Type II Topp-Leone Gumbel Type-2 distribution: Properties and Applications

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Authors’ contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/ARJOM/2021/v17i430294
Editor(s):
(1) Dr. Sheng Zhang, Bohai University, China.
Reviewers:
(1) Olaleye Olalekan Ayodeji, The Federal Polytechnic, Nigeria.
(2) S. Zimeras, University of the Aegean, Greece.
Complete Peer review History: http://www.sdiarticle4.com/review-history/69784

Received 14 April 2021
Accepted 18 June 2021
Published 21 June 2021

Abstract

In this paper, a three parameter life time model named Type II Topp-Leone Gumbel type-2 distribution which can be used to model reliability problems, fatigue life studies, and survival data has been studied. We derived explicit expressions for some of its statistical properties such as ordinary moments, generating function, incomplete moments, and order statistics. The maximum likelihood estimation technique is used to estimate the parameters of the model. The tractability of the model was illustrated by using two real life data sets. The proposed distribution provides a better fit than some well known distributions using criteria of criteria of goodness of fit.

Keywords: Type II Topp-Leone Gumbel type-2 distribution; incomplete moment; order statistics; goodness of fit.
1 Introduction

The Gumbel type-2 distribution plays an important role in the theory of extreme value distribution. The distribution can be applied in many areas most especially in extreme value events in the field of life testing, fracture roughness, seismology, reliability analysis, and meteorology. It can also be effectively used in modeling real life data set with monotonic failure (or hazard) rates most especially those one with a decreasing hazard rate. But in practice most complex phenomenon that is regularly encountered in practice are non-monotonic and cannot be modeled with Gumbel type-2(GT-2) distribution. In order to improve the fit of GT-2 distribution, Okorie et al. [1], investigated the properties of Exponentiated Gumbel type-2 distribution. The Kumaraswamy Gumbel type-2 was developed and studied by Okorie et al. [2]. Recently, Ogunde et al. [3] proposed and studied the properties of Extended Gumbel type-2 distribution and Gompertz Gumbel type-2 distribution was studied by Ogunde et al. [4]. The cumulative distribution function is given by

\[ G(x; a, b) = e^{-ax^{-b}} \quad x; a, b > 0 \]  

(1)

The associated pdf is given by

\[ g(x; a, b) = abx^{-(b+1)}e^{-ax^{-b}}, \quad x; a, b > 0 \]  

(2)

Where \( x > 0 \) and \( a \) and \( b \) are the scale and shape parameters, respectively.

We redefine the GT-2 distribution using a generalisation by Elighahy M. et al. [5]. He developed and studied a new family of distribution called the Type II Topp-Leone generated (TIITL-G) family of distributions which add a new parameter to the baseline distribution to obtain a more flexible distribution of different range of behaviour of the hazard function (decreasing, increasing, bathtub shapes). The cumulative distribution function (cdf) for (TIITL-G) family of distribution is given by

\[ F(x; a, \zeta) = 1 - [1 - G(x; \Omega)]^a, \quad x \in \mathbb{R}, \]  

(3)

Where \( a \) is a positive shape parameter and \( G(x; \Omega) \) is a cdf of a baseline continuous distribution which sometimes may depend on a parameter vector \( \Omega \). The associated pdf corresponding to (1) is given by

\[ f(x; a, \zeta) = 2ag(x; \Omega)G(x; \Omega)[1 - G(x; \Omega)]^{a-1}, \quad x \in \mathbb{R}, \]  

(4)

Where \( g(x; \Omega) \) is the pdf corresponding to \( G(x; \Omega) \). Elighahy M. et al. [5], described (TIITL-G) family as a simplified version of the Kumaraswamy-G family. The added shape parameter \( a \) is to control the tail weight and skewness of the cdf baseline distribution function. The TIITL-G family has been used by several authors to extend several distributions which includes Type II Topp-Leone generalized inverse Rayleigh distribution by Mohammed and Yahia [6], Yahia and Mohammed [7] developed and studied the properties of Type II Topp-Leone inverse Rayleigh distribution by Type II Topp-Leone Inverted Kumaraswamy distribution was studied by Zein Eldin et al. [8]. Given that a random variable \( X \) with support on the set of positive real numbers and \( TIIITLGT - 2 (a, b, x) \) distribution, the cdf of Type II Topp-Leone Gumbel Type-two (TIIITLGT-2) distribution is given by

\[ F(x; a, b, \alpha) = 1 - \left[ 1 - e^{-2ax^{-b}} \right]^a, \]  

(5)

