A note on scaling limits for truncated birth-and-death processes with interaction

Vadim Shcherbakov* and Anatoly Yambartsev†

Abstract
In this note we consider a Markov chain formed by a finite system of interacting birth-and-death processes on a finite state space. We study an asymptotic behaviour of the Markov chain as its state space becomes large. In particular, we show that the appropriately scaled Markov chain converges to a diffusion process, and derive conditions for existence of the stationary distribution of the limit diffusion process in special cases.

1 The model
Many real life systems are multicomponent, where the evolution of an isolated single component is relatively simple, but the presence of an interaction affects both the individual behaviour of a component and the collective behaviour. Also, the time evolution of many real life systems can be often described in terms of certain birth and death events. Models of interacting birth-and-death processes on integers provide a flexible mathematical framework for modelling such systems (e.g. see [9], [10], [12], [13] and references therein) that appear in biology, physics, queueing and other applications. Frequently, there are natural limitations on the system size (e.g. limited resources in biological systems, restrictions of the queue length in queueing etc.). On the other hand, a state space of finite system can be large. This motivation stimulates interest to finding an adequate asymptotic description of finite but large real life systems with the features described above.

*Department of Mathematics, Royal Holloway, University of London. Email address: vadim.shcherbakov@rhul.ac.uk
†Department of Statistics, University of Sao Paulo. Email address: yambar@gmail.com
This note concerns scaling limits for a stochastic model that describes a finite continuous Markov chain formed by interacting birth-and-death processes confined to a finite set. The system components (spins) are labelled by vertices of a finite connected graph and evolve subject to a local interaction determined by the graph. The interaction not necessarily symmetric. It should be noted that spin models with asymmetric interactions have recently been introduced for modelling interaction in biological systems (see [2] for detailed explanations and references therein). We are interested in the asymptotic behaviour of the Markov chain as the range of possible values of its components becomes large. We show that the appropriately scaled Markov chain can be approximated either by a diffusion process, or by a deterministic process depending on the scaling. Such scaling limits are widely used in queueing (e.g., see [1], [3], [11]), interacting particle systems (e.g., see [4], [6] and references therein) and in many other applications (e.g. in finance, [7] and [8]).

Let us describe the model. Let $\Lambda$ be a finite connected graph. Given integers $l \geq 0$ and $r > 0$ define $\Omega_{\Lambda,l,r} = \{-l, \ldots, r\}^\Lambda$. Denote by $\xi_x$, $x \in \Lambda$, components of $\xi \in \Omega_{\Lambda,l,r}$ and call them spins. We write $x \sim y$ to denote that vertices $x, y \in \Lambda$ are adjacent, and $x \not\sim y$ if they are not. Vertices $x$ and $y$ are called neighbours, if $x \sim y$. By convention, $x \sim x$ for all $x \in \Lambda$. A matrix $A = (\alpha_{xy})_{x,y \in \Lambda}$ is called an interaction matrix, if $\alpha_{xy} = 0$, whenever $x \not\sim y$. It is easy to see that any linear combination of interaction matrices is an interaction matrix. Given two interaction matrices $A_b$ and $A_d$ consider a continuous time birth-and-death Markov chain $\xi(t) \in \Omega_{\Lambda,l,r}$ evolving as follows. Given that $\xi(t) = \xi$ a spin $\xi_x < r$ increases by 1 at the rate $e^{b(x,\xi)}$, and a spin $\xi_x > -l$ decreases by 1 at the rate $e^{d(x,\xi)}$, where $b(x,\xi) = (A_b\xi)_x$ and $d(x,\xi) = (A_d\xi)_x$.

It is easy to see that various types of interaction between spins can be modelled by choosing appropriate interaction matrices $A_b$ and $A_d$. For example, if these matrices are diagonal then components of the Markov chain are independent truncated birth-and-death processes. In general, matrices $A_b$ and $A_d$ are not symmetric.

