Tunneling of polarized fermions in 3D double wells

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Abstract

We study the tunneling of a spin polarized Fermi gas in a three-dimensional double-well potential, focusing on the time dynamics starting from an initial state in which there is an imbalance in the number of particles in the two wells. Although fermions in different doublets of the double well tunnel with different frequencies, we point out that (incoherent) oscillations of a large number of particles can arise, as a consequence of the presence of transverse degrees of freedom. Estimates of the doublet structure and of the occupation of transverse eigenstates for a realistic experimental setup are provided.

1. Introduction

Tunneling of particles through wells and barriers is a distinctive property of quantum mechanics, commonly exploited in the realization of solid-state devices [1, 2]. The experimental realization and manipulation of ultracold atoms [3, 4] in double- and many-well potentials makes it possible to study tunneling dynamics of fermions and bosons in a highly controllable setup, in which it is possible to tune the geometrical properties of the wells, and then the tunneling rate of the particles. For ultracold bosons, the coherent Josephson oscillations of a Bose–Einstein condensate in a double-well potential were discussed [5, 6] and subsequently experimentally observed [7–9]; the dynamics of bosonic squeezed states has also been investigated [10]. For large enough barriers between the two wells, a two-mode ansatz can be used to describe the tunneling dynamics: for bosonic condensates, the two-mode equations can be mapped to a non-rigid pendulum [5] and can also be generalized to many-well potentials [11, 12]. The dynamical properties of cold bosons in double- and many-well potentials, as well as double-well arrays, in the presence of driving time-dependent modulations has been also a subject of intense study [13–22].

On the other hand, the study of ultracold Fermi gases has experienced a great advance in recent years [23–25] and the impressive experimental progresses in their control make Fermi gases very promising to realize ultracold fermionic junctions and to study tunneling phenomena. Due to the possibility of using optical lattices [26–29] and of tuning the interspecies interaction through Feshbach resonances [3, 4], the fermionic tunneling can be studied in situations having a direct counterpart in solid-state tunnel junctions. When the Fermi gases in different wells are superfluid (due to attractive interaction among species), one has the ultracold fermionic equivalent of a superconductor–insulator–superconductor junction [1], having coherent tunneling: theoretical studies of tunneling of fermionic superfluids through barriers [30–32] and in double- and many-well potentials [33–39] have been reported in literature, as well as the study of the internal Josephson tunneling between different species [40] (see more references in [32]).

At variance, one can also realize the ultracold atomic counterpart of one or more normal tunnel junctions [41] when the temperature of the Fermi mixture in a double- or many-well potential is above the superfluid critical temperature (in the presence of attractive interactions), or even at zero temperature when the Fermi gases in the wells are in the normal state. The latter situation can be obtained by polarizing a two-component Fermi mixture above a critical threshold [42, 43] or simply using spin polarized fermions [44, 45]. In [44] the center-of-mass motion of a polarized Fermi gas in a combined periodic and harmonic
potential was theoretically and experimentally investigated, showing an insulating regime when the Fermi energy lies in the bandgap of the lattice: working in the tight-binding approximation, the dynamics of the polarized gas was studied solving the classical Liouville equation, allowing for the characterization of the different regimes of the center-of-mass dynamics of the 3D Fermi gas [44, 46]. Rabi oscillations of a degenerate fermionic gas in a double-well potential have been studied in [47] in relation to the possibility of exploiting them for the interferometric measurement of external forces at micrometer length scales. The tunneling dynamics of interacting bosons in a 1D double well, from weak interactions to the fermionization (Tonks–Girardeau) limit, was studied in [48].

In this paper we study the tunneling dynamics of an ideal (spin polarized) fermionic gas in double-well potentials. Of course, unlike condensed Bose gases, where a large number of particles are in the same state moving coherently, for polarized fermions the dynamics displays in general an incoherent motion of the particles in the external potential. The situation is analogous to the incoherent tunneling of electrons between normal metals in the approximation where they are considered free particles: the application of a constant voltage creates a difference in the chemical potential and the more energetic electrons can tunnel through the barrier. For Fermi gases an imbalance in the chemical potential can be obtained by either making the double-well potential asymmetric (with the two wells having different potential minima) or creating an imbalance in the number of particles between the two wells of the system at the initial time: in this paper we will focus on the latter situation. Because of the Pauli principle, polarized fermions do not interact in s-wave and at \( T = 0 \) they occupy all the doublets of the double-well potential up to the Fermi energy (the splitting of doublets increases when the energy of the doublets increases). Since, in general, fermions in different doublets tunnel with different frequencies, a dephasing in the current flowing among the wells may arise. This is what happens in the 1D case [45]: the transverse degrees of freedom are frozen, since the distances between different doublets of the 1D double-well potential are much smaller than the confining frequencies in the transverse directions. The tunneling dynamics then shows strongly aperiodic spatiotemporal patterns [45]: for a large number of fermions, current oscillations are practically washed out.

