NON-COMMUTATIVE POLYNOMIALS WITH CONVEX LEVEL SLICES

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Abstract. The structure of symmetric polynomials $p(a, x)$ in freely noncommuting symmetric variables $a = (a_1, \ldots, a_g)$ and $x = (x_1, \ldots, x_g)$ such that the set of matrix tuples

$$\mathcal{P}_p^A := \{X : p(A, X) \text{ is positive definite}\}$$

is convex for large sets of $A$ is studied. Under fairly general hypotheses it is shown that such polynomials have degree at most two in $x$ and must have special structure. A matrix-valued version of the main result of [BM14] on quasiconvex noncommutative polynomials is obtained as a special case of the main result.

1. Introduction

The main result of this article, Theorem 2.5, is a characterization of symmetric polynomials $p(a, x) = p(a_1, \ldots, a_g, x_1, \ldots, x_g)$ in freely noncommuting variables such that the evaluations

$$p(A, X) = p(A_1, \ldots, A_g, X_1, \ldots, X_g)$$

on symmetric matrices (of the same size) belong to the set

$$\mathcal{P}_p^A = \{X = (X_1, \ldots, X_g) : p(A, X) \succ 0\}$$

is nonempty and convex for a large set of tuples of matrices $A = (A_1, \ldots, A_g)$. Under fairly general hypotheses (most of which rule out obvious pathologies) we show that such polynomials have degree at most two in $x$ and have special structure.

Here, and below, the notation $M \succ 0$ (resp., $M \succeq 0$) for $M$, a (square) symmetric matrix with real entries means that $u^T Mu > 0$ (resp., $u^T Mu \geq 0$) for all nonzero real vectors $u$. A symmetric $\kappa \times \kappa$ matrix-valued free polynomial $q$ is quasiconvex if for each $n$ and symmetric $n \kappa \times n \kappa$ matrix $M$ the set of $g$-tuples $X$ of $n \times n$ symmetric matrices such that $M - q(X) \succ 0$ is convex.

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Theorem 2.5 is illustrated by two corollaries having much simpler statements.

(1) Theorem 1.2 characterizes quasiconvex matrix free polynomials $q$ satisfying some mild additional hypotheses.

(2) Theorem 1.3 concerns a scalar polynomial $p$ of the form $p(a,x) = ar(x) + r(x)a - 2f(x)$ for a (scalar) free polynomial $r$ and a (scalar) symmetric free polynomial $f$ with $r(0) \neq 0$ and $f(0) = 0$. If for a "large" set of matrices $A$ the set $P_A$ is convex, then $p$ has a simple form that is of degree 2 in $x$. Indeed, in this setting the case $r = 1$ reduces to the case where $f$ is a scalar quasiconvex polynomial.

In the remainder of this introduction the background for and statements of Theorem 1.2 and 1.3 are introduced. It closes with a brief discussion of the motivations coming from systems engineering and operator theory (Section 1.6) and a guide to the remainder of the article (Section 1.7).

The statement of Theorems 2.5 and the prerequisite definitions are the subject of Section 2.

1.1. Free Polynomials. Fix positive integers $g$ and $\tilde{g}$ and let $g = \tilde{g} + g$. Let $\mathcal{P}$ denote the real free algebra of polynomials in the freely noncommuting symmetric variables $x = (x_1, \dots, x_g)$ and $a = (a_1, \dots, a_{\tilde{g}})$. The elements of $\mathcal{P}$ are called free polynomials, nc polynomials or often just polynomials in $(a_1, \dots, a_{\tilde{g}}, x_1, \dots, x_g)$. Thus, a free polynomial $p$ is a finite linear combination,

$p = \sum p_w w,$

of words $w$ in $(a_1, \dots, a_{\tilde{g}}, x_1, \dots, x_g)$ with coefficients $p_w \in \mathbb{R}$.

There is a natural involution $T$ on $\mathcal{P}$ given by

$p^T = \sum p_w w^T,$

where, for a word $w = z_{j_1}z_{j_2} \cdots z_{j_n}$ we have $w^T = z_{j_n} \cdots z_{j_2}z_{j_1}$, (since $x_j = x_j^T$ and $a_i = a_i^T$).

A polynomial $p$ is symmetric if it is invariant with respect to the involution, i.e., if $p = p^T$. In particular, $x_j^T = x_j$ and for this reason the variables $x_j$ are sometimes referred to as symmetric free variables. The condition $a_j = a_j^T$ was imposed above to ease the exposition. It is less essential and will be violated on occasion.

1.2. Evaluations. Given a positive integer $n$, let $\mathbb{S}_n$ denote the set of real symmetric $n \times n$ matrices and let $\mathbb{S}_n(\mathbb{R}^g)$ denote the set of $g$-tuples $X = (X_1, \dots, X_g)$ of real symmetric $n \times n$ matrices. The set of all pairs $(A, X)$ with $A \in \mathbb{S}_n(\mathbb{R}^g)$ and $X \in \mathbb{S}_n(\mathbb{R}^g)$, will be denoted $\mathbb{S}_n(\mathbb{R}^g)$.
Let $M_n$ denote the $n \times n$ matrices with real entries. A pair $(A, X) \in S_n(R^g)$ determines a mapping $e_{(A,X)} : \mathcal{P} \to M_n$ by evaluation (actually this mapping is a representation of the algebra $\mathcal{P}$). Indeed, by linearity, $e_{(A,X)}$ is determined by its action on words where $e_{(A,X)}(\emptyset) = I_n$ and, for a nonempty word $w$ in $(a, x)$ and $(A, X) \in S_n(R^g)$, the evaluation $e_{(A,X)}(w)$ is the same word in $(A, X)$. It is natural to write $p_{(A,X)}$ instead of the more formal $e_{(A,X)}(p)$.

Note that $p_{(A,X)}$ respects the involution in the sense that $p_{(A,X)}^T = p_{(A,X)}^T$. In particular, if $p$ is symmetric, then so is $p_{(A,X)}$. It is well known that, taken together, evaluations determine faithful representations of free polynomials.

1.3. Matrix-Valued Polynomials. A free $\kappa \times \kappa$ matrix-valued polynomial $p(a, x)$ is a linear combination of words in the freely noncommuting variables $a = (a_1, \ldots, a_g)$ and $x = (x_1, \ldots, x_g)$ with coefficients from $M_\kappa$. Let $\mathcal{P}^{\kappa \times \kappa'}$ denote the $\kappa \times \kappa'$ matrices with entries from $\mathcal{P}$.

For $p \in \mathcal{P}^{\kappa \times \kappa'}$, evaluation at $(A, X) \in S_n(R^g)$ is defined entrywise, leading to a $\kappa \times \kappa'$ block matrix $p_{(A,X)}$, with entries from $M_n$. Up to unitary equivalence, evaluation at $X$ is conveniently described in terms of the tensor product of matrices by writing $p$ as a finite linear combination

\begin{equation} \label{eq:1.1}
 p = \sum_w C_w w,
 \end{equation}

with coefficients $C_w \in \mathbb{R}^{\kappa \times \kappa'}$ and observing that

\[ p_{(A,X)} = \sum C_w \otimes w(A, X), \]

where $w(A, X) = e_{A,X}(w)$.

The involution $^T$ naturally extends to $\mathcal{P}^{\kappa \times \kappa}$ by

\[ p^T = \sum_w C_w^T w^T, \]

for $p$ given by equation (1.1). The matrix polynomial $p$ is symmetric if $p^T = p$. It turns out that $p$ is symmetric if and only if $p(X)^T = p^T(X)$ for every $n$ and every tuple $X \in S_n(R^g)$.

A simple method of constructing new matrix-valued polynomials from old ones is by direct sums. For instance, if $p_j \in \mathcal{P}^{\kappa_j \times \kappa_j}$ for $j = 1, 2$, then

\[ p_1 \oplus p_2 = \begin{bmatrix}
 p_1 & 0 \\
 0 & p_2
 \end{bmatrix} \in \mathcal{P}^{(\kappa_1 + \kappa_2) \times (\kappa_1 + \kappa_2)}. \]

\[ \text{In keeping with our usual usage, we write } \sum C_w w \text{ and not } \sum C_w \otimes w. \]
1.4. **Highest degree terms.** A matrix-valued nc polynomial \( p(a, x) = \sum_{w \in W} C_w w(a, x) \) of degree \( d \) in \( x \) can be expressed as a sum

\[
p = \sum_{j=0}^{d} p^j(a, x)
\]

of nc matrix-valued polynomials \( p^j(a, x) \) that are homogeneous of degree \( j \) in \( x \). Moreover, (see subsection 8.2), every homogeneous nc polynomial \( p^j(a, x) \) of degree \( j \) in \( x \) can be expressed in the form

\[
p^j(a, x) = \varphi_j^p(a) f_j(a, x),
\]

where

\[
\varphi_j^p(a) = [\varphi_{j1}(a) \cdots \varphi_{js_j}(a)]
\]

is a block row matrix that depends only upon the coefficients \( C_w \) of the words \( w \) in the polynomial \( p^j \) and the variables \( a_1, \ldots, a_g \) and \( f_j(a, x) \) is an \( s_j \times 1 \) vector polynomial in \( a \) and \( x \) that is homogeneous of degree \( j \) in \( x \) of the form,

\[
f_j(a, x) = \text{col}(x_1 f_{1,1,j}, x_1 f_{1,2,j}, \ldots, x_1 f_{1,k_1,j}, x_2 f_{2,1,j}, \ldots, x_g f_{g,k_g,j})
\]

and each \( f_{i,k,j} = f_{i,k,j}(a, x) \). We shall say that the **highest degree terms of** \( p \) **majorize at** \( A \) if

\[
\text{range } \varphi_p^j(A) \subseteq \text{range } \varphi_p^d(A) \quad \text{for } j = 1, \ldots, d - 1.
\]

The condition (1.3) is essential in Theorem 7.3, which is a key ingredient in the proof of our main result.

**Remark 1.1.** The condition (1.3) is automatically met are the following special cases.

1. If \( p \) (matrix-valued) is homogeneous in \( x \), then, as is immediate from the definitions, the highest degree terms of \( p \) majorize at each \( A \).
2. If \( \kappa = 1 \) (i.e., the polynomial \( p \) is scalar), \( p \) is of degree \( d \) in \( x \) and there is at least one term in \( p \) of degree \( d \) in \( x \) that begins with an \( x_j \) on the left, then at least one of the \( \varphi_{di}(A) \) is equal to \( cI_n \) with \( c \neq 0 \) when \( A \in \mathbb{S}_n \).

\[
\square
\]

1.5. **Free quasiconvex polynomials have degree two.** One consequence of our main result (Theorem 2.5) is the following matrix version of the result of [BM14].

**Theorem 1.2.** Suppose that \( f = f(x) = f(x_1, \ldots, x_g) \) is a \( \kappa \times \kappa \) nc matrix polynomial such that:

1. The highest degree terms of \( f \) majorize.
\[
(2) \quad f(0) = 0.
\]

(3) There exists a positive number \( \gamma \) such that for each positive integer \( n \) and each \( M \in S_{\kappa n}(\mathbb{R}) \) that meets the constraints \( M \prec \gamma I_{\kappa n} \) the set \( \{ X \in S_n(\mathbb{R}^q) : f(X) \prec M \} \) is a proper convex subset of \( S_n(\mathbb{R}^q) \).

Then there exists a symmetric polynomial \( \ell \in \mathcal{P}^{\kappa \times \kappa} \) that is affine linear in \( x \), and a finite number \( \pi \) of matrices of free polynomials \( s_j \in \mathcal{P}_{1 \times \kappa} \) each linear (and hence homogeneous of degree one) in \( x \) such that

\[
f(x) = \ell(x) + \sum_{j=1}^{\pi} s_j(x)^T s_j(x).
\]

When \( \kappa = 1 \), Theorem 1.2 is a variant of the main theorem of [BM14]. It turns out that the justification of Theorem 1.2 for the case \( \kappa > 1 \) is surprisingly involved.

Theorem 2.5 also applies to the class of free symmetric scalar polynomials of the form \( p(a, x) = r(x)a + ar(x) - f(x) \) where \( r \) and \( f \) are polynomials in one free variable \( (g = \tilde{g} = 1) \) subject to some additional constraints. Given \( \epsilon > 0 \), let \( B_{\epsilon} \) denote the sequence \( (B_{\epsilon}(n))_{n=1}^{\infty} \), where \( B_{\epsilon}(n) = \{ X \in S_n : \| X \| < \epsilon \} \), the open \( \epsilon \) ball (centered at 0) in \( S_n \).

**Theorem 1.3.** If \( p(a, x) = r(x)a + ar(x) - 2f(x) \), where \( r \) and \( f \) are scalar polynomials in a single variable \( x \) of degree \( k \) and \( d \) respectively such that

(i) \( r(0) = 1 \) and \( f(0) = 0 \).

(ii) \( fr^{-1} \) is not constant.

(iii) There exists an \( \epsilon > 0 \) such that for each \( n \) and each \( A \in S_n(\mathbb{R}^q) \) the set

\[
\mathcal{P}^A_p \cap B_{\epsilon} = \{ X \in S : p(A, X) \succ 0 \text{ and } \| X \| < \epsilon \}
\]

is convex.

Then there exists a symmetric polynomial \( \ell \in \mathcal{P} \) affine linear in \( x \) and a finite number \( \pi \) of matrices of free polynomials \( s_j \in \mathcal{P} \) each linear (and hence homogeneous) in \( x \) and independent of \( a \) such that

\[
p(a, x) = \ell(a, x) - \sum_{j=1}^{\pi} s_j(x)^T s_j(x).
\]

In particular, \( d \) is at most two.

1.6. **Related results and remarks.** Theorem 2.5 falls within the rapidly developing fields of free analysis and free semialgebraic geometry.
Free analytic functions, arise naturally in a number of contexts, including free probability [V04, V10] and multivariate systems theory. The book [KVV] provides an unparalleled introduction to the theory of free analytic functions. Important recent results can be found in [AM15a] for instance while [AM15b] develops a beautiful analog of the Oka extension theorem and related results of a free several complex variables flavor. The papers [AKV13, AKVppt, AMpptb] establish free implicit and inverse function theorems. It turns out there are serious topological subtleties. The free implicit function theorem in [Pappt] has implications for the free version of the Jacobian conjecture. The theory of Pick and matrix monotone function is extended to the free setting in [PT-Dppt] and the article [CPT-Dppt] exposes fascinating connections to representation theory. A good introduction to many of these topics can be found the survey article [AMppta]. Two related lines of development of the theory of free functions influenced by single and multivariable operator theory are the series of papers by Muhly and Solel and that of Popescu for which [MS08, MS13] and [Po10, Po11, Po13, Po15] are but just a few of the references. For a survey article see [MS11].

Free rational functions and skew fields have a long history going back to at least [S61]. A more recent development is their appearance in multivariate systems theory [BGM05, BGtH]. In this context, for a given rational function, the existence of linear fractional representations whose formal singularities and actual singularities agree is an issue [KVV09, HM14].

Free semialgebraic geometry is, by analogy to the commutative case, the study of free polynomial inequalities. An algebraic formula (certificate) equivalent to the validity of a polynomial inequality is known as a positivstellensatz. A small sample of free Positivstellensatze include [C11, KS07, KS08]. See also the references therein.

Convex inequalities (inequalities with convex solution sets) represent an important subclass of these areas and are the subject of convex real algebraic geometry, an emerging subfield of real algebraic geometry [BPT]. The results in this paper lie within the free analog of convex algebraic geometry. Its motivation comes from systems theory and control engineering as well as from the theory of operator spaces and systems and matrix convexity. See, for examples, [EW97, FP12, KPTT13].

To give a taste of the engineering connection consider the free polynomial

\[ p(a, x) = -x b b^T x + a^T x + x a + c, \quad c = c^T, \quad x = x^T \]
and the corresponding Riccati inequality

\[ 0 \preceq -XBB^TX + A^TX + XA + C, \quad C = C^T, \quad X = X^T, \]

which is ubiquitous in systems engineering. Here the matrices \( A, B, C \)
can have any compatible dimension.

Note the Riccati inequality is \textit{dimension free} in the sense that its
form does not change with the dimension of the matrices. This
independence of dimension property is typical of systems problems
described purely by a signal flow diagram and whose signals are in \( L^2 \)
and whose performance is of mean square type, cf. \textcite{OHMP95, HKM96}.
Since convexity is a highly desired property in system engineering it
is unfortunate that for such dimension free systems problems, convex-
ity is obtainable only with the, a priori more restrictive, \textit{linear matrix
inequalities} as the results here strongly suggest.

This article is a natural successor to \textcite{DHM07a, DHM07b, DGHM09, DHM11, BM14}. Here the question is what can be said of the poly-
nomial if there are sufficiently many convex level slices. The article
\textcite{HM12} characterizes the convex level sets of polynomials \( p(x) \) in the
free variables \( x = (x_1, \ldots, x_g) \). The article \textcite{HHLM08} characterizes
polynomials \( p(a, x) \) satisfying various convexity conditions in \( a \) and \( x \)
separately.

1.7. \textbf{Readers guide.} The remainder of this article is organized as fol-
lows. The central result, Theorem 2.5, appears near the end of Section
2 after the needed preliminaries are developed. Theorems 1.2 and 1.3
are shown to be a consequence of Theorem 2.5 in Sections 3 and 4 re-
spectively. The paper then turns, in the next nine sections, to proving
Theorem 2.5. This analysis hinges on the principle that the Hessian
restricted to the tangent space, a type of free second fundamental form,
is negative definite. The needed machinery is developed in Sections 5
and 6. Section 8.1 reviews a number of Kronecker product identities
for the convenience of the reader. The Hessian of a symmetric free
polynomial has a \textit{middle matrix-border vector} representation described
in Sections 8.4, 8.6 and 9. Positivity of the middle matrix of \( p \) forces
\( p \) to have degree two, a fact proved in Section 10. Section 11 provides
sufficient conditions for positivity of the middle matrix. The proof of
Theorem 2.5 culminates in Section 12. In brief, sufficient negativity
of the free second fundamental form forces positivity of the middle
matrix. Appendix 13 contains the proof of Remark 2.7. Several ad-
ditional appendices provide examples illustrating the various objects
and conditions appearing through the paper. The authors thank S.
2. The general theorem

The main result of this article, Theorem 2.5, is formulated in this section. Its statement requires a number of definitions that we now introduce and illustrate with examples.

2.1. Directional derivatives and the Hessian. Given $p \in \mathcal{P}^{\kappa \times \kappa}$ we shall compute directional derivatives of $p$ in the $x$ variable assuming the $a$ variable is fixed. While these are partial directional derivatives with respect to $x$, we shall abuse terminology and leave out the word partial. If $h = (h_1, \ldots, h_g)$ is another set of freely noncommuting variables and $t \in \mathbb{R}$,

\begin{equation}
 p(a, x + th) = \sum_{j=0}^{d} p_j(a, x)[0, h]t^j,
\end{equation}

where $p_j(a, x)[0, h]$ are polynomials in the freely noncommuting variables

$$(a, x, h) = (a_1, \ldots, a_g, x_1, \ldots, x_g, h_1, \ldots, h_g).$$

The notation indicates the different role that these variables play. Observe that $p_j(a, x)[0, h]$ is homogeneous of degree $j$ in $h$.

The polynomial $p_1(a, x)[0, h]$ is the directional derivative or simply the derivative of $p$ (in the direction $h$) and is denoted $p_x(a, x)[0, h]$; the polynomial

$$p_{xx}(a, x)[0, h] = 2p_2(a, x)[0, h]$$

is the Hessian of $p$. When there is no ambiguity, we shall write $p_x(a, x)[h]$ and $p_{xx}(a, x)[h]$ to simplify the typography.

The (partial) directional derivative $p_a(a, x)[e, 0]$ and the full directional derivative $p'(a, x)[e, h]$ are defined analogously in the spirit of (2.1) using $p(a + te, x)$ and $p(a + te, x + th)$ respectively.

2.2. Full rank conditions. Let $p$ be a symmetric $\kappa \times \kappa$-valued free polynomial. A pair $(A, X) \in \mathbb{S}_n(\mathbb{R}^g)$ is called a full rank point for $p$ if, for each positive integer $n$, the map

$$(E, H) \in \mathbb{S}_n(\mathbb{R}^g) \mapsto p'(A, X)[E, H] \in \mathbb{S}_{nk}$$

is onto $\mathbb{S}_{nk}$. Note that the full rank condition places a constraint on the relative sizes of $g$ and $\kappa$. In particular, the inequality $g(n^2 + n) \geq (nk)^2 + n\kappa$ is a necessary condition for the existence of full rank points.
2.3. **Chip sets.** For $1 \leq j \leq g$, the **right chip set** $\mathcal{RC}_w^j$ of a word $w$ in the variables $(a, x)$ is the set of words $v$ such that there exists a word $u$ (empty or not) such that

$$w = ux_jv.$$  

The left chip set is defined analogously. Thus, for example, if $w = ax_2x_1x_2a$, then

$$\mathcal{RC}_w^1 = \{x_2a\} \quad \text{and} \quad \mathcal{RC}_w^2 = \{a, x_1x_2a\},$$

whereas,

$$\mathcal{LC}_w^1 = \{ax_2\} \quad \text{and} \quad \mathcal{LC}_w^2 = \{a, ax_2x_1\}.$$  

Notice that if $w = w^T$ and the variables are symmetric, as in the present case, then the words in $\mathcal{LC}_w^j$ are the transposes of the words in $\mathcal{RC}_w^j$.

For a given $j$, the right chip set $\mathcal{RC}_p^j$ of a polynomial $p$ is the union of the right chip sets $\mathcal{RC}_w^j$ of the words $w$ appearing in $p$ (with nonzero coefficients). In particular, for a given polynomial $p$, the partial of $p$ with respect to $x$ has the form

$$(2.2) \quad p_x = \sum_j \sum_u \sum_v C_{u,v,j} uh_jv$$

where $C_{u,v,j}$ is a matrix (of appropriate size) and $u$ and $v$ are from the left and right chip sets of $p$ respectively. Similarly the Hessian of $p$ takes the form,

$$(2.3) \quad p_{xx} = \sum C_{u,v,j,\ell} uh_jwh_{\ell}v,$$

where $u$ and $v$ are from the left and right chip sets of $p$ respectively. Let

$C_p^j$ denote the chip space of polynomials in the words of $\mathcal{RC}_p^j$

$\mathcal{RC}_p$ denote the union of the sets $\mathcal{RC}_p^j$ and

$C_p$ denote the **chip space** of polynomials in the words of $\mathcal{RC}_p$.

**Example 2.1.** Let $p(a, x) = x_2^2ax_1 + x_1ax_2^2 + a^2$. Then

$$\mathcal{RC}_p^1 = \{1, ax_2^2\}, \quad \mathcal{RC}_p^2 = \{1, x_2, ax_1, x_2ax_1\}$$

and

$$\mathcal{RC}_p = \{1, x_2, ax_1, ax_2^2, x_2ax_1\}.$$  

Correspondingly,

$$p_x(a, x)[0, h] = h_2x_2ax_1 + x_2h_2ax_1 + x_2^2ah_1 + h_1ax_2^2 + x_1ah_2x_2 + x_1ax_2h_2.$$
is of the form (2.2), whereas
\[ p_{xx}(a, x)[h] = 2\{h_2ax_1 + h_2x_2ah_1 + x_2h_2ah_1 + h_1ah_2x_2 + h_1ax_2h_2 + x_1ah_2^2\} \]
\[ = 2 \begin{bmatrix} h_1 & h_2 & x_2h_2 & x_1ah_2 \end{bmatrix} \begin{bmatrix} 0 & ax_2 & a & 0 \\ x_2a & 0 & 0 & 1 \\ a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_2x_2 \\ h_2ax_1 \end{bmatrix} \]
is of the form (2.3).

The polynomial \( p(a, x) - p(a, 0) \) can also be expressed as a linear combination of terms in the right chip set \( \mathcal{C}_p \) with polynomial coefficients. In fact, \( p(a, x) - p(a, 0) \) can be recovered from the formula for \( p_x(a, x)[0, h] \) by deleting those terms in which an \( h_j \) is preceded by an \( x_i \) and then replacing \( h_j \) by \( x_j \) in the remaining terms. Thus, in the case at hand,
\[ p(a, x) - p(a, 0) = p(a, x) - a^2 = x_1(ax_2^2) + x_2(ax_2x_1). \]

\[ \blacksquare \]

**Remark 2.2.** If \( p(a, x) = a - p(x) \), then the \( a \) term will not appear in \( p_x(a, x)[0, h] \) or in \( p_{xx}(a, x)[0, h] \). Hence \( \mathcal{C}_p \) will only consist of polynomials in the \( x \) variables.

\[ \blacksquare \]

2.4. **Free sets, positivity sets and set domination.** Let \( p \) be a free \( \kappa \times \kappa \)-valued symmetric polynomial. For positive integers \( n \), let
\[ \Psi_p(n) = \{(A, X) \in S_n(\mathbb{R}^\varnothing) : p(A, X) > 0\}. \]
The **positivity set** of \( p \) is the sequence \( \Psi_p = (\Psi_p(n))_{n=1}^\infty \). The set \( \Psi_p \) naturally earns the moniker of **free open (basic) semialgebraic set**.

Given a positive integer \( n \) and \( \tilde{g} \)-tuple \( A \in S_n(\mathbb{R}^\varnothing) \), let
\[ (2.4) \quad \Psi_p^A = \{X \in S_n(\mathbb{R}^\varnothing) : p(A, X) > 0\} \]
We call \( \Psi_p^A \) the **\( A \)-cross section of \( \Psi_p \)**. Letting \( n \) denote the size of \( A \), it is a subset of \( S_n(\mathbb{R}^\varnothing) \).

Let
\[ \partial \Psi_p = \{(A, X) \in S(\mathbb{R}^\varnothing) : p(A, X) \succeq 0 \quad \text{and} \quad \ker p(A, X) \neq \{0\}\} \]
and
\[ \partial \Psi_p^A = \{X \in S(\mathbb{R}^\varnothing) : p(A, X) \succeq 0 \quad \text{and} \quad \ker p(A, X) \neq \{0\}\}. \]
We call these sets the **algebraic boundaries** of \( \Psi_p \) and \( \Psi_p^A \).
The sequence \( \partial \hat{\mathcal{P}}_p = (\partial \hat{\mathcal{P}}_p(n))_{n=1}^\infty \), where
\[
\partial \hat{\mathcal{P}}_p(n) = \{(A, X, v) \in S_n(\mathbb{R}^g) \times \mathbb{R}^{n\kappa} : p(A, X)v = 0, v \neq 0 \text{ and } p(A, X) \succeq 0 \}
\]
is the detailed algebraic boundary of the positivity set \( \mathcal{P}_p \).

2.4.1. Free sets. Positivity sets are special cases of free sets. Typically, constructions in this article are parametrized over all \( n \). Thus, sequences \( S = (S(n))_{n=1}^\infty \) where, for each \( n \), the set \( S(n) \) is a subset of \( S_n(\mathbb{R}^\ell) \) for an appropriate choice of \( \ell \) figure prominently and are called graded sets. Such a sequence is a free set if it is closed under direct sums and (simultaneous) real unitary conjugation. This last condition means, if \( U \) is an \( n \times n \) real unitary matrix and \( (A, X, v) \in S(n) \) (resp. \( (A, X, v) \in S(n) \)), then \( (U^T A U, U^T X U, Uv) \in S(n) \) too, where
\[
U^T(A_1, \ldots, A_g)U := (U^T A_1 U, \ldots, U^T A_g U) \quad \text{and} \\
U^T(X_1, \ldots, X_g)U := (U^T X_1 U, \ldots, U^T X_g U).
\]

A graded set \( S \) is a open if each \( S(n) \) is open. A graded set \( S \) is nonempty if \( S(n) \neq \emptyset \) for at least one positive integer \( n \).

The notation for projections of the graded sets \( S \subset (S_n(\mathbb{R}^g) \times \mathbb{R}^{n\kappa})_{n=1}^\infty \) (resp. \( S \subset S(\mathbb{R}^g) \))
\[
\pi_1(S) = \{A : (A, X, v) \in S\} \quad \text{(resp. } (A, X) \in S)\]
will be useful.

2.4.2. Dominating sets. A graded set \( \Omega \subset (S_n(\mathbb{R}^g) \times \mathbb{R}^{n\kappa})_{n=1}^\infty \) is said to \( \mathcal{C}_p \)-dominate a free set \( S \subset (S_n(\mathbb{R}^g) \times \mathbb{R}^{n\kappa})_{n=1}^\infty \) if \( q \in \mathcal{C}_p^{1\times \kappa} \) and \( q(A, X)v = 0 \) for all \( (A, X, v) \in \Omega \) implies \( q(A, X)v = 0 \) for all \( (A, X, v) \in S \). Note \( \Omega \) is not required to be a subset of \( S \).

Example 2.3. (Domination and Majorization) Let
\[
p(a, x) = (1 + x^k)a + a(1 + x^k) + x^d
\]
Thus \( p \) is a polynomial of the type appearing in Theorem[1.3] Observe that
\[
\mathcal{R}C_p = \{1, x, \ldots, x^{\ell-1}, a, xa, \ldots, x^{k-1}a\} \quad \text{with } \ell = \max\{k, d\}.
\]
and a polynomial \( q(x, a) \) in the linear span of the terms in the chip set \( C_p \) of \( p \) has the form

\[
q(a, x) = \sum_{j=0}^{\ell-1} \alpha_j x^j + \sum_{j=0}^{k-1} \beta_j x^j a \quad \text{with} \quad \ell = \max\{k, d\}.
\]

Because \( \kappa = 1 \) (\( p \) is scalar polynomial) and either \( x^k a \) or \( x^d \) is a highest degree word, the highest degree terms majorize at each \( A \) by Remark 1.1.

Let \( \Omega_n \) denote the set of \( (A, X, v) \in S_n(\mathbb{R}^g) \times \mathbb{R}^n \) such that

1. \( A \) and \( X \) are real \( n \times n \) diagonal matrices and \( v \in \mathbb{R}^n \).
2. \( I_n + X^k \) is invertible.
3. \( A = -\frac{1}{2}(I_n + X^k)^{-1}X^d \).

