Complete WKB asymptotics of high frequency vibrations in a stiff problem*†

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Abstract

Asymptotic behaviour of eigenvalues and eigenfunctions of a stiff problem is described in the case of the fourth-order ordinary differential operator. Considering the stiffness coefficient that depends on a small parameter $\varepsilon$ and vanishes as $\varepsilon \to 0$ on a subinterval, we prove the existence of low and high frequency resonance vibrations. The low frequency vibrations admit the power series expansions on $\varepsilon$ but this method is not applicable to the description of high frequency vibrations. However, the non-classical asymptotics on $\varepsilon$ of the high frequency vibrations were constructed using the WKB method.

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Introduction and main results.

Stiff vibrating systems belong to a class of systems with singularly perturbed potential energy. Stiff problems are known in particular as boundary value problems for differential equations with very contrasting values of coefficients in different sub-domains. They relate to modelling vibrations of elastic systems consisting of two (or more) materials with one of them being very stiff with respect to the other. For the first time the stiff problems were investigated by J. L. Lions [1].

However, it is also of interest to describe the asymptotic behaviour of spectral properties for the stiff problems. These system has two types of eigenvibrations, namely low frequency vibrations and high frequency ones. From a physical viewpoint we can postulate that two kinds of eigenvibrations can appear: one for the stiffer structure and the other for the softer structure. Different aspects of the spectral stiff problems are considered in [2]–[10] with the best general reference being [8]. The asymptotic behaviour of the low frequency vibrations has been widely studied with different techniques [2], [3], [7] and [9].

In this paper, following [10] we consider the phenomenon of high frequency vibrations. The leading terms of high frequency vibrations for different problems were constructed in [7-9]. Information on the behaviour of high frequency vibrations was also provided in [10]. Studying the stiff problem for the forth-order differential operator, we construct the complete asymptotic expansions of high frequency vibrations using the WKB technique.

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Why do high frequency vibrations appear? On the one hand, the spectrum \( \{\lambda_i^\varepsilon\}_{i=1}^\infty \) of the stiff problem is asymptotically dense in \([0, \infty)\) as \( \varepsilon \to 0 \) (see [10]). On the other hand, for fixed \( i \) the asymptotic expansions of an eigenvalue \( \lambda_i^\varepsilon \) and the corresponding eigenfunction \( u_i^\varepsilon \) are nonuniform with respect to \( i \). Hence the sequences \( u_i^\varepsilon \) with \( i(\varepsilon) \to \infty \) can support stable forms of vibration as \( \varepsilon \to 0 \). In addition, the corresponding sequences of eigenvalues \( \lambda_i^\varepsilon \) converge to some positive limit points (see Fig. 1, 2). It is also shown that the approximation of the limit form of vibrations by sequence \( \{u_i^\varepsilon\} \) has the discrete character. Therefore we construct and justify the asymptotics for a family of discrete sets of a small parameter.

Moreover, such approximation is ambiguously determined. Then we introduce a deformation parameter \( \delta \) in the asymptotics. Consequently, for each \( \delta \) we construct expansions of \( u_i^\varepsilon \), though they are asymptotically equivalent when \( \varepsilon \to 0 \).

\section{Problem statement}

Let an interval \((a, b) \subset \mathbb{R}\) contains the origin. Let us consider the eigenvalue problem

\[ \frac{d^2}{dx^2} \left( k_\varepsilon(x) \frac{d^2 u_\varepsilon}{dx^2} \right) - \lambda_\varepsilon u_\varepsilon(x) = 0, \quad x \in (a, b), \]
\[ u_\varepsilon(a) = u_\varepsilon'(a) = 0, \quad u_\varepsilon(b) = u_\varepsilon'(b) = 0, \]
\[ u_\varepsilon(-0) = u_\varepsilon(+0), \quad u_\varepsilon'(-0) = u_\varepsilon'(+0), \]
\[ (k_0(x)u_\varepsilon'')(-0) = \varepsilon^4 (k_1(x)u_\varepsilon'')(+0), \quad (k_0(x)u_\varepsilon'')'(-0) = \varepsilon^4 (k_1(x)u_\varepsilon''')'(+0). \]

Here \( \varepsilon \) is a small positive parameter and a function

\[ k_\varepsilon(x) = \begin{cases} 
  k_0(x), & a < x < 0 \\
  \varepsilon^4 k_1(x), & 0 < x < b
\end{cases} \]

is smooth and strictly positive in \((a, 0) \cup (0, b)\). We study the asymptotic behaviour of the eigenvalues \( \lambda_\varepsilon \) and the eigenfunctions \( u_\varepsilon \) of (1)–(4) as \( \varepsilon \to 0 \).
The problem models eigenvibrations of a non-homogeneous rod. The rod consists of two components with the same density of mass and strongly different elastic properties. The fourth power of \( \varepsilon \) in the definition of \( k_\varepsilon \) is suitable for the next consideration (see Section 4).

Let us introduce the Sobolev space \( H^2_0(a, b) \) as the closure of set \( C^\infty_0(a, b) \) with respect to the norm
\[
\|u\| = \left( \int_a^b |u''|^2 \, dx \right)^{1/2},
\]
and bilinear forms in \( H^2_0(a, b) \)
\[
a_0(\varphi, \psi) = \int_a^0 k_0 \varphi'' \psi'' \, dx, \quad a_1(\varphi, \psi) = \int_0^b k_1 \varphi'' \psi'' \, dx, \quad a_\varepsilon = a_0 + \varepsilon^4 a_1.
\]
For each \( \varepsilon > 0 \) we note by \( \| \cdot \|_\varepsilon \) the norm in \( H^2_0(a, b) \) associated with the form \( a_\varepsilon(\cdot, \cdot) \). It is known as an energetic norm associated with the elastic-like energy.

Let us consider a variational formulation of (1)–(4) that is to find \( \lambda_\varepsilon \) and \( u_\varepsilon \in H^2_0(a, b) \) such that
\[
a_\varepsilon(u_\varepsilon, \varphi) - \lambda_\varepsilon(u_\varepsilon, \varphi)_{L^2(a, b)} = 0, \quad \varphi \in H^2_0(a, b).
\]
Problem (5) is a standard eigenvalue problem with a real discrete spectrum. For each fixed \( \varepsilon \) let us consider an eigenvalue sequence
\[
0 < \lambda_1^\varepsilon < \lambda_2^\varepsilon < \cdots < \lambda_i^\varepsilon < \ldots, \quad \text{where} \quad \lambda_i^\varepsilon \to \infty \quad \text{as} \quad i \to \infty.
\]
Note that each eigenvalue is simple. Let the corresponding eigenfunctions \( \{u_i^\varepsilon\}_{i=1}^{\infty} \) form an orthonormal basis in \( L^2(a, b) \).

