A MODEL STRUCTURE ON INTERNAL CATEGORIES

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Abstract. We put a model structure on the category of categories internal to simplicial sets whose weak equivalences are reflected by the nerve functor to bisimplicial sets with Rezk’s model structure. This model structure is shown to be Quillen equivalent to Rezk’s model structure, thus internal category are another model for (∞,1)-categories.

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Infinity categories are category-like objects in which one can do homotopy theory. There is now a plethora of available definitions of infinity categories in the literature. The most famous are quasicategories ([Joy02]), complete Segal spaces ([Rez01]), simplicial categories ([Ber07a]), Segal categories ([Ber07b]), relative categories ([BK12b]). All these definitions actually come in the form of a model category which encodes the homotopy category of infinity categories. It has been shown by various people (Bergner, Joyal and Tierney, Barwick and Kan, Lurie) that any two of the above models are connected by a zig-zag of Quillen equivalences.

The goal of this paper is to introduce yet another model for the homotopy category of infinity category. Its underlying category is the category of categories internal to the category of simplicial sets. Equivalently it is the category of contravariant functors from the category ∆ to the category of small categories. Applying the nerve functor degree-wise, we can see the category of internal categories as a full subcategory of the category of bisimplicial set. We define a morphism between internal categories to be a weak equivalence if it is one in the model structure of complete Segal. We show that those maps are the weak equivalences of a model structure. This model structure is transferred from the model structure of projectively fibrant (as opposed to Reedy fibrant) complete Segal.

This model category inherits some of the good formal properties of the model category of complete Segal spaces. It is a left proper simplicially enriched model category. Its weak equivalences have a nice characterization as Dwyer-Kan equivalences. We show that the obvious inclusion of the category of simplicial categories in the category of internal categories is an equivalence of relative categories.

We also study the homotopy theory of internal functors over internal categories. We prove Yoneda’s lemma in this context. We define the Grothendieck construction which is a right Quillen equivalence between the category of internal presheaves over

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C and a certain localization of the category of internal categories over C whose fibrant objects are “right fibrations” over C.

There are many interesting examples of internal categories. For instance Rezk defines a nerve from relative categories to bisimplicial sets ([Rez01]) and this factors through the category of internal categories. In particular, the main result of [BK11] shows that a levelwise fibrant replacement of the Rezk nerve of a partial model category is a fibrant internal category. Simplicially enriched categories are also particular internal categories and we show that the full inclusion of the simplicial categories in the internal categories is an equivalence of ∞-categories. For a functor from a simplicial category to simplicial sets, the Grothendieck construction has a simple description as an internal category as is explained in this paper.

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Notations. The notation $S$ denotes the category of simplicial sets. We often say space instead of simplicial set. The notation $sS$ denotes the category of bisimplicial sets with the projective model structure.

When $C$ is a category and $c$ is an object of $C$, we denote by $C/c$ the category of objects of $C$ over $c$.

For $k$ a natural number, we denote by $[k]$ the poset $\{0 \leq 1 \leq \ldots \leq k\}$ seen as an object of $\text{Cat}$. The object $\Delta[k]$ in $S$ is the nerve of $[k]$. We denote by $F(k)$ the functor $\Delta^\text{op} \to S$ sending $[n]$ to $\text{Cat}([n],[k])$. In particular, $F(k)$ is levelwise discrete.

We generically denote by $Q$ a cofibrant replacement functor in the ambient model category.

The category $\text{Cat}$ is the category of small categories. The category $\text{CAT}$ is the (large) category of locally small categories. The meaning of small and large can be made precise by way of Grothendieck universes.

Nine model categories. This paper contains the definition of at least nine model categories. For the reader’s convenience, we represent them on the following diagram of right Quillen functors with reference to their definition. In this diagram, all the horizontal functors are weak equivalences reflecting right Quillen equivalences and the vertical functors are left Bousfield localizations.

\[
\begin{array}{ccc}
\text{sS}_c & \longrightarrow & \text{sS}2.1 & \text{ICat}^P5.1 \\
\uparrow & & \uparrow & & \uparrow \\
\text{sS}^S & \longrightarrow & \text{sS}^S2.2 & \text{ICat}^S5.2 \\
\uparrow & & \uparrow & & \uparrow \\
\text{CSS}_c & \longrightarrow & \text{CSS}2.3 & \text{ICat}5.3 \\
\end{array}
\]

I. A FEW FACTS ABOUT MODEL CATEGORIES

The following definition is standard terminology.

Definition 1.1. Let $X$ be a cocomplete category and $I$ a class of map in $X$. The $I$-cell complexes are the smallest class of maps in $X$ containing $I$ and closed under pushout and transfinite composition. The $I$-fibrations are the maps with the right lifting property against $I$. The $I$-cofibrations are the maps with the left lifting property against the $I$-fibrations.

Recall that the $I$-cofibrations are the closure of the $I$-cell complex under retract. One also shows that the $I$-fibrations are exactly the map with the right lifting property against the $I$-cofibrations. All these facts can be found in appendix A of [Lur09a].

For future references, we recall the following classical theorem:
**Theorem 1.2.** Let \( F : X \leftrightarrow Y : U \) be an adjunction. Assume that \( X \) has a cofibrantly generated model structure with \( I \) and \( J \) a set of generating cofibrations and trivial cofibrations respectively. Then there is a model structure on \( Y \) whose fibrations and weak equivalences are reflected by \( U \) if the following condition are satisfied

- \( U \) preserves filtered colimits.
- The functor \( U \) sends pushouts of maps in \( FI \) to \( I \)-cofibrations and pushouts of maps in \( FJ \) to \( J \)-cofibrations.

Moreover, the functor \( U \) preserves cofibrations.

**Proof.** This is proved for instance in [Pre09, Proposition 11.1.4]. □

Let us recall that for a model category \( X \) that is combinatorial and left proper and for any set \( S \) of arrows in \( X \), there is a unique model category \( L_S X \) whose underlying category is \( X \) in which the cofibrations are the cofibrations of \( X \) and the fibrant objects are the \( S \)-local objects of \( X \) that are also fibrant. See [Bar10] for more details.

For future reference we have the following easy lemma which explains how Bousfield localization interact with certain Quillen equivalences.

**Lemma 1.3.** Let \( F : X \leftrightarrow Y : G \) be a Quillen equivalence in which \( G \) reflects weak equivalences. Let \( S \) be a set of maps in \( X \) and let \( L_S X \) (resp. \( L_S Y \)) be the left Bousfield localization of \( X \) (resp. \( Y \)) with respect to \( S \) (resp. \( LFS \)). Then we have a Quillen equivalence

\[
F : L_S X \leftrightarrow L_S Y : G
\]

and the functor \( G \) reflects weak equivalences.

