Automorphic products, generalized Kac-Moody algebras and string amplitudes

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Abstract. We review automorphic products and generalized Kac-Moody algebras from a physics point of view. We discuss the appearance of automorphic products in BPS-saturated one-loop quantities in heterotic string theory. At particular points in moduli space, with enhanced gauge symmetry, these products can be used to define a generalized Kac-Moody algebra $G(g^{++})$ as an automorphic correction of the Lorentzian Kac-Moody algebra $g^{++}$, which is obtained through double extension of the complement $g = (e_8 \oplus e_8)/h$. The root multiplicities of $G(g^{++})$ are then encoded in the Fourier coefficients of certain modular forms, which appear directly in the integrand of the one-loop quantities. We review particular examples of this extension for compactifications with $\mathcal{N} = 2$ and $\mathcal{N} = 4$ space-time supersymmetry.

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1. Introduction
The BPS spectrum is a particular set of states in the Hilbert space of theories with extended supersymmetry. Especially in the context of superstring theory it has received a lot of attention during the recent years. One reason for this is that observables which receive contributions only from BPS states (BPS-saturation) are protected when extrapolated from weak to strong string coupling. The study of such quantities therefore allows invaluable insights into non-perturbative aspects of string theory and in the past has for example been the foremost arena for studying string dualities [2, 43]. Besides that, such BPS-saturated quantities are in many cases strongly related to mathematically interesting structures. Examples which have been studied extensively in the past include helicity supertraces [40, 41, 18] and topological amplitudes [1, 10, 3, 5, 6, 8]. Physically, degeneracies of BPS states can be related to the entropy of particular supersymmetric black holes (see e.g. [49] for a review). In this way, string theory is capable of a microscopic explanation of thermodynamic (macroscopic) properties of (certain) black holes. This connection is not only qualitative in nature, but in many examples precise agreement is found even to subleading order (in the black-hole charges) (see e.g. [18, 17]).

Degeneracies of BPS-states (or more precisely BPS-indices) are locally constant functions on the moduli space of string compactifications. However, they may jump at (real) co-dimension one hypersurfaces where bound states of BPS-states may decay or recombine. Such walls of marginal stability have been extensively studied in the recent past (see e.g. [48, 19, 14]).

In [34, 35] Harvey and Moore have undertaken a closer investigation of mathematical properties of the BPS-spectrum in string theory. In particular they argued that the space

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of BPS-states should form an algebra. Indeed they discovered that part of the threshold corrections to the gauge couplings in four-dimensional $\mathcal{N} = 2$ heterotic string theory can be written as infinite product representations of automorphic forms on the Grassmannian $SO(2, 2 + n)/(SO(2) \times SO(2 + n))$. The latter, through work of Borcherds [12], are in turn related to denominator formulae for Generalized Kac-Moody algebras (GKM), which was seen as evidence that the algebra of BPS-states is indeed a GKM. However, although these findings are very intriguing and suggestive, a direct connection between these infinite product formulae and the algebra of BPS-states was not established.

A different — physics inspired — approach was initiated by the work of Dijkgraaf, Verlinde and Verlinde [20], who conjectured that the degeneracies of (non-perturbative) 1/4 BPS dyon-states in four-dimensional $\mathcal{N} = 4$ heterotic compactifications are encoded in the Fourier coefficients of a certain Siegel modular form, known as the Igusa cusp form $\Phi_{10}$. As it had been shown in the mathematics literature [30], $\Phi_{10}$ has an infinite product representation which relates it to the denominator formula of a certain rank 3 (super) GKM-algebra, which was called $g_{1, II}$ in [30]. In this way, the degeneracies of dyons also become directly related to root multiplicities of a GKM-algebra. Further hints for the role of the algebra $g_{1, II}$ in physics were found in [14] (see also [15, 16] and [27, 28, 29] for related work), where it was shown that the wall-crossing behaviour of the dyon spectrum is controlled by the hyperbolic Weyl group $W(g_{1, II})$ of this GKM-algebra.

Recently, further efforts [23, 36] have been made to understand the appearance of GKM algebras in BPS-saturated couplings in string theory. In [23] a microscopic analysis of the algebraic properties of the BPS spectrum has been undertaken. Explicitly, within the framework of the $E_8 \times E_8$ heterotic string theory compactified on $T^2$, it was shown that the space of BPS states forms a representation of a certain GKM algebra $G$, which was constructed explicitly. Moreover, within the same heterotic model a particular one-loop BPS-saturated amplitude $\mathcal{F}_1$ (first discussed in [3]) was analyzed. By explicit computation it was proven that (at a generic point in moduli space) part of this amplitude encodes the root-multiplicities of a slightly larger subalgebra $G_{\text{ext}}$, of which $G$ is a subalgebra $G \subset G_{\text{ext}}$. In [36] $\mathcal{F}_1$ has been systematically studied at particular points in moduli space where the gauge group is enhanced to $h \subset e_8 \oplus e_8$. It was shown that after some proper regularization, part of $\mathcal{F}_1$ can be interpreted as the denominator formula of a GKM which is the Borcherds extension of $g^{++}$, where the latter is the hyperbolic extension of the commutant of $h$ in $e_8 \oplus e_8$.

The organization of this article is as follows: In section 2 we will review basic properties of Generalized Kac-Moody algebras. In particular, we will explain how they can be obtained through an automorphic correction of a hyperbolic Kac-Moody algebra. The key ingredients in this construction are so-called automorphic products, which in turn can be understood as Borcherds-lifts of modular forms of $SL(2, \mathbb{Z})$. Within the framework of the (singular) theta correspondence, such automorphic products appear naturally as one-loop BPS saturated quantities in string theory. As an example, we will review in section 3 the work of Harvey and Moore [34, 35] and study threshold corrections to the gauge couplings in $\mathcal{N} = 2$ heterotic string compactifications. Finally, in section 4 we will review aspects of [36] and analyze a particular one-loop topological amplitude in heterotic string theory on $T^2$, at a point in moduli space where the gauge algebra is enhanced to a Lie algebra $h$. We will extract from this amplitude the denominator formula for a GKM which can be constructed as an automorphic correction of the hyperbolic extension of the commutant of $h$ in $e_8 \oplus e_8$. For simplicity we will focus on $g$ being simple — the treatment of the semisimple case can be found in [23, 36].
2. Generalized Kac-Moody algebras

2.1. General properties

Before considering the appearance of generalized Kac-Moody algebras in the context of physical quantities, let us first review some of their properties. For any simple Lie algebra \( g \) of rank \( r \) the Cartan matrix \( C_{ij} \) (with \( i = 1, \ldots, r \)) is a matrix with the following properties

\[
C_{ii} = 2, \quad C_{ij} = 0 \Leftrightarrow C_{ji} = 0 \quad C_{ij} \in \mathbb{Z}_- \quad \forall \ i \neq j.
\] (1)

