Vacuum stress-energy tensor of a massive scalar field in a wormhole spacetime

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The vacuum average value of the stress-energy tensor of a massive scalar field with non-minimal coupling $\xi$ to the curvature on the short-throat flat-space wormhole background is calculated. The final analysis is made numerically. It was shown that the energy-momentum tensor does not violate the null energy condition near the throat. Therefore, the vacuum polarization cannot self-consistently support the wormhole.

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I. INTRODUCTION

Traversable wormholes have been introduced into physical realm by Morris and Thorne \cite{MorrisThorne1988} in 1988. At the same time they realized that a matter threading a wormhole’s throat should possess exotic properties, namely it should have a negative pressure and violate the null energy condition (NEC). Later on, this result was generalized for any traversable wormholes, both static and non-static \cite{MorrisThorneYurtsever1998}. As is known, the classical matter does satisfy the usual energy conditions, hence traversable wormholes cannot arise as solutions of classical relativity and matter. Already in 1989 Morris, Thorne, and Yurtsever \cite{MorrisThorneYurtsever1998} supposed that quantized fields could play a role of the exotic matter maintaining wormholes. Their reasons were founded on the important fact stating that quantum field theory may have states with negative energy density, thus violating the weak energy condition \cite{BirrellDavies1984} (see, also, \cite{FrolovZelnikov1985}).

In the absence of the complete theory of quantum gravity, the semi-classical approach gives the more natural way to include quantized fields in the theory of gravity. Various wormhole solutions in semi-classical gravity have been considered in the literature. For instance, semi-classical wormholes were found in the framework of the Frolov-Zelnikov approximation for $\langle T$\textsubscript{\mu\nu}\rangle\text{ren} \cite{FrolovZelnikov1985}. Analytical approximations of the stress-energy tensor of quantized fields in static and spherically symmetric wormhole space-times were also explored in Refs. \cite{Barnes1990}. Some arguments in favor of the possibility of existence of semi-classical wormholes have been given by Khatsymovsky \cite{Khatsymovsky2005}. However, the first self-consistent wormhole solution coupled to a quantum scalar field was obtained in Ref. \cite{Sushkov2005}. The ground state of a massive scalar field with a non-conformal coupling on a short-throat flat-space wormhole background was computed in Ref. \cite{Sushkov2005}, by using a zeta renormalization approach. The latter wormhole model, which was further used in the context of the Casimir effect \cite{Sushkov2005}, was constructed by excising spherical regions from two identical copies of Minkowski space-time, and finally surgically grafting the boundaries (A more realistic geometry was considered in Ref. \cite{Sushkov2005}). A in a series of works \cite{Sushkov2005, Sushkov2005} various aspects of the graviton one loop contribution to a classical energy in a wormhole background have been analyzed. The latter contribution was evaluated through a variational approach with Gaussian trial wave functionals, and the divergences were treated with a zeta function regularization. In particular, the finite one loop energy was considered as a self-consistent source for a traversable wormhole.

Note that up to now no one has succeeded in exact calculations of vacuum expectation values of the stress-energy tensor of quantized fields on the wormhole background. The reason for this state of affairs consists in considerable mathematical difficulties which one faces with trying to quantize a physical field on the wormhole background. To overcome these difficulties, in this work we will consider a simple model of wormhole space-time given in \cite{Sushkov2005}: the short-throat flat-space wormhole. The model represents two identical copies of Minkowski space with excised from each copy spherical regions, and with boundaries of those regions are to be identified. The space-time of this model is everywhere flat except a two-dimensional singular spherical surface. Due to this fact it turns out to be possible to construct the complete set of wave modes of the massive scalar field and calculate the stress energy tensor.

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The organization of the paper is as follows. In Sec. [III] we describe a space-time of wormhole in the short-throat flat-space approximation. In Sec. [IV] we analyze the solution of equation of motion for massive scalar field and obtain Euclidean Green function. Sec. [V] devoted to analysis of the vacuum expectation value of the square of field from analytical and numerical point of views. In Sec. [VI] we obtain close formulas for numerical analysis the stress-energy tensor and in Sec. [VII] discuss our results.

We use units $\hbar = c = G = 1$. The signature of the space-time, the sign of the Riemann and Ricci tensors, is the same as in the book by Hawking and Ellis [10].

