Inflationary Behaviour in Axial-symmetric Gravitational Collapse

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Abstract

We show that the interior of a charged, spinning black hole formed from a general axially symmetric gravitational collapse is unstable to inflation of both its mass and angular momentum parameters. Although our results are formulated in the context of (2 + 1)-dimensional black holes, we argue that they are applicable to (3 + 1) dimensions.
The ultimate fate of a gravitationally collapsing body remains a subject of considerable interest. The spacetime exterior to a collapsing body relaxes to that of a Kerr-Newman (KN) black hole (provided the plausible hypothesis of cosmic censorship holds) with radiative perturbations decaying as advanced time increases according to a power law. Infalling matter will either enter a region of diverging spacetime curvature (where the effects of quantum gravity are expected to dominate) or avoid this region and emerge via a ‘white hole’ into another universe, as can occur in the KN case, where the matter necessarily passes through the Cauchy horizon on the way. Since it has been shown that this surface is unstable due to the divergence of the stress-energy of massless test fields\(^2\), the question of the final fate of infalling matter crucially depends upon whether or not this instability seals off the Cauchy horizon.

New light was shed on this issue when it was demonstrated that in the presence of outflow from a collapsing body in the Reissner-Nordström geometry, the gravitational mass parameter (and thus the curvature of spacetime) must tend to infinity at the Cauchy horizon, a phenomenon called mass inflation\(^3\). Ori subsequently computed the rate of growth of mass and curvature in a simpler model; from this he argued that the tidal effects due to spacetime curvature yielded bounded distortions at the Cauchy horizon and so any object attempting to cross it will not necessarily be destroyed\(^3\). It is possible that quantum effects substantially modify this picture\(^4\), but the details of this are far from clear.

The restrictive features of these models have inspired theorists to examine other simpler models, both in (2 + 1)\(^5\) and (1 + 1) dimensions\(^6\)\(^7\)\(^8\). In these models, the phenomenon is strikingly similar to the Reissner-Nordström archetype. However, these studies have all been restricted to the case where the black holes are non-rotating, which is not general enough in practice (although suggestive arguments have been given\(^\text{[10]}\)). Progress on this front was recently made when mass inflation was also shown to take place in a (2 + 1)-dimensional black hole with constant angular momentum\(^9\) and a rotating black string in (3 + 1) dimensions\(^10\). This provides the strongest evidence to date in support of the hypothesis that mass inflation will occur in a rotating black hole in General Relativity\(^\text{[11]}\).

In this letter we shall show that the energy parameter of the black hole (which is dependent upon its mass and angular momentum) diverges at the Cauchy horizon as a consequence of the interaction between infalling matter and radiative outflow from a collapsing body. Both the mass and angular momentum parameters of the black hole can diverge at the Cauchy horizon – we argue that in physically realistic cases the former must diverge faster than the latter. Although we work in the context of the (2 + 1) dimensional black hole\(^\text{[1]}\), we argue that these features should carry over to (3 + 1) dimensions.

Einstein’s equations with cosmological constant \(\Lambda < 0\) in (2 + 1) dimensions

\[
G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}
\]

have the following exact solution\(^4\)

\[
d s^2 = - N^2(r) \, dt^2 + N^{-2} \, dr^2 + r^2 \left[ N^\phi(r) \, dt + d\phi \right]^2
\]

in the electrovacuum, where \(N^\phi(r) \equiv - J/(2 \, r^2)\) and \(N^2(r) \equiv - \Lambda \, r^2 - M + \frac{\Omega^2}{r^2} - 4 \pi G q^2 \ln \left( \frac{r_o}{r} \right)\). The constants of integration \(q, M\) and \(J\) are the static charge, mass and angular momentum of the hole respectively\(^\text{[1], [10]}\) and \(r_o\) is an arbitrary scale constant with dimension of length. If in addition to the electromagnetic stress-energy tensor, we consider a stress-energy tensor with energy density \(\hat{\rho}\) and angular momentum density \(\hat{\omega}\), that is

