FIRST CORRECTIONS TO HYPERFINE SPLITTING AND LAMB SHIFT INDUCED BY DIAGRAMS WITH TWO EXTERNAL PHOTONS AND SECOND ORDER RADIATIVE INSERTIONS IN THE ELECTRON LINE

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Abstract

Contributions to HFS and to the Lamb shift intervals of order $\alpha^2(Z\alpha)^5$ induced by graphs with two radiative photons inserted in the electron line are considered. It is demonstrated that this last gauge invariant set of diagrams which are capable of producing corrections of considered order consists of nineteen topologically different diagrams. Contributions both to HFS and Lamb shift induced by graphs containing one-loop electron self-energy as a subgraph and also by the graph containing two one-loop vertices are obtained.
1 Introduction

Recent theoretical work on high order corrections to hyperfine splitting and Lamb shift concentrated on calculation of nonrecoil contributions of order \( \alpha^2(Z\alpha)^5 \). These corrections are of immediate phenomenological interest for HFS measurements in the ground state of muonium and for \( n = 2 \) Lamb shift measurements in hydrogen. Their magnitude may run up to several kilohertz for both intervals to be compared with current experimental uncertainty equal to 0.16kHz for HFS in muonium \([1]\) and to 1.9kHz for the Lamb shift in hydrogen \([2]\). Further experimental progress is envisaged at least in the case of muonium hyperfine splitting measurements \([3]\).

The only terms, which are capable to produce corrections with magnitude about several kilohertz to muonium HFS (an order of magnitude larger than the experimental error) and are still uncalculated are of functional form \( \alpha^2(Z\alpha)E_F \) (see, e.g. \([4]\), \([5]\)). For the Lamb shift in hydrogen corrections of order \( \alpha^2(Z\alpha)^5m \) are also as large as several kilohertz but in this case there are also other sources of contributions of comparable magnitude (see, e.g. \([7]\)).

We have shown recently that there exist six gauge invariant sets of diagrams (see Fig.1), which produce corrections of order \( \alpha^2(Z\alpha)^5 \) to muonium HFS \([8]\). All these diagrams may be obtained with the help of different radiative insertions from the skeleton diagram, which contains two external photons attached to the electron line. Contributions induced by polarization operator insertions in external photons and by simultaneous insertion of radiative photon in electron line and one-loop polarization operator in external photon had been calculated in analytic form \([8]\). Correction induced by polarization operator insertions in radiative photons was obtained in semianalytic form as one-dimensional integral, where integrand is itself a complete elliptic integral \([8]\). Contribution induced by the fifth gauge invariant set of graphs containing light by light scattering insertions was reduced to the three dimensional integral over Feynman parameters which was calculated numerically \([10]\).

Contributions to the Lamb shift of order \( \alpha^2(Z\alpha)^5 \) are induced by the same six gauge invariant sets of diagrams which were just described for hyperfine structure. The only difference is that the tensor structure which is relevant

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\(^1\)Several leading logarithmic corrections of higher order which nevertheless turn out to be numerically significant were recently obtained in \([6]\).
for the Lamb shift differs from the one which was relevant for HFS. Numerical calculation of contributions induced by diagrams with polarization operator insertions in the external photons and in the radiative photons as well as by the graphs containing light by light scattering insertions was performed recently [7], [11], [12] and [13].

In the present paper we discuss all possible contributions to HFS and to the Lamb shift of order $\alpha^2(Z\alpha)^5$ which are induced by the last and most bulky set of diagrams with two radiative photons inserted in the electron line (see representative graph in Fig.1c). We present below results of calculation of all contributions both to HFS and to the Lamb shift induced by diagrams containing one-loop electron self-energy as a subgraph and also by the diagram containing two one-loop vertices.

2 General Strategy of Calculations

The set of diagrams with insertions of two radiative photons in the electron line which is relevant for calculation of corrections of order $\alpha^2(Z\alpha)^5$ to HFS and the Lamb shift contains nineteen topologically different diagrams and is presented in Fig.2. The simplest way to describe these graphs is to realize that they were obtained from three graphs for the two-loop electron self-energy by insertion of two external photons in all possible ways. Really, graphs $2a - 2c$ are obtained from the two-loop reducible electron self-energy diagram, graphs $2d - 2k$ are the result of all possible insertions of two external photons in the rainbow self-energy diagram, and diagrams $2l - 2s$ are connected with the overlapping two-loop self-energy graph.

As was discussed at length in [4] (see also [7] for the case of the Lamb shift) one has to perform a rather tedious analysis to find out which diagrams and in what kinematic conditions are relevant for calculation of the energy shifts. Happily, to this end we may use the results of just mentioned papers and we know that to obtain contribution to the energy splitting one has

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2Prof. T. Kinoshita announced recently in the talk presented at the International Workshop on Low Energy Muon Science, Santa Fe, NM, April 4 - 8, 1993 preliminary result of his calculation of the contribution to HFS of order $\alpha^2(Z\alpha)^5$ induced by graphs in Fig.1c (Preprint CLNS 93/1219, May 1993).

3Dr. K. Pachucki is also working now on calculation of contribution of order $\alpha^2(Z\alpha)^5$ induced by graphs in Fig.1c to the Lamb shift. (Private communication from K. Pachucki to M. Eides).
to calculate matrix elements of the diagrams in Fig.2 between free electron spinors with all external electron lines on mass-shell, project these matrix elements on the respective spin state and multiply the result by the square in the origin of the Schrödinger-Coulomb wave function.

It should be mentioned that some diagrams in Fig.2 contain also contributions of previous order in $Z\alpha$. Physical nature of these contributions is especially transparent in the case of HFS. As was mentioned in [14] they correspond to anomalous magnetic moment, their true order in $Z\alpha$ is lower than their apparent order and they should be subtracted from the matrix elements prior to calculation of the contributions to HFS. Analogous situation holds also in the case of the Lamb shift. The only difference is that this time not only the slope of the Pauli formfactor but also the slope of the Dirac formfactor of the electron is capable to produce lower order contribution to the splitting of the energy levels (see, e.g. [7], [11] and [12]) and all respective terms have to be subtracted. Technically cases of lower order contributions both to HFS and to the Lamb shift are quite similar. Lower order terms are each time proportional to the exchanged momentum squared and to get rid of them one has to subtract all low frequency terms proportional to the exchanged momentum squared in the asymptotes of the matrix elements when they exist.

Actual calculation of matrix elements of the diagrams in Fig.2 is impeded by the ultraviolet (UV) and infrared (IR) divergencies. To get rid of UV problems we work only with renormalized (subtracted) graphs and subgraphs and only such graphs are presented in Fig.2. This is the reason why some obvious diagrams with e.g. self-energies on the external electron lines are absent in Fig.2. IR problems are as usual more difficult to deal with than the UV ones. We choose the Fried-Yennie (FY) gauge for the radiative photons. The advantage of this choice of gauge is connected with maximally mild behavior of all matrix elements in the IR region of integration momenta in the FY gauge and with the possibility to perform on-mass-shell renormalization without introduction of the infrared photon mass (see,e.g. [4]). This last feature of the FY gauge is especially important for us as we want to push analytical calculations as far forward as possible and working without the photon mass gives us the chance to obtain much simpler analytical formulae. On the other hand working in the FY gauge without the photon mass makes it absolutely necessary to pay special attention to the infrared behavior of the integrand functions and to perform cancellation of spurious IR divergences prior to integration.
Each contribution of order $\alpha^2 (Z\alpha)^5$ arises from radiative insertions in the skeleton graph in Fig.3 with two external photons. Contribution to hyperfine splitting, produced by diagrams in Fig.2 is given by the expression (see, e.g. [4])

$$\Delta E_{HFS} = 8 \frac{Z\alpha}{\pi n^3} \left(\frac{\alpha}{\pi}\right)^2 E_F \int_0^\infty \frac{d|k|}{k^2} F(k),$$

(1)

where $k_\mu = (0, k)$ is the momentum of the external Coulomb photons measured in the electron mass units[^4] and

$$E_F = \frac{8}{3} (Z\alpha)^4 \frac{m_r^3}{mM} (1 + a_\mu),$$

(2)

where $m_r = m/(1 + m/M)$ is the reduced mass of the electron-muon system and $a_\mu$ is the anomaly of the muon magnetic moment. The function $F(k)$ is connected with the numerator structure of each particular graph and describes radiative corrections to the skeleton diagram. It is normalized on the skeleton numerator contribution.

As was already discussed above anomalous magnetic moment of the electron which is also produced by the radiative insertions in the electron line in the graphs in Fig.2 is not sufficiently mild near the lower boundary of integration region over exchanged momentum in eq.(1) and naively leads to divergence of the integral for the energy splitting. This simply means that anomalous magnetic moment produces contribution of lower order in $Z\alpha$ and respective terms should be subtracted from the expression for the electron factor $F(k)$ prior to integration.

Contribution to the Lamb shift induced by the diagrams in Fig.1 has the form (see, e.g. [7])

$$\Delta E_L = - \frac{16}{n^3} m \left(\frac{m_r}{m}\right)^3 \left(\frac{\alpha}{\pi}\right)^2 \frac{(Z\alpha)^5}{\pi} \int_0^\infty \frac{d|k|}{k^4} L(k).$$

(3)

Function $L(k)$ as in the case of the contribution to HFS is calculated for each particular graph and describes radiative corrections to the skeleton diagram. Spin structure which is relevant for the Lamb shift differs, of course, from the one relevant for the HFS. The integral for the Lamb shift interval in eq.(3) is more singular in the infrared region than the one for the HFS in eq.(1). This singularity once again indicates that some diagrams in Fig.2

[^4]: All momenta below are measured in the electron mass units if reverse is not stated explicitly.
contain not only contributions of order $\alpha^2(Z\alpha)^5m$ to the Lamb shift but also contain abundantly contributions of lower order in parameter $Z\alpha$ (compare the contribution of the anomalous magnetic moment of the electron to HFS discussed above). These contributions are connected with integration over external photon momenta of characteristic atomic order $mZ\alpha$ and they should be subtracted to make calculation of corrections of order $\alpha^2(Z\alpha)^5m$ possible [7].

