On the signed domination number of some Cayley graphs

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\section*{ABSTRACT}

A signed dominating function of graph $\Gamma$ is a function $g : V(\Gamma) \rightarrow \{ -1, 1 \}$ such that $\sum_{u \in N(v)} g(u) > 0$ for each $v \in V(\Gamma)$. The signed domination number $\gamma_{S}(\Gamma)$ is the minimum weight of a signed dominating function on $\Gamma$. Let $G = \langle S \rangle$ be a finite group such that $e \notin S = S^{-1}$. In this paper, we obtain the signed domination number of $\text{Cay}(S : G)$ based on cardinality of $S$. Also we determine the classification of group $G$ by $|S|$ and $\gamma_{S}(\text{Cay}(S : G))$.

\section*{1. Introduction}

Let $\Gamma$ be a simple graph on vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. The number of edges of the shortest walk joining $v$ and $u$ is called the distance between $v$ and $u$ and denoted by $d(v, u)$. The maximum value of the distance function in a connected graph $\Gamma$ is called the diameter of $\Gamma$ and denoted by $\text{diam}(\Gamma)$. A graph $\Gamma$ is said to be regular of degree $k$ or, $k$-regular if every vertex has degree $k$. A $t$-partite graph is one whose vertex set can be partitioned into $t$ subset, or part, in such a way that no edge has both end in a same part. A complete $t$-partite graph is a $t$-partite graph in which every pair of vertices in separate parts are adjacent and denoted by $K_{n_1,...,n_t}$ if $V = V_1 \cup \ldots \cup V_t$ and $|V_i| = n_i$. Let $G$ be a non-trivial group, $S \subseteq G, S^{-1} = \{ s^{-1} | s \in S \}$, and also $e \notin S = S^{-1}$. The Cayley graph of $G$ denoted by $\text{Cay}(S : G)$, is a graph with vertex set $G$ and two vertices $a$ and $b$ are adjacent if and only if $ab^{-1} \in S$. If $S$ generates $G$, then $\text{Cay}(S : G)$ is connected. Also Cayley graph is a simple and vertex transitive.

For a vertex $v \in V(\Gamma)$, the closed neighborhood $N[v]$ of $v$ is the set consisting of $v$ and all of its neighbors. For a function $g : V(\Gamma) \rightarrow \{ -1, 1 \}$ and a vertex $v \in V$ we define $g[v] = \sum_{u \in N[v]} g(u)$.

A signed dominating function of $\Gamma$ is a function $g : V(\Gamma) \rightarrow \{ -1, 1 \}$ such that $g[v] > 0$ for all $v \in V(\Gamma)$. The weight of a function $g$ is $\omega(g) = \sum_{v \in V(\Gamma)} g(v)$. The signed domination number $\gamma_{S}(\Gamma)$ is the minimum weight of a signed dominating function on $\Gamma$. A signed dominating function of weight $\gamma_{S}(\Gamma)$ is called a $\gamma_{S}$- function. Also $V^{-}$ is the set of vertices which assigned $-1$ by a $\gamma_{S}$- function and $V_{g}^{-} = \{ v \in \Gamma : g(v) = -1 \}$ where $g$ is a signed dominating function. The concept of signed domination number was defined in [1] and has been studied by several authors (see for instance [1, 2, 3, 4, 9, 10]). In [7], domination number as well as signed domination numbers of $\text{Cay}(S : G)$ for all cyclic group $G$ of order $n$, where $n \in \{ p^{m}, pq \}$ and $S = \{ k < n : \text{gcd}(k, n) = 1 \}$ are investigated.
Motivated by [7], in this paper we determine the signed domination number of Cayley graphs $\text{Cay}(S:G)$ for such pairs of $G$ and $S$. Also we obtain the group $G$ based on $\gamma_S(\text{Cay}(S:G))$.

**2. Computation of $\gamma_S(\text{Cay}(S:G))$**

In this section, $G = (S)$ is a finite group of order $n$ and $S$ denote an inverse closed subset of $G \setminus \{e\}$. We determine the signed domination number of $\text{Cay}(S:G)$ where $n - 4 \leq |S| \leq n - 1$. We need the following lemma and theorem.