Where \( a \) and \( b \) are positive shape parameters and \( a \) is a positive scale parameter. The graph of the cdf of \( TIIITLGT - 2 \) distribution for various values of the parameters is given by
The corresponding pdf to equation (5), is given by

\[
f(x; a, b, c) = 2abax^{-(b+1)}e^{-2ax-b}\left[1-e^{-2ax-b}\right]^{a-1}
\]  

(6)

Where \(a\) and \(b\) are positive shape parameters and \(a\) is a positive scale parameter. The graph of the pdf of \(TIITLGT-2\) distribution for various values of the parameters is given by

2 Reliability and the Hazard Function

The Reliability function of the \(TIITLGT-2\) distribution is given by:

\[
R(x) = 1 - F(x) = \left[1 - e^{-2ax-b}\right]^a
\]

(7)

Where \(a\) and \(b\) are positive shape parameters and \(a\) is a positive scale parameter. The graph of the survival function of \(TIITLGT-2\) distribution for various values of the parameters is given by
And the hazard function is given by

\[
h(x) = \frac{f(x)}{R(x)} = \frac{2abax^{-(b+1)}e^{-2ax^{-b}}}{[1 - e^{-2ax^{-b}}]^{\alpha-1}}^{\alpha-1} = \frac{2abax^{-(b+1)}e^{-2ax^{-b}}}{[1 - e^{-2ax^{-b}}]^{\alpha}}
\]

Where \(a\) and \(b\) are positive shape parameters and \(\alpha\) is a positive scale parameter. The graph of the hazard function of \(TII T L G T - 2\) distribution for various values of the parameters is given by

**Fig. 3. The graph of the survival functions of \(TII T L G T - 2\) distribution**

**Fig. 4. The graph of the survival function of \(TII T L G T - 2\) distribution**
3 Useful Expansion of the Probability Density Function

If \( m \) is a positive real non-integer and \( z \leq 1 \), we consider the power series expansion given by

\[
(1 - z)^{m-1} = \sum_{m=0}^{\infty} (-1)^i \binom{m-1}{i} z^i
\]  

(9)

Applying equation (9) in (6), we have

\[
f(x; a, b, \alpha) = 2aba \sum_{m=0}^{\infty} (-1)^i \binom{\alpha - 1}{i} x^{-(b+1)} e^{-2ax^{-b}(i+1)}
\]  

(10)

4 Statistical Properties of \( TIITLG - 2 \) distribution

Here, we studied the statistical properties of \( TIITLG - 2 \) distribution, which includes an expression for the moment and the incomplete moments.

4.1 Moments

In this section, we obtain the moment of the \( TIITLG - 2 \) distribution. Moment plays important role in statistical analysis, most especially in determining the structural properties of a distribution such as skewness, kurtosis, dispersion, mean etc.

Theorem 1. Let a random variable \( X \) follows the Type two Topp-Leone Gumbel type-2 distribution, the \( r \)th moment of \( TIITLG - 2 \) distribution is given by

\[
\mu'_r = a \sum_{i=0}^{\infty} \binom{\alpha - 1}{i} (-1)^i (2a)^r (i+1)^{r-b}/b ! \Gamma \left( 1 - r/b \right), b < r
\]

Proof: let \( X \) be a random variable from \( TIITLG - 2 \) distribution, the \( r \)th moment is given by

\[
E(X^r) = \mu'_r = \int_{-\infty}^{\infty} x^r f(x) dx
\]  

(11)

Substitute for \( f(x) \) in equation (10), we have

\[
\mu'_r = 2aba \sum_{i=0}^{\infty} (-1)^i \binom{\alpha - 1}{i} \int_{-\infty}^{\infty} x^{r-b-1} e^{-2ax^{-b}(i+1)} dx
\]  

(12)

By letting

\[
\text{Taking, } z = 2ax^{-b}(i+1), x = z^{-1/b}(2a)^{1/b}(i+1)^{1/b}, dx = -1/b z^{-1/b-1}(i+1)^{1/b}dz \text{ and substitute it in (12), we have}
\]

\[
\mu'_r = \mu'_r = a \sum_{i=0}^{\infty} \binom{\alpha - 1}{i} (-1)^i (2a)^r (i+1)^{r-b}/b ! \Gamma \left( 1 - r/b \right), b < r
\]  

(13)

Where, \( \Gamma(\cdot, \cdot, \cdot) \) is the gamma function. The \( r \)th central moment of \( TIITLG - 2 \) distribution is from
\[ \mu_r = E(X - \mu_1)^r = \sum_{i=0}^{r} (-1)^i \binom{r}{i} (\mu_1)^i \mu_{r-i} \] (14)

