ASYMPTOTIC MORPHISMS AND ELLIPTIC OPERATORS OVER $C^*$-ALGEBRAS

JODY TROiT

ABSTRACT. This paper provides an $E$-theoretic proof of an exact form, due to E. Troitsky, of the Mischenko-Fomenko Index Theorem for elliptic pseudodifferential operators over a unital $C^*$-algebra. The main ingredients in the proof are the use of asymptotic morphisms of Connes and Higson, vector bundle modification, a Baum-Douglas-type group, and a $KK$-argument of Kasparov.

1. Introduction

Let $A$ be a $C^*$-algebra with unit and $M$ be a smooth closed compact manifold. Mishchenko and Fomenko [MF80] consider an elliptic pseudodifferential $A$-operator $P : C^\infty(E_1) \to C^\infty(E_2)$ acting between the spaces of smooth sections of smooth vector $A$-bundles $E_i \to M$, whose fibers are finite projective modules over $A$. The analytic index of $P$ is the $K$-theory-valued Fredholm index $\text{Index}_a(P) \in K_0(A)$. If the kernel and cokernel of $P$ are finite projective $A$-modules, then it follows $\text{Index}_a(P) = [\text{Ker}(P)] - [\text{Coker}(P)] \in K_0(A)$. The topological index of $P$ is also an element $\text{Index}_t(P) \in K_0(A)$ defined by a Gysin map construction which embeds $M$ into Euclidean space and invokes Bott Periodicity. This uses the principal symbol $\sigma(P) : \pi^*(E_1) \to \pi^*(E_2)$ which defines an element $[\sigma(P)] \in K_0^A(T^*M) \cong K_0(C_0(T^*M) \otimes A)$, the topological $K$-theory of vector $A$-bundles on the cotangent bundle $\pi : T^*M \to M$.

Our goal is to prove that these two $K$-theory classes are actually the same

$$\text{Index}_a(P) = \text{Index}_t(P) \in K_0(A),$$

not in $K_0(A) \otimes \mathbb{Q}$ (which kills torsion) as originally proved by Mishchenko and Fomenko. If $A = \mathbb{C}$, this is the classical Atiyah-Singer Index Theorem [AS68]. This theorem has also been obtained by E. Troitsky [Tro96, Tro88, Tro93a, Tro93]. A complete proof is given in [Tro96], where he uses a generalization of the axiomatic method of Atiyah and Singer. In this paper, we prove this index theorem using new $E$-theoretic asymptotic morphism techniques, which should generalize well to equivariant and graded versions of the index theorem.

The hardest part in proving these index theorems is contained in showing that the analytic index is preserved with respect to changing the underlying base manifold from $M$ to $S$ where $S \to M$ is a smooth compact fiber bundle as contained in the multiplicative axiom B.3 [AS68]. By the Thom isomorphism $K_0^A(T^*M) \cong K_0^A(T^*S)$, it follows that the topological index is well-behaved with respect to this operation. However, the analytic index is far more delicate, even in the classical
setting. Our technique makes consistent use of this “vector bundle modification”
construction (Section 2), but bypasses the difficult calculation for the analytic index
by appealing to the asymptotic morphisms of the Connes-Higson $E$-theory [CH89].

Specifically, we use an asymptotic morphism (constructed in Appendix A)
$$\{\Phi_t\}_{t \in [1, \infty)} : C_0(T^*M) \otimes A \to K(L^2 M) \otimes A$$
which is naturally associated to $M$ and $A$ up to equivalence. If $A = \mathbb{C}$, this
asymptotic morphism is essentially the same as the one used by Higson in his
proof of the index theorem for classical first-order differential operators [Hig93]. In
Section 3, we prove that the induced map on $K$-theory,
$$\Phi_* : K_A^0(T^*M) \to K_0(A),$$
is precisely the topological index $\text{Index}_t(P) = \Phi_*([\sigma(P)])$. This is done by showing
that the topological index and this “morphism” index $\text{Index}_m = \Phi_*$ induce the
same group isomorphism
$$\text{Index}_t \simeq \text{Index}_m : \text{Ell}(A) \xrightarrow{\sim} K_0(A)$$
where Ell$(A)$ is a $K$-homological Baum-Douglas-type group [BD82] which incorpo-
rates vector bundle modification. (In Appendix B, we prove that the induced map is
Bott Periodicity if $M = \mathbb{R}^n$ by generalizing Atiyah’s elliptic operator proof [Ati68]
and relate it to the Thom isomorphism.)

In Section 4, we develop techniques that show the above asymptotic morphism,
in a sense, “quantizes” the principal symbol $\sigma(D)$ (considered as a matrix-valued
function on the phase space $T^*M$) of a self-adjoint first-order elliptic differential
$A$-operator $D$ on Euclidean space $M = \mathbb{R}^n$. The index theorem is then proven
in Section 5 as follows. First, we establish it for first-order elliptic differential $A$-
operators (on arbitrary smooth closed manifolds) by adapting Higson’s asymptotic
morphism method for $A = \mathbb{C}$ mentioned above. We then use a $KK$-theory argument
of Kasparov which says that, up to homotopy, every elliptic pseudodifferential $A$-
operator on a smooth closed spin$^c$ manifold is given by a first-order Dirac operator
$D_E$ twisted by a vector $A$-bundle $E$.

The material in this paper formed a part of my Ph.D. thesis [Trou95] at the
Pennsylvania State University. I want to thank my advisors Paul Baum and Nigel
Higson for their great help and encouragement. I also want to thank Guennadi
Kasparov and the referee for their helpful suggestions.

2. Vector Bundle Modification

For a smooth closed Riemannian manifold $M$, let $L^2(M)$ denote the Hilbert
space of square-integrable functions on $M$. If $E \xrightarrow{p} M$ is a complex Hermitian
bundle, let $L^2(M, E)$ (or $L^2(E)$ if there is no confusion) denote the Hilbert space
of square-integrable sections of $E$. Let $C^*_r(E)$ denote the $C^*$-algebra of bundle
endomorphisms of the pull-back bundle $\mathbb{E} = p^*\Lambda^* E \to E$ which vanish at infinity
on $E$, where $\Lambda^* E$ is the exterior algebra bundle. (See Definition B.8.)

Let $F \xrightarrow{p} M$ be a smooth Euclidean vector bundle on the manifold $M$. Let
$\iota : M \xrightarrow{} E$ denote the canonical embedding of $M$ into $E$ as the zero section. It
follows that $F$ is the normal bundle of this embedding. Let $s_\ast : TM \hookrightarrow TF$ denote the induced embedding. The normal bundle of this embedding is $TF$ and is just the pull-back to $TM$ of $F \oplus F$ [AS68, LM89], that is, $TF = \pi_\ast_M(F \oplus F)$, where $\pi_M : T^\ast M \rightarrow M$ denotes the projection of the cotangent bundle. This bundle $TF \rightarrow TM$ has a canonical complex structure given by

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$ 

By using the given metrics on $M$ and $F$, we identify $\pi : E = T^\ast F \rightarrow T^\ast M$ as this complex bundle. We denote $C_\tau(T^\ast F) = C_\tau(E)$ as above. Let $F_C = F \otimes_\mathbb{R} C = F \oplus iF$ denote the complexification of the bundle $F$. Let $F = p^\ast \Lambda^\ast F_C$. (Compare Definition B.8 in the appendix.) Let $\pi_F : T^\ast F \rightarrow F$ denote the cotangent bundle of $F$ considered as a manifold.

**Lemma 2.1.** $E = \pi^\ast(\Lambda^\ast E) \cong \pi_\ast^\ast(F)$. 

**Proof.** By definition, $E \cong \pi_\ast^\ast(F_C)$ as a complex vector bundle and the following diagram

$$
\begin{array}{ccc}
T^\ast F & \xrightarrow{\pi} & T^\ast M \\
\downarrow \pi_F & & \downarrow \pi_M \\
F & \xrightarrow{p} & M
\end{array}
$$

commutes. Therefore, $E \cong \pi^\ast \Lambda^\ast \pi_\ast^\ast F_C \cong \pi_\ast^\ast \Lambda^\ast p^\ast F_C = \pi_\ast^\ast F$. □

Choose a complex vector bundle $G \rightarrow F$ such that $F \oplus G \cong F \times \mathbb{C}^n$ is trivial [M158]. It follows by the previous lemma that $E \oplus \pi_\ast^\ast G \cong T^\ast F \times \mathbb{C}^n$. Therefore, we have the following isometry of Hilbert spaces

$$V : L^2(F, F) \hookrightarrow L^2(F, F \oplus G) \cong L^2(F)^n.$$ 

This induces the inclusion

$$Ad(V) : \mathcal{K}(L^2(F, F)) \hookrightarrow \mathcal{K}(L^2(F)^n) = M_n(\mathcal{K}(L^2 F)),$$

by the mapping $K \mapsto KV^\ast$. We also have the inclusion of $C^\ast$-algebras

$$J : C_\tau(T^\ast F) \hookrightarrow C_0(T^\ast F, \text{End } \mathbb{C}^n) \cong M_n(C_0(T^\ast F)).$$

**Lemma 2.2.**

1.) $J_\ast : K_0(C_\tau(T^\ast F)) \rightarrow K_0(C_0(T^\ast F))$ is an isomorphism.

2.) $Ad(V)_\ast = id : K_0(\mathcal{K}(L^2(F, F))) \rightarrow K_0(\mathcal{K}(L^2 F))$

**Proof.** 1.) If $F$ is trivial, the result holds. In general, apply a Mayer-Vietoris argument and the Five Lemma.

2.) Recall that any two isometries of a separable Hilbert space are connected by a strongly continuous path of isometries. Hence, $V$ is homotopic to a unitary isomorphism $U$. It follows that $Ad(V)$ is homotopic to $Ad(U)$. Thus, $Ad(V)_\ast = Ad(U)_\ast = id$ since it maps rank one projections to rank one projections. □

Referring to Appendix A, let

$$\{ \Phi_t^M \} : C_0(T^\ast M) \rightarrow \mathcal{K}(L^2 M)$$

be the index asymptotic morphism for $M$. The following lemma allows us to “twist” this asymptotic morphism with a complex Hermitian bundle $H \rightarrow M$. This is the “vector bundle modification” construction for the index asymptotic morphism.

Let $C_H(T^\ast M)$ denote the $C^\ast$-algebra of endomorphisms of the vector bundle $\pi_M^\ast H \rightarrow T^\ast M$ vanishing at infinity in the operator norm induced by the metrics on $H$ and $M$. 

Lemma 2.3. Let $H \to M$ be a smooth Hermitian bundle on $M$. There is an asymptotic morphism

$$\{\Phi^H_t\} : C_H(T^*M) \to \mathcal{K}(L^2(M, H))$$

such that if $\alpha \in C_H(T^*M)$ has support in $T^*U$, where $H|_U \cong U \times \mathbb{C}^n$ on the open subset $U \subset M$, then

$$\lim_{t \to \infty} \|\Phi^H_t(f) - M_n(\Phi^M_t(f))\| = 0$$

where we identify $C_H(T^*U) \cong C_0(T^*U, \text{End } \mathbb{C}^n) \cong M_n(C_0(T^*U))$.

Proof. If $H \cong M \times \mathbb{C}^n$ is trivial, then $C_H(T^*M) \cong M_n(C_0(T^*M))$ and $L^2(M, H) \cong L^2(M)^n$. Define $\{\Phi^H_t\}$ to be the $n \times n$ matrix extension of $\{\Phi^M_t\}$. Now use a partition of unity $\{\rho_j^2\}$ subordinate to an open cover $\{U_j\}_1^n$ over which $H$ trivializes $H|_{U_j} \cong U_j \times \mathbb{C}^n$ and a gluing argument. □

Considering $E = T^*F$ as a manifold and $H = F \to F$, there is, by tensoring with the identity $id_A : A \to A$, an asymptotic morphism

$$\{\Phi^{E,A}_t\} : C_\tau(T^*F) \otimes A \to \mathcal{K}(L^2(F, F)) \otimes A.$$ 

We want to use this asymptotic morphism to relate the $A$-index asymptotic morphisms associated to the manifolds $M$ and $F$,

$$\{\Phi^{F,A}_t\} : C_0(T^*F) \otimes A \to \mathcal{K}(L^2 F) \otimes A$$

$$\{\Phi^{M,A}_t\} : C_0(T^*M) \otimes A \to \mathcal{K}(L^2 M) \otimes A$$

and the Thom homomorphism (Proposition B.13)

$$\Psi = \Psi^E : C_0(\mathbb{R}) \otimes C_0(T^*M) \to C_0(\mathbb{R}) \otimes C_\tau(T^*F)$$

associated to $E = T^*F \to T^*M$. To this end, we need to construct an elliptic operator of index one acting along the fibers $F_m$ of $p : F \to M$.

Definition 2.4. For each point $m \in M$ and $t > 0$, let

$$B^m_t : \mathcal{S}(F_m, A^* F_m) \to \mathcal{S}(F_m, A^* F_m)$$

denote the operator constructed in Definition B.3. For each $m$, the kernel Ker$(B^m_t)$ is one-dimensional and spanned by the 0-form $v \mapsto e^{-t \|v\|^2}$ (Theorem B.4). Since $B^m_t$ is $O(n)$-equivariant, the collection

$$\mathbb{B}_t = \{B^m_t : m \in M\}$$

defines a smooth family of operators acting along the fibers of $F$.

Definition 2.5. Define, for each $t > 0$, the map

$$\alpha_t : C_c(M) \to \text{Ker}(\mathbb{B}_t)$$

by the following formula

$$\alpha_t(f)(v) = \left(\frac{2t}{\pi}\right)^{n/4} f(p(v)) e^{-t \|v\|^2}, \quad v \in F.$$
Thus, the diagram reduces to the following:

\[ \{ \alpha_t \} : L^2(M) \to L^2(F, \mathbb{F}) \]

which is an isomorphism onto \( \text{Ker}(\mathbb{R}_t) \).

**Corollary 2.7.** There is a continuous family of injective \(*\)-homomorphisms

\[ \{ \text{Ad}(\alpha_t) \} : \mathcal{K}(L^2M) \to \mathcal{K}(L^2(F, \mathbb{F})) \].

Define \( \{ \beta_t \} : C_0(\mathbb{R}) \otimes \mathcal{K}(L^2M) \to C_0(\mathbb{R}) \otimes \mathcal{K}(L^2(F, \mathbb{F})) \) to be the suspension \( \{ 1 \otimes \text{Ad}(\alpha_t) \} \) of \( \{ \text{Ad}(\alpha_t) \} \).

