Numerical simulation of modified Helmholtz boundary value problems for anisotropic exponentially graded materials

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Abstract. In this paper we consider the modified Helmholtz type equation governing interior two-dimensional boundary value problems (BVPs) for anisotropic functionally graded materials (FGMs) with Dirichlet and Neumann boundary conditions. Persistently spatially changing diffusivity and leakage factor coefficients are involved in the governing equation. Both the anisotropic diffusivity and leakage factor coefficients vary according to an exponential gradation function. We use a technique of transforming the variable coefficient governing equation to a constant coefficient equation for deriving a boundary integral equation. And from the boundary integral equation obtained a standard boundary element method (BEM) is constructed to find numerical solutions to the BVPs. In order to illustrate the application of the BEM, some particular examples of BVPs are solved. The results show the convergence, accuracy, consistency between the scattering and flow solutions and efficiency (less computation time) of the BEM solutions. The results also show the impact of the inhomogeneity and anisotropy of the material on the solutions.

1. Introduction

The modified Helmholtz equation appears in many kind of applications such as neutron diffusion problems [1], advection–diffusion problems [2, 3], problems governed by Laplace type equation [4], Debye–Huckel theory and the linearized Poisson–Boltzmann problems [5, 6], steady-state groundwater flow [7] and unsteady heat conduction [8].

Many works on the modified Helmholtz equation have been done, yet most of the works are limited to focusing on the case of isotropic and homogeneous media. Igarashi and Honma [9] derived axially and helically symmetric fundamental solutions to modified Helmholtz-type equations for isotropic homogeneous media. Itagaki and Brebbia [1] applied multiple reciprocity boundary element method to neutron diffusion problems. A sequence of isotropic modified Helmholtz equations with constant coefficients were considered. Singh and Tanaka [2] used the modified Helmholtz fundamental solution in the boundary element analysis to solve advection–diffusion problems after an exponential variable transformation of the dependent variable. The coefficients of the both governing equations, isotropic advection–diffusion and isotropic modified Helmholtz equations, are constant. Chen et. al [4] also transformed the
Laplace-type governing equation into a modified Helmholtz equation. Both equations are of constant coefficients. Cheng et. al [5] solved the isotropic modified Helmholtz equation with also constant coefficient. Gusyev [7] obtained solution to the steady-state flow in a semi-confined aquifer which is governed by the modified Helmholtz equation. However, in this paper, it is assumed that the aquifer is isotropic and homogeneous. Kropinski [6] used fast integral equation methods for obtaining solution to the modified Helmholtz equation. Also, in this paper the coefficients of the modified Helmholtz equation were constant. Guo [8] applied a Laplace transformation on the time variable of the unsteady heat conduction equation to obtain a modified Helmholtz equation. However the transient heat conduction equation was under the assumption that the material under consideration is isotropic and homogeneous so as the derived modified Helmholtz equation. Then a multiple reciprocity method was applied to the domain integral in the integral equation resulted from the modified Helmholtz equation. Nguyen [10] considered modified Helmholtz equation with inhomogeneous source for isotropic media. Chen [11] applied the so-called singular boundary method which is a kind of a variant of the generic method of fundamental solution, to the isotropic form of modified Helmholtz problems.

Authors commonly define a functionally graded material/medium (FGM) as an (artificial/natural) inhomogeneous material having a specific property such as thermal conductivity, hardness, toughness, ductility, corrosion resistance, etc. that changes spatially in a continuous fashion. Numerous studies on FGM for a variety of applications including wave propagation have been reported (see for example [12], [13]).

The boundary element method (BEM) has been successfully used for solving many types of problems of either homogeneous or functionally graded (inhomogeneous), and either isotropic or anisotropic materials. Some works using BEM on homogeneous anisotropic media problems have been recently reported in [14, 15, 16], in which the authors considered pollutant transport problems governed by 2D diffusion-convection. For FGMs there are two main techniques usually used. The first one uses a technique of deriving a relevant Green function or fundamental solution to the FGM problem. Cheng [17] had applied this technique. The second technique is by transforming the variable coefficient governing equation to a constant coefficient equation. Some progress on using the second technique has been made. For examples, the paper [18] considered the case for isotropic FGM. For anisotropic FGM some works have been done in [19, 20] for heat conduction problems, in [21, 22, 23] for linear static elasticity problems, in [24] for diffusion-convection-reaction equation. In addition to this, papers [25, 26, 27, 28], [29, 30, 31], [32], reported works on some different classes of elliptic, Helmholtz and modified Helmholtz problems for anisotropic FGM.

