About Hrushovski and Loeser’s work on the homotopy type of Berkovich spaces

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Introduction

At the end of the eighties, V. Berkovich suggested a new approach to non-archimedean analytic geometry ([2], [3]). One of the main advantages of his theory is that it provides spaces enjoying very nice topological properties: although they are defined over fields that are totally disconnected and often not locally compact, Berkovich spaces are locally compact and locally pathwise connected. Moreover, they have turned out to be “tame” objects – in the informal sense of Grothendieck’s *Esquisse d’un programme*; let us illustrate this rather vague assertion by several examples.

i) The generic fiber of any polystable formal scheme admits a strong deformation retraction to a closed subset homeomorphic to a finite polyhedron (Berkovich, [4]).

ii) Smooth analytic spaces are locally contractible; this is also proved in [4] by Berkovich, by reduction to i) through de Jong’s alterations.

iii) If \( X \) is an algebraic variety over a non-archimedean, complete field \( k \), then every semi-algebraic subset of \( X^{an} \) has finitely many connected components, each of which is semi-algebraic; this was proved by the author in [6]. (A subset of \( X^{an} \) is called *semi-algebraic* if it can be defined, locally for the Zariski-topology of \( X \), by a boolean combination of inequalities \( |f| \star \lambda |g| \) where \( f \) and \( g \) are regular algebraic functions, where \( \star \in \{<,>,\leq,\geq\} \), and where \( \lambda \in \mathbb{R}^+ \); it is called *strictly* semi-algebraic if the \( \lambda \)'s can be chosen in \( |k| \).

iv) Let \( X \) be a compact analytic space and let \( f \) be an analytic function on \( X \); for every \( \varepsilon \geq 0 \), denote by \( X_\varepsilon \) the set of \( x \in X \) such that \( |f(x)| \geq \varepsilon \). There exists a finite partition \( \mathcal{P} \) of \( \mathbb{R}^+ \) in intervals such that for every \( I \in \mathcal{P} \) and every \( (\varepsilon,\varepsilon') \in I^2 \) with \( \varepsilon \leq \varepsilon' \), the natural map \( \pi_0(X_{\varepsilon'}) \to \pi_0(X_{\varepsilon}) \) is bijective. This has been established by Poineau in [14] (it had already been proved in the particular case where \( f \) is invertible by Abbes and Saito in [11]).

In their recent work [12], Hrushovski and Loeser vastly improve i), ii), iii), and iv) – at least when all involved data are algebraic. More precisely, let \( k \) be a non-archimedean field, let \( X \) be a quasi-projective algebraic variety over \( k \), and let \( V \) be a semi-algebraic subset of \( X^{an} \). Hrushovski and Loeser prove the following.
A) There exists a continuous map $h : [0; 1] \times V \to V$, and a closed subset $S$ of $V$ homeomorphic to a compact polyhedral complex, such that the following hold:

- $h(0,v) = v$ and $h(1,v) \in S$ for all $v \in V$;
- $h(t,v) = v$ for all $t \in [0;1]$ and all $v \in S$;
- $h(1,h(t,v)) = h(1,v)$ for all $t \in [0;1]$ and all $v \in V$.

B) The topological space $V$ is locally contractible.

C) If $\varphi : X \to Y$ is a morphism of algebraic varieties over $k$, the set of homotopy types of fibers of the map $\varphi^{an}|_V : V \to Y^{an}$ is finite.

D) Let $f$ be a function belonging to $\Theta_X(X)$. For every $\varepsilon \geq 0$ let us denote by $V_\varepsilon$ the set of $x \in V$ such that $|f(x)| \geq \varepsilon$. There exists a finite partition $\mathcal{P}$ of $\mathbb{R}_+$ in intervals such that for every $I \in \mathcal{P}$ and every $(\varepsilon, \varepsilon') \in I^2$ with $\varepsilon \leq \varepsilon'$, the embedding $V_{\varepsilon'} \hookrightarrow V_{\varepsilon}$ is a homotopy equivalence.

Comments

- Since $V$ has a basis of open subsets which are semi-algebraic subsets of $X^{an}$, assertion B is an obvious corollary of assertion A and of the local contractibility of polyhedral complexes.
- Even when $X$ is smooth, projective and when $V = X^{an}$, assertion A was previously known only in the particular case where $X$ has a polystable model.
- The quasi-projectivity assumption on $X$ is needed by Hrushovski and Loeser for technical reasons, but everything remains likely true for general $X$.
- In [12], Hrushovski and Loeser not only prove the above statements, they also develop a new kind of geometry over valued fields. It is based upon highly sophisticated tools in model theory, and it encapsulates, in some sense, all tameness phenomena that occur in Berkovich theory. In fact, most of their work actually concerns that new geometry, and are transferred to the Berkovich framework only at the very end of the paper; for instance, they first prove counterparts of assertions A, B, C and D in their setting.

The purpose of this text is to make a quick survey\textsuperscript{1} of [12]. We will describe roughly the geometry of Hrushovski and Loeser, say a few words about the links between their spaces and those of Berkovich, and give a coarse sketch of the proof of their version of assertion A.

In the last section, we will explain how a key finiteness result of [12] has been used by the author in [9], to show the following: if $X$ is a $n$-dimensional analytic space and if $f : X \to \mathbb{G}^{nan}_{m,k}$ is any morphism, the pre-image of the “skeleton” of $\mathbb{G}^{nan}_{m,k}$ under $f$ inherits a canonical piecewise-linear structure.

\textsuperscript{1}The reader may also refer to the more detailed survey [8].
1 Model theory of valued fields

1.1 Definable functors: the embedded case

Let \( k \) be a field endowed with an abstract Krull valuation; we do not require it to have height one. We fix an algebraic closure \( \bar{k} \) of \( k \), and an extension of the valuation of \( k \) to \( \bar{k} \). Let \( \mathcal{M} \) be the category of algebraically closed valued extensions of \( k \) whose valuation is non-trivial (morphisms are isometric \( k \)-embeddings).

Definable subsets

Let \( X \) and \( Y \) be \( k \)-schemes of finite type and let \( F \in \mathcal{M} \). A subset of \( X(F) \) will be said to be \( k \)-definable if it can be defined, locally for the Zariski-topology of \( X \), by a boolean combination of inequalities of the form \( |f| \times |g| \) where \( f \) and \( g \) are regular functions, where \( \lambda \in |k| \), and where \( \times \in \{ \leq, \geq, \prec, \succ \} \). A map from a \( k \)-definable subset of \( X(F) \) to a \( k \)-definable subset of \( Y(F) \) will be said to be \( k \)-definable if its graph is a \( k \)-definable subset of \( X(F) \times Y(F) = (X \times_k Y)(F) \).

Definable sub-functors

The scheme \( X \) induces a functor \( F \mapsto X(F) \) from \( \mathcal{M} \) to \( \text{Sets} \), which we will also denote by \( X \). A sub-functor \( D \) of \( X \) will be said to be \( k \)-definable if it can be defined, locally for the Zariski-topology of \( X \), by a boolean combination of inequalities of the form \( |f| \times |g| \) where \( f \) and \( g \) are regular functions, where \( \lambda \in |k| \), and where \( \times \in \{ \leq, \geq, \prec, \succ \} \). A natural transformation from a \( k \)-definable sub-functor of \( X \) to a \( k \)-definable sub-functor of \( Y \) will be said to be \( k \)-definable if its graph is a \( k \)-definable sub-functor of \( X \times_k Y \). If \( D \) is a \( k \)-definable sub-functor of \( X \), then \( D(F) \) is for every \( F \in \mathcal{M} \) a \( k \)-definable subset of \( X(F) \); if \( D' \) is a \( k \)-definable sub-functor of \( Y \) and if \( f : D \to D' \) is a \( k \)-definable natural transformation, \( f(F) : D(F) \to D'(F) \) is for every \( F \in \mathcal{M} \) a \( k \)-definable map.

The following fundamental facts are straightforward consequences of the so-called quantifiers elimination for non-trivially valued algebraically closed fields (Robinson, cf. [5], [13] or [16]).

i) For every \( F \in \mathcal{M} \), the assignment \( D \mapsto D(F) \) establishes a one-to-one correspondence between \( k \)-definable sub-functors of \( X \) and \( k \)-definable subsets of \( X(F) \). The analogous result holds for \( k \)-definable natural transformations and \( k \)-definable maps.

ii) If \( f : D \to D' \) is a \( k \)-definable natural transformation between two \( k \)-definable sub-functors \( D \subset X \) and \( D' \subset Y \), the sub-functor of \( D' \) that sends \( F \) to the image of \( f(F) : D(F) \to D'(F) \) is \( k \)-definable.

Comments about assertion i). For a given \( F \in \mathcal{M} \), assertion i) allows to identify \( k \)-definable subsets of \( X(F) \) with \( k \)-definable sub-functors of \( X \), and the choice of one of those viewpoints can be somehow a matter of taste. But even if one is actually only interested in \( F \)-points, it can be useful to think from time to time of a \( k \)-definable subset \( D \) of \( X(F) \) as a sub-functor of \( X \). Indeed, this allows to consider points of \( D \) over valued fields larger than \( F \), which often encode in a natural and efficient way the “limit” behavior of \( D \).
1.2 Abstract definable functors

There is a natural notion of an abstract (i.e., non-embedded) $k$-definable functor: a functor $D$ from $\mathcal{M}$ to $\text{Sets}$ is $k$-definable if there exists a $k$-scheme of finite type $X$, a definable sub-functor $D_0$ of $X$ and an isomorphism $D \cong D_0$.

The functor $D_0$ is not $a \text{ priori}$ uniquely determined (up to unique definable isomorphism). But for all functors $D$ we will consider below, we will be given implicitly not only $D(F)$ for every $F \in \mathcal{M}$, but also $D(T)$ for every $F \in \mathcal{M}$ and every $F$-definable sub-functor $T$ of an $F$-scheme of finite type; otherwise said, if $D$ classifies objects of a certain kind, we not only know what such an object defined over $F$ is, but also what an embedded $F$-definable family of such objects is. Thanks to Yoneda’s lemma, these implicit data will ensure the canonicity of $D_0$ as soon as it exists (see [8], §1 for more detailed explanations about those issues).

