Complete positivity of nonlinear evolution: A case study

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Simple Hartree-type equations lead to dynamics of a subsystem that is not completely positive in the sense accepted in mathematical literature. In the linear case this would imply that negative probabilities have to appear for some system that contains the subsystem in question. In the nonlinear case this does not happen because the mathematical definition is physically unfitting as shown on a concrete example.

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I. COMPLETE POSITIVITY AND NONLINEARITY

Linear maps that are positive but not \textit{completely} positive (CP) \cite{1} \cite{2} have been shown to play an essential role in characterization of degree of entanglement of correlated quantum systems \cite{3} \cite{4}, an important issue for quantum computation and cryptography. A possibility of an experimental verification of CP of quantum evolutions was discussed in the context of the neutral kaon decay problem in \cite{5}. Although the notion of CP may seem somewhat abstract and technical, it has a simple physical interpretation for \textit{linear} maps. We begin with a system, labelled “1”, whose dynamics is given by some positive map \( \phi^1_t(a) = a(t) \), \( \phi^1_t : \mathcal{A} \rightarrow \mathcal{A} \) where \( \mathcal{A} \) is a set of bounded operators acting in a Hilbert space \( \mathcal{H}_1 \). To avoid technicalities we assume \( \mathcal{H}_1 \) is finite dimensional.

In linear quantum mechanics a reversible dynamics completely positive in the sense of \cite{21,22}. This

\[
\rho_{1+2}(0) = \begin{pmatrix}
\phi^1_{a_{11}} & \ldots & \phi^1_{a_{1m}} \\
\vdots & \ddots & \vdots \\
\phi^1_{a_{m1}} & \ldots & \phi^1_{a_{mm}}
\end{pmatrix}.
\]

(3)

It follows, the argument continues, that since the dynamics on \( \mathcal{A} \) is given by \( \phi^1_t(a_{kl}) \) and the whole density matrix is mapped into

\[
\rho_{1+2}(t) = \begin{pmatrix}
\phi^1_{a_{11}}(t) & \ldots & \phi^1_{a_{1m}}(t) \\
\vdots & \ddots & \vdots \\
\phi^1_{a_{m1}}(t) & \ldots & \phi^1_{a_{mm}}(t)
\end{pmatrix}
\]

(4)

which in linear quantum mechanics reduces to

\[
\sum_{k,l=1}^{m} a_{kl} \otimes |k\rangle\langle l| \rightarrow U_t \otimes 1_2 \left( \sum_{k,l=1}^{m} a_{kl} \otimes |k\rangle\langle l| \right) U_t^{-1} \otimes 1_2.
\]

If \( \rho_{1+2}(t) = \phi^1_{1+2}(\rho_{1+2}(0)) \) is to be a density matrix it should not lead to negative probabilities. Moreover one should be able to do the construction for any \( m \). If this is the case the map \( \phi^1_t \) is said to be CP. The dynamics one typically thinks of in quantum mechanics is linear and therefore the notion of CP was initially defined only for linear maps \cite{14}. However there are many situations in quantum physics where the dynamics is nonlinear. A nonlinear evolution of observables in Heisenberg picture is typical of quantum optics and field theory. Nonlinearly evolving states appear in mean field theories (Hartree-type equations \cite{13}), soliton theory (nonlinear Schrödinger equations), and various attempts of nonlinear generalizations of quantum mechanics. Although the latter theories do not yet correspond to any concrete physical situation they have led to some formal developments especially due to the famous “Einstein-Podolsky-Rosen malignancy” discussed by Gisin and others \cite{16} \cite{19} (see Appendix C).

The argument for CP we have presented does not seem to crucially depend on the linearity of \( \phi^1_t \). It is therefore natural to extend the above definition of CP also to maps which are not linear. This was done independently by Ando and Choi \cite{21} and Arveson \cite{24} and a general structure theorem characterizing all CP (linear and nonlinear) maps was found. Apparently the problem was solved.

A surprise came when Majewski and Alicki showed in \cite{23,24} that a simple Hartree-type nonlinear evolution of a finite-dimensional density matrix does not lead to a dynamics completely positive in the sense of \cite{21,22}. This
can be shown as follows. Consider a nonlinear equation \(i\dot{\rho} = [h(\rho), \rho]\) where \(h(\rho) = \text{Tr}(Q\rho)/\text{Tr}\rho\) is a time-independent nonlinear Hamiltonian operator. The solution of the equation is

\[\phi^t(\rho(0)) = e^{-ih(\rho)t}\rho(0)e^{ih(\rho)t}.\]  

(5)

To show that (5) is not completely positive in the sense of Ando, Choi and Arveson it is sufficient to note that \(\phi^t(\lambda\rho) = \lambda^{2t}\phi^t(\rho)\) whereas there is a theorem [21] stating that a completely positive and 1-homogeneous dynamics is linear. The result seems to imply that any mean-field nonlinear evolution of a density matrix leads to negative probabilities!