However, for a relatively large number of fermions (say \( N \gtrsim 10^5 \)), the validity of the 1D limit requires very large confining transverse frequencies: e.g., \( \omega_\perp/2\pi \gtrsim 1000 \text{ kHz} \) for \(^{40}\text{K} \) atoms with typical experimental values for the potential. Indeed, as we will discuss in section 2, if one has an energy barrier \( V_0 \) between the wells and the minima of the wells are at distance \( \lambda/2 \), then the energy difference between the doublets is \( \sim \hbar k \sqrt{V_0/m} \) (where \( k = 2\pi/\lambda \)) and \( m \) the mass of the fermionic atom). The condition for which the transverse degrees of freedom are frozen is then \( N\hbar k \sqrt{V_0/m} \lesssim \hbar \omega_\perp \). For a realistic double-well potential [7] one has \( V_0/\hbar \approx 1 \text{ kHz} \) and \( \lambda \sim 2 \mu \text{m} \): with \( m \) the mass of potassium \(^{40}\text{K} \) atoms, one gets \( \omega_\perp/(2\pi N) \gtrsim 1 \text{ kHz} \), corresponding to transverse frequencies \( \omega_\perp/2\pi \gtrsim 1000 \text{ kHz} \). One then sees that for typical transverse frequencies one has to take into account the transverse degrees of freedom.

In the following we consider a realistic 3D double well, obtained by superimposing an harmonic confinement with a periodic potential: properly tuning the parameters, one can have two wells much more populated than the others [7]. We provide estimates of the tunneling rates and of the occupation of the transverse degrees of freedom. In particular, we show that it is realistically possible to have only the first doublet occupied, showing the role played by the transverse degrees of freedom in providing a reservoir to store particles and resulting in oscillations of a large number of particles with the same frequency. We stress that these are incoherent single-particle oscillations. Then, if the 1D condition is violated, the simple observation of a sinusoidal current cannot discriminate between incoherent and coherent tunneling dynamics (the latter being expected when the weakly coupled Fermi gases are superfluids).

The plan of the paper is as follows: in section 2 we analyze the structure of levels for a 3D Fermi gas in a double-well potential obtained by the superposition of a harmonic confinement and a 1D optical lattice. In this section we also provide estimates for the number of doublets and the occupation of transverse eigenstates in a realistic setup. In section 3 we discuss the equations of motion for the dynamics in a 3D double well. In section 4 we study the dynamics of the Fermi gas in the 3D double-well potential described in section 2, showing how oscillations characterized by a single frequency can arise and studying deviations due to the occupation of higher doublets. Finally, in section 5 we present our conclusions, while in the appendix we report semiclassical estimates of the energy and the splitting of the doublets.

2. Polarized fermions in a 3D double-well potential

In this section we study the structure of levels and their filling by a polarized Fermi gas in a 3D double-well potential. We focus on the double-well potential obtained by superimposing an harmonic confinement and a 1D periodic potential: when the energy of two minima of the periodic potential is significantly lower than the energies of other minima, due to the presence of the harmonic trap, then these wells are much more populated than the others and one practically has a double well. This way to realize the double well has been used to study the tunneling dynamics of ultracold bosons in a double-well potential in [7, 10] and could also be used for Fermi gases. The energy barrier and the distance between the wells are controlled by acting on the parameters of the optical lattice [12], while the number of doublets under the barrier depends also on the trapping frequencies of the harmonic confinement. In the following we will discuss how the fermions fill the doublet structure for a realistic 3D double well, taking into account the transverse eigenstates.

The trapping potential reads

\[
V(x) = \frac{1}{2} \mu \omega^2 \left( x^2 + y^2 \right) + V_{\text{DW}}(z),
\]

(1)
where the double-well potential along the $z$-axis, $V_{DW}(z)$, has the form

$$V_{DW}(z) = \frac{1}{2}m\omega_z^2z^2 + V_0\cos^2(kz)$$  \hspace{1cm} (2)$$

(see figure 1 (left)). In (2), the 1D periodic potential is created by an optical lattice made of two counterpropagating laser beams: it is $k = 2\pi/\lambda$, where $\lambda = \lambda_{\text{laser}}\sin(\theta/2)$, $\lambda_{\text{laser}}$ being the wavelength of the lasers and $\theta$ the angle between the counterpropagating laser beams [12]. The spacing in the lattice is $\lambda/2$, and $V_0$ is proportional to the power of the laser; moreover, $\omega_z$ and $\omega_\perp$ are respectively the axial and transverse frequencies of the harmonic confinement. Typical experimental numbers are $\lambda \sim 1-10 \mu m$ and $\omega_\perp/2\pi \sim 10-100 \text{ Hz}$ for $\lambda \sim 10 \mu m$; the energy barrier (needed to have tunneling dynamics) is $V_0/h \gtrsim 500 \text{ Hz}$ [7]. In the following we set $V_0 = sE_R$, where $E_R = \hbar k_z^2/2m$ is the recoil energy, and we also introduce the ratio

$$A = \frac{\omega_{ho}}{\lambda}$$  \hspace{1cm} (3)$$

where $\omega_{ho} = \sqrt{\hbar/m\omega_z}$ is the harmonic oscillator length in the $z$-direction.

