Icosahedral coincidence rotations

Johannes Roth\(^1\) and Reinhard Lück\(^2\)

\(^1\) Institut für Theoretische und Angewandte Physik, Universität Stuttgart, Pfaffenwaldring 57, D-70550 Stuttgart, Germany
\(^2\) Max-Planck-Institut für Metallforschung, Seestr. 92, D-70174 Stuttgart, Germany

Abstract

The coincidence problem for three-dimensional discrete structures with icosahedral symmetry is reinvestigated. We present a parametric description of the coincidence rotations based on special quaternions, called icosian numbers. In particular, we give a characterization of the possible coincidence indices (\(\Sigma\)-factors) and present a complete list of the possible rotations with \(\Sigma \leq 100\).

1 Introduction

The symmetries of crystallographic lattices are well known: these are the operations that map the whole lattice onto itself, so that each lattice point coincides exactly onto another one. The symmetry operations are translations, rotations, reflections and combinations thereof. It is, however, also possible to map only a part of the lattice points onto the others: a sublattice emerges. If the sublattice has full rank (i.e. same dimension as the original lattice), but possibly lower symmetry, it is called a coincidence site lattice, or CSL for short. The CSL is characterized by its index in the original lattice. This number is also called the \(\Sigma\)-factor. CSLs are useful for several aspects.

First of all, there are mathematical reasons: the point symmetry group of the lattice \(G\) is a subgroup of the symmetry group of the isotropic Euclidean space, \(SO(n, \mathbb{R})\). The set of coincidence rotations also forms a group, and includes the group of symmetries. In particular, it contains rotations with rational angle. Therefore we have \(SO(n, \mathbb{Z}) \subset CSL \subset SO(n, \mathbb{R})\), and it would be desirable to know more about the structure of the CSL group. Since the point symmetry group may be a maximal finite subgroup of \(SO(n, \mathbb{R})\), the CSL group must have infinite order.

Second there are crystallographic reasons: A real crystal consists of several grains. It has been shown that these components are mostly not oriented arbitrarily with respect to each other, but that two adjacent grains often have a common coincidence lattice with low index in the full lattice. The CSL are important in crystallography because they allow a nontrivial classification of grain boundaries and because small-unit-cell CSL grain boundaries seem to be energetically favoured [\(\ldots\)] . It has been shown that coincidences of vertices appear also in quasicrystalline tilings [\(\ldots\)] . Quasicrystals also have grain boundaries like ordinary crystals, and one should therefore know the coincidence site quasilattices [\(\ldots\)] . Twinning has been observed in experiment [\(\ldots\)] , as well as interfaces between icosahedral and decagonal quasicrystals [\(\ldots\)] . Similar to the crystal the CSL grain boundary should be energetically favoured. To understand the occuring coincidence indices one has to enumerated the possible CSL. The grain boundary between crystals itself may be quasiperiodic as predicted by Rivier and Lawrence [\(\ldots\)] . Indirect evidence was provided already in the 1970s [\(\ldots\)] , and later observed in the growth of quasicrystalline grain at the grain boundary between two crystals [\(\ldots\)] .

Some microstructures with icosahedral (pseudo) symmetry are composed of ’approximants’ by twinning and multiple twinning. The grain boundaries of these nanodomains will also be low energy grain boundaries, but their orientation relation is different from those ones described in the present paper.

The first one who has dealt with generating functions and the enumeration of CSL seems to have been Ranganathan [\(\ldots\)] . The classification of the crystallographic CSL has been obtained by several
authors in the '70's and '80's [], and there exist tables of cubic and hexagonal CSL. But there has been no systematic mathematical treatment of the problem until recently, see [] and references therein.

The interest in CSL was renewed with the discovery of the quasicrystals. It has been found that this new type of materials also exhibits multiple grains, twin relationships and coincidence (quasi-)lattices.

In a quasicrystal the number of generating basis vectors is larger than the space dimension. This means that there exists no lattice with a minimal distance in physical space but a dense module. The CSL has to be replaced by a coincidence site module (CSM). A quasilattice is generated if only certain points are selected from the module by a window function.

Recently the complete classification of all possible CSL/CSM for crystals and quasicrystals with rotational symmetry of any order and an analysis of the structure of the CSL/CSM group has been achieved by Pleasants et al. [] with the help of number theoretical methods. This means that the CSL/CSM case is solved completely in two dimensions. In three and four dimensions, similar (but less complete) results are known through the use of quaternions, see Baake and Pleasants [] and Baake [].

In three dimensions the problem has also been addressed by Warrington and Lück [], who enumerated the icosahedral CSM with small index, and by Radulescu and Warrington []. In this paper we will generalize the systematic and appealing treatment of the crystallographic CSL in three dimensions by Grimmer [] to icosahedral quasicrystals. The method is suitable for an implementation in a computer program and allows an easy enumeration of all CSM up to high indices. Our main purpose is therefore the presentation of complete tables of icosahedral CSM representatives.

2 Formulation of the problem

A coincidence rotation $C$ maps a lattice $L$ onto a lattice $L'$:

$$L' = CL$$

(1)

The rotation $C$ is not unique, since we can apply symmetry rotations $R$ and $S$ to the lattices $L = R\bar{L}$ and $L' = S\bar{L}'$, resulting in a conjugation of $C'$:

$$S\bar{L}' = C\bar{R}\bar{L} \quad \text{or} \quad \bar{L}' = S^{-1}C\bar{R}\bar{L}$$

(2)

By comparison with Eq. 1 we may introduce a new coincidence rotation $C'$:

$$C' = S^{-1}CR$$

(3)

This means that we have to count the coincidence rotations $C$ with respect to double coset classes:

$$(S, R) \in SO(3) \times SO(3) \equiv SO(4)/Z_2$$

First of all we need a proper parametrization of the rotation matrices. A three-dimensional matrix has nine entries, but only three are independent: the axis (of length one) and the angle. Cayley’s parametrization through quaternions is useful here []. If $q = (\kappa, \lambda, \mu, \nu)$ is the quaternion, the first entry $\kappa$ parametrizes the angle:

$$\cos(\phi) = \frac{\kappa^2 - \lambda^2 - \mu^2 - \nu^2}{\kappa^2 + \lambda^2 + \mu^2 + \nu^2}$$

and the vector $(\lambda, \mu, \nu)$ represents the rotation axis. The rotation matrix is therefore given by:

$$R = \frac{1}{|q|^2} \left( \begin{array}{ccc} \kappa^2 + \lambda^2 - \mu^2 - \nu^2 & -2\kappa\nu + 2\lambda\mu & 2\kappa\mu + 2\lambda\nu \\ 2\kappa\nu + 2\lambda\mu & \kappa^2 - \lambda^2 + \mu^2 - \nu^2 & -2\kappa\lambda + 2\mu\nu \\ -2\kappa\mu + 2\lambda\nu & 2\kappa\lambda + 2\mu\nu & \kappa^2 - \lambda^2 - \mu^2 + \nu^2 \end{array} \right)$$
Figure 1: The asymmetric unit of the four-dimensional polytope \( \{5,3,3\} \). The dash-dotted line is the 5-fold axis, the long-dashed lines are 2-fold axis and the short-dashed lines 3-fold axis.

