Non-standard matrix formats of Lie superalgebras

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Abstract

The standard format of matrices belonging to Lie superalgebras consists of partitioning the matrices into even and odd blocks. In this paper, we study other possible matrix formats and in particular the so-called diagonal format which naturally occurs in various applications, e.g. in superconformal field theory, superintegrable models, for super $W$-algebras and quantum supergroups.

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1 Introduction

Lie superalgebras of matrices are usually introduced by considering a matrix representation of endomorphisms of a graded vector space. The standard matrix format consists of arranging the even and odd matrix elements (i.e. matrix elements associated to even and odd endomorphisms) into block form \[1\]-\[9\]. In references \[10, 11, 12\], we encountered a matrix format of the Lie superalgebras \(sl(n+1|n)\) and \(osp(2m \pm 1|2m)\) in which there are even and odd diagonals, i.e. alternatively even and odd elements in each row and column. (Particular examples of this arrangement also appeared in previous work concerning conformal field theory \[13, 14\] or quantum groups \[15\] and the infinite dimensional case was considered for studying Fock space representations in quantum field theory \[16\].) In references \[10\] and \[12\], this format was referred to as diagonal grading ‘representation’ or ‘realization’, but to be more precise we will rather refer to it as diagonal matrix format (as opposed to the block or standard matrix format \[5\]). For instance, a generic element of \(M \in osp(3|2)\) can be parametrized in the following way in block and diagonal format, respectively:

\[
M_{\text{block}} = \begin{bmatrix}
-\alpha & C & 0 & -\tau & -\mu \\
B & 0 & -C & \lambda & \beta \\
0 & -B & A & -\alpha & \varepsilon \\
\varepsilon & \beta & -\mu & i & -j \\
\alpha & -\lambda & \tau & -k & -i
\end{bmatrix},
M_{\text{diag.}} = \begin{bmatrix}
A & \alpha & B & \varepsilon & 0 \\
\mu & i & \beta & j & \varepsilon \\
C & \lambda & 0 & -\beta & B \\
\tau & k & \lambda & -i & -\alpha \\
0 & -\tau & C & \mu & -A
\end{bmatrix}.
\]

Here, the boldface entries (i.e. the Latin characters and zeros) represent the even part of the matrix which part belongs to the ordinary Lie algebra \(so(3) \oplus sp(2)\); the \(so(3)\)-submatrix of \(M_{\text{block}}\) (parametrized by capital Latin characters and zeros) is antisymmetric with respect to its antidiagonal for our choice of the metric (cf. eq.(A.1)).

In order to characterize the matrix \(M_{\text{diag.}}\) of eq.(1) and more generally the elements \(M \in osp(2m \pm 1|2m)\) in diagonal format, one labels the matrix diagonals by integer numbers, the main diagonal being counted as the 0-th one. Then, the even part of \(M\) only has entries on the even diagonals and these diagonals are alternatingly antisymmetric and symmetric with respect to the antidiagonal; the same applies to the odd diagonals \[11\]. Thus, the diagonal format leads to very simple and transparent expressions which are to be derived and discussed in the present paper.

Our discussion is based on references \[10, 11, 12\] which were devoted to various applications of Lie superalgebras to physics. In fact, the diagonal matrix format appears in a natural way in physics in the context of two-dimensional superconformal theories and superintegrable models and in particular in the study of \(W\)-superalgebras. The latter and the related super Toda theories are defined...
in terms of basic Lie superalgebras which admit an $osp(1|2)$ principal embedding\footnote{A classical Lie superalgebra is called basic if it admits a non-degenerate invariant bilinear form. While the study of the structure of ordinary Lie algebras and their classification can be done by investigating the embeddings of $sl(2)$ into these algebras, the one of Lie superalgebras can be performed by considering the embeddings of $osp(1|2)$.} \cite{17,18}: they are given by ($n \geq 1$)

\begin{align*}
& sl(n + 1|n) \quad , \quad sl(n|n + 1) \\
& osp(2n \pm 1|2n) \quad , \quad osp(2n|2n) \quad , \quad osp(2n + 2|2n) \\
& D(2,1;\alpha) \quad \text{with} \quad \alpha \neq 0,-1 .
\end{align*}

These algebras all admit a completely fermionic system of simple roots and are therefore referred to as fermionic Lie superalgebras \cite{18}. If the diagonal format is adopted, the Chevalley generators associated to these simple fermionic roots can be chosen to be matrices which only have non-vanishing entries on the first upper diagonal; e.g. in expression (1), the two Chevalley generators are parametrized by $\alpha$ and $\beta$, respectively. This choice of generators is quite analogous to the one that is generally considered for ordinary Lie algebras like $sl(n)$ because of its simplicity and convenience. We also note that the expressions for the supertrace, grading, etc. are easier to manipulate for the diagonal format than the standard one since they do not refer to a partitioning of matrices by blocks.

Our paper is organized as follows. After recalling some general notions concerning Lie superalgebras in section 2, we study their possible matrix formats for which we determine the corresponding expressions for the graded trace, commutator or transpose (sections 3 and 4). Thereafter, we focus on the diagonal format and discuss explicit results for the Lie superalgebras listed in equation (2) as well as for their infinite dimensional limit ($n \to \infty$). We conclude with some comments on the graded structure of the root space of $sl(n|m)$.

## 2 Lie superalgebras

In this section, we summarize some basic facts about Lie superalgebras \cite{1}-\cite{9} and we recall some definitions concerning fermionic Lie superalgebras \cite{18, 9}. No reference will be made to matrix representations which will only be discussed in the subsequent sections.

