S-PROTOTMODULARITY OF THE CATEGORY OF COCOMMUTATIVE BIALGEBRA

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Abstract. We prove that the category of cocommutative bialgebras in any symmetric monoidal category (that has equalizers) is an S-protomodular category with respect to a particular class of split extensions of cocommutative bialgebras. We also obtain the “partial” well-known Smith is Huq condition, meaning that two S-equivalence relations centralize each other as soon as the normal subobjects associated with them commute in the sense of Huq.

Introduction

The notion of protomodular category was introduced by Bourn in [2]. The categories of groups, rings, Lie algebras, crossed modules, rings, the dual of the category of sets are examples of protomodular categories. In the pointed case, a category C is protomodular if and only if the Split Short Five Lemma holds. This means that for any diagram of the form

\[
\begin{array}{ccccccc}
0 & \longrightarrow & X_1 & \xrightarrow{\kappa_1} & A_1 & \xleftarrow{e_1} & B_1 & \longrightarrow & 0 \\
& & v & \downarrow & p & \downarrow & g & & \\
0 & \longrightarrow & X_2 & \xrightarrow{\kappa_2} & A_2 & \xleftarrow{e_2} & B_2 & \longrightarrow & 0
\end{array}
\]

where \(\kappa_i\) is the kernel of \(\alpha_i\) and \(\alpha_i \cdot e_i = 1_{B_i}\) for any \(i \in \{1, 2\}\), then \(p\) is an isomorphism whenever \(v\) and \(g\) are, this property is often referred to as the “Split Short Five Lemma” holds in \(C\).

It is well-known that the category of internal groups in a finitely complete category is protomodular [2]. In particular, this result implies that the category of cocommutative Hopf algebras in a symmetric monoidal category, that has equalizers, is protomodular, seen as internal groups in the category of cocommutative coalgebras. The category of cocommutative Hopf algebras over a field is even semi-abelian [11].

In this paper, we are interested in cocommutative bialgebras in any symmetric monoidal category that has equalizers. As a matter of fact, it was proven in [10] that the category of cocommutative \(K\)-bialgebras is not a protomodular category. Similarly, the category of monoids is not a protomodular category. However, a class of split epimorphisms of monoids, called Schreier split epimorphisms, turned out to have some very interesting properties, similar to the ones of split epimorphisms of groups. For example, Schreier split epimorphisms are equivalent to the actions of monoids, the Split Short Five Lemma holds, etc. [5, 6]. The authors in [5], introduced the notion of \(S\)-protomodularity, that is the protomodularity with respect to a class \(S\) of split epimorphisms. It implies that we have the Split Short Five Lemma as in (0.1) whenever \((\alpha_1, e_1)\) and \((\alpha_2, e_2)\) are in the class \(S\). The main example is the category of monoids with the class of Schreier split epimorphisms.

Key words and phrases. Cocommutative bialgebras, Split extensions, Protomodularity, Commutators, Symmetric monoidal categories.
In [17], we defined a notion of split extensions for (non-associative) bialgebras (and non-associative Hopf algebras) such that they have “group-like” properties. More precisely, we showed that this definition of split extension of (non-associative) bialgebras is equivalent to the notion of action of (non-associative) bialgebras. Moreover, we proved the validity of the Split Short Five Lemma for these kinds of split extensions. In particular, these results restrict to the case of cocommutative bialgebras.

Hence, it is natural to hope that the category of cocommutative bialgebras in any symmetric monoidal category is \(S\)-protomodular with respect to the class introduced in [17].

In a finitely complete category, we can define two types of centrality: the centrality of equivalence relations in the sense of Smith and the centrality of normal monomorphisms in the sense of Huq. Note that these notions of centrality are not independent. If two relations of equivalence centralize each other (in the sense of Smith) then the corresponding normal monomorphisms necessarily centralize in the sense of Huq [4].

The converse is not true in general. If \(C\) is a category such that any two equivalence relations always centralize each other as soon as their normalizations centralize in the sense of Huq, one says that \(C\) satisfies the so-called Smith is Huq property [4, 16].

In [14], the authors introduced the notion of Smith is Huq for pointed \(S\)-protomodular categories. In that context, the Smith is Huq property can be expressed as follows: two \(S\)-equivalence relations centralize each other if and only if their associated normal subobjects commute (in the sense of Huq).

In this paper, we prove that the category of cocommutative bialgebras in any symmetric monoidal category is \(S\)-protomodular with respect to the class of split extensions introduced in [17], we give a description of the centrality in the sense of Huq for two subbialgebras of a cocommutative bialgebra and we examine the Smith is Huq condition for cocommutative bialgebras. The results obtained in this paper generalize results in [6] and [14] on the category of monoids.

The layout of this article is as follows: the first section contains some preliminaries on cocommutative bialgebras in a symmetric monoidal category.

In the second section, we prove that the category of cocommutative bialgebras in any symmetric monoidal category (that has equalizers) is \(S_{\text{coc}}\)-protomodular with respect to the class \(S_{\text{coc}}\) of split extensions of cocommutative bialgebras defined in [17].

In the third section, we describe the Huq commutator of two subbialgebras of a cocommutative bialgebra.

In the last section, we investigate the Smith is Huq condition for cocommutative bialgebras by using the results of [14].

1. Preliminaries

1.1. Bialgebras in a symmetric monoidal category. We recall that a monoidal category is given by a triple \((C, \otimes, I)\) where \(C\) is a category, \(\otimes: C \times C \to C\) a bifunctor and \(I\) is the identity element (we omit to explicit the three natural isomorphisms, the associator, the right unit and the left unit).