Alicki and Majewski suggestion was to investigate more precisely the problem of uniqueness of solutions leading to non-completely-positive nonlinear evolutions. In particular they pointed out that the generator given by the above nonlinear equation is not accretive and the Cauchy problem can have different solutions [23]. They did not however dare to challenge the basic definition proposed in [21].

We will now show that the Hartree-type evolution does not imply negative probabilities because it is the basic Ando-Choi-Arveson definition that is physically unfitting. To do so we shall consider an example of the Hartree-type evolution, essentially equivalent to the one discussed in [22]. The new element we introduce is a physically correct way of describing composite systems which involve nonlinear evolutions. This subtle point was clarified in the papers by Polchinski [18] and Jordan [24], and generalized in [27]. We will first show that the form of correctly extended dynamics differs from the one assumed in the discussion of complete positivity although reduces to the standard expression if the dynamics is linear. Next it will be shown that the definition of a completely positive map analyzed in [21] involves implicitly an ill defined extension of nonlinear dynamics to tensor product spaces. The physical problem turns out to be of the same type as the one with the definition of dynamics of composite systems given by Weinberg in [28]. The Weinberg definition not only led to the nowadays famous “faster-than-light telegraph” but also predicted an apparently paradoxical disagreement between the Bloch equation and Janes-Cummings approaches to two-level systems [31]. A corrected description [31] showed that the paradox is a result of a wrong formalism. It proved also that a precise way of describing noninteracting systems leads to a meaningful dynamics when the systems are coupled.

II. EXAMPLE

Consider two noninteracting systems described by Hamiltonian functions \(H_1(\rho_1) = \left(\text{Tr}h_1\rho_1\right)^2/\text{Tr}1\rho_1\) and \(H_2(\rho_2) = \text{Tr}2\rho_2\). Here \(\rho_1\) and \(\rho_2\) are, respectively, \(n \times n\) and \(m \times m\) density matrices. According to general rules the Hamiltonian function of the composite system is

\[H_{1+2}(\rho_{1+2}) = \text{Tr}_1\circ\text{Tr}_{2}(\rho_{1+2}) + \text{Tr}_2 \circ \text{Tr}_1(\rho_{1+2}) \quad (6)
\]

\[= \left(\frac{\text{Tr}_{1+2}h \otimes 1_2\rho_{1+2}}{\text{Tr}_{1+2}\rho_{1+2}}\right)^2 + \text{Tr}_{1+2}\rho_{1+2}. \quad (7)
\]

The main motivation for the definition (6) is the fact that the Lie-Poisson dynamics of density matrices generated by (6) allows for a complete separation of the two subsystems: A reduced dynamics of a subsystem is characterized entirely in terms of quantities intrinsic to this subsystem and this holds for all initial conditions for \(\rho_{1+2}\) and all Hamiltonian functions \(H_k\) (the most general discussion of this problem can be found in [24] where an extension to situations where no Hamiltonian function exists is also analyzed). The evolution is given by a Lie-Poisson equation [24]

\[i d\rho_{kk'}/dt = \{\rho_{kk'}, H\} \quad \text{involving, in this case, the Poisson bracket}
\]

\[\{A, B\} = \delta_{kk'} \frac{\partial A}{\partial \rho_{kk'}} \frac{\partial B}{\partial \rho_{kl'}} − (A \leftrightarrow B) \quad (8)
\]

which, when translated into the standard matrix notation, leads to the nonlinear Liouville-von Neumann equations \(\dot{\rho}_2 = 0\) and

\[i \dot{\rho}_1 = 2\frac{\text{Tr}_1 h\rho_1}{\text{Tr}_1 \rho_1} [h, \rho_1] \quad (9)
\]

\[i \dot{\rho}_{1+2} = 2\frac{\text{Tr}_{1+2} h \otimes 1_2\rho_{1+2}}{\text{Tr}_{1+2}\rho_{1+2}} [h \otimes 1_2, \rho_{1+2}]. \quad (10)
\]

