SOME EXTENSIONS OF THE OPEN DOOR LEMMA

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Abstract. Miller and Mocanu proved in their 1997 paper a greatly useful result which is now known as the Open Door Lemma. It provides a sufficient condition for an analytic function on the unit disk to have positive real part. Kuroki and Owa modified the lemma when the initial point is non-real. In the present note, by extending their methods, we give a sufficient condition for an analytic function on the unit disk to take its values in a given sector.

1. Introduction

We denote by $\mathcal{H}$ the class of holomorphic functions on the unit disk $D = \{z : |z| < 1\}$ of the complex plane $\mathbb{C}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let $\mathcal{H}[a, n]$ denote the subclass of $\mathcal{H}$ consisting of functions $h$ of the form $h(z) = a + c_n z^n + c_{n+1} z^{n+1} + \cdots$. Here, $\mathbb{N} = \{1, 2, 3, \ldots\}$. Let also $\mathcal{A}_n$ be the set of functions $f$ of the form $f(z) = z h(z)$ for $h \in \mathcal{H}[1, n]$.

A function $f \in \mathcal{A}_1$ is called starlike (resp. convex) if $f$ is univalent on $D$ and if the image $f(D)$ is starlike with respect to the origin (resp. convex). It is well known (cf. [1]) that $f \in \mathcal{A}_1$ is starlike precisely if $q_f(z) = z f'(z)/f(z)$ has positive real part on $|z| < 1$, and that $f \in \mathcal{A}_1$ is convex precisely if $\varphi_f(z) = 1 + z f''(z)/f'(z)$ has positive real part on $|z| < 1$. Note that the following relation holds for those quantities:

$$\varphi_f(z) = q_f(z) + \frac{z q'_f(z)}{q_f(z)}.$$

It is geometrically obvious that a convex function is starlike. This, in turn, means the implication

$$\text{Re} \left[ q(z) + \frac{z q'(z)}{q(z)} \right] > 0 \text{ on } |z| < 1 \implies \text{Re} q(z) > 0 \text{ on } |z| < 1$$

for a function $q \in \mathcal{H}[1, 1]$. Interestingly, it looks highly nontrivial. Miller and Mocanu developed a theory (now called differential subordination) which enables us to deduce such a result systematically. See a monograph [1] written by them for details.

The set of functions $q \in \mathcal{H}[1, 1]$ with $\text{Re} q > 0$ is called the Carathéodory class and will be denoted by $\mathcal{P}$. It is well recognized that the function...
\( q_0(z) = (1 + z)/(1 - z) \) (or its rotation) maps the unit disk univalently onto the right half-plane and is extremal in many problems. One can observe that the function
\[
\varphi_0(z) = q_0(z) + \frac{z q_0'(z)}{q_0(z)} = 1 + z + \frac{2z}{1 - z^2} = 1 + 4z + z^2
\]
maps the unit disk onto the slit domain \( V(-\sqrt{3}, \sqrt{3}) \), where \( V(A, B) = \mathbb{C} \{ iy : y \leq A \text{ or } y \geq B \} \) for \( A, B \in \mathbb{R} \) with \( A < B \).

Note that \( V(A, B) \) contains the right half-plane and has the “window” \( (A_i, B_i) \) in the imaginary axis to the left half-plane. The Open Door Lemma of Miller and Mocanu asserts for a function \( q \in H[1, 1] \) that, if \( q(z) + z q'(z)/q(z) \in V(-\sqrt{3}, \sqrt{3}) \) for \( z \in \mathbb{D} \), then \( q \in \mathcal{P} \). Indeed, Miller and Mocanu [3] (see also [4]) proved it in a more general form. For a complex number \( c \) with \( \text{Re } c > 0 \) and \( n \in \mathbb{N} \), we consider the positive number
\[
C_n(c) = n \text{Re } c \left[ |c| \sqrt{2 \text{Re } c / n} + 1 + \text{Im } c \right].
\]

In particular, \( C_n(c) = \sqrt{n(n + 2c)} \) when \( c \) is real. The following is a version of the Open Door Lemma modified by Kuroki and Owa [2].

