THE DIGAMMA FUNCTION, EULER-LEHMER CONSTANTS AND THEIR $p$-ADIC COUNTERPARTS

T. CHATTERJEE AND S. GUN

Abstract. The goal of this article is twofold. We first extend a result of Murty and Saradha [7] related to the digamma function at rational arguments. Further, we extend another result of the same authors [8] about the nature of $p$-adic Euler-Lehmer constants.

1. Introduction

For a real number $x \neq 0, -1, \ldots$, the digamma function $\psi(x)$ is the logarithmic derivative of the gamma function defined by

$$-\psi(x) = \gamma + \frac{1}{x} + \sum_{n=1}^{\infty} \left( \frac{1}{n + x} - \frac{1}{n} \right),$$

where $\gamma$ is Euler’s constant. Just like the case of the gamma function, the nature of the values of the digamma function at algebraic or even rational arguments is shrouded in mystery.

In the rather difficult subject of irrationality or transcendence, sometimes it is more pragmatic to look at a family of special values as opposed to a single specific value and derive something meaningful. An apt instance here is the result of Rivoal [10] about irrationality of infinitude of odd zeta values as opposed to that of a single specific odd zeta value.

In this context, Murty and Saradha [7] in a recent work have made some breakthroughs about transcendence of a certain family of digamma values. In particular, they proved the following.

Theorem 1.1 (Murty and Saradha). For any positive integer $n > 1$, at most one of the $\phi(n) + 1$ numbers in the set

$$\{\gamma\} \cup \{\psi(r/n) | 1 \leq r \leq n, (r, n) = 1\}$$

is algebraic.

In this article, we extend their result and prove the following.

\begin{itemize}
    \item \textbf{2010 Mathematics Subject Classification:} 11J91.
    \item \textbf{Key Words:} Digamma function, generalized Euler-Lehmer constants, $p$-adic Digamma function, generalized $p$-adic Euler-Lehmer constants.
\end{itemize}
Theorem 1.2. At most one element in the following infinite set
\[ \{ \gamma \} \cup \{ \psi(r/n) \mid n > 1, \ 1 \leq r < n, \ (r, n) = 1 \} \]
is algebraic.

In a recent work [3], the question of linear independence of these numbers is studied.

In another context, Lehmer [6] defined generalized Euler constants \( \gamma(r, n) \) for \( r, n \in \mathbb{N} \) with \( r \leq n \) by the formula
\[
\gamma(r, n) = \lim_{x \to \infty} \left( \sum_{m \equiv r \pmod{n}, m \leq x} \frac{1}{m} - \frac{\log x}{n} \right).
\]

Murty and Saradha, in their papers [7, 9], investigated the nature of Euler-Lehmer constants \( \gamma(r, n) \) and proved results similar to Theorem 1.1 and Theorem 1.2. For an exhaustive account of Euler’s constant, see the recent article of Lagarias [5].

From now onwards \( p \) and \( q \) will always denote prime numbers. In another work [8], Murty and Saradha investigated the \( p \)-adic analog \( \gamma_p \) of Euler’s constant as well as the generalized \( p \)-adic Euler-Lehmer constants \( \gamma_p(r, q) \). Here \( r \in \mathbb{N} \) with \( 1 \leq r < q \). They also studied the values of the \( p \)-adic digamma function \( \psi_p(r/p) \) for \( 1 \leq r < p \).

We shall give the definitions of \( \gamma_p, \gamma_p(r, q) \) and \( \psi_p(x) \) in the next section following Diamond [2]. Here are the results of Murty and Saradha [8].

Theorem 1.3 (Murty and Saradha). Let \( q \) be prime. Then at most one of
\[ \gamma_p, \ \gamma_p(r, q), \ 1 \leq r < q \]
is algebraic.

Theorem 1.4 (Murty and Saradha). The numbers \( \psi_p(r/p) + \gamma_p \) are transcendental for \( 1 \leq r < p \).

In this paper, we generalize these results. Let \( P \) denote the set of prime numbers in \( \mathbb{N} \). Here we prove the following:

Theorem 1.5. At most one number in the following set
\[ \{ \gamma_p \} \cup \{ \gamma_p(r, q) : q \in P, \ 1 \leq r < q/2 \} \]
is algebraic.

