ON THE SPECTRUM OF VOLterra-TYPE INTEGRAL OPERATORS ON FOCK–SoboLEV SPACES

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ABSTRACT. We determine the spectrum of the Volterra-type integral operators $V_g$ on the growth type Fock–Sobolev spaces $\mathcal{F}_\psi^\infty_m$. We also characterized the bounded and compact spectral properties of the operators in terms of function-theoretic properties of the inducing map $g$. As a means to prove our main results, we first described the spaces in terms of Littlewood–Paley type formula which is interest of its own.

1. Introduction

Let $m$ be a nonnegative integer and $0 < p \leq \infty$. Then the Fock–Sobolev spaces $\mathcal{F}_{(m,p)}$ consist of entire functions $f$ such that $f^{(m)}$, the $m$-th order derivative of $f$, belongs to the classical Fock spaces $\mathcal{F}_p$, which consist of all entire functions $f$ for which

$$\int_{\mathbb{C}} |f(z)|^p e^{-\frac{1}{2}|z|^2} dA(z) < \infty \quad \text{and} \quad \sup_{z \in \mathbb{C}} |f(z)| e^{-\frac{1}{2}|z|^2} < \infty,$$

respectively for finite and infinite values of the exponents $p$, and $dA$ here denotes the usual Lebesgue area measure on $\mathbb{C}$. The Fock–Sobolev spaces were introduced in [6] and studied further by several authors in different contexts for example [5, 14, 15]. Because of their Fourier characterizations in [6, Theorem A], the spaces can be simply considered as weighted Fock spaces induced by the sequence of weight functions

$$\psi_m(z) = \frac{1}{2} |z|^2 - m \log(1 + |z|).$$

In view of this, $\mathcal{F}_{(m,p)}$ are just the weighted Fock spaces $\mathcal{F}_\psi^p$ which consist of all entire functions $f$ for which

$$\int_{\mathbb{C}} |f(z)|^p e^{-p\psi_m(z)} dA(z) \asymp \|f\|_{(m,p)}^p < \infty,$$

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1The notation $U(z) \lesssim V(z)$ (or equivalently $V(z) \gtrsim U(z)$) means that there is a constant $C$ such that $U(z) \leq CV(z)$ holds for all $z$ in the set of a question. We write $U(z) \simeq V(z)$ if both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$.
for \( 0 < p < \infty \) and for \( p = \infty \), the corresponding estimate becomes

\[
\| f \|_{(m, \infty)} \simeq \sup_{z \in \mathbb{C}} |f(z)| e^{-\psi_m(z)} < \infty.
\]

See [15] for further analysis on this.

The theory of integral operators constitutes a significant part of modern functional analysis, see for example [10, 11, 18, 12] for some overviews. A typical examples of these operators includes the integral operators of Volterra. In this paper, we study some spectral properties of linear integral operators of Volterra-type. More precisely, for a holomorphic function \( g \), we consider the Volterra-type integral operator \( V_g \) defined by

\[
V_g f(z) = \int_0^z f(w) g'(w) dw.
\]

These operators have been studied for decades in the settings of various analytic function spaces. We have no intention to review a vast literature about it, but for the state of the art, one can see [1, 9, 8, 17, 13, 16, 19] and the related references therein. Very recently, some spectral properties of these operators were investigated in [15] when they act between the Fock–Sobolev spaces \( \mathcal{F}_m^p \) and \( \mathcal{F}_m^q \) in which both the exponents \( p \) and \( q \) are set to be finite. The central aim of this note is to continue that line of research and study the properties when at least one of the exponents is at infinity. Because of the monotonicity property of the spaces in the sense \( \mathcal{F}_m^p \subseteq \mathcal{F}_m^\infty \) for all exponent \( 0 < p \leq \infty \), one would expect that a fairly weaker condition on the symbol \( g \) in contrast to the finite exponent cases can give a bounded (compact) \( V_g : \mathcal{F}_m^p \rightarrow \mathcal{F}_m^\infty \). As can be seen in the next section, it turns out that this is not the case, and the conditions are the same for both the finite and the infinite exponent cases in dependent of the order \( m \) and the size of the exponents as long the exponent in the target spaces is at least as big as in the domain space. More precisely, for \( 0 < p \leq \infty \), it is proved that \( V_g : \mathcal{F}_m^p \rightarrow \mathcal{F}_m^\infty \) is bounded if and only if \( g \) is a complex polynomial of degree not bigger than two, while its compactness is characterized in terms of degree \( g \) not being bigger than one. On the other hand, it is shown that \( V_g : \mathcal{F}_m^\infty \rightarrow \mathcal{F}_m^p \) is bounded (compact) if and only if \( g \) is of degree at most one.

