Central charge and renormalization in supersymmetric theories with vortices

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Some quantum features of vortices in supersymmetric theories in 1+2 dimensions are studied in a manifestly supersymmetric setting of the superfield formalism. A close examination of the supercurrent that accommodates the central charge and super-Poincare charges in a supermultiplet reveals that there is no genuine quantum anomaly in the supertrace identity and in the supercharge algebra, with the central-charge operator given by the bare Fayet-Iliopoulos term alone. The central charge and the vortex spectrum undergo renormalization on taking the expectation value of the central-charge operator. It is shown that the vortex spectrum is exactly determined at one loop while the spectrum of the elementary excitations receives higher-order corrections.

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I. INTRODUCTION

There has recently been considerable interest in topological-charge anomalies in supersymmetric theories. In the presence of topologically stable excitations the supercharge algebra gets centrally extended and for the topological excitations that saturate the Bogomol’nyi-Prasad-Sommerfield (BPS) bound classically the spectrum is determined exactly through the central charge. Multiple shortening for BPS-saturated excitations generally reveals that saturation persists at the quantum level. It is, however, another question whether the spectrum of topological excitations acquires substantial quantum corrections. Actually it took years until it was clarified from the study of solitons in some simple supersymmetric theories that topological charges do acquire quantum anomalies, with the soliton spectrum modified accordingly. This is mainly because direct calculations of the soliton spectrum and topological charge are a delicate task that requires proper handling of supersymmetry. In calculating the soliton mass, for example, one has to handle an ill-defined sum over zero-point energies of quantum fluctuations, and may put the system in a finite box for mode regularization, which then easily spoils supersymmetry via the boundary condition.

An ideal tool for keeping supersymmetry manifest is the superfield formalism. It allows one to study various quantum aspects of supersymmetric theories in a systematic way, avoiding such subtle problems as encountered in finite-box calculations or in component-field calculations with non-supersymmetric regularizations. Calculations with superfields are sometimes much more efficient than component-field calculations.

Fujikawa and van Nieuwenhuizen developed a super-sphere approach to the central-charge anomaly. Subsequently a superfield formulation of the central-charge anomaly was put forth by making use of a supercurrent that places the topological charge, supercharges, energy and momentum in a supermultiplet.

Use of the superfield supercurrent makes manifest the supermultiplet nature of various symmetry currents, conservation laws and the associated anomalies. This in turn reveals that the problem of topological-charge anomalies is essentially the problem of the supertrace of the supercurrent. This point of view has been verified for solitons and domain walls in some low-dimensional theories, revealing a dual (bosonic/fermionic) character of the central-charge anomalies; for solitons the anomaly derives from the superconformal anomaly while for domain walls it apparently comes from induced spin.

The purpose of this paper is to extend a similar superspace analysis to vortices in 1+2 dimensions, for which somewhat nontrivial BPS saturation and finite renormalization of the spectrum have recently been reported (within the component-field formalism). We consider a one-parameter family of conserved supercurrents, which leads to a supercharge algebra of the same form and which accommodates the improved superconformal currents for three and four dimensions. It turns out that there is no genuine quantum anomaly in the supertrace of the supercurrent and in the supercharge algebra, with the central-charge operator given by the bare Fayet-Iliopoulos term alone. We discuss how the central charge and the vortex spectrum undergo renormalization on taking the expectation value of the central-charge operator. It is shown that the vortex spectrum is exactly determined at one loop while the spectrum of the elementary excitations receives higher-order corrections.

In Sec. II we review some basic features of \( N = 2 \) supersymmetric theories with classical vortices in three dimensions. In Sec. III we introduce a superfield supercurrent, and study its classical conservation laws and improvement. In Sec. IV we examine the supertrace of the supercurrent at the quantum level and show the absence of a genuine superconformal anomaly. In Sec. V we discuss renormalization of the central charge and its consequences. Section VI is devoted to a summary and discussion.
II. N=2 SUPERSYMMETRY IN THREE DIMENSIONS

Let us consider a supersymmetric version of the Abelian Higgs model with \( N = 1 \) supersymmetry in four dimensions. The action is given by

\[
S = \int d^4z \left[ \frac{1}{2} W^\alpha W_\alpha + \frac{1}{2} \nabla^2 \phi \right],
\]

where \( W^\alpha = \frac{\partial \phi}{\partial \theta^\alpha} \) and \( \phi = -\frac{1}{4} D^2 A_\alpha \). We adopt superspace notation of Ref.13 but with metric (+ − − −); \( D_\alpha = \partial / \partial \theta^\alpha - (\sigma^\mu \epsilon_\alpha) p_\mu \) and \( D^\alpha = \partial / \partial \theta^\alpha - (\theta^\mu) \epsilon_\alpha p_\mu \) with \( p_\mu = i \partial_\mu \). The Fayet-Iliopoulos (FI) term serves to introduce spontaneous breaking of U(1) gauge invariance.

We shall study vortex solutions in the Higgs model with \( N = 2 \) supersymmetry in three dimensions, which is obtained from this four-dimensional (4d) \( N = 1 \) model by dimensional reduction. The 4d model, actually, is afflicted with a gauge anomaly (because of chiral gauge coupling to matter) but this poses no problem for the reduced 3d model. One may instead start with an anomaly-free SQED (supersymmetric quantum electrodynamics) version of the model with oppositely-charged matter fields, i.e.,

\[
\Phi e^{2\epsilon V} \Phi^\dagger \Phi e^{2\epsilon V} \Phi^\dagger \Phi - e^{-2\epsilon V} \Phi^\dagger \Phi - \frac{1}{4} f_{\mu\nu} + \frac{1}{2} D^2 - D(\kappa - eA^A) + F^A F
\]
in Eq. (2.1). We study both models but handle the single-\( \Phi \) model for exposition. In addition, for conciseness, we use 4d notation until the reduction to three dimensions is really needed.

The matter supermultiplet consists of a charged scalar field \( A(x) \) and a charged fermion \( \psi_\alpha(x) = (\psi_1, \psi_2) \) along with an auxiliary field \( F(y) \), with the chiral superfield

\[
\Phi(z) = A(y) + \sqrt{\theta} \phi(y) + \theta^2 F(y),
\]

where \( \psi_\mu \equiv \psi_\alpha \delta^\alpha_\mu \). The real superfield \( Z(V) = \Psi_\alpha - \theta \phi_\alpha \), containing a gauge field \( a_\mu(x) \) and a gaugino field \( \xi_\alpha(x) \), along with an auxiliary field \( D(x) \),

\[
V_{WZ} = \theta a_\mu \Xi \alpha + \theta^2 \bar{\Psi}_\alpha + \theta^2 D + \frac{1}{2} \theta^2 D^2,
\]

where \( (\alpha_\mu)_{\alpha\alpha} = (1, \sigma^A)_{\alpha\bar{\alpha}} \). The action is invariant under gauge transformations \( \Lambda \rightarrow V + \Lambda \), \( \Phi \rightarrow e^{-2\epsilon V} \Phi \) and \( \Phi \rightarrow \Phi e^{2\epsilon A} \), with an arbitrary chiral superfield \( \Lambda \) with \( D_\alpha \Lambda = \bar{D}_\alpha \Lambda = 0 \).

