SYMMETRY AND NONEXISTENCE OF POSITIVE SOLUTIONS TO FRACTIONAL P-LAPLACIAN EQUATIONS

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Abstract. In this paper, we consider the fractional p-Laplacian equation

\[-\Delta^s_p u(x) = f(u(x)),\]

where the fractional p-Laplacian is of the form

\[-\Delta^s_p u(x) = C_{n,s,p} PV \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{n+sp}} dy.\]

By proving a narrow region principle to the equation above and extending the direct method of moving planes used in fractional Laplacian equations, we establish the radial symmetry in the unit ball and nonexistence on the half space for the solutions, respectively.

1. Introduction. In this paper, we consider the nonlinear elliptic equation involving the fractional p-Laplacian \(-\Delta^s_p\):

\[-\Delta^s_p u(x) = f(u(x)),\]

where 0 < s < 1, 2 < p < \infty,

\[-\Delta^s_p u(x) = C_{n,s,p} PV \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{n+sp}} dy,\]

PV in (2) stands for the Cauchy principal value, i.e.

\[PV \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{n+sp}} dy = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{n+sp}} dy.\]

In order the integral (3) to make sense, we require that

\[u \in C^{1,1}_{loc} \cap L^p\]

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The fractional \( p \)-Laplacian has many applications, for instance, it is used to study the non-local “Tug-of-War” game (see [1]). The interest on these nonlocal operators continues to grow in recent years. We refer to [3, 21, 22, 24] for the recent progress on these nonlocal operators.

If \( p = 2 \) in (3), then \((-\Delta)_p^s\) becomes the usual fractional Laplacian \((-\Delta)^s\). Like \((-\Delta)^s\), the operator \((-\Delta)_p^s\) is nonlocal, i.e., it does not act in the meaning of pointwise differentiation but is an integral on \(\mathbb{R}^n\) with a singular kernel, which causes the main difficulty in studying problems involving \((-\Delta)_p^s\). To investigate the fractional equations involving \((-\Delta)_p^s\), Caffarelli and Silvestre [4] introduced an extension method to overcome the difficulty induced by the nonlocal property. The idea in [14] is to localize the fractional Laplacian by constructing a Dirichlet to Neumann problem of a degenerate elliptic equation. This method has been applied successfully to study equations involving \((-\Delta)^s\), and a series of fruitful results have been obtained (see [2, 14] and the references therein). But these results are effective only for \(\frac{1}{2} \leq s < 1\). Another useful method to study the fractional Laplacian is the integral equations method, which turns a given fractional Laplacian equation into its equivalent integral equation, and then various properties of the original equation can be obtained by investigating the integral equation, see [10, 11, 12] and references therein. However, one always need to impose extra integrable conditions on the solutions to equations. For more articles concerning the method of moving planes for nonlocal equations and integral equations, see [6, 7, 15, 16, 17, 18, 19, 20, 23] and the reference therein.

Chen, Li and Li [13] developed a direct method of moving planes to the fractional Laplacian equations, which has been applied to obtain symmetry, monotonicity and nonexistence of solutions to many kinds of semi-linear equations involving \((-\Delta)^s\). Later, Chen, Li and Li in [9] refined this direct method to investigate the fully nonlinear equation

\[
F_{\alpha}(u(x)) \equiv C_{n,\alpha} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_{\varepsilon}(x)} \frac{G(u(x) - u(z))}{|x - z|^{n+\alpha}} \, dz = f(x, u) \tag{4}
\]

with \(G(\cdot)\) being a Lipschitz continuous function (see [5]), by requiring that the operator \(F_{\alpha}(\cdot)\) is non-degenerate in the sense that

\[
G'(t) \geq c_0 > 0. \tag{5}
\]

Equation (4) includes the fractional Laplacian equation when \(G(t) = t\).

Even though (1) studied in this paper is a special case of (4) with

\[
\alpha = sp \quad \text{and} \quad G(t) = |t|^{p-2} t,
\]

however, it is degenerate for \(p > 2\) and singular for \(p < 2\). Therefore, (5) is not satisfied. Note \(G'(t) = (p-1)|t|^{p-2}\) and as \(t \to 0^+\),

\[
G'(t) \to \begin{cases} 
0, & \text{if } p > 2, \\
\infty, & \text{if } 1 < p < 2.
\end{cases}
\]

The methods in either [9] or [13] relies heavily on the non-degeneracy of \(G(\cdot)\), and hence can not be applied directly to the fractional \(p\)-Laplacian equations. To
circumvent these difficulties, recently, Chen and Li in [8] introduced some new ideas and proved symmetry and monotonicity for solutions to the semi-linear equation
\[ (-\Delta)_p^s u(x) = u^q(x), \quad q \geq p - 1, \]
in which they mentioned that since the narrow region principle has not been established, their main technical lemma is the so-called key boundary estimate.