If we normalize the \( p \)-adic Euler-Lehmer constants by setting
\[ \gamma^*_p(r, q) = q \gamma_p(r, q), \]
then we have the following result:
Theorem 1.6. All the numbers in the following list
\[ \gamma_p, \quad \gamma_p^* (r, q), \quad q \in \mathcal{P}, \quad 1 \leq r < q/2 \]
are distinct.

Using this theorem, we prove the following:

Theorem 1.7. As before, let \( q \) run through the set of all prime numbers. Then there is at most one pair of repetition among the numbers
\[ \gamma_p, \quad \gamma_p (r, q), \quad 1 \leq r < q/2. \]
Also if there is such a repetition, then \( \gamma_p \) is transcendental.

Finally, we prove the following theorem.

Theorem 1.8. Fix an integer \( n > 1 \). At most one element of the following set
\[ \{ \psi_p (r/p^n) + \gamma_p : 1 \leq r < p^n, \quad (r, p) = 1 \} \]
is algebraic. Moreover, \( \psi_p (r/p) + \gamma_p \) are distinct when \( 1 \leq r < p/2 \).

2. Preliminaries

For all discussions in this section, let us fix a prime \( p \). Let \( \overline{\mathbb{Q}}_p \) be a fixed algebraic closure of \( \mathbb{Q}_p \) and \( \mathbb{C}_p \) be its completion. We fix an embedding of \( \mathbb{Q} \) into \( \mathbb{C}_p \). Thus the elements in the set \( \mathbb{C}_p \setminus \overline{\mathbb{Q}} \) are the transcendental numbers.

We begin with recalling the notion of \( p \)-adic logarithms which are of primary importance in our context. For the elements in the open unit ball around 1, that is
\[ D := \{ \alpha \in \mathbb{C}_p : |\alpha - 1|_p < 1 \}, \]
\( \log_p \alpha \) is defined using the formal power series
\[ \log(1 + X) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}X^n}{n} \]
which has radius of convergence 1. To extend this to all of \( \mathbb{C}_p^\times \), note that every element \( \beta \in \mathbb{C}_p^\times \) is uniquely expressible as
\[ \beta = p^r w \alpha \]
where \( \alpha \in D, \quad r \in \mathbb{Q} \) and \( w \) is a root of unity of order prime to \( p \). Here \( p^r \) is the positive real \( r \)-th power of \( p \) in \( \overline{\mathbb{Q}} \), embedded in \( \overline{\mathbb{Q}}_p \) from the beginning. With this, one defines
\[ \log_p \beta := \log_p \alpha. \]
Note that \( \log_p \beta = 0 \) if and only if \( \beta \) is \( p^r \) times a root of unity. We refer to Washington [11], Chapter 5 for details.
We shall now define the $p$-adic analog of the digamma function following the strategy of Diamond. The idea is to first define a suitable analog of the classical log-gamma function. This is defined, for $x \in \mathbb{C}_p$, as

$$G_p(x) = \lim_{k \to \infty} \frac{1}{p^k} \sum_{n=0}^{p^k-1} (x + n) \log_p(x + n) - (x + n).$$

This function satisfies properties analogous to the classical log-gamma function; for instance it satisfies

$$G_p(x + 1) = G_p(x) + \log_p x.$$  

It also satisfies an analog of the classical Gauss’ identity (up to a term $\log \sqrt{2\pi}$), namely

$$G_p(x) = \left( x - \frac{1}{2} \right) \log_p m + \sum_{a=0}^{m-1} G_p \left( \frac{x + a}{m} \right)$$

for a positive integer $m$ when the right side is defined.

The $p$-adic digamma function $\psi_p(x)$ is defined as the derivative of $G_p(x)$ and hence is given by (for $-x \notin \mathbb{N}$),

$$\psi_p(x) = \lim_{k \to \infty} \frac{1}{p^k} \sum_{n=0}^{p^k-1} \log_p(x + n).$$

Recall that the classical generalised Euler constant $\gamma(r, f)$ defined by Lehmer satisfies

$$\psi(r/f) = \log f - f\gamma(r, f)$$

for $1 \leq r \leq f$.