### 2.1 Generalities

We consider finite-dimensional Lie superalgebras over the field $\mathbb{C}$ of complex numbers. (The generalization from complex numbers to Grassmann numbers considered in some of the references \cite{1}-\cite{9} is fairly straightforward and will not be discussed here.) \textit{Graded} will always mean $\mathbb{Z}_2$-graded and the elements of $\mathbb{Z}_2$ will be denoted by $\bar{0}$ and $\bar{1}$.
Consider a graded, complex vector space $V = V_0 \oplus V_1$ with $\dim V_0 = n$, $\dim V_1 = m$ and $n \geq 1, m \geq 1$ finite. The elements of $V_0$ and $V_1$ (i.e. the homogeneous elements of $V$) are referred to as even and odd, respectively. This grading of $V$ amounts to the definition of an involution operator on $V$ (which is also referred to as the parity automorphism): this map is defined for the homogeneous elements of $V$ by

$$\epsilon : V \rightarrow V$$

$$v \mapsto \epsilon(v) = (-1)^{\deg v} v \quad \text{with} \quad \deg v = \begin{cases} 0 & \text{if } v \in V_0 \\ 1 & \text{if } v \in V_1 \end{cases}$$

and extended to the whole of $V$ by linearity. It satisfies $\epsilon^2 = \text{id}_V$.

A graded endomorphism $M$ of $V$ is an endomorphism which is compatible with the grading of $V$, i.e. for every $k \in \{0, 1\}$, we have

$$MV_k \subseteq V_k \quad \text{or} \quad MV_k \subseteq V_k + \bar{1}.$$ 

To these endomorphisms one assigns parities $\deg M = 0$ and $\deg M = 1$, respectively, and refers to them as even and odd. The composition of any two endomorphisms $M$ and $N$ is denoted by $MN$.

The grading of $V$ induces a similar grading of the vector space $\text{End} V$ of all endomorphisms of $V$: $\text{End} V = \text{End}_0 V \oplus \text{End}_1 V$ where $\text{End}_0 V$ and $\text{End}_1 V$ denote the vector spaces of even and odd endomorphisms, respectively (i.e. the graded endomorphisms are the homogeneous elements of $\text{End} V$). The grading corresponds to an involution operator which is defined by

$$\text{Ad}_\epsilon : \text{End} V \rightarrow \text{End} V$$

$$M \mapsto \text{Ad}_\epsilon M = \epsilon M \epsilon^{-1}.$$ 

It changes the signs of all odd elements of $\text{End} V$ and leaves the even elements invariant, henceforth $(\text{Ad}_\epsilon)^2 = \text{id}_{\text{End} V}$.

The graded commutator is introduced for graded endomorphisms $M, N$ by

$$\{M, N\} = MN - (-1)^{(\deg M)(\deg N)} NM$$

and extended to the other endomorphisms by bilinearity.

The supertrace of $M \in \text{End} V$ is defined by

$$\text{str} : \text{End} V \rightarrow \mathbb{C}$$

$$M \mapsto \text{str} M = \text{tr} \epsilon M,$$

where the ordinary trace ‘$\text{tr}$’ is defined by means of an arbitrary matrix realization. (For a coordinate-free definition, we refer to [3, 4].) The homomorphism (6) has the fundamental property

$$\text{str} \{M, N\} = 0.$$
By definition, the general linear Lie superalgebra $\text{gl}(n|m)$ is the graded vector space $\text{End} \, V$ together with the graded commutator (5). The Lie superalgebra $\text{sl}(n|m)$ consists of all elements of $\text{gl}(n|m)$ with vanishing supertrace.

The definition of the subalgebras of $\text{sl}(n|m)$ refers to the notion of supertransposition. The supertranspose $M^*$ of a graded endomorphism $M \in \text{End} \, V$ is defined by its action on homogeneous elements $\omega \in V^*$ ($V^*$ being the graded vector space which is dual to $V$):

$$
(M^*(\omega))(v) = (-1)^{(\deg M)(\deg \omega)} \omega(M(v)) \quad \text{for all } v \in V. \quad (8)
$$

### 2.2 On the structure of fermionic Lie superalgebras

#### 2.2.1 Chevalley basis

For any basic Lie superalgebra $\mathcal{G}$, there exists a unique (up to a constant factor) bilinear form $\langle \cdot , \cdot \rangle$ on $\mathcal{G}$ that has the following properties [1, 9]: it is consistent, non-degenerate, graded symmetric and invariant. For all fermionic Lie superalgebras except $\text{osp}(2n + 2|2n)$ and $D(2,1;\alpha)$, this form coincides with a multiple of the Killing form, i.e. $\langle M, N \rangle = \text{str}(\text{ad}_M \, \text{ad}_N)$.

Consider a Cartan subalgebra $H$ of $\mathcal{G}$ and a system of simple roots with respect to $H$. For fermionic Lie superalgebras to which we restrict our attention here, the simple root system can be chosen as completely fermionic, i.e. the endomorphisms associated to these roots can all be chosen as odd. We will come back to other possible choices in the concluding section.

For $i = 1, \ldots, \text{rank} \, \mathcal{G}$, let $h_i$ parametrize a basis of the Cartan subalgebra and let $e_i, f_i$ denote odd endomorphisms associated to simple roots and their negatives, respectively. A Chevalley basis of $\mathcal{G}$ consists of a set of generators $h_i, e_i, f_i$ which satisfy the commutation relations

$$
[h_i, h_j] = 0 \quad , \quad [h_i, e_j] = +a_{ij}e_j \\
\{e_i, f_j\} = \delta_{ij}h_j \quad , \quad [h_i, f_j] = -a_{ij}f_j ,
$$

(9)

where $a_{ij} \equiv \langle h_i, h_j \rangle / \langle e_j, f_j \rangle$ defines the elements of the Cartan matrix and where $\langle e_i, f_j \rangle = 0$ for $i \neq j$. The endomorphisms $h_i$ and $e_i, f_i$ are referred to as Cartan and Chevalley generators, respectively.

By the rescaling

$$
\begin{align*}
& e'_i = \alpha_i e_i \quad , \quad f'_i = \beta_i f_i \quad , \quad h'_i = \alpha_i \beta_i h_i \quad \text{with } \alpha_i \beta_i \equiv \langle e_i, f_i \rangle^{-1} ,
\end{align*}
$$

one can always achieve that the Cartan matrix is symmetric and given by $a'_{ij} = \langle h'_i, h'_j \rangle$. We note that there are other transformations which preserve the defining

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2 A bilinear form $\langle \cdot , \cdot \rangle$ on $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ is called consistent if $\langle X, Y \rangle = 0$ for all $X \in \mathcal{G}_0$ and $Y \in \mathcal{G}_1$. 

