Pulsating fields of a thin film in a static magnetic field under vertical vibrations

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Abstract. In the present paper a dynamics of a thin ferrofluid film under the vertical vibration in a static magnetic field is examined. The vibrational amplitude is assumed to be greater than film thickness so that vibrational force is greater than magnetic and gravitational forces. The pulsating part and the averaged part of the hydrodynamics fields are obtained. The solution of pulsating part for the traveling surface wave is found. The equation for the averaged surface profile is found.

1. Introduction

Magnetic fluids or ferrofluids are a colloidal dispersion of single-domain particles in nonconducting fluid. The majority of all applications of ferrofluid is based on the behaviour of film surface interface in the presence of the magnetic field [1-3].

In [4] instability of a horizontal surface interface on the solid plate in a stationary vertical magnetic field was found. This “frozen” instability can be theoretically observed based on the dispersion ratio of a nonviscous surface wave. In the [5] the parametric instability of a surface ferrofluid film subjected to vertical vibrations in a stationary magnetic field was theoretically examined. In this case, the influence of vibrations can be reduced to a periodic modulated gravity field. Therefore the initial state magnetization is dependent on time. It was shown that in a weak magnetic field the parametric (Faraday) instability occurs and in a strong magnetic field the Rosenzweig instability occurs. The experimental results can be found in [6, 7]. In [8] comparable theoretical study was carried out for the stationary horizontal magnetic field.

From the one side, the viscosity has a significant impact on the abovementioned effects that is why for high frequencies (the vibrational period is greater than viscous attenuation time) it is possible to divide flow into inviscous flow kernel and thin (Stokes) viscous layer near solid plane, but this approximation is not suitable for thin films [see 9]. From the other side, at high frequency (Ω is a vibrational frequency) time dependency of all fields can be divided into two time scales: oscillatory scale (which is proportional to Ω⁻¹) and dissipative scale (which is proportional to L²ν⁻¹, where L is a characteristic spatial scale and ν is a kinematic viscosity). According to the multiscale method idea, one can introduce two times into the initial system – the fast time and the averaged time. Therefore one can represent all the hydrodynamics fields as a sum of the pulsating part and the averaged part.

The interface thin fluid film stability is usually studied in a longwave approximation. In [10] a thin horizontal magnetic film in a stationary vertical magnetic field subjected to the vertical vibrations was considered. The vibrational amplitude was comparable to the thin film thickness like in [5, 8].
In the present article we investigate the behaviour of ferrofluid thin film under the vertical vibrations in the vertical static magnetic field. A thin film is supposed to be so thick that vibrational amplitude is greater than thickness film according to [9] and as opposed to [5, 8].

2. Problem statement
A thin magnetic film in a homogeneous vertical magnetic field subjected to vertical vibrations in a gravitational field is under consideration. The geometry of the problem is shown in figure 1.

![Figure 1. The geometry of the problem.](image)

The article aims to determine the influence of the vertical vibrations on the film instability caused by the homogeneous magnetic field.

The governing equations have the following form [9]:

\[
\nabla v = 0, \\
\frac{\partial v}{\partial t} + v\nabla v = -\frac{1}{\rho} \nabla p + \nu \Delta v + f, \\
\nabla B = 0, \nabla \times H = 0
\]

where \( v \) is the fluid velocity, \( p, \nu, \rho \) is the pressure, viscosity and density of a fluid, \( f \) is the external force, \( H \) is the magnetic field strength, \( B \) is the magnetic field induction.

There are two mechanisms in the systems responsible for fluid movement. Clearly, the first one is the vertical vibrations, the second one is the magnetic field. Both of them drive the interface and we examine the effect they have relative to each other.

Let us introduce the rectangular coordinate system with the \( i, j \) basis as shown in figure 1. The presence of the vertical vibrations allows rewriting the gravitational field in the shape of \( g = g_{eff} + a\Omega^2 \sin(\Omega t)j \), which from a physical standpoint means changing the reference frame from laboratory system to the one connected with the vessel. The equations \( y = \xi(x,t) \) and \( y = -c_0 \) represent the free interface of the moving magnetic fluid and the solid plane correspondingly.