In the $p$-adic set up, one also defines $\gamma_p(r, f)$ for integers $r, f$ with $f \geq 1$ as follows. If the $p$-adic valuation $\nu(r/f)$ of $r/f$ is negative, then

$$\gamma_p(r, f) = -\lim_{k \to \infty} \frac{1}{fp^k} \sum_{m \equiv r (\text{mod } f)}^{fp^k-1} \log_p m.$$  

On the other hand when $\nu(r/f) \geq 0$, we first write $f$ as $f = p^k f_1$ with $(p, f_1) = 1$ and then define

$$\gamma_p(r, f) = \frac{p^{\nu(f_1)}}{p^{\nu(f_1)} - 1} \sum_{n \in N(r, f)} \gamma_p(r + nf, p^{\nu(f_1)}f)$$

where

$$N(r, f) = \{n \mid 0 \leq n < p^{\nu(f_1)}, nf + r \not\equiv 0 \pmod{p^{\nu(f_1)+k}}\}.$$
Finally, we set
\[ \gamma_p = \gamma_p(0, 1) = -\frac{p}{p-1} \lim_{k \to \infty} \frac{1}{p^k} \sum_{\substack{m \geq 1, \\gcd(m, p) = 1}} \log_p m. \]

We shall need the following identity of Diamond (see p.334 of [2]).

**Theorem 2.1.** If \( q > 1 \) and \( \zeta_q \) is a primitive \( q \)-th root of unity, then
\[ q \gamma_p(r, q) = \gamma_p - \sum_{a=1}^{q-1} \zeta_q^{-ar} \log_p (1 - \zeta_q^a). \]

Let us now state the pre-requisites from transcendence theory. We shall need the following result of Baker (see p.11 of [1]) involving classical logarithms of complex numbers.

**Theorem 2.2 (Baker).** If \( \alpha_1, \cdots, \alpha_n \) are non-zero algebraic numbers and \( \beta_1, \cdots, \beta_n \) are algebraic numbers, then
\[ \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n \]

is either zero or transcendental.

We shall need analogous results for linear forms in \( p \)-adic logarithms. More precisely, we shall need the following consequence of a theorem of Kaufman [4] as noticed by Murty and Saradha (see p. 357 of [8]).

**Theorem 2.3.** Suppose that \( \alpha_1, \cdots, \alpha_m \) are non-zero algebraic numbers that are multiplicatively independent over \( \mathbb{Q} \) and \( \beta_1, \cdots, \beta_m \) are arbitrary algebraic numbers (not all zero). Further suppose that
\[ |\alpha_i - 1| < p^{-c} \quad \text{for} \quad 1 \leq i \leq m, \]

where \( c \) is a constant which depends only on the degree of the number field generated by \( \alpha_1, \cdots, \alpha_m, \beta_1, \cdots, \beta_m \). Then
\[ \beta_1 \log_p \alpha_1 + \cdots + \beta_m \log_p \alpha_m \]

is transcendental.

### 3. Proof of Theorem 1.2

In order to prove Theorem 1.2, we need the following lemmas.

**Lemma 3.1.** For all \( n > 1 \) and for all \( r \in \mathbb{N} \) with \( (r, n) = 1 \) and \( 1 \leq r < n \), all the numbers in the following list
\[ \gamma, \psi(r/n) \]

are distinct.
Proof. For a real number \( x > 0 \), we have
\[
\psi'(x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^2} > 0.
\]
Hence \( \psi(x) \) is a strictly increasing function for \( x > 0 \). \( \square \)

Lemma 3.2. For \( q > 1 \) and \( 1 \leq a < q \) with \((a,q) = 1\), one has
\[
-\psi\left(\frac{a}{q}\right) - \gamma = \log q - \sum_{b=1}^{q-1} e^{-2\pi iba/q} \log(1 - e^{2\pi ib/q}).
\]

For a proof of this lemma, see page 311 of [7].

We are now ready to complete the proof of Theorem 1.2. We prove this theorem by the method of contradiction. Suppose that the above assertion is not true. By the work of Murty and Saradha [7], it follows that \( \gamma \) and \( \psi\left(\frac{a}{q}\right) \) for some \( 1 \leq a < q \) with \((a,q) = 1\) cannot be both algebraic. So assume that there exists \( 1 \leq a_1 < q_1 \) with \((a_1,q_1) = 1\) and \( 1 \leq a_2 < q_2 \) with \((a_2,q_2) = 1\) such that both \( \psi(a_1/q_1) \) and \( \psi(a_2/q_2) \) are algebraic numbers.

