Non-Fermi Liquid Fixed Point in 2+1 Dimensions

Chetan Nayak

Department of Physics
Joseph Henry Laboratories
Princeton University
Princeton, N.J. 08544

Frank Wilczek

School of Natural Sciences
Institute for Advanced Study
Olden Lane
Princeton, N.J. 08540

* Research supported in part by a Fannie and John Hertz Foundation fellowship. nayak@puhep1.princeton.edu
† Research supported in part by DOE grant DE-FG02-90ER40542. WILCZEK@IASSNS.BITNET
ABSTRACT

We construct models of excitations about a Fermi surface that display calculable deviations from Fermi liquid behavior in the low-energy limit. They arise as a consequence of coupling to a Chern-Simons gauge field, whose fluctuations are controlled through a $\frac{1}{k^x}$ interaction. The Fermi liquid fixed point is shown to be unstable in the infrared for $x < 1$, and an infrared-stable fixed point is found in a $(1 - x)$-expansion, analogous to the $\epsilon$-expansion of critical phenomena. $x = 1$ corresponds to Coulomb interactions, and in this case we find a logarithmic approach to zero coupling. We describe the low-energy behavior of metals in the universality class of the new fixed point, and discuss its possible application to the compressible $\nu = \frac{1}{2}$ quantum Hall state and to the normal state of copper-oxide superconductors.
1. Introduction

Recent work has considerably clarified the logical foundations of Landau’s Fermi liquid theory. Fermi liquids are now well understood, using the language of the renormalization group, as interacting fermion systems whose infrared behavior is controlled by the free Fermi gas fixed point [1,2,3]. The quantitative success of Landau’s Fermi liquid theory is due to the fact that this renormalization group fixed point has (almost) no relevant perturbations so long as all interactions are local.

Thus generic fermion systems may be expected to belong to this universality class. Several known, striking exceptions occur when special kinematics emphasizes the infrared behavior of perturbations, thereby making them relevant. One such example is the Cooper pairing instability, which is central to the BCS theory of superconductivity. It is due to an interaction between electrons of momentum $k$ and $-k$, both of which lie on the Fermi surface for any time-reversal invariant system. This interaction is marginally relevant if attractive and marginally irrelevant if repulsive, and causes superconductivity in the former case. Charge- and spin-density waves are also caused by marginal interactions present for Fermi surfaces with special geometries (i.e. those which exhibit nesting). These are essentially the only known instabilities of the Fermi liquid at weak coupling, so that any new type of non-Fermi liquid behavior – i.e. any new fixed points – found in nature must be caused by the presence of additional fields and/or long-range interactions.

The classification and characterization of non-Fermi liquid metals (metals in the sense of conducting at $T = 0$) is interesting simply as a matter of principle, of course, but gains urgency from the need to understand the puzzling normal state properties of the copper-oxide superconductors and the behavior of the $\nu = \frac{1}{2}$ compressible Hall state, both of which appear to be “almost”– but definitely not – Fermi liquids.

The list of established non-Fermi liquid metals contains only two entries, not including the BCS superconductor mentioned above – both essentially one-
dimensional – namely the 1-D Luttinger liquid \([4,5,6]\) and the multi-channel Kondo effect \([7,8]\). The first of these is caused by the special kinematics which is the rule in one-dimensional systems. The interaction, which is marginal, causes the spin and charge degrees of freedom to be separated; the retarded Green function has a branch cut at the Fermi surface, rather than the pole of Fermi liquid theory. The second, the multi-channel Kondo effect for an over-screened magnetic impurity, is due to the presence of an additional field (albeit one that has only a single degree of freedom). The imaginary part of the zero-temperature electron self-energy approaches a constant value at low frequency, and does so as a non-trivial power of the frequency (hence, the spectral function is non-singular at the Fermi surface). As a result there is a non-vanishing residual resistance at zero temperature, which is approached as a power of the temperature; this behavior is to be contrasted with the usual Fermi liquid resistance which vanishes quadratically with temperature.

In the present work we add another universality class to this short list of non-Fermi liquid metals, one which is intrinsically two-dimensional and which may be a reasonable starting point for confronting the theoretical challenges posed by the aforementioned states of matter.

Our starting point is a system of fermions in 2 + 1 dimensions with a Fermi surface and low-lying excitations that interact both through a Chern-Simons gauge field and through a \(\frac{1}{k^2}\) long-range interaction. We analyze the low-energy behavior of this system, finding an infrared stable fixed point in a controlled approximation, based on the smallness of the parameter \((1 - x)\). This allows us to describe some basic physical properties of these non-Fermi liquid metals quantitatively.