2 \quad Asymptotic expansions of low frequency vibrations

We investigate the asymptotic behaviour of eigenvalues \( \lambda_i^\varepsilon \) and eigenfunctions \( u_i^\varepsilon \) for a fixed number \( i \).

**Lemma 2.1.** Each eigenvalue \( \lambda_i^\varepsilon \) is a continuous function with respect to the parameter \( \varepsilon \), \( \varepsilon \in (0, 1) \). Moreover,
\[
\lambda_i^\varepsilon \leq C_i \varepsilon^4,
\]
where constant \( C_i \) is independent of \( \varepsilon \).

**Proof.** The continuity of eigenvalues follows from the variational principle
\[
\lambda_i^\varepsilon = \sup_P \inf_{f \in P^\perp \setminus \{0\}} \frac{a_0(f, f) + \varepsilon^4 a_1(f, f)}{\|f\|^2_{L^2(a, b)}},
\]
where \( P \) is a \((i-1)\)-dimensional subspace of \( H^2_0(a, b) \) and \( P^\perp \) is the orthonormal complement of \( P \).
Suppose that the supremum in (6) is achieved on a subspace \( P_i \). Since \( P_i \) is finite dimensional, we choose a function \( f_i \in P_i^\perp \setminus \{0\} \) that vanishes in \((a,0)\). Then

\[
\lambda_i^\varepsilon = \inf_{f \in P_i^\perp \setminus \{0\}} \frac{a_0(f, f) + \varepsilon^4 a_1(f, f)}{\|f\|_{L_2((a,b))}^2} \leq \frac{a_0(f_i, f_i) + \varepsilon^4 a_1(f_i, f_i)}{\|f_i\|_{L_2((a,b))}^2} = \varepsilon^4 \frac{a_1(f_i, f_i)}{\|f_i\|_{L_2((0,b))}^2} = C_i \varepsilon^4,
\]

since \( a_1(f_i, f_i) \neq 0 \).

We postulate the expansions of the eigenvalue \( \lambda_\varepsilon = \lambda_i^\varepsilon \) and the eigenfunction \( u_\varepsilon = u_i^\varepsilon \) for given \( i \in \mathbb{N} \):

\[
\lambda_\varepsilon \sim \varepsilon^4 (\lambda_0 + \varepsilon^4 \lambda_1 + \ldots), \\
u_\varepsilon(x) \sim \begin{cases} 
\varepsilon^4 (u_0(x) + \varepsilon^4 u_1(x) + \ldots), & x \in (a,0), \\
v_0(x) + \varepsilon^4 v_1(x) + \ldots, & x \in (0,b).
\end{cases}
\tag{7}
\]

Substituting (7) into (1)–(4) we deduce that \( \lambda_0 \) is an eigenvalue and \( v_0 \) is an eigenfunction of the problem

\[
(k_1v_0''(x) - \lambda_0 v_0(x)) = 0, \quad x \in (0,b), \\
v_0(0) = v_0'(0) = 0, \quad v_0(b) = v_0'(b) = 0.
\tag{8}
\]

Note that all eigenvalues of the problem are simple.

Then the function \( u_0 \) is a solution of the boundary value problem

\[
(k_0u_0''(x)) = 0, \quad x \in (a,0), \\
u_0(a) = u_0'(a) = 0, \quad k_0(0)u_0''(0) = k_1(0)v_0''(0), \quad (k_0u_0')(0) = (k_1v_0')(0).
\]

Let us define next terms of expansions (7). The function \( v_1 \) satisfies

\[
(k_1v_1''(x) - \lambda_0 v_1(x)) = \lambda_1 v_0(x), \quad x \in (0,b), \\
v_1(0) = u_0(0), \quad v_1'(0) = u_0'(0), \quad v_1(b) = v_1'(b) = 0.
\]

Since \( \lambda_0 \) is an eigenvalue of (8), such problem has a solution under the condition

\[
\lambda_1 = \|v_0\|_{L_2((0,b)}^{-1} ((k_1v_0')(0) - k_1u_0''(0)) \bigg|_{x=0}.
\]

Further, the function \( u_1 \) is a solution of

\[
(k_0u_1''(x) = \lambda_0 u_0(x), \quad x \in (a,0), \\
u_1(a) = u_1'(a) = 0, \quad k_0(0)u_1''(0) = k_1(0)v_1''(0), \quad (k_0u_1')(0) = (k_1v_1')(0).
\]

The general terms of (7) can be found in the same way. The asymptotic expansions are justified by classical methods [14, 2].

The vibrations described above are corresponding to the low levels of potential energy, since \( a_\varepsilon(u_\varepsilon, u_\varepsilon) = o(\varepsilon^2) \) as \( \varepsilon \to 0 \). Naturally we have vibrations of the soft part of a system...
only. The soft part is clamped at $x = 0$ by the immovable stiffer structure. Note that the leading terms of expansions (7) are determined by the soft part. Since every eigenfunction $u_{i}^{\varepsilon}$ converges to an eigenfunction $v_{0}$ of (8) extended by zero to $(a, b)$, the set $\{u_{i}^{\varepsilon}\}_{i=1}^{\infty}$ is not a basis in $L_{2}(a, b)$ for $\varepsilon = 0$. In other words, the “low frequency” region does not provide a good insight on the vibration problem over all $(a, b)$. Therefore we consider here another kind of vibrations, namely, vibrations with ”finite non-vanishing energy”.

3 High frequency vibrations

On the one hand the asymptotics of $\lambda_{i}^{\varepsilon}$ and $u_{i}^{\varepsilon}$ are nonuniform with respect to $i$, on the other hand the spectrum of (1)–(4) is asymptotically dense in the positive spectral semi-axis (see Lemma 2). On account of the above remarks, we can find stable vibrations $u_{i}(\varepsilon)$ associated with certain sequences of eigenvalues $\lambda_{i}(\varepsilon)$ with $i(\varepsilon) \to \infty$.

Let us denote by $E$ the set of all pairs $(\varepsilon, \lambda(\varepsilon))$, where $\lambda(\varepsilon)$ is an eigenvalue of (1)–(4) for some $\varepsilon \in (0, 1)$.

**Lemma 3.1.** The closure of the set $E$ includes semi-axis $\{(\varepsilon, \lambda) : \varepsilon = 0, \lambda \geq 0\}$. That is to say each positive $\lambda$ can be approximated by a sequence of eigenvalues $\lambda(\varepsilon)$.