Next we recall the notion of spectrum. The spectrum \( \sigma(T) \) of a bounded operator \( T \) on a Banach space consists of all \( \lambda \in \mathbb{C} \) for which \( \lambda I - T \) is not invertible, where \( I \) is the identity operator. If \( \lambda \) is an eigenvalue of \( T \), then the operator \( \lambda I - T \) fails to be one to one and hence \( \lambda I - T \) does not have inverse. The set of all such eigenvalues is referred to as the point spectrum of \( T \) and denoted by \( \sigma_p(T) : \sigma_p(T) = \{ \lambda \in \mathbb{C} : \text{Ker}(\lambda I - T) \neq 0 \} \). It follows from this that \( \sigma_p(T) \subseteq \sigma(T) \).

In contrast to the boundedness, compactness, and Schatten class membership spectral properties of the integral operators \( V_g \), there has not been much studies on their spectra. Some recent results in this connection can be read in [7, 15]. One of our main results describes the spectrum of \( V_g \) acting on \( \mathcal{F}_m^\infty \) in terms of a closed disk of center at the origin and radius involving the coefficient of the
highest degree term in a polynomial expansion of $g$ as precisely stated in our main result below.

**Theorem 1.1.** Let $g$ be an entire function on $\mathbb{C}$ and $0 < p \leq \infty$. Then

(i) $V_g : F^p_{\psi_m} \to F^\infty_{\psi_m}$ is
   (a) bounded if and only if $g(z) = az^2 + bz + c$, $a, b, c \in \mathbb{C}$.
   (b) compact if and only if $g(z) = az + b$, $a, b \in \mathbb{C}$.

(ii) if $0 < p < \infty$, then the following statements are equivalent:
   (a) $V_g : F^\infty_{\psi_m} \to F^p_{\psi_m}$ is bounded;
   (b) $V_g : F^\infty_{\psi_m} \to F^p_{\psi_m}$ is compact;
   (c) $g(z) = az + b$, $a, b \in \mathbb{C}$ whenever $p > 2$, and $g = \text{constant otherwise}$.

(iii) if $V_g : F^\infty_{\psi_m} \to F^\infty_{\psi_m}$ is a bounded operator, i.e. $g(z) = az^2 + bz + c$, $a, b, c \in \mathbb{C}$, then we have

$$\sigma(V_g) = \{\lambda \in \mathbb{C} : |\lambda| \leq 2|a|\} = \{0\} \cup \{\lambda \in \mathbb{C} \setminus \{0\} : e^{g(z)/\lambda} \not\in F^\infty_{\psi_m}\}.$$