**Theorem A** (Open Door Lemma). *Let \( c \) be a complex number with positive real part and \( n \) be an integer with \( n \geq 1 \). Suppose that a function \( q \in H[c, n] \) satisfies the condition
\[
q(z) + \frac{z q'(z)}{q(z)} \in V(-C_n(c), C_n(\bar{c})), \quad z \in \mathbb{D}.
\]
Then \( \text{Re } q > 0 \) on \( \mathbb{D} \).

**Remark 1.1.** In the original statement of the Open Door Lemma in [3], the slit domain was erroneously described as \( V(-C_n(\bar{c}), C_n(c)) \). Since \( C_n(\bar{c}) < C_n(c) \) when \( \text{Im } c > 0 \), we see that \( V(-C_n(\bar{c}), C_n(\bar{c})) \subset V(-C_n(c), C_n(\bar{c})) \subset V(-C_n(c), C_n(c)) \) for \( \text{Im } c \geq 0 \) and the inclusions are strict if \( \text{Im } c > 0 \). As the proof will suggest us, seemingly the domain \( V(-C_n(c), C_n(\bar{c})) \) is maximal for the assertion, which means that the original statement in [3] and the form of the associated open door function are incorrect for a non-real \( c \). This, however, does not decrease so much the value of the original article [3] by Miller and Mocanu because the Open Door Lemma is mostly applied when \( c \) is real. We also note that the Open Door Lemma deals with the function \( p = 1/q \in H[1/c, n] \) instead of \( q \). The present form is adopted for convenience of our aim.

The Open Door Lemma gives a sufficient condition for \( q \in H[c, n] \) to have positive real part. We extend it so that \( |\arg q| < \pi \alpha/2 \) for a
given \(0 < \alpha \leq 1\). First we note that the Möbius transformation
\[
g_c(z) = \frac{c + \bar{c}z}{1 - z}
\]
maps \(\mathbb{D}\) onto the right half-plane in such a way that \(g_c(0) = c\), where \(c\) is a complex number with \(\text{Re} c > 0\). In particular, one can take an analytic branch of \(\log g_c\) so that \(|\text{Im} \log g_c| < \pi/2\). Therefore, the function \(q_0 = g_c^\alpha = \exp(\alpha \log g_c)\) maps \(\mathbb{D}\) univalently onto the sector \(|\text{arg} w| < \pi \alpha/2\) in such a way that \(q_0(0) = c^\alpha\). The present note is based mainly on the following result, which will be deduced from a more general result of Miller and Mocanu (see Section 2).

**Theorem 1.2.** Let \(c\) be a complex number with \(\text{Re} c > 0\) and \(\alpha\) be a real number with \(0 < \alpha \leq 1\). Then the function
\[
R_{\alpha,c,n}(z) = g_c(z)^\alpha + \frac{n\alpha z g_c'(z)}{g_c(z)} = \left(\frac{c + \bar{c}z}{1 - z}\right)^\alpha + \frac{2n\alpha \text{Re} (c) z}{(1 - z)(c + \bar{c}z)}
\]
is univalent on \(|z| < 1\). If a function \(q \in \mathcal{H}[c^\alpha, n]\) satisfies the condition
\[
q(z) + zq'(z) q(z) \in R_{\alpha,c,n}(\mathbb{D}), \quad z \in \mathbb{D},
\]
then \(|\text{arg} q| < \pi \alpha/2\) on \(\mathbb{D}\).

We remark that the special case when \(\alpha = 1\) reduces to Theorem 1 (see the paragraph right after Lemma 3.3 below. Also, the case when \(c = 1\) is already proved by Mocanu even under the weaker assumption that \(0 < \alpha \leq 2\) (see Remark 3.6). Since the shape of \(R_{\alpha,c,n}(\mathbb{D})\) is not very clear, we will deduce more concrete results as corollaries of Theorem 1.2 in Section 3. This is our principal aim in the present note.

## 2. Preliminaries

We first recall the notion of subordination. A function \(f \in \mathcal{H}\) is said to be subordinated to \(F \in \mathcal{H}\) if there exists a function \(\omega \in \mathcal{H}[0,1]\) such that \(|\omega| < 1\) on \(\mathbb{D}\) and that \(f = F \circ \omega\). We write \(f \prec F\) or \(f(z) \prec F(z)\) for subordination. When \(F\) is univalent, \(f \prec F\) precisely if \(f(0) = F(0)\) and if \(f(\mathbb{D}) \subset F(\mathbb{D})\).