The norm of a quaternion is:
\[
|q|^2 = \kappa^2 + \lambda^2 + \mu^2 + \nu^2
\]

We have a two-to-one homomorphism of the quaternions and the rotation matrices since \( R(q) = R(-q) \). In the quaternionic formulation of the coincidence problem we get \( \[ \) (small letters indicate quaternions):
\[
L = C'L \quad \rightarrow \quad L' = c'c'^{-1}
\]
\[
C' = S^{-1}CR \quad \rightarrow \quad c' = s^{-1}cr
\]

Similar to the three-dimensional Cayley parametrization there exists also a four-dimensional one:

Due to the homomorphisms between the quaternions, \( SU(2) \) and \( SO(3) \) and the homomorphism between \( SO(3) \times SO(3) \), \( SU(2) \times SU(2) \) and \( SO(4) \), we can parametrize four-dimensional rotation matrices by a pair of quaternions \( \[ \). The equation \( \[ \) is thus further transformed into
\[
c' = s^{-1}cr \quad \rightarrow \quad C' = MC
\]

Now \( C \) represents a four-dimensional vector and \( M \) a four-dimensional rotation matrix parametrized by two quaternions \( r = (k, l, m, n) \) and \( s = (a, b, c, d) \). The rotation matrix has the form:
\[
M = \frac{1}{|s|r} \begin{pmatrix}
ak + bl + cm + dn & -al + bk + cn - dm & -am - bn + ck + dl & -an + bm - cl + dk \\
al - bk + cn - dm & ak + bl - cm - dn & -an + bm + cl - dk & am + bn + ck + dl \\
am - bn - ck + dl & an + bm + cl + dk & ak - bl + cm - dn & -al - bk + cn + dm \\
an + bm - cl - dk & -an + bm - ck + dl & al + bk + cn + dm & ak - bl - cm - dn
\end{pmatrix}
\]

What is the meaning of this transformation? Equation \( \[ \) is nothing else than the symmetry description of a four-dimensional polytope! A detailed description of the polytope may be found in Ref. \( \[ \). We have transformed the problem of enumerating double coset classes into the problem of analyzing a four-dimensional polytope. If we calculate the orbit of a certain point of the polytope we find that the size of the orbit gives the number of equivalent CSM, and the type of the orbit tells us the symmetry of the CSM. Practically, we have to find a standard representation for the CSM. This is the disorientation \( \[ \): within a class of equivalent coincidence rotations there is one where the rotation angle transforming \( L \) into \( L' \) is minimal. Therefore \( \kappa \) in \( q \) is maximal. Translated into the four-dimensional language of the polytopes we get a set of linear inequalities \( \[ \) which define an asymmetric unit. Fig. \( \[ \) displays the icosahedral case. The crystallographic cases can be found in Ref. \( \[ \).
In order to count the orbits we have to fill the polytope with points. In the cubic case it has been shown \cite{ref} that it is sufficient that the entries of the quaternion \( q = (\kappa, \lambda, \mu, \nu) \) are integers. In the icosahedral case we have to use numbers of the type \( m + n\tau \) with \( m \) and \( n \) integers and \( \tau \) the golden mean \( (\sqrt{5} + 1)/2 \). These numbers are elements of the ring of icosians generated by the quaternions \( (1,0,0,0), 1/2(1,1,1,1) \) and \( 1/2(\tau,1,-1/\tau,0) \). These quaternions generate the group \( Y \) of order 120, and the ring of icosians \( I \) are all integer linear combinations.

There exists a further interesting module: It consists of all points in \( \mathbb{Z}[\tau] \), has cubic symmetry and contains the icosahedral modules as subsets (for details see \cite{ref}). The coincidence rotations are generated by the group \( SO(3, \mathbb{Q}(\tau)) \), and the quaternions have entries of type \( m + n\tau \) without any restriction.

3 Technicalities

To avoid counting equivalent quaternions several times we have to demand that three conditions are fulfilled: First, we take only quaternions whose components have no common factor: \( \gcd(\kappa, \lambda, \mu, \nu) = 1 \). Second, the entries of \( q \) are ordered with decreasing size. Third, the norm of a number \( m + n\tau \) is:

\[
N(m + n\tau) = m^2 + mn - n^2
\]

This means that

\[
N(\pm\tau^k) = (-1)^k
\]

with \( k \) an integer. To take this last condition into account it is convenient to restrict the smallest non-zero quaternion component between 1 and \( \tau \).

The index for an icosahedral CSM is given by:

\[
\Sigma = N(|q|^2)/2^\ell
\]

where \( \ell = 0 \) if one of the (coprime) quaternion components is odd and \( \ell = 2 \) if all are even. For the cubic case it is:

\[
\Sigma = |N(\text{den}(R))|
\]

where \( \text{den}(R) \) is the denominator of the rotation matrix.

4 Results

It is easy to count icosahedral CSM and the CSM of \( \mathbb{Z}[\tau] \) on a workstation by looping through the unit icosians or through the quaternions with \( m + n\tau \) entries using the algorithm described above. The correctness of the results can be checked by the generating function for the number of CSM \( \#(\Sigma) \) given in Ref. \cite{ref}. We have tabulated the CSM with a rotation axis parallel to a symmetry axis up to \( \Sigma = 5000 \), with an axis in a general up to \( \Sigma = 500 \). Some special cases of the order \( \Sigma = 20000 \) have also been checked. Summary tables of the results are given in Tab. 1-2, representatives for the icosahedral case in Tabs. 3-4. The cubic case is presented in Tabs. 5-7, and 8-10.

It is interesting to note that the number of CSM for the cubic and icosahedral case are identical if \( \Sigma \) is odd, but if \( \Sigma \) is even, \( N_{\text{cub}}(\Sigma) = 8 \times N_{\text{ico}}(\Sigma/4) \).