II. A TRAVERSABLE WORMHOLE: THE SHORT-THROAT FLAT-SPACE APPROXIMATION

In this section we briefly consider a simple model of a traversable wormhole (see Ref. [12]). Assume that the throat of the wormhole is very short, and that curvature in the regions outside the mouth of the wormhole is relatively weak. An idealized model of such a wormhole can be constructed in the following manner: Consider two copies of Minkowski space, $M_+$ and $M_-$, with the spherical coordinates $(t, r_{\pm}, \theta_{\pm}, \varphi_{\pm})$ [Notice: $M_+$ and $M_-$ have a common time coordinate $t$. One may interpret this fact as the identification $t_+ \leftrightarrow t_-$.] excise from each copy the spherical region $r_{\pm} < a$, where $a$ is a radius of sphere; and then identify the boundaries of those regions: $(t, a, \theta_+, \varphi_+) \leftrightarrow (t, a, \theta_-, \varphi_-)$. The Riemann tensor for this model is identically zero everywhere except at the wormhole mouths where the identification procedure takes place. Generically, there will be an infinitesimally thin layer of exotic matter present at the mouth of the wormhole.

Such an idealized geometry can be described by the following metric

$$ds^2 = -dt^2 + d\rho^2 + r^2(\rho) (d\theta^2 + \sin^2 \theta d\varphi^2),$$  

where $\rho$ is a proper radial distance, $-\infty < \rho < \infty$, and the shape function $r(\rho)$ is

$$r(\rho) = |\rho| + a.$$  

It is easily to see that in two regions $\mathcal{R}_+: \rho > 0$ and $\mathcal{R}_-: \rho < 0$ separately, one can introduce a new radial coordinate $r_{\pm} = \pm \rho + a$ and rewrite the metric (1) in the usual spherical coordinates:

$$ds^2 = -dt^2 + dr_{\pm}^2 + r_{\pm}^2 (d\theta^2 + \sin^2 \theta d\varphi^2),$$  

This form of the metric explicitly indicates that the regions $\mathcal{R}_+: \rho > 0$ and $\mathcal{R}_-: \rho < 0$ are flat. However, note that such the change of coordinates $r = |\rho| + a$ is not global, because it is ill defined at the throat $\rho = 0$. Hence, as was expected, the space-time is curved at the wormhole throat. To illustrate this we explicitly calculate the scalar curvature $R(\rho)$ in the metric (1):

$$R(\rho) = -8a^{-1}\delta(\rho).$$  

III. GREEN FUNCTION

Let us discuss a massive scalar field $\phi$ with non-minimal coupling to curvature. The scalar field equation of motion has the following form

$$(\Box - \xi R - m^2)\phi = 0.$$  

A corresponding equation for the Euclidean Green function reads

$$(\Box - m^2 - \xi R)G_E(x; \tilde{x}) = -\frac{\delta^{(4)}(x, \tilde{x})}{\sqrt{g}},$$  

where $\tau = -it$ is the Euclidean time. Due to spherical symmetry one can represent $G_E(x; \tilde{x})$ as follows

$$G(x; \tilde{x}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} Y_{l,m}(\Omega)Y_{l,m}(\tilde{\Omega}) \int \frac{d\omega}{2\pi} e^{i\omega \Delta \tau} g_{\omega l}(\rho, \tilde{\rho}),$$  

Here $g_{\omega l}(\rho, \tilde{\rho})$ is a radial Green function obeying the equation

$$g''_{\omega l} + \frac{2r}{r} g'_{\omega l} - \left(\lambda^2 + \frac{l(l+1)}{r^2} + \xi R\right) g_{\omega l} = -\frac{\delta(\rho - \tilde{\rho})}{r^2}.$$  

where $\lambda^2 = \omega^2 + m^2$ and a prime means the derivative with respect to $\rho$. A solution of Eq. (8) can be represented as

$$g_{\omega l}(\rho, \tilde{\rho}) = \theta(\rho - \tilde{\rho})\phi_+^{\omega}(\rho)\phi_-^{\omega}(\tilde{\rho}) + \theta(\tilde{\rho} - \rho)\phi_+^{\omega}(\tilde{\rho})\phi_-^{\omega}(\rho),$$

where $\phi_\pm(\rho)$ are two independent solutions of the radial equation of motion:

$$\phi'' + \frac{2r'}{r}\phi' - \left(\lambda^2 + \frac{l(l+1)}{r^2} + \xi R\right)\phi = 0.$$  \hfill (10)