Miller and Mocanu [3, Theorem 5] (see also [4, Theorem 3.2h]) proved the following general result, from which we will deduce Theorem 1.2 in the next section.

**Lemma 2.1** (Miller and Mocanu). Let \(\mu, \nu \in \mathbb{C}\) with \(\mu \neq 0\) and \(n\) be a positive integer. Let \(q_0 \in \mathcal{H}[c,1]\) be univalent and assume that \(\mu q_0(z) + \nu \neq 0\) for \(z \in \mathbb{D}\) and \(\text{Re}(\mu c + \nu) > 0\). Set \(Q(z) = zq_0'(z)/(\mu q_0(z) + \nu)\), and
\[
(2.1) \quad h(z) = q_0(z) + nQ(z) = q_0(z) + \frac{n\mu z q_0'(z)}{\mu q_0(z) + \nu}.
\]
Suppose further that
(a) \( \text{Re} \left[ \frac{zh'(z)/Q(z)}{h'(z)(\mu q_0(z) + \nu)/q_0(z)} \right] > 0 \), and
(b) either \( h \) is convex or \( Q \) is starlike.
If \( p \in H[c, n] \) satisfies the subordination relation
\[
(2.2) \quad q(z) + \frac{zq'(z)}{\mu q(z) + \nu} < h(z),
\]
then \( q < q_0 \), and \( q_0 \) is the best dominant. An extremal function is given by \( q(z) = q_0(z^n) \).

In the investigation of the generalized open door function \( R_{\alpha,c,n} \), we will need to study the positive solution to the equation
\[
(2.3) \quad x^2 + Ax^{1+\alpha} - 1 = 0,
\]
where \( A > 0 \) and \( 0 < \alpha \leq 1 \) are constants. Let \( F(x) = x^2 + Ax^{1+\alpha} - 1 \). Then \( F(x) \) is increasing in \( x > 0 \) and \( F(0) = -1 < 0 \), \( F(+\infty) = +\infty \). Therefore, there is a unique positive solution \( x = \xi(A, \alpha) \) to the equation. We have the following estimates for the solution.

**Lemma 2.2.** Let \( 0 < \alpha \leq 1 \) and \( A > 0 \). The positive solution \( x = \xi(A, \alpha) \) to equation (2.3) satisfies the inequalities
\[
(1 + A)^{-1/(1+\alpha)} \leq \xi(A, \alpha) \leq (1 + A)^{-1/2} \quad (< 1).
\]
Here, both inequalities are strict when \( 0 < \alpha < 1 \).

**Proof.** Set \( \xi = \xi(A, \alpha) \). Since the above \( F(x) \) is increasing in \( x > 0 \), the inequalities \( F(x_1) \leq 0 = F(\xi) \leq F(x_2) \) imply \( x_1 \leq \xi \leq x_2 \) for positive numbers \( x_1, x_2 \) and the inequalities are strict when \( x_1 < \xi < x_2 \). Keeping this in mind, we now show the assertion. First we put
\[
x_2 = (1 + A)^{-1/2}
\]
and observe
\[
F(x_2) = \frac{1}{1+A} + \frac{A}{(1+A)^{(1+\alpha)/2}} - 1 \geq \frac{1}{1+A} + \frac{A}{1+A} - 1 = 0,
\]
which implies the right-hand inequality in the assertion.
Next put \( x_1 = (1 + A)^{-1/(1+\alpha)} \). Then
\[
F(x_1) = \frac{1}{(1+A)^{2/(1+\alpha)}} + \frac{A}{1+A} - 1 \leq \frac{1}{1+A} + \frac{A}{1+A} - 1 = 0,
\]
which implies the left-hand inequality. We note also that \( F(x_1) < 0 < F(x_2) \) when \( \alpha < 1 \). The proof is now complete. \( \square \)

3. **Proof and corollaries**

Theorem 1.2 can be rephrased in the following.

