Role of maximally entangled states in the context of linear steering inequalities

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Received: 16 May 2019 / Accepted: 14 August 2019 / Published online: 24 August 2019
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Abstract
Linear steering inequalities are useful to check whether a bipartite state is steerable when both the parties are allowed to perform $n$ dichotomic measurements on their parts. In the present study, we propose the necessary and sufficient condition under which two-setting linear steering inequality will be violated for any given set of spin-$\frac{1}{2}$ observables at trusted and untrusted parties’ sides. The important result revealed by the present paper is that maximally entangled two-qubit states give the largest quantum violations of two-setting as well as three-setting linear steering inequalities attainable for any given set of spin-$\frac{1}{2}$ observables at trusted and untrusted parties’ sides (if any violation exists for that given set of spin-$\frac{1}{2}$ observables).

Keywords EPR steering · Maximally entangled state

1 Introduction

In 1935 Einstein, Podolsky and Rosen (EPR) presented an argument showing the incompleteness of quantum mechanics [1]. However, Schrodinger did not believe in that incompleteness. Rather EPR argument surprised him by the fact that an observer can control/steer a system which is not in her possession. This motivated Schrodinger to conceive the celebrated concept of ‘steering’ [2,3]. The concept of steering in the form of a task has been introduced recently [4,5]. The task of quantum steering is
to prepare different ensembles at one part of a bipartite system by performing local quantum measurements on another part in such a way that these ensembles cannot be explained by a local hidden state (LHS) model, and no-signaling condition (the probability of obtaining one party’s outcome does not depend on spatially separated other party’s setting) is always satisfied by the bipartite system. This implies that the steerable correlations cannot be reproduced by a local hidden variable-local hidden state (LHV–LHS) model. In recent years, investigations related to quantum steering have been acquiring considerable significance, as evidenced by a wide range of studies [6–18].

It is well-known that EPR steering lies in between entanglement and Bell nonlocality: Bell nonlocal states form a strict subset of EPR steerable states which also form a strict subset of entangled states [4,19]. However, unlike Bell nonlocality [20] and entanglement [21], the task of quantum steering is inherently asymmetric with respect to the observers [22]. In this case, the outcome statistics of one subsystem (which is being ‘steered’) is due to valid quantum measurements on a valid quantum state. On the other hand, there is no such constraint for the other subsystem. There exist entangled states which are one-way steerable, i.e., demonstrate steerability from one observer to the other, but not vice versa [22,23]. The study of quantum steering also finds applications in semi-device-independent scenario where the party, which is being ‘steered,’ has trust on his/her quantum device but the other party’s device is untrusted. One big advantage in this direction is that such scenarios are experimentally less demanding than fully device-independent protocols (where both of the parties distrust their devices) and, at the same time, require less assumptions than standard quantum cryptographic scenarios. Secure quantum key distribution (QKD) using quantum steering has been demonstrated [24], where one party cannot trust his/her devices.

In [25], the authors have developed a series of ‘linear steering inequalities’ which are useful to check whether a bipartite state is steerable when both the parties are allowed to perform $n$ dichotomic measurements on his or her part. Apart from that several steering inequalities have been proposed [26–37] whose violations can render a correlation to be steerable.

In case of Bell nonlocality, for an arbitrary given two-qubit state, the maximum magnitude of the left-hand side of Bell-CHSH (Bell–Clauser–Horne–Shimony–Holt) inequality [38,39] contingent upon using projective measurements of spin-$\frac{1}{2}$ observables has been studied [40]. On the other hand, in the context of EPR steering, the maximum magnitude of the left-hand side of two-setting linear steering inequality [25] as well as that of EPR-steering analog of the CHSH inequality [33] for any given two-qubit state under projective measurements of spin-$\frac{1}{2}$ observables have also been investigated [41,42]. Furthermore, it has been shown that a given two-qubit state violates two-setting linear steering inequality if and only if the given state violates Bell-CHSH inequality and this is also true for EPR-steering analog of the CHSH inequality [34,41]. In all these studies, maximum magnitudes of the left-hand sides of two-setting linear steering inequality and Bell-CHSH inequality for a given two-qubit state have been analyzed by performing the maximization over all possible measurement settings. Motivated by the above results, we investigate the maximum magnitude (maximized over all possible bipartite quantum states) of the left-hand side of two-
setting linear steering inequality attainable for any given set spin-$\frac{1}{2}$ observables in the present study. Using this, we propose the necessary and sufficient condition under which two-setting linear steering inequality will be violated for any given set of spin-$\frac{1}{2}$ observables at trusted and untrusted parties’ sides. The maximum magnitude of the left-hand side of Bell-CHSH inequality attainable for any given set of spin-$\frac{1}{2}$ observables was studied by Kar et al. [43]. By comparing these results we show that a given set of spin-$\frac{1}{2}$ observables violates two-setting linear steering inequality if and only if that given set of spin-$\frac{1}{2}$ observables violates Bell-CHSH inequality.

It was argued that for any given set of spin-$\frac{1}{2}$ observables at the two spatially separated parties’ sides, the maximum attainable quantum violation of Bell-CHSH inequality (if there exists any quantum violation for that given set of observables) is achieved if the shared state is a pure maximally entangled state [43,44]. Since, EPR steering has a vast application in semi-device-independent scenario as already discussed, it is important to study which entangled state is the most effective resource for witnessing EPR steering contingent upon using a specific set of observables. Here lies the motivation of the second part of our study. There are several inequalities to witness EPR steering [25–37]. However, in the present study, we restrict ourselves to the linear steering inequality [25] as this inequality can be used to probe EPR steering with arbitrary number of dichotomic measurements on both sides. In particular, we address the following question: which quantum states achieve the largest quantum violations of the two-setting and three-setting linear steering inequalities attainable for a given set of spin-$\frac{1}{2}$ observables (if there exists any quantum violation for that given set of observables).

The plan of the paper is as follows. In Sect. 2, the basic notions of EPR steering and linear steering inequalities have been presented for the purpose of the present study. In Sect. 3, we present the necessary and sufficient condition under which two-setting linear steering inequality will be violated for any given set of spin-$\frac{1}{2}$ observables at trusted and untrusted parties’ sides. In Sect. 4, we illustrate which quantum states provide the maximum quantum violations of the two-setting and three-setting linear steering inequalities attainable for any given set of spin-$\frac{1}{2}$ observables (if there exists any quantum violation for that given set of observables). Finally, in the concluding Sect. 5, we elaborate a bit on the significance of the results obtained.