We will show that \( \Omega = (\Omega_n)_{n=1}^{\infty} \) is \( C_p \) dominating for the free set \( S = (S_n)_{n=1}^{\infty} \) defined by

\[
S_n := \{(A, X, v) \in S_n(\mathbb{R}^g) \times \mathbb{R}^n : p(A, X)v = 0\}.
\]

If \((A, X, v) \in \Omega_n\), then \( AX =XA \) and hence \( p(A, X) = 0 \) if and only if

\[
A = -\frac{1}{2}(I_n + X^k)^{-1}X^d.
\]

For this choice of \( A \),

\[
q(A, X) = \sum_{j=0}^{\ell-1} \alpha_j X^j + \sum_{j=0}^{k-1} \beta_j X^j A
\]

\[
= \sum_{j=0}^{\ell-1} \alpha_j X^j - \frac{1}{2} \sum_{j=0}^{k-1} \beta_j X^j (I_n + X^k)^{-1}X^d
\]

\[
= \frac{1}{2}(I_n + X^k)^{-1} \left\{ 2(I_n + X^k) \sum_{j=0}^{\ell-1} \alpha_j X^j - \sum_{j=0}^{k-1} \beta_j X^j X^d \right\}.
\]

Now let \( n = \ell + k \) and \( X = \text{diag}\{\mu_1, \ldots, \mu_{\ell+k}\} \) with \( 0 < \mu_1 < \cdots < \mu_{\ell+k} \) and suppose first that \( k \leq d \) so that \( \ell = d \). Then \( q(A, X) = 0 \) if and only if

\[
\begin{bmatrix}
1 & \mu_1 & \cdots & \mu_{\ell+k-1}^d \\
\vdots & \ddots & \ddots & \vdots \\
1 & \mu_{d+k} & \cdots & \mu_{d+k}^d
\end{bmatrix}
\begin{bmatrix}
2\alpha_0 \\
\vdots \\
2\alpha_{\ell-1} \\
0_{\ell \times 1}
\end{bmatrix}
+ \begin{bmatrix}
0_{k \times 1} \\
2\alpha_0 \\
\vdots \\
2\alpha_{d-1}
\end{bmatrix}
= 0.
\]
Since the (Vandermonde) matrix on the left is invertible, it is readily seen that
\( q(A, X) = 0 \) if and only if \( \alpha_j = 0 \) for \( j = 0, \ldots, d - 1 \) and \( \beta_j = 0 \) for \( j = 0, \ldots, k - 1 \); i.e., if and only if \( q \) is the zero polynomial.

The proof for \( k > d \), i.e., for \( \ell = k \), leads to the constraint
\[
\begin{pmatrix}
1 & \mu_1 & \cdots & \mu_1^{2k-1} \\
\vdots & \ddots & & \vdots \\
1 & \mu_{2k} & \cdots & \mu_{2k}^{2k-1}
\end{pmatrix}
\begin{pmatrix}
2\alpha_0 \\
\vdots \\
2\alpha_k-1 \\
2\alpha_0 \\
\vdots \\
2\alpha_k-1
\end{pmatrix}
+ \begin{pmatrix}
0_{d \times 1} \\
-\beta_0 \\
\vdots \\
-\beta_k-1 \\
0_{(k-d) \times 1}
\end{pmatrix} = 0,
\]
which implies that \( q(A, X) = 0 \) if and only if \( \alpha_j = \beta_j = 0 \) for \( j = 0, \ldots, k - 1 \); i.e., if and only if \( q \) is the zero polynomial. \( \square \)

**Example 2.4.** With \( \tilde{g} = 1 \), suppose \( p \) is a symmetric polynomial of the form \( p(A, X) = A - f(X) \) as considered in [BM14]. Note that \( p_n(A, X)[E, 0] = E \) clearly maps \( S_n \) onto \( S_n \) independent of \( X \). Thus, in this setting every pair \( (A, X) \) is a full rank point for \( p \).

In this case, the class \( C_p \) consists of polynomials that do not depend on \( A \). In [BM14] it is also shown that for every \( X \in S_n(\mathbb{R}^q) \) there exists a matrix \( A \succ 0 \) such that \( p(A, X) \succeq 0 \) and \( \det p(A, X) = 0 \). Consequently, a polynomial \( q \in C_p \) that vanishes on \( \partial \widehat{\Psi}_p \) is singular for every \( X \in S_n(\mathbb{R}^q) \) and hence, as is well known (see e.g., [BM14]), \( q = 0 \), i.e., \( \partial \widehat{\Psi}_p \) is \( C_p \) dominating for \( S(\mathbb{R}^q) \). \( \square \)

**2.5. Statement of the main result.** Suppose \( p \in \mathcal{P}^{\kappa \times \kappa} \) is a \( \kappa \times \kappa \) symmetric matrix polynomial in \( \tilde{g} + g \) free variables of degree \( \tilde{d} \) in \( \tilde{a} \) and degree \( d \) in \( x \) and \( \mathcal{O} \subset S(\mathbb{R}^q) \) is an free open semialgebraic set. Let
\[
\widehat{\mathcal{O}}(n) = \{(A, X, v) \in S_n(\mathbb{R}^q) \times \mathbb{R}^{n\kappa} : (A, X) \in \mathcal{O}(n) \text{ and } v \neq 0\},
\]
i.e.,
\[
\widehat{\mathcal{O}} = (\widehat{\mathcal{O}}(n))_{n=1}^\infty = \{(A, X, v) \in \mathcal{O} \text{ and } v \neq 0\},
\]
and, in keeping with the notation \( \Psi_p^A = \{X \in S_n(\mathbb{R}^q) : (A, X) \in \Psi_p\} \) that was introduced in (2.3), let
\[
\mathcal{O}^A = \{X : (A, X) \in \mathcal{O}\}
\]
denote the \( A \)-cross section of \( \mathcal{O} \).

The sets
\[
\partial \Psi_p \cap \mathcal{O} = \{(A, X) \in \mathcal{O} : p(A, X) \succeq 0 \text{ and } \ker p(A, X) \neq \{0\}\}
\]
and
\[
\Psi_p^A \cap \mathcal{O}^A = \{X : (A, X) \in \mathcal{O} \text{ and } p(A, X) > 0\}
\]
play a prominent role in the formulation of the main theorem, which we are now ready to state.

**Theorem 2.5.** Suppose \( p = p(a, x) = p(a_1, \ldots, a_{\tilde{g}}, x_1, \ldots, x_\tilde{g}) \) is a \( \kappa \times \kappa \) symmetric matrix of polynomials in \( \tilde{g} + g \) free variables of degree \( \tilde{d} \) in \( a \) and degree \( d \) in \( x \), and \( \mathcal{O} \subset S(\mathbb{R}^g) \) is a free open semialgebraic set. If

(a) for each \( A \in \pi_1(\partial \mathcal{P}_p \cap \mathcal{O}) \) the set \( \mathcal{P}_p^A \cap \mathcal{O}^A \) is convex;

(b) the set of tuples \( (A, X) \in \partial \mathcal{P}_p \cap \mathcal{O} \) that are full rank for \( p \) are dense in \( \partial \mathcal{P}_p \cap \mathcal{O} \);

(c) there exists an \( N \geq \sum_{j=0}^{\tilde{d}} \tilde{g}^j \) such that \( \pi_1(\partial \mathcal{P}_p(N) \cap \mathcal{O}(N)) \) contains an open set \(^2\);

(d) for each \( A \in \pi_1(\partial \mathcal{P}_p \cap \mathcal{O}) \) the highest degree terms of \( p \) majorize at \( A \); and

(e) \( \hat{\mathcal{O}} \cap \partial \hat{\mathcal{P}}_p \) is a \( C_p \) dominating set for \( \hat{\mathcal{O}} \),

then there exists

(i) a symmetric polynomial \( \ell \in P^{\kappa \times \kappa} \) affine linear in \( x \);

(ii) a positive integer \( \rho \) and a matrix free polynomial \( R(a) \in P^{\rho \times \rho} \) that is positive semidefinite on \( \pi_1(\partial \mathcal{P}_p \cap \mathcal{O}) \);

(iii) a matrix free polynomial \( S(\in P^{\rho \times \kappa} \) linear in \( x \) such that

\[
(2.5) \quad p(a, x) = \ell(a, x) - S(a, x)^TR(a)S(a, x) .
\]

Thus, \( p \) is a weighted sum of squares of terms linear in \( x \) plus an affine linear term in \( x \). Moreover, if \( p \) does not contain any terms of the form \( c \tau(a)^T x_\tilde{g} \omega(a)x_j \sigma(a) \), for words \( \tau, \omega, \sigma \) in \( a \) with \( \omega \) not empty and a nonzero \( c \in \mathbb{R} \), then there is a choice of \( R \) that is independent of \( a \) (so that \( R \in \mathbb{R}^{\rho \times \rho} \)) and positive semidefinite.

**Remark 2.6.** The proof of Theorem 2.5 occupies Sections 5 through 12. It explicitly constructs \( S \) and \( R \). We don’t know the extent to which \( S \) and \( R \) are unique.

As a slightly weaker, but perhaps more palatable, conclusion, \( R(A) \geq 0 \) for each \( n \) and \( A \in S_n(\mathbb{R}^g) \) such that \( 0 \neq \mathcal{P}_p^A \neq S_n(\mathbb{R}^g) \). Indeed, for any such \( A \) there is an \( X \) such that \( (A, X) \in \partial \mathcal{P}_p \cap \mathcal{O} \). □

**Remark 2.7.** Suppose \( r \) is a free polynomial of degree \( \tilde{d} \) in \( \tilde{g} \) variables. It is well known that if \( r \) vanishes on an open set \( U \subset S_N(\mathbb{R}^g) \) for sufficiently large \( N \), then \( r = 0 \). A very conservative choice of \( N \) is \( N \geq k_b(\tilde{g}, \tilde{d}) \) where \( k_b \) is the number of words of length at most \( \tilde{d} \) in \( \tilde{g} \)

\(^2\)Unless otherwise indicated, open means nonempty open.
variables:

\[(2.6) \quad k_b = k_b(\tilde{g}, \tilde{d}) := \sum_{j=0}^{d} \tilde{g}^j.\]

A proof is provided in (arXiv) Appendix 13 since we could not find this statement for symmetric variables. This remark explains the choice of \(N\) in Theorem 2.5 item (c).

\[\square\]

**Remark 2.8.** In stating Theorem 2.5 the variables \(a_i\) and \(x_i\) and matrices \(A_i\) and \(X_i\) are assumed symmetric, a choice made to simplify the presentation. We see no obstruction to Theorem 2.5 holding for mixtures of symmetric and arbitrary variables and matrices. In early papers, e.g. [CHSY03], various classes of variables were treated, greatly encumbering the presentation. It was found that the proofs in other cases are essentially simplifications of the proofs in the symmetric case. Subsequent papers, including this one, typically just present the case of symmetric variables with remarks elucidating the situation for other choices of variables.

In the proof of Theorem 1.2 it is convenient to use a version of Theorem 2.5 in which the \(a\) variables are mixed, i.e., \(a = (a, b)\) where \(a = (a_1, \ldots, a_h)\) are symmetric variables and \(b = (b_1, \ldots, b_\ell)\) are non-symmetric variables. The \(x\) variables and \(X\) matrices, on the other hand, are all symmetric. In this setting Theorem 2.5 is true with little modification in its proof, since generally \(A\) is fixed and the proof uses directional derivatives in \(x\) but not in \(a\). Essentially all complication surrounds myriad properties of these derivatives. Indeed, it is only necessary to make the obvious changes to the statement of Theorem 2.5. For instance, the full rank condition in this case asks that the mapping

\[
\mathbb{S}_n(\mathbb{R}^h) \times M_n(\mathbb{R}^\ell) \times \mathbb{S}_n(\mathbb{R}^g) \ni (E, H) \mapsto p'(A, X)[E, H] \in \mathbb{S}_{n\kappa}
\]

is onto. Here \(M_n(\mathbb{R}^\ell)\) is the set of \(\ell\)-tuples of \(n \times n\) matrices with real entries.

\[\square\]

### 3. The proof of Theorem 1.2

In this section we show how to obtain Theorem 1.2 from a variant of Theorem 2.5 (see Remark 2.8). This also provides an opportunity to become familiar with the conditions of Theorem 2.5 before encountering its rather long proof.
Let $M(a)$ denote the matrix

$$M(a) = \begin{bmatrix} a_{11} & \cdots & a_{1\kappa} \\ \vdots & & \vdots \\ a_{\kappa 1} & \cdots & a_{\kappa \kappa} \end{bmatrix}$$

with $\tilde{g} = (\kappa^2 + \kappa)/2$ free noncommutative entries $a_{ij}$ subject to $a_{ij} = a_{ji}^T$. Thus, there are $\ell := \frac{\kappa(\kappa-1)}{2}$ non-symmetric variables $a_{ij}$ with $i < j$ and $\kappa$ symmetric variables $a_{jj}$. We view $a_{ii}$ as the first $\kappa$ of the $a$-variables and $a_{ij}$ with $i < j$ as the last $\ell$ of the $a$-variables. Thus, for example, if $A = (A_1, \ldots, A_\kappa, B_1, \ldots, B_\ell)$ and $\kappa = 3$, then $\ell = 3$ and

$$M(A) = \begin{bmatrix} A_1 & B_1^T & B_2^T \\ B_1 & A_2 & B_3^T \\ B_2 & B_3 & A_3 \end{bmatrix}.$$ 

Set

$$p(a, x) := M(a) - f(x).$$

Then

$$\mathcal{P}_p = \{(A, X) : M(A) - f(X) \succ 0\}$$

$$\partial \mathcal{P}_p = \{(A, X, v) : M(A) - f(X) \succeq 0, [M(A) - f(X)]v = 0 \text{ and } v \neq 0\}.$$

The set

$$\mathcal{O} := \{(A, X) : M(A) \prec \gamma I\}$$

is a nonempty free open semialgebraic set. Moreover, with this choice of $\mathcal{O}$,

$$\partial \mathcal{P}_p \cap \widehat{\mathcal{O}} = \{(A, X, v) : M(A) \prec \gamma I, M(A) - f(X) \succeq 0, [M(A) - f(X)]v = 0 \text{ and } v \neq 0\}.$$

It suffices to verify the hypotheses Theorem 2.5 are met and this we do, one by one.

(a) The set $\mathcal{P}_p^A = \mathcal{P}_p^A \cap \mathcal{O}^A$ is convex for each $A \in \pi_1(\partial \mathcal{P}_p \cap \mathcal{O})$ by hypothesis.

(b) The set of tuples $(A, X) \in \partial \mathcal{P}_p \cap \mathcal{O}$ that are full rank for $p$ are dense in $\mathcal{P}_p \cap \mathcal{O}$. Indeed, each $(A, X) \in \partial \mathcal{P}_p \cap \mathcal{O}$ is full rank for $p$ since $p_{a,x}(A, X)[E, 0] = M(E)$ and $M$ maps $\mathbb{S}_{n}(\mathbb{R}^\kappa) \times M_n(\mathbb{R}^\ell)$ onto $\mathbb{S}_{\kappa n}$.
(c) For each positive integer \(N\), the set \(\pi_1(\partial \mathfrak{P}_p(N) \cap \mathcal{O}(N))\) contains an open set. Fix \(N\). For a tuple \(A \in S_N(\mathbb{R}^\kappa) \times M_N(\mathbb{R}^t)\), \(M(A) \in S_{N\kappa}\) and all elements of \(S_{N\kappa}\) have this form. Fix \(A\) such that \(\gamma I_{kn} \succ M(A) \succ 0\). There exists an \(X\) such that \(M(A) - f(X) \not\succ 0\) by hypothesis. On the other hand, \(M(A) - f(0) \succ 0\), since \(f(0) = 0\). Hence, there exists a \(0 < t \leq 1\) and a vector \(v \neq 0\) such that \(M(A) - f(tX) \geq 0\) and \((M(A) - f(tX))v = 0\). Hence \(\pi_1(\partial \mathfrak{P}_p(N) \cap \mathcal{O}(N))\) contains the set of \(A\) satisfying \(\gamma I_{kn} \succ M(A) \succ 0\).

(d) The highest degree terms of \(f\) majorize by hypothesis.

(e) \(\partial \mathfrak{P}_p \cap \hat{\mathcal{O}}\) is a \(c_p\) dominating set for \(\hat{\mathcal{O}}\). Since elements of \(C_{f}^{1x\kappa}\) depend only on \(x\) it (more than) suffices to show, if \(q(X)v = 0\) for \((A, X, v) \in \partial \mathfrak{P}_p \cap \hat{\mathcal{O}}\), then \(q = 0\).

Suppose \(q(X)v = 0\) for all \((X, A, v) \in \partial \mathfrak{P}_p \cap \hat{\mathcal{O}}\). Fix \(n \geq k_b(d, q) = \sum_{j=0}^{d} g^j\). Suppose \(X \in S_n(\mathbb{R}^g)\) and \(f(X) \prec \gamma I\). Choose \(A \in S_n(\mathbb{R}^\kappa) \times M_n(\mathbb{R}^t)\) with \(M(A) = f(X)\). It follows that \((A, X, v) \in \partial \mathfrak{P}_p \cap \hat{\mathcal{O}}\) for all \(v \neq 0\). Hence \(q(X)v = 0\) for all such \(v\) and hence \(q(X) = 0\) for all \(X\) such that \(f(X) \prec \gamma I\). Since this is an open set of \(X\) and \(n\) is sufficiently large, \(q = 0\) by Remark 2.7.

All hypotheses of Theorem 2.5 are verified. Moreover, \(p\) contains no terms of the form \(ct(a)x_\kappa \omega(a)x_\sigma(a)\) for words \(\tau, \omega, \sigma\) in \(a\) with \(\omega\) not empty and a nonzero \(c \in \mathbb{R}\). Thus, by Theorem 2.5, \(p = M(a) - f(x)\) has the form (2.5) with \(R(a) = R\) a constant positive semidefinite matrix. Since \(R\) has a unique positive semidefinite square root that can be absorbed into the factors \(s_j(a, x)\), it can thus be assumed that \(R\) is the identity and hence

\[
    f(x) = -\ell(a, x) + \sum s_j(a, x)^T s_j(a, x).
\]

On the other hand, \(f\) does not depend upon \(a\). Hence none of the \(s_j(a, x)\) can depend upon \(a\) as otherwise the highest degree terms in \(a\) can not cancel. Thus \(s_j(a, x) = s_j(x)\). Finally, it now follows that \(\ell(a, x) = \ell(x)\) also does not depend upon \(a\). We note there is a more direct argument using Lemma 10.2.

4. The proof of Theorem 1.3

In this section we show how to obtain Theorem 1.3 from Theorem 2.5. Sample computations for the concrete special case where \(r(x) = 1 + x^k\) and \(2f(x) = -x^d\) appear in Example 2.3.
Define the free open semialgebraic set
\[ O := \{(A,X) \in S(\mathbb{R}^2) : X \in B_\epsilon \} \]
and observe
\[ \mathcal{P}_p^A = \{ X \in S : r(X)A + Ar(X) - 2f(X) \succ 0 \} \]
and, for each positive integer \( n \),
\[ \partial^\ast \mathcal{P}_p(n) = \{(A,X,v) \in S_n(\mathbb{R}^g) \times \mathbb{R}^n : r(X)A + Ar(X) - 2f(X) \succeq 0, \ [r(X)A + Ar(X) - 2f(X)]v = 0 \text{ and } v \neq 0 \} . \]

Next, we shall check that the hypotheses of Theorem 2.5 are met, one by one.

(a) For each \( A \in \pi_1(\partial^0 \mathcal{P}_p \cap O) \) the set \( \mathcal{P}_p^A \cap O^A \) is convex by assumption.

(b) \( \{(A,X) \in \partial^0 \mathcal{P}_p \cap O : p_{a,x}(A,X)[E,H] \text{ maps } (E,H) \in S_n(\mathbb{R}^g) \text{ onto } S_n \} \) is dense in \( \pi_1(\partial^0 \mathcal{P}_p \cap O) \).

Let \( (A,X) \in S(\mathbb{R}^g) \) be given. It is easily checked that
\[ p_{a,x}(A,X)[E,0] = r(X)E + Er(X) \]
and hence \( (A,X) \) is a full rank point for \( p \) if the map
\[ E \to r(X)E + Er(X) \]
is invertible, i.e., if \( \sigma(r(X)) \cap \sigma(-r(X)) = \emptyset \), where \( \sigma(\pm r(X)) \) denotes the set of eigenvalues of \( \pm r(X) \). But this can be achieved by arbitrarily small perturbations of \( X \), since \( r(x) \) is a nonzero polynomial. The desired conclusion follows.

(c) To prove for each positive integer \( n \) the set \( \pi_1(\partial^0 \mathcal{P}_p(n) \cap O(n)) \) contains an open set, choose an interval \( (u,v) \subset (-\epsilon, \epsilon) \) on which \( r \) is positive and \( f^0 = fr^{-1} \) is strictly monotonic. (This is possible because \( r(0) = 1 \) and \( fr^{-1} \) is not constant.) For simplicity, suppose \( f^0 \) is increasing on this interval and thus \( f^0((u,v)) = (f^0(u), f^0(v)) \). Given points \( f^0(u) < a_1 \leq a_2 \leq \cdots \leq a_n < f^0(v) \), there are uniquely determined points \( u < x_1 \leq x_2 \leq \cdots \leq x_n < v \) such that \( f^0(x_j) = a_j \). Let \( X \) denote the diagonal matrix with entries \( x_j \) and \( A \) the diagonal matrix with entries \( a_j = f^0(x_j) \). Then \( X \in B_\epsilon \) and \( p(A,X) = 0 \). Thus, \( A \in \pi_1(\partial^0 \mathcal{P}_p(n) \cap O(n)) \). Moreover, \( p(U^T A U, U^T X U) = 0 \) for every real unitary matrix \( U \). Therefore, \( \pi_1(\partial^0 \mathcal{P}_p(n) \cap O(n)) \) contains every \( A \in S_n \) whose spectrum lies in the interval \((f^0(u), f^0(v))\) and hence contains an open subset of \( S_n \).
(d) By Remark 1.1(2), the condition the highest degree terms of \( p \) majorize is satisfied.

(e) If \( q \) lies in the span of the right chip set of \( p \), then \( q \) has the form

\[
q(a, x) = \varphi(x)a + \psi(x),
\]

where \( \varphi(x) \) and \( \psi(x) \) are polynomials in \( x \) alone of degrees at most \( k - 1 \) and \( \max\{k - 1, d - 1\} \), respectively. To verify the domination condition of item (e) of Theorem 2.5, it suffices to show, if \( q(A, X, v) = 0 \) for each positive integer \( n \) and triple \((A, X, v) \in S_n \times B_\epsilon(n) \times \mathbb{R}^n \) such that \( v \neq 0 \), \( p(A, X)v = 0 \) and \( p(A, X) \succeq 0 \), then \( q = 0 \).

Let \( X \) be a diagonal matrix such that \( r(X) \) is invertible. The substitution \( A = B + r(X)^{-1}f(X) \) into \( p \) and \( q \) gives

\[
p(A, X) = r(X)A + Ar(X) - 2f(X) = r(X)B + Br(X)
\]

and

\[
q(A, X) = \varphi(X)A + \psi(X)
\]

\[
= \varphi(X)B + \psi(X) + \varphi(X)r(X)^{-1}f(X),
\]

respectively. Since \( r(X) \) is invertible if \( \|X\| < \epsilon \) and \( \epsilon \) is a small enough positive number, it suffices to show that, if

\[
(\varphi(X)B + \psi(X) + \varphi(X)r(X)^{-1}f(X))v = 0,
\]

for every triple \((B, X, v) \) with \( B \) real symmetric, and \( X \) real diagonal that satisfies the constraints

(4.1) \( \|X\| < \epsilon, r(X)B + Br(X) \succeq 0 \) and \((r(X)B + Br(X))v = 0 \),

then

(4.2) \( \varphi(X)B + \psi(X) + \varphi(X)r(X)^{-1}f(X) = 0 \).

The constraints in (4.3) are met for every \( v \) if \( B = 0 \) and \( \|X\| < \epsilon \), and hence in this case (4.1) implies that

\[
r(X)\psi(X) + \varphi(X)f(X) = 0.
\]

Thus, (4.1) reduces to \( \varphi(X)Bv = 0 \) and (4.3) reduces to \( \varphi(X)B = 0 \).

Next, since \( r(x) \) is a non constant polynomial with \( r(0) = 1 \), we may choose a pair of real numbers \( x \) and \( y \) such that \( |x| < \epsilon, r(x) > 0, |y| < \epsilon, r(y) > 0 \) and \( r(x) \neq r(y) \). Now let

\[
X := \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \quad \text{and} \quad B := \begin{bmatrix} a & b \\ b & c \end{bmatrix},
\]
where \( a, b, c \) are real numbers. Then \( \|X\| < \epsilon \) and
\[
 r(X)B + Br(X) = \begin{bmatrix}
 2ar(x) & b(r(x) + r(y)) \\
 b(r(x) + r(y)) & 2cr(y)
\end{bmatrix}.
\]

Now choose \( a > 0, c > 0, b = 2\sqrt{acr(x)}r(y), t = -\sqrt{ar(x)}cr(y) \) and \( v = \begin{bmatrix} 1 \\ t \end{bmatrix} \).

Then, \( \varphi(X)Bv = 0 \) for a set of triples \((B, X, v)\) of the requisite form that meet the constraints in \((4.2)\). Since \( X \) is a diagonal matrix, it is also easily seen that the constraints in \((4.2)\) are met if \( B \) and \( v \) are replaced by \( jBj \) and \( jv \), where \( j \) is the signature matrix
\[
 j = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \].

Thus,
\[
 \varphi(X)jBj^2v = \varphi(X)jBv = 0.
\]

Consequently,
\[
 \varphi(X) \begin{bmatrix} Bv \\ jBv \end{bmatrix} = \varphi(X) \begin{bmatrix} a + bt & a + bt \\ b + ct & -(b + ct) \end{bmatrix} = 0.
\]

Since
\[
 a + bt = a \frac{r(y) - r(x)}{r(x) + r(y)} \neq 0 \quad \text{and} \quad b + ct = tcr(x) \frac{r(x) - r(y)}{r(x) + r(y)} \neq 0
\]
the matrix \( \begin{bmatrix} Bv \\ jBv \end{bmatrix} \) is invertible. Thus, \( \varphi(X) = 0 \), for all \( x \) in an open interval. Hence \( \varphi = 0. \)

Since all hypotheses of Theorem 2.5 are verified, \( p \) is of the form of \((2.5)\). Moreover, as \( p \) does not contain terms of the form \( c\tau(a)x_\omega(a)x_j\sigma(a) \) for words \( \tau, \omega, \sigma \) in \( a \) with \( \omega \) not empty and a nonzero \( c \in \mathbb{R} \), Theorem 2.5 guarantees the existence of a representation \((2.5)\) with \( R \in \mathbb{R}^{\rho \times \rho} \) a positive semidefinite matrix that does not depend upon \( a \). Since \( R \) has a unique positive semidefinite square root that can be absorbed into the factors \( s_j(a, x) \), we may assume that \( R \) is the identity and follow essentially the same argument as at the end of the proof of Theorem 1.2 (see the end of Section 3) to conclude that \( s_j(a, x) = s_j(x) \) does not depend upon \( a \).

5. Boundaries and tangent spaces

Given \((A, X, v) \in S_n(\mathbb{R}^\theta) \times \mathbb{R}^n\times\mathbb{R}^\kappa \), let
\[
 \mathcal{T}(A, X, v) = \{ H \in S_n(\mathbb{R}^\theta) : p_x(A, X)[0, H]v = 0 \}.
\]
In the case \((A,X,v) \in \partial \mathcal{P}_p\), the subspace \(\mathcal{T}(A,X,v)\) is the **clamped tangent plane** at \((A,X,v)\) in the terminology of [DHM07a].

If \(p(A,X)v = 0\) and \(H \in \mathcal{T}(A,X,v)\), then, as follows easily from the general formula,

\[
p(A,X + tH) = p(A,X) + tp_x(A,X)[0,H] + \frac{t^2}{2!}p_{xx}(A,X)[0,H]
+ \frac{t^3}{3!}p_{xxx}(A,X)[0,H] + \cdots,
\]

\[
\langle p(A,X + tH)v, v \rangle = \frac{1}{2}t^2\langle p_{xx}(A,X)[0,H]v, v \rangle + t^3e(t),
\]

for some polynomial \(e(t)\). This identity provides a link between convexity and negativity of the Hessian of \(p\), much as in the commutative case.

**Proposition 5.1.** Suppose \(p \in \mathcal{P}_n^{\kappa \times \kappa}\) is symmetric, \((A,X,v) \in \mathcal{S}_n(\mathbb{R}^9)\times \mathbb{R}^{n\kappa}\) and \(v \neq 0\). If

(i) \(p(A,X) \succeq 0\);
(ii) \(p(A,X)v = 0\);
(iii) the dimension of the kernel of \(p(A,X)\) is one;
(iv) there is an open subset \(\mathcal{W}\) of \(\mathcal{S}_n(\mathbb{R}^9)\) containing \(X\) such that the open set \(\mathcal{P}_p^A \cap \mathcal{W}\) is convex;

then there exists a subspace \(\mathcal{H}\) of \(\mathcal{T}(A,X,v)\) of codimension at most one (in \(\mathcal{T}(A,X,v)\)) such that

\[
\langle p_{xx}(A,X)[0,H]v, v \rangle \leq 0 \quad \text{for} \quad H \in \mathcal{H}.
\]

**Remark 5.2.** Note that the codimension of \(\mathcal{T}(A,X,v)\) in \(\mathcal{S}_n(\mathbb{R}^9)\) is at most \(n\kappa\). Hence the inequality of Equation (5.1) holds on a subspace of \(\mathcal{S}_n(\mathbb{R}^9)\) of codimension at most \(n\kappa + 1\).

Unlike a related argument in [DHM07a], the proof here does not rely on choosing a curve lying in the boundary of a convex set, thus eliminating the need for a corresponding smoothness hypothesis. \(\square\)

**Proof.** If \(C\) is an open convex subset of \(\mathbb{R}^N\) and \(y \notin C\), then there is a vector \(w \in \mathbb{R}^N\) such that

\[
\langle y, w \rangle > \langle x, w \rangle
\]

for all \(x \in C\). This separation result, applied to the tuple \(X\) lying outside the open convex set \(\mathcal{P}_p^A \cap \mathcal{W}\) (see item (iv)), guarantees the existence of a linear functional \(\Lambda : \mathcal{S}_n(\mathbb{R}^9) \to \mathbb{R}\) such that \(\Lambda(Z) < 1\) for \(Z \in \mathcal{P}_p^A \cap \mathcal{W}\) and \(\Lambda(X) = 1\). Thus, as

\[
\dim \mathcal{T} = \dim \ker \Lambda|_{\mathcal{T}} + \dim \text{range } \Lambda|_{\mathcal{T}}
\]
and the dimension of the range of $\Lambda$ is one, the subspace
\[ \mathcal{H} = \{ H \in \mathcal{T}(A, X, v) : \Lambda(H) = 0 \} \]
has codimension one in $\mathcal{T}(A, X, v)$.

