Convex Sets of Robust Recurrent Neural Networks

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Abstract—Recurrent neural networks (RNNs) are a class of nonlinear dynamical systems often used to model sequence-to-sequence maps. RNNs have been shown to have excellent expressive power but lack stability or robustness guarantees that would be necessary for safety-critical applications. In this paper we formulate convex sets of RNNs with guaranteed stability and robustness properties. The guarantees are derived using differential IQC methods and can ensure contraction (global exponential stability of all solutions) and bounds on incremental \( \ell_2 \) gain (the Lipschitz constant of the learnt sequence-to-sequence mapping). An implicit model structure is employed to construct a jointly-convex representation of an RNN and its certificate of stability or robustness. We prove that the proposed model structure includes all previously-proposed convex sets of contracting RNNs as special cases, and also includes all stable linear dynamical systems. We demonstrate the utility of the proposed model class in the context of nonlinear system identification.

I. INTRODUCTION

Neural networks are a class of universal function approximator frequently used in machine learning. While there have been some great successes in solving complex tasks, their lack of robustness often prevents their application, particularly in a safety-critical context. This realization has motivated a large amount of research into the area of adversarial defenses [1], [2], [3]. This area seeks guarantees of robustness against adversarially chosen inputs.

Recurrent neural networks are state-space models parametrized by neural networks and are frequently used to model sequences-to-sequence mappings. Many of the concerns for ensuring robustness of learned models relate directly to traditional concerns in robust control, e.g. stability and continuity of the input-output mapping [4].

There are many definitions of stability for nonlinear systems (e.g., RNNs). Approaches that study the stability of certain equilibria are difficult to apply when synthesizing new models as the locations of equilibria are unknown for unknown inputs. Two alternatives that do not depend on prior knowledge of inputs and trajectories are incremental stability and contraction analysis [5]. These approaches define stability in terms of the distance between all trajectories, regardless of the input.

Beyond stability, a robust model must also not be critically sensitive to small changes in the input. The most commonly used method for analyzing the input-output behavior of a system is the \( \ell_2 \) gain bound. As noted in [6], however, a finite \( \ell_2 \) gain bound only guarantees only boundedness, whereas a finite incremental \( \ell_2 \) gain bound guarantees continuity. Further insight can be gained by noting that the incremental \( \ell_2 \) bound is equivalent to the Lipschitz constant of the sequence-to-sequence mapping. The Lipschitz constant of a mapping is a quantity that commonly appears in for instance proofs of generalization bounds [7], analysis of expressiveness [8] and guarantees of robustness to adversarial attacks [9], [10].

There have been a number of approaches to analyzing and training stable recurrent neural networks. Many of the approaches to analysis such as [11], [12], [13], [14], [15] ensure stability through LMI conditions. The LMIs, however, are not jointly convex in both model parameters and the stability certificate complicating their use during training. The approaches in [16] and [17] both try to learn Lyapunov functions and enforce the Lyapunov inequality via projection and penalization. These approaches, however, construct a Lyapunov function about a particular equilibrium and as such cannot be easily extended to the deal with unknown, exogenous inputs. On the other hand [18] and [19] provide conditions for contraction, easily dealing with inputs. However, these models suffer from reduced model expressiveness due to the conservatism of the stability criteria. This conservatism is caused by two issues. Firstly, the contraction metrics are limited to a diagonal structure. Secondly, they do not account for the interactions between the linear component and the nonlinearity and instead require both the system and nonlinearity to be contracting in a common metric.

One approach to reducing the conservatism of stability criteria is provided by the integral quadratic constraint (IQC) framework. This approach deals with the case of when a known, linear system is in feedback with an uncertain, nonlinear or otherwise troublesome component. By abstracting away the troublesome component in favour of quadratic descriptions of the signals it can produce, criteria such as stability, \( \ell_2 \) gain bounds and passivity can be accurately verified. In the case where there are multiple ‘troublesome’ components that share a common description, very accurate results can be obtained using the approach presented in [20] and [21]. More recently, extensions of the IQC framework were proposed in [22] lifting the IQC approach to the tangent bundle and allowing for the nominal system to be nonlinear. In this case, the stability criteria take the form of point-wise, convex LMIs. In the context of neural network analysis, the IQC approach has been recently applied to the (non-recurrent) neural networks to develop the tightest bounds on the Lipschitz constant known to date [23] and robustness guarantees for sets of input perturbations [24].

In this work, we develop a convex set of Robust RNNs. By treating the RNN as a linear system in feedback with...
its activation functions, we can apply methods from robust control to develop contraction conditions that are less conservative than prior methods. The proposed model set is the most expressive set of contracting RNNs so far and contains all previously proposed sets. The use of an implicit model structure leads to conditions that are jointly convex in both the model parameters and certificate of stability, building on the approach for polynomial models in [25], [26], [27], [28]. This greatly simplifies the problem of training new models subject to stability constraints as they can be treated using penalty, barrier or projected gradient methods. We provide a discussion of future extensions that may allow for improved expressibility and non-constant contraction metrics and discuss a number of applications of such a model set. Finally, we illustrate the efficacy of the proposed model set on a system identification task.

