A New Approach to Probing Primordial Non-Gaussianity

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Abstract
We address the dual challenge of estimating deviations from Gaussianity arising in models of the Early Universe, whilst retaining information necessary to assess whether a detection of non-Gaussianity is primordial. We do this by constructing a new statistic, the bispectrum, motivated by the fact that it contains all the information about the Early Universe, whilst retaining information necessary to assess whether a detection of non-Gaussianity is primordial. The estimator is optimised for primordial non-Gaussianity detection, but can also be useful in distinguishing primordial non-Gaussianity from secondary non-Gaussianity, such as may arise from unsubtracted point sources, or residuals from component separation. Extending earlier studies we present unbiased non-Gaussianity estimators optimised for partial sky coverage and inhomogeneous noise associated with realistic scan strategies, but which retain the ability to assess foreground contamination.

Key words: Cosmology: theory – cosmic microwave-background – large-scale structure of Universe Methods: analytical – Methods: statistical – Methods: numerical

1 INTRODUCTION
The statistical properties of fluctuations in the cosmic microwave background (CMB) radiation can be used to probe the very earliest stages of the Universe’s history, and provide valuable information on the mechanisms which ultimately gave rise to the existence of structure within the Universe. This may include evidence for inflation, the process by which the rapid expansion of the Universe is thought to have arisen. In standard inflationary models, the fields in the early Universe should be very close to random Gaussian fields, so a detection of large non-Gaussianity would be highly significant, and may indicate a different history, such as warm inflation or multiple-field inflation, or a completely different mechanism such as those arising from topological defects. A difficulty for methods designed to detect non-Gaussianity in the CMB is that other processes can contribute, such as gravitational lensing, unsubtracted point sources, and imperfect subtraction of galactic foreground emission (e.g. Goldberg & Spergel 1999; Cooray 2000; Verde & Spergel (2002); Castro (2004); Babich & Pierpaoli (2005)).

The challenge therefore is to provide evidence that any detection of non-Gaussianity is primordial in origin, and not a result of these other effects. The aim of this paper is to provide an optimised framework not just for detecting non-Gaussianity, but for assessing the contributions from various sources.

Non-Gaussianity from simplest inflationary models based on a single slowly-rolling scalar field is typically very small (Salopek & Bond 1990, 1991; Falk et al. 1993; Gangui et al. 1994; Acquaviva et al. 2003; Maldacena 2003). (see Bartolo, Matarrese & Riotto 2006 and references there in for more details). Variants of simple inflationary models such as multiple scalar fields (Lyth, Ungarelli & Wands 2003), features in the inflationary potential, non-adiabatic fluctuations, non-standard kinetic terms, warm inflation (Gupta, Berera & Heavens 2002; Moss & Xiong 2007), or deviations from Bunch-Davies vacuum can all lead to much higher level of non-Gaussianity. Early observational work on the bispectrum from COBE (Kamatsu et al. 2002) and MAXIMA (Santos et al. 2003) was followed by much more accurate analysis with WMAP (Komatsu et al. 2003; Creminelli et al. 2003; Spergel et al. 2007). With the recent claim of a detection of non-Gaussianity (Yadav & Wandell 2008) in the Wilkinson Microwave Anisotropy Probe 5-year (WMAP5) sky maps, interest in non-Gaussianity has obtained a tremendous boost. Much of the interest in primordial non-Gaussianity has focussed on a phenomenological ‘local $f_{NL}$’ parametrisation in terms of the perturbative non-linear coupling in the primordial curvature perturbation (Komatsu & Spergel 2001).

\[ \Phi(x) = \Phi_L(x) + f_{NL} \Phi_L^2(x) - \langle \Phi_L^2(x) \rangle, \]  

(1)

where $\Phi_L(x)$ denotes the linear Gaussian part of the Bardeen curvature and $f_{NL}$ is the non-linear coupling parameter. A number of models have non-Gaussianity which can be approximated by this form. The leading order non-Gaussianity therefore is at the level of the bispectrum, or in configuration space at the three-point level. Many studies involving primordial non-Gaussianity have used the bispectrum, motivated by the fact that it contains all the information about $f_{NL}$ (Babich 2005). It has been extensively studied (Komatsu, Spergel & Wandell 2005; Creminelli 2003; Creminelli et al. 2006; Medeiros & Contaldi 2006; Cabella et al. 2006; Liguori et al. 2007; Smith, Senatore & Zaldarriaga 2009, with most of these measurements providing convolved estimates of the bispectrum. Optimised 3-point estimators were introduced by Heavens (1998), and have been successively developed (Komatsu, Spergel & Wandell 2005; Creminelli et al. 2006; Creminelli, Senatore, & Zaldarriaga 2007; Smith, Zahn & Dore 2004; Smith & Zaldarriaga 2006) to the
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point where an estimator for $f_{NL}$, which saturates the Cramer-Rao bound exists for partial sky coverage and inhomogeneous noise (Smith, Senatore & Zaldarriaga 2009). Approximate forms also exist for equilateral non-Gaussianity, which may arise in models with non-minimal Lagrangian with higher-derivative terms (Chen, Huang & Kachru 2006; Chen, Easther & Lim 2007). In these models, the largest signal comes from spherical harmonic modes with $\ell_1 \simeq \ell_2 \simeq \ell_3$, whereas for the local model, the signal is highest when one $\ell$ is much smaller than the other two – the so-called squeezed configuration.

Reducing the CMB data to a single lossless estimate for $f_{NL}$ is extremely elegant, but it suffers from the disadvantage that a single number loses the ability to determine the extent to which the estimate has been contaminated by non-primordial signals. What we seek to do is to perform a less aggressive data compression of the CMB data, not to a single number, but to a function, which has known expected form for primordial models, and for which the contributions from other signals can be estimated. The purpose of this is to be able to demonstrate that a non-Gaussian signal is indeed primordial, or alternatively accounted for by non-primordial signals. We do this in a way which is still optimal for local or equilateral $f_{NL}$ models, although the formalism is general. The function we choose is the integrated cross-power spectrum of pair of maps constructed from the CMB data. Mathematically, it is closely related to previous estimators, but the interpretation of the output is different, and offers very significant advantages.

This paper is organised as follows: Section 2 provides a small review of available models of primordial non-Gaussianity. Section 3 relates the projected-bispectrum to the corresponding primordial bispectrum. Section 4 presents the optimised bispectrum-related power spectrum estimator for the idealised case of an all-sky survey with homogeneous noise. This is not optimal for partial sky coverage or inhomogeneous noise, but is straightforward and shows the connection with the $f_{NL}$ estimator of Komatsu, Spergel & Wandelt (2005). Here we present the theoretical expectation for the local $f_{NL}$ model, and show the link between the $f_{NL}$ estimator based on one-point statistics and its two-point counterpart. Section 5 provides a method which can handle partial sky coverage and non-uniform noise in an approximate way using Monte-Carlo simulations. In Section 6 we provide the full optimal weights for bispectrum analysis in the presence of sky-cuts and the inhomogeneous noise. The full inverse-covariance of the data is introduced which makes the estimator an optimal one. The final section is devoted to discussion and future plans for numerical implementation.

2 MODELS OF PRIMORDIAL NON-GAUSSIANITY

Deviations from pure Gaussian statistics can provide direct clues regarding inflationary dynamics. The single-field slow-roll model of inflation provides a very small level of departure from Gaussianity, far below present experimental detection limits (Maldacena 2003; Acquaviva et al. 2003). Many other variants however will produce a much higher-level of non-Gaussianity which will be within the reach of all-sky experiments such as Planck. In general models can be distinguished by the way they predict coupling between different Newtonian potential modes:

$$\langle \phi(k_1)\phi(k_2)\phi(k_3) \rangle = (2\pi)^3 \delta^{3D}(k_1 + k_2 + k_3) F(k_1, k_2, k_3).$$

The function $F$ encodes the information about mode-mode coupling and $\delta^{3D}(k_1 + k_2 + k_3)$ ensures triangular equality in $k$-space. Different models of inflation are typically divided into two different groups. The first class of model is known as the “local model”, where the contribution from $F(k_1, k_2, k_3)$ is largest when the wavevectors are in the so-called “squeezed” configuration, where e.g. $k_1 \ll k_2, k_3$. The local form of non-Gaussianity is predominant in models where there is non-linear coupling between the field driving inflation (the inflaton) and the curvature perturbations (Salopek & Bond 1990, 1991; Gangui et al. 1994), such as the curvaton model (Lyth, Ungarelli & Wands 2003) and the Ekpyrotic model (Koyama et al. 2007; Buchbinder, Khoury & Ovrut 2008). The primordial bispectrum in the local model can be written as:

$$f_{NL}^{loc}(k_1, k_2, k_3) = \Delta^2 f_{NL} \left[ \frac{\Delta^2}{k_1 k_2} + \text{cyc. perm.} \right],$$

where the power spectrum of inflationary curvature perturbations is given by $P_\delta = \Delta^2 / k^3$, in general for deviation from Harrison-Zeldovich power-spectra one has $P_\delta = \Delta^2 / k^{n_s}$.

