \( 2/3 \leq \alpha < 1 \) and let \( u \in C^{1,\alpha}(\Omega, \mathbb{R}^3) \) be an isometric immersion. Then for each \( m = 1, 2, 3 \), the component \( u^m \) is developable in \( \Omega \).

**Remark I.2.** Obviously this statement does not guarantee that the constancy lines or regions of \( \nabla u^m \) are the same for the three components. If this fact is independently shown (e.g. through geometric arguments) Theorem 1 will also be proved through this approach.

**Proof.** In view of Theorem 2 it suffices to show that \( \text{Det} D^2 u^m = 0 \) for each \( m \). Once again, we consider the mollifications \( u_\varepsilon = u * \phi_\varepsilon \) defined on a suitably chosen domain \( V \subset \Omega \) and use (5.1) to calculate for \( m = 1, 2, 3 \) and \( \psi \in C^\infty_c(V) \):
\[-\frac{1}{2} \text{curl}^T \text{curl}(\nabla u^m \otimes \nabla u^m)[\psi] = \lim_{\varepsilon \to 0} -\frac{1}{2} \text{curl}^T \text{curl}(\nabla u^m_\varepsilon \otimes \nabla u^m_\varepsilon)[\psi] = \lim_{\varepsilon \to 0} \nabla^2 u^m_\varepsilon[\psi] \]

\[
= \lim_{\varepsilon \to 0} \int_\Omega \det(\Gamma^k_\varepsilon \partial_k u^m_\varepsilon + A^\varepsilon N^\varepsilon,m)[\psi] \\
= \lim_{\varepsilon \to 0} \int_\Omega \det(A^\varepsilon)(N^\varepsilon,m)^2 \psi + \lim_{\varepsilon \to 0} \int_\Omega \det(\Gamma^k_\varepsilon \partial_k u^m_\varepsilon) \psi \\
+ \lim_{\varepsilon \to 0} \sum_{i,j,r,s} A^\varepsilon_{ij} N^\varepsilon,m(\Gamma^k_\varepsilon r^l_\varepsilon \partial_k u^m_\varepsilon) \psi.
\]

The second term obviously converges to 0 in view of (4.7). For the third term we use (4.1), (4.4) and (4.7) to prove:

\[
\left| \int_\Omega A^\varepsilon_{ij} N^\varepsilon,m(\Gamma^k_\varepsilon r^l_\varepsilon \partial_k u^m_\varepsilon) \psi \right| \leq C \|A^\varepsilon\|_{0;V} \|\Gamma^k_\varepsilon\|_{0;V} \|\psi\|_{L^1} \leq C \|\psi\|_{L^1} o(\varepsilon^{3\alpha - 2}) \to 0.
\]

Finally, the vanishing of the first term follows from Proposition 4.3 and (4.2) since

\[
\left| \int_\Omega \det(A^\varepsilon)(N^\varepsilon,m)^2 \psi \right| \leq C \|(N^\varepsilon,m)^2 \psi\|_{W^{1,1}} o(\varepsilon^{2\alpha - 1}) \\
\leq C \left( \|\psi\|_{W^{1,1}} o(\varepsilon^{2\alpha - 1}) + \|\nabla N^\varepsilon\|_{0;V} \|\psi\|_{L^1} o(\varepsilon^{2\alpha - 1}) \right) \\
\leq C \|\psi\|_{W^{1,1}} o(\varepsilon^{3\alpha - 2}) \to 0.
\]

**APPENDIX II. LITTLE HÖLDER SPACES AND MODULUS OF CONTINUITY**

Here we will prove two auxiliary statements regarding little Hölder spaces.

**Lemma II.1.** Let \( V \subset U \subset \mathbb{R}^n \) be open sets such that \( \text{dist}(\nabla, \partial U) > 0 \) and \( f : U \to \mathbb{R} \). If for \( 0 < \alpha < 1 \), \([f]_{0, \alpha; U|e} \) is \( o(1) \) as a function of \( r \), then \( \|f_\varepsilon - f\|_{0; V} \to 0 \) as \( \varepsilon \to 0 \).

