Multi-horizon and Critical Behavior in Gravitational Collapse of Massless Scalar

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This paper studies the whole process of gravitational collapse and accretion of a massless scalar field in asymptotically flat spacetime. Two kinds of initial configurations are considered. One is the initial data without black hole, the other contains a black hole. Under suitable initial conditions, we find that multi-horizon will appear, which means that the initial black hole formed by gravitational collapse or existing at the beginning will instantly expand and suddenly grow, rather than grows gradually in the accretion process. A new type of critical behavior is found around the instant expansion. The numerical computation shows that the critical exponents are universal.

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Introduction.— In 1993, Choptuik [1] found a quite interesting critical phenomenon in gravitational collapse of a massless scalar field in spherically symmetric, asymptotically flat spacetime with a one-parameter family of initial data parameterized by $\epsilon$. He found that there exists a critical value $\epsilon_{\ast}$, with a discrete self-similar solution, when the initial data parameter $\epsilon$ is less than the critical one, the scalar field disperses into infinity and no black hole forms; when $\epsilon$ is larger than the critical value, a black hole forms, its mass is proportional to the difference $\epsilon - \epsilon_{\ast}$ with a universal critical exponent. Since then, similar critical behaviors have been found in various systems. For a review, one can see Ref. [2]. In recent years, the gravitational collapse in the asymptotic anti-de Sitter (AdS) space-time has attracted a lot of attentions as it shows a so-called “weak turbulence” effect [3] and gapped critical behavior [4]. All these critical phenomena are related to the formation of black hole and are theoretically related to the stability of the non-black-hole states [5].

Once a black hole forms, it will accrete matters surrounding the black hole. Accretion is an ubiquitous phenomenon in our universe. The accretion will make black hole grow. Larger the accretion flow is, faster the black hole grows. One interesting question is whether the black hole always grows gradually or it is possible the accretion can make the black hole expand instantly. In this paper, we will show that the instant expansion of a black hole due to accretion may happen. Under suitable conditions, the instant expansion behavior may happen several times. And more we find that the critical behavior appears near the moment where the instant expansion just takes place. The critical behavior can be expressed as a power law as the one found by Choptuik [1]. Interestingly, the critical exponent is universal in the sense that it is the same for different configurations of initial data considered in this paper, the same for different parameters involved in the physical process, the same for different instant expansion events along a single evolution.

In particular, let us stress here that in the gravitational collapse of matter fields, when an apparent horizon forms, some matters are still outside the apparent horizon. Some of these matters will be accreted into the black hole, some will disperse into infinity if there is no reflecting wall outside the black hole. However, in the super-critical cases considered in Choptuik’s study [1] and following works, once an apparent horizon forms, the evolution is stopped due to the numerical calculation limit. In this sense the critical behavior found by Choptuik is just the first part of the whole story of gravitational collapse. In this paper, with our numerical code, we are able to study the whole process of gravitational collapse. Although the final state is a stationary black hole, the process is found rather rich unexpectedly. Multiple horizons and trapped regions will form in the gravitational collapse process. With the new horizons, we find a new critical behavior in the gravitational collapse. This critical behavior is the second part of the story of gravitational collapse. And interestingly we find that the critical exponents of these two behaviors are the same. Following [6] we can define several mutual related critical exponents for the new critical phenomenon. It is shown that these critical exponents satisfy the universal scaling laws proposed in [6].

