A note on boundedness of operators in Grand Grand Morrey spaces

Humberto Rafeiro

Dedicated with great pleasure to Stefan Samko on the occasion of his 70th birthday

Abstract. In this note we introduce grand grand Morrey spaces, in the spirit of the grand Lebesgue spaces. We prove a kind of reduction lemma which is applicable to a variety of operators to reduce their boundedness in grand grand Morrey spaces to the corresponding boundedness in Morrey spaces. As a result of this application, we obtain the boundedness of the Hardy-Littlewood maximal operator and Calderón-Zygmund operators in the framework of grand grand Morrey spaces.

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1. Introduction

In 1992 T. Iwaniec and C. Sbordone [12], in their studies related with the integrability properties of the Jacobian in a bounded open set Ω, introduced a new type of function spaces $L^p(\Omega)$, called grand Lebesgue spaces. A generalized version of them, $L^{p,\theta}(\Omega)$ appeared in L. Greco, T. Iwaniec and C. Sbordone [11]. Harmonic analysis related to these spaces and their associate spaces (called small Lebesgue spaces), was intensively studied during last years due to various applications, we mention e.g. [2, 4, 6, 7, 8, 9, 13].

Recently in [20] there was introduced a version of weighted grand Lebesgue spaces adjusted for sets $\Omega \subseteq \mathbb{R}^n$ of infinite measure, where the integrability of $|f(x)|^{p-\varepsilon}$ at infinity was controlled by means of a weight, and there grand grand

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Lebesgue spaces were also considered, together with the study of classical operators of harmonic analysis in such spaces. Another idea of introducing “bilateral” grand Lebesgue spaces on sets of infinite measure was suggested in [15], where the structure of such spaces was investigated, not operators; the spaces in [15] are two parametrical with respect to the exponent \( p \), with the norm involving \( \sup_{p_1 < p < p_2} \).

Morrey spaces \( L^{p, \lambda} \) were introduced in 1938 by C. Morrey [16] in relation to regularity problems of solutions to partial differential equations, and provided a useful tool in the regularity theory of PDE’s (for Morrey spaces we refer to books [10, 14], see also [19] where an overview of various generalizations may be found).

Recently, in the spirit of grand Lebesgue spaces, A. Meskhi [17, 18] introduced grand Morrey spaces (in [17] it was already defined on quasi-metric measure spaces with doubling measure) and obtained results on the boundedness of the maximal operator, Caldéron-Zygmund singular operators and Riesz potentials. Note that the “grandification procedure” was applied only to the parameter \( p \).

In this paper we make a further step and apply the “grandification procedure” to both the parameters, \( p \) and \( \lambda \), obtaining grand grand Morrey spaces \( L^{p, \lambda}_{\theta, \alpha}(\Omega) \). In this new framework we obtain a reduction boundedness theorem, which reduces the boundedness of operators (not necessarily linear ones) in grand Morrey spaces to the corresponding boundedness in classical Morrey spaces.

Notation
Throughout the text we use the following notation:
\( \Omega \) stands for an open set in \( \mathbb{R}^n \),
\( |A| \) denotes the Lebesgue measure of a measurable set \( A \subset \Omega \),
\( B(x, r) = \{ y \in \mathbb{R}^n : |y - x| < r \} \),
\( \bar{B}(x, r) = B(x, r) \cap \Omega \),
\( d := \text{diam } \Omega \),
\( \frac{1}{|B|} \int_B f(x) \, dx \) denotes the integral average of the function \( f \), i.e. \( \frac{1}{|B|} \int_B f(x) \, dx := \frac{1}{|B|} \int_B \frac{1}{|B|} f(x) \, dx \),
\( \hookrightarrow \) means continuous embedding.

