A Mathematical Model of Longitudinal Waves Incident at a Free Surface of a Pre-Stressed Dissipative Half-Space

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Received: 15 October 2020; Accepted: 9 November 2020; Published: 11 November 2020

Abstract: The aim of this work is to study the behavior of reflection of a longitudinal wave at a free surface of dissipative half-space under the effects of compressive initial stresses. When a longitudinal wave is incident on the free surface of an elastic dissipative half-space, two damped waves (Primary waves and secondary waves) are reflected. Among of these waves, P-waves are affected by compressional initial stresses. The governing equation and corresponding closed-form solutions are derived based on Biot's incremental deformation theory. The equations of motion are solved analytically and the influence of initial stress parameter on the reflection coefficient of P-wave incidents at the free surface of dissipative half-space is studied in detail. Numerical computations are performed for actual Earth crust and the results analyzing the incident of longitudinal waves are discussed and presented graphically. The analytical solutions and numerical results reveal that the compressive initial stress parameter has notable effects on the reflection coefficient of longitudinal wave incidents on the free surface of dissipative medium. In addition, it has been observed that the presence of compressive initial stresses increases the phase velocity of the longitudinal waves. To the authors’ best knowledge, effects of compressive initial stresses on the reflection coefficients of the incident longitudinal wave on a free surface of dissipative half-space have not been studied before. Since the actual Earth is subject to initial stresses due to different resources, understanding the influences of compressive initial stresses on the reflection coefficient of a longitudinal wave helps seismologists and earthquake engineers to get accurate results of the reflection coefficients of seismic waves propagation in the Earth. Thus, the present study would be useful for seismology and earthquake engineering fields and further study about the nature of seismic waves.

Keywords: reflection coefficient; longitudinal wave; compressive initial stresses; dissipative half-space

1. Introduction

The reflection coefficient of elastic waves is very important due to its distinguish applications in different fields for the calculation of the amplitudes of various kinds of waves [1–6]. Nemours studies were carried out about the reflection coefficients of elastic waves incident on the surfaces of homogenous and nonhomogeneous media with different kinds of phase velocity distributions, such as Tooly et al. [7], Gupta [8–10], Acharya [11], Cerveny [12], Singh et al. [13,14], Saini [15], Tomar et
The study of the reflection coefficient of elastic waves at surface boundaries is very important in seismology and earthquake engineering due to its distinguish applications in seismic prospecting and the calculation of source energy of earthquakes. Many factors influence the reflection of longitudinal waves, such as the characteristics of the reflecting surface and the elastic modulus of the medium as well as internal forces, such as compressive initial stresses present in the medium. These factors will change the spread of waves, thus affecting the phase velocity of longitudinal waves. Therefore, developing a mathematical model for longitudinal wave reflection is significant to the study of seismic plane wave propagation in the actual Earth. Longitudinal wave reflection problems in dissipative solids are of great practical importance in various technological and geophysical circumstances. The reflection of longitudinal waves in non-dissipative medium including the effect of rotation was discussed by Sing et al. [19]. More recently, Biswas and Sarkar [20] derived the solution of the steady oscillations equations in porous thermos-elastic medium and Li et al. [21] discussed the propagation of thermos-elastic waves across an interface with consideration of couple stress and second sound on the basis of Green and Naghdi GN theories. Saha et al. [22] analyzed the reflection and refraction of plane wave at the separating interface of two functionally graded incompressible monoclinic media under initial stress and gravity. The Effect of impedance on the reflection of plane waves in a rotating magneto-thermo-elastic solid half-space with diffusion was discussed by Yadav [23].

