Unsteady flows of heat-conducting non-Newtonian fluids in Musielak–Orlicz spaces

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Abstract

Our purpose is to show the existence of weak solutions for unsteady flow of non-Newtonian incompressible nonhomogeneous, heat-conducting fluids with generalised form of the stress tensor without any restriction on its upper growth. Motivated by fluids of nonstandard rheology we focus on the general form of growth conditions for the stress tensor which makes anisotropic Musielak–Orlicz spaces a suitable function space for the considered problem. We do not assume any smallness condition on initial data in order to obtain long-time existence. Within the proof we use monotonicity methods, integration by parts adapted to nonreflexive spaces and Young measure techniques.

Keywords: weak solutions, Musielak–Orlicz spaces, generalised Minty method, Young measures, nonhomogeneous fluid, heat-conducting fluids, incompressible non-Newtonian fluid

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1. Introduction and formulation of the problem

In this article we investigate mathematical model of the flow of an incompressible, non-homogeneous non-Newtonian, heat-conducting fluid governed by the following system of equations:

\[
\begin{align*}
\partial_t \varrho + \text{div}_x(\varrho \mathbf{u}) &= 0 \quad \text{in } Q, \\
\partial_t (\mathbf{u} \varrho) + \text{div}_x(\mathbf{u} \otimes \mathbf{u}) - \text{div}_x S(x, \varrho, \theta, D\mathbf{u}) + \nabla_x P &= \mathbf{f} \quad \text{in } Q, \\
\partial_t (\theta \varrho) + \text{div}_x(\mathbf{u} \theta) - \text{div}_x q(\mathbf{u}, \theta, \nabla_x \theta) &= S(x, \varrho, \theta, D\mathbf{u}) : D\mathbf{u} \quad \text{in } Q, \\
\text{div}_x \mathbf{u} &= 0 \quad \text{in } Q, \\
\mathbf{u}(0, x) &= \mathbf{u}_0 \quad \text{in } \Omega, \\
\theta(0, x) &= \theta_0 \quad \text{in } \Omega, \\
q \cdot n &= 0, \\n\mathbf{u}(t, x) &= 0 \quad \text{on } [0, T] \times \partial \Omega, 
\end{align*}
\]

where \( \varrho : Q \rightarrow \mathbb{R} \) is a mass density, \( \mathbf{u} : Q \rightarrow \mathbb{R}^3 \) stands for a velocity field, \( \theta : Q \rightarrow \mathbb{R} \) is a temperature, \( P : Q \rightarrow \mathbb{R} \) is a pressure function, \( S \) a stress tensor, \( q \) a thermal flux vector, \( \mathbf{f} : Q \rightarrow \mathbb{R}^3 \) a given outer force. The set \( \Omega \subset \mathbb{R}^3 \) is a bounded domain with a regular boundary \( \partial \Omega \) (of class, say \( C^{2+\nu}, \nu > 0 \), taken for convenience). We denote by \( Q = (0, T) \times \Omega \) the time-space cylinder with some given \( T \in (0, +\infty) \). The tensor \( D\mathbf{u} = \frac{1}{2}(\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u}) \) is a symmetric part of the velocity gradient.

For the above system we set the initial density \( \varrho \) and temperature \( \theta \) to satisfy

\[
\begin{align*}
\varrho(0, \cdot) &= \varrho_0 \in L^\infty(\Omega) \quad \text{and} \quad 0 < \varrho_* \leq \varrho_0(x) \leq \varrho^* < +\infty \quad \text{for a.a. } x \in \Omega, \\
\theta_0 &\in L^1(\Omega) \quad \text{and} \quad 0 < \theta_* \leq \theta_0 \quad \text{for a.a. } x \in \Omega.
\end{align*}
\]

In order to formulate the growth conditions of the stress tensor we use general convex function \( M \) called an \( N \)-function similarly as in [20, 21, 46, 47] (for a definition see section 2.1). We assume that stress tensor \( S : \Omega \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^{3 \times 3}_{\text{sym}} \rightarrow \mathbb{R}^{3 \times 3}_{\text{sym}} \) satisfies (\( \mathbb{R}^{3 \times 3}_{\text{sym}} \) stands for the space of \( 3 \times 3 \) symmetric matrices):

**S1.** \( S(x, \varrho, \theta, \mathbf{K}) \) is a Carathéodory function (i.e. measurable function of \( x \) for all \( \varrho, \theta > 0 \) and \( \mathbf{K} \in \mathbb{R}^{3 \times 3}_{\text{sym}} \) and continuous function of \( \varrho, \theta \) and \( \mathbf{K} \) for a.a. \( x \in \Omega \)) and \( S(x, \varrho, \theta, 0) = 0 \).

**S2.** There exists a positive constant \( c_c \in (0, 1) \), \( N \)-functions \( M \) and \( M^* \) (which denotes the complementary function to \( M \)) such that for all \( \mathbf{K} \in \mathbb{R}^{3 \times 3}_{\text{sym}}, \varrho > 0 \) and a.a. \( t, x \in Q \) holds

\[
S(x, \varrho, \theta, \mathbf{K}) : \mathbf{K} \geq c_c \{ M(x, \mathbf{K}) + M^*(x, S(x, \varrho, \theta, \mathbf{K})) \}. 
\]

**S3.** \( S \) is monotone, i.e. for all \( \mathbf{K}_1, \mathbf{K}_2 \in \mathbb{R}^{3 \times 3}_{\text{sym}}, \varrho > 0, \theta > 0 \) and a.a. \( x \in \Omega \)

\[
|S(x, \varrho, \theta, \mathbf{K}_1) - S(x, \varrho, \theta, \mathbf{K}_2)| : [\mathbf{K}_1 - \mathbf{K}_2] \geq 0.
\]

The heat flux \( q \), as usually, is set to be less general. Therefore, similarly as in [14], we expect \( q \) of the form

\[
q(\varrho, \theta, \nabla_x \theta) \approx \kappa(\varrho) \theta^\beta \nabla_x \theta = \kappa(\varrho) \frac{1}{\beta + 1} \nabla \theta^{\beta + 1} 
\]

for \( \beta \in \mathbb{R} \),

such that \( \kappa(\varrho) \) satisfies \( 0 \leq \kappa_* \leq \kappa(\varrho) \leq \kappa^* < \infty \), where \( \kappa_*, \kappa^* \) are some fixed constants. In particular, we require that \( q : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) satisfies
\[ q(\varrho, \theta, \nabla_x \theta) = \kappa_0(\varrho, \theta) \nabla_x \theta \quad \text{with } \kappa_0 \in C(\mathbb{R}^2_+) \]

and for all \( \varrho, \theta > 0, \nabla_x \theta \in \mathbb{R}^3 \)

\[ q(\varrho, \theta, \nabla_x \theta) \cdot \nabla_x \theta \geq \kappa_+ \theta^3 [\nabla_x \theta]^2 \quad \text{with } \beta \in \mathbb{R} \text{ and } \kappa_+ > 0, \]

\[ |q(\varrho, \theta, \nabla_x \theta)| \leq \kappa^* \theta^3 |\nabla \theta| \quad \text{with } \kappa^* > 0. \quad \text{(6)} \]

The main reason to investigate such a general form of stress tensor \( S \), namely satisfying (4), is the phenomena of rapidly increasing fluid viscosity under various stimuli as shear rate, electric or magnetic field. Our assumptions include power-law and Carreau-type models which are quite popular among rheologists, chemical engineering and colloidal mechanics.

In majority of publications concerning non-Newtonian fluids a \( p \)-structure for \( S \) is assumed and then typically the stress tensor can take the form \( S = \mu(\varrho, \theta)(\kappa + |Du|^2)^{p-2}Du \) or \( S = \mu(\varrho, \theta)(\kappa + |Du|^2)^{p-2}Du \) (where \( \kappa > 0 \) and \( \mu \) is a nonnegative bounded function). Then standard growth conditions of the stress tensor, namely polynomial growth: \( |S(x, \xi)| \leq c(1 + |\xi|^2)^{(p-2)/2} \xi \) and \( S(x, \xi) \cdot \xi \geq c(1 + |\xi|^2)^{(p-2)/2} |\xi|^2 \) are satisfied, see e.g. [14, 15, 31]. Unfortunately this theory is not adequate for phenomena of fluids that rapidly and significantly change their viscosity, i.e. when growth of the stress tensor may be much faster then polynomial and which may differ in various directions of shear stress or be inhomogeneous is spatial variables. Examples of these fluids are shear thickening (STF), magnetorheological (MR) and electrorheological fluids. Because of property of the changeable viscosity these fluids have applications in a variety of industry, military and natural sciences.

Firstly we would like to be able to consider flows for which constitutive relations for the stress tensor \( S \) are more general than of power-law type and, in particular, which the growth w.r.t. the shear stress may be faster than polynomial. Very promising application of this type of fluids is the one coming from military industry. STF fluids behave as a liquid until another object strike it with high kinetic energy. In this case the fluid increases its viscosity in milliseconds and behaves almost like a solid. Moreover this process is completely reversible which makes such a fluid a perfect material for military, medical and sport armours. The obtained material has high elasticity combined with protection against needles, knives and bullets [6, 11, 24, 29].

Moreover, we can study constitutive relations for fluids with dependence on outer field (magnetic or electric). Mathematical models for such fluids are considered e.g. in [37]. Governing equations are derived from motion of electrorheological fluids, taking into consideration complex interactions between electromagnetic and thermomechanical fields into consideration (see also [36]). For such general fluids, as claimed in (see [39]), the stress tensor can be written in a quite general form, which is still thermodynamically admissible (i.e. \( S : D \geq 0 \)), satisfies the principle of material frame-indifference and is monotone. But then it may appear that the standard growth conditions, i.e. \( |S(D, E)| \leq c(1 + |D|)^{(p-1)} \), \( S(D, E) : D \geq c|D|^p \) (\( E \) denotes electric flux) are not satisfied, because the tensor \( S \) may possess the growth of different powers in various directions of \( D \) (for the example see also [47]).

In our considerations we also would like to cover the case of constitutive relations which may depend on spatial variables. For example, again it may be the case of electrorheological fluids which are suspensions of extremely fine non-conducting particles in an electrically insulating fluid. Such a mode was considered e.g. in [39] where the \( N \)-function took the form: \( M(x, z) = |z|^{p(x)} \) with \( 1 < p_- \leq p(x) \leq p_+ < \infty \). The author there provided the existence theory for the case of barotropic flows without dependence on density.
The appropriate spaces to capture such formulated problem are anisotropic Musielak–Orlicz spaces. For definitions and preliminaries of $N$–functions and Orlicz spaces see section 2.1. Contrary to [32] we consider the $N$–function $M$ not dependent only on $|\xi|$, but on whole tensor $\xi$. It results from the fact that the viscosity may differ in different directions of symmetric part of velocity gradient $Du$ and the growth condition for the stress tensor dependent on the whole tensor $Du$, not only on $|Du|$. Since we allow $S$ to depend on spatial variable, $N$–function depends also on $x \in \Omega$. The general growth on $S$ is provided by quite general properties of the $N$–function $M$ defining anisotropic Musielak–Orlicz space $L_M$. Since we do not want to be restricted by any upper growth conditions on $S$, we do not assume that, so called, $\Delta_2$–condition is satisfied by $M$. The spaces with $N$-function dependent on vector-valued argument were introduced in [41, 42, 44]. Let us underline that, in general, if $M$ and $M^*$ do not satisfy a $\Delta_2$-condition the related spaces fail to be separable and reflexive, which is a source of additional difficulties arising from functional analysis. We then simply lose many facilitating properties of spaces we have to work with. The setting considered in this paper needs tools which generalise results not only of classical Lebesgue and Sobolev spaces (related to power-law fluids), but also these in variable exponent, anisotropic and classical Orlicz spaces. Most of the essential and necessary tools of functional analysis for classical Orlicz spaces (isotropic and homogenous) are already deeply understood, for example the density of smooth functions in modular topology [18] and integration by parts formula [12]. But many structures for anisotropic Musielak–Orlicz spaces have not been developed or are not understood fully yet.

