IGUSA’S CONJECTURE ON EXPONENTIAL SUMS MODULO $p$ AND $p^2$ AND THE MOTIVIC OSCILLATION INDEX

by

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Abstract. — We prove the modulo $p$ and modulo $p^2$ cases of Igusa’s conjecture on exponential sums. This conjecture predicts specific uniform bounds in the homogeneous polynomial case of exponential sums modulo $p^m$ when $p$ and $m$ vary. We introduce the motivic oscillation index of a polynomial $f$ and prove the stronger, analogue bounds for $m = 1, 2$ using this index instead of the original bounds. The modulo $p^2$ case of our bounds holds for all polynomials; the modulo $p$ case holds for homogeneous polynomials and under extra conditions also for nonhomogeneous polynomials. We obtain natural lower bounds for the motivic oscillation index by using results of Segers. We also show that, for $p$ big enough, Igusa’s local zeta function has a nontrivial pole when there are $\mathbb{F}_p$-rational singular points on $f = 0$. We introduce a new invariant of $f$, the flaw of $f$.

1. Introduction

Let $f$ be a polynomial over $\mathbb{Q}$ in $n$ variables. Consider the “global” exponential sums

$$E_f(N) := \frac{1}{N^n} \sum_{x \in \{0, \ldots, N-1\}^n} \exp \left(2\pi i \frac{f(x)}{N} \right),$$

where $N$ varies over the positive integers. Up to prime factorization of $N$, to study the dependence of $E_f(N)$ on $N$ it suffices to study $E_f(p^m)$ in terms of integers $m > 0$ and primes $p$, which requires a mixture of finite field and $p$-adic techniques.

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Igusa’s conjecture predicts sharp bounds for $|E_f(p^m)|$ in terms of integers $m > 0$ and primes $p$ when $f$ is homogeneous. Analogue conjectures for analogue sums over finite field extensions of $\mathbb{Q}$ also make sense. Fixing $m = 1$ or $m = 2$, the conjectured uniformity in $p$ gives a link between finite field and $p$-adic exponential sums.

In any case, to find sharp bounds for $|E_f(p^m)|$ for fixed $p$ uniformly in $m$ is often much more easy than bounding uniformly in $p$ and $m$. Using a resolution of singularities $\pi$ as below, Igusa defines a rational number $\alpha = \alpha(\pi) \leq 0$ such that

$$|E_f(p^m)| < c_p m^{n-1} p^{\alpha}$$

for each $m$, each prime $p$, and some constants $c_p$ depending on $p$. When $f$ is homogeneous, he conjectures in [14] that $c_p$ can be taken independently of $p$.

. — Let us define the number $\alpha(\pi)$ that is used in (1.0.1) for any polynomial $f$ over $\mathbb{Q}$, following Igusa [14], or [13, 10]. If $f$ is a constant, put $\alpha(\pi) := 0$. Otherwise, let

$$\pi_f : Y \to \mathbb{A}^n_C$$

be an embedded resolution of singularities with normal crossings of $f = 0$. Let $(N_i, \nu_i), i \in I$, be the numerical data of $\pi_f$, that is, for each irreducible component $E_i$ of $f \circ \pi_f = 0$, $i \in I$, let $N_i$ be the multiplicity of $E_i$ in $\text{div}(f \circ \pi_f)$, and $\nu_i - 1$ the multiplicity of $E_i$ in the divisor associated to $\pi_f^*(dx)$ with $dx = dx_1 \wedge ... \wedge dx_n$. The essential numerical data of $\pi_f$ are the pairs $(N_i, \nu_i)$ for $i \in J$ with $J = I \setminus I'$ and where $I'$ is the set of indices $i$ in $I$ such that $(N_i, \nu_i) = (1, 1)$ and such that $E_i$ does not intersect another $E_j$ with $(N_j, \nu_j) = (1, 1)$. Define $\alpha(\pi_f)$ as

$$\alpha(\pi_f) = -\min_{i \in J} \frac{\nu_i}{N_i}$$

when $J$ is nonempty and define $\alpha(\pi_f)$ as $-2n$ otherwise. Take a similar resolution $\pi_{f-c}$ of $f - c = 0$ for each critical value $c \in \mathbb{C}$ of $f : \mathbb{C}^n \to \mathbb{C}$ and put $\pi := (\pi_{f-c})_c$ where $c$ runs over $\{0\}$ and the critical values of $f$. Define

$$\alpha(\pi) := \max_c (\alpha(\pi_{f-c})),$$

where $c$ runs over $\{0\}$ and the critical values of $f$. Note that the number $\alpha(\pi)$ often depends on $\pi$, [17]. Namely, $\alpha(\pi)$ is independent of the choice of $\pi$ when $\alpha(\pi) \geq -1$ by properties of the logcanonical threshold of $f - c$ for all $c$, but on the other hand, when $\alpha(\pi) < -1$, then $\alpha(\pi)$ always depends on the choice of $\pi$, see Remark [16.2].

. — Now we are ready to state (a slightly generalized form of) what Igusa conjectured in the introduction of his book of 1978:

1.1. Conjecture ([14]). — Suppose that $f$ is a homogeneous polynomial over $\mathbb{Q}$ in $n$ variables. Let $\pi$ and $\alpha(\pi)$ be as above [17]. Then there exists a constant $c$ such

\[\footnote{\text{(1)} For homogeneous $f$ one has $\pi = (\pi_f)$ since $0$ is the only possible critical value of $f$ by Euler’s identity.}\]
that for each prime $p$ and each positive integer $m$ one has

$$|E_f(p^m)| < cm^{n-1}p^{\alpha_f}.$$

This conjecture might hold for a much wider class of polynomials than homogeneous ones, for example, for quasi-homogeneous polynomials. It is an open question how to generalize the conjecture optimally to the case of general $f$, cf. Example 7.2 and section 8 below.

In certain cases Conjecture 1.1 is proved. Igusa proved the case that the projective hypersurface defined by $f$ is smooth [14], by using Deligne’s bounds in [7] for finite field exponential sums. The case that $\pi$ is a toric resolution and $f$ is nondegenerate with respect to its Newton polyhedron at the origin is proven under special conditions by Denef and Sperber in [12], and in full generality by the author in [2].

In this paper we prove Igusa’s Conjecture 1.1 when we fix $m = 1$, resp. $m = 2$, see Corollary 6.2. Actually, we prove a bit more:

1) instead of using $\alpha(\pi)$, we define and use an exponent $\alpha_f$, the *motivic oscillation index* of $f$, which is intrinsically associated to $f$ and at least as sharp and sometimes sharper than $\alpha(\pi)$ since $\alpha_f \leq \alpha(\pi)$;

2) for $m = 2$ the conjecture holds for arbitrary $f$;

3) when we restrict to big enough primes $p$ we prove the analogue bounds for all finite field extensions of the fields $\mathbb{Q}_p$ and all fields $\mathbb{F}_q((t))$ of characteristic $p$.

