Mixed-state entanglement and quantum communication

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We present basics of mixed-state entanglement theory. The first part of the article is devoted to mathematical characterizations of entangled states. In second part we discuss the question of using mixed-state entanglement for quantum communication. In particular, a type of entanglement that is not directly useful for quantum communication (called bound entanglement) is analysed in detail.

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5 Volumes of entangled and separable states
Quantum entanglement is one of the most striking features of quantum formalism [1]. It can be expressed as follows: If two systems interacted in the past it is, in general, not possible to assign a single state vector to either of the two subsystems [2]. This is what is sometimes called the principle of non-separability. A common example of entangled state is the singlet state \( \psi_- = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \). (1)
One can see that it cannot be represented as a product of individual vectors describing states of subsystems. Historically, entanglement was first recognised by Einstein, Podolsky and Rosen (EPR) and by Schrödinger. In their famous paper EPR suggested a description of the world (called “local realism”) which assigns an independent and objective reality to the physical properties of the well separated subsystems of a compound system. Then EPR applied the criterion of local realism to predictions associated with an entangled state to conclude that quantum mechanics is incomplete. EPR criticism was the source of many discussions concerning fundamental differences between quantum and classical description of nature.

The most significant progress toward the resolution of the EPR problem was made by Bell who proved that the local realism implies constraints on the predictions of spin correlations in the form of inequalities (called Bell’s inequalities) which can be violated by quantum mechanical predictions for the system in the state. The latter feature of quantum mechanics called usually nonlocality is one of the most apparent manifestations of quantum entanglement.

Information theoretic aspect of entanglement was first considered by Schrödinger who wrote in the context of the EPR problem: “Thus one disposes provisionally (until the entanglement is resolved by actual observation) of only a common description of the two in that space of higher dimension. This is the reason that knowledge of the individual systems can decline to the scantiest, even to zero, while that of the combined system remains continually maximal. Best possible knowledge of a whole does not include best possible knowledge of its parts – and that is what keeps coming back to haunt us”. In this way Schrödinger recognised a profoundly non-classical relation between the information which an entangled state gives us about the whole system and the information which it gives us about the subsystems.

The recent development of quantum information theory showed that entanglement can have important practical applications (see e.g. ). In particular it turned out that entanglement can be used as a resource for communication of quantum states in astonishing process called quantum teleportation. In the latter a quantum state is transmitted by use of a pair of particles in singlet state shared by the sender and receiver (typically Alice and Bob), and two bits of classical communication. However, in real conditions, due to interaction with environment, called decoherence, we encounter mixed states rather than pure ones. They can still possess some residual entanglement. More specifically, a mixed state is considered to be entangled if it is not a mixture of product states. In mixed states the quantum correlations are weakened, hence the manifestations of mixed-state entanglement can be very subtle. Nevertheless, it appears that it can be used as a resource for quantum communication. Such possibility is due to discovery of distillation of entanglement: by manipulation over noisy pairs, involving local operations and classical communication, Alice and Bob can obtain singlet pairs, and apply teleportation. This procedure provides a powerful protection of the quantum data transmission against environment.

Consequently, the fundamental problem was to investigate the structure of mixed-state entanglement, especially in the context of quantum communication. These investigations have lead to discovery of discontinuity in the structure of mixed-state entanglement. It appeared that there are at least two qualitatively different types of entanglement: free – useful for quantum communication, and bound - a non-distillable, very weak and mysterious type of entanglement.

The present contribution is divided into two main parts. In the first one we report results of investigation of mathematical structure of entanglement. The main question is: given a mixed state, is it entangled or not? We present powerful tools that allow to obtain the answer in many interesting cases. Crucial role is here played by the connection between entanglement and theory of

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1 In fact, entangled quantum states have been used in investigations of the properties of atomic and molecular systems.
positive maps [13]. In contrast to completely positive maps [10], the positive maps were not applied in physics so far. The second part is devoted to application of the entanglement of mixed states to quantum communication. Now, the leading question is: given entangled state, can it be distilled? The mathematical tools worked out in the first part allowed to answer the question. Surprisingly, the answer did not simplify the picture, but rather revealed a new horizon including the basic question: what is the role of bound entanglement in Nature?

Since entanglement is a basic ingredient of the quantum information theory, the scope of application of the presented research goes far beyond the quantum communication problem. The insight into structure of entanglement of mixed states can be helpful in many subfields of quantum information theory, including quantum computing, quantum cryptography etc.

Finally, it must be emphasised that our approach will be basically qualitative. Thus we will not review here the beautiful work performed in the domain of quantifying entanglement [17, 18, 19, 20, 21] (we will only touch this subject in the second part). Due to limited volume of the present contribution, we will also restrict considerations to entanglement of bipartite systems, even though a number of results has been recently obtained for multipartite systems (see e.g. 22, 23).

Part II
Entanglement of mixed states: characterisation

We will deal with the states on the finite dimensional Hilbert space \( \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \). A system described by Hilbert space \( \mathcal{H}_{AB} \) we will call \( n \otimes m \) system, where \( n \) and \( m \) are dimensions of the spaces \( \mathcal{H}_A \) and \( \mathcal{H}_B \) respectively. An operator \( \varrho \) acting on \( \mathcal{H} \) is a state if \( \text{Tr} \varrho = 1 \) and if it is a positive operator i.e.

\[
\text{Tr} \varrho P \geq 0 \tag{2}
\]

for any projectors \( P \) (equivalently, positivity of operator means that it is Hermitian and has non-negative eigenvalues).

A state acting on Hilbert space \( \mathcal{H}_{AB} \) is called separable if it can be approximated in the trace norm by the states of the form

\[
\varrho = \sum_{i=1}^{k} p_i \varrho_i \otimes \tilde{\varrho}_i , \tag{3}
\]

where \( \varrho_i \) and \( \tilde{\varrho}_i \) are states on \( \mathcal{H}_A \) and \( \mathcal{H}_B \) respectively. In finite dimensions one can use simpler definition [22] (see also [14]): \( \varrho \) is separable if it is of the form (3) for some \( k \) (one can always find \( k \leq \dim \mathcal{H}_{AB} \)). Note that the property of being entangled or not does not change if one subjects the state to a product unitary transformation \( \varrho \rightarrow \varrho' = U_1 \otimes U_2 \varrho U_1^\dagger \otimes U_2^\dagger \). The states \( \varrho \) and \( \varrho' \) we call equivalent.

We will further need the following maximally entangled pure state of \( d \otimes d \) system

\[
\psi_+^d = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle \otimes |i\rangle . \tag{4}
\]

The corresponding projector we will denote by \( P_+^d \) (the superscript \( d \) will be usually omitted). Then

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2The presented definition of separable states is due to Werner [10] who called them classically correlated states.
for any state $\rho$ the quantity $F = \langle \psi_+ | \rho | \psi_+ \rangle$ is called singlet fraction. In general, by maximally entangled states we will mean vectors $\psi$ that are equivalent to $\psi_+$

$$\psi = U_1 \otimes U_2 \psi_+$$

where $U_1, U_2$ are unitary transformations. The most common two-qubit maximally entangled state is the singlet state $|1\rangle$. One can define fully entangled fraction of a state $\rho$ of $d \otimes d$ system by

$$\mathcal{F}(\rho) = \max_{\psi} \langle \psi | \rho | \psi \rangle,$$

where the maximum is taken over all maximally entangled vectors of $d \otimes d$ system.

1 Pure states

If $\rho$ is a pure state i.e. $\rho = |\psi\rangle \langle \psi|$, then it is easy to check if it is entangled or not. Indeed, the above definition implies that it is separable if and only if $\psi = \psi_A \otimes \psi_B$, i.e. if either of its reduced density matrices is pure state. Thus it suffices to find eigenvalues of either of the reductions. Equivalently one can refer to Schmidt decomposition of the state. As one knows, for any pure state $\psi$ there exist bases \{e_A^i\}, \{e_B^i\} in spaces $H_A$ and $H_B$ such that

$$\psi = \sum_{i=1}^{k} a_i |e_A^i\rangle \otimes |e_B^i\rangle, \quad k \leq \dim H_{AB}$$

with positive coefficients $a_i$ are called Schmidt coefficients. Then the state is entangled if at least two coefficients do not vanish. One finds that the positive eigenvalues of either of the reductions are equal to squares of the Schmidt coefficients. In next section we will introduce a series of necessary conditions for separability for mixed states. It turns out that all of them are equivalent to separability in the case of pure states.

2 Some necessary conditions for separability of mixed states

A condition that is satisfied by separable states will be called separability criterion. If a separability criterion is violated by state, the state must be entangled. It is important to have strong separability criteria, i.e. the ones that are violated by possibly the largest number of states.

Since violation of Bell inequalities is a manifestation of quantum entanglement, a natural separability criterion is constituted by Bell inequalities. In Werner first pointed out that separable states must satisfy all possible Bell inequalities. The common Bell inequalities derived by Clauser, Horne, Shimony and Holt (CHSH) are given by

$$\text{Tr} \rho B \leq 2,$$

where the Bell-CCHSH observable $B$ is given by

$$B = \tilde{a} \tilde{a} \tilde{\sigma} \otimes (\tilde{b} + \tilde{b}') \tilde{\sigma} + \tilde{a} \tilde{b} \tilde{\sigma} \otimes (\tilde{b} - \tilde{b}') \tilde{\sigma}$$

\[\text{In fact, the state } \psi_+ \text{ used in definition of singlet fraction is a local transformation of true singlet state. Nevertheless, we will keep the name “singlet fraction” while using the state } \psi_+ \text{ being more convenient from technical reasons.}\]

\[\text{In [10] Werner also provided a very useful criterion based on so-called flip operator (see Sect. [4]).}\]
where $\vec{a}, \vec{a}', \vec{b}, \vec{b}'$ are arbitrary unit vectors in $\mathbb{R}^3$, $\vec{a}\sigma = \sum_{i=1}^3 a_i \sigma_i$, and $\sigma_i$ are Pauli matrices. For any given set of the vectors we have a different inequality. In [29] one derived the condition for a two-qubit state equivalent to satisfying all the inequalities jointly. It has the following form

$$M(\varrho) \leq 1, \quad (9)$$

where $M$ is constructed in the following way. One considers the $3 \times 3$ real matrix $T$ with entries $T_{ij} \equiv \text{Tr} \varrho \sigma_i \otimes \sigma_j$. Then $M$ is equal to the sum of two greater eigenvalues of the matrix $T^\dagger T$. This condition, characterises states violating the most common, and so far the strongest Bell inequality for two qubits (see [31] in this context). Being interesting from the point of view of nonlocality, it appears to be not a very strong separability criterion. Indeed, there exists [10] a large class of entangled states that satisfy all standard Bell inequalities.