2 EPR steering and linear steering inequalities

Let us recapitulate the concept of EPR steering as introduced by Wiseman et. al. [4,5]. Let us consider that the joint state $\rho_{AB}$ of a pair of systems is shared between two spatially separated parties, say, Alice and Bob. Let $D_a$ and $D_\beta$ denote the sets of observables in the Hilbert space of Alice’s and Bob’s systems, respectively. An element of $D_a$ is denoted by $A$, with a set of outcomes labeled by $a \in \mathcal{L}(A)$, and similarly an element of $D_\beta$ is denoted by $B$, with a set of outcomes labeled by $b \in \mathcal{L}(B)$. The joint probability of obtaining the outcomes $a$ and $b$, when measurements $A$ and $B$ are performed locally by Alice and Bob on the joint state $\rho_{AB}$, respectively, is given by $P(a, b|A, B; \rho_{AB})$. The joint state $\rho_{AB}$ of the shared system is steerable by Alice to
Bob iff it is not the case that for all \(a \in \mathcal{L}(A), b \in \mathcal{L}(B), A \in \mathcal{D}_\alpha, B \in \mathcal{D}_\beta\), the joint probability distributions can be written in the form,

\[
P(a, b|A, B; \rho_{AB}) = \sum_{\lambda} p(\lambda) P(a|A, \lambda) P(b|B, \rho_{\lambda}),
\]

where \(p(\lambda)\) is the probability distribution over the hidden variables \(\lambda\), \(\sum_{\lambda} p(\lambda) = 1\); \(P(a|A, \lambda)\) denotes an arbitrary probability distribution and \(P(b|B, \rho_{\lambda})\) denotes the quantum probability of obtaining the outcome \(b\) when measurement \(B\) is performed on the quantum state (local hidden state) \(\rho_{\lambda}\). In other words, the joint state \(\rho_{AB}\) of the shared system will be called steerable if there is at least one measurement strategy for which the joint probability distribution does not satisfy a local hidden variable-local hidden state (LHV-LHS) model (1). One important point to be stressed here is that if for a given measurement strategy the joint probability distribution has a LHV-LHS model, this does not imply that the joint state of the shared system is not steerable, since there could be another strategy that does not.

In [25], authors have constructed the following series of steering inequalities to check whether a bipartite state is steerable from Alice to Bob when both the parties are allowed to perform \(n\) dichotomic measurements on his or her part:

\[
F_n = \frac{1}{\sqrt{n}} \left| \sum_{i=1}^{n} \langle A_i \otimes B_i \rangle \right| \leq 1.
\]

These are called \(n\)-setting linear steering inequalities. The linear steering inequalities with \(n = 2\) and \(n = 3\) (which are relevant for spin-\(\frac{1}{2}\) observables) are of the form:

\[
F_2 = \frac{1}{\sqrt{2}} \left| \sum_{i=1}^{2} \langle A_i \otimes B_i \rangle \right| \leq 1,
\]

and

\[
F_3 = \frac{1}{\sqrt{3}} \left| \sum_{i=1}^{3} \langle A_i \otimes B_i \rangle \right| \leq 1.
\]

Here, \(A_i = \hat{u}_i \cdot \sigma, B_i = \hat{v}_i \cdot \sigma, \sigma = (\sigma_1, \sigma_2, \sigma_3)\) is a vector composed of the Pauli matrices, \(\hat{u}_i \in \mathbb{R}^3\) are unit vectors, \(\hat{v}_i \in \mathbb{R}^3\) are orthonormal vectors, \(\mu_2 = \{\hat{u}_1, \hat{u}_2, \hat{v}_1, \hat{v}_2\}\) is the set of measurement directions corresponding to two-setting linear steering inequality (3), \(\mu_3 = \{\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{v}_1, \hat{v}_2, \hat{v}_3\}\) is the set of measurement directions corresponding to three-setting linear steering inequality (4), \(\langle A_i \otimes B_i \rangle = \text{Tr}(\rho A_i \otimes B_i)\), where \(\rho \in \mathcal{H}_A \otimes \mathcal{H}_B\) is some bipartite quantum state shared between two spatially separated parties (Alice and Bob). Quantum violation of any of the above inequalities implies that the shared state is steerable from Alice to Bob. Note that the trusted party’s (Bob) measurement directions are mutually orthogonal in case of two-setting and three-setting linear steering inequalities (3–4).


3 Spin-1/2 observables and two-setting linear steering inequality

In this section, we are going to investigate the necessary and sufficient condition for violating two-setting linear steering inequality for any given set of spin-1/2 observables. We start by proposing the following theorem.

**Theorem 1** For any given set of two spin-1/2 observables at Alice’s side (untrusted party) and any given set of two spin-1/2 observables in mutually orthogonal directions at Bob’s side (trusted party), there exists at least one bipartite state for which the two-setting linear steering inequality is violated iff Alice’s (untrusted party) spin-1/2 observables are non-commuting.

**Proof** The operator corresponding to two-setting linear steering inequality (3) can be written as,

\[ O_{F_2} = \frac{1}{\sqrt{2}} (A_1 \otimes B_1 + A_2 \otimes B_2), \]  

where \( A_i = \hat{u}_i \cdot \sigma, B_i = \hat{v}_i \cdot \sigma \) as discussed earlier. Hence, the square of the above steering operator is given by,

\[ O_{F_2}^2 = \frac{1}{2} (2\mathbb{I} \otimes \mathbb{I} + A_1 A_2 \otimes B_1 B_2 + A_2 A_1 \otimes B_2 B_1). \]

The above equation can be written in terms of the commutators and anti-commutators of the observables,

\[ O_{F_2}^2 = \mathbb{I} \otimes \mathbb{I} + \frac{1}{4} ([A_1, A_2] \otimes [B_1, B_2] + [A_1, A_2] \otimes [B_1, B_2]), \]

where \([A_1, A_2] = A_1 A_2 + A_2 A_1\) and \([A_1, A_2] = A_1 A_2 - A_2 A_1\). Other commutators and anti-commutators are similarly defined.