The above literatures have not studied the effects of compressive initial stresses on the reflection coefficients at the free surfaces of a dissipative half-space. Initial stresses may be present in the medium caused by various factors, such as creep, gravity, external forces, difference of temperature etc. Actually, the Earth’s crust is a pre-stressed dissipative medium and the stresses present in the medium are compressive and hydrostatic. For this reason, it is very important to conduct research and reveal theoretical results on the mathematical model for the reflection of a longitudinal wave at a free surface of a dissipative half-space incorporating initial stresses effects. As can be seen from the above literature summary, and from our best knowledge, the reflection analysis of a longitudinal wave on the free surface of dissipative media has not yet been investigated. In this sense, in this study, impacts of initial stresses on the reflection coefficient of a longitudinal wave are unique within the framework of the Biot incremental theory. To date, theoretical aspects of the reflection of plane waves in initial stress-free media have been investigated [24–29], and an inclusive knowledge of them has been provided. As can be seen from the above literatures, and from the best knowledge of the author, the effects of initial stresses on reflection coefficient of longitudinal waves have not yet been investigated. In this sense, this study is unique within the framework of Biot’s theory and thus the reflection of longitudinal waves under effects of compressive initial stresses is very important in terms of providing good information for the next-generation studies and earthquake prediction studies. The prime objective of the current work is to investigate compressive initial stresses effects on the reflection coefficient of longitudinal waves at the free surface of dissipative half-space. It is found that the presence of compressive initial stresses effects on reflection coefficients of longitudinal waves in the considered medium. In addition, it has been observed that the presence of compressive initial stresses increases the phase velocity of the longitudinal waves.

2. Model Establishment

In case of plane strain and absent of external body forces, the dynamical equations of motion for a pre-stressed dissipative half-space can be written in the following form [30]:

\[
\frac{\partial \tau_{11}}{\partial x} + \frac{\partial \tau_{12}}{\partial y} - P \frac{\partial \omega}{\partial y} = \rho \frac{\partial^2 u}{\partial t^2},
\]  

(1)

\[
\frac{\partial \tau_{21}}{\partial x} + \frac{\partial \tau_{22}}{\partial y} - P \frac{\partial \omega}{\partial x} = \rho \frac{\partial^2 v}{\partial t^2},
\]  

(2)
where \( \rho \) is the density, \( P \) represents the normal compressive initial stress along the \( y \)-axis, \( \tau_{jk} \) (\( j, m = 1, 2 \)) are the incremental stresses and \( \omega \) is the rotation component, which is given as:

\[
\omega = \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)
\]

(3)

where \( u \) and \( v \) are the displacement components in \( x \) and \( y \) directions.

The constitution relations are taken as [30]:

\[
\begin{align*}
\tau_{11} &= A_{11} e_{11} + A_{12} e_{22}, \\
\tau_{22} &= (A_{12} - P) e_{11} + A_{22} e_{22}, \\
\tau_{12} &= \tau_{21} = 2Qe_{12},
\end{align*}
\]

(4)

where \( Q \) and \( A_{jm} \) are shear modulus and incremental elastic coefficients, respectively, and \( e_{jm} \) are the incremental strain components. \( A_{jm} \) are related to Lame’s coefficients \( \lambda \) and \( \mu \) by the following relations [30]:

\[
\begin{align*}
A_{11} &= (\lambda + 2\mu + P), \quad A_{12} = (\lambda + P), \\
A_{21} &= \lambda, \quad A_{22} = \lambda + 2\mu, \quad Q = \mu.
\end{align*}
\]

(5)

In addition, the incremental strain components \( e_{jm} \) are related with the displacement components by the following relations:

\[
\begin{align*}
e_{11} &= \frac{\partial u}{\partial x}, \quad e_{22} = \frac{\partial v}{\partial y}, \\
e_{12} &= \frac{1}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right).
\end{align*}
\]

(6)

(7)

For the dissipative half-space we replace the Lame’s coefficients \( \lambda \) and \( \mu \) by the complex coefficients:

\[
\lambda = \lambda_1 + i \lambda_2, \quad \mu = \mu_1 + i \mu_2.
\]

(8)

where \( i = \sqrt{-1}, \lambda_2 \) and \( \mu_2 \) are real number and \( \lambda_2 << \lambda_1, \mu_2 << \mu_1 \).

The constitution relations for the dissipative half-space are given by Fung [31] as follows:

\[
\begin{align*}
tau_{jk} &= \tau_{jk} e^{im}, \quad e_{jk} = \bar{e}_{jk} e^{im}, \quad \bar{e}_{jk} = \frac{1}{2} \left( \frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} \right).
\end{align*}
\]

(9)

where \( u_j = \bar{u}_j e^{im}, \) \( \bar{\omega} \) being the angular frequency.

