AREA LAW AND CONTINUUM LIMIT IN
“INDUCED QCD” \[1\]

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Abstract

We investigate a class of operators with non-vanishing averages in a D-dimensional matrix model recently proposed by Kazakov and Migdal. Among the operators considered are “filled Wilson loops” which are the most reasonable counterparts of Wilson loops in the conventional Wilson formulation of lattice QCD. The averages of interest are represented as partition functions of certain 2-dimensional statistical systems with nearest neighbor interactions. The “string tension” \( \alpha' \), which is the exponent in the area law for the “filled Wilson loop” is equal to the free energy density of the corresponding statistical system. The continuum limit of the Kazakov–Migdal model corresponds to the critical point of this statistical system. We argue that in the large \( N \) limit this critical point occurs at zero temperature. In this case we express \( \alpha' \) in terms of the distribution density of eigenvalues of the matrix-valued master field. We show that the properties of the continuum limit and the description of how this limit is approached is very unusual and differs drastically from what occurs in both the Wilson theory \( (S \propto \langle \text{Tr} \prod U + c.c. \rangle) \) and in the “adjoint” theory \( (S \propto |\text{Tr} \prod U|^2) \). Instead, the continuum limit of the model appears to be intriguingly similar to a \( c > 1 \) string theory.

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1 Introduction

In a recent paper, V.Kazakov and A.Migdal proposed a new lattice gauge model where the Yang-Mills interactions are induced by minimal coupling to a scalar field which transforms in the adjoint representation of the gauge group. Their theory (to be referred below as the Kazakov-Migdal model (KMM)) resembles a D-dimensional matrix model with partition function of the form

\[ Z_{KMM} = \int [dU]d\Phi \exp\{-\sum_x \text{tr} V(\Phi(x)) + \sum_{<x,y>} \text{tr} \Phi(x)U(x,y)\Phi(y)U^\dagger(x,y)\} \] (1)

Here \( \Phi \) and \( U \) are \( N \times N \) Hermitean and unitary matrices defined on the sites and the links of a \( D \)-dimensional rectangular lattice respectively. Although the potential \( V(\Phi) \) is quite general, for the purpose of most of our work we can use the quadratic potential:

\[ V(\Phi) = m^2 \Phi^2 \] (2)

Throughout this paper we use square brackets to denote non-trivial measures of integration. Thus, for example, \( dX = \prod_{i,j} dX_{ij} \), but the Haar measure \([dU]\) is not simply \( \prod_{i,j} dU_{ij} \), there is also a non-trivial factor, see \( \{\} \). In fact \([dU]\) will always denote the Haar measure over unitary matrices. We shall denote normalized averages in the theory \( \{\} \) by \( \langle \rangle \), while unnormalized integrals over unitary matrices with Haar measure \([dU]\) will be denoted by \( < > \). Unitary matrices \( U \) are taken to be elements of the group \( SU(N) \). If we were to allow \( U \) to be in \( U(N) \) then the \( Z_N \)-gauge invariance of the KMM to be discussed throughout the text would extended to a \( U(1) \)-gauge invariance. This does not influence our results, since the model \( \{\} \) is insensitive to the \( U(1) \) factor in \( U \).

Hermitian matrices such as \( \Phi \) and \( \Psi \) are not assumed to be traceless. However if the potential is quadratic the trace part of \( \Phi(x) \) (\( \Phi(x) = \phi_t(x) \cdot I + \text{traceless part} \)) is easily separated from the traceless degrees of freedom in \( \{\} \) giving rise to an extra factor

\[ \prod_x \int d\phi_t(x) e^{-N\left(-m^2 \sum_x \phi_t^2(x) + \sum_{<x,y>} \phi_t(x)\phi_t(y)\right)} \]

The critical value of the bare mass for this abelian subsector of the KMM is obviously \( m_{\text{crit}}^2 = D \). This is the the same value of \( m_{\text{crit}}^2 \) as is obtained for the non-abelian components of \( \Phi \) in the quasi-classical approximation.
The effects of non-quadratic terms in the potential can be encoded in quantities such as the distribution function, $\rho(\phi)$, of the eigenvalues of $\Phi(x)$.

For $D = 4$ this model is a lattice approximation to a Yang-Mills theory alternate to the conventional Wilson lattice gauge theory. Recall that the partition function of the Wilson theory is given by

$$ Z = \int [dU] \exp \left( \sum_\Box \frac{N}{g^2} W(\Box) + c.c. \right), \quad (3) $$

where $W(\Box)$ is the elementary Wilson loop, i.e. the trace of the product of link operators, around an elementary plaquette which we denote by $\Box$.

The main advantage of the KMM is the existence of two reformulations of it, which, while being equivalent, look very different and emphasize different features of the theory.

One reformulation arises from integration over the $\Phi$ variables in (1) and can be written as

$$ Z_{KMM} \propto \int [dU] e^{-S_{eff}[U]} \propto \int [dU] \exp \left( \frac{1}{2} \sum_\Gamma \frac{1}{l(\Gamma)m^2(\Gamma)} |W(\Gamma)|^2 \right), \quad (4) $$

where the sum in the action is over all oriented contours $\Gamma$ on the lattice and

$$ W(\Gamma) \equiv \text{tr} \left( \prod_{<x,y> \in \Gamma} U(x, y) \right); \quad (5) $$

with

$$ W(-\Gamma) = \overline{W(\Gamma)}; \quad U(y, x) = U^\dagger(x, y). \quad (6) $$

In contrast to the Wilson theory, the effective action depends on the modulus of $W(\Gamma)$, rather than on its real part. It also depends on contours of all sizes rather than on just the elementary plaquette.

Another reformulation of the KMM arises by integrating first over $U$ rather than over $\Phi$. To do this, it is convenient also to integrate out the angular components of $\Phi$ leaving an integral over its eigenvalues $\phi_1(x),...,\phi_N(x)$. The partition function is then given by

$$ Z_{KMM} \propto \int \prod_x \prod_i d\phi_i(x) \ e^{-V(\phi_i(x))} \Delta^2(\phi(x)) \prod_{<x,y>} I[\phi(x), \phi(y)] \quad (7) $$
where

\[ \Delta(\phi) \equiv \prod_{i<j}^N (\phi_i - \phi_j), \]

and

\[ I[\phi, \psi] \equiv \int [dU] \exp \left( tr \Phi U \Psi U^\dagger \right) \quad (8) \]

The first representation (4) has been used to give naive arguments that the KMM (1) is related to QCD (or, more precisely to quantum gluodynamics) if \( D=4 \). This idea, which we shall further refer to as the “naive continuum limit”, is that the action in (4) clearly has an absolute minimum at \( U = I \). Assuming that in the continuum limit the dominant contribution to the partition function comes from the vicinity of this minimum the effective action can be computed by writing \( U = \exp(i a A) \) (where \( a \) is the lattice spacing). The result is derived in Ref. [1]:

\[ S \sim \text{const} + \frac{1}{g^2} \int d^4 x F_{\mu \nu}^2 + ... \quad (9) \]

with

\[ \frac{1}{g^2} \sim N \int \frac{d^D p}{p^4} \left. \right|_{D=4} N \log \frac{1}{a(m - m_{\text{crit}})}. \quad (10) \]

We shall argue in this paper (see especially Section 6) that this argument is too naive and that the approach to the continuum limit is much more complicated.

The second representation (7-8) can be used for a much more reliable investigation of the KMM. It may even be possible to solve the KMM exactly for any \( N \) (although most of the progress has been made for large \( N \)). The reason for these hopes is that the integrals such as (8) have a pure algebraic interpretation so that standard methods such as Fourier analysis on co-adjoint orbits or the Duistermaat-Heckmann (DH) integration formalism can be applied to them.

As argued in [1] the transformation from (4) to (6) is highly non-trivial from a conceptual point of view and is more or less equivalent to a summation of all planar diagrams in the Yang-Mills theory.
In this paper we report some preliminary considerations of correlation functions in the KMM which should correspond to the physical observables of “induced QCD”. We shall concentrate our analysis on finite values of $N$. We shall then present arguments that the features of the model for finite $N$ are sufficiently rich and interesting that they may have implications for the case of infinite $N$.

We shall see that the KMM differs from the conventional Wilson theory in many respects but in particular in the properties of the continuum limit. We show in Section 6 that the probability distribution for the functional integral near $U = I$ is much less sharp in the KMM than it is in the Wilson theory. Moreover the continuum limit in the KMM appears sensitive to the quantum measure in functional integral, not only to the form of the action. This has a crucial effect on the normalization of observable operators and probably on the other features of continuum limit as well.

Another purpose of this paper is to develop a formalism for evaluation of the expectation values of operator observables in the KMM. We shall see that the most interesting observables in the KMM are associated with 2-dimensional sublattices $S$ of the original $D$-dimensional lattice. Vacuum expectation values of these quantities can be represented as partition functions of 2$d$ statistical systems defined on $S$. We shall see that the Boltzmann factors on the links of this system are fluctuating variables so that generically they behave in a manner similar to spin glasses. In the zero-temperature limit, however, they turn into ordered magnetic-type systems similar to $N$-state Potts models. We shall argue that in the limit of large $N$ the zero-temperature limit of the 2$d$ system can be identified with the continuum limit of the KMM. Furthermore we can show using these ideas that the area law for filled Wilson Loops holds in this limit. The string tension $\alpha'$ can be expressed in terms of the density $\rho(\phi)$ of eigenvalues of the master field $\Phi$. (See Sections 5 and 7 for details.)

Note that there is an “adjoint” model “in between” the KMM and Wilson theory. Consider a model with a partition function

$$Z_A = \int [dU] \exp \sum \frac{1}{g^2} |trW(\square)|^2$$

(11)

Many of our comments in Sections 2 and 4 about the choice and the properties of observables in the KMM are a result of $Z_N$ gauge invariance of the theory. They are thus equally true for Eq. (11). The unusual behaviour of the
continuum limit which will be discussed in Sections 6 and 7 is peculiar to
the KMM and is drastically different from that in the theories (3). The
reason for this difference is that in (4) there is a sum over all contours (not
only over elementary plaquettes). In any case one must keep firmly in mind
that the KMM is significantly different from both the Wilson theory from
the model of Eq. (11).