3 Calculation of Simplest Contributions

3.1 Diagram with Two One-Loop Self-Energy Insertions

3.1.1 Contribution to HFS Interval

We begin actual calculation with the simplest possible graph in Fig.2a, containing reducible electron self-energy insertion in the skeleton graph. Explicit expression for the one-loop electron self-energy in the FY gauge has the form (see, e.g. [4])

$$\Sigma^{FY}(p-k) = (\hat{p} - \hat{k} - 1)^2 \left(\frac{-3\alpha(\hat{p} - \hat{k})}{4\pi}\right) M(k), \quad (4)$$

where

$$M(k) = \frac{1}{1-k^2} + \frac{k^2}{(1-k^2)^2} \log k^2 \quad (5)$$

and kinematic conditions are defined by relations $p_\mu = (1, 0), \; pk = 0, \; k_\mu = (0, k)$.

Substitution of this expression for the self-energy operator instead of the electron factor $F(k)$ in eq.(1) leads to the one-dimensional integral for the contribution to the HFS

$$\Delta E^a_{HFS} = \frac{\alpha^2(Z\alpha)}{\pi n^3} E_F \left( -\frac{9}{2\pi^2} \right) \int_0^\infty d|k|(1-k^2) M^2(k), \quad (6)$$

which may be easily calculated analytically

$$\Delta E^a_{HFS} = \frac{9 \alpha^2(Z\alpha)}{4 \pi n^3} E_F. \quad (7)$$
3.1.2 Contribution to the Lamb Shift

It is not difficult to see that due to the tensor structure of the electron-line factor with two one-loop electron self-energy insertions it does not lead to nonvanishing contribution to the Lamb shift of the order under consideration

\[ \Delta E_L^a = 0. \] (8)

3.2 Diagrams with Simultaneous Insertions of One-Loop Electron Self-Energy and Vertex

3.2.1 General Expression for the One-Loop Vertex with One On-Mass-Shell Leg in the Fried-Yennie Gauge

Renormalized vertex operator with on-mass-shell left leg enters as a subgraph in the diagram in Fig. 2b. We obtained general expression for those entries in such vertex operator in the FY gauge which produce contributions to the hyperfine splitting and the Lamb shift (terms which are proportional to \( k_\mu \) are omitted because they are irrelevant for our goals)

\[ \Lambda_{FY}^{\mu} = \frac{\alpha}{2\pi} \{ A(k) k^2 \gamma_\mu + B(k) \gamma_\mu (\hat{p} - \hat{k} - 1) \]

\[ + C(k) p_\mu (\hat{p} - \hat{k} - 1) + E(k) \sigma_{\mu\nu} k^\nu \}, \]

where

\[ A(k) = -\left( \frac{2}{|k|^3} + \frac{1}{2|k|} \right) \Phi(k) + \frac{2}{k^2} S(k) \]

\[ - \frac{3}{2} M(k) - 2 \frac{\log k^2}{k^2} - 3 \frac{\log k^2}{21 - k^2}, \]

\[ B(k) = -\left( \frac{1}{|k|} + \frac{|k|}{8} \right) \Phi(k) + \frac{1}{2} S(k) - \frac{5}{4} M(k) + \frac{1}{4} \]

\[ - \frac{1}{8} \log k^2 - \frac{7}{8} \frac{\log k^2}{1 - k^2}, \]

\[ C(k) = \frac{1}{|k|} \Phi(k) - S(k) - \frac{1}{2} M(k) + \frac{1}{2} \log k^2 - \frac{3}{2} \frac{\log k^2}{21 - k^2}, \]

\[ E(k) = -\frac{|k|}{8} \Phi(k) - \frac{1}{2} S(k) - \frac{1}{4} M(k) \] (13)
\[ \Phi(k) = |k| \int_0^1 \frac{dz}{1 - k^2 z^2} \log \frac{1 + k^2 z(1 - z)}{k^2 z} = \text{Li}(1 - |k|) - \text{Li}(1 + |k|) \] (14)

\[ +2 \left[ \text{Li}(1 + \sqrt{k^2 + 4 + |k|}) - \text{Li}(1 - \sqrt{k^2 + 4 + |k|}) - \frac{\pi^2}{4} \right], \]

\[ S(k) = \frac{\sqrt{k^2 + 4}}{2|k|} \log \frac{\sqrt{k^2 + 4 + |k|}}{\sqrt{k^2 + 4 - |k|}}. \]

Euler dilogarithm Li is defined here as in [4]. We would like to mention that function \( \Phi(k) \) emerged for the first time in calculation of contribution to HFS induced by the diagrams in Fig.1c [8].

### 3.2.2 Contribution to HFS Interval

It is not difficult now to obtain contribution induced by diagrams in Fig.2b to HFS. Taking into account combinatorial coefficient 2 we obtain

\[ 2\Delta E_{HFS}^b = \frac{\alpha^2(Z\alpha)}{\pi n^3} E_F \left( -\frac{6}{\pi^2} \right) \int_0^\infty d|k| |M(k)| \left[ -k^2 A(k) + B(k) + \frac{1}{2} C(k) \right] \] (15)

or numerically

\[ 2\Delta E_{HFS}^b = -6.65997(1) \frac{\alpha^2(Z\alpha)}{\pi n^3} E_F. \] (16)

### 3.2.3 Contribution to the Lamb Shift

Contribution to the Lamb shift induced by the diagrams in Fig.2b has the form

\[ 2\Delta E_L^b = m \left( \frac{m_r}{m} \right)^3 \frac{\alpha^2(Z\alpha)^5}{\pi n^3} \left( \frac{6}{\pi^2} \right) \int_0^\infty d|k| |M(k)| [-A(k) + B(k) + C(k) - E(k)] \] (17)

or numerically

\[ 2\Delta E_L^b = 2.9551(1) m \left( \frac{m_r}{m} \right)^3 \frac{\alpha^2(Z\alpha)^5}{\pi n^3}. \] (18)
3.3 Diagram with Simultaneous Insertions of Two One-Loop Vertices

3.3.1 Contribution to HFS Interval

Contribution to HFS induced by the diagram in Fig.2c has the form

\[ \Delta E_{c}^{HFS} = \frac{\alpha^2(Z\alpha)}{\pi n^3} E_F \left( \frac{2}{\pi^2} \right) \int_0^\infty d|k| \left\{ k^2 A^2(k) \right\} \]

or numerically

\[ \Delta E_{c}^{HFS} = 3.93208(1) \frac{\alpha^2(Z\alpha)}{\pi n^3} E_F. \]

3.3.2 Contribution to the Lamb Shift

Contribution to the Lamb shift induced by the diagrams in Fig.2c has the form

\[ \Delta E_{c}^{L} = m \left( \frac{m_r}{m} \right)^3 \frac{\alpha^2(Z\alpha)^5}{\pi n^3} \left( \frac{4}{\pi^2} \right) \int_0^\infty d|k| A(k) \left\{ -A(k) + B(k) \right\} \]

or numerically

\[ \Delta E_{c}^{L} = -2.2231(1) m \left( \frac{m_r}{m} \right)^3 \frac{\alpha^2(Z\alpha)^5}{\pi n^3}. \]

4 Calculation of Contributions Induced by the Diagrams with Insertions of the ”Left” Self-Energy Operator

4.1 Diagrams Containing the Two-Loop Vertex

4.1.1 General Expression for the ”Left” Vertex with Left One-Loop Self-Energy Insertion

Let us consider two-loop vertex diagram which is contained as a subgraph in Fig.2d. Insertion of the one-loop electron self-energy in the one-loop vertex corresponds in the FY gauge to the replacement

\[ \frac{\alpha^2(Z\alpha)^5}{\pi n^3} \]
\[
\frac{1}{p + \hat{q} - 1} \rightarrow \frac{3\alpha}{4\pi}(\hat{p} + \hat{q}) \int_0^1 dx \frac{x}{1 - x q^2 + 2pq - \frac{x}{1-x}} \]

(23)

of the left propagator in the one-loop vertex integrand. We used here one-dimensional integral representation for the renormalized self-energy operator in the FY gauge and took into account that the would be loop integration momentum \(q\) unlike momentum \(k\) of the external photon in eq.(4) is not orthogonal to momentum \(p\) of the external electron.