One of the essential difficulties we have to face to provide is the weak sequential stability in energy equation. Namely we need to show that $S^n := S(\cdot, g^n, \theta^n, Du^n) : Du^n \rightharpoonup S(\cdot, g, \theta, Du) : Du$ weakly in $L^1(\Omega)$, where $\{g^n\}_{n=1}^{\infty}, \{u^n\}_{n=1}^{\infty}, \{\theta^n\}_{n=1}^{\infty}$ are approximation sequences of $g$, $u$, $\theta$ and $\{S^n\}_{n=1}^{\infty} \subset L_M(\Omega)$, $\{Du^n\}_{n=1}^{\infty} \subset L_M(\Omega)$. Let us notice that if one work with reflexive spaces (such a $L^p$) the monotonicity is a sufficient argument to conclude from $(S^n - S) : (Du^n - Du) \rightharpoonup 0$ in $L^1$ that $S^n : Du^n \rightharpoonup S : Du$ weakly in $L^1$. However, once the space is not reflexive, as the case of our $L_M$-space, then the convergence may fail if one is not able to provide modular convergence of sequences $S^n$ and $Du^n$ in proper spaces. In the current paper we use bitting lemma [1, 3, 35] and methods of Young measures to show that the product of our two sequences converges weakly in $L^1$ and consequently to provide the sequential stability of the right hand side of energy equation. Similar arguments in frame of anisotropic Musielak–Orlicz spaces were used in [23] for parabolic equation and later also in [25] for the problem of thermo-visco-elasticity model.

An interesting obstacle here is the lack of the classical integration by parts formula, see [17, section 4.1]. To extend it for the case of anisotropic Musielak–Orlicz spaces we would need that $C^\infty$–functions are dense in $L_M(\Omega)$ and $L_M(\Omega) = L_M(0, T; L_M(\Omega))$. The first one only holds if $M$ satisfies $\Delta_2$-condition. The second one is not the case in Orlicz and generalized Orlicz spaces, see [10] and holds only if $M$ is equivalent to some power $p$, $1 < p < \infty$ (what provides that $L_M(\Omega)$ is separable and reflexive). In the present paper we recall the integration by parts formula obtained in [47] by adaptation of arguments from [21] and [14].

Moreover classical monotonicity methods allowing us to obtain convergence in a nonlinear viscous term in the momentum equation do not work in case of non-reflexive anisotropic Musielak–Orlicz spaces. Therefore we need to apply arguments developed in [21, 46, 47], see also [33].

Let us now recall briefly related results. The mathematical analysis of time dependent flow of homogeneous (density was assumed to be constant) non-Newtonian fluids of power-law type was initiated in [27, 28], where the global existence of weak solutions for the exponent $p \geq 1 + (2d)/(d + 2)$ ($d$ stands for space dimension) was proved for Dirichlet boundary
conditions. Later the steady flow was considered in [16], where the existence of weak solutions was established for the constant exponent $p > \frac{2d}{d+2}$, $d \geq 2$ by Lipschitz truncation methods.

In [39] generalized Lebesgue spaces $L^{p(\cdot)}$ were used to the description of flow of electrorheological fluid. The author assumed in this work that $1 < p_0 \leq p(x) \leq p_\infty < \infty$, where $p \in C^1(\Omega)$ is a function of an electric field $E$, i.e. $p = p(|E|^2)$, and $p_0 > \frac{d}{d+2}$ in case of steady flow, where $d \geq 2$ is the space dimension. The $\Delta_2$-condition is then satisfied and consequently the space is reflexive and separable (what is not the case of our work). Later in [7] the above result was improved by Lipschitz truncation methods for $L^{p(\cdot)}$ setting for $S$, where $\frac{d}{d+2} < p(\cdot) < \infty$ was log-Hölder continuous and $S$ was strictly monotone.

In [45] the author proved existence of weak solutions to unsteady motion of an incompressible homogenous fluid with shear rate dependent viscosity for $p > 2(d + 1)/(d + 2)$ without assumptions on shape and size of $\Omega$ employing an $L^{\infty}$-test function and local pressure method. Finally the existence of global weak solutions with Dirichlet boundary conditions for $p > (2d)/(d + 2)$ was achieved in [8] by Lipschitz truncation and local pressure methods.

Most of the available results concerning nonhomogeneous (without assumption that density is constant) incompressible fluids deal with the polynomial dependence between $S$ and $\text{div}U$. The analysis of nonhomogeneous Newtonian ($p = 2$ in standard growth condition) fluids was investigated in [2] in the seventies. In [30] the concept of renormalized solutions was presented what allowed to obtain convergence and continuity properties of the density.

The first result for unsteady flow of nonhomogenous non-Newtonian fluids goes back to [13], where existence of Dirichlet weak solutions was obtained for $p \geq 12/5$ if $d = 3$, later existence of space-periodic weak solutions for $p > 2$ with some regularity properties of weak solutions whenever $p \geq 20/9$ (if $d = 3$) was achieved in [19]. In [15] existence of a weak solution was showed for generalized Newtonian fluid of power-law type for $p > 11/5$. Authors also needed existence of the potential of $S$. The most related result concerning inhomogeneous, incompressible and heat-conducting non-Newtonian fluids, but of standard growth conditions of polynomial type for $p \geq 11/5$ the reader can find in [14]. The novelty of this paper w.r.t. the previously mentioned results was the consideration of the full thermodynamic model.

The analysis of non-Newtonian fluids in frame of anisotropic Musielak–Orlicz spaces have been studied with variety of approaches. Some of the considerations can be found in [20] (the case of homogeneous, incompressible non-Newtonian fluids) where $S$ was taken to be strictly monotone. The authors used Young measure technics in place of monotonicity methods. The additional assumption at strict monotonicity allows to conclude that the measure-valued solution is of the form of Dirac measure and then the system has a weak solution. Later generalisation of the Browder–Minty trick for non-reflexive anisotropic Musielak–Orlicz was used in [21, 46, 47], what allows to assume only the monotonicity of $S$.

In [22] the authors studied generalized Stokes system for the unsteady flow of homogeneous, incompressible non-Newtonian fluids of non-standard rheology. Neglecting the convective term in momentum equation they showed existence of weak solutions in anisotropic Orlicz spaces without assumption on lower bound on $N$–function $M$, what allowed them to consider also shear-thinning fluids.

In particular in [47] the author obtained existence of weak solutions to unsteady flow of non-Newtonian incompressible nonhomogeneous fluids with nonstandard growth conditions of the stress tensor assumed also in the current paper.

Summarising, our less restrictive assumptions on tensor $S$ allow to consider effects of nonhomogeneous (dependence on $x$), anisotropic behaviour of considered medium and as well more general than power-law type rheologies. In this article we focus on time dependent flow of non-Newtonian, inhomogeneous (density dependent), incompressible fluid and our main
goal is to consider also temperature and its influence on the flow. Let us emphasise that the stress tensor \( S \) may depend here not only on a shear stress but also on density and changes in temperature. To our knowledge, the existing result for such a problem has not been considered yet and our considerations extend the theory concerning thermodynamics of non-Newtonian fluids of power-law type to the case of non-standard growth conditions in Musielak–Orlicz spaces on the one hand, and the theory of non-Newtonian fluids with non-standard growth conditions to the case of heat-conducting fluids on the other.

Our paper is constructed as follows. In section 2 we recall some used facts and definition necessary for the main theorem stated in section 3. In section 4 we prove the main theorem building \( n \)-approximate solutions, providing uniform estimates and using monotonicity method, compensated compactness arguments and Young measures.

2. Preliminaries

2.1. Used notation

In the following section we introduce notation, definitions and some important properties of Orlicz spaces used in further considerations. More studies of Orlicz spaces can be found in [26, 32, 41, 42].

By \( D(\Omega) \) we mean the set of \( C^\infty \)-functions with compact support contained in \( \Omega \). Let \( \mathcal{V} \) be the set of all functions which belong to \( D(\Omega) \) and are divergence-free. Moreover, by \( L^p, W^{1,p} \) we denote the standard Lebesgue and Sobolev spaces respectively, by \( H \)–the closure of \( \mathcal{V} \) w.r.t. the \( L^2 \) norm and by \( W_{\text{div}}^{1,p} \) the closure of \( \mathcal{V} \) w.r.t. the \( \| \nabla (\cdot) \|_{L^p} \) norm. Let \( W_0^{1,p} = (W_0^{1,p})^* \) and \( W_{\text{div}}^{-1,p} = (W_{\text{div}}^{-1,p})^* \). By \( p' \) we denote a conjugate exponent to \( p \), namely \( \frac{1}{p} + \frac{1}{p'} = 1 \).

If \( X \) is a Banach space of scalar functions, then \( X^3 \) or \( X^{3 \times 3} \) denotes the space of vectoror tensor-valued functions where each component belongs to \( X \). The symbols \( L^p(0,T;X) \) and \( C([0,T];X) \) mean the standard Bochner spaces. Finally, we recall that the Nikolskii space \( N^{\alpha,p}(0,T;X) \) corresponding to the Banach space \( X \) and the exponents \( \alpha \in (0,1) \) and \( p \in [1,\infty] \) is given by

\[
N^{\alpha,p}(0,T;X) := \{ f \in L^p(0,T;X) : \sup_{0<h<T} h^{-\alpha} \| \tau_h f - f \|_{L^p(0,T-h;X)} < \infty \},
\]

where \( \tau_h f(t) = f(t+h) \) for a.a. \( t \in [0,T-h] \). By \( (a,b) \) we mean \( \int_\Omega a(x) \cdot b(x) dx \) an inner product of two vector functions or in case \( \int_\Omega a(x) : b(x) dx \) of two tensor functions and \( (a,b) \) denotes the duality pairing.

2.2. Orlicz spaces

**Definition 2.1.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^3 \), a function \( M : \Omega \times \mathbb{R}_{\text{sym}}^{3 \times 3} \rightarrow \mathbb{R}_+ \) is said to be an \( N \)-function if it satisfies the following conditions

1. \( M \) is a Carathéodory function (measurable w.r.t to the first argument and continuous w.r.t. the second one) such that \( M(x, \mathbf{K}) = 0 \) if and only if \( \mathbf{K} = 0, M(x, \mathbf{K}) = M(x, -\mathbf{K}) \) a.e. in \( \Omega \).
2. \( M(x, \mathbf{K}) \) is a convex function w.r.t. \( \mathbf{K} \).
3. \[
\lim_{|\mathbf{K}| \to 0} \frac{M(x, \mathbf{K})}{|\mathbf{K}|} = 0 \quad \text{and} \quad \lim_{|\mathbf{K}| \to \infty} \frac{M(x, \mathbf{K})}{|\mathbf{K}|} = \infty \quad \text{for a.a. } x \in \Omega.
\]
Definition 2.2. The complementary function $M^*$ to a function $M$ is defined for $L \in \mathbb{R}^{3\times3}_{\text{sym}}, x \in \Omega$ by

$$M^*(x, L) = \sup_{K \in \mathbb{R}^{3\times3}_{\text{sym}}} (K : L - M(x, K)).$$

Let us notice that the complementary function $M^*$ is also an $N$-function (see [41]).

Definition 2.3. Let $Q = (0, T) \times \Omega$. The anisotropic Musielak–Orlicz class $L_M(Q)^{3\times3}_{\text{sym}}$ is the set of all measurable functions $K : Q \to \mathbb{R}^{3\times3}_{\text{sym}}$ such that

$$\int_Q M(x, K(t, x)) \, dx \, dt < \infty.$$

Definition 2.4. The anisotropic Musielak–Orlicz space $L_M(Q)^{3\times3}_{\text{sym}}$ is defined as the set of all measurable functions $K : Q \to \mathbb{R}^{3\times3}_{\text{sym}}$ which satisfy

$$\int_Q M(x, \lambda K(t, x)) \, dx \, dt \to 0 \quad \text{as} \quad \lambda \to 0.$$

A Musielak–Orlicz space is a Banach space with Luxemburg norm given by

$$\|K\|_M = \inf \left\{ \lambda > 0 : \int_Q M \left( x, \frac{K(t, x)}{\lambda} \right) \, dx \, dt \leq 1 \right\}.$$ 

Let us denote by $E_M(Q)^{3\times3}_{\text{sym}}$ the closure of all measurable, bounded functions on $Q$ in $L_M(Q)^{3\times3}_{\text{sym}}$. Then $L_M^*(Q)^{3\times3}_{\text{sym}} = (E_M(Q)^{3\times3}_{\text{sym}})^*$ (see [46]) and we observe that $E_M \subseteq L_M \subseteq L_M^*$. The functional

$$\varrho(K) = \int_Q M(x, K(t, x)) \, dx \, dt$$

is a modular in the space of measurable functions $K : Q \to \mathbb{R}^{3\times3}_{\text{sym}}$.