From (6)–(9) in Relation (5), the constitution relations become

\[
\begin{align*}
\tau_{11} &= [(\lambda_1 + 2\mu_1 + P) + i (\lambda_2 + 2\mu_2)] \bar{\xi}_{11} + [(\lambda_1 + P) + i \lambda_2] \bar{\xi}_{22}, \\
\tau_{22} &= (\lambda_1 + i \lambda_2) \bar{\xi}_{11} + [(\lambda_1 + 2\mu_1) + i (\lambda_2 + 2\mu_2)] \bar{\xi}_{22}, \\
\tau_{12} &= 2(\mu_1 + i \mu_2) \bar{\xi}_{12}.
\end{align*}
\]

(10)

Substituting Equations (3), (5), (7) and (10), the equations of motion (1) and (2) can be written as:
\[
\left[ (\lambda + 2\mu + P) \frac{\partial^2 u}{\partial x^2} + (\lambda + \mu + P / 2) \frac{\partial^2 v}{\partial x \partial y} + (\mu + P / 2) \frac{\partial^2 u}{\partial y^2} + \rho \frac{\partial^2 \tilde{u}}{\partial t^2} \right] \\
+ i \left[ (\lambda_2 + 2\mu_i) \frac{\partial^2 \tilde{u}}{\partial x^2} + (\lambda_2 + \mu_i) \frac{\partial^2 \tilde{v}}{\partial x \partial y} + \mu_i \frac{\partial^2 \tilde{u}}{\partial y^2} \right] = 0 
\]
(11)

\[
\left[ (\lambda + 2\mu_i) \frac{\partial^2 \tilde{v}}{\partial y^2} + (\lambda_i + P / 2) \frac{\partial^2 \tilde{v}}{\partial x \partial y} + (\mu_i - P / 2) \frac{\partial^2 \tilde{v}}{\partial x^2} + \rho \frac{\partial^2 \tilde{v}}{\partial t^2} \right] \\
+ i \left[ (\lambda_2 + 2\mu_i) \frac{\partial^2 \tilde{v}}{\partial y^2} + (\lambda_2 + \mu_i) \frac{\partial^2 \tilde{v}}{\partial x \partial y} + \mu_i \frac{\partial^2 \tilde{v}}{\partial x^2} \right] = 0. 
\]
(12)

3. Analytical Solution of the Model

We assume the displacement vector \( U^{(n)} = (u^{(n)}, v^{(n)}, 0) \) is given by

\[
U^{(n)} = U_n \delta^{(n)} e^{i\delta}, 
\]
(13)

where \( U_n \) is the amplitude, the index \( n = 1, 2 \) assigns an arbitrary direction of propagation of waves, \( \delta^{(n)} = d_1^{(n)} + d_2^{(n)} \) is the unit vector of the displacement,

\[
\delta_n = k_n \left[ C_n t - \left( x \cdot \Gamma^{(n)} \right) \right], 
\]
(14)

which is the phase factor in which \( \Gamma^{(n)} = \Gamma_1^{(n)} + \Gamma_2^{(n)} \) is the unit vector of propagation, \( C_n \) is the phase velocity of propagation, \( x = (x, y, \bullet) \) represents the scalar product and \( k_n \) is the wave number, is given by

\[
\bar{\omega} = k_n C_n. 
\]
(15)

In matrix form, the displacement component (13) may be expressed as:

\[
\begin{pmatrix} u^{(n)} \\ v^{(n)} \end{pmatrix} = \begin{pmatrix} U_n & d_1^{(n)} \\ U_n & d_2^{(n)} \end{pmatrix} e^{-ik_n \left( x \Gamma_1^{(n)} + y \Gamma_2^{(n)} \right) - C_n t}. 
\]
(16)

It can be written in the form:

\[
\begin{pmatrix} u^{(n)} \\ v^{(n)} \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{-ik_n \left( x \Gamma_1^{(n)} + y \Gamma_2^{(n)} \right) - C_n t}, 
\]
(17)

where \( A = U_n d_1^{(n)} \), \( B = U_n d_2^{(n)} \).

