Riesz transforms associated to Schrödinger operators with negative potentials

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Abstract

The goal of this paper is to study the Riesz transforms $\nabla A^{-1/2}$ where $A$ is the Schrödinger operator $-\Delta - V$, $V \geq 0$, under different conditions on the potential $V$. We prove that if $V$ is strongly subcritical, $\nabla A^{-1/2}$ is bounded on $L^p(\mathbb{R}^N)$, $N \geq 3$, for all $p \in (p'_0; 2]$ where $p'_0$ is the dual exponent of $p_0$ where $2 < \frac{2N}{N-2} < p_0 < \infty$; and we give a counterexample to the boundedness on $L^p(\mathbb{R}^N)$ for $p \in (1; p'_0)\cup(p_0; \infty)$ where $p_0 := \frac{p_0N}{N+p_0}$ is the reverse Sobolev exponent of $p_0$. If the potential is strongly subcritical in the Kato subclass $K_N^{\infty}$, then $\nabla A^{-1/2}$ is bounded on $L^p(\mathbb{R}^N)$ for all $p \in (1; 2]$, moreover if it is in $L^{N/2}(\mathbb{R}^N)$ then $\nabla A^{-1/2}$ is bounded on $L^p(\mathbb{R}^N)$ for all $p \in (1; N)$. We prove also boundedness of $V^{1/2}A^{-1/2}$ with the same conditions on the same spaces. Finally we study these operators on manifolds. We prove that our results hold on a class of Riemannian manifolds.

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1 Introduction and definitions

Let $A$ be a Schrödinger operator $-\Delta + V$ where $-\Delta$ is the nonnegative Laplace operator and the potential $V: \mathbb{R}^N \to \mathbb{R}$ such that $V = V^+ - V^-$ (where $V^+$ and $V^-$ are the positive and negative parts of $V$, respectively). The operator is defined via the sesquilinear form method. We define

$$a(u, v) = \int_{\mathbb{R}^N} \nabla u(x) \nabla v(x) dx + \int_{\mathbb{R}^N} V^+(x) u(x) v(x) dx - \int_{\mathbb{R}^N} V^-(x) u(x) v(x) dx$$

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\[ D(a) = \left\{ u \in W^{1,2}(\mathbb{R}^N), \int_{\mathbb{R}^N} V^+(x)u^2(x)dx < \infty \right\}. \]

Here we assume \( V^+ \in L^1_{loc}(\mathbb{R}^N) \) and \( V^- \) satisfies (for all \( u \in D(a) \)):

\[
\int_{\mathbb{R}^N} V^-(x)u^2(x)dx \leq \alpha \left[ \int_{\mathbb{R}^N} |\nabla u|^2(x)dx + \int_{\mathbb{R}^N} V^+(x)u^2(x)dx \right] + \beta \int_{\mathbb{R}^N} u^2(x)dx \tag{1}
\]

where \( \alpha \in (0, 1) \) and \( \beta \in \mathbb{R} \). By the well-known KLMN theorem (see for example [21] Chapter VI), the form \( a \) is closed (and bounded from below). Its associated operator is \( A \). If in addition \( \beta \leq 0 \), then \( A \) is nonnegative.

We can define the Riesz transforms associated to \( A \) by

\[
\nabla A^{-1/2} := \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty \sqrt{t} \nabla e^{-tA} dt.
\]

The boundedness of Riesz transforms on \( L^p(\mathbb{R}^N) \) implies that the domain of \( A^{1/2} \) is included in the Sobolev space \( W^{1,p}(\mathbb{R}^N) \). Thus the solution of the corresponding evolution equation will be in the Sobolev space \( W^{1,p}(\mathbb{R}^N) \) for initial data in \( L^p(\mathbb{R}^N) \).

It is our aim to study the boundedness on \( L^p(\mathbb{R}^N) \) of the Riesz transforms \( \nabla A^{-1/2} \). We are also interested in the boundedness of the operator \( V^{1/2}A^{-1/2} \). If \( \nabla A^{-1/2} \) and \( V^{1/2}A^{-1/2} \) are bounded on \( L^p(\mathbb{R}^N) \), we obtain for some positive constant \( C \)

\[
\|\nabla u\|_p + \|V^{1/2}u\|_p \leq C\|(-\Delta + V)^{1/2}u\|_p.
\]

By a duality argument, we obtain

\[
\|(-\Delta + V)^{1/2}u\|_{p'} \leq C(\|\nabla u\|_{p'} + \|V^{1/2}u\|_{p'})
\]

where \( p' \) is the dual exponent of \( p \).

Riesz transforms associated to Schrödinger operators with nonnegative potentials were studied by Ouhabaz [24], Shen [27], and Auscher and Ben Ali [2]. Ouhabaz proved that Riesz transforms are bounded on \( L^p(\mathbb{R}^N) \) for all \( p \in (1, 2] \), for all potential \( V \) locally integrable. Shen and Auscher and Ben Ali proved that if the potential \( V \) is in the reverse Hölder class \( B_q \), then the Riesz transforms are bounded on \( L^p(\mathbb{R}^N) \) for all \( p \in (1, p_1) \) where \( 2 < p_1 \leq \infty \) depends on \( q \). The result of Auscher and Ben Ali generalize that of Shen because Shen has restrictions on the dimension \( N \) and on the class \( B_q \). Recently, Badr and Ben Ali [5] extend the result of Auscher and Ben Ali...
[2] to Riemannian manifolds of homogeneous type with polynomial volume growth where Poincaré inequalities hold and Riesz transforms associated to the Laplace-Beltrami operator are bounded. They also prove that a smaller range is possible if the volume growth is not polynomial.

With negative potentials new difficulties appear. If we take $V \in L^\infty(\mathbb{R}^N)$, and apply the method in [24] to the operator $A + \|V\|_\infty$, we obtain boundedness of $\nabla(A + \|V\|_\infty)^{-1/2}$ on $L^p(\mathbb{R}^N)$ for all $p \in (1; 2]$. This is weaker than the boundedness of $\nabla A^{-1/2}$ on the same spaces. Guillarmou and Hassell [17] studied Riesz transforms $\nabla (A \circ P_+)^{-1/2}$ where $A$ is the Schrödinger operator with negative potential and $P_+$ is the spectral projection on the positive spectrum. They prove that, on asymptotically conic manifolds $M$ of dimension $N \geq 3$, if $V$ is smooth and satisfies decay conditions, and the Schrödinger operator has no zero-modes nor zero-resonances, then Riesz transforms $\nabla (A \circ P_+)^{-1/2}$ are bounded on $L^p(M)$ for all $p \in (1, N)$. They also prove (see [18]) that when zero-modes are present, Riesz transforms $\nabla (A \circ P_+)^{-1/2}$ are bounded on $L^p(M)$ for all $p \in \left(\frac{N}{N-2}, \frac{N}{3}\right)$, with bigger range possible if the zero modes have extra decay at infinity.

In this paper we consider only negative potentials. From now on, we denote by $A$ the Schrödinger operator with negative potential,

$$A := -\Delta - V, \quad V \geq 0.$$ 

Our purpose is, first, to find optimal conditions on $V$ allowing the boundedness of Riesz transforms $\nabla A^{-1/2}$ and that of $V^{1/2} A^{-1/2}$ on $L^p(\mathbb{R}^N)$ second, to find the best possible range of $p$’s.

Let us take the following definition

**Definition 1.1.** We say that the potential $V$ is strongly subcritical if for some $\varepsilon > 0$, $A \geq \varepsilon V$. This means that for all $u \in W^{1,2}(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} Vu^2 \leq \frac{1}{1 + \varepsilon} \int_{\mathbb{R}^N} \nabla u^2.$$

For more information on strongly subcritical potentials see [15] and [33]. With this condition, $V$ satisfies assumption (I) where $\beta = 0$ and $\alpha = \frac{1}{1+\varepsilon}$. Thus $A$ is well defined, nonnegative and $-A$ generates an analytic contraction semigroup $(e^{-tA})_{t \geq 0}$ on $L^2(\mathbb{R}^N)$.

Since $-\Delta - V \geq \varepsilon V$ we have $(1 + \varepsilon)(-\Delta - V) \geq \varepsilon (-\Delta)$. Therefore

$$\|\nabla u\|_2^2 \leq \left(1 + \frac{1}{\varepsilon}\right)\|A^{1/2} u\|_2^2.$$  \hspace{1cm} \text{(2)}
Thus, $\nabla A^{-1/2}$ is bounded on $L^2(\mathbb{R}^N)$. Conversely, it is clear that if $\nabla A^{-1/2}$ is bounded on $L^2(\mathbb{R}^N)$ then $V$ is strongly subcritical.

We observe also that $-\Delta - V \geq \varepsilon V$ is equivalent to

$$||V^{1/2}u||_2^2 \leq \frac{1}{\varepsilon} ||A^{1/2}u||_2^2. \quad (3)$$

Thus, $V^{1/2}A^{-1/2}$ is bounded on $L^2(\mathbb{R}^N)$ if and only if $V$ is strongly subcritical.

So we can conclude that

$$||\nabla u||_2 + ||V^{1/2}u||_2 \leq C ||(-\Delta - V)^{1/2}u||_2$$

if and only if $V$ is strongly subcritical. Then by duality argument we have

$$||\nabla u||_2 + ||V^{1/2}u||_2 \approx ||(-\Delta - V)^{1/2}u||_2$$

if and only if $V$ is strongly subcritical.

