All local quantum states are mixtures of direct products

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Abstract

According to Popescu’s recent analysis [Phys. Rev. Lett. 72, 797 (1994)], nonideal measurements, rather than ideal ones, may be more sensitive to reveal nonlocal correlations between distant parts of composite quantum systems. The outcome statistics of joint nonideal measurements on local states should by definition admit local hidden variable models. We prove that the density operator of a local composite system must be convex mixture of the subsystems’ density operators. This result depends essentially on a plausible consistency condition restricting the class of admissible local hidden variable models.

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Thirty years ago John Bell’s theoretical works revealed peculiar correlations between distant subsystems of composite quantum systems. Their peculiarity means that they would be impossible between distant parts of classical statistical ensembles unless instant communications are assumed through distances separating them. Since this latter would contradict to the principle of locality, the corresponding quantum states are conventionally called nonlocal ones. On the contrary: correlations in local quantum states do admit classical statistical models called local-hidden-variable (LHV) models. Hence the notion of locality of quantum states relies upon what LHV models are. The issue of whether an arbitrary quantum state admits an LHV model can not be solved directly. For pure quantum states a trivial criterion is due to Gisin. All direct product pure states are local and all entangled pure states are nonlocal. The generalization for mixed quantum states does not exist. Quite recently, Popescu noticed that local tests based on nonideal measurements (so-called POVMs) might reveal those nonlocalities of certain mixed quantum states which would remain hidden if only ideal measurements were considered. In the present Letter we follow this idea and prove a simple criterion of locality valid for pure as well as for mixed states.

As is usual we consider a composite quantum system with Hilbert space $\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$ where the two factor Hilbert spaces belong to the two subsystems. The subsystems are separated in space. Let density operator $\rho$ stand for the quantum state of the composite system. Introduce positive-operator-valued measures (POVMs) $A = \{A_\mu\}$ and $B = \{B_\nu\}$ on $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$, respectively, where the observables $A_\mu$ and $B_\nu$ are non-negative Hermitian operators satisfying the completeness relations

$$\sum_\mu A_\mu = I^{(1)}, \quad \sum_\nu B_\nu = I^{(2)}.$$  

Here $I^{(1)}, I^{(2)}$ stand for the identity operators on $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$, respectively. For generic POVMs $A$ (or $B$) the observables $A_\mu$ (or $B_\nu$) are not necessarily independent of each other and may satisfy additional linear constraints

$$\sum_\mu f_\mu A_\mu = 0, \quad \sum_\nu g_\nu B_\nu = 0$$  

with real coefficients $f_\mu$ and $g_\nu$.  

1
Let us apply the composite POVM \{A_\mu \otimes B_\nu\} to the state \(\rho\) of the composite system. The expectation value of the local observable \(A_\mu \otimes B_\nu\) takes the standard form \[3\]

\[\langle A_\mu \otimes B_\nu \rangle = tr(\rho A_\mu \otimes B_\nu).\]

We say that the state \(\rho\) admits local hidden variable (LHV) model with nonideal measurements if, for each pair \(\mu, \nu\), the above expectation value can be expressed as a weighted sum of products of conditional expectation values of \(A_\mu\) and \(B_\nu\), respectively:

\[\langle A_\mu \otimes B_\nu \rangle = \sum_\lambda p_\lambda E_\lambda^{(1)}(A_\mu | \lambda) E_\lambda^{(2)}(B_\nu | \lambda).\] \[4\]

The \(\lambda\)'s parametrize the just mentioned ”conditions” and have given the name local hidden variables; \(p_\lambda\) is their normalized probability distribution: \(p_\lambda \geq 0\) and \(\sum_\lambda p_\lambda = 1\). The expectation values should satisfy the conditions

\[0 \leq E_\lambda^{(1)}(A_\mu | \lambda) \leq \|A_\mu\|,\quad 0 \leq E_\lambda^{(2)}(B_\nu | \lambda) \leq \|B_\nu\|,\] \[5\]

and \(\sum_\mu E_\lambda^{(1)}(A_\mu | \lambda) = \sum_\nu E_\lambda^{(2)}(B_\nu | \lambda) = 1\). Observe that, by assumption, the expectation value of \(A_\mu\) should not depend on the POVM \(B_\nu\) chosen at distance and, vice versa, the choice of the POVM \(A_\mu\) will not influence the expectation values of the \(B_\nu\)'s.

We have hitherto constructed a natural extension of the ordinary LHV models based on ideal (von Neumann) measurements, by allowing generalized (i.e. nonideal) measurements through POVMs. Now we introduce the notion of consistent LHV models: we require the conditional expectation values on the RHS of Eq. (4) to be consistent with the operator constraints (2):

\[\sum_\mu f_\mu E_\mu^{(1)}(A_\mu | \lambda) = 0,\quad \sum_\nu g_\nu E_\nu^{(2)}(A_\nu | \lambda) = 0\] \[6\]

for all \(\lambda\), whenever the Eqs.(2) hold.

