G-monopole classes, Ricci flow, and Yamabe invariants of 4-manifolds

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Abstract On a smooth closed oriented 4-manifold $M$ with a smooth action by a finite group $G$, we show that a $G$-monopole class gives the $L^2$-estimate of the Ricci curvature of a $G$-invariant Riemannian metric, and derive a topological obstruction to the existence of a $G$-invariant nonsingular solution to the normalized Ricci flow on $M$. In particular, for certain $m$ and $n$, $m\mathbb{CP}_2\#n\overline{\mathbb{CP}}_2$ admits an infinite family of topologically equivalent but smoothly distinct non-free actions of $\mathbb{Z}_d$ such that it admits no nonsingular solution to the normalized Ricci flow for any initial metric invariant under such an action, where $d > 1$ is a non-prime integer. We also compute the $G$-Yamabe invariants of some 4-manifolds with $G$-monopole classes and the orbifold Yamabe invariants of some 4-orbifolds.

Keywords Seiberg–Witten equations · Monopole class · Ricci flow · Einstein metric · Yamabe invariant

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1 Introduction

This article is a continuation of our previous paper [31] with a view to geometric applications. On a smooth closed oriented 4-manifold $M$, an element of $H^2(M; \mathbb{Z})$ is called a monopole class if it arises as the first Chern class of a Spin$^c$ structure of $M$ for which the Seiberg–Witten
equations

\[
\begin{align*}
D_A \Phi &= 0 \\
F_A^+ &= \Phi \otimes \Phi^* - \frac{1}{2} |\Phi|^2 \text{Id}
\end{align*}
\]

admit a solution \((A, \Phi)\) for every choice of a Riemannian metric on \(M\). It is well-known that a monopole class gives a lower bound of the \(L^2\)-norm of various curvatures for any Riemannian metric, and hence a necessary condition for the existence of an Einstein metric and the Yamabe invariant of the manifold can be obtained.

In order to detect a monopole class, one needs to compute Seiberg–Witten invariants gotten by the intersection theory on the moduli space of solutions of the Seiberg–Witten equations or further refined Bauer–Furuta invariant given by the stably-framed bordism class of the moduli space, which is equivalent to the homotopy class of the Seiberg–Witten equations as a map between configuration spaces with Sobolev norms. But in many important cases, those invariants are difficult to compute or get trivial.

In the meantime, sometimes we need a solution of the Seiberg–Witten equations for a specific metric rather than any Riemannian metric. In our previous paper [31], we considered the case when a 4-manifold \(M\) and its \(\text{Spin}^c\) structure \(s\) admit a smooth action by a compact Lie group \(G\), and defined a \(G\)-monopole class as an element of \(H^2(M; \mathbb{Z})\) which is the first Chern class of a \(G\)-equivariant \(\text{Spin}^c\) structure for which the Seiberg–Witten equations admit a \(G\)-invariant solution for every \(G\)-invariant Riemannian metric of \(M\).

In order to detect a \(G\)-monopole class, we need to compute \(G\)-monopole invariants obtained by the intersection theory on the moduli spaces of \(G\)-invariant solutions of the Seiberg–Witten equations, and \(G\)-Bauer–Furuta invariant given by the homotopy class of the Seiberg–Witten map between the subspaces of \(G\)-invariant configurations. We respectively denote the \(G\)-monopole invariant and the \(G\)-Bauer–Furuta invariant of \((M, s)\) by \(SW^G_{M, s}\) and \(BF^G_{M, s}\). If \(G = \{e\}\), then they are just the ordinary invariants \(SW_{M, s}\) and \(BF_{M, s}\).

In fact, \(G\)-monopole classes we have in mind in this paper are the cases when \(G\) is finite. Suppose that a compact connected Lie group \(G\) of nonzero dimension acts effectively on a smooth closed manifold \(M\). If \(G\) is not a torus \(T^k\), then \(G\) contains a Lie subgroup isomorphic to \(S^3\) or \(S^3/\mathbb{Z}_2\), and hence \(M\) admits a \(G\)-invariant metric of positive scalar curvature by the well-known Lawson–Yau theorem [18]. (In its original form, the theorem only states that \(M\) carries a metric of positive scalar curvature, but one can check that their method can yield a \(G\)-invariant such metric.) If this is the case for a 4-manifold \(M\) with \(b_2^G(M) > 1\), then \(M\) usually has no \(G\)-monopole class.

On the other hand, in case of a torus action, it is reduced to an \(S^1\) action. The Seiberg–Witten invariants of a 4-manifold with an effective \(S^1\) action were extensively studied by Baldridge [2–4]. He showed that if the action has fixed points, the Seiberg–Witten invariants vanish for all \(\text{Spin}^c\) structures, and if the action is fixed-point free, then the Seiberg–Witten invariants can be read from the those of the quotient 3-orbifold.

In our previous paper, some nontrivial examples of \(G\)-monopole classes for a finite cyclic group \(G\) were shown:

**Theorem 1.1** [31] Let \(M\) and \(N\) be smooth closed oriented 4-manifolds satisfying \(b_2^+(M) > 1\) and \(b_2^+(N) = 0\), and \(M_k\) for any \(k \geq 2\) be the connected sum \(M\# \cdots \# M \# N\) where there are \(k\) summands of \(M\).

Suppose that \(N\) admits a smooth orientation-preserving \(\mathbb{Z}_k\)-action with at least one free orbit such that there exist a \(\mathbb{Z}_k\)-invariant Riemannian metric of positive scalar curvature and a \(\mathbb{Z}_k\)-equivariant \(\text{Spin}^c\) structure \(s_N\) with \(c_1^2(s_N) = -b_2(N)\).
Define a $\mathbb{Z}_k$-action on $\tilde{M}_k$ induced from that of $N$ and the cyclic permutation of the $k$ summands of $M$ glued along a free orbit in $N$, and let $\tilde{s}$ be the Spin$^c$ structure on $\tilde{M}_k$ obtained by gluing $s_N$ and a Spin$^c$ structure $s$ of $M$.