A braided monoidal category is a 4-tuple \((C, \otimes, I, \sigma)\) where \((C, \otimes, I)\) is a monoidal category and \(\sigma\) is a braiding. A braiding consists of a family of natural isomorphisms \(\sigma_{X,Y}: X \otimes Y \to Y \otimes X\) satisfying

\[
\sigma_{X \otimes Y, Z} = (\sigma_{X, Z} \otimes 1_Y) \cdot (1_X \otimes \sigma_{Y, Z})
\]

\[
\sigma_{X, Y \otimes Z} = (1_Y \otimes \sigma_{X, Z}) \cdot (\sigma_{X, Y} \otimes 1_Z).
\]

A braided monoidal category is called symmetric when

\[(1.1)\]

\[\sigma_{Y, X}^{-1} = \sigma_{X, Y} .\]
In this paper, we omit the indexes of the braiding when this does not bring any confusion.

An algebra in a symmetric monoidal category \((\mathcal{C}, \otimes, I, \sigma)\) is given by an object \(A \in \mathcal{C}\) endowed with a morphism \(m: A \otimes A \rightarrow A\), called the multiplication. An algebra is associative and unital when there is a morphism \(u_A: I \rightarrow A\) called the unit, such that the following equalities are satisfied

\[
\begin{align*}
m \cdot (u_A \otimes 1_A) &= 1_A = m \cdot (1_A \otimes u_A) \\
m \cdot (m \otimes 1_A) &= m \cdot (1_A \otimes m)
\end{align*}
\]

\[
\begin{tikzcd}
A \arrow{r}{u_A \otimes 1_A} \arrow{d}{m} & A \otimes A \arrow{r}{1_A \otimes u_A} \arrow{d}{m} & A \\
A & A \otimes A \arrow{r}{m \otimes 1_A} \arrow{d}{1_A \otimes m} & A \otimes A \arrow{d}{m} \\
& A \otimes A \arrow{r}{m} & A
\end{tikzcd}
\]

All the algebras that we will consider in this paper are associative and unital. A morphism of algebras \(f: A \rightarrow B\) is a morphism in \(\mathcal{C}\) such that the following diagrams commute

\[
\begin{tikzcd}
A \otimes A \arrow{r}{f \otimes f} \arrow{d}{m} & B \otimes B \arrow{d}{m} \\
A \arrow{r}{f} & B
\end{tikzcd}
\]

A coalgebra is the dual notion of the notion of an algebra. In other words, a coalgebra over \((\mathcal{C}, \otimes, I, \sigma)\) is an object \(C \in \mathcal{C}\) with a comultiplication \(\Delta: C \rightarrow C \otimes C\). From now on, the coalgebras will always be coassociative, i.e. the following equality holds

\[
(\Delta \otimes 1_C) \cdot \Delta = (1_C \otimes \Delta) \cdot \Delta
\]

\[
\begin{tikzcd}
C \arrow{r}{\Delta} \arrow{d}{\Delta} & C \otimes C \arrow{d}{1_C \otimes \Delta} \\
C \otimes C \arrow{r}{\Delta \otimes 1_C} & C \otimes C \otimes C
\end{tikzcd}
\]

We will also assume that the coalgebras are counital, meaning that there exists a morphism \(\epsilon_C: C \rightarrow I\), called counit, satisfying the condition:

\[
(\epsilon_C \otimes 1_C) \cdot \Delta = 1_C = (1_C \otimes \epsilon_C) \cdot \Delta
\]

as expressed by the commutativity of the following diagram

\[
\begin{tikzcd}
C \arrow{r}{\epsilon_C \otimes 1_C} \arrow[equals]{dr} & C \otimes C \arrow{r}{1_C \otimes \epsilon_C} \arrow[equals]{d} & C \\
& C \otimes C \arrow{u}{\Delta}
\end{tikzcd}
\]

Similarly, a morphism of coalgebras \(g: C \rightarrow D\) is a morphism in \(\mathcal{C}\) such that the following two diagrams commute

\[
\begin{tikzcd}
D \otimes D \arrow{r}{g \otimes g} \arrow[equals]{d}{\Delta} & C \otimes C \arrow{d}{\Delta} \\
D \arrow{r}{g} & C \arrow{u}{\epsilon_C}
\end{tikzcd}
\]

\[
\begin{tikzcd}
I \arrow{r}{\epsilon_D} \arrow[equals]{d}{\Delta} & D \arrow{d}{\epsilon_C} \\
C \arrow{r}{g} & C
\end{tikzcd}
\]
We also recall that a bialgebra is a 5-tuple \((B, m, u_B, \Delta, \epsilon_B)\) where \((B, m, u_B)\) is an algebra, \((B, \Delta, \epsilon_B)\) is a coalgebra and \(\Delta, \epsilon_B\) are algebra morphisms (which is equivalent of asking that \(m, u_B\) are coalgebra morphisms) i.e.

\[
\Delta \cdot m = (m \otimes m) \cdot (1_B \otimes \sigma \otimes 1_B) \cdot (\Delta \otimes \Delta)
\]

\[
\Delta \cdot u_B = u_B \otimes u_B
\]

\[
\epsilon_B \cdot m = \epsilon_B \otimes \epsilon_B
\]

\[
\epsilon_B \cdot u_B = 1_I
\]

Moreover, a morphism in \(C\) is a morphism of bialgebras if it is a morphism of algebras and coalgebras.

A bialgebra is called cocommutative when its underlying coalgebra structure is cocommutative, it means that

\[
\sigma \cdot \Delta = \Delta.
\]

The category of cocommutative bialgebras in \(C\), a symmetric monoidal category, is denoted by \(\text{Bial}_{C, \text{coc}}\).

**Examples 1.1.** (1) In the symmetric monoidal category \((\text{Set}, \times, \{\ast\})\) of sets where \(\sigma\) is the twist morphism (where \(\sigma(x, y) = (y, x)\) for any element \(x\) of a set \(X\) and any element \(y\) of a set \(Y\)), every object has a coalgebra structure with \(\Delta\) being the diagonal and \(\epsilon\) the morphism sending every element to the singleton. Hence, a (cocommutative) bialgebra (or algebra) is a monoid.

