Trees with Matrix Weights: 
Laplacian Matrix and Characteristic-like Vertices

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Abstract

It is known that there is an alternative characterization of characteristic vertices for trees with positive weights on their edges via Perron values and Perron branches. Moreover, the algebraic connectivity of a tree with positive edge weights can be expressed in terms of Perron value.

In this article, we consider trees with matrix weights on their edges. More precisely, we are interested in trees with the following classes of matrix edge weights:

1. positive definite matrix weights,
2. lower (or upper) triangular matrix weights with positive diagonal entries.

For trees with the above classes of matrix edge weights, we define Perron values and Perron branches. Further, we have shown the existence of vertices satisfying properties analogous to the properties of characteristic vertices of trees with positive edge weights in terms of Perron values and Perron branches, and we call such vertices characteristic-like vertices. In this case, the eigenvalues of the Laplacian matrix are nonnegative, and we obtain a lower bound for the first non-zero eigenvalue of the Laplacian matrix in terms of Perron value. Furthermore, we also compute the Moore-Penrose inverse of the Laplacian matrix of a tree with nonsingular matrix weights on its edges.

Keywords: Tree, Laplacian Matrix, Characteristic vertices, Matrix weights, Perron values.

MSC: 05C50, 05C22

1 Introduction and Motivation

Let \( G = (V, E) \) be a simple graph, with \( V \) as the set of vertices and \( E \) as the set of edges in \( G \). For \( u, v \in V \), we write \( u \sim v \) if \( u \) and \( v \) are adjacent in \( G \), and \( u \not\sim v \) otherwise. We write, \( \deg(v) \) to denote the degree of the vertex \( v \) and \( \mathcal{P}(u, v) \) to denote the path joining vertices \( u \) and \( v \).

Given a graph \( G = (V, E) \) on \( n \) vertices, if each edge \( e \in E \) is associated with a positive number \( W(e) \), called the weight of \( e \), then the Laplacian matrix \( L(G) = [l_{uv}] \) is an \( n \times n \) matrix (we simply write \( L \) if there is no scope for confusion), and is defined as follows: for \( u, v \in V \), if \( u \not\sim v \), then \( l_{uv} \) is 0; if \( u \sim v \), then \( l_{uv} \) is \( -W(e) \) if \( u \sim v \) and \( e \) is the edge between them; finally if \( u = v \), \( l_{vv} \) is the sum of the weights of the edges in \( G \) which are incident with the vertex \( v \). It is well known that \( L(G) \) is a symmetric positive semidefinite matrix. The column vector with constant value for each of its entries (constant vector) is an eigenvector of \( L(G) \) corresponding to the smallest eigenvalue 0. In [10], Fiedler proved that the second smallest eigenvalue of \( L(G) \), say \( \mu(G) \), is positive if and only if \( G \) is connected. Since \( \mu(G) \) provides an algebraic measure of the connectivity of \( G \), it is

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named as algebraic connectivity of $G$. An eigenvector $y$ of $L(G)$, corresponding to the algebraic connectivity $\mu(G)$ is called Fiedler vector. Further, for any vertex $v \in V$, we write $y_v$ to denote the $v^{th}$ entry of $y$.

In particular, for a given tree $T$ with positive weights on its edges, there is an interesting result that gives some insight into the structure of the eigenvectors corresponding to the algebraic connectivity of $T$. This result was first proved for trees, where all the edge weights are equal to 1 in [11]. However, it is also valid for trees with positive weights.

**Proposition 1.1.** [11] Let $T = (V, E)$ be a tree with positive weights on its edges. Let $L$ be the Laplacian matrix of $T$ with algebraic connectivity $\mu(T)$ and $y$ be an eigenvector of $L$ corresponding to the algebraic connectivity $\mu(T)$. Then, exactly one of the following cases occurs:

(a) No entry of $y$ is 0. In this case, there is a unique pair of vertices $u$ and $v$ such that $u$ and $v$ are adjacent in $T$, with $y_u > 0$ and $y_v < 0$. Further, the entries of $y$ are increasing along any path in $T$ which starts at $u$ and does not contain $v$, while the entries of $y$ are decreasing along any path in $T$ which starts at $v$ and doesn’t contain $u$.

(b) Some entry of $y$ is 0. In this case, the subgraph of $T$ induced by the set of vertices corresponding to 0’s in $y$ is connected. Moreover, there is a unique vertex $x$ such that $y_x = 0$, and $x$ is adjacent to a vertex $w$ with $y_w \neq 0$. The entries of $y$ are either increasing, decreasing, or identically 0 along any path in $T$ which starts at $x$.

A tree with positive weights on its edges is said to be of Type I if (b) holds, and Type II if (a) holds. If $T$ is of Type I, Fiedler defines the characteristic vertex as the special vertex $x$ referred to in (b), whereas if $T$ is of Type II, he shows that $T$ has two characteristic vertices, namely the special vertices $u$ and $v$ referred to in (a), and we call the edge between the vertices $u$ and $v$ is the characteristic edge of $T$. In [19], it was shown that the characteristic vertex (or vertices) of $T$ is (are) independent of the choice of the eigenvector $y$ corresponding to the algebraic connectivity $\mu(T)$. The above understanding of the characteristic vertices of trees and their relations with the algebraic connectivity has created a great deal of interest amongst researchers, and many interesting results have been obtained for trees (for example, see [13, 14, 19]).

Let $T$ be a tree with positive weights on its edges and let $C_T$ denote the set of characteristic vertices of $T$. Then, $|C_T| = 1$ or 2, depending on whether $T$ contains a characteristic vertex or a characteristic edge, respectively.

Before proceeding further, we first introduce a few notations and then recall a few results from matrix theory which will be used time and again in this article. Let $1$, $I$, and $J$ denote the column vector of all ones, the identity matrix, and the matrix of all ones, respectively. We write $0_{m \times n}$ to represent the zero matrix of order $m \times n$ and simply write 0 if there is no scope for confusion with the order of the matrix. Given a matrix $A$, we use $A^T$, Range($A$) and Null($A$) to denote the transpose, range and null space of the matrix $A$, respectively. If $A$ is a square matrix, then the set of eigenvalues of $A$ is called the spectrum of $A$, denoted by $\sigma(A)$ and the spectral radius of $A$, denoted by $\rho(A)$ is defined as $\rho(A) = \max_{\lambda \in \sigma(A)} |\lambda|$. By Perron-Frobenius theory, if $A$ is an entrywise positive square matrix, then $\rho(A)$ is the largest eigenvalue of $A$. Moreover, $\rho(A)$ is a simple eigenvalue of $A$ and is called the Perron value of $A$. Next, we state a result that compares the spectral radius of two nonnegative matrices, which is an application of Perron-Frobenius theory (for details see [20]).

**Theorem 1.2.** [20, Corollary 2.2] Let $A$ be an entrywise positive square matrix and $B$ be a nonnegative matrix of the same order as $A$. If $A - B$ is a nonnegative matrix with at least one positive entry, then $\rho(A) > \rho(B)$. 
Finally, given a real symmetric matrix $A$ of order $n \times n$, we use the following convention where the eigenvalues of $A$ are in increasing order:

$$\lambda_{\min} = \lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_{n-1}(A) \leq \lambda_n(A) = \lambda_{\max}. \quad (1.1)$$

We now state a few results from matrix theory which are useful for subsequent results.

**Theorem 1.3** (Min-max Theorem). \cite{12} Let $A$ be a real symmetric matrix of order $n \times n$, and let the eigenvalues of $A$ be ordered as in Equation (1.1). Then

$$\lambda_{\max} = \lambda_n(A) = \max_{x^T x = 1} x^T A x = \max_{x \neq 0} \frac{x^T A x}{x^T x} \quad \text{and} \quad \lambda_{\min} = \lambda_1(A) = \min_{x^T x = 1} x^T A x = \min_{x \neq 0} \frac{x^T A x}{x^T x}.$$

**Theorem 1.4** (Inclusion Principle). \cite{12} Let $A$ be an $n \times n$ real symmetric matrix, let $r$ be an integer with $1 \leq r \leq n$, and let $A_r$ denote any $r \times r$ principal submatrix of $A$. For each integer $k$ such that $1 \leq k \leq r$, we have

$$\lambda_k(A) \leq \lambda_k(A_r) \leq \lambda_{k+n-r}(A).$$

**Theorem 1.5.** \cite{12, Theorem 4.3.7, Page 184} Let $A$ and $B$ be real symmetric matrices of order $n \times n$ with eigenvalues ordered as in Equation (1.1). Then for every pair of integers $j, k$ such that $1 \leq j, k \leq n$ and $j + k \geq n + 1$, we have

$$\lambda_{j+k-n}(A + B) \leq \lambda_j(A) + \lambda_k(B).$$

Let $T$ be a tree with positive weights on its edges. A branch at a vertex $v$ of $T$ is one of the connected components obtained from $T$ by deleting $v$ and all edges incident with $v$. Let $L_v$ be the principal submatrix of the Laplacian matrix $L$ obtained by deleting the row and column corresponding to the vertex $v$. It is easy to see that $L_v$ is a block diagonal invertible matrix. Hence $M_v = L_v^{-1}$ is a block diagonal matrix and each of its diagonal blocks corresponds to a branch at $v$, called the bottleneck matrix for that branch at $v$. To be precise, for a branch $B$ at $v$ consisting of $k$ vertices, the bottleneck matrix for $B$ based at $v$, denoted by $M_v(B)$ is a $k \times k$ matrix such that for $x, y \in B$, the entry at the $(x, y)^{th}$ position of $M_v(B)$ is given by

$$\sum_{e \in P(x, v) \cap P(y, v)} \frac{1}{W(e)}.$$

If $u$ and $v$ are distinct vertices of a weighted tree, we use $B_u(v)$ to denote the branch at the vertex $u$ which contains the vertex $v$. For notational convenience, we write $M_v(u)$ instead of $M_v(B_u(u))$. Note that, the bottleneck matrix for a branch $B$ at $v$ is a square entrywise positive matrix, and the Perron value of that bottleneck matrix $M_v(B)$ is $\rho(M_v(B))$, called the Perron value of $B$. Finally, a branch $B$ at $v$ is called a Perron branch, if the Perron value of $B$ is the largest amongst all the branches at $v$ and hence $\rho(M_v) = \rho(M_v(B))$.

The above notations and observations were first provided in \cite{16, 17}. Let $T = (V, E)$ be tree. For any $u, v, w \in V$, we write $B_u(w) \subseteq B_v(w)$ if $B_u(w)$ contained in $B_v(w)$, and $B_u(w) \nsubseteq B_v(w)$ if the containment is proper. The following result is a consequence of Theorems 1.2 and 1.4, and we state the result using the above notations.

**Proposition 1.6.** Let $T = (V, E)$ be a tree with positive weights on its edges. For any $u, v, w \in V$, if $B_u(w) \nsubseteq B_v(w)$, then $\rho(M_u(w)) < \rho(M_v(w))$.

We now recall an alternative characterization of the characteristic vertex and characteristic edge for trees with positive edge weights in terms of Perron branches and bottleneck matrices. As a consequence, a relation was found between the Perron values with the algebraic connectivity. The following results summarize these characterizations and some of their consequences (for details see \cite{16, 21, Chapter 6}).
Proposition 1.7. Let $T$ be a tree with positive weights on its edges. Then, the following statements are equivalent.

1. $T$ is Type II with the characteristic edge $e$ between the vertices $u$ and $v$.

2. There exists $0 < \gamma < 1$ such that $\rho(M_u(v) - \gamma(1/\theta)J) = \rho(M_v(u) - (1 - \gamma)(1/\theta)J)$, where $\theta$ is the weight of the edge $e$ between the vertices $u$ and $v$. Moreover,

$$\frac{1}{\mu(T)} = \rho(M_u(v) - \gamma(1/\theta)J) = \rho(M_v(u) - (1 - \gamma)(1/\theta)J),$$

where $\mu(T)$ is the algebraic connectivity of $T$.

3. For adjacent vertices $u$ and $v$, $B_u(v)$ is the unique Perron branch at $u$, while $B_v(u)$ is the unique Perron branch at $v$ in $T$.

Proposition 1.8. Let $T$ be a tree with positive weights on its edges. Then, $T$ is Type I with the characteristic vertex $x$ if and only if there are two or more Perron branches of $T$ at $x$. Moreover, the algebraic connectivity of $T$ is $1/\rho(M_x)$.

Proposition 1.9. Let $T$ be a tree with positive weights on its edges. If $x$ is not a characteristic vertex of $T$, then the unique Perron branch at $x$ in $T$ is the branch which contains the characteristic vertex (or vertices) of $T$.

The above characterizations for trees have provided a new direction in understanding the structure of trees using the Laplacian matrix. In this direction, several intriguing results have been obtained by various researchers (for example, see [1, 17, 18, 22, 23]). In the last decade, some interesting results were obtained by considering graphs with matrix weights on their edges (for example, see [2, 3, 5, 6, 26]). Particularly, in [2], the authors defined the Laplacian matrix analogously for graphs with matrix weights on their edges. As a special case, if the edge weights are positive definite matrices, then the Laplacian matrix is a positive semidefinite matrix. They have also proved an interesting result: Let $G$ be a connected graph on $n$ vertices with nonsingular matrix weights of order $s \times s$ on its edges and $L$ be the Laplacian matrix of $G$. Then, the Laplacian matrix $L$ is of rank $(n - 1)s$ if the graph $G$ is a tree. However, the result is not necessarily true if the graph $G$ is not a tree. These developments have encouraged us to study the Laplacian matrices of trees with matrix weights.

In this article, our objective is to consider trees with a suitable class of matrix edge weights and establish the existence of some notion of the characteristic vertex (or vertices) using characterization in terms of Perron branches and Perron values analogous to trees with positive edge weights. We refer to such vertices as characteristic-like vertex (or vertices). Moreover, we also provide a lower bound for the first non-zero eigenvalue of the Laplacian matrix. To be more specific, we are interested in trees with the following classes of matrix weights on their edges:

1. positive definite matrix weights,

2. lower (or upper) triangular matrix weights with positive diagonal entries.

This article is organized as follows. In Section 2, we consider the principal submatrix $L_v$ of the Laplacian matrix $L$ of a tree with matrix weights on its edges. We compute the determinant of $L_v$ and show that $L_v$ is an invertible matrix if and only if the edge weights are nonsingular matrices. Then, we find the inverse of $L_v$ and define the bottleneck matrix for a branch of a tree with nonsingular matrix edge weights. Further, using $L_v^{-1}$, we find the Moore-Penrose inverse of the Laplacian matrix $L$. In Section 3, we consider trees with the above class of matrix weights.
on their edges and show the existence of vertices satisfying properties analogous to the properties of characteristic vertices of trees with positive edge weights in terms of Perron values and Perron branches. Finally, in Section 4, we obtain a lower bound for the first non-zero eigenvalue of the Laplacian matrices of trees with the above classes of matrix edge weights in terms of Perron values.

2 Laplacian Matrix and Bottleneck Matrix

In this section, we consider the Laplacian matrices for trees with matrix weights on their edges and define the bottleneck matrix of a branch. As an application, we compute the Moore-Penrose inverse of the Laplacian matrix of a tree with nonsingular matrix weights.

The Laplacian matrix of a graph with matrix weights on its edges is defined analogously. For the sake of completeness we recall its definition. Let $G = (V, E)$ be a graph on $n$ vertices and for each edge $e \in E$ the associated matrix weight $W(e)$ is of order $s \times s$. The Laplacian matrix $L(G) = [l_{uv}]$ is a matrix of order $ns \times ns$ and is defined as follows: for $u, v \in V$, if $u \neq v$, then $l_{uv}$ is $0$ if $u \sim v$, and $l_{uv}$ is $-W(e)$ if $u \sim v$ and $e$ is the edge between them; finally if $u = v$, $l_{vv}$ is the sum of the weights of the edges in $G$ which are incident with the vertex $v$. We also write $L$ for the Laplacian matrix $L(G)$ if there is no scope for confusion.

Before proceeding further, we recall the definition of the Kronecker product and some of its properties.

**Remark 2.1.** The Kronecker product of matrices $A = [a_{ij}]$ of order $m \times n$ and $B$ of order $p \times q$, denoted by $A \otimes B$, is defined to be the block matrix $[a_{ij}B]$. Then the following hold true.

1. Let $A$ and $B$ be two square matrices. Let $\lambda \in \sigma(A)$ with corresponding eigenvector $x$, and let $\mu \in \sigma(B)$ with corresponding eigenvector $y$. Then $\lambda \mu$ is an eigenvalue of $A \otimes B$ with corresponding eigenvector $x \otimes y$. Moreover, any eigenvalue of $A \otimes B$ is a product of eigenvalues of $A$ and $B$.

2. Let $W$ be an $s \times s$ invertible matrix and $y_{i}^{T} \in \mathbb{R}^{s}$ for $1 \leq i \leq n$. Then, the vector $\bar{y} = (y_{1}, y_{2}, \ldots, y_{n})^{T} \in \text{Null}(J_{n} \otimes W)$ if and only if $\sum_{i=1}^{n} y_{i} = 0$.