Formal expression for the vertex has the form

\[
\Lambda^d = 3(\frac{\alpha}{4\pi})^2 \int_0^1 dx \frac{x}{1 - x} \int d^4q \frac{N_F^d + N_L^d/q^2}{i\pi^2 q^2(q^2 + 2q(p - k) + k^2)(q^2 + 2pq - \frac{x}{1-x})},
\]

(24)

where

\[
N_F^d = \gamma^\sigma (\hat{p} + \hat{q})\gamma_\mu (\hat{p} + \hat{q} - \hat{k} + 1)\gamma_\sigma,
\]

(25)

\[
N_L^d = 2\hat{q}(\hat{p} + \hat{q})\gamma_\mu (\hat{p} + \hat{q} - \hat{k} + 1)\hat{q}.
\]

(26)

It is convenient to put down second numerator in the form

\[
N_L^d = q^2 N_{L1}^d + N_{L2}^d,
\]

(27)

where

\[
N_{L1}^d = 2[\gamma_\mu (\hat{p} - \hat{k} + 1)\hat{q} + \hat{q}\gamma_\mu + q^2\gamma_\mu],
\]

(28)

\[
N_{L2}^d = 2\hat{q}\hat{p}\gamma_\mu (\hat{p} - \hat{k} + 1)\hat{q}.
\]

As a result of this transformation numerator \(N_{L1}^d\) enters all expressions below on the same footing as \(N_F^d\). Combining denominators in eq.(24)

\[
(1 - t)q^2 + t[u(q^2 + 2q(p - k) + k^2) + (1 - u)(q^2 + 2pq - \frac{x}{1-x})] = (q + Q)^2 - \Delta_t,
\]

\[
Q = pt - kut,
\]

\[
\Delta_t = t[k^2u(1 - ut) + t + \frac{x(1-u)}{1-x}] \equiv t\alpha_t,
\]

we obtain after shift of the integration variable \(q \rightarrow q - Q\) representation for the bare vertex operator.
\[ \Lambda^d_{\mu} = 3\left(\frac{\alpha}{4\pi}\right)^2 \int_0^1 dx \frac{x}{1-x} \int_0^1 dt \int_0^1 du \int \frac{d^4q}{i\pi^2} \]

\begin{align*}
\{ &2t N^d_F(q-Q) + N^d_{L1}(q-Q) \\
&+ 6t(1-t)N^d_{L2}(q-Q) \}
\end{align*}

\[ \frac{(q^2 - \Delta_l)^3}{(q^2 - \Delta_l)^4}. \]

Momentum integration leads to

\[ \Lambda^d_{\mu} = 3\left(\frac{\alpha}{4\pi}\right)^2 \int_0^1 dx \frac{x}{1-x} \int_0^1 dt \int_0^1 du \{[6t(\log \Lambda^2_2 + \Delta_l) - t \frac{4k^2u(1-ut) + 2(1-3t)}{\Delta_l}] \gamma_{\mu} + \frac{t}{\Delta_l} 2t^2 \gamma_{\mu}(\hat{p} - \hat{k} - 1) \\
+ \frac{4t}{\Delta_l}(1 - 2t + t^2 u)p_{\mu}(\hat{p} - \hat{k} - 1) + \frac{2t}{\Delta_l}((1 - t) - ut) \gamma_{\mu}\hat{k} \\
+ 2(1 - t)[-\frac{t}{\Delta_l} + \frac{t^2}{\Delta_l^2}(-2k^2ut + 2t)] \gamma_{\mu} \\
+ \frac{t^2}{\Delta_l^2} 2t(1 - t)[-k^2u(1 - u) + 1] \gamma_{\mu}(\hat{p} - \hat{k} - 1) \\
+ 4(1 - t)[-\frac{t}{\Delta_l} - \frac{t^2}{\Delta_l^2}t]p_{\mu}(\hat{p} - \hat{k} - 1) \\
+ 2(1 - t)[\frac{t}{\Delta_l} - \frac{t^2}{\Delta_l^2}ut(k^2(1 - 2u) + 2)] \gamma_{\mu}\hat{k}. \}
\]

Subtraction term is equal to

\[ \Lambda^d_{\mu}(0) = 3\left(\frac{\alpha}{4\pi}\right)^2 \int_0^1 dx \frac{x}{1-x} \int_0^1 dt \int_0^1 du \{[6t(\log \Lambda^2_2 + \Delta_{l0}) - t \frac{4k^2u(1-3t) + 2(1-3t)}{\Delta_{l0}}] \gamma_{\mu} \\
- \frac{3}{2} \frac{\Lambda^2}{\Lambda^2 + \Delta_l} - \frac{t}{\Delta_{l0}} 2(1 - 3t) + 2(1 - t)[-\frac{t}{\Delta_{l0}} + \frac{t^2}{\Delta_{l0}^2}2t] \gamma_{\mu}, \]

where

\[ \Delta_{l0} \equiv t\alpha_{l0} = \Delta_l|_{k=0}. \]
Note that we preserved term $\Delta$ when it is added to the ultraviolet cutoff $\Lambda^2$ in eq.(31) and eq.(32) in the terms which do not vanish when the cutoff goes to infinity. This was necessary to preserve convergence of the subsequent integration over the Feynman parameter $x$ in eq.(31) and eq.(32) in spite of the singular nature of the explicit integration weight $x/(1 - x)$ near $x = 1$. Really, it is easy to check that $\Delta$ itself is singular near $x = 1$

$$\Delta_{|x\to 1} \simeq \Delta_{|0|x\to 1} \simeq \frac{t(1 - u)}{1 - x}$$

and compensates would be divergence. All potential divergences in integration over $x$ vanish after subtraction and subtracted vertex admits the limit of infinite cutoff. All extra terms containing ultraviolet cutoff $\Lambda^2$ which were preserved in eq.(31) and eq.(32) to make the integration over $x$ finite vanish in the subtracted expression for the vertex at least as $(1/\Lambda^2) \log(\Lambda^2)$. Hence, subtracted expression for the vertex coincides with the naive one which would be obtained if one simply missed the problem of finiteness of integration over $x$ in eq.(31) and eq.(32)!

Last terms in eq.(31) and in eq.(32) may be simplified with the help of identity in eq.(130), where one has to substitute $n = 1$, $\eta = 1$ and $\tau = u$. We also simplify logarithmic term in eq.(31) and eq.(32) which admits (as was just explained) omission of the ultraviolet cutoff after subtraction. We perform integration by parts over $u$, under which subtracted logarithmic term transforms as

$$6t \log \frac{a_{l0}}{a_l} \to -6t(1 - u)[\frac{k^2(1 - 2ut)}{a_l} + \frac{x}{1 - x}(\frac{1}{a_{l0}} - \frac{1}{a_l})]$$

$$= 6t \frac{k^2}{a_l}[1 - 2ut - u^2t + \frac{ut(1 - ut)}{a_{l0}}].$$

Sum of the expressions in eqs.(32) and (34) and other terms in eq.(31) proportional to $\gamma_\mu$ is equal to

$$A_l = -\frac{t}{a_l}(6 + 4u - 12ut + 2u^2t)$$

$$+ \frac{6t^2u(1 - ut)}{a_l a_{l0}} - \frac{4t(1 - t)u(1 - u)}{a_l^2}.$$
\[ B_l = 2 \left\{ \frac{t}{a_l} + \frac{1 - t}{a_l^2} (1 - k^2 u(1 - u)) \right\}, \quad (36) \]

\[ C_l = 4 \left\{ \frac{2 - 3t + t^2 u}{a_l} - \frac{t(1 - t)}{a_l^2} \right\}, \]

\[ E_l = \frac{2 - 3t + t^2 - ut}{a_l} - \frac{2(1 - t)ut}{a_l^2} [k^2(1 - 2u) + 2]. \]

Then renormalized vertex with self-energy insertion has the form

\[ \Lambda_d^\mu = 3 \left( \frac{\alpha}{4\pi} \right)^2 \int_0^1 dx \frac{x}{1 - x} \int_0^1 dt \int_0^1 du \{ k^2 A_l \gamma_{\mu} \}
+ B_l m \gamma_{\mu}(\hat{p} - \hat{k} - 1) + C_l p_{\mu}(\hat{p} - \hat{k} - 1) + E_l \gamma_{\mu} \hat{k}. \]

Note that all four functions \( A_l(k), B_l(k), C_l(k) \) and \( E_l(k) \) are finite at \( k = 0 \).

### 4.1.2 Contribution to HFS Interval

It is not difficult now to obtain contribution to HFS

\[ 2 \Delta E_{\text{HFS}}^d = \frac{\alpha^2 (Z\alpha)}{\pi n^3} E_F \left( \frac{3}{\pi^2} \right) \int_0^1 dx \frac{x}{1 - x} \int_0^1 dt \int_0^1 du \int_0^\infty d|k| [A_l(k)] \quad (38) \]

\[ - \frac{E_l(k) - E_l(0)}{k^2}, \]

where

\[ A_l(k) - \frac{E_l(k) - E_l(0)}{k^2} = -\frac{2t}{a_l} (3 + 2u - 6ut + u^2t) - \frac{2ut(1 - t)}{a_l^2} \quad (39) \]

\[ + \frac{2u(1 - ut)}{a_la_{l_0}} (2 - 3t + 4t^2 - ut) - 4u^2t(1 - t)(1 - ut) \left( \frac{1}{a_l^2 a_{l_0}} + \frac{1}{a_l a_{l_0}^2} \right). \]

Subtraction of \( E_l(0) \) in eq. (38) corresponds to subtraction of the contribution to HFS induced by the second order anomalous magnetic moment which is of lower order in parameter \( Z\alpha \).
After momentum integration we obtain three-dimensional integral for the contribution to HFS induced by both diagrams with nonsymmetric self-energy insertions in Fig.2d

\[
2\Delta E_{HFS}^d = \frac{\alpha^2 (Z\alpha)}{\pi n^3} E_F \left( \frac{3}{2\pi} \right) \int_0^1 dx \frac{x}{1-x} \int_0^1 du \int_0^1 \frac{dt}{\sqrt{u} \sqrt{1-ut}} \int_0^{\infty} d|k| L^d(k) - L^d(0) \frac{k^2}{k^2},
\]

where

\[
f_{1/2}^d = -2t(3 + 2u - 6ut + u^2t),
\]

\[
f_{3/2}^d = 2u(1 - ut)(2 - 3t + 4t^2 - ut) - ut(1 - t),
\]

\[
f_{5/2}^d = -6u^2t(1 - t)(1 - ut).
\]

