Lie algebras of order $F$ and extensions of the Poincaré algebra

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Abstract. $F$–Lie algebras are natural generalisations of Lie algebras ($F = 1$) and Lie superalgebras ($F = 2$). We give finite dimensional examples of $F$–Lie algebras obtained by an inductive process from Lie algebras and Lie superalgebras. Matrix realizations of the $F$–Lie algebras constructed in this way from $osp(2|m)$ are given. We obtain a non-trivial extension of the Poincaré algebra by an Inönü-Wigner contraction of a certain $F$–Lie algebras with $F > 2$.

1. Introduction

Describing the laws of physics in terms of underlying symmetries has always been a powerful tool. Lie algebras and Lie superalgebras are central in particle physics, and the space-time symmetries can be obtained by an Inönü-Wigner contraction of certain Lie (super)algebras. $F$–Lie algebras [$1$, $2$, $3$], a possible extension of Lie (super)algebras, have been considered some times ago as the natural structure underlying fractional supersymmetry (FSUSY) [$4$, $5$, $6$] (one possible extension of supersymmetry). In this contribution we show how one can construct many examples of finite dimensional $F$–Lie algebras from Lie (super)algebras and finite-dimensional FSUSY extensions of the Poincaré algebra are obtained by Inönü-Wigner contraction of certain $F$–Lie algebras.

2. $F$–Lie algebras

The natural mathematical structure, generalizing the concept of Lie superalgebras and relevant for the algebraic description of fractional supersymmetry was introduced in [$1$] and called an $F$–Lie algebra. We do not want to go into the detailed definition of this structure here and will only recall the basic points, useful for our purpose. More details can be found in [$1$].

Let $F$ be a positive integer and $q = e^{2i\pi F}$. We consider now a complex vector space $S$ which has an automorphism $\varepsilon$ satisfying $\varepsilon^F = 1$. We set $A_k = S_{q^k}$, $1 \leq k \leq F - 1$ and $B = S_1$ ($S_{q^k}$ is the eigenspace corresponding to the eigenvalue $q^k$ of $\varepsilon$). Hence,

$$S = B \oplus A_1 \oplus \cdots \oplus A_{F-1}.$$ 

We say that $S$ is an $F$–Lie algebra if:

(i) $B$, the zero graded part of $S$, is a Lie algebra.
(ii) $A_i$ ($i = 1, \ldots, F - 1$), the $i$ graded part of $S$, is a representation of $B$.
(iii) There are symmetric multilinear $B$–equivariant maps

$$\{ , \ldots, \} : \mathcal{J}^F (A_k) \to B,$$ 

where $\mathcal{J}^F$ is the space of $F$–linear maps on $A_k$. 

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where $\mathcal{S}^F(D)$ denotes the $F$–fold symmetric product of $D$. In other words, we assume that some of the elements of the Lie algebra $B$ can be expressed as $F$–th order symmetric products of “more fundamental generators”.

(iv) The generators of $S$ are assumed to satisfy Jacobi identities $(b_i \in B, a_i \in A_k, 1 \leq k \leq F - 1)$:

\[
\sum_{i=1}^{F+1} [a_i, \{a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{F+1}\}] = 0.
\]

The first three identities are consequences of the previously defined properties but the fourth is an extra constraint.

More details (unitarity, representations, etc.) can be found in [1, 3]. Let us first note that no relation between different graded sectors is postulated. Secondly, the sub-space $B \oplus A_k \subset S$ $(k = 1, \ldots, F - 1)$ is itself an $F$–Lie algebra. From now on, $F$–Lie algebras of the types $B \oplus A_k$ will be considered.

Most of the examples of $F$–Lie algebras are infinite dimensional (see e.g. [1, 5]). However in [3] an inductive theorem to construct finite-dimensional $F$–Lie algebras was proven:

**Theorem 1** Let $g_0$ be a Lie algebra and $g_1$ a representation of $g_0$ such that

(i) $S_1 = g_0 \oplus g_1$ is an $F$–Lie algebra of order $F_1 \geq 1$ [5];

(ii) $g_1$ admits a $g_0$–equivariant symmetric form $\mu_2$ of order $F_2 \geq 1$.

Then $S = g_0 \oplus g_1$ admits an $F$–Lie algebra structure of order $F_1 + F_2$, which we call the $F$–Lie algebra induced from $S_1$ and $\mu_2$.