For suitable ranges of the parameters, the potential (1) is to a very good approximation a double-well potential (see figure 1 (left)). For large enough barriers between the two wells, a two-mode ansatz can be used to describe the tunneling dynamics of ultracold bosonic condensates [5, 49, 50]: in the two-mode approximation, the dynamics involves the lowest doublet. Similarly, in a periodic potential the dynamics of ultracold bosons in the tight-binding approximation involves only the lowest band [11, 12]. Quasi-equilibrium mixtures of itinerant and localized bosons in optical lattices were studied in [51]. At variance, polarized fermions occupy all the doublets up to the Fermi energy.

The potential $V_{DW}(z)$, rescaled in terms of $E_R$, reads

$$\frac{V_{DW}(z)}{E_R} = s\cos^2(kz) + \frac{1}{(2\pi A)^2}(kz)^2.$$  \hspace{1cm} (4)$$

One easily sees that for large barriers, $s \gg 1$, the first two minima of equation (4) are very close to $kz = \pm \frac{\pi}{2}$. Expanding the potential around these minima, we get the effective trapping frequency

$$\omega = \frac{2\sqrt{sE_R}}{h} = k \sqrt{\frac{V_0}{m}};$$  \hspace{1cm} (5)$$

the energy difference between (neighbor in energy) doublets is then $\sim \hbar \omega$. This frequency then allows us to estimate the mean energy and the number of doublets under the potential barrier $V_0$. In general the splitting of doublets increases when the energy of the doublets increases. A first estimation of the splitting of each doublet can be performed by means of a semiclassical computation [52]: in the appendix we report semiclassical estimates of the energy and the splitting of the doublets.

In figure 1 (right) we plot the number of doublets for the potential of equation (4) for different values of $V_0$ (in units of $\hbar \omega_\perp$) and $\lambda = 2\pi/k$ obtained from the solution of the time-independent Schrödinger equation. We observe that for a fixed ratio $\omega_{ho}/\lambda$ there is a finite range of values of $V_0$: for $V_0$ smaller than a critical value $V_0^{(\text{min})}$ the central barrier is not high enough to contain any doublet. For large values of $V_0$, the nearest energy minima (located, for $s \gg 1$, close to $kz = \pm \frac{\pi}{2}$) have an energy smaller than the potential energy at the top of the barrier, which is $sE_R$, and then the double-well structure disappears. From the condition $V_{DW}(z = \pm 3\pi/2k) > V_{DW}(0)$ one gets the maximum value $V_0^{(\text{max})}$, which is given by $V_0^{(\text{max})} \approx 9E_R/64\pi^2A^4 = 9m\omega_\perp^2\lambda^2/32$.

We now turn our attention to the evaluation of how the fermions fill the energy levels, in order to be able to make contact with typical ultracold atom experiments, which involve $\sim 10^3-10^5$ particles. Since typically $\bar{\omega} \gg \omega_\perp$, the system is essentially 1D when $N\bar{\omega} \ll \omega_\perp$, where $N$ is the total number of particles. In this case fermions begin to occupy the levels of the double-well potential along $z$. However, for

\[\text{The first two minima of the potential (4) which identify the position of the double-well minima are given by the solution of the transcendental equation } \sin(2kz)/2kz = 1/(2\pi A)^2 s.\]
typical experimental values (see figure 1 (right)), in a 1D setting the Fermi energy easily exceeds the barrier even for a number of fermions of a few tenths.

The opposite 3D limit is obtained when $N\omega_{\perp} \ll \tilde{\omega}$. In this limit, all the particles stay in the first doublet, occupying the transverse eigenstates associated with the doublet eigenstates. More precisely, to accommodate all the particles in the first doublet, two conditions must be satisfied at the same time: the energy difference between the first and the second doublet states ($\sim \tilde{\omega} \lambda$) must be much larger than the energy of the transverse modes ($\omega_{\perp} \lambda$) and the transverse trapping frequency $\omega_{\perp}$ must be much smaller than $\omega_z$. Indeed, if $\omega_{\perp} \sim \omega_z$, then the two conditions $\tilde{\omega} = k\sqrt{V_0/m} \gg \omega_{\perp} \sim \omega_z$ and $V_0 < V_0^{\text{max}} = (9/32)ma^2\lambda^2$ (needed to have a double-well structure) cannot be satisfied at the same time. Finally, we notice that if we relax the previous conditions then higher doublets can be occupied and situations intermediate between the 1D and 3D limits can be explored.