To study Riesz transforms on $L^p(\mathbb{R}^N)$ for $1 \leq p \leq \infty$ with $p \neq 2$ we use the results on the uniform boundedness of the semigroup on $L^p(\mathbb{R}^N)$. Taking central potentials which are equivalent to $c/|x|^2$ as $|x|$ tends to infinity where $0 < c < (\frac{N-2}{2})^2, N \geq 3$, Davies and Simon [15] proved that for all $t > 0$ and all $p \in (p'_0; p_0)$,

$$||e^{-tA}||_{p-p} \leq C$$

where $2 < \frac{2N}{N-2} < p_0 < \infty$ and $p'_0$ its dual exponent. Next Liskevich, Sobol, and Vogt [23] proved the uniform boundedness on $L^p(\mathbb{R}^N)$ for all $p \in (p'_0; p_0)$ where $2 < \frac{2N}{N-2} < p_0 = \frac{2N}{(N-2)(1-\sqrt{1-\frac{1}{p_0}})}$, for general strongly subcritical potentials. They also proved that the range $(p'_0, p_0)$ is optimal and the semigroup does not even act on $L^p(\mathbb{R}^N)$ for $p \notin (p'_0, p_0)$. Under additional condition on $V$, Takeda [31] used stochastic methods to prove a Gaussian estimate of the associated heat kernel. Thus the semigroup acts boundedly on $L^p(\mathbb{R}^N)$ for all $p \in [1, \infty]$.

In this paper we prove that when $V$ is strongly subcritical and $N \geq 3$, Riesz transforms are bounded on $L^p(\mathbb{R}^N)$ for all $p \in (p'_0; 2]$. We also give a counterexample to the boundedness of Riesz transforms on $L^p(\mathbb{R}^N)$ when $p \in (1; p'_0) \cup (p_0; \infty)$ where $2 < p_0 := \frac{p_0N}{N+p_0} < p_0 < \infty$. If $V$ is strongly subcritical in the Kato subclass $K_{N}^{\infty}, N \geq 3$ (see Section 4), then $\nabla A^{-1/2}$ is bounded on $L^p(\mathbb{R}^N)$ for all $p \in (1, 2]$. If, in addition, $V \in L^{N/2}(\mathbb{R}^N)$ then it
is bounded on $L^p(\mathbb{R}^N)$ for all $p \in (1, N)$. With the same conditions, we prove similar results for the operator $V^{1/2}A^{-1/2}$. Hence if $V$ is strongly subcritical and $V \in K^\infty_N \cap L^{N/2}(\mathbb{R}^N)$, $N \geq 3$, then

$$\|\nabla u\|_p + \|V^{1/2}u\|_p \approx \|(-\Delta - V)^{1/2}u\|_p$$

for all $p \in (N'; N)$.

In the last section, we extend our results to Riemannian manifolds. We denote by $-\Delta$ the Laplace-Beltrami operator on a complete non-compact Riemannian manifold $M$ of dimension $N \geq 3$. We prove that when $V$ is strongly subcritical on $M$, $\nabla(-\Delta - V)^{-1/2}$ and $V^{1/2}(-\Delta - V)^{-1/2}$ are bounded on $L^p(M)$ for all $p \in (p_0; 2]$ if $M$ is of homogeneous type and the Sobolev inequality holds on $M$. If in addition Poincaré inequalities hold on $M$ and $V$ belongs to the Kato class $K^\infty$ then $\nabla(-\Delta - V)^{-1/2}$ and $V^{1/2}(-\Delta - V)^{-1/2}$ are bounded on $L^p(M)$ for all $p \in (1; 2]$. When $V$ is in addition in $L^{N/2}(M)$ and the Riesz transforms associated to the Laplace-Beltrami operator are bounded on $L^r(M)$ for some $r \in (2; N]$, then $\nabla(-\Delta - V)^{-1/2}$ and $V^{1/2}(-\Delta - V)^{-1/2}$ are bounded on $L^p(M)$ for all $p \in (1; r)$.

For the proof of the boundedness of Riesz transforms we use off-diagonal estimates (for properties and more details see [1]). These estimates are a generalization of the Gaussian estimates used by Coulhon and Duong in [13] to study the Riesz transforms associated to the Laplace-Beltrami operator on Riemannian manifolds, and by Duong, Ouhabaz and Yan in [16] to study the magnetic Schrödinger operator on $\mathbb{R}^N$. We also use the approach of Blunck and Kunstmann in [8] and [9] to weak type $(p, p)$-estimates. In [1], Auscher used these tools to divergence-form operators with complex coefficients. For $p \in (2; N)$ we use a complex interpolation method (following an idea in Auscher and Ben Ali [2]).

In contrast to [17] and [18], we do not assume decay nor smoothness conditions on $V$.

In the following sections, we denote by $L^p$ the Lebesgue space $L^p(\mathbb{R}^N)$ with the Lebesgue measure $dx$, $\| . \|_p$ its usual norm, $(.,.)$ the inner product of $L^2$, $\| . \|_{p-q}$ the norm of operators acting from $L^p$ to $L^q$. We denote by $p'$ the dual exponent to $p$, $p' := \frac{p}{p-1}$. We denote by $C, c$ the positive constants even if their values change at each occurrence. Through this paper, $\nabla A^{-1/2}$ denotes one of the partial derivative $\frac{\partial}{\partial x_k} A^{-1/2}$ for any fixed $k \in \{1, ..., N\}$. 


2 Off-diagonal estimates

In this section, we show that \((e^{-tA})_{t>0}, (\sqrt{t}\nabla e^{-tA})_{t>0}\) and \((\sqrt{t}V^{1/2}e^{-tA})_{t>0}\) satisfy \(L^p - L^2\) off-diagonal estimates provided that \(V\) is strongly subcritical.

**Definition 2.1.** Let \((T_t)_{t>0}\) be a family of uniformly bounded operators on \(L^2\). We say that \((T_t)_{t>0}\) satisfies \(L^p - L^q\) off-diagonal estimates for \(p, q \in [1; \infty]\) with \(p \leq q\) if there exist positive constants \(C\) and \(c\) such that for all closed sets \(E\) and \(F\) of \(\mathbb{R}^N\) and all \(h \in L^p(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)\) with support in \(E\), we have for all \(t > 0\):

\[
\|T_t h\|_{L^q(F)} \leq Ct^{-\gamma_{pq}}e^{-\frac{cd(E,F)^2}{t}}\|h\|_p,
\]

where \(d\) is the Euclidean distance and \(\gamma_{pq} := \frac{N}{2}(\frac{1}{p} - \frac{1}{q})\).

**Proposition 2.1.** Let \(A = -\Delta - V\) where \(V \geq 0\) and \(V\) is strongly subcritical. Then \((e^{-tA})_{t>0}, (\sqrt{t}\nabla e^{-tA})_{t>0}\), and \((\sqrt{t}V^{1/2}e^{-tA})_{t>0}\) satisfy \(L^2 - L^2\) off-diagonal estimates, and we have for all \(t > 0\) and all \(f \in L^2\) supported in \(E\):

\[
(i) \quad \|e^{-ta}f\|_{L^2(F)} \leq e^{-d^2(E,F)/4t}\|f\|_2,
\]

\[
(ii) \quad \|\sqrt{t}\nabla e^{-ta}f\|_{L^2(F)} \leq Ce^{-d^2(E,F)/16t}\|f\|_2,
\]

\[
(iii) \quad \|\sqrt{t}V^{1/2}e^{-ta}f\|_{L^2(F)} \leq Ce^{-d^2(E,F)/8t}\|f\|_2.
\]

**Proof:** The ideas are classical and rely on the well known Davies perturbation technique. Let \(A_\rho := \rho\Delta - \rho\phi\) where \(\rho > 0\) and \(\phi\) is a Lipschitz function with \(|\nabla \phi| \leq 1\) a.e. Here \(A_\rho\) is the associated operator to the sesquilinear form \(a_\rho\) defined by

\[
a_\rho(u, v) := a(e^{-\rho\phi}u, e^\rho\phi v)
\]

for all \(u, v \in D(a)\).

By the strong subcriticality property of \(V\) we have for all \(u \in W^{1,2}\)

\[
((A_\rho + \rho^2)u, u) = -\int \rho^2|\nabla \phi|^2u^2 + \int |\nabla u|^2 - \int Vu^2 + \rho^2\|u\|_2^2 \geq \varepsilon\|V^{1/2}u\|_2^2.
\]

Using (2), we obtain

\[
((A_\rho + \rho^2)u, u) = -\int \rho^2|\nabla \phi|^2u^2 + \int |\nabla u|^2 - \int Vu^2 + \rho^2\|u\|_2^2 \geq \frac{\varepsilon}{\varepsilon + 1}\|\nabla u\|_2^2.
\]
In particular \((A_\rho + \rho^2)\) is a maximal accretive operator on \(L^2\), and this implies

\[
||e^{-tA_\rho}u||_2 \leq e^{t\rho^2}||u||_2.
\]  

(7)

Now we want to estimate

\[
||((A_\rho + 2\rho^2)e^{-t(A_\rho + 2\rho^2)}||_{2-2}.
\]

First, let us prove that \(A_\rho + 2\rho^2\) is a sectorial operator.