By hypothesis, there exist $g_0$–equivariant maps $\mu_1 : \mathcal{S}^{F_1}(g_1) \rightarrow g_0$ and $\mu_2 : \mathcal{S}^{F_2}(g_1) \rightarrow C$. Now, consider $\mu : \mathcal{S}^{F_1+F_2}(g_1) \rightarrow g_0 \otimes C \cong g_0$ defined by

\[
\mu(f_1, \ldots, f_{F_1+F_2}) = \frac{1}{F_1! F_2!} \sum_{\sigma \in S_{F_1+F_2}} \mu_1(f_{\sigma(1)}, \ldots, f_{\sigma(F_1)}) \otimes \mu_2(f_{\sigma(F_1+1)}, \ldots, f_{\sigma(F_1+F_2)}),
\]

where $f_1, \ldots, f_{F_1+F_2} \in g_1$ and $S_{F_1+F_2}$ is the group of permutations on $F_1 + F_2$ elements. By construction, this is a $g_0$–equivariant map from $\mathcal{S}^{F_1+F_2}(g_1) \rightarrow g_0$, thus the three first Jacobi identities are satisfied. The last Jacobi identity, is more difficult to check and is a consequence of the corresponding identity for the $F$–Lie algebra $S_1$ and a factorisation property (see [3] for more details).

### 3. Finite dimensional $F$–Lie algebras

An interesting consequence of the theorem of the previous section is that it enables us to construct an $F$–Lie algebras associated to any Lie (super)algebras.

‡ Strictly speaking this theorem is not valid for $F_1 = 1$. In this case the notion of graded 1–Lie algebra has to be introduced [3]. $S = g_0 \oplus g_1$, is a graded 1–Lie algebra if (i) $g_0$ a Lie algebra and $g_1$ is a representation of $g_0$ isomorphic to the adjoint representation, (ii) there is a $g_0$–equivariant map $\mu : g_1 \rightarrow g_0$ such that $[f_1, \mu(f_2)] + [f_2, \mu(f_1)] = 0, f_1, f_2 \in g_1$. 
3.1. Finite dimensional $F-$Lie algebras associated to Lie algebras

Consider the graded 1–Lie algebra $S = g_0 \oplus g_1$ where $g_0$ is a Lie algebra, $g_1$ is the adjoint representation of $g_0$ and $\mu : g_1 \rightarrow g_0$ is the identity. Let $J_1, \cdots, J_{\dim g_0}$ be a basis of $g_0$, and $A_1, \cdots, A_{\dim g_0}$ the corresponding basis of $g_1$. The graded 1–Lie algebra structure on $S$ is then:

$$[J_a, J_b] = f_{ab}^c J_c, \quad [J_a, A_b] = f_{ab}^c A_c, \quad \mu(A_a) = J_a,$$

where $f_{ab}^c$ are the structure constants of $g_0$. The second ingredient to construct an $F-$Lie algebra is to define a symmetric invariant form on $g_1$. But on $g_1$, the adjoint representation of $g_0$, the invariant symmetric forms are well known and correspond to the Casimir operators \cite{7}. Then, considering a Casimir operator of order $m$ of $g_1 \cong g_0$, we can induce the structure of an $F-$Lie algebra of order $m+1$ on $S_{m+1} = g_0 \oplus g_1$. One can give explicit formulae for the bracket of these $F-$Lie algebras as follows. Let $h_{a_1 \cdots a_m}$ be a Casimir operator of order $m$ (for $m = 2$, the Killing form $g_{ab} = \text{Tr}(A_a A_b)$ is a primitive Casimir of order two). Then, the $F-$bracket of the $F-$Lie algebra is

$$\{A_{a_1}, A_{a_2}, \cdots, A_{a_{m+1}}\} = \sum_{\ell=1}^{m+1} h_{a_1 \cdots a_{\ell-1} a_{\ell+1} \cdots a_{m+1}} J_{a_\ell}$$

For the Killing form this gives

$$\{A_a, A_b, A_c\} = g_{ab} J_c + g_{ac} J_b + g_{bc} J_a.$$ \hspace{1cm} (5)

If $g_0 = sl(2)$, the $F-$Lie algebra of order three induced from the Killing form is the $F-$Lie algebra of \cite{8}.

