On the lower bounds for real double Hurwitz numbers

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Abstract
As the real counterpart of double Hurwitz number, the real double Hurwitz number depends on the distribution of real branch points. We consider the problem of asymptotic growth of real and complex double Hurwitz numbers. We provide a lower bound for real double Hurwitz numbers based on the tropical computation of real double Hurwitz numbers. By using this lower bound and J. Rau’s result (Math Ann 375: 895-915, 2019), we prove the logarithmic equivalence of real and complex Hurwitz numbers.

Keywords Real enumerative geometry · Real Hurwitz numbers · Tropical Hurwitz numbers · Asymptotic growth

Mathematics Subject Classification Primary 14N10 · 14T90; Secondary 14P99

1 Introduction
The structure and approach of this paper follow [22] closely. Counting the ramified covers of $\mathbb{CP}^1$ by a genus $g$ surface with specified ramification profiles over a fixed set of points is a classical problem of enumerative geometry. The answer to such enumerative problem is called the Hurwitz number. The Hurwitz number is equivalent to enumerating the factorizations of the identity into a product of elements of the symmetric group $S_d$ with given cycle types [5, 11]. Hurwitz numbers are interesting geometric invariants connecting the geometry of algebraic curves, combinatorics, tropical geometry, the representation of symmetric groups and random matrix models [1, 3, 5, 6, 9]. In the recent two decades, many deep relationships between Hurwitz numbers and mathematical physics were also found [19, 21]. There is a particular type of Hurwitz numbers which arouses many mathematicians’ interests. The double Hurwitz number $H^\mathbb{C}_g(\lambda, \mu)$ counts covers of $\mathbb{CP}^1$ by genus $g$ surface with ramification
profiles $\lambda, \mu$ over 0, $\infty$ and simple ramification over other branch points, where $\lambda$ and $\mu$ are two partitions of an integer $d \geq 1$. For a partition $\lambda$, we denote by $l(\lambda)$ the number of parts of $\lambda$ and call it the length of $\lambda$. The sum of the parts of $\lambda$ is denoted by $|\lambda|$. There are many results about the structure of double Hurwitz numbers such as the polynomiality of the generating function and the wall-crossing formulas [4, 9, 17, 23].

In this paper, we consider the real version of double Hurwitz number. A real structure $\tau$ for a cover $\pi : C \to \mathbb{C}P^1$ is an anti-holomorphic involution such that $\pi \circ \tau = \text{conj} \circ \pi$, where conj is the standard complex conjugation. A pair $(\pi, \tau)$ consisting of a ramified cover $\pi : C \to \mathbb{C}P^1$ and a real structure $\tau$ is called a real ramified cover. The real double Hurwitz number counts real ramified covers of $\mathbb{C}P^1$ by genus $g$ surfaces with particular ramification profiles over 0, $\infty$ and simple ramification over other branch points. Note that the set of simple branch points of a real ramified cover consists of real points in $\mathbb{R}P^1 \setminus \{0, \infty\}$ and complex conjugated pairs. In the following, we only consider the case that all the simple branch points are real points in $\mathbb{R}P^1 \setminus \{0, \infty\}$, and suppose that $s$ of these simple branch points are in the positive half axis of $\mathbb{R}P^1 \setminus \{0, \infty\}$. Let $H^R_g(\lambda, \mu; s)$ denote the real double Hurwitz number counting real ramified covers of $\mathbb{C}P^1$ by genus $g$ surface with ramification profiles $\lambda, \mu$ over 0, $\infty$ and simple ramification over other branch points. The real double Hurwitz number $H^R_g(\lambda, \mu; s)$ depends on the number $s$ of positive real simple branch points. It is a common phenomenon in real enumerative geometry that the number of real solutions for a enumerative problem depends on the positions of the point constraints [12, 25, 26]. In real enumerative geometry, it is important to find the lower bounds for the real enumerative problems and to analyse the properties of these lower bounds. In the study of real algebraic curves in real surfaces passing through certain real points, signed counts of real solutions which are called the Welschinger invariants provide such lower bounds [8, 15, 18, 25, 26]. This method is also valid in the study of counting real covers. Itenberg and Zvonkine [16] found such a signed count of real polynomials and proved that the signed count of real polynomials is logarithmically equivalent to the count of complex polynomials under certain parity conditions. El Hilany and Rau [7] found that the construction of Itenberg and Zvonkine also works for counting real simple rational functions $\frac{f(x)}{x-p}$, $f(x) \in \mathbb{R}[x]$, $p \in \mathbb{R}$. How to generalize Itenberg and Zvonkine’s signed count to a more general situation is still unknown. Rau [22] found a lower bound for real double Hurwitz numbers and proved the logarithmic equivalence of real double Hurwitz numbers and complex double Hurwitz numbers under certain parity conditions.

In this paper, we continue the study on the asymptotic growth of real double Hurwitz numbers when the degree is increased and only simple ramification points are added. We prove the logarithmic equivalence of real and classical Hurwitz numbers. Let

$$h^C_{g, \lambda, \mu}(m) = H^C_{g}((\lambda, 1^m), (\mu, 1^m)),$$
$$h^R_{g, \lambda, \mu}(m) = \inf\{H^R_{g}((\lambda, 1^m), (\mu, 1^m); 0), \ldots, H^R_{g}((\lambda, 1^m), (\mu, 1^m); r(m))\},$$

where $(\lambda, 1^m)$ stands for adding $m$ ones to $\lambda$, and $r(m) = l(\lambda) + l(\mu) + 2m + 2g - 2$.  

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Theorem 1.1 Fix $g \in \mathbb{N}$, and partitions $\lambda, \mu$ with $|\lambda| = |\mu|$. Then, $h_{g,\lambda,\mu}^R(m)$ and $h_{g,\lambda,\mu}^C(m)$ are logarithmically equivalent:

$$\log h_{g,\lambda,\mu}^R(m) \sim 2m \log m \sim \log h_{g,\lambda,\mu}^C(m), \text{ as } m \to \infty.$$ 

For the enumerative problem concerning real rational algebraic curves in real algebraic surfaces, Itenberg et al. [13, 14] showed that Welschinger invariants are logarithmically equivalent to the Gromov–Witten invariants. Shustin [24] proved the logarithmic equivalence of higher genus Welschinger invariants and Gromov–Witten invariants. In the following, we recall the main asymptotic statements about real Hurwitz numbers from [7, 16, 22]. Denote by $s_{\text{pol}}^{\text{even}}(\lambda_1, \ldots, \lambda_k, 1^m)$ (resp. $s_{\text{pol}}^{\text{odd}}(\lambda_1, \ldots, \lambda_k, 1^m)$) the signed counts of real polynomials of even (resp. odd) degree with reduced ramification profiles $\lambda_1, \ldots, \lambda_k$ defined in [16].

Theorem 1.2 ([16, Theorem 5]) Assume that in each partition $\lambda_i, i \in \{1, \ldots, k\}$, every even number appears an even number of times and at most one odd number appears an odd number of times. Then, we have

$$\log |s_{\text{pol}}^{\text{even}}(m)| \sim m \log m \sim \log h_{\text{pol}}^{\text{even},C}(m), \text{ as } m \to \infty,$$

where $s_{\text{pol}}^{\text{even}}(m) = s_{\text{pol}}^{\text{even}}(\lambda_1, \ldots, \lambda_k, 1^m)$, and $h_{\text{pol}}^{\text{even},C}(m)$ is the corresponding counts of complex polynomials of even degree.

Assume that in each partition $\lambda_i, i \in \{1, \ldots, k\}$, at most one even number appears an odd number of times and at most one odd number appears an odd number of times. Then, we have

$$\log |s_{\text{pol}}^{\text{odd}}(m)| \sim m \log m \sim \log h_{\text{pol}}^{\text{odd},C}(m), \text{ as } m \to \infty,$$

where $s_{\text{pol}}^{\text{odd}}(m) = s_{\text{pol}}^{\text{odd}}(\lambda_1, \ldots, \lambda_k, 1^m)$, and $h_{\text{pol}}^{\text{odd},C}(m)$ is the corresponding counts of complex polynomials of odd degree.

Let $S_{\text{rat}}(\lambda_1, \ldots, \lambda_k)$ denote the signed counts of real simple rational functions with reduced ramification profiles $\lambda_1, \ldots, \lambda_k$ defined in [7].