A variant of this Markov chain was considered in [9], where transition rates were specified by interaction matrices $A_b = A = (\alpha_{xy})_{x,y \in \Lambda}$, such that $\alpha_{xy} \equiv \text{const}$, and $A_d \equiv 0$, and graph $\Lambda$ was a $d$-dimensional lattice cube. It was shown that a stationary distribution of the Markov chain converges to a Gibbs measure, as $\Lambda$ expands to the whole lattice, and an occupied site percolation problem was solved for the limit distribution. The long term behaviour of the Markov chain with non-negative and unbounded components (formally obtained by setting $l = 0$ and $r = \infty$) was studied in [10]. The transition rates in [10] were specified by interaction matrices $A_d \equiv 0$ (as
in [9]) and \( A_b = \alpha E + \beta I_\Lambda \), where \( \alpha, \beta \in \mathbb{R} \), \( I_\Lambda \) is the incidence matrix of graph \( \Lambda \) and \( E \) is the unit matrix. The main goal in [10] was to determine how the long term behaviour of the Markov chain depends on both the transition parameters and the structure of the underlying graph. The model in this note is somewhat intermediate between the models in [9] and in [10]. Namely, the underlying graph is fixed and we study the asymptotic behaviour of the Markov chain, as the finite range of the spin values expands. If we formally equate \( l = -\infty \) and \( r = \infty \), then the corresponding countable continuous Markov chain can be explosive (this depends on both the interaction matrices and graph \( \Lambda \)). On the other hand, if we stretch the range of the spin values and simultaneously change the model parameters, then the Markov chain can converge to a non-trivial limit under an appropriate space-time scaling.

The diffusion limit (Theorem 2) is of a particular interest as it can be interpreted in terms of a system of interacting one-dimensional diffusions. In some cases these one-dimensional diffusions are given by famous Ornstein-Uhlenbeck processes. It is known that a single Ornstein-Uhlenbeck process on the line is positive recurrent with a stationary distribution given by a Gaussian probability density. Presence of interaction can significantly change the collective behaviour of the system. Namely, the system can become transient. This effect depends on both interacting matrices and the structure of graph \( \Lambda \) as demonstrated by examples in Section 3.

We also formulate an analogue of the law of large numbers (Theorem 1), where the limit process is described by system of ordinary differential equations.

### 2 Scaling limits

Given interaction matrices \( A_b \) and \( A_d \) consider a Markov process \( u(t) = \{ u_x(t), x \in \Lambda \} \in \mathbb{R}^\Lambda \) which is a solution of the following system of stochastic differential equations

\[
du_x(t) = (b(x, u(t)) - d(x, u(t))) dt + \sqrt{2} dW_x(t), \quad x \in \Lambda, \tag{1}
\]

\[u_x(0) = u_x, \quad x \in \Lambda,
\]

where \( b(x, \xi) = (A_b \xi)_x \) and \( d(x, \xi) = (A_d \xi)_x \), and \( W_x(t) \in \mathbb{R}, \quad x \in \Lambda \), are independent one-dimensional standard Brownian motions. Define the following matrix

\[
A = A_b - A_d = (\alpha_{xy})_{x,y \in \Lambda}. \tag{2}
\]
Equations (1) can be now rewritten as follows

\[ du_x(t) = \left( \alpha_{xx}u_x(t) + \sum_{y \neq x} \alpha_{xy}u_y(t) \right) dt + \sqrt{2}dW_x(t), \quad x \in \Lambda, \]

\[ u_x(0) = u_x, \quad x \in \Lambda, \]

or, in the following vector form

\[ d\mathbf{u}(t) = A\mathbf{u}(t)dt + \sqrt{2}d\mathbf{W}(t), \quad \mathbf{u}(0) = \mathbf{u} \in \mathbb{R}^\Lambda, \]

where \( W(t) = \{ W_x(t) \in \mathbb{R}, x \in \Lambda \} \in \mathbb{R}^\Lambda. \)

