Spectral Properties of Supersymmetric Dirac-Hamiltonians in \((1 + 1)\) Dimensions

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(Dated: 4 November 2019)

Spectral properties of most general supersymmetric Dirac Hamiltonians in \((1 + 1)\) dimensions are studied. It is shown that their spectral properties are represented by those of the associated non-relativistic Witten model. Besides the general discussion we also present closed form expressions for Green function of the relativistic dirac oscillator. The supersymmetric quasi-classical approximation for the Witten model is extended to the associated relativistic model.

PACS numbers: 03.65.Pm Relativistic wave equations, 11.30.Pb Supersymmetry
Keywords: Dirac Equation, Green’s Function, Supersymmetry

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I. INTRODUCTION

Dirac’s well-known equation\(^1\)\(^2\) characterises the relativistic dynamics of spin-\(\frac{1}{2}\) particles in the framework of quantum mechanics respecting also the principles of special relativity\(^3\). This equation has been very successful in its early days by providing a clear formalism for the spin of a electron as a point particle and also lead to the postulation of the existence of its anti-particle which was discovered shortly afterwards reassuring Dirac’s interpretation\(^4\). The Dirac equation also paved the way for QED and the quantum field theory theory of electromagnetic interactions. Nowadays, the Dirac equation is also an important tool for the description of the dynamics of charge carriers in carbon nano-structures like graphene\(^5\) or more general in so-called Dirac electronic systems\(^6\). Hence exact solutions of the Dirac equation are of great interest but are rare compared to its non-relativistic counterpart, the Schrödinger equation. Here various techniques for finding exact solutions like the factorisation methods have been developed during the last century and received more recently much attention in the context supersymmetric methods\(^7\)\(^8\). Also approximations methods like the WKB approach or perturbation methods are nowadays well-established for the non-relativistic quantum mechanics.

As the Dirac Hamiltonian with supersymmetry is known to be closely related to a non-relativistic Pauli-Schrödinger type Hamiltonian\(^9\), it is natural to investigate the possibility to derive exact solution as well as approximation for a Dirac system by reducing this problem to a non-relativistic system. Whereas in the recent work\(^9\) we focused on (3 + 1)-dimensional system, we will limit ourselves in the current paper to supersymmetric Dirac Hamiltonians in (1+1) dimensions. We will show in the next section that supersymmetry puts a very strict condition on the most general Dirac Hamiltonian in (1+1) dimensions. The associated non-relativistic system is the well-studied Witten model of supersymmetric quantum mechanics\(^8\). After recalling some basic properties of supersymmetric Dirac Hamiltonians and the Witten model, we show in section 3 how the eigenvalue problem of the Dirac system can be reduced to that of the Witten model. As an explicit example the Dirac oscillator is considered and its spectral properties are derived form those of the standard harmonic oscillator on the real line. In section 4 we will study the resolvent of the Dirac system and show how this can be expressed in terms of the resolvent of the non-relativistic Witten model. Again the Dirac oscillator is chosen as an explicit example and the corresponding Green’s function is derived.
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In section 5 we will utilise the quasi-classical approximations of the Witten model to arrive at quasi-classical approximations of supersymmetric Dirac Hamiltonians. This approximation is known to respect the spectral symmetries implied by supersymmetry and this property is even respected by the derived approximation formulas of the Dirac system. Finally in the conclusions we give an outlook how that current approach can be applied to radial Dirac Hamiltonians.

II. SUPERSYMMETRIC DIRAC HAMILTONIANS IN (1 + 1) DIMENSIONS

The most general (1 + 1)-dimensional Dirac Hamiltonian, acting on Hilbert space $\mathcal{H} = L^2(\mathbb{R}) \otimes \mathbb{C}^2$, can be put into the form

$$H_D = cp\sigma_1 + W(x)\sigma_2 + [mc^2 + S(x)]\sigma_3 + eV(x)\mathbf{1}, \quad (2.1)$$

where $x$ and $p = (\hbar/i)\partial_x$ are the position and momentum operators on $L^2(\mathbb{R})$, respectively, $\{\sigma_i| i = 1, 2, 3\}$ are the Pauli matrices and $\mathbf{1}$ is the $2 \times 2$ unit matrix on $\mathbb{C}^2$. In the above $m > 0$ and $e$ stand for the mass and the charge of the Dirac particle moving along the real line $\mathbb{R}$ and $c > 0$ represents the speed of light. This particle interacts with various potentials, namely a scalar potential $S$, a pseudo-scalar potential $W$ (in $3 + 1$ dimensional models this is also called a tensor potential) and an electro-static potential $V$, which is the 0-component of an electromagnetic vector potential $(V, A)$, the 1-component of which has been omitted as it can be gauged away due to the $u(1)$ gauge-invariance.