**Proof.** The fact that \((F, G)\) forms a Quillen adjunction between the localized model structure is classical. Since \((F, G)\) was a Quillen equivalence between the non-localized model categories the maps \( LF \circ G \to id_Y \) and \( id_X \to RG \circ L \) are equivalences. Since \( G \) reflects weak equivalences, \( RG \) is equivalent to \( G \). Moreover \( LF \) is also the derived functor of \( F \) after localization (because the cofibrations are not changed). This easily implies that \((F, G)\) is a Quillen equivalence between the localized model structures. Let us prove that \( G \) reflects weak equivalences after localization. Let \( f : U \to V \) be a map in \( Y \). Then \( f \) is a weak equivalence in \( L_S Y \) if and only if for any fibrant object \( Z \) of \( L_S Y \), the induced map

\[
\text{Map}^b_Y(V, Z) \to \text{Map}^b_Y(U, Z)
\]

is a weak equivalence. Since \( G \) is fully faithful, this happens if and only if for all \( Z \) fibrant in \( L_S Y \), the map

\[
\text{Map}^b_X(GV, GZ) \to \text{Map}^b_X(GU, GZ)
\]

Since \( G \) is essentially surjective, this is equivalent to asking that for any \( T \) fibrant in \( L_S X \), the map

\[
\text{Map}^b_X(GV, T) \to \text{Map}^b_X(GU, T)
\]

is an equivalence, which is equivalent to asking that \( Gf \) be an equivalence in \( L_S X \). □

The following result will be useful to prove that certain maps are equivalence of \( \infty \)-categories.

**Proposition 1.4.** Let \((F, wF) : (C, wC) \to (D, wD)\) be a map of relative categories such that

- The functor \( F \) is fully faithful.
- The functor \( wF \) is fully faithful.
- There is a functor \( R : D \to D \) together with a zig-zag of natural weak equivalences from \( R \) to \( id_D \) which is such that \( R(d) \) is in the image of \( F \) for any \( d \).

Then \( F \) is an equivalence of relative categories.
Proof. First we claim that the map $wC \to wD$ induces an equivalence on nerve. Indeed it can be factored as

$wC \xrightarrow{wF} w \text{Im}(F) \xrightarrow{\iota} wD$

The first map is actually an equivalence of categories and we have zig-zags of natural transformations between $R\iota$ and $\text{id}_{\text{Im}(F)}$ and $\iota R$ and $\text{id}_{wD}$ which implies that $N(R)$ and $N(\iota)$ are homotopy inverses.

The hypothesis of the proposition remain true for the functor $F[n]: (C[n], wC[n]) \to (D[n], wD[n])$.

Hence according to the previous paragraph, $wC[n] \to wD[n]$ induces an equivalence on nerve. This implies that the map $N^rF: N^r(C, wC) \to N^r(D, wD)$ (where $N^r$ is Rezk’s nerve from relative categories to simplicial spaces) is a levelwise weak equivalence. □

Remark 1.5. According to [BK12a, Theorem 1.8.], a map is an equivalence of relative categories if and only if it induces a Dwyer-Kan equivalence on the hammock localizations.

2. Six model structures on simplicial spaces

2.1. The projective model structure. The category $sS$ can be given the projective model structure. In this model structure, the weak equivalences and fibrations are those maps which are weak equivalences and fibrations in each degree.

A set of generating cofibration (resp. trivial cofibration) is given by the maps $F(n) \times K \to F(n) \times L$ where $n$ can be any nonnegative integer and $K \to L$ is any generating cofibration (resp. trivial cofibration) of $S$.

This model structure is simplicial, proper, combinatorial. Its weak equivalences are stable under filtered colimits.

Let us denote by $sS_{\Sigma}$ the category of simplicial spaces equipped with the injective model structure. Then the identity map $sS \to sS_{\Sigma}$ is a left Quillen equivalence. We call a fibrant object in $sS$ (resp. $sS_{\Sigma}$) a projectively (resp. Reedy) fibrant simplicial space.

2.2. The Segal model structure. To a simplicial space $X$, we can assign the $n$-fold fiber product

$X_1 \times_{X_0} \cdots \times_{X_0} X_1$

This defines a simplicial functor from $sS$ to $S$ which is representable by a simplicial space $G(n)$. There is a map $G(n) \to F(n)$ representing the Segal map

$X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$

Definition 2.1. The category $sS^S$ is the left Bousfield localization of $sS$ with respect to the maps $G(n) \to F(n)$ for any $n \geq 1$.

This Bousfield localization exists since $sS$ is left proper and combinatorial. Moreover, this model structure is simplicial, cartesian closed, left proper, combinatorial.

If we denote by $sS^S_{\Sigma}$ the same localization on $sS_{\Sigma}$, we get by 1.3 a Quillen equivalence

$id : sS^S \simeq sS^S_{\Sigma}: id$

in which both sides have the same equivalences.

We call a fibrant object in $sS^S$ (resp. $sS^S_{\Sigma}$) a Segal fibrant simplicial space (resp. a Segal space).
2.3. The Rezk model structure. Let \( I[1] \) be the category with two objects and one isomorphism between them.

**Definition 2.2.** The model category \( CSS \) is the left Bousfield localization of \( sS^S \) with respect to the map \( \Delta[0] \to NI[1] \).

This Bousfield localization exists since \( sS^S \) is left proper and combinatorial. Moreover, this model structure is simplicial, cartesian closed, left proper, combinatorial. If we denote by \( CSS_c \) the same localization on \( sS^S_c \), we get by 1.3 a Quillen equivalence

\[
\text{id} : CSS \rightleftarrows CSS_c : \text{id}
\]

in which both sides have the same equivalences.

We call a fibrant object in \( CSS \) (resp. \( CSS_c \)) a Rezk fibrant simplicial space (resp. a complete Segal space).

### 3. Internal categories

#### 3.1. Generalities

Let \( P \) be a space, the category of \( P \)-graphs denoted \( \text{Graph}_P \) is the overcategory \( S/P \times P \). This category has a (nonsymmetric) monoidal structure given by sending \((s_A, t_A) : A \to P \times P \) and \((s_B, t_B) : B \to P \times P \) to the fiber product

\[
A \times_P B
\]

taken along the map \( t_A \) and \( s_B \).

**Remark 3.1.** If \((s_A, t_A) : A \to P \times P \) is a \( P \)-graph we will always use the following convention. A fiber product \( - \times_P A \) is taken along \( s_A \) and a fiber product \( A \times_P - \) is taken along \( t_A \).

**Definition 3.2.** The category of internal \( P \)-categories is the category of monoids in \( \text{Graph}_P \). We denote it by \( \text{ICat}_P \).

If \( u : P \to Q \) is a map of simplicial set, we get a functor \( u^* : \text{Graph}_Q \to \text{Graph}_P \) sending \( A \) to the fiber product

\[
P \times_Q A \times_Q P
\]

This functor is lax monoidal, therefore, it induces a functor

\[
u^* : \text{ICat}_Q \to \text{ICat}_P
\]

This functor is a right adjoint. We denote its left adjoint by \( u_! \).

**Definition 3.3.** The category \( \text{ICat} \) is the Grothendieck construction of the pseudo-functor \( S^{op} \to \text{CAT} \) sending \( P \) to \( \text{ICat}_P \).

More concretely, it is the category whose objects are pairs \((P, C)\) of a simplicial set \( P \) and a \( P \)-internal category \( C \) and whose morphisms \((P, C) \to (Q, D)\) are the pairs \((u, f^u)\) where \( u : P \to Q \) is a map in \( S \) and \( f^u : C \to u^*D \) is a map in \( \text{ICat}_P \).

The fact that fiber products are computed degreewise in \( S \) implies that there is an equivalence of categories

\[
\text{ICat} \to \text{Cat}^{\Delta^{op}}
\]

With this last description, it is obvious that the category \( \text{ICat} \) is locally presentable.