In [2], it was shown that if $T$ is a tree on $n$ vertices with nonsingular matrix weights of order $s \times s$, then the rank of the Laplacian matrix $L$ of $T$ is $(n - 1)s$. Thus, it is natural to study the principal matrix $L_{v}$ of $L$ obtained by deleting the row block and column block corresponding to a vertex $v$. We begin with a result on the determinant of $L_{v}$. The proof is similar to that of trees with positive edge weights.

**Theorem 2.2.** Let $L$ be the Laplacian matrix of a tree $T = (V, E)$ with matrix weights on its edges. Let $L_{v}$ be the principal submatrix of $L$ obtained by deleting the row block and column block corresponding to the vertex $v \in V$. Then

$$\det L_{v} = \prod_{e \in E} \det W(e).$$

**Proof.** We prove this result by using induction on the number of vertices $|V| = n$. The result is vacuously true for $n = 2$. Assume that the result is true for those trees whose number of vertices is strictly less than $n$.

Let $v \in V$ and $\deg(v) = r$. For $1 \leq i \leq r$, let $v$ be adjacent to the vertex $v_{i}$ via the edge $e^{(i)}$. Thus, $B_{v}(v_{i})$ for $1 \leq i \leq r$ represents all the branches at $v$ and the block matrix $L_{v}$ can be written
Further, if we add all the column blocks of \( \hat{v} \) vertex \( \left[ a_{ij} \right] \) where hypothesis, we have if \( G \) deleting the row block and column block corresponding to the vertex \( \hat{v} \) incidence matrices of graphs with positive definite matrix edge weights. The case where the edge weights are positive definite matrices. We now recall the definition of the \( L \) matrix obtained by deleting the row block of \( Q \) matrix \( v \) is the column vector of conformal order with 1 at \( v \th \) entry and 0 elsewhere.

For each \( 1 \leq i \leq r \), let \( L(B, (v_i)) \) denote the principal submatrix of \( L(B, (v_i)) \) obtained by deleting the row block and column block corresponding to the vertex \( v_i \). Using the induction hypothesis, we have

\[
\det L(B, (v_i)) = \prod_{e \in E(B, (v_i))} \det W(e).
\]  

Further, if we add all the column blocks of \( \hat{L}(B, (v_i)) \) to the column block corresponding to the vertex \( v_i \) and repeat a similar operation for row blocks, then the resulting matrix can be represented as

\[
\begin{bmatrix}
L(B, (v_i)) & 0 & \cdots & 0 \\
0 & \hat{L}(B, (v_2)) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \hat{L}(B, (v_r))
\end{bmatrix},
\]  

where \( \hat{L}(B, (v_i)) \) is the principal submatrix of \( L \) corresponding to the branch \( B, (v_i) \) for \( 1 \leq i \leq r \).

Let \( L(B, (v_i)) \) denote the Laplacian matrix of the branch \( B, (v_i) \). Then

\[
\hat{L}(B, (v_i)) = L(B, (v_i)) + e_{v_i} e_{v_i}^T \otimes W(e) \text{ for } 1 \leq i \leq r,
\]

where \( e_{v_i} \) is the column vector of conformal order with 1 at \( v_i \th \) entry and 0 elsewhere.

Using Equation (2.2), we now get

\[
\det \hat{L}(B, (v_i)) = \det W(e) \times \prod_{e \in E(B, (v_i))} \det W(e).
\]

The desired result follows from Equation (2.1). 

Under the hypothesis of the above theorem, if \( T \) is a tree with nonsingular matrix edge weights, then \( L_v \) is an invertible matrix. Our next objective is to find the inverse of \( L_v \). We first consider the case where the edge weights are positive definite matrices. We now recall the definition of the incidence matrices of graphs with positive definite matrix edge weights.

Let \( G = (V, E) \) be a graph with \( n \) vertices and \( m \) edges such that the weights associated with the edges are positive definite matrices of order \( s \times s \). We assign an orientation to each edge of \( G \). Then, the vertex-edge incidence matrix \( Q \) is a block matrix such that the row blocks are indexed by the vertex set \( V \) and the column blocks are indexed by the edge set \( E \). The vertex-edge incidence matrix \( Q = [Q_{ue}] \) is a matrix of order \( ns \times ms \), where

\[
Q_{ue} = \begin{cases} 
\sqrt{W(e)} & \text{if } u \text{ is the initial vertex of the edge } e, \\
-\sqrt{W(e)} & \text{if } u \text{ is the terminal vertex of the edge } e, \\
0 & \text{otherwise.}
\end{cases}
\]  

(2.3)

It can be seen that, for a given graph \( G = (V, E) \) with positive definite weights on its edges, the Laplacian matrix \( L \) of \( G \) is given by \( L = QQ^T \). Then \( L_v = Q_vQ_v^T \), where \( Q_v \) is the block matrix obtained by deleting the row block of \( Q \) corresponding to the vertex \( v \in V \). In particular, if \( G \) is a tree, then by Theorem 2.2 we have \( \det L_v = (\det Q_v)^2 \neq 0 \). This implies that \( Q_v \) is an invertible matrix and \( L_v^{-1} = (Q_v^{-1})^T Q_v^{-1} \). We now compute the inverse of \( Q_v \) when the graph \( G \)
is a tree. The argument used to find the inverse $Q_v^{-1}$ is similar to the proof for those trees whose edge weights are all 1 (for details, see [4, Chapter 2]).

Given a path $P$ in $G$, the incidence block vector of $P$ is an $ms \times s$ matrix (a column block indexed by the edge set $E$) and is defined as follows: for any $e \in E$, the entry corresponding to $e$ is the matrix $0$, if the path does not contain $e$. If the path contains $e$, then the entry corresponding to $e$ is $\left(\sqrt{W(e)}\right)^{-1}$ or $-\left(\sqrt{W(e)}\right)^{-1}$, depending on whether the direction of the path agrees or disagrees, respectively with $e$.

Let $T = (V, E)$ be a tree. For $v \in V$, the path matrix $P_v$ of $T$ is an $ms \times (n-1)s$ matrix ($P_v$ is a block matrix such that rows are indexed by the edge set $E$ and column blocks indexed by the vertex set $V - \{v\}$) and is defined as follows. For $u \in V - \{v\}$, the column block corresponding to the vertex $u$ of $P_v$ is the incidence vector of the path from $u$ to $v$.

**Theorem 2.3.** Let $T = (V, E)$ be a tree with positive definite matrix weights on its edges. Let $Q$ be the incidence matrix of $T$ and $Q_v$ be the block matrix obtained by deleting the row block of $Q$ corresponding to the vertex $v \in V$. Then $Q_v^{-1} = P_v$.

**Proof.** Let $X = P_v Q_v$ be an $ms \times ms$ matrix. For $i \neq j$, let $e_i$ be the edge from $x$ to $y$ and let $e_j$ be the edge from $w$ to $z$. The $(u, e_j)^{th}$ entry of the incidence matrix $Q$ is $Q_{u e_j} = 0$ unless $u = w$ or $u = z$. Thus,

$$X_{e_i e_j} = \sum_{u \in V - \{v\}} P_{e_i u} Q_{u e_j} = P_{e_i w} Q_{w e_j} + P_{e_i z} Q_{z e_j} = (P_{e_i w} - P_{e_i z}) \sqrt{W(e_j)}.$$

Note that, the path from $w$ to $v$ contains $e_i$ if and only if the path from $z$ to $v$ contains $e_i$. Moreover, if $P_{e_i w}$ and $P_{e_i z}$ are non-zero, then they share the same sign. Thus, $P_{e_i w} = P_{e_i z}$ which implies that $X_{e_i e_j} = 0$ whenever $i \neq j$.

For $i = j$, the path from $x$ to $v$ contains $e_i$ if and only if the path from $y$ to $v$ does not contain $e_i$. Thus, if $e_i$ is in the path from $x$ to $v$, then $X_{e_i e_i} = P_{e_i x} Q_{x e_i} = \left(\sqrt{W(e_i)}\right)^{-1} \sqrt{W(e_i)} = I$. Similarly, if $e_i$ is in the path from $y$ to $v$, then $X_{e_i e_i} = P_{e_i y} Q_{y e_i} = -\left(\sqrt{W(e_i)}\right)^{-1} (-\sqrt{W(e_i)}) = I$. This completes the proof.

**Corollary 2.4.** Let $T = (V, E)$ be a tree with positive definite matrix weights on its edges and $L$ be the Laplacian matrix of $T$. Let $L_v$ denote the principal submatrix of $L$ obtained by deleting the row block and column block corresponding to the vertex $v \in V$. Then for $u, w \in V - \{v\}$, the block at $(u, w)$ position of $L_v^{-1}$ is given by

$$(L_v^{-1})_{uw} = \sum_{e \in \mathcal{P}(u,v) \cap \mathcal{P}(w,v)} W(e)^{-1},$$

where $\mathcal{P}(x, y)$ denotes the path joining the vertices $x$ and $y$ in $T$.

**Proof.** Using $L_v = Q_v Q_v^T$ and by Theorem 2.3, we have $L_v^{-1} = P_v^T P_v$. Thus

$$(L_v^{-1})_{uw} = \sum_{e \in E} P_{eu} P_{ew}.$$ 

Further, $P_{eu} P_{ew}$ is non-zero if and only if the edge $e$ is in both the paths $\mathcal{P}(u, v)$ and $\mathcal{P}(w, v)$. In this case, the orientation of $e$ either agrees or disagrees simultaneously, for both paths. Thus, $P_{eu} P_{ew} = W(e)^{-1}$, if $e \in \mathcal{P}(u, v) \cap \mathcal{P}(w, v)$ and $0$ otherwise. Hence the result follows.
By Corollary 2.4, the block matrix form of \( L_v^{-1} \) for trees with positive definite matrix weights on its edges is similar to the case where trees have positive edge weights (for details, see [16, Proposition 1]). We now show that the above block matrix form is unchanged even if the weights assigned to the edges are nonsingular.

**Theorem 2.5.** Let \( L \) be the Laplacian matrix of a tree \( T = (V, E) \) with nonsingular matrix weights on its edges. Let \( L_v \) denote the principal submatrix of \( L \) obtained by deleting the row block and column block corresponding to the vertex \( v \in V \). Then for \( u, w \in V - \{v\} \), the block at \((u, w)\) position of \( L_v^{-1} \) is given by

\[
(L_v^{-1})_{uw} = \sum_{e \in \mathcal{P}(u,v) \cap \mathcal{P}(w,v)} W(e)^{-1},
\]

where \( \mathcal{P}(x,y) \) denotes the path joining the vertices \( x \) and \( y \) in \( T \).

**Proof.** For \( u, w \in V - \{v\} \), let \( B = [B_{uw}] \) be an \((n - 1)s \times (n - 1)s\) block matrix, where

\[
B_{uw} = \sum_{e \in \mathcal{P}(u,v) \cap \mathcal{P}(w,v)} W(e)^{-1}.
\]

Let \( L = [l_{xy}]_{x,y \in V} \) and \( X = L_v B = [X_{uw}] \). Then, for \( u, w \in V - \{v\} \), we have

\[
X_{uw} = l_{uu} B_{uw} + \sum_{x \neq u} l_{ux} B_{xw}. \tag{2.4}
\]

For a given \( u \in V - \{v\} \), let \( \deg(u) = r \). For \( 1 \leq i \leq r \), let \( u \) be adjacent to the vertex \( v_i \) via the edge \( e(i) \). We consider the cases \( u = w \) and \( u \neq w \) below.

**Case 1:** For \( u = w \).

If \( u = w \) and \( u \sim v \), then the path \( \mathcal{P}(u,v) \) contains exactly one vertex adjacent to \( u \). Without loss of generality, let \( v_r \in \mathcal{P}(u,v) \). Then \( B_{v_iu} = B_{uu} = \sum_{e \in \mathcal{P}(u,v)} W(e)^{-1} \) for \( 1 \leq i \leq r - 1 \), and \( B_{v_iu} = B_{uu} - W(e_r)^{-1} \). Using Equation (2.4), we have \( X_{uu} = l_{uu} B_{uu} + l_{uv_r} B_{v_iu} + \sum_{i=1}^{r-1} l_{uw_i} B_{v_iu} = (l_{uu} + \sum_{i=1}^{r-1} l_{uw_i}) B_{uu} + l_{uv_r} (W(e_r)^{-1}) \). Since the row block sum of \( L \) is zero, \( i.e., l_{uu} + \sum_{i=1}^{r} l_{uw_i} = 0 \) and \( l_{uv_r} = -W(e_r) \), so \( X_{uu} = I \).

If \( u = w \) and \( u \sim v \), then \( v = v_r \) and \( B_{v_iu} = B_{uu} = W(e_r)^{-1} \) for \( 1 \leq i \leq r - 1 \). Using Equation (2.4), we have \( X_{uu} = l_{uu} B_{uu} + \sum_{i=1}^{r-1} l_{uw_i} B_{v_iu} = (l_{uu} + \sum_{i=1}^{r-1} l_{uw_i}) W(e_r)^{-1} = (-l_{uw_r}) (W(e_r)^{-1}) = W(e_r)W(e_r)^{-1} = I \).

**Case 2:** For \( u \neq w \). We consider the following sub cases to complete the proof.

**Subcase 2.1:** For \( u \neq w \) and \( u \sim v \).

If \( u \in \mathcal{P}(w,v) \), then both the paths \( \mathcal{P}(w,u) \) and \( \mathcal{P}(u,v) \) contain exactly one vertex each, which are adjacent to \( u \). Without loss of generality, let \( v_1 \in \mathcal{P}(w,u) \) and \( v_r \in \mathcal{P}(u,v) \). Then \( B_{v_1w} = B_{uu} + W(e_1)^{-1} \), \( B_{v_2w} = B_{uu} - W(e_1)^{-1} \) and \( B_{v_rw} = B_{uu} \) for \( 2 \leq i \leq r - 1 \). By Equation (2.4), we have \( X_{uw} = (l_{uu} + \sum_{i=1}^{r} l_{uw_i}) B_{uu} + l_{uw_r} (W(e_1)^{-1}) + l_{uv_r} (W(e_r)^{-1}) = 0 + (W(e_1)^{-1}) (W(e_1)^{-1}) = I + I = 0 \).

If \( u \notin \mathcal{P}(w,v) \), then either \( \mathcal{P}(u,v) \cap \mathcal{P}(w,v) = \emptyset \) or \( w \in \mathcal{P}(u,v) \). Note that, if \( \mathcal{P}(u,v) \cap \mathcal{P}(w,v) = \emptyset \) and \( u \notin \mathcal{P}(w,v) \), we have \( B_{v_iw} = B_{uw} = 0 \) for \( 1 \leq i \leq r \), then \( X_{uw} = 0 \). For \( w \in \mathcal{P}(u,v) \), we have \( B_{v_iw} = B_{uw} \); and Equation (2.4) yields \( X_{uw} = (l_{uu} + \sum_{i=1}^{r} l_{uw_i}) B_{uw} = 0 \).
Subcase 2.2: For \( u \neq w \) and \( u \sim v \).

Let \( v = v_r \). If \( u \in \mathcal{P}(w,v) \), then the path \( \mathcal{P}(w,u) \) contains exactly one vertex \( v_1 \) (say) adjacent to \( u \). In that case, \( B_{v_1w} = W(e^{(r)})^{-1} + W(e^{(1)})^{-1} \) and \( B_{vw} = B_{uw} = W(e^{(r)})^{-1} \) for \( 2 \leq i \leq r-1 \). Using Equation (2.4), we get \( X_{uw} = \left( l_{uw} + \sum_{i=1}^{r-1} l_{uv_i} \right) W(e^{(r)})^{-1} + l_{vw} W(e^{(1)})^{-1} = (-l_{uw})(-W(e^{(r)})^{-1}) - I = 0 \). If \( u \notin \mathcal{P}(w,v) \), then \( \mathcal{P}(u,v) \cap \mathcal{P}(w,v) = \emptyset \), which implies that \( B_{v_1w} = B_{vw} = 0 \) for \( 1 \leq i \leq r-1 \) and hence the result follows.

In view of Theorems 2.2 and 2.5, for trees with nonsingular matrix edge weights, we define the bottleneck matrix of a branch as follows. Let \( L \) be the Laplacian matrix of a tree \( T = (V,E) \) with nonsingular matrix weights of order \( s \times s \) on its edges and \( L_v \) be the principal submatrix of \( L \) obtained by deleting the row block and column block corresponding to a vertex \( v \in V \). Let \( \text{deg}(v) = r \) and for \( 1 \leq i \leq r \), let \( B_i \) be the branches at \( v \) in \( T \). By Theorem 2.2, \( L_v \) is an invertible matrix. Denote \( L_v^{-1} \) by \( M_v \). Then

\[
L_v = \begin{bmatrix}
\hat{L}(B_1) & 0 & \ldots & 0 \\
0 & \hat{L}(B_2) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \hat{L}(B_r)
\end{bmatrix}
\quad \text{and} \quad
M_v = L_v^{-1} = \begin{bmatrix}
M_v(B_1) & 0 & \ldots & 0 \\
0 & M_v(B_2) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & M_v(B_r)
\end{bmatrix},
\]

where \( M_v(B_i) = \hat{L}(B_i)^{-1} \) is called the bottleneck matrix of the branch \( B_i \) at \( v \) in \( T \) for \( 1 \leq i \leq r \). Thus, by Theorem 2.5, for a branch \( B \) at \( v \) consisting of \( k \) vertices, the bottleneck matrix \( M_v(B) \) for \( B \) based at \( v \) is a \( k \times k \) matrix such that \( (M_v(B)) \) (as a block matrix) for \( x,y \in B \), the block at the \( (x,y)^{th} \) position of \( M_v(B) \) is given by

\[
\sum_{e \in \mathcal{P}(x,v) \cap \mathcal{P}(y,v)} W(e)^{-1}.
\]

As in the case of trees with positive edge weights, for notational convenience, we write \( M_v(u) \) instead of \( M_v(B_u(u)) \), whenever \( B_u(u) \) is the branch at \( v \) in \( T \) containing the vertex \( u \).