Physical Model and Numerical Method.— The physical model we are considering is the gravitational collapse of a real massless scalar field in 4-dimensional asymptotically flat space-time. The dynamics of the model is governed by Einstein-Klein-Gordon equations,

$$G_{\mu\nu} = 8\pi G [\partial_\mu \Psi \partial_\nu \Psi - \frac{1}{2} g_{\mu\nu} (\partial \Psi)^2], \quad (1)$$

$$\nabla^2 \Psi = 0, \quad (2)$$

where $\Psi$ is the massless scalar field and $G_{\mu\nu}$ is the Einstein tensor. In this paper, we will take the units with $G = 1$, and to simplify the numerical computation, we impose the spherical symmetry. The usual numerical methods in Refs. [1, 3, 6–8] and relevant papers are based on the simple spherical coordinates, which fail to
simulate the dynamics once an apparent horizon appears because of the coordinate singularity, so that the later process of gravitational collapse after the apparent horizon appears is still lack of well investigation. In order to see the whole process of gravitation collapse, we will solve the Einstein-Klein-Gordon equations (1) through the Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formulation [9, 10]. The original BSSN approach is not suitable for non-Cartesian coordinates. This issue can be remedied by introducing a covariant version of the BSSN equations [11]. This covariant BSSN formulation has been applied to study the critical gravitational collapse in Ref. [12] and to find the Choptuik’s critical exponent and self-similarity in the sub-critical case. Here we will first briefly review the covariant BSSN formulation. For more details, one can refer to Ref. [13]. The formulation employs a 3+1 decomposition of the spacetime as,

$$ds^2 = -\alpha^2 dt^2 + \gamma_{rr}(\beta dt + dr)^2 + r^2 \gamma_{\theta\theta} d\Omega^2,$$

(3)

with 3-metric components $\gamma_{ij}$, lapse function $\alpha$ and shift vector $\beta$. Differing from standard ADM formulation which rewrites Einstein’s equation as evolution equations for the 3-metric and the extrinsic curvature $\{\gamma_{ij}, K_{ij}\}$, the BSSN formulation decomposes them into conformal rescaling ADM dynamical variables,

$$\gamma_{ij} = e^{4\psi}\tilde{\gamma}_{ij}, \quad K_{ij} = e^{4\phi}(\tilde{A}_{ij} + \tilde{\gamma}_{ij} K),$$

(4)

where $e^{4\phi}$ is the conformal factor to make $\det \tilde{\gamma}_{ij} = r^2 \sin \theta$, $\tilde{A}_{ij}$ is the conformally rescaled trace-free part of the extrinsic curvature and $K = \gamma^{ij} K_{ij}$ is the trace of extrinsic curvature. Another difference is the introduction of a conformal connection $\tilde{\Lambda} = [\Lambda, 0, 0]$, a true 3-vector, as an primary dynamical variable in the covariant BSSN formulation. This variable makes the resulted partial differential equations admit strong hyperbolicity [14]. Thus in the covariant BSSN formulation, the original dynamical variables in the ADM formulation are transformed into $\{\phi, \tilde{\gamma}_{ij}, \tilde{A}_{ij}, K, \Lambda\}$.

In order to treat the physical singularity introduced by a black hole, we use the moving puncture technique [15, 16]. Together with this numerical technique, the “moving puncture” coordinate conditions are used. This gauge condition includes the “1+log” condition for the lapse function $\alpha$ and a version of a Gamma-driver condition for $\beta$ [11, 17].

In the covariant BSSN formulation in the spherical coordinates, Eqs. (1) are rewritten as 9 evolution equations for gravity field $\Psi$ and its auxiliary variable $\xi = \partial_t \Psi - \beta \partial_t \Psi$. For solving the total 11 evolutional equations, we need to specify suitable boundary conditions at $r = 0$ and $r \to \infty$, respectively. For the $r = 0$ boundary, we distinguish the evolution variables into even ones and odd ones. The even variables include $\phi, \tilde{\gamma}_{rr}, \tilde{\gamma}_{\theta\theta}, \tilde{A}_{rr}, \tilde{A}_{\theta\theta}, K, \alpha, \Psi$ and $\xi$, while the odd variables are $\Lambda$ and $\beta$. Based on this parity classification, we use buffer point method to implement the boundary condition [18]. As for the $r \to \infty$ boundary we cut out the computation at some far position and implement a Sommerfeld boundary condition there [19]. We have confirmed through numerical test that when the boundary position is far enough, it will not influence the results presented in this paper.

