One-loop Analyses of lattice QCD with the overlap Dirac operator

Masato Ishibashi*, Yoshio Kikukawa†, Tatsuya Noguchi‡
and Atsushi Yamada§

*†‡§ Department of Physics, University of Tokyo
Tokyo 113-0033, Japan

† Department of Physics, Nagoya University
Nagoya 464-8602, Japan

‡ Department of Physics, Kyoto University
Kyoto 606-8502, Japan

November, 1999

Abstract

We discuss the weak coupling expansion of lattice QCD with the overlap Dirac operator. The Feynman rules for lattice QCD with the overlap Dirac operator are derived and the quark self-energy and vacuum polarization are studied at the one-loop level. We confirm that their divergent parts agree with those in the continuum theory.

PACS: 11.15.Ha
Keywords: Lattice perturbation theory; Massless QCD; Overlap Dirac operator

*e-mail address: ishibash@hep-th.phys.s.u-tokyo.ac.jp
†e-mail address: kikukawa@eken.phys.nagoya-u.ac.jp
‡e-mail address: noguchi@gauge.scphys.kyoto-u.ac.jp
§e-mail address: atsushi@hep-th.phys.s.u-tokyo.ac.jp
1 Introduction

The overlap Dirac operator \[1\] satisfies the Ginsparg-Wilson relation \[2\] and thus keeps the lattice chiral symmetry proposed in Ref. \[3\]. This Dirac operator has no species doublers and is local (with exponentially decaying tails) for gauge fields with small field strength \[4\]. Originally the overlap Dirac operator is derived from the overlap formalism \[5, 6\]. In this formalism the axial anomaly is calculated and the renormalizability is discussed within the perturbation theory \[7, 8\]. General consequences of the analyses performed in the overlap formalism are expected to hold also in the case of the overlap Dirac operator. However the actual procedures of the perturbative analyses are so different in these two cases and the correspondence between them is not straightforward. In the overlap formalism, time independent perturbation theory using creation and annihilation operators should be invoked, and the perturbation theory is not conveniently expressed as Feynmann rules, while such diagrammatic techniques are available for the overlap Dirac operator. Moreover, the finite parts of renormalization factors on the lattice, which are necessary to extract physical observables from numerical simulations, are different \[9\].

Therefore in this paper, we discuss the weak coupling expansion of lattice QCD using the overlap Dirac operator. Our study is an extension of the previous analyses of Ref. \[10\], where only the axial anomaly is computed. \[1\] We derive Feynman rules for lattice QCD with the overlap Dirac operator and compute the quark self-energy and vacuum polarization. We demonstrate that non-local divergent terms, which are not necessarily forbidden by the Ginsparg-Wilson relation, are not in fact induced in the quark self-energy. Once non-local divergent terms are known to be absent, the Ginsparg-Wilson relation prohibits divergent mass terms, as is shown later. The divergent parts of the wave function renormalization factors agree with the continuum theory. We also consider the vacuum polarization. Again, non-local divergences are absent, and the wave function renormalization factors are correctly reproduced.

Our analyses are consistent with Ref.\[18\], where a formal argument on the proof of the renormalizability was given for a general Dirac operator satisfying the Ginsparg-Wilson relation. Supplementing their formal argument, we take the overlap Dirac operator and show explicitly the nontrivial can-

\[1\] The axial anomaly has been discussed using a different calculational scheme in Ref. \[11, 12, 13\].
cancellations necessary to ensure renormalizability at one-loop level. In Ref. [19] the finite part of the vacuum polarization at one-loop has been evaluated and the lambda parameter ratio has been obtained. But the quark self-energy, which determines the chiral property of the renormalized fermions, was not analyzed.

2 Derivation of the Feynman rules

In this section we derive the Feynman rules for lattice QCD with the overlap Dirac operator, which are necessary for the one-loop analysis. The fermion action with the overlap Dirac operator is

\[ S_F = a^4 \sum_m \bar{\psi}(m) D \psi(m), \quad (2.1) \]

where

\[ D = \frac{1}{a} \left( 1 + X \frac{1}{\sqrt{X^\dagger X}} \right). \quad (2.2) \]

Here \( X \) is the Wilson-Dirac operator defined as

\[ X_{mn} = \frac{1}{2a} \sum_{\mu=1}^{4} \left[ \gamma_\mu \left\{ \delta_{m+\bar{\mu},n} U_\mu(m) - \delta_{m,n+\bar{\mu}} U_\mu^\dagger(n) \right\} \right. \]
\[ + \left. r \left\{ 2\delta_{m,n} - \delta_{m+\mu,n} U_\mu(m) - \delta_{m,n+\mu} U_\mu^\dagger(n) \right\} \right\} + \frac{M_0}{a} \delta_{m,n}, \quad (2.3) \]

where \( U_\mu = e^{ia g A_\mu} \) is the link variable and \( r \) is the Wilson parameter. Expanding \( X_{mn} \) up to the second order in the coupling constant \( g \), we obtain

\[ X_{mn} = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} e^{ia(qm-pn)} X(q,p), \quad (2.4) \]

\[ X(q,p) = X_0(p)(2\pi)^4 \delta_P(q-p) + X_1(q,p) + X_2(q,p) + O(g^3), \quad (2.5) \]

where

\[ X_0(p) = \sum_\rho \frac{i}{a} \gamma_\rho \sin ap_\rho + \frac{r}{a} \sum_\mu (1 - \cos ap_\mu) + \frac{1}{a} M_0, \quad (2.6) \]
\[ X_1(q,p) = \sum_{A,\mu} \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} (2\pi)^4 \delta_P(q-p-k) \times g A^A_{\mu} k) T^A V_{1\mu} \left( p + \frac{k}{2} \right), \tag{2.7} \]
\[ X_2(q,p) = \sum_{A,B,\mu,\nu} \int_{-\pi}^{\pi} \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} (2\pi)^4 \delta_P(q-p-\sum k_i) \times \frac{q^2}{2} A^A_{\mu} A^B_{\nu} T^A T^B V_{2\mu} \left( p + \frac{\sum k_i}{2} \right), \tag{2.8} \]

and the index \( i \) runs 1, 2 and \( \delta_P \) is the periodic lattice delta function. \( T^A \) are \( SU(3) \) generators in the fundamental representation. The vertex functions \( V_{1\mu} \) and \( V_{2\mu} \) are given by

\[ V_{1\mu}(p + \frac{k}{2}) = i \gamma_\mu \cos \left( p + \frac{k}{2} \right)_\mu + r \sin \left( p + \frac{k}{2} \right)_\mu, \]
\[ V_{2\mu}(p + \frac{\sum k_i}{2}) = -i \gamma_\mu a \sin \left( p + \frac{\sum k_i}{2} \right)_\mu + ar \cos \left( p + \frac{\sum k_i}{2} \right)_\mu. \tag{2.9} \]