Numerically we obtain

\[
2\Delta E_{HFS}^d = -3.903(1) \frac{\alpha^2 (Z\alpha)}{\pi n^3} E_F.
\]

### 4.1.3 Contribution to the Lamb Shift

Contribution to the Lamb shift has the form

\[
2\Delta E_L^d = -m \left( \frac{m_r}{m} \right)^3 \frac{\alpha^2 (Z\alpha)^5}{\pi n^3} \left( \frac{6}{\pi^2} \right) \int_0^1 dx \frac{x}{1-x}
\]

\[
\int_0^1 dt \int_0^1 du \int_0^{\infty} d|k| \frac{L^d(k) - L^d(0)}{k^2},
\]

where

\[
L^d(k) \equiv A_d(k) - \frac{1}{2} B_d(k) - \frac{1}{2} C_d(k) + \frac{1}{2} E_d(k)
\]

and

\[
\frac{L^d(k) - L^d(0)}{k^2} = \frac{t(1-t)u^2}{a_l^2} + \frac{u(1-ut)}{a_la_l^2} \left[ 2 + 3t(1 - 4t) + ut(5 + 14t - 10ut) \right]
\]

\[
- \frac{6u^2(1-ut)^2}{a_la_l^2} + t(1-t)u(1-ut)(-1 + 6u - 4u^2)(\frac{1}{a_l^2 a_l^2} + \frac{1}{a_l^2 a_l^2}).
\]
Subtraction in eq. (43) corresponds to subtraction of the contributions to the Lamb shift induced by the slope of the Dirac formfactor and by the anomalous magnetic moment which are of lower order in parameter \( Z\alpha \).

After momentum integration we obtain three-dimensional integral for the contribution to the Lamb shift induced by both diagrams with nonsymmetric self-energy insertions in Fig.2d

\[
2\Delta E_d^L = m \left( \frac{m_r}{m} \right)^3 \frac{\alpha^2(Z\alpha)^5}{\pi n^3} \frac{3}{2\pi} \int_0^1 \frac{dx}{1-x} \int_0^1 \frac{du}{\sqrt{u}} \int_0^1 \frac{dt}{\sqrt{1-ut}}
\]

\[
\{ \frac{g_{3/2}^d}{a_{10}^{3/2}} + \frac{g_{5/2}^d}{a_{10}^{5/2}} \},
\]

where

\[
g_{3/2}^d = -2u(1-ut)(2+3t+5ut-10ut^2+2ut^2) - t(1-t)u^2,
\]

\[
g_{5/2}^d = 3ut(1-ut)[4ut(1-ut) + (1-t)(1-6u+4u^2)].
\]

Numerically we obtain

\[
2\Delta E_d^L = -5.235(2)m \left( \frac{m_r}{m} \right)^3 \frac{\alpha^2(Z\alpha)^5}{\pi n^3}.
\]

### 4.2 Spanning Photon Diagram with the ”Left” Self-Energy Insertion

#### 4.2.1 Contribution to HFS Interval

Consider now spanning photon diagram with ”left” self-energy insertion in Fig.2e. Unlike discussion of the vertex in the previous subsection we will consider here only the part of the spanning photon diagram relevant for hyperfine splitting. Insertion of the one-loop electron self-energy in the FY gauge corresponds to replacement described in eq.(23) in the left propagator in the one-loop vertex integrand.

Formal expression for the spanning photon diagram has the form

\[
\Xi_{\mu\nu}^d = 3\left( \frac{\alpha}{4\pi} \right)^2 \int_0^1 dx \frac{x}{1-x}
\]
\[ \int \frac{d^4q}{i\pi^2} \frac{N^c_F + N^c_L/q^2}{q^2(2q(p-k) + k^2)(q^2 + 2pq - \frac{x}{1-x})(q^2 + 2pq)} \]

where

\[ N^c_F = \gamma^\sigma (\hat{p} + \hat{q}) \gamma_\mu (\hat{p} + \hat{q} - \hat{k} + k + 1) \gamma_\nu (\hat{p} + \hat{q} + 1) \gamma_\sigma, \quad (50) \]

\[ N^c_L = 2\hat{q}(\hat{p} + \hat{q}) \gamma_\mu (\hat{p} + \hat{q} - \hat{k} + k + 1) \gamma_\nu (\hat{p} + \hat{q} + 1) \hat{q}. \quad (51) \]

It is convenient prior to following transformations to rewrite numerator of the longitudinal part in the form

\[ N^c_L = 2(q^2 + 2pq)[q^2 \gamma_\mu (\hat{p} + \hat{q} - \hat{k} + k + 1) \gamma_\nu + \hat{q} \hat{p} \gamma_\mu (\hat{p} + \hat{q} - \hat{k} + 1) \gamma_\nu] \quad (52) \]

\[ \equiv (q^2 + 2pq)[q^2 N^c_{L1} + N^c_{L2}]. \]

Hence, denominator factor \((q^2 + 2pq)\) in eq.(49) cancels with the same explicit factor in the longitudinal numerator in eq.(52) and we use the identity

\[ \int_0^1 dx \frac{x}{1-x} \frac{1}{(q^2 + 2pq - \frac{x}{1-x})(q^2 + 2pq)} \]

\[ = \int_0^1 dx \frac{1}{1-x} \frac{1}{(q^2 + 2pq - \frac{x}{1-x})^2} \quad (53) \]

to get rid of the same factor in the term containing numerator \(N^c_F\).

Next we combine denominators as in eq.(29) and after the shift of integration variable \(q \rightarrow q - Q\) (see eq.(29)) obtain retaining only those numerator structures which are relevant for contribution to HFS

\[ \Xi^e_{\mu\nu} = \frac{3}{8} \left(\alpha^2\right)^2 \int_0^1 dx \frac{1}{1-x} \int_0^1 du \int_0^1 dt \]

\[ \int \frac{d^4q}{i\pi^2} \{3(1-u)t^2 \frac{N^c_F(q-Q)}{(q^2 - \Delta t)^4} \]

\[ + txN^c_{L1}(q-Q) \} + \frac{3t(1-t)xN^c_{L2}(q-Q)}{(q^2 - \Delta t)^4} \} \quad (54) \]

Next we project on the spinor structure relevant for HFS splitting (see, e.g. [13] for explicit expression for the respective projector) and obtain after momentum integration scalar electron factor
\[ F^e = \frac{3}{8} k^2 \int_0^1 dx \frac{1}{1-x} \int_0^1 du \int_0^1 dt \frac{(1-u)t(1-3ut)}{a_t} \]

\[ + (1-u) \frac{k^2 u^2 t^2 (1-ut) + [-1 + t(2-t)(1-ut)]}{a_i^2} \]

\[ + \frac{(1-ut)x}{a_i} + \frac{(1-t)[1+u(1-t)]x}{a_i^2} \].

Substituting electron factor in eq.(55) in eq.(1) one obtains contribution to HFS interval

\[ 2\Delta E^e_{HFS} = 16 \frac{Z\alpha}{\pi n^3} \left( \frac{\alpha}{\pi} \right)^2 E_F \int_0^\infty \frac{d|k|}{k^2} F^e(k) \]

\[ = \frac{\alpha^2(Z\alpha)}{\pi n^3} E_F \frac{3}{2\pi} \int_0^1 \frac{dx}{1-x} \int_0^1 \frac{du}{\sqrt{u}} \int_0^1 \frac{dt}{\sqrt{1-ut}} \frac{f_{1/2}^e}{a_{i0}^{1/2}} + \frac{f_{3/2}^e}{a_{i0}^{3/2}}, \]

where

\[ f_{1/2}^e = (1-u)t[-3 + 5(1-ut)] + 2x(1-ut) \]

and

\[ f_{3/2}^e = (1-u)[-1 + t(2-t)(1-ut)] + x(1-t)[1+u(1-t)]. \]

Numerically we obtain

\[ 2\Delta E^e_{HFS} = 4.566(2) \frac{\alpha^2(Z\alpha)}{\pi n^3} E_F. \]

### 4.2.2 Contribution to the Lamb Shift

Consider now contribution induced by the spanning photon diagram with "left" self-energy insertion in Fig.2e to the Lamb shift. We will below repeat with minor changes considerations of the previous subsection.

General expression for the spanning photon diagram coincides with the one in eq.(49), only explicit expressions for the numerator structures slightly change.

