User Point Processes in Cellular Networks

Martin Haenggi
Dept. of Electrical Engineering
University of Notre Dame

Abstract

The point process of concurrent users is critical for the analysis of cellular networks, in particular for the uplink and for full-duplex communication. We analyze the properties of two popular models. For the first one, we provide an accurate characterization of the pair correlation functions from the user and the base station point of view, which are applied to approximate the user process by Poisson and Ginibre point processes. For the second model, which includes the first model asymptotically, we study the cell vacancy probability, the mean area of vacant and occupied cells, the user-base station distance, and the pair correlation function in lightly and heavily loaded regimes.

I. INTRODUCTION

Motivation. Our goal is to analyze the point process of users who are served in the same time-frequency resource block (RB) in a cellular network. These users are those who interfere with each other in the uplink and when using full-duplex transmission or co-channel D2D communication.

Prior work. There are two popular point process models for the users, which we call models of type I and II. However, no results are available for their pair correlation function (pcf) \( g(r) \), which is critical for approximations by simpler, more tractable point processes. For the pcf between the typical base station (BS) and the users in a Poisson point process (PPP) of BSs of intensity \( \lambda \), the approximation \( g_{BS}(r) \approx 1 - e^{-\lambda \pi r^2} \) is suggested in [1, Remark 1]. However, this approximation is only accurate for lightly loaded networks, where a significant fraction of BSs are idle in the RB considered. We provide tight approximations for \( g(r) \) and \( g_{BS}(r) \) for heavily and lightly loaded networks. Accordingly, PPPs with intensity functions \( \lambda g(r) \) or \( \lambda g_{BS}(r) \), respectively, provide accurate approximations for the user point processes. We also propose the Ginibre point process as an alternative model.

Notation. Let \( \mathcal{N} \) denote the space of motion-invariant counting measures (point processes) on \( \mathbb{R}^2 \). For \( \mathcal{P} \in \mathcal{N} \), \( V_{\mathcal{P}}(x), x \in \mathcal{P} \), is the Voronoi cell of \( x \) and \( \partial V_{\mathcal{P}}(x) \) is its boundary. \( \mathcal{P}_o \triangleq (\mathcal{P} \setminus o) \), and \( \mathcal{P}_o^l \triangleq \mathcal{P}_o \setminus \{o\} \), where \( o = (0, 0) \). \( \mathbb{E}_o^t \) is the expectation w.r.t. the reduced Palm measure [2]. We let \( U(B), B \subset \mathbb{R}^2 \), be a point chosen uniformly at random from \( B \), independently for different \( B \) and independent from everything else. Also we denote by \( b(x, r) \) the disk of radius \( r \) centered at \( x \in \mathbb{R}^2 \) and define

\[
S(v, r) \triangleq r^2 \cos^{-1}(v/r) - v\sqrt{r^2 - v^2}, \quad 0 \leq v \leq r,
\]

which is the area of the disk segment \( b((-v, 0), r) \cap (\mathbb{R}^+ \times \mathbb{R}) \). As usual, \( |B| \) is the area or length of \( B \), and we define \( \|B\| \triangleq \min\{\|x\| : x \in B\} \). Lastly, we use \( \simeq \) to denote an approximation that becomes better asymptotically.

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II. USER POINT PROCESS OF TYPE I

This model is suitable for a fully loaded network, where each cell has an active user in a given RB. Its definition is based on a single point process only.

**Definition 1 (User point process of type I)** For $\mathcal{P} \in \mathcal{N}$, the user point process of type I is defined as

$$
\Phi \triangleq \{x \in \mathcal{P} : U(V_\mathcal{P}(x))\}.
$$

In this model, the user point process is obtained by a uniformly random displacement of each BS $x \in \mathcal{P}$ in its Voronoi cell. By construction, $\Phi \in \mathcal{N}$, with the same density as $\mathcal{P}$. We first explore $\Phi$ when $\mathcal{P}$ is a uniform PPP.