---
relations (9) of the Chevalley basis, e.g. the rescaling \((h_i, e_i, f_i) \to (h_i, qe_i, \frac{1}{q} f_i)\) with \(q \in \mathbb{C}^*\) or the graded extension of the Chevalley involution (19, 4):

\[
\sigma(h_i) = -h_i \quad \sigma(e_i) = -f_i \quad \sigma(f_i) = e_i \quad (\sigma^4 = \text{id}) .
\]

### 2.2.2 Principal \(osp(1|2)\)-embedding

An embedding of the Lie superalgebra \(osp(1|2)\) into \(sl(n|n + 1)\) is a non-trivial homomorphism from \(osp(1|2)\) into \(sl(n|n + 1)\). The principal embedding which is denoted by \(osp(1|2)_\text{pal} \subset sl(n + 1|n)\) and defined up to conjugation, is distinguished by the fact that it leads to the so-called principal gradation (19); it is explicitly given by (18, 4, 17)

\[
J_- := \sum_i f_i \quad J_+ := \sum_{i,j} a^{ij} e_i \quad H := \{J_+, J_-\} = \sum_{i,j} a^{ij} h_i ,
\]

(10)

where the \(a^{ij}\) denote the elements of the inverse Cartan matrix. Together with the bosonic generators \(X_\pm := \frac{1}{2}\{J_\pm, J_\pm\}\), the endomorphisms (11) define a basis of the Lie superalgebra \(osp(1|2)\): in fact, the commutation relations \([H, e_i] = e_i\) and \([H, f_i] = -f_i\) imply

\[
\begin{align*}
[H, J_\pm] & = \pm J_\pm \quad \{J_+, J_-\} = H, \\
[H, X_\pm] & = \pm 2X_\pm \quad [X_+, X_-] = -H, \quad [J_+, X_\pm] = \pm J_\pm .
\end{align*}
\]

### 3 \(gl(n|m)\) in standard matrix format

#### 3.1 Definition

Consider a basis \(\{e_1, ..., e_n\}\) of \(V_0\) and a basis \(\{e_{n+1}, ..., e_{n+m}\}\) of \(V_1\). Choose an ordering \((e_1, ..., e_n)\) of the first basis and an ordering \((e_{n+1}, ..., e_{n+m})\) of the second one. Any vector \(v \in V\) can be expanded with respect to the ordered homogeneous basis \((e_1, ..., e_n, e_{n+1}, ..., e_{n+m})\) of \(V\):

\[
v = v_1 e_1 + \ldots + v_{n+m} e_{n+m} .
\]

The associated column of complex numbers will be denoted by the same symbol: \(v = [v_1, ..., v_{n+m}]^T\). (Here, the superscript ‘\(T\)’ denotes the ordinary transpose.)

Similarly, an endomorphism \(M\) of \(V\) can be expanded with respect to the ordered homogeneous basis \((e_1, ..., e_{n+m})\):

\[
M(e_j) \equiv \sum_{i=1}^{n+m} M_{ij} e_i \equiv \begin{cases} 
\sum_{i=1}^n A_{ij} e_i + \sum_{i=n+1}^{n+m} B_{ij} e_i & \text{for } j \leq n \\
\sum_{i=1}^n C_{ij} e_i + \sum_{i=n+1}^{n+m} D_{ij} e_i & \text{for } j > n .
\end{cases}
\]

3The basis vectors \(e_1, ..., e_{n+m}\) of the vector space \(V\) should not be mixed up with the Chevalley generators (introduced in the previous section and further discussed in section 6) which are endomorphisms of \(V\).
Thus, with respect to the given basis, the endomorphism $M$ corresponds to an $(n+m) \times (n+m)$ matrix with complex components which we also denote by $M$:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} + \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}.$$ \hspace{1cm} (12)

Here, the matrix involving $A_{n \times n}$ and $D_{m \times m}$ defines the even part of $M$ and the one involving $B_{n \times m}$ and $C_{m \times n}$ its odd part.

Thus, the grading of vectors $v \in V$ and the induced grading of endomorphisms $M \in \text{End} \, V$ can be implemented by even and odd blocks for the representative column vectors and matrices. This format of column vectors and matrices is called the block or standard format \cite{5}; it is defined up to even changes of basis - see section 3.3 below.

From now on, $v$ and $M$ will always denote the representative column vectors and matrices, respectively, and not the abstract vectors and endomorphisms discussed in the previous section.

### 3.2 Some explicit expressions

For convenience, one often says that the component $v_i$ of a vector is even or odd if the vector all of whose entries but $v_i$ are zero, is even or odd. (Of course, in the present context, this is an abuse of terminology since $v_i$ simply represents a complex number. However, $v_i$ has a Grassmann parity if one considers the tensor product of the complex vector space $V$ with a Grassmann algebra as often done in physics.) Similarly the element $M_{ij}$ of a matrix $M$ is said to be even or odd if it belongs to the even or odd part of the matrix. In this sense, the grading of the vector and matrix components is given in terms of the parity $\alpha(i)$ of the label $i$,

$$\alpha(i) := \deg e_i \quad \text{for} \quad i \in \{1, \ldots, n+m\},$$

by

$$\deg v_i = \alpha(i) \quad \text{deg} \ M_{ij} = \alpha(i) + \alpha(j) \mod 2.$$ \hspace{1cm} (13)

For the standard matrix format, the parity automorphism $\epsilon$ is represented by

$$\epsilon = \begin{bmatrix} 1_n & 0 \\ 0 & -1_m \end{bmatrix},$$ \hspace{1cm} (14)

the supertrace takes the explicit form

$$\text{str} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \text{tr} \, A - \text{tr} \, D \hspace{1cm} (15)$$
or
\[ \text{str } M = \sum_{i=1}^{n} M_{ii} - \sum_{i=n+1}^{n+m} M_{ii} = \sum_{i=1}^{n+m} (-1)^{\alpha(i)} M_{ii} \quad . \] (16)

The automorphism \( \text{Ad}_\epsilon \) of \( \text{End } V \) reads
\[ \text{Ad}_\epsilon M = \epsilon M \epsilon^{-1} = \begin{bmatrix} A & -B \\ -C & D \end{bmatrix} \quad , \] (17)
and thus amounts to changing the signs of all odd entries of \( M \).

