Furstenberg transformations on Cartesian products of infinite-dimensional tori

P. A. Cecchi and R. Tiedra de Aldecoa*

1 Departamento de Matemáticas, Universidad de Santiago de Chile, Av. Alameda Libertador Bernardo O’Higgins 3363, Estación Central, Santiago, Chile
2 Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Av. Vicuña Mackenna 4860, Santiago, Chile
E-mails: pcecchib@gmail.com, rtiedra@mat.puc.cl

Abstract

We consider in this note Furstenberg transformations on Cartesian products of infinite-dimensional tori. Under some appropriate assumptions, we show that these transformations are uniquely ergodic with respect to the Haar measure and have countable Lebesgue spectrum in a suitable subspace. These results generalise to the infinite-dimensional setting previous results of H. Furstenberg, A. Iwanik, M. Lemańczyk, D. Rudolph and the second author in the one-dimensional setting. Our proofs rely on the use of commutator methods for unitary operators and Bruhat functions on the infinite-dimensional torus.

2010 Mathematics Subject Classification: 28D10, 37A30, 37C40, 58J51.
Keywords: Furstenberg transformations, infinite-dimensional torus, commutator methods.

1 Introduction

We consider in this note the generalisation of Furstenberg transformations on Cartesian products $T^d$ ($d \geq 2$) of one-dimensional tori [8, Sec. 2] to the case of Cartesian products $(T^\infty)^d$ of infinite-dimensional tori. Using recent results on commutator methods for unitary operators [14, 15], we show under a $C^1$ regularity assumption on the perturbations that these transformations are uniquely ergodic with respect to the Haar measure on $(T^\infty)^d$ and are strongly mixing in the orthocomplement of functions depending only on variables in the first torus $T^\infty$ (see Assumption 3.2 and Theorem 3.3). Under a slightly stronger regularity assumption ($C^1 +$ Dini continuous derivative), we also show that these transformations have countable Lebesgue spectrum in that orthocomplement (see Theorem 3.4). These results generalise to the infinite-dimensional setting previous results of H. Furstenberg, A. Iwanik, M. Lemańczyk, D. Rudolph and the second author [8, 10, 15] in the one-dimensional setting.

Apart from commutator methods, our proofs rely on the use of Bruhat test functions [5] which provide a natural analog to the usual $C^\infty$-functions which do not exist in our infinite-dimensional setting (see Section 3 for details). We also mention that all the results of this note apply to Furstenberg transformations on Cartesian products $(T^n)^d$ of finite-dimensional tori $T^n$ of any dimension $n \geq 1$. One just has to consider the particular case where the functions defining the Furstenberg transformations on $(T^\infty)^d$ depend only on a finite number of variables.

*Supported by the Chilean Fondecyt Grant 1130168 and by the Iniciativa Cientifica Milenio ICM RC120002 "Mathematical Physics" from the Chilean Ministry of Economy.
Here is a brief description of the content of the note. First, we recall in Section 2 the needed facts on commutators of unitary operators and regularity classes associated to them. Then, we define in Section 3 the Furstenberg transformations on $(\mathbb{T}^\infty)^d$, and prove Theorems 3.3 and 3.4 on the unique ergodicity, strong mixing property and countable Lebesgue spectrum of these transformations.

We refer to [9, 11, 12, 13] and references therein for other works building on Furstenberg transformations.

Acknowledgements. The authors thank M. Măntoiu and J. Rivera-Letelier for discussions which partly motivated this note.

2 Commutator methods for unitary operators

We recall in this section some facts borrowed from [7, 14] on commutator methods for unitary operators and regularity classes associated to them.

Let $H$ be a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ antilinear in the first argument, denote by $\mathcal{B}(H)$ the set of bounded linear operators on $H$, and write $\| \cdot \|$ both for the norm on $H$ and the norm on $\mathcal{B}(H)$.

Let $A$ be a self-adjoint operator in $H$ with domain $D(A)$, and take $S \in \mathcal{B}(H)$. For any $k \in \mathbb{N}$, we say that $S$ belongs to $C_k(A)$, with notation $S \in C_k(A)$, if the map $\mathbb{R} \ni t \mapsto e^{-itA}S e^{itA} \in \mathcal{B}(H)$ (2.1) is strongly of class $C_k$. In the case $k = 1$, one has $S \in C_1(A)$ if and only if the quadratic form $D(A) \ni \varphi \mapsto \langle \varphi, SA\varphi \rangle - \langle A\varphi, S\varphi \rangle \in \mathbb{C}$ is continuous for the topology induced by $H$ on $D(A)$. We denote by $[A, S]$ the bounded operator associated with the continuous extension of this form, or equivalently $-i$ times the strong derivative of the function (2.1) at $t = 0$.