In the Wess-Zumino gauge one retains only \( V_{WZ} \) for \( V \) and sets \( \Phi e^{2\epsilon V} \rightarrow \Phi \) anew. Supertranslations

\[
x^{\mu} = x^{\mu} + i \xi^{\alpha} \delta_\alpha \Xi, \quad \theta' = \theta + \xi, \quad \delta' = \bar{\delta} + \bar{\xi}, \]

on \( V_{WZ} \) then yield the supersymmetry transformation laws of the component fields

\[
\delta a_\mu = -\xi a_\mu - \bar{\xi} \bar{a}_\mu, \quad \delta \psi_\alpha = -\xi \phi_\alpha + \frac{1}{2} (\sigma^{\mu\nu} \xi) f_{\mu\nu}, \quad \delta D = -i (\xi \sigma^{\mu\nu} \partial_\mu \Xi + \bar{\xi} \sigma^{\mu\nu} \partial_\mu \bar{\Xi}).
\]

where \( f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \). They at the same time entail gauge transformations \( i \delta V = \sigma^{\mu\nu} \Xi a_\mu(y) + \theta^2 \bar{\Xi} \bar{\Lambda}(y) \), which combine with supertranslations on \( \Phi \) to yield gauge-covariant supersymmetry transformations of the matter fields,

\[
\delta A = \sqrt{2} \phi, \quad \delta \psi_\alpha = \sqrt{2} \phi F - i \sqrt{2} (\sigma^{\mu\nu} \xi) D_\mu A, \quad \delta F = -i \sqrt{2} \bar{\phi} \sigma^{\mu\nu} (D_\mu \psi) + 2 e \bar{\phi} \bar{\Xi} A,
\]

with \( D_\mu = \partial_\mu + i e a_\mu \).

In the Wess-Zumino gauge the action reads \( S = \int d^3x \mathcal{L} \) with

\[
\mathcal{L} = \frac{1}{4} f_{\mu\nu} + \chi \bar{F} + \frac{1}{2} \bar{F}^2 - D(\kappa - eA^A) + F^A F
\]

where \( \bar{D} = \bar{\sigma} \partial_\mu D_\mu \). Eliminating the auxiliary field \( D \) yields the potential \(-1/2 (\kappa - eA^A)^2 \). For \( \kappa > 0 \) this potential has a vanishing minimum \( D = \kappa - eA^A \rightarrow 0 \) for \( \langle A \rangle_{\text{vac}} = \sqrt{\kappa/e} \), and supersymmetry is kept exact while the gauge symmetry gets spontaneously broken. For \( \kappa < 0 \), in contrast, supersymmetry is spontaneously broken while gauge invariance is kept manifest.

We now reduce the system down to three dimensions. For translation from Weyl spinors to 3d Dirac spinors we note the correspondence \( \psi \rightarrow \bar{\psi} \gamma_5 \phi \) or \( (\sigma^5)^{\alpha\alpha} = 1 - \sigma^5 \). \( \gamma \) is it convenient to take the Majorana representation for the Dirac matrices

\[
\gamma^0 = \gamma^2, \quad \gamma^1 = i \sigma^3, \quad \gamma^3 = -i \sigma^1.
\]

This yields \( \gamma^2 = -\sigma^2 \sigma^2 = -1 \), telling us to eliminate the "\( \gamma^2 \)" axis; we thus take \( x_0, x_1, x_3 \) for the 3d coordinate. We do not display the 3d form of the Lagrangian here, and simply remark that on reduction \( a_2(x) \) turns into a (pseudo)scalar field with matter coupling \(-eA^A \bar{\psi} \sigma^2 \bar{\psi} \).

We take \( \kappa > 0 \) (with \( e > 0 \)), in which case the reduced 3d model supports classical vortex solutions governed by the Bogomolny equation

\[
(D_3 \pm i D_1 A) = 0, \quad f_{13} \pm (eA^A - \kappa) = 0, \quad a_2 = 0.
\]

We take the upper sign for vortices of vorticity \( n = 1, 2, \ldots \), with the boundary condition \( A(x) \rightarrow ve^{i\theta} \) for \( r \rightarrow \infty \), where \( \theta = \arctan(x^3/x^1) \), while \( A(x) \propto r^n \) and \( a_2 \rightarrow 0 \) for \( r \rightarrow 0 \). A vortex of vorticity \( n \) carries the flux \( \int d^2x f_{13} = \frac{1}{2} \pi n/e \) and has energy \( E_{cl} \) equal to the central charge

\[
Z_{cl} = \int d^2x e^{i\theta} \partial_1 (ka_2 + iA^A D_1 A) = 2 \pi n \kappa / e
\]

with \( e^{i\theta} = 1 \).

III. SUPERFIELD SUPERCURRENT

Our main task is to study possible quantum modifications of the supercharge algebra in the presence of vortices, using the superfield formalism. In this section, as
the first step, we introduce an appropriate supercurrent and study its properties at the classical level.

The supercurrent for the present model is readily guessed from that for supersymmetric quantum electrodynamics:

$$\mathcal{R}_{\alpha\dot{\alpha}} = (\nabla_\alpha \Phi)(D_\alpha e^{2eV\Phi}) - \frac{w}{2}[D_\alpha, \bar{D}_\dot{\alpha}](\Phi e^{2eV\Phi}) - 2W_\alpha \bar{W}_\dot{\alpha},$$  \tag{3.1}

where $\nabla_\alpha = e^{-2eV}D_\alpha e^{2eV} = D_\alpha + 2e(D_\alpha V)$ is the gauge-covariant spinor derivative for $\Phi$. Here $w$ is a free (real) parameter. This gauge-invariant supercurrent $\mathcal{R}_{\alpha\dot{\alpha}}$ can also be derived from the action $S$ directly by a superspace Noether theorem using (gauge-covariant) local superconformal transformations of the form \[\delta \Phi = 4 \gamma^\alpha \Omega_\alpha(z), \delta \bar{\Omega}_{\dot{\alpha}}(z) = [\Omega_\alpha(z)]^\dagger\] for the present model, however, $\mathcal{R}_{\alpha\dot{\alpha}}$ is left arbitrary owing to percurrent or the supersymmetric trace identity, 

\[\text{term comes from the FI term and } A_\alpha, \text{ which vanishes classically, is a potential candidate for the superconformal anomaly. In contrast, } O_\alpha \text{ is not a genuine breaking term. Indeed } O_\alpha \text{ disappears for } w = 2/3 \text{ while } \partial_{\nu} \mathcal{R}^\mu = 0 \text{ holds classically for arbitrary } w. \text{ This indicates that } w \text{ represents the freedom to improve the supercurrent; this is seen explicitly from Eq. (3.1) below, where } w \text{ arises with a familiar improvement term [32] in the energy-momentum tensor.}

