Singularity formations for a surface wave model

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Abstract

In this paper we study the Burgers equation with a nonlocal term of the form \( H u \) where \( H \) is the Hilbert transform. This system has been considered as a quadratic approximation for the dynamics of a free boundary of a vortex patch (see Biello and Hunter 2010 \textit{Commun. Pure Appl. Math.} LXIII 0303–36; Marsden and Weinstein 1983 \textit{Physica D} 7 305–23). We prove blowup in finite time for a large class of initial data with finite energy. Considering a more general nonlocal term, of the form \( \Lambda^\alpha u \) for \( 0 < \alpha < 1 \), finite time singularity formation is also shown.

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1. Introduction

We shall study the formation of singularities for the equation

\[ u_t + uu_x = \Lambda^\alpha H u, \]

with \( 0 \leq \alpha < 1 \), where \( H \) is the Hilbert transform [9] defined by

\[ Hf \equiv \frac{1}{\pi} \text{P.V.} \int f(y) \frac{dy}{x-y}, \]

and \( \Lambda^\alpha \equiv (-\Delta)^{\alpha/2} \) is given by the following expression:

\[ \Lambda^\alpha f(x) = k_\alpha \int \frac{f(x) - f(y)}{|x-y|^{1+\alpha}} dy, \quad k_\alpha = \frac{\Gamma(1+\alpha) \cos((1-\alpha)\pi/2)}{\pi}. \]
The case $\alpha = 0$,  
\begin{equation}
  u_t + uu_x = Hu,
\end{equation}
was introduced by Marsden and Weinstein, in [6], as a second order approximation for the dynamics of a free boundary of a vortex patch (see [1, 3]). Recently Biello and Hunter, in [2], proposed it as a model for waves with constant nonzero linearized frequency. They gave a dimensional argument to show that it models nonlinear Hamiltonian waves with constant frequency. In addition, an asymptotic equation from (2) is derived, describing surface waves on a planar discontinuity in vorticity for a two-dimensional inviscid incompressible fluid. They also carried out numerical analysis showing evidence of singularity formation in finite time.

Let us point out that the Hamiltonian structure of equation (1) (in particular for $\alpha = 0$) comes from the representation
\begin{equation}
  u_t + \partial_x \left( \frac{\delta H}{\delta u} \right) = 0, \quad \text{where } H(u) = \int_{\mathbb{R}} \left( \frac{1}{2} u_{\Lambda}^{-1} u + \frac{1}{6} u^3 \right) \, dx.
\end{equation}

In section 2 we show that the linear term in equation (2) is too weak to prevent the singularity formation of the Burgers equation. In fact, we show that, if the $L^\infty$ norm of the initial data is large enough compared with the $L^2$ norm, the maximum of the solution has a singular behaviour during the time of existence. The proof is based on the ODE along characteristics
\begin{equation}
  \frac{d^2 u}{dt^2} + u = F,
\end{equation}
where the nonnegative ‘force’ $F$ involves a nonlocal operator which is bounded from below by $u^4$. One of the ingredients in the proof is to use the following pointwise inequality:
\begin{equation}
  u(x)^4 \leq 16\|u\|_{L^2(\mathbb{R})}^2 \int_{\mathbb{R}} \frac{(u(x) - u(y))^2}{(x - y)^2} \, dy
\end{equation}
(see lemma 2.2) which can be understood as the local version of the well-known bound
\begin{equation}
  \|u\|_{L^4(\mathbb{R})}^4 \leq C \|u\|_{L^2(\mathbb{R})}^2 \|\Lambda_{1/2} u\|_{L^2(\mathbb{R})}^2 = \frac{C}{2\pi} \|u\|_{L^2(\mathbb{R})}^2 \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(u(x) - u(y))^2}{(x - y)^2} \, dy \, dx.
\end{equation}
In the appendix we provide a generalized pointwise inequality ($n$-dimensional) in terms of fractional derivatives.

In section 3 we consider the more general family of equations, with a higher order term in derivatives, given by (1). By a different method, we prove that the blowup phenomena still arises. Let us note that, since $\Lambda H u = -u_x$, the case $\alpha = 1$ trivializes. Using the same approach as in section 2, it is possible to obtain blowup for $0 < \alpha < 1/3$. Inspired by the method used in [5], we check the evolution of the following quantity:
\begin{equation}
  J^p_q u(x) = \int_{\mathbb{R}} w^p_q(x - y) u(y) \, dy, \quad \text{where } w^p_q(x) = \begin{cases} \frac{1}{|x|^{-q} \text{sign}(x)} & \text{if } |x| < 1, \\ \frac{1}{|x|^{-q}} \text{sign}(x) & \text{if } |x| > 1, \end{cases}
\end{equation}
with $0 < q < 1$ and $p > 2$ to find a singular behaviour. Let us note that a similar approach was used by Dong et al (see [7]) to show blowup for the Burgers equation with fractional dissipation in the supercritical case ($0 < \alpha < 1$):
\begin{equation}
  u_t + uu_x = -\Lambda^\alpha u.
\end{equation}
A different method to show singularities can be found in [8].