We use the notation \( \text{Ob}(C) \) and \( \text{Mor}(C) \) to respectively denote the space of objects and morphisms of an internal category \( C \).

We see the category \( S \) as a full subcategory of \( \text{ICat} \) through the functor sending \( K \) to \((K, K)\) where both source and target are the identity map. The internal categories in the image of this functor are called discrete.

We also see the category \( \text{Cat} \) of small categories as a full subcategory of \( \text{ICat} \) in the obvious way.

**Proposition 3.4.** The category \( \text{ICat} \) is a closed symmetric monoidal category under the cartesian product.
Proof. If $C$ and $D$ are internal category, we define an internal category $C^D$ with

$$\text{Ob}(C^D)_k = \text{ICat}(D \times \Delta[k], C)$$

and

$$\text{Mor}(C^D)_k = \text{ICat}(D \times [1] \times \Delta[k], C)$$

The internal category structure is left to the reader, as well as the fact that there are natural isomorphisms

$$C^D \times E \cong (C^D)^E$$

3.2. The nerve. The main tool of this paper is the nerve functor $N : \text{ICat} \to s\mathbf{S}$. It can be defined as the ordinary nerve functor $\text{Cat} \to \mathbf{S}$ applied degreewise to simplicial objects

$$N : \text{ICat} \cong \text{Cat}^\Delta^{op} \to s\mathbf{S}^{\Delta^{op}} = s\mathbf{S}$$

Concretely $N(C)$ is the simplicial space whose space of $n$-simplices is the $n$-fold fiber product

$$\text{Mor}(C) \times_{\text{Ob}(C)} \text{Mor}(C) \times \ldots \times_{\text{Ob}(C)} \text{Mor}(C)$$

Let $S : s\mathbf{S} \to \text{ICat}$ be the left adjoint to the nerve functor

$$N : \text{ICat} \to s\mathbf{S}$$

The functor $S$ can be defined as the degreewise application of the left adjoint to the nerve functor $\text{Cat} \to \mathbf{S}$. With this description, we see that the category of $k$-simplices of $S(X)$ is the quotient of the free category on the graph $(X_1)_k \equiv (X_0)_k$ where for any point $t$ in $(X_2)_k$, we impose the relation $d_2(t) \circ d_0(t) = d_1(t)$.

Equivalently, the functor $S$ is the unique colimits preserving functor sending $\Delta[p, q]$ to $[p] \times \Delta[q]$.

Note that he functor $N$ is fully faithful. This implies that the counit

$$SN(C) \to C$$

is an isomorphism for any internal category $C$.

Lemma 3.5. The functor $N : \text{ICat} \to s\mathbf{S}$ preserves filtered colimits.

Proof. The ordinary nerve functor

$$\text{Cat} \to \mathbf{S}$$

preserves filtered colimits because each of the categories $[n]$ is a compact objects of $\text{Cat}$. The functor $N$ is the ordinary nerve applied in each degree. Since colimits in $\text{ICat}$ and $s\mathbf{S}$ are computed degreewise, we are done. □

Definition 3.6. We declare a map in $\text{ICat}$ to be a levelwise (resp. Segal, resp. Rezk) equivalence if its nerve is a levelwise (resp. Segal, resp. Rezk) weak equivalence in $s\mathbf{S}$.

3.3. An essential surjectivity result. The following lemma is a key tool in what follows. It says that any homotopy type of simplicial set can be represented as the nerve of an internal category.

Lemma 3.7. There is a functor $M$ from Reedy fibrant simplicial spaces to internal categories and a functorial zig-zag of levelwise equivalences in the category $s\mathbf{S}$

$$X \Rightarrow M(X) \Rightarrow N(M(X))$$

Proof. We use the notion of relative categories as defined in [BK11]. There is a functor

$$A : \text{RelCat} \to \text{ICat}$$

sending $(C, W)$ to the internal category whose objects is the nerve of $W$ and whose morphisms is the nerve of the category whose objects are functors $[1] \to C$ and morphisms are weak equivalences between such functors.
Rezk’s functor $N^{r 1}$ from relative categories to complete Segal space factors as $N \circ A$. According to [BK11], we can take $M(X)$ to be $A(K_{\xi}(X))$.

3.4. Mapping space. Let $C$ and $D$ be internal categories. We use the notation $\text{Map}(C, D)$ for the mapping space $\text{Map}(NC, ND)$ in the category of simplicial spaces. This mapping space has as $n$ simplices the set of maps of bisimplicial sets

$$NC \times \Delta[n] \to ND$$

but $\Delta[n]$ can be identified with the nerve of the discrete internal category $\Delta[n]$ (i.e. the internal category whose space of objects and space of morphism are both $\Delta[n]$). Therefore, the $n$ simplices of $\text{Map}(C, D)$ are the maps of internal categories

$$C \times \Delta[n] \to D$$

Hence we see that $\text{Map}(C, D)$ is the space $\text{Ob}(D^C)$. It is also clear from this description that the functor $\text{Map}(\cdot, D)$ from $\text{ICat}^{\text{op}}$ to $\mathbf{S}$ preserves limits.

4. A key lemma

As usual, the main difficulty when one tries to transfer a model structure along a right adjoint is that the right adjoint does not preserve pushouts. This case is no exception. However, in this section, we prove that certain very particular pushouts in $\text{ICat}$ are preserved by the nerve functor. In particular, the pushouts of the generating cofibrations and trivial cofibrations satisfy the hypothesis of 4.2.

The functor $\text{Set} \to \text{Cat}$ sending a set to the discrete category on that set has a left adjoint $\pi_0$. Concretely $\pi_0(C)$ is the quotient of the set $\text{Ob}(C)$ by the smallest equivalence relation containing the pairs $(c, d)$ such that at least one of $C(c, d)$ or $C(d, c)$ is non empty. We say that a category $C$ is connected if $\pi_0(C)$ consists of a single element.

Lemma 4.1. Let $A$ be an object of $\text{Cat}$ and $i : K \to L$ be a monomorphism in $\mathbf{S}$. Let

$$K \times A \to C$$

$$i \times \text{id}$$

$$L \times A \to D$$

be a pushout diagram in $\text{ICat}$. Then for each $B \in \text{Cat}$ that is connected, the induced square

$$\text{Map}(B, K \times A) \to \text{Map}(B, C)$$

$$\text{Map}(B, L \times A) \to \text{Map}(B, D)$$

is a pushout diagram in $\mathbf{S}$.

Proof. It suffices to prove that for each $k$, the square

$$\text{Map}(B, K \times A)_k \to \text{Map}(B, C)_k$$

$$\text{Map}(B, L \times A)_k \to \text{Map}(B, D)_k$$

is a pushout square of sets. Equivalently, it suffices to prove that for each $k$, the square in $\text{Set}$

$$\text{ICat}(B \times \Delta[k], K \times A) \to \text{ICat}(B \times \Delta[k], C)$$

$$\text{ICat}(B \times \Delta[k], L \times A) \to \text{ICat}(B \times \Delta[k], D)$$

\footnote{Rezk denotes this functor $N$ but we change the notation to avoid the conflict}
is a pushout square. This is equivalent to proving that
\[
\begin{array}{c}
\text{Cat}(B, K_k \times A) \longrightarrow \text{Cat}(B, C_k) \\
\downarrow \\
\text{Cat}(B, L_k \times A) \longrightarrow \text{Cat}(B, D_k)
\end{array}
\]
is a pushout square, where now each corner is just the set of functors between ordinary categories.

