VANISHING OF COHOMOLOGY ASSOCIATED TO QUANTIZED DRINFELD-SOKOLOV REDUCTION

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Abstract. We prove a vanishing theorem of the cohomology arising from the two quantized Drinfeld-Sokolov reductions ("+" and "−" reduction) introduced by Feigin-Frenkel and Frenkel-Kac-Wakimoto. As a consequence, the vanishing conjecture of Frenkel-Kac-Wakimoto is proved for the "−" reduction and partially for the "+" reduction.

1. Introduction

In this paper we study the cohomology of the BRST complex of the quantized Drinfeld-Sokolov reductions introduced by Feigin-Frenkel [6] and Frenkel-Kac-Wakimoto [9] in their study of W-algebras.

Let \( \mathfrak{g} = \bar{\mathfrak{n}}_+ \oplus \mathfrak{h} \oplus \bar{\mathfrak{n}}_+ \) be a finite-dimensional complex simple Lie algebra. Let \( \mathfrak{g} = \bar{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}] \oplus C \mathfrak{K} \oplus C \mathfrak{D} \) be the affine Lie algebra associated to \( \mathfrak{g} \). Let \( \Delta^\text{rep} \) be the set of real positive roots of \( \mathfrak{g} \), \( \bar{\Delta}^\text{rep} \subset \Delta^\text{rep} \) the set of positive roots of \( \bar{\mathfrak{g}} \), \( \mathfrak{W} \) the Weyl group of \( \mathfrak{g} \). Let \( \kappa \in \mathbb{C} \setminus \{0\} \) and let \( \bar{\mathfrak{h}}^*_\kappa \) denote the set of the weights of \( \mathfrak{g} \) of level \( \kappa - h^\vee \). Let \( O_\kappa \) be the Bernstein-Gelfand-Gelfand category of \( \mathfrak{g} \) of level \( \kappa - h^\vee \), where \( h^\vee \) is the dual Coxeter number of \( \bar{\mathfrak{g}} \). Let \( L(\Lambda), \Lambda \in \bar{\mathfrak{h}}^*_\kappa \), be the simple module of \( O_\kappa \) of highest weight \( \Lambda \).

Let \( L\bar{\mathfrak{n}}_\pm = \bar{\mathfrak{n}}_\pm \otimes \mathbb{C}[t, t^{-1}] \subset \mathfrak{g} \). Fix a nondegenerate character \( \bar{\chi}_\pm \) as in [9, 2.1]. It extends to a character \( \chi_\pm : L\bar{\mathfrak{n}}_\pm \to \mathbb{C} \) by

\[
\chi_+(X \otimes t^n) = \delta_{n, -1} \bar{\chi}_+(X) \quad (X \in \bar{\mathfrak{n}}_+, n \in \mathbb{Z}), \\
\chi_-(X \otimes t^n) = \delta_{n, 0} \bar{\chi}_-(X) \quad (X \in \bar{\mathfrak{n}}_-, n \in \mathbb{Z}).
\]

Let \( C_{\chi_\pm} \) be the one-dimensional representation of \( U(L\bar{\mathfrak{n}}_\pm) \) defined by \( \chi_\pm \). Then, the semi-infinite cohomology \( H^*_{\text{QDS}}(L\bar{\mathfrak{n}}_\pm, V) = H^*_{\text{QDS}}(L\bar{\mathfrak{n}}_+, V \otimes \mathbb{C} \chi_+) \), \( V \in O_\kappa \), is called cohomology of the BRST complex of the quantized Drinfeld-Sokolov reduction for \( L\bar{\mathfrak{n}}_\pm \) ("+" and "−" reduction) associated to \( V \) ([6] [9] [8]).

Let \( V_\kappa(\bar{\mathfrak{g}}) = U(\mathfrak{g}) \otimes U(\bar{\mathfrak{g}}) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} \mathfrak{K} \oplus \mathbb{C} \mathfrak{D} \) be the universal affine vertex algebra associated to \( \bar{\mathfrak{g}} \) of level \( \kappa - h^\vee \). Then, the 0-th cohomology \( H^0_{\text{QDS}}(L\bar{\mathfrak{n}}_+, V_\kappa(\bar{\mathfrak{g}})) \) is the Feigin-Frenkel’s W-algebra \( W_\kappa(\bar{\mathfrak{g}}) \) associated to \( \mathfrak{g} \) of level \( \kappa - h^\vee \) ([6]). Their realization of \( W_\kappa(\bar{\mathfrak{g}}) \) gives a functor

\[
V \rightsquigarrow H^i_{\text{QDS}}(L\bar{\mathfrak{n}}_\pm, V) \quad (i \in \mathbb{Z})
\]

from \( O_\kappa \) to the category of \( W_\kappa(\bar{\mathfrak{g}}) \)-modules ([6] [9] [8]).
Let us now describe our result. Let $W^\Lambda \subset W$ be the integral Weyl group of a weight $\Lambda \in \mathfrak{h}^\ast$. For $\Lambda \in \mathfrak{h}_r^\ast$, let $O_\kappa^{[\Lambda]}$ be the full subcategory of $O_\kappa$ whose objects have all their local composition factors isomorphic to $L(w \circ \Lambda)$, $w \in W^\Lambda$. Then, $O_\kappa = \bigoplus_{\Lambda \in \mathfrak{h}_r^\ast/\sim} O_\kappa^{[\Lambda]}$, where $\sim$ is the equivalent relation defined by $\lambda \sim \mu \iff \mu \in W^\Lambda \circ \lambda$. The main result of this paper is the following.

Main theorem. Let $\Lambda \in \mathfrak{h}_r^\ast$, $\kappa \in \mathbb{C}\setminus\{0\}$. Then,

1. $H^{\ast}_{\text{QDS}}(L\bar{n}_-, V) = \{0\}$ ($i \neq 0$) for all objects $V$ in $O_\kappa^{[\Lambda]}$ if $(\Lambda, \bar{\alpha}^\vee) \notin \mathbb{Z}$ for all $\bar{\alpha} \in \bar{\Delta}_+$,

2. $H^{i}_{\text{QDS}}(L\bar{n}_+, V) = \{0\}$ ($i \neq 0$) for all objects $V$ in $O_\kappa^{[\Lambda]}$ if $(\Lambda, \alpha^\vee) \notin \mathbb{Z}$ for all $\alpha \in \Delta^\vee_r \cap i\bar{\rho}(\Delta^\vee_r) = \{-\bar{\alpha} + n\bar{\delta}; \bar{\alpha} \in \bar{\Delta}_+, 1 \leq n \leq \text{ht} \bar{\alpha}\}$.

This result appears as Theorem S.3 in this paper. This shows that the correspondence $V \rightsquigarrow H^0_{\text{QDS}}(L\bar{n}_\pm, V)$ defines an exact functor from $O_\kappa^{[\Lambda]}$ to the category of $W_\kappa(\bar{g})$-modules under the condition of $\Lambda$ described as above. The irreducibility of $H^0_{\text{QDS}}(L\bar{n}_+, L(\lambda))$ will be studied in our forthcoming paper.

Frenkel-Kac-Wakimoto [2] applied the functor (1.1) to the principal admissible representations of $\mathfrak{g}$ of fractional levels $\kappa - h^\vee$. They conjectured that a vanishing of cohomology holds for that case and that the functor (1.1) sends a principal admissible representation to zero or to an irreducible “minimal” representations of $W_\kappa(\bar{g})$. Based on the vanishing conjecture, they calculated the characters and fusion coefficients for conjectural “minimal” representations of $W_\kappa(\bar{g})$. Our result settles the vanishing conjecture of Frenkel-Kac-Wakimoto for the “$-$” reduction and partially for the “$+$” reduction. Though our result for the “$+$” reduction is partial, we remark that every conjectural irreducible “minimal” representation is isomorphic to $H^0_{\text{QDS}}(L\bar{n}_+, L(\lambda))$ for some principal admissible weight $\Lambda$ which satisfies the condition of Main theorem (2), see Remark S.4 [5].

This article is organized as follows. In section 2 we collect the necessary information about the affine Lie algebra $\mathfrak{g}$ and its representations. In section 3 we prove Theorem 3.1 and Theorem 3.2 which is needed in the later arguments. In section 4 we recall the definition of the cohomology $H^\ast_{\text{QDS}}(L\bar{n}_+, V)$ and define some operators acting on the corresponding complex. In particular, we define the degree operator which acts on $H^\ast_{\text{QDS}}(L\bar{n}_+, V)$, $V \in O_\kappa$, semisimply. This is essentially the operator $-L_0^\pm$ defined in [2, 3.1]. The difficulty dealing with this cohomology arises from the fact that by construction the corresponding eigenspaces of complexes themselves are not finite-dimensional in general. The results in section 5 are straightforward generalization of [8, 14.2]. Thus, Theorem 5.7 which states the vanishing of the cohomology with coefficient in Verma modules, was essentially proved in [8]. In section 6 we prove the corresponding statement for the dual $M(\lambda)^\ast$ of Verma module $M(\lambda)$ (Theorem 6.8). Though the usual duality (3) of semi-infinite cohomology cannot be applied for this cohomology, this is done by establishing the duality

\[ H^i_{\text{QDS}}(L\bar{n}_+, M(\lambda)^\ast) \cong H^{-i}_{\text{QDS}}(L\bar{n}_-, M(t_{-\rho^\vee} \circ \lambda))^\ast, \]

\[ H^i_{\text{QDS}}(L\bar{n}_-, M(\lambda)^\ast) \cong H^{-i}_{\text{QDS}}(L\bar{n}_-, M(w_0 \circ \lambda))^\ast, \]

for $i \in \mathbb{Z}$ under the similar restriction of $\lambda$ as in the main theorem. Here, $^\ast$ is the graded dual and $w_0$ is the longest element of the Weyl group of $\bar{g}$. This result
is proved by using some spectral sequences. The duality above may explain why the “-” reduction behaves “nicer”. In section 8, we estimate the eigenvalues of degree operators on $H^*_{QDS}(L\tilde{\alpha}, V)$. The results in this section play a crucial role in proving our main theorem when $\kappa \in \mathbb{Q}_{>0}$, that is, when the objects in $\mathcal{O}_\kappa$ do not necessarily have finite length. Finally, we give the proof of our main theorem in section 9.

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2. Preliminaries

2.1. Affine Lie algebra. In the sequel, we fix a nonzero complex number $\kappa$, a simple finite-dimensional complex Lie algebra $\hat{\mathfrak{g}}$ and a Cartan subalgebra $\mathfrak{h}$. Let $\hat{\Delta}$ denote the set of roots, $\Pi$ a basis of $\hat{\Delta}$, $\hat{\Delta}_+$ the set of positive roots, and $\hat{\Delta}_-$ $= -\hat{\Delta}_+$. This gives the triangular decomposition $\hat{\mathfrak{g}} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Let $\hat{Q}$ denote the root lattice, $\hat{Q}^\vee$ the coroot lattice and $\hat{P}^\vee$ the coweight lattice. Let $\hat{\rho}$ be the half sum of positive roots, $\hat{\rho}^\vee$ the half sum of positive coroots. For $\alpha \in \hat{\Delta}_+$, the number $\langle \alpha, \hat{\rho}^\vee \rangle$ is called the height of $\alpha$ and denote by $ht \alpha$. Let $\hat{W}$ be the Weyl group of $\hat{g}$, $w_0$ the longest element of $\hat{W}$.

Let $(\ , \ )$ be the normalized invariant inner product of $\hat{\mathfrak{g}}$. Thus, $(\ , \ ) = \frac{1}{h^\vee} \text{Killing form}$, where $h^\vee$ is the dual Coxeter number of $\hat{\mathfrak{g}}$. We identify $\mathfrak{h}$ and $\mathfrak{h}^*$ using the form. Then, $\alpha^\vee = 2\alpha/(\alpha, \alpha)$, $\alpha \in \hat{\Delta}$.

Let $\mathfrak{g} = \hat{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathfrak{K} \oplus \mathfrak{D}$ be the affine Lie algebra associated to $(\hat{\mathfrak{g}}, (\ , \ ))$, where $\mathfrak{K}$ is its central element and $\mathfrak{D}$ is the degree operator (see [13]). The bilinear form $(\ , \ )$ is naturally extended from $\hat{\mathfrak{g}}$ to $\mathfrak{g}$. Set $X(n) = X \otimes t^n$, $X \in \hat{\mathfrak{g}}$, $n \in \mathbb{Z}$. The subalgebra $\hat{\mathfrak{g}} \otimes \mathbb{C} \subset \mathfrak{g}$ is naturally identified with $\hat{\mathfrak{g}}$.

Fix the triangular decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ in the standard way. Thus,

$$\mathfrak{h} = \mathfrak{h} \oplus \mathfrak{K} \oplus \mathfrak{D},$$

$$\mathfrak{g}_- = \mathfrak{n}_- \otimes \mathbb{C}[t^{-1}] \oplus \mathfrak{h} \otimes \mathbb{C}[t^{-1}] t^{-1} \oplus \mathfrak{n}_+ \otimes \mathbb{C}[t^{-1}] t^{-1},$$

$$\mathfrak{g}_+ = \mathfrak{n}_- \otimes \mathbb{C}[t] t \oplus \mathfrak{h} \otimes \mathbb{C}[t] t \oplus \mathfrak{n}_+ \otimes \mathbb{C}[t].$$

Let $\mathfrak{h}^* = \mathfrak{h}^* \oplus \mathbb{C} \Lambda_0 \oplus \mathbb{C} \delta$ be the dual of $\mathfrak{h}$. Here, $\Lambda_0$ and $\delta$ are dual elements of $\mathfrak{K}$ and $\mathfrak{D}$ respectively. For $\lambda \in \mathfrak{h}^*$, the number $\langle \lambda, K \rangle$ is called the level of $\lambda$. Let $\mathfrak{h}_\kappa^*$ denote the set of the weights of level $\kappa - h^\vee$:

$$\mathfrak{h}_\kappa^* = \{ \lambda \in \mathfrak{h}^* ; \langle \lambda + \rho, K \rangle = \kappa \},$$

where, $\rho = \hat{\rho} + h^\vee \Lambda_0 \in \mathfrak{h}^*$. Let $\tilde{\lambda}$ be the restriction of $\lambda \in \mathfrak{h}^*$ to $\tilde{\mathfrak{h}}^*$. Let $\tilde{\Delta}$ be the set of roots of $\tilde{\mathfrak{g}}$, $\tilde{\Delta}_+$ the set of positive roots, $\tilde{\Delta}_-$ $= -\tilde{\Delta}_+$. Then, $\tilde{\Delta} = \tilde{\Delta}^{re} \sqcup \tilde{\Delta}^{im}$, where $\tilde{\Delta}^{re}$ is the set of real roots and $\tilde{\Delta}^{im}$ is the set of imaginary roots. Let $\Pi$ be the basis of $\tilde{\Delta}^{re}$, $\tilde{\Delta}^{re} = \tilde{\Delta}^{re} \cap \tilde{\Delta}_+$, $\tilde{\Delta}^{im} = \tilde{\Delta}^{im} \cap \tilde{\Delta}_+$. Let $Q$ be the root lattice, $Q_+ = \sum_{\alpha \in \tilde{\Delta}_+} \mathbb{Z}_{\geq 0} \alpha \subset Q$.

Let $W \subset GL(\mathfrak{h}^*)$ be the Weyl group of $\mathfrak{g}$ generated by the reflections $s_\alpha$, $\alpha \in \tilde{\Delta}^{re}$, defined by $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$. Then, $W = \tilde{W} \ltimes \tilde{Q}^\vee$. Let $\tilde{W} = \tilde{W} \ltimes \tilde{P}^\vee$, the
extended Weyl group of $\mathfrak{g}$. For $\mu \in P^\vee$, we denote the corresponding element of $\widetilde{W}$ by $t_\mu$. Then,

$$t_\mu(\lambda) = \lambda + \langle \lambda, K \rangle \mu - \left( \langle \lambda, \mu \rangle + \frac{1}{2} |\mu|^2 \langle \lambda, K \rangle \right) \delta \quad (\lambda \in \mathfrak{h}^*) .$$

Let $\widetilde{W}_+ = \{ w \in \widetilde{W}; \Delta^\text{re}_+ \cap w^{-1}(\Delta^\text{re}) = \emptyset \}$. Then, $\widetilde{W} = \widetilde{W}_+ \times W$.

The dot action of $\widetilde{W}$ on $\mathfrak{h}^*$ is defined by $w \circ \lambda = w(\lambda + \rho) - \rho$ ($\lambda \in \mathfrak{h}^*$).