But it turns out that this notion of a $k$-definable functor is too much restrictive, for the following reason: in general, the quotient of a $k$-definable functor by a $k$-definable equivalence relation is not itself $k$-definable. For that reason, one decides to also call $k$-definable any functor isomorphic to such a quotient; from now on, we use “$k$-definable” in this new sense. One defines $k$-definable transformations between $k$-definable functors by requiring the graph to be $k$-definable.

There is no conflict of terminology: if we start with an algebraic $k$-variety $X$, a sub-functor of $X$ is abstractly $k$-definable if and only if it is $k$-definable in the original sense.

Assertions i) and ii) have natural counterparts in this context.

- **Counterpart of i).** Let $\Delta$ be a $k$-definable functor. For every $F \in \mathcal{M}$, let us say that a subset of $\Delta(F)$ is $k$-definable if it can be written $D(F)$ for $D$ a $k$-definable sub-functor of $\Delta$. The assignment $D \mapsto D(F)$ then induces a one-to-one correspondence between $k$-definable subsets of $\Delta(F)$ and $k$-definable sub-functors of $\Delta$. The analogous statement for $k$-definable transformations holds.

- **Counterpart of ii).** If $f : D \to D'$ is a $k$-definable natural transformation between two $k$-definable functors the sub-functor of $D'$ that sends $F$ to the image of $f(F) : D(F) \to D'(F)$ is $k$-definable.

**Examples**

The following functors are easily seen to be $k$-definable (one simply has to write them as nice quotients; we leave it to the reader).

- The functor $\Gamma : F \mapsto |F^\times|$.
- The functor $\Gamma_0 : F \mapsto |F|$.
- The functor $F \mapsto \tilde{F}$ (we denote by $\tilde{F}$ the residue field of $F$).
- If $a$ and $b$ are elements of $|k|$ with $a \leq b$, the functor $[a; b]$ that sends $F$ to $\{c \in |F|, a \leq c \leq b\}$. Such a functor is called a $k$-definable interval.
Remark. One can prove that none of those four functors is abstractly \( k \)-definable in the first, restrictive sense we had suggested, before allowing quotients; that is, none of them is isomorphic to a \( k \)-definable subfunctor of a \( k \)-scheme of finite type. What it means can be roughly rephrased as follows, say for \( \Gamma_0 \) (there is an analogous formulation for every of the three other functors): one can not find an algebraic variety \( X \) over \( k \) and a natural way to embed \(|F|\) in \( X(F)\) for every \( F \in M \).

Let us now mention more involved examples.

- One can concatenate \( k \)-definable intervals: one makes the quotient of their disjoint union by the identification of successive endpoints and origins; one gets that way \( k \)-definable generalized intervals. A \( k \)-definable generalized interval has itself an origin and an endpoint, but be aware that it is not in general \( k \)-definably isomorphic to a single \( k \)-definable interval. Indeed, some obstruction may occur because of the 0 element, and more precisely of the following fact: the description of a \( k \)-definable isomorphism between \( k \)-definable generalized intervals only involves monomials (with coefficients in \(|k|\)), and such a monomial sends 0 to itself. Using this remark, the reader can prove for instance that the concatenation of two copies of \([0; 1]\), where the endpoint 1 of the first one is identified with the origin 0 of the second one, is not \( k \)-definably isomorphic to a \( k \)-definable single interval.

But note that the special behavior of 0 is the only reason why such a definable isomorphism can not exist in general: the concatenation of a finite family of \( k \)-definable intervals with non-zero origins and endpoints is always \( k \)-definably isomorphic to a \( k \)-definable single interval.

- One can prove that a sub-functor of \( \Gamma_0^n \) is \( k \)-definable if and only if it can be defined by a boolean combination of monomial inequalities with coefficients in \(|k|\); we will simply call such a functor a \( k \)-definable polyhedron.

- The functor \( B \) that sends \( F \) to the set of its closed balls is \( k \)-definable. To see it, let us denote by \( T \) the \( k \)-definable functor

\[
F \mapsto \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \right\}_{a \in F^*, b \in F},
\]

and by \( T' \) its \( k \)-definable sub-functor \( F \mapsto T(F) \cap \text{GL}_2(F^*) \) (we denote by \( F^* \) the ring \( \{ x \in F, |x| \leq 1 \} \)). The quotient functor \( T/T' \) is \( k \)-definable, and the map that sends

\[
\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}
\]
to the closed ball with center \( b \) and radius \(|a|\) is easily seen to induce a functorial bijection between \( T(F)/T'(F) \) and the set of closed balls of \( F \) of positive radius. The functor \( B \) is thus isomorphic to \( F \mapsto (T(F)/T'(F)) \prod F \), and is therefore \( k \)-definable.
1.3 Hrushovski and Loeser’s fundamental construction

For every $F \in \mathcal{M}$, we denote by $\mathcal{M}_F$ the category of valued extensions of $F$ that belong to $\mathcal{M}$.

The notion of a type

Let $F \in \mathcal{M}$ and let $D$ be a $F$-definable functor. Let $L$ and $L'$ be two valued fields belonging to $\mathcal{M}_F$. Let us say that a point $x$ of $D(L)$ and a point $x'$ of $D(L')$ are $F$-indiscernible if they "satisfy the same formulas with parameters in $F$", that is, if for every $F$-definable sub-functor $\Delta$ of $D$, one has

$$x \in \Delta(L) \iff x' \in \Delta(L').$$

We denote by $S(D)$ the set of couples $(L, x)$ where $L$ belongs to $\mathcal{M}_F$ and where $x$ belongs to $D(L)$, up to $F$-indiscernibility. An element of $S(D)$ is called a type on $D$. The assignment $D \mapsto S(D)$ is functorial in $D$ with respect to $F$-definable natural transformations (apply the transformation to any representative of the type). If $\Delta$ is an $F$-definable sub-functor of $D$, then $S(\Delta)$ can be identified with the subset of $S(D)$ consisting of types admitting a representative $(L, x)$ with $x \in \Delta(L)$ (it will then be the case for all its representatives, by the very definition of $F$-indiscernibility); we simply say that such a type lies on $\Delta$. Any $x \in D(F)$ defines a type on $D$, and one can identify that way $D(F)$ with a subset of $S(D)$, whose elements will be called simple types.

Let $L \in \mathcal{M}_F$, and let $D_L$ be the restriction of $D$ to $\mathcal{M}_L$. Let $t$ be a type on $D_L$. Any representative of $t$ defines a type on $D$, which only depends on $t$ ($L$-indiscernibility is stronger than $F$-indiscernibility); one thus has a natural restriction map $S(D_L) \rightarrow S(D)$.

**Remark.** Let $F \in \mathcal{M}$ and let $x \in S(D_F)$. Let $\mathcal{U}_x$ be the set of subsets of $D(F)$ which are of the kind $\Delta(F)$, where $\Delta$ is an $F$-definable sub-functor of $D_F$ on which $x$ lies. One can rephrase the celebrated compactness theorem of model theory by saying that $x \mapsto \mathcal{U}_x$ establishes a bijection between $S(D_F)$ and the set of ultra-filters of $F$-definable subsets of $D(F)$.

Definable types

Let $t$ be a type on $D$. By definition, one knows $t$ once one knows on which $F$-definable sub-functors of $D$ it lies. We will say that $t$ is $F$-definable if, roughly speaking, the following holds: for every $F$-definable family of $F$-definable subfunctors of $D$, the set of parameters for which $t$ lies on the corresponding $F$-definable sub-functor is itself $F$-definable.

Let us be more precise. If $D'$ is an $F$-definable functor, and if $\Delta$ is an $F$-definable sub-functor of $D \times D'$, then for every $L \in \mathcal{M}_F$ and every $x \in D'(L)$, the fiber $\Delta_x$ of $\Delta_L$ over $x$ is an $L$-definable sub-functor $\Delta_x$ of $D_L$.

We will say that $t$ is $F$-definable if for every such $(D', \Delta)$, the set of $x \in D'(F)$ such that $t$ lies on $\Delta_x$ is a definable subset of $D'(F)$.

If $t$ is an $F$-definable type on $D$, then it admits for every $L \in \mathcal{M}_F$ a canonical $L$-definable pre-image $t_L$ on $S(D_L)$, which is called the canonical extension of $t$. Roughly speaking, $t_L$ is defined by the same formulas as $t$. This means
the following. Let $\Sigma$ be an $L$-definable sub-functor of $D_L$. Considering the coefficients of the formulas that define $\Sigma$ as parameters, we see that there exist an $F$-definable functor $D'$, an $F$-definable sub-functor $\Delta$ of $D \times D'$, and a point $y \in D'(L)$ such that $\Sigma = \Delta_y$.

Now since $t$ is $F$-definable, there exists an $F$-definable sub-functor $E$ of $D'$ such that for every $x \in D'(F)$, the type $t$ lies on $\Delta_x$ if and only if $x \in E(F)$. Then the type $t_L$ belongs to $\Sigma = \Delta_y$ if and only if $y \in E(L)$.

**Orthogonality to $\Gamma$**

Let $t$ be a type on $D$. We will say that $t$ is **orthogonal to $\Gamma$** if it is $F$-definable and if for every $F$-definable natural transformation $f : D \to \Gamma_{0,F}$, the image $f(t)$, which is a priori a type on $\Gamma_{0,F}$, is a a simple type, that is, belongs to $\Gamma_0(F) = |F|$. If this is the case, $t_L$ remains orthogonal to $\Gamma$ for every $L \in M_F$.

It follows from the definitions that any simple type on $D$ is orthogonal to $\Gamma$.

**The case of an algebraic variety**

Let $Y$ be an algebraic variety over $F$ and let $t$ be a type on $Y$. Let $(L,x)$ be a representative of $t$. The point $x \in Y(L)$ induces a scheme-theoretic point $y$ of $Y$ and a valuation $|.|_x$ on the residue field $F(y)$, extending that of $F$; it is easily seen that the data of the point $y$ and of (the equivalence class of) the valuation $|.|_x$ only depend on $t$, and not on the choice of $(L,x)$.

