Modular Invariance, Finiteness, and Misaligned Supersymmetry: New Constraints on the Numbers of Physical String States

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Abstract

We investigate the generic distribution of bosonic and fermionic states at all mass levels in non-supersymmetric string theories, and find that a hidden “misaligned supersymmetry” must always appear in the string spectrum. We show that this misaligned supersymmetry is ultimately responsible for the finiteness of string amplitudes in the absence of full spacetime supersymmetry, and therefore the existence of misaligned supersymmetry provides a natural constraint on the degree to which spacetime supersymmetry can be broken in string theory without destroying the finiteness of string amplitudes. Misaligned supersymmetry also explains how the requirements of modular invariance and absence of physical tachyons generically affect the distribution of states throughout the string spectrum, and implicitly furnishes a two-variable generalization of some well-known results in the theory of modular functions.

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1 Introduction: Motivation and Overview of Results

The distribution of states in string theory is an important but not particularly well-understood issue. It is well-known, for example, that string theories generically contain a variety of sectors, each contributing an infinite tower of states from the massless level to the Planck scale, and it is also a generic feature that the number of these states as a function of the worldsheet energy $n$ grows asymptotically as $e^{C\sqrt{n}}$ where $C$ is the inverse Hagedorn temperature of the theory. Beyond these gross features, however, not much is known. For example, modular invariance presumably tightly constrains the numbers of string states at all energy levels, but a precise formulation of such a constraint is still lacking. Similarly, the distribution of bosonic and fermionic string states at all energy levels is crucial in yielding the ultraviolet finiteness for which string theory is famous, yet it is not clear precisely how the actual distribution of such states level-by-level conspires to achieve this remarkable result. Of course, if the string theory in question exhibits spacetime supersymmetry, both issues are rendered somewhat trivial: there are necessarily equal numbers of bosonic and fermionic states at every energy level, the one-loop partition function vanishes, and the divergences from bosonic states are precisely cancelled by those from fermionic states. Yet how does the string spectrum manage to maintain modular invariance and finiteness in the absence of spacetime supersymmetry? Alternatively, to what extent can one break spacetime supersymmetry in string theory without destroying these desirable features?

In this paper we shall provide some answers to these questions, and in particular we shall demonstrate that even in the absence of spacetime supersymmetry, string spectra generically turn out to exhibit a residual cancellation, a so-called “misaligned supersymmetry”. In fact, this property will turn out to be completely general, and will describe the distribution of bosonic and fermionic states in any string theory which is modular-invariant and free of physical tachyons. Moreover, we will also see that this result can be interpreted as the two-variable generalization of some well-known theorems in the mathematical theory of modular functions. Misaligned supersymmetry is therefore a general result with many applications, and in the rest of this introductory section we shall outline the basic issues and results. Details can then be found in subsequent sections.

1.1 Motivation: Some Background Issues in String Theory

In order to gain insight into the relevant string issues, we shall begin by discussing some of the questions raised above.
1.1.1 How does modular invariance constrain the numbers of states in string theory?

Modular invariance is a powerful symmetry, arising at the one-loop (toroidal) level in theories with two-dimensional conformal symmetry as a consequence of the existence on the torus of “large” diffeomorphisms which are not connected to the identity. It is usually expressed as a constraint on the one-loop partition functions \( Z(\tau) \) of such theories; here \( \tau \) is the complex torus modular parameter, and modular invariance requires \( Z(\tau) = Z(\tau + 1) = Z(-1/\tau) \). This constraint is indeed quite restrictive, limiting the combinations of conformal-field-theory characters which can appear in the partition functions of modular-invariant theories. However, the partition function \( Z(\tau) \) also contains within it information concerning the net degeneracies of physical states at all energy levels in the theory. How then does modular invariance constrain these degeneracies?

In order to formulate this question more mathematically, let us first review some basic facts about one-loop partition functions. Given a torus with modular parameter \( \tau \) and a two-dimensional field theory defined on that torus with left- and right-moving Hamiltonians \( H \) and \( \tilde{H} \) respectively, the one-loop partition function is defined

\[
Z(\tau) \equiv \sum_s \text{Tr}_s (-1)^F q^H \tilde{q}^{\tilde{H}} . \tag{1.1}
\]

Here \( q \equiv e^{2\pi i \tau} \), the sum is over the different sectors \( s \) in the theory which are needed for modular invariance, the trace is over the Fock space of excitations with different \( H \) and \( \tilde{H} \) eigenvalues in that sector, and \( F \) is the spacetime fermion-number operator in the theory (= 0 for excitations which are spacetime bosons, = 1 for spacetime fermions). Since these traces are simply the various characters \( \chi_i \) and \( \chi_j \) of the relevant underlying worldsheet conformal field theories, one can typically write \( Z \) in the form

\[
Z = (\text{Im} \, \tau)^{1-D/2} \sum_{ij} N_{ij} \chi_i(q) \chi_j(\bar{q}) \tag{1.2}
\]

where \( \chi_i(q) \) and \( \chi_j(\bar{q}) \) are characters of the chiral left- and right-moving conformal field theories, \( D \) is number of uncompactified spacetime dimensions, the summation is over the various contributing sector combinations \( (i, j) \), and \( N_{ij} \) is a coefficient matrix constrained by modular invariance. Different choices for \( N_{ij} \) correspond to different theories. Since the modular transformation properties of the characters \( \chi \) are typically well-known, the modular invariance of (1.2) is usually easy to verify. However, it is often useful to expand \( Z \) as a double power series in \( q \) and \( \bar{q} \):

\[
Z(q, \bar{q}) = (\text{Im} \, \tau)^{1-D/2} \sum_{m,n} a_{mn} \bar{q}^m q^n . \tag{1.3}
\]

Written in this form, each coefficient \( a_{mn} \) is the net number of states or degrees of freedom in the theory with \( H \)-eigenvalue \( n \) and \( \tilde{H} \)-eigenvalue \( m \); typically \( m \) and \( n \)
can have both integer and non-integer values, and modular invariance requires that
\( m = n \) (modulo 1) if \( a_{mn} \neq 0 \). By “net” we mean the number of spacetime bosonic
degrees of freedom minus the number of those which are spacetime fermionic. While
states with \( m = n \) correspond to actual physical (“on-shell”) particles in spacetime,
those with \( m \neq n \) (the so-called unphysical or “off-shell” states) do not contribute
to tree-level amplitudes. Thus, it is the quantities \( \{a_{nn}\} \) for all \( n \geq 0 \) which give the
net degeneracies of physical states at all mass levels in the theory, and which form
our object of interest.

While it is usually quite straightforward to derive the constraints on \( N_{ij} \) arising
from modular invariance, it proves surprisingly difficult to see how these translate
into a constraint on the net degeneracies \( \{a_{nn}\} \). Indeed, the generic behavior of
\( \{a_{nn}\} \) required by modular invariance is almost completely unknown. Our res ult will
provide such a general constraint.

1.1.2 How does the presence of unphysical tachyons affect the balance
between bosonic and fermionic states in string theory?

A slightly more physical way of addressing the same issue is to focus instead on
the tachyonic states which generically appear in string theory. Recall that since the
worldsheet Hamiltonians \( H \) and \( \tilde{H} \) correspond to the spacetime left- and right-moving
(mass)\(^2 \) of the states, we see that states with \( m, n < 0 \) correspond to spacetime
tachyons. A theory is thus free of physical tachyons if \( a_{nn} = 0 \) for all \( n < 0 \) in
its partition function. Note, in this regard, that the statement \( a_{nn} = 0 \) for all
\( n < 0 \) actually implies the absence of all tachyons whatsoever, for there can never be
fermionic physical tachyons in a unitary string theory. By contrast, a situation with
no net unphysical tachyons merely implies that bosonic and fermionic unphysical
tachyons occur in equal numbers.

Now, it is well-known that any \( D > 2 \) string theory in which there are no net
numbers of physical or unphysical tachyons must necessarily have equal numbers of
bosonic and fermionic states at all mass levels:

\[
\text{no net physical or unphysical tachyons} \quad \iff \quad \text{equal numbers of bosons and fermions at all levels}. \quad (1.4)
\]

This is ultimately a consequence of modular invariance, which in this simple case
can be used to relate the numbers of very low energy states such as tachyons to the
numbers of states at higher mass levels. However, while the requirement that there
be no physical tachyons is necessary for the consistency of the string in spacetime,
unphysical tachyons cause no spacetime inconsistencies and are in fact unavoidable in
the vast majority of string theories (such as all non-supersymmetric heterotic strings).
This is therefore the more general case. The question then arises: how do the bosonic
and fermionic states effectively redistribute themselves at all energy levels in order to
account for these unphysical tachyons? To what extent is the delicate boson/fermion balance destroyed?

1.1.3 To what extent can one break spacetime supersymmetry without destroying the finiteness of string theory?

A third way of asking essentially the same question is within the framework of string finiteness and supersymmetry-breaking. If we start with a string theory containing an unbroken spacetime supersymmetry, then there are an equal number of bosonic and fermionic states at each mass level in the theory (i.e., $a_{mn} = 0$ for all $m$ and $n$), and consequently we find $Z = 0$. This is of course trivially modular-invariant, and the fact that such theories have $a_{nn} = 0$ for all $n < 0$ indicates that they also contain no physical tachyons. These two conditions, however, are precisely those that enable us to avoid certain ultraviolet and infrared divergences in string loop amplitudes: modular invariance eliminates the ultraviolet divergence that would have appeared as $\tau \to 0$, and the absence of physical tachyons ensures that there is no infrared divergence as $\tau \to i\infty$. For example, the one-loop vacuum energy (cosmological constant) $\Lambda$ would ordinarily diverge in field theory, but turns out to be finite in any modular-invariant, tachyon-free string theory. Indeed, these finiteness properties of string loop amplitudes are some of the most remarkable and attractive features of string theory relative to ordinary point-particle field theory.

If the spacetime supersymmetry is broken, however, the partition function $Z$ will no longer vanish, and bosonic states will no longer exactly cancel against fermionic states level-by-level in the theory. However, we would still like to retain the finiteness properties of string amplitudes that arise in the supersymmetric theory. What residual cancellation, therefore, must nevertheless survive the supersymmetry-breaking process? What weaker cancellation preserves the modular invariance and tachyon-free properties which are necessary for finiteness and string consistency?

1.2 Overview of Misaligned Supersymmetry: The Basic Ideas

It turns out that misaligned supersymmetry provides an answer to all of these questions: it yields a constraint on the allowed numbers of string states which arises from modular invariance; it describes the perturbation of the boson/fermion balance due to the presence of unphysical tachyons; and it serves as the residual cancellation which is necessary for string finiteness. Indeed, it furnishes us with a constraint on those supersymmetry-breaking scenarios which maintain string finiteness, essentially restricting us to only those scenarios in which a misaligned supersymmetry survives. In the remainder of this section we shall therefore briefly describe the basic features of this misaligned supersymmetry. Further details may be found in subsequent sections.

The basic idea behind misaligned supersymmetry is quite simple. As we have said, ordinary supersymmetry may characterized by a complete cancellation of the physical state degeneracies $a_{nn}$ for all $n$, and this in turn implies that there are equal numbers
of bosons and fermions at all mass levels in the theory. In the more general case of misaligned supersymmetry, each of these features is changed somewhat. First, the object which experiences a cancellation is no longer the actual net state degeneracies \( a_{nn} \), but rather a new object called the “sector-averaged” state degeneracies and denoted \( \langle a_{nn} \rangle \). This is will be defined below. Second, just as the cancellation of the actual net degeneracies \( a_{nn} \) implied equal numbers of bosonic and fermionic states at every mass level in the theory, the cancellation of the sector-averaged number of states \( \langle a_{nn} \rangle \) will instead turn out to imply a subtle boson/fermion oscillation in which, for example, any surplus of bosonic states at any energy level of the theory implies the existence of a larger surplus of fermionic states at the next higher level, which in turn implies an even larger surplus of bosonic states at an even higher level, and so forth. Such an oscillation is quite dramatic and highly constrained, and its precise form will be discussed below.

1.2.1 The “Sector-Averaged” Number of States \( \langle a_{nn} \rangle \)

We begin by describing the “sector-averaged” number of states \( \langle a_{nn} \rangle \) and its corresponding cancellation. In order to do this, let us first recall how states are typically arranged in string theory.

As we have mentioned, the generic string spectrum consists of a collection of infinite towers of states: each tower corresponds to a different \((i, j)\) sector of the underlying string worldsheet theory, and consists of a ground state with a certain vacuum energy \( H_{ij} \) and infinitely many higher excited states with energies \( n = H_{ij} + \ell \) where \( \ell \in \mathbb{Z} \). The crucial observation, however, is that the different sectors in the theory will in general be misaligned relative to each other, and start out with different vacuum energies \( H_{ij} \) (modulo 1). For example, while one sector may contain states with integer energies \( n \), another sector may contain states with \( n \in \mathbb{Z} + 1/2 \), and another contain states with \( n \in \mathbb{Z} + 1/4 \). Each sector thus essentially contributes a separate set of states to the total string spectrum, and we can denote the net degeneracies from each individual sector \((i, j)\) as \( \{a_{nn}^{(ij)}\} \), where \( n \in \mathbb{Z} + H_{ij} \). Of course, for each sector \((i, j)\) in the theory, these state degeneracies are simply the \( m = n \) coefficients within a power-series expansion of the corresponding partition-function characters:

\[
\chi_i(q) \overline{\chi}_j(\overline{q}) = \sum_{m,n} a_{mn}^{(ij)} \overline{q}^m \overline{q}^n .
\]  

(1.5)

Thus, the numbers of physical states \( a_{nn} \) in the entire theory can be decomposed into the separate contributions \( \{a_{nn}^{(ij)}\} \) from each relevant sector \((i, j)\) in the partition function (1.2).

Now, the number of physical states at each mass level of a theory uniquely determines many properties of that theory, and in particular one such property which may easily be determined is its central charge (or equivalently its Hagedorn temperature). Specifically, for each sector \((i, j)\), it is well-known that \( \{a_{nn}^{(ij)}\} \) must grow
exponentially with $n$,

$$a^{(ij)}_{nn} \sim A n^{-B} e^{C_{\text{tot}} \sqrt{n}} \quad \text{as} \quad n \to \infty ;$$  \hfill (1.6)

here $A$ and $B$ are constants, and $C_{\text{tot}} = 1/T_H$ is the inverse Hagedorn temperature of the theory. Note that this inverse Hagedorn temperature $C_{\text{tot}}$ receives separate contributions from the left- and right-moving underlying theories, $C_{\text{tot}} = C_{\text{left}} + C_{\text{right}}$, with each contribution directly related to the central charge of the corresponding theory via

$$C_{\text{left, right}} = 4 \pi \sqrt{c_{\text{left, right}}} / 24 .$$  \hfill (1.7)

Thus, given (1.6), the total inverse Hagedorn temperature $C_{\text{tot}}$ of the theory can be easily determined by analyzing the growth of the degeneracies $a^{(ij)}_{nn}$:

$$C_{\text{tot}} = C_{\text{left}} + C_{\text{right}} = \lim_{n \to \infty} \frac{\log a^{(ij)}_{nn}}{\sqrt{n}} .$$  \hfill (1.8)

It does not matter which of the contributing sectors $(i, j)$ is selected for this purpose, since each yields the same value $C_{\text{tot}}$.

For each sector $(i, j)$ in the theory, let us now take the next step and imagine analytically continuing the set of numbers $\{a^{(ij)}_{nn}\}$ to form a smooth function $\Phi^{(ij)}(n)$ which not only reproduces $\{a^{(ij)}_{nn}\}$ for the appropriate values $n \in \mathbb{Z} + H_{ij}$, but which is continuous as a function of $n$. Clearly these functions $\Phi^{(ij)}(n)$ must not only contain the above leading Hagedorn-type exponential dependence,

$$\Phi^{(ij)}(n) = A n^{-B} e^{C_{\text{tot}} \sqrt{n}} + \ldots ,$$  \hfill (1.9)

but must also contain all of the subleading behavior as well so that exact results can be obtained for the relevant values of $n$. Indeed, these functions $\Phi^{(ij)}(n)$ may be regarded as the complete (and exact) asymptotic expansions for the $\{a^{(ij)}_{nn}\}$, and there exist straightforward and well-defined procedures for generating these functions [1, 2]. Note that in general these functions are quite complicated, and contain infinitely many subleading terms: some of these are polynomially suppressed relative to the leading term (1.3), and others are exponentially suppressed. These functions $\Phi^{(ij)}(n)$ will be described in detail in Sect. 2.

Given that such functions exist, however, the “sector-averaged” number of states is then defined quite simply as a sum of these functions over all sectors in the theory,

$$\langle a_{nn} \rangle \equiv \sum_{ij} N_{ij} \Phi^{(ij)}(n) ,$$  \hfill (1.10)

where $N_{ij}$ are the coefficients in (1.2). This sector-averaged quantity $\langle a_{nn} \rangle$ therefore differs quite strongly from any of the actual physical-state degeneracies $a^{(ij)}_{nn}$ which

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arise within a given sector, and differs as well from the total physical-state degeneracies \( a_{nn} \) which appear in (1.3). Instead, \( \langle a_{nn} \rangle \) is a continuous function which represents their “average” as defined in (1.10).

Our main result, then, is that although each sector has a number of states \( a_{nn}^{(ij)} \) which grows in accordance with (1.6), this sector-averaged number of states \( \langle a_{nn} \rangle \) must grow \( \text{exponentially} \) more slowly. Specifically, if we define \( C_{\text{eff}} \) in analogy to \( C_{\text{tot}} \),

\[
C_{\text{eff}} \equiv \lim_{n \to \infty} \frac{\log \langle a_{nn} \rangle}{\sqrt{n}},
\]

then we must have

\[
C_{\text{eff}} < C_{\text{tot}}.
\]

(1.12)

We shall prove this result in Sect. 3. Moreover, we shall in fact conjecture that \( C_{\text{eff}} \) vanishes identically,

\[
C_{\text{eff}} \equiv 0,
\]

(1.13)

with \( \langle a_{nn} \rangle \) experiencing at most polynomial growth. This conjecture will be discussed in Sects. 3 and 5.

These, then, are the cancellations implicit in “misaligned supersymmetry”, required by modular invariance and necessary for string finiteness. Indeed, from (1.12), we see that the cancellation governing the string spectrum is sufficiently strong that not only must the leading Hagedorn terms (1.9) cancel, but so must all of the subleading terms within the \( \Phi^{(ij)}(n) \) which are polynomially suppressed. The severity of this cancellation (1.12) thus implies that all traces of the original central charge of the theory are effectively removed in this sector-averaging process, with the sector-averaged number of states \( \langle a_{nn} \rangle \) growing with \( n \) as though derived from an underlying theory with a different central charge. Furthermore, if the conjecture (1.13) is correct, then in fact all exponential growth of \( \langle a_{nn} \rangle \) must be cancelled, whether leading or subleading. This implies that \( \langle a_{nn} \rangle \) should experience at most polynomial growth, and for spacetime dimensions \( D \geq 2 \) we shall in fact argue that even this polynomial growth should be cancelled. Thus in these cases we expect \( \langle a_{nn} \rangle \) to actually \( \text{vanish} \) as \( n \to \infty \).

1.2.2 Boson/Fermion Oscillations

The cancellation (1.12) has far-reaching implications, and in particular implies a corresponding “misaligned supersymmetry” with boson/fermion oscillations. We can perhaps most easily see how this emerges by considering a particular example, a toy string theory containing only two sectors. Let us therefore focus on the following model partition function

\[
Z_{\text{toy}} = (\text{Im} \tau)^{1-D/2} \left\{ N_1 [A(q)]^* B(q) + N_2 [C(q)]^* D(q) \right\}
\]

(1.14)
where \( A, B, C, \) and \( D \) are any four characters corresponding to any chiral worldsheet conformal field theory, with chiral vacuum energies \( H_{A,B,C,D} \) respectively. For concreteness, let us assume that \( H_A, H_B \in \mathbb{Z} \), and \( H_C, H_D \in \mathbb{Z} + \frac{1}{2} \). We thus have two separate towers of states in this theory, with one sector \((AB)\) contributing states with degeneracies \( \{a_{nn}^{(AB)}\} \) situated at integer energy levels \( n \), and another sector \((CD)\) contributing states with degeneracies \( \{a_{nn}^{(CD)}\} \) at energy levels \( n \in \mathbb{Z} + 1/2 \). Let us furthermore suppose that these are the only two sectors in the theory (so that \( Z_{\text{toy}} \) is modular invariant), and that this theory has no physical tachyons (which requires that either \( H_A \) or \( H_B \) is non-negative, and that either \( H_C \) or \( H_D \) is non-negative).

Then if \( \Phi^{(AB)}(n) \) and \( \Phi^{(CD)}(n) \) are respectively the complete asymptotic expansions which correspond to the degeneracies \( a_{nn}^{(AB)} \) and \( a_{nn}^{(CD)} \) for these two sectors, then our result (1.12) asserts that a sufficient number of leading terms in \( \Phi^{(AB)} \) and \( \Phi^{(CD)} \) must cancel exactly so that the rate of exponential growth of their sum is reduced. Explicitly, denoting all of these leading terms as \( \tilde{\Phi}^{(AB)}(n) \) and \( \tilde{\Phi}^{(CD)}(n) \) respectively, we must have

\[
N_1 \tilde{\Phi}^{(AB)}(n) + N_2 \tilde{\Phi}^{(CD)}(n) = 0 .
\]

(1.15)

It is important to realize that this result does not imply any direct cancellation between bosonic and fermionic states in this theory, for (1.15) represents merely a cancellation of the functional forms \( \Phi^{(AB)}(n) \) and \( \Phi^{(CD)}(n) \). Indeed, despite the result (1.15), the total physical-state degeneracies \( \{a_{nn}\} \) for this theory do not vanish for any particular \( n \). Rather, due to the misalignment between the two sectors in this hypothetical example, the actual values taken by the total partition-function coefficients \( a_{nn} \) as \( n \to \infty \) are

\[
a_{nn} \sim \begin{cases} N_1 \tilde{\Phi}^{(AB)}(n) & \text{for } n \in \mathbb{Z} \\ N_2 \tilde{\Phi}^{(CD)}(n) & \text{for } n \in \mathbb{Z} + 1/2 . \end{cases}
\]

(1.16)

Thus we see that there exists no single value of \( n \) for which the actual physical degeneracy \( a_{nn} \) is described by the vanishing sum \( N_1 \tilde{\Phi}^{(AB)} + N_2 \tilde{\Phi}^{(CD)} \).

