Computing 2-twinless blocks

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Abstract

Let \( G = (V, E) \) be a directed graph. A 2-twinless block in \( G \) is a maximal subset \( B \subseteq V \) of size at least 2 such that for each pair of distinct vertices \( x, y \in B \), and for each vertex \( w \in V \setminus \{x, y\} \), the vertices \( x, y \) are in the same twinless strongly connected component of \( G \setminus \{w\} \). In this paper we present algorithms for computing the 2-twinless blocks of a directed graph.

Keywords: Directed graphs, Connectivity, Graph algorithms, 2-blocks, Twinless strongly connected graphs

1. Introduction

Let \( G = (V, E) \) be a directed graph. \( G \) is twinless strongly connected if it contains a strongly connected spanning subgraph \((V, E^t)\) such that \( E^t \) does not contain any pair of antiparallel edges. A twinless strongly connected component of \( G \) is a maximal subset \( C_t \subseteq V \) such that the induced subgraph on \( C_t \) is twinless strongly connected. A strong articulation point of \( G \) is a vertex whose removal increases the number of strongly connected components of \( G \). A strong bridge of \( G \) is an edge whose deletion increases the number of strongly connected components of \( G \). A strongly connected graph is 2-vertex-connected if it has at least 3 vertices and it has no strong articulation points. A 2-vertex-connected component of \( G \) is a maximal vertex subset \( C_v \subseteq V \) such that the induced subgraph on \( C_v \) is 2-vertex-connected. A 2-directed block in \( G \) is a maximal vertex subset \( B^d \subseteq V \) with \(|B^d| > 1\) such that for any distinct vertices \( x, y \in B^d \), the graph \( G \) contains two vertex-disjoint paths from \( x \) to \( y \) and two vertex-disjoint paths from \( y \) to \( x \). A 2-edge block in \( G \) is a maximal subset \( B^{eb} \subseteq V \) with \(|B^{eb}| > 1\) such that for any distinct vertices \( v, w \in B^{eb} \), there are two edge-disjoint paths from \( v \) to \( w \) and two edge-disjoint paths from \( w \) to \( v \) in \( G \). A 2-strong block in \( G \) is a maximal vertex subset \( B^s \subseteq V \) with \(|B^s| > 1\) such that for each pair of distinct vertices \( x, y \in B^s \) and for every vertex \( u \in V \setminus \{x, y\} \), the vertices \( x \) and \( y \) are in the same strongly connected component of the graph \( G \setminus \{u\} \). A twinless articulation point of \( G \) is a vertex whose removal increases the number of twinless strongly connected components of \( G \). A 2-twinless block...
in $G$ is a maximal vertex set $B \subseteq V$ of size at least 2 such that for each pair of distinct vertices $x, y \in B$, and for each vertex $w \in V \setminus \{x, y\}$, the vertices $x, y$ are in the same twinless strongly connected component of $G \setminus \{w\}$. Notice that 2-strong blocks are not necessarily 2-twinless blocks (see Figure 1).

![Graph Image](image)

Figure 1: A strongly connected graph $G$, which contains two 2-strong blocks $C_1 = \{2, 7\}, C_2 = \{12, 13, 17, 19\}$, and one 2-twinless block $B = \{2, 7\}$. Notice that the vertices 12 and 17 do not belong to the same twinless strongly connected component of $G \setminus \{13\}$.

A twinless strongly connected graph $G$ is said to be 2-vertex-twinless-connected if it has at least three vertices and it does not contain any twinless articulation point \cite{10}. A 2-vertex-twinless-connected component is a maximal subset $U^{2vt} \subseteq V$ such that the induced subgraph on $U^{2vt}$ is 2-vertex-twinless-connected. While 2-vertex-twinless-connected components have at least linear number of edges, the subgraphs induced by 2-twinless blocks do not necessarily contain edges.