Another approach originated from Schrödinger [5] observation that an entangled state gives us more information about the total system than about subsystems. This gave rise to a series of entropic inequalities of the form [32, 27]

$$S(\varrho_A) \leq S(\varrho), \quad S(\varrho_B) \leq S(\varrho), \quad (10)$$

where $\varrho_A = \text{Tr}_B \varrho$ and similarly for $\varrho_B$. The above inequalities were proven [32, 27, 33, 34] to be satisfied by separable states for four different entropies being particular cases of Rényi quantum entropies $S_\alpha = (1 - \alpha)^{-1} \log \text{Tr} \varrho^\alpha$

$$S_0 = \log R(\varrho) \quad (11)$$
$$S_1 = -\text{Tr} \varrho \log \varrho \quad (12)$$
$$S_2 = -\log \text{Tr} \varrho^2 \quad (13)$$
$$S_\infty = -\log \|\varrho\| \quad (14)$$

where $R(\varrho)$ denotes the rank of the state $\varrho$ (number of non-vanishing eigenvalues). The above inequalities are useful tools in many cases (as we will see in Sect. [10] one of them allows to obtain bound on the possible rank of the bound entangled states, still however they are not very strong criteria.

A different approach, presented in Ref. [17], is based on local manipulations of entanglement (the approach was anticipated in Ref. [12]). The main line is of the following sort: a given state is entangled, because the parties sharing many systems (pairs of particles) in this state can produce less number of pairs in highly entangled state (of easily “detectable” entanglement) by local operations and classical communication (LQCC). This approach initiated new field in quantum information theory: manipulating entanglement. The second part of this contribution will be devoted to this field. It also initiated the subject of quantification of entanglement. Still, however, the seemingly simple qualitative question of whether a given state is entangled or not was not solved.

A breakthrough was done by Peres [36] who derived a surprisingly simple but very strong criterion. He noted that a separable state remains positive operator if subjected to partial transposition (PT). We will call it positive partial transposition (PPT) criterion.

To define partial transposition, we will use matrix elements of a state in some product basis:

$$\varrho_{m\mu,n\nu} = \langle m| \otimes \langle \mu| \varrho |n\rangle \otimes |\nu\rangle, \quad (15)$$

5A qubit is the elementary unit of quantum information and denotes two level quantum system (i.e. $2 \otimes 2$ system).

6See [11, 12, 13] in the context of more sophisticated nonlocality criteria.
where the kets with Latin (Greek) letters form orthonormal basis in Hilbert space describing first (second) system. Hence the partial transposition of $\rho$ is defined as:

$$
\rho_{m\mu, n\nu}^{T_B} \equiv \rho_{m\nu, n\mu}.
$$

(16)

The form of the operator $\rho^{T_B}$ depends on the choice of basis, but its eigenvalues do not. We will say that a state is PPT if $\rho^{T_B} \geq 0$; otherwise we will say that the state is NPT. The partial transposition is easy to perform in matrix notation. Since the state of $m \otimes n$ system can be written as

$$
\rho = \begin{bmatrix}
A_{11} & \ldots & A_{1m} \\
\vdots & \ddots & \vdots \\
A_{m1} & \ldots & A_{mm}
\end{bmatrix},
$$

(17)

with $n \times n$ matrices $A_{ij}$ acting on the second ($\mathbb{C}^n$) space. They are defined by their matrix elements as $\{A_{ij}\}_{\mu\nu} \equiv \rho_{i\nu,j\mu}$. Then the partial transposition will be realised simply by transposition (denoted by $T$) of all of these matrices, namely:

$$
\rho^{T_B} = \begin{bmatrix}
A_{11}^T & \ldots & A_{1m}^T \\
\vdots & \ddots & \vdots \\
A_{m1}^T & \ldots & A_{mm}^T
\end{bmatrix}.
$$

(18)

Now for any separable state $\rho$, the operator $\rho^{T_B}$ must have still nonnegative eigenvalues. Indeed, consider partially transposed separable state:

$$
\rho^{T_B} = \sum_i p_i \tilde{\rho}_i \otimes (\tilde{\rho}_i)^T.
$$

(19)

Since the state $\tilde{\rho}_i$ remains positive under transposition, so does the total state.

Note that what distinguishes the Peres criterion from the earlier ones is that it is structural. In other words, it does not say that some scalar function of a state satisfies some inequality, but it imposes constraints on the structure of the operator resulting from PT. Thus the criterion amounts to satisfying of many inequalities at the same time. In next section we will see that there is also another crucial feature of the criterion: it involves transposition that is positive map but is not completely positive one. This feature abstracted from the Peres criterion allowed to find intimate connection between entanglement and theory of positive maps.

Finally, it should be mentioned that necessary conditions for separability have been recently developed in infinite dimensions [37, 38]. In particular, the Peres criterion was expressed in terms of Wigner representation and applied to Gaussian wave packets [38].

3 Entanglement and theory of positive maps

To describe the very fruitful connection between entanglement and theory of positive maps we will need mathematical notions like positive operators, positive maps, completely positive maps. In the following section we establish these notions. In next sections we will use them to develop characterisation of the set of separable states.
3.1 Positive and completely positive maps

We start from the following notation. By $\mathcal{A}_A$ and $\mathcal{A}_B$ we will denote the set of operators acting on $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. Recall that the set $\mathcal{A}$ of operators acting on some Hilbert space $\mathcal{H}$ constitute a Hilbert space itself (so-called Hilbert-Schmidt space) with scalar product $\langle A, B \rangle = \text{Tr} A^\dagger B$. One can consider an operator orthonormal basis in this space given by $\{|i\rangle\langle j|\}_{i,j}^{\dim \mathcal{H}}$ where $|i\rangle$ is a basis in the space $\mathcal{H}$. Since we deal with finite dimension, $\mathcal{A}$ is in fact a space of matrices. Hence we will denote it sometimes by $\mathcal{M}_d$ where $d$ is dimension of $\mathcal{H}$.

The space of the linear maps from $\mathcal{A}_A$ to $\mathcal{A}_B$ is denoted by $\mathcal{L}(\mathcal{A}_A, \mathcal{A}_B)$. We say that a map $\Lambda \in \mathcal{L}(\mathcal{A}_A, \mathcal{A}_B)$ is positive if it maps positive operators in $\mathcal{A}_A$ into the set of positive operators i.e. if $A \geq 0$ implies $\Lambda(A) \geq 0$. Finally we need the definition of completely positive (CP) map. One says \[16\] that a map $\Lambda \in \mathcal{L}(\mathcal{A}_A, \mathcal{A}_B)$ is completely positive if the induced map $\Lambda_n = \Lambda \otimes I_n : \mathcal{M}_A \otimes M_n \rightarrow \mathcal{M}_B \otimes M_n$ (20) is positive for all $n$; here $I_n$ is the identity map on the space $M_n$. Thus the tensor product of a CP map and the identity maps positive operators into positive ones. An example of CP map is $\rho \rightarrow W\rho W^\dagger$ where $W$ is an arbitrary operator. As a matter of fact, the general form of CP maps is $\Lambda(\rho) = \sum_i W_i \rho W_i^\dagger$. (21)

CP maps that do not increase trace ($\text{Tr} \Lambda(\rho) \leq \text{Tr} \rho$) correspond to the most general physical operations allowed by quantum mechanics \[16\]. If $\text{Tr} \Lambda(\rho) = \text{Tr} \rho$ for any $\rho$ (we say the map is trace preserving) then the operation can be performed with probability 1, otherwise with probability $p = \text{Tr} \Lambda(\rho)$.

It is remarkable that there are positive maps that are not CP: an example is just the transposition mentioned in the previous section. Indeed, if $\rho$ is positive, then so is $\rho^T$, because

$$\text{Tr} \rho^T P = \text{Tr} \rho P^T \geq 0$$

(22)

and $P^T$ is still some projector. We used here the fact that $\text{Tr} A^T = \text{Tr} A$. On the other hand $I \otimes T$ is no longer positive. One can easily check it, showing that $\text{Tr} (I \otimes T) P_+ \equiv P_+^{RT}$ is not a positive operator.

A positive map is called decomposable \[39\] if it can be represented in the form

$$\Lambda = \Lambda^1_{CP} + \Lambda^2_{CP} \circ T,$$

(23)

where $\Lambda^i_{CP}$ are some CP maps. For low dimensional systems ($\Lambda : M_2 \rightarrow M_2$ or $\Lambda : M_3 \rightarrow M_2$) the set of positive maps can be easily characterised. Namely it has been shown \[10, 11\] that all the positive maps are decomposable in this case. If, instead, at least one of the spaces is $M_n$ with $n \geq 4$, there exist non-decomposable positive maps \[39, 41\] (see example in Sect. \[1\]). No full characterisation of positive maps has been worked out so far in this case.

3.2 Characterisation of separable states via positive maps

The fact that complete positivity is not equivalent to positivity is crucial for the problem of entanglement we discuss here. Indeed, trivially, the product states are mapped into positive operators by the tensor product of a positive map and identity: $(\Lambda \otimes I)(\rho \otimes \tilde{\rho}) = (\Lambda \rho) \otimes \tilde{\rho} \geq 0$. Of course, the

\[7\]
same holds for separable states. Then the main idea is that this property of the separable states is essential i.e., roughly speaking, if a state \( \rho \) is entangled, then there exists a positive map \( \Lambda \) such that \( (\Lambda \otimes I) \rho \) is not positive. This means that one can seek the entangled states by means of the positive maps. Now the point is that not all the positive maps can help us to determine whether a given state is entangled. In fact, the completely positive maps do not “feel” entanglement. Thus the problem of characterisation of the set of the separable states reduces to the following: one should extract from that set of all positive maps some essential ones. As we will see further, it is possible in some cases. Namely it appears that for the \( 2 \otimes 2 \) and \( 2 \otimes 3 \) systems the transposition is the only such map. For higher dimensional systems, apart from transposition also non-decomposable maps will be relevant.