The weak coupling expansion of the overlap Dirac operator is derived in the following way [10]. Using the following identity:

\[ \frac{1}{\sqrt{X^\dagger X}} = \int_{-\infty}^{\infty} \frac{dt}{\pi} \frac{1}{t^2 + X^\dagger X}, \tag{2.10} \]

we expand the r.h.s of Eq. (2.2) up to the second order in \( g \) as

\[ aD = aD_0 + \int_{-\infty}^{\infty} \frac{dt}{\pi} \frac{1}{t^2 + X^\dagger X} \left( t^2 X_1 - X_0 X_1^\dagger X_0 \right) \frac{1}{t^2 + X_0^\dagger X_0} \]
\[ + \int_{-\infty}^{\infty} \frac{dt}{\pi} \frac{1}{t^2 + X_0^\dagger X_0} \left( t^2 X_2 - X_0 X_2^\dagger X_0 \right) \frac{1}{t^2 + X_0^\dagger X_0} \]
\[ - \int_{-\infty}^{\infty} \frac{dt}{\pi} \frac{1}{t^2 + X_0^\dagger X_0} \left( X_0^\dagger X_1 \right) \frac{1}{t^2 + X_0^\dagger X_0} \left( X_0^\dagger X_1 + X_1^\dagger X_0 \right) \frac{1}{t^2 + X_0^\dagger X_0} \]
\[ - \int_{-\infty}^{\infty} \frac{dt}{\pi} \frac{1}{t^2 + X_0^\dagger X_0} \left( X_0 X_1^\dagger \right) \frac{1}{t^2 + X_0^\dagger X_0} \left( X_1 X_1^\dagger \right) \frac{1}{t^2 + X_0^\dagger X_0} \]
\[ + \int_{-\infty}^{\infty} \frac{dt}{\pi} \frac{1}{t^2 + X_0^\dagger X_0} \left( X_0 X_1^\dagger \right) \frac{1}{t^2 + X_0^\dagger X_0} \left( X_1 X_1^\dagger \right) \frac{1}{t^2 + X_0^\dagger X_0} \]
\[ + \cdots, \tag{2.11} \]
where $D_0$ is the zeroth-order term in $g$. The inverse of $D_0$ in momentum space is

$$D_0^{-1}(p) = \frac{-i\gamma_\mu \sin ap_\mu}{2(\omega(p) + b(p))} + \frac{a}{2} = \frac{a}{2} X_0^\dagger(p) + \omega(p),$$

$$b(p) = \frac{r}{a} \sum_\mu (1 - \cos ap_\mu) + \frac{1}{a} M_0,$$

$$a\omega(p) = \sqrt{\sin^2 ap_\mu + \left(r \sum_\mu (1 - \cos ap_\mu) + M_0 \right)^2} > 0.$$

For $-2r < M_0 < 0$ the propagator $D_0^{-1}(p)$ exhibits a massless pole only when $p_\mu = 0$, and there are no doublers. In the following, we take $M_0$ in this region. We also note that the quark fields $\psi$ are not properly normalized since the continuum limit of $D_0$ is $i\not{p}/|M_0|$.

After performing Fourier transformation in Eq. (2.11), we obtain the overlap Dirac operator in momentum space up to the second order in $g$:

$$D(q, p) = D_0(p)(2\pi)^4 \delta_P(q - p) + V(q, p),$$

where

$$V(q, p) = \left\{ \frac{1}{\omega(q) + \omega(p)} \right\} \left[ X_1(q, p) - \frac{X_0(q)}{\omega(q)} X_1^\dagger(q, p) X_0(p) \right]$$

$$+ \left\{ \frac{1}{\omega(q) + \omega(p)} \right\} \left[ X_2(q, p) - \frac{X_0(q)}{\omega(q)} X_2^\dagger(q, p) X_0(p) \right]$$

$$+ \int r \left\{ \frac{1}{\omega(q) + \omega(r)} \right\} \left\{ \frac{1}{\omega(q) + \omega(r)} \right\} \left\{ \frac{1}{\omega(r) + \omega(p)} \right\}$$

$$\times \left[ -X_0(q) X_1^\dagger(q, r) X_1(r, p) - X_1(q, r) X_0^\dagger(r, p) X_1(r, p) - X_1(q, r) X_1^\dagger(r, p) X_0(p) + \frac{\omega(q) + \omega(p) + \omega(r)}{\omega(q)\omega(p)\omega(r)} X_0(q) X_1^\dagger(q, r) X_0(r) X_1^\dagger(r, p) X_0(p) \right] + \cdots.$$

(2.16)

From this weak coupling expansion we can derive the Feynman rules for lattice QCD with the overlap Dirac operator. For the quark propagator we assign the expression (2.14). For the quark-gluon three-point vertex depicted
in Fig. 4 we assign
\[- \frac{g}{a(\omega(q) + \omega(p))} T^A_{ab} \left\{ V_{1\mu} \left( p + \frac{k}{2} \right) - \frac{X_0(q)}{\omega(q)} V^\dagger_{1\mu} \left( p + \frac{k}{2} \right) \times \frac{X_0(p)}{\omega(p)} \right\}. \tag{2.17}\]