The other main class consists of models where the contribution from $F(k_1, k_2, k_3)$ is maximum for configurations where $k_1 \sim k_2 \sim k_3$. The equilateral form appears from non-canonical kinetic terms such as Dirac-Born-Infeld (DBI) action (Alishahiha et al. 2004), the ghost condensation (Akrani-Hamed et al. 2004) or various single-field models where the scalar field acquires a low sound speed (Chen, Easther & Lim 2007; Cheung et al. 2008). The equilateral model is not a separable function of the $k_i$, which complicates the analysis considerably, but it was shown by Creminelli et al. (2008) that the following approximate form can model the equilateral case very accurately:

$$f_{NL}^{eq}(k_1, k_2, k_3) = \Delta^2 f_{NL} \left[ -3 \frac{\Delta^2}{k_1 k_2 k_3} + 2 \frac{\Delta^2}{k_1^2 k_2 k_3} + 6 \frac{\Delta^2}{k_1 k_2^2 k_3} + \text{cyc. perm.} \right].$$

Secondary non-Gaussianity resulting from various sources e.g. coupling of lensing and the Sunyaev-Zel’dovich effect, or lensing and the Integrated Sachs-Wolfe effect can also contribute to the observed bispectrum (Spergel & Goldberg 1999, 2003). We will present a general analysis of secondary bispectra as well as the one induced by quasi-linear evolution of gravitational perturbations (Munshi, Souradeep & Starobinsky 1995).
3 ANGULAR CMB BISPECTRUM WITH PRIMORDIAL NON-GAUSSIANITY

The angular bispectrum can be defined as the three-point correlation in the harmonic domain. With temperature fluctuations as a function of solid angle $\Omega$, $\Delta T(\Omega)$,

$$a_{lm} \equiv \int d\Omega \frac{\Delta T(\Omega)}{T} Y_{lm}^*(\Omega)$$

and the three-point function may be written

$$(a_{lm} a_{lm'} a_{lm''}) \equiv B_{l'l' l''} \left( \begin{array}{ccc} l & m & l' \\ m & m' & l'' \end{array} \right).$$

This form preserves the rotational invariance of the three-point correlation function in the harmonic domain. The quantity in parentheses is the Wigner 3j-symbol, which is nonzero only for triplets $(l_1, l_2, l_3)$ which satisfy the triangle rule, including that the sum $l_1 + l_2 + l_3$ is even, ensuring the parity invariance of the bispectrum. The reduced bispectrum $b_{l'l' l''}$ was introduced by Komatsu, Spergel & Wandelt (2005), which will be helpful (see Babich, Creminelli & Zaldarriaga (2004) for elaborate discussion):

$$B_{l'l' l''} \equiv \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \left( \begin{array}{ccc} l & m & l' \\ m & m' & l'' \end{array} \right) b_{l'l' l''} \equiv I_{l'l' l''} b_{l'l' l''}.$$ (7)

The reduced bispectrum $b_{l'l' l''}$ can be expressed in terms of the kernel $F(k_1, k_2, k_3)$ for various models that we will be considering:

$$b_{l_1 l_2 l_3} = \left( \frac{2}{\pi} \right)^3 \int d^3 r \int k_1^2 dk_1 j_{l_1}(kr_1) \Delta_{l_1}^T(kr_1) \int k_2^2 dk_2 j_{l_2}(kr_2) \Delta_{l_2}^T(kr_2) \int k_3^2 dk_3 j_{l_3}(kr_3) \Delta_{l_3}^T(kr_3) F(k_1, k_2, k_3);$$ (8)

where $\Delta_{l}^T(k)$ denotes the transfer function which relates the inflationary potential $\Phi$ to the spherical harmonics $a_{lm}$, of the temperature perturbation in the sky (e.g. Wang & Kamionkowski (2000):

$$a_{lm} = 4\pi (-i)^l \int \frac{d^3 k}{(2\pi)^3} \Phi(k) \Delta_{l}^T(k) Y_{lm}^*(\hat{k}).$$ (9)

Using these definitions one can express the reduced bispectra for the local and equilateral cases as follows:

$$b_{l_1 l_2 l_3} = \frac{2}{\pi} \int \frac{d^3 k}{(2\pi)^3} \Phi(k) \Delta_{l_1}^T(k) j_{l_1}^2(kr_1)$$ (10)

$$b_{l_1 l_2 l_3} = 6\int \frac{d^3 k}{(2\pi)^3} \Phi(k) \Delta_{l_1}^T(k) j_{l_1}^2(kr_1)$$ (11)

We will use these forms to construct associated fields $A, B$ etc. from temperature fields with appropriate weighting to optimise our estimator. We list the explicit expressions for the functions $\alpha_i(r), \beta_i(r)$ etc. for completeness (Creminelli et al. 2006):

$$\alpha_i(r) \equiv \frac{2}{\pi} \int_0^\infty k^2 dk \Delta_i(k) j_i(kr)$$

$$\beta_i(r) \equiv \frac{2}{\pi} \int_0^\infty k^2 dk P_\Phi(k) \Delta_i(k) j_i(kr)$$

$$\gamma_i(r) \equiv \frac{2}{\pi} \int_0^\infty k^2 dk P_\Phi^{1/3}(k) \Delta_i(k) j_i(kr)$$

$$\delta_i(r) \equiv \frac{2}{\pi} \int_0^\infty k^2 dk P_\Phi^{2/3}(k) \Delta_i(k) j_i(kr)$$ (12)

Numerical evaluations of these functions can be performed by using the publicly available software such as CAMB or CMBFAST.

4 ALL SKY ANALYSIS WITH HOMOGENEOUS NOISE

In this section, we compute the main statistic which will be used to estimate primordial non-gaussianity, and which can also be used to assess whether a non-gaussian signal is indeed primordial. We call the statistic the bispectrum-related power spectrum, as it derives from the cross-power spectrum of certain maps constructed from the CMB map data, and which, as we will see, is related to the primordial non-gaussianity.

The analysis in this section is optimal for detecting primordial non-gaussianity in the case of all-sky coverage and homogeneous noise. These assumptions will not hold in practice, but we present this simpler case for clarity first, and to show the connection with previous work. We relax the assumptions later, and give optimised estimators for realistic cases in later sections.

4.1 Local Model

Following Komatsu, Spergel & Wandelt (2005), we first construct the 3D fields $A(r, \hat{\Omega})$ and $B(r, \hat{\Omega})$ from the expansion coefficients of the observed CMB map, $a_{lm}$. The harmonics here $A_{lm}(r)$ and $B_{lm}(r)$ are simply weighted spherical harmonics of the temperature field $a_{lm}$, and the cross-power spectrum of certain maps constructed from the CMB map data, and which, as we will see, is related to the primordial non-gaussianity.