**Theorem 3.1.** Let \( c \) be a complex number with \( \text{Re} \, c > 0 \) and \( \alpha \) be a real number with \( 0 < \alpha \leq 1 \). Then the function
\[
R_{\alpha,c,n}(z) = g_c(z)^\alpha + \frac{n\alpha z g_c'(z)}{g_c(z)}
\]
is univalent on $|z| < 1$. If a function $q \in \mathcal{H}[c^\alpha, n]$ satisfies the subordination condition
\[ q(z) + \frac{zq'(z)}{q(z)} < R_{\alpha,c,n}(z) \]
on $\mathbb{D}$, then $q(z) < g_c(z)^\alpha$ on $\mathbb{D}$. The function $g_c^\alpha$ is the best dominant.

**Proof.** We first show that the function $Q(z) = \alpha z g'_c(z)/g_c(z)$ is starlike. Indeed, we compute
\[
\frac{zQ'(z)}{Q(z)} = 1 - \frac{\overline{cz}}{c + \overline{cz}} + \frac{z}{1 - z} = \frac{1}{2} \left( \frac{c - \overline{cz}}{c + \overline{cz} + 1 + z} \right).
\]
Thus we can see that $\text{Re} \left[ \frac{zQ'(z)}{Q(z)} \right] > 0$ on $|z| < 1$. Next we check condition (a) in Lemma 2.1 for the functions $q_0 = g_c^\alpha, h = R_{\alpha,c,n}$ with the choice $\mu = 1, \nu = 0$. We have the expression
\[
\frac{zh'(z)}{Q(z)} = q_c(z)^\alpha + n \frac{zQ'(z)}{Q(z)}.
\]
Since both terms in the right-hand side have positive real part, we obtain (a). We now apply Lemma 2.1 to obtain the required assertion up to univalence of $h = R_{\alpha,c,n}$. In order to show the univalence, we have only to note that the condition (a) implies that $h$ is close-to-convex, since $Q$ is starlike. As is well known, a close-to-convex function is univalent (see [1]), the proof has been finished. □

We now investigate the shape of the image domain $R_{\alpha,c,n}(\mathbb{D})$ of the generalized open door function $R_{\alpha,c,n}$ given in Theorem 1.2. Let $z = e^{i\theta}$ and $c = re^{it}$ for $\theta \in \mathbb{R}, r > 0$ and $-\pi/2 < t < \pi/2$. Then we have
\[
R_{\alpha,c,n}(e^{i\theta}) = \left( \frac{re^{it} + re^{-it}e^{i\theta}}{1 - e^{i\theta}} \right)^\alpha + \frac{2\alpha e^{i\theta} \cos t}{(1 - e^{i\theta})(e^{it} + e^{-it}e^{i\theta})} \left( \frac{r \cos (t - \theta/2)}{\sin (\theta/2) i} \right)\alpha + i \frac{n \alpha}{2} \cos t \left( \frac{x}{x - 2 \sin t + 1} \right).
\]
Let $x = \cot (\theta/2) \cos t + \sin t$. When $x > 0$, we write $R_{\alpha,c,n}(e^{i\theta}) = u_+(x) + iv_+(x)$ and get the expressions
\[
\begin{cases}
  u_+(x) = a(rx)^\alpha, \\
v_+(x) = b(rx)^\alpha + \frac{n \alpha}{2 \cos t} \left( x - 2 \sin t + 1 \right),
\end{cases}
\]
where
\[
a = \cos \frac{\beta \pi}{2} \quad \text{and} \quad b = \sin \frac{\beta \pi}{2}.
\]
Taking the derivative, we get
\[
v'_+(x) = \frac{n \alpha}{2x^2 \cos t} \left[ x^2 + \frac{2br^\alpha \cos t}{n} x^{\alpha+1} - 1 \right].
\]
Hence, the minimum of $v_+(x)$ is attained at $x = \xi(A, \alpha)$, where $A = 2br^\alpha n^{-1} \cos t$. By using the relation (2.3), we obtain

$$\min_{0<x} v_+(x) = v_+(\xi) = \frac{n}{2\cos t} \left( A\xi^\alpha + \alpha\xi + \frac{\alpha}{\xi} \right) - n\alpha \tan t$$

where

$$U(x) = \frac{n}{2\cos t} \left( (\alpha - 1)x + \frac{\alpha + 1}{x} \right) - n\alpha \tan t.$$ 