2 Correlators and $Z_N$-invariance

Any lattice gauge theory possesses a gauge invariance with respect to trans-
formations specified by $P(x) \in U(N)$ at every site $x$ and which act on the $\Phi$
and $U$-fields by

$$\Phi(x) \rightarrow P(x)\Phi(x)P^\dagger(x), \quad (12)$$

$$U(x,y) \rightarrow P(x)U(x,y)P^\dagger(y). \quad (13)$$

The gauge invariant observables in lattice gluodynamics are made from traces
of products of $U$-matrices along closed oriented contours $\Gamma$ on the lattice, i.e.
conventional Wilson loops (5,6).

The first important property of the KMM (1.1), which makes it different
from Wilson’s theory, is that the space of gauge-invariant operators must be
further reduced (3). The reason for this is the presence of an additional $Z(N)$
gauge symmetry: the freedom of independent discrete transformations

$$U(x,y) \rightarrow \omega(x,y)U(x,y); \quad (14)$$

$$\omega(x,y)^N = 1,$$

so that $\omega$ is any $N$-th root of unity. This additional gauge invariance excludes
the conventional Wilson loop $W(\Gamma)$ from the set of gauge invariant observ-
able (so that $\ll W(\Gamma) \gg \equiv 0$). In fact this symmetry excludes any product
of Wilson loops as an observable in the theory unless every link involved is
passed an even number of times, equally in one and in the other direction.
To be precise, the difference between the number of $U$’s and $U^\dagger$’s at every
link should be divisible by $N$. The set of gauge invariant observables in the KMM thus consists of the products

$$W(\Gamma_1, \ldots, \Gamma_k) \equiv \frac{1}{N} W(\Gamma_1) W(\Gamma_2) \cdots W(\Gamma_k),$$

(15)

provided that

$$\Gamma_1 + \ldots + \Gamma_k = 0 \text{ [mod} N\text{]}$$

(16)

(The contours are, of course, taken to be oriented).

The simplest examples of allowed operators are the “Adjoint Wilson Loop” (Fig.1)

$$W(\Gamma, -\Gamma) = \frac{1}{N} |W(\Gamma)|^2,$$

(17)

the “Baryon Loop”

$$W_{\text{baryon}}(\Gamma) = \frac{1}{N} W(N\Gamma)$$

(18)

(in which contour wraps $N$ times around $\Gamma$) and the “Filled Wilson Loop” (FWL) \cite{3,4} (Fig.2)

$$W(S) \equiv \frac{1}{N} W(\Gamma, -\Gamma_\alpha \mid \alpha \in \text{plaquettes of } S) =$$

$$\frac{1}{N} W(\Gamma) \prod_{\text{plaquettes of } S} W(\square).$$

(19)

where $S$ is 2-dimensional surface on the lattice with a given boundary $\Gamma = \partial S$. $W$ involves the product of small Wilson loops over all the plaquettes in $S$ as well as a big Wilson loop for $\Gamma$.

If this model has anything to do with QCD it should posses an operator like the Wilson loop which obeys an area law in the confining phase. (This property need not be related to existence of quark degrees of freedom). Neither the adjoint Wilson Loop nor the Baryon Loop will suffice since they obey a perimeter rather than an area law even in the standard Wilson theory for finite $N$. The FWL (or \cite{23}, to be exact) seems to be the only reasonable
counterpart of the conventional Wilson loop in the framework of the KMM. There are several arguments to support this suggestion.

First of all FWL is a gauge-invariant operator (unlike \( W(\Gamma) \) which is not invariant under local \( Z(N) \)). It can thus have a non-vanishing average value. Secondly, in the naive continuum limit the FWL reduces to the conventional Wilson Loop

\[
W(S) \sim \text{tr} P \exp i \oint_{\Gamma} A_\mu(x) dx^\mu.
\]  
(20)

(In particular, it depends only on \( \Gamma = \partial S \).) Indeed, if \( U = \exp(iaA) \) then the plaquette loop

\[
W(\Box) = \text{tr}(I + ia^2 F - \frac{1}{2} a^4 F^2 + ...) =
\]

\[
= N - \frac{1}{2} a^4 F^2 + ..., \quad (21)
\]

where \( F \) is the field strength. The last term vanishes in the naive continuum limit and thus \( W(\Box) \) tends to a field independent constant. (Note that we have used the fact that \( tr F = 0 \) in this derivation). Therefore in this naive limit the plaquette loops do not contribute at all. Their only role is to make the entire quantity gauge invariant. (Note that along with the ansatz \( U = \exp(iaA) \) one should consider a whole class of ansatze \( U = \omega(x,y) \exp(iaA) \) with an arbitrary \( \omega(x,y) \in Z(N) \) at each link. This would lead to the cancellation of all contributions to \( \ll W(\Gamma) \gg \), but not to the average of the Filled Wilson Loop \( \ll W(S) \gg \).) We shall further justify the use of the FWL by arguing later in the paper that the expectation value of the FWL obeys the area law,

\[
\ll W(S) \gg \sim \exp (-\alpha' A(S)), \quad (22)
\]

where \( A(S) \) stands for the area of \( S \) (i.e. the number of elementary plaquettes in \( S \)).

Before proceeding we must discuss the normalization of the FWL. In the Wilson theory every loop \( W(\Gamma) \) is usually accompanied by a factor of \( 1/N \). According to this rule the FWL operator should contain an additional factor

\[
7
\]
$N^{-A(S)}$ in the definition (19). This would lead to cancellation of all the factors of $N$ in the naive continuum limit leaving precisely a conventional Wilson Loop with its correct normalization factor $1/N$. Notice however that we have used a different normalization factor in (19) — only a single factor of $1/N$. The extra factor of $1/N^A$ makes a very big difference in that it can lead to a spurious area law behaviour for the FWL. We shall argue below that this $1/N$ normalization is indeed the proper normalization. We shall see that the naive continuum limit is unrelated to the true continuum limit of the theory and that only this $1/N$ normalization leads to finite averages in the true continuum limit.

An additional complication which is introduced by the FWL and is unavoidable in the KMM is the dependence of its expectation value on the surface $S$ rather than just on its boundary $\Gamma$. It is thus nearly certain that the correct “physical” operator (i.e. the operator which is to be compared with the Wilson loop of continuum QCD) should contain an additional sum over all surfaces $S$ with the same boundary $\Gamma$:

$$\hat{\mathcal{W}}(\Gamma) \equiv \int_{S,\partial S = \Gamma} [\mathcal{D}S] \mathcal{W}(S)$$

(23)

with some (string-theory–like) measure $[\mathcal{D}S]$. This is important, for example, for the study of correlators of FWL operators (see Section 6). In this paper however we restrict ourselves to the problem of the evaluation of $\langle \mathcal{W}(S) \rangle$ for a given surface $S$. (In particular in order to derive the area law in the KMM it seems enough to take for $S$ the surface of minimal area with the given boundary $\Gamma$.)

### 3 Integrals over unitary matrices

As discussed in the previous section we are interested in evaluating the averages of operators such as filled Wilson loops. First note that averages of any operators which are invariant under the $Z(N)$ gauge transformation (13) are consistent with the integration over angular variables which is implicit in the transformation of (1) into (7). Thus the same operators should be averaged in the “Matrix Model” approach (7) as are averaged in the original formulation (4) of the KMM. To evaluate these averages one must first study
integrals over unitary matrices $U$ on a single link which are of the form

$$I_{pq}[\phi, \psi]_{i_1j_1...i_pj_p} \equiv <U_{i_1j_1}...U_{i_pj_p}U^\dagger_{k_1l_1}...U^\dagger_{k_ql_q}>$$

$$= \int [dU] \exp \sum_{i,j} \phi_i \psi_j |U_{ij}|^2 \; U_{i_1j_1}...U_{i_pj_p}U^\dagger_{k_1l_1}...U^\dagger_{k_ql_q}. \quad (24)$$

Due to the $Z(N)$ invariance under $U \rightarrow \omega U$, $U^\dagger \rightarrow \omega^{-1} U^\dagger$ under the integral differ by an integral multiple of $N$ i.e. $p - q = 0 \ [\text{mod} \ N]$. (This is the case for $U \in SU(N)$. If $U \in U(N)$ the condition is that $p = q$).

The integrals (24) are well known for $p=q=0$. In fact there is an explicit formula

$$I_{00}[\phi, \psi] = \int [dU] \exp(\sum_{i,j} \phi_i \psi_j |U_{ij}|^2) = \text{const} \; \det \; e^{\phi_i \psi_j \Delta(\phi) \Delta(\psi)}. \quad (25)$$

The standard references for this result are [5]. There are, however, two more contemporary derivations which deserve mentioning.

One such derivation starts from an interpretation of $I_{00}$ as an integral over the co-adjoint orbit of $\Psi$:

$$I_{00}[\phi, \psi] = \frac{1}{\Delta(\psi)} \int [dX] e^{tr \Phi X} \quad (26)$$

where $X = U\Psi U^\dagger$ is a generic element of the orbit of $\Psi$ under the action of $G/H$, where $H$ is the Cartan subgroup of $G$ and $[dX]$ is the standard symplectic measure on this orbit. (Note that the diagonal components of $U$ do not act on a diagonal matrix $\Psi$.) This integral was explicitly evaluated in [6] with the use of the Gelfand-Tseytlin parametrization of the orbit.

Another modern derivation of $I_{00}$ is based on application of the Duistermaat–Heckmann (DH) theorem [7]. The idea of this theorem is that (under certain very restrictive conditions) an integral can be substituted by a sum of quasi–classical contributions evaluated in the vicinities of all the extrema (not only the minima) of the action. Applicability of the DH theorem to $I_{00}$ relies on the fact that $I_{00}$ is essentially the integral over an orbit which is a symplectic and symmetric manifold.