In this manuscript, we consider a few classes of matrix weights on the edges of \( T \) so that the eigenvalues of \( L_v \) are positive. Therefore, for each \( 1 \leq i \leq r \), all the eigenvalues of the bottleneck matrices \( M_v(B_i) \) are positive and the spectral radius of \( M_v(B_i) \) is necessarily an eigenvalue. In this case, the spectral radius of \( M_v(B_i) \) need not be a simple eigenvalue, but continuing with the terminology as in the case of the trees with positive weights on edges, we call the spectral radius \( \rho(M_v(B_i)) \) as the Perron value of the bottleneck matrix \( M_v(B_i) \) for \( 1 \leq i \leq r \). Thus, the spectrum of \( M_v = L_v^{-1} \) is given by \( \sigma(M_v) = \bigcup_{i=1}^{r} \sigma(M_v(B_i)) \) and the spectral radius of \( M_v \) is given by

\[
\rho(M_v) = \max_{1 \leq i \leq r} \rho(M_v(B_i)). \tag{2.5}
\]

We also define the Perron value of a branch at \( v \) in \( T \) as the Perron value of the corresponding bottleneck matrix (or matrices) for which the maximum is attained. We call such a branch at \( v \) as a Perron branch if the Perron value of that branch is the same as the spectral radius of \( L_v^{-1} \).

Finally, we conclude this section with a few results that allow us to compute the Moore-Penrose inverse of the Laplacian matrices for trees with nonsingular matrix edge weights. We obtain this result as an application of Theorem 2.5.

If \( A \) is an \( m \times n \) matrix, then an \( n \times m \) matrix \( \Gamma \) is called a generalized inverse of \( A \) if \( A \Gamma A = A \). The Moore-Penrose inverse of \( A \), denoted by \( A^+ \), is an \( m \times n \) matrix satisfying the following equations:

\[
AA^+A = A, \quad A^+AA^+ = A^+, \quad (AA^+)^T = AA^+, \quad (A^+A)^T = A^+A.
\]
It is well known that any complex matrix admits a unique Moore-Penrose inverse, and we refer to [7, 9] for basic properties of the Moore-Penrose inverse. One such property is that the null space of $A^+$ is the same as that of $A^T$ for any matrix $A$, and we present this result as a lemma without proof.

**Lemma 2.6.** If $A$ is an $m \times n$ matrix, then for any $n \times 1$ vector $x$, $Ax = 0$ if and only if $x^TA^+ = 0$.

The following result was proved for connected graphs with positive definite matrix edge weights (see [3, Theorem 3.4]). We now show that the result holds for trees with nonsingular matrix edge weights.

**Lemma 2.7.** Let $L$ be the Laplacian matrix of a tree $T$ on $n$ vertices with nonsingular matrix weights of order $s \times s$ on its edges. Then $I_{ns} - LL^+ = J_n \otimes \frac{1}{n}I_s$.

**Proof.** Let $I_{ns} - LL^+ = [X_{ij}]$, where each $X_{ij}$ is a matrix of order $s \times s$. Since $(I - LL^+)L = 0$, each row of $I - LL^+$ belongs to the left null space of $L$. Recall that, the row and column block sum of $L$ is zero and also the null space is of dimension $s$. Thus, the rows of $1_n \otimes I_s$ generate the left null space, which implies that, any row of $I_{ns} - LL^+$ is of the form $1_n \otimes x$, for some $x \in \mathbb{R}^n$. Since $I_{ns} - LL^+ = [X_{ij}]$ is symmetric, by the above argument we get $X_{ij} = X$ for $1 \leq i, j \leq n$ and hence $I_{ns} - LL^+ = J_n \otimes X$.

Note that, $\text{Null}(L^+) \subset \text{Range}(I_{ns} - LL^+)$, this implies that $\text{rank}(I_{ns} - LL^+) \geq \text{nullity}(L^+) = \text{nullity}(L) = s$. Since $J_n$ is a rank one matrix, $\text{rank}(I_{ns} - LL^+) = \text{rank}(J_n \otimes X) = \text{rank}(X) \leq s$. Thus, $\text{rank}(I_{ns} - LL^+) = \text{rank}(J_n \otimes X) = \text{rank}(X) = s$. Hence, $X$ is a nonsingular matrix. Since $I_{ns} - LL^+$ is idempotent,

$$I_{ns} - LL^+ = J_n \otimes X = (J_n \otimes X)^2 = J_n^2 \otimes X^2 = nJ_n \otimes X^2,$$

which implies that $X = nX^2$. Since $X$ is a nonsingular matrix, $X = \frac{1}{n}I_s$. This completes the proof.

We now state a lemma that is useful in computing the Moore-Penrose inverse of the Laplacian matrix. The result is easy to verify, and hence the proof is omitted.

**Lemma 2.8.** If $M = I_{(n-1)s} - J_{n-1} \otimes \frac{1}{n}I_s$, then $M^{-1} = I_{(n-1)s} + DD^T$, where $D = 1_{n-1} \otimes I_s$.

We now compute the Moore-Penrose inverse $L^+$ of the Laplacian matrix $L$ for trees with nonsingular matrix weights.

**Theorem 2.9.** Let $T = (V, E)$ be a tree on $n$ vertices such that the weights associated with the edges are nonsingular matrices of order $s \times s$ and $L$ be the Laplacian matrix of $T$. Let $L_v$ be the principal submatrix of $L$ and $(L^+)_v$ be the principal submatrix of $L^+$, obtained by deleting the row and column corresponding to the vertex $v \in V$. Then $(L^+_v) = ML_v^{-1}M$, where $M = I_{(n-1)s} - J_{n-1} \otimes \frac{1}{n}I_s$.

Moreover, if $V = \{v_1, v_2, \ldots, v_n\}$ is the ordering of vertices in the Laplacian matrix $L$ and $v = v_n$, then

$$L^+ = \begin{bmatrix} X \\ Y \end{bmatrix},$$

where $X = \begin{bmatrix} (L^+_v)_{v_1} & (L^+_v)_{v_2}(I_{n-1} \otimes I_s) \end{bmatrix}$ and $Y = (I_{n-1} \otimes I_s)X$.

**Proof.** Let $V = \{v_1, v_2, \ldots, v_n\}$ be the ordering of the vertices in the Laplacian matrix $L$ and without loss generality, let $v = v_n$. Let us partition the Laplacian matrix as $L = \begin{bmatrix} U & V \end{bmatrix}$, where $V$ represents the column block corresponding to the vertex $v_n$. Also partition the Moore-Penrose inverse $L^+$ as $L^+ = \begin{bmatrix} X \\ Y \end{bmatrix}$, where $Y$ represents the row block corresponding to the vertex $v_n$. Since
column block sum of $L$ is zero, $\mathbf{V} = DU$, where $D = I_{n-1} \otimes I_s$. Thus $\mathbf{V} = D^T \mathbf{X}$. Using Lemma 2.7, we have

$$I_{ns} - J_n \otimes \frac{1}{n} I_s = LL^+ = UX + YY = U(I_{(n-1)s} + DD^T)X.$$ 

Thus, $M = U_n(I_{(n-1)s} + DD^T)X^n$, where $U_n$ is the matrix obtained by deleting the row block corresponding to the vertex $v_n$ and $X^n$ is the matrix obtained by deleting the column block corresponding to the vertex $v_n$. Since $U_n = L_{vn}$ and $X^n = (L^+)_{vn}$, by Lemmas 2.2 and 2.8, we have

$$(L^+)_{vn} = X^n = (I_{(n-1)s} + DD^T)^{-1}U_n^{-1}M = ML_{vn}^{-1}M.$$ 

In view of Lemma 2.6, the column block corresponding to the vertex $v_n$ of $X$ is $(L^+)_{vn}D$ and $\mathbf{Y} = D^T \mathbf{X}$. This completes the proof. □

3 Characteristic-like Vertices and Perron Values

In this section, we consider trees with the following classes of matrix weights on their edges:

1. positive definite matrix weights,
2. lower (or upper) triangular matrix weights with positive diagonal entries.

Using Perron branches, we show the existence of vertices with properties analogous to characteristic vertices of trees with positive edge weights as stated in Propositions 1.7 - 1.9. We call such vertices characteristic-like vertices. To be more precise, our objective in this section is to prove the following results.

**Result 3.1.** Let $T$ be a tree with either of the following classes of matrix weights on its edges:

1. positive definite matrix weights,  
2. lower (or upper) triangular matrix weights with positive diagonal entries. Then one of the following cases occurs:

1. There is a unique vertex $v$ such that there are two or more Perron branches at $v$ in $T$.
2. There is a unique pair of vertices $u$ and $v$ with $u \sim v$ such that the Perron branch at $u$ in $T$ is the branch containing $v$, while the Perron branch at $v$ in $T$ is the branch containing $u$.

It is easy to see that if Result 3.1 is true, then it allows us to define a notion analogous to the characteristic vertex and the characteristic edge for trees with the above classes of matrix edge weights using Perron branches. We now formally define the characteristic-like vertex and the characteristic-like edge on trees with the above-mentioned classes of matrix weights on their edges.

**Definition 3.2.** Let $T$ be a tree with either of the following classes of matrix weights on its edges:

1. positive definite matrix weights,  
2. lower (or upper) triangular matrix weights with positive diagonal entries. Then one of the following cases occurs:

1. There is a unique vertex $v$ such that there are two or more Perron branches at $v$ in $T$. In this case, the vertex $v$ is called the characteristic-like vertex of $T$.
2. There is a unique pair of vertices $u$ and $v$ with $u \sim v$ such that the Perron branch at $u$ in $T$ is the branch containing $v$, while the Perron branch at $v$ in $T$ is the branch containing $u$. In this case, we call the edge between the vertices $u$ and $v$, the characteristic-like edge of $T$. 

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It is easy to see that the notions of characteristic-like vertex and characteristic-like edge coincide with the notions of characteristic vertex and characteristic edge for trees with positive edge weights. Let $T$ be a tree with either of the above-mentioned matrix weights on its edges and let $C_T$ denote the set of characteristic-like vertices of $T$. Then, $|C_T| = 1$ or $2$, depending on whether $T$ contains a characteristic-like vertex or characteristic-like edge, respectively. We now state the second desired result.

**Result 3.3.** Let $T$ be a tree with either of the following classes of matrix weights on its edges: 1. positive definite matrix weights, 2. lower (or upper) triangular matrix weights with positive diagonal entries. If $x$ is not a characteristic-like vertex of $T$, then the unique Perron branch at $x$ in $T$ is the branch which contains the characteristic-like vertex (or vertices) of $T$.

### 3.1 Results for Positive Definite Matrix Weights

In this section, we consider trees with positive definite matrix weights on their edges. Let $L$ be the Laplacian matrix of a tree $T$ with positive definite matrix weights on its edges and $L_v$ be the principal submatrix of $L$ formed by deleting the row block and column block corresponding to the vertex $v$. By the previous section, we know that $L = QQ^T$ is a positive semidefinite matrix. In view of Theorems 1.4 and 2.2, $L_v$ is a positive definite matrix. Therefore, the definitions of the Perron value and the Perron branch (as defined in Section 2) are well-defined for trees with positive definite matrix edge weights. Before proceeding further, we prove a lemma which is useful in our subsequent proofs.

**Lemma 3.4.** Let $T = (V, E)$ be a tree with positive definite matrix weights on its edges. For any $u, v, w \in V$, if $B_u(w) \subset B_v(w)$, then $\rho(M_u(w)) \leq \rho(M_v(w))$.

**Proof.** For any $u, v, w \in V$, we have $B_u(w) \subset B_v(w)$. Thus, by renaming the vertices, the matrix $M_v(w)$ can be written as

$$M_v(w) = \begin{bmatrix} M_u(w) + J \otimes W & * \\ * & \end{bmatrix},$$

where $W = \sum_{e \in P(u,v)} W(e)^{-1}$ is a positive definite matrix. Using Theorem 1.4, we have $\rho(M_u(w)) \geq \rho(M_u(w) + J \otimes W)$. Since $J \otimes W$ is a positive semidefinite matrix, using the min-max theorem we get $\rho(M_u(w) + J \otimes W) \geq \rho(M_v(w))$. This completes the proof.

Next, we establish the existence of vertices with properties analogous to characteristic vertices in terms of Perron branches. We first show the existence of a characteristic-like edge for trees with positive definite matrix weights.

**Theorem 3.5.** Let $T$ be a tree with positive definite matrix weights on its edges such that there is a unique Perron branch at every vertex in $T$. Then, there is a unique pair of vertices $u$ and $v$ in $T$ with $u \sim v$ such that the Perron branch at $u$ is the branch containing $v$, while the Perron branch at $v$ is the branch containing $u$. Moreover, the unique Perron branch at any vertex $x$ in $T$ is the branch which contains at least one of the vertices $u$ or $v$.

**Proof.** Let $u_1$ be a vertex in $T$. For $u_2 \sim u_1$, let $B_{u_1}(u_2)$ be the unique Perron branch at $u_1$ in $T$. Proceeding inductively we find a walk $u_1 \sim u_2 \sim u_3 \sim \cdots$ such that $B_{u_i}(u_{i+1})$ is the unique Perron branch at $u_i$ in $T$ for $i = 1, 2, \ldots$. Since $T$ is acyclic and finite, there exists $i_0$ such that $u_{i_0+1} = u_{i_0-1}$, i.e., $B_{u_{i_0}}(u_{i_0-1})$ is the unique Perron branch at $u_{i_0}$, whereas $B_{u_{i_0+1}}(u_{i_0})$ is the unique Perron branch at $u_{i_0-1}$ in $T$. Let us denote $u_{i_0-1}$ by $u$ and $u_{i_0}$ by $v$. Thus, the Perron branch at $u$ is the branch $B_u(v)$ and the Perron branch at $v$ is the branch $B_v(u)$.
We prove the uniqueness of the vertices $u$ and $v$ as follows. Let $x, y$ be any two vertices in $B_u(v)$ such that $x \sim y$ and $y \notin B_x(u)$. Thus $B_x(y) \subseteq B_v(y)$ and $B_v(u) \subseteq B_x(u)$. Using Lemma 3.4, we get
\[
\rho(M_x(y)) \leq \rho(M_v(y)) \text{ and } \rho(M_x(u)) \leq \rho(M_x(u)).
\quad (3.1)
\]
Since the unique Perron branch at $v$ in $T$ is $B_v(u)$, we see that $\rho(M_v(y)) < \rho(M_v(u))$. Therefore, by Equation (3.1), we have $\rho(M_x(y)) < \rho(M_x(u))$, which implies that the unique Perron branch at $x$ in $T$ is the branch which contains at least one of the vertices $u$ or $v$. Similar assertions can be made whenever we consider $x, y \in B_v(u)$. Hence the result follows. \hfill \Box

Before proving the existence of characteristic-like vertex for trees with positive definite matrix weights, we prove the following lemma.

Lemma 3.6. Let $T$ be a tree with positive definite matrix weights on its edges. If there exists a vertex $v$ in $T$ such that there are two or more Perron branches at $v$ in $T$, then for any vertex $x$ other than $v$, the branch $B_x(v)$ is a Perron branch at $x$ in $T$, i.e., $\rho(M_x(z)) \leq \rho(M_x(v))$, whenever $z \sim x$.

Proof. Let $x$ be a vertex in $T$ other than $v$. Let $x \in B_v(u_1)$ and $u_1 \sim v$. Since there are two or more Perron branches at $v$ in $T$, there exists a vertex $u_2$ other than $u_1$ such that $u_2 \sim v$ and $B_v(u_2)$ is a Perron branch at $v$ in $T$, i.e., $\rho(M_v) = \rho(M_v(u_2))$. Thus $B_x(u_2) = B_x(v)$ and $B_v(u_2) \subseteq B_x(v)$. Next, for any $z \sim x$ with $z \notin B_x(v)$, we have $B_x(z) \subseteq B_v(z)$. By Lemma 3.4, we get
\[
\rho(M_v) = \rho(M_v(u_2)) \leq \rho(M_x(v)) \text{ and } \rho(M_z(v)) \leq \rho(M_v(u)) \leq \rho(M_v).
\]
This implies that $\rho(M_z(x)) \leq \rho(M_z(v))$, thereby completing the proof. \hfill \Box

Theorem 3.7. Let $T$ be a tree with positive definite matrix weights on its edges. If there exists a vertex $v$ in $T$ such that there are two or more Perron branches at $v$, then $v$ is a unique vertex with such a property. Moreover, if $x$ is a vertex other than $v$, then the unique Perron branch at $x$ in $T$ is the branch which contains the vertex $v$.

Proof. Let $v$ be a vertex in $T$ such that there are two or more Perron branches at $v$. We consider the following two cases to complete the proof.

