q-Deformed Conformal and Poincaré
Algebras on Quantum 4-spinors

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Abstract

We investigate quantum deformation of conformal algebras by constructing
the quantum space for $\mathfrak{sl}_q(4, \mathbb{C})$. The differential calculus on the quantum space
and the action of the quantum generators are studied. We derive deformed
$\mathfrak{su}(4)$ and $\mathfrak{su}(2, 2)$ algebras from the deformed $\mathfrak{sl}(4)$ algebra using the quantum
4-spinor and its conjugate spinor. The 6-vector in $\mathfrak{so}_q(4, 2)$ is constructed as a
tensor product of two sets of 4-spinors. The reality condition for the 6-vector
and that for the generators are found. The q-deformed Poincaré algebra is
extracted as a closed subalgebra.

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1. Introduction

Recently much attention has been paid on quantum groups and quantum algebras \([1, 2, 3, 4]\) in both theoretical physics and mathematics \([3, 4, 7, 8]\). So far most of the applications are restricted to the Lie groups or algebras corresponding to the internal symmetries. Now it would be intriguing to investigate possible quantum deformations of fundamental space-time symmetries, which might be relevant at very short distance e.g. at Planck length implied by the unification of basic interactions including gravity. In the last few years, there have been pioneering works in extending quantum-group ideas to space-time symmetries including Lorentz, Poincaré and conformal algebras \([1, 2, 11, 12, 13, 14, 15, 16]\).

The quantum group deformation can be realized on the quantum space or quantum (hyper-)plane in which the coordinates are non-commutative \([17, 17]\). The differential calculus on the non-commutative space of quantum groups has been explored by Woronowicz \([18]\), which provides us with an example for noncommutative differential geometry \([19]\). Wess and Zumino and other people developed the differential calculus on the quantum (hyper-)plane covariant with respect to quantum groups \([20]\) and multiparameter deformation of the quantum groups, especially for \(GL_q(n)\) \([21, 22, 23]\), as well as for \(SO_q(n)\) \([24]\). Based on this framework quantum deformations of Lorentz group and algebra \([2, 10, 11, 12]\) have been studied by deforming \(SL(2, C)\). The q-deformed Poincaré algebra has also been studied along this line \([13]\). In Ref. \([15, 16]\), deformation of the conformal algebra has been discussed in Drinfeld-Jimbo procedure. In addition to the algebra, a space representing the algebra is interesting. Therefore, we here study the deformed space-time representing the deformed conformal algebra, as well as the deformed conformal algebra.

In this paper we investigate q-deformation of \(D = 4\) conformal algebra based on the quantum space which realizes \(SL_q(4, C)\) where the quantum 4-spinors appears as basic ingredients for which we set up commutation relations and study the differential calculus. We then investigate the action of quantum generators of \(sl_q(4, C)\) on these quantum 4-spinors. We next introduce quantum conjugate 4-spinors, in order to derive a charge conjugation of the generators and to obtain deformed \(su(4)\) and \(su(2, 2)\) algebras. The quantum 6-vector in \(so_q(4, 2)\) is constructed as a bi-spinor of two sets of 4-spinors, and the conjugation of the 6-vector which guarantees the reality of the 6-vector is presented. For a suitable conjugation one can obtain the q-deformed
Poincaré algebra as a closed subalgebra of q-deformed conformal algebra.

In the next section we introduce the quantum space realizing $SL_q(4, C)$ and study the differential calculus on the quantum 4-spinor space. From the consistency conditions we derive the action of the generators on quantum 4-spinors. In section 3 we study conjugate spinors and q-deformed $su(4)$ algebra. We next examine the quantum 6-vector constructed as the bilinear combination of two 4-spinors in section 4. In section 5 we study the reality conditions for the 6-vector as well as for the generators and present q-deformed conformal algebra. In section 6 we extract a q-deformation of Poincaré as a subalgebra of q-deformed conformal algebra and construct a quantum Casimir invariant. The final section will be devoted to some concluding remarks.

2. Deformed $sl(4)$ algebra on quantum space

We set up the commutation relations for the coordinates $x^i$ and derivatives $\partial_i$ on the quantum space \cite{20} as follows,

\begin{align}
  x^i x^j &= q x^j x^i, \\
  \partial_i \partial_j &= \frac{1}{q} \partial_j \partial_i, \\
  \partial_i x^k &= q x^k \partial_i, \\
  \partial_i x^i &= 1 + q^2 x^i \partial_i + (q^2 - 1) \sum_{j>i} x^j \partial_j,
\end{align}

where $i < j$. These relations are governed by a $\hat{R}$-matrix of $SL_q(n)$ quantum group, which is explicitly given as

\begin{equation}
  \hat{R}_{ij}^{\ell k} = \delta^i_\ell \delta^j_k ((1 - q^{-1}) \delta^{ij} + q^{-1}) + (1 - q^{-2}) \delta^i_k \delta^j_\ell \Theta^{ij},
\end{equation}

where $\Theta^{ij}$ is equal to 1 for $i > j$, otherwise vanishes. For example, the commutation relation of coordinates is written in terms of the $\hat{R}$-matrix as $x^i x^j = \hat{R}_{ij}^{\ell k} x^k x^\ell$. The $\hat{R}$-matrix is decomposed into two projection operators, i.e., a symmetric one $S$ and an antisymmetric one $A$ as

\begin{equation}
  S = \frac{1}{1 + q^{-2}} (\hat{R} + q^{-2} 1), \quad A = \frac{-1}{1 + q^{-2}} (\hat{R} - 1).
\end{equation}

Now, let us study deformed $sl(4)$ algebra on the 4-dim quantum space, which is also called a “quantum 4-spinor” hereafter. First of all, we consider actions of
generators $T_{\ell+1}^\ell$ ($\ell = 1 \sim 3$) which correspond to $x^i \partial_{\ell+1}$ and associate with simple roots in the classical limit ($q \rightarrow 1$). Following Ref. [11], we assume the actions of $T_{\ell+1}^\ell$ on $x^i$ and $\partial_i$ as

\begin{align}
T_{\ell+1}^\ell x^i &= a^{(\ell)i}x^iT_{\ell+1}^\ell + \delta_{\ell+1}^i x^\ell, \quad (2.4.a) \\
T_{\ell+1}^\ell \partial_i &= b_i^{(\ell)}\partial_i T_{\ell+1}^\ell + \beta_i^{(\ell)}\delta_i \partial_{\ell+1}. \quad (2.4.b)
\end{align}