It is well known that the $L^p$ norms of the solutions of equation (5) are bounded for all $1 \leq p \leq \infty$. However, to the best of the authors’ knowledge, two quantities are conserved by
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The orthogonality property of the Hilbert transform provides the conservation of the $L^2$ norm, i.e.

$$
\|u(x, t)\|_{L^2(\mathbb{R})} = \|u_0\|_{L^2(\mathbb{R})}.
$$

Since the equation is given by (3), we have that

$$
\int_{\mathbb{R}} \left( \frac{1}{3} u^3(x, t) + \left( \Lambda \frac{u}{x} u(x, t) \right)^2 \right) dx = \int_{\mathbb{R}} \left( \frac{1}{3} u^3_0(x) + \left( \Lambda \frac{u}{x} u_0(x) \right)^2 \right) dx.
$$

2. Blowup for the Burgers–Hilbert equation

The purpose of this section is to show finite time formation of singularities for solutions of equation (2). The result we shall prove is the following:

**Theorem 2.1.** Let $u_0 \in L^2(\mathbb{R}) \cap C^{1+\delta}(\mathbb{R})$, with $0 < \delta < 1$, satisfying the following condition:

There exists a point $\beta_0 \in \mathbb{R}$ with

$$
Hu_0(\beta_0) > 0,
$$

such that

$$
u_0(\beta_0) \geq (32\pi \|u_0\|^2_{L^2(\mathbb{R})})^{1/3}.
$$

Then there is a finite time $T$ such that

$$
\lim_{t \to T} \|u(x, t)\|_{C^{1+\delta}(\mathbb{R})} = \infty,
$$

where $u(x, t)$ is the solution to equation (2).

**Proof.** Let us assume that there exists a solution of equation (2):

$$u(x, t) \in C([0, T), C^{1+\delta}(\mathbb{R})),$$

for all time $T < \infty$ and with $u_0$ satisfying the hypotheses.

Now, we shall define the trajectories $x(\beta, t)$ by the equation

$$
\frac{dx(\beta, t)}{dt} = u(x(\beta, t), t),
$$

$$x(\beta, 0) = \beta.
$$

Considering the evolution of the solution along trajectories, it is easy to get the identity

$$
\frac{du(x(\beta, t), t)}{dt} = u_t(x(\beta, t), t) + \frac{dx(\beta, t)}{dt} u_x(x(\beta, t), t) = Hu(x(\beta, t), t),
$$

and taking a derivative in time we obtain

$$
\frac{d^2u(x(\beta, t), t)}{dt^2} = Hu_t(x(\beta, t), t) + u(x(\beta, t), t) Hu_x(x(\beta, t), t)
$$

$$
= -H(uu_x)(x(\beta, t), t) - u(x(\beta, t), t) + u(x(\beta, t), t) Hu_x(x(\beta, t), t).
$$

Since

$$
H(uu_x)(x) = \frac{1}{2} H((u^2)_x) = \frac{1}{2} \Lambda(u^2)(x),
$$

we can write

$$
\frac{1}{2} \Lambda(u^2)(x) = \frac{1}{2\pi} \text{P.V.} \int_{\mathbb{R}} \frac{u(x)^2 - u(y)^2}{(x - y)^2} dy = u(x) \Lambda u(x) - \frac{1}{2\pi} \int_{\mathbb{R}} \frac{(u(x) - u(y))^2}{(x - y)^2} dy,
$$
and therefore it follows that
\[
\frac{d^2 u(x(\beta, t), t)}{dt^2} = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{(u(x(\beta, t), t) - u(y, t))^2}{(x(\beta, t) - y)^2} dy - u(x(\beta, t), t).
\]

(10)

In order to continue with the proof we will prove the lemma below (for similar approach see [4]):

**Lemma 2.2.** Let \( u \in L^2(\mathbb{R}) \cap C^{1+\delta}(\mathbb{R}), \) for \( 0 < \delta < 1. \) Then
\[
\frac{1}{2\pi} \int_{\mathbb{R}} \frac{(u(x) - u(y))^2}{(x - y)^2} dy \geq C u(x)^4,
\]
where
\[
C = \frac{1}{32\pi E}
\]
and
\[
E = ||u||^2_{L^2(\mathbb{R})}.
\]

**Proof of lemma 2.2.** Let us assume that \( u(x) > 0 \) (a similar proof holds for \( u(x) < 0 \)). Let \( \Omega \) be the set
\[
\Omega = \{ y \in \mathbb{R} : |x - y| < \Delta \},
\]
where \( \Delta \) will be given below. And let \( \Omega^1 \) and \( \Omega^2 \) be the subsets
\[
\Omega^1 = \left\{ y \in \Omega : u(x) - u(y) \geq \frac{u(x)}{2} \right\},
\]
\[
\Omega^2 = \left\{ y \in \Omega : u(x) - u(y) < \frac{u(x)}{2} \right\} = \left\{ y \in \Omega : u(y) > \frac{u(x)}{2} \right\}.
\]

Then
\[
\frac{1}{2\pi} \int_{\mathbb{R}} \frac{(u(x) - u(y))^2}{(x - y)^2} dy \geq \frac{u(x)^2}{8\pi \Delta^2} |\Omega^1|.
\]