Since the function $U(x)$ is decreasing in $0 < x < 1$, Lemma 2.2 yields the inequality

$$v_+(\xi) = U(\xi) \geq U((1 + A)^{-1/2})$$

where

$$U((1 + A)^{-1/2}) > U(1) = \frac{n\alpha(1 - \sin t)}{\cos t} > 0;$$

namely, $v_+(x) > 0$ for $x > 0$. When $x < 0$, letting $y = -x = -\cot(\theta/2)\cos t - \sin t$, we write $h(e^{i\theta}) = u_-(y) + iv_-(y)$. Then, with the same $a$ and $b$ as above,

$$\begin{cases}
  u_-(y) = a(ry)^\alpha, \\
  v_-(y) = -b(ry)^\alpha - \frac{n\alpha}{2\cos t} \left( y + 2\sin t + \frac{1}{y} \right),
\end{cases}$$

We observe here that $u_+ = u_- > 0$ and, in particular, we obtain the following.

**Lemma 3.2.** The left half-plane $\Omega_1 = \{w : \Re w < 0\}$ is contained in $R_{\alpha, c, n}(\mathbb{D})$.

We now look at $v_-(y)$. Since

$$v_-'(y) = -\frac{n\alpha}{2y^2\cos t} \left[ y^2 + \frac{2br^\alpha \cos t}{n} y^{\alpha+1} - 1 \right],$$

in the same way as above, we obtain

$$\max_{0<y} v_-(y) = v_-(\xi) = -\frac{n}{2\cos t} \left( (\alpha - 1)\xi + \frac{\alpha + 1}{\xi} \right) - n\alpha \tan t$$

where $\xi = \xi(A, \alpha)$ and $A = 2br^\alpha n^{-1} \cos t$. Note also that $v_-(y) < 0$ for $y > 0$.
Since the horizontal parallel strip \( v_-(\xi) < \text{Im} w < v_+(\xi) \) is contained in the image domain \( R_{\alpha,c,n}(\mathbb{D}) \) of the generalized open door function, we obtain the following.

**Lemma 3.3.** The parallel strip \( \Omega_2 \) described by

\[
|\text{Im} w + n\alpha \tan t| < \frac{n}{2\cos t} \left( \frac{\alpha - 1}{\sqrt{1 + A}} + (\alpha + 1)\sqrt{1 + A} \right)
\]

is contained in \( R_{\alpha,c,n}(\mathbb{D}) \). Here, \( t = \arg c \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) and \( A = \frac{2}{n}|c|^{\alpha} \sin \frac{n\alpha}{2} \cos t \).

When \( \alpha = 1 \), we have \( u_{\pm} = 0 \), that is, the boundary is contained in the imaginary axis. Since \( \xi(A, 1) = (1 + A)^{-1/2} \) by Lemma 2.2, the above computations tell us \( \min v_+ = (n/\cos t)(\sqrt{1 + A} - \sin t) = C_n(c) \).

Therefore, we have \( R_{1,c,n}(\mathbb{D}) = V(-C_n(c), C_n(c)) \). Note that the open door function then takes the following form

\[
R_{1,c,n}(z) = \frac{c + \bar{c}z}{1 - z} + \frac{2n(\text{Re} c)z}{(1 - z)(c + \bar{c}z)} = \frac{2\text{Re} c + n}{1 + cz/\bar{c}} - \frac{n}{1 - z} - \bar{c},
\]

which is the same as given by Kuroki and Owa [2, (2.2)]. In this way, we see that Theorem 1.2 contains Theorem 1 as a special case.

**Remark 3.4.** In [2], they proposed another open door function of the form

\[
R(z) = \frac{2n|c|}{\text{Re} c} \sqrt{\frac{2\text{Re} c}{n} + 1} \frac{(\zeta - z)(1 - \bar{\zeta}z)}{(1 - \zeta z)^2 - (\zeta - z)^2} - \frac{\text{Im} c}{\text{Re} c},
\]

where

\[
\zeta = 1 - \frac{2}{\omega}, \quad \omega = \frac{c}{|c|} \sqrt{\frac{2\text{Re} c}{n} + 1} + 1.
\]

It can be checked that \( R(z) = R_{1,c,n}(-\omega z/\bar{\omega}) \). Hence, \( R \) is just a rotation of \( R_{1,c,n} \).