Very few works on the modified Helmholtz equation have been done for the case of anisotropic FGMs. This paper discusses derivation of a boundary integral equation for numerically solving 2D interior boundary value problems governed by the modified Helmholtz type equation for anisotropic FGMs of the form

$$\frac{\partial}{\partial x_i} \left[ \lambda_{ij}(x_1, x_2) \frac{\partial \phi(x_1, x_2)}{\partial x_j} \right] - \beta^2(x_1, x_2) \phi(x_1, x_2) = 0$$

Equation (1) may be written explicitly as

$$\frac{\partial}{\partial x_1} \left( \lambda_{11} \frac{\partial \phi}{\partial x_1} \right) + 2 \frac{\partial}{\partial x_1} \left( \lambda_{12} \frac{\partial \phi}{\partial x_2} \right) + \frac{\partial}{\partial x_2} \left( \lambda_{22} \frac{\partial \phi}{\partial x_2} \right) - \beta^2 \phi = 0$$

As for the case $\beta^2 = 0$ a study has been done in [25], this paper will be restricted only for the case $\beta^2 > 0$. 

The technique of transforming (1) to constant coefficient equations will be used for obtaining a boundary integral equation for the solution of (1). It is necessary to place some constraint on the class of coefficients $\lambda_{ij}$ and $\beta$ for which the solution obtained is valid. Throughout the paper, the analysis used is purely formal; to develop a BEM for obtaining the numerical solution of BVPs of FGMs governed by equation (1) is the main purpose.

2. The boundary value problem (BVP)

The BVP is restricted to interior two-dimensional (2D) problem with boundary conditions of type Dirichlet or Neumann. Referred to a Cartesian frame $Ox_1x_2$ a solution to (1) is sought which is valid in a region $\Omega$ in $R^2$ with boundary $\partial \Omega$ consisting of a number of piecewise continuous curves. On $\partial \Omega$ either $\phi(x)$ or $P(x)$ is specified, where

$$P(x) = \lambda_{ij} \left( \frac{\partial \phi}{\partial x_j} \right) n_i$$

$x = (x_1, x_2)$ and $n = (n_1, n_2)$ is the normal vector pointing out on the boundary $\partial \Omega$. For equation (1) to be an elliptic partial differential equation throughout $\Omega$, the matrix of coefficients $[\lambda_{ij}]$ is required to be a symmetric positive definite matrix. The coefficients $\lambda_{ij}$ and $\beta$ are also required to be twice differentiable functions.

A boundary integral equation will be sought, from which numerical values of the dependent variables $\phi$ and its derivatives may be obtained for all points in $\Omega$. The analysis is in general applicable for anisotropic media but it is not excepted to isotropic materials. The analysis also applies for the case of isotropic media that is when $\lambda_{11} = \lambda_{22}$ and $\lambda_{12} = 0$.

3. The boundary integral equation

The boundary integral equation is derived by transforming the variable coefficient equation (1) to a constant coefficient equation. We restrict the coefficients $\lambda_{ij}$ and $\beta$ to be of the form

$$\lambda_{ij}(x) = \bar{\lambda}_{ij}g(x)$$
$$\beta^2(x) = \bar{\beta}^2g(x)$$

where $g(x)$ is a differentiable function and $\bar{\lambda}_{ij}$ and $\bar{\beta}^2$ are constant. Substitution of (3) and (4) into (1) gives

$$\bar{\lambda}_{ij} \frac{\partial}{\partial x_i} \left( g \frac{\partial \phi}{\partial x_j} \right) - \bar{\beta}^2 g \phi = 0$$

Assume

$$\phi(x) = g^{-1/2}(x) \psi(x)$$

therefore equation (5) can be written as

$$\bar{\lambda}_{ij} \frac{\partial}{\partial x_i} \left[ g \frac{\partial (g^{-1/2}\psi)}{\partial x_j} \right] - \bar{\beta}^2 g^{1/2} \psi = 0$$

which can be further written as

$$\bar{\lambda}_{ij} \left[ \left( \frac{1}{4} g^{-3/2} \frac{\partial g}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{1}{2} g^{-1/2} \frac{\partial^2 g}{\partial x_i \partial x_j} \right) \psi + g^{1/2} \frac{\partial^2 \psi}{\partial x_i \partial x_j} \right] - \bar{\beta}^2 g^{1/2} \psi = 0$$