Case 1: There is a unique Perron branch at $u$ in $T$, whenever $u$ adjacent to $v$.

Let $x$ be a vertex in $T$ other than $v$ and $x \in B_v(u_1)$ for $u_1 \sim v$. Then $B_{u_1}(v) \subseteq B_x(v)$. For any $y \sim x$ with $y \notin B_x(v)$, we have $B_x(y) \subseteq B_{u_1}(y)$. By Lemma 3.4, we get
\[
\rho(M_{u_1}(v)) \leq \rho(M_x(v)) \text{ and } \rho(M_{u_1}(y)) \leq \rho(M_x(y)).
\quad (3.2)
\]
Using Lemma 3.6, we see that $B_{u_1}(v)$ is a Perron branch at $u_1$ in $T$. By our assumption for this case $B_{u_1}(v)$ is the unique Perron branch at $u_1$ in $T$. Thus $\rho(M_{u_1}(v)) < \rho(M_{u_1}(v))$. Hence, by Equation (3.2), we have $\rho(M_x(y)) < \rho(M_x(v))$. Therefore, $B_x(v)$ is the unique Perron branch at $x$ in $T$ and the result follows.

Case 2: There exists a vertex $u_1$ adjacent to $v$ such that there are two or more Perron branches at $u_1$ in $T$. 

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By the hypothesis and our assumption for this case, there are two or more Perron branches of $T$ at both the vertices $v$ and $u_1$. Therefore, Lemma 3.6 shows that $B_v(u_1)$ and $B_{u_1}(v)$ are Perron branches of $T$ at $v$ and $u_1$, respectively. Thus
\[ \rho(M_v) = \rho(M_v(u_1)) \quad \text{and} \quad \rho(M_{u_1}) = \rho(M_{u_1}(v)). \] (3.3)

Since there are two or more Perron branches at $v$ in $T$, there exists $w \sim v$ ($w \neq u_1$) such that $B_v(w)$ is a Perron branch at $v$ in $T$. Similarly, there exists $u_2 \sim u_1$ ($u_2 \neq v$) such that $B_{u_1}(u_2)$ is a Perron branch at $u_1$ in $T$. Thus
\[ B_v(w) \subset B_{u_1}(w) = B_{u_1}(v) \quad \text{and} \quad B_{u_1}(u_2) \subset B_v(u_2) = B_v(u_1). \]

Hence, by Lemma 3.4 and Equation (3.3), we get
\[ \rho(M_v) = \rho(M_v(w)) \leq \rho(M_{u_1}(v)) \leq \rho(M_{u_1}) = \rho(M_{u_1}(u_2)) \leq \rho(M_v) = \rho(M_v(u_1)). \]

Therefore,
\[ \rho(M_v) = \rho(M_v(u_1)) = \rho(M_{u_1}(u_2)) = \rho(M_{u_1}). \] (3.4)

Next, we consider all the branches of $T$ at $u_2$ except for $B_{u_2}(u_1)$ and choose a branch such that the bottleneck matrix is with maximum spectral radius in the following way. Let $u_3 \sim u_2$ and $B_{u_2}(u_3)$ be a branch such that $M_{u_2}(u_3)$ is with maximum spectral radius amongst all the branches of $T$ at $u_2$ except for $B_{u_2}(u_1)$ and repeat the process until we reach a pendant vertex. Thus, there exists a path $v = u_0 \sim u_1 \sim \cdots \sim u_r$ such that $u_r$ is a pendant vertex, and $B_{u_i}(u_{i+1})$ is a branch such that $M_{u_i}(u_{i+1})$ is with maximum spectral radius amongst all the branches of $T$ at $u_i$ except for the branch $B_{u_i}(u_{i-1})$ for $1 \leq i \leq r - 1$.

For $1 \leq i \leq r - 1$, let $\tilde{M}_{u_i}$ denote the principal submatrix of $M_{u_i}$ obtained by deleting the block $M_{u_i}(u_{i-1})$ from $M_{u_i}$, i.e.,
\[ M_{u_i} = \begin{bmatrix} \tilde{M}_{u_i} & 0 \\ 0 & M_{u_i}(u_{i-1}) \end{bmatrix}, \]
and let $e_i$ denote the edge between the vertices $u_{i-1}$ and $u_i$. Then
\[ \begin{cases} M_{u_{i-1}}(u_i) = \tilde{M}_{u_i} + J \otimes [W(e_i)^{-1}] & \text{for } 1 \leq i \leq r - 1, \\ M_{u_{i-1}}(u_r) = W(e_r)^{-1}, \end{cases} \] (3.5)

where $\tilde{M}_{u_i} = \begin{bmatrix} \tilde{M}_{u_i} & 0 \\ 0 & 0 \end{bmatrix}$ if the matrix weights on the edges are of order $s \times s$.

By our construction $\tilde{M}_{u_i}$ is a block diagonal matrix and $M_{u_i}(u_{i+1})$ is of maximum spectral radius amongst all the blocks of $\tilde{M}_{u_i}$. Hence
\[ \rho(\tilde{M}_{u_i}) = \rho(\tilde{M}_{u_i}) = \rho(M_{u_i}(u_{i+1})) \quad \text{for } 1 \leq i \leq r - 1. \] (3.6)

Thus, if $x_{i+1}$ is an eigenvector of $M_{u_i}(u_{i+1})$ corresponding to $\rho(M_{u_i}(u_{i+1}))$, then the vector $\tilde{x}_{i+1} = (x_{i+1}, 0, \ldots, 0)$ of conformal order, is an eigenvector of $\tilde{M}_{u_i}$ corresponding to $\rho(\tilde{M}_{u_i})$.

For $i = r - 1$, $\rho(\tilde{M}_{u_{r-1}}) = \rho(M_{u_{r-1}}(u_r)) = \rho(W(e_r)^{-1})$. Let $x_r$ be an eigenvector of $W(e_r)^{-1}$ corresponding to $\rho(W(e_r)^{-1})$. Using $x_r^T [W(e_r)^{-1}] x_r > 0$, Equations (3.5) and (3.6), we have
\[ \tilde{x}_r^T M_{u_{r-2}}(u_{r-1}) \tilde{x}_r = \tilde{x}_r^T \tilde{M}_{u_{r-2}}(u_{r-1}) \tilde{x}_r + \tilde{x}_r^T (J \otimes [W(e_{r-1})^{-1}]) \tilde{x}_r \]
\[ = x_r^T [W(e_r)^{-1}] x_r + x_r^T [W(e_{r-1})^{-1}] x_r \]
\[ > \rho(W(e_r)^{-1}) \]
\[ = \rho(M_{u_{r-1}}(u_r)), \]
which implies that \( \rho(M_{ur-2}(u_{r-1})) > \rho(M_{ur-1}(u_r)). \)

Further, suppose \( \mathbf{x}_{r-1} \in \text{Null}(J \otimes [W(e_{r-1})^{-1}]) \). By Equation (3.5), we have

\[
x^T_{r-1}M_{ur-2}(u_{r-1})\mathbf{x}_{r-1} = \mathbf{x}^T_{r-1}\hat{M}_{ur-2}\mathbf{x}_{r-1} + \mathbf{x}^T_{r-1}(J \otimes [W(e_{r-1})^{-1}])\mathbf{x}_{r-1} \\
= \mathbf{x}^T_{r-1}\hat{M}_{ur-1}\mathbf{x}_{r-1}.
\]

The min-max theorem yields that \( \rho(M_{ur-2}(u_{r-1})) \leq \rho(M_{ur-1}(u_r)) \), which is a contradiction. Thus, \( \mathbf{x}_{r-1} \notin \text{Null}(J \otimes [W(e_{r-1})^{-1}]) \) and hence Remark 2.1 now implies \( \mathbf{x}_{r-1} \notin \text{Null}(J \otimes [W(e_{r-2})^{-1}]) \), where \( \mathbf{x}_{r-1} = (\mathbf{x}_{r-1}, \mathbf{0}, \ldots, \mathbf{0}) \) is an eigenvector of \( \hat{M}_{ur-2} \) corresponding to \( \rho(\hat{M}_{ur-2}) = \rho(M_{ur-2}(u_{r-1})). \)

Using \( \hat{x}^T_{r-1}(J \otimes [W(e_{r-2})^{-1}])\hat{x}_{r-1} > 0 \), Equations (3.5) and (3.6), we have

\[
\hat{x}^T_{r-1}M_{ur-3}(u_{r-2})\hat{x}_{r-1} = \hat{x}^T_{r-1}\hat{M}_{ur-2}\hat{x}_{r-1} + \hat{x}^T_{r-1}(J \otimes [W(e_{r-2})^{-1}])\hat{x}_{r-1} > \rho(M_{ur-2}(u_{r-1})),
\]

which implies that \( \rho(M_{ur-3}(u_{r-2})) > \rho(M_{ur-2}(u_{r-1})). \) Proceeding inductively we have

\[
\rho(M_{ur-1}(u_i)) > \rho(M_{ur}(u_{i+1})) \text{ for } 1 \leq i \leq r - 1,
\]

which is a contradiction to Equation (3.4) as \( v = u_0 \).

Therefore, the assumption for Case 2 is not valid, which implies that for any adjacent vertex \( u \) of \( v \), there is a unique Perron branch at \( u \) in \( T \). Hence, combining the conclusions of Case 1 and Lemma 3.6, the desired result follows.

In view of Lemma 3.4 and the results in Theorems 3.5 and 3.7, it is easy to see that Results 3.1 and 3.3 hold if the edge weights of the tree \( T \) are positive definite matrices.

### 3.2 Results for Lower (or Upper) Triangular Matrix Weights

In this section, we consider trees where the weights on the edges are lower (or upper) triangular matrices with positive diagonal entries. Since the arguments in the proofs for lower triangular matrix weights and upper triangular matrix weights are similar, we only provide results for lower triangular matrix weights with positive diagonal entries. We begin with a few preliminary results.

Let \( \mathcal{M}_n(\mathbb{R}) \) be the class of real matrices of order \( n \times n \) and \( A, B \in \mathcal{M}_n(\mathbb{R}) \). Matrices \( A \) and \( B \) are said to be permutation equivalent, denoted by \( A \simeq B \), if there exists a permutation matrix \( P \) such that \( A = PBPT \). We now state a result related to permutation equivalence and the Kronecker product.

**Proposition 3.8.** [15] Let \( A \in \mathcal{M}_m(\mathbb{R}) \) and \( B \in \mathcal{M}_n(\mathbb{R}) \). Let \( P \) be a permutation matrix of order \( mn \times mn \) such that

\[
P = \sum_{i=1}^{m} (e_i \otimes I_n \otimes e^T_i),
\]

where \( \{e_i : 1 \leq i \leq m\} \) is the standard basis of \( \mathbb{R}^m \). Then \( A \otimes B = P(B \otimes A)P^T \), i.e., \( A \otimes B \) and \( B \otimes A \) are permutation equivalent.

The permutation matrix \( P = \sum_{i=1}^{m} (e_i \otimes I_n \otimes e^T_i) \) is also called the vec-permutation matrix and is denoted by \( I_{m,n} \) (for details, see [15]). We now prove a lemma that plays an important role in proving Results 3.1 and 3.3. To prove this result, we use the fact that the vec-permutation matrix \( P = I_{m,n} \) does not depend on the entries of the matrices \( A \) and \( B \), but depends only on the order of these matrices.
Lemma 3.9. For $1 \leq i, j \leq m$, let $X_{ij}$ be matrices of order $s \times s$. For $1 \leq l, k \leq s$, let $X_{lk}$ be matrices of order $m \times m$ such that $(X_{lk})_{ij} = (X_{ij})_{lk}$, i.e., the $(i, j)^{th}$ entry of $X_{lk}$ is the $(l, k)^{th}$ entry of $X_{ij}$. If $X = [X_{ij}]$ and $\tilde{X} = [\tilde{X}_{lk}]$ are the block matrices of order $ms \times ms$, then $X$ and $\tilde{X}$ are permutation equivalent.

Proof. Let $\{e_i : 1 \leq i \leq m\}$ be the standard basis of $\mathbb{R}^m$ and $\{f_l : 1 \leq l \leq s\}$ be the standard basis of $\mathbb{R}^s$. For $1 \leq i, j \leq m$ and $1 \leq l, k \leq s$, let

$$E_{ij} = e_i e_j^T$$

and $F_{lk} = f_l f_k^T$.

Then, $\{E_{ij} : 1 \leq i, j \leq m\}$ is the standard basis of $\mathcal{M}_m(\mathbb{R})$ and $\{F_{lk} : 1 \leq l, k \leq s\}$ is the standard basis of $\mathcal{M}_s(\mathbb{R})$. Moreover,

$$\{E_{ij} \otimes F_{lk} : 1 \leq i, j \leq m, 1 \leq l, k \leq s\}$$

and

$$\{F_{lk} \otimes E_{ij} : 1 \leq i, j \leq m, 1 \leq l, k \leq s\}$$

are both bases of $\mathcal{M}_{ms}(\mathbb{R})$. Thus,

$$X = \sum_{i,j} X_{ij} \otimes E_{ij} = \sum_{i,j} \left( \sum_{l,k} (X_{ij})_{lk} F_{lk} \right) \otimes E_{ij} = \sum_{i,j} \sum_{l,k} (X_{ij})_{lk} (F_{lk} \otimes E_{ij}), \quad (3.7)$$

and

$$\tilde{X} = \sum_{l,k} \tilde{X}_{lk} \otimes F_{lk} = \sum_{l,k} \left( \sum_{i,j} (\tilde{X}_{lk})_{ij} E_{ij} \right) \otimes F_{lk} = \sum_{l,k} \sum_{i,j} (\tilde{X}_{lk})_{ij} (E_{ij} \otimes F_{lk}). \quad (3.8)$$

By Proposition 3.8, there exists a permutation matrix $P$ such that

$$F_{lk} \otimes E_{ij} = P (E_{ij} \otimes F_{lk}) P^T$$

for all $1 \leq i, j \leq m$ and $1 \leq l, k \leq s$. \quad (3.9)

Using $\tilde{(X_{lk})}_{ij} = (X_{ij})_{lk}$ and Equations (3.7) - (3.9), we have

$$X = \sum_{i,j} \sum_{l,k} (\tilde{X}_{lk})_{ij} P (E_{ij} \otimes F_{lk}) P^T = P \left( \sum_{i,j} \sum_{l,k} (\tilde{X}_{lk})_{ij} (E_{ij} \otimes F_{lk}) \right) P^T = P \tilde{X} P^T.$$

This completes the proof. \qed

Remark 3.10. 1. Under the hypothesis of Lemma 3.9, if $X_{ij}$’s are lower triangular matrices, then $\tilde{X}$ is a lower triangular block matrix, i.e.,

$$\tilde{X} = \begin{bmatrix}
\tilde{X}_{11} & 0 & \cdots & 0 \\
\tilde{X}_{21} & \tilde{X}_{22} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{X}_{s1} & \tilde{X}_{s2} & \cdots & \tilde{X}_{ss}
\end{bmatrix}.$$

Since $X$ and $\tilde{X}$ are permutation equivalent, $\sigma(X) = \sigma(\tilde{X}) = \bigcup_{i=1}^s \sigma(\tilde{X}_{ii})$. It is easy to see that a similar assertion can be made whenever $X_{ij}$’s are upper triangular matrices.

2. If $W = [W_{ij}]$ is an invertible lower (or upper) triangular matrix, then $W^{-1}$ is a lower (or upper) triangular matrix and the diagonal entries of $W^{-1}$ are given by $(W^{-1})_{ii} = \frac{1}{W_{ii}}$. \hfill 16
A graph $G = (V, E)$ with weights assigned to its edges can also be presented as an ordered pair $(G, \{W(e)\}_{e \in E})$, where $G$ is the underlying graph and $\{W(e)\}_{e \in E}$ is the set of weights assigned to the edges in $E$. Using this representation, we define trees with positive edge weights obtained from a tree whenever the weights assigned to its edges are lower triangular matrices with positive diagonal entries.

**Definition 3.11.** Let $T = (V, E)$ be a tree such that the weights on the edges of $T$ are $s \times s$ lower triangular matrices with positive diagonal entries. Let $W(e) = [w_{ij}(e)]$ denote the $(s \times s$ lower triangular matrix) weight on the edge $e \in E$ such that $w_{jj}(e) > 0$ for all $1 \leq j \leq s$. For $1 \leq j \leq s$, let

$$T^{(j)} = (T, \{w_{jj}(e)\}_{e \in E})$$

be the tree $T = (V, E)$ with positive edge weights $\{w_{jj}(e)\}_{e \in E}$. We say that $T^{(j)}$ is a tree with positive edge weights induced by $T = (V, E)$ with $s \times s$ lower triangular matrix edge weights.