Definition 2.5. We say that sequence $\{z^j\}_{j=1}^{\infty}$ converges modularly to $z$ in $L_M(Q)^{3\times3}_{\text{sym}}$, which is denoted by $z^j M \to z$, if there exists $\lambda > 0$ such that

$$\int_Q M \left( x, \frac{z^j - z}{\lambda} \right) \, dx \, dt \to 0 \quad \text{as} \quad j \to \infty.$$

Definition 2.6. We say that an $N$-function $M$ satisfies $\Delta_2$–condition if for some nonnegative, integrable on $\Omega$ function $g_M$ and a constant $C_M > 0$

$$M(x, 2K) \leq C_M M(x, K) + g_M(x) \quad \text{for all} \quad K \in \mathbb{R}^{d\times d}_{\text{sym}} \text{ and a.a.} \quad x \in \Omega. \quad (8)$$

This condition is crucial for the structure of $L_M(Q)^{3\times3}_{\text{sym}}$ space. It ensures that this space is separable, reflexive and that $C^\infty$ functions are dense. Whatsmore, if $\Delta_2$–condition holds, then $E_M(Q)^{3\times3}_{\text{sym}} = L_M(Q)^{3\times3}_{\text{sym}}$. Otherwise the considered space loses the above facilitating properties.

Below we recall several useful lemmas which are used within the proof of existence of weak solutions. Their proofs can be found e.g. in [20, 47].
Lemma 2.7. Let $z^j : Q \rightarrow \mathbb{R}^{3 \times 3}$ with $j = 1, \ldots, \infty$ be a measurable sequence. Then $z^j \xrightarrow{M} z$ in $L_M(Q)^3 \times 3$ modularly if and only if $z^j \rightarrow z$ in measure and there exists some $\lambda > 0$ such that the sequence $\{M(\cdot, \lambda z^j)\}_{j=1}^{\infty}$ is uniformly integrable, i.e.

$$\lim_{R \rightarrow \infty} \left( \sup_{j \in \mathbb{N}} \int_{\{(x, \lambda z^j) : |M(x, \lambda z^j)| \geq R\}} M(x, \lambda z^j) \, dx \, dt \right) = 0.$$ 

Lemma 2.8. Let $M$ be an $N$–function and for all $j \in \mathbb{N}$ let $\int_{Q} M(x, z^j) \, dx \, dt \leq c$. Then the sequence $\{z^j\}_{j=1}^{\infty}$ is uniformly integrable.

2.3. Div–Curl lemma

In further consideration we use so called Div-Curl lemma as given in [14, 43]. We denote for $a = (a_0, a_1, a_2, a_3)$

$$\text{Div}_{t, x} a := \partial_t a_0 + \sum_{i=1}^{3} \partial_x a_i \quad \text{and} \quad \text{Curl}_{t, x} a := \nabla a_t - (\nabla a, a)^T.$$  \hspace{1cm} (9)

Lemma 2.9. Let $Q = (0, T) \times \Omega \subset \mathbb{R}^4$ be a bounded set. Let $p, q, l, s \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{l}$ and vector fields $a^n, b^n$ satisfies

$$a^n \rightharpoonup a \quad \text{weakly in} \quad L^p(Q)^3 \quad \text{and} \quad b^n \rightharpoonup b \quad \text{weakly in} \quad L^q(Q)^4,$$

and $\text{Div}_{t, x} a$ and $\text{Curl}_{t, x} b$ are precompact in $W^{-1,s}(Q)$ and $W^{-1,s}(Q)^4 \times 4$ respectively. Then

$$a^n \cdot b^n \rightharpoonup a \cdot b \quad \text{weakly in} \quad L^l(Q),$$

where \cdot stands for scalar product in $\mathbb{R}^4$.

3. Main result

We start with a definition of a weak solution of the problem (1).

Definition 3.1. Let $\rho_0$ satisfy (2), $u_0 \in H(\Omega)^3$, $\theta_0$ satisfies (3) and $f \in L^p(0, T; L^p(\Omega)^3)$. Let $S$ satisfy conditions S1–S3 with an $N$–function $M$ such that

$$M(x, \xi) \geq C|\xi|^p - \tilde{C} \quad \text{with} \quad C > 0, \quad \tilde{C} \geq 0 \text{and} \quad p \geq \frac{11}{5}$$

for a.a. $x \in \Omega$ and all $\xi \in \mathbb{R}^{3 \times 3}$ and let $q$ satisfies (5) and (6) with $\beta > - \min \left\{ \frac{2}{3}, \frac{3p - 5}{2p - 3} \right\}$. We call $(\rho, u, \theta)$ a weak solution to (1) if
$0 \leq \varrho_0 \leq \varrho(t,x) \leq \varrho^*$ for a.a. $(t,x) \in Q$,

$\varrho \in C([0,T]; L^q(\Omega))$ for arbitrary $q \in [1, \infty)$,

$\partial_t \varrho \in L^{5p/3}(0,T; W^{1,5p/(5p-3)}(\Omega))$,  

$u \in L^\infty(0,T; H(\Omega)^3) \cap L^p(0,T; W_0^{1,p}_\text{div}(\Omega)^3) \cap N^{1/2,2}(0,T; H(\Omega)^3)$,  

$Du \in L_m(\Omega)^{3 \times 3}$ and $(\varrho u, \psi) \in C([0,T])$ for all $\psi \in H(\Omega)^3$,  

$\theta \in L^\infty(0,T; L_1^1(\Omega))$ and $\theta \geq \theta_0 > 0$ for a.a $(t,x) \in Q$,  

$\theta^{\alpha - \lambda - \epsilon} \in L^2(0,T; W^{1,2}(\Omega))$ for all $\lambda \in (0,1)$,  

$\partial_t (\varrho \theta) \in L^1(0,T; (W^{1,q}(\Omega))^*)$ with $q$ sufficiently large

and the following identities are satisfied: for continuity equation

$$\int_0^T (\partial_t \varrho, z) - (\varrho u, \nabla z) \, dt = 0 \quad (10)$$

for all $z \in L'(0,T; W^{1,r}(\Omega))$ with $r = 5p/(5p - 3)$, i.e.

$$\int_{t_1}^{t_2} \int_\Omega \varrho \partial_t z + (\varrho u) \cdot \nabla z \, dx \, dt = \int_{t_1}^{t_2} \int_\Omega \varrho z(s_2) - \varrho z(s_1) \, dx$$

for all $z$ smooth and $s_1, s_2 \in [0,T], s_1 < s_2$; for the momentum equation

$$- \int_0^T \int_\Omega (\varrho u \cdot \partial_t \varphi - \varrho u \otimes u : \nabla \varphi + S(x, \theta, \nabla \varphi) + D \partial_t \varphi) \, dx \, dt = \int_0^T \int_\Omega \varrho \partial_t \varphi \, dx \, dt + \int_\Omega \varrho \partial_t \varphi \, dx \, dt$$

for all $\varphi \in D((0,\infty); \mathcal{V})$; and for energy equation

$$\int_0^T (\partial_t (\varrho \theta), h) - (\varrho \theta u, \nabla h) + (q(\varrho, \theta, \nabla \varphi), \nabla h) \, dt = \int_0^T (S(x, \theta, \nabla \varphi), \nabla h) \, dt$$

for all $h \in L^\infty(0,T; W^{1,q}(\Omega))$ with $q$ sufficiently large. Moreover, initial conditions are achieved in the following way

$$\lim_{t \to 0^+} \| \varrho(t) - \varrho_0 \|_{L^q(\Omega)} + \| u(t) - u_0 \|_{L^2(\Omega)} = 0 \quad \text{for arbitrary } q \in [1, \infty),$$

$$\lim_{t \to 0^+} (\varrho_0(t), h) = (\varrho_0 \theta_0, h) \quad \text{for all } h \in L^\infty(\Omega). \quad (11)$$

**Theorem 3.2.** Let $M$ be an $N$-function satisfying for some $\xi > 0$, $\tilde{C} > 0$ and for a.a. $x \in \Omega$ and all $\xi \in \mathbb{R}^{3 \times 3}$

$$M(x, \xi) \geq \xi |\xi|^p - \tilde{C} \quad \text{with } p \geq \frac{11}{5}. \quad (12)$$

Let us assume that the complementary function

$$M^*$$

satisfies a $\Delta_2$-condition and

$$\lim \sup_{|\xi| \to \infty \forall x \in \Omega} \frac{M^*(x, \xi)}{|\xi|^p} = \infty. \quad (13)$$
Moreover, let \( S \) satisfy conditions S1–S3 and \( q \) satisfy (5) and (6) with \( \beta > - \min \left\{ \frac{3}{2}, \frac{3p-5}{2p} \right\} \).
Let \( u_0 \in H(\Omega)^3 \), \( g_0 \in L^\infty(\Omega) \) with \( 0 < \theta_* \leq g_0(x) \leq g^* < +\infty \) for a.a. \( x \in \Omega \), \( \theta \in L^1(\Omega) \), \( 0 < \theta_* \leq \theta_0 \) for a.a. \( x \in \Omega \) and \( f \in L^p(0,T;L^p(\Omega)^3) \). Then there exists a weak solution to (1).

In the following paper we consider the flow in the domain of space dimension \( d = 3 \), just for the brevity of the paper. The existence result can be extended to the case of arbitrary \( d \geq 2 \) and \( p \geq \frac{3d+2}{d+2} \).

Let us emphasise that the restriction on the exponent \( p \geq \frac{11}{2} \) allow us only to consider the case of shear-thickening fluids. Since in our approach we use as a test function an approximation of the solution in the space where we \textit{a priori} expect the solution will be, in order to proceed with the convergence in convective term the restriction (12) has to be assumed. If one is able to use method based on Lipschitz truncation, we can expect it could allow to relax this condition to \( p > \frac{5}{2} \) for dimension \( d = 3 \), see [8]. Unfortunately it seems to us that for a such general \( N \)-function as we work with (without restriction on dependence on \( x \), anisotropic, with possible exponential growth) this method is difficult, if possible, to apply. In [7], where steady flow of power-law fluids was considered, the authors studied power-law type fluids with variable exponent and in order to provide continuity of the Hardy–Littlewood maximal operator in variable exponent Lebesgue space \( L^{p(x)}(\Omega) \), they needed to assume log-Hölder continuity of \( p(\cdot) \) function and that \( 1 < p_0 \leq p(x) \leq p_\infty < \infty \). This tool was developed in [4] for the case of classical Orlicz spaces (isotropic case \( M : \mathbb{R}_+ \to \mathbb{R}_+ \)). The authors proved therein the Lipschitz approximation lemma for Bochner functions with values in Orlicz–Sobolev spaces avoiding requirement on strong continuity of maximal operator (as in [8]), but they need that isotropic \( N \)-function \( M(\cdot) \) and (consequently \( M^* \)) is bounded from below and above by some polynomial functions (what, in fact, do not provide that \( \Delta_2 \) condition is satisfied). However, for the time being, we do not see how to generalized these results for our case.

4. Proof of the theorem 3.2

To prove the theorem 3.2 we proceed with \( n \)-approximation of problem (1). First, in order to show the existence of such \( n \)-approximation we need to introduce additional two level approximation. The next step is to provide uniform estimates for \( n \)-approximate problem which allow us to pass to the limit with \( n \to \infty \) and show weak sequential stability.