Then for any $\mathbb{Z}_k$-action on $\tilde{s}$ covering the above $\mathbb{Z}_k$-action on $\tilde{M}_k$, $SW_{\tilde{M}_k, \tilde{s}}$ mod 2 is nontrivial if $SW_{M, s}$ mod 2 is nontrivial, and also $BF_{\tilde{M}_k, \tilde{s}}$ is nontrivial, if $BF_{M, s}$ is nontrivial.

LeBrun and his collaborators [15–17,20,21,29,33] used monopole classes to derive topological obstructions to the existence of an Einstein metric, improving the Hitchin–Thorpe inequality [13,34]

$$2\chi(X) \pm 3\tau(X) \geq 0$$

which holds on any smooth closed oriented Einstein 4-manifold $X$. More generally, as observed by Fang et al. [8], these topological obstructions for Einstein metrics on 4-manifolds can be extended to the obstructions to the existence of a quasi-nonsingular solution of the normalized Ricci flow

$$\frac{\partial g}{\partial t} = -2Ric_g + \frac{2\tilde{s}_g}{n} g,$$

where $\tilde{s}_g$ is the average scalar curvature $\int_M s_g d\mu_g$ of $g(t)$, and $n$ is the dimension of the manifold. Following Ishida [14], we say that a smooth solution $\{g(t)|t \in [0, T)\}$ to the normalized Ricci flow is called quasi-nonsingular if

$$T = \infty, \quad \text{and} \quad \sup_{t \in [0, \infty]} |s_{g(t)}| < \infty.$$ 

The purpose of this paper is to use $G$-monopole classes to derive a lower bound of the $L^2$-norm of the Ricci curvature of a $G$-invariant Riemannian metric, even when there is no monopole class, and give a new topological obstruction to the existence of a quasi-nonsingular solution of the normalized Ricci flow for any $G$-invariant initial metric. In particular, this implies the nonexistence of a $G$-invariant Einstein metric or an orbifold Einstein metric on its quotient orbifold.

For instance, we show that a certain connected sum $m\mathbb{C}P_2 \# n\mathbb{C}P_2$ admits an infinite family of topologically equivalent but smoothly distinct non-free actions of $\mathbb{Z}_k \oplus H$, where $k \geq 2$ is any integer, and $H$ is any nontrivial finite group acting freely on $S^3$ such that it does not admit a quasi-nonsingular solution of the normalized Ricci flow for any initial metric invariant under such an action. These examples are different from those of Ishida and Šuvaina [17,33] who showed that certain connected sums

$$m\mathbb{C}P_2 \# n\mathbb{C}P_2 \quad \text{and} \quad m(S^2 \times S^2) \# nK3$$

admit infinitely many free actions of a finite cyclic group and no Einstein metrics invariant under such an action. (In the first examples, also no nonsingular solutions to the normalized Ricci flow invariant under such an action). Their actions are free so that one can pass to their quotient manifolds and apply ordinary Seiberg–Witten invariants to show the non-existence of Einstein metrics and nonsingular solutions to the normalized Ricci flow on them.

We also apply the above theorem to compute the $\mathbb{Z}_k$-Yamabe invariant of $\tilde{M}_k$, which is roughly the $\mathbb{Z}_k$-equivariant version of the Yamabe invariant constructed using only metrics invariant under a $\mathbb{Z}_k$ action, and the orbifold Yamabe invariant of $\tilde{M}_k/\mathbb{Z}_k = M#N/\mathbb{Z}_k$.

When the $G$ action is finite with isolated fixed points, one may try to find a $G$-monopole class by searching for an ordinary monopole class on the quotient 4-orbifold. But the Seiberg–Witten theory on a 4-orbifold is not fully developed yet, and the readers are referred to [6,7].
2 Ricci flow and $G$-monopole class

In this section, $G$ denotes a compact Lie group.

**Theorem 2.1** Let $X$ be a smooth closed oriented 4-manifold with a smooth $G$-action. Suppose that $c_1(s)$ is a $G$-monopole class on $X$. Then for any $G$-invariant Riemannian metric $g$ on $X$,

$$\frac{1}{4\pi^2} \int_X \left( \frac{s_g^2}{24} + 2 |W^+_g|^2 \right) \, d\mu_g \geq \frac{2}{3} (c_1^+(s))^2,$$

and

$$\frac{1}{8\pi^2} \int_X |Ric_g|^2 \, d\mu_g \geq 2 (c_1^+(s))^2 - (2\chi(X) + 3\tau(X)),$$

where $s_g$, $W^+_g$, and $Ric_g$ are respectively the scalar curvature, self-dual Weyl curvature, and Ricci curvature of $g$, and $c_1^+$, $\chi$, and $\tau$ respectively denote the self-dual harmonic part of $c_1$ with respect to $g$, Euler characteristic, and signature.

**Proof** As usual, we denote the conformal class of $g$ by $[g]$. The proof is done using LeBrun’s Bochner-type argument in the same way as the case of $G = \{1\}$ in [21], where one needed $c_1(s)$ to be a monopole class in order to guarantee that a metric $\hat{g} \in [g]$ with constant “modified scalar curvature” $s - \sqrt{6}|W^+|$ admits a solution of the Seiberg–Witten equations for $s$.