Note that by Lemma 3.2 we have
\[
\psi\left(\frac{a_1}{q_1}\right) - \psi\left(\frac{a_2}{q_2}\right) = \log \frac{q_2}{q_1} - \sum_{b=1}^{q_2-1} e^{-2\pi iba_2/q_2} \log(1 - e^{2\pi ib/q_2})
\]
\[
+ \sum_{c=1}^{q_1-1} e^{-2\pi iba_1/q_1} \log(1 - e^{2\pi ib/q_1}).
\]

The right hand side is a algebraic linear combination of linear forms of logarithms of algebraic numbers. Also by Lemma 3.1 it is non-zero. Hence by Baker’s theorem it is transcendental, a contradiction. This completes the proof of the theorem.

4. Proofs of other theorems

Next, we prove a proposition which will play a pivotal role in proving the rest of the theorems.

Proposition 4.1. For \( p_i \in \mathcal{P} \), let \( q_i = p_i^{m_i} \), where \( m_i \in \mathbb{N} \) and \( \zeta_{q_i} \) be a primitive \( q_i \)-th root of unity. Then for any finite subset \( J \) of \( \mathcal{P} \), the numbers
\[
1 - \zeta_{q_i}, \quad \frac{1 - \zeta_{a_i}}{1 - \zeta_{q_i}}, \quad \text{where} \quad 1 < a_i < \frac{q_i}{2}; \quad (a_i, q_i) = 1 \quad \text{and} \quad p_i \in J,
\]
are multiplicatively independent.
Proof. Write I = \{i \mid p_i \in J\}. We will prove this proposition by induction on \(|I|\). First suppose that \(|I| = 1\). Then the proposition is true by the work of Murty and Saradha (see page 357 of [8]). Next suppose that the proposition is true for all I with \(|I| < n\). Now suppose that \(|I| = n\). Note that for any \(i\), the numbers

\[ \frac{1 - \zeta_{a_i}}{1 - \zeta_{q_i}}, \text{ where } 1 < a_i < \frac{q_i}{2}, \ (a_i, q_i) = 1 \]

are multiplicatively independent units in \(\mathbb{Q}(\zeta_{q_i})\) (see page 144 of [11]). Suppose that there exist integers \(\alpha_i, \beta_{a_i}\) for \(i \in I\) and with \(a_i\) as in the lemma such that

\[ \prod_{i \in I} \left\{ (1 - \zeta_{q_i})^{\alpha_i} \prod_{1 < a_i < q_i/2, (a_i, q_i) = 1} \left( \frac{1 - \zeta_{a_i}}{1 - \zeta_{q_i}} \right)^{\beta_{a_i}} \right\} = 1. \]  

(3)

Taking norm on both sides, we get

\[ \prod_{i \in I} p_i^{\alpha_i A_i} = 1, \text{ where } A_i \neq 0, \ A_i \in \mathbb{N}. \]

Since \(p_i\)'s are distinct primes, we have \(\alpha_i = 0\) for all \(i \in I\). Thus (3) reduces to

\[ \prod_{i \in I} \prod_{1 < a_i < q_i/2, (a_i, q_i) = 1} \left( \frac{1 - \zeta_{a_i}}{1 - \zeta_{q_i}} \right)^{\beta_{a_i}} = 1. \]  

(4)

Since \(|I| > 1\), there exists \(i_1, i_2 \in I\) with \(i_1 \neq i_2\) such that

\[ \prod_{i \in I, \ i \neq i_1} \prod_{1 < a_i < q_i/2, (a_i, q_i) = 1} \left( \frac{1 - \zeta_{a_i}}{1 - \zeta_{q_i}} \right)^{\beta_{a_i}} = \prod_{1 < a_{i_1} < q_{i_1}/2, (a_{i_1}, q_{i_1}) = 1} \left( \frac{1 - \zeta_{a_{i_1}}}{1 - \zeta_{q_{i_1}}} \right)^{-\beta_{a_{i_1}}} \cdot \left( \frac{1 - \zeta_{a_i}}{1 - \zeta_{q_i}} \right)^{\beta_{a_i}}. \]