Colimits in $\text{ICat}$ are computed degreewise. Hence, for each $k$, we have a pushout diagram in $\text{Cat}$
\[
\begin{array}{c}
K_k \times A \longrightarrow C_k \\
\downarrow \text{id} \downarrow f_k \\
L_k \times A \longrightarrow D_k
\end{array}
\]
The map $i_k \times \text{id}$ is split. Let us denote by $Z_k$ the set $L_k - K_k$. Then the category $D_k$ is isomorphic to $C_k \sqcup Z_k \times A$ and the map $f_k$ is the obvious inclusion.

Since the category $B$ is connected, the set $\text{Cat}(B, D_k)$ can be identified with $\text{Cat}(B, C_k) \sqcup \text{Cat}(B, A) \times Z_k$.

On the other hand, we can compute
\[
\text{Cat}(B, C_k) \sqcup \text{Cat}(B, A) \times Z_k
\]
By connectedness of $B$, this coincides with
\[
\text{Cat}(B, C_k) \sqcup \text{Cat}(B, A) \times L_k
\]
which is clearly isomorphic to $\text{Cat}(B, C_k) \sqcup (\text{Cat}(B, A) \times Z_k)$ which finishes the proof that $4.1$ is a pushout square. 

\begin{corollary}
Same notations and hypothesis as in the previous lemma. The square
\[
\begin{array}{c}
N(K \times A) \longrightarrow NC \\
\downarrow N(i \times \text{id}) \downarrow \text{nf} \\
N(L \times A) \longrightarrow ND
\end{array}
\]
is a pushout in $\text{sS}$
\end{corollary}

\begin{proof}
It suffices to check it in each degree. But the category $[n]$ is connected for all $n$, hence according to the previous proposition, the square
\[
\begin{array}{c}
N_n(K \times A) \longrightarrow N_n C \\
\downarrow N_n(i \times \text{id}) \downarrow N_nf \\
N_n(L \times A) \longrightarrow N_nD
\end{array}
\]
is a pushout square in $\text{S}$.
\end{proof}

\section{Existence of the model structure}

\subsection{The projective model structure}

Let $I_\text{S}$ and $J_\text{S}$ be a set of generating cofibrations and trivial cofibrations in $\text{S}$. The projective model structure on $\text{sS}$ admits the maps $f \times F(n)$ with $f$ in $I_\text{S}$ (resp. $f \in J_\text{S}$) and $n \in \mathbb{Z}_{>0}$ as generating cofibrations (resp. generating trivial cofibrations). We denote those sets by $I$ and $J$.

We can now prove the following:

\begin{theorem}
There is a model structure on $\text{ICat}$ whose weak equivalences and fibrations are the projective weak equivalences and fibrations. Its cofibrations are the $SI$-cofibrations and its trivial cofibrations are the $SJ$-cofibrations. Moreover the functor $N$ preserves cofibrations.
\end{theorem}
Proof. We apply 1.2. We already know that $N$ preserves filtered colimits by 3.5.

We need to check that $N$ of a pushout of a map in $SI$ is an $I$-cofibration. Let $i : K \times F(n) \to L \times F(n)$ be a map in $I$. Then $SI$ can be identified with $K \times [n] \to L \times [n]$. Let us consider a pushout square
\[\begin{array}{ccc}
K \times [n] & \xrightarrow{u} & C \\
\downarrow & & \downarrow \\
L \times [n] & \xrightarrow{t} & D
\end{array}\]

According to 4.2, the map $N(f)$ is the pushout of $N(u)$ along $N S(i) = i$. In particular, it is an $I$-cofibration. Similarly $N$ of a pushout of a map of $J$ is a $J$-cofibration. □

We denote by $\text{ICat}^P$ the category of internal categories equipped with this model structure. Note that this category is cofibrantly generated and locally presentable which implies that it is combinatorial.

**Proposition 5.2.** The category $\text{ICat}^P$ is proper.

Proof. The right properness follows from the fact that the functor $N$ preserves pullback and fibrations and reflects weak equivalences.

For the left properness, first notice that the weak equivalences in $\text{ICat}$ are preserved under filtered colimits. Indeed, weak equivalences are reflected by the nerve functor to $sS$ and the nerve functor preserves filtered colimits. The levelwise weak equivalences in $sS$ are preserved under filtered colimits because colimits are computed levelwise and the same is true in $S$.

Because of this observation, to show that $\text{ICat}$ is left proper, it suffices to prove that for any generating cofibration $K \times [n] \to L \times [n]$ and any weak equivalence $v : C \to D$ in $\text{ICat}$ fitting in a diagram
\[\begin{array}{ccc}
K \times [n] & \xrightarrow{u} & C \\
\downarrow & & \downarrow \\
L \times [n] & \xrightarrow{w} & D
\end{array}\]

where both squares are pushouts, the map $w$ is a weak equivalence. We can hit this diagram with $N$ and we get a diagram in $sS$
\[\begin{array}{ccc}
N(K \times [n]) & \xrightarrow{NC} & ND \\
\downarrow & & \downarrow \\
N(L \times [n]) & \xrightarrow{NE} & NF
\end{array}\]

Because of 4.2, the leftmost square and the total square are pushouts. This implies that the rightmost square is a pushout square. But now the result follows directly from the left properness of $sS$ and the fact that $N$ preserves cofibrations. □

**Proposition 5.3.** The functor $\text{Map} : (\text{ICat}^P)^{op} \times \text{ICat}^P \to S$ makes $\text{ICat}^P$ into a simplicial model category.

Proof. It suffices to check that for each generating cofibration $f : C \to D$ and fibration $E \to F$, the map
\[\text{Map}(D, F) \to \text{Map}(C, F) \times_{\text{Map}(C, E)} \text{Map}(D, E)\]
is a fibration. But $f : C \to D$ is $S(g)$ for $g : X \to Y$ a cofibration in $sS$.

Therefore the map
\[\text{Map}(D, F) \to \text{Map}(C, F) \times_{\text{Map}(C, E)} \text{Map}(D, E)\]
is isomorphic to
\[\text{Map}(SY, F) \to \text{Map}(SX, F) \times_{\text{Map}(SX, E)} \text{Map}(SY, E)\]
which by adjunction can be rewritten as
\[\text{Map}(Y, NF) \to \text{Map}(X, NF) \times_{\text{Map}(X, NE)} \text{Map}(Y, NE)\]

But we may now use the fact that \(sS\) is a simplicial model category and \(NE \to NF\) is a fibration to prove that this last map is a fibration. Similarly, if \(C \to D\) is a trivial cofibration or if \(E \to F\) is a trivial fibration, the map
\[\text{Map}(D, F) \to \text{Map}(C, F) \times_{\text{Map}(C, E)} \text{Map}(D, E)\]
is a trivial fibration. □

**Remark 5.4.** We do not know if the inner Hom in \(ICat\) makes it into a cartesian closed model category.

**Proposition 5.5.** The Quillen adjunction
\[sS \rightleftarrows ICat^P : N\]
is a Quillen equivalence.

**Proof.** According to 3.7, the functor \(N\) restricted to the subcategory of fibrant objects satisfies the hypothesis of 1.4 and hence induces an equivalence of relative categories. □

5.2. The Segal model structure.

**Definition 5.6.** The category \(ICat^S\) is the left Bousfield localization of \(ICat^P\) with respect to the maps \(LSG(n) \to LSF(n)\).