For convenience, we omit the bared notations of displacement components and angular frequency. When substituting Relations (15) and (17) into Equations (11) and (12), we get:

\[
\Delta \left[ (\lambda + 2\mu_i) \Gamma_1^{(n)} + (\mu_i + P / 2) \Gamma_2^{(n)} - \rho C_n^2 \right] + i \Delta \left[ (\lambda_2 + 2\mu_i) \Gamma_1^{(n)} - \mu_i \Gamma_2^{(n)} \right] \\
+ \Omega \left[ (\lambda_i + P / 2) \Gamma_1^{(n)} \Gamma_2^{(n)} + i (\lambda_2 + \mu_i) \Gamma_1^{(n)} \Gamma_2^{(n)} \right] = a_i + \Omega b_i = 0, 
\]
(18)
The two homogenous equations, (18) and (19), have a non-trivial solution:

\[ C_n^2 = C_p^2 = \frac{1}{2\rho} \left( I + \sqrt{J_n} \right), \]  

(20)

where

\[ I = \left[ \lambda_1 + \mu_1 (3 + \zeta) + i \left( \lambda_2 + 3\mu_2 \right) \right], \]  

(21)

\[ J_n = -8 \left( \lambda_2 + \mu_2 \right)^2 r_1^{(n)^4} + 8\zeta \mu_1 \left( \lambda_1 + \mu_1 (1 + \zeta) \right) \]

\[ + 8(\lambda_2 + \mu_2)^2 r_1^{(n)^2} + \left( \lambda_1 + \mu_1 (1 - \zeta) \right)^2 - (\lambda_2 + \mu_2)^2 \]

\[ + i \left( \lambda_2 + \mu_2 \right) \left( 8\zeta \mu_1 r_1^{(n)^2} + 2(\lambda_1 + \mu_1) - 2\zeta \mu_1 \right), \]  

(22)

\[ \zeta = P / 2\mu_1 \] represents the initial stress parameter and \( r_1^{(n)^2} + r_2^{(n)^2} = 1. \)

The real part of Equation (20) gives the phase velocity of longitudinal waves and the imaginary part gives the damping, respectively. It can be seen from Equation (21) that the phase velocity of a longitudinal wave \( \left( C_p^2 \right) \) depends on the compressive initial stresses, damping and the directions of propagation \( \left( r_1^{(n)}, r_2^{(n)} \right) \).

When the effects of compressive initial stresses is absent and the medium is non-dissipative, i.e., \( \left( \zeta = P / 2\mu_1 = 0, \lambda_2 = \mu_2 = 0 \right) \), Equation (20) transforms to the non-dimensional form as:

\[ \frac{C_p^2}{\alpha^2} = \frac{1}{2} \left[ 1 + \beta^2 \alpha^2 + \left( 4\zeta \left( 1 - \beta^2 \alpha^2 + \frac{\zeta}{\alpha^2} \right) \sin^2 e_1 + \left( 1 - \beta^2 \alpha^2 + \frac{\zeta}{\alpha^2} \right) \right] \]  

(23)

where \( \alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \beta = \sqrt{\frac{\mu}{\rho}} \) which agrees with the result obtained by Selim et al. [32].

In addition, from Equations (18) and (19), we can obtain the following relation:

\[ \frac{d_{1}^{(n)}}{d_{2}^{(n)}} = \frac{\Delta_n}{\rho C_n^2 - T_n} = \frac{\rho C_n^2 - \Theta_n}{\Delta_n}, \]  

(24)

where

\[ T_n = \left[ \lambda_1 + 2\mu_1 (1 + \zeta) + i(\lambda_2 + 2\mu_2) \right] r_1^{(n)^2} + \left[ \mu_1 (1 + \mu_1 \zeta) + i\mu_2 \right] r_2^{(n)^2}, \]  

(25)

\[ \Theta_n = \left[ (\lambda_1 + 2\mu_1) + i(\lambda_2 + 2\mu_2) \right] r_1^{(n)^2} + \left[ \mu_1 (1 - \mu_1 \zeta) + i\mu_2 \right] r_2^{(n)^2}, \]  

(26)

\[ \Theta_n = \left[ (\lambda_1 + 2\mu_1) + i(\lambda_2 + 2\mu_2) \right] r_1^{(n)^2} + \left[ \mu_1 (1 - \mu_1 \zeta) + i\mu_2 \right] r_2^{(n)^2}, \]  

(27)
Equations (25)–(27) may be used to find \( d^{(n)} \) in terms of \( \Gamma^{(n)} \).