4.1. Existence of the \( n \)-approximate problem

Let \( \{\omega^i\}_{i=1}^\infty \) be a basis of \( W_{0,\text{div}}^{1,p}(\Omega)^3 \) such that \( \langle \omega^i, \omega^j \rangle = \delta_{ij}, \{\omega^i\}_{i=1}^\infty \subset W_{0,\text{div}}^{1,2p}(\Omega)^3 \), and elements of the basis are constructed with the help of eigenfunctions of the problem

\[
\langle (\omega^i, \varphi) \rangle_s = \lambda_i (\omega^i, \varphi) \quad \text{for all } \varphi \in V_s,
\]

where

\[
V_s \equiv \text{the closure of } \mathcal{V} \text{w.r.t. the } W^{s,2}(\Omega) \text{-norm for } s > 3,
\]

where \( \langle \cdot, \cdot \rangle_s \) denotes the scalar product in \( V_s \). Then the Sobolev embedding theorem provides

\[
W^{s-1,2}(\Omega) \hookrightarrow L^\infty(\Omega), \quad (15)
\]
Then approximate solution is given by
\[ u^n := \sum_{i=1}^{n} \alpha_i^n(t) \omega_i \quad \text{for } i = 1, 2, \ldots, \]
with \( \alpha_i^n \in C([0, T]) \) and the triple \((g^n, u^n, \theta^n)\) satisfies
\[ \int_0^T (\partial_t g^n \cdot z) - (g^n u^n, \nabla_\omega z) \, dt = 0 \quad \text{for all } z \in L^r(0, T; W^{1,r}(\Omega)) \text{ with } r = 5p/(5p - 3), \]
where \( g^0(0, \cdot) = g_0 \) and
\[ \langle \partial_t(g^n u^n), \omega \rangle - (g^n u^n \otimes u^n, \nabla_\omega \omega) + \langle S(x, g^n, \theta^n, Du^n), Du \omega \rangle = (g^n f^n, \omega) \]
for all \( i = 1, \ldots, n \) and a.a. \( t \in [0, T] \), where \( u^n(0, \cdot) = P^n u_0 \) (\( P^n \) denotes the projection of \( H(\Omega)^3 \) onto linear hull of \( \{\omega_i\}_{i=1}^{n} \) ) and
\[ P^n u_0 \to u_0 \quad \text{strongly in } L^2(\Omega)^3, \]
\[ f^n \to f \quad \text{strongly in } L^{p'}(0, T; L^{p'}(\Omega)^3). \]
Moreover \( \theta^n \in L^\infty(0, T; L^2(\Omega)) \cup L^1(0, T; W^{1,4}(\Omega)) \) with \( s = \min \left\{ 2, \frac{5p+10}{5p+5} \right\} \) and \( \theta^n \geq \theta_* \) in \( Q \).
\[ \int_0^T \langle \partial_t(\theta^n \psi^n), h \rangle - (\theta^n \psi^n u^n, \nabla_\omega h) + \langle \kappa_0 \nabla_\omega \theta^n, \nabla_\omega h \rangle \, dt = \int_0^T \langle S(x, \theta^n, Du^n), Du \psi h \rangle \, dt \]
for all \( h \in L^\infty(0, T; W^{1,q}(\Omega)) \) for large \( q \) and where \( \theta^n(0, \cdot) = \theta_0^n \) s.t.
\[ \theta^n \to \theta_0 \quad \text{strongly in } L^1(\Omega). \]

4.1.1 Proof of the existence of the n-approximate problem. The existence of a triple \((g, u, \theta) = (g^n, u^n, \theta^n)\) given by (16)–(19) can be proven by two-steps approximation. To this end we adopt the proof for the power-law type fluid from \([14, \text{section } 6]\) where more details can be found and for isotropic case in frame of Musielak–Orlicz spaces (see \([47, \text{section } 4.1]\)). Here we present only the main steps of the reasoning for the convenience of the reader. The proof is based on standard artificial viscosity technique which combines continuous problem with two Ritz–Galerkin finite-dimensional systems.

In order to define the new two-step approximation let us take \( \{w_i\}_{i=1}^{\infty} \) a smooth basis of \( W^{1,2}(\Omega) \) orthonormal in \( L^2(\Omega) \) spanning the space where we construct a \( k \)-approximation of \( \theta \). We look for the triple \((g^{k,e}, u^{k,e}, \theta^{k,e})\) where \( u^{k,e} \) and \( \theta^{k,e} \) are defined by
\[ u^{k,e} := \sum_{i=1}^{k} \alpha_i^{k,e}(t) \omega_i \quad \text{and} \quad \theta^{k,e} := \sum_{i=1}^{k} \nu_i^{k,e}(t) w_i \]
and \((g^{k,e}, u^{k,e}, \theta^{k,e})\) satisfies the following
\[ \partial_t g^{k,e} + \text{div}(g^{k,e} u^{k,e}) - \epsilon \Delta g^{k,e} = 0 \quad \text{in } Q, \quad \nabla_x g^{k,e} \cdot n = 0 \quad \text{on } [0, T] \times \partial \Omega, \]
\[ g^{k,e} \leq g^* \leq g^{k,e} \quad \text{in } Q, \quad g^{k,e}(0, \cdot) = g_0 \quad \text{in } \Omega, \]
\[ 0 \leq g^{k,e} \leq g^* \quad \text{in } Q, \quad g^{k,e}(0, \cdot) = g_0 \quad \text{in } \Omega, \]
\[ 0 \leq g^{k,e} \leq g^* \quad \text{in } Q, \quad g^{k,e}(0, \cdot) = g_0 \quad \text{in } \Omega, \]

In section 4.1.1 we omit the superscript \( n \) to simplify the notation.
\[
(\theta^{k,e} \frac{d}{dt} u^{k,e}, \omega_i) + (\theta^{k,e} (\nabla_x u^{k,e}) u^{k,e}, \omega_i) + (S_{k,e}, D \omega_i) - \epsilon (\nabla_x \theta^{k,e}, [\nabla_x u^{k,e}] \omega_i) = (\theta^{k,e} f^n, \omega_i)
\]
in \(Q\) and for all \(i = 1, 2, \ldots, n\),

\[
u^{k,e}_i(0, \cdot) = u^{k,e}_0 = \sum_{i=1}^{n} \alpha^{k,e}_i(0) \omega_i = P^e u_0 \text{ in } \Omega,
\]

\[
(\theta^{k,e} \frac{d}{dt} \theta^{k,e}, w_i) + (\theta^{k,e} (\nabla_x \theta^{k,e}) \theta^{k,e}, w_i) + (\kappa_{k,e} \nabla_x \theta^{k,e}, \nabla_x w_i) - \epsilon (\nabla_x \theta^{k,e} \nabla_x \theta^{k,e}, w_i) = (S_{k,e} : D u^{k,e}, w_i)
\]
in \(Q\) for all \(i = 1, 2, \ldots, k\),

where \(\theta^{k,e}_{\text{max}} = \max \{\theta^{k,e}, \theta^{k,e}_{\text{max}}\}\). Here \(P^e\) denote the projection of \(H(\Omega)^n\) onto linear hull spanned by \(\{\omega_i\}_{i=1}^n\) and \(P^k\) analogously projection of \(L^2\) onto span \(\{\omega_i\}_{i=1}^k\).

In order to solve the system (20)–(25) for fixed \(k \in \mathbb{N}, \epsilon > 0\), and \(n \in \mathbb{N}\) one can apply Schauder’s fixed point theorem and basic estimates. These will appear later, therefore we skip details concerning solvability of (20)–(25). Let us notice also that we could consider, instead of a combination of characteristic methods, Schauder’s fixed point theorem, and basic estimates (for more detailed proof in barotropic case see [47]).

Multiplying the equation (20) by \(\partial^{k,e}\) leads to

\[
\sup_{t \in [0,T]} \|\theta^{k,e}(t)\|_{L^2(\Omega)}^2 + 2\epsilon \int_0^T \|\nabla_x \theta^{k,e}\|_2^2 \, dt \leq \|\theta_0\|_2^2
\]

and by minimum/maximum principle we get also that

\[\theta_* \leq \theta^{k,e} \leq \theta^*\].

Multiplying the \(i\)th equation in (22) by \(\alpha^{k,e}_i\), taking sum over \(i = 1, \ldots, n\), using \(L^2(\Omega)\) scalar product of (20) with \(|u^{k,e}|^2/2\) and integrating over \((0,t)\) we obtain that

\[
\|u^{k,e}(t)\|_{L^2(\Omega)}^2 + \|\nabla_x u^{k,e}(t)\|_{L^2(\Omega)}^2 + \int_0^t M^*(x, S_{k,e}) + M(x, D u^{k,e}) \, dx \, dt \leq C(\rho_0, u_0, f),
\]

what combined assumptions on tensor \(S\) and with the Korn inequality gives

\[
\|u^{k,e}(t)\|_{L^\infty(0,T;L^2)} + \|\nabla u^{k,e}(t)\|_{L^2(0,T;W^1_2(\Omega))} \leq C.
\]

Furthermore, multiplying the \(i\)th equation of (24) by \(\nu^{k,e}_i\), taking sum over \(i\), using \(L^2(\Omega)\) scalar product of (20) with \(|\theta^{k,e}|^2/2\) and integrating over \((0,t)\) leads to

\[
\|\theta^{k,e}(t)\|_{L^\infty(0,T;L^2(\Omega))} + \|\sqrt{\theta^{k,e}} \theta^{k,e}(t)\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla_x \sqrt{\theta^{k,e}}\|_{L^2(0,T;L^2(\Omega))} \leq C\|\sqrt{\theta_0}\|_{L^2(\Omega)}^2 + C\|S_{k,e} : D (u^{k,e})\|_{L^2(0,T;L^2(\Omega))} \leq C(n),
\]

where the last inequality holds because of (27) and the fact that \(u^{k,e}\) is a linear combination of \(n\) first elements of the basis \(\{\omega_i\}_{i=1}^\infty\) such that \(\nabla_x \omega \in L^\infty(\Omega)\) (see (15)), hence \(\|S_{k,e}\|_{L^\infty}, \|D u_{k,e}\|_{L^\infty} \leq C(n)\).
Applying once again previous reasoning but multiplying (22) by $\frac{\partial \varrho^k}{\partial x}$ and (24) by $\frac{\partial \nu_k}{\partial x}$ we conclude that

$$
\|\alpha_i^k\|_{W^{2,1}(0,T)} \leq C(n) \text{ for } i = 1, \ldots, n,
$$

$$
\|\nu_i^k\|_{W^{1,1}(0,T)} \leq C(k) \text{ for } i = 1, \ldots, k.
$$

Summarising estimates (27) and (28) together with (21) we can pass to the limit with $\varepsilon$ and obtain

$$
\partial_t \varrho^k + \text{div} (\varrho^k \mathbf{u}^k) = 0 \quad \text{in } Q, \quad \varrho_* \leq \varrho^k \leq \varrho^* \quad \text{in } Q, \quad \varrho^k(0, \cdot) = \varrho_0 \quad \text{in } \Omega,
$$

(29)

$$
(\varrho^k \frac{d}{dt} \mathbf{u}^k, \omega_i) + (\varrho^k [\nabla_s \mathbf{u}^k] \mathbf{u}^k, \omega_i) + (S_k, D \omega_i) = 0 \quad \text{in } Q \text{ for all } i = 1, 2, \ldots, n,
$$

(30)

$$
\mathbf{u}^k(0, \cdot) = P^0 \mathbf{u}_0 \quad \text{in } \Omega,
$$

$$
(\varrho^k \frac{d}{dt} \omega_i, \omega_i) + (\varrho^k [\nabla_s \mathbf{u}^k] \mathbf{u}^k, \omega_i) + (\kappa_k \nabla_s \varrho^k, \nabla_s \omega_i) = (S_k, D \omega_i) \quad \text{in } Q \text{ for all } i = 1, 2, \ldots, k,
$$

$$
\varrho^k(0, \cdot) = P^0(\varrho_0^k) \quad \text{in } \Omega.
$$

(31)

Notice that (29) may be considered as a linear equation in a strong form.