A. Pair correlation function (pcf)

We consider the usual $K$ function by Ripley [2, Def. 6.8], defined as $K(r) \triangleq \mathbb{E}_o \Phi(b(o, r))$. The pcf is $g(r) \triangleq \frac{1}{2\pi r} K'(r)$. We note that the model is scale-invariant, i.e., $g_\lambda(r) \equiv g_1(r\sqrt{\lambda})$, hence we can focus just on $g_1 = g$. For $x \in \Phi$, we let $p_x \in \mathcal{P}$ be the nucleus of the Voronoi cell that $x$ resides in, i.e., the serving BS of user $x$.

**Lemma 1** When $\mathcal{P}$ is a uniform PPP of intensity $\lambda$,

$$
g(r) = \Theta(r), \quad r \to 0.
$$

**Proof:** We consider $\Phi_o$. Let $D_V \triangleq \|\partial V(p_o)\|$ be the distance from $o$ to the nearest Voronoi cell boundary, and let $y$ be the user located in the adjacent cell across that boundary. The event that $o$ has a neighbor within distance $r$ is denoted by $E \triangleq \{\Phi_o(b(o, r) \cap V(p_y)) > 0\}$. We write $\mathbb{P}(E) = \mathbb{P}(E_0)\mathbb{P}(E \mid E_0)$, where $E_0 = \{D_V < r\} \supset E$.

Hence, for small $r$,

$$
K(r) \sim \mathbb{E}(\mathbb{P}(E_0 \mid \mathcal{P})\mathbb{P}(E \mid E_0, \mathcal{P})), \quad r \to 0
$$

$$
= \mathbb{E}\left(\frac{r\|\partial V(p_o)\|}{|V(p_o)|} S(D_V, r)\right),
$$

where $r\|\partial V(p_o)\|$ is the area of the part of the Voronoi cell that is within distance $r$ of its boundary. Similarly, $S(D_V, r)$ is the area of the part of the neighboring cell that $y$ needs to fall in. We have $\mathbb{P}(E_0 \mid \mathcal{P}) = \Theta(r)$ while $\mathbb{P}(E \mid E_0, \mathcal{P}) = \Theta(r^2)$ as the area of the disk segment grows with $r^2$. Since the expectation over $\mathcal{P}$ does not change the exponent of $r$, we have $K(r) = \Theta(r^3)$ and thus $g(r) = \Theta(r)$.

Fig. shows simulation results and approximations of $g(r)$. With some simplifying assumptions, we can give an estimate for the slope of $g(r)$ at $r = 0$, i.e., the coefficient $a$ in $g(r) \sim ar$, $r \to 0$. To estimate $a$, we assume that the Voronoi cells are square-shaped and of independent area $A = |V(p_o)|$ whose pdf follows the usual gamma approximation [3]

$$
f_A(x) = \frac{\gamma^\gamma}{\Gamma(\gamma)} x^{\gamma-1} e^{-\gamma x}, \quad \gamma \triangleq \frac{7}{2}, \quad (1)
$$

With the independence assumption, $K(r) \sim P_1 P_2$, where

$$
P_1 \simeq \mathbb{E}\left(\frac{r\|\partial V(p_o)\|}{|V(p_o)|}\right) = \mathbb{E}\frac{4r}{\sqrt{A}} = \frac{32\sqrt{14}r}{15\sqrt{\pi}} \approx 4.50r.
$$
Fig. 1. Simulated pair correlation function $g(r)$ for the type I model and analytical approximations. The prototype function is (2), $1 - e^{-3r}$ is the best fit for a simple exponential, and the fitted Ginibre pcf is (10).

For $P_2$, we note that $D_V$ given $E_0 = \{D_V < r\}$ is uniform in $[0, r)$ for small $r$. It follows that

$$P_2 \approx \mathbb{E}S(D_V, r)\mathbb{E}(A^{-1}) = \left(\frac{1}{r} \int_0^r S(v, r) dv\right) \mathbb{E}(A^{-1}) = \frac{2r^2}{3} \mathbb{E}(A^{-1}) = \frac{14}{15} r^2$$

since $\mathbb{E}(1/A) = 7/5$. Hence $P_1 P_2 \approx 4.20r^3$, and, since $3 \times 4.2(2\pi) \approx 2$, $g(r) \approx 2r$ for small $r$.