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with weights constructed from the CMB power spectrum $C_l$ and the functions $\alpha_l(r)$ and $\beta_l(r)$ respectively:

\begin{align}
A(r, \hat{\Omega}) &\equiv \sum_{lm} Y_{lm}(\hat{\Omega}) A_{lm}(r); \quad A_{lm}(r) \equiv \frac{\alpha_l(r)}{C_l} b_l a_{lm} \\
B(r, \hat{\Omega}) &\equiv \sum_{lm} Y_{lm}(\hat{\Omega}) B_{lm}(r); \quad B_{lm}(r) \equiv \frac{\beta_l(r)}{C_l} b_l a_{lm}.
\end{align}

The function $b_l$ represents beam smoothing, and from here onward we will absorb it into the harmonic transforms. Using these definitions Komatsu, Spergel & Wandelt (2005) define the one-point mixed skewness for the fields $A(r, \hat{\Omega})$ and $B(r, \hat{\Omega})$:

$$S_3^{loc} = S_{AB}^{loc} = \int r^2 \, dr \int d\hat{\Omega} A(r, \hat{\Omega}) B^2(r, \hat{\Omega}).$$

$S_3$ can be used to estimate $f_{NL}^{loc}$, but such radical data compression to a single number loses the ability to estimate contamination of the estimator by other sources of nongaussianity. As a consequence, we construct a less radical compression, to a function of $l$ which can be used to estimate $f_{NL}^{loc}$, but which can also be analysed for contamination by, for example, foregrounds. We do this by constructing the integrated cross-power spectrum of the maps $A(r, \hat{\Omega})$ and $B^2(r, \hat{\Omega})$. Expanding $B^2$ in spherical harmonics gives

$$B_{lm}^{(2)}(r) = \int d\hat{\Omega} B^2(r, \hat{\Omega}) Y_{lm}(\hat{\Omega})$$

and we define the cross-power spectrum $C_{lm}^{AB,2}$ at a radial distance $r$ as

$$C_{lm}^{AB,2}(r) = \frac{1}{2l+1} \sum_m \text{Real} \left\{ A_{lm}(r) B_{lm}^{(2)}(r) \right\}. \tag{17}$$

Integrating this over $r$ gives:

$$C_{lm}^{AB,2} = \int r^2 dr \, C_{lm}^{AB,2}(r). \tag{18}$$

This integrated cross power spectrum of $B^2(r, \hat{\Omega})$ and $A(r, \hat{\Omega})$ carries information about the underlying bispectrum $B_{l',l''}^{ll'}$, as follows:

$$C_{lm}^{AB} = \frac{1}{2l+1} \sum_{l'} \sum_{l''} \sum_{m'} \sum_{m''} \int r^2 dr \, \left\{ \frac{\alpha_l(r)}{C_l} \frac{\beta_{l'}(r)}{C_{l'}} \frac{\beta_{l''}(r)}{C_{l''}} \right\} a_{lm} a_{l'm'} a_{l''m''}$$

\begin{align}
&\times \sqrt{\frac{(2l+1)(2l'+1)(2l''+1)}{4\pi}} \left( \begin{array}{ccc} l & l' & l'' \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} l & l' & l'' \\ m & m' & m'' \end{array} \right). \tag{19}
\end{align}

Similarly we can construct the cross power spectrum of the product map $AB(r, \hat{\Omega})$ and $B(r, \hat{\Omega})$, which we denote as $C_{l}^{B,AB}$:

$$C_{lm}^{AB,B} = \frac{1}{2l+1} \sum_{l'} \sum_{l''} \sum_{m'} \sum_{m''} \int r^2 dr \, \left\{ \frac{\beta_l(r)}{C_l} \frac{\alpha_{l'}(r)}{C_{l'}} \frac{\alpha_{l''}(r)}{C_{l''}} \right\} a_{lm} a_{l'm'} a_{l''m''}$$

\begin{align}
&\times \sqrt{\frac{(2l+1)(2l'+1)(2l''+1)}{4\pi}} \left( \begin{array}{ccc} l & l' & l'' \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} l & l' & l'' \\ m & m' & m'' \end{array} \right). \tag{20}
\end{align}

Using these expressions, and the following relation, we can write this more compactly in terms of the estimated CMB bispectrum

$$B_{l''}^{ll'} = \sum_{m'm''} \left( \begin{array}{ccc} l & l' & l'' \\ m & m' & m'' \end{array} \right) a_{lm} a_{l'm'} a_{l''m''} \tag{21}$$

from which we compute our new statistic, the bispectrum-related power spectrum, $C_{lm}^{loc}$ as

$$C_{lm}^{loc} = \left( C_{lm}^{AB,2} + 2C_{lm}^{AB,B} \right) = \frac{f_{NL}^{loc}}{2l+1} \sum_{l'} \sum_{l''} \left\{ \frac{B_{l''}^{loc}}{C_{l'} C_{l''}} \frac{B_{l'}^{loc}}{C_{l''} C_{l''}} \right\}. \tag{22}$$

where $B_{l'}^{loc}$ is the bispectrum for the local $f_{NL}$ model, normalised to $f_{NL}^{loc} = 1$. We can now use standard statistical techniques to estimate $f_{NL}^{loc}$. Note that if we sum over all $l$ values then we recover the estimator $S_{\mu \nu \mu}$ of Komatsu, Spergel & Wandelt (2005), which is the cross-skewness of $ABB$:

$$S_3^{loc} = S_{AB}^{loc} = \sum_{l'} (2l+1) \left( C_{lm}^{AB,2} + 2C_{lm}^{AB,B} \right) = f_{NL}^{loc} \sum_{l'} \sum_{l''} \left\{ \frac{B_{l''}^{loc}}{C_{l'} C_{l''}} \frac{B_{l'}^{loc}}{C_{l''} C_{l''}} \right\}. \tag{23}$$
We show in Fig. 1 the form of the bispectrum-related power spectrum for the local model. If the signal observed is inconsistent with this form, it would indicate a departure from this form of primordial non-Gaussianity, and/or a significant contamination by foreground sources. In the right hand panel we show the expected form of contamination of the $C_l^{loc}$ statistic by a simple foreground source: unsubtracted point sources, randomly distributed. The contamination scales with the number density. This contamination is expected to be very low [Komatsu & Spergel 2001], but we show it as an illustration of how foreground effects may be detected. Since its structure is very different from the local primordial signal, it should be relatively easy to decouple. Note that the point source contamination here is the contribution to the local model bispectrum-related power spectrum - i.e. it is equation (22) with one local $B$ and one point source $B$ (constant $b_{l(l'*l')}$, not two point source $B$ terms.

### 4.2 Equilateral Model

The form for the reduced bispectrum for the equilateral model as mentioned above is an approximation to the real bispectrum generated in theories with higher-order derivatives in the Lagrangian. In addition to the quantities $A$ and $B$ we have constructed analogous fields $C$ and $D$ with corresponding weights $\gamma_l(r)$ and $\delta_l(r)$:

\[
C(r, \hat{\Omega}) \equiv \sum_{lm} Y_{lm}(\hat{\Omega}) C_{lm}(r); \quad C_{lm}(r) = \frac{\gamma_l(r)}{C_l^0} b_{lm}
\]

\[
D(r, \hat{\Omega}) \equiv \sum_{lm} Y_{lm}(\hat{\Omega}) D_{lm}(r); \quad D_{lm}(r) = \frac{\delta_l(r)}{C_l^0} b_{lm}.
\] (25)

From these fields and using $A$ and $B$ previously defined a one-point statistic can be constructed [Creminelli et al. 2006]:

\[
S_3^{eq} = -18 \left[ S_3^{AB^2} - \frac{2}{3} S_3^{D^3} - 2 S_3^{BCD} \right] = -18 \int r^2 dr \int d\hat{\Omega} \left[ A(r, \hat{\Omega}) B(r, \hat{\Omega})^2 + \frac{2}{3} D(r, \hat{\Omega}) D(r, \hat{\Omega}) - 2B(r, \hat{\Omega}) C(r, \hat{\Omega}) D(r, \hat{\Omega}) \right].
\] (26)

The associated power spectrum will have a composite structure with many contributing terms.

\[
C_{eq} = -18 \left\{ C_{eq}^{AB^2} + 2 C_{eq}^{AB, B} \right\} - 2 C_{eq}^{D, D^2} - 2 \left( C_{eq}^{B, CD} + C_{eq}^{C, BD} + C_{eq}^{D, BC} \right).
\] (27)