Miscellany

The results described in this paper have been presented at the XXI. Int. Coll. on Group Theoret. Methods in Physics, July 15th-19th, in Goslar, Germany. The tables listed in this paper may be obtained from the authors through email from johannes.roth@itap.physik.uni-stuttgart.de.
Acknowledgements

We thank Michael Baake for his cooperation during the preparation of this manuscript.

References

[1] A. P. Sutton, and R. W. Bullafi, “On geometric criteria for low interfacial energy”, Acta Met. 35 2177–2201 (1987).

[2] S. Ranganathan, “Coincidence-site lattices, superlattices and quasicrystals”, Trans Indian Inst. Met. 32 1–7 (1990).

[3] D. H. Warrington, “Coincidence site lattices in quasicrystal tilings”, Mater. Sci. Forum 126–128 57–60 (1993).

[4] M. X. Dai, and K. Urban, “Twins in icosahedral Al-Cu-Fe”, Phil. Mag. Lett. 67 67–71 (1993).

[5] M. Feuerbacher, M. Wollgarten, K. Urban, private communication.

[6] N. Rivier, and A. J. A. Lawrence, “Quasiperiodic ordering in grain boundaries” in: Quasicrystalline Materials, eds. C. Janot and J. M. Dubois, World Scientific, Singapore 255–263 (1990).

[7] S. L. Sass, T. Y. Tan, and R. V. Ballufi, “The detection of the periodic structure of high-angle twist boundaries. I. Electron diffraction studies”, Phil. Mag. 31 559–573 (1975).

[8] T. Y. Tan, S. L. Sass, and R. V. Ballufi, “The detection of the periodic structure of high-angle twist boundaries. II. High resolution electron microscopy study”, Phil. Mag. 31 575–585 (1975).

[9] W. A. Cassada, G. J. Shifflet, and S. J. Poon “Formation of an icosahedral phase by solid state reaction” Phys. Rev. Lett. 56 276–279 (1986).

[10] H. Siddhom, and R. Portier, “An icosahedral phase in annealed austenitic stainless steel?”, Phil. Mag. Lett. 59 131–139 (1989).

[11] S. Ranganathan, “On the geometry of coincidence-site lattices”, Acta Cryst. A21, 197–9 (1966).

[12] D. H. Warrington, and P. Bufalini, “The coincidence site lattice and grain boundaries”, Scr. Met. 5, 771–6 (1971).

[13] H. Grimmer, W. Bollmann, and D. H. Warrington, “Coincidence-site lattices and complete pattern-shift lattices in cubic crystals”, Acta Cryst. A30 197–207 (1974).

[14] H. Grimmer, “Disorientations and coincidence rotations for cubic lattices”, Acta Cryst. A 30, 685–8 (1974).

[15] H. Grimmer, “A unique description of the relative orientation of neighbouring grains”, Acta Cryst. A36 382–9 (1980).

[16] M. Baake, “Solution of the coincidence problem in dimensions $d \leq 4$”, in: The Mathematics of Aperiodic Order, ed. R. V. Moody, Kluwer, Dordrecht (1997), in press.

[17] P. A. B. Pleasants, M. Baake, and J. Roth, “Planar coincidences for $N$-fold symmetries”, J. Math. Phys. 1029–58 (1996).

[18] M. Baake, “Coincidence rotations for icosahedral patterns”, Ames preprint
[19] M. Baake, and P. A. B. Pleasants, “The coincidence problem for crystals and quasicrystals”, in: *Aperiodic 94*, eds. G. Chapuis and W. Paciorek, World Scientific, Singapore (1995), p. 25–9; and: “Solution of the coincidence problem in two and three dimensions”, *Z. Naturf.* **50a** 711–7 (1995).

[20] D. H. Warrington, and R. Lück, “Rotational space and coincidence site lattices for icosahedral symmetry”, *Mat. Sci. For.* **207–9** 825–8 (1996).

[21] R. Lück, and D. H. Warrington, “Sperical geometric representations of coincidence site lattice (CSL) rotations for icosahedral symmetry”, *J. Non-Cryst. Sol.* **205–7** 25–7(1996).

[22] O. Radulescu, and D. H. Warrington, “Arithmetic properties of module directions in quasicrystals, coincidence modules and coincidence quasicrystals”, *Acta Cryst.* **A 51** 335-343 (1996)

[23] P. Du Val, “Homographies, Quaternions and Groups”, Clarendon Press, Oxford (1964).
Table 1: Summary of the CSM orbits. I. The part left of the double line are the results for the icosahedral case, the part to the right is for the cubic module \( \mathbb{Z}[\tau] \). \( C_n \) denotes a CSM axis parallel to \( n \)-fold axis, \( M \) a rotation axis in a mirror plane and \( G \) a rotation axis in a general position. The numbers \( \#(\sigma) \) give the sum of the lengths of the orbits. Entries of type \( a* b \) denote \( a*b \) CSM with \( b \) algebraic conjugates. The symmetry groups for \( \Sigma < 10 \) are explicitly given. Primes indicate a CSM occurring in a standard cubic lattice.