![Figure 1](image)

**Example 3.12.** Let $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$. Consider the tree $T = (V, E)$, as shown in Figure (1) with the lower triangular matrix weights

$$W = \begin{cases} W(e_1) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 10 & 17 & 5 \end{bmatrix}, & W(e_2) = \begin{bmatrix} 9 & 0 & 0 \\ -3 & 4 & 0 \\ 2 & 5 & 17 \end{bmatrix}, & W(e_3) = \begin{bmatrix} 1 & 0 & 0 \\ -11 & 12 & 0 \\ 0 & -4 & 4 \end{bmatrix}, \\
W(e_4) = \begin{bmatrix} 11 & 0 & 0 \\ 2 & 1 & 0 \\ -16 & 0 & 3 \end{bmatrix}, & W(e_5) = \begin{bmatrix} 15 & 0 & 0 \\ 3 & 6 & 0 \\ 2 & 1 & 9 \end{bmatrix}, & W(e_6) = \begin{bmatrix} 7 & 0 & 0 \\ -10 & 8 & 0 \\ -9 & -1 & 6 \end{bmatrix}. \end{cases}$$

Let

$$W^{(1)} = \{W(e_1) = 3, W(e_2) = 9, W(e_3) = 1, W(e_4) = 11, W(e_5) = 15, W(e_6) = 7\},$$

$$W^{(2)} = \{W(e_1) = 2, W(e_2) = 4, W(e_3) = 12, W(e_4) = 1, W(e_5) = 6, W(e_6) = 8\},$$

$$W^{(3)} = \{W(e_1) = 5, W(e_2) = 17, W(e_3) = 4, W(e_4) = 3, W(e_5) = 9, W(e_6) = 6\}.$$ 

Let $T^{(1)} = (T, W^{(1)}), T^{(2)} = (T, W^{(2)})$ and $T^{(3)} = (T, W^{(3)})$. Thus, $T^{(1)}, T^{(2)}$ and $T^{(3)}$ are the trees with positive edge weights induced by $T = (V, E)$ with $3 \times 3$ lower triangular matrix edge weights $W$.

Let $T = (V, E)$ be a tree on $n$ vertices such that the weights on the edges of $T$ are $s \times s$ lower triangular matrices with positive diagonal entries. Let $W(e) = [w_{ij}(e)]$ denote the $(s \times s$ lower triangular matrix) weight on the edge $e \in E$ such that $w_{jj}(e) > 0$ for all $1 \leq j \leq s$. For $1 \leq j \leq s$,
let $T^{(j)}$ be the trees with positive edge weights induced by $T = (V, E)$ with $s \times s$ lower triangular matrix edge weights. However, $(T, \{W(e)\}_{e \in E})$ is the tree $T = (V, E)$ with the matrix weights $\{W(e)\}_{e \in E}$ on its edges, simply written as $T$. For $1 \leq j \leq s$, let $L(T)$ and $L(T^{(j)})$ be the Laplacian matrices of $T$ and $T^{(j)}$, respectively. Then, $L(T)$ is a matrix of order $ns \times ns$ and $L(T^{(j)})$ is a matrix of order $n \times n$ for $1 \leq j \leq s$. Thus, using Remark 3.10 (1) for the Laplacian matrix $L(T)$ of $T$, we have

$$
L(T) \simeq \widetilde{L(T)} = \begin{bmatrix}
L(T^{(1)}) & 0 & \cdots & 0 \\
* & L(T^{(2)}) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
* & * & \cdots & L(T^{(s)})
\end{bmatrix}
$$

and hence the eigenvalues of $L(T)$ are nonnegative.

**Lemma 3.13.** Let $T = (V, E)$ be a tree such that the weights on the edges of $T$ are $s \times s$ lower triangular matrices with positive diagonal entries. For $1 \leq j \leq s$, let $T^{(j)}$ be the trees with positive edge weights induced by $T = (V, E)$ with $s \times s$ lower triangular matrix edge weights. Let $v \in V$ and $B$ be a branch at $v$. For $1 \leq j \leq s$, let $M_v(B)$ and $M_v^{(j)}(B)$ be the bottleneck matrix of the branch $B$ at $v$ in $T$ and $T^{(j)}$, respectively. Then,

$$
\sigma(M_v(B)) = \bigcup_{j=1}^{s} \sigma(M_v^{(j)}(B)) \quad \text{and} \quad \rho(M_v(B)) = \max_{1 \leq j \leq s} \rho(M_v^{(j)}(B)).
$$

**Proof.** Let $T = (V, E)$ be a tree and $W(e) = [w_{ij}(e)]$ denote the $(s \times s)$ lower triangular matrix weight on the edge $e \in E$ such that $w_{jj}(e) > 0$ for all $1 \leq j \leq s$. Let $v \in V$ and $B$ be a branch at $v$ consisting of $k$ vertices. By definition $T^{(j)} = (T, [w_{ij}(e)]_{e \in E})$ for $1 \leq j \leq s$. Let $M_v(B)$ and $M_v^{(j)}(B)$ be the bottleneck matrix of the branch $B$ at $v$ in $T$ and $T^{(j)}$, respectively.

Let $x, y \in B$. For $1 \leq j \leq s$, the bottleneck matrix $M_v^{(j)}(B)$ is a $k \times k$ matrix, and the entry at the $(x, y)^{th}$ position of $M_v^{(j)}(B)$ is given by

$$
\sum_{e \in \mathcal{P}(x,v) \cap \mathcal{P}(y,v)} \frac{1}{w_{jj}(e)}. \tag{3.11}
$$

The bottleneck matrix $M_v(B)$ is a $ks \times ks$ matrix such that $(M_v(B)$ as a block matrix) the block at the $(x, y)^{th}$ position of $M_v(B)$ is an $s \times s$ matrix $W_{xy}$ (say), and is given by

$$
W_{xy} = \sum_{e \in \mathcal{P}(x,v) \cap \mathcal{P}(y,v)} W(e)^{-1}.
$$

Since the edge weights $\{W(e)\}_{e \in E}$ are lower triangular matrices, by Remark 3.10 (2), we see that $W_{xy}$ is an $s \times s$ lower triangular matrix and its diagonal entries are given by

$$(W_{xy})_{jj} = \sum_{e \in \mathcal{P}(x,v) \cap \mathcal{P}(y,v)} \frac{1}{w_{jj}(e)} \quad \text{for } 1 \leq j \leq s.
$$

Thus, using Remark 3.10 (1) for $M_v(B)$ and Equation (3.11), we have

$$
M_v(B) \simeq \widetilde{M_v(B)} = \begin{bmatrix}
M_v^{(1)}(B) & 0 & \cdots & 0 \\
* & M_v^{(2)}(B) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
* & * & \cdots & M_v^{(s)}(B)
\end{bmatrix} \tag{3.12}
$$
This completes the proof. □

**Theorem 3.14.** Let \( T = (V, E) \) be a tree such that the weights on the edges of \( T \) are \( s \times s \) lower triangular matrices with positive diagonal entries. For \( 1 \leq j \leq s \), let \( T^{(j)} \) be the trees with positive edge weights induced by \( T = (V, E) \) with \( s \times s \) lower triangular matrix edge weights.

(a) Let \( v \in V \) with \( \text{deg}(v) = r \) and \( B_i \) be the branches at \( v \) for \( 1 \leq i \leq r \). For \( 1 \leq j \leq s \), let \( M_v(B_i) \) and \( M_v^{(j)}(B_i) \) be the bottleneck matrix of the branch \( B_i \) at \( v \) in \( T \) and \( T^{(j)} \), respectively. Then,

\[
\rho(M_v) = \max_{1 \leq i \leq r} \rho(M_v(B_i)) = \max_{1 \leq j \leq s} \rho(M_v^{(j)}). 
\]

(b) The Perron value and the Perron branch are well-defined. Moreover, if \( B \) is a (unique) Perron branch at a vertex \( v \) in \( T^{(j)} \) for all \( 1 \leq j \leq s \), then \( B \) is a (unique) Perron branch at \( v \) in \( T \) and

\[
\rho(M_v) = \rho(M_v(B)) = \max_{1 \leq j \leq s} \rho(M_v^{(j)}(B)).
\]

**Proof.** Let \( v \in V \) with \( \text{deg}(v) = r \) and \( B_i \) be the branches at \( v \) for \( 1 \leq i \leq r \). For \( 1 \leq j \leq s \), let \( M_v(B_i) \) and \( M_v^{(j)}(B_i) \) be the bottleneck matrix of the branch \( B_i \) at \( v \) in \( T \) and \( T^{(j)} \), respectively. Thus, for \( 1 \leq i \leq r \), using Equation (3.12), we have

\[
M_v = \begin{bmatrix}
M_v(B_1) & 0 & \ldots & 0 \\
0 & M_v(B_2) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & M_v(B_r)
\end{bmatrix}
\quad \text{and} \quad
M_v(B_i) \simeq \begin{bmatrix}
M_v^{(1)}(B_i) & 0 & \ldots & 0 \\
* & M_v^{(2)}(B_i) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
* & * & \ldots & M_v^{(s)}(B_i)
\end{bmatrix}
\]

Then, by Lemma 3.13, we have

\[
\sigma(M_v(B_i)) = \bigcup_{j=1}^{s} \sigma(M_v^{(j)}(B_i)) \quad \text{and} \quad \rho(M_v(B_i)) = \max_{1 \leq j \leq s} \rho(M_v^{(j)}(B_i)) 
\]

and hence the spectral radius of \( M_v \) is given by

\[
\rho(M_v) = \max_{1 \leq i \leq r} \rho(M_v(B_i)) = \max_{1 \leq j \leq s} \rho(M_v^{(j)}(B_i)) = \max_{1 \leq j \leq s} \rho(M_v^{(j)}). \tag{3.14}
\]

This proves part (a).

Next, for \( 1 \leq i \leq r \) and \( 1 \leq j \leq s \), the eigenvalues of \( M_v^{(j)}(B_i) \) are positive and hence \( \rho(M_v^{(j)}(B_i)) \) is necessarily an eigenvalue of \( M_v^{(j)}(B_i) \). Therefore, by Equation (3.13), the definition of Perron value and Perron branch in Section 2 are well-defined for trees where weights on the edges are lower triangular matrices with positive diagonal entries. Hence, part (b) follows from Equation (3.14). □

We first prove results for trees where the edge weights are matrices of order \( 2 \times 2 \). Let \( T = (V, E) \) be a tree such that the weights on the edges of \( T \) are \( 2 \times 2 \) lower triangular matrices with positive diagonal entries. Let \( T^{(1)} \) and \( T^{(2)} \) be the trees with positive edge weights induced by \( T = (V, E) \) with \( 2 \times 2 \) lower triangular matrix edge weights. Let \( C_{T^{(1)}} \) and \( C_{T^{(2)}} \) denote the set of characteristic vertices of \( T^{(1)} \) and \( T^{(2)} \), respectively. The strategy adopted to achieve our goal is as follows: We consider all the possible cases for \( C_{T^{(1)}} \) and \( C_{T^{(2)}} \), and for each of these cases we use Propositions 1.7 - 1.9 for trees \( T^{(1)} \) and \( T^{(2)} \) to show that Results 3.1 and 3.3 hold true. We begin by considering \( C_{T^{(1)}} \cap C_{T^{(2)}} \neq \emptyset \) and \( C_{T^{(1)}} \cap C_{T^{(2)}} = \emptyset \) as separate cases.
Lemma 3.15. Let $T = (V, E)$ be a tree such that the weights on the edges of $T$ are $2 \times 2$ lower triangular matrices with positive diagonal entries. Let $T^{(1)}$ and $T^{(2)}$ be the trees with positive edge weights induced by $T = (V, E)$ with $2 \times 2$ lower triangular matrix edge weights such that $C_{T^{(1)}} \cap C_{T^{(2)}} \neq \emptyset$. Then, one of the following cases occurs:

1. There is a unique vertex $v$ such that there are two or more Perron branches at $v$ in $T$. Moreover, if $x$ is a vertex other than $v$, then the unique Perron branch at $x$ in $T$ is the branch which contains the vertex $v$.

2. There is a unique pair of vertices $u$ and $v$ with $u \sim v$ such that the Perron branch at $u$ in $T$ is the branch containing $v$, while the Perron branch at $v$ in $T$ is the branch containing $u$. Moreover, the unique Perron branch at any vertex $x$ in $T$ is the branch which contains at least one of the vertices $u$ or $v$.

Proof. We consider different choices of $C_{T^{(1)}}$ and $C_{T^{(2)}}$ with $C_{T^{(1)}} \cap C_{T^{(2)}} \neq \emptyset$, and prove that the result is true in each of these cases.

Case 1: Let $C_{T^{(1)}} = C_{T^{(2)}} = \{v\}$, i.e., the vertex $v$ is the characteristic vertex of $T^{(1)}$ and $T^{(2)}$. There exist branches $B_{i_1}$ and $B_{i_2}$ at $v$ such that $\rho(M_v^{(1)}) = \rho(M^{(1)}(B_{i_1})) = \rho(M^{(1)}(B_{i_2}))$. Similarly, there exist branches $B_{j_1}$ and $B_{j_2}$ at $v$ such that $\rho(M_v^{(2)}) = \rho(M^{(2)}(B_{j_1})) = \rho(M^{(2)}(B_{j_2}))$. If $\rho(M_v^{(1)}) \geq \rho(M_v^{(2)})$, then by Equation (3.13) the branches $B_{i_1}$ and $B_{i_2}$ are Perron branches at $v$ for $T$ and $\rho(M_v) = \rho(M_v(B_{i_1})) = \rho(M_v(B_{i_2}))$. Similarly, if $\rho(M_v^{(1)}) \leq \rho(M_v^{(2)})$, then $B_{j_1}$ and $B_{j_2}$ are Perron branches at $v$ for $T$ and $\rho(M_v) = \rho(M_v(B_{j_1})) = \rho(M_v(B_{j_2}))$. Further, if $x \neq v$, then $B_x(v)$ is the unique Perron branch at $x$ in $T$. Therefore, $v$ is the unique vertex such that there are two or more Perron branches at $v$ and $B_x(v)$ is the unique Perron branch at $x$ in $T$, whenever $x \neq v$.

Case 2: Let $C_{T^{(1)}} = C_{T^{(2)}} = \{u, v\}$, i.e., the edge between the vertices $u$ and $v$ is the characteristic edge of $T^{(1)}$ and $T^{(2)}$. Then, $B_u(v)$ is the unique Perron branch at $u$ in $T^{(j)}$ and $B_v(u)$ is the unique Perron branch at $v$ in $T^{(j)}$ for $j = 1, 2$. Thus,

$$\rho(M_u) = \rho(M_u(v)) = \max_{j=1,2} \rho(M_u^{(j)}(v)) \quad \text{and} \quad \rho(M_v) = \rho(M_v(u)) = \max_{j=1,2} \rho(M_v^{(j)}(u)).$$

Hence, $B_u(v)$ is the unique Perron branch at $u$ in $T$, while $B_v(u)$ is the unique Perron branch at $v$ in $T$.

Let $x \in V$ such that $x \neq u$ and $x \neq v$. If $B$ is the branch at $x$ containing $u$ and $v$, then $B$ is the unique Perron branch at $x$ in $T^{(j)}$ for $j = 1, 2$. Hence $B$ is the unique Perron branch at $x$ in $T$.

Case 3: Let $C_{T^{(1)}} = \{u, v\}$ and $C_{T^{(2)}} = \{v\}$, i.e., the edge between the vertices $u$ and $v$ is the characteristic edge of $T^{(1)}$ and $v$ is the characteristic vertex of $T^{(2)}$. Thus, $B_u(v)$ is the unique Perron branch at $u$ in $T^{(j)}$ for $j = 1, 2$. Hence

$$\rho(M_u) = \rho(M_u(v)) = \max_{j=1,2} \rho(M_u^{(j)}(v)), \quad (3.15)$$

and $B_u(v)$ is the unique Perron branch at $u$ in $T$. Further, $B_v(u)$ is the unique Perron branch at $v$ in $T^{(1)}$, i.e., $\rho(M_v^{(1)}) = \rho(M_v^{(1)}(u))$ and there exist branches $B_{j_1}$ and $B_{j_2}$ at $v$ such that $\rho(M_v^{(2)}) = \rho(M_v^{(2)}(B_{j_1})) = \rho(M_v^{(2)}(B_{j_2}))$. Therefore, the following cases arise:
• If \( \rho(M_v^{(1)}) \leq \rho(M_v^{(2)}) \), then \( \rho(M_v^{(2)}) = \rho(M_v) = \rho(M_v(B_{j_1})) = \rho(M_v(B_{j_2})) \). Thus, there are two or more Perron branches of \( T \) at \( v \) and the uniqueness of the vertex \( v \) follows from an argument similar to that in Case 1.

• If \( \rho(M_v^{(1)}) > \rho(M_v^{(2)}) \), then \( \rho(M_v^{(1)}) = \rho(M_v) = \rho(M_v(u)) \) and hence \( B_v(u) \) is the unique Perron branch at \( v \) in \( T \). By Equation (3.15), \( B_u(v) \) is the unique Perron branch at \( u \) in \( T \). The uniqueness of the vertices \( u \) and \( v \) follows from an argument similar to that in Case 2.

