Finite distance corrections to the light deflection in a gravitational field with a plasma medium

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(Dated: September 7, 2018)

The aim of the present work is twofold: first, we present general remarks about recent procedures to compute the deflection angle taking into account finite distance corrections based on the Gauss-Bonnet theorem. Second, and as the main part of our work, we apply this powerful technique to compute corrections to the deflection angle produced by astrophysical configurations in the weak gravitational regime when a plasma medium is taken into account. For applications, we apply this machinery to introduce new general formulas for the bending angle of light rays in plasma environments in different astrophysical scenarios, generalizing previously discovered results. In particular, for the case of an homogeneous plasma we study these corrections for the case of light rays propagating near astrophysical objects described in the weak gravitational regime by a Parametrized-Post-Newtonian (PPN) metric which takes into account the mass of the objects and a possible quadrupole moment. Even when our work concentrates in finite distances corrections to the deflection angle, we also obtain as particular cases of our expressions new formulas which are valid for the more common assumption of infinite distance between receiver, lens and source. We also consider the presence of an inhomogeneous plasma media introducing as particular cases of our general formulas explicit expressions for particular charge number density profiles.

PACS numbers:

\section{I. INTRODUCTION}

Gravitational lensing is a crucial tool to study the dynamics, evolution and distribution of matter in the Universe\cite{1–16}. The response of electromagnetic radiation to gravitational fields occurs at all size scales in the Universe, ranging from the size of individual black holes\cite{17} to clusters containing many individual galaxies\cite{18}. In fact, recently the first observational test of Einstein’s general relativity confirms the theory to high precision on extragalactic scales\cite{19}. At all scales, the study of the strong gravitational lensing regime gives us information about the response of electromagnetic radiation to gravitational fields and will be crucial in providing tests of gravitational theory under strong field conditions.

Typically, gravitational lens effects are considered in the vacuum. However, many compact objects are surrounded by dense, plasma-rich magnetospheres\cite{20, 21}, and even galaxies and galaxy clusters\cite{22, 23} are in general immersed in a plasma fluid. In the visible spectrum, the modification of gravitational lensing quantities due to the presence of the plasma is negligible. The same cannot be said of observations in the radio frequency spectrum where the index of refraction of the plasma causes strong frequency-dependent modifications of the usual gravitational lensing behavior. In fact, there exist some radio-telescope projects which operate in the frequency bands where plasma effects should be taken into account\cite{23–27}. For that reason, in the last years the study of the influence of plasma media on the trajectory of light rays in a external gravitational field associated to compact bodies have become a very active research area\cite{28–48}.

One of the main quantities in the study of gravitational lensing is the deflection angle. In general, expressions for this quantity are written in terms of derivatives of the various metric components. However, in \cite{49}, we introduce an expression for the deflection angle in the weak lensing regime which is written in terms of the curvature scalars. It was generalized to the cosmological context by Boero and Moreschi\cite{50} and recently by us to take into account second order corrections in perturbations of a flat metric\cite{51}. On the other hand, Gibbons and Werner have also established a new geometrical (and topological) way of studying gravitational lensing using the Gauss-Bonnet theorem and an associated optical metric\cite{52}; more precisely, they obtained an elegant relation between the deflection angle, the Gaussian curvature of the associated optical metric and the topology of the manifold. More precisely, the deflection angle can be obtained by integrating the Gaussian curvature of this metric in an appropriate two-dimensional integration region $D$.

Since the seminal work of Gibbons and Werner\cite{52}, many various applications of this method for purely gravitational lensing in astrophysical situations have

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emerged. In particular, this new technique is being used to compute gravitational lensing quantities in a variety of spacetimes including vacuum, electro-vacuum, and with a vast array of scalar fields or effective fluids at both finite and infinite distances. More recently, in , we have shown for the first time how the Gibbons-Werner method can also be applied to the study of light rays simultaneously interacting with gravitational fields and a plasma medium. It is worth noting that in this case the light rays do not propagate along null geodesics of the underlying physical spacetime. Despite this apparent difficulty, we have shown how the Gauss-Bonnet method, originally designed to study null geodesics in pure gravitational fields, can also be applied through judicious choice of optical metric to the study of timelike curves followed by light rays in a plasma environment. In fact, these results also apply to timelike geodesics followed by massive particles in pure gravitational fields. Thus, our results highlight the elegance and power of the Gibbons-Werner method by demonstrating the beautiful relationship that exists between the deflection angle, geometric and topological quantities associated with spacetime, and the implications of these relationships for both optics and mechanics.

Due to the deep connections between geometry and topology exposed by the Gibbons-Werner method, several authors have proposed alternative extensions to the Gauss-Bonnet theorem in situations where the source or the observer can not be considered to be at infinite distance to the lens. The first alternative is presented in references 65, 66, 73, 80 and the second in 64. Some remarks are in order with respect to these alternative formulations. First, even when these proposals are based in the Gauss-Bonnet theorem, they do not agree even at linear order in the mass these extra terms are given by and and in the case of Arakida it is a finite region and in the case of Ishihara et.al. it is an unbounded one. We will show below, by presenting an alternative integration region in the Ishihara et.al. definition, how the difference in the results of these expressions can be easily understood. Second, even when finite-distance corrections to the deflection angle are derived by these two alternative definitions, the authors of 64, 66, 73, 80 or 64 do not attempt to make a comparison with known expressions from the literature that were obtained using different techniques 81, 82 and which are also needed for high precision relativistic astrometry 91, 92. Due to the existence of these two incompatible definitions (as previously mentioned, they do not agree even at linear order in a Schwarschild spacetime), the comparison between these predictions and the known quantities can be used as a good test of their validity. We will carry out this comparison and we will shown that the Ishihara et.al. definition is in complete agreement with known expressions.

In addition to dealing with the technical issues around the calculation, our main motivation for the present work is to study how the consideration of finite distances between the source, lens and observer can affect the expression for the deflection angle in astrophysical situations where a plasma medium is present. The usual way to study lensing due to plasma is through the Hamiltonian equations for the timelike curves followed by light rays in the plasma environment. On the other hand, recently we presented a geometrical formulation of the problem using the Gibbons-Werner method . Therefore it is natural to use this powerful technique to study the corrections in the known expressions for the deflection angle in these situations.

Motivated by these issues, we propose a number of points to contribute to the discussion of this subject: First, we present an alternative formulation of the definition given in 65 for the bending angle at finite distances. We remark that it is not a new definition, but an equivalent formulation. Our approach is based in a finite region and allows us to compare with the expression given by Arakida in 64. Second, we fill the existent gap in the comparison with known expressions for the bending angle at finite distance and the results obtained using the definition given in 63. This comparison provides confidence in the veracity of the region definitions in that work. Finally, as the focus of our work, we apply this powerful technique to compute corrections to the deflection angle produced by astrophysical configurations in the weak gravitational field regime when a plasma medium is taken into account. In particular, for the case of a homogeneous plasma we study finite distance corrections for the case of light rays propagating through astrophysical objects described in the weak

\[ \delta \alpha = \frac{m}{b} \left( \sin^2 (\varphi_R) \cos (\varphi_R) - \sin^2 (\varphi_S) \cos (\varphi_S) \right) \] (1)

where \( \varphi_S \) and \( \varphi_R \) represent the angular coordinate of the position of the source and the receiver, respectively, and \( \delta \alpha \) is the difference between the expressions given by Arakida and Ishihara et.al. More details, in Sec. 14.
gravitational region by a Parametrized-Post-Newtonian (PPN) expansion which takes into account the mass of the objects and a possible quadrupole moment. Even when our work concentrates in finite distance corrections to the deflection angle, we also obtain as particular cases of our expressions new formulas which are valid for the more common assumption of infinite distance between receiver, lens and source generalizes previous ones.

This work is organized as follows. In Sec. II we review the definition of bending angle given by Ishihara et.al. in [64] and we propose an alternative presentation by using a finite quadrilateral region which allows us to make a comparison and remark on the difference with the Arakida definition [64]. We also present a review of known finite distance expressions for the bending angle in order to prepare for later comparison with what is obtained by the use of the Gauss-Bonnet method. In Sec. III we review the theory of light rays in cold non-magnetized plasma and the associated optical metric. In Sec. IV we study finite distance corrections to the deflection angle in astrophysical situation where the gravitational field can be represented by a PPN metric and where a homogeneous plasma medium is present. We also carry out detailed comparisons between known expressions for the bending angle and results obtained using the Ishihara et.al definition. As a by-product, we obtain several new formulas for the bending angle which generalizes previous known results in several ways. Finally in section V we briefly discuss the situation where the plasma is non-homogeneous presenting the study of a Schwarzschild spacetime surrounded by some particular cases of inhomogeneous plasma media. In particular, we study the relevance of finite distance corrections in a model for the plasma density for the solar corona. We conclude with final remarks. For completeness, in Appendix A we show how using the finite quadrilateral region (as defined in Sec. II) three different versions of the deflection angle calculation give the same result.

II. FINITE DISTANCE CORRECTIONS TO THE DEFLECTION ANGLE USING THE GAUSS-BONNET THEOREM

1. General remarks

The Gauss-Bonnet theorem provides a powerful framework to describe finite distance corrections to the gravitational lens deflection angle. A thorough discussion of this topic first requires some general discussion of definitions recently used in the literature [64, 66, 80].

Let us recall the application of the Gauss-Bonnet theorem to a two-dimensional riemannian manifold. Let $D \subset S$ be a regular domain of an oriented 2-dimensional surface $S$ with Riemannian metric $g_{ij}$, whose boundary is formed by a closed, simple, piecewise, regular and positive oriented curve $\partial D : \mathbb{R} \supset I \to D$. Then, the Gauss-Bonnet theorem states

$$
\int \int_D K dS + \int_{\partial D} \kappa_g d\sigma + \sum_i \Theta_i = 2\pi \chi(D), \quad \sigma \in I; \quad (2)
$$

where $\chi(D)$ and $K$ are the Euler characteristic and Gaussian curvature of $D$, respectively; $\kappa_g$ is the geodesic curvature of $\partial D$ and $\Theta_i$ is the exterior angle defined in the $i$th vertex, in the positive sense. Given a smooth curve $\gamma$ with tangent vector $\dot{\gamma}$ such that

$$
\dot{g}(\gamma, \dot{\gamma}) = 1, \quad (3)
$$

and acceleration vector $\ddot{\gamma}$, the geodesic curvature $\kappa_g$ of $\gamma$ can be computed as

$$
\kappa_g = \dot{g}(\nabla_\gamma \dot{\gamma}, \ddot{\gamma}), \quad (4)
$$

which is equal to zero if and only if $\gamma$ is a geodesics, because $\dot{\gamma}$ and $\ddot{\gamma}$ are orthogonal.

Let us consider a spherically symmetric spacetime,

$$
d s^2 = A(r) d t^2 - B(r) d r^2 - C(r) (d\theta^2 + \sin^2(\theta) d\varphi^2), \quad (5)
$$

and a light ray propagating from a source $S$ to a receiver $R$ on a null geodesic, which can be taken as lying in the plane defined by $\theta = \pi/2$ without a loss of generality. This null geodesic can be put in one-to-one correspondence with a spatial geodesic of the associated optical metric given by [52, 94]

$$
d\sigma^2 = \frac{B(r)}{A(r)} d r^2 + \frac{C(r)}{A(r)} d\varphi^2. \quad (6)
$$

Ishihara et.al. [65] proposed a new definition for the deflection angle at finite distance using the Gauss-Bonnet theorem, which can be written as

$$
\alpha = -\int \oint_{\Gamma \subset S} K dS. \quad (7)
$$

In order to define the integration region $\Gamma \subset S$ one starts with a region $D_r$, bounded by the geodesics $\gamma_t$ with its origin at a point $S$ and end at $R$. Let us consider two radial geodesics $\gamma_S$ and $\gamma_R$, defined by respective constants $\varphi_S$ and $\varphi_R$, pass through the points $S$ and $R$ respectively. Then, let a circular arc segment defined by $r = r_C$ = constant close the region. The arc segment is chosen to be orthogonal to the radial geodesics $\gamma_R$ and $\gamma_S$. The region $\Gamma \subset S$ is then obtained as the limit of the region $D_r$ as $r_C$ goes to infinity. For a motivation of the choice of this region see the original references [65, 66, 73, 80].

