COMBINATORIAL TOPOLOGY OF THREE-DIMENSIONAL SELF-AFFINE TILES

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ABSTRACT. We develop tools to study the topology and geometry of self-affine fractals in dimension three and higher. We use the self-affine structure and obtain rather detailed information about the connectedness of interior and boundary sets, and on the dimensions and intersections of boundary sets. As an application, we describe in algebraic terms the polyhedral structure of the six fractal three-dimensional twindragons. Only two of them can be homeomorphic to a ball but even these have faces which are not homeomorphic to a disk.

1. Introduction

When Lévy [32] introduced his famous curve in 1938, he also constructed fractal surfaces in a similar way. 70 years later, we have plenty of papers on fractal sets in the plane, and a number of general statements and constructions which hold in every dimension, but very few studies on the geometry of fractals in dimension $\geq 3$.

Visualization is certainly one reason to prefer plane sets: they can be easily shown on a computer screen. In dimension 3, visualization is more difficult, even though there are good ray-tracing programs, like chaoscope [15] and IFS builder [27], which allow to look at three-dimensional fractals from any chosen viewpoint, with prescribed light sources. Figure 1 shows two views of one of our main examples. Does it have interior points? If so, is the interior connected? Is the figure simply connected? Unlike in two dimensions, the answer to such questions can hardly be guessed from the pictures - this requires tools which we develop here.

There are several mathematical difficulties in dimension $\geq 3$. We avoid the problem that linear maps do usually not commute, by concentrating ourselves to fractals generated by a single matrix. But there is another problem which will concern us: proper self-similarity is rather an exceptional case. The moduli of all eigenvalues of a $3 \times 3$ matrix do rarely coincide while for a $2 \times 2$ matrix they coincide as soon as the eigenvalues are complex. This problem is addressed in Section 2, and Theorem 2.3 implies that in $\mathbb{R}^3$ there is essentially only one type of ‘integer’ self-similar tile with 2 up to 7 pieces.

Other difficulties concern the topology of three-dimensional fractals. Our main aim here is to study the topology of self-affine tiles, and of their boundaries which are fractal surfaces. In particular, we would like to know which of these tiles are homeomorphic to a closed ball. In two dimensions, the corresponding question for disk-like tiles was discussed in a large number of papers [3, 4, 7, 12, 19, 33, 31, 36, 38] (more references can be found in [31, 42, 43]), and Jordan curve arguments were
Figure 1. Two views of the twindragon $C$, made by chaoscope [15].
Is this set homeomorphic to a ball? Is the interior connected? We develop concepts and tools to answer such questions.

heavily used. Among others, it is easy to see that connectedness of the interior of a self-affine tile in the plane is sufficient for its disk-likeness. In $\mathbb{R}^3$ this is not the case. The topology of the boundary of a tile can be much more complicated than in $\mathbb{R}^2$. Even the connectedness of a tile is not so easy to verify [1, 26].

The geometric phenomena which we encounter in fractal tiles are not caused by knots or wild spheres [14]. It is rather the self-affine fibre structure of the boundary which makes the topology complicated. Apparently, the different eigenvalues of the generating matrix produce long and thin fibres which can pierce the interior of neighboring tiles, or distort the boundary structure.

We develop algebraic tools which describe the geometry of a self-affine tile $T$ in arbitrary dimension, in a similar way as homotopy and homology groups describe the geometry of manifolds. Our tools use self-affinity in a specific way, and they yield quite detailed information about the geometry. The basic concept is the neighbor graph $G = (V,E)$. It can be considered as a blueprint of $T$ which contains all information about the topology of $T$ in a simple scheme. In the case where $T$ tiles by translations in a unique way, the vertices $v \in V$ are those translations for which $T + v$ belongs to the tiling and intersects $T$. Such translations $v$ will be called neighbor maps, and they do not only represent the neighboring piece $T + v$ but also the boundary set $T \cap (T + v)$ of $T$. This set can be a face, an ‘edge’ or just a point of $T$, or a more complicated fractal boundary set. For a ball-like tile we expect that faces are homeomorphic to disks and edges are homeomorphic to intervals, and we have to check this idea.

Neighbors in lattice tilings were introduced by Indlekofer, Kátai and Racsko [24], and neighbor maps for self-similar sets were defined by Bandt and Graf [7]. For a self-affine tile, the boundary sets have themselves a self-affine structure: each face or edge is the union of smaller copies of other boundary sets. This is indicated by
the edges of the graph $G$. The definition of $G$ in Section 3 implies that the boundary $\partial T$ is a self-affine graph-directed construction in the sense of Mauldin and Williams [34], and the graph which directs this construction is $G$ itself. This fact was stated by Scheicher and Thuswaldner [36] for lattice tiles, and used in many other papers to determine the Hausdorff dimension and topological structure of $\partial T$ for plane self-similar tiles [21, 37, 25, 33, 31, 38, 39]. Some authors [16, 36, 31, 41, 42] prefer the contact matrix defined by Gröchenig and Haas [22] which describes a subgraph of $G$.

A first application of the neighbor graph concerns the existence of self-affine lattice tilings, a problem treated in [22] and in a series of papers by Vince [40, 41, 42]. Theorem 5.1 shows that the existence of such tilings is equivalent to the compatibility of all neighbors, which can be easily checked with $G$.

In sections 7 to 12 we study the topology of the boundary of $T$. In section 8 we define the hierarchy of different boundary sets like faces, edges, points etc. which corresponds to the hierarchy of sets with different Hausdorff dimensions in Mauldin-Williams constructions. An important problem is to determine all faces – those boundary sets which cover an open set on the boundary. Two methods are developed for this purpose. Theorem 9.2 gives an algorithmic approach which only uses the combinatorics of the neighbor graph. Theorem 10.1 provides an analytic approach based on recent work by He and Lau [23] and Akiyama and Loridant [2], connected with eigenvalues of adjacency matrices and Hausdorff dimension. In section 11 we define a polyhedral structure of fractal tiles by the intersection sets of two and more faces. There are corresponding neighbor intersection graphs $G^2, G^3, ...$ which allow to calculate details of the polyhedral structure and compare it with the structure of ordinary polyhedra.

In contrast to the contact matrix, all these tools can also be applied to tiles with incompatible neighbors as well as to fractals which are not tiles, and neighbors can be related by arbitrary isometries of $\mathbb{R}^3$ instead of translations. To illustrate our methods, we selected a small class of examples, which is introduced in section 6: the twindragons. These are the self-affine lattice tiles with two pieces. The one-dimensional twindragon is the interval. In the plane, there are three examples: rectangle or parallelogram, the twindragon and the tame twindragon. They are all disk-like [24, 7]. In three-dimensional space, there are seven examples, which we denote by letters $A$ to $G$. While $A$ is conjugate to a cube, $F$ and $G$ have an extremely intricate structure. We mention some of their properties without proof. For $D$ and $E$ we determine the boundary faces and their intersections, and we show that their interior is not simply connected. $B$ and $C$ seem to be homeomorphic to a ball. In section 12 we determine their exact polyhedral structure. We show that they have faces which are not homeomorphic to a disk, but $C$ has connected interior.

This paper focusses on concepts rather than algorithmic questions although it seems clear that the neighbor graph and related finite automata are well-suited for computerized evaluation. Section 14 gives a short account of algorithmic aspects.
2. Self-affine lattice tiles

Basic concepts. The following definitions are standard \cite{28,29,42,43}. A lattice \( L \) in \( \mathbb{R}^n \) is the set of integer linear combinations of \( n \) linearly independent vectors \( e_1, ..., e_n \). Usually we take \( L = \mathbb{Z}^n \). A regular-closed set \( T \) is called a tile if it admits a tiling of \( \mathbb{R}^n \). A tiling by the tile \( T \) is a countable union \( \bigcup_k T_k \) which covers \( \mathbb{R}^n \), such that each \( T_k \) is isometric to \( T \), and no two tiles have interior points in common. We have a lattice tiling by \( L \) if the tiles \( T_k \) are just the translates \( T + k \) with \( k \in L \).

A tile \( T \) is a self-affine lattice tile if there is an affine expanding mapping \( g \) and a lattice \( L \) such that \( g \) preserves \( L \) and maps \( T \) to a union of tiles \( T + k \) with \( k \in L \). With respect to the basis vectors \( e_i \), the map \( g \) has a matrix representation \( g(x) = Mx \) where \( M \) is an integer matrix. Expanding means that all eigenvalues of \( M \) have modulus greater one. \( T \) is determined by the map \( g \) and by the lattice coordinates \( k_1, ..., k_m \) of the tiles which form the supertile \( g(T) \).

\[
g(T) = MT = \bigcup_{j=1}^m T + k_j
\]  

For \( T \) to become a tile, we must have \( m = |\det M| \), and the set \( K = \{k_1, ..., k_m\} \) must fulfill some condition. A sufficient condition is \cite{5}

\[
g(L) = \bigcup_{j=1}^m L + k_j = L + K
\]  
in which case \( K \) is called a standard digit set or complete residue system. The latter name comes from the fact that for \( |\det g| = m \), the subgroup \( g(L) \) of the additive group \( L \) has \( m \) residue classes, so by \cite{2} each class is represented by exactly one \( k_i \).

See Lagarias and Wang \cite{28,29} who also studied non-standard digit sets. In this paper, we consider only standard digit sets.

The self-affine tile \( T \) is called self-similar if the mapping \( g \) is a similarity mapping with respect to the Euclidean metric. In this case, we must distinguish the standard basis which defines the Euclidean distance, and the basis for the lattice \( L \) which need not be orthonormal.

Conjugacy of tiles. \( T \) is said to be conjugate to a self-similar tile if there is a linear map \( h \) so that \( \tilde{T} = h(T) \) is a self-similar tile. The expanding map for \( \tilde{T} \) is \( \tilde{g} = hgh^{-1} \), the lattice is \( \tilde{L} = h(L) \), and the digits are \( \tilde{k}_i = h(k_i) \), for which \( (1) \) and \( (2) \) are easily checked.

Example 2.1. In the plane, we consider the digits \( k_1 = (0,0), k_2 = (1,0) \). The parallelogram \( T \) with vertices \( (0,0), (1,0), (1,1), (0,1) \) is a self-similar tile with respect to \( M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \) and the lattice \( L = \mathbb{Z}^2 \). The rectangle \( T' = [0,1] \times [0,\sqrt{2}] \) is also a self-similar lattice tile, with \( M' = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix} \) and \( L' = \{(m,n) \mid m,n \in \mathbb{Z}\} \). The unit square \( T'' = [0,1]^2 \) is a self-affine tile with respect to the matrix \( M'' = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \) and \( L'' = \mathbb{Z}^2 \), but not self-similar. However, \( T'' \) is conjugate to the self-similar tile \( T' \) since \( h(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}x \) fulfills \( h(T'') = T' \). Note that \( \tilde{h}(T) = T' \) for \( \tilde{h}(x) = \begin{pmatrix} 1 & -1 \\ 0 & \sqrt{2} \end{pmatrix}x \).
We shall identify conjugate tiles so that $T, T', T''$ are considered as different versions of the same tile, which is essentially self-similar. \textit{Considering tiles up to conjugacy means that we focus on the essential data: the characteristic polynomial of $g$ instead of different matrix representations.} We shall also assume that $k_1 = (0)$ since this can be obtained by conjugacy with a translation.