Eqs. (2.4) should be consistent with the commutation relations (2.1). We calculate $T_{\ell+1}^\ell(x^i x^j - q x^j x^i)$ ($i < j$) so as to derive consistency conditions as follows,

$$
\delta_{\ell+1}^i (x^i x^j - qa^{(\ell)j} x^i x^\ell) + \delta_{\ell+1}^j (a^{(\ell)i} x^i x^\ell - q x^\ell x^i) = 0. \quad (2.5)
$$

Namely, we obtain $a^{(\ell)\ell} = q$ and $a^{(\ell)j} = 1$ for $j \neq \ell, \ell + 1$. It is remarkable that if we only consider the action of the generators on $x^i$, we are not able to determine coefficients of (2.4.a) completely, i.e., $a^{(\ell)\ell+1}$ is undetermined. Consistency between eq. (2.4.b) and the commutation relation $\partial_i \partial_j - q \partial_j \partial_i = 0$ provides $b_i^{(\ell)} = q$ and $b_i^{(\ell)} = 1$ for $i \neq \ell, \ell + 1$. Further, consistency between eqs. (2.4) and the commutation relations of $x$ and $\partial$ leads to $a^{(\ell)\ell+1} = 1/q$, $b_i^{(\ell)} = 1/q$ and $\beta_i^{(\ell)} = -1/q$. Thus we obtain the action of $T_{\ell+1}^\ell$ on $x$ and $\partial$ as follows,

\begin{align}
[T_{\ell+1}^\ell,x^i] &= [T_{\ell+1}^\ell,\partial_i] = 0, \\
T_{\ell+1}^\ell x^\ell &= qx^\ell T_{\ell+1}^\ell, \\
T_{\ell+1}^\ell x^{\ell+1} &= q^{-1}x^{\ell+1} T_{\ell+1}^\ell + x^\ell, \\
T_{\ell+1}^\ell \partial_\ell &= q^{-1} \partial_\ell T_{\ell+1}^\ell - q^{-1} \partial_{\ell+1}, \\
T_{\ell+1}^\ell \partial_{\ell+1} &= q \partial_{\ell+1} T_{\ell+1}^\ell,
\end{align}

where $i \neq \ell, \ell + 1$. These relations show that $T_{1/2}$ and $T_{3/4}$ commute with each other.

Next, we define $T_{\ell+2}^\ell$ ($\ell = 1, 2$) as

$$
T_{\ell+2}^\ell = [T_{\ell+1}^\ell, T_{\ell+2}^\ell] f^\ell, \quad (2.7)
$$

where $[A,B]_p = AB - pBA$. Using (2.6), we can easily obtain actions of $T_{\ell+2}^\ell$ on $x$ and $\partial$. In terms of $T_{\ell+2}^\ell$, we can define two generators corresponding to $x^1 \partial_4$ in the classical limit as follows,

$$
T_4^1 \equiv [T_3^1, T_{4/4}^3] g, \quad T_4^{g_1} \equiv [T_2^1, T_{4/4}^2] g' . \quad (2.8)
$$

They act on $x^i$ as follows,

$$
T_4^1 x^1 = q x^1 T_4^1 .
$$
For a concrete example, let us consider a commutation relation between $T_{4}^1 x^2 = x^2 T_{4}^1 + (q - f^1)x^1[T_{3}^2, T_{4}^3]_g$,

\[ T_{4}^1 x^3 = x^3 T_{4}^1 + (1 - qf^1)(1 - \frac{g}{q})x^2 T_{2}^1 T_{4}^3 + (q - g)x^1 T_{3}^4, \]

\[ T_{4}^1 x^4 = q^{-1} x^4 T_{4}^1 + (q^{-1} - g)x^3 T_{3}^1 + (q^{-1} - f^1)x^2 T_{2}^1 + x^1, \]

\[ T_{4}^1 x^1 = qx^1 T_{4}^1, \] \hspace{1cm} (2.9)

Further, the coefficients should obey a condition that $f^1 = 1/q$ or $f^2 = q$ so that the bilinear term $T_{3}^1 T_{4}^3$ vanishes on the right hand side of eq. (2.9).

Under the above conditions, actions of the two generators $T_{4}^1$ and $T_{4}^1$ on $\partial_i$ are also identified and do not have bilinear terms of $T$.

Next, we have to investigate closure of the algebra of the generators $T_{i}^j$ ($i < j$). For a concrete example, let us consider a commutation relation between $T_{\ell+1}^\ell T_{\ell+2}^\ell$ and $T_{\ell+2}^\ell T_{\ell+1}^\ell$ on $x$ are obtained as

\[ [T_{\ell+1}^\ell T_{\ell+2}^\ell, x^i] = [T_{\ell+2}^\ell T_{\ell+1}^\ell, x^i] = 0, \hspace{1cm} (i = \ell - 1 \text{ or } \ell + 3), \]

\[ T_{\ell+1}^\ell T_{\ell+2}^\ell x^{\ell+1} = q^{-1} x^{\ell+1} T_{\ell+1}^\ell + x T_{\ell+2}^\ell + q(q - f^\ell)x^\ell T_{\ell+1}^\ell T_{\ell+2}^\ell, \]

\[ T_{\ell+1}^\ell T_{\ell+2}^\ell x^{\ell+2} = q^{-1} x^{\ell+2} T_{\ell+1}^\ell + q^{-1}(q^{-1} - f^\ell)x^{\ell+1} T_{\ell+1}^\ell T_{\ell+2}^\ell + (q + q^{-1} - f^\ell)x^\ell T_{\ell+1}^\ell, \]

\[ T_{\ell+2}^\ell T_{\ell+1}^\ell x^{\ell+1} = q^{-1} x^{\ell+1} T_{\ell+2}^\ell T_{\ell+1}^\ell + q x^\ell T_{\ell+2}^\ell T_{\ell+1}^\ell + q^{-1}(q - f^\ell)x^{\ell+1} T_{\ell+2}^\ell T_{\ell+1}^\ell, \]

\[ T_{\ell+2}^\ell T_{\ell+1}^\ell x^{\ell+2} = q^{-1} x^{\ell+2} T_{\ell+2}^\ell T_{\ell+1}^\ell + q^{-1}(q - f^\ell)x^{\ell+1} T_{\ell+1}^\ell T_{\ell+2}^\ell + x^\ell T_{\ell+2}^\ell T_{\ell+1}^\ell. \] \hspace{1cm} (2.10)