Notation. We use \( \mathbb{N}, \mathbb{R} \) to denote the set of natural and real numbers, respectively. The set of all one-side sequences \( x : \mathbb{N} \rightarrow \mathbb{R}^n \) is denoted by \( \ell_2, \ell_{2e} \). Superscript \( n \) is omitted when it is clear from the context. We use \( x_t \) to represent the value of the sequence \( x \) at time \( t \in \mathbb{N} \). The notation \( | \cdot | : \mathbb{R}^n \rightarrow \mathbb{R} \) denotes the standard 2-norm. The subset \( \ell_2 \) of all square-summable sequences, i.e., \( x \in \ell_2 \) if and only if the \( \ell_2 \)-norm \( \| x \| := \sqrt{\sum_{t=0}^{\infty} |x_t|^2} \) is finite. Given a sequence \( x \in \ell_2 \), the \( \ell_2 \)-norm of its truncation over \( [0, T] \) with \( T \in \mathbb{N} \) is written as \( \| x \|_T := \sqrt{\sum_{t=0}^{T} |x_t|^2} \). If \( f \) is differentiable, then \( D^+ f(x; v) \) to denote the one-side directional derivative of \( f(\cdot) \) at \( x \) in the direction \( v \), i.e. \( D^+ f(x; v) := \lim_{s \rightarrow 0^+} [f(x + sv) - f(x)]/s \).

II. Problem Formulation and Preliminaries

A. Problem Setup

Consider an input-output sequence mapping (operator) \( S_a : \ell_{2e}^m \rightarrow \ell_{2e} \) whose model can be described by a dynamical system with finite-dimensional state \( x_t \in \mathbb{R}^n \) and initial state \( x_0 = 0 \), driven by a known input \( u_t \in \mathbb{R}^m \) and producing an output \( y_t \in \mathbb{R}^p \). We are interested in learning RNN models parameterized by \( \theta \in \Theta \subseteq \mathbb{R}^N \):

\[
\begin{align*}
x_{t+1} &= f_\theta(x_t, u_t) \\
y_k &= g_\theta(x_k, u_k)
\end{align*}
\]

where functions \( f_\theta, g_\theta \) will be defined later.

It is often desired to learn RNNs which have predictable responses to a wide variety of inputs. One direct approach is to search for RNNs from a model set with stability and robustness guarantees. First, we introduce some stability definitions.

Definition 1. The system (1), (2) is termed incrementally \( \ell_2 \) stable if for any two initial conditions \( a \) and \( b \), given the same input sequence \( u \), the corresponding output trajectories \( y^a \) and \( y^b \) satisfy \( y^a - y^b \in \ell_2^\gamma \).

This definition implies that initial conditions are forgotten, however, the outputs can still be sensitive to small perturbations in the input. In such cases, it is natural to measure system robustness in terms of the incremental \( \ell_2 \)-gain.

Definition 2. The system (1), (2) is said to have an incremental \( \ell_2 \)-gain bound of \( \gamma \) if for all pairs of solutions with initial conditions \( a, b \in \mathbb{R}^n \) and input sequence \( u^a, u^b \in \ell_2^m \), the output sequences \( y^a, y^b \in \ell_{2e} \) satisfies

\[
\| y^a - y^b \|^2_\gamma \leq \| u^a - u^b \|^2_\gamma + d(a, b), \quad \forall T \in \mathbb{N}.
\]

Note that the above definition implies the incrementally \( \ell_2 \) stability since \( \| y^a - y^b \|^2_T \leq d(a, b) \) for all \( T \in \mathbb{N} \) when \( u^a = u^b \). It also shows that all operators defined by (1) and (2) are Lipschitz continuous with Lipschitz constant \( \gamma \), i.e. for any \( a \in \mathbb{R}^n \) and all \( T \in \mathbb{N} \)

\[
\| S_a(u) - S_b(v) \|_T \leq \gamma \| u - v \|, \quad \forall u, v \in \ell_{2n}.
\]

In this work we will address the following problems.

Problem 1. Construct a convex RNN model set \( \Theta_\gamma \) such that for any \( \theta \in \Theta_\gamma \), the system (1), (2) has a finite the incremental \( \ell_2 \)-gain.

Problem 2. Construct a convex RNN model set \( \Theta_\gamma \) such that for any \( \theta \in \Theta_\gamma \), the system (1), (2) has an incremental \( \ell_2 \)-gain bound of \( \gamma \).