The supercurrent $\mathcal{R}^\mu(z)$, expanded in components

$$\mathcal{R}^\mu = R^\mu - i\theta J^\mu + i\bar{\theta} J^{\dot{\mu}} - 2\sigma_\lambda \lambda T^{\mu \lambda} + \cdots,$$ \tag{3.8}

properly contains the R current

$$R^\mu = (w - 1)\bar{\psi} \sigma^\mu \psi + iw A^\mu D^\mu A + \bar{\chi} \sigma^\mu \chi,$$ \tag{3.9}

energy-momentum tensor $T^{\mu \lambda}$ and supersymmetry currents $J^\mu_\alpha$ and $(J^\mu)^\dagger$,

$$J^\mu_\alpha = (1 - w)\sqrt{2} \{i (\sigma^\mu \psi_\alpha) F + (\sigma^\alpha \sigma^\mu \psi_\alpha) D_\mu A^\ast \}
- 2iw (\sigma^\mu \chi_\alpha A^\ast - \sqrt{2}w A^\ast D^\mu \psi_\alpha)
- i\lambda_\mu \chi (D^\mu \psi_\alpha - \psi \sigma^\mu \phi_\alpha \phi_\alpha),$$ \tag{3.10}

where $f = \frac{2}{3} e^{\epsilon_{\mu \rho \sigma \beta}} f_{\mu \rho \sigma \beta}$ and $\epsilon^{123} = 1$. Here $A^\ast D_\mu \psi = A^\ast D_\mu \psi - (D_\mu A^\ast) \psi$ with $D_\mu = \partial_\mu + i e a_\mu$, and the conju- gate current $(J^\mu)^\dagger$ is obtained from $J^\mu_\alpha$ by substitution $i(\sigma^\mu \psi_\alpha) \rightarrow i(\bar{\sigma}^\mu \psi_\alpha), (\sigma^\alpha \sigma^\mu \psi_\alpha) \rightarrow (\sigma^\alpha \sigma^\mu \psi_\alpha)^\dagger$. Remember the presence of the "2" components such as $R^2$ and $T^{\mu \nu}$ in the 2d case as well.

The energy-momentum tensor $T^{\mu \lambda}$ is the one somewhat improved from the canonical tensor, with the symmetric component

$$T^{\mu \lambda}_{\text{sym}} = (D^\mu A^\ast) D^\lambda A + (D^\lambda A^\ast) D^\mu A - f^{\mu \alpha \lambda} f^\lambda \alpha
+ \frac{i}{2} \left( \bar{\psi} \sigma^\mu \sigma^\lambda \psi \right)_{\text{sym}} + \frac{i}{2} \left( \bar{\chi} \sigma^\mu \sigma^\lambda \chi \right)_{\text{sym}}
- g^\mu \lambda \left[ \partial_\nu A^\nu - \frac{1}{4} f_{\mu \beta \gamma} f^{\beta \gamma} - \frac{1}{2} D^2 + \Delta \right]
+ \frac{i}{4} (g^{\mu \lambda} \partial^\nu - \partial^\nu \partial^\lambda)(A^\nu A^\lambda),$$ \tag{3.11}

where $2\Delta = (1 - w)\Delta_1 - w\Delta_2$ consists of "equation-of-motion" terms

$$\Delta_1 = \psi \frac{\delta S}{\delta \psi_\alpha} + \frac{\psi^\alpha \delta S}{\delta \psi^\alpha} + F^\alpha \frac{\delta S}{\delta F^\alpha} + F^\ast \frac{\delta S}{\delta F^\ast},$$
$$\Delta_2 = A^\ast (\delta S/\delta A^\ast) + A (\delta S/\delta A).$$ \tag{3.12}

The antisymmetric component $T^{\mu \lambda}_{\text{asym}} = \frac{i}{2} (T^{\mu \lambda} - T^{\lambda \mu})$ is given by

$$T^{\mu \lambda}_{\text{asym}} = -e^{\mu \nu \rho \lambda} \left[ \frac{1}{4} \partial_\nu \partial_\rho + \frac{1}{2} f_{\nu \rho} (\epsilon A^\nu A + D) \right]
+ \Delta^{\mu \lambda}_{\psi \chi},$$ \tag{3.13}

with

$$\Sigma_\rho = (1 - 2w) \bar{\psi} \sigma_\rho \psi + \bar{\chi} \sigma_\rho \chi + 4(1 - w) i A^\nu D_\rho A,$$
$$\Delta^{\mu \lambda}_{\psi \chi} = (i/4) [\bar{\psi} \sigma^\mu \sigma^\lambda \delta S/\delta \psi + \psi \sigma^\mu \sigma^\lambda \delta S/\delta \psi].$$ \tag{3.14}
where $\triangle_{\alpha \beta}$ is defined by $\triangle_{\psi, \psi}$ with $\psi \to \chi$. Note that one can write

$$eA^*A + D = \kappa + \delta S / \delta D,$$  \hspace{1cm} (3.15)

which implies that

$$T^{\mu \lambda}_{\text{asym}} = - \epsilon^{\mu \nu \rho} \left( \frac{\kappa}{2} f_{\nu \rho} + \frac{1}{4} \partial_{\nu} \Sigma_{\rho} \right)$$  \hspace{1cm} (3.16)

classically. In this turn shows that the dimensionally-reduced 3d model has nonzero central charge

$$\int d^2 x T_{\text{asym}}^{02} = \kappa \int d^2 x f_{13} = 2\pi n k / e$$  \hspace{1cm} (3.17)

in the background of a classical vortex. Note here that $\int d^4 x \Sigma^k = 0$ since $\Sigma^k \to 0$ rapidly as $x \to \infty$.

The supertrace is, equally applicable to the 3d and 4d cases, may be further reduced to a form suited for three dimensions. Writing $\bar{D}^\alpha R_{\alpha \theta} = (\sigma \mu) D_{\nu} R^\nu$ reveals that on reduction $(\sigma \mu) D_{\nu} R^\nu$ turns into a superconformal breaking term. Note that $R^\nu$ may be rewritten as

$$2R^\nu = (w - 1) \bar{D} \sigma^2 D (\bar{\Phi} e^{2V} \varphi) + \bar{D}_{\alpha} (\bar{W}^\alpha G) + G \delta S / \delta V,$$  \hspace{1cm} (3.18)

where $G = \bar{D} \sigma^2 D V$. One can thus write the 3d form of the supertrace as

$$(\sigma \nu) \bar{D}_{\nu} R^\nu = \mathcal{O}_1 + 4k W_\alpha + A_\alpha,$$  \hspace{1cm} (3.19)

with $\mathcal{O}_1 = \mathcal{O}_0 - (\sigma_2 D) R^2$ cast in the form

$$\mathcal{O}_1 = \frac{1}{4} (1 - 2w) \bar{D}^2 D_\alpha (\bar{\Phi} e^{2V} \varphi) + \frac{1}{4} \bar{D}^2 (\sigma_2 \bar{W})_{\alpha} G - \frac{1}{2} \sigma_2 \bar{D} \left( G \delta S / \delta V \right).$$  \hspace{1cm} (3.20)

For $w = 2/3, \mathcal{O}_0 = 0$ in Eq. (3.19) and the supercurrent with $w = 2/3$ contains the improved superconformal currents for four dimensions. Similarly, for $w = 1/2$ the first term in $\mathcal{O}_1$ above disappears and this implies that $R^\nu$ with $w = 1/2$ contains the improved currents appropriate for three dimensions. The second term in $\mathcal{O}_1$ derives from the super-Maxwell term $W^2$ which breaks 3d conformal invariance while the third term, as an anomaly candidate, may better be included in $A_\alpha$.