Conversely, if $y$ is a schematic point of $Y$, any valuation on $F(y)$ extending that of $F$ is induced by an isometric embedding $F(y) \to L$ for some $L \in M_F$, hence arises from some type $t$ on $Y$. One gets that way a bijection between $S(Y)$ and the valuative spectrum of $Y$, that is, the set of couples $(y,|.|)$ with $y$ a scheme-theoretic point on $Y$ and $|.|$ a valuation on $F(y)$ extending that of $F$ (considered up to equivalence).

**The affine case.** Let us assume now that $Y = \text{Spec } B$, let $y \in Y$ and let $|.|$ be a valuation on $F(y)$ extending that of $F$. By composition with the evaluation map $f \mapsto f(y)$, we get a valuation $\varphi : B \to |F(y)|$ (the definition of a valuation on a ring is mutatis mutandis the same as on a field – but be aware that it may have a non-trivial kernel); this valuation extends that of $F$.

Conversely, let $\varphi$ be a valuation on $B$ extending that of $F$. Its kernel corresponds to a point $y$ of $Y$, and $\varphi$ is the composition of the evaluation map at $y$ and of a valuation $|.|$ on $F(y)$ extending that of $F$.

This constructions provide a bijection between the valuative spectrum of $Y$ and the set of (equivalence classes of) valuations on $B$ extending that of $F$; eventually, we get a bijection between $S(Y)$ and the set of valuations on $B$ extending that of $F$. Let $t \in S(Y)$ and let $\varphi$ be the corresponding valuation on $B$.

**Interpretation of definability.** The type $t$ is $F$-definable if and only if for every finite $F$-dimensional subspace $E$ of $B$ the following subsets of $E$ are $F$-definable:

- the set of elements $e$ such that $\varphi(e) = 0$;
- the set of elements $e$ such that $\varphi(e) \leq 1$.
Interpretation of orthogonality to $\Gamma$. The type $t$ is orthogonal to $\Gamma$ if and only if it is $F$-definable, and if $\varphi$ takes its values in $\vert F \vert$. This is equivalent to require that $\varphi$ takes its values in $\vert F \vert$ and that $\varphi\vert_E : E \to \vert F \vert$ is $F$-definable for every finite $F$-dimensional subspace $E$ of $B$.

Interpretation of the canonical extension. Assume that $t$ is $F$-definable and let $L \in \mathcal{M}_F$; we want to describe the valuation $\varphi_L$ on $B \otimes_F L$ that corresponds to the canonical extension $t_L$. It amounts to describe the set of elements $e$ of $B \otimes_k L$ such that $\varphi_L(e) = 0$, and that of elements $e$ such that $\varphi_L(e) \leq 1$. It is sufficient to describe the intersection of those subsets with $E \otimes_k L$ for any finite dimensional $F$-vector subspace $E$ of $B$.

Let us fix such an $E$. Since $t$ is definable, there exist two $k$-definable subfunctors $D$ and $D'$ of $E \otimes \bullet$ such that for all $e \in E$ one has

$$\varphi(e) = 0 \iff e \in D(F) \quad \text{and} \quad \varphi(e) \leq 1 \iff e \in D'(F).$$

Now the subset of $E \otimes_k L$ consisting of elements $e$ such that $\varphi_L(e) = 0$ (resp. $\varphi_L(e) \leq 1$) is nothing but $D(L)$ (resp. $D'(L)$).

Let us assume moreover that $t$ is orthogonal to $\Gamma$. In this situation, $\varphi$ induces a $k$-definable map $E \to \vert F \vert$. This definable map comes from a unique $k$-definable transformation $E \otimes \bullet \to \Gamma_0$. By evaluating it on $L$, one gets an $L$-definable map $E \otimes_F L \to \vert L \vert$, which coincides with the restriction of $\varphi_L$.

Definition of $\hat{\mathcal{V}}$
Let $V$ be a $k$-definable functor. Hrushovski and Loeser define a functor

$$\hat{\mathcal{V}} : \mathcal{M} \to \text{Sets}$$

as follows. If $F \in \mathcal{M}$, then $\hat{\mathcal{V}}(F)$ is the set of types on $V_F$ that are orthogonal to $\Gamma$. If $L \in \mathcal{M}_F$, the arrow $\hat{\mathcal{V}}(F) \to \hat{\mathcal{V}}(L)$ is the embedding that sends a type $t$ to its canonical extension $t_L$.

By its construction, the formation of $\hat{\mathcal{V}}$ is functorial in $V$, with respect to $k$-definable maps.

Since any simple type is orthogonal to $\Gamma$, one has a natural embedding of functors $V \hookrightarrow \hat{\mathcal{V}}$. We will therefore identify $V$ with a sub-functor of $\hat{\mathcal{V}}$. For every $F \in \mathcal{M}$, the points of $\hat{\mathcal{V}}$ that belong to $V(F)$ will be called simple points.

1.4 More concrete descriptions of $\hat{\mathcal{V}}$
The case of a polyhedron
Let $F \in \mathcal{M}$. It follows immediately from the definition that a type on $\Gamma_{0,F}$ is orthogonal to $\Gamma$ if and only if it is simple. In other words, $\widehat{\Gamma}_0 = \Gamma_0$.

This fact extends to several functors “locally modeled on $\Gamma_0$” : if $V$ is a $k$-definable polyhedron or a $k$-definable generalized interval, then the natural embedding $V \hookrightarrow \hat{\mathcal{V}}$ is a bijection.
The case of a $k$-definable sub-functor of a $k$-algebraic variety

Let $X$ be an algebraic variety over $k$, and let $V$ be a $k$-definable sub-functor of $X$.

- **First case:** $V = X$, and $X$ is affine, say $X = \text{Spec } A$. From what we have seen above, we get the following description of $\hat{X}$. Let $F \in M$.

  The set $\hat{X}(F)$ is the set of valuations $\varphi : A \otimes_k F \to |F|$ extending that of $F$ and such that for every finite dimensional $F$-vector space $E$ of $A \otimes_k F$, the restriction $\varphi|_E : E \to |F|$ is $F$-definable.

  Let $L \in M_F$ and let $\varphi \in \hat{X}(F)$. For every finite dimensional $F$-vector subspace $E$ of $A \otimes_k F$, the $F$-definable map $\varphi|_E : E \to |F|$ arises from a unique $F$-definable natural transformation $\Phi_E : E \otimes \bullet \to \Gamma_0$, which itself gives rise to a $L$-definable map $\Phi_E(L) : E \otimes_F L \to |L|$. As $E$ goes through the set of all finite dimensional $F$-vector subspaces of $A \otimes_k F$, the maps $\Phi_E(L)$'s glue and define a valuation $A \otimes_k L \to |L|$ extending that of $L$; this is precisely the image $\varphi_L$ of $\varphi$ under the natural embedding $\hat{X}(F) \to \hat{X}(L)$. Roughly speaking, $\varphi_L$ is “defined by the same formulas as $\varphi$.”

  We thus see that Hrushovski and Loeser mimic in some sense Berkovich’s construction, but with a model-theoretic and definable flavour.

- **Second case:** $X = \text{Spec } A$, and $V$ is defined by a boolean combination of inequalities of the form $|f| \ll |g|$ (with $f$ and $g$ in $A$ and $\lambda \in |k|$). The functor $\hat{V}$ is then the sub-functor of $\hat{X}$ consisting of the semi-norms $\varphi$ satisfying the same combination of inequalities.

- **The general case.** One defines $\hat{V}$ by performing the above constructions locally and glueing them.

The topology on $\hat{V}(F)$ in some special cases

Let $V$ be a $k$-definable sub-functor of an algebraic variety over $k$ and let $F \in M$. We are going to define a topology on $\hat{V}(F)$ in some particular cases; all those constructions will we based upon the order topology on $|F|$. 

- If $V$ is an embedded polyhedron, that is, $V$ is explicitly given as a $k$-definable sub-functor of $\Gamma_0^n$, then $\hat{V}(F)$ is endowed with the topology induced from the product topology on $\Gamma_0(F) = |F|^n$.

- In particular, if $V$ is a $k$-definable interval given with an explicit presentation $V \simeq [a; b]$, then $\hat{V}(F)$ inherits a topology.

- If $V$ is a $k$-definable generalized interval, again given with an explicit description, it inherits a topology using the above construction and the quotient topology.

- If $V$ is a $k$-definable sub-functor of a $k$-algebraic variety, it is given a topology as follows. We are going to describe it for $V$ being itself an
affine algebraic variety, say $V = \text{Spec } A$; the general case is obtained by restricting and glueing.

Now any element $a$ of $A \otimes_k F$ defines a map $\hat{V}(F) \to |F|$ (by applying valuations to $a$) and we endow $\hat{V}(F)$ with the coarsest topology making all those maps continuous, for $a$ going through $A \otimes_k F$.

Some comments.

• From now on, every time we will mention a $k$-definable polyhedron, interval or generalized interval, we will implicitly assume that we are given an explicit presentation of it; and we will thus see its set of $F$-points (for given $F \in M$) as a topological space.

• Let $F \in M$ and let $L \in M_L$. Be aware that in general, the topology on $|F|$ induced by the order topology on $|L|$ is not the order topology on $|F|$; it is finer. For instance, assume that there exists $\omega \in |L|$ such that $1 < \omega$ and such that there is no element $x \in |F|$ with $1 < x \leq \omega$ (in other words, $\omega$ is upper infinitely closed to 1 with respect to $|F|$). Then for every $x \in |F^*|$ the singleton $\{x\}$ is equal to $\{y \in |F|, \omega^{-1}x < y < \omega x\}$.

It is therefore open for the topology induced by the order topology on $|L|$; hence the latter induces the discrete topology on $|F^*|$.