| \( \Sigma \) | \( \#(\Sigma) \) | C5:12 | C3:20 | M:60 | C4:6 | C3:8 | C2:12 | M:24 | G:48 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | Y |  |  |  |  |  |  | O |
| 4 | 5 | T |  |  |  |  |  |  |  |
| 5 | 6 |  | D5 | 1 |  |  |  |  |  |
| 9 | 10 |  | D3 | 1 |  |  |  |  |  |
| 11 | 24 | 1*2 |  |  |  |  |  |  | D3' |
| 16 | 20 | 1*1 |  |  |  |  |  |  |  |
| 19 | 40 | 1*2 |  |  |  |  |  |  |  |
| 20 | 30 | 1*1 |  |  |  |  |  |  |  |
| 25 | 30 | 1*1 | 1' | 1 |  |  |  |  |  |
| 29 | 60 | 1*2 | 2 | 2 |  |  |  |  |  |
| 31 | 64 | 1*2 | 1*2 | 2 | 2 |  |  |  |  |
| 36 | 50 | 1*1 | 1*1 |  |  |  |  |  |  |
| 41 | 84 | 1*2 | 1*2 | 2 | 2 | 2 |  |  |  |
| 44 | 120 | 1*2 |  |  |  |  |  |  |  |
| 45 | 60 | 1*2 | 2 |  |  |  |  |  | 1 |
| 49 | 50 | 1*1 | 1*1 | 1' | 1 | 1 |  |  |  |
| 55 | 144 | 1*2 | 1*2 |  |  |  |  |  | 6 |
| 59 | 120 | 1*2 | 2 | 4 |  |  |  |  |  |
| 61 | 124 | 1*2 | 1*2 | 1*2 | 2 | 2 | 4 |  |  |
| 64 | 80 | 1*1 | 1*1 |  |  |  |  |  |  |
| 71 | 144 | 1*2 | 1*2 |  |  |  |  |  | 6 |
| 76 | 200 | 2*2 | 1*2 |  |  |  |  |  |  |
| 79 | 160 | 1*2 | 1*2 | 2 |  |  |  |  | 6 |
| 80 | 120 | 1*2 |  |  |  |  |  |  |  |
| 81 | 90 | 1*1 | 1*1 | 1 | 1' | 1 | 1 |  |  |
| 89 | 180 | 1*2 | 1*2 | 2 | 2 | 2 | 2 |  |  |
| 95 | 240 | 2*2 | 2 | 6 | 2 |  |  |  |  |
| 99 | 240 | 2*2 |  |  |  |  |  |  | 4 |
| 100 | 150 | 1*1 | 1*2 |  |  |  |  |  |  |
| 101 | 204 | 1*2 | 1*2 | 1*2 | 2 |  |  |  | 1 |
| 109 | 220 | 1*2 | 1*2 | 1*2 | 2 | 1 |  | 8 |  |
| 116 | 300 | 1*2 | 2*2 |  |  |  |  |  |  |
Table 2: Summary of the CSM orbits. II. Notation as in Tab. I.

| Σ  | #(Σ) | C5:12 | C3:20 | C2:30 | M:60 | G:120 | C4:6 | C3:8 | C2:12 | M:24 | G:48 |
|----|------|-------|-------|-------|------|-------|------|------|-------|------|------|
| 121| 408  | 2*2   | 2*2   | 2*(2+1)| 4    | 11    | 2    |      |       |      |      |
| 124| 320  | 2*2   | 1*2   | 2*2   | 4    | 10    | 2    |      |       |      |      |
| 125| 150  | 1*2   | 1*2   | 1*2   | 4    | 10    | 2    |      |       |      |      |
| 131| 264  | 1*2   | 2*2   | 2*2   | 2    | 6     | 2    |      |       |      |      |
| 139| 280  | 1*2   | 2*2   | 2*2   | 2    | 6     | 2    |      |       |      |      |
| 144| 200  | 1*1   | 1*1   | 1*1   | 1    | 1     | 1    | 2    | 6     |      |      |
| 145| 360  | 2*2   | 2*2   | 2*2   | 4    | 10    | 2    |      |       |      |      |
| 149| 300  | 1*2   | 2*2   | 2*2   | 2    | 8     | 2    |      |       |      |      |
| 151| 304  | 1*2   | 1*2   | 2*2   | 2    | 8     | 2    |      |       |      |      |
| 155| 384  | 1*2   | 1*2   | 2*2   | 2    | 8     | 4    |      |       |      |      |
| 164| 420  | 1*2   | 1*2   | 1*2   | 2    | 6     | 4    |      |       |      |      |
| 169| 170  | 1*1   | 1*1   | 2*1   | 1    | 1     | 1    | 2    | 6     |      |      |
| 171| 400  | 1*2   | 1*2   | 3*2   | 2    | 4     | 2    | 6     |      |      |      |
| 176| 480  | 4*2   |       |       |      |       |      |      |       |      |      |
| 179| 360  | 3*2   |       |       |      |       |      |      |       |      |      |
| 180| 300  | 1*2   |       |       |      |       |      |      |       |      |      |
| 181| 364  | 1*2   | 1*2   | 1*2   | 2*2   | 2    | 6     | 4    |      |      |      |
| 191| 384  | 1*2   | 3*2   |       | 2    | 6     | 4    |      |      |      |      |
| 196| 250  | 1*2   | 1*1   | 1*1   | 1    | 1     | 1    |      |       |      |      |
| 199| 400  | 1*2   | 1*2   | 1*2   | 1*2   | 12    | 2    |      |       |      |      |
| 205| 504  | 1*2   | 2*2   | 1*2   | 1*2   |       |      |      |       |      |      |
| 209| 960  | 6*2   |       |       | 1*2   |       |      |      |      |      |      |
| 211| 424  | 1*2   | 1*2   | 3*2   |       |      |      |      |      |      |      |
| 220| 720  | 4*2   |       |       | 1*2   |       |      |      |      |      |      |
| 225| 300  | 1*2   | 1*2   | 1*2   | 1*2   |       |      |      |      |      |      |
| 229| 460  | 1*2   | 1*2   | 1*2   | 1*2   |       |      |      |      |      |      |
| 236| 600  | 5*2   |       |       |       |       |      |      |      |      |      |
| 239| 480  | 2*2   |       |       | 1*2   |       |      |      |      |      |      |
| 241| 484  | 1*2   | 1*2   | 1*2   | 3*2   |       |      |      |      |      |      |
| 244| 620  | 2*2   | 1*2   | 1*2   |       |      |      |      |      |      |      |
| 245| 300  | 1*2   | 1*2   | 1*2   |       |      |      |      |      |      |      |
| 251| 504  | 1*2   | 2*2   | 1*2   |       |      |      |      |      |      |      |
| 256| 320  | 1*1   | 1*(2+1)| 1*1   |       |      |      |      |      |      |      |
Table 3: Summary of the CSM orbits. III. This table lists the coincidence rotations for the cubic module $\mathbb{Z}[\gamma]^3$ with even $\Sigma$. Notation as in Tab. I.