3.1. Reflection of Longitudinal Waves

In this case, we consider the pre-stressed dissipative half-space occupying the region \( y \geq 0 \) as shown in Figure 1. We will derive the closed-form expressions of the reflection coefficient of longitudinal waves incident at the stress-free boundary \( y = 0 \).

![Figure 1. Geometry of reflection of longitudinal Primary waves waves at the free surface of pre-stressed dissipative half-space.](image_url)

In case of plane strain, the displacement components are independent of the \( z \)-axis and are of the following form:

\[
U^{(n)} = \left( u^{(n)}(x, y, z), v^{(n)}(x, y, z), 0 \right)
\]  

(28)

Therefore, the displacement field may be represented by

\[
u(x, y, z) = \sum_{j=1}^{2} \mathcal{U}_j d^{(j)} e^{i \delta_j},
\]

(29)

\[
v(x, y, z) = \sum_{j=1}^{2} \mathcal{V}_j d^{(j)} e^{i \delta_j},
\]

(30)

where

\[
\delta_1 = k_1 \left[ C_{11} t - (x \sin e_1 - y \cos e_1) \right].
\]

(31)

\[
\delta_2 = k_2 \left[ C_{22} t - (x \sin e_2 + y \cos e_2) \right].
\]

In the plane \( y = 0 \), the displacement and stress components due to the incident of longitudinal waves \( \left( \Gamma_1^{(1)} = \sin e_1, \Gamma_2^{(1)} = -\cos e_1 \right) \) can be written as:

\[
u^{(1)} = \mathcal{U}_1 d_1^{(1)} e^{i \delta_1}, \quad v^{(1)} = \mathcal{V}_1 d_2^{(1)} e^{i \delta_1}
\]

(32)
\[ \tau_{12}^{(1)} = i \mathcal{V}_1 k_1 Q_1 \left( d_1^{(1)} \cos e_1 - d_2^{(1)} \sin e_1 \right) e^{i \delta_1}, \]
\[ \tau_{22}^{(1)} = i \mathcal{V}_1 k_1 \left( Q_3 d_2^{(1)} \cos e_1 - Q_2 d_1^{(1)} \sin e_1 \right) e^{i \delta_1}, \]

where
\[ Q_1 = \left( \mu_1 + i \mu_2 \right), \]
\[ Q_2 = \left( \lambda_1 + i \lambda_2 \right), \]
\[ Q_3 = \left[ \left( \lambda_1 + 2 \mu_1 \right) + i \left( \lambda_2 + 2 \mu_2 \right) \right]. \]

The displacement and stress components due to reflected longitudinal waves \( \left( \Gamma_1^{(2)} = \sin e_2, \Gamma_2^{(2)} = \cos e_2 \right) \) can be written as:
\[ u^{(2)} = A_{2} d_1^{(2)} e^{i \delta_2}, \quad v^{(2)} = A_{2} d_2^{(2)} e^{i \delta_2}, \]
\[ \tau_{12}^{(2)} = -i \mathcal{V}_2 k_2 Q_1 \left( d_1^{(2)} \cos e_2 + d_2^{(2)} \sin e_2 \right) e^{i \delta_2}, \]
\[ \tau_{22}^{(2)} = -i \mathcal{V}_2 k_2 \left( Q_3 d_1^{(2)} \sin e_2 + Q_2 d_2^{(2)} \cos e_2 \right) e^{i \delta_2}, \]

3.2. Boundary Conditions

There are two boundary conditions at the plane \( y = 0 \), namely,
\[ \Delta f_x = \tau_{12}^{(n)} + e^{i \delta_1} P = 0, \]
\[ \Delta f_y = \tau_{22}^{(n)} = 0. \]

These equations can be written as:
\[ \tau_{12}^{(1)} + \tau_{12}^{(2)} + 2 \mu_1 \zeta \left( e^{(1)}_{12} + e^{(2)}_{12} \right) = 0, \]
\[ \tau_{22}^{(1)} + \tau_{22}^{(2)} = 0. \]