We can estimate the number of particles which can be stored in each doublet exploiting the degeneracy of the transverse degrees of freedom. Denoting by $\epsilon_n^{S,\alpha}$ the symmetric and antisymmetric eigenvalues of the double-well potential equation (2) corresponding to the nth doublet and by $\Delta E_n$ the (average) energy difference between the $n+1$th and the $n$th doublet of the potential $V_0^l(z)$,

$$
\Delta E_n = \frac{(\epsilon_{n+1}^S + \epsilon_{n+1}^A) - (\epsilon_n^S + \epsilon_n^A)}{2}.
$$

Then the total number of particles which can be stored on one side of the 3D double well (without occupying the next one) in the nth doublet is

$$
N_3^{3D} \approx \frac{1}{2} \left( \frac{\Delta E_n}{\hbar \omega_{\perp}} \right)^2.
$$

The number of particles in the lowest doublet is then given by $N_1^{3D} \approx (\Delta E_1/\hbar \omega_{\perp})^2/2$: notice that for very large barriers $\Delta E_1 \sim \hbar \omega_{\perp}$ [52] (see also appendix). If the energy of the higher doublet is beyond the energy of barrier ($sE_k$), then $\Delta E_n$ in equation (6) must be replaced by $sE_R - (\epsilon_n^S + \epsilon_n^A)/2$. From figure 2 we observe that the number of particles in each doublet is much higher than in the 1D configuration, allowing us to reach about $10^8$ particles when a few doublets are occupied. This allows the experimental study of tunneling phenomena with polarized fermions within this setting.

We conclude this section by considering a 2D potential of the form

$$
V_{2D}(x, z) = \frac{1}{2}m\omega_x^2x^2 + \frac{1}{2}m\omega_z^2z^2 + V_0 \cos^2(kz);
$$

the number of particles which can be placed on one side of the double well in the nth doublet is then given by

$$
N_2^{2D} \approx \frac{\Delta E_n}{\hbar \omega_{\perp}}
$$

with $\Delta E_n$ given by equation (6). Of course, in the 3D case the number of particles which can be stored is larger than in the 2D double-well potential due to the higher degeneracy introduced by the transverse degrees of freedom.

3. Tunneling dynamics in a double-well potential: formalism

In this section we review the formalism to describe the dynamics of a system of polarized fermions in a 3D double-well potential of the form equation (1). The 1D limit, studied in [45], is retrieved and briefly discussed; we refer also to [45] for a study of the effects of the boson–fermion interactions when a localized Bose–Einstein condensate is placed in the wells.

The Hamiltonian for a system of polarized fermions confined in an external potential $V(x)$ reads

$$
H = \int dx \psi^\dagger(x) \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(x) \right] \psi(x).
$$

We denote by $\psi_n(x)\psi_n(y)\phi_{\gamma}(z)$ the eigenvectors of the single-particle Hamiltonian. They are characterized by three quantum numbers: $n_x, n_y = 0, 1, \ldots$ denote the transverse quantum numbers associated with the transverse harmonic oscillator, while $\gamma$ denotes the eigenstates of the double-well potential $V_{DW}(z)$, as obtained from

$$
\left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + V_{DW}(z) \right] \phi_{\gamma}(z) = \epsilon_{\gamma}\phi_{\gamma}(z).
$$

The quantum number $\gamma$ is defined by the pair $n, \alpha$, where $n = 1, 2, \ldots$ denotes the doublet, and $\alpha = S, A$ denotes the symmetric or antisymmetric state (in the following we denote $\epsilon_{\gamma}$ by $\epsilon_n^{\alpha}$). The eigenenergies of the single-particle Hamiltonian are given by $\epsilon_n^{S,\alpha} = \hbar \omega_{\perp}(n_x + n_y + 1) + \epsilon_{\gamma}$. 