In order to set up the initial data we have to solve two constraint equations (Hamiltonian and momentum). The matter field in this paper is initialized by Gaussian type distribution and zero initial velocity,

$$\Psi(0, r) = \epsilon \exp \left[ -\frac{(r - r_0)^2}{\sigma^2} \right], \quad \partial_t \Psi(0, r) = 0, \quad (5)$$

where $\epsilon, r_0$ and $\sigma$ are parameters with $\epsilon, \sigma > 0$. The conformal metric and lapse function are all initialized to unity, i.e., $\tilde{\gamma}_{rr}(0, r) = \tilde{\gamma}_{\theta\theta}(0, r) = \alpha(0, r) = 1$. We impose that the system has a time reversal symmetry, so that the extrinsic curvature, shift vector and conformal connection all vanish $\tilde{A}_{rr}(0, r) = \tilde{A}_{\theta\theta}(0, r) = K(0, r) = K(0, r) = 0$. Such a setting makes the momentum constraint be automatically satisfied. With the initial setup, it is now only the value of conformal factor unspecified. We obtain the initial value by solving the Hamiltonian constraint. We adopt the puncture idea [20] to set

$$e^{\phi(0, r)} = 1 + \frac{r_{h0}}{4r} + u(r), \quad (6)$$

and solve the regular part $u(r)$. Here $r_{h0}$ is the horizon radius of an initial black hole, which is taken to be zero if the initial configuration does not contain black hole.

Black Hole Horizon and Instant Expansion. – To locate the black hole in the spacetime, we need find horizon. We introduce two “cross normalized” null vectors $k_\alpha = \frac{1}{\sqrt{2}}[\sqrt{\gamma_{rr}} \beta - \alpha, \sqrt{\gamma_{rr}}, 0, 0]$ and $\ell_\alpha = -\frac{1}{\sqrt{2}}[\sqrt{\gamma_{rr}} \beta + \alpha, \sqrt{\gamma_{rr}}, 0, 0]$. Both null vectors are perpendicular to the symmetric sphere. And $k_\alpha$ is outward pointing and $\ell_\alpha$ is inward pointing. Let $\Theta_k$ and $\Theta_\ell$ be the expansions corresponding to $k_\alpha$ and $\ell_\alpha$, respectively. The apparent horizon (AH) is the 2-spherical surface with $\Theta_k = 0$. Note that when $\Theta_k = 0$, $\Theta_\ell = -\sqrt{2}a^{-1} \partial_r \ln(r \sqrt{\gamma_{\theta\theta}})$, which is always negative. This shows that the 2-surface AH is in fact always a marginally trapped surface and the collection of AHs forms a trapped horizon (TH). Based on this fact, the surface with $\Theta_k = 0$ is TH and its foliation at a moment is an AH.

Now we present our numerical results. We first consider the configuration with $r_{h0} = 0$, which means there is no black hole on the initial Cauchy surface. Take an example, we fix $r_0 = 10, \sigma = 1$ and take amplitude $\epsilon$ as the tuned parameter. For small value of $\epsilon$, the energy will dissipate to infinity and no black hole can form in small radius of the final state. There is a critical amplitude $\epsilon_0$, when $\epsilon > \epsilon_0$,
the black hole can form and in the case of $\epsilon \to \epsilon_0^+$, the initial apparent horizon radius (measured by the area $A$ of symmetric sphere rather than the coordinate radius $r$) $R_{AH} \equiv \sqrt{A/4\pi}$ and amplitude $\epsilon$ satisfy the relation,

$$R_{AH} \propto (\epsilon - \epsilon_0)^{\beta}$$

with $\beta \approx 0.38$. This is the critical behavior found by Choptuik [1]. In most of relevant works, the numerical simulation stops when the apparent horizon forms because of the numerical instability introduced by the singularity. In this paper the covariant BSSN formulation enables us to give the whole evolution of gravitational collapse. In the case with $\epsilon \gtrsim \epsilon_0$, we find that the apparent horizon will grow continuously and a final Schwarzschild black hole is left at last.