On the other hand,
\[
E = \int_{\mathbb{R}} u(y)^2 dy \geq \int_{\Omega^2} u(y)^2 dy \geq \frac{u(x)^2}{4} |\Omega^2|,
\]
and therefore
\[
|\Omega^2| \leq \frac{4E}{u(x)^2}.
\]

Since \( |\Omega^1| = |\Omega| - |\Omega^2| \) and \( |\Omega| = 2\Delta, \) we have that
\[
\frac{1}{2\pi} \int_{\mathbb{R}} \frac{(u(x) - u(y))^2}{(x - y)^2} dy \geq \frac{u(x)^2}{8\pi \Delta^2} \left( 2\Delta - 4E \frac{u(x)^2}{u(x)^2} \right).
\]

We arrive at the conclusion of lemma 2.2 by taking \( \Delta = \frac{4E}{u(x)^2}. \)

Next, let us define \( J(t) = u(x(\beta_0, t), t) \). Thus, applying lemma 2.2 to the expression (10), we obtain the inequality
\[
J_n(t) \geq CJ(t)^4 - J(t).
\]

(11)
Since \( H u_0(\beta_0) > 0 \) and \( J_0(t) = H u(x(\beta_0), t) \), we obtain that \( J_0(t) > 0 \) and \( J(t) > J(0) \) for \( t \in (0, t^*) \) and \( t^* \) small enough. Therefore, multiplying (11) by \( J_0(t) \) we have that

\[
\frac{1}{2} (J_0(t)^2)_t \geq \frac{C}{5} (J(t)^{5/2} - \frac{1}{2} J(t)^2), \quad \forall t \in [0, t^*).
\]

Integrating this inequality in time from 0 to \( t \) we get

\[
J_0(t) \geq \left( J_0(0)^2 + \frac{2C}{5} (J(0)^{5/2} - J(t)^{5/2}) - \frac{1}{2} J(t)^2 - J(0)^2 \right)^{1/2}, \quad \forall t \in [0, t^*). \tag{12}
\]

Now, since \( C J(0)^{5/2} - J(0)^2 \geq 0 \), by the statements of the theorem we obtain that \( J_0(t) > J_0(0) \geq 0 \) for \( t \in (0, t^*) \). Therefore, \( J_0(t) \) is an increasing function \([0, t^*)\). Thus, inequality (12) holds for all time \( t \) and we have a contradiction.

**Remark 2.3.** It is easy to check that there exists a large class of functions satisfying the requirement of theorem 2.1. For example, we can consider the function

\[
u_0(x) = \frac{-ax}{1 + (bx)^q},
\]

\[Hu_0(x) = \frac{a}{1 + (bx)^q}, \]

where \( a, b > 0 \). Choosing \( a \) and \( b \) in a suitable way we can have the norm \( ||u_0||_{L^2(\mathbb{R})} \) as small as we want and the norm \( ||u_0||_{L^\infty(\mathbb{R})} \) as large as we want.

**Remark 2.4.** We note that requirements (8) and (9) in theorem 2.1 can be replaced by

\[Hu_0(\beta_0) \geq 0,
\]

\[u_0(\beta_0) > (32\pi ||u_0||_{L^2(\mathbb{R})}^2)^{1/3},
\]

attaining the same conclusion.

3. **Blowup for the whole range \( 0 < \alpha < 1 \)**

In this section we shall show formation of singularities for equation (1), with \( 0 < \alpha < 1 \). The aim is to prove the following result:

**Theorem 3.1.** There exist initial data \( u_0 \in L^2(\mathbb{R}) \cap C^{1+\delta}(\mathbb{R}) \), with \( 0 < \delta < 1 \), and a finite time \( T \), depending on \( u_0 \), such that

\[
\lim_{t \to T} ||u(\cdot, t)||_{C^{1+\delta}(\mathbb{R})} = \infty
\]

where \( u(x, t) \) is the solution to equation (1).

**Proof.** Let us assume that there exists a solution of equation (1), \( u(x, t) \in C([0, T), C^{1+\delta}(\mathbb{R})) \), for all time \( T < \infty \). Let \( J_q^p u \) be the convolution

\[J_q^p u(x) = \int_R w_q^p(x - y) u(y) \, dy\]

where

\[w_q^p(x) = \begin{cases} \frac{1}{|x|^p} \text{sign} (x) & \text{if } |x| < 1, \\ \frac{1}{|x|^q} \text{sign} (x) & \text{if } |x| > 1, \end{cases}\]

with \( 0 < q < 1 \) and \( p > 2 \). In order to prove theorem 3.1 we shall need the following two lemmas.
Lemma 3.2. Let $f$ in $C^{1+}(\mathbb{R}) \cap L^2(\mathbb{R})$ and $0 < \alpha < 1$. Then

$$
\Lambda^\alpha Hf(x) = k_\alpha \int_\mathbb{R} \frac{f(x) - f(y)}{|x - y|^{1+\alpha}} \text{sign} (x - y) \, dy,
$$

where

$$
k_\alpha = -\frac{\Gamma(1 + \alpha) \sin((1 + \alpha)\pi/2)}{\pi}.
$$

Proof. Let $f$ be a function on the Schwartz class. The inverse Fourier transform formula yields

$$
\Lambda^\alpha Hf(x) = \frac{1}{2\pi} \int_\mathbb{R} -i \text{sign} (k) |k|^\alpha \hat{f}(k) \exp(ikx) \, dk.
$$