Equation (10) is a very natural extension of the 1-particle dynamics (4) and could be taken for granted even without the general background we have given above. Define

\[U_t(\rho_1(0)) = \exp \left[−2i\text{Tr}_1(h\rho_1(0))ht/\text{Tr}_1\rho_1(0)\right] \quad (11)
\]

All the expressions involving traces are time independent (as depending on Hamiltonian functions and \(\text{Tr}\rho\) which is a Casimir invariant). Therefore we can immediately write the solutions \(\rho_2(t) = \rho_2(0)\) and

\[\rho_1(t) = U_t(\rho_1(0))\rho_1(0)U^{-1}_t(\rho_1(0)) \quad (12)
\]

\[\rho_{1+2}(t) = U_t(\rho_1(0)) \otimes 1_2\rho_{1+2}(0)U_t(\rho_1(0)) \otimes 1_2. \quad (13)
\]

It is clear that the self-consistency condition

\[\text{Tr}_2 \circ \phi_{1+2}^t = \phi_1^t \circ \text{Tr}_2 \quad (14)
\]

typical of a well defined dynamics is fulfilled. It should be stressed that (14) is not accidental but follows from the very construction of the Lie-Poisson dynamics [27].

The dynamics given by \(\phi_1^t\) is nonlinear but 1-homogeneous. The Theorem 4 in [21] states that the dynamics can not be completely positive. It is obvious, however, that our dynamics preserves positivity of \(\rho(t)\) both for the subsystem and the composite system (this is a general property of this formalism, see [33]). The
dynamics can be uniquely extended from subsystems to the composite ones and then again reduced to subsystems giving the correct result, and this is of course valid for any \( m \). So the dynamics looks completely positive!

To understand what goes wrong consider a more detailed example. Let us take the positive matrix as a \( t = 0 \) density matrix (cf. the proof of Theorem 4 in [21]):

\[
\rho_{1+2}(0) = \begin{pmatrix}
a & a & a & a \\
a & a+b & a+b & a \\
a & a+b & a+b & a \\
a & a & a & a \\
\end{pmatrix}
\] (15)

where \( a, b \) are positive and Hermitian \( n \times n \) matrices (so here we take \( m = 4 \)). A reduced density matrix corresponding to the nonlinear subsystem is \( \rho_1(0) = \text{Tr}_2 \rho_{1+2}(0) = 4a + 2b \). The solution for the subsystem is

\[
\rho_1(t) = U_1(2a + b)(4a + 2b)U^{-1}_t(2a + b)
\]

(16)
The solution for the whole system is

\[
\rho_{1+2}(t) = \hat{U}_t(2a + b) \begin{pmatrix}
a & a & a & a \\
a & a+b & a+b & a \\
a & a+b & a+b & a \\
a & a & a & a \\
\end{pmatrix} \hat{U}^{-1}_t(2a + b),
\]

(17)

where \( \hat{U}_t(2a + b) = U_1(2a + b) \otimes 1_2 \). This dynamics is consistent with [21, 24] but is not in the form one assumes in [21, 24]. Indeed what one typically assumes would correspond to

\[
\rho_{1+2}(t) = \begin{pmatrix}
\phi_1'(a) & \phi_1'(a+b) & \phi_1'(a) & \phi_1'(a) \\
\phi_1'(a) & \phi_1'(a+b) & \phi_1'(a+b) & \phi_1'(a) \\
\phi_1'(a) & \phi_1'(a+b) & \phi_1'(a+b) & \phi_1'(a) \\
\phi_1'(a) & \phi_1'(a) & \phi_1'(a) & \phi_1'(a) \\
\end{pmatrix}
\]

(18)

It is sufficient to compare the “11” entries of (18) and (17) to see that they are different. The correct dynamics gives

\[
a \rightarrow U_1(2a + b)aU^{-1}_t(2a + b)
\]

(19)

whereas (18), which one naively expects, would give

\[
a \rightarrow U_1(2a + b)aU^{-1}_t(a)
\]

(20)

Actually, it can be shown that a physically correct dynamics cannot be in the form (18) for two essential reasons. Indeed, let us first note that the bases chosen in (18) are arbitrary. Choosing a new basis \( |\tilde{k}\rangle \) in “2” instead of \( |k\rangle \) one obtains

\[
\rho_{1+2}(0) = \sum_{ss'\tilde{k}\tilde{l}} \tilde{a}_{ss'\tilde{k}\tilde{l}} |s'\rangle \otimes |\tilde{k}\rangle \langle \tilde{l}| + \sum_m \tilde{a}_{\tilde{k}\tilde{l}} \otimes |\tilde{k}\rangle \langle \tilde{l}|
\]