For $\Lambda \in \mathfrak{h}^*$, let $R^\Lambda = \{ \alpha \in \Delta^\text{re}; (\Lambda + \rho, \alpha^\vee) \in \mathbb{Z} \}$, $R^\Lambda_+ = R^\Lambda \cap \Delta^\text{re}_+$, $\Pi^\Lambda = \{ \alpha \in R^\Lambda_+; \check{s}_\alpha(R^\Lambda_+ \setminus \{ \alpha \}) \subset R^\Lambda_+ \}$. It is known that $R^\Lambda$ is a subroot system of $\Delta^\text{re}$ with the basis $\Pi^\Lambda$ (10, 16). Let $Q^\Lambda = \sum_{\alpha \in \Pi^\Lambda} \mathbb{Z} \alpha \subset Q$, $Q^\Lambda_+ = \sum_{\alpha \in \Pi^\Lambda} \mathbb{Z}_{\geq 0} \alpha$. For $\mu = \sum_{\alpha \in \Pi^\Lambda} m_\alpha \alpha \in Q^\Lambda_+$, $m_\alpha \in \mathbb{Z}_{\geq 0}$, set $\text{ht}_\Lambda(\mu) = \sum_{\alpha \in \Pi^\Lambda} m_\alpha$. Let $W^\Lambda = \{ s_\alpha; \alpha \in R^\Lambda \}$ be the integral Weyl group corresponding to $\Lambda$. Then, $R^{w \circ \Lambda} = R^w$ for $w \in W^\Lambda$.

2.2. BGG category of $\mathfrak{g}$. For a $\mathfrak{g}$-module $V$ (or simply a $\mathfrak{h}$-module $V$), let $V^\lambda = \{ v \in V; hv = \lambda(h)v \text{ for } h \in \mathfrak{h} \}$ be the weight space of weight $\lambda$. Let $P(V) = \{ \lambda \in \mathfrak{h}^*; V^\lambda \neq \{ 0 \} \}$. If $\dim V^\lambda < \infty$ for all $\lambda$, then we set

$$V^* = \bigoplus_\lambda \text{Hom}_C(V^\lambda, C) \subset \text{Hom}_C(V, C). \quad (2.1)$$

The formal character $\text{ch} V$ of $V$ is defined as $\text{ch} V = \sum_\lambda e^{\lambda} \dim_C V^\lambda$.

Let $\mathcal{O}_\kappa$ be the full subcategory of the category of left $\mathfrak{g}$-modules consisting of objects $V$ such that (1) $V$ is locally finite over $\mathfrak{g}_+$, (2) $V = \bigoplus_{\lambda \in \mathfrak{h}^*_+} V^\lambda$ and $\dim_C V^\lambda < \infty$ for all $\lambda$, (3) there exists a finite subset $\{ \mu_1, \ldots, \mu_n \} \subset \mathfrak{h}^*_+$ such that $P(V) \subset \bigcup_\lambda \mu_i - Q_+$. The correspondence $V \mapsto V^*$ defines the duality functor in $\mathcal{O}_\kappa$. Here, $\mathfrak{g}$ acts on $V^*$ by $(X f)(v) = f(X^t v)$, where $X \mapsto X^t$ is the Chevalley antiautomorphism. For a subalgebra $\mathfrak{a} \subset \mathfrak{g}$, we set

$$\mathfrak{a}^t = \{ X^t; X \in \mathfrak{a} \} \subset \mathfrak{g} .$$

Let $M(\lambda) \in \mathcal{O}_\kappa$, $\lambda \in \mathfrak{h}^*_+$, be the Verma module of highest weight $\lambda$ and $L(\lambda)$ its unique simple quotient. Let $\mathcal{O}^{[\Lambda]}_\kappa$, $\Lambda \in \mathfrak{h}^*_+$, be the full subcategory of $\mathcal{O}_\kappa$ whose objects have all their local composition factors isomorphic to $L(w \circ \Lambda)$, $w \in W^\Lambda$. By (19), $\mathcal{O}_\kappa$ splits into the orthogonal direct sum $\mathcal{O}_\kappa = \bigoplus_{\Lambda \in \mathfrak{h}^*_+/\sim} \mathcal{O}^{[\Lambda]}_\kappa$, where $\sim$ is the equivalent relation defined by $\lambda \sim \mu$ if $\lambda - \mu \in W^\Lambda \circ \lambda$. Orthogonal here means that $\text{Ext}_{\mathcal{O}_\kappa}^1(M, N) = 0$ for $M \in \mathcal{O}^{[\Lambda]}_\kappa$, $N \in \mathcal{O}^{[\Lambda']}_\kappa$, $\Lambda \neq \Lambda'$ in $\mathfrak{h}^*_+ / \sim$.

3. Some results on $\mathcal{O}_\kappa$

3.1. For $w \in \widetilde{W}$, let $\mathfrak{g}_w = \mathfrak{g}_+ \cap w(\mathfrak{g}_-) \subset \mathfrak{g}_+$. Then, $\mathfrak{g}_w^t = \mathfrak{g}_- \cap w(\mathfrak{g}_+) \subset \mathfrak{g}_-$. In this section we shall prove the following two theorems which will be needed in the later arguments.

**Theorem 3.1.** Let $\lambda \in \mathfrak{h}^*$. Suppose that $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{\geq 1}$ for all $\alpha \in \Delta^\text{re} \cap w(\Delta^\text{re})$. Then, $M(\lambda)$ is cofree over $\mathfrak{g}_w$.

**Theorem 3.2.** Let $w \in \widetilde{W}$ and $\Lambda \in \mathfrak{h}^*_+$ such that $\langle \Lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}$ for all $\alpha \in \Delta^\text{re}_+ \cap w(\Delta^\text{re})$. Then, any object in $\mathcal{O}^{[\Lambda]}_\kappa$ is free over $\mathfrak{g}_w$. 
3.2. Let us start with the following lemma:

Lemma 3.3. Let $m$ be an ad-$\mathfrak{h}$-stable subalgebra of $\mathfrak{g}$. Let $V$ be a module over $m \oplus \mathfrak{h} \subset \mathfrak{g}$ such that $V = \bigoplus_{\lambda \in \Lambda} V^\lambda$ and $P(V) \subset \bigcup_i \mu_i - Q_+$ for some finite subset $\{\mu_1, \ldots, \mu_n\} \subset \mathfrak{h}^*$. Suppose that $V = mV$. Then, $\bar{V} = \{0\}$.

Proof. Suppose that $V \neq \{0\}$. Then, there exists $\mu \in P(V)$ such that $\mu + \alpha \notin P(V)$ for any $\alpha \in Q_+$. But this contradicts $V = mV$. □

Proposition 3.4. Let $m$ be an ad-$\mathfrak{h}$-stable subalgebra of $\mathfrak{g}$. Then, for $V \in \mathcal{O}_\kappa$, the following conditions are equivalent:

1. $V$ is free over $m$.
2. $H_1(m, V) = 0$.

Proof. Clearly (1) implies (2). Let us show (2) ⇒ (1). Let $\{\bar{v}_j; j \in J\}$ be a basis of $H_0(m, V) = V/\mathfrak{m}V$ and let $v_j, j \in J$, be an inverse image of $\bar{v}_j$ in $V$. Since $V/\mathfrak{m}V$ is naturally a $\mathfrak{h}$-module, we may suppose that each $v_j$ is a weight vector of $V$. We claim that $V = \bigoplus_{j \in J} U(m)v_j$. Indeed, $V = \bigoplus_{j \in J} U(m)v_j + \mathfrak{m}V$. Let $\bar{V} = V/\bigoplus_{j \in J} U(m)v_j$. Then, $\bar{V} = \mathfrak{m}V$ and $P(V) \subset P(\bar{V})$. Thus, $\bar{V} = \{0\}$ by Lemma 3.3.

Let $V_1 = U(\mathfrak{m})v_j$, the free $U(\mathfrak{m})$-module with a basis $\{v_j; j \in J\}$. Let $\mathfrak{h}$ act semisimply on $V_1$ so that the natural map $\pi : V_1 \rightarrow V$ is a homomorphism of $(\mathfrak{m} \oplus \mathfrak{h})$-module. Let $M = \ker \pi$. Then, $M = \bigoplus_{\lambda \in \Lambda} M^\lambda$ and $P(M) \subset P(V) + P(U(\mathfrak{m}))$. Here, $\mathfrak{h}$ acts on $U(\mathfrak{m})$ by adjoint. Now suppose that $H_1(m, V) = 0$. Then, by the long exact sequence

$$\cdots \rightarrow H_1(m, V) \rightarrow H_0(m, M) \rightarrow H_0(m, V_1) \rightarrow H_0(m, V) \rightarrow 0$$

of $m$-homology, it follows that $H_0(m, M) = 0$, that is, $M = mM$. Hence $M = \{0\}$ by Lemma 3.3. □

3.3. Arkhipov’s twisting functor. Let $S_w$ be the Arkhipov’s semiregular module corresponding to $w \in W$ ([2], see also [11, 21]). It is a $U(\mathfrak{g})$-bimodule and

$$S_w = U(\mathfrak{g}_w)^* \otimes_{U(\mathfrak{g}_w)} U(\mathfrak{g})$$

as a left $U(\mathfrak{g}_w)$-module and a right $U(\mathfrak{g})$-module.

$$= U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_w)} U(\mathfrak{g}_w)^*$$

as a left $U(\mathfrak{g})$-module and a right $U(\mathfrak{g}_w)$-module).

Here, $U(\mathfrak{g}_w)^*$ is considered to be a $U(\mathfrak{g}_w)$-bimodule by $(fx)\,(n) = f(x\,(n)), (xf)\,(n) = f(n\,x), x \in \mathfrak{g}_w, f \in U(\mathfrak{g}_w)^*, n \in U(\mathfrak{g}_w)$.

If $V$ is a $\mathfrak{g}$-module and $w \in \widehat{W}$, we obtain a new $\mathfrak{g}$-module by twisting the action on $V$ as $X \cdot v = w^{-1}(X)v, X \in \mathfrak{g}$. The module obtained in this way we shall denote by $\phi_w(V)$.

Arkhipov [2] defined a twisting functor $T_w : \mathcal{O}_\kappa^{[A]} \rightarrow \mathcal{O}_\kappa^{[w \circ A]}$, $w \in W$, by

$$T_w(V) = S_w \otimes_{U(\mathfrak{g})} \phi_w(V).$$

Let $w = s_{j_1} \ldots s_{j_t}$ be a reduced expression of $w \in W$. Then, we have

$$T_w = T_{s_{j_1}} \circ \cdots \circ T_{s_{j_t}}.$$  \hspace{1cm} (3.1)

We extend the functor $T_w : \mathcal{O}_\kappa^{[A]} \rightarrow \mathcal{O}_\kappa^{[w \circ A]}$ for $w \in \widehat{W}$ as follows: For $x \in \widehat{W}$, and a $\mathfrak{g}$-module $V$, let $T_x(V)$ be the $\mathfrak{g}$-module obtained from $\phi_x(V)$ by twisting the
action as $D \cdot v = (D + \langle x(\rho) - \rho, D \rangle) \text{id}v$ and $X \cdot v = Xv$ ($X \in [g, g], v \in \phi_{\ast}(V)$). Then, $T_w(M(\lambda)) = M(x \circ \lambda)$ ($x \in \tilde{W}_+$). Set for $\tilde{w} = xw \in \tilde{W}$ ($x \in \tilde{W}_+, w \in W$),
\[
T_w(V) = T_x(T_w(V)).
\]
Note that
\[
T_w(V) = U(g_w)^* \otimes U(g_w)\phi_w(V)
\]
as $U(g_w)$-modules.

3.4. Proof of Theorem 3.1 and Theorem 3.2

**Proof of Theorem 3.1** By the proof of Proposition 6.3 (ii), one sees that
\[
M(\lambda) = T_w(M(w^{-1} \circ \lambda)) \quad \text{if } \langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{\geq 1} \text{ for all } \alpha \in \Delta^+ \cap \Delta^w(\alpha^w) \quad (3.4)
\]
for $w \in \tilde{W}$. Since $M(\lambda), \lambda \in \mathfrak{h}^\ast$, is free over $w^{-1}(g_w) \subset g_-$, Theorem 3.1 immediately follows from (3.3) and (3.4). □

**Proof of Theorem 3.2** It is easy to see that Proposition reduces the case when $w \in \tilde{W}$. We shall proceed by induction on $\ell(w)$ for $w \in \tilde{W}$.

The case when $\ell(w) = 1$ follows from [17] Lemma 4.1. Let $w = y\gamma, \gamma \in \Pi, \ell(w) = \ell(y) + 1$. Set $\beta = y(\alpha) \in \Delta^w_t$. Then,
\[
\Delta^w_t \cap w(\Delta^w_t) = \Delta^w_t \cap y(\Delta^w_t) \cup \{\beta\}
\]
\[
g'_w = g'_y \oplus \mathbb{C}x_{-\beta}, \quad [g'_w, g'_w] \subset g'_y.
\]
Here, $x_{-\beta}$ is a root vector of $g$ of root $-\beta$.

Let $\Lambda$ be as in Theorem and $V \in \mathcal{O}_k[\Lambda]$. By (3.5) and the induction hypothesis, $V$ is free over $g'_y$. We shall show that $V/\mathfrak{g}'_y V$ is free over $\mathbb{C}x_{-\beta}$:

Let $V' = \phi_y(T_{y^{-1}}(V))$. Since $T_{y^{-1}}(V) = \phi_y^{-1} \left(U(g'_y)^* \otimes U(g'_y)^{\ast} V\right)$, it follows that $V' = U(g'_y)^* \otimes U(g'_y)^{\ast} V$. The freeness of $V$ over $g'_y$ implies that
\[
(V')^{\ast} g'_y = \mathbb{C}1^\ast \otimes U(g'_y)^{\ast} V \cong V/\mathfrak{g}'_y V
\]
where $(V')^{\ast} g'_y = H^0(\mathfrak{g}'_y, V') \subset V'$ and $1^\ast \in U(g'_y)^{\ast}$ is the dual element of $1 \in U(g'_y)$.

We claim that (3.7) is an isomorphism of $\mathbb{C}x_{-\beta}$-modules. Indeed, one can show that $x_{-\beta}$ acts on $V'$ through $U(g'_y)^* \otimes U(g'_y)^{\ast} V$ as
\[
x_{-\beta}(f \otimes v) = (\text{ad}^\ast(x_{-\beta}) f) \otimes v + f \otimes x_{-\beta} v \quad (f \in U(g'_y)^{\ast}, v \in V),
\]
where $(\text{ad}^\ast(x_{-\beta}) f) (n) = -f([x_{-\beta}, n])$.

Because $T_{y^{-1}}(V) \in \mathcal{O}_k[y^{-1} \circ \Lambda]$ and $(y^{-1} \circ \alpha, \alpha) \notin \mathbb{Z}$, it follows that $T_{y^{-1}}(V)$ is free over $\mathbb{C}x_{-\alpha}$ by [17] Lemma 4.1, and thus $V'$ is free over $\mathbb{C}x_{-\beta}$. Therefore, $(V')^{\ast} g'_y \subset V'$ is also free over $\mathbb{C}x_{-\beta}$ since $U(\mathbb{C}x_{-\beta}) = \mathbb{C}[x_{-\beta}]$ is a principal ideal domain. Hence we conclude that $V/\mathfrak{g}'_y V$ is free over $\mathbb{C}x_{-\beta}$, proving $H_i(\mathbb{C}x_{-\beta}, V/\mathfrak{g}'_y V) = 0$ for $i \neq 0$. But then the Hochschild-Serre spectral sequence for the ideal $g'_y \subset g'_w$ proves that $H_i(g'_w, V) = 0$ for $i \neq 0$. This proves Theorem 3.1 by Proposition 3.2. □

**Remark 3.5.** Let $w \in \tilde{W}$ and $\Lambda = \mathfrak{h}_k^\ast$ as in Theorem 3.2. Then, one can prove that the functor $T_w$ defines an equivalence of categories $\mathcal{O}_k[y^{-1} \circ \Lambda] \cong \mathcal{O}_k[\Lambda]$ such that $T_w(M(w^{-1} \circ \lambda)) = M(\lambda), T_w(L(w^{-1} \circ \lambda)) = L(\lambda)$ for $\lambda \in W^\wedge \circ \Lambda$. 
4. The BRST complex

In this section we collect necessary information from [16, 17, 22] about the BRST complex of the quantized Drinfeld-Sokolov reductions.

4.1. Notations. Let \( n \) denote \( \tilde{L}n_+ \) or \( L\tilde{n}_- \). Here, \( \tilde{L}n_\pm = \tilde{n}_\pm \otimes \mathbb{C}[t, t^{-1}] \subset g \) as in Introduction.

Let \( \tilde{n} = n \cap \tilde{g} \) and \( \tilde{n}' = n' \cap \tilde{g} \). Then, \( \tilde{g} = \tilde{n} \oplus \tilde{h} \oplus \tilde{n}' \). Set \( n_\pm = n \cap g_\pm \) and \( n'_\pm = n' \cap g_\pm \). Then, \( n_\pm = n_\pm \oplus n_\mp \) and \( n'_\pm = n'_\pm \oplus n'_\mp \).

Let \( \tilde{\Delta}(n) = \begin{cases} \Delta_+ & (\text{for } n = \tilde{L}n_+), \\ \Delta_- & (\text{for } n = L\tilde{n}_-). \end{cases} \) For an ad\( h \)-stable subspace \( m \) of \( g \), let \( \Delta^{re}(m) = \{ \alpha \in \Delta^{re}; g_\alpha \cap m \neq \{0\} \} \), where \( g_\alpha \subset g \) is the root space of root \( \alpha \).