We will understand the above identity as the following limit

$$
\Lambda^\alpha Hf(x) = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi} \int_\mathbb{R} -i \text{sign} (k) |k|^\alpha \exp(-\varepsilon |k|) \exp(ikx) \left( \int_\mathbb{R} f(y) \exp(-iky) \, dy \right) \, dk.
$$

Next, we can compute that

$$
\Lambda^\alpha Hf(x) = \lim_{\varepsilon \to 0^+} \frac{\Gamma(1 + \alpha)}{\pi} \int_\mathbb{R} \frac{f(y)}{\varepsilon^2 + (x - y)^2} \sin \left( (1 + \alpha) \arctan \left( \frac{x - y}{\varepsilon} \right) \right) \, dy
$$

$$
= -\lim_{\varepsilon \to 0^+} \frac{\Gamma(1 + \alpha)}{\pi} \int_\mathbb{R} \frac{f(x) - f(y)}{|x - y|^{1+\alpha}} \sin \left( (1 + \alpha) \arctan \left( \frac{x - y}{\varepsilon} \right) \right) \, dy
$$

$$
= -\frac{\Gamma(1 + \alpha) \sin((1 + \alpha)\pi/2)}{\pi} \int_\mathbb{R} \frac{f(x) - f(y)}{|x - y|^{1+\alpha}} \text{sign} (x - y) \, dy.
$$

We achieve the conclusion of lemma 3.2 by the classical density argument.

Lemma 3.3. Let $I_p^q(x)$ be the integral

$$
I_p^q(x) = \int_\mathbb{R} \frac{w_p^q(x) - w_p^q(y)}{|x - y|^{1+\alpha}} \text{sign} (x - y) \, dy,
$$

where $0 < q < 1$ and $p > 2$. Then

$$
|I_p^q(x)| \leq \begin{cases} 
K_1 \frac{1}{|x|^q} & \text{if } 0 < |x| < \frac{1}{2}, \\
K_2 \frac{1}{|x|^{2q}} & \text{if } 2 < |x| < \infty, \\
K_3 & \text{if } \frac{1}{2} \leq |x| \leq 2,
\end{cases}
$$

where $K_1$, $K_2$, and $K_3$ are universal constants depending on $q$ and $p$.

Proof. Since the function $I_p^q(x)$ is even, we can assume that $x > 0$. The constant values of $K_1$ and $K_2$ can be different along the estimates below.

First, let us consider the case $0 < x < 1/2$. We split as follows:

$$
I_p^q(x) = \int_{|y| < 1} \, dy + \int_{|y| > 1} \, dy = I_1(x) + I_2(x).$$
It yields

$$I_1(x) = \int_{|y|<1} \frac{1}{x^q} - \frac{1}{|y|^q} \frac{1}{|x-y|^{1+\alpha}} \text{sign}(x-y) \, dy$$

$$= \int_0^1 \left( \frac{1}{x^q} - \frac{1}{y^q} \frac{1}{|x-y|^{1+\alpha}} \text{sign}(x-y) + \frac{1}{x^q} + \frac{1}{y^q} \frac{1}{|x+y|^{1+\alpha}} \right) \, dy,$$

and a change of variables allow us to split further

$$I_1(x) = \frac{1}{x^{q+\alpha}} \int_0^1 \left( \frac{1 - \frac{1}{\eta^q}}{|1 - \eta|^{1+\alpha}} \text{sign}(1-\eta) + \frac{1 + \frac{1}{\eta^q}}{|1 + \eta|^{1+\alpha}} \right) \, d\eta$$

$$= \frac{1}{x^{q+\alpha}} \left( \int_0^1 \int_{1}^{\frac{1}{x^q}} + \int_{1}^{\frac{1}{x^q}} \right) = \frac{1}{x^{q+\alpha}} (F_1(x) + F_2(x)).$$

For $F_1(x)$ we find the bound

$$|F_1(x)| \leq \int_0^1 \left| \frac{1 - \frac{1}{\eta^q}}{|1 - \eta|^{1+\alpha}} \right| \, d\eta + \int_0^1 \left| \frac{1 + \frac{1}{\eta^q}}{|1 + \eta|^{1+\alpha}} \right| \, d\eta \leq K_1.$$

On the other hand

$$F_2(x) = \int_1^{\frac{1}{x^q}} \left( \frac{\frac{1}{\eta^q} - 1}{|1 - \eta|^{1+\alpha}} + \frac{\frac{1}{\eta^q} + 1}{|1 + \eta|^{1+\alpha}} \right) \, d\eta = \int_{\frac{1}{x^q}}^{\frac{3}{2}} \int_{\frac{1}{2}}^{\frac{1}{x^q}} = j_1(x) + j_2(x).$$