To see how the DH approach leads to the formula (25) note that each extrema of the action $\sum_{i,j} \phi_i \psi_j |U_{ij}|^2$ as a function of $U$ is given by a product
of any diagonal matrix $U_d$ and any matrix of permutations $P$ i.e. $U = U_d \times P$. Diagonal matrices do not contribute to the action, and thus the sum in the DH formula goes over all permutations of $N$ natural numbers $1...N$. In order to evaluate the contribution of any given extremum one needs to find the determinant of quadratic fluctuations about this extremum. To do this we write $U = P e^{iH}$ where $H$ is a Hermitean matrix. The action is then expanded to quadratic order in $H$:

$$
\sum_{i,j} \phi_i \psi_j |U_{ij}|^2 \rightarrow
$$

$$
\sum_{i} \phi_i \psi_{P(i)} + (1/2) \sum_{i,j} |H_{ij}|^2 (\phi_i - \phi_j)(\psi_{P(i)} - \psi_{P(j)}) + \ldots
$$

(27)

Then DH theorem implies that all the higher order terms can be ignored. The result of the Gaussian integration over $H$ is

$$
\int [dU] \exp \left\{ \sum_{i,j} \phi_i \psi_j |U_{ij}|^2 \right\}
$$

$$
\propto \sum_P e^{\sum_i \phi_i \psi_{P(i)}} \int_{-\infty}^{+\infty} \prod_{i,j} dH_{ij} e^{(1/2) \sum_{i,j} |H_{ij}|^2 (\phi_i - \phi_j)(\psi_{P(i)} - \psi_{P(j)})}
$$

(28)

$$
\propto \sum_P (-)^P e^{\sum_i \phi_i \psi_{P(i)}} \Delta(\phi) \Delta(\psi) \sim \text{const} \frac{\det_{ij} e^{\phi_i \psi_j}}{\Delta(\phi) \Delta(\psi)}
$$

where the sum is over permutations $P$ and the factor $(-)^P$ comes from the fact that

$$
\prod_{i<j} (\psi_{P(i)} - \psi_{P(j)}) = (-)^P \prod_{i<j} (\psi_i - \psi_j) = (-)^P \Delta(\psi)
$$

(29)

This completes the derivation of the result (25) using the DH theorem.

Unfortunately the same methods are not directly applicable to evaluation of the integrals $I_{pp}$ with $p \neq 0$. In fact the generating functional for all non-vanishing integrals of this type is given by

$$
I[A] \equiv \int [dU] \exp \sum_{i,j} A_{ij} |U_{ij}|^2.
$$

(30)
Unfortunately the DH formula is not directly applicable to this case. If it were applicable then we would find that:

$$I[A] \propto \sum_{P} \exp \sum_{k} A_{kP(k)} \prod_{i<j} [A_{iP(i)} + A_{jP(j)} - A_{iP(j)} - A_{jP(i)}] (1 + O(\frac{1}{A})).$$

(31)

with order $1/A$ corrections vanishing. (Note that the classical equations of motion are $\sum_{j} U_{ij} (A_{ji} - A_{jk}) U_{jk}^\dagger = 0$. Any permutation matrix $U = P$ is a solution to these equations.) Since the DH theorem is not directly applicable in this situation the $O(\frac{1}{A})$ corrections do not vanish in general. There are several interesting exceptions in which case they do vanish. The first case is if $A_{ij}$ is of rank 1 so that $A_{ij} = \phi_i \psi_j$ (we then recognize Eq.(28) in Eq. (31)). Another interesting case is when $N = 2$. For the case of $SU(2)$ one can check explicitly that Eq. (31) is exact so that

$$I_{SU(2)}[A] = e^{A_{11} + A_{22}} - e^{A_{12} + A_{21}}.$$

(32)

(Note that for $N = 2$ the denominator is independent of $P$). Other exceptional cases involve certain $U$-integrals arising in the study of correlators in multi-matrix models like

$$\int \int d\Phi d\Psi e^{\text{tr} \Phi \Psi} e^{\alpha \Phi + \beta \Psi}$$

(33)

The fact that the DH theorem is not directly applicable in the general case (30) is rather obvious. The action in the integral (30) is not defined on any co-adjoint orbit but on the entire group manifold which is not even symplectic. Keeping this in mind it is not too difficult to work out the corrected formula (see [8]) though we shall not use it in this paper. In what follows we shall make use of the approximate formula (31) and of the explicit expression (32) for $SU(2)$.

For our purpose which is the evaluation of the expectation value of the FWL we are mostly interested in the correlators $I_{pp}$ with $p = 1$:

$$I_{ij}^{kl} = <U_{ij} U^{\dagger kl} > \equiv \int [dU] e^{\sum_{i,j} \phi_i \psi_j [U_{ij}]^2} U_{ij} U^{\dagger kl}.$$  

(34)

It is useful to use a somewhat more explicit representation of the Haar mea-
\[ [dU] = \prod_{i,j} dU_{ij} \delta(\sum_k U_{ik} \overline{U}_{jk} - \delta_{ij}). \]  

(35)

Making use of the invariance of the measure under the transformations

\[ U_{ij} \rightarrow \omega^i U_{ij}, \quad U^{\dagger kl} \rightarrow U^{\dagger kl} \omega^{-l} \]  

(36)

and

\[ U_{ij} \rightarrow U_{ij} \omega^j, \quad U^{\dagger kl} \rightarrow \omega^{-k} U^{\dagger kl} \]  

(37)

This representation of the Haar measure suggests a possible approach to the evaluation of the integrals (30). The idea is to introduce Lagrange multipliers \( \lambda_{ij} = \lambda_{ji} \) for every one of the \( \delta \)-functions in (35). Then Gaussian integral over unconstrained \( U \)-variables is easily evaluated with the result

\[ I[A] \propto \int \frac{\prod_{i,j} d\lambda_{ij} e^{\sum_j \lambda_{jj}}}{\prod_{k=1}^N \det(\lambda_{ij} - A_{ik} \delta_{ij})}. \]

For \( N = 2 \) the integral over \( d^2 \lambda_{12} \) gives

\[ \int d\lambda_{11} d\lambda_{22} e^{\lambda_{11} + \lambda_{22}} / \lambda_{11}(A_{21} - A_{22}) + \lambda_{22}(A_{11} - A_{12}) + (A_{12} A_{22} - A_{11} A_{21}) \]

\[ [\log \frac{\lambda_{11} - A_{11}}{\lambda_{11} - A_{12}} + \log \frac{\lambda_{22} - A_{21}}{\lambda_{22} - A_{22}}] \]

Consider the first logarithmic term. The integral over \( \lambda_{22} \) is easily evaluated yielding

\[ I[A] \propto \frac{1}{A_{11} - A_{12}} \exp \frac{A_{11} A_{21} - A_{12} A_{22}}{A_{11} - A_{12}} \times \]

\[ \int d\lambda_{11} \{ \exp \lambda_{11} \frac{A_{11} + A_{22} - A_{12} - A_{21}}{A_{11} - A_{12}} \} \log \frac{\lambda_{11} - A_{11}}{\lambda_{11} - A_{12}} \]

This integral can be evaluated with the help of the formula

\[ \int d\lambda e^{\lambda} \log \frac{\lambda - \alpha}{\lambda - \beta} = \frac{1}{e} \int d\lambda e^{\lambda} \left( \frac{1}{\lambda - \alpha} - \frac{1}{\lambda - \beta} \right) = \frac{1}{e} (e^{\alpha} - e^{\beta}) \]

and is equal to (31). The contribution of the second logarithmic term is is the same as the first.
(with $\omega^N = 1$), one can concludes that $I_{ij}^{lk} = 0$ unless $i = l$ and $k = j$, so that

$$< U_{ij} U^\dagger_{kl} > \sim \delta_i^l \delta_j^k. \quad (38)$$

We now introduce the quantities

$$C_{ij} \equiv \frac{< |U_{ij}|^2 >}{< 1 >}, \quad i, j = 1...N, \quad (39)$$

which are the only non-vanishing quadratic correlators. These clearly satisfy the relations

$$C_{ij} \geq 0; \quad \sum_j C_{ij} = 1 = \sum_j C_{ji} \text{ for } \forall i; \quad C_{ij}[^φ, ψ] = C_{ji}[ψ, φ], \quad (40)$$

It thus follows that the $C_{ij}$ can be interpreted as conditional probabilities. These quantities will be used in the next section to evaluated the average of the FWL.

4 Loop averages as partition functions of statistical systems

Originally the average $< W(Γ_1...Γ_k) >$ for $Γ_1 + ... + Γ_k = 0$ can be represented as a double-loop diagram (“fat graph”). Because of the property (38) all double lines can be exchanged for single lines (Fig.3) since

$$\frac{< U_{ij} U^\dagger_{kl} >}{< 1 >} = \delta_i^l \delta_j^k C_{ij}, \quad (41)$$

Thus at every site there is only one independent value $i$ of some effective “spin”. These single lines are oriented because, in general, $C_{ij} \neq C_{ji}$. Let us denote the graph consisting of these single lines by $\tilde{Γ}$. For FWL $\tilde{Γ} = S$. Then

$$< W(Γ_1...Γ_k) > =$$

$$\int dφ \exp[-V(φ)] < 1 > \sum_{i(x)} \{ \prod_{x, y \in Γ} C_{i(x)j(y)}[φ(x), φ(y)] \} \int dφ \exp[-V(φ)] < 1 > \quad (42)$$
where the sum is over distributions \( i(x) \) of integers \( i \) at sites \( x \). The quantities on the r.h.s. are already very similar to the partition function of some statistical system with only nearest neighbor interactions. In the case of the FWL operators this system is essentially 2-dimensional (for adjoint loop it would be 1-dimensional). It is unfortunately not a very simple theory, since the statistical weights at any link depend not only on the values of the “spin” at two adjacent sites, but also on the orientation of the link \( (C_{ij} \neq C_{ji}) \) and on the field \( \phi(x) \). We can thus think of the statistical weights \( C_{ij} \) as random variables with some non-trivial distribution which is encoded in terms of some “hidden variables” \( \phi(x) \) with gaussian distribution when \( V \) is quadratic). Because of the presence of these random variables the system in question looks more like a “spin glass” than an ordinary spin system.  