Use of the identity

$$\frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} = -\frac{1}{4} g^{-3/2} \frac{\partial g}{\partial x_i} \frac{\partial g}{\partial x_j} + \frac{1}{2} g^{-1/2} \frac{\partial^2 g}{\partial x_i \partial x_j}$$
allows equation (7) to be written in the form
\[ g^{1/2} \lambda_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} - \psi \lambda_{ij} \frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} - \beta^2 g^{1/2} \psi = 0 \] (8)

If we further restrict the function \( g(x) \) to take the exponentially form
\[ g(x) = [A \exp (\alpha_m x_m)]^2 \lambda_{ij} \alpha_i \alpha_j = -k \quad k < 0 \] (9)

then
\[ \lambda_{ij} \frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} + kg^{1/2} = 0 \] (10)

Substituting (10) into (8) we obtain a constant coefficients equation
\[ \lambda_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} + (k - \beta^2) \psi = 0 \] (11)

Moreover, substitution of (3) and (6) into (2) gives
\[ P = -P_g \psi + P_\psi g^{1/2} \] (12)

where \( P_g(x) = \lambda_{ij} (\partial g^{1/2}/\partial x_j) n_i \) and \( P_\psi(x) = \lambda_{ij} (\partial \psi/\partial x_j) n_i \).

An integral equation for (11) is
\[ \eta(x_0) \psi(x_0) = \int_{\partial \Omega} \left[ \Gamma(x, x_0) \psi(x) - \Phi(x, x_0) P_\psi(x) \right] ds(x) \] (13)

where \( x_0 = (a, b) \), \( \eta = 0 \) if \((a, b) \notin \Omega \cup \partial \Omega\), \( \eta = 1 \) if \((a, b) \) lies inside the domain \( \Omega \), \( \eta = 1/2 \) if \((a, b) \) is on the boundary \( \partial \Omega \) given that \( \partial \Omega \) has a continuously turning tangent at \((a, b)\). The function \( \Phi \) in (13) is called the fundamental solution, which is any solution of the equation \( \lambda_{ij} (\partial^2 \Phi/\partial x_i \partial x_j) + (k - \beta^2) \Phi = \delta(x - x_0) \) and the \( \Gamma \) is defined as \( \Gamma(x, x_0) = \lambda_{ij} [\partial \Phi(x, x_0)/\partial x_j] n_i \) where \( \delta \) denotes the Dirac delta function. Following Azis [33], for 2-D problems \( \Phi \) and \( \Gamma \) are given by
\[ \Phi(x, x_0) = -\frac{K}{2\pi} K_0(\omega R) \] (14)
\[ \Gamma(x, x_0) = \frac{K \omega}{2\pi} K_1(\omega R) \lambda_{ij} (0) \frac{\partial R}{\partial x_j} n_i \] (15)

where \( K = \frac{\tau}{\zeta}, \omega = \sqrt{(k - \beta^2)/\zeta}, \zeta = \left[ \lambda_{11} + 2 \lambda_{12} \tau + \lambda_{22} (\tau^2 + \tau^2) \right]/2, R = \sqrt{(\hat{x}_1 - \hat{a})^2 + (\hat{x}_2 - \hat{b})^2}, \hat{x}_1 = x_1 + \tau x_2, \hat{a} = a + \tau b, \hat{x}_2 = \tau x_2 \) and \( \hat{b} = \tau b \) where \( \tau \) and \( \tau \) are respectively the real and the positive imaginary parts of the complex root \( \tau \) of the quadratic \( \lambda_{11} + 2 \lambda_{12} \tau + \lambda_{22} \tau^2 = 0 \) and \( K_0, K_1 \) denote the modified Bessel function of order zero and order one respectively and \( \tau \) represents the square root of minus one. The derivatives \( \partial R/\partial x_j \) necessary for the calculation of the \( \Gamma \) in (15) are given by \( \partial R/\partial x_1 = (\hat{x}_1 - \hat{a})/R \) and \( \partial R/\partial x_2 = \left[ \frac{\tau (\hat{x}_1 - \hat{a}) + \tau (\hat{x}_2 - \hat{b})}{R} \right] \). Use of (6) and (12) in (13) yields
\[ \eta(x_0) g^{1/2}(x_0) \phi(x_0) = \int_{\partial \Omega} \left\{ \left[ g^{1/2}(x) \Gamma(x, x_0) - P_g(x) \Phi(x, x_0) \right] \phi(x) \right. \] (16)

\[ \left. - \left[ g^{-1/2}(x) \Phi(x, x_0) \right] P(x) \right\} ds(x) \] (16)