Strongly connected components can be found in linear time \cite{25}. In 2006, Raghavan \cite{22} showed that the twinless strongly connected component of a directed graph can be found in linear time. In 2010, Georgiadis \cite{7} presented an
algorithm to check whether a strongly connected graph is 2-vertex-connected in linear time. Italiano et al. [15] gave linear time algorithms for identifying all the strong articulation points and strong bridges of a directed graph. Their algorithms are based on dominators [3, 4, 1, 2, 6, 21]. In 2014, Jaberi [17] presented algorithms for computing the 2-vertex-connected components of directed graphs in $O(nm)$ time (published in [16]). Henzinger et al. [14] gave algorithms for calculating the 2-vertex-connected components in $O(n^2)$ time. Jaberi [18] presented algorithms for computing 2-blocks in directed graphs. Georgiadis et al. [9, 10] gave linear time algorithms for determining 2-edge blocks. Georgiadis et al. [11, 12] also gave linear time algorithms for calculating 2-directed blocks and 2-strong blocks. Georgiadis et al. [13] and Luigi et al. [20] performed experimental studies of recent algorithms that calculate 2-blocks and 2-connected components in directed graphs. In 2019, Jaberi [19] presented an algorithm for computing 2-vertex-twinless-connected components. Georgiadis and Kosinas [8] gave a linear time algorithm for calculating twinless articulation points.

In the following section we show that the 2-twinless blocks of a directed graph can be calculated in $O(n^3)$ time.

2. Algorithm for computing 2-twinless blocks

In this section we present an algorithm for computing the 2-twinless blocks of a twinless strongly connected graph. Since twinless strongly connected components do not share vertices of the same 2-twinless block, we consider only twinless strongly connected graphs. Let $G = (V, E)$ be a twinless strongly connected graph. We define a relation $\sim_{2t}$ as follows. For any distinct vertices $x, y \in V$, we write $x \sim_{2t} y$ if for all vertices $w \in V \setminus \{x, y\}$, the vertices $x, y$ are in the same twinless strongly connected component of $G \setminus \{w\}$. By definition, a 2-twinless block in $G$ is a maximal subset $B_{2t} \subseteq V$ with $|B_{2t}| > 1$ such that for every two vertices $x, y \in B_{2t}$, we have $x \sim_{2t} y$.

The next lemma shows that 2-twinless blocks share at most one vertex.

**Lemma 2.1.** Let $G = (V, E)$ be a twinless strongly connected graph. Let $B_{2t}^1, B_{2t}^2$ be distinct 2-twinless blocks in $G$. Then $|B_{2t}^1 \cap B_{2t}^2| \leq 1$.

**Proof.** Suppose for the sake of contradiction that $B_{2t}^1$ and $B_{2t}^2$ have at least two vertices in common. Clearly, $B_{2t}^1 \cup B_{2t}^2$ is not a 2-twinless block in $G$. Let $x$ and $y$ be vertices belonging to $B_{2t}^1$ and $B_{2t}^2$, respectively, such that $x, y \notin B_{2t}^1 \cap B_{2t}^2$. Let $z$ be any vertex in $V \setminus \{x, y\}$. Since $|B_{2t}^1 \cap B_{2t}^2| > 1$, there is a vertex $v$ in $(B_{2t}^1 \cap B_{2t}^2) \setminus \{z\}$. Note that $x, v$ are in the same twinless strongly connected component of $G \setminus \{z\}$ since $x, v \in B_{2t}^1$. Moreover, $v$ and
y lie in the same twinless strongly connected component of $G \setminus \{z\}$. By [22], Lemma 1] $x$ and $y$ are in the same twinless strongly connected component of $G \setminus \{z\}$. Therefore, $x, y$ belong to the same 2-twinless block. □

The following lemma shows an interesting property of the relation $\leftrightarrow$.

**Lemma 2.2.** Let $G = (V, E)$ be a twinless strongly connected graph and let $\{v_0, v_1, \ldots, v_l\}$ be set of vertices of $G$ such that $v_i \leftrightarrow v_0$ and $v_{i-1} \leftrightarrow v_i$ for $i \in \{1, 2, \ldots, l\}$. Then $\{v_0, v_1, \ldots, v_l\}$ is a subset of a 2-twinless block in $G$.