A drastic simplification arises in the “mean- (or master-) field” approximation for the integral over \( \phi(x) \) in which case \( \phi(x) \) is frozen and does not depend on \( x \), i.e. \( \phi(x) = \Phi \). The \( C_{ij} \) are thus also fixed and, moreover, they are symmetric i.e. \( C_{ij} = C_{ij} [\Phi, \Phi] = C_{ji} [\Phi, \Phi] = C_{ji} \). \( W(\tilde{\Gamma}) \) is thus the partition function of a magnetic-type statistical system:

\[
\langle W(\tilde{\Gamma}) \rangle \sim \sum_{i(x)} \{ \prod_{<x,y> \in \tilde{\Gamma}} C_{i(x)j(y)} \} \quad (43)
\]

The fascinating thing about this relationship is that the area law for the FWL is just equivalent to the statement that the free energy of the statistical system is proportional to its area. This will be true provided that the free energy of this 2-dimensional system is nonzero. This seems to be a rather obvious property for a system with local (nearest neighbor) interactions. There is however one delicate point which prevents us from using this result immediately. The problem is that the “Boltzmann factors” have the rather

\footnote{Because of the presence of the factor \(<1> \) in (41) which is \( \phi \) dependent, the shape of the random distribution of the \( C_{ij} \) depends on the actual choice of contour \( \Gamma \), i.e. on the form of the 2-dimensional sample \( S = \tilde{\Gamma} \). This is a kind of a boundary effect for the statistical system and although it is not very important for the derivation of results such as the area law it is important for the study of correlators of (filled) Wilson loops. 

Note also that the r.h.s. of Eq.(42) is a ratio of averages over \( \phi \)-fields. Formally this is different from the situation in conventional spin glass systems where any back reaction of the system on the shape of the random “noise” distribution is neglected and thus the average of the ratio rather than the ratio of the average is the quantity of interest. (In practice the difference is not so drastic, since the distribution of the logarithms of Boltzmann weights \( \log C_{ij} \) in (42) is far from being Gaussian.)}
peculiar normalization \( \text{(40)} \). We now discuss the implications of this fact using the simplest example of the Ising model which arises for \( N = 2 \).

5 The case of \( N = 2 \): The Ising model

When \( N = 2 \) the diagonalized matrices \( \phi \) and \( \psi \) can be written as

\[
\phi = \left( \begin{array}{cc} \hat{\phi} & 0 \\ 0 & -\hat{\phi} \end{array} \right), \quad \psi = \left( \begin{array}{cc} \hat{\psi} & 0 \\ 0 & -\hat{\psi} \end{array} \right)
\]

\( \text{(44)} \)

It is now easy to express the \( C_{ij} \) explicitly in terms of the variable \( \gamma = 2 \hat{\phi} \hat{\psi} \).

One finds that

\[
C_{11} = C_{22} = \frac{\int_0^1 (1 - x)e^{-2\gamma x} \, dx}{\int_0^1 e^{-2\gamma x} \, dx} = \frac{1 - 1/2\gamma + e^{-2\gamma}/2\gamma}{1 - e^{-2\gamma}},
\]

\[
C_{12} = C_{21} = \frac{\int_0^1 xe^{-2\gamma x} \, dx}{\int_0^1 e^{-2\gamma x} \, dx} = \frac{1/2\gamma - e^{-2\gamma}(1 + 1/2\gamma)}{1 - e^{-2\gamma}} \quad \text{(45)}
\]

The case \( N=2 \) is mostly distinguished by the fact, that the two diagonal element \( C_{ii} \) coincide i.e. \( C_{11} = C_{22} \) (and, as a result, \( C_{12} = C_{21} \)) for any pair \( (\phi, \psi) \). This allows one to represent \( C_{ij} \) as

\[
C_{11} = C_{22} = \frac{e^{J/T}}{e^{J/T} + e^{-J/T}}
\]

\[
C_{12} = C_{21} = \frac{e^{-J/T}}{e^{J/T} + e^{-J/T}} \quad \text{(46)}
\]

so that \( \ll \mathcal{W}(S) \gg \) is proportional to the partition function \( Z_{\text{Ising}} \) of a 2-dimensional Ising model defined on the surface \( S \). In the mean–field approximation when \( \hat{\phi}(x) \) is constant, \( J \) is the same for all sites and this is

---

\( ^6 \) These formulas can be easily deduced either by an explicit evaluation of the integrals over SU(2) = S\(^3\) or by an application of the DH result \( \text{(32)} \). The simplest way to proceed with an explicit derivation is to use the representation \( \text{(35)} \) for the measure \( dU \). One writes \( U = a_0 + i\vec{a} \cdot \vec{\sigma} \). The \( \delta \)-functions in \( \text{(35)} \) then imply that \( a_0 \) and \( \vec{a} \) are real (after a \( U(1) \) factor is extracted from the measure), and the measure is simply \( \int d^4a \delta(a_0^2 + \vec{a}^2 - 1) \). In Eq.(\ref{eq:45}) \( x \equiv a_1^2 + a_2^2 = |U_{12}|^2 \) and \( 1 - x = a_0^2 + a_3^2 = |U_{11}|^2 + |U_{22}|^2 \).
just a conventional Ising model in a finite area $A(S)$. More generally there is some random distribution of the values of $J$. This distribution will not, in general, be Gaussian unless $T$ and $1/\gamma$ are small. The region of small $T$ will be of great interest to us later in this paper. In what follows we restrict ourselves to the mean-field approximation (and, without loss of generality we set $J = 1$).

The average of the FWL can now be evaluated with the help of Eq. (42)

$$\langle W(S) \rangle = \frac{Z_{\text{Ising}}}{(2 \cosh (1/T))^2 A(S)}$$

The two dimensional Ising model is exactly soluble. We can thus substitute the well known formula for the free energy of the Ising model (see, for example, §141 of [9]) to find the string tension

$$\alpha'[T] = -\log <W>$$

$$- \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} d\omega_1 d\omega_2 \log \frac{(1 + x^2)^2 - 2x(1 - x^2)(\cos \omega_1 + \cos \omega_2)}{4}$$

where

$$x = \tanh (1/T)$$

The three most interesting temperatures in the Ising model are $T = \infty$, $T = T_{\text{crit}} = 2/\log(\sqrt{2} + 1)$ and $T = 0$. $\alpha'$ is continuously decreasing with decreasing temperature $T$. At $T = 0$ $\alpha' = 0$ and at $T = T_{\text{crit}}$ $d^2 \alpha'/dT^2$ has a (logarithmic) singularity. $T = \infty$ corresponds to the value $\hat{\Phi} = \sqrt{\gamma/2} = 0$ of the mean field, while $T = 0$ corresponds to $\hat{\Phi} = \infty$. In terms of the original KMM $T = \infty$ is associated with $m = \infty$ (the strong coupling limit), while $T = T_{\text{crit}}$, where long-range correlations arise in the Ising system, should be associated with $m = m_{\text{crit}}$ (the “weak coupling limit”, where continuum-like behaviour is supposed to occur). The problem is that at $T_{\text{crit}}$ there is no reason to believe in the validity of the mean field approximation. Our main hope is that as $N$ increases, the analogue of $T_{\text{crit}}[N]$ moves toward $T=0$ so that the critical behaviour of the relevant statistical system appears in the limit $T \rightarrow 0$, $\phi \rightarrow \infty$ and $m \rightarrow m_{\text{crit}}$ at which point the mean field
approximation may be trustworthy. It is most difficult of course to prove the existence of a second order phase transition for all $N$. With this guess in mind, let us pay more attention to the point $T = 0$ in the example of the Ising model.

The crucial feature of the point $T=0$ for our considerations is that $\alpha' \to 0$ as $T \to 0$. It thus looks as if the string tension is vanishing in the continuum limit. We should be more careful, however, and consider what happens when $T$ is small, but still non-zero. The answer for $SU(2)$ is easily extracted from (48) from which one finds that

$$\alpha' = \frac{1}{2\hat{\phi}^2} + o(1/\hat{\phi}^4, e^{-4\hat{\phi}^2})$$

Approximation that $\phi$ is large near the continuum limit is self-consistent. The integral over $\phi$ is essentially

$$\int d\hat{\phi} e^{-(m^2-m_{\text{crit}}^2)\sum_x \hat{\phi}^2} e^{-\alpha' A(S)}$$

with $\alpha'$ given by (49). Thus when $m^2 \to m_{\text{crit}}^2$ the integral is dominated by $\phi^2 \sim 1/\sqrt{m^2 - m_{\text{crit}}^2}$ which is large in this limit.

We suggest that this simple example of the Ising model where everything can be calculated explicitly can serve as a prototype for the situation of $N = \infty$. We speculate that the main features of this example survive in the general situation. There are, however, several crucial differences:

(a) The phase transition occurs at a lower temperature $T_{\text{crit}}$ which approaches $T = 0$ for large $N$.

(b) The single eigenvalue $\hat{\phi}$ gets replaced by a distribution of eigenvalues with a density $\rho(\phi)$.

(c) The $\exp(-\hat{\phi}^2)$–type corrections can become relevant for smooth functions $\rho(\hat{\phi})$.

Note, that this result depends crucially on the normalization of the FWL operator. If there were an extra factor of $N^{-\text{Area}} = 2^{-\text{Area}}$ in (19), we would get $\alpha' \to \log 2$ as $T \to 0$. This would exclude any dynamics from the derivation of area law which would be valid in the continuum limit without any need for scaling arguments. This is already an argument in favor of our normalization choice. We shall return to the normalization problem in Section 6.
If this suggestion is correct then we would expect Eq. (49) to be replaced by

$$\alpha' \propto \int \int \rho(\phi)\rho(\phi')d\phi d\phi' \left( \phi - \phi' \right)^2$$ (51)

(note that $2\hat{\phi}$ in (49) is just the difference $\phi_1 - \phi_2$ of two eigenvalues of (44). It thus follows that $\alpha'$ is not necessarily vanishing in the continuum limit. Critical behaviour would arise at large but finite values of the master field $\Phi$.

6 Puzzles of the continuum limit of KMM

Before we turn to the general analysis of the $SU(N)$ case, let us examine in more detail the two limiting cases of strong and weak coupling.