**Proof.** Suppose that for a certain $\lambda > 0$ there exists a neighborhood $U$ of the point $(0, \lambda)$ such that $U \cap E = \emptyset$. Let us choose $\varepsilon^{*} > 0$ small enough for $(\varepsilon^{*}, \lambda) \in U$. Recall that $\lambda_{i}(\varepsilon)$ is a continuous function with respect to $\varepsilon$ and $\lambda_{i}(\varepsilon) \to 0$ as $\varepsilon \to 0$ (see Fig. 1). Since the neighborhood $U$ by definition does not contain points $(\varepsilon^{*}, \lambda_{i}(\varepsilon^{*}))$ then $\lambda_{i}(\varepsilon^{*}) \leq \lambda$ for all $i \in \mathbb{N}$ that is impossible. 

Let us consider a sequence of pairs $(\varepsilon_{i}, \lambda_{i}) \in E$ that converges in $\mathbb{R}^{2}$ to $(0, \omega^{4})$ with $\omega > 0$ as $i \to \infty$. Let $u_{i}$ be an eigenfunction of (1)–(4), being normalized in $L_{2}(a, b)$ and associated with $\lambda_{i}$.

**Definition 3.1.** We say that the eigenfunction sequence $\{u_{i}\}_{i=1}^{\infty}$ supports high frequency vibrations as $i \to \infty$, if the sequence of restrictions $u_{i}|_{(a, 0)}$ has a nonzero limit $v$ in $H^{2}(a, 0)$. Then $\omega$ is referred to as the limit frequency and $v$ as the limit form of these vibrations.

**Remark 3.1.** It will be shown that the functions $u_{i}$ have strongly oscillatory character in $(0, b)$. Moreover, the vibrations $u_{i}$ have the “energy” close to $\omega^{4}$ as $i \to \infty$.

**Theorem 3.1.** If a sequence $\{u_{i}\}_{i=1}^{\infty}$ supports high frequency vibrations with a limit frequency $\omega$ and a limit form $v$, then $\omega^{4}$ is an eigenvalue and $v$ is an eigenfunction of the problem

\[
\begin{align*}
(k_{0}(x)v''')'' - \omega^{4}v &= 0, \quad x \in (a, 0), \\
v(a) &= v'(a) = 0, \quad v''(0) = v'''(0) = 0
\end{align*}
\]  

(9)

and restrictions $u_{i}|_{(0, b)}$ converge to zero in the weak topology of $L_{2}(0, b)$.

**Proof.** From (5) we have $\|u_{i}\|_{c} = \lambda_{i}\|u_{i}\|_{L_{2}(a, b)} = \lambda_{i}$. Then

\[
\varepsilon_{i}^{2}a_{1}(u_{i}, \varphi) \leq \varepsilon_{i}^{2}a_{1}^{1/2}(u_{i}, u_{i})a_{1}^{1/2}(\varphi, \varphi) \leq \|u_{i}\|_{c}a_{1}^{1/2}(\varphi, \varphi) \leq \lambda_{i}a_{1}^{1/2}(\varphi, \varphi)
\]  

(10)
for any $\varphi \in H_0^2(a,b)$. Let us consider identity (5) for test functions $\varphi \in C_0^\infty(0,b)$
\[
(u_i, \varphi)_{L_2(0,b)} = \varepsilon_i^4 \lambda_i^{-1} a_1(u_i, \varphi).
\]
Using (10) we obtain
\[
|(u_i, \varphi)_{L_2(0,b)}| \leq \varepsilon_i^2 a_1^{1/2}(\varphi, \varphi).
\]
It follows immediately that $u_i \to 0$ in $L_2(0,b)$ weakly.

Throughout the proof, $\mathcal{H}_a$ denotes the space $\{v \in H^2(a,0) : v(a) = v'(a) = 0\}$. Passing to the limit as $\varepsilon_i \to 0$ in (5) we obtain
\[
a_0(v, \varphi) - \lambda(v, \varphi)_{L_2(a,0)} = 0, \quad \varphi \in H_0^2(a,b). \quad (11)
\]
Since the restriction $\varphi|_{(0,a)}$ is an element of $\mathcal{H}_a$, the identity (11) corresponds to the eigenvalue problem (9). This finishes the proof, because $v$ is a nonzero function. $\square$

Consequently, the limit frequencies $\omega$ are generated by means of stiffer part of a vibrating system.

4 Asymptotic expansions of high frequency vibrations. The leading terms

By $\omega_\varepsilon$ we denote an eigenfrequency $\lambda_\varepsilon^{1/4}$ of vibrating system (1)–(4). We search for $\omega_\varepsilon$ with the asymptotic expansion
\[
\omega_\varepsilon \sim \sum_{k=0}^{\infty} \varepsilon^k \omega_k, \quad \omega_0 \neq 0.
\]
Let us consider a one-parameter family of the following vector-functions
\[
N(\kappa, x) = \left( \cos \kappa S(x), \sin \kappa S(x), e^{-\kappa S(x)}, e^{\kappa(S(x)-S(b))} \right), \quad x \in (0,b), \quad \kappa \in \mathbb{R},
\]
where
\[
S(x) = \omega_0 \int_0^x k_1^{-1/4}(t) \, dt.
\]
Let $\langle \cdot, \cdot \rangle$ be an inner product in $\mathbb{R}^4$. We postulate the expansions of $u_\varepsilon$ in the form
\[
u_\varepsilon(x) \sim \begin{cases}
\sum_{k=0}^{\infty} \varepsilon^k v_k(x), & x \in (a,0), \\
\sum_{k=0}^{\infty} \varepsilon^k < f_k(x), N(\varepsilon^{-1}, x) >, & x \in (0,b),
\end{cases}
\]
with $f_k : (0,b) \to \mathbb{R}^4$.

**Remark 4.1.** On the interval $(0,b)$ we have the equation
\[
\varepsilon^4 \frac{d^2}{dx^2} \left( k_1 \frac{d^2 u_\varepsilon}{dx^2} \right) - \omega_\varepsilon^4 u_\varepsilon = 0
\]
with a small parameter at the highest order derivative. Thus the construction of expansions (13) on the interval $(0, b)$ was motivated by [13] with the method of WKB-approximations, which are also known as short-wave approximations. Hence, we can find a solution $u_\varepsilon$ in the form

$$e^{-\frac{S(x)}{\varepsilon}} \sum_{k=0}^{\infty} \varepsilon^k a_k(x),$$

where function $S(x)$ has to satisfy the eikonal equation $k_1(x)S'(x)^4 - \omega_0^4 = 0$. Since there exist 4 different solutions of the eikonal equation with respect to $S'$, we have introduced the function $N(x, \cdot)$ combining in it’s components a fundamental set of solutions.