For the quark-gluon four-point vertex depicted in Fig. (2), we assign
\[- \frac{g^2}{2a} \left\{ T^A, T^B \right\}_{ab} \left[ \frac{1}{\omega(q) + \omega(p)} \left\{ V_{2\mu} \left( p + \frac{k}{2} \right) \delta_{\mu\nu} - \frac{X_0(q)}{\omega(q)} V^\dagger_{2\mu} \left( p + \frac{k}{2} \right) \times \frac{X_0(p)}{\omega(p)} \delta_{\mu\nu} \right\} \right.\]
\[- \frac{1}{\{\omega(q) + \omega(p)\}\{\omega(p) + \omega(p + k_1)\}\{\omega(p + k_1) + \omega(q)\}} \times \left\{ X_0(q) V^\dagger_{1\mu} \left( p + k_2 + \frac{k_1}{2} \right) V_{1\nu} \left( p + \frac{k_2}{2} \right) + V_{1\mu} \left( p + k_2 + \frac{k_1}{2} \right) X_0(p) \right\} \]
\[- \frac{\omega(q) + \omega(p) + \omega(p + k_2)}{\omega(q)\omega(p)\omega(p + k_2)} \left\{ X_0(q) V^\dagger_{1\nu} \left( p + k_2 + \frac{k_1}{2} \right) X_0(p + k_2) V^\dagger_{1\mu} \left( p + \frac{k_2}{2} \right) X_0(p) \right\} \]
\[- \frac{\omega(q) + \omega(p) + \omega(p + k_1)}{\omega(q)\omega(p)\omega(p + k_1)} \left\{ X_0(q) V^\dagger_{1\mu} \left( p + k_1 + \frac{k_2}{2} \right) V_{1\nu} \left( p + \frac{k_1}{2} \right) + V_{1\nu} \left( p + k_1 + \frac{k_2}{2} \right) X_0(p) \right\} \]
\[- \frac{1}{\omega(q)\omega(p)\omega(p + k_1)} \times \left\{ X_0(q) V^\dagger_{1\nu} \left( p + k_1 + \frac{k_2}{2} \right) X_0(p + k_1) V^\dagger_{1\mu} \left( p + \frac{k_1}{2} \right) X_0(p) \right\} \] \tag{2.18}

The gauge field action is chosen as Wilson’s plaquette action, where
\[ S_G = \frac{1}{g^2} \sum_m \sum_{\mu\nu} \text{Tr} (1 - U_{\mu\nu}(m)), \tag{2.19} \]

where \( U_{\mu\nu}(m) = U_{\mu}(m)U_{\nu}(m+\hat{\mu})U_{\mu}(m+\hat{\nu})U_{\nu}(m) \). The gluon propagator is given by
\[ G_{\mu\nu}(k) = \frac{1}{k^2} \left( \delta_{\mu\nu} - (1 - \alpha) \frac{k_{\mu} k_{\nu}}{k^2} \right), \tag{2.20} \]

\(^2\) Lüscher proposed other action for gauge field in the abelian gauge theory, which seems to be more appropriate in view of the locality properties.
where $\tilde{k}_\mu = (2/a) \sin \alpha k_\mu / 2$, and $\alpha$ is the gauge fixing parameter.

In the following sections we calculate the quark self-energy and the vacuum polarization at one-loop level using these Feynman rules.

Figure 1: Gluon-fermion three-point vertex

Figure 2: Gluon-fermion four-point vertices (a) and (b)

3 Evaluation of the quark self-energy

In this section we evaluate the one-loop quark self-energy of the lattice QCD with the overlap Dirac operator and we show that the structure of divergence is equal to that in the continuum theory. In the lattice QCD with the overlap Dirac operator, there exists the chiral symmetry based on the Ginsparg-Wilson relation as $\{\gamma_5, D\} = aD\gamma_5D$. First, let us investigate what the Ginsparg-Wilson relation tells us about the structure of divergence. Up to one-loop order the Ginsparg-Wilson relation is written as
\begin{equation}
\gamma_5, S_F^{-1}(p) + \Sigma(p) = a(S_F^{-1}(p) + \Sigma(p)) \gamma_5 (S_F^{-1}(p) + \Sigma(p)), \tag{3.1}
\end{equation}

where $S_F(p)$ is the quark propagator and $\Sigma(p)$ is the one-loop quark self-energy. Ignoring the second-order term and using the Ginsparg-Wilson relation at the tree level, we obtain

\begin{equation}
\{ \gamma_5, \Sigma(p) \} = a \Sigma(p) \gamma_5 S^{-1}_F(p) + a S_F^{-1}(p) \gamma_5 \Sigma(p). \tag{3.2}
\end{equation}

Since $\Sigma(p)$ has dimension 1, the possible structure of divergence is

\begin{equation}
\Sigma(p) = \sum_{n=1}^{\infty} \left\{ \frac{i p C_{1n}}{(a^2 p^2)^n} + \frac{a p^2 C_{2n}}{(a^2 p^2)^{n+1}} \right\} + \frac{C_3}{a} + \sum_{n=1}^{\infty} C_{4n} i p (\log a^2 p^2)^n + \text{finite terms}, \tag{3.3}
\end{equation}

where $C_{1n}, C_{2n}, C_3$ and $C_{4n}$ is generally a linear combination of constants and the power of $\log a^2 p^2$. In the above equation the first and second terms are non-local divergences and the third and fourth terms are local divergences. In general, the Ginsparg-Wilson relation does not exclude non-local divergent terms, which spoil the renormalizability of the theory. Now we compute the quark self-energy $\Sigma$ with the overlap Dirac operator and we show that non-local divergent terms don’t appear. If such non-local terms are absent, the linearly divergent terms $C_3$ in Eq. (3.3) should also vanish due to the Ginsparg-Wilson relation, which we elucidate in our calculation. We also compute the logarithmic divergent part and show that the divergent part of the quark field renormalization factor is equal to that in the continuum theory.