For any two unit vectors \( \hat{a} \) and \( \hat{b} \), the commutation relation \([\hat{a} \cdot \sigma, \hat{b} \cdot \sigma] \) is given by,

\[ [\hat{a} \cdot \sigma, \hat{b} \cdot \sigma] = 2i (\hat{n}_{ab} \cdot \sigma) \sin \theta_{ab}, \]

where \( \hat{n}_{ab} = \frac{\hat{a} \times \hat{b}}{|\hat{a} \times \hat{b}|} \), is an unit vector perpendicular to the plane containing \( \hat{a} \) and \( \hat{b} \); \( \theta_{ab} \) is the angle between the unit vectors \( \hat{a} \) and \( \hat{b} \). On the other hand, the anti-commutation relation \( [\hat{a} \cdot \sigma, \hat{b} \cdot \sigma] \) is given by,

\[ [\hat{a} \cdot \sigma, \hat{b} \cdot \sigma] = 2 \cos \theta_{ab} \mathbb{I}. \]

Using the above relations, Eq. (7) can be rewritten as,

\[ O_{F_2}^2 = \mathbb{I} \otimes \mathbb{I} + \cos \theta_{u_1u_2} \cos \theta_{v_1v_2} \mathbb{I} \otimes \mathbb{I} - \sin \theta_{u_1u_2} \sin \theta_{v_1v_2} (\hat{n}_{u_1u_2} \cdot \sigma) \otimes (\hat{n}_{v_1v_2} \cdot \sigma). \]
where $\theta_{u_1 u_2}, \theta_{v_1 v_2}, \hat{n}_{u_1 u_2}, \hat{n}_{v_1 v_2}$ are defined similarly.

Since the two spin-$\frac{1}{2}$ observables at trusted party’s side are defined to be in orthogonal direction [25,41], i.e., $\theta_{u_1 u_2} = \frac{\pi}{2}$, we have,

$$O_{F_2}^2 = \mathbb{I} \otimes \mathbb{I} - \sin \theta_{u_1 u_2} (\hat{n}_{u_1 u_2} \cdot \sigma) \otimes (\hat{n}_{v_1 v_2} \cdot \sigma). \quad (11)$$

Hence, the largest eigenvalue ($\lambda$) of $O_{F_2}^2$ is,

$$\lambda = 1 + |\sin \theta_{u_1 u_2}| \quad (12)$$

Corresponding to the eigenvalue (12), the largest eigenvalue ($\mu$) of the steering operator ($O_{F_2}$) is,

$$\mu = \sqrt{1 + |\sin \theta_{u_1 u_2}|} \quad (13)$$

From Eq. (13) it is evident that for any given set of two spin-$\frac{1}{2}$ observables at Alice’s side (untrusted party) and any given set of two spin-$\frac{1}{2}$ observables in mutually orthogonal directions at Bob’s side (trusted party), there exists at least one bipartite qubit state for which the two-setting linear steering inequality is violated iff the largest eigenvalue $\mu$ of the steering operator $O_{F_2}$ is greater than 1, i.e., $|\sin \theta_{u_1 u_2}| > 0$, i.e., $\sin \theta_{u_1 u_2} \neq 0$. Hence, we can conclude that for any given set of two spin-$\frac{1}{2}$ observables at Alice’s side (untrusted party) and any given set of two spin-$\frac{1}{2}$ observables in mutually orthogonal directions at Bob’s side (trusted party), there exists at least one bipartite qubit state for which the two-setting linear steering inequality is violated iff Alice’s (untrusted party) two spin-$\frac{1}{2}$ observables are non-commuting. □

**Corollary 1** The largest quantum mechanical violation of two-setting inequality is given by $\sqrt{2}$ as the maximum value of the largest eigenvalue $\mu$ of the steering operator ($O_{F_2}$) is $\sqrt{2}$ when Alice’s (untrusted party) spin-$\frac{1}{2}$ observables are in mutually orthogonal directions.

**Corollary 2** In Ref. [43], the maximum magnitude of the left-hand side of CHSH inequality was derived. For any given set of two spin-$\frac{1}{2}$ observables ($A_1, A_2, B_1, B_2$; $A_i = \hat{u}_i \cdot \sigma, B_i = \hat{v}_i \cdot \sigma$) at the spatially separated two party’s (Alice and Bob) side, CHSH inequality will be violated iff $|\sin \theta_{u_1 u_2} \sin \theta_{v_1 v_2}| > 0$ [43]. Hence, one can conclude that for any given set of two spin-$\frac{1}{2}$ observables at Alice’s side (untrusted party) and any given set of two spin-$\frac{1}{2}$ observables in mutually orthogonal directions ($\theta_{u_1 u_2} = \frac{\pi}{2}$) at Bob’s side (trusted party), the two-setting linear steering inequality will be violated iff CHSH inequality is violated with that given set of spin-$\frac{1}{2}$ observables at two spatially separated party’s side.
4 Spin-\(\frac{1}{2}\) observables and states which give maximum violations of \(n\)-setting linear steering inequalities

In this section, we are going to address the following question: which quantum states produce the maximum quantum violations of two-setting linear steering inequality (3) and three-setting linear steering inequality (4) attainable for any given set of spin-\(\frac{1}{2}\) observables. We start by proposing the following theorem:

**Theorem 2** If there exists any quantum violation of the two-setting linear steering inequality (3) for any two given spin-\(\frac{1}{2}\) observables at Alice’s side and any two given spin-\(\frac{1}{2}\) observables in mutually orthogonal directions at Bob’s side, then that quantum violation reaches the maximum value (attainable for that given set of observables) when the shared state is maximally entangled two-qubit state.

**Proof** For any set of given spin-\(\frac{1}{2}\) observables at Alice’s and Bob’s sides, the maximum attainable magnitude of the left-hand side of two-setting linear steering inequality given by (3) must be achieved by some pure state as this state is the eigenstate corresponding to the largest eigenvalue of the operator (5) associated with two-setting linear steering for the given set of spin-\(\frac{1}{2}\) observables. Furthermore, this eigenstate must be a two-qubit state as the operator (5) associated with two-setting linear steering for the given set of spin-\(\frac{1}{2}\) observables belongs to the Hilbert space in \(\mathbb{C}^2 \otimes \mathbb{C}^2\). Any pure two-qubit state can be written in the following form, called the Schmidt decomposition [45,46]:

\[
|\psi\rangle = \cos \alpha |00\rangle + \sin \alpha |11\rangle, \tag{14}
\]

where \(0 \leq \alpha \leq \frac{\pi}{2}\), \(|\{0\}, \{1\}\rangle\) is an orthonormal basis in the Hilbert space in \(\mathbb{C}^2\). Let the operators corresponding to the given two spin-\(\frac{1}{2}\) observables at Alice’s side are \(\hat{a}_i \cdot \sigma\) and \(\hat{a}_2 \cdot \sigma\) and that at Bob’s side are \(\hat{b}_1 \cdot \sigma\) and \(\hat{b}_2 \cdot \sigma\). Any spin-\(\frac{1}{2}\) observable can always be written as a linear combination of any three spin-\(\frac{1}{2}\) observables in mutually orthogonal directions in the following way:

\[
\hat{a}_i \cdot \sigma = \sin \theta_i \cos \phi_i (\hat{m}_1 \cdot \sigma) + \sin \theta_i \sin \phi_i (\hat{m}_2 \cdot \sigma) + \cos \theta_i (\hat{m}_3 \cdot \sigma), \tag{15}
\]

and

\[
\hat{b}_j \cdot \sigma = \sin \theta_j \cos \phi_j (\hat{m}_1 \cdot \sigma) + \sin \theta_j \sin \phi_j (\hat{m}_2 \cdot \sigma) + \cos \theta_j (\hat{m}_3 \cdot \sigma), \tag{16}
\]

where \((\hat{m}_1 \cdot \sigma), (\hat{m}_2 \cdot \sigma)\) and \((\hat{m}_3 \cdot \sigma)\) are three spin-\(\frac{1}{2}\) observables in mutually orthogonal directions. \(0 \leq \theta_i \leq \pi, 0 \leq \phi_i \leq 2\pi, 0 \leq \theta_j \leq \pi, 0 \leq \phi_j \leq 2\pi\). Let us construct the above three spin-\(\frac{1}{2}\) observables in mutually orthogonal directions in the following way,

\[
(\hat{m}_1 \cdot \sigma) = |0\rangle \langle 0| - |1\rangle \langle 1|, \tag{17}
\]

\[
(\hat{m}_2 \cdot \sigma) = |+\rangle \langle +| - |–\rangle \langle –|, \tag{18}
\]
and

\[(\hat{m}_3 \cdot \sigma) = | \uparrow \rangle \langle \uparrow | - | \downarrow \rangle \langle \downarrow |. \]  \quad (19)

Here, \{\ket{+}, \ket{-}\} is an orthonormal basis in the Hilbert space in \( \mathbb{C}^2 \) given by,

\[|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \]  \quad (20)

and

\[|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle). \]  \quad (21)

\{\ket{\uparrow}, \ket{\downarrow}\} is another orthonormal basis in the Hilbert space in \( \mathbb{C}^2 \) given by,

\[\ket{\uparrow} = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle), \]  \quad (22)

and

\[\ket{\downarrow} = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle). \]  \quad (23)

Now, for any two spin-\( \frac{1}{2} \) observables \( \hat{u} \cdot \sigma \) and \( \hat{v} \cdot \sigma \), the anti-commutation relation is given by,

\[\{\hat{u} \cdot \sigma, \hat{v} \cdot \sigma\} = 2\hat{u} \cdot \hat{v}\mathbb{I}. \]  \quad (24)

From above equation, it is clear that \(\{\hat{u} \cdot \sigma, \hat{v} \cdot \sigma\} = 0\) iff \( \hat{u} \) and \( \hat{v} \) are mutually orthogonal, i.e., \( \hat{u} \cdot \sigma \) and \( \hat{v} \cdot \sigma \) are two spin-\( \frac{1}{2} \) observables in mutually orthogonal direction. Now, from Eqs. (17), (18), (19), it can easily be checked that \( \{\hat{m}_1 \cdot \sigma, \hat{m}_2 \cdot \sigma\} = \{\hat{m}_1 \cdot \sigma, \hat{m}_3 \cdot \sigma\} = 0 \). Hence, \( \hat{m}_1 \cdot \sigma, \hat{m}_2 \cdot \sigma \) and \( \hat{m}_3 \cdot \sigma \) given by Eqs. (17), (18) and (19), respectively, are indeed three spin-\( \frac{1}{2} \) observables in mutually orthogonal directions.

Now consider that Alice and Bob share an arbitrary pure two-qubit state (14) as only a pure two-qubit state will achieve the maximum quantum violation of two-setting linear steering inequality (3) attainable for any given set of spin-\( \frac{1}{2} \) observables. With these, the left-hand side of the two-setting linear steering inequality given by (3) becomes,

\[
F_2 = \frac{1}{\sqrt{2}} \left[ \cos \theta_1^a \cos \theta_1^b + \cos \theta_2^a \cos \theta_2^b + \cos(\phi_1^a + \phi_1^b) \sin \theta_1^a \sin \theta_1^b 
+ \cos(\phi_2^a + \phi_2^b) \sin \theta_2^a \sin \theta_2^b \sin(2\alpha) \right].
\]  \quad (25)
Note that \( \sin(2\alpha) \geq 0 \) as \( 0 \leq \alpha \leq \frac{\pi}{4} \). There are the following possible cases, one of which will appear for any given set of observables:

**Case I:** \( \theta_i^a, \theta_j^b, \phi_i^a, \phi_j^b \) (i, j = 1, 2) are such that \( \cos \theta_i^a \cos \theta_j^b + \cos \theta_i^b \cos \theta_j^a \geq 0 \) and \( \cos(\phi_i^a + \phi_j^b) \sin \theta_i^a \sin \theta_j^b + \cos(\phi_i^b + \phi_j^a) \sin \theta_i^b \sin \theta_j^a \geq 0 \). In this case, from Eq. (25), it is clear that for any fixed values of \( \theta_i^a, \theta_j^b, \phi_i^a, \phi_j^b \) (i, j = 1, 2), the left-hand side of the two-setting linear steering inequality given by (3) will be maximized if \( \alpha = \frac{\pi}{4} \).

**Case II:** \( \theta_i^a, \phi_j^b, \phi_i^a, \phi_j^b \) (i, j = 1, 2) are such that \( \cos \theta_i^a \cos \theta_j^b + \cos \theta_i^b \cos \theta_j^a \leq 0 \) and \( \cos(\phi_i^a + \phi_j^b) \sin \theta_i^a \sin \theta_j^b + \cos(\phi_i^b + \phi_j^a) \sin \theta_i^b \sin \theta_j^a \leq 0 \). In this case also, for any fixed values of \( \theta_i^a, \phi_j^b, \phi_i^a, \phi_j^b \) (i, j = 1, 2), the left-hand side of the two-setting linear steering inequality given by (3) will be maximized if \( \alpha = \frac{\pi}{4} \).