**Proposition 2.2. (The class of self-similar tiles)**

A self-affine lattice tile $T$ with respect to $g(x) = Mx$ is conjugate to a self-similar tile if and only if all eigenvalues of $M$ have the same modulus.

**Proof.** Necessity of the condition follows from the fact that eigenvalues are not changed under conjugacy. On the other hand, if the eigenvalues have equal modulus, there is a matrix $B$ such that $\tilde{g}(x) = BMB^{-1}x$ is a similarity map. Thus $h(x) = Bx$ maps $T$ to a self-similar tile. 

**Theorem 2.3. (Very few self-similar lattice tiles in dimension 3)**

If in $\mathbb{R}^3$ a self-affine lattice tile $T$ with respect to $g(x) = Mx$ is conjugate to a self-similar tile, then one of the following two conditions is true.

(i) $m = \lvert \det M \rvert$ is a cubic number - in particular $m \geq 8$,

(ii) $M$ is conjugate to $\tilde{M} = \begin{pmatrix} 0 & 0 & \pm m \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

**Proof.** With respect to the lattice base, $g$ has an integer matrix, and so the characteristic polynomial $p(\lambda)$ of $M$ has integer coefficients. We express them in terms of the eigenvalues $\lambda_i$.

\[
p(\lambda) = -\lambda^3 + (\lambda_1 + \lambda_2 + \lambda_3) \cdot \lambda^2 - (\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3) \cdot \lambda + \lambda_1 \lambda_2 \lambda_3
\]

where $\lambda_1 \lambda_2 \lambda_3 = \det M = \pm m$. Since $T$ is conjugate to a self-similar tile, all $\lambda_i$ have equal modulus: $|\lambda_i| = r > 1$.

If the eigenvalues are real, then $\lambda_i = \pm r$ and $m = \pm r^2$. In this case $\lambda_1 + \lambda_2 + \lambda_3 \in \{ \pm r, \pm 2r, \pm 3r \}$, so $r$ must be an integer and $m$ a cubic number.

Now let us assume $\lambda_1, \lambda_2$ are complex eigenvalues. Then $\lambda_1 = r(\cos \alpha + i \sin \alpha)$ and $\lambda_2 = r(\cos \alpha - i \sin \alpha)$ for some $\alpha$, so

\[
p(\lambda) = -\lambda^3 + 2r(\cos \alpha + \frac{s}{2}) \cdot \lambda^2 - (r^2 + 2sr^2 \cos \alpha) \cdot \lambda + r^3 s
\]

where $s = \pm 1$ is the sign of $\lambda_3$. The coefficient of $\lambda$ can be written as $-2sr^2(\cos \alpha + \frac{s}{2})$. Since the coefficients are integers, either $r$ must be a rational number - and hence an integer, and $m$ a cubic number. Or $\cos \alpha + \frac{s}{2}$ must be zero. In this case, $p(\lambda) = -\lambda^3 + sm$ which is the characteristic function of $\tilde{M}$. 

**Remark 2.4.** The orthogonal part of $\tilde{M}$ is a rotation around $120^\circ$ for $s = +1$, and a rotation around $60^\circ$ composed with a reflection at the plane of rotation for $s = -1$. Even in the case where $r$ is an integer and $m$ a cubic number, $2r \cos \alpha$ must be an integer and $\alpha$ can assume only few values. Thus there are really very few self-similar lattice tiles in $\mathbb{R}^3$. 

3. Neighbors of tiles and self-affine sets

When we want to study the boundary $B$ of a lattice tile $T$, it is quite natural to consider the neighbors in the tiling - those tiles $T + k$ for which $B_k = T \cap (T + k)$ is non-empty. The $B_k$ can be considered as the faces of $T$, and $B$ is the union of the $B_k$. Obviously, the number of neighbors is finite. This idea was introduced by Gilbert [21] and Indlekofer, Kátai and Racsko [24] and used in many other papers.

**Neighbors in fractals.** There is a related concept of neighbor in self-similar fractals which appeared first in [2]. Suppose we have a fractal $A$ which consists of two copies or itself: $A = f_1(A) \cup f_2(A)$, where the $f_i$ denote contracting similarity maps. Then the geometry and topology is determined by the structure of the intersection set $C = f_1(A) \cap f_2(A)$. Only at points of $C$ it makes sense to zoom into the picture. If $x$ is in $f_1(A)$, say, and $U$ is a neighborhood of $x$ in $A$ which does not intersect $C$, then the magnified copy $f_1^{-1}(U)$ of $U$ does already exist in $A$.

The sets $f_1^{-1}(C)$ and $f_2^{-1}(C)$ can be considered as boundary sets of $A$, where $A$ intersects a potential neighbor $f_1^{-1}f_2(A)$ or $f_2^{-1}f_1(A)$. These need not be the only boundary sets, however, since $C$ consists of intersections of smaller pieces like $f_1f_2(A) \cap f_2f_2(A)$, and these may be at other positions when they are pulled back to $A$. This argument shows that the boundary sets have a self-similar structure. The number of boundary sets obtained by going to smaller and smaller pieces need not be finite, but in most common examples it is.

Let us define neighbors in this general sense. Let $f_1, \ldots, f_m$ denote contractive affine mappings on $\mathbb{R}^n$. There is a unique compact non-empty set $A$ with

$$A = \bigcup_{j=1}^m f_j(A),$$

the self-affine set with respect to the $f_i$, cf. [17]. For each integer $q$ the set $A$ splits into $m^q$ small copies $f_j(A) = A_j$, where $j = j_1\cdots j_q$ denotes a word of length $q$ from the alphabet $I = \{1, \ldots, m\}$, and $f_j = f_{j_1} \cdots f_{j_q}$.

A potential neighbor of $A$ has the form $h(A) = f_i^{-1}f_j(A)$ where $i, j \in I^*$ are any words over $I$ such that the pieces $A_i = f_i(A)$ and $A_j = f_j(A)$ intersect. The idea is that $f_i^{-1}$ maps $A_i$ to $A$ and $A_j$ to $h(A)$, a neighbor set which intersects $A$ and should be of comparable size. $h$ is called a neighbor map and the set $B = A \cap h(A)$ the corresponding boundary set of $A$. To get ‘comparable size’, we confine ourselves to words $i, j$ of equal length and make some assumptions concerning the $f_i$.

**Proposition 3.1. (Neighbor maps which are isometries)**

Let $A \subset \mathbb{R}^n$ be a self-affine set of the form (3). Let $i, j$ be words of the same length $q$, and $h = f_i^{-1}f_j$.

(i) If all $f_j$ are similarity maps with the same similarity factor $r$, then $h$ is a Euclidean isometry.

(ii) If $f_j(x) = M^{-1}(x + k_j)$ where $M$ is an expanding matrix and $k_1, \ldots, k_m \in \mathbb{R}^n$, then $h$ is a translation.
Consider the boundary sets $B_r$ (ii) is first proved for $q = 1$. Since $f_i^{-1}(y) = My - k_i$, we have

$$f_i^{-1}f_j(x) = x + k_j - k_i$$

Moreover, if $h(x) = x + v$ is a translation then

$$f_i^{-1}h f_j(x) = x + Mv + k_j - k_i$$

Now induction on $q$ shows that for $h = f_1^{-1}f_2$, $h(x) = f_i^{-1} f_j^{-1} f_{j_1} \ldots f_{j_q}(x) = x + M^{q-1}(k_{j_1} - k_{i_1}) + M^{q-2}(k_{j_2} - k_{i_2}) + \ldots + (k_{j_q} - k_{i_q})$

Lattice tiles as a special case. The case (ii) includes our self-affine lattice tiles since the defining equation (1) can be rewritten as

$$T = \bigcup_{j=1}^{m} g^{-1}(T) + g^{-1}(k_j) = \bigcup_{j=1}^{m} M^{-1}(T + k_j)$$

We get $f_j(x) = M^{-1}(x + k_j)$ as contracting maps for the self-affine set $T$.

At this point it is possible to explain the concept of standard digit set $k_1, \ldots, k_m$ which says that $k_{j_q} - k_{i_q}$ is not in $ML$ for $j_q \neq i_q$ (see section 2). This property implies that the translation vector in (6) can never be zero if $j_q \neq i_q$ since $k_{j_q} - k_{i_q}$ cannot cancel with the other terms in (6). As a consequence, the inequality $h \neq id$ then holds whenever $i \neq j$. From the results in [7] then follows the so-called open set condition which says that the pieces $f_i(T)$ and $f_j(T)$ have no interior points in common. (In [5], this was shown by Baire’s category theorem.) In other words, $T$ and a translate $h(T)$ in (6) have no common interior points.

4. The Neighbor Graph

Self-similarity of boundary sets. To explain the self-similar structure of boundary sets, we return to the general case. Consider intersecting pieces $A_i$ and $A_j$ in the self-affine set $A$. Since all pieces divide into $m$ subpieces, like $A$ in (3),

$$A_i \cap A_j = \bigcup_{i,j=1}^{m} A_{ii} \cap A_{jj}$$

Now consider the boundary sets $B = f_i^{-1}(A_i \cap A_j)$ and $B_{ij} = f_i^{-1}(A_{ii} \cap A_{jj})$. Then

$$B = f_i^{-1} \left( \bigcup_{i,j=1}^{m} A_{ii} \cap A_{jj} \right) = \bigcup_{i,j=1}^{m} f_i f_i^{-1} f_i^{-1}(A_{ii} \cap A_{jj}) = \bigcup_{i,j=1}^{m} f_i(B_{ij})$$

At $B, B_{ij}$ we suppressed the subscripts $i, j$ since such an equation holds for each possible boundary set. In other words, the subdivision of pieces induces self-similar representations of the type (3) for all boundary sets - not with one type of set, but with several types. We must assume now that we have only finitely many possible boundary sets, which is true for lattice tiles with standard digit sets. The unions
on the right contain a lot of empty terms, so we introduce a graph which better describes the system of equations.

**Concept and properties of neighbor graph.** The neighbor graph $G = (V, E)$ of a self-affine fractal $A$ has as vertex set all neighbor maps $h = f_i^{-1} f_j$, and a directed edge marked with $i$ goes from $h$ to $h' = f_i^{-1} f_j f_j$. Loops and multiple edges (even with the same label) are possible [10]. For a formal definition, we focus on self-affine tiles.

**Definition 4.1.** Let $T = \bigcup f_j(T)$ be a self-affine lattice tile with $f_j(x) = M^{-1}(x+k_j)$ so that the neighbor maps have the form $h(x) = x + k$. We identify $h$ with the translation vector $k$, and associate it with the boundary set $B_k = T \cap (T+k)$. An edge from $k$ to $k'$ with label $i$ is drawn when relation (5) holds:

$$G = (V, E) \quad \text{with} \quad V = \{ k \mid T \cap (T+k) \neq \emptyset \} \quad \text{and}$$

$$E = \bigcup_{i=1}^{m} E_i \quad \text{with} \quad E_i = \{ (k, k', i) \mid k' = Mk + k_j - k_i \} \quad (8)$$

Here $E_i$ denotes the set of edges with label $i$. It should be mentioned that $V$ contains a root vertex $k = 0$ (or $h_0 = id$ in the more general notation) which does not correspond to a boundary set. There are edges $(0, k_j - k_i, i)$ from the root which correspond to the first step (4) in the calculation of neighbor maps. The loops $(0, 0, i)$ will not be drawn, however. For the calculation of $G$, see section [14, 10, 11] and the example below.