(2) In the symmetric monoidal category \((\text{Vect}_K, \otimes, K)\) of vector spaces over a field \(K\) where \(\sigma\) is the twist morphism (defined by \(\sigma(x \otimes y) = y \otimes x\) for any \(x \otimes y \in X \otimes Y\)), we recover the notion of \(K\)-algebra, \(K\)-coalgebra and \(K\)-bialgebra.

(3) In [8], a symmetric monoidal category was introduced such that Hom-algebras, Hom-coalgebras and Hom-bialgebras (see [13]) coincide with the algebras, coalgebras and bialgebras in this symmetric monoidal category.

1.2. Adjunction cocommutative bialgebras and cocommutative coalgebras. It is well-known that we have the following adjunction between cocommutative bialgebras and cocommutative coalgebras.

\[
\begin{array}{c}
\text{BiAlg}_{C, \text{coc}} \\
\downarrow \quad F
\end{array}
\begin{array}{c}
\text{CoAlg}_{C, \text{coc}}
\end{array}
\]

where \(U\) is the forgetful functor and \(F\) is the free algebra functor.

In particular, this adjunction implies that \(U\) preserves the limits and that a monomorphism of bialgebras is in particular also a monomorphism of coalgebras. This observation will be useful several times in this paper.

1.3. Limits in the category of cocommutative bialgebras. We recall the constructions of products, equalizers and pullbacks in the category of cocommutative bialgebras in a symmetric monoidal category that has equalizers. It is interesting to recall that the categorical product of cocommutative bialgebras is given by the monoidal product.

**Proposition 1.2.** In the category \(\text{BiAlg}_{C, \text{coc}}\), the categorical product of two cocommutative bialgebras \(A\) and \(B\) is given by the monoidal product \((A \otimes B, \pi_A, \pi_B)\) where the projections \(\pi_A: A \otimes B \to A\) and \(\pi_B: A \otimes B \to B\) are defined by \(\pi_A := 1_A \otimes \epsilon_B\) and \(\pi_B := \epsilon_A \otimes 1_B\).
From now on, we are considering a symmetric monoidal category $\mathcal{C}$ that has equalizers.

The equalizers are defined as in [1]. Let $f, g: A \to B$ be two morphisms of bialgebras, then the following construction in $\mathcal{C}$ is the equalizer of $f$ and $g$.

$$
\begin{array}{c}
E \xrightarrow{\varepsilon} A \xrightarrow{(g \otimes 1_A) \cdot \Delta} B \otimes A .
\end{array}
$$

where $\varepsilon$ is the equalizer of $(f \otimes 1_A) \cdot \Delta$ and $(g \otimes 1_A) \cdot \Delta$ in $\mathcal{C}$. We can easily check that this construction is the equalizer of $f$ and $g$ in $\text{BiAlg}_{\mathcal{C}, \text{cocom}}$.

Via the definitions of products and equalizers in $\text{BiAlg}_{\mathcal{C}, \text{cocom}}$, we define the pullback in $\text{BiAlg}_{\mathcal{C}, \text{cocom}}$ of two morphisms of bialgebras $f: A \to B$ and $g: C \to B$ as follows

$$
\begin{array}{c}
A \otimes_B C \xrightarrow{p_C} C \\
\downarrow p_A \quad \quad \quad \quad \quad \downarrow g \\
A \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad B
\end{array}
$$

where $A \otimes_B C$ is the object in the following equalizer

$$
\begin{array}{c}
A \otimes_B C \xrightarrow{\varepsilon} A \otimes C \xrightarrow{(1_A \otimes g \otimes 1_C) \cdot (1_A \otimes \Delta)} A \otimes B \otimes C .
\end{array}
$$

and the two projections are defined as $p_A := (1_A \otimes \varepsilon_C) \cdot \varepsilon$ and $p_C := (\varepsilon_A \otimes 1_C) \cdot \varepsilon$.

1.4. **Split extensions of cocommutative bialgebras.** In [17], we introduced a notion of split extensions of (non-associative) bialgebras and we showed several properties. Here, we recall this notion and some results in the case of cocommutative bialgebras that will be useful later on.

**Definition 1.3.** A *split extension of cocommutative bialgebras* is given by a diagram

$$
\begin{array}{c}
\xymatrix{ X & A \ar[l]_{\kappa} \ar[r]^{\lambda} & B } \\
\ar[rr]_{\varepsilon} & & \ar[ll]_{\alpha}
\end{array}
$$

where $X, A, B$ are cocommutative bialgebras, $\kappa, \alpha, e$ are morphisms of bialgebras, such that

1. $\lambda \cdot \kappa = 1_X, \alpha \cdot e = 1_B$ ,
2. $\lambda \cdot e = u_X \cdot \varepsilon_B, \alpha \cdot \kappa = u_B \cdot \varepsilon_X$,
3. $m \cdot ((\kappa \cdot \lambda) \otimes (e \cdot \alpha)) \cdot \Delta = 1_A$,
4. $\lambda \cdot m \cdot (\kappa \otimes e) = 1_X \otimes \varepsilon_B$,
5. $\lambda$ is a morphism of coalgebras preserving the unit.