Note that the left hand side of the above equation belongs to the number field \(\mathbb{Q}(\zeta_{\delta})\), where \(\delta = \prod_{i \in I \setminus \{i_1\}} q_i\) whereas the right hand side belongs to \(\mathbb{Q}(\zeta_{q_{i_1}})\). Since

\[ \mathbb{Q}(\zeta_{\delta}) \cap \mathbb{Q}(\zeta_{q_{i_1}}) = \mathbb{Q}, \]

we see that both sides of the above equation is a rational number having norm 1. Thus we have

\[ \prod_{i \in I \setminus \{i_1\} \ 1 < a_i < q_i/2, (a_i, q_i) = 1} \left( \frac{1 - \zeta_{a_i}}{1 - \zeta_{q_i}} \right)^{\beta_{a_i}} = \prod_{1 < a_{i_1} < q_{i_1}/2, (a_{i_1}, q_{i_1}) = 1} \left( \frac{1 - \zeta_{a_{i_1}}}{1 - \zeta_{q_{i_1}}} \right)^{-\beta_{a_{i_1}}} \pm 1. \]
Squaring both sides, we get
\[
\prod_{i \in \Gamma \setminus \{i_1\}} \prod_{1 < a_i < q_i/2} \frac{1 - \zeta_{a_i}}{1 - \zeta_{q_i}} = \prod_{1 < a_1 < q_1/2} \frac{1 - \zeta_{a_1}}{1 - \zeta_{q_1}} = 1.
\]

Using (2), we see that \( \beta_{a_1} = 0 \) for all \( 1 < a_1 < q_1/2 \) and \( (a_1, q_1) = 1 \). Then (4) reduces to
\[
\prod_{i \in \Gamma \setminus \{i_1\}} \prod_{1 < a_i < q_i/2} \frac{1 - \zeta_{a_i}}{1 - \zeta_{q_i}} = 1.
\]

Now by induction hypothesis, we have \( \beta_{a_i} = 0 \) for all \( a_i \) and \( i \in \Gamma \setminus \{i_1\} \). This completes the proof of the lemma.

Using the above Proposition 4.1 and Theorem 2.3, we can prove the following statement.

**Lemma 4.2.** Let \( J \) be any finite subset of \( \mathcal{P} \). For \( q \in J \) and \( 1 < a < q/2 \), let \( s_q, t^a_q \) be arbitrary algebraic numbers, not all zero. Further, let \( t^a_q \) be not all zero when \( p \in J \). Then
\[
\sum_{q \in J} s_q \log_p(1 - \zeta_q) + \sum_{1 < a < q/2} t^a_q \log_p \left( \frac{1 - \zeta^a_q}{1 - \zeta_q} \right)
\]
is transcendental.

**Proof.** Write \( \delta = \prod_{q \in J} q \). For any \( \alpha \in \mathbb{Z}[\zeta_{\delta}] \) with \( p \nmid \alpha \) and \( M \in \mathbb{N} \), one has
\[
|\alpha^A - 1| < p^{-M}
\]
for some \( A \in \mathbb{N} \). By choosing \( M \) sufficiently large and using Theorem 2.3 and Proposition 4.1, we get the lemma.

**Lemma 4.3.** Let \( q_1, q_2 \) be two distinct prime numbers and \( 1 \leq r_i < q_i \) for \( i = 1, 2 \). Then
\[
\sum_{b=1}^{q_2-1} \zeta_{q_2}^{-br_2} \log_p(1 - \zeta_{q_2}^b) - \sum_{a=1}^{q_1-1} \zeta_{q_1}^{-ar_1} \log_p(1 - \zeta_{q_1}^a)
\]
is transcendental.

**Proof.** For any \( q > 1 \) and \( (r, q) = 1 \), we know that
\[
\sum_{a=1}^{q-1} \zeta_q^{-ar} = -1.
\]
Hence (3) can be written as
\[
\log_p \left( \frac{1 - \zeta_{q_1}}{1 - \zeta_{q_2}} \right) - \sum_{a=1}^{q_1-1} \zeta_{q_1}^{-ar_1} \log_p \left( \frac{1 - \zeta_{q_1}^a}{1 - \zeta_{q_1}} \right) + \sum_{b=1}^{q_2-1} \zeta_{q_2}^{-br_2} \log_p \left( \frac{1 - \zeta_{q_2}^b}{1 - \zeta_{q_2}} \right). \tag{6}
\]
It is because
\[
(1 - \zeta_q^{-t}) = -\zeta_q^{-t}(1 - \zeta_q^t)
\]
for any \( t \in \mathbb{N} \) and the \( p \)-adic logarithm is zero on roots of unity, we have
\[
\log_p (1 - \zeta_q^{-t}) = \log_p (1 - \zeta_q^t).
\]
Note that the summands in (3) for \( a = 1, b = 1 \) and \( a = q_1 - 1, b = q_2 - 1 \) are zero. Now pairing up \( a \) with \(-a\) and \( b \) with \(-b\) in (6), we get
\[
\log_p \left( \frac{1 - \zeta_{q_1}}{1 - \zeta_{q_2}} \right) = \sum_{1 < a < q_1/2} \alpha_a \log_p \left( \frac{1 - \zeta_{q_1}^a}{1 - \zeta_{q_1}} \right) + \sum_{1 < b < q_2/2} \beta_b \log_p \left( \frac{1 - \zeta_{q_2}^b}{1 - \zeta_{q_2}} \right),
\]
where \( \alpha_a = (\zeta_{q_1}^{-ar_1} + \zeta_{q_1}^{ar_1}) \), \( \beta_b = (\zeta_{q_2}^{-br_2} + \zeta_{q_2}^{br_2}) \) are non-zero algebraic numbers. Hence using Lemma 4.2 we deduce that (6) is transcendental. \( \square \)