Representing the Pauli matrices in the standard form

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.2)$$

the above Dirac Hamiltonian can be put into supersymmetric matrix form

$$H_D = \begin{pmatrix} M_+ & A \\ A^\dagger & -M_- \end{pmatrix}, \quad (2.3)$$

where

$$M_\pm := mc^2 + S(x) \pm eV(x) \quad \text{and} \quad A := cp - iW(x). \quad (2.4)$$
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In order to represent a supersymmetric Dirac Hamiltonian these operators are required to obey the conditions

\[ AM_- = M_+ A. \] (2.5)

This condition turns out to be very restrictive and leads to constraints on the potentials with which the Dirac particle interacts. To be more explicit the constraints are \( V(x) = 0 \) and \( S(x) = \text{const.} \), that is, the electro-static potential must vanish and the scalar potential has to be constant. Without loss of generality, a constant \( S \) can always be absorbed by the mass term \( mc^2 \), we also set \( S = 0 \). Hence, the most general supersymmetric Dirac Hamiltonian in \((1 + 1)\) dimensions is of the form

\[
H_D = \begin{pmatrix}
mc^2 & cp - iW(x) \\
cp + iW(x) & -mc^2
\end{pmatrix}.
\] (2.6)

That is, it is of the supersymmetric form (2.3) with

\[
M_\pm = mc^2 \quad \text{and} \quad A = cp - iW(x).
\] (2.7)

The \( N = 2 \) SUSY structure becomes explicit by introducing the supersymmetric Hamiltonian

\[
H_{\text{SUSY}} := \frac{1}{2mc^2} (H_D^2 - m^2 c^4) = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}
\] (2.8)

with partner Hamiltonians

\[
H_+ := \frac{1}{2mc^2} AA^\dagger, \quad H_- := \frac{1}{2mc^2} A^\dagger A
\] (2.9)

and the SUSY charges

\[
Q := \frac{1}{2mc^2} \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \quad Q^\dagger = \frac{1}{2mc^2} \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix}.
\] (2.10)

These operators obey the \( N = 2 \) SUSY algebra

\[
H_{\text{SUSY}} = \{Q, Q^\dagger\}, \quad Q^2 = 0 = (Q^\dagger)^2, \quad \{Q, \sigma_3\} = 0,
\] (2.11)

where \( \sigma_3 \) plays the role of the grading (or Witten) operator. In fact, the partner Hamiltonians have the explicit form

\[
H_\pm = \frac{p^2}{2m} + \Phi^2(x) \pm \frac{\hbar}{\sqrt{2m}} \Phi'(x),
\] (2.12)
which is the well-known one-dimensional non-relativistic Witten model of supersymmetric quantum mechanics. Note that in the above we have rescaled the pseudo-scalar potential $W(x) =: \sqrt{2mc^2} \Phi(x)$, where $\Phi$ may now be identified with the SUSY potential of the Witten model.

This Witten model has been studied extensively in the last 25 years and finds many applications in various fields of physics. Here let us summarize the most essential properties of this model. Obviously, both partner Hamiltonians are non-negative and furthermore are essential isospectral, that means, there strictly positive eigenvalues are identical. Let us assume that both have a pure discrete spectrum represented by the eigenvalues $\varepsilon_n$ with associated eigenstates $\phi_n^\pm$, we have

$$H_n^\pm \phi_n^\pm = \varepsilon_n \phi_n^\pm, \quad \varepsilon_n > 0, \quad n = 1, 2, 3, \ldots.$$ (2.13)