| $\Sigma$ | #(\Sigma) | C3:8 | M:24 | G:48 |
|----------|-----------|------|------|------|
| 4        | 8         | 1    |      |      |
| 16       | 32        | 1    | 1    |      |
| 20       | 48        |      | 2    |      |
| 36       | 80        | 1    | 1    | 1    |
| 44       | 192       |      | 8    |      |
| 64       | 128       | 1    | 3    | 1    |
| 76       | 320       | 4    | 4    | 4    |
| 80       | 192       |      | 4    | 2    |
| 100      | 240       |      | 4    | 3    |
| 116      | 480       |      | 8    | 6    |
| 124      | 512       | 4    | 12   | 4    |
| 144      | 320       | 1    | 7    | 3    |
| 164      | 672       |      | 12   | 8    |
| 176      | 768       |      | 8    | 12   |
| 180      | 480       |      | 8    | 6    |
| 196      | 400       | 2    | 4    | 6    |
| 220      | 1152      |      | 16   | 16   |
| 236      | 960       |      | 8    | 16   |
| 244      | 992       | 4    | 16   | 12   |
| 256      | 512       | 1    | 7    | 7    |
Table 4: Representatives for coincidence rotations with $\Sigma < 100$ for the icosahedral module. I. The two sections indicate rotation axis along 5- and 3-fold symmetry axis, respectively. The coordinates are given as $m+\tau n$. The first column lists the conjugate pairs and the last column the disorientation angle.

| conjugates | $\kappa$ | $\lambda$ | $\mu$ | $\nu$ | $\Sigma$ | angle   |
|------------|----------|----------|-------|-------|----------|---------|
| a a        | 0 1      | 1 -1     | 0 0   | 1 2   | 5        | 36.000000 |
| a b        | 0 1      | 1 -1     | 0 0   | 3 2   | 11       | 19.464600 |
| b a        | 0 1      | 1 -1     | 0 0   | 3 0   | 11       | 27.227642 |
| a b        | 0 1      | 1 -1     | 0 0   | 5 2   | 31       | 13.290735 |
| b a        | 0 1      | 1 -1     | 0 0   | 1 4   | 31       | 23.637167 |
| a b        | 0 1      | 1 -1     | 0 0   | 5 4   | 41       | 11.107199 |
| b a        | 0 1      | 1 -1     | 0 0   | 3 4   | 41       | 15.125998 |
| a b        | 0 1      | 1 -1     | 0 0   | 7 4   | 55       | 8.772344  |
| b a        | 0 1      | 1 -1     | 0 0   | 5 0   | 55       | 16.535397 |
| a b        | 0 1      | 1 -1     | 0 0   | -1 6  | 61       | 30.034790 |
| b a        | 0 1      | 1 -1     | 0 0   | 5 -4  | 61       | 32.070736 |
| a b        | 0 1      | 1 -1     | 0 0   | 5 -2  | 71       | 21.850590 |
| b a        | 0 1      | 1 -1     | 0 0   | 2 1   | 71       | 31.020212 |
| a a        | 0 1      | 1 -1     | 0 0   | 1 1   | 4        | 44.477505 |
| a a        | 0 1      | 1 -1     | 0 0   | 3 3   | 9        | 60.000000 |
| a a        | 0 1      | 1 -1     | 0 0   | 3 -1  | 16       | 31.049971 |
| a b        | 0 1      | 1 -1     | 0 0   | 3 1   | 19       | 20.724964 |
| b a        | 0 1      | 1 -1     | 0 0   | 1 3   | 19       | 26.101496 |
| a b        | 0 1      | 1 -1     | 0 0   | -1 5  | 31       | 35.127438 |
| b a        | 0 1      | 1 -1     | 0 0   | -3 7  | 31       | 53.023972 |
| a a        | 0 1      | 1 -1     | 0 0   | 3 3   | 36       | 15.522505 |
| a a        | 0 1      | 1 -1     | 0 0   | 5 5   | 49       | 38.213223 |
| a b        | 0 1      | 1 -1     | 0 0   | 5 3   | 61       | 11.026676 |
| b a        | 0 1      | 1 -1     | 0 0   | 0 2   | 61       | 56.314327 |
| a a        | 0 1      | 1 -1     | 0 0   | 5 1   | 64       | 13.432528 |
| a b        | 0 1      | 1 -1     | 0 0   | 1 5   | 76       | 18.376030 |
| b a        | -1 3     | 3 -4     | 0 0   | 3 -2  | 76       | 54.797520 |
| a b        | 0 1      | 1 -1     | 0 0   | 5 -3  | 76       | 23.752563 |
| b a        | 0 1      | 1 -1     | 0 0   | 7 -9  | 76       | 49.421051 |
| a b        | 0 1      | 1 -1     | 0 0   | 5 -1  | 79       | 17.171223 |
| b a        | 0 1      | 1 -1     | 0 0   | 2 -1  | 79       | 51.163399 |
Table 5: Representatives for coincidence rotations with $\Sigma < 100$ for the icosahedral module. II. Same as Tab. 4 for rotations along 2-fold symmetry axis.

| conjugates | $\kappa$ | $\lambda$ | $\mu$ | $\nu$ | $\Sigma$ | angle     |
|------------|----------|----------|-------|-------|---------|-----------|
| a a        | 0 1      | 0 0      | 0 0   | 0 1   | 4       | 90.000000 |
| a a        | 0 1      | 0 0      | 0 0   | 1 0   | 5       | 63.434948 |
| a a        | 0 1      | 0 0      | 0 0   | 1 1   | 9       | 41.810310 |
| a a        | 0 1      | 0 0      | 0 0   | 2 1   | 20      | 26.565052 |
| a a        | 0 1      | 0 0      | 0 0   | 0 2   | 25      | 53.130100 |
| a b        | 0 1      | 0 0      | 0 0   | 2 0   | 29      | 34.344078 |
| b a        | 0 1      | 0 0      | 0 0   | 2 -2  | 29      | 77.946892 |
| a a        | 0 1      | 0 0      | 0 0   | 2 -1  | 36      | 48.189682 |
| a b        | 0 1      | 0 0      | 0 0   | 1 2   | 41      | 30.900869 |
| b a        | 0 1      | 0 0      | 0 0   | -1 3  | 41      | 71.779510 |
| a b        | 0 1      | 0 0      | 0 0   | 2 2   | 45      | 21.624634 |
| b a        | -4 8     | 0 0      | 0 0   | 0 2   | 45      | 74.754700 |
| a a        | 0 1      | 0 0      | 0 0   | 3 2   | 49      | 16.601551 |
| a b        | 0 1      | 0 0      | 0 0   | 3 1   | 61      | 19.387444 |
| b a        | -1 3     | 0 0      | 0 0   | 1 0   | 61      | 81.001404 |
| a a        | -4 8     | 0 0      | 0 0   | 6 -8  | 81      | 83.620689 |
| a b        | 0 1      | 0 0      | 0 0   | 4 2   | 89      | 13.463411 |
| b a        | -4 8     | 0 0      | 0 0   | 2 0   | 89      | 50.547329 |
| a a        | 0 1      | 0 0      | 0 0   | 0 3   | 100     | 36.869896 |
Table 6: Representatives for coincidence rotations with $\Sigma < 100$ for the icosahedral module. III. Same as Tab. 4 for rotations in mirror planes or general positions (last entry only).