The Feynman diagrams which contribute to the one-loop quark self-energy are shown in Fig. 3. The amplitude corresponding to Fig. 3(a) is

\begin{equation}
\Sigma^{(a)}(p) = \frac{g^2}{a^2} \sum_A (T^A)_{ba} \sum \sum_{\mu \nu} \int \frac{d^4 k}{(2\pi)^4} G_{\mu \nu}(p - k) \frac{1}{\{\omega(k) + \omega(p)\}^2} \left\{ V_{1\mu} \left( \frac{p + k}{2} \right) - \frac{X_0(p)}{\omega(p)} V_{1\mu}^\dagger \left( \frac{p + k}{2} \right) \frac{X_0(k)}{\omega(k)} \right\} \times \left\{ -i \gamma_\rho \sin ak_\rho + a \frac{1}{2} \right\} \left\{ V_{1\nu} \left( \frac{p + k}{2} \right) - \frac{X_0(k)}{\omega(k)} V_{1\nu}^\dagger \left( \frac{p + k}{2} \right) \frac{X_0(p)}{\omega(p)} \right\}. \tag{3.4}
\end{equation}
Figure 3: Feynman diagrams for the quark self-energy

The amplitude corresponding to Fig.3(b) is

\[
\Sigma^{(b)}(p) = -g^2 \frac{2}{a^2} \sum_A \langle T^A \rangle_{b,a} \sum_{\mu \nu} \int_{\frac{-\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4} G_{\mu\nu}(k) \\
\times \left[ \frac{1}{2\omega(p)} \left\{ V_{2\mu}(p) - \frac{X_0(p)}{\omega(p)} V_{2\mu}^\dagger(p) \frac{X_0(p)}{\omega(p)} \right\} \delta_{\mu\nu} - \frac{1}{\omega(p) \{ \omega(p) + \omega(p + k) \}^2} \right] \\
\times \left\{ X_0(p) V_{1\mu}^\dagger \left( p + \frac{k}{2} \right) V_{1\nu} \left( p + \frac{k}{2} \right) + V_{1\mu} \left( p + \frac{k}{2} \right) X_0^\dagger(p + k) V_{1\nu} \left( p + \frac{k}{2} \right) \\
+ V_{1\mu} \left( p + \frac{k}{2} \right) V_{1\nu}^\dagger \left( p + \frac{k}{2} \right) X_0(p) \\
- \frac{2\omega(p) + \omega(p + k)}{\omega(p)^2 \omega(p + k)} X_0(p) V_{1\mu}^\dagger \left( p + \frac{k}{2} \right) X_0(p + k) V_{1\nu}^\dagger \left( p + \frac{k}{2} \right) X_0(p) \right\}.
\]

(3.5)

To evaluate the divergent parts of \( \Sigma^{(a)} \) and \( \Sigma^{(b)} \), we rescale the integration momentums \( k_\mu \rightarrow k_\mu / a \) in each amplitude and we obtain

\[
\Sigma^{(a)}(p) = g^2 \frac{2}{a^2} \sum_{\mu \nu} \int_k G_{\mu\nu}(pa - k) \\
\times \left\{ \frac{1}{\{ \omega(k) + \omega(pa) \}^2} \left\{ V_{1\mu} \left( \frac{pa + k}{2} \right) - \frac{X_0(pa)}{\omega(pa)} V_{1\mu}^\dagger \left( \frac{pa + k}{2} \right) \frac{X_0(k)}{\omega(k)} \right\} \\
\times \left\{ \frac{-i \gamma_\rho \sin k_\rho}{(\omega(k) + b(k))} + 1 \right\} \left\{ V_{1\nu} \left( \frac{pa + k}{2} \right) - \frac{X_0(k)}{\omega(k)} V_{1\nu}^\dagger \left( \frac{pa + k}{2} \right) \frac{X_0(pa)}{\omega(pa)} \right\} \right\},
\]

(3.6)
and

\[
\Sigma^{(i)}(p) = \frac{-g^2}{2\alpha} \sum_{\mu\nu} \int \frac{1}{\omega(pa)} G_{\mu\nu}(k) \delta_{\mu\nu} - \frac{1}{\{\omega(pa) + \omega(pa + k)\}^2} \delta_{\mu\nu} - \frac{1}{\{\omega(pa) + \omega(pa + k)\}^2} \delta_{\mu\nu}
\]

\[
\times \left\{ X_0(pa) V_{1\mu} \left( pa + \frac{k}{2} \right) V_{1\nu} \left( pa + \frac{k}{2} \right) + V_{1\mu} \left( pa + \frac{k}{2} \right) X_0(pa + k) V_{1\nu} \left( pa + \frac{k}{2} \right) + V_{1\mu} \left( pa + \frac{k}{2} \right) X_0(pa + k) V_{1\nu} \left( pa + \frac{k}{2} \right) X_0(pa) \right\},
\]

(3.7)

where \( \sigma_{\mu\nu}^{(i)} \equiv g^2 \sum_{\alpha}(T^A)_{\alpha\beta} \delta_{\alpha\beta} \) and \( \int \equiv \int_{-\pi}^{\pi} d^4k/(2\pi)^4 \). In the above two equations \( \omega, X_0, V_{1\mu}, V_{2\mu} \) and \( G_{\mu\nu} \) are appropriately redefined according to the rescaling of \( k_{\mu} \). For example \( \omega(k) = \sqrt{\sin^2 k_{\mu} + \left( r \sum_{i=1}^r (1 - \cos k_{\mu}) + M_0 \right)^2} \).

First, we show that there are not the nonlocal divergences of the forms \( i\theta C_{1n}/(a^2\beta)^n \) and \( ap^2C_{2n}/(a^2\beta)^{n+1} \). For this purpose we see that Eq. (3.8) and (3.7) are not singular for \( p = 0 \). Setting \( p = 0 \) in them, we have

\[
\Sigma^{(i)}(0) = \frac{-g^2}{2\alpha} \sum_{\mu\nu} \int \frac{1}{\omega(k) - M_0} \sigma_{\mu\nu}^{(i)}(k), \quad i = a, b.
\]

where

\[
\sigma_{\mu\nu}^{(i)}(k) = \left\{ V_{1\mu} \left( \frac{k}{2} \right) + V_{1\nu} \left( \frac{k}{2} \right) \right\} \left\{ \frac{\omega(k) + X_0^\dagger(k)}{\omega(k) + b(k)} \right\} \left\{ V_{1\nu} \left( \frac{k}{2} \right) + \frac{X_0(k)}{\omega(k)} V_{1\nu} \left( \frac{k}{2} \right) \right\},
\]

(3.8)

\[
\sigma_{\mu\nu}^{(b)}(k) = - \left\{ V_{1\mu} \left( \frac{k}{2} \right) V_{1\nu} \left( \frac{k}{2} \right) + \frac{1}{M_0} V_{1\mu} \left( \frac{k}{2} \right) X_0^\dagger(k) V_{1\nu} \left( \frac{k}{2} \right) \right\} + V_{1\mu} \left( \frac{k}{2} \right) V_{1\nu} \left( \frac{k}{2} \right) - \left( \frac{1}{M_0} - \frac{2}{\omega(k)} \right) V_{1\mu} \left( \frac{k}{2} \right) X_0(k) V_{1\nu} \left( \frac{k}{2} \right),
\]