The motivic oscillation index $\alpha_f$ is defined and studied in section 3; this notion is based on a suggestion made to us by Jan Denef. In particular, we show that (1.0.1) holds with $\alpha = \alpha_f$ instead of $\alpha = \alpha(\pi)$.

For arbitrary polynomials $f$ in $n$ variables over some number field, we establish in Theorem 5.1 the lower bound

$$\frac{-n + \delta_f}{2} \leq \alpha_f,$$

where $\delta_f$ is the dimension of the locus of grad $f$, where we say that the empty scheme has dimension $-\infty$. It is proven by showing in Theorem 2.2 that for big enough $p$ Igusa’s local zeta function has a nontrivial pole if and only if there is a $\mathbb{F}_p$-rational point on grad $f = f = 0$ (Theorem 2.2) and by using the lower bounds of Segers [18] for the isolated singularity case in combination with a Cartesian product argument on integrals.

Having this lower bound for $\alpha_f$, Conjecture 1.1 with $m = 1$ fixed follows from upper bounds given by Katz [15] for exponential sums over finite fields of large characteristic. The case $m = 2$ follows elementarily from Hensel’s Lemma and bounds on the number of $\mathbb{F}_q$-rational points on grad $f = 0$. 
The analogue results over the fields $\mathbf{F}_q((t))$ of big enough characteristic just follow from a comparison principle coming from the theory of motivic integration, named transfer principle, cf. \cite{4}, \cite{6}, or with some work from \cite{10}.

In Theorem \cite{7} we generalize our results for homogeneous polynomials to arbitrary polynomials satisfying Condition \cite{7.0.1} which relates the singularities at infinity in $\mathbb{P}^n$ with the singularities in affine space.

In section \cite{8} we slightly sharpen Igusa’s conjecture by using $\alpha_f$ instead of $\alpha(\pi)$. In general, $\alpha_f$ might not be the only ingredient for sharp bounds, so we introduce a new invariant of a polynomial $f$, the flaw of $f$, for the study of uniform bounds for $|E_f(p^m)|$.

1.2. Remark. — When $\alpha(\pi) < -1$, then $\alpha(\pi)$ depends on the choice of $\pi$, as is communicated to the author by Segers and Veys \cite{17}. Namely, let $E$ be an exceptional irreducible component (for $\pi$) with numerical data $(N, \nu)$ which intersects an irreducible component $S$ of the strict transform. The numerical data of $S$ is automatically of the form $(N_S, 1)$, since it is a component of the strict transform. Since $\alpha(\pi) < -1$, we find that $N_S = 1$ and that $-\nu/N < -1$. Performing one more blowing up with center the intersection of $E$ and $S$ yields a new exceptional component with numerical data $(N + 1, \nu + 1)$, which still intersects an irreducible component of the strict transform. Repeating this process with finitely many steps yields an alternative tuple of resolutions $(\pi')$ with $\alpha(\pi) < \alpha(\pi') < -1$, and with moreover $\alpha(\pi')$ as close to $-1$ as one wants.

Some notation. — Let $k$ be a number field and let $f$ be a polynomial over $k$ in $n$ variables. Define $\alpha(\pi)$ for a tuple $\pi$ of embedded resolutions of polynomials $f - c$ as in the introduction. Let $\mathcal{A}_k$ be the collection of all finite field extensions of $p$-adic completions of $k$. For $K \in \mathcal{A}_k$, let $\mathcal{O}_K$ be its valuation ring with maximal ideal $\mathfrak{m}_K$ and residue field $k_K$ with $q_K$ elements, and let $\psi_K$ be any additive character\footnote{For example, one can take the standard nontrivial additive character $\psi_K : K \to \mathbb{C} : x \mapsto \exp(2\pi i (\text{Trace}_{K, \mathbb{Q}_p}(x) \mod \mathbb{Z}_p))$ for $K$ a finite field extension of $\mathbb{Q}_p$, and where $\text{Trace}_{K, \mathbb{Q}_p}$ is the trace of $K$ over $\mathbb{Q}_p$.} from $K$ to $\mathbb{C}^\times$ which is trivial on $\mathcal{O}_K$ and nontrivial on some element of order $-1$. Write $| \cdot |_K$ for the norm on $K$ and $v_K : K^\times \to \mathbb{Z}$ for the valuation.

A Schwartz-Bruhat function $K^n \to \mathbb{C}$ is a locally constant function with compact support. For $K$ in $\mathcal{A}_k$, $\varphi$ a Schwartz-Bruhat function on $K^n$, and $y \in K$, consider the exponential integral

$$E_{f, K, \varphi}(y) := \int_{K^n} \varphi(x) \psi_K(yf(x))|dx|_K,$$
with \(|dx|_K\) the normalized Haar measure on \(K^n\), and write \(E_{f,K}\) for \(E_{f,K,\varphi_{\text{triv}}}\) with \(\varphi_{\text{triv}}\) the characteristic function of \(\mathcal{O}_K^n\).

Let \(\mathcal{B}_k\) be the collection of all fields \(K\) of the form \(\mathbb{F}_q((t))\) such that \(K\) is an algebra over \(\mathcal{O}_k\). Let \(\mathcal{C}_k\) be the union of \(\mathcal{A}_k\) and \(\mathcal{B}_k\). Write \(\mathcal{C}_{k,M}\) for the set of fields in \(\mathcal{C}_k\) with residue field of characteristic \(> M\) and write similarly for \(\mathcal{A}_{k,M}\) and \(\mathcal{B}_{k,M}\). For \(K\) in \(\mathcal{B}_k\), use the notations \(\mathcal{O}_K\), \(\mathcal{M}_K\), \(\psi_K\), etc., similarly as for \(K\) in \(\mathcal{A}_k\). We can consider \(f\) as a polynomial over \(\mathcal{O}_K[1/N]\) for some large enough integer \(N\), and hence, for \(K \in \mathcal{B}_{k,N}\), the polynomial \(f\) makes sense as a polynomial over \(K\) and \(E_{f,K,\varphi}\) is defined as the corresponding integral for any Schwartz-Bruhat function \(\varphi\) on \(K^n\); for \(K \in \mathcal{B}_k\) of characteristic \(\leq N\), \(E_{f,K,\varphi}\) is defined to be zero.

The Vinogradov symbol \(\ll\) has its usual meaning, namely that for complex valued functions \(f\) and \(g\) with \(g\) taking non-negative real values \(f \ll g\) means \(|f| \leq cg\) for some constant \(c\).