We can distinguish three different cases, depending on whether \( C \) is

- positive definite: \( g \) is a finite dimensional Lie algebra falling into the Cartan-Killing classification of finite simple Lie algebras (see e.g. [39] for an overview)
- positive semi definite: \( g \) is an affine Kac-Moody algebra; affine KM can be constructed as extensions of finite dimensional Lie algebras \( h \) (to indicate this we will mostly write \( g = h^+ \))
- indefinite: in this work we focus on the case that \( C \) is of Lorentzian signature such that \( g \) is a Lorentzian Kac-Moody algebra; a full classification exists only for particular subclasses e.g. hyperbolic Kac-Moody algebras. In these notes we will denote with \( g = h^{++} \) the hyperbolic extension of a finite dimensional Lie algebra \( h \) (as an example we have depicted the Dynkin diagram of \( a^{++}_1 \) in figure 1)

The algebras that we will be concerned with in this review are of a more general type (i.e. their Cartan matrices generalize (1)). A Generalized Kac-Moody algebra (GKM)\(^1\) \( G \) has finite rank and relaxes the first condition of (1). In particular, \( G \) is characterized by the fact that some of the diagonal entries \( C_{ii} < 0 \). If we view the Cartan matrix as the inner product matrix of the simple roots \( \alpha_i \) of \( G \), this entails that \( G \) has two types of simple roots

\[
(\alpha_i|\alpha_i) > 0 \quad \ldots \quad \text{real simple roots}
\]

\[
(\alpha_i|\alpha_i) \leq 0 \quad \ldots \quad \text{imaginary simple roots}
\]

For further convenience, we will denote the set of all roots of \( G \) as \( \Delta(G) \). GKMs share in common with finite dimensional Lie algebras the existence of a Weyl group \( W(G) \) defined as the group of reflections in the simple roots. \( I.e. \) if \( \{\alpha_I\} \) (with \( I = 1, \ldots, n \)) denotes the set of simple real roots of \( G \), \( W(G) \) is generated by \( n \) fundamental reflections

\[
w_I : x \mapsto x - 2 \frac{(x|\alpha_I)}{(\alpha_I|\alpha_I)} \alpha_I \quad \text{with} \quad \in \Lambda G \otimes \mathbb{C}.
\]

GKMs also posses a Weyl vector \( \rho \) which satisfies

\[
(\rho|\alpha) \leq -\frac{1}{2}(\alpha|\alpha) \quad \text{for} \quad \alpha \in \Delta(G),
\]

and equality holds if and only if \( \alpha \) is a simple root of \( G \). The existence of \( W(G) \) and \( \rho \) also allows to formulate a generalization of the Weyl-Kac-character formula. For any (integrable) lowest weight representation \( R(\lambda) \) of \( G \) we have [39, 11]

\[
\text{ch} \ R(\lambda) = \sum_{w \in W} \epsilon(w) w(S) e^{-\rho} \prod_{\alpha \in \Delta^+} (1 - e^\alpha)^{\text{mult} \alpha}.
\] (2)

Here \( \epsilon(w) = (-1)^{\ell(w)} \), where \( \ell(w) \) denotes the length of the Weyl element \( w \) (see e.g. [38]). Notice that the product runs over the positive roots \( \Delta^+ \) of \( G \) and \( \text{mult} \alpha \) denotes the multiplicity of

\(^1\) Some authors also use the name Borcherds-Kac-Moody algebra (BKM) or just Borcherds algebra.
α ∈ Δ⁺. The main difference of (2) to the ‘standard’ Weyl-Kac character formula (for e.g. finite Lie algebras) is the correction factor $S$ which is modified due to the presence of imaginary simple roots in $G$ [11]:

$$S = e^{\lambda + \rho} \sum_{\alpha \in \Lambda^+_G} \xi(\alpha) e^\alpha,$$

where $\Lambda^+_G$ is the positive part of the root lattice of $G$ and $\xi(\alpha) = (-1)^m$ if $\alpha$ is a sum of $m$ distinct pairwise orthogonal imaginary simple roots which are orthogonal to $\lambda$, and $\xi(\alpha) = 0$ otherwise. Thus, if $G$ contains no imaginary roots, we recover the familiar expression [39]. In this review we will mainly be interested in (2) for the trivial representation with $\lambda = 0$ and $R(0) = 1$. For this particular case (2) reduces to the so-called denominator formula

$$\sum_{w \in W} c(w)w(S) = e^\rho \prod_{\alpha \in \Delta_+} \left(1 - e^{\alpha}\right)^{\text{mult} \cdot \alpha}. \quad (3)$$

This formula relates a sum over the Weyl group $W(G)$ to an infinite product over all positive roots of $G$.

2.2. Automorphic correction

An important question is how to construct GKM. A method which will be very important for the further sections of this review is the automorphic correction of a hyperbolic Kac-Moody algebra. Morally speaking, this method extends the root system of a hyperbolic Kac-Moody algebra $g^{++}$ by an infinite set of imaginary simple roots. To guarantee consistency, a further ingredient is needed [12, 30], namely a weak Jacobi form $f(\tau, z)$ of $SL(2, \mathbb{Z})$ with Fourier expansion

$$f(\tau, z) = \sum_{\lambda \in \Lambda_{g^{++}}} c(-\lambda^2 / 2) q^{\frac{1}{2} (\lambda | \lambda)} e^{2\pi i (z | \lambda)} , \quad (4)$$

where $\Lambda_{g^{++}}$ is the root lattice of $g^{++}$. If the dimension $\text{dim}(\Lambda_{g^{++}}) = r + 2$, then $f(\tau, z)$ has weight $-r/2$. Notice that these modular properties guarantee that all Fourier coefficients $c(-\lambda^2 / 2)$ of (4) are integers, which allows us to define the modular product (with respect to $O(2, 2 + r)$) [12]

$$e^{-2\pi i (\rho | y)} \prod_{\lambda \in \Lambda_{g^{++}}} \left(1 - e^{-2\pi i (\lambda | y)}\right)^{c(-\lambda^2 / 2)} , \quad y \in \Lambda_{g^{++}} \otimes \mathbb{C} , \quad (5)$$

where $\rho$ is the lattice Weyl-vector of $g^{++}$. The idea of constructing a GKM is to identify (5) with the right hand side of (3) and to interpret the additional terms in (5) as new roots on top of the ones already present in $\Delta_{g^{++}}$. Because of the crucial minus sign in the exponent of $q$ in (4), these additional roots are generically imaginary. Moreover, since the product in (5) is infinite, we generically add an infinite set of additional simple roots. It was shown in [11, 12] that there exists indeed a GKM $G$ with these roots. In order to emphasize that the latter was constructed from $g^{++}$ we will in many cases write $G \equiv G(g^{++})$. It is, however, important to realize that the extension of $g^{++}$ is not unique, since different modular products (i.e. different $f(\tau, z)$ in (4)) will lead to different algebras $G$.