![Figure 2. Plot of the maximum number of particles $N_1, N_2$ and $N_3$ which can be stored in the first, second and third doublet respectively as a function of the strength of the optical lattice potential in equation (2) with $\nu_\perp = 10$ Hz, $v_c = 300$ Hz and $A = 0.15$. For these values of parameters up to three doublets have energy lower than $V_0$. The circle and the star correspond to $V_0/\hbar \omega_\perp = 7.55, N = 9.8 \times 10^9$ (circle) and $V_0/\hbar \omega_\perp = 10, N = 2.4 \times 10^9$ (triangle); in both cases the first and second doublets are completely occupied and the second one is almost completely occupied. Inset: the maximum number of particles which can occupy the first doublet as a function of the strength of the optical lattice potential equation (2) with $\nu_\perp = 10$ Hz, $v_c = 300$ Hz and $A = 0.25$. For these values of parameters, there is only a doublet under the barrier.](image-url)
The fermionic field can be written as \( \psi (x) = \sum_{n,n',y} \phi_{n}(x) \phi_{n'}(y) c_{n,n',y} \). Since we want to study the dynamics of particles in the doublet well potential we decompose each pair of eigenfunctions corresponding to the same doublet into its right and left components:

\[
\psi (x) = \sum_{n,n',y} \phi_{n}(x) \phi_{n'}(y) \left[ \phi_{R}^{n}(z)c_{n,n',y}^{R} + \phi_{L}^{n}(z)c_{n,n',y}^{L} \right]
\] (12)

where the (Wannier) wavefunctions \( \phi_{R}^{n} \) centered in the wells R, L are given as usual by

\[
\phi_{R}^{n}(z) = \frac{\phi_{n,S}^{R}(z) + \phi_{n,A}^{R}(z)}{\sqrt{2}};
\]

\[
\phi_{L}^{n}(z) = \frac{\phi_{n,S}^{L}(z) - \phi_{n,A}^{L}(z)}{\sqrt{2}}.
\]

We will also use the notations \( \phi_{n,n,a}(x) = \phi_{n}(x) \phi_{n}(y) \phi_{n,a}(z) \) for the eigenfunctions of the single-particle Hamiltonian and \( \phi_{n,n,a}^{R}(x) = \phi_{n}(x) \phi_{n}(y) \phi_{n}^{R}(z) \) for the 3D Wannier wavefunctions. The fermionic operators \( c_{n,n',y}^{R,L} \) are defined as

\[
c_{n,n',y}^{R} = \frac{c_{n,n',y}^{R} + c_{n,n',y}^{L} \epsilon_{n}^{A}}{\sqrt{2}};
\]

\[
c_{n,n',y}^{L} = \frac{c_{n,n',y}^{R} - c_{n,n',y}^{L} \epsilon_{n}^{A}}{\sqrt{2}}.
\] (13)

The Hamiltonian \( H \) then reads

\[
H = \sum_{n,n',y} \left\{ \hbar \omega_{e} (n_{e} + 1) + \frac{\epsilon_{n}^{S} + \epsilon_{n}^{A}}{2} \right\} \times (c_{n,n',y}^{R} c_{n,n',y}^{R}^{\dagger} + c_{n,n',y}^{L} c_{n,n',y}^{L}^{\dagger}) + \frac{\epsilon_{n}^{S} - \epsilon_{n}^{A}}{2} \left( c_{n,n',y}^{R} c_{n,n',y}^{L} + c_{n,n',y}^{L} c_{n,n',y}^{R} \right).
\] (14)

The density operator of particles along the direction of the double well is given by

\[
n(z) = \int dx \psi^{\dagger}(x) \psi (x) = \sum_{n,n',y} \sum_{y'} \phi_{n}^{*}(y') \phi_{n'}(y) c_{n,n',y}^{R,L} c_{n,n',y'}^{R,L}.
\] (15)

Clearly, when we average the density operator over a generic state with a definite number of particles on the left and right parts of the barrier the off-diagonal elements of the density operator give a null contribution. We introduce the quantity

\[
\Delta N_{n,n,n}(t) = \langle (c_{n,n',y}^{R} c_{n,n',y}^{R}^{\dagger}) - (c_{n,n',y}^{L} c_{n,n',y}^{L}^{\dagger}) \rangle
\]

which is the particle difference in the doublet \( n_{e}, n_{y}, n \) between the right and the left sides of the double well for a generic initial state \( \psi_{0} \) (given an operator \( \mathcal{O} \), we use the notation \( \langle \mathcal{O} \rangle = \langle \psi_{0}(\mathcal{O}(t)) \psi_{0} \rangle \)). Next we define the total population imbalance \( \Delta N_{n}(t) \) for the \( n \)th doublet:

\[
\Delta N_{n}(t) = \sum_{n,n'} \left\{ \langle c_{n,n',y}^{R} c_{n,n',y}^{R}^{\dagger} \rangle - \langle c_{n,n',y}^{L} c_{n,n',y}^{L}^{\dagger} \rangle \right\}
\] (16)

which evolves according to

\[
\frac{d^{2}}{dt^{2}} \Delta N_{n,n,n}(t) = -\left( \frac{\epsilon_{n}^{A} - \epsilon_{n}^{S}}{\hbar} \right)^{2} \Delta N_{n,n,n}(t).
\] (17)

This equation has the straightforward solution

\[
\Delta N_{n,n,n}(t) = \Delta N_{n,n,n}(0) \cos(2\Omega_{n} t + \varphi_{n,n,n}),
\] (18)

where \( \Omega_{n} = \frac{\epsilon_{n}^{A} - \epsilon_{n}^{S}}{\hbar} \) is the Rabi frequency for the oscillation of a particle in the same doublet and \( \varphi_{n,n,n} \) depend in general on the initial conditions.