By [13], Proposition 2.7, one has for any \(K\) in \(\mathcal{A}_k \cup \mathcal{B}_{k,M}\) for big enough \(M\) that
\[
|E_{f,K,\varphi}(y)| \ll |v_K(y)|^{n-1}|y|^{\alpha(\pi)}
\]
for any \(\pi\) as above, where both sides are considered as functions in \(y \in K^\times\) and \(|v_K(y)|\) denotes the real absolute value of \(v_K(y)\).

Let \(C_f\) be the closed subscheme of \(\mathbb{A}_k^n\) given by \(0 = \text{grad } f\) and let \(\delta_f\) be the dimension of \(C_f\), where we say that the empty scheme has dimension \(-\infty\). Let \(S_f\) be the closed subscheme of \(\mathbb{A}_k^n\) given by \(f = 0 = \text{grad } f\) and let \(\varepsilon_f\) be the dimension of \(S_f\). If \(f\) is a polynomial over \(\mathcal{O}_k[1/N]\) for some \(N\), then for any ring \(R\) which is an algebra over \(\mathcal{O}_k[1/N]\), denote the set of \(R\)-rational points on \(C_f\) by \(C_f(R)\), and likewise on \(S_f\). Use similar notation for polynomials over other fields than \(k\).

2. **Igusa’s local zeta function has nontrivial poles**

Following Shavarevitch, Borevitch, Igusa, and others, for \(f\) a nonconstant polynomial in \(n\) variables over \(\mathcal{O}_k\) and for \(K\) in \(\mathcal{C}_k\), consider the Poincaré series
\[
P_{f,K}(T) := \sum_{m \in \mathbb{N}} N_{f,K,m} T^m
\]
with
\[
N_{f,K,m} := \frac{1}{q_K^{mn}} \# \{ x \in (\mathcal{O}_K \mod \mathcal{M}_K^m)^n \mid f(x) \equiv 0 \mod \mathcal{M}_K^m \}.
\]

For \(K \in \mathcal{A}_k\) it is known that \(P_{f,K}(T)\) is rational in \(T\), see [14] and [8]. Also for \(K \in \mathcal{B}_{k,M}\) for \(M\) big enough, \(P_{f,K}(T)\) is rational in \(T\), shown using resolutions with good reduction [9], [10], or by insights of motivic integration, cf. [11] or [4].

\(^{(3)}\)For \(k = \mathbb{Q}\), \(K = \mathbb{Q}_p\), and \(\psi_K\) the standard nontrivial additive character, one thus has \(E_{f,K}(p^{-m}) = E_f(p^m)\).

\(^{(4)}\)It is conjectured that \(P_{f,K}(T)\) is rational for all \(K \in \mathcal{B}_k\).
2.1. Definition. — For $K$ in $C_\mathbb{Q}$ and $Q_K(T)$ a rational function over $\mathbb{Q}$ in $T$ define the nontrivial poles of $Q_K(T)$ as the poles of $Q_K(T)$ without the pole $T = q_K$ if this is a pole of multiplicity exactly 1.

2.2. Theorem. — Let $f$ be a nonconstant polynomial in $n$ variables over $k$. There exists a number $M$ such that for all $K \in C_{k,M}$, the rational function $P_{f,K}(T)$ has a nontrivial pole if and only if $S_f(k_K)$ is nonempty, with $S_f(k_K)$ as in the section on notation.

Proof. — For $M$ big enough and for $K$ in $A_{k,M}$, if $S_f(k_K)$ is nonempty, then $S_f(\mathcal{O}_K)$ is nonempty and thus $N_{f,K,m} \neq 0$ for all $m$. Hence, $P_{f,K}(T)$ has at least one pole.

In [9] it is shown that for $M$ big enough and $K \in A_{k,M}$, the degree of $P_{f,K}(T)$ in $T$ is $\leq 0$. Hence, the same holds for $K \in B_{k,M}$ with $M$ big enough. So, in the case that $P_{f,K}(T)$ has only $T = q_K$ as pole, if this pole has multiplicity 1, and if $K$ lies in $C_{k,M}$ with $M$ big enough, we can write

$$P_{f,K}(T) = (A_K + B_K T)/(1 - q_K^{-1} T),$$

for some rational numbers $A_K$ and $B_K$ depending on $K$. By developing the right hand side into series we get

$$N_{f,K,1} = q_K^{-1} A_K + B_K$$

and

$$N_{f,K,2} = q_K^{-2} A_K + q_K^{-1} B_K = q_K^{-1} N_{f,K,1}.$$

We will only use that $q_K^{-1} N_{f,K,1} = N_{f,K,2}$. Let $\bar{f}$ be the reduction of $f$ modulo $\mathcal{M}_K$. Put

$$Y_{K,f} := \{ x \in k^n_K \mid \bar{f}(x) = 0 \neq (\text{grad } \bar{f})(x) \}.$$

Suppose now that $S_f(k_K)$ is nonempty. Count points to get

$$N_{f,K,1} = \frac{1}{q_K^n}(\sharp Y_{K,f} + \sharp S_f(k_K))$$

and

$$N_{f,K,2} = \frac{1}{q_K^{n+1}} \sharp Y_{K,f} + \frac{1}{q_K^n} \sharp S_f(k_K),$$

by Taylor expansion of $f$ around points with residue in $S_f(k_K)$ resp. in $Y_{K,f}$, and since the projection

$$\{ x \in (\mathcal{O}_k/\mathcal{M}_K^2 \mathcal{O}_k)^n \mid f(x) \equiv 0 \mod \mathcal{M}_K^2 \} \rightarrow \{ x \in (\mathcal{O}_k/\mathcal{M}_K \mathcal{O}_k)^n \mid f(x) \equiv 0 \mod \mathcal{M}_K \}$$

is surjective for $K$ in $A_{k,M}$ and $M$ big enough.[5]. This gives a contradiction with $q_K^{-1} N_{f,K,1} = N_{f,K,2}$.

[5] This is so because the projection $\{ x \in (k[t]/(t^2))^n \mid f(x) \equiv 0 \mod t^2 \} \rightarrow \{ x \in k^n \mid f(x) = 0 \}$ is surjective for $k$ any field of characteristic zero. Alternatively, this follows from Greenleaf’s theorem, cf. [1].
2.3. Remark. — By an observation of Igusa, cf. [10], section 1.2, one has

\[ P_{f,K}(T) = \frac{1 - TZ_{f,K}(s)}{1 - T}, \]

with \( T = q_K^{-s} \) and \( Z_{f,K}(s) \) Igusa’s local zeta function defined for real \( s > 0 \) by

\[ Z_{f,K}(s) := \int_{O_K^n} |f(x)|^s |dx|, \]

with \( |dx| \) the normalized Haar measure on \( O_K^n \). Since \( Z_{f,K}(0) = 1 \), \( Z_{f,K}(s) \) and \( P_{f,K}(T) \) have exactly the same poles in \( T = q_K^{-s} \) and thus Theorem 2.2 also applies to \( Z_{f,K}(s) \).