**Proof.** Assume for the purpose of contradiction that there are two vertices $v_r$ and $v_q$ in $G$ such that $v_r$ and $v_q$ are in distinct 2-twinless blocks of $G$ and $r, q \in \{0, 1, \ldots, l\}$. Suppose without loss of generality that $r < q$. Then there is a vertex $z \in V \setminus \{v_r, v_q\}$ such that $v_r$ and $v_q$ are in distinct twinless strongly connected components of $G \setminus \{z\}$. We distinguish two cases.

1. $z \in \{v_{r+1}, v_{r+2}, \ldots, v_{q-1}\}$. In this case, the vertices $v_{r-1}, v_i$ belong to the same twinless strongly connected component of $G \setminus \{z\}$ for each $i \in \{1, 2, \ldots, r\} \cup \{q + 1, q + 2, \ldots, l\}$. Moreover, the vertices $v_0, v_l$ are in the same twinless strongly connected component of $G \setminus \{z\}$ because $v_0 \leftrightarrow v_l$. Therefore, the vertices $v_r, v_q$ are in the same twinless strongly connected component of the graph $G \setminus \{z\}$, a contradiction.

2. $z \notin \{v_{r+1}, v_{r+2}, \ldots, v_{q-1}\}$. Then, for each $i \in \{r + 1, r + 2, \ldots, q\}$, the vertices $v_{i-1}, v_i$ lie in the same twinless strongly connected component of $G \setminus \{z\}$. Consequently, the vertices $v_r, v_q$ belong to the same twinless strongly connected component of the graph $G \setminus \{z\}$, a contradiction. □

Let $G = (V, E)$ be a twinless strongly connected graph. We construct the 2-twinless block graph $G^{2t} = (V^{2t}, E^{2t})$ of $G$ as follows. For every 2-twinless block $B_i$, we add a vertex $v_i$ to $V^{2t}$. Moreover, for each vertex $v \in V$, if $v$ belongs to at least two distinct 2-twinless blocks, we add a vertex $v$ to $V^{2t}$. For any distinct 2-twinless blocks $B_i, B_j$ with $B_i \cap B_j = \{v\}$, we put two undirected edges $(v_i, v), (v, v_j)$ into $E^{2t}$.

**Lemma 2.3.** The 2-twinless block graph of a twinless strongly connected graph is a forest.

**Proof.** This follows from Lemma 2.2 and Lemma 2.1. □

**Lemma 2.4.** Let $G = (V, E)$ be a twinless strongly connected graph and let $x, y$ be distinct vertices in $G$. Suppose that $v \in V \setminus \{x, y\}$ is not a twinless articulation point. Then $x, y$ are in the same twinless strongly connected component of $G \setminus \{v\}$.
Algorithm 2.5.

Input: A twinless strongly connected graph $G = (V, E)$.

Output: The 2-twinless blocks of $G$.

1. if $G$ is 2-vertex-twinless connected then
   2. Output $V$.
else
   4. Let $S$ be an $n \times n$ matrix.
   5. Initialize $S$ with 1s.
   6. determine the twinless articulation points of $G$.
   7. for each twinless articulation point $z$ of $G$ do
      8. Identify the twinless strongly connected components of $G \setminus \{z\}$.
      9. for each pair $(v, w) \in (V \setminus \{z\}) \times (V \setminus \{z\})$ do
         10. if $v, w$ in different twinless strongly connected components of $G \setminus \{z\}$ then
             11. $S[v, w] \leftarrow 0$.
      12. $E^b \leftarrow \emptyset$.
      13. for each pair $(v, u) \in V \times V$ do
         14. if $S[v, u] = 1$ and $S[u, v] = 1$ then
            15. $E^b \leftarrow E^b \cup \{(v, u)\}$.
   16. calculate the blocks of size $\geq 2$ of $G^b = (V, E^b)$ and output them.

The correctness of Algorithm 2.5 follows from the following lemma.

Lemma 2.6. A vertex subset $B \subseteq V$ is a 2-twinless block of $G$ if and only if $B$ is a block of the undirected graph $G^b = (V, E^b)$ which is constructed in lines 12–15 of Algorithm 2.5.

Proof. It follows from Lemma 2.2 and Lemma 2.4. □

Theorem 2.7. Algorithm 2.5 runs in $O(n^3)$ time.

