ON WEAK*-CONVERGENCE IN THE LOCALIZED HARDY SPACES $H^1_{\rho}(\mathcal{X})$ AND ITS APPLICATION

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Abstract. Let $(\mathcal{X}, d, \mu)$ be a complete RD-space. Let $\rho$ be an admissible function on $\mathcal{X}$, which means that $\rho$ is a positive function on $\mathcal{X}$ and there exist positive constants $C_0$ and $k_0$ such that, for any $x, y \in \mathcal{X}$,

$$\rho(y) \leq C_0[\rho(x)]^{1/(1+k_0)}[\rho(x) + d(x, y)]^{k_0/(1+k_0)}.$$ 

In this paper, we define a space $V_{MO, \rho}(\mathcal{X})$ and show that it is the predual of the localized Hardy space $H^1_{\rho}(\mathcal{X})$ introduced by Yang and Zhou [14]. Then we prove a version of the classical theorem of Jones and Journé [7] on weak*-convergence in $H^1_{\rho}(\mathcal{X})$. As an application, we give an atomic characterization of $H^1_{\rho}(\mathcal{X})$.

1. Introduction

It is a well-known and classical result (see [2]) that the space $BMO(\mathbb{R}^n)$ is the dual of the Hardy space $H^1(\mathbb{R}^n)$ one of the few examples of separable, nonreflexive Banach space which is a dual space. In fact, let $C_c(\mathbb{R}^n)$ be the space of all continuous functions with compact support and denote by $VMO(\mathbb{R}^n)$ the closure of $C_c(\mathbb{R}^n)$ in $BMO(\mathbb{R}^n)$, Coifman and Weiss showed in [2] that $H^1(\mathbb{R}^n)$ is the dual space of $VMO(\mathbb{R}^n)$, which gives to $H^1(\mathbb{R}^n)$ a richer structure than $L^1(\mathbb{R}^n)$. For example, the classical Riesz transforms $\nabla(-\Delta)^{-1/2}$ are not bounded on $L^1(\mathbb{R}^n)$, but are bounded on $H^1(\mathbb{R}^n)$. In addition, the weak*-convergence is true in $H^1(\mathbb{R}^n)$ (see [7]), which is useful in the application of Hardy spaces to compensated compactness (see [1]) and in the study of commutators of singular integral operators (see [8, 10]). Let $L = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^n$, $n \geq 3$, where $V$ is a nonnegative function, $V \neq 0$, and belongs to the reverse Hölder class $RH_{n/2}(\mathbb{R}^n)$. The Hardy space associated with the Schrödinger operator $L$, $H^1_L(\mathbb{R}^n)$, is then defined as the set of functions $f \in L^1(\mathbb{R}^n)$ such that $\|f\|_{H^1_L} := \|\mathcal{M}_L f\|_{L^1} < \infty$, where $\mathcal{M}_L f(x) := \sup_{t>0}|e^{-tL}f(x)|$. Recently, Ky [9] established that the weak*-convergence is true in $H^1_L(\mathbb{R}^n)$, which is useful in studying the endpoint estimates for commutators of singular integral operators related to $L$ (see [10]).

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Let \((\mathcal{X}, d, \mu)\) be an RD-space, which means that \((\mathcal{X}, d, \mu)\) is a space of homogeneous type in the sense of Coifman-Weiss with the additional property that a reverse doubling property holds in \(\mathcal{X}\) (see Section 2). Typical examples for such RD-spaces include Euclidean spaces, Heisenberg groups, Lie groups of polynomial growth, or more generally, Carnot-Carathéodory spaces with doubling measures. We refer to the seminal paper of Han, Müller and Yang \([4]\) for a systematic study of the theory of function spaces in harmonic analysis on RD-spaces. Recently, Yang and Zhou \([14]\) introduced and studied the theory of localized Hardy spaces \(\text{H}^1_{\rho}(\mathcal{X})\) related to the admissible functions \(\rho\). There, they showed that this theory has a wide range of applications in studying the theory of Hardy spaces associated with Schrödinger operators or degenerate Schrödinger operators on \(\mathbb{R}^n\), or associated with sub-Laplace Schrödinger operators on Heisenberg groups or connected and simply connected nilpotent Lie groups, see \([14, \text{Section 5}]\) for details.