Since the simulated curve in Fig. 1 shows a maximum at $r = 1.2$ before descending to 1, a natural candidate for an approximating analytical function is

$$g_p(r) \triangleq 1 - e^{-ar} + br^2 e^{-cr^2}$$

for $a, b, c > 0$, henceforth referred to as the prototype function. $a$ denotes the slope at 0, as above. Fitting yields $a = 9/4$, $b = 1/2$, $c = 5/4$, hence a very good approximation is (see Fig. 1)

$$g(r) \approx 1 - e^{-(9/4)r} + (1/2)r^2 e^{-(5/4)r^2}. \quad (2)$$

This shows that the relatively simple estimate above for the slope is only 12% off.

B. Base station/user pair correlation function

Here we explore the point process of users as seen from a base station. Formally, the point process of interfering users at the typical BS, assumed at 0, is

$$\Phi_{BS} \triangleq \{x \in \mathcal{P}_o: U(V_{P_o}(x))\}.$$

Since $\Phi_{BS}$ is only isotropic but not stationary, we need to specify that the pcf of interest is the one with respect to the origin, hence we define $K_{BS}(r) \triangleq \mathbb{E}\Phi_{BS}(b(o, r))$ and, as before, $g_{BS}(r) \triangleq K_{BS}^\prime(r)/(2\pi r)$. Again we can focus on $\lambda = 1$ due to scale-invariance.
**Lemma 2** When $\mathcal{P}$ is a uniform PPP of intensity $\lambda$,

$$g_{BS}(r) = \Theta(r^2), \quad r \to 0.$$  

*Proof:* Let $R = \|P\|$, which is the distance of the BS at $o$ to its nearest neighbor $y$. If $R \geq 2r$, then the nearest interfering user is farther than $r$. If $R < 2r$, then the user in BS $y$’s cell has probability $\mathbb{E}(S(R/2,r)/A_y)$ to be within distance $r$ of $o$, where $A_y$ is the area of the cell of $y$. We have

$$\mathbb{E} S(R/2,r) = \int_0^{2r} S(u/2,r)2\pi u e^{-\pi u^2} du \sim \frac{1}{2}\pi^2 r^4, \quad r \to 0,$$

since $1 - e^{-u^2} \sim \pi u^2$, $u \to 0$. $A_y$ does not depend on $r$, so $K_{BS}(r) = \Theta(r^4), r \to 0$.  

As in the previous case, we can obtain an estimate for the pre-constant. When approximating $\mathbb{E}(A_y^{-1})$, we need to take into account that $A_y$ is not the area of the typical cell but that of a cell whose BS is very close to a boundary. Intuitively, the mean area of such a skewed cell is about half the mean area of the typical cell. Detailed simulations confirm that this is indeed the case. In fact, if $\rho$ is the distance to the nearest neighbor, $\mathbb{E}A(r) \approx 1/2 + \rho/3$, for small $\rho$, see Fig.2. Hence, assuming the shape of the cell is the same as that of the typical cell, we can use (1) with an adjusted mean and obtain $\mathbb{E}(A_y^{-1}) \approx 14/5$.

Combining the results for $\mathbb{E}S(R/2,r)$ and $\mathbb{E}(A_y^{-1})$, we obtain $K_{BS}(r) \approx \frac{1}{2}\pi^2 r^4$ and thus, for small $r$, $g_{BS}(r) \approx \frac{14}{5}\pi r^2 \approx 8.8r^2$. Hence we expect

$$g_{BS}(r) \approx 1 - e^{-14\pi r^2/5}$$  

(3)

to be near-exact for small $r$ and a decent approximation for all $r$. In view of Lemma 2 and the simulation result in Fig.3, the natural prototype function here is of the form

$$g_{p,BS}(r) \triangleq 1 - e^{-ar^2} + br^2 e^{-cr^2}, \quad a, b, c > 0.$$  

(4)
Fig. 3. Simulated base station/user pair correlation function $g_{BS}(r)$ for the type I model and approximations. The “analytical approximation” is (3), the prototype function is (4), and the best exponential is (5). The last curve, for comparison, is the approximation $1 - e^{-\pi r^2}$ from [1].