**Lemma 2.8.** The induced map \( \beta_* = \text{id} \) on \( K \)-theory.

**Proof.** \( \{ \beta_t \} \) is homotopic to an isomorphism since \( \{ \text{Ad}(\alpha_t) \} \) is homotopic to an isomorphism. \( \square \)

We come to the main result of this section. Consider the following suspended diagram of \( C^* \)-algebras and asymptotic morphisms:

\[
\begin{array}{ccc}
C_0(\mathbb{R}) \otimes M_n(C_0(T^*F)) & \xrightarrow{1 \otimes \Phi_F^t} & C_0(\mathbb{R}) \otimes M_n(\mathcal{K}(L^2F)) \\
\uparrow 1 \otimes J & & \uparrow 1 \otimes \text{Ad}(V) \\
C_0(\mathbb{R}) \otimes C_\tau(T^*F) & \xrightarrow{1 \otimes \Phi_\tau^t} & C_0(\mathbb{R}) \otimes \mathcal{K}(L^2(F, \mathbb{F})) \\
\uparrow \Psi & & \uparrow \{ \beta_t \} \\
C_0(\mathbb{R}) \otimes C_0(T^*M) & \xrightarrow{1 \otimes \Phi_1^M} & C_0(\mathbb{R}) \otimes \mathcal{K}(L^2M)
\end{array}
\]

**Theorem 2.9.** The diagram above commutes up to homotopy.

**Proof.** 1.) If \( M = \{ pt \} \), the theorem is true by Corollary B.21.

2.) If \( F = M \times \mathbb{C}^n \), then we have that

\[ C_\tau(T^*F) = C_0(T^*M \times T^*\mathbb{R}^n, \text{End} \Lambda^* \mathbb{C}^n) \cong M_n(C_0(T^*\mathbb{R}^n)) \otimes C_0(T^*M). \]

Thus, the diagram reduces to the following:

\[
\begin{array}{ccc}
C_0(\mathbb{R}) \otimes M_nC_0(T^*\mathbb{R}^n) \otimes C_0(T^*M) & \xrightarrow{1 \otimes \Phi_F^t \otimes \Phi_1^M} & C_0(\mathbb{R}) \otimes M_n\mathcal{K}(L^2\mathbb{R}^n) \otimes \mathcal{K}(L^2M) \\
\uparrow 1 \otimes \Psi & & \uparrow \beta_t \otimes 1 \\
C_0(\mathbb{R}) \otimes C_0(T^*M) & \xrightarrow{1 \otimes \Phi_1^M} & C_0(\mathbb{R}) \otimes \mathcal{K}(L^2M)
\end{array}
\]

The homotopy is given by the formula

\[ f \otimes g \mapsto f(\epsilon x + s^{-1} \bar{B}_t) \otimes \Phi_1^M(g), \ 0 \leq s \leq 1, \]

and is the tensor product of the homotopy in Lemma B.20 with the index asymptotic morphism for the manifold \( M \).

3.) In general, the homotopy in the previous case patches together via a partition of unity from \( M \) to form the desired homotopy since it is diffeomorphism invariant by Theorem B.4 (6) and Corollary A.12. \( \square \)
Corollary 2.10. For any $C^*$-algebra $A$, the following diagram commutes:

$$
\begin{array}{ccc}
K_0(C_0(T^*F) \otimes A) & \xrightarrow{\phi_{F,A}} & K_0(A) \\
\uparrow\{\Psi \otimes id_A\}, & & \uparrow= \\
K_0(C_0(T^*M) \otimes A) & \xrightarrow{\phi_{M,A}} & K_0(A)
\end{array}
$$

Proof. Tensor the diagram prior to Theorem 2.9 with $id_A : A \to A$. The resulting diagram commutes up to homotopy by tensoring the homotopy from Theorem 2.9 with $id_A$. Now take the induced maps and invoke homotopy invariance. \hfill \square

3. The Topological and Morphism Indices

Let $A$ be a $C^*$-algebra with unit. In this section, we assemble (the symbols of) all elliptic pseudodifferential $A$-operators on all manifolds $M$ into an abelian group $\text{Ell}(A)$ using $K$-homological ideas of Baum and Douglas [BD82]. The topological index and “morphism” index will define two group homomorphisms

$$\text{Ind}_t, \text{Ind}_m : \text{Ell}(A) \to K_0(A).$$

By checking examples for $M = \{pt\}$ we will show that they are both group isomorphisms. Bott Periodicity will then show that they are the same $\text{Ind}_m = \text{Ind}_t$.

Let $M$ be a smooth compact manifold. Recall that a vector $A$-bundle $E \to M$ is a locally trivial fiber bundle whose fibers $E_p$ for $p \in M$ are given by a finite projective $A$-module $P$ [Kar78, MF80]. (The structure group is the automorphism group $\text{Aut}_A(P)$ of $P$.) Denote by $K_A^0(M)$ the Grothendieck group of all (isomorphism classes of) vector $A$-bundles on $M$ under direct sum. (If $A = \mathbb{C}$, this is the ordinary topological $K$-theory of $M$.) If $M$ is only locally compact, then we can identify $K_A^0(M) = \text{def} \ker\{K_A^0(M^+) \to K_A^0(\{\infty\}) = K_0(A)\}$ as the abelian group generated by vector $A$-bundle homomorphisms $\sigma : E \to F$ with compact support under direct sum, where $\text{supp}(\sigma) = \{p \in M \mid \sigma_p : E_p \to F_p \text{ is not a module isomorphism}\}$.

There are natural notions of isomorphism and pull-backs as in the classical case $A = \mathbb{C}$. Since $M$ is a smooth manifold, we may take $E, F$, and $\sigma$ to also be smooth.

There is the Mingo-Serre-Swan isomorphism $K_A^0(M) \cong K_0(C_0(M) \otimes A)$, which is induced by taking sections as in the classical case [Min82, Swa62].

A Hermitian $A$-metric on $E$ is a smooth choice $\langle \cdot, \cdot \rangle_p : E_p \times E_p \to A$ of Hilbert $A$-module structures on the fibers of $E$. Every $A$-bundle has a Hermitian $A$-metric by a smooth partition of unity argument and using the fact that any finite projective $A$-module $P$ has a canonical Hilbert $A$-module structure (up to unitary isomorphism) [W093]. Moreover, any two such Hermitian $A$-metrics are homotopic to each other via the straight line homotopy.

Let $\sigma : E \to F$ be a homomorphism of vector $A$-bundles equipped with Hermitian $A$-metrics. There is an adjoint homomorphism $\sigma^* : F \to E$ such that

$$\langle \sigma_p(e_p), f_p \rangle_p = \langle e_p, \sigma^*_p(f_p) \rangle_p$$

for all $e_p \in E_p$ and $f_p \in F_p$. Furthermore, $\text{supp}(\sigma^*) = \text{supp}(\sigma)$ and $\sigma^*$ is well-defined up to homotopy.
Let $\pi : V \to M$ be a complex vector bundle. Let $c : V \to \text{End}(V)$ denote the canonical section of the bundle $V = \pi^*(\Lambda^*V) = \mathcal{V}^{even} \oplus \mathcal{V}^{odd} \to V$. Put a Hermitian metric on $V$. Define a bundle morphism $\lambda_V : \mathcal{V}^{even} \to \mathcal{V}^{odd}$ by the formula

$$\lambda_V(\omega_v) = c(v)\omega_v = v \wedge \omega_v - v_v \omega_v,$$

It follows that $\text{supp}(\lambda_V)$ is the zero section $M$ of $V$. (Compare Definition B.8 and Lemma B.9.)

**Definition 3.1.** (Sharp Product) For an $A$-bundle homomorphism $\sigma : E \to F$ on $M$, define the $A$-bundle homomorphism on $(T^*M)$

$$\pi^*(\sigma)\#\lambda_V : \pi^*E \otimes \mathcal{V}^{even} \oplus \pi^*F \otimes \mathcal{V}^{odd} \to \pi^*F \otimes \mathcal{V}^{odd} \oplus \pi^*E \otimes \mathcal{V}^{even}$$

by the formula

$$\pi^*(\sigma)\#\lambda_V = \begin{pmatrix} \pi^*(\sigma) \otimes 1 & -1 \otimes \lambda^*_V \\ 1 \otimes \lambda_V & \pi^*(\sigma^*) \otimes 1 \end{pmatrix},$$

where $\pi^*(\sigma) : \pi^*E \to \pi^*F$ denotes the pull-back of $\sigma$. Note that

$$\text{supp}(\pi^*(\sigma)\#\lambda_V) = \pi^{-1}(\text{supp}(\sigma)) \cap \text{supp}(\lambda_V) = \text{supp}(\sigma)$$

is a compact subset of $V$, where we identify $M$ as the zero section of $V$.

**Proposition 3.2.** (Thom Isomorphism [Troi88, Troi93]) The map $\Theta : K^0_A(M) \to K^0_A(V)$ defined by

$$[\sigma] \mapsto [\pi^*(\sigma)\#\lambda_V]$$

is an isomorphism.

**Definition 3.3.** Define $\text{Ell}(A)$ to be the Grothendieck group of the semigroup generated by all pairs of the form $(M, \sigma)$, where $M$ is a smooth, second-countable, Hausdorff manifold without boundary and $\sigma : E \to F$ is a homomorphism of smooth vector $A$-bundles on the cotangent bundle $T^*M$ with compact support, subject to the following relations:

1. **Isomorphism:** $(M, \sigma) = (M', \sigma')$ if there is a diffeomorphism $\phi : M \to M'$ such that $\sigma \cong \phi^*(\sigma')$, where $\phi : T^*M \to T^*M'$ denotes the induced map.$^1$

2. **Homotopy:** $(M, \sigma_0) = (M, \sigma_1)$ if $\sigma_0$ and $\sigma_1$ are homotopic ($\sigma_0 \sim_h \sigma_1$). That is, there is a pair $(M \times [0,1], \hat{\sigma})$ such that $\hat{\sigma}|_{T^*M \times 0} \cong \sigma_0$ and $\hat{\sigma}|_{T^*M \times 1} \cong \sigma_1$.

3. **Excision:** $(M, \sigma) = (U, \sigma|_{T^*U})$ if $U \subset M$ is open with $\text{supp}(\sigma) \subset T^*U$.

4. **Direct Sum - Disjoint Union:**

   $$(M, \sigma) + (M', \sigma') = (M \amalg M', \sigma \amalg \sigma')$$

   $$(M, \sigma) + (M, \tau) = (M, \sigma \oplus \tau)$$

5. **Vector Bundle Modification:** If $F \to M$ is a smooth Euclidean vector bundle, then

   $$(M, \sigma) = F \#(M, \sigma) =_{\text{def}} (F, \pi^*(\sigma)\#\lambda_V)$$

   where $V = T^*F \xrightarrow{\pi} T^*M$ has the complex bundle structure from the beginning of Section 2 and $\pi^*(\sigma)\#\lambda_V$ is the sharp product in Definition 3.1.

$^1$See the discussion preceding Proposition A.7.
Note that Ell(A) is abelian by the isomorphism relation. A pair \((M, \sigma)\) in Ell(A) will be called a symbol pair and \(\sigma\) will be called an A-symbol. If \(\text{supp}(\sigma) = \emptyset\), then \((M, \sigma)\) will be called a trivial pair. From the definition of Ell(A), we see that \((M, \sigma) \in \text{Ell}(A)\) if and only if \([\sigma] \in K^0_A(T^*M)\).

**Lemma 3.4.** The following are true:

1. If \((M, \sigma)\) is trivial, then \((M, \sigma) = 0 \in \text{Ell}(A)\).
2. If \((M, \sigma_0) = (M, \sigma_1)\) via homotopy, then \([\sigma_0] = [\sigma_1] \in K^0_A(T^*M)\).
3. If \([\sigma] = 0 \in K^0_A(T^*M)\), then \((M, \sigma) = 0 \in \text{Ell}(A)\).

**Proposition 3.5.** For every symbol pair \((M, \sigma)\), there is a pair \((\mathbb{R}^n, \tau)\) such that \((M, \sigma) = (\mathbb{R}^n, \tau) \in \text{Ell}(A)\), for some \(n\).

**Proof.** Let \(\phi : M \hookrightarrow \mathbb{R}^n\) be a smooth embedding of the manifold into Euclidean space, which exists by the Whitney Embedding Theorem. By Isomorphism (1), we may identify \(M\) with its image in \(\mathbb{R}^n\). Let \(N\) be an open tubular neighborhood of \(M\) in \(\mathbb{R}^n\). Then \(M \subset N \subset \mathbb{R}^n\) and \(N\) has the structure of an \(\mathbb{R}\)-vector bundle \(\pi : N \to M\). By Vector Bundle Modification (5), we have that

\[(M, \sigma) = (N, \hat{\sigma})\]

where \(\hat{\sigma} = p^*(\sigma) \circ \lambda_{T^*N} : E \to F\) (\(E\) and \(F\) are vector \(A\)-bundles over \(T^*N\)) and \(p : T^*N \to T^*M\). Choose an \(A\)-bundle \(G\) such that \(F \oplus G \cong T^*N \times A^m\). Then

\[\hat{\sigma} \oplus 1_G : E \oplus G \to F \oplus G \cong T^*N \times A^m\]

and has compact \(\text{supp}(\hat{\sigma} \oplus 1_G) = \text{supp}(\sigma) \subset T^*N\). By Lemma 3.4 and Direct Sum - Disjoint Union (4), we have

\[(N, \hat{\sigma}) = (N, \hat{\sigma}) + (N, 1_G) = (N, \hat{\sigma} \oplus 1_G)\]

Let \(H = T^*\mathbb{R}^n \times A^m\). Since \(\hat{\sigma} \oplus 1_G\) is an isomorphism off the compact set \(K = \text{supp}(\hat{\sigma})\), we can use a clutching construction with \(1_H : H \to \mathcal{H}\) to obtain the \(A\)-symbol

\[\tau = (\hat{\sigma} \oplus 1_G) \amalg_{T^*N \setminus K} 1_H : (E \oplus G) \amalg_{T^*N \setminus K} H \to H\]

Since \(\tau|_{T^*N} = \hat{\sigma} \oplus 1_G\) and \(\text{supp}(\tau) \subset T^*N\), we have by Excision (3) that

\[(M, \sigma) = (N, \hat{\sigma}) = (N, \hat{\sigma} \oplus 1_G) = (N, \tau|_{T^*N}) = (\mathbb{R}^n, \tau)\]

as was desired. \(\square\)

Although the Gysin construction is well-known in the classical (complex bundle) setting [AS68, LM89], we will define it, since we will need to refer to the construction later.

**Lemma 3.6.** Let \(g : M \hookrightarrow M'\) be a proper smooth embedding of smooth manifolds. There is a canonical functorial homomorphism

\[g_* : K^0_A(T^*M) \to K^0_A(T^*M')\]

That is, if \(h : N \hookrightarrow Z\) is an embedding, then \((h \circ g)_* = h_* \circ g_*\).

The induced map \(\hat{g} : T^*M \to T^*M'\) embeds \(T^*M\) as an open submanifold of \(T^*M'\). Let \(N(T^*M)\) be the normal bundle of this embedding (defined by pulling back \(T^*M\) from \(M\) to \(N\)). Let \(E = T^*M \times \mathbb{R}^n\) and \(F = T^*M \times \mathbb{R}^m\). If \(\phi : M \hookrightarrow \mathbb{R}^n\) is a smooth embedding of \(M\) into \(\mathbb{R}^n\) such that \(T^*M \subset E\), then \(\phi_* : K^0_A(E) \to K^0_A(F)\) is the homomorphism induced by \(\phi\).
back the normal bundle $N(M)$ of $M$ in $M'$. It has the structure of a smooth complex vector bundle $N(T^*M) \to T^*M$ and is an open subset of $T^*M'$. The map $g_*$ is defined as the composition

$$K^0_A(T^*M) \xrightarrow{\Theta} K^0_A(N(T^*M)) \xrightarrow{i_*} K^0_A(T^*M')$$

where $\Theta$ is the Thom isomorphism and $i : N(T^*M) \hookrightarrow T^*M'$ is the inclusion.

**Remark 3.7.** If $g$ is the inclusion of $M$ as an open submanifold of $N$, then $g_*$ is the map induced by the open inclusion $T^*M \subset T^*N$, since the normal bundle is the zero vector bundle $T^*M \times 0 = T^*M$.

**Proposition 3.8.** If $f : M \to N$ is a smooth map of smooth manifolds, there is a canonical homomorphism $f_1 : K^0_A(T^*M) \to K^0_A(T^*N)$ depending only on the homotopy class of $f$.