Since all the topologies we have considered above ultimately rely on the order topology, this phenomenon also holds for them. That is, let $V$ be a $k$-definable functor belonging to one of the class for which we have defined topologies. The natural embedding $\hat{V}(F) \hookrightarrow \hat{V}(L)$ is not a topological embedding in general; the topology on $\hat{V}(F)$ is coarser than the topology induced from that of $\hat{V}(L)$.

• Let $V$ be a $k$-definable sub-functor of a $k$-algebraic variety and let $F \in M$. The topology on $V(F)$ induced by that of $\hat{V}(F)$ is the natural one, that is, the topology inherited from the topology on the valued field $F$. Moreover, $V(F)$ is a dense subset of $\hat{V}(F)$.

• Let $V$ and $W$ be two $k$-definable functors of one of the aforementioned classes; both inherit a topology, and we will say that a natural transformation $f : \hat{V} \to \hat{W}$ is continuous if $f(F) : \hat{V}(F) \to \hat{W}(F)$ is continuous for every $F \in M$; we then define in the obvious way the fact for $f$ to be a homeomorphism, or to induce a homeomorphism between $\hat{V}$ and a sub-functor of $\hat{W}$, etc.

A fundamental example

Let $F \in M$. For every $a \in F$ and $r \in |F|$, the map $\eta_{a,r,F}$ from $F[T]$ to $|F|$ that sends $\sum a_i(T-a)^i$ to $\max |a_i| \cdot r^i$ is a multiplicative semi-norm on $F[T]$ which belongs to $\hat{A}_1^k(F)$. Note that for every $a \in F$, the semi-norm $\eta_{a,0,F}$ is nothing but $P \mapsto |P(a)|$; hence it is equal to the simple point $a \in F = \hat{A}_1^k(F)$.

If $L \in M_L$, the natural embedding $\hat{A}_1^k(F) \hookrightarrow \hat{A}_1^k(L)$ sends the semi-norm $\eta_{a,r,F}$
(for given \( a \in F \) and \( r \in |F| \)) to the semi-norm on \( L[T] \) “that is defined by the same formulas”, that is, to \( \eta_{a,r,L} \).

**Remark.** If the ground field \( F \) is clear from the context, we will sometimes write simply \( \eta_{a,r} \) instead of \( \eta_{a,r,F} \).

For every \((a, b) \in F^2\) and \((r, s) \in |F|^2\), an easy computation shows that the semi-norms \( \eta_{a,r} \) and \( \eta_{b,s} \) are equal if and only if \( r = s \) and \(|a - b| \leq r\), that is, if and only if he closed balls \( B(a, r) \) and \( B(b, s) \) of \( F \) are equal. Moreover, one can prove that every semi-norm belonging to \( \hat{A}^1_k(F) \) is of the form \( \eta_{a,r} \) for suitable \((a, r) \in F \times |F|\). Therefore we get a functorial bijection between \( \hat{A}^1_k(F) \) and the set of closed balls of \( F \); in particular, the functor \( \hat{A}^1_k \) is \( k \)-definable.

This is also the case of \( \hat{P}^1_k(F) \), which is simply obtained by adjoining the simple point \( \infty \) to \( \hat{A}^1_k(F) \).

**Pro-definability of \( \hat{V} \)**

We fix a \( k \)-definable sub-functor of an algebraic \( k \)-variety \( X \). Hrushovski and Loeser prove that \( \hat{V} \) is pro-\( k \)-definable, and \( k \)-definable if \( \dim X \leq 1 \).

**Some comments.**

- A functor is pro-\( k \)-definable if it is isomorphic to a projective limit of \( k \)-definable functors \(^2\).
- The pro-definability of \( \hat{V} \) comes from general arguments of model-theory, which hold in a very general context, and not only in the theory of valued fields.
- One can in fact prove that \( \hat{V} \) is strictly pro-\( k \)-definable: this means that for every pro-\( k \)-definable natural transformation \( f \) from \( \hat{V} \) to a \( k \)-definable functor \( D \), the direct image functor \( f(\hat{V}) \) is a \( k \)-definable sub-functor of \( D \) (it could \textit{a priori} be only pro-\( k \)-definable, e.g. a countable intersection of \( k \)-definable sub-functors). This comes from model-theoretic properties of the theory of algebraically closed fields, used at the “residue field” level.
- The definability of \( \hat{V} \) in dimension 1 is far more specific. It ultimately relies on Riemann-Roch’s theorem for curves, through the following consequence of the latter: if \( X \) is a projective, irreducible, smooth curve of genus \( g \) over an algebraically closed field \( F \), the group \( F(X)^* \) is generated by rational functions with at most \( g + 1 \) poles (with multiplicities).

**Definable compactness**

Since \( \hat{V} \) is \( k \)-pro-definable, there is a natural notion of a \textit{relatively \( k \)-definable} sub-functor of \( \hat{V} \): this is a sub-functor \( D \) such that there exists a pro-\( k \)-definable natural transformation \( f : \hat{V} \to \Delta \) for \( \Delta \) a \( k \)-definable functor, and a \( k \)-definable sub-functor \( \Delta' \) of \( \Delta \) such that \( D = f^{-1}(\Delta') \).

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\(^2\) Let us quickly mention two issues we will not discuss in full detail.

- If one wants this isomorphism with a projective limit to be canonical, one has to be implicitly given a definition of this functor “in families” (see the corresponding comment for \( k \)-definable functors at section [1,2]); this is the case here: one knows what a \( k \)-definable family of types orthogonal to \( \Gamma \) is.
- The set of indices of the projective system has to be of small enough (infinite) cardinality; in the case of \( \hat{V} \), it can be taken to be countable.
By looking only at $F$-points, one gets the notion of a relatively $k$-definable subset of $\hat{V}(F)$, and more generally of a relatively $F_0$-definable subset of $\hat{V}(F)$ for $F_0$ any valued field lying between $k$ and $F$.

We have defined for every $F \in M$ a topology $\mathcal{T}_F$ on $\hat{V}(F)$. The system $(\mathcal{T}_F)_F$ is by construction definable in the following sense. For every $F \in M$, there exists a set $\mathcal{O}_F$ of relatively $F$-definable sub-functors of $\hat{V}_F$ such that:

- the set $\{D(F)\}_{D \in \mathcal{O}_F}$ is a basis of open subsets of $\hat{V}(F)$;
- for every $D \in \mathcal{O}_F$ and every $L \in M_F$, the sub-functor $D_L$ of $\hat{V}_L$ belongs to $\mathcal{O}_L$ (in particular, $D(L)$ is an open subset of $\hat{V}(L)$).

Replacing $F$-definable sub-functors by relatively $F$-definable ones, in the definitions of section 1.3 one gets the notion of a type on $\hat{V}_F$, and of an $F$-definable type on $\hat{V}_F$. There is a natural bijection between the set of types on $\hat{V}_F$ and that of ultra-filters of relatively $F$-definable subsets of $\hat{V}(F)$.

Let $F \in M$ and let $t$ be a type on $\hat{V}_F$. We say that a point $x \in \hat{V}(F)$ is a limit of $t$ if the following holds: for every $D \in \mathcal{O}_F$ such that $x \in D(F)$, the type $t$ lies on $D$. If $X$ is separated $\hat{V}(F)$ is easily seen to be Hausdorff, and the limit of a type is then unique provided it exists.

We will say that $\hat{V}$ is definably compact if for every $F \in M$, every $F$-definable type on $\hat{V}_F$ has a unique limit in $\hat{V}(F)$.

Remark. Viewing types as ultra-filters, we may rephrase the above definition by saying that $\hat{V}$ is definably compact if for every $F \in M$, every $F$-definable ultra-filter of relatively $F$-definable subsets of $\hat{V}(F)$ converges.

Hrushovski and Loeser prove that $\hat{V}$ is definably compact as soon as it “has to” be so: if $X$ is projective and admits a finite affine covering $(X_i)$ such that $V \cap X_i$ can be defined for every $i$ by non-strict inequalities, $\hat{V}$ is definably compact.

2 Homotopy type of $\hat{V}$ and links with Berkovich spaces

2.1 The link with Berkovich theory

For this paragraph, we assume that the valuation $|.|$ of $k$ takes real values, and that $k$ is complete.

Let $X$ be an algebraic variety over $k$, and let $V$ be a $k$-definable sub-functor of $X$. The inequalities that define $V$ also define without any ambiguity a strict semi-algebraic subset $V^{an}$ of $X^{an}$ (and any strict semi-algebraic subset of $X^{an}$ is of that kind).

Let $F \in M$ be such that $|F| \subset \mathbb{R}_+$. Any point of $\hat{V}(F)$ can be interpreted in a suitable affine chart as a valuation with values in $|F| \subset \mathbb{R}_+$; it thus induces a point of $V^{an}$. We get that way a map $\pi : \hat{V}(F) \to V^{an}$, which is easily seen to be continuous, by the very definitions of the topologies involved. One can say a bit more about $\pi$ in some particular cases.
• The case where \( F = k \). The map \( \pi \) induces then a homeomorphism onto its image.

Let us describe this image for \( V = X = \mathbb{A}^1_k \). By the explicit description of \( \mathbb{A}^1_k \), it consists precisely of the set of points \( \eta_{a,r} \) with \( a \in k \) and \( r \in |k| \), that is, of the set of points of type 1 and 2 (according to Berkovich’s classification).

This generalizes as follows: if \( X \) is any curve, then \( \pi(\hat{V}(k)) \) is precisely the set of points of \( V^{an} \) that are of type 1 or 2.

Comments. The fact that Hrushovski and Loeser’s theory, which focuses on definability, only sees points of type 1 and type 2, encodes the following (quite vague) phenomenon: when one deals only with algebraic curves and scalars belonging to the value groups of the ground field, points of type 1 and type 2 are the only ones at which something interesting may happen.

Of course, considering all Berkovich points is useful because it ensures good topological properties (like pathwise connectedness and compactness if one starts from a projective, connecter variety). Those properties are lost when one only considers points of type 1 and type 2: Hrushovski and Loeser remedy it by introducing the corresponding model-theoretic properties, like definable compactness, or definable arcwise connectedness (see below for the latter).

