A CONSTRUCTIVE APPROACH TO ROBUST CHAOS USING INVARIANT MANIFOLDS AND EXPANDING CONES

PAUL A. GLENDINNING

Department of Mathematics
University of Manchester, Oxford Road
Manchester, M13 9PL, UK

DAVID J. W. SIMPSON*

School of Fundamental Sciences
Massey University, Colombo Road
Palmerston North, 4410, New Zealand

(Communicated by Rafael de la Llave)

Abstract. Chaotic attractors in the two-dimensional border-collision normal form (a piecewise-linear map) can persist throughout open regions of parameter space. Such robust chaos has been established rigorously in some parameter regimes. Here we provide formal results for robust chaos in the original parameter regime of [S. Banerjee, J.A. Yorke, C. Grebogi, Robust Chaos, Phys. Rev. Lett. 80(14):3049–3052, 1998]. We first construct a trapping region in phase space to prove the existence of a topological attractor. We then construct an invariant expanding cone in tangent space to prove that tangent vectors expand and so no invariant set can have only negative Lyapunov exponents. Under additional assumptions we characterise an attractor as the closure of the unstable manifold of a fixed point and prove that it satisfies Devaney’s definition of chaos.

1. Introduction. A fundamental difference between smooth and piecewise-smooth dynamical systems is the possibility of robust chaos. This refers to the existence of a chaotic attractor throughout open regions of parameter space. This cannot happen, for instance, in typical families of smooth one-dimensional maps because in this case periodic windows are typically dense in parameter space [35]. Robust chaos is highly desirable in applications that use chaos. In chaos-based cryptography [21], for example, robust chaos is preferred because periodic windows in ‘key space’ can be usurped by a hacker to decipher the encryption [1].

One of the most widely studied families of piecewise-smooth maps is the two-dimensional border-collision normal form

---

2020 Mathematics Subject Classification. Primary: 37G35; Secondary: 39A28.

Key words and phrases. Piecewise-linear, piecewise-smooth, border-collision bifurcation, Lyapunov exponent, robust chaos.

The authors were supported by Marsden Fund contract MAU1809, managed by Royal Society Te Apírangi.

*Corresponding author: D. J. W. Simpson.
\[
\left[\begin{array}{c}
x \\
y
\end{array}\right] \mapsto f(x, y) = \left\{
\begin{array}{ll}
\tau_L & 1 \\
-\delta_L & 0 & x & \leq 0, \\
\tau_R & 1 & x & > 0
\end{array}
\right.
\]

where \(\tau_L, \delta_L, \tau_R, \delta_R \in \mathbb{R}\) are parameters. This was introduced in [26], except in (1) the constant term is \([1, 0]^T\) instead of \([\mu, 0]^T\), where \(\mu \in \mathbb{R}\). Via a linear rescaling, \(\mu \neq 0\) can be transformed to \(\mu = \pm 1\), and the choice \(\mu = 1\) can be made by interchanging the roles of \(x < 0\) and \(x > 0\). The border-collision normal form arises by transforming and truncating a piecewise-smooth map that has a border-collision bifurcation at \(\mu = 0\) [30]. In this way any border-collision bifurcation (that satisfies certain non-degeneracy conditions) can be understood using the normal form and this has been demonstrated in diverse applications [13]. Many groups have described non-chaotic dynamics of (1) in detail, see for instance [2, 32, 34, 39].

In a highly influential paper, Banerjee, Yorke, and Grebogi [3] considered (1) in a certain parameter regime \(\mathcal{R}\) where \(f\) is orientation-preserving (i.e. \(\delta_L > 0\) and \(\delta_R > 0\)). Based on the intersections of the stable and unstable manifolds of two fixed points, they argued heuristically that \(f\) has a unique chaotic attractor. Their arguments apply throughout \(\mathcal{R}\), so suggest robust chaos. Although their arguments are incomplete, their conclusions have been well supported by numerical investigations.

In this paper we prove for the first time that \(f\) has an attractor that is chaotic, in a certain sense, throughout \(\mathcal{R}\). Subject to additional restrictions on the parameter values we also characterise the attractor and show that it satisfies Devaney’s definition of chaos [12]: transitivity, dense periodic orbits, and sensitive dependence on initial conditions. Note that sensitive dependence is actually implied by transitivity and dense periodic orbits [4, 17], and we exploit this in our proof. We also use methods developed by Misiurewicz [25] for the Lozi map (given by (1) with \(\tau_L = -\tau_R\) and \(\delta_L = \delta_R\)), and Benedicks and Carleson [6] for smooth maps.

Much has been done on chaos in the Lozi map. For an orientation-reversing parameter regime Misiurewicz identified a hyperbolic cone structure and proved the existence of a unique attractor on which the map is transitive. Subsequently Collet and Levy [9] showed that this attractor supports an SRB measure (and so has many nice ergodic properties [20]), while Rychlik [29] obtained similar results for orientation-preserving parameters (see also [8]). However, the Lozi map was created as a piecewise-linear approximation to the Hénon map [23] and serves primarily as an example of a multi-dimensional chaotic map that is reasonably tractable. Its analysis provides a first step towards understanding chaos in the Hénon map and smooth systems more generally. To understand chaos associated with two-dimensional border-collision bifurcations it is necessary to work with (1) as this map has sufficiently many parameters.

Chaos in (1) was studied by Kowalczyk [22] in the case \(\delta_R = 0\) for which one-dimensional techniques suffice. With \(\delta_L \delta_R \leq 0\) (where \(f\) is non-invertible) Glendinning [18] identified parameter regimes where \(f\) has a (necessarily chaotic) two-dimensional attractor by using general results on piecewise-expanding maps. Recently Glendinning [19] used Young’s theorem [38] to prove that in certain subsets of \(\mathcal{R}\) there exists an attractor with an SRB measure.

The remainder of this paper is organised as follows. We first define \(\mathcal{R}\) and state our main results in §2. In §3 we identify a trapping region \(\Omega_{\text{trap}}\) that necessarily
contains a topological attractor. Then in §4 we study the evolution of tangent vectors and identify a cone in tangent space that is forward invariant and expanding under $Df$. On the invariant expanding cone, tangent vectors expand under every iteration of $f$. Thus if an attractor has well-defined Lyapunov exponents, one of these exponents must be positive, §5. One could also construct an invariant expanding cone for $f^{-1}$ to establish uniform hyperbolicity as in [25, 29] but we do not do this here as it is not needed for our results.

In subsequent sections we seek to make more precise statements, and to this end assume that both fixed points have an eigenvalue with absolute value greater than $\sqrt{2}$. In §6 we analyse the closure of the unstable manifold of one fixed point, and in §7 we show that on this set $f$ is transitive and has dense periodic orbits, thus also exhibits sensitive dependence on initial conditions. Finally, §8 provides a discussion and outlook for future studies.

2. Preliminaries and main results.

2.1. The fixed points and their invariant manifolds. Let

$$A_L = \begin{bmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{bmatrix}, \quad A_R = \begin{bmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{bmatrix}, \quad (2)$$

denote the matrices in (1). As in [3], throughout this paper we assume

$$\delta_L > 0, \quad \delta_R > 0,$$

$$\tau_L > \delta_L + 1, \quad \tau_R < -(\delta_R + 1). \quad (3)$$

This is equivalent to assuming that $A_L$ has eigenvalues $0 < \lambda^s_L < 1 < \lambda^u_L$ and $A_R$ has eigenvalues $\lambda^s_R < -1 < \lambda^u_R < 0$. Then $f$ has two fixed points:

$$Y = (Y_1, Y_2) = \left( \frac{-1}{\tau_L - \delta_L - 1}, \frac{\delta_L}{\tau_L - \delta_L - 1} \right), \quad (4)$$

$$X = (X_1, X_2) = \left( \frac{1}{\delta_R + 1 - \tau_R}, \frac{-\delta_R}{\delta_R + 1 - \tau_R} \right), \quad (5)$$

where $Y_1 < 0$ and $X_1 > 0$. These are saddle-type fixed points because the eigenvalues associated with $Y$ and $X$ are simply those of $A_L$ and $A_R$, respectively.

As with smooth maps, the stable and unstable subspaces of $Y$ and $X$ are lines intersecting $Y$ and $X$ and with slopes matching those of the eigenvectors of $A_L$ and $A_R$. Since $f$ is piecewise-linear, the stable and unstable manifolds of $Y$ and $X$ initially coincide with their corresponding subspaces as they emanate from $Y$ and $X$. Globally the stable and unstable manifolds have a complicated piecewise-linear structure due to the piecewise-linear nature of $f$. Strictly speaking they are not manifolds in view of their ‘kinks’, but this is just a matter of notation.