Perhaps even more interestingly, this result implies that we cannot even pair the states situated at corresponding levels in the \((AB)\) and \((CD)\) sectors, for while the net number of states at the \( \ell \)th level of the \((AB)\) sector is given by \( N_1 \Phi^{(AB)}(\ell) \), the net number of states at the \( \ell \)th level of the \((CD)\) sector is given by \( N_2 \Phi^{(CD)}(\ell + \frac{1}{2}) = -N_1 \Phi^{(AB)}(\ell + \frac{1}{2}) \). The two sectors thus “sample” these cancelling functions at different energies \( n = H_{ij} + \ell \), and it is only by considering these state degeneracies as general functions of \( n \) — or equivalently by considering the “sector averaged” quantity \( \langle a_{nn} \rangle \) — that the cancellation indicated in (1.15) becomes apparent.

In Fig. 1, we have sketched a likely scenario for this toy model, plotting (as functions of energy \( n \)) both the physical-state degeneracies \( a_{nn} \), and their “sector-average” \( \langle a_{nn} \rangle \). Note that for the actual degeneracies \( a_{nn} \), we are plotting \( \pm \log_{10}(|a_{nn}|) \) where the minus sign is chosen if \( a_{nn} < 0 \) (i.e., if there is a surplus of fermionic states over bosonic states at energy \( n \)). Although \( a_{nn} \) takes values only at the discrete energies
$n \in \mathbb{Z}/2$, we have connected these points in order of increasing $n$ to stress the fluctuating oscillatory behavior that $a_{nn}$ experiences as the energy $n$ is increased. Note that for $n \in \mathbb{Z}$, the values of $a_{nn}$ are all positive; these are the state degeneracies $a_{nn}^{(AB)}$ from the $(AB)$ sector alone. Similarly, the negative values of $a_{nn}$ appear for $n \in \mathbb{Z} + 1/2$, and represent the individual contributions $a_{nn}^{(CD)}$ from the $(CD)$ sector. We have also superimposed the typical behavior of $\langle a_{nn} \rangle$, which, given its definition as a sum of asymptotic forms, is a continuous function of $n$. While the values of the individual $a_{nn}$ experience exponential growth with $C = C_{\text{tot}}$, the cancellation (1.12) guarantees that $\langle a_{nn} \rangle$ grows exponentially more slowly than $a_{nn}$, with a rate $C_{\text{eff}} < C_{\text{tot}}$. The sketch in Fig. 1 assumes that $C_{\text{eff}} = C_{\text{tot}}/4$, but if (1.13) is true, then of course $\langle a_{nn} \rangle$ experiences no exponential growth at all and remains flat.

As can be seen from Fig. 1, the immediate consequence of the cancellation (1.12) is that the net number of actual physical states $a_{nn}$ can at most oscillate around zero as the energy $n$ increases and as the contributions from different sectors with differently aligned values of $n$ contribute either positively or negatively to the partition function $Z$. The wavelength of this oscillation is clearly $\Delta n = 1$, corresponding to the energy difference between adjacent states in the same sector, while the amplitude of this oscillation of course grows exponentially with $n$ (since each individual $\Phi^{(ij)}(n)$ retains its leading Hagedorn-like behavior). Thus, although the net number of states $a_{nn}$ diverges as $n \to \infty$, it can do so only in a very tightly constrained manner, so that the functional forms which describe this leading asymptotic behavior have a sum which vanishes.

We will see that this oscillatory behavior is in fact a generic consequence of our result, and appears in any modular-invariant theory which is free of physical tachyons. Thus, in analogy to (1.4), we now have

\[ \text{no physical tachyons} \iff \text{boson/fermion oscillations} \quad \text{(1.17)} \]

Conversely, the existence of such oscillations in the spectrum of a given unknown theory may well be taken as a signature of an underlying modular invariance [8]. Moreover, as we shall demonstrate in Sect. 5, these boson/fermion oscillations also serve as the general mechanism by which the finiteness of modular-invariant, tachyon-free theories is reflected level-by-level in the degeneracies of physical states. Of course, a non-trivial corollary of the result (1.17) is that the absence of physical tachyons requires the presence of spacetime fermions. This explains why the bosonic string was doomed to have physical tachyons in its spectrum, and why the cure to this problem (a GSO projection) could only be implemented in the context of an enlarged theory (e.g., the superstring or heterotic string) which also contained spacetime fermions.

Finally, of course, this result can also be interpreted as providing a tight constraint on the general pattern of supersymmetry-breaking in string theory. As we discussed above, in a spacetime-supersymmetric (and therefore tachyon-free) theory, bosonic and fermionic sectors contribute precisely equal numbers of states at each individual energy $n$; consequently we have $a_{nn} = 0$ level-by-level, and the amplitude of our “os-
cillation” in this special case is zero. However, if the supersymmetry is to be broken in such a way that physical tachyons are not introduced and modular invariance is to be maintained (as we would demand in any physically sensible theory), then our result implies that one can at most “misalign” this bosonic and fermionic cancellation, introducing a mismatch between the bosonic and fermionic state degeneracies at each level in such a way that any surplus of bosonic states at any energy level is compensated for by an even greater surplus of fermionic states at a neighboring higher energy level, leading to an even greater bosonic surplus at an even higher energy, and so forth.

The magnitudes of these surpluses are of course tightly constrained, since the cancellation of the corresponding leading functional forms must be preserved. Thus, we see that this residual cancellation, this hidden “misaligned” supersymmetry, is another unavoidable consequence of modular invariance, and it would be interesting to see which classes of physical supersymmetry-breaking scenarios are thereby precluded. For example, we can already rule out any supersymmetry-breaking scenario in which the energies of, say, the fermionic states are merely shifted relative to those of their bosonic counterparts by an amount $\Delta n$; rather, we require a mechanism which somehow also simultaneously creates (or eliminates) a certain number $\Phi(n + \Delta n) - \Phi(n)$ of states at each level $n$ so that the state degeneracies at the shifted energies are still described the same (cancelling) functional forms. Such a mechanism would clearly be highly non-trivial. In particular, “misaligned supersymmetry” is as yet only a result concerning the numbers of bosonic and fermionic states in string theory, and no dynamical symmetry operators or currents relating these misaligned states have been constructed.

We have now completed our overview of our main results; the rest of this paper will provide details and examples, and is organized as follows. In Sect. 2 we shall first review the asymptotic expansions upon which our results rest, and in Sect. 3 we will prove our main theorem (1.12) and demonstrate that (1.12) in fact serves as the two-variable generalization of a well-known theorem in modular function theory. In Sect. 4 we shall then provide some explicit examples of these cancellations and the corresponding “misaligned supersymmetry”, and in Sect. 5 we shall discuss how this phenomenon may ultimately be responsible for the finiteness of string loop amplitudes by considering the case of the one-loop cosmological constant. We will also discuss our conjecture that in fact $C_{\text{eff}} = 0$. Our final comments concerning various extensions and applications will be presented in Sect. 6.

Because our proofs and calculations will be presented in great detail, this paper is somewhat lengthy. We have therefore organized the rest of this paper in such a way that the reader unconcerned with the details of the asymptotic expansions $\Phi^{(ij)}(n)$ and willing to accept the result (1.12) can omit Sects. 2–4 without loss of continuity, and proceed directly to Sect. 5.
2 Asymptotic Expansions for the Numbers of States

In this section we shall briefly review the methods (originally due to Hardy and Ramanujan [1] and recently generalized by Kani and Vafa [2]) for deriving the asymptotic functions describing the physical-state degeneracies. We will concentrate on only those broad features of the derivation which will be relevant for our later work, leaving many of the technical details to be found in [2].

The problem may be stated mathematically as follows. We are given a set of functions \( \chi_i(\tau) \), \( i = 1, ..., N \), forming an \( N \)-dimensional representation of the modular group with modular weight \( k \in \mathbb{Z}/2 \). This means that for any transformation \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \) and any \( \tau \) in the fundamental domain \( \mathcal{F} \equiv \{ \tau : |\text{Re} \tau| \leq 1/2, \text{Im} \tau > 0, |\tau| \geq 1 \} \) of the modular group, we have

\[
\chi_i(M\tau) = (c\tau + d)^k \sum_{j=1}^N M_{ij} \chi_j(\tau) \quad (2.1)
\]

where \( M\tau \equiv (a\tau + b)/(c\tau + d) \) and where \( M_{ij} \) is an \( N \times N \) matrix in the representation space. We assume each function \( \chi_i \) to have a \( q \)-expansion of the form

\[
\chi_i(q) = q^{H_i} \sum_{n=0}^{\infty} a_n^{(i)} q^n \quad (2.2)
\]

where \( q \equiv e^{2\pi i \tau} \) and all \( a_n^{(i)} \geq 0 \); thus each \( \chi_i \) is an eigenfunction of the transformation \( T : \tau \rightarrow \tau + 1 \) with eigenvalue \( \exp(2\pi i H_i) \). We will also assume that our functions \( \chi_i \) are normalized so that each \( a_0^{(i)} = 1 \). Such functions \( \chi_i \) arise, for example, as the characters \( \text{Tr} q^{H} \) of the various highest-weight sectors of conformal field theories with Hamiltonians \( H \); the quantities \( H_i \) in (2.2) are then interpreted as the sector vacuum energies, which are related to the central charge \( c \) of the conformal field theory and its various highest weights \( h_i \geq 0 \) via

\[
H_i = h_i - c/24 . \quad (2.3)
\]

The \( a_n^{(i)} \), on the other hand, are interpreted as the state degeneracies of the \( i \)th sector at excitation number \( n \), and our goal is to derive asymptotic expansions for these degeneracies \( a_n^{(i)} \) as functions of \( n \). It is well-known that the leading asymptotic behavior of these degeneracies is of the Hagedorn form

\[
a_n^{(i)} \sim A_i n^{-B_i} e^{C_i \sqrt{n}} \quad \text{as } n \rightarrow \infty \quad (2.4)
\]

where \( A_i, B_i, \) and \( C_i \) are constants, and in fact we will see that

\[
C_i = \sqrt{\frac{2c}{3} \pi} , \quad B_i = \frac{3}{4} - \frac{1}{2} k , \quad A_i = \frac{1}{\sqrt{2}} (e^{\pi i k/2} S_{11}) \left( \frac{c}{24} \right)^{B_i - 1/2} . \quad (2.5)
\]
Here $S_{i1}$ is the $(i, 1)$ element of the representation matrix [as defined in (2.1)] corresponding to the modular transformation $S \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, where $j = 1$ denotes the vacuum-sector or identity-sector character for which $H_j = -c/24$ (or $h_j = 0$). Note that the product $e^{\pi ik/2}S_{i1}$ is always real and non-zero. There are, however, an infinite number of additional subdominant and subleading terms in the complete (and often exact) asymptotic expansions of the physical-state degeneracies, and we will be seeking these complete expansions.

The derivation proceeds in two basic steps. The first is to invert (2.2), extracting the degeneracies $a^{(i)}_n$ through a contour integral:

$$a^{(i)}_n = \frac{1}{2\pi i} \oint_C dq \frac{\chi_i(q)}{q^{n+H_{i+1}}}$$

(2.6)

where the contour $C$ is any closed counter-clockwise loop encircling the origin once and remaining entirely within the unit disk $|q| \leq 1$ (so that $\chi_i(q)$ remains convergent). It is convenient to take $C$ to be a circle of radius $1$. While this integral can in principle be evaluated for any radius, it proves advantageous to take the radius near $1$, for in this limit the contour integral will be dominated by contributions from the regions near certain points on the unit circle $|q| = 1$, and these contributions will be relatively simple to evaluate. In particular, we see from the general forms (2.2) that the characters $\chi_i(q)$ develop essential singularities at all points on the unit circle for which $\tau \in \mathbb{Q}$, for at these points there exist an infinite number of values of $n$ for which $q^n = 1$, causing $\chi(q)$ to diverge. This can often also be seen by writing the infinite sums (2.2) as infinite products, for such product representations (when they exist) typically include the factor

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

(2.7)

which diverges whenever there exists an $n \geq 1$ such that $q^n = 1$. Thus, we can evaluate the contour integral (2.6) in the radius $= 1$ limit by carefully dissecting our contour into arcs near each rational point on the unit circle, and summing the separate arc integrals.

The second step involves performing this dissection most efficiently. While the divergences at each rational point $q_{\alpha\beta} \equiv \exp(2\pi i \beta/\alpha)$ with $\alpha > \beta \geq 0$ are necessarily of infinite degree, we see that if $\alpha$ and $\beta$ are chosen relatively prime (so that $\beta/\alpha \equiv \tau_{\alpha\beta}$ is expressed in lowest form), the divergences will be “stronger” at points for which $\alpha$ is smaller. For example, the divergence of $\chi(q)$ at $q_{\alpha\beta} \equiv \exp(2\pi i \beta/\alpha)$ is “twice” as strong as that at $q_{2\alpha\beta}$ in the sense that there are twice as many values of $n$ in (2.2) or (2.7) for which $q^n = 1$. Indeed, the dominant contribution to the integral (2.6) is that near the point $q_{1,0} = 1$, and we will see that this contribution alone is sufficient to yield (2.4).

We can thus dissect our contour most efficiently by summing the contributions near points $q_{\alpha\beta}$ in order of increasing $\alpha \geq 1$, choosing a terminating maximum value
\( \alpha_{\text{max}} \) depending on the overall accuracy desired. Then, for each value \( \alpha, 1 \leq \alpha \leq \alpha_{\text{max}} \), we consider those \( \beta \) \((1 \leq \beta < \alpha)\) which are relatively prime to \( \alpha \), allowing \( \beta = 0 \) only for \( \alpha = 1 \). For example, we have \( \beta = 1, 3 \) for \( \alpha = 4 \), and \( \beta = 1, 2, \ldots, 6 \) for \( \alpha = 7 \). This procedure ensures that we have organized the rational points according to their relative contributions to the integral (2.7).

Finally, in order to evaluate these contributions from each such rational point \( q_{\alpha, \beta} \), we make use of the fact that these points can be transformed to \( q'_{\alpha, \beta} = 0 \) by modular transformations. Thus while it may at first seem difficult to evaluate the numerator \( \chi_i(q \approx q_{\alpha, \beta}) \), modular transformations allow us to relate this to the much simpler quantities \( \chi_j(q' \approx 0) \). In particular, let \( M_{\alpha, \beta} \in SL(2, \mathbb{Z}) \) be a modular transformation transforming \( q = q_{\alpha, \beta} \) to \( q' = 0 \). Although such a transformation is not unique (because the phase of the resulting \( q' = 0 \) is unspecified), any such chosen transformation can be written in the form

\[
M_{\alpha, \beta} \equiv \begin{pmatrix} -\beta' & r \\ -\alpha & \beta \end{pmatrix}, \quad r = (1 + \beta'\alpha) \alpha^{-1} \tag{2.8}
\]

where \( \beta' \in \mathbb{Z} \) is the free parameter indicating our specific choice. One can then use (2.7) to relate \( \chi_i(q \approx q_{\alpha, \beta}) \) to the various \( \chi_j(q' \approx 0) \), each of which behaves as \( (q')^H_j \). Explicitly,

\[
\chi_i(q \approx q_{\alpha, \beta}) \approx (-\alpha \tau + \beta)^{-k} \sum_{j=1}^{N} (M_{\alpha, \beta}^{-1})_{ij} (q' \approx 0)^H_j. \tag{2.9}
\]

This result can then be substituted into each separate arc integral near each rational point, and in this form these arc integrals can be evaluated (ultimately yielding Bessel functions). These separate contributions from each rational point can then be summed to yield an approximation for the whole contour integral (2.6).

The details of this analysis are somewhat intricate, and are given in Refs. [1] and [2]. The result one finds, however, is relatively simple:

\[
q_{n}^{(i)} = \sum_{\alpha=1}^{\alpha_{\text{max}}} \left( \frac{2\pi}{\alpha} \right) \sum_{j=1}^{N} Q_{ij}^{(\alpha)} f_j^{(\alpha)}(n) \tag{2.10}
\]

where

\[
Q_{ij}^{(\alpha)} \equiv e^{\pi i k/2} \sum_{\beta} (M_{\alpha, \beta}^{-1})_{ij} \exp \left[ 2\pi i \left( \frac{\beta'}{\alpha} H_j - \frac{\beta}{\alpha} H_i \right) \right] \exp \left( -2\pi i \frac{\beta}{\alpha} n \right) \tag{2.11}
\]

and where

\[
f_j^{(\alpha)}(n) \equiv \left( \sqrt{\frac{H_j}{n + H_i}} \right)^{1-k} J_{k-1} \left( 4\pi \frac{H_j}{\alpha} \sqrt{n + H_i} \right). \tag{2.12}
\]

* This freedom to adjust the phase of \( q_{\alpha, \beta} \), or equivalently to adjust \( \text{Re} \tau'_{\alpha, \beta} \), corresponds to the freedom to multiply \( M_{\alpha, \beta} \) from the left by arbitrary additional factors of \( T \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \). Thus we see that all of the allowed values of \( \beta' \) are equal modulo \( \alpha \). All final results will nevertheless be invariant under such shifts in \( \beta' \).
Here the sum in (2.11) is over those values of $\beta$, $1 \leq \beta < \alpha$ (with $\beta = 0$ only for $\alpha = 1$), such that $\alpha$ and $\beta$ are relatively prime. The modular transformations $M_{\alpha \beta}$ are defined in (2.8), and their representation matrices are defined generally in (2.1). It is in fact straightforward to show [2] that the quantity $Q_{ij}^{(\alpha)}$ (as well as $a_n^{(i)}$ itself) is not only real for all $i$, $j$, and $\alpha$, but also independent of the particular choices for $\beta'$ in each term of (2.11), as asserted earlier. Indeed, different values of $\beta'$ correspond to extra factors of $T$ in $M_{\alpha \beta}$, and these are offset by the explicit $\beta'$-dependent phase in (2.11). The functions $J_\nu(x)$ appearing in (2.12) are the Bessel functions of the first kind.

While (2.10) is written in a form suitable for those values of $j$ for which $H_j \geq 0$, there are often other values of $j$ for which $H_j < 0$; indeed, if the $\chi_j$ are the characters of the different highest-weight sectors of a conformal field theory with positive central charge, then the existence of an identity-sector character with $H_j < 0$ is guaranteed. In these cases, the argument of the Bessel function becomes imaginary. However, introducing the Bessel function of first kind with imaginary argument, $I_\nu(x) \equiv e^{-i\pi \nu/2} J_\nu(e^{i\pi/2}x)$, (2.13)

we find that for these cases we can rewrite (2.12) as

$$H_j > 0 : \quad f_j^{(\alpha)}(n) = \left( \sqrt{|H_j|} \right)^{1-k} I_k-1 \left( \frac{4\pi}{\alpha} \sqrt{|H_j|(n+H_i)} \right).$$

Thus the reality of the $a_n^{(i)}$ is preserved. Similarly, there may also exist cases for which $H_j = 0$. In these cases $f_j^{(\alpha)}(n)$ takes the simple form

$$H_j = 0 : \quad f_j^{(\alpha)}(n) = \left[ \frac{2\pi}{\alpha} (n+H_i) \right]^{k-1},$$

and once again the contribution to $a_n^{(i)}$ is real.

The foregoing discussion has only touched upon the basic method of derivation, and in particular there are many places in the complete derivation where approximations are introduced in order to achieve the above result. This implies, of course, that the above asymptotic expansions are at best only approximate. A detailed analysis of the error terms has been performed in Ref. [2], however, and we shall present here only the final results. First, it is shown in [2] that the minimum error is achieved if, for a given value of $n$, the $\alpha$-summation in (2.10) is terminated at values

$$\alpha_{\text{max}} \sim \mathcal{O}(\sqrt{n}).$$

As is typical with such asymptotic expansions, subsequent “higher-order” terms only increase the error. Given this termination, then, it is shown in [2] that asymptotic expansions are correct to within

$$\text{error} \sim \mathcal{O}(n^p) \quad \text{where} \quad p = \begin{cases} k/2 & \text{for } k \leq 0 \\ k & \text{for } k > 0. \end{cases}$$

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Thus, since we know that \( a_n^{(i)} \) is always an integer, we see that the truncated asymptotic expansion (2.10) becomes exact for \( k \leq 0 \) and sufficiently large \( n \): we simply take the integer closest to the sum in (2.10). For \( k = 0 \), on the other hand, the sum (2.10) can differ from the true \( a_n^{(i)} \) by only an overall additive constant. Such agreement is indeed remarkable, indicating that in most physically relevant cases (as we shall see), the asymptotic series (2.10) can be indeed taken to truly represent the physical-state degeneracies \( a_n^{(i)} \).

Let us now get some feeling for how the result (2.10) works in practice by demonstrating that (2.10) implies the leading Hagedorn exponential behavior (2.4). The Bessel function with imaginary argument \( I_\nu(x) \) has the asymptotic \(|x| \to \infty\) exponentially-growing behavior [4]:

\[
I_\nu(x) = \frac{e^x}{\sqrt{2\pi x}} \left[ 1 - \frac{(\mu - 1)(\mu - 9)}{8x} \frac{2!(8x)^2}{3!(8x)^3} + \ldots \right]
\]

(2.18)

with \( \mu \equiv 4\nu^2 \), whereas the ordinary Bessel function \( J_\nu(x) \) has the damped asymptotic behavior

\[
J_\nu(x) = \sqrt{\frac{2}{\pi x}} \left\{ \left[ 1 - \frac{(\mu - 1)(\mu - 9)}{2!(8x)^2} + \ldots \right] \cos \theta - \left[ \frac{\mu - 1}{8x} - \frac{(\mu - 1)(\mu - 9)(\mu - 25)}{3!(8x)^3} + \ldots \right] \sin \theta \right\}
\]

(2.19)

with \( \theta \equiv x - \frac{1}{2} \pi \nu - \frac{1}{4} \pi \). Thus, we see that the dominant, exponentially-growing contributions to \( a_n^{(i)} \) come from those terms in (2.10) for which \( H_j < 0 \). In particular, the minimum value of \( H_j \) is \( H_j = -c/24 \) (which occurs for \( h_j = 0 \), i.e., for the identity sector), and thus the strongest exponential growth generally arises for this value of \( j \) and for \( \alpha = 1 \). We will henceforth denote this sector as \( \beta' = 0 \) and \( M_{10}^{-1} = S \). Hence \( Q_{ij}^{(1)} = e^{\pi i k / 2} \mathcal{S}_{ij} \) (a general result), and for \( n \gg H_i \) we find

\[
a_n^{(i)} = 2\pi \left( \frac{24n}{c} \right)^{k-1} \left( e^{\pi i k / 2} \mathcal{S}_{11} \right) I_{k-1} \left( 4\pi \sqrt{\frac{c}{24n}} \right) + \ldots .
\]

(2.20)

Use of (2.18) then yields (2.4) and (2.5). Note that since \( j = 1 \) in (2.20) corresponds to the identity sector with \( h_j = 0 \), we are guaranteed that \( S_{i1} \neq 0 \) for all \( i \). Perhaps the easiest way to see this is via the general relationship between the \( S \)-matrix and the fusion rules of any conformal field theory [3]; since each highest-weight sector must fuse with the identity to give back itself, we must have \( S_{i1} \neq 0 \). Indeed, the fusion rules of a given conformal field theory

\[
[\phi_i] \times [\phi_j] = \sum_k N_{ijk} [\phi_k]
\]

(2.21)
can be deduced from this $S_{ij}$-matrix directly via the Verlinde formula
\[ N_{ijk} = \sum_{\ell} \frac{S_{i\ell} S_{j\ell} S_{\ell k}}{S_{\ell \ell}} , \]  
(2.22)
and we see that we must have $S_{i1} \neq 0$ for all $i$ in order for sensible fusion rules to be obtained.