The next step is to pass with $k \to \infty$. We proceed as in [14]. From (29) we find that

$$
\varrho_* \leq \varrho^k \leq \varrho^* \quad \text{and} \quad \int_0^T \|\partial_t \varrho^k\|_{W^{1,1}}^s \, dt \leq C \quad \text{with arbitrary } s \in (1, \infty).
$$

(32)

Repeating procedures (27) and (28) we obtain the following estimates

$$
\|\sqrt{\varrho^k} \mathbf{u}^k\|_{L^\infty(0,T;L^2(\Omega))} \leq C,
$$

$$
\|\mathbf{u}^k\|_{L^\infty(0,T;L^2(\Omega))} + \|\mathbf{u}^k\|_{L^p(0,T,W^{1,2}(\Omega))} \leq C,
$$

$$
\|\varrho^k\|_{L^\infty(0,T,L^2(\Omega))} + \|\sqrt{\varrho^k} \mathbf{u}^k\|_{L^\infty(0,T;L^2(\Omega))} \leq C,
$$

$$
\|\varrho^k\|_{W^{1,1}(0,T)} \leq C(n) \text{ for } 1 = 1, \ldots, n,
$$

where $\kappa_k(\theta) := \kappa_0(\theta^k \theta_{\text{max}})$. Setting $\kappa(\theta) := \theta$ if $\theta \geq \theta_*$, $\kappa(\theta) := \theta^2$ if $\theta < \theta_*$, and $K(\theta) = \theta^2 \theta$ for $\theta < \theta_*$, $K(\theta) = \frac{\theta^2}{\theta^2 + 1} + \frac{\theta^2}{\theta^2 + 1}$ for $\theta \geq \theta_*$ we may infer that

$$
\|\tilde{K}(\varrho^k)\|_{L^\infty(0,T;L^2(\Omega))} + \|\tilde{K}(\varrho^k)\|_{L^2(0,T;W^{1,2}(\Omega))} \leq C(n),
$$

(33)
\[ \| \tilde{\kappa} \nabla \theta^k \|_{L^p_t(0,T;L^q_s(\Omega))} \leq C(n) \quad \text{with} \quad m_1 = 2 \beta \leq 0 \quad \text{and} \quad m_1 = (3 \beta + 10)/(3 \beta + 5) \quad \text{for} \quad \beta > 0, \]

\[ \left\| \frac{\tilde{\kappa}}{\sqrt{\kappa(\theta^k)}} \right\|_{L^2(\Omega)} \quad \text{with} \quad m_2 = \infty \quad \text{for} \quad \beta \leq 0 \quad \text{and} \quad m_2 = (6 \beta + 20)/(3 \beta) \quad \text{for} \quad \beta > 0, \]

\[ \| \nabla \theta^k \|_{L^p_t(0,T;L^q_s(\Omega))} \leq C(n) \quad \text{with} \quad m_3 = (5 \beta + 10)/(\beta + 5) \quad \text{for} \quad \beta \leq 0 \quad \text{and} \quad m_3 = 2 \quad \text{for} \quad \beta > 0, \]

\[ \| \partial_t (\theta^k) \|_{L^p_t(0,T;W^{-1,\infty}(\Omega))} \leq C(n) \quad \text{where} \quad m_4 = \min \{2, (3 \beta + 10)/(3 \beta + 5)\}. \]

In particular the above implies

\[ u^k \to u \quad \text{strongly in} \quad L^{2p}(0,T; W^{1,2p}_n(\Omega)). \]

Using the theory of renormalised solutions as in [30] we conclude that for all \( p \in [1, \infty) \) holds

\[ \theta^k \to \theta \quad \text{strongly in} \quad C(0,T; L^p(\Omega)) \quad \text{and} \quad \text{a.e. in} \quad Q. \]

As in [5] (more details therein), for selected subsequence, we can obtain that

\[ \theta^k \to \theta \quad \text{strongly in} \quad L^2(\Omega) \quad \text{and} \quad \text{a.e. in} \quad Q \]

and

\[ \tilde{\kappa} \nabla \theta^k \to \tilde{\kappa} \nabla \theta \quad \text{weakly in} \quad L^\gamma(0,T; W^{1,\gamma}(\Omega)) \quad \text{for} \quad \gamma = \min \{2, 1+5/(3 \beta + 5)\}, \]

where \( \tilde{\kappa} := \kappa_0(\rho_\max) \) with \( \rho_\max = \max \{\rho, \theta^\nu\}. \)

Summarising (38) and (39) we pass to the limit in the system (29)–(31) obtaining (17)–(19). In addition from the minimum principle (see [14, section 6.4]) we get

\[ 0 < \theta_\ast \leq \theta^\nu \quad \text{a.a. in} \quad Q. \]

### 4.2. Uniform estimates

Let solutions obtained in section 4.1 be denoted by \( \{ (\theta^\nu_n, u^\nu_n, \theta^\nu_n) \}_n \) with \( n = 1, 2, \ldots. \) Now we concentrate on passing with \( n \to \infty. \) In the first several steps we adopt reasoning from [47] and [14]. By standard method of characteristics for the transport equation (the reader can find more details in [9, 47]) we obtain

\[ 0 < \theta_\ast \leq \theta^\nu_n(t,x) \leq \theta^\ast \leq +\infty \quad \text{for} \quad \text{a.a.} \quad (t,x) \in Q. \]

Multiplication (18) by \( \alpha^\nu_n, \) taking sum up over \( i = 1, \ldots, n \) and use of (17) leads to

\[ \frac{1}{2} \frac{d}{dt} \int_\Omega |\theta^\nu_n|^2 dx + (S(x, \theta^\nu_n, Du^\nu_n), Du^\nu_n) = (\theta^\nu_n, Du^\nu_n). \]

Using the Hölder, the Korn and the Young inequalities, assumption (12) and inequality (41) we are able to estimate the right-hand side of (42) in the following way

\[ |(\theta^\nu_n, Du^\nu_n)| \leq C(\Omega, c_\ast, L^\nu_n, \rho^\ast, p) \| f^\nu_n \|_{L^p(\Omega)} + \frac{c_\ast}{2} \int_\Omega M(x, Du^\nu_n) dx + C(\Omega, c_\ast, \tilde{C}). \]
Integrating (42) over the time interval \((0, s_0)\), using estimates (43), (41), the coercivity conditions \(S_2\) on \(S\), uniform continuity of \(P^n\) w.r.t. \(n\) and strong convergence \(f^n \to f\) in \(L^{p'}(0, T; L^{p'}(\Omega)^3)\) we obtain
\[
\int_{\Omega} \frac{1}{2} \theta^n(s_0) |u^n(s_0)|^2 \, dx + \int_0^{s_0} \int_{\Omega} \frac{c_1}{2} M(x, Du^n) + c_2 M^+(x, S(t, g^n, Du^n)) \, dx \, dt \\
\leq C(\Omega, c_1, \xi, \theta^n, p, \tilde{C}, \|f\|_{L^{p'}(0, T; L^{p'}(\Omega))}) + \frac{1}{2} \theta^n u^n_0^2 \|u^n\|^2_{L^2(\Omega)},
\]
(44)
where \(C\) is a nonnegative constant independent of \(n\) and dependent on the given data.

By (44), the condition (12) provides that \({\{Du^n\}}_{n=1}^{\infty}\) is uniformly bounded in the space \(L^p(Q)^{3 \times 3}\), i.e.
\[
\int_0^T \|Du^n\|^p_{L^p(\Omega)} \, dt \leq C.
\]
(45)

From the Korn inequality it is straightforward to show that
\[
\int_0^T \|\nabla u^n\|^p_{L^p(\Omega)} \, dt \leq C.
\]
(46)

Using (44) one can deduce that
\[
\|S(x, g^n, \theta^n, Du^n) : Du^n\|_{L^1(Q)} \leq C,
\]
(47)
\[
\|S(x, g^n, \theta^n, Du^n)\|_{L^1(Q)} \leq C.
\]
(48)

Whatsmore, the sequence \({\{S(x, g^n, \theta^n, Du^n)\}}_{n=1}^{\infty}\) is uniformly bounded in Orlicz class \(L_{\mathcal{M}}(Q)^{3 \times 3}\).

Furthermore, (44) and (41) provide
\[
\sup_{t \in [0, T]} \|u^n(t)\|^2_{L^2(\Omega)} \leq C \quad \text{and} \quad \sup_{t \in [0, T]} \|g^n(t)\|u^n(t)\|^2_{L^2(\Omega)} \leq C,
\]
(49)
where \(C\) is a positive constant dependent on the size of data, but independent of \(n\). Since the sequence \({u^n}\}_{n=1}^{\infty}\) is uniformly bounded in \(L^p(0, T; W_{0, div}^{1,p}(\Omega)^3)\) the Gagliardo–Nirenberg–Sobolev inequality gives us uniform bound in \(L^p(0, T; L^{3p/(3-p)}(\Omega))\). Interpolation (see e.g. [38, proposition 1.41]) between spaces \(L^\infty(0, T; L^2)\) and \(L^p(0, T; L^{3p/(3-p)}(\Omega))\) provides
\[
\int_0^T \|u^n\|^r_{L^r(\Omega)} \, dt \leq C \\quad \text{for} \quad 1 \leq r \leq 5p/3
\]
(50)

(the above particular argument deals with the case \(p < 3\), the case \(p \geq 3\) can be treated easier, e.g. with the Poincaré or the Morrey inequalities). Therefore from (41) and (50) we infer also
\[
\int_0^T \|g^n u^n\|^5_{L^{5p/3}(\Omega)} \, dt \leq C.
\]
(51)

Use of (41), (46) and (50) combined with the Hölder inequality leads to
\[
\int_0^T |(g^n u^n \otimes u^n, \nabla u^n)| \, dt \leq C \quad \iff \quad p \geq \frac{11}{5}.
\]

One can notice that here restriction for the exponent \(p\) stated in (12) is given. By (51) and (41) we obtain from (17) that
\[ \int_0^T \| \partial_t u^n \|_{W^{1,3}((0,T)\setminus\{0\})}^{5p/3} \, dt \leq C. \]  

(52)

**Lemma 4.1.** The sequence \( \{u^n\}_{n=1}^{\infty} \) is uniformly bounded w.r.t. \( n \) in Nikolskii space \( N^{1/2,2}(0,T;H(\Omega)^3) \), namely

\[ \sup_{0 < \delta < T} \delta^{\beta/2} \left( \int_0^{T-\delta} \| u^n(s + \delta) - u^n(s) \|_{L^2(\Omega)}^2 \, ds \right)^{1/2} < C. \]  

(53)

The proof of the above lemma can be found in [47, section 3.1] (equations (55)–(62) therein).

It is based on reasoning from [2, chapter 3, lemma 1.2] with modifications concerning a change of \( L^2 \) to \( L^p \) structure and due to the nonlinear term controlled by the non-standard conditions (4). We notice that the presence of temperature does not influence this proof. The above lemma leads to the conclusion that \( u \in N^{1/2,2}(0,T;H(\Omega)^3) \).

Finally we need to provide estimates concerning energy equation and the temperature. First we notice that taking \( h = 1 \) in (19), by the Fenchel–Young inequality, (44) and (41) one gets

\[ \sup_{t \in [0,T]} \| \theta^n \|_{L^1(\Omega)} \leq C \quad \text{and} \quad \sup_{t \in [0,T]} \| \theta^n \|_{L^1(\Omega)} \leq C. \]  

(54)

Now, let us take \( h = - (\theta^n)^{-\lambda} \) with \( \lambda \in (0,1) \) in (19). As \( \theta^n \geq \theta_0 \) (see (40)) we have that

\[ \| - (\theta^n)^{-\lambda} \|_{L^\infty(\Omega)} \leq C \]  

and from this substitution we obtain

\[ \int_0^T \| (\theta^n)^{\frac{3-\lambda}{2}} \nabla x \theta^n \|_{L^2(\Omega)}^2 \, dt = C \int_0^T \| \nabla x ((\theta^n)^{\frac{3-\lambda}{2}}) \|_{L^2(\Omega)}^2 \, dt \leq C_2, \]  

(55)

which provides

\[ \int_0^T \| (\theta^n)^{\frac{3-\lambda}{2}} \|_{W^{1,2}(\Omega)}^2 \, dt \leq \int_0^T \| (\theta^n)^{\frac{3-\lambda}{2}} \|_{L^2(\Omega)}^2 \, dt + \int_0^T \| \nabla x (\theta^n)^{\frac{3-\lambda}{2}} \|_{L^2(\Omega)}^2 \, dt \]

\[ \leq \int_0^T \| (\theta^n)^{\frac{3-\lambda}{2}} \|_{L^2(\Omega)}^2 \, dt + C_2 \left[ \int_0^T \| (\theta^n)^{\frac{3-\lambda}{2}} \|_{L^2(\Omega)}^2 \, dt + \int_0^T \| \nabla x (\theta^n)^{\frac{3-\lambda}{2}} \|_{L^2(\Omega)}^2 \, dt \right] \]

\[ + C_1 \leq C_3. \]  