\[
[T_{\mu\nu}] = \begin{bmatrix}
\hat{\rho}(v, r) & 0 & -\hat{\omega}(v, r) \\
0 & 0 & 0 \\
-\hat{\omega}(v, r) & 0 & 0 
\end{bmatrix},
\]

we obtain

\[
d s^2 = \left[ \Lambda \, r^2 + m(v) + 4 \pi G q^2 \ln(r/r_o) \right] \, dv^2 + 2 \, dv \, dv - j(v) \, dv \, d\theta + r^2 \, d\theta^2,
\]

as an exact solution to\(^4\)\(^\text{[1], [11]}\). Here \(m(v)\) and \(j(v)\) satisfy the differential equations

\[
\frac{d m(v)}{d v} = 16 \pi G \rho(v) \quad \text{and} \quad \frac{d j(v)}{d v} = 16 \pi G \omega(v)
\]

and \(\hat{\rho}(v, r) = \rho(v)/r + j(v) \omega(v)/(2 \, r^2)\) and \(\hat{\omega}(v, r) = \omega(v)/r\), as dictated by the conservation laws, with \(\rho, j\) and \(\omega\) arbitrary functions of \(v\). It is furthermore straightforward to show from\(^\text{[2]}\) that the null geodesic equations simplify to

\[
\frac{d}{d \lambda} \left[ \frac{2}{\tilde{v}(\lambda)} \right] = \partial_r N^2(v, r),
\]

where \(N^2(v, r) \equiv \alpha(v, r) + j^2(v)/(4 \, r^2)\) with \(\alpha(v, r) = - g_{vv}\), and

\[
2 \dot{v} \dot{r} = N^2(v, r) \, \tilde{v}^2
\]

so that \(v\) is a null ingoing coordinate.
Consider a pulse, $S$, of outgoing null radiation between the Cauchy and outer horizons in the background spacetime [2]. We can model this by matching two patches of solution [3] with different $m$ and $j$ along $S$ as shown in Figure 1. We denote the region enclosed by the ring $S$ as II and its complement as region I. Each region is characterized by its own $m_a(v_a)$ and $j_a(v_a)$, where $a$ has value of either 1 or 2 for corresponding region. Since we assume that the two regions have different masses, the Cauchy horizon cannot coincide with the inner horizon as shown in Figure 3. If $j_1 = j_2$, the null ring will rotate at the same pace as the spacetime in regions I and II. However, when $j_1 \neq j_2$, $S$ will carry intrinsic spin.

\[
\text{Figure 1: Matching two pieces of non-stationary BTZ solution. Region I has mass } m_1 \text{ and angular momentum } j_1 \text{ and Region II is characterized by } m_2 \text{ and } j_2. \]

Without loss of generality, we suppose the affine parameter $\lambda$ to be zero at the Cauchy horizon and positive behind that, and in addition take $M$ and $J$ to be the respective asymptotic values of $m_1$ and $j_1$. Defining a function $R(\lambda)$ such that $2 \pi R$ is the perimeter of $S$, continuity of inflow along a null curve with tangent vector

\[
l_a = \left( \frac{2}{N^2}, 1, \frac{j_a}{r N^2} \right)
\]

yields (using the null condition [3])

\[
\frac{dm_1 - d(j_1^2)/(4 R^2)}{N_1^2} = \frac{dm_2 - d(j_2^2)/(4 R^2)}{N_2^2}. \tag{5}
\]

When the ring is close to the Cauchy horizon $r_c$, the left side of equation (5) can be approximated as

\[
\frac{dm_1 - d(j_1^2)/(4 R^2)}{N_1^2} \approx - \frac{dm_1 + d(j_1^2)/(4 r_c^2)}{N_1^2(v_1, r_c)} \]

which tends to negative infinity at the Cauchy horizon (where $v_1 \to \infty$).