Interesting thing happens when we increase the amplitude to strong enough. We find that the black hole will expand instantly at some time. This instant expansion is the result that a new AH appears outside the original AH. Since we treat the outmost AH as the boundary of a black hole, the black hole looks instant expansion, namely the black hole grows suddenly. One numerical result of expansion $\Theta_k$ in the equatorial plane at different times is shown in Fig. 1. It clearly shows that, after the first AH appears, a new AH shows up outside the original AH at some time. Furthermore, this new AH splits into two AHs at some time late as shown in the bottom left panel of Fig. 1. There are two trapped regions (the yellow shell and the inner region of the innermost circle) and a “normal region” (the red shell) between them. With further evolution, the “normal region” will shrink and disappear, which corresponds to the time that the two inner AHs coincide with each other and they disappear. Finally only the outermost AH is left.

Unlike event horizon, the area of AH can increase or decrease during the gravitational collapse. As shown in Fig. 2, the area of AH will increase when $\mathcal{L}_\epsilon \Theta_k < 0$, while decrease when $\mathcal{L}_\epsilon \Theta_k > 0$. This can be understood as follows. In dynamical spacetime, AH can be the foliation of future outer trapped horizon (FOTH) for negative $\mathcal{L}_\epsilon \Theta_k$ or future inner trapped horizon (FITH) for positive $\mathcal{L}_\epsilon \Theta_k$. Since the massless scalar field satisfies the null energy condition, the area will increase for FOTHs but decrease for FITHs in the spherically symmetric spacetime [21, 22]. We verify this property numerically as an example shown in Fig. 2.

Now let us consider the case with the initial Cauchy surface containing a black hole. As expected, we find that instant expansion of black hole can also happen. At first glimpse, this conclusion can be naively deduced from the results for the case with no black hole in the initial configuration we just discussed above. Because once a black hole forms in gravitational collapse, some matter is still outside the black hole, the configuration can be viewed as the initial configuration with a black hole. But in detail there is a difference between these two initial configurations. For the configuration which does not contain a black hole at the beginning, the size of the formed black hole depends on the initial matter setting. Stronger the initial matter is, bigger the formed black hole is and fewer matter is left outside the black hole. But for the configuration that the initial data contains a black hole, the size of the black hole and the content of the matter can be tuned freely. More explicitly, the configuration which does not contain a black hole admits three parameters $\{\epsilon, r_0, \sigma\}$, while the configuration which contains a black hole initially admits four parameters $\{r_{h0}, \epsilon, r_0, \sigma\}$. Our numerical calculations show that instant expansion of the
black hole may happen when we adjust any one of the four parameters. And the behavior is similar to the one shown in Fig. 1.

Many times of instant expansions may happen during a given evolution process if the initial setting is suitably chosen. For example, if we continuously increase the amplitude $\epsilon$, then two times or three times instant expansions and multiple apparent horizons can appear. However, we cannot obtain infinite times instant expansion by increasing the amplitude in general, because new horizons may appear in the initial Cauchy surface. When we adjust other parameters, we get similar results.

**Critical Behavior due to Accretion.**—After the first apparent horizon appears (formed from the gravitational collapse or imposed in the initial data), by adjusting parameters involved in the initial configuration, a new horizon can appear. When the instant expansion just happens (e.g., the moment shown in the up-right panel of Fig. 1), we find that the area radius difference between the two apparent horizons $\Delta R_{AH}$ obeys a universal scaling law. Let $p$ stand for any tuning parameter in the initial configuration and $p_*$ denote the critical value that a new horizon can just appear when we fix the other parameters, the power law can be expressed as,

$$\Delta R_{AH} = a_p |p - p_*|^{\beta_p}, \text{ when } p \rightarrow p_*^+. \quad (8)$$