Fix $H \in \mathcal{H}$ and define $F : \mathbb{R} \to \mathbb{S}_{nn}$ by $F(t) = p(A, X + tH)$. Thus, $F$ is a matrix-valued polynomial in the real variable $t$. Let $U$ be an orthogonal matrix in $\mathbb{R}^{nn \times nn}$ with its last column proportional to $v$ and write
\[ F(t) = U \begin{bmatrix} Q(t) & g(t) \\ g(t)^T & f(t) \end{bmatrix} U^T, \]
where $Q(t) \in \mathbb{S}_{nn-1}$, $g(t) \in \mathbb{R}^{nn-1}$ and $f(t) \in \mathbb{R}$ are polynomials of degree at most $d$ in $t$. Thus,
\[ F(t)v = U \begin{bmatrix} g(t) \\ f(t) \end{bmatrix} \|v\| \]
and the assumption $p(A, X)v = 0$ of item (iii) implies that $f$ and $g$ vanish at 0. The supplementary assumption that $H \in \mathcal{T}(A, X, v)$ implies that
\[ F'(0)v = U \begin{bmatrix} Q'(0) & g'(0) \\ g'(0)^T & f'(0) \end{bmatrix} U^T v = U \begin{bmatrix} g'(0) \\ f'(0) \end{bmatrix} \|v\| = 0, \]
i.e., $f$ and $g$ vanish to second order at 0. Therefore, there are polynomials $\beta$ and $\gamma$ such that $g(t) = t^2\beta(t)$ and $f(t) = t^2\gamma(t)$. Moreover,
\[ \langle p_{xx}(A, X)[0, H]v, v \rangle = \langle F''(0)v, v \rangle = \|v\|^2 f''(0) = 2\|v\|^2\gamma(0). \]
Thus, to complete the proof of the proposition it suffices to use the choice of $\Lambda$ (and thus the convexity hypothesis on $\mathfrak{P}_p^A \cap \mathscr{W}$) and the assumption on the dimension of the kernel of $F(0) = p(A, X)$ to show that $\gamma(0) \leq 0$. Indeed, since the kernel of $F(0)$ has dimension one and $F'(0) \succeq 0$ by item (i), it follows that $Q(0) \succeq 0$. Therefore, $Q(t) \succeq 0$ for $|t|$ sufficiently small. On the other hand, since $H \in \mathcal{H}$,
\[ \Lambda(X + tH) = \Lambda(X) + t\Lambda(H) = \Lambda(X) = 1 \]
for all $t$. Thus, $X + tH \not\in \mathfrak{P}_p^A \cap \mathscr{W}$. On the other hand, for $|t|$ sufficiently small, $X + tH \in \mathscr{W}$ and hence $X + tH \not\in \mathfrak{P}_p^A$. Thus for $|t|$ sufficiently small both $F(t) = p(A, X + tH) \neq 0$ and $Q(t) \succeq 0$. Hence, for such $t$, the Schur complement of $Q$ is nonpositive; i.e.,
\[ t^2[\gamma(t) - t^2\beta^T(t)Q^{-1}(t)\beta(t)] \leq 0. \]
Therefore,
\[ \gamma(t) \leq t^2\beta^T(t)Q^{-1}(t)\beta(t) \quad \text{for } t \in (0, \epsilon) \]
\[ \text{If } \mathfrak{P}_p^A \text{ is replaced by } \mathcal{D}_p^A, \text{ the component of 0 of } \mathfrak{P}_p^A, \text{ the failure of positivity of } F(t) \text{ would not guarantee that } X + tH \not\in \mathcal{D}_p(A). \]
and hence $\gamma(0) \leq 0$. □

5.1. Kernel of Dimension 1. The main result of this subsection, Proposition 5.3, extends the applicability of Proposition 5.1. A full rank assumption, similar to condition (b) in Theorem 2.5, is imposed to justify an application of the implicit function theorem.

First observe that if $(A, X) \in S_n(\mathbb{R}^g)$, then the matrix $p(A, X)$ belongs to $S_n^\kappa$ and can be identified as an $s \times 1$ vector with $s = (m^2 + m)/2$ and $m = n\kappa$. Likewise by identifying $S_n(\mathbb{R}^g)$ with $\mathbb{R}^r$ for $r = g(n^2 + n)/2$, the mapping

$$S_n(\mathbb{R}^g) \ni (A, X) \mapsto p(A, X)$$

can be identified with a mapping $f : \mathbb{R}^r \to \mathbb{R}^s$,

$$f(y) = \begin{bmatrix} f_1(y_1, \ldots, y_r) \\ \vdots \\ f_s(y_1, \ldots, y_r) \end{bmatrix}$$

Moreover, the Jacobian matrix

$$J_f(y) = \begin{bmatrix} \frac{\partial f_1}{\partial y_1}(y) & \cdots & \frac{\partial f_1}{\partial y_r}(y) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_s}{\partial y_1}(y) & \cdots & \frac{\partial f_s}{\partial y_r}(y) \end{bmatrix}$$

can be identified with

$$[p_{a,x}^1(Y)(G_1) \cdots p_{a,x}^s(Y)(G_r)]$$

for an appropriate choice of points $G_j \in S_n(\mathbb{R}^g)$, for $j = 1, \ldots, r$. Thus, the statement that

$$p_{a,x}(A, X)[E, H] \text{ maps } Y = (E, H) \in S_n(\mathbb{R}^g) \text{ onto } S_{n\kappa}$$

is equivalent to the statement that the rank of the Jacobian matrix is equal to the dimension of $S_{n\kappa}$.

**Proposition 5.3.** Suppose $(A^o, X^o, v) \in S_n(\mathbb{R}^g) \times \mathbb{R}^{n\kappa}$, $v \neq 0$, and $p(a, x)$ is a symmetric $\kappa \times \kappa$ matrix polynomial in the noncommuting variables $a$ and $x$. If

1. $p(A^o, X^o)v = 0$ and
2. $p'(A^o, X^o)[E, H]$ maps $(E, H) \in S_n(\mathbb{R}^g)$ onto $S_{n\kappa}$ (i.e., $(A^o, X^o)$ is a full rank point for $p$),

then, for each $\varepsilon > 0$, there is a full rank point $(A, X) \in S_n(\mathbb{R}^g)$ such that $\|A - A^o\| < \varepsilon, \|X - X^o\| < \varepsilon$ and the kernel of $p(A, X)$ is spanned by $v$.

Furthermore, if $p(A^o, X^o) \succeq 0$, then $(A, X, v)$ can be chosen so that $p(A, X) \succeq 0$ too.
Proof. For $A, X \in S_n(\mathbb{R}^g)$ and $C \in S_{n\kappa}$, let

$$f(A, X, C) := [p(A, X) - C]v \quad \text{and} \quad C^o = p(A^o, X^o).$$

Then $f(A^o, X^o, C^o) = 0$ and, as the derivative of $f$ with respect to the variables $A$ and $X$ has full rank, the implicit function theorem implies that there is a neighborhood $\mathcal{N}$ of $C^o$ and a neighborhood $\mathcal{N}'$ of $(A^o, X^o)$ and a continuous mapping $g : \mathcal{N} \to \mathcal{N}'$ such that $f(g(C), C) = 0$. Let $U \in \mathbb{R}^{n\kappa \times n\kappa}$ be a unitary matrix with its last column proportional to $v$. Thus,

$$C^o = U \begin{bmatrix} Q^o & \beta^o \\ \gamma^o & 0 \end{bmatrix} U^T \quad \text{with } \beta^o \in \mathbb{R}^{n\kappa - 1} \text{ and } \gamma^o \in \mathbb{R}.$$

As

$$C^o v = U \begin{bmatrix} \beta^o \\ \gamma^o \end{bmatrix} \|v\| = 0,$$

it follows that $\beta^o = 0$ and $\gamma^o = 0$ and hence that

$$C^o = U \begin{bmatrix} Q^o & 0 \\ 0 & 0 \end{bmatrix} U^T.$$

Choose $C \in \mathcal{N}$ with

$$C = U \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} U^T, \quad Q = Q^T \text{ invertible and } \|Q - Q^o\| \text{ small}.$$

If $(A, X) = g(C)$, then

$$p(A, X)v = Cv = U \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} U^Tv = U \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0_{(n\kappa - 1) \times 1} \\ 1 \end{bmatrix} = 0$$

and, since $Q$ is invertible, the dimension of the kernel of $p(A, X)$ is equal to one. If $p(A^o, X^o) \succeq 0$, then $Q^o \succeq 0$ and $Q = Q^T$ may be chosen positive definite. \qed

5.2. The Hessian on a Tangent Plane vs. the Relaxed Hessian.

Our main tool for analyzing the curvature of noncommutative real varieties is a variant of the Hessian for symmetric $\kappa \times \kappa$-valued polynomials $p$ of degree $d$ in $g$ noncommuting variables. The curvature of an nc variety $\mathcal{V}(p)$ is defined in terms of the Hessian of $p$ compressed to tangent planes, for each dimension $n$. Since this compression of the Hessian is awkward to work with directly, we introduce a quadratic polynomial $F(a, x)[h]$ (defined for all $H \in S_n(\mathbb{R}^g)$, not just for $H \in T(A, X, v)$) called the relaxed Hessian. This approach is taken in \cite{DHM11} and used in \cite{BM14}; §3 of \cite{DHM11} should be referred to for motivation and more details; see especially Example 3.1.
Given a symmetric nc matrix polynomial \( p \in \mathcal{P}^{\kappa \times \kappa} \) with right chip sets \( \mathcal{R}P^j, j = 1, \ldots, g \), let \( \mathcal{U}(a, x)[h] \) denote the column vector with entries \( h_jw(a, x) \) for \( 1 \leq j \leq g \) and \( w \in \mathcal{R}P^j \). The relaxed Hessian of \( p \) is defined to be the polynomial\(^4\)

\[
(5.2) \quad p''_{\lambda,\delta}(a, x)[h] := p_{xx}(a, x)[h] + \delta I_\kappa \otimes \mathcal{U}(a, x)[h]^T \mathcal{U}(a, x)[h] + \lambda p_x(a, x)[h]^T p_x(a, x)[h].
\]

Suppose \( (A, X) \in S_n(\mathbb{R}^g) \) and \( v \in \mathbb{R}^{n\kappa} \). We say that the relaxed Hessian is positive at \( (A, X, v) \) if for each \( \delta > 0 \) there is a \( \lambda_\delta > 0 \) so that for all \( \lambda > \lambda_\delta \)

\[
0 \leq \langle p''_{\lambda,\delta}(A, X)[h]v, v \rangle
\]

for all \( H \in S_n(\mathbb{R}^g) \). Correspondingly we say that the relaxed Hessian is negative at \( (A, X, v) \) if for each \( \delta < 0 \) there is a \( \lambda_\delta < 0 \) so that for all \( \lambda \leq \lambda_\delta \),

\[
0 \leq -\langle p''_{\lambda,\delta}(A, X)[h]v, v \rangle
\]

for all \( H \in S_n(\mathbb{R}^g) \). Given a sequence \( \mathcal{S} = (S_n)_{n=1}^\infty \), with \( S_n \subseteq (S_n(\mathbb{R}^g) \times \mathbb{R}^{n\kappa}) \), we say that the relaxed Hessian is positive (resp., negative) on \( \mathcal{S} \) if it is positive (resp., negative) at each \( (A, X, v) \in \mathcal{S} \).

Suppose \( f(a, x)[h] \) is a \( \kappa \times \kappa \)-valued free symmetric polynomial in the \( \bar{g}+2g \) symmetric variables \( a, x \) and \( h \) of degree \( s \) in \( x \) and homogeneous of degree two in \( h \). Given a subspace \( \mathcal{H} \) of \( S_n(\mathbb{R}^g) \), let

\[
(5.3) \quad e_n^\pm(A, X, v; f, \mathcal{H})
\]

denote the maximum dimension of a strictly positive/negative subspace of \( \mathcal{H} \) with respect to the quadratic form

\[
\mathcal{H} \ni H \mapsto \langle f(A, X)[h]v, v \rangle.
\]

Here strictly positive (resp., negative) subspace \( \mathcal{H} \) means

\[
\langle f(A, X)[h]v, v \rangle > 0 \text{ (resp. } < 0 \text{) for } H \in \mathcal{H}, H \neq 0.
\]

Lemma 5.4 below provides a link between the signature of the clamped second fundamental form (i.e., the Hessian compressed to the clamped tangent space \( \mathcal{T} \)) and that of the relaxed Hessian (a quadratic form on all of \( S_n(\mathbb{R}^g) \)).

\(^4\)The tensor product of the matrix \( I \) with the vector (column matrix) nc polynomial is defined either abstractly or concretely by the Kronecker product just as for the tensor product of matrices. The Kronecker product of two matrix nc polynomials requires more care as discussed later in Subsection 8.1.
Lemma 5.4. Suppose $p$ is a $\kappa \times \kappa$ symmetric nc matrix polynomial of degree $\tilde{d}$ in $a$, degree $d$ in $x$ and $(A, X, v) \in \mathbb{S}_n(\mathbb{R}^\sigma) \times \mathbb{R}^{n\kappa}$. Then there exists an $\varepsilon < 0$ such that for each $\delta \in [\varepsilon, 0)$ there exists a $\lambda_\delta < 0$ so that for every $\lambda \leq \lambda_\delta$,
\[ e_+^n(A, X, v; p_{\lambda, \delta}, \mathbb{S}_n(\mathbb{R}^\sigma)) = e_+^n(A, X, v; p_{xx}, \mathbb{T}(A, X, v)). \]

Similarly, there exists an $\varepsilon > 0$ such that for each $\delta \in (0, \varepsilon]$ there exists a $\lambda_\delta > 0$ so that for every $\lambda \geq \lambda_\delta$,
\[ e_-^n(A, X, v; p_{\lambda, \delta}, \mathbb{S}_n(\mathbb{R}^\sigma)) = e_-^n(A, X, v; p_{xx}, \mathbb{T}(A, X, v)). \]

Remark 5.5. The lemma does not require $p(A, X)v = 0$ or even $p(A, X) \geq 0$.

The proof of Lemma 5.4 is postponed in favor of two preliminary lemmas.

Lemma 5.6. Suppose $p$ is a $\kappa \times \kappa$ symmetric nc matrix polynomial of degree $\tilde{d}$ in $a$, degree $d$ in $x$ and $(A, X, v) \in \mathbb{S}_n(\mathbb{R}^\sigma) \times \mathbb{R}^{n\kappa}$. If $H \in \mathbb{S}_n(\mathbb{R}^\sigma)$ and
\[ (5.4) \quad \left\{ I_{n} \otimes (\tilde{\mathbf{u}}(A, X)[H]^T \tilde{\mathbf{u}}(A, X)[H]) \right\} v = 0, \]
then $p_{xx}(A, X)[h]v = 0$ and $p_x(A, X)[h]v = 0$.

Proof. Write $v = \oplus_{j=1}^n v_j$ with $v_j \in \mathbb{R}^n$. Since
\[ I_{n} \otimes (\tilde{\mathbf{u}}(A, X)[H]^T \tilde{\mathbf{u}}(A, X)[H]) = (I_{n} \otimes \tilde{\mathbf{u}}(A, X)[H])^T (I_{n} \otimes \tilde{\mathbf{u}}(A, X)[H]), \]
the assumption that $(I_{n} \otimes \{\tilde{\mathbf{u}}(A, X)[H]\})^T (\tilde{\mathbf{u}}(A, X)[H]) v = 0$, the hypothesis (5.4) implies that
\[ (5.5) \quad H_j w(A, X)v_k = 0 \text{ for each } j, w \in C_p^j \text{ and } 1 \leq k \leq \kappa. \]

By the definition of the right chip sets for $p$, each word in $p - p(0)$ is of the form $uw_j w$, for some $j$, some word $w \in C_p^j$ and some word $u$. Correspondingly, each word in the polynomial $p_x$ is of the form $uh_j w$ and each word in the polynomial $p_{xx}$ is of the form $vh_j w$ where $v$ is a polynomial in $a$, $x$ and $h$. Hence each term in $p_x(A, X)[h]v$ has the form $u(A, X)H_j w(A, X)v_k$ which, by (5.5) is 0, implying $p_x(A, X)[h]v$. Likewise for $p_{xx}$. \hfill \Box

Let $\mu_+(R)$ (resp., $\mu_-(R)$) denote the number of positive (resp., negative) eigenvalues of a real symmetric matrix $R$.

Lemma 5.7. Let $E, F, G \in \mathbb{S}_n$ be given and let $P$ denote the orthogonal projection of $\mathbb{R}^n$ onto $\ker(F)$, the kernel of $F$. If
(i) $F \neq 0$;
(ii) $F \succeq 0$, $G \succeq 0$; and
(iii) $\ker(G) \subseteq \ker(F) \cap \ker(E)$,
then there exists a number $\varepsilon < 0$ such that for each $\delta \in [\varepsilon, 0)$, there exists a $\lambda_\delta$ such that for each $\lambda \leq \lambda_\delta$,

$$\mu_+(E + \lambda F + \delta G) = \mu_+(PEP).$$

An analogous result holds for $\mu_-$.

Proof. By restricting to the orthogonal complement of $\ker(G)$, we assume, without loss of generality, that $G \succ 0$.

Suppose first that

$$\dim \ker(F) = k \quad \text{and} \quad k \geq 1.$$ 

Then there exists a real unitary matrix

$$U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$$

with $U_1 \in \mathbb{R}^{n \times k}$ and $U_2 \in \mathbb{R}^{n \times (n-k)}$ such that

$$FU_1 = 0 \quad \text{and} \quad U_2^T FU_2 \succ 0.$$ 

Let

$$E_{ij} = U_i^T E U_j, \quad F_{ij} = U_i^T F U_j \quad \text{and} \quad G_{ij} = U_i^T G U_j$$

for $i, j = 1, 2$ and note that

$$F_{11} = 0_{k \times k}, \quad F_{12} = 0_{k \times (n-k)} \quad \text{and} \quad F_{21} = 0_{(n-k) \times k}.$$ 

Thus,

$$\mu_+(E + \lambda F + \delta G) = \mu_+(U^T [E + \lambda F + \delta G] U) = \mu_+(Q_{\lambda, \delta})$$

with

$$Q_{\lambda, \delta} = \begin{bmatrix} E_{11} + \delta G_{11} & E_{12} + \delta G_{12} \\ E_{21} + \delta G_{21} & E_{22} + \lambda F_{22} + \delta G_{22} \end{bmatrix}.$$ 

Since $G_{11} \succ 0$, the additive perturbation $\delta G_{11}$ with $\delta < 0$ shifts the eigenvalues of $E_{11}$ to the left. However, if $\delta \in [\varepsilon, 0)$ and $|\varepsilon|$ is sufficiently small, then the positive eigenvalues of $E_{11}$ will stay positive, whereas, the nonpositive eigenvalues of $E_{11}$ become negative. Consequently,

$$\mu_+(E_{11} + \delta G_{11}) = \mu_+(E_{11}) \quad \text{and} \quad E_{11} + \delta G_{11} \quad \text{is invertible if} \ \delta \in [\varepsilon, 0).$$

Thus, for such $\varepsilon$ and $\delta$,

$$\mu_+(Q_{\lambda, \delta}) = \mu_+(E_{11} + \delta G_{11}) + \mu_+(S_{\lambda, \delta}) = \mu_+(E_{11}) + \mu_+(S_{\lambda, \delta}),$$

where $S_{\lambda, \delta}$ denotes the Schur complement of $E_{11} + \delta G_{11}$ in $Q_{\lambda, \delta}$, i.e.,

$$S_{\lambda, \delta} = E_{22} + \lambda F_{22} + \delta G_{22} - (E_{12} + \delta G_{12})^T (E_{11} + \delta G_{11})^{-1} (E_{12} + \delta G_{12}).$$
Therefore, since $F_{22} \succ 0$, there exists a number $\lambda_\delta < 0$ such that

$$\mu_+(S_{\lambda,\delta}) = 0 \quad \text{for } \lambda \leq \lambda_\delta.$$ 

Thus, to this point we have established the equality

$$\mu_+(E + \lambda F + \delta G) = \mu_+(U_1^T E U_1) \quad \text{when } \delta \in [\varepsilon, 0), \lambda \leq \lambda_\delta$$

and $k \geq 1$. To complete the proof in this case, note first that

$$P = U_1 U_1^T$$

and hence, by a well known theorem of Sylvester,

$$\mu_+(P E P) = \mu_+(U_1 U_1^T E U_1 U_1^T) \leq \mu_+(U_1^T E U_1)$$

and, as $U_1^T U_1 = I_k$,

$$\mu_+(U_1^T E U_1) = \mu_+(U_1^T U_1 U_1^T E U_1 U_1^T U_1) \leq \mu_+(U_1^T U_1 U_1^T E U_1 U_1^T) = \mu_+(P E P).$$

The proof of (5.6) is now complete when $k \geq 1$, i.e., when $\ker(F) \neq 0$.

However, if $\ker(F) = 0$, then $P = 0_{n \times n}$ and $F \succ 0$. Therefore, $\mu_+(P E P) = 0$ and it is readily checked that for any $\delta \leq 0$ there exists a $\lambda_\delta$ such that the equality in (5.6) holds for $\lambda \leq \lambda_\delta$. \hfill $\square$

The proof of Lemma 5.4 employs a bilinear variant of the Hessian that we now introduce. Let $p_{xx}(A, X)[h][0, K]$ denote the matrix obtained by differentiating $p_x(A, X)[h]$ in the direction $(0, K)$ for $K \in S_n(R^g)$; i.e.,

$$p''(A, X)[h][0, K] = \lim_{t \to 0} \frac{1}{t} (p_x(A, X + tK)[h] - p_x(A, X)[h]).$$

In particular,

$$p_{xx}(A, X)[h] = p_{xx}(A, X)[h][h]$$

and

(5.7) \hspace{1cm} p_{xx}(A, X)[h][0, K] = p_{xx}(A, X)[0, K][h].$$

Proof of Lemma 5.4. Again, we shall only consider the case of positive signatures, since the case of negative signature is similar (and may be obtained by considering $-P$ in place of $P$). Recall that $g$ denotes the number of noncommutative symmetric variables $x$ and $d$ the degree of $p$ in $x$. Let $H = S_n(R^g)$ endowed with the Hilbert Schmidt norm:

$$\langle H, K \rangle_H = \text{trace } K^T H = \sum_{j=1}^g \text{trace } K_j H_j.$$

By the Reisz representation theorem any continuous symmetric bilinear form $Q : H \times H \to \mathbb{R}$ can be represented as

$$Q(H, K) = \langle RH, K \rangle_H$$
where $R$ is a bounded selfadjoint operator on $\mathcal{H}$.

The mapping $\mathcal{B} : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ defined by

$$\mathcal{B}(H, K) = \langle p''(X)[H][K]v, v \rangle_{\mathbb{R}^{n\kappa}}$$

is bi-linear and by Equation (5.7) it is symmetric. Thus, there is a bounded selfadjoint operator $E$ on $\mathcal{H}$ so that

$$\langle EH, K \rangle_{\mathcal{H}} = \mathcal{B}(H, K).$$

Similarly, there are bounded linear selfadjoint operators $F$ and $G$ on $\mathcal{H}$ so that

$$\langle FH, K \rangle_{\mathcal{H}} = \sum_{j,w \in \mathbb{R}C_p^j} \langle H_j w(A, X)v, K_j w(A, X)v \rangle_{\mathbb{R}^{n\kappa}}.$$

In particular,

$$\langle GH, H \rangle_{\mathcal{H}} = \langle I_{\kappa} \otimes \tilde{U}(A, X)[H]v, I_{\kappa} \otimes \tilde{U}(A, X)[H]v \rangle_{\mathbb{R}^{n\kappa}} = \langle (E + \lambda F + \delta G)H, H \rangle_{\mathcal{H}}$$

for $H \in S_n(\mathbb{R}^g)$.

Note that $H \in \ker(G)$ if and only if $H_j w(A, X)v = 0$ for all $j$ and $w \in C_p^j$. Thus, by Lemma 5.6, $\ker(G) \subset \ker(F) \cap \ker(E)$. Since the kernel of $F$ is exactly $T(A, X, v)$, an application of Lemma 5.7 completes the proof. □

6. Direct sums and linear independence

Recall that $p(a, x)$ is a $\kappa \times \kappa$ matrix-valued symmetric polynomial of degree $\tilde{d}$ in $a$ and $d$ in $x$ and that $C_p$ is the chip space of $p$. Let $C_p^{\kappa' \times \kappa}$ denote the $\kappa' \times \kappa$ matrices with entries from $C_p$.

6.1. Dominating points. A point $(\hat{A}, \hat{X}, \hat{v}) \in S_n(\mathbb{R}^g) \times \mathbb{R}^{n\kappa}$ is a $C_p$-dominating point for a free set $S \subset (S_n(\mathbb{R}^g) \times \mathbb{R}^{n\kappa})_{n=1}^\infty$, if the set $W = \{(\hat{A}, \hat{X}, \hat{v})\}$ is $C_p$-dominating for $S$, i.e., if $q \in C_p^{1\times \kappa}$ and $q(\hat{A}, \hat{X})\hat{v} = 0$ implies $q(A, X)v = 0$ for all $(A, X, v) \in S$. As before, note that $(\hat{A}, \hat{X}, \hat{v})$ is not required to be in $S$.

**Lemma 6.1.** Suppose $p \in \mathcal{P}^{\kappa \times \kappa}$ is symmetric nc matrix polynomial and $(\hat{A}, \hat{X}, \hat{v})$ is a $C_p$-dominating point for a free set $S \subset (S_n(\mathbb{R}^g) \times \mathbb{R}^{n\kappa})_{n=1}^\infty$. If $(B, Y, w)$ is a point in $S$ that is sufficiently close to $(\hat{A}, \hat{X}, \hat{v})$, then it is also a $C_p$-dominating point for $S$.

**Proof.** Let

$$S^o = \{ r \in C_p^{1\times \kappa} : r(A, X)v = 0, \text{ for all } (A, X, v) \in S \}.$$
Choose a subspace $B$ of $C_p^{1\times \kappa}$ that is complementary to $S^o$, i.e.,
$$C_p^{1\times \kappa} = B + S^o,$$
(where the symbol $+$ means that $C_p^{1\times \kappa} = B + S^o$ and $B \cap S^o = \{0\}$) and let $(\hat{A}, \hat{X}, \hat{v})$ be a $C_p$-dominating point for $S$. It is readily seen that the linear mapping $B \to \mathbb{R}^n$ defined by
$$q \mapsto q(\hat{A}, \hat{X})\hat{v}$$
is one-one, because if $q$ and $q_0$ both belong to $B$ and $q(\hat{A}, \hat{X})\hat{v} = q_0(\hat{A}, \hat{X})\hat{v}$, then $q - q_0 \in B \cap S^o = \{0\}$.

Hence, if $(B, Y, w) \in S$ is sufficiently close to $(\hat{A}, \hat{X}, \hat{v})$, then the mapping
$$(6.1) \quad B \ni q \mapsto q(B, Y)w$$is also one-one.

If $q \in C_p^{1\times \kappa}$, then $q = q_1 + q_2$ with $q_1 \in S^o$ and $q_2 \in B$. Thus,
$$q(B, Y)w = q_1(B, Y)w + q_2(B, Y)w = q_2(B, Y)w,$$since $(B, Y, w) \in S$. Consequently,
$$q(B, Y)w = 0 \implies q_2(B, Y)w = 0 \implies q_2 = 0,$$since $q_1 \in S^o$. Therefore, $(B, Y, w)$ is a $C_p$ dominating point for $S$. \hfill \Box

**Lemma 6.2.** Let $p \in P^{\kappa \times \kappa}$ be a symmetric nc matrix polynomial of degree $\tilde{d}$ in $a$ and degree $d$ in $x$ and let $S = (S(k))_{k=1}^\infty$ be a free nonempty set such that
$$S(n) \subseteq \{(A, X, v) \in S_n(\mathbb{R}) \times \mathbb{R}^n : p(A, X)v = 0\},$$then for every positive integer $N$ there exists an integer $n \geq N$ and a triple $(\hat{A}, \hat{X}, \hat{v}) \in S(n)$ that is a $C_p$ dominating point for $S$.

**Proof.** For a given $(A, X, v)$ in $S$, define
$$\mathcal{I}(A, X, v) = \{q \in C_p^{1\times \kappa} : q(A, X)v = 0\}$$and
$$\mathcal{I}(S) = \bigcap\{\mathcal{I}(A, X, v) : (A, X, v) \in S\}.$$Thus, $\mathcal{I}(S)$ consists of all polynomials in $C_p^{1\times \kappa}$ that vanish on $S$. 

Thus, if $q \in C_{p}^{1 \times \kappa}$ and $q(A^j, X^j)v^j = 0$ for $j = 1, \ldots, t$, then $q \in \mathcal{I}(\mathcal{S})$ and hence $q = 0$ on $\mathcal{S}$, i.e., if

$$[q_{11}(A^j, X^j)v^j \cdots q_{1\kappa}(A^j, X^j)v^j] = 0 \quad \text{for } j = 1, \ldots, t,$$

then

$$[q_{11}(A, X)v \cdots q_{1\kappa}(A, X)v] = 0 \quad \text{for every point } (A, X, v) \in \mathcal{S}.$$

Let

$$A' = \text{diag}\{A^1, \ldots, A^t\}, \quad X' = \text{diag}\{X^1, \ldots, X^t\} \text{ and } v' = \text{col}\{v^1, \ldots, v^t\}.$$

Then $A' \in \mathcal{S}_{n'}(\mathbb{R}^{q})$, $X' \in \mathcal{S}_{n'}(\mathbb{R}^{q})$ and $v' \in \mathbb{R}^{n'\kappa}$, where $n' = n_1 + n_2 + \cdots + n_t$, and $q(A', X')v' = 0$ if and only if $q(A^j, X^j)v^j = 0$ for $j = 1, \ldots, t$. Thus, if $q \in C_{p}^{1 \times \kappa}$ and $q(A', X')v' = 0$, then $q = 0$ on $\mathcal{S}$. If $n' \geq N$, then the construction stops here. If not, then choose a positive integer $k$ such that $n = kn' \geq N$ and consider the $k$-fold direct sums

$$\hat{A} = \text{diag}\{A', \ldots, A'\}, \quad \hat{X} = \text{diag}\{X', \ldots, X'\} \quad \text{and } \hat{v} = \text{col}\{v', \ldots, v'\}.$$

The triple $(\hat{A}, \hat{X}, \hat{v})$ is a $C_p$-dominating point for $\mathcal{S}$: if $q \in C_{p}^{1 \times \kappa}$ and $q(\hat{A}, \hat{X})\hat{v} = 0$, then $q = 0$ on $\mathcal{S}$. \hfill \Box

7. **The middle matrix representation and its properties**

In this section we develop a representation for nc polynomials $q(a, x, h)$ in the nc variables $a = (a_1, \ldots, a_2)$, $x = (x_1, \ldots, x_q)$ and $h = (h_1, \ldots, h_q)$ that are homogeneous of degree two in $h$ with particular attention given to the Hessian $p_{xx}(a, x)[h]$ of an nc polynomial $p(a, x)$ and two of its relatives.

7.1. **Definition of the middle matrix.** We begin with the case of scalar-valued polynomials, before turning to the matrix case with its additional bookkeeping overhead.