We recall that the conditions $\lambda \cdot \kappa = 1_X, \lambda \cdot e = u_X \cdot \varepsilon_B$ and the preservation of the unit by $\lambda$ are consequences of the axiom (4).
Definition 1.4. A morphism of split extensions from the split extension \( X \xleftarrow{\lambda \kappa} A \xrightarrow{e \alpha} B \) to the split extension \( X' \xleftarrow{\lambda' \kappa'} A' \xrightarrow{e' \alpha'} B' \) is given by 3 morphisms of bialgebras \( g: B \to B' \), \( v: X \to X' \) and \( p: A \to A' \) such that the following diagram commutes

\[
\begin{array}{ccc}
X & \xleftarrow{\lambda \kappa} & A & \xrightarrow{e \alpha} & B \\
\downarrow v & & \downarrow p & & \downarrow g \\
X' & \xleftarrow{\lambda' \kappa'} & A' & \xrightarrow{e' \alpha'} & B'.
\end{array}
\]

The two definitions above give rise to the category \( \text{SplitExt}(\text{BiAlg}_{\mathcal{C}, \text{coc}}) \) of split extensions of bialgebras. In this paper, we will denote this class of split extensions of cocommutative bialgebras by \( S_{\text{coc}} \).

We recall a convenient equality.

Lemma 1.5. Let \( X \xleftarrow{\lambda \kappa} A \xrightarrow{e \alpha} B \) be a split extension of bialgebras, then the following identities hold,

\[
\lambda \cdot m = m \cdot (\lambda \otimes \lambda) \cdot (1_A \otimes (1_A \otimes (e \cdot \alpha) \otimes (\kappa \otimes \lambda))) \cdot (\Delta \otimes 1_A),
\]

where \( \triangleright = \lambda \cdot m \cdot (e \otimes \kappa) \).

Even if the category of cocommutative bialgebras is not protomodular (see [10]), we have an interesting result for the split extensions defined above: a relative form of the Split Short Five Lemma holds in \( \text{BiAlg}_{\mathcal{C}, \text{coc}} \).

Theorem 1.6. Let \( (g, v, p) \) be a morphism of split extensions of bialgebras in a symmetric monoidal category \( \mathcal{C} \)

\[
\begin{array}{ccc}
X & \xleftarrow{\lambda \kappa} & A & \xrightarrow{e \alpha} & B \\
\downarrow v & & \downarrow p & & \downarrow g \\
X' & \xleftarrow{\lambda' \kappa'} & A' & \xrightarrow{e' \alpha'} & B'.
\end{array}
\]

then \( p \) is an isomorphism whenever \( v \) and \( g \) are.

In [17], it was proved that there is an equivalence between the categories of actions and the one of split extensions of (cocommutative) bialgebras in a symmetric monoidal category. We recall this equivalence and the definition of the objects and morphisms of the category of actions of bialgebras.

Definition 1.7. Let \( X \) and \( B \) be cocommutative bialgebras in a symmetric monoidal category \( (\mathcal{C}, \otimes, I, \sigma) \). An action of bialgebras is a morphism in \( \mathcal{C}, \triangleright: B \otimes X \to X \), such that

\[
\begin{align*}
\triangleright \cdot (u_B \otimes 1_X) &= 1_X, \\
\triangleright \cdot (1_B \otimes \triangleright) &= \triangleright \cdot (m \otimes 1_X), \\
\triangleright \cdot (1_B \otimes u_X) &= u_X \cdot e_B,
\end{align*}
\]
is defined as a pair of morphisms of bialgebras \( \Delta \) and the category of bialgebras.

**Theorem 1.9.** There is an equivalence between the category \( \text{SplitExt}(\text{BiAlg}_{\text{coc}}) \) of split extensions of cocommutative bialgebras and the category \( \text{Act}(\text{BiAlg}_{\text{coc}}) \) of actions of cocommutative bialgebras.

**Proof.** The functor \( F: \text{SplitExt}(\text{BiAlg}_{\text{coc}}) \to \text{Act}(\text{BiAlg}_{\text{coc}}) \) is defined as

\[
F \begin{pmatrix}
X \xrightarrow{\lambda} A \\
\alpha \downarrow \\
B
\end{pmatrix}
\xrightarrow{\beta} 
F \begin{pmatrix}
X' \xrightarrow{\lambda'} A' \\
\alpha' \downarrow \\
B'
\end{pmatrix}
\]

where \( \beta = \lambda \cdot m \cdot (e \otimes \kappa) \). The functor \( G: \text{Act}(\text{BiAlg}_{\text{coc}}) \to \text{SplitExt}(\text{BiAlg}_{\text{coc}}) \) is defined as

\[
G \begin{pmatrix}
B \otimes X \xrightarrow{\beta} X \\
g \otimes v \downarrow \\
B' \otimes X'
\end{pmatrix}
\xrightarrow{v} 
G \begin{pmatrix}
X \xrightarrow{i_1} X \otimes B \\
i_2 \downarrow \\
B
\end{pmatrix}
\]

where \( i_1 = 1_X \otimes u_B, i_2 = u_X \otimes 1_B \), \( \pi_1 = 1_X \otimes \epsilon_B, \pi_2 = \epsilon_X \otimes 1_B \) and \( X \otimes B \) is the object \( X \otimes B \) where the bialgebra structure is given by the following morphisms of \( C \)

\[
m_{X \otimes B} = (m \otimes m) \cdot (1_X \otimes \Delta \otimes 1_B) \cdot (1_X \otimes \Delta \otimes 1_X \otimes 1_B)
\]

\[
u_{X \otimes B} = u_X \otimes u_B,
\]

\[
\Delta_{X \otimes B} = (1_X \otimes \sigma \otimes 1_B) \cdot (\Delta \otimes \Delta),
\]

\[
\epsilon_{X \otimes B} = \epsilon_X \otimes \epsilon_B.
\]

Let \( B \) and \( X \) be two cocommutative bialgebras, since the trivial action \( \epsilon_B \otimes 1_X : B \otimes X \to X \) is an action of bialgebras as defined in Definition 1.7, we obtain the following corollary.
Corollary 1.10. Let $B$ and $X$ be two cocommutative bialgebras, the projection $\pi_2$ of the product

$$X \overset{\pi_1}{\leftarrow} X \otimes B \overset{i_2}{\leftrightarrow} B \overset{\pi_2}{\leftarrow}$$

where $i_1 = 1_X \otimes u_B$, $i_2 = u_X \otimes 1_B$, $\pi_1 = 1_X \otimes \epsilon_B$ and $\pi_2 = \epsilon_X \otimes 1_B$, belongs to $S_{\text{coc}}$.