Given a complete RD-space \((\mathcal{X}, d, \mu)\) and an admissible function \(\rho\), we denote by \(\text{BMO}^\rho(\mathcal{X})\) the dual space of \(\text{H}^1_{\rho}(\mathcal{X})\) (see Section 2) and \(\text{VMO}^\rho(\mathcal{X})\) the closure in the \(\text{BMO}^\rho\)-norm of the space \(\text{C}_c(\mathcal{X})\) of all continuous functions with compact support.

The aim of the present paper is to show that \(\text{H}^1_{\rho}(\mathcal{X})\) is a dual space and that the weak\(\ast\)-convergence is true in \(\text{H}^1_{\rho}(\mathcal{X})\). Our main results can be read as follows:

**Theorem 1.1.** The space \(\text{H}^1_{\rho}(\mathcal{X})\) is the dual of the space \(\text{VMO}^\rho(\mathcal{X})\).

**Theorem 1.2.** Suppose that \(\{f_j\}_{j \geq 1}\) is a bounded sequence in \(\text{H}^1_{\rho}(\mathcal{X})\), and that \(f_j(x) \to f(x)\) for almost every \(x \in \mathcal{X}\). Then, \(f \in \text{H}^1_{\rho}(\mathcal{X})\) and \(\{f_j\}_{j \geq 1}\) weak\(\ast\)-converges to \(f\), that is, for every \(\varphi \in \text{VMO}^\rho(\mathcal{X})\), we have

\[
\lim_{j \to \infty} \int_{\mathcal{X}} f_j(x) \varphi(x) d\mu(x) = \int_{\mathcal{X}} f(x) \varphi(x) d\mu(x).
\]

It should be pointed out that when \(\mathcal{X} \equiv \mathbb{R}^n, n \geq 3\), and \(\rho(x) \equiv \sup\{r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y) dy \leq 1\}\), where \(V\) is in the reverse Hölder class \(RH_{n/2}(\mathbb{R}^n)\), Theorem 1.2 is just the main theorem in the paper of Ky \([9, \text{Theorem 1.1}]\).

Throughout the whole paper, \(C\) denotes a positive geometric constant which is independent of the main parameters, but may change from line to line.

## 2. Preliminaries

Let \(d\) be a quasi-metric on a set \(\mathcal{X}\), that is, \(d\) is a nonnegative function on \(\mathcal{X} \times \mathcal{X}\) satisfying

(a) \(d(x, y) = d(y, x)\),
(b) \(d(x, y) > 0\) if and only if \(x \neq y\),
(c) there exists a constant \(\kappa \geq 1\) such that for all \(x, y, z \in \mathcal{X}\),

\[
d(x, z) \leq \kappa (d(x, y) + d(y, z)).
\]
A trip \((\mathcal{X}, d, \mu)\) is called a space of homogeneous type in the sense of Coifman-Weiss if \(\mu\) is a regular Borel measure satisfying doubling property, i.e., there exists a constant \(C > 1\) such that for all \(x \in \mathcal{X}\) and \(r > 0\),

\[
\mu(B(x, 2r)) \leq C\mu(B(x, r)).
\]

**Remark 2.1.** By [2 Theorem (3.2)], we see that if \((\mathcal{X}, d, \mu)\) is a complete space of homogeneous type, then the closure of \(B\) is a compact set for all ball \(B \subset \mathcal{X}\).