2. Preliminaries
Everywhere in the sequel, \( \Omega \) is supposed to be a bounded open set.

2.1. Grand Lebesgue spaces
For \( 1 < p < \infty \), \( \theta > 0 \) and \( 0 < \varepsilon < p - 1 \) the grand Lebesgue space is the set of measurable functions for which
\[
\| f \|_{L^{p, \varepsilon}(\Omega)} := \sup_{0 < \varepsilon < p - 1} \left( \frac{\varepsilon}{\varepsilon - \varepsilon^p} \right) \| f \|_{L^{p-\varepsilon}(\Omega)} < \infty.
\]  
In the case \( \theta = 1 \), we simply denote \( L^{p, \theta}(\Omega) := L^p(\Omega) \).
When $|\Omega| < \infty$, then for all $0 < \varepsilon \leq p - 1$ we have
\[ L^p(\Omega) \hookrightarrow L^{p\varepsilon}(\Omega) \hookrightarrow L^{p-\varepsilon}(\Omega). \]

For more properties of grand Lebesgue spaces, see [13].

2.2. Morrey spaces
For $1 \leq p < \infty$ and $0 \leq \lambda < 1$, the usual Morrey space $L^{p,\lambda}(\Omega)$ is introduced as the set of all measurable functions such that
\[ \|f\|_{L^{p,\lambda}(\Omega)} := \sup_{x \in \Omega} \left( \frac{1}{|B(x,r)|^\lambda} \int_{B(x,r)} |f(y)|^p \, dy \right)^{\frac{1}{p}} < \infty \]
where $d := \text{diam} \, \Omega$.

3. Grand grand Morrey spaces and the reduction lemma
For $\theta > 0$, $\alpha \geq 0$, $1 < p < \infty$ and $0 \leq \lambda < 1$, we consider the functional
\[ \Phi_{p,\lambda}^{\theta,\alpha}(f, s) := \sup_{0 < \varepsilon < s} \varepsilon^{p-1} \|f\|_{L^{p-\varepsilon,\lambda-\alpha\varepsilon}(\Omega)}, \tag{2} \]
where $0 < s < \min\{p-1, \lambda/\alpha\}$.

**Remark 3.1.** We make a convention that the quotient $\lambda/\alpha$ when $\alpha = 0$ is always $\lambda/\alpha := \infty$ even if $\lambda = 0$.

**Definition 3.2 (Grand grand Morrey spaces).** Let $1 < p < \infty$, $\theta > 0$, $\alpha \geq 0$ and $0 \leq \lambda < 1$. By $L^{p,\lambda}_{\theta,\alpha}(\Omega)$ we denote the space of measurable functions having the finite norm
\[ \|f\|_{L^{p,\lambda}_{\theta,\alpha}(\Omega)} := \Phi_{p,\lambda}^{\theta,\alpha}(f, s_{\max}), \quad s_{\max} = \min\left\{ p - 1, \frac{\lambda}{\alpha} \right\}. \tag{3} \]

**Remark 3.3.** In the case $\alpha = 0, \lambda > 0$ we recover the Grand Morrey spaces introduced in [18], and when $\lambda = \alpha = 0$, by the convention in Remark 3.1 we have the grand Lebesgue spaces introduced in [11] (and in [12] in the case $\theta = 1$).

For fixed $p, \theta, \lambda, \alpha, f$ we have that $s \mapsto \Phi_{p,\lambda}^{\theta,\alpha}(f, s)$ is a non-decreasing function, but it is possible to estimate $\Phi_{p,\lambda}^{\theta,\alpha}(f, s)$ via $\Phi_{p,\lambda}^{\theta,\alpha}(f, \sigma)$ with $\sigma < s$ as follows.

**Lemma 3.4.** Let $\Omega$ be a bounded open set. For $0 < \sigma < s < \min\{p-1, \frac{\lambda}{\alpha}\}$ we have that
\[ \Phi_{p,\lambda}^{\theta,\alpha}(f, s) \leq Cs^{p-1} \sigma^{-\alpha} \Phi_{p,\lambda}^{\theta,\alpha}(f, \sigma), \tag{4} \]
where $C$ depends on $n$, the parameters $p, \lambda, \theta, \alpha$ and the diameter $d$, but does not depend on $f, s$ and $\sigma$. 

\[ \Phi_{\theta, \alpha}^{p, \lambda}(f, s) = \max \left\{ \Phi_{\theta, \alpha}^{p, \lambda}(f, \sigma), \sup_{\sigma \leq s} \frac{\varepsilon}{\sigma \left\| f \right\|_{L^{p_{\sigma}, \lambda-\alpha}(\Omega)}} \right\} \]  \tag{5}