• The case where \(|F| = \mathbb{R}_+\) and where \( F \) is maximally complete (i.e. it does not admit any proper valued extension with the same value group and the same residue field). In this case, the continuous map \( \pi \) is a proper surjection.

• The case where \( F = k \), where \(|k| = \mathbb{R}_+\) and where \( k \) is maximally complete. Fitting together the aforementioned facts, we see that \( \pi \) establishes then a homeomorphism \( \hat{V}(k) \simeq V^{an} \).

Suppose for the sake of simplicity that \( V = X = \text{Spec } A \). By definition, a point of \( X^{an} \) is a valuation \( \varphi : A \to \mathbb{R}_+ = |k| \) extending that of \( k \). Therefore the latter assertion says that for any finite dimensional \( k \)-vector space \( E \) of \( A \) the restriction \( \varphi|_E \) is automatically \( k \)-definable, because of the maximal completeness of \( k \). This “automatic definability” result comes from the previous work \([11]\) by Haskell, Hrushovski and Macpherson about the model theory of valued fields (which is intensively used by Hrushovski and Loeser throughout their paper); the reader who would like to have a more effective version of it (with kind of an explicit description of the formulas describing \( \varphi|_E \)) may refer to the recent work \([13]\) by Poineau.

Hrushovski and Loeser therefore work purely inside the “hat” world; they transfer thereafter their results in the Berkovich setting using the aforementioned map \( \pi \) and its good properties in the maximally complete case.

For that reason, we will not deal anymore with Berkovich spaces. We are now going to describe the “hat-world” avatar of assertion A of the introduction, and sketch its proof very roughly.

In fact, let \( A_{str} \) be the same assertion as A, except than it only concerns strict semi-algebraic subsets of Berkovich spaces. For the sake of simplicity, the
assertion we will focus on is the avatar of $A_{str}$ and not of $A$. But this is not a serious restriction: indeed, the avatar of $A$ is reduced to that of $A_{str}$ through a nice trick, consisting more or less in “seeing bad scalars as extra parameters” (the reader may refer to [8], §4.3 for detailed explanations).

2.2 The “hat-world” avatar of assertion $A_{str}$

We do not assume anymore that the valuation $|.|$ of $k$ takes real values.

**Theorem.** Let $X$ be a quasi-projective $k$-variety, and let $V$ be a $k$-definable sub-functor of $X$. Let $G$ be a finite group acting on $X$ and stabilizing $V$, and let $E$ be a finite set of $k$-definable transformations from $V$ to $\Gamma_0$ (if $f \in E$, we still denote by $f$ the induced natural transformation $\hat{V} \to \hat{\Gamma}_0 = \Gamma_0$; if $g \in G$, we still denote by $g$ the induced automorphism of $\hat{V}$). There exists:

- a $k$-definable generalized interval $I$, with endpoints $o$ and $e$;
- a $k$-definable sub-functor $S$ of $\hat{V}$;
- a polyhedron $P$ of dimension $\leq \dim X$;
- a $k'$-definable homeomorphism $S \simeq P$, for a suitable finite extension $k'$ of $k$ inside $\overline{k}$;
- a continuous $k$-definable map $h : I \times \hat{V} \to S$ satisfying the following properties for every $F \in M$, every $x \in \hat{V}(F)$, every $t \in I(F)$, every $g \in G$ and every $f \in E$:
  - $h(o,x) = x$, and $h(e,x) \in S(F)$;
  - $h(t,x) = x$ if $x \in S(F)$;
  - $h(e,h(t,x)) = h(e,x)$;
  - $f(h(t,x)) = f(x)$;
  - $g(h(t,x)) = h(t,g(x))$.

**Comments.**

- We will refer to the existence of $k'$, of $P$, and of the $k'$-definable homeomorphism $S \simeq P$ by saying that $S$ is a twisted polyhedron. The finite extension $k'$ of $k$ cannot be avoided; indeed, it reflects the fact that the Galois action on the homotopy type of $\hat{V}$ is non necessarily trivial. In the Berkovich language, think of the $\mathbb{Q}_3$-elliptic curve $E : y^2 = x(x-1)(x-3)$. The analytic curve $E_{\mathbb{Q}_3(i)}$ admits a Galois-equivariant deformation retraction to a circle, on which the conjugation exchanges two half-circles; it descends to a deformation retraction of $E^\mathrm{an}$ to a compact interval.
- We will sum up the three first properties satisfied by $h$ by simply calling $h$ a homotopy with image $S$, and the two last ones by saying that $h$ preserves the functions belonging to $E$ and commutes with the elements of $G$.
- The quasi-projectivity assumption can likely be removed, but it is currently needed in the proof for technical reasons.
• The theorem does not simply assert the existence of a (model-theoretic) deformation retraction of $\hat{V}$ onto a polyhedron; it also ensures that this deformation retraction can be required to preserve finitely many arbitrary natural transformations from $V$ to $\Gamma_0$, and to commute with an arbitrary algebraic action of a finite group. This strengthening of the expected statement is of course intrinsically interesting, but this is not the only reason why Hrushovski and Loeser have decided to prove it. Indeed, even if one only wants to show the existence of a deformation retraction onto a polyhedron with no extra requirements, there is a crucial step in the proof (by induction) at which one needs to exhibit a deformation retraction of a lower dimensional space to a polyhedron which preserves some natural transformations to $\Gamma_0$ and the action of a suitable finite group.

2.3 The case of $\hat{\mathbb{P}}^1_k$

$\hat{\mathbb{P}}^1_k$ is a tree

Let $F \in \mathbb{M}$ and let $x$ and $y$ be two points of $\hat{\mathbb{P}}^1_k(F)$. One proves the following statement, which one can roughly rephrase by saying that $\hat{\mathbb{P}}^1_k$ is a tree: there exists a unique $F$-definable sub-functor $[x; y]$ of $\hat{\mathbb{P}}^1_F$ homeomorphic to a generalized interval with endpoints $x$ and $y$.

Let us describe $[x; y]$ explicitly. If $x = y$ there is nothing to do; if not, let us distinguish two cases. In each of them, we will exhibit an $F$-definable generalized interval $I$ and an $F$-definable natural transformation $\varphi : I \to \hat{\mathbb{P}}^1_F$ inducing a homeomorphism between $I$ and a sub-functor of $\hat{\mathbb{P}}^1_F$, and sending one of the endpoints of $I$ to $x$, and the other one to $y$.

- The case where, say, $y = \infty$. One then has $x = \eta_{a,r}$ for some $a \in F$ and some $r \in |F|$. We take for $I$ the generalized interval $[r; +\infty]$ defined in the obvious way (if $r > 0$ is is homeomorphic to the interval $[0; 1/r]$; if not, one has to concatenate two intervals), and for $\varphi$ the natural transformation given by the formula $t \mapsto \eta_{a,t}$, with the convention that $\eta_{a, +\infty} = \infty$.

- The case where $x = \eta_{a,r}$ and $y = \eta_{b,s}$ for $a, b \in F$ and $r, s \in |F|$. We may assume that $r \leq s$.

  - If $|a - b| \leq s$ then $y = \eta_{a,s}$. We take for $I$ the $F$-definable interval $[r; s]$ and for $\varphi$ the natural transformation given by the formula $t \mapsto \eta_{a,t}$.

  - If $|a - b| > s$ we take for $I$ the concatenation of $I_1 := [r; |a - b|]$ and of $I_2 = [|a - b|; s]$, and for $\varphi$ the natural transformation given by the formulas $t \mapsto \eta_{a,t}$ for $t \in I_1$ and $t \mapsto \eta_{b,t}$ for $t \in I_2$.

If $\Delta$ is a finite, $k$-definable (that is, Galois-invariant) subset of $\hat{\mathbb{P}}^1_k$, the sub-functor $\bigcup_{x,y \in \Delta}[x; y]$ of $\hat{\mathbb{P}}^1_k$ is called the convex hull of $\Delta$. It is a $k$-definable “twisted finite sub-tree” of $\hat{\mathbb{P}}^1_k$. 

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The homotopy type of $\hat{\mathbb{P}}^1_k$

Let $U$ and $V$ be the $k$-definable sub-functors of $\mathbb{P}^1_k$ respectively described by the conditions $|T| \leq 1$ and $|T| > 1$. Note that $V$ is $k$-definably isomorphic to $U$ through $\psi : T \mapsto 1/T$; we still denote by $\psi$ the induced isomorphism $\hat{V} \cong \hat{U}$.

Let $F \in \mathcal{M}$ and let $x \in \hat{U}(F)$. One has $x = \eta_{a,r}$ for some $a \in F^\times$ and some $r \in |F^n|$. For every $t \in |F^n|$, set

$$h(t, x) = \eta_{a, \max(t, r)},$$

and for every $y \in \hat{V}(F)$, set

$$h(t, y) = \psi^{-1}(h(t, \psi(y))).$$

One immediately checks that both definitions of $h$ agree on $\hat{U} \cap \hat{V}$, and we get by gluing a homotopy $h : [0; 1] \times \hat{\mathbb{P}}^1_k \to \hat{\mathbb{P}}^1_k$ with image $\{\eta_{0,1}\}$. Hence $\hat{\mathbb{P}}^1_k$ is “$k$-definably contractible”.

Now let $\Delta$ be a finite, $k$-definable subset of $\hat{\mathbb{P}}^1_k$, and let $D$ be the convex hull of $\Delta \cup \{\eta_{0,1}\}$. Define $h_\Delta : [0; 1] \times \hat{\mathbb{P}}^1_k \to \hat{\mathbb{P}}^1_k$ as follows: for every $x$, we denote by $\tau_x$ the smallest time $t$ such that $h(t, x) \in D$, and we set $h_\Delta(t, x) = h(t, x)$ if $t \leq \tau_x$, and $h_\Delta(t, x) = h(\tau_x, x)$ otherwise. The natural transformation $h_\Delta$ is then a homotopy, whose image is the twisted polyhedron $D$.