Then, \( \Delta^{re} = \Delta^{re}(n) \cup \Delta^{re}(n') \), \( \Delta^{re}_\pm = \Delta^{re}(n_\pm) \cup \Delta^{re}(n'_\pm) \).

Let \( \widetilde{h} = h \oplus \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D \) be the Heisenberg subalgebra of \( g \). Let \( b = n \oplus \widetilde{h} \subset g \). Then, \( g = n' \oplus b = n \oplus b' \).

Set \( b_- = b \cap g_- \), \( b_+ = b \cap (h \oplus g_+) \) so that \( b = b_- \oplus b_+ \). Similarly, let \( b'_- = b' \cap g_- \) and \( b'_+ = b' \cap (h \oplus g_+) \). Then, \( g_- = n_- \oplus b'_- = n'_- \oplus b_- \) and \( h \oplus g_+ = n_+ \oplus b'_+ = n'_+ \oplus b_+ \) (see Table 1).

Set \( B = U(b) \otimes \Lambda(n) \), where \( \Lambda(n) \) is the Grassmann algebra of \( n \). We regard \( B \) as a \( \mathbb{C} \)-algebra containing \( U(b) \) and \( \Lambda(n) \) as its subalgebras such that \( [X, \omega] = \text{ad}(X)(\omega) = \Lambda(n) \) for \( X \in b \) and \( \omega \in \Lambda(n) \). Then, \( B = U(b) \Lambda(n) = \Lambda(n) \cdot U(b) \). Let \( N = U(n) \otimes \Lambda(n) = U(n \cdot U(n)) \subset B \). Similarly, we define algebras \( B_{\pm} = U(b_{\pm}) \otimes U(n_{\pm}) \subset B \), \( B' = U(b') \otimes \Lambda(n') \), \( B'_\pm = U(b'_\pm) \otimes \Lambda(n'_\pm) \), \( N'_\pm = U(n'_\pm) \otimes \Lambda(n'_\pm) \), \( w(B) = U(w(b)) \otimes \Lambda(w(n)) \) \( \omega \in \tilde{W} \) and so on.

Let \( t : B \to B^t \) be the algebra anti-isomorphism induced by the Chevalley anti-isomorphism of \( g \).

4.2. The Clifford algebra. Let \( \tilde{I} = \{ 1, 2, \ldots, \text{rank } \tilde{g} \} \). Choose a basis \( \{ J_\alpha; a \in \tilde{I} \cup \tilde{\Delta} \} \) of \( \tilde{g} \) such that \( J_\alpha \in \tilde{g}_\alpha \), \( (J_\alpha, J_{-\alpha}) = 1 \) and \( (J_\alpha)^t = J_{-\alpha} \). Let \( c_{\alpha, \beta} \) be the structure constant with respect to this basis. Then, \( c_{\alpha, \beta} = -c_{-\alpha, -\beta} \). In the sequel, we identify \( n^* \) with \( n^{**} \) via \( (, ) \) (observe \( (L\tilde{n}_\pm)^t = L\tilde{n}_\mp \)).

Let \( C \) be the Clifford algebra associated to \( n \oplus n^* = n \oplus n' \) and its natural symmetric bilinear form. Denote by \( \psi_\alpha(n) \), \( \alpha \in \tilde{\Delta} \), \( n \in \mathbb{Z} \), the generators of \( C \) which correspond to the elements \( J_\alpha(n) = (J_{-\alpha}(-n))^* \). Then, \( \{ \psi_\alpha(m), \psi_\beta(n) \} = \delta_{\alpha + \beta, 0} \delta_{m+n, 0} \). Here, \( \{ x, y \} = xy + yx \). The algebra \( C \) contains \( \Lambda(n), \Lambda(n') \) as its subalgebras and \( C \) as \( \mathbb{C} \)-vector spaces.
Let $\mathcal{F}(n)$ be the irreducible representation of $Cl$ generated a vector $1$ such that $\psi_\alpha(n)1 = 0$ ($\alpha \in \Delta, n \in \mathbb{Z}, \alpha + n\delta \in \Delta^\text{red}$). Then, $\mathcal{F}(n) = \Lambda(n^+) \otimes \Lambda(n_-)$ as $\mathbb{C}$-vector spaces. Let

$$\mathcal{F}^p(n) = \sum_{i-j=p} \Lambda^i(n^+) \otimes \Lambda^j(n_-) \subset \mathcal{F}(n) \quad (p \in \mathbb{Z}).$$

Then, $\mathcal{F}(n) = \bigoplus_{p \in \mathbb{Z}} \mathcal{F}^p(n)$. By definition,

$$\psi_\alpha(n)\mathcal{F}^i(n) \subset \mathcal{F}^{i-1}(n) \text{ and } \psi_{-\alpha}(n)\mathcal{F}^i(n) \subset \mathcal{F}^{i+1}(n) \quad (\alpha \in \Delta(n), n \in \mathbb{Z}).$$

Obviously, $\mathcal{F}(n) \cong F(n^\dagger)$ as $Cl$-modules, but their gradings are opposite.

Let $\iota : Cl \to Cl$ be the algebra anti-isomorphism defined by $\psi_\alpha(n) \mapsto \psi_{-\alpha}(-n)$ ($\alpha \in \Delta, n \in \mathbb{Z}$). Then, there is a unique non-degenerate bilinear form $\langle \cdot, \cdot \rangle_F : \mathcal{F}(n) \times \mathcal{F}(n) \to \mathbb{C}$ such that $\langle 1, 1 \rangle_F = 1, \langle \psi, \psi' \rangle_F = 0, \psi \in Cl, \psi \in \mathcal{F}(n), \psi' \in \mathcal{F}(n)$. It is non-degenerate on $\mathcal{F}^1(n) \times \mathcal{F}^1(n)$.

### 4.3. The complex $C(n, V)$

For $V \in \mathcal{O}_\kappa$, let

$$C(n, V) = V \otimes \mathcal{F}(n) = \bigoplus_{i \in \mathbb{Z}} C^i(n, V), \quad \text{where } C^i(n, V) = V \otimes \mathcal{F}^i(n).$$

Define the operator $d^\text{st}_n$ on $C(n, V)$ by

$$d^\text{st}_n = \sum_{\alpha \in \Delta(n), n \in \mathbb{Z}} J_\alpha(-n) \psi_{-\alpha}(n) - \frac{1}{2} \sum_{\alpha, \beta \in \Delta(n), k+l+m=0} c^{\beta}_{\alpha, \beta} \psi_{-\alpha}(k) \psi_{-\beta}(l) \psi_{-\gamma}(m)$$

(4.2)

Here, $J_\alpha(-n)$ acts on the first factor $V$ and $\psi_\alpha(n)$ acts on the second factor $\mathcal{F}(n)$. Then, $(d^\text{st}_n)^2 = 0, d^\text{st}_n C^i(n, V) \subset C^{i+1}(n, V)$. The cohomology $H^{\text{st}+\bullet}(n, V) = H^{\bullet}(C(n, V), d^\text{st}_n)$.

is called the semi-infinite cohomology of $n$ with coefficients in $V$ (4).

Define $\chi_n \in n^* \subset Cl$ by

$$\chi_n = \sum_{\alpha \in \mathfrak{fl}} \psi_{-\alpha}(1) \quad (\text{for } n = L\mathfrak{n}_+), \quad \chi_n = \sum_{\alpha \in \mathfrak{fl}} \psi_{\alpha}(0) \quad (\text{for } n = L\mathfrak{n}_-).$$

($\chi_n$ was denoted by $\chi_\pm$ in Introduction). Let $d_n = d^\text{st}_n + \chi_n$. Then, $\chi_n^2 = 0, \{\chi_n, d_n\} = 0$ and $\chi_n C^i(n, V) \subset C^{i+1}(n, V)$. In particular, $d_n^2 = 0$. Define

$$H^\text{st}_\kappa(n, V) = H^\bullet(C(n, V), d_n).$$

(4.3)

It is called the cohomology of the BRST complex of the quantized Drinfeld-Sokolov reduction for $n$ associated to $V$ (6).

### 4.4. The weight space decomposition

The space $C(n, V)$ is naturally a $\mathfrak{g}$-module, see (1). Let $C(n, V) = \bigoplus_{\lambda \in \mathfrak{h}^*} C(n, V)^\lambda$ be its weight space decomposition. Then, $\dim C(n, V)^\lambda < \infty$ for all $\lambda \in \mathfrak{h}^*$ and $V \in \mathcal{O}_\kappa$. We have:

$$d^\text{st}_n C(n, V)^\lambda \subset C(n, V)^\lambda,$$

(4.4)

$$\chi_n C(n, V)^\lambda \subset \sum_{\alpha \in \mathfrak{fl}} C(n, V)^{\lambda - \alpha - \delta} \quad (\text{for } n = L\mathfrak{n}_+),$$

(4.5)

$$\chi_n C(n, V)^\lambda \subset \sum_{\alpha \in \mathfrak{fl}} C(n, V)^{\lambda + \alpha} \quad (\text{for } n = L\mathfrak{n}_-).$$

(4.6)
By (4.4), the complex \((C(n, V), d_n^*\)) is a direct sum of finite-dimensional subcomplexes \(C(n, V)^\lambda, \lambda \in \mathfrak{h}^*\). Therefore, \(H^{\text{top}}(n, V), V \in \mathcal{O}_\kappa, \) admits a weight space decomposition: \(H^{\text{top}}(n, V) = \bigoplus_{\lambda \in \mathfrak{h}^*} H^{\text{top}}(n, V)^\lambda\), \(\dim H^{\text{top}}(n, V)^\lambda < \infty (\lambda \in \mathfrak{h}^*).\)

Note that \(\langle \cdot, \cdot \rangle\) induces a non-degenerate paring \(C(n, V^*) \times C(n, V) \rightarrow \mathbb{C}\) which is non-degenerate on \(C^i(n, V^*)^\lambda \times C^i(n, V)^\lambda, i \in \mathbb{Z}, \lambda \in \mathfrak{h}^*\). Thus,

\[
C^i(n, V^*) = C^i(n, V)^* \tag{4.7}
\]

as \(\mathbb{C}\)-vector spaces, where * is defined in (4.4). Let

\[
d_n^* = d_n^* + \chi_n \in \text{End} C(n, V). \tag{4.8}
\]

Here, \(d_n^*\) acts on \(C(n, V)\) by the identification \(C(n, V) = C(n^t, V)\), and

\[
\chi_n = \sum_{\alpha \in \Pi} \psi_\alpha (-1) \quad \text{(for } n = L\tilde{n}^+_t), \quad \chi_n^t = \sum_{\alpha \in \Pi} \psi_-\alpha (0) \quad \text{(for } n = L\tilde{n}^-_t).
\]

Then, \((d_n^*)^2 = 0, d_n^* C^i(n, V) \subset C^{i-1}(n, V),\) and

\[
(d_n^* f)(v) = f(d_n^* v) \quad (f \in C(n, V^*), v \in C(n, V)) \tag{4.9}
\]

under the identification (4.7).

### 4.5. The action of \(B\) on \(C(n, V)\)

In the sequel we follow [9] for the definition of the normal ordering \(\cdot\cdot\). Thus, \(\psi_\alpha(n) \psi_\beta(m) := \left\{
\begin{array}{ll}
\psi_\alpha(n) \psi_\beta(m) & (\alpha + n\delta \in \Delta^e), \\
-\psi_\beta(m) \psi_\alpha(n) & (\alpha + n\delta \in \Delta^e).
\end{array}
\right.
\]

and so on. We have:

\[
: \psi_\alpha(n) \psi_\beta(m) := - : \psi_\beta(m) \psi_\alpha(n): \quad (\alpha, \beta \in \tilde{\Delta}, n, m \in \mathbb{Z})
\]

Let

\[
\hat{J}_\alpha(n) = J_\alpha(n) + \sum_{\beta, n \in \tilde{\Delta}(n), \ k \in \mathbb{Z}} c_{\alpha, \beta, n} \psi_\gamma(n - k) \psi_-\beta(k) : \quad (a \in \tilde{\Delta}(n) \cup \tilde{I}, n \in \mathbb{Z}),
\]

\[
\hat{D} = D + \sum_{\alpha \in \tilde{\Delta}(n), \ n \in \mathbb{Z}} n : \psi_\alpha(n) \psi_-\alpha(-n):. \tag{4.10}
\]

Then, for \(V \in \mathcal{O}_\kappa,\) the correspondences

\[
\pi : B = U(\mathfrak{b}) \otimes \Delta(n) \rightarrow \text{End}_\mathbb{C} C(n, V)
\]

\[
\begin{array}{ll}
J_\alpha(n) & \mapsto \hat{J}_\alpha(n) \quad (a \in \tilde{\Delta}(n) \cup \tilde{I}, n \in \mathbb{Z}) \\
K & \mapsto \kappa \text{id} \\
D & \mapsto \hat{D} \\
\psi_\alpha(n) & \mapsto \psi_\alpha(n) \quad (\alpha \in \tilde{\Delta}(n), n \in \mathbb{Z})
\end{array}
\tag{4.12}
\]

defines a representation of \(B\) on \(C(n, V)\). We have:

\[
\hat{J}_\alpha(n) = \{d_n^* \psi_\alpha(n)\} \quad (\alpha \in \tilde{\Delta}(n), n \in \mathbb{Z}), \tag{4.13}
\]

\[
[d_n^*, \hat{J}_\alpha(n)] = 0 \quad (a \in \tilde{\Delta}(n) \cup \tilde{I}, n \in \mathbb{Z}), \tag{4.14}
\]

\[
[\chi_n, \hat{J}_\alpha(n)] = 0 \quad (\alpha \in \tilde{\Delta}(n), n \in \mathbb{Z}), \tag{4.15}
\]

\[
C(n, V)^\lambda = \{v \in C(n, V); \pi(h) v = \langle \lambda + h^\vee A_0, h \rangle v \} \quad (h \in \mathfrak{h}). \tag{4.16}
\]
Let $B^t$ act on $C(n, V)$ via the identification $C(n, V) = C(n^t, V)$. The representation of $B^t$ obtained in this way we shall denote by $\pi^t$. Set $\tilde{\mathcal{J}}_{-\alpha}(n) = \pi^t(J_{-\alpha}(n))$ $(\alpha \in \tilde{\Delta}(n), n \in \mathbb{Z})$. Observe $\pi_{\tilde{\mathcal{J}}} = \pi_{\tilde{\mathcal{J}}}^t$, and under the identification (4.4),

$$(\pi(b)f)(v) = f(\pi^t(b')v) \quad (b \in B, f \in C(n, V^*), v \in C(n, V)). \quad (4.17)$$

We have:

$$[d_n^{\pi^t}, \tilde{\mathcal{J}}_{-\alpha}(n)] = \sum_{\beta \in \tilde{\Delta}(n), h \in \tilde{\Delta}(n^t)} e_{\beta, -\alpha}^h : \psi_{-\beta}(k)J_{h}(n - k) : -nk_{\alpha}\psi_{-\alpha}(n) \quad (4.18)$$

for $\alpha \in \tilde{\Delta}(n), n \in \mathbb{Z}$, where $k_{\alpha} = h^{\vee} - \sum_{\beta, \gamma \in \tilde{\Delta}(n)} c_{\alpha, \beta}^\gamma c_{-\gamma, -\beta}^h \in \mathbb{C}$, and

$$[\chi_n, \tilde{\mathcal{J}}_a(n)] = \sum_{\beta, \gamma \in \tilde{\Delta}(n)} c_{\alpha, \beta}^\gamma \chi_n(J_{\gamma}(-1))\psi_{-\beta}(n + 1) \quad (for \ n = L\tilde{n}_+) \quad (4.19)$$

$$[\chi_n, \tilde{\mathcal{J}}_a(n)] = \sum_{\beta, \gamma \in \tilde{\Delta}(n)} c_{\alpha, \beta}^\gamma \chi_n(J_{\gamma}(0))\psi_{-\beta}(n) \quad (for \ n = L\tilde{n}_-) \quad (4.20)$$

for $a \in \tilde{\Delta}(n^t) \sqcup \tilde{I}$ and $n \in \mathbb{Z}$.

4.6. **The degree operator $D_n^{\pi^t}$.** Define

$$D_n^{\pi^t} = \hat{D} + \pi(\hat{\rho}^{\vee}) \quad (for \ n = L\tilde{n}_+),$$

$$D_n^{\pi^t} = \hat{D} + \left(\frac{1}{2}\hat{\rho}^{\vee}|^2 \kappa - \langle \hat{\rho}, \hat{\rho}^{\vee} \rangle \right) \text{id} \quad (for \ n = L\tilde{n}_-).$$

Set

$$h_n^\lambda = \langle \lambda, \hat{\rho}^{\vee} + D \rangle \quad (for \ n = L\tilde{n}_+), \quad h_n^\lambda = \langle t_{\hat{\rho}^{\vee}} \circ \lambda, \hat{\rho}^{\vee} + D \rangle \quad (for \ n = L\tilde{n}_-).$$

Then, $D_n^{\pi^t}$ acts as the multiplication by $h_n^\lambda \in \mathbb{C}$ on the weight space $C(n, V)^\lambda$, $\lambda \in h_n^{\lambda^*}$. Let $C(n, V)_a = \{v \in C(n, V); D_n^{\pi^t}v = av\}$ for $a \in \mathbb{C}$. Clearly, $C(n, V) = \bigoplus_{a \in \mathbb{C}} C(n, V)_a$ and

$$C(n, V)_a = \bigoplus_{\lambda \in \hat{h}^{\lambda^*}_n \cap a} C(n, V)^\lambda. \quad (4.21)$$

By (4.12), (4.13) and (4.16), it follows that $d_n^{\pi^t}C(n, V)_a \subset C(n, V)_a$ and $\chi_n C(n, V)_a \subset C(n, V)_a$. Therefore,

$$H_{\text{QDS}}^n(n, V) = \bigoplus_{a \in \mathbb{C}} H_{\text{QDS}}^n(n, V)_a, \quad (4.22)$$

where $H_{\text{QDS}}^n(n, V)_a = H^*(C(n, V)_a, d_n)$. 