This Bousfield localization exists since \(ICat^P\) is left proper and combinatorial. We observe by 1.3, that we have a Quillen equivalence
\[S : sS \rightleftarrows ICat^S : N\]
Moreover, both sides are left proper and simplicial. Indeed, the simplicial enrichment is preserved under Bousfield localization since we are localizing a tractable model category (see [Bar10]).

5.3. The Rezk model structure.

**Definition 5.7.** The model category \(ICat\) is the left Bousfield localization of \(ICat^R\) with respect to any of the maps \(LS[0] \to LSF[1]\).

The Bousfield localization exists since \(ICat^S\) is left proper and combinatorial. Moreover, by 1.3, we have a Quillen equivalence
\[S : CSS \rightleftarrows ICat : N\]
Again both sides of this equivalence are left proper and simplicial.

5.4. The fibrant objects. We have the following characterization of fibrant objects.

**Proposition 5.8.** The fibrant objects in \(ICat^S\) and \(ICat\) are reflected by the functor \(N\) to \(sS\) and \(CSS\) respectively.

**Proof.** We do the case of \(ICat^S\). The other case is entirely analogous.

Let \(C\) be an internal category. Then \(C\) is fibrant in \(ICat^S\) if and only if it is fibrant in \(ICat^P\) and for each \(n\) the composite
\[\text{Map}(SQF(n), C) \to \text{Map}(SQG(n), C)\]
is a weak equivalence. This is equivalent to asking for \(NC\) to be projectively fibrant and the maps
\[\text{Map}(QF(n), NC) \to \text{Map}(QG(n), NC)\]
to be weak equivalences, which is exactly requiring for \(NC\) to be fibrant in \(sS\). □

Recall from [Rez01] the following definition
Definition 5.9. A Segal space is a Reedy fibrant simplicial space satisfying the Segal condition. A complete Segal space is a Segal space for which the map \( X_0 \to X^\equiv_1 \) is a weak equivalence (see [Rez01] for more details).

Proposition 5.10. The fibrant objects of \( sS^S \) are the levelwise fibrant simplicial spaces which are levelwise equivalent to Segal spaces. The fibrant objects in \( CSS \) are the simplicial spaces that are levelwise fibrant and levelwise equivalent to a complete Segal space.

Proof. We treat the case of \( sS^S \), the other being entirely analogous.

An object \( X \) is fibrant in \( sS^S \) if and only if it is fibrant in \( sS \) and the composite map
\[
X_n \to \text{Map}(G(n), X) \to \text{Map}(QG(n), X)
\]
is a weak equivalence. We claim that the Segal spaces are fibrant in \( sS^S \). Indeed for a Reedy fibrant simplicial space, the map \( \text{Map}(G(n), X) \to \text{Map}(QG(n), X) \) is automatically a weak equivalence. Moreover the property of being local in \( sS^S \) is clearly preserved under weak equivalences between fibrant objects of \( sS \).

Conversely, if \( X \) is fibrant in \( sS^S \), then a Reedy fibrant replacement of \( X \) is a Segal space which is levelwise weakly equivalent to \( X \). □

6. Weak equivalences

6.1. The Dwyer-Kan equivalences.

Definition 6.1. Let \( U \) be a fibrant object of \( sS^S \). For any two points \( x \) and \( y \) in \( U_0 \), we define \( \text{map}_U(x, y) \) to be the homotopy fiber
\[
\text{map}_U(x, y) = \text{hofib}_{(x,y)}(U_1 \xrightarrow{(d_0,d_1)} U_0 \times U_0)
\]

One can show exactly as in [Rez01] that there is a well-defined category \( \text{Ho}(U) \) whose objects are the 0-simplices of \( U_0 \) and whose set of morphisms from \( x \) to \( y \) is \( \pi_0(\text{map}_U(x, y)) \). Note that when \( U \) is Reedy fibrant, then our definition coincides with Rezk’s. If \( C \) is an internal category we denote by \( \text{Ho}(C) \) the homotopy category of \( NC \).

Definition 6.2. A map \( f: U \to V \) in \( sS \) is said to be a Dwyer-Kan equivalence if

- The induced map \( \text{Ho}(U) \to \text{Ho}(V) \) is essentially surjective.
- For each pair \( (x, y) \) of 0-simplices of \( U_0 \), the induced map
  \[
  \text{map}_U(x, y) \to \text{map}_U(f(x), f(y))
  \]
  is a weak equivalence.

A map \( f: C \to D \) between Segal fibrant internal category is said to be a Dwyer-Kan equivalence if its nerve is one.

Proposition 6.3. A levelwise weak equivalence between Segal fibrant simplicial spaces is a Dwyer-Kan equivalence.

Proof. For \( X \) a simplicial space, \( \pi_0(X) \) is a simplicial set. If \( X \) happens to be Segal fibrant, then \( \pi_0(X) \) is the nerve of a category (because \( \pi_0 \) sends homotopy fiber products to fiber products). We denote this category \( \text{ho}(X) \). There is clearly a fully faithful map
\[
\text{Ho}(X) \to \text{ho}(X)
\]
Moreover this map is surjective on objects. Hence \( \text{Ho}(X) \to \text{ho}(X) \) is an equivalence.

Now let \( f: X \to Y \) be a levelwise weak equivalence between Segal spaces. By the previous discussion, \( \text{Ho}(X) \to \text{Ho}(Y) \) is essentially surjective. Moreover the maps on mapping spaces are weak equivalences by definition of a homotopy fiber. □

Theorem 6.4. Let \( f: U \to V \) be a map in \( sS \) where \( U \) and \( V \) are Segal fibrant. Then \( f \) is a weak equivalence in \( CSS \) if and only if it is a Dwyer-Kan equivalence.
Proof. The strategy is to reduce this to the case where \( U \) and \( V \) are Reedy fibrant in which case the theorem is proved by Rezk (see [Rez01, Theorem 7.7]).

Let \( U \to U' \) and \( V \to V' \) be fibrant replacements in \( s \mathbb{S}_c \). Since the fibrant replacement can be made functorial, we can assume that there is a commutative diagram

\[
\begin{array}{ccc}
U & \xrightarrow{f} & V \\
\downarrow & & \downarrow \\
U' & \xrightarrow{f'} & V'
\end{array}
\]

The vertical maps are levelwise weak equivalences. Therefore they are Rezk equivalences. Moreover they are Dwyer-Kan equivalences by the previous proposition. Hence the top horizontal map is a Rezk or Dwyer-Kan equivalence if and only if the bottom horizontal map is one. But the bottom horizontal map is a map between Segal spaces, for which the two notion coincides by [Rez01, Theorem 7.7]. □

Corollary 6.5. Between Segal fibrant internal categories, the Rezk equivalences coincide with the Dwyer-Kan equivalences. □

6.2. A convenient subcategory. In this subsection, we define a particularly nice class of objects of \( \text{ICat} \).

Definition 6.6. An internal category \( C \) such that \( \text{Ob}(C) \) is fibrant and such that the source and target maps are fibrations in \( \mathbb{S} \) is called a strongly Segal internal category.