2. $S_{\text{coc}}$-Protomodularity

It is known that the category of cocommutative bialgebras is not protomodular. However, we have interesting results with respect to the class $S_{\text{coc}}$ of split extensions of cocommutative bialgebras, as the Split Short Five Lemma (Theorem 1.6) and the equivalence with the actions (Theorem 1.9).

These results are the motivation to prove that the category is $S_{\text{coc}}$-protomodular, which means protomodular with respect to $S_{\text{coc}}$.

Let $C$ be a pointed category, and $S$ be a class of split epimorphisms with their kernels (also called points), to recall the definition of an $S$-protomodular category we give the definition of a strong point.

**Definition 2.1.** A split epimorphism with its kernel $X \xrightarrow{\kappa} A \xleftarrow{\alpha} B$ is called a strong point whenever the kernel $\kappa$ and the splitting $\alpha$ are jointly strongly epimorphic.

**Definition 2.2.** The morphisms $v: X \to A$ and $w: B \to A$ are jointly strongly epimorphic if for any morphism $\mu: M \to A$ that factors through $v$ and $w$, $\mu$ is an isomorphism.

**Definition 2.3.** [5] Let $C$ be a pointed finitely complete category and $S$ a class of split epimorphisms stable under pullbacks. $C$ is said to be $S$-protomodular when the class $S$ is closed under finite limits and any point in $S$ is a strong point.

2.1. Pullback stable. We prove that the class $S_{\text{coc}}$ of split extensions of cocommutative bialgebras is pullback stable.

We consider the pullback of $X \xleftarrow{\lambda} A \xrightarrow{\alpha} B$ along the morphism of bialgebras $g: C \to B$.

$$X \xleftarrow{(\kappa, u_{C \cdot \epsilon_X})} A \otimes_B C \xrightarrow{(e \cdot g, 1_C)} C$$

(2.1)

where $(\kappa, u_{C \cdot \epsilon_X})$ and $(e \cdot g, 1_C)$ are the morphisms induced by the universal property of the pullback.
Proposition 2.4. Let (2.1) be the pullback of $X \xleftarrow{\lambda} A \xrightarrow{\kappa} B$ along a morphism of bialgebras $g: C \to B$, then the upper row of (2.1):

$$X \xleftarrow{\lambda \cdot p_A} \left((\kappa, \epsilon_X) \cdot p_C\right) \xrightarrow{(e \cdot g, 1_C)} A \otimes_B C \xrightarrow{p_C} C,$$

belongs to $S_{coc}$.

Proof. The conditions (1), (2), (4) and (5) of Definition 1.14 are easily checked. We give the details of the condition (3). To show that

$$m_{A \otimes_B C} \cdot ((\kappa, \epsilon_X) \cdot \lambda \cdot p_A \otimes (e \cdot g, 1_C) \cdot p_C) \cdot \Delta_{A \otimes_B C} = 1_{A \otimes_B C},$$

we compose the two sides of the equality with $p_A$ and $p_C$.

Since $p_A$ and $p_C$ are jointly monic in $\text{BiAlg}_{coc}$ (and then also jointly monic in $\text{CoAlg}_{coc}$ thanks to the adjunction (1.11)) and the cocommutativity implies that $m_{A \otimes_B C} \cdot ((\kappa, \epsilon_X) \cdot \lambda \cdot p_A \otimes (e \cdot g, 1_C) \cdot p_C) \cdot \Delta_{A \otimes_B C}$ is a morphism of coalgebras, we can conclude that the condition (3) holds. Hence, the class $S_{coc}$ of split extensions of cocommutative bialgebras is stable under pullbacks. \(\square\)

2.2. Closure under finite limits. To prove that the split extensions in $S_{coc}$ are closed under finite limits, we prove that they are closed under products and equalizers.

Proposition 2.5. The class $S_{coc}$ of split extensions of bialgebras is closed under finite limits.

Proof. Let $X \xleftarrow{\lambda} A \xrightarrow{\kappa} B$ and $X' \xleftarrow{\lambda'} A' \xrightarrow{\kappa'} A'$ two split extensions of bialgebras. We can prove that

$$X \otimes X' \xleftarrow{\lambda \otimes \lambda'} A \otimes A' \xrightarrow{\kappa \otimes \kappa'} B \otimes B'$$
belongs to $S_{\text{coc}}$. Since the construction is made component wise it is clear that (2.2) belongs to $S_{\text{coc}}$ as we can explicitly see in the proof of condition (3) via the commutativity of the following diagram:

\[
\begin{array}{ccccccccc}
A \otimes A' & \xrightarrow{\Delta^2} & A^2 \otimes A'^2 & \xrightarrow{1_A \otimes \sigma \otimes 1_A'} & (A \otimes A')^2 & \xrightarrow{\kappa \cdot \lambda \otimes \kappa' \cdot \lambda' \otimes e \cdot \alpha \otimes e' \cdot \alpha'} & (A \otimes A')^2 \\
& & & & \downarrow & & \\
1_A \otimes 1_A' & & & & \downarrow & & \\
A \otimes A' & & & & \downarrow & & \\
\end{array}
\]

The other conditions hold via similar computations.