Theorem 1.3 [7, Theorem 1.3] Assume that in each partition $\lambda_i, i \in \{1, \ldots, k\}$, at most one even number appears an odd number of times and at most one odd number appears an odd number of times. Then, we have

$$\log |s_{\text{rat}}(m)| \sim m \log m \sim \log h_{\text{rat}}^C(m), \text{ as } m \to \infty, \text{ and } \sum |\lambda_i| + m \equiv 0 \text{mod}2,$$

where $s_{\text{rat}}(m) = S_{\text{rat}}(\lambda_1, \ldots, \lambda_k, 1^m)$, and $h_{\text{rat}}^C(m)$ is the corresponding counts of complex simple rational functions. When $\sum |\lambda_i| + m \equiv 1 \text{mod}2$, if the partitions $\lambda_i, i \in \{1, \ldots, k\}$, satisfy the above condition and an extra parity condition, $|s_{\text{rat}}(m)|$ is also logarithmically equivalent to $h_{\text{rat}}^C(m)$. 

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A tropical cover is a continuous map from a connected metric graph to \( \mathbb{R} \cup \{ \pm \infty \} \) satisfying certain conditions (see Definition 2.7 for more details). There is a multiplicity associated with each tropical cover in [10, 20]. Markwig and Rau [20] gave a tropical interpretation of real double Hurwitz numbers via the weighted count of tropical covers. That is the tropical cover with odd multiplicity which is called zigzag cover. Rau [22] observed that the number of zigzag covers, \( Z_g(\lambda, \mu) \), is independent of the number of real positive simple branch points. From Markwig and Rau’s tropical computation of real double Hurwitz numbers in [20], \( Z_g(\lambda, \mu) \) is a lower bound for real double Hurwitz numbers.

**Theorem 1.4** [22, Theorem 5.10] Fix \( g \in \mathbb{N} \), and partitions \( \lambda, \mu \) with \( |\lambda| = |\mu| \). Assume that the number of odd elements which appear an odd number of times in \( \lambda \) plus the number of odd elements which appear an odd number of times in \( \mu \) is 0 or 2. Then, \( z_{g,\lambda,\mu}(2m) \) and \( h_{g,\lambda,\mu}^C(2m) \) are logarithmically equivalent,

\[
\log z_{g,\lambda,\mu}(2m) \sim 4m \log m \sim \log h_{g,\lambda,\mu}^C(2m), \text{ as } m \to \infty,
\]

where \( z_{g,\lambda,\mu}(2m) = Z_g((\lambda, 1^{2m}), (\mu, 1^{2m})) \).

**Remark 1.5** The assumption on odd elements in \( \lambda \) and \( \mu \) in [22, Theorem 5.10] is a necessary condition to guarantee the existence of zigzag covers.

We provide a new lower bound for real double Hurwitz numbers to prove our main result. A real tropical cover is a pair consisting of a tropical cover and a colouring of it. In the correspondence theorem [20], the real double Hurwitz number \( H_R^g(\lambda, \mu; s) \) is expressed as a weighted sum over isomorphism classes of real tropical covers. From [22, Proposition 4.8 and Lemma 4.14], we know that the set of zigzag covers is exactly the set of tropical covers admitting a colouring compatible with any splitting of real branch points. We have an observation that \( H_R^g(\lambda, \mu; s) = H_R^g(\lambda, \mu; r-s) \), where \( r \) is the number of real simple branch points of the ramified covers counted by \( H_R^g(\lambda, \mu; s) \) (see Proposition 2.5). The integer \( r = l(\lambda) + l(\mu) + 2g - 2 \) is determined by Riemann–Hurwitz formula. In order to find a lower bound for \( H_R^g(\lambda, \mu; s) \), \( 0 \leq s \leq r \), we only need to find a lower bound for \( H_R^g(\lambda, \mu; s) \), \( \left\lceil \frac{r}{2} \right\rceil \leq s \leq r \). Our idea is to characterize a set of tropical covers such that for any \( s \geq \left\lceil \frac{r}{2} \right\rceil \) the tropical cover in this set admits a colouring with \( s \) positive branch points. From [20], we know that the number of these covers is a lower bound for real double Hurwitz numbers. It is easy to see that this lower bound is bigger than the number of zigzag covers, because some tropical covers with even multiplicity are also counted. We call this new lower bound the effective number, and denote it by \( E_g(\lambda, \mu) \) (see Definition 3.8). We have

\[
Z_g(\lambda, \mu) \leq E_g(\lambda, \mu) \leq H_R^g(\lambda, \mu; s) \leq H_C^g(\lambda, \mu).
\]

**Theorem 1.6** Fix \( g \in \mathbb{N} \), and partitions \( \lambda, \mu \) with \( |\lambda| = |\mu| \). Suppose that the sum of odd numbers which appear odd number of times in \( \lambda \) is greater than or equal to the sum of odd numbers which appear odd number of times in \( \mu \), then \( e_{g,\lambda,\mu}(m) \) and
$h_{g,\lambda,\mu}(m)$ are logarithmically equivalent for even $m$:

$$\log e_{g,\lambda,\mu}(m) \sim 2m \log m \sim \log h_{g,\lambda,\mu}(m), \text{ as } m \to \infty \text{ for even } m,$$

where $e_{g,\lambda,\mu}(m) = E_g((\lambda, 1^m), (\mu, 1^m))$.

**Remark 1.7** Theorem 1.1 is a straightforward application of Theorem 1.6 and the symmetry of $H^R_g(\lambda, \mu; s)$ in $\lambda$ and $\mu$ (see Proposition 2.6).

## 2 Double Hurwitz numbers

In this section, we recall some facts about double Hurwitz numbers. The readers may refer to [3, 5, 20] for more details.

### 2.1 Complex double Hurwitz numbers

We fix two integers $d \geq 1$, $g \geq 0$, and let $\lambda$ and $\mu$ be two partitions of $d$. Fix a collection of $r = l(\lambda) + l(\mu) + 2g - 2$ points $p = \{p_1, \ldots, p_r\} \subset \mathbb{C}P^1 \setminus \{0, \infty\}$.

**Definition 2.1** A complex Hurwitz cover of type $(g, \lambda, \mu, p)$ is a degree $d$ holomorphic map $\pi : C \to \mathbb{C}P^1$ such that:

- $C$ is a connected Riemann surface of genus $g$;
- $\pi$ ramifies with profiles $\lambda$ and $\mu$ over $0$ and $\infty$, respectively;
- all the points in $p$ are simple branch points of $\pi$;
- $\pi$ is unramified everywhere else.

An isomorphism of two complex Hurwitz covers $\pi_1 : C_1 \to \mathbb{C}P^1$ and $\pi_2 : C_2 \to \mathbb{C}P^1$ is an isomorphism of Riemann surfaces $\varphi : C_1 \to C_2$ such that $\pi_1 = \pi_2 \circ \varphi$. The complex double Hurwitz number is

$$H^C_g(\lambda, \mu) = \sum_{[\pi]} \frac{1}{|\text{Aut}^C(\pi)|},$$

where we sum over all isomorphism classes of complex Hurwitz covers of type $(g, \lambda, \mu, p)$. It is a classical result that this number does not depend on the positions of $p$ [5, 11].

There is also an equivalent way to define complex double Hurwitz number via symmetric groups. Let $S_d$ denote the symmetric group of order $d$. We denote by $C(\sigma) \vdash S_d$ the cycle type of $\sigma \in S_d$. Let $d$, $g$, $\lambda$ and $\mu$ be as above.

**Definition 2.2** A factorization of type $(g, \lambda, \mu)$ is a tuple $(\sigma_1, \tau_1, \ldots, \tau_r, \sigma_2)$ of elements of $S_d$ such that:

- $\sigma_2 \cdot \tau_r \cdot \cdots \cdot \tau_1 \cdot \sigma_1 = \text{id}$;
- $r = l(\lambda) + l(\mu) + 2g - 2$;
\( C(\sigma_1) = \lambda, C(\sigma_2) = \mu, C(\tau_i) = (2, 1, \ldots, 1), i = 1, \ldots, r; \)

the subgroup generated by \( \sigma_1, \sigma_2, \tau_1, \ldots, \tau_r \) acts transitively on the set \( \{1, \ldots, d\} \).

We denote by \( \mathcal{F}(g, \lambda, \mu) \) the set of all factorizations of type \((g, \lambda, \mu)\).