Our construction of a non-Fermi liquid fixed point helps provide a firm basis for some previous work of Halperin, Lee, and Read \([9]\) and of Polchinski \([10]\). It can also be regarded as a concrete realization of the concept of a marginal Fermi liquid, which was discussed from a phenomenological standpoint by Varma, et al. \([11]\). Let us briefly recall some central results of these papers, which provided much of the motivation for the present work. Halperin, Lee, and Read considered the same
model that we do, with an eye to describing the $\nu = \frac{1}{2}$ Hall state. In the spirit of earlier work on the phase diagram for anyons in a background magnetic field [14] and the influential formulations of Jain [15] regarding the odd-denominator Hall states, they exploited the observation that attaching two flux tubes of a fictitious magnetic field to an electron – as may be accomplished most elegantly using a Chern-Simons gauge field construction – leaves it completely unchanged, since it can be effected with a (singular) gauge transformation. For electrons in a spatially constant magnetic field at half-filling there are two units of quantized flux per electron. Thus one can imagine attaching two units of fictitious magnetic field to each electron in such a way that the total net flux – and hence the average field experienced by each electron – vanishes. In this situation one might anticipate the possibility that the electrons behave to a first approximation as if there were no field at all – and in particular, exhibit a Fermi surface. It is not at all obvious or trivial, however, that it is valid to replace the fictitious magnetic field by its average: in fact this field fluctuates with the density. Halperin, Lee, and Read calculated the one-loop self-energy correction due to these gauge field fluctuations, allowing for the $\frac{1}{k^2}$ non-local interaction. They found that this one-loop self-energy exhibits non-Fermi liquid properties; specifically, the Green function has a branch cut at the Fermi surface, rather than a pole. (The fermion Green function is similar to that of the Luttinger liquid, but there is no spin-charge separation). However, their self-energy calculation was uncontrolled. The higher-order contributions are not small; indeed, as we shall see, it is crucial to include radiative corrections to the fermion-gauge field vertex in order to arrive at a consistent picture. Polchinski considered a related system, that of fermions with a Fermi surface interacting through a gauge field with a Maxwell action. (We shall see that this lies in the same universality class as the $x = 0$ case of Halperin, Lee, and Read’s model.) He was motivated by the possibility that this model might describe the dynamics of spinons in undoped copper-oxides. He solved the coupled fermion and gauge field self-energy equations self-consistently, assuming that vertex corrections could be neglected, and thereby obtained results similar to those of Halperin, Lee, and Read.
In order to justify his neglect he invoked a large \( n \) approximation, where \( n \) is the number of species of spinons, that we shall argue is not quite valid. Varma, \textit{et al.} simply assumed a certain form for the charge- and spin-polarizability fluctuations exchanged between electrons, and showed that they could lead to a reasonable semi-quantitative account of anomalous behaviors in the copper oxide superconductors. They did not discuss the possible origin of these fluctuations, which in our picture arise naturally as gauge fluctuations. The one-loop self-energy which results from the exchange of these fluctuations leads to a Green function which, again, has a branch cut rather than a pole at the Fermi surface.

In this paper, we find a non-trivial weak-coupling fixed point for small values of \((1-x)\). This allows us to justify lower order calculations such as those of Halperin, Lee, and Read and of Varma, \textit{et al.} in the regime \((1-x) \ll 1\), since the running coupling becomes small in the infrared. We cannot shed much quantitative light on Polchinski’s results, since he considers the case \((1-x) = 1\); however one might be encouraged to think that a non-trivial fixed point exists even in this case, as the weak coupling fixed point might well evolve into a strong coupling fixed point as \((1-x)\) increases, rather than disappearing.