Notice that identifying the modular product (5) with the right hand side of (3) suggests that it has an alternative presentation as an infinite sum (corresponding to the left hand side of (3)). While the presentation (5) is usually called the multiplicative (or Borcherds) lift [30, 32, 33, 11] of the seed $f(\tau, z)$, the infinite sum representation can be thought of as an additive (or arithmetic) lift [44, 31] of some different automorphic form.
2.3. Automorphic products as one-loop amplitudes in string theory

All ingredients for the construction of a GKM that we have mentioned in the previous paragraph—in particular the automorphic products (5)—arise naturally in one-loop expressions in string theory. In toroidally compactified heterotic string theory, the latter are typically of the following form

\[ I = \int_{\mathbb{H}} \frac{d^2 \tau}{\tau_2} \frac{1}{\tau_2^{m/2}} f(\tau, \bar{\tau}) \sum_{p \in \Gamma^{m,n}} q^{\frac{1}{2}|p|_{\mathbb{H}}^2} q^{\frac{1}{2}|p_R|_{\mathbb{H}}^2}, \quad \text{with} \quad m, n \in \mathbb{N} \quad \text{and} \quad q = e^{2\pi i \tau}. \]  

(6)

Here the integral is over the modular parameter of the world-sheet torus \( \tau = \tau_1 + i\tau_2 \in \mathbb{H}^+ \) [here \( \mathbb{H}^+ \) is the upper half-plane] with the integration region \( \mathbb{H} \) the fundamental domain of \( SL(2, \mathbb{Z}) \)

\[ \mathbb{H} = \{ \tau \in \mathbb{H}^+ | \tau_1^2 + \tau_2^2 \geq 1 \text{ and } -\frac{1}{2} \leq \tau_1 < \frac{1}{2} \} \]  

(7)

\( f(\tau, \bar{\tau}) \) is an (almost) modular form of weight \( \frac{m-n}{2} \) while \( \Gamma^{m,n} \) denotes the Narain lattice of signature \((m, n)\) with the left- and right-moving momenta \((p^L, p^R)\). The latter also encode the full (target space) moduli dependence of \( I \). Recalling that the modular invariant integral measure of \( SL(2, \mathbb{Z}) \) is \( d^2 \tau/\tau_2^{3/2} \) one can readily check that the integrand in (6) is modular invariant, which is necessary for the physical consistency of the amplitude \( I \). In the following we will mainly be interested in the case \( m = 2 \) but \( n \) generic. In this case, \( f \) has weight \( 1 - n/2 \) and we will see in many cases that it has exactly the right properties to define an automorphic product as in (5).

From a more mathematical point of view, integrals of the type (6) fall into the category of so-called \( (\text{singular}) \) theta correspondences. To understand this, we recall that the summation over \( \Gamma^{m,n} \) in (6) is a particular class of theta-function\(^2\) \( \Theta_n(\tau, v) \) with

\[ v \in \frac{O(2, n)}{O(2) \times O(n)}, \]

representing the moduli of the toroidal heterotic compactification (we recall that \( m = 2 \)). \( O(2, n) \) and \( SL(2, \mathbb{R}) \) form a dual reductive pair\(^3\) inside the metaplectic group (the double cover of the symplectic group \( Sp(2, n) \)). The decomposition of representations of the metaplectic group into tensor representations of the dual reductive pair induces a correspondence between representations of the two subgroups. In the case of \( SL(2, \mathbb{R}) \) and \( O(2, n) \) this correspondence can also be extended to automorphic forms associated with these representations [37, 24, 47], which concretely takes the form

\[ \Pi(v, f) = \int_{\mathbb{H}} \frac{d^2 \tau}{\tau_2} \Theta_n(\tau, v)f(\tau), \]

with \( \Pi \) a modular form of \( O(2, n, \mathbb{Z}) \). Since in general the integral in (6) will have singularities at particular points in moduli space (see e.g. section 4.4), we are usually dealing in string theory with a singular theta correspondence [12, 13].

3. Automorphic products with \( \mathcal{N} = 2 \) supersymmetry and the BPS algebra

3.1. \( \mathcal{N} = 2 \) threshold corrections

As a first appearance of automorphic products in string theory we will briefly review the results of [34, 35]. We consider the \( E_8 \times E_8 \) heterotic string compactified on \( K3 \times T^2 \) which preserves

\(^2\) In the literature it is indeed often called \textit{Siegel-Narain theta-function}.

\(^3\) Two subgroups of a group \( G \) form a dual reductive pair if each of them is the commutant of the other in \( G \). Moreover, at the level of representation theory, we are thinking in terms of the continuous group \( SL(2, \mathbb{R}) \).
\( \mathcal{N} = 2 \) space-time supersymmetry. Furthermore, we choose to work at a point in the moduli space where the gauge group is given by

\[
G = [U(1)^2]_{\text{left}} \times [E_7 \times U(1)^{10}]_{\text{right}},
\]

such that the original \( \Gamma_{6,22} \) Narain lattice decomposes as

\[
\Gamma_{6,22} = \Gamma_{4,4} \oplus \Gamma_{0,8} \oplus \Gamma_{2,10}.
\]

In this setting it was found in \([34, 35]\) that the one-loop threshold correction to the gauge coupling contains a term which can be written as

\[
\frac{1}{g^2} = \int \frac{d^2 \tau}{\tau_2} \frac{E_4^2(\tau)}{\eta^{24}(\tau)} \sum_{p \in \Gamma_{2,10}} q^{\frac{1}{2}|p^L|^2 q^{\frac{1}{2}|p^R|^2} + \ldots . \tag{8}
\]

Comparing this with (6) we have \( m = 2 \) and \( n = 10 \) and we extract the modular form

\[
f(\bar{\tau}) = \frac{E_4^2(\bar{\tau})}{\eta^{24}(\bar{\tau})} = \sum_{n=-1}^{\infty} c(n) q^n = \bar{q}^{-1} + 504 + 73764 \bar{q} + 2695040 \bar{q}^2 + \ldots , \tag{9}
\]

which has weight \(-4\). Here \( E_4(\tau) \) is an Eisenstein series of weight 4 and \( \eta(\tau) \) is the Dedekind eta-function. The integration over \( \tau \) in (8) is rather non-trivial, however, can be performed explicitly using e.g. methods first developed in \([21]\) and the explicit result can be written as \([34]\)

\[
\frac{1}{g^2} \sim -2 \log ||\Phi||^2 - c(0) \left( \log(-\langle \Re y \rangle^2) + \text{const.} \right) + \ldots , \tag{10}
\]

with the following modular form of \( O(2,10) \)

\[
\Phi(y) = e^{-2\pi \rho \cdot y} \prod_{r>0} \left( 1 - e^{-2\pi r \cdot y} \right)^{c(-r^2/2)} \quad \text{with} \quad y \in \mathbb{R}^{1,11} \otimes \mathbb{C}.
\]