**Case III-A:** \( \theta_i^a, \theta_j^b, \phi_i^a, \phi_j^b \) (i, j = 1, 2) are such that \( \cos \theta_i^a \cos \theta_j^b + \cos \theta_i^b \cos \theta_j^a \geq 0; \cos(\phi_i^a + \phi_j^b) \sin \theta_i^a \sin \theta_j^b + \cos(\phi_i^b + \phi_j^a) \sin \theta_i^b \sin \theta_j^a \leq 0 \) and \( \cos \theta_i^a \cos \theta_j^b + \cos \theta_i^b \cos \theta_j^a \geq 0; \cos(\phi_i^a + \phi_j^b) \sin \theta_i^a \sin \theta_j^b + \cos(\phi_i^b + \phi_j^a) \sin \theta_i^b \sin \theta_j^a \leq 0 \). In this case, from Eq. (25), we get \( F_2 = \frac{1}{\sqrt{2}} \left[ \cos \theta_i^a \cos \theta_j^b + \cos \theta_i^b \cos \theta_j^a \right] \). Hence, here for any fixed values of \( \theta_i^a, \theta_j^b, \phi_i^a, \phi_j^b \) (i, j = 1, 2), the left-hand side of the two-setting linear steering inequality given by (3) will be maximized if \( \alpha = 0 \). But \( \alpha = 0 \) implies that the shared state is separable. Therefore, this maximum magnitude of left-hand side of the two-setting linear steering inequality (3) does not lead to any quantum violation (i.e., this maximum magnitude is less than or equal to 1) as any separable state is unsteerable, and it cannot violate the two-setting linear steering inequality (3).

**Case III-B:** \( \theta_i^a, \theta_j^b, \phi_i^a, \phi_j^b \) (i, j = 1, 2) are such that \( \cos \theta_i^a \cos \theta_j^b + \cos \theta_i^b \cos \theta_j^a \geq 0; \cos(\phi_i^a + \phi_j^b) \sin \theta_i^a \sin \theta_j^b + \cos(\phi_i^b + \phi_j^a) \sin \theta_i^b \sin \theta_j^a \leq 0 \) and \( \cos \theta_i^a \cos \theta_j^b + \cos \theta_i^b \cos \theta_j^a \geq 0; \cos(\phi_i^a + \phi_j^b) \sin \theta_i^a \sin \theta_j^b + \cos(\phi_i^b + \phi_j^a) \sin \theta_i^b \sin \theta_j^a \leq 0 \). In this case, from Eq. (25), we get \( F_2 = \frac{1}{\sqrt{2}} \left[ \cos \theta_i^a \cos \theta_j^b + \cos \theta_i^b \cos \theta_j^a \right] \). Hence, here for any fixed values of \( \theta_i^a, \theta_j^b, \phi_i^a, \phi_j^b \) (i, j = 1, 2), the left-hand side of the two-setting linear steering inequality given by (3) will be maximized if \( \alpha = \frac{\pi}{4} \).

**Case IV-A:** \( \theta_i^a, \theta_j^b, \phi_i^a, \phi_j^b \) (i, j = 1, 2) are such that \( \cos \theta_i^a \cos \theta_j^b + \cos \theta_i^b \cos \theta_j^a \leq 0; \cos(\phi_i^a + \phi_j^b) \sin \theta_i^a \sin \theta_j^b + \cos(\phi_i^b + \phi_j^a) \sin \theta_i^b \sin \theta_j^a \leq 0 \) and \( \cos \theta_i^a \cos \theta_j^b + \cos \theta_i^b \cos \theta_j^a \geq 0; \cos(\phi_i^a + \phi_j^b) \sin \theta_i^a \sin \theta_j^b + \cos(\phi_i^b + \phi_j^a) \sin \theta_i^b \sin \theta_j^a \leq 0 \). Following the argument presented in Case III-A, we can state that for any fixed values of \( \theta_i^a, \theta_j^b, \phi_i^a, \phi_j^b \) (i, j = 1, 2), the left-hand side of the two-setting linear steering inequality...
given by (3) will be maximized if \( \alpha = 0 \), which does not correspond to any quantum violation of the two-setting linear steering inequality (3).

**Case IV-B:** \( \theta^a_i, \theta^b_i, \phi^a_i, \phi^b_i \) \((i, j = 1, 2)\) are such that \[ \cos(\theta^a_1 \cos \theta^b_1 + \cos \theta^a_2 \cos \theta^b_2) \leq 0; \cos(\phi^a_1 + \phi^b_1) \sin \theta^a_1 \sin \theta^b_1 + \cos(\phi^a_2 + \phi^b_2) \sin \theta^a_2 \sin \theta^b_2 \geq 0 \] and \[ \cos(\phi^a_3 + \phi^b_3) \sin \theta^a_3 \sin \theta^b_3 + \cos(\phi^a_4 + \phi^b_4) \sin \theta^a_4 \sin \theta^b_4 \leq \cos(\phi^a_5 + \phi^b_5) \sin \theta^a_5 \sin \theta^b_5 \]. In this case, we follow the argument presented in Case III-B. Here for any fixed values of \( \theta^a_i, \theta^b_i, \phi^a_j, \phi^b_j \) \((i, j = 1, 2)\), the left-hand side of the two-setting linear steering inequality given by (3) will be maximized if \( \alpha = \frac{\pi}{4} \).

Note that in Eq. (25), we have not assumed that the two spin-\( \frac{1}{2} \) observables at Bob’s side are in mutually orthogonal directions. Hence, the above result holds even if we assume that \( \hat{b}_1 \cdot \hat{b}_2 = 0 \). \( \square \)

Now we are going to address the aforementioned question in the context of three-setting linear steering inequality (4). In this case, we propose the following theorem:

**Theorem 3** If there exists any quantum violation of the three-setting linear steering inequality (4) for any three given spin-\( \frac{1}{2} \) observables at Alice’s side and any three given spin-\( \frac{1}{2} \) observables in mutually orthogonal directions at Bob’s side, then that quantum violation reaches the maximum value (attainable for that given set of observables) when the shared state is maximally entangled two-qubit state.