Let $g : M \hookrightarrow \mathbb{R}^n$ be any smooth embedding. Then $f \times g : M \to N \times \mathbb{R}^n$ is also an embedding. The Gysin map $f_1$ is defined to be the composition

$$K^0_A(T^*M) \xrightarrow{(f \times g)_*} K^0_A(T^*N \times T^*\mathbb{R}^n) \xrightarrow{\Theta^{-1}} K^0_A(T^*N)$$

and is independent of the choice of embedding $g$.

Let $f^M : M \to \{pt\}$ denote the unique map to a point. The previous results imply that $f_1^M = \alpha_A \circ g_*$, where $g : M \hookrightarrow \mathbb{R}^n$ is any embedding of $M$ into Euclidean space and $\alpha_A : K_0(C_0(T^*\mathbb{R}^n) \otimes A) \xrightarrow{\mathbb{R}} K_0(A)$ is Bott Periodicity (Theorem B.1).

**Definition 3.9.** (The Topological Index)
For each $(M, \sigma) \in \text{Ell}(A)$, define the **topological index** of $(M, \sigma)$ by the formula

$$\text{Ind}_t(M, \sigma) = f^M_1([\sigma]) \in K^0(A)$$

where $f^M_1 : K^0_A(T^*M) \to K^0_A(\{pt\}) = K_0(A)$ is the associated Gysin map.

**Theorem 3.10.** $\text{Ind}_t : \text{Ell}(A) \to K_0(A)$ induces an isomorphism of abelian groups.

**Proof.** First, we must show that $\text{Ind}_t$ is well-defined, i.e., it respects the equivalence relations in Definition 3.3 used to define the group $\text{Ell}(A)$.

1. **Isomorphism:** Obvious.
2. **Homotopy:** Follows from Lemma 3.4 (2). If $\sigma_0 \sim_h \sigma_1$ are homotopic then $[\sigma_0] = [\sigma_1] \in K^0_A(T^*M)$ and so

$$\text{Ind}_t(M, \sigma_0) = f^M_1[\sigma_0] = f^M_1[\sigma_1] = \text{Ind}_t(M, \sigma_1).$$

3. **Excision:** Suppose $U \subset M$ is open and $\text{supp}(\sigma) \subset T^*U$. It follows (Remark 3.7) that since $U$ is open and the inclusion $i : U \hookrightarrow M$ is an embedding, that the normal bundle $N(T^*U) = T^*U$ is the zero vector bundle over $T^*U$. Thus, the Thom isomorphism $\Theta$ is the identity map on $K^0_A(T^*U)$.

Let $g : M \hookrightarrow \mathbb{R}^n$ be an embedding of $M$ into $\mathbb{R}^n$. Then $g \circ i : U \hookrightarrow \mathbb{R}^n$ is an embedding. Hence we have (by Lemma 3.6) that

$$f^U = \Theta^{-1} \circ (g \circ i) = \Theta^{-1} \circ g \circ i = f^M \circ i.$$
which implies
\[
\text{Ind}_t(U, \sigma|_{T^* U}) = f^M_t(i_*(\sigma|_{T^* U})) = f^M_t(i_* \circ i^*(\sigma)) = f^M_t(\sigma) = \text{Ind}_t(M, \sigma).
\]

(4) Direct Sum-Disjoint Union: Let \((M, \sigma)\) and \((M, \tau)\) be given. Then we have that
\[
\text{Ind}_t((M, \sigma) + (M, \tau)) = f^M_t(\sigma \oplus \tau) = f^M_t(\sigma) + f^M_t(\tau) = \text{Ind}_t(M, \sigma) + \text{Ind}_t(M, \tau).
\]

Suppose \((N, \rho)\) is given. Since \(T^*(M \amalg N) = T^*M \amalg T^*N\), we have that
\[
K_0^A(T^*(M \amalg N)) = K_0^A(T^*M) \oplus K_0^A(T^*N).
\]

Let \(g : M \amalg N \hookrightarrow \mathbb{R}^n\) be an embedding. Consider the diagram of inclusions
\[
\begin{array}{ccc}
M & \xrightarrow{i} & M \amalg N \\
\downarrow & & \downarrow j \\
\mathbb{R}^n & \xleftarrow{\varphi} & N
\end{array}
\]

It follows by functoriality (Lemma 3.6) that
\[
\text{Ind}_t(M \amalg M, \sigma \amalg \rho) = \Theta^{-1} \circ g_*(i_*[\sigma] + j_*[\rho]) = f^M_t([\sigma]) + f^M_t([\rho]) = \text{Ind}_t(M, \sigma) + \text{Ind}_t(N, \rho).
\]

(5) Vector Bundle Modification: Let \(F \to M\) be a smooth \(\mathbb{R}\)-vector bundle. Let \(s : M \hookrightarrow F\) denote the inclusion of \(M\) as the zero section. Choose an embedding \(g : F \hookrightarrow \mathbb{R}^n\). Then \(g \circ s : M \hookrightarrow \mathbb{R}^n\) is an embedding of \(M\) as a closed submanifold. By construction \(s_\ast : K_0^A(T^*M) \to K_0^A(T^*F)\) is the Thom Isomorphism for the complex bundle \(T^*F \to T^*M\), since \(F = N\) is the normal bundle of \(M\). Thus,
\[
\text{Ind}_t(F, \pi^*(\sigma) \# \lambda_{T^*F}) = \Theta^{-1} \circ g_*(\pi^*(\sigma) \# \lambda_{T^*F}) = g_*(s_\ast([\sigma])) = \Theta^{-1} \circ (g \circ s)_\ast([\sigma]) = f^M_t([\sigma]) = \text{Ind}_t(M, \sigma).
\]

Therefore, \(\text{Ind}_t : \text{Ell}(A) \to K_0(A)\) induces a well-defined homomorphism.\(^2\)

Let \(M = \{pt\}\) denote the unique 0-dimensional manifold. Then \(f^M : \{pt\} \to \{pt\}\) is the identity and so, by definition,
\[
f^M_t = \text{id} : K_0^A(\{pt\}) = K_0(A) \to K_0(A)
\]
is the identity. Thus, \(\text{Ind}_t\) is surjective. Now to show it is also injective.

\(^2\)Recall that if \(\phi : S \to G\) is a homomorphism from an abelian semigroup \(S\) to an abelian group \(G\) there is a canonical extension \(\hat{\phi} : G(S) \to G\) or the Grothendieck group \(G(S)\).
Suppose \( \text{Ind}_t(M, \sigma) = 0 \). By Proposition 3.5, we may assume that \((M, \sigma) = (\mathbb{R}^n, \tau)\). By construction
\[
f_!^{\mathbb{R}^n} : K_0^{\mathbb{T}}(T^*\mathbb{R}^n) \to K_0(A)
\]
is the (inverse) of the Thom isomorphism for the bundle \( C^n \cong T^*\mathbb{R}^n \to \{pt\} \). This implies that \([\tau] = 0 \in K_0^{\mathbb{T}}(T^*\mathbb{R}^n)\). By Lemma 3.4 (3), we have that
\[
(M, \sigma) = (\mathbb{R}^n, \tau) = 0 \in \text{Ell}(A).
\]
This completes the proof that the topological index is an isomorphism.

For any manifold \( M \), let \( \{\Phi^{M,\mathcal{A}}_t\} : C_0(T^*M) \otimes \mathcal{A} \to \mathcal{K}(L^2M) \otimes \mathcal{A} \) denote the \( \mathcal{A} \)-index asymptotic morphism of \( M \) from Appendix A. Let
\[
\Phi^{M,\mathcal{A}}_* : K_0^\mathbb{T}(T^*M) \to K_0(A)
\]
denote the induced map, where we identify \( K_0^\mathbb{T}(T^*M) \cong K_0(C_0(T^*M) \otimes \mathcal{A}) \) and \( K_0(\mathcal{K} \otimes \mathcal{A}) = K_0(A) \).

**Definition 3.11.** (The Morphism Index)
For each \((M, \sigma) \in \text{Ell}(\mathcal{A})\), define the **morphism index** of \((M, \sigma)\) by the formula
\[
\text{Ind}_m(M, \sigma) = \Phi^{M,\mathcal{A}}_*([\sigma]) \in K_0(A).
\]

**Proposition 3.12.** \( \text{Ind}_m : \text{Ell}(\mathcal{A}) \to K_0(A) \) induces an isomorphism of abelian groups.

The proof proceeds as for the topological index.

**Proof.** First, we show that \( \text{Ind}_m \) respects the equivalence relations in \( \text{Ell}(\mathcal{A}) \).

1. **Isomorphism:** Follows from diffeomorphism invariance (Corollary A.12).
2. **Homotopy:** Obvious.
3. **Excision:** Follows from the restriction property (Corollary A.11).
4. **Direct-Sum Disjoint-Union:** The relation \( \text{Ind}_m(M, \sigma \oplus \tau) = \text{Ind}_m(M, \sigma) + \text{Ind}_m(M, \tau) \)

   follows since \( \Phi^{M,\mathcal{A}}_* \) is a group homomorphism. The disjoint union relation follows from the (compatible) canonical isomorphisms
   \[
   C_0(T^*M \sqcup T^*N) \otimes \mathcal{A} = (C_0(T^*M) \otimes \mathcal{A}) \oplus (C_0(T^*N) \otimes \mathcal{A})
   \]
   \[
   \mathcal{K}(L^2(M \sqcup N)) \otimes \mathcal{A} = (\mathcal{K}(L^2M) \otimes \mathcal{A}) \oplus (\mathcal{K}(L^2N) \otimes \mathcal{A})
   \]
   and so \( \{\Phi^{M\sqcup N,\mathcal{A}}_t\} = \{\Phi^{M,\mathcal{A}}_t\} \oplus \{\Phi^{N,\mathcal{A}}_t\} \).

   And last but not least, the very important:
5. **Vector Bundle Modification:** Follows from Corollary 2.10 and Theorem B.22.

Hence, \( \text{Ind}_t \) induces a well-defined group homomorphism. If \( M = \{pt\} \), then \( \{\Phi^{M,\mathcal{A}}_t\} \cong id_\mathcal{A} : \mathcal{A} \cong A \to A \) and so \( \text{Ind}_t \) is surjective. If \( M = \mathbb{R}^n \), then Bott Periodicity (Theorem B.7) again shows that it is injective.

The Bott Periodicity arguments in Theorem 3.10 and Proposition 3.12 show that these two isomorphisms are, in fact, the same.

**Theorem 3.13.** For any \((M, \sigma) \in \text{Ell}(\mathcal{A})\), the topological and morphism indices are equal, i.e.,
\[
\text{Ind}_t(M, \sigma) = \text{Ind}_m(M, \sigma) \in K_0(A).
\]
4. Elliptic $A$-Operators on Euclidean Space

Let $A$ be a $C^*$-algebra with unit. Let $D$ be a differential $A$-operator of order one on $\mathbb{R}^n$ acting on $A^k$-valued functions. Thus, $D$ has the form

$$D = \sum_{j=1}^{n} a_j(x) \frac{\partial}{\partial x_j} + b(x),$$

where $a_j, b : \mathbb{R}^n \to M_k(A)$ are smooth matrix $A$-valued functions on $\mathbb{R}^n$. Initially, $\text{Domain}(D) = C^\infty_c(\mathbb{R}^n, A^k)$ is the pre-Hilbert $A$-module of smooth, compactly supported vector $A$-valued functions on $\mathbb{R}^n$, where $A^k$ has the standard Hilbert $A$-module structure arising from $A$.

If $D$ is formally self-adjoint on $\text{Domain}(D)$, this implies that the principal symbol

$$\sigma(D)(x, \xi) = \sqrt{-1} \sum_{j=1}^{n} a_j(x) \xi_j \in M_k(A),$$

where $\xi = \sum_{j=1}^{n} \xi_j dx_j \in T^*_x \mathbb{R}^n \cong \mathbb{R}^n$, is a self-adjoint matrix for all $(x, \xi) \in T^* \mathbb{R}^n$. $D$ is called elliptic if $\sigma(D)(x, \xi)$ is an invertible matrix for all $\xi \neq 0$. We will also need the total symbol $\text{sym}(D) = \sigma(D) + b$.

Consider $D$ as a densely defined unbounded symmetric operator on the Hilbert $A$-module $H = L^2(\mathbb{R}^n, A^k)$ of square-integrable vector $A$-valued functions. It follows that $D$ has a densely defined adjoint $D^*$ and $(D \pm i)$ are injective with closed ranges. The operator $D$ is called regular if $(1 + D^* D)$ has dense range. Since $D$ is symmetric, $D$ is regular if and only if $\text{Range}(D \pm i)$ is complemented in $H$. (For a review of unbounded operators on Hilbert $A$-modules, see Lance [Lan95].)

**Theorem 4.1 (Functional Calculus).** If $D$ is a self-adjoint regular $A$-operator on the Hilbert $A$-module $H$, there is a unique $*$-homomorphism

$$\phi_D : C_0(\mathbb{R}) \to \mathcal{L}(H) : f \mapsto f(D)$$

defined by sending $(x \pm i)^{-1} \mapsto (D \pm i)^{-1}$.

**Definition 4.2.** Let $D$ be a formally self-adjoint elliptic partial differential $A$-operator of order one on $\mathbb{R}^n$. We shall call $D$ an essential $A$-operator if the closure $\bar{D}$ is self-adjoint and $(D \pm i)$ have dense range. This implies that $\bar{D}$ is also regular.

Let $\phi$ be a smooth function on $\mathbb{R}^n$. Considered as a multiplication operator on the Hilbert $A$-module $\mathcal{H} = L^2(\mathbb{R}^n, A^k)$ of square-integrable vector $A$-valued functions, $M_\phi$ maps $\text{Domain}(D)$ into itself, so the commutator $[D, M_\phi]$ is defined on the domain of $D$. We have the symbol identity:

$$\sqrt{-1}[D, M_\phi] f(x) = \sigma(D)(x, d\phi(x)) f(x) = \sum_{j=1}^{n} a_j(x) \frac{\partial \phi}{\partial x_j} f(x)$$

where $f \in \text{Domain}(D)$. Thus, the commutator $[D, M_\phi]$ is a pointwise skew-adjoint multiplication operator with same domain as $D$.

Using the symbol identity and the fact that if $\phi$ is a smooth and compactly supported function then $d\phi = \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i} dx_i$ is a compactly supported one-form, we obtain the following.
Lemma 4.3. For any smooth, compactly supported function \( \phi \) on \( \mathbb{R}^n \), the commutator \([D, M_\phi]\) extends to a bounded adjointable \( A \)-operator on \( \mathcal{H} \). In fact,

\[
\|[D, M_\phi]\| \leq \sup\{\|d\phi(x)\| \text{Prop}(D, x) : x \in \mathbb{R}^n\}
\]

where \( \text{Prop}(D, x) = \sup\{\|\sigma(D)(x, \xi)\| : (x, \xi) \in T^*\mathbb{R}^n, \|\xi\| = 1\} \).

The proof of the next result is the same as in the classical case by using the generalized Rellich Lemma (Lemma 3.3 [MF80]). (The case for \( A = \mathbb{C} \) is proved in Lemma 3.5 [Gue98].)

Lemma 4.4. Let \( D \) be an essential \( A \)-operator on \( \mathbb{R}^n \). For any \( f \in C_0(\mathbb{R}) \) and \( \phi \in C_0(\mathbb{R}^n) \), the operator \( M_\phi f(D) \) is a compact \( A \)-operator on \( \mathcal{H} \).