**Theorem 2.3** (Hurwitz [5, 11]) Let \( d \geq 1, g \geq 0 \) be two integers, \( \lambda \) and \( \mu \) be two partitions of \( d \). Then,

\[
H^g_C(\lambda, \mu) = \frac{1}{d!} |\mathcal{F}(g, \lambda, \mu)|.
\]

**2.2 Real double Hurwitz numbers**

Let \( g, d, \lambda \) and \( \mu \) be as above. In the rest of this paper, we assume that the set of simple branch points \( p = \{p_1, \ldots, p_r\} \) is a subset of \( \mathbb{R}P^1 \setminus \{0, \infty\} \), and satisfies \( p_1 < \ldots < p_r \).

**Definition 2.4** A real Hurwitz cover of type \((g, \lambda, \mu, p)\) is a tuple \((\pi, \tau)\) such that

\[ \pi : C \rightarrow \mathbb{C}P^1 \]

\[ \tau : C \rightarrow C \]

is an anti-holomorphic involution such that \( \pi \circ \tau = \text{conj} \circ \pi \).

An isomorphism of two real Hurwitz covers \((\pi_1 : C_1 \rightarrow \mathbb{C}P^1, \tau_1)\) and \((\pi_2 : C_2 \rightarrow \mathbb{C}P^1, \tau_2)\) is an isomorphism of complex Hurwitz covers \( \varphi : C_1 \rightarrow C_2 \) such that \( \varphi \circ \tau_1 = \tau_2 \circ \varphi \). Let \( s = |p \cap \mathbb{R}^+| \). The real double Hurwitz number is

\[
H^g_R(\lambda, \mu; s) = \sum_{[(\pi, \tau)]} \frac{1}{|\text{Aut}^R(\pi, \tau)|},
\]

where we sum over all isomorphism classes of real Hurwitz covers of type \((g, \lambda, \mu, p)\).

Note that the integer \( H^g_R(\lambda, \mu; s) \) depends on the positions of points in \( p \).

The symmetric group can also be used to study real double Hurwitz number [2, 10]. In the Appendix, we give an equivalent description of real double Hurwitz number using symmetric group (see Lemma A.2).

**Proposition 2.5** Let \( d \geq 1, g \geq 0 \) be two integers, and \( \lambda, \mu \) be two partitions of \( d \). Suppose that \( 0 \leq s \leq r \), where \( r = l(\lambda) + l(\mu) + 2g - 2 \). Then,

\[
H^g_R(\lambda, \mu; s) = H^g_R(\lambda, \mu; r - s).
\]

**Proof** For any real Hurwitz cover \((\pi, \tau)\) of type \((g, \lambda, \mu, p)\), we compose the cover map \( \pi \) with the map

\[
-1 : \mathbb{C}P^1 \rightarrow \mathbb{C}P^1
\]

\[ z \mapsto -z, \]

then we obtain a real Hurwitz cover \((\pi', \tau)\) of type \((g, \lambda, \mu, -p)\), where \( -p = \{-p_1, -p_2, \ldots, -p_r\} \). Suppose that \( s = |p \cap \mathbb{R}^+| \). It is easy to see that |\text{Aut}^R(\pi, \tau)| =
Since the map $-1 : \mathbb{C}P^1 \to \mathbb{C}P^1$ induces a bijection between the set of all isomorphism classes of real Hurwitz covers of type $(g, \lambda, \mu, p)$ and the set of type $(g, \lambda, \mu, -p)$, we obtain that $H^R_g(\lambda, \mu; s) = H^R_g(\lambda, \mu; r - s)$.

**Proposition 2.6** Let $d, g, \lambda, \mu, s$ and $r$ be the same as Proposition 2.5. Then,

$$H^R_g(\lambda, \mu; s) = H^R_g(\mu, \lambda; s).$$

**Proof** We compose any real Hurwitz cover $(\pi, \tau)$ of type $(g, \lambda, \mu, p)$ with the map

$$\psi : \mathbb{C}P^1 \to \mathbb{C}P^1 \quad \quad z \mapsto \frac{1}{z},$$

then we obtain a real Hurwitz cover $(\pi', \tau)$ of type $(g, \mu, \lambda, p')$, where $p' = \left\{ \frac{1}{p_1}, \ldots, \frac{1}{p_r} \right\}$. A same argument as the proof of Proposition 2.5 implies that $H^R_g(\lambda, \mu; s) = H^R_g(\mu, \lambda; s)$.

**2.3 Tropical double Hurwitz numbers**

Let us recall some definitions first. The readers may refer to [3, 10, 20, 22] for more details. Let $\Gamma$ be a connected graph without 2-valent vertices. We call 1-valent vertices of $\Gamma$ the leaves, and the higher-valent vertices are called the inner vertices. The edges adjacent to a 1-valent vertex are called ends. Edges which are not ends are called inner edges. By $\Gamma^\circ$ we denote the subgraph obtained by removing the 1-valent vertices of $\Gamma$. The number $g = b_1(\Gamma)$, the first Betti number, is called the genus of $\Gamma$. A tropical curve $C$ is a connected metric graph without 2-valent vertices such that the length of an end is $\infty$, and the length $\ell(e) \in \mathbb{R}$ of an inner edge $e$ is finite. Note that the tropical projective line $T\mathbb{P}^1$ is considered as $\mathbb{R} \cup \{\pm \infty\}$. Throughout this paper, except $T\mathbb{P}^1$, we only consider graph without two-valent vertices. An isomorphism $\Phi : C_1 \to C_2$ of two tropical curves is an isometric homeomorphism $\Phi : C_1^\circ \to C_2^\circ$.

**Definition 2.7** A tropical cover $\varphi : C \to T\mathbb{P}^1$ is a continuous map satisfying:

- The image of any inner vertex of $C$ under $\varphi$ is contained in $\mathbb{R}$. Let $\mathcal{X}$ denote the set of images of inner vertices of $C$, and we call $\mathcal{X}$ the inner vertices of $T\mathbb{P}^1$.
- $\varphi^{-1}(\infty) \neq \emptyset$, $\varphi^{-1}(-\infty) \neq \emptyset$, and $\varphi^{-1}(\infty) \cup \varphi^{-1}(-\infty)$ is the set of leaves of $C$.
- $\varphi$ is a piecewise linear map: for any edge $e$ of $C$, we interpret $e$ as an interval $[0, \ell(e)]$, then there is a positive integer $\omega(e)$ such that $\varphi(t) = \pm \omega(e)t + \varphi(0)$, $\forall t \in [0, \ell(e)]$. The integer $\omega(e)$ is called the weight of $e$. 

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• For any vertex \( v \in C \), we choose an edge \( e' \subset T \mathbb{P}^1 \) adjacent to \( \varphi(v) \). Then, the integer
\[
\deg(\varphi, v) := \sum_{e \text{ edge of } C, v \in e, \varphi(e) = e'} \omega(e)
\]
does not depend on the choice of \( e' \). This is called the balancing or harmonicity condition.

Let \( \varphi : C \rightarrow T \mathbb{P}^1 \) be a tropical cover. The sum
\[
\deg(\varphi) := \sum_{e \text{ edge of } C, \varphi(e) = e'} \omega(e)
\]
is independent of \( e' \). The integer \( \deg(\varphi) \) is the degree of \( \varphi \). Let \( d \geq 1 \), \( g \geq 0 \), and \( \lambda, \mu \) be two partitions of \( d \). Suppose \( r = l(\lambda) + l(\mu) + 2g - 2 > 0 \). Fix \( r \) points \( \underline{x} = \{x_1, \ldots, x_r\} \subset \mathbb{R} \) satisfying \( x_1 < \ldots < x_r \).

