Regime of small number of photons in the cavity for a single-emitter laser

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Abstract. The model of a single-emitter laser generating in the regime of small number of photons in the cavity mode is theoretically investigated. Based on a system of equations for different moments of the field operators the analytical expressions for mean photon number and photon number variance are obtained. Using the master equation approach the differential equation for the phase-averaged quasi-probability $Q$ is derived. For some limiting cases the exact solutions of this equation are found.

1. Introduction

One of the fundamental models of quantum optics is the model of a single-emitter laser (SEL). Its simplest implementation is represented by a two-level atom placed inside a single-mode cavity and incoherently pumped to its upper level. The quantum theory of the SEL was first considered in [1] and since then this model and related ones have been studied by many authors (see references in [2-9]). In addition to the theoretical works there are various experimental realizations of a SEL (see e.g. [10]).

A significant contribution to the development of the quantum theory of a SEL was made by a scientific group headed by professor S. Ya. Kilin [4,5]. One of the theoretical approaches used by this group is based on the analysis of the master equation written for such quasi-probability distribution functions as $P$ and $Q$ (the Glauber–Sudarshan $P$ representation and the Husimi $Q$ representation).

In [6], for the SEL operating in steady state, we derived a second-order linear homogeneous differential equation for the phase-averaged $P$-function. In the limiting case, when the atom-field coupling is stronger than the coupling of the field to the reservoir that provides its decay, an approximate solution of this equation was obtained. This solution, which is unperturbed solution in the problem of a small parameter at the highest derivative, agrees well with numerical simulation of the master equation and, moreover, contains some solutions previously found in [4,5]. But for some values of the laser parameters the atom Fermi statistics begins to manifest itself strongly and the $P$-function begins to demonstrate non-classical behavior. In this case, the approximate solution does not work, and the above equation becomes difficult to analyze. Therefore, there is a natural desire to obtain a similar equation, but for a "good" quasi-probability.

In this paper, we consider the SEL generating in steady state and in the regime of small number of photons in the cavity mode, accompanied by strong-coupling regime. Using the approach based on an infinite system of equations for different moments of the field operators [7,11] the analytical expressions for mean photon number and photon number variance are obtained. Unlike our previous paper [7], where the specific example was considered, these expressions are obtained in the more
general case. Based on the master equation for density operator the equation for the phase-averaged $Q$-function for a SEL is derived. For the limiting cases in which the pumping tends to zero, two simple exact solutions of this equation are found.

2. Model of a SEL. Equation for the phase averaged $Q$-function. System of equations for different moments of the field operators

The simplest model of a SEL is represented by two-level atom interacting with a single damping cavity mode. The atom is characterized by two rate constants: $\gamma / 2$ - the spontaneous emission rate from the upper atomic level $|2\rangle$ to the lower atomic level $|1\rangle$; $\Gamma / 2$ - the incoherent pumping rate from the level $|1\rangle$ to the level $|2\rangle$. The cavity mode decay rate is denoted as $\kappa / 2$ and the coupling constant between atom and cavity mode is denoted as $g$.

The master equation for density operator $\hat{\rho}$ is

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar} [\hat{V}, \hat{\rho}] + \frac{\kappa}{2} (2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) + \frac{\gamma}{2} (2\hat{\sigma}\hat{\rho}\hat{\sigma}^\dagger - \hat{\sigma}^\dagger\hat{\sigma}\hat{\rho} - \hat{\rho}\hat{\sigma}^\dagger\hat{\sigma}) + \frac{\Gamma}{2} (2\hat{\sigma}\hat{\rho}\hat{\sigma}^\dagger - \hat{\sigma}^\dagger\hat{\sigma}\hat{\rho} - \hat{\rho}\hat{\sigma}^\dagger\hat{\sigma}), \quad \hat{V} = \hbar g (\hat{a}^\dagger \hat{\sigma} - \hat{\sigma}^\dagger \hat{a}),$$

(1)

where $\hat{a}^\dagger, \hat{a}$ are the photon annihilation and creation operators in the cavity mode, $\hat{\sigma} = |1\rangle\langle 2|$ ($\hat{\sigma}^\dagger = |2\rangle\langle 1|$) is the operator of polarization of the two-level atom and $\hat{V}$ is the interaction operator between atom and cavity mode.