**Proof.** Let \( \varepsilon < \text{dist}(\nabla, \partial U) \). We first note that by Lemma 3.1(i) we have

\[
(II.1) \quad \|f_\varepsilon - f\|_{0; V} \leq c[f]_{0, \alpha; U|e} \varepsilon^\alpha.
\]

We therefore estimate

\[
[f_\varepsilon - f]_{0, \alpha; V} \leq [f_\varepsilon - f]_{0, \alpha; V|e} + \sup_{x, y \in V} \frac{|f_\varepsilon(x) - f(x) + f_\varepsilon(y) - f(y)|}{\varepsilon < |x - y|} \\
\leq [f_\varepsilon - f]_{0, \alpha; V|e} + c[f_\varepsilon - f]_{0; V} \varepsilon^{-\alpha} \\
\leq [f_\varepsilon]_{0, \alpha; V|e} + [f]_{0, \alpha; V|e} + c[f]_{0, \alpha; U|e} \\
\leq [f_\varepsilon]_{0, \alpha; V|e} + c[f]_{0, \alpha; U|e}.
\]
We now observe that
\[
[f_\varepsilon]_{0,\alpha;U;\varepsilon} = \sup_{x, y \in V} \left| \frac{\int_{\mathbb{R}^n} (f(x - z) - f(y - z)) \phi_\varepsilon(z) \, dz}{|x - y|^{\alpha}} \right|
\]
\[
\leq \sup_{x, y \in V} \int_{\mathbb{R}^n} \frac{|f(x - z) - f(y - z)|}{|x - y|^{\alpha}} |\phi_\varepsilon(z)| \, dz
\]
\[
\leq c[f]_{0,\alpha;U;\varepsilon}.
\]
Combining the latter two estimates we hence obtain
\[
[f_\varepsilon - f]_{0,\alpha;V} \leq c[f]_{0,\alpha;U;\varepsilon}.
\]
In view of (II.1) and the main assumption, we conclude with the desired approximation of $f$ by $f_\varepsilon \in C^\infty(V)$ in $C^0,\alpha$.

**Proposition II.2.** Let $U \subset \mathbb{R}^n$ be a bounded weakly Lipschitz domain and let $f : U \to \mathbb{R}$ be uniformly continuous. Then $f$ admits an extension $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ with $\omega_{\tilde{f},\mathbb{R}^n} \leq c(U)\omega_{f,U}$. 

**Proof.** $U$ is a bounded weakly Lipschitz domain, i.e. by the standard definition, there is a covering of $\overline{U}$ of open sets $U_i$ and charts $\psi_i : U_i \to \mathbb{R}^n$, $0 \leq i \leq N$ such that $U_0 \subset U$ and for $0 \leq i \leq N$, $\psi_i$ is a bilipschitz diffeomorphism between $U_i$ and an open cylinder $V_i := B_i \times (-1, 1) \subset \mathbb{R}^n$ such that $\psi_i(U \cap U_i) = V_i^+ := B_i \times (0, 1)$ and $\psi_i(\partial U \cap U_i) = B_i \times \{0\}$, where $B_i \subset \mathbb{R}^{n-1}$ is the unit ball.

For each $1 \leq i \leq N$, consider the function $h_i := f \circ \psi_i^{-1} : V_i^+ \to \mathbb{R}$. We can estimate the modulus of continuity of each $h_i$ from above by the modulus of continuity of $f$ using the maximum bilipschitz constant of the $\psi_i$, which depends on $U$:

\[
\omega_i := \omega_{h_i;V_i^+} \leq c(U)\omega_{f,U}.
\]

Note that $\omega_i$ is necessarily nondecreasing by definition. Since each $V_i^+$ is convex, $\omega_i$ is also subadditive, i.e. for all $r_1, r_2 \geq 0$,

\[
\omega_i(r_1 + r_2) \leq \omega_i(r_1) + \omega_i(r_2).
\]

Moreover the uniform continuity of $f$ implies that

\[
\lim_{r \to 0} \omega_i(r) = 0.
\]