2. Construction of the Fixed Point.

We begin with the following effective action:

\[
S = \int d\omega \, d^2k \left\{ \psi^\dagger \left( i\omega - \epsilon(k) \right) \psi \right\} + \int d\omega \, d^2k \, a_0 \epsilon_{ij} k_i a_j \\
+ g \int d\omega \, d\omega' \, d^2k \, d^2q \left\{ \psi^\dagger (k + q, \omega + \omega') \psi(k, \omega) \left( a_i (q, \omega') \frac{\partial}{\partial k_i} \epsilon(q + 2k) + a_0 (q, \omega') \right) \right\} \\
+ V_0 \int d\omega \, d\omega' \, d\omega'' \, d^2k \, d^2k' \, d^2k'' \psi^\dagger (k + k', \omega + \omega') \psi(k', \omega') \frac{1}{k^x} \psi^\dagger (-k + k'', -\omega + \omega'') \psi(k'', \omega'') 
\]

The first two terms are the free, kinetic terms that describe, respectively, fermionic excitations about a Fermi surface and a Chern-Simons gauge field whose origin is...
left unspecified at this time. This Chern-Simons field is not to be identified with the electromagnetic field. \( \epsilon(k) \), to be further specified later, is the single-particle energy. It is proportional to the Fermi velocity, \( v_F \). The next two terms are the fermion-gauge field interaction, with coupling constant \( g \), and the non-local four-fermion interaction. In a non-relativistic system, there is also a term of the form \( a_x a_x \psi^\dagger \psi \) in the fermion-gauge field interaction, but this term is unimportant according to arguments given in the appendix. We include, as well, a term in the action of the form \( a_0 a_0 \); such a term is allowed by the power counting arguments below and will arise anyway as a result of fermion loops.

The fixed point governing the low-energy behavior of this model will be found in a \((1 - x)\)-expansion, analogous to the \( \epsilon \)-expansion of critical phenomena. According to this analogy, Fermi liquid theory plays the role of mean field theory.

The \( a_0 \) equation of motion is a constraint,

\[
\epsilon_{ij} k_i a_j(k) = g \int d^2 q d\omega' \psi^\dagger(k + q, \omega + \omega') \psi(q, \omega) .
\]  

(2.2)

If we substitute this constraint back into the non-local four-fermion interaction, the important role of this interaction becomes clear. It takes the form

\[
S_a = \int d\omega d^2 k \epsilon_{ij} \epsilon_{mn} k_i k_m k^{-x} a_j(k, \omega) a_n(-k, -\omega) .
\]  

(2.3)

Thus as \( x \) is increased, long-range fluctuations of the gauge field are suppressed. This is quite natural since the non-local \( \frac{1}{k^x} \) interaction suppresses density fluctuations (as may be seen from a calculation of the compressibility), and the Chern-Simons term enslaves gauge field fluctuations to density fluctuations. Note that when \( x = 0 \) \( S_a \) is of the Maxwell form \( k^2 a_y a_y \) It will turn out that for \( x > 1 \) the gauge field fluctuations are so strongly suppressed that the fermion-gauge field interaction is irrelevant, and the theory is controlled in the infrared by the Fermi liquid fixed point. For \( x < 1 \) the interaction is relevant, so the stable fixed point is the new one we construct perturbatively in \((1 - x)\). Whereas all previous non-Fermi liquids owed their solubility to special kinematic constraints, this model is
soluble as a result of the enslavement of transverse gauge field fluctuations to density fluctuations, whose magnitude is directly controlled via $x$. This possibility is unique to two spatial dimensions, where transverse gauge fields have only a single component.

As Polchinski pointed out, the significant interaction between fermions arises for those which are near the same point on the Fermi surface. To analyze this, we consider a renormalization group transformation that scales the system towards a single point on the Fermi surface. Near a given point (which we will align on the $k_y$-axis for simplicity) on the Fermi surface the single-particle energy, $\epsilon(k)$, has the form:

\[ \epsilon(k) = v_F(k_y + ak_x^2) \]  \hspace{1cm} (2.4)

(see Figure 1). Here $a = 1/(2k_F)$. The correct scaling, which scales $\epsilon(k)$ with $\omega$ and leaves the free action invariant, is the following:

\[ k_x \to s^{1/2}k_x \]  \hspace{1cm} (2.5)

\[ k_y \to sk_y \]  \hspace{1cm} (2.6)

\[ \omega \to s\omega \]  \hspace{1cm} (2.7)

Under this scaling, the naive (tree-level) dimensions of the fields and coupling constants are ($a_x$ will be discussed in the appendix):

\[ [\psi] = -\frac{7}{4} \]  \hspace{1cm} (2.8)

\[ [a_y] = -\left(\frac{7 - x}{4}\right) \]  \hspace{1cm} (2.9)

\* We emphasize that there is nothing anisotropic here; this is simply the form of $\epsilon(k)$ expanded to lowest order in $k_x$ and $k_y$ about a point on the $k_y$-axis.
\[ [v_F] = 0 \]  
\[ [g] = -\left(\frac{1-x}{2}\right). \]  