Unlike transformations in the previous sections we have to preserve temporarily small nonvanishing virtuality \( \rho = 1 - p^2 \) of external electron lines while combining denominators. This virtuality will be put to be equal to zero in the final formulae but it is necessary to preserve it on intermediate
stages to qualify spurious infrared divergences which appear in the subtraction term below and cancel one another. Hence, we use instead of eq. (29) slightly modified formulae

$$\Delta_l = t[k^2u(1-ut) + t + \frac{x(1-u)}{1-x} + \rho] \equiv ta_l$$  \hspace{1cm} (59)$$

Explicit expression for the term $N_L^e$ differs from the one used in the previous subsection only due to the change of free Lorentz indices

$$N^e_L = \gamma(\hat{p} + \hat{q})\gamma_0(\hat{p} + \hat{q} - \hat{k} + 1)\gamma_0(\hat{p} + \hat{q} + 1)\gamma_\sigma,$$  \hspace{1cm} (60)$$

while we use another separation of different entries in the term $N^e_L$ here

$$N^e_L = 2\hat{q}(\hat{p} + \hat{q})\gamma_0(\hat{p} + \hat{q} - \hat{k} + 1)\gamma_0(\hat{p} + \hat{q} + 1)\hat{q}$$  \hspace{1cm} (61)$$

$$= (q^2 + 2pq)[2q^2\gamma_0(\hat{p} + \hat{q} - \hat{k} + 1)\gamma_0 + 2\hat{q}\hat{p}\gamma_0(\hat{p} + \hat{q} + 1)\gamma_0]$$

$$+ q^2\hat{q}\hat{p}\gamma_0(\hat{p} + 1)\gamma_0 + 4(pq)\hat{q}\hat{p}\gamma_0(\hat{p} + 1)\gamma_0$$

$$\equiv (q^2 + 2pq)[q^2N^e_{L1} + N^e_{L2}] + q^2N^e_{L3} + N^e_{L4}.$$

Repeating calculations performed in the previous section we obtain (compare eq. (54))

$$\Xi^e_{00} = \frac{3}{8} \left( \frac{\alpha}{\pi} \right)^2 \int_0^1 dx \int_0^1 du \int_0^1 dt$$

$$\int \frac{d^4q}{i\pi^2} \left\{ 3(1-u)t^2\frac{N^e_F(q-Q) + N^e_{L3}(q-Q)}{(q^2 - \Delta_l)^4} \right. $$

$$+ \frac{txN^e_{L1}(q-Q)}{(q^2 - \Delta_l)^3} + \frac{3t(1-t)xN^e_{L2}(q-Q)}{(q^2 - \Delta_l)^4}$$

$$\left. + \frac{12t(1-u)t^2N^e_{L4}(q-Q)}{(q^2 - \Delta_l)^5} \right\}.$$  \hspace{1cm} (62)$$

Performing next integration over momentum we obtain (compare eq. (55)) for the electron factor

$$L^e = \frac{3}{16} \int_0^1 dx \int_0^1 du \int_0^1 dt \left\{ \frac{t(1-u)(3t-5) + x(t-3)}{a_l} \right.$$ $$+ \frac{k^2}{a_l^2}u[1(1-t) + (1-u)t^2(3ut - 3u - 2) + \frac{1}{a_l t}(1-u)(t^2 + t - 6)$$

$$\right\}.$$  \hspace{1cm} (63)$$
+2(1 - u)[\frac{(2 - t)}{a_i^2} - \frac{4t(1 - t)}{a_i^3}]].

Next it is necessary to perform subtraction of the part proportional to $k^2$ from this expression. Note that the separate terms in the last brackets in eq.(63) would lead to infrared divergent integrals at $k^2 = 0$ if one omits small virtuality of the external electron line introduced in the beginning of this subsection. However, all integrals are perfectly convergent with nonvanishing virtuality and we may use auxiliary identity from eq.(130) (for $n = 2$ and $\eta = 1$) to simplify the integral representation. After this simplification integral representation admits vanishing virtuality even at $k^2 = 0$ and we obtain after subtraction

$$\frac{L_e^e}{k^4} \equiv \frac{\mathcal{L}^e(k) - \mathcal{L}^e(0)}{k^2}$$

$$= \frac{3}{16} \left( \int_0^1 dx \int_0^1 du \int_0^1 dt \left\{ \frac{u(1 - ut)}{a_i a_{i0}} [x(3 - t) + (1 - u)t(5 - 3t)] 
-t^2(1 + t)u(1 - ut)(1 - ut)(\frac{1}{a_i^2 a_{i0}} + \frac{1}{a_3 a_{i0}^2}) + \frac{u}{a_i^2} [x(1 - t) + (1 - u)t^2(-2 - 3u + 3ut)] 
- \frac{8}{a_i^3} u^2(1 - u)t(1 - t) \right\}.$$  

Substituting subtracted electron factor in eq.(3) and integrating over $k$ we obtain

$$2\Delta E_{Lamb}^e = m \left( \frac{m_r}{m} \right)^3 \frac{\alpha^2 (Z \alpha)^5}{\pi n^3} \left( \frac{3}{2\pi} \right) \int_0^1 dx \int_0^1 du \int_0^1 dt \left\{ \frac{g_{3/2}^e}{a_{i0}^2} + \frac{g_{5/2}^e}{a_{i0}^3} \right\},$$

where

$$g_{3/2}^e = xu[1 - t + 2(1 - ut)(3 - t)] + ut(1 - u)[t(-2 - 3u + 3ut) + 2(1 - ut)(5 - 3t)],$$

$$g_{5/2}^e = -3u(1 - u)t[2u(1 - t) + t(1 - ut)(1 + t)].$$

Numerically we have

$$2\Delta E_{Lamb}^e = 5.056(1) m \left( \frac{m_r}{m} \right)^3 \frac{\alpha^2 (Z \alpha)^5}{\pi n^3}$$

$$= 18.$$
5 Calculation of Contributions Induced by the Diagrams with Insertions of the ”Right” Self-Energy Operator

5.1 Diagrams Containing the Two-Loop Vertex

5.1.1 General Expression for the ”Left” Vertex with Right One-Loop Self-Energy Insertion

Let us consider vertex diagram which is contained as a subgraph in Fig.2f. Insertion of the one-loop electron self-energy in the FY gauge corresponds to replacement (compare eq.(23))

\[
\frac{1}{\hat{p} + \hat{q} - k - 1} \rightarrow \frac{3\alpha}{4\pi} (\hat{p} + \hat{q} - \hat{k}) \int_0^1 dx \frac{x}{1 - x} \frac{1}{q^2 + 2q(p - k) - \frac{x}{1-x}}
\]

Formal expression for the vertex has then the form

\[
\Lambda_F^f = 3(\frac{\alpha}{4\pi})^2 \int_0^1 dx \frac{x}{1 - x} \int \frac{d^4q}{i\pi^2} \frac{N_F^f + N_L^f/q^2}{q^2 + 2qp(q^2 + 2q(p - k) + k^2 - \frac{x}{1-x})},
\]

where

\[
N_F^f = \gamma^\sigma (\hat{p} + \hat{q} + 1)\gamma_\mu (\hat{p} + \hat{q} - \hat{k}) \gamma_\sigma,
\]

\[
N_L^f = 2\hat{q}(\hat{p} + \hat{q} + 1)\gamma_\mu (\hat{p} + \hat{q} - \hat{k}) \hat{q}.
\]

It is convenient to put down second numerator in the form

\[
N_L^f = (q^2 + 2pq)[q^2 N_{L1}^f + N_{L2}^f],
\]

where

\[
N_{L1}^f = 2\gamma_\mu,
\]

\[
N_{L2}^f = 2\gamma_\mu (\hat{p} - \hat{k}) \hat{q}.
\]

Combining denominators in eq.(69) (compare eq.(29))

\[
(1 - t)q^2 + t[(1 - u)(q^2 + 2qp) + u(q^2 + 2q(p - k) - \frac{x}{1-x})] = (q + Q)^2 - \Delta_r,
\]
\[ Q = pt - kut, \]
\[ \Delta_r = t[k^2u(1 - ut) + t + \frac{xu}{1 - x}] \equiv ta_r, \]

we obtain representation for the bare vertex operator in the form
\[
\Lambda^f_\mu = 3(\frac{\alpha}{4\pi})^2 \int_0^1 dx \frac{x}{1 - x} \int_0^1 du \int_0^1 dt \int_0^{\infty} dq q^2 \frac{2t}{(q^2 - \Delta_r)^3} \tag{75}
\]
\[ + \frac{N^f_{L1}(q - Q)}{(q^2 - \Delta_r)^2} + \frac{2(1 - t)N^f_{L2}(q - Q)}{(q^2 - \Delta_r)^3}, \]

where
\[
\Delta_{r1} = \Delta_r(u = 1) = t[k^2(1 - t) + t + \frac{x}{1 - x} + \rho] \equiv ta_{r1}. \tag{76}
\]

Next we perform shift of integration variable over momentum and euclidean rotation (and we slightly change notation below as \( q \) further means euclidean shifted momentum) and obtain
\[
\Lambda^f_\mu = 3(\frac{\alpha}{4\pi})^2 \int_0^1 dx \frac{x}{1 - x} \int_0^1 du \int_0^1 dt \int_0^{\infty} dq q^2 \frac{2t}{(q^2 + \Delta_r)^3} \tag{77}
\]
\[ + \frac{2}{(q^2 + \Delta_{r1})^2} \gamma_\mu - \frac{4t(1 - t)(k^2 - 1)}{(q^2 + \Delta_{r1})^3} \gamma_\mu \}
\]

Momentum integration leads to
\[
\Lambda^f_\mu = 3(\frac{\alpha}{4\pi})^2 \int_0^1 dx \frac{x}{1 - x} \int_0^1 du \int_0^1 dt \{ [2t(\log \frac{\Lambda^2 + \Delta_r}{\Delta_r} - \frac{3}{2} \frac{\Lambda^2}{\Delta_r + \Delta_r}) - \frac{2k^2ut(1 - ut) + 2(1 - t^2)}{a_r} \gamma_\mu - \frac{2t(1 - t)}{a_r} \gamma_\mu(\hat{p} - \hat{k} - 1)]
\]
\[ + \frac{4(1 - t)(1 - ut)}{a_r} p_\mu(\hat{p} - \hat{k} - 1) + \frac{2(1 - t)^2}{a_r} \gamma_\mu(\hat{k})
\]
\[ + 2t(\log \frac{\Lambda^2 + \Delta_{r1}}{\Lambda_{r1}} - \frac{\Lambda^2}{\Lambda^2 + \Delta_{r1}}) \gamma_\mu + \frac{2(1 - t)(1 - k^2)}{a_{r1}} \gamma_\mu \} \tag{78} \]
Subtraction term is equal to

\[
\Lambda^f_\mu(0) = 3\left(\frac{\alpha}{4\pi}\right)^2 \int_0^1 dx \frac{x}{1-x} \int_0^1 du \int_0^1 dt \left\{2t(\log \frac{\Lambda^2 + \Delta_{r0}}{\Delta_{r0}} - \frac{3}{2} \frac{\Lambda^2}{\Lambda^2 + \Delta_{r0}}) \right. \\
- \frac{2(1-t^2)}{a_{r0}} + 2 \left[ \log \frac{\Lambda^2 + \Delta_{r10}}{\Delta_{r10}} - \frac{\Lambda^2}{\Lambda^2 + \Delta_{r10}} \right] + \frac{2(1-t)}{a_{r10}} \left\} \gamma_\mu,
\]

where

\[
a_{r0} = a_r|_{k=0},
\]
\[
a_{r10} = a_{r1}|_{k=0}.
\]