In general, Eq. (10) cannot be solved in an explicit form. However, this is possible in the particular case $r(\rho) = |\rho| + a$. In this case $R = -8\delta(\rho)/a$ and Eq. (10) yields

$$\phi'' + \frac{2}{\rho \pm a}\phi' - \left(\lambda^2 + \frac{l(l+1)}{(\rho \pm a)^2} - \frac{8\xi}{a}\delta(\rho)\right)\phi = 0.$$  \hfill (11)

Note that Eq. (11) is an ordinary second-order differential equation with a delta-like coefficient, $\delta(\rho)$. A common treatment of such equations is to solve them separately in regions with $\rho \neq 0$ and then match the obtained solutions and their first derivatives at $\rho = 0$. In the regions $\rho \neq 0$ a general solution of Eq. (11) reads

$$\phi(\rho) = \sqrt{\frac{\pi}{2r}}\left[C_1 I_\nu(\lambda r) + C_2 K_\nu(\lambda r)\right],$$

where $C_1$ and $C_2$ are constants of integration and $\nu = l + 1/2$. Integrating Eq. (11) around $\rho = 0$ gives the following matching conditions at the throat

$$\phi(+0) - \phi(-0) = 0,$$

$$\phi'(+0) - \phi'(-0) = -\frac{8\xi}{a}\phi(+0).$$  \hfill (13)

Using the formula (12) and the relations (13) one may define constants of integration $C_1$ and $C_2$ and then construct the radial Green function $g(\rho, \tilde{\rho})$ given by Eq. (8). As the result, one obtains (see for details Ref. [24])

$$g_{\omega l}(\rho, \tilde{\rho}) = g_{\omega l}^M(\rho, \tilde{\rho}) - \frac{\lambda a(I_\nu K_\nu' + I_\nu' K_\nu + (8\xi - 1)I_\nu K_\nu')}{2\lambda a K_\nu K_\nu' + (8\xi - 1)K_\nu^2}|_{\lambda a} \frac{K_\nu(\lambda r)K_\nu(\lambda \tilde{r})}{\sqrt{r\tilde{r}}},$$  \hfill (14)

if $\rho$ and $\tilde{\rho}$ have the same signs, and

$$g_{\omega l}(\rho, \tilde{\rho}) = -\frac{1}{2\lambda a K_\nu K_\nu' + (8\xi - 1)K_\nu^2}|_{\lambda a} \frac{K_\nu(\lambda r)K_\nu(\lambda \tilde{r})}{\sqrt{r\tilde{r}}},$$  \hfill (15)

if $\rho$ and $\tilde{\rho}$ have different signs. Here

$$g_{\omega l}^M(\rho, \tilde{\rho}) = \frac{K_\nu(\lambda r)I_\nu(\lambda \tilde{r})}{\sqrt{r\tilde{r}}}$$

is the radial Green function of the Minkowski spacetime.

### IV. VACUUM POLARIZATION $\langle \phi^2 \rangle$

A vacuum of any quantized physical field is polarized in curved space-time. A vacuum polarization $\langle \phi^2 \rangle$ of the massive scalar field $\phi$ in a wormhole space-time has been discussed in Refs. [4, 5] in the WKB approximation. In this section we represent exact calculations for $\langle \phi^2 \rangle$ in a wormhole space-time with the metric (1). The renormalized expression for the vacuum polarization is defined as follows

$$\langle \phi^2 \rangle = \lim_{\varepsilon \to 0} \left[ G_E(x; \tilde{x}) - G_{DS}(x; \tilde{x}) \right],$$

where $G_{DS}(x; \tilde{x})$ are the well-known DeWitt-Schwinger counterterms [20]:

$$G_{DS}(x; \tilde{x}) = \frac{\Delta^{1/2}}{8\pi^2} \left\{ a_0 \left[ \frac{1}{\sigma} + m^2 L \left( 1 + \frac{1}{4}m^2 \sigma + \cdots \right) - \frac{1}{2}m^2 - \frac{5}{16}m^4 \sigma + \cdots \right] 
- a_1 L \left( 1 + \frac{1}{2}m^2 \sigma + \cdots \right) - \frac{1}{2}m^2 \sigma - \cdots 
+ a_2 \left[ L \left( 1 + \frac{1}{2}m^2 \sigma + \cdots \right) - \frac{1}{4} \cdots \right] + \cdots 
+ \frac{1}{m^2}[a_2 + \cdots] + \frac{1}{2m^4}[a_3 + \cdots] + \cdots \right\},$$  \hfill (18)
where $\sigma$ is half the squared distance between the points $x$ and $\bar{x}$ along the shortest geodesic connecting them, $L = \gamma + \frac{1}{2} \ln \frac{m^2|\sigma|}{2}$ and $\gamma$ is Euler’s constant. In the above expression we have to take all terms which survive after taking two derivatives and going to the coincidence limit. Here $a_k$ are the heat kernel coefficients (see, for example [21]).

