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Donghun Park (박동훈 ), Jaeyoung Park (박재영 ), Minwoo Kim (김민우 ), Jiseop Lim (임지섭 ), Seungtae Kim (김승태 ), and Solkeun Jee (지솔근 )

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Influence of initial phase on subharmonic resonance in an incompressible boundary layer

Donghun Park (朴东勋), Jaeyoung Park (朴재영), Minwoo Kim (김민우), Jiseop Lim (임지섭), Seungtae Kim (김승태), and Solkeun Jee (지솔근)

AFFILIATIONS
1Department of Aerospace Engineering, Pusan National University, Busan 46241, South Korea
2School of Mechanical Engineering, Gwangju Institute of Science and Technology, Gwangju 61005, South Korea

ABSTRACT
The influence of the initial phase of fundamental and subharmonic waves on subharmonic resonance is investigated for an incompressible boundary layer with zero and adverse pressure gradients. Parabolized stability equation analyses are carried out for various combinations of the initial phases of fundamental and subharmonic waves. The amplification of subharmonic and higher modes is found to depend strongly on the initial phases, and this dependence is consistent with observations from previous experimental studies. There exists a certain combination of initial phases that leads to resonance or anti-resonance condition (i.e., maximum or minimum growth, respectively). For all combinations of the initial phases, the phase dependence appears to be a function of a single parameter that represents the initial phase difference between the fundamental and subharmonic waves. The amplification in the subharmonic resonant interaction depends on the initial phase difference rather than the individual initial phase of the fundamental or subharmonic wave. In the downstream direction, the phase difference changes from the initial value and eventually converges to a specific value approximately ranging from 80° to 90°, regardless of the initial phase difference. This transient behavior does not start until the subharmonic wave enters the parametric resonant stage, which yields double-exponential growth. The qualitative characteristic of the phase dependence remains unchanged for the fundamental frequencies and spanwise wavenumbers as well as for the pressure gradients studied. The method of analysis and results contribute to the physical foundations of controlling boundary-layer transition dominated by the subharmonic resonance.

I. INTRODUCTION
The boundary-layer transition is directly related to the aerothermodynamic performance of flight vehicles, as it causes considerable changes in skin-friction and heat-transfer characteristics. Physical understanding of transition phenomena and the associated mechanisms is an essential prerequisite for improving the performance and design. A typical process of the natural transition consists of receptivity, linear amplification of instability waves, nonlinear evolution of instabilities or secondary instability, and final breakdown.5,7 The transition process can be classified into several types depending on the physical mechanism that dominates the nonlinear stage and final breakdown process. Thus, studying the nonlinear stage can provide useful information for the physical understanding of transition phenomena.

Several experimental, theoretical, and numerical studies have been conducted to investigate the nonlinear stage of the transition that leads to breakdown through the interaction of instability waves. As described by Borodulin et al., some resonant interactions of instability modes usually dominate the initial nonlinear stage.5 In early research, the importance of resonant wave interactions was recognized in theoretical studies, including those of the theory of secondary instability.6,8 Craik9 developed the theory of resonant interaction of three waves in boundary-layer flows on the basis of a weakly nonlinear approach. Craik's wave triad consists of a two-dimensional (2D) instability wave, nonlinear evolution of instabilities or secondary instability, and final breakdown.5,7 The transition can be classified into several types depending on the physical mechanism that dominates the nonlinear stage and final breakdown process. Thus, studying the nonlinear stage can provide useful information for the physical understanding of transition phenomena.

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low-speed boundary layers. The fundamental breakdown is a process that is dominated by the interaction of 2D and 3D waves of the same frequency. By contrast, the subharmonic breakdown is a process that is initiated by the nonlinear interaction of the 2D wave of the fundamental frequency and 3D waves of a subharmonic frequency. For low-speed boundary layers, the subharmonic breakdown has attracted considerable research interest, because it can be regarded as a possible transition scenario in a low-disturbance environment. More recently, the fundamental breakdown also has attracted research interest again as an important mechanism for the transition of high-speed boundary layers at hypersonic Mach numbers.10

After the first experimental observation made by Kachanov et al., many experimental studies have been conducted to investigate the subharmonic resonant interaction in boundary layers. Theoretical studies based on the Floquet theory and weakly nonlinear theory have provided suitable explanations for the experimental observations. All theoretical approaches have predicted the possible rapid growth of oblique subharmonic waves. Numerical studies using direct numerical simulation (DNS) and the resonant interaction including the post-parametric stage. The nonlinear instability of boundary layers continues to be studied using theoretical, numerical, and experimental approaches. Borodulin et al. thoroughly reviewed theoretical, numerical, and experimental studies related to subharmonic resonance; hence, previous studies are not reviewed here. Instead, the typical characteristics and aspects of subharmonic resonance revealed by previous studies are briefly summarized as follows for introductory purposes.

The “natural” subharmonic disturbances, which are selected and amplified from background fluctuations by resonance, consist of a pair of oblique instability waves. Phase synchronism conditions (i.e., \( \omega_1 = \omega_2 \) and \( \chi_2 = \chi_1 \)) are required for the resonant amplification of these subharmonics. The subharmonic (parametric) resonant interaction of quasi-subharmonic 3D disturbances with the 2D fundamental wave results in the amplification of a broad continuous spectrum of low frequency. The resonance occurs in a wide frequency spectrum; hence, quasi-subharmonic waves with frequencies of \( \omega' = \omega_{12} + \Delta \omega \) can amplify even with detuning \( (\Delta \omega) \) greater than half of the subharmonic frequency. The resonance is also found to occur for the large width of the wavenumber spectrum and the propagation angle of the most amplified subharmonic modes is found to depend on the fundamental wave amplitude. It is well known that subharmonic resonances exhibit linear characteristics in their parametric stage. Thus, there is no back-influence of subharmonics on the fundamental wave when the disturbance amplitudes are small. The parametric stage is found to extend up to the situation in which the subharmonic amplitude reaches values greater than twice the amplitude of the fundamental wave. This catalytic role of the fundamental wave for energy transfer from the mean flow to the subharmonics has been explained by several studies. This observation supports the adequacy of using the Floquet theory for examining the subharmonic resonant interaction up to late stages before the post-parametric stage. During subharmonic resonance, the amplitude of the subharmonic wave grows double-exponentially. In the later stage (i.e., post-parametric stage), although several theoretical studies based on the weakly nonlinear theory and asymptotic theory predicted “explosive” growth in which both fundamental and subharmonic waves amplify, experimental and DNS studies have shown amplitude saturation. As the streamwise pressure gradient affects the transition onset location significantly, studies have been extended to the investigation of subharmonic resonance in the boundary layer with pressure gradients. It has been revealed that the streamwise pressure gradient does not yield a noticeable qualitative change in the general characteristics of subharmonic resonance. The double-exponential growth of subharmonic waves is more pronounced in the case of an adverse pressure gradient (APG). Further studies on the boundary layer over an airfoil have been conducted to investigate the influence of the variable streamwise pressure gradient. Research on the secondary instability in high-speed boundary layers is still being actively conducted. Recently, Xu et al. and Liu et al. investigated the effects of micro-porous surface and pressure gradient on the secondary instability of hypersonic boundary layers, respectively. By considering thermochemical equilibrium conditions, Kumar and Prakash studied the effect of chemical reaction of air on the secondary instability of hypersonic boundary layer.

Most previous studies have investigated and examined the effects of frequency, spanwise wavenumber, amplitude of instability waves, and pressure gradient on the subharmonic resonant interaction. Apart from the above-mentioned parameters, several experimental studies have found that the resonant interaction also strongly depends on the phase relationship between the fundamental and subharmonic waves in a Blasius boundary layer and in one under adverse and variable pressure gradients. By contrast, few theoretical or numerical studies have investigated the effect of the phases of fundamental and subharmonic waves. Craik provided a simple explanation for the dependency on the initial phase in the subharmonic resonance within wave packets observed by Medeiros and Gaster. However, theoretical or numerical studies involving detailed investigations or parameter studies on the dependency on the phase are scarce in the literature. To the best of the authors’ knowledge, no theoretical or numerical study has examined the effect of controlled initial phases of fundamental and subharmonic waves on subharmonic resonance over a wide range of parameters.

The objective of the present study is to investigate the effect of the initial phases of both fundamental and subharmonic waves on the evolution of instability waves during the subharmonic resonant interaction. The analysis is carried out using nonlinear parabolized stability equations (PSE). As described by Borodulin et al., PSE analysis is a simple and efficient theoretical approach for predicting all main properties of the resonant interactions of instability waves, including basic-flow non-parallelism. However, detailed parameter studies of the subharmonic resonant interaction using PSE or other numerical methods are rarely found in the literature.