**Proposition 4.2.** *(The boundary equations)*

Let $T = \bigcup_{j=1}^{m} f_j(T)$ be a self-affine lattice tile with neighbor graph $G = (V, E)$. The boundary sets $B_k$ corresponding to the $k \in V \setminus \{0\}$ fulfil the following equations.

$$B_k = \bigcup \{ f_j(B_{k'}) \mid j \in \{1, ..., m\}, (k, k', j) \in E \} \quad (9)$$

This is an immediate consequence of Definition 4.1 and of the discussion above. Mauldin and Williams [34] called families of such fractals $B_k$ *graph-directed constructions*, and proved that the $B_k$ are uniquely determined by the graph $G$ and the maps $f_j$. At the end of section 10 we briefly discuss the open set condition of (9).

**Proposition 4.3.** *(The basic symmetry of $G$)*

If $h$ is a neighbor map, then $h^{-1}$ is also a neighbor map. For translation this means that to every $k$ there is a $-k$, and each boundary set $B_k = T \cap (T+k)$ has an opposite set

$$B_{-k} = (T-k) \cap T = B_k - k.$$  

This symmetry of $V$ will extend as a graph isomorphism to the edges, but not to the labels. If $k' = Mk + k_j - k_i$ then $(k, k')$ has label $i$ and $(-k, -k')$ has label $j$.

In a previous publication with M. Mesing [10], we wrote both labels $i, j$ at the edge $(k, k')$. This is not necessary here since we can always recover the second label from the opposite edge.

**A two-dimensional example.** In our examples, neighbors will be denoted by lower-case Roman letters, and $-b$ will denote the opposite vertex of $b$. We start
with a two-dimensional tile [5][12] which can be considered as a modification of the square, and as an extension of the Sierpiński gasket. The construction of neighbor graphs for three-dimensional tiles proceeds in the same way as demonstrated here.

**Example 4.4.** Let $f_j(x) = (x + k_j)/2$ for $j = 1, ..., 4$ where $k_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $k_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $k_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $k_4 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$. The tile $T$ is shown in Figure [2]. Clearly, $T \subset [-1, 1]^2$. Thus a translation $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ with $T \cap (T + x) \neq \emptyset$ must fulfill $|x_1| \leq 2$ and $|x_2| \leq 2$. Moreover, $L$ is the integer lattice, so $x_1, x_2 \in \{-2, -1, 0, 1, 2\}$. In this way we prove that for an arbitrary self-affine lattice tile, the set $V$ is finite.

The tile $T$ has six neighbors which intersect $T$ in a single point, and six neighbors which intersect $T$ in an uncountable set which is in fact a Sierpiński gasket.

**Proof.** We start with the translation vectors $k = k_j - k_i$ where $i \neq j$, see the right-hand part of Figure [2]. If $k$ is in $V$, there is the edge $(0, k, i)$. Since our matrix is $M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, all these points do belong to $V$, with a loop $(k, k, j)$ at each point $k$, because $k_j - k_i = 2(k_j - k_i) + k_i - k_j$.

Moreover, these will be the only points of $V$. For the other points $x$ with $|x_1| \leq 2, |x_2| \leq 2$ the recursion (5) will lead to vectors $k'$ which are outside the range of $x_1, x_2$ and thus are no neighbor maps. A simple calculation [10] shows that once we are outside the range, we can never come in again by the recursion (5).

Thus the translations $\pm a = \pm \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\pm b = \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\pm c = \pm \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ marked by $\bullet$, and $\pm \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, $\pm \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$, $\pm \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$ marked by $\circ$ are the vertices of our neighbor graph. The last six vertices have only the loop and no further outgoing edges. These six vectors are the translations along the sides of the big triangle in Figure [2]. They translate $T$ to a neighbor $h(T)$ which has a single point as intersection with $T$. These neighbors are called point neighbors.

To find point neighbors, we need only look at the graph, not at the picture. The loop at $k = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ with label 2 says that the boundary set has address $2222... = \overline{2}$,
Figure 3. The reduced neighbor graph for the Sierpiński tile.

and the opposite vector \((-\frac{2}{1})\) has address \(4\). This means that \(T \cap (T + k)\) is exactly
one point, which has address \(2\) in \(T\) and address \(4\) in \(T + k\). Indeed the intersection
is the point \(k_2 = (\frac{1}{0})\) which is the fixed point of \(f_2\), and \(T + k\) will meet this point
with the translate of \(k_4\), the fixed point of \(f_4\). The translate of \(k_3\) along the vector
\(k' = (\frac{-1}{1})\) is also \(k_2\). This way it turns out that two of the six point neighbors will
meet \(T\) at each vertex of the triangle.

We now forget the point neighbors \(\circ\). The graph of the remaining six \(\bullet\) neighbors
\(\pm a, \pm b, \pm c\) and the root, shown in Figure 3, will be called the reduced neighbor
graph \([10]\). On the left of Figure 2 the corresponding boundary sets are indicated.
As mentioned above, each vertex \(k = k_j - k_i\) has a loop with label \(i\). The edge
\((-b, a)\) has label 2 and the opposite edge has label 3 because \(a = 2 \cdot (-b) + k_3 - k_2\).
In a similar way, all other edges were determined.

The graph in Figure 2 is irreducible which means that all corresponding boundary
sets have a similar structure. This also follows from the equations of the graph-
directed system \([9]\):

\[
B_a = f_1(B_a) \cup f_2(B_{-c}) \cup f_3(B_{-b}), \quad B_{-b} = f_2(B_a \cup B_{-b} \cup B_c), \quad B_{-c} = f_3(B_a \cup B_b \cup B_{-c}).
\]

All these boundary sets are in fact small Sierpiński gaskets, as can be seen in the
figure. The theorem below shows that they are uncountable. \(\square\)

**Topology and address map.** A sequence \(\beta = s_1 s_2 ... \in I^\infty\) is called address of
the point \(x\) in a self-affine tile \(T\) if it is contained in the pieces \(T_{s_1...s_n}\) for \(n = 1, 2, ...
\) (cf. \([13]\), Chapter IV, \([17]\), Section 9.1). The points of the intersection sets \(T_i \cap T_j\)
have two addresses, one starting with \(i\) and the other one with \(j\). The address map
\(\pi : I^\infty \rightarrow T\) is a quotient map, it determines the topology of \(T\). For that reason it is
important to know which addresses will be identified. The neighbor graph provides
this information, as can be easily shown (\([10]\), sections 4 and 5). A path in the
directed graph \(G\) is a finite or infinite sequence \(e_1 e_2 \ldots\) such that the terminal vertex
of \(e_k\) coincides with the initial vertex of \(e_{k+1}\). A finite path \(e_1 \ldots e_n\) is a cycle if the
terminal vertex of \(e_n\) is the initial vertex of \(e_1\).
Theorem 4.5. (Neighbor graph and addresses of points, see \cite{10})

Let $T$ be a self-affine tile with the address map $\pi : \mathcal{I}^\infty \to T$ and the neighbor graph $G = (V, E)$.

(i) Two sequences $\beta = s_1s_2...$ and $t = t_1t_2...$ are addresses of the same point if and only if there is a path $e_1e_2...$ starting in the root with labels $s_1s_2...$, such that the opposite path $-e_1-e_2...$ has labels $t_1t_2...$

(ii) The addresses of the points of a boundary set $B_k$ coincide with label sequences of paths starting in the vertex $k$. They form a regular language $L_k$.

(iii) If only one cycle can be reached from the vertex $k$ by a directed path, then $B_k$ is a singleton.

(iv) If two different cycles can be reached from $k$, and these cycles can be reached from each other, then the set $B_k$ is uncountable.

Note that a cycle with a diagonal, or with a loop at one of its points counts as two different cycles which can reach each other.

5. Existence of self-affine lattice tilings

Before we can go on, we must settle one difficulty with neighbors. They need not be compatible: $h(T)$ and $\tilde{h}(T)$ can be neighbors of $T$, but not of each other. In that case, different tiles in a tiling will have different neighborhoods.

It turns out that the compatibility of neighbors is connected with another question which was studied by Gröchenig and Haas \cite{22} and in several papers of Vince \cite{10, 41, 42} as a "central" problem in the field. A self-affine lattice tile with standard digit set will always admit self-affine tilings. It is a non-trivial result that it will also admit lattice tilings \cite{30} but these need not be self-affine in the sense that tiles assemble to form supertiles (\cite{42}, Example 4.2). Vince (\cite{41, 42}, Theorem 4.3) gave 10 equivalent conditions for the existence of a self-affine lattice tiling. Two of them are of an algorithmic nature.

Here we extend Vince’s list by proving that a self-affine lattice tiling exists if and only if all neighbors are compatible with each other, and we show that this property can be easily decided with the neighbor graph. The following theorem also shows that all self-affine sets with mappings $f_j(x) = M^{-1}(x + k_j)$ which are not lattice tiles must have incompatible neighbors.

Theorem 5.1. (Compatible neighbors and self-affine lattice tilings)

Let $A \subset \mathbb{R}^n$ be a connected self-affine set with respect to mappings $f_j(x) = M^{-1}(x + k_j)$, $j = 1, ..., m$ where $M$ is an expanding matrix and the open set condition holds. Then the following conditions are equivalent.

(i) All potential neighbors appear together at one piece $A_i$.

(ii) $A$ is a self-affine lattice tile which admits a self-affine tiling by a lattice.

(iii) In the neighbor graph, each vertex $k \neq 0$ has at least one incoming edge with each of the labels $j = 1, ..., m$.

Proof. (i)⇒(ii): We assume $A$ is a self-affine set, and $A_i$ has all possible neighbors. Then the number of neighbors must be finite, at most $m^n$ if $i = i_1...i_n$. Moreover,
$f^{-1}_i(A)$ consists of copies of $A$ which include all possible neighbors $h(A)$. Since the maps are translations $h(x) = x + k$ by Proposition 3.1, let again $K$ denote the set of all these translation vectors. We define an infinite pattern of translates of $A$ as a union of increasing compact sets:

$$\bigcup_{q=1}^{\infty} f^{-q}_i(A).$$

All translates $A'$ of $A$ in this pattern have the form $f^{-q}_i(A_{ij})$ for some number $q$ and some word $j$. This implies that they all have the same neighbors as $A$. To see this, consider the (potential) boundary of a subpiece $A_{ij}$ with $j \in \{1, \ldots, m\}$, defined as the union of all its intersections with potential neighbors. This boundary is contained in the union of the boundary of $A_i$ and the intersections $A_{ij} \cap A_{ii}$ with $i = 1, \ldots, m, i \neq j$. So by assumption the boundary of $A_{ij}$ is contained in $A$, and by induction on the length of $j$ this is proved for subpieces $A_{ij}$.

Thus all ‘atoms’ $A'$ of our infinite pattern have a maximal set of neighbors which covers their boundary completely. But there is only one maximal set: the complete set of translates $A' + k$ with $k \in K$.