The quantity $E_a \equiv m_a - j_a^2/(4 r_a^2)$ appearing in $N_a^2$ is proportional to the total energy of spacetime at large $r$, neglecting electromagnetic contributions [5]. It is straightforward to show [6] that $E$ is positive when the weak energy condition [18] on null geodesics is satisfied. One can further show that $E$ is also a geometrically invariant quantity, since

\[
E = -g^{\mu\nu} \nabla_\mu L \nabla_\nu L - \Lambda L^2 \quad \tag{6}
\]

where $L^2 = g(\zeta, \zeta)$ with $\zeta$ a spacelike Killing vector that obeys the algebra of $SO(2)$ [19]. Since the generalized DTR relation has the same form as in $(3 + 1)$ dimensions [20], (except that the dilation rate $K_i$ is half as large), we find for two colliding null shells

\[
E_D = E_A + E_B + \frac{E_A E_B}{\Lambda r^2}, \quad \tag{7}
\]

where $E_D$ is the energy parameter in region $D$ as shown in Figure 2, showing that the energy parameters effectively add together for large $r$. This is the analogue of the $(3 + 1)$-dimensional result in ref. [4].

\[
\text{Figure 2: The spacetime diagram for the generalized DTR relation. Since the two null rings collide, this event divides the spacetime into regions A, B, C and D.}
\]

For constant $q$ the electromagnetic contributions drop out of (5), and so we shall refer to $E_a$ as the energy parameter in region $a$. The smaller roots of the
transcendental equations $N^2_a = 0$ cannot be the same if the energy content differs between regions; hence if $S$ has energy $E^\text{ring} = E_2 - E_1 \neq 0$ when it reaches the Cauchy horizon, $d m_2 - d(j^2_2/(4 r^2))$ will diverge there because $N^2_a \neq 0$ at $r_c$. Positivity of energy in region II (i.e. $m_2 > j^2_2/(4 r^2)$) forces $m_2$ to have the faster growth rate, forcing the inner horizon to recede behind the Cauchy horizon as shown in Figure 3.

![Figure 3: Horizon locations of the matched spacetime.](image)

We can approximate the rate of energy inflation as follows. From (8) and (9) we have

$$m_a(v_a(\lambda)) - \frac{j^2_a(v_a(\lambda))}{4 R^2(\lambda)} = \mathcal{H}(R(\lambda)) - \dot{R}(\lambda) \frac{z_a(\lambda)}{R(\lambda)},$$

$$v_a(\lambda) = 2 \int_\lambda^\infty \frac{R(\zeta)}{z_a(\zeta)} d\zeta,$$

$$\frac{z_a(\lambda)}{R(\lambda)} = Z_a + \int_0^\lambda \partial_r N^2_a(v_a(\zeta), r)|_{r=\zeta} d\zeta,$$

where $\mathcal{H}(R) \equiv -\Lambda R^2 - 4 \pi G q^2 \ln(R/r_o)$ has the limit $M - J^2/(4 r^2)$ when $R$ tends to $r_c$ and $Z_a$ are constants. Since $N^2_1$ and $N^2_2$ have finite slope at the Cauchy horizon, $z_a(\lambda)$ can be approximated as

$$\frac{z_1(\lambda)}{R(\lambda)} \approx Z_1 - 2 k_o \lambda$$

and

$$\frac{z_2(\lambda)}{R(\lambda)} \approx Z_2 - K \lambda$$

for small $\lambda$, where $k_o$ and $K$ are constants. The constant $k_o$ must be positive since the slope of $N^2_1$ at the Cauchy horizon is always negative. (See Figure 3.)