It was shown in \([34]\) that \( \rho \) can be identified with the lattice Weyl vector of \( \varepsilon_{10} = \varepsilon_8^+ \) (which is the hyperbolic extension of \( \varepsilon_8 \)) and the range of the infinite product can be written as a product over \( \Lambda^+_{10} \)

\[
\Phi(y) = e^{-2\pi i(\rho \cdot y)} \prod_{\lambda \in \Lambda^+_{10}} \left( 1 - e^{-2\pi i(\lambda \cdot y)} \right)^{c(-\lambda^2/2)} . \tag{11}
\]

The \( || \cdot ||^2 \) in (10) takes into account a similar contribution with \( r < 0 \) which can be rewritten as a product over \( \Lambda^-_{10} \). Comparing (11) with (3) we see that \( \Phi(y) \) can be identified with the infinite product part of the denominator formula. Therefore, \( \Phi \) can be used to extend \( \varepsilon_{10} \) to the GKM \( G(\varepsilon_{10}) \) whose root multiplicities are encoded in the Fourier coefficients \( c(-\lambda^2/2) \): The simple real roots have \( \lambda^2 = 2 \), such that their multiplicity is \( c(-1) = 1 \) (see (9)). However, in addition to the latter there is an infinite set of further simple imaginary roots, which are the hallmark of a GKM as discussed in section 2.1.

\(^4\) For simplicity, we are focusing here on the special case \( s = 8 \) of the slightly more general class of examples discussed in \([34]\). Note that for later convenience we have exchanged left- and right moving side compared to \([34]\).
3.2. Algebra of BPS states
As already alluded to in the introduction, the result presented in the previous section has been interpreted in [34, 35] as an indication for particular algebraic properties of the spectrum of BPS states in string theory. Indeed, it has been argued that BPS states [which form a very particular class of states within the spectrum of string theory] should form an algebra. Moreover, threshold corrections of the type considered in (8) are BPS-saturated in the sense that they only receive contributions from such BPS states. The fact that part of these corrections can be written as the denominator formula of a GKM (10) was interpreted as a sign that this BPS algebra is a GKM. This line of argumentation was extended in [35] where it was attempted to construct this algebra microscopically and to identify a particular inner product structure which represents the corresponding GKM. An interesting question, however, is in how far these results depend on the particular model we have chosen. It is interesting to study different BPS-saturated objects than $\mathcal{N} = 2$ threshold corrections in theories with more supersymmetry and to explore the particular moduli dependence of such objects. In the following section we will therefore consider a particular class of so-called topological amplitudes and show their relation to particular GKM algebras.

4. Automorphic products with $\mathcal{N} = 4$ supersymmetry
4.1. Topological amplitudes
The first step in extending the results of [34, 35] to theories with more supersymmetry is to identify suitable BPS-saturated one-loop quantities. We will consider toroidally compactified heterotic string theory where in the recent years a number of so-called topological amplitudes has been found and studied extensively [9, 3, 5, 7] (see also [4] for a review). These are string theory amplitudes which can also be written as correlation functions in suitably twisted two-dimensional topological theories (for further examples in the context of $\mathcal{N} = 2$ supersymmetry see also [1, 6, 8]). From a field-theoretic perspective, they only receive contributions from BPS states and are therefore BPS-saturated which makes them the ideal candidate to play a similar role as the $\mathcal{N} = 2$ threshold corrections in the study of GKM algebras within string theory. To be precise, we will focus on a particular class of $\mathcal{N} = 4$ topological amplitudes which has first been discovered and extensively described in [3]. These quantities compute BPS couplings in the string effective action, which in harmonic superspace take the form (see [5] for further details)

$$S = \int d^4x \int du \int d^4\theta^- \int d^4\bar{\theta}^- (K_{\mu
u}^{++} K_{\mu
u}^{++,\mu\nu})^{g+1} \mathcal{F}_g(Y_A^{++}, u).$$

(12)

Here $u_{IA}^{\pm a}$ are the (bosonic) harmonic coordinates which parameterize the coset

$$u_{IA}^{\pm a} \in \frac{SU(4)}{SU(2) \times U(2)}.$$

We will use the notation that $A = 1, \ldots, 22$ is an index of $SO(22)$, $I = 1, \ldots, 4$ is an index of $SU(4) \cong SO(6)$, while $a = 1, 2$ and $\dot{a} = 1, 2$ are indices of either of the two $SU(2)$’s, and the $\pm$ signs denote the charge with respect to the diagonal $U(1)$. Moreover $K_{\mu\nu}^{IJ}$ is a particular super-descendant of the (linearized) $\mathcal{N} = 4$ supergravity multiplet, while $Y_A^{IJ}$ is a linearized $\mathcal{N} = 4$ vector-multiplet, whose lowest components $y_A^{IJ}$ form the moduli space

$$\mathcal{M}_{(6,22)} = \frac{SO(6,22)}{SO(6) \times SO(22)},$$

of the $\mathcal{N} = 4$ string compactification. $K_{\mu\nu}^{++}$ and $Y_A^{++}$ are respective projections with the harmonic coordinates.
The class of amplitudes $F_g$ in (12) corresponds to a series of $g$-loop amplitudes in type II string theory compactified on $K3 \times T^2$, while their dual counterparts in heterotic string theory compactified on $T^6$ start receiving contributions at the one-loop level. However, the latter will generically receive perturbative and non-perturbative corrections. Focusing on the leading contribution in the (heterotic) perturbative regime it was found in [3, 5] to be given by

$$F_g(y) = \int_F \frac{d^2 \tau}{\eta(\tau)^2} \frac{1}{2} G_{g+1}(\tau, \bar{\tau}) \Theta^{(6,22)}_g(\tau, \bar{\tau}, y),$$

where the integral is over the fundamental domain $F$ of $SL(2, \mathbb{Z})$ as defined in (7). Moreover, the expression

$$\Theta^{(6,22)}_g(\tau, \bar{\tau}, y) = \begin{cases} \sum_{p \in \Gamma^{6,22}} \frac{(p^L_{+,+})^{2g-2}}{q^{\frac{1}{2}|p^L|^2}} q^{\frac{1}{2}|p^R|^2} & g > 1 \\ \sum_{p \neq 0} \frac{1}{q^{\frac{1}{2}|p^L|^2}} & g = 1 \end{cases}$$

is again a Siegel-Narain theta-function of the even unimodular lattice $\Gamma^{6,22}$ with harmonically projected momentum insertions

$$p^L_{++}(y) = \frac{1}{2} e^{a_b u_{+a} J_{+b}^L} y^I, \quad \text{with} \quad y \in \mathcal{M}_{(6,22)}.$$ 

Finally, as above, $y^I_A$ are the moduli of the heterotic compactification and span $\mathcal{M}_{(6,22)}$ of the $N = 4$ string compactification (for details see [5]). For $g = 1$ we do not sum over $p = 0$ in the definition (14): since the amplitude $F_1$ has no explicit $p^L$-insertions it would receive contributions from $p = 0$. However, integrating this contribution over $\tau$ will diverge for any generic value of $y^I_A$, thereby rendering $F_1$ infinite. In order to avoid this singularity, we have chosen to regularize $F_1$ by performing the summation in (14) only over $p \neq 0$.