The strong coupling limit of the model corresponds to $m \to \infty$. The $\exp -m^2\phi^2$ factor in the integrand then damps all fluctuations of the $\phi$ fields, which are thus restricted to vanish. It follows that the action in the integral (44) over $U$ vanishes and what remains in the evaluation of expectation values are just integrals with the conventional Haar measure, of the form

$$\frac{<U_{ij}U_{kl}^\dagger>}{<1>}_{\phi=0} = \frac{\int[dU]U_{ij}U_{kl}^\dagger}{\int[dU]} = \frac{1}{N} \delta_{il} \delta_{jk}$$ (52)

so that

$$C_{ij}[\phi = \psi = 0] = \frac{1}{N}$$ (53)

is equal for all $i$ and $j$. (One can similarly derive expressions such as

$$\frac{<U_{ij}U_{mn}U_{kl}^\dagger U_{pq}^\dagger>}{<1>}_{\phi=0} = \frac{1}{N^2 - 1} \left( \delta_{il} \delta_{jk} \delta_{mq} \delta_{np} + \delta_{iq} \delta_{jp} \delta_{ml} \delta_{nk} \right) - \frac{1}{N(N^2 - 1)} \left( \delta_{jk} \delta_{lm} \delta_{np} \delta_{qi} + \delta_{jp} \delta_{qm} \delta_{nk} \delta_{li} \right)$$. (54)
In this strong coupling approximation any average of a gauge invariant product of Wilson loops (2.4) is simply equal to

$$\ll W(\Gamma_1...\Gamma_k) \gg |_{\text{strong coupling}} = \frac{1}{N} \frac{N^\# \text{vertices}}{N^\# \text{links}} = \frac{1}{N^\# \text{loops}} = \frac{1}{N^{k-1}}$$

(55)

provided $\Gamma_1 + ... + \Gamma_k = 0$. For the FWL operator $k - 1 = \#$(of plaquettes in S) = $A(S)$, and we obtain:

$$\ll W(\Gamma_1...\Gamma_k) \gg |_{\text{strong coupling}} = \frac{1}{N \text{Area}}$$

(56)

The area law is thus trivially true in the strong coupling limit. Of course, the really interesting question is whether it is also true in the “weak coupling” (continuum) limit.

Naively the “weak coupling limit” corresponds to $m \to 0$, where integrals over $\phi$ are dominated by large values of the $\phi$-fields (i.e. all the eigenvalues $\phi_i$ are large). In fact integrals over $\phi$ diverge already for a finite value of $m = m_{\text{crit}}$ (quasi-classically $m_{\text{crit}}^2 = D$), and the relevant limit is $m \to m_{\text{crit}} + 0$. The fact that integrals over $\phi$ diverge still allows interesting quantities such as $<|U_{ij}|^2>$, which are ratios of divergent integrals, to be well defined. In fact, when all $\phi_i$’s and $\psi_i$’s are large (and positive) the integral

$$\int [dU] \exp \sum_{i,j} \phi_i \psi_j |U_{ij}|^2$$

(57)

is clearly dominated by diagonal matrices

$$U_{ij} = e^{i\theta_i} \delta_{ij}, \quad |U_{ij}|^2 = 1$$

(58)

and thus

$$\frac{< U_{ij} U_{kl}^\dagger >}{<1>} \xrightarrow{\phi,\psi \gg 1} \delta_{ij} \delta_{jk} \delta_{ij}, \quad C_{ij} \xrightarrow{\phi,\psi \gg 1} \delta_{ij}$$

(59)

In this approximation it is clear that all the spins in the statistical system take on the same value at each site. (This is clearly the ordered phase which occurs in the spin system at zero temperature.) There is just an overall degeneracy factor of $N$, which cancels the only $N$ in the normalization in (19):

$$\ll W(\Gamma_1...\Gamma_k) \gg |_{m=m_{\text{crit}}} = 1 \times \delta(\Gamma_1 + ... + \Gamma_k)$$

(60)
In particular in this approximation the FWL is given by

\[ \ll W(S) \gg |_{m=m_{\text{crit}}} = 1 \] (61)

and, as explained at the end of the previous section, the vicinity of the point \( m = m_{\text{crit}} \) should be investigated in order to explain whether and why \( \alpha' \) may be non-vanishing in the continuum limit.

This will be our purpose in the next section. In this section we discuss what happens in the “naive continuum limit”. As we already mentioned in Section 2, evaluation of \( W(S) \) in the naive continuum limit would give a factor \( N^{k-1} \) instead of 1 (there is \( k-1 \) instead of \( k \) in the exponent is because of normalization factor \( 1/N \) in (19)). This estimate was however based on the implicit suggestion that

\[ \langle U_{ij} U_{kl}^\dagger \rangle _{ncl} = \delta_{ij} \delta_{kl} \] (62)

which is obviously inconsistent with the basic property (41) of the correlators of \( U \) (which can of course be traced back to the \( Z(N) \) gauge symmetry). This argument is already sufficient to demonstrate the failure of naive continuum limit. We shall now present a somewhat more constructive argument against the naive continuum limit.

The discrepancy between the naive continuum limit (which we claim is incorrect for the KMM) and the limit of \( m \to m_{\text{crit}} \) for the eigenvalue model (7) (which we claim is the correct prescription for the KMM) appears already in a very simple situation, which can be examined exactly. Indeed consider a lattice which is just a closed 1-dimensional chain of \( L \) links. All the techniques and results which we derived for a generic D-dimensional lattices is applicable to this simplified model. The set of (gauge-invariant) observables is very small. In fact only adjoint Wilson loops and “Baryon” loops along the closed chain and their "powers" (the same contour passed several times) are nonzero. Such adjoint Wilson loops are given by

\[ \mathcal{O}_k = \frac{1}{N} |W(k\Gamma)|^2. \] (63)

The predictions of our above analysis are that in the strong coupling limit

\[ \ll \mathcal{O}_k \gg |_{m=\infty} = \frac{1}{N} \frac{N\# \text{ of vertices}}{N\# \text{ of links}} = \frac{1}{N} \frac{N^k L}{N^k L} = \frac{1}{N} \] (64)
while in the weak coupling limit (near $m_{\text{crit}}$)

$$\ll \mathcal{O}_k \gg \big|_{m=m_{\text{crit}}} = \frac{1}{N} \cdot N = 1 \quad \text{in the limit} \quad m \to m_{\text{crit}} \quad (65)$$

On the other hand the “naive continuum limit” prescription (62) would give

$$\ll \mathcal{O}_k \gg \big|_{\text{ncl}} = \frac{1}{N} (\text{tr} I)^2 = N, \quad (66)$$

which differs from (65).

We shall now solve the original KMM (1) exactly for this particular lattice and prove that while (64) and (65) are correct predictions the “prediction” (66) is incorrect. We shall also find a reason for the failure of (66) which can serve as a general explanation of why the naive continuum limit is wrong in the KMM in higher dimensions. Recall that the argument in support of (66) was that the action in (4) diverges as $m \to m_{\text{crit}}$ and it has a single maximum as a function of $U$ which occurs when $U = I$. This is absolutely true but what is unusual is that in the KMM is that the peak of the action at $U = I$ is very moderate. In fact the action grows only logarithmically as $U$ approaches $I$. This makes it necessary to take the measure of integration $[dU]$ into consideration. The vicinity of $U = I$ where action is big has, of course, a small volume and it appears that this is enough to compensate completely for the peak of the action. In the particular example of the one dimensional chain we shall see that when $m = m_{\text{crit}}$ the action and the measure cancel each other completely over the entire range of integration over $U$. Once observed, this phenomenon is clearly a very general feature of the KMM. The integrand of the partition function is very smooth as the continuum limit is approached so that even quasi–classically it has is no dominant peak. The continuum limit is thus far from naive.

Let us now turn to our calculation for the one dimensional chain. The quantity to compute is

$$\ll \mathcal{O}_k \gg \equiv$$

$$\frac{1}{Z} \int d\Phi_1 \prod_{x=1}^{L} dU_{x,x+1} e^{-m^2 \text{tr} \Phi_1^2 + \text{tr} \Phi U_{x,x+1} \Phi^\dagger U_{x,x+1}^\dagger} |\text{tr}[\prod_{x=1}^{L} U_{x,x+1}]|^2 \quad (67)$$
with

\[ Z = \int \prod_{x=1}^{L} d\Phi_x [dU_{x,x+1}] e^{-m^2 \text{tr} \Phi_x^2} + \text{tr} \Phi_x U_{x,x+1} \Phi_{x+1} U_{x,x+1}^\dagger \]

\((x=L+1\text{ is identified with } x=1)\). This one dimensional case is particularly simple since it is possible to perform a gauge transformation \((13)\) which eliminates (i.e. makes equal to \(I\)) all the matrices \(U_{x,x+1}\) except for a single one, say \(U \equiv U_{L,1}\). In other words

\[ Z = \int [dU] \prod_{x=1}^{L} d\Phi_x e^{-m^2 \text{tr} \Phi_x^2} \prod_{x=1}^{L-1} e^{\text{tr} \Phi_x U_{x,x+1} \Phi_{x+1} U_{x,x+1}^\dagger} \]

\((68)\)

and \(< O_k >\) is simply the normalized average of \(|\text{tr} U^k|\) with this generating functional.

The next step is to integrate first over \(\Phi_2, ..., \Phi_{L-1}\) and then over \(\Phi_1\). All these integrals are Gaussian and are thus easy to perform. The result can be expressed in terms of the determinant \(\mu_l\) of the \(l \times l\) matrix

\[ \mathcal{M}_l \equiv \begin{pmatrix} 2m^2 & -1 & 0 & 0 & 0 \\ -1 & 2m^2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2m^2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2m^2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2m^2 \end{pmatrix} \]

\((69)\)

The determinant \(\mu_l \equiv \det \mathcal{M}_l\) satisfies the recursion relation \(\mu_{l+1} = 2m^2 \mu_l - \mu_{l-1}\) with “initial conditions” \(\mu_0 = 1, \mu_1 = 2m^2\) (in this respect they are somewhat similar to Fibonacci numbers \(f_{l+1} = f_l + f_{l-1}, f_0 = f_1 = 1\)). Since the matrix \(\mathcal{M}_l\) is simply a Laplacian the \(\mu_l\) can also be represented as

\[ \mu_l = 2^l \prod_{k=1}^{l} (m^2 - \cos \frac{\pi k}{l+1}) = \]

\[ \frac{(1 - q)(1 - \nu q) \cdots (1 - \nu^{2l+1} q)}{q^l (1 - q^2)} = \frac{1 - q^{2l+2}}{q^l (1 - q^2)} \]

\((70)\)

where

\[ q = q_\pm \equiv m^2 \pm \sqrt{m^4 - 1} \]

\((71)\)
and \( \nu = \nu_l \equiv e^{\frac{2\pi i}{n}} \).