The skewness and kurtosis can be obtained from equation (14)

### 4.2 Moment generating function

The moment generating function of T11T1G2 distribution is given in the following theorem. Theorem 2. Let \( X \) follows the T11T1G2 distribution, the moment generating function, \( M_X(t) \) is

\[ M_X(t) = \alpha \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{t^j}{j!} \binom{\alpha - 1}{i} (-1)^i (2a)^r (i + 1)^{r-b} b^r \Gamma \left( 1 - \frac{r}{b} \right), \quad t \in \mathbb{R}, \quad b < r \]

Proof the moment generating function of a random variable \( X \) is given by

\[ M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} w(x) dx \] (15)

Where \( f(x) \) is given in (). Using series expansion for \( e^{tx} \) given by

\[ e^{tx} = \sum_{r=0}^{\infty} \frac{t^r}{r!} x^r \] (16)

we can re-write equation (15) using (16) as follows

\[ M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r) \]

Using \( E(X^r) \) given in equation (13), and putting it in equation (17), we have

\[ M_X(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{t^j}{j!} \binom{\alpha - 1}{i} (-1)^i (2a)^r (i + 1)^{r-b} b^r \Gamma \left( 1 - \frac{r}{b} \right), \quad t \in \mathbb{R}, \quad b < r \] (18)

Where, \( \Gamma(\cdot, \cdot, \cdot) \) is the gamma function

### 4.3 Incomplete moment

The \( r^{th} \) lower incomplete moment, say \( \overline{\Phi} (t) \), of T11T1G2 distribution is given by

\[ \overline{\Phi} (t) = 2aba \sum_{m=0}^{\infty} (-1)^i \binom{\alpha - 1}{i} \int_{0}^{t} x^{r-b-1} e^{-2ax-2b(i+1)} dx \] (19)

Taking, \( z = 2ax^{-b}(i + 1) \), \( x = z^{-1/b}(2a)^{1/b}(i + 1)^{1/b} dx = -1/b z^{-1/b-1}(i + 1)^{1/b}dz \) and substitute it in (18), we have

\[ \overline{\Phi} (t) = a \sum_{i=0}^{\infty} \binom{\alpha - 1}{i} (-1)^i (2a)^r(i + 1)^{r-b} b^r \Gamma \left( 1 - \frac{r}{b} \right)(1 - \frac{r}{b}, 2at^{-b}(i + 1)), \quad b < r \] (20)
The element of the score vector is given by the maximum likelihood estimates (MLEs) of the sample data, called the maximum likelihood estimation. The maximum likelihood estimation is used in determining the parameters that maximize the likelihood function. Equation (23) is an expression for the probability density function of the order statistics for the TIITLGT-2 distribution.

\[ f_r(x; \Omega) = \frac{1}{B(r, n - r + 1)} F_{\text{TIITLGT-2}}(x; \Omega)^{r-1} \left[ 1 - F_{\text{TIITLGT-2}}(x; \Omega) \right]^{n-r} f_{\text{TIITLGT-2}}(x; \Omega) \]  

(21)

Then by applying the series expansion given in equation (9), we have

\[ f_r(x; \xi) = \frac{f_{\text{TIITLGT-2}}(x; \xi)}{B(r, n - r + 1)} \sum_{i=1}^{n-r} (-1)^i \binom{n-r}{i} F_{\text{TIITLGT-2}}(x; \xi)^{r+i-1} \]  

(22)

Now, by substituting equation (5) and (6) in \( f_r(x; \Omega) \), we have

\[ f_r(x; \xi) = \frac{2abax^{-(b+1)}e^{-2ax^{-b}}}{B(r, n - r + 1)} \sum_{i=1}^{n-r} (-1)^i \binom{n-r}{i} \left( 1 - \left[ 1 - e^{-2ax^{-b}} \right]^\alpha \right)^{r+i-1} \]  

(23)

Also by applying series expansion in (22), we have

\[ f_r(x; \xi) = \frac{2abax^{-(b+1)}e^{-2ax^{-b}}}{B(r, n - r + 1)} \sum_{i=1}^{n-r} \sum_{j=0}^{\infty} \sum_{k=0}^{j} (-1)^{i+k} \binom{n-r}{i} \binom{r+i-1}{j} \left( \frac{\alpha j}{k} \right) e^{-2ax^{-b}(k+1)} \]  

(24)

Equation (23) is an expression for the \( r \)th order statistics for the TIITLGT-2 distribution.