**Remark 4.1.** (1) The operator $D_n^{\pi^t}$ is the semisimplification of $-L_n^\lambda$ defined in [9 3.1] up to constant shift.

(2) The eigenspace $C(n, V)_a$ of $D_n^{\pi^t}$ is not necessarily finite-dimensional.
4.7. The Weyl group action. The group \( \hat{W} \) acts naturally on \( \mathcal{C} \). Let \( \hat{W} \) act on \( U(\mathfrak{g}) \otimes \mathcal{C} \) by \( uw \otimes \omega = u(w) \otimes \omega(w) \).

Let \( w \in \hat{W} \). If \( V \) is a \( U(\mathfrak{g}) \otimes \mathcal{C} \)-module, we obtain a new \( U(\mathfrak{g}) \otimes \mathcal{C} \)-module by twisting its action on \( V \) as \( A \cdot v = w^{-1}(A) v \). The module obtained in this way we shall also denote by \( \phi_w(V) \). Then, the action \( \pi_{\hat{W}} \) of \( \hat{W} \) is well-defined on \( \phi_w(C(\mathfrak{n}, V)) \).

By direct calculation, one gets the following proposition.

**Proposition 4.2.** For \( w \in \hat{W} \) and \( v \in C(\mathfrak{n}, V) \),

\[
\pi(h) \cdot \phi_w(v) = \phi_w(\pi(w^{-1}(h))v + \rho(w^{-1}(h) - h)v) \quad (h \in \mathfrak{h}),
\]

\[
\hat{h}(n) \cdot \phi_w(v) = \phi_w(\hat{h}(n)v) \quad (h \in \mathfrak{h}, n \in \mathbb{Z} \setminus \{0\})
\]

where \( \phi_w(v) \) denotes the image of \( v \) in \( \phi_w(C(\mathfrak{n}, V)) \).

5. Cohomology associated to Verma modules

In this section we review the results obtained in [8, 14.2] for our case.

5.1. The decomposition of \( C(\mathfrak{n}, M(\lambda)) \). Fix \( \lambda \in \mathfrak{h}^* \). Let \( v_\lambda \) be the highest weight vector of \( M(\lambda) \). Let \( |\lambda\rangle = v_\lambda \otimes 1 \in C(\mathfrak{n}, M(\lambda)) \). Then,

\[
d_n^\alpha |\lambda\rangle = 0, \quad \chi_n|\lambda\rangle = 0. \quad (5.1)
\]

**Proposition 5.1.** The map defines by

\[
\begin{align*}
N_- \otimes B_+^l & \longrightarrow C(\mathfrak{n}, M(\lambda)) \\
n \otimes b & \longmapsto \pi(n) \cdot \pi^b(\lambda)
\end{align*}
\]

(5.2)

gives an isomorphism of \( \mathbb{C} \)-vector spaces.

**Proof.** Observe that \( M(\lambda) = U(\mathfrak{n}_-)U(\mathfrak{b}_+^l)v_\lambda \). Thus, by comparing the dimension of weight spaces of both sides, it follows that it is sufficient to show that \( \pi(n) \) is surjection. By definition,

\[
\pi(N_-)\pi(B_-)|\lambda\rangle = U(\mathfrak{n}_-) \cdot \Lambda(\mathfrak{n}_-) \cdot U(\mathfrak{b}_+^l) \cdot \Lambda(\mathfrak{n}_-^l)|\lambda\rangle.
\]

But we have \( U(\mathfrak{b}_+^l) \cdot \Lambda(\mathfrak{n}_-) = \Lambda(\mathfrak{n}_-) \cdot U(\mathfrak{b}_+^l) \). This can be seen by the commutation relations

\[
[\hat{J}_a(z), \psi_\beta(z)] = - \sum_{\beta, \gamma \in \hat{\Delta}(n)} c_{\alpha, \gamma}^{-\beta} \psi_\gamma(n + m) \quad (a \in \hat{\Delta}(n) \cap I, \beta \in \hat{\Delta}(n), n, m \in \mathbb{Z}).
\]

On the other hand, \( \Lambda(\mathfrak{n}_-) \cdot \Lambda(\mathfrak{n}_-^l)|\lambda\rangle = (\mathbb{C}v_\lambda) \otimes \mathcal{F}(n) \). Thus, it is enough to show that

\[
U(\mathfrak{n}_-) \cdot U(\mathfrak{b}_+^l) \cdot ((\mathbb{C}v_\lambda) \otimes \mathcal{F}(n)) = M(\lambda) \otimes \mathcal{F}(n).
\]

But \( U(\mathfrak{b}_+^l)v_\lambda \) is a free \( \mathfrak{b}_+^l \)-submodule of \( M(\lambda) \). Thus, \( U(\mathfrak{b}_+^l) \cdot ((\mathbb{C}v_\lambda) \otimes \mathcal{F}(n)) = (U(\mathfrak{b}_+^l)v_\lambda) \otimes \mathcal{F}(n) \). Similarly, we get \( U(\mathfrak{n}_-) \cdot (U(\mathfrak{b}_+^l)v_\lambda) \otimes \mathcal{F}(n) = M(\lambda) \otimes \mathcal{F}(n) \). \( \square \)
5.2. **The subcomplex** $C(n, \lambda)_0$. We define the subspace $C(\lambda)_0 = C(n, \lambda)_0$ of $C(n, M(\lambda))$, $\lambda \in h^*_\lambda$, by

$$C(\lambda)_0 = \pi^t(B^t|\lambda)$$  \hspace{1cm} (5.3)

Since $\pi^t$ defines one-dimensional representation of $B^t_+$ on $C(\lambda)_0$, it follows that

$$C(\lambda)_0 = \pi^t(B^t_+)|\lambda) = U(b^t_+ \cdot \Lambda(n^t_+) \cdot |\lambda) = \Lambda(n^t_-) \cdot U(b^t_-) \cdot |\lambda).$$

**Proposition 5.2.** $C(\lambda)_0 = B^t \otimes_{B^t_+} C(\lambda)_0$.

**Proof.** It is sufficient to show that the multiplication map $B^t_+ \otimes C(\lambda)_0 \rightarrow \pi^t(B^t_+)|\lambda) = C(\lambda)_0$ is an injection. But this easily follows form Proposition 5.1. \hfill $\square$

By [8, 14.2], $C(\lambda)_0$ is a subcomplex of $C(n, M(\lambda))$. This can be seen from (5.1) and the commutation relations (14.1), (14.9), (14.20).

Let $(C(\lambda)_0)_a = C(\lambda)_0 \cap C(n, M(\lambda))_a$, $a \in C$. The following Proposition is easy to see.

**Proposition 5.3.** For $\lambda \in h^*$, $C(\lambda)_0 = \bigoplus_{\alpha \in h^*_\lambda + Z \leq 0} (C(\lambda)_0)_a$ and $(C(\lambda)_0)_a$ is finite-dimensional for all $a$.

Define the subspace $C(n_-)'$ of $C(n, M(\lambda))$ by

$$C(n_-)' = \pi(N_-)|\lambda) = U(n_-) \cdot \Lambda(n_-) \cdot |\lambda) = \Lambda(n_-) \cdot U(n_-) \cdot |\lambda).$$

Then, $C(n_-)' \cong N_- \subset C(\lambda)_0$ as $C$-vector spaces. This can be seen in the same way as Proposition 5.2. By (5.1), (14.9), (14.10) and the fact

$$\{\chi_n, \psi_\alpha(n)\} = \chi_n(J_\alpha(n)) \hspace{1cm} (\alpha \in \Lambda(n), n \in Z),$$

(5.4) it follows that $C(n_-)'$ is a subcomplex of $C(n, M(\lambda))$. It is easy to see that this complex does not depend on $\lambda \in h^*_\lambda$.

**Proposition 5.4.** [8, 14.2] The map in Proposition 5.2 defines an isomorphism

$$C(n, M(\lambda)) \cong C(n_-)' \otimes C(\lambda)_0$$

of complexes.

Though the following Proposition is proved in [8, 14.2] in slightly different setting, the same proof applies.

**Proposition 5.5** ([8, 14.2]).

1. $H^i(C(\lambda)_0) = \{0\}$ for $i \neq 0$.

2. $H^i(C(n_-)') = H^i(C(n_-)', d^t_{n+}) = \begin{cases} C & \text{if } i = 0 \\ \{0\} & \text{if } i \neq 0. \end{cases}$

**Remark 5.6.** In the proof of Proposition 5.5 (1), one uses the fact that

$$P(C(\lambda)_0) \subset \{\mu \in h^*_\lambda : (\lambda - \mu, \bar{\rho}^\vee) \in Z \geq 0\} \hspace{1cm} \text{for } n = L_{\lambda}^+,$$

$$P(C(\lambda)_0) \subset \{\mu \in h^*_\lambda : (\mu - \lambda, \bar{\rho}^\vee) \in Z \geq 0\} \hspace{1cm} \text{for } n = L_{\lambda}^-,$$

to assure the converygence of the spectral sequence described in [8, 14.2.8].

Proposition 5.4 and Proposition 5.5 imply:

**Theorem 5.7.** For $\lambda \in h^*$, $H^i_{\text{QDS}}(L_{\lambda}^\pm, M(\lambda)) = \{0\}$ for $i \neq 0$. 

Remark 5.8. Set $\text{ch} H^i_{\text{QDS}}(n, V) = \text{tr} H^i_{\text{QDS}}(n, V)^q D^v$ when $\dim H^i_{\text{QDS}}(n, V)_a < \infty$ for all $a$. It is easy to see that $\text{ch} H^0_{\text{QDS}}(n, M(\lambda)) = \frac{q^{n^2}}{1-1/(1-q^{n^2})}$ for $\lambda \in \mathfrak{h}^*$, see [8 14.2].

6. COHOMOLOGY ASSOCIATED TO DUALS OF VERMA MODULES

In this section we prove the vanishing of $H^i_{\text{QDS}}(n, M(\lambda)^*)$ for $i \neq 0$ under the certain restriction of $\lambda$.

6.1. Relative complex. For $\lambda \in \mathfrak{h}^*$, define the subspace $C(\lambda)^0 = C(n, M(\lambda)^*)$ of $C(n, M(\lambda)^*)$ by

$$C(\lambda)^0 = \left\{ v \in C(n, M(\lambda)^*) : \hat{J}_n(n)v = \psi_\alpha(n)v = 0 \quad (\alpha \in \Delta(n), n \in \mathbb{Z}, \alpha + n\delta \in \Delta^\circ(n_+)) \right\}.$$

It is the relative complex (with respect to the differential $d^R_n$) considered in [10]. By (4.13), (4.14) and (5.4), it follows that $d_n V_n \subset ( \lambda \in C(n, M(\lambda)^*)$.

Proposition 6.1. For $\lambda \in \mathfrak{h}^*$, $C(\lambda)^0 = \left( M(\lambda)^* \otimes \Lambda(n_-) \right)^{n_+} \subset C(n, M(\lambda)^*)$ and the restriction of $d_n$ on $C(\lambda)^0$ is $d_n = d^R_n + \chi_n$. Here, $n_-$ acts on $\Lambda(n_-)$ via the identification $n_- = n/n_+$, and $d^R_n$ is the differential of $n_-$-homology, that is,

$$d^R_n = \sum_{\alpha + n\delta \in \Delta^\circ(n_-)} J_\alpha(n)\psi_{-\alpha}(-n) - \frac{1}{2} \sum_{\alpha + k\delta + \lambda \delta \in \Delta^\circ(n_-)} c_{\alpha,\beta,\gamma}^\delta \psi_{-\alpha}(-k)\psi_{-\beta}(-l)\psi_{\gamma}(k + l).$$

Proof. Clearly, $C(\lambda)^0$ is contained in the subspace

$$M(\lambda)^* \otimes \Lambda(n_-) = \left\{ v \in C(n, M(\lambda)^*); \psi_\alpha(n)v = 0 \quad (\alpha \in \Delta(n), n \in \mathbb{Z}, \alpha + n\delta \in \Delta^\circ(n_+)) \right\}.$$

It is easy to see that the operators $J_\alpha(n)$ $(\alpha \in \Delta(n), n \in \mathbb{Z}, \alpha + n\delta \in \Delta^\circ(n_+))$ preserve this subspace and their action coincide with the one via the identification $n_- = n/n_+$. Hence, it follows that $C(\lambda)^0 = \left( M(\lambda)^* \otimes \Lambda(n_-) \right)^{n_+}$ as $\mathbb{C}$-vector spaces. But then, by the proof of [22 Theorem 2.2], it follows that the restriction of $d^R_n$ to this subspace is $d^R_n$. \hfill \Box

Define a subspace $C(\lambda)^0 = C(n, M(\lambda)^*)$ of $C(n, M(\lambda))$ by

$$C(\lambda)^0 = B \otimes_{B_+} \mathbb{C}[\lambda],$$

see Proposition 5.2. Then, $\chi^4_n C(\lambda)^0 \subset C(\lambda)^0$. We view $C(\lambda)^0$ as a complex with differential $d^R_n$, where $d^R_n$ is defined in 4.3.

Proposition 6.2. For $\lambda \in \mathfrak{h}^*$, $C(\lambda)^0 = (C(\lambda)^0)^*$ as a complex.

Proof. follows from (4.7), (4.17) and Proposition 5.1. \hfill \Box

Proposition 6.3. For $\lambda \in \mathfrak{h}^*$,

$$H^\bullet_{\text{QDS}}(n, M(\lambda)^*) = H^\bullet(C(\lambda)^0) \left( = H^\bullet \left( (C(\lambda)^0)^* \right) \right).$$
Proof: The proof can be done using the corresponding statement to Proposition 5.3. Or one can apply [22, Theorem 2.2]. Indeed, by Proposition 6.1, the complex $C(\lambda)^0$ is nothing but the $E_1^{*,0}$-row of the Hochschild-Serre spectral sequence for $n_+ \subset n$ in [22, Theorem 2.2]. But since $M(\lambda)^*$ is a cofree $n_+$-module, it follows that $E_1^{*,q} = 0$ for $q \neq 0$. Thus this spectral sequence collapses at $E_2 = H^*(C(\lambda)^0) = E_\infty$. □

We have:

$$(D_n^W f)(v) = f(D_n^W v) \quad (f \in C(\lambda)^0 = (C(\lambda)^0)^*, v \in C(\lambda)^0).$$

Let $(C(\lambda)^0)_a = \{ v \in C(\lambda)^0; D_n^W v = av \}$. Then, $d_n^*(C(\lambda)^0)_a \subset (C(\lambda)^0)_a$ and $C(\lambda)^0 = \bigoplus_{a \in h^+_n + Z} (C(\lambda)^0)_a$. Observe that the eigenspace $(C(\lambda)^0)_a$ is not finite-dimensional in general (compare Proposition 5.3). Below we shall define a subspace $\mathcal{N}(\lambda)^0_0 = \mathcal{N}(n, \lambda)^0_0 \subset C(\lambda)^0$ so that the quotient $C(\lambda)^0_0 / \mathcal{N}(\lambda)^0_0$ is a direct sum of finite-dimensional eigenspaces of $D_n^W$. The definition is different for $n = \tilde{n}_+$ and $n = \tilde{n}_-$. \[6.2. The subspace $\mathcal{N}(\lambda)^0_0$ for $n = \tilde{n}_+$. Let $n = \tilde{n}_+$. Observe that $n \supset t_{\tilde{b}^\vee}(n_+) \supset n_+$. Define $t_{\tilde{b}^\vee}(B_+) = \tilde{n}_+ \oplus t_{\tilde{b}^\vee}(n_+) \supset B_+$.