We extract explicit dependence on \( k^2 \) in eq.(78) after subtraction with the help of integration by parts

\[
\int_0^1 du \log \frac{a_{r0}}{a_r} = k^2 \int_0^1 du \frac{t(1-u)}{a_r} \left( u - \frac{1-ut}{a_{r0}} \right),
\]
\[
\int_0^1 dt \log \frac{a_{r10}}{a_{r1}} = -k^2 \int_0^1 dt \frac{t}{a_{r1}x + t(1-x)}.
\]

Then renormalized vertex with the right self-energy insertion acquires the form (compare eq.(37))

\[
\Lambda^f_\mu = 3\left(\frac{\alpha}{4\pi}\right)^2 \int_0^1 dx \frac{x}{1-x} \int_0^1 du \int_0^1 dt \left\{k^2 A_r \gamma_\mu \right. \\
+ B_r \gamma_\mu (\hat{p} - \hat{k} - 1) + C_r p_\mu (\hat{p} - \hat{k} - 1) + E_r \gamma_\mu \hat{k} \left\},
\]

where

\[
A_r = -\frac{2ut(1-t)}{a_r} + 2 \frac{(1-ut)(u-t^2)}{a_r a_{r0}} - \frac{2}{a_{r1}[x + t(1-x)]},
\]
\[
B_r = -\frac{2t(1-t)}{a_r},
\]
\[
C_r = \frac{4(1-t)(1-ut)}{a_r},
\]
\[
E_r = \frac{2(1-t)^2}{a_r}.
\]

Note that as well as in the case of the left self-energy insertion in eq.(33) and eq.(34) all four functions \( A_r(k), B_r(k), C_r(k) \) and \( E_r(k) \) are finite at \( k = 0 \).
5.1.2 Contribution to HFS Interval

It is not difficult now to obtain contribution to HFS

\[ 2\Delta E_{HFS}^f = \frac{\alpha^2(Z\alpha)}{\pi n^3} E_F \left( \frac{3}{\pi^2} \right) \int_0^1 dx \frac{x}{1-x} \int_0^1 du \int_0^1 dt \int_0^\infty d|k| \left[ A_r(k) \right] \]

\[ -\frac{\mathcal{E}_r(k) - \mathcal{E}_r(0)}{k^2}, \]

where

\[ A_r(k) - \frac{\mathcal{E}_r(k) - \mathcal{E}_r(0)}{k^2} = 2\left\{ -\frac{ut(1-ut)}{a_r} + \frac{(1-ut)[2u(1-t) - (1-u)t^2]}{a_r a_{r0}} \right\} \]

\[ -\frac{1}{a_{r1} (x + t - xt)} \}. \]

Subtraction of \( \mathcal{E}_r(0) \) in eq.(83) corresponds to subtraction of the contribution to HFS induced by the second order anomalous magnetic moment which is of lower order in parameter \( Z\alpha \).

After momentum integration we obtain three-dimensional integral for the contribution to HFS induced by both diagrams with nonsymmetric self-energy insertions in Fig.2f

\[ 2\Delta E_{HFS}^f = \frac{\alpha^2(Z\alpha)}{\pi n^3} E_F \left\{ -3 + \frac{3}{\pi} \int_0^1 dx \frac{x}{1-x} \int_0^1 du \int_0^1 dt \frac{1}{\sqrt{u}} \frac{1}{\sqrt{1-ut}} \right\} \]

\[ \left[ \frac{f_{1/2}^f}{a_{r0}^2} + \frac{f_{3/2}^f}{a_{r0}^2} \right], \]

where

\[ f_{1/2}^f = -ut(1-t), \]

\[ f_{3/2}^f = (1-ut)[2u(1-t) - (1-u)t^2]. \]

Numerically we obtain

\[ 2\Delta E_{HFS}^f = -3.401(1) \frac{\alpha^2(Z\alpha)}{\pi n^3} E_F. \]
5.1.3 Contribution to the Lamb Shift

Contribution to the Lamb shift induced by the diagram in Fig.2f has the same form as in eq.(43), where

\[
\frac{\mathcal{L}_f(k) - \mathcal{L}_f(0)}{k^2} = \frac{u(1-t)(1-ut)}{a_r a_{r0}} \quad (88)
\]

\[- \frac{2u(1-ut)^2(u-t^2)}{a_r a_{r0}^2} + \frac{2(1-x)(1-t)}{a_{r1}[x+t(1-x)]^2}.
\]

After momentum integration we obtain respective contribution to the Lamb shift in the form of the three-dimensional integral

\[
2\Delta E^f_L = m \left( \frac{m_r}{m} \right) \frac{3\alpha^2(Z\alpha)^5}{\pi n^3} \left\{ -2 + \frac{3}{\pi} \int_0^1 dx \frac{x}{1-x} \int_0^1 du \int_0^1 dt \left[ g_{3/2}^f \frac{g_{3/2}^f}{a_{r0}} + g_{5/2}^f \frac{g_{5/2}^f}{a_{r0}} \right] \right\},
\]

where

\[
g_{3/2}^f = -1 + t, \quad (90)
\]

\[
g_{5/2}^f = 2(1-ut)(u-t^2).
\]

Numerically we obtain

\[
2\Delta E^f_L = -1.017(1)m \left( \frac{m_r}{m} \right) \frac{3\alpha^2(Z\alpha)^5}{\pi n^3}. \quad (91)
\]

5.2 Spanning Photon Diagram with Symmetrical Self-Energy Insertion

5.2.1 Contribution to HFS Interval

Consider now spanning photon diagram with symmetrical self-energy insertion in Fig.2g. Unlike discussion of the vertex in the previous subsection we will consider here only the part of the spanning photon diagram relevant for hyperfine splitting. Insertion of the one-loop electron self-energy in the FY gauge corresponds to replacement described in eq.(83) in the central propagator in the spanning photon diagram.

Formal expression for the spanning photon diagram has the form
\[ \Xi_{\mu\nu}^g = 3\left(\frac{\alpha}{4\pi}\right)^2 \int_0^1 dx \frac{x}{1-x} \]  
\[ \int \frac{d^4q}{i\pi^2 q^2 (q^2 + 2qp - \rho)^2 [q^2 + 2q(p - k) + k^2 - \frac{x}{1-x} - \rho]} \]

where

\[ N^g_F = \gamma^\sigma (\hat{p} + \hat{q} + 1) \gamma_\mu (\hat{p} + \hat{q} - \hat{k}) \gamma_\nu (\hat{p} + \hat{q} + 1) \gamma_\sigma, \]  
\[ N^g_L = 2\hat{q} (\hat{p} + \hat{q} + 1) \gamma_\mu (\hat{p} + \hat{q} - \hat{k}) \gamma_\nu (\hat{p} + \hat{q} + 1) \hat{q} \]
\[ = 2(q^2 + 2pq)^2 \gamma_\mu (\hat{q} - \hat{p}) \gamma_\nu. \]  

Note that we temporarily preserve in eq.(92) nonvanishing virtuality \( \rho = 1 - p^2 \) of the external fermion lines. This virtuality will be put to be equal to zero in the final formulae but it is necessary to preserve it on intermediate stages to qualify spurious infrared divergences which cancel one another. The problem of infrared divergences is even more acute here than in discussion of the Lamb shift contribution induced by the diagram in Fig. 2e, since integral in eq.(92) contains more powers of integration momentum \( q \) in the denominator. We will see below that as a result separate parametric integrals giving contributions to the electron factor diverge (for vanishing virtuality) even in the case of nonvanishing exchanged momentum \( k \) and only total contribution to the electron factor turns out to be finite.