Recall (see [4]) that a space of homogeneous type \((\mathcal{X}, d, \mu)\) is called an RD-space if it satisfies reverse doubling property, i.e., there exists a constant \(C > 1\) such that

\[
\mu(B(x, 2r)) \geq C\mu(B(x, r))
\]

for all \(x \in \mathcal{X}\) and \(r \in (0, \text{diam}(\mathcal{X})/2)\), where \(\text{diam}(\mathcal{X}) := \sup_{x, y \in \mathcal{X}} d(x, y)\).

Here and what in follows, for \(x, y \in \mathcal{X}\) and \(r > 0\), we denote \(V_r(x) := \mu(B(x, r))\) and \(V(x, y) := \mu(B(x, d(x, y)))\).

**Definition 2.1.** Let \(x_0 \in \mathcal{X}, r > 0, 0 < \beta \leq 1\) and \(\gamma > 0\). A function \(f\) is said to belong to the space of test functions, \(\mathcal{G}(x_0, r, \beta, \gamma)\), if there exists a positive constant \(C_f\) such that

\[
\begin{align*}
(i) \quad |f(x)| & \leq C_f V_{r(x_0) + V(x_0, x)} \left(\frac{r}{r + d(x_0, x)}\right)^\gamma \text{ for all } x \in \mathcal{X}; \\
(ii) \quad |f(x) - f(y)| & \leq C_f \left(\frac{d(y, x)}{r + d(x_0, x)}\right)^\beta V_{r(x_0) + V(x_0, x)} \left(\frac{r}{r + d(x_0, x)}\right)^\gamma \text{ for all } x, y \in \mathcal{X}.
\end{align*}
\]

For any \(f \in \mathcal{G}(x_0, r, \beta, \gamma)\), we define

\[
\|f\|_{\mathcal{G}(x_0, r, \beta, \gamma)} := \inf\{C_f : (i) \text{ and (ii) hold}\}.
\]

Let \(\rho\) be a positive function on \(\mathcal{X}\). Following Yang and Zhou [14], the function \(\rho\) is said to be admissible if there exist positive constants \(C_0\) and \(k_0\) such that, for any \(x, y \in \mathcal{X}\),

\[
\rho(y) \leq C_0 \rho(x)^{1/(1+k_0)} \rho(x + d(x, y))^{k_0/(1+k_0)}.
\]

Throughout the whole paper, we always assume that \(\mathcal{X}\) is a complete RD-space with \(\mu(\mathcal{X}) = \infty\), and \(\rho\) is an admissible function on \(\mathcal{X}\). Also we fix \(x_0 \in \mathcal{X}\).

In Definition 2.1 it is easy to see that \(\mathcal{G}(x_0, 1, \beta, \gamma)\) is a Banach space. For simplicity, we write \(\mathcal{G}(\beta, \gamma)\) instead of \(\mathcal{G}(x_0, 1, \beta, \gamma)\). Let \(\epsilon \in (0, 1]\) and \(\beta, \gamma \in (0, \epsilon]\), we define the space \(\mathcal{G}_0^\beta(\beta, \gamma)\) to be the completion of \(\mathcal{G}(\epsilon, \epsilon)\) in \(\mathcal{G}(\beta, \gamma)\), and denote by \((\mathcal{G}_0^\beta(\beta, \gamma))'\) the space of all continuous linear functionals on \(\mathcal{G}_0^\beta(\beta, \gamma)\). We say that \(f\) is a distribution if \(f\) belongs to \((\mathcal{G}_0^\beta(\beta, \gamma))'\).

Remark that, for any \(x \in \mathcal{X}\) and \(r > 0\), one has \(\mathcal{G}(x, r, \beta, \gamma) = \mathcal{G}(x_0, 1, \beta, \gamma)\) with equivalent norms, but of course the constants are depending on \(x\) and \(r\).