For \(u\) complex-valued,

\[
a_\rho(u, u) := a(u, u) + \rho \int u \nabla \phi \nabla \overline{u} - \rho \int \overline{u} \nabla \phi \nabla u - \rho^2 \int |\nabla \phi|^2 |u|^2.
\]

Then

\[
a_\rho(u, u) + 2\rho^2 ||u||_2^2 \geq a(u, u) + \rho \int u \nabla \phi \nabla \overline{u} - \rho \int \overline{u} \nabla \phi \nabla u + \rho^2 ||u||_2^2
\]

\[
= a(u, u) + 2i\rho \text{Im} \int u \nabla \phi \nabla \overline{u} + \rho^2 ||u||_2^2.
\]

This implies that

\[
\Re(a_\rho(u, u) + 2\rho^2 ||u||_2^2) \geq a(u, u),
\]  

(8)

and

\[
\Re(a_\rho(u, u) + 2\rho^2 ||u||_2^2) \geq \rho^2 ||u||_2^2.
\]  

(9)

On the other hand,

\[
a_\rho(u, u) = a(u, u) + \rho \int u \nabla \phi \nabla \overline{u} - \rho \int \overline{u} \nabla \phi \nabla u - \rho^2 \int |\nabla \phi|^2 |u|^2
\]

\[
= a(u, u) + 2i\rho \text{Im} \int u \nabla \phi \nabla \overline{u} - \rho^2 \int |\nabla \phi|^2 |u|^2.
\]

So

\[
|\text{Im}(a_\rho(u, u) + 2\rho^2 ||u||_2^2)| = 2|\rho| \int |u||\nabla \phi||\nabla u|
\]

\[
\leq 2|\rho||u||_2||\nabla u||_2.
\]

Using (2) we obtain that

\[
|\text{Im}(a_\rho(u, u) + 2\rho^2 ||u||_2^2)| \leq 2|\rho||u||_2c_\epsilon \frac{a}{2}(u, u)
\]

\[
\leq c_\epsilon^2 a(u, u) + \rho^2 ||u||_2^2.
\]
where $c_\varepsilon = (1 + \frac{1}{\varepsilon})^{\frac{1}{2}}$. Now using estimates (8) and (9), we deduce that there exists a constant $C > 0$ depending only on $\varepsilon$ such that

$$|\Im(a_\rho(u, u) + 2\rho^2\|u\|_2^2)| \leq C\Re(a_\rho(u, u) + 2\rho^2\|u\|_2^2).$$

We conclude that (see [21] or [24])

$$\|e^{-z(A_\rho + 2\rho^2)}\|_{2-2} \leq 1$$

for all $z$ in the open sector of angle $\arctan(1/C)$. Hence by the Cauchy formula

$$\|(A_\rho + 2\rho^2)e^{-t(A_\rho + 2\rho^2)}\|_{2-2} \leq \frac{C}{t}. \quad (10)$$

The constant $C$ is independent of $\rho$.

By estimate (5) and (6) we have

$$( (A_\rho + 2\rho^2)u, u ) \geq ( (A_\rho + \rho^2)u, u ) \geq \varepsilon\|V^{1/2}u\|_2^2,$$

and

$$( (A_\rho + 2\rho^2)u, u ) \geq ( (A_\rho + \rho^2)u, u ) \geq \frac{\varepsilon}{\varepsilon + 1}\|\nabla u\|_2^2.$$ Setting $u = e^{-t(A_\rho + 2\rho^2)}f$ and using (10) and (7) we obtain

$$\|\sqrt{t}\nabla e^{-tA_\rho}f\|_2 \leq Ce^{2t\rho^2}\|f\|_2. \quad (11)$$

and

$$\|\sqrt{t}V^{1/2}e^{-tA_\rho}f\|_2 \leq Ce^{2t\rho^2}\|f\|_2. \quad (12)$$

Let $E$ and $F$ be two closed subsets of $\mathbb{R}^N$, $f \in L^2(\mathbb{R}^N)$ supported in $E$, and let $\phi(x) := d(x, E)$ where $d$ is the Euclidean distance. Since $e^{\rho\phi}f = f$, we have the following relation

$$e^{-tA}f = e^{-\rho\phi}e^{-tA_\rho}f.$$ Then

$$\nabla e^{-tA}f = -\rho\nabla \phi e^{-\rho\phi}e^{-tA_\rho}f + e^{-\rho\phi}\nabla e^{-tA_\rho}f,$$

and

$$V^{1/2}e^{-tA}f = e^{-\rho\phi}V^{1/2}e^{-tA_\rho}f.$$
Now taking the norm on \( L^2(F) \), we obtain from (7), (11) and (12)
\[
\| e^{-tA} f \|_{L^2(F)} \leq e^{-\rho d(E,F)} e^{\rho^2 t} \| f \|_2,
\]
(13)
\[
\| \nabla e^{-tA} f \|_{L^2(F)} \leq \rho e^{-\rho d(E,F)} e^{\rho^2 t} \| f \|_2 + \frac{C}{\sqrt{t}} e^{-\rho d(E,F)} e^{2\rho^2 t} \| f \|_2,
\]
and
\[
\| V^{1/2} e^{-tA} f \|_{L^2(F)} \leq \frac{C}{\sqrt{t}} e^{-\rho d(E,F)} e^{2\rho^2 t} \| f \|_2.
\]
(15)
We set \( \rho = d(E,F)/2t \) in (13) and \( \rho = d(E,F)/4t \) in (15), then we get the \( L^2 - L^2 \) off-diagonal estimates (i) and (iii).
We set \( \rho = d(E,F)/4t \) in (14), we get
\[
\| \nabla e^{-tA} f \|_{L^2(F)} \leq C \sqrt{t} \left( 1 + \frac{d(E,F)}{4\sqrt{t}} \right) e^{-d^2(E,F)/8t} \| f \|_2.
\]
This gives estimate (ii).

Now, we study the \( L^p - L^2 \) boundedness of the semigroup, of its gradient, and of \( (V^{1/2} e^{-tA})_{t>0} \).

**Proposition 2.2.** Suppose that \( A \geq \epsilon V \), then \( (e^{-tA})_{t>0}, (\sqrt{t} \nabla e^{-tA})_{t>0} \) and \( (\sqrt{t} V^{1/2} e^{-tA})_{t>0} \) are \( L^p - L^2 \) bounded for all \( p \in (p'_0; 2] \). Here \( p'_0 \) is the dual exponent of \( p_0 \) where \( p_0 = \frac{2N}{(N-2)(1-\sqrt{1-\frac{N}{N-1}})} \), and the dimension \( N \geq 3 \). More precisely we have for all \( t > 0 \):

i) \( \| e^{-tA} f \|_2 \leq C t^{-\gamma_p} \| f \|_p \),

ii) \( \| \sqrt{t} \nabla e^{-tA} f \|_2 \leq C t^{-\gamma_p} \| f \|_p \),

iii) \( \| \sqrt{t} V^{1/2} e^{-tA} f \|_2 \leq C t^{-\gamma_p} \| f \|_p \),

where \( \gamma_p = \frac{N}{2} \left( \frac{1}{p} - \frac{1}{2} \right) \).

**Proof.** i) We apply the Gagliardo-Nirenberg inequality
\[
\| u \|_2^2 \leq C_{a,b} \| \nabla u \|_2^a \| u \|_p^{2b},
\]
where \( a + b = 1 \) and \( (1 + 2\gamma_p) a = 2\gamma_p \), to \( u = e^{-tA} f \) for all \( f \in L^2 \cap L^p \), all \( t > 0 \), and all \( p \in (p'_0; 2] \). We obtain
\[
\| e^{-tA} f \|_2^2 \leq C_{a,b} \| \nabla e^{-tA} f \|_2^a \| e^{-tA} f \|_p^{2b}.
\]
At present we use the boundedness of the semigroup on $L^p$ for all $p \in (p'_0; 2]$ proved in [23], and the fact that $||\nabla u||^2 \leq (1 + 1/\varepsilon)(Au, u)$ from the strong subcriticality condition, then we obtain that

$$||e^{-tA}f||_{2/a}^{2/a} \leq -C\psi'(t)||f||_{p}^{2b/a}$$

where $\psi(t) = ||e^{-tA}f||_2^2$. This implies

$$||f||_{p}^{-2b/a} \leq C(\psi(t)^{a-1}a').$$

Since $\frac{2b}{a} = \frac{1}{\gamma_p}$ and $\frac{a-1}{a} = -\frac{1}{2\gamma_p}$, integration between 0 and $t$ yields

$$t||f||_{p}^{-1/\gamma_p} \leq C||e^{-tA}f||_2^{1/\gamma_p},$$

which gives $i$).

We obtain $ii$ by using the following decomposition:

$$\sqrt{t}\nabla e^{-tA} = \sqrt{t}\nabla A^{-1/2}A^{1/2}e^{-tA/2}e^{-tA/2},$$

the boundedness of $\nabla A^{-1/2}$ and of $(\sqrt{t}A^{1/2}e^{-tA})_{t>0}$ on $L^2$, and the fact that $(e^{-tA})_{t>0}$ is $L^p - L^2$ bounded for all $p \in (p'_0; 2]$ proved in $i$).