In addition, in case of an unbroken SUSY, there exists a zero energy ground state, which for convenience is then assumed to belong to $H_-$. This can always be achieved with a change of sign in the SUSY potential. Hence in the case of unbroken SUSY we have the ground state $\phi_n^-$ associated with $\varepsilon_0 = 0$ defined via $A\phi_0^- = 0$ leading to the explicit form

$$\phi_0^-(x) = N \exp \left\{ -\frac{\sqrt{2m}}{\hbar} \int dx \Phi(x) \right\} = N \exp \left\{ -\frac{1}{\hbar c} \int dx W(x) \right\}$$ (2.14)

with $N$ denoting a normalisation constant. The eigenstates belonging to the strictly positive spectrum are related to each other via the SUSY transformations

$$A\phi_n^- = \sqrt{2mc^2} \varepsilon_n \phi_n^+, \quad A^\dagger \phi_n^+ = \sqrt{2mc^2} \varepsilon_n \phi_n^-.$$ (2.15)

**III. SPECTRAL PROPERTIES OF THE HAMILTONIAN**

In this section we will study the spectral properties of the most general supersymmetric Dirac Hamiltonian in $(1 + 1)$ dimensions, which is fully characterized by a pseudo-scalar potential as shown in the previous section, cf. eq. (2.6). Such supersymmetric Dirac Hamiltonians are known to be block-diagonalizable via a unitary transformation separating positive and negative energy eigenspaces. Indeed, it is possible to show that there exists a unitary matrix $U$, see for example ref. 8, which transforms the Dirac Hamiltonian (2.6) into the form

$$\tilde{H}_D := UH_DU^\dagger = \begin{pmatrix} \sqrt{2mc^2}H_+ + m^2c^4 & 0 \\ 0 & -\sqrt{2mc^2}H_- + m^2c^4 \end{pmatrix}. \quad (3.1)$$
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Hence the positive and negative energy spectrum of \(H_D\) is fully determined by that of \(H_+\) and \(H_-\), respectively. As both partner Hamiltonians are essential isospectral, the spectrum of \(H_D\) is in fact symmetric about the origin,

\[
E^\pm_n = \pm \sqrt{2mc^2E_n + m^2c^4}, \quad n > 0.
\]  

In addition, in the case of an unbroken SUSY the eigenvalue \(E_0^- = -mc^2\) belongs to the spectrum of \(H_D\). The corresponding eigenstates are given by

\[
\psi^\pm_n = U^\dagger \tilde{\psi}^\pm_n, \quad \text{with} \quad \tilde{\psi}^+ = \begin{pmatrix} \phi^+_n \\ 0 \end{pmatrix}, \quad \tilde{\psi}^- = \begin{pmatrix} 0 \\ \phi^-_n \end{pmatrix}.
\]  

(3.3)

In other words the spectral properties of \(H_D\) are fully characterized by those of \(H_{\text{SUSY}}\):

\[
H_D\psi^\pm_n = E^\pm_n \psi^\pm_n.
\]  

(3.4)

In case of unbroken SUSY in addition we have

\[
H_D\tilde{\psi}_0^- = -mc^2\psi^-_0, \quad \tilde{\psi}_0^- = U^\dagger \begin{pmatrix} 0 \\ \phi^-_0 \end{pmatrix} = \begin{pmatrix} 0 \\ \phi^-_0 \end{pmatrix}.
\]  

(3.5)

Note that \(U = 1\) on \(\text{ker} \ H^-\) as will be shown below.

Let us now study the unitary transformation matrix \(U\) in more detail. According to the general SUSY approach\(^8\) this matrix is given by

\[
U := a_+ + \sigma_3 \text{sgn} Q_1 a_- \quad \text{with} \quad a_\pm := \sqrt{\frac{|H_D| \pm mc^2}{2|H_D|}}.
\]  

(3.6)

Here the self-adjoint supercharge \(Q_1\) is defined as follows

\[
Q_1 := \sqrt{2mc^2} \left( Q + Q^\dagger \right) = \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix}
\]  

(3.7)

and the definition of the sign function is

\[
\text{sgn} x := \begin{cases} +1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}.
\]  

(3.8)

To make things a bit more explicit let us first look into the spectral properties of \(Q_1\). It is straightforward to show with the help of the SUSY transformations\(^{(2.15)}\) that the states

\[
\chi^\pm_n := \frac{1}{\sqrt{2}} \begin{pmatrix} \phi^+_n \\ \pm \phi^-_n \end{pmatrix}
\]  