Therefore, we have inclusions of algebras $B \supset t_{\tilde{b}^\vee}(B_+) \supset B_+$. Notice that $\chi^t_n \in t_{\tilde{b}^\vee}(B_+)$. Define

$$\tilde{C}(\lambda)^t = t_{\tilde{b}^\vee}(B_+) \otimes_{B_+} \mathbb{C}[\lambda].$$

It is a subspace of $C(\lambda)^0$ spanned by the vectors of the form

$$\psi_{\alpha_r}(-m_1) \ldots \psi_{\alpha_s}(-m_p) \tilde{J}_{\alpha_n}(-n_1) \ldots \tilde{J}_{\alpha_s}(-n_s)[\lambda]$$

with $\alpha_r, \alpha_s \in \Delta_+$, $1 \leq m_i \leq \text{ht} \alpha_r$, $1 \leq n_i \leq \text{ht} \alpha_s$. By definition, $\chi^t_n \tilde{C}(\lambda)^t \subset \tilde{C}(\lambda)^t$ and

$$\tilde{C}(\lambda)^t = \bigoplus_{a \in h^+_n + Z \geq 0} \tilde{C}(\lambda)^t_a,$$

where $\tilde{C}(\lambda)^t_a = \tilde{C}(\lambda)^t \cap (C(\lambda)^0)_a$. \[6.1\]

Define the subspace $\tilde{\mathcal{N}}(\lambda)^t_0$ of $\tilde{C}(\lambda)^t$ by

$$\tilde{\mathcal{N}}(\lambda)^t_0 = \sum_{\mu, \lambda, \rho^\vee} (\tilde{C}(\lambda)^t)^\mu.$$ \[6.2\]

Then, $t_{\tilde{b}^\vee}(B_+) \cdot \tilde{\mathcal{N}}(\lambda)^t \subset \tilde{\mathcal{N}}(\lambda)^t$. In particular, $\chi^t_n \tilde{\mathcal{N}}(\lambda)^t \subset \tilde{\mathcal{N}}(\lambda)^t$. Define

$$\mathcal{N}(\lambda)^0_0 = B \otimes_{t_{\tilde{b}^\vee}(B_+)} \tilde{\mathcal{N}}(\lambda)^t \subset C(\lambda)^0.$$ \[6.2\]

Then,

$$\chi^t_n \mathcal{N}(\lambda)^0_0 \subset \mathcal{N}(\lambda)^0_0.$$

Observe that $\tilde{C}(\lambda)^t / \tilde{\mathcal{N}}(\lambda)^t$ is spanned by the image $[\lambda]$ of $[\lambda]$. We have

$$d_n^*(\lambda) = 0,$$ \[6.3\]

$$J_i(n)[\lambda] = 0 \quad (i \in \tilde{I}, n > 0), \quad J_a(n)[\lambda] = \psi_{\alpha}(n)[\lambda] = 0 \quad (\alpha \in \Delta_+, n \geq \text{ht} \alpha),$$ \[6.4\]

$$C(\lambda)^0_0 / \mathcal{N}(\lambda)^0_0 = B \otimes_{t_{\tilde{b}^\vee}(B_+)} \mathbb{C}[\lambda].$$ \[6.5\]
6.3. The subspace $\mathcal{N}(\lambda)_0^t$ for $n = Ln$. Let $n = Ln$ and $w_0$ be the longest element of $\hat{W}$. Then,

$$n \supset w_0(n^*_+ \supset n_+, \quad b \supset w_0(b^*_+) \supset b_+.$$ 

Thus, $B \supset w_0(B^*_+) \supset B_+$. Notice $\chi_n^t \in w_0(B_+)$. Define

$$\bar{C}(\lambda)^t = w_0(B^*_+)\bar{\cap}(\lambda) \subset C(\lambda)_0^t.$$ 

It is the span of the vectors of the form

$$\psi_{-\alpha_{r_1}}(0) \ldots \psi_{-\alpha_{r_2}}(0)\hat{J}_{-\alpha_s}(0) \ldots \hat{J}_{-\alpha_{s'}}(0)|\lambda)$$

with $\alpha_{r_1}, \alpha_{s} \in \Delta$. We have:

$$C(\lambda)_0^t = B\otimes w_0(B^*_+)\bar{C}(\lambda)^t, \quad (6.7)$$

$$\chi_n^t \bar{C}(\lambda)^t \subset \bar{C}(\lambda)^t$$

and

$$\bar{C}(\lambda)^t = \bar{C}(\lambda)|h^t_n.$$ 

(6.8)

Define the subspace $\mathcal{N}(\lambda)^t$ of $\bar{C}(\lambda)^t$ by

$$\mathcal{N}(\lambda)^t = \sum_{(\lambda - \mu, \rho') > 0} (\bar{C}(\lambda)^t)^\mu$$

Then, $w_0(B^*_+)\cdot\mathcal{N}(\lambda)^t \subset \mathcal{N}(\lambda)^t, \text{ in particular, } \chi_n^t\mathcal{N}(\lambda)^t \subset \mathcal{N}(\lambda)^t$. Define

$$\mathcal{N}(\lambda)_0^t = B\otimes w_0(B^*_+)\mathcal{N}(\lambda)^t \subset C(\lambda)_0^t. \quad (6.9)$$

We have

$$\chi_n^t\mathcal{N}(\lambda)_0^t \subset \mathcal{N}(\lambda)_0^t, \quad (6.10)$$

and

$$C(\lambda)_0^t/\mathcal{N}(\lambda)_0^t = B\otimes w_0(B^*_+)|\bar{\cap}(\lambda). \quad (6.11)$$

Here, $\bar{\cap}(\lambda)$ is the image of $|\lambda|$ in $\bar{C}(\lambda)^t/\mathcal{N}(\lambda)^t$:

$$d_n^t|\bar{\cap}(\lambda) = 0, \quad \hat{J}_i(n)v = 0 \quad (i \in \bar{I}, n > 0),$$

$$\hat{J}_{-\alpha}(0)|\bar{\cap}(\lambda) = \psi_{-\alpha}(0)|\bar{\cap}(\lambda) = 0 \quad (\alpha \in \Delta_+).$$

6.4. $\mathcal{N}(\lambda)_0^t$ is a null subcomplex. Let $\left( C(\lambda)_0^t/\mathcal{N}(\lambda)_0^t \right)_a$ be the $D_n^W$-eigenspace of $\mathcal{N}(\lambda)_0^t/\mathcal{N}(\lambda)_0^t$ of eigenvalue $a \in \mathbb{C}$. The following proposition is easy to see by (6.8) and (6.11).

**Proposition 6.4.** For $\lambda \in \mathfrak{h}^+$, $C(\lambda)_0^t/\mathcal{N}(\lambda)_0^t = \bigoplus_{a \in h^+_n + \mathbb{Z} \leq 0} \left( C(\lambda)_0^t/\mathcal{N}(\lambda)_0^t \right)_a$ and $\left( C(\lambda)_0^t/\mathcal{N}(\lambda)_0^t \right)_a$ is finite-dimensional for all $a \in \mathbb{C}$.

The proof of the following proposition will be given in (6.6) and (6.7).

**Proposition 6.5.** (1) Let $n = Ln$. For $\lambda \in \mathfrak{h}^+$, $d_n^t\mathcal{N}(\lambda)_0^t \subset \mathcal{N}(\lambda)_0^t$. Moreover, if $\langle \lambda + \rho, \alpha' \rangle \notin \mathbb{Z} \geq 1$ for all $\alpha \in \Delta^+ \cap t_{\rho'}(\Delta^+)$, then $H_*(\mathcal{N}(\lambda)_0^t, d_n^t) \equiv 0$. (2) Let $n = Ln$. For $\lambda \in \mathfrak{h}^+$, $d_n^t\mathcal{N}(\lambda)_0^t \subset \mathcal{N}(\lambda)_0^t$. Moreover, if $\langle \lambda + \rho, \alpha' \rangle \notin \mathbb{Z} \geq 1$ for all $\alpha \in \Delta^+$, then $H_*(\mathcal{N}(\lambda)_0^t, d_n^t) \equiv 0$. 
By Proposition 6.6, we have an exact sequence $0 \to \mathcal{N}(\lambda)^0 \to C(\lambda)^0 \to \mathcal{N}(\lambda)^0 \to 0$ of complexes. Therefore, we get the following exact sequence of complexes:

$$0 \to \left(\frac{C(\lambda)^0}{\mathcal{N}(\lambda)^0}\right)^* \to (C(\lambda)^0)^* \to \left(\frac{\mathcal{N}(\lambda)^0}{\mathcal{N}(\lambda)^0}\right)^* \to 0,$$

where $^*$ is defined in (6.12).

**Proposition 6.6.** Let $\lambda \in \mathfrak{h}^*$ be as in Proposition 6.3. Then,

$$H^i_{\text{QDS}}(n, M(\lambda))^* = \text{Hom}_\mathbb{C} \left(H_i \left(\frac{C(\lambda)^0}{\mathcal{N}(\lambda)^0}\right)_a, \mathbb{C}\right)$$

for all $i$ and $a \in \mathbb{C}$.

**Proof.** We first claim that

$$H^* \left(\frac{\mathcal{N}(\lambda)^0}{\mathcal{N}(\lambda)^0}\right)^* \equiv 0. \tag{6.13}$$

Considering the spectral sequence described in [2, 3.2], it is enough to show that $H^* \left(\frac{\mathcal{N}(\lambda)^0}{\mathcal{N}(\lambda)^0}\right)^*, d_{\mathfrak{n}}^t \equiv 0$. But since the action of $d_{\mathfrak{n}}^t$ is compatible with the weight space decomposition, this is equivalent to $H^* \left(\frac{\mathcal{N}(\lambda)^0}{\mathcal{N}(\lambda)^0}\right) \equiv 0$. Hence Proposition 6.3 proves (6.13).

Now consider the long exact sequence induced by (6.12). Then, by (6.13), we get $H^* \left(\frac{C(\lambda)^0}{\mathcal{N}(\lambda)^0}\right)^* \equiv H^* \left(\frac{C(\lambda)^0}{\mathcal{N}(\lambda)^0}\right)^*$. But Proposition 6.4 implies

$$H^i \left(\frac{C(\lambda)^0}{\mathcal{N}(\lambda)^0}\right)_a = \text{Hom}_\mathbb{C} \left(H_i \left(\frac{C(\lambda)^0}{\mathcal{N}(\lambda)^0}\right)_a, \mathbb{C}\right),$$

for $a \in \mathbb{C}$. Thus, Proposition 6.3 proves the proposition. \qed

6.5. **The cohomology $H^*_{\text{QDS}}(n, M(\lambda))^*$**.

**Proposition 6.7.** Let $\lambda \in \mathfrak{h}^*$.

1. Let $n = L\mathfrak{n}^+$. For all $i \in \mathbb{Z}$ and $a \in \mathbb{C}$,

$$H_i \left(\frac{C(\lambda)^0}{\mathcal{N}(\lambda)^0}\right)_a \cong H^i_{\text{QDS}}(L\mathfrak{n}^+, M(t_{-\rho^\vee} \circ \lambda))_a.$$

2. Let $n = L\mathfrak{n}^-$. For all $i \in \mathbb{Z}$ and $a \in \mathbb{C}$,

$$H_i \left(\frac{C(\lambda)^0}{\mathcal{N}(\lambda)^0}\right)_a \cong H^i_{\text{QDS}}(L\mathfrak{n}^-, M(t \circ \lambda))_a.$$

**Proof.** (1) By Proposition 5.3 and Proposition 6.3 (2), we have

$$H^*_{\text{QDS}}(n^t, M(t_{-\rho^\vee} \circ \lambda)) = H^* (B \otimes_{B_B} \mathbb{C}[t_{-\rho^\vee} \circ \lambda], d_{n^t}).$$

Observe that

$$d_{n^t} \phi_{t_{\rho^\vee}} (v) = \phi_{t_{\rho^\vee}} (d_{n^t} v), \quad D_{n^t}^W \phi_{t_{\rho^\vee}} (v) = \phi_{t_{\rho^\vee}} (D_{n^t}^W v) \tag{6.14}$$

for $v \in C(n^t, M(t_{-\rho^\vee} \circ \lambda))$, see Proposition 1.2. Therefore,

$$H^*_{\text{QDS}}(n^t, M(t_{-\rho^\vee} \circ \lambda))_a \cong H^* \left(\phi_{t_{\rho^\vee}} \left(B \otimes_{B_B} \mathbb{C}[t_{-\rho^\vee} \circ \lambda]\right), d_{n^t}^a\right).$$

for $a \in \mathbb{C}$. Moreover, the action of $\tilde{J}_\alpha(n)$, $\alpha \in \Delta_+$, $n \in \mathbb{Z}$, is well-defined on $\phi_{t_{\rho^\vee}} (C(n^t, M(t_{-\rho^\vee} \circ \lambda)))$, and we have

$$\tilde{J}_\alpha(n) \phi_{t_{\rho^\vee}} (v) = \phi_{t_{\rho^\vee}} (\tilde{J}_\alpha(n + \text{ht} \alpha) v) \quad (\alpha \in \Delta, n \in \mathbb{Z}). \tag{6.15}$$
Thus, by Proposition 6.12 and (6.15), it follows that
\[ \phi_{t_{\rho'}}(B \otimes B_+ C[t_{-\beta'} \circ \lambda]) = B \otimes t_{\rho'}(B_+ C\overline{\lambda}). \]
Here, $C\overline{\lambda}$ is the one-dimensional representation of $t_{\rho'}(B_+)$ appeared in (6.6).

Then, by (6.4) and (6.6), we conclude $C(\lambda)_0^\mu / \mathcal{N}(\lambda)_0^\mu = t_{\rho'}(B \otimes B_+ C[t_{-\beta'} \circ \lambda])$ as complexes. (2) can be similarly proved using
\[
\begin{align*}
&d_n^\gamma \phi_{w_0}(v) = \phi_{w_0}(d_n v), \quad D_n^\gamma \phi_{w_0}(v) = \phi_{w_0}(D_n^\gamma v), \quad (6.16) \\
&\tilde{J}_\alpha(n) \phi_{w_0}(v) = \phi_{w_0}(\tilde{w}_0(J_n)(n)v) \quad (\alpha \in \Delta, n \in \mathbb{Z}). \quad (6.17)
\end{align*}
\]
for $v \in C(n, M(w_0 \circ \lambda))$. 

\[ \square \]

**Theorem 6.8.** Let $\lambda \in \mathfrak{h}^*$.

1. Suppose $\langle \lambda + \rho, \alpha \rangle \notin \mathbb{Z}_{\geq 1}$ for all $\alpha \in \Delta^e \cap t_{\rho'}(\Delta^e)$. Then, for all $a \in \mathbb{C}$,
\[
H^1_{\text{QDS}}(\bar{L}_{\mu}^+, M(\lambda)^a) \cong \begin{cases} \text{Hom}_C(H^0_{\text{QDS}}(\bar{L}_{\mu}^+, M(t_{-\beta'} \circ \lambda))_a, \mathbb{C}) & (i = 0) \\
\{0\} & (i \neq 0). \end{cases}
\]

2. Suppose $\langle \lambda + \rho, \alpha \rangle \notin \mathbb{Z}_{\geq 1}$ for all $\alpha \in \bar{\Delta}_+$. Then, for all $a \in \mathbb{C}$,
\[
H^i_{\text{QDS}}(\bar{L}_{\mu}^+, M(\lambda)^a) \cong \begin{cases} \text{Hom}_C(H^0_{\text{QDS}}(\bar{L}_{\mu}^+, M(w_0 \circ \lambda))_a, \mathbb{C}) & (i = 0) \\
\{0\} & (i \neq 0). \end{cases}
\]

**Proof.** follows from Proposition 6.6, Proposition 6.7 and Theorem 5.7. \[ \square \]

**6.6. Proof of Proposition 6.5 (1).**

**Step 1** Define the subspace $F^p \bar{C}(\lambda)^t$, $p \leq 0$, of $\bar{C}(\lambda)^t$ by
\[
F^{-p} \bar{C}(\lambda)^t = \bigoplus_{a \geq h_2^* + p} \bar{C}(\lambda)^t_a \subset \bar{C}(\lambda)^t.
\]
Then, by (6.11),
\[
\cdots \subset F^{-p} \bar{C}(\lambda)^t \subset \cdots \subset F^0 \bar{C}(\lambda)^t = \bar{C}(\lambda)^t, \quad \bigcap_p F^p \bar{C}(\lambda)^t = \{0\}.
\]
Notice that $P\{F^p \bar{C}(\lambda)^t\} \subset \{\mu \in \mathfrak{h}_k^*; \langle \mu - \lambda, \bar{\rho} \rangle \geq -p\}$. Thus,
\[
F^p \bar{C}(\lambda)^t \subset \mathcal{N}(\lambda)^t \quad \text{for } p \leq -1, \\
\mathcal{N}(\lambda)^t = (\mathcal{N}(\lambda)^t \cap F^{-1} \bar{C}(\lambda)^t) \oplus \sum_{(\lambda - \mu, \bar{\rho}) < 0} (\bar{C}(\lambda)^t)_a^{\mu}, 
\]
where $(\bar{C}(\lambda)^t)_a^{\mu} = (\bar{C}(\lambda)^t)_a \cap (\bar{C}(\lambda)^t)_a^{\mu}$. Define the subspace $F^p C(\lambda)_0^\mu$, $p \leq 0$, of $C(\lambda)_0^\mu$ by
\[
F^p C(\lambda)_0^\mu = B \otimes_{t_{\rho'}(B)} F^p \bar{C}(\lambda)^t.
\]
Then,
\[
\cdots \subset F^{-p} C(\lambda)_0^\mu \subset \cdots \subset F^0 C(\lambda)_0^\mu = C(\lambda)_0^\mu, \quad \bigcap_p F^p C(\lambda)_0^\mu = \{0\},
\]
\[
P\{F^p C(\lambda)^t\} \subset \{\mu \in \mathfrak{h}_k^*; \langle \mu - \lambda, \bar{\rho} \rangle \geq -p\}, \quad P\{F^p C(\lambda)_0^\mu\} \subset C(\lambda)_0^\mu, \quad (6.20)
\]
\[
F^p C(\lambda)^t \subset \mathcal{N}(\lambda)_0^\mu \quad \text{for } p \leq -1. \quad (6.21)
\]
Proposition 6.11. Let \( \lambda \in \mathfrak{h}^* \).