The class $S_{\text{coc}}$ is also stable under equalizers. We construct the equalizer of two morphisms of split extensions of bialgebras $(g, v, p)$ and $(g', v', p')$.

\[
\begin{array}{ccccccccc}
\hat{E} & \xleftarrow{\hat{\lambda}} & E & \xleftarrow{\hat{\epsilon}} & E' \\
\hat{\epsilon} & & & & \hat{\alpha} & & \\
\hat{\kappa} & & & & \hat{\alpha'} & & \\
X' & \xleftarrow{g'} & A' & \xleftarrow{p'} & B' \\
& & & & \downarrow g & & \\
X & \xleftarrow{\lambda} & A & \xleftarrow{\alpha} & B \\
\end{array}
\]

where $\hat{\alpha}, \hat{\epsilon}, \hat{\kappa}$ and $\hat{\lambda}$ are induced by the universal properties of the equalizers. Note that $\hat{\lambda}$ is induced by the universal property of $\hat{\epsilon}$ seen as a coequalizer in $\text{CoAlg}_{\text{coc}}$ via (1.11). By using the fact that $\epsilon, \epsilon'$ and $\hat{\epsilon}$ are monomorphisms of bialgebras (and hence also of coalgebras) we can conclude that

(2.3) \[
\begin{array}{ccccccccc}
\hat{E} & \xleftarrow{\hat{\lambda}} & E & \xleftarrow{\hat{\epsilon}} & E' \\
\hat{\kappa} & & & & \hat{\alpha} & & \\
\end{array}
\]

belongs to $S_{\text{coc}}$. We give an explicit proof of the condition (4) via the commutativity of this diagram:

\[
\begin{array}{ccccccccc}
\hat{E} \otimes E' & \xrightarrow{\hat{k} \otimes \hat{\epsilon}} & E^2 & \xrightarrow{m} & E & \xrightarrow{\hat{\lambda}} & \hat{E} \\
\hat{\epsilon} \otimes \epsilon' & & & & \epsilon^2 & & \\
X' \otimes B' & \xrightarrow{\kappa' \otimes \epsilon'} & A'^2 & \xrightarrow{m} & A' & \xrightarrow{\lambda'} & X' \\
1_{\hat{E}} \otimes \epsilon & & & & 1_{X'} \otimes \epsilon_{B'} & & \\
\hat{E} & & & & \hat{\epsilon} & & \\
\end{array}
\]
Since $\hat{\varepsilon}$ is a monomorphism of coalgebras and $\tilde{\lambda} \cdot m \cdot (\tilde{\kappa} \otimes \tilde{\varepsilon})$ is a coalgebra morphism, we can conclude that (2.3) satisfies condition (4).

2.3. Strong points. It was proven in [17] that for a split extension of (cocommutative) bialgebras

\[ X \leftarrow \underline{\lambda} \kappa A \leftarrow \underline{e} \alpha B \]

$\kappa$ and $e$ are jointly epimorphic. Now we prove that for cocommutative bialgebras, $\kappa$ and $e$ are jointly strongly epimorphic and hence any split extension of cocommutative bialgebras is a strong point.

**Proposition 2.6.** Any split extension of cocommutative bialgebras is a strong point.

**Proof.** Let $X \leftarrow \underline{\lambda} \kappa A \leftarrow \underline{e} \alpha B$ be a split extension of cocommutative bialgebras and $\mu: M \to A$ be a monomorphism of bialgebras. Let $\kappa$ and $e$ factor through $\mu$:

\[
\begin{array}{ccc}
X & \xrightarrow{\kappa} & A \\
\downarrow{\delta} & & \downarrow{\mu} \\
M & \xrightarrow{\gamma} & B \\
\end{array}
\]

We can form the following commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\lambda \cdot \mu} & M \\
\downarrow{\delta} & & \downarrow{\alpha \cdot \mu} \\
X & \xleftarrow{\kappa} & A \\
\end{array}
\]

We prove that the upper row is in the class $S_{coc}$ by checking all the conditions of Definition 1.14. In particular, via the commutativity of the following diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\Delta} & M^2 \\
\downarrow{\mu^2} & & \downarrow{\mu^2} \\
A^2 & \xrightarrow{1_A} & A^2 \\
\end{array}
\]

we conclude that $X \leftarrow \underline{\lambda} \mu M \leftarrow \underline{\gamma} \alpha \mu B$ satisfies condition (3) since $\mu$ is a monomorphism of bialgebras (and then also a monomorphism of coalgebras (1.11)) and $m \cdot (\delta \cdot \lambda \cdot \mu \otimes \gamma \cdot \alpha \cdot \mu) \cdot \Delta$ is a coalgebra morphism thanks to the cocommutativity.

Thanks to the previous results we obtain the following theorem:
**Theorem 2.7.** The category of cocommutative bialgebras in any symmetric monoidal category is an $S_{\text{coc}}$-protomodular category with respect to the class $S_{\text{coc}}$ of split extensions of cocommutative bialgebras defined in Definition 1.14.

*Proof.* This theorem holds thanks to Proposition 2.4, Proposition 2.5 and Proposition 2.6. □

**Remark 2.8.** This theorem generalizes the result of [5] saying that the category of monoids is $S$-protomodular with respect to the class of Schreier split epimorphisms. Indeed, if $C$ is the symmetric monoidal category $(\text{Set}, \times, \{\ast\})$, then $S_{\text{coc}}$ becomes the class of the Schreier split epimorphisms and $\text{BiAlg}_{\text{Set}, \text{coc}}$ the category of monoids.

## 3. Huq commutator

In this section, we recall the notion of centrality in the sense of Huq in a pointed category with binary products. Moreover, we give an explicit description of this centrality for two subbialgebras of a cocommutative bialgebra.

Let $C$ be a pointed category with binary products, we say that two subobjects $x: X \to A$ and $y: Y \to A$ commute (or centralize) in the sense of Huq [12] if there exists a morphism $p: X \times Y \to A$ such that the following diagram commutes

$$
\begin{array}{ccc}
X & \xymatrix{& X \times Y \ar[dl]_x \ar[dr]^y} & Y \\
& A & \\
\end{array}
$$

Then we denote by $[X,Y] = 0$ the fact that the subobjects $X$ and $Y$ commute.