Here special remarks are in order as to the combination $G = \bar{D} \sigma^2 D V = -D \varphi^2 D V \sim -2a^2$, associated with the "reduced" dimension $x^2$. It forms a linear supermultiplet $(D^2 G = D^2 G = 0)$, is gauge invariant, and is odd under parity. Let us remember that parity in 1+2 dimensions corresponds to inversions about one of the spatial axes, $x^1$ or $x^3$. Under inversion $x^1 \to x^1 = -x^1$, e.g., spinors undergo the change $\theta_\alpha = (\sigma_1 \beta) \theta_\beta = (\sigma^3 \beta) \theta_\beta$, so that one has $\theta = - (\theta')^* \theta' \sigma^2 \theta' \theta' \sigma^2 \theta = - \theta' \sigma^2 \theta' \theta' \sigma^2 \theta$, etc. The component-field action (2.8) thereby tells us that $a_0$ and $D$ are even under parity while $a_2$ and $A$ are odd. Hence $V(z)$ is even under parity while $\Phi(x, \theta, \theta') = - \Phi(x', \theta', \theta')$ is odd, and the superfield action (2.9) is parity-invariant. [The parity-violating Chern-Simons term is contained in the combination $V D^2 D^2 V$. A direct one-loop calculation (though we omit the details) suggests that no Chern-Simons term is induced in the present 3d model.]

We have so far handled the single-$\Phi$ Higgs model. For its SQED generalization (2.2), one can define the supercurrent $\mathcal{O}_{\theta}$ by including the oppositely-charged partner $\Phi_-$ as well, with the conservation laws modified in an obvious fashion. The model still supports the same classical vortex solutions (3.13) governed by the Bogomolnyi equation (2.10) with $\Phi_- = 0$.

**IV. Trace Identities**

In this section we promote the supercurrent to the quantum level. Let us first note that $\mathcal{A}_0$ in the supertrace identity (3.17) consists of products of the form (fields)×(equations of motion). Such operator products vanish trivially at the tree level. They, however, are potentially singular and, when properly regulated, may not vanish at the quantum level, leading to anomalies, as is familiar from Fujikawa’s method (37, 38). Accordingly we examine $\mathcal{A}_0$ and determine the form of the supertrace identity at the quantum level; see Ref. (34) for an early study of the superconformal anomaly along this line.

For actual calculations we employ the superspace background-field method (34) and expand $V = V_c + U$ and $\Phi = \Phi_c + \eta$ around the classical fields $V_c$ and $\Phi_c$. The quantum fluctuations at the one-loop level, in particular, are then governed by the action quadratic in $(U, \eta, \eta)$.

$$S_q = \int d^3 z \left[ \bar{\eta} \Xi \eta + 2e(\bar{\Phi}_c \Xi \eta + \bar{\eta} \Xi \Phi_c) U + \frac{1}{2} U M U \right],$$

$$M = 2 (-p^2 + 2e \bar{\Phi}_c \Xi \Phi_c),$$  \hspace{1cm} (4.1)

with $\Xi = e^{2V_c}$. Here we have taken the superfield Feynman gauge by including the gauge-fixing term $- (1/8) (\bar{D}^2 U)(\bar{D}^2 U)$ in $S_q$: $- p^2$ in $M$ has been $(1/8) D^\alpha D^\beta D_{\alpha \beta}$ before this gauge fixing. The Feynman gauge is suited for formal reasoning because one can avoid inessential complications due to ghost fields (although $U \eta$ mixing leads to a set of Feynman rules not very efficient for practical calculations).

The original action $S$ as well as the one-loop action $S_q$ clearly has background gauge invariance

$$(\Phi_c, \eta) \to e^{-2i e \Lambda}(\Phi_c, \eta), (\bar{\Phi}_c, \eta) \to e^{2i e \Lambda}(\bar{\Phi}_c, \eta),$$

$$V_c \to V_c + i \Lambda - i \bar{\Lambda}, U \to U,$$  \hspace{1cm} (4.2)

with arbitrary chiral phases $\bar{D}_{\lambda} \Lambda = D_{\lambda} \bar{\Lambda} = 0$. They also have invariance under parity. The background gauge invariance and parity therefore are exact symmetries of the effective action calculated from $S$, although both eventually get spontaneously broken in the physical vacuum.
where $\langle \Phi \rangle_{\text{vac}} \neq 0$. We shall regularize the theory so that both of them are kept manifest for the effective action.

See Appendix A for regularization and Feynman rules. Actually the anomaly $\mathcal{A}_\alpha$ of Eq. (3.3) is somewhat modified via gauge fixing; see Appendix B for details. Its exact form, however, is not needed for our discussion below, which relies on the fact that the modified $\mathcal{A}_\alpha$ still preserves both background gauge invariance and parity; our analysis thus applies to any gauges as long as they respect such invariance.

The potentially anomalous product $\mathcal{A}_\alpha$, being of short-distance origin, should consist of local polynomials of $\Xi = e^{2\varphi} \phi$, $e\bar{\phi}(\Xi)$, and $e\Xi \Phi_c$, which are (i) background gauge invariant, (ii) neutral under $U(1)$, and are (iii) spinor superfields of dimension 5/2 or less in units of mass and of order $\hbar^0$ or higher. As for possible gauge-invariant combinations, one may think of, e.g., $e^2 \bar{\Phi}_c \Xi \Phi_c$ of dimension 2, $e(W_c)_\alpha \equiv -(e/4)\bar{D}^2 D_\alpha V_c$ of dimension 3/2, and $eD^2 D\bar{V}_c$ of dimension 1.

The structure of possible superconformal anomalies is not quite arbitrary and is constrained owing to boson-fermion balancing [22]. One can form chiral or real superfields out of quantum corrections. If the anomalies form a chiral supermultiplet, their contribution to the supertrace identity $\bar{D}^3 \mathcal{R}_{\alpha\dot{\alpha}}$ is of the form $\mathcal{A}_\alpha \sim D_\alpha$(chiral superfield), as seen explicitly from the $D_\alpha(\Phi \delta S/\delta \Phi)$ term in Eq. (3.3). This possibility, however, is excluded in the present 3d case (where $\bar{D}^3 \mathcal{R}_{\alpha\dot{\alpha}}$ has dimension 5/2). This is simply because there is no candidate of (scalar) chiral superfields of dimension 2 or less. (Note that $e^2 W_c W_c$ and $e^2 D^2(\Phi_c \Xi \Phi_c)$, e.g., have dimension 3.)