A **middle matrix representation** or **border vector-middle matrix representation** of a scalar nc polynomial $q(a, x, h)$ that is homogeneous of degree two in $h$ is a representation of the form

$$q(a, x, h) = \sum_{i,j=0}^{\ell} B_i(a, x)[h]^T M_{ij}(a, x) B_j(a, x)[h],$$

(7.1)
in which $B_j(a,x)[h]$ is a column vector with entries of the form $h_k w(a,x)$, where $w(a,x)$ is a word in $a,x$ that is homogeneous of degree $j$ in $x$ and $M_{ij}(a,x)$ is a matrix polynomial in $a$ and $x$ (and not $h$). A middle matrix representation for $q(a,x,h) = p_{xx}(a,x)[h]$, the Hessian of the polynomial $p(a,x) = x_2^2 ax_1 + x_1 ax_2^2 + a^2$, appears in Example 2.1. There $\ell = 1$, $g = 2$. The block entries (by degree in $x$) of the border vector are $B^T_0 = \begin{bmatrix} h_1 & h_2 \end{bmatrix}$, $B^T_1 = \begin{bmatrix} x_2 h_2 & x_1 ah_2 \end{bmatrix}$, and

$$M_{00}(a,x) = \begin{bmatrix} 0 & ax_2 \\ x_2 a & 0 \end{bmatrix}, \quad M_{01} = M_{10} = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, \quad M_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$  

The middle matrix construction extends naturally to the case of matrix polynomials. If $q$ is a $\kappa \times \kappa$ matrix valued polynomial, its middle matrix representation has the more general but similar form

$$q(a,x,h) = \sum_{i,j=0}^\ell (I_\kappa \otimes B_i(a,x)[h]^T)\mathcal{M}_{ij}(a,x)(I_\kappa \otimes B_j(a,x)[h]).$$

Note that the middle matrix for $q = C_w w(a,x,h)$ is simply $C_w \otimes M$, where $M$ is the middle matrix for $w$.

7.1.1. **Uniqueness of the middle matrix.** The middle matrix depends upon $q$, but once the border vector $B$ is fixed (so a choice of list of monomials, sorted by degree in $x$, is made), the middle matrix $M = (M_{ij})$ (resp. $\mathcal{M}$) is uniquely determined, justifying the terminology the middle matrix. For instance, for a word $w = w(a,x,h)$ that is homogeneous of degree two in $h$,

$$w = w_L(a,x)h_i \, cw_M(a,x) \, h_j w_R(a,x),$$

(7.2)

the border vector for any middle matrix representation of $w$ must include the words $h_i w_L(a,x)^T$ and $h_j w_R(a,x)$.

To illustrate the extent of the uniqueness of the middle matrix and its dependence on the border vector, consider middle matrix representations for the (scalar polynomial) word $w$ of equation (7.2). Using a border vector $B$ with just the two words $h_i w_L(a,x)^T$ and $h_j w_R(a,x)$, the corresponding middle matrix representation is

$$w = \begin{bmatrix} w_L(a,x)h_i & w_R(a,x)^T h_j \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} h_i w_L(a,x)^T \\ h_j w_R(a,x) \end{bmatrix}.$$  

If the degrees of $w_L(a,x)$ and $w_R(a,x)$ in $x$ are the same, then there is a choice of order (a permutation) in constructing this $B$. Of course, one could have chosen a border vector $B$ with more words.
But then the middle matrix would have more zeros to ensure that the superfluous words are not counted, e.g.,

\[
\begin{bmatrix}
w_L(a, x)h_i & w_R(a, x)^T h_j & * \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
h_iw_L(a, x)^T \\
0 \\
0 \\
0
\end{bmatrix}.
\]

7.2. **Middle matrix representations for Hessians.** Our ultimate goal is to describe the middle matrix representation for the relaxed Hessian in sufficiently fine detail to make it a powerful tool. Throughout the remainder of this section \(d\) and \(\tilde{d}\) are fixed positive integers and our polynomials are assumed to have degree at most \(d\) in \(x\) and \(\tilde{d}\) in \(a\). Given a word \(w = w(a, x)\), let \(Z^w(a, x)\) denote the middle matrix of its Hessian \(w_{xx}\)

\[
w_{xx}(a, x) = \sum_{i,j=0}^{d-2} V_i(a, x)[h]^T Z^w_{ij}(a, x) V_j(a, x)[h],
\]

based upon the **full border vector**

\[
V(a, x)[h] = \begin{bmatrix} V_0 \\ \vdots \\ V_{d-2} \end{bmatrix},
\]

where the \(V_j\) lists all words \(h_k f(a, x)\), for \(f(a, x)\) of the form

\[
f = u_0(a)x_{i_1}u_1(a)x_{i_2}\cdots u_{j-1}(a)x_{i_j}u_j(a),
\]

and the \(u_j\) are words of length at most \(\tilde{d}\). This choice of border vector works over all such choices of \(w\).

Given \(C_w \in \mathbb{R}^{\kappa \times \kappa}\),

\[
C_w w_{xx}(a, x)[h]
\]

\[
= \sum_{i,j=0}^{d-2} (I_{\kappa} \otimes V_i(a, x)[h]^T) (C_w \otimes Z^w_{ij}(a, x)) (I_{\kappa} \otimes V_j(a, x)[h]).
\]

In particular, for \(p = C_w w\), the entries of its middle matrix based on \(V\) are \(Z_{ij} = C_w \otimes Z^w_{ij}\). (We use \(Z_{ij}(a, x)\) to denote the \((\kappa \times \kappa\) block) entries of the middle matrix for matrix-valued polynomials and \(Z_{ij}(a, x)\) in the case of scalar-valued polynomials.)
Given a polynomial $p$ expressed as in equation (1.1), its Hessian has the middle matrix representation

$$p_{xx}(a, x)[h] = \sum_{i,j=0}^{d-2} (I_\kappa \otimes V_i(a, x)[h]^T) \left( \sum_w C_w \otimes Z_{ij}^w(a, x) \right) (I_\kappa \otimes V_j(a, x)[h]).$$

Thus the middle matrix $Z$ of the Hessian of $p$ based on the border vector $V$ has (block) entries $Z_{ij} = \sum_w C_w \otimes Z_{ij}^w(a, x)$.

### 7.3. Middle matrices for the modified Hessian

The modified Hessian of $p$ is, by definition,

$$p_{xx}(a, x)[h] + \lambda p_x(a, x)[h]^T p_x(a, x)[h].$$

The middle matrix of $p_x(a, x)[h]^T p_x(a, x)[h]$ is obtained by expressing the derivative $p_x(a, x)[h]$ in terms of the extended full border vector

(7.7) $$\tilde{V}(a, x)[h] = \begin{bmatrix} V_0(a, x)[h] \\ \vdots \\ V_{d-1}(a, x)[h] \end{bmatrix}.$$ 

If $p = \sum_{w \in W} C_w \otimes w(a, x)$ is a sum of words with coefficients $C_w \in \mathbb{R}_{\kappa \times \kappa}$, then

$$p_x(a, x)[h] = \sum_{j=0}^{d-1} \Phi_j(a, x) \tilde{V}^\kappa_{\deg}(a, x)[h],$$

where

$$\Phi(a, x) = \begin{bmatrix} \Phi_0(a, x) & \cdots & \Phi_{d-1}(a, x) \end{bmatrix},$$

is a block row matrix with $\kappa$ rows and

(7.8) $$\tilde{V}_{\deg}^\kappa(a, x)[h] = \begin{bmatrix} I_\kappa \otimes V_0(a, x)[h] \\ \vdots \\ I_\kappa \otimes V_{d-1}(a, x)[h] \end{bmatrix}.$$ 

This notation yields the formula

(7.9) $$p_x(a, x)[h]^T p_x(a, x)[h] = \tilde{V}_{\deg}^\kappa(a, x)[h]^T \left( \Phi(a, x)^T \Phi(a, x) \right) \tilde{V}_{\deg}^\kappa(a, x)[h]$$

and thus the middle matrix, in block form, is $(\Phi_k(a, x) \Phi_j(a, x))_{j,k}$. 
Putting things together gives the middle matrix representation

\[ p_{xx}(a, x)[h] + \lambda p_{x}(a, x)[h]^T p_{x}(a, x)[h] \]

\[ = \sum_{i,j=0}^{d-1} (I_\kappa \otimes V_i(a, x)[h])^T (3_\lambda)_{ij}(a, x)(I_\kappa \otimes V_j(a, x)[h]), \]

where

\[ (3_\lambda)_{ij}(a, x) = \begin{cases} 3_{ij}(a, x) + \lambda \Phi_i(a, x)^T \Phi_j(a, x) & i, j \leq d - 2 \\ \lambda \Phi_i(a, x)^T \Phi_j & \text{otherwise} \end{cases} \]

and the \( 3_{ij}(a, x) \) are the block entries of the middle matrix of the Hessian of \( p \) with respect to the full border vector \( \tilde{V} \). As with the Hessian, the middle matrix for the modified Hessian is uniquely determined by the choice of border vector and is symmetric if \( p \) is.

Finally, the **middle matrix for the relaxed Hessian**, denoted \( 3_{\lambda, \delta} \) (the notational conflict between \( 3_{ij} \), the \( (i, j) \) block entry of the middle matrix \( 3 \) of the Hessian of \( p \), and \( 3_{\lambda, \delta} \) should cause no confusion) is obtained from the middle matrix for the relaxed Hessian by simply adding \( \delta I \),

\[ 3_{\lambda, \delta} = 3_\lambda + \delta I. \]

**7.4. The reduced border vector.** The middle matrix for the modified and relaxed Hessians based upon the full border vector \( \tilde{V} \) has rows and columns of zeros corresponding to words that do not appear in the right chip set of \( p \). Let \( \mathcal{U}_j(a, x)[h] \) denote the (column) vector of words \( h_k f(a, x) \) for \( 1 \leq k \leq g \) and \( f \) in the right chip set of \( p \) of degree \( j \) in \( x \) and let

\[ \tilde{\mathcal{U}}(a, x)[h] = \begin{bmatrix} \mathcal{U}_0 \\ \vdots \\ \mathcal{U}_{d-1} \end{bmatrix}. \]

We refer to \( \tilde{\mathcal{U}} \) also as the **reduced border vector**\(^5\) (Compare with equation (5.2).) Indeed, \( \tilde{\mathcal{U}} \) includes only those words needed to construct a middle matrix-border vector representation for the modified and relaxed Hessians of \( p \). The middle matrix, still denoted \( 3 \), for the reduced border vector is obtained from the middle matrix associated to \( V \) by removing rows and columns of zeros - and applying a permutation if needed. For instance, the border vector for the border vector middle

\(^5\)Really it is a reduced border vector as any two reduced border vectors are related by a permutation. The vector obtained by excluding the words of degree \( d - 1 \) in \( x \) is also called the reduced border vector.
matrix representation appearing in Example 2.1] is the reduced border vector based on the right chip set of \( p(a, x) = x_2^3 a x_1 + x_1 a x_2^2 + a^2 \).

If the aim is to construct the most parsimonious middle matrix representation for the Hessian of \( p \), then one is led to use a border vector built from the secondary right chip set of \( p \) i.e., the set of words \( v(a, x) \) that appear to the right of at least two \( x \)'s in a word

\[
w(a, x) = w_L(a, x) x_j w_M(a, x) x_i v(a, x).
\]

that appears in \( p \). Thus, for example, the secondary right chip set of the word \( w(a, x) = a_1 x_1^2 a_2 x_2 x_1 a_1 \) is the set of words \( \{a_1, x_1 a_1, a_2 x_2 x_1 a_1\} \).

As a mnemonic, the chip set is associated to the first derivative and the secondary chip set the second derivative.

The proof of the following proposition is immediate from the preceding discussion.

**Proposition 7.1.** If \( 3_\lambda, 3_\lambda, 3_\lambda, 3_\lambda \) are the middle matrices for the Hessian, modified Hessian and relaxed Hessian of \( p \) based upon two different border vectors and if \( (A, X) \in S(\mathbb{R}^g) \), then \( 3(A, X) \) and \( \tilde{3}(A, X) \) (resp., \( 3_\lambda(A, X) \) and \( \tilde{3}_\lambda(A, X) \); \( 3_\lambda, 3_\lambda \) and \( \tilde{3}_\lambda, \tilde{3}_\lambda \)) have the same number of (strictly) positive and (strictly) negative eigenvalues. Thus, \( 3(A, X) \) (resp. \( 3_\lambda(A, X) \), \( \tilde{3}_\lambda(A, X) \)) is positive semi-definite if and only if \( \tilde{3}(A, X) \) (respectively \( \tilde{3}_\lambda(A, X) \), \( \tilde{3}_\lambda(A, X) \)) is.

**Remark 7.2.** In view of Proposition 7.1, we often do not make a notational distinction between choices of the middle matrix based on different choices of border vector. In what follows, typically we prove results first for the full border vector(s) where the bookkeeping is more easily automated and then establish the result for other choices, most notably the reduced border vector(s).

7.5. An example of a middle matrix representation for the modified Hessian. Suppose that \( C \in \mathbb{R}^{k \times k} \) and let

\[
p(a, x) = C a_1 x_1 a_2 x_2^2 + C^T x_2^2 a_2 x_1 a_1.
\]

Its right chip set is given by

\[
\mathcal{RC}_p^1 = \{a_1, a_2 x_2^2\} \quad \text{and} \quad \mathcal{RC}_p^2 = \{1, x_2, a_2 x_1 a_1, x_2 a_2 x_1 a_1\}.
\]

and thus in this case

\[
\tilde{\mathcal{U}}(a, x)[h] = \text{col}(h_1 a_1, h_2, h_2 x_2, h_2 a_2 x_1 a_1, h_1 a_2 x_2^2, h_2 x_2 a_2 x_1 a_1),
\]

\[
\mathcal{U}(a, x)[h] = \text{col}(h_1 a_1, h_2, h_2 x_2, h_2 a_2 x_1 a_1),
\]

are the reduced border vectors.
By direct computation,
\[ p_x(a, x)[h] = C \otimes \{ a_1 h_1 a_2 x_2^2 + a_1 x_1 a_2 h_2 x_2 + a_1 x_1 a_2 h_2^2 \} \]
\[ + C^T \otimes \{ h_2 x_2 a_2 x_1 a_1 + x_2 h_2 a_2 x_1 a_1 + x_2^2 a_2 h_1 a_1 \} \]
\[ = (C \otimes a_1)(I_\kappa \otimes h_1 a_2 x_2^2) + (C \otimes a_1 x_1 a_2)(I_\kappa \otimes h_2 x_2) \]
\[ + (C \otimes a_1 x_1 a_2 x_2)(I_\kappa \otimes h_2) + (C^T \otimes 1)(I_\kappa \otimes h_2 x_2 a_2 x_1 a_1) \]
\[ + (C^T \otimes x_2)((I_\kappa \otimes h_2 a_2 x_2 a_1) + (C^T \otimes x_2^2 a_2)(I_\kappa \otimes h_1 a_1) \]

which is of the form
\[ p_x(a, x)[h] = [G_1(a, x) \cdots G_6(a, x)] (I_\kappa \otimes \tilde{\Sigma}(a, x)[h]). \]

Correspondingly \( p_x(A, X)[h] \) can be written as
\[
(7.11) \\
p_x(A, X)[h] = \begin{bmatrix}
I_\kappa \otimes I_n & I_\kappa \otimes A_1 & I_\kappa \otimes X_2 & I_\kappa \otimes A_1 X_1 A_2 & I_\kappa \otimes X_2^2 A_2 & I_\kappa \otimes A_1 X_1 A_2 X_2 \\
0 & 0 & 0 & 0 & C \otimes I_n & 0 \\
0 & 0 & 0 & C \otimes I_n & C^T \otimes I_n & 0 \\
0 & 0 & C \otimes I_n & 0 & 0 & 0 \\
C \otimes I_n & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix}
I_\kappa \otimes H_2 \\
I_\kappa \otimes H_1 A_1 \\
I_\kappa \otimes H_2 X_2 \\
I_\kappa \otimes H_2 A_2 X_1 A_1 \\
I_\kappa \otimes H_1 A_2 X_2^2 \\
I_\kappa \otimes H_2 A_2 X_1 A_1 \\
\end{bmatrix}
\]

and
\[
(7.12) \\
p_{xx}(A, X)[h] = 2 \begin{bmatrix}
I_\kappa \otimes H_2 & I_\kappa \otimes A_1 H_1 & I_\kappa \otimes X_2 H_2 & I_\kappa \otimes A_1 X_1 A_2 H_2 \\
C \otimes A_2 X_2 & 0 & C \otimes A_2 & 0 \\
0 & C \otimes A_2 & 0 & 0 \\
C \otimes I_n & 0 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix}
I_\kappa \otimes H_2 \\
I_\kappa \otimes H_1 A_1 \\
I_\kappa \otimes H_2 X_2 \\
I_\kappa \otimes H_2 A_2 X_1 A_1 \\
\end{bmatrix}
\]

Thus the middle matrix for the Hessian of \( p \) (based on the reduced border vector) is the square matrix in formula (7.12). Likewise, the middle matrix for the modified Hessian of \( p \) corresponding to \( \tilde{\Sigma} \) is
\[
\begin{bmatrix}
0 & C^T \otimes X_2 A_2 & C^T \otimes I_n & 0 & 0 \\
C \otimes A_2 X_2 & 0 & C \otimes A_2 & 0 & 0 \\
0 & C \otimes A_2 & 0 & 0 & 0 \\
C \otimes I_n & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} + \lambda \Phi^T \Phi,
\]
where Φ is the product of the first two matrices on the right hand side of formula (7.11).

7.6. A Structure theorem for the middle matrix of the Hessian. In this subsection, we state our main result describing the middle matrix for the Hessian and modified Hessian needed for the proof of Theorem 2.5.

**Theorem 7.3.** Suppose \( p \) is a \( \kappa \times \kappa \) matrix polynomial and let \( Z \) and \( Z_\lambda \) denote the middle matrices of the Hessian of \( p \) and the modified Hessian of \( p \) with respect to (the same) fixed choice of border vector. Given \( (A,X) \in S_n(\mathbb{R}^g) \), there exists an invertible matrix \( S \) such that for all \( \lambda \in \mathbb{R} \),

\[
Z_\lambda(A,X) = \lambda \alpha + S^T \beta S
\]

where

\[
\alpha = \begin{bmatrix} U(I - \Pi_{W^T W}) U^T & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix} Z(A,0) & 0 \\ 0 & \lambda W W^T \end{bmatrix}
\]

and \( \Pi_{W^T W} \) is the projection onto the range of \( W^T W \). In particular, if the highest degree terms of \( p \) majorize at \( A \), then range of \( U^T \) is contained in the range of \( W^T \) and therefore \( Z_\lambda(A,X) \) is similar to

\[
\begin{bmatrix} Z(A,0) & 0 \\ 0 & \lambda W W^T \end{bmatrix}.
\]

**Proof.** The proof of this theorem occupies Subsection 8.7.

Since it is of independent interest, illustrates both the structure of the middle matrix \( a \) and provides the opportunity to introduce notations used in the remainder of this article, we conclude this section with the statement of the following scalar version of a theorem from Section 8. It is a variation on Theorem 7.3. To state the result, let \( k_b \) denote the number of words in \( a \) of length at most \( \tilde{d} \), \( t_j \) is the number of terms (words) in \( V_j \) (namely \( g \) times the number of words of the form (7.5)), and \( t \) is the number of words of the form (7.5) when \( j = d \). Thus,

\[
(7.13) \quad k_b = 1 + \tilde{g} + \cdots + \tilde{g}^{\tilde{d}}, \quad t_j = (k_b g)^{j+1}, \quad t = k_b t_{d-1}.
\]

We note that \( t_j t_\ell = t_{j+\ell+1} \).

Let \( K_j = c \otimes I_j \), where \( c \) denotes the column vector \((x_i u(a))\) parametrized over \( 1 \leq i \leq g \) and \( u \in U \), the collection of words in \( a \) of length at most \( \tilde{d} \). Thus the size of \( K_j \) is \( t_{j+1} \times t_j \).
(7.14) \[ K(a, x) = \begin{bmatrix}
I_{s_0} & 0 & \cdots & 0 & 0 \\
-K_0 & I_{t_1} & \cdots & 0 & 0 \\
0 & -K_1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & I_{u_{d-3}} & 0 \\
0 & 0 & \cdots & -K_{d-3} & I_{t_{d-2}}
\end{bmatrix}, \]

**Theorem 7.4.** Let \( \ell = d - 2 \). The middle matrix of the Hessian \( p_{xx}(a, x)[h] = \sum_{i,j=0}^{d-2} V_i(a, x)[h]Z_{ij}(a, x)V_j(a, x)[h] \) of \( p \) is of the form

\[ Z(a, x) = \begin{bmatrix}
Z_{00}(a, x) & Z_{01}(a, x) & \cdots & Z_{0,\ell-1}(a, x) & Z_{0\ell}(a, 0) \\
Z_{10}(a, x) & Z_{11}(a, x) & \cdots & Z_{1,\ell-1}(a, 0) & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
Z_{\ell-1,0}(a, x) & Z_{\ell-1,1}(a, 0) & \cdots & 0 & 0 \\
Z_{00}(a, 0) & 0 & \cdots & 0 & 0
\end{bmatrix}
\]

\[ = Z(a, 0)K(a, x)^{-1}. \]

From Theorem 7.4 it is evident that definiteness (either positive or negative) of the middle matrix of a Hessian imposes serious restrictions on the middle matrix that we will later exploit.

7.7. The polynomial congruence for the middle matrix of the Hessian. The proof of Theorem 7.4 depends essentially upon the following congruence result, which also plays an essential role in the proof of Theorem 2.5.

Given a matrix nc polynomial \( Y = (Y_{ij})_{i,j=1}^{s,t} \) where the \( Y_{ij} \) are \( n_j \times m_j \) matrices and a positive integer \( \kappa \), consistent with the usage in equation (7.8),

\[ Y^\kappa_{\text{deg}} = (I_\kappa \otimes Y_{ij})_{i,j=1}^{s,t}. \]

We note that \( I_\kappa \otimes Y \) differs from \( Y^\kappa_{\text{deg}} \) by a (block matrix) permutation as \( I_\kappa \otimes Y \) is the block \( \kappa \times \kappa \) diagonal block matrix with diagonal entries \( Y \), whereas \( Y^\kappa_{\text{deg}} \) is a block \( s \times t \) matrix with \( \kappa n_j \times \kappa m_j \) entries \( I_\kappa \otimes Y_{ij} \). Moreover, in the case of the full border vectors \( V \) and \( \tilde{V} \), the block matrices \( V^\kappa_{\text{deg}}(a, x)[h] \) and \( \tilde{V}^\kappa_{\text{deg}}(a, x)[h] \) are sorted by (increasing) degree in \( h \) (whereas \( I_\kappa \otimes \tilde{V} \) is the block diagonal \( \kappa \times \kappa \) matrix with \( \tilde{V} \) as the
diagonal entries). For instance, in the case \( d = 4 \) so that \( d - 2 = 2 \),

\[
I_2 \otimes V = \begin{pmatrix}
(V_0 & 0 \\
(V_1 & 0 \\
(V_2 & 0 \\
(0 & V_0 \\
(0 & V_1 \\
(0 & V_2 \\
\end{pmatrix},
\]

and

\[
V^2 = \begin{pmatrix}
(I_2 \otimes V_0 \\
(I_2 \otimes V_1 \\
(I_2 \otimes V_2 \\
\end{pmatrix} = \begin{pmatrix}
(V_0 & 0 \\
(0 & V_0 \\
(0 & V_1 \\
(0 & V_1 \\
(V_0 & 0 \\
(V_1 & 0 \\
(V_2 & 0 \\
(0 & V_2 \\
\end{pmatrix}.
\]

**Theorem 7.5.** Fix a collection of words \( W \) of degree at most \( d - 2 \) in \( x \) and \( \hat{a} \) in \( a \). Let \( B \) denote a border vector determined by \( W \); that is, \( B_j, j = 0, \ldots, d-2 \), the \( j \)-th block entry of \( B \) lists all words of the form \( h_k f(a,x) \) for \( 1 \leq k \leq g \) and \( f \in W \) of degree \( j \) in \( x \). Let \( \nu_j \) denote the length of \( B_j \).

There exists a matrix polynomial \( Y(a,x) = (Y_{ij})_{i,j=0}^{d-2} \) with entries \( Y_{ij} \) of size \( \nu_i \times \nu_j \) such that

(i) \( Y_{ii} = I_{\nu_i} \);
(ii) \( Y_{ij} = 0 \) for \( i < j \);
(iii) \( Y \) is invertible;
(iv) \( Y^{-1} \) is a polynomial; and
(v) if \( p \in \mathcal{P}^{\kappa \times \kappa} \) is a nc matrix polynomial whose right chip set is a subset of \( W \), then the middle matrix \( Z \) for the Hessian \( p_{xx} \) based upon \( B \) satisfies

\[
Z(a,x) = Y_{\deg(x)}(a,x)^T Z(a,0) Y_{\deg(x)}(a,x);
\]

i.e., \( Z(a,x) \) is polynomially congruent to \( Z(a,0) \) via a polynomial that depends only upon the choices \( W \) and \( B \).

**Proof.** Items (iii) and (iv) follow immediately from items (i) and (ii). The construction of \( Y \) and the proof of the remaining items is carried out in Section 8 and concludes in Subsection 8.6. \( \square \)

Theorem 7.5 and Theorem 7.3 are needed for the proof of Theorem 2.5. Their proofs proceed by direct calculation and appear in Section 8. The reader who is willing to accept the theorems in this section can skip to Section 9.
8. The proofs of Theorems 7.5 and 7.3

This section begins by introducing a Kronecker product formalism for free polynomials. (See Subsection 8.1.) In terms of this formalism, clean and useful formulas for derivatives and Hessians and their middle matrix representations are developed in Subsections 8.2 and 8.4. These formulas are applied in Subsections 8.6 and 8.7 to prove Theorems 7.5 and 7.3 respectively.

8.1. Extending the Kronecker product. Recall the Kronecker product of matrices $A$ and $B$ with real entries is, by definition,

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1q}B \\ \vdots & \vdots & & \vdots \\ a_{p1}B & a_{p2}B & \cdots & a_{pq}B \end{bmatrix}.$$  

In particular, the Kronecker product is a coordinate dependent concrete interpretation of the abstract construction of the tensor product of matrices. It will be convenient to extend the Kronecker product to allow one or both of the matrices to have free polynomials as entries. A number of the resulting identities that will play a central role in later calculations are collected in this section for easy future access.

If $a$ is a free polynomial and $B = (b_{jk})$ is a matrix whose entries are free polynomials, interpret $aB$ as the matrix $(ab_{jk})$. With this convention the Kronecker product extends, via equation (8.1) to matrices whose entries are free polynomials.

Because of the lack of commutativity, the Kronecker product of matrices of polynomials does not enjoy all the properties of the Kronecker product of ordinary matrices. For instance, if $C \in \mathbb{R}^{p \times q}$, $D \in \mathbb{R}^{q \times r}$ and $Y$ and $Z$ are matrix polynomials of sizes $s \times t$ and $t \times u$, respectively, then

$$CD \otimes YZ$$

These identities fail, however, if $C, D, Y$ and $Z$ are all arrays of nc variables. Indeed, for free scalar polynomials $c, y, d, z$, the polynomials $(c \otimes y)(d \otimes z)$ is $cydz$, but the polynomial $cd \otimes yz$ is $cdyz$.

As for the transpose, consider the example

$$\left( \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \otimes \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \right)^T = \begin{bmatrix} z_1^T y_1^T & z_2^T y_1^T & z_3^T y_1^T \\ z_1^T y_2^T & z_2^T y_2^T & z_3^T y_2^T \end{bmatrix}.$$  

6The exposition here is for the case of symmetric variables.
whereas
\[
\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix}^T \otimes \begin{bmatrix}
z_1 \\
z_2 \\
z_3
\end{bmatrix}^T = \begin{bmatrix}
y_1^T z_1^T & y_1^T z_2^T & y_1^T z_3^T \\
y_2^T z_1^T & y_2^T z_2^T & y_2^T z_3^T
\end{bmatrix}.
\]

On the other hand,
\[
(8.3) \quad \left(\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix} \otimes \begin{bmatrix}
z_1 \\
z_2 \\
z_3
\end{bmatrix}\right)^T = \left(\begin{bmatrix}
z_1 \\
z_2 \\
z_3
\end{bmatrix}^T \otimes \begin{bmatrix}
y_1 \\
y_2
\end{bmatrix}\right) \Pi
\]
for a suitably chosen $6 \times 6$ permutation matrix $\Pi$.

Two other useful formulas for nc polynomials $y_1, \ldots, y_s, z_1, \ldots, z_t$ are:
\[
\begin{bmatrix}
y_1 & \cdots & y_s
\end{bmatrix} \otimes \begin{bmatrix}
z_1 & \cdots & z_r
\end{bmatrix} = \begin{bmatrix}
y_1 & \cdots & y_s
\end{bmatrix} \left(I_s \otimes \begin{bmatrix}
z_1 & \cdots & z_r
\end{bmatrix}\right)
\]
and
\[
(8.4) \quad \begin{bmatrix}
y_1 \\
\vdots \\
y_s
\end{bmatrix} \otimes \begin{bmatrix}
z_1 \\
\vdots \\
z_r
\end{bmatrix} = \begin{bmatrix}
y_1 \\
\vdots \\
y_s
\end{bmatrix} \left(I_r \otimes \begin{bmatrix}
z_1 \\
\vdots \\
z_r
\end{bmatrix}\right).
\]

If $j, k, \ell$ and $m$ are positive integers, $u_1, \ldots, u_j, y_1, \ldots, y_\ell$ are nc polynomials and $c_1, \ldots, c_k$ are real numbers, then
\[
(8.5) \quad \left(\begin{bmatrix}
u_1 \\
\vdots \\
u_j
\end{bmatrix} \otimes I_m\right) \left(\begin{bmatrix}
y_1 \\
\vdots \\
y_\ell
\end{bmatrix} \otimes I_m\right) = \begin{bmatrix}
u_1 \\
\vdots \\
u_j
\end{bmatrix} \otimes I_m \begin{bmatrix}
y_1 \\
\vdots \\
y_\ell
\end{bmatrix} \otimes I_m
\]
and
\[
(8.6) \quad \begin{bmatrix}
c_1 & \cdots & c_m
\end{bmatrix} \left(\begin{bmatrix}
y_1 \\
\vdots \\
y_\ell
\end{bmatrix} \otimes I_m\right) = \begin{bmatrix}
y_1 & \cdots & y_\ell
\end{bmatrix} \begin{bmatrix}
c_1 & \cdots & c_m \\
c_{m+1} & \cdots & c_{2m} \\
\vdots & \ddots & \vdots \\
c_v & \cdots & c_{m+\ell}
\end{bmatrix}
\]
with $v = (\ell - 1)m + 1$. Given the column vector $y$ as above, let
\[
\mathrm{row}(y) = \begin{bmatrix}
y_1 \\
\vdots \\
y_\ell
\end{bmatrix}
\]
and, given positive integers $m$ and $\ell$, let $\text{mat}(\ell, m; \cdot)$ denote the linear map from row vectors $r = \mathrm{row}(r_1, \ldots, r_{\ell})$ with components $r_j$ that are row vectors of length $m$, to matrices of size $\ell \times m$
\[
(8.7) \quad \text{mat}(\ell, m; r) = \begin{bmatrix}
r_1 \\
\vdots \\
r_{\ell}
\end{bmatrix}.
\]
With these notations, formula (8.6) can be expressed in terms of the row vector $c = [c_1 \cdots c_ml]$ as

$$c \left( \begin{bmatrix} y_1 \\ \vdots \\ y_t \end{bmatrix} \otimes I_m \right) = \text{row}(y)\text{mat}(\ell, m; c).$$

As a special case of the identity of formula (8.2), if $C \in \mathbb{R}^{n \times n}$ and $X$ is an $n \times n$ array of nc polynomials, then

$$(C \otimes I_n)(I_n \otimes X) = (I_n \otimes X)(C \otimes I_n) = C \otimes X.$$  

If $X_{ij}$ are compatibly sized arrays of nc polynomials, then

$$I_n \otimes \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \end{bmatrix} = \Pi \begin{bmatrix} I_n \otimes X_{11} & I_n \otimes X_{12} & I_n \otimes X_{13} \\ I_n \otimes X_{21} & I_n \otimes X_{22} & I_n \otimes X_{23} \end{bmatrix} \Pi'$$

for suitably chosen permutations $\Pi$ and $\Pi'$ (that serve to interchange block rows and block columns, respectively).