Substituting (13) for $u_\varepsilon$ in conditions (4), we obtain

$$k_0(0) v_0''(0) + \ldots + \varepsilon \sum_{k=0}^{\infty} k_0(0) S'(0)^2 < f_0(0), T^2 N (\varepsilon^{-1}, 0) > + \ldots, \quad (14)$$

$$(k_0v_0'')'(0) + \ldots + \varepsilon k_1(0) S'(0)^3 < f_0(0), T^3 N (\varepsilon^{-1}, 0) > + \ldots. \quad (15)$$

Above we take into account the equality

$$\frac{d}{dx} N (\varepsilon^{-1}, x) = \varepsilon^{-1} S'(x) TN (\varepsilon^{-1}, x)$$

with an orthonormal matrix

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \quad \text{where} \quad T_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$  

By (14) and (15), we obtain $v_0''(0) = 0$ and $v_0'''(0) = 0$. Substituting expansions (12), (13) for $\omega_\varepsilon$ and $u_\varepsilon$ in equation (1) and boundary value conditions (2), we have the eigenvalue problem

$$\frac{d^2}{dx^2} \left( k_0(x) \frac{d^2 v_0}{dx^2} \right) - \omega_0^4 v_0(x) = 0, \quad x \in (a, 0),$$

$$v_0(a) = v_0'(a) = 0, \quad v_0''(0) = v_0'''(0) = 0. \quad (16)$$

Let $\omega_0$ and $v_0$ be an eigenfrequency and an eigenfunction of the problem (compare with Th.1). Note that every eigenvalue $\omega_0$ is simple. Suppose that the function $v_0$ satisfies condition $\|v_0\|_{L_2(a, 0)} = 1$.

In order to find the leading term $f_0$ of expansion (13), we substitute the second of series (13) in equation (1). In particular, we obtain

$$(k_1 S'^4 - \omega_0^4) f_0 = 0, \quad (17)$$

$$4k_1 S'^3 T f_0' + (2k_1 S'^3 T + 6k_1 S'^2 S'' T - 4\omega_0^3 \omega_1) f_0 = -(k_1 S'^4 - \omega_0^4) f_1. \quad (18)$$

Since the function $S$ satisfies the eikonal equation $k_1 S'^4 - \omega_0^4 = 0$ then (17) holds. Hence (18) is a homogeneous system of linear differential equations with respect to the function $f_0$.

Taking into account conditions (3) and boundary value conditions (2) at $x = b$, we can write

$$f_0' = A(x) f_0, \quad x \in (0, b),$$

$$\langle f_0(0), N(\varepsilon^{-1}, 0) \rangle = v_0(0), \quad \langle f_0(0), T N(\varepsilon^{-1}, 0) \rangle = 0, \quad (19)$$

$$\langle f_0(b), N(\varepsilon^{-1}, b) \rangle = 0, \quad \langle f_0(b), T N(\varepsilon^{-1}, b) \rangle = 0.$$
with a matrix

\[
A = \begin{pmatrix}
-\frac{k_1}{\sqrt{k_1}} & \frac{\omega_1}{\sqrt{k_1}} & 0 & 0 \\
-\frac{\omega_1}{\sqrt{k_1}} & \frac{k_1}{\sqrt{k_1}} & 0 & 0 \\
0 & 0 & -\frac{k_1}{\sqrt{k_1}} & -\frac{\omega_1}{\sqrt{k_1}} \\
0 & 0 & 0 & -\frac{k_1}{\sqrt{k_1}} + \frac{\omega_1}{\sqrt{k_1}}
\end{pmatrix}.
\]

Note that the matrix \( A \) depends on a parameter \( \omega_1 \). We shall define \( \omega_1 \) below.

On the one hand problem (19) depends on \( \varepsilon \) by means of the boundary value conditions, on the other hand that one is an ill-posed problem. To resolve both these problems, we consider a discrete set of small parameter \( \varepsilon \). We shall choose below a specific sequence \( \varepsilon_p \rightarrow 0 \) of a small parameter and asymptotic expansions will have a discrete character with respect to \( \varepsilon \). That agrees with a discrete phenomenon of high frequency vibrations.

**Lemma 4.1.** Let \( w : (0, b) \rightarrow \mathbb{R}^4 \) be a smooth vector-function and \( \sigma \) be a vector in \( \mathbb{R}^4 \). There exists an infinitely small sequence \( \{\varepsilon_p\}_{p=1}^{\infty} \) such that the problem

\[
y'(\varepsilon, x) = A(x)y(\varepsilon, x) + w(x), \quad x \in (0, b),
\]

\[
\langle y(\varepsilon, 0), N(\varepsilon^{-1}, 0) \rangle = \sigma_1, \quad \langle y(\varepsilon, 0), TN(\varepsilon^{-1}, 0) \rangle = \sigma_2,
\]

\[
\langle y(\varepsilon, b), N(\varepsilon^{-1}, b) \rangle = \sigma_3, \quad \langle y(\varepsilon, b), TN(\varepsilon^{-1}, b) \rangle = \sigma_4,
\]

has a unique solution \( y(\varepsilon_p, \cdot) \) for \( p \in \mathbb{N} \). The family of solutions \( \{y(\varepsilon_p, \cdot)\}_{p=1}^{\infty} \) holds the inequality

\[
\|y(\varepsilon_p, \cdot) - y_*\|_{C^1} \leq Ce^{-\frac{M}{\varepsilon_p}}
\]

for a smooth function \( y_* : [0, b] \rightarrow \mathbb{R}^4 \). The constants \( C \) and \( M \) are independent of parameter \( \varepsilon \).