We prove $iii$ by using the following decomposition:

$$\sqrt{t}V^{1/2}e^{-tA} = \sqrt{t}V^{1/2}A^{-1/2}A^{1/2}e^{-tA/2}e^{-tA/2},$$

the boundedness of $V^{1/2}A^{-1/2}$ and of $(\sqrt{t}A^{1/2}e^{-tA})_{t>0}$ on $L^2$, and the fact that $(e^{-tA})_{t>0}$ is $L^p - L^2$ bounded for all $p \in (p'_0; 2]$ proved in $i$). \QED \QED

We invest the previous results to obtain:

**Theorem 2.1.** Assume that $A \geq \varepsilon V$ then $(e^{-tA})_{t>0}$, $(\sqrt{t}\nabla e^{-tA})_{t>0}$ and $(\sqrt{t}V^{1/2}e^{-tA})_{t>0}$ satisfy $L^p - L^2$ off-diagonal estimates for all $p \in (p'_0; 2]$. Here $p'_0$ is the dual exponent of $p_0$ where $p_0 = \frac{2N}{(N-2)(1-\sqrt{1-1/\varepsilon})}$, and the dimension $N \geq 3$.

Then we have for all $t > 0$, all $p \in (p'_0; 2]$, all closed sets $E$ and $F$ of $\mathbb{R}^N$ and all $f \in L^2 \cap L^p$ with supp$f \subseteq E$

$$i)$$

$$||e^{-tA}f||_{L^2(F)} \leq Ct^{-\gamma_p}e^{-c\overline{d^2(E,F)}t}||f||_p,$$

(16)

$$ii)$$

$$||\sqrt{t}\nabla e^{-tA}f||_{L^2(F)} \leq Ct^{-\gamma_p}e^{-c\overline{d^2(E,F)}t}||f||_p,$$

(17)

10
\[ \| \sqrt{V^{1/2}} e^{-tA} f \|_{L^2(F)} \leq C t^{-\gamma_p} e^{-\alpha_2^2(E,F) t} \| f \|_p, \]  

(18)

where \( \gamma_p = \frac{N}{2} \left( \frac{1}{p} - \frac{1}{2} \right) \) and \( C, c \) are positive constants.

**Remark:** By duality, we deduce from (16) an \( L^2 - L^p \) off-diagonal estimate of the norm of the semigroup for all \( p \in [2, p_0) \), but we cannot deduce from (17) and (18) the same estimate of the norm of \( \sqrt{V^{1/2}} e^{-tA} f \) and of \( \sqrt{V^{1/2}} e^{-tA} f \) because they are not selfadjoint. This affects the boundedness of Riesz transforms and of \( V^{1/2} A^{-1/2} \) on \( L^p \) for \( p > 2 \).

**Proof.** i) In the previous proposition we have proved that

\[ \| e^{-tA} f \|_2 \leq C t^{-\gamma_p} \| f \|_p \]

for all \( p \in (p'_0, 2) \). This implies that for all \( t > 0 \)

\[ \| \chi_F e^{-tA} \chi_E f \|_2 \leq C t^{-\gamma_p} \| f \|_p \]

where \( \chi_M \) is the characteristic function of \( M \). The \( L^2 - L^2 \) off-diagonal estimate proved in the Proposition 2.1 implies that

\[ \| \chi_F e^{-tA} \chi_E f \|_2 \leq e^{-d^2(E,F)/4t} \| f \|_2. \]

Hence we can apply the Riesz-Thorin interpolation theorem and we obtain the off-diagonal estimate (16).

Assertions ii) and iii) are proved in a similar way. We use \( L^2 - L^2 \) off-diagonal estimates of Proposition 2.1 and assertions ii) and iii) of Proposition 2.2.

\[ \square \]

## 3  Boundedness of \( \nabla A^{-1/2} \) and \( V^{1/2} A^{-1/2} \) on \( L^p \) for \( p \in (p'_0, 2] \)

This section is devoted to the study of the boundedness of \( V^{1/2} A^{-1/2} \) and Riesz transforms associated to Schrödinger operators with negative and strongly subcritical potentials. We prove that \( \nabla A^{-1/2} \) and \( V^{1/2} A^{-1/2} \) are bounded on \( L^p(\mathbb{R}^N), N \geq 3 \), for all \( p \in (p'_0, 2] \), where \( p'_0 \) is the exponent mentioned in Theorem 2.1.
**Theorem 3.1.** Assume that $A \geq \varepsilon V$, then $\nabla A^{-1/2}$ is bounded on $L^p(\mathbb{R}^N)$ for $N \geq 3$, for all $p \in (p'_0; 2]$ where $p'_0 = \left(\frac{2N}{(N-2)(1-\frac{3}{1+p})}\right)'$.

To prove Theorem 3.1, we prove that $\nabla A^{-1/2}$ is of weak type $(p, p)$ for all $p \in (p'_0; 2]$ by using the following theorem of Blunck and Kunstmann [8]. Then by the boundedness of $\nabla A^{-1/2}$ on $L^2$, and the Marcinkiewicz interpolation theorem, we obtain boundedness on $L^p$ for all $p \in (p'_0; 2]$. This result can also be deduced from Theorem 2.1 together with Theorem 2.1 of [9].

**Theorem 3.2.** Let $p \in [1; 2)$. Suppose that $T$ is sublinear operator of strong type $(2, 2)$, and let $(A_r)_{r>0}$ be a family of linear operators acting on $L^2$.

Assume that for $j \geq 2$

$$
\left(\frac{1}{|2^{j+1}B|} \int_{C_j(B)} |T(I - A_r(B))f|^2 \right)^{1/2} \leq g(j) \left(\frac{1}{|B|} \int_B |f|^p \right)^{1/p},
$$

(19)

and for $j \geq 1$

$$
\left(\frac{1}{|2^{j+1}B|} \int_{C_j(B)} |A_r(B)f|^2 \right)^{1/2} \leq g(j) \left(\frac{1}{|B|} \int_B |f|^p \right)^{1/p},
$$

(20)

for all ball $B$ with radius $r(B)$ and all $f$ supported in $B$. If $\Sigma := \sum g(j)2^{Nj} < \infty$, then $T$ is of weak type $(p, p)$, with a bound depending only on the strong type $(2, 2)$ bound of $T$, $p$, and $\Sigma$.

Here $C_1 = 4B$ and $C_j(B) = 2^{j+1}B \setminus 2^jB$ for $j \geq 2$, where $\lambda B$ is the ball of radius $\lambda r(B)$ with the same center as $B$, and $|\lambda B|$ its Lebesgue measure.

Let $T = \nabla A^{-1/2}$. We prove assumptions (19) and (20) with $A_r = I - (I - e^{-r^2A})^m$ for some $m > N/4 - \gamma_p$, using arguments similar to Auscher [11] Theorem 4.2.

Let us prove (20). For $f$ supported in a ball $B$ (with radius $r$),

$$
\frac{1}{|2^{j+1}B|^{1/2}} \|A_rf\|_{L^2(C_j(B))} \leq \frac{1}{|2^{j+1}B|^{1/2}} \left\| \sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} e^{-k \nabla A} f \right\|_{L^2(C_j(B))} \\
\leq \frac{1}{|2^{j+1}B|^{1/2}} \sum_{k=1}^{m} \binom{m}{k} C(kr)^{-\gamma_p} e^{-c \rho_{(B,C_j(B))}} \|f\|_p.
$$
for all \( p \in (p'_0; 2) \) and all \( f \in L^2 \cap L^p \) supported in \( B \). Here we use the \( L^p - L^2 \) off-diagonal estimates (16) for \( p \in (p'_0; 2] \). Since \( \gamma_p = \frac{N}{2} (\frac{1}{p} - \frac{1}{2}) \) we obtain

\[
\left( \frac{1}{|2^{j+1}B|} \int_{C_j(B)} |A_r f|^2 \right)^{1/2} \leq C \frac{\gamma_p^{-2} e^{-cd(B,C_j(B))}}{|2^{j+1}B|^{1/2}} \|f\|_p \leq C 2^{-jN/2} e^{-cd(B,C_j(B))} \left( \frac{1}{|B|} \int_{B} |f|^p \right)^{1/p}.
\]

This yields, for \( j = 1 \),

\[
\left( \frac{1}{|4B|} \int_{4B} |A_r f|^2 \right)^{1/2} \leq C 2^{-N/2} \left( \frac{1}{|B|} \int_{B} |f|^p \right)^{1/p},
\]

and for \( j \geq 2 \)

\[
\left( \frac{1}{|2^{j+1}B|} \int_{C_j(B)} |A_r f|^2 \right)^{1/2} \leq C 2^{-jN/2} e^{-cd} \left( \frac{1}{|B|} \int_{B} |f|^p \right)^{1/p}.
\]

Thus assumption (20) of Theorem 3.2 holds with \( \sum_{j \geq 1} g(j) 2^j N < \infty \).