| conjugates | $\kappa$ | $\lambda$ | $\mu$ | $\nu$ | $\Sigma$ | angle |
|------------|----------|-----------|-------|-------|----------|-------|
| a b        | -3 -4    | 1 -1      | 2 1   | 0 0   | 44       | 27.951245 |
| b a        | -3 5     | 0 1       | 1 2   | 0 0   | 44       | 31.212158 |
| a b        | -1 3     | 2 -3      | 3 0   | 0 0   | 55       | 32.220027 |
| b a        | 3 -4     | 1 -1      | 4 -3  | 0 0   | 55       | 33.780077 |
| a b        | -1 2     | -1 3      | 2 3   | 0 0   | 59       | 25.896548 |
| b a        | 0 1      | 3 -4      | 1 3   | 0 0   | 59       | 31.791262 |
| a a        | -1 2     | -1 3      | 4 -1  | 0 0   | 64       | 29.364595 |
| a b        | 2 -2     | -2 4      | 2 2   | 0 0   | 71       | 31.020216 |
| b a        | 1 0      | -3 5      | 4 -3  | 0 0   | 71       | 50.149411 |
| a b        | -1 2     | -1 3      | 0 5   | 0 0   | 76       | 32.001147 |
| b a        | -2 4     | -2 4      | 0 4   | 0 0   | 76       | 30.228843 |
| a b        | -3 5     | 0 1       | 3 0   | 0 0   | 79       | 23.521099 |
| b a        | -3 6     | 1 -1      | 2 1   | 0 0   | 79       | 34.169244 |
| a b        | -1 2     | -1 3      | 4 1   | 0 0   | 80       | 21.724197 |
| b a        | 2 -2     | -2 4      | 4 -2  | 0 0   | 80       | 36.000000 |
| a a        | 0 1      | 3 -4      | 3 1   | 0 0   | 81       | 25.322158 |
| a b        | -1 3     | 2 -3      | 1 4   | 0 0   | 89       | 28.024012 |
| b a        | 2 -3     | -3 6      | 3 -1  | 0 0   | 89       | 33.772669 |
| a d        | -1 3     | 2 -3      | 3 2   | 0 0   | 95       | 23.120394 |
| b c        | -1 3     | 2 -3      | 5 -4  | 0 0   | 95       | 37.840255 |
| e b        | 3 -4     | 1 -1      | 4 -1  | 0 0   | 95       | 21.809873 |
| d a        | -1 3     | 2 -3      | -3 8  | 0 0   | 95       | 48.040716 |
| a c        | -2 4     | -2 4      | 2 2   | 0 0   | 99       | 23.316582 |
| b d        | -2 4     | -2 4      | 4 -2  | 0 0   | 99       | 27.162221 |
| c a        | -1 3     | 2 -3      | -1 6  | 0 0   | 99       | 35.483519 |
| d b        | -3 5     | 0 1       | -3 8  | 0 0   | 99       | 35.618152 |
| a b        | 3 -4     | 1 -1      | 2 3   | 0 0   | 100      | 19.191145 |
| b a        | 3 -4     | 1 -1      | -2 7  | 0 0   | 100      | 31.294523 |
| a b        | 2 -2     | -2 4      | 4 -2  | 0 0   | 80       | 36.000000 |
| b a        | 0 1      | 5 -8      | -3 5  | 2 0   | 124      | 34.815460 |
Table 7: Representatives for coincidence rotations with odd $\Sigma < 100$ for the cubic module $\mathbb{Z}[\tau]^3$. I. The coordinates are given as $m + \tau n$. The sixth column gives the orbit length. The last column lists the disorientation angle.

| $\kappa$ | $\lambda$ | $\mu$ | $\nu$ | $\Sigma$ | orbit | angle   |
|-------|---------|-------|-------|-------|-------|--------|
| 1 0   | 0 0     | 0 0   | 0 0   | 1     | 1     | 0.00000|
| 0 1   | 1 0     | 0 0   | 0 0   | 5     | 6     | 26.56504|
| 1 1   | 1 1     | 0 0   | 0 0   | 9     | 6     | 41.81032|
| 0 1   | 1 0     | 1 0   | 0 0   | 9     | 4     | 60.00000|
| 0 1   | 0 1     | 1 0   | 0 0   | 11    | 12    | 47.21220|
| 0 1   | 1 0     | 1 0   | 0 0   | 11    | 12    | 61.03971|
| 0 1   | 0 1     | 0 1   | 1 0   | 19    | 8     | 20.72495|
| 0 1   | 1 0     | 1 0   | 1 0   | 19    | 8     | 26.10149|
| 1 1   | 1 1     | 1 0   | 0 0   | 19    | 12    | 30.22885|
| 1 1   | 1 0     | 1 0   | 0 0   | 19    | 12    | 56.75419|
| 2 0   | 1 0     | 0 0   | 0 0   | 25    | 6     | 36.86989|
| 1 1   | 0 1     | 1 0   | 1 0   | 25    | 24    | 37.39882|
| -2 2  | 1 0     | 0 0   | 0 0   | 29    | 6     | 12.05311|
| 0 2   | 1 0     | 0 0   | 0 0   | 29    | 6     | 34.34407|
| -1 0  | 1 1     | 1 0   | 1 0   | 29    | 24    | 38.72486|
| 1 0   | 1 0     | 1 0   | 1 0   | 29    | 24    | 45.97971|
| 1 1   | 1 1     | 1 0   | 0 0   | 31    | 24    | 34.81546|
| 1 1   | 1 1     | 1 1   | 1 0   | 31    | 8     | 35.62743|
| 1 2   | 0 1     | 1 0   | 0 0   | 31    | 24    | 48.36283|
| 1 1   | 1 0     | 1 0   | 1 0   | 31    | 8     | 53.02398|
| 3 -1 | 1 0     | 0 0   | 0 0   | 41    | 6     | 18.22049|
| 1 2   | 1 2     | 1 0   | 0 0   | 41    | 12    | 18.95350|
| 2 1   | 1 0     | 0 0   | 0 0   | 41    | 6     | 30.90087|
| 1 2   | 1 0     | 1 0   | 0 0   | 41    | 12    | 36.92323|
| 2 0   | 0 1     | 1 0   | 0 0   | 41    | 24    | 44.08459|
| 0 2   | 0 1     | 1 0   | 0 0   | 41    | 24    | 47.55893|
| 0 1   | -2 2    | 0 0   | 0 0   | 45    | 6     | 15.24529|
| 2 2   | 1 0     | 0 0   | 0 0   | 45    | 6     | 21.62464|
| 1 2   | 1 1     | 0 1   | 1 0   | 45    | 48    | 49.21481|
| 2 3   | 1 0     | 0 0   | 0 0   | 49    | 6     | 16.60156|
| 0 2   | 1 1     | 1 0   | 0 0   | 49    | 24    | 29.38548|
| 2 0   | 1 0     | 1 0   | 1 0   | 49    | 8     | 38.21321|
| -1 2  | 1 0     | 1 0   | 0 0   | 49    | 12    | 60.26300|
Table 8: Representatives for coincidence rotations with odd $\Sigma < 100$ for the cubic module $\mathbb{Z}[\tau]$. II. Notation as in Tab. 7.