Let \(f\) be a distribution in \((\mathcal{G}_0^\beta(\beta, \gamma))'\). We define the grand maximal functions \(\mathcal{M}(f)\) and \(\mathcal{M}_\rho(f)\) as following

\[
\begin{align*}
\mathcal{M}(f)(x) := & \sup\{\langle f, \varphi \rangle : \varphi \in \mathcal{G}_0^\beta(\beta, \gamma), \|\varphi\|_{\mathcal{G}(x, r, \beta, \gamma)} \leq 1 \text{ for some } r > 0\}, \\
\mathcal{M}_\rho(f)(x) := & \sup\{\langle f, \varphi \rangle : \varphi \in \mathcal{G}_0^\beta(\beta, \gamma), \|\varphi\|_{\mathcal{G}(x, r, \beta, \gamma)} \leq 1 \text{ for some } r \in (0, \rho(x))\}.
\end{align*}
\]
Definition 2.2. Let $\varepsilon \in (0,1)$ and $\beta, \gamma \in (0,\varepsilon)$.

(i) The Hardy space $H^1(\mathcal{X})$ is defined by

$$H^1(\mathcal{X}) = \{ f \in (G_0(\beta,\gamma))' : \| f \|_{H^1} := \| \mathcal{M}(f) \|_{L^1} < \infty \}.$$ 

(ii) The Hardy space $H^1_\rho(\mathcal{X})$ is defined by

$$H^1_\rho(\mathcal{X}) = \{ f \in (G_0(\beta,\gamma))' : \| f \|_{H^1_\rho} := \| \mathcal{M}_\rho(f) \|_{L^1} < \infty \}.$$ 

Remark 2.2. It was established in [3] that the space $H^1(\mathcal{X})$ coincides with the atomic Hardy $H^1_{at}(\mathcal{X})$ of Coifman and Weiss [2]. Moreover, for all $f \in H^1(\mathcal{X})$,

$$\| f \|_{L^1} \leq C \| f \|_{H^1} \leq C \| f \|_{H^1}.$$ 

Recall (see [2]) that a function $f \in L^1_{loc}(\mathcal{X})$ is said to be in $\text{BMO}(\mathcal{X})$ if

$$\| f \|_{\text{BMO}} := \sup_B \frac{1}{\mu(B)} \int_B | f(x) - \frac{1}{\mu(B)} \int_B f(y) d\mu(y) | d\mu(x) < \infty,$$ 

where the supremum is taken all over balls $B \subset \mathcal{X}$. Denote by $\text{VMO}(\mathcal{X})$ the closure in $\text{BMO}$ norm of $C_c(\mathcal{X})$. The following is well-known (see [2]).

Theorem 2.1. (i) The space $\text{BMO}(\mathcal{X})$ is the dual space of $H^1(\mathcal{X})$.

(ii) The space $H^1(\mathcal{X})$ is the dual space of $\text{VMO}(\mathcal{X})$.

Definition 2.3. Let $\rho$ be an admissible function and $\mathcal{D} := \{ B(x, r) \subset \mathcal{X} : r \geq \rho(x) \}$. A function $f \in L^1_{loc}(\mathcal{X})$ is said to be in $\text{BMO}_\rho(\mathcal{X})$ if

$$\| f \|_{\text{BMO}_\rho} := \| f \|_{\text{BMO}} + \sup_{B \in \mathcal{D}} \frac{1}{\mu(B)} \int_B | f(x) | d\mu(x) < \infty.$$ 

It was established in [13] that

Theorem 2.2. The space $\text{BMO}_\rho(\mathcal{X})$ is the dual space of $H^1_\rho(\mathcal{X})$.

3. Proof of Theorems 1.1 and 1.2

We begin by recalling the following (see [13, Proposition 3.1]).