Fitting yields $a = 13/2$, $b = 2/7$, and $c = 13/9$. The best exponential fit, also obtained numerically, is

$$g_{BS}(r) \approx 1 - e^{-12\pi r^2/5}.$$  (5)

C. Mean interference

The mean interference at the typical user from other users is $E[I] = 2\pi \int_0^\infty g(r)r^{-\alpha}dr$ [4]. If $g(r) = \Theta(r)$, a necessary condition for the mean interference to be finite is $\alpha < 3$, since $\int_0^\epsilon r^{2-\alpha}dr$ for $\epsilon > 0$ is finite only if $\alpha < 3$. Hence Lemma 1 implies that the mean interference is finite only for $2 < \alpha < 3$. For the mean interference at a base station (uplink), $g_{BS}(r) = \Theta(r^2)$, $r \to 0$, thus the range of $\alpha$ that results in finite mean interference is $2 < \alpha < 4$. In this case, for $g_{BS}(r) = 1 - e^{-a\pi r^2}$, $E(I_{BS}) = \pi^{\alpha/2}a^{\alpha/2-1}\Gamma(1-\alpha/2)$. Therefore, for $3.6 < \alpha < 4$, if the approximation in [1] was used for heavily loaded networks, it would underestimate the mean interference by more than a factor 2.

D. Approximation with PPP

Here we consider the user point process $\Phi$ of general intensity $\lambda$. We would like to find a PPP $\Phi_{PPP}$ such that for all $f: \mathbb{R}^2 \to \mathbb{R}^+$,

$$E[I] = \lambda \int_{\mathbb{R}^2} f(x)g(\|x\|\sqrt{\lambda})dx.$$  (6)

Expanding both the left and right sides, using Campbell’s theorem on the right and its Palm version (where the reduced second factorial moment measure replaces the intensity measure [2]) on the left, the identity can be formulated as

$$\lambda \int_{\mathbb{R}^2} f(x)g(\|x\|\sqrt{\lambda})dx \equiv \int_{\mathbb{R}^2} f(x)\lambda_{PPP}(\|x\|)dx,$$
which is satisfied if we set the intensity function to
\[ \lambda_{PPP}(r) := \lambda g(r \sqrt{\lambda}). \]  
(7)

The same idea was put forth in [1, Assumption 1]. In principle any point process with intensity function \( \lambda_{PPP} \) could be used, but the PPP is a natural choice due its unparalleled tractability. Accordingly, for the user point process,
\[ \lambda_{PPP}(r) := \lambda \left( 1 - e^{-(9/4)r} + (1/2)\lambda r^2 e^{-(5/4)\lambda r^2} \right) \]

or, for a coarser but simpler approximation,
\[ \lambda_{PPP}(r) := \lambda \left( 1 - e^{-3\sqrt{\lambda}r} \right). \]  
(8)

From the perspective of the typical BS, we replace \( g \) by \( g_{BS} \) in (7) and set
\[ \lambda_{BS,PPP}(r) := \lambda \left( 1 - e^{-(13/2)\lambda r^2} + (2/7)\lambda r^2 e^{-(13/9)\lambda r^2} \right) \]

or
\[ \lambda_{BS,PPP}(r) := \lambda \left( 1 - e^{-(12/5)\pi \lambda r^2} \right). \]  
(9)

E. Approximation with Ginibre point process

The \( \beta \)-Ginibre point process\[^1\] has the pcf \( g_{gin}(r) = 1 - e^{-\pi r^2/\beta} \), which is strikingly similar to that of the interfering users at a base station. Hence the \( \beta \)-Ginibre process with \( \beta = 5/12 \), denoted by \( \Phi_{gin} \), has \( g_{gin}(r) \approx g_{BS}(r) \). Since the Ginibre process also models the repulsion between users, not just the repulsion between the BS and the users, it is a more accurate approximation than the PPP, albeit at reduced tractability. It also provides an efficient way of simulating \( \Phi_{BS} \) especially if only the distances matter, since the moduli of the points of the Ginibre process are equal in distribution to a set of independent gamma random variables [5, Sec. III.A].