Assuming that \( u, v \) are the \( x \)- component and \( y \)- components of the velocity, respectively, the boundary conditions in the abovementioned terms can be written as:

\[
y = -c_0 : u = \nu = 0, \\
y = \xi(x,t) : \frac{\partial \xi}{\partial t} + v \nabla \xi = \nu_y,
\]

\[
p - p_a = \frac{2\rho v}{1 + \left(\frac{\partial \xi}{\partial x}\right)^2} \left[ \left(\frac{\partial^2 \xi}{\partial x^2}\right)^2 - 1 \right] - \left[ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \frac{\partial \xi}{\partial x} \right] - \frac{\alpha}{\rho} \frac{\partial^2 \xi}{\partial x^2} \left[ 1 + \left(\frac{\partial \xi}{\partial x}\right)^2 \right]^{-\frac{3}{2}} - \frac{\mu_a}{2} \left[ \frac{M_y - M_x}{M_x} \left(\frac{\partial \xi}{\partial x}\right) \right]^2
\]
where \( p_a \) is the pressure amplitude on the curved interface, \( \alpha \) is the coefficient of the surface tension, \( \mu_s \) is the magnetic constant, \( M_x \) and \( M_y \) are the \( x \)– and \( y \)– components of the magnetization vector, respectively. The first boundary condition (4) is no-slip condition on a solid plate. The second condition (5) is the kinematic condition corresponding to the surface velocity depending on the fluid velocity. The third boundary condition (6) is the condition of the continuity of normal stresses and consists of three terms - the continuity of viscous stresses, the contribution of the surface curvation and the jump of the magnetic field value from one medium to another. The third condition can be derived from the expression

\[
p - p_a = \rho(\frac{\partial v}{\partial x_j} + \frac{\partial v}{\partial x_i})n_i n_j + \alpha n \nabla n \cdot \frac{1}{2} \mathbf{M} n. \quad \mathbf{n} = \frac{\nabla F}{|\nabla F|},
\]

where summation over repeated subscripts \( i, j \) is assumed, \( \mathbf{n} \) is a normal vector to the surface, \( F = y - \xi(x, t) \).

3. The asymptotic analyse in the thin film approximation

Since we consider a thin film, where longitudinal size \( (L) \) is much greater than transverse size \( (c_0) \) \( \varepsilon = c_0 / L \ll 1 \) it is possible to significantly simplify the initial differential system by the multiscale method. Having estimated the orders of magnitude in the equations, we proceed to dimensionless variables, setting \( c_0, c_0^2 v^{-1}, v c_0^{-1}, \rho v^2 c_0^{-2} \) as the scales of length, time, velocity and pressure respectively. The existence of a thin film allows the introduction of scaling by spatial variables and therefore one can rewrite: \( X = \varepsilon x, Y = y, A = \varepsilon A, u = \varepsilon U \) and \( v = \varepsilon^2 V \), where capital letters stand for new variables.

The presence of two different characteristic times in the system (the time of averaged processes and the characteristic time of pulsations) makes it possible to divide the fields into averaged and pulsating ones. Thus, the velocity components, the pressure field and the surface film coordinate can be represented as:

\[
\begin{align*}
u &= \langle u \rangle + \frac{1}{\varepsilon} \tilde{u}, \quad v &= \langle v \rangle + \frac{1}{\varepsilon} \tilde{v}, \\
p &= \langle p \rangle + \frac{1}{\varepsilon} \tilde{p}, \\
\xi &= \langle \xi \rangle + \varepsilon \tilde{\xi},
\end{align*}
\]

where brackets mean averaged value and tilde means the pulsating part. The magnetic field is represented as

\[
\mathbf{H} = H_0 \mathbf{j} + \varepsilon \langle \mathbf{h} \rangle + \tilde{\mathbf{h}},
\]

where \( H_0 \) is a stationary solution. Pulsating time has a scale of \( \omega t \), whereas average time has a scale of \( \varepsilon^2 t \).

The introduction of two-time scales together with a linearization allows dividing the governing equations into pulsation part and the averaged flow. The pulsating part of the problem after linearization has the following form:

\[
\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2}, \quad \omega t = \frac{\partial p}{\partial x} + \alpha \omega^2 \sin t = 0, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,
\]

the boundary conditions have the form of:

\[
y = -1: u = v = 0,
\]
\[
y = \langle \xi(x,t) \rangle > : \frac{\partial u}{\partial y} = 0, \ p = 0, \ \omega \frac{\partial \xi}{\partial t} + u \frac{\partial < \xi >}{\partial x} = v, \quad (11)
\]
where prime symbols are omitted, \( \omega \) is dimensionless pulsating frequency, \( \frac{\partial}{\partial t} \) is the derivative with respect to pulsating time, \( \frac{\partial}{\partial T} \) is the derivative with respect to averaged time. The averaged part of the initial system has the shape of:

\[
\begin{align*}
\omega \frac{\partial < u >}{\partial t} + < u \frac{\partial u}{\partial x} > + < v \frac{\partial u}{\partial y} > &= - \frac{\partial < p >}{\partial x} + \frac{\partial^2 < u >}{\partial y^2} + < F_m >, \\
\omega \frac{\partial < v >}{\partial t} &= - \frac{\partial < p >}{\partial y} - g + V \bar{h}_v \frac{\partial \bar{h}_x}{\partial y}, \\
\frac{\partial < u >}{\partial x} + \frac{\partial < v >}{\partial y} &= 0,
\end{align*}
\]

\[
y = -1: < u >= < v > = 0, \quad (15)
\]

\[
y = \langle \xi(x,t) \rangle > : \frac{\partial < u >}{\partial y} = 0, \ p = G < \xi > - \frac{S}{2} H_m, \ \frac{\partial < \xi >}{\partial T} + < u > \frac{\partial < \xi >}{\partial x} = < v >, \quad (16)
\]

where \( V = \mu_0 \chi \rho^{-1} v^{-2} H_0^2 c_0^2 \), \( G = g c_0^2 v^{-2} \) is the gravitational number, \( S = \mu_0 c_0^2 \rho^{-1} v^{-2} \), \( H_m \) is a contribution of the magnetic field to the averaged flow. Therefore, the pulsating fields do not affect the magnetic field and the impact of the magnetic field is essential only on the averaged flow.

**Figure 2.** Amplitude \( A \) of the longitudinal component of the pulsating velocity. Real (a) and imaginary (b) parts.

\( \omega = 0.01 \) – red line, \( \omega = 0.1 \) – blue, \( \omega = 1 \) – green, \( \omega = 10 \) – black.

The main interest is the influence of pulsation amplitudes on the average flow, so the solution of the pulsating system is presented in the exponential form:

\[
u = a \omega \text{Re} \left( A(x,y) e^{i\phi} \right), \quad p = a \omega \text{Re} \left( B(x,y) e^{i\phi} \right), \quad \xi = a \omega \text{Re} \left( D(x) e^{i\phi} \right).
\]
Figure 3. Stream function $\psi$ of pulsations at $t=0$, $0.25\pi$, $0.5\pi$, $0.75\pi$ (from top to bottom).

(a) $\omega = 1$, (b) $\omega = 10$.

4. The solution of pulsating part
The solution for the amplitude part is as follows:

$$A = -\left(\frac{\text{ch}(\lambda (y - \xi))}{\text{ch}(\lambda (1 + \xi))}\right) \frac{\partial \xi}{\partial x}, \quad \lambda^2 = i\omega, \quad C = i\omega (y - \xi),$$

$$B = -\frac{1}{\lambda \text{ch}(\lambda (1 + \xi))} \left[\left(\text{sh}(\lambda (1 + \xi)) + \text{sh}(\lambda (y - \xi))\right) \frac{\partial^2 \xi}{\partial x^2} + \lambda \frac{1 - \text{sh}(\lambda (y + 1))}{\text{ch}(\lambda (1 + \xi))} \left(\frac{\partial \xi}{\partial x}\right)^2\right] + (y + 1) \frac{\partial^2 \xi}{\partial x^2},$$
\[ D = -\frac{i}{\omega} \left( B - A \frac{\partial \xi}{\partial x} \right). \]

where \( \xi \) is regarded as the averaged value.

Examples of the distribution of \( \text{Re}(A) \) and \( \text{Im}(A) \) across the ferrofluid film are given in figure 2 for different value of \( \omega \). Since \( A \) is proportional to \( \partial \xi / \partial x \), the value of the latter derivative only rescales the longitudinal velocity and for simplicity we assume \( \partial \xi / \partial x = 1 \). To make more clear the behavior of the pulsation velocity we show in figure 3 the isolines of the pulsation stream function \( \psi \) at four progressive moments of oscillations time-period, assuming \( \xi = 0.1 \cos(x) \).

5. Conclusion

In this paper an investigation of ultrathin ferrofluid behaviour in a homogeneous magnetic field under the vertical vibrations was begun. The vibrational amplitude is of a great scale than the film thickness. The initial equations were divided into two parts, the hydrodynamics fields were represented as a sum of the pulsating part and the averaged part according to oscillatory scale and the viscous attenuation scale respectively. The solution of pulsating part for the traveling surface wave is found. The equation for the averaged surface profile is found.

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