As $\lambda \to 0$ we expect $\lim_{\lambda \to 0} \dot{v}_1(\lambda) = 2/|Z_1| = \infty$ but that $\dot{v}_2$ is finite, implying that $Z_1 = 0$ and $Z_2 \neq 0$. Thus

$$v_1(\lambda) \approx -\frac{1}{k_o} \ln |\lambda|$$

and

$$v_2(\lambda) \approx \frac{2}{Z_2} \lambda.$$

Since $N^2_1 = -|\mathcal{H}(r) - m_1 + j^2_1/(4 r^2)| \approx -|M - J^2/(4 r^2) - m_1 + j^2_1/(4 r^2)|$ just outside the Cauchy horizon, we have $\mathcal{R}(\lambda) \approx \delta E(\lambda)/(2 k_o \lambda)$ from equation (3), where

$$\delta E(\lambda) = M - m_1(\lambda) - \frac{J^2}{4 r_c^2} + \frac{j^2_2(\lambda)}{4 r_c^2}$$

is the tail of the total energy at late time. Thus equation (8) implies that the energy of the ring is

$$E^\text{ring}(\lambda) \approx -\frac{Z_2}{2 k_o \lambda} \delta E(\lambda),$$

where $Z_2$ must be positive so that $E^\text{ring} > 0$. If we assume that $\delta E(\lambda)$ decays to zero via a power law $h v_1^{-p}$, we obtain

$$E_2(v_2) \approx M - \frac{J^2}{4 r_c^2}$$

$$- \frac{h}{v_2} k_o p^{-1} \ln \left( \frac{Z_2 v_2}{2} \right)^{1-p}. \quad (12)$$

As a result, $E_2(v_2)$ goes to infinity while $S$ approaches the Cauchy horizon because $v_2$ tends to zero from below at that instant.

We have shown that the phenomenon of mass inflation is replaced by the more general phenomenon of inflation of the energy parameter in the case of collapse of a null fluid with non-zero angular momentum density. Although this conclusion arises from an analysis of $2 + 1$-dimensional black holes, we expect that the qualitative features of this phenomenon (namely that both mass and angular momentum parameters inflate) carry over to $3 + 1$ dimensions for several reasons. First of all, the results of this paper carry over straightforwardly to the case of a charged, spinning black string in $(3 + 1)$ dimensions. Second, the result (12) for constant angular momenta is qualitatively similar to the situation hypothesized for axisymmetric collapse to a Kerr black hole as the mass parameter inflates, the angular momentum per unit mass $J/m$ becomes negligible. We expect that when fluids with non-vanishing angular momentum density are included, the angular momentum parameter will inflate in a manner consistent with the results of ref. [10].

It is also possible to consider the results of a fluid with vanishing $\rho(v)$, yielding constant $m$’s in each of regions I and II. We find that the growth rate of $j^2_2$ is qualitatively similar to that of $E_2$ in (12) but is opposite in sign, diverging to negative infinity. This unphysical situation is associated with the breakdown of the positivity of energy due to the vanishing of $\rho(v)$.
We close by considering tidal distortions at the horizon. In the triad frame, the relevant components of the Riemann tensor look like the following:

\[ R^{1001} = \frac{\partial_{\alpha} \alpha - \partial_{r} \alpha}{2r} - \frac{\partial_{r}(j^2)}{8r^4} - \frac{j^2}{8r} \partial_{r} \left( \frac{\partial_{r} \alpha}{r} \right), \]
\[ R^{1002} = R^{2001} = \frac{j}{4} \partial_{r} \left( \frac{\partial_{r} \alpha}{r} \right) + \frac{\partial_{r} j}{2r^2}, \]
\[ R^{2002} = - \frac{1}{2} \partial_{tr} \alpha. \]

It is clear that the most divergent component is \( R^{1001} \). The tidal distortion is finite since one can approximate the distortion by integrating the above components twice with respect to \( v \) [3,22] and obtain a finite result. Furthermore the Kretschmann scalar of the BTZ solution is given by

\[ R_{abcd} R^{abcd} = \frac{2}{r^2} \left[ \partial_{r} \alpha(v,r) \right]^2 + \left[ \partial_{tr} \alpha(v,r) \right]^2 \]

which is obviously bounded at Cauchy horizon. As a result, we see no reason to terminate the classical extension of the spacetime beyond the Cauchy horizon, as Ori has suggested [3,10].

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