A condition slightly stronger than the inclusion $S \in C_1(A)$ is provided by the following definition: $S$ belongs to $C^{1+0}(A)$, with notation $S \in C^{1+0}(A)$, if $S \in C_1(A)$ and if $[A, S]$ satisfies the Dini-type condition

$$ \int_0^1 \frac{dt}{t} \left\| e^{-itA}[A, S] e^{itA} - [A, S] \right\| < \infty. $$

As banachisable topological vector spaces, the sets $C^2(A)$, $C^{1+0}(A)$, $C_1(A)$ and $C^0(A) \equiv \mathcal{B}(H)$ satisfy the continuous inclusions [2, Sec. 5.2.4]

$$ C^2(A) \subset C^{1+0}(A) \subset C_1(A) \subset C^0(A). $$

Now, let $U \in C_1(A)$ be a unitary operator with (complex) spectral measure $E^U(\cdot)$ and spectrum $\sigma(U) \subset S^1 := \{ z \in \mathbb{C} \mid |z| = 1 \}$. If there exist a Borel set $\Theta \subset S^1$, a number $a > 0$ and a compact operator $K \in \mathcal{B}(H)$ such that

$$ E^U(\Theta) U^*[A, U] E^U(\Theta) \geq a E^U(\Theta) + K, \quad (2.2) $$

then one says that $U$ satisfies a Mourre estimate on $\Theta$ and that $A$ is a conjugate operator for $U$ on $\Theta$. Also, one says that $U$ satisfies a strict Mourre estimate on $\Theta$ if (2.2) holds with $K = 0$. One of the consequences of a Mourre estimate is to imply spectral properties for $U$ on $\Theta$. We recall here these spectral results in the case $U \in C^{1+0}(A)$. We also recall a result on the strong mixing property of $U$ in the case $U \in C_1(A)$ (see [7, Thm. 2.7 & Rem. 2.8] and [14, Thm. 3.1] for more details).
**Theorem 2.1** (Absolutely continuous spectrum). Let $U$ and $A$ be respectively a unitary and a self-adjoint operator in a Hilbert space $\mathcal{H}$, with $U \in C^{1,0}(A)$. Suppose there exist an open set $\Theta \subseteq S^1$, a number $a > 0$ and a compact operator $K \in \mathcal{B}(\mathcal{H})$ such that

$$E^U(\Theta) U^*[A, U] E^U(\Theta) \geq a E^U(\Theta) + K.$$  

(2.3)

Then, $U$ has at most finitely many eigenvalues in $\Theta$, each one of finite multiplicity, and $U$ has no singular continuous spectrum in $\Theta$. Furthermore, if (2.3) holds with $K = 0$, then $U$ has only purely absolutely continuous spectrum in $\Theta$ (no singular spectrum).

**Theorem 2.2** (Strong mixing). Let $U$ and $A$ be respectively a unitary and a self-adjoint operator in a Hilbert space $\mathcal{H}$, with $U \in C^{1}(A)$. Assume that the strong limit

$$D := \lim_{N \to \infty} \frac{1}{N} \sum_{\ell=0}^{N-1} U^\ell ([A, U] U^{-\ell})$$

exists, and suppose that $\eta(D) D(A) \subset D(A)$ for each $\eta \in C_\infty(\mathbb{R} \setminus \{0\})$. Then,

(a) $\lim_{N \to \infty} \langle \varphi, U^N \psi \rangle = 0$ for each $\varphi \in \ker(D)^\perp$ and $\psi \in \mathcal{H}$,

(b) $U|_{\ker(D)^\perp}$ has purely continuous spectrum.

### 3 Furstenberg transformations on Cartesian products of infinite-dimensional tori

Let $\mathbb{T}^\infty \simeq \mathbb{R}^\infty/\mathbb{Z}^\infty$ be the infinite-dimensional torus with elements $x \equiv \{x_k\}_{k=1}^\infty$, and let $\mathbb{T}^\infty$ be the dual group of $\mathbb{T}^\infty$. For each $n \in \mathbb{N}_1$, let $\mu_n$ be the normalised Haar measure on $(\mathbb{T}^\infty)^n$, and let $\mathcal{H}_n := L^2((\mathbb{T}^\infty)^n, \mu_n)$ be the corresponding Hilbert space. In analogy to Furstenberg transformations on Cartesian products of one-dimensional tori (see [8, Sec. 2.3]), we consider Furstenberg transformations on Cartesian products of infinite-dimensional tori $T_d : (\mathbb{T}^\infty)^d \to (\mathbb{T}^\infty)^d$, $d \in \mathbb{N}_2$, given by

$$T_d(x_1, x_2, \ldots, x_d) := (x_1 + \alpha, x_2 + \phi_1(x_1), \ldots, x_d + \phi_{d-1}(x_1, x_2, \ldots, x_{d-1})).$$