(56)

From continuous embedding of \( W^{1,2} \) in \( L^6 \) we obtain \( \int_0^T \| (\theta^n)^{\frac{3-\lambda}{2}} \|_{L^6(\Omega)}^2 \, dt \leq C \). By interpolation with (54) we conclude that

\[ \int_0^T \| (\theta^n)^{s} \|_{L^s(\Omega)}^2 \, dt \leq C \quad \text{for all} \ s \in \left[ 1, \frac{5}{3} + \beta \right]. \]  

(57)

By the assumption made on heat flux (6) we have

\[ |\kappa_0 \nabla_x \theta^n| \leq \kappa^* (\theta^n)^{\frac{3-\lambda}{2}} |\nabla \theta^n| (\theta^n)^{\frac{3+\lambda}{2}}. \]

Let us notice that \( \| (\theta^n)^{\frac{3-\lambda}{2}} \|_{L^2(\Omega)}^2 \) by (56) and \( (\theta^n)^{\frac{3-\lambda}{2}} \in L^{\frac{5+3\beta}{1+3\beta}}(\Omega) \) by (57) with arbitrary small \( \epsilon > 0 \). Thus we see that

\[ \int_Q |\kappa_0 \nabla_x \theta^n|^m \, dx \, dt \leq C \quad \text{for all} \ m \in \left[ 1, \frac{5+3\beta}{4+3\beta} \right]. \]  

(58)
Now we are ready to estimate the last term in energy equation. By the Sobolev embedding, the Riesz–Torin interpolation theorem, the Hölder inequality and above considerations we obtain

$$\int_0^T \| \vartheta^n \varphi^n \|_1^\gamma \, dt \leq \varrho^* \int_0^T \| \varphi^n \|_1^\gamma \, dt \leq C_1 \int_0^T \| \varphi^n \|_1^\gamma \, dt \leq C_1 \int_0^T \| \varphi^n \|_1^\gamma \, dt \leq C_1 \int_0^T \| \varphi^n \|_1^\gamma \, dt \leq C_2. \tag{59}$$

The parameter $\alpha$ is taken such that

$$\frac{(3 + \gamma) p - 3 \gamma}{3 p \gamma} = \frac{1 - \alpha}{1 + \alpha}, \tag{60}$$

Moreover, the last inequality in (59) gives constrains combining values of $\beta$, $\alpha$, $\lambda$, $p$ and $\gamma$, i.e. $\frac{(\beta - \lambda + 1) \gamma}{\alpha \gamma} = p$. Using formula (60) we claim that $\gamma > 1 \leftrightarrow \beta > \frac{3 - \alpha}{3 p - 3 \alpha}$ which is the restriction we demand in theorem 3.2. To sum up we obtain that for $p < 3$ and appropriate $\beta$ there exist $\gamma > 1$ such that

$$\| \varphi^n \varphi^n \|_{L^1(\Omega)} < C.$$

The above result holds also for $p \geq 3$ because of embedding properties of $W^{1,p}$.

In the end we obtain from (19) and estimates (47) and (58) and the fact that $\varphi^n \varphi^n \in L^1(0, T; L^\gamma)$ that

$$\| \partial_t (\varphi^n \varphi^n) \|_{L^1(0, T; (W^{1,\gamma}))} < C \quad \text{for } s \text{ sufficiently large.}$$

The above considerations provide all necessary uniform estimates.

### 4.3. Weak limits

Uniform estimates obtained in previous section together with the Banach–Alaoglu theorem ensure existence of subsequences selected from $\{ \varphi^n \}_{n=1}^\infty$, $\{ \varphi^n \}_{n=1}^\infty$, $\{ \theta^n \}_{n=1}^\infty$ such that

$$\varphi^n \rightarrow \varphi \quad \text{weakly in } L^q(Q) \text{ for any } q \in [1, \infty) \text{ and weakly-(*-in \textit{a.}} \text{)} \text{ in } L^\infty(Q), \tag{61}$$

$$\partial_t \varphi^n \rightarrow \partial_t \varphi \quad \text{weakly in } L^{5p/3}(0, T; W^{1,5p/(5p-3)})^*, \tag{62}$$

$$\theta^n \rightarrow \theta \quad \text{weakly in } L^1(Q) \text{ for any } q \in [1, 5/3 + \beta), \tag{63}$$

$$\theta^n \geq \theta_0 > 0 \text{ for a.a. } (t, x) \in Q. \tag{64}$$

In addition, there exist $\overline{\varphi} \in L^{5p/(3)}(Q)^3$ and $\overline{\varphi} \in L^2(0, T; W^{1,2}(Q))$ such that

$$\varphi^n \varphi^n \rightarrow \overline{\varphi} \quad \text{weakly in } L^5(Q)^3, \tag{66}$$

$$\overline{(\varphi^n)^\alpha} \rightarrow \overline{(\varphi)^\alpha} \quad \text{weakly in } L^2(0, T; W^{1,2}(\Omega)) \text{ for } \alpha \in (0, (\beta + 1)/2). \tag{67}$$
What'smore, noticing that $E_M$ and $E_M^*$ are separable spaces and $(E_M)^* = L_{M^*}$, $(E_{M^*})^* = L_M$ we obtain

$$Du^n \rightharpoonup Du \quad \text{weakly-}(*) \quad \text{in} \quad L_M(Q)$$

(68)

$$S(\cdot, \phi^n, \theta^n, Du^n) \rightharpoonup S \quad \text{weakly-}(*) \quad \text{in} \quad L_{M^*}(Q)^{3 \times 3}_{\text{sym}}$$

(69)

where $S \in L_{M^*}(Q)^{3 \times 3}_{\text{sym}}$. Applying lemma 2.8 we conclude the uniform integrability of $\{S(\cdot, \phi^n, \theta^n, Du^n)\}_{n=1}^\infty$. Consequently there exists a tensor $S \in L^1(Q)^{3 \times 3}$ such that

$$S(\cdot, \phi^n, \theta^n, Du^n) \rightarrow S \quad \text{weakly in} \quad L^1(Q)^{3 \times 3}.$$  

(70)

### 4.4. Strong convergence

In this section we will prove strong convergence of the triple $(\phi^n, u^n, \theta^n)$ using the Aubin–Lions lemma and lemma 2.9.

Let us start with the velocity field. Since (49), (46) and (53) hold, due to [40, theorem 3] we have

$$u^n \rightarrow u \quad \text{strongly in} \quad L^2(Q)$$

(71)

and by (63)

$$u^n \rightarrow u \quad \text{strongly in} \quad L^q(Q) \quad \text{with} \quad q \in [1, 5p/3).$$

(71)

Using (41) and (62) and the Aubin–Lions argument (see [40]) we obtain

$$\phi^n \rightarrow \phi \quad \text{strongly in} \quad C([0, T]; (W^{1,5p/(5p-3)}(\Omega))^*).$$

The concept of the renormalised solutions (see [30] for details) leads to

$$\phi^n \rightarrow \phi \quad \text{strongly in} \quad C([0, T]; L^q(\Omega)) \quad \text{for all} \quad q \in [1, \infty) \quad \text{and a.e. in} \quad Q.$$  

(72)

Moreover, one can show also that

$$\lim_{t \to 0^+} \|\phi(t) - \phi(0)\|_{L^q(\Omega)} = 0 \quad \text{for all} \quad q \in [1, \infty)$$

(what gives the first part of (11)) and

$$\lim_{t \to 0^+} (\phi(t), |u(t)|^2) = (\phi_0, |u_0|^2),$$

(73)

for details see [47, section 3.4]. The above strong convergence of the velocity field provides

$$\sqrt{\phi^n}u^n \rightarrow \sqrt{\phi}u \quad \text{strongly in} \quad L^2(Q)$$

(74)

and

$$\sqrt{\phi^n}u^n(t) \rightarrow \sqrt{\phi}u(t) \quad \text{strongly in} \quad L^2(\Omega)^3 \quad \text{for almost all} \quad t < T.$$

Arguments (71), (72) together with (41) leads to

$$\phi^n u^n \otimes u^n \rightarrow \phi u \otimes u \quad \text{weakly in} \quad L^\gamma(0, T; L^\gamma(\Omega))$$

for $\gamma$ sufficiently large, i.e. $\frac{1}{2} + \frac{6}{5p} < \frac{1}{7} < 1$ with arbitrary $q \in [1, \infty)$. Density argument together with (63) provides for $p \geqslant \frac{11}{3}$ that

$$\phi^n u^n \otimes u^n \rightarrow \phi u \otimes u \quad \text{weakly in} \quad L^p(0, T; L^p(\Omega)).$$
Now we show that (71) together with (72) we conclude that
\[ \varrho^{n}u^{n} \rightharpoonup \varrho u \quad \text{weakly in } L^{q}(Q) \quad \text{for all } q \in [1, 5p/3]. \]  
(75)

Using the Div-Curl lemma \(9\) the convergence of the \(\{\varrho^{n} \theta^{n}u^{n}\}_{n=1}^{\infty}\) can be shown. To this end we set
\[ a^{n} = (\varrho^{n} \theta^{n}, \varrho^{n} \theta^{n}u_{1}^{n} + \kappa_{0} \nabla \theta^{n}, \varrho^{n} \theta^{n}u_{2}^{n} + \kappa_{0} \nabla \theta^{n}, \varrho^{n} \theta^{n}u_{3}^{n} + \kappa_{0} \nabla \theta^{n}) \]
and \(b^{n} = ((\theta^{n})^{\alpha}, 0, 0)\) with \(\alpha \in (0, (\beta + 1)/2).\) Inequalities (58), (59), (65) and (72) ensure that \(a^{n}\) converges weakly to \(a\) in \(L^{1}(Q)\) for some \(s > 1\) close to 1 and \(b \rightharpoonup (\overline{\theta}^{\alpha}, 0, 0)\) in \(L^{1}(Q)\) for \(r\) such that \((\frac{1}{2} + \frac{1}{r}) < 1\) (which is possible for small \(\alpha\) and due to condition (64)). In view of the energy equation it holds that
\[ \text{Div}_{x}a = \partial_{t}(\varrho^{n} \theta^{n}) + \text{div}_{x}(\varrho^{n} \theta^{n}u^{n} + \kappa_{0} \nabla \theta^{n}) = \mathbf{S}(x, \varrho^{n}, \theta^{n}, D\mathbf{u}^{n}) : D\mathbf{u}^{n} \in L^{1}(Q) \rightharpoonup W^{-1,\tilde{r}}(Q), \]
where \(\tilde{r} \in (1, 4/3).\) On the other hand
\[ \text{Curl}_{x}b^{n} = \begin{pmatrix} 0 \\ -\nabla(\theta^{n})^{\alpha}^{T} \end{pmatrix} \in L^{2}(Q)^{4 \times 4} \rightharpoonup (W^{-1,2}(Q)^{4 \times 4})^{*}. \]
(\(Q\) denotes here zero \(3 \times 3\) matrix). The statement of lemma \(2.9\) provides that
\[ \varrho^{n}(\theta^{n})^{\alpha+1} \rightharpoonup \varrho \overline{\theta}^{\alpha} \quad \text{weakly in } L^{1+\eta}(Q) \text{ for some } \eta > 0. \]
The above combined with (57) and (72) gives
\[ \varrho(\theta^{n})^{\alpha+1} \rightharpoonup \varrho \overline{\theta}^{\alpha} \quad \text{weakly in } L^{1+\zeta}(Q) \text{ for some } \zeta > 0. \]  
(76)

The next step is to show that \(\overline{\theta}^{\alpha} = \theta^{\alpha}\) a.e. in \(Q.\) To do so we employ Minty’s trick. For \(y \in \mathbb{R}^{+}\) and \(\alpha > 0, y^{\alpha}\) is an increasing function which leads to
\[ 0 \leq \int_{0}^{T} (\varrho(\theta^{n})^{\alpha} - h^{\alpha}, \theta^{n} - h) \, dt \quad \text{for all } h \in L^{1+\eta}(Q). \]
Passing to the limit with \(n \to \infty\) and using (76) we obtain
\[ 0 \leq \int_{0}^{T} (\varrho(\overline{\theta}^{\alpha} - h^{\alpha}, \theta - h) \, dt \quad \text{for all } h \in L^{1+\eta}(Q). \]
By setting \(h = \theta - \lambda \varphi\) for \(\lambda > 0, \varphi \in L^{1+\eta}(Q)\) and \(h = \theta + \lambda \varphi\) then passing to the limit with \(\lambda \to 0\) we conclude that
\[ 0 = \int_{0}^{T} (\varrho(\overline{\theta}^{\alpha} - \varphi^{\alpha}), \varphi) \, dt \quad \text{for all } \varphi \in L^{1+\eta}(Q). \]
Therefore as \(\varrho > \varrho_{*}\) we deduce
\[ \overline{\theta}^{\alpha} = \theta^{\alpha} \quad \text{a.e. in } Q. \]