(21)

The new \( \mathcal{A} \)-valued matrix

\[
\begin{pmatrix}
\tilde{a}_{11} & \ldots & \tilde{a}_{1m} \\
\vdots & \ddots & \vdots \\
\tilde{a}_{m1} & \ldots & \tilde{a}_{mm}
\end{pmatrix}
\]

(23)

is related to (3) by a similarity transformation \( \tilde{a}_{kk} = \sum_{k'k''} U_{kk'}a_{kk''}U^{-1}_{k'l} \) where \( U_{kk'} \) is a \( \mathcal{C} \)-valued unitary \( m \times m \) matrix.

In the generic case the choice of bases is arbitrary and no physically meaningful quantity in the subsystem “1” can depend on the choices made in “2”. Mathematically this means that all physical quantities in “1” should be invariant under unitary similarity transformations of (3) or, which is equivalent, changes from (4) to (22). If the latter condition is not satisfied the density matrix \( \phi_{1+2}(\rho_{1+2}(0)) \) is mathematically ill defined for \( t > 0 \). A knowledge of \( \rho_{1+2}(0) \) is insufficient for predicting \( \rho_{1+2}(t) \): One has to additionally fix a basis. It is obvious that this problem does not occur for the “correctly” extended dynamics.

(18) implies that after time \( t \) the reduced density matrix is

\[
\rho_1(t) = \phi_1'(a) + \phi_1'(a + b) + \phi_1'(a + b) + \phi_1'(a)
\]

(24)

Assume that at \( \tau = 0 \) we change the basis in “2” by the unitary transformation

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}
\]

(25)

Let us stress again the important fact that this unitary transformation represents a passive modification of coordinates of \( \rho_{1+2}(0) \) resulting from the change of basis. This should not be confused with the active transformation \( \rho_{1+2}(0) \rightarrow 1 \otimes U \rho_{1+2}(0) 1 \otimes U^{-1} \). Such a transformation would in general change \( \rho_{1+2}(0) \) into a new density matrix \( \rho_{1+2}'(0) \) and it might not be very surprising that two different density matrices evolve differently. The transformations we discuss leave \( \rho_{1+2}(t) \) unchanged and still change the dynamics. Indeed, the reduced dynamics of “1” becomes

\[
\tilde{\rho}_1(t) = \phi_1'(a) + \phi_1'(a + b) + \phi_1'(b + \frac{b}{2}) + \phi_1'(2a + \frac{b}{2})
\]

(26)

Obviously the reduced dynamics of “1” is now different and this is essentially the celebrated “faster-than-light telegraph” of Gisin [21, 24]. In the original Gisin telegraph described in [16] one performs the change of basis in “2” by changing the direction of a Stern-Gerlach device. Assuming that each measurement of spin in “2” reduces the two-particle entangled state to a concrete eigenstate in “1” one nonlocally decomposes the beam of particles into two sub-beams which are assumed to evolve
independently. Mathematically this amounts to assuming that the reduced density matrix in “1” is a convex combination of projectors corresponding to the chosen basis. The effect is based on the fact that for a nonlinear map \( \phi_1 \) and two different ways of writing the density matrix of “1” as convex combinations \( \rho_1(0) = \sum_k p_k \phi_k = \sum_k \tilde{p}_k \tilde{\phi}_k \) one has \( \sum_k p_k \phi_k^2(\phi_k) \neq \sum_k \tilde{p}_k \tilde{\phi}_k^2(\tilde{\phi}_k) \). The reduced density matrices of “1” we obtain by the reductions \([24,26]\) have these properties. To explicitly see that they are different take \( h = \sigma_z, a = \frac{\sqrt{t}}{2} (1 + \sigma_x), b = \frac{1}{\sqrt{t}} (1 + \sigma_z), \) where \( \sigma_k \) are the Pauli matrices. Then

\[
\rho_1(t) - \tilde{\rho}_1(t) = -\frac{1}{4} \sin^2 \frac{2}{3} t \left[ \cos \frac{4}{3} t \sigma_x + \sin \frac{4}{3} t \sigma_y \right]. \quad (27)
\]

The correct dynamics is free of this problem because the nonlinear terms occurring in the reduced density matrix are basis independent.

Paraphrasing Gisin’s statement \([24]\) one can say that a nonlinear evolution which is completely positive in the sense of \([21,22]\) is physically relevant if and only if it is linear. A few years ago this might be a perfect argument against nonlinear quantum mechanics.