with $\alpha \in \mathbb{T}^\infty$ such that $\{n \alpha\}_{n \in \mathbb{Z}}$ is dense in $\mathbb{T}^\infty$ and $\phi_j \in C((\mathbb{T}^\infty)^j; \mathbb{T}^\infty)$ for each $j \in \{1, \ldots, d-1\}$. Since $T_d$ is invertible and preserves the measure $\mu_d$, the Koopman operator

$$W_d : \mathcal{H}_d \to \mathcal{H}_d, \quad \varphi \mapsto \varphi \circ T_d,$$

is a unitary operator. Furthermore, $W_d$ is reduced by the orthogonal decompositions

$$\mathcal{H}_d = \mathcal{H}_1 \bigoplus_{j \in \{2, \ldots, d\}} (\mathcal{H}_j \otimes \mathcal{H}_{j-1}) = \mathcal{H}_1 \bigoplus_{j \in \{2, \ldots, d\}} \mathcal{H}_{j,x}, \quad \mathcal{H}_{j,x} := \{\eta \otimes x \mid \eta \in \mathcal{H}_{j-1}\},$$

and the restriction $W_d|_{\mathcal{H}_{j,x}}$ is unitarily equivalent to the unitary operator

$$U_{j,x} \eta := (\chi \circ \phi_{j-1}) \mathcal{W}_{j-1} \eta, \quad \eta \in \mathcal{H}_{j-1}.$$

In order to define later a conjugate operator for $U_{j,x}$, we first define a suitable group of translations on $(\mathbb{T}^\infty)^{j-1}$. For this, we choose $\{y_n\}_{n \in \mathbb{N}}$ an ergodic continuous one-parameter subgroup of $\mathbb{T}^\infty$ such that each map

$$\mathbb{R} \ni t \mapsto ((y_1)_n, \ldots, (y_n)_n) \in \mathbb{T}^n, \quad n \in \mathbb{N}_1,$$

is of class $C^1$. An example of such a subgroup $\{y_t\}_{t \in \mathbb{R}}$ is for instance given by the formula

$$(y_t)_k = y^k t \mod \mathbb{Z}, \quad t \in \mathbb{R}, \quad k \in \mathbb{N}_1.$$
with \( y \in \mathbb{R} \) a transcendental number (see [6, Ex. 4.1.1]). Then, we associate to \( \{ y_t \}_{t \in \mathbb{R}} \) the translation flow
\[
F_{j-1,t}(x_1, \ldots, x_{j-1}) := (x_1, \ldots, x_{j-1} + y_t), \quad t \in \mathbb{R}, \quad (x_1, \ldots, x_{j-1}) \in (T^\infty)^{j-1},
\]
and the translation operators \( V_{j-1,t} : \mathcal{H}_{j-1} \to \mathcal{H}_{j-1} \) given by \( V_{j-1,t} \eta := \eta \circ F_{j-1,t} \). Due to the continuity of the map \( \mathbb{R} \ni t \mapsto y_t \in T^\infty \) and of the group operation, the family \( \{ V_{j-1,t} \}_{t \in \mathbb{R}} \) defines a strongly continuous unitary group in \( \mathcal{H}_{j-1} \) with self-adjoint generator
\[
\mathcal{H}_{j-1} \eta := \lim_{t \to 0} it^{-1}(V_{j-1,t} - 1) \eta, \quad \eta \in \mathcal{D}(\mathcal{H}_{j-1}) := \left\{ \eta \in \mathcal{H}_{j-1} \mid \lim_{t \to 0} |t|^{-1} \|(V_{j-1,t} - 1) \eta\| < \infty \right\}.
\]

When dealing with differential operators on compact manifolds, one typically does the calculations on an appropriate core of the operators such as the set of \( C^\infty \)-functions. But here, the are no such functions on \( (T^\infty)^{j-1} \), since \( T^\infty \) is not a manifold \( (T^\infty \) does not admit any differentiable structure modeled on locally convex spaces, see [4, Sec. 10.2] for details). So, we use instead the set \( B_{j-1} \) of Bruhat test functions on \( (T^\infty)^{j-1} \), whose definition is the following (see [3, Sec. 2.2] or [5] for details). Set \( T^0 := \{0\} \), and for each \( n \in \mathbb{N} \) let \( \pi_n : (T^\infty)^{j-1} \to (T^n)^{j-1} \) be the projection given by
\[
\pi_n(x_1, \ldots, x_{j-1}) := \begin{cases} (0, \ldots, 0) & \text{if } n = 0 \\ ((x_1)_1, \ldots, (x_1)_n, (x_{j-1})_1, \ldots, (x_{j-1})_n) & \text{if } n \geq 1. \end{cases}
\]