B. Contraction Analysis

Contraction analysis [5] studies the incremental stability properties of system (1), (2) based on its associated differential dynamics (a.k.a. variational, linearized, prolonged):

\[
\begin{align*}
\delta x_{t+1} &= D^+ f_\theta(x_t, u_t; \delta x_t, \delta u_t) \\
\delta y_t &= D^+ g_\theta(x_t, u_t; \delta x_t, \delta u_t)
\end{align*}
\]

where the sequence \( (\delta x, \delta u, \delta y) \) can be interpreted as the infinitesimal displacements between two neighboring trajectories of the original system (1), (2).

A system (1), (2) is said to have differential \( \ell_2 \)-gain bound of \( \gamma \) if for all \( T \in \mathbb{N} \)

\[
\| \delta y \|^2_T \leq \gamma^2 \| \delta u \|^2_T + b(x_0, \delta x_0)
\]

where \( b(x, \delta x) \geq 0 \) with \( b(x, 0) = 0 \). For smooth systems the differential \( \ell_2 \)-gain bound is equivalent to the incremental \( \ell_2 \)-gain bound [29].

A sufficient, and in some cases necessary, condition for (7) is the existence of a storage function \( V(x, \delta x) \geq 0 \) with \( V(x, 0) = 0 \) such that the following dissipation inequality holds for all \( t \in \mathbb{N} \)

\[
V(x_{t+1}, \delta x_{t+1}) - V(x_t, \delta x_t) \leq \gamma |\delta u_t|^2 - \frac{1}{\gamma}|\delta y_t|^2.
\]

In general there are many forms of differential storage functions. One common choice is the quadratic form \( V(\delta x) = \delta x^T M \delta x \) where \( M > 0 \).
C. Bounds and Identities for Quadratic Matrix Functions

Throughout the paper we will frequently use the following simple property. The matrix functions $g(E,P) = -E^T P^{-1} E$ with $E \in \mathbb{R}^{n \times n}$, $P > 0$ obey the upper bound:

$$-E^T P^{-1} E \leq P - E^T - E$$  \hspace{1cm} (9)

where the right-hand side is convex in $E,P$. Inequality (9) follows directly from the expansion

$$P - E^T - E + E^T P^{-1} E = (E - P)^T P^{-1} (E - P) \geq 0.$$  \hspace{1cm} (10)

From this expansion it is also clear that the bound (9) is tight if $E = P$.

III. ROBUST RNNs

A. Model Set

Our work can be applied to the models with multi-layer network. To streamline the presentation, we will focus on the case with one-layer network.

As shown in Fig. 1 we treat the RNN as a feedback interconnection of a linear system $G$ and a static, memoryless nonlinear operator $\Phi$ of the form

$$w = \Phi(v),$$  \hspace{1cm} (11)

where $\Phi(v) = [\phi(v_1) \cdots \phi(v_i)]^T$ with $v_i$ as the $i$th component of the vector $v$. We assume that the slope of $\phi$ is restricted to the interval $[\alpha, \beta]$:

$$\alpha \leq \frac{\phi(y) - \phi(x)}{y-x} \leq \beta, \quad \forall x,y \in \mathbb{R}, \ x \neq y.$$  \hspace{1cm} (12)

In the neural network literature, such functions are referred to as “activation functions”, and common choices such as (e.g. tanh, ReLU, sigmoid) are slope restricted [30].

We will parameterize the linear system $G$ using the following implicit, redundant parametrization:

$$\begin{align*}
Ex_{t+1} &= Fx_t + B_1w_t + B_2u_t \\
y_t &= C_1x_t + D_{11}w_t + D_{12}u_t \\
v_t &= C_2x_t + b + D_{22}u_t
\end{align*}$$  \hspace{1cm} (13)

where $\theta = (E,F,B_1,B_2,b,C_1,D_{11},D_{12})$ is the model parameter with $E$ invertible. The choice of $C_2$ and $D_{22}$ is discussed later in Remark 2.

Note that by setting $F = 0$, $E^{-1}B_1 = A$, $E^{-1}B_2 = B$, $C_1 = 0$, $D_{11} = C$, $D_{12} = D$, $C_2 = I$ and $D_{22} = 0$, the above implicit RNN (12), (10) is reduced to the conventional RNN [31] of the form:

$$\begin{align*}
z_{t+1} &= \Phi(Az_t + Bu_t + b) \\
y_t &= Cz_t + Du_t
\end{align*}$$  \hspace{1cm} (14)

where $z_t = \Phi(x_t + b)$.