If the Ginibre process was used to approximate the user process \( \Phi \) (well aware that the slope at 0 does not match), the best fit is obtained for
\[ g_{gin}(r) = 1 - e^{-\pi (12/5) r^2}, \]  
(10)

which, remarkably, is exactly the same approximation as for \( \Phi_{BS} \). As seen in Fig. 1, the gap to the true curve here is bigger, but the \( \beta \)-Ginibre process still is a decent substitute for the user point process of type I. The Ginibre approximation also shows that independent thinning does not affect the pcfs \( g \) and \( g_{BS} \).

F. Nearest-neighbor distance distribution

Let \( D = \| \Phi_0 \| \). From the resemblance to the Ginibre process and [5, Prop. 1], an educated guess for the nearest-neighbor distance distribution \( f_D \) is that \( D^2 \) follows a gamma distribution with parameters 2 and 1/5, since simulation results indicate \( \mathbb{E}(D^2) \approx 2/5 \). It follows that
\[ f_D(r) \approx 50r^3 e^{-5r^2}, \]  
(11)

which results in a mean \( \mathbb{E}(D) = 0.594 \), in perfect agreement with simulation results. The same approximation can be used also for the distribution of the nearest interfering user distance \( \| \Phi_{BS} \| \).

\[^1\]This process is obtained by independent thinning of a Ginibre process of intensity 1 with retention probability \( \beta \) and subsequent rescaling to preserve the intensity.
G. User-base station distance distribution

Since the typical user resides in the typical cell, the distance \( R = ||p_o|| \) to its serving BS is not the standard Rayleigh distribution with mean \( 1/2 \) (for \( \lambda = 1 \)), but a similarly-shaped distribution with (empirical) mean 0.439, corresponding to a correction factor 13/10 in the density. Hence a very good approximation for general \( \lambda \) is

\[
f_R(r) \approx 2(13/10)\pi r \lambda e^{-(13/10)\lambda \pi r^2}.
\]  

(12)

H. Users in lattice processes

Lemma 3 For the stationary square lattice \( \mathcal{P} = U([0, 1]^2) + \mathbb{Z}^2 \), the pcf of the user process \( \Phi \) satisfies

\[
g(r) \sim \frac{8}{2\pi} r, \quad r \to 0.
\]

Proof: Following the same procedure as for the Poisson case, we can determine the behavior of \( g(r) \) for small \( r \). In this case, \( K(r) \sim 4r \mathbb{E}(D_V, r) = \frac{8}{3} r^3, \quad r \to 0. \) Due to the concavity of the pcf, \( g(r) \lesssim \min\{8r/(2\pi), 1\} \) is a good upper bound, and \( g(r) \gtrsim 1 - e^{-8r/(2\pi)} \) is the corresponding lower bound. Both are asymptotically exact.

III. USER POINT PROCESS OF TYPE II

A. Definition

Definition 2 (User point process of type II) For independent \( \Phi_0, \mathcal{P} \in \mathcal{N} \), the user point process of type II is defined as

\[
\Phi \triangleq \{ x \in \mathcal{P}: U(V_\mathcal{P}(x) \cap \Phi_0) \}.
\]

Again \( \mathcal{P} \) models the BSs. Here the user in each cell \( V_\mathcal{P}(x) \) is chosen only from the countable set \( V_\mathcal{P}(x) \cap \Phi_0 \), where \( \Phi_0 \) can be viewed as the entire user population, while \( \Phi \) are those served in the RB under consideration.