Nonlinear PSE (NPSE) analysis was carried out to simulate the subharmonic resonant interaction in a boundary layer with the adverse pressure gradient measured in the experiment of Borodulin et al. The PSE results on the downstream evolution of instability waves and the influence of the initial amplitude were carefully evaluated and compared with the measurement data. Then, various combinations of phase angles were imposed for the fundamental and subharmonic waves as the initial condition for PSE analysis. The downstream evolutions of the amplitudes of the instability waves and the phase difference between fundamental and subharmonic waves were analyzed in terms of the imposed initial phases. From the results, the features and general relationship for the initial phase dependency...
on the subharmonic resonant interaction were identified. The general characteristics of the phase dependency were also examined for subharmonic resonance with different frequencies and spanwise wave-numbers. Further analyses were carried out for boundary layers over a flat plate with a zero pressure gradient following a previous experimental study.22

II. METHOD OF ANALYSIS

A. Parabolized stability equations

For an instability wave of a specific frequency or wavenumber propagating convectively through a given laminar base flow, information about its stability (e.g., amplification rate and amplitude evolution) can be obtained by stability analysis as well as numerical simulation. Alongside linear stability theory (LST),12 PSE13 is one of the most widely known methods for the stability analysis of shear flows. In contrast to LST, PSE reflects the flow non-parallelism and curvature effect on its results. In the case of linear analysis, PSE is computationally more efficient than LST as it performs stability analysis by downstream marching from the initial condition. Furthermore, it enables us to carry out non-linear stability analysis, which is the main objective of the present study. The PSE has widely been used to study the linear and non-linear stability of boundary-layer flows over simple geometries.33–45

The PSE formulation and solution technique can be found in many studies.42,46 Thus, the details are not provided here. The derivation of the nonlinear PSE used in this study starts from the non-dimensionalized 3D Navier–Stokes (N–S) equations. The instantaneous flow variables are decomposed into a mean and fluctuating component, \( q = Q + q' \), where \( q \) is any instantaneous flow variable. This expression is substituted into the N–S equation and the equation for the mean flow is subtracted, resulting in the disturbance equations. The disturbance equations are transformed from the Cartesian coordinate system \((x, y, z)\) to a generalized coordinate system \((x_1, x_2, x_3)\), where \( x_1, x_2, \) and \( x_3 \) represent the streamwise, outward, and spanwise coordinates, respectively. The disturbances, \( \phi = (\phi', \alpha', v', w', T') \), are assumed in the form of a wave, that is, periodic in time and the spanwise direction. The disturbance vector can then be expressed by the following truncated discrete Fourier series:

\[
\phi(x_1, x_2, x_3, t) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} \psi_{mn}(x_1, x_2) \exp \left[ i \int_{x_3}^{x_3_0} \psi_{mn}(\tilde{x}_3) d\tilde{x}_3 \right] \times \exp \left[ i(nf\tilde{x}_3 - mot) \right],
\]

where \( \beta \) and \( \omega \) are the fundamental spanwise wavenumber and fundamental frequency of disturbance, respectively. The subscripts \( m \) and \( n \), represent the temporal and spanwise mode number, respectively. The variable, \( \psi_{mn} \) is the streamwise wavenumber for mode \((m, n)\). By using Eq. (1), the disturbance of each mode is modeled as a combination of a fast-oscillatory wave part (exponential terms) and a slowly varying shape function. The shape function, \( \psi_{mn} = (\tilde{p}_{mn}, \tilde{u}_{mn}, \tilde{v}_{mn}, \tilde{w}_{mn}, \tilde{T}_{mn})' \), represents the complex amplitude for mode \((m, n)\). \( M \) and \( N \) are the numbers of temporal (i.e., frequency) and spatial (i.e., spanwise wavenumber) modes, respectively, which are kept in the truncated Fourier series.

The expression for disturbance is substituted into the transformed disturbance equations. Collecting all the terms corresponding to \( \exp[i(nf\tilde{x}_3 - mot)] \) yields the equation for the Fourier mode \((m, n)\). By dividing the equations by \( \lambda_{mn} \exp[i(nf\tilde{x}_3 - mot)] \), we have the governing equation for the shape function. In the PSE approach, the decomposition of \( \psi_{mn} \) and \( \tilde{\psi}_{mn} \) is assumed to be chosen such that the change in the shape function along the streamwise direction is of the order of \( 1/R_0 \) and the second derivative \( (\partial^2 \tilde{\psi}_{mn}/\partial x_3^2) \) is negligible. Under this assumption, by neglecting the terms of \( O(1/R_0^2) \) and smaller, we get the PSE, which can be written compactly as follows:

\[
D_{mn}\tilde{\psi}_{mn} + \Lambda_{mn} \frac{\partial \tilde{\psi}_{mn}}{\partial x_1} + B_{mn} \frac{\partial \tilde{\psi}_{mn}}{\partial x_2} = V_{22, mn} \frac{\partial^2 \tilde{\psi}_{mn}}{\partial x_3^2} + \frac{F_{mn}}{\Lambda_{mn}},
\]

where

\[
D_{mn} = -i\omega \tilde{\Gamma} + i\sigma_{mn} \tilde{A} + i\beta \tilde{C} + \tilde{D} + \sigma_{mn}^2 \tilde{V}_{11} + n^2 \beta^2 \tilde{V}_{33} + n^2 \beta^2 \tilde{V}_{11},
\]

\[
A_{mn} = \tilde{A} - i\sigma_{mn} \tilde{V}_{11},
\]

\[
B_{mn} = \tilde{B} - i\sigma_{mn} \tilde{V}_{12} - i\beta \tilde{V}_{23},
\]

\[
V_{22, mn} = \tilde{V}_{22},
\]

\[
\Lambda_{mn} = \exp \left[ i \int_{x_3}^{x_3_0} \psi_{mn}(\tilde{x}_3) d\tilde{x}_3 \right],
\]

\[
F_{mn}(x_1, x_2, x_3, t) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} F_{mn}(x_1, x_2) e^{i(nf\tilde{x}_3 - mot)}.
\]

The matrices with the overbar in Eq. (1) are 5 × 5 matrices whose elements correspond to the linear part of the disturbance equations, which are composed of the mean-flow data and fluid properties. \( F \) represents the sum of all non-linear terms in the disturbance equations, whereas \( F_{mn} \) represents the Fourier components of \( F \). Specifically, this general formulation of PSE that includes nonlinear terms \((F_{mn})\) is referred to as nonlinear PSE (NPSE). On the other hand, if only a single-mode \((m, n)\) is considered while neglecting the nonlinear term, the resulting equation is referred to as linear PSE (LPSE). The LPSE cannot account for the nonlinear effect of the disturbances. Generally, PSE includes both NPSE and LPSE. In this study, however, PSE represents NPSE unless otherwise noted. PSE is referred to as NPSE only when it needs to be distinguished from LPSE or compared with the result from LPSE.

For discrete modes, Eq. (2) is subjected to the Dirichlet boundary condition such that the velocity and temperature disturbances vanish at both the solid surface and the free-stream boundary except for the mean flow distortion (MFD) [i.e., mode \((0, 0)\)]. For MFD, the Neumann boundary condition for the transverse velocity component is used at the free-stream boundary to take into account the change in the displacement thickness, owing to the non-linear interaction. Another additional condition, known as the normalizing condition, is imposed to determine the unique combination of \( \lambda_{mn} \) and \( \tilde{\psi}_{mn} \) that satisfies the basic assumptions of the PSE. For this, the marching procedure of the PSE involves an iteration procedure to update \( z \) at each streamwise location. In this study, \( z \) is updated on the basis of the integrated disturbance kinetic energy which is a general approach used in previous studies.44–46

Computational grid transformation is introduced to numerically solve the PSE. The fourth-order central differencing scheme is used for
all derivatives except for the streamwise derivative of the shape function for which the second-order backward difference scheme is adopted. To efficiently alleviate the numerical instability caused by the weak ellipticity of the PSE, we follow the approach of Chang et al., which suppresses the term corresponding to the streamwise derivative of the shape function for pressure by weighting the factor in terms of the local Mach number. Owing to the very low-speed of flow for which the Mach number is nearly zero, the term is eliminated almost completely for the cases examined in this study. The non-linear terms for non-linear stability are simply treated as explicit source terms and evaluated in a pseudo-spectral manner. The initial condition that locally satisfies the stability equation is required to initiate the marching procedure. This initial condition is obtained by solving the eigenvalue problem for \( \lambda \), which is similar to LST but includes non-parallel terms. Because the spatial stability is considered in the present study, the nonlinear eigenvalue problem is formulated and solved as described in Ref. 47.

The PSE analysis code based on a generalized coordinate system was developed by Park and used to investigate the effect of a smooth roughness element on the linear and non-linear instability of the boundary layer from subsonic to hypersonic speeds. Further details on the formulation and numerical method can be found in Ref. 45. We note that the code was developed for enabling analysis for flows of all speeds ranging from incompressible flows to hypersonic Mach numbers. Indeed, the density and temperature disturbances are also accounted for in the analysis no matter how low the flow speed may be. For the case of very low-speed flows, which is the main concern of the present study (Mach number is \( \sim 0.03 \)), the analysis yields nearly zero values for the density and temperature disturbances as well as the change in mean density and temperature.