We now do the integral over \( \Phi_x \) in (68):

\[
\int d\Phi_1 d\Phi_L e^{-m^2 \text{tr}(\Phi_1^2 + \Phi_L^2) + \text{tr} \Phi_L U \Phi_1 U^\dagger} \prod_{x=2}^{L-1} \int d\Phi_x e^{-m^2 \text{tr} \Phi_x^2} e^{\text{tr} \Phi_1 \Phi_2 + \text{tr} \Phi_2 \Phi_3 + \ldots + \text{tr} \Phi_L} \]  

(72)

\( \Phi_1 \) and \( \Phi_L \) can be considered as “source terms” in the Gaussian integral over \( \Phi_2, \ldots, \Phi_{L-1} \) with the action

\[
\frac{1}{2} \sum_{x,y=2}^{L-1} \text{tr} \Phi_x (M_{L-2})_{xy} \Phi_y =
\]

(73)

\[
\mu_{L-2}^{-N^2/2} \exp \frac{1}{2} \left[ \Phi_1^2 (M_{L-2})_{22} + 2 \Phi_1 \Phi_L (M_{L-2})_{2L-1} + \Phi_L^2 (M_{L-2})_{L-1,L-1} \right]
\]

The relevant matrix elements are

\[
(M_{L-2}^{-1})_{22} = (M_{L-2}^{-1})_{L-1,L-1} = \frac{\mu_{L-3}}{\mu_{L-2}} \\
\text{and} \quad (M_{L-2}^{-1})_{2,L-1} = \frac{1}{\mu_{L-2}}
\]

(74)

Since \( m^2 - \mu_{L-3}/2\mu_{L-2} = \mu_{L-1}/2\mu_{L-2} \) we get for (72):

\[
\mu_{L-2}^{-N^2/2} \int d\Phi_1 d\Phi_L \exp \left[ -\frac{\mu_{L-1}}{2\mu_{L-2}} \text{tr} (\Phi_1^2 + \Phi_L^2) + \frac{1}{\mu_{L-2}} \text{tr} \Phi_1 \Phi_L + \text{tr} \Phi_L U \Phi_1 U^\dagger \right].
\]

(75)

The remaining integration over \( \Phi_1 \) is done by rescaling \( \Phi_L \to \Phi \equiv \Phi_L/\sqrt{\mu_{L-1}} \),

The result is:

\[
Z \sim \mathcal{N}_L \int [dU] \int d\Phi \ e^{-M_2^2 \text{tr} \Phi^2 + \text{tr} \Phi U \Phi U^\dagger}
\]

(76)

with \( \mathcal{N}_L = 1 \) and “renormalized” (generically lattice and contour-dependent) mass parameter

\[
M_2^2 = \frac{\mu_{L-1}^2 - \mu_{L-2}^2 - 1}{2\mu_{L-2}} = \frac{\mu_{2L-2} - 1}{2\mu_{L-2}} = 1 + q^{2L}.
\]

(77)
In terms of the new parameter $q_L \equiv M_L^2 - \sqrt{M_L^4 - 1}$ (i.e. $M_L^2 = (1 + q_L^2)/2q_L$), Eq.(77) turns into a simple relation:

$$q_L = (q_{\pm})^\mp L$$

(78)

where $q_{\pm}$ is defined in (71).

The integral for $Z$ and for $\ll O_k \gg$ is easily evaluated by performing a gauge rotation on $U$ and $\Phi$ so as to make $U$ diagonal ($U = \text{diag}(e^{i\theta_1}, ..., e^{i\theta_N})$).

It follows that

$$\text{tr}U^k = \sum_{j=1}^N e^{ik\theta_j}$$

while the Haar measure is

$$[dU] \sim \prod_{j=1}^N d\theta_j \Delta^2 (e^{i\theta}) = \prod_{j=1}^N d\theta_j \prod_{i<j} (e^{i\theta_i} - e^{i\theta_j})^2 \sim \prod_{j=1}^N d\theta_j \prod_{i<j} \sin^2 \frac{\theta_i - \theta_j}{2}.$$  

(79)

For diagonal $U$ the action in (76) becomes

$$\sum_{i,j} |\Phi_{ij}|^2 (M_L^2 \delta_{ij} - e^{i(\theta_i - \theta_j)}) =$$

$$(M_L^2 - 1) \sum_i |\Phi_{ii}|^2 (M_L^2 - \cos(\theta_i - \theta_j))$$

(80)

The integral over $\Phi$ is now easily evaluated. Using (79) one finds

$$Z \sim \frac{1}{M_L^{N^2-N}(M_L^2-1)^{N/2}} \int \prod_{j=1}^N d\theta_j \prod_{i<j} \frac{\sin^2 \frac{\theta_i - \theta_j}{2}}{1 - \frac{\cos(\theta_i - \theta_j)}{M_L^2}} =$$

$$= 2^{N^2/2} \int \prod_{j=1}^N d\theta_j \prod_{i<j} \sin^2 \frac{\theta_i - \theta_j}{2} \prod_{i,j} \frac{q^L/2}{1 - q^L e^{i(\theta_i - \theta_j)}}$$

(81)
The critical value of $M$ is clearly $M_{\text{crit}} = 1$. At this critical point the integrand in (81) becomes equal to unity. The Jacobian factor in the measure, $\prod_{i<j} \sin^2 \frac{\theta_i - \theta_j}{2}$, completely cancels the contribution from the action, $\prod_{i<j} (1 - \cos(\theta_i - \theta_j))$. Note that the contribution of the action is in denominator and is indeed singular as $M \to M_{\text{crit}}$ but, as stated above, the singularity is very moderate. We may now evaluate the average of $|\text{tr}U^k|^2$ at $M = M_{\text{crit}}$. Since the measure of integration is just $\prod_{j} N d\theta_j$ we find

$$\ll \mathcal{O}_k \gg |_{m=m_{\text{crit}}} = \frac{1}{N} \frac{\int \prod_{j} N d\theta_j | \sum_{j=1}^{N} e^{ik\theta_j}|^2}{\int \prod_{j} N d\theta_j} = \frac{N}{N} = 1$$

in accordance with (55) (and contrary to (56)). This completes the proof.

The conclusions which we deduce from this result have been discussed previously.

In the remainder of this section we use the opportunity provided by this calculation to make another important observation. Note, that there is a somewhat different way to evaluate the integral (76). It can be represented as the determinant of an $N^4 \times N^4$ matrix,

$$M^{-N^2/2} \exp\{-\frac{1}{2} \text{tr} \log[I \otimes I - \frac{1}{2} (U \otimes U^\dagger + U^\dagger \otimes U)]\} =$$

$$M^{-N^2/2} \exp\left\{\frac{1}{2} \sum_{l=1}^{\infty} \frac{1}{l M^2 l} \text{tr} \left(\frac{U \otimes U^\dagger + U^\dagger \otimes U}{2}\right)^l\right\} \sim$$

$^{8}$Eq. (77) implies that $M_{\text{crit}}^2 = 1$ ($q_{\text{crit}} = 1$) simply corresponds $m_{\text{crit}}^2 = 1$, which is the same for all values of $L$. (This is easy to check, since for $m^2 = 1$ the formula for $\mu_l$ is very simple: $\mu_l = l + 1$.) This value of $m_{\text{crit}}^2$ is also in accordance with the simple quasi–classical estimate $m_{\text{crit}}^2 = D$ (see footnote 3 above).

Eq. (51) was also deduced in [10] and in a very recent paper [11], where also the result of integration over all $\theta$-variables was derived:

$$(68) \sim (51) \sim \frac{q_L^{N^2/2}}{(1 - q_L)(1 - q_L^2) \cdots (1 - q_L^N)}; \quad q_L = (q_\pm)^{2L}.$$
If we now substitute $U$ in the diagonal form this expression becomes

$$M^{-N^2/2} \exp \left\{ \frac{1}{2} \sum_{l=1}^{\infty} \frac{1}{l M^{2l}} \sum_{i,j=1}^{N} \left( \frac{e^{i(\theta_i - \theta_j)} + e^{-i(\theta_i - \theta_j)}}{2} \right)^l \right\} =$$

$$M^{-N^2/2} \exp \left\{ \frac{N}{2} \sum_{l=1}^{\infty} \frac{1}{l M^{2l}} + \sum_{i<j}^{N} \sum_{l=1}^{\infty} \frac{1}{l} \left( \frac{\cos(\theta_i - \theta_j)}{M^2} \right)^l \right\} = \tag{84}$$

$$\frac{1}{M^{N^2-N}} \frac{1}{(M^2-1)^{N/2}} \prod_{i<j}^{N} \frac{1}{1 - \frac{\cos(\theta_i - \theta_j)}{M^2}}$$

i.e. we reproduce the contribution of the action to (79).

The exponent at the r.h.s. of (84) can now be rewritten as

$$\frac{1}{2} \sum_{l=1}^{\infty} \frac{1}{l (M^2)^l} \sum_{k=0}^{l} \frac{l!}{2^l k! (l-k)!} |\text{tr}U^l|^{2k}.$$  \tag{85}

In this expression one can easily recognize the $D = 1$ version of (4).

We now review our conclusions from this section. Our main claim is that the action grows only logarithmically in the vicinity of the critical point $m = m_{\text{crit}}$. This fact is of course related to the occurrence of the factor $1/l$ in the sum over Wilson loops in Eqs. (1, 84). However, the paths which are summed over in these formulae are allowed to have steps which “double back” on themselves (Fig.4). Sums over such inverse steps introduce complicated combinatorial factors such as the binomial coefficients in (85). The above calculation serves as an illustration that these complicated corrections do not necessarily destroy the logarithmic behaviour but rather renormalize the bare mass parameter.

### 7 The area law in the case of arbitrary N

So far we have examined two exactly solvable examples. We have solved the KMM for the group $SU(2)$ and for 1–dimensional lattices. We return now to the general case where $N$ is arbitrary and the lattice is $D$–dimensional for which an exact solution is not yet available. We now suggest a qualitative
picture which we expect to be valid in the generic situation. This would include the following ingredients.

(a) Gauge invariant observables in the KMM are associated with either one or two dimensional sublattices \( \tilde{\Gamma} \) of original lattice. The corresponding average is given by the partition function of a 1\( d \) or of a 2\( d \) statistical system on \( \tilde{\Gamma} \) (which is defined by the Boltzmann weights \( C_{ij}[\phi, \psi] \) with randomly distributed \( \phi, \psi \)). FWL operators are associated with two dimensional lattices \( \tilde{\Gamma} \) and thus with two dimensional statistical systems.