It remains to check the assumption (19):

We know that

\[
\nabla A^{-1/2} f = C \int_0^\infty \nabla e^{-tA} f \frac{dt}{\sqrt{t}}
\]

then, using the Newton binomial, we get

\[
\nabla A^{-1/2} (I - e^{-r^2 A})^m f = C \int_0^\infty \nabla e^{-tA} (I - e^{-r^2 A})^m f \frac{dt}{\sqrt{t}}
\]

\[
= C \int_0^\infty g_r(t) \nabla e^{-tA} f dt
\]

where

\[
g_r(t) = \sum_{k=0}^m \binom{m}{k} (-1)^k \frac{\chi(t-kr^2 > 0)}{\sqrt{t-kr^2}}.
\]

Hence, using the \( L^p - L^2 \) off-diagonal estimate (17), we obtain for all \( p \in (p'_0; 2) \), all \( j \geq 2 \), and all \( f \in L^2 \cap L^p \) supported in \( B \)

\[
\| \nabla A^{-1/2} (I - e^{-r^2 A})^m f \|_{L^2(C_j(B))} \leq C \int_0^\infty |g_r(t)| t^{-\gamma_p - 1/2} e^{-cd r^2 / t} dt \|f\|_p.
\]

We observe that (see [1] p. 27)

\[
|g_r(t)| \leq \frac{C}{\sqrt{t-kr^2}} \text{ if } kr^2 < t \leq (k+1)r^2 \leq (m+1)r^2
\]
In the preceding equation and by the Laplace transform formula, this yields

\[ \| \nabla A^{\frac{1}{2}} (I - e^{-r^2 A})^m f \|_{L^2(C_j(B))} \leq C \sum_{k=0}^{m} \int_{kr^2}^{(k+1)r^2} \frac{t^{-\gamma_p-1/2}}{\sqrt{t-kr^2}} e^{-\frac{c_k r^2}{t}} dt \| f \|_p \]

\[ + \quad C \int_0^{\infty} r^{2m} t^{-\gamma_p-1-m} e^{-\frac{c_f r^2}{t}} dt \| f \|_p \]

\[ \leq I_1 + I_2. \]

We have

\[ I_2 := C \int_{(m+1)r^2}^{\infty} r^{2m} t^{-\gamma_p-1-m} e^{-\frac{c_f r^2}{t}} dt \| f \|_p \leq C r^{-2\gamma_p} 2^{-2j(m+\gamma_p)} \| f \|_p \]

by the Laplace transform formula, and

\[ I_1 := C \| f \|_p \sum_{k=0}^{m} \int_{kr^2}^{(k+1)r^2} \frac{t^{-\gamma_p-1/2}}{\sqrt{t-kr^2}} e^{-\frac{c_k r^2}{t}} dt \]

\[ = C \| f \|_p \left( \sum_{k=1}^{m} \int_{kr^2}^{(k+1)r^2} \frac{t^{-\gamma_p-1/2}}{\sqrt{t-kr^2}} e^{-\frac{c_k r^2}{t}} dt + \int_0^{r^2} t^{-\gamma_p-1} e^{-\frac{c_f r^2}{t}} dt \right) \]

\[ = J_1 + J_2. \]

In the preceding equation

\[ J_1 := C \| f \|_p \sum_{k=1}^{m} \int_{kr^2}^{(k+1)r^2} \frac{t^{-\gamma_p-1/2}}{\sqrt{t-kr^2}} e^{-\frac{c_k r^2}{t}} dt \]

\[ \leq C \| f \|_p e^{-\frac{c_f r^2}{2(m+1)}} \sum_{k=1}^{m} (kr^2)^{-\gamma_p-1/2} \int_{kr^2}^{(k+1)r^2} (t-kr^2)^{-1/2} dt \]

\[ \leq C r^{-2\gamma_p} 2^{-2j(m+\gamma_p)} \| f \|_p, \]

and

\[ J_2 := C \int_0^{r^2} t^{-\gamma_p-1} e^{-\frac{c_f r^2}{t}} dt \| f \|_p \]

\[ \leq C \| f \|_p e^{-\frac{c_f r^2}{2(m+1)}} \int_0^{r^2} t^{-\gamma_p-1} e^{-\frac{c_f r^2}{t}} dt \]

\[ \leq C \| f \|_p 2^{-2jm} \int_0^{r^2} t^{-1-\gamma_p} C(2^{-2j} r^{-2t})^\gamma_p e^{-\frac{c_f r^2}{4t}} dt \]

\[ \leq C \| f \|_p 2^{-2j(m+\gamma_p)} r^{-2\gamma_p} \int_0^{r^2} t^{-1} e^{-\frac{c_f r^2}{4t}} dt \]

\[ \leq C r^{-2\gamma_p} 2^{-2j(m+\gamma_p)} \| f \|_p. \]
Here, for the last inequality, we use the fact that \( j \geq 2 \) to obtain the convergence of the integral without dependence on \( r \) nor on \( j \).

We can therefore employ these estimates in \((21)\) to conclude that

\[
\| \nabla A^{-1/2}(I - e^{-r^2A})^m f \|_{L^2(C_j(B))} \leq C r^{-2\gamma_p / 2} 2^{-2j(m + \gamma_p)} \| f \|_p,
\]

which implies

\[
\left( \frac{1}{|2j+1|} \right) \int_{C_j(B)} |\nabla A^{-1/2}(I - e^{-r^2A})^m f|^2 \right)^{1/2} \leq C 2^{-2j(m + \gamma_p + \gamma_p)} \left( \frac{1}{|B|} \int_B |f|^p \right)^{1/2},
\]

where \( \sum g(j)2^{jN} < \infty \) because we set \( m > N/4 - \gamma_p \).

**Proposition 3.1.** Assume that \( A \geq \varepsilon V \), then \( V^{1/2}A^{-1/2} \) is bounded on \( L^p(\mathbb{R}^N) \) for \( N \geq 3 \), for all \( p \in (p_0', 2] \) where \( p_0' \) is the dual exponent of \( p_0 \) with \( p_0 = \frac{2N}{(N-2)(1-\sqrt{1-1/p})} \).

**Proof.** We have seen in \((3)\) that the operator \( V^{1/2}A^{-1/2} \) is bounded on \( L^2 \). To prove its boundedness on \( L^p \) for all \( p \in (p_0', 2] \) we prove that it is of weak type \((p, p)\) for all \( p \in (p_0', 2] \) by checking assumptions \((19)\) and \((20)\) of Theorem 3.2, where \( T = V^{1/2}A^{-1/2} \). Then, using the Marcinkiewicz interpolation theorem, we deduce boundedness on \( L^p \) for all \( p \in (p_0', 2] \).

We check assumptions of Theorem 3.2 similarly as we did in the proof of Theorem 3.1 using the \( L^p - L^2 \) off-diagonal estimate \((18)\) instead of \((17)\).

\( \square \)

Let us now move on, setting \( V = c|x|^{-2} \) where \( 0 < c < (\frac{N-2}{2})^2 \), which is strongly subcritical thanks to the Hardy inequality, we prove that the associated Riesz transforms are not bounded on \( L^p \) for \( p \in (1; p_0') \) neither for \( p \in (p_0'; \infty) \). Here \( p_0' = \frac{p_0N}{N + p_0} \) is the reverse Sobolev exponent of \( p_0 \).

**Proposition 3.2.** Set \( V \) strongly subcritical and \( N \geq 3 \). Assume that \( \nabla A^{-1/2} \) is bounded on \( L^p \) for some \( p \in (1; p_0') \). Then there exists an exponent \( q_1 \in [p; p_0') \) such that \( (e^{-tA})_{t>0} \) is bounded on \( L^q \) for all \( r \in (q_1; 2) \).

Consider now \( V = c|x|^{-2} \) where \( 0 < c < (\frac{N-2}{2})^2 \). It is proved in \([23]\) that the semigroup does not act on \( L^p \) for \( p \notin (p_0'; p_0) \). Therefore we obtain from this proposition that the Riesz transform \( \nabla A^{-1/2} \) is not bounded on \( L^p \) for \( p \in (1; p_0') \).
Proof. Assume that $\nabla A^{-1/2}$ is bounded on $L^p$ for some $p \in (1; p_0')$. By the boundedness on $L^2$ and the Riesz-Thorin interpolation theorem, we get the boundedness of $\nabla A^{-1/2}$ on $L^q$ for all $q \in [p; 2]$. Now we apply the Sobolev inequality

$$\|f\|_{q^*} \leq C\|\nabla f\|_q$$

where $q^* = \frac{Nq}{N-q}$ if $q < N$ to $f := A^{-1/2}u$, so we get

$$\|A^{-1/2}u\|_{q^*} \leq C\|\nabla A^{-1/2}u\|_q \leq C\|u\|_q$$

for all $q \in [p; 2]$. In particular, $\|A^{-1/2}\|_{q_1-q_1^*} \leq C$ where $p \leq q_1 < p_0$ such that $q_1^* > p_0$.

Decomposing the semigroup as follows

$$e^{-tA} = A^{1/2}e^{-tA/2}e^{-tA/2}A^{-1/2}$$

where $A^{-1/2}$ is $L^{q_1} - L^{q_1^*}$ bounded , $e^{-tA/2}$ has $L^{q_1^*} - L^2$ norm bounded by $Ct^{-\gamma q_1}$ (Proposition 2.2) and $A^{1/2}e^{-tA/2}$ is $L^2 - L^2$ bounded by $Ct^{-1/2}$ because of the analyticity of the semigroup on $L^2$. Therefore, we obtain

$$\|e^{-tA}\|_{q_1-2} \leq C t^{-\gamma q_1-1/2} = C t^{-\gamma q_1}.$$

We now interpolate this norm with the $L^2 - L^2$ off-diagonal estimate of the norm of $e^{-tA}$, as we did in the proof of Theorem 2.1, so we get a $L^r - L^2$ off-diagonal estimate for all $r \in (q_1; 2)$. Then Lemma 3.3 of [1] yields that $(e^{-tA})_{t>0}$ is bounded on $L^r$ for all $r \in (q_1; 2)$ for $q_1 \in [p; p_0')$ such that $q_1^* > p_0'$.