The spacetime under consideration [11] possesses the singular curvature [11] and therefore we have a spacetime with the singular surface, $\rho = 1$, with codimension one. In this case we have to use expressions for heat kernel coefficients obtained in Ref. [22] (see also Ref. [21]). The main result of the paper [22] is that the heat kernel coefficients in this case may be represented as a sum of standard expressions for heat kernel coefficients calculated without a singular surface and surface terms:

$$a_n(M, \Sigma) = a_n(M \setminus \Sigma) + a_n^\Sigma(\Sigma),$$

(19)

where $M$ is a manifold, $\Sigma$ is a singular surface and $a_n^\Sigma(\Sigma) = 0$. The spacetime given by the metric [11] is flat and therefore $a_0(M \setminus \Sigma) = 1$, $a_{n>1}(M \setminus \Sigma) = 0$. All surface terms $a_n^\Sigma(\Sigma)$ are localized at the singular surface. For example

$$a_1^\Sigma(\Sigma) = -\left(\frac{1}{6} - \xi\right) \frac{8}{a} \delta(\rho) = \left(\frac{1}{6} - \xi\right) R,$$

(20a)

$$a_2^\Sigma(\Sigma) = \frac{256}{3a^3} \xi^3 \delta(\rho).$$

(20b)

Therefore for $\rho \neq 0$ all singular contributions are zero and we find

$$G_{DS}(x; \bar{x}) = \frac{1}{8\pi^2\sigma} + \frac{m^2}{8\pi^2} \left[ \gamma + \frac{2}{1} \ln \frac{m^2|\sigma|}{2} \right] - \frac{m^2}{16\pi^2}.$$  

(21)

In fact, this expression for $G_{DS}(x; \bar{x})$ does not contain any curvature terms and therefore coincides with that of Minkowski space-time. Because of this fact the renormalization procedure reduces to discarding the term $g_{ij}^M(\rho, \bar{\rho})$ in Eq. (14). Now, by using the formulas (7), (14), (17) and (21), we obtain

$$\langle \phi^2(\rho) \rangle = -\frac{1}{2\pi^2} \int_0^\infty \frac{1}{2\pi^2} \int_0^{\infty} d\omega \sum_{\lambda} \frac{\lambda a(\nu K^\nu + I^\nu K^\nu) + (8\xi - 1) I^\nu K^\nu}{2\lambda a K^\nu K^\nu + (8\xi - 1) K^\nu_2 K^\nu} |\lambda\rangle K^\nu_2(\lambda r).$$

(22)

For $p = -(1 + \xi)/2 = 4\xi - 1$ we observe singularity in integrand for $l = 0$ for $z^2 = (4\xi - 1)^2 - m^2a^2$. It is possible for $\xi > 1/4$. For $\xi \leq 1/4$ there is no singularity. We will assume $\xi \leq 1/4$.

Another singularity is for $\rho = 0$. Indeed for $\rho = 0$ the uniform expansion of integrand gives

$$\langle \phi^2(\rho) \rangle = -\frac{1}{2\pi^2} \int_0^\infty \frac{1}{2\pi^2} \int_0^{\infty} d\omega \sum_{\lambda} \left\{ \frac{1}{4} t^2(1 + \xi - t^2) - \frac{1}{8\nu} t^3(1 + \xi - t^2)^2 + O(\nu^{-2}) \right\},$$

(23)

with $t = 1/\sqrt{1 + z^2}$. The first two terms of the series are divergent.

The result of numerical analysis of $\langle \phi^2 \rangle$ given by Eq. (22) is shown in Figs. 1 and 2.