Consider the lemma \[15\] that will lead us to the basic theorem relating entanglement and positive maps

**Lemma 1** A state \( \rho \in \mathcal{A}_A \otimes \mathcal{A}_B \) is separable if and only if

\[
\text{Tr}(A\rho) \geq 0
\]

for any operator \( A \) satisfying \( \text{Tr}(AP \otimes Q) \geq 0 \), for all pure states \( P \) and \( Q \) acting on \( \mathcal{H}_A \) and \( \mathcal{H}_B \) respectively.

**Remark.** Note that operator \( A \) that is positive on product states (i.e. satisfying \( \text{Tr}(AP \otimes Q) \geq 0 \)) is automatically Hermitian.

The lemma is a reflection of the fact that in real Euclidean space, a convex set and a point lying outside it can always be separated by a hyper-plane. Here, the convex set is the set of separable states, while the point is the entangled state. The hyper-plane is determined by the operator \( A \). The operator that is positive on product states but is not positive has been called “entanglement witness” \[12\], as it indicates entanglement of some state (first entanglement witness was provided in \[11\], see Sect. 3). Now, to pass to positive maps, we will use isomorphism between entanglement witnesses and positive non-CP maps \[13\]. Note that if we have any linear operator \( A \in \mathcal{A}_A \otimes \mathcal{A}_B \), we can define a map \( \Lambda : \mathcal{A}_B \to \mathcal{A}_A \) by

\[
\langle k| A(|i\rangle \langle j|)|l\rangle = \langle i| \otimes \langle k| A|j\rangle \otimes |l\rangle
\]

which can be rephrased as follows

\[
\frac{1}{d} A = (I \otimes \Lambda) P^d_+
\]

with \( d = \dim \mathcal{H}_A \). Conversely, given a map, the above formula allows to obtain a corresponding operator. It turns out, that this formula gives also one-to-one correspondence between entanglement witnesses and positive non-CP maps \[13\]. Applying this fact one can prove \[13\] the following theorem

**Theorem 1** Let \( \rho \) act on Hilbert space \( \mathcal{H}_A \otimes \mathcal{H}_B \). Then \( \rho \) is separable if and only if for any positive map \( \Lambda : \mathcal{A}_B \to \mathcal{A}_A \) the operator \( (I \otimes \Lambda) \rho \) is positive.

As we mentioned, the relevant positive maps are here the ones that are not completely positive. Indeed, for CP map \( \Lambda \) we have \( (I \otimes \Lambda) \rho \geq 0 \) for any state \( \rho \), hence CP maps are of no use here. The above theorem presents, to authors knowledge, the first application of the theory of positive maps in physics. So far, only completely positive ones were of interest for physicists. As we will see the theorem proved fruitful both for mathematics (theory of positive maps) and for physics (theory of entanglement).

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\[8\]For infinite dimensions one must invoke Hahn-Banach theorem, geometric form of which is generalisation of this fact.
3.3 Operational characterisation of entanglement in low dimensions (2 ⊗ 2 and 2 ⊗ 3 systems)

The first conclusion from the theorem is operational characterisation of the separable states in low dimensions (2 ⊗ 2 and 2 ⊗ 3). It follows from the mentioned fact that positive maps in low dimensions are decomposable. Then the condition \((\mathbb{I} \otimes \Lambda)\varrho \geq 0\) reads as \((\mathbb{I} \otimes \Lambda_1^{CP})\varrho + (\mathbb{I} \otimes \Lambda_2^{CP})\varrho^{T_B}\).

Now, since \(\varrho\) is positive and \(\Lambda_1^{CP}\) is CP, the first term is always positive. If \(\varrho^{T_B}\) is positive, then also the second term is positive, hence their sum is a positive operator. Thus to check whether for all positive maps we have \((\mathbb{I} \otimes \Lambda)\varrho \geq 0\) it suffices to check only transposition. One obtains \([15]\) (see \([44]\) in this context)

**Theorem 2** A state \(\varrho\) of 2 ⊗ 2 or 2 ⊗ 3 system is separable if and only if its partial transposition is a positive operator.

**Remark.** Equivalently one can use the partial transposition with respect to the first space.

The above theorem is an important result, as it allows to determine unambiguously whether a given quantum state of 2 ⊗ 2 (2 ⊗ 3) system can be written as mixture of product states or not. The necessary and sufficient condition for separability is here surprisingly simple, hence it found many applications. In particular, it was applied in the context of broadcasting entanglement \([13]\), quantum information flow in quantum copying networks \([40]\), disentangling machines \([47]\), imperfect two-qubit gate \([18]\), analysis of volume of the set of entangled states \([49, 50]\), decomposition of separable states into minimal ensembles or pseudo-ensembles \([51]\), entanglement splitting \([52]\), analysis of entanglement measures \([53, 20, 54]\).

In Sect. \([55]\) we describe the first application \([15]\): by use of the theorem we show that any entangled two-qubit system can be distilled, hence is useful for quantum communication.

3.4 Higher dimensions - entangled states with positive partial transposition

Since the Størmer-Woronowicz characterisation of positive maps applies only to low dimensions, it follows that for higher dimensions partial transposition will not constitute necessary and sufficient condition for separability. Thus there exist states that are entangled, but are PPT (see Fig. \([24]\)). First explicit examples of a entangled but PPT state were provided in \([24]\). Later on it appeared, that the mathematical literature concerning non-decomposable maps contains examples of matrices that can be treated as prototypes of PPT entangled states \([14, 54]\).

We will now describe the way of obtaining them presented in \([24]\), as it proved to be a fruitful direction in searching for PPT entangled states. Chapter \([11]\) will provide motivation for undertaking the very tedious task of the search – the states will represent a curious type of entanglement – bound entanglement.

To find desired examples we must take a PPT state and somehow show that it is entangled. Of course, we cannot use the strongest so far tool, i.e. PPT criterion, just because the state is to be PPT. So we must derive a criterion that would be stronger in some cases. It appears that the very range of the state can say us much about its entanglement in some cases. This is contained in the following theorem, derived in \([24]\) on the basis of analogous condition for positive maps considered in \([41]\).

**Theorem 3** \(\textit{(range criterion)}\)

\(^9\) The range of an operator \(A\) acting on the Hilbert space \(\mathcal{H}\) is given by \(R(A) = \{A(\psi) : \psi \in \mathcal{H}\}\). If \(A\) is Hermitian operator then the range is equivalent to the support, i.e., the space spanned by its eigenvectors with nonzero eigenvalues.
Figure 1: Structure of entanglement of mixed states for $2 \otimes 2$ and $2 \otimes 3$ system (a) and for higher dimensions (b)
If a state $\varrho$ acting on the space $\mathcal{H}_{AB}$ is separable, then there exists a family of product vectors $\psi_i \otimes \phi_i$ such that

a) they span the range of $\varrho$

b) the vectors $\{\psi_i \otimes \phi_i^*\}_{i=1}^k$ span the range of $\varrho^{TB}$ (where $*$ denotes complex conjugation in the basis in which partial transposition was performed).

In particular, any of the vectors $\psi_i \otimes \phi_i^*$ belongs to the range of $\varrho$.

Now, in [24] there were presented two examples of PPT states violating the above criterion. We will present the example for $2 \otimes 4$ case\[^{10}\]. The matrix is written in the standard product basis $\{|ij\rangle\}$

$$\varrho_b = \frac{1}{7b+1} \begin{bmatrix}
    b & 0 & 0 & 0 & 0 & b & 0 & 0 \\
    0 & b & 0 & 0 & 0 & 0 & b & 0 \\
    0 & 0 & b & 0 & 0 & 0 & 0 & b \\
    0 & 0 & 0 & b & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & \frac{1+b}{2} & 0 & 0 & \sqrt{1-b^2} \\
    b & 0 & 0 & 0 & 0 & b & 0 & 0 \\
    0 & b & 0 & 0 & 0 & 0 & b & 0 \\
    0 & 0 & b & 0 & \sqrt{1-b^2} & 0 & 0 & \frac{1+b}{2}
\end{bmatrix}, \tag{27}$$

where $0 < b < 1$. Now performing PT, as shown by formula (18) we can check that $\varrho_b^{TB}$ remains positive operator. By a tedious calculations, one can check, that none of the product vectors belonging to the range of $\varrho_b$, if partially conjugated (as stated in theorem), belong to the range of $\varrho_b^{TB}$. Thus the condition stated in theorem is drastically violated, hence the state is entangled. As we will see further, the entanglement is masked so subtly, that it cannot be distilled at all!

### 3.4.1 Range criterion and positive non-decomposable maps

The separability criterion given by the above theorem has been fruitfully applied in search for PPT entangled states [23, 57, 58]. The Theorem 3 was applied in [59] where a technique of subtraction of product vectors from the range of state was used to get best separable approximation (BSA) of the state. As a tool, the authors considered subspaces containing no product vectors. Note that the (normalised) projector onto such a subspace must be entangled, as the condition a) from the theorem is not satisfied. This approach was successfully applied in Ref. [23] (see also [57, 42]) and, in connection with seemingly completely different concept of unextendible product bases, produced an elegant, and so far the most transparent way of construction of PPT entangled states.