**Proof** Following the similar argument presented in the proof of Theorem 2, we can state that the largest magnitude of the left-hand side of three-setting linear steering inequality (4) attainable for any given set of spin-\( \frac{1}{2} \) observables on the spatially separated two parties’ sides must be achieved by some pure two-qubit state. Let us consider that Alice and Bob share an arbitrary pure two-qubit state (14). The operators corresponding to the given three spin-\( \frac{1}{2} \) observables at Alice’s side are \( \hat{a}_i \cdot \sigma \), \( \hat{a}_j \cdot \sigma \) and \( \hat{a}_k \cdot \sigma \), where \( \hat{a}_i \cdot \sigma \) \((i = 1, 2, 3)\) is given by Eq. (15). On the other hand, the operators corresponding to the given three spin-\( \frac{1}{2} \) observables at Bob’s side are \( \hat{b}_1 \cdot \sigma \), \( \hat{b}_2 \cdot \sigma \) and \( \hat{b}_3 \cdot \sigma \), where \( \hat{b}_j \cdot \sigma \) \((j = 1, 2, 3)\) is given by Eq. (16). With these the left-hand side of the three-setting linear steering inequality given by (4) becomes,

\[
F_3 = \frac{1}{\sqrt{3}} \left| \cos \theta^a_1 \cos \theta^b_1 + \cos \theta^a_2 \cos \theta^b_2 + \cos \theta^a_3 \cos \theta^b_3 \right| \\
+ \left[ \cos(\phi^a_1 + \phi^b_1) \sin \theta^a_1 \sin \theta^b_1 + \cos(\phi^a_2 + \phi^b_2) \sin \theta^a_2 \sin \theta^b_2 \right. \\
+ \left. \cos(\phi^a_3 + \phi^b_3) \sin \theta^a_3 \sin \theta^b_3 \right] \sin(2\alpha) \right|.
\] (26)
In this case, also the following cases appear:

**Case I:** \( \theta_i^a, \theta_j^b, \phi_i^a, \phi_j^b \) (\( i, j = 1, 2, 3 \)) are such that \( \cos \theta_i^a \cos \theta_j^b + \cos \theta_i^b \cos \theta_j^a \), \( \cos \theta_i^a \cos \theta_j^b + \cos \theta_i^b \cos \theta_j^a \) \( \geq 0 \) and \( \cos \phi_i^a + \cos \phi_i^b \sin \theta_i^a \sin \theta_j^b + \cos \phi_i^b + \cos \phi_i^a \sin \theta_i^a \sin \theta_j^b \). If and only if linear inequality \( \cos \theta_i^a \cos \theta_j^b + \cos \theta_i^b \cos \theta_j^a \) \( \leq 0 \) and \( \cos \phi_i^a + \cos \phi_i^b \sin \theta_i^a \sin \theta_j^b + \cos \phi_i^b + \cos \phi_i^a \sin \theta_i^a \sin \theta_j^b \). If and only if linear inequality \( \cos \theta_i^a \cos \theta_j^b + \cos \theta_i^b \cos \theta_j^a \) \( \geq 0 \).

**Case II:** \( \theta_i^a, \theta_j^b, \phi_i^a, \phi_j^b \) (\( i, j = 1, 2, 3 \)) are such that \( \cos \theta_i^a \cos \theta_j^b + \cos \theta_i^b \cos \theta_j^a \), \( \cos \theta_i^a \cos \theta_j^b + \cos \theta_i^b \cos \theta_j^a \) \( \leq 0 \) and \( \cos \phi_i^a + \cos \phi_i^b \sin \theta_i^a \sin \theta_j^b + \cos \phi_i^b + \cos \phi_i^a \sin \theta_i^a \sin \theta_j^b \). If and only if linear inequality \( \cos \theta_i^a \cos \theta_j^b + \cos \theta_i^b \cos \theta_j^a \) \( \geq 0 \).

**Case III-A:** \( \theta_i^a, \theta_j^b, \phi_i^a, \phi_j^b \) (\( i, j = 1, 2, 3 \)) are such that \( \cos \theta_i^a \cos \theta_j^b + \cos \theta_i^b \cos \theta_j^a \) \( \geq 0 \), \( \cos \phi_i^a + \cos \phi_i^b \sin \theta_i^a \sin \theta_j^b + \cos \phi_i^b + \cos \phi_i^a \sin \theta_i^a \sin \theta_j^b \) \( \leq 0 \) and \( \cos \theta_i^a \cos \theta_j^b + \cos \theta_i^b \cos \theta_j^a \) \( \geq 0 \).

**Case III-B:** \( \theta_i^a, \theta_j^b, \phi_i^a, \phi_j^b \) (\( i, j = 1, 2, 3 \)) are such that \( \cos \theta_i^a \cos \theta_j^b + \cos \theta_i^b \cos \theta_j^a \) \( \geq 0 \), \( \cos \phi_i^a + \cos \phi_i^b \sin \theta_i^a \sin \theta_j^b + \cos \phi_i^b + \cos \phi_i^a \sin \theta_i^a \sin \theta_j^b \) \( \leq 0 \) and \( \cos \theta_i^a \cos \theta_j^b + \cos \theta_i^b \cos \theta_j^a \) \( \geq 0 \).

**Case IV-A:** \( \theta_i^a, \theta_j^b, \phi_i^a, \phi_j^b \) (\( i, j = 1, 2, 3 \)) are such that \( \cos \theta_i^a \cos \theta_j^b + \cos \theta_i^b \cos \theta_j^a \) \( \geq 0 \), \( \cos \phi_i^a + \cos \phi_i^b \sin \theta_i^a \sin \theta_j^b + \cos \phi_i^b + \cos \phi_i^a \sin \theta_i^a \sin \theta_j^b \) \( \leq 0 \) and \( \cos \theta_i^a \cos \theta_j^b + \cos \theta_i^b \cos \theta_j^a \) \( \geq 0 \).

**Case IV-B:** \( \theta_i^a, \theta_j^b, \phi_i^a, \phi_j^b \) (\( i, j = 1, 2, 3 \)) are such that \( \cos \theta_i^a \cos \theta_j^b + \cos \theta_i^b \cos \theta_j^a \) \( \leq 0 \), \( \cos \phi_i^a + \cos \phi_i^b \sin \theta_i^a \sin \theta_j^b + \cos \phi_i^b + \cos \phi_i^a \sin \theta_i^a \sin \theta_j^b \) \( \geq 0 \) and \( \cos \theta_i^a \cos \theta_j^b + \cos \theta_i^b \cos \theta_j^a \) \( \leq 0 \).

In all these cases, we follow the same argument presented in the proof of Theorem 2. For any given set of spin-\( \frac{1}{2} \) observables, we obtain the following: either (i) maximum quantum violation of three-setting linear steering inequality (4) is achieved when the shared two-qubit state is maximally entangled, or (ii) there does not exist any violation of the three-setting linear steering inequality (4). Note that in Eq. (26), we have not assumed that the three spin-\( \frac{1}{2} \) observables at Bob’s side are in mutually orthogonal directions. Hence, the above result holds even if we assume that \( \hat{b}_1 \cdot \hat{b}_2 = 0; \hat{b}_2 \cdot \hat{b}_3 = 0; \hat{b}_1 \cdot \hat{b}_3 = 0 \). \( \square \)

### 5 Conclusion

In the present study, we have calculated the maximum magnitude (maximum over all possible two-qubit states) of the left-hand side of two-setting linear steering inequality attainable for any given set of spin-\( \frac{1}{2} \) observables on the spatially separated two parties’ sides. It has also been shown that a given set of spin-\( \frac{1}{2} \) observables violates two-setting linear inequality if and only if the given set of spin-\( \frac{1}{2} \) observables violates Bell-CHSH.
inequality. Note that it was earlier shown that any given two-qubit state violates two-setting linear steering inequality \(\text{if and only if}\) that state violates Bell-CHSH inequality [41]. Hence, the result presented in this study complements the result obtained in the previous studies [34,41].