**Remark 1.** Note that if diagonal elements of matrix \( A \) are negative, i.e. \( \alpha_{xx} < 0 \) for all \( x \in \Lambda \), then diffusion process \( \mathbf{u}(t) \) can be interpreted as a system of locally interacting Ornstein-Uhlenbeck processes with individual drifts \( \alpha_{xx}, \quad x \in \Lambda \) and constant diffusion coefficients equal to \( \sqrt{2}. \)

**Theorem 1.** Given interaction matrices \( A_b \) and \( A_d \), and sequences of positive numbers \( \varepsilon_n, l_n, r_n, n \in \mathbb{N} \), consider a sequence of Markov chains \( \xi^{(n)}(t) \in \Omega_{\Lambda, l_n, r_n}, n \in \mathbb{N} \), whose transition rates are specified by interaction matrices \( \varepsilon_n^2A_b \) and \( \varepsilon_n^2A_d \). Suppose that

\[ \lim_{n \to \infty} \varepsilon_n = 0, \quad \lim_{n \to \infty} l_n\varepsilon_n = \lim_{n \to \infty} r_n\varepsilon_n = \infty, \quad \lim_{n \to \infty} \varepsilon_n\xi^{(n)}(0) = \mathbf{u} \in \mathbb{R}^\Lambda. \]

Under these conditions the sequence of rescaled Markov chains \( \varepsilon_n\xi^{(n)}(t\varepsilon_n^{-2}) \) converges as \( n \to \infty \) to a Markov process \( \mathbf{u}(t) \in \mathbb{R}^\Lambda \) which is a unique solution of equation (3) with initial condition \( \mathbf{u}(0) = \mathbf{u} \). The convergence is understood in a sense of the weak convergence of the corresponding semigroups.

The proof of Theorem 1 is given in Section 4.

**Remark 2.** Both the Markov chain and the diffusion limit are Feller Markov processes. For Feller Markov processes weak convergence is equivalent to convergence of the corresponding semigroups (Theorem 2.5, Chapter 4, [1]).

The following theorem is an analogue of the law of large numbers.

**Theorem 2.** Given interaction matrices \( A_b \) and \( A_d \), and sequences of positive numbers \( \varepsilon_n, l_n, r_n, n \in \mathbb{N} \), consider a sequence of Markov chains \( \xi^{(n)}(t) \in \Omega_{\Lambda, l_n, r_n}, n \in \mathbb{N} \), whose transition rates are specified by interaction matrices \( \varepsilon_n^2A_b \) and \( \varepsilon_n^2A_d \). Suppose that

\[ \lim_{n \to \infty} \varepsilon_n = 0, \quad \lim_{n \to \infty} l_n\varepsilon_n = \lim_{n \to \infty} r_n\varepsilon_n = \infty, \quad \lim_{n \to \infty} \varepsilon_n\xi^{(n)}(0) = \mathbf{u} \in \mathbb{R}^\Lambda. \]

Under these conditions the sequence of rescaled Markov chains \( \varepsilon_n\xi^{(n)}(t\varepsilon_n^{-2}) \) converges as \( n \to \infty \) to a Markov process \( \mathbf{u}(t) \in \mathbb{R}^\Lambda \) which is a unique solution of equation (4) with initial condition \( \mathbf{u}(0) = \mathbf{u} \). The convergence is understood in a sense of the weak convergence of the corresponding semigroups.