Due to we are interested in the comparison of this formula with the Arakida proposal which is based in a different quadrilateral (and finite) region [64], we will give an alternative presentation of (7) which also makes use of a finite quadrilateral region.

Of course, when we talk about the bending angle, we are referring to how the path of light rays are curved
with respect to a flat spacetime. Therefore, it is natural that we relate the behavior of null geodesics in the two spacetimes. Consider a two dimensional space with a Euclidean metric written in a standard polar coordinate system \( \{r, \varphi\} \). In this space let \( D_r \) be a region with boundaries formed by two straight line segments defined by \( \varphi = \varphi_S = \text{constant} \) and \( \varphi = \varphi_R = \text{constant} \) and such that their ends farthest from the origin are connected by a circular arc segment \( \gamma_C \) defined by \( r = r_C = \text{constant} \) and the two ends nearest the origin connected by a straight line segment \( \gamma_\ell \) (see Fig. 1).

If we apply the Gauss-Bonnet theorem in this region, we obtain the following relation for the sum of the interior angles \( \epsilon_i \) of the region \( D_r \) (which are related to the exterior angles \( \Theta_i \) by \( \Theta_i = \pi - \epsilon_i \)):

\[
\sum_i \epsilon_i = \int_{\gamma_C} \kappa \, d\sigma + 2\pi, \quad \sigma \in I.
\]  

(8)

Of course,

\[
\int_{\gamma_C} \kappa \, d\sigma = \varphi_R - \varphi_S,
\]  

(9)

but it will be not relevant for us.

In a similar way, let us consider a Riemannian two-dimensional space defined in a region \( \mathbb{R}^2/B \) with \( B \) a compact set, such that it allows a SO(2) symmetry group and it is also asymptotically Euclidean. The metric associated with this Riemannian manifold can be written as \( d\tilde{s}^2 = a(\tilde{r})d\tilde{r}^2 + r^2b(\tilde{r})d\tilde{\varphi}^2 \), with \( a(\tilde{r}) \) and \( b(\tilde{r}) \) going to 1 as \( \tilde{r} \) goes to infinity. As this metric is asymptotically Euclidean and therefore tends to the Euclidean metric \( dr^2 + r^2 d\varphi^2 \) as \( r \) goes to infinity, we can make an identification between the coordinates \( \{r, \varphi\} \) used in the polar coordinates system of the Euclidean space where the region \( D_r \) was defined and the new coordinate \( \tilde{r}, \tilde{\varphi} \) of the Riemannian manifold.

Now let \( \tilde{D}_r \) be a slightly modified region in this Riemannian manifold chosen such that three of its sides are defined in a similar way as were \( \gamma_R, \gamma_S \) and \( \gamma_C \) and with the remaining boundary chosen as the geodesic \( \tilde{\gamma}_\ell \) which coincides with the spatial geodesic associated with the spatial orbit of the null geodesic followed by a light ray connecting \( S \) with \( R \) in the physical curved spacetime. See Fig. (2). Therefore, for this region we obtain

\[
\sum \tilde{\epsilon}_i = \int_{\tilde{D}_r} Kd\tilde{s} + \int_{\tilde{\gamma}_C} \tilde{\kappa} \, d\tilde{\sigma} + 2\pi, \quad \tilde{\sigma} \in I.
\]  

(10)

Note that by construction the following crucial property is satisfied \( \epsilon_3 = \tilde{\epsilon}_3 = \epsilon_4 = \tilde{\epsilon}_4 = \pi/2 \), and therefore the difference between the sum of inner angles for the regions \( D_r \) and \( \tilde{D}_r \) is only related to the difference in the angles that the geodesic \( \gamma_R \) and \( \gamma_S \) make with the radial geodesics \( \gamma_R \) and \( \gamma_S \). Motivated by the last fact, we propose the following expression as the definition of the deflection angle \( \alpha \):

\[
\alpha = \sum_i (\epsilon_i - \tilde{\epsilon}_i).
\]  

(11)

Therefore, taking into account the equations (8) and (10) we obtain the alternative expression:

\[
\alpha = - \int_{\tilde{D}_r} Kd\tilde{s} - \int_{\tilde{\gamma}_C(S \rightarrow R)} \tilde{\kappa} \, d\tilde{\sigma} + \int_{\gamma_C(S \rightarrow R)} \kappa \, d\sigma.
\]  

(12)
Where the notation $\tilde{\gamma}_C(S \to R)$ is to recall that the integration must be done on the circular arc segment $\gamma_C$ in the direction from $S$ to $R$. Alternatively, as the other three curves in the quadrilateral region are geodesics, Eq. (12) can be written as

$$\alpha = - \int \int_{D^*_r} K dS - \int_{\partial D^*_r} \tilde{k} \, d\tilde{\sigma} + \int_{\partial D^*_r} \kappa \, d\sigma, \quad (13)$$

with the line integrals made on the respective boundaries $\partial D_r$ an $\partial D^*_r$ of the regions $D_r$ and $D^*_r$ in a counter-clockwise direction. By construction the right hand side of Eq. (12) gives the same result for any curve $\gamma_C$ defined by $r_C = \text{constant}$. This definition is an alternative presentation of the proposed definition of Ishihara et.al. (13).

In particular, as the metric is assumed to be asymptotically Euclidean, we can take the limit of $r_C$ going to infinity, in which case $\int_{C} \kappa \, d\sigma \to \int_C \kappa \, d\sigma$, resulting in an expression for the angle $\alpha$ which reduces to the formula (7) as given by Ishihara et.al. (14).

In fact, we can repeat the same procedure but without the assumption that the curve $\tilde{\gamma}_C$ is geodesic. In this case, even when the region $D_r$ remains unchanged, we obtain a new region $\tilde{D}^*_r$ and Eq. (12) is modified to:

$$\alpha = - \int \int_{\tilde{D}^*_r} K dS - \int_{\tilde{\gamma}_{\text{Euclidean}}(\tilde{S} \to \tilde{R})} \tilde{k} \, d\tilde{\sigma} - \int_{\tilde{\gamma}_C(\tilde{S} \to \tilde{R})} \kappa \, d\sigma \nonumber + \int_{\tilde{\gamma}_C(\tilde{S} \to \tilde{R})} \kappa \, d\sigma \nonumber$$

$$= - \int \int_{\tilde{D}^*_r} K dS - \int_{\partial \tilde{D}^*_r} \tilde{k} \, d\tilde{\sigma} + \int_{\partial D^*_r} \kappa \, d\sigma. \quad (14)$$

If we assume a region $\tilde{D} \equiv \tilde{D}^* \bigcap \tilde{S} \in\tilde{R}$ obtained from $\tilde{D}^*_r$ in the limit of $r_C$ going to infinity, it is easy to see that the relation (14) reduces formally to the expression found in Ref. (15) for the deflection angle at finite distances valid for a general stationary and axially-symmetric spacetime (Note that in such cases, as explained in detail in Ref. (16) a modification of the form of the optical metric is needed).

With the expression (12), we are ready to compare with the Arakida definition (17). In that reference the author also takes a finite quadrilateral region but instead of using the circular curve $\gamma_C$, a new curve $\gamma_T$ is chosen which is identified with the spatial geodesic associated to a light ray connecting $R$ and $S$ if the spacetime were flat, that is, in the Euclidean space it is in fact a straight line.

Keeping the definition (11) for the deflection angle with these new regions, and noting that for a quadrilateral trapezoid in Euclidean space, the sum of interior angles always equal to $2\pi$, we obtain a new deflection angle,

$$\tilde{\alpha} = 2\pi - \sum_{i} \tilde{\epsilon}_i; \quad (15)$$

which exactly agrees with the definition of Arakida (18) (in that reference the interior angles are denoted $\beta_i$.) Equivalently, for this new choice of regions, the integration around the curve $\gamma_T$ (which replaces the curve $\gamma_C$) in the last term in (12) is exactly zero, because it is computed in the Euclidean background spacetime and $\gamma_T$ is a geodesic of the Euclidean space by construction, and therefore only the first two terms in (12) survive, and we arrive at an expression with exactly the same form as found in (19) (Equation (35) of that reference). Therefore, it should appear at first sight that the definition (11) also contains as particular case the definition for the deflection angle given by Arakida. However, note that in the motivation for (11) the equality between the interior angles $\tilde{\epsilon}_3$ and $\tilde{\epsilon}_4$ and between $\epsilon_4$ and $\tilde{\epsilon}_4$ was crucial. Note, that we could have written the expression for the deflection angle as the difference between the sum of $\epsilon_1$ and $\epsilon_2$ and their tilde version, emphasizing in this way that it only depends on the angles that the null geodesic connecting $S$ with $R$ makes with the radial curves in the curved space as compared to the similar angles defined in the background. However, this is not the case if we replace the curve $\gamma_C$ by the new curve $\gamma_T$. Therefore, the use of the equation (14) seems not to be well motivated. In this case the inner angles that the curve $\gamma_T$ forms with the radial curves will not be the same in the curved space and in the Euclidean space. This means the addition of the inner angles in the Arakida region in the curved space has not only information of the new angles formed by the geodesic $\gamma_T$ with the radial curves but also of the angles that the new curve $\gamma_T$ forms with these curves. In fact, as we mentioned in the introduction, comparison of the expressions found in (17) and (18) for the deflection angle at finite distances in a Schwarzschild background do not agree even at first order in the mass. For this example, it is easy to check that the origin of the difference between the Ishihada et.al. expression for the deflection angle and the Arakida expression originates in the difference between the values of of the inner angles that the curve $\gamma_T$ makes with the radial curves in both the Euclidean and the curved spaces. More precisely, as follows from eq.(44) of (19) at linear order in the mass the difference between $\epsilon_3$ and $\tilde{\epsilon}_3$ is

$$\epsilon_3 - \tilde{\epsilon}_3 = \frac{m}{b} \sin^2(\varphi_S \cos(\varphi_S)). \quad (16)$$

A similar expression follows for the difference of the angles $\epsilon_4$ and the tilde version. This difference will contribute to the deflection angle (even at linear order in $m$). Hence, from the two definitions, we arrive at a difference between the formulas for the deflection. In particular, using the definition of (20), the deflection angle at linear order in the mass, denoted as $\alpha_{\text{Arakida}}$ reads (See Eq.(A.3) of Ref. (21) in terms of the angular coordinates with $\alpha = 0$, or Eq.(37) of Ref. (22) with $\Lambda = 0$):

$$\alpha_{\text{Arakida}} = \frac{2m}{b} \left( \cos(\varphi_S) - \cos(\varphi_R) \right). \quad (17)$$

2 In the Arakida notation our inner angle $\epsilon_3$ is denoted as $\beta_2$, and in particular $\beta_2 = E$ in his notation. See Eq.(44) of (19).
In comparison, the expression \( \alpha_{\text{Arakida}} \) given by Arakida (Eq. (54) of \[64\]) reads:

\[
\alpha_{\text{Arakida}} = \alpha_{\text{RS}} + \delta \alpha; \quad (18)
\]

with

\[
\delta \alpha = \frac{m}{b} \left( \sin^2(\varphi_R) \cos(\varphi_R) - \sin^2(\varphi_S) \cos(\varphi_S) \right). \quad (19)
\]

As anticipated, these extra terms are originated by the relations like (10) and a similar formula for \( \epsilon_{4} - \epsilon_{4} \).