Envisaging problems with infrared divergences it is convenient prior to integration to separate in the numerators \( N^g_F \) and \( N^g_L \) infrared safe terms \( N^g_{F1} \) and \( N^g_{L2} \)

\[ N^g_F = N^g_{F1} + \frac{\gamma^\sigma (\hat{p} + \hat{q} + 1) \gamma_\mu (\hat{p} + \hat{q} - \hat{k}) \gamma_\nu (\hat{p} + \hat{q} + 1) \gamma_\sigma}{N^g_{F1} + 4 \gamma_\mu (\hat{q} - \hat{k}) \gamma_\nu} \equiv N^g_{F1} + N^g_{F2}, \]  
and

\[ N^g_L = q^2 N^g_{L1} + 8(pq)^2 \gamma_\mu (\hat{q} - \hat{k}) \gamma_\nu \equiv q^2 N^g_{L1} + N^g_{L2}, \]  

Next we combine denominators as in eq.(74) but preserving nonvanishing virtuality of external electron lines so that explicit expression for \( \Delta_r \) slightly changes and below
\[ \Delta_r = t[k^2u(1 - ut) + t + \frac{ux}{1 - x} + \rho] \equiv ta_r. \quad (97) \]

After shift of integration variable \( q \rightarrow q - Q \) (see eq.(74)) we obtain retaining only those numerator structures which are relevant for contribution to HFS

\[ \Xi^{g}_{\mu\nu} = \frac{3}{16} \frac{\alpha}{\pi^2} \int_0^1 dx \frac{x}{1 - x} \int_0^1 du (1 - u) \int_0^1 dt t^2 \]
\[ \int \frac{d^4q}{i\pi^2} \left\{ \frac{N_{F2}^g(q - Q) + [N_{F1}^g(q - Q) + N_{L1}^g(q - Q)]}{(q^2 - \Delta_r)^4} \right\} \]
\[ + 4(1 - t) \frac{N_{L2}^g(q - Q)}{(q^2 - \Delta_r)^5}. \]

After integration over momentum we obtain

\[ F^g = \frac{3}{4} \int_0^1 dx \frac{x}{1 - x} \int_0^1 du (1 - u) \int_0^1 dt \left\{ (1 - 2ut) \left[ \frac{2 - t}{a_r^2} - \frac{4t(1 - t)}{a_r^3} \right] \right\} \]
\[ + \frac{3t(1 - 2ut)}{a_r} + \frac{2t(1 - ut)(3 - t + k^2u^2t)}{a_r^2}. \]

infrared unsafe terms in the central brackets in eq.(99) may be easily exorcised with the help of the identity ineq.(130). Substituting then expression electron factor in eq.(99) in eq.(1) and integrating over \( k \) one obtains contribution to HFS interval

\[ \Delta E^{g}_{HFS} = \frac{\alpha^2(Z\alpha)}{\pi n^3} - E_F \left( \frac{3}{2\pi} \right) \int_0^1 dx \frac{x}{1 - x} \int_0^1 du \frac{u}{\sqrt{1 - u}} \int_0^1 dt \frac{t}{\sqrt{1 - ut}} \left\{ \frac{f_{1/2}^g}{a_r} \right\} \]
\[ + \frac{f_{3/2}^g}{a_r^2}, \]

where
\[ f_{1/2}^g = 3 - 5ut \quad (101) \]

and
\[ f_{3/2}^2 = -u(1 - t) - t(1 - ut). \]

Numerically we obtain
\[ \Delta E_{HF}^g = 2.682(1) \frac{\alpha^2 (Z \alpha)}{\pi n^3} E_F. \quad (102) \]

### 5.2.2 Contribution to the Lamb Shift

Consider now contribution induced by the spanning photon diagram with symmetrical self-energy insertion in Fig. 2g to the Lamb shift. We will closely follow considerations performed in the previous subsection.

Formal expression for the spanning photon diagram has the same form as in eq. (92), the only difference is connected with the free Lorentz indices.

We separate infrared safe terms in the numerators in the same way as in the previous subsection (see eq. (95) and eq. (96)) and obtain repeating all by now standard steps

\[ L^g = \frac{3}{8} \int_0^1 dx \frac{x}{1 - x} \int_0^1 du (1 - u) \int_0^1 dt \left\{ \frac{3(2t - 1)}{2a_r} \right\} \]

This expression may be further simplified with the help of identity similar to the one in eq. (103) but obtained with the help of integration by parts of the fraction \( t^2(1 - t)/a_r \) instead of the fraction in eq. (128). We then obtain

\[ L^g = \frac{3}{8} \int_0^1 dx \frac{x}{1 - x} \int_0^1 du (1 - u) \int_0^1 dt \left\{ \frac{3(2t - 1)}{2a_r} \right\} \]

This expression is very convenient for subtraction of the low frequency asymptote since almost all terms in it contain factor \( k^2 \) explicitly. Performing subtraction we obtain

\[ \frac{L^g}{k^4} = \frac{L^g(k) - L^g(0)}{k^2} = \frac{3}{8} \int_0^1 dx \frac{x}{1 - x} \int_0^1 du (1 - u) \int_0^1 dt \quad (105) \]
\[ \left\{ \frac{-3u(1-ut)(2t-1)}{2aa_r0} + \frac{u(2t^2u-t-1)}{a_r^2} - \frac{4u^2(1-t)^2}{a_r^3} \right\}. \]

Substituting now subtracted electron factor in eq. (3) and performing integration over \( k \) we obtain contribution to the Lamb shift

\[ \Delta E_{Lamb}^g = m \left( \frac{m_r}{m} \right)^3 \frac{\alpha^2(Z\alpha)^5}{\pi n^3} \left( \frac{3}{\pi} \right) \int_0^1 dx \frac{x}{1-x} \int_0^1 du \sqrt{u(1-u)} \int_0^1 dt \frac{t}{\sqrt{1-ut}} \right\} \]

\[ \left\{ \frac{g_{3/2}^g}{a_r0} + \frac{g_{5/2}^g}{a_r0} \right\}, \]

where

\[ g_{3/2}^g = 2 - 7t - 3ut + 8u^2, \]

\[ g_{5/2}^g = -3u(1-t)^2. \]

Numerically we obtain

\[ \Delta E_{Lamb}^g = -0.14601(4) m \left( \frac{m_r}{m} \right)^3 \frac{\alpha^2(Z\alpha)^5}{\pi n^3}. \]

6 Calculation of Contribution Induced by Insertion of the Rainbow Self-Energy Operator

6.1 General Expression for the Rainbow Self-Energy Diagram

Let us construct renormalized two-loop rainbow contribution to the electron self-energy operator in the FY gauge. To this end we take general expression for the one-loop bare self-energy operator in the FY gauge

\[ \Sigma(p) = \frac{\alpha}{4\pi} \int \frac{d^4q}{i\pi^2 q^2} (g^{\alpha\beta} + 2q^\alpha q^\beta \gamma_\alpha \gamma_\beta \frac{1}{p+q-m\gamma_\beta} \right) \]

\[ \frac{g_{3/2}^g}{a_r0} + \frac{g_{5/2}^g}{a_r0} \]

\[ G_{3/2}^g = 2 - 7t - 3ut + 8u^2, \]

\[ G_{5/2}^g = -3u(1-t)^2. \]

Numerically we obtain

\[ \Delta E_{Lamb}^g = -0.14601(4) m \left( \frac{m_r}{m} \right)^3 \frac{\alpha^2(Z\alpha)^5}{\pi n^3}. \]

6 Calculation of Contribution Induced by Insertion of the Rainbow Self-Energy Operator

6.1 General Expression for the Rainbow Self-Energy Diagram

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and perform substitution similar to the one in eq. (68), which describes insertion of the renormalized self-energy operator in the FY gauge on the internal line. We then obtain

\[
\Sigma^R_B(p) = 3 \left( \frac{\alpha}{4\pi} \right)^2 \int_0^1 dx \frac{x}{1 - x} \int \frac{d^4 q}{i\pi^2} \frac{N^h_F + N^h_L/q^2}{q^2[(p + q)^2 - m^2] + 1 - x},
\]

(110)

where

\[
N^h_F = \gamma^\sigma (\hat{p} + \hat{q}) \gamma^\sigma = -2(\hat{p} + \hat{q}),
\]

(111)

\[
N^h_L = 2\hat{q}(\hat{p} + \hat{q}) = q^2(2\hat{q} - \hat{p}) + (2\hat{q}\hat{p}\hat{q} + q^2\hat{p}) \equiv q^2(N^h_{L1} + N^h_{L2}).
\]

Combining denominators with the help of the Feynman parameters and performing shift of integration momentum \[^6\] Wick rotation and momentum integration we obtain explicit expression for the unrenormalized rainbow self-energy operator

\[
\Sigma^R_B(p) = 3 \left( \frac{\alpha}{4\pi} \right)^2 \int_0^1 dx \frac{x}{1 - x} \int_0^1 dt \left[ -3\hat{p}H(p) - \frac{3\hat{p}\hat{p}t^2(1 - t)}{\Delta} \right],
\]

(112)

where

\[
H(p) = \log \frac{\Lambda^2 + \Delta_h}{\Delta_h} - \frac{\Lambda^2}{\Lambda^2 + \Delta_h},
\]

(113)

\[
\Delta_h = t[-p^2(1 - t) + \frac{m^2}{1 - x}] - \frac{m^2}{1 - x}.
\]

Term \( \Delta_h \) is preserved on the background of the UV cutoff in eq. (113) due to the same reasons which were explained after eq. (32). It is convenient to rewrite \( H(p) \) in the form

\[
H(p) = H(m) + [H(p) - H(m)],
\]

(114)

where first term is the momentum independent constant which in any case disappears after subtraction and the second term already admits limit of the infinite cutoff. We integrate the this second term over \( t \) by parts to get rid of logarithm and obtain

\[^6\] Note that \( N^h_F + N^h_{L1} \) does not contain linear in the integration momentum term, while integral depending on \( N^h_{L2} \) is UV finite, and, hence, no surface terms appear under shift of the integration momentum.
\[ \Sigma_R^R(p) = -9(\frac{\alpha}{4\pi})^2 \hat{p} \int_0^1 dx \frac{x}{1-x} \int_0^1 dt \left[ H(m) + \frac{p^2 t^2 (2-t)}{\Delta_h} \right]. \quad (115) \]

After two standard subtractions on the mass-shell followed by change of variables \( v = 1-t, \xi = v(1-x) \) and integration by parts over new variable \( v \) we come to the representation for the FY gauge renormalized rainbow mass-operator which is the most convenient one for subsequent calculation of the contributions to the energy splitting

\[ \Sigma_R(p) = -9(\frac{\alpha}{4\pi})^2 (\hat{p} - m)^2 \int_0^1 dv \frac{1}{v} (1-v)^2 \int_0^v \frac{d\xi}{(1-\xi)^2} \frac{\hat{p}(1+\xi) + 2m}{m^2 - p^2 \xi}. \quad (116) \]