The proof of Theorem 2 is given in Section 4.
$\Omega_{\Lambda, l, n}, \ n \in \mathbb{N}$, whose transition rates are specified by interaction matrices $\varepsilon_n A_b$ and $\varepsilon_n A_d$. Suppose that

$$\lim_{n \to \infty} \varepsilon_n = 0, \lim_{n \to \infty} l_n \varepsilon_n = \lim_{n \to \infty} r_n \varepsilon_n = \infty, \lim_{n \to \infty} \varepsilon_n \xi^{(n)}(0) = u \in \mathbb{R}^\Lambda. \quad (5)$$

Under these conditions for every $t \geq 0$

$$\lim_{n \to \infty} \sup_{s \leq t} |\varepsilon_n \xi^{(n)}(t \varepsilon_n^{-1}) - \gamma(t)| = 0, \ a.s. \quad (6)$$

where deterministic process $\gamma(t) = (\gamma_x(t), x \in \Lambda)$ solves the following system of non-linear differential equations

$$\dot{\gamma}_x(t) = e^{b(x, \gamma(t))} - e^{d(x, \gamma(t))}, \ x \in \Lambda, \quad (7)$$

with initial conditions $\gamma(0) = u$.

**Remark 3.** Note that assumptions (4) and (5) of Theorem 1 and Theorem 2 respectively can be modified as follows

$$\lim_{n \to \infty} \varepsilon_n = 0, \lim_{n \to \infty} l_n \varepsilon_n = a, \lim_{n \to \infty} r_n \varepsilon_n = b, \lim_{n \to \infty} \varepsilon_n \xi^{(n)}(0) = u(0) \in [-a, b]^{\Lambda},$$

where both $a$ and $b$ can be either finite, or infinite.

### 3 Invariant measures and the diffusion limit in the reversible case

In this section we consider invariant measures of both the Markov chain and the diffusion limit in the reversible case. Let us assume, throughout the section, that matrix $A = A_b - A_d = (\alpha_{xy})_{x,y \in \Lambda}$ is symmetric (the symmetric case). In the symmetric case the Markov chain is reversible with the following stationary distribution

$$\mu_\Lambda(\xi) = Z_\Lambda^{-1} e^{\frac{1}{2} \sum_x \alpha_{xx} \xi_x (\xi_x^{-1}) + \sum_{x \sim y} \alpha_{xy} \xi_x \xi_y}, \ \xi \in \Omega_{\Lambda, l, r}, \quad (8)$$

where

$$Z_\Lambda = \sum_{\xi \in \Omega_{\Lambda, l, r}} e^{\frac{1}{2} \sum_x \alpha_{xx} \xi_x (\xi_x^{-1}) + \sum_{x \sim y} \alpha_{xy} \xi_x \xi_y}.$$ 

Indeed, it is easy to see that measure (8) satisfies the following detailed balance equation

$$e^{(A_b \xi)} \mu_\Lambda(\xi) = \mu_\Lambda(\xi + e^{(x)}) e^{(A_d \xi)x},$$
or, equivalently,
\[ e^{((A_\lambda - A_d) \xi)x} \mu_\Lambda (\xi) = \mu_\Lambda (\xi + e^{(x)}) , \]
where addition of configuration is understood component-wise and \( e^{(x)} \in \Omega_{\lambda,t,r} \) is the configuration such that \( e^{(x)}_y = 0 \) if \( y \neq x \), and \( e^{(x)}_x = 1 \). In vector notation \( \mu_\Lambda (\xi) = Z_\Lambda^{-1} e^{\frac{1}{2}[\langle A_\xi, \xi \rangle - \langle \alpha, \xi \rangle]} \) and \( Z_\Lambda = \sum_{\xi \in \Omega_{\lambda,t,r}} e^{\frac{1}{2}[\langle A_\xi, \xi \rangle - \langle \alpha, \xi \rangle]} \),
where \( \xi \in \Omega_{\lambda,t,r} \), \( \alpha = (\alpha_{xx}, x \in \Lambda) \) is a vector formed by diagonal elements of matrix \( A \), and \( (\xi', \xi'') \) is the Euclidean scalar product of vectors \( \xi', \xi'' \in \Omega_{\lambda,t,r} \) (considered as elements of \( \mathbb{R}^\Lambda \)).

**Remark 4.** If \( l = 0 \), \( r = 1 \) and \( \alpha_{xy} \equiv \text{const} \), then probability distribution \( \mathbb{P} \) corresponds to a particular case of the celebrated Ising model.