Note that in $\text{BiAlg}_{C, \text{coc}}$, if it exists, $p$ is unique thanks to Corollary 1.10 and Proposition 2.6.

**Proposition 3.1.** In $\text{BiAlg}_{C, \text{coc}}$, the following are equivalent for $x: X \to A$ and $y: Y \to A$ two subobjects of a cocommutative bialgebra $A$:

(i) $m \cdot \sigma \cdot (x \otimes y) = m \cdot (x \otimes y)$

(ii) $[X,Y] = 0$, i.e there exists a (unique) morphism of bialgebras $p: X \otimes Y \to A$ such that the following diagram commutes

$$
\begin{array}{ccc}
X & \xymatrix{& X \otimes Y \ar[dl]_x \ar[dr]^y} & Y \\
& A & \\
\end{array}
$$

(3.1)

*Proof.* $(i) \Rightarrow (ii)$, we define $p: X \otimes Y \to A$ by $p := m \cdot (x \otimes y)$. It is easy to see that this map makes the diagram (3.1) commute. Moreover, thanks to $(i)$ we can prove that $p$ is a morphism of bialgebras (this proof is straightforward and is left to the reader). On the other way around, since $p$ is a morphism of bialgebras which makes (3.1) commute, we can make the following diagram commute
and conclude that (i) holds. □

Note that we obtained a similar result in the case of cocommutative Hopf algebras in the paper [11].

4. Smith is Huq

Now we would like to compare the notion of centrality in the sense of Huq and in the sense of Smith.

We first recall the centrality of two equivalence relations in the sense of Smith in any category with pullbacks [4].

**Definition 4.1.** Let \((R, r_0, r_1)\) and \((S, s_0, s_1)\) be two equivalence relations over the same object \(X\).

We denote the pullback of \(s_0\) along \(r_1\) by \(R \times_X S\) and \(p_2\):

\[
\begin{array}{ccc}
R \times_X S & \rightarrow & S \\
p_1 & & s_0 \\
R & \rightarrow & X \\
\end{array}
\]

A connector between \(R\) and \(S\) is an arrow \(\hat{p}: R \times_X S \rightarrow X\) such that

- \(x \hat{p}(x, y, z)\) and \(z \hat{R}\hat{p}(x, y, z)\),
- \(\hat{p}(x, x, y) = y\) and \(\hat{p}(x, y, y) = x\),
- \(\hat{p}(x, y, \hat{p}(y, u, v)) = \hat{p}(x, u, v)\) and \(\hat{p}(\hat{p}(x, y, u), u, v) = \hat{p}(x, y, v)\),

where the above expressions are defined. When such an arrow exists, we say that the relations \(R\) and \(S\) centralize (in the sense of Smith).

Note that this notion of centrality of equivalence relations is not independent of the centrality in the sense of Huq. If two equivalence relations centralize each other (in the sense of Smith) then it is true that the normal subobjects associated with them, called normalizations [2], centralize in the sense of Huq [4].

The converse is not true in general. If \(C\) is a category such that any two equivalence relations always centralize each other as soon as their normalizations centralize in the sense of Huq, will say that \(C\) satisfies the **Smith is Huq** property [4].

For example, the categories of groups and cocommutative Hopf algebras over a field satisfy this condition [11].

In the paper [14], the authors studied this condition in the context of \(S\)-protomodular categories.
They considered pointed $S$-protomodular categories with respect to a class of points $S$ that are stable under composition and such that any product projection, i.e. any such diagram

$$
\begin{array}{c}
X & \xrightarrow{\pi_1} & X \times B & \xrightarrow{i_2} & B,
\end{array}
$$

where $i_1 := (1_X, 0)$ and $i_2 := (0, 1_B)$, belongs to the class $S$.

In that context, the condition Smith is Huq means that two $S$-equivalence relations centralize each other (in the sense of Smith) if and only if their normalization commute in the sense of Huq. The category of monoids (but also the category of monoids with operations [15]) satisfies this property, with respect to the class of Schreier split epimorphisms.

First, we prove that the class $S_{\text{coc}}$ is stable under composition.

**Proposition 4.2.** The class $S_{\text{coc}}$ of split extensions of cocommutative bialgebras is closed under composition.

**Proof.** Let $X \xleftarrow{\lambda} A \xrightarrow{\alpha} B$ and $Y \xleftarrow{\lambda'} B \xrightarrow{\alpha'} C$ be two composable split extensions. We can build the following diagram

$$
\begin{array}{ccc}
Z & \xleftarrow{\hat{\kappa}} & A & \xrightarrow{\alpha' \cdot \alpha} C
\end{array}
$$

where $\hat{\kappa} : Z \to A$ is the kernel of $\alpha' \cdot \alpha$ and $\hat{\lambda}$ is the factorization through $\hat{\kappa}$ of the morphism of coalgebras $m \cdot (\kappa \cdot \lambda \otimes e \cdot \kappa' \cdot \lambda' \cdot \alpha) \cdot \Delta$. In particular, we have the following equality

$$
\hat{\kappa} \cdot \hat{\lambda} = m \cdot (\kappa \cdot \lambda \otimes e \cdot \kappa' \cdot \lambda' \cdot \alpha) \cdot \Delta.
$$

The condition (4) holds thanks to the commutativity of the two diagrams in Figure 1, where $\alpha \cdot \hat{\kappa}$ is the factorization of $\alpha \cdot \hat{\kappa}$ through the kernel $\kappa'$ of $\alpha'$:

$$
\kappa' \cdot \alpha \cdot \hat{\kappa} = \alpha \cdot \hat{\kappa},
$$

and the fact that $\hat{\lambda} \cdot m \cdot (\hat{\kappa} \otimes e \cdot e')$ and $(1_Z \otimes \epsilon_C)$ are coalgebras morphisms and $\hat{\kappa}$ is a monomorphism of coalgebras. □
Thanks to this proposition and Corollary 1.10, BiAlg$_{C,coc}$ is a category in which we can apply the results obtained in [14]. To apply the results of [14], we recall the notions of $S$-equivalence relation and reflexive-multiplicative graph.