**Definition 2.8** A tropical cover \( \varphi : C \rightarrow T \mathbb{P}^1 \) of type \( (g, \lambda, \mu, \underline{x}) \) is a tropical cover of degree \( d \) such that
\begin{itemize}
  \item \( C \) is a tropical curve of genus \( g \);
  \item The tuple of weights of ends adjacent to leaves mapping to \(-\infty\) is \( \lambda \), the tuple of weights of ends adjacent to leaves mapping to \(+\infty\) is \( \mu \);
  \item Each \( x_i \in \underline{x} \) is the image of an inner vertex of \( C \).
\end{itemize}

An isomorphism of two tropical covers \( \varphi_1 : C_1 \rightarrow T \mathbb{P}^1 \) and \( \varphi_2 : C_2 \rightarrow T \mathbb{P}^1 \) is an isomorphism \( \psi : C_1 \rightarrow C_2 \) of tropical curves such that \( \varphi_1 = \varphi_2 \circ \psi \). The multiplicity of a tropical cover \( \varphi \) is defined to be
\[
\mult^C(\varphi) := \frac{1}{|\text{Aut}(\varphi)|} \prod_{e \text{ inner edge of } C} \omega(e).
\]

A symmetric cycle is a pair of inner edges of the same weight and adjacent to the same two vertices. A symmetric fork is a pair of ends of the same weight adjacent to a same vertex. Note that the group of automorphisms of a tropical cover \( \varphi : C \rightarrow T \mathbb{P}^1 \) is induced by the interchange of two edges in symmetric cycles and symmetric forks.

**Theorem 2.9** [3, Theorem 5.28] The complex Hurwitz number \( H^C_\mathbb{C}(\lambda, \mu) \) is equal to
\[
H^C_\mathbb{C}(\lambda, \mu) = \sum_{[\varphi]} \mult^C(\varphi),
\]
where we sum over all isomorphism classes \([\varphi]\) of tropical covers of type \( (g, \lambda, \mu, \underline{x}) \).
2.4 Real tropical double Hurwitz numbers

In this subsection, we will review some basic facts about real tropical double Hurwitz numbers. Our notations follow closely with [22, Section 3]. Let \( \varphi : C \to T \mathbb{P}^1 \) be a tropical cover. An edge of even or odd weight is called even or odd edge, respectively. A symmetric cycle (resp. fork) consisting of a pair of even or odd inner edges (resp. ends) is called an even or odd symmetric cycle (resp. fork). The following notations will be used in the rest of this paper.

- \( \text{Sym}(\varphi) \) denotes the set of symmetric cycles and symmetric odd forks.
- \( \text{SymC}(\varphi) \subset \text{Sym}(\varphi) \) is the set of symmetric cycles.
- For \( T \subset \text{Sym}(\varphi) \), \( C \setminus T^\circ \) is the subgraph of \( C \) obtained by removing the interior of the edges contained in \( T \).
- \( E(T) \) is the set of even inner edges in \( C \setminus T^\circ \).

**Definition 2.10** [22, Definition 3.3] A colouring \( \rho \) of a tropical cover \( \varphi : C \to T \mathbb{P}^1 \) consists of a choice of subset \( T_\rho \subset \text{Sym}(\varphi) \) and a choice of a colour red or blue for every component of the subgraph of even edges of \( C \setminus T_\rho^\circ \).

A real tropical cover is a tuple \( (\varphi, \rho) \) of a tropical cover \( \varphi \) and a colouring \( \rho(\varphi) \) of \( \varphi \). An isomorphism of two real tropical covers is an isomorphism of these two tropical covers that respects the colouring. The real multiplicity of a real tropical cover is

\[
\text{mult}^R(\varphi, \rho) := 2|E(T_\rho)| - |\text{Sym}(\varphi)| \prod_{c \in T_\rho \cap \text{SymC}(\varphi)} \omega(c),
\]

where \( \omega(c) \) is the weight of one edge of the symmetric cycle \( c \).

**Remark 2.11** The multiplicity introduced here equals the one in [22, Definition 3.3]. In [22, Definition 3.3], the symmetric cycle \( c \) was allowed to be chosen in \( \text{SymC}(\varphi) \setminus T_\rho \). We compensate the contribution of even cycles in \( \text{SymC}(\varphi) \setminus T_\rho \) by multiplying \( 2|E(\text{Sym}(\varphi))| \) by a factor:

\[
2|E(T_\rho)| = 2|E(\text{Sym}(\varphi))| + 2k,
\]

where \( k \) is the number of even symmetric cycles in \( \text{SymC}(\varphi) \setminus T_\rho \).

Let \( (\varphi, \rho) \) be a real tropical cover. An inner vertex \( x_i \in x \) is called a positive or negative point if it is the image of a 3-valent vertex of \( C \) which is depicted in Fig. 1 or Fig. 2, respectively, up to reflection along a vertical line. We denote by \( x^+ \) and \( x^- \) the collection of positive and negative points in \( x \), respectively. Note that a colouring \( \rho(\varphi) \) of a tropical cover \( \varphi \) induces a splitting of \( x = x^+ \sqcup x^- \) into positive and negative branch points.

**Theorem 2.12** [20, Corollary 5.9] Let \( d, g, \lambda, \mu, x \subset \mathbb{R} \) be as above. Suppose \( x = x^+ \sqcup x^- \) is a splitting such that \( |x^+| = s \). Then,

\[
H^x_g(\lambda, \mu; s) = \sum_{[(\varphi, \rho)]} \text{mult}^R(\varphi, \rho),
\]
where we sum over all isomorphism classes \( [(\varphi, \rho)] \) of real tropical covers of type \((g, \lambda, \mu, x)\) whose positive and negative branch points reproduce the splitting \(x^+, x^-\).

**Remark 2.13** We use the real multiplicity introduced in [22] which differs from the one in [20]. The subset \( T_\rho \) of a colouring \( \rho \) of a tropical cover \( \varphi : C \to \mathbb{T}P^1 \) was allowed to contain even symmetric forks in [20]. The readers may refer to [22, Remark 3.5] for more detailed analysis on the difference between these two definitions.

### 3 Zigzag covers and effective non-zigzag covers

From Proposition 2.5, we know that if \( H \) is a lower bound for \( H^R_g(\lambda, \mu; s), \left\lceil \frac{r}{2} \right\rceil \leq s \leq r \), \( H \) is also a lower bound for \( H^R_g(\lambda, \mu; s), 0 \leq s \leq r \). In the rest of this paper, we find some tropical covers with even weight which contribute to \( H^R_g(\lambda, \mu; s) \) for \( \left\lceil \frac{r}{2} \right\rceil \leq s \leq r \).

Let us recall the lower bound established in [22] and the properties of it. A *string* \( S \) in a tropical curve \( C \) is a connected subgraph such that \( S \cap C^\circ \) is a closed submanifold of \( C^\circ \).

**Definition 3.1** [22, Definition 4.4] A zigzag cover is a tropical cover \( \varphi : C \to \mathbb{T}P^1 \) if there is a subset \( S \subset C \setminus \text{Sym}(\varphi) \) satisfying

- \( S \) is either a string of odd edges or consists of a single inner vertex;

- \( S \) turns or not here. The number of cycles in the first two types can be arbitrary.
Fig. 4 Bridge edges connecting two strings. The bending behaviour of the strings matters here.

- the connected components of $C \setminus S$ are of the type depicted in Fig. 3. In Fig. 3, all the cycles and forks are symmetric and of odd weight.

**Remark 3.2** For the convenience, in Fig. 3 and in the rest of this paper, we use the following notations: the variables $o, o_1, o_2, \ldots$ are used to denote odd integers, and $e, e_1, e_2, \ldots$ are used to denote even integers.

**Lemma 3.3** [22, Lemma 4.3] For any real tropical cover $(\phi, \rho)$ the multiplicity $\operatorname{mult}^\mathbb{R}(\phi, \rho)$ is an integer whose parity is independent of the colouring $\rho$.

**Proposition 3.4** [22, Proposition 4.7] The real tropical cover $(\phi, \rho)$ is of odd multiplicity if and only if $\phi$ is a zigzag cover.

The tropical covers with even weight which we are looking for are characterized by the following definition.

**Definition 3.5** An effective non-zigzag cover of type $(g, \lambda, \mu, x)$ is a tropical cover $\phi : C \to \mathbb{TP}^1$ of type $(g, \lambda, \mu, x)$ satisfying the following conditions:

- there are $n$ strings $S_1, \ldots, S_n \subset C \setminus \operatorname{Sym}(\phi)$ of odd edges, $n > 1$.
- the connected components of $C \setminus (\bigcup_{i=1}^n S_i)$ are of the type depicted in Fig. 3 and Fig. 4. In Fig. 3, all the cycles and forks are symmetric of odd weight. In Fig. 4, two strings are connected by exactly one bridge edge.
- Every inner vertex of the bridge edges depicted in Fig. 4 is mapped to some $x_j$ by $\phi$ with $j \leq \lceil \frac{r}{2} \rceil$.