Let $L$ denote the lattice generated by $K$. Since $A$ was connected, any two pieces of the same level are connected by a chain of neighboring pieces (cf. [11], 8.2.1), and each atom has the form $A' = A + k$ with $k \in L$. Moreover, $L$ has the same rank as the linear subspace generated by $A$, which we will now assume is $\mathbb{R}^n$.

Since all $A'$ have the neighbor vectors $K$, the infinite pattern coincides with $A + L$. As a consequence, the pattern must be a tiling of $\mathbb{R}^n$: If there was a small open set $U$ outside $A + L$ then there would be arbitrary large copies $f^{-q}_i(U)$ outside $A + L$ which would contradict the finite mesh size of $L$. Thus we have constructed a self-affine lattice tiling by copies of $A$.

(ii)$\Rightarrow$(iii): Let $\mathbb{R}^n = A + L$ be a self-affine lattice tiling by the tile $A$. Then each tile of the tiling has the same set of neighbor translations $K$. Thus if $k$ is in $K$, then $k$ appears as a neighbor translation of any piece $T_j$ of a supertile $T$, where $j \in \{1, \ldots, m\}$. If the neighbor is another piece $T_i$ of $T$, there is the edge $(0, k, j)$ in $G$. If the neighbor of $T_j$ is in another supertile $T'$ then we have the edge $(k', k, j)$ where $k'$ is the neighbor map between $T$ and $T'$. Thus for each vertex $k$ of the vertex set $K$, there is an incoming edge with label $j$.

(iii)$\Rightarrow$(i): We assume that each vertex $k$ in the neighbor graph $G = (V, E)$ has incoming edges with each label $j$. We have to find a word $i = i_1 \ldots i_N$ such that $A_1$ has all neighbors $k \in V$. For each $k$ there must be a suffix $i_{n(k)} \ldots i_N$ of $i$ which consists of labels of a path from 0 to $k$.

It is not difficult to construct such a word $i$ by induction, going paths backwards (against the direction of edges) towards 0. Start with a vertex $k_1$ and let $j_1 = 1 \ldots j_{q_1}$ be the labels of a backward path from $k_1$ to 0. According to our assumption, there are also backward paths with label sequence $j_1$ from all other vertices. Their endpoints, which are different from 0, will form a set $V_1$.

Next, take a $k_2 \in V_1$ and a backward path $j_2 = j_{q_1+1} \ldots j_{q_2}$ from $k_2$ to 0. Also take backward paths from all other $k \in V_1$ with label sequence $j_2$ and denote the set of
their non-zero endpoints by \( V_2 \). Continue with \( k_3 \in V_2 \), and so on. Since \( V_{n+1} \) has less points than \( V_n \), we will have \( V_n = \emptyset \) for some \( n \). The sequence \( i = j_{q_n}...j_{q_{n-1}}j_1 \) will have all required suffixes.

All examples considered in this paper will fulfil the condition of theorem 5.1. For Example 4.4 this can be seen in Figure 3, for the twindragons in the figures below.

6. The seven three-dimensional twindragons

**Twindragons and their symmetry.** We now define the family of examples we are going to study here. A twindragon is a self-affine lattice tile with \( m = 2 \) pieces. The expanding matrix \( M \) has determinant \( \pm 2 \). In dimension 1, a twindragon is an interval. In \( \mathbb{R}^2 \), there are three examples: the rectangle (Example 2.1), the ordinary twindragon and the tame twindragon, and they are all disk-like [7].

Since we identified conjugate tiles, we could take \( k_1 = 0 \) in section 2. For twindragons in \( \mathbb{R}^n \), we can also choose \( k_2 \) arbitrarily, as long as \( k_2 \) is not in an invariant linear subspace of \( M \). For the basis \( \{k_2, g(k_2),..., g^{n-1}(k_2)\} \) the matrix \( M \) of \( g \) will always be the same, determined by the characteristic polynomial of \( g \) (see [20] and the proof of Theorem 6.2 below). For our twindragons in \( \mathbb{R}^3 \), we choose another normalization:

\[
k_1 = \left(-\frac{1}{2}, 0, 0\right)', \quad k_2 = \left(\frac{1}{2}, 0, 0\right)' \tag{10}
\]

This has the effect that 0 is the symmetry center of \( T \), and in the neighbor graph, an edge with label 1 passes from the root to \( a = (1, 0, 0)' \).

**Proposition 6.1.** (Symmetry of twindragons)
Each twindragon \( T \) in \( \mathbb{R}^n \) has a symmetry center at \( c = \frac{1}{2}(k_1 + k_2) \). The point reflection at \( c \) interchanges the two pieces \( T_1, T_2 \), and each boundary set \( B_k \) with the opposite set \( B_{-k} \). Moreover, each boundary set \( B_k \) of \( T \) also has a symmetry center at \( c_k = c + \frac{k}{2} \).

**Proof.** We can take \( k_1, k_2 \) from (10) since the symmetry is not changed by an affine conjugacy. Thus \( f_1(x) = M^{-1}x + k_1, f_2(x) = M^{-1}x - k_1 \). With \( \phi(x) = -x \) we obtain \( \phi f_1 = f_2 \phi \) or \( f_i = \phi f_{3-i} \phi^{-1} \) since \( \phi^{-1} = \phi \). For any word \( i = i_1...i_n \) on \( I = \{1, 2\} \), we have \( f_i = \phi f_{j_i} \phi^{-1} \), where each \( j_i = 3 - i_i \) is the opposite symbol. Since the point of \( T \) with address \( b = i_1i_2... \) can be represented as \( x = \lim_{n \to \infty} f_{i_1}f_{i_2}...f_{i_n}(z) \) where \( z \) is any starting point [17], this implies that \( \phi \) transforms this point into the point with the opposite address. In particular \( \phi(T) = T \) and \( \phi(T_i) = T_{3-i} \) from which it follows that \( \phi \) maps \( T_1 \cap T_2 \) onto itself.

For each boundary set \( B_k = T \cap (T + k) \), the map \( \phi \) transforms \( T + k \) to \( \phi(T) - k = T - k \), thus \( \phi(B_k) = B_{-k} \). On the other hand we had \( B_{-k} = B_k - k \), so \( \psi(x) = -x + k \) maps \( B_k \) to itself. \( \psi \) is the point reflection at \( c_k = \frac{k}{2} \). \( \square \)

**The seven twindragons.** The last proposition shows that the tile generated by \( f_1, f_2 \) can also be generated by \( f_1 \phi \) and \( f_2 \phi \), where \( \phi \) is the point reflection at \( c \). Since in \( \mathbb{R}^3 \) the map \( \phi \) has determinant -1, this has the consequence that we need only consider the matrices \( M \) with \( \det M = 2 \). With -2 we get the same twindragons.
Theorem 6.2. (List of three-dimensional twindragons, cf. [26 1])

Up to conjugacy, there are exactly seven twindragons, each of which is uniquely determined by the pair of coefficients \((a, b)\) of its characteristic polynomial

\[
p(\lambda) = -\lambda^3 + a\lambda^2 + b\lambda + 2 :\]

| Twindragon | A | B | C | D | E | F | G |
|------------|---|---|---|---|---|---|---|
| Parameter  | (0, 0) | (-1, 1) | (1, -1) | (0, 1) | (2, -2) | (1, 0) | (0, 2) |

Proof. These are the only integers \((a, b)\) for which \(p(\lambda)\) has only roots of modulus > 1 [26 1]. For \(k_1 = 0, k_2 = (1, 0, 0)\) and the basis \(\{k_2, Mk_2, M^2k_2\}\) the affine map

\[
g(x) = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & b \\ 0 & 1 & a \end{pmatrix} \cdot x
\]

has characteristic polynomial \(p(\lambda)\), and yields a twindragon. The standardization (10) translates this twindragon by \((-\frac{1}{2}, 0, 0)\) so that its center is 0. \(\square\)

Example 6.3. (The non-fractal twindragon A)

A, the self-affine cube with 2 pieces, is the three-dimensional analogue of Example 2.1. When considered as a rectangular parallelepiped with side lengths \(1, \sqrt{2}, \sqrt{4}\), it is the self-similar exception mentioned in Theorem 2.3, (ii). The similarity map \(g\) then is a 120° rotation composed with a homothety by the factor \(\sqrt{2}\). Of course, this tile has 26 neighbors: 6 faces, 12 edge neighbors and 8 point neighbors.

Number of neighbors of twindragons. The twindragons form a nice little family with seven examples on which we can test our neighbor methods. However, the six fractal examples are truly self-affine, which makes their topology intricate. As a first overview, we compare the moduli of eigenvalues of \(M\) - the inverses of the contraction factors of the \(f_j\) along different directions - and the number of
neighbors. If the contraction factors differ much, there are lots of neighbors. We also determine the number of infinite boundary sets, by deleting from the neighbor graph the vertices representing finite boundary sets, as was explained in Example 4.4 and Theorem 4.5. It turns out that all infinite boundary sets are uncountable. The number of faces will be determined in the sections below.

**Proposition 6.4. (First neighbor calculation for the seven twindragons)**

| Twindragon | A       | B       | C       | D       | E       | F       | G       |
|------------|---------|---------|---------|---------|---------|---------|---------|
| Parameter  | (0,0)   | (-1,1)  | (1,-1)  | (0,1)   | (1,0)   | (2,-2)  | (0,2)   |
| Complex eigenvalues’ modulus | 1.26    | 1.29    | 1.22    | 1.15    | 1.14    | 1.09    | 1.06    |
| Real eigenvalue     | 1.26    | 1.21    | 1.35    | 1.52    | 1.54    | 1.70    | 1.77    |
| Number of neighbors | 26      | 18      | 20      | 34      | 34      | 48      | 76      |
| Uncountable boundary sets | 18      | 14      | 12      | 14      | 12      | 16      | ≥22     |
| Faces     | 6       | 14      | 12      | 14      | 12      | 16      | ≥22     |

For $B$, $C$ and $D$ the number of infinite boundary sets is smaller than for the cube, so that we can expect them to be more or less ball-like. $E$ has medium complexity, and the last two examples have many infinite neighbor sets – they could be three-dimensional counterparts of Example 4.4.

When we compare with the figures, we see that the twindragons possess a kind of fibre structure along the direction of the smaller eigenvalues of $M$, that is, the larger eigenvalues of $M^{-1}$. For $B$, the pieces drawn in Figure 4 are stretched along the direction of the real eigenvalue of $M^{-1}$ which is the largest contraction factor. So roughly the pieces look like ‘cigars’. In all other examples, the larger contraction factor is given by the complex eigenvalues, and the pieces look like leaves or plates. For $C$ in Figure 5, the leaves still have some thickness, while for $E$ illustrated in Figure 5, the plates are already very thin. For the other three twindragons, the plates look similar or still thinner.
How to draw neighbor graphs of twindragons. Let us mention some other nice properties of our family and fix some notation for the neighbor graphs. First, all twindragons are connected. For two pieces this follows from $T_1 \cap T_2 \neq \emptyset$ (see Barnsley [13], 8.2.1). If this relation would not hold, $T$ would be a Cantor set and would have no interior points.