2.4 The case of a general algebraic curve

We are going to sketch the proof of the theorem when $X = \hat{V}$ is a projective algebraic curve. There exists then a finite $G$-equivariant map $f : X \to \mathbb{P}^1_k$, inducing a natural transformation $\hat{f} : \hat{X} \to \hat{\mathbb{P}}^1_k$.

The key point is the following: there exists a finite $k$-definable subset $\Delta_0$ of $\mathbb{P}^1_k$ (or, in other words, a divisor on $\mathbb{P}^1_k$) such that for every finite $k$-definable subset $\Delta$ of $\mathbb{P}^1_k$ containing $\Delta_0$, the homotopy $h_\Delta$ lifts uniquely to a homotopy $h_\Delta^\times : [0; 1] \times \hat{X} \to \hat{X}$.

Hrushovski and Loeser prove it by carefully analyzing the behavior of the cardinality of fibers of the fibers $\hat{f}$, as a function from $\hat{\mathbb{P}}^1_k$ to $\mathbb{N}$. The definability of $\hat{X}$ and $\hat{\mathbb{P}}^1_k$ plays a crucial role for that purpose.

Let us now explain why it allows to conclude. Let $\Delta$ be as above, let $D$ be the convex hull of $\Delta \cup \{\eta_{0,1}\}$, and set $D' = \hat{f}^{-1}(D)$.

- By a definability argument, $D'$ is a twisted polyhedron (this is even a “twisted locally finite sub-graph” of $\hat{X}$).
- By choosing $\Delta$ sufficiently big, one may ensure that every function belonging to the orbit of $E$ under $G$ is locally constant outside $D'$; this comes from the fact that every $k$-definable natural transformation from $\hat{X}$ to $\Gamma_0$ is locally constant outside a finite subgraph of $\hat{X}$ (indeed, any $k$-definable function can be described by using only norms of regular functions; and the result for such a norm is deduced straightforwardly from the behavior of $|T|$ on $\hat{\mathbb{P}}^1_k$, which is locally constant outside $[0; \infty]$).
This implies that $h_X$ preserves the functions belonging to the orbit of $E$ under $G$. By uniqueness, $h_X$ commutes with the elements of $G$, and we are done.

A consequence: arcwise connectedness of “hat” spaces

Let $W$ be any $k$-definable sub-functor of a $k$-algebraic variety. We say that $\hat{W}$ is definably arcwise connected if for every $F \in M$ and every $(x, y) \in \hat{W}(F)^2$ there exists an $F$-definable generalized interval $I$ with endpoints $a$ and $e$, and an $F$-definable continuous natural transformation $\varphi : I \to \hat{W}_F$ such that $h(a) = x$ and $h(e) = y$.

It follows easily from the above that if $W$ is a projective curve, then $\hat{W}$ has finitely many arcwise connected components, each of which is $k$-definable (indeed, this holds for any twisted polyhedron). Starting from this result, Hrushovski and Loeser prove in fact that if $Y$ is a $k$-algebraic variety, $\hat{Y}$ is definably arcwise connected as soon as $Y$ is geometrically connected; in general, there is a Galois-equivariant bijection between the set of geometrically connected components of $Y$ and that of arcwise connected components of $\hat{Y}$.

2.5 The general case

Preliminaries

Using some elementary geometrical trick one reduces to the case where the quasi-projective variety $X$ is actually projective of pure dimension $n$ for some $n$; by extending the scalars to the perfect closure of $k$ and by replacing $X$ with its underlying reduced sub-scheme, one can also assume that it is generically smooth. One proceeds then by induction on $n$.

The case $n = 0$ is obvious; now one assumes that $n > 0$ and that the theorem holds for smaller integers.

We will explain how to build a homotopy from the whole space $\hat{X}$ to a twisted polyhedron $S$ which commutes with the elements of $G$ and preserves the characteristic function of $V$ and the functions belonging to $E$; this will be sufficient, since this homotopy will stabilize $\hat{V}$ and send it onto a $k$-definable subset of $S$, hence a twisted polyhedron.

In other words, by adding the characteristic function of $V$ to the set $E$, we can assume $V = X$.

One can blow up $X$ along a finite set of closed points so that the resulting variety $X'$ admits a morphism $X' \to \mathbb{P}^{n-1}_{\bar{k}}$ whose generic fiber is a curve; this is the point where we need $X$ to be projective. By proceeding carefully, one can even ensure the following.

- The action of $G$ on $X$ extends to $X'$, and $X' \to \mathbb{P}^{n-1}_{\bar{k}}$ is $G$-equivariant.
- There exists a $G$-equivariant divisor $D_0$ on $X'$, finite over $\mathbb{P}^{n-1}_{\bar{k}}$ and containing the exceptional divisor $D$ of $X' \to X$, and a $G$-equivariant étale map from $X' \setminus D_0$ to $\mathbb{A}^1_{\bar{k}}$ (in particular, $D_0$ contains the singular locus of $X$).
- There exists a non-empty open subset $U$ of $\mathbb{P}^{n-1}_{\bar{k}}$, whose pre-image on $X'$ is the complement of a divisor $D_1$, and a factorization of $X' \setminus D_1 \to U$ through a finite, $G$-equivariant map $f : X' \setminus D_1 \to \mathbb{P}^1_{\bar{k}} \times_{\bar{k}} U$. 

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If one builds a homotopy from $\hat{X}'$ to a twisted polyhedron which commutes with the action of $G$, preserves the functions belonging to the (pullback of) $E$, and the characteristic function of the exceptional divisor $D$, it will descend to a homotopy on $\hat{X}$ satisfying the required properties, because every connected component of $\hat{D}$ collapses to a point. Hence we reduce to the case where $X' = X$.

**Concatenating a homotopy on the base and a fiberwise homotopy**

This step is the core of the proof. It first consists in applying the relative version of the general construction we have described in §2.4, to the finite map $f$. It provides a divisor $\Delta$ on $\mathbb{P}^1_k \times_k U$ finite over $U$ and a “fiberwise” homotopy $h_\Delta$ on $\mathbb{P}^1_k \times_k U$ which lifts uniquely to a homotopy $h^X_\Delta$ on $\hat{X} \setminus \hat{D}_1$ whose image is a relative (at most) one-dimensional twisted polyhedron over $\hat{U}$. By taking $\Delta$ big enough, one ensures that $h^X_\Delta$ preserves the functions belonging to $E$ and commutes with the elements of $G$. Moreover, we can also assume that the pre-image of $\Delta$ on $X \setminus D_1$ contains $D_0 \setminus D_1$. Under this last assumption, $D_0 \setminus D_1$ is pointwise fixed under $h^X_\Delta$ at every time; therefore $h^X_\Delta$ extends to a homotopy (which is still denoted by $h^X_\Delta$) on $\hat{X} \setminus D_1 \cup \hat{D}_0$ which fixes $\hat{D}_0$ pointwise at every time. This homotopy preserves the functions belonging to $E$ and commutes to the action of $G$, and its image $\Upsilon$ is a relative (at most) one-dimensional twisted polyhedron over $\mathbb{P}^{n-1}_k$. Indeed, this is true by construction over $\hat{U}$; and over the complement of $\hat{U}$, the fibers of $\Upsilon$ coincide with those of $\hat{D}_0$, which are finite.

Our purpose is now to exhibit a homotopy $h^\Upsilon$ on $\Upsilon$, preserving the functions belonging to $E$ and commuting to the action of $G$, whose image is a twisted polyhedron of dimension at most $n$. The key point allowing such a construction is a proposition by Hrushovski and Loeser which ensures that this problem is under algebraic control, in the following sense: there exists a finite quasi-Galois cover $Z \to \mathbb{P}^{n-1}_k$, with Galois group $H$, and a finite family $F$ of $k$-definable natural transformation from $Z$ to $\Gamma_0$, such that for every homotopy $\lambda$ on $\mathbb{P}^{n-1}_k$, the following are equivalent:

i) $\lambda$ lifts to a homotopy on $\hat{Z}$ preserving the functions belonging to $F$;

ii) $\lambda$ lifts to a homotopy on $\Upsilon$ preserving the functions belonging to $E$ and commuting to the action of $G$.

Now by induction hypothesis there exists a homotopy $\lambda'$ on $\hat{Z}$, preserving the functions belonging to $F$ and commuting to the action of $H$, and whose image is a twisted polyhedron of dimension $\leq n - 1$. Since $\lambda'$ commutes with the action of $H$, it descends to a homotopy $\lambda$ on $\mathbb{P}^{n-1}_k$. By its very definition, $\lambda$ satisfies condition i) above, hence also condition ii). Let $h^\Upsilon$ be a homotopy on $\Upsilon$ lifting $\lambda$, preserving the functions belonging to $E$ and commuting to the action of $G$. Since the image of $\lambda'$ is a twisted polyhedron of dimension $\leq n - 1$, so is the image of $\lambda$ (because it is the quotient of that of $\lambda'$ by the action of the finite group $H$). As a consequence, the image $\Sigma$ of $h^\Upsilon$ is a relative (at most) one dimensional twisted polyhedron over a twisted polyhedron of dimension $\leq n - 1$; therefore $\Sigma$ is a twisted polyhedron of dimension $\leq n$.

By making $h^\Upsilon$ follow $h^X_\Delta$, we get a homotopy $h^0$ on $\hat{X} \setminus \hat{D}_1 \cup \hat{D}_0$, preserving the functions belonging to $E$ and commuting to the action of $G$, whose image
is the twisted polyhedron \( \Sigma \).

**Fleeing away from \( \overline{D}_1 \): the inflation homotopy**

Hrushovski and Loeser then define a “inflation” homotopy \( h^\text{inf} : [0; 1] \times \hat{X} \to \hat{X} \)
which fixes pointwise \( \overline{D}_0 \) at every time, preserves the functions belonging to \( E \)
and commutes with the action of \( G \), and which is such that \( h^\text{inf}(t, x) \in \hat{X} \setminus \overline{D}_1 \)
for every \( x \notin \overline{D}_0 \) and every \( t > 0 \).