3. Definition and basic properties of the oscillation indices

3.1. — Let \( f \) be a polynomial in \( n \) variables over a number field \( k \), let \( K \) be in \( C_k \), and let \( \varphi : K^n \to \mathbb{C} \) be a Schwartz-Bruhat function. Define \( \alpha_{f,K,\varphi} \) as the infimum in \( \mathbb{R} \cup \{-\infty\} \) of all real numbers \( \alpha \) satisfying

\[ E_{f,K,\varphi}(y) \ll |y|^\alpha_K, \]

where both sides are considered as functions in \( y \in K^\times \). Note that \( \alpha_{f,K,\varphi} \leq 0 \) because \( \alpha = 0 \) always satisfies the previous Vinogradov inequality. Define \( \alpha_{f,K} \) as the supremum of the \( \alpha_{f,K,\varphi} \) when \( \varphi \) varies over all Schwartz-Bruhat functions on \( K^n \). Call \( \alpha_{f,K} \) the \( K \)-oscillation index of \( f \).

3.2. Definition. — Define the motivic oscillation index of \( f \) (over \( k \)) as

\[ \alpha_f := \lim_{M \to +\infty} \sup_{K \in \mathcal{A}_{k,M}} \alpha_{f,K}. \]

3.3. Lemma. — For \( K \) in \( \mathcal{A}_k \cup \mathcal{B}_{k,M} \) with \( M \) big enough and \( \varphi \) a Schwartz-Bruhat function on \( K^n \), the number \( \alpha_{f,K,\varphi} \) equals \( -\infty \) or one of the finitely many quotients \(-\nu_i/N_i \) with \((N_i, \nu_i)\) the essential numerical data of the tuple of resolutions \( \pi = (\pi_{f,-})_c \) chosen as in the introduction.

Proof. — This follows from Proposition 2.7 of [13], cf. (3.5.1) below.

3.4. Corollary. — For all \( K \) in \( \mathcal{A}_k \cup \mathcal{B}_{k,M} \) with \( M \) big enough there exists \( \varphi \) such that

\[ \alpha_{f,K,\varphi} = \alpha_{f,K}, \]

and for all \( N \) there exists \( K \) in \( \mathcal{A}_{k,N} \) and \( L \) in \( \mathcal{B}_{k,N} \) such that

\[ \alpha_f = \alpha_{f,K} = \alpha_{f,L}. \]

Moreover, the following list of a priori inequalities holds

\[ -\infty \leq \alpha_{f,K,\varphi} \leq \alpha_{f,K} \leq \alpha_f \leq \alpha(\pi) \leq 0 \]

for big \( M \), any \( K \) in \( \mathcal{C}_{k,M} \) and any Schwartz-Bruhat function \( \varphi \) on \( K^n \).
Proof. — The existence of $L$ follows from the motivic understanding of exponential integrals over $K^n$ for $K$ varying in $C_{k,M}$ for $M$ big enough, as established in [4] (equivalently, one can use [10]). The inequality $\alpha_f \leq \alpha(\pi)$ follows from (1.2.1). □

Although the $\alpha_{f,K,\varphi}$ are defined as infima, they yield actual bounds, analogous to (1.0.1) and (1.2.1):

3.5. Lemma. — For $K$ in $A_k \cup B_{k,M}$ for sufficiently big $M$ and $\varphi : K^n \to \mathbb{C}$ a Schwartz-Bruhat function, either

$$\alpha_{f,K,\varphi} = -\infty$$

and $E_{f,K,\varphi}(y) = 0$ for all $y$ with $|y|_K$ sufficiently big,

or,

$$\alpha_{f,K,\varphi} > -\infty$$

and $E_{f,K,\varphi}(y) \ll |v_K(y)|^{n-1}|y|_K^{\alpha_{f,K,\varphi}}$ as functions in $y \in K^\times$,

where $|v_K(y)|$ denotes the real absolute value of $v_K(y)$.

Proof. — The statements for $K$ in $A_k$ are immediate corollaries of the asymptotic expansions of $E_{f,K,\varphi}(y)$ given by Igusa [14] and generalized by Denef and Veys in [13], for $|y|$ going to $\infty$. Namely, by [13], Proposition 2.7, for $K \in A_k$ and for $y$ with $|y|$ big enough, one can write $E_{f,K,\varphi}(y)$ as a finite $C$-linear combination of terms of the form

\begin{equation}
\chi(y)|y|^\lambda v_K(y)\beta \psi(cy)
\end{equation}

with $c \in K$ a critical value of $f$, $\chi$ a character with finite image of $K^\times$ into the complex unit circle, $\lambda$ a complex number with negative real part, and $\beta$ an integer between 0 and $n - 1$, where real parts of the occurring $\lambda$ are among the quotients $-\nu_i/N_i$ with $(N_i, \nu_i)$ the essential numerical data of some $\pi$ as above. From this the statements follow for $K \in A_k$.

For $K$ in $B_{k,M}$ one then uses the transfer principle of [4]. Namely, by this transfer principle it follows that for $M$ big enough and for $K \in B_{k,M}$, $E_{f,K,\varphi}(y)$ can be written as a similar linear combination as given higher up in the proof for $K$ in $A_k$. (The statement about $B_{k,M}$ also follows from [10] and the existence of embedded resolutions of $f = 0$ with good reduction modulo $p$ for $p$ any prime bigger than some $M$, see also [9].) □

The following Proposition essentially follows from Theorem 2.2 and [13], with $C_f$ as in the section on notation.

3.6. Proposition (Nontriviality of $\alpha_f$). —

(i) For $K$ in $A_k$ or in $B_{k,M}$ for $M$ big enough, $\alpha_{f,K} = 0$ if and only if $f$ is a constant.

(ii) If $f$ is not a constant, then there exists $M > 0$ such that, for $K$ in $C_{k,M}$,

$$\alpha_{f,K,\varphi_{\text{equiv}}} = -\infty$$

if and only if $C_f(k_K)$ is empty.

By consequence, $-\infty < \alpha_f < 0$ if and only if $f$ is nonconstant and $\text{grad } f = 0$ has a solution in $\mathbb{C}^n$. 