In this paper, we affirm that the narrow region principle is valid for the fractional \( p \)-Laplacian, which plays the role of Hopf lemma in the second step of the method of moving planes. Compared with the key boundary estimate in [8], the narrow region principle is more convenient since it can start moving the plane directly. We believe that this principle and the idea behind the proof will be useful in studying other nonlinear nonlocal problems. As its applications, we prove symmetry and monotonicity for solutions to (1) in the unit ball and nonexistence for solutions to (1) on the half space respectively.

Before state our main results, we introduce some notations. For \( x = (x_1, x') \) with \( x' = (x_2, ..., x_n) \in \mathbb{R}^{n-1} \), and \( \lambda \in \mathbb{R} \), let
\[ T_\lambda = \{ x \in \mathbb{R}^n | x_1 = \lambda \} \]
be the moving plane and
\[ \Sigma_\lambda = \{ x \in B_1(0) | x_1 < \lambda \}. \]
Denote by
\[ x^\lambda = (2\lambda - x_1, x') \]
the reflection of \( x \) with respect to the hyperplane \( T_\lambda \) and by \( u_\lambda(x) = u(x^\lambda) \) the reflection of function and introduce a function
\[ w_\lambda(x) = u_\lambda(x) - u(x). \]
We will use \( C \) for a general various positive constant which is usually different in different contexts.

Our main results are

**Theorem 1.1. (Narrow region principle)** Let \( \Omega \) be a bounded narrow region in \( \Sigma \subset \mathbb{R}^n \) (where \( \Sigma \) is an open set of \( \{ x | x_1 < \lambda \} \)), such that it is contained in
\[ \{ x | \lambda - \delta < x_1 < \lambda \} \]
with a small \( \delta \). Suppose that \( u \in L_{sp} \cap C^{1,1}_{loc}(\Omega) \), and \( w_\lambda \) is lower semi-continuous on \( \Omega \). If \( c(x) \) is bounded from below in \( \Omega \),
\[
\begin{cases}
(-\Delta)_p^s (u_\lambda(x)) - (-\Delta)_p^s (u(x)) + c(x)w_\lambda(x) \geq 0, & \text{in } \Omega, \\
w_\lambda(x) \geq 0, & \text{in } \Sigma \setminus \Omega, \\
w_\lambda(x^\lambda) = -w_\lambda(x), & \text{in } \Sigma
\end{cases}
\tag{6}
\]
and there exists \( y^0 \in \Sigma \) such that \( w_\lambda(y^0) > 0 \), then for sufficiently small \( \delta \), we have
\[ w_\lambda(x) \geq 0 \text{ in } \Omega. \]
Furthermore, if \( w_\lambda(x) = 0 \) at some point in \( \Omega \), then
\[ w_\lambda(x) \equiv 0 \text{ in } \mathbb{R}^n. \]
These conclusions hold for an unbounded region \( \Omega \) if we further assume that
\[ \lim_{|x| \to \infty} w_\lambda(x) \geq 0. \]
Remark 1. Compared with the narrow region principle for fractional Laplacian equations in [13], we need to add the condition that there exists \( y^0 \in \Sigma \) such that \( w_{\lambda}(y^0) > 0 \) in Theorem 1.1. However, this condition is automatically satisfied which can be easily seen from the proof of Theorem 1.2 and Theorem 1.3.

Theorem 1.2. (Radial symmetry) Assume that \( u \in L^{sp} \cap C^{1,1}_{loc}(B_1(0)) \) is a positive solution to the problem

\[
\begin{cases}
(-\Delta)^s_p u(x) = f(u(x)), & x \in B_1(0), \\
u(x) = 0, & x \notin B_1(0),
\end{cases}
\]

with \( f(\cdot) \) being a Lipschitz continuous function. Then \( u \) must be radially symmetric and monotone decreasing about the origin.