Lemma 4.5. Let \( D_1 \) and \( D_2 \) be essential \( A \)-operators on \( \mathbb{R}^n \). Let \( \phi \in C_0(\mathbb{R}^n), f \in C_0(\mathbb{R}) \). Then we have that:

1. \( \lim_{t \to \infty} \|[M_\phi, f(t^{-1}D_1)]\| = 0. \)
2. If \( D_2 - D_1 \) is order zero, then \( \lim_{t \to \infty} \|[M_\phi f(t^{-1}D_1) - M_\phi f(t^{-1}D_2)]\| = 0. \)
3. If \( D_1 = D_2 \) near \( \text{supp}(\phi) \), then \( \lim_{t \to \infty} \|[M_\phi f(t^{-1}D_1) - f(t^{-1}D_2)M_\phi]\| = 0. \)

Proof. The collection of \( f \in C_0(\mathbb{R}) \) for which the Lemma holds is a \( C^* \)-subalgebra of \( C_0(\mathbb{R}) \). Thus, we need only check on the resolvent functions \( r_\pm(x) = (x \pm i)^{-1} \). Note that \( \|r_\pm(t^{-1}D_1)\| \leq 1 \) independently of \( t \).

1. From the commutator identity

\[
[M_\phi, r_\pm(t^{-1}D_1)] = [M_{\phi - \phi_n}, r_\pm(t^{-1}D_1)] + [M_{\phi_n}, r_\pm(t^{-1}D_1)]
\]

we obtain the inequality

\[
\|[M_\phi, r_\pm(t^{-1}D_1)]\| \leq 2\|\phi - \phi_n\||f|| + \|[M_{\phi_n}, r_\pm(t^{-1}D_1)]\|
\]

and so we may assume that \( \phi \in C_0^\infty(\mathbb{R}^n) \). Now, the commutator identity

\[
[M_\phi, r_\pm(t^{-1}D_1)] = t^{-1}r_\pm(t^{-1}D_1)[D_1, M_\phi]r_\pm(t^{-1}D_1)
\]

implies that

\[
\|[M_\phi, r_\pm(t^{-1}D_1)]\| \leq t^{-1}\|[D_1, M_\phi]\| \to 0
\]

as \( t \to \infty \) and the result follows.

(2) Suppose \( D_2 - D_1 = b(x) \), where \( b \) is a smooth function. Then we have that

\[
M_\phi r_\pm(t^{-1}D_1) - r_\pm(t^{-1}D_2)M_\phi = t^{-1}r_\pm(t^{-1}D_2)(D_2M_\phi - M_\phi D_1)r_\pm(t^{-1}D_1)
\]

\[
= t^{-1}r_\pm(t^{-1}D_2)(M_{\phi b} + [D_2, M_\phi])r_\pm(t^{-1}D_1) \to 0 \text{ as } t \to \infty
\]

since \( M_{\phi b} \) and \([D_2, M_\phi]\) are bounded. The result follows.

(3) Suppose \( D_2 - D_1 = 0 \) near \( \text{supp}(\phi) \). Then \( M_\phi(D_2 - D_1) = 0 \) and the result follows from the previous computation. \( \square \)
Theorem 4.6. Let $D$ be an essential $A$-operator on $\mathbb{R}^n$. There is an asymptotic morphism $\mathcal{E}^D_t : C_0(\mathbb{R}) \otimes C_0(\mathbb{R}^n) \to \mathcal{K}(\mathcal{H})$ which is determined (up to asymptotic equivalence) by the maps
\[ f \otimes \phi \mapsto M_\phi f(t^{-1}D). \]

The bounded operator $f(t^{-1}D)$ on $\mathcal{H}$ is defined via the functional calculus.

Let $\{\Phi^A_t\}_{t \in [1, \infty)} : C_0(\mathbb{R}^n) \otimes A \to \mathcal{K}(L^2(\mathbb{R}^n)) \otimes A$ be the $A$-index asymptotic morphism defined in Appendix A. Recall that Banach $A$-module $L^1(\mathbb{R}^n, A)$ is an algebra under convolution
\[ f \ast g(x) = \int_{\mathbb{R}^n} f(x - y)g(y)dy. \]

It also acts as adjointable $A$-operators via this formula on the Hilbert $A$-module $L^2(\mathbb{R}^n, A)$. Let $C^*(\mathbb{R}^n, A)$ denote the $C^*$-algebra completion of $L^1(\mathbb{R}^n, A)$ in the operator norm of $\mathcal{L}(L^2(\mathbb{R}^n, A))$. The generalized Fourier Transform

\[ \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x)e^{-ix\xi}dx \]

determines an isomorphism $\wedge : C^*(\mathbb{R}^n, A) \to C_0(\mathbb{R}^n, A) \cong C_0(\mathbb{R}^n) \otimes A$. We can define a continuous family of $*$-homomorphisms
\[ C^A_t : C_0(\mathbb{R}^n, A) \to \mathcal{L}(L^2(\mathbb{R}^n, A)) : [C^A_t(f)]\eta = \hat{f} * \eta. \]

The next result follows easily from Lemma A.1 since $C^A_t = C_t \otimes id_A$ with respect to the isomorphism $C_0(\mathbb{R}^n, A) \cong C_0(\mathbb{R}^n) \otimes A$.

Lemma 4.7. For any $\phi \in C_0(\mathbb{R}^n)$ and $f \in C_0(\mathbb{R}^n, A)$, we have
\[ \lim_{t \to \infty} \|\Phi^A_t(\phi \otimes f) - M_\phi C^A_t(f)\| = 0. \]

Extend $\{\Phi^A_t\}$ to $k \times k$ matrices
\[ \{\Phi^A_t\}_{t \in [1, \infty)} : M_k(C_0(\mathbb{R}^n) \otimes A) \to M_k(\mathcal{K}(L^2(\mathbb{R}^n)) \otimes A) \cong \mathcal{K}(L^2(\mathbb{R}^n, A^k)) \]
by applying element-wise. Also extend $C^A_t : M_k(C_0(\mathbb{R}^n) \otimes A) \to \mathcal{L}(L^2(\mathbb{R}^n, A^k))$ in a similar manner.

The following is the most basic result which relates the operator $D$, its principal symbol $\sigma(D)$, and this asymptotic morphism.

Lemma 4.8. If $D$ is a formally self-adjoint elliptic differential $A$-operator of order one on $\mathbb{R}^n$ with constant coefficients, then $D$ is essential and for every $\phi \in C_0(\mathbb{R}^n)$ and $f \in C_0(\mathbb{R})$, we have that
\[ \lim_{t \to \infty} \|\Phi^A_t(\phi f(\sigma(D))) - M_\phi f(t^{-1}D)\| = 0. \]

Proof. The fact that $D$ is essentially self-adjoint follows from the Fourier identity
\[ \widehat{\Phi_t(\xi)} = \text{sym}(D)(\xi)\hat{\phi}(\xi). \]
since the total symbol \( \text{sym}(D)(\xi) \) is a self-adjoint matrix for \( \xi \in T^*_x \mathbb{R}^n \cong \mathbb{R}^n \). For any \( \eta \in \text{Domain}(D) = C^\infty_c(\mathbb{R}^n, A^k) \) we have the estimate

\[
\|(1 + D^2)\eta\| \geq \|\eta\|
\]

from which it easily follows that \( (1 + D^2) \) has dense range. Thus, \( D \) is essential.

We define

\[
[\phi f(\sigma(D))](x, \xi) = \phi(x)f(\sigma(D)(\xi))
\]

via the \( C^* \)-functional calculus. Since \( f \in C_0(\mathbb{R}) \) and \( D \) is elliptic, \( f(\sigma(D)) \in M_k(C_0(\mathbb{R}^n) \otimes A) \) and so

\[
\phi f(\sigma(D)) = \phi \otimes f(\sigma(D)) \in M_k(C_0(\mathbb{R}^{2n}) \otimes A).
\]

From the definition of \( C^A_t : C_0(\mathbb{R}^n, A) \to \mathcal{L}(L^2(\mathbb{R}^n, A)) \) we see that

\[
[C^A_t(\phi f(\sigma(D)))\eta](\xi) = \phi(\sigma(D)(t^{-1}\xi))\eta(\xi)
\]

where \( \eta \in L^2(\mathbb{R}^n, A^k) \). Let \( D_1 = D - b \) be the first-order part of \( D \). It follows by Lemma 4.7 that

\[
\lim_{t \to \infty} \|\Phi^A_t(\phi f(\sigma(D))) - M_\phi f(t^{-1}D_1)\| = 0
\]

since \( D_1 \) has constant coefficients and the total symbol of \( t^{-1}D_1 \) is

\[
\text{sym}(t^{-1}D_1)(\xi) = \sigma(D)(t^{-1}\xi) = t^{-1}\sigma(D)(\xi)
\]

for all \( t \geq 1 \). Now \( D - D_1 = b \) is order zero and so the result follows by part 2 of Lemma 4.5. \( \square \)

**Lemma 4.9.** Let \( K \) be a compact subset of \( \mathbb{R}^n \). Let \( \phi \in C^\infty_c(\mathbb{R}^n) \) with \( \text{supp}(\phi) \) contained in \( K \) and let \( f \in C_0(\mathbb{R}) \). Let \( D \) be as in the previous lemma. Then for every \( \epsilon > 0 \), there is a \( \delta > 0 \) such that if

\[
B = \sum_{1}^{n} c_j(x) \frac{\partial}{\partial x_j} + d(x)
\]

is an essential \( A \)-operator on \( \mathbb{R}^n \) and the coefficients of \( D \) and \( B \) differ in the uniform norm on \( K \) by \( \delta \), then

\[
\|M_\phi f(t^{-1}D) - M_\phi f(t^{-1}B)\| < \epsilon
\]

for \( t \) large enough.

**Proof.** The set \( C \) of all \( f \in C_0(\mathbb{R}) \) for which the Lemma holds is clearly closed under addition (by the triangle inequality) and is closed under operator adjoints. Suppose \( \{f_n\} \) is a sequence in \( C \) such that \( f_n \to f \in C_0(\mathbb{R}) \). Consider the following inequality

\[
\|M_{\phi_n} f(t^{-1}D) - M_{\phi_n} f(t^{-1}B)\| \leq 2\|\phi_n - \phi\| \|f_n - f\| + \|M_{\phi_n} f(t^{-1}D) - M_\phi f(t^{-1}D)\| + \|M_{\phi_n} f(t^{-1}B) - M_\phi f(t^{-1}B)\|
\]

for all \( n \).
which follows by the triangle inequality and the Spectral Theorem. Given \( \epsilon > 0 \), choose \( n > 0 \) such that \( \| f - f_n \| < \epsilon/(2\|\phi\|) \). Choose \( \delta > 0 \) such that the Lemma is true for \( f_n \) with \( \epsilon/2 \). It follows that the Lemma is true for \( f \) with this \( \delta \) and so \( C \) is closed. Now suppose the Lemma is true for \( f \) and \( g \). Consider the inequality

\[
\| M_\phi(fg)(t^{-1}\bar{D}) - M_\phi(f)(t^{-1}\bar{B}) \| \\
\leq \| M_\phi f(t^{-1}\bar{D})g(t^{-1}\bar{D}) - M_\phi f(t^{-1}\bar{D})g(t^{-1}\bar{B}) \| \\
+ \| M_\phi f(t^{-1}\bar{D})g(t^{-1}\bar{B}) - M_\phi f(t^{-1}\bar{B})g(t^{-1}\bar{B}) \| \\
\leq 2\| [M_\phi, f(t^{-1}\bar{D})] \| \| g \| + \| f \| \| M_\phi g(t^{-1}\bar{D}) - M_\phi g(t^{-1}\bar{B}) \| \\
+ \| g \| \| M_\phi f(t^{-1}\bar{D}) - M_\phi f(t^{-1}\bar{B}) \| .
\]

It follows that the Lemma is true for \( fg \) by part 1 of Lemma 4.5. Thus, \( C \) is a closed \( C^* \)-subalgebra of \( C_0(\mathbb{R}) \).

By the Stone-Weierstrass Theorem, it suffices to prove that \( C \) contains the resolvent functions. Let \( \epsilon > 0 \) be given. Consider

\[
M_\phi r_\pm(t^{-1}\bar{D}) - M_\phi r_\pm(t^{-1}\bar{B}) = M_\phi (r_\pm(t^{-1}\bar{D}) - r_\pm(t^{-1}\bar{B})) \\
= t^{-1}M_\phi r_\pm(t^{-1}\bar{B})(B-D)r_\pm(t^{-1}\bar{D}) \\
= t^{-1}[M_\phi, r_\pm(t^{-1}\bar{B})](B-D)r_\pm(t^{-1}\bar{D}) \\
+ t^{-1}r_\pm(t^{-1}\bar{B})M_\phi(B-D)r_\pm(t^{-1}\bar{D})
\]

By Lemma 4.5 again, we have that \( \lim_{t \to \infty} \| [M_\phi, r_\pm(t^{-1}\bar{B})] \| = 0 \). From elliptic operator theory [MF80], \( M_\phi(B-D) : H^1(\mathbb{R}^n, A^k) \to L^2(K, A^k) \) is bounded by a constant times the sum of terms involving

\[
\sup \| \phi(x)(c_j(x) - a_j) \|, \quad \sup \| \phi(x)(d(x) - b) \|, \\
\sup \| \frac{\partial}{\partial x_i}(\phi(x)(c_j(x) - a_j)) \|, \quad \text{and} \quad \sup \| \frac{\partial}{\partial x_i}(\phi(d(x) - b)) \|.
\]

The result now follows by choosing \( t \) large enough so that the first term in (1) above has norm less than \( \epsilon/2 \) and \( \delta > 0 \) small enough so the second term has norm less than \( \epsilon/2 \). \( \square \)

Finally, we come to the main theorem of this sections which relates the principal symbol \( \sigma(D) \) and the A-index asymptotic morphism \( \{ \Phi_t^A \} \) to the asymptotic morphism \( \{ \mathcal{E}_t^D \} \) determined by \( D \) in Theorem 4.6.

**Theorem 4.10.** Let \( D \) be an essential A-operator on \( \mathbb{R}^n \). Then for all \( f \in C_0(\mathbb{R}) \) and \( \phi \in C_0(\mathbb{R}^n) \), we have

\[
\lim_{t \to \infty} \| \Phi_t^A(\phi f(\sigma(D))) - M_\phi f(t^{-1}D) \| = 0.
\]

**Proof.** Consider the inequality

\[
\| \Phi_t^A(\phi f(\sigma(D))) - M_\phi f(t^{-1}\bar{D}) \| \leq \| \Phi_t^A(\phi f(\sigma(D))) - \Phi_t^A(\psi f(\sigma(D))) \| \\
+ \| \Phi_t^A(\psi f(\sigma(D))) - M_\phi f(\sigma(D)) \| + \| M_\phi f(\sigma(D)) \| + \| M_\phi f(t^{-1}\bar{D}) - M_\phi f(t^{-1}\bar{D}) \|.
\]
This implies the inequality
\[
\limsup_t \| \Phi_t^A(\phi f(\sigma(D))) - M\phi f(t^{-1}D) \| \leq 2\| \phi - \psi \| \| f \| + \limsup_t \| \Phi_t^A(\psi f(\sigma(D))) - M\psi f(t^{-1}D) \|.
\]

Hence, we may assume that $\phi$ is smooth and compactly supported.