Then,
\[
B_{j-1} := \bigcup_{n \in \mathbb{N}} \{ \zeta \circ \pi_n : \zeta \in C^\infty((T^n)^{j-1}) \},
\]
that is, \( B_{j-1} \) is the set of all functions on \( (T^\infty)^{j-1} \) that are obtained by lifting to \( (T^\infty)^{j-1} \) any \( C^\infty \)-function on one of the Lie groups \( (T^n)^{j-1} \). The set \( B_{j-1} \) is dense in \( \mathcal{H}_{j-1} \), it is left invariant by the group \( \{ V_{j-1,t} \}_{t \in \mathbb{R}} \), and satisfies the inclusion \( B_{j-1} \subset \mathcal{D}(\mathcal{H}_{j-1}) \) (to show the latter, one has to use the \( C^1 \)-assumption (3.1)). Therefore, it follows from Nelson's theorem [1, Prop. 5.3] that \( \mathcal{H}_{j-1} \) is essentially self-adjoint on \( B_{j-1} \).

Now, if the (multiplication operator valued) map \( \mathbb{R} \ni t \mapsto \chi \circ \phi_{j-1} \circ F_{j-1,t} \in \mathcal{B}(\mathcal{H}_{j-1}) \) is strongly of class \( C^1 \), then \( \chi \circ \phi_{j-1} \in C^1(\mathcal{H}_{j-1}) \) since
\[
\chi \circ \phi_{j-1} \circ F_{j-1,t} = V_{j-1,t}(\chi \circ \phi_{j-1}) V_{j-1,-t} = e^{-it\mathcal{H}_{j-1}} (\chi \circ \phi_{j-1}) e^{it\mathcal{H}_{j-1}}.
\]
Thus, the operator
\[
g_{j,\chi} := [\mathcal{H}_{j-1}, (\chi \circ \phi_{j-1})] (\chi \circ \phi_{j-1}) \equiv \frac{d}{dt} \left( i \chi \circ \phi_{j-1} \circ F_{j-1,t} \right) (\chi \circ \phi_{j-1}) \bigg|_{t=0} = \frac{d}{dt} i \chi \circ (\phi_{j-1} \circ F_{j-1,t} - \phi_{j-1}) \bigg|_{t=0}
\]
is a bounded self-adjoint multiplication operator in \( \mathcal{H}_{j-1} \).

**Lemma 3.1.** Take \( j \in \{2, \ldots, d\} \) and \( \chi \in \overline{T^\infty} \setminus \{1\} \), and assume that the map \( \mathbb{R} \ni t \mapsto \chi \circ \phi_{j-1} \circ F_{j-1,t} \in \mathcal{B}(\mathcal{H}_{j-1}) \) is strongly of class \( C^1 \). Then, \( U_{j,\chi} \in C^1(\mathcal{H}_{j-1}) \) with \( [\mathcal{H}_{j-1}, U_{j,\chi}] = g_{j,\chi} U_{j,\chi} \).

**Proof.** Take \( \eta \in \mathcal{B}_{j-1} \). Then, the commutation of \( V_{j-1,t} \) and \( W_{j-1} \) and the differentiability assumption
\begin{align*}
&\langle H_{j-1}\eta, U_jx\eta\rangle_{\mathcal{H}_{j-1}} - \langle \eta, U_jxH_{j-1}\eta\rangle_{\mathcal{H}_{j-1}} \\
&= i \frac{d}{dt} \left\{ -\langle V_{j-1}, t\eta, (\chi \circ \phi_{j-1}) W_{j-1}\eta \rangle_{\mathcal{H}_{j-1}} - \langle \eta, (\chi \circ \phi_{j-1}) W_{j-1} V_{j-1}, t\eta \rangle_{\mathcal{H}_{j-1}} \right\}_{t=0} \\
&= i \frac{d}{dt} \langle \eta, (\chi \circ \phi_{j-1} - F_{j-1}, t) - (\chi \circ \phi_{j-1}) W_{j-1} V_{j-1}, t\eta \rangle_{\mathcal{H}_{j-1}}}_{t=0} \\
&= \frac{d}{dt} \langle \eta, \left(i\chi \circ \phi_{j-1} - F_{j-1}, t\right) - (\chi \circ \phi_{j-1}) W_{j-1} V_{j-1}, t\eta \rangle_{\mathcal{H}_{j-1}}}_{t=0} \\
&= \langle \eta, g_jx U_jx\eta\rangle_{\mathcal{H}_{j-1}},
\end{align*}

and thus the claim follows from the boundedness of $g_jx$ and the density of $\mathcal{B}_{j-1}$ in $\mathcal{D}(\mathcal{H}_{j-1})$.