The function $G_g(\tau, \bar{\tau})$ in (13) is a weight 2$g$ non-antiholomorphic modular form and can be obtained as the coefficient of $\lambda^{2g}$ of the generating functional [2] (see also [8])

$$G(\lambda, \tau, \bar{\tau}) = \left( \frac{2 \pi i \lambda \eta^3}{\theta(\lambda, \bar{\tau})} \right)^2 e^{-\frac{\pi s^2}{2}} = \sum_{g=0}^{\infty} \lambda^{2g} G_g(\tau, \bar{\tau}).$$

$G_g$ can be written in terms of Eisenstein series in the following manner [45]

$$S_k(x_1, \ldots, x_k) = x_k + \cdots + x_k^{k-1}/(k!);$$

$$G_g(\tau, \bar{\tau}) = -S_g \left( \frac{\hat{E}_2}{2}, \frac{1}{2} \hat{E}_4, \ldots, \frac{1}{2g} \hat{E}_{2g} \right), \quad \text{with} \quad \hat{E}_2(\tau, \bar{\tau}) = \frac{\pi^2}{3} \left( \bar{E}_2(\bar{\tau}) - \frac{3}{\pi^2} \right),$$

$$\hat{E}_{2k}(\tau) = 2\zeta(2k) E_{2k}(\tau),$$

where $S$ are the Schur-polynomials and $E_{2k}$ the Eisenstein series of weight $2k$. Notice that $\hat{E}_2$ does not only transform with a weight under modular transformations but receives an additional anomalous shift-term. We have therefore introduced the quantity $\hat{E}_2$ which only transforms with weight 2 but is non-antiholomorphic in $\tau$.

4.2. Harmonicity relation and analytic amplitude

Upon inspecting the superspace coupling (12) we see that it only depends on one particular harmonic projection of the vector multiplets, which is compatible with the Grassmann
component of the integral measure. As was discussed in [5], for $g > 1$ this particular analytic dependence ('G-analyticity') manifests itself as two second-order differential equations for $F_g$.

$$\epsilon_{abc}^{JKL} \frac{\partial}{\partial u_{a+b}} D_{KL,A} F_g = (2g - 2)\bar{u}_a^I D_{++A} F_{g-1},$$

$$\left(\epsilon^{JKL} D_{IJ,A} D_{KL,B} + 4(g + 1)\delta_{AB}\right) F_g = 4D_{++A} D_{++B} F_{g-1},$$

where $D_{++A}$ are harmonic projections of the covariant derivatives $D_{IJ,A}$ in the moduli space $\mathcal{M}_{(6,22)}$. We will refer to these equations as the harmonicity and second order relation, respectively. In both equations the amplitude $F_{g-1}$ appears on the right hand side which from a field-theoretic perspective is an anomalous contribution in string theory, which generalizes the holomorphic anomaly [10] in $\mathcal{N} = 2$. It can be understood as an anomalous violation of the above mentioned G-analyticity of $F_g$ at the quantum level. Moreover, as has been pointed out in [23], analyzing the explicit computation of the anomaly in [5], it follows that its origin can be traced back to the holomorphic $\tau$-dependence of $G_g(\tau, \bar{\tau})$ in (15). We can use this circumstance to split $G_g$ into an analytic (anomaly-free) and a non-analytic (anomalous) part

$$G_g(\tau, \bar{\tau}) = G^{\text{analy}}_g(\tau) + G^{\text{non-analy}}_g(\tau, \bar{\tau})$$

with the explicit expressions (using (16))

$$G^{\text{analy}}_g(\tau) = -S_g\left(0, \frac{1}{2} \bar{\theta}_4, \ldots, \frac{1}{2g} \bar{\theta}_{2g}\right),$$

$$G^{\text{non-analy}}_g(\tau, \bar{\tau}) = -S_g\left(\bar{\theta}_2, \frac{1}{2} \bar{\theta}_4, \ldots, \frac{1}{2g} \bar{\theta}_{2g}\right) + S_g\left(0, \frac{1}{2} \bar{\theta}_4, \ldots, \frac{1}{2g} \bar{\theta}_{2g}\right),$$

which both have weight $2g$ under modular transformations. It is therefore consistent to define the purely analytic contribution to the amplitude as

$$F^{\text{analy}}_g(y) = \int_{\mathcal{M}} d^2\tau \frac{2g^{-1}G^{\text{analy}}_g(\tau)}{\eta^{24}} \Theta^{(6,22)}_g(\tau, \bar{\tau}, y),$$

which yields a vanishing anomaly when inserted into the harmonicity and second order relation. In the following we will study more carefully $F^{\text{analy}}_g(y)$ to highlight its connection to the denominator formula of a GKM.

4.3. Gauge group and Wilson line
In this section we will follow the discussion of [36]. In order to simplify the expression for $F_1$, we will first split the internal $T^4$ of the heterotic $\mathcal{N} = 4$ compactification into $T^3 \times T^2$ and take the large volume limit of $T^4$. The moduli space of such a compactification is described by the Kähler ($T$) and complex structure ($U$) moduli of $T^2$, as well as by two real Wilson lines $\bar{v}_{1,2} \in \mathbb{R}^{16}$ which we combine into a single complex vector $\bar{V} = \bar{v}_1 + i\bar{v}_2$. For the one-loop integrand of (17) this assumption implies that the Siegel-Narain theta function of the original $\Gamma^{6,22}$ Narain-lattice will be decomposed as

$$\frac{G^{\text{analy}}_2(\bar{\tau}) \tau^2}{\eta^{24}} \Theta^{(6,22)}_{g=1} \sim \text{Vol} \frac{G^{\text{analy}}_2(\bar{\tau})}{\eta^{24}} \Theta^{(2,18)}_{g=1},$$

where Vol is the volume of $T^4$ and $\Theta^{(2,18)}_{g=1}$ is the corresponding Siegel-Narain theta function for the lattice $\Gamma^{2,18}$. For later use we will denote the inner product on $\Gamma^{2,18}$ as $\langle \cdot | \cdot \rangle$. Moreover, in
the following we will consider the subspace of the moduli space where $\vec{V}$ breaks the $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ gauge symmetry to

$$\mathfrak{e}_8 \oplus \mathfrak{e}_8 \rightarrow \mathfrak{h}, \quad \text{with } \mathfrak{h} \oplus \mathfrak{g} \subset \mathfrak{e}_8 \oplus \mathfrak{e}_8,$$