Let $K = \text{supp}(\phi) \subset \mathbb{R}^n$, which is compact, and let $\epsilon > 0$. Since $D$ is elliptic and self-adjoint $(\sigma(D)(x,\xi)^* = \sigma(D)(x,\xi))$, it follows that
\[
\phi f(\sigma(D)) \in M_k(C_0(T^*\mathbb{R}^n) \otimes A).
\]

For each $x \in K$, let $D^x$ be the constant coefficient operator defined by “freezing” the coefficients of $D$ at $x$, i.e.,
\[
D^x = \sum_{1}^{n} a_j^x \frac{\partial}{\partial x_j} + b^x,
\]
where $a_j^x = a_j(x)$ and $b^x = b(x)$. Thus, the principal symbol of $D^x$ is
\[
\sigma(D^x)(\xi) = \sigma(D)(x, \xi).
\]

By the compactness of $K$ and the finite covering dimension [Mun75] of $\mathbb{R}^n$, we may choose $\delta > 0$ small enough and cover $K$ regularly by finitely many small balls
\[
B_\delta(x_1), \ldots, B_\delta(x_N), \quad x_i \in K,
\]
so that no more than $d = n + 1$ balls intersect at any point in $K$ and a subordinate partition of unity $\{\phi_i\}$ such that
\[
\| M\phi_i \Phi_t^A(\phi f(\sigma(D))) - M\phi_i \Phi_t^A(\phi f(\sigma(D^i))) \| \leq \frac{\epsilon}{2d}
\]
\[
\| M\phi_i M\phi f(\tilde{D}^i) - M\phi_i M\phi f(t^{-1}D) \| \leq \frac{\epsilon}{2d}
\]
for all $i$ and $t$ large enough (using the previous lemma).

We wish to show that $\limsup_t \| \Phi_t^A(\phi f(\sigma(D))) - M\phi f(t^{-1}D) \| \leq \epsilon$.

Partition $\{1, 2, \ldots, N\}$ into $d$ sets $C_k$ such that for all $i, j \in C_k$ with $i \neq j$, we have
\[
B_\delta(x_i) \cap B_\delta(x_j) = \emptyset.
\]
Let $A_t = \Phi_t^A(\phi f(\sigma(D))) - M\phi f(t^{-1}D)$. It follows that
\[
A_t = \sum_{i=1}^{N} \phi_i A_t \sim_n \sum_{i=1}^{N} \phi_i^{1/2} A_t \phi_i^{1/2} = \sum_{k=1}^{d} \sum_{i \in C_k} \phi_i^{1/2} A_t \phi_i^{1/2}
\]

Hence, we see that
\[
\limsup_t \| A_t \| \leq \limsup_t \sum_{k=1}^{d} \sum_{i \in C_k} \| \phi_i^{1/2} A_t \phi_i^{1/2} \|
\]
\[
= \limsup_t \sum_{i \in C_k} \max_{\phi_i^{1/2} A_t \phi_i^{1/2}} \| \phi_i^{1/2} A_t \phi_i^{1/2} \| \leq d \limsup_t \max_{\phi_i^{1/2} A_t \phi_i^{1/2}} \| \phi_i^{1/2} A_t \phi_i^{1/2} \|
\]
Since $\phi_t^{1/2}A_t\phi_t^{1/2} \sim_a \phi_t A_t$, we have $\limsup_t \|\phi_t^{1/2}A_t\phi_t^{1/2}\| \leq \limsup_t \|\phi_t A_t\|$. But, for all $t$ and large enough values of $t$,

$$
\|\phi_t A_t\| = \|\phi_t \Phi_t^A(\phi f(\sigma(D))) - \phi_t \phi f(t^{-1}D)\|
\leq \|\phi_t \Phi_t^A(\phi f(\sigma(D))) - \phi_t \Phi_t^A(\phi f(\sigma(D)))\| + \|\phi_t \phi f(t^{-1}D) - \phi_t \phi f(t^{-1}D)\|
\leq \frac{\epsilon}{2d} + \frac{\epsilon}{2d} < \epsilon
$$

since $\Phi_t^A(\phi f(\sigma(D))) \sim_a M\phi f(t^{-1}D)$ by Lemma 4.8. The result follows. \(\square\)

5. The Exact Mishchenko-Fomenko Index Theorem

Let $A$ be a unital $C^*$-algebra. Let $M$ be a smooth closed Riemannian manifold and let $D : C^\infty(E) \to C^\infty(F)$ be an elliptic differential $A$-operator of order one on $M$. The following is the main example in the theory, which we will need in the proof of the index theorem. For the basic theory of (pseudo)differential $A$-operators on vector $A$-bundles see [MF80, Mis78].

Example 5.1. Let $S$ be a smooth Clifford bundle on $M$ with Clifford multiplication $c : T^*M \to \text{End}(S)$. Let $D : C^\infty(S) \to C^\infty(S)$ be the associated Dirac-type operator [LM89] defined via a compatible connection $\nabla^S : C^\infty(S) \to C^\infty(T^*M \otimes S)$. Let $E$ be a smooth vector $A$-bundle on $M$ also equipped with a connection $\nabla^E : C^\infty(E) \to C^\infty(T^*M \otimes E)$. Let $\nabla = \nabla^S \otimes 1 + 1 \otimes \nabla^E$ denote the tensor product connection on the vector $A$-bundle $S \otimes E$. The associated generalized Dirac operator $D_E : C^\infty(S \otimes E) \to C^\infty(S \otimes E)$ is an elliptic differential $A$-operator of order one on $M$. In local coordinates,

$$D_E = \sum_{j=1}^n c(e_j)\nabla_{e_j}$$

where $\{e_j\}$ is a local orthonormal frame for $TM \cong T^*M$.

Since $D$ is elliptic, we may assume $F = E$. Put a smooth Hermitian $A$-metric on the vector $A$-bundle $E$. Consider $D$ as a densely-defined, unbounded $A$-operator on the Hilbert $A$-module $L^2(E)$ with dense Domain$(D) = C^\infty(E)$. If $D$ is formally self-adjoint, then the closure of the graph of $D$ in $L^2(E) \oplus L^2(E)$ is the graph of an unbounded $A$-operator $\bar{D}$, called the closure of $D$. The basic elliptic estimate [MF80] shows that Domain$(\bar{D})$ is the Sobolev space $H^1(E)$. We refer the reader to Definition 4.2 for the following.

Lemma 5.2. If $D$ is a formally self-adjoint elliptic differential $A$-operator of order one on $M$, then $D$ is essential and $\bar{D}$ has compact resolvents. Moreover, the restriction $D|_U : C^\infty(U, E) \to C^\infty(U, E)$ to any open subset $U$ of $M$ is also essential.

Proof. The fact that $\bar{D}$ is self-adjoint follows from ellipticity and the existence of a parametrix [MF80] as in the classical case. The estimate [MP93]

$$\|(1 + D^2)\eta\| \geq \|\eta\|$$

shows that $(1 + D^2)$ has dense range. From the identity $(D + i)(D - i) = (1 + D^2)$ we have that $(D + i)$ have dense range and so $D$ is essential. The generalized
Rellich Lemma and basic elliptic estimate then prove that \((D \pm i)^{-1}\) are compact \(A\)-operators on \(L^2(E)\). Note that each resolvent is the adjoint of the other. Using the fact that \(\text{supp}(D\eta) \subset \text{supp}(\eta)\) for differential operators and the decomposition \(L^2(E) \cong L^2(U, E) \oplus L^2(M \setminus U, E)\), the last statement follows. \(\square\)

Consider the formally self-adjoint \(A\)-operator

\[
\mathbb{D} = \begin{pmatrix} 0 & D^i \\ D & 0 \end{pmatrix} : C^\infty(E \oplus E) \to C^\infty(E \oplus E)
\]

where \(D^i\) is the formal adjoint of \(D\). The principal symbol of \(D\) is the self-adjoint homomorphism

\[
\sigma = \sigma(\mathbb{D}) = \begin{pmatrix} 0 & \sigma(D)^* \\ \sigma(D) & 0 \end{pmatrix} : \pi^*(E \oplus E) \to \pi^*(E \oplus E),
\]

where \(\sigma(D) : \pi^*E \to \pi^*E\) is the principal symbol of \(D\).

**Lemma 5.3.** The resolvents

\[(\sigma \pm i)^{-1} : \pi^*(E \oplus E) \to \pi^*(E \oplus E)\]

are \(A\)-homomorphisms which vanish at infinity on \(T^*M\) in the operator norm induced by the Hermitian \(A\)-metrics on \(E\).

**Proof.** Follows from homogeneity \(\sigma(x, t\xi) = t\sigma(x, \xi)\) and ellipticity. \(\square\)

Form the Cayley transform \([\text{Qui}88]\)

\[
u = (\sigma + i)(\sigma - i)^{-1} = 1 + 2i(\sigma - i)^{-1}.
\]

By complementing the vector \(A\)-bundle \(E\), we may embed \(\pi^*(E \oplus E)\) in a trivial \(A\)-bundle

\[\mathbb{A} = T^*M \times (A^n \oplus A^n)\]

Now extend the automorphism \(\nu\) to the \(A\)-bundle \(\mathbb{A}\) by defining it to be equal to the identity on the complement of \(\pi^*E \oplus \pi^*E\) in \(\mathbb{A}\). From the lemma above, it follows that \(\nu\) extends continuously to the trivial \(A\)-bundle on the one-point compactification \((T^*M)^+\) by setting \(\nu(\infty) = I\).

Let

\[\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\]

be the grading of the trivial \(A\)-bundle \((T^*M)^+ \times (A^n \oplus A^n)\). Since \(\epsilon\sigma = -\sigma\epsilon\) it follows that \((\nu \epsilon)^2 = 1\). A simple calculation also shows that \((\nu \epsilon)^* = \nu \epsilon\) is self adjoint. We also have obviously that \(\epsilon^* = \epsilon\) and \(\epsilon^2 = 1\).

Recall that for every self adjoint involution \(x\) there is an associated projection \(p(x) = \frac{1}{2}(x + 1)\). In our case, we obtain two projection-valued functions \(p(\epsilon)\) and \(p(\nu \epsilon)\) on \((T^*M)^+\) which are equal at infinity. Both define elements in

\[K_0(C(T^*M^+) \otimes A) = K_0^A(T^*M^+)\]

and so their difference defines an element in

\[\sigma_D = [p(\epsilon)] - [p(\nu \epsilon)] \in K_0(C_0(T^*M) \otimes A) = K_0^A(T^*M)\]

This is the symbol class of the elliptic \(A\)-operator \(D\) as constructed in \([\text{Hig}93]\) for \(A = C_0\). (See also Quillen \([\text{Qui}88]\).)
Lemma 5.4. \( \sigma_D = [\sigma(D)] \in K^0_\partial(T^*M) \).

For each \( t \geq 1 \), we can form the compact \( A \)-operators\(^3\)
\[
 f(t^{-1} \mathbb{D}) : L^2(M, E \oplus E) \to L^2(M, E \oplus E)
\]
by Lemma 5.3 and the functional calculus. We can extend \( f(t^{-1} \mathbb{D}) \) to an operator on the Hilbert \( A \)-module \( L^2(M, A^n \oplus A^n) \) by defining it to be zero on the complement of the \( A \)-submodule \( L^2(M, E \oplus E) \) (which exists by complementing \( E \) as above) in \( L^2(M, A^n \oplus A^n) = L^2(M, A^{2n}) \). This then defines a continuous family of \( * \)-homomorphisms
\[
 C_0(\mathbb{R}) \to \mathcal{K}(L^2(M, A^{2n})) : f \mapsto f(t^{-1} \mathbb{D}).
\]

Likewise, we may apply \( f \) to the symbol \( \sigma \) of \( \mathbb{D} \). This \( A \)-homomorphism \( f(\sigma) \) of the vector \( A \)-bundle \( \pi^*(E \oplus F) \) also vanishes at infinity (compare Lemma 5.3). We may then extend \( f(\sigma) \) to an \( A \)-homomorphism of the trivial \( A \)-bundle \( \mathbb{A} \) by setting it equal to zero on the complement of \( \pi^*(E \oplus F) \).

Let \( \{ \Phi_t^{M,A} \} : M_{2n}(C_0(T^*M) \otimes A) \to \mathcal{K}(L^2(M, A^{2n})) \) denote the \( A \)-index asymptotic morphism for \( M \) from Appendix A. By thinking of \( f(\sigma) \) as a matrix of \( A \)-valued functions on \( T^*M \) vanishing at infinity, we obtain a \( * \)-homomorphism
\[
 C_0(\mathbb{R}) \to C_0(T^*M, M_{2n}(A)) = M_{2n}(C_0(T^*M) \otimes A) : f \mapsto f(\sigma).
\]
The following result relates the spectral theory of \( D \) to the principal symbol \( \sigma(\mathbb{D}) \) and is central to the index theorem.

Lemma 5.5. If \( D \) is an elliptic differential \( A \)-operator of order one on \( M \) with symbol \( \sigma \), then for every \( f \in C_0(\mathbb{R}) \),
\[
 \lim_{t \to \infty} \| \Phi_t^{M,A}(f(\sigma(\mathbb{D}))) - f(t^{-1} \mathbb{D}) \| = 0
\]

Proof. By complementing the vector \( A \)-bundle \( E \) we may assume that \( E = M \times A^n \) is actually trivial. Let \( \{ U_j \}_{j=1}^m \) be an open cover of \( M \) by coordinate charts. Let \( \mathbb{D}_j = \mathbb{D}|_{U_j} \) be the restriction to \( U_j \). Choose a smooth partition of unity \( \{ \rho_j^2 \} \) subordinate to the cover \( \{ U_j \} \). Let
\[
 \{ \Phi_t^j \} = \{ \Phi_t^{M,A}|_{U_j} \} : M_{2n}(C_0(T^*U_j) \otimes A) \to \mathcal{K}(L^2(U_j, A^{2n}))
\]
denote the restriction to \( U_j \) as in Corollary A.11. By Lemma 5.2 and Theorem 4.10, we have for each \( 1 \leq j \leq m \)
\[
 \lim_{t \to \infty} \| \Phi_t^j(\rho_j f(\sigma(\mathbb{D}_j)))\rho_j - M_{\rho_j} f(t^{-1} \mathbb{D}_j) M_{\rho_j} \| = 0.
\]
Consider the following asymptotic equivalences:
\[
 \Phi_t^{M,A}(f(\sigma(\mathbb{D}))) = \Phi_t^{M,A}(\sum_{1}^{m} \rho_j f(\sigma(\mathbb{D})))\rho_j) \sim_a \sum_{1}^{m} \Phi_t^{M,A}(\rho_j f(\sigma(\mathbb{D})))\rho_j)
\]
\[
 = \sum_{1}^{m} \Phi_t^{M,A}(\rho_j f(\sigma(\mathbb{D})))\rho_j) \sim_a \sum_{1}^{m} \Phi_t^j(\rho_j f(\sigma(\mathbb{D}_j)))\rho_j)
\]
\[
 \sim_a \sum_{1}^{m} M_{\rho_j} f(t^{-1} \mathbb{D}_j) M_{\rho_j} = \sum_{1}^{m} M_{\rho_j} f(t^{-1} \mathbb{D}) M_{\rho_j}
\]
\[
 \sim_a \sum_{1}^{m} M_{\rho_j} M_{\rho_j} f(t^{-1} \mathbb{D}) = f(t^{-1} \mathbb{D})
\]

\(^3\)We will write \( \mathbb{D} \) instead of the closure \( \bar{\mathbb{D}} \) from now on.
This completes the proof. □

**Proposition 5.6.** Let $D$ be an elliptic differential $A$-operator of order one on the smooth closed manifold $M$. The analytic and topological indices of $D$ are equal, that is,

\[
\text{Index}_a(D) = \text{Index}_t(D) \in K_0(A).
\]