We next study the argument of the boundary curve of \( R_{\alpha,c,n}(\mathbb{D}) \). We will assume that \( 0 < \alpha < 1 \) since we have nothing to do when \( \alpha = 1 \).

As we noted above, the boundary is contained in the right half-plane \( \text{Re} w > 0 \). When \( x > 0 \), we have

\[
v_+(x)/u_+(x) = \frac{b}{a} + \frac{n\alpha}{2\text{ar}^{\alpha}x^{\alpha}\cos t} \left[ x + \frac{1}{x} - 2\sin t \right].
\]

We observe now that \( v_+(x)/u_+(x) \to +\infty \) as \( x \to 0^+ \) or \( x \to +\infty \). We also have

\[
\left( \frac{v_+}{u_+} \right)'(x) = \frac{n\alpha}{2\text{ar}^{\alpha}x^{\alpha+2}\cos t} \left[ (1 - \alpha)x^2 + 2\alpha x \sin t - (1 + \alpha) \right].
\]
Therefore, \( v_+(x)/u_+(x) \) takes its minimum at \( x = \xi \), where

\[
\xi = \frac{-\alpha \sin t + \sqrt{1 - \alpha^2 \cos^2 t}}{1 - \alpha}
\]

is the positive root of the equation \((1 - \alpha)x^2 + 2\alpha x \sin t - (1 + \alpha) = 0\). It is easy to see that \(1 < \xi\) and that

\[
T_+ := \min_{0 < x} \frac{v_+(x)}{u_+(x)} = \frac{v_+(\xi)}{u_+(\xi)} = \frac{b}{a} + \frac{n\alpha}{2r^\alpha \xi^\alpha \cos t} \left[ \xi + \frac{1}{\xi} - 2 \sin t \right]
\]

When \( x = -y < 0 \), we have

\[
\frac{v_-(y)}{u_-(y)} = -\frac{b}{a} - \frac{n\alpha}{2r^\alpha y^{\alpha+2} \cos t} \left[ y + \frac{1}{y} + 2 \sin t \right]
\]

and

\[
\left(\frac{v_-}{u_-}\right)'(y) = \frac{-n\alpha}{2r^\alpha y^{\alpha+2} \cos t} \left[ (1 - \alpha)y^2 - 2\alpha y \sin t - (1 + \alpha) \right].
\]

Hence, \( v_-(y)/u_-(y) \) takes its maximum at \( y = \eta \), where

\[
\eta = \frac{\alpha \sin t + \sqrt{1 - \alpha^2 \cos^2 t}}{1 - \alpha}
\]

Note that

\[
T_- := \max_{0 < y} \frac{v_-(y)}{u_-(y)} = \frac{v_-(\eta)}{u_-(\eta)} = -\tan \frac{\pi \alpha}{2} - \frac{n(\eta - \eta^{-1})}{2r^\alpha \eta^{\alpha} \cos t}.
\]

Therefore, the sector \( \{ w : T_- < \arg w < T_+ \} \) is contained in the image \( h(\mathbb{D}) \). It is easy to check that \( T_- < -\tan(\pi \alpha/2) < \tan(\pi \alpha/2) < T_+ \). In particular \( T_- < \arg e^\alpha = \alpha t < T_+ \). We summarize the above observations, together with Theorem 1.2, in the following form.

**Corollary 3.5.** Let \( 0 < \alpha < 1 \) and \( c = re^{it} \) with \( r > 0, -\pi/2 < t < \pi/2 \), and \( n \) be a positive integer. If a function \( q \in \mathcal{H}[e^\alpha, n] \) satisfies the condition

\[
-\Theta_- < \arg \left( \frac{q(z) + zq'(z)}{q(z)} \right) < \Theta_+
\]

on \(|z| < 1\), then \(|\arg q| < \pi \alpha/2\) on \( \mathbb{D} \). Here,

\[
\Theta_\pm = \arctan \left[ \tan \frac{\pi \alpha}{2} + \frac{n(\xi_\pm - \xi_\pm^{-1})}{2r^\alpha \xi_\pm^{\alpha} \cos(\pi \alpha/2) \cos t} \right],
\]

and

\[
\xi_\pm = \frac{\mp \alpha \sin t + \sqrt{1 - \alpha^2 \cos^2 t}}{1 - \alpha}.
\]
It is a simple task to check that $x^{1-\alpha} - x^{-1-\alpha}$ is increasing in $0 < x$. When $\text{Im} \, c > 0$, we see that $\xi_- > \xi_+$ and thus $\Theta_- > \Theta_+$. It might be useful to note the estimates $\xi_- < \sqrt{(1 + \alpha)/(1 - \alpha)} < \xi_+$ and $\xi_- < 1/\sin t$ for $\text{Im} \, c > 0$.