**Lemma 2.1.** [4] If $\Gamma$ is a complete graph of order $n$, then

$$\gamma_S(\Gamma) = \begin{cases} 1 & \text{if } n \text{ is odd}, \\ 2 & \text{if } n \text{ is even}. \end{cases}$$

**Theorem 2.2.** [1, 6] Let $\Gamma$ be a $k$-regular graph of order $n$. If $k$ is odd, then $\gamma_S(\Gamma) \geq \frac{2n}{k+1}$ and if $k$ is even, then $\gamma_S(\Gamma) \geq \frac{n}{k+1}$.

**Theorem 2.3.** Let $G$ be a group of order $n$. Then $\gamma_S(\text{Cay}(S:G)) = 1$ if and only if $S = G \setminus \{e\}$ and $n$ is odd.

**Proof.** $\gamma_S(\text{Cay}(S:G)) = 1$. We know that $\text{Cay}(S:G)$ is a $|S|$-regular graph. If $|S|$ is odd, then $\gamma_S((\text{Cay}(S:G)) \geq \frac{2n}{|S|+1}$ by Theorem 2.2 and so $|S| \geq (2n-1)$. This is impossible. Hence, $|S|$ is even and by Theorem 2.2, $\gamma_S(\text{Cay}(S:G)) \geq \frac{n}{|S|+1}$. So $|S| = n-1$ and $n$ is odd. The converse is clear.

**Theorem 2.4.** If $G$ is a group of order $n$ and $|S| = n-2$, then $\gamma_S(\text{Cay}(S:G)) = 2$.

**Proof.** Since $G$ is $|S|$-regular, so by Theorem 2.2, $\gamma_S(\text{Cay}(S:G)) \geq \frac{n}{|S|+1}$ and so $|V^-| \leq \frac{n}{2} - 1$. Let $S = \{v_1, \ldots, v_{n-2}\}$ and $G = S \cup \{e, a\}$.

Define $f : V(\text{Cay}(S:G)) \to \{-1, 1\}$ such that

$$f(v_i) = \begin{cases} -1 & i = 1, \ldots, \frac{n}{2} - 1, \\ 1 & \text{otherwise}. \end{cases}$$

Since $\deg(a) = |S|$ and $a \notin S$, so $N(a) = S$ and also $f[e] = f[a] = 1$. For every $v_i \in S$ there is exactly one $v_j \in S$ such that $v_i$ and $v_j$ are not adjacent. Suppose that $f(v_i) = -f(v_j) = -1$, then $f[v_i] = 1$. Hence, $f[v] \geq 1$ for every $v \in V(\text{Cay}(S:G))$. Since $|V^-| = \frac{n}{2} - 1$, $f$ is a $\gamma_S$-function and so $\gamma_S(\text{Cay}(S:G)) = \omega(f) = 2$.

**Theorem 2.5.** Let $G$ be a group of order $n$ and $|S| = n-3$. Then

$$\gamma_S(\text{Cay}(S:G)) = \begin{cases} 3 & \text{if } n \text{ is odd}, \\ 4 & \text{if } n \text{ is even}. \end{cases}$$

**Proof.** Let $S = \{v_1, \ldots, v_{n-3}\}$ and $G \setminus S = \{e, x, y\}$. If $n$ is odd, then $|S| = n-3$ is even. By Theorem 2.2, $\gamma_S(\text{Cay}(S:G)) \geq \frac{n}{|S|+1}$ and so $|V^-| \leq \frac{n-3}{2}$. It is sufficient to define a signed dominating function $g$ such that $V_g^- \subseteq S$ and $|V_g^-| = \frac{n-3}{2}$. Hence, for each $x \in G, g[x] = |S| + 1 - 2|N(x) \cap V_g^-| \geq n - 2 - |V_g^-| = 1$. Thus
\[ \gamma_S(Cay(S : G)) = \omega(g) = 3 \]

Now, if \( n \) is even, then \(|S|\) is odd and so by Theorem 2.2, \( \gamma_S(Cay(S : G)) \geq \frac{4n}{n-2} \). Hence, \(|V^-| \leq \frac{n-4}{2} \) and so each signed dominating function that gives label \(-1\) to exactly \( \frac{n-4}{2} \) vertices is a \( \gamma_S \)-function. Thus \( \gamma_S(Cay(S : G)) = 4 \).

**Lemma 2.6.** Let \( G \) be a group of order \( n \) and \( S = G \setminus \{e,a,b,c\} \). Then \( n \) is even and the induced subgraph on \( \{a,b,c\} \) in \( Cay(S : G) \) is \( K_3 \), \( P_3 \) or empty graph.