Remark 2. In [13], Chen, Li and Li considered a semilinear fractional Laplacian equation \((-\Delta)^s_p u(x) = f(u(x)).\)

Under the similar assumptions on \( u \) and \( f(\cdot) \) in Theorem 1.2, they obtained the radially symmetry and monotonicity in the unit ball for positive solutions. Theorem 1.2 here generalizes their result to the fractional \( p \)-Laplacian equation. In addition, our Theorem 1.1 contains the result on symmetry of the solution to \((-\Delta)^s_p u(x) = u^q(x), q \geq p - 1\) considered in [8].

Theorem 1.3. (Nonexistence) Assume that \( u \in L^{sp} \cap C^{1,1}_{loc}(\mathbb{R}_+^n) \) is a nonnegative solution to the problem

\[
\begin{cases}
(-\Delta)^s_p u(x) = g(u(x)), & x \in \mathbb{R}_+^n, \\
u(x) = 0, & x \notin \mathbb{R}_+^n
\end{cases}
\]

with

\[
\lim_{|x| \to \infty} u(x) = 0,
\]

and \( u \) is continuous on \( \mathbb{R}_+^n \). Suppose \( g(0) = 0, g'(\cdot) \) is locally bounded, and \( g'(t) \leq 0 \) for \( t > 0 \) sufficiently small. Then \( u \equiv 0 \).

Remark 3. Chen and Li [8] proved symmetry of solutions to (1) in \( \mathbb{R}^n \). We can get the same conclusion again by Theorem 1.1 and the direct method of moving planes.

The article is organized as follows: Section 2 is devoted to the proof of Theorem 1.1. In Section 3, we prove Theorem 1.2. The proof of Theorem 1.3 is given in Section 4.

2. Narrow region principle.

Proof of Theorem 1.1. Assume that the conclusion is wrong, then for any \( \delta > 0 \), there exists \( x_\delta \in \Omega_\delta \), such that

\[
w_\lambda(x_\delta) = \min_{\Omega_\delta} w_\lambda < 0.
\]

Then for \( \delta_k = \frac{1}{k}, k = 1, 2, \ldots \), there exist \( x_{\delta_k} \) and \( \Omega_{\delta_k} \), simply denoted as \( x_k \) and \( \Omega_k, k = 1, 2, \ldots \), such that

\[
w_\lambda(x_k) = \min_{\Omega_k} w_\lambda < 0.
\]
Note that $G(t) = |t|^{p-2}t$ is a strictly increasing function in $t$, and $G'(t) = (p - 1)|t|^{p-2} \geq 0$. A direct calculation gives

\[
(-\Delta)_p^s u_\lambda(x_k) - (-\Delta)_p^s u(x_k) = C_{n,s,p} PV \int_{\Sigma} \frac{G(u_\lambda(x_k) - u_\lambda(y)) - G(u(x_k) - u(y))}{|x_k - y|^{n+sp}} dy
\]

\[
= C_{n,s,p} PV \int_{\Sigma} \frac{G(u_\lambda(x_k) - u_\lambda(y)) - G(u(x_k) - u(y))}{|x_k - y|^{n+sp}} dy
\]

\[
+ C_{n,s,p} PV \int_{\Sigma} \frac{G(u_\lambda(x_k) - u_\lambda(y)) - G(u(x_k) - u_\lambda(y))}{|x_k - y|^{n+sp}} dy
\]

\[
= \left[ C_{n,s,p} PV \int_{\Sigma} \frac{G(u_\lambda(x_k) - u_\lambda(y)) - G(u(x_k) - u(y))}{|x_k - y|^{n+sp}} dy
\]

\[
+ C_{n,s,p} PV \int_{\Sigma} \frac{G(u_\lambda(x_k) - u_\lambda(y)) - G(u(x_k) - u_\lambda(y))}{|x_k - y|^{n+sp}} dy
\]

\[
\right]

\[
P V \int_{\Sigma} \left[ \frac{1}{|x_k - y|^{n+sp}} - \frac{1}{|x_k - y^\lambda|^{n+sp}} \right] [G(u_\lambda(x_k) - u_\lambda(y)) - G(u(x_k) - u(y))] dy
\]

\[
:= C_{n,s,p}(I_1 + I_2).
\]