**Case 4:** Let \( C_{T(1)} = \{u,v\} \) and \( C_{T(2)} = \{v,w\} \), i.e., the edge between the vertices \( u \) and \( v \) is the characteristic edge of \( T^{(1)} \) and the edge between the vertices \( v \) and \( w \) is the characteristic edge of \( T^{(2)} \). Observe that, \( B_u(v) \) is the unique Perron branch at \( u \) in \( T^{(1)} \) and \( T^{(2)} \). Similarly, \( B_w(v) \) is the unique Perron branch at \( w \) in \( T^{(1)} \) and \( T^{(2)} \). Hence, \( B_u(v) \) is the unique Perron branch at \( u \) in \( T \), while \( B_w(v) \) is the unique Perron branch at \( w \) in \( T \)

\[
\rho(M_u) = \rho(M_u(v)) \quad \text{and} \quad \rho(M_w) = \rho(M_w(v)). \tag{3.16}
\]

Further, \( B_v(u) \) is the unique Perron branch at \( v \) in \( T^{(1)} \) and \( B_v(w) \) is the unique Perron branch at \( v \) in \( T^{(2)} \). Hence

\[
\rho(M_v) = \max\{\rho(M_v(u)), \rho(M_v(w))\}. \tag{3.17}
\]

Therefore, the following cases arise:

• If \( \rho(M_v(u)) = \rho(M_v(w)) \), then by Equation (3.17), \( \rho(M_v) = \rho(M_v(u)) = \rho(M_v(w)) \). Thus, there are two or more Perron branches at \( v \) in \( T \) and the uniqueness of the vertex \( v \) follows from an argument similar to that in Case 1.

• If \( \rho(M_v(u)) > \rho(M_v(w)) \), then Equations (3.16) and (3.17) yield that \( B_u(v) \) is the unique Perron branch at \( u \) in \( T \) and \( B_v(u) \) is the unique Perron branch at \( v \) in \( T \). The uniqueness of the vertices \( u \) and \( v \) follows from an argument similar to that in Case 2.

• If \( \rho(M_v(u)) < \rho(M_v(w)) \), then Equations (3.16) and (3.17) yield that \( B_v(w) \) is the unique Perron branch at \( v \) in \( T \) and \( B_w(v) \) is the unique Perron branch at \( w \) in \( T \). The uniqueness of the vertices \( v \) and \( w \) follows from an argument similar to that in Case 2.

This completes the proof. \( \square \)

**Lemma 3.16.** Let \( T = (V, E) \) be a tree such that the weights on the edges of \( T \) are \( 2 \times 2 \) lower triangular matrices with positive diagonal entries. Let \( T^{(1)} \) and \( T^{(2)} \) be the trees with positive edge weights induced by \( T = (V, E) \) with \( 2 \times 2 \) lower triangular matrix edge weights such that \( C_{T(1)} \cap C_{T(2)} = \emptyset \). Then one of the following cases occurs:

1. There is a unique vertex \( v \) such that there are two or more Perron branches at \( v \) in \( T \). Moreover, if \( x \) is a vertex other than \( v \), then the unique Perron branch at \( x \) in \( T \) is the branch which contains the vertex \( v \).

2. There is a unique pair of vertices \( u \) and \( v \) with \( u \sim v \) such that the Perron branch at \( u \) in \( T \) is the branch containing \( v \), while the Perron branch at \( v \) in \( T \) is the branch containing \( u \). Moreover, the unique Perron branch at any vertex \( x \) in \( T \) is the branch which contains at least one of the vertices \( u \) or \( v \).

**Proof.** Let \( C_{T(1)} = \{v\} \) and \( C_{T(2)} = \{x, y\} \), where \( v \neq x \) and \( v \neq y \). Without loss of generality, assume that \( y \notin P(v, x) \), where \( P(v, x) \) is the path joining the vertices \( v \) and \( x \) such that \( P(v, x) : v = v_1 \sim v_2 \sim \cdots \sim v_{p-1} \sim v_p = x \). Since \( v \) is the characteristic vertex of \( T^{(1)} \), there exists a
vertex \( u \) adjacent to \( v \) with \( u \neq v_1 \) such that \( B_v(u) \) is a Perron branch at \( v \) in \( T^{(1)} \) and \( \rho(M_v^{(1)}) = \rho(M_v^{(1)}(u)) \). Thus, \( B_v(u) \) is the unique Perron branch at \( v_i \) in \( T^{(1)} \), while \( B_v(y) \) is the unique Perron branch at \( v_i \) in \( T^{(2)} \) for \( 1 \leq i \leq p \). Hence

\[
\rho(M_v^{(1)}) = \rho(M_v^{(1)}(u)) \quad \text{and} \quad \rho(M_v^{(2)}) = \rho(M_v^{(2)}(y)) \quad \text{for} \quad 1 \leq i \leq p. \tag{3.18}
\]

Next, since \( B_v(u) \subsetneq B_{v_{i+1}}(u) \) and \( B_v(y) \supseteq B_{v_{i+1}}(y) \) for \( 1 \leq i \leq p - 1 \), using Proposition 1.6 and Equation (3.18), we have

\[
\begin{align*}
\rho(M_v^{(1)}) & = \rho(M_v^{(1)}(u)) < \rho(M_v^{(1)}(v_2)) < \cdots < \rho(M_v^{(1)}(v_{p-1})) < \rho(M_v^{(1)}(v_1)) = \rho(M_v^{(1)}), \\
\rho(M_v^{(2)}) & = \rho(M_v^{(2)}(y)) > \rho(M_v^{(2)}(v_2)) > \cdots > \rho(M_v^{(2)}(v_{p-1})) > \rho(M_v^{(2)}(v_1)) = \rho(M_v^{(2)}).
\end{align*}
\tag{3.19}
\]

Therefore, the following cases arise.

**Case 1:** Let \( \rho(M_v^{(1)}) \geq \rho(M_v^{(2)}) \). Using Equation (3.19), we have

\[
\rho(M_v) = \max\{\rho(M_v^{(1)}), \rho(M_v^{(2)})\} = \rho(M_v^{(1)}) \quad \text{for} \quad 1 \leq i \leq p. \tag{3.20}
\]

Since \( v \) is the characteristic vertex of \( T^{(1)} \), there exist branches \( B_{v_1} \) and \( B_{v_2} \) at \( v \) such that \( \rho(M_v^{(1)}) = \rho(M_v^{(1)}(B_{v_1})) = \rho(M_v^{(1)}(B_{v_2})) \). By Equation (3.20), we have \( \rho(M_v) = \rho(M_v^{(1)}) \) and hence

\[
\rho(M_v) = \rho(M_v(B_{v_1})) = \rho(M_v(B_{v_2})).
\]

To show the uniqueness of the vertex \( v \), let us consider the branch \( B_w(v) \), where \( w \neq v \). If \( w \neq v_i \) for \( i = 2, 3, \ldots, p \), then \( x \in B_w(v) \). Hence \( B_w(v) = B_w(x) \). Thus, \( B_w(v) \) is the unique Perron branch at \( w \) in \( T^{(1)} \) and \( T^{(2)} \), and therefore, \( B_w(v) \) is the unique Perron branch at \( w \) in \( T \). Next, if \( w = v_i \) for \( i = 2, 3, \ldots, p \), using Equation (3.19) and the assumption for this case, we have

\[
\rho(M_v^{(1)}) > \rho(M_v^{(1)}(1)) \geq \rho(M_v^{(2)}) > \rho(M_v^{(1)}). \tag{3.21}
\]

This implies that \( \rho(M_w) = \max\{\rho(M_v^{(1)}), \rho(M_w^{(2)})\} = \rho(M_v^{(1)}(v)) = \rho(M_w(v)) \). Thus, \( B_w(v) \) is the unique Perron branch at \( w \) in \( T \).

Therefore, \( v \) is the unique vertex of \( T \) such that there are two or more Perron branches at \( v \) in \( T \) and for any \( w \neq v \), \( B_w(v) \) is the unique Perron branch at \( w \) in \( T \).

**Case 2:** Let \( \rho(M_v^{(2)}) \geq \rho(M_v^{(1)}) \). Using Equation (3.19), we have

\[
\rho(M_v) = \max\{\rho(M_v^{(1)}), \rho(M_v^{(2)})\} = \rho(M_v^{(2)}) \quad \text{for} \quad 1 \leq i \leq p. \tag{3.21}
\]

Since the edge between the vertices \( x \) and \( y \) is the characteristic edge of \( T^{(2)} \), \( B_x(y) \) is the unique Perron branch at \( x \) in \( T^{(2)} \) and hence \( \rho(M_v^{(2)}) = \rho(M_v^{(2)}(y)) \). By Equation (3.21), \( \rho(M_x) = \rho(M_v^{(2)}) = \rho(M_v^{(2)}(y)) \) which implies that \( B_x(y) \) is the unique Perron branch at \( x \) in \( T \). Since \( v \in B_y(x) \), \( B_y(x) \) is the unique Perron branch at \( y \) in \( T^{(j)} \) for \( j = 1, 2 \) and hence \( B_y(x) \) is the Perron branch at \( y \) in \( T \).

Let \( w \) be a vertex such that \( w \neq x \) and \( w \neq y \). If \( B \) is a branch at \( w \) in \( T \) containing \( x \) and \( y \), it can be shown that \( B \) is the unique Perron branch at \( w \) in \( T \) by an argument similar to that in Case 1.

**Case 3:** Let \( \rho(M_v^{(1)}) < \rho(M_v^{(2)}) \) and \( \rho(M_v^{(1)}) > \rho(M_v^{(2)}) \). By Equation (3.19), \( \rho(M_v^{(1)}) \) is increasing and \( \rho(M_v^{(1)}) \) is decreasing with respect to \( i = 1, 2, \ldots, p \). Then, one of the following cases occurs:
(a) There exists a unique vertex \( v_{i_0} \) for some \( 2 \leq i_0 \leq p - 1 \) such that \( \rho(M^{(1)}_{v_{i_0}}) = \rho(M^{(2)}_{v_{i_0}}) \).

(b) There exists a unique pair of vertices \( v_{i_0} \) and \( v_{i_0+1} \) for some \( 1 \leq i_0 \leq p - 1 \) such that
\[
\rho(M^{(1)}_{v_{i_0}}) < \rho(M^{(2)}_{v_{i_0}}) \quad \text{and} \quad \rho(M^{(1)}_{v_{i_0+1}}) > \rho(M^{(2)}_{v_{i_0+1}}).
\]

For case (a), let \( B_{j_1} = B_{v_{i_0}}(v) \) and \( B_{j_2} = B_{v_{i_0}}(x) \). Then \( B_{j_1} \neq B_{j_2} \) and \( B_{j_1} \) is the unique Perron branch at \( v_{i_0} \) in \( T^{(1)} \), while \( B_{j_2} \) is the unique Perron branch at \( v_{i_0} \) in \( T^{(2)} \). Thus, \( \rho(M^{(1)}_{v_{i_0}}) = \rho(M^{(1)}_{v_{i_0}}(B_{j_1})) \) and \( \rho(M^{(1)}_{v_{i_0}}(B_{j_2})) \) and hence by the hypothesis, we get \( \rho(M^{(1)}_{v_{i_0}}) = \rho(M^{(2)}_{v_{i_0}}) = \rho(M^{(2)}_{v_{i_0}}(B_{j_1})) = \rho(M^{(2)}_{v_{i_0}}(B_{j_2})). \) Therefore, by Equation (3.14), we have
\[
\rho(M^{(1)}_{v_{i_0}}) = \rho(M^{(2)}_{v_{i_0}}(B_{j_1})) = \rho(M^{(2)}_{v_{i_0}}(B_{j_2})).
\]

Now we show that \( B_{v_{i_0}}(v_{i_0}) \) is the unique Perron branch of \( T \) at \( v \), whenever \( w \neq v_{i_0} \). If \( w \neq v_i \) for \( i = 1, 2, \ldots, p \), then \( v, x \in B_{v_{i_0}}(v_{i_0}) \). Thus, \( B_{w}(v_{i_0}) \) is the unique Perron branch at \( w \) in \( T^{(1)} \) and \( T^{(2)} \), and hence \( B_{w}(v_{i_0}) \) is the unique Perron branch at \( v \) in \( T \). If \( w = v_i \) for \( 1 \leq i < i_0 \), then \( \rho(M^{(1)}_{v_i}) < \rho(M^{(2)}_{v_i}) \). Thus \( \rho(M_{v_i}) = \rho(M^{(2)}_{v_i}) \). Since \( B_{v_i}(v_{i_0}) = B_{v_i}(x) \) is the unique Perron branch at \( v_i \) in \( T^{(2)} \), \( \rho(M^{(2)}_{v_i}) = \rho(M^{(2)}_{v_i}(v_{i_0})) \) and hence
\[
\rho(M^{(1)}_{v_i}) = \rho(M^{(2)}_{v_i}) = \rho(M^{(2)}_{v_i}(v_{i_0})) \quad \text{for} \quad 1 \leq i < i_0.
\]

Therefore, \( B_{v_i}(v_{i_0}) \) is the unique Perron branch at \( v_i \) in \( T \) for \( 1 \leq i < i_0 \). It is easy to see that a similar assertion can be made for \( w = v_i \) for \( i_0 < i \leq p \).

For case (b), using \( \rho(M^{(1)}_{v_{i_0}}) < \rho(M^{(2)}_{v_{i_0}}) \) and that \( B_{v_{i_0}}(v_{i_0+1}) = B_{v_{i_0}}(x) \) is the unique Perron branch at \( v_{i_0} \) in \( T^{(2)} \), we have
\[
\rho(M^{(1)}_{v_{i_0}}) = \rho(M^{(2)}_{v_{i_0}}) = \rho(M^{(2)}_{v_{i_0}}(v_{i_0+1})).
\]

Hence \( B_{v_{i_0}}(v_{i_0+1}) \) is the unique Perron branch at \( v_{i_0} \) in \( T \). Similarly, using \( \rho(M^{(1)}_{v_{i_0+1}}) > \rho(M^{(2)}_{v_{i_0+1}}) \) and the fact that \( B_{v_{i_0+1}}(v_{i_0}) = B_{v_{i_0+1}}(v) \) is the unique Perron branch at \( v_{i_0+1} \) in \( T^{(1)} \), we have \( B_{v_{i_0+1}}(v_{i_0}) \) is the unique Perron branch at \( v_{i_0+1} \) in \( T \). Further, if \( w \) is a vertex other than \( v_{i_0} \) and \( v_{i_0+1} \), it can be seen that the unique Perron branch at \( w \) in \( T \) is the branch which contains \( v_{i_0} \) and \( v_{i_0+1} \) by proceeding in a manner similar to that in case (a).

The other possible cases are (i) \( C^{(1)} = \{v\} \) and \( C^{(2)} = \{x\} \), (ii) \( C^{(1)} = \{u, v\} \) and \( C^{(2)} = \{x, y\} \). It can be seen that the proof follows analogously to the above cases and hence we omit the details. Combining the conclusion from all the above cases, the desired result follows. \( \square \)

By combining the results of Lemmas 3.15 and 3.16, we have shown that Results 3.1 and 3.3 hold if the edge weights of a tree are \( 2 \times 2 \) lower triangular matrices with positive diagonal entries. Before proceeding further, we state a few observations from the proof of Lemmas 3.15 and 3.16 in the following remark.

**Remark 3.17.** 1. The Results 3.1 and 3.3 are valid, if the weights on the edges of the tree \( T \) are \( 2 \times 2 \) lower triangular matrices with positive diagonal entries.

2. The characteristic-like vertex (or vertices) of \( T \) lie in the path joining characteristic vertices of \( T^{(1)} \) and \( T^{(2)} \).

3. The arguments used to prove Lemmas 3.15 and 3.16 are summarized as follows:

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(a) For any \( v \in V \), if \( B \) is a branch of \( T \) at \( v \), then \( M_v(B) \) is a \( 2 \times 2 \) lower triangular block matrix, i.e.,

\[
M_v(B) = \begin{bmatrix}
M_v^{(1)}(B) & 0 \\
* & M_v^{(2)}(B)
\end{bmatrix}.
\]

Hence, \( \rho(M_v(B)) = \max\{\rho(M_v^{(1)}(B)), \rho(M_v^{(2)}(B))\} \).

(b) Results 3.1 and 3.3 are true for both \( T^{(1)} \) and \( T^{(2)} \).

(c) For any \( u, v, w \in V \), if \( B_u(w) \subsetneq B_v(w) \), then \( \rho(M_u^{(j)}(w)) < \rho(M_v^{(j)}(w)) \) for \( j = 1, 2 \).

Before proving the results for the general case, we prove a lemma analogous to Proposition 1.6.

**Lemma 3.18.** Let \( T = (V, E) \) be a tree such that weights on the edges of \( T \) are lower triangular matrices with positive diagonal entries. For any \( u, v, w \in V \), if \( B_u(w) \subsetneq B_v(w) \), then \( \rho(M_u(w)) < \rho(M_v(w)) \).

**Proof.** Let the weights on the edges of \( T \) be \( s \times s \) lower triangular matrices with positive diagonal entries. For each \( 1 \leq j \leq s \), let \( T^{(j)} \) be the tree with positive edge weights induced by \( T = (V, E) \) with \( s \times s \) lower triangular matrix edge weights. For each \( 1 \leq j \leq s \), using Proposition 1.6 for the tree \( T^{(j)} \) if \( B_u(w) \subsetneq B_v(w) \) then \( \rho(M_u^{(j)}(w)) < \rho(M_v^{(j)}(w)) \). Hence, the result follows from Equation (3.14).

**Theorem 3.19.** Let \( T = (V, E) \) be a tree such that the weights on the edges of \( T \) are lower triangular matrices with positive diagonal entries. Then, one of the following cases occurs:

1. There is a unique vertex \( v \) such that there are two or more Perron branches at \( v \) in \( T \). Moreover, if \( x \) is a vertex other than \( v \), then the unique Perron branch at \( x \) in \( T \) is the branch which contains the vertex \( v \).