It is plausible to call a given quantum state \(\rho\) of the composite system local if and only if consistent LHV models are admitted for each choice of the POVMs \(A\) and \(B\). In the remaining part of our Letter we prove a simple mathematical consequence of such a narrower definition of locality.

For simplicity’s sake, we assume that \(\dim \mathcal{H}^{(1)} = \dim \mathcal{H}^{(2)} = 2\). First we construct a subtle POVM \(A\) on \(\mathcal{H}^{(1)}\). We introduce the pure state projectors
\( A_m = \frac{1}{2}(1 + m\sigma) \) where \( m \) is unit vector in \( \mathbb{R}^3 \) and \( \sigma \) is the 3-vector of Pauli-matrices. Let us make the following specific choice for the POVM on \( \mathcal{H}^{(1)} \):

\[
\mathcal{A} = \{ A_m; m \in \mathbb{R}^3, |m| = 1 \},
\]

(7)
i.e. we include all pure state projectors into the set of observables. A simple choice for completeness relation (1) has the integral form \( \sum_m A_m d\Omega = 1 \), where \( d\Omega \) is the solid angle element of \( m \) \[6\]. The set of observables is overcomplet. Introduce Descartes–coordinates on \( \mathbb{R}^3 \), then \( m = (m_1, m_2, m_3) \). Let us denote the orthonormal basis vectors \((1, 0, 0), (0, 1, 0), (0, 0, 1)\) by \( e_1, e_2, e_3 \), respectively. The observables in the POVM \( A \) (7) satisfy the constraints

\[
A_m - \sum_{i=1}^{3} m_i A_{e_i} - \frac{1}{2}(1 - \sum_{i=1}^{3} m_i) I^{(1)} = 0
\]

(8)
for all unit vectors \( m \in \mathbb{R}^3 \). These constraints could be rewritten into the general forms (2) if we replaced \( I^{(1)} \) by \( \sum_{m'} A_{m'} \) from the normalization condition.

In a similar way we introduce the POVM \( \mathcal{B} = \{ B_n; n \in \mathbb{R}^3, |n| = 1 \} \) on \( \mathcal{H}^{(2)} \). Then it follows from our definition of locality proposed earlier in this Letter that, if the state \( \rho \) is local, the LHV model (4) must exist for the above choices of the POVMs \( \mathcal{A} \) and \( \mathcal{B} \), i.e.:

\[
\langle A_m \otimes B_n \rangle = \sum_{\lambda} p_{\lambda} E^{(1)}_{\mathcal{A}}(A_m|\lambda) E^{(2)}_{\mathcal{B}}(B_n|\lambda)
\]

(9)
where \( 0 \leq E^{(1)}_{\mathcal{A}}(A_m|\lambda) \leq 1, 0 \leq E^{(2)}_{\mathcal{B}}(B_n|\lambda) \leq 1 \), and \( \sum_m E^{(1)}_{\mathcal{A}}(A_m|\lambda) = \sum_n E^{(2)}_{\mathcal{B}}(B_n|\lambda) = 1 \), for all \( \lambda \)'s. Let us concentrate on the features of the expectation values \( E^{(1)}_{\mathcal{A}}(A_m|\lambda) \) at fixed hidden parameter \( \lambda \). Given the operator constraints (8), the consistency condition (6) will take the following form:

\[
E^{(1)}_{\mathcal{A}}(A_m|\lambda) - \sum_{i=1}^{3} m_i E^{(1)}_{\mathcal{A}}(A_{e_i}|\lambda) - \frac{1}{2}(1 - \sum_{i=1}^{3} m_i) = 0.
\]

(10)
It must be satisfied for all unit vectors \( m \in \mathbb{R}^3 \). For notational convenience, we introduce the ”polarization vector” \( s = (s_1, s_2, s_3) \in \mathbb{R}^3 \) with
\[ s_i = 2E^{(1)}_A(A_{e_i} | \lambda) - 1 \] for \( i = 1, 2, 3 \). One can define the corresponding conditional “density operator”:
\[
\rho^{(1)}_\lambda \equiv \frac{1}{2}(1 + s \sigma).
\]

(11)
The quotation marks are still reminding us that we have not yet proved the fulfilment of the standard relation \(|s| \leq 1\) assuring the crucial property \(\rho^{(1)}_\lambda \geq 0\). If, however, we substitute \(m_1, m_2, m_3\) in Eq. (10) via the trivial relations \(m_i = tr(\sigma_i A_m)\) the expectation value of \(A_m\) can be expressed in the following form:
\[
E^{(1)}_A(A_m | \lambda) = tr(A_m \rho^{(1)}_\lambda).
\]

(12)
This equation holds for all pure state projectors \(A_m\) on \(\mathcal{H}^{(1)}\). Since the LHS is by definition non-negative the operator \(\rho^{(1)}_\lambda\) must be non-negative as well. Hence one can indeed interpret it as the conditional density operator of the first subsystem at the given value of the hidden parameter \(\lambda\).