Here $c_1(s)$ is assumed to be a $G$-monopole class, and hence it’s enough to prove that $\hat{g}$ is also $G$-invariant. Noting that $\hat{g}$ is a minimizer for the Yamabe-type functional

$$\mathcal{Y}(\hat{g}) := \frac{\int_X (s_{\hat{g}} - \sqrt{6} |W^+_{\hat{g}}|) \, d\mu_{\hat{g}}}{(\text{Vol}_{\hat{g}})^{\frac{1}{2}}}$$

defined on $[g]$ of $g$, and the modified scalar curvature of $\hat{g}$ is nonpositive, $\hat{g}$ is unique up to constant multiplication, as shown in [30]. Since $g$ is invariant under the $G$-action, $\hat{g}$ is pulled-back under the $G$-action only to a constant multiple of $\hat{g}$, which should be $\hat{g}$ itself, because the total volume remains unchanged under the group action. \[\square\]

**Corollary 2.2** Let $X$ be a smooth closed oriented 4-manifold with a smooth $G$-action. Suppose that $c_1(s)$ is a $G$-monopole class on $X$, and $X$ admits a $G$-invariant Einstein metric $g$. Then

$$2\chi(X) + 3\tau(X) \geq \frac{2}{3} (c_1^+(s))^2 \geq \frac{2}{3} c_1^2(s).$$

**Proof** Using the Chern–Gauss–Bonnet theorem and the fact that the trace-free part $\tilde{r}_g$ of $Ric_g$ is zero,

$$2\chi(X) + 3\tau(X) = \frac{1}{4\pi^2} \int_X \left( \frac{s_g^2}{24} + 2 |W^+_g|^2 - \frac{\tilde{r}_g}{2} \right) \, d\mu_g$$

$$= \frac{1}{4\pi^2} \int_X \left( \frac{s_g^2}{24} + 2 |W^+_g|^2 \right) \, d\mu_g$$
\[ \geq \frac{2}{3} (c_1^+(s))^2 \]
\[ \geq \frac{2}{3} c_1^2(s), \]

where the first inequality is due to Theorem 2.1, and the second one obviously comes from that any 2-form \( \alpha \) on \( X \) has an orthogonal decomposition \( \alpha^+ + \alpha^- \) into self-dual and anti-self-dual forms so that
\[ \int_X \alpha \wedge \alpha = \int_X (\alpha^+ \wedge \alpha^+ + \alpha^- \wedge \alpha^-) = \|\alpha^+\|_{L^2}^2 - \|\alpha^-\|_{L^2}^2. \]

**Theorem 2.3** Let \( X \) be a smooth closed oriented 4-manifold with a smooth \( G \)-action. Suppose that \( c_1(s) \) is a \( G \)-monopole class on \( X \), and \( X \) admits a quasi-nonsingular solution \( \{g(t)\mid t \geq 0\} \) of the normalized Ricci flow for a \( G \)-invariant initial metric such that
\[ \liminf_{t \to \infty} (c_1^{+t}(s))^2 > 0. \]

Then
\[ 2\chi(X) + 3\tau(X) \geq \frac{2}{3} \liminf_{t \to \infty} (c_1^{+t}(s))^2 \geq \frac{2}{3} c_1^2(s), \]
where \( c_1^{+t} \) is the self-dual harmonic part of \( c_1 \) with respect to \( g(t) \).

**Proof** We claim that there exists a constant \( c > 0 \) such that
\[ \tilde{s}_{g(t)} := \min_{x \in X} s_{g(t)}(x) < -c. \] (2.1)

Suppose not. Then for any \( \epsilon > 0 \), there exists \( t_\epsilon > 0 \) such that
\[ \tilde{s}_{g(t_\epsilon)} \geq -\epsilon. \]

Note that \( g(t) \) for any \( t \) is also \( G \)-invariant by the uniqueness of the Ricci flow. Thus, there exists a solution \( (A_t, \Phi_t) \) of the Seiberg–Witten equations for \( (X, s) \) with respect to \( g(t) \). Thus
\[ \left( c_1^{+t}(s) \right)^2 \leq \frac{1}{4\pi^2} \int_X \left| F_{A_t}^{+} \right|^2 d\mu_{g(t)} \]
\[ = \frac{1}{4\pi^2} \int_X \left| \Phi_t \otimes \Phi_t^* - \frac{\|\Phi_t\|^2}{2} \text{Id} \right|^2 d\mu_{g(t)} \]
\[ = \frac{1}{4\pi^2} \int_X \frac{\|\Phi_t\|^4}{8} d\mu_{g(t)}. \]

By the well-known Weitzenböck argument,
\[ |\Phi_t| \leq \max\{-\tilde{s}_{g(t)}, 0\}, \]
and hence \( |\Phi_{t_\epsilon}| \leq \epsilon \). Therefore,
\[ \left( c_1^{+t_\epsilon}(s) \right)^2 \leq \frac{1}{4\pi^2} \int_X \frac{\epsilon^4}{8} d\mu_{g(t)} = \frac{\epsilon^4}{32\pi^2} \int_X d\mu_{g(0)}, \]
because the normalized Ricci flow preserves the volume. Since \( \epsilon > 0 \) is arbitrary, this yields a contradiction to \( \liminf_{t \to \infty} (c_1^+(s))^2 > 0 \).

By Lemma 3.1 of [8], any quasi-nonsingular solution satisfying (2.1) on a closed manifold must have that

\[
\int_0^\infty \int_X |\nabla g(t)|^2 d\mu_g(t) < \infty. \tag{2.2}
\]

Then by the Chern–Gauss–Bonnet theorem combined with Theorem 2.1,

\[
2\chi(X) + 3\tau(X) = \int_m^{m+1} (2\chi(X) + 3\tau(X)) dt
\]

\[
= \frac{1}{4\pi^2} \int_m^{m+1} \int_X \left( \frac{s^2_g(t)}{24} + 2 |W^+_g(t)|^2 - \frac{\varphi g(t)}{2} \right) d\mu_g(t) dt
\]

\[
\geq \liminf_{m \to \infty} \frac{1}{4\pi^2} \int_m^{m+1} \int_X \left( \frac{s^2_g(t)}{24} + 2 |W^+_g(t)|^2 - \frac{\varphi g(t)}{2} \right) d\mu_g(t) dt
\]

\[
\geq \liminf_{m \to \infty} \int_m^{m+1} \frac{2}{3} (c_1^+(s))^2 dt
\]

\[
= \liminf_{m \to \infty} \int_0^1 \frac{2}{3} (c_1^{+m+t}(s))^2 dt
\]

\[
\geq \int_0^1 \frac{2}{3} \liminf_{m \to \infty} (c_1^{+m+t}(s))^2 dt
\]

\[
\geq \frac{2}{3} \liminf_{t \to \infty} (c_1^+(s))^2,
\]

where the 2nd inequality from the last is due to Fatou’s lemma. \( \square \)

**Remark** The assumption that \( \liminf_{t \to \infty} (c_1^+(s))^2 > 0 \) was needed only to get (2.1), and so it can be replaced by the condition

\[ Y_G(X) < 0 \]

on the \( G \)-Yamabe invariant of \( X \), which will be introduced in the following section.