It is easy to see that under conditions of Theorem 1 if a sequence of states \( \xi^{(n)} \in \Omega_{\lambda,n,r,n} \), \( n \in \mathbb{N} \), is such that \( \varepsilon_n \xi^{(n)} \to u \in \mathbb{R}^\Lambda \) as \( n \to \infty \), then
\[ e^{\frac{1}{2} \left[ (e^{2A_\lambda n} \xi^{(n)}) - e^{2\alpha_n \xi^{(n)}} \right]} \to e^{\frac{1}{2} \langle A_\lambda, u \rangle}, \quad u \in \mathbb{R}^\Lambda, \]
where the function in the right side should be a density of an invariant measure of the diffusion limit. If this density is integrable, then the properly normalised invariant measure is the stationary distribution of the diffusion limit and, hence, the latter is positive recurrent. In what follows we are going to consider conditions for existence of the stationary distribution of the diffusion limit and, therefore, existence of the stationary distribution.

Note first that a unique strong solution of equation (3), regardless of symmetry of \( A \), is given by the following formula (e.g. Section 5.6 in [5])
\[ u(t) = e^{At}u(0) + \sqrt{2} \int_0^t e^{A(t-s)}dW_s, \]
where \( e^{At} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n \). If all eigenvalues of \( A \) have negative real parts, then the process is positive recurrent and its stationary distribution is a zero mean Gaussian distribution with the covariance that can be expressed in terms of matrix \( A \) (e.g. Theorem 6.7, Section 5.6, [5]).

If \( A \) is symmetric, then equation (3) can be written in the following gradient form
\[ du(t) = \frac{1}{2} \nabla (Au(t), u(t))dt + \sqrt{2}dW(t), \]
which is a particular case of the Langevin equation. It follows from the general theory of Langevin equations that the diffusion process \( u(t) \) is reversible and the following function (the same as in (9), as it should be)
\[ e^{\frac{1}{2} \langle Au, u \rangle} = e^{\frac{1}{2} \langle \lambda, u \rangle} = e^{\frac{1}{2} \sum_x \alpha_{xx} u_x^2 + \sum_{x \sim y} \alpha_{xy} u_x u_y}, \quad u = \{ u_x, x \in \Lambda \} \in \mathbb{R}^\Lambda, \]
is a density of an invariant measure of the process. This density is integrable if and only if matrix $\hat{A} = -A$ is positive definite, in which case a stationary distribution of diffusion process $u(t)$ is a multivariate normal distribution with zero mean and covariance matrix $(\hat{A})^{-1}$. In the rest of the section we are going to obtain conditions of positive definiteness of symmetric $-A$ in special cases.

We start with noticing that for matrix $-A$ to be positive the diagonal elements must be positive, which means that bounds $\alpha_{xx} < 0$ must hold for all $x \in \Lambda$. Note that in this case the diffusion limit can be interpreted in terms of interacting Ornstein-Uhlenbeck processes (see Remark 1).

It is known from algebra, that a diagonally dominant symmetric matrix with positive elements on the main diagonal is positive definite. This implies in our case that if $\alpha_{xx} < 0$ and $\alpha_{xx} + \sum_{y \sim x} |\alpha_{xy}| < 0$ for all $x \in \Lambda$, then matrix $\hat{A}$ is positive definite. In turn, this fact implies the following proposition.

**Proposition 1.** Let $\Lambda$ be an arbitrary finite connected graph and let $I_\Lambda$ be the incidence matrix of $\Lambda$. If $A = \alpha E + \beta I_\Lambda$, where $\alpha, \beta \in \mathbb{R}$, and $E$ is the unit matrix, and $\alpha < 0$, $\alpha + |\beta| \max_{x \in \Lambda} \nu(x) < 0$, where $\nu(x)$ is the degree of vertex $x$ (i.e. the number of edges incident to the vertex), then $\hat{A}$ is positive definite.