**Proof.** Since \( S \) is an inverse closed subset of \( G \), at least one of vertices \( a \), \( b \) or \( c \) as an element of group \( G \) has order two and so \( n \) is even. Let \( A = \{a,b,c\} \) and consider two following cases.

Case 1. Let \( O(x) = 2 \) for each \( x \in A \). If \( a \in N(b) \), then \( ab \in S \) and so \( ab \neq c \). Hence, \( cb \neq a \) and \( ac \neq b \) and so \( bc,ac \in E(Cay(S : G)) \). Therefore, the induced subgraph on \( A \) is \( K_3 \). If \( a \not\in N(b) \), then \( ab = c \) and so \( cb = a \) and \( ac = b \). Thus the induced subgraph on \( A \) has no edge.

Case 2. Let \( O(a) = 2 \) and \( c = b^{-1} \). There are two cases. Suppose that \( 4|n \) and \( O(b) = 4 \). If \( a = b^2 \), then the induced subgraph on \( A \) is empty. If \( a \neq b^2 \), then \( c^2 = b^2 \in S \). So \( ba,ca,bc \in S \). Hence, the induced subgraph on \( A \) is \( K_3 \). If \( 4|n \) or \( O(b) \neq 4 \), then \( ab,ac \not\in A \cup \{e\} \). So \( a \in N(b) \cap N(c) \). If \( O(b) = 3 \), then \( bc \not\in E(Cay(S : G)) \) and so the induced subgraph on \( A \) is \( P_3 \). Otherwise the induced subgraph on \( A \) is \( K_3 \).

**Lemma 2.7.** [7] Let \( G \) be a group and \( H \neq G \) be a subgroup of \( G \) such that \([G:H] = t\). If \( S = G \setminus H \), then \( Cay(S : G) \) is a complete \( t \)-partite graph.

**Theorem 2.8.** Let \( G = \langle S \rangle \) be a finite group of order \( n \) and \( e \not\in S = S^{-1} \) and \(|S| = n - 4\). Then \( \gamma_S(Cay(S : G)) = 4 \).

**Proof.** Let \( G = S \cup \{e,a,b,c\} \). Since \( S \) is an inverse closed subset of \( G \), at least one of elements \( a \), \( b \) or \( c \) has order two, so both of \( n \) and \(|S|\) are even. By Theorem 2.2, \( \gamma_S(Cay(S : G)) \geq \frac{n}{n-3} \). So \(|V^-| \leq \frac{n}{2} - 1 \). If \(|V^-| = \frac{n}{2} - 1 \), then there is vertex \( u \) such that \( f[u] = -1 \). This is contradiction. Hence, \(|V^-| \leq \frac{n}{2} - 2 \). Let \( S = \{v_1, \ldots, v_{n-4}\} \). Define \( f : V(Cay(S : G)) \to \{-1, 1\} \) such that \( f(v_i) = -1 \) if and only if \( 1 \leq i \leq \frac{n}{2} - 2 \). Since \(|V^-| = \frac{n}{2} - 1 \), so \( f \) is a \( \gamma_S^- \) function. Therefore, \( \gamma_S(Cay(S : G)) = \omega(f) = 4 \).

3. Determining the group \( G \) based on \( \gamma_S(Cay(S : G)) \)

Let \( \Gamma_G \) be the Cayley graph \( Cay(S : G) \) where \( S \) is an inverse closed subset of \( G \setminus \{e\} \). In this section we determine the finite group \( G \) based on \( \gamma_S(\Gamma_G) \). In the following there are some remarks for characterizing all cubic Cayley graphs where \( n \in \{8, 10, 12\} \).

**Remark 3.1.** Let \( G \) be a group of order \( n \). By Lemma 1 of [4], \( \gamma_S(\Gamma_G) = n \) if and only if \( G = \mathbb{Z}_2 \). Also for graph \( \Gamma \) of order \( n \), \( \gamma_S(\Gamma) \neq n - t \), where \( t \) is odd.

**Remark 3.2.** Let \( \Gamma_1 \) and \( \Gamma_2 \) be graphs in Figure 1. If \(|S| = 3 \) and \( O(s) = 2 \) for every \( s \in S \), then \( Cay(S : D_8) \simeq \Gamma_1 \) or \( \Gamma_2 \) and \( Cay(S : \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4) \simeq \Gamma_2 \).

