QUANTUM MECHANICS AND RELATIVITY
— THEIR UNIFICATION BY LOCAL TIME —

HITOSHI KITADA

Department of Mathematical Sciences
University of Tokyo
Komaba, Meguro, Tokyo 153, Japan
e-mail: kitada@ms.u-tokyo.ac.jp

Abstract. In a framework of a stationary universe, time is defined as a local and quantum-mechanical notion in the sense that it is defined for each local and quantum-mechanical system consisting of a finite number of particles. In this context, the total universe consisting of an infinite number of particles has no time associated, and quantum mechanics and general theory of relativity are united consistently. Relativistic Hamiltonians including gravitation are derived as a consequence of our treatment of observation. Related open problems in mathematical physics are presented.

Introduction

Physics is a work to explain phenomena, i.e. a job to give a description of visible events. Insofar as we understand physics as such activities, it is neither surprise nor ridiculous thing if one takes other ways in explaining phenomena than the present physics: The problem of combining relativity and quantum theories, which has been an old and difficult one, might be able to be considered from a different viewpoint than the present trends where relativity theory is tried to be quantized or quantum mechanics is tried to be modified relativistically. It is enough if one can explain relativistic quantum-mechanical phenomena or observations of them, in a systematic way. The purpose of the present paper is to give an attempt in this direction to explain relativistic quantum-mechanical phenomena. To make clear the contrast of our approach to the current physics, we briefly review the problems of physics in relation with relativity and quantum theories.

As is well-known, the solution \( \psi = \psi(t) = \psi(x,t) \ (x \in \mathbb{R}^{3N}) \) of the Schrödinger equation for \( N(\geq 1) \) particles, numbered as \( \ell = 1, 2, \cdots, N \), with positions \( x_\ell = \)
(x_{\ell 1}, x_{\ell 2}, x_{\ell 3}) \in \mathbb{R}^3_{x_{\ell}} and masses \( m_\ell > 0 \):

\[
\frac{1}{i} \frac{d\psi}{dt}(t) + H\psi(t) = 0, \quad \psi(0) = \phi, \quad \phi \in \mathcal{D}(H) \subset L^2(\mathbb{R}^{3N}),
\]

\[
H = -\sum_{\ell=1}^{N} \sum_{k=1}^{3} \frac{1}{2m_\ell} \frac{\partial^2}{\partial x_{\ell k}^2} + \sum_{1 \leq i < j \leq N} V_{ij}(x_i - x_j),
\]

is invariant with respect to the Galilei transformation:

\[
x'_\ell = x_\ell - vt, \quad \ell = 1, 2, \cdots, N
\]

\[
t' = t
\]

up to a factor of absolute value 1:

\[
\exp \left[ i \sum_{\ell=1}^{N} \left( \frac{1}{2} m_\ell v^2 t - m_\ell v \cdot x_\ell \right) \right],
\]

consistently with Born’s interpretation ([Bo]). Here we adopted the unit system such that \( \hbar = \frac{h}{2\pi} = 1; v = (v_1, v_2, v_3) \in \mathbb{R}^3 \) denotes the velocity between two inertial frames of reference; and \( v \cdot x_\ell = \sum_{k=1}^{3} v_k x_{\ell k} \) is the inner product of \( v \) and \( x_\ell \). This implies that the Schrödinger equation is not invariant under Lorentz transformation: \( x'^{\mu'} = a_{\mu'}^\nu x^\nu \) from \( \mathbb{R}^4 \times \mathbb{R}^3_{x_{\ell}} \) to \( \mathbb{R}^4 \times \mathbb{R}^3_{x'_{\ell}} \) with \( x'^0 = ct' \), \( x'^k = x'_{\ell k} \), \( x^0 = ct \), and \( x^k = x_{\ell k} \). Here \( c \) is the speed of light in vacuum and the coefficients \( a_{\mu'}^\nu \) are independent of \( \ell \) and determined by the following condition with some extra informations:

\[
(x^{1'})^2 + (x^{2'})^2 + (x^{3'})^2 - (x'^0)^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 - (x^0)^2.
\]

Therefore the quantum mechanics which is described by Schrödinger equation is understood, in the current physical context, as incompatible with special theory of relativity. Even if the free energy part in \( (S) \) (i.e. the sum of the differential operators in \( (S) \)) is replaced by a Lorentz invariant one as we will do in \( (QMG) \) of subsection I.3.2, the Schrödinger equation is not Lorentz invariant. This fact is known as that the instantaneous force among particles, which depends only on the locations of those particles, is not relativistic, i.e. is not Lorentz invariant.

One of the features of the Schrödinger equation is that it yields the stability of matter (of the first kind, see [Li]), which is violated in the classical framework of Maxwell’s equations and Rutherford model of atoms, where atoms collapse by the continuous radiation of light from the electrons around the nucleus according to classical electromagnetism so that the electrons fall into the nucleus. However the Schrödinger operator \( H \) defined in \( (S) \) is bounded from below by some constant \( -L > -\infty \) in the sense that \( L^2 \)-inner product \( (Hf, f) \) satisfies

\[
(Hf, f) \geq -L(f, f)
\]

for any \( f \) belonging to the domain \( \mathcal{D}(H) \) of \( H \) under a suitable assumption on the pair potentials \( V_{ij} \). This means that the total energy of the quantum system does not
decrease below $-L$, therefore the system does not collapse. In this respect, quantum mechanics remedies the difficulty of classical theory, while it is not Lorentz invariant.

In 1928, Dirac [Di] introduced a system of equations, which is invariant under Lorentz transformation, and could explain some of the relativistic quantum-mechanical phenomena. However, Dirac operator is not bounded from below, and Dirac equation does not imply the stability of matter unlike the Schrödinger equation.

Dirac thus proposed an idea that the vacuum is filled with electrons with negative energy so that the electrons around the nucleus cannot fall into the negative energy anymore by the Pauli exclusion principle, which explains the stability of matter. However, if one has to consider plural kinds of elementary particles at a time, one has to introduce the vacuum which is filled with those plural kinds of particles, and the vacuum can depend on the number of the kinds of particles which one takes into account. The vacuum then may not be determined to be unique. Further if one has to include Bosons into consideration, the Pauli exclusion principle does not hold and the stability of matter does not follow. In this sense, the idea of “‘filling the negative energy sea;” unfortunately,” “is ambiguous in the many-body case,” as Lieb writes in [Li, p.33].

Quantum field theory is introduced (see [St] for a review) to overcome this difficulty as well as to explain the annihilation-creation phenomena of particles, which are familiar in elementary particle physics. Quantum field theory is a theory of infinite degrees of freedom. In the case of the Schrödinger equation (S), the degree of freedom is $3N$, the number of coordinates $x_{11}, x_{12}, x_{13}, \ldots, x_{N1}, x_{N2},$ and $x_{N3}$ of $N$ particles. Contrary to this, quantum field theory deals with the infinite number of particles, which makes it possible to discuss creation-annihilation processes inside the theory. However, since it deals with infinite number of freedom, even at the first step of the definition of the Hamiltonian of the system obtained by second quantization, there is a difficulty, the difficulty of divergence. This sort of difficulty appears at almost every stage of the development of the theory, and physicists had to find clever ways to avoid the difficulties at each step after the theory was introduced. Mathematically, the difficulty of divergence has not been overcome yet at all. Physicists however noticed that if one could get finite quantities in a systematic way by extracting some infinite quantities from the divergent quantities, then those finite quantities might express the reality. Actually in their explanation of Lamb shift, they seemed to have succeeded going in this way and to have been able to give predictions outstandingly close to experiments. However, the calculation done is up to the 6th or 8th order of a series giving Lamb shift or anomalous magnetic moment of electrons ([K-L]). Dyson noticed ([Dy]) that the series has symptom to diverge to infinity.

The procedure mentioned in the above to yield finite quantities from infinite ones is called process of “renormalization,” and still forms active areas of researches in theoretical physics. In the mathematical attempt, called “axiomatic quantum field theory,” which was planned to clarify the meaning of quantum field theory and construct the theory consistently, it is known that in some mathematical but important examples (see, e.g., [Fr]), renormalizability conditions and the axioms of quantum field theory yield that the theory must not involve interaction terms inside the theory. I.e., the theory is void as a physics.

These are the situation currently understood as an incompatibility problem between
quantum theory and special theory of relativity. In the case of general theory of relativity and quantum mechanics, the situation seems similar or no better (see, e.g., [Ish] for a review of the current approaches). The traditional attempts toward the unification of quantum theory and general relativity, like quantum gravity, superstring theory, and so on, are trying to find a way to unify them in a single layered theory where these two difficult theories should admit each other.

We present below an attempt in a different direction, where general theory of relativity and quantum mechanics are considered as independent aspects of nature, but as playing complementary roles to each other. Our approach may be called a two-layered theory, where these two theories have their own residences and they interfere only when observation is done. A procedure which describes the interference between them at observation will be our basis of explanation of relativistic quantum-mechanical phenomena.

Our spirit behind the procedure we will introduce below for that interference is that what is intrinsic is the quantum-mechanical aspect of nature, while relativity plays a role of glasses to see nature. This attitude is contrary to the one adopted by current physics, which in origin comes from the spirit of Einstein [Ein]:

Thus, according to the general theory of relativity, gravitation occupies an exceptional position with regard to other forces, particularly the electromagnetic forces, since the ten functions representing the gravitational field at the same time define the metrical properties of the space measured.

His position is that the metrical properties of space-time are intrinsic for nature, and space-time is a vessel of nature, into which other forces should be incorporated. In the framework of classical theory, electromagnetic forces can be treated in this direction in the sense that the equation for electromagnetic fields can be written as a tensor equation. In the framework of quantum theory, the characteristic of traditional approach is to treat gravity as a one which should be quantized, and the inclusion of other forces is a problem which is treated only after gravity is quantized successfully. In such attempts to quantize gravity or general theory of relativity, the canonical formalism of general theory of relativity is assumed usually, and this means that one has to introduce some global time coordinate which is common throughout the total universe. This itself produces a problem incompatible with the spirit of general theory of relativity that time is a local notion. If one would admit of introducing such a global time, it is difficult to reformulate general theory of relativity into canonical formalism even if gravitation is weak (see, e.g., [Ish]), and the quantization of gravity or space-time remains as a difficult problem even if one would defer to the global time.

To overcome these difficulties, we introduce a notion of local time $t_L$ which is proper to each local system $L$ consisting of a finite number of quantum-mechanical particles. Our local time is a quantum-mechanical notion inasmuch as it is defined in each quantum-mechanical system as a parameter $t_L$ in the exponent of the propagator $\exp(-it_L H_L)$ describing the propagation of the local system $L$. It is a local notion defined for each local system $L$ with a local Hamiltonian $H_L$, and this will enable us to regard the time $t_L$ as a classical general relativistic local time, proper to the center of mass of
the local system $L$, by identifying the classical particles with those centers of mass of local systems. We will show that these classical local times proper to the centers of mass of local systems constitute general relativistic notion of local times, compatible with the quantum mechanics inside each local system. The proof is, in part, a recall of the inclusion/exclusion assumption which has been adopted in physics, that the time $t_B$ of a bigger system $L_B$ which includes a smaller system $L_S$ dominates the time $t_S$ of $L_S$, i.e. the assumption that $t_S$ must be equal to $t_B$ if the system $L_B$ includes $L_S$. Apart from this traditional position on which physics has been founded, we retrieve the independence of each local system and its time coordinate among local systems, and liberate them from the bondage of inclusion/exclusion relation, which has been implicitly assumed for systems of physical particles. Geometrically expressed, our position may be formulated as a vector bundle with base space $X$ representing the Riemannian manifold consisting of classical particles, identified with the centers of mass of local systems, and with the local system $L$ which obeys the quantum mechanics on its own geometry being associated as a fibre to each point $x \in X$, which is identified with the center of mass of the local system $L$.

There is a theory by Prugovečki [Pru] successful, in a sense, in quantizing general relativity, where he modifies quantum mechanics and general relativity so that the usual results are obtained as limits of his theory. His approach looks similar to ours in that he associates a Lorentzian quantum-mechanical world to each point of a Riemannian manifold as a fibre, regarding the total universe as a vector or fibre bundle equipped with connections compatible with the Riemannian metric of the base Riemannian space. Our approach differs from his in the following points:

1. Each quantum-mechanical world associated to a point of a Riemannian manifold is Euclidean;
2. We do not introduce any connections among those Euclidean quantum mechanics;
3. We treat electromagnetic forces and gravity on the same level in our explanation of observation, under the assumption that gravitation is weak; and
4. Quantum mechanics and general theory of relativity are intact in our formulation.