From (1), using well-known rules, one can obtain the following system for the anti normally-ordered representation of the density matrix of our laser $\rho_\delta(z,z',t) = \langle i|\hat{\rho}(z,z',t)|k\rangle$ over the coherent states $|z\rangle$ of the field and over the projections on the atomic states $|i\rangle, |k\rangle$, $i,k = 1,2$

$$\frac{dQ}{dz} = \frac{\partial}{\partial z} \left[ \frac{\kappa}{2}(zQ + \frac{\partial Q}{\partial z}) - g\rho_{21} \right] + \frac{\partial}{\partial z} \left[ \frac{\kappa}{2}(z'Q + \frac{\partial Q}{\partial z'}) - g\rho_{12} \right],$$

$$\frac{dD}{dz} = (\Gamma - 2\gamma)Q - (\Gamma + 2\gamma)D + \frac{\partial}{\partial z} \left[ \frac{\kappa}{2}(zD - g\rho_{21}) \right] + \frac{\partial}{\partial z'} \left[ \frac{\kappa}{2}(z'\rho_{21} + z\rho_{12}) \right] - 2g(z\rho_{21} + z\rho_{12}) + \frac{\kappa}{2}(z^2 + z^2D) \frac{\partial^2 D}{\partial z^2},$$

$$\frac{d\rho_{21}}{dt} = -\frac{(\Gamma + 2\gamma)}{2} \rho_{21} + \frac{\kappa}{2} \left[ \frac{\partial}{\partial z} (z\rho_{21} + z\rho_{21}) + \frac{\partial}{\partial z'} (z'\rho_{21} + z\rho_{21}) \right] + g \left[ \frac{\partial}{\partial z} (D - Q) + \frac{\kappa}{2}(z^2 + z^2D) \frac{\partial^2 D}{\partial z^2} \right],$$

(2)

where we introduce the following quasi-probabilities: the $Q$-function $Q = \rho_{11} + \rho_{22}$ and the difference $D = \rho_{22} - \rho_{11}$, where $\rho_i = \rho_i(z,z',t)$; the coherence $\rho_{i\delta} = \rho_{i\delta}(z,z',t)$ for $i \neq k$. The physical meaning of the additional two quasiprobabilities $D$ and $\rho_{i\delta}$ follows from theirs mean values:

$$\langle D \rangle = \langle \hat{\sigma} \rangle = \int D dz dz^\prime$$

is the mean value for the atomic inversion; $\langle \rho_{21} \rangle = \langle \hat{\sigma} \rangle = \int \rho_{21} dz dz^\prime$ - the mean value of the atomic polarization.

The first equation in the system (2) can be written in the continuity equation form [6]:

$$\frac{\partial Q}{\partial t} + \text{div} \vec{J} = q,$$

where $\text{div} = \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial z'}, \frac{\partial}{\partial z'} \right)$ and quasi-probability current is defined as

$$\vec{J} = \left( J_1, J_2 \right)$$

with $J_1 = -\kappa / 2 \left( z + \frac{\partial}{\partial z'} \right) Q$; the source $q = -g \left( \frac{\partial \rho_{21}}{\partial z} + \frac{\partial \rho_{12}}{\partial z'} \right)$ can be also represented as divergence of some vector.

In the stationary regime, from above continuity equation it is easy to obtain the following relation between phase-averaged $Q$-function $Q(I)$ and sum of phase-averaged coherencies $\rho_{z}(I) = \rho_{12}(I) + \rho_{21}(I)$.
\[ \rho_z(I) = \frac{\kappa}{g} I^{1/2} \left[ Q(I) + \frac{dQ(I)}{dI} \right], \]  

where \[ Q(I) = \frac{1}{2\pi} \int_0^{2\pi} Q(I, \varphi) d\varphi, \quad \rho_{z_1}(I) = \rho_{z_1}^2(I) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\varphi} \rho_{z_2}(I, \varphi) d\varphi \]

and we introduced the polar coordinates \[ I^{1/2}, \varphi \] of the complex variable \[ z = I^{1/2} e^{i\varphi} \].