Here we see, as claimed, that \( g \) is a relevant coupling for \((1-x)>0\), so \( g^* = 0 \) is no longer an infrared stable fixed point. We will find the new fixed point, \( g^* \), in an expansion in \((1-x)\). The calculation may be done with Wilsonian recursion relations which are closer in spirit to the effective field theory language. In this method, recursion relations are derived for the flow of the couplings under the elimination of high-energy modes followed by rescalings of the fields and coordinates. Instead we shall proceed within the physically equivalent field-theoretic method more convenient for higher-order calculations, by introducing renormalization functions \( Z, Z_{v_F}, Z_g \) and calculating the \( \beta \)-function. In either case, the three diagrams which must be calculated in order to obtain the fixed point and scaling exponents to lowest order in \((1-x)\) are the self-energy diagrams and the vertex correction shown in Figure 2.

To be more explicit, we take the action,

\[
S = \int d\omega \, d^2k \left\{ \psi^\dagger (iZ_\omega - Z_{v_F} \epsilon(k)) \psi \right\} + \int d\omega \, d^2k a_0 \epsilon_{ij} k_i a_j \\
+ V_0 \int d\omega \, d^2k \epsilon_{ij} \epsilon_{mn} k_i k_m k^{-x} a_j(k, \omega) a_n(-k, -\omega) \\
+ \mu \frac{1-x}{2} g Z_g \int d\omega \, d\omega' \, d^2k \, d^2q \left\{ \psi^\dagger(k + q, \omega + \omega') \psi(k, \omega) \left( a_i(q, \omega') \frac{\partial}{\partial k_i} \epsilon(q + 2k + a_0(q, \omega')) \right) \right\} 
\]  

(2.12)

where the renormalization functions \( Z, Z_{v_F}, \) and \( Z_g \) relate the bare and physical (i.e. low-energy) quantities,

\[
\psi_0 = Z^{1/2} \psi \]  
\[
v_F 0 = Z_{v_F} v_F \]  
\[
g_0 = \mu \frac{1-x}{2} g Z_g \frac{Z_g}{Z v_F} \].  

(2.13)

(2.14)

(2.15)

We will adopt a regularization scheme that is analogous to dimensional regulariza-
tion. That is, $Z$, $Z_{vF}$, and $Z_g$ are chosen to cancel the pole parts in $(1 - x)$ of the integrals corresponding to the diagrams of figure 2.

The integrals are elementary (see the Appendix for details). We obtain the renormalization group functions in terms of the expansion parameter $\alpha = \frac{g^2 v_F}{2\pi}$

$$\beta(\alpha) = -(1 - x)\alpha + 8\alpha^2 + O(\alpha^3)$$

$$\eta_{vF}(\alpha) = \beta(\alpha) \frac{\partial}{\partial \alpha} \ln Z_{vF} = 4\alpha + O(\alpha^2).$$

$Z Z_{vF} = 1$ since the one-loop self energy depends only on the frequency and not on the momentum. As a result, $\eta = -\eta_{vF}$. The gauge field does not receive any anomalous dimension at one loop because this diagram gives a contribution $\sim k^2$ which is subleading compared to $k^{2-x}$ in the infrared; the operator dimensionality of the gauge field is, as we noted earlier, controlled at tree level by varying $x$.

The fixed point (i.e. zero of the $\beta$-function) occurs at

$$\alpha^* = \frac{1}{8} (1 - x) + O((1 - x)^2)$$

and hence,

$$\eta_{vF}(\alpha^*) = \frac{1}{2} (1 - x) + O((1 - x)^2).$$

Note that for $n$ species of fermions (with an explicit factor of $n$ in front of the gauge-field kinetic term), the $\beta$-function is:

$$\beta(\alpha) = -(1 - x)\alpha + \frac{8}{n}\alpha^2 + O(\alpha^3)$$

and the fixed point is at

$$\alpha^* = \frac{n}{8} (1 - x) + O((1 - x)^2).$$

For large $n$ the fixed point occurs at large coupling; all terms in the $\beta$-function are $O(n)$, so it cannot be truncated at finite order.
At \((1 - x) = 0\) (Coulomb interaction!), the fermion-gauge field interaction is marginal and Fermi liquid theory is approached logarithmically. As \((1 - x)\) is increased from zero, the interaction becomes relevant – just as, in the case of critical phenomena, the \(\phi^4\) interaction is marginal in \(d = 4\) and relevant for \(\epsilon = 4 - d > 0\).