To describe the construction\[^{11}\], one needs the following definition [23]

**Definition 1** A set of product orthogonal vectors in $\mathcal{H}_{AB}$

a) that has less elements than the dimension of the space

b) such that there does not exist any product vector orthogonal to all of them is called unextendible product basis.

\[^{10}\] It bases on an example concerning positive maps [41].

\[^{11}\] The construction applies to multipartite case [24], in the present review we consider only bipartite systems.
Here we recall an example of such basis in $3 \otimes 3$ system:

$$
|v_0\rangle = \frac{1}{\sqrt{2}} |0\rangle |0 - 1\rangle, \quad |v_2\rangle = \frac{1}{\sqrt{2}} |2\rangle |1 - 2\rangle, \\
|v_1\rangle = \frac{1}{\sqrt{2}} |0 - 1\rangle |2\rangle, \quad |v_3\rangle = \frac{1}{\sqrt{2}} |1 - 2\rangle |0\rangle \\
|v_4\rangle = \frac{1}{3} |0 + 1 + 2\rangle |0 + 1 + 2\rangle.
$$

(28)

Of course, the above five vectors are orthogonal to each other. However, any subset of three vectors on either side spans the full three-dimensional space. This prevents from existence of a sixth product vector that would be orthogonal to all five of them. How to connect this with the problem we deal in this section? The answer is: via the subspace complementary to the one spanned by these vectors. Indeed, suppose that $\{w_i = \phi_i \otimes \psi_i\}_{i=1}^{k}$ is UPB. For $d \otimes d$ system. Consider projector $P = \sum_{i=1}^{k} |w_i\rangle\langle w_i|$ onto the subspace $\mathcal{H}$ spanned by the vectors $w_i$ (dim $\mathcal{H} = k$) Now, consider the state uniformly distributed on its orthogonal complement $\mathcal{H}_\perp$ (dim $\mathcal{H}_\perp = d^2 - k$)

$$
\varrho = \frac{1}{d^2 - k} (I - P).
$$

(29)

The range of the state ($\mathcal{H}_\perp$) contains no product vectors: otherwise one would be able to extend the product basis $\{w_i\}$. Then by Theorem 3 the state must be entangled. Let us now calculate $\varrho^{TB}$. Since $w_i = \phi_i \otimes \psi_i$ then $(|w_i\rangle\langle w_i|)^{TB} = |\tilde{w}_i\rangle\langle \tilde{w}_i|$ where $\tilde{w}_i = \phi_i \otimes \psi_i^*$. The vectors $\tilde{w}_i$ are orthogonal to each other so that the operator $P^{TB} = \sum_{i} |\tilde{w}_i\rangle\langle \tilde{w}_i|$ is a projector. Consequently, the operator $(I - P)^{TB} = I - P^{TB}$ is also projector, hence it is positive. We conclude that $\varrho$ is PPT.

A different way of obtaining examples of PPT entangled states can be inferred from the papers devoted to search for non-decomposable positive maps in mathematical literature [44, 56] (see Sect. 3.1). A way to find non-decomposable map is the following. One constructs some map $\Lambda$ and proves somehow that it is positive. Thus one must guess some (possibly unnormalised) state $\varrho$ that is PPT. Now, if $(I \otimes \Lambda)\varrho$ is not positive, then $\Lambda$ cannot be decomposable, as shown in the discussion preceding Theorem 2. At the same time, the state must be entangled. In Sect. 4 we present an example of PPT entangled state (based on [40]) found in this way. Thanks to its symmetric form the state allowed to reveal the first quantum effect produced by bound entanglement (see Sect. 12.2).

Thus a possible direction of exploring the “PPT region” of entanglement is to develop the description of non-decomposable maps. However, it appears that there can be also a “back-reaction”: exploration of PPT region, allowed to obtain new results on non-decomposable maps. It turns out that just the UPB method described before allows for easy construction of new non-decomposable maps [60]. The interested reader we direct to the original article as well as [42]. We only note that to find a non-decomposable map, one needs only to construct some UPB. Then the procedure is automatic, as the one described above. To our knowledge, this is the first systematic way of finding non-decomposable maps.

### 4 Examples

We present here a couple of examples, illustrating the results contained in previous sections. In particular we introduce two families of states that play important role in the problem of distillation of entanglement.
4.1 Reduction criterion for separability

As mentioned in Sect. 3.2, if Λ is a positive map then for separable states we have

\[(I \otimes \Lambda) \varrho \geq 0,\]  

(30)

If the map is not CP, then this condition is not trivial, i.e. for some states \((I \otimes \Lambda) \varrho\) is not positive. Consider the map given by \(\Lambda(A) = (\text{Tr} A)I - A\). The eigenvalues of the resulting operator \(\Lambda(A)\) are given by \(\lambda_i = \text{Tr} A - a_i\) where \(a_i\) are eigenvalues of \(A\). If \(A\) is positive, then \(a_i \geq 0\). Now, since \(\text{Tr} A = \sum a_i\) then also \(\lambda_i\) are nonnegative. Thus the map is positive. Now, the formula (30) and the dual one \((\Lambda \otimes I) \varrho \geq 0\) applied to this particular map implies that separable states must satisfy the following inequalities

\[I \otimes \varrho_B - \varrho \geq 0, \quad \varrho_A \otimes I - \varrho \geq 0,\]  

(31)

The two conditions taken jointly are called reduction criterion [33, 61]. One can check that it implies the entropic inequalities (hence it is better in “detecting” entanglement). From the reduction criterion it follows that states \(\varrho\) of \(d \otimes d\) system with \(F(\varrho) > 1/d\) must be entangled (this was originally argued in [17]). Indeed, from the above inequalities it follows that for separable state \(\sigma\) and a maximally entangled state \(\psi_{me}\) one has \(\langle \psi_{me}\sigma_B \otimes I \varrho - \sigma | \psi_{me}\rangle \geq 0\). Since the reduced density matrix \(\varrho_A^{\psi_{me}}\) of the state \(\psi_{me}\) is proportional to identity we obtain \(\langle \psi_{me}\sigma_A \otimes I | \psi_{me}\rangle = \text{Tr}(\varrho_A^{\psi_{me}} \sigma_A) = 1/d\). Hence we get \(F \leq 1/d\). We conclude that the latter condition is separability criterion.

Let us finally note [33], that for \(2 \otimes 2\) and \(2 \otimes 3\) systems the reduction criterion is equivalent to PPT criterion, hence is equivalent to separability.

4.2 Strong separability criteria from entanglement witness

Consider unitary flip operator \(V\) on \(d \otimes d\) system defined by \(V \psi \otimes \phi = \phi \otimes \psi\). Note that it can be written as \(V = P_S - P_A\) where \(P_S\) (\(P_A\)) is projector onto symmetric (antisymmetric) subspace of the total space. Hence \(V\) is dichotomic observable (with eigenvalues \(\pm 1\)). One can check that \(\text{Tr} VA \otimes B = \text{Tr} AB\) for any operators \(A, B\). Then \(V\) is an entanglement witness, so that \(\text{Tr} \varrho V \geq 0\) is separability criterion [10]. Now, let us find the corresponding positive map via the formula (26). One easily gets that it is transposition (up to an irrelevant factor). Remarkably, in this way, given entanglement witness, one can find the corresponding map, to obtain much stronger criterion given by formula (30).

4.3 Werner states

In [10] Werner considered states that do not change if subjected to the same unitary transformation to both subsystems:

\[\varrho = U \otimes U \varrho U^\dagger \otimes U^\dagger\]  

for any unitary \(U\).

(32)

He showed that such states (called Werner states) must be of the following form

\[\varrho_W(d) = \frac{1}{d^2 - \beta d}(I + \beta V), \quad -1 \leq \beta \leq 1,\]  

(33)

where \(V\) is flip operator defined above. Other form for \(\varrho_W\) is [13]

\[\varrho_W(d) = p \frac{P_A}{N_A} + (1 - p) \frac{P_S}{N_S}, \quad 0 \leq p \leq 1,\]  

(34)
where \( N_A = (d^2 - d)/2 \) (\( N_S = (d^2 + d)/2 \)) is the dimension of the antisymmetric (symmetric) subspace. It was shown \(^{10}\) that \( \rho_W \) is entangled if and only if \( \text{Tr} V \rho_W < 0 \). Equivalent conditions are: \( \beta < -1/d, \ p > 0 \) or \( \rho \) is NPT. Thus \( \rho_W \) is separable if and only if it is PPT. For \( d = 2 \) (two-qubit case) the state can be written as (see \(^{11}\) in this context)

\[
\rho_W(2) = p|\psi_\perp\rangle\langle\psi_\perp| + (1 - p)\frac{\mathbb{I}}{4}, \quad -\frac{1}{3} \leq p \leq 1.
\] (35)

Note that any state \( \rho \) if subjected to random transformation of the form \( U \otimes U \) (call such operation \( U \otimes U \) twirling) becomes Werner state

\[
\int dU \rho U \otimes U \rho U^\dagger \otimes U^\dagger = \rho_W.
\] (36)

Moreover \( \text{Tr} \rho V = \text{Tr} \rho_W V \) (i.e. \( \text{Tr} \rho V \) is invariant of \( U \otimes U \) twirling).

4.4 Isotropic state

If we apply local unitary transformation to the state \(^{33}\), changing \( \psi_\perp \) into \( \psi_\perp \), we can generalise its form to higher dimension as follows \(^{33}\)

\[
\rho(p, d) = pP_+ + (1 - p)\frac{\mathbb{I}}{d^2}, \quad \text{with} \quad -\frac{1}{d^2 - 1} \leq p \leq 1.
\] (37)

The state will be called isotropic \(^{33}\). For \( p > 0 \) it is interpreted as mixture of maximally entangled state \( P_+ \) with a completely chaotic noise represented by \( \mathbb{I}/d \). It was shown that it is the only state invariant under \( U \otimes U^\ast \) transformations\(^{13}\). If we use singlet fraction \( F = \text{Tr} \rho P_+ \) as a parameter, we obtain

\[
\rho(F, d) = \frac{d^2}{d^2 - 1} \left( (1 - F)\frac{\mathbb{I}}{d^2} + (F - \frac{1}{d^2})P_+ \right), \quad 0 \leq F \leq 1.
\] (38)

The two parameters are related via \( p = (d^2 F - 1)/(d^2 - 1) \). The state is entangled if and only if \( F > 1/d \), or equivalently if it is NPT. Similarly as for Werner states, a state subjected to \( U \otimes U^\ast \) twirling (random \( U \otimes U^\ast \) operations) becomes isotropic, and the parameter \( F(\rho) \) is invariant under this operation.