To understand this structure, observe that $f$ is continuous but non-differentiable on $x = 0$, the switching manifold. The image of the switching manifold is $y = 0$. Thus if $\alpha \subseteq \mathbb{R}^2$ is a line segment that intersects $x = 0$ transversally, then $f(\alpha)$ is the union of two line segments that meet at a point on $y = 0$. Thus the unstable manifolds have kinks at points on $y = 0$, and on the forward orbits of these points. Similarly the stable manifolds have kinks at points on $x = 0$, and on the backward orbits of these points.

Since the eigenvalues associated with $Y$ are positive, the stable and unstable manifolds of $Y$, $W^s(Y)$ and $W^u(Y)$, each have two dynamically independent branches. In the direction of decreasing $x$ they simply coincide with the stable and unstable
subspaces of \( Y \): \( E^s(Y) \) and \( E^u(Y) \). In the direction of increasing \( x \), let \( D = (D_1, 0) \) and \( S = (0, S_2) \) denote the first kinks of \( W^s(Y) \) and \( W^u(Y) \) as we follow these manifolds outwards from \( Y \), see Fig. 1. By using the fact that the line segments \( YD \) and \( YS \) are contained within \( E^s(Y) \) and \( E^u(Y) \), it is a simple exercise to obtain
\[
D_1 = \frac{1}{1 - \lambda_L^s}, \tag{6}
\]
\[
S_2 = \frac{-\lambda_L^u}{\lambda_L^u - 1}. \tag{7}
\]
Notice \( D_1 > 1 \) and \( S_2 < -1 \).

2.2. The parameter regime \( \mathcal{R} \). As we continue to follow the stable manifold \( W^s(Y) \) outwards from \( Y \), the manifold has its second kink at \( f^{-1}(S) \). Due to the constraints \( (3) \), the point \( f^{-1}(S) \) lies in the first quadrant \( x, y > 0 \). Let \( C = (C_1, 0) \) denote the intersection of \( SF^{-1}(S) \) with \( y = 0 \). If \( C_1 > D_1 \), that is, \( C \) lies to the right of \( D \), then the quadrilateral \( YDCS \) is forward invariant under \( f \) (see Lemma 1 of \cite{19} and compare Lemma 3.1 below). If instead \( C_1 < D_1 \), then \( f(D) \) lies outside \( YDCS \) and so this quadrilateral is not forward invariant. Numerical explorations suggest that \( f \) has no attractor in this case.

From \( (1) \) we immediately obtain
\[
C_1 = \frac{-S_2}{\delta_R - \tau_R + \frac{\delta_L}{S_2}}. \tag{8}
\]
By then combining \( (6) - (8) \) we obtain, after much simplification,
\[
C_1 - D_1 = \frac{\phi(\tau_L, \delta_L, \tau_R, \delta_R)}{(\tau_L - \delta_L - 1)(\delta_R - \tau_R \lambda_L^u)}.
\tag{9}
\]
where
\[
\phi(\tau_L, \delta_L, \tau_R, \delta_R) = \delta_R - (\tau_R + \delta_L + \delta_R - (1 + \tau_R)\lambda_L^u)\lambda_L^u. \tag{10}
\]

\textbf{Figure 1.} Initial portions of the stable and unstable manifolds of the fixed point \( Y \). Throughout this paper stable and unstable manifolds are coloured blue and red respectively.
Since the denominator of (9) is positive by (3), the condition $\phi > 0$ ensures that $C_1 > D_1$. The parameter region $\mathcal{R}$ of [3] is defined by the constraints (3) and $\phi > 0$, see Fig. 2.

2.3. Lyapunov exponents. Let $\Sigma_\infty \subseteq \mathbb{R}^2$ be the set of points whose forward orbits intersect $x = 0$. Then the Jacobian matrix $Df^n(z)$ is well-defined for all $z \in \mathbb{R}^2 \setminus \Sigma_\infty$ and all $n \geq 1$. The Lyapunov exponent of a point $z \in \mathbb{R}^2 \setminus \Sigma_\infty$ in a direction $v \in \mathbb{T}\mathbb{R}^2$ is defined as

$$\lambda(z, v) = \lim_{n \to \infty} \frac{1}{n} \ln(\|Df^n(z)v\|),$$

assuming this limit exists. Throughout this paper $\| \cdot \|$ denotes the Euclidean norm on $\mathbb{R}^2$. Oseledets’ theorem [5, 15, 36] gives conditions under which (11) is well-defined for almost all points in an invariant set. The Lyapunov exponent represents the asymptotic rate of expansion in the direction $v$. For bounded invariant sets positive Lyapunov exponents are part of many definitions of chaos. The following theorem uses Lyapunov exponents to demonstrate robust chaos throughout $\mathcal{R}$.

**Theorem 2.1.** Suppose (3) is satisfied and $\phi > 0$. Then (1) has a topological attractor $\Lambda$ with the property that for any $z \in \Lambda \setminus \Sigma_\infty$, if the limit (11) exists with $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $\lambda(z, v) > 0$.

We have not been able to show that the conditions of Oseledets’ theorem are satisfied, or verify that the limit (11) exists directly. However, below we actually show that the infimum limit of the right hand-side of (11) is positive, thus even if the limit does not exist nearby orbits still diverge exponentially. Although the two-dimensional Lebesgue measure of $\Sigma_\infty$ is zero (because it is a countable union of measure zero sets), we do not know that $\mu(\Sigma_\infty) = 0$, where $\mu$ is the invariant probability measure associated with $\Lambda$. Also, it is not known whether or not $\Lambda$ is unique, although numerical simulations by several authors have failed to find parameter values in $\mathcal{R}$ for which $f$ has multiple attractors.

![Figure 2](image-url)  
**Figure 2.** The parameter region $\mathcal{R}$: (3) and $\phi > 0$, where $\phi$ is given by (10). The striped region indicates parameter values valid for Theorem 2.3. (This figure was created using $\delta_L = 0.2$ and $\delta_R = 0.4$.)
2.4. A homoclinic connection and chaos in the sense of Devaney. Next we describe $W^s(X)$ and $W^u(X)$ in more detail. Since the eigenvalues associated with $X$ are negative, $W^s(X)$ and $W^u(X)$ each have one dynamically independent branch. Let $T = (T_1, 0)$ denote the intersection of $E^u(X)$ with $y = 0$, and let $V = (0, V_2)$ denote the intersection of $E^s(X)$ with $x = 0$, see Fig. 3. Then $W^u(X)$ coincides with $E^u(X)$ on $T f(T)$, and $W^s(X)$ coincides with $E^s(X)$ on $V f^{-1}(V)$ where $f^{-1}(V)$ lies in the first quadrant (it is not visible in Fig. 3 as it lies far outside the scale shown).

As we follow $W^u(X)$ outwards, the first part of $W^u(X)$ that does not coincide with $E^u(X)$ is the line segment $T f^2(T)$. Let

$$Z = T f^2(T) \cap E^s(X),$$

if this point of intersection exists. The point $Z$ corresponds to a transverse intersection between the stable and unstable manifolds of $X$ and implies there exists a chaotic orbit. This transverse intersection exists if and only if $f^2(T)$ lies to the left of $E^s(X)$, which can be equated to a condition on the parameter values of $f$ (see Lemma 2 of [19]).

Assuming $Z$ exists, let $\Delta_0$ be the (compact filled) triangle $XTZ$. Then $\Delta = \bigcup_{n=0}^{\infty} f^n(\Delta_0)$ is forward invariant. Also let $\tilde{\Delta} = \bigcap_{n=0}^{\infty} f^n(\Delta)$. The next result gives conditions under which $\tilde{\Delta}$ is well-defined, equal to the closure of $W^u(X)$, and satisfies the following well-known definition of chaos:

**Definition 2.2.** An invariant set $\Lambda \subseteq \mathbb{R}^2$ of $f$ is **chaotic in the sense of Devaney** [12] if, on $\Lambda$, $f$

i) is transitive,

ii) has dense periodic points, and

iii) exhibits sensitive dependence on initial conditions.
Condition (iii) is actually a consequence of (i) and (ii) \cite{4,17} but we have retained (iii) in the definition to emphasize that sensitive dependence on initial conditions has been established and to be consistent with \cite{12}.

**Theorem 2.3.** Suppose (3) is satisfied, \( \delta_L < 1, \delta_R < 1, \phi > 0 \), and

\[
\tau_L > \frac{\delta_L + 2}{\sqrt{2}}, \quad \tau_R < \frac{-\delta_R + 2}{\sqrt{2}}. \tag{13}
\]

Then

i) \( f^2(T) \) lies to the left of \( E^s(X) \) (so \( Z \) exists),

ii) \( \Delta = \text{cl}(W^u(X)) \), and

iii) \( f \) is chaotic in the sense of Devaney on \( \Delta \).