| $\kappa$ | $\lambda$ | $\mu$ | $\nu$ | $\Sigma$ | orbit | angle |
|----------|-----------|-------|-------|----------|-------|-------|
| 2 0      | 0 1       | 1 0   | 1 0   | 55       | 24    | 33.78008 |
| -1 2     | 0 1       | 1 0   | 0 0   | 55       | 24    | 43.64693 |
| 0 2      | 1 2       | 1 1   | 1 0   | 55       | 24    | 47.86958 |
| 1 2      | 0 1       | 1 0   | 1 0   | 55       | 24    | 47.86958 |
| 2 1      | 0 1       | 1 0   | 0 0   | 55       | 24    | 50.14141 |
| 1 2      | 0 1       | 1 0   | 1 0   | 55       | 24    | 53.79745 |
| 2 1      | 1 0       | 1 0   | 0 0   | 59       | 12    | 42.69896 |
| 3 -1     | 1 0       | 1 0   | 0 0   | 59       | 12    | 62.43954 |
| 2 0      | 0 1       | 1 0   | 1 0   | 59       | 24    | 25.89653 |
| 1 2      | 1 2       | 0 1   | 1 0   | 59       | 24    | 35.23041 |
| 0 2      | 0 1       | 1 0   | 1 0   | 59       | 24    | 55.84497 |
| 1 2      | 1 1       | 1 0   | 1 0   | 59       | 24    | 41.14530 |
| 0 1      | 3 -1      | 0 0   | 0 0   | 61       | 6     | 8.99856  |
| -2 2     | 1 0       | 1 0   | 1 0   | 61       | 8     | 11.02669 |
| 1 3      | 1 0       | 0 0   | 0 0   | 61       | 6     | 19.38743 |
| 1 1      | 2 0       | 1 0   | 0 0   | 61       | 24    | 36.95959 |
| 2 2      | 0 1       | 1 0   | 0 0   | 61       | 24    | 39.92927 |
| 2 2      | 1 1       | 1 0   | 0 0   | 61       | 24    | 41.49912 |
| 0 1      | -2 2      | 1 0   | 0 0   | 61       | 24    | 54.33899 |
| 0 2      | 1 0       | 1 0   | 1 0   | 61       | 8     | 56.31433 |
| 2 3      | 1 2       | 1 0   | 0 0   | 71       | 24    | 30.03838 |
| 2 3      | 0 1       | 1 0   | 0 0   | 71       | 24    | 31.02022 |
| -1 2     | 0 1       | 1 0   | 1 0   | 71       | 24    | 38.59492 |
| 0 2      | 1 1       | 1 1   | 1 0   | 71       | 24    | 38.59492 |
| 0 2      | 1 1       | 1 0   | 1 0   | 71       | 24    | 39.35215 |
| 2 1      | 0 1       | 0 1   | 1 0   | 71       | 24    | 53.42054 |
| 3 -1     | 1 0       | 1 0   | 1 0   | 79       | 8     | 17.17122 |
| 0 1      | -2 2      | 1 0   | 1 0   | 79       | 24    | 23.52110 |
| 1 1      | -1 2      | 1 0   | 0 0   | 79       | 24    | 33.58591 |
| 1 3      | 1 1       | 1 0   | 0 0   | 79       | 24    | 45.25742 |
| 1 1      | 2 0       | 1 0   | 1 0   | 79       | 24    | 45.27756 |
| 2 2      | 0 1       | 0 1   | 1 0   | 79       | 24    | 50.99529 |
| 2 1      | 1 0       | 1 0   | 1 0   | 79       | 8     | 51.16339 |
| 2 2      | 1 1       | 1 1   | 1 0   | 79       | 24    | 55.79540 |
Table 9: Representatives for coincidence rotations with odd $\Sigma < 100$ for the cubic module $\mathbb{Z}[\tau]$. III. Notation as in Tab. 7.