For $j_1(x)$ it is easy to obtain

$$|j_1(x)| \leq \int_{\frac{1}{2}}^{\frac{3}{2}} \left| \frac{\frac{1}{\eta^q} - 1}{|1 - \eta|^{1+\alpha}} \right| \, d\eta + \int_{\frac{1}{2}}^{\frac{3}{2}} \left| \frac{\frac{1}{\eta^q} + 1}{|1 + \eta|^{1+\alpha}} \right| \, d\eta \leq K_1.$$

For $j_2(x)$ we decompose as follows:

$$j_2(x) = \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{\eta^q} \left( \frac{1}{|1 - \eta|^{1+\alpha}} + \frac{1}{|1 + \eta|^{1+\alpha}} \right) \, d\eta - \int_{\frac{1}{2}}^{\frac{3}{2}} \left( \frac{1}{|1 - \eta|^{1+\alpha}} - \frac{1}{|1 + \eta|^{1+\alpha}} \right) \, d\eta.$$

Thus, since $0 < q < 1$ and

$$\left| \frac{1}{|1 - \eta|^{1+\alpha}} + \frac{1}{|1 + \eta|^{1+\alpha}} \right| \leq \frac{K_1}{\eta}$$

we have that

$$|j_2(x)| \leq K_1 \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{\eta^{q+1}} \, d\eta + \int_{\frac{1}{2}}^{\frac{3}{2}} \left| \frac{1}{|1 - \eta|^{1+\alpha}} - \frac{1}{|1 + \eta|^{1+\alpha}} \right| \, d\eta \leq K_1.$$
Let us continue with $I_2(x)$ which can be written in the form

$$I_2(x) = \int_{|y|>1} \frac{1}{x^q} - \text{sign}(y) \frac{1}{|y|^p} \frac{1}{|x-y|^{1+\alpha}} \text{sign}(x-y) \, dy = \int_1^\infty \left( -\frac{1}{x^q} - \frac{1}{|y|^p} \frac{1}{|x-y|^{1+\alpha}} + \frac{1}{x^q} + \frac{1}{|y|^p} \frac{1}{|x+y|^{1+\alpha}} \right) \, dy$$

$$= \frac{1}{x^{p+\alpha}} \int_1^{\infty} \left( \frac{1}{x^{p+\alpha}} - \frac{1}{|y|^p} \frac{1}{(1+\eta)^{1+\alpha}} + \frac{1}{x^{p+\alpha}} + \frac{1}{|y|^p} \frac{1}{(1+\eta)^{1+\alpha}} \right) \, d\eta,$$

The following decomposition

$$I_2(x) = \frac{1}{x^{p+\alpha}} \int_1^{\infty} \left( \frac{1}{|1+\eta|^{1+\alpha}} - \frac{1}{|1-\eta|^{1+\alpha}} + \frac{1}{|1+\eta|^{1+\alpha}} + \frac{1}{|1-\eta|^{1+\alpha}} \right) \, d\eta$$

yields

$$|I_2(x)| \leq \frac{K^1}{x^{q+\alpha}} + \frac{1}{x^{p+\alpha}} \int_1^{\infty} \left( \frac{1}{|1+\eta|^{1+\alpha}} - \frac{1}{|1-\eta|^{1+\alpha}} + \frac{1}{|1+\eta|^{1+\alpha}} + \frac{1}{|1-\eta|^{1+\alpha}} \right) \, d\eta$$

$$\leq \frac{K^1}{x^{q+\alpha}} + \frac{K^1}{x^{p+\alpha}} \int_1^{\infty} \frac{1}{\eta^{p+1}} \, d\eta \leq \frac{K^1}{x^{q+\alpha}} \left( \frac{1}{|x|^\alpha} + \frac{1}{|x|^\alpha} \right) \leq \frac{K^1}{x^{q+\alpha}}.$$

Next, we consider the case $2 < x < \infty$ taking

$$I_2^p(x) = \int_\mathbb{R} \frac{1}{|x-y|^{1+\alpha}} \text{sign}(x-y) \, dy = \int_{|y|<1} \, dy + \int_{|y|>1} \, dy = J_1(x) + J_2(x).$$

For $J_2(x)$ we have that

$$J_2(x) = \int_{|y|<1} \frac{1}{x^p - w(y)} \frac{1}{|x-y|^{1+\alpha}} \text{sign}(x-y) \, dy = \int_1^{\infty} \left( \frac{1}{x^p} - \frac{1}{|y|^p} \frac{1}{|x-y|^{1+\alpha}} \text{sign}(x-y) + \frac{1}{x^p} + \frac{1}{|y|^p} \frac{1}{|x+y|^{1+\alpha}} \right) \, dy$$

and a change of variables provides

$$J_2(x) = \frac{1}{x^{p+\alpha}} \int_1^{\infty} \left( \frac{1 - \frac{1}{\eta^p}}{|1-\eta|^{1+\alpha}} \text{sign}(1-\eta) + \frac{1 + \frac{1}{\eta^p}}{|1+\eta|^{1+\alpha}} \right) \, d\eta$$

$$= \frac{1}{x^{p+\alpha}} \left( \int_1^{1} + \int_1^{\infty} \right) = \frac{1}{x^{p+\alpha}} (H_1(x) + H_2(x)).$$