We now produce some non-existence examples of \( G \)-invariant Einstein metrics or more generally quasi-nonsingular solutions of the normalized Ricci flow, where the Hitchin–Thorpe inequality is satisfied while there may not exist any monopole class.
Theorem 2.4 Let $M$, $N$, and $\tilde{M}_k$ be as in Theorem 1.1. Suppose that $M$ has nonzero mod 2 Seiberg–Witten invariant for a Spin$^c$ structure $s$, and

$$0 < 2\chi(M) + 3\tau(M) < \frac{1}{k}(12(k - 1) + 12b_1(N) + 3b_2(N)).$$

Then $\tilde{M}_k$ does not admit a quasi-nonsingular solution to the normalized Ricci flow for any $\mathbb{Z}_k$-invariant initial metric. In particular, $\tilde{M}_k$ never admits an orbifold Einstein metric, and $M\#N/\mathbb{Z}_k$ never admits an orbifold Einstein metric.

Proof First note that $\tilde{M}_k$ admits a $\mathbb{Z}_k$-invariant Einstein metric iff $\tilde{M}_k/\mathbb{Z}_k = M\#N/\mathbb{Z}_k$ admits an orbifold Einstein metric, when $N/\mathbb{Z}_k$ is an orbifold. Because an Einstein metric is a static solution of the normalized Ricci flow, we will prove only the first statement.

Think of $\tilde{M}_k$ as the connected sum $kM\#N$, and let $s_1$ and $s_2$ be the restriction of $\tilde{s}$ to $kM - B^4$ and $N - B^4$ respectively, where $B^4$ is a small open ball for the connected sum operation. Then

$$c_1(\tilde{s}) = c_1(s_1) + c_1(s_2) \in H^2(kM - B^4) \oplus H^2(N - B^4) = H^2(\tilde{M}_k),$$

and with respect to any Riemannian metric on $\tilde{M}_k$

$$(c_1^+(\tilde{s}))^2 = (c_1^+(s_1) + c_1^+(s_2))^2$$

$$= (c_1^+(s_1))^2 + 2c_1^+(s_1) \cdot c_1^+(s_2) + (c_1^+(s_2))^2$$

$$\geq (c_1^+(s_1))^2 + 2c_1^+(s_1) \cdot c_1^+(s_2).$$

Lemma 2.5 $-s_N := s_N \otimes (-\det(s_N))$ is also $\mathbb{Z}_k$-equivariant.

Proof Since $s_N$ is $\mathbb{Z}_k$-equivariant, so is its associated determinant line bundle $\det(s_N).$ Therefore, $s_N \otimes (-\det(s_N))$ is also $\mathbb{Z}_k$-equivariant. \qed

Let $\tilde{s}'$ be the Spin$^c$ structure on $\tilde{M}_k$ replacing $s_N$ in $\tilde{s}$ by $-s_N$, and $s_1'$ and $s_2'$ be defined as above. Then $s_1' = s_1$ and $c_1(s_2') = -c_1(s_2).$ Therefore, we have either

$$c_1^+(s_1) \cdot c_1^+(s_2) \geq 0,$$

or

$$c_1^+(s_1') \cdot c_1^+(s_2') \geq 0.$$

In the first case,

$$(c_1^+(\tilde{s}))^2 \geq (c_1^+(s_1))^2$$

$$\geq c_1^2(s_1)$$

$$= k c_1^2(s)$$

$$\geq k(2\chi(M) + 3\tau(M)),$$

where the last inequality holds because the Seiberg–Witten moduli space of $s$ on $M$ has nonnegative dimension. Likewise in the second case,

$$(c_1^+(\tilde{s}'))^2 \geq k(2\chi(M) + 3\tau(M)).$$

Now let’s assume to the contrary that $\tilde{M}_k$ does admit such a solution $\{g(t)\} t \geq 0$ of the normalized Ricci flow. We claim that there exists a constant $c > 0$ such that

$$\tilde{s}_{g(t)} := \min_{x \in X} s_{g(t)}(x) < -c.$$
By Theorem 1.1, both $\tilde{s}$ and $\tilde{s}'$ are $\mathbb{Z}_k$-monopole classes on $\tilde{M}_k$. If there does not exist such $c > 0$, then by the same method as in Theorem 2.3, for any $\epsilon > 0$, there exists $t_\epsilon > 0$ such that

$$\left( c_1^{+t_\epsilon} (\tilde{s}) \right)^2 \leq \frac{\epsilon^4}{32\pi^2} \int_X d\mu_{g(0)},$$

and

$$\left( c_1^{+t_\epsilon} (\tilde{s}') \right)^2 \leq \frac{\epsilon^4}{32\pi^2} \int_X d\mu_{g(0)},$$

both of which together imply

$$k(2\chi(M) + 3\tau(M)) \leq \frac{\epsilon^4}{32\pi^2} \int_X d\mu_{g(0)}.$$

By the assumption $2\chi(M) + 3\tau(M) > 0$, the claim is justified, and we obtain (2.2) by Lemma 3.1 of [8].