Equation (16) provides a boundary integral equation which is the starting point of BEM construction for determining the numerical solutions of \( \phi \) and its derivatives at all points of \( \Omega \).
4. Numerical examples
To illustrate the use of BEM some examples of BVPs governed by (1) for FGMs are considered. The coefficients $\lambda_{ij}$ and $\beta^2$ in (1) are required to take the forms (3), (4) respectively with $g(x)$ taking the exponential form (9). The modified Bessel functions in (14) and (15) are approximated by their series, and the integral in (16) is evaluated using Gaussian quadrature (see Abramowitz and Stegun [34]). A FORTRAN script is developed to compute the solutions and a unique FORTRAN command is imposed to calculate the elapsed CPU computation time for obtaining the results.

After all, the main purpose of this section is to verify the validity of analysis used to derive the boundary integral equation in the previous sections and to examine the developed FORTRAN script.

4.1. A test problem

![Figure 1. The domain Ω](image)

The aim of this problem is to show the convergence, accuracy, efficiency and consistency of the BEM solutions. For a simplicity, the domain $\Omega$ is taken to be a unit square and the boundary conditions are as depicted in Figure 1. A number of elements of equal length on each side of the unit square are used. We take the constant coefficients $\lambda_{ij}, \beta^2$ and the parameter $k$

$$\bar{\lambda}_{ij} = \begin{bmatrix} 1 & 1.5 \\ 1.5 & 2 \end{bmatrix} \quad \bar{\beta}^2 = 3 \quad k = -0.25$$

The inhomogeneity function $g(x)$ taking exponential form (9) and satisfying (10) is assumed to be

$$g(x) = [1.5 \exp (0.25x_1 + 0.2057x_2)]^2$$

The analytical solution is

$$\phi(x) = 0.6667 \exp (1.75x_1 - 0.4151x_2)$$

Figure 2 shows the absolute errors of BEM $\phi$ solutions when using total number of 160, 320 and 640 elements. It indicates the accuracy and convergence of the BEM solution. Table 1 shows the efficiency of BEM for obtaining solutions at $19 \times 19$ interior points. Figure 3 shows consistency
between the scattering and flow solutions. Figure 2 shows the absolute errors of BEM $\phi$ solutions when using total number of 160, 320 and 640 elements. It indicates the accuracy and convergence of the BEM solution.

4.2. A problem without simple analytical solutions
A layered material consisting of eight layers of the same size as depicted in Figure 4 is under consideration. Each layer is supposed to be homogeneous, but from layer to layer the coefficients $\lambda_{ij}$ and $\beta^2$ vary as smoothly as the variability can be fitted to an exponential function

$$g(x) = [A \exp(\alpha_2 x_2)]^2$$
As an illustration, suppose that we have a set of values of the coefficients $\lambda_{ij}$ and $\beta^2$ at center point of each layer as shown in Table 2. And we also have reference values of constant coefficients $\lambda_{ij} = [0.35, 0.5]$ $\beta = 2$.

Fitting the data in Table 2 to the function $g(x) = [A \exp(\alpha_2 x_2)]^2$ we will get the values of the parameters $\alpha_0$ and $\alpha_2$

$$A = 0.5 \quad \alpha_2 = 0.483$$

Therefore we can then approximate the layered material as a sole material with continuously varying coefficients. So we may now use the analysis in Section 2 to solve the problem numerically.

The boundary conditions are depicted in Figure 4. We take the parameter $k = -0.35$ and the inhomogeneity function $g(x)$ takes the exponential form in equation (9). Again, a number of 640 elements of equal length, namely 160 elements on each side of the unit square, are used. As shown in Figure 5 for the given constant orthotropic coefficient $\lambda_{ij}$ above the solution $\phi$ resembles the nature of the considered medium as a layered material.