Then by (76), weak convergence in \(L^{1+\alpha}(Q)\) of \(\{\varrho(\overline{\theta}^{\alpha})\}_{n=1}^{\infty}\) to \(\varrho(\overline{\theta}^{\alpha})\) and convergence of \(\|\varrho(\overline{\theta}^{\alpha})\|_{L^{1+\alpha}(Q)}\) to \(\|\varrho(\overline{\theta}^{\alpha})\|_{L^{1+\alpha}(Q)}\) are provided. Consequently
\[ \varrho(\overline{\theta}^{\alpha}) \to \varrho(\overline{\theta}^{\alpha}) \quad \text{strongly in } L^{1+\alpha}(Q), \]
which combined with (64) and (61) leads to
\[ \theta^{n} \to \theta \quad \text{strongly in } L^{q} \text{ for all } q \in [1, 5/3 + \beta) \text{ and a.e. in } Q. \]  
(77)
The above strong convergence together with (67) ensure that
\[(\theta^p)_{\alpha}^\rightarrow \theta^p \quad \text{weakly in } L^2(0, T; W^{1,2}(\Omega)) \quad \text{for all } \alpha \in (0, (\beta + 1)/2). \tag{78}\]

Using the same arguments as in (59), (71), (72) and (77) we conclude that
\[\theta^p \theta^p u^p \rightarrow \theta \theta u \quad \text{strongly in } L^1(\Omega)^3.\]

Next step is to establish convergence of \(q(\theta^p, \theta^p, \nabla \theta^p)\). According to (5)
\[q(\theta^p, \theta^p, \nabla \theta^p) = \kappa_0(\theta^p, \theta^p)\nabla \theta^p = \frac{2}{\beta - \lambda + 1} (\theta^p)^{-\frac{\beta + \lambda - 1}{\beta - 1}} \kappa_0(\theta^p, \theta^p)^{-\frac{\beta - \lambda + 1}{\beta - 1}}. \tag{79}\]

Inequality (57) can be used to provide
\[\int_{0}^{T} \left\| (\theta^p)^{-\frac{\beta + \lambda - 1}{\beta - 1}} \kappa_0(\theta^p, \theta^p) \right\|_{L^2(\Omega)}^2 \mathrm{d}t \leq \int_{0}^{T} (\theta^p)^{(\beta + \lambda - 1)} \mathrm{d}t \leq C \tag{80}\]
for \(r \text{s.t. } r(\beta + \lambda + 1) = 5/3 + \beta - \lambda\). Notice that \(r > 1\) for \(\lambda\) small enough. Then almost everywhere convergence of \(\{\theta^p\}_{m=1}^{\infty}\) showed in (72) and (77) combined with Vitali’s convergence theorem and (80) leads to
\[(\theta^p)^{-\frac{\beta + \lambda - 1}{\beta - 1}} \kappa_0(\theta^p, \theta^p) \rightarrow \theta^{-\frac{\beta - \lambda + 1}{\beta - 1}} \kappa_0(\theta, \theta) \quad \text{strongly in } L^2(0, T; L^2(\Omega)).\]

However, convergence proved in (78) gives us
\[\nabla \theta^p (-\frac{\beta - \lambda + 1}{\beta - 1}) \rightarrow \nabla \theta (-\frac{\beta - \lambda + 1}{\beta - 1}) \quad \text{weakly in } L^2(0, T; L^2(\Omega)^3),\]
which applied to (79) and by (58) leads to
\[q(\theta^p, \theta^p, \nabla \theta^p) \rightarrow q(\theta, \theta, \nabla \theta) \quad \text{weakly in } L^1(\Omega)^3 \quad \text{for all } s \in \left(1, \frac{5 + 3\beta}{4 + 3\beta}\right). \tag{79}\]

Arguments established in this section allows us to pass to the limit in system (17) and (18). It remains only to characterise the nonlinear term and to show convergence in the the RHS of the energy equations (19).

4.5. Integration by parts

Let us notice that classical integration by parts formula does hold for our considered problem since Orlicz spaces are not reflexive and smooth functions are not dense if \(\Delta_2\)–condition is not satisfied. In general also there is no equivalence between Bochner type space \(L^M(0, T; L^M(\Omega))\) and \(L^M(\Omega)\), which holds only in case \(N\)–function \(M\) is of polynomial type (see [10]). Therefore let us recall the following result from [47]:

**Lemma 4.2.** Let exponent \(p\), function \(f\), \(N\)–functions \(M\) and \(M^*\) be as in theorem 3.2. We assume that
\[0 < g^* \leq g(t, x) \leq g^* \quad \text{for a.a. } (t, x) \in Q, \quad g \in C([0, T]; L^q(\Omega)) \quad \text{for arbitrary } q \in [1, \infty), \]
\[\partial_t g \in L^{5p/3}(0, T; (W^{1,5p/(5p-3)}(\Omega))^*), \]
\[u \in L^\infty(0, T; H(\Omega)^3) \cap L^p(0, T; W^3_{0,\text{div}}(\Omega)^3) \cap N^{1/2,2}(0, T; H(\Omega)^3), \]
\[Du \in L^3(\Omega)^{3 \times 3}_{\text{sym}} \quad \text{and} \quad (\theta u, \psi) \in C([0, T])\text{for all } \psi \in H(\Omega)^3, \]
\[S \in L^M(\Omega)^{3 \times 3}_{\text{sym}} \]

\[720\]
and the couple \((\varrho, u)\) is a weak solution of \(\partial_t \varrho + \text{div}_x (\varrho u) = 0\) (see definition 3.1) and satisfies
\[
- \int_0^T \int_\Omega \varrho u \cdot \partial_t \varphi - \varrho u \otimes u : \nabla \varphi + \mathcal{S} : D\varphi \, dx \, dt = \int_0^T \int_\Omega \varrho f \cdot \varphi \, dx \, dt
\]
for all \(\varphi \in \mathcal{D}((0, T); \mathcal{V})\). Then for a.a. \(s_0\) and s.s.t. \(0 < s_0 \leq s\) it holds that
\[
\frac{1}{2} \int_\Omega \varrho(s, x) |u(s, x)|^2 \, dx + \int_{s_0}^s \int_\Omega \mathcal{S} : D\varphi \, dx \, dt = \int_{s_0}^s \int_\Omega \varrho f \cdot u \, dx \, dt + \frac{1}{2} \int_\Omega \varrho(s_0, x) |u(s_0, x)|^2 \, dx.
\]
(81)

The detailed proof can be found in [47, section 3.3] and it is based on a proper choice of a test function in (81) and goes via Steklov regularisation with respect to the time variable.

Let us remark that now the assumptions of lemma 4.5 are satisfied due to the previous subsections, in particular (82) holds for a sufficiently rich family of test functions by density arguments.

4.6. Monotonicity method

In this section we investigate the weak limit \(\mathcal{S}\) and we show that \(\mathcal{S} = \mathcal{S}(x, \varrho, \theta, Du)\) a.e. in \(Q\). The proof is based on monotonicity method adopted to nonreflexive anisotropic Musielak–Orlicz spaces. Reasoning follows the one presented in [21, 47] with modifications which allow to deal with dependence of \(\mathcal{S}\) on density and temperature and we recall it here for the convenience of the reader.

Using integration by parts formula (see lemma 4.5) and letting \(s_0 \to 0\) (see (73)) we obtain that
\[
\frac{1}{2} \int_\Omega \varrho(s, x) |u(s, x)|^2 \, dx + \int_0^s \int_\Omega \mathcal{S} : D\varphi \, dx \, dt = \int_0^s \int_\Omega \varrho f \cdot u \, dx \, dt + \frac{1}{2} \int_\Omega \varrho(0, x) |u_0(x)|^2 \, dx.
\]
(82)

After integrating equation (42) over the interval \((0, s)\), passing with \(n \to \infty\) and comparing the result with the above one may conclude that
\[
\limsup_{n \to \infty} \int_0^s \int_\Omega \mathcal{S}(x, \varrho^n, \theta^n, Du^n) : Du^n \, dx \, dt \leq \int_0^s \int_\Omega \mathcal{S} : Du \, dx \, dt.
\]
(83)

Denoting by \(Q^s\) time-space cylinder \((0, s) \times \Omega\) and using monotonicity of \(\mathcal{S}\) (see condition S3) one obtains that
\[
\int_{Q_s} \left( \mathcal{S}(x, \varrho^n, \theta^n, w) - \mathcal{S}(x, \varrho^n, \theta^n, Du^n) \right) : (w - Du^n) \, dx \, dt \geq 0
\]
(84)

holds for all \(w \in L^\infty(Q)^{3\times 3}\). Let us notice that \(\mathcal{S}(x, \varrho^n, \theta^n, w) \in L^\infty(Q)^{3\times 3}\). This statement can be proven by contradiction. To do so suppose that \(\mathcal{S}(x, \varrho^n, \theta^n, w)\) is unbounded. Since \(M\) is nonnegative, by coercivity condition (4), it holds that
\[
|w| \geq M^*(x, \varrho^n, \theta^n, w)\frac{\mathcal{S}(x, \varrho^n, \theta^n, w)}{\mathcal{S}(x, \varrho^n, \theta^n, w)}.
\]

Then the right-hand side tends to infinity as \(|\mathcal{S}(x, \varrho^n, \theta^n, w)| \to \infty\) by (7), which contradicts that \(w \in L^\infty(Q)^{3\times 3}\).

Employing continuity of \(\mathcal{S}\) w.r.t. second and third argument and a.e. convergence of \(\varrho^n \to \varrho\) \((\theta^n \to \theta)\) we obtain a.e. convergence of the \(\{\mathcal{S}(x, \varrho^n, \theta^n, w)\}_{n=1}^\infty\) to \(\mathcal{S}(x, \varrho, \theta, w)\). Since
\(\{S(x, \theta^n, \theta^n, w)\}_{n=1}^{\infty} \subset L^\infty(Q^3)\) we obtain uniform integrability of \(\{M^* (S(x, \theta^n, \theta^n, w))\}_{n=1}^{\infty}\). Therefore by lemma 2.7 modular convergence of the sequence \(\{S(x, \theta^n, \theta^n, w)\}_{n=1}^{\infty}\) is provided in \(L^\infty(Q^3)\). As \(M^*\) satisfies the \(\Delta_2\)-condition, then the modular and strong convergence in \(L^*\) coincide (see [26]) and \(S(x, \theta^n, \theta^n, w) \rightarrow S(x, \theta^n, \theta^n, w)\) strongly in \(L^\infty\). Therefore by (68) we deduce

\[
\lim_{n \to \infty} \int_Q S(x, \theta^n, \theta^n, w) : Du \, dx \, dt = \int_Q S(x, \theta^n, \theta^n, w) : Du \, dx \, dt.
\]  

(85)

Passing to the limit with \(n \to \infty\), by (63), (69), (83), (85) we obtain from (84) that

\[
\int_Q S(x, \theta^n, \theta^n, w) : Du \, dx \, dt \geq \int_Q \tilde{S} : w \, dx \, dt + \int_Q S(x, \theta^n, \theta^n, w) : (Du - w) \, dx \, dt
\]

(86)

and consequently

\[
\int_Q (S(x, \theta^n, \theta^n, w) - \tilde{S}) : (w - Du) \, dx \, dt \geq 0.
\]  

(87)