Proof. — The first statement for $K$ in $A_k$ follows from (1.2.1), (3.4.1) and the definition of $\alpha(\pi)$ in the introduction. The second statement follows from Theorem 2.2 applied to some $K$ and one of the polynomials $f - c$ with $c \in \mathbb{C}$ a critical value of $f$ with $c \in K$ and from the fact that any nontrivial pole of $P_{f,K}(T)$ effectively appears as $\lambda$ in one of the terms (3.5.1) when writing $E_{f,K}(y)$ as a linear combination of terms as (3.5.1), by Proposition 2.7 of [13].

4. A lemma on parameterized integrals

Let $\mathcal{L}_{DP}$ be the language of Denef-Pas, consisting of three sorts: valued field, residue field and value group, and the language of fields for the residue field, the language of fields for the valued field, the language of ordered additive groups for the value group, and the extra symbols $\overline{ac}$ and $\text{ord}$.

Let $k$ be a number field. Consider the language $\mathcal{L}_{DP}(k)$, which is the language $\mathcal{L}_{DP}$ together with coefficients for $k$ in the valued and residue field sort. On a field $K \in \mathcal{A}_k$, the symbols of $\mathcal{L}_{DP}(k)$ have their natural meaning, where $\overline{ac}(x)$ for nonzero $x \in K$ is interpreted as $x\pi_K^{-\text{ord}x} \mod M_K$ in the residue field $k_K$, for some (consequent) choice of uniformizer $\pi_K$ of $\mathcal{O}_K$ and where $\overline{ac}(0) = 0$. Note that the language $\mathcal{L}_{DP}$ can be naturally interpreted in $\mathcal{L}((t))$ for any field $L$ in a similar way.

Then, by quantifier elimination results by Pas [16], for any given $\mathcal{L}_{DP}(k)$-formula $\phi$ there exists a $\mathcal{L}_{DP}(k)$-formula $\phi'$ without valued field quantifiers such that for all $K \in \mathcal{A}_{k,M}$ with $M$ big enough, $\phi$ and $\phi'$ determine the same set over $K$.

By a basic step function on $K^n$ with $K$ in $\mathcal{A}_k$ we mean a characteristic function of a Cartesian product of factors of the form $\mathcal{O}_K$ and $\mathfrak{a} + M_K$ for $\mathfrak{a} \in \mathcal{O}_K$. For example, the characteristic function of $\mathcal{O}_K \times \mathcal{M}_K \times (1 + \mathcal{M}_K)^{n-2}$ is a basic step function on $K^n$.

The goal of this section is to prove the following lemma.

4.1. Lemma. — Let $k$ be a number field and let $f$ be a polynomial in the variables $x_1, \ldots, x_n$ over $k$. Write $\hat{x}$ for $x_2, \ldots, x_n$. Then there are $M > 0$ and a polynomial $g$ in one variable over $k$ such that for all $a \in k$ with $g(a) \neq 0$, all $K \in \mathcal{A}_{k,M}$, and all basic step functions $\phi$ on $K^{n-1}$, one has that

$$I_K(b, s, \phi) := \int_{\mathcal{O}_K^{n-1}} \phi(\hat{x})|f(b, \hat{x})|^s|d\hat{x}|,$$

with $s > 0$ a real variable and $|d\hat{x}|$ the normalized Haar measure, is independent of $b \in a + \mathcal{M}_K$.

Proof. — We give a proof based on motivic integration and constructible functions as developed in [5]. By [5] (or already with some work by Pas [16]), there are $\mathcal{L}_{DP}(k)$-formulas such that their interpretation in $K$ with $K$ a field in $\mathcal{A}_{k,M}$ with $M$ big enough yields functions

$$\alpha_i(K), \beta_i(K) : K \to \mathbb{Z}$$
and definable subsets

\[ A_i(K) \subset K \times k_K^{(n-1)+\ell} \]

of \( K \) times a power of the residue field of \( K \) such that the following holds for \( K \) in \( A_{k,M} \) with \( M \) big enough: with \( \varphi_z \) the characteristic function of the product of the residue classes mod \( \mathcal{M}_K \) of \( z = (z_2, \ldots, z_n) \in k_K^{n-1} \), one has that \( I_K(b, s, \varphi_z) \) for \( b \in K \) is a \( \mathbb{Q} \)-linear combination of products of factors of the form

\[ q_K^{\alpha_i(K)s + \beta_i(K)}, \]

\[ \frac{1}{1 - q_K^{a_i+b}}, \]

and

\[ \beta_i(K), \]

with integers \( (a, b) \neq (0, 0) \). We show that the \( a_{iz}(K), \alpha_i(K), \) and \( \beta_i(K) \), are constant on \( a + \mathcal{M}_K \) for all \( a \) with \( g(a) \neq 0 \) for some polynomial \( g \) over \( k \), uniformly in \( K \) in \( A_{k,M} \). But this follows at once from Lemma 4.4. For basic step functions which also involve the characteristic function of Cartesian products with \( \mathcal{O}_K \) one works similarly, say, with some more (witnessing) \( z \)-variables. The Lemma then follows.

**4.2. Definition (One dimensional Cells).** — Let \( L \) be any field of characteristic zero. Let \( X \subset L((t)) \) be a \( \mathcal{L}_{DP}(k) \)-definable set and

\[ f : X \to S \]

\[ c : S \to L((t)) \]

\( \mathcal{L}_{DP}(k) \)-definable functions, with \( S \) a Cartesian product of some power of \( L \) and some power of \( \mathbb{Z} \). The tuple \( (X, f, c) \) is called a \((1)\)-cell with presentation \( f \) and center \( c \) when for each \( s \in f(X) \), there exists \( \xi \in L^x \) and \( a \in \mathbb{Z} \) such that the fiber \( f^{-1}(s) \) is an open ball of the form

\[ \{ x \in L((t)) \mid \pi(x - c(s)) = \xi \land \text{ord}(x - c(s)) = a \}. \]

The tuple \( (X, f, c) \) is called a \((0)\)-cell with presentation \( f \) and center \( c \) when \( c \circ f \) is the identity on \( X \).

**4.3. Lemma ([16], [3], One Dimensional Cell decomposition)**

Let \( k \) be a number field and \( L \) a field over \( k \). Let \( g_j \) be polynomials in 1 variable over \( k \). Then \( L((t)) \) can be written as the disjoint union of finitely many \( \mathcal{L}_{DP}(k) \)-definable cells \( X_i \) with \( \mathcal{L}_{DP}(k) \)-definable presentations \( f_i \) and \( \mathcal{L}_{DP}(k) \)-definable centers \( c_i \) such that for each \( i, j \) the restrictions \( \text{ord}_{g_j|_{X_i}} \) and \( \pi_{g_j|_{X_i}} \) factorize through \( f_i \). Moreover, the formulas describing the \( X_i, f_i, \) and \( c_i \) can be taken independently of \( L \).
Proof. — This form of Denef - Pas cell decomposition is proven in [3] with \( k = \mathbb{Q} \), where it is a relative cell decomposition relative to \( \mathbb{Z}^l \times L^l \) under the map \((\operatorname{ord}(g_j), \operatorname{ac}(g_j))_j\). The Lemma for general \( k \) follows easily from it by first replacing the parameters from \( k \) by variables, then applying the form of [3] to all the variables, and then plugging in the parameters for the variables again.