### 8.2. NC polynomials in Kronecker notation.

For a column vector $y = \text{col}(y_1, \ldots, y_k)$, let

$$\begin{bmatrix} y_1 \\ \vdots \\ y_k \end{bmatrix}_0 = 1 \quad \text{and} \quad \begin{bmatrix} y_1 \\ \vdots \\ y_k \end{bmatrix}_j = \begin{bmatrix} y_1 \\ \vdots \\ y_k \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} y_1 \\ \vdots \\ y_k \end{bmatrix}.$$

Thus $\begin{bmatrix} y_1 \end{bmatrix}_j$ is a $j$-fold product for $j = 1, 2, \ldots$. The notation reflects the fact that the product(s) are associative. In particular $\begin{bmatrix} y_1 \end{bmatrix}_j \otimes \begin{bmatrix} y_1 \end{bmatrix}_k = \begin{bmatrix} y_1 \end{bmatrix}_{j+k}$.

Let

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_g \end{bmatrix}, \quad b = \text{col} \left\{ 1, \begin{bmatrix} a_1 \\ \vdots \\ a_\bar{g} \end{bmatrix}, \begin{bmatrix} a_1 \\ \vdots \\ a_\bar{g} \end{bmatrix}_2, \ldots, \begin{bmatrix} a_1 \\ \vdots \\ a_\bar{g} \end{bmatrix}_d \right\}.$$

In particular, $\begin{bmatrix} x \otimes b \end{bmatrix}_{j+1}$ has length $t_j$, where $t_j$ (and also $t$ appearing below) is defined in equation (1.13). If $p(a, x) = p^d(a, x)$ is an nc polynomial of degree at most $\bar{d}$ in $a$ and homogeneous of degree $d$ in $x$, then it admits a representation of the form

$$p^d(a, x) = c_p^d(b \otimes (x \otimes b)_d) = \begin{bmatrix} c_1 \cdots c_l \end{bmatrix} \begin{bmatrix} b \otimes (x \otimes b)_d \end{bmatrix},$$

where $c_p^d$ is a row vector of length $t = k_b(k_bg)^d$ (the number of words (entries) in $b \otimes (x \otimes b)_d$).

Letting

$$\varphi_p^d(a) = c_p^d(b \otimes I_{t_d-1}) = \begin{bmatrix} c_1 \cdots c_l \end{bmatrix} \begin{bmatrix} b \otimes I_{t_d-1} \end{bmatrix}$$
(a polynomial in \(a\) alone), formula (8.9) can be re-expressed as

\[
(8.10) \quad p^d(a, x) = \varphi^d_p(a) (x \otimes b)_d.
\]

Let

\[
h = \text{col}(h_1, \ldots, h_g).
\]

With this notation, the vector \(V_j\) (the portion of the full border vector homogeneous of degree \(j\) in \(x\)) is given by

\[
(8.11) \quad V_j = p^d(a, x)[h] = h \otimes b \otimes (x \otimes b)_j.
\]

Correspondingly, from the representation (8.9) and the product rule,

\[
(8.12) \quad p^d(a, x)[h] = c_p^d \left( \sum_{j=0}^{d-1} b \otimes (x \otimes b)_{d-1-j} \otimes V_j(a, x)[h] \right).
\]

8.3. Computing the middle matrix of the Hessian. In this section we shall present a transparent example that exhibits the main features of the calculation of the middle matrix representation of \(p_{xx}^d(a, x)[h]\). Suppose \(p(a, x)\) is an nc polynomial that is homogeneous of degree \(d\) in \(x\) and that \(p\) is expressed in the form (8.9) (see also (8.10)).

**Example 8.1.** Suppose that \(c \in \mathbb{R}\) and

\[
p(a, x) = c a_1 x_1 a_2 x_2^2.
\]

Since \(p\) is a homogeneous polynomial of degree \(d = 3\) in \(x\) and is of degree \(\tilde{d} = 2\) in \(a\) and there are no consecutive strings of \(a\), it suffices to choose\(^7\)

\[
b = \begin{bmatrix} 1 \\ a_1 \\ a_2 \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.
\]

In this case \(k_b\), the number of entries in \(b\), is equal to 3 and \(p(a, x)\) is \(c\) times one of the entries in the vector polynomial \(b \otimes (x \otimes b)_3\) of height \(t = k_b(k_b g)^d = 3(6^3) = 648\), i.e.,

\[
p(a, x) = \begin{bmatrix} c_1 & \cdots & c_t \end{bmatrix} b \otimes (x \otimes b)_3,
\]

\(^7\)Here, because there are no consecutive strings of \(a\) of length \(> 1\), it is enough to use the full border vector based on words in \(a\) of degree at most one, rather than that based on words in \(a\) of degree at most two.
where one of the coefficients $c_1, \ldots, c_t \in \mathbb{R}$ is equal to $c$ and the remaining $t - 1$ coefficients are equal to zero. It is readily checked that

$$p_x(a, x)[h] = [c_1 \cdots c_t] b \otimes \{(h \otimes b) \otimes (x \otimes b)\}
+ (x \otimes b) \otimes (h \otimes b) \otimes (x \otimes b) + (x \otimes b) \otimes (h \otimes b).$$

and

$$p_{xx}(a, x)[h] = 2 [c_1 \cdots c_t] b \otimes \{(h \otimes b) \otimes (x \otimes b)
+ (x \otimes b) \otimes (h \otimes b) \otimes (x \otimes b) + (h \otimes b) \otimes (x \otimes b) \otimes (h \otimes b)\}.$$

Thus, in terms of the notation $V_0 = h \otimes b$ and $V_j = (h \otimes b) \otimes (x \otimes b)$ for $j = 1, 2, \ldots,$

$$p_x(a, x)[h] = [c_1 \cdots c_t] 
[b \otimes \{V_2 + (x \otimes b) \otimes V_1 + (x \otimes b) \otimes V_0\}]$$

and

$$p_{xx}(a, x)[h] = 2 [c_1 \cdots c_t] 
[b \otimes \{V_0 \otimes V_1 + (x \otimes b) \otimes V_0 \otimes V_1 + V_0 \otimes V_0\}].$$

(8.13)

The first step in the computation of the blocks $Z_{ij}(a, x)$ in the middle matrix representation (7.1) of the Hessian is to invoke formula (8.4) in order to re-express the term $b \otimes \{V_0 \otimes V_1 + (x \otimes b) \otimes V_0 \otimes V_1 + V_0 \otimes V_0\}.$

in formula (8.13) as

$$[b \otimes V_0 \otimes I_{t_1}] V_1 + [b \otimes x \otimes b \otimes V_0 \otimes I_{t_0}] V_0 + [V_1 \otimes I_{t_0}] V_0.$$

The contributions of each of these three summands will be evaluated separately:

1. **Verify the formula**

(8.14)

$$2 [c_1 \cdots c_t] [b \otimes V_0 \otimes I_{t_1}] = V_0^T Z_{01}$$

where

$$Z_{01} = 2 \Pi_0 \begin{bmatrix}
  c_1 & \cdots & c_{108} \\
  c_{109} & \cdots & c_{216} \\
  \vdots & \ddots & \vdots \\
  c_{541} & \cdots & c_{648}
\end{bmatrix} (b \otimes I_{t_1}),$$

$$\Pi_0 \in \mathbb{R}^{6 \times 6} \text{ is the permutation matrix defined by } b^T \otimes h^T = V_0^T \Pi_0,$$

$t = 648 \text{ and } t_1 = 36.$

To verify (8.14), note first that in view of formula (8.4),

$$b \otimes V_0 \otimes I_{t_1} = b \otimes h \otimes b \otimes I_{t_1} = (b \otimes h \otimes I_{k_{t_1}}) (b \otimes I_{t_1}).$$
To ease the notation, set
\[ b \otimes h = \begin{bmatrix} u_1 \\ \vdots \\ u_{k_b g} \\ u_6 \end{bmatrix} = \begin{bmatrix} u_1 \\ \vdots \\ u_6 \end{bmatrix}, \]
where, in the case at hand, \( t = 3 \times 6^3, t_1 = 6^2 \) and \( s = t/(k_b g) = k_b t_1 = 108 \). Since \( c_i u_j = u_j c_i \),
\[
\begin{bmatrix} c_1 & \cdots & c_t \end{bmatrix} (b \otimes h \otimes I_s) = \begin{bmatrix} c_1 & \cdots & c_t \end{bmatrix} \begin{bmatrix} u_1 I_{108} \\ \vdots \\ u_{6 I_{108}} \end{bmatrix} = \begin{bmatrix} u_1 & \cdots & u_6 \end{bmatrix} \begin{bmatrix} c_1 & \cdots & c_{108} \\ c_{109} & \cdots & c_{216} \\ \vdots & \vdots & \vdots \\ c_{541} & \cdots & c_{648} \end{bmatrix}
\]
\[
= b^T \otimes h^T \begin{bmatrix} c_1 & \cdots & c_{108} \\ c_{109} & \cdots & c_{216} \\ \vdots & \vdots & \vdots \\ c_{541} & \cdots & c_{648} \end{bmatrix} = V_1^T \Pi_0 \begin{bmatrix} c_1 & \cdots & c_{108} \\ c_{109} & \cdots & c_{216} \\ \vdots & \vdots & \vdots \\ c_{541} & \cdots & c_{648} \end{bmatrix},
\]
which, upon combining terms, leads easily to (8.14).

2. Verify the formula
\[
2 \begin{bmatrix} c_1 & \cdots & c_t \end{bmatrix} b \otimes ((x \otimes b) \otimes V_0 \otimes I_{t_0}) = V_1^T Z_{10}.
\]
where
\[
Z_{10} = 2\Pi_1 \begin{bmatrix} c_1 & \cdots & c_{18} \\ c_{19} & \cdots & c_{36} \\ \vdots & \vdots & \vdots \\ c_{631} & \cdots & c_{648} \end{bmatrix} (b \otimes I_{t_0}),
\]
\( \Pi_1 \in \mathbb{R}^{36 \times 36} \) is the permutation matrix defined by the formula
\[
b^T \otimes x^T \otimes b^T \otimes h^T = V_1^T \Pi_1
\]
and \( t_0 = 6 \).

To verify (8.15), first invoke (8.4) to obtain
\[
b \otimes x \otimes b \otimes V_0 \otimes I_{t_0} = [(b \otimes x) \otimes (b \otimes h) \otimes I_{k_b t_0}] [b \otimes I_{t_0}]
\]
and set
\[
(b \otimes x) \otimes (b \otimes h) = \begin{bmatrix} u_1 \\ \vdots \\ u_{(k_b g)^2} \end{bmatrix} = \begin{bmatrix} u_1 \\ \vdots \\ u_{36} \end{bmatrix}.
\]
where again \( t = 3 \times 6^3, t_1 = 6^2 \) and \( s = t/(k_0 g)^2 = k_0 t_0 = 18 \). Then, since \( c_i u_j = u_j c_i \),

\[
\begin{bmatrix}
c_1 & \cdots & c_{648}
\end{bmatrix}
\begin{bmatrix}
u_1 \\
\vdots \\
u_{36}
\end{bmatrix} \otimes I_{18} =
\begin{bmatrix}
u_1 & \cdots & v_{18} \\

v_{19} & \cdots & v_{36} \\
\vdots & & \vdots \\
v_{631} & \cdots & v_{648}
\end{bmatrix}
\begin{bmatrix}
c_1 \\
\vdots \\
c_{18}
\end{bmatrix}
\begin{bmatrix}
c_{19} \\
\vdots \\
c_{36}
\end{bmatrix}
\begin{bmatrix}
c_{631} \\
\vdots \\
c_{648}
\end{bmatrix}.
\]

The verification of (8.15) is completed by noting that

\[
\begin{bmatrix}
u_1 & \cdots & v_{36}
\end{bmatrix} = b^T \otimes x^T \otimes b^T \otimes h^T = V_1^T \Pi_1
\]

and combining formulas.

3. Verify the formula

\[(8.16)\]

\[
2 \begin{bmatrix}
c_1 & \cdots & c_t
\end{bmatrix}
\begin{bmatrix}
u \otimes V_1 \otimes I_{t_0}
\end{bmatrix} = V_0^T Z_{00},
\]

where

\[
Z_{00} = 2 \Pi_0 \begin{bmatrix}
c_1 & \cdots & c_{648}
\end{bmatrix}
\begin{bmatrix}
u_1 I_{108} \\
\vdots \\
u_{6} I_{108}
\end{bmatrix} (b \otimes x \otimes b \otimes I_6).
\]

The verification of formula (8.16) is similar to the verification of (8.14). The main new ingredients are the formulas

\[
b \otimes V_1 \otimes I_{t_0} = (b \otimes h \otimes b \otimes x \otimes b) \otimes I_{t_0}
\]

\[
= (b \otimes h \otimes I_{k_0 t_0})(b \otimes x \otimes b \otimes I_{t_0})
\]

and, upon setting

\[
\begin{bmatrix}
u_1 \\
\vdots \\
u_{6}
\end{bmatrix} = b \otimes h,
\]

\[
\begin{bmatrix}
c_1 & \cdots & c_t
\end{bmatrix}
\begin{bmatrix}
u_1 I_{108} \\
\vdots \\
u_{6} I_{108}
\end{bmatrix} =
\begin{bmatrix}
c_1 & \cdots & c_{648}
\end{bmatrix}
\begin{bmatrix}
u_1 I_{108} \\
\vdots \\
u_{6} I_{108}
\end{bmatrix}
\begin{bmatrix}
c_1 \\
\vdots \\
c_{108}
\end{bmatrix}
\begin{bmatrix}
c_{541} & \cdots & c_{648}
\end{bmatrix},
\]

which lead easily to (8.16).
8.4. Formulas for the middle matrix of the Hessian of a homogeneous $p$. This section is devoted to the detailed computation of the middle matrix $(Z_{ij})$ of the Hessian (see equation (7.6)) of a scalar nc polynomial $p(a,x)$ that is homogeneous of degree $\ell$ in $x$.

In view of formulas (8.11) and (8.3),

\[
\text{row}(b \otimes (x \otimes b)_i \otimes h) = b^T \otimes (x^T \otimes b^T)_i \otimes h^T \Pi_i
\]

\[
= ((h \otimes b) \otimes (x \otimes b)_i)^T \Pi_i = V_i(a,x)[h]^T \Pi_i,
\]

(8.17)

for a suitably chosen permutation matrix $\Pi_i$ of size $t_i$.

If $p$ is homogeneous of degree $\ell$ in $x$, then the constant row vector $c^\ell_p$ defined in formula (8.9) has length $k_b t_{\ell-1} = k_b t_{\ell-i}$. Moreover, in terms of the notation $t^j = (x \otimes b)_j$,

\[
p_{xx}(a,x)[h] = 2c^\ell_p \left( b \otimes \left( \sum_{i+j+k=\ell-2} f^i \otimes (h \otimes b) \otimes f^k \otimes (h \otimes b) \otimes f^j \right) \right).
\]

Fix $i, j$ and $k = \ell - 2 - i - j$, set $s = k_b t_{\ell-i-2}$ and compute, using $\text{mat}$ as defined in equation (S.7) and formulas (8.17), (8.8) and (8.17) in that order,

\[
c^\ell_p \left( b \otimes f^i \otimes (h \otimes b) \otimes f^k \otimes (h \otimes b) \otimes f^j \right)
\]

\[
= c^\ell_p ((b \otimes f^i \otimes h) \otimes I_{k_b t_{\ell-2-i}}) ((b \otimes f^k) \otimes I_j) ((h \otimes b) \otimes f^j)
\]

\[
= \text{row}(b \otimes f^i \otimes h) \text{mat}(t_i, s; c^\ell_p) \left( (b \otimes f^k) \otimes I_j \right) ((h \otimes b) \otimes f^j)
\]

\[
= V_i(a,x)[h]^T \left[ (\Pi, \text{mat}(t_i, s; c^\ell_p)) \left( (b \otimes f^k) \otimes I_j \right) \right] V_j(a,x)[h].
\]

Hence, the $(i, j)$ block entry of the middle matrix of $p$ is given by

\[
Z_{i,j} = 2(\Pi, \text{mat}(t_i, s; c^\ell_p)) \left( (b \otimes f^{\ell-2-i-j}) \otimes I_j \right).
\]

On the other hand, the $(i, j+1)$ block is (so long as $\ell - 2 - i - j - 1 \geq 0$)

\[
Z_{i,j+1} = 2(\Pi, \text{mat}(t_i, s; c^\ell_p)) \left( (b \otimes f^{\ell-3-i-j}) \otimes I_{t_{j+1}} \right).
\]

Using formula (8.5) and the relation $t_{j+1} = t_j(gk_b)$ to justify the identity

\[
((b \otimes f^{\ell-3-i-j}) \otimes I_{t_{j+1}}) ((x \otimes b) \otimes I_{t_j}) = b \otimes f^{\ell-2-i-j} \otimes I_{t_j},
\]

it now follows that

\[
Z_{i,j}(a,x) = Z_{i,j+1}(a,x) ((x \otimes b) \otimes I_{t_j}) = Z_{i,j+1}(a,x)K_j(a,x).
\]

(8.18)

when $i + j < \ell - 2$, where

\[
K_j(a,x) = (x \otimes b) \otimes I_{t_j}
\]

is a matrix polynomial of size $t_{j+1} \times t_j$. 
The following proposition summarizes and expands a bit on this discussion.

**Proposition 8.2.** If $\ell \geq 2$, then

$$Z_{ij}(a, x) = \begin{cases} 0 & \text{if } i + j > \ell - 2 \\ Z_{ij}(a, 0) & \text{if } i + j = \ell - 2 \\ Z_{i,j+1}(a, x)K_j(a, x) & \text{if } i + j < \ell - 2 \end{cases}$$

Moreover, if $i + j = \ell - 2$, then

$$Z_{ij}(a, 0) = 2\Pi_{\text{mat}}(t_i, s; c_p^\ell)(b \otimes I_t) = 2\Pi_i \begin{bmatrix} c_1 & \cdots & c_s \\ c_{s+1} & \cdots & c_{2s} \\ \vdots & \vdots & \vdots \\ c_u & \cdots & c_t \end{bmatrix} (b \otimes I_t)$$

with $u = t - s + 1$ and $s = k_b t_{\ell-1-i}$.

The formulas in Proposition 8.2 can be expressed conveniently in terms of the matrix $K(a, x)$ defined in (7.14) as

$$Z(a, x) K(a, x) = Z(a, 0),$$

where $Z_{ij}(a, 0) = 0$ if $i + j \neq \ell - 2$.

If $i + j < \ell - 2$, then the formula for $Z_{ij}(a, x)$ in Proposition 8.2 may be iterated to obtain

$$Z_{ij}(a, x) = Z_{i,j+1}(a, x)K_j(a, x) = Z_{i,j+2}(a, x)K_{j+1}(a, x)K_j(a, x) = \cdots = Z_{i,\ell-2-i}(a, 0)K_{\ell-1-i}(a, x)\cdots K_j(a, x).$$

These formulas can be expressed in terms of the block matrix $L(a, x) = (L_{ij}(a, x))_{i,j=0}^{d-2}$ with entries

$$L_{ij} = \begin{cases} I_j & \text{if } i = j \\ K_{i-1} \cdots K_j = (x \otimes b)_{i-j} \otimes I_j & \text{if } i > j \\ 0_{t_i \times t_j} & \text{if } i < j \end{cases}$$

of size $t_i \otimes t_j$. In fact

(8.19) \hspace{1cm} L(a, x) K(a, x) = I.$$

In terms of this notation, formula (8.12) can be expressed as

$$p^d_x(a, x)[h] = \varphi^d_p(a) \left( \sum_{j=0}^{d-1} L_{d-1,j}(a, x) V_j(a, x)[h] \right).$$

Let $(K^\kappa_{\text{deg}})_{ij} = I_{\kappa} \otimes K_{ij}$. 
Theorem 8.3. Suppose \( d \geq \ell \geq 2 \). If \( p \) is a \( \kappa \times \kappa \) nc polynomial homogeneous of degree \( \ell \) in \( x \) and if \( \mathcal{Z} \) is the middle matrix of \( p \) with respect to the full border vector of formula (7.4), then
\[
\mathcal{Z}(a,x)K_{\deg}(a,x) = \mathcal{Z}(a,0),
\]
where \( K(a,x) \) is defined in formula (7.14).

Proof. Observe that since \( K(a,x) \) depends only upon the degree \( d \) of the polynomial \( p \) in \( x \) and not upon the words in \( p \), it suffices to prove the result for \( p = C \otimes w \), for \( C \) a \( \kappa \times \kappa \) matrix and \( w \) a word of length \( \ell \leq d \) in \( x \). Using the notation introduced in equation (7.3), the entries of the middle matrix of such a \( p \) have the form
\[
\mathcal{Z}_{ij} = C \otimes \mathcal{Z}_{ij}^w.
\]
Now apply the recursion (8.18) and the relation \( Z_{i,\ell-i-2}(a,x) = Z_{i,\ell-i-2}(a,0) \) from Proposition 8.2 to conclude that the result holds for \( C \otimes w \). \( \square \)

Remark 8.4. Theorem 8.3 is the matrix version of Theorem 7.4. Note too that \( K(a,x) \) is lower triangular with the identity on the diagonal and depends only upon \( d \) (and not on \( p \)). \( \square \)

8.5. Beyond homogeneous. In this section we illustrate how to extend Theorem 8.3 to polynomials that are not necessarily homogeneous. The main points and structure involved in this process are present in the scalar case (\( \kappa = 1 \)). Accordingly suppose \( p(a,x) \) is a polynomial that is homogeneous of degree 5 in \( x \). Proposition 8.2 implies that the entries \( Z_{ij}(a,x) \) in the middle matrix \( Z(a,x) \) in formula (7.6) with \( i + j \leq 2 \) can be expressed as
\[
Z_{00} = Z_{03}K_2K_1K_0, \ Z_{01} = Z_{03}K_2K_1, \ Z_{02} = Z_{03}K_2, \\
Z_{10} = Z_{12}K_1K_0, \ Z_{11} = Z_{12}K_1, \ Z_{20} = Z_{21}K_0.
\]
Equivalently,
\[
Z(a,x) = Z(a,0) \begin{bmatrix}
I_0 & 0 & 0 & 0 \\
K_0 & I_1 & 0 & 0 \\
K_1K_0 & K_1 & I_2 & 0 \\
K_2K_1K_0 & K_2K_1 & K_2 & I_3
\end{bmatrix},
\]
with \( K_j = K_j(a,x) \) as in (8.4) and
\[
Z(a,0) = \begin{bmatrix}
0 & 0 & 0 & Z_{03}(a,0) \\
0 & 0 & Z_{12}(a,0) & 0 \\
0 & Z_{21}(a,0) & 0 & 0 \\
Z_{30}(a,0) & 0 & 0 & 0
\end{bmatrix}.
\]
Moreover,

\[
L(a, x) = \begin{bmatrix}
I_0 & 0 & 0 & 0 \\
K_0 & I_1 & 0 & 0 \\
K_1K_0 & K_1 & I_{t_2} & 0 \\
K_2K_1K_0 & K_2K_1 & K_2 & I_{t_3}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
I_0 & 0 & 0 & 0 \\
-K_0 & I_{t_1} & 0 & 0 \\
0 & -K_1 & I_{t_2} & 0 \\
0 & 0 & -K_2 & I_{t_3}
\end{bmatrix}^{-1}
= K(a, x)^{-1}.
\]

Next we indicate how the basic recursion formulas (8.18) are extended to the middle matrix \(Z(a, x)\) in the representation of the Hessian \(p_{xx}(a, x)[h]\) of polynomials \(p(a, x)\) not constrained to be homogeneous in \(x\). We suppose now that \(p\) has degree four and express it as a sum of its homogeneous of degree \(0 \leq j \leq 4\) in \(x\) parts,

\[
p(a, x) = \sum_{j=0}^{4} p^j(a, x).
\]

In particular,

(8.20) \(p^j\) is a linear combination of the \(k_b(k_bg)^j\) entries in \(b \otimes (x \otimes b)_j\).

Next apply Proposition 8.2 to each term \(p^j(a, x)\) separately. The condition (8.20) insures that the same border vectors intervene in the representation of the Hessian \(p^j_{xx}\) for each choice of \(j\) and hence that the entries \(Z_{ij}(a, x)\) in the middle matrix satisfy the recursion (8.18) of equation (8.18). In particular,

\[
p_{xx}(a, x)[h] = p_{xx}^2(a, x)[h] + p_{xx}^3(a, x)[h] + p_{xx}^4(a, x)[h]
\]

\[
= V_0^T Z_0^2 V_0 + \begin{bmatrix} V_0^T \\ V_1^T \end{bmatrix} \begin{bmatrix} Z_0^3 & Z_0^3 \\ Z_{10}^3 & 0 \end{bmatrix} \begin{bmatrix} V_0 \\ V_1 \end{bmatrix}
\]

\[
+ \begin{bmatrix} V_0^T \\ V_1^T \end{bmatrix} \begin{bmatrix} Z_0^4 & Z_0^4 \\ Z_{10}^4 & Z_{11}^4 \\ Z_{20}^4 & 0 \end{bmatrix} \begin{bmatrix} V_0 \\ V_1 \\ V_2 \end{bmatrix}
\]

\[
= \begin{bmatrix} V_0^T \\ V_1^T \\ V_2^T \end{bmatrix} \{ \cdots \} \begin{bmatrix} V_0 \\ V_1 \\ V_2 \end{bmatrix},
\]
where the term in curly brackets is
\[
\{\cdots\} = \begin{bmatrix}
Z_{00}^2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
Z_{00}^3 & Z_{01}^3 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
Z_{00}^4 & Z_{01}^4 & Z_{02}^4 \\
0 & 0 & 0 \\
Z_{10}^4 & Z_{11}^4 & 0 \\
Z_{20}^4 & 0 & 0
\end{bmatrix}.
\]

On the other hand, the recursion (8.18) gives
\[
\{\cdots\} \begin{bmatrix}
1 & 0 & 0 \\
-K_0 & 1 & 0 \\
0 & -K_1 & 1
\end{bmatrix}^{-1} = \begin{bmatrix}
Z_{00}^2 & Z_{01}^2 & Z_{02}^2 \\
Z_{10}^2 & Z_{11}^2 & 0 \\
Z_{20}^2 & 0 & 0
\end{bmatrix} = Z(a, 0).
\]

8.6. **Polynomial congruence.** A pair of \(p \times p\) polynomial matrices \(F(a, x)\) and \(G(a, x)\) are **polynomially congruent**, denoted \(F \sim_p G\), if there exists a \(p \times p\) polynomial matrix \(R(a, x)\) such that \(R(a, x)^{-1}\) is a polynomial matrix and
\[
F(a, x) = R(a, x)G(a, x)R(a, x)^T.
\]

**Lemma 8.5.** Suppose \(X\) is an \(n \times n\) matrix-valued free polynomial and \(X^k = 0\) for some positive integer \(k\).

(1) The matrix polynomial
\[
Y = I_n + \sum_{j=1}^{k-1} \left(\frac{1}{j!}\right) X^j
\]
has a polynomial inverse.

(2) \(Y^2 = (I_n + X)\).

(3) If \(M\) is also a \(n \times n\) matrix-valued free polynomial and \(M = M^T\) and \(MX = X^TM\), then \(MY = Y^TM\).

(4) If \(N = XM\) is symmetric (\(N = N^T\)), then \(MY^2 = Y^TMY\). In particular, \(M\) and \(N\) are congruent.

**Proof.** That \(Y^2 = (I_n + X)\) follows from Newton’s generalized binomial theorem (and the fact that \(X^k = 0\)). Since \(X\) is nilpotent, \((I_n + X)\) has a polynomial inverse and thus \(Y(I + X)^{-1}\) is a polynomial inverse for \(Y\).

If \(MX = X^TM\), then
\[
MY = M(I_n + \sum_{j=1}^{k-1} \left(\frac{1}{j!}\right) X^j) = (I_n + \sum_{j=1}^{k-1} \left(\frac{1}{j!}\right) X^j)M = Y^TM
\]
and hence \(MY^2 = Y^TMY\). \qed

The next result is adapted from [DHM07a].
Theorem 8.6. If $K(a,x)$ is defined by formula (7.14), then $N(a,x) = K(a,x) - I$ is nilpotent of order $d - 1$ and the matrix polynomial
\[ Y(a,x) := I + \sum_{j=1}^{d-2} \left( \frac{1}{2} \right)^j N(a,x)^j \]
satisfies the conditions of Theorem 7.5 based upon the full border vector $V$. In particular $Y$ is invertible and its inverse is a polynomial.