The conditions (13) on the parameters of \( f \) are equivalent to the following conditions on the eigenvalues of \( A_L \) and \( A_R \):

\[
\lambda^u_L > \sqrt{2}, \quad \lambda^u_R < -\sqrt{2}. \tag{14}
\]

Certainly the conclusions of Theorem 2.3 may be false if (13) is not satisfied. For instance \( f^2(T) \) may lie to the right of \( E^s(X) \) (see Figure 1 of \cite{19} for an example) in which case \( \text{cl}(W^u(X)) \) has a fundamentally different character. The conditions \( \delta_L < 1 \) and \( \delta_R < 1 \) are used at one place below to show that the area of \( f^n(\Delta_0) \) decreases with \( n \), but we believe these conditions are actually unnecessary.

Theorem 2.3 tells us that in \( \Delta \) the map \( f \) has a unique chaotic attractor equal to the closure of \( W^u(X) \). We have not proved that the quadrilateral \( YDCS \) doesn’t contain other attractors. Certainly \( YDCS \) may contain other invariant sets. As an example, Fig. 4 shows all periodic solutions of \( f \) (except \( Y \)) with period \( \leq 20 \) for the parameter values

\[
\tau_L = 1.6, \quad \delta_L = 0.4, \quad \tau_R = -1.6, \quad \delta_R = 0.4. \tag{15}
\]

This numerical result suggests that periodic solutions are dense in \( \text{cl}(W^u(X)) \) and form a Cantor set bounded away from \( \text{cl}(W^u(X)) \). The Cantor set seems to be formed from the stable manifold of a period-3 solution (not shown). We have observed a similar partition of the periodic solutions of \( f \) for other parameter values including those that satisfy the conditions of Theorem 2.3. This shows that \( \bigcap_{n=0}^\infty f^n(\Omega_{\text{trap}}) \), where \( \Omega_{\text{trap}} \) is defined by (18) and illustrated in Fig. 6, is not always equal to \( \text{cl}(W^u(X)) \) which is different to the analogous situation in the orientation-reversing case \cite{25}.

3. **A forward invariant region and a trapping region.** Throughout this section we study \( f \) subject to (3) and \( \phi > 0 \). This is the parameter region \( R \) of \cite{3} shown in Fig. 2.

As illustrated in Fig. 5, we let \( B \in \overline{YD} \) be such that \( Bf(D) \) is parallel to \( \overline{YS} \) and define the triangle

\[
\Omega = BDf(D). \tag{16}
\]

Below we show that \( \Omega \) is forward invariant under \( f \).

Given \( \varepsilon > 0 \), let

\[
B_{\varepsilon} = B - \varepsilon(D - Y) - \varepsilon^2(S - Y). \tag{17}
\]

As illustrated in Fig. 6, let \( D_{\varepsilon} \) be the point on \( y = 0 \) for which \( B_{\varepsilon}D_{\varepsilon} \) is parallel to \( \overline{YD} \), and let \( F_{\varepsilon} \) be the point on \( x = 0 \) for which \( B_{\varepsilon}F_{\varepsilon} \) is parallel to \( \overline{YS} \). Now define the triangle

\[
\Omega_{\text{trap}} = B_{\varepsilon}D_{\varepsilon}F_{\varepsilon}. \tag{18}
\]
Below we show that if $\varepsilon > 0$ is sufficiently small then $\Omega_{\text{trap}}$ is a trapping region for $f$, i.e., $\Omega_{\text{trap}}$ maps to its interior. This ensures the existence of a topological attractor: $\bigcap_{n=0}^{\infty} f^n(\Omega_{\text{trap}})$ is an attracting set by definition. In (17) the $(S-Y)$-term is smaller than the $(D-Y)$-term to ensure that $D_\varepsilon$ maps inside $\Omega_{\text{trap}}$.

Our proofs use the following elementary principle that motivates our definitions of $\Omega$ and $\Omega_{\text{trap}}$. If $\alpha \subseteq \mathbb{R}^2$ is a line segment in $x \leq 0$ that is parallel to either $Y$ or $D$.

![Figure 4](image-url)  
**Figure 4.** A phase portrait of (1) using the parameter values (15). This shows all periodic solutions (except $Y$) up to period 20 (as black dots). These were computed via a brute-force search and the algorithm of [14] to generate all possible symbolic itineraries. The unstable manifold $W^u(X)$ (coloured red but mostly obscured by the periodic solutions) was computed numerically by following it outwards from $X$ until no further growth could be discerned.

![Figure 5](image-url)  
**Figure 5.** The forward invariant region $\Omega$ and its image $f(\Omega)$. 
or \( YS \), then \( f(\alpha) \) is parallel to \( \alpha \). This is because the directions of \( YD \) and \( YS \) are those of the eigenvectors of \( A_L \).

**Lemma 3.1.** Suppose (3) is satisfied and \( \phi > 0 \). Then \( f(\Omega) \subseteq \Omega \).

*Proof.* We have \( f(D) = (\tau_R D_1 + 1, -\delta_R D_1) \), thus \( f(D) \) lies in the quadrant \( x, y < 0 \) (because \( D_1 > 1, \tau_R < -1, \) and \( \delta_R > 0 \)). Also from (1) we have
\[
 f(C) - f(D) = (\tau_R(C_1 - D_1) + 1, -\delta_R(C_1 - D_1)),
\]
thus \( f(D) \) lies above \( f(C) \) (because \( C_1 > D_1 \) by (9)). Also \( f(C) \in YS \) (because \( f^{-1}(S) \) lies in \( x, y > 0 \)), thus \( f(D) \) lies above \( YS \).

Consequently \( B \) lies between \( Y \) and \( f^{-1}(D) \), where \( f^{-1}(D) \) is the intersection of \( YD \) with \( x = 0 \). Let \( U \) be the intersection of \( Df(D) \) with \( x = 0 \), see Fig. 5.

Write \( \Omega = \Omega_L \cup \Omega_R \), where \( \Omega_L \) and \( \Omega_R \) are the parts of \( \Omega \) in \( x \leq 0 \) and \( x \geq 0 \) respectively. Notice \( \Omega_L \) is the quadrilateral \( Uf(D)Bf^{-1}(D) \), and \( \Omega_R \) is the triangle \( DUf^{-1}(D) \). Then \( f(\Omega) = f(\Omega_L) \cup f(\Omega_R) \), where \( f(\Omega_L) \) is the quadrilateral \( f(U)f^2(D)f(B)D \), and \( f(\Omega_R) \) is the triangle \( f(D)f(U)D \). Since \( \Omega \) is convex, to complete the proof it suffices to show that each vertex of \( f(\Omega_L) \) and \( f(\Omega_R) \) belongs to \( \Omega \).

The point \( f(B) \) lies between \( B \) and \( D \), thus \( f(B) \in \Omega \). Since \( BF(D) \) is parallel to \( YS \), \( f(B)f^2(D) \) is also parallel to \( YS \). Furthermore, since \( BF(D) \) is located above \( YS \), \( f(B)f^2(D) \) is located above \( BF(D) \) (because \( \lambda^x > 1 \)). Also \( f^2(D) \) lies below \( YD \), and \( f^2(D)_2 > 0 \) because \( f(D)_1 < 0 \). Thus \( f^2(D) \in \Omega \). Finally, \( U \) lies above the line that passes through \( B \) and \( f(D) \), thus \( f(U) \) lies on \( y = 0 \), above the line through \( B \) and \( f(D) \), and to the left of \( D \), thus \( f(U) \in \Omega \). This shows that all vertices of \( f(\Omega_L) \) and \( f(\Omega_R) \) belong to \( \Omega \). \( \square \)

**Lemma 3.2.** Suppose (3) is satisfied and \( \phi > 0 \). Then \( f(\Omega_{\text{trap}}) \subseteq \text{int}(\Omega_{\text{trap}}) \), for sufficiently small \( \varepsilon > 0 \).

*Proof.* Let \( G_\varepsilon \) be the intersection of \( B_\varepsilon D_\varepsilon \) with \( x = 0 \). Then \( f(\Omega_{\text{trap}}) \) is the union of the triangles \( f(B_\varepsilon)f(G_\varepsilon)f(F_\varepsilon) \) and \( f(G_\varepsilon)f(D_\varepsilon)f(F_\varepsilon) \). Since \( \Omega_{\text{trap}} \) is convex, to

![Figure 6. The trapping region \( \Omega_{\text{trap}} \).](image-url)
complete the proof it suffices to show that the vertices of these triangles belong to \( \operatorname{int}(\Omega_{\text{trap}}) \).