It is also a straightforward matter to classify the sorts of subleading terms which appear in addition to the dominant Hagedorn term (2.4). First there are the subleading terms within the dominant exponential term
\[ Q e^{C_{\text{max}} \sqrt{n + H}} \left[ A_1 (n + H)^{-B_1} + A_2 (n + H)^{-B_2} + A_3 (n + H)^{-B_3} + \ldots \right] ; \]  
(2.23)
here $Q$ signifies the coefficients defined in (2.11), the values of $B_n$ increase by half-integer steps, and this series terminates only if $2k$ is an odd integer. This dominant exponential term arises from the identity sector, as noted above, and will appear for the character of any sector connected to the identity sector via the $S$ modular transformation. Next, there are the subdominant exponentially growing terms; these are again of the same form as (2.23), but have exponential growth parameters $C''_{\text{max}}$, $C'''_{\text{max}}$, etc. which are smaller than $C_{\text{max}}$ and which depend on the spectrum of vacuum energies $H_i$. Such terms arise from sectors with $-c/24 < H_i < 0$. Then there are the sets of terms, again of the form (2.23), with exponential growth parameters $C = \frac{1}{2}C_{\text{max}}, \frac{1}{3}C_{\text{max}}, \ldots, \frac{1}{2}C''_{\text{max}}, \frac{1}{3}C'''_{\text{max}}, \ldots$, etc.; these terms arise from the $H_i < 0$ sectors as well. Finally there are the non-growing terms which arise from those sectors with $H_i \geq 0$: these vanish in the $n \to \infty$ limit, and are otherwise essentially negligible as contributors to $a_n^{(i)}$. The remarkable fact, however, is that for $k < 0$ and large $n$, the sum of all of these terms reproduces the values of $a_n^{(i)}$ exactly.

This concludes our review of the derivation of the asymptotic expansions describing the physical-state degeneracies of conformal field theories. In the remainder of this section, we shall, for completeness, evaluate the first few coefficients $Q_{ij}^{(\alpha)}$ and demonstrate that they are indeed real. We will also discuss what modifications are necessary to the above formulas if our characters $\chi_i$ are not normalized with $a_0^{(i)} = 1$.

The fundamental observation in evaluating the coefficients $Q_{ij}^{(\alpha)}$ is to recognize that although the general formula for $Q_{ij}^{(\alpha)}$ is given in (2.11), certain choices for $\beta'$ within each term of (2.11) will enable the $Q_{ij}^{(\alpha)}$'s to assume particularly simple forms. If we define the modular transformations
\[ S \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -S^{-1} \quad \text{and} \quad T \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} , \]  
(2.24)
then any modular-transformation $M_{\alpha \beta}$ can be expressed as a product of $S$ and $T$ transformations. Note that the representation matrices as defined in (2.1) satisfy $(AB)_{ij} = \sum_k A_{ik} B_{kj}$; furthermore, $T_{ij}$ is diagonal in the space of characters: $T_{ij} =
\[ \exp(2\pi i H_j) \delta_{ij}. \] Also note that \((-1)_{ij} = e^{\pi i} \delta_{ij},\] so that while the two \(SL(2, \mathbb{Z})\) matrices \(M\) and \(-M\) both describe the same modular transformation, they in general have different representation matrices in the space of characters: \((-M)_{ij} = e^{\pi i} M_{ij}.\] This ambiguity is avoided by ensuring that \(M_{\alpha \beta}\) is of the form \((2.8).\) Now, for \(\alpha = 1,\) we have \(\beta = 0,\) and we have already seen that the simplest choice \(\beta' = 0\) can be accommodated by choosing \(M_{10} = -S.\) This then leads, as before, to the general result

\[ Q_{ij}^{(1)} = e^{\pi i/2} S_{ij}. \] (2.25)

Note that this quantity is manifestly real, since under the \(S\)-transformation one finds that modular functions transform as

\[ \chi_i \left( -1/\tau \right) = e^{-\pi i/2} \tau^k \sum_{ij} \tilde{S}_{ij} \chi_j(\tau) \] (2.26)

with real coefficients \(\tilde{S}_{ij},\) and according to the definitions \((2.1)\) and \((2.24)\) we find that \(S_{ij} = e^{-\pi i/2} \tilde{S}_{ij}.\) Proceeding to the \(\alpha = 2\) case, we have only \(\beta = 1,\) and by choosing \(M_{21} = T^{-1} ST^{-2} ST^{-1}\) we have \(\beta' = -1.\) Thus \(M_{21}^{-1} = TST^2 ST,\) and since in general we have \((TXT)_{ij} = \exp[2\pi i (H_i + H_j)] X_{ij},\) we obtain the result

\[ Q_{ij}^{(2)} = e^{\pi i/2} (ST^2 S)_{ij} \exp \left[ \pi i (n + H_i + H_j) \right] = e^{\pi i/2} (ST^2 S)_{ij} \exp \left[ \pi i (H_i + H_j) \right] \left( -1 \right)^n. \] (2.27)

In order to see that \((2.27)\) is manifestly real, we note that this same choice for \(M_{21}\) can also be written as \(-ST^2 S.\) Thus \((2.27)\) can equivalently be written as

\[ Q_{ij}^{(2)} = e^{\pi i/2} (-ST^{-2} S)_{ij} \exp \left[ -\pi i (H_i + H_j) \right] \left( -1 \right)^n, \] (2.28)

and since we find

\[ e^{\pi i/2} (-ST^{-2} S)_{ij} = e^{\pi i/2} e^{\pi i} \sum_k e^{-4\pi i H_k} S_{ik} S_{kj} = e^{-\pi i/2} \left[ \sum_k e^{4\pi i H_k} S_{ik} S_{kj} \right]^* = \left[ e^{\pi i/2} (ST^2 S)_{ij} \right]^* \] (2.29)

[where we have used the fact that \(S_{ij}^* = e^{\pi i} S_{ij},\)] we see that \((2.28)\) is merely the complex conjugate of \((2.27).\) Now, for \(\alpha = 3,\) there are two values of \(\beta\) to consider: \(\beta = 1\) and \(\beta = 2.\) Choosing the transformations \(M_{31} = -ST^3 S\) and \(M_{32} = T^{-1} ST^{-2} ST^{-1}\) respectively yields \(\beta' = -1\) and \(\beta' = -2,\) giving

\[ Q_{ij}^{(3)} = e^{\pi i/2} (-ST^{-3} S)_{ij} \exp \left[ -\frac{2\pi i}{3} (n + H_i + H_j) \right] \]

\[ + e^{\pi i/2} (TST^3 ST)_{ij} \exp \left[ -\frac{4\pi i}{3} (n + H_i + H_j) \right]. \] (2.30)
Similar manipulations show, however, that these two terms are complex conjugates of each other. We thus obtain the simpler result

\[ Q^{(3)}_{ij} = 2 \text{Re} \left\{ e^{\pi i k/2} (ST^3 S)_{ij} \exp \left[ \frac{2\pi i}{3} (n + H_i + H_j) \right] \right\} . \] (2.31)

The final results for the \( \alpha = 4, 5, \) and 6 cases can be determined in the same manner:

\[ Q^{(4)}_{ij} = 2 \text{Re} \left\{ e^{\pi i k/2} (ST^4 S)_{ij} \exp \left[ \frac{\pi i}{2} (n + H_i + H_j) \right] \right\} , \]

\[ Q^{(5)}_{ij} = 2 \text{Re} \left\{ e^{\pi i k/2} (ST^5 S)_{ij} \exp \left[ \frac{2\pi i}{5} (n + H_i + H_j) \right] \right\} + 2 \text{Re} \left\{ e^{-\pi i k/2} (ST^2 ST^{-2} S)_{ij} \exp \left[ \frac{4\pi i}{5} (n + H_i - H_j) \right] \right\} , \]

\[ Q^{(6)}_{ij} = 2 \text{Re} \left\{ e^{\pi i k/2} (ST^6 S)_{ij} \exp \left[ \frac{\pi i}{3} (n + H_i + H_j) \right] \right\} . \] (2.32)

Finally, let us consider what modifications must be made to these asymptotic expansion formulas in the case that our characters \( \chi_i \) are not normalized to \( a_0^{(i)} = 1 \) for all \( i \). Of course, given a particular expansion derived for a normalized character \( \chi_i \), we need simply multiply this result by \( a_0^{(i)} \) in order to obtain the expansion for the corresponding unnormalized character. However, this assumes that the above values of \( Q^{(\alpha)}_{ij} \) are taken to be those calculated for the normalized characters; in particular, the representation matrices \( M_{ij} \) for modular transformations \( M \) which appear in the above results for \( Q^{(\alpha)}_{ij} \) are assumed to be those calculated for the normalized characters. The new matrices \( \hat{M}_{ij} \) for the unnormalized characters have, however, elements which are trivially rescaled relative to those for the normalized characters:

\[ \hat{M}_{ij} = a_0^{(i)} M_{ij} (a_0^{(j)})^{-1} . \] (2.33)

This implies

\[ Q_{ij} = (a_0^{(i)})^{-1} \hat{Q}_{ij} a_0^{(j)} , \] (2.34)

and thus, rewriting the new result in terms of appropriately unnormalized quantities, we see that the overall factor \( a_0^{(i)} \) cancels and leaves only an extra factor of \( a_0^{(j)} \) inside the \( j \)-summation. We therefore have two equivalent options when working with unnormalized characters \( \chi_i \): we can either calculate an asymptotic expansion by using normalized matrices \( M_{ij} \) in the definition for \( Q^{(\alpha)}_{ij} \) and multiplying the entire result by \( a_0^{(i)} \), or we can use unnormalized matrices in the definition for \( Q^{(\alpha)}_{ij} \) and insert an extra factor of \( a_0^{(j)} \) inside the summation over \( j \). We shall use both approaches in Sect. 4.
3 The Main Theorem

In this section we shall prove our main result, given in (1.12). It turns out that this result can be viewed as the two-variable generalization of a well-known theorem in modular function theory which applies to functions \( f(q) \) of a single modular parameter \( q \). We shall therefore first provide a proof of this one-variable theorem which makes use of the asymptotic expansions discussed in Sect. 2. The proof of the two-variable case will then be relatively straightforward.

3.1 The One-Variable Case

The one-variable theorem that we wish to prove states that any function

\[
f(\tau) = q^H \sum_{n=0}^{\infty} a_n q^n, \quad q \equiv e^{2\pi i \tau}
\]

(3.1)

which transforms as an eigenfunction with modular weight \( k \) under the \( S \) and \( T \) modular transformations

\[
\begin{align*}
f(\tau + 1) &= \exp(2\pi i H) f(\tau), \\
f(-1/\tau) &= \sigma \tau^k f(\tau), \quad |\sigma| = 1
\end{align*}
\]

(3.2)

must vanish identically (i.e., \( a_n = 0 \) for all \( n \)) if

\[
k < 12 H.
\]

(3.3)

In general, \( k \in \mathbb{Z}/2 \) and \( \sigma^8 = 1 \). Thus, since the Dedekind eta-function \( \eta = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \) satisfies (3.2) with \( H = 1/24, k = 1/2, \) and \( \sigma = \exp(-\pi i/4) \), one can always multiply or divide \( f \) by a sufficient number of factors of \( \eta \) to cast this theorem into its more familiar form: any function \( f \) which is modular-invariant (i.e., \( H \geq 0, H \in \mathbb{Z}, \sigma = 1 \)) must vanish identically if it has negative modular weight \( k \).

An example of such a function meeting all of these conditions is

\[
J \equiv \vartheta_3^4 - \vartheta_2^4 - \vartheta_4^4
\]

(3.4)

where the \( \vartheta_i \) are the classical Jacobi \( \vartheta \)-functions; since \( J \) has \( H = 1/2, k = 2, \) and \( \sigma = -1 \) (or equivalently, since \( J/\eta^{12} \) is modular-invariant with negative modular weight \( k = -4 \)), \( J \) vanishes identically. While the proof of this general theorem is standard (see, for example, [3]), our goal here is to see how this “cancellation” can be understood from the point of view of the asymptotic expansions presented in Sect. 2.

Note that since \( f \) is assumed to satisfy (3.2), \( f \) forms a one-dimensional representation of the modular group, with \( T_{ij} = \exp(2\pi i H) \) and \( S_{ij} = \sigma \). Using the defining relations of the modular group \( S^2 = (ST)^3 = -1 \) and the fact that \( (-1)_{ij} = e^{\pi i k} \delta_{ij} \), it is easy to show that any such one-dimensional representation “matrices” \( S_{ij} \) and
$T_{ij}$ can satisfy the defining relations only if $k - 12H \in 4\mathbb{Z}$. This is therefore a general constraint on any such modular-eigenfunctions $f$ (such as $\eta$), and the theorem applies only to those special cases (such as $J$) for which $k - 12H < 0$.

In order to use the machinery of the asymptotic expansions presented in Sect. 2, we shall first explicitly organize $f$ as a linear combination of non-vanishing, linearly-independent characters $\chi_i(q)$:

$$f = \sum_i c_i \chi_i = \sum_i c_i q^{H_i} \sum_{n=0}^{\infty} a_n^{(i)} q^n .$$

(3.5)

Here the $\chi_i$ are members of a single system of characters closed under modular transformations, and therefore satisfy (2.1) and (2.2) with normalizations $a_0^{(i)} = 1$. It is necessary to proceed in this manner because $f$ by itself will be shown to vanish, and therefore $f$ itself cannot be properly normalized (there exists no $a_0 \neq 0$). We can also assume that $f$ contains a sufficient number of $\eta$-function factors so that it is modular-invariant: this implies that the vacuum energies are all integers, $H_i \in \mathbb{Z}$, and that the coefficients satisfy

$$\sum_{i=1}^{N} c_i M_{ij} = c_j$$

(3.6)

for all matrices $M_{ij}$ which represent modular transformations in the $\chi_i$ system of characters. Furthermore, as discussed above, the condition (3.3) also allows us to choose the number of $\eta$-function factors so that, without loss of generality,

$$k < 0 \quad \text{and} \quad H_i \geq 0 \quad \text{if} \quad c_i \neq 0 .$$

(3.7)

Given this decomposition of $f$, then, the procedure for our proof will be quite straightforward: we shall simply calculate the asymptotic expansions of each $\chi_i$ individually, and add them together.

The asymptotic expansions for each $a_n^{(i)}$ in (3.3) are given in (2.10), but before these expansions can be added together to form an asymptotic expansion for the combined coefficients $a_n$ in (3.1), we need to make one important adjustment: we first need to take into account the fact that each character $\chi_i$ in principle has a different vacuum energy $H_i$, and that therefore the degeneracy of the $n$th level in

* It is possible, of course, that several of the $\chi_i$ appearing in (3.3) correspond to sectors with vacuum energies $H_i$ which are equal modulo 1 and negative; the resulting function $f$ would still have a positive overall vacuum energy $H \geq 0$ provided that all tachyonic contributions from these sectors cancel in the sum (3.5). However, in such cases we can always form linear combinations of these characters $\chi_i$ so that the new “characters” are still eigenfunctions of $T$ and have positive net vacuum energies $H_i$. Thus, even in such cases, a decomposition into characters (or linear combinations of characters) exists for which (3.3) is satisfied. The effects of working with such linear combinations of characters will be discussed in Sect. 4, and we shall assume for the remainder of this section that no such linear combinations are necessary.
a sector with vacuum energy $H_i$ should really be added to the degeneracy of the $(n-1)^{\text{th}}$ level in a sector which vacuum energy $H_i + 1$, etc. Indeed, it is only the combination $n' \equiv n + H_i$ which has physical meaning as the “energy” of the state, invariant across all sectors of the theory. We can therefore, as a first step, rewrite the asymptotic expansions from Sect. 2 in terms of this properly shifted variable $n'$ (and henceforth drop the prime on $n$):

$$a_n^{(i)} = \sum_{\alpha = 1}^{\alpha_{\text{max}}} \left( \frac{2\pi}{\alpha} \right) \sum_{j=1}^{N} Q_{ij}^{(\alpha)} f_j^{(\alpha)}(n)$$

where

$$Q_{ij}^{(\alpha)} \equiv e^{\pi i k / 2} \sum_{\beta} (M_{\alpha \beta}^{-1})_{ij} \exp \left[ \frac{2\pi i}{\alpha} (\beta' H_j - \beta n) \right]$$

and where

$$f_j^{(\alpha)}(n) = \begin{cases} 
(\sqrt{H_j/n})^{1-k} J_{k-1} \left( 4\pi \sqrt{H_j n/\alpha} \right) & \text{for } H_j > 0 \\
(\sqrt{H_j/n})^{1-k} I_{k-1} \left( 4\pi \sqrt{|H_j| n/\alpha} \right) & \text{for } H_j < 0 \\
(2\pi n/\alpha)^{k-1} & \text{for } H_j = 0.
\end{cases}$$

While this variable shift $n \rightarrow n' \equiv n + H_i$ is somewhat trivial for the present one-variable case (in particular, both the original and final $n$’s are integers since each $H_i \in \mathbb{Z}$), we will see that this step is far more subtle for the two-variable case.

In terms of these shifted variables, then, the $q$-expansion for $f$ becomes $f = \sum_n a_n q^n$ where $a_n$ is now the number of states at energy $n$, and the asymptotic expansion for $a_n$ is simply given by

$$a_n = \sum_{i,\alpha,j} \left( \frac{2\pi}{\alpha} \right) c_i Q_{ij}^{(\alpha)} f_j^{(\alpha)}(n).$$

However, having performed the shift of variables, we now see from (3.10) that $f_j^{(\alpha)}(n)$ is independent of $i$. We therefore find that the sum over $i$ factors,

$$a_n = \sum_{\alpha,j} \left( \frac{2\pi}{\alpha} \right) \left( \sum_i c_i Q_{ij}^{(\alpha)} \right) f_j^{(\alpha)}(n),$$

and that within this factored sum there is yet another factorization:

$$\sum_i c_i Q_{ij}^{(\alpha)} = e^{\pi i k / 2} \sum_{\beta} \left( \sum_i c_i (M_{\alpha \beta}^{-1})_{ij} \right) \exp \left[ \frac{2\pi i}{\alpha} (\beta' H_j - \beta n) \right].$$

However, we recall from Sect. 2 that $M_{\alpha \beta}^{-1}$ is a modular transformation, and that $(M_{\alpha \beta}^{-1})_{ij}$ is the representation matrix corresponding to this transformation in the space of characters $\chi_i$. From (3.6), therefore, we see that

$$\sum_i c_i (M_{\alpha \beta}^{-1})_{ij} = c_j$$

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for all $\alpha$ and $\beta$, yielding simply

$$\sum_i c_i Q^{(\alpha)}_{ij} = e^{\pi ik/2} c_j \sum_{\beta} \exp \left[ \frac{2\pi i}{\alpha} (\beta' H_j - \beta n) \right]. \quad (3.15)$$

It may seem strange at this point that $\beta'$ still appears in (3.15), since a priori all knowledge of the original choice of $\beta'$ [which was implicit in our choice of modular transformation $M_{\alpha \beta}$ in (2.8)] has been removed. The important point to realize is that for any given values of $\alpha$ and $\beta$, there are only certain allowed values of $\beta'$ which may be chosen, and (as discussed in the footnote in Sect. 2) these values of $\beta'$ are all equal modulo $\alpha$. Thus, since $H_j \in \mathbb{Z}$ for all $j$, the result in (3.15) is independent of the particular choice of $\beta'$, and depends instead on only the value of $\beta'$ modulo $\alpha$. This value is uniquely determined from knowledge of $\alpha$ and $\beta$ alone, and therefore does not require knowledge of the chosen modular transformation $M_{\alpha \beta}$.

Thus, combining the above results, we have

$$a_n = \sum_{\alpha,j} \left( \frac{2\pi}{\alpha} \right) e^{\pi ik/2} \left\{ \sum_{\beta} \exp \left[ \frac{2\pi i}{\alpha} (\beta' H_j - \beta n) \right] c_j f_j^{(\alpha)}(n) \right\}. \quad (3.16)$$

However, since the $c_j$ are the original coefficients of $f$, we recognize that $c_j \neq 0$ only if $H_j \geq 0$. Thus, the only functions $f_j^{(\alpha)}(n)$ which contribute to the asymptotic expansion of $a_n$ are those for which $H_j \geq 0$. However, for $k < 0$, all of these functions vanish in the $n \to \infty$ limit: if $H_j > 0$, then $f_j^{(\alpha)}(n) \sim n^{k/2-3/4}$ [recall (3.10) and (2.19)], while if $H_j = 0$, then $f_j^{(\alpha)}(n) \sim n^{-1}$. We therefore find

$$a_n \to 0 \quad \text{as } n \to \infty, \quad (3.17)$$

and since the error in the asymptotic expansion also vanishes for $k < 0$ [recall (2.17)], for sufficiently large $n$ this can be taken to be an exact result:

$$a_n = 0 \quad \text{for sufficiently large } n. \quad (3.18)$$

It is then a simple matter, using the modular transformation $\tau \to -1/\tau$ and a Poisson resummation, to demonstrate that all coefficients $a_n$ must vanish exactly.

As an example, let us consider the function $J$ given in (3.4). As indicated in (3.5), this function can be written as a sum of characters which meet all of the necessary conditions:

$$\frac{J}{\eta^{12}} = \frac{1}{\eta^8} \left\{ (\chi_0)^7 \chi_{1/2} + 7 (\chi_0)^5 (\chi_{1/2})^3 + 7 (\chi_0)^3 (\chi_{1/2})^5 + \chi_0 (\chi_{1/2})^7 - (\chi_{1/16})^8 \right\}. \quad (3.19)$$

Here $\chi_h$ with $h \in \{0, 1/2, 1/16\}$ are the characters of the $c = 1/2$ Ising model, and recalling that $1/\eta$ is the character of an uncompactified boson with $c = 1$, we see that each of the five terms in the sum (3.19) is therefore a distinct character in
the large $c = 12$ conformal field theory which consists of a tensor product of eight uncompactified bosons and eight Ising models. Indeed, each of these characters has $k = -4$, with individual vacuum energies $H_i = \sum h - c/24 = 0, 1, 2, 3, 0$ respectively. The conditions (3.7) are therefore satisfied. Indeed, since each of the terms in (3.19) is the character $\chi_i$ of a $c = 12$ conformal field theory, their separate degeneracies $a_n^{(i)}$ each grow asymptotically $\sim \exp(C\sqrt{n})$ with $C = 2\sqrt{2}\pi$, in accordance with (2.3).