Proposition 6.7. A strongly Segal internal category is fibrant in \( \text{ICat}^{S} \)

Proof. Let \( C \) be a strongly Segal internal category. We write \( C_n \) for the space of \( n \)-simplices of \( NC \). We need to check that \( NC \) is levelwise fibrant. It suffices to prove that the source and target maps from \( C_n \) to \( C_0 \) are fibrations. We do that by induction on \( n \). This is by assumption true for \( n = 1 \). Now we have a diagram

\[
\begin{array}{ccc}
C_1 \times_{C_0} C_{n-1} & \xrightarrow{q} & C_{n-1} \\
\downarrow & & \downarrow \\
C_1 & \xrightarrow{p} & C_0
\end{array}
\]

\[
\begin{array}{ccc}
C_1 & \xrightarrow{s} & C_0 \\
\downarrow & & \downarrow \\
C_0 & &
\end{array}
\]

By induction hypothesis all maps but possibly \( p \) and \( q \) are fibrations. Since the square is a pullback, \( p \) and \( q \) must be fibrations as well.

We also see from this diagram that \( C_n \) is the homotopy pullback of \( C_1 \) and \( C_{n-1} \) over \( C_0 \). Hence \( NC \) is fibrant in \( s \mathbb{S}^{S} \). □

A map between strongly Segal internal categories is a weak equivalence in \( \text{ICat}^{S} \) (resp. \( \text{ICat} \)) if and only if it is an equivalence on objects and morphisms (resp. it is a Dwyer-Kan equivalence). Moreover, there are enough strongly Segal internal categories by the following proposition.

Proposition 6.8. Let \( \text{ICat}^{ss} \) be the full relative subcategory of \( \text{ICat} \) on strongly Segal internal categories. Then the inclusion

\[
\text{ICat}^{ss} \to \text{ICat}
\]

is an equivalence of relative categories.

Proof. This will be proved at the end of the next section. □
7. Comparison with simplicial categories

The category $\text{Cat}_\Delta$ of simplicial categories is the full subcategory of $\text{ICat}$ on objects whose space of objects is discrete. The inclusion $\text{Int}$ (for internalization) is a left adjoint. Let $u : \text{Ob}(C)_0 \to \text{Ob}(C)$ be the obvious map, the right adjoint $\Delta$ of $\text{Int}$ takes an internal category $(\text{Ob}(C), C)$ to the category $(\text{Ob}(C)_0, u^*C)$.

Recall from [Ber07a] that the category $\text{Cat}_\Delta$ has a model structure in which the weak equivalences are the Dwyer-Kan equivalences and the fibrant objects are the simplicial categories whose mapping spaces are fibrant simplicial sets.

**Proposition 7.1.** The functor $\text{Int}$ reflects weak equivalences.

**Proof.** For $C$ a simplicial category, there is a fibrant replacement $C'$ with same set of objects (one can apply a product preserving fibrant replacement in $S$ to each mapping space). The map $C \to C'$ is a levelwise equivalence on nerve, hence $\text{Int}(C) \to \text{Int}(C')$ is a weak equivalence in $\text{ICat}$. Let $f : C \to D$ be a map in $\text{Cat}_\Delta$. We can include it in a diagram

$$
\begin{array}{ccc}
C & \xrightarrow{f} & D \\
\downarrow & & \downarrow \\
C' & \xrightarrow{f'} & D'
\end{array}
$$

in which each of the vertical maps is a fibrant replacement as above. By the previous discussion, $\text{Int}(f)$ is a Rezk equivalence if and only if $\text{Int}(f')$ is a Rezk equivalence. Observe that if $C'$ is a fibrant simplicial category, then its internalization is Segal fibrant. Therefore using 6.5, we see that $\text{Int}$ sends weak equivalences between simplicial categories that are fibrant to weak equivalences in $\text{ICat}$. In particular, if $f$ is a weak equivalence, then $\text{Int}(f')$ is a weak equivalence which implies that $\text{Int}(f)$ is a weak equivalence. Conversely, if $\text{Int}(f)$ is a weak equivalence, $\text{Int}(f')$ is a Dwyer-Kan equivalence by 6.5. This implies that $f$ is a Dwyer-Kan equivalence. □

**Proposition 7.2.** A map between strongly Segal internal category is a Rezk equivalence if and only if the induced map $\Delta(C) \to \Delta(D)$ is a weak equivalence in Bergner’s model structure.

**Proof.** We know that between Segal fibrant internal categories, Rezk and Dwyer-Kan equivalences coincide. If $C$ is a strongly Segal internal category, the homotopy category and mapping spaces of $C$ coincide with those of $\Delta(C)$. Using this observation, the proposition is obvious. □

**Proposition 7.3.** The functor $\text{Int}$ induces an equivalence of relative categories $\text{Int} : \text{Cat}_\Delta \to \text{ICat}$

**Proof.** Let $\text{Cat}_\Delta^f$ be the full subcategory of $\text{Cat}_\Delta$ on fibrant simplicial categories. It suffices to check that $\text{Int} : \text{Cat}_\Delta^f \to \text{ICat}$ is an equivalence of relative categories. To do that it suffices to check that the composite

$$N \circ \text{Int} : \text{Cat}_\Delta^f \to \text{CSS}$$

is an equivalence of relative categories. But we can also write $N \circ \text{Int}$ as the composite of the two obvious relative inclusions

$$(\text{Cat}_\Delta^f, DK) \to (\text{SegCat}^f, DK) \to (\text{CSS}, v\text{CSS})$$

where $\text{SegCat}^f$ is the category of levelwise fibrant Segal categories. These two maps are equivalences by results of Bergner together with the fact that between Segal categories, the weak equivalences in Bergner’s model structure are the Dwyer-Kan equivalences ([Lur09b, 2.2.19]). □

**Corollary 7.4.** The inclusion $\text{ICat}^{ss} \to \text{ICat}$ is an equivalence of relative categories.
Proof. We know from the previous proposition that the composite
\[ \text{Cat}_\Delta \to \text{ICat}^{ss} \to \text{ICat} \]
is an equivalence of relative categories, hence it suffices to check that
\[ \text{Cat}_\Delta \to \text{ICat}^{ss} \]
is an equivalence of relative categories. But we know that $\text{Int} \Delta(C) \to C$ is an equivalence for any strongly Segal internal categories, therefore, we can apply 1.4 with $R = \text{Int} \Delta$. □

8. Categories of internal functors

8.1. Construction. Let $C$ be an internal category with space of object $P$. The category $S/C$ of spaces over $P$ is tensored over the monoidal category $S/P \times P$. The category $S^C$ is the category of right $C$-modules in $S/P$. More concretely, an object of $S^C$ is a map $F \to P$ together with an action $F \times P C \to F$ which is associative and unital.

The overcategory model structure on $S/P$ is the category whose cofibrations, fibrations and weak equivalences are reflected by the forgetful functor $S/P \to S$. A set of generating (trivial) cofibrations is obtained by taking all the triangles

\[
\begin{array}{ccc}
K & \to & L \\
\downarrow & & \downarrow \\
P & \to & P
\end{array}
\]

with $K \to L$ a generating (trivial) cofibration in $S$.