For \((1 - x)\) small, we can find a new stable fixed point at weak coupling. As we noted above, one would expect a new fixed point, even for \((1 - x)\) not very small. None of its properties are reliably given to low order in \((1 - x)\), but one might hope that a higher-order calculation combined with Borel summation techniques might prove successful, as it has in computations of critical exponents in three dimensions.

A basic property we wish to calculate at the new fixed point is the time rescaling which corresponds to a given spatial rescaling, or, simply, the anomalous dimension of the Fermi velocity. Many of the simple physical properties of the fixed point follow from the value of this anomalous dimension. In particular, the Green function acquires a branch cut when it is non-zero.

Important results may be obtained through study of the fermion 2-point function,

\[
G(\omega, r) = G(\omega, v_F(\mu)r, \alpha(\mu), \mu) \equiv \langle \psi(k, \omega) \psi(k, \omega) \rangle .
\]  
(2.22)

Here \(\mu\) is the energy scale, \(r = k_y + a k_z^2\) encodes the momentum dependence of \(\epsilon(k)\) near the point under consideration, and \(\alpha(\mu)\) is the coupling constant at the scale \(\mu\). Rescaling by \(\mu\),

\[
G(\omega, v_F(\mu)r, \alpha(\mu), \mu) = \mu^{-1 - \eta} G\left(\frac{\omega}{\mu}, \frac{v_F(1)r}{\mu^{1 - \eta}}, \alpha(\mu), 1\right) .
\]  
(2.23)

Taking \(\mu = \omega\),

\[
G(\omega, v_F r, \alpha(\omega), \omega) = \omega^{-1 - \eta} G\left(1, \frac{v_F r}{\omega^{1 - \eta}}, \alpha(\omega), 1\right) .
\]  
(2.24)

At low energy, \(\alpha(\omega) \to \alpha^*\), so

\[
G(\omega, v_F r) = \omega^{-1 + \eta} G\left(1, \frac{v_F r}{\omega^{1 - \eta}}, \alpha^*, 1\right) ,
\]  
(2.25)
where we have substituted $\eta_{v_F} = -\eta$ into this last equation. The phase of the self-energy may also be obtained from the one-loop diagram of Figure 2. When this calculation is done with the one-loop corrected gauge field propagator we obtain a phase $e^{-\frac{i\pi}{2}(\frac{1-x}{3-x})}$. Combining these observations, we finally arrive at the following form for the fermion Green function:

$$\langle \psi^\dagger(k,\omega)\psi(k,\omega) \rangle \sim \frac{1}{e^{-\frac{i\pi}{2}(\frac{1-x}{3-x})}(i\omega)^{1-\eta_{v_F}} - \epsilon(k)}.$$  \hspace{1cm} (2.26)

This form for the Green function implies the existence of a branch cut rather than a pole at the Fermi surface, so long as $\eta_{v_F} > 0$. Said differently, the quasiparticle weight vanishes,

$$Z \sim \lim_{\omega \to 0} \omega^{\eta_{v_F}} = 0.$$ \hspace{1cm} (2.27)

Here $Z$ can be defined alternatively as $\left(1 - \frac{\partial}{\partial \omega} Re \Sigma\right)^{-1}$, where $\Sigma$ is the self-energy, or simply as the wavefunction renormalization of the $\psi$ field at the Fermi surface.

For the marginal case $(1-x) = 0$ – which is, remarkably, the case of Coulomb interactions between fermions – there are only logarithmic corrections to the mean-field Fermi liquid behavior, just as arise for critical phenomena in four dimensions or for the ultraviolet behavior of QCD. The $\beta$-function equation may be integrated,

$$\beta(\alpha) = 4\alpha^2 \Rightarrow \alpha(\mu) \sim \frac{1}{8 \ln \mu}.$$ \hspace{1cm} (2.28)

As a result the Fermi velocity receives logarithmic scaling corrections:

$$G(\omega, v_{F\mu}) = \frac{1}{\omega \ln \omega^{1/2}} G\left(\frac{v_{F\mu}}{\omega \ln \omega^{1/2}}\right).$$ \hspace{1cm} (2.29)

Note that this one-loop renormalization group calculation sums the leading logarithms in all orders of perturbation theory.
3. Physical Properties at the New Fixed Point

With the scaling exponents of this theory – and the fermion Green function, in particular – in hand, we may extract the basic physical properties of these metals. In this analysis we follow closely the analysis of Varma, et al., who described the phenomenology resulting from the Green functions of the marginal Fermi liquid. As we shall mention further below a fully realistic description of the normal state of copper oxide superconductors may well involve a more complicated model featuring spin-charge separation, so this section is meant to be illustrative rather than definitive for that application.