Additional integration by parts over \( v \) leads to the representation in the form of one-dimensional integral

\[ \Sigma_R(p) = 9(\frac{\alpha}{4\pi})^2 (\hat{p} - m)^2 \int_0^1 dv \left[ \frac{1 - v + \log v}{(1-v)^2} + \frac{1}{2} \right] \frac{\hat{p}(1+v) + 2m}{m^2 - p^2 v} \quad (117) \]

\[ \equiv (\hat{p} - m)^2 (\hat{p}\sigma_p + m\sigma_m). \]

Even remaining final integration may be performed analytically and one can obtain representation of the rainbow contribution to the electron self-energy in the closed form

\[ \Sigma_R^R(p) = \frac{3\alpha}{8\pi} \Sigma_{FY}^R(p) + \frac{9}{16} \left( \frac{\alpha}{\pi} \right)^2 \frac{(\hat{p} - m)^2}{m^2} \{ 2(\hat{p} + m)F_1(\rho) \}
\]

\[ -(\hat{p} + 2m)F_2(\rho) - m \frac{\log \rho}{1 - \rho} \}, \]

where \( \rho \) is the virtuality of the electron line (momentum \( p \) is arbitrary in this subsection)

\[ \rho = \frac{m^2 - p^2}{m^2}, \quad (119) \]

\( \Sigma_{FY}^R(p) \) is the one-loop electron self-energy (compare eq.(11))

\[ \Sigma_{FY}^R(p) = -\frac{3\alpha}{4\pi} (\hat{p} - m)^2 \left[ \frac{1}{1 - \rho} + \frac{\rho}{(1-\rho)^2} \log \rho \right] \]

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and functions $F_i$ are defined as follows

\[
F_1(\rho) = \frac{1}{\rho^2} \left( -\frac{\pi^2}{6} - \text{Li}_2(1 - \rho) + \rho(\log \rho - 1) \right),
\]

\[
F_2(\rho) = \frac{1}{\rho} \left( -\frac{\pi^2}{6} - \text{Li}_2(1 - \rho) \right).\]

### 6.1.1 Contribution to HFS Interval

For final calculation of the contribution to the HFS interval we use representation (116)

\[
\Delta E_{\text{HFS}} = \alpha^2 \left( \frac{Z\alpha}{\pi n^3} \right) \frac{9}{2\pi^2} \int_0^\infty dk \int_0^1 dv \frac{dv}{v(1-v)^2} \int_0^v \frac{d\xi}{(1-\xi)^2} \frac{1+\xi}{1-\xi + k^2}\xi
\]

\[
= \alpha^2 \left( \frac{Z\alpha}{\pi n^3} \right) \frac{9}{2\pi^2} \int_0^1 dv \frac{dv}{v(1-v)^2} \int_0^v \frac{d\xi}{(1-\xi)^2} \left( \frac{3+\xi}{1-\xi + k^2\xi} - \frac{3+\xi}{1-\xi} \right)
\]

(121)

### 6.1.2 Contribution to the Lamb Shift

With the help of representation (116) we obtain contribution to the Lamb shift interval

\[
\Delta E_{\text{L}} = m \left( \frac{m_r}{m} \right)^3 \frac{16(Z\alpha)^5}{\pi n^3} \int_0^\infty dk \frac{dk}{k^2} \left[ \sigma_p(k) + \sigma_m(k) \right] - \left[ \sigma_p(0) + \sigma_m(0) \right]
\]

\[
= m \left( \frac{m_r}{m} \right)^3 \frac{16(Z\alpha)^5}{\pi n^3} \left( -\frac{9}{\pi^2} \right) \int_0^1 dv \frac{dv}{v(1-v)^2} \int_0^v \frac{d\xi}{(1-\xi)^2} \left[ \frac{3+\xi}{1-\xi + k^2\xi} - \frac{3+\xi}{1-\xi} \right]
\]

\[
= \frac{153}{40} \left( \frac{m_r}{m} \right)^3 \frac{\alpha^2(Z\alpha)^5}{\pi n^3}.
\]

Footnote 7: We use standard Euler dilogarithm $\text{Li}_2(\rho)$ here unlike function $\text{Li}(\rho)$ in eq.(14) which was defined as the real part of the Euler dilogarithm.
7 Discussion of Results

We presented above results of calculation of contributions to the Lamb shift and HFS of order \(\alpha^2(Z\alpha)^5\) induced by all two-loop insertions in the electron line containing as a block one-loop electron self-energy graph. Calculations were performed in the FY gauge which is the most suitable one due to its infrared smoothness. The formulae for different contributions obtained above admit numerical calculation with arbitrary accuracy. We consider it a bit premature to present here the sum of all contributions obtained above since considered set of graphs is not gauge invariant and comparison with the experimental data have to await until all other contributions would be obtained. Nevertheless we have chosen to present above calculations with sufficient details both because of their volume and to present main technical tricks used in our work.

Results of calculation of other contributions will be published separately.

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A

Calculations performed in the main body of this paper were greatly facilitated by the infrared smoothness of the FY gauge. We would like to display in this appendix how cancellation of infrared divergences in the FY gauge occurs and derive a useful identity widely used above.

Typical integral for the contribution to the electron factor with the worst infrared behavior has the form

\[
I = \int \frac{d^4q}{(2\pi)^2} \frac{1}{q^2} \left( g^{\alpha\beta} + \xi q^\alpha q^\beta - \frac{q^2 q^2}{q^2} \right) \int_0^1 dx \ldots \int_0^1 dz f(x, \ldots, z) \frac{\gamma_\alpha(\hat{p} + m)\hat{A}(\hat{p} + m)\gamma_\beta}{[q^2 + 2q(p\eta - k\tau) - \omega]^{n+1}},
\]

(123)
where $p$ is the on-shell electron momentum $\hat{p} = m$ and matrix $\hat{A}$ and function of the Feynman parameters $\omega$ do not depend on $q$.

Taking into account mass-shell condition for the vector $p$ one easily obtains

$$I = 4\hat{A}\int_0^1 dx \ldots \int_0^1 dz f(x, \ldots, z) \int \frac{d^4q}{\pi^2i} \left\{ \frac{p^2}{q^2[q^2 + 2q(p\eta - k\tau) - \omega]^{n+1}} + \frac{\xi(pq)^2}{q^4[q^2 + 2q(p\eta - k\tau) - \omega]^{n+1}} \right\}. \tag{124}$$

We omit below unessential for further considerations integration over the Feynman parameters and obtain after combining denominators, shift of integration variable $q \rightarrow q - (p\eta - k\tau)t$ and the Wick rotation

$$I = (n + 1)(-1)^{n+2}p^2\int_0^1 dt^n \int \frac{d^4q}{\pi^2} \left\{ \frac{1}{(q^2 + \Delta)^{n+2}} + \xi(n + 2)(1-t)\frac{q^2/4 - p^2\eta^2t^2 - k^2\tau^2t^2}{(q^2 + \Delta)^{n+3}} \right\}, \tag{125}$$

where

$$\Delta = (p^2\eta^2t + k^2\tau^2t + \omega)t \equiv ta. \tag{126}$$

Integrating over $q$ one obtains

$$I = \frac{(-1)^n}{n}m^2\int_0^1 dt \left\{ \frac{1 + (\xi/2)(1-t)}{a^n} - n\xi t(1-t)(m^2\eta^2 + k^2\tau^2) \right\}. \tag{127}$$

Integration over $t$ is in the general case infrared unsafe and may lead to infrared divergences if, e.g. variable $\omega$ vanishes for some reason. Hence, it would be helpful to gain additional power of $t$ in the numerator of the integrand. To this end it is useful to observe validity of the simple identity

$$\frac{\partial}{\partial t} \frac{t(1-t)}{a^n} = \frac{1 - 2t}{a^n} - n\frac{t(1-t)}{a^{n+1}} \frac{\partial a}{\partial t} \equiv \frac{1 - 2t}{a^n} - n\frac{t(1-t)(m^2\eta^2 + k^2\tau^2)}{a^{n+1}}. \tag{128}$$

Integrating the last term in eq.(127) with the help of the identity in eq.(128) one obtains
\[ I = \frac{(-1)^n}{n} m^2 \int_0^1 \frac{dt}{a^n} [1 - \frac{\xi}{2} + \frac{3}{2} \xi t]. \] (129)

We see that terms with minimal power of \( t \) in the numerator disappear when \( \xi = 2 \), i.e. in the FY gauge, thus making integration over \( t \) more smooth for low \( t \). We used this trick with integration by parts abundantly in this paper and we put it down here for references in the FY gauge (i.e. when \( \xi = 2 \))

\[
\int_0^1 dt \{ \frac{2 - t}{a^n} - 2n \frac{t(1 - t)(m^2 \eta^2 + k^2 \tau^2)}{a^{n+1}} \} = 3 \int_0^1 dt \frac{t}{a^n}. \] (130)

Note that identity in eq.(130) is only a representative of a large family of identities which may be obtained in the same way but choosing different numerators in eq.(128).
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Figure Captions

Fig.1. Six gauge-invariant sets of diagrams producing contributions of order $\alpha^2(Z\alpha)^5$ to HFS and Lamb shift.

Fig.2. All graphs with two radiative photons producing contributions of order $\alpha^2(Z\alpha)^5$ to HFS and Lamb shift.