Moreover, if $p$ is a symmetric $\kappa \times \kappa$ matrix polynomial of degree at most $d$ in $x$ and $Z$ is the middle matrix of its Hessian, then
\[ (Y_{\deg(a,x)}^\kappa (a,x))^T Z(a,x) (Y_{\deg(a,x)}^\kappa (a,x)) = Z(a,0) \]

Proof. We prove the case $\kappa = 1$. Since $p$ is assumed symmetric $Z(a,x) = Z(a,x)^T$. Hence, using Theorem 7.4 (see also Theorem 8.3),
\[ Z(a,x) K(a,x) = Z(a,0) = Z(a,0)^T = K(a,x)^T Z(a,x). \]

Therefore,
\[ Z(a,x) N(a,x) = N(a,x)^T Z(a,x). \]

Consequently, by Lemma 8.5 $Y^2(a,x) = I + N(a,x) = K(a,x)$,
\[ Z(a,0) = Z(a,x) Y(a,x)^2 = Y(a,x)^T Z(a,x) Y(a,x) \]

and $Y$ has a polynomial inverse. □

Remark 8.7. Theorem 8.6 covers Theorem 7.5 for the case of the full border vector $V$. □

Proof of Theorem 7.5. Let $V$ denote the full border vector and
\[ p_{xx}(a,x)[h] = V_{\deg(a,x)}^\kappa (a,x)[h]^T Z(a,x) V_{\deg(a,x)}^\kappa (a,x) \]
\[ = \sum_{j,k} (I_\kappa \otimes V_j(a,x)[h]^T) Z_{j,k}(a,x) (I_\kappa \otimes V_k(a,x)[h]) \]
the border-middle matrix representation for the Hessian of $p$. Fix a set of words $W$ and a corresponding border vector $B$ with (block) entry $B_j$ consisting of precisely of all the words of the form $h_i f(a,x)$ for $1 \leq i \leq g$ and $f \in W$ of degree $j$ in $x$. For each $k$ there is a natural inclusion map $\iota_k$ from the words of degree $k$ in $x$ in $W$ to the words of degree $k$ in $x$ in the full border vector $V_k$ such that
\[ B_k(a,x) = \Pi_k \iota_k^T V(a,x), \]
where $\Pi_k$ is a permutation matrix. Thus, writing $V_j$ for $V_j(a, x)[h]$ and letting $Q_k = \Pi_k^{j T}$,
\[
p_{xx}(a, x)[h] = \sum_{j,k} (I_\kappa \otimes V_j^T)(I_\kappa \otimes \iota_{j,k}^{j T})3_{j,k}(a, x)(I_\kappa \otimes \iota_{j,k}^{j T})(I_\kappa \otimes V_k)
\]
\[
= \sum_{j,k} (I_\kappa \otimes V_j^T)(I_\kappa \otimes Q_j^T)3_{j,k}(a, x)(I_\kappa \otimes Q_j^T Q_k)(I_\kappa \otimes V_k)
\]
\[
= \sum_{j,k} (I_\kappa \otimes V_j^T Q_j^T)(I_\kappa \otimes Q_j)3_{j,k}(a, x)(I_\kappa \otimes Q_j^T)(I_\kappa \otimes Q_j V_k)
\]
\[
= \sum_{j,k} (I_\kappa \otimes B_j(a, x)[h])\tilde{3}_{j,k}(a, x)(I_\kappa \otimes B_k(a, x)[h])
\]
where
\[
\tilde{3}_{j,k}(a, x) = (I_\kappa \otimes Q_j)3_{j,k}(a, x)(I_\kappa \otimes Q_j^T).
\]
Hence, letting $Q$ denote the block diagonal matrix with diagonal entries $I_\kappa \otimes Q_j$,
\[
\tilde{3}(a, x) = Q \tilde{3}(a, x)Q^T
\]
is the middle matrix in the border vector-middle matrix representation of $p_{xx}(a, x)[h]$ with respect to the border vector $B$.

Letting $\tilde{Y}_{j,k} = \Pi_j^{T}Y_{j,k}(a, x)\iota_{j}\Pi_k^{T} = Q_jY_{j,k}(a, x)Q_j^T$ and using Theorem 8.6,
\[
\tilde{3}(a, x) = Q3(a, x)Q^T
\]
\[
= QY^\kappa_{deg}(a, x)^T3(a, 0)Y^\kappa_{deg}(a, x)Q^T
\]
\[
= QY^\kappa_{deg}(a, x)^TQ^T(Q3(a, 0)Q^T)QY^\kappa_{deg}(a, x)Q^T
\]
\[
= \tilde{Y}^\kappa_{deg}(a, x)^T3(a, 0)\tilde{Y}^\kappa_{deg}(a, x),
\]
which is item (ii) in Theorem 7.5. Since items (i) and (ii) are evident from the construction, the proof of Theorem 7.5 is complete. □

8.7. The proof of Theorem 7.3. Recall that matrices $A, B \in \mathbb{R}^{n \times n}$ are congruent if there exists an invertible matrix $C \in \mathbb{R}^{n \times n}$ such that $A = C^TBC$. The matrix $C$ (or $C^{-1}$, or $C^T$ or $(C^{-1})^T$) is often referred to as the congruence.

Lemma 8.8. If $M \in \mathbb{R}^{s \times s}, U \in \mathbb{R}^{s \times t}, W \in \mathbb{R}^{t \times t}, \lambda \in \mathbb{R}$ and $\Pi_{R_{wT}W}$ denotes the orthogonal projection of $\mathbb{R}^{s+t}$ onto the range of $W^T$, then there exists an invertible matrix $S \in \mathbb{R}^{(s+t) \times (s+t)}$, independent of $\lambda$, such that
\[
\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} U & W \end{bmatrix} \begin{bmatrix} \Pi_{R_{wT}W}U^T & UW^T \\ WU^T & WW^T \end{bmatrix} = S^T \begin{bmatrix} M & 0 \\ 0 & \lambda WW^T \end{bmatrix} S.
\]
In particular, if
\begin{equation}
\text{range } U^T \subseteq \text{range } W^T,
\end{equation}
then
\begin{equation}
[M 0 \ 0 0] + \lambda \begin{bmatrix} UU^T & UW^T \\ WU^T & WW^T \end{bmatrix} = S^T \begin{bmatrix} M & 0 \\ 0 & \lambda WW^T \end{bmatrix} S.
\end{equation}

\textbf{Proof.} If $W = 0$, then, in view of the constraint \(8.21\), the matrix $U = 0$ and the asserted conclusion is self-evident. Thus, it suffices to consider the case $W \neq 0$.

Let $(WW^T)^\dagger$ denote the Moore-Penrose inverse of $WW^T$ and, for $K \in \mathbb{R}^{p \times q}$, let $\Pi_K$ denote the orthogonal projection of $\mathbb{R}^q$ onto $\mathcal{R}_K$, the range of $K$. As is well known,
\begin{align*}
(WW^T)(WW^T)^\dagger &= \Pi_{WW} = \Pi_{WW^T} \\
(W^TWW^T)^\dagger W &= \Pi_{WW^T} = \Pi_{WW^T}
\end{align*}
is the orthogonal projection onto the range of $W$ and
\begin{align*}
W^T(WW^T)^\dagger &= \Pi_{WW^T} = \Pi_{WW^T}
\end{align*}
is the orthogonal projection onto the range of $W^TW$. In particular,
\begin{align*}
W^T(WW^T)^\dagger W &= W^T, \\
(WW^T)(WW^T)^\dagger W &= W.
\end{align*}
Hence
\begin{equation}
\begin{bmatrix} I_s & UW^T(WW^T)^\dagger \\ 0 & I_t \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & \lambda WW^T \end{bmatrix} \begin{bmatrix} I_s & \lambda UU^T \\ 0 & \lambda WW^T \end{bmatrix} = \begin{bmatrix} M + \lambda U\Pi_{WW^T}U^T & \lambda UU^T \\ \lambda WW^T & \lambda WW^T \end{bmatrix}.
\end{equation}

Thus the matrices
\begin{align*}
\begin{bmatrix} M & 0 \\ 0 & \lambda WW^T \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} M + \lambda U\Pi_{WW^T}U^T & \lambda UU^T \\ \lambda WW^T & \lambda WW^T \end{bmatrix}
\end{align*}
are congruent and, as follows from \(8.23\), the congruence is independent of $\lambda$.

If the constraint \(8.21\) is in force, then $U \Pi_{WW^T} U^T = UU^T$ and consequently \(8.22\) holds. \hfill \Box

The following proposition contains Theorem \(7.3\).

\textbf{Proposition 8.9.} Let $p$ denote a $\kappa \times \kappa$ symmetric matrix nc polynomial of degree $d$ in $x$ as in equation \(1.1\).

There exists matrix nc polynomials $f(x)$ and $g(a,x)$ such that the middle matrix $3_\lambda$ in the representation of the modified Hessian (with
respect to the full border vector) of equation (7.10) evaluated at a tuple 
\((A, X) \in S_n(\mathbb{R}^g)\) is of the form

\[
Z_{\lambda}(A, X) := \begin{bmatrix} Z(A, X) & 0 & 0 & 0 \\ 0 & U & 0 & \lambda W \end{bmatrix} + \lambda \begin{bmatrix} U & U & 0 & W \\ W & U & 0 & W \\ 0 & W & W & 0 \\ 0 & 0 & 0 & W \end{bmatrix},
\]

where \(W = f(A)\) and \(U = g(A, X)\).

Let \(\Pi_{W^T}\) denote the orthogonal projection onto the range of \(W^T\) and suppose \(\lambda \in \mathbb{R} \setminus \{0\}\).

1. There exists an invertible matrix \(S\), depending on \((A, X)\) and \(d\), but not upon \(p\) or \(\lambda\), such that

\[
\begin{bmatrix} Z(A, X) & 0 & 0 & 0 \\ 0 & U & 0 & \lambda W \end{bmatrix} + \lambda \begin{bmatrix} U & U & 0 & W \\ W & U & 0 & W \\ 0 & W & W & 0 \\ 0 & 0 & 0 & W \end{bmatrix} = S^T \begin{bmatrix} Z(A, X) & 0 \\ 0 & \lambda WW \end{bmatrix} S.
\]

2. If the range of \(U^T\) is contained in the range of \(W^T\), then the left hand side of (8.25) is equal to \(Z_{\lambda}(A, X)\) and hence

\[
\mu_+(Z(A, X)) = \mu_+(Z_{\lambda}(A, X)),
\]

for all \(\lambda < 0\).

3. The range inclusion condition of item (2) holds if and only if the highest degree terms of \(p\) majorize at \(A\).

Recall,

\[
\tilde{V}_\kappa^{\text{deg}} = \begin{bmatrix} I_\kappa \otimes V_0 \\ \vdots \\ I_\kappa \otimes V_{d-1} \end{bmatrix},
\]

from the definition of \(\tilde{V}\) in equation (7.7) and the discussion preceding.

Proof. Observe that the \(f_j\) in equation (1.2) are precisely (up to, as usual, a permutation), \((x \otimes b)_j\) and

\[
p(a, x) = \sum_{j=0}^d \varphi^j_p(a) f_j(a, x) = \sum_{j=0}^d \varphi^j_p(a)(x \otimes b)_j.
\]

Thus the derivative \(p_x\) is

\[
p_x(a, x)[h] = [\varphi^1_p(a) \quad \cdots \quad \varphi^d_p(a)] [f_1(a, x)[h] \quad \cdots \quad (f_d)(a, x)[h]],
\]
which, can be rewritten as

\[ p_x(a, x)[h] = \begin{bmatrix} \varphi_p^1(a) & \cdots & \varphi_p^d(a) \end{bmatrix} L(a, x) \begin{bmatrix} V_0(a, x)[h] \\ \vdots \\ V_{d-1}(a, x)[h] \end{bmatrix}, \]

where \( L(a, x) \) is the block lower triangular matrix polynomial with \( L_{ii} = I_{t_j} \) that was defined in terms of the blocks of \( K(a, x) \) just above (8.19).

Thus, upon writing

\[ L(a, x) = \begin{bmatrix} M_{11}(a, x) & 0 \\ M_{21}(a, x) & I_{d-1} \end{bmatrix} \]

and setting

\[ w^T = \phi_1^d(a) \quad \text{and} \quad u^T = \begin{bmatrix} \varphi_1^1(a) & \cdots & \varphi_1^{d-1}(a) \end{bmatrix} M_{11}(a, x) + \varphi_p^d(a) M_{21}(a, x), \]

it follows that

\[ p_x(a, x)[h] = \begin{bmatrix} u^T & w^T \end{bmatrix} \tilde{V}(a, x)[h] \]

and hence that (see equation (7.9))

\[ p_x(a, x)[h]^T p_x(a, x)[h] = (\tilde{V}_{\text{deg}}^\kappa)(a, x)[h]^T \begin{bmatrix} uu^T & uw^T \\ wu^T & ww^T \end{bmatrix} (\tilde{V}_{\text{deg}}^\kappa)(a, x)[h], \]

which serves to explain the formula (8.24).

In view of equation (8.24), an application of Lemma 8.8 gives item (1); item (2) follows immediately from (1). Let \( U = u(A, X) \) and \( W = w(A, X) \). Since \( M_{11} \) is invertible, it is readily seen that

\[ \text{range } U^T \subseteq \text{range } W^T \]

if and only if (1.3) holds. Hence the range inclusion \( \text{range}(U^T) \subseteq \text{range}(W^T) \) holds if and only if the highest degree terms majorize, completing the proof of (3).

\[ \square \]

**Corollary 8.10.** If \( p \in \mathcal{P}^{n \times n} \) is an nc matrix polynomial that is homogeneous of degree \( d \) in \( x \), then the range inclusion condition in Theorem 7.3 is met for every choice of \( A \in S_n(\mathbb{R}^g) \).

**Proof.** If \( p \) is homogeneous of degree \( d \) in \( x \), then the range inclusion condition of Proposition 8.9 item (2) is automatically met, since \( \varphi_p^j = 0 \) for \( j = 1, \ldots, d - 1 \). Compare with Remark 1.1. \[ \square \]
9. The CHSY lemma

Let \( \{w_{i1}, \ldots, w_{i\beta_i}\} \) denote the set of words in the right chip set \( \mathcal{RC}_p^i \) of the nc polynomial \( p \) of degree at most \( d \) in \( x \). Given a positive integer \( \kappa \), a triple \( (A, X, v) \in S_n(\mathbb{R}^\kappa) \times \mathbb{R}^{n\kappa} \) with \( v \neq 0 \), an \( H \in S_n \) and a subspace \( \mathcal{H} \) of \( S_n \), let

\[
\mathcal{R}_\kappa^i(H) := \begin{bmatrix}
I_\kappa \otimes Hw_{i1}(A, X) \\
\vdots \\
I_\kappa \otimes Hw_{i\beta_i}(A, X)
\end{bmatrix}
\]

and let \( \mathcal{R}^i(\mathcal{H}) = \{ \mathcal{R}^i(H) : H \in \mathcal{H} \} \). The number \( \beta_i \) of distinct words in \( \mathcal{RC}_p^i \) is equal to the dimension of the chip space \( \mathcal{C}_p^i \), which is equal the span of \( \mathcal{RC}_p^i \) in the space of nc polynomials. For \( H = (H_1, \ldots, H_g) \in S_n(\mathbb{R}^g) \), let

\[
\mathcal{R}_\kappa^p(H) := \begin{bmatrix}
\mathcal{R}_\kappa^1(H_1) \\
\vdots \\
\mathcal{R}_\kappa^g(H_g)
\end{bmatrix}
\]

and let \( \mathcal{R}^p(\mathcal{H}) = \{ \mathcal{R}^p(H) : H \in \mathcal{H} \} \). Note that the words \( w_{ij} \) in (9.1) all have degree at most \( d - 1 \) in \( x \) and that

\[\mathcal{R}^p(\mathcal{H})v \subseteq \mathbb{R}^{n\beta} \quad \text{with} \quad \beta = \beta_1 + \cdots + \beta_g.\]

The border vector obtained by using just the words from \( \mathcal{C} \) will be called the reduced border vector based on \( \mathcal{C} \) and is generally much smaller than the border vector based on all words of degree at most \( \tilde{d} \) in \( a \) and \( d - 1 \) in \( x \). The reduced border vector evaluated at \( (A, X, H) \) is \( \mathcal{R}^p(\mathcal{H}) \).

The main tool developed here, Theorem 9.3 below, is an elaboration of the following result.

Lemma 9.1. (The CHSY lemma) If \( \{u_1, \ldots, u_k\} \) is a set of linearly independent vectors in \( \mathbb{R}^n \), then

\[
\dim \text{span} \left\{ \begin{bmatrix} Hu_1 \\ \vdots \\ Hu_k \end{bmatrix} : H \in S_n \right\} = kn - \frac{k(k - 1)}{2}.
\]

Therefore, the codimension of the space on the left hand side of (9.2) in \( \mathbb{R}^{kn} \) is equal to \( k(k - 1)/2 \), independently of \( n \).

Proof. This result is Lemma 9.5 of CHSY03.

If \( w \) is a word in \( (a, x) \), \( (A, X) \in S_n(\mathbb{R}^\kappa) \), \( C \in \mathbb{R}^{\kappa \times \kappa} \) and \( H \in S_n \), then

\[C \otimes Hw(A, X) = (C \otimes I_\kappa)(I_\kappa \otimes Hw(A, X)).\]
Thus, if \( Hw(A, X) \) belongs to the reduced border vector based on the chip set of \( p = \sum_{w \in W} c_w w \), then \( I_\kappa \otimes Hw(A, X) \) will belong to the reduced border vector for the matrix polynomial \( p = \sum_{w \in W} \mathbf{C}_w \otimes w \) with \( \mathbf{C}_w \in \mathbb{R}^{\kappa \times \kappa} \); the matrix \( \mathbf{C}_w \) gets absorbed into the middle matrix of the Hessian.

**Lemma 9.2.** Set \( v = \text{col}(v_1, \ldots, v_\kappa) \) with components \( v_1, \ldots, v_\kappa \in \mathbb{R}^n \) and let \( \{w_1, \ldots, w_s\} \) be a given set of words in \( a \) and \( x \). If \( (A, X) \in S_n(\mathbb{R}^g) \) and if the set of \( \kappa \) vectors \( \{w_i(A, X)v_k : 1 \leq i \leq s, \ 1 \leq k \leq \kappa\} \) is linearly independent, then the codimension of

\[
\text{span} \left\{ \begin{bmatrix} [I_\kappa \otimes Hw_1(A, X)]v \\ \vdots \\ [I_\kappa \otimes Hw_s(A, X)]v \end{bmatrix} : H \in S_n \right\}
\]

in \( \mathbb{R}^{s\kappa n} \) is equal to \( s\kappa(s\kappa - 1)/2 \), independently of \( n \).

**Proof.** This lemma follows from the CHSY lemma (Lemma 9.1), by choosing an enumeration \( \{u_1, \ldots, u_{s\kappa}\} \) of \( \{w_i(A, X)v_k : 1 \leq i \leq s, \ 1 \leq k \leq \kappa\} \) and observing for \( H \in S_n \) and up to permutation,

\[
\begin{bmatrix} [I_\kappa \otimes Hw_1(A, X)]v \\ \vdots \\ [I_\kappa \otimes Hw_s(A, X)]v \end{bmatrix} = \begin{bmatrix} Hu_1 \\ \vdots \\ Hu_{s\kappa} \end{bmatrix}.
\]

\( \square \)

**Theorem 9.3.** Fix \( 1 \leq j \leq g \) and \( (A, X, v) \in S_n(\mathbb{R}^g) \times \mathbb{R}^{\kappa n} \). Let \( v_k \) for \( 1 \leq k \leq \kappa \) denote the entries of \( v \). If

\[
\{w(A, X)v_k : w \in \mathcal{RC}_p^j, \ 1 \leq k \leq \kappa\}
\]

is a linearly independent subset of \( \mathbb{R}^n \) or, equivalently, the mapping

\[
(C_p^j)^\kappa \ni q \mapsto q(A, X)v \in \mathbb{R}^n
\]
given by

\[
(q_1, \ldots, q_\kappa) \mapsto \sum_{k=1}^\kappa q_k(A, X)v_k
\]

is one-one, then

\[
\text{codim} \mathcal{R}_p^j(S_n)v = \frac{\kappa \beta_j(\kappa \beta_j - 1)}{2}
\]
as a subspace of \( \mathbb{R}^{n\kappa \beta_j} \).
If for each \( j \), the set \( \{ w(A, X)v_k : w \in \mathcal{R}C_p^j, 1 \leq k \leq \kappa \} \) is linearly independent, then

\[
\text{codim} \mathcal{R}_\kappa^C(\mathbb{S}_n(\mathbb{R}^g))v = \sum_{i=1}^g \kappa \beta_i (\kappa \beta_i - 1) / 2
\]

as a subspace of \( \mathbb{R}^{n\kappa^2} \).

Proof. The first conclusion is immediate from Lemma 9.2. The second is its manifestation in the direct sum \( C_p \) of the \( C_p^j \) and is thus an immediate consequence of the first statement. \( \square \)

10. Positivity of the middle matrices forces \( d \leq 2 \)

Lemma 10.1. If \( p(a, x) \) is an nc symmetric matrix-valued polynomial of degree \( d \geq 2 \) in \( x \), then the \( 3_{0,d-2} \) entry in the middle matrix \( 3 \) of its Hessian with respect to \( x \) is nonzero and depends only on \( a \).

Proof. Write

\[
p(a, x) = \sum_{w \in W} C_w \otimes w(a, x),
\]

with \( C_w \neq 0 \) for every \( w \in W \). Here \( W \) is a set of words \( w(a, x) \) of degree at most \( d \) in \( x \) and \( C_w \in \mathbb{R}^{\kappa \times \kappa} \).

A word \( w \in W \) of degree \( d \geq 2 \) in \( x \) must be of the form

\[
w(a, x) = u(a)x_i v(a)x_j f(a, x) \quad \text{for some } i, j \in \{1, \ldots, g\},
\]

where \( u(a) \) and \( v(a) \) are words that depend only upon \( a \) and \( f(a, x) \) is a word of degree \( d - 2 \) in \( x \) that may depend upon both \( a \) and \( x \). Let

\[
W' = W'_{u, i, j, f}
\]

denote the set of words in \( W \) that begin with \( u(a)x_i \) and end with \( x_j f(a, x) \) and, assuming that \( W' \neq \emptyset \), let

\[
q(a, x) = \sum_{w \in W'} C_w \otimes w(a, x)
\]

\[
= \sum_{w \in W'} C_w \otimes u(a)x_i v_w(a)x_j f(a, x)
\]

\[
= (I_\kappa \otimes u(a)x_i) \left( \sum_{w \in W'} C_w \otimes v_w(a) \right) (I_\kappa \otimes x_j f(a, x)).
\]
Since, for \( w \in \mathcal{W}_{u,i,j,f}' \),

\[
\begin{align*}
w_{xx}(a, x)[h] &= 2u(a)h_i v(a)h_j f(a, x) \\
&\quad + 2u(a)h_i v(a)x_j f_x(a, x)[h] + 2u(a)x_i v(a)h_j f_x(a, x)[h] \\
&\quad + u(a)x_i v(a)x_j f_{xx}(a, x)[h],
\end{align*}
\]

the \( \kappa \times \kappa \) subblock of \( Z_{0,d-2} \) that is specified by the \( I_\kappa \otimes (u(a)h_i) \) and \( I_\kappa \otimes h_j f(a, x) \) entries of the border vectors \( V_{\deg}(x, a)[h]^T \) and \( V_{\deg}(a, x)[h] \) respectively is equal to

\[
2 \left( \sum_{w \in \mathcal{W}'} C_w \otimes v_w(a) \right).
\]

Thus, if \( Z_{0,d-2} = 0 \), then

\[
2 \left( \sum_{w \in \mathcal{W}'} C_w \otimes v_w(a) \right) = 0
\]

and hence \( q(a, x) = 0 \). Since the same argument applies for every such subblock of \( Z_{0,d-2} \), and each such subblock depends only upon \( a \) and not upon \( x \), for each word \( w \) of degree \( d \) in \( x \) the coefficient \( C_w = 0 \) and we have reached a contradiction. \( \square \)

One last piece of notation is needed to state our next lemma. Let \( p_2(a, x) \) denote the homogeneous of degree two in \( x \) portion of the nc symmetric polynomial \( p \). The polynomial \( p_2 \) can be expressed as

\[
p_2(a, x) = \sum_{\sigma, \tau, j, k} \sigma^T(a)x_k r_{\sigma, \tau, j, k}(a)x_j \tau(a),
\]

where \( \sigma, \tau \) are words in \( a \) and \( r_{\sigma, \tau, j, k}(a) \) is a \( \kappa \times \kappa \) matrix polynomial in \( a \) such that the sum of the degrees of \( \sigma, \tau, r_{\sigma, \tau, j, k} \) is at most the degree of \( p_2 \) in \( a \). Let \( R^p(a) \) denote the matrix indexed by \( 1 \leq j \leq g \) and words of length at most that of the degree of \( p_2 \) in \( a \), with \( ((\sigma, k), (\tau, j)) \) entry,

\[
r_{\sigma, \tau, j, k}(a)
\]

and let \( S^p \) denote the vector polynomial with \( (\tau, j) \) entry equal to the \( \kappa \times \kappa \) matrix,

\[
S^p_{\tau,j}(a, x) = x_j \tau(a)I_\kappa.
\]

Thus,

\[
p_2(a, x) = -S^p(a, x)^T R^p(a) S^p(a, x).
\]

**Lemma 10.2.** Let \( Z \) denote the middle matrix of the Hessian of a nonzero nc symmetric polynomial \( p \) of degree \( d \) in \( g \) variables \( x \) and...
degree \(d\) in \(\tilde{q}\) variables \(a\) and suppose \(N \geq \sum_{j=0}^{d} \tilde{q}^j\). For positive integers \(n\), let \(U(n)\) denote the set of those \(A \in \mathcal{S}_n(\mathbb{R}^9)\) for which there is an \(X\) such that \(\mathfrak{Z}(A, X) \preceq 0\). If \(U(N)\) has nonempty interior, then

(i) \(p(a, x)\) has degree at most two in \(x\);

(ii) there exists a polynomial \(\ell(a, x)\) that is affine linear in \(x\) such that

\[
(10.1) \quad p(a, x) = \ell(a, x) - S^p(a, x)^T R^p(a) S^p(a, x);
\]

(iii) \(R^p(A) \succeq 0\) for each \(A \in U\).

**Proof.** Suppose \(d > 2\). In this case, \(\mathfrak{Z}_{0,d-2}(a, x) = \mathfrak{Z}_{0,d-2}(a)\) is not zero and depends only upon \(a\) by Lemma 10.1. Given \((A, X)\), if \(\mathfrak{Z}(A, X) \preceq 0\), then the submatrix

\[
M = \begin{bmatrix}
\mathfrak{Z}_{00}(A, X) & \mathfrak{Z}_{0,d-2}(A) \\
\mathfrak{Z}_{d-2,0}(A) & 0
\end{bmatrix} \preceq 0,
\]

therefore, \(\mu_+(M) = 0\). The inequality

\[
\mu_{\pm}(M) \geq \text{rank}(\mathfrak{Z}_{0,d-2}(A))
\]

now implies that \(\mathfrak{Z}_{0,d-2}(A) = 0\). Therefore, \(\mathfrak{Z}_{0,d-2}(A) = 0\) on an open set in \(\mathcal{S}_N(\mathbb{R}^9)\) and hence that \(\mathfrak{Z}_{0,d-2} = 0\), contradicting Lemma 10.1. Consequently, \(d \leq 2\) and in particular \(\mathfrak{Z}_{00}(a, x) = \mathfrak{Z}_{00}(a)\) depends only upon \(a\).

Next, to verify item (iii), write

\[
p(a, x) = \sum_{w \in \mathcal{W}} C_w \otimes w + \ell(a, x),
\]

where \(\mathcal{W}\) is a set of words \(w(a, x)\) of degree two in \(x\) and \(\mathfrak{Z}_{0,0}(A, X) \preceq 0\) and \(\ell(a, x)\) is affine linear in \(x\). Then

\[
p_{xx}(a, x)[0, h] = \sum_{w \in \mathcal{W}} C_w \otimes w_{xx}(a, x)[h] = 2[p(a, h) - \ell(a, h)].
\]

On the other hand,

\[
p_{xx}(a, x)[h] = 2(I_\kappa \otimes V_0(a, x)[h])^T \mathfrak{Z}_{0,0}(a, 0)(I_\kappa \otimes V_0(a, x)[h]),
\]

\(I_\kappa \otimes V_0(a, x)[h] = I_\kappa \otimes (h \otimes b)\) depends only on \(h\) and \(a\) (and not on \(x\)). Hence,

\[
p(a, x) = \frac{1}{2} 2(I_\kappa \otimes V_0(a, x)[x])^T \mathfrak{Z}_{0,0}(a, 0)(I_\kappa \otimes V_0(a, x)[x]) + \ell(a, x)
\]

with \(I_\kappa \otimes V_0(a, x)[x] = I_\kappa \otimes (x \otimes b)\) linear in \(x\). In particular, \(p(A, X) \preceq 0\) whenever \(\mathfrak{Z}_{0,0}(A, 0) \preceq 0\). Choosing \(R(a) = -\mathfrak{Z}_{00}(a, 0), S = I_\kappa \otimes V_0(a, x)[x]\) produces the representation of equation (10.1) from which item (iii) immediately follows. \(\square\)
11. Positivity of Middle Matrices

This section focuses on the positivity of middle matrices for the Hessians, the relaxed Hessians and the positivity of various second derivatives. The results will be used in the proof of Theorem 2.5. Throughout the polynomial $p$ is fixed.

The following consequence of Theorem 7.3 generalizes Proposition 5.2 in [DHM11].

**Lemma 11.1.** Let $(A, X) \in S_n(\mathbb{R}^g)$ be given and let $3_{\lambda, \delta}(A, X)$ denote the middle matrix for the relaxed Hessian of a symmetric nc matrix polynomial $p \in P^{\kappa \times \kappa}$. There exists an $\epsilon < 0$ such that if $\epsilon \leq \delta \leq 0$ and $\lambda \leq 0$, then

\[
\mu_+(3_{\lambda, \delta}(A, X)) \leq \mu_+(3(A, 0))
\]

with equality if the matrices $U$ and $W$ in formula (8.24) for the middle matrix of the modified Hessian,

\[
p_{xx}(A, X)[h] + \lambda p_x(A, X)[h]^{T} p_x(A, X)[h],
\]

meet the range inclusion condition (8.21) (equivalently the highest degree terms of $p$ majorize at $A$).

**Proof.** In view of Theorem 7.3,

\[
3_{\lambda, \delta}(A, X) = \lambda \alpha + S^T \beta S + \delta I,
\]

where

\[
\alpha = \begin{bmatrix} U(I - \Pi_{R_{wT_w}}) U^T & 0 \\ 0 & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} 3(A, 0) & 0 \\ 0 & \lambda W W^T \end{bmatrix}
\]

and $S$ is an invertible matrix that depends on $(A, X)$, but not on $\lambda$. Since the additive perturbation by the negative definite matrix $\delta I$ with $\delta < 0$ shifts the eigenvalues of the matrix $S^T \beta S$ to the left, the nonpositive eigenvalues of $S^T \beta S$ become negative but the positive eigenvalues will stay positive if $\delta \in [-\epsilon, 0]$ and $\epsilon > 0$ is small enough. For such $\delta$ and each $\lambda \leq 0$,

\[
\mu_+(S^T \beta S + \delta I) = \mu_+(S^T \beta S) = \mu_+(\beta)
\]

\[
= \mu_+(\begin{bmatrix} 3(A, 0) & 0 \\ 0 & \lambda W W^T \end{bmatrix})
\]

\[
= \mu_+(3(A, 0)).
\]

The inequality (11.1) follows from the fact that, for $\lambda \leq 0$,

\[
\mu_+(\lambda \alpha + S^T \beta S) \leq \mu_+(S^T \beta S) = \mu_+(\beta).
\]

Equality prevails in (11.1) under the added assumption that the range of $U^T$ is contained in the range of $W^T$ because in that case $\alpha = 0$. □
The next result is a variant of Proposition 6.7 of [DHM11]. It provides a bound on $\mu_+(3(A,0))$ which, as will be seen later, turns out to be independent of $n$, the size of $A$.