The lesson we are taught by the example is the following. First, to speak about the composition problem in nonlinear theories, one has to specify the way the subsystems “1” and “2” evolve. This concerns not only the subsystem “1” we are interested in, but also the “rest” (this, in principle, can also be a nonlinear evolution). Then one has to specify the dynamics of the composite “1+2” system. This is the most delicate point and one cannot just take any linear definition and use it for a nonlinear system. One must make sure the definitions are basis independent if the choice of bases is physically irrelevant (i.e., if there is no superselection rule). For a nonlinear positive dynamics the condition \( \text{Tr} \rho_1 \phi_1 \phi_2 = \phi_1 \circ \text{Tr} \phi_2, \text{Tr} \phi_1 \circ \phi_1 \phi_2 = \phi_1 \circ \text{Tr} \phi_2, \) \( \text{Tr} \phi_1 \circ \phi_1 + \phi_2 = \phi_1 \circ \text{Tr} \phi_2, \) play a physical role analogous to the requirement of CP for linear maps. The definition of CP accepted in \([21,22]\) does not satisfy these requirements and therefore the fundamental problem of a general characterization of physically relevant nonlinear completely positive maps is still open.

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III. APPENDIX: CONTROVERSIAL ISSUES

In our analysis we have used several techniques and made some statements that may appear controversial. The Appendix addresses three mutually related groups of such problems: Definition of mixed states, projection postulate and faster-than-light effects.

A. Mixed states

In linear quantum mechanics mixed states are defined in several equivalent ways. In nonlinear quantum mechanics the definitions are no longer equivalent.

According to the first definition a mixed state is a probability measure on the set of pure states. This definition is nonunique when one switches to nonlinear observables. To see this consider a state vector \(|\psi\rangle\) representing a spin-1/2 particle. Its Hilbert space is \( C^2 \) and the basis vectors are denoted by \(|0\rangle\) and \(|1\rangle\). A state of mixed polarization can be represented by a \( C^2 \)-valued random variable

\[
|\psi\rangle = \psi_0|0\rangle + e^{i\theta} \psi_1|1\rangle, \quad (28)
\]

where \( \theta \in [0, 2\pi) \) is a random phase. In linear quantum mechanics the average of an observable \( \hat{A} \) could be calculated as follows

\[
\frac{1}{2\pi} \int_0^{2\pi} d\theta \langle \psi \mid \hat{A} |\psi\rangle = |\psi_0|^2 \langle 0 \mid \hat{A} |0\rangle + |\psi_1|^2 \langle 1 \mid \hat{A} |1\rangle, \quad (29)
\]

which is equivalent to representing the state by the projector-valued random variable: With probability \(|\psi_0|^2\) one finds \(|0\rangle\langle 0|\), and \(|1\rangle\langle 1|\) with probability \(|\psi_1|^2\). When it comes to nonlinear quantum mechanics the two approaches are inequivalent. Consider a generalized average (i.e. observable)

\[
A(|\psi\rangle, \langle \psi|) = (|\psi_1 + \psi_1^*|)^5. \quad (30)
\]

A mixture of pure states can be represented by the “|\psi\rangle-valued” random variable \( \theta \mapsto |\psi\rangle \) and the average by

\[
\frac{1}{2\pi} \int_0^{2\pi} d\theta A(|\psi\rangle, \langle \psi|). \quad (31)
\]

However, the mixture cannot be represented by a “|\psi\rangle\langle \psi|”-valued” random variable because \( A(|\psi\rangle, \langle \psi|) \) cannot be written as a function of \( |\psi\rangle\langle \psi| \). [Proof: Assume there exists a function \( B(|\psi\rangle \langle \psi|) = A(|\psi\rangle, \langle \psi|) \) for any \( \psi \). But \( B(|\psi\rangle \langle \psi|) = B(|e^{i\alpha} \psi\rangle \langle e^{i\alpha} \psi|) \) for any \( \alpha \), and \( A(|\psi\rangle, \langle \psi|) \neq A(|e^{i\alpha} \psi\rangle, \langle e^{i\alpha} \psi|) \) for almost all \( \alpha \). Contradiction.] The assumption that all nonlinear observables on pure states can be written as functions of \( |\psi\rangle\langle \psi| \) is therefore a serious restriction (see below). The approach to mixtures via probability measures on pure states was developed the works of Michik \([30]\) who was also the first to seriously address the question of mixtures vs. nonlinearity (see also \([27]\)).