**Proposition 3.3.** Set $V$ strongly subcritical and $N \geq 3$. Assume that $\nabla A^{-1/2}$ is bounded on $L^p$ for some $p \in (p_0; \infty)$. Then there exists an exponent $q_2 > p_0^*$ such that the semigroup $(e^{-tA})_{t>0}$ is bounded on $L^s$ for all $s \in (2; q_2^*)$. Here $q_2^* > p_0$.

Consider now $V = c|x|^{-2}$ where $0 < c < (\frac{N-2}{2})^2$. It is proved in [23] that the semigroup does not act on $L^p$ for $p \notin (p_0'; p_0)$. Therefore we obtain from this proposition that the Riesz transforms $\nabla A^{-1/2}$ are not bounded on $L^p$ for $p \in (p_0; \infty)$.

Proof. Assume that $\nabla A^{-1/2}$ is bounded on $L^p$ for some $p \in (p_0; \infty)$. Then by interpolation we obtain the boundedness of $\nabla A^{-1/2}$ on $L^q$ for all $q \in [2; p]$. In particular,

$$\|\nabla A^{-1/2}\|_{q_2-q_2} \leq C$$
where \( p_0^* < q_2 < p_0, q_2 \leq p, q_2 < N \). Using the Sobolev inequality (22), we obtain that \( A^{-1/2} \) is \( L^q_2 - L^{q_2}_2 \) bounded where \( q_2^* > p_0 \).

Now we decompose the semigroup as follows

\[
e^{-tA} = A^{1/2} e^{-tA/2} A^{1/2} e^{-tA/2}.
\]

(24)

Thus we remark that it is \( L^2 - L^{q_2^*}_2 \) bounded where \( q_2^* > p_0 \).

Then, using similar arguments as in the previous proof, we conclude that \( (e^{-tA})_{t>0} \) is bounded on \( L^s \) for all \( s \in (2; q_2^*) \) for \( p_0^* < q_2 < \inf(p_0, p, N) \).

\[
4 \text{ Boundedness of } \nabla A^{-1/2} \text{ and } V^{1/2} A^{-1/2} \text{ on } L^p \\
	ext{for all } p \in (1; N)
\]

In this section we assume that \( V \) is strongly subcritical in the Kato subclass \( K_N^\infty, N \geq 3 \). Following Zhao [33], we define

\[
K_N^\infty := \left\{ V \in K_N^{loc}; \lim_{B \to \infty} \sup_{x \in \mathbb{R}^N} \int_{|y| \geq B} \frac{|V(y)|}{|y - x|^{N-2}} dy = 0 \right\},
\]

where \( K_N^{loc} \) is the class of potentials that are locally in the Kato class \( K_N \). For necessary background of the Kato class see [29] and references therein.

We use results proved by stochastic methods to deduce a \( L^1 - L^\infty \) off-diagonal estimate of the norm of the semigroup which leads to the boundedness of \( \nabla A^{-1/2} \) and \( V^{1/2} A^{-1/2} \) on \( L^p \) for all \( p \in (1; N) \).

**Theorem 4.1.** Let \( A \) be the Schrödinger operator \( -\Delta - V, V \geq 0 \). Assume that \( V \) is strongly subcritical in the class \( K_N^\infty, (N \geq 3) \), then \( \nabla A^{-1/2} \) and \( V^{1/2} A^{-1/2} \) are of weak type \((1,1)\), they are bounded on \( L^p \) for all \( p \in (1; 2] \).

If in addition \( V \in L^{N/2} \), then \( \nabla A^{-1/2} \) and \( V^{1/2} A^{-1/2} \) are bounded on \( L^p \) for all \( p \in (1; N) \).

**Proof.** We assume that \( V \) is strongly subcritical in the class \( K_N^\infty \). Therefore \( V \) satisfies assumptions of Theorem 2 of [31] (The classes \( K_\infty \) and \( S_\infty \) mentioned in [31] are equivalent to the class \( K_N^\infty \) (see Chen [11] Theorem 2.1 and Section 3.1)). Thus the heat kernel associated to \( (e^{-tA})_{t>0} \) satisfies a Gaussian estimate. Therefore \( (e^{-tA})_{t>0}, (\sqrt{t} \nabla e^{-tA})_{t>0}, \) and \( (\sqrt{t} V^{1/2} e^{-tA})_{t>0} \) satisfy \( L^1 - L^2 \) off-diagonal estimates. Arguing now as in the proof of Theorem [31] (or using Theorem 5 of [28]) we conclude that \( \nabla A^{-1/2} \) and \( V^{1/2} A^{-1/2} \) are of weak type \((1,1)\) and they are bounded on \( L^p \) for all \( p \in (1; 2] \).
To prove the boundedness of $\nabla A^{-1/2}$ on $L^p$ for higher $p$ we use the Stein complex interpolation theorem (see [30] Section V.4). Let us first mention that $D := R(A) \cap L^1 \cap L^\infty$ is dense in $L^p$ for all $p \in (1; \infty)$ provided that $V$ is strongly subcritical in $K_N^\infty$, $N \geq 3$. We prove the density as in [2], where in our case we have the following estimate

$$|f_k - f| \leq k(c(-\Delta) + k)^{-1}f$$

(25)

where $f_k := A(A + k)^{-1}f$ and $c$ is a positive constant. This estimate holds from the Gaussian estimate of the heat kernel associated to the semigroup $(e^{-tA})_{t > 0}$.

Set $F(z) := \langle (-\Delta)^z f, g \rangle$ where $f \in D$, $g \in C^\infty_0(\mathbb{R}^N)$ and $z \in S := \{x + iy \text{ such that } x \in [0; 1] \text{ and } y \in \mathbb{R}\}$. $F(z)$ is admissible. Indeed, the function $z \mapsto F(z)$ is continuous in $S$ and analytic in its interior. In addition

$$|F(z)| = |\langle (-\Delta)^z f, (-\Delta)^z g \rangle| \leq ||A^{-z} f||_2 ||(-\Delta)^z g||_2.$$  

(26)

For $\Re z \in (0; 1), D(-\Delta) \subset D((-\Delta)^z)$, so

$$||(-\Delta)^z g||_2 \leq C||g||_{W^{2,2}}$$

(27)

for all $z \in S$.

When $V$ is strongly subcritical, $A$ is non-negative self-adjoint operator on $L^2$, hence $||A^{iy}||_{2-2} \leq 1$ for all $y \in \mathbb{R}$. Therefore for all $z = x + iy \in S$ and $f = Au \in R(A)$ we have

$$||A^{-z} f||_2 \leq ||A^{-iy}||_{2-2} ||A^{1-x} u||_2 \leq C(||u||_2 + ||Au||_2).$$

(28)

Here we use $D(A) \subset D(A^{1-x})$ because $(1 - x) \in (0; 1)$.

Now we employ (27) and (28) in (26) to deduce the admissibility of $F(z)$ in $S$. Thus we can apply the Stein complex interpolation theorem to $F(z)$.

Since $V$ is strongly subcritical and belongs to the class $K_N^\infty$, $N \geq 3$, we obtain a Gaussian estimate of the heat kernel of $A$. Thus $A$ has a $H^\infty$-bounded calculus on $L^p$ for all $p \in (1; \infty)$ (see e.g. [8] Theorem 2.2). Hence

$$|F(iy)| \leq ||A^{-iy} f||_{p_0} ||(-\Delta)^{-iy} g||_{p_0'} \leq C_{\gamma, p_0} e^{2\gamma|y|} ||f||_{p_0} ||g||_{p_0'}$$

for all $\gamma > 0$, all $p_0 \in (1; \infty)$.

Let us now estimate $||VA^{-1}||_{p_1-p_1}$. By Hölder’s inequality

$$||VA^{-1}||_{p_1} \leq ||V||_{N/2} ||A^{-1} u||_q$$

(29)
where $p_1 < N$ and $\frac{1}{p_1} = \frac{1}{q} + \frac{2}{N}$. As mentioned above we have a Gaussian upper bound for the heat kernel. In particular

$$\|e^{-tA}\|_{1-\infty} \leq Ct^{-N/2}$$

for all $t > 0$. Therefore $A^{-1}$ extends to a bounded operator from $L^s$ to $L^q$ such that $s < \frac{N}{2}$ and $\frac{1}{s} = \frac{1}{q} + \frac{2}{N}$, and we have

$$\|A^{-1}u\|_q \leq C\|u\|_s.$$

(see Coulhon [12]). Thus $s = p_1$, $D(A) \subseteq D(V)$ and (29) implies

$$\|VA^{-1}\|_{p_1-p_1} \leq C$$

where $C$ depends on $\|V\|_{N/2}$. Hence we can estimate

$$\|(-\Delta)A^{-1}u\|_{p_1} = \|(-\Delta - V + V)A^{-1}u\|_{p_1} \leq \|u\|_{p_1} + \|VA^{-1}u\|_{p_1} \leq C\|u\|_{p_1}$$

(30)

where $C$ depends on $\|V\|_{N/2}$. We return to $F(z)$,

$$|F(1 + iy)| \leq \|((-\Delta)^{-1/2} - i\gamma g)\|_{p_1} \leq \|(-\Delta)^{-1/2} \|_{p_1-p_1} \|(-\Delta)^{-iy}f\|_{p_1} \|(-\Delta)^{-iy}g\|_{p_1} \leq C_{\gamma,p_1}\|V\|_{N/2}e^{2\gamma|y|} \|f\|_{p_1}\|g\|_{p_1}$$

for all $p_1 \in (1; N/2)$ and all $\gamma > 0$.