The total fractional population imbalance \( z(t) \) is defined as

\[
z(t) = \sum_{n} \frac{\Delta N_{n}(t)}{N}.
\] (19)

For a single particle in the doublet \( n_{e}, n_{y}, n \), the initial state \( \psi_{0}(x) \) can be in general be written in the Bloch sphere as

\[
\psi_{0}(x) = \cos \left( \frac{\theta_{n,n,n}}{2} \right) \phi_{n,n,n,S}(x) + e^{i\omega_{n,n,n}} \sin \left( \frac{\theta_{n,n,n}}{2} \right) \phi_{n,n,n,A}(x).
\] (20)

where \( \theta_{n,n,n} \) and \( \varphi_{n,n,n} \) are the coordinates on the Bloch sphere: when the particle is initially in the state \( \phi_{n,n,n}(x) \), then \( \theta_{n,n,n} = \pi/2 \) and \( \varphi_{n,n,n} = 0 \). The energy of such a state is given by

\[
\langle \psi_{0}|H|\psi_{0} \rangle = \epsilon_{n}^{S} \cos^{2} \left( \frac{\theta_{n,n,n}}{2} \right) + \epsilon_{n}^{A} \sin^{2} \left( \frac{\theta_{n,n,n}}{2} \right).
\] (21)

It is then easy to see that the population imbalance for this state evolves according to (18), where

\[
\Delta N_{n,n,n}(0) = \sin \theta_{n,n,n}.
\] (22)

Let consider now many fermions in the double-well potential. For a generic initial state with the fermions in the same \( n \)th doublet with different initial phases \( \varphi_{n,n,n} \), then oscillations in the population imbalance are rapidly washed out: indeed, since the fermions are polarized, the dynamics in different doublets are independent and dephasing of the different particle differences \( \Delta N_{n,n,n}(t) \) in general occurs. However, the experimental situation we have in mind is a system of two identical wells initially practically decoupled (i.e., with a very high energy barrier between them) with a definite number of particles (say \( N_{L} \) and \( N_{R} \), with \( N_{L} < N_{R} \)) on each side in their ground state. Then, at the initial time \( t = 0 \) the barrier along the \( z \) direction is lowered and the system is left to evolve freely with the Hamiltonian (10). Alternatively, one could start from the ground state of a symmetric 3D double well and tilt the double well for some time, so that the energy minima of the double-well potential are different and particles flow toward the energetically favored well: removing the tilted potential abruptly, one has an initial state with a different number of particles in the two wells.

For sufficiently high barriers the first \( N_{L} \) particles will be frozen in the two wells, while the others will start to tunnel. Take as initial state for one such particle on the right-hand side
ψ_{n_i,n_f}, where \( n_i \) and \( n_f \) are the quantum numbers along the transverse directions which remain unperturbed and \( n_z \) labels the states along the longitudinal \( z \)-axis before the barrier gets lowered; then the initial state \( \psi_0(x) = \psi_{n_i,n_f,(x)} \) can be well approximated by

\[
\psi_0(x) \approx \phi_{n_i,n_f}(x).
\]

Therefore we decide to study the evolution of the system of an arbitrary number of fermions under the conditions that the initial state of each particle is of the type \( \psi_{n_i,j}(x)\psi_{n_f,j}(x) \frac{e^{i\phi_{n_i,j}\pm i\phi_{n_f,j}}}{\sqrt{2}} \) to have a reasonable description of the dynamics.

If, for instance, we suppose all the particles in the \( n \)th doublet initially in the right-hand well, the phases \( \psi_{n_i,n_f} \) can be taken equal to zero and the numbers of particles on each side of the barrier read

\[
\begin{align*}
n^R_n &\equiv \sum_{n_i,n_f} \langle \epsilon_{n_i}^R | n_i,n_f, n_i,n_f \rangle^R = \Delta N_n(0) \cos^2(\Omega_n t) \quad (24) \\
n^L_n &\equiv \sum_{n_i,n_f} \langle \epsilon_{n_i}^L | n_i,n_f, n_i,n_f \rangle^L = \Delta N_n(0) \sin^2(\Omega_n t). \quad (25)
\end{align*}
\]

We observe that in the 1D limit the transverse degrees of freedom are frozen and fermions belonging to different doublets oscillate with different frequencies: this case has been studied in detail in [45]. Oscillations follow an aperiodic time dependence due to the superposition of different incommensurate oscillations: for typical experimental values one sees that already for a few hundred particles the current oscillations are practically washed out for a not too small initial imbalance. Moreover, since Rabi frequencies for distant levels can be very different, the timescale required for a complete oscillation of the population imbalance \( z(t) \) can be much larger than the experimental observation times.