As pointed earlier, when the operator $V_g$ acts between the Fock-Sobolev spaces $F^p_{\psi_m}$ and $F^q_{\psi_m}$ for finite exponents $p$ and $q$, the analogous results were proved in [15]. More specifically, it was proved that $V_g : F^p_{\psi_m} \to F^q_{\psi_m}$ for $0 < p \leq q < \infty$ is bounded if and only if $g$ is a complex polynomial of degree not exceeding. Compactness was described in terms of the degree of $g$ being at most one. On the other hand, if $0 < q < p < \infty$, then $V_g : F^p_{\psi_m} \to F^q_{\psi_m}$ is bounded(compact) if and only if $g$ is a polynomial of degree not bigger than one. The same conclusion as above holds with respect to the spectrum of the operators $V_g$. Thus, our main results now could be seen as a completion of the missing gap in [15] when at least one of the exponent is at infinity. From this and the corresponding results in [15], we now, in addition, conclude that the boundedness, compactness, and spectrum of $V_g$ are independent of the order $m$ of the Fock–Sobolev spaces.

We may now note that if we set $\psi_{(\alpha,t)}(z) = \alpha|z|^t$, $t > 0, \alpha > 0$, then the boundedness and compactness properties of $V_g$ acting on the growth type weighted spaces $F^\infty_{\psi_{(\alpha,t)}}$ has been described in [2] while its spectrum was identified later in [3] 2. In context of the works in these two articles, our results in Theorem 1.1 for the case when $p = \infty$ can be considered as extension results obtained by making logarithmic perturbation of the weight function $\psi_{(1/2,2)}(z) = \frac{1}{2}|z|^2$ into $\psi_m(z) = \frac{1}{2}|z|^2 - m \log(1 + |z|)$ for all positive integers $m$. It follows that the results are preserved under such perturbation. We also note in passing that most of the techniques used to prove our results in here are different from those used in [2, 3].

2. Preliminaries

Note that for each nonnegative integer $m$, the spaces $F^p_{\psi_m}$ are reproducing kernel Hilbert spaces with kernel $K_{(w,m)}$ and normalized reproducing kernel functions

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2We would like to thank the anonymous reviewer for bringing the works in [2] and [3] to our attention.
Observe that when \(m = 0\), the space \(F^2_m\) reduces to the classical Fock space \(F^2\), and in this case we precisely have \(\|K_{(w,0)}\|_{(2,0)}^2 = e^{\|z\|^2}\) and \(K_{(w,0)}(z) = e^{\|z\|^2}z\). For other values of \(p\), by Corollary 14 of [6], we only have a one sided estimate

\[
\|K_{(w,m)}\|_{(p,m)} \lesssim e^{\psi_m(w)}. \tag{2.2}
\]

Studying Volterra-type integral operators in normed spaces gets handy when the norms in the target spaces of the operators are described in terms of Littlewood–Paley type formula. These operators have been extensively studied in the spaces where such formulas are happened to be known. Our next key lemma does this job by characterizing the growth type Fock–Sobolev spaces in terms of derivatives which is interest of its own.

**Lemma 2.1.** Let \(f\) be an entire function on \(\mathbb{C}\). Then \(f\) belongs to the spaces \(F^\infty_{\psi_m}\) if and only if

\[
\sup_{z \in \mathbb{C}} \frac{|f'(z)|e^{-\psi_m(z)}}{1 + \psi'_m(z)} < \infty,
\]

and in this case we estimate the \(F^\infty_{\psi_m}\)-norm of \(f\) as

\[
\|f\|_{(m, \infty)} \simeq |f(0)| + \sup_{z \in \mathbb{C}} \frac{|f'(z)|e^{-\psi_m(z)}}{1 + \psi'_m(z)}. \tag{2.3}
\]

**Proof.** For each function \(f\) in \(F^\infty_{\psi_m}\), we make the estimate

\[
|f(z) - f(0)| \leq \int_0^1 |z||f'(t\,z)|\,dt
\]

\[
= \int_0^1 \frac{|f'(t\,z)|e^{-\psi_m(t|z|)}}{1 + \psi'_m(t|z|)}|z|(1 + \psi'_m(t|z|))e^{\psi_m(t|z|)}\,dt
\]

\[
\leq \left(\sup_{z \in \mathbb{C}} \frac{|f'(z)|e^{-\psi_m(z)}}{1 + \psi'_m(z)}\right) \int_0^1 |z|(1 + \psi'_m(t|z|))e^{\psi_m(t|z|)}\,dt. \tag{2.4}
\]