Then proceeding as in the last part in the proof of Theorem 2.3 we get

$$2\chi(\tilde{M}_k) + 3\tau(\tilde{M}_k) \geq \liminf_{m \to \infty} \frac{1}{4\pi^2} \int_X \left( \frac{s_\tilde{s}(t)}{24} + 2 \left| W_{\tilde{g}(t)}^+ \right|^2 \right) d\mu_{\tilde{g}(t)} dt$$

$$\geq \liminf_{m \to \infty} \int \max_{m} \left( \frac{2}{3} \left( c_1^{+t} (\tilde{s}) \right)^2 + \frac{2}{3} \left( c_1^{+t} (\tilde{s}') \right)^2 \right) dt$$

$$\geq \frac{2}{3} k(2\chi(M) + 3\tau(M)).$$

A simple computation gives

$$2\chi(\tilde{M}_k) + 3\tau(\tilde{M}_k) = k(2\chi(M) + 3\tau(M)) + 2\chi(N) + 3\tau(N) - 4k$$

$$= k(2\chi(M) + 3\tau(M)) + 4(1 - k) - 4b_1(N) - b_2(N).$$

Plugging this into the above gives

$$k(2\chi(M) + 3\tau(M)) + 4(1 - k) - 4b_1(N) - b_2(N) \geq \frac{2}{3} k(2\chi(M) + 3\tau(M))$$

which simplifies to

$$\frac{k}{3} (2\chi(M) + 3\tau(M)) \geq 4(k - 1) + 4b_1(N) + b_2(N),$$

yielding a contradiction. $\Box$

The following lemma is a slight generalization of Sung [31, Theorem 7.1].

**Lemma 2.6** Let $M$ be a smooth closed 4-manifold and $\{M_i | i \in I\}$ be a family of smooth 4-manifolds such that every $M_i$ is homeomorphic to $M$ and the numbers of mod 2 basic classes of $M_i$’s are all mutually different, but each $M_i \# l_i (S^2 \times S^2)$ is diffeomorphic to $M \# l_i (S^2 \times S^2)$ for an integer $l_i \geq 1$.

If $l_{\text{max}} := \sup_{i \in I} l_i < \infty$, then for any integers $k \geq 2$, $n \geq 0$, and $l \geq l_{\text{max}} + 1$,

$$X := klM \# kl\mathbb{C}P_2^\#(l - 1)(S^2 \times S^2)$$

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admits an $\mathcal{I}$-family of topologically equivalent but smoothly distinct non-free actions of $\mathbb{Z}_k \oplus H$ where $H$ is any group of order $l$ acting freely on $S^3$.

**Proof** The following proof for $n \geq 1$ cases is almost parallel to the $n = 0$ case of [31], and first recall that $(l - 1)(S^2 \times S^2)$ admits an $H$ action defined as the deck transformation map of the $l$-fold covering map onto $S^1 \times L$ for $L = S^3 / H$, where $S^1 \times L$ is the manifold obtained from the surgery on $S^1 \times L$ along an $S^1 \times \{\text{pt}\}$.

Think of $X$ as

$$klM_i \# kn\mathcal{CP}_2 \# (l - 1)(S^2 \times S^2),$$
on which $H$ acts as the deck transformation map of the $l$-fold covering map onto

$$\tilde{M}_{i,k} := klM_i \# kn\mathcal{CP}_2 \# S^1 \times L.$$To define a $\mathbb{Z}_k$-action, note that $\tilde{M}_{i,k}$ has a $\mathbb{Z}_k$-action coming from the $\mathbb{Z}_k$-action of $kn\mathcal{CP}_2 \# S^1 \times L$ defined in Sung [31, Theorem 6.4], which is basically a rotation along the $S^1$-direction, and whose fixed point set is $\{0\} \times S^2$ in the attached $D^2 \times S^2$. This $\mathbb{Z}_k$ action is obviously lifted to the above $l$-fold cover, and it commutes with the above defined $H$ action. Thus, we have an $\mathcal{I}$-family of $\mathbb{Z}_k \oplus H$ actions on $X$, which are all topologically equivalent by using the homeomorphism between each $M_i$ and $M$.

Recall from Sung [31, Theorem 6.4] that all the Spin$^c$ structures on $S^1 \times L$ are $\mathbb{Z}_k$-equivariant and satisfy $c_1^{2} = -b_2(S^1 \times L) = 0$. Let $\tilde{s}_i$ be the $\mathbb{Z}_k$-equivariant Spin$^c$ structure on $\tilde{M}_{i,k}$ obtained by gluing a Spin$^c$ structure $s_i$ of $M_i$ and a $\mathbb{Z}_k$-equivariant Spin$^c$ structure $s_N$ on $S^1 \times L \# kn\mathcal{CP}_2$ satisfying

$$c_1^{2}(s_N) = -b_2(S^1 \times L \# kn\mathcal{CP}_2) = -kn.$$By Sung [31, Theorem 4.2] and the fact that $b_1(S^1 \times L \# kn\mathcal{CP}_2) = 0$, for any Spin$^c$ structure $s_i$ on $M_i$,

$$SW_{\mathbb{Z}_k, \tilde{M}_{i,k}, \tilde{s}_i} = SW_{M_i, s_i} \text{ mod } 2,$$and hence the corresponding Seiberg–Witten polynomials satisfy

$$SW_{\mathbb{Z}_k, \tilde{M}_{i,k}} = SW_{M_i} \sum_{[\alpha],[\beta]} [\alpha][\beta]$$mod 2, where $[\alpha]$ runs through any element of $H^2(S^1 \times L; \mathbb{Z}) = H_1(L; \mathbb{Z})$ and $[\beta]$ runs through any $\mathbb{Z}_k$-equivariant element of $H^2(kn\mathcal{CP}_2; \mathbb{Z})$ satisfying that $[\beta]^2 = -kn$, and $[\beta]$ restricts to a generator of the 2nd cohomology in each $\mathcal{CP}_2$-summand. Therefore, $SW_{\mathbb{Z}_k, \tilde{M}_{i,k}}$ mod 2 for all $i$ have mutually different numbers of monomials, and hence the $\mathcal{I}$-family of $\mathbb{Z}_k \oplus H$ actions on $X$ cannot be smoothly equivalent, completing the proof. \hfill \Box