2. There is a unique pair of vertices \( u \) and \( v \) with \( u \sim v \) such that the Perron branch at \( u \) in \( T \) is the branch containing \( v \), while the Perron branch at \( v \) in \( T \) is the branch containing \( u \). Moreover, the unique Perron branch at any vertex \( x \) in \( T \) is the branch which contains at least one of the vertices \( u \) or \( v \).

**Proof.** Let the edges of \( T \) be assigned with lower triangular matrix weights of order \( s \times s \) with positive diagonal entries. We prove this result using induction on \( s \). By Lemmas 3.15 and 3.16, the result is true for \( s = 2 \). Let us assume that the result is true whenever matrix weights are of order \( (s-1) \times (s-1) \).

Let \( \{W(e)\}_{e \in E} \) denote the lower triangular matrix weights on \( T \) of order \( s \times s \) with positive diagonal entries and let \( W^*(e) \) denote the principal submatrix of \( W(e) \) corresponding to the indices \( 1, 2, \ldots, s-1 \). Let \( T^* \) denote the tree \( T = (V, E) \) with the matrix weights \( \{W^*(e)\}_{e \in E} \) of order \( (s-1) \times (s-1) \). Then, by the induction hypothesis Results 3.1 and 3.3 hold true for the tree \( T^* \).

Now, we consider matrix weights of order \( s \times s \). For \( v \in V \) with \( \deg(v) = r \) and for \( 1 \leq i \leq r \), let \( B_i \) be the branches at \( v \). Then, by Equation (3.12), we have

\[
M_v^*(B_i) \simeq M_v^{(r)}(B_i) = \begin{bmatrix}
M_v^{(1)}(B_i) & 0 & \ldots & 0 \\
* & M_v^{(2)}(B_i) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
* & * & \ldots & M_v^{(s-1)}(B_i)
\end{bmatrix}.
\]
and

\[
M_v(B_i) \simeq \tilde{M}_v(B_i) = \begin{bmatrix}
M_v^{(1)}(B_i) & 0 & \cdots & 0 \\
* & M_v^{(2)}(B_i) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
* & * & \cdots & M_v^{(s)}(B_i)
\end{bmatrix} = \begin{bmatrix}
\tilde{M}_v^{(1)}(B_i) & 0 \\
* & \tilde{M}_v^{(s)}(B_i)
\end{bmatrix}.
\]

This implies that \( \rho(M_v(B_i)) = \max\{\rho(M_v^{(i)}(B_i)), \rho(M_v^{(s)}(B_i))\} \) for all \( 1 \leq i \leq r \). Therefore, in view of Remark 3.17, Lemma 3.18 and the induction hypothesis, the desired result follows by proceeding in exactly the same manner as Lemmas 3.15 and 3.16.

**Corollary 3.20.** Let \( T = (V, E) \) be a tree such that the weights on the edges of \( T \) are \( s \times s \) lower triangular matrices with positive diagonal entries. For \( 1 \leq j \leq s \), let \( T^{(j)} \) be the trees with positive edge weights induced by \( T = (V, E) \) with \( s \times s \) lower triangular matrix edge weights. Then, the characteristic-like vertex (or vertices) of \( T \) lies in the minimal sub tree of \( T \) containing the characteristic vertices of \( T^{(j)} \) for \( 1 \leq j \leq s \).

**Proof.** We use induction on \( s \) to prove the result. The result is true for \( s = 2 \) (see Remark 3.17 (2)). Let us assume that the result is true whenever matrix weights are of order \( (s-1) \times (s-1) \). Let \( T^* \) be the tree \( T \) with weights of order \( (s-1) \times (s-1) \) as defined in proof of Theorem 3.19. Then, by the induction hypothesis, characteristic-like vertex (or vertices) of \( T^* \) lie in the minimal sub tree of \( T \) containing all the characteristic vertices of \( T^{(j)} \) for \( 1 \leq j \leq (s-1) \). Thus, proceeding in a manner similar to that in Lemmas 3.15 and 3.16 for the trees \( T^* \) and \( T^{(s)} \), we see that the characteristic-like vertex (or vertices) of \( T \) lie in the path joining the characteristic-like vertex (or vertices) of \( T^* \) and the characteristic vertex (or vertices) of \( T^{(s)} \). Hence, the desired result follows.

From Propositions 1.7 and 1.8, we have seen that the first non-zero eigenvalue of the Laplacian matrix (algebraic connectivity) of a tree with positive weights can be expressed in terms of Perron values. In the next section, we attempt to find a similar relation for trees with matrix weights. However, here we obtain an inequality instead.

## 4 Lower Bound on the First Non-zero Laplacian Eigenvalue

In the literature, the algebraic connectivity plays an important role in understanding the geometry of a tree with positive edge weights (for example, see [1, 14, 17, 18, 22, 23, 24, 25]). In particular, the understanding of the characteristic vertex (vertices) via Perron value and Perron branch, and the representation of the algebraic connectivity in terms of Perron values yielded several interesting results related to the structure of a tree (for example, see [1, 17, 18, 22, 23]). Given Section 3, we are interested in a similar representation for the first non-zero eigenvalue of Laplacian matrices via Perron values for trees with matrix edge weights. However, we obtain a lower bound involving a similar expression on Perron values.

Let \( T \) be a tree on \( n \) vertices with either of the following classes of matrix weights on its edges: (1) positive definite matrix weights, (2) lower (or upper) triangular matrix weights with positive diagonal entries. From the previous sections, we know that the eigenvalues of the Laplacian matrix \( L(T) \) are nonnegative. Moreover, if the matrix weights assigned to the edges of \( T \) are of order \( s \times s \), then by [2, Theorem 2.4], we have \( \text{rank}(L(T)) = (n-1)s \). Therefore, if the eigenvalues of \( L(T) \) are ordered as in Equation (1.1), then \( \lambda_{s+1}(L(T)) \) is the first non-zero eigenvalue of \( L(T) \). For notational consistency, we denote the first non-zero eigenvalue \( \lambda_{s+1}(L(T)) \) as \( \mu(T) \) (similar to
the case of trees with positive edge weights). In this section, we provide a lower bound for $\mu(T)$ in terms of Perron value. Before proceeding further, using arguments similar to the proof of [16, Theorem 1], we extend the result for trees with the above classes of matrix edge weights.

**Lemma 4.1.** Let $T$ be a tree with either of the following classes of matrix weights on its edges:

1. positive definite matrix weights,
2. lower (or upper) triangular matrix weights with positive diagonal entries.

If $T$ has a characteristic-like edge $e$ between the vertices $u$ and $v$, then $\exists 0 < \nu < 1$ such that

$$\rho(M_u(v)) - \nu(J \otimes [W(e)^{-1}]) = \rho(M_v(u)) - (1 - \nu)(J \otimes [W(e)^{-1}]),$$

where $W(e)$ denotes the matrix weight on the edge $e$.

**Proof.** Let $\widehat{M}_u$ denote the principal submatrix of $M_u$ obtained by deleting the block $M_u(v)$ (the block corresponding to the unique Perron branch $B_u(v)$ at $u$ in $T$) from $M_u$. Similarly, let $\widehat{M}_v$ denote the principal submatrix of $M_v$ obtained by deleting the block $M_v(u)$ (the block corresponding to the unique Perron branch $B_v(u)$ at $v$ in $T$) from $M_v$, i.e.,

$$M_u = \begin{bmatrix} M_u(v) & 0 \\ 0 & \widehat{M}_u \end{bmatrix} \quad \text{and} \quad M_v = \begin{bmatrix} \widehat{M}_v & 0 \\ 0 & M_v(u) \end{bmatrix}. $$

Then

$$\rho(M_u(v)) > \rho(\widehat{M}_u) \quad \text{and} \quad \rho(M_v(u)) > \rho(\widehat{M}_v). \quad (4.1)$$

Further,

$$\widehat{M}_u = M_u(v) - J \otimes [W(e)^{-1}] \quad \text{and} \quad \widehat{M}_v = M_v(u) - J \otimes [W(e)^{-1}], \quad (4.2)$$

where

$$\widehat{M}_u = \begin{bmatrix} O_{s \times s} & 0 \\ 0 & \widehat{M}_u \end{bmatrix} \quad \text{and} \quad \widehat{M}_v = \begin{bmatrix} \widehat{M}_v & 0 \\ 0 & 0_{s \times s} \end{bmatrix},$$

when the matrix weights on edges are of order $s \times s$. Thus $\rho(\widehat{M}_u) = \rho(\widehat{M}_u)$ and $\rho(\widehat{M}_v) = \rho(\widehat{M}_v)$. Using Equations (4.1) and (4.2), we have

$$\rho(M_u(v)) > \rho(M_v(u) - J \otimes [W(e)^{-1}]) \quad \text{and} \quad \rho(M_v(u)) > \rho(M_u(v) - J \otimes [W(e)^{-1}]). \quad (4.3)$$

For $0 \leq t \leq 1$, let

$$\begin{cases} f(t) = \rho(M_u(v) - t(J \otimes [W(e)^{-1}]), \\ g(t) = \rho(M_v(u) - (1 - t)(J \otimes [W(e)^{-1}]). \end{cases}$$

Then,

- for positive definite matrix weights, $J \otimes [W(e)^{-1}]$ is a positive semidefinite matrix. Then using the min-max theorem we see that $f(t)$ is a continuous, decreasing function and $g(t)$ is a continuous, increasing function.

- for lower triangular matrix weights of order $s \times s$ with positive diagonal entries, let $W(e) = [W_{ij(e)}]$. By Equation (3.14), we have

$$f(t) = \rho(M_u(v) - t(J \otimes [W(e)^{-1}])) = \max_{1 \leq j \leq s} \rho(M_u^{(j)}(v)) - t(1/W_{jj(e)})(J).$$

Since $\rho(M_u^{(j)}(v)) - t(1/W_{jj(e)})(J)$ is a decreasing function with respect to $t$ for all $0 \leq t \leq 1$ and $1 \leq j \leq s$, we see that $f(t)$ is a continuous, decreasing function. Similarly, it can be seen that $g(t)$ is a continuous, increasing function.
Note that, in the above cases the continuity of functions \( f(t) \) and \( g(t) \) for \( 0 \leq t \leq 1 \), follows from [8, Corollary VI.1.6]. Further, \( f(t) \) decreases from \( \rho(M_u(v)) \) to \( \rho(M_u(v) - J \otimes [W(e)^{-1}]) \) and \( g(t) \) increases from \( \rho(M_u(u) - J \otimes [W(e)^{-1}]) \) to \( \rho(M_v(u)) \). By Equation (4.3), \( f(t) \) and \( g(t) \) must intersect, and hence the result follows.

In view of Result 3.1 and Lemma 4.1, we now define a constant in terms of Perron values for trees with a suitable class of matrix edge weights.

**Definition 4.2.** Let \( T \) be a tree with either of the following classes of matrix weights on its edges: (1) positive definite matrix weights, (2) lower (or upper) triangular matrix weights with positive diagonal entries. We define a constant \( \kappa(T) \) as follows:

\[(a) \text{ If } T \text{ has a characteristic-like vertex } v, \text{ then } \kappa(T) = \frac{1}{\rho(M_v)}.
\]

\[(b) \text{ If } T \text{ has a characteristic-like edge } e \text{ between the vertices } u \text{ and } v, \text{ then}
\]

\[
\kappa(T) = \frac{1}{\rho(M_u(v) - \nu(J \otimes [W(e)^{-1}]))} = \frac{1}{\rho(M_v(u) - (1 - \nu)(J \otimes [W(e)^{-1}]))},
\]

where \( 0 < \nu < 1 \) as defined in Lemma 4.1 and \( W(e) \) denotes the matrix weight on edge \( e \).

To obtain a lower bound on \( \mu(T) \) for any tree \( T \) with positive definite matrix edge weights, we first prove the following lemmas.

**Lemma 4.3.** Let \( T \) be a tree with nonsingular matrix weights on its edges. If \( e \) is an edge between the vertices \( u \) and \( v \), then for \( 0 < \alpha < 1 \), we have

\[
\left[ M_u(v) - \alpha(J \otimes [W(e)^{-1}]) \right]^{-1} = M_u(v)^{-1} + e_v e_v^T \otimes \left[ \frac{\alpha}{1 - \alpha} W(e) \right],
\]

where \( e_v \) is the column vector of conformal order with 1 at the \( v \)th entry and 0 elsewhere, and \( W(e) \) is the weight on the edge \( e \).

**Proof.** Let \( L(T) \) be the Laplacian matrix of \( T \) and \( \hat{L}(B_u(v)) \) be the principal submatrix of \( L(T) \) corresponding to the vertices in the branch \( B_u(v) \). By Theorem 2.5, we know that \( \hat{L}(B_u(v)) = M_u(v)^{-1} \). Let

\[
X = M_u(v)^{-1} + e_v e_v^T \otimes [-W(e)].
\]

Then, the row and column block sums of \( X \) are zero. Thus, from Remark 2.1 (2), we have

\[
X(J \otimes [W(e)^{-1}]) = 0. \tag{4.4}
\]

Further note that, the column block of \( M_u(v) \) (by Theorem 2.5) and \( J \otimes [W(e)^{-1}] \) corresponding to the vertex \( v \) is \( I \otimes [W(e)^{-1}] \). Hence,

\[
\begin{cases}
(e_v e_v^T \otimes W(e))M_u(v) = e_v 1^T \otimes 1, \\
(e_v e_v^T \otimes W(e))(J \otimes [W(e)^{-1}]) = e_v 1^T \otimes [-1].
\end{cases} \tag{4.5}
\]

This implies that

\[
XM_u(v) = I + e_v e_v^T \otimes [-W(e)] M_u(u) = I + e_v 1^T \otimes [-I]. \tag{4.6}
\]
Now,
\[
\left( M_u(v)^{-1} + e_v e_v^T \otimes \left[ \left( \frac{\alpha}{1 - \alpha} \right) W(e) \right] \right) \left( M_u(v) - \alpha (J \otimes [W(e)^{-1}]) \right)
\]
\[
= \left( X + e_v e_v^T \otimes \left[ \left( \frac{1}{1 - \alpha} \right) W(e) \right] \right) \left( M_u(v) - \alpha (J \otimes [W(e)^{-1}]) \right)
\]
\[
= XM_u(v) - \alpha X (J \otimes [W(e)^{-1}]) + \left( e_v e_v^T \otimes \left[ \left( \frac{1}{1 - \alpha} \right) W(e) \right] \right) M_u(v) - \alpha \left( e_v e_v^T \otimes \left[ \left( \frac{1}{1 - \alpha} \right) W(e) \right] \right) (J \otimes [W(e)^{-1}]).
\]
Using Equations (4.4), (4.5) and (4.6), the above equation reduces to
\[
\left( M_u(v)^{-1} + e_v e_v^T \otimes \left[ \left( \frac{\alpha}{1 - \alpha} \right) W(e) \right] \right) \left( M_u(v) - \alpha (J \otimes [W(e)^{-1}]) \right)
\]
\[
= I + \left( -1 + \frac{1}{1 - \alpha} - \frac{\alpha}{1 - \alpha} \right) e_v I^T \otimes I = I.
\]
Hence, the desired result follows. □

**Lemma 4.4.** Let \( T \) be a tree on \( n \) vertices with nonsingular matrix weights on its edges and \( L(T) \) be the Laplacian matrix of \( T \). If \( e \) is an edge between the vertices \( u \) and \( v \), then for \( 0 < \alpha < 1 \), we have
\[
L(T) + E \otimes W(e) = \begin{bmatrix} \left[ M_u(v) - \alpha (J \otimes [W(e)^{-1}]) \right]^{-1} & 0 \\ 0 & \left[ M_v(u) - (1 - \alpha) (J \otimes [W(e)^{-1}]) \right]^{-1} \end{bmatrix}, \tag{4.7}
\]
where \( W(e) \) is the weight on edge \( e \) and \( E = [E_{xy}]_{x,y \in V} \) is an \( n \times n \) matrix with
\[
E_{xy} = \begin{cases} \frac{\alpha}{1 - \alpha} & \text{if } x = y = v, \\ \frac{1 - \alpha}{\alpha} & \text{if } x = y = u, \\ 1 & \text{if } x = u, y = v \text{ or } x = v, y = u, \\ 0 & \text{otherwise.} \end{cases}
\]

**Proof.** Let \( B_u(v) \) be the branch consisting of \( k \) vertices and \( B_v(u) \) be the branch consisting of \( (n - k) \) vertices. By suitable rearrangement of the vertex ordering and from Theorem 2.5, the Laplacian matrix \( L(T) \) of \( T \) can be written as
\[
L(T) = \begin{bmatrix} M_u(v)^{-1} & E_{k1} \otimes [-W(e)] \\ E_{1k} \otimes [-W(e)] & M_v(u)^{-1} \end{bmatrix}, \tag{4.8}
\]
where \( E_{k1} \) is the \( k \times (n - k) \) matrix with 1 at the \((k, 1)\)th position and 0 elsewhere, and \( E_{1k} \) is its transpose. Observe that, the partitioning here is such that the last row of \( M_u(v)^{-1} \) corresponds to the vertex \( v \), whereas the first row of \( M_v(u)^{-1} \) corresponds to the vertex \( u \).