(3.9)

\( ^3 \)When \( T^A \) are the \( SU(N) \) generators in the fundamental representation, \( g^2 \) is \( \frac{N^2-1}{2N} g^2 \).
Now by the expressions of $X_0$, $V_{1\mu}$ and $\omega$ and the fact that there are not doublers, singularity may occur around the region $k \simeq 0$ in each integral. Only the quark and gluon propagators can exhibit singular behaviour for $k \simeq 0$. The leading order part in $k$ in $\Sigma^{(a)}(0)$ vanishes as

$$\frac{g^2}{\alpha M_0} \sum_{\mu \nu} \int_{k^2 < \delta^2} \frac{(\alpha - 3)k}{k^4} = 0. \quad \delta \ll 1.$$  \hspace{1cm} (3.10)

Higher order terms in $k$ are obviously non-singular. In $\Sigma^{(b)}(0)$ the part which can behave singular around $k \simeq 0$ is only the part of a gluon propagator and then the four momentum integral is non-singular. Since both $\Sigma^{(a)}(0)$ and $\Sigma^{(b)}(0)$ is non-singular, $\Sigma(p)$ does not have the divergences of the forms $ipC_{1n}/(a^2 p^2)^n$ and $ap^2C_{2n}/(a^2 p^2)^{n+1}$. Now using Eq.(3.3) we can show that there is not the $1/a$ divergence in $\Sigma(p)$, and thus the $1/a$ divergence in $\Sigma^a$ and $\Sigma^b$ should cancel. Since the highest order divergence is the $1/a$ divergence and derivatives with respect to momentum $p$ decrease the degree of divergence, the $1/a$ divergent parts are given by Eq.(3.8) and Eq.(3.9).

We evaluate $\sigma^{(a)}_{\mu\nu}(k)$ and $\sigma^{(b)}_{\mu\nu}(k)$, noting that odd functions of $k$ in them vanish. First, we investigate $\sigma^{(b)}_{\mu\nu}(k)$. Because $V_{1\mu}^\dagger(-k) = -V_{1\mu}(k)$ and $X_0^\dagger(-k) = X_0(k)$, we find $V_{1\mu}\left(\frac{k}{2}\right)X_0^\dagger(k)V_{1\nu}\left(\frac{k}{2}\right) - V_{1\mu}^\dagger\left(\frac{k}{2}\right)X_0(k)V_{1\nu}^\dagger\left(\frac{k}{2}\right)$ is odd. Therefore

$$\sigma^{(b)}_{\mu\nu}(k) = -\left(V_{1\mu}^\dagger V_{1\nu} + V_{1\mu}V_{1\nu}^\dagger + \frac{2}{\omega}V_{1\mu}^\dagger X_0 V_{1\nu}^\dagger\right).$$  \hspace{1cm} (3.11)

We next evaluate $\sigma^{(a)}_{\mu\nu}(k)$. Using the relations $X_0X_0^\dagger = \omega^2$ and $X_0^2 = -\omega^2 + 2bX_0$, we obtain

$$\sigma^{(a)}_{\mu\nu}(k) = V_{1\mu}V_{1\nu}^\dagger + V_{1\mu}^\dagger V_{1\nu} + \frac{1}{\omega + b}\left\{V_{1\mu}X_0 V_{1\nu}^\dagger + V_{1\mu}^\dagger X_0 V_{1\nu}\right\} + \omega\left(V_{1\mu}V_{1\nu}^\dagger - V_{1\mu}^\dagger V_{1\nu}\right) + \frac{2b}{\omega}V_{1\mu}X_0 V_{1\nu}^\dagger + V_{1\mu}^\dagger i\gamma_\rho \sin k_\rho V_{1\nu}^\dagger + V_{1\mu}^\dagger i\gamma_\rho \sin k_\rho V_{1\nu}.$$  \hspace{1cm} (3.12)

Now $V_{1\mu}i\gamma_\rho \sin k_\rho V_{1\nu}^\dagger + V_{1\mu}^\dagger i\gamma_\rho \sin k_\rho V_{1\nu}$, $V_{1\mu}V_{1\nu} - V_{1\mu}^\dagger V_{1\nu}$ and $V_{1\mu}X_0 V_{1\nu}^\dagger + V_{1\mu}^\dagger X_0 V_{1\nu}^\dagger$ are odd. Therefore

$$\sigma^{(a)}_{\mu\nu}(k) = V_{1\mu}V_{1\nu}^\dagger + V_{1\mu}V_{1\nu}^\dagger + \frac{2}{\omega}V_{1\mu}^\dagger X_0 V_{1\nu}^\dagger.$$  \hspace{1cm} (3.13)
Thus $\Sigma^{(a)}(0)$ and $\Sigma^{(b)}(0)$ exactly cancel and there is no $1/a$ divergence in
$\Sigma(p)$. This analysis shows that the massless pole of the Dirac operator at
tree level is stable against one-loop radiative correction \[15\,].

Next, we investigate the logarithmic divergence. From the above analysis
$\Sigma(p)$ does not have the divergences of the power of $a$. Therefore if there is
the logarithmic divergence in $\Sigma(p)$, it appears from the singular part in
the integral for $a \to 0$ and the singular part is the integration region around
$k \simeq 0$. $\Sigma^{(b)}(p)$ from which was subtracted the $1/a$ divergence is non-singular
for $k \simeq 0$ and $a \to 0$ as it has only a gluon propagator. Accordingly we
evaluate only $\Sigma^{(a)}(p)$. There are some ways for taking out the logarithmic
divergence [9, 17]. Here we use the procedure discussed in the paper by
Karsten and Smit. First, the denominators of the propagator are combined
using Feynman’s parameter as follows,