However, the sum of these asymptotic expansions, and indeed each total degeneracy $a_n$, vanishes identically. The physical interpretation of this fact is that the final term in (3.19) is the contribution of a fermionic Ramond sector (as indicated by the fact that each factor $\chi_{1/16}$ corresponds to an $h = 1/16$ Ising-model spin field), and thus cancels the contributions of the four bosonic (Neveu-Schwarz) sectors to yield a supersymmetric theory. The result $J = 0$ is often called the Jacobi identity, and the partition functions of supersymmetric theories are usually proportional to $J$.

An interesting corollary of this proof concerns the case of functions $f$ which are modular-invariant with $H \geq 0$ but which have positive modular weight $k$. Such functions of course do not vanish, and we would a priori expect exponential growth in their coefficients. However, as we can see from the above derivation, this is not the case: the relevant functions $f_j^{(a)}(n)$ can grow at most polynomially with $n$, and moreover the error in the asymptotic expansion for a given $a_n$ can also grow at most polynomially ($\sim n^k$). Thus, surprisingly, the maximum possible growth for such cases is polynomial rather than exponential. For example, the $E_8$ character $\text{ch}(E_8) = \vartheta_2^8 + \vartheta_3^4 + \vartheta_4^8$ has $H = 0$ and $k = 4$, and indeed the coefficients in its $q$-expansion grow only as fast as $n^4$.

A similar but more delicate situation exists for modular-invariant functions which have $k = 0$. In this case the relevant functions $f_j^{(a)}(n)$ all vanish in the asymptotic limit $n \to \infty$, and indeed one again obtains (3.17). However, the error in the asymptotic expansions can be as great as an overall additive constant. Thus we have an interesting situation in which the values of $a_n$ experience no growth at all, but are not necessarily zero. However, one can show that if the $a_n$’s experience no growth and $f(-1/\tau) = f(\tau)$, then the only possible solution is $a_n \propto \delta_{n,0}$ if $H = 0$, or all $a_n = 0$ if $H > 0$. Thus, if $k = 0$ and $H \geq 0$, we have $f$ = constant, and this constant is non-zero only if $H = 0$. An example of the latter situation is the modular-invariant function

$$K \equiv \frac{\vartheta_2 \vartheta_3 \vartheta_4}{2\eta^3} = \left[(\chi_0)^2 - (\chi_{1/2})^2\right]^2 (\chi_{1/16})^2$$

with $H = k = 0$; by the above result this must equal a constant, and indeed one finds $K = 1$. These results are also in accord with known theorems in modular function theory.
3.2 The Two-Variable Case

We shall now proceed to examine the more physically relevant case of modular functions of two variables, \( Z(q, \bar{q}) \), to see if an analogous result can be obtained.

At the outset, there are a number of differences between this and the simpler one-variable case. The fact that we have both a holomorphic variable \( q \) and an anti-holomorphic variable \( \bar{q} \) means that modular-invariant functions \( Z(q, \bar{q}) \) can in principle be formed with \( \) two separate systems of characters, \( \chi \) and \( \bar{\chi} \):

\[
Z = (\text{Im} \, \tau)^k \sum_{i, \bar{i}} \chi^*_i \chi_i = (\text{Im} \, \tau)^k \sum_{i, \bar{i}} N_{\bar{i}i} q^{H_i} \bar{q}^{\bar{H}_i} \sum_{m,n} \bar{\chi}^{(i)}_m q^m \bar{q}^n .
\]

(3.21)

In fact, partition functions of this general type appear for any string theory (such as all heterotic string theories) with unequal underlying left- and right-moving worldsheet conformal field theories, and the \((\bar{i}, i)\) summation within (3.21) will indeed be finite if these worldsheet theories are rational. It is therefore necessary, in general, to keep track of \( \) two separate systems of characters, \( \chi \) and \( \bar{\chi} \), related to each other in (3.21) only through the requirement of modular invariance. Hence, throughout this section, we shall use the following notation: a bar above any variable will refer to the right-moving system of characters \( \bar{\chi} \), and complex conjugation will instead be indicated explicitly with an asterisk. Of course, \( \bar{q} = q^* \).

As in the one-variable case, we will demand that our function \( Z \) is modular-invariant with modular weight \( k \). Explicitly, this means that \( Z(\tau + 1) = Z(-1/\tau) = Z(\tau) \) where the individual characters \( \chi \) and \( \bar{\chi} \) satisfy (2.1) with modular weight \( k \). [Note that the factor \((\text{Im} \, \tau)^k\) in (3.21) is necessary so that \( Z(-1/\tau) = Z(\tau) \).] It is then straightforward to show that the coefficients \( N_{\bar{i}i} \) in (3.21) must satisfy

\[
\sum_{i, \bar{i}} M_{\bar{i}i}^* N_{\bar{i}i} M_{ij} = N_{\bar{j}j} \quad \text{(3.22)}
\]

for all pairs of matrices \( M_{ij} \) and \( M_{\bar{i}j} \) which respectively represent the same modular transformation \( M \) in the \( \chi_i \) and \( \bar{\chi}_{\bar{i}} \) systems. This condition is of course the two-variable analogue of (3.6). Note that for the special case \( M = T \), (3.22) implies the so-called level-matching condition

\[
H_i - \bar{H}_\bar{i} \in \mathbb{Z} \quad \text{if} \quad N_{\bar{i}i} \neq 0 .
\]

(3.23)

This condition is the source of the second fundamental difference from the one-variable case. For the one-variable case, the analogous modular-invariance condition required simply that all \( H_i \in \mathbb{Z} \); essentially the “anti-holomorphic characters” were all \( \bar{\chi} = 1 \), with \( \bar{H}_i = 0 \). Now, however, we see that we need no longer have integer vacuum energies \( H_i \) and \( \bar{H}_i \), and in fact for most physical situations non-integer values of \( H_i \) and \( \bar{H}_i \) do appear. Indeed, all that is required is that any pairs \( H_i \) and \( \bar{H}_i \) coupled together via the \( N_{\bar{i}i} \) matrix be equal modulo 1. It is this fact which
permits a “misalignment” of the various sectors in such a left/right theory. Recall that each character $\chi_i$ and $\bar{\chi}_i$ corresponds to an entire tower of chiral states, with integer spacing between adjacent levels in each tower. Thus, while the level-matching condition (3.23) guarantees that a given pair of left- and right-moving characters are properly aligned relative to each other (yielding a set of properly constructed left/right states), a given pair of characters corresponding to a given sector of the theory can nevertheless be “misaligned” relative to another pair. Thus, the different sectors will in general exhibit a whole spectrum of alignments, each corresponding to a different value of $H_i = \Pi_\tau$ (modulo 1), and any analogous cancellation which we expect to observe between the different sectors of the theory must now somehow take into account the fact that these sectors are misaligned.

Given the modular-invariance condition (3.22), we now must impose the conditions analogous to those of the one-variable case in (3.7). The first of these is straightforward: we shall once again consider those partition functions $Z$ for which the modular weight $k$ is negative. Indeed, in most physical situations the weight $k$ is related to the number of uncompactified spacetime dimensions $D$ via

$$k = 1 - D/2,$$

(3.24)

so this condition corresponds to the physically interesting cases with $D > 2$. (We will nevertheless show at the end of this section that our results apply for all spacetime dimensions $D$, and thereby include the cases with non-negative values of $k$ as well.) Generalizing the second condition in (3.7) is more subtle, however, for two distinct possibilities present themselves:

$$H_i \geq 0 \quad \text{and} \quad \Pi_\tau \geq 0 \quad \text{if} \quad N_\tau \neq 0,$$

(3.25)

or

$$H_i \geq 0 \quad \text{or} \quad \Pi_\tau \geq 0 \quad \text{if} \quad N_\tau \neq 0.$$

(3.26)

The first of these two options is the strictest generalization of the one-variable condition (3.7), and for $k < 0$ leads to the analogous result $Z = 0$ (i.e., equal numbers of bosonic and fermionic states at all mass levels). This is therefore the appropriate condition for theories with spacetime supersymmetry, for such theories, in agreement with (3.25), necessarily lack both physical and unphysical tachyons. By contrast, the second option (3.26) is far weaker, and merely requires that there exist no physical tachyons; unphysical tachyons are still permitted. As we will see, this case corresponds to theories without spacetime supersymmetry, theories in which the numbers of bosons and fermions at all mass levels are a priori unequal. The condition (3.26) nevertheless still prohibits the appearance of physical tachyons: these are the tachyons which cause fundamental physical inconsistencies in spacetime, and

† Similarly, for $k = 0$ we would actually have $Z = \text{constant}$, with a net number of physical states surviving at the massless level only.
which lead to divergent amplitudes. Thus (3.26) is the weakest condition that can be imposed for physically sensible theories, and while the existence of unphysical tachyons will of course prevent $Z$ from vanishing altogether, our goal is to determine if the absence of physical tachyons nevertheless leads to any (weaker) cancellation.

Our derivation now proceeds exactly as for the one-variable case: we shall derive an asymptotic expansion for each term in the partition function (3.21), corresponding to a different sector of the theory, and then add these asymptotic expansions together.

Our first step is to adjust our summation variables $m$ and $n$ in (3.21) so that they correspond to the invariant energy of the corresponding state, and not just the number of integer-excitations above the varying vacuum energies $H_i$ and $\overline{H}_\tau$. This will enable us to expand the partition function (3.21) in the simple form

$$Z = (\text{Im } \tau)^k \sum_{m,n} a_{mn} \overline{q}^m q^n \quad (3.27)$$

and so read off the total degeneracy $a_{nn}$ of physical states with a given energy $n$. As in the one-variable case, this is trivially accomplished by shifting the summation variables $m$ and $n$ in each $(\tau, i)$ sector so that these different vacuum energies are properly incorporated: $n \rightarrow n + H_i$, $m \rightarrow m + \overline{H}_\tau$. Unlike the one-variable case, however, this operation has drastic consequences. Of course, since the values of $H_i$ and $\overline{H}_\tau$ are non-integral, the new summation variables $m$ and $n$ in (3.27) are also non-integral, and each sector $(\tau, i)$ will contribute states to only those degeneracy counts $a_{mn}$ with $m = n = H_i = \overline{H}_\tau$ (modulo 1). This much is non-problematic. However, it is then no longer appropriate to simply add the asymptotic expansions as functions of $n$ in order to obtain a value for a given $a_{nn}$; indeed, for any $n$, one should properly add together only those asymptotic expansions which correspond to sectors with vacuum energies $H_i$ and $\overline{H}_\tau$ satisfying $H_i = \overline{H}_\tau = n$ (modulo 1).

Despite this fact, we shall nevertheless proceed to add together the asymptotic expansions from all the sectors, just as was done in the one-variable case. While the physical significance of this addition may not yet be apparent, we will find that a powerful result with important physical consequences can nevertheless be obtained. We will refer to this addition operation as “sector-averaging”, and denote the resulting “sector-averaged” degeneracy at energy $n$ as $\langle a_{nn} \rangle$. Our results from this point forward shall therefore apply to $\langle a_{nn} \rangle$ rather than to any particular value of $a_{nn}$ in (3.27), and after obtaining our ultimate result for $\langle a_{nn} \rangle$ we shall discuss the consequences for the true state degeneracies $a_{nn}$.

† As in the one-variable case, we note that a decomposition of the partition function $Z$ into characters satisfying (3.26) exists even in those cases in which certain sectors are individually tachyonic, provided their tachyonic contributions are cancelled (or “GSO-projected”) in the sum (3.21). Thus (3.26) indeed represents a general condition for functions $Z$ without physical tachyons, and in these particular cases linear combinations of characters may be necessary when constructing a suitable character-decomposition of $Z$. In Sect. 4 we shall discuss in detail the effects of working with such linear combinations, and in the remainder of this section we shall assume for simplicity that no such linear combinations are needed.
Thus, explicitly, our calculation will be formulated as follows. First, for each sector \((i, \bar{i})\) in (3.21), we shift the variables \(m\) and \(n\) so that the vacuum energies \(H_i\) and \(\bar{H}_{\bar{i}}\) are properly incorporated and
\[
\chi_{\bar{i}} \chi_i = \sum_{m,n} a_{mn}^{(i)} q^m q^n. \tag{3.28}
\]
The quantities \(a_{mn}^{(i)}\) then represent the net degeneracies of physical and unphysical states in the \((\bar{i}, i)\) sector, related to the separate chiral degeneracies \(a_n^{(i)}\) and \(\bar{a}_m^{(i)}\) via
\[
a_{mn}^{(i)} = \bar{a}_m^{(i)} q_n^{(i)}. \tag{3.29}
\]
Let us now draw a notational distinction between a particular degeneracy \(a_n^{(i)}\) and its corresponding asymptotic expansion, denoting this latter function [given on the right side of (3.8)] as \(\phi_n^{(i)}(n)\). Thus, further denoting by \(\Phi_{\bar{i}}(n)\) the function describing the physical degeneracies \(a_n^{(i)}\), we have
\[
\Phi_{\bar{i}}(n) = [\phi_{\bar{i}}^{(i)}(n)]^* \phi_n^{(i)}(n). \tag{3.30}
\]
The operation of “sector averaging” is then rigorously defined via the modular-invariant summation of these functional forms:
\[
\langle a_{mn} \rangle \equiv \sum_{i, \bar{i}} N_{\bar{i}} \Phi_{\bar{i}}(n) = \sum_{i, \bar{i}} N_{\bar{i}} [\phi_{\bar{i}}^{(i)}(n)]^* \phi_n^{(i)}(n). \tag{3.31}
\]
Although the asymptotic expansions \(\phi_n^{(i)}(n)\) are \(a priori\) real, we have explicitly indicated their complex conjugations (where appropriate) in (3.30) and (3.31).
Substituting the asymptotic expansions given in (3.8), we then find
\[
\langle a_{mn} \rangle = \sum_{i, \bar{i}} N_{\bar{i}} \left(\frac{4\pi^2}{\alpha \bar{\alpha}}\right) \sum_{j, \bar{j}} Q_{ij}^{(a)} \left(\bar{Q}_{\bar{i}}^{(\bar{a})}\right)^* f_{\bar{j}}^{(\bar{a})}(n) f_{\bar{j}}^{(\bar{a})}(n), \tag{3.32}
\]
and once again the summations over \(i\) and \(\bar{i}\) factor explicitly:
\[
\langle a_{mn} \rangle = \sum_{a, \bar{a}, i, \bar{i}} \left(\frac{4\pi^2}{\alpha \bar{\alpha}}\right) \left\{ \sum_{j, \bar{j}} \left(\bar{Q}_{i\bar{j}}^{(\bar{a})}\right)^* N_{\bar{i}} Q_{ij}^{(a)} \right\} f_{\bar{j}}^{(\bar{a})}(n) f_{\bar{j}}^{(\bar{a})}(n). \tag{3.33}
\]
Indeed, just as in the one-variable case, there is now an additional factorization within the \((i, \bar{i})\) summation:
\[
\sum_{i, \bar{i}} \left(\bar{Q}_{i\bar{j}}^{(\bar{a})}\right)^* N_{\bar{i}} Q_{ij}^{(a)} = \sum_{\beta, \bar{\beta}} \exp \left[\frac{2\pi i}{\alpha} (\beta' H_j - \beta n) - \frac{2\pi i}{\alpha} (\beta' \bar{H}_{\bar{j}} - \bar{\beta} n)\right] \times
\]
\[
\times \left\{ \sum_{i, \bar{i}} \left(\bar{M}_{i\bar{j}}^{(\bar{a}\bar{\beta})}\right)^* N_{\bar{i}} (M_{a\beta})_{ij} \right\}. \tag{3.34}
\]
Note that these two last steps implicitly assume that the variable $n$ is independent of $i$, which is the essence of the “sector averaging” discussed above. Even though different sectors $(\tau, i)$ correspond to different values of $n$ modulo 1, we are here adding the sum of functional forms of $n$.

The above factorization now enables us to make one important simplification. Using (3.22), we can rewrite the second line of (3.34):
\[
\sum_{i, \tau} (M_{\alpha\beta}^{-1})^{\tau \tau} N_{\alpha\tau} (M_{\alpha\beta}^{-1})_{ij} = \left( N M_{\alpha\beta} M_{\alpha\beta}^{-1} \right)_{\tau j}; \quad (3.35)
\]
such rewritings are useful in that they generally allow us to focus on the modular-transformation representation matrices in only one character system (e.g., the holomorphic system, as formulated above) rather than on both simultaneously. Thus, collecting our results in this final form, we have the following total expansion for $\langle a_{nn} \rangle$:
\[
\langle a_{nn} \rangle = \sum_{\alpha, \alpha', j, \bar{j}} \left( \frac{4\pi^2}{\alpha \alpha' j} \right) P_{\alpha j}^{(\pi, \alpha)} f_j^{(\alpha)} (n) f_{\bar{j}}^{(\alpha)} (n) \quad (3.36)
\]
where
\[
P_{\alpha j}^{(\pi, \alpha)} = \sum_{\beta, \beta'} \exp \left[ \frac{2\pi i}{\alpha} (\beta' H_j - \beta n) - \frac{2\pi i}{\alpha} (\bar{\beta} \bar{H}_{\bar{j}} - \bar{\beta} n) \right] \left( N M_{\alpha\beta} M_{\alpha\beta}^{-1} \right)_{\tau j}. \quad (3.37)
\]

Let us now examine the leading term in this expansion for $\langle a_{nn} \rangle$. Since the leading term for each chiral character is obtained for $\alpha = 1$, the leading term in (3.36) is that for which $\alpha = 1$ and $\bar{\alpha} = 1$. However, in this case we find that $P_{\alpha j}^{(\pi, \alpha)}$ assumes a particularly simple form: since the only allowed values of $\beta$ and $\bar{\beta}$ are $\beta = \bar{\beta} = 0$, we have simply
\[
P_{1 j}^{(1, 1)} = (NM_{10} M_{10}^{-1})_{\tau j} = N_{\tau j}. \quad (3.38)
\]
Thus, we find that the leading term in the expansion of $\langle a_{nn} \rangle$ is simply
\[
\langle a_{nn} \rangle = 4\pi^2 \sum_{j, \bar{j}} N_{\tau j} f^{(1)}_j (n) f^{(1)}_{\bar{j}} (n) + \ldots \quad (3.39)
\]

This result is sufficient to prove our main theorem. Since the partition function $Z$ in (3.21) is assumed to satisfy (3.26), we see that the only values of $j$ and $\bar{j}$ which contribute to the sum in (3.39) are those for which either $H_j \geq 0$ or $\bar{H}_{\bar{j}} \geq 0$. This means that there exists no term in (3.39) which contains a product of two functions $f^{(1)}_j (n)$ and $f^{(1)}_{\bar{j}} (n)$ with both experiencing maximum growth. In particular, the term in the asymptotic expansion with the greatest possible growth would a priori have been given by $f^{(1)}_{j=1} (n) f^{(1)}_{\bar{j}=1} (n)$ where $j = 1$ and $\bar{j} = 1$ respectively indicate the vacuum sectors of the separate chiral theories, yet we see that such a maximally-growing term is necessarily absent in (3.39). The absence of such a term in the expansion
for the sector-averaged \( \langle a_{nn} \rangle \) is of course the direct consequence of the absence of a corresponding physical tachyon in the partition function \( Z \).

By contrast, let us consider the degeneracies \( a_{nn}^{(\tau)} \) of each individual sector \((\tau, i)\) before sector-averaging, as given in (3.28) and (3.29). In analogy to (3.33), we find that these \( a_{nn}^{(\tau)} \) have expansions

\[
a_{nn}^{(\tau)} = \sum_{\alpha, \alpha', j} \left( \frac{4\pi^2}{\alpha^2} \right) \left( Q_{ij}^{(\tau)} \right)^* Q_{ij}^{(\alpha)} f_j^{(\alpha)}(n) \overline{f_j^{(\tau)}(n)},
\]

and using (2.25), we see that the leading terms in (3.40) are

\[
a_{nn}^{(\tau)} = \sum_{j} 4\pi^2 S_{ij} S_{ij} f_j^{(\alpha)}(n) \overline{f_j^{(\tau)}(n)} + ...
\]

The terms in (3.40) with maximal growth are again \textit{a priori} given by \( f_j^{(1)}(n) \overline{f_j^{(1)}(n)} \) where \( j = \tilde{j} = 1 \) indicate the vacuum sectors of the separate chiral theories. However, recall that for valid conformal field theory characters, \( S_{11} \) and \( S_{\tilde{1}} \) are necessarily non-zero [5]. Thus, for each of the individual sectors \((\tau, i)\), this term with maximal growth must exist in the expansion for the degeneracies, and it is only in the process of “sector averaging” that this term is cancelled.

This is precisely the content of our main result, as expressed in (1.12). Since the leading term in (3.41) contains the factor \( f_j^{(1)}(n) \overline{f_j^{(1)}(n)} \), the growth of each \( a_{nn}^{(\tau)} \) as \( n \to \infty \) is

\[
a_{nn}^{(\tau)} \sim \exp \left\{ 4\pi \left( \sqrt{|H_1|n} + \sqrt{|H_{\tilde{1}}|n} \right) \right\} \equiv \exp \left( C_{\text{tot}} \sqrt{n} \right)
\]

where

\[
C_{\text{tot}} = C_{\text{left}} + C_{\text{right}} \equiv 4\pi \left( \sqrt{\frac{c_{\text{left}}}{24}} + \sqrt{\frac{c_{\text{right}}}{24}} \right).
\]

However, the growth in \( \langle a_{nn} \rangle \) is necessarily less rapid, since the rate of exponential growth in (3.42) can come only from the dominant term \( f_j^{(1)}(n) \overline{f_j^{(1)}(n)} \). Thus, defining the total effective “sector-averaged” value of \( C \) through

\[
\langle a_{nn} \rangle \sim \exp \left( C_{\text{eff}} \sqrt{n} \right) \quad \text{as} \quad n \to \infty,
\]

we have

\[
C_{\text{eff}} < C_{\text{tot}}.
\]

Since the exact value of \( C_{\text{eff}} \) in (3.44) is in principle determined by the largest remaining \textit{subdominant} term in the complete \( \langle a_{nn} \rangle \) expansion, let us now focus briefly on those subdominant terms which might be relevant. Such terms can have a variety of origins, and we shall outline them below.
• First, there are of course subdominant terms within (3.39), for the presence of unphysical tachyons within the partition function $Z$ requires that there exist non-vanishing terms in (3.39) for which $H_j < 0$ and $\overline{H}_\tau \geq 0$ (or vice versa). It is in fact a generic property that there exist such unphysical tachyons in non-supersymmetric theories, for modular invariance alone can be used to demonstrate that any $D > 2$ theory which lacks both physical and unphysical tachyons will have a vanishing partition function. For example, in the case of generic non-supersymmetric string theories containing gravitons, it is not difficult to show that such unphysical tachyons will appear in sectors with $H_j = -\frac{c_{left}}{24}$ and $\overline{H}_\tau = 0$ (or vice versa). These unphysical tachyons then imply exponential growth (3.44) with $C_{eff} = C_{left} \equiv 4\pi \sqrt{\frac{c_{left}}{24}}$ (or $C_{eff} = C_{right} \equiv 4\pi \sqrt{\frac{c_{right}}{24}}$) only. Note that this is, in general, the greatest exponential growth that can arise from the leading terms (3.39).