4. Dynamics in a three-dimensional double-well potential

In this section we consider the dynamics of polarized fermions in the 3D confining potential (1) when the 1D validity condition is violated, as happens for realistic transverse confining potentials. As discussed in section 2, the number of particles which can be stored in one doublet can be very much increased by taking a weak confinement in the transverse direction thanks to the possibility to fill the transverse states before reaching the next doublet. In the inset of figure 2 we consider the range of parameters \( V_{0}^{\min} < V_{0} < V_{0}^{\max} \) such that the double-well potential (4) has only a doublet under the barrier and we plot in this range the maximum number \( N_{\text{max}} \) of particles which can be stored in that (first) doublet: in other words, for a number of atoms \( N > N_{\text{max}} \), they start to occupy levels above the top of the barrier. One sees that \( N_{\text{max}} \approx 10^3 \) for reasonable experimental parameters of the potential. Since in this case there is only one doublet with energy below \( V_{0} \) then the number of particles scales as \( \sqrt{V_{0}} \) (the energy of the particle in the double-well direction, \( \sqrt{\epsilon_{n_{\text{min}}}^d} \), grows as \( \sqrt{V_{0}} \)).

The number of particles can be further increased by decreasing the ratio \( \Lambda = \frac{\epsilon_{n_{\text{min}}}^d}{\lambda} \), as shown in figure 2. In this figure we consider values of the parameters such that the double-well potential (4) can have one, two or three doublets: we plot the maximum number of particles \( N_{\text{max}} \) which can be stored in the doublet \( n = 1, 2 \) or 3. The maximum number of particles in the first level can reach \( N_{\text{max}} \approx 3.5 \times 10^3 \).

The dynamics for the fractional population imbalance \( z(t) \) is illustrated in figure 3; here for simplicity we show the results for the evolution of particles initially distributed on the right-hand side of the barrier. In the left-hand part of the figure we considered \( N = 9.8 \times 10^3 \) atoms distributed between the first two doublets, filling the first one completely and the second almost totally at \( V_{0} = 7.5\hbar \omega_{z} \) (corresponding to the circle in figure 2): we see that the oscillation of \( z(t) \) contains only two Rabi frequencies. The (normalized) particle density at different times is plotted in figure 4. In the right-hand part of figure 3 we considered \( N = 2.4 \times 10^4 \) atoms distributed among the first (completely filled) and the second doublet at \( V_{0} = 10\hbar \omega_{z} \) (corresponding to the triangle in figure 2), and the particle density at different times is plotted in figure 5. In this case, the profile of the densities has more pronounced secondary peaks due to the contribution from particles in the second doublet.

Finally we observe that a way to observe the relative number of particles in each doublet at the initial time \( t = 0 \) is through the analysis of the Fourier transform of the fractional population imbalance. Since our considerations are done at \( T = 0 \) the motion of the particles in the system is undamped, so the Fourier transform of \( z(t) \) is simply given by a sum of \( \delta \)-functions, each one localized at the characteristic Rabi frequency of each doublet: for the cases considered in figures 3–5, where only two doublets are occupied, one has

\[
z(\omega) = \int_{-\infty}^{+\infty} e^{i\omega t} z(t) \, dt = 2\pi \left( \frac{N_1}{2N} [\delta(\omega - \Omega_1) + \delta(\omega + \Omega_1)] + \frac{N_2}{2N} [\delta(\omega - \Omega_2) + \delta(\omega + \Omega_2)] \right) \quad (26)
\]
Table A.1. Semiclassical estimation of the splitting energy using the value given by (5) for the energy $E$ ($E^1$) or the numerical value for the average energy of the doublet ($E^0$). Energies are in units of $E_R$. We use here $s = 76$ and $A = 0.1$ corresponding to four doublets.

| Doublet | $\Delta E_{\text{Exact}}$ | $\Delta E_{\text{semi}}^1$ | $\frac{E_{\text{semi}}^1 - E_{\text{Exact}}}{E_{\text{Exact}}}$ | $\Delta E_{\text{semi}}^0$ | $\frac{E_{\text{semi}}^0 - E_{\text{Exact}}}{E_{\text{Exact}}}$ |
|---------|-----------------|-----------------|------------------------|-----------------|------------------------|
| 1       | $5.44 \times 10^{-5}$ | $4.95 \times 10^{-5}$ | $-0.09$ | $4.93 \times 10^{-5}$ | $-0.094$ |
| 2       | $3.34 \times 10^{-3}$ | $3.70 \times 10^{-3}$ | $0.11$ | $3.19 \times 10^{-3}$ | $-0.045$ |
| 3       | $8.75 \times 10^{-2}$ | $0.142$ | $0.62$ | $8.85 \times 10^{-2}$ | $0.011$ |
| 4       | $1.12$ | $2.80$ | $1.50$ | $1.14$ | $0.018$ |

where $N_1$ and $N_2$ are the number of particles respectively in the first and second doublets. At the bottom of figure 3 we plot the coefficients of the Fourier transform for the two cases described above.