$$
\Sigma^{(a)}(p) = \frac{g^2}{2a} \sum_{\mu \nu} \int_0^1 dx \int_k \left[ \frac{-2M_0 \delta_{\mu\nu}}{\left\{ -2M_0 (1-x)(\omega(k) + b(k)) + x(pa-k)^2 \right\}^2} \right. \\
\left. - (1-\alpha) \frac{-4M_0 x(p\bar{a}-k)_\mu (p\bar{a}-k)_\nu}{\left\{ -2M_0 (1-x)(\omega(k) + b(k)) + x(p\bar{a}-k)^2 \right\}^3} \right] \\
\times \frac{1}{\left\{ \omega(k) + \omega(pa) \right\}^2} \left\{ V_{1\mu} \left( \frac{pa+k}{2} \right) - \frac{X_0(pa)}{\omega(pa)} V_{1\mu}^\dagger \left( \frac{pa+k}{2} \right) \frac{X_0(k)}{\omega(k)} \right\} \\
\times \left\{ V_{1\nu} \left( \frac{pa+k}{2} \right) - \frac{X_0(k)}{\omega(k)} V_{1\nu}^\dagger \left( \frac{pa+k}{2} \right) \frac{X_0(pa)}{\omega(pa)} \right\}. 
$$
\(3.14\)

And the integration variables are shifted $k_\mu \to k_\mu + ax p_\mu$. Then we split the
integral into two regions as follows,

$$
\int k^2 = \int_{k^2<\delta^2} + \int_{k^2>\delta^2} \delta \ll 1, \quad (3.15)
$$

and we evaluate the $k^2 < \delta^2$ part in the continuum limit, ignoring the
$k^2 > \delta^2$ part which does not have the infrared divergence. Eq.[3.14] is the
complicated integral over the sines and cosines. However for $k^2 < \delta^2$ and
$a \to 0$ we can expand both the denominator and the numerator of Eq.[3.14]
in $k$ and $a$. The singular part corresponding to the logarithmic divergence is

\[4\]This part is studied previously in Ref. [8] and in Ref. [16].
obtained from the leading part in $k$ and $a$. Noting the spherical symmetry of the integral, the singular part is given by

$$ 2i\bar{g}^2\bar{p} \left[ \int_0^1 dx x \int_{k^2 < \delta^2} \frac{1}{k^2 + p^2 a^2 x(1 - x)} \right]^2 $$

$$ + (1 - \alpha) \int_0^1 dx \left( -2x + \frac{3}{2} x^2 \right) \int_{k^2 < \delta^2} \frac{k^2}{\{k^2 + p^2 a^2 x(1 - x)\}^3} \right]. $$

After some calculations, the part proportional to $\log a^2 p^2$ is

$$ -i\bar{g}^2\bar{p} \frac{1}{16\pi^2 M_0} \left\{ 1 - (1 - \alpha) \right\} \log a^2 p^2. \quad (3.16) $$

Now we pay attention to the normalization of quark fields. Since the continuum limit of $D_0$ is $i\bar{p}/|M_0|$, the effective action is

$$ S_{eff} = \int_p \bar{\psi}(p) \left\{ i\bar{p} \frac{\bar{g}^2}{|M_0|} \left( 1 - \frac{g^2}{16\pi^2} \alpha \log a^2 p^2 + \text{const} \right) \right\} \psi(p) + \cdots. \quad (3.17) $$

The coefficient $1/|M_0|$ can be absorbed by the redefinition of quark fields. Finally, the divergent part of the quark field renormalization factor $Z_\psi$ is

$$ Z_\psi = 1 + \frac{\bar{g}^2}{16\pi^2} \alpha \log a^2 \mu^2, \quad (3.18) $$

where $\mu$ is the renormalization scale. This factor agrees with the continuum theory.

### 4 Evaluation of the vacuum polarization

In this section we calculate the one-loop vacuum polarization with quark loop $\Pi_{\mu\nu}$ and then we show that there are not the divergences of the forms $p^2/(a^2 p^2)^n \delta_{\mu\nu}$ and $p_{\mu}p_{\nu}/(a^2 p^2)^n$ ($n \geq 1$) and that the divergent part of the gluon field renormalization factor is equal to that in the continuum theory.

Feynman diagrams for the vacuum polarization with quark loop are shown in Fig.4.

The amplitude corresponding to Fig.4(a) is
The amplitude corresponding to Fig.4(b) is

$$\Pi^{(b)}_{\mu\nu}(p) = \frac{g^2}{a^2} \text{tr}(T^A T^B) \int \frac{d^4k}{(2\pi)^4} \frac{1}{\{\omega(k) + \omega(k+p)\}^2} \text{tr}\left[ \left\{ \frac{-i\gamma_\rho \sin a(k+p)\rho}{2(\omega(k) + b(k+p))} + \frac{a}{2} \right\} \right] \times \left\{ V_{1\nu} \left( k + \frac{p}{2} \right) - \frac{X_0(k+p)}{\omega(k+p)} V_{1\nu}^\dagger \left( k + \frac{p}{2} \right) X_0(k) \right\} \times \left\{ V_{1\mu} \left( k + \frac{p}{2} \right) - \frac{X_0(k)}{\omega(k)} V_{1\mu}^\dagger \left( k + \frac{p}{2} \right) X_0(k+p) \right\} \right].$$

(4.1)
\[- \frac{2\omega(k) + \omega(k-p)}{\omega(k)^2 \omega(k-p)}X_0(k)V_{1\nu}^\dagger \left( k - \frac{p}{2} \right) X_0(k-p)V_{1\mu}^\dagger \left( k - \frac{p}{2} \right) X_0(k) \right] .\]

(4.2)

Rescaling the integration momentums as before \( k_\mu \to k_\mu / a \) and noting that for \( N_f \) flavor QCD, \( tr \left( T^A T^B \right) = N_f \delta^{AB} \), we obtain

\[
\Pi^{(a)}_{\mu\nu}(p) = -\frac{N_f g^2}{4a^2} \int_k \frac{1}{\omega(k) + \omega(k + pa)} \text{tr} \left[ \left\{ \frac{-i\gamma_\rho \sin (k + pa)_{\rho}}{(\omega(k + pa) + b(k + pa))} + 1 \right\} \right.
\]
\[
\times \left\{ V_{1\nu} \left( k + \frac{pa}{2} \right) - X_0(k + pa) \omega(k + pa) V_{1\nu}^\dagger \left( k + \frac{pa}{2} \right) X_0(k) \right\} \left\{ \frac{-i\gamma_\rho \sin k_\rho}{(\omega(k) + b(k))} + 1 \right\}
\]
\[
\times \left\{ V_{1\mu} \left( k + \frac{pa}{2} \right) - X_0(k) \omega(k) V_{1\mu}^\dagger \left( k + \frac{pa}{2} \right) X_0(k + pa) \right\} \right],
\]