### 5.1 Maximum likelihood estimation method

The maximum likelihood estimation is used in determining the parameters that maximize the likelihood function of the sample data, called the maximum likelihood estimates (MLEs). Suppose we obtain a random sample \( x_1, ..., x_n \) from TIITLGT-2 distribution, the corresponding likelihood function is given by

\[ L(x; a, b, \alpha) = \prod_{i=1}^{n} f(x_i; a, b, \alpha) \]  

(25)

\[ = \prod_{i=1}^{n} \left[ 2abax_i^{-(b+1)}e^{-2ax_i^{-b}} \left[ 1 - e^{-2ax_i^{-b}} \right]^{(\alpha-1)} \right] \]

The MLEs of \( a, b, \) and \( \alpha \) are denoted by \( \hat{a}, \hat{b}, \) and \( \hat{\alpha} \), respectively. The values can be obtained by maximizing the log-likelihood function given by

\[ LogL = l = \log(2ab) - (b+1) \sum_{i=1}^{n} x_i - 2\alpha \sum_{i=1}^{n} x_i^{-b} + (\alpha - 1) \sum_{i=1}^{n} \left( 1 - e^{-2ax_i^{-b}} \right) \]  

(26)

The element of the score vector is given by
\[
\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \left( 1 - e^{-2ax_i^{-b}} \right)
\]

(27)

\[
\frac{\partial l}{\partial a} = \frac{n}{\alpha} - 2 \sum_{i=1}^{n} x_i^{-b} + (\alpha - 1) \sum_{i=1}^{n} \left( \frac{2x_i^{-b}a^{-2ax_i^{-b}}}{1 - e^{-2ax_i^{-b}}} \right)
\]

(28)

\[
\frac{\partial l}{\partial b} = \frac{n}{b} + 2a \sum_{i=1}^{n} x_i^{-b} log x - (\alpha - 1) \sum_{i=1}^{n} \left( \frac{2a e^{-2ax_i^{-b}}x_i^{-b} log x}{1 - e^{-2ax_i^{-b}}} \right)
\]

(29)

By setting the above partial equations above to zero, the equations obtained are not in closed form and values of the parameters \(\alpha, a, \text{ and } b\) must be found by iterative methods. The maximum likelihood estimates of the parameters, denoted by \(\hat{\Omega}\), is obtained by solving nonlinear equation \(\left( \frac{\partial l}{\partial \alpha} \frac{\partial l}{\partial a} \frac{\partial l}{\partial b} \right)^{\top} = 0\)

6 Applications

In this section, we prove the effectiveness of TIITLG\(\text{T}\)-2 distribution by use of two well known datasets: the glass fibre data set and the pigs data, presented in more detail below. we also make comparison with other distributions making a formative evaluation of the goodness of fit of the models. We considered the following goodness of fit measures: the Akaike information criterion (AIC), Bayesian information criterion and the Hannan-Quinn information criterion. It should be noted that the smaller the values of these statistics is, the better the fit to the data is.

6.1 Glass fibre data

The first data set is obtained from Smith and Naylor [9] and consists of strengths of 1.5 cm glass fibers measured at the National Physical Laboratory. The data are: 0.55, 0.74, 0.77, 0.81, 0.84, 1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25, 1.27, 1.28, 1.29, 1.48, 1.36, 1.39, 1.42, 1.48, 1.51, 1.49, 1.49, 1.50, 1.50, 1.55, 1.52, 1.53, 1.54, 1.55, 1.61, 1.58, 1.59, 1.60, 1.61, 1.63, 1.61, 1.61, 1.62, 1.62, 1.67, 1.64, 1.66, 1.66, 1.66, 1.70, 1.68, 1.68, 1.69, 1.70, 1.78, 1.73, 1.76, 1.76, 1.77, 1.89, 1.81, 1.82, 1.84, 1.84, 2.00, 2.01, 2.24. The exploratory data analysis of the data is given below in Table 1. We observe that the data is negatively skewed and mesokurtic. The total time on test plot in Fig. 5 shows a concave transform, indicating that the hrf is possibly increasing.

![Glass fibre data](image-url)
Thus for the glass fibre data we compare its fits with the following distribution: Topp-Leone Bur XII (TLBXII) distribution by Reyad and Othman [10], Beta Burr XII (BBXII) distribution by Paranaiba et al. [11]. The MLEs of the parameters of these distributions considered and their corresponding standard errors (SEs) for glass fibre data are provided in Table 2. The statistics AIC, BIC, CAIC and HQIC are listed in Table 3.