**Remark 3.3.** The Cayley graph of groups \( D_8 \) and \( \mathbb{Z}_2 \times \mathbb{Z}_4 \) with \( S = \{s_1, s_2, s_1^{-1}\} \), is isomorphic to \( \Gamma_1 \) and \( Cay(\{s_1, s_2, s_1^{-1}\} : \mathbb{Z}_8) \simeq \Gamma_2 \) in Figure 1.

**Remark 3.4.** Any cubic Cayley graphs of order 10 is isomorphic to one of the graphs in Figure 2.
Remark 3.5. Any cubic Cayley graph of groups $D_{12}, \mathbb{Z}_{12}$ and $\mathbb{Z}_2 / C_2 \mathbb{Z}_6$ is isomorphic to one of the graphs in Figure 3. Moreover let $S = \{s_1, s_2, s_1^{-1}\}$. If $G \simeq D_{12}, \mathbb{Z}_2 \times \mathbb{Z}_6$, then $\text{Cay}(S : G) \simeq \Gamma_2$, otherwise $\text{Cay}(S : \mathbb{Z}_{12}) \simeq \Gamma_1$.

Theorem 3.6. [5] For $n \geq 3$, $\gamma_S(C_n) = n - 2 \left\lfloor \frac{n}{3} \right\rfloor$.

Theorem 3.7. [8] Let $\Gamma$ be a graph with $\Delta \leq 3$, $g$ be a signed dominating function of $\Gamma$ and $u, v \in V(\Gamma)$. If $g(u) = g(v) = -1$, then $d(u, v) \geq 3$.

Theorem 3.8. Let $G$ be a group of order $n$. If $\gamma_S(\Gamma_G) = n - 2$, then $G \simeq \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_5, S_3, \mathbb{Z}_6, \mathbb{Z}_8$ or $D_8$.

Proof. Since $\gamma_S(\Gamma_G) = n - 2$, so $|V^-| = 1$. Let $f$ be a $\gamma_S$-function. Since Cayley graph is a vertex transitive graph, we can suppose that $f(e) = -1$ and $f(x) = 1$ for all $x \in G \setminus \{e\}$. If $|S| = 2$, then $\Gamma$ is a cycle. By Theorem 3.6, $n \in \{3, 4, 5\}$. So $G \simeq \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_5$.

If $|S| \geq 4$, then $n \geq 5$. If the induced subgraph on $S$ is $K_{n-1}$, then $\Gamma_G \simeq K_n$. Thus $\gamma_S(K_n) = 1$ or 2 and so $n = 3$ or 4, respectively, which is impossible. So the induced subgraph on $S$ is not a complete graph and so there are $s, s' \in S$ such that $ss' \notin E(\Gamma_G)$. Since $\Gamma_G$ is $|S|$-regular, there is $t \in N(S) \setminus (S \cup \{e\})$. Also the label of $t$ can be $-1$ (because $|S| \geq 4$). Thus $|V^-_t| \geq 2$. This is contradiction by $\gamma_S(\Gamma_G) = n - 2$. Therefore, $|S| = 3$ and $n$ is even. Since $\gamma_S(\Gamma_G) = n - 2$, by Theorem 3.7, $\text{diam}(\Gamma_G) \leq 2$. Let $S = \{s_1, s_2, s_3\}$ and consider two general cases.