• Similarly, there may be other strong subdominant exponentials which arise from terms in the general expansion (3.36) with either $\alpha > 1$ or $\tau > 1$. The greatest of these is a priori the case when $(\tau, \alpha) = (1, 2)$ or $(2, 1)$. Since $M^{-1}_{10}$ and $M^{-1}_{21}$ are different modular transformations, a simplification analogous to that in (3.38) is not possible, and therefore in principle any combination of functions $f^{(1)}_j(n)f^{(2)}_\tau(n)$ or $f^{(2)}_j(n)f^{(1)}_\tau(n)$ is possible. Whether such terms actually appear in general depends on the details of the types of unphysical tachyons that a given partition function contains, and therefore, unlike the first class of subleading exponentials discussed above, such terms are not necessarily universal. Note that these terms may nevertheless yield stronger exponential growth than those in the first class discussed above. In particular, general terms such as $f^{(\alpha)}_j(n)f^{(\tau)}_\tau(n)$ with $H_j < 0$ and $\overline{H}_\tau < 0$ yield values

$$C_{eff} = 4\pi \left( \frac{\sqrt{|H_j|}}{\alpha} + \frac{\sqrt{|\overline{H}_\tau|}}{\tau} \right),$$

and, depending on the particular values of $H_j$ and $\overline{H}_\tau$ involved, such values of $C_{eff}$ may exceed the values $C_{left}$ or $C_{right}$ which arise from from the first class

---

$b$ The proof runs as follows. Such unphysical tachyonic states are bosonic, and are essentially the graviton state without the excitation of the left- (or right-) moving mode of the coordinate boson. Since the constraint equations for these unphysical tachyons are therefore those of the graviton itself, such a state is guaranteed to survive all Fock-space projections. The only way to cancel the contribution of this state to the partition function $Z$ is to have a corresponding fermionic state with the same (unphysical tachyonic) left/right energy distribution. However, such fermionic states are likewise related to the physical gravitino states, and therefore any constraint equations which project out the gravitino state (thereby rendering the theory non-supersymmetric) must project out this fermionic tachyon as well. Thus, for non-supersymmetric theories containing a graviton, there always exist sectors in the partition function with $H_i = -\frac{c_{left}}{24}$ and $\overline{H}_\tau = 0$ (or vice versa) whose contributions to $Z$ are not cancelled.
of exponentials. Of course, terms leading to (3.46) are possible only if $\mathcal{P}_{ij}$ in (3.37) is non-zero for appropriate values of $(\bar{\mathfrak{j}}, j)$. We shall discuss these subleading terms in more detail in Sect. 4.3.

- Finally, there is another potential source of exponential growth, this one arising from the asymptotic expansion error terms. Recall that in the one-variable case, no exponential growth could ever arise from the error terms, for these terms are at most polynomial. In the present two-variable case, however, there is a separate error term from each of the left and right-moving asymptotic expansions, and thus the error term from the left-moving expansion can multiply an exponential term from the right-moving expansion, and vice versa. The maximum exponential growth this can produce, however, is $C = C_{\text{left}}$ or $C_{\text{right}}$ respectively, and this is the same growth which arises from the subdominant terms in the first class discussed above.

Thus, we see that the largest value of $C_{\text{eff}}$ depends crucially on the details of the underlying non-supersymmetric theory, and is determined by the structure and energy distribution of its unphysical tachyons. In fact, we shall discuss the precise value of $C_{\text{eff}}$ at several points in this and later sections. The important point, however, is that none of these subdominant terms can ever reproduce the leading exponential growth that is experienced by each $a_{n\bar{n}\mathfrak{i}}$ individually. Thus the existence of these subdominant terms can never alter our main result (3.45).

As discussed in the Introduction, the result (3.45) for $\langle a_{nn} \rangle$ has profound implications for the values of the actual degeneracies $a_{n\bar{n}\mathfrak{i}}$. We have already seen that each $a_{n\bar{n}\mathfrak{i}}$ necessarily experiences stronger exponential growth than does the sector-averaged quantity $\langle a_{nn} \rangle$, and indeed we have

$$\lim_{n \to \infty} \frac{\langle a_{nn} \rangle}{a_{n\bar{n}\mathfrak{i}}} = 0.$$  (3.47)

This of course implies that the dominant exponentials in the different $a_{n\bar{n}\mathfrak{i}}$ expansions must exactly cancel in the sector-averaging summation, with some sectors having positive dominant terms and others negative (respectively, with some bosonic and others fermionic). However, this does not necessarily imply any cancellation in the net number of actual states in the theory. Recall that each sector $(\bar{\mathfrak{i}}, i)$ in principle has a different alignment, and contributes states to the spectrum at only those energies $n$ satisfying $n = H_i = \Pi_\tau$ (modulo 1). Thus two sectors — for example, one with states at $n = 0$ (modulo 1) and the other with $n = 1/2$ (modulo 1) — can have equal and opposite dominant terms in their asymptotic expansions without having any cancellation between their actual states at any given mass level. Indeed, even the number of states at the different mass levels will not be equal: if $\Phi(n)$ represents this leading asymptotic term, then while the degeneracies of the first sector will be given asymptotically by the values $\{\Phi(\ell), \ell \in \mathbb{Z}\}$, those of the second sector will be given
asymptotically by the different values \(\{-\Phi(\ell + \frac{1}{2}), \ \ell \in \mathbb{Z}\}\). It is only by considering the functional forms \(\Phi\) that cancellations are apparent.

Our result thus implies that as the invariant energy \(n\) is increased, the net number of states \(a_{nm}\) at energy \(n\) exhibits an oscillation: first this number will be positive (implying more bosonic states than fermionic states), then negative (implying more fermionic states than bosonic states at the next mass level), and then positive again. Indeed, the “wavelength” of this repeating oscillation is \(\Delta n = 1\), corresponding to energy difference between adjacent states in the same sector. While the amplitude of this oscillation grows exponentially, the oscillation asymptotically becomes symmetric between positive and negative values. In general, of course, there are more than two sectors in the theory, and while some groups of sectors will be aligned relative to each other, others will be misaligned. Thus there may exist a potentially large set of values of \(n\) (modulo 1) at which states will be found, and the pattern of oscillation which we have described may be quite complicated within each “wavelength” \(\Delta n\). We will see explicit examples of this in Sect. 4. What is guaranteed by our result, however, is that the leading terms must cancel as we sum over sectors, and that therefore the behavior of the net number of states at energy \(n\) must execute this increasingly symmetric oscillatory behavior as \(n\) is increased.

In physical terms, this result amounts to a strong constraint on the degree to which supersymmetry may be broken in a modular-invariant theory without introducing damaging physical tachyons. Quite simply, the supersymmetry can be at most “misaligned”: introducing any surplus of bosonic states at any mass level in the theory necessarily implies the simultaneous creation a larger surplus of fermionic states at a higher mass level, which in turn implies an even larger surplus of bosonic states at an even higher mass level, etc. Since modular invariance and the absence of physical tachyons are precisely the conditions for finite string amplitudes, this scenario now provides a glimpse of how this finiteness arises level-by-level in the actual spectrum of states that contribute to single- and multi-loop processes. Indeed, we shall discuss the relationship between this “misaligned supersymmetry” and the value of one such one-loop amplitude, namely the one-loop cosmological constant, in Sect. 5.

Finally, let us discuss the role played in the two-variable theorem by the modular weight \(k\). As we recall from Sect. 2, the most important part of the asymptotic expansions which depend on the modular weight \(k\) are the error terms: for \(k < 0\) they asymptotically vanish like a (negative) power of \(n\), and for \(k > 0\) they grow at most polynomially with \(n\). In the one-variable case, this was sufficient to imply that any tachyon-free modular-invariant function \(f(q)\) would have to vanish if \(k < 0\), and contain at most polynomially growing coefficients if \(k > 0\). In the two-variable case, however, no such implication follows, primarily because the error terms from each chiral half of the theory multiply the exponentially growing terms from the other half of the theory. This has the benefit of effectively rendering our result independent of the value of \(k\). Thus, our theorem concerning the reduction of the effective growth rate from \(C_{\text{tot}}\) to \(C_{\text{eff}}\) is unaffected by the actual value of the modular weight \(k\),
and our two-variable theorem (as well as its primary consequence, the misaligned supersymmetry) remain valid regardless of the spacetime dimension $D$.

This theorem is therefore quite powerful from a purely physical standpoint, implying a misaligned supersymmetry for all spacetime dimensions $D$. From a mathematical standpoint, however, we see that its robustness essentially results from the weakness of its assertion. Indeed, our theorem asserts merely that $C_{\text{eff}} < C_{\text{tot}}$, yet the true analogue of the one-variable result would instead be the more powerful claim that

$$C_{\text{eff}} \leq 0.$$  \hspace{1cm} (3.48)

Note that this would imply that all exponential growth must vanish in the sum \((3.36)\), with the contributions from subleading terms cancelling the contributions from unphysical tachyons. Moreover, in analogy with the one-variable result, we would assert that in the limit $n \to \infty$, the behavior of the asymptotic expansions $\langle a_{nn} \rangle$ must be

$$\langle a_{nn} \rangle \sim \begin{cases} 0 & \text{for } k < 1 \\ n^{k-1} & \text{for } k \geq 1 \end{cases}, \hspace{1cm} (3.49)$$

since the strongest growth that can arise from any of the functions $f_j^{(\alpha)}(n)$ with $H_j \geq 0$ is $\sim n^{k-1}$ for $k > 1/2$. For $k \leq 1/2$, by contrast, all such functions $f_j^{(\alpha)}$ vanish as $n \to \infty$.

Surprisingly, it turns out that there is independent evidence for both of these conjectures \((3.48)\) and \((3.49)\), evidence which comes from an analysis of the finiteness properties of certain one-loop amplitudes in string theories which lack physical tachyons. This evidence will be discussed in Sect. 5. Therefore, while we have proven that $C_{\text{eff}} < C_{\text{tot}}$, we shall in fact conjecture that \((3.48)\) and \((3.49)\) hold as well.

It is nevertheless evident from the above discussion we can never prove these conjectures by following the procedure presented this section, a procedure which consists of multiplying together the separate chiral asymptotic expansions for $a_{n}^{(i)}$ and $a_{n}^{(\overline{j})}$ in order to obtain an expansion for their product $a_{nm}^{(i\overline{j})}$. Indeed, we shall see in Sect. 4.3 that even these separate chiral asymptotic expansions have significant shortcomings when applied to situations (like those encountered for $\langle a_{nn} \rangle$) in which the energy $n$ is treated as a continuous variable. Thus, what is necessary is a fundamentally new type of asymptotic expansion, one which is calculated directly for $\langle a_{nn} \rangle$. We shall briefly outline how this might be derived in Sect. 6.
4 Two Examples

In this section, we provide two examples to illustrate the general results proven in Sect. 3. We will also illustrate certain techniques which often enable great simplifications when analyzing specific partition functions and their misaligned supersymmetry properties. While the first example is chosen for its simplicity, the second is more typical of the sorts of partition functions which arise for heterotic string theories compactified to four dimensions. This second example will also be relevant to Sect. 5, where we relate “misaligned supersymmetry” to the one-loop cosmological constant.

4.1 First Example

As our first simple example, let us consider the modular-invariant, tachyon-free, purely real partition function:

$$Z \equiv \frac{1}{128} (\text{Im } \tau)^{-1/2} |\eta|^{-12} \sum_{i,j,k=2}^4 (-1)^{i+1} |\vartheta_i|^2 \left[ \vartheta_j^2 \vartheta_k^2 + (-1)^{i+1} \vartheta_i^2 \vartheta_j^2 \right]^2.$$  \hspace{1cm} (4.1)

Like the typical partition functions of string theory, this function has negative modular weight ($k = -1/2$), and contains both unphysical tachyons and massless states. Here the $\vartheta$-functions are the classical Jacobi theta functions, which are related to the Ising-model characters $\chi_0$, $\chi_{1/2}$, and $\chi_{1/16}$ via

$$\chi_0 = \frac{1}{2} \left( \sqrt{\frac{\vartheta_3}{\eta}} + \sqrt{\frac{\vartheta_4}{\eta}} \right)$$

$$\chi_{1/2} = \frac{1}{2} \left( \sqrt{\frac{\vartheta_3}{\eta}} - \sqrt{\frac{\vartheta_4}{\eta}} \right)$$

$$\chi_{1/16} = \sqrt{\frac{\vartheta_2}{2\eta}}.$$  \hspace{1cm} (4.2)

Recall that the $c = 1/2$ Ising model is the conformal field theory corresponding to a single real (Majorana) worldsheet fermion.

Our first step is to cast this partition function $Z$ in the form (3.21), so that we can determine the relevant set of characters and their mixings under modular transformations. Note that our partition function (4.1) consists of a sum of terms, each of which contains five $\vartheta$-factors and five $\bar{\vartheta}$-factors. Using the relations (4.2) and recalling that $\eta^{-1}$ is the character of an uncompactified $c = 1$ boson, this would suggest that the relevant characters $\chi_i$ and $\overline{\chi}_i$ are those of a $c = 6$ theory formed as a tensor product of one uncompactified boson and ten (chiral) Ising models (or ten Majorana fermions). Since each Ising model factor in this tensor-product has three separate sectors, there are in principle $3^{10}$ separate sectors in this $c = 6$ theory, and therefore $3^{10}$ individual characters $\chi_i$ and $\overline{\chi}_i$. Even if we do not distinguish between the ordering of the factors, we still face the possibility of working with a very large system.
of characters. Expressed as an expansion in terms of these characters, our partition function \((4.1)\) would therefore have many terms whose properties would not be immediately transparent. This is unfortunate, especially since \((4.1)\) was constructed for its relative simplicity, and does not approach the complexity to be found in more realistic situations.

It is nevertheless possible to bypass these difficulties by recognizing that the only properties of the “characters” \(\chi_i\) and \(\chi_i^2\) demanded in the derivations in Sects. 2 and 3 is that they satisfy \((2.1)\) and \((2.2)\): they must have \(q\)-expansions with non-negative coefficients, they must transform covariantly under modular transformations, and in particular they must transform as eigenfunctions under \(T : \tau \rightarrow \tau + 1\). Indeed, it is not necessary that they be true conformal-field-theoretic characters at all, and in particular linear combinations of such true characters can also serve as “characters” for the purposes of our analyses, provided that they meet the above conditions.

Thus, by appropriately choosing linear combinations of the many characters of the above \(c = 6\) conformal field theory, we will be able to find a relatively small number of “pseudo-characters” which close into each other under modular transformations and which thereby serve as a reduced set of “characters” with which to work. Such a decomposition into pseudo-characters is not unique, and in principle there exist many different sets of such pseudo-characters in terms of which a given partition functions may be expressed. The particular choice of pseudo-characters will not affect our final results, of course, and merely amounts to reorganizing the calculation in different ways. There are some minor consequences of working with this reduced set of pseudo-characters, however, and we shall point them out as they arise.

Thus, our first step is to find a reduced set of pseudo-characters which meet the above conditions, and in terms of which our partition function \((4.1)\) manifestly satisfies the condition \((3.26)\). Since our partition function \((4.1)\) is devoid of physical tachyons, such a set of pseudo-characters must always exist, and indeed it turns out that the following reduced set of only nine chiral pseudo-characters suffices:

\[
\begin{align*}
A &\equiv \frac{1}{8} \eta^{-6} \varphi_2^2 \varphi_3 \varphi_4 (\varphi_3 + \varphi_4) \\
B_1 &\equiv \frac{1}{32} \eta^{-6} \varphi_2^4 (\varphi_3 + \varphi_4) \\
B_2 &\equiv \frac{1}{12} \eta^{-6} \varphi_3 \varphi_4 (\varphi_3^3 - \varphi_4^3) \\
C &\equiv \frac{1}{32} \eta^{-6} \varphi_2 (\varphi_3^4 - \varphi_4^4) \\
D &\equiv \frac{1}{16} \eta^{-6} \varphi_2^2 \varphi_3 \varphi_4 (\varphi_3 - \varphi_4) \\
E_1 &\equiv \frac{1}{2} \eta^{-6} \varphi_2^4 (\varphi_3 - \varphi_4) \\
E_2 &\equiv \frac{1}{2} \eta^{-6} \varphi_3 \varphi_4 (\varphi_3^3 + \varphi_4^3) \\
F_1 &\equiv \frac{1}{258} \eta^{-6} \varphi_2 (\varphi_3^2 - \varphi_4^2)^2 \\
F_2 &\equiv \frac{1}{8} \eta^{-6} \varphi_2 (\varphi_3^2 + \varphi_4^2)^2.
\end{align*}
\]

We have defined these pseudo-characters in such a way that they are each normalized, with their first non-vanishing \(q\)-expansion coefficients equal to one. Note that the
names of these characters indicate their respective vacuum energies $H_i$, with the letters $A$ through $F$ respectively signifying vacuum energies $H_1 = 0, 1/4, 3/8, 1/2, 3/4$, and $7/8$ (modulo 1). Indeed, these characters have the following explicit $q$-expansions:

\[
A = q^0 \left( 1 + 4q + 14q^2 + 40q^3 + 100q^4 + 232q^5 + 480q^6 + \ldots \right)
\]

\[
B_1 = q^{1/4} \left( 1 + 10q + 59q^2 + 270q^3 + 1044q^4 + 3572q^5 + 11111q^6 + \ldots \right)
\]

\[
3B_2 = q^{3/4} \left( 3 + 10q + 41q^2 + 94q^3 + 260q^4 + 548q^5 + 1173q^6 + \ldots \right)
\]

\[
C = q^{3/8} \left( 1 + 11q + 67q^2 + 308q^3 + 1190q^4 + 4059q^5 + 12574q^6 + \ldots \right)
\]

\[
D = q^{1/2} \left( 1 + 4q + 12q^2 + 32q^3 + 77q^4 + 172q^5 + 340q^6 + \ldots \right)
\]

\[
E_1 = q^{3/4} \left( 1 + 10q + 57q^2 + 250q^3 + 931q^4 + 3082q^5 + 9308q^6 + \ldots \right)
\]

\[
E_2 = q^{-1/4} \left( 1 + 14q + 37q^2 + 134q^3 + 305q^4 + 786q^5 + 1594q^6 + \ldots \right)
\]

\[
F_1 = q^{7/8} \left( 1 + 7q + 37q^2 + 154q^3 + 557q^4 + 1806q^5 + 5367q^6 + \ldots \right)
\]

\[
F_2 = q^{-1/8} \left( 1 + 15q + 113q^2 + 590q^3 + 2467q^4 + 8908q^5 + 28877q^6 + \ldots \right)
\]

(4.4)

from which we see that only the pseudo-characters $E_2$ and $F_2$ are tachyonic. Indeed, $E_2$ serves as the effective vacuum character in this reduced system, with vacuum energy $H_i = -1/4 = -c/24$.

Under the $S$ modular transformation, these pseudo-characters have the following mixing matrix:

\[
S_{ij} = \frac{e^{i\pi/4}}{4} \begin{pmatrix}
  2 & 0 & 0 & -4 & 0 & 0 & -16 & 1 \\
  0 & 0 & -\frac{3}{4} & -2 & 0 & 0 & \frac{1}{8} & 4 \\
  0 & -\frac{16}{7} & 0 & \frac{16}{7} & 0 & -\frac{32}{7} & 0 & \frac{32}{7} \\
  0 & -2 & \frac{3}{4} & 0 & 0 & 4 & \frac{1}{8} & 0 \\
 -1 & 0 & 0 & 0 & 2 & 0 & 0 & -8 & \frac{1}{2} \\
  0 & 0 & -\frac{3}{8} & 1 & 0 & 0 & \frac{1}{16} & -2 & \frac{1}{4} \\
  0 & 32 & 0 & 32 & 0 & 64 & 0 & 64 & 4 \\
 -\frac{1}{4} & \frac{1}{2} & \frac{3}{16} & 0 & -\frac{1}{2} & -1 & \frac{1}{32} & 0 & 0 \\
  4 & 8 & 3 & 0 & 8 & -16 & \frac{1}{2} & 0 & 0
\end{pmatrix}
\]

(4.5)

Thus, since $\langle 1, 1 \rangle$ indicates that each pseudo-character is an eigenfunction of $T$, we see that this set of pseudo-characters is closed under all modular transformations.

Written in terms of these nine pseudo-characters, our modular-invariant partition function (4.1) now takes the simple form

\[
Z = (\text{Im } \tau)^{-1/2} \left\{ |A|^2 - 3 \text{ (} B_1^* B_2 + B_2^* B_1 \text{)} + 8 |C|^2 + 4 |D|^2 + (E_1^* E_2 + E_2^* E_1) - 4 (F_1^* F_2 + F_2^* F_1) \right\}.
\]

(4.6)

Note that since only the pseudo-characters $E_2$ and $F_2$ are tachyonic with $H_i < 0$, the condition $\langle 3, 20 \rangle$ is indeed satisfied. Indeed, from (4.4) we see that this partition
function $Z$ contains only unphysical tachyons, and that these come in two distinct
sets: those with $m \geq 3/4$ and $n = -1/4$ (and vice versa) which arise from the $E$
terms, and those with $m \geq 7/8$ and $n = -1/8$ (and vice versa) which arise from the
$F$ terms.