Following the same procedure outlined above one can now write:

\[
C_{eq} = \frac{\tilde{f}_{NL}^{eq}}{(2l + 1)} \sum_{l'} \sum_{l''} \left\{ \frac{B_{l' l''}^{eq} \tilde{B}_{l' l''}^{eq}}{C_{l'} C_{l''}} \right\}
\] (28)

and finally we can recover the one-point statistic or the cross-skewness of [Komatsu, Spergel & Wandelt 2005].

\[
S_3^{eq} = \sum_l (2l + 1) C_{eq} = \tilde{f}_{NL}^{eq} \sum_l \sum_{l'} \sum_{l''} \left\{ \frac{B_{l' l''}^{eq} \tilde{B}_{l' l''}^{eq}}{C_{l'} C_{l''}} \right\}.
\] (29)

Individual contributions to the final skewness following relations, which can be useful diagnostics for numerical checks:

\[
\sum_l (2l + 1) C_{eq}^{AB^2} = \sum_l (2l + 1) C_{eq}^{AB, B}; \quad \sum_l (2l + 1) C_{eq}^{B, CD} = \sum_l (2l + 1) C_{eq}^{C, BD} = \sum_l (2l + 1) C_{eq}^{D, BC}.
\] (30)
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Given the signal-to-noise ratio of current estimates of $S_\delta$ or equivalently $f_{NL}$ from WMAP surveys, it may not be possible to use many narrow bins to evaluate the $C_2$s associated with the primordial bispectra. However, with the increase in experimental sensitivity of future CMB experiments, it will be possible to divide the $l$ range in narrower bins. The important point here is that the data must show consistency with the theoretical $C_l^{AA,eq}$ to make a convincing case that the non-gaussianity is primordial. However, these expressions are only optimal for all-sky coverage and homogeneous noise; we relax these assumptions in the next sections.

5 PARTIAL SKY COVERAGE AND NON-UNIFORM NOISE: AN APPROXIMATE TREATMENT

It was pointed out in [Babich (2005), Creminelli et al. (2006), Yadav et al. (2008)] that in the presence of partial sky coverage, e.g. due to the presence of a mask or because of galactic foregrounds and bright point sources, as well as, in the case of non-uniform noise, spherical symmetry is destroyed. The estimator introduced above will then have to be modified by adding terms linear in the observed map. The linear terms therefore are constructed from correlating the Monte-Carlo (MC) averaged $\langle A(n, r)B(n, r)\rangle_{\text{sim}}$ product maps with the input $B$ map. The mask and the noise that are used in constructing the Monte-Carlo averaged product map are exactly same as the observed maps and the ones derived from them such as $A$ or $B$.

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$$S_{\text{loc}}^{\text{linear}} = -\frac{1}{f_{\text{sky}}} \int d\Omega \left\{ 2B(r, \Omega)\langle A(r, \Omega)B(r, \Omega)\rangle_{\text{sim}} + A(r, \Omega)\langle B^2(r, \Omega)\rangle_{\text{sim}} \right\}$$

(31)

The estimator includes a cubic term, which by matched-filtering maximises the response for a specific type of input map bispectrum. The general expression for the bispectrum estimator was developed by Babich (2005) for arbitrary sky coverage and inhomogeneous noise.

$$C_l^{\text{loc}} = \frac{1}{f_{\text{sky}}} \left\{ C_l^{A^2} - 2C_l^{(A,B)B} - C_l^{A,(B^2)} \right\} + \frac{2}{f_{\text{sky}}} \left\{ C_l^{AB,B} - C_l^{(AB),B} - C_l^{B,(AB)} - C_l^{A(B,B)} \right\}$$

(33)

where $f_{\text{sky}}$ is the sky fraction observed.

The $C_2$s such as $C_l^{(AB),B}$ describe the cross-power spectra associated with Monte-Carlo averaged product maps $\langle A(n, r)B(n, r)\rangle_{\text{sim}}$ constructed with the same mask and the noise model as the the observed map $B$. Likewise, the term $C_l^{(AB),B}$ denotes the average cross-correlation computed from MC averaging, of the product map constructed from the observed map $A(\Omega, r)$ multiplied with a MC realisation of map $B(\Omega, r)$ against the same MC realisation $B(\Omega, r)$.

$$C_l^{AB} = -\frac{18}{f_{\text{sky}}} \left\{ C_l^{A^2,B} - 2C_l^{(A,B),B} - C_l^{A,(B^2)} \right\} + \frac{2}{f_{\text{sky}}} \left\{ C_l^{AB,B} - C_l^{(AB),B} - C_l^{B,(AB)} - C_l^{A(B,B)} \right\}$$

$$+ \frac{2}{f_{\text{sky}}} \left\{ C_l^{D^2} - 2C_l^{(D,D),D} - C_l^{D,(D^2)} \right\} - \frac{2}{f_{\text{sky}}} \left\{ C_l^{B,(CD)} - 2C_l^{(B,C),D} - C_l^{C,(BD)} \right\}$$

$$- \frac{2}{f_{\text{sky}}} \left\{ C_l^{D,C,B} - 2C_l^{(D,C),B} - C_l^{D,(CB)} \right\} - \frac{2}{f_{\text{sky}}} \left\{ C_l^{C,B,D} - 2C_l^{(C,B),D} - C_l^{C,(BD)} \right\}.$$  

(34)

Creminelli et al. (2006) showed via numerical analysis that the linear terms are less important in the equilateral case than in the local model. The use of such Monte-Carlo maps to model the effect of mask and noise greatly improves the speed compared to full bispectrum analysis.

The use of linear terms was found to greatly reduce the scatter of the estimator, thereby improving its optimality. The estimator was used in [Yadav & Wandelt (2008)] also to compute the $f_{NL}$ from combined $T$ and $E$ maps. The analysis presented above for both the one-point statistics and the power-spectral analysis is approximate, because it uses a crude $f_{\text{sky}}$ approximation to deconvolve the estimated power spectrum to compare with analytical prediction. A more accurate analysis should take into account the mode-mode coupling which can dominate at low $l$. The general expression which includes the mode-mode coupling will be presented in the next section. However it was found out by [Yadav & Wandelt (2008)] that removing low $l$s from the analysis can be efficient way to bypass the mode-mode coupling. A complete numerical treatment for the case of two-point statistics such as $C_l^{A,B^2}$ will be presented elsewhere.

6 GENERALISATION OF OPTIMAL ESTIMATORS FOR REALISTIC SURVEY STRATEGY: EXACT ANALYSIS

The general expression for the bispectrum estimator was developed by Babich (2005) for arbitrary sky coverage and inhomogeneous noise. The estimator includes a cubic term, which by matched-filtering maximises the response for a specific type of input map bispectrum. The linear terms vanish in the absence of anisotropy but should be included for realistic noise to reduce the scatter in the estimates (see Babich 2005 for details). We define the estimator as optimal:

$$\hat{E}_L[a] = \sum_{L'} \left\{ \left[ N^{-1} \right]_{LL'} \right\} \sum_{M'M''} \sum_{l'm'lm} B_{L'M''} \left( L' M' \ l' m' \ l \ m \right)$$

$$\times \left\{ (C_l^{-1})_{L'M',l'm} a_{l'1}(m_1) (C_m^{-1})_{l'm'2} a_{l2}(m_2) (C_n^{-1})_{l'm'm3} a_{l3}(m_3) \right\}$$
where $N_{LL'}$ is a normalization constant to be discussed later. A factor of $1/(2l+1)$ can be introduced with the sum $\sum_{M}$, if we choose not to introduce the $N_{LL'}$ normalization constant. This will make the estimator equivalent to the one introduced in the previous section. Clearly as the data is weighted by $C^{-1} = (S + N)^{-1}$, or the inverse covariance matrix, addition of higher modes will reduce the variance of the estimator. In contrast, the performance of sub-optimal estimators can degrade with resolution, due to the presence of inhomogeneous noise or a galactic mask. However, a wrong noise covariance matrix can not only make the estimator sub-optimal but it will make the estimator biased too. The noise model will depend on the specific survey scan strategy. Numerical implementation of such inverse-variance weighting or multiplication of a map by $C^{-1}$ can be carried out by conjugate gradient inversion techniques. Taking clues from Smith and Zaldarriaga (2006), we extend their estimators for the case of the bispectrum-related power spectrum. We will be closely following their notation whenever possible. First we define $\text{QL}[a]$ and its derivative $\partial_{\text{QL}}[a]$. The required input harmonics $a_{lm}$ are denoted as $a_{l}^{m}$.