The implicit model (12), (10) is called a Robust RNN if its differential $\ell_2$-gain is bounded by some constant $\gamma$. To characterize the set of Robust RNNs, we first derive the associated differential dynamics, which can also be expressed as a feedback interconnection of a linear system $\delta G$ and a differential operator $\delta \Phi$:

$$\begin{align*}
\delta G : \begin{cases}
E\delta x_{t+1} &= F\delta x_t + B_1\delta w_t + B_2\delta u_t \\
\delta y_t &= C_1\delta x_t + D_{11}\delta w_t + D_{12}\delta u_t \\
\delta v_t &= C_2\delta x_t + D_{22}\delta u_t \\
\delta \Phi : \begin{bmatrix}
\delta x_t \\
\delta w_t \\
\delta u_t
\end{bmatrix} &= D^+ \Phi(v_t; \delta v_t).
\end{cases}
\end{align*}$$  \hspace{1cm} (15)

Analysis of the above system is primarily complicated by the presence of the nonlinear activation function in $\Phi$. We will simplify the analysis by replacing $\delta \Phi$ with differential integral quadratic constraints ($\delta$-IQC).

B. Description of $\Phi$ by $\delta$-IQC

A $\delta$-IQC for the operator $\Phi$ can be viewed as an IQC ((32)) for the differential operator $\delta \Phi$ [22].

Definition 3. The operator $\Phi$ satisfies the differential integral quadratic constraint defined by a multiplier $M = M^T \in \mathbb{R}^{2q \times 2q}$ if for any $v, \delta v, \delta w \in \mathbb{R}^q$ satisfying (16), the following inequality holds for all $T \in \mathbb{N}$:

$$\sum_{t=0}^{T} \begin{bmatrix}
\delta v_t \\
\delta w_t
\end{bmatrix}^T M \begin{bmatrix}
\delta v_t \\
\delta w_t
\end{bmatrix} \geq 0.$$  \hspace{1cm} (17)

For the nonlinear operator (10) where each component is slope-restricted to the same interval $[\alpha, \beta]$, we can obtain a simpler quadratic constraint

$$\begin{bmatrix}
\delta v_t \\
\delta w_t
\end{bmatrix}^T \begin{bmatrix}
-2\alpha \beta T(\lambda) & (\alpha + \beta) T(\lambda) \\
(\alpha + \beta) T(\lambda) & -2T(\lambda)
\end{bmatrix} M(\lambda) \begin{bmatrix}
\delta v_t \\
\delta w_t
\end{bmatrix} \geq 0$$  \hspace{1cm} (18)

for all $t \in \mathbb{N}$, where $T(\lambda) = \text{diag}(\lambda_1, \ldots, \lambda_{2q}) \in \mathbb{R}^{2q}$. Note that the above $\delta$-IQC is a conic combination of quadratic constraints for each activation function $\phi$, i.e.,

$$\sum_{i=1}^{q} \lambda_i \begin{bmatrix}
\delta v_{i,t} \\
\delta w_{i,t}
\end{bmatrix}^T \begin{bmatrix}
-2\alpha \beta & \alpha + \beta \\
\alpha + \beta & -2
\end{bmatrix} \begin{bmatrix}
\delta v_{i,t} \\
\delta w_{i,t}
\end{bmatrix} \geq 0$$

where $v_{i,t}, w_{i,t}$ are the $i$th elements of $v_t$ and $w_t$, respectively.

C. Convex Parametrization of Robust RNNs

To construct the set of stable models with finite differential $\ell_2$-gain, we use the following constraint:

$$\begin{bmatrix}
E + E^T & -P & 0 \\
0 & E^T & -P & 0 \\
B_1^T & -P & F^T & -P & 0 \\
B_2^T & -P & F^T & -P
\end{bmatrix} - \begin{bmatrix}
\tilde{C}^T_2 \\
\tilde{C}^T_1
\end{bmatrix} M(\lambda) \begin{bmatrix}
\tilde{C}_2^T \\
\tilde{C}_1^T
\end{bmatrix} \geq 0$$  \hspace{1cm} (19)
where \( \hat{C}_2^T = [C_2^T \ 0] \) and \( \hat{D}_{21}^T = [D_{21}^T \ 0] \). We define the set of Robust RNNs with finite differential \( \ell_2 \)-gain as follows

\[
\Theta_\gamma := \{ \theta : \exists P \succ 0, \lambda \in \mathbb{D}_+ \text{ s.t. } E + E^T \succ 0, \ (19) \}.
\]

To obtain a Robust RNN with differential \( \ell_2 \)-gain bound of \( \gamma \), we propose to use the following constraint:

\[
\begin{bmatrix}
E & E^T - P & 0 & 0 \\
0 & 0 & 0 & \gamma I
\end{bmatrix} - \begin{bmatrix}
F^T & \beta_1^T \\
\beta_2^T & \beta_3^T
\end{bmatrix} P^{-1} \begin{bmatrix}
F^T \\
\beta_1 \\
\beta_2 \\
\beta_3
\end{bmatrix} ^T > 0 \quad (20)
\]

where \( P \succ 0 \) and \( \hat{D}_{22}^T = [D_{22}^T \ 0] \). Note that if the LMI condition (19) is satisfied, there exists a sufficiently large \( \gamma \) such that (20) holds for any choice of \( B_2, C_1, D_{11}, D_{12} \) and \( D_{22} \). We define the set of Robust RNNs with differential \( \ell_2 \)-gain bound of \( \gamma \) as follows

\[
\Theta_\gamma := \{ \theta : \exists P \succ 0, \lambda \in \mathbb{D}_+ \text{ s.t. } E + E^T \succ 0, \ (20) \}.
\]