From the coordinate-free definition (8) of the supertranspose \( M^* \) of an endomorphism \( M \), one can easily deduce a matrix expression (8). However, the usual definition of the supertranspose \( M^{sT} \) of a matrix \( M \) differs from this expression by the automorphism \( \text{Ad}_\epsilon \), i.e. \( M^{sT} = \text{Ad}_\epsilon M^* \); more explicitly,
\[ M^{sT} = \begin{bmatrix} A^T & C^T \\ -B^T & D^T \end{bmatrix} \] (18)
or
\[ (M^{sT})_{ij} = M_{ji} (-1)^{\alpha(i)(\alpha(j)+1)} \quad . \] (19)
This operation satisfies
\[ (M^{sT})^{sT} = \text{Ad}_\epsilon M \] (20)
and, for the homogeneous elements,
\[ (MN)^{sT} = (-1)^{(\deg M)(\deg N)} N^{sT} M^{sT} \quad , \] (21)
which relation implies
\[ [M, N]^{sT} = -[M^{sT}, N^{sT}] \quad . \] (22)

In summary, the standard or block format of \( \mathfrak{gl}(n|m) \) consists of matrices (12), the grading, supercommutator, supertrace and supertranspose being defined by eqs.(14),(5), (15) and (18), respectively.

### 3.3 Formats and changes of formats

In order to define matrix formats in general terms, we consider once more the standard format for column vectors and matrices discussed in the previous section. Under a change of basis, the vector \( v \) transforms according to \( v \mapsto v' = Fv \) where \( F \) denotes a non-singular matrix (i.e. the superdeterminant of \( F \) does not vanish). The induced change of matrices is a similarity transformation, \( M \mapsto M' = FMF^{-1} \), in particular \( \epsilon \mapsto \epsilon' = F\epsilon F^{-1} \).

Obviously, the only changes of basis which do not modify the partitioning of matrices into even and odd blocks are separate changes of basis in \( V_0 \) and \( V_1 \).
other words, the block format is preserved by non-singular matrices \( F \) which are even in the sense of block format matrices (and only by these).

Henceforth, we define a matrix format to be a choice of ordered homogeneous basis of the graded vector space \( V \), modulo even changes of basis. For each format, the involution matrix is diagonal with entries \( \pm 1 \) and an even change of basis leaves this matrix invariant (i.e. \( \epsilon' = \epsilon \)).

A generic change of basis (e.g. leading to a non homogeneous basis) leads to another matrix realization, but in general not to another matrix format. In the following, we will study format changing transformations, i.e. changes of basis which relate different matrix formats. They simply permute the diagonal elements of the involution matrix.

4 Passage from the block format of \( gl(n|m) \) to other formats

Starting from the block format, we consider a different ordering for the homogeneous basis \( \{ e_1, \ldots, e_n \} \cup \{ e_{n+1}, \ldots, e_{n+m} \} \) of \( V \): instead of arranging the even and odd components of a vector into blocks, we arrange them in a different way by considering a permutation \( P \) of the ordered set \( (1, \ldots, n+m) \) which mixes \( (1, \ldots, n) \) and \( (n+1, \ldots, n+m) \): in matrix notation, this linear transformation of vectors reads as

\[
v = [v_1, v_2, \ldots, v_{n+m}]^T \xrightarrow{F} v' = [v_{P(1)}, v_{P(2)}, \ldots, v_{P(n+m)}]^T
\]

or

\[
v' = Fv \quad \text{with} \quad F_{ij} = \delta_{P(i)j}.
\]

The mapping \( F \) (which satisfies \( F^{-1} = F^T \)) represents the simplest example of a format changing transformation. It induces the following similarity transformation for the matrices (12):

\[
M' = FMF^{-1} \quad \text{i.e.} \quad M'_{ij} = M_{P(i)P(j)}.
\]

The involution \( \epsilon' = F\epsilon F^{-1} \) differs from \( \epsilon \) and the grading of \( M' \) is related to the one of \( M \) by

\[
\deg M'_{ij} = \deg M_{P(i)P(j)}.
\]

Since the trace is coordinate-independent, we have \( \tr \epsilon M = \tr \epsilon' M' \), henceforth

\[
\str M' = \tr \epsilon' M'.
\]

According to equation (25), the principal diagonal elements remain principal diagonal elements and therefore this expression still represents a sum of such elements with certain signs.
The supercommutator of the matrices $M'$ and $N'$ is related to the one of $M$ and $N$ by
\[ [M', N'] := F[M, N]F^{-1} \] (28)
and the supertranspose of $M'$ is defined by
\[ (M')^{sT} := FM^{sT}F^{-1} . \] (29)

Just as for matrices in block format, the format of the matrices $M'$ is preserved by non-singular endomorphisms which are even in the sense of the new format.

5 $\text{gl}(n + 1|n)$ in diagonal format

Consider $\dim V_0 = n + 1$ and $\dim V_1 = n$ with $n \geq 1$. The even and odd components of a column vector $[v_1, \ldots, v_{2n+1}]^T$ in block format can be rearranged in an alternating way (even/odd/even/odd/...). The simplest such arrangement is the one which respects the order:
\[ v = [v_1, \ldots, v_{n+1}, v_{n+2}, \ldots, v_{2n+1}]^T \overset{F}{\rightarrow} v' = [v_1, v_{n+2}, v_2, v_{n+3}, \ldots, v_{2n+1}, v_{n+1}]^T . \] (30)

In terms of the notation (23)(24), this transformation corresponds to the permutation
\[ P(2i + 1) = i + 1 \quad \text{for } 0 \leq i \leq n \]
\[ P(2i) = n + i + 1 \quad \text{for } 1 \leq i \leq n . \]