In the same stationary regime, after averaging all over the phase, the last two equations from system (2) can be written as

\[
\begin{align*}
(\Gamma - \gamma)Q(I) - (\Gamma + \gamma)D(I) - 2gI^{1/2} \rho_z(I) &= \frac{d}{dI} \left[ gI^{1/2} \rho_z(I) - \kappa ID(I) - \kappa I \frac{dD(I)}{dI} \right], \\
(\Gamma + \gamma) \rho_z(I) + \frac{\kappa}{2I} \rho_z^2(I) - 2gI^{1/2} \left[ 2D(I) + \frac{d}{dI} \left(D(I) - Q(I)\right)\right] &= 2\kappa \frac{d}{dI} \left[ \rho_z(I) + \frac{d}{dI} \rho_z(I) \right],
\end{align*}
\]

where \[ D(I) = (1/2\pi) \int_0^{2\pi} D(I, \varphi) d\varphi \] is the phase-averaged difference.

A system of \( n \) coupled differential equations, as is well known from the corresponding theory, can be reduced to one differential equation of an order higher than \( n \) or equal to \( n \). In our case, the system (4) together with relation (3) is equivalent to a single 5th-order differential equation for function \( Q(I) \)

\[ \sum_{\nu=0}^{5} f_{\nu}(I) Q^{(\nu)}(I) = 0, \]

\[ f_1(I) = b_{02}I^2 + b_{01}I^3, \quad f_2(I) = b_{11} + b_{12}I + b_{13}I^3, \quad f_3(I) = b_{20} + b_{21}I + b_{22}I^2 + b_{23}I^3, \]

where \( Q^{(\nu)}(I) \equiv d^{\nu}Q(I)/dI^{\nu} \) and coefficients \( b_k = b_k(\Gamma, \gamma, \kappa, g) \) are written out in the appendix.

Note that to exclude the phase-averaged difference and its derivatives from system (4), another equation was obtained by differentiating the second equation in this system.

Equation (5) is the first main result of this section.

Next thing we need is the infinite system of coupled equations for different moments of the field operators \( \langle \hat{a}^\dagger k \hat{a}^k \rangle \). This system of equations for a SEL was first derived by G. S. Agarwal and S. Dutta Gupta [11]. In [7] we introduced one simple way to obtain this equation for phase-averaged \( P \)-function. We also showed there that in the regime of small number of photons in the cavity \( \langle n \rangle \equiv \langle \hat{a}^\dagger \hat{a} \rangle \approx 1 \), when \( \Gamma \approx \kappa \), our laser can be adequately described by the first three reduced equations from this infinite system. Reducing of these equations means retaining only the terms arising from moments \( \langle \hat{a}^\dagger k \hat{a}^k \rangle \) with \( 0 \leq k \leq 3 \). Here, without detailed derivation, we present this finite system of equations and its solutions, obtained using the Cramer's rule

\[
\begin{align*}
\{ c_{11} \langle n \rangle + c_{12} \langle n^2 \rangle \} &= B, \\
\{ c_{21} \langle n \rangle + c_{22} \langle n^2 \rangle + c_{23} \langle n^3 \rangle \} &= a_{10}, \\
\{ c_{31} \langle n \rangle + c_{32} \langle n^2 \rangle + c_{33} \langle n^3 \rangle \} &= 0, \\
\{ B(c_{23}c_{33} - c_{23}c_{32}) - a_{10}c_{33} \} &= \Delta, \\
\{ B(c_{23}c_{31} - c_{21}c_{33}) + a_{10}c_{11}c_{33} \} &= \Delta, \\
\{ B(c_{23}c_{32} - c_{22}c_{33}) + a_{10}c_{11}c_{32} \} &= \Delta, \\
\end{align*}
\]
where coefficients $c_0, a_{ik}, B$ are also taken out at the appendix and $\Delta$ is a determinant.

Expressions for $\langle n \rangle, \langle n^2 \rangle, \langle n^4 \rangle$ (6) are the second main result in this section.

In the next section results (5), (6) will be used for analyzing the statistical properties of a SEL.

3. Results

From now on, the following three dimensionless constants will be used: $\omega = \Gamma / 2g, \eta = \gamma / 2g, \tau = k / 2g$. Using results (6) we plot the mean number of photons in the cavity mode $\langle n \rangle$ and the Mandel $Q$-parameter $Q = (\langle n^3 \rangle - \langle n^2 \rangle) / \langle n \rangle - 1$ as functions of $\omega$ (see figure 1).