For $H_2(x)$ one could bind as follows:

$$|H_2(x)| \leq \int_1^{\infty} \left| \frac{1 - \frac{1}{\eta^p}}{|1-\eta|^{1+\alpha}} \right| \, d\eta + \int_1^{\infty} \left| \frac{1 + \frac{1}{\eta^p}}{|1+\eta|^{1+\alpha}} \right| \, d\eta \leq K^1.$$
On the other hand, in $H_1(x)$ we split further

$$H_1(x) = \int_{\frac{1}{2}}^{1} \left( \frac{1 - \frac{1}{\eta} \frac{1}{|1 - \eta|^{1+\alpha}}}{|1 - \eta|^{1+\alpha}} + \frac{1 + \frac{1}{\eta} \frac{1}{|1 + \eta|^{1+\alpha}}}{|1 + \eta|^{1+\alpha}} \right) d\eta = \int_{\frac{1}{2}}^{\frac{1}{2}} d\eta + \int_{\frac{1}{2}}^{1} d\eta = h_1(x) + h_2(x).$$

The term $h_2(x)$ is bounded by

$$|h_2(x)| \leq \int_{\frac{1}{2}}^{1} \left| \frac{1 - \frac{1}{\eta} \frac{1}{|1 - \eta|^{1+\alpha}}}{|1 - \eta|^{1+\alpha}} \right| d\eta + \int_{\frac{1}{2}}^{1} \left| \frac{1 + \frac{1}{\eta} \frac{1}{|1 + \eta|^{1+\alpha}}}{|1 + \eta|^{1+\alpha}} \right| d\eta \leq K^2.$$

We reorganize $h_1(x)$ so that

$$h_1(x) = \int_{\frac{1}{2}}^{\frac{1}{2}} \left( \frac{1 - \frac{1}{\eta} \frac{1}{|1 - \eta|^{1+\alpha}}}{|1 - \eta|^{1+\alpha}} \right) d\eta + \int_{\frac{1}{2}}^{1} \frac{1}{\eta} \frac{1}{|1 - \eta|^{1+\alpha}} - \frac{1}{|1 + \eta|^{1+\alpha}} d\eta.$$

Since $p > 2$ and

$$\left| \frac{1}{|1 - \eta|^{1+\alpha}} - \frac{1}{|1 + \eta|^{1+\alpha}} \right| \leq K^2 \eta \quad \text{for } \eta \in [0, 2/3],$$

we obtain that

$$|h_1(x)| \leq \int_{\frac{1}{2}}^{\frac{1}{2}} \left( \frac{1 - \frac{1}{\eta} \frac{1}{|1 - \eta|^{1+\alpha}}}{|1 - \eta|^{1+\alpha}} \right) d\eta + K^2 \int_{\frac{1}{2}}^{1} \frac{1}{\eta^{p-1}} d\eta \leq K^2(1 + x^{p-2}).$$

Therefore,

$$|J_2(x)| \leq K^2 \left( \frac{1}{x^{p+\alpha}} + \frac{1}{x^{2+\alpha}} \right) \leq K^2 \frac{1}{x^{2+\alpha}}.$$

Next, we deal with $J_1$ given by

$$J_1(x) = \int_{|y| < 1} \frac{1}{x^p - \text{sign}(y) \frac{1}{|y|^{q}} \frac{1}{|x - y|^{1+\alpha}}} d\eta = \int_{0}^{1} \left( \frac{1}{x^p - \frac{1}{|y|^{q}} \frac{1}{|x - y|^{1+\alpha}}} + \frac{1}{x^p + \frac{1}{|y|^{q}} \frac{1}{|x + y|^{1+\alpha}}} \right) dy$$

$$= \frac{1}{x^{p+\alpha}} \int_{0}^{\frac{1}{2}} \left( \frac{1 - x^{p-q} \frac{\eta^{q}}{|1 - \eta|^{1+\alpha}}}{|1 - \eta|^{1+\alpha}} + \frac{x^{p-q} \frac{\eta^{q}}{|1 + \eta|^{1+\alpha}}}{|1 + \eta|^{1+\alpha}} \right) d\eta.$$

Hence

$$J_1(x) = \frac{1}{x^{p+\alpha}} \int_{0}^{\frac{1}{2}} \frac{1}{|1 - \eta|^{1+\alpha}} + \frac{1}{|1 + \eta|^{1+\alpha}} d\eta$$

$$+ \frac{1}{x^{q+\alpha}} \int_{0}^{\frac{1}{2}} \frac{1}{\eta^{q}} \frac{1}{|1 + \eta|^{1+\alpha}} - \frac{1}{|1 - \eta|^{1+\alpha}} d\eta.$$

Since $p > 2$ and

$$\left| \frac{1}{|1 + \eta|^{1+\alpha}} - \frac{1}{|1 - \eta|^{1+\alpha}} \right| \leq K^2 \eta \quad \text{for } \eta \in [0, 1/2],$$

we have

$$|J_1(x)| \leq K^2 \frac{1}{x^{2+\alpha}}.$$
we obtain that

\[ |J_1(x)| \leq \frac{K_2}{x^{p+\alpha}} + \frac{K_2}{x^{q+\alpha}} \int_0^x \eta^{1-q} \, d\eta \leq K_2 \left( \frac{1}{x^{p+\alpha}} + \frac{1}{x^{2+\alpha}} \right) \leq \frac{K_2}{x^{2+\alpha}}. \]