**Remark 3.6** From Lemma 3.3 and Proposition 3.4, we know that the multiplicity of an effective non-zigzag cover is a positive even integer.

**Proposition 3.7** Let $\phi$ be an effective non-zigzag cover simply branched at $x$. If $x = x^+ \sqcup x^-$ is a splitting such that $\{x_1, \ldots, x_{\lceil \frac{r}{2} \rceil}\} \subset x^+$. Then, there is a unique colouring $\rho$ of $\phi$ such that the real tropical cover $(\phi, \rho)$ has positive and negative branch points as the splitting $x = x^+ \sqcup x^-$. 

**Proof** Suppose that $v$ is an inner vertex of $S_i$, $i = 2, \ldots, n$. If the even edge $E_i$ adjacent to $v$ is the first type depicted in Fig. 4, the colouring of $E_i$ is blue. Otherwise, the colouring of $E_i$ is red.

Assume that $v_1 \in S_1$ is a vertex from which a given tail $C_1$ depicted in Fig. 3 emanates. The colour rules from Figs. 1 and 2 induce a unique colouring of even edges of this tail. The method to impose colouring is described in the proof of [22, Proposition 4.8], so we omit it here. \qed
The zigzag number $Z_g(\lambda, \mu)$ is the number of zigzag covers of type $(g, \lambda, \mu, x)$ [22, Definition 4.9].

**Definition 3.8** The effective non-zigzag number $Z'_g(\lambda, \mu)$ is twice the number of effective non-zigzag covers of type $(g, \lambda, \mu, x)$, and the sum $E_g(\lambda, \mu) = Z_g(\lambda, \mu) + Z'_g(\lambda, \mu)$ is called the effective number.

From Remark 3.6, the multiplicity of an effective non-zigzag cover is at least 2. Therefore, we take twice the number of effective non-zigzag covers as the effective non-zigzag number.

**Remark 3.9** In [22, Remark 5.3], a combinatorial description of tropical covers was given. Since we assume the points $x$ satisfying $x_1 < \ldots < x_r$, the fourth condition of Definition 3.5 is actually a condition on the total order of inner vertices of $C$. Let $t_C$ be the number of total orders, satisfying the fourth condition of Definition 3.5 and compatible with the partial order induced by the orientation, of the inner vertices of $C$. It follows from [22, Remark 5.3] that $Z'_g(\lambda, \mu)$ depends on the number of tropical curves $C$ satisfying the first three conditions of Definition 3.5 and $t_C$. So $Z'_g(\lambda, \mu)$ does not depend on the choice of $x \subset \mathbb{R}$. Hence, the effective number $E_g(\lambda, \mu)$ does not depend on $x$.

**Proposition 3.10** Fix an integer $g \geq 0$, and two partitions $\lambda, \mu$ such that $|\lambda| = |\mu|$, $\left.\{\lambda, \mu\}\right\} \not\in \{(2k), (k, k)\}$ and $r = l(\lambda) + l(\mu) + 2g - 2 > 0$. Then, the number of real ramified covers is bounded from below by the effective number and they have the same parity:

$$
Z_g(\lambda, \mu) \leq E_g(\lambda, \mu) \leq H^R_g(\lambda, \mu; s) \leq H^C_g(\lambda, \mu),
$$

$$
Z_g(\lambda, \mu) \equiv E_g(\lambda, \mu) \equiv H^R_g(\lambda, \mu; s) \equiv H^C_g(\lambda, \mu) \mod 2.
$$

**Proof** It is straightforward from [22, Theorem 4.10] and Definition 3.8. □

**Remark 3.11** Note that $\{\lambda, \mu\} \not\in \{(2k), (k, k)\}$ and $l(\lambda) + l(\mu) + 2g - 2 > 0$ is a sufficient condition to guarantee that real and complex double Hurwitz numbers are actual counts of covers (see [22, Remark 2.5, Remark 2.6] for more details).

We end this subsection by providing some examples of zigzag covers and effective non-zigzag covers.

**Example 3.12** We present two different kinds of zigzag covers in Fig. 5, two types of effective non-zigzag covers in Fig. 6, and a tropical cover that is not zigzag, nor effective non-zigzag cover in Fig. 7.

### 4 Asymptotic behaviour of effective non-zigzag covers

In this section, we follow closely the arguments of [22, Section 5] to investigate the asymptotic behaviour of effective non-zigzag covers. A sufficient condition to
guarantee the existence of effective non-zigzag covers is given, and the asymptotic behaviour of effective numbers is also considered.

We first recall the definition of tail decomposition of a partition $\lambda$ introduced in [22]. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_m)$ be two partitions of $d$. We use $\lambda_i \in \lambda$ to mean that $\lambda_i$ is a part of $\lambda$. We denote

\[
2\lambda := (2\lambda_1, 2\lambda_2, \ldots, 2\lambda_n),
\]

\[
\lambda^2 := (\lambda_1, \lambda_1, \ldots, \lambda_n, \lambda_n),
\]

\[
(\lambda, \mu) := (\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m).
\]

For any partition $\lambda$, there is a unique decomposition $\lambda = (2\lambda_{2e}, 2\lambda_{2o}, \lambda_{o,o}, \lambda_o)$ such that

- every part $\lambda_{2e}^i$ in $\lambda_{2e}$ is an even number;
- every part $\lambda_{2o}^i$ or $\lambda_{o,o}^i$ in $\lambda_{2o}$ or $\lambda_{o,o}$, respectively, is an odd number;
- every part $\lambda_o^i$ in $\lambda_o$ is an odd number and the odd number $\lambda_o^i$ does not appear more than once in $\lambda_o$.

Such a decomposition is called the tail decomposition of $\lambda$. Let $l(\lambda_{1,1})$ denote the number of ones which appear in $\lambda_{o,o}$. 
**Example 4.1** Assume $\lambda = (7, 6, 4, 5, 3, 1, 1, 1)$, then the tail decomposition of $\lambda$ is $\lambda = (2\lambda_{2e}, 2\lambda_{2o}, \lambda_{o, o}, \lambda_{o})$, where $\lambda_{2e} = (2), \lambda_{2o} = (3), \lambda_{o, o} = (5, 1), \lambda_{o} = (7, 3, 1)$.

**Definition 4.2** An effective non-zigzag cover $\varphi : C \to T\mathbb{P}^1$ of type $(g, \lambda, \mu, x)$ is properly mixed if all tails of the type depicted in Fig. 3 are attached to a unique string $S$ such that the branch points $x_1 < x_2 < \ldots < x_r$ grouped in subsets of respective sizes $f, t_1, b_1, u - 1, b_r, 2g, t_r$ occur as images of

- The vertices of the bridge edges of $C$;
- The symmetric fork vertices of tails of type $o, o$ labelled by $\lambda$ attaching to $S$;
- Bends of $S$ with peaks pointing to the left;
- Unbent vertices of $S$ other than the intersection point of the bridge edge $E$ and the string $S$;
- Bends of $S$ with peaks pointing to the right;
- The vertices of symmetric cycles located on tails labelled by $\mu$;
- The symmetric fork vertices of tails of type $o, o$ labelled by $\mu$ attaching to $S$.

Here, $f$ is the number of vertices of the bridge edges and $f \leq \left\lceil \frac{r}{2} \right\rceil$, $t_1$ or $t_r$ is the number of tails of the first type depicted in Fig. 3 labelled by $\lambda_{o, o}$ or $\mu_{o, o}$, respectively, $b_1$ or $b_r$ is the number of bends of the string $S$ with the peaks pointing to the left or right, respectively, and $u$ is the number of unbent vertices of $S$.

**Example 4.3** The effective non-zigzag cover depicted in Fig. 8 is properly mixed. In this effective non-zigzag cover, $f = 6$.

**Proposition 4.4** Fix $g \in \mathbb{N}$, partitions $\lambda, \mu$ with $|\lambda| = |\mu|$. Suppose that $l(\lambda_{o, o}) > 2$ and $\sum_{i=1}^{l(\lambda_{o, o})} \lambda_{o, o}^i \geq \sum_{i=1}^{l(\mu_{o, o})} \mu_{o}^i$.

(1) For sufficiently large even integer $m$, properly mixed effective non-zigzag covers $\varphi : C \to T\mathbb{P}^1$ of type $(g, (\lambda, 1^m), (\mu, 1^m), x)$ exist as in Definition 4.2, where $f = 2 \max(l(\lambda_{o, o}), l(\mu_{o, o})) - 2$. 