**Lemma 11.2.** Suppose $(A, X, v) \in S_n(\mathbb{R}^g) \times \mathbb{R}^{n\beta}$ and there exists an $\varepsilon < 0$ such that equality holds in (11.1) for all $\varepsilon \leq \delta \leq 0$ and $\lambda \leq 0$. If $\varepsilon \leq \delta \leq 0$ and $\lambda \leq 0$ and if $H_+^{\lambda,\delta}$ is a maximal strictly positive subspace for the quadratic form

$$S_n(\mathbb{R}^g) \ni H \mapsto \langle p''_{\lambda,\delta}(A, X)[h]v, v \rangle,$$

then

$$\mu_+(3(A,0)) \leq \dim H_+^{\lambda,\delta} + \text{codim} \mathcal{R}^C_p(S_n(\mathbb{R}^g)),$$

where the codimension is computed in the space $\mathbb{R}^{n\beta}$, $\beta = \beta_1 + \cdots + \beta_g$ and $\beta_j = \dim C_j^p$ for $j = 1, \ldots, g$.

**Proof.** Given $\lambda, \delta \leq 0$, let $H_+^{\lambda,\delta}$ be a maximal strictly positive subspace of $S_n(\mathbb{R}^g)$ with respect to the quadratic form

$$\langle p''_{\lambda,\delta}(A, X)[h]v, v \rangle = v^T(\tilde{V}_\text{deg}^\kappa)(A, X)[H]^T \tilde{3}_{\lambda,\delta}(A, X)(\tilde{V}_\text{deg}^\kappa)(A, X)[H]v.$$

Let $\mathcal{K}^{\lambda,\delta}$ be a complementary subspace of $H_+^{\lambda,\delta}$ of $S_n(\mathbb{R}^g)$ on which this quadratic form is nonpositive. Recall

$$\langle \tilde{V}_\text{deg}^\kappa(A, X)[H]v, v \rangle \in \mathbb{R}^{n\nu}$$. It follows that

$$v^T(\tilde{V}_\text{deg}^\kappa)(A, X)[H]^T \tilde{3}_{\lambda,\delta}(A, X)(\tilde{V}_\text{deg}^\kappa)(A, X)[H]v$$

$$= v^T \tilde{V}_\text{deg}^\kappa(A, X)[H]^T u^T \tilde{3}_{\lambda,\delta}(A, X) u^T \tilde{V}_\text{deg}^\kappa(A, X)[H]v$$

$$= v^T \mathcal{R}^C_p(H)^T u^T \tilde{3}_{\lambda,\delta}(A, X) u^T \mathcal{R}^C_p(H)v.$$

It is now readily checked that

$$\mathcal{R}^C_p(H_+^{\lambda,\delta})v \cap \mathcal{R}^C_p(\mathcal{K}^{\lambda,\delta})v = \{0\},$$
because if \( y \in \mathcal{R}_\kappa^C(\mathcal{H}_+^{\lambda,\delta}) \), then either \( y = 0 \) or \( y^T T \mathcal{H}_+^{\lambda,\delta}(A, X)y > 0 \), whereas if \( y \in \mathcal{R}_\kappa^C(\mathcal{K}_+^{\lambda,\delta}) \), then \( y^T T \mathcal{H}_+^{\lambda,\delta}(A, X)y \leq 0 \). Thus, the sum \( \mathcal{R}_\kappa^C(\mathcal{H}_+^{\lambda,\delta})v + \mathcal{R}_\kappa^C(\mathcal{K}_+^{\lambda,\delta})v \) is direct. Consequently, letting \( \mathcal{H}_+^{\lambda,\delta} \) denote the algebraic direct sum, there exists a subspace \( \mathcal{Y} \) of \( \mathbb{R}^{n\beta\kappa} \) such that

\[
\mathbb{R}^{n\beta\kappa} = \mathcal{R}_\kappa^C(\mathcal{H}_+^{\lambda,\delta})v + \mathcal{R}_\kappa^C(\mathcal{K}_+^{\lambda,\delta})v + \mathcal{Y}.
\]

Therefore,

\[
n\beta\kappa = \dim \mathcal{R}_\kappa^C(\mathcal{H}_+^{\lambda,\delta})v + \dim \mathcal{R}_\kappa^C(\mathcal{K}_+^{\lambda,\delta})v + \dim \mathcal{Y},
\]

and

\[
n\beta\kappa - \dim \mathcal{R}_\kappa^C(\mathcal{K}_+^{\lambda,\delta})v = \dim \mathcal{R}_\kappa^C(\mathcal{H}_+^{\lambda,\delta})v + \dim \mathcal{Y} \leq \dim \mathcal{H}_+^{\lambda,\delta} + \dim \mathcal{R}_\kappa^C(\mathcal{S}_n(\mathbb{R}^g))v.
\]

The next step is to verify the bound

\[
\mu_+(\mathcal{H}_+^{\lambda,\delta}(A, X)) = \mu_+(v^T \mathcal{H}_+^{\lambda,\delta}(A, X)v) \leq n\beta\kappa - \dim \mathcal{R}_\kappa^C(\mathcal{K}_+^{\lambda,\delta})v.
\]

To this end, let \( U \in \mathbb{R}^{n\beta\kappa \times n\beta\kappa} \) be an invertible matrix in which the first \( k \) columns is a basis for \( \mathcal{R}_\kappa^C(\mathcal{K}_+^{\lambda,\delta})v \) and let \( \mu_+(M) \) denote the dimension of the space spanned by the eigenvectors of a real symmetric matrix \( M \) corresponding to its nonpositive eigenvalues. Then, as \( \mu_+(v^T \mathcal{H}_+^{\lambda,\delta}(A, X)v) = \mu_+(U^T v^T \mathcal{H}_+^{\lambda,\delta}(A, X)vU) \geq k = \dim \mathcal{R}_\kappa^C(\mathcal{K}_+^{\lambda,\delta})v \),

it follows that

\[
\mu_+(\mathcal{H}_+^{\lambda,\delta}(A, X)) = \mu_+(v^T \mathcal{H}_+^{\lambda,\delta}(A, X)v) = n\beta\kappa - \mu_+(v^T \mathcal{H}_+^{\lambda,\delta}(A, X)v) \leq n\beta\kappa - \dim \mathcal{R}_\kappa^C(\mathcal{K}_+^{\lambda,\delta})v.
\]

By assumption, \( \mu_+(\mathcal{H}_+^{\lambda,\delta}(A, X)) = \mu_+(\mathcal{H}(A, 0)) \) for appropriate choices of \( \delta \) and \( \lambda \). Hence,

\[
\mu_+(\mathcal{H}(A, 0)) \leq n\beta\kappa - \dim \mathcal{R}_\kappa^C(\mathcal{K}_+^{\lambda,\delta})v.
\]

An application of the inequality in (11.3) completes the proof.\[\square\]

The next proposition is a variant of Lemma 7.2 from [DHM11] and follows the strategy of Proposition 5.4 of [BM14].

**Proposition 11.3.** Let \( p \in \mathcal{P}^{n \times n} \) be a symmetric nc matrix polynomial of degree \( \tilde{d} \) in \( \alpha \) and degree \( d \) in \( x \). Assume that \( (A, X, v) \in \mathbb{S}_n(\mathbb{R}^g) \otimes \mathbb{R}^{\alpha n} \) with \( v = \text{col}(v_1, \ldots, v_\alpha) \) meets the following conditions:

1. There exists a subspace \( \mathcal{H} \) of \( \mathcal{T}(A, X, v) \) of codimension at most one in \( \mathcal{T}(A, X, v) \) such that

   \[
   \langle p_{XX}(A, X)[h]v, v \rangle \leq 0 \quad \text{for each } H \in \mathcal{H};
   \]

   ...
(2) The set
\[ \{ w(A, X)v_k : 1 \leq k \leq \kappa, w \in C_p^i \} \]
is linearly independent for each choice of \( i = 1, \ldots, g \); and

(3) The highest degree terms of \( p \) majorize at \( A \).

Then there exists an integer \( \gamma \) that depends on the number of variables and the degree of the polynomial \( p(a, x) \) but is independent of \( n \) such that the middle matrix \( \mathfrak{F}(A, X) \) of the Hessian \( p_{xx}(A, X)[h] \) is subject to the constraint
\[ \mu_+(\mathfrak{F}(A, 0)) \leq \gamma. \]

Proof. Recall the definition of \( e_n^a \); it is given just below (5.3). In view of assumption (i), \( e_n^a(A, X, v, p_{xx}, T(A, X, v)) \leq 1 \). Therefore, by Lemma 5.4, there is a \( \delta_0 < 0 \) such that for each \( \delta_0 \leq \delta < 0 \) there exists a \( \lambda < 0 \) such that
\[ e_n^a(A, X, v; p''_{\lambda, \delta}, S_n(R^g)) \leq 1. \]

By item (3), all the conditions of Lemma 11.1 are met. Hence the hypotheses of Lemma 11.2 are met. By (11.2),
\[ \mu_+(\mathfrak{F}(A, 0)) \leq \dim H_{+}^{\lambda, \delta} + \text{codim} R_{\kappa}^{C_p}(S_n(R^g))v \]
\[ = e_n^a(A, X, v; p''_{\lambda, \delta}, S_n(R^g)) + \text{codim} R_{\kappa}^{C_p}(S_n(R^g))v \]
\[ \leq 1 + \text{codim} R_{\kappa}^{C_p}(S_n(R^g))v = \gamma. \]

The hypothesis in item (2) justifies the use of formula (9.3) of Theorem 9.3 to conclude that \( \gamma \) is independent of \( n \). \( \square \)

Remark 11.4. In the setting of Proposition 11.3, item (2) can be replaced with the condition

(2') if \( q \in C_p^{1 \times \kappa} \) and \( q(A, X)v = 0 \), then \( q = 0 \).

To see that (2') implies (2), fix an \( i \) and suppose, for \( c_w \in \mathbb{R} \),
\[ \sum_{w \in C_p^i} c_w w(A, X)v_i = 0. \]

Let \( q \in C_p^i \) denote the polynomial \( q = \sum_{w \in C_p^i} c_w w \otimes e_i \) (so the only non-zero entry of the vector polynomial \( q \) is in the \( i \)-th position). With this choice of \( q \),
\[ q(A, X)v = \sum_{w \in C_p^i} c_w w(A, X)v_i \otimes e_i = 0. \]

Thus, by hypothesis, \( q = 0 \) and therefore \( c_w = 0 \) for each \( w \in C_p^i \). \( \square \)
12. Proof of the Main Result, Theorem 2.5

Recall $p \in \mathcal{P}^{n \times n}$ is a symmetric nc matrix polynomial in $a$ and $x$, $\mathcal{O} \subset S(\mathbb{R}^n)$ is a free open semialgebraic set and

$$\partial \hat{\mathcal{P}}_p \cap \hat{\mathcal{O}}(n) = \{(A, X, v) \in S_n(\mathbb{R}^n) \times \mathbb{R}^{n \times n} : (A, X) \in \mathcal{O}, p(A, X) \succeq 0, v \neq 0 \text{ and } p(A, X)v = 0\}.$$ 

We will first show, if $(\hat{A}, \hat{X}, \hat{v}) \in \partial \hat{\mathcal{P}}_p \cap \hat{\mathcal{O}}$ is a $\mathcal{C}_p$ dominating point for $\partial \hat{\mathcal{P}}_p \cap \hat{\mathcal{O}}$, then $Z(A, 0)$, its middle matrix evaluated at $(A, 0)$, is positive semidefinite. Accordingly, given a positive integer $m$, let $(B, Y, w) := (I_m \otimes \hat{A}, I_m \otimes \hat{X}, \text{col}_m\{\hat{v}, 0, \ldots, 0\})$ with $m - 1$ zero vectors of the same height as $\hat{v}$ in the last entry and proceed in steps.

1. There exists a point $(\tilde{B}, \tilde{Y}, \tilde{w}) \in \partial \hat{\mathcal{P}}_p \cap \hat{\mathcal{O}}$ arbitrarily close to $(B, Y, w)$ that is $\mathcal{C}_p$ dominating for $\partial \hat{\mathcal{P}}_p \cap \hat{\mathcal{O}}$ such that the pair $(\tilde{B}, \tilde{Y})$ is full rank and
   (1) $p(\tilde{B}, \tilde{Y}) \succeq 0$
   (2) $p(\tilde{B}, \tilde{Y})\tilde{w} = 0$.
   (3) dim kernel $p(\tilde{B}, \tilde{Y}) = 1$.
   (4) $\mathcal{P}_p^B \cap \mathcal{W}$ is convex for some open subset $\mathcal{W}$ of $S(\mathbb{R}^n)$ containing $\tilde{Y}$.

The point $(B, Y, w) \in \partial \hat{\mathcal{P}}_p \cap \hat{\mathcal{O}}$ and is a $\mathcal{C}_p$ dominating point for $\partial \hat{\mathcal{P}}_p \cap \hat{\mathcal{O}}$. By hypotheses (b) of Theorem 2.5 and Lemma 6.1 there exists a point $(\tilde{B}, \tilde{Y}, \tilde{w}) \in \partial \hat{\mathcal{P}}_p \cap \hat{\mathcal{O}}$ that is $\mathcal{C}_p$ dominating for $\partial \hat{\mathcal{P}}_p \cap \hat{\mathcal{O}}$ such that the pair $(\tilde{B}, \tilde{Y})$ is full rank. By Proposition 5.3 and another application of Lemma 6.1 it can be assumed that the kernel of $p(\tilde{B}, \tilde{Y})$ is spanned by multiples of a single nonzero vector $\tilde{w}$ and that $(\tilde{B}, \tilde{Y}, \tilde{w})$ is still $\mathcal{C}_p$ dominating for $\partial \hat{\mathcal{P}}_p \cap \hat{\mathcal{O}}$. It remains only to verify that item (4) in the list is in force. But this is covered by hypothesis (a) of Theorem 2.5. □

2. There exists a positive integer $\gamma$ that is independent of the integer $m$ appearing in the construction of the tuple $(\tilde{B}, \tilde{Y}, \tilde{w})$ such that

$$\mu_+(\mathcal{F}(\tilde{B}, 0)) \leq \gamma.$$ 

The proof rests heavily on Proposition 11.3. Thus, our first task is to show that it applies to $(\tilde{B}, \tilde{Y}, \tilde{w})$. In view of Step 1, we may apply
Proposition 5.1 to the point \((\tilde{B}, \tilde{Y}, \tilde{w})\) to guarantee the existence of a subspace \(\mathcal{H}\) of \(\mathcal{T}(\tilde{B}, \tilde{Y}, \tilde{w})\) of codimension at most one in \(\mathcal{T}(\tilde{B}, \tilde{Y}, \tilde{w})\) such that

\[
\langle p_{xx}(\tilde{B}, \tilde{Y})|H\rangle \tilde{w} \leq 0 \quad \text{for } H \in \mathcal{H},
\]
i.e., the first condition in Proposition 11.3 is met.

The validity of the second condition in Proposition 11.3 follows from the fact that \((\tilde{B}, \tilde{Y}, \tilde{w})\) is a \(C_p\) dominating point for \(\partial \hat{\mathcal{P}}_p \cap \hat{\mathcal{O}}\). Thus, if \(q \in C_p^{1\times \kappa}\) and \(q(\tilde{B}, \tilde{Y})\tilde{w} = 0\), then \(q(A, X)v = 0\) for all \((A, X, v) \in \partial \hat{\mathcal{P}}_p \cap \hat{\mathcal{O}}\). Since \(\partial \hat{\mathcal{P}}_p \cap \hat{\mathcal{O}}\) is a \(C_p\) dominating set for \(\mathcal{O}\) by hypothesis (ii) of Theorem 2.5, it follows that \(q(A, X)v = 0\) for all \((A, X, v) \in \hat{\mathcal{O}}\). Thus \(q(A, X) = 0\) for \((A, X) \in \mathcal{O}\) and since \(\mathcal{O}\) is open, \(q = 0\). The desired conclusion now follows from Remark 11.4.

The third condition is hypothesis (i) of Theorem 2.5. Thus, Proposition 11.3 is applicable. \(\square\)

3. \(3(\hat{A}, 0) \leq 0\).

By Theorem 7.3 the middle matrices \(3(B, Y)\) and \(3(\tilde{B}, \tilde{Y})\) for the Hessian of \(p\) are polynomially congruent to the middle matrices \(3(B, 0)\) and \(3(\tilde{B}, 0)\), of the Hessian of \(p\), respectively. Thus, by choosing \((\tilde{B}, \tilde{Y})\) sufficiently close to \((B, Y)\), it can be assumed that

\[
\mu_+(3(B, 0)) = \mu_+(3(B, Y)) \leq \mu_+(3(\tilde{B}, \tilde{Y})) = \mu_+(3(\tilde{B}, 0)).
\]

(The middle inequality in (12.2) holds because the strictly positive eigenvalues of \(3(B, 0)\) will stay positive under small perturbations of \(B\).) Combining Equations (12.2) and (12.1) and the evident fact that \(\mu_+\) is additive with respect to direct sums,

\[
m \mu_+(3(\hat{A}, 0)) = \mu_+(3(B, 0)) \leq \mu_+(3(\tilde{B}, 0)) \leq \gamma.
\]

Since the right hand side of the last inequality is independent of \(m\), it follows that \(3(\hat{A}, 0) \leq 0\). \(\square\)

To complete the proof, let \(\tilde{W} = \pi_1(\partial \hat{\mathcal{P}}_p \cap \mathcal{O})\), let \(A \in \tilde{W}\) be given and let \(X\) and \(v\) be such that \((A, X, v) \in \partial \hat{\mathcal{P}}_p \cap \hat{\mathcal{O}}\). By Lemma 6.2 there exists a \(C_p\) dominating point \((A_*, X_*, v_*) \in \partial \hat{\mathcal{P}}_p \cap \hat{\mathcal{O}}\) for \(\partial \hat{\mathcal{P}}_p \cap \hat{\mathcal{O}}\).

Let \((\hat{A}, \hat{X}, \hat{v}) = (A, X, v) \oplus (A_*, X_*, v_*)\) and note that \((\hat{A}, \hat{X}, \hat{v}) \in \partial \hat{\mathcal{P}}_p \cap \hat{\mathcal{O}}\) and is a \(C_p\) dominating point for \(\partial \hat{\mathcal{P}}_p \cap \hat{\mathcal{O}}\). Hence, by what is already proved, \(3(\hat{A}, 0) \leq 0\) and therefore \(3(0, 0) \leq 0\). Hence,

\[
\tilde{W} \subset \mathcal{W} = \{A : \text{there is an } X \text{ such that } 3_{00}(A, X) \geq 0\}.\
\]
Since, by assumption $\tilde{\mathcal{W}}(N)$, and therefore $\mathcal{W}(N)$, has nonempty interior for a sufficiently large $N$, an application of Lemma 10.2 completes the proof.

**Remark 12.1.** We make a final remark about the overall structure of the proof of Theorem 2.5. Proposition 5.3 gives the existence of sufficiently many points $(A, X)$, independent of the size $n$, in the boundary of $\mathfrak{Q}_p$, for which $p(A, X)$ has a one-dimensional kernel. An application of Proposition 5.1 produces a subspace of codimension one (again regardless of the size of $(A, X)$) in the tangent space on which the *second fundamental form* of equation (5.1) is negative. The CHSY lemma, Theorem 9.3 is then used to show that the middle matrix $Z(A, X)$ is positive semidefinite on a subspace of this tangent space with a small codimension independent of $n$ and thus $Z(A, X)$ is positive semidefinite sufficiently often imply it has a simple structure. Likely the one dimensional kernel and subspace of codimension one in Propositions 5.3 and 5.1 could be relaxed to requiring only some upper bound on these dimensions independent of $n$. The hypothesis on $\mathcal{O}$ in Theorem 2.5, namely that it is a free open semialgebraic set can be relaxed considerably. □

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Appendix: Faithfulness of finite dimensional representations

This section provides a proof of Remark 2.7. Namely, any free polynomial vanishing on a large enough set for sufficiently large matrices is zero.

Lemma 13.1. Suppose \( r(a) \) is a free (not necessarily symmetric) polynomial in \( \tilde{g} \) variables of degree \( \tilde{d} \) and let \( N \geq \sum_{j=0}^{\tilde{d}} \tilde{g}^j \) (see \( k_{b} \) defined in equation (2.6)). If \( r(A) = 0 \) for all \( A \in \mathcal{S}_N(\mathbb{R}^g) \), then \( r = 0 \).

Proof. It suffices to prove the result for scalar-valued polynomials homogeneous of degree \( \tilde{d} \) and for \( N = \sum_{j=0}^{\tilde{d}} \tilde{g}^j \). Let \( \tilde{d} \) denote the degree of \( r \). Let \( \mathcal{H} \) denote the Hilbert space with orthonormal basis words \( m(a) \) of degree at most \( \tilde{d} \). Thus the dimension of \( \mathcal{H} \) is \( N \). Define the tuple \( S = (S_1, \ldots, S_{\tilde{g}}) \) on \( \mathcal{H} \) by \( S_jw = a_jw \) for words \( w \) of degree (length) at
most \( \tilde{d} - 1 \) and \( S_j w = 0 \) for words \( w \) of degree \( \tilde{d} \). It is routine to verify that \( S \) extends, by linearity, to an operator on \( \mathcal{H} \). It is also routine to verify, for words \( w \) of degree at most \( \tilde{d} - 1 \),

\[
S_j^* a_k w = \begin{cases} 
0 & \text{if } j \neq k \\
w & \text{if } j = k 
\end{cases}
\]

and \( S^* \emptyset = 0 \).

Of course the \( S_j \) are not symmetric. Let \( T_j = S_j + S_j^* \). Thus \( T = (T_1, \ldots, T_\tilde{g}) \in \mathbb{S}_N(\mathbb{R}^{\tilde{g}}) \). By hypothesis \( r(T) = 0 \). Let \( v \) denote a word of degree \( \tilde{d} \) and observe

\[ v(T) \emptyset = v + h_v, \]

where \( h_v \) is a linear combination of words of degree at most \( \tilde{d} - 2 \). In particular, the vectors \( v \) and \( h_v \) are orthogonal. Writing \( r = \sum r_v v \), it follows that

\[ 0 = r(T) \emptyset = \sum_v r_v v + \sum_v r_v h_v. \]

Using orthogonality of words once again, we conclude \( r_v = 0 \) for each \( v \) and therefore \( r = 0 \).

**Proof of Remark 2.7.** Let \( U \) be an open set on which \( r \) vanishes and \( Y \) a given point in \( U \). Let \((A, X)\) be a given tuple in \( \mathbb{S}_N(\mathbb{R}^{\tilde{g}}) \). The matrix polynomial of a single real variable, \( r_{Y,Z}(t) = p(Y + tZ) \) has entries which are ordinary polynomials which vanish on an open interval containing 0 and hence vanish everywhere. It follows that \( r(X) = 0 \) for all \( X \in \mathbb{S}_N(\mathbb{R}^{\tilde{g}}) \) and therefore \( r(X) = 0 \) for all \( X \in \mathbb{S}_m(\mathbb{R}^{\tilde{g}}) \) for all \( m \leq N \). Lemma 13.1 now implies \( r = 0 \).

**14. Appendix: Illustrating the basic calculations**

**Note to the referee:** We defer to you opinion on whether to keep or not this, and the following, appendices.

Fix \( c, d \in \mathbb{R} \) and let

\[ p_3(a, x) = ca_1 x_1 a_2 x_2^2 + dx_2^2 a_2 x_1 a_1. \]

In this section we compute explicitly many of the objects appearing in this article such as the border vector and middle matrix, the various Hessians and spaces appearing in the CHSY lemma and the chip set representations. Since \( p_3 \) is a homogeneous polynomial of degree \( \tilde{d} = 3 \) in \( x \) and is of degree \( \tilde{d} = 2 \) in \( a \) and there are no consecutive strings of
a, it suffices to choose

\[(14.1) \quad b = \begin{bmatrix} 1 \\ a_1 \\ a_2 \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.\]

Then \(k_b\), the number of entries in \(b\), is equal to 3 and \(p_3(a, x)\) is a linear combination of two of the entries in the vector polynomial \(b \otimes (x \otimes b)_3\) of height \(t = k_b(k_b g)^d = 3(6^3) = 648\), i.e.,

\[p_3(a, x) = \begin{bmatrix} c_1 & \cdots & c_t \end{bmatrix} b \otimes (x \otimes b)_3\]

where all but two of the coefficients \(c_1, \ldots, c_t \in \mathbb{R}\) are equal to zero.

By successive applications of formula (8.4), Kronecker products can be reexpressed as ordinary matrix products:

\[b \otimes (x \otimes b)_3 = (b \otimes I_{t_0})(x \otimes b)_3 \]
\[= (b \otimes I_{t_0})(x \otimes b \otimes I_{t_1})(x \otimes b \otimes I_{t_0})(x \otimes b) \]
\[= (b \otimes I_{t_0})K_1K_0W_0,\]

where
\[W_0 = x \otimes b, \quad t_j = (k_b g)^{j+1} \quad \text{for} \quad j = 0, \ldots, 2 \quad \text{(since} \quad d = 3\).

and
\[K_j = K_j(a, x) = (x \otimes b) \otimes I_{t_j} \quad \text{is a matrix polynomial of size} \quad t_{j+1} \times t_j.\]

Thus, upon setting
\[\varphi_2 = \begin{bmatrix} c_1 & \cdots & c_t \end{bmatrix} (b \otimes I_{t_0}),\]

it is readily seen that
\[p_3(a, x) = \varphi_2K_1K_0W_0 \quad \text{with} \quad W_0 = x \otimes b\]

and, analogously, since \(\varphi_2\) is independent of \(x\),

\[(p_3(a, x))_x[h] = \varphi_2\{(h \otimes b) \otimes (x \otimes b)_2 + (x \otimes b) \otimes (h \otimes b) \otimes (x \otimes b)\]
\[+ (x \otimes b)_2 \otimes (h \otimes b)\}
\[= \varphi_2\{V_2 + (x \otimes b) \otimes V_1 + (x \otimes b)_2 \otimes V_0\}
\[= \varphi_2\{V_2 + K_1V_1 + K_1K_0V_0\}.\]

in which
\[V_0 = h \otimes b \quad \text{and} \quad V_j = (h \otimes b) \otimes (x \otimes b)_j \quad \text{for} \quad j = 1, 2.\]

Thus,
\[p_3(a, x) = \begin{bmatrix} 0 & 0 & \varphi_2 \end{bmatrix} \begin{bmatrix} I_{t_0} & 0 & 0 \\ K_0 & I_{t_1} & 0 \\ K_1K_0 & K_1 & I_{t_2} \end{bmatrix} \begin{bmatrix} W_0 \\ 0 \\ 0 \end{bmatrix} \]
and
\[(p_3(a, x))_x[h] = \begin{bmatrix} I_0 & 0 & 0 \\ K_0 & I_1 & 0 \\ K_1K_0 & K_1 & I_{t_2} \end{bmatrix} \begin{bmatrix} V_0 \\ V_1 \\ V_2 \end{bmatrix}.
\]

Similarly, the homogeneous polynomials (in \(x\))
\[p_2(a, x) = ea_2x_2a_1x_1 + fx_1a_1x_2a_2 \quad \text{and} \quad p_1(a, x) = ga_1x_2a_2 + ma_2x_2a_1\]
can be written as
\[p_2(a, x) = \begin{bmatrix} d_1 & \cdots & d_s \end{bmatrix} b \otimes (x \otimes b)_2
= \begin{bmatrix} d_1 & \cdots & d_s \end{bmatrix} (b \otimes I_{t_1})(x \otimes b)_2
= \varphi_1K_0W_0\]

and
\[p_1(a, x) = \begin{bmatrix} e_1 & \cdots & e_r \end{bmatrix} b \otimes (x \otimes b)
= \begin{bmatrix} e_1 & \cdots & e_r \end{bmatrix} (b \otimes I_{t_0})(x \otimes b)
= \varphi_0W_0,\]

with the same choice of \(b\) and \(x\) as in \([14.1]\), \(d_1, \ldots, d_s; e_1, \ldots, e_r \in \mathbb{R}, \ s = k_b(k_bg)^2 = 3(6^2) = 108, \ r = k_b(k_bg) = 3(6) = 18, \ \varphi_1 = \begin{bmatrix} d_1 & \cdots & d_s \end{bmatrix} (b \otimes I_{t_1}) \quad \text{and} \quad \varphi_0 = \begin{bmatrix} e_1 & \cdots & e_r \end{bmatrix} (b \otimes I_{t_0}).\]

Since \(\varphi_1\) and \(\varphi_0\) are independent of \(x\), it is readily checked that
\[(p_2)_x(a, x)[h] = \varphi_1\{(h \otimes b) \otimes (x \otimes b) + (x \otimes b) \otimes (h \otimes b)\}
= \varphi_1\{V_1 + K_0V_0\}\]

and
\[(p_1)_x(a, x)[h] = \varphi_0\{(h \otimes b)\}
= \varphi_0\{V_0\}.
\]

Consequently, the polynomial
\[p(a, x) = p_1(a, x) + p_2(a, x) + p_3(a, x)\]

admits the representation
\[p(a, x) = \begin{bmatrix} \varphi_0 & \varphi_1 & \varphi_2 \end{bmatrix} \begin{bmatrix} I_0 & 0 & 0 \\ K_0 & I_1 & 0 \\ K_1K_0 & K_1 & I_{t_2} \end{bmatrix} \begin{bmatrix} W_0 \\ V_1 \\ V_2 \end{bmatrix}\]

and
\[p_x(a, x)[h] = \begin{bmatrix} \varphi_0 & \varphi_1 & \varphi_2 \end{bmatrix} \begin{bmatrix} I_0 & 0 & 0 \\ K_0 & I_1 & 0 \\ K_1K_0 & K_1 & I_{t_2} \end{bmatrix} \begin{bmatrix} V_0 \\ V_1 \\ V_2 \end{bmatrix}.\]
14.1. **Range inclusions.** If \( p_x(a, x)[h] \) is expressed in the form

\[
p_x(a, x)[h] = U^T \begin{bmatrix} V_0 \\ V_1 \end{bmatrix} + W^T V_2,
\]

then

\[
U^T = \begin{bmatrix} \varphi_0 & \varphi_1 & \varphi_2 \\
I_0 & K_0 & I_t \\
K_1 K_0 & I_t \\
\end{bmatrix}
\]

and

\[
W^T = \varphi_2.
\]

Thus,

\[
\text{range } U^T \subseteq \text{range } W^T
\]

if and only if

\[
\text{range } \{ \begin{bmatrix} \varphi_0 & \varphi_1 \\ I_0 & K_0 \end{bmatrix} + \varphi_2 \begin{bmatrix} K_1 K_0 & K_1 \end{bmatrix} \} \subseteq \text{range } \varphi_2.
\]

Thus, as the matrix \( \begin{bmatrix} I_0 & 0 \\ K_0 & I_t \end{bmatrix} \) is invertible,

\[
\text{range } \subseteq \text{range } W^T \iff \text{range } \begin{bmatrix} \varphi_0 & \varphi_1 \end{bmatrix} \subseteq \text{range } \varphi_2.
\]

14.2. **Chip set representation.** The chip sets for \( p_1, p_2, p_3 \) are:

- \( C_{p_1} = \{ a_2, a_1 \} \),
- \( C_{p_2} = \{ 1, a_2, a_1 x_1, a_1 x_2 a_2 \} \) and
- \( C_{p_3} = \{ 1, a_1, x_2, a_2 x_1 a_1, a_2 x_2, x_2 a_2 x_1 a_1 \} \),

respectively. Upon setting

\[
w_1 = 1, \ w_2 = a_1, \ w_3 = a_2, \ w_4 = a_1 x_1, \ w_5 = a_1 x_1 a_2, \ w_6 = x_2, \ w_7 = a_2 x_1 a_1, \ w_8 = x_2 a_2 x_1 a_1 \text{ and } w_9 = a_2 x_2^2,
\]

it is readily checked that

\[
\begin{align*}
(p_1)_x(a, x)[h] &= ma_2(h_2 w_2) + ga_1(h_2 w_3), \\
(p_2)_x(a, x)[h] &= f x_1 a_1(h_2 w_3) + ea_2 x_2 a_1(h_1 w_1) + ea_2(h_2 w_4) + f(h_1 w_5) \text{ and} \\
(p_3)_x(a, x)[h] &= dx_2^2 a_2(h_1 w_2) + ca_1 x_1 a_2 x_2(h_2 w_1) + dx_2(h_2 w_7) + ca_1 x_1 a_2(h_2 w_6) + ca_1(h_1 w_9) + d(h_2 w_8).
\end{align*}
\]
Since $w_8$ and $w_9$ are of degree $d - 1 = 2$ in $x$, the majorization condition in §1.4 is the ranges of

$M \otimes A_2, G \otimes A_1, E \otimes A_2, F \otimes I_n$

are included in the space

$\text{span}\{\text{range } C \otimes A_1, \text{range } D \otimes I_n\}$.