To estimate $I_1$, we apply the mean value theorem to derive

\[
I_1 = \int_{\Sigma} \frac{|G(u_\lambda(x_k) - u_\lambda(y)) - G(u(x_k) - u_\lambda(y))|}{|x_k - y|^{n+sp}} dy
\]

\[
+ \int_{\Sigma} \frac{|G(u_\lambda(x_k) - u(y)) - G(u(x_k) - u(y))|}{|x_k - y^\lambda|^{n+sp}} dy
\]

\[
= w_\lambda(x_k) \int_{\Sigma} \frac{G'(\xi(y)) + G'(\eta(y))}{|x_k - y|^{n+sp}} dy
\]

\[
\leq 0,
\]

where $\xi(y)$ is between $u_\lambda(x_k) - u_\lambda(y)$ and $u(x_k) - u_\lambda(y)$, and $\eta(y)$ is between $u_\lambda(x_k) - u_\lambda(y)$ and $u(x_k) - u_\lambda(y)$.

Now let us estimate $I_2$. Denote

\[
\delta_{x_k} = \text{dist} \{x_k, \{x_1 = \lambda\}\},
\]

obviously, $\delta_{x_k} = \lambda - x_{k,1}$, where $x_{k,1}$ denote the first component of $x_k$. Consider the function $f(t) = t^{-(n+sp)/2}$ over $[t_1, t_2]$ with $t_1 = |x_k - y|^2$ and $t_2 = |x_k - y^\lambda|^2$.

Using

\[
t_1 - t_2 = |x_k - y|^2 - |x_k - y^\lambda|^2
\]

\[
= [(x_{k,1} - y_{1})^2 + \ldots + (x_{k,n} - y_{1})^2] - [(x_{k,1} - (2\lambda - y_{1}))^2 + \ldots + (x_{k,n} - y_{1})^2]
\]

\[
= (x_{k,1} - y_{1})^2 - (x_{k,1} - (2\lambda - y_{1}))^2
\]

\[
= -4(\lambda - x_{k,1})(\lambda - y_{1})
\]

\[
= -4\delta_{x_k}(\lambda - y_{1})
\]
and the mean value theorem, it implies
\[
\frac{1}{|x_k - y|^{n+sp}} - \frac{1}{|x_k - y^\lambda|^{n+sp}} = f(t_1) - f(t_2) = \frac{2(n + sp)(\lambda - y_1)}{|x_k - \zeta(y)|^{n+sp+2}} \delta x_k,
\]

where \(\zeta(y)\) is some point on the line segment between \(y\) and \(y^\lambda\). Hence
\[
I_2 = \int_{\Sigma} \left[ \frac{1}{|x_k - y|^{n+sp}} - \frac{1}{|x_k - y^\lambda|^{n+sp}} \right] [G(u_\lambda(x_k) - u_\lambda(y)) - G(u(x_k) - u(y))] \, dy
\]
\[
= \delta x_k \int_{\Sigma} \frac{2(n + sp)(\lambda - y_1)}{|x_k - \zeta(y)|^{n+sp+2}} [G(u_\lambda(x_k) - u_\lambda(y)) - G(u(x_k) - u(y))] \, dy
\]
\[
:= \delta x_k F(x_k). \tag{12}
\]
We will show that there exists a positive constant \(C_0\) such that for a sufficiently small \(\delta x_k\) (or a sufficiently large \(k\)), it holds
\[
F(x_k) \leq -\frac{C_0}{2}. \tag{13}
\]
To prove (13), let us first claim
\[
F(x_k) < 0.
\]
In fact, we first notice that
\[
\frac{\lambda - y_1}{|x_k - \zeta(y)|^{n+sp+2}} < 0, x, y \in \Sigma.
\]
While for the second part in the integral, due to the monotonicity of \(G\) and the fact that
\[
[u_\lambda(x_k) - u_\lambda(y)] - [u(x_k) - u(y)] = w_\lambda(x_k) - w_\lambda(y) \leq 0
\]
but not identically 0 in \(\Sigma\), we have
\[
G(u_\lambda(x_k) - u_\lambda(y)) - G(u(x_k) - u(y)) \leq 0
\]
but not identically 0 in \(\Sigma\). This derives \(F(x_k) < 0\).
We continue to check (13). If (13) is not true, then there exists a subsequence of \(\{x_k\}\), we might as well assume that it is still \(\{x_k\}\), which satisfies
\[
F(x_k) \to 0 \text{ as } \delta x_k \to 0 (\text{or } k \to \infty),
\]
and therefore,
\[
G(u_\lambda(x_k) - u_\lambda(y)) - G(u(x_k) - u(y)) \to 0, \text{ for any } y \in \Sigma,
\]
which shows
\[
w_\lambda(x_k) - w_\lambda(y) \to 0, y \in \Sigma.
\]
Since \(w_\lambda(x_k) \to 0\) as \(\delta x_k \to 0\), we have
\[
w_\lambda(y) \equiv 0, y \in \Sigma,
\]
which is a contradiction with \(w_\lambda(y^0) > 0\). Thus there exists \(C_0 > 0\) such that
\[
F(x_k) \to -C_0 \text{ as } \delta x_k \to 0.
\]
By the continuity of \(F(x_k)\) with respect to \(x_k\), we obtain (13).
It follows from (12) and (13) that
\[ I_2 \leq -\frac{C_0}{2} \delta x_k. \]  
(14)