(3.9)
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are eigenstates of \(Q_1\), that is,

\[
Q_1 \chi_n^\pm = \pm \sqrt{2mc^2 \varepsilon_n} \chi_n^\pm .
\] (3.10)

In addition, for unbroken SUSY, \(Q_1\) has a zero eigenvalue as

\[
Q_1 \chi_n^- = 0 \quad \text{for} \quad \chi_0^- := \begin{pmatrix} 0 \\ \phi_0^- \end{pmatrix} ,
\] (3.11)

which implies \(U = 1\) on the \(\ker H_- = \text{span} \left| \phi_0^- \right\rangle \left\langle \phi_0^- \right|\). Having this in mind it is obvious that the operator \(\text{sgn} Q_1\) can explicitly be written as

\[
\text{sgn} Q_1 = \begin{pmatrix} 0 & (AA^\dagger)^{-1/2} A \\ (A^\dagger A)^{-1/2} A^\dagger & 0 \end{pmatrix} .
\] (3.12)

Note that \(Q_1 \chi_n^\pm = \pm \chi_n^\pm\).

Together with the diagonal matrix

\[
|H_D| = \begin{pmatrix} \sqrt{AA^\dagger + m^2c^4} & 0 \\ 0 & \sqrt{A^\dagger A + m^2c^4} \end{pmatrix}
\] (3.13)

we have with the dimension less operator \(a := A/mc^2\)

\[
a_\pm = \begin{pmatrix} \sqrt{\frac{aa^\dagger + 1}{2\sqrt{aa^\dagger + 1}}} & 0 \\ 0 & \sqrt{\frac{a^\dagger a + 1}{2\sqrt{a^\dagger a + 1}}} \end{pmatrix}
\] (3.14)

leading to the explicit but rather complicated expression

\[
U = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{aa^\dagger + 1}{\sqrt{aa^\dagger + 1}}} & (aa^\dagger)^{-1/2} a \sqrt{\frac{a^\dagger a + 1}{a^\dagger a + 1}} \\ -(a^\dagger a)^{-1/2} a^\dagger \sqrt{\frac{aa^\dagger + 1}{\sqrt{aa^\dagger + 1}}} & \sqrt{\frac{a^\dagger a + 1}{\sqrt{a^\dagger a + 1}}} \end{pmatrix} .
\] (3.15)

Noting that

\[
a \phi^-_n = \sqrt{\frac{2\varepsilon_n}{mc^2}} \phi^+_n , \quad a^\dagger \phi^+_n = \sqrt{\frac{2\varepsilon_n}{mc^2}} \phi^-_n ,
\] (3.16)

which implies

\[
a^\dagger (aa^\dagger)^{-1/2} \phi^+_n = \phi^-_n , \quad a(a^\dagger a)^{-1/2} \phi^-_n = \phi^+_n ,
\] (3.17)
we obtain explicit expression for the eigenfunctions [3.3] of the original Dirac Hamiltonian (2.6) in terms of the eigenfunctions of the corresponding non-relativistic Witten model

$$\psi^+_n = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1 + (1 + 2\varepsilon_n/mc^2)^{-1/2}} \phi^+_n \\ \sqrt{1 - (1 + 2\varepsilon_n/mc^2)^{-1/2}} \phi^-_n \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1 + \frac{mc^2}{E_n}} \phi^+_n \\ \sqrt{1 - \frac{mc^2}{E_n}} \phi^-_n \end{pmatrix},$$

$$\psi^-_n = \frac{1}{\sqrt{2}} \begin{pmatrix} -\sqrt{1 - (1 + 2\varepsilon_n/mc^2)^{-1/2}} \phi^+_n \\ \sqrt{1 + (1 + 2\varepsilon_n/mc^2)^{-1/2}} \phi^-_n \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -\sqrt{1 - \frac{mc^2}{E_n}} \phi^+_n \\ \sqrt{1 - \frac{mc^2}{E_n}} \phi^-_n \end{pmatrix},$$

$$\psi^-_0 = \begin{pmatrix} 0 \\ \phi^-_0 \end{pmatrix}.$$  

Hence, we have reduced the eigenvalue problem of a general supersymmetric Dirac Hamiltonian (2.6) to that of the associated non-relativistic Witten model (2.12).