To estimate
\[ I = \sup_{\sigma \leq \varepsilon < s} \sup_{0 < r \leq d} |B(x, r)| \frac{\varepsilon^{\frac{\sigma}{p-\sigma}}}{\frac{\sigma}{p-\sigma}} \left\| f \right\|_{L^{p_{\sigma}, \lambda-\alpha}(\Omega)} \]

note that the function \( g(\varepsilon) := \varepsilon^{\frac{\sigma}{p-\sigma}} \) is increasing in \( 0 < \varepsilon < \min\{ p-1, \frac{1}{\alpha} \} \), so that
\[ I \leq \frac{s^{\theta}}{\sigma^{\sigma}} \sup_{\sigma < \varepsilon < \min\{ p-1, \frac{1}{\alpha} \}} \sup_{0 < r \leq d} |B(x, r)| \frac{1+\alpha \varepsilon - \lambda}{p-\varepsilon} \left( \int_{B(x, r)} |f(y)|^{p-\sigma} \, dy \right)^{\frac{1}{p-\sigma}} \]

where \( \Delta(\varepsilon) := \frac{1+\alpha \varepsilon - \lambda}{p-\varepsilon} - \frac{1+\alpha \sigma - \lambda}{p-\sigma} \).

Observe that
\[ \Delta(\varepsilon) = \frac{1+\alpha \varepsilon - \lambda}{p-\varepsilon} - \frac{1+\alpha \sigma - \lambda}{p-\sigma} = \frac{(1-\lambda+\alpha p)(\varepsilon - \sigma)}{(p-\sigma)(p-\varepsilon)} \geq 0, \]

and for \( 0 \leq \varepsilon < \min\{ p-1, \frac{\lambda}{\alpha} \} \) we have \( \frac{1}{p-\varepsilon} \leq \frac{1+\alpha \varepsilon - \lambda}{p-\varepsilon} \leq 1 \), so that \( 0 \leq \Delta(\varepsilon) \leq 1 \). Then
\[ |B(x, r)| \frac{1+\alpha \varepsilon - \lambda}{p-\varepsilon} \leq C \max\{1, d^n\} \]

and we obtain
\[ I \leq C \frac{s^{\theta}}{\sigma^{\sigma}} \sup_{\sigma < \varepsilon \min\{ p-1, \frac{1}{\alpha} \}} \sup_{0 < r \leq d} \varepsilon^{\frac{\sigma}{p-\sigma}} \left( \frac{\sigma^{\theta}}{|B(x, r)|^{\lambda-\alpha}} \int_{B(x, r)} |f(y)|^{p-\sigma} \, dy \right)^{\frac{1}{p-\sigma}} \]

\[ = C \frac{s^{\theta}}{\sigma^{\sigma}} \cdot \frac{\sigma}{p-\sigma} \cdot \Phi_{\theta, \alpha}^{p, \lambda}(f, \sigma). \]

From Lemma 3.5 we immediately have

**Lemma 3.5.** For \( 0 < \sigma < \min\{ p-1, \frac{1}{\alpha} \} \), the norm defined in (3) has the following dominant
\[ \left\| f \right\|_{L^{p_{\sigma}, \lambda-\alpha}(\Omega)} \leq C \frac{\Phi_{\theta, \alpha}^{p, \lambda}(f, \sigma)}{\sigma^{\frac{\sigma}{p-\sigma}}}, \]  \tag{6}
where $C$ depends on $n, p, \lambda, \theta, \alpha$ and $d$, but does not depend on $f$ and $\sigma$.

**Lemma 3.6 (Reduction lemma).** Let $U$ be an operator (not necessarily linear) bounded in the usual Morrey spaces $L^{p, \lambda}_{\sigma, \alpha}(\Omega)$:

$$
\|Uf\|_{L^{p, \lambda}_{\sigma, \alpha}(\Omega)} \leq C_{p, \lambda, \sigma, \alpha} \|f\|_{L^{p, \lambda}_{\sigma, \alpha}(\Omega)}
$$