The resistivity, $\rho$, is inversely proportional to the fermion mean free path, $ho \sim (v_F \tau)^{-1}$, as may be seen from the Kubo formula or from a simple Drude picture [6]. Since $\tau^{-1} = \text{Im} \Sigma \sim \omega^{1-\eta_{v_F}}$, or, at finite temperature, $\tau^{-1} \sim T^{1-\eta_{v_F}}$, and $v_F \sim T^{\eta_{v_F}}$, the temperature dependence of the resistivity is

$$\rho \sim T^{1-2\eta_{v_F}}.$$  (3.1)

This should be compared with the usual Fermi liquid form, $\rho \sim T^2$.

This analysis may be a little too quick. Strictly speaking, the resistance always vanishes in a translationally invariant system. A finite resistance is obtained only when the underlying lattice – umklapp processes in particular – is taken into account. Also, the transport lifetime, not simply the lifetime, must be used in the formula for the resistivity. The resistance is then given by

$$\rho \sim \frac{1}{v_F \tau} \left( \frac{q}{2k_F} \right)^2$$  (3.2)

$q$ is the transverse momentum exchanged in the relevant scattering processes. Typically there is a compensation of errors, and the naive formula gives the temperature dependence since $q$ is a reciprocal lattice vector rather than a temperature
dependent quantity in an umklapp process. In our model, scattering is strong for $q^2 \sim \omega^{1-\eta_{vF}}$, so one might be led to conclude, instead, that

$$\rho \sim T^{2-3\eta_{vF}}. \quad (3.3)$$

However, umklapp processes should be responsible for the scattering leading to electrical resistance, as in the Fermi liquid case, and we expect (3.1) to hold in practice. In future work, we will attempt to address this problem more rigorously.

The tunneling conductance between one of these metals and a conventional Fermi liquid metal is

$$g(V) = 4\pi e^2 |M|^2 N_2(0) \int d^2q A(k, -eV). \quad (3.4)$$

where $A(k, -eV)$ is the spectral function (i.e. the imaginary part of the Green function) of the non-Fermi liquid, $M$ is the tunneling matrix element (assumed to be approximately constant near the Fermi surface) and $N_2(0)$ is the slowly varying density of states of the conventional metal. Since $A(k, \omega) \sim \omega^{1-\eta_{vF}}$ we expect, quite generally,

$$g(V) \sim g_0 + |V|^{1-\eta_{vF}}. \quad (3.5)$$

The NMR relaxation rate, $T_1^{-1}$, is given by

$$T_1^{-1} \sim \lim_{\omega \to 0} \frac{T}{\omega} \int d^2q \text{Im} \chi(q, \omega), \quad (3.6)$$

where $\chi(q, \omega)$ is the spin susceptibility. The lowest order bubble diagram for $\chi(q, \omega)$ may be calculated using the full fermion propagator (2.26). The result is $T_1^{-1} \sim T$ just as in Fermi liquid theory, independent of $\eta_{vF}$. However, there are also vertex corrections which must be taken into account. These may be obtained at low
energy from the $\beta$-function. Linearizing the $\beta$-function near the fixed point and integrating, we obtain

$$\alpha(\omega) = \alpha^* \left(1 + \left(\frac{\alpha_0 - \alpha^*}{\alpha^*}\right) \left(\frac{\omega}{\mu_0}\right)^{8\alpha^*}\right).$$

(3.7)

Since the vertex goes as $\mu^{1-x} g v_F \sim \mu^{1-x} (\alpha v_F)^{1/2}$, $\chi \rightarrow \chi \omega^{(1-x)/2} (1 + a\omega^{8\alpha^*})$, or, for $\omega < T$, $\chi \rightarrow \chi T^{(1-x)/2} (1 + aT^{8\alpha^*})$. Therefore the Fermi liquid NMR relaxation rate is replaced by

$$T_1^{-1} \sim T^{(1+ \frac{1-x}{2} + \eta v_F/2)} (1 + aT^{8\alpha^*}).$$

(3.8)

At $x = 1$ the coupling is marginal and flows, instead, as in (2.28), so that

$$T_1^{-1} \sim T(\ln T)^{-3/4}.$$

(3.9)

A number of other properties of these metals may be obtained by standard methods. These, too, exhibit a characteristic pattern of deviations from Fermi liquid behavior.