A. The Dirac oscillator in (1 + 1) dimensions

As an illustrative example let us consider the (1 + 1)-dimensional version of the Dirac oscillator which is characterized by the pseudo-scalar potential

$$W(x) = mc\omega x, \quad \omega > 0.$$  

(3.19)

Obviously the two partner Hamiltonians are represented by a shifted harmonic oscillator

$$H_\pm = \frac{p^2}{2m} + \frac{m}{2}\omega^2 x^2 \pm \hbar \omega \frac{\hbar}{2}.$$  

(3.20)

The corresponding spectral properties of these Hamiltonians are well known and given by

$$\varepsilon_n = \hbar \omega n, \quad \phi^+_n = \ket{n - 1}, \quad \phi^-_n = \ket{n}. \quad \text{ (3.21)}$$

Here $n \in \mathbb{N}$ for $H_+$, whereas $n \in \mathbb{N}_0$ for $H_-$, and $\ket{n}$ denotes a standard harmonic oscillator eigenstate which obeys the relations

$$b\ket{n} = \sqrt{n}\ket{n - 1}, \quad b^\dagger\ket{n} = \sqrt{n + 1}\ket{n + 1},$$  

(3.22)

where $b := ip/m\omega + x$ and $b^\dagger = -ip/m\omega + x$ are the standard harmonic oscillator annihilation and creation operators. They are related to the dimensionless operators $a$ and $a^\dagger$ introduced above as follows:

$$a = -i\sqrt{\frac{2\hbar \omega}{mc^2}} b, \quad a^\dagger = i\sqrt{\frac{2\hbar \omega}{mc^2}} b^\dagger.$$  

(3.23)
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Hence via (3.2) and (3.18) we immediately find that the eigenvalues and eigenstates of the Dirac oscillator Hamiltonian

\[
H_D = \begin{pmatrix} mc^2 & cp - imc\omega x \\ cp + imc\omega x & -mc^2 \end{pmatrix}
\]

are given by

\[
E^+_n = mc^2 \sqrt{1 + 2n \frac{\hbar \omega}{mc^2}}, \quad \psi^+_n = \begin{pmatrix} \sqrt{1 + (1 + 2n \frac{\hbar \omega}{mc^2})^{-1/2}} |n - 1\rangle \\ \sqrt{1 - (1 + 2n \frac{\hbar \omega}{mc^2})^{-1/2}} |n\rangle \end{pmatrix},
\]

\[
E^-_n = -mc^2 \sqrt{1 + 2n \frac{\hbar \omega}{mc^2}}, \quad \psi^-_n = \begin{pmatrix} -\sqrt{1 - (1 + 2n \frac{\hbar \omega}{mc^2})^{-1/2}} |n - 1\rangle \\ \sqrt{1 + (1 + 2n \frac{\hbar \omega}{mc^2})^{-1/2}} |n\rangle \end{pmatrix},
\]

where \(n = 0\) is only allowed for the second line with \(\psi^-_0\) having only a lower component, cf. eq. (3.18).

IV. SPECTRAL PROPERTIES OF THE RESOLVENT

In this section we will study the resolvent of a general supersymmetric Dirac Hamiltonian defined by

\[
G_D(z) := \frac{1}{H_D - z}, \quad z \in \mathbb{C}\setminus\text{spec } H_D
\]

which can be expressed in terms of the so-called iterated resolvent \(g\) as follows

\[
G_D(z) = (H_D + z) g(z^2), \quad g(z^2) := \frac{1}{H_D^2 - z^2}.
\]

As \(H_D^2\) is block-diagonal,

\[
H_D^2 = 2mc^2 \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} + m^2c^4,
\]

so is

\[
g(z^2) = \frac{1}{2mc^2} \begin{pmatrix} G_+(\zeta(z)) & 0 \\ 0 & G_-(\zeta(z)) \end{pmatrix},
\]

where

\[
G_{\pm}(\zeta) := \frac{1}{H_{\pm} - \zeta}
\]
denotes the resolvent for the SUSY partner Hamiltonians \((2.12)\) and \(\zeta(z) := \frac{z^2}{2mc^2} - \frac{mc^2}{2}\).