**Proof.** The fundamental matrix of (20) is

\[
\Phi(x) = k_1^{-1/8}(x) \begin{pmatrix}
\cos \frac{\omega_1}{\omega_0} S(x) & \sin \frac{\omega_1}{\omega_0} S(x) & 0 & 0 \\
-\sin \frac{\omega_1}{\omega_0} S(x) & \cos \frac{\omega_1}{\omega_0} S(x) & 0 & 0 \\
0 & 0 & e^{-\frac{\omega_1}{\omega_0} S(x)} & 0 \\
0 & 0 & 0 & e^{-\frac{\omega_1}{\omega_0} (S(x) - S(b))}
\end{pmatrix}.
\]

Therefore we have a representation of the general solution

\[
y(x) = \Phi(x)(\beta + h(x)),
\]

where \( \beta \) is a constant vector and \( h(x) = \int_0^x \Phi^{-1}(t)w(t) \, dt \). Suppose a vector-function \( y(\varepsilon, x) = \Phi(x)(\beta_\varepsilon + h(x)) \) is a solution of (20)–(22). Setting \( \gamma_\varepsilon = \varepsilon^{-1} + \omega_1\omega_0^{-1} \), it is easy to check that

\[
\Phi'(x)N(\varepsilon^{-1}, x) = k_1^{-1/8}(x)N(\gamma_\varepsilon, x),
\]

where \( \Phi' \) is a transposed matrix. Then

\[
<y(\varepsilon, x), N(\varepsilon^{-1}, x)> = k_1^{-1/8}(x) < \beta_\varepsilon + h(x), N(\gamma_\varepsilon, x)>,
\]
and we can write (21), (22) in the form
\[
\langle \beta_\varepsilon, \ N(\gamma_\varepsilon, 0) \rangle = m_0 \sigma_1,
\langle \beta_\varepsilon, TN(\gamma_\varepsilon, 0) \rangle = m_0 \sigma_2,
\langle \beta_\varepsilon, N(\gamma_\varepsilon, b) \rangle = m_1 \sigma_3 < h(b), N(\gamma_\varepsilon, b) \rangle,
\langle \beta_\varepsilon, TN(\gamma_\varepsilon, b) \rangle = m_1 \sigma_4 - \langle h(b), TN(\gamma_\varepsilon, b) \rangle,
\]
where \( m_0 = k_1^{1/8}(0) \) and \( m_1 = k_1^{1/8}(b) \). Note that the matrix \( \Phi^t \) commutes with \( T \) and, moreover, \( h(0) = 0 \).

Hence, the vector \( \beta_\varepsilon \) is a solution of a linear algebraic system with a matrix
\[
G(\gamma_\varepsilon) = \begin{pmatrix}
1 & 0 & 1 & e^{-\gamma_\varepsilon S(b)} \\
0 & 1 & -1 & e^{-\gamma_\varepsilon S(b)} \\
\cos \gamma_\varepsilon S(b) & \sin \gamma_\varepsilon S(b) & e^{-\gamma_\varepsilon S(b)} & 1 \\
-\sin \gamma_\varepsilon S(b) & \cos \gamma_\varepsilon S(b) & -e^{-\gamma_\varepsilon S(b)} & 1
\end{pmatrix}.
\]
Since \( S(b) \neq 0 \), the determinant
\[
\det G(\gamma_\varepsilon) = -2 \cos \gamma_\varepsilon S(b) + 2e^{-\gamma_\varepsilon S(b)}(2 - e^{-\gamma_\varepsilon S(b)} \cos \gamma_\varepsilon S(b))
\]
does not vanish for all \( \varepsilon > 0 \).

Let us fix \( \delta \in [0, 2\pi) \) and choose a sequence \( \varepsilon_p \) from the set of conditions \( \gamma_\varepsilon_p S(b) = \delta + 2\pi p \) for \( p = 1, 2, \ldots \). Namely, let
\[
\varepsilon_p = \frac{\omega_0 S(b)}{\omega_0 (\delta + 2\pi p) - \omega_1 S(b)},
\]
for all \( p \geq p_0 \). Here \( p_0 \) is the smallest natural number such that the denominator of (25) is positive. Without restriction of generality we set \( p_0 = 1 \). We denote \( \gamma_p = \gamma_{\varepsilon_p} \). Since \( \gamma_p \to \infty \) as \( \varepsilon_p \to 0 \), the matrix \( G(\gamma_p) \) is an exponentially small perturbation of a matrix
\[
G_0 = \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 \\
\cos \delta & \sin \delta & 0 & 1 \\
-\sin \delta & \cos \delta & 0 & 1
\end{pmatrix}.
\]
Since \( N(\gamma_p, b) = (\cos \delta, \sin \delta, e^{-\gamma_p S(b)}, 1) \) then the right-hand side of (24) is an exponentially small perturbation of vector
\[
g = (m_0 \sigma_1, m_0 \sigma_2, m_1 \sigma_3 - \langle h(b), N_\delta \rangle, m_1 \sigma_4 - \langle h(b), TN_\delta \rangle),
\]
where \( N_\delta \) differs from vector \( N(\gamma_p, b) \) by the third component only.

Let us suppose that \( \delta \) is not equal to \( \pi/2 \) and \( 3\pi/2 \). Then the matrix \( G_0 \) is non-degenerate. From the theory of finite-dimensional perturbations we obtain
\[
\|\beta_{\varepsilon_p} - \beta_*\|_{\mathbb{R}^4} \leq Ce^{-\gamma_p S(b)},
\]
where \( \beta_* \) is a solution of \( G_0 \beta = g \). Let \( y_* (x) = \Phi(x)(\beta_* + h(x)) \), then
\[
\|y(\varepsilon_p, \cdot) - y_*\|_{C^1} \leq \|\Phi\|_{C^1} \|\beta_{\varepsilon_p} - \beta_*\|_{\mathbb{R}^4},
\]
where \( \|\Phi\|_{C^1} = \max_{x \in (0,b)} (\|\Phi(x)\| + \|\Phi'(x)\|) \). Note that \( \gamma_\varepsilon \geq c_0 \varepsilon^{-1} \) with a positive constant \( c_0 \), which proves the Lemma. \( \square \)
Remark 4.2. From now on, we say that $y_*$ is a solution of (20)–(22) with neglect of exponentially small terms. However the choice of a sequence $\varepsilon_p$ is non-unique and depends on $\delta$ and $\omega_1$. We shall define $\omega_1$ at the next step but, on the other hand, we shall keep the dependence on $\delta$, which will be a deformation parameter. The approximation of the limit function $v_0$ by eigenfunctions $u_{n(\varepsilon)}$ is ambiguously determined. Hence we can not define $\delta$ uniquely. Consequently, for each $\delta$ we construct the expansions of $u^{\varepsilon}$, though they are asymptotically equivalent as $\varepsilon \to 0$.

Let us return to the study of problem (19), that is a sub-case of (20)–(22) with the right-hand side $w = 0$ and $\sigma = (v_0(0), 0, 0, 0)$. Since (19) is a homogeneous system, we obtain $f_0 = \Phi \beta_0$, where $\beta_0 = \frac{1}{2} k_1^{1/8}(0) v_0(0) \left(1 - \tan \delta, 1 + \tan \delta, 1 + \tan \delta, -\frac{1}{\cos \delta}\right)$ is a solution of the corresponding system with matrix $G_0$.