Note that these differences are more than relevant with respect to the actual observability of the finite distance corrections. Let us consider, for example, the deflection produced by our Sun when the light rays of a far away source graze the surface and reach us on Earth. For such situation we can make the following approximations: \( \varphi_S = 0, \varphi_R = \pi - \delta \varphi \), with \( \delta \varphi \approx b/r_o \approx 4 \times 10^{-3} \), where \( r_o \) is the distance from the Sun to the Earth, and \( b \) equal to the radius of the Sun. Then, as proved by Ishihara et al., the difference between the infinite distance expression and \( \alpha_{\text{RS}} \) is of the order of \( 10^{-3} \) arcsec. More precisely, by doing a Taylor expansion of (17) we obtain

\[
\alpha_{\text{RS}} \approx \frac{4m}{b} \frac{m \delta \varphi^2}{b} - \frac{m \delta \varphi^4}{4b} + \mathcal{O}(\delta \varphi^5). \quad (20)
\]

Hence, the first correction to the infinite distance expression is approximately given by \( \delta \alpha \approx 10^{-5} \) arcsec, which is within the capabilities of actual observations. Even when Arakida do not compute the numerical correction for this example, we can do the same exercise. The new terms contribute as

\[
\delta \alpha \approx \frac{m \delta \varphi^2}{b} - \frac{m \delta \varphi^4}{2b} + \mathcal{O}(\delta \varphi^5). \quad (21)
\]

Surprisingly, as the Arakida expression is obtained by the addition of (20) and (21), we note that there exists a cancellation between the quadratic terms in \( \delta \varphi \), resulting in a final expression given by

\[
\alpha_{\text{Arakida}} \approx \frac{4m}{b} - \frac{3m \delta \varphi^4}{4b}. \quad (22)
\]

Hence, the correction to the usual Schwarzschild expression is of the order of \( 10^{-5} \) μsec, a value undetectable with the actual technology. Therefore, the difference in predictions of these two formulations in not only of academic interest, but also practical.

On the other hand, as we will show below (Secs. 1.2 and (IVC)), the eq. (7) or its equivalent (12) yield the same results when they are compared with other well-known expressions obtained using post-Newtonian techniques, even in more general situations, taking into account possible quadrupole aspect of a central body and second order corrections in the mass.

All these discussed issues give us confidence in the expression defined by Ishihara et al. in \[65\] and given by (7) or its equivalent versions (12) and (13).

Note that even when (11) and (12) are equivalent to the original version given by equation (7) as presented by Ishihara et al., they were not presented before in the literature. In particular, (11) has a clear geometrical meaning. As a useful test, in Appendix (A) we will show using an explicit example how the original version (7), or its equivalent new finite region versions (11) and (12) give the same result. Of course, Eq. (7) is more easy to use, because one does not need to compute the geodesic curvatures. Therefore, from now on we will continue using this last expression.

2. Relation between the Ishihara et al. definition of deflection angle at finite distance and some known expressions in the literature using a post-Newtonian approach.

The deflection angle for a Schwarzschild metric and for a Kottler spacetime were calculated using equation (4) in \[65\] and using the version (10) in \[64\]. In particular, the possibility of observing these finite corrections the angle of deviation for a Schwarzschild spacetime were discussed \[66, 80\]. Despite these results, it is worthwhile to mention that the computation of corrections at finite distances for the angle of deviation has been done by different authors even for more general situations and also well discussed in textbooks for many years. Recently these calculations have been completed using different techniques and methods, for example using post-Newtonian methods, solving explicitly the geodesic equation in particular spacetimes etc\[81–90\]. In fact, such expressions are needed in high-precision astrometry\[91–93\]. Unfortunately, the authors of \[65\] or \[64\] do not try to make a compression with these different results.

We fill this gap, showing that the deflection angle which follows from (7) is in complete agreement with some known finite distance expressions even considering second order effects and more general metrics than the Schwarzschild solution. In particular, we are interested in the comparison of the finite distance expression for the deflection angle as computed by Richter and Matzner for a parametrized-post-Newtonian (PPN) metric \[82\].

A detailed discussion of the PPN metric first requires some review of basic facts and assumptions. Let us recall the form of the general PPN metric that represents the exterior of a static and axially symmetric compact body

\[3\] In references \[62, 64, 72, 81\] there is an alternative presentation for the deflection angle \( \alpha \) as a particular sum of three angles, which is written in terms of two geometrical angles that \( \varphi_R \) form with the radial curves and a coordinate angle \( \varphi_{RS} \), but as the authors claim, this definition seems to rely on a choice of the angular coordinate \( \varphi_{RS} \), however they show that this definition is equivalent to the geometrical invariant version (7). More details about the comparison between (11) and the angular definition in terms of the sum of three angles \[65\] are presented in the last part of Appendix (A).
with mass $m$ and multipole moments $J_n$. For this case the metric is described by an expression similar to the expression given by Eq. (4) but now with the associated metric functions $	ilde{A}, 	ilde{B}$ and $	ilde{C}$ depending on the coordinates $r$ and $\vartheta$:

$$
\begin{align*}
\tilde{A}(r, \vartheta) &= 1 + 2\tilde{U}(r, \vartheta) + 2\beta U^2(r, \vartheta), \\
\tilde{B}(r, \vartheta) &= 1 - 2\mu\tilde{U}(r, \vartheta) + \frac{3}{2}\nu U^2(r, \vartheta), \\
\tilde{C}(r, \vartheta) &= B(r, \vartheta)r^2,
\end{align*}
$$

where the potential $\tilde{U}$ reads

$$
\tilde{U}(r, \vartheta) = -\frac{m}{r}\left[1 - \sum_{n=2}^{\infty} \left(\frac{R}{r}\right)^n P_n(\cos(\vartheta))\right],
$$

with $P_n(x)$ the Legendre polynomials. Here $\beta$, $\mu$, and $\nu$ are three parameters which take the value 1 in the Einstein general relativity theory. In that case, if $J_n = 0$, this metric represents the second order version in Schwarzschild metric. Let us also assume that in addition to the mass $m$, the only non-vanishing multipole is the quadrupole moment, $J_2$.

Of course, this metric is not spherically symmetric. However, if we restrict our study to the propagation of light rays in the plane defined by $\vartheta = \pi/2$, the PPN metric to this plane has an SO(2) symmetry and the metric functions are given by

$$
\begin{align*}
A(r) &:= \tilde{A}(r, \pi/2) = 1 + 2U(r) + 2\beta U^2(r), \\
B(r) &:= \tilde{B}(r, \pi/2) = 1 - 2\mu U(r) + \frac{3}{2}\nu U^2(r), \\
C(r) &:= \tilde{C}(r, \pi/2) = B(r)r^2,
\end{align*}
$$

with

$$
U(r) = -\frac{m}{r}\left(1 + \frac{R^2 J_2}{2r^2}\right).
$$

Let a gravitational compact object be represented in the weak gravitational field region outside the object by the previous metric, and let us assume a lens $L$, receiver $R$ and a source $S$ configuration as shown in Fig. 3. For the moment, we also assume that the source is far away from the lens and take $\varphi_S = 0$, referring to Fig. 3. However, the receiver is assumed to be at a finite distance from the lens. In this case, the standard operational way to define the deflection angle is through the observable

$$
\delta \theta = \theta_I - \theta',
$$

where $\theta_I$ is the angle between the image of the source as seen by the receiver and the receiver-lens axis, and $\theta'$ is the value that this angle should take if the lens were absent $^{83,86}$. If we were to assume that the receiver $R$ is at infinite distance from the lens, then $\delta \theta$ should agree with the asymptotic deflection angle $\alpha_\infty$. However, due to the finite distance location of the receiver there exists a disagreement between these two angles in general.

As mentioned previously, different authors using different techniques have computed the deflection angle $\delta \theta$ in terms of the parameters of the compact object and the observable angle $\theta_I$. These expressions are also found in two alternate ways: in terms of the impact parameter (which at finite distance is not an observable) or in terms of the radial coordinate $r_o$ between the receiver and the lens. If we consider only the computation of $\delta \theta$ at first order in $m$ and in $J_2$, the relation between the impact parameter and the radial coordinate $r_o$ is simply

$$
b = r_o \sin(\varphi_R)$$

which must be corrected at second order.

Before continuing, let us remark on the notation: even when $\delta \theta$ is a commonly used notation for the deflection angle at finite distances, we will continue denoting $\delta \theta$ as $\alpha_\infty$ (we use the suffix $S_\infty$ in $\alpha$ as a reminder that the source is assumed to be placed at infinite distance from the lens). On the other hand in the case of infinite distances (for both, the receiver and the source) the deflection angle will be denoted as $\alpha_\infty$.

More than three decades ago it was shown by Richter and Matzner in $^{83}$ that the deflection angle for the previous configuration of receiver, lens and source in a gravitational field represented by a PPN metric given by (25), (26), and (27), can be written in terms of the observable angle $\theta_I$ and the (non-observable) impact parameter $b$ as

$$
\delta \theta = \alpha_\infty = \alpha^{(1)} + \alpha^{(2)},
$$

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$$
\delta \theta = \alpha_\infty = \alpha^{(1)} + \alpha^{(2)},
$$

FIG. 3: A light ray travels from a far away source $S$ to the receiver $R$ through a region where a lens $L$ is present. The angle $\theta_I$ is defined by the angle between the lens, the receiver and the angular position of the image $S'$. The angle $\theta'$ is the angular position of the source in the absence of the lens as it would be seen by the receiver if the source were considered to be far away. The difference between these two angles is defined to be $\delta \theta$. 
where $\alpha^{(1)}_{S,\infty}$ and $\alpha^{(2)}_{S,\infty}$ are the linear and quadratic terms in the mass of the deflection angle:

$$\alpha^{(1)}_{S,\infty}(b, \vartheta_1) = \frac{m}{b} (1 + \mu) (1 + \cos(\vartheta_1)) \left[ 1 + \frac{J_2 R^2}{2b^2} \left( 2 + \cos(\vartheta_1) - \cos^2(\vartheta_1) \right) \right], \quad (31)$$

$$\alpha^{(2)}_{S,\infty}(b, \vartheta_1) = \frac{m^2}{b^2} \left( 2 - \beta + 2\mu + \frac{3}{4} \nu \right) \frac{\pi - \theta_1 + \sin(\theta_1) \cos(\theta_1)}{\sin^2(\theta_1)} + (1 + \mu)^2 \frac{1 + \cos(\theta_1)}{\sin(\theta_1)} \right]. \quad (32)$$

In fact, a more general metric which admits rotation of the gravitational object and more general energy-momentum distributions has been studied [83], but for our purposes it is sufficient to restrict the study to the considered case.