We begin with \( f(B_x) \). Assume \( \varepsilon > 0 \) is sufficiently small that \( B_x \) lies above \( YS \).
Since \( B_x D_x \) and \( B_x F_x \) are parallel to the eigenvectors of \( A_L \) corresponding to the eigenvalues \( \lambda^+_2 > 1 \) and \( 0 < \lambda^-_1 < 1 \), respectively, the point \( f(B_x) \) lies below \( B_x D_x \) and above \( B_x F_x \). Also \( B_x \) lies to the left of \( x = 0 \), thus \( f(B_x) \) lies above \( y = 0 \).
These three constraints on \( f(B_x) \) ensure \( f(B_x) \in \operatorname{int}(\Omega_{\text{trap}}) \).

For similar reasons \( f(F_x) \) lies above \( B_x F_x \) and below \( B_x D_x \). Since \( f(F_x) \) lies on \( y = 0 \) to the left of \( D \), we have \( f(F_x) \in \operatorname{int}(\Omega_{\text{trap}}) \). Also \( f(G_x) \) lies between \( D \) and \( D_x \), thus \( f(G_x) \in \operatorname{int}(\Omega_{\text{trap}}) \).

Finally, in view of the definition of \( B_x \) (17), the point \( D_x \) is an order \( \varepsilon^2 \) distance from \( D \). Thus \( f(D_x) \) is an order \( \varepsilon^2 \) distance from \( f(D) \). But \( f(D) \) lies above \( B_x F_x \) by a distance \( k_1 \varepsilon + k_2 \varepsilon^2 \), where \( k_1 > 0 \). Thus, for sufficiently small \( \varepsilon > 0 \), \( f(D_x) \) lies above \( B_x F_x \), and so \( f(D_x) \in \operatorname{int}(\Omega_{\text{trap}}) \).

4. Invariant expanding cones. We first define invariant expanding cones for arbitrary \( 2 \times 2 \) matrices.

**Definition 4.1.** Let \( A \) be a real-valued \( 2 \times 2 \) matrix and let \( K \subseteq \mathbb{R} \) be a closed interval. The cone

\[
\Psi_K = \left\{ a \begin{bmatrix} 1 \\ m \end{bmatrix} \mid a \in \mathbb{R}, m \in K \right\},
\]

is said to be

i) **invariant** if \( Av \in \Psi_K \) for all \( v \in \Psi_K \), and

ii) **expanding** if there exists \( c > 1 \) such that \( \|Av\| \geq c\|v\| \) for all \( v \in \Psi_K \).

In [25], Misiurewicz identified invariant expanding cones for the Jacobian matrices of the Lozi map and its inverse. This was done to demonstrate hyperbolicity and as part of his proof of transitivity. Many groups have studied the linear algebra problem of the existence of a cone that is invariant for a finite collection of matrices, see for instance [16, 27, 28]. Invariant expanding cones have also been used to give bounds on Lyapunov exponents for maps on tori [10, 11, 37].

**Proposition 4.1.** Suppose (3) is satisfied. Let

\[
q_L = -\frac{\tau_L}{2} \left( 1 - \sqrt{1 - \frac{4\delta}{\tau_L}} \right), \quad q_R = -\frac{\tau_R}{2} \left( 1 - \sqrt{1 - \frac{4\delta}{\tau_R}} \right),
\]

and let \( K = [q_L, q_R] \). Then \( \Psi_K \) is an invariant expanding cone for both \( A_L \) and \( A_R \). If (13) is also satisfied, then the expansion condition is satisfied for some \( c > \sqrt{2} \).

For the remainder of this section we work towards a proof of Proposition 4.1. Let

\[
A = \begin{bmatrix} \tau & 1 \\ -\delta & 0 \end{bmatrix},
\]

where \( \tau, \delta \in \mathbb{R} \). Given \( m \in \mathbb{R} \), the slope of \( v = \begin{bmatrix} 1 \\ m \end{bmatrix} \) is \( m \), and the slope of \( Av = \begin{bmatrix} \tau + m \\ -\delta \end{bmatrix} \) is

\[
G(m) = \frac{-\delta}{\tau + m},
\]
assuming \( m \neq -\tau \). The fact that \( G \) is undefined at \( m = -\tau \) will not be a problem below because an infinite slope corresponds to a vector in direction \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \). This vector cannot belong to an invariant expanding cone because \( A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \), hence the direction \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) is not of interest to us.

We have chosen to characterise the direction of tangent vectors by their slope, rather than by an angle, because slopes are easier to deal with than angles algebraically. Indeed the fixed point equation \( G(m) = m \) is quadratic, and the fixed points are

\[
q(\tau, \delta) = -\frac{\tau}{2} \left( 1 - \sqrt{1 - \frac{4\delta}{\tau^2}} \right), \quad (23)
\]
\[
r(\tau, \delta) = -\frac{\tau}{2} \left( 1 + \sqrt{1 - \frac{4\delta}{\tau^2}} \right), \quad (24)
\]

assuming \( \tau^2 > 4\delta \).

Notice that \( q_L = q(\tau_L, \delta_L) \) and \( q_R = q(\tau_R, \delta_R) \), see (20). Notice also that \( q(\tau, \delta) \) and \( r(\tau, \delta) \) are the slopes of the eigenvectors of \( A \). If the eigenvalues of \( A \) are real and distinct, call them \( \lambda^s \) and \( \lambda^u \), then the slopes of the eigenvectors are \(-\lambda^u\) (corresponding to \( \lambda^s \)) and \(-\lambda^s\) (corresponding to \( \lambda^u \)). It follows that \( q_L = -\lambda^u_L \in (-1, 0) \) and \( q_R = -\lambda^u_R \in (0, 1) \).

For \( v = \begin{bmatrix} 1 \\ m \end{bmatrix} \) we have

\[
\|v\| = \sqrt{1 + m^2}, \quad (25)
\]
\[
\|Av\| = \sqrt{(\tau + m)^2 + \delta^2}, \quad (26)
\]
as we are using the Euclidean norm. Solving \( \|v\| = \|Av\| \) gives \( m = p(\tau, \delta) \) where

\[
p(\tau, \delta) = -\frac{\tau^2 + \delta^2 - 1}{2\tau}, \quad (27)
\]

assuming \( \tau \neq 0 \). We first show that \( p, q, \) and \( r \) appear as in Fig. 7.

![Figure 7](image_url)

**Figure 7.** The functions \( p (27) \), \( q (23) \), and \( r (24) \) for \( \tau > \delta + 1 \) and a fixed value of \( \delta \in (0, 1) \).
Lemma 4.2. Suppose $\delta > 0$ and $|\tau| > \delta + 1$. Then
\[ |q(\tau, \delta)| < |p(\tau, \delta)| < |r(\tau, \delta)|. \] (28)

Proof. Observe:
\[
\tau^2 \sqrt{1 - \frac{4\delta}{\tau^2}} = |\tau| \sqrt{\tau^2 - 4\delta} > (\delta + 1) \sqrt{(\delta + 1)^2 - 4\delta} = (\delta + 1)|\delta - 1|.
\]
Thus
\[
|p(\tau, \delta)| - |q(\tau, \delta)| = \frac{1}{2|\tau|} (\tau^2 + \delta^2 - 1) - \frac{|\tau|}{2} \left( 1 - \sqrt{1 - \frac{4\delta}{\tau^2}} \right) > \frac{\delta + 1}{2|\tau|} (\delta - 1 + |\delta - 1|) \geq 0.
\]
Similarly,
\[
|p(\tau, \delta)| - |r(\tau, \delta)| = \frac{1}{2|\tau|} (\tau^2 + \delta^2 - 1) - \frac{|\tau|}{2} \left( 1 + \sqrt{1 - \frac{4\delta}{\tau^2}} \right) < \frac{\delta + 1}{2|\tau|} (\delta - 1 - |\delta - 1|) \leq 0.
\]

Lemma 4.3. Suppose $\delta > 0$ and $|\tau| > \delta + 1$. Then $\frac{dG}{dm} > 0$ for all $m \neq -\tau$, and $\frac{dG}{dm}(q(\tau, \delta)) < 1$.