If, on the other hand, the anomalies are associated with a real superfield, they form a linear supermultiplet of the form $\bar{D}^2 D_\alpha$(real superfield) in the 4d case [22]. In the dimensionally-reduced 3d case, there is one more possibility: Anomalies of the form $(\sigma_2 D_\alpha) X$ are possible, as seen from Eq. (3.19), but such a superfield $X$ must be odd under parity just like $\mathcal{R}^2$. The simplest choice $X \sim V_c$ therefore is excluded. Instead, a possible candidate of least dimension is $(\sigma_2 D_\alpha) D_\alpha \bar{D}^2 D$(real superfield), which is rewritten as $-(1/2)\bar{D}^2 D_\alpha$(real superfield), thus reducing to the anomaly multiplet of the 4d case.

The search for a possible linear anomaly multiplet is simplified if one notes that $e^2 D^2 D_\alpha(\Phi_c \Xi \Phi_c)$ has dimension 7/2 so that it does not arise as a short-distance anomaly. This implies that $\mathcal{A}_\alpha$ is formed of $V_c$ alone. Then there are two candidates, $e(W_c)_\alpha$ of dimension 3/2 and $e\bar{D}^2 D_\alpha(D_\alpha \bar{D}^2 D\bar{V}_c)$ of dimension 5/2; of these the latter is excluded because of a parity mismatch. Therefore the only possibility is

$$\mathcal{A}_\alpha = c e W_\alpha \tag{4.3}$$

in operator form, with a coefficient $c = c_1 + e^2 c_2$ necessarily divergent at one loop and terminating at two-loop order. Indeed, a direct one-loop calculation yields a linearly-divergent coefficient $c_1 = 1/(2\pi \sqrt{\pi t})$ (with a UV cutoff $\tau \to 0_+$), which derives from

$$\langle \nabla_\alpha \Phi \rangle \frac{\delta S}{\delta \Phi} = \frac{1}{2} c_1 e W_\alpha; \tag{4.4}$$

see Appendix C for details.

Fortunately an anomaly of the form (1.3) is harmless and is effectively removed from the supertrace identity (3.3) by redefining the bare F1 coupling $\kappa \to \kappa' = \kappa - ec/4$, without causing any change in observables. In this sense there is no genuine anomaly, $\mathcal{A}_\alpha = 0$ to all orders in $\hbar$.

Actually, the situation is much clearer for the SQED$_3$ version of the model. There the matter fields ($\Phi, \bar{\Phi}$) are oppositely charged and they give a vanishing contribution to the anomaly, $\mathcal{A}_\alpha = 0$. The supertrace identity thus takes the naive form for both models,

$$\bar{D} \mathcal{R}_{\alpha\dot{\alpha}} = \mathcal{O}_\alpha + 4\kappa W_\alpha, \tag{4.5}$$

along with its conjugate $D^\alpha \mathcal{R}_{\alpha\dot{\alpha}} = \bar{\mathcal{O}}_\alpha + 4\kappa \bar{W}_\alpha$. Note here that $\mathcal{O}_\alpha$ may effectively be absorbed into $W_\alpha = -1/4 \bar{D}^2 D_\alpha V$ on the right-hand side by the replacement

$$V \to V + \gamma \bar{\Phi}_c e^{2\varphi} \Phi, \tag{4.6}$$

with $\gamma = (1/4\kappa)(3w - 2)$.

The supertrace (1.3) implies that $\bar{D}^3 \mathcal{R}_{\alpha\dot{\alpha}}$ is a chiral (spinor) superfield. This property fixes the higher components in terms of $R^\alpha$ and $J^\mu$ uniquely,

$$\mathcal{R}^\mu = R^\mu - i\theta \bar{J}^\mu + i\bar{\theta} J^\mu - 2\Theta_{\alpha} \Theta^T \kappa^\mu \lambda \left(1 + \frac{1}{2} \theta^2 \bar{\theta} \bar{J}^\mu - \frac{1}{2} \bar{\theta}^2 \theta J^\mu + \frac{1}{4} \theta^2 \bar{\theta}^2 \bar{\theta} J^\mu \right), \tag{4.7}$$

and at the same time yields the conservation law

$$\partial_\lambda T^{\mu\lambda} = 0. \tag{4.8}$$

Equation (4.5) accommodates superpartners of the trace identity, which, for $w = 2/3$, read (in 4d notation)

$$i(\sigma_\mu \bar{J}^\mu)_\alpha = 4\kappa \chi_\alpha; \tag{4.9}$$

$$T^\mu = 2\kappa D, \tag{4.10}$$

$$T_{\mu}^{\alpha\lambda} = -\epsilon^{\mu\nu\rho} \kappa^\lambda \left[\frac{1}{2} f_{\nu\rho} + \frac{1}{4} \partial_\nu R_\rho\right]. \tag{4.11}$$

where we have used the formula $\epsilon^{\mu\nu\rho\sigma} \sigma_{\rho\sigma} = 2\Theta_{\omega} \Theta^T \kappa$. The effect of $\Theta_{\omega}$ is readily recovered if one notes Eq. (4.6): The $\Theta_{\omega} \Theta^T \kappa$ component of $\Phi e^{2\varphi} \Phi$, in particular, is nothing but the gauge current

$$j^{\text{gauge}}_\mu = \bar{\psi} \sigma_\mu \psi + i A^* \bar{A}_\mu A, \tag{4.12}$$

and, on setting $a_\mu \to a_\mu - \gamma j^{\text{gauge}}_\mu$ in $f_{\mu\nu}$ of Eq. (4.11), $T_{\mu}^{\alpha\lambda}$ precisely agrees with the classical expression (4.10).

Letting $D^\mu$ act on Eq. (4.11) reveals that the supercurrent $R^\mu$ is conserved at the quantum level

$$\partial_\mu R^\mu = 0. \tag{4.13}$$
This leads to a conserved-charge superfield (in obvious 3d notation)

\[
\int d^2x R^0 = Q_R - i\theta Q + i\bar{\theta}Q - 2\theta\sigma_{\mu\nu} \partial \mu P^\nu - 2\theta\sigma_{\lambda\delta} \partial \lambda Z,
\]

(4.14)

consisting of the R charge \(Q_R\), supercharges \((Q_\alpha, \bar{Q}_{\bar{\alpha}})\), three-momenta \(P^\nu = \int d^2x T^\nu_{asym}\) and central charge \(Z = \int d^2x T^{02}_{asym}\), with other charges vanishing. This charge superfield, upon supertranslations \([2,3]\), gives rise to the supercharge algebra

\[
\{Q_\alpha, \bar{Q}_{\bar{\alpha}}\} = 2(\sigma^\nu)_\alpha\bar{\alpha} P^\nu + 2(\sigma_2)_{\alpha\bar{\alpha}} Z,
\]

(4.15)

and \(\{Q_\alpha, Q_\beta\} = 0\). Note here that, as seen from Eq. (4.11), \(\int d^2x T^0_{asym} = 0\) for \(k = 1, 3\) [since \(a_2 \rightarrow 0\) and \(R_2 \rightarrow 0\) for \(x \rightarrow \infty\)] while the central charge \(Z = \int d^2x T^{02}_{asym}\) is sensitive to the asymptotic nontrivial winding of the vortex field \(A(x) \sim ue^{i\theta}\).

\[
Z = \kappa \int dx_k a^k = \kappa \int d^2x f_{13}.
\]

(4.16)

As noted earlier, \(\int dx_k \Sigma^k = 0\) for \(|x| \rightarrow \infty\) and \(\Sigma_\rho\) does not contribute to \(Z\). There is a general way to conclude this. One may imagine piercing the vacuum state with a vortex adiabatically. The flux is thereby detected by \(\int dx_k a^k\). In contrast, \(\Sigma_\rho\), being gauge-invariant, is single-valued and \(\int dx_k \Sigma^k\) stays vanishing.