(4.3)

and

\[
\Pi^{(b)}_{\mu\nu}(p) = \frac{N_f g^2}{4a^2} \int_k \frac{1}{\omega(k)} \text{tr} \left[ \left\{ \frac{-i\gamma_\rho \sin k_\rho}{(\omega(k) + b(k))} + 1 \right\} \right.
\]
\[
\times \left\{ \delta_{\mu\nu} \left\{ V_{2\mu} \left( k \right) - \frac{X_0(k)}{\omega(k)} V_{2\mu}^\dagger \left( k \right) \frac{X_0(k)}{\omega(k)} \right\} \right.
\]
\[
- \frac{1}{(\omega(k) + \omega(k + pa))^2} \left\{ X_0(k) V_{1\mu}^\dagger \left( k + \frac{pa}{2} \right) V_{1\nu} \left( k + \frac{pa}{2} \right) X_0(k) \right\}
\]
\[
+ V_{1\nu} \left( k + \frac{pa}{2} \right) X_0(k + pa) V_{1\nu} \left( k + \frac{pa}{2} \right) + V_{1\mu} \left( k + \frac{pa}{2} \right) V_{1\nu}^\dagger \left( k + \frac{pa}{2} \right) X_0(k) \right]
\]
\[
- \frac{2\omega(k) + \omega(k + pa)}{\omega(k)^2 \omega(k + pa)} X_0(k) V_{1\mu}^\dagger \left( k + \frac{pa}{2} \right) X_0(k + pa) V_{1\nu}^\dagger \left( k + \frac{pa}{2} \right) X_0(k) \right]
\]
\[
- \frac{1}{(\omega(k) + \omega(k - pa))^2} \left\{ X_0(k) V_{1\nu}^\dagger \left( k - \frac{pa}{2} \right) V_{1\mu} \left( k - \frac{pa}{2} \right) X_0(k) \right\}
\]
\[
+ V_{1\nu} \left( k - \frac{pa}{2} \right) X_0(k - pa) V_{1\nu} \left( k - \frac{pa}{2} \right) + V_{1\mu} \left( k - \frac{pa}{2} \right) V_{1\nu}^\dagger \left( k - \frac{pa}{2} \right) X_0(k) \right]
\]
\[
- \frac{2\omega(k) + \omega(k - pa)}{\omega(k)^2 \omega(k - pa)} X_0(k) V_{1\nu}^\dagger \left( k - \frac{pa}{2} \right) X_0(k - pa) V_{1\mu}^\dagger \left( k - \frac{pa}{2} \right) X_0(k) \right] ,
\]

(4.4)

where \( \omega, X_0, V_{1\mu}, V_{2\mu} \) and \( G_{\mu\nu} \) are appropriately redefined as done in the previous section. First, we show that \( \Pi^{(a)}_{\mu\nu}(p) \) and \( \Pi^{(b)}_{\mu\nu}(p) \) are constants for \( p = 0 \) and thus the divergent terms of the forms \( p^2 / (a^2 p^2)^{n+1} \delta_{\mu\nu} \) and
$p_\mu p_\nu / (a^2 p^2)^n$ do not appear. Setting $p = 0$ in them, we have

$$
\Pi^{(a)}_{\mu\nu}(0) = -\frac{N_f g^2}{16a^2} \int \frac{1}{\omega(k)^2} \text{tr} \left[ \left\{ -i\gamma_\rho \sin k_\rho \right\} \left\{ (\omega(k) + b(k)) + 1 \right\} \right]
\times \left\{ V_{1\nu}(k) - \frac{X_0(k)}{\omega(k)} V_{1\nu}^\dagger(k) \right\}
\times \left\{ V_{1\mu}(k) - \frac{X_0(k)}{\omega(k)} V_{1\mu}^\dagger(k) \right\},
$$

and

$$
\Pi^{(b)}_{\mu\nu}(0) = \frac{N_f g^2}{4a^2} \int \frac{1}{\omega(k)} \text{tr} \left[ -i\gamma_\rho \sin k_\rho \left( \omega(k) + b(k) \right) + 1 \right]
\times \left[ \delta_{\mu\nu} \left\{ V_{2\mu}(k) - \frac{X_0(k)}{\omega(k)} V_{2\mu}^\dagger(k) \right\} \right]
- \frac{1}{4\omega(k)^2} \left\{ X_0(k) V_{1\mu}^\dagger(k) V_{1\nu}(k) \right\}
+ V_{1\mu}(k) X_0^\dagger(k) V_{1\nu}(k) + V_{1\nu}(k) V_{1\mu}^\dagger(k) X_0(k)
- \frac{3}{\omega(k)^2} X_0(k) V_{1\mu}^\dagger(k) X_0(k) V_{1\nu}^\dagger(k) X_0(k)
- \frac{1}{4\omega(k)^2} \left\{ X_0(k) V_{1\nu}^\dagger(k) V_{1\mu}(k) \right\}
+ V_{1\nu}(k) X_0^\dagger(k) V_{1\mu}(k) + V_{1\mu}(k) V_{1\nu}^\dagger(k) X_0(k)
- \frac{3}{\omega(k)^2} X_0(k) V_{1\nu}^\dagger(k) X_0(k) V_{1\mu}^\dagger(k) X_0(k) \right].
$$

In $\Pi^{(b)}_{\mu\nu}(0)$ the part which can be singular is only a quark propagator which behaves $i\vec{k}/k^2$ at $k \approx 0$ and the four momentum integral is non-singular, and thus the results of the $k$ integration is a constant. In $\Pi^{(a)}_{\mu\nu}(0)$ the leading part in $k$ is

$$
\frac{N_f g^2}{4a^2} \int_{k^2 < \delta^2} \frac{\text{tr} \left[ k(2i\gamma_\nu) [k(2i\gamma_\mu)] \right]}{k^4}
\sim \frac{1}{a^2} \int_0^\delta d\vec{k}.
$$