Here $a_p$ is some constant and $\beta_p$ is the critical exponent which is independent of the choice of $p$, independent of the initial configurations and independent of which instant expansion during the given evolution process. We have computed $a_p$ and $\beta_p$ for different tuning parameters in both two configurations. For the initial configuration which does not contain a black hole, we find a set of critical values $\{r_{ho}, \epsilon_*, \sigma_*\}$. After tuning any one of them but fixing others, we can obtain the results of $\Delta R_{AH}$. The results are shown in Fig. 3(a). On the other hand, for the initial configuration which contains a black hole, we also find a set of critical values $\{r_{ho}, \epsilon_*, \sigma_*\}$. A similar relation between $\Delta R_{AH}$ and tuned parameters is shown in Fig. 3(b). For both configurations, we find an universal exponent $\beta_p \approx 0.38$. Besides the first instant expansion, for the case that the two times or three times instant expansion can happen during the accretion, we also find a similar power law behavior for the first horizon and its nearest horizon when the outmost horizon just appears. The critical exponents are the same up to the numerical error.

As a self-consistency check, we compute the ‘susceptibility’ between any two tuning parameters $p$ and $p'$ as $\chi_{pp'} = (\partial \delta p' / \partial p)_{\Delta R_{AH} = 0} = c_{p'p} |p - p_*|^{|\beta_{p'}|}$. We find $c_{p'p} \approx 0$ and exponents and coefficients in Eq. (8) satisfy following two equations proposed in Ref. [6],

$$\beta_p (\delta_{pp'} + 1) \approx \beta_{p'}, \quad a_p |c_{p'p}|^{\beta_p} / a_{p'} \approx 1. \quad (9)$$

Although our new scaling behavior Eq. (8) looks similar to the one found by Choptuik, there is an essential difference between them. In the Choptuik’s case, the scalar curvature diverges at the critical point. However, the new critical behavior found here happens in a regular region, where all the geometrical quantities are finite. In other words, the Choptuik’s scaling law is applicable for a small black hole case, while ours is also valid for a finite mass black hole.

**Discussion.**—Our computations in this paper are done in the asymptotically flat spacetime, it can be straightforwardly generalized to the asymptotic AdS case in principle. Due to the “weak turbulence” effect and the existence of the reflection boundary of AdS space, we expect that rich structures in the inner region of black hole and criticality between multiple horizons can appear in the gravitational collapse in AdS spacetime.

Note that the Choptuik’s critical exponent can be obtained by analyzing the Lyapunov’s exponents and renormalization group of the system [23], it is interesting to consider if the scaling behaviors in Eq. (8) can also be obtained in a similar way. As we have mentioned, since there is an essential difference between this case and Choptuik’s case, such a work will not be a trivial generalization. Numerically, we find that critical exponents $\beta_n$ here is roughly the same as the Choptuik’s critical exponent [12]. Do they really take the same value? If yes, why do such different critical behaviors have the same

![FIG. 3: The universal power law (8) due to the critical accretion for the first instant expansion during the given evolution process. (a) For the initial configuration which does not contain black hole. (b) For the initial configuration which contains a black hole. Here $d_i$ are imposed so that the intercepts coincide with each other.](image-url)
critical exponent? We hope all these interesting questions could be addressed in future.

To conclude, we have numerically investigated the accretion process of scalar field by a black hole which is formed through gravitational collapse or exists at the beginning. With the covariant BSSN formulation, we are able to study the whole process of the gravitational collapse. For both configurations, we have found that the black hole will undergo instant expansion during the accretion under very wide conditions. The appearance of the new apparent horizons is due to the weight enough of the mater outside the original horizon to form new horizon. A new universal power law behavior was found in the instant expansions. The critical exponent is the same not only for different one-parameter initial configurations but also for all the instant expansions. Since accretion phenomenon is quite ubiquitous in the universe, it would be quite interesting to see whether the existence of the intermediate and supermassive black holes is related to the critical behavior found in this paper.

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