In order to calculate the first-order correction $\omega_1$, we consider the problem for $v_1$

$$
\frac{d^2}{dx^2} \left( k_0(x) \frac{d^2 v_1}{dx^2} \right) - \omega_0^4 v_1 = 4 \omega_1 \omega_0^3 v_0, \quad x \in (a, 0),
$$

$$
v_1(a) = v_1'(a) = 0, \quad v_1''(0) = 0, \quad v_1'''(0) = -\frac{k_1^{1/4}(0)}{k_0(0)} \omega_0^3 v_0(0)(1 + \tan \delta).
$$

Since $\omega_0^4$ is a simple eigenvalue of (16), there exists a solution of (27) under the condition $\omega_1 = -\frac{1}{4} k_1^{1/4}(0) v_0^2(0)(1 + \tan \delta)$. Let us choose a solution $v_1$ such that $(v_1, v_0)_{L^2(a, 0)} = 0$. Now by (25), it follows that

$$
\varepsilon_p(\delta) = \frac{4 \omega_0 \omega_1}{4 \omega_0(\delta + 2 \pi p) + k_1^{1/4}(0)v_0^2(0)S(b)(1 + \tan \delta)},
$$

where $\delta \in [0, 2\pi) \setminus \{\pi/2, 3\pi/2\}$. Let us denote by $\mathcal{E}_\delta$ the sequence (28) for fixed $\delta$. We shall construct the asymptotic expansions only for $\varepsilon \in \mathcal{E}_\delta$. Hence, we have just found $\omega_0$, $v_0$, $f_0$, $\omega_1$, $v_1$ and the sequence $\mathcal{E}_\delta$. Recall that $f_0$, $\omega_1$ and $v_1$ depend on $\delta$.

5 Complete asymptotics of high frequency vibrations

Let us find the general terms $f_k$, $\omega_{k+1}$ and $v_{k+1}$ of expansions (12), (13) for $k \geq 1$. In the same way as for the vector $f_0$, we obtain the boundary value problem for $f_k$. With neglect of the exponentially small terms in the boundary conditions, we can write the problem in the form

$$
\begin{align*}
&f'_k = A(x) f_k + w_k, \quad x \in (0, b), \\
&\langle f_k(0), N(\varepsilon^{-1}, 0) \rangle = v_k(0), \quad (f_k(b), N(\varepsilon^{-1}, b)) = 0, \\
&\langle f_k(0), TN(\varepsilon^{-1}, 0) \rangle = -\omega_0^{-1}(k_1^{1/4}(0)v_{k-1}'(0) - k_1^{1/8}(0)(\Phi^{-1}(0)f_{k-1}'(0), N_0)), \\
&\langle f_k(b), TN(\varepsilon^{-1}, b) \rangle = -\omega_0^{-1}(k_1^{1/8}(b)(\Phi^{-1}(b)f_{k-1}'(b), N_0),)
\end{align*}
$$

where $N_0 = (1, 0, 1, 0)$ and the $N_k$ is defined in the proof of Lemma 3.

For each $k$ the right-hand side $w_k$ of (29) is a smooth function, namely,

$$
w_k = \frac{1}{4k_1S^2} \left( \sum_{m=0}^{k-1} \lambda_{k-m} T^3 f_m - S'T^2 P_k - \frac{d}{dx}(k_1'(S')^2 T f_{k-1} + 3k_1 S' S'' T f_{k-1} + 3k_1(S')^2 T f_{k-1}''' + T^3 P_{k-1}) \right),
$$

10
where \( P = k_1S'T^3f_i'' + \frac{d}{dx}(2k_1S'T^3f_i' + k_1S''T^3f_i + k_1f_i'''). \) The proof is by induction on \( k. \) According to Lemma 3 there exists a solution of (29) for \( \varepsilon \in \mathcal{E}_\delta. \)

Now we can find \( \omega_{k+1} \) and \( v_{k+1}: \)

\[
\frac{d^2}{dx^2} \left( k_0(x) \frac{d^2v_{k+1}}{dx^2} \right) - \omega_0^4v_{k+1} = \sum_{m=1}^{k+1} \lambda_m v_{k-m+1}, \quad x \in (a, 0),
\]

\[
v_{k+1}(a) = 0, \quad v'_{k+1}(a) = 0,
\]

\[
(k_0v''_{k+1})(0) = \langle \Phi^{-1}(\omega_0^3k_1^{1/8}T^2f_{k-1} + k_1^{-1/8}Q_{k-2}), N_0 \rangle \bigg|_{x=0},
\]

\[
(k_0v''_{k+1})'(0) = \langle \Phi^{-1}(\omega_0^3k_1^{1/8}Tf_k + k_1^{-1/8}R_{k-1}), N_0 \rangle \bigg|_{x=0},
\]

where

\[
Q_k = k_1\frac{d}{dx}(S'T^3f_k + f_{k-1}') + S'T^3f_k',
\]

\[
R_k = S'T^3Q_{k-1} + \frac{d}{dx}(\omega_0^3k_1^{1/2}T^2f_k + Q_{k-1})
\]

and \( \lambda_m = \sum \omega_i \omega_j \omega_k \omega_s \) with \( i + j + l + s = m. \) Since \( \omega_0^4 \) is an eigenvalue of (16), boundary value problem (30) has no solution for an arbitrary right-hand side. We can write the existence condition in the form

\[
\omega_{k+1} = \frac{1}{4\omega_0^8} < v_0 \Phi^{-1}(\omega_0^3k_1^{1/8}T^2f_{k-1} + k_1^{-1/8}Q_{k-2}) - v_0 \Phi^{-1}(\omega_0^3k_1^{1/8}Tf_k + k_1^{-1/8}R_{k-1}), N_0 > \bigg|_{x=0}.
\]

Let us choose a solution \( v_{k+1} \) of (30) such that \( (v_{k+1}, v_0)_{L^2(a,0)} = 0. \)

Hence, the algorithm scheme of asymptotics (12), (13) is

\[
\omega_0 \to v_0 \to f_0 \to \omega_1 \to \mathcal{E}_\delta \to v_1 \to \cdots \to f_k \to \omega_{k+1} \to v_{k+1} \to \cdots
\]

Note that all terms, except for \( \omega_0 \) and \( v_0, \) depend on parameter \( \delta. \)

6 Justification of asymptotics

The nonstandard object of investigation, namely, the sequences of eigenfunctions \( u^{\varepsilon}_{k(\varepsilon)} \), changes the classical scheme of justification. Note that in Sections 4 and 5 we have constructed series (12), (13), however we have not defined the object that is approximated by ones yet.