A straightforward integration by substitution shows that

\[
\int_0^1 |z|(1 + \psi'_m(t|z|))e^{\psi_m(t|z|)}\,dt \lesssim e^{\psi_m(z)}.
\]

Taking into account this and estimate \((2.4)\), we obtain

\[
\|f - f(0)\|_{(m, \infty)} \lesssim \sup_{z \in \mathbb{C}} \frac{|f'(z)|e^{-\psi_m(z)}}{1 + \psi'_m(z)},
\]

from which and triangle inequality we deduce the one sided inequality

\[
\|f\|_{(m, \infty)} \lesssim |f(0)| + \sup_{z \in \mathbb{C}} \frac{|f'(z)|e^{-\psi_m(z)}}{1 + \psi'_m(z)}. \tag{2.5}
\]
To prove the converse inequality, we first observe that by subharmonicity of \(|f|\),
\[
|f(z)| \lesssim \int_{D(z,1)} |f(w)|dA(w) = \int_{D(z,1)} e^{\psi_m(w)} \left(|f(w)|e^{-\psi_m(w)}\right)dA(w),
\]
(2.6)
where \(D(z, \delta)\) refers to a disk of center at \(z\) and radius \(\delta\). On the other hand, for \(w \in D(z, 1)\), the estimate \(\psi_m(w) \simeq \psi_m(z)\) holds. Thus, taking this into account in (2.6) leads
\[
|f'(z)| \lesssim \frac{d}{dz} \left( \int_{D(z,1)} e^{\psi_m(w)} \left(|f(w)|e^{-\psi_m(w)}\right)dA(w) \right)
\]
\[\simeq \frac{d}{dz} e^{\psi_m(z)} \int_{D(z,1)} |f(w)|e^{-\psi_m(w)}dA(w)\]
\[\lesssim e^{\psi_m(z)} |f'(z)| f_{m,\infty} \|f\|_{m,\infty} \lesssim \|f\|_{m,\infty},\]
and from which we obtain the remaining estimate
\[
|f(0)| + \sup_{z \in \mathbb{C}} \frac{|f'(z)|e^{-\psi_m(z)}}{1 + \psi'_m(z)} \lesssim |f(0)|e^{-\psi_m(0)} + \|f\|_{m,\infty} \lesssim \|f\|_{m,\infty},
\]
as desired and completes the proof.

As pointed, the approximation formula (2.3) is in the spirit of the famous Littlewood–Paley formula for entire functions in the growth type space \(F_{\psi_m}^\infty\). The corresponding formula in \(F_{\psi_m}^p\) for finite \(p\) was obtained in [15] and reads as
\[
\|f\|_{m,p}^p \simeq |f(0)|^p + \int_{\mathbb{C}} |f'(z)|^p \frac{e^{-p\psi_m(z)}}{(1 + \psi'_m(z))^p} dm(z).
\]
(2.7)
Both formulas (2.3) and (2.7) will be used repeatedly in our subsequent considerations.

**Lemma 2.2.** Let \(g(z) = az^2 + bz + c\) and \(|\lambda| > 2|a|\). Then for any entire function \(f\) on \(\mathbb{C}\) it holds that
\[
\sup_{z \in \mathbb{C}} \left| e^{\frac{g(z)}{\lambda}} f(z) e^{-\psi_m(z)} \right| \lesssim |f(0)| + \sup_{z \in \mathbb{C}} \left| e^{\frac{g(z)}{\lambda}} \left| f'(z) \right| e^{-\psi_m(z)} \right| \left( \frac{1 + \psi'_m(z)}{1 + \psi'_m(z)} \right). \tag{2.8}
\]

This lemma provides a key tool to prove our main result on the spectrum of \(V_g\) in the next section.