The matrix $M'$ given by eq.(25) then represents a $(2n + 1) \times (2n + 1)$ matrix with alternating even and odd diagonals (alternating even and odd elements in the rows and columns). The different diagonals are labeled by $i$ where $-n \leq i \leq n$ and where the main diagonal is counted as the 0-th one; a matrix which has entries $a_1, a_2, \ldots$ along the $i$-th diagonal, all other entries being zero, is denoted by $\text{diag}_i(a_1, a_2, \ldots)$. We will say for short that such a matrix only has entries on the $i$-th diagonal. We have the explicit expressions
\[ \deg M'_{ij} = i + j \mod 2 \]
\[ e' = \text{diag}_0 (1, -1, 1, -1, \ldots, 1) \]
\[ \text{str } M' = \sum_{i=1}^{2n+1} (-1)^{i+1} M'_{ii} \] (31)
\[ [M', N']_{ik} = \sum_{j=1}^{2n+1} \left( M'_{ij}N'_{jk} - (-1)^{(i+j)(j+k)} N'_{ij}M'_{jk} \right) \]
\[ (M'^{sT})_{ij} = (-1)^{(i+j)} M'_{ji} . \]
This format of representative column vectors and matrices will be called the **diagonal format**. Thus, \( gl(n + 1|n) \) in diagonal format is the Lie superalgebra of complex \((2n + 1) \times (2n + 1)\) matrices with a grading, super-commutator, -trace and -transpose defined by eqs. (31). The non-singular matrices all of whose odd diagonals vanish are those endomorphisms which preserve the diagonal format - see next section for examples.

Note that these expressions (31) are easier to manipulate than those for the block format since the sign factors refer directly to the indices rather than to their parity, e.g.

\[
\epsilon'_{ij} = (-1)^{i+1} \delta_{ij}, \quad (M')^{sT}_{ij} = (-1)^{i+j} M'_{ij} = (\text{Ad}_{e'} M')_{ij}.
\]

Obviously, the alternating partitioning can be performed in a similar way in the case of the Lie superalgebras \( gl(n|n + 1) \) or \( gl(n|n) \).

### 6 Fermionic superalgebras in diagonal format

In this section, we discuss the diagonal format of different subalgebras of \( gl(n + 1|n) \), namely of \( sl(n + 1|n) \) and of the orthosymplectic algebras \( osp(2m \pm 1|2m) \) with \( 4m \pm 1 = 2n + 1 \). We do so by elaborating on the expressions encountered in the study of \( W \) superalgebras in reference [10].

#### 6.1 \( sl(n + 1|n) \)

A general element of \( sl(n + 1|n) \) in diagonal format is simply a \((2n + 1) \times (2n + 1)\) matrix of numbers together with an even (odd) grading for the even (odd) diagonals, the only restriction being the vanishing of the supertrace.

The Cartan matrix elements \( a_{ij} \) of \( sl(n + 1|n) \) are independent of the chosen format and define a \( 2n \times 2n \) matrix. If all simple roots are chosen to be fermionic, this matrix reads as [24]

\[
(a_{ij}) = \sum_{k=\pm 1} \text{diag}_k (1, -1, 1 - 1, \ldots)
\]

and its inverse

\[
(a^{ij}) = \sum_{k=\pm 1,\pm 3,\ldots} \text{diag}_k (1, 0, 1, 0, \ldots)
\]

has the property

\[
\sum_j a^{ij} = \begin{cases} i/2 & \text{for } i \text{ even} \\ n - i - 1/2 & \text{for } i \text{ odd} \end{cases}
\]  

(32)
Let $E_{ij}$ denote the matrix with elements $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. As Cartan and Chevalley generators of $sl(n+1|n)$, we can take the matrices $h_i = (-1)^{i+1}(E_{ii} + E_{i+1,i+1})$, $e_i = (-1)^{i+1}E_{i,i+1}$, and $f_i = E_{i+1,i}$, i.e. the Cartan generator $h_i \equiv \{e_i, f_i\}$ only has entries on the main diagonal while $e_i$ and $f_i$ only have entries on the first upper and lower diagonals, respectively.

According to equations (10) and (32), the principal embedding $osp(1|2)_{pal} \subset sl(n+1|n)$ is generated by the $(2n+1) \times (2n+1)$ matrices

\[
J_- = \text{diag}_{-1}(1, \ldots, 1) \\
J_+ = \text{diag}_{+1}(n, -1, n - 1, -2, \ldots, -n) \\
H = \text{diag}_{0}(n, n - 1, \ldots, -n)
\]

which are, respectively, symmetric and antisymmetric with respect to their antidiagonal.

**Highest weights** The highest weight generators of $osp(1|2)_{pal}$ are defined as those $M_k \in sl(n+1|n)$ (with $k = 1, \ldots, 2n$) which satisfy

\[
[H, M_k] = kM_k \quad \text{and} \quad [J_+, M_k] = 0.
\]

For the solutions of these conditions one finds $M_k = M_1^k$ with

\[
M_1 = \text{diag}_{+1}(1, 1, n - 1, 2, \ldots, n) \quad \text{for} \quad osp(2m + 1|2m)
\]

\[
M_1 = \text{diag}_{+1}(1, 1, n - 1, 2, \ldots, n) \quad \text{for} \quad osp(2m - 1|2m)
\]

Thus, $M_k$ is a matrix which only has entries on the $k$-th diagonal and which is symmetric with respect to the antidiagonal.

### 6.2 The subalgebras $osp(2m \pm 1|2m)$

Consider the graded algebras $osp(2m \pm 1|2m) \subset sl(n+1|n)$ with $4m \pm 1 = 2n + 1$. Their rank is $2m$ for $osp(2m + 1|2m)$ and $2m - 1$ for $osp(2m - 1|2m)$.

By definition, the superalgebra $osp(2m \pm 1|2m)$ consists of supermatrices $M$ which satisfy $M^{\dagger} G + GM = 0$ where the supermetric $G = G_{so(2m \pm 1)} \oplus G_{sp(2m)}$ is a direct sum of $so(2m \pm 1)$ and $sp(2m)$ metrics. The latter represent symmetric, respectively antisymmetric, non-degenerate bilinear forms on a complex vector space.