Equations (31) and (32) can also be rewritten in terms of the radial coordinate $r_0$ of the receiver which is related to $b$ (see [88]) by:

$$\frac{1}{b} = \frac{1}{r_0 \sin(\vartheta_1)} - (1 + \mu) \frac{m}{r_0^2 \sin(\vartheta_1)} + O(m^2, mJ_2). \quad (33)$$

In terms of $r_0$, the relations (31) and (32) read [88]

$$\alpha^{(1)}_{S,\infty}(r_0, \vartheta_1) = \frac{m}{r_0} (1 + \mu) \left[ \frac{1 + \cos(\vartheta_1)}{\sin(\vartheta_1)} \right] + \frac{J_2 R^2}{2r_0^2 \sin^3(\vartheta_1)} \left( 2 + 3 \cos(\theta_1) - \cos^3(\theta_1) \right)], \quad (34)$$

$$\alpha^{(2)}_{S,\infty}(r_0, \vartheta_1) = \frac{m^2}{r_0^2} \left( 2 - \beta + 2\mu + \frac{3}{4} \nu \right) \frac{\pi - \theta_1 + \sin(\theta_1) \cos(\theta_1)}{\sin^2(\theta_1)} - (1 + \mu)^2 \frac{1 + \cos(\theta_1)}{\sin(\theta_1)} \right]. \quad (35)$$

A natural question arises: Can these finite distance relations for the bending angle, (31) and (32) or their equivalent (34) and (35), be recovered from the proposal given by the formula (7) of [65] or the other inequivalent alternative given by (13) of [64]? We will show in Section IV.C that the answer to this question is affirmative for the version given by the Ishihara et al. definition, giving us much more confidence to this geometrical way to compute the deflection angle at finite distances. Moreover, this non trivial result will follow as a particular case of the study of more general astrophysical situations, where the gravitational objects described by the PPN metric given by Eqs. (25), (26) and (27) are now surrounded by a plasma medium. That is, we will obtain expressions which can be used for a variety of spacetimes into the framework of gravitational metric theories which contains as a particular case the Einstein general relativity theory and which does not only represent the spacetime of a central spherical body mass but also it allows a the body with a nontrivial quadrupole moment and that can be immersed in a plasma environment. Of course, in that case, the light rays do not follow null geodesics of the PPN metric; however their dynamics is such that there exists an associated two-dimensional optical metric $g^{opt}_{ij}$ where the spatial orbits of the light rays in the physical metric can also be considered to be spatial geodesics of $g^{opt}_{ij}$ allowing us to use the Gibbons-Werner techniques.

In particular, we will show for the first time that the relations (31) and (32) or (34) and (35) can be recovered and successfully derived from the simple and geometrical relation (7) and, what is more important, they can be generalized to more general scenarios taking into account the presence of a homogeneous plasma environment. However, let us first review the behavior of light rays in the presence of plasma and how they can be studied using the Gauss-Bonnet theorem.

### III. THE OPTICAL METRIC AND THE GAUSS-BONNET THEOREM IN A PLASMA ENVIRONMENT

#### A. The optical metric associated to a plasma medium in an external gravitational field

Let us consider a static spacetime $(M, g_{\alpha\beta})$ filled with a cold non-magnetized plasma described by the refractive index $n$ [33, 34],

$$n^2(x, \omega(x)) = 1 - \frac{\omega^2(x)}{\omega^2_e(x)}, \quad (36)$$

where $\omega(x)$ is the photon frequency measured by a static observer while $\omega_e(x)$ is the electron plasma frequency,

$$\omega^2_e(x) = \frac{4\pi e^2}{m_e} N(x) = K_e N(x), \quad (37)$$

with $e$ and $m_e$ the charge of the electron and its mass, respectively; and $N(x)$ is the number density of electrons in the plasma.
We are interested in the deflection of the light path when rays travel through a gravitational field in a plasma filled environment. The dynamics of the light rays are usually described through the Hamiltonian [33,95],

$$H(x,p) = \frac{1}{2} \left( g^{\alpha\beta}(x)p_\alpha p_\beta + \omega_\alpha^2(x) \right),$$  
(38)

where light rays are solutions of the Hamilton’s equation

$$\ell^\alpha := \frac{dx^\alpha}{ds} = \frac{\partial H}{\partial p_\alpha}, \quad \frac{dp_\alpha}{ds} = -\frac{\partial H}{\partial x^\alpha},$$  
(39)

with the constraint

$$H(x,p) = 0,$$  
(40)

and $s$ is a curve parameter along the light curves.

From (10) in can be shown that in general light rays do not follow timelike or null geodesics with respect to $g_{\alpha\beta}$. Instead, they describe timelike curves with the exception of a homogeneous plasma medium where light rays follow timelike geodesics of $g_{\alpha\beta}$. Note that only light rays with $\omega(x) > \omega_c(x)$ propagate through the plasma.

On the other hand, for the case of static spacetimes, even considering dispersive media one can use a Fermat-like principle [94], where the spatial projections of the light rays on the slices $t =$ constant which solve the Hamilton’s equations are also spacelike geodesics of the following Riemannian optical metric,

$$g_{ij}^{\text{opt}} = -\frac{n^2}{g_{00}} g_{ij}.$$  
(41)

It was precisely this last fact that recently allowed us to study the deflection of light in plasma environments using the Gauss-Bonnet theorem [17].

From now on, we will restrict our attention to static and axially symmetric metrics surrounded by a cold non-magnetized plasma, that is, the physical spacetime is assumed to be described by a metric of the form

$$ds^2 = -\tilde{A}(r,\vartheta)dt^2 + \tilde{B}(r,\vartheta)dr^2 + \tilde{C}(r,\vartheta)(\Theta(r,\vartheta)d\vartheta^2 + \sin^2 \vartheta d\varphi^2),$$  
(42)

and with a dependence of the plasma frequency on the coordinates $r$ and $\vartheta$, $\omega_c = \omega_c(r,\vartheta)$. Note that we are neglecting the self-gravitation of the plasma. We also assume asymptotic flatness and that the plasma medium is static with respect to observers following integral curves of the timelike Killing vector field $\xi^\alpha = (\frac{dr}{d\theta})^\alpha$. Due to the gravitational redshift, the frequency of a photon at a given radial position $r$ is given by:

$$\omega(r, \vartheta) = \frac{\omega_\infty}{\sqrt{\tilde{A}(r, \vartheta)}},$$  
(43)

where $\omega_\infty$ is the photon frequency measured by an observer at infinity. Now we will restrict to the study of light propagation in the plane defined by $\vartheta = \pi/2$. If the spacetime under consideration is spherically symmetric this restriction does not constitute any loss of generality. However, for the axially-symmetric case we should keep in mind that our results will be only valid for light propagation on this plane. Restricted to $\vartheta = \pi/2$, all variables only have a radial dependence and the metric functions will be written without a tilde in a similar way as was done in [25], [26], [27].

As we are interested in the application of the Gauss-Bonnet theorem to the determination of the bending angle, following our previous work [47], we will make use of the associated 2-dimensional Riemannian manifold $(\mathcal{M}^\text{opt}, g_{ij}^{\text{opt}})$ with the $SO(2)$ optical metric [11] (restricted to the plane $\vartheta = \pi/2$),

$$d\sigma^2 = g_{ij}^{\text{opt}} dx^i dx^j = \frac{n^2(r)}{A(r)} \left( B(r) dr^2 + C(r) d\varphi^2 \right).$$  
(44)

This metric is conformally related to the induced metric on the spatial section $t =$ constant, $\vartheta = \pi/2$, of the physical spacetime, and therefore it preserves the angles formed between two curves at a given point.

The geodesic motion follows from the Lagrangian,

$$\mathcal{L} = \frac{1}{2} \left[ \frac{n^2(r)}{A(r)} \left( B(r) \left( \frac{dr}{d\sigma} \right)^2 + C(r) \left( \frac{d\varphi}{d\sigma} \right)^2 \right) \right],$$  
(45)

with the constraint:

$$\frac{n^2(r)}{A(r)} \left[ B(r) \left( \frac{dr}{d\sigma} \right)^2 + C(r) \left( \frac{d\varphi}{d\sigma} \right)^2 \right] = 1.$$  
(46)

In the case of a homogeneous plasma ($\omega_c =$ constant, see below), it follows from (45) and (46), the orbital equation is given by [17],

$$\left( \frac{dn}{d\varphi} \right)^2 = \frac{C(r)}{B(r)} \left[ \frac{C(r) n^2(r)}{A(r)} n_0^2 b^2 - 1 \right].$$  
(47)

where $n_0^2 = 1 - \frac{\omega_\infty^2}{\omega_c^2 A(r_o)}$, and with $\omega_c$ the frequency of the light ray measured by the receiver in $r_o$ (related to $\omega_\infty$ by $\omega_\infty = \omega_c \sqrt{A(r_o)}$).

Defining $u = \frac{1}{n}$, the above equation reduces to,

$$\left( \frac{du}{d\varphi} \right)^2 = u^4 C(u) \left[ \frac{C(u) n^2(u)}{B(u)} \left( \frac{n_0^2 u_0 b^2 A(u)}{n_0^2 u_0 b^2 A(u)} - 1 \right) \right].$$  
(48)

In terms of the curvature tensor associated with the optical metric, the Gaussian curvature $K$ can be computed from

$$K = \frac{R_{r_\varphi r_\varphi} (g^{\text{opt}})}{\det(g^{\text{opt}})}.$$  
(49)

### IV. FINITE DISTANCE CORRECTIONS FOR THE LIGHT DEFLECTION IN A HOMOGENEOUS PLASMA MEDIUM: PPN METRIC

Let us consider a gravitational lens surrounded by a ho-
homogeneous plasma whose electron number density reads,

\[ N(r, \vartheta) = N_0 = \text{constant}. \]  

(50)

A. Parametrized-post-Newtonian (PPN) metric

As an initial example, we study the light propagation in the equatorial plane of a astrophysical object surrounded by an homogeneous plasma medium and whose gravitational field is described by Eqs. \[24, 26\] and \[27\]. In the following we assume that \( J_2 \ll 1 \) such that we will also neglect terms of order \( O(J_2 \times m^2) \).

Due to the gravitational redshift and considering that both the source and the observer at a finite distance from the lens, the refractive index reads

\[ n(r) = \sqrt{1 - \frac{\omega_o^2 A(r)}{\omega_o^2 A(r_o)}}. \]  

(51)

where \( r_o \) is the radial position of the observer from the lens. The associated optical metric at the considered order follows from using the relation [49] and reads,

\[ ds^2 = \Omega^2 (dr^2 + r^2 d\varphi^2); \]  

(52)

with

\[ \Omega^2 = \frac{\omega_o^2 - \omega_e^2}{\omega_o^2} + \frac{m}{\omega_o^2 r^3 r_o} \left[(\mu + 1)\omega_o^6 r^3(J_2 R^2 + 2r^2) - \omega_o^2 \left(J_2 R^2 (\mu r_o^3 + r^3) + 2r^2 r_o^3(\mu r_o + r)\right)\right] + \frac{m^2}{2 \omega_o^2 r^2 r_o^2} \left[\omega_o^4 r_o^2 (8\mu - 4\beta + 3\nu + 8) + \omega_o^2 (4(\beta - 2)r^2 - 8\mu r r_o - 3\nu r_o^2)\right] + \mathcal{O}(m^3, m^2 \times J_2). \]  

(53)

In order to implement the method described in Section [11] for calculating the bending angle at finite distances, first we need to solve the equation \[48\] at first order, which explicitly reads,

\[ \left(\frac{du}{d\varphi}\right)^2 = \frac{1}{b^2} - u^2 + \frac{m u}{b^2} (2 + J_2 R^2 u^2) \left(\mu + \frac{1}{1 - \omega_e^2/\omega_o^2}\right), \]  

(54)

with the asymptotic condition,

\[ \lim_{\varphi \to 0} u(\varphi) = 0. \]  

(55)

Then, assuming a solution of the form,

\[ u(\varphi) = \frac{1}{b}[\sin(\varphi) + mu_1(\varphi)], \]  

(56)

we obtain at first order in \( m \),

\[ u(\varphi) = \frac{\sin(\varphi)}{b} + \frac{m(1 - \cos(\varphi))}{2b^4} \left(\mu + \frac{1}{1 - \omega_e^2/\omega_o^2}\right) \times \left(2b^2 + J_2 R^2 (1 - \cos(\varphi))\right) + \mathcal{O}(m^2). \]  

(57)

For completeness, in eq. \[57\] we have written explicitly terms of order \( \mathcal{O}(m \times J_2) \). From this expression it is worthwhile to emphasize that since the Gaussian curvature is order \( \mathcal{O}(m) \) (see \[55\] below), it follows that terms of the form \( \mathcal{O}(m \times J_2) \) in \( u \) will contribute to the deflection angle with corrections of order \( \mathcal{O}(m^2 \times J_2) \) which are of higher order than considered. Therefore, they are not necessary in the computation of the bending angle.