3.2. Finite dimensional $F-$Lie algebras associated to Lie superalgebras

The construction of $F-$Lie algebras associated to Lie superalgebras is more involved. We just give here a simple example (for more details see \cite{8}): the $F-$Lie algebra of order 4 $S = g_0 \oplus g_1$ induced from the (i) Lie superalgebra $osp(2|2m) = (so(2) \oplus sp(2m)) \oplus \mathbb{C}^2 \otimes \mathbb{C}^{2m}$, and (ii) the quadratic form $\varepsilon \otimes \Omega$, where $\varepsilon$ is the invariant symplectic form on $\mathbb{C}^2$ and $\Omega$ the invariant symplectic form on $\mathbb{C}^{2m}$. Let $\{S_{\alpha \beta} = S_{\beta \alpha}\}_{1 \leq \alpha \leq 2m \atop 1 \leq \beta \leq 2m}$ be a basis of $sp(2m)$ and $\{h\}$ be a basis of $so(2)$. Let $\{F_{q \alpha}\}_{q = -1 \atop 1 \leq \alpha \leq 2m}$ be a basis of $\mathbb{C}^2 \otimes \mathbb{C}^{2m}$. Then the four brackets of $S$ take the following form

$$\{F_{q_1 \alpha_1}, F_{q_2 \alpha_2}, F_{q_3 \alpha_3}, F_{q_4 \alpha_4}\} = \varepsilon_{q_1 q_3} \Omega_{\alpha_1 \alpha_3} \left(\delta_{q_2 + q_4} S_{\alpha_2 \alpha_4} + \varepsilon_{q_2 q_4} \Omega_{\alpha_2 \alpha_4} h\right) + \text{perm.}$$

(6)

It is interesting to notice that this $F-$Lie algebra admits a simple matrix representation \cite{8}: $g_0 = \left\{ \begin{pmatrix} q & 0 & 0 \\ 0 & -q & 0 \\ 0 & 0 & S \end{pmatrix}, q \in \mathbb{C}, S \in sp(2n) \right\} \cong so(2) \oplus sp(2n)$ and $g_1 = \left\{ \begin{pmatrix} 0 & 0 & F_+ \\ 0 & 0 & F_- \\ -\Omega F^t & i\Omega F^t & 0 \end{pmatrix}, F_\pm \in \mathcal{M}_{1,2n}(\mathbb{C}) \right\}$. 


4. Finite-dimensional FSUSY extensions of the Poincaré algebra

It is well known that supersymmetric extensions of the Poincaré algebra can be obtained by Inönü-Wigner contraction of certain Lie superalgebras. In fact, one can also obtain some FSUSY extensions of the Poincaré algebra by Inönü-Wigner contraction of certain \( F \)-Lie algebras as we now show with one example [3]. Let \( S_3 = sp(4) \oplus \text{ad} \, sp(4) \) be the real \( F \)-Lie algebra of order three induced from the real graded \( 1 \)-Lie algebra \( S_1 = sp(4) \oplus \text{ad} \, sp(4) \) and the Killing form on \( \text{ad} \, sp(4) \) (see eq. 5). Using vector indices of \( so(1,3) \) coming from \( so(1,3) \subset so(2,3) \cong sp(4) \), the bosonic part of \( S_3 \) is generated by \( M_{\mu \nu}, M_4 \), with \( \mu, \nu = 0, 1, 2, 3 \) and the graded part by \( J_{\mu \nu}, J_4 \). Letting \( \lambda \to 0 \) after the Inönü-Wigner contraction,

\[
M_{\mu \nu} \to L_{\mu \nu}, \quad M_4 \to \frac{1}{\lambda} P_\mu \\
J_{\mu \nu} \to \frac{1}{\sqrt{\lambda}} Q_{\mu \nu}, \quad J_4 \to \frac{1}{\sqrt{\lambda}} Q_\mu,
\]

(7)

one sees that \( L_{\mu \nu} \) and \( P_\mu \) generate the \((1+3)D\) Poincaré algebra and that \( Q_{\mu \nu}, Q_\mu \) are the fractional supercharges in respectively the adjoint and vector representations of \( so(1,3) \). This \( F \)-Lie algebra of order three is therefore a non-trivial extension of the Poincaré algebra where translations are cubes of more fundamental generators. The subspace generated by \( L_{\mu \nu}, P_\mu, Q_\mu \) is also an \( F \)-Lie algebra of order three extending the Poincaré algebra in which the trilinear symmetric brackets have the simple form:

\[
\{ Q_\mu, Q_\nu, Q_\rho \} = \eta_{\mu \nu} P_\rho + \eta_{\mu \rho} P_\nu + \eta_{\nu \rho} P_\mu,
\]

(8)

where \( \eta_{\mu \nu} \) is the Minkowski metric.

5. Conclusion

In this paper a sketch of the construction of \( F \)-Lie algebras associated to Lie (super)algebras were given. More complete results, such as a criteria for simplicity, representation theory, matrix realizations etc., was given in [3].

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