Next, the only possible differences $k_j - k_i$ are $k_2 - k_1 = (1,0,0)'$, $k_1 - k_2 = (-1,0,0)'$, and $k_i - k_i = 0$. The corresponding labels $i$ of the edges are 1 and 2 and where the last case means two edges labelled with 1 and 2. We shall draw edges for the last case as double arrows, and we distinguish labels 1 and 2 by assigning a fat tip to the arrows with label 2. This way we save the labelling of edges. The label of the opposite edge is $j = 3 - i$ for simple arrows, and $j = i$ for double arrows.

In view of the symmetry, we can simplify the neighbor graph further by drawing only one vertex from each pair of opposite vertices $k, -k$ and introducing the convention that a wavy arrow $\rightsquigarrow$ to $k$ denotes an arrow to $-k$. This will also simplify the equation systems (9) for the boundary sets. It really matters whether one has to work with 20 or 40 vertices! Note that the opposite vertex of $k$ is $-k$ not only as a vector, but by Proposition 6.1 also as a boundary set: $B_{-k} = -B_k$.

For sake of brevity, we shall provide no details for the complicated twindragons $F$ and $G$, but we give full arguments for the other four examples.

To verify the condition of Theorem 5.1, we check in Figures 6, 7, 8, and 9 that each vertex is reached either by a double arrow, or by two ordinary arrows with different tips indicating label 1 and 2, or by an ordinary and a wavy arrow with equal tips. In the last case the wavy arrow leads to $-k$, so we have to count the opposite edge.
Example 6.5. We derive the equations (9) of the boundary sets from Figure 6. We neglect all point neighbors, given by vertices below the thin line.

$B_a = f_1(B_b) \cup f_2(B_b) \cup f_2(B_c), \quad B_b = f_1(B_d) \cup f_2(B_d), \quad B_d = f_2(B_t),$

$B_c = f_1(B_e) \cup f_2(B_e) \cup f_1(B_t), \quad B_e = f_2(-B_c), \quad B_t = f_2(B_a).$

By substitution we reduce the system to two equations:

$B_a = f_2(B_c) \cup \bigcup_{i,j=1}^2 f_{ij22}(B_a), \quad B_c = f_{12}(B_a) \cup \bigcup_{i=1}^2 f_{i2}(-B_c).$

7. Identifying faces: the simple case

Defining faces. The topological boundary of a set $T$ will be denoted by $\partial T = T \setminus \text{int} T$. If $T$ is a tile in a tiling, then $\partial T$ must be covered by the boundary of neighboring tiles: $\partial T = \bigcup_{k \in V} B_k$.

Theorem 4.5 says that we can determine the cardinality of a boundary set $B_k$ by counting the number of paths starting from $k$ in the neighbor graph $G$. It is much more complicated to determine the topology of $B_k$. In three-dimensional self-affine tiles, we would like to distinguish boundary sets which are two-dimensional fractals, one-dimensional curves, Cantor sets or just finite sets.

Let us say that the boundary set $B_k = T \cap (T + k)$ is a face of $T$ if there is a point $x \in B_k$ and an open neighborhood $U$ of $x$ with $U \cap \partial T \subset B_k$ and $U \subset T \cup (T + k)$. In that case, all boundary points of $U \cap \text{int} T$ must belong to $B_k$. Thus $B_k$ has topological dimension 2 since the boundary of each open set in $\mathbb{R}^n$ has dimension $n - 1$. The following statement shows that we have six faces in Example 4.4.

Proposition 7.1. Let $k \neq 0$ be a vertex in $G$ such that for each other vertex $k' \neq 0$, either $k'$ or the opposite vertex $-k'$ can be reached by a path from $k$. Then $B_k$ is a face of $T$. 
Proof. Each tile must have faces. By (9), any path from $k$ to $k'$ implies the existence of a small copy of $B_{k'}$ in $B_k$. Thus our assumption implies that $B_k$ contains copies of faces, so it must be a face (cf. Proposition 8.1).

For twindragons, there is always an edge with label 1 from the root to the first other vertex $a = k_2 - k_1 = (1,0,0)'$, and this must be a face by Proposition 7.1. It turns out that for the twindragons $B$, $C$ and $D$, all infinite boundary sets fulfil the assumption of the proposition since from their vertices in $G$, there is a path to $a$. The neighbor graphs are shown in Figures 6, 7 and 8. The point neighbors are separated from the faces by a thin line. They are found by Theorem 4.5, (iii).

For $B$ this can be seen in Figure 7. For $C$ we have the loop at $g$ with address $\overline{2}$, and the 2-cycle $\{i, h\}$ in Figure 6 which denotes a 4-cycle $\{i, -h, -i, h\}$ in the complete neighbor graph, with address $\overline{2112}$. (The wavy arrow indicates that we go to the opposite sets, and there the labels 1 and 2 have to be interchanged. A second wavy arrow reestablishes the original labels.) In Figure 8 we have a similar 4-cycle $\{q, s, j, n\}$ which denotes an 8-cycle of point neighbors in the complete neighbor graph with address $\overline{21221211}$. Also $h$ and $m$ are point neighbors, while $i, r, p$ and $k$ describe finite boundary sets with more than one point. We summarize:

**Proposition 7.2. (Faces of $B$, $C$ and $D$)**

The twindragon $C$ has 12 faces and 8 point neighbors. $B$ has 14 faces and 4 point neighbors. The twindragon $D$ has 14 faces, 12 point neighbors and 8 further finite boundary sets.
The partial order for boundary sets. To compare different infinite boundary sets, we define an order on the vertices of $G$. For $u, v \in V$ we write $u \succ v$ if there is a directed path from $u$ to $v$. We say that the vertices are equivalent, and write $u \sim v$ if either $u = v$ or both $u \succ v$ and $v \succ u$. For the boundary sets, $u \succ v$ means that $B_u \supset f_i(B_v)$ for some word $i = i_1...i_q$.

**Proposition 8.1.** If $u \succ v$ then $\dim B_u \geq \dim B_v$ holds for the topological dimension, the Hausdorff dimension and the box dimension. Moreover, if $B_v$ is a face, then also $B_u$ will be a face.

**Proof.** The dimension concepts mentioned here (and also some others [17]) have the following properties.

(i) If $A \subseteq B$ then $\dim A \leq \dim B$.

(ii) If $h$ is an affine map then $\dim h(B) = \dim B$.

Together with the remark above this proves the first assertion.

If $B_v = T \cap T + v$ is a face, there is $x \in B_v$ with an open neighborhood $U \subset T \cup T + v$. Now $u \succ v$ means that $B_u \supset f_i(B_v)$ for some word $i$ where $T + u \supset f_i(T + v)$. So $y = f_i(x)$ and its neighborhood $f_i(U)$ can be taken to show that $B_u$ is a face. □

Strong components of the neighbor graph. We can now classify the vertices of $G$ in a similar way as states of a Markov chain. Clearly $\sim$ is an equivalence relation. The topological and Hausdorff dimension, as well as the property of describing a face, are invariant with respect to $\sim$. Let $\tilde{V}$ be the set of equivalence classes with respect to $\sim$ which are called strong components of $G$. Then we get a new graph $\tilde{G}$ by drawing an edge from $(\tilde{u}, \tilde{v})$ if there is $u \in \tilde{u}$ and $v \in \tilde{v}$ such that $(u, v)$ is an edge in $G$.

The graph $\tilde{G}$ compresses the structure of $G$. If two vertices $u, v \in \tilde{G}$ belong to the same class, this means that $B_u$ contains a copy of $B_v$ and conversely, so both boundary sets have essentially the same structure. The root $0 \in \tilde{V}$ would also be the root of $\tilde{G}$ but it has no meaning and is omitted.

The relation $\succ$ is a partial order on $\tilde{V}$. If $\tilde{u} \succ \tilde{v}$ then all boundary sets $B_u$ contain copies of all boundary sets $B_v$, and $\dim B_u \geq \dim B_v$ for all choices $u \in \tilde{u}, v \in \tilde{v}$. Thus the largest classes of boundary sets are the faces, then we have one-dimensional sets, Cantor sets, and as terminal classes countable and finite sets. Boundary sets from classes $\tilde{u}, \tilde{v}$ with $\tilde{u} \succ \tilde{v}$ can also have equal dimensions, however.

**Remark 8.2.** Consider a class $\tilde{u} \in \tilde{V}$, as a subset of $V$. There are three cases.

(i) $\tilde{u}$ contains only one vertex. Then the boundary set $B_{\tilde{u}}$ has the same dimension as the successor vertex $v$ (in case of several successors, the union of the successor boundary sets). Namely $B_{\tilde{u}} = f_i(B_v)$.

(ii) $\tilde{u}$ is a cycle in $G$, without loops and diagonals. In that case the boundary sets $B_u$ with $u \in \tilde{u}$ are countable unions of copies of the successor boundary sets.

(iii) $\tilde{u}$ contains two cycles. Only in this case $\tilde{u}$ produces its own Cantor structure, and the $B_{\tilde{u}}$ can have larger dimension than the successor sets.
In Figures 6, 7, and 8 we have one class of faces, a number of singleton classes and a terminal cycle which is a class of two, one or four points. In Figure 9 we have a class of six faces, and a class of eight infinite neighbors h.r on the right, as well as three singleton classes. The vertex s describes a point neighbor, the vertices g and p correspond to infinite boundary sets with equal dimensions as h.r, by the above remark, (i). We now have to decide whether g,p and h.r are faces.

9. Identifying faces: the general case

An algorithmic approach to find the faces.

When we consider one equivalence class of the neighbor graph which contains a face, then it consists of faces only.

Problem 9.1. Must all faces belong to one equivalence class of G?

This problem also arises when we work with the contact matrix. In [2], irreducibility was just assumed. We present methods to decide the problem algorithmically. We start with a simple topological observation.

Proposition 9.2. The faces cover the boundary of a self-affine lattice tile T.

Proof. Let x be a boundary point of T and U an open neighborhood of x. Let $E = U \cap \partial T$ and let $T_k, k = 1, ..., q$ denote the neighbor tiles of T. The sets $T \cap T_k$ cover E, and by the Baire category theorem one of these sets, say $T \cap T_1$, must contain an open subset V of E. Each $y \in V$ has a neighborhood $W \subset T \cup T_1$ of x. Since this holds for any U, the point x is an accumulation point of the faces. Since we have only finitely many faces, x must belong to one of them. □

Theorem 9.3. (The neighbor graph decides which boundary sets are faces)

Let us assume that all neighbors of the tile T are compatible. Then a boundary set $B_u$ is a face if it is not contained in the union of all other boundary sets.

For the neighbor graph this means that there is a word i which belongs to the language $L_u$ of labels of paths starting in u, but not to $L_v$ for any other $v \neq u$.

Proof. Consider the union of the tile T and all its neighbor tiles. If $B_u$ is not a face, then by the proposition it is contained in the union of the remaining boundary sets. If $B_u$ is a face, there is $x \in B_u$ and a neighborhood $U \subset T \cup T_u$ of x. Then U cannot intersect any other neighbor $T + u$ since neighbors must have no common interior points. The first part of the theorem is proved.

Now x has an address: $x = \pi(\beta)$ where $\pi$ is the projection from the symbol space $I^\infty$ to T. Since $\pi$ is continuous [11, 17], there is a prefix i of x such that all addresses t which begin with i will fulfil $\pi(t) \in U$. By Theorem 4.5 (iii), this shows that for a face $B_u$ there is a word i which is contained in $L_u$ and in no other $L_v$.