Lemma 3.1. Let $\rho$ be an admissible function. Then, there exists a function $K_\rho : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ and a positive constant $C$ such that

(i) $K_\rho(x, y) \geq 0$ for all $x, y \in \mathcal{X}$, and $K_\rho(x, y) = 0$ if $d(x, y) > C \min\{\rho(x), \rho(y)\}$;

(ii) $K_\rho(x, y) \leq C \frac{1}{\mu(B(x, \rho(y)))} \frac{1}{\mu(B(y, \rho(y)))}$ for all $x, y \in \mathcal{X}$;

(iii) $K_\rho(x, y) = K_\rho(y, x)$ for all $x, y \in \mathcal{X}$;

(iv) $| K_\rho(x, y) - K_\rho(x, y') | \leq C \frac{1}{\mu(B(x, \rho(x)))} \frac{1}{\mu(B(y, \rho(y)))}$ for all $x, y, z \in \mathcal{X}$ with $d(y, y') \leq \rho(x) + d(x, y)/2$;
for all \( f \in X \) satisfying \( d(x, x') \leq [\rho(y) + d(x, y)] / 3 \) and \( d(y', y) \leq [\rho(x) + d(x, y)] / 3 \), we have

\[
\left| [K_{\rho}(x, y) - K_{\rho}(x, y')] - [K_{\rho}(x', y) - K_{\rho}(x', y')] \right| \leq C \frac{d(x, x')}{\rho(y)} \frac{d(y, y')}{\rho(x)} \frac{1}{\mu(B(x, \rho(x))) + \mu(B(y, \rho(y)))};
\]

(vi) \( \int_X K_{\rho}(x, y) d\mu(x) = 1 \) for all \( y \in X \).

Given a function \( f \) in \( L^1(X) \), following [14], we define

\[
K_{\rho}(f)(x) = \int_X K_{\rho}(x, y) f(y) d\mu(y)
\]

for all \( x \in X \). It follows directly from Lemma 3.1 that

\[
\int_X K_{\rho}(f)(x) g(x) d\mu(x) = \int_X K_{\rho}(g)(x) f(x) d\mu(x)
\]

for all \( f \in L^1(X) \) and \( g \in L^\infty(X) \). Moreover, by Remark 2.1

(3.2) \( K_{\rho}(\phi) \in C_c(X) \) for all \( \phi \in C_c(X) \),

and, for any \( x \in X \), the function \( K_{\rho}(x, \cdot) : X \to \mathbb{R} \), defined by

\[
\mathbb{K}_{\rho}(x, z) := \int_X K_{\rho}(x, y) K_{\rho}(y, z) d\mu(y),
\]

is in \( C_c(X) \). Remark that \( K_{\rho}(K_{\rho}(f))(x) = \int_X \mathbb{K}_{\rho}(x, z) f(z) d\mu(z) \).

The following lemma is due to Yang and Zhou [14].

**Lemma 3.2.** There exists a positive constant \( C \) such that

(i) for any \( f \in L^1(X) \),

\[
\|K_{\rho}(f)\|_{H^1_\rho} \leq C \|f\|_{L^1};
\]

(ii) for any \( g \in H^1_\rho(X) \),

\[
\|g - K_{\rho}(g)\|_{H^1} \leq C \|g\|_{H^1_\rho}.
\]

As a consequence of Lemma 3.2 and (3.1), for any \( \phi \in C_c(X) \),

(3.4) \( \|\phi - K_{\rho}(K_{\rho}(\phi))\|_{BMO} \leq C \|\phi\|_{BMO} \).

Now we are ready to give the proofs of Theorems 1.1 and 1.2

**Proof of Theorem 1.1.** Since \( VMO_\rho(X) \) is a subspace of \( BMO_\rho(X) \), which is the dual space of \( H^1_\rho(X) \), every function \( f \) in \( H^1_\rho(X) \) determines a bounded linear functional on \( VMO_\rho(X) \) of norm bounded by \( \|f\|_{H^1_\rho} \).