4.1. **Proof of Theorem 1.5.** We now complete the proof of Theorem 1.5. Suppose that two of the numbers from the above set are algebraic. It follows from the works of Murty and Saradha (see page 351 of [8]) that one of them cannot be equal to \( \gamma_p \) or both of them can not be of the form \( \gamma_p(r_1, q) \) and \( \gamma_p(r_2, q) \).

Without loss of generality, we can assume that these two numbers are of the form \( \gamma_p(r_1, q_1) \), where \( 1 \leq r_1 < q_1 \) and \( \gamma_p(r_2, q_2) \), where \( 1 \leq r_2 < q_2 \) and \( q_1 \neq q_2 \). Then \( q_1 \gamma_p(r_1, q_1) - q_2 \gamma_p(r_2, q_2) \) is algebraic. Now by Diamond’s theorem (see theorem 18 of [2]), we have
\[
q_1 \gamma_p(r_1, q_1) - q_2 \gamma_p(r_2, q_2) = -\sum_{a=1}^{q_1-1} \zeta_{q_1}^{-ar_1} \log_p (1 - \zeta_{q_1}^a) + \sum_{b=1}^{q_2-1} \zeta_{q_2}^{-br_2} \log_p (1 - \zeta_{q_2}^b). \tag{7}
\]
The left hand side is algebraic by assumption whereas the right hand side is transcendental by Lemma 4.3 a contradiction. This completes the proof of the theorem.

4.2. **Proof of Theorem 1.6.** It follows from Diamond’s theorem [2] that
\[
\gamma_p^*(r_1, q) - \gamma_p = -\sum_{a=1}^{q-1} \zeta_q^{-ar} \log_p (1 - \zeta_q^a)
\]
and
\[
\gamma_p^*(r_1, q) - \gamma_p^*(r_2, q) = -\sum_{a=1}^{q-1} (\zeta_q^{-ar_1} - \zeta_q^{-ar_2}) \log_p (1 - \zeta_q^a) \tag{8}
\]
where \(1 \leq r_1, r_2 < q/2\) and \(r_1 \neq r_2\). Transcendence of the first number follows from the works of Murty and Saradha (see page 358 of [9]) while that of the second one follows from the fact that \(\zeta^{-ar_1} + \zeta^{ar_1} \neq \zeta^{-ar_2} + \zeta^{-ar_2}\) when \(1 \leq a, r_1, r_2 < q/2\) with \(r_1 \neq r_2\). Again by Diamond’s theorem, we have

\[
\gamma_p^*(r_1, q_1) - \gamma_p^*(r_2, q_2) = -\sum_{a=1}^{q_1-1} \zeta_{q_1}^{-ar_1} \log_p(1 - \zeta_{q_1}^a) + \sum_{b=1}^{q_2-1} \zeta_{q_2}^{-br_2} \log_p(1 - \zeta_{q_2}^b),
\]

where \(q_1 \neq q_2\). By Lemma [4.3] we know that this number is transcendental and hence non-zero. This completes the proof.

4.3. Proof of Theorem [1.7] We will prove this theorem by contradiction. First note that it is impossible to have

\[
\gamma_p(r_1, q) = \gamma_p = \gamma_p(r_2, q),
\]

as otherwise \(\gamma_p^*(r_1, q) = \gamma_p^*(r_2, q)\), a contradiction to Theorem [1.6] Next assume that

\[
\gamma_p(r_1, q_1) = \gamma_p = \gamma_p(r_2, q_2),
\]

where \(q_1 \neq q_2\) and \(1 \leq r_i < q_i/2\). Using Diamond’s theorem, we can write