Let us set

\[
w = (Du) \mathbb{1}_{Q_i} + h \nu \mathbb{1}_{Q_i},
\]

with \(Q_i = \{(t, x) \in Q : |Du(t, x)| \leq k \text{ a.e. in } Q_i\}\) and where \(k > 0, 0 < j < i, h > 0\) and \(\nu \in L^\infty(Q^3)\) are arbitrary. As \(S(x, \theta^n, \theta^n, 0) = 0\), from (87) we have

\[
- \int_{Q_i \setminus Q_j} (S(x, \theta^n, \theta^n, 0) - \tilde{S}) : Du \, dx \, dt + h \int_{Q_j} (S(x, \theta^n, \theta^n, Du + h \nu) - \tilde{S}) : \nu \, dx \, dt \geq 0
\]  

(88)

and obviously

\[
\int_{Q_i \setminus Q_j} \tilde{S} : Du \, dx \, dt = \int_Q \tilde{S} : Du \, dx \, dt.
\]

Since \(\tilde{S} \in L^\infty(Q^3), Du \in L^\infty(Q^3)\) (which is a consequence of convexity, nonnegativity of \(M^*, M\) and of weak lower semi-continuity and estimate (44)) by the Fenchel–Young inequality we obtain that \(\int_Q \tilde{S} : Du \, dx \, dt < \infty\) and consequently

\[
(\tilde{S} : Du) \mathbb{1}_{Q_i \setminus Q_j} \to 0 \quad \text{a.e. in } Q_i \text{ for } i \to \infty.
\]

Hence by the Lebesgue dominated convergence theorem

\[
\lim_{i \to \infty} \int_{Q_i \setminus Q_j} \tilde{S} : Du \, dx \, dt = 0.
\]

Letting \(i \to \infty\) in (88) and dividing by \(h\), we get

\[
\int_{Q_j} (S(x, \theta^n, \theta^n, Du + h \nu) - \tilde{S}) : \nu \, dx \, dt \geq 0.
\]

Since \(Du + h \nu \to Du\) a.e. in \(Q\) when \(h \to 0^+\) and as \(\{S(x, \theta^n, \theta^n, Du + h \nu)\}_{h>0} \subset L^\infty(Q^3)\), \(|Q_j| < \infty\), by the Vitali lemma we conclude

\[
S(x, \theta^n, \theta^n, Du + h \nu) \to S(x, \theta^n, \theta^n, Du) \quad \text{in } L^1(Q^3) \quad \text{as } h \to 0^+
\]  

and

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\[
\begin{align*}
\int_{Q_t} (S(x, \varrho, \theta, Du + i\nu) - \mathcal{S}) : v dx dt \to \int_{Q_t} (S(x, \varrho, \theta, Du) - \mathcal{S}) : v dx dt \quad \text{as } h \to 0^+.
\end{align*}
\]

Consequently,
\[
\int_{Q_t} (S(x, \varrho, \theta, Du) - \mathcal{S}) : v dx dt \geq 0 \quad \text{for all } v \in L^\infty(Q)^{3 \times 3}.
\]

The choice of \( v \) s.t. \( v = -\frac{S(x, \varrho, \theta, Du) - \mathcal{S}}{|S(x, \varrho, \theta, Du) - \mathcal{S}|} \) if \( S(x, \varrho, \theta, Du) \neq \mathcal{S} \) and \( v = 0 \) if \( S(x, \varrho, \theta, Du) = \mathcal{S} \) yields
\[
\int_{Q_t} |S(x, \varrho, \theta, Du) - \mathcal{S}| dx dt \leq 0.
\]

Hence \( S(x, \varrho, \theta, Du) = \mathcal{S} \) a.e. in \( Q_t \) and as \( j \) is arbitrary it holds also a.e. in \( Q^t \) for almost all \( s \) such that \( 0 < s < T \). Finally we conclude that
\[
\mathcal{S} = S(x, \varrho, \theta, Du) \quad \text{a.e. in } Q.
\]  \( \tag{89} \)

### 4.7 Convergence of \( S(x, \varrho^n, \theta^n, Du^n) : Du^n \)

The next crucial part of the proof is to establish convergence of the sequence \( \{S(x, \varrho^n, \theta^n, Du^n) : Du^n \}_{n=1}^\infty \). The idea follows [23] (later used also in [25]) and is based on the concept of biting convergence (see [3]) and the theory of Young measures (for details see [34]). Let us start with recalling definition of biting limit (see [3]).

**Definition 4.3 (Biting limit).** Let \( \{a_n\}_{n=1}^\infty \) be a bounded sequence in \( L^1(Q) \). We say that \( a \in L^1(Q) \) is a biting limit of subsequence of \( \{a_n\}_{n=1}^\infty \) if there exists nonincreasing sequence \( \{E_k\}_{k=1}^\infty, E_k \subset Q \) satisfying \( \lim_{k \to \infty} |E_k| = 0 \) such that \( a^n \) converge weakly to \( a \) in \( L^1(Q \setminus E_k) \). We denote biting convergence with \( \to_b \).

**Lemma 4.4.** Let \( \{a_n\}_{n=1}^\infty \) be a bounded sequence in \( L^1(Q) \) and let \( 0 \leq a_0 \in L^1(Q) \). If assumptions

- (A1) \( a_n \geq -a_0 \) for all \( n = 1, \ldots, \infty \),
- (A2) \( a_n \to_b a \) as \( n \to \infty \),
- (A3) \( \limsup_{n \to \infty} \int_Q a_n dx dt \leq \int_Q adx dt \),

hold, then
\[
a_n \to a \quad \text{weakly in } L^1(Q) \text{ as } n \to \infty.
\]

For the proof see [23, 35].

We show now that for \( \{a_n\}_{n=1}^\infty = \{S(x, \varrho^n, \theta^n, Du^n) : Du^n\}_{n=1}^\infty \) the assumptions of lemma 4.4 are fulfilled which lead to a weak convergence of \( a_n \) in the \( L^1(Q) \) space. Assumption A1 is fulfilled due the coercivity condition S3 namely \( a_n \geq 0 \). Then A3 is a straightforward consequence of (83). What is left, is the A2-biting convergence of \( a_n = S(x, \varrho^n, \theta^n, Du^n) : Du^n \) to \( a = S(x, \varrho, \theta, Du) : Du \).
Using the monotonicity of $S$ (see S3) we can write down that

$$0 \leq (S(x, \varrho^n, \theta^n, Du^n) - S(x, \varrho^n, \theta^n, Du)) : (Du^n - Du). \quad (90)$$

The right hand side of the above inequality is uniformly bounded in $L^1(\Omega)$, which holds due to the Hölder inequality for Orlicz spaces and uniform estimates obtained in (44). In particular, the uniform boundedness of $S(x, \varrho^n, \theta^n, Du)$ in $L_{M^*}$ is a consequence of the following reasoning: by coercivity condition S2 the Fenchel–Young inequality and convexity of the $N$–function $M^*$ we can deduce that

$$cM(x, Du) + \frac{2c - d}{d} M^*(x, S(x, \varrho^n, \theta^n, Du)) \leq M(x, \frac{2}{d} Du) \quad (91)$$

with $d = \min\{c, 1\}$. Then since $Du \in L_M(\Omega)^{3\times 3}$, we obtain that $\{S(x, \varrho^n, \theta^n, Du)\}_{n=1}^\infty$ is uniformly bounded in $L_{M^*}(\Omega)^{3\times 3}$. Since RHS of (90) is uniformly bounded in $L^1(\Omega)$, there exists a Young measure $\mu_{x, l}(\cdot, \cdot, \cdot)$ (see [34, theorem 3.1]) satisfying

$$(S(x, \varrho^n, \theta^n, Du^n) - S(x, \varrho^n, \theta^n, Du)) : (Du^n - Du) \rightarrow_{b} \int_{\mathbb{R}^n} (S(x, l, s, \lambda) - S(x, l, s, Du)) : (\lambda - Du) d\mu_{x, l}(l, s, \lambda)$$

as $n \to \infty$. Using [34, corollary 3.4] and (72) together with (77) we have that in fact $\mu_{x, l}(\cdot, \cdot, \cdot)$ can be written down in a form $\delta_{\theta, \varrho}(l, s) \otimes \nu_{x, l}(\lambda)$. This leads to

$$\int_{\mathbb{R}^n} (S(x, l, s, \lambda) - S(x, l, s, Du)) : (\lambda - Du) d\mu_{x, l}(l, s, \lambda)$$

$$= \int_{\mathbb{R}^n} (S(x, \varrho, \theta, \lambda) - S(x, \varrho, \theta, Du)) : (\lambda - Du) d\nu_{x, l}(\lambda)$$

$$= \int_{\mathbb{R}^n} S(x, \varrho, \theta, \lambda) : (\lambda - Du) d\nu_{x, l}(\lambda) - \int_{\mathbb{R}^n} S(x, \varrho, \theta, Du) : (\lambda - Du) d\nu_{x, l}(\lambda). \quad (92)$$

On the other hand

$$\int_{\mathbb{R}^n} S(x, \varrho, \theta, Du) : (\lambda - Du) d\nu_{x, l}(\lambda) = S(x, \varrho, \theta, Du) : \left( \int_{\mathbb{R}^n} \lambda d\nu_{x, l}(\lambda) - Du \right) = 0. \quad (93)$$

Indeed, the above holds since $S(x, \varrho, \theta, Du)$ is independent of $\lambda$. Moreover $\int_{\mathbb{R}^1} \lambda d\nu_{x, l}(\lambda) = Du$ for a.e. $(t, x) \in \Omega$ by [34, theorem 3.1] and $Du^n \rightharpoonup Du$ in $L^1(\Omega)^{3\times 3}$ (consequence of (68)). Then the second term of RHS of (92) disappears and (92) becomes

$$\int_{\mathbb{R}^n} (S(x, l, s, \lambda) - S(x, l, s, Du)) : (\lambda - Du) d\mu_{x, l}(l, s, \lambda) = \int_{\mathbb{R}^n} S(x, \varrho, \theta, \lambda) : (\lambda - Du) d\nu_{x, l}(\lambda). \quad (94)$$

Furthermore, as $\{a_n\}_{n=1}^\infty$ is uniformly bounded in $L^1(\Omega)$ (by Fenchel–Young inequality and (44)) we get that

$$S(x, \varrho^n, \theta^n, Du^n) : Du^n \rightharpoonup b \int_{\mathbb{R}^n} S(x, l, s, \lambda) : \lambda d\mu_{x, l}(l, s, \lambda)$$

$$= \int_{\mathbb{R}^n} S(x, \varrho, \theta, \lambda) : \lambda d\nu_{x, l}(\lambda).$$

Then as $a_n \geq 0$ for $n = 1, \ldots, \infty$, by [34, corollary 3.3] and by (83) and (89), we obtain that
\[
\int_Q S(x, \varrho, \theta, Du) : Du \, dx \geq \liminf_{n \to \infty} \int_Q S(x, \varrho^n, \theta^n, Du^n) : Du^n \, dx \, dt \\
\geq \int_Q \int_{\mathbb{R}^{3 \times 3}} S(x, \varrho, \theta, \lambda) : \lambda d\nu_{\varrho, \theta}(\lambda) \, dx \, dt.
\]

(95)

However, \( S(x, \varrho, \theta, Du) = \int_{Q^{3 \times 3}} S(x, l, s, \lambda) d\nu_{\varrho, \theta}(\lambda) \) as \( S(x, \varrho^n, \theta^n, Du^n) \to S(x, \varrho, \theta, Du) \) in \( L^1(Q^{3 \times 3}) \) so by (94) and (95) the RHS (90) is non-positive as well as RHS of (92). That implies that

\[
(S(x, \varrho^n, \theta^n, Du^n) - S(x, \varrho, \theta^n, Du^n)) : (Du^n - Du) \to 0.
\]

(96)

Whatsmore in the similar way as (93) we obtain that

\[
S(x, \varrho^n, \theta^n, Du) : (Du^n - Du) \to 0,
\]

(97)

and one can infer also

\[
S(x, \varrho^n, \theta^n, Du^n) : Du \to S(x, \varrho, \theta, Du) : Du.
\]

(98)

Summarising (96)–(98) we provide that \( a_n \to a \) so assumption A2 of lemma 4.4 holds. From the statement of lemma 4.4 we conclude that

\[
S(x, \varrho^n, \theta^n, Du^n) : Du^n \to S(x, \varrho, \theta, Du) : Du \quad \text{weakly in } L^1(Q).
\]

This finishes the proof of theorem 3.2.

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