Case 1. Let $O(s_i) = 2$ for every $1 \leq i \leq 3$. Suppose that the induces subgraph on $S$ has the edge $s_is_2$. Hence, $s_is_2 = s_3$ and so $s_is_3 = s_2$ and $s_3s_2 = s_1$. Thus $\langle S \rangle \simeq K_3$, so $\Gamma_G \simeq K_4$ and so
G \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. If $\langle S \rangle$ is isomorphic to an empty graph, then $\text{diam}(\Gamma_G) = 2$ and also $N(s_i) = \{e, x_i, y_i\}$ for $1 \leq i \leq 3$. If $N(s_1) = N(s_2) = N(s_3)$, then $|V(\Gamma_G)| = 6$. Since $\mathbb{Z}_6$ does not have three elements of order two, $G \not\simeq \mathbb{Z}_6$. Let $G \simeq S_3$ and $S = \{(1 \ 2), (1 \ 3), (2 \ 3)\}$. Then $\gamma_S(\text{Cay}(S : S_3)) = n - 2$. Now let $N(s_1) = N(s_2)$ and $N(s_1) \cap N(s_3) = \{e\}$. Let $x_3 \not\in N(y_3)$. So there is a vertex $b$ such that $b \in N(x_3) \setminus S \cup N(S)$. So $f(b)$ can be $-1$. This is impossible. Hence, $x_3y_3 \in E(\Gamma_G)$ and so $n = 8$. Since $G \simeq (S)$ and $O(s) = 2, G \not\simeq \mathbb{Z}_8, Q_8$ and $\mathbb{Z}_2 \times \mathbb{Z}_4$. Also $\Gamma_G \not\simeq \Gamma_1, \Gamma_2$ in Remark 3.2, and so $G \not\simeq \{D_8, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\}$. Since $\Gamma_G$ is a cubic graph and $\text{diam}(\Gamma_G) = 2$, so it is impossible that $N(s_1) = N(s_2)$ and $N(s_1) \cap N(s_3) = \{e, x_1\}$. If $N(s_i) \neq N(s_j)$ for $1 \leq i, j \leq 3$, then there are five cases.

(i) If $N(s_1) \cap N(s_2) = \{e, x\}$ and $N(s_2) \cap N(s_3) = \{e, y\}$, then there are $z \in N(s_1)$ and $w \in N(s_3)$.

(ii) If $N(s_1) \cap N(s_2) \cap N(s_3) = \{e, x, y\}$, then $N(s_i) = \{e, x_i, y_i\}$ for $1 \leq i \leq 3$. So $y_1s_1 \neq s_1$ and so $y_1s_1 = s_2s_2, y_2s_2 = s_3$ and $y_3s_3 = s_1$. Thus $y_1 = s_2s_1, y_2 = s_3s_2$ and $y_3 = s_3s_1$. Also $s_1s_2, s_2s_1 = s_1$ and $s_3s_2 = s_2$. Hence, $s_1s_2 = s_2s_3 = s_3s_1$. On the other hand, since $\text{diam}(\Gamma_G) = 2$, so $n = 8$. Thus $\langle S \rangle \simeq K_3$. But we have $y_3y_2^{-1} = s_1s_3(s_2s_2)^{-1} = e$. This is impossible.

(iii) Let $N(s_i) \cap N(s_j) = \{e\}$ for $i \neq j$. Since $\gamma_S(\text{Cay}(S : \Gamma_G)) = n - 2$, the induced subgraph on $N(S) \setminus \{e\}$ is 2–regular. Regular. So $n = 9$. But group $\mathbb{Z}_9$ does not have three elements of order two. Let $G \simeq \mathbb{Z}_9 = \{(a, b : a^2 = b^3 = (ab)^2 = e)\}$. We know that $\{a, ab, ab^2, ab^3, ab^4\}$ are all elements of $D_9$ of order two. So $N(S) \setminus \{e\} = \{b^i : 1 \leq i \leq 4\}$. Hence, $\text{Cay}(S : D_9) \not\simeq \Gamma_G$ and so $G \not\simeq D_9$.

(iv) If $N(s_1) \cap N(s_2) = \{e, x\}$ and $N(s_1) \cap N(s_3) = N(s_2) \cap N(s_3) = \{e\}$. Since $\text{diam}(\Gamma_G) = 2$, $n = 9$. This is impossible to have a cubic graph of order 9.

(v) If $N(s_1) \cap N(s_2) = \{e, x\}, N(s_1) \cap N(s_3) = \{e, y\}$ and $N(s_2) \cap N(s_3) = \{e, z\}$. Since $G$ is a cubic graph, so there exists $t \in G \setminus N[S]$ and so $d(e, t) = 3$. This is contradiction by $\text{diam}(\Gamma_G) = 2$.

Case 2. Let $s_1 = s_2^{-1}$ and $O(s_2) = 2$. If $s_1s_2 \in E(\Gamma_G)$, then $s_1s_2 = s_3$. So $s_1^2 = s_2$. Thus $O(s_1) = 4$. Hence, $G \simeq \mathbb{Z}_4$. The similar argument applies when $s_2s_2 \in E(\Gamma_G)$. Let $s_1, s_3 \not\in N(s_2)$. Then $N(s_2) = \{e, x_2, y_3\}$. If $O(s_1) = 3$, then $s_3 = s_1^2$ and $s_1 \in N(s_3)$. If $\{s_1, s_3, x_2, y_3\}$, then $n = 6$. Let $G \simeq \mathbb{Z}_6$ and $S = \{a^2, a^3, a^4\}$ or $G \simeq \mathbb{Z}_3$ and $S = \{(1 \ 2), (1 \ 2 \ 3), (1 \ 3 \ 2)\}$. Then the result holds.