The explanation of observation stated in the third item is our point and is realized by a procedure which yields a quantized Hamiltonian including gravity and electric forces on the same level.

In our explanation of observation, we appeal to a procedure which transforms quantum-mechanical quantities into the classical quantities which obey the relativistic change of coordinates among the Euclidean quantum-mechanical worlds, which we call local systems. The quantum-mechanical world associated to each point of a Riemannian manifold has no relation with the Riemannian metric of the base space of our vector bundle, for we do not define any connections among the quantum-mechanical worlds. The procedure which transforms quantum-mechanical quantities into classical quantities is consistent with the two aspects of nature, i.e. with the quantum-mechanical aspect inside local systems and the general relativistic aspect outside local systems, because the results obtained by transformations are just concerned with observed facts. The
relativity appears in our theory as “glasses,” which deform quantum-mechanical quantities into classical relativistic quantities at each step of quantum-mechanical evolution, so that the resultant classical quantities accord with the observation. The intrinsic for our theory is quantum mechanics inside local systems, and relativity modifies quantum-mechanical calculations to accord with observations.

Summing up, our point is in the liberation of local systems from the inclusion/exclusion relation which has been an implicit assumption of physics. Instead of the inclusion/exclusion relation, we introduce a relation which transforms the quantum-mechanical values to classical relativistic values, as a procedure describing the interference between the two aspects of nature, the general relativistic aspect and the quantum-mechanical aspect.

In Part I of the paper, we give a presentation of our theory without using the notion of vector bundle. We first recall in section I.1 the basic notions related with the definition of local times from [Ki]. This notion of local times is a quantum-mechanical one defined in each local system consisting of a finite number of quantum-mechanical particles. In the sense that the local time is a local notion, it serves an ingredient which adheres the two layers: general theory of relativity and quantum mechanics. A result in many body quantum scattering is used to assign the usual meaning of time to our notion of local times. In section I.2, we review the proof of the consistency of the notion of local times with general theory of relativity. We give, in section I.3, a procedure of interpreting observation of quantum-mechanical process through the glasses of the relativity, yielding a relativistic quantum-mechanical Hamiltonian which explains gravitation and electric forces in quantum-mechanical way. In Part II, we treat two examples following the spirit of Part I. We present some open problems related with our formulation of physics in Part III.

Part I. Local Time and Observation

I.1. Local Time

To state our definition of quantum-mechanical local times, we begin with introducing a stationary universe \( \phi \). What we adopt here for the universe may be called a closed universe, within which is all and which has a definite property specified by a certain quantum-mechanical condition.

Let \( \mathcal{H} \) be a separable Hilbert space, and set

\[
U = \{ \phi \} = \bigoplus_{n=0}^{\infty} \left( \bigoplus_{\ell=0}^{\infty} \mathcal{H}^{n} \right) \quad (\mathcal{H}^{n} = \mathcal{H} \otimes \cdots \otimes \mathcal{H} \text{ with } n \text{ factors}).
\]

\( U \) is called a Hilbert space of possible universes. An element \( \phi \) of \( U \) is called a universe and is of the form of an infinite matrix \( (\phi_{n\ell}) \) with components \( \phi_{n\ell} \in \mathcal{H}^{n} \). \( \phi = 0 \) means \( \phi_{n\ell} = 0 \) for all \( n, \ell \).
Let $\mathcal{O} = \{A\}$ be the totality of the selfadjoint operators $A$ in $\mathcal{U}$ of the form $A\phi = (A_{n\ell} \phi_{n\ell})$ for $\phi = (\phi_{n\ell}) \in \mathcal{D}(A) \subset \mathcal{U}$, where each component $A_{n\ell}$ is a selfadjoint operator in $\mathcal{H}^n$. We assume the following condition for our universe $\phi$.

**Axiom 1.** There is a selfadjoint operator $H \in \mathcal{O}$ in $\mathcal{U}$ such that for some $\phi \in \mathcal{U} - \{0\}$ and $\lambda \in \mathbb{R}$

$$H\phi = \lambda\phi \quad \text{(U)}$$

in the following sense: Let $F_n$ be a finite subset of $\mathbb{N} = \{1, 2, \cdots \}$ with $\sharp(F_n) (= \text{the number of elements in } F_n) = n$ and let $\{F_n\}_{n=0}^\infty$ be a countable set of such $F_n$. Then the formula (U) in the above means that there are integral sequences $\{n_k\}_{k=1}^\infty$ and $\{\ell_k\}_{k=1}^\infty$ and a real sequence $\{\lambda_{n_k\ell_k}\}_{k=1}^\infty$ such that $F_{n_k}^{\ell_k} \subset F_{n_{k+1}}^{\ell_{k+1}} \cup \bigcup_{k=1}^\infty F_{n_k}^{\ell_k} = \mathbb{N}$;

$$H_{n_k\ell_k} \phi_{n_k\ell_k} = \lambda_{n_k\ell_k} \phi_{n_k\ell_k}, \quad \phi_{n_k\ell_k} \neq 0, \quad k = 1, 2, 3, \cdots; \quad \text{(Eigen)}$$

and

$$\lambda_{n_k\ell_k} \to \lambda \quad \text{as} \quad k \to \infty.$$

$H$ is an infinite matrix $(H_{n\ell})$ of selfadjoint operators $H_{n\ell}$ in $\mathcal{H}^n$. Axiom 1 asserts that this matrix converges in the sense of (U) on our universe $\phi$. We remark that our universe $\phi$ is not determined uniquely by this condition.

The universe as a state $\phi$ is a whole, within which is all. As such a whole, the state $\phi$ can follow the two ways: The one is that $\phi$ develops along a global time $T$ in the grand universe $\mathcal{U}$ under a propagation $\exp(-iTH)$, and another is that $\phi$ is a bound state of $H$. If there were such a global time $T$ as in the first case, all phenomena had to develop along that global time $T$, and the locality of time would be lost. We could then not construct a notion of local times compatible with general theory of relativity. The only one possibility is therefore to adopt the stationary universe $\phi$ of Axiom 1.

The following axiom asserts the existence of configuration and momentum operators and that the canonical commutation relation between them holds. This is a basis of our definition of time, where configuration and momentum are given first, and then local times are defined in each local system of finite number of quantum-mechanical particles.

**Axiom 2.** Let $n \geq 1$ and $F_{n+1}$ be a finite subset of $\mathbb{N} = \{1, 2, \cdots \}$ with $\sharp(F_{n+1}) = n + 1$. Then for any $j \in F_{n+1}$, there are selfadjoint operators $X_j = (X_{j1}, X_{j2}, X_{j3})$ and $P_j = (P_{j1}, P_{j2}, P_{j3})$ in $\mathcal{H}^n$, and constants $m_j > 0$ such that

$$[X_{j\ell}, X_{km}] = 0, \quad [P_{j\ell}, P_{km}] = 0, \quad [X_{j\ell}, P_{km}] = i\delta_{jk}\delta_{\ell m},$$

$$\sum_{j \in F_{n+1}} m_j X_j = 0, \quad \sum_{j \in F_{n+1}} P_j = 0.$$
What we want to mean by the \((n, \ell)\)-th component \(H_{n\ell}\) \((n, \ell \geq 0)\) of \(H = (H_{n\ell})\) in Axiom 1 is the usual \(N = n + 1\) body Hamiltonian with center of mass removed in accordance with the requirement \(\sum_{j \in F_{n+1}} m_j x_j = 0\) in Axiom 2. For the local Hamiltonian \(H_{n\ell}\) we thus make the following postulate.

**Axiom 3.** The component Hamiltonian \(H_{n\ell}\) \((\ell \geq 0)\) of \(H\) in Axiom 1 is of the form

\[
H_{n\ell} = H_{n\ell 0} + V_{n\ell}, \quad V_{n\ell} = \sum_{\alpha=(i,j)} V_\alpha(x_\alpha) \quad 1 \leq i < j < \infty, i,j \in F^\ell_N
\]
on \(C^\infty_0(\mathbb{R}^3n)\), where \(x_\alpha = x_i - x_j\) with \(x_i\) being the position vector of the \(i\)-th particle, and \(V_\alpha(x_\alpha)\) is a real-valued measurable function of \(x_\alpha \in \mathbb{R}^3\) which is \(H_{n\ell 0}\)-bounded with \(H_{n\ell 0}\)-bound of \(V_{n\ell}\) less than 1. \(H_{n\ell 0} = H_{(N-1)\ell 0}\) is the free Hamiltonian of the \(N\)-particle system, whose concrete form is similar to the interaction-free part of \(H\) in \((S)\) of the introduction.

This axiom implies that \(H_{n\ell} = H_{(N-1)\ell}\) is uniquely extended to a selfadjoint operator bounded from below in \(\mathcal{H}^n = \mathcal{H}^{N-1} = L^2(\mathbb{R}^{3(N-1)})\) by the Kato-Rellich theorem.

We do not include vector potentials in the Hamiltonian \(H_{n\ell}\) of Axiom 3, for we take the position that what is elementary is the electronic charge, and the magnetic forces are the consequence of the motions of charges. Thus when we restrict our attention to a system consisting of the \(N\) number of particles, the vector potential is redundant to our argument. It would be, however, a good approximation to introduce vector potentials, when we consider a subsystem of a bigger system, and we concentrate on the analysis of the behavior of that subsystem inside the bigger system.

Let \(P_H\) denote the orthogonal projection onto the space of bound states for a self-adjoint operator \(H\). We recall that a state orthogonal to the space of bound states is called a scattering state. Let \(\phi = (\phi_{n\ell})\) with \(\phi_{n\ell} = \phi_{n\ell}(x_1, \ldots, x_n) \in L^2(\mathbb{R}^{3n})\) be the universe in Axiom 1, and let \(\{n_k\}\) and \(\{\ell_k\}\) be the sequences specified there. Let \(x^{(n,\ell)}\) denote the relative coordinates of \(n + 1\) particles in \(F^\ell_{n+1}\).

**Definition 1.**

1. We define \(\mathcal{H}_{n\ell}\) as the sub-Hilbert space of \(\mathcal{H}^n\) generated by the functions \(\phi_{n_k \ell_k}(x^{(n,\ell)}_N, y)\) of \(x^{(n,\ell)}_N \in \mathbb{R}^{3n}\) with regarding \(y \in \mathbb{R}^{3(n_k - n)}\) as a parameter, where \(k\) moves over a set \(\{k \mid n_k \geq n, F_{n+1} \subset F^\ell_{n_k+1}, k \in \mathbb{N}\}\).
2. \(\mathcal{H}_{n\ell}\) is called a local universe of \(\phi\).
3. \(\mathcal{H}_{n\ell}\) is said to be non-trivial if \((I - P_{\mathcal{H}_{n\ell}})\mathcal{H}_{n\ell} \neq \{0\}\).

The total universe \(\phi\) is a single element in \(\mathcal{U}\). The local universe \(\mathcal{H}_{n\ell}\) may be richer and may have elements more than one. This is because we consider the subsystems of the universe consisting of a finite number of particles. These subsystems receive the influence from the other particles of infinite number outside the subsystems, and may vary to constitute a non-trivial subspace \(\mathcal{H}_{n\ell}\).
in mind in Axiom 3. Consider, e.g., a system $H_{3\ell}$ consisting of four particles, the one of which has positive charge, and other three have negative charge. Then this system tends to scatter, i.e. it is probable that this system is in a scattering state with respect to the Hamiltonian $H_{3\ell}$ (see, e.g., [Cy, p.50] for a theorem asserting the absence of eigenvalues for a similar case). Add one particle with positive charge to this system to constitute a system $H_{n_k\ell_k}$ ($n_k = 4$). Then this new system may be in an eigenstate $\phi_{n_k\ell_k} = \phi_{4\ell_k}$ with respect to the extended Hamiltonian $H_{n_k\ell_k} = H_{4\ell_k}$ for some eigenvalue $\lambda_{n_k\ell_k} = \lambda_{4\ell_k}$ so that it satisfies the condition (Eigen) in the above for a $k$, while the restriction $\phi_{4\ell_k}(x^{(3,\ell)}, y)$ to $\mathbb{R}^9_{x^{(3,\ell)}}$, with $y \in \mathbb{R}^3$ arbitrary but fixed, of the bound state $\phi_{4\ell_k}$ of $H_{4\ell_k}$ is a scattering state of the original system $H_{3\ell}$. Here $y \in \mathbb{R}^3$ is the intercluster coordinates between the added particle of positive charge and the center of mass of the four particles in the system $H_{3\ell}$. Namely, the extended system $H_{4\ell_k}$ is in a bound state $\phi_{4\ell_k}$, while the restriction $\phi_{4\ell_k}(x^{(3,\ell)}, y)$ moves over the scattering states of $H_{3\ell}$ belonging to the Hilbert space $L^2(\mathbb{R}^9_{x^{(3,\ell)}})$ of the state vectors for the system $H_{3\ell}$, and constitutes a nontrivial subspace $\mathcal{H}_{3\ell}$ of $\mathcal{H}^3$ when $y$ varies.