4. Remarks on Applications

In this paper, we have explicitly constructed a non-Fermi liquid fixed point for fermions interacting with gauge fields in two spatial dimensions. This fixed point, which may with some justification be called a marginal Fermi liquid, implicitly underlies the work of Halperin, Lee, and Read and of Polchinski. These authors have attempted to obtain the behavior of the model by summing certain classes of diagrams; we have, instead, used the language and methods of the renormalization group. By showing that the physical, renormalized coupling – the running coupling – approaches an infrared stable fixed point value which is small when $x$ is near 1, we
have justified low-order perturbation theory in the renormalized coupling. Given this result, the striking semi-quantitative success of Halperin, Lee, and Read’s theory of the $\nu = \frac{1}{2}$ metallic Hall state becomes less surprising. For while the bare coupling in the model they use is $2\pi$, there is a fixed point for the low-energy behavior in the same universality class (specifically, the same model!) at weak coupling. If the theory at $g = 2\pi$ is in the basin of attraction of this fixed point, then use of low-order perturbation theory is approximately valid at low energy.

It is natural to consider the generalization of the $\nu = \frac{1}{2}$ state of fermions to a state of anyons in a magnetic field which cancels their Chern-Simons mean field, say half-fermions at filling fraction $\nu = 2$, or perhaps even bosons at filling fraction $\nu = 1$! These states should be in the universality class that we have described; again, although the bare coupling is large, the effective coupling at low energies is not. Experimental studies of the $\nu = \frac{1}{2}$ state – or one of these more exotic systems if they can be realized in the laboratory or numerically – should uncover the characteristic physical properties of this non-Fermi liquid universality class, including the logarithmic approach to free behavior.

Another possible application of this universality class is to the description of the normal state of the copper-oxide superconductors. Anderson and collaborators have argued that the correct theory for the copper-oxides is a theory of fermionic spinons, bosonic holons, and gauge fields, where the “confining” gauge fields are present to eliminate the redundancy in the spinon-holon description [12]. At half-filling (no doping), the holon spectrum has a large gap, so the low-energy theory is a theory of spinons and gauge fields, which presumably falls into the $x = 0$ case of the model considered here. Polchinski considered this theory and presented evidence for the existence of a strong-coupling fixed point, which is the $x \to 0$ extrapolation of our fixed point, as we discussed earlier. Unfortunately these ideas, taken at face value, lead to Bose condensation of the holons and to a strong coupling theory for the spinons, with phenomenological implications that are respectively problematic and difficult to assess.
It is interesting to speculate, as well, on the relevance of our fixed point at $x = 1$ to the doped copper-oxides. The Green functions at $x = 1$ are so similar to those of the marginal Fermi liquid of Varma et al. that provides a good phenomenological description of these substances, that one might be led to guess that copper-oxides are really systems of electrons, rather than just spinons, interacting with dynamically-generated gauge fields.

Things may not be quite so simple, however. As Anderson has argued forcefully, spin-charge separation may occur in these substances – i.e. spinons and holons might be independent excitations which propagate at different velocities in the low-energy theory – as it does in the one-dimensional Luttinger liquid [13]. Greiter, Wilczek, and Zou [16] have put forward a line of thought that ameliorates some of the difficulties of the earlier spinon-holon theories, and in which Chern-Simons fields play an important role. Their idea is that the quantum statistics of the electron separates at the same time as its charge and spin, so that both the spinons and the holons are half-fermions. They further suppose that it is valid to expand these species around Fermions, so that each is represented by a Fermion field interacting with a Chern-Simons field with $|g| = \frac{\pi}{2}$. If the mean fields of these anyons are cancelled – perhaps most plausibly by a non-zero vacuum expectation value of the “confining” gauge field mentioned above – then the spinons and holons are described at low-energies by the fixed point of this paper. It is plausible that the holons, being charged, have $x = 1$. The phenomenology of the charge carriers is therefore governed by the weak-coupling marginal Fermi liquid fixed point, as elaborated above. If for some reason the Coulomb interaction does not survive to low energies (where it would be dynamically screened), but is completely screened by high-energy processes, then the holons belong to the $x = 0$ universality class.