If $x_i \in N(s_i) \setminus N(s_2)$ for $i = 1, 3$, then $x_1x_2, x_3y_2 \in E(\Gamma_G)$. So $|G| = 8$. Since $O(s_1) = 3$, this is impossible and if $x_1 \in N(s_1) \setminus N(s_2)$ and $s_3y_2 \in E(\Gamma_G)$, then this graph is not cubic. Now, let $O(s_1) \neq 3$. Then $\langle S \rangle$ is an empty graph. The argument is likewise Case 1 when $\langle S \rangle \simeq \{e\}$. If $N(s_1) = N(s_2) = N(s_3)$, then $n = 6$. For $G \simeq \mathbb{Z}_6$ and $S = \{1, 3, 5\}$ the result holds. Suppose that $N(s_1) = N(s_2), N(s_1) \cap N(s_3) = \{e\}$. Hence, $n = 8$. But all elements of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ have order two. Also $Q_8$ does not have any generator of order 3 and for each generator of $\mathbb{Z}_6, D_8$ or $\mathbb{Z}_2 \times \mathbb{Z}_4$ of cardinality 3, $\text{Cay}(S : G) \not\simeq \Gamma_G$.

Likewise Case 1, we can see $N(s_1) \cap N(s_3) = \{e, x_1\}$ and $N(s_1) = N(s_2)$ and Cases (iv) and (v) are not happened. To complete the proof it is sufficient to consider Cases (i)–(iii).

(i) Let $i \neq j$ and $N(s_i) \neq N(s_j)$. Then $\text{Cay}(\{a, a^2, b, D_8\})$ is isomorphic to $\Gamma_G$.

(ii) If $N(s_1) \cap N(s_2) \cap N(s_3) = \{e, x\}$, then $x_3 \in \{s_2, s_1^{-1}\}, x_5 \in \{s_1, s_1^{-1}\}$ and $x_3^{-1} \in \{s_1, s_2\}$. It is not hard to see that all of them are impossible.

(iii) Let $N(s_i) \cap N(s_j) = \{e\}$ for $i \neq j$. Then $G \in \{Z_{10}, D_{10}\}$. But $\text{Cay}(S : G) \not\simeq \Gamma_G$ when $|S| = 3$.

\[ \square \]

**Theorem 3.9.** Let $G \simeq \langle S \rangle$ be a group of order $n$ and $|S| = 2$. Then $\gamma_S(\text{Cay}(S : G)) = n - 4$ if and only if $G \simeq \mathbb{Z}_6, \mathbb{Z}_7, \mathbb{Z}_8, S_3$ and $D_8$. \[ \square \]
Figure 3. The cubic Cayley graphs where $G \in \{D_{12}, \mathbb{Z}_{12}, \mathbb{Z}_2 \times \mathbb{Z}_6\}$.

**Proof.** By assumption $|S| = 2$ and so the graph $\gamma_S(Cay(S : G))$ is $2$-regular and so is isomorphic to $C_n$. Let $\gamma_S(Cay(S : G)) = n - 4$. By Theorem 3.6, $6 \leq n \leq 8$. Since $S$ is an inverse closed subset of $G$ of order two, $\gamma_S(Cay(S : G) \notin \{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_4, Q_8\}$ but the result is reached for other groups with following generators: $\mathbb{Z}_6 = \langle 1,5 \rangle, S_3 = \langle (1,2),(1,3) \rangle, \mathbb{Z}_7 = \langle 1,6 \rangle, \mathbb{Z}_8 = \langle 1,7 \rangle$ and $D_8 = \langle a^2b, a^3b \rangle$. □

**Theorem 3.10.** Let $G = \langle S \rangle$ be a group of order $n$ where $e \notin S = S^{-1}$ and $|S| = 3$. Then $\gamma_S(Cay(S : G)) = n - 4$ if and only if $G \simeq D_8, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_{10}, D_{10}, \mathbb{Z}_{12}, D_{12}, \mathbb{Z}_2 \times \mathbb{Z}_6$ and $A_4$.