(1) \( d_{n_t}^t \bar{F}^p \bar{C}(\lambda)^t \subset \bar{F}^p \bar{C}(\lambda)^t + F^{p-1}C(\lambda)_0^t \).
(2) \( d_{n_t}^t F^pC(\lambda)_0^t \subset F^pC(\lambda)_0^t \).

Proof. (1) follows from the commutativity of \( \mathcal{D}_n^W \) and the fact that the operators \( J_\alpha(-n) \) and \( \psi_\alpha(-n) \) have negative eigenvalues with respect to the adjoint action of \( \mathcal{D}_n^W \). (2) follows from (1) and the definition of \( F^pC(\lambda)_0^t \). \( \square \)

Consider the spectral sequence \( E^r \Rightarrow H_\bullet(C(\lambda)_0^t, d_{n_t}^t) \) corresponding to the filtration \( \{F^pC(\lambda)_0^t\} \). We have: \( E_1^{1, \bullet} = H_\bullet(F^pC(\lambda)_0^t/F^{p-1}C(\lambda)_0^t, d_{n_t}^t) \).

Let

\[
C(\lambda)' = \sum_p F^pC(\lambda)_0^t/F^{p-1}C(\lambda)_0^t,
\]

\[
\mathcal{N}(\lambda)' = \text{Im} : \sum_p F^p\mathcal{N}(\lambda)_0^t/F^{p-1}\mathcal{N}(\lambda)_0^t \hookrightarrow C(\lambda)',
\]

where \( F^p\mathcal{N}(\lambda)_0^t = \mathcal{N}(\lambda)_0^t \cap F^pC(\lambda)_0^t \).

Proposition 6.10. \( d_{n_t}^t \mathcal{N}(\lambda)' \subset \mathcal{N}(\lambda)' \), and if \( \lambda \in \mathfrak{h}^* \) satisfies \( \langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{\geq 1} \) for all \( \alpha \in \Delta^\sigma \cap \Delta^w(\Delta^\sigma) \), then \( H_\bullet(\mathcal{N}(\lambda)', d_{n_t}^t) \equiv 0 \).

Proposition \( \ref{6.10} \) will be proven in Step 3. Note that Proposition \( \ref{6.10} \) implies Proposition \( \ref{6.6} \) (1). Indeed, by \( \ref{6.10} \), \( \ref{6.24} \) and Proposition \( \ref{6.9} \) \( d_{n_t}^t \mathcal{N}(\lambda)' \subset \mathcal{N}(\lambda)' \) implies \( d_{n_t}^t \mathcal{N}(\lambda)_0^t \subset \mathcal{N}(\lambda)_0^t \), and by \( \ref{6.24} \) again, \( H_\bullet(\mathcal{N}(\lambda)', d_{n_t}^t) \equiv 0 \) implies that \( E^r \) degenerates at the \( E_2 \)-term itself and that \( H_\bullet(C(\lambda)_0^t, d_{n_t}^t) = H_\bullet(C(\lambda)_0^t/F(\lambda)_0^t, d_{n_t}^t) \), that is, \( H_\bullet(\mathcal{N}(\lambda)^t, d_{n_t}^t) \equiv 0 \).

Step 2 Let

\[
\bar{C}(\lambda)' = \text{Im} : \sum_p \bar{F}^p \bar{C}(\lambda)^t/\bar{F}^{p-1} \bar{C}(\lambda)^t \hookrightarrow C(\lambda)'.
\]

Then, \( C(\lambda)' = B \otimes_{\mathfrak{g}^\vee (B)} \bar{C}(\lambda)' \) and \( \bar{C}(\lambda)' \) is a subcomplex of \( C(\lambda)' \) by Proposition \( \ref{6.14} \)(1). Observe that by definition, it is the following quotient complex of \( C(\lambda)^t \):

\[
\bar{C}(\lambda)' = C(\lambda)^t/\text{span} \left\{ j_\alpha(-n)v, \psi_\alpha(-n)v, j_i(-m)v, \; ; \alpha \in \Delta^+, n > \text{ht} \alpha, i \in I, m > 0, v \in C(\lambda)^t \right\}.
\]

Proposition 6.11. \( H_i(\bar{C}(\lambda)', d_{n_t}^t) = H^{\bar{C}(\lambda)'^\vee} t_{\bar{p}^\vee}(\mathfrak{g}^-, M(\lambda)) \).

Proof. By the duality of the standard semi-infinite cohomology \( \mathcal{C}^\vee \), we have

\[
H^{\bar{C}(\lambda)'^\vee} t_{\bar{p}^\vee}(\mathfrak{g}^-, M(\lambda)) = H^{\bar{C}(\lambda)'^\vee} (t_{\bar{p}^\vee}(\mathfrak{g}^+, M(\lambda))^\vee).
\]

(6.23)

Thus, it is sufficient to show that \( H^\bullet((\bar{C}(\lambda)^t)^\vee) = H^{\bar{C}(\lambda)^t} (t_{\bar{p}^\vee}(\mathfrak{g}^+), M(\lambda))^* \).
Let $m = t_{\tilde{\rho}^\vee}(g_+) \cap g_- = t_{\tilde{\rho}^\vee}(n_+) \cap n_-$. Since $M(\lambda)^*$ is $t_{\tilde{\rho}^\vee}(g_+) \cap g_+$-cofree, by Theorem 2.11 it follows that

$$H^{\Xi+i}(t_{\tilde{\rho}^\vee}(g_+), M(\lambda)^*)$$

where $t_{\tilde{\rho}^\vee}(g_+) \cap g_+$ acts on $\Lambda(m)$ via the identification $m = t_{\tilde{\rho}^\vee}(g_+) / (t_{\tilde{\rho}^\vee}(g_+) \cap g_+)$ and $d^*_{m}$ is the differential of m-homology, i.e,

$$d^*_{m} = \sum_{\alpha \in \Delta_+ - \text{ht } \alpha \leq n \geq 0} J_\alpha(n) \psi_\alpha(-n) - \frac{1}{2} \sum_{\alpha, \beta \in \Delta_+ - \text{ht } \alpha \leq k \geq 0, \text{ht } \beta \leq l} c_{\alpha, \beta}^\gamma \psi_\alpha(-k) \psi_\beta(-l) \psi_\gamma(k + l).$$

On the other hand, by (6.22) and (6.17), we have

$$(\tilde{C}(\lambda)^*)^* = (M(\lambda)^* \otimes \Lambda(m))^\ast_{t_{\tilde{\rho}^\vee}(g_+) \cap n^+} \subset (M(\lambda)^* \otimes \Lambda(n_-))^n_+.$$ (6.24)

This can be proved in the same way as Proposition 6.12. Therefore, by Proposition 6.12 it is now sufficient to check that $d^*_{m}$ acts as $d^*_{n}$ on the right-hand-side of (6.24). But this is easy to see.

**Step 3** Define the subspace $F^{p}C(\lambda)^{\prime}, p \leq 0$, of $C(\lambda)^{\prime}$ by

$$F^{p}C(\lambda)^{\prime} = B \otimes t_{\tilde{\rho}^\vee}(B_+^{\ast}) F^{p} \tilde{C}(\lambda)^{\prime},$$

where

$$F^{p} \tilde{C}(\lambda)^{\prime} = \bigoplus_{(\mu - \lambda, \rho^\vee) \geq -p} F^{p} (\tilde{C}(\lambda)^{\prime})^{\mu} \subset \tilde{C}(\lambda)^{\prime}.$$ Then, similarly as in the step 1, we have:

$$N(\lambda)^{\prime} = F^{-1} C(\lambda)^{\prime}$$

$$\cdots \subset F^{-p} C(\lambda)^{\prime} \subset \cdots \subset F^{0} C(\lambda)^{\prime} = C(\lambda)^{\prime}, \quad \bigcap F^{p} C(\lambda)^{\prime} = \{0\},$$

$$F^{p} C(\lambda)^{\prime} = \{0\} \quad (p \ll 0).$$ (6.27)

Since $d^*_{n} F^{p} \tilde{C}(\lambda)^{\prime} \subset F^{p} \tilde{C}(\lambda)^{\prime}$ and $t_{\tilde{\rho}^\vee}(B_+^{\ast}) F^{p} \tilde{C}(\lambda)^{\prime} \subset F^{p} \tilde{C}(\lambda)^{\prime}$, it follows that $d^*_{n} F^{p} C(\lambda)^{\prime} \subset F^{p} C(\lambda)^{\prime}$. In particular, $d^*_{n} N(\lambda)^{\prime} \subset N(\lambda)^{\prime}$.

Let $E^{p} \rightarrow H_{\bullet}(C(\lambda)^{\prime}, d^*_{n})$ be the corresponding spectral sequence. We have:

$$E_{1}^{p, \bullet} = H_{\bullet} \left( F^{p} C(\lambda)^{\prime} / F^{p-1} C(\lambda)^{\prime}, d^*_{n} \right).$$

Notice that $F^{p} \tilde{C}(\lambda)^{\prime} / F^{p-1} \tilde{C}(\lambda)^{\prime}$ is a subcomplex of $F^{p} C(\lambda)^{\prime} / F^{p-1} C(\lambda)^{\prime}$ and that

$$F^{p} \tilde{C}(\lambda)^{\prime} / F^{p-1} \tilde{C}(\lambda)^{\prime} = \bigoplus_{(\mu - \lambda, \rho^\vee) \geq -p} (\tilde{C}(\lambda)^{\prime})^{\mu}$$

as a complex.

Consider $\phi_{t_{\tilde{\rho}^\vee}} \subseteq B \otimes B_+ \mathbb{C} \mathbb{C}(\mu)$ as a complex with differential $d^*_{n}$ as in the proof of Proposition 6.12.

**Proposition 6.12.** Let $\lambda \in \mathfrak{h}^\ast$.

$$F^{p} C(\lambda)^{\prime} / F^{p-1} C(\lambda)^{\prime} = \bigoplus_{(\mu - \lambda, \rho^\vee) \geq -p} \phi_{t_{\tilde{\rho}^\vee}} \left( B \otimes B_+ \mathbb{C} (t_{\tilde{\rho}^\vee} \circ \mu) \right) \otimes (\tilde{C}(\lambda)^{\prime})^{\mu}$$
as a complex.

Proof. is similar to that of Proposition 6.7. Indeed, we have

\[
F^{p} C(\lambda)^{\prime} / F^{p-1} C(\lambda)^{\prime} = B \otimes_{t_{\bar{\rho}^{\prime}}(B_{+})} \left( \bar{F}^{p} C(\lambda)^{\prime} / \bar{F}^{p-1} C(\lambda)^{\prime} \right)
\]

\[
= \bigoplus_{\mu} B \otimes_{t_{\bar{\rho}^{\prime}}(B_{+})} \left( \bar{F}^{p} C(\lambda)^{\prime} / \bar{F}^{p-1} C(\lambda)^{\prime} \right)^{\mu},
\]

and \( \left( \bar{F}^{p} C(\lambda)^{\prime} / \bar{F}^{p-1} C(\lambda)^{\prime} \right)^{\mu} \) is a direct sum of copies of \( C[\mu] \) as a \( t_{\bar{\rho}^{\prime}}(B_{+}) \)-module.

\[ \square \]

**Proposition 6.13.** Let \( \lambda \) as in Proposition 6.5 (1). Then,

\[ H_{\bullet} \left( F^{p} C(\lambda)^{\prime} / F^{p-1} C(\lambda)^{\prime}, d^{\mu}_{n} \right) = \{ 0 \} \quad (p \neq 0). \]

Proof. By Theorem 5.1 the assumption on the weight \( \lambda \) implies \( M(\lambda) \) is cofree over \( t_{\bar{\rho}^{\prime}}(g_{-}) \cap g_{+} \). Since \( M(\lambda) \) is obviously free over \( t_{\bar{\rho}^{\prime}}(g_{-}) \cap g_{+} \), Theorem 2.1 implies

\[ H^{\pm i} \left( t_{\bar{\rho}^{\prime}}(g_{-}), M(\lambda) \right)^{\mu} = \begin{cases} C_{\lambda} & (i = 0 \text{ and } \mu = \lambda) \\ \{ 0 \} & (\text{otherwise}). \end{cases} \]

Thus Proposition 6.11 and Proposition 6.12 prove the proposition. \[ \square \]

By Proposition 6.13 \( E^{\sigma} \) degenerates at the \( E_{1} \)-term itself, i.e,

\[ H_{\bullet} \left( C(\lambda)^{\prime}, d^{\mu}_{n} \right) = H_{\bullet} \left( F^{0} C(\lambda)^{\prime} / F^{1-1} C(\lambda)^{\prime}, d^{\mu}_{n} \right) = H_{\bullet} \left( C(\lambda)^{\prime} / N(\lambda)^{\prime}, d^{\mu}_{n} \right). \]

Here the last equality follows from (6.27). This implies \( H_{\bullet} (N(\lambda)^{\prime}, d^{\mu}_{n}) = 0 \). Proposition 6.10 is proved. Thus, Proposition 6.5 (1) is proved. \[ \square \]

**6.7. Proof of Proposition 6.5 (2).** We omit the most of the proof of (2). Indeed, its proof is simpler than (1): By 6.6, step 1 in the previous section is not needed for this case and the argument in step 2 is replaced by the following proposition.

**Proposition 6.14.** Let \( \lambda \in \mathfrak{b}^{\ast} \).

1. \( \bar{C}(\lambda)^{t} = \bar{M}(\lambda) \otimes \Lambda(n_{\bar{\lambda}})^{\ast} \subset M(\lambda) \otimes \mathbb{F} \). Here, \( \bar{M}(\lambda) \) is the Verma module of \( \mathfrak{g} \) of highest weight \( \bar{\lambda} \) identified with \( U(n_{\bar{\lambda}})v_{\lambda} \subset M(\lambda) \).

2. \( d^{\mu}_{n} \bar{C}(\lambda)^{t} \subset \bar{C}(\lambda)^{t} \) and

\[ H_{i} \left( \bar{C}(\lambda)^{t}, d^{\mu}_{n} \right) = H^{-i} \left( n_{+}, \bar{M}(\lambda) \right), \]

where \( \bar{C}(\lambda)^{t} = \sum \bar{C}^{i}(\lambda)^{t}, \bar{C}^{i}(\lambda)^{t} = \bar{C}(\lambda)^{t} \cap C(\lambda)^{t} \).

Proof. (1) follows from the fact that \( \bar{J}_{\bar{\alpha}}(0) \) acts as \( J_{\alpha}(0) \) on \( M(\lambda) \otimes \mathbb{F} \). (2) easily follows form (1). Indeed,

\[ d^{\mu}_{n} \mid _{\bar{C}(\lambda)^{t}} = \sum_{\alpha \in \Delta_{t}} J_{\alpha}(0) \psi_{-\alpha}(0) - \frac{1}{2} \sum_{\alpha, \beta, \gamma \in \Delta_{t}} c_{\alpha, \beta}^{\gamma} \psi_{-\alpha}(0) \psi_{-\beta}(0) \psi_{-\gamma}(0). \]

Thus, by [3] Proposition 4.7, it follows that \( H_{i} \left( \bar{C}(\lambda)^{t}, d^{\mu}_{n} \right) = H^{-i} \left( n_{+}, \bar{M}(\lambda) \right). \] 

\[ \square \]
7. Estimate on $D_n^W$-eigenvalues

In this section we shall give an estimate of $D_n^W$-eigenvalues of $H^\bullet_{QDS}(n, V)$ for $V \in O^A_n$ under the restriction on $\Lambda$ as in Introduction. The results in this section will be needed when $\kappa \in \mathbb{Q}_{>0}$. Let $C_{+, \kappa} = \{ \Lambda \in h^*_\mathfrak{g}; \langle \Lambda+\rho, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in R^+_A \}$, the set of dominant weights of level $\kappa - h^\vee$. Then, $O_{\kappa} = \bigoplus_{\Lambda \in C_{+, \kappa}} O^A_{\kappa}$ if $\kappa \notin \mathbb{Q}_{\leq 0}$.

The following proposition is clear.

Proposition 7.1. $H^\bullet_{QDS}(n, V)_a \neq \{0\}$ only if $H^\bullet_{\mathfrak{h}^+}(n, V)_a \neq \{0\}$ ($V \in O_{\kappa}, i \in \mathbb{Z}, a \in \mathbb{C}$).