### B. Base Flow

Throughout the present study, the similarity solutions of an incompressible boundary layer over a flat plate with the pressure gradient in the streamwise direction are used as the laminar mean flow data for the PSE analysis. The similarity solutions are obtained from the boundary-layer code, which is fourth-order accurate in the surface normal direction following the work of Iyer. For a given pressure gradient parameter, \( \beta_1 = 2m/(m + 1) \), where \( u_e = C_x \), the boundary-layer equation is solved at each streamwise station. The mean flow data include not only the streamwise velocity but also the transverse velocity component.

### C. Validation of PSE

In the previous study by Park and Park, a PSE calculation was carried out for the case of an experimental study conducted by Kachanov and Levchenko to examine the capability and validity of the PSE for analyzing subharmonic resonance. A 2D fundamental wave (primary mode) and a pair of subharmonic oblique waves (secondary mode) were initiated together in an incompressible flat plate boundary layer with a zero pressure gradient. The fundamental wave was a typical Tollmien–Schlichting (T–S) wave with frequency \( f_1 \) where \( 2\pi f_1 U_e^2 \times 10^6 = 124 = 2F \) (\( F = 62 \)), and the subharmonic waves were a pair of oblique waves with half of the fundamental frequency, \( f_{12} = (0.5f_1) \) and \( \beta/R \times 10^3 = \pm 0.317 \). The number of modes kept in the truncated Fourier series for the disturbance was chosen as \( M = 6 \) and \( N = 3 \), respectively (see Eq. (1)). This means that higher modes up to the third harmonics of both the fundamental frequency and spanwise wavenumber were kept in the analysis. The normal (discrete) mode solutions from the eigenvalue problem similar to the LST were used as the initial condition for both fundamental and subharmonic waves. The solution obtained individually for each mode was imposed together to consider a controlled experiment in which two instability waves were imposed independently. The initial amplitudes were set to 0.48% and 0.02% root mean square (RMS) for the fundamental and subharmonic modes, respectively, at \( R = \sqrt{Re} = 393 \). The initial amplitudes were determined such that the resulting amplitudes from the PSE analysis were close to the experimental data of Kachanov and Levchenko at the streamwise location of the first available data \( (R \approx 440) \) for both modes. The initial amplitudes for all modes other than the fundamental and subharmonic ones were set as zero at the initial location. Although only the fundamental and subharmonic modes were initiated, the higher modes were generated with corresponding amplitudes immediately after the initial location via their interaction. The RMS amplitude of the streamwise velocity disturbance at \( y/\delta = 0.26 \) obtained from the PSE was compared with the experimental data and the DNS results of Fasel (see Fig. 4 in Ref. 49) where \( \delta \) denotes the thickness of the boundary layer. The results qualitatively and quantitatively showed satisfactory agreement with the other data, although all three cases were subjected to disturbances generated in different ways. In the DNS study by Fasel et al., the disturbance was introduced into the boundary layer by blowing and suction through the wall. In the experiment, a vibrating ribbon was used to impose the disturbances. For these disturbances to be adapted to the instability mechanism to form instability waves, a certain length of streamwise distance would be necessary. The profiles of the RMS amplitude for various modes were also compared together with those of the numerical and measurement data at \( R = 608 \) (see Fig. 5 in Ref. 49). The distributions were confirmed to be quite similar to one another and the orders of magnitude were the same for all the modes.

Joslin et al. made a similar comparison of the results from PSE and DNS studies with the measurement data. Their DNS and PSE results showed excellent agreement, as the same initial condition was adopted for both the methods in their study. Additional calculations were also carried out for high-speed boundary layers including the oblique mode breakdown in supersonic boundary layers and the non-linear evolution of Mack’s second mode in hypersonic boundary layers. The results showed excellent agreement with those from previous studies. However, they are not included here, as the incompressible boundary layer is the main focus of the present study.

Recently, the present authors carried out a similar calculation for the comparison of PSE results and numerical simulation results using wall-resolved LES (large eddy simulation). The details of the formulation and numerical methods for LES calculation can be found in the authors’ previous studies. For this computation, the fundamental wave and subharmonic oblique wave were imposed at \( R = 400 \). The initial amplitudes were set to 0.46% and 0.01% for the fundamental and subharmonic modes, respectively, as the maximum amplitude rather than the RMS amplitude described above. Because the maximum amplitudes are considered, this computation corresponds to the case where the initial amplitudes are rather small compared to the former case. The same disturbance profiles from the normal-mode eigen-
solution are imposed at the inlet for both LES and PSE calculations. Figure 1 compares the amplitudes of several Fourier modes along the streamwise direction (see also Fig. 5 in Ref. 54). The amplitudes are evaluated at the location of the maximum amplitude of the subharmonic wave. The first and second indices represent the temporal and spanwise modes, respectively. The fundamental wave corresponds to \((f_1, 0)\), and the subharmonic wave corresponds to \((f_1/2, \beta)\). We confirmed a reasonable agreement with the wall-resolved LES results. The comparison of the results confirms that the PSE suitably simulates the non-linear evolution of the instability waves. Therefore, the PSE of the present study is suitable and reliable for parametric studies of subharmonic resonance.

III. RESULTS AND DISCUSSION

The evolution of instability waves in subharmonic resonance is analyzed for a flat plate boundary layer with and without a pressure gradient. The growth of modes along the downstream direction is examined for various initial phase relationships between the primary and secondary modes. Several experimental studies have measured the downstream evolution of instability waves in the subharmonic resonant interaction. The work of Borodulin et al. was chosen as the reference case, as it is a recent publication and contains detailed measurement data including the influence of the initial phase difference.

A. Mean flow data and basic validation

1. Mean flow properties

The Falkner–Skan solution is used as the mean flow data for PSE analysis. The boundary-layer edge velocity (i.e., potential flow velocity) distribution along the streamwise direction must be chosen to approximate the actual boundary layer generated in the experiment. Borodulin et al. placed a Plexiglas sheet on top of a flat plate model to adjust the desired streamwise pressure gradient. Figure 2(a) shows the edge velocity measured from the experiment (reproduced from Fig. 3 in Ref. 5). The theoretical velocity distribution of the form, \(U_n(x) = Cx^n\), is considered where \(m = \beta H/(2 - \beta H)\) and \(\beta H\) is the pressure-gradient parameter. They suggested that the distribution with \(C = 8.36\) and \(\beta H = -0.115\) best represents the measurement data for the entire streamwise extent, which covers all locations for pressure measurement \((x = 200–750\) mm). The curve for the corresponding parameters is shown together as the solid line in Fig. 2(a).

However, we note that the actual measurement of instability waves in the resonant interaction was carried out only within a streamwise extent of \(x = 350–600\) mm, as indicated by the vertical dashed lines. The disturbance generator that imposes the instability waves into the boundary-layer boundary is located at 300 mm. We examined the four additional pressure-gradient parameters of \(\beta H = -0.118, -0.122, -0.126,\) and \(-0.130\) to represent the velocity distribution for this region. For each parameter value, the coefficient, \(C\), was determined to minimize the RMS error with respect to the measurement data within the region of our interest \((x = 350–600\) mm). The values of \(C\) and the corresponding RMS errors are listed in Table I. The curves for five pressure-gradient parameters are plotted together in Fig. 2(b) with an enlarged view. We see from the figure that the fitted curves do not exhibit considerable differences. The minimum RMS error occurs for \(\beta H = -0.118\), and the errors for \(-0.115\) and \(-0.122\) are similar albeit slightly higher.

Figure 3(a) shows the velocity profiles from the Falkner–Skan solution for several values of the pressure-gradient parameters. The measurement data at \(x = 560\) mm given by Borodulin et al. (see also Fig. 4 in Ref. 5) and the Blasius profile are plotted together for comparison. For each profile, the wall-normal coordinate is normalized by the local displacement thickness \((\delta_x)\). All velocity profiles of non-zero pressure-gradient parameters are different from the Blasius profile and exhibit reasonable agreement with the measurement data. This implies that the experimental boundary layer is clearly under the state of the adverse pressure gradient. However, we observe that the profiles of \(\beta H = 0.115\) to \(\beta H = -0.130\) are close to each other and do not exhibit noticeable discrepancies. In addition, the streamwise distribution of the boundary-layer displacement thickness is shown in Fig. 3(b). We see that for all the values of \(\beta H\) considered, the tendency of the displacement thicknesses well represents that of the measurement data. The difference between them is barely discernible over a streamwise extent of \(x = 350–600\) mm, which is the main measurement region of the instability wave. Therefore, only a comparison of the mean flow data is not sufficient to determine the proper value of the pressure gradient parameter for stability analysis of the boundary layer in the experiment.

2. Growth of instability waves

Because the boundary-layer instability is very sensitive to the mean flow, we can easily expect the results of the stability analysis to change considerably with respect to a small change in the mean flow. To examine the sensitivity of the stability results with respect to the pressure-gradient parameter, linear PSE analyses were carried out for the subharmonic wave. Following the experiment of Borodulin et al., the physical frequency and spanwise wavenumber of the subharmonic wave were set to \(f_{1/2} = 54.5\) Hz and \(\beta = \pm 0.131\) rad mm\(^{-1}\),...
respectively. Linear PSE analysis was initiated at $x = 300$ mm, where the disturbance generator is located, and the initial amplitude was chosen to be similar to the measurement data at $x = 350$ mm, which was the first location of the available data. Figure 4(a) shows the linear growth of the subharmonic wave amplitude under five different pressure gradient parameters. As expected, all results exhibit exponential growth, and more rapid growth is achieved for higher adverse pressure gradients (lower value of $b_H$).