**Definition 2.**

1. The restriction of $H$ to $\mathcal{H}_{n\ell}$ is also denoted by the same notation $H_{n\ell}$ as the $(n, \ell)$-th component of $H$.
2. We call the pair $(H_{n\ell}, \mathcal{H}_{n\ell})$ a local system.
3. The unitary group $e^{-iH_{n\ell}}$ ($t \in \mathbb{R}^1$) on $\mathcal{H}_{n\ell}$ is called the proper clock of the local system $(H_{n\ell}, \mathcal{H}_{n\ell})$, if $\mathcal{H}_{n\ell}$ is non-trivial: $(I - P_{H_{n\ell}})\mathcal{H}_{n\ell} \neq \{0\}$. (Note that the clock is defined only for $N = n + 1 \geq 2$, since $H_{0\ell} = 0$ and $P_{H_{0\ell}} = I$.)
4. The universe $\phi$ is called rich if $\mathcal{H}_{n\ell}$ equals $\mathcal{H}^n = L^2(\mathbb{R}^{3n})$ for all $n \geq 1, \ell \geq 0$. For a rich universe $\phi$, $H_{n\ell}$ equals the $(n, \ell)$-th component of $H$.

**Definition 3.**

1. The parameter $t$ in the exponent of the proper clock $e^{-iH_{n\ell}} = e^{-iH_{(N-1)\ell}}$ of a local system $(H_{n\ell}, \mathcal{H}_{n\ell})$ is called the (quantum-mechanical) proper time or local time of the local system $(H_{n\ell}, \mathcal{H}_{n\ell})$, if $(I - P_{H_{n\ell}})\mathcal{H}_{n\ell} \neq \{0\}$.
2. This time $t$ is denoted by $t(H_{n\ell}, \mathcal{H}_{n\ell})$ indicating the local system under consideration.

This definition is a one reverse to the usual definition of the motion or dynamics of the $N$-body quantum systems, where the time $t$ is given a priori and then the motion of the particles is defined by $e^{-itH_{(N-1)\ell}}f$ for a given initial state $f$ of the system.

*Time* is thus defined only for local systems $(H_{n\ell}, \mathcal{H}_{n\ell})$ and is determined by the associated proper clock $e^{-itH_{n\ell}}$. Therefore there are infinitely many number of times $t = t(H_{n\ell}, \mathcal{H}_{n\ell})$ each of which is proper to the local system $(H_{n\ell}, \mathcal{H}_{n\ell})$. In this sense time is a local notion. There is no time for the total universe $\phi$ in Axiom 1, which is a bound state of the total Hamiltonian $H$ in the sense specified by the condition (U) of Axiom 1.

To see the meaning of our definition of time, we quote a theorem from [En]. To state the theorem we make some notational preparation concerning the local system
(\(H_{n\ell}, \mathcal{H}_{n\ell}\)), assuming that the universe \(\phi\) is rich: Let \(b = (C_1, \cdots, C_{\sharp(b)})\) be a decomposition of the set \(\{1, 2, \cdots, N\}\) \((N = n + 1)\) into \(\sharp(b)\) number of disjoint subsets \(C_1, \cdots, C_{\sharp(b)}\) of \(\{1, 2, \cdots, N\}\). \(b\) is called a cluster decomposition. \(H_b = H_{n\ell,b} = H_{n\ell} - I_b = H_{n\ell}^{b} + T_{n\ell,b} = H^{b} + T_{b}\) is the truncated Hamiltonian for the cluster decomposition \(b\) with \(1 \leq \sharp(b) \leq N\), where \(I_b\) is the sum of the intercluster interactions between various two different clusters in \(b\), and \(T_b\) is the sum of the intercluster free energies among various clusters in \(b\). \(x_b \in \mathbb{R}^{3(\sharp(b) - 1)}\) is the intercluster coordinates among the centers of mass of the clusters in \(b\), while \(x^b \in \mathbb{R}^{3(N - \sharp(b))}\) denotes the intraccluster coordinates inside the clusters of \(b\) so that \(x \in \mathbb{R}^{3n} = \mathbb{R}^{3(N - 1)}\) is expressed as \(x = (x_b, x^b)\). Note that \(x^b\) is decomposed as \(x^b = (x^b_1, \cdots, x^b_{\sharp(b)})\), where each \(x^b_j \in \mathbb{R}^{3(\sharp(C_j) - 1)}\) is the internal coordinate of the cluster \(C_j\), describing the configuration of the particles inside \(C_j\). The operator \(H^b\) is accordingly decomposed as \(H^b = H_1 + \cdots + H_{\sharp(b)}\), and each component \(H_j\) is defined in the space \(\mathcal{H}^b_j = L^2(\mathbb{R}^{3(\sharp(C_j) - 1)})\), whose tensor product \(\mathcal{H}_1^b \otimes \cdots \otimes \mathcal{H}_{\sharp(b)}^b\) is the internal state space \(\mathcal{H}^b = L^2(\mathbb{R}^{3(N - \sharp(b))})\). The free energy \(T_b\) is defined in the external space \(\mathcal{H}_b = L^2(\mathbb{R}^{3(\sharp(b) - 1)})\), and the truncated Hamiltonian \(H_b = H^{b} + T_{b} = I \otimes H^{b} + T_{b} \otimes I\) is defined in the total space \(\mathcal{H}_{n\ell} = \mathcal{H}_b \otimes \mathcal{H}^b = L^2(\mathbb{R}^{3(N - 1)})\). \(v_b\) is the velocity operator conjugate to the intercluster coordinates \(x_b\). \(P_b = P_{H^b}\) is the eigenprojection associated with the subsystem \(H^b\) of \(H\), i.e. the orthogonal projection onto the eigenspace of \(H^b\), defined in \(\mathcal{H}^b\) and extended obviously to the total space \(\mathcal{H}_{n\ell}\). \(P^M_{b}\) is the \(M\)-dimensional partial projection of this eigenprojection \(P_b\). We define for a \(k\)-dimensional multi-index \(M = (M_1, \cdots, M_k)\), \(M_j \geq 1\) and \(k = 1, \cdots, N - 1\),

\[
\hat{P}^M_k = \left(I - \sum_{\sharp(b) = k} P^M_{b}\right) \cdots \left(I - \sum_{\sharp(d) = 2} P^M_{d}\right) (I - P^M_{1}),
\]

where note that \(P^M_{1} = P^M_{a1} = P_{H}^{M_1}\) for \(\sharp(a) = 1\) is uniquely determined. We also define for a \(\sharp(b)\)-dimensional multi-index \(M_b = (M_1, \cdots, M_{\sharp(b) - 1}, M_{\sharp(b)}) = (\tilde{M}_b, M_{\sharp(b)})\)

\[
\hat{P}^M_b = P^M_{b1(\sharp(b))} \hat{P}^{\tilde{M}_b}_{\sharp(b) - 1}, \quad 2 \leq \sharp(b) \leq N.
\]

It is clear that

\[
\sum_{2 \leq \sharp(b) \leq N} \hat{P}^M_b = I - P^M_1,
\]

provided that the component \(M_k\) of \(M_b\) depends only on the number \(k\) but not on \(b\). In the following we use such \(M_b\)’s only. Under these circumstances, the following is known to hold.

**Theorem 1** ([En]). Let \(N = n + 1 \geq 2\) and let \(H_{N-1} = H_{n\ell}\) be the Hamiltonian for a local system \((H_{n\ell}, \mathcal{H}_{n\ell})\). Let suitable conditions on the decay rate for the pair potentials \(V_{ij}(x_{ij})\) be satisfied (see, e.g., Assumption 1 in [Ki(N)]). Let \(\|x^a\|^2 P^M_a < \infty\) be
satisfied for any integer \( M \geq 1 \) and cluster decomposition \( a \) with \( 2 \leq \sharp(a) \leq N - 1 \). Let \( f \in \mathcal{H}^{N-1} \). Then there is a sequence \( t_m \rightarrow \pm \infty \) (as \( m \rightarrow \pm \infty \)) and a sequence \( M_b^m \) of multi-indices whose components all tend to \( \infty \) as \( m \rightarrow \pm \infty \) such that for all cluster decompositions \( b, 2 \leq \sharp(b) \leq N, \) and \( \varphi \in C_0^\infty (\mathbb{R}^{3(\sharp(b) - 1)}) \)

\[
\| \{ \varphi(x_m/t_m) - \varphi(v_b) \} P_b^{M_b} e^{-it_m H_N^{-1} f} \| \rightarrow 0
\]  

(A)

as \( m \rightarrow \pm \infty \).

The asymptotic relation (A) roughly means that, if we restrict our attention to the part \( \tilde{P}_b M_b^{\infty} \) of the evolution \( e^{-it H_N^{-1} f} \), in which the particles inside any cluster of \( b \) are bounded while any two different clusters of \( b \) are scattered, then the quantum-mechanical velocity \( v_b = m_b^{-1} p_b \), where \( m_b \) is some diagonal mass matrix, is approximated by a classical value \( v_b^{(c)} = \lim_{m \rightarrow \pm \infty} (v_b) \tilde{P}_b^{M_b^e} e^{-it_m H_N^{-1} f}, \tilde{P}_b^{M_b^e} e^{-it_m H_N^{-1} f} \) asymptotically as \( m \rightarrow \pm \infty \) and the local time \( t \) of the \( N \) body system \( H_{N-1} = H_{n\ell} \) is asymptotically equal to the quotient of the configuration by the velocity of the scattered particles (or clusters, exactly speaking):

\[
\frac{|x_b|}{v_b^{(c)}}.
\]  

(Q)

This means by \( v_b = m_b^{-1} p_b \) that the local time \( t \) is asymptotically and approximately measured if the values of the configurations and momenta for the scattered particles of the local system \( (H_{N-1}, H_{N-1}) = (H_{n\ell}, H_{n\ell}) \) are given.

We note that the time measured by (Q) is independent of the choice of cluster decomposition \( b \) according to Theorem 1. This means that \( t \) can be taken as a common parameter of motion inside the local system, and can be called time of the local system in accordance with the notion of ‘common time’ in Newton’s sense: “relative, apparent, and common time, is some sensible and external (whether accurate or unequable) measure of duration by the means of motion, ...” ([New], p.6). Once we take \( t \) as our notion of time for the system \( (H_{n\ell}, H_{n\ell}) \), \( t \) recovers the usual meaning of time, by the identity for \( e^{-it H_{n\ell} f} \) known as the Schrödinger equation:

\[
\left( \frac{1}{i} \frac{d}{dt} + H_{n\ell} \right) e^{-it H_{n\ell} f} = 0.
\]

Time \( t = t_{(H_{n\ell}, H_{n\ell})} \) is a notion defined only in relation with the local system \( (H_{n\ell}, H_{n\ell}) \). To other local system \( (H_{mk}, H_{mk}) \), there is associated other local time \( t_{(H_{mk}, H_{mk})} \), and between \( t = t_{(H_{n\ell}, H_{n\ell})} \) and \( t_{(H_{mk}, H_{mk})} \), there is no relation, and they are completely independent notions. In other words, \( H_{n\ell} \) and \( H_{mk} \) are different spaces unless \( n = m \) and \( \ell = k \). And even when the two local systems \( (H_{n\ell}, H_{n\ell}) \) and \( (H_{mk}, H_{mk}) \) have a non-vanishing common part: \( F_{n+1}^k \cap F_{m+1}^k \neq \emptyset \), the common part constitutes its own local system \( (H_{pj}, H_{pj}) \), and its local time cannot be compared with those of the two bigger systems \( (H_{n\ell}, H_{n\ell}) \) and \( (H_{mk}, H_{mk}) \), because these three systems have different base spaces, Hamiltonians, and clocks. More concretely speaking, the times
are measured through the quotients (Q) for each system. But the $L^2$-representations of the base Hilbert spaces $\mathcal{H}_{n\ell}, \mathcal{H}_{mk}, \mathcal{H}_{pj}$ for those systems are different unless they are identical with each other, and the quotient (Q) has incommensurable meaning among these representations.