Let us roughly sketch its construction. One first defines a homotopy \( \alpha \) on \( \hat{A}^n_k \)
by “making the radii of balls increase” (or, in other words, by generalizing to
the higher dimension case the formulas we have given for the affine line). It
has the following genericity property: if \( x \in \hat{A}^n_k(F) \) for some \( F \in \mathcal{M} \), if \( Y \) is a \((n - 1)\)-dimensional Zariski-closed subset of \( \hat{A}^n_k \) and and if \( t \) is a non-
zero element of \( |F^*| \), then \( \alpha(t, x) \notin \hat{Y}(F) \). The homotopy \( \alpha \) is lifted to a
homotopy \( \mu \) on \( \hat{X} \setminus \overline{D}_0 \) thanks to the \( \acute{e} \)tale \( G \)-equivariant map \( X \setminus D_0 \to \hat{A}^n_k \),
and the genericity property of \( \alpha \) is transferred to \( \mu \). The homotpy \( h^\text{inf} \) is then
defined using a suitable “stopping time function” \( x \mapsto \tau_x \) from \( \hat{X} \setminus \overline{D}_0 \) to \( \Gamma \) by
the formulas:

- \( h^\text{inf}(t, x) = \mu(t, x) \) if \( x \notin \overline{D}_0 \) and if \( t \leq \tau_x \);
- \( h^\text{inf}(t, x) = \mu(\tau_x, x) \) if \( x \notin \overline{D}_0 \) and if \( t \geq \tau_x \);
- \( h^\text{inf}(t, x) = x \) if \( x \in \overline{D}_0 \).

Let us quickly explain why \( h^\text{inf} \) satisfies the required properties.

- Its continuity comes from the choice of the stopping time \( \tau_x \): the closer \( x \)
is to \( \overline{D}_0 \), the smaller is \( \tau_x \).
- The \( G \)-equivariance of \( h^\text{inf} \) comes from its construction, and from the \( G \)-
equivariance of \( X \setminus D_0 \to \hat{A}^n_k \).
- The fact that \( h^\text{inf}(t, x) \in \hat{X} \setminus \overline{D}_1 \) for every \( x \notin \overline{D}_0 \) and every \( t > 0 \) is a
particular case of the aforementioned genericity property of \( \mu \).
- The fact that \( h^\text{inf} \) preserves the functions belonging to \( E \) goes as follows.

Those functions are defined using norms of regular functions; now if a regular
function is invertible at a point \( x \) of \( \hat{X} \setminus \overline{D}_0 \), then its norm is constant in a
neighborhood of \( x \), hence will be preserved by \( h^\text{inf}(\cdot, x) \) if \( \tau_x \) is small enough;
this is also obviously true if the function vanishes in the neighborhood of \( x \).

Of course, it may happen that the zero locus of a regular function involved
in the description of \( E \) has some \((n - 1)\)-dimensional irreducible component \( Y \),
and if \( x \in \hat{Y} \setminus \overline{D}_0 \), none of the above arguments will apply around it. One
overcomes this forthcoming issue by an additional work at the very beginning
of the proof: one simply includes such bad components in the divisor \( D_0 \).

**The tropical homotopy**

By first applying \( h^\text{inf} \), and then \( h^0 \), one gets a continuous natural transformation \( h^1 : J \times \hat{X} \to \hat{X} \) for some \( k \)-definable generalized interval \( J \). Its image is
a \( k \)-definable sub-functor \( \Sigma' \) of \( \Sigma \), hence is in particular a twisted polyhedron
of dimension \( \leq n \).

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It follows from its construction that $h^1$ enjoys all required properties, except (possibly) one of them: there is no reason why $\Sigma'$ should be pointwise fixed at every time, because $h_{\text{inf}}$ could disturb it.

Nevertheless, all the above construction can be made in such a way that there exists a “big” $G$-equivariant twisted polyhedron $S \subset \Sigma'$ which remains pointwise fixed at every time under $h^1$ (to ensure the latter property, it is sufficient that $S$ be purely $n$-dimensional, and that $h^1$ preserve a continuous, $\bar{\mathbb{k}}$-definable embedding $S \hookrightarrow \Gamma^m_0$ for some $m$). The last step of the proof then consists in building a homotopy $h^{\text{trop}}$ on $\Sigma'$, preserving the functions belonging to $E$ and commuting to the action of $G$, and whose image is precisely $S$. The construction is purely tropical (that is, piecewise monomial) and rather technical, and we will not give any detail here.

The expected homotopy $h$ on $\hat{X}$ is now defined by applying first $h^1$, and then $h^{\text{trop}}$.

3 An application of the definability of $\hat{C}$ for $C$ a curve

We again assume that the valuation of $k$ takes real values, and that $k$ is complete. For every $n$, denote by $S_n$ the “skeleton” of $G^n_{\text{an}}$. This is the set of semi-norms of the form $n_{r_1,\ldots,r_n} := \sum a_I T_I \mapsto \max |a_I|^r I$. It is naturally homeomorphic to $(\mathbb{R}_+^\times)^n$, hence to $\mathbb{R}^n$ through a logarithm map. This provides it with a rational piecewise-linear structure.

Now let $X$ be a $k$-analytic space of dimension $n$, and let $f : X \to G^n_{\text{an}}$ be a morphism. In [9], the author has proven that $f^{-1}(S_n)$ inherits a canonical rational piecewise-linear structure, with respect to which the restriction of $f$ is a piecewise immersion. This generalizes his preceding work [7], in which some additional assumptions on $X$ were needed, and in which the canonicity of the PL structure (answering a question by Temkin) had not been addressed.

Moreover, the proof given in [7] used de Jong’s alterations, while that given in [9] replaces it by the definability of $\hat{C}$ for $C$ an algebraic curve, which we have mentioned above – and which simply comes from Riemann-Roch theorem for curves –, together with some general results about model theory of valued fields that are established by Haskell, Hrushovski and Macpherson in [11].

The purpose of this section is to give a very rough sketch of this proof.

3.1 Canonical PL subsets of Berkovich spaces

Let $X$ be a (Hausdorff) $k$-analytic space of dimension $n$, and let $P$ be a locally closed subset of $X$. Let us say that $P$ is a $\text{chop}$ if it consists of Abhyankar points of rank $n$ and if it admits a rational PL structure satisfying the following conditions.

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3 Our original terminology was “skeleton”. During the Simons’s symposium, Temkin suggested to rather call it a “chop”, and to use “skeleton” only for chops to which the whole space admits a deformation retraction.
• For every analytic domain $Y$ of $X$ and every invertible function $f$ on $Y$, the intersection $Y \cap P$ is a PL subspace of $P$, and the restriction of $|f|$ to $Y \cap P$ is PL.

• There exist a G-covering $(P_i)$ of $P$ by compact PL subspaces, and for every $i$ an analytic domain $Y_i$ of $X$ containing $P_i$ and invertible functions $f_{i,1}, \ldots, f_{i,n_i}$ on $Y_i$ such that the $|f_{i,j}|$’s induce a PL-isomorphism between $P_i$ and a compact polyhedron of $(\mathbb{R}_+)^{n_i}$.

If such a structure exists on $P$, it is easily seen to be unique.

A simple criterion for being a chop

The conditions for being a chop may seem slightly complicated to check. But in fact, in practical this is not so difficult. Indeed, let $X$ be an irreducible affinoid space of dimension $n$, and let $P$ be a compact subset of $X$ consisting of Abhyankar points of rank $n$. The latter property ensures that every point of $P$ is Zariski-generic. In particular, any non-zero analytic function on $X$ is invertible on $P$.

Now assume that for every finite family $(f_1, \ldots, f_m)$ of non-zero analytic functions on $X$, the image $(|f_1|, \ldots, |f_m|)(P)$ is a polyhedron of $(\mathbb{R}_+)^m$, and that there exists such a family $(f_1, \ldots, f_m)$ with $(|f_1|, \ldots, |f_m|)|_P$ injective. Under this assumption, $P$ is a chop; the two main ingredients of the proofs are the Gerritzen-Grauert theorem, and the density of rational functions on $X$ inside the ring of analytic functions of any rational affinoid domain of $X$.

Rephrasing the main theorem of [9]

Now the theorem we want to prove simply asserts that if $X$ is an $n$-dimensional $k$-analytic space and if $f : X \to \mathbb{G}_m^{\text{an}}$ is a morphism, then $f^{-1}(S_n)$ is a chop.

3.2 The general strategy: algebraization and choice of a faithful tropicalization

Algebraization

To prove our result, we first algebraize the situation by standard arguments. More precisely, one first extends the scalars to the perfect closure of $k$, and then replaces $X$ with its underlying reduced space (the expected assertion is insensitive to radical extensions and nilpotents). Now $X$ is generically quasi-smooth (quasi-smoothness is the Berkovich version of rig-smoothness, see [10], §4.2 et sq.). Since every point of $f^{-1}(S_n)$ is Abhyankar of maximal rank, it is quasi-smooth, and one can thus shrink $X$ so that it is quasi-smooth. Krasner’s lemma then ensures that $X$ is G-locally algebraizable, which eventually allows (because being a chop is easily seen to be a G-local property) to reduce to the case where $X$ is a connected, irreducible rational affinoid domain of $X^{\text{an}}$ for $X$ an irreducible, normal (and even smooth) algebraic variety over $k$.

By density arguments, we also can assume that $f$ is induced by a dominant, generically finite algebraic map from $X$ to $\mathbb{G}_m^n$ (which we still denote by $f$). Now let $x \in f^{-1}(S_n)$. It is Abhyankar of rank $n$, hence Zariski-generic. By openness of finite, flat morphisms, it therefore admits a connected affinoid
neighborhood $U_x$ in $\mathcal{X}^{\text{an}}$ such that $f$ induces a finite and flat map from $U_x$ to an affinoid domain $V$ of $\mathbb{G}_m^{n,\text{an}}$ with $V \cap S_n$ being a non-empty $n$-dimensional simplex. Now there are finitely many such $U_x$’s covering $X \cap f^{-1}(S_n)$; if we prove that $U_x \cap f^{-1}(S_n)$ is a chop for every $x \in S_n$, it will follow that $X \cap f^{-1}(S_n)$ itself is a chop. Therefore we may assume that $f$ induces a finite and flat map from $X$ to an affinoid domain $V$ of $\mathbb{G}_m^{n,\text{an}}$ with $V \cap S_n$ being a non-empty $n$-dimensional simplex.