**Remark 1.** Note that the LMI (20) is not jointly convex in the weights \( C_2, D_{22} \) and the \( \delta \)-IQC multipliers \( \lambda \). We will see that fixing \( C_2 \) and \( D_{22} \) and optimizing over the remaining parameters still leads to highly expressive models. Fixing \( C_2 = I \) and \( D_{22} = 0 \), the model set still contains the set of ci-RNNs, see Theorem 4 below. Alternatively, expressibility can be improved at the cost of computational complexity by fixing \( C_2 \) and \( D_{22} \) as random wide layers (i.e. with large \( q \)). This is similar to the approach taken in the echo-state network [33].

**Theorem 1.** Suppose that \( \theta \in \Theta_\gamma \), then the Robust RNN (12), (10) has a differential \( \ell_2 \)-gain bound of \( \gamma \).

**Proof.** To establish the differential \( \ell_2 \)-gain bound, we first left and right multiply (20) by the vectors \([\delta_{x_t}, \delta_{u_t}, \delta_{w_t}]^T\) and \([\delta_{x_t}^T, \delta_{u_t}^T, \delta_{w_t}^T]^T\). Then, applying the bound (9) and taking summation over \( [0, T] \) gives

\[
V_T - V_0 \leq \gamma \|\delta_u \|^2_T - \frac{1}{\gamma} \|\delta_y \|^2_T - T \sum_{t=0}^{T} \left[ \delta_{ct} \delta_{ct}^T \right] M(\lambda) \left[ \delta_{ct} \delta_{ct}^T \right]^T
\]

where \( V_t = \delta_x E P^{-1} E \delta_x \). From (17) the differential \( \ell_2 \)-gain condition (7) follows with \( b(x_0, x_{v0}) = \gamma V_0 \).

**Theorem 2.** Suppose that \( \theta \in \Theta_\gamma \), then the Robust RNN (12), (10) has a finite differential \( \ell_2 \)-gain bound of \( \gamma \).

**Proof.** Since (19) implies (20) for some sufficiently large \( \gamma \), from Theorem 1 the Robust RNN (12), (10) has a finite differential \( \ell_2 \)-gain bound of \( \gamma \).

**D. Expressivity of the model set**

Some recent works show that the model sets with additional stability constraints can improve generalizability and trainability, e.g. contracting implicit RNNs (ci-RNNs) [19] and stable linear time-invariant (LTI) models [34]. To be able to learn models for a wide class of systems, it is of course beneficial to have as expressive a model set as possible. The main result regarding expressivity is that the Robust RNN set \( \Theta_\gamma \) contains all ci-RNNs and stable LTI models.

To explain this result, we first introduce the concept of input/output equivalence, i.e., two RNNs with parameter \( \theta_1 \) and \( \theta_2 \) is said to be input/output equivalent (denoted \( \theta_1 \sim \theta_2 \)) if they admit the same input/output trajectory set. A model set \( \Theta_\gamma \) is said to be a subset of another model set \( \Theta_\gamma \) (denoted \( \Theta_\gamma \subset \Theta_\gamma \)) if for any \( \theta \in \Theta_\gamma \), there exists \( \theta_2 \in \Theta_\gamma \) such that \( \theta_1 \sim \theta_2 \). And \( \Theta_\gamma \subset \Theta_\gamma \) means \( \Theta_\gamma \) is a strict subset of \( \Theta_\gamma \).

The set of all stable LTI systems will be denoted \( \Theta_{LTI} \) and can be described by the state-space model:

\[
x_{t+1} = Ax_t + Bu_t, \quad y_t = Cx_t + Du_t \quad (21)
\]

with a necessary and sufficient condition for stability given by the LMI:

\[
\mathcal{P} - A^T \mathcal{P} A > 0, \quad (22)
\]

for some \( \mathcal{P} > 0 \).

**Theorem 3.** The Robust RNN set \( \Theta_\gamma \) contains all stable LTI models, i.e. \( \Theta_{LTI} \subset \Theta_\gamma \).