\begin{assumption}
For each $j \in \{2, \ldots, d\}$, the map $\phi_{j-1} \in C((\mathbb{T}^\infty)^{j-1}; \mathbb{T}^\infty)$ satisfies $\phi_{j-1} = \xi_{j-1} + \eta_{j-1}$, where
\begin{enumerate}[(i)]
\item $\xi_{j-1} \in C((\mathbb{T}^\infty)^{j-1}; \mathbb{T}^\infty)$ is a group homomorphism such that $\mathbb{T}^\infty \ni x_{j-1} \mapsto (\chi \circ \xi_{j-1})(0, \ldots, 0 + x_{j-1}) \in S^1$ is nontrivial for each $\chi \in \mathbb{T}^\infty \setminus \{1\}$,
\item $\eta_{j-1} \in C((\mathbb{T}^\infty)^{j-1}; \mathbb{R}^\infty)$ is such that there exists $\bar{\eta}_{j-1} \in C((\mathbb{T}^\infty)^{j-1}; \mathbb{R}^\infty)$ with $\eta_{j-1}(x_1, \ldots, x_{j-1}) = (\bar{\eta}_{j-1}(x_1, \ldots, x_{j-1}) \ (\text{mod } \mathbb{Z}^\infty))$ for each $(x_1, \ldots, x_{j-1}) \in (\mathbb{T}^\infty)^{j-1}$, and with $\mathbb{R} \ni t \mapsto \bar{\eta}_{j-1,k} \circ F_{j-1,t} \in \mathcal{B}(\mathcal{H}_{j-1})$ strongly of class $C^1$ for each $k \in \mathbb{N}_{\geq 1}$.
\end{enumerate}

In the next theorem, we use the fact that the map $\mathbb{R} \ni t \mapsto (\chi \circ \xi_{j-1})(0, \ldots, 0 + y_t) \in S^1$ is a character on $\mathbb{R}$, and thus of class $C^\infty$. We also use the notation $\xi_{j-1}^{(x)} := \frac{d}{dt} (\chi \circ \xi_{j-1})(0, \ldots, 0 + y_t)_{t=0} \in i\mathbb{R}$.

\begin{theorem}[Strong mixing and unique ergodicity]
Suppose that Assumption 3.2 is satisfied. Then, $W_d$ is strongly mixing in $\mathcal{H}_d \oplus \mathcal{H}_1$ and $T_d$ is uniquely ergodic with respect to $\mu_d$.
\end{theorem}

These results of strong mixing and unique ergodicity are an extension to the case of infinite-dimensional tori of results previously obtained in the case of one-dimensional tori (see [8, Thm. 2.1] for the unique ergodicity and [10, Rem. 1] for the strong mixing property). For instance, if the functions $\eta_{j-1}$ in Assumption 3.2 were to depend only on a finite number of variables, then the strong $C^1$ regularity condition on $\eta_{j-1}$ in Assumption 3.2(ii) would reduce to a uniform Lipschitz condition of $\eta_{j-1}$ along the flow $\{F_{j-1,t}\}_{t \in \mathbb{R}}$, as in the one-dimensional case treated by Furstenberg in [8, Thm. 2.1].

\begin{proof}
(i) Take $j \in \{2, \ldots, d\}$, $\chi \in \mathbb{T}^\infty \setminus \{1\}$ and $t \in \mathbb{R}$. Then, there exist $k_\chi \in \mathbb{N}_{\geq 1}$ and $n_1, \ldots, n_{k_\chi} \in \mathbb{Z}$ such that $\chi(x_{j-1}) = e^{2\pi i \sum_{k=1}^{k_\chi} n_k x_{j-1,k}}$.
\end{proof}
with \( x_{j-1} \in \mathbb{T}^\infty \) and \( x_{j-1,k} \in [0, 1) \) the \( k \)-th cyclic component of \( x_{j-1} \). Therefore, we infer from Assumption 3.2 that

\[
\chi \circ (\phi_{j-1} \circ F_{j-1,t} - \phi_{j-1}) = \chi \circ (\xi_{j-1} \circ F_{j-1,t} - \xi_{j-1}) = (\chi \circ \xi_{j-1})(0, \ldots, 0 + y_k) \cdot e^{2\pi i \sum_{k=1}^{\infty} n_k (\eta_{j-1} \circ F_{j-1,t} - \eta_{j-1})},
\]

and Lemma 3.1 implies that \( U_{j,x} \in C^1(H_{j-1}) \) with \( [H_{j-1}, U_{j,x}] = g_{j,x} U_{j,x} \) and

\[
g_{j,x} = \lim_{t \to 0} \frac{d}{dt} \chi \circ (\phi_{j-1} \circ F_{j-1,t} - \phi_{j-1})|_{t=0} = i \xi_{j-1}^{(x)} - 2\pi \sum_{k=1}^{\infty} n_k ((-1)^k \frac{d}{dt} \eta_{j-1,k} \circ F_{j-1,t} |_{t=0})
\]

\[
= i \xi_{j-1}^{(x)} + 2\pi i \sum_{k=1}^{\infty} n_k (H_{j-1} \tilde{\eta}_{j-1,k}).
\]