From the Stein interpolation theorem it follows that for all $t \in [0; 1]$ there exists a constant $M_t$ such that

$$|F(t)| \leq M_t\|f\|_{p_1}\|g\|_{p_1'}$$

where $\frac{1}{p_1} = \frac{1-t}{p_0} + \frac{t}{p_1}$. Setting $t = \frac{1}{2}$ and using a density argument we conclude that $\nabla A^{-1/2}$ is bounded on $L^p$ for all $p \in (1; N)$.

To prove boundedness of $V^{1/2}A^{-1/2}$ on $L^p$ we use the following decomposition

$$V^{1/2}A^{-1/2} = V^{1/2}(-\Delta)^{-1/2}(-\Delta)^{1/2}A^{-1/2}.$$

Assuming $V \in L^{N/2}$ we have by Hölder’s inequality

$$\|V^{1/2}u\|_p \leq \|V^{1/2}\|_N\|u\|_q$$
where \( p < N \) and \( \frac{1}{p} - \frac{1}{q} = \frac{1}{N} \). Then by Sobolev inequality and the boundedness of Riesz transforms associated to the Laplace operator we obtain

\[
\|V^{1/2}u\|_p \leq C_{p,N}\|V\|_{N/2}\|\nabla u\|_p \leq C_{p,N}\|V\|_{N/2}\|(-\Delta)^{1/2}u\|_p
\]

for all \( p \in (1; N) \). Thus if \( V \in L^{N/2} \) we have for all \( p \in (1; N) \)

\[
\|V^{1/2}(-\Delta)^{-1/2}\|_{p-p} \leq C.
\]

Using the boundedness of Riesz transforms associated to the Schrödinger operator \( A \) we have

\[
\|(-\Delta)^{1/2}A^{-1/2}u\|_p \leq C\|u\|_p
\]

for all \( p \in (1; N) \).

Therefore \( V^{1/2}A^{-1/2} \) is bounded on \( L^p \) for all \( p \in (1; N) \) provided that \( V \) is strongly subcritical in the class \( K_N^\infty \cap L^{N/2}, N \geq 3 \).

**Example:** Set \( N \geq 3 \), and let us take potentials \( V \) in the Kato subclass \( K_N \cap L^{N/2} \) such that \( V \sim c|x|^{-\alpha} \) when \( x \) tends to infinity, where \( \alpha > 2 \). Suppose that \( \|V\|_N \) is small enough. Let us prove that these potentials are strongly subcritical, so we should prove that

\[
\|V^{1/2}u\|_2^2 \leq C\|\nabla u\|_2^2
\]

where \( C < 1 \). This is (31) where \( p = 2 \), and \( C < 1 \) for \( \|V\|_N \) is small enough. Hence these potentials are strongly subcritical. Z. Zhao [33] proved that they are in the subclass \( K_N^\infty \). Hence they satisfy the assumptions of Theorem 4.1. Then \( \nabla(-\Delta-V)^{-1/2} \) and \( V^{1/2}(-\Delta-V)^{-1/2} \) are bounded on \( L^p \) for all \( p \in (1; N) \).

**Remarks:** 1) The proof of the previous theorem shows that

\[
\|Vu\|_{p_1} \leq C\|Au\|_{p_1}
\]

and

\[
\|\Delta u\|_{p_1} \leq C\|Au\|_{p_1}
\]

for all \( p_1 \in (1; N/2) \).

2) If we consider \( H = -\Delta + V \) a Schrödinger operator with non-negative potential \( V \in L^{N/2} \), we obtain by the previous arguments the \( L^{p_1} \)-boundedness of \( VH^{-1} \) and \( \Delta H^{-1} \) for all \( p_1 \in (1; N/2) \), and the \( L^p \)-boundedness of \( V^{1/2}H^{-1/2} \) and \( \nabla H^{-1/2} \) for all \( p \in (1; N) \).
5 Schrödinger operators on Riemannian manifolds

Let $M$ be a non-compact complete Riemannian manifold of dimension $N \geq 3$. Denote by $d\mu$ the Riemannian measure, $\rho$ the geodesic distance on $M$ and $\nabla$ the Riemannian gradient. Denote by $|.|$ the length in the tangent space, and by $\|\cdot\|_p$ the norm in $L^p(M, d\mu)$. Let $-\Delta$ be the positive self-adjoint Laplace-Beltrami operator on $M$. Take $V$ a strongly subcritical positive potential on $M$, which means that there exists an $\varepsilon > 0$ such that

$$\int_MVu^2 d\mu \leq \frac{1}{1+\varepsilon} \int_M |\nabla u|^2 d\mu. \quad (32)$$

and set $A := -\Delta - V$ the associated Schrödinger operator on $M$. By the sesquilinear form method $A$ is well defined, non-negative, and $-A$ generates a bounded analytic semigroup $(e^{-tA})_{t>0}$ on $L^2(M)$.

As in $\mathbb{R}^N$, we have the $L^2(M)$-boundedness of $V^{1/2}A^{-1/2}$ and of the Riesz transforms $\nabla A^{-1/2}$ if and only if $V$ is strongly subcritical.

We remark that methods used in [23] hold in manifolds. The semigroup $(e^{-tA})_{t>0}$ can be extrapolated to $L^p(M)$, and it is uniformly bounded for $p \in \left( \left( \frac{2}{1-\sqrt{1-\frac{1}{1+\varepsilon}}} \right)'; \left( \frac{2}{1-\sqrt{1-\frac{1}{1+\varepsilon}}} \right) \right)$. If in addition the Sobolev inequality

$$\|f\|_{L^{\frac{2N}{N-2}}(M)} \leq C \|\nabla f\|_{L^2(M)} \quad (33)$$

for all $f \in C_0^\infty(M)$ holds on $M$, then we obtain for all $t > 0$

$$\|e^{-tA}\|_{L^p(M) \to L^{\frac{2N}{N-2}}(M)} \leq Ct^{-1/p}$$

for all $p \in \left( \left( \frac{2}{1-\sqrt{1-\frac{1}{1+\varepsilon}}} \right)'; \left( \frac{2}{1-\sqrt{1-\frac{1}{1+\varepsilon}}} \right) \right)$. Using the $L^2(M) - L^2(M)$ off-diagonal estimate we obtain as in [23] the fact that $(e^{-tA})_{t>0}$ is bounded on $L^p(M)$ for all $p \in (p'_0; p_0)$ where $p_0 := \frac{2N}{N-2} \frac{1}{\sqrt{1-\frac{1}{1+\varepsilon}}}$.

For classes of manifolds satisfying (33) see [26]. Note that (33) is equivalent to the following Gaussian upper bound of the heat kernel $p(t,x,y)$ of the Laplace-Beltrami operator (see [32] and [14])

$$p(t,x,y) \leq Ct^{-N/2}e^{-\rho^2(x,y)/t} \quad \forall x, y \in M, t > 0. \quad (34)$$

We say that $M$ is of homogeneous type if for all $x \in M$ and $r > 0$

$$\mu(B(x, 2r)) \leq C \mu(B(x, r)) \quad (35)$$
where \( B(x, r) := \{ y \in M \text{ such that } \rho(x, y) \leq r \} \).

We say that the \( L^2 \)-Poincaré inequalities hold on \( M \) if there exists a positive constant \( C \) such that

\[
\int_{B(x, r)} |f(y) - f_t(x)|^2 d\mu(y) \leq Cr^2 \int_{B(x, r)} |\nabla f(y)|^2 d\mu(y) \tag{36}
\]

for all \( f \in C^\infty_0(M) \), \( x \in M \), \( r > 0 \), where \( f_t(x) := \frac{1}{\mu(B(x, r))} \int_{B(x, r)} f(y) d\mu(y) \).