**Proof.** For any stable LTI system \( \theta' \in \Theta_{LTI} \), it is easy to verify that \( \theta' \sim \theta \) where \( \theta \) represents an implicit model with \( E^{-1}F = A, B_1 = 0, E^{-1}B_2 = B, C = C \) and \( D = D \). We will show that \( \theta \in \Theta_\gamma \). Let \( P = E^T \mathcal{P} E \) we have

\[
\begin{align*}
\Rightarrow & E^T P^{-1} E - F^T E^{-1} E^T P^{-1} EE^{-1} F > 0 \\
\Rightarrow & E^T - E - P - F^T - P^T E > 0 \\
\Rightarrow & \begin{bmatrix}
E + E^T & -P & -F^T -P^{-1} F \\
-P & -2T
\end{bmatrix} \succ 0
\end{align*}
\]

where \( T \in \mathbb{D}_+ \) satisfies \( T/2 \leq E + E^T - P - F^T P^{-1} F \).

A ci-RNN [19] is an implicit model of the form:

\[
E_{x_{t+1}} = \Phi(F_{x_{t+1}} + Bu_t + b), \quad y_t = C_{x_t} + Du_t \quad (23)
\]

such that the following contraction condition holds

\[
\begin{bmatrix}
E + E^T & -P & -F^T -P^{-1} F \\
-P & -2T
\end{bmatrix} \succ 0 \quad (24)
\]

where \( \mathcal{P} \in \mathbb{D}_+ \). We define a convex set of ci-RNNs as

\[
\Theta_{ci} := \{ \theta : \exists \mathcal{P} \in \mathbb{D}_+ \text{ s.t. } E + E^T > 0, \ (25) \}.
\]

Note that \( \Theta_{ci} \) does not contain all stable LTI systems. For example, the system \( x_{t+1} = 0.5x_t + u_t, \ y_t = x_t \) cannot be converted into the form (25) via coordinate transformation.

**Theorem 4.** The Robust RNN set \( \Theta_\gamma \) contains all ci-RNNs, i.e. \( \Theta_{ci} \subset \Theta_\gamma \).

**Proof.** For any ci-RNN \( \theta' \in \Theta_{ci} \), we first show that \( \theta' \sim \theta \) where \( \theta \) represents the implicit model with \( F = 0, E = \mathcal{P}^{-1}, B_1 = \mathcal{P}^{-1} FE^{-1}, B_2 = EB, C_1 = 0, D_{11} = CE^{-1}, D_{12} = D, C_2 = I \) and \( D_{22} = 0 \). By substituting \( \theta \) into (12) and (10), the implicit model can be rewritten as

\[
x_{t+1} = FE^{-1} \Phi(x_t + b) + Bu_t, \quad y_t = CE^{-1} \Phi(x_t + b) + Du_t.
\]
Note that the above system is equivalent to \( \Theta \) via the coordinate transformation \( z_t = \mathcal{E}^{-1} \Phi(x_t + b) \) since
\[
\mathcal{E} z_{t+1} = \Phi(x_{t+1} + b) = \Phi(\mathcal{F} \mathcal{E}^{-1} \phi(x_t + b) + B u_t + b) = \Phi(\mathcal{F} z_t + B u_t + b).
\]

Second, we will prove that \( \theta \in \Theta_\ast \). Let \( P = T = \mathcal{P}^{-1} \) we have
\[
\Theta \Rightarrow \mathcal{E} + \mathcal{E}^\top - \mathcal{P} - \mathcal{F}^\top \mathcal{P}^{-1} \mathcal{F} > 0
\]
\[
\Rightarrow \mathcal{E}^\top \mathcal{P}^{-1} \mathcal{E} - \mathcal{F}^\top \mathcal{P}^{-1} \mathcal{F} > 0
\]
\[
\Rightarrow \mathcal{P}^{-1} - \mathcal{E}^{-\top} \mathcal{F}^\top \mathcal{P}^{-1} \mathcal{P} \mathcal{F} \mathcal{E}^{-1} > 0
\]
\[
\Rightarrow 2P^{-1} - \mathcal{E}^{-\top} \mathcal{F}^\top \mathcal{P}^{-1} \mathcal{P} \mathcal{F} \mathcal{E}^{-1} - \mathcal{P}^{-1} \mathcal{P} > 0
\]
\[
\Rightarrow 2T - B_1^\top P^{-1} B_1 - T(E + E^\top - P)^{-1} T > 0
\]
\[
\Rightarrow \begin{bmatrix} E + E^\top - P & -T \\ -T & 2T - B_1^\top P^{-1} B_1 \end{bmatrix} > 0 \Rightarrow \Theta_\ast
\]

And \( \Theta_\ast \) is a strict subset of \( \Theta_\ast \) as \( \Theta_\ast \) does not contain all stable LTI systems.

We also note that, unlike \( \Theta_\ast \), the proposed model set does not require a diagonal metric. This may further reduce the conservatism of \( \Theta_\ast \) compared to \( \Theta_\ast \).

IV. DISCUSSION

We have focused on the case where \( G \) is a linear system and \( \Phi \) is described by simple sector bound quadratic constraints, however both of these assumptions can be relaxed with some additional technical considerations.