Conversely, suppose $B_u$ is not a face and i $\in L_u$. There is an address $\beta$ which begins with i such that $\pi(\beta) = x$ is not on the intersection of different pieces $f_j(T)$. (It is known that the intersections have Lebesgue measure 0 [7], and since the normalized
Figure 9. Neighbor graph of twindragon $E$. The strong component of a..f represents faces, s is a point neighbor. Vertices h..r form a strong component representing infinite boundary sets. g and p are components with boundary sets of the same dimension as h..r.

Lebesgue measure on $T$ can be considered as the image measure of the product measure $\{\frac{1}{m},...,\frac{1}{m}\}$ on $I^\infty$, no cylinder can be mapped completely into the intersections.) Since $x$ is also in a face $B_v$ and has only one address, $L_v$ contains the sequence $\beta$ starting with $i$. □

Examples of address calculations for boundary sets. We shall apply our theorem to Figure 9. Address calculations are boring and best left to the computer, but they can provide valuable information about the intersection of pieces. We start with simple examples.

Example 9.4. (Point neighbors in $C$ and $B$)
Which faces contain the point neighbors of $C$ and $B$? In Figure 6 we find
$$B_g = \pi(2) = B_a \cap B_b \cap B_c \cap B_d,$$  
$$B_h = \pi(2211) \subseteq B_c \cap B_d \cap B_e,$$  
$$B_i = \pi(2112) = B_{-c} \cap B_e \cap B_f,$$  
$$B_j = \pi(21221) \subseteq B_a \cap B_{-c},$$
so $g,h,i$ seem proper vertices but $j$ seems to be on an edge. On the other hand, both point neighbors of Figure 7 coincide with the intersection of only two faces:
$$B_i = \pi(2) = B_c \cap B_e,$$  
$$B_h = \pi(12) = B_a \cap B_c.$$

Already these calculations show the strange polyhedral structure of twindragons. For Figure 9 the boundary sets look much more complicated than for ordinary polyhedra where an edge is covered by two faces. The proof of the next proposition will also reveal hidden symmetry behind the apparently orderless structure of $G$.

Proposition 9.5. (Boundary structure of $E$)
The vertices $g$, $p$, $h..r$ in the neighbor graph of the twindragon $E$ do not represent faces. The boundary set $i$ is contained in the union of the four faces $e$, $-f$, $c$, $-a$ and not in a union of three of them.
Proof. We determine the language $L_i$ from the three elementary cycles which lead in $G$ from $i$ to either $i$ or $-i$ (cf. section 7 and Example 9.4).

$$L_i = *2111L_i \cup 12L_{-i} \cup **1122L_{-i} = C_1L_i \cup C_2L_i \cup \bigcup_{j=0}^{4} D_jL_{-i}$$ (11)

where $*$ is considered as a wildcard for both 1 and 2, and $C_1 = 12111$, $C_2 = 22111$, $D_0 = 12$, $D_1 = 1*11122$, $D_2 = 1*21122$, $D_3 = 2*11122$, $D_4 = 2*21122$. We now consider the set of faces $J = \{e, -f, c, -a\}$. We will show

$$L_i \subset L_J = L_e \cup L_{-f} \cup L_c \cup L_{-a}$$ (12)

First we study the action of $C_1$ and $C_2$ on the vertices of $J$. In the following table $+$ means there is path from $x$ to $y$ with label $C_1$, and $*$ means the same for $C_2$.

| x | y | e | -f | c | -a |
|---|---|---|----|---|----|
| e | * | * |    | * |    |
| -f| + | + |    |   |    |
| c | * | * |    |   |    |
| -a| + | + |    |   |    |

Claim 1: any word $C \in \{C_1, C_2\}^*$ is in $L_J$, and it can be realized with any prescribed terminal point in $J$.

This claim can be proved by induction on the number $t$ of terms in $C = C_{q_1}...C_{q_t}$, using the table. The starting point always depends on the word. For instance, it must be $-f$ for $C = C_1^3$.

Now we study how the $D_j$, $j = 0, ..., 4$ lead from $J$ to $-J$. In $G$ it can be seen that $D_0$ leads from $e$ to $-e$, $-f$ to $f$, $c$ to $a$, and $-a$ to $-c$, respectively. Moreover, $D_1$ leads from $-a$ to all vertices of $-J$, and from no other vertex of $J$ to any vertex of $-J$. Similarly, $D_2$ leads exclusively from $-f$ to all of $-J$, and $D_3$ leads exclusively from $e$ to all of $-J$, and $D_4$ leads exclusively from $c$ to all of $-J$. From this observation it follows that all vertices of $J$ are needed to obtain $L_i \subset L_J$. Moreover, it is clear that

Claim 2: any word $D_j$, $j = 0, ..., 4$ can be realized by a path from $J$ to $-J$ with any prescribed terminal point in $-J$, provided we can select the starting point.

Applying Claims 1 and 2 alternatingly, to $J$ and its opposite set $-J$, we complete the proof of (12). The rest follows from Theorem 9.3. □

10. Dimensions and eigenvalues

The modified Hausdorff dimension. In this section we give an alternative proof for the fact that $E$ has only 12 faces. It is based on recent work of He and Lau [23] and of Akiyama and Loridant [2]. Let $H$ denote the adjacency matrix of the graph $G \setminus \{0\}$. For two non-zero vertices $u, v$ the entry $h_{uv}$ denotes the number of edges leading from $u$ to $v$. It is also possible to take the adjacency matrix of our simplified graphs, where only one vertex from each pair $\{k, -k\}$ is taken into account and $h_{uv}$ counts the edges from $u$ to $\{v, -v\}$. Since $H$ is a non-negative matrix it has a real eigenvalue of maximum modulus which we call $\lambda$. 

When the matrix $M$ defining the affine maps $f_j$ is conjugate to a similarity matrix with expanding factor $R$ (see Proposition 2.2) it was shown by Mauldin and Williams [34] that the Hausdorff dimension of the sets $B_k$ from (9) fulfills
\[
\dim B_k \leq \frac{\log \lambda}{\log R},
\] (13)
and equality holds if an open set condition is satisfied, see the discussion at the end of this section.

For self-affine sets, however, calculation of dimensions is much more complicated then for self-similar ones. However, for our case where all $f_j$ are defined by one matrix $M$, He and Lau [23] have defined a pseudo-norm $w$ on $\mathbb{R}^n$ which turns the $f_j$ into similarity mappings, with factor $r = 1/\sqrt{\det M}$. Moreover, $w$ fulfills a weak type of triangle inequality which is sufficient to develop Hausdorff measure and dimension as usual. And $w$ generates the Euclidean topology even though it drastically distorts the geometry of $\mathbb{R}^n$. The value of the modified Hausdorff dimension $\dim w B_v$ was used in [23] to give estimates of the real Hausdorff dimension in terms of eigenvalues of $M$. Akiyama and Loridant [2] applied the modified Hausdorff dimension to boundary sets of tiles. Here we shall compare it with the topological dimension.

**Theorem 10.1.** (Upper estimate of topological dimension of boundary sets)

Suppose that $T$ is a self-affine tile with $m$ pieces in $\mathbb{R}^n$, and all neighbors are compatible. Let $W$ denote a strong component of the graph $G$, and let $\lambda_W$ denote the largest eigenvalue of the adjacency matrix $H_W$ of $W$ as a subgraph of $G$. If for some integer $q < n$,
\[
\frac{n \log \lambda_W}{\log m} < q
\]
then the topological dimension of all boundary sets $B_v$ with $v \in W$ is strictly smaller than $q$.

**Proof.** The matrix $H_W$ is irreducible, and $W$ is non-periodic (i.e. the g.c.d. of the lengths of cycles from a point $v \in W$ to itself is 1) because of the compatibility of neighbors. So $\lambda_W$ is the unique largest eigenvalue. The Hausdorff dimension of the sets $B_v$ with $v \in W$ with respect to the He-Lau pseudo-norm is \( \leq \frac{\log \lambda_W}{\log R} = \frac{n \log \lambda_W}{\log m} \) since $R = \sqrt{m}$ [2, 23]. The pseudo-norm generates the Euclidean topology, and the topological dimension is always smaller than the Hausdorff dimension. □

**Example 10.2.** (Boundary sets of the twindragon $E$)

For $E$ we determine the dimension of the boundary sets $B_v$ with $v \in \{g, p, h, r\}$. We obtain $\lambda_W \approx 1.554$. A small polynomial for $\lambda_W$ can be obtained from equation (11) for $B = B_i$ which expresses $B$ as a self-affine set
\[
B = \bigcup_{i=1}^{2} f_{i2111}(B) \cup f_{12}(-B) \cup \bigcup_{i,j,k=1}^{2} f_{ijkl22}(-B).
\]
The characteristic equation is $1 = 2z^5 + z^2 + 8z^7$, which for $\lambda = 1/z$ becomes $\lambda^7 - \lambda^5 - 2\lambda^2 - 8 = 0$.
The modified Hausdorff dimension is $\log \lambda_w / \log \sqrt{2} \approx 1.908$. Thus the topological dimension of the $B_v$ is $\leq 1$. So the $B_v$ cannot be faces. A calculation for the faces a.e. gives modified dimension 2.67.

For the complicated twindragons $F$ and $G$ we also computed the strong components and their modified dimensions. For $G$, the above theorem does not apply since beside the irreducible component of 22 faces with modified dimension 2.87, there is another component of 20 boundary sets, plus 2 singleton components, all with modified dimension 2.13. All the remaining 32 boundary sets have modified dimension 1.52. This boundary seems really complicated! The other example is less intricate. We omit the details of the following example.

**Example 10.3. (Boundary sets of the twindragon $F$)**

For $F$, the neighbor graph $G \setminus \{0\}$ has an irreducible component $V_1$ of 16 faces with modified dimension 2.78, and the other 32 neighbors are not faces. 10 of them form a cycle without diagonals as second component $V_2$, and there are two singleton components between $V_1$ and $V_2$. Since the cycle $V_2$ consists of six double arrows and 4 simple arrows, $\lambda_2 = 2^{3/5}$ and $3 \log \lambda_2 / \log 2 = 1.8$. Due to the theorem, the topological dimension is at most 1.

Moreover, there is another 10-cycle $V_3$ with only two double arrows and $\lambda_3 = 2^{1/5}$, and 10 singleton components between $V_2$ and $V_3$. So the 20 corresponding boundary sets have modified dimension $3 \log \lambda_3 / \log 2 = 0.6$ and by Theorem 10.1 must be Cantor sets.

When one considers the labels along the cycles $V_2$ and $V_3$, one obtains the languages

$$L_2 = 11 \ast 1 \ast 2 \ast 2 \ast 2$$

and

$$L_3 = 11 \ast 1122 \ast 22$$

and their cyclic permutations as address sets of the corresponding boundary sets. From this it is easy to conclude that each Cantor set associated to a vertex in $V_3$ is the intersection of two respective boundary sets in $V_2$.