**Proof.** Let $\Gamma_G$ be a cubic Cayley graph and $\gamma_G(\Gamma_G) = n - 4$. Then $n$ is even and $|V^-| = 2$, where $f$ is a $\gamma_S-$ function. By Theorem 2.2, $n \geq 8$. Let $S = \{s_1, s_2, s_3\}$. Since Cayley graph is a vertex transitive graph, so we can assume that $f(e) = -1$. So $f(s_1) = f(x) = 1$ for every $i \in \{1,2,3\}$ and every $x \in N[S] \setminus \{e\}$. Hence, there is vertex $a \in G \setminus N[S]$ such that $f(a) = -1$. Now there are two general cases.

Case 1. If $O(s_i) = 2$ for every $1 \leq i \leq 3$, then the induced subgraph on $S$ is $K_3$ or empty. If the induced subgraph on $S$ is $K_3$, then $Cay(S : G) = K_4$. This is contradiction by $n \geq 8$, so the induced subgraph on $S$ has no edges.

If $N(s_1) = N(s_2) = \{e,x,y\}$, then $x = s_3s_2 = s_3s_1$ and $y = s_1s_2 = s_3s_1$. It implies that $s_3 = s_1s_2s_1 = s_2s_1s_2$. If $x' \in N(s_3)$, then $x's_3 \in \{s_1s_2\}$ and so $x' \notin \{x,y\}$. Hence, $N(s_1) = N(s_2) = N(s_3)$, i.e. $n = 6$. This is impossible. Now suppose that $N(s_i) \neq N(s_j)$ for $1 \leq i \neq j \leq 3$. Let $x \in \bigcap_{i=1}^3 N(s_i)$. Then $x = s_1s_3 = s_3s_2 = s_2s_1$. If $y \in N(s_1), z \in N(s_2)$, then $y = s_3s_1 = x^{-1} = s_1s_2 = z$. So again $n = 6$ which is impossible. Now we consider the following four cases.

(i) Let $N(s_1) \cap N(s_2) = \{e,x\}, N(s_2) \cap N(s_3) = \{e,y\}$ and $N(s_1) \cap N(s_3) = \{e\}$. Then $|N[S]| = 8$. Since there is $a \in G \setminus N[S]$ such that $f(a) = -1$, so $n \geq 10$. If $n \geq 14$, then $|V^-| \geq 3$. Hence, $n \in \{10,12\}$. The groups $\mathbb{Z}_{10}, \mathbb{Z}_{12}, \mathbb{Z}_2 \times \mathbb{Z}_6, A_4$ and $T = \langle \{a,b : a^4 = b^3 = e, a^{-1}ba = b^{-1}\} \rangle$ do not have such this generator. But $Cay(\{a, ab^2, ab^3\}, D_{10}), Cay(\{a, ab, b^3\}, D_{12}) \simeq \Gamma_G$.

(ii) If $N(s_i) \cap N(s_j) = \{e\}$ for every $i,j \in \{1,2,3\}$, then $|N[S]| = 10$. Since $n$ is even and $|V^-| = 2$, so $n = 12$. But for all groups of this order $Cay(S : G) \neq \Gamma_G$.

(iii) Let $N(s_1) \cap N(s_2) = \{e,x\}$ and $N(s_2) \cap N(s_3) = N(s_1) \cap N(s_3) = \{e\}$. Likewise case (i), $n \in \{10,12\}$. For any inverse closed subset of $D_{10}, \mathbb{Z}_{10}$ of cardinality $3$, $Cay(S : D_{10}), Cay(S : \mathbb{Z}_{10}) \neq \Gamma_G$ but $Cay(\{a, ab, ab^3\}, D_{12}) \simeq \Gamma_G$. 

On the other hand, loss of generality, suppose that \( N(\ell_1) \cap N(\ell_2) = \{e, x\}, N(\ell_1) \cap N(\ell_3) = \{e, y\} \) and \( N(\ell_2) \cap N(\ell_3) = \{e, z\} \). Then \( |N(S)| = 7 \). Thus \( n \in \{8, 10, 12\} \). By similar argument in Case (i), \( n \neq 10, 12 \). The groups \( Q_8, \mathbb{Z}_8 \) and \( \mathbb{Z}_2 \times \mathbb{Z}_4 \) do not have this generator. The Cayley graph of group \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \) with any inverse closed subsets of cardinality 3 is isomorphic to \( \Gamma_G \) and \( \text{Cay}(\{a, b, ab\}, D_8) \cong \Gamma_G \).