**Theorem 2.7** Let $k \geq 2$ be an integer and $H$ be a finite group of order $l \geq 2$ acting freely on $S^3$. Then for infinitely many $m \in \mathbb{Z}^+$ and any integer $n > \frac{4m - 2}{3}$, the manifold

$$(km + l - 1)\mathcal{CP}_2 \# (kl(m + n) + l - 1)\mathcal{CP}_2$$admits an infinite family of topologically equivalent but smoothly distinct non-free actions of $\mathbb{Z}_k \oplus H$ such that it admits no quasi-nonsingular solution to the normalized Ricci flow for any initial metric invariant under such an action.
Proof Note that the above manifold is diffeomorphic to
\[ Y := (klm + l - 1)(S^2 \times S^2)\# k\mathbb{C}P_2. \]

As in [31, Corollary 7.2], we use the construction of Hanke et al. [11], which shows that
\( m(S^2 \times S^2) \) for infinitely many \( m \) has the property of \( M \) in Lemma 2.6 with each \( l_i = 1 \) and \( |\mathcal{I}| = \infty \). Using Lemma 2.6, \( Y \) admits such \( \mathbb{Z}_k \oplus H \) actions so that
\[ Y/H = kM_i\# k\mathbb{C}P_2\# \hat{S}^1 \times L \]
for \( \{M_i| i \in \mathcal{I}\} \).

Thus, we only need to show that \( Y/H \) does not admit a quasi-nonsingular solution to the normalized Ricci flow invariant under the \( \mathbb{Z}_k \) action. As proven in Sung [31, Theorem 6.4], \( k\mathbb{C}P_2\# \hat{S}^1 \times L \) is an example of \( N \) in Theorem 1.1, and hence we can apply Theorem 2.4 to \( Y/H \).

Using \( b_1(S^1 \times L) = b_2(S^1 \times L) = 0 \), a simple computation gives
\[ \frac{12(k-1) + 12b_1(N) + 3b_2(N)}{k} \geq 6 + 3n \]
\[ > 4m + 4 \]
\[ = 2\chi(M_i) + 3\tau(M_i), \]
which completes the proof. \( \square \)

Remark Just for a simple remark, the above manifold of Theorem 2.7 satisfies the Hitchin–Thorpe inequality when \( n \leq 4(m + \frac{1}{k}) \).

Also note that it obviously admits a metric of positive scalar curvature. But such metrics are never invariant under those \( \mathbb{Z}_k \oplus H \) actions, because it has nontrivial \( \mathbb{Z}_k \oplus H \) monopole invariant. LeBrun [23] was the first who discovered that a finite group acts freely on certain connected sums \( m\mathbb{C}P_2\# n\mathbb{C}P_2 \) so that there exist no metrics of positive scalar curvature invariant under the action, and moreover the quotient manifolds have negative Yamabe invariants. On the other hand, Ruberman [24] showed that for any \( m \geq 2 \) and \( n > 10m \), the space of positive scalar curvature metrics on \( 2m\mathbb{C}P_2\# n\mathbb{C}P_2 \) has infinitely many components.

For an example of \( H \), one can take \( \mathbb{Z}_l \) for \( (k, l) = 1 \) so that \( \mathbb{Z}_k \oplus H \) is isomorphic to \( \mathbb{Z}_{kl} \).

3 Computation of G-Yamabe invariant and orbifold Yamabe invariant

When a smooth closed \( n \)-manifold \( X \) admits a smooth group action by a compact Lie group \( G \), the \( G \)-Yamabe invariant can be defined in an analogous way to the ordinary Yamabe invariant. For a \( G \)-invariant Riemannian metric \( g \) on \( X \), we let \( [g]_G \) be the set of smooth \( G \)-invariant metrics conformal to \( g \). Following [26], define the \( G \)-Yamabe constant of \( (X, [g]_G) \) as
\[ Y(X, [g]_G) := \inf_{\tilde{g} \in [g]_G} \frac{\int_X s_{\tilde{g}} dV_{\tilde{g}}}{(\int_X dV_{\tilde{g}})^\frac{n-2}{n}}, \]
and the \( G \)-Yamabe invariant of \( X \) as
\[ Y_G(X) := \sup_{[g]_G} Y(X, [g]_G). \]
When the $G$-action is trivial, $Y(X, [g]_G)$ and $Y_G(X)$ are obviously the ordinary Yamabe constant $Y(X, [g])$ and the ordinary Yamabe invariant $Y(X)$ respectively.

By the result of Hebey and Vaugon [12], the $G$-equivariant Yamabe problem can be solved for $n \geq 3$ by minimizing the Yamabe functional defined on each $[g]_G$. The minimizers have constant scalar curvature, and there exists the Aubin-type inequality

$$Y(X, [g]_G) \leq \frac{\Lambda_n}{\sqrt[n]{n}} \inf_{\tilde{g} \in [g]_G} \left( \int_X |s_{\tilde{g}}|^{-\frac{n}{2}} d\mu_{\tilde{g}} \right)^{\frac{1}{2}} \left( \text{Vol}_{\tilde{g}} \right)^{\frac{2}{n} - \frac{1}{r}},$$

where $\Lambda_n$ defined as $n(n - 1)(\text{Vol}(S^n(1)))^{\frac{1}{n}}$ is the Yamabe invariant $Y(S^n)$ of $S^n$, and $|Gx|$ denotes the cardinality of the orbit of $x$.

When $Y(X, [g]_G) \leq 0$, by definition

$$Y(X, [g]) \leq Y(X, [g]_G) \leq 0,$$

and hence an ordinary Yamabe minimizer in $[g]$ must also be a $G$-Yamabe minimizer, because the metrics with nonpositive constant scalar curvature are unique up to constant in a conformal class so that they are also $G$-invariant. Thus, in that case

$$Y(X, [g]_G) = Y(X, [g]).$$

We present some practical formulae for computing $Y(X, [g]_G)$ and $Y_G(X)$, which are exactly the same forms as the ordinary Yamabe case.