$$Q_{L}[a] \equiv \sum_{M} a_{LM} \sum_{(l',m')}(B_{LM'}^{l'})(m' m')(\sum_{(l''m'')}a_{l'm'}a_{l''m''})$$

(36)

$$\partial_{\text{QL}} Q_{L}[a] \equiv \delta_{LL'} \sum_{(l',m')} (B_{LM'}^{l'})(m' m')(\sum_{(l''m'')}a_{l'm'}a_{l''m''} + 2 \sum_{M} a_{LM} \sum_{(l',m')} B_{LM'}^{l'} a_{l'm'} a_{l'm'})$$

(37)

These expressions differ from that of one-point estimators by the absence of an extra summation index. $Q_{L}[a]$ therefore represents a map as well as $\partial_{\text{QL}} Q_{L}[a]$, however $Q_{L}[a]$ is cubic in input maps $a_{lm}$, whereas as $\partial_{\text{QL}} Q_{L}[a]$ is quadratic in input.

The bispectrum-related power spectrum can then be written as (summation convention for the next two equations):

$$\tilde{E}_{L}[a] = \langle N^{-1}_{LL'} \rangle \{Q_{L}[C^{-1} a] - \langle C^{-1} a \rangle \langle \partial_{l} Q_{L}[C^{-1} a] \rangle \}_{M C}$$

(38)

Here $\langle \ldots \rangle_{M C}$ denotes the Monte-Carlo averages. The inverse covariance matrix in harmonic domain $C_{-1,m_1,2,m_2}^{-1} \equiv \langle a_{1,m_1} a_{2,m_2} \rangle^{-1}$ encodes the effects of noise and the mask. For all sky and signal-only limit, it reduces to the usual $C^{-1}_{1,m_1,2,m_2} = \frac{1}{l(l+1)} \delta_{lm}$. The normalisation of the estimator which ensures unit response can be written as:

$$N_{LL'} = \frac{1}{3} \left\{ \langle \partial_{l_1} Q_{L}[C^{-1} a] \rangle \right\}_{m_2}^{-1} C_{1,m_1,2,m_2}^{-1} \left\{ \partial_{l_2} Q_{L}[C^{-1} a] \right\}_{m_2}^{-1} C_{1,m_1,2,m_2}^{-1} \left\{ \partial_{l_3} Q_{L}[C^{-1} a] \right\}_{m_2}^{-1} C_{1,m_1,2,m_2}^{-1} \left\{ \partial_{l_4} Q_{L}[C^{-1} a] \right\}_{m_2}^{-1} C_{1,m_1,2,m_2}^{-1}$$

(39)

We will be using the following identity in our derivation:

$$\langle [C^{-1} a]_{1,m_1} [C^{-1} a]_{2,m_2} \rangle = C_{-1,m_1,2,m_2}^{-1} C_{1,m_1,2,m_2}^{-1}$$

(40)

The Fisher matrix, encapsulating the errors and covariances on the $E_{L}$ for a general survey associated with a specific form of bispectrum can finally be written as:

$$F_{LL'} = \sum_{M' M} \sum_{l_1 m_1} B_{LM}^{l_1} B_{LM'}^{l_1} \left( \begin{array}{ccc} \frac{1}{6} \{ C_{-1,m_1,2,m_2}^{-1} C_{1,m_1,2,m_2}^{-1} \} C_{2,m_1,2,m_2}^{-1} + 4 C_{-1,m_1,2,m_2}^{-1} C_{1,m_1,2,m_2}^{-1} C_{2,m_1,2,m_2}^{-1} \} \right)$$

(41)

Using the following expressions which are extension of Smith and Zaldarriaga (2006), we find that the Fisher matrix can be written as sum of two $\alpha$ terms $\alpha_{PP}$ and $\alpha_{QQ}$. The alpha terms correspond to coupling only of modes that appear in different $3j$ symbols. Self couplings are represented by the beta terms. The subscripts describes the coupling of various $l$ and $L$ indices. The subscript $PP$ correspond to coupling of free indices, i.e one free index $L_1$ with another free index $L_2$ and similar coupling for indices that are summed over such as $l_1$, $l_2$ etc. Similarly for subscript $QQ$ the free indices are coupled with summed indices. Couplings are represented by the inverse covariance matrices in harmonic domain e.g. $C^{-1}_{1,m_1,2,m_2}$ denotes coupling of mode $LM$ with $lm$.

$$\alpha_{L_1 L_2}^{PP} = \sum_{M_{1}, M_{2}} \sum_{l_{1}, l_{1}'} B_{L_{1} l_{1} l_{1}'} B_{L_{2} l_{2} l_{2}'} \left( \begin{array}{ccc} \frac{1}{6} \{ C_{-1,m_1,2,m_2}^{-1} C_{1,m_1,2,m_2}^{-1} \} C_{2,m_1,2,m_2}^{-1} + 4 C_{-1,m_1,2,m_2}^{-1} C_{1,m_1,2,m_2}^{-1} C_{2,m_1,2,m_2}^{-1} \} \right)$$

(42)

$$\alpha_{L_1 L_2}^{QQ} = \sum_{M_{1}, M_{2}} \sum_{l_{1}, l_{1}'} B_{L_{1} l_{1} l_{1}'} B_{L_{2} l_{2} l_{2}'} \left( \begin{array}{ccc} \frac{1}{6} \{ C_{-1,m_1,2,m_2}^{-1} C_{1,m_1,2,m_2}^{-1} \} C_{2,m_1,2,m_2}^{-1} + 4 C_{-1,m_1,2,m_2}^{-1} C_{1,m_1,2,m_2}^{-1} C_{2,m_1,2,m_2}^{-1} \} \right)$$

(43)

$$\alpha_{L_1 L_2}^{PP} = \langle L_{1} l_{1} l_{1}' \rangle \langle L_{2} l_{2} l_{2}' \rangle; \quad \alpha_{L_1 L_2}^{QQ} = \langle L_{1} l_{1} l_{1}' \rangle \langle L_{2} l_{2} l_{2}' \rangle$$

(44)

These results will reduce to those of Smith and Zaldarriaga (2006) when further summations over $L_1$ and $L_2$ are introduced to collapse the two-point object to the corresponding one-point quantity. The beta terms that denote cross-coupling can be written as:

$$\beta_{L_1 L_2}^{PP} = \sum_{M_{1}, M_{2}} \sum_{l_{1}, l_{1}'} B_{L_{1} l_{1} l_{1}'} B_{L_{2} l_{2} l_{2}'} \left( \begin{array}{ccc} \frac{1}{6} \{ C_{-1,m_1,2,m_2}^{-1} C_{1,m_1,2,m_2}^{-1} \} C_{2,m_1,2,m_2}^{-1} + 4 C_{-1,m_1,2,m_2}^{-1} C_{1,m_1,2,m_2}^{-1} C_{2,m_1,2,m_2}^{-1} \} \right)$$

(45)
The optimal weighting can be replaced by an arbitrary weight or no weighting at all ($R = I$). However, in this case the estimator though unbiased

\[ \text{No summation over repeated indices is assumed. Using these expressions one can finally show that} \]

\[ [F^{-1}]_{LL'} = \langle \hat{E}_L \hat{E}_{L'} \rangle - \langle \hat{E}_L \rangle \langle \hat{E}_{L'} \rangle = \langle AA + BB + CC + 2AB + 2BC + 2AC \rangle_{LL'} \]

where

\[ A_{LL'} = \left\{ \frac{2}{36} \alpha_{PP}^{PO} + \frac{4}{36} \beta_{PP}^{PO} + \frac{2}{36} \beta_{PP}^{PO} + \frac{4}{36} \beta_{PP}^{PO} + \frac{4}{36} \beta_{QQ}^{PO} \right\}; \]

\[ BB_{LL'} = \beta_{QQ}^{PP}; \quad CC_{LL'} = 4 \beta_{QQ}^{PP} \]  

\[ 2AB_{LL'} = -2(\beta_{PP}^{PP} + 2 \beta_{QQ}^{PP}); \quad 2AC_{LL'} = -4(2 \beta_{QQ}^{PP} + \beta_{LL'}^{PP}); \quad 2BC_{LL'} = 4 \beta_{PP}^{PP} \}

The final expression can be written in terms of only $\alpha$ terms as the $\beta$ terms cancel out:

\[ F_{LL'} = \left\{ \frac{2}{36} \alpha_{PP}^{PP} + \frac{4}{36} \alpha_{PP}^{QQ} \right\}. \]

If we sum over $LL'$ the Fisher matrix reduces to a scalar $F = \sum_{LL'} F_{LL'}$ with, $\alpha_{PP}^{PO} = \alpha_{PP}^{QQ} = \alpha$ and $\beta_{PP}^{PO} = \beta_{PP}^{QQ} = \beta_{QQ}^{QQ} = \beta$, where $\alpha$, $\beta$ and $F$ are exactly the same as introduced in [Smith & Zaldarriaga (2006)].