Hence we can express the resolvent \((4.1)\) in terms of the resolvents \((4.5)\) as follows

\[
G_D(z) = \frac{1}{2mc^2} \begin{pmatrix}
(z + mc^2)G_+(\zeta(z)) & AG_-(\zeta(z)) \\
A^\dagger G_+(\zeta(z)) & (z - mc^2)G_-(\zeta(z))
\end{pmatrix}.
\]  

(4.6)

Utilising the spectral representations

\[
G_{\pm}(\zeta) = \sum_{\varepsilon_n \geq 0} \frac{|\phi_n^\pm\rangle\langle\phi_n^\pm|}{\varepsilon_n - \zeta},
\]

(4.7)

where the \(\varepsilon_0 = 0\) term is only present in \(G_-\) in case of an unbroken SUSY, we arrive with the help of the SUSY transformations \((2.15)\) at the spectral representation of \((4.1)\)

\[
G_D(z) = \left( \begin{array}{cc}
0 & 0 \\
0 & \frac{|\phi_0^+\rangle\langle\phi_0^-|}{z + mc^2}
\end{array} \right)
+ \sum_{\varepsilon_n > 0} \frac{1}{2mc^2\varepsilon_n + m^2c^4 - z^2} \begin{pmatrix}
(z + mc^2)|\phi_n^+\rangle\langle\phi_n^-| & \sqrt{2mc^2\varepsilon_n}|\phi_n^+\rangle\langle\phi_n^-| \\
\sqrt{2mc^2\varepsilon_n}|\phi_n^-\rangle\langle\phi_n^+| & (z - mc^2)|\phi_n^-\rangle\langle\phi_n^+|
\end{pmatrix}.
\]

(4.8)

Again the first term, which has a pole at \(E_0 = -mc^2\), is only present in case of an unbroken SUSY. The poles of the second term reflect the energy eigenvalues as given in \((3.2)\) as expected.

A. The Green’s function of the Dirac oscillator

Let us reconsider the Dirac oscillator in \((1 + 1)\) dimensions. As we have seen in the previous section the partner Hamiltonians of the Dirac oscillator are related to the standard harmonic oscillator Hamiltonian

\[
H_0 := \frac{\mathbf{p}^2}{2m} + \frac{m}{2}\omega^2x^2
\]

(4.9)

via constants shifts given by \(H_{\pm} = H_0 \pm \hbar\omega/2\). Hence the two partner resolvents \((4.1)\) can be obtained from the usual harmonic oscillator resolvent

\[
G_0(\zeta) := \frac{1}{H_0 - \zeta}
\]

(4.10)

via the relation \(G_{\pm}(\zeta) = G_0(\zeta \mp \hbar\omega/2)\). Following a recent work by Glasser and Nieto\textsuperscript{10} the coordinate representation of \((4.10)\) can be given in closed form

\[
G_0(x''; x'; \hbar\omega\epsilon) := \langle x''|G_0(\hbar\omega\epsilon)|x'\rangle = \sqrt{\frac{m}{\pi\omega\hbar^3}} \Gamma \left( \frac{1}{2} - \epsilon \right) D_{\epsilon-1/2}(\mu x_+)D_{\epsilon-1/2}(-\mu x_-).
\]

(4.11)
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In the above $D_\nu$ stands for the parabolic cylinder functions, $\Gamma$ is Euler’s gamma function, $\mu := \sqrt{2m\omega/\hbar}$ and $x_\pm$ stands for the maximum and minimum of $x''$ and $x'$, respectively, that is, $x_+ = \max(x'', x')$ and $x_- = \min(x'', x')$. For simplicity we have introduced the dimensionless parameter $\epsilon = \zeta/\hbar \omega = \frac{x^2}{2mc^2\hbar} - \frac{mc^2}{2\hbar\omega}$. With this explicit expression it is now also possible to find the corresponding closed-form expression for the Green’s function of the Dirac oscillator. In fact, with the help of the relations

$$D'_\nu(y) + (y/2)D_\nu(y) = \nu D_{\nu-1}(y), \quad -D'_\nu(y) + (y/2)D_\nu(y) = D_{\nu+1}(y) \quad (4.12)$$

one finds following useful relations for the creation and annihilation operators \[3.23],

$$bD_\nu(\mu x) = \nu D_{\nu-1}(\mu x), \quad bD_\nu(\mu x) = D_{\nu+1}(\mu x), \quad b^b bD_\nu(\mu x) = \nu D_\nu(\mu x). \quad (4.13)$$