We did not use the open set condition of the boundary equations (9), and this condition need not always be fulfilled. In the cube (Example 6.3), for instance, some of the equations (9) contain equal terms since an edge is counted two times. However, the open set condition will be true if we remove such double sets and if the neighbors are all compatible. In particular it is true when we restrict ourselves to faces $[2]$. When the graph is a cycle, as $V_2$ and $V_3$ in the last example, the open set condition also holds.

11. Polyhedral structure of tiles

**Homeomorphism with a polyhedron.** We have seen that the structure of boundary sets can be quite complicated. We are interested in the presence of simple structure, however. Let us say that $T \subset \mathbb{R}^3$ has the structure of polyhedron if $T$ is homeomorphic to a ball in $\mathbb{R}^3$, the faces are homeomorphic to disks in $\mathbb{R}^2$, and their non-empty intersections are either singletons or homeomorphic to an interval - in such a way that $T$ together with its boundary structure becomes homeomorphic to a polyhedron in Euclidean $\mathbb{R}^3$. 
This condition is stronger than just requiring that $T$ is homeomorphic to a ball. However, this is the structure which crystallographers expect of a tiling: “We define a tiling as a periodic subdivision of space into bounded, connected regions without holes, which we call tiles. If two tiles meet along a surface, we call the surface a face. If three or more faces meet along a curve, we call the curve an edge. Finally, if at least three edges meet at a point, we call that point a vertex” [18].

**Definition of edges and vertices.** We follow this quotation and define a polyhedral structure on every self-affine tile. We remove from $G$ all vertices which do not represent faces. Then we define the graph $G^2$ of edges - that is, of intersection sets $B_k \cap B_{k'}$. For these, it is not so difficult to check whether they are homeomorphic to intervals [10]. If necessary, we can also determine the graph $G^3$ of vertices of the tiling. Euler’s polyhedra formula can then be used to check whether $T$ is homeomorphic to an ordinary polyhedron. The method is quite fast in showing that certain tiles are not ball-like.

A definition similar to the following was given by Thuswaldner and Scheicher [36] but it was not applied to examples.

**Definition 11.1.** Let $T \subset \mathbb{R}^n$ be a self-affine lattice tile with neighbor graph $G = (V, E = \bigcup_{i=1}^{m} E_i)$. For $\ell = 2, 3$ the graph of $\ell$-intersections of boundary sets of $T$ is $G^\ell = (V^\ell, E^\ell)$ with $V^\ell = \{K = \{k_1, \ldots, k_\ell\} \mid \text{all } k_i \text{ different, } B_{k_1} \cap \ldots \cap B_{k_\ell} \neq \emptyset\}$

$$E^\ell = \bigcup_{i=1}^{m} E_i^\ell$$ (14)

$E_i^\ell = \{(K, K', i) \mid \text{there is a 1-1-map } \phi : K \to K' \text{ with } (k, \phi(k), i) \in E_i \text{ for } k \in K\}$

The idea is that a neighbor intersection $(T + k) \cap (T + k')$ has a piece which is at the boundary of $T_i$ if and only if both neighbor sets have such a piece, and the pieces do intersect. The algorithmic check for non-empty intersection is the same as for $G$. We start with all $\ell$-subsets of $V$, and step by step we remove those which have no outgoing vertices to the remaining vertex set. Since for $\ell = 2, 3$ we must have non-empty intersections, we end with a graph where each set has outgoing vertices, and hence ‘infinite paths’ of edges (counting repeated use of cycles). The properties of $G^\ell$ are similar to those of $G$.

**Proposition 11.2.** $G^\ell$ provides a graph-directed fractal structure on the $\ell$-intersections of neighbors. For each vertex $K \in V^\ell$, the labels of infinite paths in $G^\ell$ with starting point $K$ are the addresses of the corresponding boundary intersection $\bigcap_{k \in K} B_k$. The cardinality of this intersection can be determined as in Theorem 4.5.

**Example 11.3.** We continue Example 4.4 by determining $G^2$ for the six faces of the Sierpiński tile, see Figure 2. Since the outgoing edges from $-a, -b, -c$ all start with 4, 2, and 3, respectively, the corresponding boundary sets have pairwise empty intersection. The intersection of a boundary set with its opposite is also empty. We get edges

$$\{a, -b\} \xrightarrow{2} \{a, -c\} \xrightarrow{3} \{a, -b\}$$
so \(B_a \cap B_{-b}\) is a singleton with address 2323... = \(\overline{23}\) and \(B_a \cap B_{-c}\) has address \(\overline{32}\). The vertices \(a, b, c\) all have a loop with label 1, so the corresponding three boundary sets meet in the point with address \(\overline{1}\). Altogether, \(G^2\) consists of three cycles of length 2, and three isolated loops. \(G^3\) has only one vertex with a loop, representing the center point of the tile.

12. Polyhedral structure of twindragons

Intersections between antipodal surfaces. The first information which the face intersection graph \(G^2\) provides is a list of non-empty intersections of pairs of faces. As in Theorem 4.5, it is easy to decide whether such an intersection consists just of one point.

**Proposition 12.1.** Two faces \(B_u, B_v\) with \(u, v \in V\) intersect each other if and only if \(\{u, v\}\) is a vertex in \(G^2\). Moreover, if there is only one infinite directed path in \(G^2\) starting in the vertex \(\{u, v\}\), then \(B_u \cap B_v\) is one point, and the address of this point can be read from the labels of the path.

For our twindragons we check whether there is a face \(B_k\) which intersects its opposite face \(B_{-k}\). If such intersections exist, it is unlikely that our tile has the structure of a polyhedron. Actually, this happens for four of our twindragons and provides an argument to show that \(D\) and \(E\) are not homeomorphic to a ball.

A set \(A \subset \mathbb{R}^n\) is called simply connected if it is connected and its homotopy group is trivial. The latter means that \(A\) has “no holes” – any closed curve \(C \subset A\) can be contracted within \(A\) to a point.

**Theorem 12.2.** (Tiles for which the interior is not simply connected) Let \(T\) be a connected self-affine lattice tile which has a symmetry center \(c\) which is not an interior point of \(T\). Then the interior of \(T\) is not simply connected. In particular, the interior of \(D\) and \(E\) is not simply connected. These twindragons are therefore not homeomorphic to a ball.

**Proof.** We can assume \(c = 0\) and thus \(T = -T\) by changing the maps as in Proposition 6.1. If 0 is not an interior point of \(T\), there is some tile \(T + k\) in the tiling which contains 0, and by symmetry also \(-(T + k) = T - k\) does not intersect \(\text{int} T\) and contains 0. Take a big sphere \(S\) around 0 which contains the three tiles in its interior. Let \(z \in S\) be of the form \(z = t \cdot k, t > 0\). There is a line segment \(\alpha\) from \(z\) to the point \(y \in T + k\) which maximizes the scalar product \(<y, k>\). Since \(T\) is connected, it is arcwise connected, and there is an arc \(\beta \subset T + k\) from \(y\) to 0. The union \(\varepsilon\) of the arcs \(\alpha, \beta, -\beta, -\alpha\) is an arc in \(S\) from \(z\) to \(-z\) which does not intersect \(\text{int} T\).

Now assume \(\text{int} T\) is connected and take a point \(x \in \text{int} T\). The opposite point \(-x\) is also in \(\text{int} T\). So there is a connecting arc from \(x\) to \(-x\). Covering this arc by finitely many open balls inside \(\text{int} T\), we see that we can replace it by a union \(\gamma\) of finitely many line segments, and we can choose them so that no two segments are parallel. Then the union \(\gamma \cup -\gamma\) is a closed curve from \(x\) through \(-x\) to \(x\) with a finite number of self-intersections. If \(x_1\) is the first self-intersection point, we find a
closed arc from \( x_1 \) through \(-x_1\) to \( x_1 \) with two self-intersection points less. After a finite number of steps we get a simple closed curve \( \delta \) in \( \text{int} \ T \) from some \( x_n \) through \(-x_n\) to \( x_n \). By construction, \( \delta \) surrounds the arc \( \varepsilon \) and thus cannot be contracted within \( \text{int} \ T \) to a single point. So \( \text{int} \ T \) is not simply connected. The second assertion will be proved in the example below. \( \square \)

**Proposition 12.3.** If an infinite path in the neighbor graph \( G \) of a twindragon starts in the root and contains no double arrow, it defines an address of the center 0 of \( T \).

**Proof.** For a path of single arrows which addresses a point \( x \), the opposite path is obtained by just interchanging labels 1 and 2. In the proof of Proposition 6.1, it was shown that the address of the opposite path corresponds to the point \(-x\). For paths starting in the root of \( G \), Theorem 4.5 (i) says that both addresses belong to the same point. \( x = -x \) implies \( x = 0 \). \( \square \)

**Example 12.4.** (Faces intersecting their opposite face in \( D \) and \( E \))
For \( D \), Figure 8 shows only one path from the root without double arrows, given by the cycle acfd which describes an 8-cycle in the complete neighbor graph. The associated address 12211112 corresponds to the center 0. As can be directly seen in Figure 8, this address is also obtained from a path starting in vertex \( c \), going through the cycle cefg and the opposite vertices. Thus the center 0 belongs to the boundary set \( B_c \), hence not to the interior of \( T \).

For \( E \), two root addresses of the center 0 are visible in Figure 9: 1212 from abs, and 1221221 from abim. The second address is also obtained when we start at \(-e\) and go -ebcef-e-fef... Thus \( 0 \in B_{-e} \), the center is not in \( \text{int} \ T \).

The addresses were found from a calculation of \( G^2 \) although \( G \) is sufficient to check them. We determined \( G^2 \) also for the more complicated twindragons and found that for \( F \) there are three faces \( B_k \) which intersect their opposite faces \( B_{-k} \), not in a point, but in a Cantor set with \( \dim^w B_k \cap B_{-k} = 0.6 \). It is not clear whether any of the \( B_k \) contains \( c \), so the above argument fails, but it is obvious that \( T \) has not the structure of a polyhedron. In an ordinary polyhedron, given topologically as a planar map on the sphere, at most one pair of opposite faces can intersect. For \( G \) the situation is similar: we have even five pairs of opposite faces \( B_k \) which intersect each other, not in a point, but in an uncountable set.

**The polyhedral structure of \( B \) and \( C \).** Only the two twindragons shown in Figures 1 and 4 can still have the structure of a polyhedron. We shall prove that this is not quite the case but we expect that

**Conjecture 12.5.** The twindragons \( B \) and \( C \) are homeomorphic to a ball.

It will not be possible to settle this question here, but we shall establish the polyhedral structure which leads us to the conjecture. We start with the list of non-empty intersections of faces which was established by computing the graph \( G^2 \).
Example 12.6. (Polyhedral structure of $C$)

In $C$, the following faces have uncountable intersection:

- $a$ with $b,f,-c,-e$
- $b$ with $a,c,d,-e$
- $c$ with $b,d,-a,-f$
- $d$ with $b,c,e,f$
- $e$ with $d,f,-a,-b$
- $f$ with $a,d,e,-c$

Moreover, there are one-point intersections:

- $a,-a$ with $d,-d$
- $b,-b$ with $f,-f$
- $c,-c$ with $e,-e$

and two-point intersections $a$ with $c$ and $-a$ with $-c$. The two points coincide, however, since the corresponding addresses are identified by Theorem 4.5 (i).