If, however, we change the constant orthotropic coefficient $\lambda_{ij}$ to an anisotropic coefficient $\overline{\lambda}_{ij} = \begin{bmatrix} 0.35 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}$
Figure 5. The BEM scattering $\phi$ and flow vector $\left( \frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2} \right)$ solutions of the orthotropic inhomogeneous medium

Figure 6. The BEM scattering $\phi$ and flow vector $\left( \frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2} \right)$ solutions of the anisotropic inhomogeneous medium

(therefore the values of $\lambda_{12}$ in Table 2 are not appropriate anymore) by keeping the values of $k, \beta^2, A, \alpha_2$ then we will obtain a significantly different solution $\phi$ as shown in Figure 6. This means that the anisotropy of the medium gives an impact on the solution. Therefore in application it is necessary for the anisotropy to be taken into account.

Now, if we assume that the medium is anisotropic homogeneous with

$$\bar{\lambda}_{ij} = \begin{bmatrix} 0.35 & 0.5 \\ 0.5 & 1.5 \end{bmatrix} \quad \alpha_2 = 0 \quad k = 0$$

then we will obtain solutions $\phi$ as shown in Figure 7 which are different with those for the previous case of anisotropic inhomogeneous as shown in Figure 6. It indicates that the impact of material’s inhomogeneity is also evident. This suggests to put the inhomogeneity in consideration for any application.

5. Conclusion

It is apparently possible to find numerical solutions of BVPs governed by an equation of variable coefficients such as the modified Helmholtz type equation (1) by using a standard BEM. In this work, a transformation is used to transform the variable coefficient equation into a constant coefficient equation from which a boundary integral equation can be derived. A BEM may then be constructed from the boundary integral equation obtained. The standard BEM provides an ease of implementation, timeless computation and accurate solutions.

Modeling physical application for an anisotropic FGM always involves a variable coefficients governing equation such as (1). In this paper, exponentially graded materials are considered as the FGMs.
In addition to its efficiency and accuracy, when the BEM works properly then consistency between the flow vectors and scattering solutions will be evident. Moreover, it is also observed that the anisotropy and inhomogeneity of the material effect the results. This suggests both anisotropy and inhomogeneity should be taken into account in applications.

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References
[1] Itagaki M and Brebbia C A 1993 Multiple reciprocity boundary element formulation for one-group fission neutron source iteration problems Engineering Analysis with Boundary Elements 11 39
[2] Singh K M and Tanaka M 2000 On exponential variable transformation based boundary element formulation for advection–diffusion problems Engineering Analysis with Boundary Elements 24 225
[3] Solekhudin I and Ang K-C 2012 A DRBEM with a predictor-corrector scheme for steady infiltration from periodic channels with root-water uptake Engineering Analysis with Boundary Elements 36 1199
[4] Chen K H, Chen J T, Chou C R and Yueh C Y 2002 Dual boundary element analysis of oblique incident wave passing a thin submerged breakwater Engineering Analysis with Boundary Elements 26 917
[5] Cheng H, Huang J and Leiterman T J 2006 An adaptive fast solver for the modified Helmholtz equation in two dimensions Journal of Computational Physics 211 616
[6] Kropinski M C A and Quaife B D 2011 Fast integral equation methods for the modified Helmholtz equation Journal of Computational Physics 230 425
[7] Gusyev M A and Haitjema H M 2011 An exact solution for a constant-strength line-sink satisfying the modified Helmholtz equation for groundwater flow Advances in Water Resources 34 519
[8] Guo S, Zhang J, Li G and Zhou F 2013 Three-dimensional transient heat conduction analysis by Laplace transformation and multiple reciprocity boundary face method Engineering Analysis with Boundary Elements 37 15
[9] Igarashi H and Honma T 1992 On axially and helically symmetric fundamental solutions to modified Helmholtz-type equations Applied Mathematical Modelling 16(6) 314
[10] Nguyen H T, Tran Q V and Nguyen V T 2013 Some remarks on a modified Helmholtz equation with inhomogeneous source Applied Mathematical Modelling 37 793
[11] Chen W, Zhang J-Y and Fu Z-J 2014 Singular boundary method for modified Helmholtz equations Engineering Analysis with Boundary Elements 44 112
[12] Zhang G M and Batra R C 2007 Wave propagation in functionally graded materials by modified smoothed particle hydrodynamics (MSPH) method Journal of Computational Physics 222 374
[13] Helayatrasa S, Bui T Q, Zhang C and Lim C W 2014 Numerical modeling of wave propagation in functionally graded materials using time-domain spectral Chebyshev elements Journal of Computational Physics 258 381