Because of condition $\omega \approx \tau$ we consider two special cases: $\omega = \tau, \omega = \tau / 2$, i.e. the dimensionless cavity decay rate $\tau$ is changed together with the dimensionless pumping rate $\omega$.

The obtained dependences of $\langle n \rangle$ and $Q$ on the pumping rate is well known: self-quenching effect when $\omega \to \infty$; characteristic peak for the Mandel $Q$-parameter indicates on transition from regime when the field statistics is determined by atomic spontaneous relaxation out of the cavity mode to the regime when the atom Fermi statistics starts to show up better; the negative value of the Mandel $Q$-parameter corresponds to photon antibunching effect which manifest itself better for strong-coupling regime ($\eta << 1$) and for certain ratio between $\omega$ and $\tau$. If we equate $\eta = 0$ we get the results obtained in [7].

From presented calculation one can see that analytical results are in a very good agreement with numerical simulation of the master equation (we presented numerical simulation only for the case $\eta = 0.5$, for other case $\eta = 0.1$ we have the same excellent coincidence between analytical and numerical results). Thus, the following assumption is confirmed: for a complete description of the statistical properties of the SEL operating in the regime when $\omega \approx \tau$, it is sufficient to limit ourselves to the system of equations for the field moments $(\hat{a}^\dagger \hat{a}^k)$ with $k < 4$ [7].

Let us use obtained equation for phase-averaged $Q$-function (5) to find analytical results for two simple limiting cases: 1) $\eta \neq 0, \omega = \tau \to 0$ and 2) $\eta = 0, \omega = \tau \to 0$. In the first and second cases from (5) we obtain the following simple equations

$$Q_1(I) + Q(I) = 0, \quad \omega = \tau \to 0; \quad Q_2(I) + 3 + 4\eta I^2 Q(I) = 0, \quad \eta = 0, \quad \omega = \tau \to 0.$$

and the corresponding solutions are

$$Q_1(I) = e^{-I}, \quad Q_2(I) = e^{-I} \frac{\cosh[(2I)^{1/2}] + (2I)^{1/2} \sinh[(2I)^{1/2}]}{1 + (2\pi)^{1/2} \text{erf}(2^{1/2})}.$$

Solution for the first case is a well-known $Q$-function for a vacuum field state: $\langle n \rangle = \int_0^\infty Q_1(I) I dI - 1 = 0$. From solution for the second case one can obtain the same results for $\langle n \rangle$ and $\langle n^2 \rangle$ as in [7]: $\langle n \rangle = \int_0^\infty Q_1(I) I dI - 1 = 0.630843, \quad \langle n^2 \rangle = \int_0^\infty Q_2(I) I^2 dI - 3 \langle n \rangle - 2 = 1$.

Note that the function $Q_1(I)$ can be easily obtained using the phase-averaged $P$-function $P(I)$ derived in [7]

$$P(I) = C_0 \frac{1 - e^{I}}{(1 - 2I)^{1/2}}, \quad Q_1(I) = \int P(I') e^{-i(I' - I)} I_0 \left[2(I')^{1/2}\right] dI'.$$

where $I_0[x]$ is the modified Bessel function of the first kind, $C_0$ is the normalization constant and we used the well-known rules for relation between $Q(I)$ and $P(I)$ distribution functions. Due to the
singularity of the $P$-function at a point $I = 1/2$ the Cauchy principal value should be defined for the integral (9).

![Figure 1](image_url)

**Figure 1.** The mean number of photons in the cavity mode (a), (c) and corresponding Mandel $Q$-parameter (b), (d) vs $\omega$. Solid black lines (a), (b) – numerical simulations of equation (1); Dashed colour lines – analytical results obtained with the help of (6); $\tau = \omega$ – dashed red lines, $\tau = 2\omega$ – dashed blue lines.

4. Conclusion

In this paper, we considered the simple model of a single-emitter laser generating in the strong coupling regime $\eta < 1$, accompanied by the regime of small number of photons in the cavity mode $\omega \tau \approx 1$. Based on the system of coupled equations for certain moments of the field operators $(\hat{a}^\dagger)^k \hat{a}^k$ with $k < 4$, derived in [7], the analytical expressions for mean photon number and photon number variance was obtained. These analytical results demonstrate excellent agreement with the results of numerical simulation of the master equation.