(b) The area law for the average of the FWL is simply a reflection of the fact that the free energy of a 2\( d \) statistical system is proportional to its size (area).

(c) In the strong coupling limit \( m \to \infty \) the string tension \( \alpha' = \frac{1}{a^2} \log N \).

(d) Long range correlations occur in 2\( d \) statistical system, if they have a second order phase transition, at a certain temperature \( T[N; m] = T_{\text{crit}}[N] \) corresponding to a certain value of \( m = \tilde{m}_{\text{crit}}[N] \). It seems that a necessary condition for the KMM to have a continuum limit is the presence of long range correlations in the 2\( d \) subsystem and thus the phase transition at \( T_{\text{crit}} \) must be second order.

(e) The partition function of the 2\( d \) statistical system can easily be evaluated in the vicinity of an a–priori different point \( T[N; m] = 0 \) when \( m = m_{\text{crit}}[N] \). (The partition function of the KMM diverges at this point. This is why the subscript “crit” seems reasonable.) When \( T \) is precisely equal to zero the string tension \( \alpha' = 0 \). However, in the vicinity of the point \( T = 0 \) it obeys some scaling law such as \( \alpha' \sim \frac{1}{a^2 N^\gamma} \). It can thus be finite in the appropriate limit \( a \to 0, N \to \infty \). The region of small temperatures is easy to analyze since fluctuations of the Boltzmann weights are suppressed i.e. the \( \phi \)-fields are “frozen” and the statistical system is in its “ordered” phase.

(f) It may be hard to find the critical temperature \( T_{\text{crit}}[N] \) for the actual system (14). If, however, (14) is evaluated in the (unjustifiable) approximation of frozen \( \phi \)-fields then \( T_{\text{crit}}[N] > 0 \), and, moreover, the phase transition is generically first order. However, in the limit \( N \to \infty \) both \( T_{\text{crit}}[N] \to 0 \) and the latent heat \( \Delta \to 0 \). The reason for this is that the number of degrees of freedom and thus the fluctuations are increasing as \( N \to \infty \). This increase in the fluctuations should also occur for finite \( N < \infty \) if one goes beyond the frozen-\( \phi \) approximation. Thus in the large \( N \) limit the two interesting values of \( T \) namely \( T = T_{\text{crit}} \) from d) and \( T = 0 \) from e) seem to approach each other. Thus the continuum limit of the KMM (if it exists at all) can
be examined in terms of the phase transitions of 2d statistical systems. We thus expect the continuum limit to occur at small “temperatures”.

(g) Our description of the continuum limit is somewhat more intricate than the usual description. This is a peculiarity of the KMM where the “naive continuum limit” which is used in the Wilson theory \((\int [dU] \exp(\frac{N}{g^2} \sum W(\Box) + c.c.))\) and in the “adjoint” theory \((\int [dU] \exp(\frac{1}{m^2} \sum |W(\Box)|^2))\) \((\text{without a sum over non-elementary contours!})\) does not exist. Although the most natural assumption after the failure of the “naive continuum limit” would be that there is no continuum limit at all, the above scenario implies that it still can exist but that the approach to this limit is more sophisticated and interesting in that it is described in terms of 2d systems!

The remainder of this section is devoted to various comments on this scenario. Most of them concern notes d)-f) above. The note g) was already discussed in some detail (though not exhaustively) in Section 6. Notes a)-c) were discussed in Sections 2 and 4 and illustrated by the example of the Ising model in Section 5. Our goal now is to try to generalize some results of that section to \(N > 2\).

We begin by deriving the literal analogue of Eq.(49) for the average of the FWL for the case of generic \(N\). We shall subsequently discuss the validity of the result. For this derivation we simply need to find the first order correction to the free energy of our statistical system at low temperature in the master-field approximation for \(\phi\) assuming that the system is in its ordered phase. Since at \(T = 0\) \(C_{ij} = \delta_{ij}\) we introduce a new variable \(\xi_{ij}\) where \(C_{ij} = \delta_{ij} + \xi_{ij}\). In the ordered phase the “spins” at all the sites of the sublattice \(\tilde{\Gamma} = S\) have the same value \(i\). Taking into account the elementary fluctuations (i.e. the flip of a “spin” \(i \to j\) at one point of the lattice) as well as the deviation of \(C_{ii}\) from unity, we get:

\[
\alpha' = -\frac{\log \ll W(S) \gg}{a^2 A(S)} = \\
- \frac{1}{a^2 N A(S)} \sum_{i=1}^{N} \left( \sum_{\text{links} \in S} (\xi_{ii} + \sum_{j \neq i} \xi_{ij}^2) \right) + \text{higher order corrections} \quad (86)
\]

Due to the normalization condition (100) \(\xi_{ii} = -\sum_{j \neq i} \xi_{ij}\), while the second
term in (86) can be neglected since $2D > 1$. Thus

$$\alpha' = \frac{2}{a^2 N} \sum_{i \neq j} \xi_{ij}. \quad (87)$$

In order to estimate $\xi_{ij}$ in the limit of large $\phi$ (which is the same as “low temperature” and $m \to m_{\text{crit}}$), one can make use of the formula (31) ($O(1/A)$ corrections being unessential in this limit).

The leading contribution to (31) comes from the identity permutation $P = I$. The corresponding contribution to $\xi_{ij}$ is

$$\frac{\partial}{\partial A_{ij}} \log \frac{e^{\sum_k A_{kk}}}{\prod_{k<l}(A_{kk} + A_{ll} - A_{kl} - A_{lk})} \left(1 + O \left(\frac{1}{A}\right)\right) \bigg|_{A_{ij} = \phi_i \phi_j} \quad (88)$$

Since we are only interested in the case $i \neq j$, this gives

$$\frac{1}{A_{ii} + A_{jj} - A_{ij} - A_{ji}} \left(1 + O \left(\frac{1}{A}\right)\right) \bigg|_{A_{ij} = \phi_i \phi_j} =$$

$$\frac{1}{(\phi_i - \phi_j)^2} \left(1 + O \left(\frac{1}{\phi^2}\right)\right) \quad (89)$$

In the approximation where all the $\phi$’s are large and different, the $O(1/\phi)$-corrections can be neglected. The same is true about the contributions of other permutations $P \neq I$, which are suppressed exponentially: the contribution of $P_{kl}$, which permutes $k$ and $l$, is $O \left(e^{-\left(\phi_k - \phi_l\right)^2}\right)$. Therefore the literal analogue of (49) for generic $N$ is

$$\alpha' = \frac{2}{a^2 N} \sum_{i \neq j} \frac{1}{(\phi_i - \phi_j)^2} \left(1 + O \left(\frac{1}{\phi^2}; e^{-\left(\phi_k - \phi_l\right)^2}\right)\right) \quad (90)$$

However as $N \to \infty$ the most interesting phase, associated with the continuum limit of the KMM, seems to occur when eigenvalues $\phi_i$, though large ($\phi_i \gg 1$), are not all very different i.e. $\frac{(\phi_i - \phi_j)^2}{\phi_i^2} \ll 1$. In other words the eigenvalues should be smoothly distributed with some density $\rho(\phi)$ ($\int \rho(\phi) d\phi = N$), so that $\sum_i F(\phi_i) \sim \int \rho(\phi) d\phi F(\phi)$. If expressed in terms of $\rho(\phi)$, Eq.(91) looks like

$$\alpha' = \frac{2}{a^2 N} \int \frac{\rho(\phi)\rho(\phi')d\phi d\phi'}{(\phi - \phi')^2} + \text{corrections}. \quad (91)$$
For smooth distributions $\rho(\phi)$ the corrections, neglected in (88), can become essential. First of all there can be corrections which contain negative powers of the differences $(\phi_i - \phi_j)$ which are enhanced in integrals like (91). Indeed the $O(1/A)$-corrections in (31) contain terms with negative powers $(A_iP(i) - A_jP(j) - A_jP(i))^{-k}$ with $k = 1 \cdots \text{rank}(SU(N))$. If it were not the $k = 1$ term (as supposed in (88)) but the $k = N - 1$ term which dominated then we would get an extra factor of $(N - 1)$ on the r.h.s. of (90). It may however happen that the contributions of all $k$ are equally important in which case the exact counterpart of (31) should be used (see [8]). Secondly when $(\phi_k - \phi_l)^2$ is not usually large the exponential corrections, (arising from non-trivial permutations $P$), are also important. Expressed in terms of the density $\rho(\phi)$ they have the form

$$\int e^{(\phi - \phi')^2} G(\phi, \phi') \rho(\phi) \rho(\phi') d\phi d\phi'$$

where $G$ is some rational function of $\phi$ and $\phi'$.

We shall not go into more detail about the evaluation of these corrections because the very approximation (the low temperature expansion for the ordered phase) may be unreliable in the situation of interest. As we have stated previously, in order for a non-trivial continuum limit to occur there are several different phenomena which must occur simultaneously.

(a) The auxiliary 2-dimensional statistical system must have long range correlations. This is most probably realized at the critical point $T_{\text{crit}}$ which should be (nearly) second order.

(b) The master-field approximation for the $\phi$-fields must be applicable. This is a reasonable approximation when $N \to \infty$ and it leads to the occurrence of magnetic-type statistical systems.

(c) The “temperature” $T$ which characterizes the behaviour of the 2-dimensional system should be close to zero. At $T = 0$ the string tension vanishes, $\alpha' = 0$. This is a normalization condition for the “Boltzman weights” $C_{ij}$ related to (40). For the statistical system it corresponds to subtracting the ground state energy and is independent of other details of the system. On the other hand the area law which occurs at small but non-vanishing temperature depends on the the actual dynamics of the statistical system, i.e. on whether there is a scaling limit at $T \sim 0$.

These requirements are mutually consistent at large $N$, when $T_{\text{crit}}[N]$ is assumed to approach zero. This in turn implies that $T = 0$ is in fact a
critical point and thus one cannot rely on the low temperature expansion in
the vicinity of this point (as we have done for the Ising model). One should
go beyond this approximation and apply some formalism adequate for the
study of phase transitions. This should be, however, a solvable problem, since
it concerns a phase transition in a 2-dimensions (though the KMM itself is
formulated in $D$ dimensions).