### 6.1 Joint Estimation of Multiple bispectrum-related Power-Spectra

The estimation technique described above can be generalised to take cover the bispectrum-related power spectrum associated with different set of bispectra ($X$, $Y$):

\[ \hat{E}_L[a] = [F^{-1}]_{XX'} \left\{ Q_L^Y | [C^{-1} a] - [C^{-1} a]|_{\text{m}} \langle \partial_{m} Q_{L'}^Y | [C^{-1} a] \rangle_{\text{MC}} \right\}. \]

The associated Fisher matrix now will consist of sectors $F_{XX'}^{YY} F_{XX'}^{YY}$ and $F_{XX'}^{YY}$. The sector $XX$ and $YY$ will in general be related to errors associated with estimation of bispectra of $X$ and $Y$ types, whereas the sector $XY$ will correspond to their cross-correlation.

\[ F_{XX'}^{XY} = \left\{ \frac{2}{36} [\alpha_{PP}^{PO}]^{XY} + \frac{4}{36} [\alpha_{PP}^{QQ}]^{XY} \right\} \]

where we have:

\[ [\alpha_{PP}^{PO}]^{XY} = \sum_{MM'} \sum_{l_1, l_1'} B_{L_1, l_1}^B B_{L_1', l_1'}^B \left( \frac{L}{M} \frac{l_1}{m_1} \frac{l_1'}{m_1'} \right) \left( \frac{L'}{M'} \frac{l_2}{m_2} \frac{l_2'}{m_2'} \right) C_{L,MM',l_2m_2, l_2'm_2'}^{LM,L'M',l_1m_1, l_1'm_1'} \]

and a similar expression holds for $[\alpha_{QQ}^{QQ}]^{XY}$.

### 6.2 Generalisation to non-optimal weights

Although the estimator as constructed is fully optimal - its true usefulness is determined by the affordability of the construction of the $C^{-1}$ matrix as well as the availability of a fast method to multiply it with CMB maps in a Monte Carlo chain. A more general class of estimator which is sub-optimal can be constructed by replacing the inverse covariance weighting of the data $[C^{-1} a]$ by $[Ra]$, where $[R]$ is an arbitrary filter function. In this case the estimator with unit response can be written as:

\[ \hat{E}_L[R] = \sum_{LL'} [F^{-1}]_{LL'} \left\{ Q_L[R] - [Ra]_{\text{m}} \langle \partial_{m} Q_{L'}[Ra] \rangle_{\text{MC}} \right\} \]

where the normalisation and the variance of the estimator can be constructed in a similar manner:

\[ F_{LL'} = \langle \hat{E}_L \rangle \langle \hat{E}_{L'} \rangle^{-1} = \sum_{LL'} [\langle \hat{E}_L \hat{E}_{L'} \rangle^{-1}]_{\text{MC}} \]

\[ = \frac{1}{3} \left\{ \langle \partial_{l_1m_1} Q_l[R] \rangle \langle FCF \rangle_{l_1m_1,l_2m_2} \langle \partial_{l_1m_1} Q_{l'}[Ra] \rangle \right\} \sum_{LL'} [\langle \hat{E}_L \hat{E}_{L'} \rangle^{-1}]_{\text{MC}} \]

The optimal weighting can be replaced by an arbitrary weight or no weighting at all ($R=I$). However in this case the estimator though unbiased clearly becomes a sub-optimal one.
6.3 Recovery of all-sky homogeneous noise model

In the all-sky limit we recover the usual expression:

\[ F_{LL'} = \frac{1}{36} \left\{ 2 \sum_{ll'} \frac{B_{LL'}^{2}}{C_l C_{l'} C_{l''}} \delta_{LL'} + 4 \sum_{l} \frac{B_{LL'l}^{2}}{C_l C_{l'} C_{l''}} \right\}. \] (58)

In the case of joint analysis as before we can write down the off-diagonal blocks of the Fisher matrix as:

\[ F_{XY}^{LL'} = \frac{1}{36} \left\{ 2 \sum_{ll'} \frac{B_{XY}^{2}}{C_l C_{l'} C_{l''}} \delta_{LL'} + 4 \sum_{l} \frac{B_{XY}^{2}}{C_l C_{l'} C_{l''}} \right\}. \] (59)

For \( X = Y \) we recover the diagonal blocks of the Fisher matrix for the independent estimations derived before. The errors for independent estimates are given by \( \sqrt{\langle F_{XX}^{-1} \rangle} \), where as for the joint estimation the errors are \( \sqrt{\langle F_{XY}^{-1} \rangle} \).

6.4 More general bispectra

A more general bispectrum can be written as a sum of individual product terms:

\[ b_{l_1 l_2 l_3} = \frac{1}{6} \sum_{i} A_{l_1}^{i} B_{l_2}^{i} C_{l_3}^{i} + \text{symm.perm.} \] (60)

This can be seen as a generalization of the type of bispectra introduced for the equilateral case. It may also possible to approximate the contribution of all terms will constitute the final Fisher matrix:

\[ \sum_{i} w_i A_{l_1}^{i} B_{l_2}^{i} C_{l_3}^{i} + \text{symm.perm.} \] (61)

Clearly significant computational gain can only be achieved if \( N_{opt} \ll N_{fact} \). In the following discussion we generalise the description in previous sections to such composite bispectra. Following the same analytical reasoning we can show that the Fisher matrix elements for such a composite bispectrum can be written as:

\[ [F_{XY}^{XX}]_{ij} = \frac{1}{36} \left\{ 2 \sum_{ll'} \frac{(2L+1)(2l+1)(2l'+1)}{144\pi} \left( \begin{array}{ccc} L & l & l' \\ 0 & 0 & 0 \end{array} \right) \right\}^{2} \frac{1}{C_l C_{l'} C_{l''}} \left[ A_{l_1}^{i} B_{l_2}^{i} C_{l_3}^{i} + \ldots \right]^{X} \left[ A_{l_1}^{j} B_{l_2}^{j} C_{l_3}^{j} + \ldots \right]^{Y} 
+ 4 \sum_{l} \left( \frac{(2L+1)(2L'+1)(2l+1)}{144\pi} \left( \begin{array}{ccc} L & l' & l \\ 0 & 0 & 0 \end{array} \right) \right)^{2} \frac{1}{C_l C_{l'} C_{l''}} \left[ A_{l_1}^{i} B_{l_2}^{i} C_{l_3}^{i} + \ldots \right]^{X} \left[ A_{l_1}^{j} B_{l_2}^{j} C_{l_3}^{j} + \ldots \right]^{Y}. \] (62)

The symbols \( A^i, B^j \) etc denotes the \( i \)-th term in the factorised representation of a specific type of the bispectrum of type \( X \) or \( Y \). The total contribution of all terms will constitute the final Fisher matrix:

\[ F_{XY}^{XX} = \sum_{i j} [F_{XY}^{XX}]_{ij}; \quad F^{XY} = \sum_{LL'} F_{XY}^{XX}. \] (63)