7.2. The formal character. For $\lambda \in h^*$, let $I(\lambda)$ be the irreducible representation of $\mathfrak{h}$ of highest weight $\lambda + h^\vee \Lambda_0$. Since the category of highest weight $\mathfrak{h}$-modules is completely reducible, $C(n, V), V \in O_{\kappa}$, decomposes into a direct sum of $I(\lambda)$: $C(n, V) \cong \bigoplus_{\lambda \in h^*_\mathfrak{g}} B^\bullet_{\mathfrak{g}}(n, V) \otimes I(\lambda)$. Here,

$$B^\bullet_{\mathfrak{g}}(n, V) = \{ v \in C^i(n, V); \hat{h}(n) \cdot v = 0, \hat{h}(0) \cdot v = \lambda(h)v, (h \in h, n > 0), \hat{D} \cdot v = \lambda(D)v \},$$

$$= \{ v \in C^i(n, V); \hat{h}(n) \cdot v = 0, \hat{h}(0) \cdot v = 0, (h \in h, n > 0) \}. \quad (7.2)$$

Note that $\dim B^\bullet_{\mathfrak{g}}(n, V) < \infty$ by definition. By the commutativity of $d^\mathfrak{h}_n$ with the action of $\mathfrak{h}$, it follows that

$$H^\bullet_{\mathfrak{h}^+}(n, V) \cong \bigoplus_{\lambda \in h^*_\mathfrak{g}} H^\bullet(B^\bullet_{\mathfrak{g}}(n, V)) \otimes I(\lambda) \quad (V \in O_{\kappa}). \quad (7.3)$$

Here, $H^\bullet(B^\bullet_{\mathfrak{g}}(n, V)) = H^\bullet(B^\bullet_{\mathfrak{g}}(n, V), d^\mathfrak{h}_n)$. $d^\mathfrak{h}_n$.

Remark 7.2. By [7.2], it follows that the sum in the right-hand side in (7.3) is taken over $\lambda \in h^*_\mathfrak{g}$ such that $|\lambda + \rho|^2 = |\Lambda + \rho|^2$ for $V \in O^A_{\kappa}$.

The following is clear by (7.4), (7.2), and Proposition 7.1.

Lemma 7.3. The $D_n^W$-eigenvalues of $H^\bullet_{QDS}(n, V), V \in O_{\kappa}$, are contained in the set

$$\bigcup_{\lambda \in h^*_\mathfrak{g}^*} h^\lambda \cap \mathbb{Z}_{\geq 0}.$$

Define $ch H^\bullet(B^\bullet_{\mathfrak{g}}(n, V)) = \sum_{i \in \mathbb{Z}} z^i \dim H^i(B^\bullet_{\mathfrak{g}}(n, V))$. Then, by (7.3),

$$\sum_\lambda ch H^\bullet(B^\bullet_{\mathfrak{g}}(n, V)) e^\lambda = \prod_{\alpha \in \Delta^+_n} (1 - e^{-\alpha})^{\dim \mathfrak{h}_\alpha} ch H^\bullet_{\mathfrak{h}^+}(n, V), \quad (7.4)$$

These results will be used in the following sections.
where \( \text{ch} H^{\Lambda+\bullet}(n, V) = \sum_{i \in \mathbb{Z}} z^i \sum_{\Lambda \in \mathfrak{h}^*} e^\Lambda \dim H^{\Lambda+\bullet}(n, V)^\Lambda \).

7.3. The estimate on \( D^W_n \)-eigenvalues.

**Lemma 7.4.** Let \( \Lambda \in \mathfrak{h}^* \). For \( w \in \widehat{W} \), the following conditions are equivalent:

1. \( \langle \Lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z} \) for all \( \alpha \in \Delta^w_+ \cap w(\Delta^w_+) \).
2. \( w^{-1}(R^A_+) \subset \Delta^w_+ \).

**Proof.** (1) is equivalent to \( R^A_+ \cap \Delta^w_+ \cap w(\Delta^w_+) = \emptyset \). On the other hand, (2) is equivalent to \( R^A_+ \subset \Delta^w_+ \cap w(\Delta^w_+) \). But these two conditions are equivalent. \( \square \)

**Lemma 7.5.** Let \( \Lambda \in \mathfrak{h}^*_n \) such that

\[
\begin{cases}
\langle \Lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z} & \text{for all } \alpha \in \Delta^w_+ \cap t_{\bar{\rho}^w}(\Delta^w_+) \text{ (if } n = L\tilde{n}_+) \\
\langle \Lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z} & \text{for all } \alpha \in \Delta_+ \text{ (if } n = L\tilde{n}_-) 
\end{cases}
\]

Then, \( h_n^\Lambda - h_n^\mu \in \mathbb{Z}_{\bar{\text{ht}}A}(\Lambda - \mu) \) for all \( \mu \in \Lambda - Q^A_+ \).

**Proof.** Let \( n = L\tilde{n}_+ \). Notice that \( t_{\bar{\rho}^w}(\Delta^w_+) \cap \Delta_+ = \emptyset \). Thus, by the assumption and Lemma 7.4, \( t_{\bar{\rho}^w}(\alpha) \in \Delta^w_+ \cap t_{\bar{\rho}^w}(\Delta^w_+) \) for all \( \alpha \in \Pi^A \). Hence, \( \langle t_{\bar{\rho}^w}(\alpha), D \rangle \in \mathbb{Z}_{\geq 1} \) for all \( \alpha \in \Pi^A \), and thus, \( \langle t_{\bar{\rho}^w}(\mu), D \rangle \in \mathbb{Z}_{\geq \bar{\text{ht}}A(\mu)} \) for \( \mu \in Q^A_+ \). But

\[
h_n^\Lambda - h_n^\mu = \langle \Lambda - \mu, \bar{\rho}^w + D \rangle = \langle t_{\bar{\rho}^w}(\Lambda - \mu), D \rangle. \quad (7.5)
\]

Therefore the assertion follows. The \( n = L\tilde{n}_- \) case follows from the formula

\[
h_n^\Lambda - h_n^\mu = \langle \Lambda - \mu, D \rangle. \quad (7.6)
\]

and the fact that \( R^A_+ \subset \Delta^w_+ \cap \Delta_+ \). \( \square \)

**Proposition 7.6.** Let \( \Lambda \in \mathfrak{h}^*_n \) be as in Lemma 7.5. Then, for all \( i \in \mathbb{Z} \) and \( V \in \mathcal{O}_\kappa^A \), \( D^w_n \)-eigenvalues of \( H^i_{\text{QDS}}(n, V) \) is contained in the set

\[
\bigcup_{\mu \in W^A \cup \Lambda \atop \langle V : L(\mu) \rangle \neq 0} h_n^\mu - \mathbb{Z}_{\geq |i|}.
\]

Here, \( [V : L(\mu)] \) is the multiplicity of \( L(\mu) \) in \( V \) in the sense of 3.

Proof of Proposition 7.6 is given at the end of this section.

**Lemma 7.7** and **Proposition 7.8** imply:

**Corollary 7.7.** Let \( \kappa \in \mathbb{Q}_{> 0} \). Suppose that \( \Lambda \in \mathcal{C}_{\kappa,+} \) satisfies the condition in Lemma 7.3. Then, for all \( i \in \mathbb{Z} \) and \( V \in \mathcal{O}_\kappa^A \), \( D^w_n \)-eigenvalues of \( H^i_{\text{QDS}}(n, V) \) is contained in the set \( h^\Lambda_n - \mathbb{Z}_{\geq |i|} \).

**Lemma 7.8.** Let \( \Lambda \in \mathcal{C}_{\kappa,+}, \kappa \in \mathbb{Q}_{> 0} \). Then, for a given \( N \in \mathbb{Z}_{> 0} \) and \( V \in \mathcal{O}_\kappa^A \), there exists a finitely generated submodule \( M \) of \( V \) such that \( [V : M : L(\mu)] = 0 \) if \( \text{ht}_A(\Lambda - \mu) \leq N \).

**Proof.** Let \( \{0\} = V_0 \subset V_1 \subset V_2 \subset \ldots \) be a highest weight series of \( V \), that is, a filtration of \( V \) such that (1) \( V = \bigcup V_i \), (2) Each submodule \( V_i/V_{i-1} \) is a quotient of \( M(\mu_i) \) for some \( \mu_i \in \mathfrak{h}^* \), and (3) \( \mu_j - \mu_i \notin Q^+ \) for \( i < j \). Since \( V \in \mathcal{O}_\kappa^A \), it follows that \( \mu_i \in W^A \circ \Lambda \) for all \( i \). We may assume that \( V_i \neq \{0\} \) for all \( i \), because there is nothing to show if \( V \) is not finitely generated. Since \( \{\lambda \in W^A \circ \Lambda ; \text{ht}_A(\Lambda - \lambda) \leq N\} \) is a finite set, there exists an integer \( k \) such that \( \text{ht}_A(\Lambda - \mu_i) > N \) for all \( i > k \).
Let \( M = V_k \). Then, \( P(V/M) \subset \bigcup_{i > k} \mu_i - Q_+ \), and therefore, \([V/M : L(\mu)] = 0 \) if \( \text{ht}_{\lambda}(\Lambda - \mu) \leq N \). \( \square \)

**Proposition 7.9.** Let \( \Lambda \in \mathcal{C}_{\kappa, +}, \kappa \in \mathbb{Q}_{>0} \). Suppose that \( \Lambda \) satisfies satisfies the condition in Lemma 7.5. Let \( V \in \mathcal{O}^{[\Lambda]}_{\kappa} \) and suppose \( a \in \mathbb{C} \) is given.

1. There exists a finitely generated submodule \( M \) of \( V \) such that \( \mathcal{H}_{\text{qds}}^\bullet(n, V)_a \cong \mathcal{H}_{\text{qds}}^\bullet(n, M)_a \).
2. There exists a quotient \( M' \) of \( V \) such that \( (M')^* \) is finitely generated and \( \mathcal{H}_{\text{qds}}^\bullet(n, V)_a \cong \mathcal{H}_{\text{qds}}^\bullet(n, M')_a \).

**Proof.** By Corollary 7.7 we may assume that \( a \in \mathfrak{h}_\kappa^\lambda - \mathbb{Z}_{\geq 0} \). Let \( N = \mathfrak{h}_\kappa^\lambda - a \).

1. By Lemma 7.8 there exists a finitely generated submodule \( M \) of \( V \) such that \([V/M : L(\mu)] = 0 \) if \( \text{ht}_{\lambda}(\Lambda - \mu) \leq N \). Then, by Lemma 7.5 and Proposition 7.6 it follows that

\[
\mathcal{H}_{\text{qds}}^\bullet(n, V/M)_a = \{0\} \quad (a \geq \mathfrak{h}_\kappa^\lambda - N) \quad (7.7)
\]

Consider the exact sequence \( 0 \to M \to V \to V/M \to 0 \) in \( \mathcal{O}^{[\Lambda]}_{\kappa} \). It induces the long exact sequence of semi-infinite cohomology. Clearly, its restriction to a \( \mathbb{D}_n^W \)-eigenspace remains exact. Thus, (1) follows from (7.7). (2) is similarly proved as (1). Indeed, let \( M \) be a finitely generated submodule of \( V^* \) such that \([V^*/M : L(\mu)] = 0 \) if \( \text{ht}_{\lambda}(\Lambda - \mu) \leq N \). Then, \( 0 \to (V^*/M)^* \to V \to M^* \to 0 \) and \([V^*/M]^* : L(\mu)] = 0 \) if \( \text{ht}_{\lambda}(\Lambda - \mu) \leq N \). \( \square \)

### 7.4. Proof of Proposition 7.6

Let \( w_n = \begin{cases} t_{\bar{n}^\lambda} & \text{if } n = \bar{n}^\lambda_+ \\ w_0 & \text{if } n = \bar{n}^\lambda_- \end{cases} \) and \( m = g_{w_n}^\lambda \).

Thus, \( m = \begin{cases} t_{\bar{n}^\lambda}(g_+) \cap g_- & \text{if } n = \bar{n}^\lambda_+ \\ w_0 & \text{if } n = \bar{n}^\lambda_- \end{cases} \). Note \( m \subset n_- \) for the either case.

Let \( \Lambda \in \mathfrak{h}_\kappa^\lambda \) be as in Lemma 7.3. Then, any objects in \( \mathcal{O}^{[\Lambda]}_{\kappa} \) is free over \( m \) by Theorem 3.2. Therefore,

\[
\text{ch}(V/mV) = \prod_{\alpha \in \Delta^\omega(m)} (1 - e^\alpha) \text{ch}\ V \quad (V \in \mathcal{O}^{[\Lambda]}_{\kappa}). \quad (7.8)
\]

For \( V \in \mathcal{O}^{[\Lambda]}_{\kappa}, \Lambda \in \mathfrak{h}_\kappa^\lambda \), define \([V : M(\mu)] \in \mathbb{Z}, \mu \in \mathbb{W}^\Lambda \circ \Lambda \), by

\[
\text{ch} V = \sum_{\mu \in \mathbb{W}^\Lambda \circ \Lambda} [V : M(\mu)] \text{ch} M(\mu).
\]

Recall

\[
\text{ch} M(\lambda) = \frac{e^\lambda}{\prod_{\alpha \in \Delta^\omega(1 - e^\alpha)} \dim g_\alpha} \prod_{\alpha \in \Delta^\omega(1 - e^\alpha)}.
\]
Proposition 7.10. Let \( \Delta \in \mathfrak{h}^* \) be as in Lemma 7.9. Then, for any \( V \in \mathcal{O}_{\Lambda}^{[\lambda]} \),
\[
\sum_{\lambda \in \mathfrak{h}^*} \text{ch} \, H^* (B_3^*(n, V)) e^\lambda \leq \prod_{\alpha \in \Delta^m} \left( 1 - e^\alpha \right)^{\dim \mathfrak{g}_0} \text{ch}(V/mV) \prod_{\alpha \in \Delta^{n} \setminus \Delta^m} (1 + z^{-1} e^\alpha) \prod_{\alpha \in \Delta^{n_\perp}} (1 + z e^\alpha) \\
= \prod_{\alpha \in \Delta^{n} \setminus \Delta^m} (1 + z^{-1} e^\alpha) \prod_{\alpha \in \Delta^{n_\perp}} (1 + z e^\alpha) \sum_{\mu \in W^\Lambda \circ \Lambda} [V : M(\mu)] e^\mu
\]
where inequity \( \leq \) means that each coefficient of \( z^i e^\lambda \) of the left-hand-side is smaller than or equal to that of the right-hand-side.

Proof. Consider the (obvious semi-infinite analogue of) Hochschild-Serre spectral sequence for the subalgebra \( m \subset n \). It is easy to check that the corresponding filtration is bounded upper on each \( C(n, V)^\lambda, \lambda \in \mathfrak{h}^* \). By definition,
\[
E_{i=1}^{p,q} = H_{-q} (m, V \otimes \Lambda^{\pm p} (n/m)) .
\]
Here, \( \Lambda^{\pm p} (n/m) = \sum_{i-j=p} \Lambda^i n^j \otimes \Lambda^j (n_-/m) \) and \( m \) acts on \( \Lambda^i n^j \) via the identification \( n^j = n^*_{\perp} = (n/n_-)^j \). Clearly, we have
\[
\text{ch} \, H^{\pm p} (n, V) \leq \sum_{p,q} z^{p+q} \sum_{\lambda} \text{dim} \, H_{-q} (m, V \otimes \Lambda^{\pm p} (n/m)) e^\lambda .
\]
(7.9)
Since \( V \) is a free \( m \)-module, so is \( V \otimes \Lambda^{\pm p} (n/m), p \in \mathbb{Z} \). Thus,
\[
E_{i=1}^{p,q} = \begin{cases} \text{ch}(V \otimes \Lambda^{\pm p} (n/m)) / m & (q = 0) \\ \{0\} & (q \neq 0). \end{cases}
\]
(7.10)
By (7.9) and (7.10), we get
\[
\text{ch} \, H^{\pm p} (n, V) \leq \text{ch}(V/mV) \cdot \text{ch} \, \Lambda^{\pm p} (n/m) \cdot \prod_{\alpha \in \Delta^{n \perp}} (1 - e^\alpha).
\]
Here, we have set \( \text{ch} \, \Lambda^{\pm p} (n/m) = \sum_{i} z^i \sum_{\lambda} \text{dim} \, (\Lambda^{\pm i} (n/m)) e^\lambda \). It is easy to see that
\[
\text{ch} \, \Lambda^{\pm i} (n/m) = \prod_{\alpha \in \Delta^{n \perp}} (1 + z^{-1} e^\alpha) \prod_{\alpha \in \Delta^{n \perp}} (1 + z e^\alpha).
\]
Therefore, \( \text{ch} \) and (7.8) prove the Proposition. \( \square \)

Proof of Proposition 7.8. Suppose \( H^i (B_3 (n, V)) \neq \{0\} \) for some \( \lambda \in \mathfrak{h}^* \). Since \( \Delta^{m} (m) = w_n (\Delta^m) \cap \Delta^m \), we have \( \Delta^m \setminus \Delta^{n_\perp} (m) = \Delta^m \cap w_n (\Delta^m) = - \Delta^m \cap w_n (\Delta^m) \). Therefore, by Proposition 7.10 \( \lambda \) has the form as
\[
\lambda = \mu - \sum_{\alpha \in \Delta^{m} \cap w_n (\Delta^m)} m_\alpha \alpha, \quad \text{with} \quad \sum m_\alpha \geq |i|,
\]
(7.11)
with \( \mu \in W^\Lambda \circ \Lambda \) such that \( [V : M(\mu)] \neq 0 \). We claim that (7.11) implies
\[
h^*_n \leq \frac{h^*_n}{2} - |i|.
\]
(7.12)
Indeed, for the \( n = L\bar{n}_{-} \) case \((\ref{12})\) easily follows from \((\ref{10})\) and the fact that \( \Delta_{\bar{\rho}}^{\mu} \cap \mu_{0}(\Delta_{\bar{\rho}}^{\mu}) \cap \Delta_{\bar{\rho}} = 0 \). To see \((\ref{12})\) for the \( n = L\bar{n}_{+} \) case, notice that \( t_{-\bar{\rho}}(\Delta_{\bar{\rho}}^{\mu}) \cap \Delta_{\bar{\rho}} = 0 \), and, therefore, \( \langle t_{-\bar{\rho}}(\alpha), D \rangle \geq 1 \) for any \( \alpha \in \Delta_{\bar{\rho}}^{\mu} \cap t_{\bar{\rho}}(\Delta_{\bar{\rho}}^{\mu}) \). Then, \((\ref{12})\) follows from \((\ref{12})\).