4.5 A two-qubit state

Consider the following two-qubit state

\[
\rho = p|\psi_\perp\rangle\langle\psi_\perp| + (1 - p)|00\rangle\langle00|.
\] (39)

By formula \(^{9}\) we obtain that for \( p \leq 1/\sqrt{2} \) CHSH-Bell inequalities are satisfied. A little bit stronger is the criterion involving fully entangled fraction, we have \( F \leq 1/2 \) for \( p \leq 1/2 \). Entropic inequalities, apart from the one involving \( S_0 \), are equivalent to each other for this state and give again \( p \leq 1/2 \). Thus they reveal entanglement for \( p > 1/2 \). Applying the partial transposition one can convince, that the state is entangled for all \( p > 0 \) (for \( p = 0 \) it is manifestly separable).

\(^{12}\) In \(^{33}\) it was called noisy singlet.

\(^{13}\) The star denotes complex conjugation.
4.6  Entangled PPT state via non-decomposable positive map

Consider the following state (constructed on the basis of Størmer matrices [40]) of $3 \otimes 3$ system

$$\sigma_\alpha = \frac{2}{7} |\psi_+\rangle\langle\psi_+| + \frac{\alpha}{7} \sigma_+ + \frac{5 - \alpha}{7} \sigma_- \quad 2 \leq \alpha \leq 5 .$$  \hfill (40)

with

$$\sigma_+ = \frac{1}{3} (|0\rangle|1\rangle\langle0| + |1\rangle|2\rangle\langle1| + |2\rangle|0\rangle\langle2|) ,$$

$$\sigma_- = \frac{1}{3} (|1\rangle|0\rangle\langle1| + |2\rangle|1\rangle\langle2| + |0\rangle|2\rangle\langle0|) .$$  \hfill (41)

Using the formulas (18) one easily finds that for $\alpha \leq 4$ the state is PPT. Consider now the following map [39]:

$$\Lambda \left( \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \right) = \left[ \begin{bmatrix} a_{11} & -a_{12} & -a_{13} \\ -a_{21} & a_{22} & -a_{23} \\ -a_{31} & a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} a_{33} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{22} \end{bmatrix} \right] .$$  \hfill (42)

This map was shown to be positive [39]. Now, one can calculate the operator $(\mathbb{I} \otimes \Lambda) \rho$ and find that one of its eigenvalues is negative for $\alpha > 3$ (explicitly $\lambda_- = (3 - \alpha)/2$). This implies that

- the state is entangled (for separable state we would have $(\mathbb{I} \otimes \Lambda) \rho \geq 0$)
- the map is non-decomposable (for decomposable map and PPT state we also would have $(\mathbb{I} \otimes \Lambda) \rho \geq 0$).

For $2 \leq \alpha \leq 3$ it is separable, as it can be written as a mixture of other separable states $\sigma_\alpha = \frac{6}{7} \rho_1 + \frac{\alpha - 2}{7} \sigma_+ + \frac{5 - \alpha}{7} \sigma_-$ with $\rho_1 = (|\psi_+\rangle\langle\psi_+| + \sigma_+ + \sigma_-)/3$. The latter state can be written as an integral over product states:

$$\rho_1 = \frac{1}{8} \int_{0}^{2\pi} |\psi(\theta)\rangle\langle\psi(\theta)| \otimes |\psi(-\theta)\rangle\langle\psi(-\theta)| \frac{d\theta}{2\pi} ,$$

with $|\psi(\theta)\rangle = \frac{1}{\sqrt{3}} (|0\rangle + e^{i\theta}|1\rangle + e^{-2i\theta}|2\rangle)$ (there exists also finite decomposition exploiting phases of roots of unity [62]).

5  Volumes of entangled and separable states

The question of volume of the set of separable or entangled states in the set of all states raised in [13] is important for several reasons. First one could be interested in the following basic question: Is the world more classical or more quantum? Second, the size of the volume would reflect the fact important for numerical analysis of entanglement, to what extent the separable or entangled states are typical. Later it appeared that the considerations on volume of separable states lead of important results concerning the question of relevance of entanglement in quantum computing [63].

We will mainly consider qualitative question: is the volume of separable ($V_s$), entangled ($V_e$) or PPT entangled ($V_{pe}$) states nonzero? All these problems can be solved by the same method [13]: one picks a suitable state from either of the sets, and tries to show that some (perhaps small) ball round the state is still contained in the set.
For separable states one takes the ball round the maximally mixed state: one needs a number \( p_0 \) such that for any state \( \tilde{\rho} \) the state

\[
\rho = p\mathbb{I}/N + (1 - p)\tilde{\rho} \tag{43}
\]
is separable for all \( p \leq p_0 \) (here \( N \) is the dimension of the total system). In [49] it was shown that in the general case of multipartite systems of any finite dimension, such \( p_0 \) exists. Note that, in fact, we obtained sufficient condition for separability: if the eigenvalues of a given state do not differ too much from the uniform spectrum of the maximally mixed state, then the state must be separable. One would like to have some concrete values of \( p_0 \) that constitute the condition (the larger \( p_0 \), the stronger the condition).

Consider, for example, \( 2 \otimes 2 \) system. Here one can provide the largest possible \( p_0 \), as there exists the necessary and sufficient condition for separability (PPT criterion). Consider eigenvalues \( \lambda_i \) of the partial transposition of state \([43]\). They are of the form \( \lambda_i = (1 - p)/N + p\tilde{\lambda}_i \) where \( \tilde{\lambda}_i \) are eigenvalues of \( \tilde{\rho}^{TB} \) (in our case \( N = 4 \)). One easily can see (basing on the Schmidt decomposition) that partial transposition of a pure state cannot have eigenvalues smaller than \(-1/2\). Hence the same is true for mixed states. In conclusion we obtain that if \((1 - p)/N - p/2 \geq 0\) then the eigenvalues \( \lambda_i \) are nonnegative for arbitrary \( \tilde{\rho} \). Thus for \( 2 \otimes 2 \) system one can take \( p_0 = 1/3 \) to obtain sufficient condition for separability. Concrete values of \( p_0 \) for the case of n-partite systems each of dimension \( d \) were obtained in \([27]\):

\[
p_0 = \frac{1}{(1 + \frac{2}{d})^{n-1}}. \tag{44}
\]

These considerations proved to be crucial for analysis of the experimental implementation of quantum algorithm in high temperature systems via nuclear magnetic resonance (NMR) methods. This is because a generic state used in this approach is the maximally mixed one with a small admixture of some pure entangled state. In [63] the sufficient conditions of the above sort were further developed and it was concluded, that in all the NMR quantum computing experiments performed so far the admixture of the pure state was too small. Thus the total state used in these experiments was separable: it satisfied condition sufficient for separability. This raised an interesting discussion to what extent entanglement is necessary for quantum computing \([64, 65]\) (see also \([66]\) in this context). Even though there is still no general answer, it was shown \([64]\) that Shor algorithm \([67]\) requires entanglement.

Let us now turn back to the question of volumes of \( V_e \) and \( V_{pe} \). If one takes \( \psi_+ \) of a \( d \otimes d \) system for simplicity, it is easy to see that a not very large admixture of any state will keep \( F > 1/d \). Thus any state belonging to the neighbourhood must be entangled. Showing that the volume of PPT entangled states is nonzero is a bit more involved \([49] \).

In conclusion, all the three types of states are not atypical in the set of all states of a given system. However, it appears that the ratio of the volume set of PPT states \( V_{PPT} \) (hence also separable states) to the volume of the total set of states goes down exponentially with the dimension of the system (see Fig. 4). This result was obtained numerically \([49]\), and still awaits for analytical proof\([14]\). However it is compatible with the rigorous result in \([68]\) that in infinite dimension the set of separable states is nowhere dense in the total set of states. Then the generic infinite-dimensional state is entangled.

\[14\] The result could rely on the chosen measure of the volume \([68]\). In \([50]\) two different measures were compared and produced similar results.
Figure 2: The ratio $P_N = V_{PPT}/V$ of the volume of PPT states to the volume of the set of all states versus dimension $N$ of the total system. Different symbols distinguish different sizes of one subsystem ($k = 2$ (⋄), $k = 3$ (∆)). (This figure is reproduced from Phys. Rev. A 58, 883 (1998) by permission of authors.)
Part III

Mixed-state entanglement as a resource for quantum communication

As one knows, if two distant observers (one usually calls them Alice and Bob) share a pair of particles in singlet state $\psi_-^+$ then they can send a quantum state to one another by use of only additional two classical bits. This is called quantum teleportation \[9\]. If the classical communication is free of charge (since it is much cheaper than communication of quantum bits), one can say that a singlet pair is a resource equivalent to sending one qubit. In the following it will be shown that mixed-state entanglement can also be a resource for quantum communication. The quantum communication via mixed entangled states will require, apart from teleportation, the action called distillation. It will be also shown that there exists a peculiar type of entanglement (bound entanglement) that is a surprisingly weak resource.

6 Distillation of entanglement: counterfactual error correction

Now we will attempt to describe an ingenious concept of distillation of entanglement introduced in Ref. \[13\] and developed in \[17, 70\] (see also \[71\]). To this end let us first briefly describe the idea of classical and quantum communication via noisy channel. As one knows \[72\], the central idea of classical information theory pioneered by Shannon is that one can send information reliably and with nonzero rate via noisy information channel. This is achieved by coding: the input $k$ bits of information is encoded into a larger number of $n$ bits. Such a package is sent down the noisy channel. Then, the receiver performs decoding transformation, recovering the input $k$ bits with asymptotically (in the limit of large $n$ and $k$) perfect fidelity. Moreover, the asymptotic rate of information transmission $\frac{k}{n}$ is nonzero.