Only by using the formal series (12) we shall define a sequence \( u^{\varepsilon}_{k(\varepsilon)} \) that supports the high frequency vibrations with the limit frequency \( \omega_0 \) and the limit form \( v_0. \)

Let us fix \( n \in \mathbb{N} \) and introduce a sequence of real numbers \( \{\lambda^{(n)}_\varepsilon\} \) \( \varepsilon \in \mathcal{E}_\delta \) and sequence of functions \( \{u^{(n)}_\varepsilon\} \) \( \varepsilon \in \mathcal{E}_\delta \) in \( H^2(a, b). \) Namely, for each \( \varepsilon \in \mathcal{E}_\delta \) we set

\[
\lambda^{(n)}_\varepsilon = \lambda_0 + \varepsilon \lambda_1 + \ldots + \varepsilon^n \lambda_n, \quad \lambda_n = \sum_{i+j+k+l=s} \omega_i \omega_j \omega_k \omega_l, \quad (31)
\]

\[
u^{(n)}_\varepsilon(x) = \begin{cases} v_0(x) + \varepsilon v_1(x) + \cdots + \varepsilon^n v_n(x), & x \in (a, 0), \\ \sum_{i=0}^n \varepsilon^i < f_i(x), N(\varepsilon^{-1}, x) >, & x \in (0, b), \end{cases} \quad (32)
\]
where the numbers $\omega_k$, the functions $v_k$, the vectors $f_k$ and the set $E_\delta$ are defined in Sections 4 and 5.

**Lemma 6.1.** There exists a sequence $\{\lambda^\varepsilon\}_{\varepsilon \in E_\delta}$ of eigenvalues for (1)--(4) such that

$$
|\lambda^\varepsilon_{i}(\varepsilon) - \lambda^\varepsilon| \leq C_n \varepsilon^{n+1}, \quad \varepsilon \in E_\delta.
$$

**Proof.** Denoting by $H_\varepsilon$ the self-adjoint and compact for all $\varepsilon > 0$. Then we can write problem (1)--(4) in the form

$$
A_\varepsilon u_\varepsilon - (\lambda^\varepsilon)^{-1} u_\varepsilon = 0.
$$

Substituting sequences (31), (32) in problem (1)--(4), we obtain

$$
\begin{align*}
\frac{d^2}{dx^2} \left( k_\varepsilon(x) \frac{d^2}{dx^2} u_\varepsilon(n) \right) - \lambda^\varepsilon_{i}(\varepsilon) u_\varepsilon(n) &= F_n(\varepsilon, x), \quad x \in (a, b), \\
u_\varepsilon(n)(a) &= 0, \quad \frac{d}{dx} u_\varepsilon(n)(a) = 0, \quad u_\varepsilon(n)(b) = g^{(1)}_n(\varepsilon), \quad \frac{d}{dx} u_\varepsilon(n)(b) = g^{(2)}_n(\varepsilon), \\
\left[ u_\varepsilon(n) \right]_0 &= g^{(3)}_n(\varepsilon), \\
\left[ \frac{d}{dx} u_\varepsilon(n) \right]_0 &= h_n(\varepsilon),
\end{align*}
$$

where $[f]_0$ is a jump of a function $f$ at $x = 0$. The right-hand sides of problem (34) satisfy the inequalities

$$
\begin{align*}
\|F_n(\varepsilon, x)\|_{C^0(a,b)} &\leq C_n \varepsilon^{n+1}, \quad |g^{(i)}_n(\varepsilon)| \leq C_n \varepsilon^{-\frac{\omega_i}{2}}, \quad i = 1, 2, 3, \\
|h_n(\varepsilon)| &\leq C_n \varepsilon^n, \quad |z^{(i)}_n(\varepsilon)| \leq C_n \varepsilon^{n+1}, \quad i = 1, 2.
\end{align*}
$$

The function $u_\varepsilon(n)$ does not belong to the space $H_\varepsilon$, because it has a point of discontinuity at $x = 0$. Taking into account (35) we can choose a function $\varphi_\varepsilon(n)$ such that $u_\varepsilon(n) + \varphi_\varepsilon(n) \in H_\varepsilon$ and

$$
\max_{x \in (a,b)} \left( |\varphi_\varepsilon(n)| + \left\| \frac{d}{dx} \varphi_\varepsilon(n) \right\| + \left\| \frac{1}{k_\varepsilon^{1/2}} \frac{d^2}{dx^2} \varphi_\varepsilon(n) \right\| \right) \leq C_n \varepsilon^n.
$$

Let $V_\varepsilon(n) = \varphi_\varepsilon(u_\varepsilon(n) + \varphi_\varepsilon(n))$, where $\varphi_\varepsilon$ is a normalizing constant such that $\|V_\varepsilon(n)\|_{H_\varepsilon} = 1$. It is easy to check that $\varphi_\varepsilon \geq \varphi_0 > 0$. It follows from (35), (36) that

$$
\left\| \left( A_\varepsilon - (\lambda^\varepsilon)^{-1} I \right) V_\varepsilon(n) \right\|_{E_\delta} \leq K_n \varepsilon^n, \quad \varepsilon \in E_\delta,
$$

where $K_n$ is a constant independent of $\varepsilon$. Hence, according to the Vishik-Lusternik lemma [14] there exists the eigenvalue $(\lambda^\varepsilon)^{-1}$ of the operator $A_\varepsilon$ such that

$$
\left| \frac{1}{\lambda^\varepsilon} - \frac{1}{\lambda^\varepsilon_{i}(\varepsilon)} \right| \leq K_n \varepsilon^n.
$$
Applying this inequality for the value \( n + 1 \) instead of \( n \), we obtain
\[
|\lambda^\varepsilon - \lambda^{(n)}_\varepsilon| \leq C_n \varepsilon^{n+1}, \quad \varepsilon \in E_\delta.
\]

\[\square\]

Lemma 6.2. There exists a positive number \( d \) such that for each \( \varepsilon \in E_\delta \) the interval \( I_\varepsilon = (\lambda^{(n)}_\varepsilon - d\varepsilon, \lambda^{(n)}_\varepsilon + d\varepsilon) \) contains exactly one eigenvalue \( \lambda^{\varepsilon}_{k(\varepsilon)} \) of problem (1)–(4). Moreover, the eigenvalue number \( k(\varepsilon) \) satisfies the inequality
\[
a_0\varepsilon^{-1} \leq k(\varepsilon) \leq a_1\varepsilon^{-1},
\]
where \( a_0, a_1 \) are constants independent of \( \varepsilon \).