Proof. We have
\[
\frac{dG}{dm} = \frac{\delta}{(\tau + m)^2}, \tag{29}
\]
which is evidently positive for all $m \neq -\tau$. The function $q(\tau, \delta)$ is a root of $m^2 + \tau m + \delta = 0$, thus to evaluate $\frac{dG}{dm}(q(\tau, \delta))$ we can replace one of the $(\tau + m)$'s in the denominator of (29) with $-\frac{m}{\tau}$ to obtain
\[
\frac{dG}{dm}(q(\tau, \delta)) = \frac{-m}{\tau + m},
\]
where $m = q(\tau, \delta)$, and so
\[
\frac{dG}{dm}(q(\tau, \delta)) = \frac{-1}{q(\tau, \delta) + 1}.
\]
Notice $\frac{q(\tau, \delta)}{\tau} = -\frac{1}{2} + \sqrt{1 - \frac{4\delta}{\tau^2}} > -\frac{1}{2}$. Thus $\frac{\tau}{q(\tau, \delta) + 1} < -1$, hence $\frac{dG}{dm}(q(\tau, \delta)) < 1$, as required.

Lemma 4.4. Suppose $\delta > 0$ and $|\tau| > \delta + 1$. If $m \in \mathbb{R}$ is such that $\tau m > \tau p(\tau, \delta)$, then $\|Av\| > \|v\|$, where $v = \left[ \frac{1}{m} \right]$. 

Proof. We have
\[
\| Av \|^2 - \| v \|^2 = (\tau + m)^2 + \delta^2 - (1 + m^2)
\]
\[
= \tau^2 + \delta^2 - 1 + 2\tau m
\]
\[
> \tau^2 + \delta^2 - 1 + 2\tau p(\tau, \delta).
\]
The last expression is zero by (27), thus \( \| Av \| > \| v \| \), as required. \( \Box \)

**Lemma 4.5.** Suppose \( \delta > 0 \), \( |\tau| > \delta + 1 \), and \( |\tau| > \frac{\delta^2 + 2}{\sqrt{2}} \). If \( m \in \mathbb{R} \) is such that \( |m - \tau| \leq |q(\tau, \delta) - \tau| \), then \( \| Av \| > \sqrt{2} \| v \| \), where \( v = \left[ \frac{1}{m} \right] \).

**Proof.** Let
\[
H(m) = \| Av \|^2 - 2\| v \|^2 = -m^2 + 2\tau m + \tau^2 + \delta^2 - 2.
\]
We only need to show \( H(q(\tau, \delta)) > 0 \), because \( H(m) \) is a concave down parabola that achieves its maximum value at \( m = \tau \).

By substituting (23) into (30) we obtain
\[
H(q(\tau, \delta)) = \delta^2 + \delta - 2 + \frac{\tau^2}{2} \left( -1 + 3\sqrt{1 - \frac{4\delta}{\tau^2}} \right).
\]
(31)
For any fixed \( \delta > 0 \), this is an increasing function of \( |\tau| \) because
\[
\frac{\partial H(q(\tau, \delta))}{\partial (\tau^2)} = 1 + \frac{3\left( \sqrt{1 - \frac{4\delta}{\tau^2}} - 1 \right)^2}{4\sqrt{1 - \frac{4\delta}{\tau^2}}},
\]
which is evidently positive. Thus \( H(q(\tau, \delta)) \) is strictly greater than its value at \( |\tau| = \frac{\delta^2 + 2}{\sqrt{2}} \). From (31), we obtain, after simplification,
\[
H\left( q\left( \pm \frac{\delta^2 + 2}{\sqrt{2}}, \delta \right) \right) = \frac{3}{4}(\delta + 2)(\delta - 2 + |\delta - 2|) \geq 0.
\]
Thus \( H(q(\tau, \delta)) > 0 \), which completes the proof. \( \Box \)

We are now ready to prove Proposition 4.1. Let
\[
G_L(m) = \frac{-\delta_L}{\tau_L + m}, \quad G_R(m) = \frac{-\delta_R}{\tau_R + m}.
\]
(32)
be the ‘slope maps’ for \( A_L \) and \( A_R \). Lemma (4.3) has shown that these maps are increasing and have stable fixed points \( q_L \) and \( q_R \), respectively. Consequently they appear as in Fig. 8, from which we see that \( K = [q_L, q_R] \) is forward invariant under both \( G_L \) and \( G_R \) (this is proved carefully below). That \( \Psi_K \) is expanding follows from Lemmas 4.2 and 4.4, and the strong expansion \( (c > \sqrt{2}) \) follows from Lemma 4.5.

**Proof of Proposition 4.1.** We first show that \( \Psi_K \) is expanding. Choose any \( v \in \Psi_K \), and let \( m \) be its slope. By linearity it suffices to consider \( v = \left[ \frac{1}{m} \right] \).

Since \( \tau_L > 0 \), we have \( p(\tau_L, \delta_L) < q_L \) by Lemma 4.2. Thus \( m > p(\tau_L, \delta_L) \), and so \( \| A_L v \| > \| v \| \) by Lemma 4.4. Similarly, since \( \tau_R < 0 \), we have \( p(\tau_R, \delta_R) > q_R \). Thus \( m < p(\tau_R, \delta_R) \), and so \( \| A_R v \| > \| v \| \). Since \( K \) is compact, the set \( \left\{ \frac{\| A_{L,R} \|}{\| v \|} : J \in \{ L, R \}, v \in \Psi_K \right\} \) has a minimum, call it \( c \), and \( c > 1 \) as required.
Next we show that $\Psi_K$ is invariant. To do this we show that $G_J(K) \subseteq K$, for both $J = L$ and $J = R$. The function $G_J$ has fixed points $q_J$ and $r_J = r(\tau_J, \delta_J)$, where $r_J \notin K$ by Lemma 4.2. Thus, by Lemma 4.3, for all $m \in K$ we have $G_L(m) \geq G_L(q_L) = q_L$, and $G_L(m) \leq m \leq q_R$. Similarly, for all $m \in K$ we have $G_R(m) \geq m \geq q_L$, and $G_R(m) \leq G_R(q_R) = q_R$. This shows that $G_J(K) \subseteq K$, for both $J = L$ and $J = R$. Thus $\Psi_K$ is an invariant expanding cone for both $A_L$ and $A_R$.

Now suppose (13) is also satisfied. By Lemma 4.5 and since $K$ is compact, to verify the strong expansion property we just need to show that for any $m \in K$ we have

$$|m - \tau_L| \leq |q_L - \tau_L|,$$

and

$$|m - \tau_R| \leq |q_R - \tau_R|. \tag{34}$$

Since $q_L = -\lambda^*_L$ and $q_R = -\lambda^*_R$ (as explained in the text) we have $-1 < q_L < 0 < q_R < 1$, and so

$$q_R < 1 < 2 - q_L < 2\tau_L - q_L.$$

Thus $K \subseteq [q_L, 2\tau_L - q_L]$, and so (33) is satisfied. For similar reasons $K \subseteq [2\tau_R - q_R, q_R]$, which implies (34).

5. **Consequences of invariant expanding cones.** In this section we use the existence of an invariant expanding cone (see Proposition 4.1) to prove Theorem 2.1 and show that all periodic solutions are unstable. This includes periodic solutions with points on $x = 0$ for which $Df$ is undefined. The stability of such periodic solutions can be extremely complicated [31], but here a lack of stability follows simply from the definition of Lyapunov stability.

**Proof of Theorem 2.1.** By Lemma 3.2, $f$ has a trapping region $\Omega_{\text{trap}}$. Thus $f$ has a topological attractor $\Lambda \subseteq \Omega_{\text{trap}}$.

![Diagram](image-url)  
**Figure 8.** The slope maps (32). $G_L(m)$ and $G_R(m)$ are the slopes of $A_Lv$ and $A_Rv$, respectively, where $v$ has slope $m$.  


By Proposition 4.1, there exists an invariant expanding cone \( \Psi_K \), for both \( A_L \) and \( A_R \), and \( v = v_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \Psi_K \) (because \( q_L < 0 < q_R \)). For all \( i \geq 0 \), let

\[
v_{i+1} = \frac{Df(f^i(z))v_i}{\|Df(f^i(z))v_i\|}, \tag{35}
\]

so that

\[
\|Df^n(z)v\| = \prod_{i=0}^{n-1} \|Df(f^i(z))v_i\|. \tag{36}
\]

That the \( v_i \) are well-defined is easily established inductively: Each derivative is well-defined because \( z \notin \Sigma_\infty \). Also \( v_i \in \Psi_K \) implies that the denominator in (35) is non-zero by the expansion property, and \( v_{i+1} \in \Psi_K \) by invariance.