There are, however, two obvious features which indicate that these pseudo-
characters are not true characters in and of themselves. The first appears, for ex-
ample, in the $q$-expansion of $B_2$ in (4.4): not all of its coefficients are integers (or
equivalently, trivially rescaling $B_2 \to 3B_2$, we find that not all of its coefficients are di-
visible by its first non-zero coefficient). For a true character, this divisibility property
is essential, since the first non-zero coefficient is the multiplicity of the vacuum state
in that sector of the conformal field theory, and all higher states in that sector must
share that multiplicity. The breaking of this property in our case, however, simply
reflects the fact that $B_2$ represents the added contributions of many such characters
whose highest weights may indeed differ by integers. Thus this divisibility condition,
valid for each character individually, need no longer hold for the sum. This does not
affect the validity of the asymptotic expansions, however.

The second feature which indicates that our pseudo-characters are not true char-
acters is the appearance of certain vanishing elements in the $S_{ij}$-matrix (4.5). As
we have stated in previous sections, one must have $S_{ij} \neq 0$ for all $i$ where $j$ repre-
sents the vacuum sector; such a condition is necessary in order to obtain meaningful
fusion rules in which any sector fused with the identity (vacuum) sector reproduces
that sector [5]. For our pseudo-characters, the effective vacuum sector corresponds
to $E_2$, yet we see that several of the corresponding matrix elements in (4.5) vanish.
This again represents the fact that we have taken linear combinations when form-
ing our pseudo-characters, and causes no fundamental problem. The effects of these
vanishing matrix elements will be discussed below.

Given our partition function decomposed into products of pseudo-characters as in
(4.6), it is now straightforward to examine the behavior of the state degeneracies $a_{im}^{(m)}$
in each of its six sectors ($A$ through $F$). Let us first focus on the expected asymptotic
behavior of each of the nine chiral pseudo-characters individually, and determine the
values of the inverse Hagedorn temperature $C$ that they each separately exhibit in
(4.4). Since $E_2$ and $F_2$ are the only tachyonic pseudo-characters, with $H_{E_2} = -1/4$
and $H_{F_2} = -1/8$, we see that the dominant growth in the $q$-expansion coefficients
of each character $\chi_i$ depends on the elements of the matrices $Q_{ij}^{(\alpha=1)}$ which couple
the character $\chi_i$ to the tachyonic sectors $E_2$ and $F_2$. The strongest growth will come
from the coupling to $E_2$, since $E_2$ serves as the identity (vacuum) pseudo-sector with
$H = -c/24 = -1/4$; such growth will be exponential with rate $C = 4\pi\sqrt{1/4} = 2\pi$.
Couplings to $F_2$, by contrast, produce exponential growth with $C = 4\pi\sqrt{1/8} = \sqrt{2\pi}$.
From (2.27), we see that $Q_{ij}^{(1)} \propto S_{ij}$, and thus, denoting $S(\chi_i, \chi_j) \equiv S_{ij}$, we find that
this growth for any character $\chi_i$ is determined by the values of the elements $S(\chi_i, E_2)$
and $S(\chi_i, F_2)$. Specifically, the $q$-expansion coefficients of a given $\chi_i$ will experience
exponential growth with $C = 2\pi$ if $S(\chi_i, E_2) \neq 0$, and $C = \sqrt{2}\pi$ only if $S(\chi_i, E_2) = 0$ and $S(\chi_i, F_2) \neq 0$. We thus immediately find that

$$\{ B_1, C, E_1, F_1, F_2 \} \iff C = 2\pi , \quad (4.7)$$

whereas

$$\{ A, B_2, D, E_2 \} \iff C = \sqrt{2}\pi . \quad (4.8)$$

This division into “strongly” and “weakly” growing pseudo-characters is actually already evident in their $q$-expansions (4.4). Note that this division, however, is ultimately a consequence of the fact that we are dealing with pseudo-characters, and not true conformal field theory characters. If these had been true characters, the $S_{ij}$-matrix elements $S(\chi_i, E_2)$ would all have been non-vanishing, and all characters would consequently experience the same maximum asymptotic growth. Physically, this implies that all sectors of a given conformal field theory necessarily have the same Hagedorn temperature, and it is only because we have here taken linear combinations of these characters in forming our pseudo-characters and pseudo-sectors that this dominant exponential has occasionally cancelled. Thus, organizing our calculation in terms of pseudo-characters rather than true characters has already yielded an early cancellation of the sort that we are investigating.

Combining these chiral pseudo-characters together to form the full left/right partition function (4.6), we can now easily predict the rates of growth of the physical degeneracies $a_{nn}^{(A)}$ for the six different groupings of sectors [i.e., for the $A$-terms contributing states with $n = 0$ (modulo 1), for the combined $B$-terms contributing states with $n = 1/4$ (modulo 1), for the $C$-terms with $n = 3/8$ (modulo 1), and so forth]. Indeed, using (4.7) and (4.8), we can separate these six groupings of sectors according to the rates of exponential growth for their separate degeneracies $a_{nn}^{(A)}$ through $a_{nn}^{(F)}$:

$$\{ a_{nn}^{(C)}, a_{nn}^{(F)} \} \iff C_{tot} = 4\pi$$

$$\{ a_{nn}^{(B)}, a_{nn}^{(E)} \} \iff C_{tot} = (2 + \sqrt{2})\pi \approx 3.41\pi$$

$$\{ a_{nn}^{(A)}, a_{nn}^{(D)} \} \iff C_{tot} = 2\sqrt{2}\pi \approx 2.83\pi . \quad (4.9)$$

These results are also easy to verify in terms of explicit $q$- and $q$-expansions. In Figs. 2 and 3 we have plotted the net physical-state degeneracies $a_{nn}$ in this example as functions both of energy $n$ and spacetime mass $\sqrt{n}$, where $Z = (\text{Im} \tau)^{-1/2} \sum_{m,n} a_{mn} q^m q^n$. Those values of $a_{nn}$ with $n \in \mathbb{Z}$ of course arise from the $A$ sector, those with $n \in \mathbb{Z} + 1/4$ from the $B$ sectors, etc. As in Fig. 1, we have plotted $\pm \log(|a_{nn}|)$ where the sign chosen is the sign of $a_{nn}$ itself; note that base-10 logarithms are used for Fig. 2, while natural logarithms are used for Fig. 3. From these figures we see that this partition function indeed exhibits the expected oscillations between bosonic and fermionic surpluses as the energy $n$ is increased, with wavelength $\Delta n = 1$ (as is evident from Fig. 2). Note, however, that the pattern of oscillation is now far more complex than that for the toy example of Fig. 1; this is
due to the presence of six sectors $A$ through $F$ in this example, as opposed to the mere two of the toy model. Plotting these degeneracies versus $\sqrt{n}$ as in Fig. 3, we see that the asymptotic exponential behavior for the separate sector degeneracies $a_{nn}^{(A,\ldots,F)}$ begins quite early, and indeed straight-line fits on this logarithmic plot are accurate down to remarkably small values of $n$. From the different slopes of these lines, we can easily verify (4.9); indeed, the fastest rates of growth are experienced by the $C$- and $F$-sectors, and the slowest by the $A$- and $D$-sectors. The signs of these individual sector contributions $a_{nn}^{(A,\ldots,F)}$ are of course given by the signs in the partition function (4.6). Note that since the $A$- and $D$-sectors have both the same rate of growth and the same sign, they are fit remarkably well by the same asymptotic function.

Once again, we stress that each $a_{nn}^{(F)}$ must grow exponentially with rate $C_{\text{tot}} = 4\pi$ when $\tau$ and $i$ refer to true conformal field theory characters. Thus, given the results (4.9), we see that our organization in terms of pseudo-characters has already enabled certain cancellations to become evident. In particular, the cancellations in the $A$, $B$, $D$, and $E$ sectors can be viewed as a sort of “aligned supersymmetry”, since they represent the cancellations between different conformal field theory sectors whose highest weights and vacuum energies are aligned modulo 1. Indeed, this is the alignment which enabled us to take their linear combinations and thereby produce pseudo-characters which were also eigenfunctions of the $T$ modular transformation.

Our goal, of course, is to verify analytically that in fact all vestiges of the dominant exponential $C_{\text{tot}} = 4\pi$ behavior are cancelled in the full partition function — in this case by virtue of a “misaligned supersymmetry” between the misaligned $C$ and $F$ sectors. While this certainly appears likely given the logarithmic plots in Figs. 2 and 3, we must in fact verify that all terms which have this exponential growth are precisely cancelled when their appropriate prefactors are taken into account. This is not difficult, however. From the results (4.9), we see that such terms can only arise in the $C$ and $F$ sectors. Using the explicit forms of the asymptotic expansions, we see that the relevant exponentially-growing terms arising from the $C$-sector are given by

$$a_{nn}^{(C)} \sim N_{CC} \left| 2\pi Q_{C,E_2}^{(1)} f_{E_2}(n) \right|^2 = 8 \left| 2\pi S(C, E_2) (4n)^{-3/4} I_{-3/2}(2\pi \sqrt{n}) \right|^2 ,$$  

(4.10)

while those from the $F$-sector are given by

$$a_{nn}^{(F)} \sim N_{F_1 F_2} \left[ 2\pi Q_{F_1,E_2}^{(1)} f_{E_2}(n) \right] \left[ 2\pi Q_{F_2,E_2}^{(1)} f_{E_2}(n) \right]^* + N_{F_2 F_1} \left[ 2\pi Q_{F_1,E_2}^{(1)} f_{E_2}(n) \right] \left[ 2\pi Q_{F_2,E_2}^{(1)} f_{E_2}(n) \right]^* = -8 \text{ Re} \left\{ S(F_1, E_2) [S(F_2, E_2)]^* \right\} \left| 2\pi (4n)^{-3/4} I_{-3/2}(2\pi \sqrt{n}) \right|^2 .$$  

(4.11)

These expressions will be equal and opposite provided

$$|S(C, E_2)|^2 = \text{ Re} \left\{ S(F_1, E_2) [S(F_2, E_2)]^* \right\} ,$$  

(4.12)
and consulting the $S_{ij}$-matrix in (4.3) we see that (4.12) is indeed satisfied. Thus
we have verified the cancellation of all terms with $C = 4\pi$ exponential growth, and
as a consequence the sector-averaged number of states $\langle a_{nn} \rangle$ for this example must
indeed grow more slowly than either of the separate $C$- or $F$-sectors (or indeed than
any other true conformal field theory sector of the theory).

The above cancellation is of course predicted by our general theorem, since $C_{\text{tot}} = 4\pi$
for the present example, and $C_{\text{eff}}$ must be less than $C_{\text{tot}}$. However, our theorem
can actually be used to predict much more. We have just seen that the $C$- and $F$-
sectors have leading $C = 4\pi$ terms which cancel. However, from (4.9) we see that the
next-largest growth comes from the $B$- and $E$-sectors, with $C = (2 + \sqrt{2})\pi$, and these
sectors also appear to cancel in Fig. 2. We might then wonder whether all exponential
growth also cancels for $C = (2 + \sqrt{2})\pi$. Such a cancellation would in principle be
far more subtle than the leading cancellation, for we have only demonstrated that the
$C$- and $F$-sector cancellation removes the $C = 4\pi$ terms; in particular, residual
subleading $C = (2 + \sqrt{2})\pi$ terms might nevertheless survive from these sectors. These
residual terms would then have to combine with the leading $C = (2 + \sqrt{2})\pi$ terms
from the $B$- and $E$-sectors in order to produce the next level of cancellation.

However, using our theorem, it is straightforward to demonstrate that this sec-
dondary cancellation also takes place. Note that the partition function $Z$ in this
example contains unphysical tachyons in the $E$-sectors; these sectors have $H_i = -1/4, \overline{H}_i > 0$ and vice versa. From this and the result (3.33) we deduce that the
maximum possible growth that is allowed from the leading term with $\alpha = \overline{\alpha} = 1$
is $C = 2\pi$. We can now quickly survey the possible rates of growth that can arise
from subleading terms with general values of $\alpha$ and $\overline{\alpha}$; these values of $C$ are given
in (3.46), and whether or not such rates of growth actually appear is determined by
the corresponding values of $P_{ij}^{(\alpha,\overline{\alpha})}$ in (3.37). It turns out that we do not even need to
explicitly calculate $P_{ij}^{(\alpha,\overline{\alpha})}$. In the present example, growth can occur only with values
$H_j, \overline{H}_j \in \{-1/4, -1/8, 0\}$. (The value 0 corresponds to all $H, \overline{H} > 0$ situations and
also to all possible error terms, as discussed in Sect. 3; note that for $\alpha \neq \overline{\alpha}$ we must
of course permit even the possibilities with $H_i = \overline{H}_i$.) We easily find that there exists
no combination of values of $\alpha, \overline{\alpha}, H_i$, and $\overline{H}_i$ for which $C$ in (3.46) exceeds $3\pi$. Thus
our theorem in this case actually precludes all exponential growth with $C > 3\pi$, and
as a consequence any remaining subleading $C = (2 + \sqrt{2})\pi$ terms from the $C$- and
$F$-sectors must cancel exactly against the leading $C = (2 + \sqrt{2})\pi$ terms from the $B$-
and $D$-sectors.

We have therefore been able to demonstrate that although $C_{\text{tot}} = 4\pi$ for this
example, cancellations bring this down to $C_{\text{eff}} \leq 3\pi$, in accordance with our theorem.
This amounts to a remarkable total cancellation, reducing the sector-averaged number
of states $\langle a_{nn} \rangle$ by more than six orders of magnitude for energies as low as $n \approx 20$. 

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4.2 Second Example

Our second example is more complicated than the first, but in many ways more realistic. In particular, this partition function resembles those found for actual non-supersymmetric heterotic string theories compactified to four spacetime dimensions, and contains precisely the types of unphysical tachyons that such strings necessarily contain. This function is special, however, in that it has an exactly vanishing one-loop cosmological constant — i.e., even though this function is non-zero, its integral over the fundamental domain of the modular group vanishes identically \[7\]. We will discuss the relation between misaligned supersymmetry and the cosmological constant in Sect. 5.

This function is as follows \[7\]:

\[
Z \equiv \frac{1}{2} (\text{Im } \tau)^{-1} \eta^{-24} \tau^{-12} \sum_{i,j,k=1}^{4} |\vartheta_{i}|^{4} \left\{ \vartheta_{i}^{4} \vartheta_{j}^{4} \vartheta_{k}^{4} \left[ 2 |\vartheta_{j} \vartheta_{k}|^{8} - \vartheta_{j}^{8} \vartheta_{k}^{8} - \vartheta_{j}^{8} \vartheta_{k}^{8} \right] \\
+ \vartheta_{i}^{12} \left[ 4 \vartheta_{i}^{8} \vartheta_{j}^{4} \vartheta_{k}^{4} + (-1)^{i} 13 |\vartheta_{j} \vartheta_{k}|^{8} \right] \right\}.
\]

(4.13)

Given the relations (4.12), we immediately see that this function corresponds to left and right worldsheet theories which are tensor products of two uncompactified bosons and 44 or 20 Majorana fermions respectively. These worldsheet theories thus have total central charges \(c_{\text{left}} = 24\) and \(c_{\text{right}} = 12\), and correspond to those of lightcone ten-dimensional heterotic strings compactified to four dimensions. We thus have \(C_{\text{tot}} = 2(2 + \sqrt{2})\pi\) for this example (and indeed for all heterotic superstrings, regardless of spacetime dimension). The modular weight is \(k = -1\). A decomposition into pseudo-characters satisfying all of the conditions of Sect. 3 and Sect. 4 can be achieved by defining the nine holomorphic pseudo-characters:

\[
\begin{align*}
A_{1} & \equiv \eta^{-24} \vartheta_{2}^{4} \vartheta_{3}^{6} \vartheta_{4}^{6} (\vartheta_{3}^{6} - \vartheta_{4}^{6}) \\
A_{2} & \equiv \eta^{-24} \vartheta_{2}^{12} \vartheta_{3}^{4} \vartheta_{4}^{4} (\vartheta_{3}^{2} - \vartheta_{4}^{2}) \\
A_{3} & \equiv \eta^{-24} \left\{ 2 \vartheta_{2}^{8} \vartheta_{3}^{6} \vartheta_{4}^{4} (\vartheta_{3}^{2} + \vartheta_{4}^{2}) + 4 (\vartheta_{3}^{22} + \vartheta_{4}^{22}) \\
& \quad - 13 (\vartheta_{2} \vartheta_{3} \vartheta_{4})^{4} (\vartheta_{3}^{10} - \vartheta_{4}^{10}) \right\} \\
B & \equiv \eta^{-24} \vartheta_{2}^{6} \vartheta_{3}^{4} \vartheta_{4}^{4} (\vartheta_{3}^{8} - \vartheta_{4}^{8}) \\
C_{1} & \equiv \eta^{-24} \left\{ -2 \vartheta_{2}^{8} \vartheta_{3}^{6} \vartheta_{4}^{4} (\vartheta_{3}^{2} - \vartheta_{4}^{2}) + 4 (\vartheta_{3}^{22} - \vartheta_{4}^{22}) \\
& \quad - 13 (\vartheta_{2} \vartheta_{3} \vartheta_{4})^{4} (\vartheta_{3}^{10} + \vartheta_{4}^{10}) + 2 \vartheta_{2}^{4} \vartheta_{3}^{6} \vartheta_{4}^{4} (\vartheta_{3}^{6} + \vartheta_{4}^{6}) \right\} \\
C_{2} & \equiv \eta^{-24} \vartheta_{2}^{12} \vartheta_{3}^{4} \vartheta_{4}^{4} (\vartheta_{3}^{2} - \vartheta_{4}^{2}) \\
C_{3} & \equiv \eta^{-24} \vartheta_{2}^{4} \vartheta_{3}^{6} \vartheta_{4}^{4} (\vartheta_{3}^{6} + \vartheta_{4}^{6}) \\
D_{1} & \equiv \eta^{-24} \left\{ 2 \vartheta_{2}^{6} \vartheta_{3}^{8} \vartheta_{4}^{4} + 4 \vartheta_{2}^{22} + 13 \vartheta_{2}^{14} \vartheta_{3}^{4} \vartheta_{4}^{4} \\
& \quad - \vartheta_{2}^{6} \vartheta_{3}^{4} \vartheta_{4}^{4} (\vartheta_{3}^{8} + \vartheta_{4}^{8}) \right\} \\
D_{2} & \equiv \eta^{-24} \left\{ 2 \vartheta_{2}^{6} \vartheta_{3}^{8} \vartheta_{4}^{4} + 4 \vartheta_{2}^{22} + 13 \vartheta_{2}^{14} \vartheta_{3}^{4} \vartheta_{4}^{4} \\
& \quad + \vartheta_{2}^{6} \vartheta_{3}^{4} \vartheta_{4}^{4} (\vartheta_{3}^{8} + \vartheta_{4}^{8}) \right\}.
\end{align*}
\]

(4.14)
and the nine anti-holomorphic pseudo-characters:

\[
\begin{align*}
A_1 & \equiv \eta^{-12} \partial_2^8 (\partial_3^2 - \partial_4^2) \\
A_2 & \equiv \eta^{-12} (\partial_3 \partial_4)^2 (\partial_3^6 - \partial_4^6) \\
A_3 & \equiv \eta^{-12} \partial_2^4 \partial_3^2 \partial_4^2 (\partial_3^2 + \partial_4^2) \\
B & \equiv \eta^{-12} \partial_2^2 (\partial_3^8 - \partial_4^8) \\
C_1 & \equiv \eta^{-12} \partial_2^4 \partial_3^2 \partial_4^2 (\partial_3^2 - \partial_4^2) \\
C_2 & \equiv \eta^{-12} \{\partial_2^8 (\partial_3^2 + \partial_4^2) - 2 \partial_2^4 \partial_3^2 \partial_4^2 (\partial_3^2 - \partial_4^2)\} \\
C_3 & \equiv \eta^{-12} (\partial_3 \partial_4)^2 (\partial_3^6 + \partial_4^6) \\
D_1 & \equiv \eta^{-12} \partial_2^2 (\partial_3^4 - \partial_4^4)^2 \\
D_2 & \equiv \eta^{-12} \partial_2^2 (\partial_3^4 + \partial_4^4)^2 \quad (4.15)
\end{align*}
\]

Unlike the pseudo-characters of the previous example, we have not chosen these to be normalized. As discussed at the end of Sect. 2, this will be ultimately reflected as a rescaling of the elements of the modular-transformation representation matrices, and our above normalizations will render these matrices particularly simple.

Like the pseudo-characters of the previous example, these pseudo-characters have been organized according to their vacuum energies \(H\) (modulo 1), so that those lettered \(A\) through \(D\) have vacuum energies \(H_i = 0, 1/4, 1/2,\) and \(3/4\) (modulo 1) respectively. Explicit \(q\)-expansions for these pseudo-characters are as follows:

\[
\begin{align*}
A_1 &= 128 (3 + 52q + 292q^2 + 1440q^3 + ...) \\
A_2 &= 32768 q (1 + 20q + 216q^2 + ...) \\
A_3 &= 8 q^{-1} (1 + 36q + 78720q^2 + 6803824q^3 + 230743038q^4 + ...) \\
B &= 2048 q^{1/4} (1 + 42q + 633q^2 + ...) \\
C_1 &= 512 q^{1/2} (53 + 13464q + 669874q^2 + ...) \\
C_2 &= 8192 q^{1/2} (1 + 24q + 298q^2 + ...) \\
C_3 &= 32 q^{-1/2} (1 + 64q + 510q^2 + 2688q^3 + ...) \\
D_1 &= 65536 q^{3/4} (3 + 322q + 12541q^2 + ...) \\
D_2 &= 256 q^{-1/4} (1 + 894q + 85251q^2 + 3243130q^3 + ...) \\
\overline{A}_1 &= 2048 q (1 + 20q + 216q^2 + ...) \\
\overline{A}_2 &= 8 (3 + 52q + 292q^2 + 1440q^3 + ...) \\
\overline{A}_3 &= 32 (1 + 12q + 76q^2 + 352q^3 + ...) \\
\overline{B} &= 128 q^{1/4} (1 + 42q + 633q^2 + 6042q^3 + ...) \\
\overline{C}_1 &= 128 q^{1/2} (1 + 8q + 42q^2 + 176q^3 + ...) \\
\overline{C}_2 &= 256 q^{1/2} (1 + 40q + 554q^2 + 4976q^3 + ...) \\
\overline{C}_3 &= 2 q^{-1/2} (1 + 64q + 510q^2 + 2688q^3 + 11267q^4 + ...) \\
\overline{D}_1 &= 1024 q^{3/4} (1 + 22q + 255q^2 + ...) \\
\overline{D}_2 &= 16 q^{-1/4} (1 + 62q + 1411q^2 + 16314q^3 + ...) \quad (4.16)
\end{align*}
\]
Note that the pseudo-characters which are tachyonic are \{A_3, C_3, D_2, \overline{C_3}, \overline{D_2}\}; indeed, the pseudo-characters corresponding to the “vacuum” sectors for left- and right-moving systems are \(A_3\) and \(\overline{C_3}\) respectively.