Hence $\Pi^{(a)}_{\mu\nu}(0)$ is a constant as well. The above analysis exhibits that there is not the divergences of the forms $p^2/(a^2 p^2)^n+1 \delta_{\mu\nu}$ and $p_\mu p_\nu/(a^2 p^2)^n$ in
\( \Pi_{\mu\nu}(p) \). Accordingly assuming that \( SO(4) \) symmetry restores in the continuum limit, the structure of the vacuum polarization amplitude is

\[
\Pi_{\mu\nu}(p) = \frac{1}{a^2} C_0 \delta_{\mu\nu} + C_{1\mu\nu}(p) + O(a^2). \quad C_0 : \text{constant.} \quad (4.7)
\]

However we take the gauge invariant formulation and the Ward identity determines the structure of \( \Pi_{\mu\nu}(p) \) as follows,

\[
\Pi_{\mu\nu}(p) = (p^2 \delta_{\mu\nu} - p_\mu p_\nu) \Pi(p^2). \quad (4.8)
\]

From the above two equations we conclude that there is not also the \( 1/a^2 \) divergence in \( \Pi_{\mu\nu}(p) \).

Next, we evaluate the logarithmic divergent part in the vacuum polarization. The procedure which taking out the logarithmic divergent part is same as we used in the previous section. Since \( \Pi^{(b)}_{\mu\nu}(p) \) from which was subtracted the \( 1/a^2 \) divergence is not singular for \( k \approx 0 \) and \( a \to 0 \) from the same reason for \( \Sigma^{(b)}(p) \), we consider only \( \Pi^{(a)}_{\mu\nu}(p) \). In Eq.(4.3), using Feynman’s parameter and shifting the integration variables \( k_\mu \to k_\mu - axp_\mu \) and then we evaluate the part of the integration region \( k^2 < \delta^2, \delta \ll 1 \) in the continuum limit. The singular part for \( k \approx 0 \) and \( a \to 0 \) is the leading part in \( k \) and \( a \). Noting the spherical symmetry of the integral, its part is given by

\[
\frac{N_f g^2}{a^2} \int_0^1 dx \int_{k^2 < \delta^2} \frac{2k^2 \delta_{\mu\nu} - 4a^2 x(1-x)(p^2 \delta_{\mu\nu} - 2p_\mu p_\nu)}{(k^2 + p^2 a^2 x(1-x))^2}. \quad (4.9)
\]

After some calculations, the term proportional to \( \log a^2 p^2 \) is extracted as follows,

\[
\frac{N_f g^2}{12\pi^2} (p^2 \delta_{\mu\nu} - p_\mu p_\nu) \log a^2 p^2. \quad (4.10)
\]

Hence the divergent part of the gluon field renormalization factor due to the quark loop is written as \( Z_A = 1 + (N_f g^2/12\pi^2) \log a^2 \mu^2 \), which agrees with the continuum theory.

## 5 Summary and discussion

In this paper, we discussed the weak coupling expansion of massless QCD defined with the overlap Dirac operator at one-loop. In the weak coupling
expansion of the overlap Dirac operator, the fermion propagator has the single pole for the physical mode and is free from species doublers. The quark-gluon vertices are regular functions in momenta and thus the gauge interaction is local. With these properties of the quark propagator and the quark-gluon vertices, we have studied the fermion self-energy and vacuum polarization and confirmed that non-local divergent terms are not in fact induced, which are not necessarily forbidden by the Ginsparg-Wilson relation. Then the Ginsparg-Wilson relation prohibits linearly divergent mass terms. The divergent parts of the wave function renormalization factors for quarks and gluons agree with the continuum theory.

References

[1] H. Neuberger, Phys. Lett. B417 (1998) 141; Phys. Lett. B427 (1998) 353.

[2] P. H. Ginsparg and K. G. Wilson, Phys. Rev. D25 (1982) 2649.

[3] M. Lüscher, Phys. Lett. B428 (1998) 342.

[4] P. Hernández, K. Jansen and M. Lüscher, Nucl. Phys. B552 (1999) 363.

[5] R. Narayanan and H. Neuberger, Nucl. Phys. B412 (1994) 574; Phys. Rev. Lett. 71 (1993) 3251; Nucl. Phys. B443 (1995) 305.

[6] R. Narayanan and J. Nishimura, Nucl. Phys. B508 (1997) 371; Y. Kikukawa and H. Neuberger, Nucl. Phys. B513 (1998) 735.

[7] S. Randjbar-Daemi and J. Strathdee, Nucl. Phys. B443 (1995) 386; Nucl. Phys. B461 (1996) 305; Nucl. Phys. B466 (1996) 335; Phys. Lett. B402 (1997) 134.

[8] A. Yamada, Phys. Rev. D57 (1998) 1433; Nucl. Phys. B514 (1998) 399; Nucl. Phys. B529 (1998) 483.

[9] H. Kawai, R. Nakayama and K. Seo, Nucl. Phys. B189 (1981) 40.

[10] Y. Kikukawa and A. Yamada, Phys. Lett. B448 (1999) 265.

[11] K. Fujikawa, Nucl. Phys. B546 (1999) 480; Phys. Rev. D60 (1999) 74505.
[12] D. H. Adams, hep-lat/9812003.

[13] H. Suzuki, hep-lat/9812019. (to appear in Prog. Theor. Phys)

[14] M. Lüscher, Nucl. Phys. B549 (1999) 295

[15] T. W. Chiu, C. W. Wang and S. V. Zenkin, Phys. Lett. B438 (1998) 321.

[16] S. Aoki and Y. Taniguchi, Phys. Rev. D59 (1999) 54510;
    Y. Kikukawa, H. Neuberger and A. Yamada, Nucl. Phys. B526 (1998) 572.

[17] L.H.Karsten and J.Smit, Nucl. Phys. B183 (1981) 103.

[18] T. Reitz and H.J. Rothe, “Renormalization of lattice gauge theories
    with massless Ginsparg-Wilson fermions”, hep-lat/9908013.

[19] C. Alexandrou, H. Panagopoulos and E. Vicari, “A-parameter of lattice
    QCD with the overlap-Dirac operator”, hep-lat/9909158