FIG. 1: Plots of $\langle \phi^2 \rangle a^2$ for $\xi = 0, 0.1, 0.2, 0.4$ from bottom to top. The value of the field mass $m$ is fixed: $m = 1$. The $\langle \phi^2 \rangle a^2$ behaviour is shown on the figure b for small distances at the throat. All lines fall down to minus infinity near the throat.
To calculate the stress-energy tensor one may use the standard formula

\[
\langle T_{\mu\nu} \rangle_{\text{ren}} = \lim_{x \to \infty} \left\{ \frac{1}{2} - \xi \right\} \left[ g_{\mu}^{\alpha'} G_{\alpha'\nu} + g_{\nu}^{\alpha'} G_{\mu\alpha'} + \left( 2\xi - \frac{1}{2} \right) g_{\mu\nu} g^{\alpha\beta'} G_{\alpha\beta'} - \xi (G_{\mu\nu} + g_{\mu}^{\alpha'} g_{\nu}^{\beta'} G_{\alpha\beta'}) \right] + 2\xi g_{\mu\nu} (m^2 + \xi R) G + \xi \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) G - \frac{1}{2} m^2 g_{\mu\nu} G ,
\]

where \( G \) is the renormalized Euclidean Green function. For above function it is easy to show that

\[
\langle \Phi^2 \rangle_{\text{ren}} = \left[ (1 - 2\xi) G_{\mu\nu} - 2\xi G_{\mu\nu} + \left( 2\xi - \frac{1}{2} \right) g_{\mu\nu} (G_{\alpha\alpha'} + (m^2 + \xi R) G) + \xi R_{\mu\nu} G \right].
\]

For this reason we may rewrite the expression for EMT in the following form

\[
\langle T_{\mu\nu} \rangle = \left[ (1 - 2\xi) G_{\mu\nu} - 2\xi G_{\mu\nu} + \left( 2\xi - \frac{1}{2} \right) g_{\mu\nu} (G_{\alpha\alpha'} + (m^2 + \xi R) G) + \xi R_{\mu\nu} G \right].
\]

For calculation the EMT we have to take into account that \( P_l(1) = 0 \) and \( P_{l'}(1) = l(l+1)/2 \). For brevity let us define operator \( \hat{L} \) and the function \( f \)

\[
\hat{L}\{ \cdot \} = -\frac{1}{2\pi^2} \int_0^\infty d\omega \sum_{l=0}^{\infty} \frac{\lambda a(I_\nu K'_\nu + I'_\nu K_\nu) + (8\xi - 1)I_\nu K_\nu}{2\lambda a K'_\nu + (8\xi - 1)K_\nu^2} \left| _{\lambda a} \right. \{ \cdot \}
\]

\[
f = \frac{K_\nu(\lambda r)}{\sqrt{r}}.
\]

The components of the EMT have the following form

\[
\langle T_{l}^l \rangle = \hat{L} \left\{ \omega^2 f^2 + \left( \frac{2\xi}{2} - \frac{1}{2} \right) \left( \lambda^2 + \xi R - \frac{l(l+1)}{r^2} \right) f^2 + \frac{f^4}{2} \right\},
\]

\[
\langle T_{p}^p \rangle = \hat{L} \left\{ (1 - 2\xi) f^2 - 2\xi \hat{f} f + \left( \frac{2\xi}{2} - \frac{1}{2} \right) \left( \lambda^2 + \xi R - \frac{l(l+1)}{r^2} \right) \hat{f}^2 + \frac{f^2}{2} \right\},
\]

\[
\langle T_{l}^\theta \rangle = \hat{L} \left\{ \frac{l(l+1)}{2r^2} f^2 - \frac{\xi}{r} \hat{f} f + \left( \frac{2\xi}{2} - \frac{1}{2} \right) \left( \lambda^2 + \xi R + \frac{l(l+1)}{r^2} \right) \hat{f}^2 + \frac{f^2}{2} \right\},
\]

\[
\langle T_{\phi}^\phi \rangle = \langle T_{\theta}^\theta \rangle,
\]

where an overdot notes the derivative with respect to \( \rho \). The function \( f \) obeys to the equation

\[
\hat{f} + \frac{2}{r} \hat{f} - \left( \frac{l(l+1)}{r^2} + \lambda^2 \right) f = 0.
\]
VI. NUMERICAL ANALYSIS

The expressions (28)–(31) for components of $\langle T_{\mu\nu}\rangle$ are not much suitable for an analytical consideration, therefore, we have applied numerical methods for their analysis. In this section we will discuss results of numerical analysis.