In this sense, local systems are independent mutually. Also they cannot be decomposed into pieces in the sense that the decomposed pieces constitute different local systems.

I.2. Relativity

We note that the center of mass of a local system $(H_{n\ell}, \mathcal{H}_{n\ell})$ is always at the origin of the space coordinate system $x_{(H_{n\ell}, \mathcal{H}_{n\ell})} \in \mathbb{R}^3$ for the local system by the requirement: $\sum_{j \in F_{n,1}} m_j x_j = 0$ in Axiom 2, and that the space coordinate system describes just the relative motions inside a local system by our formulation. The center of mass of a local system, therefore, cannot be identified from the local system itself, except that it is at the origin of the coordinates.

Moreover, just as we have seen in the previous section, we see that, not only the time coordinates $t_{(H_{n\ell}, \mathcal{H}_{n\ell})}$ and $t_{(H_{mk}, \mathcal{H}_{mk})}$, but also the space coordinates $x_{(H_{n\ell}, \mathcal{H}_{n\ell})}$ and $x_{(H_{mk}, \mathcal{H}_{mk})}$ of these two local systems are independent mutually. Thus the space-time coordinates $(t_{(H_{n\ell}, \mathcal{H}_{n\ell})}, x_{(H_{n\ell}, \mathcal{H}_{n\ell})})$ and $(t_{(H_{mk}, \mathcal{H}_{mk})}, x_{(H_{mk}, \mathcal{H}_{mk})})$ are independent between two different local systems $(H_{n\ell}, \mathcal{H}_{n\ell})$ and $(H_{mk}, \mathcal{H}_{mk})$. In particular, insofar as the systems are considered as quantum-mechanical ones, there is no relation between their centers of mass. In other words, the center of mass of any local system cannot be identified by other local systems quantum-mechanically.

Summing these two considerations, we conclude:

(1) The center of mass of a local system $(H_{n\ell}, \mathcal{H}_{n\ell})$ cannot be identified quantum-mechanically by any local system $(H_{mk}, \mathcal{H}_{mk})$ including the case $(H_{mk}, \mathcal{H}_{mk}) = (H_{n\ell}, \mathcal{H}_{n\ell})$.

(2) There is no quantum-mechanical relation between any two local coordinates $(t_{(H_{n\ell}, \mathcal{H}_{n\ell})}, x_{(H_{n\ell}, \mathcal{H}_{n\ell})})$ and $(t_{(H_{mk}, \mathcal{H}_{mk})}, x_{(H_{mk}, \mathcal{H}_{mk})})$ of two different local systems $(H_{n\ell}, \mathcal{H}_{n\ell})$ and $(H_{mk}, \mathcal{H}_{mk})$.

Utilizing these properties of the centers of mass and the coordinates of local systems, we may make any postulates concerning

(1) the motions of the centers of mass of various local systems, and

(2) the relation between two local coordinates of any two local systems.

In particular, we may impose classical postulates on them as far as the postulates are consistent in themselves.

Thus we assume an arbitrary but fixed transformation:

$$y_2 = f_{21}(y_1) \quad (\text{Tr})$$
between the coordinate systems \( y_j = (y^i_j) = (ct_j, x_j) \) for \( j = 1, 2 \), where \( c \) is the speed of light in vacuum and \((t_j, x_j)\) is the space-time coordinates of the local system \( L_j = (H_{n_j}, \mathcal{H}_{n_j}) \). We regard these coordinates \( y_j = (ct_j, x_j) \) as classical coordinates, when we consider the motions of centers of mass and the relations of coordinates of various local systems. We now postulate the general principle of relativity on the physics of the centers of mass:

**Axiom 4.** The laws of physics which control the relative motions of the centers of mass of local systems are covariant under the change of the coordinates from \((ct(H_{mk}, \mathcal{H}_{mk}), x(H_{mk}, \mathcal{H}_{mk}))\) to \((ct(H_{nt}, \mathcal{H}_{nt}), x(H_{nt}, \mathcal{H}_{nt}))\) of the reference frame local systems for any pair \((H_{mk}, \mathcal{H}_{mk})\) and \((H_{nt}, \mathcal{H}_{nt})\) of local systems.

We note that this axiom is consistent with the Euclidean metric adopted for the quantum-mechanical coordinates inside a local system, because Axiom 4 is concerned with classical motions of the centers of mass outside local systems, and we are dealing here with a different aspect of nature from the quantum-mechanical one inside a local system.

Axiom 4 implies the invariance of the distance under the change of coordinates between two local systems. Thus the metric tensor \( g_{\mu\nu}(ct, x) \) which appears here satisfies the transformation rule:

\[
g^1_{\mu\nu}(y_1) = g^2_{\alpha\beta}(f_{21}(y_1)) \frac{\partial f_{21}^\alpha}{\partial y_1^\mu}(y_1) \frac{\partial f_{21}^\beta}{\partial y_1^\nu}(y_1),
\]

where \( y_1 = (ct_1, x_1) \); \( y_2 = f_{21}(y_1) \) is the transformation (Tr) in the above from \( y_1 = (ct_1, x_1) \) to \( y_2 = (ct_2, x_2) \); and \( g_{\mu\nu}(y_j) \) is the metric tensor expressed in the classical coordinates \( y_j = (ct_j, x_j) \) for \( j = 1, 2 \).

The second postulate is the principle of equivalence, which asserts that the classical coordinate system \((ct(H_{nt}, \mathcal{H}_{nt}), x(H_{nt}, \mathcal{H}_{nt}))\) is a local Lorentz system of coordinates, insofar as it is concerned with the classical behavior of the center of mass of the local system \((H_{nt}, \mathcal{H}_{nt})\):

**Axiom 5.** The metric or the gravitational tensor \( g_{\mu\nu} \) for the center of mass of a local system \((H_{nt}, \mathcal{H}_{nt})\) in the coordinates \((ct(H_{nt}, \mathcal{H}_{nt}), x(H_{nt}, \mathcal{H}_{nt}))\) of itself are equal to \( \eta_{\mu\nu} \), where \( \eta_{\mu\nu} = 0 \) for \( \mu \neq \nu, = 1 \) for \( \mu = \nu = 1, 2, 3 \), and \(-1\) for \( \mu = \nu = 0 \).

Since, at the center of mass, the classical space coordinates \( x = 0 \), Axiom 5 together with the transformation rule (Me) in the above yields

\[
g^1_{\mu\nu}(f_{21}^{-1}(ct_2, 0)) = \eta_{\alpha\beta} \frac{\partial f_{21}^\alpha}{\partial y_1^\mu}(f_{21}^{-1}(ct_2, 0)) \frac{\partial f_{21}^\beta}{\partial y_1^\nu}(f_{21}^{-1}(ct_2, 0)).
\]

Also by the same reason: \( x = 0 \) at the center of mass, the relativistic proper time \( d\tau = \sqrt{-g_{\mu\nu}(ct, 0)dy^\mu dy^\nu} = \sqrt{-\eta_{\mu\nu}dy^\mu dy^\nu} \) at the origin of a local system is equal to \( c \) times the quantum-mechanical proper time \( dt \) of the system.
By the fact that the classical Axioms 4 and 5 of physics are imposed on the centers of mass which are uncontrollable quantum-mechanically, and on the relation between the coordinates of different, therefore quantum-mechanically non-related local systems, the consistency of classical relativistic Axioms 4 and 5 with quantum-mechanical Axioms 1–3 is clear:

**Theorem 2.** *Axioms 1 to 5 are consistent.*

I.3. Observation

Thus far, we did not mention any about the physics which is actually observed. We have just given two aspects of nature which are mutually independent. We will introduce a procedure to yield what we observe when we see nature. This procedure will not be contradictory with the two aspects of nature which we have discussed, as the procedure is concerned solely with “*how nature looks, at the observer,*” i.e. it is solely concerned with “*at the place of the observer, how nature looks,*” with some abuse of the word “place.” The validity of the procedure should be judged merely through the comparison between the observation and the prediction given by our procedure.

We note that we can observe only a finite number of disjoint systems, say $L_1, \ldots, L_k$ with $k \geq 1$ a finite integer. We cannot grasp an infinite number of systems at a time. Further each system $L_j$ must have only a finite number of elements by the same reason. Thus these systems $L_1, \ldots, L_k$ may be identified with local systems in the sense of section I.1.

Local systems are quantum-mechanical systems, and their coordinates are confined to their insides insofar as we appeal to Axioms 1–3. However we postulated Axioms 4 and 5 on the classical aspects of those coordinates, which make the local coordinates of a local system a classical reference frame for the centers of mass of other local systems. This leaves us the room to define observation as the *classical* observation of the centers of mass of local systems $L_1, \ldots, L_k$. We call this an observation of $L = (L_1, \ldots, L_k)$ inquiring into sub-systems $L_1, \ldots, L_k$, where $L$ is a local system consisting of the particles which belong to one of the local systems $L_1, \ldots, L_k$.

When we observe the sub-local systems $L_1, \ldots, L_k$ of $L$, we observe the relations or motions among these sub-systems. Internally the local system $L$ behaves following the Hamiltonian $H_L$ associated to the local system $L$. However the actual observation differs from what the pure quantum-mechanical calculation gives for the system $L$. For example, when an electron is scattered by a nucleus with relative velocity close to that of light, the observation is different from the pure quantum-mechanical prediction.

In the usual explanation of this phenomenon, one introduces Dirac equation, and calculates differential cross section. However, the calculation only applies to that experiment or to the case which can be described by the Dirac equation, and no gravity is included.

We propose below a procedure which explains gravity as well as quantum-mechanical forces in one framework.
The quantum-mechanical process inside the local system \( L \) is described by the evolution
\[
\exp(-it_L H_L) f,
\]
when the initial state \( f \) of the system and the local time \( t_L \) of the system are given. The Hamiltonian \( H_L \) is decomposed as follows in virtue of the local Hamiltonians \( H_1, \ldots, H_k \), which correspond to the sub-local systems \( L_1, \ldots, L_k \):
\[
H_L = H^b + T + I, \quad H^b = H_1 + \cdots + H_k.
\]
Here \( b = (C_1, \ldots, C_k) \) is the cluster decomposition corresponding to the decomposition \( L = (L_1, \ldots, L_k) \) of \( L \); \( H^b = H_1 + \cdots + H_k \) is the sum of the internal energies \( H_j \) inside \( L_j \), and is defined in the internal state space \( \mathcal{H}^b = \mathcal{H}^b_1 \otimes \cdots \otimes \mathcal{H}^b_k \); \( T = T_b \) denotes the intercluster free energy among the clusters \( C_1, \ldots, C_k \) defined in the external state space \( H_b \); and \( I = I_b = I_b(x) = I_b(x_b, x^b) \) is the sum of the intercluster interactions between various two different clusters in the cluster decomposition \( b \) (cf. the explanation after Definition 3 in section I.1).

The main concern in this process would be the case that the clusters \( C_1, \ldots, C_k \) form asymptotically bound states as \( t_L \to \infty \), since other cases are hard to be observed along the process if the observer’s concern is upon the final state of the sub-systems \( L_1, \ldots, L_k \).

The evolution \( \exp(-it_L H_L) f \) behaves asymptotically as \( t_L \to \infty \) as follows for some bound states \( g_1, \ldots, g_k \) (\( g_j \in \mathcal{H}_j^b \)) of local Hamiltonians \( H_1, \ldots, H_k \) and for some \( g_0 \) belonging to the external state space \( \mathcal{H}_b \):
\[
\exp(-it_L H_L) f \sim \exp(-it_L h_b) g_0 \otimes \exp(-it_L H_1) g_1 \otimes \cdots \otimes \exp(-it_L H_k) g_k, \quad k \geq 1, \quad (P)
\]
where \( h_b = T_b + I_b(x_b, 0) \). It is easy to see that \( g = g_0 \otimes g_1 \otimes \cdots \otimes g_k \) is given by
\[
g = g_0 \otimes g_1 \otimes \cdots \otimes g_k = \Omega^{++}_b f = P_b \Omega^{++}_b f,
\]
provided that the decomposition of the evolution \( \exp(-it_L H_L) f \) is of the simple form as in (P). Here \( \Omega^{++}_b \) is the adjoint of a canonical wave operator ([De]) corresponding to the cluster decomposition \( b \):
\[
\Omega^+_b = \lim_{t \to \infty} \exp(it H_L) \cdot \exp(-ith_b) \otimes \exp(-it H_1) \otimes \cdots \otimes \exp(-it H_k) P_b,
\]
where \( P_b \) is the eigenprojection onto the eigenspace of the Hamiltonian \( H^b = H_1 + \cdots + H_k \). The process (P) just describes the quantum-mechanical process inside the local system \( L \), and does not specify any meaning related with observation up to the present stage.