(18)

where $\mathfrak{h}$ and $\mathfrak{g}$ are fixed subalgebras\(^5\) of $\mathfrak{e}_8 \oplus \mathfrak{e}_8$. This breaking of the gauge groups can best be understood through the decomposition of the Narain lattice $\Gamma^{2,18}$ (for more details see [23]): given $\vec{V}$ we denote by $\Lambda_\mathfrak{h}$ the sublattice of $\Lambda_{\mathfrak{e}_8} \oplus \Lambda_{\mathfrak{e}_8}$ consisting of all vectors that are orthogonal to the complex Wilson line $\vec{V}$

$$\Lambda_\mathfrak{h} = \{ \vec{d} \in \Lambda_{\mathfrak{e}_8} \oplus \Lambda_{\mathfrak{e}_8} : \vec{d} \cdot \vec{V} = 0 \}.$$

The vectors of length squared two in $\Lambda_\mathfrak{h}$ are the roots of what we would like to call the unbroken Lie algebra $\mathfrak{h}$. The commutant of $\mathfrak{h}$ in $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ defines the Lie algebra $\mathfrak{g}$, whose root lattice is spanned by the roots of $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ that are orthogonal to $\Lambda_\mathfrak{h}$. Adding to the corresponding root lattice the $T^2$ torus directions in $\Gamma^{2,18}$ leads to the sublattice $\Gamma_\mathfrak{g} \subseteq \Gamma^{2,18}$. Since $\Lambda_\mathfrak{h}$ is naturally a sublattice of $\Gamma^{2,18}$, we have

$$\Gamma_\mathfrak{g} \oplus \Lambda_\mathfrak{h} \subseteq \Gamma^{2,18},$$

and the sublattice on the left hand side is of maximal rank. Generically, $\Gamma_\mathfrak{g} \oplus \Lambda_\mathfrak{h}$ is a proper sublattice of $\Gamma^{2,18}$ with index $s$, thus we can write the decomposition

$$\Gamma^{2,18} = (\Gamma_\mathfrak{g} \oplus \Lambda_\mathfrak{h}) \oplus \bigoplus_{\mu=1}^{s-1} \left( \lambda_\mu + \Gamma_\mathfrak{g} \oplus \Lambda_\mathfrak{h} \right),$$

(19)

where $\lambda_\mu$ denotes the different cosets.

### 4.4. Singularities of the one-loop integral

Before explicitly performing the $\tau$ integration for the choice of gauge group (18), it is instructive to first consider possible singularities of $\mathcal{F}_{g=1}^{\text{analy}}(y)$. As we will see, this will already provide us some hints at interesting algebraic structures in the integral. To begin, we will write the BPS-integral in the following manner

$$\mathcal{F}_{g=1}^{\text{analy}} = \int_{\mathcal{F}} \frac{d^2 \tau}{\eta^2 \tau^2} \tau_2^{\text{analy}}(\bar{\tau}) \sum_{p \in \Gamma^{2,18}} \bar{q}^D e^{-\pi \tau_2 |p^L|^2},$$

(20)

where $D = \frac{1}{2} (|p^R|^2 - |p^L|^2)$. To understand the mechanism by which a singularity might occur in (20) (see also [12, 34, 13] for related discussions), we observe that the upper boundary of the $\tau_2$ integration is at infinity. At a generic point in moduli space, the integrand is rendered finite for $\tau_2 \rightarrow \infty$ due to the exponential suppression of the Narain momenta. However, at particular points in moduli space this damping might fail, thus leading to a logarithmic divergence. A necessary condition for such a divergence to appear is

$$|p^L| = 0,$$

(21)

\(^5\) For simplicity we will assume in this review that $\mathfrak{g}$ is simple. For a discussion of the semi-simple case see [23].
since then the suppression induced by the factor $e^{-\pi \tau_{1} |p|^2}$ in (20) is absent. However, condition (21) alone is not sufficient. Indeed, for $|p\|^2 = 0$ the $\tau_{1}$-integration will pick the coefficient of $q^{0}$ in the expansion

$$
\frac{G_{2}^{\text{analy}}(\bar{\tau})}{\eta^{24}(\bar{\tau})} \sum_{p \in \Gamma^{2,18}} q^{\frac{1}{2} |p|^2} = \sum_{p \in \Gamma^{2,18}} \sum_{n=-1}^{\infty} d(n) q^{n+D}.
$$

Therefore, if $d(-D) \neq 0$ for some vector in the sum over $\Gamma^{2,18}$, we indeed find a logarithmic divergence. Following [36], we will focus on the singularities with $D = 1$, which have an interesting algebraic interpretation. To see this, we fix a vector $p \in \Gamma^{2,18}$ with $\langle p|p \rangle = 2$, and determine those values of the moduli $y = (U,T,\bar{V})$ such that $p\bar{L} = 0$. As it was explained in [23], due to $SO(2,18)$ invariance of the integral [this is just the familiar T-duality action] it is in fact sufficient to focus on vectors $\hat{p} \in \Gamma^{1,17} \subset \Gamma^{2,18}$. Recalling moreover the choice of gauge group (18), out of these only the vectors $\hat{p} \in \Gamma^{1,17} \cap \Gamma_{\bar{g}} = \Lambda_{\bar{g}^{++}}$ will be of interest. Here $\Lambda_{\bar{g}^{++}}$ is the root lattice of the hyperbolic extension of $\bar{g}$ (see section 2.1; as an example, the Dynkin diagram of $a_{k}^{++}$ is depicted in figure 1). The norm condition

$$
D = 1 \iff \langle \hat{p}|\hat{p} \rangle = 2,
$$

then implies that only the real roots of $\bar{g}^{++}$ are of importance, which means that we can identify the singular loci of $F_{\gamma=1}^{\text{analy}}(y)$ with fixed points of the Weyl reflections $w_{\alpha} \in \mathcal{W}(\bar{g}^{++})$. The latter act as

$$
w_{\alpha} : y \mapsto y - \langle y|\alpha \rangle \alpha,
$$

on the moduli, where $\langle \cdot|\cdot \rangle$ is the inner product on the lattice $\Lambda_{g^{++}}$, which is inherited from $\Gamma^{2,18}$. The group $\mathcal{W}(\bar{g}^{++})$ is generated by the $k + 2$ reflections $w_{I}$, where $I = -1,0,\ldots,k = \text{rk}(\bar{g})$ and $w_{I}$ is the reflection associated to the simple root $\alpha_{I}$ of $\bar{g}^{++}$. The latter may be taken to be

$$
\alpha_{-1} = (1,-1;\bar{\theta}), \quad \alpha_{0} = (-1,0;-\bar{\theta}), \quad \alpha_{i} = (0,0;\bar{e}_{i}), \quad i = 1,\ldots,k,
$$

where $\bar{\theta}$ is the highest root of $\bar{g}$, and $\bar{e}_{i}$ are the simple roots of $\bar{g}$. The Weyl reflection associated to any real root in $\bar{g}^{++}$ can then be written as a finite product of the Weyl reflections associated to the simple roots. With the basis given above, the singular divisors can then be explicitly given to be

$$
\mathcal{D}_{-1} = \{ y \in \mathcal{M}_{2,2+k} \mid \langle y|\alpha_{-1} \rangle = U - T = 0 \}
$$

$$
\mathcal{D}_{0} = \{ y \in \mathcal{M}_{2,2+k} \mid \langle y|\alpha_{0} \rangle = T - \bar{\theta} \cdot \bar{V} = 0 \}
$$

$$
\mathcal{D}_{i} = \{ y \in \mathcal{M}_{2,2+k} \mid \langle y|\alpha_{i} \rangle = \bar{e}_{i} \cdot \bar{V} = 0 \}.
$$