**Remark 3.6.** When $c = 1$ and $n = 1$, we have $\xi := \xi_\pm = \sqrt{(1 + \alpha)/(1 - \alpha)}$, $\xi_- \xi^{-1} = 2\alpha/\sqrt{1 - \alpha^2}$, and thus

$$
\Theta_\pm = \arctan \left[ \tan \frac{\pi \alpha}{2} + \frac{\xi - \xi^{-1}}{2\xi^\alpha \cos \frac{\pi \alpha}{2}} \right] = \arctan \left[ \tan \frac{\pi \alpha}{2} + \frac{\alpha}{\cos \frac{\pi \alpha}{2} (1 - \alpha) \frac{1 - \alpha}{2} (1 + \alpha) \frac{1 + \alpha}{2}} \right] = \frac{\pi \alpha}{2} + \arctan \left[ \frac{\alpha \cos \frac{\pi \alpha}{2}}{(1 - \alpha) \frac{1 - \alpha}{2} (1 + \alpha) \frac{1 + \alpha}{2} + \alpha \sin \frac{\pi \alpha}{2}} \right].
$$

Therefore, the corollary gives a theorem proved by Mocanu [6].

Since the values $\Theta_+$ and $\Theta_-$ are not given in an explicitly way, it might be convenient to have a simpler sufficient condition for $|\arg q| < \pi \alpha/2$.

**Corollary 3.7.** Let $0 < \alpha \leq 1$ and $c$ with $\text{Re} \, c > 0$ and $n$ be a positive integer. If a function $q \in H[c^\alpha, n]$ satisfies the condition

$$
q(z) + \frac{zq'(z)}{q(z)} \in \Omega,
$$

then $|\arg q| < \pi \alpha/2$ on $\mathbb{D}$. Here, $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$, and $\Omega_1$ and $\Omega_2$ are given in Lemmas 3.2 and 3.3, respectively, and $\Omega_3 = \{w \in \mathbb{C} : |\arg w| < \pi \alpha/2\}$.

**Proof.** Lemmas 3.2 and 3.3 yield that $\Omega_1 \cup \Omega_2 \subset R_{\alpha,c,n}(\mathbb{D})$. Since $\Theta_\pm > \pi \alpha/2$, we also have $\Omega_3 \subset R_{\alpha,c,n}(\mathbb{D})$. Thus $\Omega \subset R_{\alpha,c,n}(\mathbb{D})$. Now the result follows from Theorem 1.2. □

See Figure 1 for the shape of the domain $\Omega$ together with $R_{\alpha,c,n}(\mathbb{D})$. We remark that $\Omega = R_{\alpha,c,n}(\mathbb{D})$ when $\alpha = 1$.

**References**

[1] Duren, P. L., *Univalent Functions*, Springer-Verlag, 1983.

[2] Kuroki, K. and Owa, S., *Notes on the open door lemma*, Rend. Semin. Mat. Univ. Padova, to appear.

[3] Miller, S. S. and Mocanu, P. T., *Briot-Bouquet differential equations and differential subordinations*, Complex Variables Theory Appl. 33 (1997), 217–237.

[4] ______, *Differential subordinations. Theory and applications*, Marcel Dekker, Inc., New York, 2000.

[5] Mocanu, P. T., *On strongly-starlike and strongly-convex functions*, Studia Univ. Babeş-Bolyai Math. 31 (1986), 16–21.
Figure 1. The image $R_{\alpha,c,n}(\mathbb{D})$ and $\Omega$ for $\alpha = 1/2, c = 4 + 3i, n = 2$. 

[6] ______, Alpha-convex integral operator and strongly starlike functions, Studia Univ. Babeş-Bolyai, Math. 34 (1989), 18–24.

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