In order to compute the bending angle using \[7\] we must integrate the Gaussian curvature \( K \) over \( \hat{n} \hat{\sigma} \). For this, we need to compute \( K \) using the relation \[59\] for the optical metric \[52\] at second order in \( m \) neglecting terms of order \( \mathcal{O}(m^2 \times J_2) \). The result reads:

\[ K = \frac{m(2r^2 + 9J_2 R^2)}{2r^5} \frac{\omega_o^4(\mu \omega_o^2 - (\mu + 1)\omega_e^2)}{(\omega_o^2 - \omega_e^2)^2} \]  

\[ + \frac{m^2 \omega_o^2}{r_o \mu r^4(\omega_o^2 - \omega_e^2)^3} \left[r_o \omega_o^4 \left(4\beta - 2 + 4\mu + 6\mu^2 - 3\nu\right) - 2\omega_o^2 \omega_e^2 \left(2 + \mu\right) r + 2r_o(\beta - 2 + \mu + 3\mu^2) - 3r_o \nu\right] \]  

\[ + \omega_o^4 \left[2\mu r + 6\mu^2 r_o - 3\nu r_o\right] + \mathcal{O}(m^3, m^2 \times J_2). \]  

(58)

The two-form \( KdS \) reads:

\[ KdS = \left\{-\frac{m(9J_2 R^2 + 2r^2)(\mu \omega_o^2 - (\mu + 1)\omega_e^2)}{2r^4(\omega_o^2 - \omega_e^2)} + \frac{m^2}{r_o \mu r^4(\omega_o^2 - \omega_e^2)^2} \left[r_o \omega_o^4 \left(4\beta + \mu^2 - 1\right) - 3\nu\right] - 2\omega_o^2 \omega_e^2 \left[r_o \left(2\beta + 4\mu^2 - 3\nu - 4\right) + r\right] + r_o \omega_o^4 \left(4\mu^2 - 3\nu\right)\right\} dr d\varphi + \mathcal{O}(m^3, m^2 \times J_2). \]  

(59)

Finally, after doing the corresponding integral \[14\], the deflection angle follows:

\[ \alpha = -\int_{\varphi_R}^{\varphi_L} \int_{r_{\gamma_L}}^{r_{\gamma_R}} KdS = \alpha^{(1)} + \alpha^{(2)}, \]  

(60)

where

\[ r_{\gamma_R} = \frac{1}{u(\varphi)} = \frac{b}{\sin(\varphi)} \frac{1 - \cos(\varphi)}{\sin^2(\varphi)} \left(\mu + \frac{1}{1 - \omega_e^2/\omega_o^2}\right) m \]  

\[ + \mathcal{O}(m^2, m \times J_2). \]  

(61)
and with
\[
\alpha^{(1)} = \frac{m}{b} \left( \cos(\varphi_S) - \cos(\varphi_R) \right) \left( \mu + \frac{1}{1 - \omega_e^2/\omega_o^2} \right) \\
\times \left[ 1 + \frac{J_2 R^2}{4b^2} \left( 4 - \cos(2\varphi_S) - \cos(\varphi_R - \varphi_S) \\
- \cos(2\varphi_R) - \cos(\varphi_S + \varphi_R) \right) \right],
\]
(62)

the linear term in \( m \) and the second order correction,
\[
\alpha^{(2)} = \frac{m^2}{4b^2(\omega_o^2 - \omega_e^2)^2} \left( \varphi_S - \varphi_R \right) \left( \omega_o^2 - \omega_e^2 \right) \\
\times \left( \omega_o^2(4\beta - 8 - 8\mu - 3\nu) + 3\nu \omega_e^2 \right) \\
+ 4(\omega_o^2(1 + \mu) - \omega_e^2\mu) \left( \sin(\varphi_S) - \sin(\varphi_R) \right) \\
+ \frac{1}{2} \omega_o^4 \left( 4(\beta - 1 + \mu^2) - 3\nu \right) + \omega_e^4(4\mu^2 - 3\nu) \\
- 2\omega_o^2\omega_e^2(2\beta + 4\mu^2 - 3\nu) \left( \sin(2\varphi_R) - \sin(2\varphi_S) \right) \\
+ 8\omega_o^2\omega_e^2 \cos(\varphi_S) \left( \sin(\varphi_R) - \sin(\varphi_S) \right) \right). 
\]
(63)

In [53], we have used the approximation \( r_o \approx b/\sin(\varphi_R) \) which can be safely used at the considered order.

Expressions [62] and [63] generalize previous known results in several ways. In particular, for the bending angle in a plasma environment these expressions which take into account finite distance corrections, as well as second order effects in the mass and linear in the quadrupole moment. We are not aware of any previous derivations of these general expressions.

Now, we will study some special cases of the above expressions which help to test their validity and to give new relevant formulas for describing the lensing effects of the astrophysical objects under consideration.

### B. Special cases of [62] and [63]

1. **Infinite distances case**

Let us consider the limit where the source and the observer are far away from the lens. In such a situation we may take
\[
\varphi_R \rightarrow \pi \text{ and } \varphi_S \rightarrow 0
\]
(64)

into [62] and [63], such that the deflection angle for an astrophysical object described in the weak gravitational field for the PPN metric which takes into account the monopole and quadrupole gravitational moments is given by Eqs. [25], [26] and [27] reduces to:

\[
\alpha = \frac{2m(b^2 + J_2 R^2)}{b^3} \left( \mu + \frac{1}{1 - \omega_e^2/\omega_o^2} \right) + \frac{\pi m^2}{b^3} \left( \frac{2 - \beta + 2\mu}{1 - \omega_e^2/\omega_o^2} + \frac{3\nu}{4} \right). 
\]
(65)

Despite the simplicity of this expression, it generalizes many recent results. We have no knowledge of a previous presentation of this general formula.

In particular, in the absence of plasma (\( \omega_e = 0 \) or equivalently \( \omega_e/\omega_o \ll 1 \)) the previous equation reduces to,
\[
\alpha = 2(\mu + 1) \frac{m}{b} + \pi(2 - \beta + 2\mu + \frac{3\nu}{4}) \frac{m^2}{b^2}, 
\]
(66)

which coincides with the expression found in [51] [96].

On the other hand, even considering the presence of the plasma, if the object under study is a spherical mass (\( J_2 = 0 \)) and the gravitational field is described by the Einstein general relativity theory (\( \mu = \nu = \beta = 1 \)), then the equation [55] reduces to,
\[
\alpha = \frac{2m}{b} \left( 1 + \frac{1}{1 - \omega_e^2/\omega_o^2} \right) + \frac{3\pi}{4} \left( 1 + \frac{4}{1 - \omega_e^2/\omega_o^2} \right) \frac{m^2}{b^2}. 
\]
(67)

The first term of the previous expression agrees with the formula obtained for the first time by Bisnovatyi-Kogan and Tsupko in [28] [29] and including the second term coincides with the result recently found by us in [47].

2. **Schwarzschild metric at finite distances**

The finite distance contributions for the bending angle in the presence of an homogeneous plasma in a Schwarzschild background follows by setting \( \mu = \beta = \nu = 1 \) and \( J_2 = 0 \) into equations [62] and [63]:
\[
\alpha^{(1)} = \frac{m}{b} \left( 1 + \frac{1}{1 - \omega_e^2/\omega_o^2} \right) \left( \cos(\varphi_S) - \cos(\varphi_R) \right); 
\]
(68)
\[
\alpha^{(2)} = \frac{m^2}{8b^2(\omega_v^2 - \omega_e^2)^2} \left[ 6(\varphi_R - \varphi_S)(5\omega_v^4 - 6\omega_v^2\omega_e^2 + \omega_e^4) \\
+ 16\omega_v^2\omega_e^2 \cos(\varphi_S) \sin(\varphi_R) - (\omega_v^2 + \omega_e^2)^2 \sin(2\varphi_S) \\
+ 8(\omega_e^4 - 2\omega_v^2)^2 \left( \sin(\varphi_S) - \sin(\varphi_R) \right) \\
+ (\omega_v^4 - 6\omega_v^2\omega_e^2 + \omega_e^4) \sin(2\varphi_R) \right].
\]

(69)

These expressions generalize the relations describing light deflection in a vacuum Schwarzschild spacetime recently found by Ishihara et al. in [63] at first order in the mass and extended to second order by Ono et al. in [80]. In particular, in the absence of plasma or where the effect of the plasma environment is negligible \((\omega_e/\omega_v \ll 1)\) these expressions reduce to the following vacuum values:

\[
\alpha^{(1)}_{\text{vac}} = \frac{2m}{b} \left( \cos(\varphi_S) - \cos(\varphi_R) \right); \\
\alpha^{(2)}_{\text{vac}} = \frac{m^2}{8b^2} \left[ 30(\varphi_R - \varphi_S) + \sin(2\varphi_R) - \sin(2\varphi_S) \\
+ 32 \left( \sin(\varphi_S) - \sin(\varphi_R) \right) \right].
\]

(70)

(71)

The expression (70) is in complete agreement with the first order computation of the deflection angle derived in reference [63]. The analogous expression of (71) has been computed in the appendix of reference [80]. However, note that even when there is perfect agreement between our first order expression (70) and the corresponding formula given by the authors in [63], it seems on first sight that there is an inconsistency between our second order correction as given by (71) and the expression from the appendix of the article of [80], which for the convenience of the reader and in order to differentiate from our expression (71) we reproduce here under the alternative name of \(\tilde{\alpha}^{(2)}_{\text{vac}}\) and also with a tilde in their angular variable \(\varphi\):

\[
\tilde{\alpha}^{(2)}_{\text{vac}} = \frac{m^2}{8b^2} \left[ 30(\varphi_R - \varphi_S) + \sin(2\varphi_R) - \sin(2\varphi_S) \right].
\]

(72)

It seems that an apparent discrepancy between (72) and (71) appears, because of the following missing terms in (71):

\[
\delta = 32 \left( \sin(\varphi_S) - \sin(\varphi_R) \right); \\
\delta = 32 \left( \sin(\varphi_S) - \sin(\varphi_R) \right);
\]

(73)

which is however present in (71). The difference is only apparent because the angular coordinate \(\tilde{\varphi}\) used by the authors of [80] is related to our \(\varphi\) by

\[
\tilde{\varphi} = \varphi - \frac{\alpha_\infty}{2} \approx \varphi - \frac{2m}{b} + O(m^2).
\]

(74)

The transformation (74) follows from the fact that we have chosen the polar axis such that the orbit followed by a light ray which reaches the asymptotic region \(r \to \infty\) (or, equivalently \(u \to 0\)) has the following angular coordinate behavior in this limit: \(\varphi \to 0\) or \(\varphi \to \alpha_\infty\) (as can be seen from Eq. (71) with \(\mu = 1\) and \(\omega_e = 0\)). On the other hand, the authors of [80] choose the polar axis such that the closest approach of the light ray to the lens occurs when their angular coordinate \(\tilde{\varphi}\) takes the value \(\tilde{\varphi} = \pi/2\), resulting in a corresponding orbit which is symmetric with respect to the radial direction defined by \(\tilde{\varphi} = \pi/2\). As the total deflection angle at infinite distance is \(\alpha_\infty\), the asymptotic points of the orbit occur when \(\tilde{\varphi} \to -\alpha_\infty/2\) (the position of an asymptotic source) or when \(\tilde{\varphi} \to \pi + \alpha_\infty/2\) (the position of an asymptotic receiver). Note that the difference between \(\varphi\) and \(\tilde{\varphi}\) is \(O(m)\), and therefore \(\alpha^{(2)}\) as given by Eq. (71) preserves its form in terms of \(\tilde{\varphi}\). However, it also follows from the relation (74) that at first order in \(m\) we have

\[
\cos(\varphi) \approx \cos(\tilde{\varphi}) - \frac{2m}{b} \sin(\tilde{\varphi}) + O(m^2).
\]

(75)

Hence, if we replace Eq. (75) into Eq. (71), it can be seen that new quadratic terms in \(m\) appear as functions of the variable \(\tilde{\varphi}\) which exactly cancel the apparent discrepant terms \(\delta\) present in \(\alpha^{(2)}_{\text{vac}}\). Therefore, when our expressions for the deflection angle are written in terms of the the angular coordinate \(\tilde{\varphi}\) of Ono et al. [80] the relation (72) is recovered.