![Figure 1. Dynkin diagram of the double extensions $a_{k}^{++}$ of $a_{k}$. The nodes corresponding to the roots $\alpha_{0}$ and $\alpha_{-1}$ are called affine- and hyperbolic node respectively and are the key-features of the hyperbolic extension.](image-url)
Note that the divisor $D_I$ describes a boundary component of $\mathcal{M}_{2,2+k}$ since for $y \in D_I$ a root of $\mathfrak{g}$ becomes orthogonal to $\bar{V}$, and hence should have been part of $\mathfrak{h}$. Put differently, for a given point $y$ in the moduli space $\mathcal{M}_{2,2+k}$ the divisors $D_I$ represent the ‘dominant’ walls of the complexified Weyl chamber of $\mathfrak{g}^{\mathbb{C}}$:

$$C_C = \{ y \in \mathcal{M}_{2,2+k} \, | \, D_I \geq 0, \, I = -1, 0, 1, \ldots k \},$$

and the only singularities appear at the boundary of $C_C$.

### 4.5. Torus integral and denominator formula

After having discussed the singularities of the integral (17) which has given us first hints to a connection with a GKM algebra, we will now demonstrate the appearance of a denominator formula. To see the latter, one has to explicitly perform the world-sheet torus integration. This has been done in [23, 36] using the method of orbits first introduced in [21] which have been further developed in [34].

The first step is to implement the choice of gauge group in (18) as

$$\frac{G_2^{\text{analy}}}{\bar{q}^{24}}(\tau) \Theta_{g=1}^{(2,18)} = \sum_{\mu=0}^{r-1} P_{\mu}^{(k)}(\bar{\tau}) \Theta_{\mu}^{(2,2+k)}(\tau, \bar{\tau}, y),$$

where $\Theta_{\mu}^{(2,2+k)}(\tau, \bar{\tau}, y)$ is the theta series associated to the $\Gamma_g$ coset in the decomposition of (19)

$$\Theta_{\mu}^{(2,2+k)}(y) = \sum_{x \in \Gamma_g + \lambda_{\mu}^0} q^{\frac{1}{2} \langle x | x \rangle} e^{2 \pi i \tau \sum_{(\mu)(\eta_{(\mu)(\eta)}^{(2,2+k)} = 1}. \]

while $P_{\mu}^{(k)}(\bar{\tau})$ captures the contributions from $\Lambda_h$ (as well as the parts coming from $G_2^{\text{analy}}$ and the powers of the $\bar{q}$-function).

$$P_{\mu}^{(k)}(\bar{\tau}) = \frac{G_2^{\text{analy}}}{\bar{q}^{24}}(\bar{\tau}) = \frac{G_2^{\text{analy}}}{\bar{q}^{24}}(\bar{\tau}) \sum_{\ell \in \Lambda_h + \lambda_{\mu}^0},$$

where $\lambda_{\mu}^0$ and $\lambda_{\mu}^b$ are the projections of the glue vector $\lambda_{\mu}$ in (19) onto $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Since $\Lambda_h$ is a sublattice of $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ that is orthogonal to $\bar{V}$, $P_{\mu}^{(k)}(\bar{\tau})$ does not depend on the moduli $y = (U, T; \bar{V})$. Using (22), the full integral (20) was given explicitly in [36] (see also [23])

$$F_{g=1}^{\text{analy}} = 8 \sum_{\mu=1}^{s} \sum_{\ell \in \Lambda_h + \lambda_{\mu}^0} \frac{2\pi Y}{3U_2} \left( c_{\mu}(0, \ell) - 24c_{\mu}(-1, \ell) \right) + 2 \log \left| 1 - e^{2\pi i \bar{\ell} \cap \bar{V}} \right| c_{\mu}(0, \bar{\ell})$$

$$+ 2 \log \prod_{n', r \in \mathbb{Z}, \ell' > \ell} \left| 1 - e^{2\pi i (rT + n'TU + \ell' \cap \bar{V})} \right| c_{\mu}(n', \ell') + 2 \log \prod_{n=1}^{\infty} \left| 1 - e^{2\pi i nU \bar{\ell} \cap \bar{V}} \right| c_{\mu}(0, \bar{\ell})$$

$$+ c_{\mu}(0, \bar{0}) \left( \frac{\pi U_2}{3} - \ln Y + K \right) + 2 \log \prod_{n=1}^{\infty} \left| 1 - e^{2\pi inU \bar{\ell} \cap \bar{V}} \right| c_{\mu}(0, 0)$$

$$+ \frac{2U_2}{3\pi} + \frac{2\pi}{U_2} (\bar{\ell} \cap \Im \bar{V}) \left( (\bar{\ell} \cap \Im \bar{V}) + U_2 \right),$$