The bound for \(1/2 < x < 2\) is obvious, which allow us to conclude the proof. ■

In order to prove theorem 3.1, we shall study the evolution of \(J(t) = J^p_\alpha u(x_b(t), t)\), where \(x_b\) is the trajectory \(x_b(t) = x(0, t)\). Hence

\[
\frac{dJ(t)}{dt} = -\frac{1}{2} J^p_\alpha (u^2)_x(x_b(t), t) + J^p_\alpha \Lambda^\alpha H u(x_b(t), t) + u(x_b(t), t)(\partial_x J^p_\alpha u)(x_b(t), t).
\]

We can write

\[
J^p_\alpha ((u^2)_x) = \int_{\mathbb{R}} (u(x)^2 - u(y)^2) W^p_\alpha (x - y) \, dy
\]

and

\[
\partial_x (J^p_\alpha u)(x) = \int_{\mathbb{R}} (u(x) - u(y)) W^p_\alpha (x - y) \, dy,
\]

where

\[
W^p_\alpha = \begin{cases} 
q & \text{if } |x| < 1, \\
p & \text{if } |x| > 1.
\end{cases}
\]

Then, it is easy to check that

\[
-\frac{1}{2} J^p_\alpha ((u^2)_x) + u(x)(\partial_x J^p_\alpha u)(x) = \frac{1}{2} \int_{\mathbb{R}} (u(x) - u(y))^2 W^p_\alpha (x - y) \, dy,
\]

and therefore

\[
\frac{dJ(t)}{dt} = \frac{1}{2} \int_{\mathbb{R}} (u(x_b(t)) - u(y))^2 W^p_\alpha (x_b(t) - y) \, dy + J^p_\alpha \Lambda^\alpha H u(x_b(t), t). \tag{13}
\]

Using lemma (3.2), the linear term becomes

\[
J^p_\alpha \Lambda^\alpha H u(x) = k_\alpha \int_{\mathbb{R}} w^p_\alpha (x - y) \int_{\mathbb{R}} \frac{u(y) - u(s)}{|y - s|^{1+\alpha}} \text{sign} (y - s) \, ds \, dy,
\]

and a wise use of the principal value provides

\[
J^p_\alpha \Lambda^\alpha H u(x) = k_\alpha \int_{\mathbb{R}} w^p_\alpha (x - y) P.V. \int_{\mathbb{R}} \frac{-u(s)}{|y - s|^{1+\alpha}} \text{sign} (y - s) \, ds \, dy \]

\[
= k_\alpha \int_{\mathbb{R}} w^p_\alpha (x - y) P.V. \int_{\mathbb{R}} \frac{u(x) - u(s)}{|y - s|^{1+\alpha}} \text{sign} (y - s) \, ds \, dy \]

\[
= k_\alpha \int_{\mathbb{R}} (u(x) - u(s)) P.V. \int_{\mathbb{R}} \frac{w^p_\alpha (x - y)}{|y - s|^{1+\alpha}} \text{sign} (y - s) \, dy \, ds \]

\[
= k_\alpha \int_{\mathbb{R}} (u(x) - u(s)) \int_{\mathbb{R}} \frac{w^p_\alpha (x - s) - w^p_\alpha (x - y)}{|y - s|^{1+\alpha}} \text{sign} (s - y) \, dy \, ds
\]

to find finally

\[
J^p_\alpha \Lambda^\alpha H u(x) = k_\alpha \int_{\mathbb{R}} (u(x) - u(s)) J^p_\alpha (x - s) \, ds.
\]
Therefore,
\[ |J^p_q \Lambda^w Hu(x)| \leq |k_w| \int_{\mathbb{R}} |u(x) - u(y)||P^p_q(x - y)| \ dy \]
\[ \leq |k_w| \left( \int_{\mathbb{R}} (u(x) - u(y))^2 W^p_q(x - y) \ dy \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} \frac{I^p_q(x)^2}{W^p_q(x)} \ dx \right)^{\frac{1}{2}}. \]

Since
\[ \frac{I^p_q(x)^2}{W^p_q(x)} \leq \begin{cases} 
\frac{C}{|x|^{2a+q-1}} & \text{when } |x| \to 0, \\
\frac{C}{|x|^{3+2a-p}} & \text{when } |x| \to \infty,
\end{cases} \]
by taking \(2 < p < 2 + 2\alpha\) and \(q < 2(1 - \alpha)\), we obtain that
\[ |J^p_q \Lambda^w Hu(x)| \leq C(p, q) \left( \int_{\mathbb{R}} (u(x) - u(y))^2 W^p_q(y) \ dy \right)^{\frac{1}{2}} \]
\[ \leq \frac{1}{4} \int_{\mathbb{R}} (u(x) - u(y))^2 W^p_q(x - y) \ dy + C. \]

This inequality in equation (13) yields
\[ \frac{dJ(t)}{dt} \geq \frac{1}{4} \int_{\mathbb{R}} (u(x_b(t)) - u(y))^2 W^p_q(x_b(t) - y) \ dy - C(p, q). \]