One might worry that the Chern-Simons term leads to $PT$-violating effects which are not supported by light scattering experiments. However, $PT$-violation should be more subtle to detect since the mean-field – which leads to $PT$ violation at tree-level – is cancelled. Said differently, the ground state possesses a circular Fermi surface, which is $PT$-invariant, so $PT$-violation can only appear in radiative corrections, which become small in the low-energy limit.
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APPENDIX

Calculation of the Renormalization Group Functions

The three diagrams to be calculated are shown in Figure 2. The gauge field self-energy diagram is not logarithmically divergent at $x = 1$, so there is no wave-function renormalization for the gauge field. Said differently, this diagram gives a contribution,

$$g^2 v_F^2 \int \frac{d\epsilon \, d^2 p}{(2\pi)^3} \frac{i}{i\omega + i\epsilon - \epsilon(k + p)} \frac{i}{i\epsilon - \epsilon(p)} \sim \alpha \left( q^2 - (\text{const.}) \frac{i |\omega|}{q} \right)$$  \hspace{1cm} (A.1)

which is subleading compared to $q^{2-x}$.

The fermion self-energy diagram is:

$$\frac{g^2 v_F^2}{(2\pi)^3} \int d\epsilon \, dq_x dq_y \frac{1}{q^{2-x}} \frac{1}{i\omega - i\epsilon - \epsilon(k - q)}.$$  \hspace{1cm} (A.2)

This is the contribution resulting from the exchange of transverse gauge bosons. There is only one of these in 2+1 dimensions, so in Coulomb gauge (for instance) one may solve for $a_x$ in terms of $a_y$, $a_x = -\frac{q_y}{q_x} a_y$. In the kinematic region of interest, $q_y \sim q_x^2/k_F$ – as enforced by the pole at this value in the $q_y$ integral – so the $a_x - a_x$ propagator is suppressed by a factor of $q_x^2/k_F^2$, and the $a_y - a_y$ propagator is all that needs to be considered. The contribution from the $a_0 - a_0$ and $a_0 - a_i$ propagators is subleading for a similar reason, namely factors of $\omega$ and $q$ in the numerator. The $dq_y$ integral may be done by contour integration since $q_y$ appears linearly in the denominator of the the fermion propagator. Then $\epsilon$ disappears from the integrand, and the $d\epsilon$ integral may be done, leaving

$$2\omega \alpha \int \frac{dq_x}{q_x^{2-x}} = 4\omega \alpha \left( \frac{1}{1 - x} \right) + \text{finite part}$$  \hspace{1cm} (A.3)

where the divergent part of the integral has been evaluated by taking the pole part in $(1 - x)$ in analogy with dimensional regularization. The $\omega$ integral is actually
not quite well defined, but if we use the one-loop corrected gauge-field propagator, this is remedied, with the same result for the divergent piece, (A.3). Since the self-energy contribution depends only on $\omega$, we may conclude that $ZZ_{vF} = 1$ and:

$$Z = Z_{vF}^{-1} = 1 + 4\alpha \left( \frac{1}{1-x} \right) + O(\alpha^2) . \quad (A.4)$$

Finally, we turn to the vertex correction:

$$\left(gvF\right)^2 \int \frac{d\epsilon \, d^2k}{(2\pi)^3} \frac{1}{i\omega_1 + i\epsilon - \epsilon(p_1 + k)} \frac{1}{i\omega_2 - i\epsilon - \epsilon(p_2 - k)} \frac{1}{k^2 - x} = \alpha \int \frac{dk_x}{k_x^2 - x}, \quad (A.5)$$

where the $dk_x$ and $d\epsilon$ integrals have been done as in the self-energy integral. Again, the renormalization counterterm is chosen to cancel the pole part in $1 - x$,

$$Z_g = 1 + 2\alpha \left( \frac{1}{1-x} \right) + O(\alpha^2) . \quad (A.6)$$

This yields a $\beta$-function,

$$\beta(\alpha) = -(1-x) \left( \frac{\partial}{\partial \alpha} \ln(\alpha Z_g^2 / Z_{vF}) \right)^{-1}$$

$$= -(1-x) \alpha + 8\alpha^2 + O(\alpha^3) \quad (A.7)$$

and hence

$$\eta_{vF}(\alpha) = \beta(\alpha) \frac{\partial}{\partial \alpha} \ln Z_{vF} = 4\alpha + O(\alpha^2) . \quad (A.8)$$
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Figure 1. Near a given point on the Fermi surface, the distance from the Fermi surface, and hence the energy, has the anisotropic form shown.
Figure 2. Gauge field self-energy, fermion self-energy, and the vertex correction.