Figure 4(b) shows the results from nonlinear PSE analyses for the case of resonance growth in the experiment. The fundamental wave having a frequency of $f_1 = 109$ Hz is imposed at the initial location ($x = 300$ mm) with the subharmonic wave. The detailed setup for the

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**TABLE I.** Edge velocity distribution parameters for several pressure-gradient parameters.

| $b_H$ | $m$     | $C$     | RMS error |
|-------|---------|---------|-----------|
| -0.115 | -0.054 37 | 8.3527 | 0.0121 |
| -0.118 | -0.054 37 | 8.3435 | 0.0118 |
| -0.122 | -0.057 49 | 8.3374 | 0.0121 |
| -0.126 | -0.059 27 | 8.3191 | 0.0133 |
| -0.130 | -0.061 03 | 8.3069 | 0.0151 |

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**FIG. 2.** Distribution of boundary-layer edge velocity over (a) entire region and (b) only the measurement region.

**FIG. 3.** Boundary-layer (a) velocity profile and (b) displacement thickness distribution.

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initial amplitudes and phases are described in Sec. III B 1. The growth of both fundamental and subharmonic waves shows reasonable agreement with the measurement data. In their experiment, Borodulin et al. observed that the periodicity of the subharmonic wave in the spanwise direction and the two-dimensionality of the fundamental wave were sustained up to $x = 510–550$ mm. Those structures began to collapse further downstream and eventually reached breakdown (see Fig. 16 in Ref. 5). This implies that the parametric resonant amplification can be expected up to around $x \approx 550$ mm. For the PSE result corresponding to $\beta_H = -0.122$ to $-0.126$, the overall growth of the fundamental wave amplitude matches well with the measurement data for the streamwise extent from $x = 350$ to $550$ mm. The growth of the subharmonic wave amplitude matches well with the results corresponding to $\beta_H = -0.126$ to $-0.130$. Additional parameter studies found that it is difficult to achieve perfect agreements for both waves simultaneously with a specific pressure gradient parameter. For the remainder of the present study, we chose $\beta_H = -0.126$ for the APG case of Bordulin et al. because the PSE results showed reasonable agreement for amplification of both fundamental and subharmonic waves.

B. Case of adverse pressure gradient for $\beta_H = -0.126$

1. Amplitude and phase evolution

First, an analysis was conducted for the baseline case, which is the magnetoresistance (MR) case following the notation used by Borodulin et al. As mentioned, the fundamental wave having $f_1 = 109$ Hz and subharmonic waves with $f_1/2 = 54.5$ Hz and $\beta = \pm 0.131$ rad mm$^{-1}$ were used. The nonlinear PSE was initiated at $x = 300$ mm, where the disturbance generator was located in the experiment. Again, the numbers of modes kept in the analysis were chosen as $M = 6$ and $N = 3$. From preliminary tests, additional calculations with $M = 8$ and $N = 4$, and $M = 10$ and $N = 5$ revealed that keeping the higher mode does not exhibit noticeable discrepancies in the results. Only a slight difference was observed in the far-downstream region, where the amplitudes are largely amplified such that PSE analysis has difficulty reaching a converged solution. Because there is no difference in the results for the main region of interest (i.e., region of subharmonic resonance), the analysis with $M = 6$ and $N = 3$ is used throughout considering the computational cost for analysis of the many cases considered in this study. The initial conditions for the PSE were obtained by conducting a local stability analysis, which is similar to LST but constitutes the nonlinear eigenvalue problem for $x$ including non-parallel terms. Here, we note that the measurements for instability waves start from $x = 350$ mm in the experiment. Because the disturbances were imposed by blowing/suction through a slit at $x = 300$ mm in the experiment, a certain streamwise distance was required in the downstream direction to settle the disturbances into the boundary layer as instability waves. The initial amplitudes for the PSE analysis ($x = 300$ mm) were chosen so that they matched the first available measurement data ($x = 350$ mm). For the MR case, the initial amplitude was chosen as $A_{1,\text{inital}} = 0.0241\%$ and $A_{1/2,\text{inital}} = 0.00581\%$ in RMS for the fundamental and subharmonic wave, respectively. The chosen amplitudes were multiplied with the corresponding normalized eigenvector obtained from the local stability analysis. Thus, they represent the maximum RMS amplitudes in the wall-normal direction at $x = 300$ mm, and the corresponding heights of the maximum amplitude differ for the two waves. In addition to the initial amplitude, we focus on the initial phase in the present study. We define an initial phase angle ($\theta_{\text{initial}}$) of instability waves as the phase angle at the height of the maximum amplitude for each mode at the initial location. We note that although the normalized eigenvector possesses a profile of complex numbers in the wall-normal direction, it has a maximum value of 1.0 at the height of the maximum amplitude as it is normalized by its maximum amplitude. Thus, $\exp(i \theta_{\text{initial}})$ is together with the initial amplitude to the normalized eigenvector to determine the initial condition for PSE analysis. As is shown in Secs. III C 1–3, the evolution of a subharmonic wave is considerably affected by initial phase angles. As a preliminary
test, we carried out PSE analyses for several initial phase angles for the fundamental wave (\(\varphi_{1\text{ initial}}\)) and a fixed phase angle for the subharmonic wave (\(\varphi_{1/2\text{ initial}}\)) of 0°. The calculations were conducted for phase angles between 0° and 360° at an interval of 30°. The results showed that the initial phase angle of 240° led to the largest amplification of the subharmonic wave. Borodulin et al. also determined the MR case as the one having the maximum attainable resonance amplification by changing the phase of the signal input to the disturbance generator for the fundamental wave (which is denoted as \(\varphi_{1\text{ initial}}\)). Therefore, the initial phase angle of the fundamental wave (\(\varphi_{1\text{ initial}} = 240°\)) corresponds to the largest amplification of the subharmonic wave, which is temporarily regarded as the MR case in the present study. The results given in Fig. 4(b) with several pressure-gradient parameters also correspond to the initial phase of the MR case (\(\varphi_{1\text{ initial}} = 240°\) and \(\varphi_{1/2\text{ initial}} = 0°\)). For the initial amplitudes and phase angles described above, the nonlinear PSE results are shown in Fig. 5. For all PSE calculations of the present study, 221 grid points with clustering are used in the wall-normal direction within a height approximately 90 times the local displacement thickness. The grid spacing in the streamwise direction was set as 5 mm. Following the experiment in Ref. 5, the amplitude represents the value relative to the local boundary-layer edge velocity (\(U_e\)). The amplitudes of the fundamental (\(f_1\)) and subharmonic (\(f_{1/2}\)) waves are obtained at the maximum location across the boundary layer, which is denoted as \(y_{\text{max}}\) for \(f_1\) and \(f_{1/2}\), respectively. The results of \(f_1\) and \(f_{1/2}\) are the same as those in Fig. 4(b) for \(\beta_H = -0.126\). The measurement data are plotted with filled symbols. We see from the figure that the growth of amplitudes shows good agreement with the measurement data not only for the fundamental and subharmonic waves but also for higher modes of \(f_{3/2}\) and \(f_{5/2}\) which are produced by the nonlinear interaction of the two initial waves. The amplitudes for the higher modes are evaluated at \(y_{\text{max}}\). For the measurement data for higher modes (\(f_{3/2}\)) at streamwise locations less than \(x = 450\) mm, the amplitude less than \(10^{-5}\) (denoted by the horizontal dashed line in the figure) seems to be affected by the measurement uncertainty and noise level in the signal. The results are in good agreement with the measurement data up to \(x \approx 570\) mm, for which parametric resonance is dominant. Further downstream, they exhibit some discrepancies in \(x = 550–590\) mm, where the experiment observed the collapse of the periodicity and uniformity in the spanwise distribution of the amplitudes. This implies that highly nonlinear interaction and spectrum broadening begin to dominate in this region, and PSE cannot completely capture the process. Despite the difference in the later stage, the comparison of the results confirms that PSE analysis is a reliable method for parametric studies on the evolution of instability waves in parametric resonance and nonlinear interaction.

Figure 6 compares the results for the case of the subharmonic resonant interaction shown in Fig. 5 and the linear evolution of the individual wave. The experimental data obtained for the case of generating the disturbance for fundamental or subharmonic wave individually are shown with empty symbols. The linear PSE results obtained for each fundamental and subharmonic wave are plotted with dashed lines. It is clearly seen from the figure that the subharmonic wave amplifies linearly (exponential growth) without the presence of the fundamental wave. However, with the presence of the fundamental wave, the subharmonic wave grows more rapidly and shows double-exponential growth, as discussed by Borodulin et al. Meanwhile, in the early stage, the growth of the fundamental wave is not considerably affected by the existence of the subharmonic wave. This observation confirms the well-known parametric resonance of the subharmonic wave owing to the fundamental wave, with no significant back-influence on the fundamental wave.