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(2) For a properly mixed effective non-zigzag cover \( \varphi : C \to T \mathbb{P}^1 \) obtained above, there exist constants \( N_1 \) and \( N_2 \) which are not dependent on \( m \) such that

\[
\begin{align*}
t_l & \geq \frac{m}{2} - N_1, \quad t_r \geq \frac{m}{2} - N_1, \\
b_l & \geq \frac{m}{2} - N_2, \quad b_r \geq \frac{m}{2} - N_2.
\end{align*}
\]

Proof (1) Since \( l(\lambda_o, \mu_o) = |\lambda| + |\mu| = 2d \mod 2 \), the number \( l(\lambda_o, \mu_o) \) is even. Suppose that \( l(\lambda_o) - l(\mu_o) = 2a \). We construct a properly mixed zigzag cover of type \((g, (\lambda, 1^m), (\mu, 1^m), x)\) by considering the three cases: \( a = 0, a > 0 \) and \( a < 0 \).

Case \( a \geq 0 \): We assume that \( m \) is a sufficiently large even integer, so when \( a > 0 \) it is possible to take \( 2a \) ones from \( (1^m) \) and add them to the partition \( \mu_o \). We consider the partitions \( \lambda_o \) and \( (\mu_o, 1^{2a}) \) which have the same length \( n = l(\lambda_o) \). Since the partition \( \lambda_o \) does not contain repeated elements, we can find an ordering of the elements \( \lambda_1, \ldots, \lambda_n \) of \( \lambda_o \) and \( \mu_1, \ldots, \mu_n \) of \( (\mu_o, 1^{2a}) \) such that \( w_k := \sum_{i=1}^{k} (\lambda_i - \mu_i) \) is different from zero for all \( k = 1, \ldots, n - 1 \), and \( w = \sum_{i=1}^{n} \lambda_i - \sum_{i=1}^{n-1} \mu_i > 0 \). The existence of such an ordering is guaranteed by the following reasons. Because there are always at least two distinct elements in \( \lambda_o \) to choose from, it is possible to choose an ordering which guarantees that \( w_k \) is nonzero for all \( k = 1, \ldots, n - 1 \). Moreover, once the maximal odd integer in \( \mu_o \) is chosen to be \( \mu_n \) in the ordering \( \mu_1, \ldots, \mu_n \), we obtain that \( w > 0 \) from the assumption that \( \sum_{i=1}^{l(\lambda_o)} \lambda_i \geq \sum_{i=1}^{l(\mu_o)} \mu_i \).

We construct a weighted and oriented graph \( C \) by the following recursive procedure:

(i) We start with a string \( S_1 \) with two ends which are labelled by \( \lambda_1 \) and \( \mu_1 \). The rule of equipping orientation on an end is that any end labelled by a part of \((\lambda, 1^m)\) (resp. \((\mu, 1^m)\)) is oriented inwards (resp. outwards). Therefore, the string \( S_1 \) is oriented from left to right.

(ii) We add a bridge edge \( E_k \) connecting \( S_k \) to the next string \( S_{k+1} \), where \( k = 1, \ldots, n - 1 \). Suppose that \( E_k \) intersects with \( S_k \) at \( u_k \) and intersects with \( S_{k+1} \) at \( v_{k+1} \), where \( k = 1, \ldots, n - 1 \). The weight of \( E_k \) is \( |w_k| \). If \( w_k > 0 \), the bridge edge \( E_k \) is oriented such that it points from \( u_k \) to \( v_{k+1} \) (i.e. \( E_k \) directs to the right when coming from \( S_k \)). Otherwise, \( E_k \) is oriented such that it points from \( v_{k+1} \) to \( u_k \) (i.e. \( E_k \) directs to the left when coming from \( S_k \)).

(iii) The next string \( S_{k+1} \) has two ends which are labelled by \( \lambda_{k+1} \) and \( \mu_{k+1} \), \( k = 1, \ldots, n - 1 \). Now, we choose the position of \( u_{k+1} \). If \( w_k > 0 \), we choose \( u_{k+1} \) on the right of \( v_{k+1} \). Otherwise, we add \( u_{k+1} \) on the left of \( v_{k+1} \) (see Fig. 9 for a local picture of the bridge vertices \( v_{k+1} \) and \( u_{k+1} \)). Note that this construction ensures that the internal bounded edge of \( S_{k+1} \), where \( k = 1, \ldots, n - 2 \), carries a weight and is oriented from left to right by the balancing condition at the inner vertices of \( S_{k+1} \). Hence, all strings \( S_1, \ldots, S_{n-1} \) are oriented from left to right without bends. Moreover, all bridge edges are of the first type depicted in Fig. 4.

(iv) We attach tails of the first, second or third type depicted in Fig. 3 which correspond to each part of \((\lambda, 1^m)_{o,o}, (\mu, 1^{m-2a})_{o,o}, (\lambda_2o, \mu_2o) \) or \((\lambda_2e, \mu_2e) \), respectively, to the final string \( S_n \) on the right of the final bridge vertex \( v_n \). Note that \( w = \sum_{i=1}^{n} \lambda_i - \sum_{i=1}^{n} \mu_i > 0 \), so attaching the tails on the right of the final bridge vertex \( v_n \) is possible (see Fig. 10 for a local picture of the bridge vertex \( v_n \)).
At last, we obtain a graph $C$. Since $m$ is large enough, there is at least one tail of the first type depicted in Fig. 3, on which we place $g$ balanced cycles. By the balancing condition, we extend the orientation and the weight function to all the edges of $C$.

(v) The orientation of edges of $C$ induces a partial order on the set of inner vertices. We now choose a total order of the inner vertices of the graph $C$ extending the partial order such that the set of bridge vertices $\{u_1, v_2, u_2, \ldots, v_{n-1}, u_{n-1}, v_n\}$ contains exactly the first $2n - 2$ vertices. By our construction, it is clear that this is possible.

For sufficiently large even integer $m$, it is obvious that $2n - 2 < \frac{\ell_2}{2} = \frac{l(\lambda_1^m) + l(\mu_1^m) + 2g - 2}{2}$, so the resulting tropical cover $\varphi : C \to \mathbb{T}P^1$ is properly mixed. See Fig. 11 for an example in the case $a = 0$, and see Fig. 12 for an example in the case $a > 0$.

Case $a < 0$: Since $m$ is large enough, we can add $-2a$ ones to $\lambda_o$ and consider the partitions $(\lambda_o, 1^{-2a})$ and $\mu_o$. They have the same length $n = l(\mu_o)$. Up to a reordering of the elements of $(\lambda_o, 1^{-2a})$ and $\mu_o$, we assume that $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ are two orderings of elements of $(\lambda_o, 1^{-2a})$ and $\mu_o$, respectively, such that $w_k := \sum_{i=1}^{k} (\lambda_i - \mu_i)$ is different from zero for all $k = 1, \ldots, n - 1$, and $w = \sum_{i=1}^{n} \lambda_i - \sum_{i=1}^{n-1} \mu_i > 0$. The existence of such orderings is clear.

We construct a weighted and oriented graph $C$ by the same recursive procedure given in the case $a \geq 0$. The only difference is that the tails of the first
An example of effective non-zigzag cover constructed in the case when $l(\lambda_0) = 3$, $l(\mu_0) = 1$

An example of effective non-zigzag cover constructed in the case when $l(\lambda_0) = 1$, $l(\mu_0) = 3$

type depicted in Fig. 3 attached to the final string $S_n$ corresponds to parts of $((\lambda, 1^m+2a)_{o,o}, (\mu, 1^m)_{o,o})$. The resulting tropical cover $\varphi : C \to T^{\mathbb{P}^1}$ is properly mixed of type $(g, (\lambda, 1^m), (\mu, 1^m), x)$ for sufficiently large $m$. See Fig. 13 for an example in the case $a < 0$.