The unique feature of the FI term \(2\kappa V\) is that it is gauge variant by itself, and, in view of Eq. (4.10), it is precisely this gauge-variant portion of \(V\) that detects the vortex flux. This explains why the central charge \(Z\) derives from the bare FI term alone in Eq. (4.16).

In the present 3d models gauge invariance is kept exact (though spontaneously broken). This is seen from

\[
\Phi \delta S/\delta \Phi = - (1/4) \bar{D}^2(\bar{\Phi}e^{2\kappa V}\Phi) = 0,
\]

(4.17)

which implies exact conservation of the gauge current, \(\partial_{\mu\nu}^\text{gauge} = 0\), and which follows from the absence of (gauge-invariant) chiral superfields of dimension two. In this connection, we remark that \(\Phi \delta S/\delta \Phi = 0\) implies \(\Delta_1 + \Delta_2 = 0\) in Eq. (3.12) while Eq. (4.14) implies \(\Delta^{\lambda\alpha}_{\nu\bar{\nu}} = \frac{1}{8} c_1 e^{\kappa \lambda\nu \rho} f_{\nu\rho\mu}\) in Eq. (3.19).

V. RENORMALIZATION

In Eq. (4.16) we have seen that there is no genuine quantum anomaly in the central charge \(Z = \kappa \int dx_k a^k\). It takes the same form as the classical expression \(\Delta^{\lambda\alpha}_{\nu\bar{\nu}}\).

The key difference is that it is promoted to an operator, and one now has to determine the physical central charge from the expectation value \(\langle Z \rangle = \langle \text{vor} | Z | \text{vor} \rangle\). In this section we discuss how the central charge undergoes renormalization through this process.

As for renormalization, power counting reveals that only the FI coupling \(\kappa\) requires infinite renormalization for the single-\(\Phi\) Higgs model. In addition, a nonrenormalization theorem \([10]\) tells us that the quantum correction to the FI term arises only at the one-loop level. It is illuminating to calculate such a quantum correction over the vacuum (with spontaneous parity breaking \(\langle \Phi \rangle = v\), using the Feynman rules given in Appendix A. Combining \(e^{2V_c}(\bar{\eta}\eta), 2e^2v^2e^{2V_c}(UU)\) and \(2euv^{2kV_c}((U\eta)+(U\bar{\eta}))\) one obtains the one-loop correction to the FI term \(-2\kappa V_c\).

\[
2e(x)\int \frac{i e^\tau r^2}{p^2 - m^2} |x| V_c = \frac{e}{2\pi} \left[ \frac{1}{\sqrt{\pi \tau}} - m \right] V_c,
\]

(5.1)

where \(\tau\) is an UV cutoff and \(m = \sqrt{2}e v\). The divergent term \(\propto 1/\sqrt{\pi \tau}\) can be removed on setting \(\kappa = \kappa_{\text{ren}} + \delta\kappa\) and choosing the counterterm \(\delta\kappa = \epsilon/(4\pi \sqrt{\pi \tau})\), with the renormalized FI coupling \(\kappa_{\text{ren}}\).

In the SQED\(_3\) version of the model the divergent term is precisely canceled by an analogous contribution from the (massless) partner \(\Phi_-\) and one is left only with the finite correction \(-\langle ev \rangle/2\pi\) \(V_c\). This SQED\(_3\) model therefore is a finite theory.

One might be tempted to regard the \(-\langle ev \rangle/2\pi\) \(V_c\) term in both models as contributing to finite renormalization of the FI term, which, however, is not quite right. The meaning of this term becomes clear if one notes that Eq. (5.3) should come from a gauge-invariant one-loop effective action of the form

\[
\Gamma^{(1)} = \int d^7z \left[ \frac{e}{2\pi} \frac{1}{\sqrt{\pi \tau}} V_c - \frac{1}{2\pi} \sqrt{2e^2\Phi_+V_c e^{2V_c}} \right],
\]

(5.2)

where \(d^7z = d^3x d^2\theta d\bar{\theta}\). Indeed, a direct calculation yields precisely this form of the action if one retains only terms with no \(D_\alpha\) or \(\bar{D}_{\bar{\alpha}}\) acting on \((\Phi_+, V_c)\), i.e., the leading long-wavelength components; see Appendix A. The first term of Eq. (5.2) acts to renormalize the FI coupling while the second term, being gauge-invariant [just like \(\Sigma_\rho\) of Eq. (3.16)], does not contribute to the central charge \(Z\). It is therefore not legitimate to treat this second term or the \(-\langle ev \rangle/2\pi\) \(V_c\) term as finite renormalization of the FI term. It derives from long-wavelength quantum fluctuations and rather contributes to a finite shift of energy: Minimizing the effective action with respect to \(V_c\) within the asymptotic (vacuum) region yields

\[
\kappa_{\text{ren}} = e\{v^2 - m^2/(4\pi)\},
\]

(5.3)

which relates \(v = \langle \Phi \rangle_{\text{vac}}\) to the central charge \(\propto \kappa_{\text{ren}}\).

Renormalization of the FI term takes place only at the one-loop level. This means, in particular, that the one-loop results \(\kappa = \kappa_{\text{ren}} + \delta\kappa\) with \(\delta\kappa = \epsilon/(4\pi \sqrt{\pi \tau})\) for the single-\(\Phi\) model and \(\kappa = \kappa_{\text{ren}}\) for the SQED\(_3\) version are actually exact. Thus the bare FI term is a finite operator under renormalization, and the Heisenberg operator \(\kappa V\) equals \(\kappa_{\text{ren}}V_{\text{in}}\) in the interaction picture. As a result, the central charge, upon taking the expectation value \(\langle Z \rangle = \langle \text{vor} | Z | \text{vor} \rangle\), undergoes renormalization

\[
\langle Z \rangle = \kappa_{\text{ren}} \int dx_k \langle \phi_{\text{in}}^k \rangle = 2\pi n \kappa_{\text{ren}}/e.
\]

(5.4)
This, together with $\kappa_{\text{ren}}$ of Eq. (5.3), confirms an earlier result and reveals that this apparently one-loop result is actually exact to all orders in perturbation theory. Remember, however, that Eq. (5.3) which relates $v = \langle \Phi \rangle_{\text{vac}}$ to $\kappa_{\text{ren}}$, in general, has higher-order corrections [in powers of $e^3/\kappa_{\text{ren}} \sim O(\hbar)$].