All of our results can be extended to the case where the nominal system is a smooth nonlinear state space model. When \( G \) is no longer a linear system however, the stability conditions take the form of pointwise LMIs. This allows us to deal with state dependent contraction metrics as \( E \) now becomes a state dependent function and the contraction metric takes the form \( E(x)^\top P^{-1} E(x) \) [26]. For the case where \( G \) is a polynomial state space model, sum of squares programming can be used to enforce the LMIs.

The conservatism of the IQC framework is largely associated with the accuracy of the IQC descriptors used for the activation function. For the non-incremental case, extensive libraries of IQC descriptors have been developed, e.g. the Zames-Falb, Popov and RL/RC multipliers. In the incremental case was shown that these descriptors cannot be easily used as they do not preserve incremental positivity [35]. To the authors’ knowledge, it remains an open problem to to find more expressive IQCs for activation functions that are valid for incremental and differential analysis.

V. NUMERICAL EXAMPLE

We will compare the proposed Robust RNN with the commonly used (Elman) RNN [31] and LSTM [36], along with two stable model sets, the stable RNN (sRNN) [18] and contracting implicit RNN (ci-RNN) [19]. All models have a state dimension of 10 and all models except for the LSTM use a ReLU activation function. The LSTM uses a tanh activation.

In order to generate data, we will use a simulation of a series of four coupled mass spring dampers. The goal is to identify a mapping from the force on the initial mass to the position of the final mass. Nonlinearity is introduced through the springs’ piecewise linear force profile described by:
\[
F_{spring,i}(d) = k_i \Gamma(d)
\]
where \( \Gamma(d) \) is the piecewise linear function depicted in Figure 2 and \( k_i \) is the spring constant for that spring.

We excite the system using a pseudo-random binary sequence that changes values after a an interval distributed uniformly in \([0, \tau]\) and takes values that are normally distributed with stand deviation \( \sigma_u \). To generate a training batch we simulate the system for 200 seconds and sample the system at 5Hz to generate 1000 datapoints with an input signal characterized by \( \tau = 20s \) and \( \sigma_u = 3N \). The training data consists of 100 training batches. We also generate validation \( (\tau = 20s \) and \( \sigma_u = 3N \) and a test set with \( \tau = 20s \) and \( \sigma_u = 3N \) by simulating the system for 1000 seconds and sampling at 5 Hz.

A. Training Procedure

Fitting Robust RNNs requires a constrained optimization problem to be solved subject to a number of LMI constraints. I.e. we are interested in solving the following optimization problem:
\[
\min_{\theta \in \Theta_\gamma} ||\tilde{y} - S(\tilde{u})||^2
\]
where \( \Theta_\gamma \) is defined in \[21\]. In this work, we enforce the constraint \( \theta \in \Theta_\gamma \) using an interior point method where we minimize a series of objective functions of the following form:
\[
J = MSE(\tilde{y}, S(\tilde{u})) - \sum_i \beta \log \det(M_i) - \sum_j \beta \log \lambda_j.
\]
where \( M_i \) are the LMIs to be satisfied and \( \lambda_j \) are the IQC multipliers. As \( \beta \to 0 \), this approaches the solution of the problem \[28\]. We minimize \[29\] using the ADAM optimizer \[37\] for stochastic gradient descent with an initial learning rate of \( 1E^{-3} \). A backtracking line search is used to ensure strict feasibility though out the optimization. When
we observe more than 10 epochs without an improvement in performance on a validation, we decrease the learning rate by a factor of 0.25 and decrease the barrier parameter $\beta$ by a factor of 10 until a final learning rate of 1E-6 is achieved.

We initialize all Robust RNNs by first solving a linear subspace identification problem using N4SID to find matrices $A, B, C, D$ and using the resulting matrices to initialize $E^{-1} F = A$, $E^{-1} B_2 = B$, $C_1 = C$ and $D_{12} = D$.

### B. Model Evaluation

We will compare models on a number of different metrics measuring nominal quality of fit and robustness. Quality of fit will be compared in terms of the normalized simulation error:

$$\text{NSE} = \frac{||\hat{y} - y||}{||y||}$$

where $\hat{y}, y \in \ell^2$ are the simulated and true system outputs respectively. In order to study the robustness of the systems, we will study the the Lipschitz constant estimated via the following:

$$\hat{\gamma} = \max_{u \neq v} \frac{||S(u) - S(v)||}{||u - v||}$$

While solving (31) exactly is complicated by non-convexity, an approximate solution can be found using gradient ascent. The values of $\hat{\gamma}$ mentioned in this work are thus a lower bound on the true Lipschitz constant of the model, which can be interpreted as a measure of worst case sensitivity to input perturbations.

### C. Results

The performance of a number of the models on the various test sets and estimates of the Lipschitz constants are shown in Table 1. There are a number of apparent benefits from the proposed model set. Firstly, if we compare the nominal performance of the stable models, we can see that the Robust RNN ($\Theta_s$) outperforms all other stable models. This is due to the reduced conservatism of the proposed stability constraint.