**Proposition 3.1** Let $r \in [\frac{n}{2}, \infty]$. Then

$$|Y(X, [g]_G)| = \inf_{\tilde{g} \in [g]_G} \left( \int_X |s_{\tilde{g}}|^{-\frac{n}{2}} d\mu_{\tilde{g}} \right)^{\frac{1}{2}} \left( \text{Vol}_{\tilde{g}} \right)^{\frac{2}{n} - \frac{1}{r}}$$

if $Y(X, [g]_G) \leq 0$,

where the infimums are realized only by the minimizer in (3.1), and $s_{\tilde{g}}$ is defined as $\min(s_{\tilde{g}}, 0)$.

If $Y_G(X) \leq 0$,

$$Y_G(X) = -\inf_{g \in \mathcal{M}_G} \left( \int_X |s_g|^{-\frac{n}{2}} d\mu_g \right)^{\frac{1}{2}} \left( \text{Vol}_g \right)^{\frac{2}{n} - \frac{1}{r}}$$

where $\mathcal{M}_G$ is the space of all smooth $G$-invariant Riemannian metrics on $X$.

**Proof** If $Y(X, [g]_G) > 0$, then it can be proved in the same way as the ordinary Yamabe case. (For a proof, see Sung [27].)
If $Y(X, [g]_G) \leq 0$, then

$$Y(X, [g]_G) = Y(X, [g])$$

$$= - \inf_{\tilde{g} \in [g]} \left( \int_X \left| s_{\tilde{g}} \right|^r d\mu_{\tilde{g}} \right)^{1/r} \left( \text{Vol}_{\tilde{g}} \right)^{2/r - 1}$$

$$= - \inf_{\tilde{g} \in [g]} \left( \int_X \left| s_{\tilde{g}}^{-1} \right|^r d\mu_{\tilde{g}} \right)^{1/r} \left( \text{Vol}_{\tilde{g}} \right)^{2/r - 1},$$

and these infimums are realized by the Yamabe minimizers, which are $G$-invariant. Therefore, it’s enough to take the infimums on a smaller set $[g]_G$.

The formulae for $Y_G(X)$ are now straightforward. \( \square \)

One of the important facts about $G$-Yamabe constant and $G$-Yamabe invariant is that they are basically equivalent to the orbifold Yamabe constant and the orbifold Yamabe invariant of the quotient manifold, when $G$ is finite and $X/G$ is an orbifold.

Let $V$ be a closed orbifold of dimension $n$. For an orbifold Riemannian metric $g$ on $V$, $[g]_{\text{orb}}$ denotes the set of orbifold Riemannian metrics conformal to $g$. In the same as the ordinary Yamabe problem, Akutagawa and Botvinnik [1] defined the orbifold Yamabe constant $Y(V, [g]_{\text{orb}})$ of $[g]_{\text{orb}}$ as the infimum of the normalized Einstein-Hilbert functional on $[g]_{\text{orb}}$, and the orbifold Yamabe invariant

$$Y_{\text{orb}}(V) := \sup_{[g]_{\text{orb}}} Y(V, [g]_{\text{orb}}).$$

They also obtained the Aubin-type inequality

$$Y(V, [g]_{\text{orb}}) \leq \min_{1 \leq i \leq m} \frac{\Lambda_n}{|\Gamma_i|^2/2},$$

where \{($\tilde{p}_1$, $\Gamma_1$), ..., ($\tilde{p}_m$, $\Gamma_m$)\} is the singularity of $V$.

A group action is called pseudo-free, if non-free orbits are isolated. For a smooth pseudo-free action on a smooth manifold by a finite group, its quotient space has a natural orbifold structure.

**Theorem 3.2** Let $X$ be a smooth closed $n$-manifold with smooth pseudo-free action by a finite group $G$. Then for an orbifold Riemannian metric $g$ on $X/G$,

$$Y_{\text{orb}}(X/G, [g]_{\text{orb}}) = \frac{Y_G(X, [\pi^*g])}{|G|^2/2}, \quad \text{and} \quad Y_{\text{orb}}(X/G) = \frac{Y_G(X)}{|G|^2/2},$$

where $\pi : X \to X/G$ is the quotient map.

**Proof** The proof is obvious from the observation that $[\pi^*g]_G = \pi^*[g]_{\text{orb}}$ and $\pi$ is a branched $|G|$-fold covering. \( \square \)

In Sung [26], we obtained gluing formulae for the $G$-Yamabe invariant for the surgery of codimension 3 and more, which made it possible to compute some $G$-Yamabe invariants of products of spheres and their connected sums. Here, the existence of a $\mathbb{Z}_k$-monopole class on $\bar{M}_k$ enables us to compute its $\mathbb{Z}_k$-Yamabe invariant:
**Theorem 3.3** Let $M$ be a smooth closed oriented $4$-manifold with a Spin$^c$ structure $s$ satisfying

$$Y(M) = -4\sqrt{2\pi}\sqrt{c^2_1(s)},$$

and $N, \bar{M}_k$ be as in Theorem 1.1. Suppose that $s$ has nonzero mod $2$ Seiberg–Witten invariant or nontrivial Bauer–Furuta invariant, and the $\mathbb{Z}_k$-action on $N$ is pseudo-free. Then

$$Y_{\mathbb{Z}_k}(\bar{M}_k) = \sqrt{k}Y(M),$$

and

$$Y_{orb}(M\#N/\mathbb{Z}_k) = Y(M).$$

**Proof** First, we show that

$$Y_{\mathbb{Z}_k}(\bar{M}_k) \geq \sqrt{k}Y(M)$$

by using the standard gluing method of the ordinary Yamabe invariant.