Putting (11) and (14) into (10), it implies
\[ (-\Delta)_{p_k} u_{\lambda}(x_k) - (-\Delta)_{p_k} u(x_k) \leq -C\delta x_k. \]  
(15)

On the other hand, since
\[ \nabla w_{\lambda}(x_k) = 0, \]
and
\[ w_{\lambda}(\bar{x}_k) = w_{\lambda}(x_k) + \nabla w_{\lambda}(x_k)(\bar{x}_k - x_k) + o(|\bar{x}_k - x_k|) \]
where \( \bar{x}_k = (\lambda, (x_k)') \in T_{\lambda}, \) then
\[ w_{\lambda}(x_k) = o(1) \delta x_k, \]
and
\[ c(x_k)w_{\lambda}(x_k) \leq o(1) \delta x_k. \]  
(16)

Using (15) and (16), it infers
\[ (-\Delta)_{p_k} u_{\lambda}(x_k) - (-\Delta)_{p_k} u(x_k) + c(x_k)w_{\lambda}(x_k) \leq (-C + o(1)) \delta x_k < 0. \]

But this contradicts the equation and so Theorem 1.1 is proved.

3. Symmetry of solutions in the unit ball.

Proof of Theorem 1.2. Let \( x_{\lambda}, u_{\lambda}, T_{\lambda}, \Sigma_{\lambda} \) and \( w_{\lambda} \) be defined in Section 1. It is easy to verify that
\[ (-\Delta)_{p_k} u_{\lambda}(x) - (-\Delta)_{p_k} u(x) + c_{\lambda}(x)w_{\lambda}(x) = 0, x \in \Sigma_{\lambda}, \]
where
\[ c_{\lambda}(x) = \frac{f(u(x)) - f(u_{\lambda}(x))}{u_{\lambda}(x) - u(x)}. \]

The Lipschitz continuity of \( f \) guarantees that \( c_{\lambda}(x) \) is uniformly bounded from below. Now we concluded that for \( \lambda > -1 \) sufficiently closing to \(-1, \)
\[ w_{\lambda}(x) \geq 0, x \in \Sigma_{\lambda}, \]  
(17)

by applying Theorem 1.1 to \( \Omega = \Sigma = \Sigma_{\lambda} = \{ x \in B_1(0)| x_1 < \lambda \}, \) where \( \Sigma_{\lambda} \) is a narrow region.

Define
\[ \lambda_0 = \sup \{ \lambda | w_{\mu}(x) \geq 0, x \in \Sigma_{\mu}, \mu \leq \lambda \}, \]
then we must have
\[ \lambda_0 = 0. \]

Otherwise, if \( \lambda_0 < 0, \) one will have

Assertion. the plane can be moved to the right a little more and (17) is still valid, i.e., there exists a small \( \varepsilon > 0, \) such that for all \( \lambda \in [\lambda_0, \lambda_0 + \varepsilon], \) (17) holds.