4.4. Corollary. — Let the functions \( a_{iz}(K), \alpha_i(K), \) and \( \beta_i(K) \) be as in the proof of Lemma 4.1. Then there are \( M > 0 \) and a polynomial \( g \) in one variable over \( k \) such that for all \( a \in k \) with \( g(a) \neq 0 \), all \( K \in A_{k,M} \) and all \( z \in k^{n-1}_K \) one has that these functions \( a_{iz}(K), \alpha(K), \) and \( \beta(K) \) are constant functions in the variable \( b \) when \( b \) runs over \( a + M^K \).

Proof. — Let \( g_\ell \) be the polynomials in one valued field variable that occur in the valued field quantifier free formulas that describe the \( \alpha_i, \beta_i, \) and \( A_i \). Apply Lemma 4.3 for the polynomials \( g_\ell \) to find cells \( X_j \) with centers \( c_j \) and presentations \( f_j \). By quantifier elimination in the language \( L_{DP}(k) \) and by examining valued field quantifier free formulas in one free valued field variables, it follows that the images of the centers \( c_j \) are contained in the zero set of a polynomial \( g \) over \( k \). Now the corollary follows from the definition of cells and the evident ultraproduct argument to go to \( K \) in \( A_{k,M} \) for \( M \) big enough. Namely, for \( a \in k \) with \( g(a) \neq 0 \), all \( K \in A_{k,M} \) for \( M \) big, the set \( a + M_K \) is contained in \( X_j(K) \) for some unique \( j \), and \( f_j(K)(a + M_K) \) is a singleton, where \( X_j(K) \) and \( f_j(K) \) are the \( K \)-interpretations of \( X_j \) and \( f_j \).

5. Estimation of the motivic oscillation index

Let \( f \) be a polynomial in \( n \) variables over a number field \( k \). Let \( C_f, S_f, \delta_f, \) and \( \varepsilon_f \) be as in the section on notation. Let \( \alpha_f \) be the motivic oscillation index as before.

5.1. Theorem. — Let \( f \) be a polynomial in \( n \) variables over a number field \( k \). Then

\[
\frac{-n + \delta_f}{2} \leq \alpha_f.
\]

Proof. — Since one can always replace \( k \) by a finite field extension and \( f \) by \( f - a \) for some critical value \( a \in k \), it is clear that it is enough to prove that

\[
\frac{-n + \varepsilon_f}{2} \leq \alpha_f.
\]

We may suppose that \( \varepsilon_f \geq 0 \) since otherwise \( \varepsilon_f = -\infty \) and then (5.1.2) is trivial. We then prove the following statement by induction on \( n \) when \( \varepsilon_f \geq 0 \).

For each \( M \) there exist \( L \) in \( A_{k,M} \), a basic step function \( \varphi_L \) on \( L^n \) (in the sense of section [3]), a real number \( s_0 \) with

\[
\frac{-n + \varepsilon_f}{2} \leq s_0 \leq 0,
\]
and a complex number \( t_0 \) with real part \( q_L^{-s_0} \) such that \( t_0 \) is a nontrivial pole (in the sense of Definition \( \text{(2.1)} \)) of the rational function in \( T = q_L^{-s} \) defined for \( s > 0 \) by

\[
Z_{f,L,\varphi_L}(s) := \int_{L^n} \varphi_L(x)|f(x)|^s dx.
\]

This statement implies \( \text{(5.1.2)} \) (since a nontrivial pole of \( Z_{f,L,\varphi_L}(s) \) corresponds to an effectively appearing \( \lambda \) in \( \text{(5.1.1)} \) by \( \text{[13]} \), Proposition 2.7).

The case that \( n = 1 \) is easy by explicit calculation using factorization of \( f \) into irreducible polynomials over the algebraic closure of \( k \).

For \( n > 1 \), first suppose that \( \varepsilon_f = 0 \). Let \( M \) be big enough and choose \( L \) in \( \mathcal{A}_{k,M} \) such that \( S_f(k_L) \) is nonempty. By Theorem \( \text{[2.2]} \) and Remark \( \text{[2.3]} \) it follows that \( Z_{f,L,\varphi_{\text{triv}}}(s) \) has a nontrivial pole in \( q_L^{-s} \), with \( \varphi_{\text{triv}} \) the basic step function which is the characteristic function of \( \mathcal{O}_k^n \). Since each complex pole \( t_0 \) of \( Z_{f,L,\varphi_{\text{triv}}}(s) \) with real part \( q_L^{-s_0} \) satisfies \( -n/2 \leq s_0 \leq 0 \) when \( n > 1 \) by results by Segers \( \text{[18]} \), the case that \( \varepsilon_f = 0 \) and \( n > 1 \) follows.

Now we treat the case \( n > 1 \) and \( \varepsilon_f > 0 \). There is a coordinate, say \( x_1 \), such that for infinitely many points \( a \) in the algebraic closure of \( k \), \( \varepsilon_{f_a} \geq \varepsilon_f - 1 \), with \( f_a \) the polynomial in \( \hat{x} := (x_2, \ldots, x_n) \) over \( k(a) \) defined by \( f_a(\hat{x}) = f(a, \hat{x}) \). Indeed, since \( \varepsilon_f > 0 \) there is a coordinate, say \( x_1 \), to which an irreducible component of \( S_f \) of dimension \( \varepsilon_f \) projects dominantly, and then one compares the equations defining \( S_f \) and the \( S_{f_a} \). Let \( g \) be a polynomial in one variable as in Lemma \( \text{[4.1]} \) for the polynomial \( f \). Then, by replacing \( k \) by a bigger field extension, we can suppose that there is \( c \in k \) such that \( g(c) \neq 0 \) and such that \( \varepsilon_{f_c} \geq \varepsilon_f - 1 \). Note that \( f_c \) is a polynomial in \( n - 1 \) variables. By the induction hypothesis, there exist \( L \) in \( \mathcal{A}_{k,M} \), a basic step function \( \psi_L \) on \( L^{n-1} \) and a real number \( s_0 \) with

\[
\frac{-n + \varepsilon_f}{2} = \frac{-(n-1) + (\varepsilon_f - 1)}{2} \leq \frac{-(n-1) + \varepsilon_{f_c}}{2} \leq s_0 \leq 0
\]

and a complex number with real part \( q_L^{-s_0} \) which is a nontrivial pole of \( Z_{f_c,L,\psi_L}(s) \). Now let \( \varphi_L \) be the basic step function on \( L^n \) which is the product of the characteristic function of \( c + \mathcal{M}_L \) with \( \psi_L \). Then \( t_0 \) is also a nontrivial pole of \( Z_{f,L,\varphi_L}(s) \), since \( Z_{f,L,\varphi_L}(s) \) is a product integral by the above application of Lemma \( \text{[4.1]} \) to \( f \) and since \( g(c) \neq 0 \). Namely, \( Z_{f,L,\varphi_L}(s) = q_L^{-1}Z_{f_c,L,\psi_L}(s) \). This proves the above statement and thus the theorem.