| $\kappa$ | $\lambda$ | $\mu$ | $\nu$ | $\Sigma$ | orbit | angle   |
|----------|----------|-------|-------|----------|-------|---------|
| 3       | -2       | 2     | 0     | 0        | 81    | 6 6.37933 |
| 2 0     | 2 0      | 1 0   | 0 0   | 81       | 12    | 38.94244 |
| 2 1     | 1 1      | 0 1   | 1 0   | 81       | 48    | 48.42724 |
| 2 3     | 1 1      | 1 1   | 1 0   | 81       | 24    | 58.45731 |
| 2 4     | 1 0      | 0 0   | 0 0   | 89       | 6     | 13.46340 |
| 0 2     | 0 2      | 1 0   | 0 0   | 89       | 12    | 24.65171 |
| 1 1     | -2       | 2     | 0     | 0 0      | 89    | 6 39.45264 |
| 2 1     | 1 1      | 1 0   | 1 0   | 89       | 24    | 39.45264 |
| 2 1     | 1 1      | 1 1   | 1 0   | 89       | 24    | 41.70632 |
| 2 3     | 1 2      | 1 1   | 1 0   | 89       | 48    | 46.37142 |
| 2 3     | 1 1      | 0 1   | 1 0   | 89       | 48    | 50.54736 |
| -2 2    | -2       | 2     | 1     | 0 0      | 89    | 12 59.54441 |
| 1 3     | 1 2      | 1 0   | 0 0   | 95       | 24    | 24.04593 |
| 1 2     | 2 1      | 1 0   | 0 0   | 95       | 24    | 22.23439 |
| 0 1     | 0 1      | -2    | 2     | 1 0      | 95    | 24 21.89087 |
| 2 3     | 0 1      | 0 1   | 1 0   | 95       | 24    | 40.03725 |
| 1 2     | 0 2      | 0 1   | 1 0   | 95       | 48    | 42.02099 |
| 2 2     | 0 1      | 1 0   | 1 0   | 95       | 24    | 44.62797 |
| 2 2     | 1 2      | 1 1   | 1 0   | 95       | 48    | 46.68387 |
| 2 3     | 1 2      | 1 2   | 1 0   | 95       | 24    | 53.90720 |
| 2 3     | 2 3      | 1 0   | 0 0   | 99       | 12    | 11.78025 |
| 2 3     | 1 0      | 1 0   | 0 0   | 99       | 12    | 23.31658 |
| 1 3     | 1 0      | 1 0   | 0 0   | 99       | 12    | 27.16223 |
| 2 2     | 1 2      | 0 1   | 1 0   | 99       | 48    | 33.71138 |
| 1 1     | 1 1      | 2 0   | 1 0   | 99       | 24    | 35.61815 |
| 1 1     | -1       | 2 0   | 1 0   | 99       | 24    | 45.51314 |
| 2 2     | 1 1      | 1 0   | 1 0   | 99       | 24    | 45.51314 |
| 1 2     | 0 2      | 1 1   | 1 0   | 99       | 48    | 45.93451 |
| 0 1     | 0 1      | 3 -1  | 1 0   | 99       | 24    | 60.51462 |
| 1 3     | 1 1      | 1 1   | 1 0   | 99       | 24    | 60.51462 |
### Table 10: Representatives for coincidence rotations with even $\Sigma < 100$ for the cubic module $\mathbb{Z}[^\tau]$. IV.

Notation as in Tabl. 7.

| $\kappa$ | $\lambda$ | $\mu$ | $\nu$ | $\Sigma$ | orbit | angle   |
|--------|----------|-------|-------|---------|-------|---------|
| 1      | 1        | 0     | 0     | 4       | 8     | 44.7752 |
| 2      | 1        | 1     | 0     | 0       | 16    | 8       | 31.04498 |
| 23     | 1        | 1     | 0     | 0       | 16    | 24      | 44.7752 |
| 13     | 0        | 1     | 0     | 0       | 20    | 24      | 36.00000 |
| 01     | 3        | -1    | 1     | 0       | 0     | 20      | 51.08879 |
| 23     | 1        | 3     | 0     | 1       | 0     | 24      | 15.52248 |
| 03     | 1        | 1     | 0     | 0       | 36    | 24      | 38.99632 |
| 11     | 0        | 1     | -2    | 2       | 1     | 36      | 41.81032 |
| 35     | 0        | 1     | 0     | 0       | 44    | 24      | 19.46549 |
| 33     | 0        | 1     | 0     | 0       | 44    | 24      | 27.22766 |
| 35     | 2        | 3     | 0     | 0       | 44    | 24      | 27.95126 |
| 13     | 2        | 1     | 0     | 0       | 44    | 24      | 31.21215 |
| 31     | 0        | 1     | 0     | 0       | 44    | 24      | 44.77236 |
| -33    | 0        | 1     | 0     | 0       | 44    | 24      | 44.87530 |
| -13    | 0        | 1     | 0     | 0       | 44    | 24      | 51.60709 |
| -12    | 0        | 1     | 3     | -1      | 0     | 44      | 56.01215 |
| 33     | 2        | 3     | 1     | 0       | 0     | 64      | 13.43254 |
| -23    | 1        | 1     | 0     | 0       | 64    | 8       | 29.36458 |
| 11     | -12      | 0     | 1     | -2      | 2     | 64      | 31.04498 |
| 21     | 3        | -1    | 1     | 0       | 64    | 24      | 50.48567 |
| 13     | 2        | 0     | 1     | 1       | 0     | 64      | 50.48567 |
| 13     | 0        | 3     | 1     | 0       | 0     | 76      | 8       | 18.37603 |
| 47     | 1        | 1     | 0     | 0       | 76    | 24      | 20.72495 |
| 31     | 2        | 1     | 0     | 0       | 76    | 8       | 23.75257 |
| 45     | 1        | 1     | 0     | 0       | 76    | 24      | 26.10149 |
| 11     | 4        | -1    | 1     | 0       | 0     | 76      | 24      | 32.00114 |
| 31     | 0        | 2     | 1     | 0       | 1     | 76      | 48      | 42.11836 |
| 21     | 1        | 1     | -2    | 2       | 1     | 76      | 48      | 43.06977 |
| 41     | 3        | -1    | 1     | 0       | 0     | 76      | 8       | 49.42100 |
| 12     | 0        | 1     | 3     | -1      | 0     | 76      | 24      | 53.34293 |
| -13    | 2        | 0     | 1     | 1       | 0     | 76      | 48      | 53.34293 |
| 30     | 0        | 1     | 3     | -1      | 0     | 76      | 8       | 54.79753 |
| 24     | 2        | 1     | 1     | 1       | 0     | 76      | 48      | 54.79753 |
| 12     | 1        | 1     | 0     | -2      | 2     | 80      | 48      | 51.08879 |
| -12    | 0        | 1     | 3     | -1      | -2    | 2     | 80      | 26.56504 |
| -13    | 2        | 1     | 1     | 0       | 0     | 80      | 24      | 21.72421 |
| 33     | 2        | 1     | 0     | 0       | 80    | 24      | 42.54018 |
| -23    | 3        | -1    | 1     | 0       | 0     | 80      | 24      | 51.45900 |
| 3-1    | 6        | -3    | 1     | 0       | 0     | 80      | 24      | 59.12300 |
| 47     | 3        | 5     | 1     | 0       | 0     | 100     | 24      | 19.19113 |
| 25     | 1        | 3     | 1     | 0       | 0     | 100     | 24      | 31.29453 |
| 30     | 1        | 1     | 0     | -2      | 2     | 100     | 24      | 37.39882 |
| 35     | 0        | 2     | 0     | 1       | 0     | 100     | 48      | 37.39882 |
| 25     | 2        | 2     | 1     | 1       | 0     | 100     | 48      | 44.47752 |
| 04     | 1        | 1     | 0     | 1       | 0     | 100     | 48      | 53.13010 |
| 5-2    | 0        | 1     | 3     | -1      | 0     | 100     | 24      | 60.18418 |