Conversely, given a bounded linear functional \( L \) on \( VMO_\rho(X) \). Then,

\[
|L(\phi)| \leq \|L\| \|\phi\|_{VMO_\rho} \leq \|L\| \|\phi\|_{L^\infty}
\]
for all $\phi \in C_c(\mathcal{X})$. This implies (see \cite{12}) that there exists a finite signed Radon measure $\nu$ on $\mathcal{X}$ such that, for any $\phi \in C_c(\mathcal{X})$,

$$L(\phi) = \int_{\mathcal{X}} \phi(x) d\nu(x),$$

moreover, the total variation of $\nu$, $|\nu|(\mathcal{X})$, is bounded by $\|L\|$. Therefore,

$$(3.5) \quad \|K_\rho(K_\rho(\nu))\|_{H^1_\rho} \leq C\|K_\rho(\nu)\|_{L^1} \leq C|\nu|(\mathcal{X}) \leq C\|L\|$$

by Lemma 3.2, where $K_\rho(\nu)(x) := \int_{\mathcal{X}} K_\rho(x, y) d\nu(y)$ for all $x \in \mathcal{X}$.

On the other hand, by (3.4) and (3.2), we have

$$\|\rho \circ L - K_\rho(K_\rho(L))(\phi)\| = \|L(\rho \circ L - K_\rho(K_\rho(\phi)))\|$$

$$\leq \|L\|\|\phi - K_\rho(K_\rho(\phi))\|_{VMO_\rho}$$

$$\leq C\|L\|\|\phi\|_{BMO}$$

for all $\phi \in C_c(\mathcal{X})$, where $K_\rho(K_\rho(L))(\phi) := \int_{\mathcal{X}} K_\rho(K_\rho(\nu))(x) \phi(x) d\mu(x)$. Consequently, by Theorem 2.1(ii), there exists a function $h$ belongs $H^1(\mathcal{X})$ such that $\|h\|_{H^1} \leq C\|L\|$ and

$$(L - K_\rho(K_\rho(L))(\phi) = \int_{\mathcal{X}} h(x) \phi(x) d\mu(x)$$

for all $\phi \in C_c(\mathcal{X})$. This, together with (3.5), allows us to conclude that

$$L(\phi) = \int_{\mathbb{R}^d} f(x) \phi(x) d\mu(x)$$

for all $\phi \in C_c(\mathcal{X})$, where $f := h + K_\rho(K_\rho(\nu))$ is in $H^1_\rho(\mathcal{X})$ and satisfies that $\|f\|_{H^1_\rho} \leq \|h\|_{H^1_\rho} + \|K_\rho(K_\rho(\nu))\|_{H^1_\rho} \leq C\|L\|$. The proof of Theorem 1.1 is thus completed. \hfill \Box

**Proof of Theorem 1.2.** Let $\{f_{n_k}\}_{k=1}^\infty$ be an arbitrary subsequence of $\{f_n\}_{n=1}^\infty$. As $\{f_{n_k}\}_{k=1}^\infty$ is a bounded sequence in $H^1_\rho(\mathcal{X})$, by Theorem 1.1 and the Banach-Alaoglu theorem, there exists a subsequence $\{f_{n_{k_j}}\}_{j=1}^\infty$ of $\{f_{n_k}\}_{k=1}^\infty$ such that $\{f_{n_{k_j}}\}_{j=1}^\infty$ weak* converges to $g$ for some $g \in H^1_\rho(\mathcal{X})$. Therefore, by (3.3), for any $x \in \mathcal{X}$,

$$\lim_{j \to \infty} K_\rho(K_\rho(f_{n_{k_j}}))(x) = \lim_{j \to \infty} \int_{\mathcal{X}} K_\rho(x, z) f_{n_{k_j}}(z) d\mu(z)$$

$$= \int_{\mathcal{X}} K_\rho(x, z) g(z) d\mu(z) = K_\rho(K_\rho(g))(x).$$

This implies that $\lim_{j \to \infty} [f_{n_{k_j}}(x) - K_\rho(K_\rho(f_{n_{k_j}}))(x)] = f(x) - K_\rho(K_\rho(g))(x)$ for almost every $x \in \mathcal{X}$. Hence, by Lemma 3.2 and \cite{3} Theorem 1.1,