To see what we observe at actual observations, let us reflect a process of observation of scattering phenomena. We note that the observation of scattering phenomena is concerned with their initial and final stages by what the scattering itself means. At the final stage of observation of scattering processes, the quantities observed are firstly the
points hit by the scattered particles on the screen stood against them. If the circumstances are properly set up, one can further indicate the momentum of the scattered particles at the final stage to the extent that the uncertainty principle allows. Consider, e.g., a scattering process of an electron by a nucleus. Given the magnitude of initial momentum of an electron relative to the nucleus, one can infer the magnitude of momentum of the electron at the final stage as being equal to the initial one by the law of conservation of energy, since the electron and the nucleus are far away at the initial and final stages so that the potential energy between them can be neglected compared to the relative kinetic energy. The direction of momentum at the final stage can also be indicated, up to the error due to the uncertainty principle, by setting a sequence of slits toward the desired direction at each point on the screen so that the observer can detect only the electrons scattered to that direction. The magnitude of momentum at initial stage can be selected in advance by applying a uniform magnetic field to the electrons, perpendicularly to their momenta, so that they circulate around circles with the radius proportional to the magnitude of momentum, and then by setting a sequence of slits midst the stream of those electrons. The selection of magnitude of initial momentum makes the direction of momentum ambiguous due to the uncertainty principle, since the sequence of slits lets the position of electrons accurate to some extent. To sum up, the sequences of slits at the initial and final stages necessarily require to take into account the uncertainty principle so that some ambiguity remains in the observation.

However, in the actual observation of a single particle, we have to decide at which point on the screen the particle hits and which momentum the particle has, using the prepared apparatus like the sequence of slits located at each point on the screen. Even if we impose an interval for the observed values, we have to assume that the boundaries of the interval are sharply designated. These are the assumption which we always impose on “observations” implicitly. I.e., we idealize the situation in any observation or in any measurement of a single particle so that the observed values for each particle are sharp for both of the configuration and momentum. In this sense, the values observed actually for each particle must be classical. We have then necessary and sufficient conditions to make predictions about the differential cross section, as we will see in subsection I.3.1.

Summarizing, we observe just the classical quantities for each particle at the final stage of all observations. In other words, we have to presuppose that the values observed for each particle have sharp values, even if we cannot know the values actually. We can apply to this fact the remark stated in the third paragraph of this section about the possibility of defining observation as that of the classical centers of mass of local systems, and may assume that the actually observed values follow the classical Axioms 4 and 5. Those sharp values actually observed for each particle give, when summed over the large number of particles, the probabilistic nature of physical phenomena, i.e. that of scattering phenomena.

Theoretically, the quantum-mechanical, probabilistic nature of scattering processes is described by differential cross section, defined as the square of the absolute value of the scattering amplitude obtained from scattering operators \( S_{bd} = W_b^+W_d^- \), where \( W_b^\pm \) are usual wave operators. Given the magnitude of the initial momentum of the incoming particle and the scattering angle, the differential cross section gives a prediction about the probability at which point and to which direction on the screen each particle hits.
on the average. However, as we have remarked, the idealized point on the screen hit by each particle and the scattering angle given as an idealized difference between the directions of the initial and final momenta of each particle have sharp values, and the observation at the final stage is classical. We are then required to supplement these classical observations with taking into account the classical relativistic effects on those classical quantities, e.g., on the configuration and the momentum of each particle.

I.3.1. As the first step of the relativistic modification of the scattering process, we consider the scattering amplitude \( S(E, \theta) \), where \( E \) denotes the energy level of the scattering process and \( \theta \) is a parameter describing the direction of the scattered particles. Following our remark made in the previous paragraph, we make the following postulate on the scattering amplitude observed in actual experiment:

**Axiom 6.1.** When one observes the final stage of scattering phenomena, the total energy \( E \) of the scattering process should be regarded as a classical quantity and is replaced by a relativistic quantity, which obeys the relativistic change of coordinates from the scattering system to the observer’s system.

Since it is not known much about \( S(E, \theta) \) in the many body case, we consider an example of the two body case: Consider a scattering phenomenon of an electron by a Coulomb potential \( Z e^2/r \), where \( Z \) is a real number, \( r = |x| \), and \( x \) is the position vector of the electron relative to the scatterer. We assume that the mass of the scatterer is large enough compared to that of the electron and that \( |Z|/137 \ll 1 \). Then quantum mechanics gives the differential cross section in a Born approximation:

\[
\frac{d\sigma}{d\Omega} = |S(E, \theta)|^2 \approx \frac{Z^2 e^4}{16E^2 \sin^4(\theta/2)},
\]

where \( \theta \) is the scattering angle and \( E \) is the total energy of the system of the electron and the scatterer. We assume that the observer is stationary with respect to the center of mass of this system of an electron and the scatterer. Then, since the electron is far away from the scatterer after the scattering and the mass of the scatterer is much larger than that of the electron, we may suppose that the energy \( E \) in the formula in the above can be replaced by the classical kinetic energy of the electron by Axiom 6.1. Then, assuming that the speed \( v \) of the electron relative to the observer is small compared to the speed \( c \) of light in vacuum and denoting the rest mass of the electron by \( m \), we have by Axiom 6.1 that \( E \) is observed to have the following relativistic value:

\[
E' = c\sqrt{p^2 + mc^2} - mc = \frac{mc^2}{\sqrt{1 - (v/c)^2}} - mc^2 \approx \frac{mv^2}{2\sqrt{1 - (v/c)^2}},
\]

where \( p = mv/\sqrt{1 - (v/c)^2} \) is the relativistic momentum of the electron. Thus the differential cross section should be observed approximately equal to

\[
\frac{d\sigma}{d\Omega} \approx \frac{Z^2 e^4}{4m^2v^4 \sin^4(\theta/2)} (1 - (v/c)^2). \tag{RDC}
\]
This coincides with the usual relativistic prediction obtained from the Klein-Gordon equation by a Born approximation. See [Ki, p.297] for a case which involves the spin of the electron.

Before proceeding to the inclusion of gravity in the general $k$ cluster case, we review this two body case. We note that the two body case corresponds to the case $k = 2$, where $L_1$ and $L_2$ consist of single particle, therefore the corresponding Hamiltonians $H_1$ and $H_2$ are zero operators on $\mathcal{H}^0 = \mathbb{C}$ = the complex numbers. The scattering amplitude $S(E, \theta)$ in this case is an integral kernel of the scattering matrix $\hat{S} = \mathcal{F}SF^{-1}$, where $S = W^+W^-$ is a scattering operator; $W^\pm = s \text{-lim}_{t \to \pm \infty} \exp(itH_L) \exp(-itT)$ are wave operators ($T$ is negative Laplacian for short-range potentials under an appropriate unit system, while it has to be modified when long-range potentials are included); and $\mathcal{F}$ is Fourier transformation so that $FTF^{-1}$ is a multiplication operator by $|\xi|^2$ in the momentum representation $L^2(\mathbb{R}_3^3)$. By definition, $S$ commutes with $T$. This makes $\hat{S}$ decomposable with respect to $|\xi|^2 = FFT^{-1}$: For $a.e. \ E > 0$, there is a unitary operator $S(E)$ on $L^2(S^2)$, $S^2$ being two dimensional sphere with radius one, such that for $a.e. \ E > 0$ and $\omega \in S^2$

$$(\hat{S}h)(\sqrt{E}\omega) = \left(S(E)h(\sqrt{E}\cdot)\right)(\omega), \quad h \in L^2(\mathbb{R}_3^3) = L^2((0, \infty), L^2(S^2_\omega), |\xi|^2d|\xi|).$$

Thus $\hat{S}$ can be written as $\hat{S} = \{S(E)\}_{E>0}$. It is known [I-Ki] that $S(E)$ can be expressed as

$$(S(E)\varphi)(\theta) = \varphi(\theta) - 2\pi i\sqrt{E} \int_{S^2} S(E, \theta, \omega)\varphi(\omega)d\omega$$

for $\varphi \in L^2(S^2)$. The integral kernel $S(E, \theta, \omega)$ with $\omega$ being the direction of initial wave, is the scattering amplitude $S(E, \theta)$ stated in the above and $|S(E, \theta, \omega)|^2$ is called differential cross section. These are the most important quantities in physics in the sense that they are the only quantities which can be observed in actual physical observation.

The energy level $E$ in the previous example thus corresponds to the energy shell $T = E$, and the replacement of $E$ by $E'$ in the above means that $T$ is replaced by a classical relativistic quantity $E' = c\sqrt{p^2 + m^2c^2} - mc^2$. We have then seen that the calculation in the above gives a correct relativistic result, which explains the actual observation.

Axiom 6.1 is concerned with the observation of the final stage of scattering phenomena. To include the gravity into our consideration, we extend Axiom 6.1 to the intermediate process of quantum-mechanical evolution. The intermediate process cannot be an object of any actual observation, because the intermediate observation would change the process itself, consequently the result observed at the final stage would be altered. Our next Axiom 6.2 is an extension of Axiom 6.1 from the actual observation to the ideal observation in the sense that Axiom 6.2 is concerned with such invisible intermediate processes and modifies the ideal intermediate classical quantities by relativistic change of coordinates. The spirit of the treatment developed below is to trace the quantum-mechanical paths by ideal observations so that the quantities will be transformed into classical quantities at each step, but the quantum-mechanical paths will not be altered owing to the ideality of the observations. The classical Hamiltonian obtained
at the last step will be “requantized” to recapture the quantum-mechanical nature of the process, therefore the ideality of the intermediate observations will be realized in the final expression of the propagator of the observed system.

I.3.2. With these remarks in mind, we return to the general $k$ cluster case, and consider a way to include gravity in our framework.

In the scattering process into $k \geq 1$ clusters, what we observe are the centers of mass of those $k$ clusters $C_1, \cdots, C_k$, and of the combined system $L = (L_1, \cdots, L_k)$. In the example of the two body case of the previous subsection, only the combined system $L = (L_1, L_2)$ appears due to $H_1 = H_2 = 0$, therefore the replacement of $T$ by $E'$ is concerned with the free energy between two clusters $C_1$ and $C_2$ of the combined system $L = (L_1, L_2)$.

Following this treatment of $T$ in subsection I.3.1, we replace $T = T_b$ in the exponent of $\exp(-it_L h_b) = \exp(-it_L (T_b + I_b(x_b,0)))$ on the right hand side of the asymptotic relation (P) by the relativistic kinetic energy $T_b'$ among the clusters $C_1, \cdots, C_k$ around the center of mass of $L = (L_1, \cdots, L_k)$, defined by

$$T_b' = \sum_{j=1}^{k} \left( c\sqrt{p_j^2 + m_j^2 c^2} - m_j c^2 \right).$$  \hspace{1cm} (Energy 1)

Here $m_j > 0$ is the rest mass of the cluster $C_j$, which involves all the internal energies like the kinetic energies inside $C_j$ and the rest masses of the particles inside $C_j$, and $p_j$ is the relativistic momentum of the center of mass of $C_j$ inside $L$ around the center of mass of $L$. For simplicity, we assume that the center of mass of $L$ is stationary relative to the observer. Then we can set in the exponent of $\exp(-it_L (T_b' + I_b(x_b,0)))$

$$t_L = t_O,$$  \hspace{1cm} (Time 1)

where $t_O$ is the observer’s time.

For the factors $\exp(-it_L H_j)$ on the right hand side of (P), the object of the (ideal) observation is the centers of mass of the $k$ number of clusters $C_1, \cdots, C_k$. These are the ones which now require the relativistic treatment. Since we identify the clusters $C_1, \cdots, C_k$ as their centers of mass moving in a classical fashion, $t_L$ in the exponent of $\exp(-it_L H_j)$ should be replaced by $c^{-1}$ times the classical relativistic proper time at the origin of the local system $L_j$, which is equal to the quantum-mechanical local time $t_j$ of the sub-local system $L_j$. By the same reason and by the fact that $H_j$ is the internal energy of the cluster $C_j$ relative to its center of mass, it would be justified to replace the Hamiltonian $H_j$ in the exponent of $\exp(-it_j H_j)$ by the classical relativistic energy inside the cluster $C_j$ around its center of mass

$$H_j' = m_j c^2,$$  \hspace{1cm} (Energy 2)

where $m_j > 0$ is the same as in the above.