In quantum domain one would like to communicate quantum states instead of classical messages. It appears, that here the analogous scheme can be applied \[73, 74\]. The input $k$ qubits of quantum information are supplemented with additional qubits in some standard initial state, and the total system of $n$ qubits is subjected to some quantum transformation. Now the package can be sent down the channel. After decoding operation, the state of $k$ qubits is recovered with asymptotically perfect fidelity \[30\] (now it is quantum fidelity - characterising how close is the output state to the input one) regardless of the particular form of the state. The discovery of the above possibility (called quantum error correction; we will call it here direct error correction) initiated, in particular, extensive studies of quantum error correcting codes (see \[75\] and references therein), as well as capacities of quantum channel\[15\] (see \[76\] and references therein). A common example of a quantum channel is the one-qubit quantum depolarising channel: here an input state is undisturbed with probability $p$ and subjected to a random unitary transformation with probability $1-p$. It can be described by the following completely positive map $\varrho \rightarrow \Lambda(\varrho) = p\varrho + (1-p)\frac{I}{2}$, (45)

where $I/2$ is the maximally mixed state of one qubit. This channel has been thoroughly investigated \[13, 17, 77, 78\]. What is important here, it was shown \[78\] that for $p \leq \frac{2}{3}$ the above method of error correction does not work. In classical domain it would mean that the channel is useless.

\[15\] Capacity $Q$ of a quantum channel is the greatest ratio $\frac{k}{n}$ of reliable transmission down the given channel.
Here, surprisingly, there is a trick that allows to beat this limit, even down to $p = \frac{1}{3}$! The scheme that realizes it is quite mysterious. In direct error correction we deal directly with the systems carrying information to be protected. Now, it appears that using entanglement, one can remove the results of action of noise even without having the information to be sent. Therefore, it can be called counterfactual error correction.

How does it work? The very idea is not complicated. Alice (the sender) instead of the qubits of information, sends to Bob particles from entangled pairs (in state $\psi^-$), keeping one particle from each pair. The pairs get disturbed by the action of the channel, so that their state turns into mixture$^{16}$ that still possesses some residual entanglement. Now it turns out that by local quantum operations (including collective actions over all members of pairs in each lab) and classical communication (LQCC) between Alice and Bob, they are able to obtain less number of pairs in nearly maximally entangled state $\psi^-$ (see Fig. 3). Such a procedure proposed in $^{13}$ is called distillation. As in the case of direct error correction, one can achieve finite asymptotic rate $k$ of the distilled pairs per input pair, and the fidelity that now denotes similarity of the distilled pairs to the product of singlet pairs, is asymptotically perfect. Now, the distilled pairs can be used for teleportation of quantum information. The maximal possible rate achievable within the above framework is called entanglement of distillation of the state $\rho$, and denoted by $D(\rho)$. Thus, if Alice and Bob share $n$ pairs each in state $\rho$, they can faithfully teleport $k = nD(\rho)$ qubits.

As we have mentioned, the error correction stage and the transmission stage are here separated in time; the error correction can be performed even before the information to be sent was produced. Using terminology of $^{79}$ one can say, that Alice and Bob operate on potentialities (entangled pair represents a potential communication) and correct potential error, so that, when the actual information is coming, it can be teleported without any additional action.

The above scheme is not only mysterious. It is also much more powerful from the direct method. In next section we describe a distillation protocol, that allows to send quantum information reliably via the channel with $p > 1/3$. A general question is: where are the limits of distillation? As we have seen, the basic action refer mixed bipartite state, so that instead of saying about channels, we can concentrate on bipartite states. The question can be formulated: which states $\rho$ can be distilled by the most general LQCC actions? Here, saying that a state $\rho$ can be distilled, we mean that Alice and Bob can obtain singlets from the initial state $\rho \otimes n$ of $n$ pairs (thus we will work with memoryless channel).

One can easily see, that separable states cannot be distilled: they contain no entanglement, so it is impossible to convert them into entangled ones by LQCC operations. Then the final form of our question is: Can all entangled states can be distilled? Before answer the to this question was provided, the default was “yes”, and the problem was: how to prove it. Now, one knows that the

$^{16}$If the channel is memoryless, it factorises into states $\rho$ of individual pairs.
answer is “no”, so that the structure of entanglement of bipartite states is much more puzzling than one could suspect.

Finally, one should mention that for pure states the problem of conversion into singlet pairs is solved. Here there is no surprise: all entangled pure states can be distilled [84] (see also [80] in this context). What is especially important, this distillation can be made reversibly: from the obtained singlet pairs, we can recover (asymptotically) the same number of input pairs [84]. As we will see, for mixed states it is not the case.

7 Distillation of two-qubit states

In this section we will describe historically the first distillation protocol for two-qubit states devised by Bennett, Brassard, Popescu, Schumacher, Smolin and Wootters (BBPSSW) [13]. Then we will show that a more general protocol can distill any entangled two-qubit state [17].

7.1 BBPSSW distillation protocol

The BBPSSW distillation protocol still remains the most transparent example of distillation. It works for the two-qubit states $\rho$ with fully entangled fraction satisfying $F > 1/2$. Such states are equivalent to the ones with $F > 1/2$, so that we can restrict ourselves to the latter states. Hence we assume that Alice and Bob share initially a huge number of pairs, each in the same state $\rho$ with $F > 1/2$, so that the total state is $\rho^\otimes n$. Now, they aim to obtain a smaller number of pairs of higher singlet fraction $F$. To this end they iterate the following steps:

1. They take two pairs, apply to each of them $U \otimes U^*$ twirling i.e. random unitary transformation of the form $U \otimes U^*$ (Alice picks at random a transformation $U$, applies it, and communicate Bob, which transformation she chose; then he applies $U^*$ to his particle). Thus one has transformation from two copies of $\rho$ to two copies of isotropic state $\rho_F$ with unchanged $F$:

$$\rho \otimes \rho \rightarrow \rho_F \otimes \rho_F.$$ (46)

2. Each party performs unitary transformation XOR on their members of pairs (see Fig. 4). The transformation is given by

$$U_{XOR} |a\rangle |b\rangle = |a\rangle (a + b) \mod 2.$$ (47)

(the first qubit is called source, the second one - target). They obtain some complicated state $\tilde{\rho}$ of two pairs.

3. The pair of target qubits is locally measured in the basis $|0\rangle, |1\rangle$ and it is discarded. If the results agree (success), the source pair is kept and has a greater singlet fraction. Otherwise (failure) the source pair is discarded, too.

If the results in step 3 agreed the final state $\rho'$ of the kept source pair can be calculated from the formula

$$\rho' = \text{Tr}_{H_t} (P_t \otimes I_s \tilde{\rho} P_t \otimes I_s),$$ (48)

\[17\] In this contribution we restrict ourselves to distillation by means of perfect operations. The more realistic case of imperfections of the quantum operations performed by Alice and Bob is considered in [81].

\[18\] Quantum XOR gate is the most common quantum two-qubit gate introduced in [82].
where the partial trace is performed over the Hilbert space $\mathcal{H}(t)$ of the target pair, $I_s$ is identity on the space of source pair (because it was not measured), while $P_t = |00\rangle\langle00| + |11\rangle\langle11|$ acts on target pair space and corresponds to the case “results agree”.

Subsequently, one can calculate the singlet fraction of the survived pair as a function of the singlet fraction of the two initial ones obtaining

$$F'(F) = \frac{F^2 + \frac{1}{2}(1 - F)^2}{F^2 + \frac{2}{3}F(1 - F) + \frac{5}{9}(1 - F)^2}.$$  \hspace{1cm} (49)

Since the function $F(F')$ is continuous, $F'(F) > F$ for $F > \frac{1}{2}$ and $F'(1) = 1$, we obtain that iterating the procedure Alice and Bob can obtain state with arbitrarily high $F$. Of course, the larger $F$ is required the more pairs must be sacrificed, and the less the probability $p$ of the success is. Thus if they start with some $F_{in}$ and would like to end up with some higher $F_{out}$, the number of final pairs will be on average $k = np/2^{l}$, where $l$ and $p$ depend on $F_{in}$ and $F_{out}$, and denote respectively the number of iterations of the function $F'(F)$ required to reach $F_{out}$ starting from $F_{in}$, and the probability of the string of $l$ successful operations.

The above method allows to obtain arbitrarily high $F$, but the asymptotic rate is zero. However, if $F$ is high enough that $1 - S > 0$ where $S$ is von Neumann entropy of the state $\varrho$, then there exists a protocol (so called hashing), that gives nonzero rate \cite{Bennett}. We will not describe it here, but we note that for any state with $F > \frac{1}{2}$ Alice and Bob can start by the recurrence method to obtain $1 - S > 0$, and then apply the hashing protocol. This gives nonzero rate for any state with $F > 1/2$. This means that quantum information can be transmitted via depolarising channel \cite{Huang} if only $p > 1/3$. Indeed, one can check that if Alice send one of the particles from a pair in a state $\psi_+$ via the channel to Bob, then the final state shared by them will be isotropic one with $F > 1/2$. Repeating this process, Alice and Bob can obtain many such pairs. Then distillation will allow to use them for asymptotically faithful quantum communication.

### 7.2 All entangled two-qubit states are distillable

As it was mentioned in Sect. 4, there exist entangled two-qubit states with $F < 1/2$, so that no product unitary transformation can produce $F > 1/2$. Thus the BBPSSW protocol does not apply to all entangled two-qubit states. Below we will show that, nevertheless, all such states are distillable \cite{Bennett}. The problem was possible to solve mainly due to the characterisation of the entangled states as discussed in Sect. 3.