Eq. (2.10) shows that if $f^\ell$ is equal to $q$ or $1/q$, $f^\ell T_{\ell+1}^\ell T_{\ell+2}^\ell$ is identified with $T_{\ell+2}^\ell T_{\ell+1}^\ell$. Similarly the algebra of the other generators close only if $f^\ell = q$ or $1/q$. Here, we choose a basis of the algebra where $f^1 = f^2 = q$. Under the basis, we can summarize the actions of $T_{k}^\ell$ ($\ell < k$) as follows,

\[ T_{k}^\ell x^\ell = qx^\ell T_{k}^\ell. \]
They act on 

\[ T^\ell_k x^k = q^{-1} x^k T^\ell_k + x^\ell + (q^{-1} - q) \sum_{j=\ell+1}^{k-1} x^j T^\ell_j , \]

\[ T^\ell_k x^j = x^j T^\ell_k, \quad (i \neq \ell, k), \]  \hspace{1cm} (2.11)

\[ T^\ell_k \partial_i = \partial_i T^\ell_k, \quad (i < \ell \text{ or } k < i), \]

\[ T^\ell_k \partial_\ell = q^{-1} \partial_\ell T^\ell_k - q^{-1} \partial_k, \]

\[ T^\ell_k \partial_k = q \partial_k T^\ell_k, \]

\[ T^\ell_k \partial_j = \partial_j T^\ell_k + (q - q^{-1}) \partial_k T^\ell_j, \quad (\ell < j < k). \]

Further, the algebra of these generators is obtained as follows,

\[ [T^\ell_j, T^\ell_i]_q = [T^\ell_k, T^\ell_i]_q = 0, \quad (i < j), \]

\[ [T^1_2, T^3_4] = [T^1_4, T^2_3] = 0, \]

\[ [T^\ell_j, T^\ell_i]_q = T^\ell_k, \]  \hspace{1cm} (2.12)

\[ [T^2_4, T^3_1] = (q - q^{-1}) T^1_4 T^2_3. \]

Similarly we can obtain actions of \( T^k_\ell (\ell < k) \), and algebra of them. The results are summarized in Appendix.

Now let us define three Cartan elements \( H_\ell (\ell = 1 \sim 3) \) in terms of commutation relations between \( T^{\ell+1}_\ell \) and \( T^{\ell+1}_{\ell+1} \). Making use of (2.11) and (A.1), we can easily find actions of \( T^{\ell+1}_\ell T^{\ell}_\ell \) and \( T^{\ell+1}_{\ell+1} T^{\ell}_{\ell+1} \) on \( x^i \) as follows,

\[ T^{\ell+1}_\ell T^{\ell}_\ell x^\ell = q^2 x^\ell T^{\ell+1}_\ell T^{\ell}_{\ell+1} + q^{-1} x^{\ell+1} T^{\ell}_{\ell+1} + x^\ell, \]

\[ T^{\ell+1}_\ell T^{\ell+1}_\ell x^{\ell+1} = q^{-2} x^{\ell+1} T^{\ell+1}_\ell T^{\ell}_{\ell+1} + q^{-1} x^\ell T^{\ell+1}_{\ell+1}, \]  \hspace{1cm} (2.13)

\[ T^{\ell+1}_\ell T^{\ell+1}_\ell x^{\ell} = q^2 x^\ell T^{\ell+1}_\ell T^{\ell}_{\ell+1} + q x^{\ell+1} T^{\ell}_{\ell+1}, \]

\[ T^{\ell+1}_\ell T^{\ell+1}_\ell x^{\ell+1} = q^{-2} x^{\ell+1} T^{\ell+1}_\ell T^{\ell}_{\ell+1} + q x^\ell T^{\ell+1}_{\ell+1} + x^{\ell+1}. \]

Following Ref. [14], we define \( H_\ell \) by linear combination of \( T^{\ell+1}_\ell T^{\ell}_\ell \) and \( T^{\ell+1}_{\ell+1} T^{\ell}_{\ell+1} \) in order to eliminate the linear term of \( T \) on the right hand side of (2.13)

\[ H_\ell \equiv q^{-1} T^{\ell+1}_\ell T^{\ell}_{\ell+1} - q T^{\ell+1}_{\ell+1} T^{\ell}_{\ell+1}. \]  \hspace{1cm} (2.14)

They act on \( x \) and \( \partial \) as

\[ H_\ell x^\ell = q^2 x^\ell H_\ell - q x^\ell, \]

\[ H_\ell x^{\ell+1} = q^{-2} x^{\ell+1} H_\ell + q^{-1} x^{\ell+1}, \]
\[ H_\ell \partial_\ell = q^{-2} \partial_\ell H_\ell + q^{-1} \partial_\ell, \quad (2.15) \]

\[ H_\ell \partial_{\ell+1} = q^2 \partial_{\ell+1} H_\ell - q \partial_{\ell+1}, \]

\[ [H_\ell, x^i] = [H_\ell, \partial_i] = 0, \quad (i \neq \ell, \ell + 1). \]

Using (2.15), we can easily find that the Cartan elements \( H_\ell \) commute with each other, i.e.,

\[ [H_\ell, H_k] = 0. \quad (2.16) \]

Making use of (2.11) and (A.1), we can calculate commutation relations between \( T^\ell_{\ell+1} \) and \( T^{k+1}_k \) \( (\ell \neq k) \) as follows,

\[ [T^\ell_{\ell+1}, T^{\ell+2}_{\ell+1}] = [T^{\ell+1}_{\ell+2}, T^\ell_\ell] = 0, \]

\[ [T^1_2, T^4_3] = [T^3_4, T^2_1] = 0. \quad (2.17) \]

Moreover, we can obtain commutation relations between \( T^i_j \) for \( k = j + 1 \) or \( j - 1 \) as follows,

\[ q^{-2} H_\ell T^\ell_{\ell+1} - q^2 T^\ell_{\ell+1} H^\ell = -(q + q^{-1}) T^\ell_{\ell+1}, \]

\[ q H_\ell T^{\ell+1}_{\ell+2} - q^{-1} T^{\ell+1}_{\ell+2} H^\ell = T^{\ell+1}_{\ell+2}, \]

\[ q H^{\ell+1}_\ell T^\ell_{\ell+1} - q^{-1} T^\ell_{\ell+1} H^{\ell+1} = T^\ell_{\ell+1}, \]

\[ q^2 H^\ell T^{\ell+1}_\ell - q^{-2} T^{\ell+1}_\ell H^\ell = (q + q^{-1}) T^{\ell+1}_\ell, \]

\[ q^{-1} H^\ell T^{\ell+2}_{\ell+1} - q T^{\ell+2}_{\ell+1} H^\ell = -T^{\ell+2}_{\ell+1}, \]

\[ q^{-1} H^{\ell+1} T^{\ell+1}_\ell - q T^{\ell+1}_\ell H^{\ell+1} = -T^{\ell+1}_\ell. \quad (2.18) \]

Using all the above algebra, we can find the other commutation relations of the generators, which are shown in Appendix. The above approach could be extended to obtain deformed \( sl(n) \) \( (n > 4) \) algebra on the \( n \)-dim quantum space.