Summing up, we arrive at the following postulate, which has the same spirit as in Axiom 6.1 and includes Axiom 6.1 as a special case concerned with actual observation:
**Axiom 6.2.** In either actual or ideal observation, the space-time coordinates \( (ct_L, x_L) \) and the four momentum \( p = (p^\mu) = (E_L/c, p_L) \) of the observed system \( L \) should be replaced by classical relativistic quantities, which are transformed into the classical quantities \( (ct_O, x_O) \) and \( p = (E_O/c, p_O) \) in the observer’s system \( L_O \) according to the relativistic change of coordinates specified in Axioms 4 and 5. Here \( t_L \) is the local time of the system \( L \) and \( x_L \) is the internal space coordinates inside the system \( L \); and \( E_L \) is the internal energy of the system \( L \) and \( p_L \) is the momentum of the center of mass of the system \( L \).

In the case of the present scattering process into \( k \) clusters, the system \( L \) in this axiom is each of the local systems \( L_j \) \( (j = 1, 2, \ldots, k) \) and \( L \).

We continue to consider the \( k \) centers of mass of the clusters \( C_1, \ldots, C_k \). At the final stage of the scattering process, the velocities of the centers of mass of the clusters \( C_1, \ldots, C_k \) would be steady, say \( v_1, \ldots, v_k \), relative to the observer’s system. Thus, according to Axiom 6.2, the local times \( t_j \) \( (j = 1, 2, \ldots, k) \) in the exponent of \( \exp(-it_j H'_j) \), which are equal to \( c^{-1} \) times the relativistic proper times at the origins \( x_j = 0 \) of the local systems \( L_j \), are expressed in the observer’s time coordinate \( t_O \) by

\[
t_j = t_O \sqrt{1 - (v_j/c)^2} \approx t_O \left( 1 - v_j^2/(2c^2) \right), \quad j = 1, 2, \ldots, k, \tag{Time 2}
\]

where we have assumed \( |v_j/c| \ll 1 \) and used Axioms 4 and 5 to deduce the Lorentz transformation:

\[
t_j = \frac{t_O - (v_j/c^2)x_O}{\sqrt{1 - (v_j/c)^2}}, \quad x_j = \frac{x_O - v_j t_O}{\sqrt{1 - (v_j/c)^2}}.
\]

(For simplicity, we wrote the Lorentz transformation for the case of 2-dimensional space-time.)

Inserting (Energy 1-2) and (Time 1-2) into the right-hand side of (P), we obtain a classical approximation of the evolution:

\[
\exp \left( -it_O \left[ (T'_b + I_b(x_b, 0) + H'_1 + \cdots + H'_k) - (m_1 v_1^2/2 + \cdots + m_k v_k^2/2) \right] \right) \tag{AP}
\]

under the assumption that \( |v_j/c| \ll 1 \) for all \( j = 1, 2, \ldots, k \).

What we want to clarify is the final stage of the scattering process. Thus as we have mentioned, we may assume that all clusters \( C_1, \ldots, C_k \) are far away from any of the other clusters and moving almost in steady velocities \( v_1, \ldots, v_k \) relative to the observer. We denote by \( r_{ij} \) the distance between two centers of mass of the clusters \( C_i \) and \( C_j \) for \( 1 \leq i < j \leq k \). Then, according to our spirit that we are observing the behavior of the centers of mass of the clusters \( C_1, \ldots, C_k \) in classical fashion following Axioms 4 and 5, the clusters \( C_1, \ldots, C_k \) can be regarded to have gravitation among them. This gravitation can be calculated if we assume Einstein’s field equation, \( |v_j/c| \ll 1 \), and certain conditions that the gravitation is weak (see [M, section 17.4]), in addition to our Axioms 4 and 5. As an approximation of the first order, we obtain the gravitational potential of Newtonian type for, e.g., the pair of the clusters \( C_1 \) and \( U_1 = \bigcup_{i=2}^k C_i \):

\[
-G \sum_{i=2}^k m_i m_i/r_{1i},
\]
where $G$ is Newton’s gravitational constant.

Considering the $k$ body classical problem for the $k$ clusters $C_1, \cdots, C_k$ moving in the sum of these gravitational fields, we see that the sum of the kinetic energies of $C_1, \cdots, C_k$ and the gravitational potentials among them is constant by the classical law of conservation of energy:

$$m_1v_1^2/2 + \cdots + m_kv_k^2/2 - G \sum_{1 \leq i < j \leq k} m_im_j/r_{ij} = \text{constant}.$$ 

Assuming that $v_j \to v_{j\infty}$ as time tends to infinity, we have constant $= m_1v_{1\infty}^2/2 + \cdots + m_kv_{k\infty}^2/2$. Inserting this relation into (AP) in the above, we obtain the following as a classical approximation of the evolution (P):

$$\exp\left(-it\mathcal{O}\begin{bmatrix} T'_b + I_b(x_b, 0) + \sum_{j=1}^{k} (m_jc^2 - m_jv_{j\infty}^2/2) - G \sum_{1 \leq i < j \leq k} m_im_j/r_{ij} \end{bmatrix} \right). \quad (CP)$$

What we do at this stage are ideal observations, and these observations should not give any sharp classical values. Thus we have to consider (CP) as a quantum-mechanical evolution and we have to recapture the quantum-mechanical feature of the process. To do so we replace $p_j$ in $T'_b$ in (CP) by a quantum-mechanical momentum $D_j$, where $D_j$ is a differential operator $-i\partial/\partial x_j = -i \left(\partial/\partial x_{j1}, \partial/\partial x_{j2}, \partial/\partial x_{j3}\right)$ with respect to the 3-dimensional coordinates $x_j$ of the center of mass of the cluster $C_j$. Thus the actual process should be described by (CP) with $T'_b$ replaced by a quantum-mechanical Hamiltonian

$$\tilde{T}_b = \sum_{j=1}^{k} \left(c\sqrt{D_j^2 + m_j^2c^2} - m_jc^2\right).$$

This procedure may be called “requantization,” and is summarized as the following axiom concerning the ideal observation.

**Axiom 6.3.** In the expression describing the classical process at the time of the ideal observation, the intercluster momentum $p_j = (p_{j1}, p_{j2}, p_{j3})$ should be replaced by a quantum-mechanical momentum $D_j = -i \left(\partial/\partial x_{j1}, \partial/\partial x_{j2}, \partial/\partial x_{j3}\right)$. Then this gives the evolution describing the intermediate quantum-mechanical process.

We thus arrive at an approximation for a quantum-mechanical Hamiltonian including gravitational effect up to a constant term, which depends on the system $L$ and its decomposition into $L_1, \cdots, L_k$, but not affecting the quantum-mechanical evolution, therefore can be eliminated:

$$\tilde{H}_L = \tilde{T}_b + I_b(x_b, 0) - G \sum_{1 \leq i < j \leq k} m_im_j/r_{ij}$$

$$= \sum_{j=1}^{k} \left(c\sqrt{D_j^2 + m_j^2c^2} - m_jc^2\right) + I_b(x_b, 0) - G \sum_{1 \leq i < j \leq k} m_im_j/r_{ij}. \quad \text{(QMG)}$$
We remark that the gravitational terms here come from the substitution of local times $t_j$ to the time $t_L$ in the factors $\exp(-it_L H_j)$ on the right-hand side of (P). This form of Hamiltonian in (QMG) is actually used in [Li] with $I_b = 0$ to explain the stability and instability of cold stars of large mass, showing the effectiveness of the Hamiltonian.

Summarizing these arguments from (P) to (QMG), we have obtained the following interpretation of the observation of the quantum-mechanical evolution: To get our prediction for the observation of local systems $L_1, \cdots, L_k$, the quantum-mechanical evolution of the combined local system $L = (L_1, \cdots, L_k)$

$$\exp(-it_L H_L)f$$

should be replaced by the following evolution, in the approximation of the first order under the assumption that $|v_j/c| \ll 1$ ($j = 1, 2, \cdots, k$) and the gravitation is weak,

$$(\exp(-it_O \tilde{H}_L) \otimes I \otimes \cdots \otimes I)P_b \Omega_b^{++}f,$$  \hspace{1cm} (GP)

provided that the original evolution $\exp(-it_L H_L)f$ decomposes into $k$ number of clusters $C_1, \cdots, C_k$ as $t_L \to \infty$ in the sense of (P). Here $b$ is the cluster decomposition $b = (C_1, \cdots, C_k)$ that corresponds to the decomposition $L = (L_1, \cdots, L_k)$ of $L$; $t_O$ is the observer’s time; and

$$\tilde{H}_L = \tilde{T}_b + I_b(x_b, 0) - G \sum_{1 \leq i < j \leq k} m_i m_j/r_{ij}$$  \hspace{1cm} (RH)

is the relativistic Hamiltonian inside $L$ given by (QMG), which describes the motion of the centers of mass of the clusters $C_1, \cdots, C_k$.

We remark that (GP) may produce a bound state combining $C_1, \cdots, C_k$ as $t_O \to \infty$ therefore for all $t_O$, due to the gravitational potentials in the exponent. Note that this is not prohibited by our assumption that $\exp(-it_L H_L)f$ has to decompose into $k$ clusters $C_1, \cdots, C_k$, because the assumption is concerned with the original Hamiltonian $H_L$ but not with the resultant Hamiltonian $\tilde{H}_L$.

Extending our primitive assumption Axiom 6.1, which was valid for an example stated in subsection I.3.1, we have arrived at a relativistic Hamiltonian $\tilde{H}_L$, which would describe approximately the intermediate process, under the assumption that the gravitation is weak and the velocities of the particles are small compared to $c$, by using the Lorentz transformation. We note that, since we started our argument from the asymptotic relation (P), which is concerned with the final stage of scattering processes, we could assume that the velocities of particles are almost steady relative to the observer in the correspondent classical expressions of the processes, therefore we could appeal to the Lorentz transformations when performing the change of coordinates in the relevant arguments.

The final values of scattering amplitude should be calculated by using the Hamiltonian $\tilde{H}_L$. Then they would explain actual observations. This is our prediction for
the observation of relativistic quantum-mechanical phenomena including the effects by gravity and quantum-mechanical forces.

In the example discussed in subsection I.3.1, this approach gives the same result as (RDC) in the approximation of the first order, showing the consistency of our spirit. This can be seen by a representation formula of the scattering matrix similar to (3.7) in [I-Ki] for the present Hamiltonian \( \tilde{H}_L = \tilde{T} + V - GmM/r \) with \( V = Ze^2/r \) and \( M \) being the mass of the scatterer: Define \( \mu = (E + mc^2)^2/c^2 - m^2c^2 = c^{-2}E(E + 2mc^2) \) for \( E > 0 \), and set

\[
(F_0(E)f)(\omega) = c^{-1}\sqrt{2(E + mc^2)}\mu^{1/4}(\mathcal{F}f)(\sqrt{\mu}\omega), \quad f \in C_0^\infty(\mathbb{R}^3),
\]

where \( \mathcal{F} \) is Fourier transformation, so that \( F_0(E) \) decomposes \( \tilde{T} = c\sqrt{D^2 + m^2c^2} - mc^2 \)

\[
\mathcal{F}_0(E)\tilde{T}f = EF_0(E)f,
\]

and satisfies

\[
\int_0^\infty \|F_0(E)f\|^2_{L^2(S^2)}dE = \|f\|^2_{L^2(\mathbb{R}^3)}.
\]

The scattering matrix \( S(E) \) is then given as follows, in Born approximation of the first order under the assumption that \( |Z|/137 \ll 1 \), as in (3.7) of [I-Ki]:

\[
S(E) \approx I - 2\pi iF_0(E)\tilde{V}F_0(E)^*,
\]

where \( \tilde{V} \) is a modified Coulomb potential obtained from \( V = Ze^2/r \) in accordance with the long-range tail of \( V \), and we omitted the gravitational potential in \( \tilde{H}_L \), since it is small compared to \( V \). Then, calculating by using oscillatory integrals, we obtain the differential cross section \( |S(E, \theta)|^2 \) equal to (RDC), if we replace the quantum-mechanical quantity \( \tilde{T} = E \) by the corresponding classical quantity \( E' = c\sqrt{p^2 + m^2c^2} - mc^2 \approx mv^2/(2\sqrt{1 - (v/c)^2}) \), assuming that the speed \( v \) of the electron is small compared to \( c \).