In a basis of the root space with only fermionic simple roots, the algebras $osp(2m \pm 1|2m)$ are characterized by Cartan matrices whose only non-zero elements are $a_{ii} = 1$, $a_{i,i+1} = a_{i+1,i} = (-1)^i$ with

\[
a_{11} = 1, \quad a_{i,i+1} = a_{i+1,i} = (-1)^i \quad \text{with} \quad \begin{cases}
    i = 1, \ldots, 2m - 2 & \text{for} \quad osp(2m - 1|2m) \\
    i = 1, \ldots, 2m - 1 & \text{for} \quad osp(2m + 1|2m).
\end{cases}
\]

(37)
The inverse matrix for \( osp(2m - 1|2m) \) is given by
\[
(a^{ij}) = \sum_{k=0,\pm 2,\ldots} \text{diag}_k(1,0,1,0,\ldots) + \sum_{k=\pm 1,\pm 3,\ldots} \text{diag}_k(0,1,0,1,\ldots)
\]
and for \( osp(2m + 1|2m) \) it reads
\[
(a^{ij}) = \sum_{k=0,\pm 2,\ldots} \text{diag}_k(0,-1,0,-1,\ldots) + \sum_{k=\pm 1,\pm 3,\ldots} \text{diag}_k(-1,0,-1,0,\ldots) .
\]

We will now discuss the two cases \( osp(2m - 1|2m) \) and \( osp(2m + 1|2m) \) in turn. Their block format is summarized in appendix A. The transition between matrices in block and diagonal format is mediated by a similarity transformation, \( M_{\text{diag.}} = L^{-1} M_{\text{block}} L \). We will not choose the matrix \( L \) according to the simplest alternating arrangement as given by equation (30), but rather according to a more complicated arrangement which leads to diagonal format and which allows us to recover the expression (34) for the generators of \( osp(1|2)_{\text{pal}} \). This requirement determines the form of the Chevalley generators of \( osp(2m \pm 1|2m) \), see below. By expanding with respect to the corresponding Chevalley basis, it follows that a general \( osp(2m \pm 1|2m) \) element in diagonal format is a \((4m \pm 1) \times (4m \pm 1)\) matrix for which the even and odd elements satisfy the following conditions, respectively:
\[
M_{i,i+2k} = (-1)^{k+1} M_{p+1-i-2k,p+1-i} \\
M_{i,i+2k+1} = (-1)^{k+1} M_{p-i-2k,p+1-i}
\]
(\( p = 4m \pm 1 \), \( i \in \{1,\ldots,p\} \), \( k \in \{0,\pm 1,\ldots\} \)).

Thus, the even diagonals of \( M \) are alternatingly antisymmetric and symmetric with respect to the antidiagonal and so are the odd diagonals, e.g. see expression (1) for \( M \in osp(3|2) \).

6.2.1 \( osp(2m - 1|2m) \)

The choice of \( L \) that leads to the expression (34) for the generators of \( osp(1|2)_{\text{pal}} \) is
\[
L = \sum_{i=0}^{m-1} (-1)^{i+1} E_{2m+i,2i+1} + \sum_{i=1}^{m} (-1)^{i} E_{2m-i,2i} \\
+ (-1)^{m+1} \sum_{i=1}^{m} E_{4m-i,2m+2i-1} + (-1)^{m} \sum_{i=1}^{m-1} E_{m-i,2m+2i}
\]
(39)

which matrix satisfies \( L^{-1} = L^T \). This endomorphism describes a permutation of basis vectors together with some changes of signs and thus defines a format changing transformation followed by a format preserving endomorphism. It leads to diagonal format with an involution given by \( \epsilon_{ij} = (-1)^i \delta_{ij} \), i.e. the diagonal
format matrices have even and odd elements on the even and odd diagonals, respectively.

The Chevalley generators can be represented as \((i = 1, \ldots, 2m - 1)\)

\[
\begin{align*}
    h_i & = (-1)^{i+1}(E_{2m-i,2m-i} - E_{2m+i,2m+i} + E_{2m+1-i,2m+1-i} - E_{2m-1+i,2m-1+i}) \\
    e_i & = (-1)^i(E_{2m-1+i,2m+i} - E_{2m+i,2m+1-i}) \\
    f_i & = E_{2m+i,2m-1+i} + E_{2m+1-i,2m-i}
\end{align*}
\] (40)

i.e. the Cartan generators only have entries on the main diagonal while \(e_i (f_i)\) only has entries on the first upper (lower) diagonal.

The metric matrix takes the form

\[
G = (-1)^m \sum_{i=1}^{4m-1} (-1)^{[i+1/2]} E_{i,4m-i} = (-1)^m \text{adiag}(1, -1, -1, 1, -1, -1, \ldots),
\] (41)

where \([i]\) denotes the integer part of \(i\) and where \(\text{adiag}(a_1, a_2, \ldots)\) represents a matrix which only has entries \(a_1, a_2, \ldots\) on the antidiagonal. The fact that the factors 1 and \(-1\) occur by pairs in the metric (41), reflects itself in the matrices (38) of \(osp(2m-1|2m)\) by the fact that the symmetric and antisymmetric diagonals occur by pairs (cf. eq.(1) for the \(osp(2m+1|2m)\) elements which have the same symmetry properties).

For the even and odd elements of \(M^{sT}\), we find

\[
\begin{align*}
    (M^{sT})_{i,i+2k} & = M_{i+2k,i} \\
    (M^{sT})_{i,i+2k+1} & = (-1)^i M_{i+2k+1,i} \\
    (i & \in \{1, \ldots, 4m-1\}, \; k \in \{0, \pm 1, \ldots\})
\end{align*}
\] (42)

i.e. supertransposition coincides with ordinary transposition apart from the fact that one also has to change the signs on all odd diagonals in an alternating way.

6.2.2 \(osp(2m+1|2m)\)

The results have the same form and characteristics as those for \(osp(2m-1|2m)\) and therefore we will only list the expressions.