Substituting Equations (32)–(34) and (36) into Equations (39) and (40), we obtain
\[ i \mathcal{V}_1 k_1 Q_1 \left( d_1^{(1)} \cos e_1 - d_2^{(1)} \sin e_1 \right) e^{i \delta_1} \]
\[ - i \mathcal{V}_2 k_2 Q_1 \left( d_1^{(2)} \cos e_2 + d_2^{(2)} \sin e_2 \right) e^{i \delta_1} \]
\[ + i \mu_1 \zeta \left( \mathcal{V}_1 k_1 \left( d_1^{(1)} \cos e_1 - d_2^{(1)} \sin e_1 \right) e^{i \delta_1} \right) = 0, \]
and
\[ i \mathcal{U}_1 k_1 \left( Q_3 d_2^{(1)} \cos e_1 - Q_2 d_1^{(1)} \sin e_1 \right) e^{i \delta_1} - i \mathcal{U}_2 k_2 \left( Q_2 d_1^{(2)} \sin e_2 + Q_3 d_2^{(2)} \cos e_2 \right) e^{i \delta_2} = 0. \]  

(42)

Since Equations (41) and (42) are satisfied for all values of \( x \) and \( t \), then

\[ \delta_1(x, 0) = \delta_2(x, 0), \]  

(43)

which means

\[ k_1 \left( C_p^{in} t - x \sin e_1 \right) = k_2 \left( C_p^r t - x \sin e_2 \right). \]  

(44)

where \( C_p^{in} \) and \( C_p^r \) represent the phase velocities of incident and reflected waves, respectively.

Equations (43) and (44) give

\[ k_1 \sin e_1 = k_2 \sin e_2, \]  

(45)

and

\[ k_1 C_p^{in} = k_2 C_p^r. \]  

(46)

From Relation (15), Equations (45) and (46) can be written as

\[ \frac{\sin e_1}{C_p^{in}} = \frac{\sin e_2}{C_p^r} = \frac{1}{C_a}, \]  

(47)

where \( C_a \) is the apparent velocity.

Relation (47) represents the well-known Snell’s Law for orthotropic medium.

Substituting Relations (43)–(47) into Equations (41) and (42), we get

\[ \mathcal{U}_1 \mathcal{S}_1 + \mathcal{U}_2 \mathcal{S}_2 = 0, \]  

(48)

\[ \mathcal{U}_1 \mathcal{S}_3 + \mathcal{U}_2 \mathcal{S}_4 = 0, \]  

(49)

where

\[ \mathcal{S}_1 = k_1 L \left( d_1^{(1)} \cos e_1 - d_2^{(1)} \sin e_1 \right), \]

\[ \mathcal{S}_2 = -k_2 L \left( d_1^{(2)} \cos e_2 + d_2^{(2)} \sin e_2 \right), \]

\[ \mathcal{S}_3 = k_1 \left( Q_3 d_2^{(1)} \cos e_1 - Q_2 d_1^{(1)} \sin e_1 \right), \]

\[ \mathcal{S}_4 = -k_2 \left( Q_2 d_1^{(2)} \sin e_2 + Q_3 d_2^{(2)} \cos e_2 \right), \]

\[ L = (Q_1 + 2 \zeta \mu_1). \]  

(50)

Solving Equations (48)–(50), we get the amplitude ratios in the form

\[ \frac{\mathcal{U}_2}{\mathcal{U}_1} = -\frac{(\mathcal{S}_1 + \mathcal{S}_3)}{(\mathcal{S}_2 + \mathcal{S}_4)}. \]  

(51)
Equation (51) contains real and imaginary parts. Real part allows us to determine the reflection coefficients of longitudinal waves at a given angle of incident and the imaginary part corresponding to the damping.