Using the following identity we can project this expression onto real-space:

\[ \int_{-1}^{1} dz P_{1}(z) P_{2}(z) P_{3}(z) = 2 \left( \begin{array}{ccc} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{array} \right)^{2} \] (64)

\[ [F_{LL'}]_{ij} = \int_{-1}^{1} dz \left[ \xi_{L}^{A^{i} A^{j}}(z) \xi_{L}^{B^{i} B^{j}}(z) \xi^{C^{i} C^{j}}(z) + \ldots \right] \delta_{LL'} + \left[ \xi_{L}^{A^{i} A^{j}}(z) \xi_{L}^{B^{i} B^{j}}(z) \xi^{C^{i} C^{j}}(z) + \ldots \right] \] (65)

The first term describes the diagonal entries of the Fisher matrix and the second term relates to the off-diagonal terms.

\[ \xi_{L}^{A^{i} A^{j}}(z) = \sum_{l} \xi_{l}^{A^{i} A^{j}}(z); \quad \text{where} \quad \xi_{l}^{A^{i} A^{j}}(z) = \frac{2l + 1}{4\pi} \frac{A_{l}^{A^{i} A^{j}}}{C_{l}^{2}} P_{l}(z). \] (66)

In case of cross-correlational studies the \( A^{i} \) and \( A^{j} \) will come from two different factorization of distinct bispectra denoted before by \( X \) and \( Y \). The case of weighted sum can also be derived in an exactly similar manner.

7 MODELS FOR NON-GAUSSIANITY

We will use two commonly-used specific models for the primordial non-Gaussianity as well as one foreground source of contamination i.e. extra-galactic point sources to demonstrate the power of our statistics in this section.
7.1 Local or squeezed Model

Using the specific form for $b_{l_1l_2l_3}^{\alpha\beta}$, the Fisher matrix element for the local model can be expressed in terms of the functions $\alpha$ and $\beta$ and the power spectrum $C_l$.

\[ \alpha_{L_1L_2}^{PP} = \frac{f_{NL}^2}{4\pi} (2L_1 + 1)(2L_2 + 1) \sum_l \begin{pmatrix} L_1 & L_2 & l \\ 0 & 0 & 0 \end{pmatrix}^2 \frac{1}{C_{L_1}C_{L_2}C_l} \times \left\{ \int r^2 dr (2\alpha_{L_1}(r)\beta_{L_2}(r)\beta_l(r) + \alpha_l(r)\beta_{L_1}(r)\beta_{L_2}(r)) \right\}^2 \]

\[ \alpha_{L_1L_2}^{QQ} = \frac{f_{NL}^2}{4\pi} (2L_1 + 1)(2L_2 + 1) \sum_{l_1 l_2} \begin{pmatrix} L_1 & l_1 & l_2 \\ 0 & 0 & 0 \end{pmatrix}^2 \frac{1}{C_{L_1}C_{L_1}C_{L_2}} \times \left\{ \int r^2 dr (\alpha_{L_1}(r)\beta_{l_1}(r)\beta_{l_2}(r) + \alpha_{l_1}(r)\beta_{l_2}(r)\beta_{L_1}(r) + \alpha_{l_2}(r)\beta_{l_1}(r)\beta_{L_2}(r)) \right\}^2 \]

7.2 Equilateral Model

Using the equilateral model the contribution to the Fisher matrix can be expressed as:

\[ \alpha_{L_1L_2}^{PP} = \frac{f_{NL}^2}{4\pi} (2L_1 + 1)(2L_2 + 1) \sum_l (2l + 1) \begin{pmatrix} L_1 & l_1 & l_2 \\ 0 & 0 & 0 \end{pmatrix}^2 \frac{1}{C_{L_1}C_{L_2}C_l} \times \left\{ \int r^2 dr (-\alpha_{L_1}(r)\beta_{L_2}(r)\beta_l(r) + \delta_{L_1}(r)\delta_{L_2}(r)\delta_l(r) + \beta_{L_1}(r)\gamma_{L_2}(r)\delta_l(r) + \text{cyc.perm.}) \right\}^2 \]

\[ \alpha_{L_1L_2}^{QQ} = \frac{f_{NL}^2}{4\pi} \sum_{l_1 l_2} (2l_1 + 1)(2l_2 + 1) \begin{pmatrix} L_1 & l_1 & l_2 \\ 0 & 0 & 0 \end{pmatrix}^2 \frac{1}{C_{L_1}C_{L_1}C_{L_2}} \times \left\{ \int r^2 dr (-\alpha_{L_1}(r)\beta_{l_1}(r)\beta_{l_2}(r) + \delta_{L_1}(r)\beta_{l_2}(r)\delta_{l_1}(r) + \beta_{L_1}(r)\gamma_{l_1}(r)\gamma_{l_2}(r) + \text{cyc.perm.}) \right\}^2 \]

The power spectra $C_l$ appearing in the denominator take contributions both from the pure signal or CMB and the detector noise. It is possible to bin the estimates in large enough bins to report uncorrelated estimates which may be possible for an experiment such as Planck with very high sky-coverage. A detailed analysis of the singularity structure of the error-covariance matrix will be presented elsewhere.

7.3 Point Sources

The bispectrum from residual point-sources which are assumed random can be modelled as $b_{l_1l_2l_3} = b_{ps}$. The exact value depends on the flux limit and the mask used in the survey. The accuracy of such an approximation can indeed be extended by adding contributions from correlation terms.

\[ \alpha_{L_1L_2}^{PP} = \frac{b_{ps}^2}{4\pi} (2L_1 + 1)(2L_2 + 1) \sum_l \begin{pmatrix} L_1 & L_2 & l \\ 0 & 0 & 0 \end{pmatrix}^2 \frac{1}{C_{L_1}C_{L_2}C_l} \]

\[ \alpha_{L_1L_2}^{QQ} = \frac{b_{ps}^2}{4\pi} (2L_1 + 1) \sum_{l_1 l_2} \begin{pmatrix} L_1 & l_1 & l_2 \\ 0 & 0 & 0 \end{pmatrix}^2 \frac{1}{C_{L_1}C_{L_1}C_{L_2}} \]

Similar computations can be done for cross-correlation among various contributions, e.g. contamination due to point sources or estimation of a specific type of non-Gaussianity. A joint estimation is useful for finding out also the level of cross-contamination from one theoretical model while another is being estimated.

8 CONCLUSIONS

We have addressed the problem of finding an estimator of primordial non-Gaussianity from microwave background data. The new feature of this analysis is that the technique presented here allows one to make an assessment of whether any non-Gaussian signal is primordial or not. The issue here is that if one finds an estimate of a level of primordial non-Gaussianity which is inconsistent with zero, then it is very difficult at present to make a convincing case that it is indeed primordial and not simply contamination by any number of other effects which might lead to a non-Gaussian CMB map. The method does this by performing a less aggressive data compression than previous analyses. Rather than compressing the data to a single number (typically an estimate of $f_{NL}$), it reduces the data to a function, the bispectrum-related power spectrum. This is an average cross-power spectrum of certain maps constructed from the CMB data. By doing this construction, one retains the ability to assess the contributions from different sources, such as residual point sources, incomplete foreground subtraction...
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and so on (see e.g. Serra & Cooray (2008)). As an example, we have computed the expected bispectrum-related power spectrum optimised for local non-gaussianity (we also consider the equilateral type), and calculated the contribution expected from an unclustered population of point sources. Indeed, one can use standard statistical methods to estimate the amplitude of components of non-Gaussianity, and since these contributors have quite different harmonic dependences, the estimators will be largely decoupled. The power of the technique will depend on the level of the primordial signal, but if it is at the level claimed by Yadav & Wandelt (2008), then it will be possible with Planck data to construct a large number of band-power estimates of the bispectrum-related power spectrum to see if it is primordial. We also include polarisation in the analysis (see appendix). The work draws extensively on previous studies, in particular generalising the Komatsu, Spergel & Wandelt (2005) analysis for the all-sky, homogeneous noise case. For the more realistic case of partial sky coverage and inhomogeneous noise, we present optimised estimators of the bispectrum-related power spectrum, including linear terms and extending the work of Babich (2008); Smith & Zaldarriaga (2006); Smith, Senatore & Zaldarriaga (2009). For studies such as this, it is normal to assume the background cosmology is known from the power spectrum, but uncertainties will propagate into the $f_{NL}$ estimates (Liguori & Riotto 2008) and will be investigated elsewhere. Note that the techniques here could be generalised to higher-order statistics such as the trispectrum, should the bispectrum vanish for some symmetry reason.