**Proof.** For all $t > 0$ form the Cayley transform

\[
U_t = (t^{-1}D + i)(t^{-1}D - i)^{-1} = I + 2i(t^{-1}D - i)
\]

which defines a unitary $A$-operator on $L^2(M, A^{2n})$ which is equal to the identity on the complement of $L^2(M, E \oplus E)$. It follows from Lemma 5.5 that

\[
\lim_{t \to \infty} \|U_t - \Phi^{M,A}_t(u)\| = 0
\]

(where we extended the result by adjoining a unit.) It follows from this that $\Phi^{M,A}_t([p(eu)]) = [p(eU_1)]$ and so the morphism index can be written as

\[
\text{Ind}_m(M, \sigma(D)) = \Phi^{M,A}_t([\sigma(D)]) = \Phi^{M,A}_t(\sigma_D) = [p(e)] - [p(eU_1)] \in K_0(A).
\]

Let us now ponder what happens as $t \to 0$. By homotopy invariance, we may assume that the kernel and cokernel of $D$ are finitely generated and projective $A$-modules and that $D$ is a Fredholm $A$-operator with closed range. Thus, we have the Hilbert $A$-module decomposition

\[
L^2(M, A^n \oplus A^n) = \text{Ker}(D) \oplus \mathcal{H} = \text{Ker}(D) \oplus \text{Ker}(D^*) \oplus \mathcal{H}.
\]

For all $t$, we have that

\[
U_t|_{\text{Ker}(D)} = I + 2i(0 - iI)^{-1} = I - 2I = -I
\]

on the kernel of $D$. Thus,

\[
eU_t|_{\text{Ker}(D)} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
\]

with respect to the decomposition $\text{Ker}(D) = \text{Ker}(D) \oplus \text{Ker}(D^*)$. On the orthogonal complement of $\text{Ker}(D)$, $U_t$ converges in norm to the identity as $t \to \infty$, since $D$ is invertible off $\text{Ker}(D)$ and has a gap in the spectrum. Thus, $U_t$ converges in norm to $U_0$ which is $-I$ on $\text{Ker}(D)$ and $I$ on the complement. A simple calculation then shows that

\[
[p(e)] - [p(eU_1)] = [p(e)] - [p(eU_0)] = [P_{\text{Ker}(D)}] - [P_{\text{Ker}(D^*)}]
\]

where $P_{\text{Ker}(D)}$ and $P_{\text{Ker}(D^*)}$ are the projections onto the kernel and cokernel of $D$. Therefore, we have by Theorem 3.13 that

\[
\text{Index}_a(D) = \text{Ind}_m(M, \sigma(D)) = \text{Ind}_m(M, \sigma(D)) = \text{Index}_t(D). \quad \square
\]
This establishes the exact version of the Mishchenko-Fomenko Index Theorem for first-order elliptic differential $A$-operators on arbitrary smooth closed manifolds. It is not known if these techniques can be extended verbatim to higher-order differential (or even pseudodifferential) $A$-operators since the results in Section 4, especially the estimates in Lemma 4.9, were specific to first-order operators via the symbol identity. However, using Kasparov's $KK$-theory \cite{kas81}, we can derive the full Mishchenko-Fomenko Index Theorem for spin$^c$ manifolds using twisted Dirac operators \cite{LM89}. See the helpful discussions in Section 24 of Blackadar \cite{Bla86}.

Let $M$ be a smooth compact spin$^c$-manifold without boundary and let $D$ be the Dirac operator of $M$. Let $\partial$ denote the Dolbeaut operator on the cotangent bundle $T^*M$. These two elliptic operators determine $KK$-classes

$$[D] \in KK^0(C(M), \mathbb{C})$$

$$[\partial] \in KK^0(C_0(T^*M), \mathbb{C})$$

The Thom isomorphism determines a class

$$x = [\Psi] \in KK^0(C(M), C_0(T^*M)).$$

This element relates these two elliptic classes (Lemma 24.5.1 \cite{Bla86}) by

$$x \otimes_{C_0(T^*M)} [\partial] = [D],$$

where $\otimes$ denotes Kasparov’s intersection product (and corresponds to composition of asymptotic morphisms).

Let $P : C^\infty(E_1) \to C^\infty(E_2)$ be an elliptic pseudodifferential $A$-operator of order $m$ on $M$. $P$ determines a $KK$-class $[P] \in KK^0(C(M), A)$. (Compare Theorem 4.6). The principal symbol $\sigma(P)$ of $P$ determines two $KK$-classes

$$[[\sigma(P)]] \in KK^0(C(M), C_0(T^*M) \otimes A)$$

$$[\sigma(P)] \in KK^0(\mathbb{C}, C_0(T^*M) \otimes A) = K^0_A(T^*M).$$

They are related by $f^M_*[[\sigma(P)]] = [\sigma(P)]$ where $f^M : M \to \{pt\}$ is the collapsing map.

**Lemma 5.7.** (Theorem 5 \cite{kas84}) $[P] = [[\sigma(P)]] \otimes_{C_0(T^*M)} [\partial]$.

The *analytic index* of $D$ is given by the formula

$$\text{Index}_a(D) = f^M_*([P]) \in KK^0(\mathbb{C}, A) = K_0(A).$$

The *topological index* of $P$ is defined to be

$$\text{Index}_t(P) = \text{Ind}_t(M, \sigma(P)) \in K_0(A)$$

where $\sigma(P) : \sigma^!(E_1) \to \sigma^!(E_2)$ is the principal symbol of $P$. 

Theorem 5.8 (Exact Mishchenko-Fomenko Index Theorem). If $P$ is an elliptic pseudodifferential $A$-operator on the smooth closed spin$^c$-manifold $M$, then the analytic and topological indices of $P$ are equal, that is,

$$\text{Index}_a(P) = \text{Index}_t(P) \in K_0(A).$$

Proof. We may suppose $M$ is even-dimensional. (If dim($M$) = $n$ is odd, then replace $M$ by $M \times \mathbb{T}^N$.)

By the Thom isomorphism, there is a vector $A$-bundle $E$ on $M$ such that

$$[E] \otimes_{C(M)} x = [\sigma(P)],$$

where $[E] \in K^0_A(M) = KK^0(C, C(M) \otimes A)$ is the associated $K$-theory class. It also determines a class $[[E]] \in KK^0(C(M), C(M) \otimes A)$, where $f^M_*([E]) = [E]$.

Let $D_E$ denote the Dirac operator of $M$ twisted by the vector $A$-bundle $E$ (as in Example 5.1). Note that $D_E$ is a first-order elliptic differential $A$-operator on $M$ and so

$$\text{Index}_a(D_E) = \text{Index}_t(D_E)$$

by the previous theorem. By Lemma 24.5.3 [Bla86],

$$[D_E] = [[E]] \otimes_{C(M)} [D].$$

The $A$-index asymptotic morphism of $M$ determines a class

$$[\Phi^M_{t,A}] \in E^0(C_0(T^*M) \otimes A, A) \cong KK^0(C_0(T^*M) \otimes A, A).$$

We then compute that:

$$\text{Index}_a(P) = f^M_*(P) = f^M_*([[\sigma(P)]] \otimes_{C_0(T^*M)} [\theta])$$

$$= [\sigma(P)] \otimes_{C_0(T^*M)} [\theta] = [E] \otimes_{C(M)} x \otimes_{C_0(T^*M)} [\theta]$$

$$= [E] \otimes_{C(M)} [D] = f^M_*([[E]] \otimes_{C(M)} [D]) = f^M_*(D_E)$$

$$= \text{Index}_a(D_E) = \text{Index}_t(D_E) = \text{Ind}_m(M, \sigma(D_E))$$

$$= [\sigma(D_E)] \otimes_{C_0(T^*M) \otimes A} [\Phi^M_{t,A}] = [E] \otimes_{C(M)} x \otimes_{C_0(T^*M) \otimes A} [\Phi^M_{t,A}]$$

$$= [\sigma(P)] \otimes_{C_0(T^*M) \otimes A} [\Phi^M_{t,A}] = \text{Ind}_m(M, \sigma(P)) = \text{Index}_t(P)$$

and we are done. \hfill \Box

**Appendix A: The Index Asymptotic Morphism**

We will construct for each smooth Riemannian manifold $M$ without boundary and $C^*$-algebra $A$ a natural asymptotic morphism [CH89]

$$\{\Phi^A_t\}_{t \in [1, \infty)} : C_0(T^*M) \otimes A \to \mathcal{K}(L^2M) \otimes A.$$

When $A = \mathbb{C}$ is the complex numbers, this asymptotic morphism is equivalent to the one originally described by Connes and Higson [CH89] and elaborated on by Higson [Hig93], although we are presenting it from a different viewpoint.
Fix an integer \( n \geq 0 \). Denote by \( \mathcal{L}(L^2(\mathbb{R}^n)) \) the \( C^* \)-algebra of bounded linear operators on \( L^2(\mathbb{R}^n) \) and let \( C_0(\mathbb{R}^n) \) denote the \( C^* \)-algebra of continuous functions vanishing at infinity. Let \( C^*(\mathbb{R}^n) \) be the \( C^* \)-algebra completion of the convolution algebra \( L^1(\mathbb{R}^n) \) in the operator norm of \( \mathcal{L}(L^2(\mathbb{R}^n)) \). The Fourier Transform gives a continuous algebra homomorphism \( L^1(\mathbb{R}^n) \xrightarrow{\hat{\cdot}} C_0(\mathbb{R}^n) \) which extends to a \( C^* \)-algebra isomorphism \( C^*(\mathbb{R}^n) \xrightarrow{\hat{\cdot}} C_0(\mathbb{R}^n) \) (called the Gel’fand Transform [Rud91].) For \( t \geq 1 \) and \( f \in C_0(\mathbb{R}^n) \) with \( \hat{f} \in L^1(\mathbb{R}^n) \), we define \( C_t(f) : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n) \) by
\[
(C_t(f)\phi)(x) = \hat{f} * \phi(x)
\]
where \( \phi \in L^2(\mathbb{R}^n) \) and \( f_t(x) = f(t^{-1}x) \) for \( x \in \mathbb{R}^n \).

**Lemma A.1.** \( C_t \) extends to a continuous family of \( * \)-monomorphisms
\[
\{C_t\}_{t \in [1, \infty)} : C_0(\mathbb{R}^n) \rightarrow \mathcal{L}(L^2(\mathbb{R}^n)).
\]

For \( f \in C_0(\mathbb{R}^n) \) let \( M_f \in \mathcal{B}(L^2(\mathbb{R}^n)) \) denote multiplication by \( f \). This defines a \( * \)-homomorphism \( M : C_0(\mathbb{R}^n) \rightarrow \mathcal{B}(L^2(\mathbb{R}^n)) \).

**Proposition A.2.** For all functions \( f, g \in C_0(\mathbb{R}^n) \),
\[
\lim_{t \rightarrow \infty} \|M_f C_t(g) - C_t(g)M_f\| = 0.
\]

**Proof.** We may assume that \( f \) is real-valued and compactly supported. The set \( \mathcal{A} \) of all \( g \) which satisfy the conclusion above forms a \( C^* \)-subalgebra of \( C_0(\mathbb{R}) \).

(a.) If \( n = 1 \), then we need only to check that \( \mathcal{A} \) contains the generators \( r_\pm(x) = (x \pm i) \) of \( C_0(\mathbb{R}) \). We have that
\[
C_t(r_\pm) = r_\pm(t^{-1}D) = (t^{-1}D \pm i)^{-1}
\]
where \( D = -\sqrt{-1}\frac{\partial}{\partial x} \) is the (closure of the) Dirac operator. The result now follows from the commutator identity
\[
[M_f, r_\pm(t^{-1}D)] = r_\pm(t^{-1}D)[t^{-1}D, M_f]r_\pm(t^{-1}D)
\]
and noting that the bounded operators \( [t^{-1}D, M_f] = t^{-1}[D, M_f] \rightarrow 0 \) as \( t \rightarrow \infty \).

(b.) If \( n > 1 \), then \( C_0(\mathbb{R}^n) \cong C_0(\mathbb{R}) \otimes C_0(\mathbb{R}) \otimes \cdots \otimes C_0(\mathbb{R}) \). If \( g \in C_0(\mathbb{R}^n) \) has the form \( g(\xi) = g_1(\xi_1) \otimes \cdots \otimes g_n(\xi_n) \) then it follows that \( C_t(g) = g_1(t^{-1}D_1) \cdots g_n(t^{-1}D_n) \) where \( D_j = -\sqrt{-1}\frac{\partial}{\partial x_j} \). Apply part (a.) inductively. \( \square \)

The previous result says that the continuous family \( \{C_t\} \) and the constant family \( \{M_t = M\} \) “asymptotically commute” with each other.

**Definition A.3.** Using the canonical identification \( T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n \), which identifies \( C_0(T^*\mathbb{R}^n) \cong C_0(\mathbb{R}^n) \otimes C_0(\mathbb{R}^n) \), we define the asymptotic morphism
\[
\{\Phi_t\}_{t \in [1, \infty)} : C_0(T^*\mathbb{R}^n) \rightarrow \mathcal{L}(L^2(\mathbb{R}^n))
\]
to be (induced by) the tensor product \( f \otimes g \mapsto M_f \circ C_t(g) \). This asymptotic morphism is only well-defined up to asymptotic equivalence.

We now want to show that the image of each \( \Phi_t \) actually lies in the \( C^* \)-algebra of compact operators \( \mathcal{K}(L^2(\mathbb{R}^n)) \).
Theorem A.4. \( \{ \Phi_t \}_{t \in [1, \infty)} : C_0(T^*\mathbb{R}^n) \to \mathcal{K}(L^2\mathbb{R}^n) \)

Proof. Let \( F = f \otimes g \) where \( f, g \) are compactly supported. For \( \eta \in L^2(\mathbb{R}^n) \):

\[
[\Phi_t(F)\eta](x) = [M_f \circ C_t(g)\eta](x) = \int_{\mathbb{R}^n} k_t^F(x, y)\eta(y) \, dy,
\]

where the kernel

\[
k_t^F(x, y) = \frac{1}{(2\pi)^{n/2}} f(x)g(y) = \left(\frac{t}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f(x)g(\xi)e^{-\text{it}(x-y)\xi} \, d\xi
\]

is in \( L^2(\mathbb{R}^n \times \mathbb{R}^n) \). Thus, \( \Phi_t(F) \) is an integral operator with square-integrable kernel and so is a compact operator on \( L^2(\mathbb{R}^n) \). Now take a norm limit. \( \square \)

Lemma A.5. For all \( \rho \in C_0(\mathbb{R}^n) \) and \( F \in C_0(T^*\mathbb{R}^n) \), we have that:

1. \( \Phi_t(\rho F) = M_\rho \Phi_t(F) \), for all \( t \geq 1 \);
2. \( \lim_{t \to \infty} \| M_\rho \Phi_t(f) - \Phi_t(f)M_\rho \| = 0 \).

Proof. Part (1) follows from the fact that \( M_\rho M_f = M_\rho f \). The second follows from a similar argument in Proposition A.2. \( \square \)

This asymptotic morphism also “restricts” to open subsets. Let \( U \) be an open subset of \( \mathbb{R}^n \). Then \( T^*U \cong U \times \mathbb{R}^n \subset T^*\mathbb{R}^n \) is also open and we have the natural inclusion \( C_0(T^*U) \subset C_0(T^*\mathbb{R}^n) \) and decomposition \( L^2(\mathbb{R}^n) = L^2(U) \oplus L^2(\mathbb{R}^n \setminus U) \).

Lemma A.6. For all \( t \geq 1 \), \( \Phi_t|_{C_0(T^*U)} : C_0(T^*U) \to \mathcal{K}(L^2U) \).