\[
(q_i - 1)\gamma_p = \log_p(1 - \zeta_{q_i}) + \sum_{1 < a < q_i/2} (\zeta_{q_i}^{ar_i} + \zeta_{q_i}^{-ar_i}) \log_p\left(\frac{1 - \zeta_{q_i}^a}{1 - \zeta_{q_i}^b}\right),
\]

where \(i = 1, 2\). From this, we get

\[
0 = (q_2 - 1) \log_p(1 - \zeta_{q_1}) - (q_1 - 1) \log_p(1 - \zeta_{q_2})
- \sum_{1 < a < q_2/2} (q_2 - 1)(\zeta_{q_2}^{ar_2} + \zeta_{q_2}^{-ar_2}) \log_p\left(\frac{1 - \zeta_{q_2}^a}{1 - \zeta_{q_2}^b}\right)
+ \sum_{1 < a < q_2/2} (q_1 - 1)(\zeta_{q_1}^{ar_1} + \zeta_{q_1}^{-ar_1}) \log_p\left(\frac{1 - \zeta_{q_1}^a}{1 - \zeta_{q_1}^b}\right)
\]

which is transcendental by Lemma [4.2] a contradiction. Now suppose that

\[
\gamma_p(r_1, q_1) = \gamma_p(r_2, q_2) \text{ and } \gamma_p(r_3, q_3) = \gamma_p(r_4, q_4)
\]

for some \(1 \leq r_i < q_i/2, 1 \leq i \leq 4\). We may assume that \(q_1 \neq q_2\). For if \(q_1 = q_2 = q\) (say), then \(\gamma_p(r_1, q) = \gamma_p(r_2, q)\) implies \(\gamma_p^*(r_1, q) = \gamma_p^*(r_2, q)\), a contradiction to Theorem [1.6] Similarly, we may assume that \(q_3 \neq q_4\). Now
from equation $\gamma_p(r_1, q_1) = \gamma_p(r_2, q_2)$, we deduce that

$$(q_1 - q_2)\gamma_p = q_2 \log p(1 - \zeta_{q_1}) - q_1 \log p(1 - \zeta_{q_2}) - q_2 \sum_{1 < a_1 < q_1/2} (\zeta^{a_1 r_1}_{q_1} + \zeta^{-a_1 r_1}_{q_1}) \log p \left( \frac{1 - \zeta^{a_1}_{q_1}}{1 - \zeta_{q_1}} \right) + q_1 \sum_{1 < a_2 < q_2/2} (\zeta^{a_2 r_2}_{q_2} + \zeta^{-a_2 r_2}_{q_2}) \log p \left( \frac{1 - \zeta^{a_2}_{q_2}}{1 - \zeta_{q_2}} \right).$$

Similarly from equation $\gamma_p(r_3, q_3) = \gamma_p(r_4, q_4)$, we deduce that

$$(q_3 - q_4)\gamma_p = q_4 \log p(1 - \zeta_{q_3}) - q_3 \sum_{1 < a_3 < q_3/2} (\zeta^{a_3 r_3}_{q_3} + \zeta^{-a_3 r_3}_{q_3}) \log p \left( \frac{1 - \zeta^{a_3}_{q_3}}{1 - \zeta_{q_3}} \right) - q_3 \log p(1 - \zeta_{q_4}) + q_3 \sum_{1 < a_4 < q_4/2} (\zeta^{a_4 r_4}_{q_4} + \zeta^{-a_4 r_4}_{q_4}) \log p \left( \frac{1 - \zeta^{a_4}_{q_4}}{1 - \zeta_{q_4}} \right).$$

Eliminating $\gamma_p$ from the above two equations, we get

$$0 = (q_3 - q_4)q_2 \log p(1 - \zeta_{q_1}) - (q_3 - q_4)q_1 \log p(1 - \zeta_{q_2}) - (q_1 - q_2)q_4 \log p(1 - \zeta_{q_3}) + (q_1 - q_2)q_3 \log p(1 - \zeta_{q_4}) + \sum_{i=1}^4 \sum_{1 < a_i < q_i/2} c_{a_i} \log p \left( \frac{1 - \zeta^{a_i}_{q_i}}{1 - \zeta_{q_i}} \right),$$

where $c_{a_i}$ are algebraic numbers for all $1 \leq i \leq 4$ and $1 < a_i < q_i/2$. Again using Lemma 4.2 the number is transcendental. This completes the proof.