Proof. The proof is a simple but lengthy calculation. We sketch some facts which can be seen directly from inspection of $G$ in Figure 6.

The address 2 belongs to $L_a, L_b, L_d, L_f$ and it is clear that it is the only address in $L_a \cap L_d$ as well as in $L_b \cap L_f$.

For the opposite faces we have address 1. Moreover, $L_c \cap L_e = \overline{2211} = L_d \cap L_{-a}$ and $L_c \cap L_{-c} = \overline{2221} = L_f \cap L_{-b}$. We can consider the languages restricted to the vertices of faces, due to Proposition 9.2. Nevertheless, we see that these addresses also appear for the point neighbors $g, h$ and $-j$, respectively, so they should really represent corner points in the tiling by $T$.

It is obvious from $G$ that $L_f \cap L_{-f} = L_f \cap L_{-e} = \emptyset$. From this we conclude

$$L_b \cap L_e = 2(L_d \cap L_{-c}) = 22(L_f \cap L_{-f}) \cup 22(L_f \cap L_{-e}) = \emptyset.$$ Now we can determine

$$L_a \cap L_c = *(L_b \cap L_e) \cup 2(L_c \cap L_e) \cup 1(L_b \cap L_f) = \overline{2221} \cup \overline{12},$$

and these two addresses are equivalent since they label opposite paths from the root of $G$.

Proposition 12.7. (Equivalent convex polyhedron for $C$)

The polyhedron on the left of Figure 10 realizes exactly all intersections of the faces of $C$, except for $B_a \cap B_c$ and $B_{-a} \cap B_{-c}$. It is an octahedron truncated at four vertices,
in the middle of the corresponding edges, and its faces are rhombi.
Nevertheless, \( C \) has not the structure of an ordinary polyhedron because the interior of the face \( B_b \) is not connected and so \( B_b \) is not homeomorphic to a disk.

**Proof.** The first part is done by checking all pairs of faces. For the second part becomes clear when we prove that \( B_a \cap B_b \) lies on the surface of \( T \), in \( B_b \). However, it can be seen in \( G \) that \( 1 \over 2 \) is an address in \( L_b \).

Here is another argument. From \( G \) we have the equation \( B_b = f_1(B_d) \cup f_2(B_d) \). On the other hand \( f_1(T) \cap f_2(T) = f_1(B_a) = f_2(B_{-a}) \). We have seen that \( B_a \) and \( B_{-a} \) intersect \( B_d \) in a single point, and this also holds for their images under \( f_1 \) or \( f_2 \). Thus \( f_1(B_d) \) and \( f_2(B_d) \) have only one common point. So \( B_b \) consists of two isometric pieces which intersect in a singleton, and cannot be homeomorphic to a disk.

It should be mentioned that the polyhedron in Figure 10 describes \( C \) only topologically - the metric structure is completely different. We have \( B_c = f_2(B_a) \) and \( B_d = f_2(B_t) \) so these faces are of different size. The intersection of the faces \( B_a, B_b, B_d, B_t \) is not a corner as in Figure 10 the point \( \pi(\overline{2}) = (\overline{1}, 0, \overline{1})' = \frac{a+d}{2} = \frac{b+f}{2} \) is on the middle between the centers of opposite adjoining faces (see the table at Figure 6). A similar calculation shows that \( B_a \cap B_c \) is in fact the center point of \( B_b \).

Now let us study \( B \) in the same way as \( C \).

**Example 12.8. (Polyhedral structure of \( B \))**
The following faces have uncountable intersection in \( B \):

- a with c,d,e,g,b,f
- b with c,f,a,d
- c with a,b,e,f,g,d
- d with a,e,b,c,g
- e with a,c,d,f
- f with b,c,d,e,a,g
- g with a,c,d,f

Moreover, there are one-point intersections

\( B_c \cap B_{-e} = \pi(\overline{2}) \), \( B_c \cap B_{-a} = \pi(2\overline{1}) \), and their opposites.

**Proposition 12.9. (Equivalent convex polyhedron for \( B \))**
The polyhedron on the right of Figure 10 realizes exactly all intersections of the faces of \( B \), except for the point neighbors. It is an octahedron truncated at all vertices, at one third of the corresponding edges. The faces are regular hexagons and squares.

Nevertheless, \( B \) has not the structure of an ordinary polyhedron because the interior of the faces \( B_b \) and \( B_d \) is not connected and so these faces are not homeomorphic to a disk.

This is proved similarly as for \( C \). Note that faces of the truncated octahedron never meet in a single point. Using Figure 7 we see that \( B_c \cap B_{-a} \) is on \( B_b \) and \( B_c \cap B_{-e} \) on \( B_{-d} \).

**13. Connectedness of the interior**

In this section, we go a small step towards proving our conjecture, by proving that the interior of \( C \) is connected.

**Proposition 13.1.** Suppose \( T = \bigcup_{j=1}^{m} f_j(T) \) is a self-affine tile and there is a connected set \( E \subset \text{int } T \) such that \( E \cap f_j(E) \neq \emptyset \) for \( j \in I = \{1, \ldots, m\} \). Then the interior of \( T \) is connected.
Proof. \( D_1 = \bigcup_{j \in I} f_j(E) \) is a connected subset of \( \text{int} \, T \). By induction we show that \( D_n = \bigcup_{|w| \leq n} f_w(E) \) is connected, where \(|w|\) denotes the length of the word \( w \). Then \( D = \bigcup_{n=1}^{\infty} D_n = \bigcup_{w \in I^*} f_w(E) \) is connected, and this set is dense in \( \text{int} \, T \). Hence \( \text{int} \, T \) is connected. \( \square \)

**Proposition 13.2.** \( \text{int} \, C \) is connected.

**Proof.** A piece \( T_w \) belongs to \( \text{int} \, T \) if it is surrounded in \( T \) by all possible neighbors. \( T_w \) has all possible neighbors within \( T \) if a path labelled with a suffix of \( w \) leads from the root to each vertex of \( G \) (cf. proof of Theorem 5.1). When we require this for faces only, \( w = 121212 \) and \( v = 2121121 \) fulfil this condition, as one can check with Figure 6. We put

\[
E = T'_w \cup T'_v \cup T''_w \cup T''_v, \quad \tilde{w} = 212121, \quad \tilde{v} = 121221.
\]

where \( T'_w \) means \( T_w \) minus its one-point boundary sets. Then \( E \subset \text{int} \, T \). Since \( E \) contains \( \pi(12) \) and \( \pi(21) \), the set \( E \) intersects \( f_1(E) \) and \( f_2(E) \). To apply Proposition 13.1, it only remains to verify that \( E \) is connected. Since twindragons are connected (cf. section 7), each \( T'_w \) is connected. Moreover, there are pairs of opposite paths from the root in \( G \) with labels \( w \) and \( v \), hence also with \( \tilde{w}2 \) and \( \tilde{v} \), and also with \( v \) and \( \tilde{v} \), which all end in a vertex of \( G \) corresponding to a face. Thus \( T_{w1} \) and \( T_v \), \( T_{\tilde{w}2} \) and \( T_{\tilde{v}} \), as well as \( T_v \) and \( T_{\tilde{v}} \) have a face in common. So \( E \) is connected, and the proposition applies. \( \square \)

\( w \) and \( v \) were obtained in a straightforward way. \( w \) is the shortest word for which \( T_w \) has all possible face neighbors (we used a computer search). \( v \) is the shortest word with all face neighbors which contains the center point 0 by Proposition 12.3 (this is easy: the address of 0 is 12121). In fact, 0 is an interior point of \( T_v \cup T_{\tilde{v}} \). For \( B \) we also determined such \( v \). The address of 0 is 1122, which gives \( v = 112212122 \). This indicates that the interior of \( B \) is more fragmented than that of \( C \) and, to let the above method work, we would need a longer chain of pieces.

**14. Remarks on algorithms**

**Use of computer.** We have shown that the topology of self-affine tiles can be studied by rather simple methods: finite automata and regular languages. To answer concrete questions, however, we have to perform quite a number of elementary logical operations. Computer assistance seems necessary and convenient to study more complicated examples.

Although all results in this paper have been checked by hand, interactive computer work was essential to obtain them. Let us briefly document the main algorithms we have used.

**Construction of the neighbor graph.** To construct \( G \), three steps are performed, as indicated in Example 4.4.

(i) Lower and upper bounds \( l_q, b_q \) for \( \{x_q \mid x \in T\} \) are derived for each coordinate \( q = 1, 2, 3 \). They need not be sharp, but should be taken rather generously (we work with integers anyway). If \( b_q \) is too large, the computer will work
a little longer – if it is too small, only part of the neighbor graph will be determined. For examples like $B$ and $C$, $100000$ points of an IFS algorithm \cite{11} are completely sufficient to find $l_q$ and $b_q$. For $F$ and $G$ there are rare outliers on the thin fibres, and an exact estimate is needed.

(ii) Starting with $k = 0$, a list of new vectors is calculated by the recursion

\[ k' = Mk + k_j - k_i \]

where $i, j$ take all values in $\{1, \ldots, m\}$. The index of $k$ is listed together with the labels $i, j$ at $k'$ as the predecessor of $k'$, and the number of successors of $k$ is updated at $k$. Of course, $k'$ is only processed if it is within the bounds $l_q$ and $b_q$. Moreover, it must always be checked whether $k'$ has already been listed before. In that case, the new predecessor is added at the old place.

Since the number of possible $k$ is finite, the whole list will be completed after finite time, and describes a graph containing $G$.

(iii) All $k$ without successors are removed from the list, and for each of their predecessors, the number of successors is updated. This step is repeated until each $k$ has at least one successor (or, in case of a Cantor set, $G$ is empty).

When the resulting $G$ has predecessors of the root, the open set condition is not fulfilled. This cannot happen if the $k_j$ form a complete residue system for $M$.

Other operations with graphs. For constructing the intersection graph $G^2$ from $G$, we have to consider $G \times G$, identify $(u, v)$ with $(v, u)$ and define the edges according to Definition 11.1. Then we apply step (iii) as above, to exclude empty intersections.

To identify point neighbors in $G$, we use the adjacency matrix $H$ defined in section 10. If $q$ is the number of non-zero vertices in $G$, then $k$ is a point neighbor if the sum of the row associated with $k$ in $H, H^2, \ldots, H^q$ always equals 1. When we repeatedly apply a procedure like (iii) to the point neighbors, we also remove the finite boundary sets. One-point intersection sets in $G^2$ are found in the same way as point neighbors in $G$.

The partial order of boundary sets (section 8) is also determined with $H$. Let the matrix $N$ be the maximum of $H + H^2 + \ldots + H^q$ and 1, taken in each cell. Then $n_{uv} = 1$ if and only if $u \succ v$. In the row of $u$ there are the elements which can be reached from $u$, in the column of $u$ there are the $v$ from which $u$ can be reached. The strong components of $G$ can be found by properly ordering the rows and columns of $N$.

To deal with languages $L_u$, we introduce adjacency matrices $H_i$ for the edges with label $i$. Then the matrix $H_1H_2H_2$, for instance, tell us between which vertices there is a path labelled $122$. This fact was used to find the proof of Proposition 9.5.

There are many similar tools waiting to be developed and tested to help reveal the geometric structure of fractal tiles.
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