- Format changing matrix \((L^{-1} = L^T)\) :

\[
L = \sum_{i=0}^{m} (-1)^i E_{2m+1-i,2i+1} + \sum_{i=1}^{m} \left[ (-1)^i E_{2m+1+i,2i} \\
+ (-1)^{m+1} E_{4m+2-i,2m+2i} + (-1)^m E_{m+1-i,2m+1+2i} \right]
\] (43)

- Involution :

\[
\epsilon_{ij} = (-1)^{i+1} \delta_{ij}
\] (44)
- Cartan-Chevalley generators \((i = 1, \ldots, 2m)\):

\[
\begin{align*}
  h_i &= (-1)^{i+1} \left( E_{2m+1-i,2m+1-i} - E_{2m+1-i,2m+1+i} + E_{2m-1-i,2m+1-i} - E_{2m+i,2m+i} \right) \\
  e_i &= (-1)^i \left( E_{2m+i,2m+1+i} - E_{2m+i,2m+1-i} \right) \\
  f_i &= E_{2m+1+i,2m+i} + E_{2m+2-i,2m+1-i}
\end{align*}
\] (45)

- Supermetric:

\[ G = (-1)^m \sum_{i=1}^{4m+1} (-1)^{\frac{i}{2}} E_{i,4m+2-i} = (-1)^m \text{adig}(1,1,-1,-1,1,1,...) \] (46)

- Supertranspose:

\[
\begin{align*}
  (M^{sT})_{i,i+2k} &= M_{i+2k,i} \\
  (M^{sT})_{i,i+2k+1} &= (-1)^{i+1} M_{i+2k+1,i} \\
  & \quad (i \in \{1, \ldots, 4m+1\}, \ k \in \{0, \pm 1, \ldots\})
\end{align*}
\] (47)

**Principal \(osp(1|2)\)-embedding and highest weights** The basis (40) and (45) have been chosen such as to lead to the same explicit expressions for the principal embedding \(osp(1|2)_{pal} \subset osp(2m \pm 1|2m)\) as in the \(sl(n+1|n)\) case, see equation (34). The highest weight generators are also the same, except that we have to exclude those \(M_k\) which do not belong to \(osp(2m \pm 1|2m)\). This leaves us with \(M_{2+4q}\) and \(M_{3+4q}\) where \(q = 0, 1, \ldots\)

### 6.3 Infinite dimensional limit

For \(n \to \infty\) and \(m \to \infty\), the algebras \(sl(n+1|n)\) and \(osp(2m \pm 1|2m)\) of diagonal format matrices give rise to infinite dimensional Lie superalgebras denoted by \(sl_{\infty}\) and \(osp_{\infty}\), respectively. Similar (and closely related) algebras occur in the study of superintegrable models (KP-hierarchy,...) and have been introduced and discussed in references \[16\].

In order to derive infinite dimensional algebras from our expressions, we label the matrix entries \(M_{ij}\) by \(i, j \in \mathbb{Z}\) and consider the diagonal format grading defined by \(\text{deg} M_{ij} = i + j\). Then, expressions (33) with \(i \in \mathbb{Z}\) represent a Chevalley basis of \(sl_{\infty}\). A basis of \(osp_{\infty}\) is given by the matrices

\[ E_{i,j} - (-1)^{\lfloor k/2 \rfloor} E_{-j,-i}, \] (48)

where \(\lfloor k \rfloor\) denotes the smallest integer greater or equal to \(k\). (This result simply amounts to enlarging \(ad infinitum\) the matrix \(M_{\text{diag}}\) of eq.(1).) A Chevalley basis of \(osp_{\infty}\) reads \((i = 0, 1, 2, ...)\)

\[
\begin{align*}
  e_i &= E_{i,i+1} - E_{-i-1,-i} \\
  f_i &= E_{i+1,i} + E_{-i,-i-1} \\
  h_i &= E_{i+1,i+1} + E_{i,i} - E_{-i,-i} - E_{-i-1,-i-1}
\end{align*}
\] (49)

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These results are equivalent to those given and applied in references [13].

7 Concluding remarks

In the previous section, we have not discussed all the fermionic Lie superalgebras listed in eq.(3). For osp(2m|2m), the diagonal format can be considered, but due to the fact that 2m + 2m \neq 2n + 1, it is not possible to recover the expressions (3.4) for the generators of osp(1|2) \subset sl(n + 1|n) (as one might wish for the applications to W-superalgebras). For osp(2m + 2|2m), alternating formats can also be introduced for vectors and matrices; the simplest and most symmetric such arrangement is obtained by putting the two extra even entries in the middle, e.g. for the involution of osp(6|4):

$$\epsilon = \text{diag}_0 (1, -1, 1, -1, 1, -1, 1, -1, 1) .$$

The matrices of osp(2m + 2|2m) in this format then have a symmetry structure which is quite similar to the one for the diagonal format of osp(2m ± 1|2m).

We note that one can also introduce the superprincipal embedding of sl(2|1) in sl(n + 1|n) and derive from it the one of osp(1|2): osp(1|2) \subset sl(2|1) \subset sl(n + 1|n). Explicit formulas have been given in references [18, 9] and diagonal format expressions have been applied to superintegrable models and W-algebras in [11, 12].

Among the various inequivalent simple root systems (SRS’s) of a basic Lie superalgebra, there exists a canonical one, the so-called distinguished SRS [9, 1]: it is characterized by the fact that it contains the smallest possible number of odd roots, e.g. for sl(n + 1|n), this system only contains one odd root. The so-called fermionic SRS (which exists for the Lie superalgebras listed in eq.(4) and which has been considered in the present paper) represents the other extreme where all simple roots are chosen to be odd. Both SRS’s can be related by a generalized Weyl transformation [9]. In order to get a better understanding of the graded structure of the root space of sl(n + 1|n) (which is of dimension 2n), we consider sl(3|2) as an example. To start with, we do not specify the matrix format and we assume that the generators \(e_i\) associated to a SRS only have entries on the first upper diagonal, i.e. \(e_i = E_{i,i+1}\) with \(i = 1, 2, 3, 4\). If the block matrix format is considered, one of the generators is odd (namely \(e_3 = E_{3,4}\)) while all the others are even. A generic element of the Cartan subalgebra is then parametrized by \(h_{\text{block}} = \text{diag}_0 (\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_1, \delta_2)\) with \(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \delta_1 + \delta_2\) and from the eigenvalue equation