Proof. If \( F = f \otimes g \in C_0(T^*U) = C_0(U) \otimes C_0(\mathbb{R}^n) \), then for all \( \eta \in L^2(U) \), it follows that

\[
\text{supp}(\Phi_t(F)\eta) = \text{supp}(M_f C_t(g)\eta) \subset \text{supp}(f) \subset U.
\]

The result now follows by an approximation argument since each \( \Phi_t \) is linear. \( \square \)

Suppose that \( \psi : U \to W \) is a diffeomorphism of open subsets of \( \mathbb{R}^n \). Denote by \( \hat{\psi} : T^*U \to T^*W \) the induced diffeomorphism of cotangent bundles which is defined by

\[
\hat{\psi}(x, \xi) = (\psi(x), d\psi^{-t}\xi),
\]

where \( d\psi : TU \to TW \) denotes the derivative of \( \psi \), mapping tangent vectors at \( x \) to tangent vectors at \( \psi(x) \), and \( d\psi^{-t} \) denotes the inverse of the transpose, mapping cotangent vectors at \( x \) to cotangent vectors at \( \psi(x) \). Let \( T_\psi : L^2(W) \to L^2(U) \) denote the induced unitary isomorphism of Hilbert spaces defined by

\[
T_\psi \eta(x) = \eta(\psi(x)) \, J^{1/2}(x), \, \eta \in L^2(W),
\]

where \( J(x) \) denotes the absolute value of the Jacobian determinant of \( \psi \) at \( x \in U \).

Proposition A.7. (Lemma 8.7 [Hig93]) If a function \( F \in C_0(T^*\mathbb{R}^n) \) has support in the open subset \( T^*W \), then

\[
\lim_{t \to \infty} \| \Phi_t(F \circ \hat{\psi}) - T_\psi \Phi_t(F)T_\psi^{-1} \| = 0.
\]

Now let \( M \) be a smooth Riemannian \( n \)-manifold without boundary. Cover \( M \) with open charts \( U \), each diffeomorphic to an open subset \( W \) of \( \mathbb{R}^n \) via the
diffeomorphism $\psi_\alpha : U_\alpha \rightarrow W_\alpha$. Let $\tilde{\psi}_\alpha = (\psi_\alpha^*)^t : T^*W_\alpha \rightarrow T^*U_\alpha$ be the induced diffeomorphism of cotangent bundles and let $T_\alpha : L^2(W_\alpha) \rightarrow L^2(U_\alpha)$ be the induced unitary isomorphism of Hilbert spaces. Define $\Phi^\alpha_t : C_0(T^*U_\alpha) \rightarrow \mathcal{K}(L^2U_\alpha)$ by the following

$$\Phi^\alpha_t(f) = T_\alpha \Phi_t(f \circ \tilde{\psi}_\alpha)T_\alpha^{-1}.$$ 

(We include the appropriate Radon-Nikodym derivative in the definition of $T_\alpha$.) That is, we define $\Phi^\alpha_t$ so that the following diagram commutes asymptotically:

$$\begin{array}{ccc}
C_0(T^*U_\alpha) & \xrightarrow{\Phi^\alpha_t} & \mathcal{K}(L^2U_\alpha) \\
\Downarrow \cong & & \Downarrow \cong \\
C_0(T^*W_\alpha) & \xrightarrow{\Phi_t|_{W_\alpha}} & \mathcal{K}(L^2W_\alpha) \\
\Downarrow & & \Downarrow \\
C_0(T^*\mathbb{R}^n) & \xrightarrow{\Phi_t} & \mathcal{K}(L^2\mathbb{R}^n)
\end{array}$$

where the bottom diagram commutes by Lemma A.6.

**Corollary A.8.** If $f \in C_0(T^*M)$ has support in $U_\alpha \cap U_\beta$, then

$$\lim_{t \rightarrow \infty} \|\Phi^\alpha_t(f) - \Phi^\beta_t(f)\| = 0$$

in the operator norm on $L^2(M)$.

Now let $\{\rho_\alpha^2\}$ be a smooth partition of unity subordinate to the open cover $\{U_\alpha\}$. Define the family of functions

$$\{\Phi^M_t\}_{t \in [1, \infty)} : C_0(T^*M) \rightarrow \mathcal{K}(L^2M)$$

in the following way:

$$\Phi^M_t(f)\eta = \sum_\alpha \Phi^\alpha_t(\rho_\alpha f)(\rho_\alpha \eta),$$

where $f \in C_0(T^*M)$ and $\eta \in L^2(M)$.

**Theorem A.9.** $\{\Phi^M_t\}_{t \in [1, \infty)} : C_0(T^*M) \rightarrow \mathcal{K}(L^2M)$ is an asymptotic morphism which is asymptotically independent of the choice of open cover and partition of unity. Moreover, if $\psi : U \rightarrow W$ is a diffeomorphism from an open subset in $M$ to an open subset in $\mathbb{R}^n$ then for all $f \in C_c^\infty(T^*W)$

$$\lim_{t \rightarrow \infty} \|\Phi^M_t(f \circ \tilde{\psi}) - T_\psi \Phi_t(f)T_\psi^{-1}\| = 0$$

**Definition A.10.** Let $A$ be a $C^*$-algebra. Define the $A$-index asymptotic morphism for $M$,

$$\{\Phi^{M,A}_t\}_{t \in [1, \infty)} : C_0(T^*M) \otimes A \rightarrow \mathcal{K}(L^2M) \otimes A,$$

to be the (asymptotic morphism) tensor product of the index asymptotic morphism $\{\Phi^M_t\} : C_0(T^*M) \rightarrow \mathcal{K}(L^2M)$ above with the identity morphism $id_A : A \rightarrow A$.

The following properties follow easily from the previous results.
Corollary A.11. Let $U \subset M$ be an open subset of $M$. Then $C_0(T^*U)$ is an ideal in $C_0(T^*M)$ and $\mathcal{K}(L^2 U) \hookrightarrow \mathcal{K}(L^2 M)$. The following diagram commutes asymptotically

\[
\begin{array}{ccc}
C_0(T^*M) \otimes A & \xrightarrow{\{\Phi_t^{M,A}\}} & \mathcal{K}(L^2 M) \otimes A \\
\uparrow & & \uparrow \\
C_0(T^*U) \otimes A & \xrightarrow{\{\Phi_t^{M,A}|_U\}} & \mathcal{K}(L^2 U) \otimes A
\end{array}
\]

Corollary A.12. Let $\psi : M \to M$ be a smooth diffeomorphism. The following diagram commutes asymptotically:

\[
\begin{array}{ccc}
C_0(T^*M) \otimes A & \xrightarrow{\{\Phi_t^{M,A}\}} & \mathcal{K}(L^2 M) \otimes A \\
\uparrow_{\psi^* \otimes id_A} & & \uparrow_{Ad(\psi^*) \otimes id_A} \\
C_0(T^*U) \otimes A & \xrightarrow{\{\Phi_t^{M,A}|_U\}} & \mathcal{K}(L^2 U) \otimes A
\end{array}
\]

Appendix B: Bott Periodicity and Thom Isomorphism

Let $A$ be a $C^\ast$-algebra. In this section, we prove that the asymptotic morphism

\[
\{\Phi_t\}_{t \in [1, \infty)} : C_0(\mathbb{R}^{2n}) \otimes A \to \mathcal{K}(L^2 \mathbb{R}^n) \otimes A,
\]

constructed in Appendix A, induces the (inverse of the) Bott Periodicity isomorphism $K_0(C_0(\mathbb{R}^{2n}) \otimes A) \cong K_0(A)$. The proof is a direct generalization of Atiyah’s functorial proof of Bott Periodicity in topological $K$-theory [Ati68].

Suppose $A$ has a unit. If $B$ is another unital $C^\ast$-algebra, there is a well-defined map

\[
\mu : K_0(A) \otimes K_0(B) \to K_0(A \otimes B) \\
[p] \otimes [q] \mapsto [p \otimes q]
\]

where $p$ is a projection over $A$ and $q$ is a projection over $B$. (If $A$ or $B$ has no unit, then adjoin one and note that the above map for the unitized algebras restricts as needed.)

Theorem B.1. Suppose for each $C^\ast$-algebra $A$ there is a homomorphism

\[
\alpha_A : K_0(C_0(T^*\mathbb{R}^n) \otimes A) \to K_0(A)
\]

which satisfies the following properties:

1. $\alpha_A$ is natural in $A$, i.e. for every morphism $\psi : A \to B$ the following diagram commutes:

\[
\begin{array}{ccc}
K_0(C_0(T^*\mathbb{R}^n) \otimes A) & \xrightarrow{\alpha_A} & K_0(A) \\
\psi_* \downarrow & & \psi_* \downarrow \\
K_0(C_0(T^*\mathbb{R}^n) \otimes B) & \xrightarrow{\alpha_B} & K_0(B)
\end{array}
\]
(2) For any $C^*$-algebra $B$, the diagram below commutes:

$$K_0(C_0(T^*\mathbb{R}^n) \otimes A) \otimes K_0(B) \xrightarrow{\alpha_A \otimes id_B} K_0(A) \otimes K_0(B)$$

$$\mu \downarrow \quad \mu \downarrow$$

$$K_0(C_0(T^*\mathbb{R}^n) \otimes A \otimes B) \xrightarrow{\alpha_A \otimes B} K_0(A \otimes B)$$

(3) There is an element $b \in K_0(C_0(T^*\mathbb{R}^n))$ such that $\alpha_C(b) = 1 \in K_0(C) = \mathbb{Z}$. Then $\alpha_A$ is an isomorphism for all $A$.

Proof. The inverse map $\beta_A : K_0(A) \rightarrow K_0(C_0(T^*\mathbb{R}^n) \otimes A)$ is defined by the formula

$$\beta_A(x) = \mu(b \otimes x),$$

for $x \in K_0(A)$. Now use properties (1) - (3) to verify this. □

Lemma B.2. $\Phi_A^T$ satisfies properties (1) and (2).

Proof. Property (1) follows from the fact, since $\Phi_t$ is linear, the diagram

$$C_0(T^*\mathbb{R}^n) \otimes A \xrightarrow{\Phi_t \otimes id_A} K \otimes A$$

$$\downarrow id \otimes \psi \quad \downarrow id \otimes \psi$$

$$C_0(T^*\mathbb{R}^n) \otimes B \xrightarrow{\Phi_t \otimes id_B} K \otimes B$$

commutes. Thus, upon completion, it commutes asymptotically.

Property (2) follows because $\{\Phi_t^A \otimes B\} = \{\Phi_t^A\} \otimes \{id_B\}$ as asymptotic morphisms. □

To finish the proof that $\Phi_A^T$ is an isomorphism for all $A$, we only need to construct an element $b \in K_0(C_0(T^*\mathbb{R}^n))$ such that $\Phi_C^T(b) = 1$.

Let $E = \Lambda_n^\mathbb{R}$ denote the complexified exterior algebra of Euclidean $n$-space. We have that the $C^*$-algebra of endomorphisms of $E$ is $\text{End}(E) \cong M_2^\mathbb{C}$ since $\dim(E) = 2^n$. Let $\delta$ be the grading operator of $E = E_{even} + E_{odd}$ into even and odd forms. For $v \in \mathbb{R}^n$, let $c(v) : E \rightarrow E$ denote the operation defined by

$$c(v)\omega = d_v\omega - \delta_v\omega, \quad \omega \in E,$$

where $d_v = v \wedge$ denotes exterior multiplication by $v$ and $\delta_v = -d_v^* = v \hook$ denotes interior multiplication by $v$. Define an $\mathbb{R}$-linear map (also denoted by $c$)

$$c : T^*\mathbb{R}^n \rightarrow \text{End}(E)$$

by the formula

$$c(v, \xi) = c(\sqrt{-1}\xi) + c(v) = \sqrt{-1}(d_\xi + \delta_\xi) + (d_v - \delta_v),$$

where we identify $T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$. This “Clifford multiplication” $c$ has the following properties: For all $(v, \xi) \in T^*\mathbb{R}^n$:

1. $c(v, \xi)^* = c(v, \xi)$ is self-adjoint.
2. $c(v, \xi)\epsilon = -\epsilon c(v, \xi)$.
3. $c(v, \xi)^2 = \|v\|^2 I + \|\xi\|^2 I.$
Our proposed element $b$ will be induced by the following. Define the graded $\ast$-homomorphism
\[
\Psi : C_0(\mathbb{R}) \to C_0(T^*\mathbb{R}^n) \otimes \text{End}(E)
\]
\[f \mapsto f(c),\]
where $f(c)(v, \xi) = f(c(v, \xi))$ is defined via the $C^\ast$-algebra functional calculus. Since
\[c(v, \xi)^2 = \|v\|^2 + \|\xi\|^2, \quad (v, \xi) \in T^*\mathbb{R}^n,
\]
it follows that the function
\[(v, \xi) \mapsto f(c(v, \xi))
\]
vanishes at infinity on $T^*\mathbb{R}^n$, and so
\[\Psi(f) \in C_0(T^*\mathbb{R}^n, \text{End}(E)) \cong C_0(T^*\mathbb{R}^n) \otimes \text{End}(E)
\]
as desired.

Letting $A$ be trivially graded, $K_0(A)$ is isomorphic to the group of graded homotopy classes of graded $\ast$-homomorphisms from $C_0(\mathbb{R})$ (with the even/odd grading) to the graded tensor product $A \hat{\otimes} M_2(K)$ (with the standard grading) where the addition is given by direct sum (Theorem 4.7 [Trou98]). Thus, this graded $\ast$-homomorphism defines a $K$-theory class
\[b = [\Psi] \in K_0(C_0(T^*\mathbb{R}^n)).
\]

**Definition B.3.** Let $\{e_j\}_{j=1}^n$ be an orthonormal basis for $\mathbb{R}^n$. For each $t > 0$, define the first order differential operator $B_t : S(\mathbb{R}^n, E) \to S(\mathbb{R}^n, E)$ on the Schwartz space of rapidly decreasing smooth $E$-valued functions by
\[B_t = \sum_{j=1}^n t^{-1}(d e_j + \delta e_j) \frac{\partial}{\partial x_j} + c(v).
\]
The operator $B_t$ is formally self-adjoint and we consider it as an unbounded operator on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, E)$ of square-integrable, $E$-valued functions. The definition of $B_t$ is independent of the basis $\{e_j\}$.

We collect the facts we will need about the the operator $B_t$ and its spectral theory in the following theorem [Hig93, Roe88].

**Theorem B.4.** $B_t$ is an essentially self-adjoint elliptic operator on $\mathcal{H}$. Moreover,
1. $B_t \epsilon = -\epsilon B_t$. Thus, with respect to the decomposition $E = E^{\text{even}} \oplus E^{\text{odd}}$, $B_t = \begin{pmatrix} 0 & B_t^- \\ B_t^+ & 0 \end{pmatrix}$.
2. The spectrum of $B_t$ is discrete with real eigenvalues $\lambda_n = \pm \sqrt{2nt^{-1}}$, where $n \geq 0$.
3. There is an orthonormal basis of $\mathcal{H}$ consisting of eigenfunctions of $B_t$.
4. $\text{Ker}(B_t)$ is 1-dimensional and spanned by the 0-form $f(x) = e^{-t\|x\|^2/2}$.
5. The total symbol of $B_t$ is $\text{sym}_{B_t}(v, \xi) = c(v, t^{-1}\xi)$.
6. $B_t$ is $O(n)$-equivariant.
7. $\text{Index}(B_t^+) = +1$.