In the mathematics literature it is usually known as the Rankin-Selberg method.
where \( K = \gamma_E - 1 - \ln \frac{8\pi}{3\sqrt{3}} \), with \( \gamma_E \) being the Euler-Mascheroni constant and \( Y = (3y|3y) \). Furthermore, we have introduced the shorthand notation for the modified scalar-product
\[
\ell \odot \bar{V} = \ell \cdot \Re V + i |\ell \cdot \Im V|.
\]
The coefficients \( c_\mu(n',\ell) \) arise from the Fourier expansion
\[
\sum_{\mu=0}^{s-1} P_\mu^{(k)}(\bar{\tau}) \sum_{\ell \in \Lambda_g + \lambda^g_{\mu}} q^{1/2 \ell \cdot \ell} e^{2\pi i \ell \cdot z} = \sum_{\mu=0}^{s-1} \sum_{n=-1}^{\infty} \sum_{\ell \in \Lambda_g + \lambda^g_{\mu}} c_\mu(n,\ell) q^n e^{2\pi i \ell \cdot z}.
\]
By construction the left hand side of (25) transforms as a weak Jacobi form under \( SL(2,\mathbb{Z}) \), and in fact every summand in the sum over \( \mu \) transforms as a weak Jacobi form under some congruence subgroup. Thus the coefficients \( c_\mu(n,\ell) \) only depend on \( (n,\ell) \) through the combination [22]
\[
c_\mu(n,\ell) \equiv c_\mu \left(n - \frac{1}{2} \ell \cdot \ell \right).
\]
Moreover, by inspection of (23) it is clear that \( P_\mu^{(k)}(\bar{\tau}) \) has a simple pole at \( \bar{\tau} \to i\infty \), and hence
\[
c_\mu \left(n - \frac{1}{2} \ell \cdot \ell \right) = 0 \quad \forall n - \frac{1}{2} \ell \cdot \ell < -1.
\]
In the following we shall mainly be interested in the contribution of the trivial conjugacy class labeled by \( \mu = 0 \). We will show that this contribution can be identified with the infinite product side of the denominator formula for the Borcherds extension \( G(g^{++}) \), where \( g^{++} \) is the double extension of the unbroken gauge algebra \( g \). Moreover, within the \( \mu = 0 \) contribution we will concentrate on the logarithmic terms. Most of the non-logarithmic terms contribute to the Weyl vector \( \rho \), appearing in the exponential prefactor of the denominator formula, and ensure that the whole denominator formula will be invariant under a congruence subgroup of \( SO(2,2+r,\mathbb{Z}) \). Thus, the relevant part of (24) can be written as
\[
\log ||\Phi_g(y)||^2,
\]
where \( \Phi_g(y) = \prod_{(r,n';\ell)>0} \left(1 - e^{2\pi i (rT + n'U + \ell \cdot \bar{V})} \right)^{c_\alpha(n',\ell)} \),
\]
where the range of the product \( (r,n';\ell) > 0 \) is defined by
\[
n'\ell - \frac{1}{2} \ell \cdot \ell \geq -1, \quad \text{and} \quad \begin{cases} r > 0, \ n' \in \mathbb{Z}, \ \ell \in \Lambda_g & \text{or} \\ r = 0, \ n' > 0, \ \ell \in \Lambda_g^+ & \text{or} \\ r = n', \ \ell \in \Lambda_g^+ & \end{cases}
\]
The norm \( || \cdot ||^2 \) in (26) takes into account that there are contributions with \( (r,n';\ell) > 0 \) and contributions with \( (r,n';\ell) < 0 \). It was shown in [36] that (27) are just the conditions that characterize the elements \( \alpha \) of \( \Lambda_{g^{++}}^+ \) with norm \( \alpha^2 \leq 2 \). (Note that the correspondence only holds provided that we require this norm condition.) Thus we can then write \( \Phi_g(y) \) as a product over the positive roots \( \Delta_{g^+}^+ \) of the GKM-algebra \( G_g \)
\[
\Phi_g(y) = \prod_{\alpha \in \Delta_{g^+}^+} \left(1 - e^{2\pi i (\alpha|y)} \right)^{c_\alpha(-\alpha^2/2)}.
\]
where we have used that $c_0(n) = 0$ for $n < -1$. Following the reasoning of [11, 12], the automorphic correction $\mathcal{G}(q^{++})$ is by construction a generalized Kac-Moody algebra. This implies in particular that the product includes imaginary simple roots. While the multiplicity of all real positive roots is given by $c_0(-1) = 1$, the multiplicities of the imaginary roots are encoded via (28) in the remaining Fourier coefficients $c_0\left(n'\ell - \frac{\ell \cdot r}{2}\right)$. Finally, the norm $\| \cdot \|^2$ in (26) is now interpreted as splitting the infinite product into contributions from positive and negative roots.

5. Conclusions
In this work we have reviewed the appearance of automorphic products in BPS-saturated one-loop quantities in heterotic string theory with $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetry. Following the work of Borcherds [11, 12], these products can be used to construct generalized Kac-Moody algebras as automorphic corrections of hyperbolic Kac-Moody algebras, which are related in a natural manner to the gauge group of the heterotic model. At a particular point in moduli space, the Weyl group of this Borcherds algebra appears directly in the physical quantities, since it governs the singular loci in moduli space.

Here we have focused on two particular examples, namely the extension of $e_{10}$ in the case of $\mathcal{N} = 2$ supersymmetry and the extension of $q^{++}$ (with $q \subset e_8$) in the case of $\mathcal{N} = 4$. Additional examples can be found in the original works, i.e. [34, 35] and [23, 36] respectively. In the latter case, the analysis was performed for an arbitrary choice of the gauge group (determined by the choice of Wilson-line moduli) and thereby a full family of Borcherds algebras was constructed.

For the future it would be interesting to further extend this analysis: Particularly in the $\mathcal{N} = 4$ example, the limit of large $T^4$-volume in the four-dimensional heterotic compactification has been assumed. As was explained in [36] this is the reason why the Borcherds algebra which we have constructed is an extension of a hyperbolic Kac-Moody algebra. It would be interesting to repeat the analysis with finite $T^4$ volume and evaluate the integral $\mathcal{F}_1$ for the full Narain lattice of the six-torus $T^6$. In this case one might expect that the $\mathcal{F}_1$ integral encodes the denominator formula of a GKM which corresponds to the automorphic correction of a more general indefinite Kac-Moody algebra. However, since the Narain moduli space $SO(6,22)/(SO(6) \times SO(22))$ is then no longer a hermitian symmetric domain and it is therefore unclear whether the theta correspondence affords an infinite product representation which can be related to a denominator formula [13] (see also [42, 46] for related discussions).

Moreover, it would be highly desirable to have a further interpretation of the Fourier coefficients which enter into the automorphic product. From the algebraic point of view they encode the root-multiplicities of the imaginary roots of the GKM algebras. However, since the coupling $\mathcal{F}_1$ is topological, one would expect that they can be understood as some generalization of Gopakumar-Vafa invariants from the $\mathcal{N} = 2$ context (see [25, 26]). In particular, since heterotic string theory is compactified on $T^6$ is dual to type II string theory on $K3 \times T^2$, one might wonder whether these coefficients encode (basic) topological data of $K3$.

Finally, in this review we have focused on extracting denominator formulae from one-loop objects which appear in the heterotic effective action. Our approach was therefore in a sense macroscopical and in particular did not involve an understanding of the GKM algebra at the level of the BPS-(micro)states, as originally envisaged in the work of Harvey and Moore [34, 35]. A closer study of this aspect has been attempted in [23] (see also [36], where explicitly a Borcherds algebra $\mathcal{G}$ was constructed which acts naturally on BPS states. Put differently, BPS states fall into particular representations of $\mathcal{G}$. Notice that this is a slightly modification of the original idea of Harvey and Moore [34, 35]: There it was proposed that the full space of BPS-states would be made into a Lie algebra. In connection with the physical amplitudes it was moreover shown in [23, 36] that (at a generic point in moduli space) $\mathcal{F}_1$ encodes the root multiplicities.
of some slightly bigger GKM-algebra, of which $\mathcal{G}$ is a subalgebra. It will be clearly interesting to obtain a better understanding of the connection between the microscopic construction of the algebra and further macroscopic quantities. For example it might be interesting to consider the appearance of $\mathcal{G}$ in different BPS-saturated objects than $\mathcal{F}_1$ (see e.g. [3]).

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