From these relation it is straight-forward to obtain the closed-form expressions

$$\frac{1}{2mc^2}\langle x''|A \Gamma_{-}(\zeta)|x'\rangle = \frac{i}{\hbar c} \frac{\text{sgn}(x'' - x')}{\sqrt{2\pi}} \Gamma(1 - \epsilon) D_{\epsilon-1}(\mu x) D_{\epsilon}(-\mu x_-)$$

$$\frac{1}{2mc^2}\langle x''|A b \Gamma_{-}(\zeta)|x'\rangle = \frac{i}{\hbar c} \frac{\text{sgn}(x'' - x')}{\sqrt{2\pi}} \Gamma(1 - \epsilon) D_{\epsilon}(\mu x) D_{\epsilon-1}(-\mu x_-)$$

$$\frac{z + mc^2}{2mc^2}\langle x''|G_{+}(\zeta)|x''\rangle = \frac{1}{\hbar c} \frac{z + mc^2}{\sqrt{2\pi mc^2\hbar\omega}} \Gamma(1 - \epsilon) D_{\epsilon}(\mu x) D_{\epsilon-1}(-\mu x_-) \quad (4.14)$$

$$\frac{z - mc^2}{2mc^2}\langle x''|G_{-}(\zeta)|x''\rangle = \frac{1}{\hbar c} \frac{z - mc^2}{\sqrt{2\pi mc^2\hbar\omega}} \Gamma(-\epsilon) D_{\epsilon-1}(\mu x) D_{\epsilon-1}(-\mu x_-)$$

which constitute the four components of the resolvent \[4.6] in the coordinate representation, that is $\langle x''|G_\nu(z)|x'\rangle$, for the Dirac oscillator. Obviously the pole at $\epsilon = 0$ leads to the eigenvalue $z = E_0^- = -mc^2$ whereas the other poles at $\epsilon = n \in \mathbb{N}$ result in the eigenvalues as given in \[3.23].

V. QUASI-CLASSICAL APPROXIMATION

In the previous section we have seen that the supersymmetric Dirac problem in (1+1) dimensions can be reduce to the corresponding non-relativistic supersymmetric Witten model. That is, the spectral properties of $H_{\mathcal{D}}$ are fully deduced from those of the partner Hamiltonians $H_\pm$. In particular, all exactly solvable Witten models, e.g. those which are shape-invariance, immediately result in the exact solutions of the corresponding Dirac system. Even closed form expression of the resolvent $G_\nu$ can be obtained when the corresponding resolvents $G_\pm$ are given in closed form.
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However, the above general discussion of section 2 is not limited to only exactly solvable Witten models. It may also be applied to approximation methods. Here we consider the so-called quasi-classical approximations to the energy eigenvalues of the Witten model. In fact, this quasi-classical approximation is in general better than the usual WKB approximation as it respects the SUSY induced symmetry between the eigenvalues of the two partner Hamiltonians and is exact for the ground-state energy in case of unbroken SUSY. To be more explicit, in case of unbroken SUSY with the ground state belonging to \(H_-\) the eigenvalues \(\varepsilon_n\) for the SUSY partners \(H_\pm\) are obtained via the so-called CBC formula

\[
\int_{x_L}^{x_R} dx \sqrt{2m (\varepsilon - \Phi^2(x))} = \hbar \pi n, \quad n \in \mathbb{N}_0, \tag{5.1}
\]

where the left and right turning points \(x_L \leq x_R\) are defined by \(\Phi^2(x_L/R) = \varepsilon\). Note that as before \(n = 0\) is only allowed for \(H_-\) and obviously implies \(\varepsilon_0 = 0\), which is exact. For \(n \in \mathbb{N}\) above CBC formula provides a quasi-classical approximation to the joint eigenvalues of \(H_\pm\). In case of shape-invariant systems this approximation is even exact for any \(n\).