A direct application of [59, Theorem 13.16] implies that $h_i$ admits an extension $\tilde{h}_i$, given by

\[
\tilde{h}_i(x) := \sup \{ h_i(y) - \omega_i(|x - y|); \; y \in V_i^+ \},
\]

to the whole $\mathbb{R}^n$ also satisfying

\[
\omega_{\tilde{h}_i;\mathbb{R}^n} \leq \omega_i \leq c(U)\omega_{f,U}.
\]

Let $\{\theta_i\}_{0 \leq i \leq N}$ be a partition of unity corresponding to the covering $U_i$, i.e.

\[
\forall \; 0 \leq i \leq N \quad \theta_i \in C^\infty_c(U_i, [0, 1]) \quad \text{and} \quad \sum_{i=0}^N \theta_i = 1.
\]
We extend $\theta_0 f : U_0 \to \mathbb{R} \to f_0 : \mathbb{R}^n \to \mathbb{R}$, and $\theta_i (\tilde{h}_i \circ \psi_i) : U_i \to \mathbb{R}$ to $\tilde{f}_i : \mathbb{R}^n \to \mathbb{R}$ by the trivial zero extension and write

$$f := \sum_{i=0}^N \tilde{f}_i.$$ 

The conclusion follows.

References

[1] D.N. Arnold, L.R. Scott and M. Vogelius, Regular inversion of the divergence operator with Dirichlet boundary conditions on a polygon, Ann. Scuola Norm. Sup. Pisa Cl. Sci. Serie IV, 15 no. 2, (1988), 169–192.

[2] H. Bahouri, J.Y. Chemin and R. Danchin, Fourier analysis and nonlinear partial differential equations, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 343. Springer, Heidelberg, 2011.

[3] V. Borrelli, S. Jabrane, F. Lazarus and B. Thibert, Flat tori in three-dimensional space and convex integration, Proc. Natl. Acad. Sci. USA 109 (2012), no. 19, 7218–7223.

[4] V. Borrelli, S. Jabrane, F. Lazarus and B. Thibert, Isometric embeddings of the square flat torus in ambient space, Ensaios Matemáticos, Brazilian Mathematical Society, 24(2013), pp.1–91.

[5] Yu. F. Borisov, The parallel translation on a smooth surface. III., Vestnik Leningrad. Univ. 14 no. 1, (1959), 34–50.

[6] Yu. F. Borisov, Irregular surfaces of the class $C^{1,\beta}$ with an analytic metric, (Russian) Sibirsk. Mat. Zh. 45 no. 1, (2004), 25–61; translation in Siberian Math. J. 45 no. 1, (2004), 19–52.

[7] F. Cajori, Generalisations in Geometry as Seen in the History of Developable Surfaces, Math. Monthly 36, (1929), 431–437.

[8] S. Campanato, Proprietà di h"olderianità di alcune classi di funzioni, Ann. Scuola Norm. Sup. Pisa (3), 17 (1963), 175–188.

[9] W. Cao and L. Székelyhidi Jr, Very weak solutions to the two-dimensional Monge-Ampère equation, Science China Mathematics, 62 (6), (2019),1041–1056.

[10] W. Cao and L. Székelyhidi Jr, Global Nash-Kuiper theorem for compact manifolds, Preprint (2019), [arXiv:1906.08608](https://arxiv.org/abs/1906.08608).

[11] M.P. do Carmo, Differential geometry of curves and surfaces, Translated from the Portuguese. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1976.

[12] S.S. Chern and R.K. Lashof, On the Total Curvature of Immersed Manifolds, American Journal of Mathematics, 79 No. 2., (1957), 306–318.

[13] P. Constantin, W. E and E. S. Titi, Onsager’s conjecture on the energy conservation for solutions of Euler’s equation, Comm. Math. Phys. 165 no. 1, (1994), 207–209.

[14] S. Conti, C. De Lellis and L. Székelyhidi Jr., $h$-principle and rigidity for $C^{1,\alpha}$ isometric embeddings, Nonlinear Partial Differential Equations, Proceedings of the Abel Symposium, Abel Symposia Volume 7, (2012), 83–116.

[15] S. Conti, C. De Lellis and L. Székelyhidi Jr., Errata on “$h$-principle and rigidity for $C^{1,\alpha}$ isometric embeddings”, [https://www.math.ias.edu/delellis/sites/math.ias.edu.delellis/files/Errata-iso.pdf](https://www.math.ias.edu/delellis/sites/math.ias.edu.delellis/files/Errata-iso.pdf) 2020.