A definition of a mixed state which is widely used in the literature is the following: A mixed state is an operator \( \rho \) which is Hermitian, positive, trace-class and normalized \((\text{Tr} \rho = 1)\), and which is not a projector \((\rho^2 \neq \rho)\). This is the definition we use in the paper. In the nonlinear framework we use a density matrix \( \rho \) is a fully quantum object and has an ontological status analogous to this of a wave function in standard quantum mechanics. For
a modern discussion of density matrices from such a perspective see [39].

From what we have written it does not yet follow how to introduce dynamics. Starting with pure states $|\psi\rangle$ one can take a nonlinear Schrödinger dynamics $t \mapsto |\psi(t)\rangle$. Starting with pure states $|\psi\rangle\langle\psi|$ one can take a nonlinear Liouville-von Neumann dynamics $t \mapsto |\psi(t)\rangle\langle\psi(t)|$. These approaches are, in general, inequivalent because not all Hamiltonian functions $H(|\psi\rangle, \langle\psi|)$ can be written as $H(|\psi\rangle\langle\psi|)$. However, once we have an $H = H(|\psi\rangle\langle\psi|)$ we can treat it as a restriction to projectors of a more general $H = H(\rho)$ and define a Lie-Poisson dynamics in terms of the Bôna-Jordan Poisson bracket. This is what we do in the paper.

Suppose now we have a dynamics of $\rho$ which preserves Hermiticity, trace-class property, and positivity of $\rho(t)$. The eigenvalues of $H(t)$ play then a role of probabilities analogous to those we discussed above. In spite of this expected property of $\rho(t)$ our density matrix cannot satisfy an ordinary convexity principle: A convex combination of two solutions is no longer a solution of the nonlinear evolution equation. Typically this is regarded as an argument against nonlinearly evolving $\rho$. This problem was discussed by Bôna and Jordan. They proposed the following interpretation. There are two kinds of density matrices in quantum mechanics. One class corresponds to a situation where an experimentalist controls the mixture by, for example, introducing the random phase. Then the pure-state components of the mixture should be treated separately and the dynamics is nonlinear at the pure state level. This also assumes that there exists a privileged set of observables which is controlled during an experiment.

As a result what one gets is a kind of a superselection principle. The second class of density matrices consists of those that arise because of some reduction procedure and entanglement. The typical example is a one-particle subsystem of an EPR pair. An observer at one side of this experiment has no way to control the mixture at the other side (see below). One can add that there exists a third class of mixtures that cannot be controlled either: These are simply very large systems. It is impossible to control the pure-state components of nonlinearly evolving mixtures that occur in Bose-Einstein condensation of atomic clouds or chemical reactions described by nonlinear thermodynamics.

Finally, let us give the fourth example. The nonlinear gauge transformations introduced by Doebner and Goldin [45] are based on the assumption that all actual measurements are based on those of the position observable. All theories that lead to the same probability densities in position space at any time and for all physical situations are therefore regarded as physically equivalent. Nonlinear gauge transformations do not change the position space probability density and although transform a linear Schrödinger dynamics into a nonlinear one, they nevertheless do not generate any new physics [40]. An extension of Doebner-Goldin transformations to density matrices [41] leads to the requirement that the diagonal elements of density matrices in position space, $\rho(x, x)$, must be unchanged by gauge transformations. Repeating the Doebner-Goldin argument one obtains a class of nonlinear theories that are fully equivalent to the linear Liouville-von Neumann dynamics. Such theories do not satisfy the ordinary convexity principle but only a “convexity principle on the diagonal” in position space.

B. Projection postulate

Consider a solution $|\psi\rangle$ of a linear Schrödinger equation. Projection of $|\psi\rangle$ on an eigenstate of a self-adjoint operator $A$ does not pose any problem since the projected state is again a solution of the same equation. When one tries to perform the same operation with a nonlinear Schrödinger equation one immediately faces two difficulties. First, the projected state is no longer a solution of the same equation and one has to add something not to leave the Hilbert space. This property was even used to test the logarithmic nonlinearity [42–44]. This “something” one has to add may be highly nontrivial (for example, a nonlinear gauge transformation [45]). Second, if the observable one measures is nonlinear the notion of an eigenstate is ambiguous [41,44]. To make an argument based on the projection postulate physically sound one has to explicitly address these issues. Otherwise the argument is hand-waving and cannot be conclusive.