To prove the second part of the theorem, suppose that

$$\gamma_p(r_1, q_1) = \gamma_p(r_2, q_2)$$

where $q_1 \neq q_2$ and $1 \leq r_i < q_i/2$ for $i = 1, 2$. Again, we deduce the result from (9) or (10) using Lemma 4.2.

4.4. **Proof of Theorem 1.8** Write

$$S = \{ \psi_p(r/p^n) + \gamma_p : 1 \leq r < p^n, \ (r, p) = 1 \}.$$

Suppose that $a, b \in S$ be distinct and algebraic. Suppose that

$$(a, b) = (\psi_p(r_1/p^n) + \gamma_p, \ \psi_p(r_2/p^n) + \gamma_p).$$
Using Diamond’s theorem, we have
\[
\psi_p\left(\frac{r_1}{p^n}\right) - \psi_p\left(\frac{r_2}{p^n}\right)
= \sum_{a=1}^{p^n-1} \zeta^{-ar_1} \log_p \left(1 - \zeta_p^a\right) - \sum_{a=1}^{p^n-1} \zeta^{-ar_2} \log_p \left(1 - \zeta_p^a\right)
= \sum_{1 < a < p^n/2 \atop (a,p)=1} \alpha_a \log_p \left(\frac{1 - \zeta_p^a}{1 - \zeta_p}\right),
\]
where \(\alpha_a\)'s are algebraic numbers. But by Lemma 4.2 this is necessarily transcendental, a contradiction.

Moreover when \(n = 1\), we have
\[
\psi_p\left(\frac{r_1}{p}\right) - \psi_p\left(\frac{r_2}{p}\right) = \sum_{1 < a < p/2} \left(\zeta^{-ar_1} + \zeta^{ar_1} - \zeta^{-ar_2} - \zeta^{ar_2}\right) \log_p \left(\frac{1 - \zeta_p^a}{1 - \zeta_p}\right).
\]
Since \(1 \leq r_1, r_2 < p/2\), the above linear form in logarithm is transcendental by Lemma 4.2 and hence non-zero. This completes the proof.

Acknowledgements. It is our pleasure to thank Ram Murty for going through an earlier version of the paper. We would also like to thank Purusottam Rath for many helpful discussions. The second author would like to thank ICTP for the hospitality where the final part of the work was done.

References

[1] A. Baker, Transcendental number theory, 2nd edn, Cambridge University Press, Cambridge, 1990.
[2] J. Diamond, The \(p\)-adic log gamma function and the \(p\)-adic Euler constants, Trans. Amer. Math. Soc. 233 (1977), 321–337.
[3] S. Gun, M. R. Murty and P. Rath, Linear independence of digamma function and a variant of a conjecture of Rohrlich, J. Number Theory 129 (2009), no. 8, 1858–1873.
[4] R. M. Kaufman, An estimate of a linear form of logarithms of algebraic numbers with \(p\)-adic metric, Vestnik Moskov. Univ. Ser. I , 26 (1971), no. 2, 3–10.
[5] J. C. Lagarias, Euler’s constant: Euler’s work and modern developments, preprint, math arXiv:1303.1856
[6] D. H. Lehmer, Euler constants for arithmetical progressions, Acta Arith. 27 (1975), 125–142.
[7] M. Ram Murty and N. Saradha, Transcendental values of the digamma function, J. Number Theory 125 (2007), 298–318.
[8] M. Ram Murty and N. Saradha, Transcendental values of the \(p\)-adic digamma function, Acta. Arith. 133 (2008), no.4, 349–362.
[9] M. Ram Murty and N. Saradha, Euler-Lehmer constants and a conjecture of Erdős, J. Number Theory 130 (2010), 2671–2682.
[10] T. Rivoal, *La fonction zêta de Riemann prend une infinité de valeurs irrationnelles aux entiers impairs*, C. R. Acad. Sci. Paris Sér. I Math. **331** (2000), no. 4, 267–270.

[11] L. C. Washington, *Introduction to Cyclotomic fields*, 2nd edn, Graduate Texts in Mathematics 83, *Springer-Verlag*, New York, 1997.

T. Chatterjee,  
Department of Mathematics, Queen’s University, Kingston, ON K7L3N6, Canada.  
*E-mail address*: tapas@mast.queensu.ca

S. Gun,  
Institute for Mathematical Sciences, C.I.T Campus, 4th Cross street, Taramani, Chennai, 600 113, Tamil Nadu, India.  
*E-mail address*: sanoli@imsc.res.in