Since we are not interested in the value of asymptotic rate, it suffices to show that starting with pairs in an entangled state, Alice and Bob are able to obtain a fraction of them in a new state with $F > 1/2$ (then the BBPSSW protocol will do the job). Our main tool will be the so called filtering operation \cite{Bennett, Peres} that involves generalised measurement performed by one of the parties.
(say, Alice) on individual pairs. It will consists of two outcomes \( \{1, 2\} \), associated with operators \( W_1 \) and \( W_2 \) satisfying
\[
W_1^\dagger W_1 + W_2^\dagger W_2 = \mathbb{I}_A \tag{50}
\]
(\( \mathbb{I}_{A(B)} \) denote identity on Alice (Bob) system). After such measurement, the state becomes
\[
\varrho \to \frac{1}{p_i} W_i \otimes \mathbb{I}_B \varrho W_i^\dagger \otimes \mathbb{I}_B, \quad i = 1, 2 \tag{51}
\]
with probability \( p_i = \text{Tr}(W_i \varrho W_i^\dagger) \). The condition (50) ensures \( p_1 + p_2 = 1 \). Now Alice will be interested only in one outcome (say, 1). If it occurs, Alice and Bob keep the pair, otherwise they discard it (this requires communication from Alice to Bob). Then we are only interested in the operator \( W_1 \). If its norm does not exceed 1, one can always find suitable \( W_2 \), so that the condition (50) is satisfied. Now, since either the form of the final state (51) or the fact whether \( p_1 \) is zero or not, do not depend on the positive factor multiplying \( W_1 \), we are free to consider completely arbitrary filtering operators \( W_1 \). In conclusion, for any entangled state \( \varrho \) we must find such an operator \( W \) that the state resulting from filtering \( W \otimes \mathbb{I} \varrho W^\dagger \otimes \mathbb{I} / \text{Tr}(W \otimes \mathbb{I} \varrho W^\dagger \otimes \mathbb{I}) \) has \( F > 1/2 \).

Consider then an arbitrary two-qubit entangled state \( \varrho \). From the Theorem 2 we know that \( \varrho^{TB} \) is not a positive operator, hence there exists a vector \( \psi \) for which
\[
\langle \psi | \varrho^{TB} | \psi \rangle < 0 \tag{52}
\]
Now let us note that any vector \( \phi \) of \( d \otimes d \) system can be written as \( \phi = A_\phi \otimes \mathbb{I}_+ \), where \( A_\phi \) is some operator. Indeed, write \( \phi \) in product basis: \( \phi = \sum_{i,j=1}^d a_{ij} |i\rangle \otimes |j\rangle \). Then the matrix elements of the operator \( A_\phi \) are given by \( \langle i | A_\phi | j \rangle = \sqrt{d} a_{ij} \) (in our case \( d = 2 \)). Therefore the formula (52) can be rewritten the form
\[
\text{Tr}[(A_\psi^\dagger \otimes \mathbb{I} \varrho A_\psi \otimes \mathbb{I})^{TB} P_+] < 0 \tag{53}
\]
Using identity \( \text{Tr}A^{TB} B = \text{Tr}A B^{TB} \) valid for any operators \( A, B \) and the fact that \( P_+^{TB} = \frac{1}{4} V \) (where \( V \) is the flip operator, see Sect. 3) we obtain
\[
\text{Tr}[(A_\psi^\dagger \otimes \mathbb{I} \varrho A_\psi \otimes \mathbb{I}) V] < 0 \tag{54}
\]
We conclude that \( A_\psi^\dagger \otimes \mathbb{I} \varrho A_\psi \otimes \mathbb{I} \) cannot be equal to null operator, hence we can consider the following state
\[
\tilde{\varrho} = \frac{A_\psi^\dagger \otimes \mathbb{I} \varrho A_\psi \otimes \mathbb{I}}{\text{Tr}(A_\psi^\dagger \otimes \mathbb{I} \varrho A_\psi \otimes \mathbb{I})}.
\]
Now it is clear that the role of filter \( W \) will be played by operator \( A_\psi^\dagger \). We will show that
\[
\langle \psi_- | \tilde{\varrho} | \psi_- \rangle > 1/2 \quad \text{where} \quad \psi_- = (|01\rangle - |10\rangle) / \sqrt{2} \tag{55}
\]
Then a suitable Alice’s unitary transformation can convert \( \tilde{\varrho} \) into a state \( \tilde{\varrho}' \) with \( F > 1/2 \).

From the inequality (54) one obtains
\[
\text{Tr} \tilde{\varrho} V < 0 \tag{55}
\]
If we use product basis \( |1\rangle = |00\rangle, |2\rangle = |01\rangle, |3\rangle = |10\rangle, |4\rangle = |11\rangle \) inequality (55) writes as
\[
\tilde{\varrho}_{11} + \tilde{\varrho}_{44} + \tilde{\varrho}_{23} + \tilde{\varrho}_{32} < 0 \tag{56}
\]
The above inequality, together with the trace condition \( \text{Tr} \tilde{\varrho} = \sum_i \tilde{\varrho}_{ii} = 1 \) gives
\[
\langle \psi_- | \tilde{\varrho} | \psi_- \rangle = \frac{1}{2} (\tilde{\varrho}_{22} + \tilde{\varrho}_{33} - \tilde{\varrho}_{23} - \tilde{\varrho}_{32}) > \frac{1}{2} \tag{57}
\]
To summarise, given a large supply of pairs, each in entangled state $\rho$ Alice and Bob can distill maximally entangled pairs in the following way. First Alice applies filtering determined by the operator $W = A^\dagger_\psi$ described above. Then Alice and Bob obtain on average a supply of $np$ survived pairs in the state $\tilde{\rho}$ (here $p = \text{Tr}W \otimes \mathbb{1} W^\dagger \otimes \mathbb{1}$ is the probability that the outcome of Alice measurement will be the one associated with the operator $W^\dagger$). Now Alice applies an operation $i\sigma_y$ to obtain the state with $F > 1/2$. Then they can use the BBPSSW protocol to distill maximally entangled pairs useful for quantum communication. Note that we assumed that Alice and Bob know the initial state of the pairs. It can be shown that if they do not know, they still can do the job (in the two-qubit case) sacrificing $\sqrt{n}$ pairs to estimate the state $[86]$. The above protocol can easily be shown to work in $2 \otimes 3$ case. The protocol can be also fruitfully applied for the system $2 \otimes n$ if the state is NPT $[87]$.

8 Examples

Consider the state (39) from Sect. $[46] \rho = p |\psi_-\rangle\langle\psi_-| + (1 - p) |00\rangle\langle00|$. It is entangled for all $p > 0$. In matrix notation we have

$$\rho = \begin{bmatrix} 1 - p & 0 & 0 & 0 \\ 0 & \frac{p}{2} & -\frac{p}{2} & 0 \\ 0 & -\frac{p}{2} & \frac{p}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \rho^{TB} = \begin{bmatrix} 1 - p & 0 & 0 & -\frac{p}{2} \\ 0 & \frac{p}{2} & 0 & 0 \\ 0 & 0 & \frac{p}{2} & 0 \\ -\frac{p}{2} & 0 & 0 & 0 \end{bmatrix}. \quad (58)$$

The negative eigenvalue of $\rho^{TB}$ is $\lambda_- = \frac{1}{2} (1 - p - \sqrt{(1 - p)^2 + p^2})$ with the corresponding (unnormalised) eigenvector $\psi = \lambda_- |00\rangle - \frac{p}{2} |11\rangle$, hence we can take the filter of the form $W = \text{diag}[\lambda_-, -\frac{p}{2}]$. The new state $\tilde{\rho}$ resulting from filtering is of the form

$$\tilde{\rho} = \frac{1}{N} \begin{bmatrix} \lambda_-^2 (1 - p) & 0 & 0 & 0 \\ 0 & \frac{p^3}{8} & \frac{p^2}{4} \lambda_- & 0 \\ 0 & \frac{p^2}{4} \lambda_- & \frac{p^2}{2} \lambda_- & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (59)$$

where $N = \lambda_-^2 (1 - p) + p^2/8 + \lambda_-^2 p/2$. Now the overlap with $\psi_-$ given by $\langle\psi_- | \tilde{\rho} | \psi_- \rangle = (p^3/8 + \lambda_-^2 p/2 - \lambda_- p^2/2)/N$ is greater than $1/2$ if only $p > 0$. The new state can be distilled by BBPSSW protocol.

Below we will prove that some states of higher dimensional systems are distillable. We will do it by showing that some LQCC operation can convert them (possibly with some probability) into entangled two-qubit state.

8.1 Distillation of isotropic state for $d \otimes d$ system.

For $F > 1/d$ isotropic state can be distilled $[33, 63]$. If both Alice and Bob apply the projector $P = |0\rangle\langle0| + |1\rangle\langle1|$ where $|0\rangle, |1\rangle$ are vectors from the local basis, then the isotropic state will be converted into two-qubit isotropic state. (Note that the projectors play the role of filters; also, the success of the filtering is if both Alice and Bob obtain outcome corresponding to $P$). Now, if the initial state satisfied $F > 1/d$ then the final one, as a two qubit one, will have $F > 1/2$. Thus, it is entangled hence can be distilled.
8.2 Distillation and reduction criterion

Any state $\rho$ of $d \otimes d$ system that violates reduction criterion (see Sect. 4) can be distilled \cite{33}. Indeed, take the vector $\psi$ for which $\langle \psi | \rho_A \otimes I - \rho | \psi \rangle < 0$. Now, it is easy to see that applying the filter $W$ given by $\psi = W \otimes I \psi_+$, one obtains state with $F > \frac{1}{4}$. Now, the random $U \otimes U^*$ transformations will convert it into isotropic state with the same $F$. As shown above the latter one is distillable.

9 Bound entanglement

In the light of the result for two qubits, one naturally expected that any entangled state can be distilled. It was a great surprise when it appeared that it is not the case. In \cite{14} it has been shown that there exist entangled states, that cannot be distilled. The following theorem provides necessary and sufficient condition for distillability of mixed states \cite{14}.

**Theorem 4** A state $\rho$ is distillable if and only if for some two-dimensional projectors $P, Q$ and for some number $n$, the state $P \otimes Q \rho^{\otimes n} P \otimes Q$ is entangled.

**Remarks.** (1) Note that the state $P \otimes Q \rho^{\otimes n} P \otimes Q$ is effectively two-qubit one as its support is contained in the $C_2 \otimes C_2$ subspace determined by the projectors $P, Q$. This means that the distillable entanglement is two-qubit entanglement. (2) One can see that the theorem is compatible with the fact \cite{84} that any pure state can be distilled.