Now let us prove this assertion. First, since \( w_{\lambda_0}(x) \) is not identically zero, we have by Theorem 1.1 that
\[ w_{\lambda_0}(x) > 0, x \in \Sigma_{\lambda_0}. \]  
(18)

Thus for any \( \delta > 0, \)
\[ w_{\lambda_0}(x) > c_{\delta} > 0, x \in \Sigma_{\lambda_0 - \delta}. \]

By the continuity of \( w_{\lambda} \) in \( \lambda, \) there exists \( \varepsilon > 0, \) such that
\[ w_{\lambda}(x) \geq 0, x \in \Sigma_{\lambda_0 - \delta}, \lambda \in [\lambda_0, \lambda_0 + \varepsilon). \]  
(19)
We take $\Sigma = \Sigma_\lambda$ and the narrow region $\Omega = \Sigma_\lambda \setminus \Sigma_{\lambda_0 - \delta}$ and get by Theorem 1.1, 

$$w_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda \setminus \Sigma_{\lambda_0 - \delta}.$$ 

Combining this with (19), 

$$w_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda, \quad \lambda \in [\lambda_0, \lambda_0 + \varepsilon).$$ 

and the assertion is proved.

But it will contradict the definition of $\lambda_0$. Therefore, we must have $\lambda_0 = 0$.

It follows from $\lambda_0 = 0$ that 

$$w_0(x) \geq 0, \quad x \in \Sigma_0$$ 

and then 

$$u(-x_1, x_2, ..., x_n) \leq u(x_1, x_2, ..., x_n), \quad 0 < x_1 < 1.$$ 

(20)

Since the $x_1$ direction can be chosen arbitrarily, we see that (20) implies that $u$ is radially symmetric about the origin. The monotonicity is a consequence of the fact that 

$$w_\lambda(x) > 0, \quad x \in \Sigma_\lambda, -1 < \lambda \leq 0.$$

4. Symmetry of solutions in the unit ball.

Proof of Theorem 1.3. Based on (9) and $g(0) = 0$, one can claim that 

either $u(x) > 0$ or $u(x) \equiv 0, x \in \mathbb{R}_n^+$. (21)

Indeed, if there exists $x^0$ in $\mathbb{R}_n^+$ such that 

$$u(x^0) = \min_{\mathbb{R}_n^+} u = 0,$$

then 

$$(-\Delta)_p^* u(x^0) = C_{n,s,p} \text{PV} \int_{\mathbb{R}_n^+} \frac{|u(x^0) - u(y)|^{p-2}(u(x^0) - u(y))}{|x^0 - y|^{n+sp}} dy$$

$$= C_{n,s,p} \text{PV} \int_{\mathbb{R}_n^+} \frac{|u(x^0) - u(y)|^{p-2}(u(x^0) - u(y))}{|x^0 - y|^{n+sp}} dy$$

$$= C_{n,s,p} \text{PV} \int_{\mathbb{R}_n^+} -|u(y)|^{p-2} u(y) \frac{|x^0 - y|^{n+sp}}{|x^0|^n} \frac{|x^0 - y|^{n+sp}}{|x^0|^n} dy$$

$$= g(u(x^0)) = 0.$$ 

Therefore, $\int_{\mathbb{R}_n^+} -|u(y)|^{p-2} u(y) \frac{|y|^n}{|x^0 - y|^{n+sp}} dy = 0$ and then $u \equiv 0$ which proves (21).

In the sequel, we always assume $u > 0$ in $\mathbb{R}_n^+$.

Now let us use the method of moving planes to the solution $u$ along the $x_n$ direction. Denote 

$$T'_\lambda = \{ x \in \mathbb{R}_n | x_n = \lambda \}, \lambda > 0,$$

$$\Sigma'_\lambda = \{ x \in \mathbb{R}_n | 0 < x_n < \lambda \}$$

and 

$$x^\lambda = (x_1, ..., x_{n-1}, 2\lambda - x_n)$$

which is the reflection of $x$ about the $T'_\lambda$. Let 

$$w_\lambda(x) = u(x^\lambda) - u(x).$$
Step 1. We point out that for \( \lambda \) sufficiently small, it holds

\[ w_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda, \quad (22) \]

In fact, taking \( \Sigma = \Sigma'_\lambda \cup \mathbb{R}^n_{-}, \mathbb{R}^n_{-} = \{ x \in \mathbb{R}^n | x_n \leq 0 \} \), and using Theorem 1.1, we arrive at (22).

Step 2. Since (22) provides a starting point, we can move the plane \( T^\prime_\lambda \) up as long as (22) holds to its limiting position and show that \( u \) is symmetric about the limiting plane. More precisely, let \( \lambda_0 = \sup \{ \lambda | w_\mu(x) \geq 0, \quad x \in \Sigma'_\mu, \mu \leq \lambda \} \), we claim that

\[ \lambda_0 = \infty. \quad (23) \]

If \( \lambda_0 < \infty \), then we are able to show that the plane \( T^\prime_\lambda \) can still be moved further up, i.e., there exists \( \delta_0 > 0 \) such that for all \( 0 < \delta < \delta_0 \),

\[ w_{\lambda_0 + \delta}(x) \geq 0, \quad x \in \Sigma'_{\lambda_0 + \delta}. \quad (24) \]

This will contradict the definition of \( \lambda_0 \), and hence (23) must be true.