These two sets of pseudo-characters close separately under the \(S\) modular transformation, with mixing matrices

\[
S_{ij} = \frac{i}{4} \begin{pmatrix}
0 & -2 & 0 & 2 & 0 & -2 & 0 & -1 & 1 \\
-2 & 0 & 0 & 2 & 0 & 0 & 2 & 1 & -1 \\
0 & 2 & 0 & 0 & 0 & -4 & 2 & 2 & 0 \\
2 & 2 & 0 & 0 & 0 & -2 & 2 & 0 & 0 \\
0 & 4 & 2 & 4 & 2 & 4 & -4 & -4 & 0 \\
-2 & 0 & 0 & -2 & 0 & 0 & 2 & -1 & 1 \\
0 & 2 & 0 & 2 & 0 & 2 & 0 & -1 & 1 \\
-2 & 2 & 2 & 0 & -2 & -2 & 2 & 0 & 0 \\
2 & -2 & 2 & 0 & -2 & 2 & 6 & 0 & 0
\end{pmatrix}
\] (4.17)

and

\[
S_{\overline{ij}} = \frac{i}{4} \begin{pmatrix}
0 & -2 & 0 & 2 & 0 & 0 & 2 & -1 & -1 \\
-2 & 0 & 0 & 2 & -4 & -2 & 0 & 1 & 1 \\
0 & 0 & 2 & 0 & -2 & 0 & 0 & -1 & 1 \\
2 & 2 & 0 & 0 & -4 & -2 & 2 & 0 & 0 \\
0 & 0 & -2 & 0 & 2 & 0 & 0 & -1 & 1 \\
0 & -2 & 4 & -2 & -4 & 0 & 2 & 3 & -1 \\
2 & 0 & 0 & 2 & 4 & 2 & 0 & 1 & 1 \\
-2 & 2 & -4 & 0 & 0 & 2 & 2 & 0 & 0 \\
-2 & 2 & 4 & 0 & 8 & 2 & 2 & 0 & 0
\end{pmatrix}
\] (4.18)

Thus, since each pseudo-character is also an eigenfunction of \(T\), each set of pseudo-characters is separately closed under all modular transformations. Note that the relative simplicity of the matrices (4.17) and (4.18) is due to the chosen normalizations of the pseudo-characters.

In terms of these pseudo-characters, our modular-invariant partition function (4.13) now takes the relatively simple form

\[
Z = (\text{Im} \tau)^{-1} \left\{ \frac{1}{2} \left( \overline{A_1}^* A_1 + \overline{A_2}^* A_2 + \overline{A_3}^* A_3 \right) + \frac{1}{2} \overline{B}^* B \\
- \frac{1}{2} \left( \overline{C_1}^* C_1 + \overline{C_2}^* C_3 + \overline{C_3}^* C_2 \right) + \frac{1}{4} \left( \overline{D_2}^* D_1 - \overline{D_1}^* D_2 \right) \right\} .
\] (4.19)

In this form it is easy to see that \(Z\) contains unphysical tachyons but lacks physical tachyons, and indeed the condition (3.26) is satisfied.

As in the previous example, we can now easily determine the rates of exponential growth for each separate pseudo-character, and then for each of the four combined sectors (\(A\) through \(D\)) in this partition function. The results for the separate pseudo-characters are:

\[
\{A_3, C_1, D_1, D_2\} \leftrightarrow C = 4\pi
\]
\{A_2, B, C_2, \overline{A_1}, \overline{B}, \overline{C_2}, \overline{D_1}, \overline{D_2}\} \iff C = 2\sqrt{2} \pi
\{A_1, C_3, \overline{A_2}, \overline{A_3}, \overline{C_1}, \overline{C_3}\} \iff C = 2\pi , \hspace{1cm} (4.20)

and once again this division into separate groups (which arises from our use of pseudo-characters rather than true characters) is reflected in the actual \(q\)-expansions (4.16). From (4.20), then, we can immediately conclude that the combined \(A\)-, \(B\)-, and \(C\)-sectors of the partition function experience the following exponential rates of growth:

\{a_{nn}^{(A)}, a_{nn}^{(C)}\} \iff C = 6\pi
\{a_{nn}^{(B)}\} \iff C = 4\sqrt{2} \pi . \hspace{1cm} (4.21)

Indeed, it is the \(\overline{A_3}^*A_3\) and \(\overline{C_1}^*C_1\) terms in the partition function (4.19) which dominate in producing the above asymptotic behavior.

Determining the rate of growth for the \(D\)-sector degeneracies \(a_{nn}^{(D)}\) is a bit more subtle, however, since the two \(D\)-sector terms \(\overline{D_2}^*D_1\) and \(\overline{D_1}^*D_2\) each separately have \(C = 2(2 + \sqrt{2})\pi\) growth but appear in the partition function (4.19) with opposite signs. Thus, a cancellation within the \(D\)-sector terms may take place which entirely removes this leading exponential behavior for \(a_{nn}^{(D)}\). The fastest way to see that this indeed occurs (and to simultaneously determine the rate of the largest non-cancelling exponential growth) is to define \(D_1, 2 \equiv D_2 \mp D_1\) and \(D_1, 2 \equiv D_2 \pm D_1\) respectively.

The \(D\)-sector terms in the partition function then become

\[\frac{1}{4} (\overline{D_2}^*D_1 - \overline{D_1}^*D_2) = \frac{1}{8} (\overline{D_2}^*D_1 - \overline{D_1}^*D_2) , \hspace{1cm} (4.22)\]

but consulting the matrices (4.17) and (4.18), we now find

\[S(D_1, A_3) \neq 0 , \]
\[S(D_2, A_3) = 0 , \quad S(D_2, C_3) \neq 0 , \]
\[S(D_1, C_3) \neq 0 , \]
\[S(D_2, C_3) = 0 , \quad S(D_1, D_2) = 0 . \hspace{1cm} (4.23)\]

This means that these new \(D\) and \(\overline{D}\) combinations have the following exponential rates of growth:

\[D_1 \iff C = 4\pi \]
\[\{D_2, \overline{D_1}\} \iff C = 2\sqrt{2} \pi \]
\[\overline{D_2} \iff C \leq \sqrt{2} \pi . \hspace{1cm} (4.24)\]

Note that the growth for \(\overline{D_2}\) is extraordinarily suppressed relative to the others, with all leading \(\overline{\sigma} = 1\) terms in its expansion vanishing. We thus easily find that the dominant growth comes from the \(\overline{D_1}^*D_2\) term in (4.22), with

\[\{a_{nn}^{(D)}\} \iff C = 4\sqrt{2} \pi . \hspace{1cm} (4.25)\]

* The reader may therefore wonder why we did not define \(D_{1,2}\) and \(\overline{D_{1,2}}\) to be our pseudo-characters originally. There are two reasons. The first is that if we had expressed our partition
These rates of growth for the individual sector degeneracies \( a_{nn}^{(A,B,C,D)} \) which are given in (4.21) and (4.25) can also be verified by explicitly expanding \( Z \) as a function of \( q \) and \( \bar{q} \), \( Z = (\text{Im} \tau)^{-1} \sum a_{mn} \bar{q}^m q^n \). In Fig. 4 we have plotted the resulting physical degeneracies \( a_{nn} \) as a function of \( n \), following the conventions of our earlier figures. We once again observe the appearance of a “misaligned supersymmetry”, with alternating signs for the net numbers of physical states. The largest rate of growth comes from the \( A \)- and \( C \)- sectors, in agreement with (4.21), and the smaller rate of growth is experienced by the \( B \)- and \( D \)- sectors, in agreement with (4.21) and (4.25). The presence of only four sectors in this example renders the pattern of oscillation significantly simpler than that for the first example we considered.

Let us now determine the extent to which the “misaligned supersymmetry” in this example implies an exact cancellation of functional forms for the sector-averaged number of states \( \langle a_{nm} \rangle \). As we have seen, this partition function corresponds to a worldsheet conformal field theory with \( c_{\text{left}} = 24 \) and \( c_{\text{right}} = 12 \), and therefore each individual conformal field theory sector \((\bar{n}i)\) of the theory separately experiences leading exponential growth with \( C_{\text{tot}} = 2(2 + \sqrt{2})\pi \approx 6.83\pi \). We thus easily see that the \( C_{\text{eff}} \) for this theory is less than \( C_{\text{tot}} \), for our organization in terms of the above pseudo-characters has made the cancellation of this leading \( C = 2(2 + \sqrt{2})\pi \) term manifest within each of the four sector-groupings, \( A \) through \( D \). Hence, the strictest prediction of our theorem is in fact already trivially verified for this example, with \( C_{\text{tot}} = 2(2 + \sqrt{2})\pi \) and \( C_{\text{eff}} \leq 6\pi \).

However, just as in the previous example, we might suspect that even further cancellations necessarily follow as a result of our theorem, and this is indeed the case. The next largest rates of growth are, of course, \( C = 6\pi \) from the \( A \)- and \( C \)-sector groupings, and \( C = 4\sqrt{2}\pi \approx 5.66\pi \) from the \( B \)- and \( D \)-sector groupings; cancellations of these respective rates of growth would require first that the leading terms within the \( A \)- and \( C \)-sectors cancel directly, and then that the remaining subleading terms with \( C = 4\sqrt{2}\pi \) from these sectors cancel against the leading terms from the \( B \)- and \( D \)-sectors. However, our theorem can also be used to demonstrate that both of these cancellations also occur, for the theorem guarantees that the largest contribution from the \( \alpha = \bar{n} = 1 \) leading terms is that due to the unphysical tachyon in the \( A_3^* A_3^* \) term, yielding only \( C_{\text{eff}} = 4\pi \), while the largest subdominant term with \( \alpha > 1 \) and/or function solely in terms of the \( D \) pseudo-characters, the condition \( (3.29) \) would not have been satisfied, since all of the \( D \) pseudo-characters are tachyonic with \( H < 0 \) (as opposed to only two of the \( D \) pseudo-characters). Thus, the structure of the physical and unphysical tachyons in this example would have been less apparent, and one would have needed to verify explicitly via \( q \)- and \( \bar{q} \)-expansions of the partition function that all physical tachyons are indeed cancelled. The second reason concerns the \( D \)-combinations themselves, for their \( q \)-expansion coefficients are not all of the same sign for all \( n \) (indeed, both \( D_2 \) and \( D_2^* \) have \( q \)-expansion coefficients which are negative at small \( n \) but positive for large \( n \)). This is related to the fact that the leading asymptotic terms vanish for these \( D \) combinations, and that the various subleading terms have different signs. It thus requires greater values of \( n \) for one of these subleading terms to become dominant, and thereby fix a sign for the coefficients as \( n \to \infty \).
$\alpha > 1$ is easily found to be that with $(\alpha, \overline{\alpha}) = (1, 2)$ and $(H_j, \overline{H}_j) = (-1, -1/2)$, yielding only $C = (4 + \sqrt{2})\pi \approx 5.41\pi$.

Thus we conclude that the leading terms from all four of our sectors cancel completely in this case, with the contributions from the $A$-, $B$-, $C$-, and $D$-sectors all cancelling with each other as a result of the misaligned supersymmetry. Indeed, despite the separate rates of growth given in (4.21) and (4.25), we find that $C_{\text{eff}} \leq (4 + \sqrt{2})\pi$. This is an extraordinary cancellation for $\langle a_{nn} \rangle$, amounting to nearly eight orders of magnitude for energies as small as $n \approx 10$.

4.3 $C_{\text{eff}}$ and the Subleading Terms

In the previous two examples, we witnessed some remarkable cancellations, with all of the leading $\alpha = \overline{\alpha} = 1$ terms cancelling in the summation over sectors leading to $\langle a_{nn} \rangle$. Indeed, in each case we were able to explicitly demonstrate that the value of $C_{\text{eff}}$ was at most that of the largest subleading term with either $\alpha > 1$ or $\overline{\alpha} > 1$. Given these results, then, it is naturally tempting to take the next step, and determine whether any further cancellations occur between the purely subdominant contributions with either $\alpha$ or $\overline{\alpha}$ greater than 1. Indeed, we shall see in Sect. 5 that this second example is somewhat special by virtue of its vanishing one-loop cosmological constant [7], and we would therefore anticipate that many more cancellations should occur for this case in particular. (This function is special for other reasons as well; a detailed discussion of this function and its properties can be found in Ref. [7].) Moreover, we conjectured at the end of Sect. 3 that all subleading terms should cancel in general, resulting in an ultimate value $C_{\text{eff}} = 0$. This would of course require an infinite number of additional cancellations, for there are an infinite number of subleading terms contributing smaller and smaller values of $C$.

Regrettably, however, it is not possible to proceed any further for these examples, and check explicitly whether such additional subleading cancellations actually occur using the asymptotic-expansion formalism presented in Sects. 2 and 3. The reason, as briefly mentioned at the end of Sect. 3, concerns the suitability of these asymptotic expansions for situations (such as our calculation of $\langle a_{nn} \rangle$) in which the energy $n$ is treated as a continuous variable. In particular, recall the form of the final result (3.36): the coefficients $P_{ij}^{(\alpha, \overline{\alpha})}$ with either $\alpha > 1$ or $\overline{\alpha} > 1$ are precisely the coefficients which determine whether such subleading terms survive the sector-averaging process. It turns out, however, that the definition of these coefficients given in (3.37) is ultimately unsuitable for the $\alpha, \overline{\alpha} > 1$ cases.

In order to see why this is so, let us first recall the original derivation of the asymptotic expansions in Sect. 2. The asymptotic expansions of the coefficients of any individual character $\chi_i(q)$ are given in (2.10), and in particular the coefficients $Q_{ij}^{(\alpha)}$ are given in (2.11). Later in Sect. 2, we evaluated these coefficients explicitly, with the results listed in (2.25), (2.27), (2.31), and (2.32). It was important for the consistency of these results and for the validity of the asymptotic expansions in
general that these coefficients \(Q_{ij}^{(\alpha)}\) be real quantities; since each value of \(\alpha\) ultimately corresponds to a different rate of exponential growth for the chiral degeneracies \(a_n^{(i)}\), there is no way that complex coefficients \(Q_{ij}^{(\alpha)}\) can combine to yield a value of \(a_n^{(i)}\) which is real for each value of \(n\). However, while the dominant coefficient \(Q_{ij}^{(1)}\) is indeed real for all values of \(n\), the results given for all of the \(Q_{ij}^{(\alpha)}\) with \(\alpha > 1\) implicitly assumed that \(n \in \mathbb{Z}\). For example, tracing again the steps leading to \(Q_{ij}^{(3)}\) and not assuming that \(n \in \mathbb{Z}\), we actually obtain

\[
Q_{ij}^{(3)} = e^{-\pi i n} 2 \text{Re} \left\{ e^{\pi i k/2} (ST^3 S)_{ij} \exp \left[ \frac{2\pi i}{3} \left( n + H_i + H_j \right) \right] e^{-\pi i n} \right\}
\]  

(4.26)

instead of (2.31). While this result of course reduces to (2.31) for \(n \in \mathbb{Z}\), for other values of \(n\) this quantity is actually complex, with phase \(e^{-i\pi n}\). This phase is in fact the same for all of the \(\alpha > 1\) coefficients, with the modifications to the other values of \(Q_{ij}^{(\alpha)}\) taking the same general form as (4.26).

This observation is very important, because ultimately these coefficients \(Q_{ij}^{(\alpha)}\) became the building blocks of the coefficients \(P_{\bar{j}j}^{(\alpha,\bar{\alpha})}\) defined in (3.37), and these are precisely the quantities whose values determine whether or not the subleading terms cancel. There is thus no guarantee that the coefficients \(P_{\bar{j}j}^{(\alpha,\bar{\alpha})}\) with \(\alpha > 1\) and/or \(\bar{\alpha} > 1\) will be real, and in fact the first non-cancelling subleading term in the second example we examined turns out to be complex. We thus see that the very definitions of \(P_{\bar{j}j}^{(\alpha,\bar{\alpha})}\) for \(\alpha > 1\) and/or \(\bar{\alpha} > 1\) are unsuitable as coefficients in the asymptotic expansions for the sector-averaged quantities \(\langle a_{nn} \rangle\), and that an intrinsically different sort of asymptotic expansion is necessary. Note that this is the same conclusion we reached at the end of Sect. 3. Indeed, what is needed is an asymptotic expansion which is calculated directly for sector-averaged quantities such as \(\langle a_{nn} \rangle\) in which the energy \(n\) is to be regarded as a continuous or “sector-averaged” variable. Such an expansion would also hopefully have many of these cancellations built in at an early stage, and thereby enable us to efficiently determine the precise value of \(C_{\text{eff}}\) and the behavior of \(\langle a_{nn} \rangle\) as \(n \to \infty\). We shall outline the steps by which such an expansion might be obtained in Sect. 6.

Note, however, that the lack of such a suitable asymptotic expansion does not affect the primary cancellations of the leading terms with \((\alpha, \bar{\alpha}) = (1, 1)\), and in particular the validity of our theorem relies upon cancellations between only these terms. Thus, our result that \(C_{\text{eff}} < C_{\text{tot}}\), and its implications concerning “misaligned supersymmetry”, remain unaltered.
5 Finiteness and the Cosmological Constant

As discussed in the Introduction, modular invariance and the absence of physical tachyons are the conditions which guarantee finite loop amplitudes in string theory. Since these are also the conditions which yield the “misaligned supersymmetry”, it is natural to interpret the resulting boson/fermion oscillation as the mechanism by which the net numbers of states in string theory distribute themselves level-by-level so as to produce finite amplitudes. In this section we will provide some evidence for this by focusing on the simplest loop amplitude in string theory, namely the one-loop vacuum polarization amplitude or cosmological constant, defined as

$$\Lambda \equiv \int_{F} \frac{d^{2}\tau}{(\text{Im}\,\tau)^{2}} Z(\tau, \bar{\tau}) = \int_{F} \frac{d^{2}\tau}{(\text{Im}\,\tau)^{2}} (\text{Im}\,\tau)^{k} \sum_{m,n} a_{mn} q^{m} q^{n}. \quad (5.1)$$

Here $F$ is the fundamental domain of the modular group,

$$F \equiv \left\{ \tau \mid \text{Im}\,\tau > 0, -\frac{1}{2} \leq \text{Re}\,\tau \leq \frac{1}{2}, |\tau| \geq 1 \right\}, \quad (5.2)$$

and one is instructed to integrate over $\tau_{1} \equiv \text{Re}\,\tau$ in (5.1) before integrating over $\tau_{2} \equiv \text{Im}\,\tau$ in the $\tau_{2} > 1$ region.

The first thing we notice about (5.1) is that contributions to $\Lambda$ come from both the physical states with $m = n$, and the unphysical states with $m \neq n$. This arises because the fundamental domain $F$ consists of two distinct regions in the complex $\tau$-plane: an infinite rectangular-shaped region $F_{1}$ with $\text{Im}\,\tau \geq 1$, and a curved region $F_{2}$ with $\text{Im}\,\tau \leq 1$. Integrating over the rectangular region $F_{1}$, we see that only the physical states with $m = n$ contribute, for the contributions from terms in (5.1) with $m \neq n$ are cancelled in the integration over $\tau_{1}$. However, since the curved second region $F_{2}$ does not extend the over the full range $-\frac{1}{2} \leq \tau_{1} \leq \frac{1}{2}$, the unphysical contributions from this part of the integration are not completely cancelled, and unphysical (or “off-shell”) states with $m \neq n$ thereby contribute to loop amplitudes.

This would in principle cause a problem for us, since our result concerning misaligned supersymmetry applies to only the physical states with $m = n$. Furthermore, even after the $\tau_{1}$-integration is performed in (5.1), one is left with the integration over $\tau_{2}$, and it is not readily clear how our result concerning the coefficients $a_{nn}$ will then translate into a result concerning the finiteness of $\Lambda$. In particular, note that (5.1) is manifestly finite if the spectrum is free of physical tachyons, since the presumed modular invariance of the theory has already been used to truncate the region of $\tau$-integration to the fundamental domain $F$ and thereby avoid the dangerous ultraviolet $\tau \to 0$ region. Thus the finiteness of (5.1) only implicitly rests on the behavior of the state degeneracies, and it is difficult to see how to directly relate the two.

* It is a trivial exercise, however, to extend the result in Sect. 3 to apply to unphysical states as well; indeed, these unphysical states also experience analogous “misaligned supersymmetries” with analogous boson/fermion oscillations.
All of these problems can be circumvented, however, due to a remarkable result of Kutasov and Seiberg \[8\] which expresses the one-loop cosmological constant directly in terms of only the physical-state degeneracies, and which does so without subsequent $\tau_2$-integrations. Their result is as follows. Let us first define the quantity

$$g(\tau_2) \equiv \int^{+1/2}_{-1/2} d\tau_1 Z(\tau_1, \tau_2).$$

(5.3)

Substituting the general $q$-expansion for $Z$ given in (5.1) and performing the $\tau_1$-integration, we easily find that $g(\tau_2)$ receives contributions from only the physical states:

$$g(\tau_2) = \tau_2^k \sum_n a_{nn} \exp(-4\pi n \tau_2).$$

(5.4)

Indeed, we can interpret $g(\tau_2)$ as being a regulated measure of the total number of physical states in the theory, with $\tau_2$ serving as the cutoff which regulates this divergent quantity. Therefore $\lim_{\tau_2 \to 0} g(\tau_2)$ might be interpreted as giving the total number of states without any cutoff. One would naively expect this quantity to diverge, since $k$ is negative in most situations of interest and since $\sum_n a_{nn}$ is formally a divergent quantity. However, Kutasov and Seiberg show \[8\] that this limit is actually finite in any modular-invariant theory which is free of physical tachyons, and moreover

$$\lim_{\tau_2 \to 0} g(\tau_2) = \frac{3}{\pi} \Lambda.$$

(5.5)

This result is quite general, and applies to all partition functions $Z$ which are free of physical tachyons (i.e., which have $a_{mn} = 0$ for all $n < 0$), and whose unphysical tachyons are not too tachyonic. Explicitly, this latter condition states that $Z$ must have non-zero values of $a_{mn}$ only for $m \geq m_0$ and $n \geq n_0$ where $m_0, n_0 > -1$. Results similar to (5.5) can nevertheless be obtained for cases in which this last condition is violated \[8\].