Figure 7 shows the amplitude and phase profiles at \(x = 450\) mm for both fundamental and subharmonic waves. The amplitude and wall-normal distance are normalized by the maximum value and local displacement thickness (\(d_h\)), respectively. Figures 7(a) and 7(c) show reasonably good agreements between the PSE results and the

![FIG. 5. Amplification of wave amplitudes in baseline case (case MR).](image)

![FIG. 6. Comparison of amplitude evolution in subharmonic resonant interaction and individual (linear) growth.](image)
measurement data except for a small difference in the fundamental wave at around a height slightly above the maximum amplitude \( \frac{y}{\delta_1} \approx 0.4-1.0 \). The location of the local minimum (i.e., corresponding to the phase shift of \( \pi/2 \)) at around \( \frac{y}{\delta_1} \approx 1.8 \), and amplitude and location of the secondary peak are also in good agreement with the experimental data. The comparison of the phase distribution is given in Figs. 7(b) and 7(d). The PSE results were shifted suitably to match the experimental data, because the phase itself is a relative property. Reasonably good agreement of the phase distribution is observed, including the rapid \( 180^\circ \) phase change of the fundamental wave near \( \frac{y}{\delta_1} \approx 1.8 \). Figure 7 shows that the correlation between PSE calculation and the experiment is quite reliable for both amplitude and phase distributions across the boundary layer.

Figure 8(a) shows the downstream evolution of the phase of instability waves. Following Borodulin et al., the phase, \( \phi \), is introduced, which is defined as \( \phi_{n2} = (2/n) \phi_{n2} \). In contrast to the definition of the initial phase angle \( (\phi_{\text{initial}}) \) mentioned previously, following the experiment, \( \phi_{n2} \) is evaluated at the height of the maximum amplitude of the subharmonic wave \( (y_{\text{max}}) \) for both modes \( (n = 1 \text{ and } 2) \) for subharmonic and fundamental waves, respectively. The measurement data are plotted together. Again, we observe good agreement of the downstream evolution of the phase. As with the amplitude, a slight deviation from the measurement data are observed beyond \( x \approx 560 \, \text{mm} \), which is the post-parametric (fully nonlinear) stage described above. The phase difference between the fundamental and subharmonic waves is plotted in Fig. 8(b) with the measurement data. They are also in good agreement with each other, and the phase difference is found to be \( \sim 130^\circ-140^\circ \) at the early stage, and it gradually increases to \( 180^\circ-190^\circ \) at the end of the region of interest \( (x \approx 600 \, \text{mm}) \). Figures 7 and 8 indicate that PSE is an appropriate method for studying not only the amplification of the instability waves but also the amplitude and phase distributions across the boundary layer and the downstream evolution of their phase.

2. Influence of initial amplitude

Several calculations were conducted further to examine the influence of the initial amplitude on the subharmonic resonance. The MR case described in Sec. III B 1 was considered to be the baseline case. Additional cases having twice the amplitude of the fundamental wave and four times the amplitude of the subharmonic wave were considered. The cases investigated in the present study are summarized in Table II. The notations follow those used by Borodulin et al.\(^5\) The letter "R" represents the case of resonance amplification when the fundamental and subharmonic waves are imposed together.
Figure 9 shows the downstream evolution of amplitudes for the 2FR and 4SR cases. The PSE results are given for the fundamental and subharmonic waves. The measurement data for the subharmonic wave are plotted together. As the amplitude of the fundamental wave is doubled [Fig. 9(a), 2FR case], the growth rate of the subharmonic wave during the parametric resonance increases compared with the baseline case. As observed and discussed by Borodulin et al., the 2FR case also exhibits double-exponential growth but with a larger value of the growth rate. The change in growth depending on the fundamental wave amplitude is in good agreement with the measurement data. As the amplitude of the subharmonic wave is increased by four times [Fig. 9(b), 4SR case], the amplitude remains four times larger with respect to the MR case throughout the streamwise extent up to \( x = 550 \) mm. This indicates that the growth rate of the subharmonic wave remains the same, and the amplitude of the subharmonic wave does not affect its growth during subharmonic resonance. Additionally, the amplification of the fundamental wave remains the same for the larger subharmonic wave amplitude. This indicates again that there is no back influence of the subharmonic wave on the fundamental wave during the subharmonic resonance, which implies that it is parametric resonance.

Figure 10 shows the results for the case of 4S2FR in which there is the joint influence of fundamental and subharmonic initial amplitudes. Again, the results are in good agreement with the measurement data. The results for case 4S are given together for comparison. As expected, the subharmonic amplitude grows faster than that of the 4SR case, owing to the larger amplitude of the fundamental wave. From the results of Figs. 9 and 10, we conclude that PSE reflects well the influence of the initial amplitude on the subharmonic resonance observed in the experiment.

C. Influence of initial phase

1. Baseline case

The influence of the initial phase on the subharmonic resonant interaction is investigated in detail. The effect of the initial phase of both fundamental and subharmonic waves is first analyzed for the baseline case (Sec. III A). Calculations are carried out for various combinations of the initial phases of the fundamental (\( \phi_{1 \text{ initial}} \)) and subharmonic (\( \phi_{1/2 \text{ initial}} \)) waves. The initial phase combinations and their designations used in the present study are summarized in Table III. The initial phase of the fundamental wave is varied with a fixed initial phase of the subharmonic wave at 0° or 60° (case F0SV or F60SV, respectively). By contrast, the initial phase of the subharmonic wave is varied with a fixed initial phase of the fundamental wave at 0° or 60° (case FV50 or F60SV, respectively). Finally, the initial phases of both fundamental and subharmonic waves are varied simultaneously with a fixed difference of 30° (case D30).

| Case | Fundamental wave initial amplitude | Subharmonic wave initial amplitude | Remarks |
|------|-----------------------------------|-----------------------------------|---------|
| MR   | \( A_{1 \text{ initial}} \)     | \( A_{1/2 \text{ initial}} \)      | Baseline case |
| 2FR  | \( 2A_{1 \text{ initial}} \)     | \( A_{1/2 \text{ initial}} \)      | Twice the initial amplitude of fundamental wave |
| 4SR  | \( A_{1 \text{ initial}} \)     | \( 4A_{1/2 \text{ initial}} \)    | Four times the initial amplitude of subharmonic wave |
| 4S2FR| \( 2A_{1 \text{ initial}} \)     | \( 4A_{1/2 \text{ initial}} \)    | Combined case of 2FR and 4SR |
| F    | \( A_{1 \text{ initial}} \)     | \( \ldots \)                      | Fundamental wave only |
| 2F   | \( 2A_{1 \text{ initial}} \)     | \( \ldots \)                      | Fundamental wave only with twice the amplitude |
| S    | \( \ldots \)                      | \( A_{1/2 \text{ initial}} \)      | Subharmonic wave only |
| 4S   | \( \ldots \)                      | \( 4A_{1/2 \text{ initial}} \)    | Subharmonic wave only with four times the amplitude |
Figure 11(a) shows the selected results from the FVS0 case ($u_{1/2}$ initial fixed at 0°) for several initial phases of the fundamental wave at an interval of 30°C14. The initial amplitudes are set to be the same as those of the MR case (Sec. III B, Table II). We can see from the figure that the amplification of the subharmonic wave varies considerably, depending on the initial phase of the fundamental wave, $u_{1initial}$. Among the results in the figure, the case of $u_{1initial} = 240°$ exhibits the largest amplification and corresponds to the result given in Sec. III B for comparison with experimental data of the MR case. The initial phase of $u_{1initial} = 60°$ yields the minimum amplification as shown in the figure. We note that the amplification of the fundamental wave is not affected by its initial phase. Similar calculations are also conducted for the initial amplitudes corresponding to the 4SR case, and the corresponding results are shown in Fig. 11(b). We observe the same tendency and dependency on the initial phase. Again, $u_{1initial} = 240$ and 60 correspond to the most and least amplified cases, respectively. This indicates that the dependency on the initial phase is not affected by the initial amplitude of the subharmonic wave.