To conclude this section, we mention that it would be interesting to study in the future the strongly driven dynamics of polarized fermions in 3D double wells in the presence of time-dependent potentials [13, 53], as well as their dynamics in the presence of interparticle (eventually repulsive) interactions in order to investigate in this experimental setup the issues of equilibration and thermalization [54–62].

5. Conclusion

We studied the tunneling dynamics of a spin polarized Fermi gas in a three-dimensional double-well potential at zero temperature. We focused in particular on the time dynamics starting from an initial state in which there is an imbalance between the number of particles in the two wells. Although fermions in different doublets of the double-well tunnel with different frequencies, we point out that (incoherent) oscillations of a large number of particles can arise, as a consequence of the presence of transverse degrees of freedom.

Estimates of the doublet structure and of the occupation of transverse eigenstates for a realistic experimental setup are provided. In the 1D limit the current oscillations are washed out, as a result of the dephasing, but for not too large confining transverse frequencies or not too small number of particles the fermions can occupy only the first doublets (using the transverse eigenstates), resulting in oscillations of a large number of fermions. We stress that these are incoherent single-particle oscillations. We can conclude that, if the 1D condition is violated, then the simple observation of a sinusoidal current cannot in general simply discriminate between incoherent and coherent (Josephson) tunneling dynamics at zero temperature.

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Appendix. Semiclassical estimation of the splittings

In this appendix we report a semiclassical estimation of the energy splitting among the levels of the double-well potential (2). The semiclassical formula for the splitting of the nth energy level is given by [52]

$$E_n^A - E_n^S = \frac{\hbar \omega_{cl}}{\pi} e^{-\frac{1}{2}} \int_{-a}^{a} dx |p|^2,$$  \hspace{1cm} (A.1)

where $a, -a$ denote the turning points at the central barrier of the potential (classical motion is inhibited there), $|p| = \sqrt{2m(V(z) - E)}$ and $\omega_{cl}$ is the frequency of the classical

Table A.1. Semiclassical estimation of the splitting energy using the value given by (5) for the energy $E$ ($E^1$) or the numerical value for the average energy of the doublet ($E^0$). Energies are in units of $E_R$. We use here $s = 76$ and $A = 0.1$ corresponding to four doublets.

| Doublet | $\Delta E_{\text{Exact}}$ | $\Delta E_{\text{semi}}^1$ | $\frac{E_{\text{semi}}^1 - E_{\text{Exact}}}{E_{\text{Exact}}}$ | $\Delta E_{\text{semi}}^0$ | $\frac{E_{\text{semi}}^0 - E_{\text{Exact}}}{E_{\text{Exact}}}$ |
|---------|-----------------|-----------------|------------------------|-----------------|------------------------|
| 1       | $5.44 \times 10^{-5}$ | $4.95 \times 10^{-5}$ | $-0.09$ | $4.93 \times 10^{-5}$ | $-0.094$ |
| 2       | $3.34 \times 10^{-3}$ | $3.70 \times 10^{-3}$ | $0.11$ | $3.19 \times 10^{-3}$ | $-0.045$ |
| 3       | $8.75 \times 10^{-2}$ | $0.142$ | $0.62$ | $8.85 \times 10^{-2}$ | $0.011$ |
| 4       | $1.12$ | $2.80$ | $1.50$ | $1.14$ | $0.018$ |
motion between the turning points where classical motion is allowed.

In Table A.1 we compute the semiclassical splitting energy in two cases: in the computation of $E_{\text{semi}}^{II}$ we used the estimation for the average energy $E = h\tilde{\omega}(n + \frac{1}{2})$ of the doublet based on the effective frequency (5); in $E_{\text{semi}}^{I}$ we used instead the average value of the energy of each doublet obtained from the numerical solution of the Schrödinger equation for the average value of the energy of each doublet obtained from the numerical solution of the Schrödinger equation for the potential (2). As we might expect the agreement of the semiclassical estimation with the numerical values is much better in the second case; we also note that in the first one there is a good agreement (the error is around 10%) for the first two doublets due to the validity of using the effective frequency (5) for the lowest energy levels.

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