The result (5.5) is quite powerful, since it enables us to formally calculate the complete one-loop cosmological constant (5.1) given knowledge of only the physical state degeneracies. Indeed, this implies that the assumed modular invariance of $Z$ is sufficiently strong a constraint that these physical degeneracies themselves determine the contributions to $\Lambda$ from the unphysical states as well. However, (5.5) can also be interpreted as a severe constraint on the distributions of the physical states in any tachyon-free modular-invariant theory, for somehow the net degeneracies $\{a_{nn}\}$ in (5.4) must arrange themselves in such a way that $\lim_{\tau_2 \to 0} g(\tau_2)$ is finite. Indeed, such an arrangement is evidently precisely what is required to yield finite amplitudes. Thus, via this result (5.5), we are furnished with a mathematical condition on a modular-invariant set of degeneracies $\{a_{nn}\}$ which is necessary and sufficient to yield

\[1\] Note that this does not fix the actual number and distribution of unphysical states, but instead determines only their total integrated contribution to $\Lambda$. 

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a finite one-loop cosmological constant:

$$\lim_{\tau_2 \to 0} g(\tau_2) < \infty ,$$

(5.6)

or explicitly,

$$\lim_{\tau_2 \to 0} \tau_2^k \sum_n a_{nn} \exp(-4\pi \tau_2 n) < \infty .$$

(5.7)

For physical situations in which the modular weight $k$ is negative (i.e., for $D > 2$), this in fact requires

$$\lim_{\tau_2 \to 0} \sum_n a_{nn} \exp(-4\pi \tau_2 n) = 0 .$$

(5.8)

A priori, there are any number of conceivable distributions $\{a_{nn}\}$ which might satisfy the finiteness condition (5.8), and this condition alone is therefore not sufficiently restrictive to predict the resulting behavior for the actual distribution of physical states. For example, simple distributions which trivially satisfy (5.8) with $n \in \mathbb{Z}$ include $a_{nn} = (n^2 - n - \frac{1}{4})^2$ for $n \geq 1$, or $a_{nn} = r^n - (1 - r)^{-1} \delta_{n,0}$ for all $n \geq 0$ and $r < 1$. The condition (5.8) can nevertheless be used to rule out certain behavior for the net degeneracies $a_{nn}$. For example, it is straightforward to show that if $a_{nn} \sim n^{-B} e^{C\sqrt{n}}$ with $C > 0$ as $n \to \infty$, then

$$g(\tau_2) \sim (\tau_2)^{k+2B-3/2} \exp\left(\frac{C^2}{16\pi \tau_2}\right)$$

as $\tau_2 \to 0$.

Thus the finiteness condition (5.8) cannot be satisfied for any $B$ and $C$, and as a consequence all direct exponential growth for the net degeneracies $a_{nn}$ is prohibited.

Without knowledge of misaligned supersymmetry and the resulting boson/fermion oscillation, this last result would seem quite remarkable, since we know that each individual sector contributes a set of degeneracies $\{a_{nn}^{(\pi)}\}$ which does grow exponentially with $C = C_{\text{tot}}$. Indeed, $C_{\text{tot}}$ is the inverse Hagedorn temperature, and it is precisely this exponential growth which is responsible for the famous Hagedorn phenomenon which is thought to signal a phase transition in string theory.

However, misaligned supersymmetry and the resulting boson/fermion oscillation now provide a natural alternative solution which reconciles the finiteness condition (5.8) with growing behavior for which $|a_{mn}| \to \infty$ as $n \to \infty$. Indeed, we can easily see that such oscillations permit many growing solutions to (5.8); for example, simple distributions such as $a_{nn} = (-1)^n n^2$, $a_{nn} = (-1)^n n^4$, $a_{nn} = (-1)^n (n - n^5)$, and $a_{nn} = (-1)^n (2n^3 + n^5)$ all non-trivially satisfy (5.8), where the factor of $(-1)^n$ is meant to illustrate the alternating-sign behavior for $a_{nn}$ which is implicit in the boson/fermion oscillations. In fact, we can easily show that (5.8) is satisfied for any $a_{nn} = (-1)^n f(n)$ provided the function $f(n)$ is even in $n$, with a Taylor-expansion of the form $f(n) = \sum_{k=1}^{\infty} c_k n^{2k}$ with no constant term. The argument goes as follows.

Since

$$\sum_{n=0}^{\infty} (-1)^n e^{-4\pi \tau_2 n} = \frac{1}{1 + e^{-4\pi \tau_2}} = \frac{1}{2} + \frac{1}{2} \left( \frac{\sinh 4\pi \tau_2}{1 + \cosh 4\pi \tau_2} \right) ,$$

(5.10)
we have
\[
g(\tau_2) = \sum_{n=0}^{\infty} (-1)^n f(n) e^{-4\pi \tau_2 n} = \sum_{k=1}^{\infty} c_k \left( \frac{-1}{4\pi} \frac{d}{d\tau_2} \right)^{2k} \left\{ \frac{1}{2} + \frac{1}{2} \left( \frac{\sinh 4\pi \tau_2}{1 + \cosh 4\pi \tau_2} \right) \right\}. \tag{5.11}
\]
But now we see that the derivatives remove the contributions from the additive factor of $\frac{1}{2}$ within the braces, and of course the remaining term within the braces is odd in $\tau_2$. Thus we find that $g(\tau_2)$ itself is an odd function of $\tau_2$, and since $g(\tau_2)$ is continuous at $\tau_2 = 0$, we have $\lim_{\tau_2 \to 0} g(\tau_2) = 0$.

This heuristic argument demonstrates that the non-trivial critical ingredient in the success of all of these functions is their pattern of oscillation which is represented schematically by $(-1)^n$. Indeed, it is precisely this oscillation which permits solutions for which $|a_{nn}| \to \infty$ as $n \to \infty$ to satisfy (5.8), and which therefore renders the Hagedorn-type growth of the separate numbers of bosonic and fermionic states in string theory consistent with the finiteness condition (5.8).

Of course, our complete result predicts much more than this, yielding not only the overall boson/fermion oscillation, but also detailed information concerning both the actual asymptotic forms of the degeneracies $a_{nn}$ as $n \to \infty$, and their cancellations upon sector-averaging. Indeed, our result demands that all traces of the leading exponential behavior with $C = C_{\text{tot}}$ must cancel, and we have seen that our theorem very often implies that many subleading terms with smaller values of $C$ must cancel as well. The examples presented in Sect. 4 showed that these cancellations can in fact be quite subtle, with the subleading terms from one sector cancelling against the leading terms from another sector, and we found that the combined effect of these successive cancellations was often quite dramatic.

However, the result (5.8) now presents us with a slightly different tool in interpreting these cancellations. The vanishing of the limit (5.8) depends crucially on two seemingly-separate properties of the degeneracies $a_{nn}$: their high-energy behavior for large $n$ must be such that this limit converges, and then the value at which this convergence takes place must exactly balance the contributions to the limit which come from the low-energy states. Let us focus for the moment on the high-energy states, and consider the question of convergence. We have seen that oscillatory asymptotic behavior for the $\{a_{nn}\}$ generically yields convergence, with the rapid fluctuations in some sense cancelling each other and yielding no net contribution to (5.8). This is precisely the content of our main result, which asserts that all traces of the $C = C_{\text{tot}}$ behavior of each individual sector are removed in the sector-averaged quantity $\langle a_{nn} \rangle$. In this sense, then, we might approximate the $a_{nn}$ for asymptotically high values of $n$ in (5.8) by $\langle a_{nn} \rangle$, and work only with the net functional form that this quantity represents.

However, assuming that this replacement is justified at high $n$, we now reach an interesting conclusion. The asymptotic behavior of $\langle a_{nn} \rangle$ is governed, of course, not by $C_{\text{tot}}$, but by $C_{\text{eff}}$. Thus, if $C_{\text{eff}} \neq 0$, then even the value of $\langle a_{nn} \rangle$ experiences
exponential growth, for there must remain subleading terms in $\langle a_{nn} \rangle$ with $C = C_{\text{eff}}$ whose contributions remain uncancelled. But even this residual exponential growth would be dangerous for the convergence of \((5.8)\), for we have seen in \((5.9)\) that any net exponential growth causes $g(\tau_2)$ to diverge as $\tau_2 \to 0$. Thus we would conjecture that we must in fact have

$$C_{\text{eff}} \not= 0 \tag{5.12}$$

for any modular invariant theory which is free of physical tachyons. Indeed, this would require that all exponential growth cancel in the sector-averaging process, whether this growth is leading or subleading. This would of course necessitate an infinite number of cancellations, for there are indeed an infinite number of such subleading terms whose cancellations would necessarily become more and more intricate and inter-related.

Given the conjecture $C_{\text{eff}} \not= 0$, let us now carry this argument one step further, and proceed to examine the situation with polynomial growth,

$$\langle a_{nn} \rangle \sim n^\epsilon , \quad \epsilon \geq 0 . \tag{5.13}$$

For theories without physical tachyons, the cosmological constant $\Lambda$ is finite, and thus the finiteness condition \((5.7)\) implies

$$\lim_{\tau_2 \to 0} \sum_n a_{nn} \exp(-4\pi \tau_2 n) \sim \tau_2^{-k} . \tag{5.14}$$

Once again assuming that we can replace $a_{nn}$ by the sector-average $\langle a_{nn} \rangle$, we thereby obtain the constraint

$$\lim_{\tau_2 \to 0} \sum_n n^\epsilon \exp(-4\pi \tau_2 n) \sim \tau_2^{-k} , \tag{5.15}$$

and since

$$\lim_{\tau_2 \to 0} \left( \frac{-1}{4\pi} \frac{d}{d\tau_2} \right)^\epsilon \sum_n \exp(-4\pi \tau_2 n) \sim \left( \frac{-1}{4\pi} \frac{d}{d\tau_2} \right)^\epsilon \left( \frac{1}{4\pi \tau_2} \right) \sim (\tau_2)^{-(1+\epsilon)} , \tag{5.16}$$

we find $\epsilon = k - 1$. Similar arguments can also be used to show that for $k \leq 0$, we must have $\langle a_{nn} \rangle \to 0$ as $n \to \infty$ (indeed, for $k = 0$ the fastest allowed growth for $\langle a_{nn} \rangle$ is $\sim r^n$ with $r < 1$). Thus, as $n \to \infty$, we have

$$\langle a_{nn} \rangle \not\sim \begin{cases} 0 & \text{for } k \leq 0 \\ n^{k-1} & \text{for } k \geq 1 . \end{cases} \tag{5.17}$$

These conjectures \((5.12)\) and \((5.17)\) are certainly appealing on aesthetic grounds. Moreover, these are in fact the same conjectures that we made at the end of Sect. 3 on the basis of a comparison between our two-variable theorem and its one-variable counterpart. This remarkable agreement indicates that there exists a certain logical internal consistency to these conjectures, and it would be interesting to see if a proof could be constructed using what would necessarily be a fundamentally different type of asymptotic expansion for $\langle a_{nn} \rangle$. We shall briefly outline the initial steps by which such an expansion might be obtained in Sect. 6.
6 Concluding Remarks

In this paper we have been able to prove a general theorem concerning the distributions of physical bosonic and fermionic states in any theory which is modular-invariant and which contains no physical tachyons. This result is therefore especially applicable to string theories lacking spacetime supersymmetry, and we demonstrated that despite the absence of such supersymmetries, a “misaligned supersymmetry” must nevertheless survive in which any surplus of bosonic states at any mass level implies a surplus of fermionic states at a higher mass level, which in turn implies a surplus of bosonic states at a still higher level, etc. We demonstrated that this oscillation may be responsible for the finiteness of string amplitudes by considering the case of the one-loop cosmological constant, and also showed that our general theorem is the natural mathematical generalization of some powerful one-variable theorems in modular-function theory to the physically relevant cases of partition functions $Z(q, \bar{q})$ of two variables. Our analysis also introduced a new quantity, the so-called “sector-averaged” number of states $\langle a_{nm} \rangle$, which may well be relevant for the asymptotic behavior of string amplitudes at high energy, and we made a conjecture concerning its behavior as $n \to \infty$.

There are nevertheless a number of possible extensions to our results which we shall now briefly indicate; these will hopefully become the subjects of future research. First, we have only considered the general question of finiteness as it pertains to the case of the one-loop cosmological constant, yet a demonstration that our misaligned supersymmetry implies finiteness to all orders for all $n$-point functions (or even finiteness for all one-loop amplitudes) would be a far more difficult task. In particular, such functions depend on knowledge of the interactions, and not just the numbers of physical states. Second, such investigations would almost certainly also involve the examination of the behavior of the degeneracies of the unphysical (or so-called “off-shell”) states in string theory; indeed, our focus in this paper has been on the physical states, and it was only due to the result of Kutasov and Seiberg discussed in Sect. 5 that this was sufficient for discussing the finiteness of the one-loop cosmological constant. We expect that the unphysical states also experience analogous asymptotic cancellations and misaligned supersymmetries, however, for the degeneracies and distributions of the unphysical states are closely tied by modular invariance to those of the physical states.

A third issue which we have mentioned at various points in this paper concerns the suitability of the asymptotic expansions presented in Sect. 2 as building blocks for the asymptotic expansions of the sector-averaged number of states $\langle a_{nm} \rangle$; not only are there problems with the behavior of certain subleading coefficients, but the forms of these expansions themselves are somewhat cumbersome for our purposes, with successive cancellations occurring in highly non-trivial ways which do not lend themselves to general analysis. The structure of the error terms in these expansions also prohibited a more powerful conclusion concerning the ultimate value of $C_{\text{eff}}$, and
in particular we have conjectured on other grounds that in fact $C_{\text{eff}} = 0$. One might therefore hope instead for a reformulation of these asymptotic expansions, for a new derivation which would proceed directly from the definition of $\langle a_{nn} \rangle$ in a manner analogous to that of Refs. [1] and [3]. We can in fact see fairly quickly how such a derivation might be formulated. Starting directly from the contour integrals in (2.4), we would immediately combine the holomorphic and anti-holomorphic sectors of the theory,

$$a_{nn}^{(\tau)} = a_{nn}^{(\bar{\tau})} = \frac{-1}{4\pi^2} \int d\tau \int dq \frac{\chi_i(q)}{(\tau - \bar{\tau})^{n+1}}.$$  \hspace{1cm} (6.1)

Note we are here using the shifted variables $n$ from the beginning, as discussed in Sect. 3, and therefore $n$ is not necessarily an integer. Also note that $q$ and $\bar{q}$ are to be regarded as independent variables in this analysis, with each independently taking values along its respective contour. We would then multiply (6.1) by $N_{\tau}$ and sum over $(\tau, i)$ in order to build an expansion for $\langle a_{nn} \rangle$ directly, yielding

$$\langle a_{nn} \rangle = \frac{-1}{4\pi^2} \int d\tau \int dq \sum_{\tau} N_{\tau} \frac{\chi_i(q)}{(\tau - \bar{\tau})^{n+1}} = \frac{-1}{4\pi^2} \int d\tau \int dq \frac{\sum_{\tau} N_{\tau} \chi_i(q)}{(\tau - \bar{\tau})^{n+1}}.$$  \hspace{1cm} (6.2)

Note that the second equality above follows from the presumed independence of $n$ and $i$, which is the essence of the sector-averaging. However, we now recognize that the numerator of the final expression in (6.2) is nothing but the partition function $Z(q, \bar{q})$ without its factor $(\text{Im} \tau)^k$. We thus simply have

$$\langle a_{nn} \rangle = \frac{-1}{4\pi^2} \int d\tau \int dq \frac{Z(q, \bar{q})}{(\tau - \bar{\tau})^{n+1}[\tau - \bar{\tau}]/(2i)^k}.$$  \hspace{1cm} (6.3)

where we have generalized $\text{Im} \tau \rightarrow (\tau - \bar{\tau})/(2i)$ (as is necessary for the invariance of $Z$ under simultaneous identical modular transformations of $q$ and $\bar{q}$). Thus, we see already that the behavior of $\langle a_{nn} \rangle$ now directly depends on the behavior of the partition function $Z(q, \bar{q})$ as $q$ and $\bar{q}$ approach the singular points on their respective unit circles, and all cancellations between the expansions of the individual characters comprising $Z$ have already been incorporated. One could therefore hope to begin afresh from (6.3), and derive an asymptotic expansion directly for $\langle a_{nn} \rangle$ in which a minimum of unnecessary cancellations appear and the error terms are minimized. Such a derivation would presumably parallel the steps and analyses given in Ref. [2] for the simpler case of the one-variable function $\chi_i(q)$, although the appearance of two independent variables $q$ and $\bar{q}$ will of course introduce new subtleties.

Finally, let us mention some potential applications of our results. Perhaps the most important concerns the general question of supersymmetry-breaking in string theory; our results concerning a residual misaligned supersymmetry are of course quite general, and thus it should be possible to use them to constrain the possible supersymmetry-breaking scenarios in string theory. For example, it would be
interesting to understand on a deeper level the relation between our misaligned-supersymmetry theorem and the various soft-supersymmetry breaking theorems in string theory. This would no doubt entail developing a dynamical understanding of the types of symmetry-breaking terms which, while breaking supersymmetry, nevertheless preserve the cancellation of the functional forms which describe the separate distributions of bosonic and fermionic states. Note, in this regard, that our results clearly preclude any supersymmetry-breaking scenario in which, for example, the energies of fermionic states are merely shifted (even infinitesimally) relative to those of their bosonic counterparts. Rather, misaligned supersymmetry requires that any such energy shifts must be simultaneously accompanied by the introduction or removal of a certain number $\Phi(n + \Delta n) - \Phi(n)$ of extra states, where $\Phi(n)$ is the asymptotic function describing the density of bosonic or fermionic states and $\Delta n$ is magnitude of the induced energy shift.

Another application of our results concerns string theory at the hadronic scale, and the implications of our results as they relate to the effective “QCD strings” which model hadron dynamics. There have in fact been some recent developments in this area. Interpreting the various bosonic states in such hadronic string theories as individual mesons, and assuming that one can similarly model the corresponding fermionic string states as baryons [1], it has recently been demonstrated [3] that the actual numbers and distributions of the experimentally observed meson and baryon states are not in disagreement with the oscillations resulting from misaligned supersymmetry. Indeed, since it is never possible to experimentally survey the infinite range of energies necessary in order to test modular invariance directly, such oscillations generally serve as the only profound yet indirect experimental “signature” of modular invariance which is “local” in energy (acting level-by-level) and thereby experimentally accessible. It has even been possible, using both the Kutasov-Seiberg result and the misaligned supersymmetry, to make concrete predictions [3, 9] concerning the appearance and structure of new hadronic states.

The forms of these asymptotic expansions themselves have also played an important role in recent analyses of the hadronic spectrum [10]. It has long been known that the Hagedorn-type rise of the number of meson states as $n \to \infty$ is indicative of an effective string-like picture underlying the color flux tube in mesons, and by fitting the simple Hagedorn exponential form $a_n \sim n^{-b} e^{C\sqrt{n}}$ to the meson spectrum one obtains [11] the well-known result $T_H \equiv C^{-1} \approx 160$ MeV [where $\alpha'$, the Regge slope, is taken to be 0.85 (GeV)$^{-2}$]. From this one can directly calculate the central charge of the effective “QCD string” underlying meson spectroscopy, obtaining $c \approx 7$. However, this result is in strong disagreement with the majority of QCD string proposals which have $c \approx 2$, and indeed alternative analyses of the central charge of the QCD string which compare the results of the effective static-quark potential with data also predict $c \approx 2$. It has recently been demonstrated [10], however, that the conflict between these two results arises from the incorrect application of the simple Hagedorn form $n^{-b} e^{C\sqrt{n}}$ to data which is not sufficiently asymptotic in energy.
Indeed, using the complete asymptotic forms discussed in this paper (most notably, the replacement of the above Hagedorn exponentials by Bessel functions) profoundly alters the results, and yields new estimates of the Hagedorn temperature which are now in agreement with the result \( c \approx 2 \). Thus these asymptotic expansions, derived from the principles of modular invariance and conformal invariance, also turn out to have wide-ranging applications beyond their primarily theoretical interest. Taken together, then, these recent results demonstrate that the observed hadronic spectrum is consistent with an underlying string theory in which modular invariance plays a significant role.

There have also been other recent efforts to compare the rigorous predictions of string theory with the predictions of QCD. Those which are most closely related to the ideas of this paper include attempts to construct a toy model of QCD which analytically exhibits an infinite number of Regge trajectories, an exponential rise in the number of meson states, and a Hagedorn deconfining transition. It has recently been shown [12] that two-dimensional QCD coupled to adjoint matter has precisely these properties, and this has sparked efforts [12] to determine whether the spectrum of states predicted in such models is consistent with the Kutasov-Seiberg constraint (5.8). Since this is in general a difficult constraint to verify, it might be interesting (and simpler) to determine whether the spectrum of this model exhibits a misaligned supersymmetry. This would indeed be powerful evidence for the applicability of string-like ideas to the realm of QCD.

Finally, of course, we remark that our result concerning misaligned supersymmetry and the cancellation of the separate bosonic and fermionic functional forms is, due to its generality, relevant for all physical situations in which \( \langle a_{nn} \rangle \), rather than \( a_{nm} \), plays a role. This thereby includes all areas, such as string thermodynamics or the high-energy behavior of string scattering amplitudes, in which the physics is determined by the high-energy asymptotic behavior of the physical-state degeneracies. Indeed, the full consequences of the reduction of the effective degeneracy exponential growth rate from \( C_{\text{tot}} \) to \( C_{\text{eff}} \) are likely to be significant and far-reaching. An investigation of some of these issues is currently underway.

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Fig. 1: The net number of physical states $a_{nn}$ for the toy model (1.14), plotted versus the energy $n$ [equivalently the spacetime $(\text{mass})^2$]. Negative values of $\pm \log_{10}(|a_{nn}|)$ are plotted for $a_{nn} < 0$. Also sketched is the sector-averaged number of states $\langle a_{nn} \rangle$, assuming a cancellation scenario with $C_{\text{eff}} = C_{\text{tot}}/4$. 

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Fig. 2: The net number of physical states $a_{nn}$ for the partition function (4.6), plotted versus worldsheet energy $n$ as in Fig. 1. The complex pattern of oscillation is due to the presence of six sectors in this theory; the “wavelength” of this oscillation is nevertheless $\Delta n = 1$. 
Fig. 3: The net number of physical states $a_{nn}$ for the partition function (4.6), now plotted versus spacetime mass $\sqrt{n}$. Note that the asymptotic exponential growth for each sector begins quite early, with the sign of each $a_{nn}$ determined from the partition function (4.6).
Fig. 4: The net number of physical states $a_{nn}$ for the partition function (4.19), plotted versus energy $n$ as in Fig. 1. This partition function has a one-loop cosmological constant which vanishes identically (see Ref. [7]), and — like the partition functions of heterotic string theories built from periodic/anti-periodic worldsheet fermions — contains exactly four sectors.
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