Then (36) and the expansion property give \( \|Df^n(z)v\| \geq c^n \), for some \( c > 1 \), and so

\[
\frac{1}{n} \ln(\|Df^n(z)v\|) \geq \ln(c), \tag{37}
\]

for all \( n \geq 1 \). Therefore

\[
\liminf_{n \to \infty} \frac{1}{n} \ln(\|Df^n(z)v\|) > 0,
\]

and thus \( \lambda(z, v) > 0 \), if the limit (11) exists. \( \square \)

**Proposition 5.1.** Suppose (3) is satisfied. Then all periodic solutions of \( f \) are unstable.

**Proof.** Let \( z \in \mathbb{R}^2 \) be a point of a period-\( n \) solution of \( f \). Let \( \mathcal{I} \) be the set of all \( i \in \{0, \ldots, n-1\} \) for which \( f^i(z) \) does not lie on \( x = 0 \). Let

\[
\varepsilon = \min_{i \in \mathcal{I}} |f^i(z)_1|,
\]

and \( \varepsilon = 1 \) if \( \mathcal{I} = \emptyset \).

Choose any \( \delta \in (0, \varepsilon) \), and let \( z_\delta = z + \begin{bmatrix} \delta \\ 0 \end{bmatrix} \). For each \( i \geq 0 \), let \( v_i = f^i(z_\delta) - f^i(z) \).

Notice \( \|v_0\| = \delta \leq \varepsilon \), and \( v_0 \in \Psi_K \) (the cone defined in Proposition 4.1).

For any \( i \geq 0 \), if \( \|v_i\| \leq \varepsilon \) then \( f^i(z_\delta) \) and \( f^i(z) \) do not lie on different sides of \( x = 0 \) and so there exists \( J \in \{L, R\} \) such that

\[
f^{i+1}(z_\delta) = A_J f^i(z_\delta) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad f^{i+1}(z) = A_J f^i(z) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{38}
\]

Consequently \( v_{i+1} = A_J v_i \). Thus if we also have \( v_i \in \Psi_K \), then \( v_{i+1} \in \Psi_K \) and \( \|v_{i+1}\| \geq c\|v_i\| \) (where \( c > 1 \)).

This shows that we cannot have \( \|v_i\| \leq \varepsilon \) for all \( i \geq 0 \) because, by induction, this would imply \( \|v_i\| \geq c^i\delta \) for all \( i \geq 0 \). Hence \( \|v_i\| > \varepsilon \) for some \( i \geq 0 \). That is, the forward orbit of \( z_\delta \) escapes an \( \varepsilon \)-neighbourhood of the periodic solution. Since we have allowed arbitrary values of \( \delta > 0 \), this shows that the periodic solution is not Lyapunov stable. \( \square \)

6. The unstable manifold \( W^u(X) \). Here we prove the first two parts of Theorem 2.3. Part (i) is proved via direct calculations. Our proof of part (ii) mimics arguments used to prove Theorem 2 of [25] and requires the assumption \( \delta_L, \delta_R < 1 \).

**Lemma 6.1.** Suppose (3) and (13) are satisfied and \( \phi > 0 \). Then \( f^2(T) \) lies to the left of \( E^u(X) \).
Proof. For any \( z \in E^u(X) \) with \( z_1 \geq 0 \), we have \( f(z) - X = \lambda_R^u(z - X) \). Using \( z = f^{-1}(T) \) and just taking the first components, we obtain

\[
T_1 - X_1 = |\lambda_R^u|X_1.
\]

(39)

With instead \( z = T \) we obtain

\[
X_1 - f(T)_1 = |\lambda_R^u|(T_1 - X_1).
\]

(40)

Combining these gives

\[
|f(T)_1| = \left( |\lambda_R^u| - \frac{1}{|\lambda_R^u|} \right)(T_1 - X_1).
\]

Then by (14),

\[
|f(T)_1| > \left( \sqrt{2} - \frac{1}{\sqrt{2}} \right)(T_1 - X_1) = \frac{1}{\sqrt{2}}(T_1 - X_1).
\]

(41)

From (1) we have \( T_1 = \tau_L f^{-1}(T)_1 + f^{-1}(T)_2 + 1 = f^{-1}(T)_2 + 1 \), and \( f^2(T)_1 = \tau_L f(T)_1 + f(T)_2 + 1 \). Subtracting these gives

\[
T_1 - f^2(T)_1 = -\tau_L f(T)_1 + f^{-1}(T)_2 - f(T)_2
\]

\[
> -\tau_L f(T)_1
\]

\[
> \sqrt{2}|f(T)_1|
\]

\[
> T_1 - X_1.
\]

Thus \( f^2(T) \) lies to the left of \( X \). Also \( f^2(T) \) lies in \( y > 0 \) (because \( f(T)_2 < 0 \)), so certainly \( f^2(T) \) lies to the left of \( E^s(X) \). \( \square \)

Lemma 6.2. \textit{Suppose (3) and (13) are satisfied, \( \delta_L < 1, \delta_R < 1, \) and \( \phi > 0 \). Then \( \Delta = \text{cl}(W^u(X)) \).}

Proof. First we show that \( \text{cl}(W^u(X)) \subseteq \Delta \). Choose any \( z \in \text{cl}(W^u(X)) \). Then there exist \( z_k \in W^u(X) \) with \( k \to z \) as \( k \to \infty \). For each \( k \), the backward orbit of \( z_k \) converges to \( X \). The convergence eventually occurs on the unstable subspace \( E^u(X) \) and includes points on both sides of \( X \) because \( \lambda_R^u < 0 \). Thus there exists \( n_k \geq 0 \) such that \( f^{-n_k}(z_k) \in X \Delta_0 \). Thus \( z_k \in f^{n_k}(\Delta_0) \), and so \( z_k \in \Delta \). Hence \( \text{cl}(W^u(X)) \subseteq \Delta \). Since \( \text{cl}(W^u(X)) \) is invariant we must also have \( \text{cl}(W^u(X)) \subseteq \Delta \).

Second we show that \( \Delta \subseteq \text{cl}(W^u(X)) \). Choose any \( z \in \Delta \). Then \( z \in f^n(\Delta) \) for all \( n \geq 0 \). Let \( \text{Area}(-) \) denote the two-dimensional Lebesgue measure and let \( \delta_{\text{max}} = \max(\delta_L, \delta_R) \). Then

\[
\text{Area}(f^n(\Delta)) \leq \delta_{\text{max}}^n \text{Area}(\Delta),
\]

which converges to 0 as \( n \to \infty \) because we have assumed \( \delta_L, \delta_R < 1 \). Thus the distance of \( z \) to the boundary of \( f^n(\Delta) \) goes to 0 as \( n \to \infty \).

The boundary of \( \Delta_0 \) is contained in \( X \Delta \cup W^u(X) \), so the boundary of \( f^n(\Delta_0) \) is contained in \( X f^n(Z) \cup W^u(X) \). Thus the boundary of \( \Delta \) is contained in \( Z f^n(Z) \cup W^u(X) \), so the boundary of \( f^n(\Delta) \) is contained in \( f^n(Z) f^{n+1}(Z) \cup W^u(X) \). But \( f^n(Z) f^{n+1}(Z) \) converges to \( X \) as \( n \to \infty \). Hence the distance of \( z \) to \( W^u(X) \) goes to 0 as \( n \to \infty \). Thus \( z \in \text{cl}(W^u(X)) \) which shows that \( \Delta \subseteq \text{cl}(W^u(X)) \). \( \square \)
7. Proof of chaos in the sense of Devaney. Here we provide three results that combine to complete the proof of Theorem 2.3. First we use direct calculations to show that the point $U$ lies above the point $V$, as in Fig. 5. This requires significant effort because the required assumption $\phi > 0$ (equivalently $C_1 > D_1$) does not relate to the points $U$ and $V$ in a simple way.

Given that $U$ lies above $V$, it follows that, as in Fig. 5, any line segment in $f(\Omega)$ that intersects $x = 0$ and $y = 0$ must also intersect $E^u(X)$. This is the key step to establishing transitivity and is also based on the ideas in [25]. The strong expansion ($c > \sqrt{2}$) of Proposition 4.1 is used below in the proof of Lemma 7.2.

**Lemma 7.1.** Suppose (3) and (13) are satisfied and $\phi > 0$. Then $U_2 > V_2$.

**Proof.** Similar to $S$, see (7), the point $V$ has $y$-component

$$V_2 = \frac{-\lambda_R^u}{\lambda_R^u - 1}.$$  \hfill (42)

The point $U$ is defined as the intersection of $Df(D)$ with $x = 0$. From $f(D) = (\tau_RD_1 + 1, -\delta_RD_1)$, we obtain

$$U_2 = \frac{-\lambda_R^u\lambda_R^uD_1}{1 - \lambda_R^u - \lambda_R^u - \delta_R^u}.$$  \hfill (43)

Upon substituting (6) into (43), subtracting (42), and carefully factorising, we obtain

$$U_2 - V_2 = \frac{-\lambda_R^u(1 - \lambda_R^u + \lambda_R^u)(\lambda_R^u - \lambda_R^u)}{(1 - \lambda_R^u)(1 - \lambda_R^u)(\lambda_R^u - \lambda_R^u - \lambda_R^u)},$$  \hfill (44)

Each factor in (44) is evidently positive, except possibly the middle factor in the numerator. Thus it remains to show that $1 - \lambda_R^u + \lambda_R^u > 0$.