Inequality $|\beta| \max_{x \in \Lambda} \nu(x) < -\alpha$ in the above proposition means that interaction (specified by parameter $\beta$ and graph $\Lambda$) is sufficiently small, so that the collective behaviour of the system of interacting Ornstein-Uhlenbeck processes (with individual drifts equal $\alpha$) is still positive recurrent.

An additional information about graph $\Lambda$ allows to improve this result, which we are going to demonstrate in the case of the following graphs.

1. A constant vertex degree graph is a graph such that $\nu(x) \equiv \nu$, for some integer $\nu > 0$, where $\nu(x)$ is the degree of vertex $x$.

2. A star graph with $m + 1$ vertices is a graph with central vertex $x$ and its neighbouring vertices $y_1, \ldots, y_m$, i.e. $x \sim y_i$, $i = 1, \ldots, m$, so that $x$ is the only neighbour for each of $y_i$, $i = 1, \ldots, m$.

3. A unary tree of length $n + 2$, where $n \in \mathbb{Z}_+$, is a graph which vertices can be enumerated by natural numbers $1, \ldots, n + 2$, and such that $1 \sim 2 \sim \cdots \sim n + 1 \sim n + 2$.

The following theorem gives criteria for positive definiteness of matrix $\hat{A}$ and, hence, for positive recurrence of the corresponding diffusion limit in the case of these graphs.
Theorem 3. Suppose that $A = \alpha E + \beta I_\Lambda$, where $\alpha, \beta \in \mathbb{R}$, $E$ is the unit matrix and $I_\Lambda$ is the incidence matrix of $\Lambda$.

1) If $\Lambda$ is a graph with constant vertex degree $\nu(x) \equiv \nu \geq 1$, then $\tilde{A}$ is positive definite if and only if $\alpha < 0, \alpha + |\beta|\nu < 0$.

2) If $\Lambda$ is a star-like graph with $(m+1)$ vertices, then $\tilde{A}$ is positive definite if and only if $\alpha < 0, \alpha + |\beta|\sqrt{m} < 0$.

3) If $\Lambda$ is a unary tree of length $n+2$, where $n \in \mathbb{Z}_+$, then $\tilde{A}$ is positive definite if and only if $\alpha < 0, \alpha + 2\beta \cos\left(\frac{\pi n}{n+3}\right) < 0$.

Proof of Part 1) of Theorem 3. The “if” part of the statement is implied by Proposition 1. To show that the sufficient condition is also a necessary one it suffices to notice that all eigenvalues of matrix $\tilde{A}$ lie, by the Gershgorin circle theorem, within the closed interval $[-\alpha - |\beta|\nu, -\alpha + |\beta|\nu]$.

Proof of Part 2) of Theorem 3. Let $\Lambda$ be a star graph with a central vertex $x$ and its neighbouring vertexes $y_1, \ldots, y_m$, i.e. $x \sim y_i, i = 1, \ldots, m$, and $x$ is the only neighbour for each of $y_i, i = 1, \ldots, m$. Denote by $D_m(\mu)$ the characteristic polynomial of matrix $\hat{A}$ corresponding to the graph. It was shown in [10] that

$$D_m(\mu) = (-\alpha - \mu)^{m-1}(-\alpha - \beta\sqrt{m} - \mu)(-\alpha + \beta\sqrt{m} - \mu),$$

so that $-\alpha > 0$ is the matrix eigenvalue of order $m-1$ and $-\alpha \pm \beta\sqrt{m} > 0$ are two remaining eigenvalues, each of order 1 and the result follows.