$$\|f - K_\rho(K_\rho(g))\|_{H^1} \leq \sup_{j \geq 1} \|f_{n_{k_j}} - K_\rho(K_\rho(f_{n_{k_j}}))\|_{H^1} \leq C \sup_{j \geq 1} \|f_{n_{k_j}}\|_{H^1_\rho} < \infty,$$
moreover,
\[ \lim_{j \to \infty} \int_{\mathcal{X}} [f_{n_{k_j}}(x) - K_{\rho}(K_{\rho}(f_{n_{k_j}}))(x)] \phi(x) d\mu(x) = \int_{\mathcal{X}} [f(x) - K_{\rho}(K_{\rho}(g))(x)] \phi(x) d\mu(x) \]

for all \( \phi \in C_c(\mathcal{X}) \). As a consequence, we obtain that
\[ \|f\|_{H^1_\rho} \leq \|f - K_{\rho}(K_{\rho}(g))\|_{H^1_\rho} + \|K_{\rho}(K_{\rho}(g))\|_{H^1_\rho} \]
\[ \leq C \|f - K_{\rho}(K_{\rho}(g))\|_{H^1_\rho} + C \|g\|_{H^1_\rho} \]
\[ \leq C \sup_{j \geq 1} \|f_{n_{k_j}}\|_{H^1_\rho} < \infty, \]

moreover, by \( \{f_{n_{k_j}}\}_{j=1}^{\infty} \) weak*-converges to \( g \) in \( H^1_\rho(\mathcal{X}) \), (3.1) and (3.2),
\[ \lim_{j \to \infty} \int_{\mathcal{X}} f_{n_{k_j}}(x) \phi(x) d\mu(x) = \lim_{j \to \infty} \int_{\mathcal{X}} [f_{n_{k_j}}(x) - K_{\rho}(K_{\rho}(f_{n_{k_j}})))(x)] \phi(x) d\mu(x) + \lim_{j \to \infty} \int_{\mathcal{X}} f_{n_{k_j}}(x) K_{\rho}(K_{\rho}(\phi))(x) d\mu(x) \]
\[ = \int_{\mathcal{X}} [f(x) - K_{\rho}(K_{\rho}(g))(x)] \phi(x) d\mu(x) + \int_{\mathcal{X}} g(x) K_{\rho}(K_{\rho}(\phi))(x) d\mu(x) \]
\[ = \int_{\mathcal{X}} f(x) \phi(x) d\mu(x). \]
This, by \( \{f_{n_{k_j}}\}_{k=1}^{\infty} \) be an arbitrary subsequence of \( \{f_n\}_{n=1}^{\infty} \), allows us to complete the proof of Theorem 1.2. \( \square \)

4. An Application

The purpose of this section is to give an atomic characterization of \( H^1_\rho(\mathcal{X}) \) by using Theorems 1.1 and 1.2. First, we define the concept of atoms of log-type.

Definition 4.1. Given \( 1 < q \leq \infty \). A measurable function \( a \) is called an \((H^1_\rho, q)\)-atom of log-type related to the ball \( B(x_0, r) \) if
(i) \( \text{supp } a \subset B(x_0, r) \),
(ii) \( \|a\|_{L^q(\mathcal{X})} \leq \mu(B(x_0, r))^{1/q-1} \),
(iii) \( \int_{\mathcal{X}} a(x) d\mu(x) \leq \frac{1}{\log(e + \frac{\mu(B(x_0, r))}{r})} \).

The main result in this section can be read as follows:

Theorem 4.1. Let \( 1 < q \leq \infty \). A function \( f \) is in \( H^1_\rho(\mathcal{X}) \) if and only if it can be written as \( f = \sum_j \lambda_j a_j \), where \( a_j \) are \((H^1_\rho, q)\)-atoms of log-type and \( \sum_j |\lambda_j| < \infty \).
Moreover, there exists a constant $C > 0$ such that, for any \( f \in H^1_\rho(\mathcal{X}) \),

\[
\|f\|_{H^1_\rho} \leq C \inf \left\{ \sum_j |\lambda_j| : f = \sum_j \lambda_j a_j \right\} \leq C \|f\|_{H^1_\rho},
\]

Before giving the proof of Theorem 4.1 let us recall the definition of $H^1_\rho$ atoms introduced by Yang and Zhou [14].