(ii) We now proceed by induction on \( d \) to prove the claims. For the case \( d = 2 \), we take \( \chi \in \mathbb{T}^\infty \setminus \{1\} \) and note that point (i) implies

\[
D_{2,x} := \lim_{N \to \infty} \frac{1}{N} \sum_{\ell = 0}^{N-1} (U_{2,x})^{\ell} ([H_1, U_{2,x}] (U_{2,x})^{-1}) (U_{2,x})^{-\ell}
\]

\[
= \lim_{N \to \infty} \frac{1}{N} \sum_{\ell = 0}^{N-1} g_{2,x} \circ (T_1)^{\ell}
\]

\[
= i \xi_1^{(x)} + 2\pi i \sum_{k=1}^{\infty} n_k \left( \lim_{N \to \infty} \frac{1}{N} \sum_{\ell = 0}^{N-1} (H_1 \tilde{\eta}_{1,k}) \circ (T_1)^{\ell} \right).
\]

Since \( T_1 \) is ergodic and \( H_1 \tilde{\eta}_{1,k} \in L^\infty(\mathbb{T}^\infty) \), it follows by Birkhoff’s pointwise ergodic theorem and Lebesgue dominated convergence theorem that

\[
D_{2,x} = i \xi_1^{(x)} + 2\pi i \sum_{k=1}^{\infty} n_k \int_{\mathbb{T}^\infty} d\mu_1 (H_1 \tilde{\eta}_{1,k}) = i \xi_1^{(x)} + 2\pi i \sum_{k=1}^{\infty} n_k \int_{\mathbb{T}^\infty} d\mu_1 (1, H_1 \tilde{\eta}_{1,k}) = i \xi_1^{(x)}.
\]

Now, since the character \( \mathbb{T}^\infty \ni x_{j-1} \mapsto (\chi \circ \xi_1)(0, \ldots, 0 + x_{j-1}) \in S^1 \) is nontrivial and the subgroup \( \{y_k \}_{k \in \mathbb{R}} \) ergodic, we have \( \xi_1^{(x)} \neq 0 \) (see [6, Thm. 4.1.1]). Thus, \( D_{2,x} \neq 0 \), and we deduce from Theorem 2.2(a) that \( U_{2,x} \) is strongly mixing. Since this is true for each \( \chi \in \mathbb{T}^\infty \setminus \{1\} \), and since \( U_{2,x} \) is unitarily equivalent to \( W_2 |_{\mathcal{U}_{2,x}} \), we infer that \( W_2 \) is strongly mixing in \( \bigoplus_{x \in \mathbb{T}^\infty \setminus \{1\}} \mathcal{H}_{2, \mathcal{U}_2} = \mathcal{H}_2 \oplus \mathcal{H}_1 \).

To show that \( T_2 \) is uniquely ergodic with respect to \( \mu_2 \), we take an eigenvector of \( W_2 \) with eigenvalue 1, that is, a vector \( \varphi \in \mathcal{H}_2 \) such that \( W_2 \varphi = \varphi \). Since \( W_2 \) is strongly mixing in \( \mathcal{H}_2 \oplus \mathcal{H}_1 \), \( W_2 \) has purely continuous spectrum in \( \mathcal{H}_2 \oplus \mathcal{H}_1 \) (see Theorem 2.2(b)), and thus \( \varphi = \eta \oplus 1 \) for some \( \eta \in \mathcal{H}_1 \). So,

\[
W_2 \varphi = \varphi \iff W_2 (\eta \oplus 1) = \eta \oplus 1 \iff W_1 \eta = \eta,
\]

and thus \( \eta \) is an eigenvector of \( W_1 \) with eigenvalue 1. It follows that \( \eta \) is constant \( \mu_1 \)-almost everywhere due to the ergodicity of \( T_1 \). Therefore, \( \varphi \) is constant \( \mu_2 \)-almost everywhere, and \( T_2 \) is ergodic. This implies that \( T_2 \) is uniquely ergodic because ergodicity implies unique ergodicity for transformations such as \( T_2 \) (see [6, Thm. 4.2.1]).