The stationary differential equation for the phase-averaged $Q$-function $Q(I)$ was derived from the master equation. In the two limiting cases 1) $\eta \neq 0$, $\omega = \tau \to 0$ and 2) $\eta = 0$, $\omega = \tau \to 0$ this differential equation was resolved and clear analytical expressions was obtained. In the first limiting case we obtained the explicit result – the $Q$-function for the vacuum state of the cavity mode. The expression for the $Q$-function obtained in the second limiting case is in agreement with the result from [7], where this problem was considered with the help of phase-averaged $P$-function.

The derived equation for $Q(I)$ is a fifth-order differential equation and that is the problem. The corresponding equation for phase-averaged $P$-function [6] is a second-order differential equation. The
high order of the differential equation for \( Q(I) \) is the price we pay to get away from the problems inherent in the \( P \)-function.

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Appendix
The coefficients in the equation (5)

\[
b_{32} = -2\tau^2 (8\eta\tau - 3\eta^2 + 15\tau^2 - 6\eta\omega + 8\tau\omega - 3\omega^2);
\]

\[
b_{33} = 4\tau^4;\]

\[
b_{04} = \eta^4 \tau^4 (8\tau^2 - 1 - 3\omega) - \tau (3\tau + 24\tau^3 + 3\omega - \omega^2 - 8\tau\omega + \omega^3) + \eta (\tau^3 - \omega + 16\tau\omega - \omega (4 + 3\omega^2)),
\]

\[
b_{41} = \tau (5\eta \tau + 15\tau^3 - 4 - 20\tau^2 \omega - 2\eta (1 + 10\tau - 5\omega) + \tau (5\omega^2 - 2)), b_{52} = 2\tau^2 (4 - 3\eta \tau + 7\tau^2 - 3\omega^2);\]

\[
b_{30} = \eta^4 \tau^4 (8\tau^2 - 1 - 3\omega) + \eta (\tau^3 - \omega + 16\tau\omega - \omega (4 + 3\omega^2) + \tau^2 (2\omega^2 - 3),
\]

\[
b_{31} = 2\tau (\eta^2 \tau + 3\tau^3 - 2 - 4\tau^2 \omega + \omega^2 + 2\tau \omega (\omega - 2)).
\]

The coefficients in the system (6)

\[
c_{12} = A = \frac{\omega + \eta + \tau}{2(\omega + \eta)} (\frac{\omega - \eta + \tau}{2})^2, c_{12} = 1, B = \frac{\omega + \eta + \tau}{2(\omega + \eta)};
\]

\[
c_{21} = (6a_{10} - 12a_{11} - 2a_{12} + 2a_{20} + 2a_{21}), c_{22} = (12a_{10} - 3a_{12} + a_{21} - 3a_{22}), c_{23} = a_{22};
\]

\[
c_{31} = (40a_{10} + 3a_{11} - 8a_{12} - 2a_{20} + 2a_{21} - 12a_{10} - 2a_{12} + 2a_{20} - 3a_{21} + 12a_{22}),
\]

\[
c_{32} = (20a_{10} - 4a_{12} + a_{21});
\]

\[
a_{01} = \frac{\tau^5}{2} (\tau - \omega - \eta), a_{02} = \frac{\tau^4}{2} (\tau - \omega - \eta), a_{10} = \frac{\tau^4}{2} (\tau - \omega - \eta), a_{11} = \frac{\tau^4}{2} (\tau - \omega - \eta),
\]

\[
a_{12} = \frac{\tau^5}{2} (7\tau^2 - 3\eta^2 - 3\omega - 2), a_{20} = \frac{1}{4} (6\tau^2 + \omega^2 - \eta^2 \tau - 11\tau^2 \omega - \tau \omega^2 + \eta^2 (6\tau^2 - 3\omega^2 - 1 + \eta \tau (12\tau^2 + 4 + 11\tau^3 - 3\omega^2) + \tau^2 (6\omega^2 - 3)), a_{21} = \frac{\tau^4}{2} (\eta^2 \tau + 3\tau^3 - 2 - 4\tau^2 \omega + \omega^2 + 2\eta \tau (\omega - 2)), a_{22} = \tau^2.
\]

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