for all sufficiently small $\varepsilon \in [0, \sigma]$, where $0 < \sigma < \min\{p - 1, \frac{\lambda}{\alpha}\}$. If we have $\sup_{0 < \varepsilon < \sigma} C_{p, \lambda, \sigma, \alpha} < \infty$, then it is also bounded in the grand grand Morrey space $L^{p, \lambda}_{\sigma, \alpha}(\Omega)$:

$$
\|Uf\|_{L^{p, \lambda}_{\sigma, \alpha}(\Omega)} \leq C \|f\|_{L^{p, \lambda}_{\sigma, \alpha}(\Omega)}
$$

with

$$
C = \frac{C_0}{\sigma^{-\frac{\alpha}{\sigma}}} \sup_{0 < \varepsilon < \sigma} C_{p, \lambda, \sigma, \alpha},
$$

where $C_0$ may depend on $n, p, \lambda, \theta, \alpha$ and $d$, but does not depend on $\sigma$.

**Proof.** By (6), we have

$$
\|Uf\|_{L^{p, \lambda}_{\sigma, \alpha}(\Omega)} \leq \frac{C}{\sigma^{-\frac{\alpha}{\sigma}}} \Phi_{p, \lambda}(Uf, \sigma).
$$

The estimation of $\Phi_{p, \lambda}(Uf, \sigma)$ by $\|f\|_{L^{p, \lambda}_{\sigma, \alpha}(\Omega)}$ is direct:

$$
\Phi_{p, \lambda}(Uf, \sigma) = \sup_{0 < \varepsilon < \sigma} \varepsilon^{-\frac{\alpha}{\sigma}} \|Uf\|_{L^{p, \lambda}_{\sigma, \alpha}(\Omega)}
\leq \sup_{0 < \varepsilon < \sigma} \varepsilon^{-\frac{\alpha}{\sigma}} \cdot C_{p, \lambda, \sigma, \alpha} \cdot \|f\|_{L^{p, \lambda}_{\sigma, \alpha}(\Omega)}
\leq \sup_{0 < \varepsilon < \sigma} C_{p, \lambda, \sigma, \alpha} \cdot \|f\|_{L^{p, \lambda}_{\sigma, \alpha}(\Omega)}
$$

which completes the proof. □

### 4. On boundedness of operators in the grand grand Morrey spaces

#### 4.1. Maximal operator in grand grand Morrey spaces

Let

$$
Mf(x) = \sup_{0 < r < d} \int_{B(x, r)} |f(y)| \, dy, \quad x \in \Omega
$$

be the usual centered maximal operator. The Hardy-Littlewood-Wiener theorem regarding the boundedness of the maximal operator in Lebesgue spaces is a well-known result, see e.g. [5]. A similar result is valid in the framework of Morrey spaces, namely

**Lemma 4.1.** Let $1 < p < \infty$ and let $0 \leq \lambda < 1$. Then

$$
\|Mf\|_{L^{p, \lambda}(\Omega)} \leq \left(2^{\frac{\alpha}{p}} c_0(p')^\frac{\lambda}{p} + 1\right) \|f\|_{L^{p, \lambda}(\Omega)}.
$$

(12)
Remark 4.2. The above explicit evaluation of the constant in Lemma 4.1 is the one obtained in [17] (see also [13, 18]). For another approach with slightly different evaluation of the constant, see [1, 3].

Theorem 4.3. Let \( 1 < p < \infty, \theta > 0, \alpha \geq 0 \) and \( 0 \leq \lambda < 1 \). Then the Hardy-Littlewood maximal operator \( \mathcal{M} \) is bounded in grand grand Morrey spaces \( L_{\theta, \alpha}^{(p), \lambda}(\Omega) \).

Proof. By the reduction lemma 3.6 and (12), we only need to show the finiteness of
\[
\sup_{0 < \epsilon \leq \sigma} C_{p-\epsilon, \lambda-\alpha \epsilon} = \sup_{0 < \epsilon \leq \sigma} \left( 2^{\frac{\epsilon(\lambda-\alpha \epsilon)}{p-\epsilon \lambda}} c_0 \left( \frac{p-\epsilon}{p-\epsilon \lambda} \right)^{\frac{1}{p-\epsilon \lambda}} + 1 \right)
\]
which holds if we choose \( \sigma < p - 1 \). Note that the use of the reduction lemma in this proof is not necessary when the grand grand space \( L_{\theta, \alpha}^{(p), \lambda}(\Omega) \) is considered with \( \alpha > \lambda \frac{p}{p-1} \).