Case 2. Let \( s_1^{-1} = s_3 \) and \( O(s_2) = 2 \). We show that \( s_1s_3 \notin E(\Gamma_G) \). If not, \( s_3 = s_1s_2 \) and so \( s_1 = s_3s_2 \).

Hence, \( n = O(s_1) = 4 \). This is contradiction by \( n \geq 8 \). Thus \( N(s_2) = \{e, x = s_1s_2, y = s_3s_2\} \).

Let \( O(s_1) = 3 \). Then \( s_3 = s_1^2 \) and \( s_1s_3 \in E(\Gamma_G) \). If \( x \) and \( s_1 \) are adjacent, then \( s_1s_2 = x = s_2s_1 \).

On the other hand, \( s_3y^{-1} = s_2 \). So \( s_3y \in E(\Gamma_G) \) and so \( s_3s_2 = y = s_2s_3 \). Also \( xy^{-1} = s_3 \), i.e. \( x, y \in E(\Gamma_G) \). This means \( n = 6 \) which is impossible. The same argument applies when \( s_3 \in N(x) \) and \( x \in N(y) \). Hence, \( |N(S)| = 8 \). Since \( G \) has some elements of order 2 and 3 and also \( |V^-| = 2 \), \( n = 12 \). The Cayley graph of group \( A_4 = \langle \{a, b : a^2 = b^3 = (ab)^3 = e\} \rangle \) with inverse closed subset \( S = \{a, b, b^2\} \) is isomorphic to \( \Gamma_G \) (Figure 4).

If \( O(s_1) \neq 3 \), then the induced subgraph on \( S \) has at least one edge. If \( N(s_1) = N(s_2) = \{e, x, y\} \), then \( x = s_1s_2 \) and \( y = s_1^{-1}s_2 \). On the other hand, \( ys_1^{-1} = s_1 \) and \( xs_1^{-1} = s_2 \). Thus \( s_2 = s_1^2 \) and \( O(s_1) = 6 \). So \( n = 6 \) which is a contradiction. With similar argument, we can see \( N(s_1) \neq N(s_3) \). Hence, for each \( i \neq j, N(s_i) \neq N(s_j) \). If \( \cap_{i=1}^{3} N(s_i) = \{e, x\} \), then \( xs_2 \in \{s_1, s_1^{-1}\} \). Without loss of generality, suppose that \( s_2x = s_1 \). So \( xs_1^{-1} = s_2 \) and \( s_2s_1 = x = s_1s_2 \). Also since \( x \in N(s_1^{-1}) \), \( s_1 \in \{s_2, s_1^{-1}\} \). If \( x = s_2 \), then \( s_2s_2^{-1} = s_2s_1 \) and if \( x = s_1^{-1} \), then \( s_2 = s_1^3 \). Both of them are impossible. Now we have four cases likewise Case 1. Because of Remarks 3.4 and 3.5, Cases (ii) and (iii) can not be happened.

(i) With similar argument we see that \( n \in \{10, 12\} \). The groups \( A_4 \) and \( T = \langle \{a, b : a^4 = b^3 = e, a^{-1}ba = b^{-1}\} \rangle \) and \( A_4 = \langle \{a, b : a^2 = b^3 = (ab)^3 = e\} \rangle \) do not have such this generator but for other groups \( \Gamma_G \) is isomorphic to following Cayley graphs:

\[
\text{Cay}(\{2, 5, 6\} : \mathbb{Z}_{10}), \text{Cay}(\{a, b, b^4\} : D_{10}), \text{Cay}(\{a, b^3, ab\} : D_{12}), \\
\text{Cay}(\{0, 1\}, (1, 3), (0, 5) : \mathbb{Z}_2 \times \mathbb{Z}_6), \text{Cay}(\{1, 5, TT\} : \mathbb{Z}_{12}).
\]

(iv) In this case, \( |N(S)| = 4 \). Because of properties of the groups of order 10 and 12, \( n \notin \{10, 12\} \) and also \( G \neq Q_8 \). So \( G \cong D_8, \mathbb{Z}_2 \times \mathbb{Z}_4 \). For any generator of these groups \( (S = \{s_1, s_2, s_1^{-1}\}) \), \( \text{Cay}(S : G) \cong \Gamma_G \) and this completes the proof.

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