[16] C. De Lellis, D. Inauen, $C^{1,\alpha}$ isometric embeddings of polar caps, Preprint (2018), [arXiv:1809.04161](https://arxiv.org/abs/1809.04161).

[17] C. De Lellis, D. Inauen and L. Székelyhidi Jr., A Nash-Kuiper theorem for $C^{1,\frac{1}{2} - \varepsilon}$ immersions of surfaces in 3 dimensions, Revista matemática iberoamericana, 34 (3), (2018) 1119–1152.

[18] C. De Lellis and L. Székelyhidi Jr., Dissipative continuous Euler flows, Invent. Math. 193 no. 2, (2013), 377–407.

[19] C. De Lellis and L. Székelyhidi Jr., High dimensionality and $h$-principle in PDE, Bull. Amer. Math. Soc. (N.S.) 54 no. 2 (2017), 247–282.

[20] C. De Lellis and L. Székelyhidi Jr., On turbulence and geometry: from Nash to Onsager, Notices Amer. Math. Soc. 66 no. 5, (2019), 677–685.

[21] G. L. Eyink, Energy dissipation without viscosity in ideal hydrodynamics. I. Fourier analysis and local energy transfer. Phys. D, 78 (1994) no. 3-4, 222–240.

[22] I. Fonseca and J. Malý, From Jacobian to Hessian: distributional form and relaxation, Riv. Mat. Univ. Parma 7 4*, (2005), 45–74.

[23] M. Friedman, A history of folding in mathematics: Mathematizing the margins, Science Networks. Historical Studies, 59. Birkhäuser/Springer, Cham, 2018.

[24] P. Friz and M. Hairer, A course on rough paths. With an introduction to regularity structures, Universitext. Springer, Cham, 2014.
30