For a static spherically symmetric configuration one has $\langle T^t_t\rangle = -\varepsilon$, $\langle T^\rho_\rho\rangle = p$, and $\langle T^\theta_\theta\rangle = \langle T^\varphi_\varphi\rangle = p_t$, where $\varepsilon$ is the energy density, $p$ is the radial pressure, and $p_t$ is the transverse pressure. Values of $\varepsilon$, $p$, and $p_t$ are connected by the conservation law $\langle T_{\mu\nu}\rangle_{;\mu} = 0$, which for the Minkowsky metric (3) takes the simple form:

$$p_t = p + \frac{1}{a}\dot{p}.$$  

From here one may easily find $p_t$ provided $p$ is found.

![Graph](image)

FIG. 3: Plots of $p a^4$ for $m = 1$ and $\xi = 0, \frac{1}{8}, \frac{1}{6}, 0.2, 0.3$. The thicker is the line, the grater is the value of $\xi$.

We compute numerically the energy density $\varepsilon = -\langle T^t_t\rangle$ and the radial pressure $p = \langle T^\rho_\rho\rangle$ using Eqs. (28), (29). Results of numerical computations are given in Figs. (3, 4). Let us discuss them in details. In Fig. 3 plots of $p(\rho)$ are shown for a fixed value of $m$ and various values of the curvature coupling parameter $\xi$. It is seen that $p$ is everywhere negative, and $p \to -\infty$ in the limit $\rho \to 0$. Thus, a vacuum polarization leads to an infinitely negative radial pressure $p$ at the wormhole’s throat.

![Graph](image)

FIG. 4: Plots of $\varepsilon a^4$ for $m = 1$ and $\xi = 0, \frac{1}{8}, \frac{1}{6}, 0.2, 0.3$. The thicker is the line, the grater is the value of $\xi$. The $\varepsilon a^4$ behaviour is shown on the figure b for small distances at the throat. All lines fall down to infinity near the throat.

Plots of energy density $\varepsilon(\rho)$ are shown in Fig. 4. It is seen that a qualitative behavior of $\varepsilon$ depends on $\xi$. Provided $\xi < 1/8$ or $\xi > 0.2$, the function $\varepsilon(\rho)$ reaches a negative minimum at some $\rho$, and tends to zero far from the throat. In case $1/8 < \xi < 0.2$ $\varepsilon$ is everywhere positive monotonically decreasing function. It is worth noting that in both cases the vacuum energy density goes to infinity at the wormhole’s throat, i.e., $\varepsilon \to \infty$ in the limit $\rho \to 0$ (see Fig. 4a).

It is particularly important for a wormhole geometry to check whether the vacuum stress energy tensor obeys the usual energy conditions. In particular, the null energy condition (NEC) reads $\langle T_{\mu\nu}\rangle k^\mu k^\nu \geq 0$, where $k^\mu$ is an arbitrary null vector. In a static spherically symmetric case the NEC reduces to $\varepsilon + p \geq 0$. Graphs for the combination $\varepsilon + p$ are given in Fig. 5. It is seen that the combination $\varepsilon + p$ behaves similarly to the energy density $\varepsilon$. In particular, $\varepsilon + p \to \infty$ in the limit $\rho \to 0$ (see Fig. 5b). Thus, the vacuum stress energy tensor $\langle T_{\mu\nu}\rangle$ does not violate the NEC in the vicinity of the wormhole’s throat.
FIG. 5: Plots of $(\varepsilon + p)a^4$ for $m = 1$ and $\xi = 0, \frac{1}{8}, 0.2, 0.3$. The thicker is the line, the greater is the value of $\xi$. The $(\varepsilon + p)a^4$ behaviour is shown on the figure $b$ for small distances at the throat. All lines fall down to infinity near the throat.

VII. CONCLUSION

We have calculated the vacuum polarization $\langle \phi^2 \rangle$ and components of the vacuum stress energy tensor $\langle T_{\mu\nu} \rangle$ of the massive scalar field in a wormhole spacetime using the short-throat flat-space approximation for the wormhole geometry. The most important result obtained consists in the fact that $\langle T_{\mu\nu} \rangle$ does not violate the NEC in the vicinity of the wormhole’s throat. As a consequence, this implies that the vacuum polarization cannot self-consistently support the wormhole. Of course, it is necessary to emphasize that this conclusion has been obtained for the very simple model of wormhole. To make more general conclusions one should consider more realistic models of wormholes.

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