Figure 7. The BEM scattering $\phi$ and flow vector $\left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}\right)$ solutions of the anisotropic homogeneous medium
[14] Haddade A, Salam N, Khaeruddin and Azis M I 2017 A boundary element method for 2D diffusion-convection problems in anisotropic media Far East Journal of Mathematical Sciences 102(8) 1593
[15] Azis M I, Kasbawati, Haddade A and Thamrin S A 2018 On some examples of pollutant transport problems solved numerically using the boundary element method Journal of Physics: Conference Series 979 012075
[16] Azis M I, Asrul L, Khaeruddin and Paharuddin 2018 BEM solutions for unsteady transport problems in anisotropic media JP Journal of Heat and Mass Transfer 15(4) 915
[17] Cheng A H-D 1984 Darcy’s Flow with Variable Permeability: A Boundary Integral Solution Water Resources Research 20 980
[18] Clements D L and Azis M I 2000 A Note on a Boundary Element Method for the Numerical Solution of Boundary Value Problems in Isotropic Inhomogeneous Elasticity Journal of the Chinese Institute of Engineers 23(3) 261
[19] Azis M I and Clements D L 2008 Nonlinear transient heat conduction problems for a class of inhomogeneous anisotropic materials by BEM Engineering Analysis with Boundary Elements 32(12) 1054
[20] Azis M I and Clements D L 2014 A Boundary Element Method for Transient Heat Conduction Problem of Nonhomogeneous Anisotropic Materials Far East Journal of Mathematical Sciences 89(1) 51
[21] Azis M I and Clements D L 2014 On some problems concerning deformations of functionally graded anisotropic elastic materials Far East Journal of Mathematical Sciences 87(2) 173
[22] Azis M I, Toaha S, Bahri M and Ilyas N 2018 A boundary element method with analytical integration for deformation of inhomogeneous elastic materials Journal of Physics: Conference Series 979 012072
[23] Hamzah S, Azis M I and Syamsuddin E 2019 On some examples of BEM solution to elasticity problems of isotropic functionally graded materials IOP Conference Series: Materials Science and Engineering 619 012018
[24] Azis M I 2019 Standard-BEM solutions to two types of anisotropic-diffusion convection reaction equations with variable coefficients Engineering Analysis with Boundary Elements 105 87
[25] Salam N, Haddade A, Clements D L and Azis M I 2017 A boundary element method for a class of elliptic boundary value problems of functionally graded media Engineering Analysis with Boundary Elements 84(3) 186
[26] Azis M I 2019 Numerical solutions to a class of scalar elliptic BVPs for anisotropic exponentially graded media Journal of Physics: Conference Series 1218 012001
[27] Haddade A, Azis M I, Djafar Z, Jabir St N and Nurwahyu B 2019 Numerical solutions to a class of scalar elliptic BVPs for anisotropic quadratically graded media IOP Conference Series: Earth and Environmental Science 279 012007
[28] Jabir St N, Azis M I, Djafar Z and Nurwahyu B 2019 BEM solutions to a class of elliptic BVPs for anisotropic trigonometrically graded media IOP Conference Series: Materials Science and Engineering 619 012059
[29] Azis M I 2019 BEM solutions to exponentially variable coefficient Helmholtz equation of anisotropic media Journal of Physics: Conference Series 1277 012036
[30] Nurwahyu B, Abdullah B, Massinai A and Azis M I 2019 Numerical solutions for BVPs governed by a Helmholtz equation of anisotropic FGM IOP Conference Series: Earth and Environmental Science 279 012008
[31] Hamzah S, Azis M I and Amir A K 2019 Numerical solutions to anisotropic BVPs for quadratically graded media governed by a Helmholtz equation IOP Conference Series: Materials Science and Engineering 619 012060
[32] La Nafie N, Azis M I and Fahruddin 2019 Numerical solutions to BVPs governed by the anisotropic modified Helmholtz equation for trigonometrically graded media IOP Conference Series: Materials Science and Engineering 619 012058
[33] Azis M I 2017 Fundamental solutions to two types of 2D boundary value problems of anisotropic materials Far East Journal of Mathematical Sciences 101(11) 2405
[34] Abramowitz M and Stegun I A 1972 Handbook of mathematical functions: with formulas, graphs and mathematical tables, Dover Publications, Washington