**Proof.** Arguing as in the proof of the above lemma, we make the pointwise estimate
\[
|f(z) - f(0)||e^{\frac{g(z)}{\lambda}}| \leq e^{\frac{g(z)}{\lambda}} \int_0^1 |z| |f'(tz)| dt
\]
\[\leq \left( \sup_{z \in \mathbb{C}} \left| e^{\frac{g(z)}{\lambda}} f'(z) e^{-\psi_m(z)} \right| \left( \frac{1 + \psi'_m(z)}{1 + \psi'_m(z)} \right) \right) \int_0^t |z|(1 + \psi'_m(t|z|))e^{\psi_m(t|z|)} dt
\]
\[\lesssim e^{\psi_m(z)} \left( \sup_{z \in \mathbb{C}} \left| e^{\frac{g(z)}{\lambda}} f'(z) e^{-\psi_m(z)} \right| \left( \frac{1 + \psi'_m(z)}{1 + \psi'_m(z)} \right) \right).
\]
From this and triangle inequality, we deduce
\[
\sup_{z \in \mathbb{C}} |f(z)e^{\frac{g(z)}{z}}e^{-\psi_m(z)}| \leq \sup_{z \in \mathbb{C}} |f(0)e^{\frac{g(0)}{z}}e^{-\psi_m(z)}| + \sup_{z \in \mathbb{C}} \frac{|e^{\frac{g(z)}{z}}f'(z)|e^{-\psi_m(z)}}{1 + \psi'_m(z)} \\
\leq |f(0)| + \sup_{z \in \mathbb{C}} \frac{|e^{\frac{g(z)}{z}}f'(z)|e^{-\psi_m(z)}}{1 + \psi'_m(z)},
\]
where for the last inequality we used the assumption that $|\lambda| > 2|a|$ and
\[
\sup_{z \in \mathbb{C}} |e^{\frac{g(z)}{z}}e^{-\psi_m(z)}| = \sup_{z \in \mathbb{C}} e^{\frac{z^2 + ka + c}{2} - \psi_m(z)} \\
\lesssim \sup_{z \in \mathbb{C}} e^{\left(\frac{|a|}{\lambda} - \frac{1}{2}\right)|z|^2 + \frac{|a|}{\lambda} + m \log(1+|z|)} \lesssim 1. \tag{2.9}
\]

3. Proof of the main result

In this section we prove our main result. We begin with the proof of part (a) of (i). If $g$ is a polynomial of degree not exceeding two, then $|g'(z)| \lesssim 1 + \psi'_m(z)$ for all $z \in \mathbb{C}$. Taking this into account and applying Lemma 2.1 leads to
\[
\|V_g f\|_{(m,\infty)} \simeq \sup_{z \in \mathbb{C}} \left|\frac{g'(z)}{1 + \psi'_m(z)}\right| \lesssim \sup_{z \in \mathbb{C}} \left|\frac{g'(z)}{1 + \psi'_m(z)}\right| \lesssim \|f\|_{(m,\infty)},
\]
where for the last inequality we used the monotonicity property $F_{\psi_m}^p \subseteq F_{\psi_m}^\infty$, and from which boundedness of $V_g$ follows.

To prove the converse, for each point $w \in \mathbb{C}^n$ we consider the sequence of functions $\xi_{(w,m)}(z) = e^{-\psi_m(w)}K_{(w,m)}(z)$. Then,
\[
\|\xi_{(w,m)}\|_{(m,p)} \lesssim 1 \tag{3.1}
\]