Take a $\mathbb{Z}_k$-invariant metric of positive scalar curvature on $N$, and make $k$ cylindrical ends in a $\mathbb{Z}_k$-symmetric way keeping the positivity of scalar curvature by performing the Gromov–Lawson surgery [10]. On each $M$ we take a metric $g$ which approximates the Yamabe invariant of $M$, and also make a cylindrical end likewise. By gluing these pieces, we have a $\mathbb{Z}_k$-invariant metric on $\bar{M}_k$, denoted by $h$. For any $\varepsilon > 0$, we can arrange the Gromov–Lawson surgery so that $h$ depending $\varepsilon$ satisfies

$$\int_{\bar{M}_k} (s_h^{-})^2 d\mu_h \leq k \int_M (s_{\bar{g}}^{-})^2 d\mu_{\bar{g}} + \frac{\varepsilon}{2} \leq k(Y(M))^2 + \varepsilon. \quad (3.2)$$

Since $\varepsilon > 0$ is arbitrary, the application of Proposition 3.1 with $r = 2$ yields

$$Y_{\mathbb{Z}_k}(\bar{M}_k) \geq \sqrt{k}Y(M).$$

To prove the reverse inequality we will show

$$\int_{\bar{M}_k} s_{\bar{g}}^2 d\mu_{\bar{g}} \geq k(Y(M))^2$$

for any $\mathbb{Z}_k$-invariant metric $\bar{g}$ on $\bar{M}_k$. Since $c_1(\bar{s})$ is a $\mathbb{Z}_k$-monopole class of $\bar{M}_k$, there exists a solution of the Seiberg–Witten equations of $\bar{s}$ for $\bar{g}$. Then LeBrun’s Weitzenböck-type argument [22] gives

$$\int_{\bar{M}_k} s_{\bar{g}}^2 d\mu_{\bar{g}} \geq 32\pi^2(c_1^+(\bar{s}))^2.$$

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1 For this, one may consult a refined way of Gromov–Lawson surgery as in [25]. Another easy way suggested by LeBrun in [19] is as follows. Let $W \subset M$ be a small ball around the point where the connected sum is performed. One can take a conformal change $\varphi g$ of $g$ such that $\varphi \equiv 1$ outside of $W$ and the scalar curvature of $\varphi g$ is positive on a much smaller open subset $W'$ of $W$, and

$$\int_M (s_{\varphi g}^{-})^2 d\mu_{\varphi g} \leq \int_M (s_{\bar{g}}^{-})^2 d\mu_{\bar{g}} + \varepsilon^2.$$
Using \((c_1^+(\tilde{s}))^2 \geq k c_1^2(s)\) (or \((c_1^+(\tilde{s}'))^2 \geq k c_1^2(s)\)) proved in Theorem 2.4, we get desired

\[
\int_{M_k} s^2 g \, d\mu_{\tilde{g}} \geq 32 \pi^2 k \, c_1^2(s) = k(Y(M))^2,
\]

which completes the proof of the first statement. Then the second statement follows from Theorem 3.2.

In fact, one can easily generalize the above theorem to the statement that for any blow-up \(M'\) of such \(M\),

\[
Y_{\mathbb{Z}_k}(\tilde{M}'_k) = \sqrt{k} Y(M') = \sqrt{k} Y(M),
\]

and

\[
Y_{\text{orb}}(M'\# N/\mathbb{Z}_k) = Y(M') = Y(M).
\]

**Example** For such an example of \(M\) in the above theorem which has nonzero mod 2 Seiberg–Witten invariant, there exists a minimal compact Kähler surface of nonnegative Kodaira dimension with \(b_2^+(M) > 1\). Certain surgeries along tori in product manifolds of two Riemann surfaces of genus \(> 1\) also have such a property. For details, the readers are referred to [28].

But such examples of \(M\) with nontrivial Bauer–Furuta invariant are not well understood enough. According to Bauer’s computation [5], if \(X_j\) for \(j = 1, \ldots, 4\) are minimal compact Kähler surfaces satisfying

\[
b_1(X_j) = 0, \quad b_2^+(X_j) \equiv 3 \mod 4, \quad \sum_{j=1}^{4} b_2^+(X_j) \equiv 4 \mod 8,
\]

then \(#_{j=1}^m X_j\) for each \(m = 1, \ldots, 4\) is such an example of \(M\).

Applying the above theorem to such an \(M\) and \(N = S^4\), we obtain

\[
Y_{\text{orb}}(M\# S(L(p; q))) = Y_{\text{orb}}(M\# S^4/\mathbb{Z}_p) = Y(M),
\]

where \(S(L(p; q))\) is the suspension of the Lens space \(L(p; q) = S^3/\mathbb{Z}_p\) with the \(\mathbb{Z}_p\)-action given by \((z_1, z_2) \sim (e^{\frac{2\pi i}{p}} z_1, e^{\frac{2\pi iq}{p}} z_2) \in \mathbb{C}^2\) for coprime integers \(p\) and \(q\).

More examples of \(N\) are given in [31].

**Remark** Just as the ordinary Yamabe invariant is a smooth topological invariant, the orbifold Yamabe invariant can distinguish differential structures of orbifolds. For example, let \(M\) be as in the above example and \(N\) be as in the above theorem. Suppose further that \(M\) is simply connected. The above theorem asserts that

\[
Y_{\text{orb}}(M\# \overline{CP}_2\# N/\mathbb{Z}_k) = Y(M\# \overline{CP}_2) = Y(M) \leq 0.
\]

On the other hand, \(M\# \overline{CP}_2\) is nonspin, and hence by Freedman’s theorem [9], \(M\# \overline{CP}_2\# N/\mathbb{Z}_k\) is homeomorphic to

\[
b_2^+(M)\overline{CP}_2\# (b_2^+(M) + 1)\overline{CP}_2\# N/\mathbb{Z}_k,
\]

whose orbifold Yamabe invariant is positive. Therefore, they are not diffeomorphic as orbifolds.
The ordinary Yamabe invariant $Y(\bar{M}_k)$ of $\bar{M}_k$ is hardly known except for very special cases [16]. It seems plausible that it is equal to $Y_{Z_k}(\bar{M}_k)$ under the assumption of Theorem 3.3.

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