$$[h, e_i] = \alpha_i e_i$$

we obtain the distinguished SRS of sl(3|2):

$$\alpha_1 = \varepsilon_1 - \varepsilon_2 , \quad \alpha_2 = \varepsilon_2 - \varepsilon_3 , \quad \alpha_3 = \varepsilon_3 - \delta_1 , \quad \alpha_4 = \delta_1 - \delta_2 .$$
On the other hand, if the diagonal matrix format is chosen, the generators $e_i$ are all odd. The Cartan subalgebra is then parametrized by $h_{\text{diag.}} = \text{diag}_0(\varepsilon_1, \delta_1, \varepsilon_2, \delta_2, \varepsilon_3)$ with $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \delta_1 + \delta_2$ and it follows from eq.(50) that the fermionic SRS of $\text{sl}(3|2)$ reads as

$$\alpha_1 = \varepsilon_1 - \delta_1 \quad , \quad \alpha_2 = \delta_1 - \varepsilon_2 \quad , \quad \alpha_3 = \varepsilon_2 - \delta_2 \quad , \quad \alpha_4 = \delta_2 - \varepsilon_3 \quad .$$

From this point of view, the number of odd roots in a SRS of $\text{sl}(n+1|n)$ (and more generally of $\text{sl}(m|n)$) simply coincides with the number of sign changes in the involution defining the format under consideration, e.g. for the block and diagonal formats of $\text{sl}(3|2)$, we respectively have one and four such changes of signs:

$$M_{\text{block}} = \begin{pmatrix} + & + & + \\ - & + & - \\ - & - & - \end{pmatrix} \quad , \quad M_{\text{diag.}} = \begin{pmatrix} + & - & - \\ - & + & + \\ - & - & + \end{pmatrix} \quad .$$

This argument also explains why it is not possible to find a fermionic SRS for $\text{sl}(m|n)$ for $m \neq n + 1$.

For the sake of clarity, we should emphasize that the choice of a specific SRS does not impose or imply a choice of matrix format, e.g. once one has chosen the fermionic SRS and the diagonal matrix format, one can go over to the block (or any other) matrix format by virtue of a similarity transformation, the consequence being that the root generators $e_i$ loose their simple form (of matrices with entries on the first upper diagonal only). The previous discussion confirms that the diagonal matrix format (for which there is a maximal number of even/odd alternations in column vectors and matrices) is best adapted to fermionic SRS’s while the block format (for which there is no even/odd alternation at all) is best suited for the distinguished SRS. For intermediate SRS’s, an intermediate matrix format is appropriate and can be introduced along the lines described in the text.

In summary, we have studied the general structure of specific supermatrix arrangements which have previously been encountered in the literature. We hope that our systematic presentation elucidates these examples and that it will prove to be useful for further applications in physics and mathematics.

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A  \( osp\)-superalgebras in block format

If the standard matrix format is chosen for \( osp(2m \pm 1|2m) \), the supertranspose of \( M \) is defined by equation (A.8) and the basis of \( V_0 \) and \( V_1 \) can be chosen in such a way that the metrics are given by

\[
G_{so(2m \pm 1)} = \begin{bmatrix}
1 & \\
& \ddots & 1 \\
& & 1
\end{bmatrix}, \quad G_{sp(2m)} = \begin{bmatrix}
0 & -1_m \\
1_m & 0
\end{bmatrix}.
\] (A.1)

This form implies that the matrices belonging to \( so(2m \pm 1) \) are antisymmetric with respect to the antidiagonal (i.e. \( A_{ij} = -A_{p+1-j,p+1-i} \) with \( p = 2m \pm 1 \)) and that the Cartan subalgebra of \( osp(2m \pm 1|2m) \) consists of diagonal matrices.

\( osp(2m - 1|2m) \) Among the independent matrix entries, there are \( 4m^2 - 2m + 1 \) even and \( 2m(2m - 1) \) odd elements. The Chevalley basis in block format is represented by the matrices [10]

\[
e_{2i-1} = E_{m+1-i,4m-i} + E_{3m-i,m-1+i} \quad (i = 1, \ldots, m) \\
e_{2i} = E_{m+i,3m-i} - E_{3m-i,m-i} \quad (i = 1, \ldots, m - 1)
\]

\[
f_{2i-1} = E_{m-1+i,3m-i} - E_{4m-i,m+1-i} \\
f_{2i} = -E_{m-i,4m-i} - E_{3m-i,m+i}
\] (A.2)

\[
h_{2i-1} = E_{3m-i,3m-i} - E_{4m-i,4m-i} - E_{m+1-i,m+1-i} + E_{m-1+i,m-1+i} \\
h_{2i} = E_{m-i,m-i} - E_{m+i,m+i} - E_{3m-i,3m-i} + E_{4m-i,4m-i}
\]

\( osp(2m + 1|2m) \) There are \( 2m(2m + 1) \) even and as many odd independent matrix elements. The Chevalley basis in block format is [10] \( (i = 1, \ldots, m) \)

\[
e_{2i-1} = E_{m+2-i,4m+2-i} + E_{3m+2-i,m+i} \\
e_{2i} = E_{m+1+i,3m+2-i} - E_{4m+2-i,m+i-1-i}
\]

\[
f_{2i-1} = E_{m+i,3m+2-i} - E_{4m+2-i,m+2-i} \\
f_{2i} = -E_{m+1-i,4m+2-i} - E_{3m+2-i,m+1+i}
\] (A.3)

\[
h_{2i-1} = E_{3m+2-i,3m+2-i} - E_{4m+2-i,4m+2-i} - E_{m+2-i,m+2-i} + E_{m+i,m+i} \\
h_{2i} = E_{m+1-i,m+1-i} - E_{m+1+i,m+1+i} - E_{3m+2-i,3m+2-i} + E_{4m+2-i,4m+2-i}
\]
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