On the other hand,
\[ J(t) = \int_{\mathbb{R}} u(y) w^p_q(x_b(t) - y) \ dy = \int_{\mathbb{R}} (u(y) - u(x_b(t))) w^p_q(x_b(t) - y) \ dy \]
\[ \leq \left( \int_{\mathbb{R}} (u(x_b(t)) - u(y))^2 W^p_q(x_b(t) - y) \ dy \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} \frac{w^p_q(x)^2}{W^p_q(x)} \ dx \right)^{\frac{1}{2}}. \]

The following bound
\[ \frac{w^p_q(x)^2}{W^p_q(x)} \leq \begin{cases} 
\frac{C}{|x|^{q-1}} & \text{when } |x| \to 0, \\
\frac{C}{|x|^{p-1}} & \text{when } |x| \to \infty,
\end{cases} \]
allows us to obtain, for \(2 < p < 2 + 2\alpha\) and \(0 < q < 1\),
\[ \int_{\mathbb{R}} (u(x_b(t)) - u(y))^2 W^p_q(x_b(t) - y) \ dy \geq c(q, p) J(t)^2. \]

Therefore, we obtain a quadratic evolution equation
\[ \frac{dJ(t)}{dt} \geq c(q, p) J(t)^2 - C(q, p) \]
and by taking \(c(q, p) J(0)^2 - C(q, p) > 0\), we find a contradiction for the mere fact that
\[ J(t) \leq C(q, p) ||u||_{L^\infty}. \]
Appendix

In this section we generalize the pointwise inequality (4) involving the nonlocal operator
\( 2 f \Lambda^a f - \Lambda^a (f^2) \). Some simple applications to Gagliardo–Nirenberg–Sobolev inequalities are also shown.

**Lemma 4.1.** Consider a function \( f : \mathbb{R}^n \to \mathbb{R} \) in the Schwartz class, \( 0 < \alpha < 2 \) and \( 0 < p < \infty \). Then
\[
|f(x)|^{2 + \frac{2}{p}} \leq C(\alpha, p, n) ||f||^\frac{2}{p}_{L^p(\mathbb{R}^n)} (2 f(x) \Lambda^a f(x) - \Lambda^a (f^2)(x))
\] (14)
for any \( x \in \mathbb{R}^n \).

**Proof.** The formula for the operator \( \Lambda^a \) in \( \mathbb{R}^n \)
\[
\Lambda^a f(x) = k_{a,n} \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x - y|^{n+\alpha}} \, dy
\]
and \( 0 < \alpha < 2 \), allows us to find
\[
2 f(x) \Lambda^a f(x) - \Lambda^a (f^2)(x) = k_{a,n} \int_{\Omega} \frac{(f(y) - f(x))^2}{|x - y|^{n+\alpha}} \, dy.
\]
We consider \( f(x) > 0 \), being the case \( f(x) < 0 \) analogous. Let \( \Omega, \Omega^1 \) and \( \Omega^2 \) be the sets
\[
\Omega = \{ y \in \mathbb{R} : |x - y| < \Delta \},
\]
\[
\Omega^1 = \{ y \in \Omega : f(x) - f(y) \geq f(x)/2 \},
\]
\[
\Omega^2 = \{ y \in \Omega : f(x) - f(y) < f(x)/2 \} = \{ y \in \Omega : f(y) > f(x)/2 \}.
\]
Then,
\[
2 f(x) \Lambda^a f(x) - \Lambda^a (f^2)(x) \geq k_{a,n} \int_{\Omega} \frac{(f(y) - f(x))^2}{|x - y|^{n+\alpha}} \, dy \geq k_{a,n} \frac{f(x)^2}{4\Delta^{n+\alpha}} |\Omega^1|.
\] (15)
On the other hand,
\[
||f||^p_{L^p(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |f(y)|^p \, dy \geq \frac{f(x)^p}{2p} |\Omega^2|,
\]
therefore
\[
2 f(x) \Lambda^a f(x) - \Lambda^a (f^2)(x) \geq k_{a,n} \frac{f(x)^2}{4\Delta^{n+\alpha}} (|\Omega| - |\Omega^2|)
\]
\[
\geq k_{a,n} \frac{f(x)^2}{4\Delta^{n+\alpha}} \left( c_n \Delta^n - \frac{2p ||f||^p_{L^p(\mathbb{R}^n)}}{f(x)^p} \right),
\]
where \( c_n = 2\pi^{n/2}/(n\Gamma(n/2)) \). By choosing
\[
\Delta^n = \frac{(n + \alpha)2^p ||f||^p_{L^p(\mathbb{R}^n)}}{ac_n f(x)^p}
\]
we obtain the desired inequality. ■

**Remark 4.2.** Inequality (14) allows us to get easily the following Gagliardo–Nirenberg–Sobolev estimate:
\[
||f||^{2 + \frac{2}{p}}_{L^{2 + \frac{2}{p}}(\mathbb{R}^n)} \leq 2C(\alpha, p, n) ||f||^\frac{2}{p}_{L^p(\mathbb{R}^n)} ||\Lambda^a f||^2_{L^2(\mathbb{R}^n)}.
\]
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