### 3. Conjugate spinor and deformed \( su(4) \) algebra

Here we introduce another spinor \( \pi_i \) conjugate to the quantum 4-spinor \( x^i \), in order to obtain deformed \( su(m, n) \) \( (m + n = 4) \) algebra. The conjugate spinor is required to satisfy a commutation relation as

\[ \pi_i \pi_j = q^{-1} \pi_j \pi_i, \quad (i < j). \quad (3.1) \]
Further we define a commutation relation between $x_i$ and $x^j$ as follows,

$$x_i x^j = \hat{R}^{jk}_{\ell i} x^\ell x_k. \quad (3.2)$$

This commutation relation is explicitly written as

$$x_i x^j = q^{-1} x^j x_i, \quad (i \neq j), \quad (3.3)$$

$$x_i x^i = x^i x_i + (1 - q^{-2}) \sum_{i<j} x^j x_j.$$

The above commutation relation has a center $c = x^i x_i$.

Furthermore, we assume that the generators act on $\xi_i$ in the same way as on $\partial_i$, (2.11),(2.15) and (A.1). For example the generator $T^\ell_{\ell+1}$ acts on $\xi_{\ell+1}$ as

$$T^\ell_{\ell+1} \xi_{\ell+1} = q \xi_{\ell+1} T^\ell_{\ell+1} - q \xi_{\ell}. \quad (3.4)$$

We now consider conjugation between $x^i$ and $\xi_i$. The conjugation should be consistent with the commutation relations, (2.1), (3.1) and (3.3). We can find two types of consistent charge conjugations depending on values of $q$. If $q$ is real, the conjugation includes reversing the order, i.e., $ab = b a$. On the other hand, the conjugation for $|q| = 1$ is simply written as $ab = a b$. Under both conjugations, the spinor and conjugate spinor are related as

$$\xi_i = \eta^i (x^i)^*, \quad (3.5)$$

where * implies the complex conjugate and $\eta^i$ is a metric of the $su(m, n)$ algebra. The center $c$ is written in terms of $x^i$ and their complex conjugate as $\eta^i x^i (x^i)^*$. Thus, deformed $su(4)$ and $su(2, 2)$ algebras have $\eta^i = (1, 1, 1, 1)$ and $(1, 1, -1, -1)$, respectively.

Now we consider charge conjugation of the generators under the above conjugations. At first, we study the case with real $q$, where we also have to reverse the order of the generators and the spinors when taking the conjugation, i.e., $\overline{T x} = \overline{\xi T}$. For example, we take the conjugation of the commutation relation between $T^\ell_{\ell+1}$ and $x^\ell_{\ell+1}$ (2.6), so that we obtain

$$\overline{T^\ell_{\ell+1} \xi_{\ell+1}} = q \overline{x_{\ell+1}} \overline{T^\ell_{\ell+1}} - q \eta^\ell \eta^\ell_{\ell+1} \overline{x_{\ell}}. \quad (3.6)$$

Comparing (3.6) with (3.4), we find charge conjugation of the generator $T^\ell_{\ell+1}$ as follows,

$$\overline{T^\ell_{\ell+1}} = \eta^\ell \eta^\ell_{\ell+1} T^\ell_{\ell+1}. \quad (3.7)$$
Similarly we find charge conjugation of the other generators as

\[ \overline{T^\ell_k} = \eta^\ell \eta^k T^\ell_k, \quad \overline{H_\ell} = H_\ell. \] (3.8)

On the other hand, in the case with \(|q| = 1\), we need not reverse the order, i.e., \( \overline{T^\ell x} = T^\ell \overline{x} \). We take the conjugation of the commutation relation between \( T^\ell_{\ell+1} \) and \( x \), so as to obtain

\[ \overline{T^\ell_{\ell+1} x_{\ell+1}} = q x_{\ell+1} T^\ell_{\ell+1} + \eta^\ell \eta^{\ell+1} x_\ell. \] (3.9)

Thus we find the charge conjugation of \( T^\ell_{\ell+1} \) as follows,

\[ \overline{T^\ell_{\ell+1}} = -q^{-1} \eta^\ell \eta^{\ell+1} T^\ell_{\ell+1}. \] (3.10)

Similarly we obtain the charge conjugation of the other generators as,

\[ \overline{T^\ell_k} = -q^{2(\ell-k)+1} \eta^\ell \eta^k T^\ell_k, \quad \overline{H_\ell} = -H_\ell. \] (3.11)

At last we obtain two types of deformed \( su(m, n) \) \((m + n = 4)\) algebra as the deformed \( sl(4) \) algebra with two types of the conjugations, (3.8) and (3.11). Especially, we obtain deformed conformal algebra represented on the quantum 4-spinor and the conjugate spinor. Here, Lorentz generators are assigned to \( T^1_2, T^2_1, H_1, T^3_4, T^4_3, H_3 \) and the other Cartan element \( H_2 \) corresponds to degree of freedom of dilatation. Further, the other generators of the \( su(2, 2) \) algebra correspond to linear combinations of translation and conformal boost.

4. Quantum 6-vector

Classical algebras \( su(4) \) and \( su(2, 2) \) are isomorphic to \( so(6) \) and \( so(4, 2) \), respectively. In Ref. [25], quantum generalization of the isomorphism between \( sl(4) \) and \( so(6) \) has been discussed. Here we construct “quantum 6-vector” from two copies of quantum 4-spinors in a similar way to Ref. [26, 10], to represent the deformed conformal algebra on the quantum 6-vector. In addition to \( x^i \), the other quantum 4-spinor is denoted by \( y^i \), which satisfy the same commutation relation as one of \( x^i \). Further, the generators have the same actions on \( y^i \) as those on \( x^i \). We assume that commutation relations between \( x^i \) and \( y^i \) are obtained as

\[ x^i y^j = \hat{R}_{ij}^{\ell} k_x y^k x^\ell, \] (4.1)
Eq. (4.1) is explicitly written as
\begin{align*}
x^j y^i &= q^{-1} y^i x^j, \quad x^i y^j = y^j x^i, \\
x^i y^j &= q^{-1} y^j x^i + (1 - q^{-2}) y^i x^j, \quad (4.2)
\end{align*}
where \( i < j \).