To do this we first show that $C_1 < \frac{1}{\lambda_R^u}$. Suppose for a contradiction that $C_1 \geq \frac{1}{\lambda_R^u}$.

By (8) we have

$$\frac{-S_2}{-\lambda_R^u + \lambda_R^u(\lambda_R^u - 1 + \frac{S_2}{S_2})} \geq \frac{-1}{\lambda_R^u}.$$  

But $\lambda_R^u < -\sqrt{2}$, see (14), thus

$$\frac{-S_2}{-\lambda_R^u - \sqrt{2}(\lambda_R^u - 1 + \frac{S_2}{S_2})} > \frac{-1}{\lambda_R^u},$$  

which is equivalent to

$$S_2 + 1 + \frac{\sqrt{2}}{S_2} > \frac{\sqrt{2}}{\lambda_R^u}.$$  

But $\lambda_R^u > -1$, thus

$$S_2 + 1 + \frac{\sqrt{2}}{S_2} > -\sqrt{2},$$  

which is equivalent to

$$\left(S_2 + 2 + \sqrt{2}\right)\left(S_2 + \sqrt{2} - 1\right) > 0.$$  \hfill (45)

However, $\lambda_L^u > \sqrt{2}$, see (14), thus by (7) we have $-(2 + \sqrt{2}) < S_2 < -1$, which contradicts (45).

Therefore $C_1 < \frac{1}{\lambda_R^u}$. The assumption $\phi > 0$ implies $D_1 < C_1$, thus $D_1 < \frac{1}{\lambda_R^u}$. By (6), this is equivalent to $1 - \lambda_L^u + \lambda_R^u > 0$, which completes the proof. \qed
Lemma 7.2. Suppose (3) and (13) are satisfied and \( \phi > 0 \). Let \( \alpha \subset \Omega \) be a line segment with slope \( m \in K = [q_L, q_R] \). Then there exists \( n \geq 1 \) and points \( P \) on \( x = 0 \) and \( Q \) on \( y = 0 \) such that \( PQi \subseteq f^n(\alpha) \).

Proof. Let \( \alpha_0 = \alpha \). We iteratively construct a sequence of line segments \( \{\alpha_i\} \) in \( \Omega \) with slopes in \( K \) and lengths \( \alpha_i \), as follows. For each \( i \geq 0 \) suppose \( \alpha_i \) and \( f(\alpha_i) \) do not both intersect \( x = 0 \). Then \( f^2(\alpha_i) \) is a union of at most two line segments (and belongs to \( \Omega \) because \( \Omega \) is forward invariant, Lemma 3.1). The line segments comprising \( f^2(\alpha_i) \) have slopes in \( K \) because \( \Psi_K \) is invariant (see Proposition 4.1). Also \( \Psi_K \) is expanding with some \( c > \sqrt{2} \), thus the length of \( f^2(\alpha_i) \) is at least \( c^2 \alpha_i \). Thus \( f^2(\alpha_i) \) contains a line segment, \( \alpha_{i+1} \), with \( \alpha_{i+1} \geq \frac{\alpha_i}{c^2} \).

This gives \( \alpha_n \geq \frac{\alpha_0}{c^{2n}} \to \infty \) as \( n \to \infty \) because \( c^2 > 2 \). But \( \Omega \) is bounded, so this is not possible. Thus there exists \( k \geq 0 \) such that \( \alpha_k \) and \( f(\alpha_k) \) both intersect \( x = 0 \). Notice \( f(\alpha_k) \) is a union of at most two line segments, both of which intersect \( y = 0 \). Thus there exists a line segment \( PQi \subseteq f(\alpha_k) \subseteq f^{2k+1}(\alpha) \) with \( P \) on \( x = 0 \) and \( Q \) on \( y = 0 \).

Proposition 7.1. Suppose (3) and (13) are satisfied and \( \phi > 0 \). Then, on \( \text{cl}(W^u(X)) \), \( f \) is transitive, has dense periodic points, and exhibits sensitive dependence on initial conditions.

Proof. Choose any open \( M_0, N_0 \subset \mathbb{R}^2 \) with \( M = M_0 \cap \text{cl}(W^u(X)) \neq \emptyset \) and \( N = N_0 \cap \text{cl}(W^u(X)) \neq \emptyset \). To verify dense periodic orbits we show that \( f \) has a periodic point in \( M \). To verify transitivity we show there exists \( n \geq 0 \) such that \( f^n(M) \cap N \neq \emptyset \). Sensitive dependence then follows from the result of [4].

Since \( M_0 \) is open there exists \( u \in M_0 \cap W^u(X) \). Since \( W^u(X) = \bigcup_{k \geq 0} f^k(Xr) \), where each \( f^k(Xr) \) is a union of line segments with slopes in \( K = [q_L, q_R] \), there exists a line segment \( \alpha \subset M_0 \cap W^u(X) \) with slope in \( K \) (and \( u \in \alpha \)). By Lemma 7.2, there exists \( n_1 \geq 1 \) such that \( f^{n_1}(\alpha) \) contains a line segment \( PQi \) with \( P \) on \( x = 0 \) and \( Q \) on \( y = 0 \). Since \( \alpha \subset W^u(X) \subset \Omega \) we have \( PQi \subseteq f(\Omega) \) because \( n_1 \geq 1 \) and \( \Omega \) is forward invariant. Thus \( P \) lies on or above \( U \), see Fig. 5. Since \( V_2 < U_2 \) (see Lemma 7.1), \( P \) lies above \( E^s(X) \). Also, \( Q \) lies on or to the right of \( f(U) \). Since \( f(V) < f(U) \), \( Q \) lies below \( E^s(X) \). Thus \( PQi \) intersects \( E^s(X) \) transversely at some \( z_{\text{int}} \in \mathbb{R}^2 \). The point \( w_{\text{int}} = f^{-n_1}(z_{\text{int}}) \in M_0 \) is also a transverse intersection of \( W^s(\chi) \) and \( W^u(X) \) (although \( W^s(\chi) \) may be non-differentiable here [7]). Arbitrarily close to \( w_{\text{int}} \) there exists a non-wandering set associated with a Smale horseshoe [24, 33]. In the non-wandering set periodic points of \( f \) are dense. Thus \( f \) has a periodic point \( p \in M \) where \( p \in \text{cl}(W^u(X)) \) because \( W^u(X) \) is also dense in the non-wandering set.

Now let \( z \in N_0 \cap W^u(X) \). Since \( f^{-n}(z) \to X \) as \( n \to \infty \), there exists \( n_2 \geq 0 \) such that \( f^{-n}(z) \) lies in \( x > 0 \) for all \( n \geq n_2 \). Then there exists open \( N_0 \subset N \), with \( z \in N_0 \), such that \( f^{-n_2}(N_0) \) lies in \( x > 0 \). Iteratively define \( N_k \subset N_{k-1} \) as the maximal open set for which \( f^{-(n_2+k)}(N_k) \) lies in \( x > 0 \). Since \( f^{-1} \) is affine in \( x > 0 \) with saddle-type fixed point \( X \), as \( k \to \infty \) the sets \( f^{-(n_2+k)}(N_k) \) approach \( E^s(X) \) and stretch across \( \Omega \) for sufficiently large values of \( k \). Thus there exists \( n_3 \geq 0 \) such that \( f^{-(n_2+n_3)}(N_{n_3}) \) intersects \( PQi \). Thus there exists \( w \in M \) such that \( f^{n_3}(w) \in f^{-(n_2+n_3)}(N_{n_3}) \). Let \( n = n_1 + n_2 + n_3 \). Then \( f^n(w) \in N \) and hence \( f^n(M) \cap N \neq \emptyset \) which verifies transitivity.\( \square \)
8. **Discussion.** We have used invariant expanding cones to prove that, throughout the parameter region $\mathcal{R}$ of [3], no invariant set of (1) can have only negative Lyapunov exponents, Theorem 2.1. In fact we have shown that for any $n \geq 1$ the average expansion after $n$ iterations is at least $\text{ln}(c)$ for some $c > 1$, see (37). Thus $\text{ln}(c)$ may be used as a lower bound on the maximal Lyapunov exponent, assuming the Lyapunov exponents are well-defined. One could also identify an invariant expanding cone for $f^{-1}$, as done in [25, 29] for the Lozi map, to obtain an upper bound on the minimal Lyapunov exponent.