4. Numerical Calculation of the Model

In this section, numerical computations are carried out to show the effects of compressive initial stresses on the reflection coefficients and phase velocity of longitudinal waves incidents on the free surface of dissipative half-space using the constants presented in Table 1 [33].

| ρ (g/cm³) | μ₁ (dyne/cm²) | μ₂ (dyne/cm²) | α (cm/s) |
|-----------|---------------|---------------|-----------|
| 2.15      | 1.90930 × 10¹¹ | 0.436 × 10¹¹  | 5.3 × 10⁵ |
| β         | λ₁            | λ₂            |           |
| 2.98 × 10⁵ | 2.22075 × 10¹¹ | 0.305 × 10¹¹  |           |

In order to show the effects of compressive initial stresses on the phase velocity of longitudinal waves incident on the surface boundary of dissipative half-space, the relation between the dimensionless phase velocity of longitudinal waves and the direction of propagation (θ) are considered in Figure 2. Moreover, to observe the impacts of compressive initial stress on the reflection coefficient of a longitudinal wave at any angle of incidence, reflection coefficients of longitudinal waves are computed for a certain range of angles of incidence with different values of initial stress parameter (ζ = 0.0, 0.2, 0.4, 0.6) as shown in Figure 3.

Figure 2 shows the variations of dimensionless phase velocity \( \left( \frac{C_p^2}{\alpha^2} \right) \) of a longitudinal wave vs. the direction of propagation θ for different values of compressive initial stress parameter (ζ = 0.0, 0.2, 0.4, 0.6). From this figure, it is can be seen that the phase velocity of a longitudinal wave is clearly dependent on the compressive initial stresses present in the medium. In addition, it is observed that the phase velocity of longitudinal waves increases considerably as the initial stress parameter increases.
Figure 2. Variation of dimensionless phase velocity \( \left( \frac{C_p^2}{\alpha^2} \right) \) of a longitudinal wave vs. the direction of propagation \( (\theta) \) for different values of compressive initial stress parameter \( (\zeta = 0.0, 0.2, 0.4, 0.6) \).

The variations of the reflection coefficients for the incident longitudinal waves as the absolute values of the real part of Equation (51) vs. the angle of incidence with different values of initial stress parameter \( (\zeta = 0.0, 0.2, 0.4, 0.6) \) are shown in Figure 3. From Figure 3, it can be observed that the presence of compressive initial stresses considerably affects the reflection coefficient of longitudinal waves incident on the free surface of dissipative half-space.

Figure 3. Variation of reflection coefficients \( \left( \frac{\mathcal{U}_2}{\mathcal{U}_1} \right) \) of a longitudinal wave vs. the angle of incidence \( (\theta) \) for different values of compressive initial stress parameter \( (\zeta = 0.0, 0.2, 0.4, 0.6) \).
5. Conclusions

In this article, we have reported a novel equation of motion of longitudinal waves incident on a pre-stressed dissipative half-space based on Biot’s incremental deformation theory. The noteworthy component of the present exploration is the examination of the impacts of compressive initial stresses as well as dissipation on the reflection coefficient of longitudinal waves on the surface of half-space. The numerical results have been shown through graphs to illustrate the dependence of the reflection coefficient on the compressive initial stresses present in the medium. It has been detected that the presence of compressive initial stresses considerably affects the reflection coefficient of longitudinal waves. The analytical solutions and numerical results reveal that the compressive initial stress parameter has notable effects on the reflection coefficient of longitudinal wave incidents on the free surface of dissipative half-space. In addition, it has been observed that the presence of compressive initial stresses increases the phase velocity of longitudinal waves.

Author Contributions: Conceptualization, M.M.S. and T.A.N.; methodology M.M.S. and T.A.N.; software, M.M.S. and T.A.N.; validation, M.M.S. and T.A.N.; writing—original draft preparation, M.M.S.; writing—review and editing, M.M.S. and T.A.N.; supervision, M.M.S.; project administration, T.A.N.; funding acquisition, T.A.N. All authors have read and agreed to the published version of the manuscript.

Funding: Taif University Researches Supporting Project number (TURSP-2020/031), Taif University, Taif, Saudi Arabia.

Acknowledgments: Taif University Researches Supporting Project number (TURSP-2020/031), Taif University, Taif, Saudi Arabia. The first author would like to acknowledge the supports provided by the Deanship of Scientific Research of Prince Sattam bin Abdulaziz university during this research work.

Conflicts of Interest: The authors declare no conflict of interest.

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