In the classical \( su(4) \) algebra, a product of two fundamental quartet representations, which are spinor representations of \( so(6) \), is decomposed into \( 10 \) and \( 6 \) representations. The latter is a vector representation of \( so(6) \) and antisymmetric for two quartets. Thus we need quantum generalization of the decomposition of the representations, in order to construct the quantum 6-vector from the two copies of the quantum 4-spinors. A condition of the decomposition is closure of commutation relation algebra. Namely, from linear combinations of \( x^i y^j \), we have to pick up six elements whose commutation relations close themselves. For that purpose, it is adequate to use the antisymmetric projection operator \( A \). Commutation relations of \( A_{ij}^{\ell} x^k y^\ell \) are close themselves. Six independent elements of \( A_{ij}^{\ell} x^k y^\ell \) are proportional to \( x^i y^j - qx^j y^i \) (\( i < j \)) and hereafter we use
\begin{align*}
a^{ij} &\equiv x^i y^j - qx^j y^i, \quad (i < j),
\end{align*}
as the quantum 6-vector instead of \( A_{ij}^{\ell} x^k y^\ell \). Actually, they satisfy commutation relations as follows,
\begin{align*}
a^{ij} a^{ik} &= qa^{ik} a^{ij}, \\
a^{ij} a^{jk} &= qa^{jk} a^{ij}, \\
a^{ik} a^{jk} &= qa^{jk} a^{ik}, \\
[a^{14}, a^{23}] &= 0, \\
[a^{13}, a^{24}] &= (q - q^{-1}) a^{14} a^{13}, \\
[a^{12}, a^{34}] &= q(-q + q^{-1}) a^{14} a^{23} + (q - q^{-1}) a^{13} a^{24}, \quad (4.4)
\end{align*}
where \( i < j < k \). The commutation relations are nothing but those of the quantum space for \( SO_q(6) \). Further, this algebra has a center given by
\begin{align*}
L &\equiv qa^{14} a^{23} - a^{13} a^{24} + \frac{1}{q} a^{12} a^{34}.
\end{align*}
Similarly in terms of two copies of the quantum conjugate 4-spinors, we can construct another quantum 6-vector \( \tilde{a} \), which has a similar commutation relation to (4.4).
Actions of the generators on the 6-vector are obtained through (2.11), (2.15) and (A.1). For example, the actions of the Cartan elements on the 6-vector are obtained as

\[ [H_\ell, a^\ell \ell+1] = [H_\ell, a^{ij}] = 0, \]

\[ H_\ell a^\ell j = q^2 a^\ell i H_\ell - qa^\ell j, \]

\[ H_\ell a^{i \ell +1} = q^2 a^i \ell +1 H_\ell + q^{-1} a^i \ell +1, \]

\[ H_\ell a^i \ell = q^2 a^i \ell H_\ell - qa^i \ell, \]

\[ H_\ell a^{\ell+1} j = q^2 a^{\ell+1} j H_\ell + q^{-1} a^{\ell+1} j, \]

where \( i < \ell \) and \( \ell + 1 < j \), and \( T^\ell_k (\ell < k) \) acts on the 6-vector as

\[ T^\ell_k a^{\ell k} = q^{-1} a^{\ell k} T^\ell_k + a^{\ell j} + (q^{-1} - q) \sum_{m=\ell+1}^{k-1} a^{m j} T^\ell_m, \]

\[ [T^\ell_k, a^{ij}] = 0, \]

\[ T^\ell_k a^{jk} = a^{jk} T^\ell_k + a^{\ell j} + (q^{-1} - q) \sum_{m=\ell+1}^{k-1} a^{mj} T^\ell_m, \]

\[ T^\ell_{\ell+2} a^{\ell+1 \ell+2} = q^{-1} a^{\ell+1 \ell+2} T^\ell_{\ell+2} - qa^{\ell+1 \ell+1}, \]

\[ T^4 a^{24} = q^{-1} a^{24} T^4 + (q^{-1} - q) a^{23} T^3 - qa^{12}, \]

\[ T^4 a^{34} = q^{-1} a^{34} T^4 + (q^2 - 1) a^{23} T^2 - qa^{13}, \]

where in the last equation each index differs from the others. Actions of \( T^\ell_k (\ell < k) \) on \( a^{ij} \) are shown in Appendix.

In order to obtain four-dimensional space from six-dimensional one, we have to compactify two coordinates. In the classical coordinates, we can choose arbitrarily two compactified coordinates, but in the quantum coordinates all compactification are not consistent with commutation relations (4.4). Note that commutation relations of \( a^{13}, a^{14}, a^{23} \) and \( a^{24} \) close themselves. Thus, we should choose \( a^{12} \) and \( a^{34} \) as the compactified coordinates. That is the remarkable fact in “quantum compactification”. 
5. Conjugation and q-deformed conformal algebras

In this section we first consider another type of conjugation which acts on 6-vectors as well as on the generators. Through this conjugation procedure, the two sets of quantum 4-spinors are related with each other.

As discussed in the previous section, the 6-vectors are represented as tensor products of two quantum 4-spinors, $x$ and $y$ as

$$a^{ij} \equiv x^i \otimes y^j - qx^j \otimes y^i, \quad (i < j), \quad (5.1)$$

where we put $\otimes$ between $x$ and $y$ to show the tensor product more explicitly. We now introduce a new type of conjugation denoted by $\mathcal{C}^{\ast}$ in which we exchange the 4-spinors $x$ and $y$ and subsequently take complex conjugates as follows:

$$(a^{ij})^{\mathcal{C}} \equiv (y^i)^{\ast} \otimes (x^j)^{\ast} - q^{\ast}(y^j)^{\ast} \otimes (x^i)^{\ast}, \quad (i < j). \quad (5.2)$$

As we have mentioned in the previous section, a bi-spinor of two conjugate 4-spinors leads to another quantum 6-vector $\tilde{a}$ which can now be identified with $a^{\mathcal{C}}$ defined in the above equation.