**Definition 4.3.** An equivalence relation

\[
A \xleftrightarrow{\alpha} e \xrightarrow{\beta} B
\]

is an $S$-equivalence relation if the point $(\alpha, e)$ is in $S$.

**Definition 4.4.** [9] In a category $C$ with pullbacks, a reflexive graph, denoted by $A \xleftrightarrow{\delta} \iota \xrightarrow{\gamma} A_0$, together with a morphism $c: A_1 \times_{A_0} A_1 \to A_1$ is called a reflexive-multiplicative graph

\[
(4.3) \\
A_1 \times_{A_0} A_1 \xrightarrow{c} A_1 \xleftrightarrow{\delta} \iota \xrightarrow{\gamma} A_0,
\]

where $A_1 \times_{A_0} A_1$ is the (object part of the) following pullback

\[
\begin{array}{ccc}
A_1 \times_{A_0} A_1 & \xrightarrow{p_2} & A_1 \\
\downarrow{p_1} & & \downarrow{\gamma} \\
A_1 & \xrightarrow{\delta} & A_0
\end{array}
\]

and $c$ is a multiplication that is required to satisfy the identities

\[
(4.4) \\
c \cdot (1_{A_1}, \iota \cdot \delta) = 1_{A_1} = c \cdot (\iota \cdot \gamma, 1_{A_1}),
\]

where $(1_{A_1}, \iota \cdot \delta): A_1 \to A_1 \times_{A_0} A_1$ and $(\iota \cdot \gamma, 1_{A_1}): A_1 \to A_1 \times_{A_0} A_1$ are induced by the universal property of the pullback $A_1 \times_{A_0} A_1$.

To apply Theorem 4.2 in [14] we need the following result.

**Proposition 4.5.** Every reflexive graph

\[
\begin{array}{ccc}
X' & \xrightarrow{\lambda'} & X \\
\downarrow{\kappa'} & & \downarrow{\lambda} \\
A & \xrightarrow{e} & B
\end{array}
\]

such that $X \xleftarrow{\lambda} \kappa \xrightarrow{\alpha} A \xleftarrow{e} \beta \xrightarrow{\gamma} B$ and $X' \xleftarrow{\lambda'} \kappa' \xrightarrow{\alpha} A \xleftarrow{e} \beta \xrightarrow{\gamma} B$ belong to $S_{coc}$ and $[X, X'] = 0$, is a reflexive-multiplicative graph.
Proof. Let consider the following pullback

\[
\begin{array}{ccc}
A \otimes_B A & \xrightarrow{p_2} & A \\
\downarrow p_1 & & \downarrow \beta \\
A & \xrightarrow{\alpha} & B
\end{array}
\]

where \( A \otimes_B A \) is the object in the following equalizer

\[
A \otimes_B A \xrightarrow{\varepsilon} A \otimes A \xrightarrow{(1_A \otimes \beta \otimes 1_C) \cdot (1_A \otimes \Delta)} (1_A \otimes \alpha \otimes 1_A) \cdot (\Delta \otimes 1_A) A \otimes B \otimes A.
\]

By defining \( c : A \otimes_B A \to A \) by

\[
c = m \cdot (\kappa \cdot \lambda \otimes 1_A) \cdot (\varepsilon \otimes \varepsilon)
\]

we can verify that this reflexive graph is multiplicative thanks to Figure 2 and Figure 3. □

This result and Theorem 4.2 in [14], implies the following result

Theorem 4.6. In \( \text{BiAlg}_{\text{cocom}} \), let \( S_{\text{cocom}} \) be the class of split extensions of cocommutative bialgebras as defined in Definition 1.14, two \( S_{\text{cocom}} \)-equivalence relations centralize each other (in the sense of Smith) if and only if their normalization commute in the sense of Huq.

By restricting the above theorem to the symmetric monoidal category \((\text{Set}, \times, \{\cdot\})\), we obtain the result on monoids of [14]. Note that, in [14], they also prove this result for every category of monoids with operations.

5. Conclusion

In [17], we introduced a class of split extensions of (cocommutative) bialgebras, called \( S_{\text{cocom}} \), such that their category is equivalent to the category of actions of (cocommutative) bialgebras and we proved the Split Short Five Lemma when we restrict it to the split extensions of (cocommutative) bialgebras. In this paper, we use this class \( S_{\text{cocom}} \), to prove that \( \text{BiAlg}_{\text{cocom}} \) is \( S_{\text{cocom}} \)-protomodular. It implies that we can now apply the theory and results of [7, 3] to obtain new results for cocommutative bialgebras. For example, it follows that \( \text{BiAlg}_{\text{cocom}} \) is a \( S_{\text{cocom}} \)-Mal’tsev category [7, 3], hence any \( S_{\text{cocom}} \)-reflexive relation is transitive. Another result of this paper is the description of the notion of centrality in the sense of Huq for cocommutative bialgebras. Moreover, we prove that \( \text{BiAlg}_{\text{cocom}} \) satisfies the “partial” Smith is Huq condition, meaning that two \( S_{\text{cocom}} \)-equivalence relations centralize each other as soon as their normalization commute in the sense of Huq. Note that Theorem 4.6 and Theorem 2.7 generalize results in [5] and [14] on the category of monoids.
Figure 1. Condition (4) of the composition of two split extensions
Figure 2. The morphism $c$ is a morphism of bialgebras (part 1)
Figure 3. The morphism $c$ is a morphism of bialgebras (part 2)
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