Moreover, $p_x(a, x)[h]$ can be expressed as

$p_x(a, x)[h] = [U^T \ W^T] \begin{bmatrix} Q_1(a, x, h) \\ Q_2(a, x, h) \end{bmatrix}$,

in which

$U^T = \begin{bmatrix} dx_2^2a_2 & ma_2 & fx_1a_1 & ga_1 & ea_2x_1a_1 & ca_1x_1a_2 & ea_2 & dx_2 & ca_1x_1a_2 & f1 \end{bmatrix}$,

and the entries in

$W^T = \begin{bmatrix} ca_1 & d1 \end{bmatrix}$,

and the entries in

$Q_1(a, x, h) = \text{column}(h_1w_2, h_2w_2, h_1w_3, h_2w_3, h_1w_1, h_2w_1, h_1w_4, h_2w_7, h_2w_6, h_1w_5)$

are a subcollection of the entries in $V_0$ and $V_1$, whereas the entries in

$Q_2(a, x, h) = \text{column}(h_1w_9, h_2w_8)$

are a subcollection of the entries in $V_2$. Thus if the majorization condition of §1.4 is met, then the range of $U^T$ is a subspace of the range of $W^T$.

### 15. Appendix: A Range Inclusion Example

In Example 2.3 it was shown that if $1 \leq k \leq d$, then the highest degree terms of the polynomial $p(a, x) = a(1 + x^k) + (1 + x^k)a + x^d$ majorize. Thus, in view of item (3) of Theorem 7.3, the range inclusion condition Theorem 7.3 (2) must hold. This is confirmed in this example. To ease the exposition only the case $k = d$ will be discussed.

Since

$p(a, x) = (1 + a)x^d + x^d a + p(a, 0)$,

it is readily checked that

$p_x(a, x)[h] = (1 + a)(hx^{d-1} + xh x^{d-2} + \cdots + x^{d-1}h) + (hx^{d-1} + xh x^{d-2} + \cdots + x^{d-1}h)a$. 

Thus, if \((A, X) \in S_n(\mathbb{R}^g)\), then

\[
p_x(A, X)[h] = [U^T \quad W^T] \begin{bmatrix} H \\ HA \\ HX \\ HXA \\ \vdots \\ HX^{d-1} \\ HX^{d-1}A \end{bmatrix},
\]

with

\[
W^T = [(I_n + A) \quad I_n]
\]

and

\[
U^T = [(I_n + A)X^{d-1} \quad X^{d-1} \quad (I_n + A)X^{d-2} \quad X^{d-2} \cdots (I_n + A)X \quad X].
\]

Consequently,

\[
U^T = [(I_n + A) \quad I_n] \begin{bmatrix} X^{d-1} & 0 & X^{d-2} & 0 & \cdots & X & 0 \\ 0 & X^{d-1} & 0 & X^{d-2} & \cdots & X & 0 \end{bmatrix} = W^T \begin{bmatrix} X^{d-1} & 0 & X^{d-2} & 0 & \cdots & X & 0 \\ 0 & X^{d-1} & 0 & X^{d-2} & \cdots & X & 0 \end{bmatrix},
\]

which clearly displays the fact that \(\text{range } U^T \subseteq \text{range } W^T\).

Moreover,

\[
p(A, X) = [U_1^T \quad W^T] \begin{bmatrix} X \\XA \\ XX \\ XXA \\ \vdots \\ XX^{d-1} \\ XX^{d-1}A \end{bmatrix} + p(A, 0),
\]

with \(U_1^T = 0_{n \times 4n}\). Therefore,

\[
\text{range } U_1^T \subseteq \text{range } W^T,
\]

i.e., the highest degree terms in \(p, W^T\), clearly majorize \(U_1^T\), the lower degree terms in \(p\). \(\square\)

16. Appendix: Examples of chip set representations

In this appendix chips sets are calculated for several polynomials.
Example 16.1. Let
\[ p(a, x) = cx_1ax_2^3 + dx_2^3ax_1 \]
with \( c, d \in \mathbb{R} \). Then
\[ C^1_p = \{1, ax_2^3\} \quad \text{and} \quad C^2_p = \{1, x_2, ax_1, x_2ax_1, x_2^2ax_1\} \]
and, by direct computation, it is readily checked that
\[ C^1_p = \left\{ 1, ax_2^3 \right\} \quad \text{and} \quad C^2_p = \left\{ 1, x_2, x_2^2ax_1 + x_2^3ax_1 \right\}, \]
and hence that \( p_x(a, x)[h] \) is a linear combination of the 8 words in \( C_p \) with polynomial coefficients as it should be. By contrast, the general representation (8.10) for nc polynomials \( p(a, x) \) that are homogeneous in \( x \) of degree 4 yields the formula
\[
p_x(a, x)[h] = \varphi_3 \left\{ \sum_{j=0}^{3} (x \otimes b)_j \otimes V_{3-j} \right\}
\]
with
\[
b = \begin{bmatrix} 1 \\ a \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.
\]
This exhibits \( p_x(a, x)[h] \) as a linear combination of \( t_0 + \cdots + t_3 = 2^2 + 2^4 + 2^6 + 2^8 = 340 \) terms in which at least 332 have zero coefficients.

The general middle matrix representation
\[
p_{xx}(a, x)[h] = \sum_{i,j=0}^{2} V_i(a, x)[h]^T Z_{ij}(a, x) V_j(a, x)[h]
\]
based on the representation (8.10) involves a middle matrix \( Z(a, x) \) with \((t_0 + t_1 + t_2)^2 = 84^2\) blocks. However, in fact for the matrix polynomial under consideration, direct computation yields the formula
\[
p_{xx}(a, x)[h] = \begin{bmatrix} 0 & dax_2^2 & dax_1 & 0 & da & 0 \\ cax_2^2 & 0 & 0 & cax_2 & 0 & c \\ cx_2A & 0 & 0 & c & 0 & 0 \\ 0 & d & 0 & 0 & d & 0 \\ ca & 0 & 0 & 0 & 0 & 0 \\ 0 & d & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_2x_2 \\ h_2ax_1 \\ h_2^2x_2^2 \\ h_2^2ax_1^2 \end{bmatrix}
\]
Thus, only two of the four entries in $V_0 = h \otimes b$, two of the sixteen entries in $V_1 = (h \otimes b) \otimes (x \otimes b)$ intervene and two of the sixty four entries in $V_2 = (h \otimes b) \otimes (x \otimes b)_2$ come into play in this representation.

**Remark 16.2.** Since this polynomial is homogeneous of degree 4 in $x$, it can also be expressed as

$$p(a, x) = [c_1 \cdots c_4] b_0 \otimes x \otimes b_1 \otimes x \otimes b_2 \otimes x \otimes b_3 \otimes x \otimes b_4$$

with

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b_1 = b_3 = \begin{bmatrix} 1 \\ a \end{bmatrix} \quad \text{and} \quad b_0 = b_2 = b_4 = 1.$$ 

This is more efficient than the general representation (8.10), but less efficient than the representation based on the chip set $\mathcal{RC}_p$. 

**Example 16.3.** Let $\tilde{g} = 1$, $a = a_1$, $g = 2$, $C, D \in \mathbb{R}^{k \times k}$ and

$$p(a, x) = C \otimes x_1 ax_3 + D \otimes x_2 ax_1.$$ 

Then

$$p_{xx}(a, x)[h] =$$

$$2C \otimes \{h_1 a(h_2 x_2^2 + x_2 h_2 x_2 + x_2^3 h_2) + x_1 a(h_2 x_2 + h_2 x_2 h_2 + x_2 h_2^3)\}$$

$$2D \otimes \{(h_2 x_2 + h_2 x_2 h_2 + x_2^2 h_2)ax_1 + (h_2 x_2^2 + x_2 h_2 x_2 + x_2^2 h_2)ah_1\}.$$ 

Correspondingly, if $A \in S_n$ and $X \in S_n(\mathbb{R}^2)$, then

$$p_{xx}(A, X)[h] =$$

$$2C \otimes \{H_1 A(H_2 x_2^2 + X_2 x_2 h_2 + X_2^2 h_2) + X_1 a(H_2 x_2^2 + X_2 x_2 h_2 + X_2^2 h_2)\}$$

$$2D \otimes \{(H_2 x_2 + H_2 x_2 h_2 + X_2^2 h_2^2)AX_1 + (H_2 x_2^2 + X_2 x_2 h_2 + X_2^2 h_2)AH_1\}.$$ 

Thus, as

$$C \otimes (M_i H_j M_0 H_j M_j) = (I_\kappa \otimes M_i H_i)(C \otimes M_0)(I_\kappa \otimes H_j M_j)$$

for monomials $M_i$, $M_0$ and $M_j$ in $A$ and $X$,

$$p_{xx}(A, X)[h] =$$

$$2 [\begin{array}{cccccc} I_\kappa \otimes H_1 & I_\kappa \otimes H_2 & I_\kappa \otimes X_2 H_2 & I_\kappa \otimes X_1 AH_2 & I_\kappa \otimes X_2 H_2 & I_\kappa \otimes X_1 AX_2 H_2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & D \otimes A & 0 & 0 & 0 & 0 \\ C \otimes X_2 H_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & D \otimes I_\kappa & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}]$$
Let $M$ denote the $6 \times 6$ block matrix with blocks of size $kn \times kn$ in the last formula for $p_{xx}(A, X)[h]$. If $A \in \mathbb{S}_n(\mathbb{R}^g)$, $X \in \mathbb{S}_n(\mathbb{R}^g)$ and $D = C^T$, then $M = M^T$. Moreover, if $M$ is partitioned as a $3 \times 3$ block matrix with blocks of size $2kn \times 2kn$, then

\[
\begin{bmatrix}
M_{00} & M_{01} & M_{02} \\
M_{10} & M_{11} & 0 \\
M_{20} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
I_{2kn} & 0 & 0 \\
-K_0 & I_{2kn} & 0 \\
0 & -K_1 & I_{2kn}
\end{bmatrix}
= 
\begin{bmatrix}
0 & 0 & M_{02} \\
0 & M_{11} & 0 \\
M_{20} & 0 & 0
\end{bmatrix},
\]

where

\[
\widetilde{K}_0 = \begin{bmatrix} 0 & I_\kappa \otimes X_2 \end{bmatrix} \quad \text{and} \quad \widetilde{K}_1 = \begin{bmatrix} I_\kappa \otimes X_2 & 0 \\
0 & I_\kappa \otimes X_2 \end{bmatrix}.
\]

Correspondingly, $p_x$ can be expressed as

\[
p_x(a, x)[h] = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix}
I_{2kn} & 0 & 0 \\
\widetilde{K}_0 & I_{2kn} & 0 \\
\widetilde{K}_1 \widetilde{K}_0 & \widetilde{K}_1 & I_{2kn}
\end{bmatrix}
\begin{bmatrix}
G_0 \\
G_1 \\
G_2 \end{bmatrix}
\]

with $Q = \begin{bmatrix} C \otimes I_n & D \otimes I_n \end{bmatrix}$,

\[
G_0 = \begin{bmatrix} I_\kappa \otimes H_1 \\
I_\kappa \otimes H_2 \end{bmatrix}, \quad G_1 = \begin{bmatrix} I_\kappa \otimes H_2 X_2 \\
I_\kappa \otimes H_2 A X_1 \end{bmatrix}, \quad G_2 = \begin{bmatrix} I_\kappa \otimes H_2 X_2^2 \\
I_\kappa \otimes H_2 A X_1 \end{bmatrix},
\]

\[
G_3 = \begin{bmatrix} I_\kappa \otimes H_2 X_2^2 \\
I_\kappa \otimes H_2 A X_1 \end{bmatrix} \quad \text{and} \quad \widetilde{K}_2 = \begin{bmatrix} I_\kappa \otimes X_1 A & 0 \\
0 & I_\kappa \otimes X_2 \end{bmatrix}.
\]

Thus, the entries $U$ and $W$ in the middle matrix (8.24) in the representation of

\[
p_{xx}(A, X)[h] + \lambda p_x(A, X)[h]^T p_x(A, X)[h]
\]

are equal to

\[
U^T = Q \begin{bmatrix} \widetilde{K}_2 \widetilde{K}_1 \widetilde{K}_0 & \widetilde{K}_2 \widetilde{K}_1 & \widetilde{K}_2 \end{bmatrix} \quad \text{and} \quad W^T = Q.
\]

Therefore, the range inclusion condition Theorem 7.3 (2)) is met and hence Theorem 7.3 guarantees that the matrix (8.24) is congruent to the matrix on the right hand side of (8.25). Compare with Remark 11.1.

In Example 16.3

\[
W^T W = (C^T C + D^T D) \otimes I_n,
\]

is invertible if and only if $C^T C + D^T D$ is invertible. If $D = C^T$, as is the case for symmetric $\rho$, then $W^T W$ is invertible if and only if $C^T C + C C^T$...
is invertible. This will fail for every symmetric real matrix with zero eigenvalues.

**Remark 16.4.** The condition that the range of $U^T$ is contained in the range of $W^T$ is necessary for the nice factorization in [8.23] to hold. In this example $W$ is independent of $A$ because the words in the polynomial under consideration are all of the form $x_i \cdots x_j$ (i.e., they begin and end with one of the $x$ variables). □

**Remark 16.5.** The $6 \times 6$ block matrix with blocks of size $kn \times kn$ in the last formula for $p_{xx}(A,X)[h]$ can also be rewritten in terms of the permutation matrix $\Pi$ that is defined by the formula

$$
\Pi = \begin{pmatrix} I \otimes H_1 & 0 & 0 & 0 \\
0 & I \otimes H_2 & 0 & 0 \\
0 & 0 & H_2X_2 & 0 \\
0 & 0 & 0 & H_2AX_1 \\
H_2X_2^2 & 0 & 0 & 0 \\
H_2X_2AX_1 & 0 & 0 & 0
\end{pmatrix}
$$

then, since $\Pi^T\Pi = I$,

$$
p_{xx}(A,X)[h] = 2(I_\kappa \otimes [H_1 \ H_2 \ X_2H_2 \ X_1AH_2 \ X_2^2H_2 \ X_1AX_2H_2])
\times (\Pi^T M \Pi)
$$

**Example 16.6.** Suppose that $c, d \in \mathbb{R}$ and

$$
p(a, x) = ca_1x_1a_2x_2^2 + dx_2^2a_2x_1a_1.
$$

Then

$$
\mathcal{R}_p^1 = \{a_1, a_2x_2^2\} \quad \text{and} \quad \mathcal{R}_p^2 = \{1, x_2, a_2x_1a_1, x_2a_2x_1a_1\}
$$

and, by direct computation,

$$
p_x(a, x)[h] = c\{a_1h_1a_2x_2^2 + a_1x_1a_2h_2x_2 + a_1x_1a_2x_2h_2\} + d\{h_2x_2a_2x_1a_1 + x_2h_2a_2x_1a_1 + x_2^2a_2h_1a_1\}$$
is a linear combination of the words in $\mathcal{RC}_p$ with polynomial coefficients, as it should be and can also be expressed as

$$p_x(a, x)[h] = [1 \quad a_1 \quad x_2 \quad a_1x_1a_2 \quad x_2^2a_2 \quad a_1x_1a_2x_2]$$

(16.1)

Moreover,

$$p_{xx}(a, x)[h] = 2\begin{bmatrix} h_2 & a_1h_1 & x_2h_2 & a_1x_1a_2h_2 \\ 0 & dx_2a_2 & 0 & d \\ ca_2x_2 & 0 & ca_2 & 0 \\ 0 & da_2 & 0 & 0 \\ c & 0 & 0 & 0 \end{bmatrix}$$

The reduced middle matrix will be Hermitian if $c = d$.

Correspondingly, if

$$p(A, X) = C \otimes (A_1X_1A_2X_2^2) + D \otimes (X_2^2A_2X_1A_1)$$

with $C, D \in \mathbb{R}^{n \times n}$ and $A, X \in \mathbb{S}_n(\mathbb{R}^2)$. Then, by straightforward computation,

$$p_{xx}(A, X)[h] = 2\begin{bmatrix} I_n \otimes H_2 & I_n \otimes A_1H_1 & I_n \otimes X_2H_2 & I_n \otimes A_1X_1A_2H_2 \\ 0 & D \otimes X_2A_2 & 0 & D \otimes I_n \\ C \otimes A_2X_2 & 0 & C \otimes A_2 & 0 \\ 0 & D \otimes A_2 & 0 & 0 \\ C \otimes I_n & 0 & 0 & 0 \end{bmatrix}$$

Corresponding to (16.1)

$$p_x(A, X)[h] = [I_n \quad A_1 \quad X_2 \quad A_1X_1A_2 \quad X_2^2A_2 \quad A_1X_1A_2X_2]$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & dI_n \\ 0 & 0 & 0 & cI_n & 0 \\ 0 & 0 & 0 & dI_n & 0 \\ 0 & 0 & cI_n & 0 & 0 \\ 0 & dI_n & 0 & 0 & 0 \\ cI_n & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} H_2 \\ H_1A_1 \\ H_2X_2 \\ H_2A_2X_1A_1 \\ H_1A_2X_2 \end{bmatrix}$$

In view of (8.2), the derivative with respect to $x$ of the matrix valued polynomial

$$p(a, x) = C \otimes a_1x_1a_2x_2^2 + D \otimes x_2^2a_2x_1a_1$$
can be expressed as

\[ p_x(A, X)[h] \]

\[
= \begin{bmatrix}
I_\kappa \otimes I_n & I_\kappa \otimes A_1 & I_\kappa \otimes X_2 & I_\kappa \otimes A_1X_1A_2 & I_\kappa \otimes X_2^2A_2 & I_\kappa \otimes A_1X_1A_2X_2 \\
0 & 0 & 0 & 0 & 0 & D \otimes I_n \\
0 & 0 & 0 & D \otimes I_n & C \otimes I_n & 0 \\
0 & 0 & C \otimes I_n & 0 & 0 & 0 \\
C \otimes I_n & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
= U^T \begin{bmatrix}
I_\kappa \otimes H_2 \\
I_\kappa \otimes H_1A_1 \\
I_\kappa \otimes H_2X_2 \\
I_\kappa \otimes H_2A_2X_1A_1 \\
\end{bmatrix}
+ W^T \begin{bmatrix}
I_\kappa \otimes H_1A_2X_2^2 \\
I_\kappa \otimes H_2X_2^2A_2X_1A_1 \\
\end{bmatrix}
\]

with

\[ W^T = [C \otimes A_1 \ D \otimes I_n] \]

and

\[
U^T = \begin{bmatrix}
C \otimes A_1X_1A_2X_2 & D \otimes X_2^2A_2 & C \otimes A_1X_1A_2 & D \otimes X_2 \\
\end{bmatrix}
\]

\[
= W^T \begin{bmatrix}
I_\kappa \otimes X_1A_2X_2 & 0 & I_\kappa \otimes X_1A_2 & 0 \\
0 & I_\kappa \otimes X_2^2A_2 & 0 & I_\kappa \otimes X_2 \\
\end{bmatrix}
\]

Thus, condition Theorem 7.3 (2) is met, whereas

\[
W^T W = \begin{bmatrix}
C \otimes A_1 & D \otimes I_n \\
\end{bmatrix}
\begin{bmatrix}
C^T \otimes A_1 \\
D^T \otimes I_n \\
\end{bmatrix} = CC^T \otimes A_1^2 + DD^T \otimes I_n
\]

is not necessarily invertible, even if \( D = C^T \).

The polynomial \( p \) will be symmetric if and only if \( C = D^T \).

17. Appendix: Illustrating the CHSY Lemma

Let

\[
p(a, x) = ca_1x_1a_2x_2^2 + dx_2^2a_2x_1a_1.
\]

Let \( A_1 = A_2 = X_1 = X_2 = I_n \) and \( c = d = 1 \). Thus,

\[
p_{xx}(A, X)[h] = 2 \begin{bmatrix}
H_2 & H_1 & H_2 & H_2 \\
H_2 & H_1 & H_2 & H_2 \\
H_2 & H_1 & H_2 & H_2 \\
H_2 & H_1 & H_2 & H_2 \\
\end{bmatrix}
\]
and
\[
p_x(A, X)[h] = [I_n \ I_n \ I_n \ I_n \ I_n \ I_n]
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & I_n \\
0 & 0 & 0 & 0 & I_n & 0 \\
0 & 0 & I_n & 0 & 0 & I_n \\
0 & I_n & 0 & 0 & 0 & I_n \\
I_n & 0 & 0 & 0 & 0 & I_n
\end{bmatrix}
\begin{bmatrix}
H_2 \\
H_1 \\
H_2 \\
H_1 \\
H_2
\end{bmatrix}
\]
\[
= [I_n \ I_n \ I_n \ I_n \ I_n \ I_n]
\begin{bmatrix}
H_2 \\
H_1 \\
H_2 \\
H_1 \\
H_2
\end{bmatrix}
\]

The last formula identifies the blocks \( U \) and \( W \) that are introduced in Theorem 6.3 as
\[
U^T = [I_n \ I_n \ I_n \ I_n] \quad \text{and} \quad W^T = [I_n \ I_n].
\]
Thus, as \( U^T U = 4I_n \) and \( W^T W = 2I_n \),
\[
\mathcal{R}_{U^T} = \mathcal{R}_{U^T U} = \mathcal{R}_{W^T W} = \mathcal{R}_{W^T}.
\]

It is convenient to write
\[
\begin{bmatrix}
H_2 \\
H_1 \\
H_2 \\
H_1 \\
H_2
\end{bmatrix} = Q_n \begin{bmatrix}
H_1 \\
H_2
\end{bmatrix}
\text{with } Q_n =
\begin{bmatrix}
0 & I_n \\
I_n & 0 \\
0 & I_n \\
I_n & 0 \\
0 & I_n
\end{bmatrix}
\]
and to choose \( v = e_1 \), where \( \{e_1, \ldots, e_n\} \) is the standard orthonormal basis for \( \mathbb{R}^n \). Then
\[
p_x(A, X)[h] = [I_n \ I_n \ I_n \ I_n \ I_n] Q_n \begin{bmatrix}
H_1 \\
H_2
\end{bmatrix}.
\]

Moreover, the linear transformation \( T \) from \( S_n(\mathbb{R}^2) \) into \( \mathbb{R}^{6n} \) that is defined by the formula
\[
(17.1) \quad TH =
\begin{bmatrix}
H_2 \\
H_1 \\
H_2 \\
H_1 \\
H_2
\end{bmatrix} e_1 =
\begin{bmatrix}
0 & I_n \\
I_n & 0 \\
0 & I_n \\
I_n & 0 \\
0 & I_n
\end{bmatrix} e_1
\]

\begin{bmatrix}
H_1 \\
H_2
\end{bmatrix} e_1
maps the $2(n^2 + n)/2$ dimensional space $\mathbb{S}_n(\mathbb{R}^2)$ onto the $2n$ dimensional subspace

$$\mathcal{R}_T = \left\{ Q_n \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} : u_1 \in \mathbb{R}^n \text{ and } u_2 \in \mathbb{R}^n \right\}$$

of $\mathbb{R}^{6n}$. Correspondingly, the inner products based on the terms in the relaxed Hessian

$$p_{xx}(A, X)[h] + \lambda p_x(A, X)[h]^T p_x(A, X)[h] + \delta \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} Q_n^T Q_n \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}$$

can be calculated as follows:

$$e_1^T p_{xx}(A, X)[h] e_1 = 2y^T \begin{bmatrix} 0 & I_n & 0 & 0 & 0 \\ I_n & 0 & I_n & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} Q_n^T Q_n \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}$$

$$= y^T \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} y,$$

with

$$y = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} e_1,$$

$$e_1^T p_x(A, X)[h]^T p_x(A, X)[h] e_1 = y^T \begin{bmatrix} 4I_n & 8I_n \\ 8I_n & 16I_n \end{bmatrix} y$$

and

$$e_1^T \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} Q_n^T Q_n \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} e_1 = y^T \begin{bmatrix} 2I_n & 0 \\ 0 & 4I_n \end{bmatrix} y.$$

Combining terms (with $A$, $X$ and $v$ as specified above) yields the formula

$$\langle p''_{\lambda, \delta}(A, X)v, v \rangle = 2y^T \begin{bmatrix} (2\lambda + \delta)I_n & (2 + 4\lambda)I_n \\ (2 + 4\lambda)I_n & (2 + 8\lambda + 2\delta)I_n \end{bmatrix} y.$$

Now, for $i, j = 1, \ldots, n$, let

$$\Sigma_{ij} = \frac{e_i e_j^T + e_j e_i^T}{\sqrt{2}} \quad \text{if } i \neq j \quad \text{and} \quad \Sigma_{ii} = e_i e_i^T$$

and let $\mathcal{N} = \mathcal{N}_T$ denote the nullspace of $T$. The set of $(n^2 + n)/2$ matrices

$$\{\Sigma_{ij} : 1 \leq i \leq j \leq n\}$$

is an orthonormal basis for $\mathbb{S}_n$ equipped with the inner product

$$\langle H, H' \rangle = \text{trace}\{(H_1')^T H_1 + (H_2')^T H_2\}.$$
Moreover,
\[ N^\perp = \text{span}\{ (\Sigma_{11}, 0), \ldots, (\Sigma_{n1}, 0), (0, \Sigma_{11}), \ldots, (0, \Sigma_{n1}) \} \]
is the orthogonal complement of \( N \) with respect to this inner product and the linear transformation \( T \) defined by formula (17.1) maps \( N^\perp \) injectively onto \( R_T \). Thus, the dimensions of the maximal positive and negative subspaces of \( S_n(\mathbb{R}^2) \) with respect to the quadratic form \( \langle p''_{\lambda,\delta}(A,X)v,v \rangle \) may be evaluated from the signature of the matrix
\[
\begin{pmatrix}
(2\lambda + \delta)I_n & (2 + 4\lambda)I_n \\
(2 + 4\lambda)I_n & (2 + 8\lambda + 2\delta)I_n
\end{pmatrix},
\]
i.e., from the signature of \((2\lambda + \delta)I_n\) and, assuming that \(2\lambda + \delta \neq 0\), the signature of its Schur complement
\[
\left\{(2 + 8\lambda + 2\delta) - \frac{(4\lambda + 2)^2}{2\lambda + \delta}\right\}I_n = 2\left\{\frac{(6\lambda + \delta + 2)(\delta - 1)}{2\lambda + \delta}\right\}I_n.
\]
If \(\lambda \leq 0\) and \(\delta < 0\), then \(2\lambda + \delta < 0\) and \(\delta - 1 < 0\) and hence the Schur complement will be positive definite if \(6\lambda + \delta + 2 > 0\) and negative definite if \(6\lambda + \delta + 2 < 0\). Consequently,
\begin{equation}
(17.3)
\end{equation}
\[ e^+_n(A,X,v; p''_{\lambda,\delta}, S_n(\mathbb{R}^2)) = 0 \quad \text{if } \delta < 0, \lambda \leq 0 \text{ and } 6\lambda < -\delta - 2. \]

By similar calculations,
\begin{equation}
(17.4)
\end{equation}
\[ e^+_n(A,X,v; p''_{\lambda,\delta}, S_n(\mathbb{R}^2)) = n \quad \text{if } 0 < \delta < 1 \text{ and } \lambda \geq 0. \]

Now, as
\[ p_x(A,X)[h]v = (2H_1 + 4H_2)e_1, \]
it is readily seen that if \((H_1,H_2)\) is restricted to \( N^\perp \), then
\[ p_x(A,X)[h]v = 0 \iff H_2 = -\frac{1}{2}H_1, \]
i.e., if and only if \(H = (H_1,H_2)\) belongs to the space
\[ \mathcal{M} = \text{span}\{ (\Sigma_{11}, -\frac{1}{2}\Sigma_{11}), \ldots, (\Sigma_{n1}, -\frac{1}{2}\Sigma_{n1}) \}. \]
Therefore, as follows easily from (17.2),
\[ H_2 = -\frac{1}{2}H_1 \implies \langle p_x(A,X)[0,H)e_1, e_1 \rangle = -3e_1^TH_1^2e_1. \]

Consequently,
\[ e^+_n(A,X,v; p_x, T(A,X,v)) = 0, \]
which coincides with (17.3), and
\[ e^-_n(A,X,v; p_x, T(A,X,v)) = n, \]
which coincides with (17.4). Moreover,

\[ \mathcal{N}^\perp = \mathcal{M} \oplus \mathcal{L} \]

with

\[ \mathcal{L} = \text{span}\{(\Sigma_{11}, 2\Sigma_{11}), \ldots, (\Sigma_{n1}, 2\Sigma_{n1})\}. \]
18. Not for publication

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