Proposition 8.1. Let $C$ be an internal category for which the source map is a fibration. There is a model structure on $S^C$ whose fibrations and weak equivalences are reflected by the forgetful functors $S^C \to S/P$. Moreover, the forgetful functor $S^C \to S$ preserves cofibrations.

Proof. The left adjoint to the forgetful functor $U : S^C \to S/P$ sends $Y \to P$ to $Y \times P C$. We apply 1.2. Let $K \to L$ be a cofibration in $S/P$, then $K \times P C \to L \times P C$ is a cofibration in $S/P$. Indeed, it suffices to check that the underlying map in $S$ is a monomorphism which is trivial. If $K \to L$ is a trivial cofibration then $K \times P C \to L \times P C$ is a cofibration by what we have just said and is a weak equivalence because the source map $C \to P$ is a fibration. Since the functor $U$ preserves colimits, we are done. □

8.2. A cofibrant replacement functor on $S^C$. We have the bar resolution
\[ F \times P C \Leftarrow F \times P C \times P C \Leftarrow F \times P C \times P C \times P C \ldots \]
It is a classical fact that the realization of this simplicial object is weakly equivalent to $F$. Note that the forgetful functors $S^C \to S/P$ and $S/P \to S$ preserve colimits and the tensor product with spaces, hence it does not make a difference to compute the realization in any of these three categories.

Moreover, the realization is cofibrant by the following proposition:

Proposition 8.2. The bar resolution is Reedy cofibrant in $S^C$.

Proof. We proceed as in [BdBW13, Lemma A.4]. It suffices to check that the degeneracy diagram
\[ F \to C \times P F \Leftarrow C \times P C \times P F \ldots \]
is Reedy cofibrant in functors $\Delta_{s}^{op} \to S/P$. Since colimits are preserved under $S/P \to S$ it suffices to check that the underlying diagram $\Delta_{s}^{op} \to S$ is Reedy cofibrant which is the case for any functor $\Delta_{s}^{op} \to S$. □
8.3. **Yoneda’s lemma.** Let $C$ be an internal category whose source and target map are fibrations, then the category $S^{C^{op}}$ is a simplicial model category. Let $c \in \text{Ob}(C)$ be an object. We define the internal functor $y_c \in S^{C^{op}}$. It is given by

$$y_c = C \times_{\text{Ob}(C)} \{c\} \to \text{Ob}(C)$$

where the map $y_c \to \text{Ob}(C)$ is given by the source map $C \to \text{Ob}(C)$.

Note that if $C$ is actually a simplicial category, then $y_c$ is exactly the presheaf on $C$ represented by $c$.

We now have an internal version of Yoneda’s lemma:

**Proposition 8.3.** Let $F$ be any object of $S^{C^{op}}$. Then we have

$$\mathbb{R} \text{Map}_{S^{C^{op}}}(y_c, F) \simeq \text{hofib}_c F$$

**Proof.** The functor $y_c$ is cofibrant in $S^{C^{op}}$. Both sides of the equations preserve equivalences in the $F$ variable, therefore, it suffices to prove the proposition for $F$ fibrant in $S^{C^{op}}$, then

$$\mathbb{R} \text{Map}_{S^{C^{op}}}(y_c, F) \simeq \text{Map}_{S^{C^{op}}}(y_c, F) \simeq \text{hofib}_c F$$

$\square$

8.4. **Base change adjunction.** Let $u : P \to Q$ be a map in $S$, we have a functor $u^* : S_{/Q \times Q} \to S_{/P}$ and $u^* : S_Q \to S_P$. We have already observed that the functor $u^*$ on $S_{/Q \times Q}$ is monoidal. Now we claim that the following diagram in which the vertical maps are the tensor

$$\begin{array}{ccc}
S_{/Q \times Q} \times S_{/Q \times Q} & \xrightarrow{u^* \times u^*} & S_{/P} \times S_{/P} \\
\downarrow \quad & & \downarrow \\
S_{/Q} & \xrightarrow{u^*} & S_{/P}
\end{array}$$

is commutative. This is because for $C \in S_{/Q \times Q}$ and $F \in S_{/Q}$ we have the equation

$$F \times_Q C \times_Q P \simeq (F \times_Q P) \times_P (P \times_Q C \times_Q P)$$

Let $\alpha : C \to D$ be a morphism of internal categories whose source and target maps are fibrations. We want to extract from it a Quillen adjunction

$$\alpha_! : S^C \rightleftarrows S^D : \alpha^*$$

We denote by $P$ the space of objects of $C$ and $Q$ the space of objects of $D$ and by $u : P \to Q$ the value of $\alpha$ on objects.

The previous observation, implies that $u^*F$ has an action of $u^*D$. We can pullback this action along the map $C \to u^*D$ to construct an action of $C$ on $u^*F$. The object $u^*F$ with this action of $C$ is defined to be $\alpha^*F$.

The functor $\alpha_!$ is the left adjoint of $\alpha^*$. It is the unique colimit preserving functor sending an internal functor of the form $F \times_P C$ to $F \times_Q D$.

If $F \to G$ is a (trivial) fibration in $S^D$, then $P \times_Q F \to P \times_Q G$ is a (trivial) fibration in $S^C$. Thus $\alpha^*$ is a right Quillen functor.

This allows us to define the homotopy colimit and more generally the homotopy left Kan extension of an internal functor.

**Definition 8.4.** Let $F \in S^C$ and $\alpha : C \to D$, then the homotopy left Kan extension of $F$ along $\alpha$ is the left derived functor of $\alpha_!$ applied to $F$.

Explicitly if we write $P = \text{Ob}(C)$ and $Q = \text{Ob}(D)$, then $\mathbb{L}\alpha_!F$ can be computed as the realization of the following simplicial object in $S^D$

$$F \times_Q D \simeq F \times_P C \times_Q D \simeq F \times_P C \times_P C \times_Q D \ldots$$
Remark 8.5. Note that if $C$ is a simplicial category, then $S^C$ coincides with the category of simplicial functors $C \rightarrow S$. Moreover, if $C$ is fibrant as a simplicial category, then the source and target maps $C \rightarrow \text{Ob}(C)$ are fibrations and the model structure we get on $S^C$ is exactly the projective model structure. In the case of a simplicial category, the simplicial object we constructed to compute the left Kan extension coincides with the usual bar construction.

Unsurprisingly, if $C \rightarrow D$ is a Rezk equivalence between strongly Segal internal categories, then $S^C \equiv S^D$ is a Quillen equivalence. To prove this fact, we can reduce to the case where $C = \Delta(D)$. Indeed we have a square of left Quillen functors

$$
\begin{array}{ccc}
S^{\Delta(C)} & \to & S^{\Delta(D)} \\
\downarrow & & \downarrow \\
S^C & \to & S^D
\end{array}
$$

Since $C$ and $D$ are assumed to be strongly Segal, $\Delta(C) \rightarrow \Delta(D)$ is an equivalence of simplicial categories. Hence, by a classical result about presheaves categories, the top horizontal map is a Quillen equivalence. Therefore, the bottom functor will be an equivalence if the two vertical functors are equivalences which we prove in the next proposition:

Proposition 8.6. Let $D$ be a strongly Segal internal category and $\alpha : C = \Delta(D) \rightarrow D$ be the counit. The Quillen adjunction

$$
\alpha^! : S^C \rightleftarrows \alpha^* : S^D
$$

is a Quillen equivalence.