Having determined the renormalized central charge, let us finally look into some consequence of the supercharge algebra (5.14). In terms of $Q_{\pm} = (Q_1 \pm iQ_2)/2$ and $Q_{\pm}^1 = (Q_1 \mp iQ_2)/2$, it reads

$$\{Q_{\pm}, Q_{\pm}^1\} = P^0 \pm Z, \quad \{Q_-, Q_-^1\} = -P^3 + iP^1. \quad (5.5)$$

The static vortex solution (2.10) is invariant under supertranslations caused by $Q_-$, realizing BPS saturation $Q_-|\text{vor}\rangle = 0$ classically. The saturation persists at the quantum level. This is because the vortex belongs to a short multiplet, realizing only half of the original $N = 2$ supersymmetry. Then $Q_-|\text{vor}\rangle = 0$ implies that the vortex spectrum is determined by the central charge or $\kappa_{\text{ren}}$ exactly,

$$\langle \text{vor}|P^0|\text{vor}\rangle = \langle \text{vor}|Z|\text{vor}\rangle = 2\pi n\kappa_{\text{ren}}/e. \quad (5.6)$$

Remember here that formal reasoning based on the algebra is justified since we have verified that the supercharge algebra has no quantum anomaly. In contrast to the exact vortex spectrum, the spectra $m \sim \sqrt{2\kappa e} \sim \sqrt{2\kappa_{\text{ren}}}$ of the elementary excitations $\eta$ and $U$, degenerate owing to supersymmetry, are determined only approximately through $v$, which, as in Eq. (5.3), is affected by higher-order long-wavelength fluctuations.

VI. SUMMARY AND DISCUSSION

In this paper we have studied some quantum aspects of vortices in the supersymmetric Higgs model in 1+2 dimensions and its SQED$_3$ generalization, by a close examination of the supercurrent and associated conservation laws within the superfield formalism. The central-charge operator $Z = \kappa \oint dx^6 d^6k \mp$ turns out to derive from the bare FI term alone and makes the vortex flux. It acquires no (genuine) quantum anomaly but undergoes renormalization so that the observable central charge, given by the expectation value $\langle \text{vor}|Z|\text{vor}\rangle$, is an integral multiple of the renormalized FI coupling $\times$ flux quantum, $n\kappa_{\text{ren}}2\pi/e$, which, via BPS saturation, equals the vortex spectrum. Renormalization of the FI coupling $\kappa$ takes place only at the one-loop level and the vortex spectrum $n\kappa_{\text{ren}}2\pi/e$ is exactly known. In contrast, the spectrum $m \sim \sqrt{2\kappa e}$ of the elementary excitations $\eta$ and $U$ receives corrections from higher-order quantum fluctuations.

Our analysis makes heavy use of the superfield supercurrent $R^\mu$, but the choice of such a conserved supercurrent is not unique (because of the simultaneous presence of gauge invariance and R-symmetry). We have considered a family of supercurrents parameterized by the R-weight $w$ of the matter superfield, and noted that it leads to a supercharge algebra of the same form. This $w$ represents the freedom to improve the supercurrent. The improved superconformal currents thereby emerge as components precisely when one assigns the canonical R-weight, i.e., $w = 1/2$ in 1+2 dimensions and $w = 2/3$ in 1+3 dimensions.

Finally, we wish to emphasize that use of superfields (+ supersymmetric regularization) allows one not only to keep supersymmetry manifest in actual calculations but also to substantiate formal reasoning based on supersymmetry. With such firm control on supersymmetry, in particular, persistence of expectation values at the quantum level is a concrete consequence of the supercharge algebra once multiplet shortening for BPS-saturated excitations is concluded algebraically. The excitation spectrum is thereby determined from the central-charge operator, with the effect of renormalization entering on taking the expectation value. The superfield formalism can thus neatly avoid some subtle problems encountered in component-field calculations.

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APPENDIX A: CALCULATIONS

In this appendix we outline some one-loop calculations in the text, based on the superfield action (3.11). Let us first consider the simplified action (in 4d notation)

$$S_q = \int d^8z \bar{\eta} \Xi \eta + \int d^6z \eta j + \int d^6\bar{z} \bar{\eta} \bar{j} \quad \text{(A1)}$$

with $\Xi = e^{2eV_c}$ and source terms $j(z)$ and $\bar{j}(z)$. We solve the equation of motion $\not{\partial} \Xi \eta + j = 0$ in such a way that background gauge invariance is kept manifest:

$$\eta = -D^{-1} 1_+ \Xi^{-1} \bar{j}, \quad \text{(A2)}$$

with $D = 1_- \Xi^{-1} \Xi_+\Xi_-, 1_+ = -\frac{1}{4} D^2$ and $1_- = -\frac{1}{4} \bar{D}^2$. Note here that $D$ acting on $1_-$ is invertible so that

$$D^{-1} 1_- = [\Pi^2 - eW_c^\alpha \nabla_\alpha - (e/2)(DW_c)]^{-1} 1_-, \quad \text{(A3)}$$

with $\Pi_\mu = p_\mu - e\alpha_\mu$. Substituting Eq. (A2) back into $S_q$ yields a stationary action of the form $i \int d^6z d^6z' \langle \eta \rangle_0 \langle \bar{j} \rangle_0 \bar{j}(z')$, from which one can read off the propagator

$$\langle \eta(z) \bar{\eta}(z') \rangle^{(0)} = i |z| \frac{e^{D}}{D} 1_+ \Xi^{-1} 1_+ |z'|, \quad \text{(A4)}$$

where for regularization we have introduced an UV cutoff $e^{D}$ with $\tau \to 0_+$ which respects background gauge invariance, parity and supersymmetry.
Let us now include the remaining terms of Eq. (4.1). The effect of $\eta U$ coupling is readily taken care of by the replacement $j \rightarrow j + 2e\Phi_\tau \Xi U$ and $\bar{j} \rightarrow \bar{j} + 2e\Phi_\tau \Xi U$. This yields a $U^2$ term, which modifies the kinetic term $M$ of $U$. Minimizing the action with respect to $U$ then yields a set of relevant propagators. We omit the details here and simply remark that the $\eta$ propagator thereby acquires corrections, $\langle \eta \bar{\eta} \rangle = \langle \eta \bar{\eta} \rangle^{(0)} + O(\Phi_\tau \Phi_\tau)$.