C. Deflection angle in terms of the observable \(\theta_i\) and comparison with previous particular known expressions

Let us now compare between our finite distance results and the well known expressions from the literature [83, 84]. In order to do that we will assume the source is at infinite distance from the lens. In this case the deflection angle \(\alpha^{(1)}\) and \(\alpha^{(2)}\) take the following limits:

\[
\alpha^{(1)}_{\tilde{\varphi} \to \infty}(b, \varphi_R) := \lim_{\varphi_S \to 0} \alpha^{(1)}(b) = \frac{m}{b} \left( 1 - \cos(\varphi_R) \right) \left( \mu + \frac{1}{1 - \omega_v^2/\omega_e^2} \right) \times \left[ 1 + \frac{J_2 R^2}{4b^2} \left( 3 - 2 \cos(\varphi_R) - \cos(2\varphi_R) \right) \right].
\]

(76)
\[
\alpha_{S_\infty}^{(2)} (b, \varphi_R) := \lim_{\varphi_R \to 0} \alpha^{(2)} = \frac{m^2}{4b^2 (\omega_o^2 - \omega_e^2)^2} \left\{ \varphi_R (\omega_e^2 - \omega_o^2) \left( \omega_o^2 (4\beta - 8 - 8\mu - 3\nu) + 3\nu \omega_o^2 \right) \right.
\]
\[
- 4 \sin(\varphi_R) (\omega_e^2 (1 + \mu) - \omega_o^2 \mu)^2 + \frac{1}{2} \left[ \omega_o^4 \left( 4(\beta - 1 + \mu^2) - 3\nu \right) + \omega_e^4 (4\mu^2 - 3\nu) \right]
\]
\[
- 2\omega_o^2 \omega_e^2 (2\beta + 4\mu^2 - 3\nu) \left\{ \sin(2\varphi_R) + 8\omega_o^2 \omega_e^2 \sin(\varphi_R) \right\}.
\]

As seen from Fig. [3], the following relation follows between the angular position of the receiver \( \varphi_R \) and the angles \( \theta_I \) and \( \delta \theta \):
\[
\varphi_R = \pi - \theta_I + \delta \theta
\]
\[
= \pi - \theta_I + \alpha_{S_\infty}^{(1)} + \mathcal{O}(m^2).
\]  

Finally, by replacing this relation into Eqs. [76] and [77] we obtain

\[
\alpha_{S_\infty}^{(1)} (b, \theta_I) = \frac{m}{b} \left( 1 + \cos(\theta_I) \right) \left( \mu + \frac{1}{1 - \omega_o^2 / \omega_e^2} \right) \left[ 1 + \frac{J_2 R^2}{2b^2} \left( 2 + \cos(\theta_I) - \cos^2(\theta_I) \right) \right],
\]

\[
\alpha_{S_\infty}^{(2)} (b, \theta_I) = \frac{m^2}{8b^2 (\omega_o^2 - \omega_e^2)^2} \left[ 16\omega_o^2 \omega_e^2 \sin(\theta_I) + (\omega_o^2 - \omega_e^2)^2 \left( 2(\pi - \theta_I) + \sin(2\theta_I) \right) \right]
\]
\[
\times \left( \omega_o^2 (-4\beta + 8\mu + 3\nu + 8) - 3\nu \omega_o^2 + 8 \sin(2\theta_I) \omega_o^2 \right).
\]  

Equations [79], and [80] are the generalization of the relations [31] and [32] to the case of a PPN spacetime surrounded by a homogeneous plasma.

Alternatively, if we take into account the following relation between the impact parameter \( b \) and the coordinate \( r_o \) which follows from [77] and [78] and generalizes the relation [53]:
\[
\frac{1}{b} = \frac{1}{r_o \sin(\theta_I)} - \frac{m}{r_o^2 \sin(\theta_I)} \left( \mu + \frac{1}{1 - \omega_o^2 / \omega_e^2} \right) + \mathcal{O}(m \times J_2, m^2),
\]

then Eqs. [79], and [80] can be rewritten as:
\[
\alpha_{S_\infty}^{(1)} (r_o, \theta_I) = \frac{m}{r_o} \frac{1 + \cos(\theta_I)}{\sin(\theta_I)} \left( \mu + \frac{1}{1 - \omega_o^2 / \omega_e^2} \right)
\]
\[
\times \left[ 1 + \frac{J_2 R^2}{2r_o^2} \left( 2 + \cos(\theta_I) - \cos^2(\theta_I) \right) \right],
\]

\[
\alpha_{S_\infty}^{(2)} (r_o, \theta_I) = \frac{m^2}{r_o^2} \left[ \frac{1}{8(\omega_o^2 - \omega_e^2) \sin^2(\theta_I)} \left( 16\omega_o^2 \omega_e^2 \sin(\theta_I) \right)
\]
\[
+ (\omega_o^2 - \omega_e^2) \left( 2(\pi - \theta_I) + \sin(2\theta_I) \right) \right]
\]
\[
\times \left( \omega_o^2 (-4\beta + 8\mu + 3\nu + 8) - 3\nu \omega_o^2 + 8 \sin(2\theta_I) \omega_o^2 \right)
\]
\[
- \frac{1 + \cos(\theta_I)}{\sin(\theta_I)} \left( \mu + \frac{1}{1 - \omega_o^2 / \omega_e^2} \right)^2 \right].
\]  

In particular, it is easy to check that if \( \omega_e = 0 \), or alternatively \( \omega_e / \omega_o \ll 1 \) then the equations [79], and [80] or their alternative versions [82], [83] reduce to the known expressions [31] and [32] by Richter and Matzner [33]. The advantage of relations [82] and [83] is that they are written in terms of physical observables.

It is very nice to see how starting with an elegant, geometrical and compact expression for the deflection angle as given by Eq. [7] well known formulas like [31] and [32] can be recovered. Moreover, we have not only confirmed for the first time the success of the Gauss-Bonnet formula [74] to recover known results of the angle deflection at finite distances (giving us confidence in that definition), but also we have been able to generalize these results to
more general astrophysical environments. In particular, from \cite{22} we see that the correction produced by a homogeneous plasma in the deflection angle even considering finite distances, is given by a global factor $\mu + \frac{1}{1-\omega_e^2/\omega_o^2}$.

This peculiar characteristic however does not remain if we consider the second order terms, in which case the plasma contribution is much more complicated. In particular, neglecting the quadrupole moment, and considering the validity of the Einstein equations we obtain that reduces to

$$\alpha^{(1)}(r_o, \theta_f) = \frac{m}{r_o} \frac{1 + \cos(\theta_f)}{\sin(\theta_f)} \left(1 + \frac{1}{1 - \omega_e^2/\omega_o^2}\right). \quad (84)$$

This expression can be compared with a similar relation obtained for the first time by Bisnovatyi-Kogan and Tsupko \cite{28, 29}, which reads

$$\alpha^{(1)} = \frac{2m}{b} \left(1 + \frac{1}{1 - \omega_e^2/\omega_o^2}\right). \quad (85)$$

The formula \cite{33} was obtained under the more common assumption of infinite distances. The advantage of \cite{34}, is that it is written in terms of the observable quantity $\theta_f$ and the coordinate distance $r_o$.

V. INHOMOGENEOUS PLASMA MEDIUM

In this section, we focus on finite distance corrections to the deflection angle for light rays propagating in a non-uniform plasma. In this case, the steps that lead to the final expression for the deflection angle are basically the same which we applied in our previous article \cite{17}, therefore we skip the intermediate computations and only present the essential steps.

Let us consider an asymptotically flat and spherically symmetric gravitational lens surrounded by an inhomogeneous plasma whose electron number density $N(r)$ is a decreasing function of the radial coordinate $r$, and such that its radial derivative $N'(r)$ is also decreasing and smaller than $N(r)$. In isotropic coordinates, the components of the metric in the physical spacetime are codified in the following expressions:

$$A(r) = 1 - \mu h_{00}(r), \quad B(r) = 1 + \gamma h_{rr}(r), \quad C(r) = r^2 B(r). \quad (86)$$

The refractive index reads,

$$n(r) = \sqrt{1 - \frac{\omega_e^2(1 - \mu h_{00}(r))}{\omega_o^2}}, \quad (87)$$

where as before, $\omega_o$ is related to the detected frequency by a receiver by $\omega_o = \omega_\infty \sqrt{A(r_o)}$.

The associated optical metric is given by,

$$\mathrm{d}s^2 = \left(\frac{1 + \gamma h_{rr}(\omega_o^2 - \omega_e^2 + \mu \omega_e^2 h_{00})}{\omega_o^2(1 - \mu h_{00})}\right) \left(\mathrm{d}r^2 + r^2 \mathrm{d}\phi^2\right). \quad (88)$$

In general, the change in the deflection angle due to the presence of the refractive index is smaller than the main part due to the purely gravitational effect. We will assume as in \cite{24, 47} that the deflection angle is small and therefore as a first approximation the path followed by the light ray can be taken as the straight line geodesic of the flat euclidean space. We also neglect all higher order terms of the form $O(N'^2, \mu N', \mu N''', N''', \gamma N'^2, \gamma N''').$

Working at linear order in $\mu$ and $\gamma$, and following the same steps explained in detail in \cite{17}, we obtain for $K\ell S$ (expressed in terms of the detected frequency $\omega_o$ by the receiver):

$$K\ell S = \frac{1}{2} \int K_{\epsilon} r_N' \left(\frac{\omega_o^2}{\omega_o^2 - \omega_e^2} - \frac{\omega_e^2}{\omega_o^2 - \omega_e^2}\right) \mu - (r h_{rr})' \gamma \mathrm{d}r \mathrm{d}\varphi. \quad (89)$$

By inserting this expression into Eq. (17), we find that the deflection angle in this approximation is given by

$$\alpha \approx - \lim_{R \to \infty} \int_{\mathcal{D}_r} K\ell S = - \int_{\varphi_S}^{\varphi_S} \int_{b/\sin \varphi}^{\infty} \frac{1}{2} \left(\frac{K_{\epsilon} r_N'}{\omega_o^2 - \omega_e^2} - \frac{\omega_e^2}{\omega_o^2 - \omega_e^2}\right) \mu - (r h_{rr})' \gamma \mathrm{d}r \mathrm{d}\varphi. \quad (90)$$

Using integration by parts in the first two terms of the radial integral and neglecting terms of order $O(N'^2, h_{00} N')$, we obtain the final expression:

$$\alpha \approx \int_{\varphi_S}^{\varphi_S} \frac{1}{2} \left[ K_{\epsilon} (r N') - \frac{\omega_e^2}{\omega_o^2 - \omega_e^2} \mu - (r h_{rr})' \gamma \right] \mathrm{d}r \mathrm{d}\varphi. \quad (91)$$

This equation gives us a general formula to compute the deflection angle in a spherically symmetric spacetime when an inhomogeneous plasma medium is present taking into account finite distance corrections. Note that this expression can also be derived with the technique used by Bisnovatyi-Kogan and Tsupko in \cite{29}, where they find the deflection angle considering infinite distances by solving the Hamilton equations perturbatively for a not necessarily spherically symmetric metric of the form $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$.