Let $\varphi : C \to T^{\mathbb{P}^1}$ be a properly mixed tropical cover of type $(g, (\lambda, 1^m), (\mu, 1^m), x)$ obtained in (1) for sufficiently large even integer $m$. From the construction of graph $C$ in (1), the tails in Fig. 3 attached to the final string $S_n$ are labelled by $(2\lambda_2, 2\lambda_2, (\lambda, 1^{m-m_0})_{o,o})$ and $(2\mu_2, 2\mu_2, (\mu, 1^{m-m_1})_{o,o})$, where

$$m_0 = \begin{cases} 0, & \text{if } l(\lambda_0) \geq l(\mu_0); \\ l(\mu_0) - l(\lambda_0), & \text{if } l(\lambda_0) < l(\mu_0), \end{cases} \quad \text{and} \quad m_1 = \begin{cases} 0, & \text{if } l(\lambda_0) \leq l(\mu_0); \\ l(\lambda_0) - l(\mu_0), & \text{if } l(\lambda_0) > l(\mu_0). \end{cases}$$

When $m$ is even and large enough, it is easy to see that

$$t_l \geq \frac{m}{2} - N_1 \quad \text{and} \quad t_r \geq \frac{m}{2} - N_1,$$

where $N_1 = |l(\lambda_0) - l(\mu_0)|$.

Now, we analyse the asymptotic behaviours of $b_l, b_r$. Recall that $v_n$ is assumed to be the intersection point of $E_{n-1}$ and $S_n$ in the recursive procedure in (1). Let $e_1$ be the inner edge of $S_n$ which is an outgoing edge adjacent to the vertex $v_n$. Suppose that $\omega(e_1) = a$ and the right end of string $S_n$ is weighted by $b \in (\mu, 1^m).$ Let $\lambda' = (2\lambda_2, 2\lambda_2, 2(\lambda, 1^{m-m_0})_{o,o}, a)$ and $\mu' = (2\mu_2, 2\mu_2, 2(\mu, 1^{m-m_1})_{o,o}, b).$ We consider a set $S(\lambda', \mu')$ consisting of all sequences of the form $(k_0, k_1, k_2, \ldots, k_h)$, where $k_0 = a, k_h = b, k_{i+1} = k_i + \lambda'_s$ or $k_{i+1} = k_i - \mu'_t$ (each part of $(\lambda', \mu')$
is used exactly once here) for \(i \in \{1, \ldots, h - 1\}\). Let \(B(\lambda', \mu')\) denote the maximal number of sign changes that occur in such sequences. The integer \(B(\lambda', \mu')\) is the maximal number of bends of \(S_n\) that we can create in the construction of \(C\) in (1), so the weighted graph \(C\) can be chosen such that \(S_n \subset C\) has \(B(\lambda', \mu')\) bends. Then, we obtain that

\[
bl \geq \left\lfloor \frac{B(\lambda', \mu')}{2} \right\rfloor, \quad br \geq \left\lfloor \frac{B(\lambda', \mu')}{2} \right\rfloor.
\]

Let \((a, k_1, \ldots, k_{h-1}, b)\) be a sequence such that the first \(s\) entries \(k_1, \ldots, k_s\) are obtained by either adding a part of \((2\lambda_2e, 2\lambda_2o)\) to the previous entry or subtracting a part of \((2\mu_2e, 2\mu_2o)\) from the previous entry, and the last \(h - s - 1\) entries \(k_{s+1}, \ldots, k_{h-1}\) are obtained by either adding a part of \(2(\lambda, 1m-m_0)_{o,o}\) to the previous entry or subtracting a part of \(2(\mu, 1m-m_1)_{o,o}\) from the previous entry. Moreover, every element in \((\lambda', \mu')\) is used only once here. From the above construction, we get that \(s\) depends only on \((\lambda_2e, \lambda_2o)\) and \((\mu_2e, \mu_2o)\), and it does not depend on \(m\). Obviously, this sequence is contained in \(S(\lambda', \mu')\). By arranging the order on \(+2\) and \(-2\) suitably, we assume that there is a continuous segment \(k_j, k_{j+1}, \ldots, k_{j+m-N}\), where \(j > s + 1\) is an integer, in this sequence \((a, k_1, \ldots, k_{h-1}, b)\) which has the form

\[
\pm 1 \rightarrow \mp 1 \rightarrow \cdots \rightarrow \pm 1 \rightarrow \mp 1,
\]

for a fixed integer \(N > 0\). Note that \(N\) is not dependent on \(m\). Therefore, we get

\[
B(\lambda', \mu') \geq (m - N),
\]

for sufficiently large \(m\). Hence,

\[
bl \geq \frac{m}{2} - N_2, \quad br \geq \frac{m}{2} - N_2,
\]

where

\[
N_2 = \left\lfloor \frac{N}{2} \right\rfloor + 1.
\]

\textbf{Lemma 4.5} Given a properly mixed effective non-zigzag cover \(\varphi : C \rightarrow T\mathbb{P}^1\) of type \((g, \lambda, \mu, x)\), the number of properly mixed covers of that type is bounded from below by:

\[
t_1! \cdot b_1! \cdot b_r! \cdot t_r!.
\]

\textbf{Proof} We consider certain symmetric group actions on the inner vertices of \(C\) which are mapped to the last \(r - n_1\) simple branch points \(x_{n_1+1} < \cdots < x_r\) by the cover map \(\varphi\). Let \(OS\) be the ordered set of vertices of \(C\) which are mapped to the segment \(x_{n_1+1} < \cdots < x_{n_1+t_1}\). Any permutation on the vertices in \(OS\) produces a new order on these vertices, so any element in \(S_t\) gives a new properly mixed cover map. Similarly, any permutation on the subgroups of vertices which are mapped to segments in \(x_{n_1+1} < \cdots < x_r\) of length \(b_l, b_r\) and \(t_r\), respectively, produces a new properly mixed cover.
Therefore, the number of properly mixed cover is bounded from below by \(t_1! \cdot t_r! \cdot b_l! \cdot b_r!\). \(\square\)

**Proposition 4.6** Fix \(g \in \mathbb{N}\), partitions \(\lambda, \mu\) with \(|\lambda| = |\mu|\). Suppose that \(\sum_i \lambda'_i \geq \sum_i \mu'_o\) and \(l(\lambda_o, \mu_o) > 2\). Then, the logarithmic asymptotics for \(z'_{g,\lambda,\mu}(m)\) as \(m \to \infty\) for \(m\) even is at least \(2m \log m\), where \(z'_{g,\lambda,\mu}(m) = Z'_{g}((\lambda, 1^m), (\mu, 1^m))\).

**Proof** From Proposition 4.4 (1), there is a properly mixed effective non-zigzag cover \(\varphi : C \to T\mathbb{P}^1\) of type \((g, (\lambda, 1^m), (\mu, 1^m), x)\) for sufficiently large even integer \(m\). It follows from Lemma 4.5 and Proposition 4.4 (2) that

\[z'_{g,\lambda,\mu}(m) \geq \left(\frac{m}{2} - N_1\right)! \cdot \left(\frac{m}{2} - N_2\right)!^2.\]

Since \(\log((\frac{m}{2} - N_1)! \cdot (\frac{m}{2} - N_2)!^2) \sim 2m \log m\) as \(m \to \infty\), we obtain the required estimate of the logarithmic asymptotics for \(z'_{g,\lambda,\mu}(m)\). \(\square\)

**Proof of Theorem 1.6** Let \(H^C_g(m) = H^C_g((1^m), (1^m))\). From [6, Equation 5] and the proof of [22, Theorem 5.10], we know \(\log h^C_{g,\lambda,\mu}(m) \leq \log H^C_g(m) \sim 2m \log m\).

We assume that the partitions \(\lambda\) and \(\mu\) satisfy \(\sum_i \lambda'_i \geq \sum_i \mu'_o\). When \(m\) is even, the statement follows from [22, Theorem 5.10] for \(l(\lambda_o, \mu_o) \leq 2\) and from Proposition 4.6 for \(l(\lambda_o, \mu_o) > 2\).

**Proof of Theorem 1.1** Because of Proposition 2.6, one can assume without loss of generality that \(\sum_i \lambda'_i \geq \sum_i \mu'_i\). The statement for even \(m\) follows from Theorem 1.6. Applying the same argument to the partitions \((\lambda, 1)\) and \((\mu, 1)\), the statement also follows for odd \(m\) and hence completes the proof. \(\square\)

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**Data availability** Data sharing is not applicable to this article as no data sets were generated or analysed during the current study.

**Appendix A. Real double Hurwitz numbers via factorization**

In this appendix, we give an equivalent description of real double Hurwitz number via symmetric group.