[25] J.H.G. Fu, Monge-Ampère functions I, Indiana Univ. Math. Jour., 38 no. 3, (1989), 745–771.
[26] M. Gromov, Partial differential relations, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 9. Springer-Verlag, Berlin, 1986.
[27] M. Gromov, Geometric, algebraic, and analytic descendants of Nash isometric embedding theorems, Bull. Amer. Math. Soc., 54 (2017), 173–245.
[28] Q. Han and J-X. Hong, Isometric embedding of Riemannian manifolds in Euclidean spaces, Mathematical Surveys and Monographs, 130, American Mathematical Society, Providence, RI, (2006).
[29] P. Hartman and L. Nirenberg, On spherical image maps whose Jacobians do not change sign, Amer. J. Math., 81 (1959), 901–920.
[30] P. Hornung, Fine level set structure of flat isometric immersions, Arch. Rational Mech. Anal., 199 (2011), 943–1014.
[31] P. Hornung, Approximation of flat $W^{2,2}$ isometric immersions by smooth ones, Arch. Rational Mech. Anal., 199 (2011), 1015–1067.
[32] P. Hornung and I. Velčić, Regularity of intrinsically convex $H^2$ surfaces and the derivation of homogenized bending shell models, Journal de Mathématiques Pures et Appliquées 115, (2018), 1–23.
[33] P. Isett, A proof of Onsager’s conjecture Annals of Mathematics, 188 No. 3, (2018), 871–963.
[34] T. Iwaniec, On the concept of weak Jacobian and Hessian, Report Univ. Jyväskylä 83, (2001), 181–205.
[35] R.L. Jerrard, Some remarks on Monge-Ampère functions, Singularities in PDE and the calculus of variations, CRM Proc. Lecture Notes, 44, (2008), 89–112.
[36] R.L. Jerrard, Some rigidity results related to Monge-Ampère functions, Canad. J. Math. 62 no. 2, (2010), 320–354.
[37] R.L. Jerrard and M.R. Pakzad, Sobolev Spaces of isometric immersions of arbitrary dimension and co-dimension, Ann. Mat. Pura Appl. (4), 196 no. 2, (2017), 687–716.
[38] B. Kirchheim, Geometry and rigidity of microstructures, habilitation thesis. University of Leipzig, (2001). (MPI-MIS Lecture Notes 16/ 2003 http://www.mis.mpg.de/preprints/ln/index.html).
[39] M.V. Korobkov, Properties of the $C^1$-smooth functions with nowhere dense gradient range, Sibirsk. Mat. Zh. 48 no. 6, (2007), 1272–1284; translation in Siberian Math. J. 48 no. 6, (2007), 1019–1028.
[40] M.V. Korobkov, Properties of $C^1$-smooth mappings with one-dimensional gradient range Sibirsk. Mat. Zh. 50 no. 5, (2009), 874–886; translation in Siberian Math. J. 50 no. 5, (2009), 874–886.
[41] N. H. Kuiper, On $C^1$-isometric imbeddings. I, II., Nederl. Akad. Wetensch. Proc. Ser. A. 58, (1955) 545–556, 683–689.
[42] H. Lebesgue, Sur quelques surfaces non réglées applicables sur le plan, Comptes Rendus de l’Académie des Sciences CXVIII, (1899), 1502–1505.
[43] M. Lewicka and M.R. Pakzad, Convex integration for the Monge-Ampère equation in two dimensions, Analysis and PDE, 10 No. 3, (2017), 695–727.
[44] S. Li, M.R. Pakzad and A. Schikorra, Fractional Sobolev isometric immersions of planar domains, arXiv e-print (2021), arXiv:2103.01723 to appear in Ann. Sc. Norm. Sup. Pisa - Classe di Scienze.
[45] Z. Liu and M.R. Pakzad, Regularity, rigidity and density of smooth mappings for co-dimension 1 Sobolev isometric immersions, Ann. Sc. Norm. Sup. Pisa - Classe di Scienze, Vol. XIV (3), (2015), 76–817.
[46] J. Malý and O. Martio, Lusin’s condition (N) and mappings of the class $W^{1,n}$, J. Reine Angew. Math., 458, (1995), 19–36.
[47] J. Marschall, Nonregular Pseudo-Differential Operators, Zeitschrift für Analysis und ihre Anwendungen, Journal for Analysis and its Applications, Volume 15 No. 1, (1996), 109–148.
[48] S. Müller and M.R. Pakzad, Regularity properties of isometric immersions, Math. Zeit. 251 (2), (2005), 313–331.
[49] J. Nash, The imbedding problem for Riemannian manifolds, Ann. Math., 63, (1956), 20–63.
[50] J. Nash, $C^1$ isometric imbeddings, Ann. Math., 60, (1954), 383–396.
[51] L. Onsager, Statistical hydrodynamics. II, Nuovo Cimento (1943-1954), 6 (2), (1949), 279–287.
[52] M.R. Pakzad, On the Sobolev space of isometric immersions, Journal of Differential Geometry, 66, (2004), 47–69.
[53] E. Picard, Traité d’analyse, 3rd ed. Paris: Gauthier-Villars et fils, 1922.
[54] A.V. Pogorelov, Extrinsic geometry of convex surfaces, Translations of mathematical monographs vol. 35, American Math. Soc., 1973.
[55] T. Runst and W. Sickel, Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations, De Gruyter Series in Nonlinear Analysis and Applications, 3. Walter de Gruyter & Co., Berlin, 1996.
[56] D. Spring, *Convex integration theory. Solutions to the h-principle in geometry and topology*, Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel, 2010.

[57] L. Székelyhidi, *From Isometric Embeddings to Turbulence*, MPI-MIS Lecture note 41/2012, http://www.mis.mpg.de/publications/other-series/ln/lecturenote-4112.html

[58] S. C. Venkataramani, T. A. Witten, E. M. Kramer, and R. P. Geroch, *Limitations on the smooth confinement of an unstretchable manifold*, J. Math. Phys. 41 (7), (2000), 5107–5128.

[59] J.H. Wells and L.R. Williams *Embeddings and Extensions in Analysis*, 1st Ed., Ergebnisse der Mathematik und ihrer Grenzgebiete Band 84, Springer-Verlag Berlin Heidelberg, 1975.

[60] L.C. Young, *An inequality of the Hölder type, connected with Stieltjes integration*, Acta Math. 67 no. 1, (1936), 251–282.

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