Now it turns out that there exist two possibilities for this conjugation in order for the six-vector to transform properly under the conjugation.

i) Conjugation 1

For real $q$, we take

$$(x^i)^{\ast} = \eta^i y^{5-i} \quad (i = 1, \cdots, 4), \quad (5.3)$$

where $\eta^i = \pm 1$ depending upon metric for $su(m, n)$. For $su(2, 2)$ we take $\eta^1 = \eta^2 = +1, \eta^3 = \eta^4 = -1$. This conjugation leads to the reality condition:

$$(a^{ij})^{\mathcal{C}} = \eta^i \eta^{5-j} a^{5-j, 5-i} \quad (i < j). \quad (5.4)$$

Explicitly we have $(a^{14})^{\mathcal{C}} = a^{14}, (a^{23})^{\mathcal{C}} = a^{23}, (a^{12})^{\mathcal{C}} = \eta^1 \eta^3 a^{34} = -a^{34}, (a^{13})^{\mathcal{C}} = \eta^1 \eta^2 a^{24} = a^{24}$.

ti) Conjugation 2

ii) Conjugation 2
For $|q| = 1$ or $q^* = 1/q$, we set the following relation
\[
(x^i)^* = q^i \sqrt{q} y^i, \quad (y^j)^* = q^j \sqrt{q} x^i \quad (i = 1, \cdots, 4).
\]
(5.5)

The conjugation of the 6-vector reads
\[
(a^{ij})^\oplus = -\eta^i \eta^j a^{ij} \quad (i < j),
\]
(5.6)
namely we obtain the following reality condition:
\[
(a^{12})^\oplus = -a^{12}, \quad (a^{13})^\oplus = a^{13}, \quad (a^{14})^\oplus = a^{14} \quad (i = 1, \cdots, 4).
\]
(5.7)

We now turn to the transformation of the generators under the conjugation $\oplus$. The transformations are obtained in a procedure similar to the approach discussed in section 3, (3.6) and (3.7).

For the conjugation 1, we find
\[
(T^\ell_{\ell+1})^\oplus = -q^\ell \eta^\ell \eta^\ell+1 T^\ell_{(\ell+1)'} \quad (\ell' = 5 - \ell)
\]
\[
(T^\ell_{\ell+2})^\oplus = -q^{3} \eta^\ell \eta^\ell+2 T^\ell_{(\ell+2)'} \quad (T^1_4)^\oplus = -q^{5} \eta^1 \eta^4 T^4_1
\]
\[
(H_1)^\oplus = H_3, \quad (H_2)^\oplus = H_2.
\]
(5.8)

Or generally we have
\[
(T^i_k)^\oplus = -q^{2(k-i)-1} \eta^i \eta^k T^i_k.
\]
(5.9)

For conjugation 2 we have
\[
(T^\ell_{\ell+1})^\oplus = -\frac{1}{q} \eta^\ell \eta^\ell+1 T^\ell_{(\ell+1)}, \quad (T^\ell_{\ell+1})^\oplus = -q^\ell \eta^\ell \eta^\ell+1 T^\ell_{(\ell+1)'}
\]
\[
(T^\ell_{\ell+2})^\oplus = -q^{-3} \eta^\ell \eta^\ell+2 T^\ell_{(\ell+2)} \quad (T^{\ell+2}_{\ell+2})^\oplus = -q^3 \eta^\ell \eta^\ell+2 T^{\ell+2}_{(\ell+2)'}
\]
\[
(T^4_1)^\oplus = -q^{-5} \eta^1 \eta^4 T^4_1 \quad (T^1_4)^\oplus = -q^{5} \eta^1 \eta^4 T^4_1
\]
\[
(H_\ell)^\oplus = -H_\ell.
\]
(5.10)

Generally we have
\[
(T^i_k)^\oplus = -q^{2(i-k)+1} \eta^i \eta^k T^i_k.
\]
(5.11)

The above conjugations are important in the case when we consider a subalgebra of $su(2, 2)$. Namely, the $\oplus$-conjugations should close in the subalgebra.
Let us now assign the $su_q(2, 2)$ generators to the q-deformed conformal algebra. We first consider the case of conjugation 2. Before going to the q-deformed case, we recapitulate the representation for the the generators of classical 4D conformal group, namely translation operators $P_\mu$, conformal boost operators $K_\mu$, Lorentz rotation operators $M_{\mu\nu}$ and dilatation operator $D$ as $su(2, 2)$ generators (See e.g. ref.[27]):

$$P_\mu = \begin{pmatrix} 0 \\ \sigma_\mu \\ 0 \end{pmatrix}, \quad K_\mu = \begin{pmatrix} 0 \\ \sigma_\mu \\ 0 \end{pmatrix}$$

$$M_{\mu\nu} = \begin{pmatrix} \sigma_{\mu\nu} \\ 0 \\ \bar{\sigma}_{\mu\nu} \end{pmatrix}, \quad D = \begin{pmatrix} -\frac{i}{2}1 \\ 0 \\ \frac{1}{2}1 \end{pmatrix}$$

(5.12)

where

$$\sigma_\mu = (1, \sigma^1, \sigma^2, \sigma^3), \quad \bar{\sigma}_\mu = (1, -\sigma^1, -\sigma^2, -\sigma^3)$$

$$\sigma_{\mu\nu} = \frac{1}{4}(\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu), \quad \bar{\sigma}_{\mu\nu} = \frac{1}{4}(\bar{\sigma}_\mu \sigma_\nu - \bar{\sigma}_\nu \sigma_\mu).$$

(5.13)

and $\sigma^i$ ($i = 1, 2, 3$) are Pauli matrices. For example $P_\mu$ and $K_\mu$ correspond to the generators $T^i_k$ as follows

$$P_0 = T^1_3 + T^2_4, \quad P_1 = T^1_4 + T^2_3$$

$$P_2 = -iT^1_4 + iT^2_3, \quad P_3 = T^1_3 - T^2_4$$

$$K_0 = T^3_1 + T^4_2, \quad K_1 = -T^1_4 - T^2_3$$

$$K_2 = -iT^4_1 + iT^3_2, \quad K_3 = -T^3_1 + T^4_2.$$  

(5.14)

For the Lorentz generators $M_{\mu\nu}$ and the dilatation operator $D$ we take the following assignment:

$$M_+ = M_{23} + iM_{31} = -iT^1_2 - iT^3_4, \quad M_- = M_{23} - iM_{31} = -iT^2_1 - iT^4_3$$

$$M_3 = M_{21} = \frac{i}{2}(H_1 + H_3)$$

$$L_+ = M_{20} + iM_{01} = -iT^1_2 + iT^3_4, \quad L_- = M_{20} - iM_{01} = iT^2_1 - iT^4_3$$

$$L_3 = M_{03} = \frac{i}{2}(H_1 - H_3)$$

$$D = \frac{1}{2}(H_1 + 2H_2 + H_3).$$

(5.15)

Now we take the same assignment of the generators for the q-deformed case. The q-deformed conformal algebra can be read off from the commutation relations among the $sl_q(4, C)$ generators given in section 2 together with Appendix. What is remarkable
here is that the Poincaré generators assigned above form a closed subalgebra of the q-deformed conformal algebra in the case of conjugation 2.