We remark that our stand does not require the resultant Hamiltonian \( \tilde{H}_L \) to satisfy the Lorentz invariance or other kinds of invariance under transformations among coordinate systems, unlike the usual attempts require in constructing relativistic quantum theories. We have just given a procedure to predict what we actually observe, but did not propose a physical law. Usual attempts identify physics with observation, and require such kind of invariance of observation. We separate observation from physics, allowing asymmetry to observation, but with preserving two mutually incompatible invariances for physics: Galilei invariance for internal quantum mechanics and general relativistic invariance for external classical physics. This becomes possible by our position that relativity is concerned with the external world outside local systems, but not with the internal physics, which is ruled by quantum mechanics. In fact, we postulated relativity as concerned with the centers of mass of local systems in Axioms 4 and 5, and in Axioms 6.1 and 6.2 we clarified the role the relativity plays when observing the centers of mass. We refer the reader to [Ki-FI] for further philosophical position of ours.
Part II. Examples

In this Part II we consider two examples of human size and of cosmological size following the spirit of the previous Part, both of which involve the quantum-mechanical aspects and relativistic aspects simultaneously.

II.1. Scattering of one neutron in a uniform gravitational field

Consider the experiment done by Collela et al. [Co] of measuring the interference of one neutron. This experiment is described in some simplification as in the following Figure 1:

A neutron beam emitted at S is split into two beams by an interferometer at A, and the two beams are recombined at point D by other interferometers or mirrors B and C. The height $L$ of the line BD on the earth can be varied. The dependence on $L$ of the relative phase difference is given as follows, according to the experiment of [Co], up to the error of about 1%:

$$\hbar^{-1}mgLT,$$

where $m$ is the mass of the neutron, $g$ is the acceleration by gravity, and $T$ is the (observed) time that the beams travel from C to D or A to B. This experiment shows that quantum mechanics and gravity play important roles \textit{simultaneously} in the size of desktop environment. In fact, the lengths of the lines AB and BD are less than 10 cm in [Co].

This experiment can be explained in our context, if we see it as a 3-body scattering process of a neutron N by two mirrors B and C as in Figure 2.
We denote the local system of the three bodies N, B, and C by $L$. Let the masses of mirrors B and C be $M$, the neutron mass be $m$, and assume $0 < m \ll M$. Let $x$, $X_B$, and $X_C$ denote the 3-dimensional coordinates of N, B and C. Then the Hamiltonian of this system $L$ with $\hbar = 1$ is

$$H = \frac{D_x^2}{2m} + \frac{D_B^2}{2M} + \frac{D_C^2}{2M},$$

where $D_x$, $D_B$ and $D_C$ are the momentum operators $\frac{1}{i} \frac{\partial}{\partial x} = \frac{1}{i} \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right)$, $\frac{1}{i} \frac{\partial}{\partial X_B}$, and $\frac{1}{i} \frac{\partial}{\partial X_C}$ for N, B and C. To separate the center of mass, we introduce the two sets of Jacobi coordinates:

$$\begin{cases} x^{(1)} = x - X_C, \\ y^{(1)} = X_B - \frac{mx + MX_C}{m + M}, \end{cases} \quad (C1)$$

and

$$\begin{cases} x^{(2)} = x - X_B, \\ y^{(2)} = X_C - \frac{mx + MX_B}{m + M}. \end{cases} \quad (C2)$$

These choices of Jacobi coordinates $(x^{(j)}, y^{(j)})$ $(j = 1, 2)$ correspond to two cluster decompositions $b^{(j)} = (C_1^{(j)}, C_2^{(j)})$ of $L$ such that $C_1^{(1)} = \{N, C\}$ and $C_2^{(1)} = \{B\}$, or $C_1^{(2)} = \{N, B\}$ and $C_2^{(2)} = \{C\}$. In either case, $x^{(j)}$ is the internal coordinate inside the cluster $C_i^{(j)}$, and $y^{(j)}$ is the intercluster coordinate between two clusters $C_1^{(j)}$ and $C_2^{(j)}$.

Using these coordinates, we remove the center of mass of the system $L$. Then we obtain the Hamiltonian $H$ which has the same form for both coordinates:

$$H = H^{(1)} + T^{(1)} = H^{(2)} + T^{(2)},$$

$$H^{(j)} = \frac{(D_{x^{(j)}})^2}{2\mu}, \quad T^{(j)} = \frac{(D_{y^{(j)}})^2}{2\nu}.$$
Here \( D_x^{(j)} = \frac{1}{i} \frac{\partial}{\partial x^{(j)}} \) and \( D_y^{(j)} = \frac{1}{i} \frac{\partial}{\partial y^{(j)}} \) are momentum operators conjugate to \( x^{(j)} \) and \( y^{(j)} \) (\( j = 1, 2 \)), and \( \mu, \nu \) are the reduced masses:

\[
\mu^{-1} = m^{-1} + M^{-1}, \quad \nu^{-1} = M^{-1} + (m + M)^{-1}.
\]

Note that the operators \( D_x^{(j)} \) and \( D_y^{(j)} \) are mutually independent, therefore \( H^{(j)} \) commutes with \( T^{(j)} \). Thus the propagation of the 3-body system \( L \) is given by

\[
\exp(-iT) f = \exp(-iT^{(1)}) \exp(-iT^{(2)}) f,
\]

where \( f = f(x^{(j)}, y^{(j)}) \) is the initial wave function at time \( t = 0 \), just after the neutron has been split into two beams by the interferometer A. Here the \emph{time} \( t \) is the local time determined by the Hamiltonian \( H \) or the correspondent local system \( L \).

\( x^{(j)} \) is the distance vector between \( N \) and \( C \), or between \( N \) and \( B \), and \( y^{(j)} \) is the distance vector between \( B \) and the center of mass of the system \( N + C \), or between \( C \) and the center of mass of the system \( N + B \). Therefore, as seen from the formula for \( y^{(j)} \) in (C1) or (C2), we may regard it as

\[
y^{(1)} = X_B - X_C \quad \text{or} \quad y^{(2)} = X_C - X_B,
\]

for \( M \) is larger enough than \( m \). We can thus regard \( y^{(j)} \) as constant during the scattering process, hence \( f(x^{(j)}, y^{(j)}) \) can be regarded as a function of \( x^{(j)} \) only. (Exactly speaking, \( f(x^{(j)}, y^{(j)}) \) can be written as \( f(x^{(j)}) F(y^{(j)}) \) with \( F(y^{(j)}) \) close to the delta function \( \delta_{+BC} \) having support at \( y^{(j)} = X_B - X_C \) or \( y^{(j)} = X_C - X_B \). But as we will see, the factor \( F(y^{(j)}) \) does not play any essential role in our argument, and we can simply omit it from \( f(x^{(j)}, y^{(j)}) \).)

Namely \( f(x^{(j)}, y^{(j)}) \) can be regarded as the wave function of the neutron \( N \), and is split into two wave packets \( f_1(x^{(j)}), f_2(x^{(j)}) \) at time \( t = 0 \) by the interferometer A:

\[
f = f_1 + f_2.
\]

\( f_1 \) is the wave packet moving to the direction from A to C, and \( f_2 \) is the one from A to B. (E) can then be rewritten as follows:

\[
\exp(-iT) f = \exp(-iT^{(1)}) \exp(-iT^{(2)}) f_1 + \exp(-iT^{(2)}) \exp(-iT^{(1)}) f_2.
\]

As remarked in the above, we can regard \( y^{(1)} = X_B - X_C \) or \( y^{(2)} = X_C - X_B \), therefore we may set \( T^{(1)} = T^{(2)} = T \). We thus have

\[
\exp(-iT) f = \exp(-IT) \{ \exp(-iT^{(1)}) f_1 + \exp(-iT^{(2)}) f_2 \}.
\]

(E1)

The description up to here is by the local time \( t \) determined by the local system \( L \).

\( H^{(1)} \) is the Hamiltonian of the local system consisting of \( N \) and \( C \), and the center of mass of \( N \) and \( C \) is regarded, by \( m \ll M \), as located at \( C \) with the same height as the
observer. Hence, corresponding to (Time 1) in subsection I.3.2, we can set in the first term \( \exp(-itH^{(1)})f_1 \) of (E1):

\[
t = t_O.
\]

\( H^{(2)} \) is the Hamiltonian consisting of N and B, and its center of mass is regarded as located at B by \( m \ll M \). Therefore that local system has a lower gravitational potential in amount \( gL \) compared to the observer O, hence the local time \( t \) of the local system \( H^{(2)} \) is related with the observer’s time \( t_O \) as in (Time 2) of subsection I.3.2:

\[
t = t_O \sqrt{1 - (2gL)/c^2} \approx t_O(1 - (gL)/c^2).
\]

Therefore

\[
\exp(-itH^{(2)})f_2 \approx \exp(-it_O \cdot H^{(2)}) \exp(it_O \cdot (gL/c^2)H^{(2)})f_2.
\]

We note that we can regard \( H^{(1)} = H^{(2)} \) by \( H = H^{(1)} + T^{(1)} = H^{(2)} + T^{(2)} \) and \( T^{(1)} = T^{(2)} = T \). As in (Energy 2) of subsection I.3.2, the internal energy \( H^{(1)} = H^{(2)} \) of the system N+C or N+B is then approximated by a classical quantity \( \mu c^2 = m \left( 1 - \frac{m}{m+M} \right) \approx mc^2 \).

For the first factor \( \exp(-iT) \) on the right hand side of (E1), the time \( t \) in the exponent is the local time of the local system \( L \) as in (Time 1) of subsection I.3.2, because \( T = T^{(j)} \) is the total intercluster free energy \( T^{(j)} \) corresponding to the cluster decomposition \( b^{(j)} \) of the local system \( L \). Since the center of mass of the local system \( L \) is at the middle height between B and C, the time \( t \) in \( \exp(-iT) \) is thus related with \( t_O \) as follows:

\[
t = t_O \sqrt{1 - (gL)/c^2}.
\]

From these, we have the following decomposition of the observed wave function for this 3-body system:

\[
\exp(-iH)f \approx \exp(-it_O \sqrt{1 - (gL)/c^2}T) \exp(-it_O mc^2) \{f_1 + \exp(it_O \cdot gLm)f_2\}.
\]

Setting

\[
h_k(t_O) = \exp(-it_O \sqrt{1 - (gL)/c^2}T) \exp(-it_O mc^2)f_k, \quad (k = 1, 2)
\]

we then have

\[
\exp(-iH)f \approx h_1(t_O) + \exp(it_O \cdot gLm)h_2(t_O).
\]

Therefore at the time \( t_O \) of observation, there remains the desired phase difference, which explains the interference observed in [Co]. Note that \( T = T^{(j)} = \frac{(D^{(j)})^2}{2\nu} \) does not play any essential role in this argument, therefore we did not replace it by a classical quantity.

II.2. Hubble’s law
Hubble’s law is a phenomenon that appears when one observes the light emitted from stars and galaxies far away from the earth. The emission of light itself is a quantum-mechanical phenomenon that could be explained by the nonrelativistic quantum field theory as in [Ki], Section 11-(2). The observation or reception of this emission of light on the earth is explained as a classical observation according to our postulate Axiom 6.2, by introducing Robertson-Walker metric.

Robertson-Walker metric is the metric derived from the assumptions of homogeneity and isotropy of the large scale structure of the universe. We refer the reader to [M], Chap. 27 for the details, and we here only outline the argument.