\section{5.2. Remark.} — A similar but refined proof than that of Theorem \( \text{[5.1]} \) might give the stronger result

\[
\text{(5.2.1)} \quad \frac{-n + \delta_f(K)}{2} \leq \alpha_{f,K}
\]

for all \( K \in \mathcal{C}_{k,M} \) with \( M \) sufficiently big and \( \delta_f(K) \) the Zariski dimension of the Zariski closure of \( C_f(K) \) in \( \mathbb{A}_{k}^n \). Note that this implies \( \text{(5.1.1)} \) since there are enough \( K \) with \( \delta_f(K) = \delta_f \).
6. Igusa’s conjecture modulo $p$ and modulo $p^2$

6.1. Theorem. — Let $f$ be a homogeneous polynomial in $n$ variables over the number field $k$. Then there exists $M > 0$ and a constant $c$ such that for each $K$ in $C_{k,M}$ and each $y \in K$ with $v_K(y) = -1$ or $v_K(y) = -2$ one has

$$|E_{f,K}(y)| < c |y|_{K}^{\alpha_f}.$$  

Moreover, the statement for $y$ with $v_K(y) = -2$ holds for all polynomials over $k$.

Proof. — Let $\delta_f$ be as in the section on notation. First note that for homogeneous $f$ one has $\delta_f = -\infty$ if and only if $f$ is linear, and, in the linear case one has $\alpha_f = -\infty$ and $E_{f,K}(y) = 0$ for all $K \in C_{k,M}$ for $M$ big enough and all $y \in K$ with $v_K(y) < 0$.

The case that $f$ is a constant is trivial, cf. Proposition 3.6.

Suppose now that the degree of $f$ is $\geq 2$. Thus $n > \delta_f > -\infty$. Let $H_f$ be the (not necessarily reduced) projectivization in $P^n_k$ of the closed subscheme of $A^n_k$ given by $f = 0$, and let $\delta$ be the (projective) dimension of the singular locus of the intersection of $H_f$ with the hyperplane at infinity. Then

$$\delta + 1 = \delta_f.$$  

Then, by Katz [15], Theorem 4, and by (6.1.2), there exist constants $C$ and $M$ such that for all $K$ in $C_{k,M}$ and all $y$ in $K$ with $v_K(y) = -1$ one has

$$|E_{f,K}(y)| < C q_K^{-n+\delta_f},$$  

with $q_K$ the number of elements of the residue field of $K$. Together with (5.1.1) of Theorem 5.1 this implies the statement for $y$ with $v_K(y) = -1$.

Now let $f$ be a general polynomial over $k$ in $n$ variables. For $K \in C_{k,M}$ with $M$ big enough, define

$$A_f(K) := \{ x \in O^n_K \mid \text{grad } f(x) \equiv 0 \mod M_K \}$$  

and

$$B_f(K) := \{ x \in O^n_K \mid \text{grad } f(x) \not\equiv 0 \mod M_K \}.$$  

By Noether’s Normalization Theorem, the number of elements of the set $C_f(k_K)$ is $\leq D q_K^{\delta_f}$ for some $D$ independent of $q_K$ when the characteristic of $k_K$ is big enough. Hence, there exists some $M$ and $D$ such that for all $K$ in $A_{k,M}$ the measure of $A_f(K)$ against the normalized Haar measure $|dx|$ satisfies

$$\int_{A_f(K)} |dx| \leq D q_K^{-n+\delta_f}.$$  

On the other hand, for $M$ big enough, $K$ in $A_{k,M}$, and $y \in K$ with $v_K(y) = -2$,

$$0 = \int_{B_f(K)} \psi_K(y f(x)) |dx|_K,$$  

by Hensel’s Lemma and the fact that the sum

$$\sum_{z \in F_q} \psi_q(z)$$
equals zero for any nontrivial additive character $\psi_q$ on $F_q$. Hence, for such $M$ and $K$, and for all $y$ in $K$ with $v_K(y) = -2$,

$$|E_{f,K}(y)| = \left| \int_{A_f(K)} \psi_K(yf(x))|dx|_K + \int_{B_f(K)} \psi_K(yf(x))|dx|_K \right|$$

$$= \left| \int_{A_f(K)} \psi_K(yf(x))|dx|_K \right|$$

$$\leq \int_{A_f(K)} |dx|_K$$

$$\leq Dq_K^{n+\delta_f}$$

(6.1.4)

where $D$ is independent of $K$ and where inequality (6.1.4) follows from $|y| = q_K^2$ and (5.1.1).

6.2. Corollary. — Igusa’s Conjecture 1.1 with argument of order $-1$ or $-2$ holds for all homogeneous polynomials $f$ over $k$. Namely, for $f$ a homogeneous polynomial in $n$ variables over $k$ and $\pi = \pi_f$ a resolution as in the introduction, there exists a constant $c$ such that for each $p$-adic completion $K$ of $k$ and each $y \in K$ with $v_K(y) = -1$ or $v_K(y) = -2$ one has

$$|E_{f,K}(y)| < c |y|^{\alpha_f}.$$  

Proof. — This follows from Theorem 6.1 and (3.4.1) for big residue characteristics, and from (1.2.1) for small residue characteristics. 

6.3. Remark. — The factor $|v_K(y)|^{n-1}$ featuring in the original conjecture clearly becomes unnecessary when one focuses on $y$ with $v_K(y) = -1$ and $v_K(y) = -2$. When $v_K(y)$ varies over $\mathbb{Z}$, the factor $|v_K(y)|^{n-1}$ can in general not be omitted.