It deserves mentioning that in contrast with the situation in the Wilson
theory nothing special happens to the quantities of interest when we go
through the (2-dimensional!) critical point. As we saw in the examples
for finite $N$ the area law for the average of a FWL is equally valid both
above and below the phase transition: the averages in the KMM are related
not to correlators of the 2d system, but to extensive quantities (like density
of free energy).

It remains to comment on the very notion of “temperature” for these two
dimensional systems. As is already clear from the example of the Ising model
in Section 5, this “temperature” is nothing but some function of the “frozen”
$\phi$-fields. In the case of $N = 2$ it is simply related to the Boltzmann weights
$C_{ij}$ by Eq.(46). Denominators in this formula account for the normalization
condition (40) for the Boltzmann weights, which is peculiar for the KMM
(and in fact guarantees that $\alpha' = 0$ for $T = 0$). This condition cannot
however be taken into account in the same simple way when $N \neq 2$ since
in the latter case the “bond energies” $J_{ij}$ are $T$-dependent. One can deal
with this by assuming that the bond energies also depend on some other
external parameters (like magnetic fields) and the KMM is associated with
some particular line in the space of all of these parameters, labeled by $T$.
This remark is intended to alleviate confusion resulting from the counter–
intuitive properties of the system which arise from the normalization of the
Boltzman weights. The constraints (44) guarantee that they are conditional
probabilities which is unusual for Gibbs-like distributions.

Thus we see, that much remains to be done in order to study the contin-
uum limit of the KMM, both for finite $N$ (where it may also exist!) and for
infinite $N$. In this paper we have tried to show that this study is very much
in line with modern mathematical physics and it should deal with familiar
2-dimensional systems, their critical behaviour (i.e. conformal models and
their perturbations) and string–like models.
8 Conclusion

In our preliminary investigation of gauge invariant observables in the KMM (1) we find the model to be very interesting on its own (irrespective of whether it is related to 4–dimensional gluodynamics). It is related in a non-trivial fashion to various statistical systems and a deeper understanding of these connections seems desirable. Moreover the study of the KMM for finite values of $N$ seems to be already giving us non-trivial information about the probable large $N$ behaviour – a fact which is rather common in modern approaches to matrix models.

As to the relation of the KMM to gluodynamics we emphasize primarily the drastic difference between the behaviour of the KMM as the continuum limit is approached and that of the conventional Wilson lattice QCD. The KMM approaches this limit in a much smoother way. This may be an advantage of the model, at least from the viewpoint of exact solvability. It however implies that the transition to the continuum limit is far less naive than is usually believed. (We illustrated this point with an example of the normalization of observable operators.) This phenomenon is related to the relevance of the angular variables (rather than only the eigenvalues) of the $\Phi$ field in the KMM and it can thus be of general importance for all $c \geq 1$ models.

Another difference between the KMM and the Wilson theory – the occurrence of extra $Z(N)$ gauge symmetry – may somehow be related to the previous difference. In fact whenever this $Z(N)$ symmetry is unbroken it comes very close to guaranteeing the existence of a confinement phase all by itself since all operators with any color properties, even locally, are simply not gauge invariant. The operators which can be considered look like nets of double lines imitating planar–diagram structures. Despite this essentially confinement–like description, the area law in the continuum limit seems to arise in a somewhat non-trivial manner which accounts for the actual dynamics of the theory.

We argued that the dynamical behaviour of the KMM can be studied in terms of 2–dimensional systems. This fact reflects the already mentioned difference between the KMM (and, in fact, the “intermediate” model (11)) and the Wilson theory which makes KMM especially attractive, namely that the string–like description is essentially built into the theory. We have already seen this in several places. Elementary gauge-invariant observables like the
FWL depend on a *surface* rather than just on a contour. Moreover, in reasonable approximations (like that of the master-field approximation for $\phi$) their averages can be evaluated within the framework of auxiliary 2–*dimensional* models.

The operators related to the FWL which are presumably the physically relevant ones were suggested in (23) to be of the form $\hat{W}(\Gamma) = \int [DS] W(S)$ and thus to include sums over all surfaces with a given boundary. Note, that the averaging procedure, $\ll \gg$, prescribed by (1) does not involve any summation of this kind. It comes entirely from the definition of the observables. Conversely the “stringy” measure $[dS]$ need not involve the usual $e^{-\text{Area}}$ factor since it appears from the area law behaviour of $\ll W(S) \gg$.

If we now consider the correlator of two Wilson loops the quantity of interest is

$$G = \ll \hat{W}(\Gamma_1; \Gamma_2) \gg - \ll \hat{W}(\Gamma_1) \gg \ll \hat{W}(\Gamma_2) \gg =$$

$$\int_{\partial S = \Gamma_1 + \Gamma_2} [DS] \ll W(S) \gg -$$

$$\int_{\partial S_1 = \Gamma_1} [DS_1] \int_{\partial S_2 = \Gamma_2} [DS_2] \ll W(S_1) \gg \ll W(S_2) \gg.$$

The first term on the r.h.s. contains sums over connected and disconnected surfaces (Fig.5). and

$$G = \ll \hat{W}(\Gamma_1; \Gamma_2) \gg_{\text{connected}} +$$

$$\ll \hat{W}(\Gamma_1) \hat{W}(\Gamma_2) \gg - \ll \hat{W}(\Gamma_1) \gg \ll \hat{W}(\Gamma_2) \gg$$

The last two terms on the r.h.s. are equal to

$$\int [DS_1][DS_2] (\ll W(S_1)W(S_2) \gg - \ll W(S_1) \gg \ll W(S_2) \gg).$$

The expression in the brackets obviously vanishes in the master–field approximation for $\phi$ unless the surfaces $S_1$ and $S_2$ are intersecting. However as soon as $D > 4$ two 2–dimensional surfaces in general locations do not intersect at
all. For $D = 4$ they intersect only at isolated points. It follows that disconnected surfaces do not contribute to the correlators (perhaps, up to contact terms), which are thus represented as sums over connected 2-dimensional surfaces with a fixed boundary distributed according to the area law. (At large distances the minimal–area surfaces will dominate.)

Another example of the same type arises when the KMM theory is considered at finite temperature $\beta^{-1}$ (not to be confused with the “temperature” in our analysis of auxiliary 2-dimensional systems). What is peculiar to this situation in Yang-Mills theory is the occurrence of uncontractable closed contours $C$ which wrap around the periodic imaginary time $\tau$ with a period $\beta$. This contour has an associated Polyakov loop (PL) operator:

$$L(C) \equiv W(C) \xrightarrow{\text{ncI}} \text{trP} \exp i \oint_0^\beta A_0 d\tau$$

(96)

Usually this operator is the relevant order parameter for the confinement–deconfinement phase transition. It is however not suitable for KMM since it is not invariant under the $\mathbb{Z}(N)$ gauge symmetry so that $\ll L \gg = 0$ for all temperatures. This is, of course, contrary to the usual case where in the high-temperature phase $\ll L \gg \neq 0$, signaling the breakdown of a global $\mathbb{Z}_N$-symmetry. Our standard trick of “filling” the Polyakov loop is not applicable since the contour in uncontractable. What can be considered instead in the KMM is the counterpart of a correlator of two Polyakov loops (Fig.6):

$$\ll \mathcal{L}(C, C') \gg \equiv L(C) L(C') \prod_{\Box \in S(C, C')} W(\Box)$$

(97)

One can consider the behaviour of this “Filled Polyakov Loop” (FPL) when the distance $R$ between $C$ and $C'$ is large. In analogy with Polyakov’s result for the ordinary QCD [12] one can suggest that there are two possible regimes:

$$\ll \mathcal{L}(C, C') \gg \rightarrow e^{-m(\beta)R}; \quad \beta > \beta_{\text{crit}}$$
$$\ll \mathcal{L}(C, C') \gg \rightarrow \text{const}; \quad \beta < \beta_{\text{crit}}$$

(98)

and there is a confinement deconfinement transition at some critical inverse temperature $\beta_{\text{crit}}$, such that $m(\beta_{\text{crit}}) = 0$. [9]

[9] What is amusing is that precisely the same configuration was suggested in [10] to describe the vortex-antivortex configuration in the matrix model for the $d = 1$ string. One
Thus, if one accepts that observables in KMM can be represented as sums over fluctuating 2d surfaces, then their contractable boundaries (Wilson Loops) play the role of the physical states (gluonia), while uncontractable boundaries (Polyakov Lines) describe (world-sheet) vortices. We see, that in contrast to the Wilson theory the string–like description is somehow built into the KMM. It just cannot be treated in any other terms. This may be the main reason why this model should be studied in more detail and why one may ultimately believe in its relevance for the theoretical analysis of realistic QCD.

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can consider the boundary Polyakov lines $L(C)$ and $L^\dagger(C')$ as a vortex and antivortex on the world sheet given by the net $\prod_{D\in S(C,C')} W(\Box)$. This coincidence is not accidental and is connected to the fact that the confinement–deconfinement (or Hagedorn) transition is equivalent to the Berezinski-Kosterlitz-Thouless (BKT) transition on the world-sheet which involves condensation of the vortex-antivortex pairs. The critical phase transition occurs when the vortex-antivortex configurations become unsuppressed, i.e. precisely when $\ll L(C,C') \gg \neq 0$. Analogy between the confinement-deconfinement transition in large $N$ QCD and the BKT transition on the world sheet was also considered in [14].
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Figure Captions:

Fig. 1  Adjoint Wilson Loop for a rectangular contour $\Gamma$.

Fig. 2  Filled Wilson Loop for a rectangular surface $S$.

Fig. 3  Example of the substitution of the “fat graph” $\Gamma_1 + \Gamma_2 + \ldots \Gamma_6 = 0$ by an ordinary graph $\tilde{\Gamma}$ according to the identity (41).

Fig. 4  Examples of contours with “inverse steps” which are included in the sums (4) and (85). Only contours of the type (b) exist in the case when the lattice is a one dimensional chain.

Fig. 5  The identity $\ll \hat{W}(\Gamma_1; \Gamma_2) \gg = \ll \hat{W}(\Gamma_1, \Gamma_2) \gg_{\text{connected}} + \ll \hat{W}(\Gamma_1)\hat{W}(\Gamma_2) \gg$.

Fig. 6  The surface $S$ arising in the definition of the filled Polyakov Line $L(C,C')$. 

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