By assuming the condition $\omega_e/\omega_o \ll 1$ and motivated by the decomposition presented by Bisnovatyi-Kogan and Tsupko in \cite{29} for the deflection angle, Eq. (91) can be
decomposed in terms of the form:

\[ \alpha = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \]  

(92)

where,

\[ \alpha_1 = -\frac{1}{2} \int_{\varphi_0}^{\varphi_1} r_{\varphi} \left( h'_{rr}(r_\varphi) + h'_{00}(r_\varphi) \right) d\varphi, \]  

(93)

\[ \alpha_2 = -\frac{\mu}{2\omega_0^2} \int_{\varphi_0}^{\varphi_1} r_{\varphi} h'_{00}(r_\varphi) \omega_0^2(r_\varphi) d\varphi, \]  

(94)

\[ \alpha_3 = \frac{K e}{2\omega_0^2} \int_{\varphi_0}^{\varphi_1} r_{\varphi} N'(r_\varphi) d\varphi, \]  

(95)

\[ \alpha_4 = \frac{K e}{2\omega_0^2} \int_{\varphi_0}^{\varphi_1} r_{\varphi} N'(r_\varphi) \omega_0^2(r_\varphi) d\varphi, \]  

(96)

and \( r_\varphi = b / \sin(\varphi) \). These expressions are the finite distance counterparts of the expressions found by [29]. In particular the first term \( \alpha_1 \) is the pure gravitational deflection angle; the second term \( \alpha_2 \) is a correction of the first due to the presence of the plasma, the third term is the pure refractive angle (present even without a gravitational field), and the last term is a correction to the third term. As explained by Bisnovatyi-Kogan and Tsukpo in [29] in general astrophysical situations the first and the third terms in (92) make the main contribution to the deflection angle, where in general \( \alpha_3 < \alpha_1 \).

### A. Diverging lensing due solely to an inhomogeneous plasma medium

An interesting example arises when we consider the limit in which gravity does not affect the deflection at all, \( m \to 0 \). In this case, spacetime is Minkowski and the perturbed components of the metric \( h_{ij} \) vanish, which causes Eqs. (93) and (94) to vanish. This leaves only \( \alpha_3 \) and \( \alpha_4 \) to contribute to the deflection angle. These contributions are in the opposite sense to the gravitational deflection from \( \alpha_1 \) and \( \alpha_2 \), and therefore the lensing effect due to inhomogeneous plasma in the absence of gravitational is diverging, rather than the usual converging behavior of a gravitational lens [40].

Lensing due to plasma is relevant to the observation of radio sources through the intervening interstellar medium (ISM), which can contain inhomogeneities in the electron density. These density perturbations can act as diverging lenses, causing frequency-dependent dimming of radio sources observed through the ISM. A number of astrophysical phenomena are associated with this type of lensing, including extreme scattering events [97–100] as well as pulsar scintillation [101,102]. Furthermore, it has recently been suggested that plasma lensing may play a role in the mechanism responsible for generating fast radio bursts [103].

In terms of the derivation presented here, the inhomogeneous plasma case is particularly interesting due to the fact that it depends only on the Minkowski metric. In this case, the effect of gravitation is totally removed from the problem, which ultimately illustrates the elegant utility of the Gauss-Bonnet method when coupled to a Riemann optical metric representation.

### B. Schwarzschild spacetime with plasma density profile of the form \( N(r) = N_0 r^{-h} \)

Now we will apply this general result to the case of Schwarzschild spacetime with the density profile

\[ N(r) = N_0 r^{-h}, \quad h > 0. \]  

(97)

For this, we make the following identification,

\[ \gamma = \mu = m, \quad h_{00} = h_{rr} = \frac{2}{r}. \]  

(98)

For completeness we write the expression for each individual term [93], [94], [95], [96] but, as discussed, the main contribution to the deflection angle is given by \( \alpha_1 \) and \( \alpha_3 \). Explicitly, these terms read:

\[ \alpha_1 = \frac{2m}{b} \left( \cos(\varphi_S) - \cos(\varphi_R) \right), \]  

(99)

\[ \alpha_2 = \frac{mK eN_0}{\omega_0^2 b^{h+1}} \left[ \cos(\varphi_S) 2F_1 \left( \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \cos^2(\varphi_S) \right) - \cos(\varphi_R) 2F_1 \left( \frac{1}{2}, \frac{h}{2}; \frac{3}{2}; \cos^2(\varphi_R) \right) \right], \]  

(100)

\[ \alpha_3 = \frac{K eN_0 h}{2\omega_0^2 b^h} \left[ \cos(\varphi_S) 2F_1 \left( \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \cos^2(\varphi_S) \right) - \cos(\varphi_R) 2F_1 \left( \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \cos^2(\varphi_R) \right) \right], \]  

(101)

\[ \alpha_4 = \frac{K eN_0^2 h}{2\omega_0^2 b^{3h}} \left[ \cos(\varphi_S) 2F_1 \left( \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \cos^2(\varphi_S) \right) - \cos(\varphi_R) 2F_1 \left( \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \cos^2(\varphi_R) \right) \right], \]  

(102)

with \( 2F_1(a; b; c; x) \) the ordinary hypergeometric function [104].
These analytical and closed expressions generalize the known equivalent formulas for the infinite distance case. In particular we can also recover the expressions for the infinite distance case by taking the limit of the previous expressions in the limit of \( \varphi_S \to 0 \) and \( \varphi_R \to \pi + \mathcal{O}(m) \).

In this case, they reduce to:

\[
a_1^\infty = \frac{4m}{b},
\]

\[
a_2^\infty = \frac{\sqrt{\pi}mK_eN_o\Gamma\left(\frac{b}{2} + 1\right)}{b^{h+1}\omega_o^2\Gamma\left(\frac{b+3}{2}\right)},
\]

\[
a_3^\infty = -\frac{\sqrt{\pi}K_eN_o\Gamma\left(\frac{b+1}{2}\right)}{b^2\omega_o^3\Gamma\left(\frac{b}{2}\right)},
\]

\[
a_4^\infty = -\frac{\sqrt{\pi}K_e^2N_o^2b^{-2b}\Gamma\left(h + \frac{1}{2}\right)}{2\omega_o^4\Gamma(h)}.
\]

Now we analyze the difference in magnitude between these expressions and those obtained considering infinite distances. In particular, we consider that the source is at infinite distance from the lens but that the receiver is at a finite (but large) distance. Hence, we consider a Taylor expansion of the previous expressions around \( \varphi_R = \pi \). That is, we take \( \phi_R = \pi - \delta \phi \) and \( \varphi_S = 0 \) where \( \delta \phi \approx \sin(\delta \phi) = b/r_o \).

The difference between the main contributions to the deflection angle given by \( \alpha_1 \) and \( \alpha_3 \) and their respective values \( \alpha_1^\infty \) considering infinite distances defined by

\[
\delta \alpha_i = \alpha_i - \alpha_i^\infty
\]

reads:

\[
\delta \alpha_1 = \frac{m}{b}(\delta \varphi)^2 + \mathcal{O}(\delta \varphi^3), \quad (111)
\]

\[
\delta \alpha_3 = \frac{3}{7} \frac{\omega_o^2(b)}{\omega_o^2} (\delta \varphi)^7 + \mathcal{O}(\delta \varphi^9), \quad (112)
\]

where \( \Gamma(x) \) is the Gamma function. Expression (103) was found by first time by Bisnovatyi-Kogan and Tsupko in [29]. We will study in the next subsection the particular plasma density profile \( h = 6 \) which describes the plasma around the corona of our Sun.

1. Particular case \( h = 6 \)

As an application, we ask how relevant could be these finite distance contributions in the near future observation for the deflection angle of a light ray passing through the plasma region of the corona of our Sun. For the particular choice of \( h = 6 \) we have a good model of the electronic density profile of solar corona [105]. Then the expressions (100), (101) and (102), written in terms of standard functions, are given by,

\[
\alpha_2 = \frac{mK_eN_o}{b\omega_o^2}\left[\cos(\varphi_S) - \cos(\varphi_R) + \cos^3(\varphi_R) - \cos^3(\varphi_S) + \frac{21}{35}\left(\cos^5(\varphi_S) - \cos^5(\varphi_R)\right) + \frac{1}{7}\left(\cos^7(\varphi_R) - \cos^7(\varphi_S)\right)\right],
\]

\[
\alpha_3 = \frac{K_eN_o}{640b\omega_o^5}\left[60(\varphi_S - \varphi_R) + 45\left(\sin(2\varphi_R) - \sin(2\varphi_S)\right) + 9\left(\sin(4\varphi_S) - \sin(4\varphi_R)\right) + \sin(6\varphi_R) - \sin(6\varphi_S)\right],
\]

\[
\alpha_4 = \frac{K_e^2N_o^2}{40960b^2\omega_o^7}\left[27720(\varphi_S - \varphi_R) + 23760\left(\sin(2\varphi_R) - \sin(2\varphi_S)\right) + 7425\left(\sin(4\varphi_S) - \sin(4\varphi_R)\right)
+ 2200\left(\sin(6\varphi_R) - \sin(6\varphi_S)\right) + 495\left(\sin(8\varphi_S) - \sin(8\varphi_R)\right) + 72\left(\sin(10\varphi_R) - \sin(10\varphi_S)\right)
\right.
\]

\[
+ \left.5\left(\sin(12\varphi_S) - \sin(12\varphi_R)\right)\right].
\]
we conclude that these last corrections (at the difference of the finite vacuum corrections) should not be necessary to take into account for the near future radio-frequency antennas projects. We must wait for a more advanced generation of radio-telescopes to observe these differences in the case of the Sun. In Appendix [13] we present analytic expression for a plasma density profile with $h = 2$.

VI. FINAL REMARKS

In conclusion, our calculations achieve three main goals. First, by carefully constructing a finite quadrilateral region to apply the Gibbons-Werner method, we have resolved an apparent contradiction in the literature. Second, our results are derived in terms of observable quantities that facilitate comparison with previous, well-studied cases in the literature. Third, by making use of the Gibbons-Werner approach of coupling a Riemann optical metric to the Gauss-Bonnet method, we have expanded on well-known cases in the literature. For example, by including the effects of the PPN expansion and a possible quadrupole moment into the case of a homogeneous plasma in a gravitational field, as well as including the corrections arising from consideration of the finite distance between source, lens and observer. This work demonstrates the utility and elegance of the Gauss-Bonnet theorem and the Gibbons-Werner method and their relevance for all forms of lensing - both gravitational (converging) and plasma (diverging).