Proof. Georgiadis and Kosinas [8] showed that the twinless articulation points can be computed in linear time. The initialization of matrix $S$ takes $O(n^2)$ time. The number of iterations of the for-loop in lines 7–11 is at most $n$ because the number of twinless articulation points is at most $n$. Consequently, lines 7–11 require $O(n^3)$. Furthermore, the blocks of an undirected graph can be found in linear time [25, 24]. □
Lemma 2.8. The graph $G^b$ which is constructed in lines 12–15 of Algorithm 2.5 is chordal.

Proof. It follows from Lemma 2.2. □

By Lemma 2.8, one can calculate the maximal cliques of $G^b$ instead of blocks. The maximal cliques of a chordal graph can be calculated in linear time [3, 23].

3. An improved version of Algorithm 2.5

In this section we present an improved version of Algorithm 2.5.

The following lemma shows a connection between 2-twinless blocks and 2-strong blocks.

Lemma 3.1. Let $G = (V, E)$ be a twinless strongly connected graph. Suppose that $B_t$ is a 2-twinless block in $G$. Then $B_t$ is a subset of a 2-strong block in $G$.

Proof. Let $v$ and $w$ be distinct vertices in $B_t$, and let $x \in V \setminus \{v, w\}$. By definition, the vertices $v, w$ belong to the same twinless strongly connected component $C$ of $G \setminus \{x\}$. Since $C$ is a subset of a strongly connected component of $G$, the vertices $v, w$ also lie in the strongly connected component of $G \setminus \{x\}$. Consequently, $v, w$ are in the same 2-strong block in $G$. □

Algorithm 3.2 describes this improved version which is based on Lemma 3.1 and Lemma 2.2.

Theorem 3.3. The running time of Algorithm 3.2 is $O(t(\frac{s^2}{2} + n^2))$, where $s = |A|$ and $t$ is the number of twinless articulation points of $G$.

Proof. The 2-strong blocks of a directed graph can be computed in linear time [11]. Furthermore, the twinless articulation points of a directed graph can be identified in linear time using the algorithm of Georgiadis and Kosinas [8]. Since the number of iterations of the for-loop in lines 16–20 is at most $t$, lines 16–20 take $O(t(\frac{s^2}{2} + m))$ time. □

Let $G = (V, E)$ be a twinless strongly connected graph. If the refine operation defined in [20] is used to refine the 2-strong blocks of $G$ for all twinless articulation points, then the 2-twinless blocks of a directed graph $G = (V, E)$ can be computed in $O(tm)$ time, where $t$ is the number of twinless articulation points of $G$.

We leave as open problem whether the 2-twinless blocks of a directed graph can be calculated in linear time.
Algorithm 3.2.
Input: A twinless strongly connected graph $G = (V, E)$.
Output: The 2-twinless blocks of $G$.
1 if $G$ is 2-vertex-twinless connected then
2 Output $V$.
3 else
4 find the 2-strong blocks of $G$
5 Let $S$ be an $n \times n$ matrix.
6 Initialize $S$ with 0.
7 $A \leftarrow \emptyset$.
8 for each 2-strong block $B$ of $G$ do
9 for each pair of vertices $v, w \in B$ do
10 $S[v, w] \leftarrow 1$
11 $S[w, v] \leftarrow 1$
12 for each vertex $v \in B$ do
13 if $v \notin A$ then
14 add $v$ to $A$
15 determine the twinless articulation points of $G$.
16 for each twinless articulation point $z$ of $G$ do
17 Identify the twinless strongly connected components of $G \setminus \{z\}$.
18 for each pair $(v, w) \in (A \setminus \{z\}) \times (A \setminus \{z\})$ do
19 if $v, w$ in different twinless strongly connected components of $G \setminus \{z\}$ then
20 $S[v, w] \leftarrow 0$.
21 $E^b \leftarrow \emptyset$.
22 for each pair $(v, u) \in A \times A$ do
23 if $S[v, u] = 1$ and $S[u, v] = 1$ then
24 $E^b \leftarrow E^b \cup \{(v, u)\}$.
25 calculate the blocks of size $\geq 2$ of $G^b = (A, E^b)$ and output them.

Acknowledgements.

The author would like to thank the anonymous reviewers for their helpful comments and suggestions.

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