Subject to additional constraints on the parameter values, we have shown that on $\text{cl}(W^u(X))$ (1) is transitive and has dense periodic orbits, thus also exhibits sensitive dependence on initial conditions, Theorem 2.3. We have also identified a forward invariant set $\Delta \subseteq \Omega_{\text{trap}}$ with the property that $\bigcap_{n=0}^{\infty} f^n(\Delta) = \text{cl}(W^u(X))$. We have not proved that there do not exist other attractors in $\Omega_{\text{trap}}$; certainly there may be other invariant sets as in Fig. 4.

It remains to extend Theorems 2.1 and 2.3 to larger regions of parameter space. For instance we believe the constraint in Theorem 2.3 that both pieces of $f$ are area-contracting is unnecessary. It also remains to extend the ergodic theory results of [9, 29] for the Lozi map to the more general border-collision normal form, and extend results to higher dimensions.

Finally we discuss consequences for border-collision bifurcations. The border-collision normal form contains the leading order terms of a piecewise-smooth map in the neighbourhood of a border-collision bifurcation. Assuming the bifurcation occurs when a parameter $\mu$ is zero, and with $\mu > 0$ a scaling has been done such that the constant term $[\mu, 0]^T$ is transformed to $[1, 0]^T$, then the nonlinear terms that have been neglected to produce (1) are order $\mu$ (assuming the map is piecewise-$C^2$). In this way the effect of the nonlinear terms increases as the value of $\mu$ increases to move away from the border-collision bifurcation at $\mu = 0$. We believe that the features we have used to construct robust chaos are also robust to these nonlinear terms. This is because small nonlinear terms will not destroy transverse intersections of invariant manifolds, the existence of trapping region, or the existence of an invariant expanding cone.

**REFERENCES**

[1] G. Álvarez, F. Montoya, M. Romera and G. Pastor, Cryptanalysis of a discrete chaotic cryptosystem using external key, Phys. Lett. A, 319 (2003), 334–339.  
[2] S. Banerjee and C. Grebogi, Border collision bifurcations in two-dimensional piecewise smooth maps, Phys. Rev. E, 59 (1999), 4052–4061.  
[3] S. Banerjee, J. A. Yorke and C. Grebogi, Robust chaos, Phys. Rev. Lett., 80 (1998), 3049–3052.  
[4] J. Banks, J. Brooks, G. Cairns, G. Davis and P. Stacey, On Devaney’s definition of chaos, Amer. Math. Monthly, 99 (1992), 332–334.  
[5] L. Barreira and Y. Pesin, Nonuniform Hyperbolicity. Dynamics of Systems with Nonzero Lyapunov Exponents, volume 115 of Encyclopedia of Mathematics and Its Applications, Cambridge University Press, Cambridge, 2007.  
[6] M. Benedicks and L. Carleson, The dynamics of the Henon map, Ann. Math., 133 (1991), 73–169.  
[7] Y. Cao and Z. Liu, The geometric structure of strange attractors in the Lozi map, Commun. Nonlin. Sci. Numer. Simul., 3 (1998), 119–123.  
[8] Y. Cao and Z. Liu, Strange attractors in the orientation-preserving Lozi map, Chaos Solitons Fractals, 9 (1998), 1857–1863.
[9] P. Collet and Y. Levy, Ergodic properties of the Lozi mappings, *Commun. Math. Phys.*, 93 (1984), 461–481.

[10] E. Corneliş and M. Wojtkowski, A criterion for the positivity of the Liapunov characteristic exponent, *Ergod. Th. & Dynam. Sys.*, 4 (1984), 527–539.

[11] S. Das and J. A. Yorke, Multichaos from quasiperiodicity, *SIAM J. Appl. Dyn. Syst.*, 16 (2017), 2196–2212.

[12] R. L. Devaney, *An Introduction to Chaotic Dynamical Systems*, Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1989.

[13] M. di Bernardo, C. J. Budd, A. R. Champneys and P. Kowalczyk, *Piecewise-smooth Dynamical Systems. Theory and Applications*, Springer-Verlag, London, Ltd., London, 2008.

[14] J.-P. Duval, Génération d’une section des classes de conjugaison et arbre des mots de Lyndon de longueur bornée, *Theoret. Comput. Sci.*, 60 (1988), 255–283. In French.

[15] J.-P. Eckmann and D. Ruelle, Ergodic theory of chaos and strange attractors, *Rev. Mod. Phys.*, 57 (1985), 617–650.

[16] R. Edwards, J. J. McDonald and M. J. Tsatsomeros, On matrices with common invariant cones with applications in neural and gene networks, *Linear Algebra Appl.*, 398 (2005), 37–67.

[17] E. Glasner and B. Weiss, Sensitive dependence on initial conditions, *Nonlinearity*, 6 (1993), 1067–1075.

[18] P. Glendinning, Bifurcation from stable fixed point to 2D attractor in the border collision normal form, *IMA J. Appl. Math.*, 81 (2016), 699–710.

[19] P. Glendinning, Robust chaos revisited, *Eur. Phys. J. Special Topics*, 226 (2017), 1721–1738.

[20] B. R. Hunt, J. A. Kennedy, T.-Y. Li and H. E. Nusse, SLYRB measures: natural invariant measures for chaotic systems, *Phys. D*, 170 (2002), 50–71.

[21] L. Kocarev and S. Lian, *Chaos-Based Cryptography. Theory, Algorithms and Applications*, Springer, New York, 2011.

[22] P. Kowalczyk, Robust chaos and border-collision bifurcations in non-invertible piecewise-linear maps, *Nonlinearity*, 18 (2005), 485–504.

[23] R. Lozi, Un attracteur étrange(?) du type attracteur de Hénon, *J. Phys. (Paris)*, 39 (1978), 9–10. In French.

[24] R. S. MacKay, *Renormalisation in Area-preserving Maps*, World Scientific, World Scientific Publishing Co., Inc., River Edge, NJ, 1993.

[25] M. Misiurewicz, Strange attractors for the Lozi mappings, In R.G. Helleman, editor, *Nonlinear Dynamics, Annals of the New York Academy of Sciences*, New York, Wiley, 1980, 348–358.

[26] H. E. Nusse and J. A. Yorke, Border-collision bifurcations including “period two to period three” for piecewise smooth systems, *Phys. D*, 57 (1992), 39–57.

[27] V. Yu. Protasov, When do several linear operators share an invariant cone?, *Linear Algebra Appl.*, 433 (2010), 781–789.

[28] L. Rodman, H. Seyalioglu and I. M. Spitkovsky, On common invariant cones for families of matrices, *Linear Algebra Appl.*, 432 (2010), 911–926.

[29] M. Rychlik, Invariant measures and the variational principle for Lozi mappings, *Ergod. Th. & Dynam. Sys.*, 5 (1985), 145–161.

[30] D. J. W. Simpson, Border-collision bifurcations in $\mathbb{R}^n$, *SIAM Rev.*, 58 (2016), 177–226.

[31] D. J. W. Simpson, The stability of fixed points on switching manifolds of piecewise-smooth continuous maps, *J. Dyn. Diff. Equat.*, 32 (2020), 1527–1552.

[32] D. J. W. Simpson and J. D. Meiss, Neimark-Sacker bifurcations in planar, piecewise-smooth, continuous maps, *SIAM J. Appl. Dyn. Sys.*, 7 (2008), 795–824.

[33] S. Smale, Differentiable dynamical systems, *Bull. Amer. Math. Soc.*, 73 (1967), 747–817.

[34] I. Sushko and L. Gardini, Center bifurcation for two-dimensional border-collision normal form, *Int. J. Bifurcation Chaos*, 18 (2008), 1029–1050.

[35] S. van Strien, One-parameter families of smooth interval maps: Density of hyperbolicity and robust chaos, *Proc. Amer. Math. Soc.*, 138 (2010), 4443–4446.

[36] M. Viana, *Lectures on Lyapunov Exponents*, volume 145 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, 2014.

[37] M. Wojtkowski, Invariant families of cones and Lyapunov exponents, *Ergod. Th. & Dynam. Sys.*, 5 (1985), 145–161.

[38] L.-S. Young, Bowen-Ruelle measures for certain piecewise hyperbolic maps, *Trans. Amer. Math. Soc.*, 287 (1985), 41–48.
Z. T. Zhuzubaliyev, E. Mosekilde, S. Maity, S. Mohanan and S. Banerjee, Border collision route to quasiperiodicity: Numerical investigation and experimental confirmation, Chaos, 16 (2006), 023122, 11 pp.

Received July 2020; revised November 2020.

E-mail address: p.a.glendinning@manchester.ac.uk
E-mail address: d.j.w.simpson@massey.ac.nz