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APPENDIX A: JOINT ANALYSIS OF TEMPERATURE AND POLARIZATION

Most current constraints on non-Gaussianity still come from temperature maps, but with WMAP and Planck, the situation is changing. Similar calculations can be performed for joint temperature and $E$-type polarisation analysis, and the estimators discussed above can be generalised to include $E$-type polarisation to tighten the constraints. The functions $\alpha_l$ and $\beta_l$ that we discussed in the main text needs to be generalised for both $T$, temperature and $E$-type polarisation. We follow the discussion in Yadav, Komatsu, & Wandelt (2007), see also Babich & Zaldarriaga (2004), Liguori et al. (2007) for related discussions.

\[
\alpha_l^X (r) \equiv \frac{2}{\pi} \int_0^\infty k^2 dk P(k) \Delta^X_l(k) j_l(kr) \quad \beta_l^X (r) \equiv \frac{2}{\pi} \int_0^\infty k^2 dk \Delta^X_l(k) j_l(kr). \tag{A1}
\]

The index $X$ can be $T$ or $E$. Similar calculations can in principle be performed for the equilateral case. We will focus however here only on the local or squeezed model. The power spectra now can be a temperature $(TT)$-only power spectrum or an Electric-Electric $(EE)$ polarization spectra which we define below, with specific choice of $\Delta_l$.

\[
C_l^{XY} (r) \equiv \frac{2}{\pi} \int_0^\infty k^2 dk P(k) \Delta^X_l(k) \Delta^Y_l(k). \tag{A2}
\]

We arrange the all-sky covariance matrix (in the harmonic domain) in a matrix, which takes the following form in terms of the $C_l$ defined above:

\[
[C]_l = \begin{bmatrix} C_l^{TT} & C_l^{TE} \\ C_l^{TE} & C_l^{EE} \end{bmatrix}
\]

(A3)

We can now construct the $A_l(r, \hat{\Omega})$ and $B_l(r, \hat{\Omega})$ fields now can be constructed using the full covariance matrix instead of only temperature data.

\[
A_l(r, \hat{\Omega}) = \sum_{l_m} \sum_{l_p} \left[ C_l^{-1} \right]^{ij}_p a_{l_m}^i \alpha_l^j (r) Y_{l_m}(\hat{\Omega}) \tag{A4}
\]

\[
B_l(r, \hat{\Omega}) = \sum_{l_m} \sum_{l_p} \left[ C_l^{-1} \right]^{ij}_p a_{l_m}^i \beta_l^j (r) Y_{l_m}(\hat{\Omega}). \tag{A5}
\]

The construction now follows exactly the same steps as depicted for the temperature case. We compute the cross-correlation of $B_l^2(r, \hat{\Omega})$ with the $A_l(r, \hat{\Omega})$. The $B_l^2(r, \hat{\Omega})$ can be decomposed in terms of the $T$ and $E$ harmonics both $a_{l_m}^i$ ($j$ here takes values $T$ or $E$).

\[
B_l^{(2)}(r) = \int d\Omega B_l^2(r, \hat{\Omega}) Y_{l_m}(\hat{\Omega}) \]

\[
= \sum_{l_{i}', m_{i}'} \sum_{l_{i}'' m_{i}''} \beta_{l_{i}'}(r) \left[ C_{l_{i}'}^{-1} \right]^{ij}_{i'} \beta_{l_{i}''}(r) \left[ C_{l_{i}''}^{-1} \right]^{ kr}_{i''} \sqrt{\frac{(2l_{i}')!(2l_{i}' + 1)(2l_{i}'')!(2l_{i}'' + 1)}{4\pi}} \left( \begin{array}{ccc} l_{i}' & l_{i}'' & 0 \\ m & m' & m'' \end{array} \right) a_{l_{i}'}^i a_{l_{i}''}^j a_{l_{i}'}^k a_{l_{i}''}^l. \tag{A6}
\]

The cross-correlation of the associated $A$ and $B$ field will contain information both from temperature and polarization maps.

\[
C_l^{AB} = \int r^2 dr C_l^{A} r^2 dr B_l^2(r) = \int \frac{1}{2l + 1} \sum_{l_{i}' m_{i}'} \sum_{l_{i}'' m_{i}''} \int r^2 dr \alpha_{l_{i}'}(r) \left[ C_{l_{i}'}^{-1} \right]^{ij}_{i'} \beta_{l_{i}''}(r) \left[ C_{l_{i}''}^{-1} \right]^{ kr}_{i''} \sqrt{\frac{(2l_{i}')!(2l_{i}' + 1)(2l_{i}'')!(2l_{i}'' + 1)}{4\pi}} \left( \begin{array}{ccc} l_{i}' & l_{i}'' & 0 \\ m & m' & m'' \end{array} \right) a_{l_{i}'}^i a_{l_{i}''}^j a_{l_{i}'}^k a_{l_{i}''}^l. \tag{A7}
\]

\[
B_{l_{i}'}^{ij} = \sum_{m_{i}'} \left( \begin{array}{ccc} l_{i}' & l_{i}'' & m_{i}'' \\ m & m' & m'' \end{array} \right) a_{l_{i}'}^i a_{l_{i}''}^j a_{l_{i}'}^k a_{l_{i}''}^l \tag{A8}
\]
The mixed-bispectrum $B_{l_0l_1l_2}^{pqr}$, contains information about three-point correlation in harmonic space with the possibility that the harmonics $a_{im}^r$ can either be of $T$ or temperature type or $E$ or electric-polarisation type. The theoretical model for such bispectrum depends on functions $\alpha^r_l(r)$ and $\beta^r_l(r)$ where again depending on the superscript the functions can be of temperature or the electric type.

$$B_{l_0l_1l_2}^{pqr} = \sqrt{(2l_0 + 1)(2l_1 + 1)(2l_2 + 1)} \frac{4\pi}{4\pi} \left( \sum_{l_0} \sum_{l_1} \sum_{l_2} \right) \int r^2 dr \left\{ \beta^p_l(r) \beta^q_{l_1}(r) \alpha^r_{l_2}(r) + \beta^p_l(r) \alpha^q_{l_1}(r) \beta^r_{l_2}(r) + \alpha^p_l(r) \beta^q_{l_1}(r) \beta^r_{l_2}(r) \right\} . \tag{A9}$$

Finally in an analogous way we can express the power-spectrum related to the mixed-bispectrum as follows:

$$(2l + 1)(C_{AA,B}^{l} + 2C_{AB,B}^{l}) = \bar{f}_{NL} \sum_{l_0} \sum_{l_1} \sum_{l_2} \left\{ B_{l_0l_1l_2}^{ijk} \left[ C^{-1}_{l_0} \right]_{l_0} \left[ C^{-1}_{l_1} \right]_{l_1} \left[ C^{-1}_{l_2} \right]_{l_2} B_{l_0l_1l_2}^{pqr} \right\} . \tag{A10}$$

We have assumed summation of repeated indices which denote the polarisation types, i.e. $i, j, k$ and $p, q, r$ in the above expression. The one-point mixed skewness which is related to the above power spectrum is analogously written as follows:

$${S}_{T,E}^{prim} = \sum_{l}(2l + 1)(C_{AA,B}^{l} + 2C_{AB,B}^{l}) = \bar{f}_{NL} \sum_{l_0} \sum_{l_1} \sum_{l_2} \left\{ B_{l_0l_1l_2}^{ijk} \left[ C^{-1}_{l_0} \right]_{l_0} \left[ C^{-1}_{l_1} \right]_{l_1} \left[ C^{-1}_{l_2} \right]_{l_2} B_{l_0l_1l_2}^{pqr} \right\} . \tag{A11}$$

This generalises the temperature-only power spectrum estimator introduced in the text of this paper and can be extended to take into account other models, sky-coverage and secondary anisotropies in an analogous manner.