Figure 12 shows amplitudes of several modes for the FVS0 case at $x = 510$ mm. This streamwise location is denoted by the vertical dashed line in Fig. 11(b). The PSE results are plotted with respect to the initial phase, $u_{1initial}$. We see that the amplitude of the fundamental wave ($f_1$) is independent of the initial phase. By contrast, the subharmonic wave ($f_{1/2}$) and higher modes (both $f_{3/2}$ and $f_{5/2}$) exhibit clear dependency on the initial phase of the fundamental wave. This tendency is repeated with a cycle of 360°C14. The minimum amplification is found to occur at approximately $u_{1initial} = 70°$. The measurement data [Fig. 25(a) in Ref. 5] are also plotted together. We note here that the definition of $u_{1initial}$ in the present study is different from $u_{1i}$ in the experiment. Specifically, $u_{1initial}$ is the phase of the complex $u$-velocity profile imposed as the initial condition of the PSE at $x = 300$ mm at the height of the maximum amplitude, whereas $u_{1i}$ is the phase of the input signal to the disturbance generator for the fundamental wave in the experiment. Because the values of the initial phase have different meanings, we focus only on the tendency with respect to the initial phase. Therefore, $u_{1initial}$ is suitably shifted in Fig. 12 such that the phase angle at which the minimum amplitudes occur coincides with

**TABLE III.** List of cases for initial phase combination.

| Case  | $\varphi_1$ initial | $\varphi_{1/2}$ initial |
|-------|---------------------|-------------------------|
| FVS0  | $0° - 355°$ ($\Delta \varphi = 5°$) | $0°$ (fixed) |
| FVS60 | $0° - 150°$ ($\Delta \varphi = 5°$) | $60°$ (fixed) |
| F0SV  | $0°$ (fixed)         | $0° - 355°$ ($\Delta \varphi = 5°$) |
| F60SV | $60°$ (fixed)        | $0° - 355°$ ($\Delta \varphi = 5°$) |
| D30   | $0° - 355°$ ($\Delta \varphi = 5°$) | $\varphi_1$ initial + 30° |
the measurement data (shifted by 170°). The results are in good agreement in terms of both the amplitudes of the modes and the behavior of the initial phase dependency.

We designate the case of the initial phase, \( \varphi_{1\text{initial}} = 70° \) and \( \varphi_{1/2\text{initial}} = 0° \), which results in the minimum amplification of the subharmonic wave as the anti-resonance (AR) case. In Fig. 13, the evolution of the amplitudes for the AR case is compared with that for the MR case. The same initial amplitudes are used for both cases. The measurement data [Fig. 26(a) in Ref. 5] for the AR case are plotted using filled symbols. The results of the fundamental and subharmonic waves for the MR case (Fig. 5) are plotted again for comparison. We see that the growth of the subharmonic wave is considerably suppressed in the AR case. The amplitudes of the modes also show reasonably good qualitative and quantitative agreement with the measurement data.

Figure 14(a) shows the amplitude of the subharmonic wave \((f_{1/2})\) at \( x = 510 \text{ mm} \) for initial amplitudes of the MR, 4SR, and 4S2FR cases. For a given initial phase, the 4S2FR and MR cases exhibit the largest and smallest amplitudes, respectively. The results for the 4S2FR and 4SR cases seem like the result of the MR case shifted upward, as observed in Fig. 9. The results in Fig. 14(a) indicate that the phase dependency of the subharmonic resonance is not influenced by the amplitude of both fundamental and subharmonic waves. Figure 14(b) shows the results of the subharmonic wave amplitude at several

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**FIG. 11.** Influence of initial phase of fundamental wave for (a) MR and (b) 4SR cases.

**FIG. 12.** Amplitudes of several modes at \( x = 510 \text{ mm} \) with respect to initial phase of fundamental wave.

**FIG. 13.** Evolution of amplitudes for anti-resonance (AR) case.
streamwise locations for the MR case. As observed previously, the subharmonic wave is largely amplified as it proceeds downstream, whereas this amplification is considerably suppressed for the initial phases in the vicinity of the AR condition ($\varphi_{\text{initial}} \approx 70^\circ$).

The results for the FVS0 case given in Figs. 11–14 reveal that the initial phase of the fundamental wave has significant influences on the amplification of instability waves during the subharmonic resonant process. As the fundamental and subharmonic waves are imposed simultaneously, there is another degree of freedom for the initial phase of the subharmonic instability wave. Figure 15 shows the results for the F0SV case in which the initial phase of the subharmonic wave varies while that of the fundamental wave is fixed at $0^\circ$. The dependency of the subharmonic and higher modes with respect to the initial phase of the subharmonic wave is found to be similar to that observed for the FVS0 case (Figs. 12 and 14) except the periodic pattern, that is, repeated every $180^\circ$. As with the FVS0 case, the amplitude of the fundamental wave appears to be independent of the initial phase of the subharmonic wave. There is also a certain range of the initial phase that results in a considerable suppression of growth of the subharmonic wave. However, the values of the initial phase of the subharmonic wave, $\varphi_{1/2 \text{ initial}}$, corresponding to the minimum amplitude appear to be $145^\circ$ and $325^\circ$, which are different from the FVS0 case.

From the results given above, it is easy to speculate that the initial phases of both fundamental and subharmonic waves can significantly affect the subharmonic resonance process. If the amplitude of instability waves at a specific streamwise location is plotted with respect to the initial phase of the fundamental or subharmonic wave, the resulting plot for all the cases listed in Table III does not seem to have any regular or simple dependency. To seek a more regular dependency, we consider the relative phase difference between the fundamental and subharmonic waves as a major parameter. Following Borodulin et al., we introduce an initial phase difference defined as $\Delta \varphi_{\text{initial}} = \varphi_{1/2 \text{ initial}} - (1/2) \varphi_{\text{initial}}$. It represents the relative phase difference in terms of the subharmonic wave phase. Any combination of the initial phases of the fundamental and subharmonic waves will have a corresponding value of the initial phase difference. The results for all the cases that are given in Table III are shown in Fig. 16, in which the amplitudes of three modes at $x = 510$ mm are plotted with respect to the initial phase difference. The results of all the combinations considered in this study appear to lie on a single curve. This implies that the dependency of the subharmonic resonance on the initial phase can be regarded as a function of a single parameter (i.e., the initial phase difference, $\Delta \varphi_{\text{initial}}$). The dependency with respect to $\Delta \varphi_{\text{initial}}$ exhibits a periodic behavior, that is, repeated every $180^\circ$. It can be concluded that the subharmonic resonance depends on the relative difference of the initial phase rather
than the initial phase of the individual fundamental or subharmonic wave. In other words, the results in Fig. 16 indicate that the relative phase of the fundamental and subharmonic waves is a key parameter for determining the amplification of instability waves in subharmonic resonance.

Furthermore, a downstream evolution of the phase difference between the fundamental and subharmonic waves is investigated for the MR case. The phase difference, \( \Delta \phi_{1/2} \), is defined as \( \phi_{1/2} - \phi_{1} \), where \( \phi_{1} = (1/2) \phi_{1} \). As with Fig. 8, the phase is evaluated at \( y_{max} \) for both modes. The phase difference is calculated at each streamwise location for all cases investigated. The results for several selected cases of the initial phase (from the FVS0 case) are shown in Fig. 17. The phase difference near the initial location (\( x \approx 350 \) mm) is varied with the initial phase of the imposed fundamental wave. We see that not only the growth of the subharmonic wave amplitude (see Fig. 8) but also the evolution of the phase difference along the streamwise direction differs considerably depending on the initial phase. The phase difference does not change significantly during the early stage. However, it starts to change downstream and eventually converges toward a certain value regardless of the initial phase difference. The value where the phase difference converges appears to be approximately 85° at around \( x = 570 \) mm.

Borodulin et al. observed that the phase difference did not change until the subharmonic wave enters double-exponential growth [see Fig. 26(b) in Ref. 5]. When we compare the results in Figs. 11 and 17, the same characteristics are identifiable from the PSE results. For the initial phases close to the AR case (\( \phi_{1 initial} \approx 70° \)), the start of double-exponential growth of the subharmonic wave is delayed to downstream locations (see Fig. 11). For the case of \( \phi_{1 initial} = 60° \) (closest to the AR case in Figs. 11 and 17), the subharmonic wave does not grow double-exponentially up to \( x \approx 450 \) mm, and in this region, the phase difference remains nearly unchanged (see Fig. 17). Beyond this region, the phase difference starts to change rapidly to the converging value (i.e., \( \approx 85° \)) with a slight overshoot. As the phase difference becomes close to the converging value (\( x \approx 480 \) mm), the subharmonic wave starts to grow double-exponentially. Meanwhile, for the case of \( \phi_{1 initial} = 240° \), which is the nearest condition to the MR case, the phase difference changes only slightly and remains around 90° for the entire streamwise extent. Furthermore, the subharmonic wave grows double-exponentially nearly immediately from the initial state. Therefore, it is easily conjectured that a certain phase difference (\( \approx 85° \)) for the results given in Fig. 17 can lead to subharmonic resonance. This phase difference value corresponds to the condition of the MR case and is expected to be close to that obtained when the Floquet analysis is conducted. As shown in the figure, all the phase differences eventually tend to converge to the resonance condition. For the cases of \( \phi_{1 initial} = 60° \) and 90° in Fig. 17 (nearest conditions to the AR case), the phase difference near the initial location is the farthest from the value for the resonance condition and starts to change at the most downstream location.

2. Dependency on frequency and spanwise wavenumber

Thus far, the effect of the initial phase difference has been investigated for a fixed frequency and spanwise wavenumber. In this section, the dependency on the initial phase difference is examined for other frequencies and wavenumbers. First, two additional fundamental frequencies of \( f_{1} = 119.9 \) and 98.1 Hz are considered. They are 1.1 and 0.9 times, respectively, the baseline frequency (109 Hz) studied previously. The frequency of the subharmonic wave is halved correspondingly. The mean flow and spanwise wavenumber remain unchanged. Analyses are carried out for the initial phases corresponding to FVS0 while the initial amplitudes are fixed as in the MR case.