There are several inequalities which are useful for showing EPR steering [25–37]. Since the entangled states that demonstrate EPR steering are proved to be useful resources for various semi-device-independent quantum informational tasks, it is important to investigate which state maximally violates a steering inequality for any given set of observables on the spatially separated two parties’ sides. In the present study, restricting ourselves to the linear steering inequalities [25] and spin-\(\frac{1}{2}\) observables, we have addressed the above issue. In particular, we have shown that the pure maximally entangled two-qubit states give the largest attainable quantum violations of two-setting and three-setting linear steering inequalities among all possible quantum states for any given set of spin-\(\frac{1}{2}\) observables (if there exists any violation for that given set of spin-\(\frac{1}{2}\) observables).

It is well-known that Bell nonlocality and EPR steering are operationally inequivalent. However, for a given two-qubit state, these two notions are equivalent in \(2 - 2 - 2\) experimental scenario (involving two parties, two measurement settings per party, two outcomes per setting) in the sense that a given two-qubit state demonstrates steering in this scenario \(\text{if and only if}\) the given two-qubit state shows Bell nonlocality in this scenario [34,41]. Motivated by these facts, we have investigated in the present study whether there exists any non-equivalence between Bell nonlocality and EPR steering when the set of spin-\(\frac{1}{2}\) observables on the spatially separated two parties’ sides is fixed in the above scenario. However, the present results demonstrate the equivalence between Bell nonlocality and EPR steering in the above context.

Addressing the above questions in the context of other steering criterion is worth for future research. We have seen that maximally entangled states give largest violation of linear steering inequality attainable for any given set of spin-\(\frac{1}{2}\) observables at trusted and untrusted parties’ sides (if there exists any violation for that given set of spin-\(\frac{1}{2}\) observables). Moreover, it can also be conjectured that any steering inequality whose left-hand side is a linear function of correlations will show the same feature. However, there are lots of linear local realist inequalities which are optimally violated by some pure states other than maximally entangled states [47,48]. Hence, it will be interesting to find out steering inequalities which are maximally violated by some pure states other than maximally entangled states for any given set of spin-\(\frac{1}{2}\) observables at trusted and untrusted parties’ sides. Another direction for future research will be posing the same question for nonlinear steering inequalities and entropic steering inequalities [33,34,36,37] proposed so far. Moving beyond spin-\(\frac{1}{2}\) observables and two-qubit states, it is legitimate to ask which quantum state maximally violates (if there exists a violation) a steering inequality for any set of observables on both sides. Though the results presented in this paper have shown similarities between Bell nonlocality and EPR steering in the context of state space structure, the above-mentioned questions may demonstrate non-equivalence between Bell nonlocality and EPR steering.
Acknowledgements  We would like to gratefully acknowledge fruitful discussions with Prof. Guruprasad Kar and Some Sankar Bhattacharya. DD acknowledges the financial support from University Grants Commission (UGC), Government of India. SS acknowledges the financial support from INSPIRE programme, Department of Science and Technology, Government of India.