To avoid such dilemmas it is best to use any interpretation of quantum mechanics which is not based on the postulate. This is clearly acceptable and is not a peculiarity of nonlinear quantum mechanics. Similar problems occur in quantum cosmology.

C. Faster-than-light signals

The problem with faster than light signals was independently discovered by Gisin [16], Polchinski (see the footnote in [31]), Svetlichny [47] and one of us [48]. The first paper where a possibility of a conflict between locality and linearity was mentioned is the work of Haag and Bannier [20].

The original formulation due to Gisin made an explicit use of the projection postulate and was apparently model independent. An alternative “model-independent” version was given in [48]. The argument of Polchinski referred to the concrete version of nonlinear quantum mechanics proposed by Weinberg. The Weinberg model can serve as an illustration of all the three ways of generating the effect. It simultaneously shows what can be done to avoid it.

Consider two separated systems described by Hamiltonian functions $H_1(|\phi\rangle\langle\phi|) = E_1|\phi\rangle\langle\phi|$ and

$$H_2(|\chi\rangle\langle\chi|) = E_2|\chi\rangle\langle\chi| + c|\chi|^{2}x^{2}(|\chi|^{2}).$$
The element which is responsible for the faster-than-light effects is the particular form of Hamiltonian function of the composite system which is chosen as

$$H_{1+2}(\ket{\psi}, \langle \psi \rangle) = \sum_l H_1(\ket{\varphi_l} \langle \varphi_l \rangle) + \sum_k H_2(\ket{\chi_k} \langle \chi_k \rangle),$$  \hspace{1cm} (32)

where \( \ket{\varphi_l} = \sum_k \psi_{lk} \ket{k} \), \( \ket{\chi_k} = \sum_l \psi_{kl} \ket{l} \). Notice that \( H_{1+2}(\ket{\psi}, \langle \psi \rangle) \neq H_{1+2}(\bra{\psi} \langle \psi \rangle) \). There exist entangled solutions of the corresponding nonlinear 2-particle equation

$$\psi = \ket{\varphi_1} \otimes \ket{\chi_1} + \ket{\varphi_2} \otimes \ket{\chi_2}$$  \hspace{1cm} (33)

where \( \ket{\varphi_1} \) and \( \ket{\chi_1} \) are some solutions of the 1-particle Schrödinger equations corresponding to the subsystems. The nonlinearity in “2” can make \( \langle \chi_1 | \chi_2 \rangle \) time dependent and proportional to \( \sin(4\epsilon \sigma_z t) \), where \( \sigma_z \) is, in general, nonvanishing. Now consider the linear subsystem. Its reduced density matrix obtained from \( (33) \) contains \( \langle \chi_1 | \chi_2 \rangle \) which depends on \( \epsilon \). As a result there exists an observable in the linear system whose average value depends on \( \epsilon \). For example \( \langle \sigma_y \rangle = \text{Im} \langle \chi_1 | \chi_2 \rangle \). The telegraph so obtained allows one to send information from the nonlinear system to the linear one. A physical interpretation of this phenomenon was given in detail in \[14\]. Technically the effect follows from the fact that an appropriate 2-particle Poisson bracket does not vanish: \( \{ \langle \sigma_y \rangle, H_2 \} \neq 0 \). The two functions appearing in this bracket correspond to different subsystems. An observation that in Weinberg’s nonlinear quantum mechanics such brackets may not be unique is due to Polchinski. The proof given in \[15\] was based on the observation that entangled solutions of 2-particle nonlinear Schrödinger equations may involve states whose scalar product \( \langle \chi_1 | \chi_2 \rangle \) is not conserved by the dynamics. As such it did not, apparently, refer to a concrete model. It turned out, however, that although the nonconservation of scalar products is a general property of nonlinear evolutions in Hilbert spaces \[14\] the existence of appropriate entangled solutions is not at all general.