Now, assume the claims are true for \( d - 1 \geq 1 \). Then, \( W_{d-1} \) is strongly mixing in \( \mathcal{H}_{d-1} \oplus \mathcal{H}_1 \). Furthermore, a calculation as in the case \( d = 2 \) shows that \( W_d \) is strongly mixing in \( \bigoplus_{x \in \mathbb{T}^\infty \setminus \{1\}} \mathcal{H}_{d \mathcal{U}_2} = \mathcal{H}_d \oplus \mathcal{H}_{d-1} \). This implies that \( W_d \) is strongly mixing in

\[
(\mathcal{H}_{d-1} \oplus \mathcal{H}_1) \oplus (\mathcal{H}_d \oplus \mathcal{H}_{d-1}) = \mathcal{H}_d \oplus \mathcal{H}_1.
\]

This, together with the unique ergodicity of \( T_{d-1} \), allows us to show that \( T_d \) is uniquely ergodic as in the case \( d = 2 \). \( \square \)
We know from the proof of Theorem 3.3 that if Assumption 3.2 is satisfied, then \( U_{j,x} \in C^1(H_{j-1}) \) with \([H_{j-1}, U_{j,x}] = g_{j,x} U_{j,x}\). So, it follows from [7, Sec. 4] that the operator
\[
A_{j,x}^{(N)} \eta := \frac{1}{N} \sum_{\ell=0}^{N-1} (U_{j,x})^\ell H_{j-1}(U_{j,x})^{-\ell} \eta, \quad N \in \mathbb{N}_1, \quad \eta \in \mathcal{D}(A_{j,x}^{(N)}) := \mathcal{D}(H_{j-1}),
\]
is self-adjoint, and that \( U_{j,x} \in C^1(A_{j,x}^{(N)}) \) with
\[
[A_{j,x}^{(N)}, U_{j,x}] = g_{j,x}^{(N)} U_{j,x} \quad \text{and} \quad g_{j,x}^{(N)} := \frac{1}{N} \sum_{\ell=0}^{N-1} g_{j,x} \circ (T_{j-1})^\ell.
\]

**Theorem 3.4 (Countable Lebesgue spectrum).** Suppose that Assumption 3.2 is satisfied, and assume for each \( j \in \{2, \ldots, d\} \) and \( k \in \mathbb{N}_1 \) that \( H_{j-1} \eta_{j-1,k} \in C((\mathbb{T}^\infty)^{-1}) \) and that
\[
\int_0^1 \frac{dt}{t} \left \| (H_{j-1} \eta_{j-1,k}) \circ F_{j-1,t} - (H_{j-1} \eta_{j-1,k}) \right \|_{L^\infty((\mathbb{T}^\infty)^{-1})} < \infty. \tag{3.3}
\]
Then, \( W_d \) has countable Lebesgue spectrum in \( \mathcal{H}_d \oplus \mathcal{H}_1 \).

**Proof.** Take \( j \in \{2, \ldots, d\} \) and \( x \in \mathbb{T}^\infty \setminus \{1\} \). Then, we know from Lemma 3.1 that \( U_{j,x} \in C^1(H_{j-1}) \) with \([H_{j-1}, U_{j,x}] = g_{j,x} U_{j,x}\). Furthermore, (3.2) and (3.3) imply that
\[
\begin{align*}
\int_0^1 \frac{dt}{t} \left \| e^{itH_{j-1}} g_{j,x} e^{itH_{j-1}} - g_{j,x} \right \|_{\mathcal{B}(H_{j-1})} & = \int_0^1 \frac{dt}{t} \left \| g_{j,x} \circ F_{j-1,t} - g_{j,x} \right \|_{L^\infty((\mathbb{T}^\infty)^{-1})} \\
& \leq 2\pi \sum_{k=1}^{k_0} |n_k| \int_0^1 \frac{dt}{t} \left \| (H_{j-1} \eta_{j-1,k}) \circ F_{j-1,t} - (H_{j-1} \eta_{j-1,k}) \right \|_{L^\infty((\mathbb{T}^\infty)^{-1})} \\
& < \infty.
\end{align*}
\]
So, we obtain that \( U_{j,x} \in C^{1+0}(H_{j-1}) \) with \([H_{j-1}, U_{j,x}] = g_{j,x} U_{j,x}, \) and thus deduce from [7, Sec. 4] that \( U_{j,x} \in C^{1+0}(A_{j,x}^{(N)}) \) with \([A_{j,x}^{(N)}, U_{j,x}] = g_{j,x}^{(N)} U_{j,x}\). Now, \( H_{j-1} \eta_{j-1,k} \in C((\mathbb{T}^\infty)^{-1}) \) and \( T_{j-1} \) is uniquely ergodic with respect to \( \mu_{j-1} \) due to Theorem 3.3. Thus, we infer from (3.2) that
\[
\lim_{N \to \infty} g_{j,x}^{(N)} = i \xi_{j-1} + 2\pi i \sum_{k=1}^{k_0} n_k \frac{1}{N} \sum_{\ell=0}^{N-1} (H_{j-1} \eta_{j-1,k}) \circ (T_{j-1})^\ell \\
= i \xi_{j-1} + 2\pi i \sum_{k=1}^{k_0} n_k \int_{(\mathbb{T}^\infty)^{-1}} d\mu_{j-1} \langle 1, H_{j-1} \eta_{j-1,k} \rangle_{\mathcal{H}_{j-1}} \\
= i \xi_{j-1}
\]
uniformly on \((\mathbb{T}^\infty)^{-1}\). Since \( \xi_{j-1} \neq 0 \) by the proof of Theorem 3.3, one has \( |g_{j,x}^{(N)}| > 0 \) if \( N \) is large enough. So, \( |g_{j,x}^{(N)}| > a \) with \( a := \inf_{x \in (\mathbb{T}^\infty)^{-1}} |g_{j,x}^{(N)}(x)| > 0 \), and \( U_{j,x} \) satisfies the following strict Mourre estimate on \( S^1 \):
\[
(U_{j,x})^* \left \| g_{j,x}^{(N)} \right \|_{\mathcal{B}(S^1)} (U_{j,x}) = (U_{j,x})^* \left \| g_{j,x}^{(N)} \right \|_{\mathcal{B}(S^1)} \geq a.
\]
Therefore, all the assumptions of Theorem 2.1 are satisfied, and thus \( U_{j,x} \) has purely absolutely continuous spectrum. Since this occurs for each \( j \in \{2, \ldots, d\} \) and \( x \in \mathbb{T}^\infty \setminus \{1\} \), and since \( U_{j,x} \) is unitarily equivalent to \( W_d \mid_{\mathcal{H}_{j,x}} \), one infers that \( W_d \) has purely absolutely continuous spectrum in \( \bigoplus_{j \in \{2, \ldots, d\}, \chi \in \mathbb{T}^\infty \setminus \{1\}} \mathcal{H}_{j,x} = \mathcal{H}_d \oplus \mathcal{H}_1 \). Finally, the fact that \( W_d \) has has countable Lebesgue spectrum in \( \mathcal{H}_d \oplus \mathcal{H}_1 \) can be shown in a similar way as in [10, Lemmas 3 & 4].
The result of Theorem 3.4 is an extension to the case of infinite-dimensional tori of results previously obtained in the case of one-dimensional tori (see [10, Cor. 3] or [15, Thm. 5.3]).