Using the Spectral Theorem [Gue98], define a family of $\ast$-homomorphisms
\[
\{\mathcal{E}_t^B\} : C_0(\mathbb{R}) \to \mathcal{B}(\mathcal{H}) : f \mapsto f(\bar{B}_t).
\]
Lemma B.5. \( \{ \mathcal{E}^B_t \} : C_0(\mathbb{R}) \to \mathcal{K}(\mathcal{H}) \) defines a continuous family of graded \(*\)-homomorphisms.

Extend the index asymptotic morphism \( \{ \Phi_t \} : C_0(T^*\mathbb{R}^n) \to \mathcal{K} \) on \( \mathbb{R}^n \) to \( 2^n \times 2^n \) matrices

\[
\{ \Phi_t \} : M_{2^n}(C_0(T^*\mathbb{R}^n)) \to M_{2^n}(\mathcal{K})
\]

by applying element-wise. Recall that \( \text{End}(E) \cong M_{2^n}(\mathbb{C}) \) under the identification \( E = \Lambda^*_c \mathbb{R}^n \cong \mathbb{C}^{2^n} \).

Proposition B.6. \( \Phi_c^* (b) = +1 \)

Proof. Let \( \psi \in C_c(\mathbb{R}^n) \). Since \( B_t \) is essentially self-adjoint on \( \mathcal{H} = L^2(\mathbb{R}^n, E) \), we have, by an approximation argument similar to the proof of Theorem C.10, that

\[
\lim_{t \to \infty} \| \Phi_t(\psi f(c)) - M_{\psi} f(\bar{B}_t) \| = 0
\]

for any \( f \in C_0(\mathbb{R}) \) because the total symbol of \( B_1 \) is \( c \). (See also Lemma 9.4 of [Hig93].)

Now, for any \( \epsilon > 0 \), there is a \( \psi \in C_c(\mathbb{R}^n) \) such that \( \| (1 - \psi) f(\bar{B}_t) \| < \epsilon \). It follows that for all \( f \in C_0(\mathbb{R}) \),

\[
\lim_{t \to \infty} \| \Phi_t(f(c)) - f(\bar{B}_t) \| = 0
\]

Thus, we have the asymptotic equivalence \( \Phi_t \circ \Psi \sim_a \mathcal{E}^B_t \). The result now follows from Corollary 4.8 [Trou98] since \( \text{Index}(B^+_t) = +1 \). □

Theorem B.7 (Bott Periodicity). \( \Phi^A_c : K_0(C_0(T^*\mathbb{R}^n) \otimes A)) \to K_0(A) \) is an isomorphism of abelian groups.

We now turn to the Thom isomorphism.

Let \( \pi : E \to X \) be a Hermitian complex vector bundle on the locally compact topological space \( X \). In this section, we will associate to \( E \xrightarrow{\pi} X \) an injective \(*\)-homomorphism

\[
\psi^E : C_0(\mathbb{R}) \otimes C_0(X) \to C_0(\mathbb{R}) \otimes C_\tau(E),
\]

where \( C_\tau(E) \) is a \( C^* \)-algebra Morita equivalent to \( C_0(E) \), such that the induced map on \( K \)-theory is the topological Thom isomorphism \( K^0(X) \xrightarrow{\pi} K^0(E) \) [Kar78].

Definition B.8. Let \( E = \pi^*(\Lambda^* E) \) denote the pull-back over \( E \) of the exterior algebra bundle \( \Lambda^* E \). Let \( \epsilon \) denote the grading of \( E \) into even and odd forms

\[
E = E^{\text{even}} \oplus E^{\text{odd}} = \pi^*(\Lambda^{\text{even}} E) \oplus \pi^*(\Lambda^{\text{odd}} E).
\]

Let \( c : E \to \text{End}(E) \) denote the canonical section

\[
c(\epsilon) = d_\epsilon - \delta_\epsilon = \epsilon \wedge -e_\perp
\]

of the endomorphism bundle \( \text{End}(E) \to E \), where \( d_\epsilon \) denotes exterior multiplication and \( \delta_\epsilon = -d^*_\epsilon \) interior multiplication by \( \epsilon \).
Lemma B.9. For all \( e \in E \), \( c(e) : \mathbb{E}_e \to \mathbb{E}_e \) satisfies the following:

1. \( c(e)^* = c(e) \) is self-adjoint.
2. \( c(e)^2 = \|e\|^2 \).
3. \( c(e)\epsilon = -\epsilon e c(e) \).

Let \( C\tau(E) \) denote the \( C^* \)-algebra of bundle endomorphisms \( \alpha : \mathbb{E} \to \mathbb{E} \) which vanish at infinity on \( E \), under the pointwise supremum operator norm

\[
\|\alpha\| = \sup_{e \in E} \|\alpha(e)\|.
\]

Lemma B.10. \( C\tau(E) \) is Morita equivalent to \( C_0(E) \).

Corollary B.11. \( K_j(C\tau(E)) \cong K_j(C_0(E)) \cong K^j(E) \) for all \( j \).

Definition B.12. Define \( \Psi^E : C_0(\mathbb{R}) \otimes C_0(X) \to C_0(\mathbb{R}) \otimes C\tau(E) \) on elementary tensors \( f \otimes g \) by the following

\[
f \otimes g \mapsto f(\epsilon x + c)\pi^*(g),
\]

where \( \pi^*(g) = g \circ \pi \) is the pull-back of \( g \) to \( E \), and extend linearly. That is, we have

\[
\Psi^E(f \otimes g)(x, e) = f(\epsilon x + c(e))g(\pi(e)),
\]

for all \( x \in \mathbb{R} \) and \( e \in E \), where the endomorphism \( f(\epsilon x + c) \) is defined via the functional calculus.

The proof of the following is left to the reader.

Proposition B.13. \( \Psi^E \) extends to an injective \( * \)-homomorphism

\[
\Psi^E : C_0(\mathbb{R}) \otimes C_0(X) \to C_0(\mathbb{R}) \otimes C\tau(E).
\]

Example B.14. Suppose \( E = X \times \mathbb{C}^n \) is a trivial bundle. Then

\[
\mathbb{E} = X \times \mathbb{C}^n \times \Lambda^*\mathbb{C}^n.
\]

Thus, \( C\tau(E) \cong C_0(X) \otimes C_0(\mathbb{C}^n), \text{End } \Lambda^*\mathbb{C}^n \cong C_0(X) \otimes M_{2^n}(C_0(\mathbb{R}^{2n})) \) and it follows that

\[
\Psi^E : f \otimes g \mapsto f(\epsilon x + c(v, \xi)) \otimes g
\]

where \( (v, \xi) = v + i\xi \in \mathbb{C}^n \cong \mathbb{R}^{2n} \).

The next two lemmas contain the functorial properties of \( \Psi^E \) we will need below.

Lemma B.15. If \( F \) is a Hermitian bundle on \( X \) isomorphic to \( E \), then the following diagram commutes:

\[
\begin{array}{ccc}
C_0(\mathbb{R}) \otimes C_0(X) & \xrightarrow{\Psi^E} & C_0(\mathbb{R}) \otimes C\tau(E) \\
\downarrow & & \downarrow \\
C_0(\mathbb{R}) \otimes C_0(X) & \xrightarrow{\Psi^F} & C_0(\mathbb{R}) \otimes C\tau(F)
\end{array}
\]
Lemma B.16. If \( f : Y \to X \) is a continuous proper map, then the following diagram commutes:

\[
\begin{array}{ccc}
C_0(\mathbb{R}) \otimes C_0(X) & \xrightarrow{\Psi E} & C_0(\mathbb{R}) \otimes C_\tau(E) \\
\downarrow f^* & & \downarrow f^* \\
C_0(\mathbb{R}) \otimes C_0(Y) & \xrightarrow{\Psi f^* E} & C_0(\mathbb{R}) \otimes C_\tau(f^*E)
\end{array}
\]

Let \( \{B_t\}_{t \in [1, \infty)} : \mathcal{S}(\mathbb{R}^n, \Lambda^* \mathbb{C}^n) \to \mathcal{S}(\mathbb{R}^n, \Lambda^* \mathbb{C}^n) \) be the family of operators considered in Theorem B.4. Let \( \epsilon \) be the grading operator of \( \Lambda^* \mathbb{C}^n \). Using the Spectral Theorem [Gue98] again, define a family of \( * \)-homomorphisms

\[
\{A^B_t\} : C_0(\mathbb{R}) \to C_0(\mathbb{R}) \otimes \mathcal{B}(\mathcal{H}) : f \mapsto f(\epsilon x + B_t).
\]

Note that for each \( x \in \mathbb{R} \), the operator \( \epsilon x + B_t \) is essentially self-adjoint.

Lemma B.17. \( \{A^B_t\} \) is a continuous family of \( * \)-homomorphisms.

Lemma B.18. Suppose \( E = \{pt\} \times \mathbb{C}^n \). Then for all \( f \in C_0(\mathbb{R}) \), we have that

\[
\lim_{t \to \infty} \|\Phi_t(f(\epsilon x + c)) - f(\epsilon x + B_t)\| = 0.
\]

Proof. This follows from the discussion in Proposition B.6. \( \Box \)

Corollary B.19. If \( E = \{pt\} \times \mathbb{C}^n \), then the following diagram asymptotically commutes:

\[
\begin{array}{ccc}
C_0(\mathbb{R}) \otimes M_{2^n}(C_0(T^*\mathbb{R}^n)) & \xrightarrow{\mathcal{L} \Phi_t} & C_0(\mathbb{R}) \otimes M_{2^n}(\mathcal{K}) \\
\uparrow \Psi E & & \uparrow A^B_t \\
C_0(\mathbb{R}) & \xrightarrow{=} & C_0(\mathbb{R})
\end{array}
\]

Let \( P_t : \mathcal{H} \to \text{Ker}(B_t) \) denote the orthogonal projection onto the kernel of the operator \( B_t \), where \( \mathcal{H} = L^2(\mathbb{R}^n, \Lambda^* \mathbb{C}^n) \). By Theorem 3.7, \( \{P_t\} \) is a continuous family of rank one projections in \( \mathcal{K}(\mathcal{H}) \), the \( \mathcal{C}^* \)-algebra of compact operators on \( \mathcal{H} \).

Lemma B.20. The family \( \{A^B_t\} \) is homotopic to the family \( f \mapsto f \otimes P_t \).

Proof. The homotopy is given by

\[
f \mapsto f(\epsilon x + s^{-1}B_t), \quad 0 \leq s \leq 1.
\]

First, we need to check continuity in \( s \). Considering the factorization

\[
r_\pm(\epsilon x + s^{-1}B_t) - r_\pm(\epsilon x + r^{-1}B_t) = r_\pm(\epsilon x + s^{-1}B_t)(r^{-1} - s^{-1})B_tr_\pm(\epsilon x + r^{-1}B_t)
\]

\[
= r_\pm(\epsilon x + s^{-1}B_t)(1 - rs^{-1})(\epsilon x + r^{-1}B_t)r_\pm(\epsilon x + r^{-1}B_t) + r_\pm(\epsilon x + s^{-1}B_t)(1 - rs^{-1})(-\epsilon x)r_\pm(\epsilon x + r^{-1}B_t)
\]

we see that for each \( x \in \mathbb{R} \)

\[
\|r_\pm(\epsilon x + s^{-1}B_t) - r_\pm(\epsilon x + r^{-1}B_t)\| \leq (1 - rs^{-1})(1 + |x|)
\]
Thus, as \( r \to s \), the operator-valued functions \( r_{\pm}(\epsilon x + r^{-1}B_t) \) on the real line converge uniformly to \( r_{\pm}(\epsilon x + s^{-1}\bar{B}_t) \) on compact subsets. From the inequality
\[
\|r_{\pm}(\epsilon x + s^{-1}\bar{B}_t)\| \leq \frac{1}{\sqrt{x^2 + 1}}
\]
these operators are uniformly small on the complement of \([-x, x]\) for \( x \geq 0 \) large. Therefore, they converge uniformly on \( \mathbb{R} \).

Since \((\epsilon x + s^{-1}B_t)^2 = x^2 + s^{-2}B_t^2\), it follows that the eigenvalues of \((\epsilon x + s^{-1}B_t)\) are of the form
\[
\lambda_n = \pm \sqrt{x^2 + 2ns^{-2}t^{-1}}
\]
by Theorem B.4. Thus, as \( s \to 0 \), the spectrum of \((\epsilon x + s^{-1}B_t)\) corresponding to \( n > 0 \) goes to infinity. This implies that
\[
f(\epsilon x + s^{-1}\bar{B}_t) \to f(x)P_t
\]
in norm as \( s \to 0 \). Note that if \( \text{supp}(f) \subset [-a, a] \), then for small enough \( s \),
\[
f(\epsilon x + s^{-1}\bar{B}_t) = f(x)P_t.
\]
The result follows. \( \square \)

Combining the previous two results we obtain the following.

**Corollary B.21.** The following diagram commutes up to homotopy:
\[
\begin{array}{ccc}
C_0(\mathbb{R}) \otimes M_{2^n}(C_0(T^*\mathbb{R}^n)) & \xrightarrow{1 \otimes \Phi_t} & C_0(\mathbb{R}) \otimes M_{2^n}(K) \\
\Psi^E \uparrow & & \uparrow 1 \otimes P_t \\
C_0(\mathbb{R}) & \xrightarrow{=} & C_0(\mathbb{R})
\end{array}
\]

Let \( E \to X \) be a Hermitian complex vector bundle on \( X \). Let
\[
\Psi^E_* : K^0(X) \to K^0(E)
\]
denote the mapping induced on \( K \)-theory by the \( * \)-homomorphism
\[
\Psi^E : C_0(\mathbb{R}) \otimes C_0(X) \to C_0(\mathbb{R}) \otimes C_7(E),
\]
where we invoke the isomorphisms \( K_1(C_0(\mathbb{R}) \otimes C_0(Y)) \cong K_0(C_0(Y)) \cong K^0(Y) \) for any locally compact space \( Y \).

**Theorem B.22 (Thom Isomorphism).** \( \Psi^E_* : K^0(X) \to K^0(E) \) is an isomorphism.

**Proof.** Our proof is divided into the following cases.

1.) If \( X = \{pt\} \), then \( E = \mathbb{C}^n \cong \mathbb{R}^{2n} \), \( \mathbb{E} = \Lambda^* \mathbb{C}^n \) and
\[
\Psi^E : C_0(\mathbb{R}) \to C_0(\mathbb{R}) \otimes M_{2^n}(C_0(\mathbb{C}^n))
\]
is the map \( f \mapsto f(\epsilon x + c) \). Thus, by Corollary B.21, the induced map on \( K \)-theory is the Bott Periodicity isomorphism from Theorem B.7.
2.) Suppose $E = X \times \mathbb{C}^n$ is trivial. Example B.14 above then shows that $
abla^E = \nabla \otimes id_{C_0(X)}$ where $\nabla$ is the map in the previous case. The induced map is then seen to be $\nabla^E_* = \beta_{C_0(X)}$ which is the Bott Periodicity map constructed in Theorem B.1.

3.) If $E$ is trivializable, the result follows from Lemma B.15 and case 2.

4.) Now use the Mayer-Vietoris sequence and the Five Lemma for $X = X_1 \cup X_2$ where $E$ trivializes over $X_1$ and $X_2$.

5.) In general, cover $X$ with open sets $\{X_j\}$ such that over each $X_j$, $E$ trivializes as $E_j = E|_{X_j} \cong X_j \times \mathbb{C}^n$. An induction argument using the previous case and the continuity of $K$-theory

$$K^0(X) = \lim\{K^0(X_j) : X_j \subset X \text{ is open}\}$$

finishes the proof. □

It follows from the proof that $\nabla^E_* : K^0(X) \to K^0(E)$ is, in fact, the topological Thom Isomorphism [Kar78].

REFERENCES