References

1. Einstein, A., Podolsky, B., Rosen, N.: Can quantum-mechanical description of physical reality be considered complete? Phys. Rev. 47, 777 (1935)
2. Schrodinger, E.: Discussion of probability relations between separated systems. Proc. Cambridge Philos. Soc. 31, 555 (1935)
3. Schrodinger, E.: Probability relations between separated systems. Proc. Cambridge Philos. Soc. 32, 446 (1936)
4. Wiseman, H.M., Jones, S.J., Doherty, A.C.: Steering, entanglement, nonlocality, and the Einstein–Podolsky–Rosen paradox. Phys. Rev. Lett. 98, 140402 (2007)
5. Jones, S.J., Wiseman, H.M., Doherty, A.C.: Entanglement, Einstein–Podolsky–Rosen correlations, Bell nonlocality, and steering. Phys. Rev. A 76, 052116 (2007)
6. Wittmann, B., Remalow, S., Steinlechner, F., Langford, N.K., Brunner, N., Wiseman, H., Ursin, R., Zeilinger, A.: Loophole-free Einstein–Podolsky–Rosen experiment via quantum steering. New. J. Phys. 14, 053030 (2012)
7. He, Q.Y., Reid, M.D.: Genuine multipartite Einstein–Podolsky–Rosen steering. Phys. Rev. Lett. 111, 250403 (2013)
8. Milne, A., Jevtic, S., Jennings, D., Wiseman, H., Rudolph, T.: Quantum steering ellipsoids, extremal physical states and monogamy. New J. Phys. 16, 083017 (2014)
9. Jevtic, S., Pusey, M., Jennings, D., Rudolph, T.: Quantum steering ellipsoids. Phys. Rev. Lett. 113, 020402 (2014)
10. Evans, D.A., Wiseman, H.M.: Optimal measurements for tests of Einstein–Podolsky–Rosen steering with no detection loophole using two-qubit Werner states. Phys. Rev. A 90, 012114 (2014)
11. Skrzypczyk, P., Navascues, M., Cavalcanti, D.: Quantifying Einstein–Podolsky–Rosen steering. Phys. Rev. Lett. 112, 180404 (2014)
12. Jevtic, S., Hall, M.J.W., Anderson, M.R., Zwierz, M., Wiseman, H.M.: Einstein–Podolsky–Rosen steering and the steering ellipsoid. JOSA B 32, A40 (2015)
13. Piani, M., Watrous, J.: Necessary and sufficient quantum information characterization of Einstein–Podolsky–Rosen steering. Phys. Rev. Lett. 114, 060404 (2015)
14. Hsieh, C.-Y., Liang, Y.-C., Lee, R.-K.: Quantum steerability: characterization, quantification, superactivation, and unbounded amplification. Phys. Rev. A 94, 062120 (2016)
15. Gallego, R., Aolita, L.: Resource theory of steering. Phys. Rev. X 5, 041008 (2015)
16. Cavalcanti, D., Skrzypczyk, P.: Quantum steering: a review with focus on semidefinite programming. Rep. Prog. Phys. 80, 024001 (2017)
17. Das, D., Datta, S., Jafari, C., Majumdar, A.S.: Cost of Einstein–Podolsky–Rosen steering in the context of extremal boxes. Phys. Rev. A 97, 022110 (2018)
18. Ku, H.-Y., Chen, S.-L., Budroni, C., Miranowicz, A., Chen, Y.-N., Nori, F.: Einstein–Podolsky–Rosen steering: its geometric quantification and witness. Phys. Rev. A 97, 022338 (2018)
19. Quintino, M.T., Vertesi, T., Cavalcanti, D., Augusiak, R., Demianowicz, M., Acin, A., Brunner, N.: Inequivalence of entanglement, steering, and Bell nonlocality for general measurements. Phys. Rev. A 92, 032107 (2015)
20. Brunner, N., Cavalcanti, D., Pironio, S., Scarani, V., Wehner, S.: Bell nonlocality. Rev. Mod. Phys. 86, 419 (2014)
21. Horodecki, R., Horodecki, P., Horodecki, M., Horodecki, K.: Quantum entanglement. Rev. Mod. Phys 81, 865 (2014)
22. Bowles, J., Vertesi, T., Quintino, M.T., Brunner, N.: One-way Einstein–Podolsky–Rosen steering. Phys. Rev. Lett. 112, 200402 (2014)
23. Chen, J.L., Ye, X.J., Wu, C.F., Su, H.Y., Cabello, A., Kwek, L.C., Oh, C.H.: All-versus-nothing proof of Einstein–Podolsky–Rosen steering. Sci. Rep. 3, 2143 (2013)
24. Branciard, C., Cavalcanti, E.G., Walborn, S.P., Scarani, V., Wiseman, H.M.: One-sided device-independent quantum key distribution: security, feasibility, and the connection with steering. Phys. Rev. A 85, 010301(R) (2012)
25. Cavalcanti, E.G., Jones, S.J., Wiseman, H.M., Reid, M.D.: Experimental criteria for steering and the Einstein–Podolsky–Rosen paradox. Phys. Rev. A 80, 032112 (2009)
26. Reid, M.D.: Demonstration of the Einstein–Podolsky–Rosen paradox using nondegenerate parametric amplification. Phys. Rev. A 40, 913 (1989)
27. Ou, Z.Y., Pereira, S.F., Kimble, H.J., Peng, K.C.: Realization of the Einstein–Podolsky–Rosen paradox for continuous variables. Phys. Rev. Lett. 68, 3663 (1992)
28. Walborn, S.P., Salles, A., Gomes, R.M., Toscano, F., Souto Ribeiro, P.H.: Revealing hidden Einstein–Podolsky–Rosen nonlocality. Phys. Rev. Lett. 106, 130402 (2011)
29. Schneeloch, J., Broadbent, C.J., Walborn, S.P., Cavalcanti, E.G., Howell, J.C.: Einstein–Podolsky–Rosen steering inequalities from entropic uncertainty relations. Phys. Rev. A 87, 062103 (2013)
30. Pramanik, T., Kaplan, M., Majumdar, A.S.: Fine-grained Einstein–Podolsky–Rosen steering inequalities. Phys. Rev. A 90, 050305(R) (2014)
31. Chowdhury, P., Pramanik, T., Majumdar, A.S., Agarwal, G.S.: Einstein–Podolsky–Rosen steering using quantum correlations in non-Gaussian entangled states. Phys. Rev. A 89, 012104 (2014)
32. Chowdhury, P., Pramanik, T., Majumdar, A.S.: Stronger steerability criterion for more uncertain continuous-variable systems. Phys. Rev. A 92, 042317 (2015)
33. Cavalcanti, E.G., Foster, C.J., Fuwa, M., Wiseman, H.M.: Analog of the Clauser–Horne–Shimony–Holt inequality for steering. J. Opt. Soc. Am. B 32, A74 (2015)
34. Girdhar, P., Cavalcanti, E.G.: All two-qubit states that are steerable via Clauser–Horne–Shimony–Holt-type correlations are Bell nonlocal. Phys. Rev. A 94, 032317 (2016)
35. Sasmal, S., Pramanik, T., Home, D., Majumdar, A.S.: A tighter steering criterion using the Robertson–Schrodinger uncertainty relation. Phys. Lett. A 382, 27 (2018)
36. Costa, A.C.S., Uola, R., Guhne, O.: Steering criteria from general entropic uncertainty relations. Phys. Rev. A 98, 050104(R) (2018)
37. Costa, A.C.S., Uola, R., Guhne, O.: Entropic steering criteria: applications to bipartite and tripartite systems. Entropy 20(10), 763 (2018)
38. Bell, J.S.: On the Einstein–Podolsky–Rosen paradox. Physics 1, 195 (1965)
39. Clauser, J.F., Horne, M.A., Shimony, A., Holt, R.A.: Proposed experiment to test local hidden-variable theories. Phys. Rev. Lett. 23, 880 (1969)
40. Horodecki, R., Horodecki, P., Horodecki, M.: Violating Bell inequality by mixed spin-1/2 states: necessary and sufficient condition. Phys. Lett. A 200, 340 (1995)
41. Costa, A.C.S., Angelo, R.M.: Quantification of Einstein–Podolsky–Rosen steering for two-qubit states. Phys. Rev. A 93, 020103(R) (2016)
42. Mal, S., Das, D., Sasmal, S., Majumdar, A. S.: Necessary and sufficient state condition for two-qubit steering using two measurement settings per party and monogamy of steering. arXiv:1711.00872 [quant-ph] (2017)
43. Kar, G.: Noncommuting spin-1/2 observables and the CHSH inequality. Phys. Lett. A 204, 99 (1995)
44. Cereceda, J.L.: Maximally entangled states and the Bell inequality. Phys. Lett. A 212, 123 (1996)
45. Peres, A.: Quantum Theory: Concepts and Methods. Kluwer, Dordrecht (1993)
46. Hughston, L.P., Jozsa, R., Wootters, W.K.: A complete classification of quantum ensembles having a given density matrix. Phys. Lett. A 183, 14 (1993)
47. Collins, D., Gisin, N., Linden, N., Massar, S., Popescu, S.: Bell inequalities for arbitrarily high-dimensional systems. Phys. Rev. Lett. 88, 040404 (2002)
48. Acin, A., Massar, S., Pironio, S.: Randomness versus nonlocality and entanglement. Phys. Rev. Lett. 108, 100402 (2012)

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