Under the hypotheses of homogeneity and isotropy, the metric is given in general as follows:

$$ds^2 = -(dx^0)^2 + d\sigma^2 = -(dx^0)^2 + a(x^0)^2 \gamma_{ij}(x^k)dx^idx^j,$$

where $x^0$ is the time parameter that ‘slices’ the spacetime by means of a one parameter family of some spacelike surfaces, and $(x^1, x^2, x^3)$ is the ‘comoving, synchronous space coordinate system’ for the universe, in the sense of [M], sections 27.3–27.4. $a(x^0)$ is the so-called “expansion factor” that describes the ratio of expansion of the universe in the usual context of general theory of relativity. A consideration by the use of homogeneity and isotropy yields ([M], section 27.6) that for some functions $f(r)$ ($r = |(x^1, x^2, x^3)|$) and $h(x^0)$

$$ds^2 = -(dx^0)^2 + e^{f(r)} e^{h(x^0)} \{(dx^1)^2 + (dx^2)^2 + (dx^3)^2\},$$

Assuming Einstein field equation $G^\mu_\nu - \lambda \delta^\mu_\nu = \kappa T^\mu_\nu$ and calculating, we get with replacing $e^{h(x^0)}$ by a constant times $e^{h(x^0)}$

$$ds^2 = -(dx^0)^2 + e^{h(x^0)} \left(1 + k \frac{r^2}{4r_0^2}\right)^{-2} \{(dx^1)^2 + (dx^2)^2 + (dx^3)^2\},$$

where $k = -1, 0, \text{ or } +1$. This is called Robertson-Walker metric. Using the polar coordinates $(r, \theta, \varphi)$ and setting $t = x^0$ and

$$\frac{r}{r_0} = u, \quad R(t) = r_0e^{h(t)/2},$$

one can rewrite $ds^2$ as follows:

$$ds^2 = -(dt)^2 + R(t)^2 \left(1 + \frac{k}{4} u^2\right)^{-2} [du^2 + u^2 \{(d\theta)^2 + (\sin \theta d\varphi)^2\}].$$

Suppose $k = +1$, and consider a 3-dimensional sphere of radius $A$ in a 4-dimensional Euclidean space

$$A^2 = (y^4)^2 + \sum_{k=1}^{3} (y^k)^2.$$

The metric on this sphere is

$$d\sigma^2 = \sum_{k=1}^{3} (dy^k)^2 + (dy^4)^2.$$
This is rewritten by using the equation of the sphere in the above as follows:

\[
d\sigma^2 = \sum_{k=1}^{3} (dy^k)^2 + A^2 - \sum_{k=1}^{3} (y^k)^2 \left( \sum_{\ell=1}^{3} y^\ell dy^\ell \right)^2.
\]

Set \( \rho^2 = \sum_{k=1}^{3} (y^k)^2 \), and define \( v \) by

\[
\rho = A \left( 1 + \frac{v^2}{4} \right)^{-1} v.
\]

Using polar coordinates \((\rho, \theta, \varphi)\) instead of \((y^1, y^2, y^3)\), and rewriting \( \rho \) by the use of \( v \), we have

\[
d\sigma^2 = A^2 \left( 1 + \frac{v^2}{4} \right)^{-2} \left[ (dv)^2 + v^2 (d\theta)^2 + (\sin \theta d\varphi)^2 \right].
\]

If we set \( A = R(t) \), and identify \( v \) as \( u \), this formula coincides with the space part \( d\sigma^2 \) of the aforementioned Robertson-Walker metric \( ds^2 \).

In this sense, the space part slice \( t = \text{constant} \) of the spacetime can be regarded as a 3-dimensional sphere of radius \( R(t) \) in a 4-dimensional Euclidean space, hence \( R(t) = r_0 e^{h(t)/2} \) can be regarded as the radius of the universe and may expand as \( t \) grows. The cosmological redshift observed by Hubble [Hu] gives in this context that \( R(t) \) is growing at present (see section 29.2 of [M]), and this is interpreted as a proof of ‘expansion’ of the universe. However, as we have seen, the ‘expansion’ is a consequence of the identification of \( R(t) \) in the Robertson-Walker metric \( ds^2 \) with the radius of a sphere in a virtual 4-dimensional Euclidean space. In this sense, the growth of \( R(t) \) in \( ds^2 \) does not imply the expansion of the universe in any other senses than it is an ‘interpretation.’

The ‘expansion’ of this type does not contradict the stationary universe \( \phi \) in quantum-mechanical sense specified in Axiom 1. The ‘expansion’ is an interpretation of the observation with one observer’s coordinate system fixed. The quantum-mechanical stationary universe \( \phi \) is the inner structure of its own and is independent of the observer’s coordinate system. In this sense, the ‘expansion’ is an ‘appearance,’ which the universe takes under the ‘interference’ of the observer to try to reveal its morphology. More philosophically stating, the past and the future do not exist unless one fixes a time coordinate. The ‘Big Bang’ is an imagination under the presumption that the time coordinate exists a priori. Unless it is observed with assuming the existence of a time coordinate, the universe can be a stationary state.

Our theory is a reflection and a clarification of this supposition of the existence of time coordinate, adopted implicitly in almost all physical theories today.

Example of the previous section is an experiment of human size, and the one in this section is an observation of cosmological size. These two examples together with the one in subsection I.3.2 would indicate a unified treatment of physical phenomena from the microscopic size to the cosmological size.

Part III. Open Problems
In this Part III we state some open problems related with our formulation of physics. Some of them are known problems, but do not seem to have been given solutions. We conclude with stating a final goal of our formulation.

III.1. Stability of Matter

As we have seen in Part I, the relativistic Hamiltonian considered by E. H. Lieb and others (see [Li] and the references therein) has reasonable grounds under the assumption that gravitation is weak. It has been thought that the non-invariance with respect to Lorentz transformation is its fault. However, according to our formulation, the non-invariance is not a fault but has natural foundations as a Hamiltonian which describes observational facts.

It is therefore meaningful to research the related spectral and scattering theory for the relativistic Hamiltonian

$$\tilde{H}_L = \sum_{j=1}^{k} \left( c\sqrt{D_j^2 + m_j^2c^2} - m_jc^2 \right) + I_b(x_b, 0) - G \sum_{1 \leq i < j \leq k} m_i m_j / r_{ij}, \quad \text{(QMG)}$$

which includes the electric potentials and gravitational potentials simultaneously.

The first problem to be treated in this field is the problem of the stability of matter. It is known certain facts about this problem unless the electric potentials and gravitational ones are present simultaneously: If gravitation is absent and \(I_b\) is of the form

$$I_b = -e^2 \sum_{j=1}^{k} z|x_j - R|^{-1} + e^2 \sum_{1 \leq i < j \leq k} |x_i - x_j|^{-1},$$

where \(R\) is the position of the nucleus with \(z\) number of protons, it is known that the stability of the first kind is equivalent to the stability of the second kind, and that atoms are stable when \(z \leq 87\). If electric potentials are absent, it is shown ([Li-Ya]) that Thomas-Fermi theory is asymptotically exact for fermions. However, nothing seems known for the case which includes both of the electric potentials and gravitational ones as Lieb [Li] writes. The research to include both of electric and gravitational potentials would lead us to a deeper understanding of the nature of matter, since any matter includes both kinds of internal forces. E.g., the stability and instability of stars with large number of particles would be understood in a more satisfactory manner than in the present understanding.

III.2. Scattering Theory

The second problem to be considered would be the scattering theory for the Hamiltonian in the formula (QMG). This Hamiltonian has the potentials which are of Coulomb type, therefore, of critical singularity with respect to the free part:

$$\tilde{H}_{L0} = \sum_{j=1}^{k} \left( c\sqrt{D_j^2 + m_j^2c^2} - m_jc^2 \right).$$
Hence the problem of self-adjointness arises in the first place, and this is closely related with the problem of stability proposed in the previous section. The point is to what extent the free part $\tilde{H}_{L0}$ and the positive part of the sum of the potentials suppress the bad behavior of the negative parts. The gravitational potentials are quite small compared to the electric part and is negligible in the usual human size. But in the size of stars they cannot be neglected, and we have to develop some method which is able to treat the electric and gravitational parts at a time. This is the first problem which we should research in the scattering theory for (QMG).

If some conditions are established for the self-adjointness of (QMG), we should go on to the scattering phenomena governed by the Hamiltonian (QMG). This would give us an image about the phenomena which would be observed when the gravity and electrical forces are present simultaneously.

At first glance, it looks as if there were a problem in our formulation in the point that the approximate relativistic Hamiltonian (QMG) would lose the self-adjointness and stability when the number of particles becomes large. This should, however, be taken as an evidence of the success of our formulation to include gravity. The fact that the universe does not seem to be subject to Boltzmann’s heat death, but it, which is in the usual physical context supposed to have been in an equilibrium originally, could develop hot stars, is owing to the instability of gravitation. Our Hamiltonian (QMG) explains this fact so that our inclusion of gravity into (QMG) would be a reasonable one to that extent.

The final problem in this direction is to find a full general relativistic Hamiltonian, which explains the observation without assuming that the gravitation is weak. This seems difficult seeing the present stage of the theory, but we hope that this end would be accomplished in the future.

References

[A-M] R. Abraham and J. E. Marsden, *Foundations of Mechanics*, The Benjamin/Cummings Publishing Company, 2nd ed., London-Amsterdam-Don Mills, Ontario-Sydney-Tokyo, 1978.

[Bo] M. Born, *Zur Quantenmechanik der Stossvorgänge*, Zeitschrift für Physik 37 (1926), 863-867.

[Co] R. Collela, A. W. Overhauser and S. A. Werner, *Observation of gravitationally induced quantum mechanics*, Phys. Rev. Lett. 34 (1975), 1472-1474.

[Cy] H. L. Cycon et al., *Schrödinger Operators*, Springer-Verlag, 1987.

[De] J. Dereziński, *Asymptotic completeness of long-range N-body quantum systems*, Annals of Math. 138 (1993), 427-476.

[Di] P. A. M. Dirac, Proc. Roy. Soc. A117, (1928), 610.

[Dy] F. J. Dyson, *Divergence of perturbation theory in quantum electrodynamics*, Phys. Rev. 75 (1953), 486.

[Ein] A. Einstein, *The foundation of the general theory of relativity*, Translated by W. Perrett and G. B. Jeffery, The Principle of Relativity, Dover, 1923, pp. 111-164.

[En] V. Enss, *Introduction to asymptotic observables for multiparticle quantum scattering*, Schrödinger Operators, Aarhus 1985, Ed. by E. Balslev, Lect. Note in Math., vol. 1218, Springer-Verlag, 1986, pp. 61-92.

[Fr] J. Fröhlich, *On the triviality of $\lambda \Phi^4_3$ theories and the approach to the critical point in $d \leq 4$ dimensions*, Nucl. Phys. B 200 [FS4] (1982), 281-296.

[Hu] E. P. Hubble, *A relation between distance and radial velocity among extragalactic nebulae*, Proc. Nat. Acad. Sci. U.S. 15 (1929), 169-173.
[Ish] C. J. Isham, *Canonical quantum gravity and the problem of time*. Proceedings of the NATO Advanced Study Institute, Salamanca, June 1992, Kluwer Academic Publishers, 1993.

[I-Ki] H. Isozaki and H. Kitada, *Scattering matrices for two-body Schrödinger operators*, Scientific Papers of the College of Arts and Sciences, The University of Tokyo 35 (1986), 81-107.

[K-L] T. Kinoshita and W. B. Lindquist, *Eighth-order magnetic moment of the electron*, Phys. Rev. D27 (1983), 866.

[Ki] H. Kitada, *Theory of local times*, Il Nuovo Cimento 109 B, N. 3 (1994), 281-302.

[Ki(N)] H. Kitada, *Asymptotic completeness of N-body wave operators II. A new proof for the short-range case and the asymptotic clustering for long-range systems*, Functional Analysis and Related Topics, 1991, Ed. by H. Komatsu, Lect. Note in Math., vol. 1540, Springer-Verlag, 1993, pp. 149-189.

[Ki-Fl] H. Kitada and L. Fletcher, *Local time and the unification of physics, Part I: Local time*, Apeiron 3 (1996), 38-45.

[Li] E. H. Lieb, *The stability of matter: From atoms to stars*, Bull. Amer. Math. Soc. 22 (1990), 1-49.

[Li-Ya] E. H. Lieb and H-T. Yau, *The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics*, Commun. Math. Phys. 112 (1987), 147-174.

[M] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*, W. H. Freeman and Company, New York, 1973.

[New] I. Newton, *Sir Isaac Newton Principia, Vol. I The Motion of Bodies*, Motte’s translation Revised by Cajori, Tr. Andrew Motte ed. Florian Cajori, Univ. of California Press, Berkeley, Los Angeles, London, 1962.

[Pru] E. Prugovecki, *Quantum Geometry, A Framework for Quantum General Relativity*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1992.

[St] F. Streater, *Why should anyone want to axiomatize quantum field theory?* Philosophical Foundations of Quantum Field Theory, Ed. by H. R. Brown and R. Harré, Clarendon, Oxford University Press, 1990, pp. 137-148.