Proof of Part 3) of Theorem 3. If $n = 0$, then this is the simplest case of a constant degree graph (see Part 1)). If $n = 1$, then this is the simplest case of a star graph (see Part 2)). In what follows we assume that $n \geq 2$. It is easy to see that matrix $\tilde{A}$ is the following tridiagonal symmetric Toeplitz matrix

$$\tilde{A} = \begin{bmatrix} -\alpha & -\beta \\ -\beta & -\alpha & -\beta \\ & & \ddots & \ddots \\ & & & -\beta \\ 0 & & & -\beta & -\alpha \end{bmatrix}_{(n+2) \times (n+2)}$$

The well known results for tridiagonal symmetric Toeplitz matrices yield that eigenvalues of matrix $\tilde{A}$ are simple and given by the following equations

$$\lambda_k = -\alpha - 2\beta \cos\left(\frac{k\pi}{n+3}\right), \ k = 1, \ldots, n+2,$$
where $\lambda_1 = -\alpha - 2\beta \cos \left( \frac{\pi}{n+3} \right)$ is the minimal eigenvalue. This finishes the proof of Part 3) of the theorem.

4 Proof of Theorem 1

The proof consists in showing that a sequence of generators of rescaled Markov chains converges in a certain sense (explained below) to the generator of the limit diffusion process \([\Pi]\). Solution of \([\Pi]\) is a Feller Markov process and its generator is

$$L f(u) = \sum_{x \in \Lambda} f''(u) + \sum_{x \in \Lambda} (b(x, u) - d(x, u)) f'(u), \quad u \in \mathbb{R}^\Lambda,$$

(12)

where we denoted $f''(u) = \frac{\partial^2 f(u)}{\partial u^2}$ and $f'(u) = \frac{\partial f(u)}{\partial u}$ for the ease of notation. Generator $L$ is defined on twice continuously differentiable functions that vanish at infinity together with their first and second order derivatives. By Theorem 6.1, Chapter 1, \([\Pi]\), in order to prove the convergence of semigroups it is sufficient to prove that for each $f \in D(L)$, where $D(L)$ is a core of the limit generator $L$, there exists $f_n \in B(\varepsilon_n \Lambda, n)$, $n \geq 1$, such that

$$\sup_{\xi(n) \in \Omega_{\Lambda, n}} |f_n(\varepsilon_n \xi^{(n)}) - f(\varepsilon_n \xi^{(n)})| \to 0$$

and

$$\sup_{\xi(n) \in \Omega_{\Lambda, n}} |L_n f_n(\varepsilon_n \xi^{(n)}) - L f(\varepsilon_n \xi^{(n)})| \to 0$$

as $n \to \infty$, where $L_n$ is the generator of the rescaled Markov chain $\varepsilon_n \xi^{(n)}(\varepsilon_n^{-2} t)$. Theorem 2.5, Chapter 8, \([\Pi]\), yields that the set $C^\infty_c(\mathbb{R}^d)$ of infinitely differentiable functions with a compact support is a core for generator \([\Pi]\).

Denote $\Omega_{\Lambda, n} = \Omega_{\Lambda, l_1, r_n}$ to ease notation. Let $\xi^{(n)}(t) \in \Omega_{\Lambda, n}$, $n \in \mathbb{Z}_+$, be the Markov chain whose transition rates are specified by interaction matrices $\varepsilon_n A_b$ and $\varepsilon_n A_d$. Given $f : C^\infty_c(\mathbb{R}^\Lambda) \to \mathbb{R}$ and $n$ define $f_n : \Omega_{\Lambda, n} \to \mathbb{R}$ as $f_n = f|_{\Omega_{\Lambda, n}}$, i.e. as a restriction of $f$ on $\Omega_{\Lambda, n}$. If $L_n$ is the generator of the rescaled Markov chain $\varepsilon_n \xi^{(n)}(\varepsilon_n^{-2} t)$, then we have that

$$L_n f_n(\varepsilon_n \xi) = \varepsilon_n^{-2} \sum_{x \in \Lambda} (f(\varepsilon_n (\xi + e(x))) - f(\varepsilon_n \xi)) b_n(x, \xi) 1_{\{\xi < r_n\}}$$

$$+ \varepsilon_n^{-2} \sum_{x \in \Lambda} (f(\varepsilon_n (\xi - e(x))) - f(\varepsilon_n \xi)) d_n(x, \xi) 1_{\{\xi > -l_n\}},$$

where we denoted $b_n(x, \xi) = \varepsilon_n^2 b(x, \xi)(= \varepsilon_n^2 (A_b \xi)_x)$, $d_n(x, \xi) = \varepsilon_n^2 d(x, \xi)(= \varepsilon_n^2 (A_d \xi)_x)$ and $1_B$ is an indicator of set $B$. 