independent of $p$ and $w$ which follows from (2.2) for $p < \infty$ and from a simple argument for $p = \infty$. Applying $V_g$ to such a sequence yields
\[
\|V_g\| \gtrsim \|V_g \xi_{(w,m)}\|_{(m,\infty)} \simeq \sup_{z \in \mathbb{C}} \frac{|g'(z)||\xi_{(w,m)}(z)|e^{-\psi_m(z)}}{1 + \psi'_m(z)} \\
\gtrsim \frac{|g'(z)||\xi_{(w,m)}(z)|e^{-\psi_m(z)}}{1 + \psi'_m(z)}
\]
for all points $w$ and $z$ in $\mathbb{C}$. In particular, setting $w = z$ and applying (2.1) gives
\[
\|V_g\| \gtrsim \frac{|g'(w)||\xi_{(w,m)}(w)|e^{-\psi_m(w)}}{1 + \psi'_m(w)} \simeq \frac{|g'(w)|}{1 + \psi'_m(w)},
\]
which holds only when $|g'(w)| \lesssim 1 + \psi'_m(w)$ for all $w$. This again holds if and only if $g$ is a complex polynomial of degree not exceeding two as asserted, and completes the proof of part (a).

To prove part (b) of (i), we may first assume that $V_g$ is compact, and observe that the sequence $\xi_{(w,m)}$ converges to zero as $|w| \to \infty$, and the convergence is uniform on compact subset of $\mathbb{C}$. Then, arguing as in the previous part, compactness
of $V_g$ implies
\[
0 = \limsup_{|w| \to \infty} \|V_g\zeta_{(w,m)}\|_{(m,\infty)} \simeq \limsup_{|w| \to \infty} \left( \sup_{z \in \mathbb{C}} \frac{|g'(z)| |\zeta_{(w,m)}(z)| e^{-\psi_m(z)}}{1 + \psi'_m(z)} \right) \geq \limsup_{|w| \to \infty} \frac{|g'(w)| |\zeta_{(w,m)}(w)| e^{-\psi_m(w)}}{1 + \psi'_m(w)} \simeq \limsup_{|w| \to \infty} \frac{|g'(w)|}{1 + \psi'_m(w)},
\]
from which it follows that $g$ is a polynomial of degree not exceeding one.

Conversely, assume that $g(z) = az + b$, $a, b \in \mathbb{C}$. Then, obviously, $g$ belongs to the space $\mathcal{F}_{\psi_m}^\infty$. We aim to show that $V_g : \mathcal{F}_{\psi_m}^p \to \mathcal{F}_{\psi_m}^\infty$ is compact. To this end, let $f_j$ be a sequence of functions in $\mathcal{F}_{\psi_m}^p$ such that $\sup_j \|f_j\|_{(m,p)} < \infty$ and $f_j$ converges uniformly to zero on compact subsets of $\mathbb{C}$ as $j \to \infty$. Since $\psi_m(z) \to \infty$ as $|z| \to \infty$, for each $\epsilon > 0$ there exists a positive $N_1$ such that
\[
\frac{1}{1 + \psi'_m(z)} < \epsilon
\]
for all $|z| > N_1$. From this and Lemma 2.1, we obtain
\[
|V_g f_j(z)| e^{-\psi_m(z)} \lesssim \frac{|af_j(z)| e^{-\psi_m(z)}}{1 + \psi'_m(z)} \lesssim \|f_j\|_{(p,m)} \frac{|a|}{1 + \psi'_m(z)} \lesssim \epsilon
\]
for all $|z| > N_1$ and all $j$. On the other hand, if $|z| \leq N_1$, then
\[
|V_g f_j(z)| e^{-\psi_m(z)} \lesssim \frac{|g'(z)f_j(z)| e^{-\psi_m(z)}}{1 + \psi'_m(z)} \leq \|g\|_{(\infty,m)} \sup_{|z| \leq N_1} |f_j(z)| \lesssim \sup_{|z| \leq N_1} |f_j(z)| \to 0 \text{ as } j \to \infty.
\]