Now in the case of conjugation 1, we make the following assignment for the generators. For translation and conformal boost we take

\[ P^1_3 \equiv T^1_3 + q^3 T^4_2 \, , \, P^1_4 \equiv T^1_4 + q^5 T^4_1 \, , \, P^2_3 \equiv T^2_3 + q T^3_2 \, , \, P^2_4 \equiv T^2_4 + q^3 T^3_1 \]

\[ K^1_3 \equiv T^1_3 - q^3 T^4_2 \, , \, K^1_4 \equiv T^1_4 - q^5 T^4_1 \, , \, K^2_3 \equiv T^2_3 - q T^3_2 \, , \, K^2_4 \equiv T^2_4 - q^3 T^3_1 \]

and the same assignment for the Lorentz and the dilatational generators as given in (5.15). In this case, however, the q-deformed conformal algebra does not include Poincaré algebra as a closed subalgebra. It turns out that there is no possible contraction procedure which leads to the closed Poincaré algebra.

6. q-deformed Poincaré algebra and Casimir invariant

Poincaré algebra, which is not a simple Lie algebra, cannot be deformed in the standard prescription. The conceivable method for obtaining Poincaré algebra is through a contraction of anti-de Sitter algebra \( so(3,2) \) or conformal algebra \( so(4,2) \) \[14, 13, 12, 28\]. In this paper we extract it from the conformal algebra as a closed subalgebra. Here we shall restrict ourselves to the case of conjugation 2.

For this choice we have q-deformed Lorentz algebra generated by \( T^1_2 \, , \, T^2_1 \, , \, H_1 \) together with \( T^3_4 \, , \, T^4_3 \) and \( H_3 \), which are related to the conventional Lorentz generators as given in (5.15). They satisfy the following commutation relation:

\[ q^{-1} T^{\ell+1}_{\ell} T^{\ell}_{\ell+1} - q T^{\ell}_{\ell+1} T^{\ell+1}_{\ell} = H_\ell \]

\[ q^{-2} H_\ell T^{\ell}_{\ell+1} - q^2 T^{\ell}_{\ell+1} H_\ell = -(q + q^{-1}) T^{\ell}_{\ell+1} \]

\[ q^2 H_\ell T^{\ell+1}_{\ell} - q^{-2} T^{\ell+1}_{\ell} H_\ell = (q + q^{-1}) T^{\ell+1}_{\ell} \quad (\ell = 1, 3). \]

From (5.14) we can form light-cone combinations which are associated with single
generators as
\[ P_\pm = \frac{1}{2}(P_0 \pm P_3) \quad \tilde{P}_\pm = \frac{1}{2}(P_1 \pm iP_2) \]
\[ P_+ = T^1_3 \quad P_- = T^2_4 \quad \tilde{P}_+ = T^1_4 \quad \tilde{P}_- = T^2_3 \]
\[ K_\pm = \frac{1}{2}(K_0 \pm K_3) \quad \tilde{K}_\pm = \frac{1}{2}(K_1 \pm iK_2) \]
\[ K_+ = T^4_2 \quad K_- = T^3_1 \quad \tilde{K}_+ = -T^3_2 \quad \tilde{K}_- = -T^4_1. \] (6.2)

In this base the q-deformed commutation relations read
\[ [\tilde{P}_+, \tilde{P}_-] = 0 \quad [P_+, P_-] = -(q - q^{-1})\tilde{P}_+\tilde{P}_- \]
\[ [\tilde{P}_-, P_+] = 0 \quad [\tilde{P}_+, P_+] = 0 \]
\[ [P_-, \tilde{P}_+] = 0 \quad [P_-, \tilde{P}_+] = 0. \] (6.3)

The commutation relations between the translational operators \( P_\mu \) and the Lorentz generators \( M_{\mu\nu} \) can be read off from the commutation relations of \( T^{ik} \) given in section 2 and Appendix.

Here we find the quadratic Casimir invariant for q-deformed Poincaré group is given by
\[ C = qT^1_3T^2_4 - T^1_4T^2_3. \] (6.4)

In terms of translational operators (5.15), the above Casimir can be written as
\[ C = qP_+P_- - \tilde{P}_+\tilde{P}_-, \] (6.5)
which can be seen to commute with all the generators of the q-deformed Poincaré algebra, and
\[ 4C = (P_0^2 - P_3^2 - P_1^2 - P_2^2) - \frac{q}{2}(q - q^{-1})(P_1^2 + P_2^2) + (q - 1)(P_0^2 - P_3^2). \] (6.6)

 corresponds to the q-deformed version of the relativistic mass squared operator. It would be interesting to investigate the q-deformed Klein-Gordon equation for the q-deformed d’Alembertian corresponding to (6.6).

7. Concluding remarks
In this paper we investigated the q-deformed conformal and Poincaré algebras in the framework of the quantum space for $sl_q(4, C)$. We set up the non-commutative relation for the quantum 4-spinors and analyzed the differential calculus as well as the action of the generators on this quantum space. Through the charge conjugations, we obtain two types of the deformed $su(2,2)$ algebras and assign their generators to the elements of the deformed conformal algebra. The 6-vectors were constructed out of two sets of 4-spinors as tensor products and their commutation relations were obtained. We derived the q-deformed conformal algebra by setting up proper conjugations of 4-spinors and 6-vectors. The q-deformed Poincaré algebra was extracted as the closed subalgebra for the suitable choice of the conjugation.

Now some remarks on the possible extensions are in order. The present analysis of the one-parameter deformation could be extended to multi-parameter deformations. In the present paper we studied the deformation of the 4D conformal algebra corresponding to the deformed $sl(4, C)$, which could be extended to higher dimensional algebras based on the present method. It would be extremely interesting to extend this formalism to 4D superconformal algebra on the quantum superspace\cite{29}, which might shed some light on the quantum deformation of the super-Poincaré algebra, which is now under investigation.