For \( 0 < \alpha < 1 \), using Lemma 4.3, we have
\[
\begin{align*}
M_u(v)^{-1} &= \left[ M_u(v) - \alpha (J \otimes [W(e)^{-1}]) \right]^{-1} - e_v e_v^T \otimes \left[ \left( \frac{\alpha}{1 - \alpha} \right) W(e) \right], \\
M_v(u)^{-1} &= \left[ M_v(u) - (1 - \alpha) (J \otimes [W(e)^{-1}]) \right]^{-1} - e_v e_v^T \otimes \left[ \left( \frac{1 - \alpha}{\alpha} \right) W(e) \right].
\end{align*}
\]
Proof. Let $T$ be a tree on $n$ vertices and the weights on the edges of $T$ be $s \times s$ positive definite matrices. Then $L(T)$ is a symmetric matrix of order $n s \times n s$. Let the eigenvalues of $L(T)$ be ordered as in Equation (1.1). Thus $\mu(T) = \lambda_{s+1}(L(T))$.

If $T$ has a characteristic-like vertex, say $v$, then $\kappa(T) = 1/\rho(M_v)$. Let $L_v$ be the principal submatrix of $L(T)$ obtained by deleting the row block and column block corresponding to the vertex $v$. Let the eigenvalues of $L_v$ be ordered as in Equation (1.1). Since $L_v = M_v^{-1}$, we see that $\lambda_1(L_v) = \kappa(T)$. Using Theorem 1.4, for the principal submatrix $L_v$ of order $(n-1)s \times (n-1)s$, we have

$$0 = \lambda_1(L(T)) \leq \lambda_1(L_v) \leq \lambda_{1+ns-(n-1)s}(L(T)) = \lambda_{s+1}(L(T)).$$

Hence, $\kappa(T) \leq \mu(T)$.

If $T$ has a characteristic-like edge $e$ between the vertices $u$ and $v$, then $\exists 0 < \nu < 1$ such that

$$\kappa(T) = \frac{1}{\rho(M_u(v) - \nu(J \otimes [W(e)^{-1}]))} \leq \frac{1}{\rho(M_u(u) - (1 - \nu)(J \otimes [W(e)^{-1}]))}.$$ (4.9)

By Lemma 4.4, Equation (4.7) holds true for $\alpha = \nu$. Since the edges of $T$ are assigned with positive definite matrices, both $L(T)$ and $E \otimes W(e)$ are real symmetric matrices. Using Theorem 1.5, we have

$$\lambda_1(L(T) + E \otimes W(e)) \leq \lambda_{s+1}(L(T)) + \lambda_{(n-1)s}(E \otimes W(e)).$$ (4.10)

For $\alpha = \nu$, from Equations (4.7) and (4.9), we have $\lambda_1(L(T) + E \otimes W(e)) = \kappa(T)$. Further, note that $E$ is a rank one matrix. Since $0 < \nu < 1$, from Remark 2.1 we see that $E \otimes W(e)$ is a positive semidefinite matrix with $\text{rank}(E \otimes W(e)) = \text{rank}(W(e)) = s$. This implies that $\lambda_i(E \otimes W(e)) = 0$ for all $1 \leq i \leq (n-1)s$. Hence, Equation (4.10) reduces to $\kappa(T) \leq \mu(T)$ and this completes the proof.

We now prove the result that gives a lower bound on $\mu(T)$ whenever the edges of the tree $T$ are assigned lower triangular matrix weights with positive diagonal entries.

Theorem 4.7. Let $T = (V, E)$ be a tree such that the weights on the edges of $T$ are lower triangular matrices with positive diagonal entries. Let $L(T)$ be the Laplacian matrix of $T$ and $\mu(T)$ be the first non-zero eigenvalue of $L(T)$. Then $\kappa(T) \leq \mu(T)$.

Proof. Let the weights on the edges of $T$ be $s \times s$ lower triangular matrices with positive diagonal entries. For $1 \leq j \leq s$, let $T^{(j)}$ be the trees with positive edge weights induced by $T = (V, E)$ with $s \times s$ lower triangular matrix edge weights. For $1 \leq j \leq s$, let $L(T^{(j)})$ be the Laplacian matrix of $T^{(j)}$. Then, using Equation (3.10), we have

$$\sigma(L(T)) = \bigcup_{j=1}^{s} \sigma(L(T^{(j)})) \text{ and } \mu(T) = \min_{1 \leq j \leq s} \mu(T^{(j)}),$$

Substituting the above values in Equation (4.8), the desired result follows. □

Remark 4.5. 1. From the proofs of Lemmas 4.3 and 4.4, it is easy to see that the result also applies for any real $\alpha$, where $\alpha \neq 1$ and $\alpha \neq 0$.

2. In Lemma 4.4, the matrix $E$ is a rank one matrix, and for $0 < \alpha < 1$, its only non-zero eigenvalue is positive.
where \( \mu(T^{(j)}) \) denotes the algebraic connectivity of \( T^{(j)} \).

Without loss of generality, let us assume \( \mu(T) = \mu(T^{(1)}) \). We now consider the following cases to complete the proof.

**Case 1:** Let \( T \) have a characteristic-like vertex, say \( v \). Then, \( \kappa(T) = \frac{1}{\rho(M_v)} \).

**Subcase 1.1:** Let \( T^{(1)} \) have a characteristic vertex, say \( x \). By Proposition 1.8, there are two or more Perron branches at \( x \) in \( T^{(1)} \) and hence there exists a vertex \( y \) adjacent to \( x \) (and \( y \) is not in the path \( P(v, x) \) if \( v \neq x \)) such that \( B_x(y) \) is a Perron branch of at \( x \) in \( T^{(1)} \). Thus,

\[
B_x(y) \subseteq B_v(y) \text{ and } \rho(M_x^{(1)}) = \rho(M_x^{(1)}(y)).
\]

Using Proposition 1.8, Lemma 3.18 and Equation (3.13), we have

\[
\frac{1}{\kappa(T)} = \rho(M_v) \geq \rho(M_v(y)) \geq \rho(M_x(y)) = \rho(M_x^{(1)}(y)) = \rho(M_x^{(1)}) = \frac{1}{\mu(T^{(1)})} = \frac{1}{\mu(T)}.
\]

**Subcase 1.2:** Let \( T^{(1)} \) have a characteristic edge \( e \) between the vertices \( x \) and \( y \). Using Proposition 1.7, there exists \( 0 < \gamma < 1 \) such that

\[
\frac{1}{\mu(T^{(1)})} = \rho(M_x^{(1)}(y)) - \gamma (1/\theta)J = \rho(M_x^{(1)}(x)) - (1 - \gamma)(1/\theta)J,
\]

where \( \theta \) is the positive weight assigned to the edge \( e \) in \( T^{(1)} \). Without loss of generality, let \( y \) not be in the path \( P(v, x) \). Here \( B_x(y) \subseteq B_v(y) \). Using Lemma 3.18, Equations (3.13) and (4.11), we have

\[
\frac{1}{\kappa(T)} = \rho(M_v) \geq \rho(M_v(y)) \geq \rho(M_x(y)) \geq \rho(M_x^{(1)}(y)) > \rho(M_x^{(1)}(y)) - \gamma (1/\theta)J = \frac{1}{\mu(T^{(1)})} = \frac{1}{\mu(T)}.
\]

**Case 2:** Let \( T \) have a characteristic-like edge \( e \) between the vertices \( u \) and \( v \). For \( 0 \leq t \leq 1 \), let

\[
\begin{align*}
&f(t) = \rho(M_u(v)) - t(J \otimes [W(e)^{-1}]), \\
g(t) = \rho(M_u(v)) - (1-t)(J \otimes [W(e)^{-1}]), \\
h(t) = \min\{f(t), g(t)\}.
\end{align*}
\]

From the proof of Lemma 4.1, we know that \( f(t) \) is a continuous, decreasing function and \( g(t) \) is a continuous, increasing function. Hence, there exists \( 0 < \nu < 1 \) such that \( f(\nu) = g(\nu) \), i.e.,

\[
\rho(M_u(v) - \nu(J \otimes [W(e)^{-1}]) = \rho(M_u(v) - (1 - \nu)(J \otimes [W(e)^{-1}]),
\]

where \( W(e) \) is the matrix weight on the edge \( e \). Therefore,

\[
h(\nu) = \max_{0 \leq t \leq 1} h(t) = \max_{0 \leq t \leq 1} \min\{f(t), g(t)\} = f(\nu) = g(\nu).
\]

By definition,

\[
\kappa(T) = \frac{1}{\rho(M_u(v) - \nu(J \otimes [W(e)^{-1}]))} = \frac{1}{\rho(M_u(v) - (1 - \nu)(J \otimes [W(e)^{-1}]))},
\]

and hence by Equation (4.13), we have

\[
h(\nu) = \max_{0 \leq t \leq 1} h(t) = \max_{0 \leq t \leq 1} \min\{f(t), g(t)\} = \frac{1}{\kappa(T)}.
\]
Let \( \hat{M}_v \) denote the principal submatrix of \( M_v \) obtained by deleting the block \( M_v(u) \) (the block corresponding to the unique Perron branch \( B_v(u) \) at \( v \) in \( T \)) from \( M_v \). Then, from Equation (4.2) we see that \( \hat{M}_v = M_v(v) - J \otimes [W(e)^{-1}] \) and \( \rho(\hat{M}_v) = \rho(\hat{M}_v) \). Hence,

\[
\frac{1}{\kappa(T)} = \rho(M_v(v) - J \otimes [W(e)^{-1}]) > \rho(M_v(v) - J \otimes [W(e)^{-1}]) = \rho(\hat{M}_v). \tag{4.15}
\]

**Subcase 2.1:** Let \( T^{(1)} \) have a characteristic vertex, say \( x \). Without loss of generality, let us assume \( x \in B_u(v) \). By Proposition 1.8, there are two or more Perron branches at \( x \) in \( T^{(1)} \). Hence, there exists a vertex \( y \) adjacent to \( x \) (and \( y \) is not in the path \( P(v, x) \) if \( v \neq x \)) such that \( B_x(y) \) is a Perron branch at \( x \) in \( T^{(1)} \). Thus,

\[ B_x(y) \subseteq B_v(y) \text{ and } \rho(M_{x}^{(1)}) = \rho(M_{y}^{(1)}). \]

Note that, \( \hat{M}_v \) is a block diagonal matrix and \( M_v(y) \) is one of its blocks. Thus, \( \rho(\hat{M}_v) \geq \rho(M_v(y)) \). Hence, using Lemma 3.18, Equations (3.13) and (4.15), we have

\[
\frac{1}{\kappa(T)} \geq \rho(\hat{M}_v) \geq \rho(M_v(y)) \geq \rho(M_{x}^{(1)}(y)) = \rho(M_{y}^{(1)}). \tag{4.16}
\]

**Subcase 2.2:** Let \( T^{(1)} \) have a characteristic edge \( \hat{e} \) between the vertices \( x \) and \( y \). Thus, Equation (4.11) is valid.

Let \( e \neq \hat{e} \). Without loss of generality, let \( x, y \in B_u(v) \) and \( y \) not be in the path \( P(v, x) \) if \( v \neq x \). Here, \( B_x(y) \subseteq B_u(y) \). Using Lemma 3.18, Equations (3.13), (4.11) and (4.15), we have

\[
\frac{1}{\kappa(T)} \geq \rho(\hat{M}_v) \geq \rho(M_v(y)) \geq \rho(M_{x}^{(1)}(y)) > \rho(M_{x}^{(1)}(y) - \gamma(1/\theta)J) = \frac{1}{\mu(T^{(1)})} = \frac{1}{\mu(T)}. \tag{4.17}
\]

Let \( e = \hat{e} \). Without loss of generality, let us assume that \( u = x \) and \( v = y \). Thus, Equation (4.11) can be rewritten as

\[
\frac{1}{\mu(T^{(1)})} = \rho(M_{u}^{(1)}(v) - \gamma(1/\theta)J) = \rho(M_{v}^{(1)}(u) - (1 - \gamma)(1/\theta)J) \text{ for some } 0 < \gamma < 1. \tag{4.18}
\]

Using Equations (4.12) and (4.16), we have

\[
f(\gamma) = \rho(M_u(v) - \gamma(J \otimes [W(e)^{-1}])) \geq \rho(M_{u}^{(1)}(v) - \gamma(1/\theta)J) = \frac{1}{\mu(T^{(1)})}. \]

Similarly, \( g(\gamma) \geq \frac{1}{\mu(T^{(1)})} \). Therefore, \( h(\gamma) = \min\{f(\gamma), g(\gamma)\} \geq \frac{1}{\mu(T^{(1)})} \). Using Equation (4.14), we have

\[
\frac{1}{\kappa(T)} = h(\nu) = \max_{0 \leq t \leq 1} h(t) \geq h(\gamma) \geq \frac{1}{\mu(T^{(1)})} = \frac{1}{\mu(T)}. \tag{4.19}
\]

This completes the proof. \( \square \)

From [16], we know that the equality is attained in Theorems 4.6 and 4.7 for trees with positive edge weights, but in general, this may not be true. This is illustrated in the following examples.

**Example 4.8.** Let \( V = \{v_1, v_2, v_3, v_4, v_5, v_6\} \) and \( E = \{e_1, e_2, e_3, e_4, e_5\} \). Consider the tree \( T = (V, E) \), as shown in Figure (2a) with the matrix weights

\[
W = \left\{ W(e_1) = W(e_2) = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}, W(e_3) = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}, W(e_4) = W(e_5) = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix} \right\}.
\]
Let 
\[ W^{(1)} = \{ W(e_1) = W(e_2) = 1, W(e_3) = 10, W(e_4) = W(e_5) = 10 \}, \]
\[ W^{(2)} = \{ W(e_1) = W(e_2) = 10, W(e_3) = 10, W(e_4) = W(e_5) = 1 \}. \]

Let \( T^{(1)} = (T, W^{(1)}) \) and \( T^{(2)} = (T, W^{(2)}) \). Then, \( v_3 \) is the characteristic vertex of \( T^{(1)} \) with \( \mu(T^{(1)}) = 1 \), while \( v_4 \) is the characteristic vertex of \( T^{(2)} \) with \( \mu(T^{(2)}) = 1 \). Whereas, \( e_3 \) is the characteristic-like edge of \( T \) with \( \mu(T) = 1 \). Thus,
\[
\frac{1}{\kappa(T)} = \rho(M_{v_3}(v_4) - 0.5(J \otimes [W(e_3)^{-1}])) = \rho(M_{v_4}(v_3) - 0.5(J \otimes [W(e_3)^{-1}])) = 1.104741,
\]
and hence \( \kappa(T) < \mu(T) \).

**Example 4.9.** Let \( V = \{ v_1, v_2, v_3, v_4, v_5 \} \) and \( E = \{ e_1, e_2, e_3, e_4 \} \). Consider the tree \( T = (V, E) \), as shown in Figure (2b) with the matrix weights
\[
W = \left\{ W(e_1) = W(e_2) = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}, W(e_3) = W(e_4) = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} \right\}.
\]

Let 
\[ W^{(1)} = \{ W(e_1) = W(e_2) = 10, W(e_3) = W(e_4) = 1 \}, \]
\[ W^{(2)} = \{ W(e_1) = W(e_2) = 1, W(e_3) = W(e_4) = 10 \}. \]

Let \( T^{(1)} = (T, W^{(1)}) \) and \( T^{(2)} = (T, W^{(2)}) \). Then, \( e_3 \) is the characteristic edge of \( T^{(1)} \) with \( \mu(T^{(1)}) = 0.58963 \), while \( e_2 \) is the characteristic edge of \( T^{(2)} \) with \( \mu(T^{(2)}) = 0.58963 \). Whereas, \( v_3 \) is the characteristic-like vertex of \( T \) with \( \mu(T) = 0.58963 \). Thus, \( \frac{1}{\kappa(T)} = \rho(M_{v_3}) = 2.618034 \), and hence \( \kappa(T) < \mu(T) \).

**5 Conclusion**

In this manuscript, we have studied the Laplacian matrices for trees with matrix weights on their edges. We consider the principal submatrix \( L_v \) of the Laplacian matrices for trees with matrix weights on their edges. We first computed the determinant of \( L_v \) and proved that \( L_v \) is an invertible matrix if and only if the edge weights are nonsingular matrices. Then, we found the inverse of \( L_v \) and defined the bottleneck matrix for a branch of a tree with nonsingular matrix edge weights. In this case, we defined Perron values and Perron branches whenever the eigenvalues of \( L_v \) are nonnegative. Using \( L_v^{-1} \), we obtained the Moore-Penrose inverse of the Laplacian matrix \( L \). We then considered trees with the following classes of matrix edge weights:

1. positive definite matrix weights,
2. lower (or upper) triangular matrix weights with positive diagonal entries.

For trees with the above classes of matrix edge weights, we found that the eigenvalues of $L_v$ are nonnegative, and we have shown the existence of vertices satisfying properties analogous to the properties of characteristic vertices of trees with positive edge weights in terms of Perron values and Perron branches. We call such vertices characteristic-like vertices.

For trees with positive edge weights, it is known that the algebraic connectivity (first non-zero eigenvalue of the Laplacian matrix) can be expressed in terms of the Perron value. We attempted to find a similar relation for trees with the above class of matrix edge weights. However, here we obtained an inequality instead and hence provided a lower bound for the first non-zero eigenvalue of the Laplacian matrix.

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