Proof. We denote by $Q$ the set of objects of $D$, then the set of objects of $C$ is $Q_0$.

First we claim that the unit map $F \to \mathbb{R}\alpha^* \text{Lan}_\alpha F$ is an equivalence. The functor $\text{Lan}_\alpha F$ is given as the realization of

$$
F \times Q = \{ f \} \times_D F \times Q = F \times Q_0 C \times Q D \ldots
$$

Since we have assumed $D$ to be strongly Segal, this simplicial object is levelwise fibrant in $\mathbf{S}/Q$.

The map $F \to \mathbb{R}\alpha^* \text{Lan}_\alpha F$ is an equivalence if and only if it is a fiberwise equivalence. We know that in $\mathbf{S}$, the geometric realization commutes with base change. Hence, if $q$ is a 0-simplex of $Q$, then the fiber at $q$ of $\mathbb{R}\alpha^* \text{Lan}_\alpha F$ is the realization of

$$
F \times Q y_q = \{ f \} \times_D F \times Q_0 C \times Q y_q \equiv F \times Q_0 C \times Q y_q \ldots
$$

This coincides with the realization of

$$
F \times Q_0 Q_0 \times Q y_q = \{ f \} \times_D F \times Q_0 C \times Q_0 Q_0 \times Q y_q \equiv F \times Q_0 C \times Q_0 Q_0 \times Q y_q \ldots
$$

which is nothing but the fiber at $q$ of the realization of

$$
F \times Q_0 C \equiv \{ f \} \times_D F \times Q_0 C \times Q_0 C \equiv F \times Q_0 C \times Q_0 C \ldots
$$

which is equivalent to $F_q$.

Now let $F \to Q$ be an object of $S^D$. We want to prove that $\text{Lan}_\alpha \alpha^* F \to F$ is an equivalence. Again it suffices to do it fiberwise. Let $z_q = \{ q \} \times_D D$ be the functor corepresented by $q$. By the dual of Yoneda’s lemma, it suffices to check that

$$
\mathbb{R}\text{Map}_{S^D}(z_q, \text{Lan}_\alpha \alpha^* F) \to \mathbb{R}\text{Map}_{S^D}(z_q, F)
$$

is an equivalence for any $q$ in $Q_0$.

Let $z_q'$ be the functor corepresented by $q$ in $C$. That is $z_q' = \{ q \} \times_C C$. Then $\alpha z_q' = z_q$, moreover, since $z_q'$ is cofibrant, $\alpha z_q'$ coincides with $\text{Lan}_\alpha z_q'$. Hence it suffices to check that for any $q$ in $Q_0$, we have an equivalence

$$
\mathbb{R}\text{Map}_{S^D}(\text{Lan}_\alpha z_q', \alpha^* \text{Lan}_\alpha \alpha^* F) \to \mathbb{R}\text{Map}_{S^D}(\text{Lan}_\alpha z_q', F)
$$

by adjunction, this is equivalent to proving that

$$
\mathbb{R}\text{Map}_{S^C}(z_q', \alpha^* \text{Lan}_\alpha \alpha^* F) \to \mathbb{R}\text{Map}_{S^C}(z_q', \alpha^* F)
$$
is an equivalence. But we have proved that the unit of the derived adjunction is an equivalence, which concludes the proof. □

Example 8.7. In this example we allow ourselves to treat topological spaces as simplicial sets. The reader is invited to apply the functor $\text{Sing}$ as needed.

Let $(K, k)$ be a connected based topological space. There is a strongly Segal internal category $\text{Path}(K)$ described in [And10, Example II.3.4] whose objects are points of $K$ and morphisms are Moore paths between points. There is an obvious map

$$(K, K) \to \text{Path}(K)$$

where $(K, K)$ is the subcategory of constant paths. This map is a levelwise equivalence. On the other hand, there is a Dwyer-Kan equivalence

$$\Omega K \to \text{Path}(K)$$

where $\Omega K$ is the monoid of endomorphisms of $k$ in $\Delta(\text{Path}(K))$ seen as a simplicial category. It is a strict monoid with the homotopy type of $\Omega K$. Hence we have a zig-zag of left Quillen equivalences

$$\mathbf{S}_{|K} \to \mathbf{S}^{\text{Path}(K)} \leftarrow \mathbf{S}^{\Omega K}$$

8.5. Grothendieck construction. The proofs in this subsection are a bit sketchy since they will appear in greater generality in Pedro Boavida de Brito’s thesis ([BdB14]).

Let $C$ be an internal category and $W \to P = \text{Ob}(C)$ be an object of $\mathbf{S}^{C^{\text{op}}}$. We define $\mathcal{G}(W)$ to be the internal category whose space of objects is $W$ and space of morphism is $C \times_P W$. The source is the projection

$$C \times_P W \to W$$

and the target is the action map

$$C \times_P W \to W$$

In particular, if $C$ is a simplicial category, $W$ is just an ordinary simplicial presheaf on $C$ and $\mathcal{G}(W)$ is the internal category with

$$\text{Ob}(\mathcal{G}(W)) = \bigsqcup_{x \in \text{Ob}(C)} W(x)$$

and

$$\text{Mor}(\mathcal{G}(W)) = \bigsqcup_{x,y \in \text{Ob}(C)} \text{Map}_C(x, y) \times W(y)$$

If $c$ is an object of an internal category $C$, we denote by $C/c$ the internal category $\mathcal{G}(y_c)$.

The Grothendieck construction defines a functor

$$\mathcal{G} : \mathbf{S}^{C^{\text{op}}} \to \mathbf{ICat}/C$$

This functors preserves limits and hence must have a left adjoint.

There is a model structure $\mathbf{ICat}^{RF}/C$ which is obtained as the Bousfield localization of $\mathbf{ICat}/C$ in which the fibrant objects are the right fibrations $E \to C$. That is, the fibrations that are such that

$$\text{Mor}(E) \xrightarrow{L} \text{Ob}(E)$$

$$\text{Mor}(C) \xrightarrow{L} \text{Ob}(C)$$

is a homotopy pullback.

Theorem 8.8. Let $C$ be a strongly Segal internal category, then

$$\mathcal{G} : \mathbf{S}^{C^{\text{op}}} \to \mathbf{ICat}^{RF}/C$$

is a right Quillen equivalence.
Proof. One can show easily that the Grothendieck construction defines a right Quillen functor.

There is a Bousfield localization $\text{CSS}_{/NC}^{RF}$ of $\text{CSS}_{/NC}$ described in [BdB14] whose fibrant objects are the right fibrations over $NC$. An easy application of 1.3 shows that there is a Quillen equivalence

$$S : \text{CSS}_{/NC}^{RF} \rightleftarrows \text{ICat}_{/C}^{RF} : N$$

On the other hand, the functor $N : \text{ICat}_{/C}^{RF} \to \text{CSS}_{/NC}^{RF}$ composed with $G$ coincides with what Boavida de Brito defines as the Grothendieck construction of a presheaf. This composite is shown to be a right Quillen equivalence in [BdB14], hence $G$ must be a right Quillen equivalence as well. □

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