Saloff-Coste [25] proved that (35) and (36) hold if and only if the heat kernel \( K \) exists a compact set \( \{ x \in M : \mu(B(x, r)) > 0 \} \). Assume (32) and (33). Then we have for all \( t > 0 \), \( x, y \in M \), \( r > 0 \), where

\[
B \subset M, r > 0 \quad \text{such that} \quad \{ x \in M : \mu(B(x, r)) > 0 \}.
\]

Arguing as in the Euclidean case we obtain the following theorem:

**Theorem 5.1.** Let \( M \) be a non-compact complete Riemannian manifold of dimension \( N \geq 3 \). Assume (32) and (33). Then \((e^{-tA})_{t>0}, (\sqrt{t}\nabla e^{-tA})_{t>0} \) and \((\sqrt{t}V^{1/2} e^{-tA})_{t>0} \) satisfy \( L^p(M) - L^2(M) \) off-diagonal estimates for all \( p \in (p'_0; 2] \). Here \( p'_0 \) is the dual exponent of \( p_0 \) where

\[
p_0 = \frac{2N}{(N-2)(1-\sqrt{1-\frac{N}{N+4}})}.
\]

Then we have for all \( t > 0 \), all \( p \in (p'_0; 2] \), all closed sets \( E \) and \( F \) of \( M \), and all \( f \in L^2(M) \cap L^p(M) \) with \( \text{supp} f \subseteq E \)

\[
i) \quad \| e^{-tA} f \|_{L^2(F)} \leq Ct^{-\gamma_p} e^{-\frac{c^2(E,F)}{p}} \| f \|_{p},
\]

\[
\tag{37}
ii) \quad \sqrt{t}\nabla e^{-tA} f \|_{L^2(F)} \leq Ct^{-\gamma_p} e^{-\frac{c^2(E,F)}{p}} \| f \|_{p},
\]

\[
iii) \quad \sqrt{t}V^{1/2} e^{-tA} f \|_{L^2(F)} \leq Ct^{-\gamma_p} e^{-\frac{c^2(E,F)}{p}} \| f \|_{p},
\]

where \( \gamma_p = \frac{N}{2}(\frac{1}{p} - \frac{1}{2}) \) and \( C, c \) are positive constants.

We invest these off-diagonal estimates as in the proof of Theorem 3.1 to obtain the following result:

**Theorem 5.2.** Let \( M \) be a non-compact complete Riemannian manifold of dimension \( N \geq 3 \). Assume (32), (33) and (35). Then \( V^{1/2}A^{-1/2} \) and \( \nabla A^{-1/2} \) are bounded on \( L^p(M) \) for all \( p \in (p'_0; 2] \) where

\[
p'_0 = \left( \frac{2N}{(N-2)(1-\sqrt{1-\frac{N}{N+4}})} \right)'
\]

We say that the potential \( V \) is in the class \( K_\infty(M) \), if for any \( \varepsilon > 0 \) there exists a compact set \( K \subset M \) and \( \delta > 0 \) such that

\[
\sup_{x \in M} \int_{K^c} G(x, y)|V(y)|d\mu(y) \leq \varepsilon
\]

22
where $K^c := M \setminus K$, and for all measurable sets $B \subset K$ with $\mu(b) < \delta$,

$$\sup_{x \in M} \int_B G(x,y)|V(y)|d\mu(y) \leq \varepsilon.$$ 

Here $G(x,y) := \int_0^\infty p(t,x,y)dt$ is the Green function, and $p(t,x,y)$ is the heat kernel of the Laplace-Beltrami operator. This class is the generalization of $K_\infty$ to manifolds (see [11] Section 2).

Since (35) and (36) imply the Li-Yau estimate (37), we can use Theorem 2 of [31] and obtain a Gaussian upper bound of the heat kernel of $-\Delta - V$.

Thus arguing as in the Euclidean case, we obtain the following result.

**Theorem 5.3.** Let $M$ be a non-compact complete Riemannian manifold of dimension $N \geq 3$, and let $A$ be the Schrödinger operator $-\Delta - V, 0 \leq V \in L^{N/2}(M) \cap K_\infty$. Assume that for all ball $B$, $\mu(B(x,r)) \geq Cr^N$. Assume (32), (35) and (36). Then $\Delta(-\Delta - V)^{-1}$ and $V(-\Delta - V)^{-1}$ are bounded on $L^p(M)$ for all $p \in (1; N/2)$.

Now using Theorem 2 of [31] and Theorem 5 of [28], then arguing as in the Euclidean case we obtain the following.

**Theorem 5.4.** Let $M$ be a non-compact complete Riemannian manifold of dimension $N \geq 3$, and let $A$ be the Schrödinger operator $-\Delta - V, 0 \leq V \in K_\infty$. Assume (32), (35) and (36). Then $\nabla A^{-1/2}$ and $V^{1/2}A^{-1/2}$ are of weak type $(1,1)$, thus they are bounded on $L^p(M)$ for all $p \in (1; 2]$.

If in addition we assume that for all ball $B$ $\mu(B(x,r)) \geq Cr^N$, and for some $r \in (2; N]$, the Riesz transforms $\nabla(-\Delta)^{-1/2}$ bounded on $L^r(M)$ then $\nabla A^{-1/2}$ and $V^{1/2}A^{-1/2}$ are bounded on $L^p(M)$ for all $p \in (1; r)$ provided that $V \in L^{N/2}(M)$.

**Remark** Let $M$ be a non-compact complete Riemannian manifold of dimension $N \geq 3$. Let $H = -\Delta + V$ be a Schrödinger operator with non-negative potential $V \in L^{N/2}(M)$. Assume that for some $r \in (2; N]$, the Riesz transforms $\nabla(-\Delta)^{-1/2}$ are bounded on $L^p(M)$ for all $p \in (2; r)$ or for $p = r$. Assume also (33). Then the heat kernel associated to $H$ satisfies (34). Hence we obtain by the previous argument the $L^p$-boundedness of $V^{1/2}H^{-1/2}$ and $\nabla H^{-1/2}$ for all $p \in (1; r)$.

Note that (35) and (36) hold on manifolds with non-negative Ricci curvature (see [22]) as well as the boundedness on $L^p(M)$ for all $p \in (1, \infty)$ of Riesz transforms associated to the Laplace-Beltrami operator (see [6]).
Sobolev inequality (33) is valid on manifolds with Ricci curvature bounded from below satisfying
\[ \inf_{x \in M} \mu(B(x, 1)) > 0 \]
(see [19] Theorem 3.14). Therefore manifolds with non-negative Ricci curvature satisfying \( \inf_{x \in M} \mu(B(x, 1)) > 0 \) are a class of manifolds where Theorem 5.4 holds.

We mention that Carron, Coulhon and Hassell [10] proved that the Riesz transforms \( \nabla (\Delta^{-1/2}) \) are bounded on \( \mathcal{L}^p(M) \) for all \( p \in (2; N) \) on smooth complete Riemannian manifolds of dimension \( N \geq 3 \) which are the union of a compact part and a finite number of Euclidean ends. Ji, Kunstmann and Weber [20] proved that this boundedness holds for all \( p \in (1; \infty) \), on the complete connected Riemannian manifolds whose Ricci curvature is bounded from below, if there is a constant \( a > 0 \) with \( \sigma(\Delta) \subset \{0\} \cup [a; \infty) \). They also give examples of manifolds that satisfy their conditions. Auscher, Coulhon, Duong and Hofmann [3] proved that on complete non-compact Riemannian manifolds satisfying assumption (37), the uniform boundedness of \( (\sqrt{t} \nabla e^{-t(\Delta)})_{t>0} \) on \( \mathcal{L}^q(M) \) for some \( q \in (2; \infty) \) implies the boundedness on \( \mathcal{L}^p(M) \) of \( \nabla(\Delta^{-1/2}) \) for all \( p \in (2; q) \). And we have equivalence if \( (\sqrt{t} \nabla e^{-t(\Delta)})_{t>0} \) is uniformly bounded on \( \mathcal{L}^r(M) \) for all \( r \in (2; q) \).

Therefore we deduce the following propositions using our previous theorem and the criterion of [3]. We also use the fact that the semigroup \( (e^{-t(\Delta - V)})_{t>0} \) is bounded analytic on \( \mathcal{L}^p(M) \) for all \( p \in (1; \infty) \). This is true on manifolds where assumptions (35) and (36) hold and when \( V \in \mathcal{K}_\infty \) satisfying (32) (see e.g. [7] Theorem 1.1).

**Proposition 5.1.** Let \( M \) be a non-compact complete Riemannian manifold of dimension \( N \geq 3 \). Assume that for all ball \( B \) \( \mu(B(x, r)) \geq Cr^N \), assume (32), (35) and (36), and assume that \( V \in \mathcal{K}_\infty \cap \mathcal{L}^{N/2}(M) \). If for some \( r \in (2; N] \)
\[ |||\nabla e^{-t(\Delta)}|||_{\mathcal{L}^r(M) - \mathcal{L}^r(M)} \leq C/\sqrt{t} \]
for all \( t > 0 \), then
\[ |||\nabla e^{-t(\Delta - V)}|||_{\mathcal{L}^p(M) - \mathcal{L}^p(M)} \leq C/\sqrt{t} \]
for all \( t > 0 \), all \( p \in (1; r) \).

**Proposition 5.2.** Let \( M \) be a non-compact complete Riemannian manifold of dimension \( N \geq 3 \). Assume (33) and assume that \( V \in \mathcal{L}^{N/2}(M) \). If for some \( r \in (2; N] \)
\[ |||\nabla e^{-t(\Delta)}|||_{\mathcal{L}^r(M) - \mathcal{L}^r(M)} \leq C/\sqrt{t} \]
for all \( t > 0 \), all \( p \in (1; r) \).
for all $t > 0$, then

$$\|\nabla e^{-t(-\Delta + V)}\|_{L^p(M)\to L^p(M)} \leq C/\sqrt{t}$$

for all $t > 0$, all $p \in (1,r)$.

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