By repeating the same proof for the expectation values \(E^{(2)}_B(B_n | \lambda)\), too, the RHS of Eq. (9) can be rewritten as follows:
\[
\langle A_m \otimes B_n \rangle = \sum_\lambda p_\lambda tr(A_m \rho^{(1)}_\lambda)tr(B_n \rho^{(2)}_\lambda).
\]

(13)
This equation is valid for all pure state projectors \(A_m, B_n\) which leads \([7]\) to the following form for the state \(\rho\) of the composite system:
\[
\rho = \sum_\lambda p_\lambda \rho^{(1)}_\lambda \otimes \rho^{(2)}_\lambda,
\]

(14)
with conditional density operators \(\rho^{(1)}_\lambda, \rho^{(2)}_\lambda\) and with positive normalized probability distribution \(p_\lambda\) of the hidden parameters \(\lambda\). This is our central result: local states are mixtures of product states.

The theorem (14) can also be derived for local states in higher dimensional Hilbert spaces. We only outline the proof. If, for instance, \(dim \mathcal{H}^{(1)} = 3\) then the “subtle” choice of POVM \(\mathcal{A}\) means including all one-dimensional Hermitian projectors \(A_\psi = |\psi\rangle \langle \psi|\) and all two-dimensional Hermitian projectors \(A_{\psi\phi} = |\psi\rangle \langle \psi| + |\phi\rangle \langle \phi|\) (\(\psi, \phi \in \mathcal{H}^{(1)}; \langle \psi| \phi \rangle = 0\)) into \(\mathcal{A}\). There exists the following class of constraints (2):
\[
A_\psi + A_\phi - A_{\psi\phi} = 0.
\]

If a given state \(\rho\) is local then, according to our definitions, a consistent LHV model exists and the corresponding expectation values are consistent with the above constraints:
\[
E^{(1)}_{\mathcal{A}}(A_\psi | \lambda) + E^{(1)}_{\mathcal{A}}(A_\phi | \lambda) - E^{(1)}_{\mathcal{A}}(A_{\psi\phi} | \lambda) = 0.
\]

(15)
This equation holds for all pairs of orthogonal vectors $\psi, \phi$ of the three-dimensional Hilbert space $\mathcal{H}^{(1)}$. Then, Gleason’s theorem \cite{8, 9} implies the existence of the density operator $\rho^{(1)}_\lambda$ such that $E^{(1)}_\mathcal{A}(A_\psi|\lambda) = tr(A_\psi \rho^{(1)}_\lambda)$ and $E^{(1)}_\mathcal{A}(A_\psi\phi|\lambda) = tr(A_\psi\phi \rho^{(1)}_\lambda)$. The proof can obviously be generalized for any finite dimensions of $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$. Hence the RHS of Eq. (4) can generally be written as

$$\langle A_\mu \otimes B_\nu \rangle = \sum_\lambda p_\lambda tr(A_\mu \rho^{(1)}_\lambda)tr(B_\nu \rho^{(2)}_\lambda), \quad (16)$$

where $A_\mu$ and $B_\nu$ may be any Hermitian projectors on $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$, respectively. This completes the proof of Eq. (14).

We have pointed out that, according to a plausible definition of local hidden variable models based on generalized measurements, all local density operators can be decomposed as convex mixtures of the subsystems’ conditional density operators, see Eq. (14). If a given state $\rho$ can not be expressed in the form (14) at all then there exist certain POVMs $\mathcal{A}, \mathcal{B}$ which do not admit any consistent LHV model. Logically it means that the statistics of joint nonideal measurements on $A_\mu$ and $B_\nu$ show nonlocal correlations and/or violate the linear constraints (6) of consistency. In fact, however, we have not yet obtained any constructive algorithm to find these POVMs for a generic non-local (i.e. non-product) state though the problem has been solved earlier for pure \cite{2} as well as for a special class of mixed non-product states \cite{3}. In both cases the solutions have essentially been based on testing standard Bell-operators. Further investigations are needed to see what nonideal measurements can reveal nonlocality of generic non-product mixed states. The merit of our Letter is the proof of existence of such sensitive generalized measurements.

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[6] Though we use the $SU(2)$-covariant POVM including continuum number of observables, forthcoming derivations could equally be done with countable or finite number of them, due to the compactness of the unitary groups on finite-dimensional Hilbert spaces. Proofs, however, would then become more circuitous.

[7] The proof relies upon the fact that all pure state projectors on $\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$ lay in the (complex) linear space spanned by the local pure state projectors $A_m \otimes B_n$. The corresponding statement is valid for higher dimensional Hilbert spaces as well.

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