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Appendix

In a similar way to (2.11), we find actions of the generators $T^k_\ell$ ($\ell < k$) as follows,

$$T^k_\ell x^i = x^i T^k_\ell, \quad (i < \ell \text{ or } k < i),$$

$$T^k_\ell x^\ell = qx^\ell T^k_\ell + x^k,$$

$$T^k_\ell x^j = x^j T^k_\ell + (q^{-1} - q)x^k T^j_\ell, \quad (\ell < j < k),$$

$$T^k_\ell x^k = q^{-1} x^k T^k_\ell,$$

$$T^k_\ell \partial_\ell = q^{-1} \partial_\ell T^k_\ell,$$

$$T^k_\ell \partial_k = q \partial_k T^k_\ell - q^2(k-\ell)-1 \partial_\ell + (q - q^{-1}) \sum_{j=\ell+1}^{k-1} q^2(k-j) \partial_j T^j_\ell,$$

$$T^k_\ell \partial_i = \partial_i T^k_\ell, \quad (i \neq \ell, k).$$

Further, they satisfy commutation relations as

$$[T^\ell_\ell, T^\ell_\ell]_q = [T^\ell_i, T^\ell_j]_q = 0, \quad (i < j),$$

$$[T^3_3, T^2_1] = [T^4_1, T^3_2] = 0,$$

$$[T^k_\ell, T^j_\ell]_q = T^k_\ell,$$

$$[T^3_1, T^4_2] = (q - q^{-1}) T^4_1 T^3_2.$$

In section 2, we have shown the basic commutation relations from which the other commutation relations can be derived. The remaining commutation relations are obtained as

$$[H_\ell, T^\ell_k]_q^2 = -qT^\ell_k, \quad [H_{k-1}, T^\ell_k]_q^2 = -qT^\ell_k, \quad (k > \ell + 1 \text{ or } k < \ell - 1),$$

$$[H_k, T^\ell_k]_{q^{-2}} = q^{-1}T^\ell_k, \quad [H_{\ell-1}, T^\ell_k]_{q^{-2}} = q^{-1}T^\ell_k, \quad (k > \ell + 1 \text{ or } k < \ell - 1),$$

$$[H_2, T^1_4] = [H_2, T^4_1] = 0,$$

$$[T^\ell_j, T^{\ell+1}_j]_{q^{-1}} = (1 - q^{-2})T^{\ell+1}_j H_\ell - q^{-1}T^{\ell+1}_j, \quad (\ell + 1 < j)$$

$$[T^{\ell+1}_j, T_j]_{q^{-1}} = (q^2 - 1)T^{\ell+1}_j H_\ell - qT^{\ell}_j, \quad (\ell + 1 < j)$$

$$[T^i_k, T^j_k]_{q^{-1}} = T^i_j, \quad (i < j, j = k - 1) \text{ or } (j < i, i = k - 1),$$

$$[T^1_4, T^1_3]_q = [T^1_\ell, T^1_\ell]_q = 0, \quad (\ell = 2, 3),$$

$$[T^4_4, T^3_3] = [T^3_3, T^4_2] = 0.$$
\[ [T^3_3, T^1_4] = [T^4_4, T^3_1] = 0, \]
\[ [T^4_4, T^1_4]_{q^{-1}} = T^i_{j} - \lambda T_j^3 T_i^3, \quad ((i,j) = (1, 2) \text{ or } (2, 1)), \]
\[ [T^1_4, T^j_4]_{q^{-1}} = -q^{-i+2} T^j_{i} + \lambda T_j^3 T_i^2 + \lambda q^{-i+2} T^j_{i}(H_1 + H_2) \]
\[-\lambda^2 T_j^2 T_i^2 H_1 - \lambda^2 q^{-i+2} T^j_{i} H_1 H_2, \quad ((i, j) = (3, 4) \text{ or } (4, 3)), \]
\[ [T^\ell_4, T^{\ell+2}_4]_{q^{-2}} = -q^{-1} H_{\ell} - qH_{\ell+1} - q^{-1} \lambda T^{\ell+1}_4 T^{\ell+1}_{\ell+1} + q^{-1} \lambda T^{\ell+2}_4 T^{\ell+1}_{\ell+2} \]
\[ + q^{-1} \lambda H_{\ell} H_{\ell+1} - q^2 \lambda T^{\ell+1}_{\ell+2} T^{\ell+2}_{\ell+1} H_{\ell}, \]
\[ [T^4_4, T^1_4]_{q^{-2}} = -qH_1 - q^{-1}(2 - q^2) H_2 - q(2 - q^2) H_3 - q^{-1} \lambda T^3 T_4^2 + q\lambda T^3 T_1^2 \]
\[-\lambda^2 T^3 T_3^2 + q^{-1} (2 - q^2) \lambda T^3 T_4^3 + (1 + q^{-2}) \lambda T^4 T_4^2 H_1 - \lambda q H_1 H_2 \]
\[-\lambda q^{-1} (2 - q^2) \lambda T^2 T_3^2 + q + q^{-1}) \lambda^2 T^3 T_2^3 H_1 + \lambda^2 T^3 T_2^2 \]
\[-q^{-1}(2 - q^2) \lambda^2 T^3 T_4^4 H_2 + q\lambda^2 T^4 T_4^3 H_1 - q^2 \lambda T^3 T_4^4 H_1 H_2, \]

where \( \lambda = q - 1/q \). At last, actions of \( T^k_{\ell} (\ell < k) \) on the 6-vector are obtained as follows

\[ T^{k}_{\ell} a^{\ell k} = a^{\ell k} T^{k}_{\ell}, \]
\[ T^{k}_{\ell} a^{ij} = qa^{ij} T^{k}_{\ell} + a^{kj}, \quad (k < j), \]
\[ T^{k}_{\ell} a^{ik} = q^{-1} a^{ik} T^{k}_{\ell}, \quad (i \neq \ell), \]
\[ T^{k}_{\ell} a^{i\ell} = qa^{i\ell} T^{k}_{\ell} + a^{ik}, \]
\[ T^{k}_{\ell} a^{kj} = q^{-1} a^{kj} T^{k}_{\ell}, \quad (k < j < k), \]
\[ T^{k}_{\ell} a^{ij} = qa^{ij} T^{k}_{\ell} + (1 - q^2) a^{\ell k} T^{k}_{\ell} - qa^{jk}, \quad (\ell < j < k), \]
\[ T^4_{2} a^{13} = a^{13} T^4_{2} - \lambda a^{14} T^2_{3}, \]
\[ T^3_{1} a^{24} = a^{24} T^3_{1} - \lambda a^{34} T^2_{1}, \]
\[ T^4_{1} a^{23} = a^{23} T^4_{1} - \lambda a^{24} T^3_{1} + q\lambda a^{34} T^2_{1}, \]
\[ T^{k}_{\ell} a^{ij} = a^{ij} T^{k}_{\ell}, \quad (k < i \text{ or } j < \ell). \]
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