**Definition 4.2.** Given \( 1 < q \leq \infty \). A measurable function \( a \) is called an \((H^1_\rho, q)\)-atom related to the ball \( B(x_0, r) \) if \( r < \rho(x_0) \) and

1. \( \text{supp} \ a \subset B(x_0, r) \),
2. \( \|a\|_{L^q(\mathcal{X})} \leq [\mu(B(x_0, r))]^{1/q-1} \),
3. If \( r < \rho(x_0)/4 \), then \( \int_{B(x_0, r)} a(x) d\mu(x) = 0 \).

**Remark 4.1.** If \( a \) is an \((H^1_\rho, q)\)-atom, then \( \frac{1}{\log(\epsilon+1)} a \) is an \((H^1_\rho, q)\)-atom of log-type.

**Proof of Theorem 4.1.** By Remark 4.1 and [14, Theorems 3.2], it suffices to prove that there exists a constant \( C > 0 \) such that if \( f \) can be written as \( f = \sum_j \lambda_j a_j \), where \( a_j \) are \((H^1_\rho, q)\)-atoms of log-type related to the balls \( B(x_j, r_j) \) and \( \sum_j |\lambda_j| < \infty \), then \( \|f\|_{H^1_\rho} \leq C \sum_j |\lambda_j| \). Since Theorem 1.2 we only need to prove that

\[
\|a_j\|_{H^1_\rho} \leq C
\]

for all \( j \). This is reduced to showing that, for any \( \phi \in C_c(\mathcal{X}) \),

\[
(4.1) \quad \left| \int_{\mathcal{X}} a_j(x) \phi(x) d\mu(x) \right| \leq C \|\phi\|_{BMO_\rho}
\]

by Theorem 1.1. To prove (4.1), let us consider the following two cases:

(a) The case: \( r_j \geq \rho(x_j) \). Then, by the Hölder inequality and [14, Lemma 2.2],

\[
\left| \int_{\mathcal{X}} a_j(x) \phi(x) d\mu(x) \right| \leq \|a_j\|_{L^q(B(x_j, r_j))} \|\phi\|_{L^{q'}(B(x_j, r_j))} \\
\leq \left[ \mu(B(x_j, r_j)) \right]^{1/q-1} C \left[ \mu(B(x_j, r_j)) \right]^{1/q'} \|\phi\|_{BMO_\rho} \\
\leq C \|\phi\|_{BMO_\rho}
\]

where and hereafter \( 1/q' + 1/q = 1 \).
ON WEAK*-CONVERGENCE IN H₁(χ)

(b) The case: \( r_j < \rho(x_j) \). Then, by the Hölder inequality, \([14, Lemma 2.2]\) and \([11, Lemma 2.1]\),

\[
\left| \int_{\chi} a_j(x)\phi(x) d\mu(x) \right| \leq \left| \int_{\chi} a_j(\phi - \phi_{B(x_j,r_j)}) d\mu \right| + \left| \phi_{B(x_j,r_j)} \right| \left| \int_{\chi} a_j d\mu \right|
\]

\[
\leq \|a_j\|_{L^q(B(x_j,r_j))} \|\phi - \phi_{B(x_j,r_j)}\|_{L^{q'}(B(x_j,r_j))} + C\|\phi\|_{BMO}\rho
\]

\[
\leq [\mu(B(x_j,r_j))]^{1/q-1} C [\mu(B(x_j,r_j))]^{1/q'} \|\phi\|_{BMO}\rho + C\|\phi\|_{BMO}\rho
\]

where \( \phi_{B(x_j,r_j)} := \frac{1}{\mu(B(x_j,r_j))} \int_{B(x_j,r_j)} \phi d\mu \). This ends the proof of Theorem 4.1.

\[\Box\]

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