The amplitude evolution for the case of \( \phi_{1 initial} = 240° \) is plotted for three frequency cases in Fig. 18(a). We see that within a streamwise extent up to \( x \approx 500 \) mm, the growth rate of the subharmonic wave is greater for the case of high frequency (\( f_{1} = 119.9 \) Hz). This rapid growth of the subharmonic wave is a consequence of the large amplitude of the fundamental wave in this region [see Fig. 9(a) and 10],
Moreover, the generation of large amplitude of the fundamental wave is caused by a large growth rate in the upstream region. In terms of the linear stability for the individual fundamental wave, the streamwise extent of the unstable region, including neutral points and the location of the maximum growth rate, depends on the frequency and is located further downstream as the frequency decreases. Among the three frequencies considered, the frequency of $f_1 = 119.9$ Hz is found to exhibit the largest growth rate in the upstream region, which leads to the largest amplitude. However, the growth rate of low frequencies becomes larger than that of high frequencies further downstream. Consequently, the amplitude of the lower frequency eventually exceeds that of the higher frequency at a certain streamwise location downstream. At $x = 500$ mm, the amplitude of the fundamental wave of the $f_1 = 109$ Hz case already exceeded that of the 119.9 Hz case.

The amplitudes at $x = 510$ mm with respect to the initial phase difference ($\Delta \varphi_{\text{initial}}$) are plotted in Fig. 18(b). At this streamwise location, the fundamental wave has reached nearly the same amplitude for three frequency cases, as confirmed in Fig. 18(a). Meanwhile, the subharmonic wave amplitudes of the three cases have different amplitudes for the same initial phase difference. Over most of the initial phase difference, the largest amplitude is reached for the case of higher frequency (119.9 Hz). The results indicate that the growth of the subharmonic wave in subharmonic resonance depends not only on the initial amplitudes and phases of the instability waves but also on the frequency. As can be seen in the figure, the initial phase differences leading to resonance and anti-resonance conditions vary with the frequency. The initial phase differences corresponding to anti-resonance are $-25^\circ$, $-35^\circ$, and $-30^\circ$ for the cases of 98.1, 109, and 119.9 Hz, respectively. The initial phase difference that leads to resonance and anti-resonance does not appear to considerably vary with the fundamental frequency. As shown in Fig. 17, the evolution of the phase difference ($\Delta \varphi_{1/2}$) was also evaluated and compared for three frequency cases. The results (although not shown here) indicated that the value at which the phase difference converges was nearly unchanged between approximately 85° and 90°, regardless of the fundamental frequencies considered.

To examine the effect of the spanwise wavenumber, further analyses are carried out for several fundamental spanwise wavenumbers. Three spanwise wavenumbers [i.e., 0.9-, 1.25-, and 1.5-times that of baseline case ($\beta = 0.131$ rad/mm)] are additionally chosen. The corresponding physical values of the spanwise wavenumbers are 0.1179, 0.1638, and 0.1965 rad/mm, respectively. The analyses are performed for the initial phases corresponding to FVS0, and the initial amplitudes are fixed at those of the MR case. The mean flow and fundamental frequency remain unchanged from the baseline case. Figure 19 shows the evolution of amplitudes for four spanwise wavenumbers with $\varphi_1_{\text{initial}} = 240^\circ$ and $60^\circ$. For the case of $\varphi_1_{\text{initial}} = 240^\circ$ [Fig. 19(a)], it seems that the results for the 0.9$\beta$, 1.0$\beta$, and 1.25$\beta$ cases exhibit small differences in the subharmonic wave amplitude, and a considerable difference is observed only for the case of 1.5$\beta$. Among all the cases, the 1.5$\beta$ case shows the smallest amplitude at $x = 510$ mm (indicated with a vertical dashed line). Meanwhile, for the case of $\varphi_1_{\text{initial}} = 60^\circ$ [Fig. 19(b)], the subharmonic wave amplitude varies with the spanwise wavenumber, and the 1.5$\beta$ case shows the largest amplitude at $x = 510$ mm. A comparison of the results of Figs. 19(a) and 19(b) implies that the initial phase dependency of the subharmonic resonance varies considerably with the spanwise wavenumber.

This feature can be clearly confirmed in Fig. 20, which shows the amplitudes of the fundamental and subharmonic waves at $x = 510$ mm for the cases of various initial phase differences. We see that the dependency on the initial phase difference considerably differs with the spanwise wavenumber. The initial phase difference corresponding to the anti-resonance condition is found to change with the spanwise wavenumber. The values for the 0.9$\beta$, 1.0$\beta$, 1.25$\beta$, and 1.5$\beta$ cases appear to be $\Delta \varphi_{\text{initial}} = -50^\circ$, $-35^\circ$, $-25^\circ$, and $-152.5^\circ$ ($-27.5^\circ$), respectively. Noting that the analysis is carried out for discrete values of the initial phase, we can regard the change in the initial phase difference for anti-resonance as nearly proportional to the change in the
spanwise wavenumber. The overall pattern of dependency on the initial phase difference is found to be shifted to the right as the spanwise wavenumber increases. In Figs. 18–20, the effect of the spanwise wavenumber is found to be much greater than that of the fundamental frequency on the initial phase difference that leads to the resonance or anti-resonance condition.

As with Fig. 17, the evolution of the phase difference is also investigated for fundamental frequency and spanwise wavenumber cases other than the baseline case. For example, the results for four spanwise wavenumber cases (Figs. 19 and 20) are plotted together in Fig. 21. The phase difference is plotted only for selected cases of the initial phase. We see from the figure that the initial tendency of the evolution of the phase difference varies with the spanwise wavenumber. For the cases of higher spanwise wavenumbers (1.25β and 1.5β), the phase difference initially exhibits a tendency to decrease, whereas it initially increases for the cases of lower spanwise wavenumbers (0.9β and 1.0β). For a given initial phase, the phase difference shows different transient behaviors, depending on the spanwise wavenumber. As observed in Fig. 20, the initial phase difference that leads to the anti-resonance condition depends on the spanwise wavenumber. The results indicate that the initial phase difference corresponding to anti-resonance yields the largest oscillatory amplitude and streamwise extents of transient behavior. Meanwhile, after the transient behavior, the phase difference eventually converges to values in a specific range (i.e., 80°–90°), regardless of the spanwise wavenumber. Although it is not shown in the figure repeatedly, the same characteristics are also observed from the results for the cases of different fundamental frequencies (Fig. 18). These observations indicate that the subharmonic resonance stage can be achieved at a certain phase difference between the fundamental and subharmonic waves. The phase difference representing the subharmonic resonance appears to be between 85° and 90° for the boundary layer in the adverse pressure gradient considered in this study. The results reveal that the subharmonic resonance is reached eventually regardless of the imposed initial phase. The initial phase difference only causes the location of the initiation of the subharmonic resonance to vary.

3. Case of zero pressure gradient

Additional investigations are carried out for the flat plate at the zero pressure gradient (i.e., the Blasius boundary layer). The wave properties (i.e., frequency and spanwise wavenumber) and mean flow are the same as those in the case of Kachanov and Levchenko, which
were used for the validation of the PSE in Sec. II C. The initial amplitudes are set as the maximum amplitude of 0.46% and 0.01% at \( R = 400 \) for the fundamental and subharmonic waves, respectively. These amplitudes were used for comparison with the LES results (see Fig. 1). Similar to the previous study, several combinations of the initial phases of the fundamental and subharmonic waves (\( \phi_1 \) initial and \( \phi_{1/2} \) initial) are considered, as summarized in Table IV.

The downstream evolution of the maximum amplitude of the representative modes along the downstream direction is shown in Fig. 22 for several initial phase angles selected from the F0SV case. As with the preceding observation, the amplitude of the subharmonic wave in the early stage (up to \( R \approx 550 \)) exhibits various tendencies according to the initial phase. Among the results shown in Fig. 22, the case of the initial phase of 90\(^\circ\) corresponds to the fastest amplification of the subharmonic wave. Meanwhile, the case of the initial phase of 30\(^\circ\) represents the lowest amplification of the subharmonic wave, owing to the initial attenuation up to \( R \approx 500 \). We see that the initial phase does not yield a noticeable influence on the amplification of the fundamental wave (\( f_1 \)) and its second harmonic (\( f_2 \)). When the subharmonic wave amplitude exceeds a certain level, the interaction between the modes enters a post-parametric or fully nonlinear stage in which the amplitudes of all the modes including the fundamental wave and its second harmonic are increased together, which leads to the final breakdown. Depending on the initial phase, it can be seen that the post-parametric stage starts at different streamwise locations. The larger the subharmonic wave amplitude, the faster is the initiation of the fully nonlinear stage.

The LES results shown in Fig. 1 correspond to the case of an initial phase of 0\(^\circ\) and are plotted in Fig. 22 for comparison. Additional LES analyses are carried out for several initial phases to clarify the validity of using PSE for the parametric study conducted in the present study. The LES results for the case of an initial phase of 60\(^\circ\) are plotted together. The results from PSE and LES show fairly good agreement in terms of the different behavior in the amplitude evolution according to the imposed initial phase. This confirms that the PSE effectively captures the evolution of instability waves in subharmonic resonance considering the phase difference effect comparable to the high-fidelity numerical simulation.