Acknowledgments

E.G. acknowledges support from CONICET and SeCyT-UNC. A.R. acknowledges support from Dr. Samar Safi-Harb at the University of Manitoba.

Appendix A

1. Explicit comparison between formulas [7], [11] and [12]

In this Appendix we will use a particular example to illustrate how the alternative expressions for the deflection angle given by Eqs. [7], [11] and [12] give the same results. Let us focus on a Schwarzschild metric written in isotropic coordinates:

$$ds^2 = -\left(1 - \frac{m}{2r}\right)^2 dt^2 + \left(1 + \frac{m}{2r}\right)^4 \left[dr^2 + r^2 (d\psi^2 + \sin^2(\psi) d\varphi^2)\right].$$

(A1)

Let us focus on the plane defined by $\psi = \pi/2$. We shall calculate the deflection angle to second order precision in the mass $m$. However, for the moment we will write the exact relationships. The associated optical metric is given by:

$$d\sigma^2 = \frac{(1 + \frac{m}{2r})^6}{(1 - \frac{m}{2r})^2} \left(dr^2 + r^2 d\varphi^2\right):$$

(A2)

and the associated Gaussian curvature reads:

$$\mathcal{K} = -\frac{128mr^3(4r^2 - 2rm + m^2)}{(2r + m)^8}.$$  

(A3)

The surface element is

$$dS = \sqrt{\det(g^\text{opt})} dr d\varphi = \frac{(2r + m)^6}{16r^5(2r - m)^2} dr d\varphi.$$  

(A4)

Therefore at second order in the mass we obtain

$$KdS = \left(-\frac{2m}{r^2} + \frac{m^2}{r^3}\right) dr d\varphi + \mathcal{O}(m^3),$$

(A5)

which can be rewritten in terms of the variable $u = 1/r$ as

$$KdS = \left(2m - m^2 u\right) du d\varphi + \mathcal{O}(m^3).$$

(A6)

Now, we will use equation [12] to compute the deflection angle. To do that, we must integrate over a region $D_r$ bounded by the radial curves $\gamma_R, \gamma_S$, the geodesic $\tilde{\gamma}_t$ and the arc of circle segment $\gamma_C$. $\gamma_R$ and $\gamma_S$ are given by $\varphi = \varphi_R$ and $\varphi = \varphi_S$, respectively. In terms of the coordinate $u, \gamma_C$ is defined by $u = 1/r_C = \text{constant,}$ and finally the spatial geodesic $\tilde{\gamma}_t$ describing the orbit of a light ray between $S$ and $R$ is given by:

$$u_t = \frac{\sin(\varphi)}{b} + \frac{2m(1 - \cos(\varphi))}{b^2} + \frac{15m^2 \cos(\varphi)(\tan(\varphi) - \varphi)}{b^3} + \mathcal{O}(m^3).$$

(A7)

This expression for $u_t$ follows by solving Eq. (A5) at second order in the mass,

$$\left(\frac{du_t}{d\varphi}\right)^2 = \frac{1}{b^2} - u_t^2 + \frac{4nu_t}{b^2} + \frac{15m^2 u_t^2}{2b^2},$$

(A8)

with the asymptotic condition,

$$\lim_{\varphi\to 0} u_t(\varphi) = 0.$$  

(A9)

Note however, that in order to compute the integral of the Gaussian curvature we only need to consider the first two terms in (A1) because $KdS$ is already order $m$. Hence we obtain for the first term of (12).
We must also compute the other two terms of (12). The last term is computed in the Euclidean metric and it is simply given by Eq.(6). In order to compute the second term we need first to compute the geodesic curvature of \( \tilde{\gamma}_C \) defined by \( r = r_C = \text{constant} \). The exact value of this curvature in the optical metric (A2) is given by

\[
\tilde{\kappa}_C = \frac{4r_C(4r_C - 2m) + m^2}{(2r_C + m)^4}
\]  

(A11)

and therefore we obtain for the second term of (12)

\[
- \int_{\tilde{\gamma}_C(\varphi - R)} \tilde{\kappa} d\tilde{\sigma} = \int_{\varphi_R}^{\varphi_S} \sqrt{g_{\varphi \varphi}^{\text{opt}}} d\varphi = \int_{\varphi_R}^{\varphi_S} \left( 1 - \frac{2m^2}{2r_C^2} \right) d\varphi + O(m^3)
\]

(A12)

Taking cognizance of Eqs.(9), (A10) and (A12), and replacing these expressions into the formula (12) we obtain:

\[
\alpha = \frac{2m}{b} \left[ \cos(\varphi_S) - \cos(\varphi_R) \right] + \frac{m^2}{8b^2} \left[ 30(\varphi_R - \varphi_S) + \sin(2\varphi_R) - \sin(2\varphi_S) + 32(\sin(\varphi_S) - \sin(\varphi_R)) + \frac{4b^2}{r_C^2} (\varphi_R - \varphi_S) \right] + O(m^3),
\]

(A13)

which agrees with our previous expression given by (7) and (11) obtained directly using Eq.(7).

Now we will repeat the computation, but using the expression given by (11). To do that we must compute the sum of angles in the regions \( D_r \) on the Euclidean space, and also in the region \( \tilde{D}_r \) of the optical space.

In the Euclidean space, it is easy to see that the sum of the interior angles is:

\[
\sum_i \epsilon_i = 2\pi + \varphi_R - \varphi_S.
\]

(A14)

The angle \( \tilde{\epsilon}_1 \) can be computed from the following relation (see Fig.2)

\[
\tan \tilde{\epsilon}_1 = \left[ \sqrt{g_{\varphi \varphi}^{\text{opt}}} \right]_{\tilde{\gamma}_t(\varphi_R)} = - \left[ \left. r \frac{d\varphi}{du} \right|_{\tilde{\gamma}_t(\varphi_R)} \right] = \frac{u(\varphi_R)}{2 \cos^2(\varphi_R)} - u(\varphi_R),
\]

(A16)

and similarly, \( \tilde{\epsilon}_2 = \pi - \tilde{\chi}_2 \) with \( \tilde{\chi}_2 \) the supplementary angle to \( \tilde{\epsilon}_2 \) which satisfies:

\[
\tan \tilde{\chi}_2 = \left[ \sqrt{g_{\varphi \varphi}^{\text{opt}}} \right]_{\tilde{\gamma}_t(\varphi_S)} = - \left[ \left. r \frac{d\varphi}{du} \right|_{\tilde{\gamma}_t(\varphi_S)} \right] = \frac{u(\varphi_S)}{2 \cos^2(\varphi_S)} - u(\varphi_S).
\]

(A17)

Using (A7), we obtain that

\[
\tan \tilde{\epsilon}_1 = u(\varphi_R) \frac{d\varphi}{du} \bigg|_{\varphi = \varphi_R} = \tan(\varphi_R) - \frac{2m}{b} \frac{1 - \cos(\varphi_R)}{\cos^2(\varphi_R)} + \frac{m^2}{8b^2} \frac{15(2\varphi_R - 2\varphi_R) \cos(\varphi_R) - 16 \sin(2\varphi_R) + 32 \sin(\varphi_R)}{\cos^3(\varphi_R)}
\]

(A18)

Hence, to the considered order we obtain:

\[
\tilde{\epsilon}_1 = \varphi_R - \frac{2m}{b} \left( 1 - \cos(\varphi_R) \right) - \frac{m^2}{8b^2} (30\varphi_R + \sin(2\varphi_R) - 32 \sin(\varphi_R)) + O(m^3).
\]

(A19)

Repeating for \( \tilde{\chi}_2 \), which is obtained from (A17) we find:

\[
\tilde{\chi}_2 = \varphi_S - \frac{2m}{b} \left( 1 - \cos(\varphi_S) \right) - \frac{m^2}{8b^2} (30\varphi_S + \sin(2\varphi_S) - 32 \sin(\varphi_S)) + O(m^3).
\]

(A20)
Therefore, we arrive at
\[ \sum_i \hat{\alpha}_i = 2\pi + \hat{\alpha}_1 - \hat{\chi}_2 \]
\[ = 2\pi + \varphi_R - \varphi_S + \frac{2m}{b}(\cos(\varphi_R) - \cos(\varphi_S)) \]
\[ + \frac{m^2}{8b^2} \left[ 30(\varphi_S - \varphi_R) + \sin(2\varphi_S) - \sin(2\varphi_R) \right. \]
\[ + 32(\sin(\varphi_R) - \sin(\varphi_S)) \right]. \quad \text{(A21)} \]
Finally, by replacing (A14) and (A21) into the angular expression reads
\[ \Psi_R = \Psi_S + \varphi_RS. \quad \text{(A22)} \]

On the other hand, the inner angles \( \hat{\alpha}_1 \) and \( \hat{\alpha}_2 \) used in this work are related to the angles \( \Psi_S \) and \( \Psi_R \) by:
\[ \hat{\alpha}_1 = \pi - \Psi_R, \quad \text{(A23)} \]
\[ \hat{\alpha}_2 = \Psi_S. \quad \text{(A24)} \]

Then, taking into account the relations (A14) and (A15), it is easy to see that the definition (11) which was based in the sum of the inner angles of finite quadrilateral regions agrees with the expression as \( \text{(A22)} \) (which does not make mention to any region).

**Appendix B**

1. **Particular case of inhomogeneous plasma density profile \( N(r) = N_0 / r^2 \)**

If \( h = 2 \), Eqs. (100), (101) and (102) can be written explicitly in terms of standard trigonometric functions:

\[ \alpha_2 = \frac{mK_eN_o}{3b^4\omega_o^2} \left[ 3 \left( \cos(\varphi_S) - \cos(\varphi_R) \right) - \cos^3(\varphi_S) + \cos^3(\varphi_R) \right], \quad \text{(B1)} \]
\[ \alpha_3 = \frac{K_eN_o}{2b^2\omega_o^2} \left[ \varphi_S - \varphi_R + \cos(\varphi_R) \sin(\varphi_R) - \cos(\varphi_S) \sin(\varphi_S) \right], \quad \text{(B2)} \]
\[ \alpha_4 = \frac{K_e^2N_o^2}{32b^4\omega_o^4} \left[ 12(\varphi_S - \varphi_R) + 8 \left( \sin(2\varphi_R) - \sin(2\varphi_S) \right) + \sin(4\varphi_S) - \sin(4\varphi_R) \right]. \quad \text{(B3)} \]

The difference between each of the quantities \( \alpha_i \) and their respective values \( \alpha_i^\infty \) considering infinite distances read
\[ \delta\alpha_1 = -\frac{m}{b}(\delta\varphi)^2 + \mathcal{O}(\delta\varphi^3), \quad \text{(B4)} \]
\[ \delta\alpha_2 = -\frac{m}{4b} \frac{\omega_o^2(b)}{\omega_o^2} (\delta\varphi)^4 + \mathcal{O}(\delta\varphi^6), \quad \text{(B5)} \]
\[ \delta\alpha_3 = \frac{1}{3} \frac{\omega_o^2(b)}{\omega_o^2} (\delta\varphi)^3 + \mathcal{O}(\delta\varphi^5), \quad \text{(B6)} \]

\[ \delta\alpha_4 = \frac{1}{5} \frac{\omega_o^4(b)}{\omega_o^4} (\delta\varphi)^5 + \mathcal{O}(\delta\varphi^7), \quad \text{(B7)} \]

where
\[ \omega_o^2(b) = \frac{K_eN_o}{b^2} \quad \text{(B8)} \]
is the value of the square of the plasma frequency at \( r = b \).

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