In Eq. (5.4) we make a one-loop calculation over the ordinary vacuum (with $\langle \Phi \rangle = v$), based on the zeroth-order action

$$S_q = \int d^8z \left[ \bar{\eta} \eta + 2ev(\eta + \bar{\eta})U + \frac{1}{2} UMU \right]$$

with $M = 2(-p^2 + m^2)$ and $m^2 = 2v^2$. In this case, the $O(U^2)$ term mentioned above modifies the kinetic term $M$ into $M' = M - 2m^2(1,1) / p^2 = -2(p^2 + (m^2/p^2)Y)$ with $Y = (1/8)DD^2D$, where we have used the identity $(1,1) = p^2 + Y$. One can readily invert $M'$ by noting $Y^2 = -p^2Y$ and $Y_{1+} = Y_{1-} = 0$. Here we quote the Feynman rules constructed in this way:

$$\langle \eta \bar{\eta} \rangle = \langle z \frac{i}{p^2} (1 - \frac{m^2}{p^2})1_{1+}|z'\rangle,$$

$$\langle UU \rangle = -\langle z \frac{i}{2p^2} \left[ 1 - \frac{m^2}{p^2(p^2 - m^2)} \right] |z'\rangle,$$

$$\langle U\eta \rangle = ev\langle z | \frac{i}{p^2} \left[ 1_{1+} |z'\rangle,$$

$$\langle U\bar{\eta} \rangle = ev\langle z | \frac{i}{p^2} \left[ 1_{1-} |z'\rangle.$$

In actual calculations one handles $\theta$-diagonal elements, which can be read off by using $\langle \theta, \bar{\theta}|1_{1+} |\theta, \bar{\theta}\rangle = 1$ and $\langle \theta, \bar{\theta}|Y|\theta, \bar{\theta}\rangle = 2$.

In Eq. (6.2), we construct the one-loop effective action. There we retain only terms with no $D_\alpha$ and $\bar{D}_\alpha$ acting on $(\Phi_\tau, V_c)$. Let us first consider the contribution from the $\eta \bar{\eta}$ loop, which is best calculated by first differentiating it with respect to $V_c$. Evaluating $2e \int d^8z \langle \eta \bar{\eta} \rangle^{(0)}$ using Eq. (A3), and functionally-integrating back over $V_c$ then yields, in our approximation,

$$2e\langle x \rangle |e^{\tau p^2 / p^2}|x\rangle V_c.$$  \hfill (A7)

This gives the UV divergent term in Eq. (5.2).

The effective action coming from the $U$ loop is written as $(i/2) \int d^8z \langle |M'|^2 \rangle$. In the present approximation, $M' = -2(p^2 + (\mu^2/p^2)Y)$ with $\mu^2 = 2e^2\Phi_\tau \Xi \Phi_\tau$. One can calculate $(i/2)\langle |M'|^2 \rangle$ by first differentiating with respect to $\mu^2$, with the result

$$\langle x | i \{ p^2(p^2 - \mu^2) \} | x \rangle.$$

This is convergent in $(1+2)$ dimensions and, upon integrating over $\mu^2$, yields $-|\mu|/(2\pi)$, which gives the second term in Eq. (5.2).

**APPENDIX B: GAUGE FIXING**

In this appendix we show, for completeness, that the supertrace identity (25) is promoted to a background gauge invariant form via gauge fixing. The gauge-fixing term $S_{gf} = \int d^8z (-\frac{1}{4}) (D^2U)D^2U$ leads to an additional contribution to the supercurrent $R_{\alpha\bar{\alpha}}$ of Eq. (3.7), $R_{\alpha\bar{\alpha}}^{gf} = (1/24)[-\langle D_\alpha \bar{D}_\bar{\alpha} U \rangle D_\bar{\alpha}D^2U + \cdots]$; see Ref. 22 for the explicit form which is rather involved. Its supertrace is written as $\bar{D}^\alpha R_{\alpha\bar{\alpha}}^{gf} = C_{\alpha\bar{\alpha}} + A_{\alpha\bar{\alpha}}$ with

$$C_{\alpha\bar{\alpha}} = -(1/48)D_\alpha D_2U[D^2U, \bar{D}^2U],$$

$$A_{\alpha\bar{\alpha}} = 2W_\alpha U S_{gf}/\delta U - D_\alpha U p^2 \bar{D}^2U,$$  \hfill (B1)

where $W_\alpha U = -\frac{1}{4} \bar{D}^2D_\alpha U$. Here $C_{\alpha\bar{\alpha}}$ is an explicit conformal breaking term (which, on replacing $D_\alpha \bar{D}^2D_\alpha$, may partly be absorbed into $R_{\alpha\bar{\alpha}}^{gf}$). For the full supercurrent $R_{\alpha\bar{\alpha}}^{full} = R_{\alpha\bar{\alpha}} + R_{\alpha\bar{\alpha}}^{gf}$ the anomaly $A_{\alpha\bar{\alpha}}^{full}$ is cast in a neat form

$$A_{\alpha\bar{\alpha}}^{full} = 2 (W_\alpha)_{\alpha} \delta S' / \delta c + 2W_\alpha U \delta S'/\delta U + \frac{1}{2} (D_\alpha U) \bar{D}^2 \delta S'/\delta U + \cdots.$$  \hfill (B2)

where the omitted terms are the same as those in Eq. (3.7), except that $\nabla V[T]$ there is replaced by the background covariant derivative $\nabla_\alpha V[T]$; $S' = S[V] + S_{gf}[U]$ stands for the full action. We note that $A_{\alpha\bar{\alpha}}^{full}$ as well as the supercurrent $R_{\alpha\bar{\alpha}}^{full}$ preserves background gauge invariance and parity, which is crucial for determining its quantum form in Sec. IV.

**APPENDIX C: ANOMALY**

In this appendix we verify the discussion of Sec. IV by a direct one-loop calculation of the anomaly $A_{\alpha\bar{\alpha}}$ in Eq. (3.7). As noted in Sec. IV, one may set $\Phi_\tau, \Phi_\tau \to 0$ in calculating $A_{\alpha\bar{\alpha}}$. The $U$ sector in the one-loop action $S_q$ thereby becomes a free field and does not contribute to the quantum anomaly at one loop. Thus $W_\alpha \delta S / \delta V = 0$ in Eq. (3.7). Similarly, we conclude that $G(\delta S / \delta V) = 0$. In Eq. (3.8).

The possible one-loop anomaly derives from the matter $(\eta, \bar{\eta})$ sector alone. Let us first consider $(\nabla_\alpha \Phi)(\delta S / \delta \Phi)$, which, using Eq. (A4), is cast in the form

$$i \langle z | D_\alpha e^{\tau p^2 1_{-}} | z \rangle \approx -i (W_\alpha)_{\alpha} \tau |e^{\tau p^2}|x\rangle,$$  \hfill (C1)

for $\tau \to 0_+$, where one reaches the right-hand side by expanding the heat kernel in powers of $W_\alpha \nabla V_c$ and on taking the $\theta$-diagonal matrix element. This leads to Eq. (4.4). Similarly, one can verify $\Phi(\delta S / \delta \Phi) = 0$ by a direct calculation of $i \langle z | e^{\tau p^2 1_{-}} | z \rangle$.  

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