Figure 23 shows the amplitudes of several modes with respect to the initial phase of the subharmonic wave (\( \phi_{1/2} \) initial) at two streamwise locations marked with vertical dashed lines in Fig. 22 (\( R \approx 601 \) and 675). As has been observed, the subharmonic and higher modes (i.e., \( f_{3/2}, f_{4/2}, \) and \( f_{5/2} \)) show strong dependency on the initial phase. The results exhibit repeatability with a cycle of 180\(^\circ\). The influence of the initial phase does not appear on the fundamental wave and its second harmonic at \( R \approx 601 \) [see Fig. 23(a)]. Further downstream, as discussed previously, the subharmonic wave amplitude exceeds the fundamental wave amplitude, and all modes enter the fully nonlinear stage. At \( R \approx 675 \) [see Fig. 23(b)], the dependency on the initial phase

| Case       | \( \phi_1 \) initial       | \( \phi_{1/2} \) initial   | Remarks                  |
|------------|---------------------------|----------------------------|--------------------------|
| F0SV       | 0\(^\circ\) (fixed)       | 0\(^\circ\)–360\(^\circ\) (\( \Delta \phi = 5\(^\circ\)\)) | Results in Figs. 22 and 23 |
| F60SV      | 60\(^\circ\) (fixed)      | 0\(^\circ\)–225\(^\circ\) (\( \Delta \phi = 15\(^\circ\)\)) |                          |
| FVS60      | 0\(^\circ\)–150\(^\circ\) (\( \Delta \phi = 30\(^\circ\)\)) | 60\(^\circ\) (fixed)     |                          |
| D30        | 0\(^\circ\)–330\(^\circ\) (\( \Delta \phi = 30\(^\circ\)\)) | \( \phi_1 \) initial + 30\(^\circ\) | \( \phi_{1/2} \) initial – \( \phi_1 \) initial = 30\(^\circ\) |
appears even for the fundamental wave ($f_1$) and its second harmonic ($f_2$). This indicates that there are back-influences on the fundamental wave from the subharmonic wave, and purely parametric resonant amplifications have been ended.

Based on the PSE results shown in Fig. 23, the initial phase corresponding to resonance and anti-resonance is found to be $\phi_{1/2 \text{ initial}} = 105^\circ$ and $15^\circ$, respectively. Further comparisons of PSE and LES results for these two conditions are made in Fig. 24. Good agreement is confirmed for all modes considered in the case of the resonance condition. Although relatively notable discrepancies for the amplitudes of the subharmonic wave and higher mode ($f_{3/2}$) arise, fair agreement, in general, behavior is still identifiable for the case of the anti-resonance condition ($\phi_{1/2 \text{ initial}} = 15^\circ$). This discrepancy is mainly due to the very high sensitivity on the initial phase angle in the vicinity of the anti-resonance condition. In contrast to the region around the resonance condition, a slight change in the initial phase angle can yield a considerable change in the amplitude near the anti-resonance condition (see Fig. 23). Further investigations for other initial phase angles reveal that the PSE and LES results have slightly different values of the initial phase angle corresponding to the anti-resonance condition. The initial phase of the anti-resonance condition identified from the LES calculations is found to differ by approximately $5^\circ$ compared with that of PSE results. The comparison of the results for $\phi_{1/2 \text{ initial}} = 15^\circ$ (see Fig. 24) is consistent with this observation in the sense that the LES result represents the simulation of the initial phase, which is not exact but close to the anti-resonance condition.

To confirm the general dependency of the subharmonic resonance on the initial phase, we conducted additional calculations for all the cases listed in Table IV. The results are plotted in Fig. 25. The amplitudes for the subharmonic and higher modes at $R \approx 601$ are compared. The amplitudes are plotted with $\Delta \phi_{\text{initial}}$ as the abscissa and the results for all combinations considered fall on a single curve. This confirms that the dependency of subharmonic resonance on the initial phase can be regarded as a function of a single parameter [i.e., the initial phase difference ($\Delta \phi_{\text{initial}}$)]. It can be concluded from the figure that the subharmonic resonant interaction is affected by the relative initial phase difference between the fundamental and subharmonic waves rather than the initial phase of only the fundamental or subharmonic wave.

As in Secs. III C 1–2, the downstream evolution of the phase difference $\Delta \phi_{1/2}$, is also examined for the present case of the zero pressure gradient. The results for several initial phases of the subharmonic wave from the F0SV case are presented in Fig. 26. The phase angle for each wave is evaluated at the height of the maximum amplitude of the subharmonic wave ($y_{\text{max}}$).

The downstream evolution of the phase difference shows various behaviors depending on the initial phase. The variety of the initial behavior is apparent in the early region up to $R \approx 560$. In this region, the phase difference seems to eventually change toward a specific value. Meanwhile, beyond the region, the phase difference nearly remains at a specific value of approximately $82^\circ$, regardless of the initial phase. This tendency is nearly the same as the observation made...
for the case of the adverse pressure gradient (see Figs. 17 and 21). From a comparison of the results given in Figs. 17, 21, and 26, we can readily conclude that the phase difference converges to approximately $80^\circ$--$90^\circ$ after entering the parametric resonance stage for both zero and adverse pressure gradients.

IV. CONCLUSION

We thoroughly investigated and assessed the influence of the initial phase of fundamental and subharmonic waves on the subharmonic resonant interaction. Nonlinear analysis PSE was conducted to calculate the downstream nonlinear evolution of the instability waves imposed at the initial location. The subharmonic resonance in a flat plate boundary layer with zero and adverse pressure gradient was considered following previous experimental studies. Blasius and Falkner–Skan solutions were used as the mean flow for PSE analysis for the cases of zero and adverse pressure gradients, respectively. The pressure-gradient parameter for the adverse pressure-gradient case was determined to match the amplification of both fundamental and subharmonic waves with the available experimental data. The results on the downstream evolutions of the amplitude of the instability waves showed good agreement with the measurement data and LES results, not only for the fundamental and subharmonic waves but also for higher modes. The double-exponential growth of the subharmonic wave during the parametric stage with no back-influence was confirmed in line with observations from previous studies. The dependence on the initial amplitudes was also confirmed in line with the experimental observations. The downstream evolution of the phase difference between the fundamental and subharmonic waves during the resonant interaction coincided well with the experimental data.

Analyses were carried out for various combinations of initial phases of fundamental and subharmonic waves. As with observations made from the experimental studies, the amplification of the subharmonic wave and related higher modes shows a strong dependence on the initial phases. The amplitude variation with respect to the initial phase also matched well with the experimental data obtained by imposing controlled disturbance. There were certain combinations of phase angles that led to the resonance or anti-resonance (maximum or minimum growth of the subharmonic wave, respectively) condition. The initial amplitude of neither the fundamental wave nor the subharmonic wave affected the phase dependence within the amplitude range considered. From the examination of the results for various combinations of the initial phase, the phase dependence was confirmed to be a function of a single independent parameter $[\Delta \phi_{\text{initial}} = \varphi_{1/2 \text{ initial}} - 0.5 \varphi_{1 \text{ initial}}]$ that represents the initial phase difference between the fundamental and subharmonic waves. In other words, the growth in the subharmonic resonant interaction depends on the relative initial phase rather than the initial phase of only the fundamental or subharmonic wave. Proceeding downstream, the phase difference started to change from the initial imposed value and eventually converged to a specific value regardless of the initial phase difference. The phase difference did not begin to change until the subharmonic resonant interaction (parametric stage) was entered. This observation indicated that the transient behavior of the phase difference occurs in the region in the streamwise direction in which the subharmonic wave grows double-exponentially (i.e., during the subharmonic resonant interaction). Through the parametric stage, the value of the phase difference that eventually converged was identified to be approximately between $80^\circ$ and $90^\circ$. The general characteristic of the phase dependence mentioned above remained unchanged even for other frequencies and spanwise wavenumbers studied. The value of the initial phase difference that leads to the anti-resonance condition was found to depend largely on the spanwise wavenumber. The dependence on the initial phase was the same for the boundary layer at zero and adverse pressure gradients. This indicates that the streamwise pressure gradient does not affect the qualitative feature, basic mechanisms, and properties of the subharmonic resonant interaction.

The analysis framework used in this study can be extended to studies of other configurations, flow conditions, and flow types. Detailed investigations on boundary layers over airfoils can be
considered as future work to examine the influence of the initial phase on resonant interaction of instability waves in variable pressure gradient conditions. The initial phase influence on fundamental or subharmonic breakdowns in hypersonic boundary layers is also a good candidate for further study. Accumulation results can contribute to a database or to physical foundations to develop a methodology for the control of stability and transition of boundary layers. The method of analysis can also be used to investigate and explore the possible dominant resonance interaction mechanisms and phase dependencies in free shear layers, such as mixing layer and wake flow.

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DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon request.

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