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Suppose that \( n \) is sufficiently large so that support \( \text{Supp}(f) \) of \( f \) is covered by \([-\varepsilon_n b, \varepsilon_n r_n]^\Lambda\). Then
\[
L_n f_n(\varepsilon_n \xi) = \varepsilon_n^{-2} \sum_{x \in \Lambda} \left( f(\varepsilon_n (\xi + e^{(x)})) - f(\varepsilon_n \xi) \right) e^{b_n(x, \xi)}
+ \varepsilon_n^{-2} \sum_{x \in \Lambda} \left( f(\varepsilon_n (\xi - e^{(x)})) - f(\varepsilon_n \xi) \right) e^{d_n(x, \xi)},
\]
in other words, we can remove the indicators \( 1_{\{\xi, < r_n\}} \) and \( 1_{\{\xi, > l_n\}} \). Note that due to linearity we have that \( b_n(x, \xi) = \varepsilon_n b(x, \varepsilon_n \xi) \) and \( d_n(x, \xi) = \varepsilon_n d(x, \varepsilon_n \xi) \). If a sequence of states \( \xi^{(n)} \in \Omega_{\Lambda,n} \), \( n \geq 1 \), is such that \( \varepsilon_n \xi^{(n)} \rightarrow u_x \) for every \( x \in \Lambda \) as \( n \rightarrow \infty \), then by Taylor’s formula with the reminder term we can write that
\[
e^{\varepsilon_n b(x, \varepsilon_n \xi)} = (1 + \varepsilon_n b(x, u)) + R_{n,1}(x, u),
\]
\[
e^{\varepsilon_n d(x, \varepsilon_n \xi)} = (1 + \varepsilon_n d(x, u)) + R_{n,2}(x, u)
\]
where \( |R_{n,i}(x, u)| < C_i \varepsilon_n^2 \), for some \( C_i = C_i(f) \), \( i = 1, 2 \), as \( n \rightarrow \infty \). Also,
\[
f(\varepsilon_n (\xi + e^{(x)})) - f(\varepsilon_n \xi) = f'_x(u)\varepsilon_n + \frac{1}{2} f''_x(u)\varepsilon_n^2 + R_{n,3}(x, u),
\]
where \( |R_{n,3}(x, u)| \leq C_3 \varepsilon_n^3 \), with some \( C_3 = C_3(f) \), and
\[
f(\varepsilon_n (\xi - e^{(x)})) - f(\varepsilon_n \xi) = -f'_x(u)\varepsilon_n + f''_x(u)\varepsilon_n^2 + R_{n,4}(x, u),
\]
where \( |R_{n,4}(x, u)| \leq C_4 \varepsilon_n^3 \), with some \( C_4 = C_4(f) \). Equations \((13), (14), (15)\) and \((16)\) yield that
\[
L_n f_n(\varepsilon_n \xi) = f''_x(u) + (b(x, u) - d(x, u)) f'_x(u) + J_n(x, u),
\]
where \( |J_n(x, u)| \leq C \varepsilon_n^3 \), \( C = C(f) \). Thus we have that
\[
L_n f(\varepsilon_n \xi^{(n)}) \rightarrow \sum_{x \in \Lambda} \left( f''_x(u) + (b(x, u) + d(x, u)) f'_x(u) \right),
\]
uniformly over \( u \in \text{Supp}(f) \). Now, Theorem 6.1, Chapter 1, \((11)\) applies and the convergence of \( u \in \text{Supp}(f) \). References

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