By Proposition \((\ref{9})\) we have shown that \( \mathbf{D}_{n}^{\lambda_{-}} \)-eigenvalues of \( \mathbf{H}_{QDS}^{i}(\mathfrak{n}, V) \) is contained in the set \( \bigcup_{\mu' \in W^{\Lambda} \circ \Lambda} \bar{h}_{n}^{\mu'} - Z_{\geq |i|} \). But \([V : M(\mu)] \neq 0\) implies there exists \( \mu' \in W^{\Lambda} \circ \Lambda \) such that \([V : L(\mu')] \neq 0\) and \( \mu' - \mu \in Q_{\Lambda}^{+} \). Thus, Proposition follows from Lemma \((\ref{8})\).

8. Vanishing of cohomology

8.1. Vanishing of cohomology associated to projective modules and injective modules. For a given \( \Lambda \in \mathfrak{h}_{\Lambda}^{*} \), let \( \mathcal{O}_{\Lambda}^{[\leq \Lambda]} \) be the full subcategory of \( \mathcal{O}_{\Lambda}^{[\Lambda]} \) consisting of module \( V \) such \( V^{\Lambda} = \{0\} \) unless \( \lambda \in \Lambda - Q_{\Lambda}^{+} \). Then, every finitely generated object of \( \mathcal{O}_{\Lambda}^{[\leq \Lambda]} \) is an image of some projective object of \( \mathcal{O}_{\Lambda}^{[\leq \Lambda]} \) by \((\ref{20})\) \(2.10\). Let \( \Delta \mathcal{O}_{\Lambda}^{[\leq \Lambda]} \) be the full subcategory of \( \mathcal{O}_{\Lambda}^{[\leq \Lambda]} \) consisting of modules \( V \) that admits a Verma flag, i.e., a finite filtration

\[
V = V_{0} \supset V_{1} \supset \cdots \supset V_{k} = \{0\}
\]

such that each successive subquotient \( V_{i}/V_{i+1} \) is isomorphic to some Verma module. It is known that an object \( V \) in \( \mathcal{O}_{\Lambda}^{[\leq \Lambda]} \) belongs to \( \Delta \mathcal{O}_{\Lambda}^{[\leq \Lambda]} \) if and only if \( \text{Ext}^{1}_{\mathcal{O}}(V, M(\lambda)^{*}) = \{0\} \) for all \( M(\lambda)^{*} \in \mathcal{O}_{\Lambda}^{[\leq \Lambda]} \). In particular, projective objects in \( \mathcal{O}_{\Lambda}^{[\leq \Lambda]} \) are objects in \( \Delta \mathcal{O}_{\Lambda}^{[\leq \Lambda]} \).

**Theorem 8.1.** For a given \( \Lambda \in \mathfrak{h}_{\Lambda}^{*}, H_{QDS}^{i}(\mathfrak{n}, V) = \{0\} \) \((i \neq 0)\) for all \( V \in \Delta \mathcal{O}_{\Lambda}^{[\leq \Lambda]} \). In particular, \( H_{QDS}^{i}(\mathfrak{n}, P) = \{0\} \) \((i \neq 0)\) for all projective objects in \( \mathcal{O}_{\Lambda}^{[\leq \Lambda]} \).

**Proof.** We prove by induction on the length \( k(V) \) of the Verma flag of \( V \). We have already proved the \( k(V) = 1 \) case in Theorem \((\ref{7})\). Let \( k(V) \geq 2 \). Then, there exits an exact sequence \( 0 \rightarrow V_{1} \rightarrow V \rightarrow M(\mu) \rightarrow 0 \) \((\mu \in \mathfrak{h}_{\Lambda}^{*}) \) in \( \Delta \mathcal{O}_{\Lambda}^{[\leq \Lambda]} \). Thus, the corresponding long exact sequence and the induction hypothesis prove the proposition. \( \square \)

Similarly, let \( \nabla \mathcal{O}_{\Lambda}^{[\leq \Lambda]} \) be the full subcategory of \( \mathcal{O}_{\Lambda}^{[\leq \Lambda]} \) consisting of modules \( V \) such that \( V^{*} \in \Delta \mathcal{O}_{\Lambda}^{[\leq \Lambda]} \).

**Theorem 8.2.** Let \( \Lambda \in \mathfrak{h}_{\Lambda}^{*} \) such that

\[
\begin{cases}
\langle \Lambda + \rho, \alpha \rangle \notin \mathbb{Z} \text{ for all } \alpha \in \Delta_{\bar{\rho}}^{\mu} \cap t_{\bar{\rho}'}(\Delta_{\bar{\rho}}^{\mu}) & \text{if } n = L\bar{n}_{+}, \\
\langle \Lambda + \rho, \alpha \rangle \notin \mathbb{Z} \text{ for all } \alpha \in \Delta_{\bar{\rho}} & \text{if } n = L\bar{n}_{-}.
\end{cases}
\]

Then, \( H_{QDS}^{i}(\mathfrak{n}, V) = \{0\} \) \((i \neq 0)\) for all \( V \in \nabla \mathcal{O}_{\Lambda}^{[\leq \Lambda]} \). In particular, \( H_{QDS}^{i}(\mathfrak{n}, I) = \{0\} \) \((i \neq 0)\) for all injective objects in \( \mathcal{O}_{\Lambda}^{[\leq \Lambda]} \).

**Proof.** The assumption on \( \Lambda \) implies \( H_{QDS}^{i}(\mathfrak{n}, M(\lambda)^{*}) = \{0\} \) \((i \neq 0)\) for \( \lambda \in W^{\Lambda} \circ \Lambda \) by Theorem \((\ref{6})\). Thus, the theorem can be proved similarly as Theorem \((\ref{8})\). \( \square \)
8.2. Main theorem.

**Theorem 8.3.** Let \( \kappa \in \mathbb{C}\setminus\{0\} \) and \( \Lambda \in \mathfrak{h}_\kappa^* \).

1. Suppose that \( \langle \Lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z} \) for all \( \alpha \in \Delta^\vee \cap t_{\rho^+}(\Delta^\vee) \). Then,
   \[
   H_{QDS}^i(L\bar{n}_+, V) = \{0\} \quad (i \neq 0)
   \]
   for all \( V \in \mathcal{O}_\kappa^{[\Lambda]} \).

2. Suppose that \( \langle \Lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z} \) for all \( \alpha \in \bar{\Delta}_+ \). Then,
   \[
   H_{QDS}^i(L\bar{n}_-, V) = \{0\} \quad (i \neq 0)
   \]
   for all \( V \in \mathcal{O}_\kappa^{[\Lambda]} \).

*Proof of Theorem 8.3 when \( \kappa \in \mathbb{C}\setminus\mathbb{Q}_{\geq 0} \).* We may assume that \( V \in \mathcal{O}_\kappa^{[\leq \Lambda]} \). Since \( \kappa \in \mathbb{C}\setminus\mathbb{Q}_{\geq 0} \), the cohomological dimension of \( V \in \mathcal{O}_\kappa^{[\leq \Lambda]} \) is finite, that is, there exists a projective resolution
\[
0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to V \to 0
\]
of \( V \) in \( \mathcal{O}_\kappa^{[\leq \Lambda]} \). Let \( N_k = \text{Im} \partial_k \). Then, \( 0 \to N_{k+1} \to P_k \to N_k \to 0 \). Thus, by the long exact sequence of semi-infinite cohomology, we get \( H^i(N_k) \cong H^{i+1}(N_{k+1}) \) for \( i > 0 \) by Theorem 8.1. This implies \( H^i(V) = H^{i+1}(P_n) = \{0\} \) for all \( i > 0 \). The proof of \( H^i(V) = \{0\} \) for \( i < 0 \) is similar.

When \( \kappa \in \mathbb{Q}_{>0} \), some modification of the proof is needed:

**Proposition 8.4.** Let \( \kappa \in \mathbb{Q}_{>0} \) and let \( \Lambda \in \mathfrak{h}_\kappa^* \) be as in Theorem 8.3. Let \( V \in \mathcal{O}_\kappa^{[\Lambda]} \).

1. For a given \( \lambda \in \mathbb{C} \), there exist an object \( N \in \mathcal{O}_\kappa^{[\Lambda]} \) such that \( H_{QDS}^i(n, V)_a = H_{QDS}^{i+1}(n, N)_a \) for all \( i > 0 \).

2. For a given \( \lambda \in \mathbb{C} \), there exist an object \( N' \) in \( \mathcal{O}_\kappa^{[\Lambda]} \) such that \( H_{QDS}^i(n, V)_a = H_{QDS}^{i+1}(n, N')_a \) for all \( i < 0 \).

*Proof.* (1) By Proposition 7.9, there exists finitely generated submodule \( V' \) of \( V \) such that
\[
H_{QDS}^i(n, V)_a \cong H_{QDS}^i(n, V')_a.
\]
for the given \( \lambda \). Since \( V' \) is finitely generated, there exists some projective object \( P \) of \( \mathcal{O}_\kappa^{[\Lambda]} \) and an exact sequence \( 0 \to N \to P \to V' \to 0 \) in \( \mathcal{O}_\kappa^{[\Lambda]} \). Therefore, we get \( H_{QDS}^i(n, V') \cong H_{QDS}^{i+1}(n, N) \) for all \( i > 0 \) by Theorem 8.1. By 8.1, this implies \( H_{QDS}^i(n, V)_a \cong H_{QDS}^{i+1}(n, N)_a \) for all \( i > 0 \). (2) can be similarly proved by using Theorem 8.2.

*Proof of Theorem 8.3 when \( \kappa \in \mathbb{Q}_{>0} \).* We may assume \( \Lambda \in \mathcal{C}_{\kappa,+} \). It is sufficient to show that \( H_{QDS}^i(n, V)_a = \{0\} \) (\( i \neq 0 \)) for all \( V \in \mathcal{O}_\kappa^{[\Lambda]} \) and \( \lambda \in \mathbb{C} \).

Fix \( \lambda \in \mathbb{C} \). By applying Proposition 8.4 (1) repeatedly, it follows that, for any \( r > 0 \), there exists an object \( N_r \) of \( \mathcal{O}_\kappa^{[\Lambda]} \) such that
\[
H_{QDS}^i(n, V)_a \cong H_{QDS}^{i+r}(n, N_r)_a \quad (i > 0).
\]
This forces \( H_{QDS}^i(n, V)_a = \{0\} \) for \( i > 0 \) by Corollary 8.1. The proof for \( i < 0 \) is similar.

The following is straightforward from Theorem 8.3 and Remark 8.8.
Corollary 8.5. Let $\Lambda$ be as in Theorem \[8.5\]. Then, the correspondence $V \rightsquigarrow H^0_{\text{QDS}}(n, V)$ defines an exact functor from $O^{[\Lambda]}_\kappa$ to the category of $W_\kappa(\mathfrak{g})$-modules. In particular,

$$\text{ch} H^0_{\text{QDS}}(n, V) = \sum_{\mu \in W^\Lambda \circ \Lambda} [V : M(\mu)] \frac{q^{h^\mu}}{\prod_{i \geq 1} (1 - q^i)^{\text{rank} \mathfrak{g}}}.$$ 

for $V \in O^{[\Lambda]}_\kappa$.

Remark 8.6.

1. Let $\kappa \in \mathbb{C} \setminus \mathbb{Q}$. Then, any $\Lambda \in \mathfrak{h}_\kappa^*$ such that $\bar{\Lambda} \in \bar{P}$ satisfies the condition of Theorem \[8.3\] (1).
2. It was proved in \[9\] that $H^0_{\text{QDS}}(L\bar{\Lambda}_-, L(\lambda)) = 0$ if $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for some $\alpha \in \bar{P}$.
3. Suppose that $\langle \Lambda + \rho, \alpha^\vee \rangle \not\in \mathbb{Z}$ for all $\alpha \in \bar{\Delta}_+$. Then, by Corollary \[8.5\] and Remark \[8.5\] it follows that

$$\text{ch} H^0_{\text{QDS}}(L\bar{\Lambda}_-, L(\lambda)) = \text{ch} H^0_{\text{QDS}}(L\bar{\Lambda}_-, L(w \circ \lambda))$$ 

for $w \in \bar{W}$ and $\lambda \in W^\Lambda \circ \Lambda$.
4. A principal admissible weight $\Lambda$ (\[13\]) is called non-degenerate if $\langle \Lambda + \rho, \alpha^\vee \rangle \not\in \mathbb{Z}$ for all $\alpha \in \bar{\Delta}_+$. By Theorem \[8.3\] (2), it follows that $H^0_{\text{QDS}}(L\bar{\Lambda}_-, L(\lambda)) = \{0\}$ ($i \neq 0$) if $\Lambda$ is a non-degenerate principal admissible weight. This was conjectured by Frenkel-Kac-Wakimoto (\[9, \text{Conjecture 3.4}\]).
5. Let $\kappa = p/q$, $p \in \mathbb{Z}_{\geq h^\vee}$, $q \in \mathbb{Z}_{\geq h}$, $(p, q) = 1$, $(q, r^\vee) = 1$, where $h$ is the Coxeter number of $\bar{\mathfrak{g}}$ and $r^\vee = \max\{2/(\alpha, \alpha) ; \alpha \in \bar{P}\}$. Then, $\kappa - h^\vee$ is a principal admissible number (\[13\]). Set

$$\Lambda_{\lambda, \mu} = \lambda - \kappa \mu + (\kappa - h^\vee)\Lambda_0 \quad (\lambda \in \bar{P}_+^{p-h^\vee}, \mu \in \bar{P}_+^{p-h^\vee}),$$

where $\bar{P}_+^{p-h^\vee} = \{\lambda \in \bar{P} \mid 0 \leq \langle \lambda, \alpha^\vee \rangle \leq p - h^\vee \ (\forall \alpha \in \bar{\Delta}_+)\}$ and $\bar{P}_+^{\rho-q-h} = \{\mu \in \bar{P}^\vee \mid 0 \leq \langle \alpha, \mu \rangle \leq q - h \ (\forall \alpha \in \bar{\Delta}_+)\}$. Let

$$\bar{P}_\kappa = \{\Lambda_{\lambda, \mu} ; (\lambda, \mu) \in \bar{P}_+^{p-h^\vee} \times \bar{P}_+^{\rho-q-h}\} \subset \mathfrak{h}_\kappa^*.$$ 

Then, $\bar{P}_\kappa$ is a subset of the set of principal admissible weights of $\mathfrak{g}$ of level $\kappa - h^\vee$. Note that $(\kappa - h^\vee)\Lambda_0 = \Lambda_0, 0 \in \bar{P}_\kappa$.

It is easy to see that any element of $\bar{P}_\kappa$ satisfies the condition of Theorem \[8.3\] (1). Thus,

$$H^0_{\text{QDS}}(L\bar{\Lambda}_+, L(\Lambda)) = \{0\} \quad (i \neq 0) \quad \text{for} \quad \Lambda \in \bar{P}_\kappa \quad (8.2)$$

by Theorem \[8.3\] (1). This proves the conjecture of Frenkel-Kac-Wakimoto \[9, \text{Conjecture 3.4}\] partially. Note that \[8.2\] in particular implies

$$H^i_{\text{QDS}}(L\bar{\Lambda}_+, L((\kappa - h^\vee)\Lambda_0)) = \{0\} \quad (i \neq 0).$$

It is expected that $H^0_{\text{QDS}}(L\bar{\Lambda}_+, L((\kappa - h^\vee)\Lambda_0))$ is a rational VOA and that the modules $\{H^0_{\text{QDS}}(L\bar{\Lambda}_+, L(\lambda)) ; \Lambda \in \bar{P}_\kappa\}$ exhaust the simple objects of the vertex operator algebra $H^0_{\text{QDS}}(L\bar{\Lambda}_+, L((\kappa - h^\vee)\Lambda_0))$, see \[9\].
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