**Definition A.1** A real factorization of type \((g, \lambda, \mu; s)\) is a tuple \((\gamma, \sigma_1, \tau_1, \ldots, \tau_r, \sigma_2)\) of elements of the symmetric group \(S_d\) satisfying:

- \(\sigma_2 \cdot \tau_r \cdot \cdots \cdot \tau_1 \cdot \sigma_1 = \text{id}\);
- \(r = l(\lambda) + l(\mu) + 2g - 2\).
The classical Hurwitz construction (see [11] or [5, Chapter 7]), we represent by
that $g$, $λ$, $μ$, and the monodromy actions of the loops $l_0, l_1, \ldots, l_r$ generate the fundamental group $π_1(\mathbb{C}P^1 \setminus \{0, \infty, p_1, \ldots, p_r\}, p_0)$. The action of complex conjugation on $π_1(\mathbb{C}P^1 \setminus \{0, \infty, p_1, \ldots, p_r\}, p_0)$ is determined by:

$$\text{conj}(l_i \cdots l_0) = (l_i \cdots l_0)^{-1}, \quad 0 \leq i \leq s;$$

$$\text{conj}(l_j \cdots l_{s+1}) = (l_j \cdots l_{s+1})^{-1}, \quad s + 1 \leq j \leq r. \quad (1)$$

A real factorization $(γ, σ_1, τ_1, \ldots, τ_r, σ_2)$ of type $(g, λ, μ; s)$ induces a real cover as follows: From the classical Hurwitz construction (see [11] or [5, Chapter 7]), we know that a tuple $(σ_1, τ_1, \ldots, τ_r, σ_2)$ satisfying the first four conditions in Definition A.1 induces a cover $π : C \to \mathbb{C}P^1$ with ramification profiles $λ$ and $μ$ over 0 and $∞$, respectively, and simple ramification over $p$. Moreover, $π^{-1}(p_0)$ are labelled, i.e. $π^{-1}(p_0) = \{q_1, \ldots, q_d\}$, and the monodromy actions of the loops $l_0, \ldots, l_r$ are represented by $σ_1, τ_1, \ldots, τ_r$, respectively. Suppose that $p ∈ C$ is an unramified point. Choose a path $α$ in $\mathbb{C}P^1 \setminus \{0, \infty, p_1, \ldots, p_r\}$ from $p_0$ to $π(p)$. Lift $α$ to a path $\tilde{α}$ in $C$ with endpoint $p$. Let $k$ be the starting point of $\tilde{α}$. Let $β = \text{conj}(α)$ be the conjugated path of $α$. Then, lift $β$ to a path $\tilde{β}$ with starting point $q_{γ(k)}$. Let $\tilde{p}$ be the

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**Fig. 14** Generators of $π_1(\mathbb{C}P^1 \setminus \{0, \infty, p_1, \ldots, p_r\}, p_0)$
endpoint of \( \tilde{\beta} \). We define \( \tau(p) = \bar{p} \). The fifth condition in Definition A.1 implies that \( \tau(p) \) is well defined. Then, one can extend \( \tau \) to \( C \) by standard arguments. From the construction, we know \( \pi \circ \tau = \text{conj} \circ \pi \). Actually, this construction gives a map \( \psi: \mathcal{F}_R(g, \lambda, \mu; s) \to \mathcal{R} \), where \( \mathcal{R} \) is the set of isomorphism classes of real Hurwitz covers of type \( (g, \lambda, \mu, \bar{p}) \). By a similar argument to the proof of [10, Lemma 2.3], we have \( \psi: \mathcal{F}_R(g, \lambda, \mu; s)/S_d \to \mathcal{R} \) is bijective, and \( \text{Stab}_{S_d}(T) = \text{Aut}(T) \), where \( T \in \mathcal{F}_R(g, \lambda, \mu; s) \) is a factorization. Then, we get Lemma A.2.

References

1. Bertrand, B., Brugallé, E., Mikhalkin, G.: Tropical open Hurwitz numbers. Rend. Semin. Mat. Univ. Padova 125, 157–171 (2011)
2. Cadoret, A.: Counting real Galois covers of the projective line. Pacific J. Math. 219(1), 53–81 (2005)
3. Cavalieri, R., Johnson, P., Markwig, H.: Tropical Hurwitz numbers. J. Algebraic Combin. 32(2), 241–265 (2010)
4. Cavalieri, R., Johnson, P., Markwig, H.: Wall crossings for double Hurwitz numbers. Adv. Math. 228(4), 1894–1937 (2011)
5. Cavalieri, R., Miles, E.: Riemann surfaces and algebraic curves: a first course in Hurwitz theory. London Mathematical Society Student Texts, vol. 87. Cambridge University Press, Cambridge (2016)
6. Dubrovin, B., Yang, D., Zagier, D.: Classical Hurwitz numbers and related combinatorics. Mosc. Math. J. 17(4), 601–633 (2017)
7. El Hilany, B., Rau, J.: Signed counts of real simple rational functions. J. Algebraic Combin. 52(3), 369–403 (2020)
8. Georgieva, P., Zinger, A.: Real Gromov-Witten theory in all genera and real enumerative geometry: construction. Ann. Math. (2) 188(3), 685–752 (2018)
9. Goulden, I.P., Jackson, D.M., Vakil, R.: Towards the geometry of double Hurwitz numbers. Adv. Math. 198(1), 43–92 (2005)
10. Guay-Paquet, M., Markwig, H., Rau, J.: The combinatorics of real double Hurwitz numbers with real positive branch points. Int. Math. Res. Not. IMRN 2016(1), 258–293 (2016)
11. Hurwitz, A.: Ueber Riemann'sche Flächen mit gegebenen Verzweigungspunkten. Math. Ann. 39(1), 1–60 (1891)
12. Itenberg, I., Kharlamov, V., Shustin, E.: Welschinger invariant and enumeration of real rational curves. Int. Math. Res. Not. 49, 2639–2653 (2003)
13. Itenberg, I., Kharlamov, V., Shustin, E.: Logarithmic equivalence of the Welschinger and the Gromov-Witten invariants. Russian Math. Surv. 59(6), 1093–1116 (2004)
14. Itenberg, I., Kharlamov, V., Shustin, E.: New cases of logarithmic equivalence of Welschinger and Gromov-Witten invariants. Proc. Steklov Inst. Math. 258(1), 65–73 (2007)
15. Itenberg, I., Kharlamov, V., Shustin, E.: Welschinger invariants of real del Pezzo surfaces of degree \( \geq 3 \). Math. Ann. 355(3), 849–878 (2013)
16. Itenberg, I., Zvonkine, D.: Hurwitz numbers for real polynomials. Comment. Math. Helv. 93(3), 441–474 (2018)
17. Johnson, P.: Double Hurwitz numbers via the infinite wedge. Trans. Amer. Math. Soc. 367(9), 6415–6440 (2015)
18. Kharlamov, V., Răsdeaconu, R.: Counting real rational curves on K3 surfaces. Int. Math. Res. Not. IMRN 2015(14), 5436–5455 (2015)
19. Li, A.-M., Zhao, G., Zheng, Q.: The number of ramified covering of a Riemann surface by Riemann surface. Comm. Math. Phys. 213(3), 685–696 (2000)
20. Markwig, H., Rau, J.: Tropical real Hurwitz numbers. Math. Z. 281(1–2), 501–522 (2015)
21. Okounkov, A., Pandharipande, R.: Gromov-Witten theory, Hurwitz theory, and completed cycles. Ann. Math. (2) 163(2), 517–560 (2006)
22. Rau, J.: Lower bounds and asymptotics of real double Hurwitz numbers. Math. Ann. 375(1–2), 895–915 (2019)
23. Shadrin, S., Shapiro, M., Vainshtein, A.: Chamber behavior of double Hurwitz numbers in genus 0. Adv. Math. 217(1), 79–96 (2008)
24. Shustin, E.: On higher genus Welschinger invariants of del Pezzo surfaces. Int. Math. Res. Not. IMRN 16, 6907–6940 (2015)
25. Welschinger, J.-Y.: Invariants of real symplectic 4-manifolds and lower bounds in real enumerative geometry. Invent. Math. 162(1), 195–234 (2005)
26. Welschinger, J.Y.: Spinor states of real rational curves in real algebraic convex 3-manifolds and enumerative invariants. Duke Math. J. 127(1), 89–121 (2005)

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