Part (ii): Applying the estimate in (2.7), $V_g : \mathcal{F}_{\psi_m}^\infty \to \mathcal{F}_{\psi_m}^p$ is bounded if and only if the inequality
\[
\|V_g f\|_{(m,p)}^p = \int_{\mathbb{C}} \frac{|f(z)|^p |g'(z)|^p e^{-p\psi_m(z)} dA(z)}{(1 + \psi'_m(z))^p} = \int_{\mathbb{C}} \frac{|f(z)|^p |g'(z)|^p (1 + |z|)^{mp} e^{-2\psi_m(z)} dA(z)}{e^{2\psi_m(z)} (1 + \psi'_m(z))^p} \lesssim \|f\|_{(m,\infty)}^p \tag{3.2}
\]
holds for each $f \in \mathcal{F}_{\psi_m}^\infty$. It means that if we set
\[
d\mu_{(g,p)}(z) = \frac{|g'(z)|^p (1 + |z|)^{mp}}{(1 + \psi'_m(z))^p} dA(z),
\]
then the inequality in (3.2) holds if and only if $\mu_{(g,p)}$ is an $(\infty, p)$ Fock–Carleson measure. Then an application of Theorem 2.4 of [14] immediately gives that the statements in (a) and (b) are equivalent, and any of these holds if and only if $\tilde{\mu}_{(t,mp)}$ belongs to $L^1(\mathbb{C}, dA)$ for some or any positive $t$ where
\[
\tilde{\mu}_{(t,mp)}(w) = \int_{\mathbb{C}} \frac{e^{-\frac{t}{2} |z-w|^2} d\mu_{(g,p)}(z)}{(1 + |z|)^{mp} (1 + \psi'_m(z))^p}.
\]
Having singled out this equivalent reformulation, our next task will be to investigate further the new formulation. Let us first assume $\tilde{\mu}_{(p,mp)}$ belongs to $L^1(\mathbb{C}, dA)$.
and plan to show $g$ is a complex polynomial of degree not exceeding one. Then, using the fact that $1 + \psi'(z) \simeq 1 + \psi'(w)$ and whenever $w$ belongs to the disk $D(z, 1)$, and subharmonicity of $|g'|^p$, we infer

$$\frac{|g'(z)|^p}{(1 + \psi'(z))^p} \lesssim \int_{D(z,1)} \frac{|g'(w)|^p}{(1 + \psi'(w))^p} dA(w) \lesssim \int_{D(z,1)} \frac{e^{-\frac{|z-w|^2}{2}}}{(1 + |w|)^{mp}} d\mu(g,p)(w) \lesssim \tilde{\mu}(p,mp)(z).$$

Integrating the above shows that

$$\int_{C} \frac{|g'(z)|^p}{(1 + \psi'(z))^p} dA(z) \lesssim \int_{C} \tilde{\mu}(t,mp)(w)dA(w) < \infty$$

holds only if $g'$ is a constant for $p > 2$, and $g' = 0$ whenever $p \leq 2$. On the other hand if $g' = c =$constant, then

$$\int_{C} \tilde{\mu}(p,mp)(w)dA(w) = \int_{C} \left( \int_{C} \frac{e^{-\frac{|z-w|^2}{2}}}{(1 + |z|)^{mp}} d\mu(g,p)(z) \right) dA(w)$$

$$= \int_{C} \int_{C} \frac{e^{-\frac{|z-w|^2}{2}}|c|^p}{(1 + \psi'(z))^p} dA(z)dA(w)$$

$$\simeq \int_{C} \frac{|c|^p}{(1 + \psi'(z))^p} dA(w) < \infty,$$

as $c = 0$ whenever $p \leq 2$ by our assumption, and $\psi'_m$ is $L^p$ integrable for all $p > 2$. This completes the proof of part (ii) of the main result.