Proof. According to Section 2, we have
\[
\lambda^\varepsilon_m = \varepsilon^4 \mu_m (1 + \alpha_m(\varepsilon)), \quad \varepsilon \to 0,
\]
where \( \mu_m \) is an eigenvalue of (8) and \( \alpha_m(\varepsilon) = o(1) \) as \( \varepsilon \to 0 \). Moreover, the following asymptotics hold
\[
\mu_m = m^4(c_0 + \tau(m)) \quad \text{with} \quad \tau(m) = o(1) \quad \text{as} \quad m \to \infty.
\]
Then
\[
\lambda^\varepsilon_m = (\varepsilon m)^4(c_0 + \tau(m))(1 + \alpha_m(\varepsilon)).
\]
By Lemma 4, for each \( \varepsilon \in E_\delta \) there exists a number \( k = k(\varepsilon) \) such that
\[
\lambda_0 - b_0\varepsilon \leq \lambda^{\varepsilon}_{k(\varepsilon)} \leq \lambda_0 + b_0\varepsilon
\]
with \( b_0 > 0 \). We conclude from (39) that
\[
\lambda_0 - b_0\varepsilon \leq (\varepsilon k)^4(c_0 + \tau(k))(1 + \alpha_k(\varepsilon)) \leq \lambda_0 + b_0\varepsilon,
\]
and finally (38) holds. According to (39) we have
\[
\lambda^{\varepsilon}_{k(\varepsilon)} = c_0 \varepsilon^4 k(\varepsilon)^4 + o(\varepsilon^4), \quad E_\delta \ni \varepsilon \to 0.
\]
Then the distance between the two neighboring eigenvalues that are the closest to the point \( \lambda_0 \) admits the estimate
\[
|\lambda^{\varepsilon}_{k(\varepsilon)+1} - \lambda^{\varepsilon}_{k(\varepsilon)}| = 4c_0 \varepsilon^4 k(\varepsilon)^3 + o(\varepsilon^4), \quad E_\delta \ni \varepsilon \to 0.
\]
Taking into account the asymptotic behaviour of \( k(\varepsilon) \), we obtain the existence of the interval \( I_\varepsilon \) of the length \( 2d\varepsilon \) such that exactly one eigenvalue \( \lambda^\varepsilon \) of problem (1)–(4) belongs to it. The case of the point \( \lambda^{(n)}_\varepsilon \) being a midpoint of \( I_\varepsilon \) is impossible since this contradicts (33) and (40).

Now we can introduce the object that is approximated by formal series (13) constructed in Sections 4 and 5. Let \( \{\lambda^{\varepsilon}_{k(\varepsilon)}\}_{\varepsilon \in E_\delta} \) be the sequence which is defined in Lemma 5, where \( \lambda^{\varepsilon}_{k(\varepsilon)} \) is the nearest eigenvalue to the value \( \lambda^{(n)}_\varepsilon \). Let \( \{u^{\varepsilon}_{k(\varepsilon)}\}_{\varepsilon \in E_\delta} \) be the sequence of the corresponding eigenfunctions, \( \|u^{\varepsilon}_{k(\varepsilon)}\|_{\mathcal{H}_\varepsilon} = 1 \).
Theorem 6.1. As $E_\delta \ni \varepsilon \to 0$ the sequence $\{u^\varepsilon_{k(\varepsilon)}\}_{\varepsilon \in E_\delta}$ supports the high frequency vibrations with the limit frequency $\omega_0$ and the limit form $v_0$. Moreover,

$$\|u^\varepsilon_{k(\varepsilon)} - V^{(n)}_\varepsilon\|_\varepsilon \leq C_n \varepsilon^{n+1}, \quad \varepsilon \in E_\delta,$$

for $n = 0, 1, \ldots$. In particular, the following inequalities hold

$$\left\|u^\varepsilon_{k(\varepsilon)}(x) - \sum_{k=0}^{n} v_k(x)\varepsilon^k\right\|_{C^1(a,0)} \leq C_n \varepsilon^{n+1},$$

$$\left\|u^\varepsilon_{k(\varepsilon)}(x) - \sum_{k=0}^{n} \langle f_k(x), N(\varepsilon^{-1}, x)\rangle \varepsilon^k\right\|_{C^1(0,b)} \leq C_n \varepsilon^{n-1},$$

where $\varepsilon = \|V^{(n)}_\varepsilon\|_\varepsilon^{-1}$ is a normalizing constant and $\varepsilon \geq \varepsilon_0 > 0$.

Proof. By Lemma 7, we choose $d_0 > 0$ such that $d_0 \varepsilon$-neighborhood of the point $(\lambda^{(n)}_k)^{-1}$ contains exactly one eigenvalue $(\lambda^{\varepsilon}_{k(\varepsilon)})^{-1}$ of the operator $A_\varepsilon$. From inequality (37), we obtain (see [14])

$$\|u_\varepsilon - V^{(n)}_\varepsilon\|_\varepsilon \leq K_n d_0^{-1} \varepsilon^{n-1},$$

where $u_\varepsilon$ is a normalized eigenfunction of $A_\varepsilon$ associated with the eigenvalue $(\lambda^{\varepsilon}_{k(\varepsilon)})^{-1}$. Hence, $u_\varepsilon = \pm u^\varepsilon_{k(\varepsilon)}$. There is no loss of generality in assumption that (43) holds for the number $n + 2$, we obtain (41).

In addition, by (36) and the definition of norm, we obtain

$$\|u^\varepsilon_{k(\varepsilon)} - \varepsilon u^{(n)}_\varepsilon\|_{H^2(a,0)} \leq C_n \varepsilon^{n+1}, \quad \|u^\varepsilon_{k(\varepsilon)} - \varepsilon u^{(n)}_\varepsilon\|_{H^2(0,b)} \leq C_n \varepsilon^{n-1}.$$ 

Estimate (42) follows from the Sobolev embedding theorem.

It is easy to check that the constant $\varepsilon$ has a nonzero limit as $\varepsilon \to 0$. Then sequence $\{u^\varepsilon_{k(\varepsilon)}\}_{\varepsilon \in E_\delta}$ converges to the eigenfunction $\varepsilon v_0$ of problem (8) in $(a, 0)$, namely, this sequence supports the high frequency vibrations. \qed

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