In the case of a broken SUSY the corresponding quasi-classical approximation is given by the so-called EIJ formula and reads

\[
\int_{x_L}^{x_R} dx \sqrt{2m (\varepsilon - \Phi^2(x))} = \hbar \pi \left( n - \frac{1}{2} \right), \quad n \in \mathbb{N}. \tag{5.2}
\]

Again this formula reflects the strict iso-spectral property of the partner Hamiltonians as well as the strict positivity \(\varepsilon_n > 0\) for all \(n\). Furthermore, this approximation results in exact eigenvalues in case of shape-invariant systems.

Let us now utilize the relation \((3.2)\) between the eigenvalues of the Dirac Hamiltonian and the Witten Hamiltonians. For unbroken SUSY the relativistic version of the quasi-classical approximation then reads

\[
\int_{x_L}^{x_R} dx \sqrt{E^2 - m^2 c^4 - W^2(x)} = \hbar \pi n, \quad n \in \mathbb{N}_0, \tag{5.3}
\]

resulting in the exact ground-state energy \(E^0_- = -mc^2\) and the approximate eigenvalues given by \(E^\pm_n = \pm \sqrt{E^2}\) for \(n \in \mathbb{N}\). Similar, for broken SUSY we have

\[
\int_{x_L}^{x_R} dx \sqrt{E^2 - m^2 c^4 - W^2(x)} = \hbar \pi \left( n - \frac{1}{2} \right), \quad E^\pm_n = \pm \sqrt{E^2}, \quad n \in \mathbb{N}. \tag{5.4}
\]

Again, as in the non-relativistic case, whenever the SUSY potential \(W\) is shape-invariant both formulas \((5.3)\) and \((5.4)\) reproduce the exact spectrum of corresponding Dirac Hamiltonian.
VI. CONCLUSIONS

In the current paper we have studied the most general supersymmetric Dirac Hamiltonian in (1 + 1) space-time dimensions. It has been shown that the spectral properties, that is, of the relativistic Hamiltonian $H_D$ as given in eq (2.6) can be reduced to the non-relativistic spectral problem (2.13) of the Witten model. Hence the main results are represented by eq (3.2) and eq (3.19). In addition, we have shown that the relativistic resolvent kernel (4.1) can be expressed in terms of those of the Witten model (4.5) via eq (4.6). As an explicit example we have chosen the Dirac oscillator for which the closed-form expression (4.14) of the Green’s function was obtained. Obviously, any of the shape-invariant non-relativistic SUSY models, which exhibit exact solutions, immediately result in exact solutions of the corresponding relativistic model. With the discussion of section 5 we may also consider not exactly solvable systems by applying supersymmetric quasi-classical approximations. Here for example, one could study anharmonic oscillator system being characterised by a SUSY potential $\Phi(x) = |x|^d$ and $\Phi(x) = \text{sgn}x|x|^d$ exhibiting broken and unbroken supersymmetry, respectively. It shall be noted that in the limit $d \to \infty$ theses system simulate a particle in the box with various boundary conditions and the quasi-classical approximation is known\textsuperscript{8} to become exact in that limit, too. Hence, this will provide another route towards the study of a Dirac particle in a box, a topic being still of interest\textsuperscript{11}.

Despite the fact that the current paper focuses on the (1 + 1) dimensional Dirac systems so of the present results are valid in the more general case of arbitrary supersymmetric Dirac Hamiltonians. For example, the results of section 3 and in particular the explicit expression (3.15) for the unitary transformation matrix $U$ is valid for an arbitrary SUSY Hamiltonian (2.3) as long as $M_+ = M_- = mc^2$, which is the case for almost all supersymmetric Dirac Hamiltonians\textsuperscript{8}.

Finally let us mention the the present results also apply to radial symmetry Dirac Hamiltonians being supersymmetric. In fact, due the spherical symmetry the radial Dirac Hamiltonian is of the same form as given in (2.6), cf eq (9.102) in\textsuperscript{8}, with a pseudo-scalar potential being of the form $W(r) = \kappa/r + \Phi(r)$ now acting on the positive half-line $r \in \mathbb{R}^+$ and $\kappa$ denotes the eigenvalues of the spin-orbit operator.
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