References

[1] W. O. Amrein. *Hilbert space methods in quantum mechanics*. Fundamental Sciences. EPFL Press, Lausanne, 2009.

[2] W. O. Amrein, A. Boutet de Monvel, and V. Georgescu. *C₀-groups, commutator methods and spectral theory of N-body Hamiltonians*, volume 135 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1996.

[3] A. Bendikov and L. Saloff-Coste. On the sample paths of Brownian motions on compact infinite dimensional groups. *Ann. Probab.*, 31(3):1464–1493, 2003.

[4] V. I. Bogachev. Differentiable measures and the Malliavin calculus. *J. Math. Sci. (New York)*, 87(4):3577–3731, 1997. Analysis, 9.

[5] F. Bruhat. Distributions sur un groupe localement compact et applications à l’étude des représentations des groupes p-adiques. *Bull. Soc. Math. France*, 89:43–75, 1961.

[6] I. P. Cornfeld, S. V. Fomin, and Y. G. Sinai. *Ergodic theory*, volume 245 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1982. Translated from the Russian by A. B. Sosinskiĭ.

[7] C. Fernández, S. Richard, and R. Tiedra de Aldecoa. Commutator methods for unitary operators. *J. Spectr. Theory*, 3(3):271–292, 2013.

[8] H. Furstenberg. Strict ergodicity and transformation of the torus. *Amer. J. Math.*, 83:573–601, 1961.

[9] G. Greschonig. Nilpotent extensions of Furstenberg transformations. *Israel J. Math.*, 183:381–397, 2011.

[10] A. Iwanik, M. Lemańczyk, and D. Rudolph. Absolutely continuous cocycles over irrational rotations. *Israel J. Math.*, 83(1-2):73–95, 1993.

[11] R. Ji. *ON THE CROSSED PRODUCT C*-ALGEBRAS ASSOCIATED WITH FURSTENBERG TRANSFORMATIONS ON TORI*. ProQuest LLC, Ann Arbor, MI, 1986. Thesis (Ph.D.)–State University of New York at Stony Brook.

[12] H. Osaka and N. C. Phillips. Furstenberg transformations on irrational rotation algebras. *Ergodic Theory Dynam. Systems*, 26(5):1623–1651, 2006.

[13] K. Reihani. *K*-theory of Furstenberg transformation group C*-algebras. *Canad. J. Math.*, 65(6):1287–1319, 2013.

[14] R. Tiedra de Aldecoa. Commutator criteria for strong mixing. to appear in *Ergodic Theory Dynam. Systems*, preprint on http://arxiv.org/abs/1406.5777.

[15] R. Tiedra de Aldecoa. Commutator methods for the spectral analysis of uniquely ergodic dynamical systems. *Ergodic Theory and Dynamical Systems*, FirstView:1–24, 10 2014.