B. Properties for the Poisson/Poisson case

The simplest model here is that \( \Phi_0 \) and \( \mathcal{P} \) are independent uniform PPPs of intensities \( \lambda_0 \) and \( \lambda_\mathcal{P} \). In this case, a fraction \( \nu \triangleq \mathbb{E}(e^{-\lambda_0 A}) \) of cells is vacant, where \( A \) is the area of the typical Voronoi cell of \( \mathcal{P} \), and the density of the user process \( \Phi \) is \( \lambda = \lambda_\mathcal{P}(1 - \nu) \). Using the approximation (11),

\[
\nu \approx (\gamma \lambda_\mathcal{P})^\gamma (\gamma \lambda_\mathcal{P} + \lambda_0)^{-\gamma} = (\gamma/(\gamma + \eta))^{\gamma},
\]

(13)

where \( \eta = \lambda_0/\lambda_\mathcal{P} \). We distinguish three regimes:

1) \( \eta \ll 1 \) (low user density): Here, \( \nu \approx 1 \), and most cells are empty, i.e., \( \lambda \approx \lambda_0 \ll \lambda_\mathcal{P} \). Those that are occupied are, on average, larger than the typical cell, i.e., they are Crofton cells (or 0-cells) [6], which have a mean area \( \mathbb{E}(A_0) = \mathbb{E}(A^2)/\mathbb{E}(A) \). For the PPP, \( \mathbb{E}(A_0) \approx (9/7) \lambda_\mathcal{P}^{-1} \). The link distance \( R \) follows the standard Rayleigh distribution \( f_R(r) = 2\lambda_\mathcal{P} \pi r e^{-\lambda_\mathcal{P} \pi r^2} \) with mean \( 1/(2\sqrt{\lambda_\mathcal{P}}) \), and the BS/user pcf \( g_{BS}(r) = 1 - e^{-\lambda_\mathcal{P} \pi r^2} \) from (1) becomes accurate since \( \Phi \approx \Phi_0 \) is essentially still a PPP.

2) \( \eta \gg 1 \) (high user density): In this regime, the model behaves as the model of type I (essentially no cells are empty).
Fig. 4. Properties of the user model of type II as a function of the density ratio $\eta$ (dB) for $\lambda_P = 1$. The left $y$ axis applies to the fraction of vacant cells given in (13) and the mean areas of occupied and vacant cells $\mathbb{E}(A_{occ})$ and $\mathbb{E}(A_{vac})$. The right axis applies to the mean distance $\mathbb{E}(R)$ between user and serving BS. Interestingly, $\mathbb{E}(A_{occ})/\mathbb{E}(R)^2 \approx 5$ for all values of $\eta$.

3) $\eta \approx 1$ (intermediate regime): For $\eta \in [1/2, 2]$, 60%-20% of the cells are empty, and the mean link distance falls in the regime $[0.46, 0.48]/\sqrt{\lambda_P}$. The mean area of the occupied cell is in the range $[1.1, 1.2]/\lambda_P$.

Fig. 4 summarizes and illustrates these properties. Note that for the user-BS distance, the right $y$ axis applies. In general, since the average cell size is $\lambda_P^{-1}$, the mean areas of vacant and occupied cells are related as $1/\lambda_P = \nu\mathbb{E}(A_{vac}) + (1 - \nu)\mathbb{E}(A_{occ})$.

IV. CONCLUSIONS

Even in cellular networks whose BSs are modeled as a PPP, active users that are served in the same RB do not form a PPP but a soft-core process. Indeed, in the case where BSs are heavily loaded, the user point process shows interesting connections to the Ginibre point process. This case is naturally represented using models of type I but can also be achieved with models of type II if the ratio parameter $\eta \gg 1$. Lightly loaded cellular networks can be modeled by applying independent thinning to the type I model, which does not affect the pcfs, or using the type II model whose pcfs depend on the density ratio $\eta$.

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