The telegraph described by Gisin in \[16,17\] works in the opposite direction and its mathematical origin is different. The element which is technically responsible for the Gisin effect is the basis dependence of the Hamiltonian function \([12]\). To see this consider two bases in the linear system: A basis \( | \pm \rangle \) (spin “up” or “down”), and some other basis \( | \alpha, \beta, \pm \rangle = U(\alpha, \beta) | \pm \rangle \) obtained from the “up-down” one by means of an SU(2) transformation \( U(\alpha, \beta) = \begin{pmatrix} \alpha & \beta \\ -\beta & \bar{\alpha} \end{pmatrix} \). The 2-particle solution which is the singlet at \( t = 0 \), written in the basis \( | \alpha, \beta, r \rangle | s \rangle \), where \( r, s = \pm \), is

$$|\psi\rangle = \frac{1}{\sqrt{2}} e^{-i(E_1+E_2-\epsilon X^2)t} \begin{pmatrix} -\beta e^{-i2\epsilon X t} \\ \alpha e^{i2\epsilon X t} \end{pmatrix} \begin{pmatrix} -\beta e^{-i2\epsilon X t} \\ \alpha e^{i2\epsilon X t} \end{pmatrix}$$

where \( X = |\beta|^2 - |\alpha|^2 \). The reduced density matrix of the nonlinear system “2” is

$$\rho_2 = \frac{1}{2} \left[ 1 + \text{Re}(\bar{\alpha}\beta)\sin(4\epsilon(|\alpha|^2 - |\beta|^2)t)\sigma_y + \text{Im}(\bar{\alpha}\beta)\sin(4\epsilon(|\alpha|^2 - |\beta|^2)t)\sigma_z \right].$$  \hspace{1cm} (34)

The average of \( \sigma_y \) in the nonlinear system is

$$\langle \sigma_y \rangle = 2\text{Re}(\bar{\alpha}\beta)\sin(4\epsilon(|\alpha|^2 - |\beta|^2)t)$$  \hspace{1cm} (35)

and, hence, depends on the choice of basis made in the linear one.

As we can see we have obtained the telegraphs without any use of the projection postulate. Interpreting \( H_{1+2} \) as an average energy, and taking into account that its value is basis dependent, we can conclude that this kind of description cannot correspond to a closed system.

It is obvious how to eliminate both phenomena. First, one has to guarantee that the Hamiltonian function is basis independent (this eliminates the Gisin effect). The other effect is eliminated if any two functions corresponding to the two subsystems commute with respect to the 2-particle Poisson bracket.

To make sure that the first condition is satisfied one can require that each subsystem observable is a function of the reduced density matrix of this subsystem. This leads naturally to a density matrix formalism but can be done also for pure states by restricting all 2-particle density matrices to projectors. It is quite remarkable that the restriction of all 1-particle observables to functions of local density matrices turns out to automatically guarantee the commutability of separated observables. Denote by \( \text{Tr}_1 \) and \( \text{Tr}_2 \) the partial traces. Consider two functions

$$A = A(\rho_{1+2}) = A_1 \circ \text{Tr}_2(\rho_{1+2}) = A_1(\rho_1),$$

$$B = B(\rho_{1+2}) = B_2 \circ \text{Tr}_1(\rho_{1+2}) = B_2(\rho_2),$$

and let \( \{ \cdot, \cdot \} \) be a 2-particle bracket. Polchinski and Jordan noticed that then \( \{ A, B \} = 0 \). Introducing the Casimir invariant \( C_2 = \text{Tr} (\rho^2) \) one can show that the Poisson bracket \( \{ \cdot, \cdot \} \) is a particular case of a Nambu-type 3-bracket \[16,27\]: \( \{ \cdot, \cdot, \cdot \} = \frac{1}{2} C_2 \). The most general version of the Polchinski-Jordan theorem was proved in \[27\] in the following form:

**Theorem:** Assume \( A \) and \( B \) are observables which are functions of reduced density matrices of two separated \( N \)- and \( M \)-particle systems. Then \( \{ A, B, \cdot \} = 0 \), where the bracket corresponds to the composite \( (N+M) \)-particle system.

This result allows for an extension of the above construction to a more general class of “Lie-Nambu” theories where instead of \( C_2 \) one puts a general nonlinear function \( F \) (the commutability of observables is independent of the choice of \( F \)).

In this context we would like to add two comments. First, the formalism of Polchinski, Bóna and Jordan leads ultimately to integro-differential equations, which contradicts the belief that local physics must involve “local” equations (for a discussion see \[29\]). Second, there exist global phenomena which do not necessarily lead to
faster-than-light telegraphs but have no counterpart in liner quantum mechanics (e.g. the threshold effects discussed in \cite{51} and the “Big Brother effect” occuring for a class of Lie-Nambu equations \cite{52}.)

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$$A \otimes M_m(C) \xrightarrow{I_B} M_m(A) \xrightarrow{\phi} M_m(A) \xrightarrow{I_m^{-1}} M_m(A) \xrightarrow{\phi} M_m(A) \xrightarrow{I_m} A \otimes M_m(C)$$

is noncommutative.

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