We have also plotted the performance on the second test set versus the observed Lipschitz constant in Figure 2. From this, we can see the Robust RNNs have the best trade-off between nominal performance and robustness signified by the fact that they lie further in the lower left corner. For instance if we compare the LSTM with the Robust RNN ($\Theta_s$), we observe similar nominal performance, however the Robust RNN has a much smaller Lipschitz constant.

Another interesting observation is that all models with stability constraints have significantly reduced Lipschitz constant when compared to those without. This provides additional support for the observation that stability constraints can improve robustness and generalizability of models [19], [34].

In Figure 4 we present boxplots showing the performance of each of the models for a number of realizations of the input signal with varying $\sigma_u$. Note that we only present the results for a subset of the models due to space restrictions. We can see that for all plots, there is a trough corresponding to where the training data was drawn with $\sigma_u = 3$. Note however, that for the LSTM and the RNN, the performance of the models quickly degrades as the amplitude of the input data distribution increases. For the remaining models, however, we can see that this loss in performance is much more gradual. We interpret this as an improvement in model generalizability, supporting the assertion that the Lipschitz constant is a fundamental quantity affecting a models ability to generalize. Examples of the outputs and errors for these models are shown in Figure 5 for $\sigma_u = 3$ and for $\sigma_u = 10$. Figure 6 shows the medians of the same dataset for the models with and without stability constraints. Comparing the stable models, we can see that the nominal NSE is improves as the stability condition used becomes less conservative. Additionally, comparing the unstable models with the stable models, we can see that the stable models have better generalization, signified by the reduced slope after $\sigma_u = 3$.

### References

[1] H. Salman, M. Sun, G. Yang, A. Kapoor, and J. Z. Kolter, “Blackbox smoothing: A provable defense for pretrained classifiers,” arXiv preprint arXiv:2003.01908, 2020.
[2] I. J. Goodfellow, J. Shlens, and C. Szegedy, “Explaining and harnessing adversarial examples,” arXiv preprint arXiv:1412.6572, 2014.
[3] M. Cheng, J. Yi, P.-Y. Chen, H. Zhang, and C-J. Hsieh, “Seq2stick: Evaluating the robustness of sequence-to-sequence models with adversarial examples,” arXiv preprint arXiv:1803.01128, 2018.
[4] G. Zames, “On the input-output stability of time-varying nonlinear feedback systems part one: Conditions derived using concepts of loop gain, concity, and positivity,” IEEE Transactions on Automatic Control, vol. 11, no. 2, pp. 228–238, 1966.
[5] W. Lohmiller and J.-J. E. Slotine, “On contraction analysis for nonlinear systems,” Automatica, vol. 34, pp. 683–696, 1998.
[6] C. A. Desoer and M. Vidyasagar, Feedback systems: input-output properties. Siam, 1975, vol. 55.
[7] P. L. Bartlett, D. J. Foster, and M. J. Telgarsky, “Spectrally-normalized margin bounds for neural networks,” in Advances in Neural Information Processing Systems, 2017, pp. 6240–6249.
[8] S. Zhou and A. P. Schoellig, “An analysis of the expressiveness of deep neural network architectures based on their lipchitz constants,” arXiv preprint arXiv:1912.11511, 2019.
[9] T. Huster, C.-Y. J. Chiang, and R. Chadha, “Limitations of the Lipschitz constant as a defense against adversarial examples,” pp. 16–29, 2019.
Fig. 4: Boxplots showing model performance for varying input data distribution amplitude $\sigma_u$. Each boxplot shows NSE over for 300 input realizations.

(a) LSTM

(b) RNN

(c) ci-RNN ($\Theta_{ci}$)

(d) Robust RNN ($\Theta_3$)

(e) Robust RNN ($\Theta_8$)

(f) Robust RNN ($\Theta_\ast$)

Fig. 5: Simulation and performance for two example inputs with different amplitudes.

(a) $\sigma_u = 3$

(b) $\sigma_u = 10$
Table I: Performance of models on coupled mass spring damper example.

| Test NSE | RNN | LSTM | $\Theta_3$ | $\Theta_0$ | $\Theta_8$ | $\Theta_{10}$ | $\Theta_2$ | cr-RNN | s-RNN |
|----------|-----|------|-----------|-----------|-----------|-----------|---------|--------|--------|
| $\gamma$ | 0.132 | 0.097 | 0.2573 | 0.1940 | 0.102 | 0.0964 | 0.0677 | 0.1031 | 0.1044 |
| 7.84 | 5.082 | 2.9753 | 4.6359 | 7.03 | 7.84 | 11.449 | 5.994 | 10.182 |

Fig. 6: Model performance versus input distribution parameter.