The remaining is to verify (24). To do so, we first observe that for any \( 0 < \lambda < \infty \) and \( |x| \) sufficiently large,

\[ w_\lambda(x) \geq 0, \quad x \in \Sigma'_\lambda. \quad (25) \]

Indeed, it implies from (5) that

\[ (-\Delta)^{s}_\mu u_\lambda(x) - (-\Delta)^{s}_\mu u(x) = g'(\psi(x))w_\lambda(x), \]

where \( \psi(x) \) is between \( u_\lambda(x) \) and \( u(x) \). If (25) is violated, then by the condition \( \lim_{|x| \to \infty} u(x) = 0 \), there exists a point \( \hat{x} \in \Sigma'_\lambda \) such that

\[ w_\lambda(\hat{x}) = \min_{\Sigma'_\lambda} w_\lambda < 0, \]

and consequently,

\[ u_\lambda(\hat{x}) \leq \psi(\hat{x}) \leq u(\hat{x}). \]

For sufficiently large \( |x| \), \( u(\hat{x}) \) is small, hence so is \( \psi(\hat{x}) \), and therefore \( g'(\psi(\hat{x})) \leq 0 \) due to the condition on \( g' \). It follows

\[ (-\Delta)^{s}_\mu u_\lambda(\hat{x}) - (-\Delta)^{s}_\mu u(\hat{x}) \geq 0. \quad (26) \]

On the other hand, we derive from the process in the proof of Theorem 1.1 that

\[ (-\Delta)^{s}_\mu u_\lambda(\hat{x}) - (-\Delta)^{s}_\mu u(\hat{x}) < 0. \quad (27) \]

This contradicts to (26). Therefore (25) must be true.

Note that by Theorem 1.1, we have

\[ w_{\lambda_0}(x) \equiv 0, \quad x \in \Sigma'_{\lambda_0}. \quad (28) \]

or

\[ w_{\lambda_0}(x) > 0, \quad x \in \Sigma'_{\lambda_0}. \quad (29) \]

If (28) holds, then

\[ u(x_1, ..., x_{n-1}, 2\lambda_0) = u(x_1, ..., x_{n-1}, 0) = 0, \]

this is impossible, because we assume that \( u > 0 \) in \( \mathbb{R}^n_+ \). Therefore, It follows from (29) that there exists \( R_0 > 0 \) such that for any positive number \( \sigma \),

\[ w_{\lambda_0}(x) \geq c_0 > 0, \quad x \in \Sigma'_{\lambda_0 - \sigma} \cap B_{R_0}(0). \quad (30) \]
Since \(w_\lambda\) depends on \(\lambda\) continuously, we have for \(\delta\) positive small,\[w_{\lambda_0+\delta}(x) \geq 0, \quad x \in \Sigma_{\lambda_0-\sigma} \cap B_{R_0}(0).\]

Let us prove (24) by a contradiction. Suppose (24) is false, then there exists \(\bar{x} \in \Sigma_{\lambda_0+\delta}\), such that\[w_{\lambda_0+\delta}(\bar{x}) = \min_{\Sigma_{\lambda_0+\delta}} w_{\lambda_0+\delta} < 0.\]

From (25), we must have\[\bar{x} \in (\Sigma_{\lambda_0+\delta} \setminus \Sigma_{\lambda_0-\sigma}) \cap B_{R_0}(0).\] (31)

Notice that \((\Sigma_{\lambda_0+\delta} \setminus \Sigma_{\lambda_0-\sigma}) \cap B_{R_0}(0)\) is a narrow region for sufficiently small \(\sigma\) and \(\delta\), it implies by Theorem 1.1 that \(w_{\lambda_0+\delta}\) cannot attain its negative minimum, which contradicts to (31). Hence (24) holds.

Consequently, the solution \(u(x)\) is monotonely increasing with respect to \(x_n\) by (23). This contradicts to (9), which shows that \(u(x) > 0\) is not true. Then \(u(x) \equiv 0.\)

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