7. Nonhomogeneous polynomials

Let $f$ be a polynomial in $n$ variables over a number field $k$. Let $H_f$ be the (not necessarily reduced) projectivization in $\mathbb{P}_k^n$ of the scheme $f = 0$ in $\mathbb{A}_k^n$. Let $X_f$ be the scheme-theoretic intersection of $H_f$ with the hyperplane at infinity. Let $\delta$ be the dimension of the singular locus of $X_f$ when this singular locus is nonempty, and put $\delta = -1$ when $X_f$ is smooth. We say that $f$ satisfies Tameness Condition (7.0.1) if

$$\delta_f \geq \delta + 1.$$  

Note that by (6.1.2) a homogeneous polynomial of degree $\geq 2$ always satisfies Tameness Condition (7.0.1). Under this Tameness Condition we get the analogue of Theorem 6.1

7.1. Theorem. — Let $f$ be a polynomial in $n$ variables over a number field $k$. Suppose that $f$ satisfies Tameness Condition (7.0.1). Then there exists a number
and a constant $c$ such that for each $K \in \mathcal{C}_{k,M}$ and each $y \in K$ with $v_K(y) = -1$ one has

$$|E_{f,K}(y)| < c |y|^{\alpha_f}, \quad (7.1.1)$$

**Proof.** — Same proof as for Theorem 6.1, using $\delta + 1$ instead of $\delta_f$ in (6.1.3), which holds by the same Theorem 4 of [15]. \hfill $\square$

**7.2. Example.** — For $f(x_1, x_2) = x_1^2x_2 - x_1$, one has $\alpha(\pi) = \alpha_f = -\infty$, but $E_{f,K}(y) \neq 0$ for any $K \in \mathcal{C}_{\mathcal{Q}^2}$ when $v_K(y) = -1$. Hence, a literal analogue of Conjecture [11] would not make sense for general polynomials. Note that $f$ does not satisfy Tameness Condition (7.0.1).

**8. Igusa’s conjecture revisited**

We come back to the sums $E_f(N)$ from the introduction and study the growth of $|E_f(N)|$ with $N$. As noted before, it is sufficient to bound $|E_f(p^m)|$ in terms of primes $p$ and positive integers $m$. We focus on $m > 1$ since the case $m = 1$ is rather well understood by work by Deligne, Laumon, Katz, and others.

In general one has the following situation which follows from [10] or [4]. Let $f$ be any polynomial over $\mathcal{Q}$ in $n$ variables. Then there exist constants $\beta, \gamma \geq 0$ such that for all big enough primes $p$ and all $m \geq \gamma$, one has

$$|E_f(p^m)| < p^\beta m^{n-1} p^{\alpha_f/m}, \quad (8.0.1)$$

with $\alpha_f$ the motivic oscillation index.

A natural sharpening of Igusa’s Conjecture [11] is the conjecture that one can take $\beta = \gamma = 0$ in (8.0.1) when $f$ is homogeneous. (Conjecture [11] follows from this sharpening by (1.2.1) and by Corollary 3.4.) An alternative sharpening of Igusa’s conjecture by Denef and Sperber in [12] is based on complex oscillation indices and might be equivalent with our sharpening (that is, the supremum of the complex oscillation indices of $f$ around each of the points of $\mathcal{C}^n$ might equal $\alpha_f$).

We introduce the following mysterious invariant of $f$ which might play a role in understanding $E_f(p^m)$ for general $f$:

**8.1. Definition.** — Let $f$ be a polynomial in $n$ variables over a number field $k$. Define $\beta_f$ as the limit over $M$ and $\gamma$ of the infima of all real $\beta \geq 0$ such that

$$|E_{f,K}(y)| < q_K^\beta |v_K(y)|^{n-1} |y|^{\alpha_f}_K$$

for all $y \in K$ with $v_K(y) < -\gamma$ and all $K \in \mathcal{A}_{k,M}$. Call $\beta_f$ the flaw of $f$ w.r.t. $\alpha_f$.

For non-homogeneous $f$, $\beta_f$ is not even conjecturally understood.

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References

[1] B. J. Birch and K. McCann, *A criterion for the *p*-adic solubility of diophantine equations*, Q. J. Math. 18 (1967), no. 1, 59–63.

[2] R. Cluckers, *Igusa - Denef - Sperber conjecture on nondegenerate *p*-adic exponential sums*, Accepted if revised for publication in Duke Math. J., arXiv:math.NT/0606269.

[3] R. Cluckers and F. Loeser, *b-minimality*, Submitted to Journal of Mathematical Logic, arXiv:math.LO/0610183.

[4] ———, *Constructible exponential functions, motivic Fourier transform and transfer principle*, Submitted, arXiv:math.AG/0512022.

[5] ———, *Constructible motivic functions and motivic integration*, Submitted, arxiv:math.AG/0410203.

[6] ———, *Fonctions constructibles exponentielles, transformation de Fourier motivique et principe de transfert*, Comptes Rendus Mathématique. Académie des Sciences 341 (2005), no. 12, 741–746, arXiv:math.NT/0509723.

[7] P. Deligne, *La conjecture de Weil. I. (French) [Weil’s conjecture. I]*, Inst. Hautes Études Sci. Publ. Math. 43 (1974), 273–307.

[8] J. Denef, *The rationality of the Poincaré series associated to the *p*-adic points on a variety*, Inventiones Mathematicae 77 (1984), 1–23.

[9] ———, *On the degree of Igusa’s local zeta function*, American Journal of Mathematics 109 (1987), 991–1008.

[10] ———, *Report on Igusa’s local zeta function*, Séminaire Bourbaki Vol. 1990/91, Exp. No.730-744 (1991), 359–386, Astérisque 201-203, http://wis.kuleuven.be/algebra/denef.html.$\oplus$D2.

[11] J. Denef and F. Loeser, *Definable sets, motives and *p*-adic integrals*, Journal of the American Mathematical Society 14 (2001), no. 2, 429–469.

[12] J. Denef and S. Sperber, *Exponential sums mod *p*^n and Newton polyhedra*, Bull. Belg. Math. Soc. Simon Stevin suppl. (2001), 55–63.

[13] J. Denef and W. Veys, *On the holomorphy conjecture for Igusa’s local zeta function*, Proc. Amer. Math. Soc. 123 (1995), no. 10, 2981–2988.

[14] J. Igusa, *Lectures on forms of higher degree (notes by S. Raghavan)*, Lectures on mathematics and physics, Tata institute of fundamental research, vol. 59, Springer-Verlag, 1978.

[15] N. Katz, *Estimates for “singular” exponential sums*, Internat. Math. Res. Notices (1999), no. 16, 875–899.

[16] J. Pas, *Uniform *p*-adic cell decomposition and local zeta functions*, Journal für die reine und angewandte Mathematik 399 (1989), 137–172.

[17] D. Segers and W. Veys, *Private communication*, (2006), Leuven.

[18] Dirk Segers, *Lower bound for the poles of Igusa’s *p*-adic zeta functions*, Math. Ann. 336 (2006), no. 3, 659–669.