The key result is now the following.

**Finite separation theorem.** There exist finitely many non-zero rational functions $g_1, \ldots, g_r$ on $X$ whose norms separate the pre-images of $x$ under $f$ for every $x \in S_n$.

Before saying some words about its proof, let us quickly explain why it allows to conclude.

**Pre-image of $S_n$ and tropical dimension**

Let $f_1, \ldots, f_n$ be the invertible functions on $X$ that define $f$. If $x \in \mathcal{X}^{\text{an}}$, the tropical dimension of $f$ at $x$ is the infimum of the dimensions of the polyhedra $([f_1], \ldots, [f_n])(Y)$ for $Y$ going through the set of compact analytic neighborhoods of $x$. One can then characterize $f^{-1}(S_n)$ as the subset of $\mathcal{X}^{\text{an}}$ consisting of points at which the tropical dimension of $f$ is exactly $n$.

**The compact $f^{-1}(S_n) \cap X$ is a chop**

Since every point of $f^{-1}(S_n)$ is Zariski-generic, the functions $g_i$’s are invertible on $f^{-1}(S_n)$; hence we can shrink $X$ so that the $g_i$’s are invertible on it. Let $h_1, \ldots, h_m$ be arbitrary non-zero analytic functions on $X$ and let us denote by $\pi$ the tropicalization

$$(|f_1|, \ldots, |f_n|, |g_1|, \ldots, |g_r|, |h_1|, \ldots, |h_m|) : X \to (\mathbb{R}_+^*)^{n+m+r}.$$

Note that $\pi|_{f^{-1}(S_n) \cap X}$ is injective, because so is already $(|g_1|, \ldots, |g_r|)|_{f^{-1}(S_n) \cap X}$. We will prove that $\pi(f^{-1}(S_n) \cap X)$ is a polyhedron; this will imply (by projection) that so is $(|h_1|, \ldots, |h_m|)(f^{-1}(S_n) \cap X)$, and then allow to conclude that $f^{-1}(S_n) \cap X$ is a chop, in view of the aforementioned criterion.

Let us choose a triangulation of the compact polyhedron $\pi(X)$ by convex compact polyhedra, and let $P$ be the union of the $n$-dimensional closed cells $C$ such that the following holds: the restriction of $(|f_1|, \ldots, |f_n|)$ to $C$ is injective.

Using the characterization of $f^{-1}(S_n)$ through tropical dimension, the openness of finite, flat morphism and the fact that $V \cap S_n$ is non-empty of pure dimension $n$, one proves that $\pi(f^{-1}(S_n) \cap X) = P$, which ends the proof.

**3.3 Definability of “hat-curves” and separation theorem**

The separation theorem asserts that there exist finitely many elements $g_1, \ldots, g_r$ of the function field $k(\mathcal{X})$ which separate the extensions of every real-valued Gau norm on $k(T_1, \ldots, T_n)$. In fact, we establish the more general, purely valuation-theoretic following theorem. Let $k$ be an arbitrary valued field, let $n \in$
Let $L$ be a finite extension of $k(T_1, \ldots , T_n)$. There exists a finite subset $E$ of $L$ such that the following hold: for every ordered abelian group $G$ containing $|k|$ and any $n$-tuple $\mathfrak{r}$ of elements of $G$, the elements of $E$ separate the extensions of the valuation $\eta_\mathfrak{r}$ of $k(T_1, \ldots , T_n)$ to $L$.

**A particular case**

To illustrate the general idea the proof is based upon, let us explain what is going on in a simple particular case, namely if $k$ is non-trivially valued and algebraically closed, and if $n = 1$; we write $T$ instead of $T_1$. The field $L$ can then be written $k(C)$ for a suitable projective, smooth, irreducible curve $C$ over $k$, equipped with a finite morphism $C \to \mathbb{P}^1_k$ inducing the extension $k(T) \hookrightarrow L$.

The map $C \to \mathbb{P}^1_k$ induces a $k$-definable natural transformation between hat spaces $f : \hat{C} \to \mathbb{P}^1_k$; since we work in dimension 1, the source and the target of $f$ are not only pro-$k$-definable, but $k$-definable.

The map $r \mapsto \eta_r$ defines a $k$-definable natural embedding $\Gamma \hookrightarrow \mathbb{P}^1_k$; let $\Delta$ be the pre-image of $\Gamma$ under $f$. This is a sub-functor of $\hat{C}$ which is $k$-definable by its very definition, and the fibers of $\Delta \to \Gamma$ are finite.

Let $F$ be an algebraically closed valued extension of $k$, and let $r \in |F|$. The pre-image $D$ of $\eta_{r,F}$ inside $\Delta(F)$ is by construction definable over the set of parameters $k \cup \{r\}$, and is finite. By a result proven in [11], this implies that every element of $D$ is individually $(k \cup \{r\})$-definable. In other words, for every $d \in D$, there exists a $k$-definable natural transformation $\sigma : \Gamma \to \Delta$ such that $d = \sigma(\eta_{r,F})$.

As this holds for arbitrary $F$ and $r$, the celebrated compactness theorem of model theory ensures the existence of finitely many sections $\sigma_1, \ldots , \sigma_m$ of $\Delta \to \Gamma$ such that $\Delta = \bigcup \sigma_i(\Gamma)$. Every sub-functor $\sigma_i(\Gamma)$ is $k$-definable, and $k$-definably isomorphic to $\Gamma$ through $\sigma_i$. One then easily builds, starting from the equality $\Delta = \bigcup \sigma_i(\Gamma)$, a $k$-definable isomorphism between $\Delta$ and a suitable polyhedron $P \subset \Gamma^N$ (for some big enough $N$ – one simply has to be able to realize the “coincidence diagram” of the $\sigma_i$’s inside $\Gamma^N$).

Now Hrushovski and Loeser have proven that is $S$ is a $k$-definable subfunctor of $\hat{C}$ such that there exists a $k$-definable abstract isomorphism $S \simeq Q$ with $Q$ a polyhedron, then there exists such an isomorphism induced by the norms of finitely many $k$-rational functions on $\hat{C}$; this comes from the explicit description of $\hat{C}$ as a $k$-definable functor, and from general results established in [11]. Applying this to $\Delta$, we get the existence of finitely many rational functions on $C$ whose norms separate points of $\Delta$; in particular those functions separate for every algebraically closed valued extension $F$ of $k$ and every $r \in |F^*|$ the pre-images of $\eta_{r,F}$ in $\hat{C}(F)$. They a fortiori separate the extensions of the valuation $\eta_{r,k}$ to the field $L$ (indeed, it is easily seen that such an extension is always the restriction of a valuation on $F(C_F) = L \otimes_{k(T)} F(T)$ inducing $\eta_{r,F}$ on $F(T)$).

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It is crucial for this result that the additional parameter $r$ belongs to the value group of $F$, and not to $F$ itself. For instance, let $a$ be an element of $F$ which is transcendental over $k$, assume that char. $k \not= 2$ and let $E$ be the two-element set of square roots of $a$. Then $E$ is globally definable over $k \cup \{a\}$, but this is not the case of any of those two square roots: one cannot distinguish between them by a formula involving only $a$ and elements of $k$. 

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The general case

The proof goes by induction on \( n \). The crucial step is of course the one that consists in going from \( n - 1 \) to \( n \), and it roughly consists of a relative version of what we have done above.

**Erratum concerning [9].** As we have seen, one of the key points of the proof is the fact that abstract (relative) polyhedra embedded into (relative) hat curves can be actually described concretely using norms of rational functions. This argument is unfortunately not mentioned in [9], where it is used only implicitly — it should have been written down on page 36, in the paragraph beginning with “Fixons \( i \). Par ce qui précède, il existe...”.

### 3.4 Some complements

Let us again assume that \( k \) is complete for a non-archimedean real valuation, and let \( X \) be an \( n \)-dimensional \( k \)-analytic space.

**Finite union of pre-image of skeleta**

Let \( f_1, \ldots, f_m \) be morphisms from \( X \) to \( \mathbb{G}^{n,an}_m \).

As we have explained, \( f_i^{-1}(S_n) \) is a chop for every \( i \). We in fact also prove in [9] that the union \( \bigcup_i f_i^{-1}(S_n) \) is still a chop; this essentially means that for every \( (i, j) \), the intersection of the chops \( f_i^{-1}(S_n) \) and \( f_j^{-1}(S_n) \) is PL in both of them. The proof consists in exhibiting in a rather explicit way an integer \( N \), a “Shilov section with non-constant radius” \( \sigma : X \to \mathbb{A}^N_X \), and a chop \( P \subset \mathbb{A}^N_X \) (described as the pre-image of the skeleton under a suitable map) such that for every \( i \), the section \( \sigma \) identifies \( f_i^{-1}(S_n) \) with a PL subspace of \( P \).

**Stabilization after a finite, separable extension**

For every complete extension \( F \) of \( k \), let \( \Sigma_F \subset X_F \) be the chop \( \bigcup_i f_i^{-1}(S_n,F) \), where \( S_n,F \) is the skeleton of \( \mathbb{G}^{n,an}_m,F \).

If \( F \hookrightarrow L \), there is a natural surjection \( \Sigma_L \to \Sigma_F \), which is a piecewise immersion of PL spaces. We also prove in [9] that there exists a finite separable extension \( F \) of \( k \) such that \( \Sigma_L \to \Sigma_F \) is a homeomorphism for every complete extension \( L \) of \( k \). To see it, one essentially refines the proof of the “separation theorem” on Gauss norms that is sketched in §3.3 to show that after making a finite, separable extension of the ground field, one can exhibit a finite family of functions separating universally (that is, after any extension of the ground field) the extensions of Gauss norms.

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