**Part(iii):** We assume that $V_g$ is bounded on $F^\infty_{\psi_m}$ and hence $g(z) = az^2 + bz + c$. Then, by linearity of the integral we may write $\lambda I - V_g = (\lambda I - V_{g_1}) - V_{g_2}$ where $g_1(z) = az^2$ and $g_2(z) = bz + c$. A simple analysis shows that $\lambda I - V_g$ and $\lambda I - V_{g_1}$ are injective maps. On the other hand, by part (b) of (i) in the theorem, $V_{g_2}$ is a compact operator. Thus $\sigma(V_{g_2}) = \{0\}$. We shall proceed to consider the corresponding case with $g_1$. We may first observe that if $\lambda \neq 0$, then the equation $\lambda f - V_g f = h$ has the unique analytic solution

$$f(z) = (\lambda I - V_{g_1})^{-1} h(z) = \frac{1}{\lambda} h(0) e^{\frac{g_1(z)}{\lambda}} + \frac{1}{\lambda} e^{\frac{g_1(z)}{\lambda}} \int_0^z e^{-\frac{g_1(w)}{\lambda}} h'(w) dA(w). \quad (3.3)$$

This can easily be seen by solving an initial valued first order linear ordinary differential equation

$$\lambda y' - g_1' y = h', \quad \lambda f(0) = h(0).$$

Recall that $(\lambda I - V_{g_1})^{-1} h = R_{(g_1,\lambda)} h$ is the Resolvent operator of $V_{g_1}$ at $\lambda$. It follows that $\lambda \in C$ belongs to the resolvent of $V_{g_1}$ whenever $R_{(g_1,\lambda)}$ is a bounded operator. Since we assumed that $V_{g_0}$ is bounded and as $F^p_{\psi_m}$ contain the constants, setting $h = 1$ in (3.3) shows that $R_{(g_1,\lambda)} 1 = \frac{1}{\lambda} e^{g(z)/\lambda} \in F^\infty_{\psi_m}$ for each $\lambda$ in the resolvent set of $V_{g_1}$. On the other hand, if $|\lambda| > 2|a|$ then, from the estimation in (2.9) we have

$$\sup_{z \in C} \left| e^{g_1(z)/\lambda} e^{-\psi_m(z)} \right| < \infty,$$
and from which we conclude that $|\lambda| > 2|a|$ is a sufficient condition for the boundedness of $R((g,\lambda))$ on $F_{\psi_m}^\infty$. We aim to show that the condition is in fact necessary as well. To this end, let $f_1(z) = \int_0^z e^{-\frac{g_1(w)}{\lambda}} f'(w) dA(w)$, and for $2|a| < |\lambda|$, Lemma 2.2 and then Lemma 2.1 implies

$$\| R((g,\lambda)) \|_{m,\infty} \leq \frac{|f(0)|}{|\lambda|} \| e^{\frac{g_1}{\lambda}} \|_{m,\infty} + \frac{1}{|\lambda|} \sup_{z \in \mathbb{C}} \frac{e^{\frac{g_1(z)}{\lambda}} |f'_1(z)| e^{-\psi_m(z)}}{1 + \psi'_m(z)} \lesssim \frac{|f(0)|}{|\lambda|} + \frac{1}{|\lambda|} \sup_{z \in \mathbb{C}} \frac{e^{\frac{g_1(z)}{\lambda}} |f'(z)| e^{-\psi_m(z)}}{1 + \psi'_m(z)} \lesssim \frac{1}{|\lambda|} \| f \|_{m,\infty}.$$ 

We have now proved that for a nonzero $\lambda$, the resolvent operator $R((g,\lambda))$ is bounded if and only if $e^{g(z)/\lambda}$ belongs to $F_{\psi_m}^\infty$, and this holds if and only if $|\lambda| > 2|a|$. From this our assertion

$$\sigma(V_g) = \{0\} \cup \{ \lambda \in \mathbb{C} \setminus \{0\} : e^{g(z)/\lambda} \notin F_{\psi_m}^p \} = \{ \lambda \in \mathbb{C} : |\lambda| \leq 2|a| \}$$

immediately follows.

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