### Table 2. MLEs and SEs (in parenthesis) for the survival times of Glass fibre data

| Model       | \(a\) (SE) | \(b\) (SE) | \(\alpha\) (SE) | \(\theta\) (SE) |
|-------------|------------|------------|-----------------|-----------------|
| TLTXGT − 2  | 2.64(0.31) | 1.35(0.18) | 16.44(7.73)     | −(−)            |
| TLBXII      | 2.65(0.87) | 0.94(0.40) | 7.59(5.02)      | −(−)            |
| BBXII       | 0.94(1.61) | 1.68(1.49) | 14.85(23.83)    | 8.86(12.712)    |
| GT − 2      | 1.96(0.25) | 2.89(0.24) | −(−)            | −(−)            |

### Table 3. The AIC, BIC, and HQIC for the Glass fibre data

| Model       | AIC        | BIC        | HQIC       |
|-------------|------------|------------|------------|
| TLTXGT − 2  | 60.523     | 66.952     | 63.052     |
| TLBXII      | 77.114     | 83.543     | 79.642     |
| BBXII       | 67.174     | 75.746     | 70.545     |
| GT − 2      | 97.707     | 101.993    | 99.393     |

### 6.2 Survival time of pig data

The second data set (data set 2) represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal et al. [12]. The observations are as follows: 0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55. The exploratory data analysis of the data is given below in Table 4. We observe that the data is negatively skewed and mesokurtic. The total time on test plot in Fig. 6 shows a concave transform, indicating that the hrf is possibly increasing.

![Survival times of pigs data](image-url)

**Fig. 6.** Plots TTT and Kernel density plot for the survival times of pigs data
Table 4. Exploratory data analysis of the pig's data

| min | Q₁ | median | mean | Q₃ | max | kurtosis | skew. | range |
|-----|----|--------|------|----|-----|----------|-------|-------|
| 0.10| 1.08| 1.50   | 1.77 | 2.24| 5.56| 2.2      | 1.4   | 5.46  |

Thus for the glass fibre data we compare its fits with the following distribution: Exponentiated Gumbel Type-2 (ExGT-2) distribution by Okorie et al. [1], Extended Gumbel Type-2 (EGT-2) distribution by Ogunde et al. [3]. The MLEs of the parameters of these distributions considered and their corresponding standard errors (SEs) for survival times of pig’s data are provided in Table 5 the statistics AIC, BIC, CAIC and HQIC are listed in Table 6.

Table 5. MLEs and SEs (in parenthesis) for the survival times of pig’s data

| Model   | α         | b         | α         | θ         |
|---------|-----------|-----------|-----------|-----------|
| THITLGT - 2 | 1.96(0.30)| 0.55(0.08)| 16.19(8.87)| (−)       |
| ExGT - 2 | 1.30(1.80)| 1.17(0.08)| 0.82(1.14)| (−)       |
| EGT - 2  | 14.06(8.72)| 1.15(0.73)| 0.56(0.15)| 3.60(1.05)|
| GT - 2   | 1.07(0.13)| 1.17(0.08)| (−)       | (−)       |

Table 6. The AIC, BIC, and HQIC for the Pig’s data

| Model   | l         | AIC      | BIC      | HQIC     |
|---------|-----------|----------|----------|----------|
| THITLGT - 2 | 97.60   | 201.20   | 208.03   | 203.92   |
| EGT - 2   | 118.167  | 242.333  | 249.163  | 245.052  |
| ExGT - 2  | 98.063   | 204.126  | 213.331  | 207.758  |
| GT - 2    | 118.17   | 240.33   | 244.15   | 242.15   |

7 Conclusion

In this research, we studied a three-parameter distribution named the Type II Topp-Leone Gumbel type-2 distribution which is an extension of Gumbel type-2 distribution. We also provide some mathematical properties of the Type II Topp-Leone Gumbel Type-2 distribution including explicit expression for the density function, ordinary moments, incomplete moments, and order statistics. We make use of maximum likelihood estimation method to estimate the model parameters. In order to evaluate the performance of the new distribution, two real life data applications is used and we found out that Type II Topp-Leone Gumbel Type-2 distribution shows a reasonable fit with the two life data than other competing distributions in terms of the values of AIC, BIC, and HQIC. Therefore it could be considered to produce the best fit in the class of other distribution considered based on the data application. Since it possesses the lowest AIC, BIC, and HQIC.

Competing Interests

Authors have declared that no competing interests exist.

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