\[ \square \]

4.2. Singular integral operators in Grand Grand Morrey spaces

We follow [18] in this section, in particular, making use of the following definition of the Calderón-Zygmund singular operators. Namely, the Calderón-Zygmund operator is treated as the integral operator
\[
Tf(x) = \text{p.v.} \int_{\Omega} K(x, y) f(y) \, dy
\]
with the kernel \( K : \Omega \times \Omega \setminus \{(x, x) : x \in \Omega\} \to \mathbb{R} \) satisfying the conditions:
\[
|K(x, y)| \leq \frac{C}{|x - y|^n}, \quad x, y \in \Omega, \quad x \neq y;
\]
\[
|K(x_1, y) - K(x_2, y)| + |K(y, x_1) - K(y, x_2)| \leq C w \left( \frac{|x_2 - x_1|}{|x_2 - y|} \right) \frac{1}{|x_2 - y|^n}
\]
fors all \( x_1, x_2 \) and \( y \) with \( |x_2 - y| > C|x_2 - x_1| \), where \( w \) is a positive non-decreasing function on \( (0, \infty) \) which satisfies the doubling condition \( w(2t) \leq cw(t) \) and the Dini condition \( \int_0^t w(t)/t \, dt < \infty \). In the case where \( w \) is a power function, this goes back to Coifman-Meyers version of singular operators with standard kernel. We also assume that \( Tf \) exists almost everywhere on \( \Omega \) in the principal value sense for all \( f \in L^2(\Omega) \) and that \( T \) is bounded in \( L^2(\Omega) \).

The boundedness of such Calderón-Zygmund operators in Morrey spaces is valid, as can be seen in the following Proposition, proved in [18]

Proposition 4.4. Let \( 1 < p < \infty, 0 \leq \lambda < 1 \). Then
\[
\|Tf\|_{L^{p, \lambda}(\Omega)} \leq C_{T, p, \lambda} \|f\|_{L^{p, \lambda}(\Omega)}
\]
where
\[
C_{T, p, \lambda} \leq c \left\{ \begin{array}{ll}
\frac{p}{p-1} + \frac{p}{2-p} + \frac{p-\lambda+1}{1-\lambda} & \text{if } 1 < p < 2; \\
\frac{p}{p-2} + \frac{p-\lambda+1}{1-\lambda} & \text{if } p > 2.
\end{array} \right.
\]
with \( c \) not depending on \( p \) and \( \lambda \).
Theorem 4.5. Let $1 < p < \infty$, $\theta > 0$ and $0 < \lambda < 1$. Then the Calderón-Zygmund operator $T$ is bounded in grand grand Morrey spaces $L^{\theta,\lambda}_{p,\alpha}(\Omega)$.

Proof. Keeping in mind that by the reduction lemma, we are interested only in small values of $\epsilon$, from (13), we deduce that

$$
C_{T,p-\epsilon,\lambda-\alpha \epsilon} \leq c \begin{cases} 
\frac{p}{p-\epsilon-1} + \frac{p-\epsilon}{2p+\epsilon} + \frac{p-\epsilon-\lambda+\alpha+1}{1-\lambda+\alpha} & \text{if } p \leq 2 \text{ and } 0 < \epsilon < p-1; \\
\frac{p-\epsilon}{p-\epsilon-2} + \frac{p-\epsilon-\lambda+\alpha+1}{1-\lambda+\alpha} & \text{if } p > 2 \text{ and } 0 < \epsilon < p-2.
\end{cases}
$$

so that when applying the reduction lemma, it suffices to take $\sigma < \min \{p-1, \frac{\lambda}{\alpha}\}$ when $p \leq 2$ and $\sigma < \min \{p-2, \frac{\lambda}{\alpha}\}$ when $p > 2$. \qed

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Humberto Rafeiro
Instituto Superior Técnico,
Departamento de Matemática,
Centro CEAF, Av. Rovisco Pais,
1049-001 Lisboa, PORTUGAL
e-mail: hrafeiro@math.ist.utl.pt