As a consequence of this theorem we obtain the following one \cite{14}:

**Theorem 5** A PPT state cannot be distilled.

**Proof.** We will give here a proof independent of the Theorem 4. As a matter of fact, we will show that the set of PPT states is (i) invariant under LQCC operations \cite{14} and (ii) it is bounded away from maximally entangled state \cite{88, 89}. Then, since $(\rho^{\otimes n})^{TB} = (\rho^{TB})^{\otimes n}$ we obtain the proof of the theorem. To prove (i) note that any LQCC operation can be written as

$$\rho \rightarrow \rho' = \frac{1}{p} \sum_i A_i \otimes B_i \rho A_i^\dagger \otimes B_i^\dagger,$$  \hspace{1cm} (60)

where $p$ is a normalisation constant interpreted as probability of realization of the operation, and the map $\rho \rightarrow \sum_i A_i \otimes B_i \rho A_i^\dagger \otimes B_i^\dagger$ does not increase trace (this ensures $p \leq 1$). Suppose now that $\rho$ is PPT, i.e. $\rho^{TB} \geq 0$ and examine partial transposition of the state $\rho'$. We will use the following property of partial transpose

$$(A \otimes B \otimes C \otimes D)^{TB} = A \otimes D^T \rho^{TB} C \otimes B^T$$ \hspace{1cm} (61)

for any operators $A, B, C, D$ and $\rho$. Then we obtain

$$(\rho')^{TB} = \sum_i A_i \otimes (B_i^T)^T \rho^{TB} A_i \otimes (B_i)^T.$$ \hspace{1cm} (62)

Thus $(\rho')^{TB}$ is a result of action of some completely positive map on operator $\rho^{TB}$ that by assumption is positive. Then also the operator $(\rho')^{TB}$ must be positive. Thus LQCC map do not move outside the set of PPT states.
To prove (ii) let us now show that PPT states can never have high singlet fraction $F$. Consider a PPT state $\rho$ of a $d \otimes d$ system. We obtain

$$\text{Tr}\rho P_+ = \text{Tr}\rho^{T_B} P_+^{T_B}.$$ \hfill (63)

Now, it is easy to check that $P_+ = \frac{1}{d} V$, where $V$ is the flip operator described in section [4]. Note that $V$ is Hermitian and has eigenvalues $\pm 1$. Since $\rho$ is PPT then $\tilde{\rho} = \rho^{T_B}$ is a legitimate state, and the above expression can be rewritten in terms of mean value of the observable $V$

$$\text{Tr}\rho P_+ = \frac{1}{d} \text{Tr}\tilde{\rho} V.$$ \hfill (64)

The mean value of dichotomic observable cannot exceed 1 so that we obtain

$$F(\rho) \leq \frac{1}{d}.$$ \hfill (65)

Thus the maximal possible singlet fraction that can be attained by PPT states is the one that can be obtained without any prior entanglement between the parties. Indeed, a product state $|00\rangle$ has singlet fraction $1/d$ (if it belongs to the Hilbert space $C^d \otimes C^d$). Consequently, for however large amount of PPT pairs, even a single two-qubit pair with $F > 1/2$ cannot be obtained by LQCC actions.

Now, one can appreciate the results presented in the first part of this contribution. From Sect. [3] we know that there exist entangled states that are PPT. So far, the question of whether there exist entangled states that are PPT was merely a technical one. At present, since the above theorem implies that PPT states are non-distillable, we can draw a remarkable conclusion: there exist non-distillable entangled states. Since in the process of distillation no entanglement can be liberated to the useful singlet form, they have been called bound entangled. Thus there exist at least two qualitatively different types of entanglement: apart from the free entanglement that can be distilled, there is a bound one that cannot be distilled and seems to be completely useless for quantum communication. This discontinuity of the structure of entanglement of mixed states was considered to be possible for multipartite systems, but it was completely surprising for bipartite systems. It should be emphasised here, that the BE states are not atypical in the set of all possible states: as we have mentioned in Sect. [4] the volume of the PPT entangled states is nonzero. One of the main consequence of existence of BE is revealing a transparent form of irreversibility in entanglement processing. If Alice and Bob share pairs in pure state, then to produce BE state they need some prior entanglement. However once they produced the BE states, they would not be able to recover the pure entanglement back from them. It is entirely lost. This is a qualitative irreversibility that is probably a source of the quantitative irreversibility [14, 17] due to the fact that we need more pure entanglement to produce some mixed states than we can then distill back from them [19].

To analyse the phenomenon of bound entanglement, one needs as many examples of BE states as possible. Then there is a very exciting physical motivation for search for PPT entangled states. In Sect. [3.4] we discussed different methods of the search. As a result we have a couple of examples of BE states obtained via the separability criterion given by Theorem [3] from the mathematical literature on non-decomposable maps, and via unextendible product bases method.

The examples produced via UPB are extremely interesting from the physical point of view. It is because UPB is not only a mathematical object: as shown in [23] it produces a very curious

\footnote{The quantitative irreversibility was rigorously proved in [23]. There is still no fully rigorous proof for qualitative one (see [24]).}
physical effect \cite{24} called “nonlocality without entanglement”. Namely, suppose that Alice and Bob share a pair in one of the states from the UPB, but they do not know which one it is. It appears that by LQCC operations (with finite resources), they are not able to read the identity of the state. However, if the particles were together, then, since the states are orthogonal, they can be perfectly distinguished from each other. Thus we have a highly non classical effect produced by ensemble of separable states. On the other hand, the BE state associated with the given UPB (the uniform state on the complementary subspace, see (29)) presents opposite features: it is entangled but, since its entanglement is bound, it ceases to behave quantumly. Moreover in both situations we have a kind of irreversibility. As it was mentioned, BE states are reflection of the formation–distillation irreversibility: to create them by LQCC from singlet pairs, Alice and Bob need to non-zero amount of the latter. However, once they were created, there is no way to distill singlets out of them. On the other hand, UPB exhibits preparation–measurement irreversibility: any of the states belonging to UPB can be prepared by LQCC operations, but once Alice and Bob forgot the identity of the state, they cannot recover it by LQCC. This surprising connection between some BE states and bases that are not distinguishable by LQCC implies many interesting questions concerning future unification of our knowledge about nature of quantum information.

Finally, we will mention about the result concerning rank of the BE state. In numerical analysis of BE states (especially their tensor products) it is very convenient to have examples with low rank. However, in \cite{34} the following bound on the rank of the BE state $\varrho$ was derived

\[ R(\varrho) \geq \max\{R(\varrho_A), R(\varrho_B)\}. \tag{66} \]

(recall that $R(\varrho)$ denotes the rank of $\varrho$). Note that the above inequality is nothing but the entropic inequality \cite{14} with entropy \cite{13}. Thus it appears that the latter inequality is not only necessary condition for separability, but also for non-distillability. The proof bases on the fact \cite{33} that any state violating reduction criterion (see Sect. 4 and 8) can be distilled. It can be shown, that if a state violate the above equation, then it must also violate reduction criterion, hence can be distilled. Then it follows that there do not exist BE states of rank two \cite{34}. Indeed, if it existed, then its local ranks must have not exceed two. Hence the total state would be effectively two-qubit one. However, from Sect. 7.2 we know that two-qubit bound entangled states do not exist.

10 Do there exist bound entangled NPT states?

So far we have considered BE states due to the Theorem 5 which says that NPT condition is necessary for distillability. As mentioned in Sect. 7.2, for $2 \otimes n$ systems all NPT states can be distilled \cite{57}, hence the condition is also sufficient in this case. However, it is not known whether it is sufficient in general. The necessary and sufficient condition is given by the Theorem 4. To find if the condition is equivalent to PPT one, it must be determined if there exists an NPT state, such that, nevertheless, for any number of copies $n$, the state $\varrho \otimes^n$ will not have an entangled two-qubit “substate” (i.e. the state $P \otimes Q \varrho \otimes^n P \otimes Q$). In \cite{33} it was pointed out that one can reduce the problem by means of the following observation.

Proposition 1 The following statements are equivalent:

1. Any NPT state is distillable.

2. Any entangled Werner state (eq. 33) is distillable.

Proof. The proof of the implication $(1) \Rightarrow (2)$ is immediate, as Werner states are entangled if and only if they are NPT. Then if we can distill any NPT state, then also Werner entangled states are
distillable. To obtain (2) \( \Rightarrow \) (1) note that the reasoning of Sect. 7.2, from formula (52) to (55), is insensitive to the dimension \( d \) of the problem. Consequently, from any NPT state a suitable filtering produces a state \( \tilde{\rho} \) satisfying \( \text{Tr}\tilde{\rho}V < 0 \). As mentioned in Sect. 4, the parameter \( \text{Tr}\rho V \) is invariant under \( U \otimes U \) twirling, so that applying the latter (which is LQCC operation) Alice and Bob obtain Werner state \( \rho_W \) satisfying \( \text{Tr}\rho_W V < 0 \). Thus any NPT state can be converted by means of LQCC operations into entangled Werner state, which completes the proof.

The above proposition implies that to determine if there exist NPT bound entangled states, one can restrict to the family of Werner states which is one parameter family of very high symmetry. Even after such a reduction of the problem, the latter remains extremely difficult. In [93, 94] the authors examine the \( n \)-th tensor power of Werner states (in [94] a larger, two-parameter family is considered). The results, though not conclusive yet, strongly suggest that there exist NPT bound entangled states (see Fig. 5).

Thus it is likely that the characterisation of distillable states is not so simple to reduce to NPT condition. Possible existence of the NPT bound entanglement would make the total picture much more obscure (hence much more interesting). Among others, there would arise a question: for two
distinct BE states \( \rho_1 \) and \( \rho_2 \) is the state \( \rho_1 \otimes \rho_2 \) also BE? (if BE would equal PPT, this question has immediate answer “yes”, because the PPT property is additive, i.e. if two states are PPT, then so their tensor product does [36]). Recently, negative answer to this question was obtained in Ref. [95] in the case of multipartite system. For bipartite states the answer is still unknown.

11 Example

Consider the family of states (40) considered in Sect. 4. One obtains the following classification: \( \rho \) is

- separable for \( 2 \leq \alpha \leq 3 \)
- BE for \( 3 < \alpha \leq 4 \)
- FE for \( 4 < \alpha \leq 5 \)

Separability was shown in Sect. (4). It was also shown there that for \( 3 < \alpha \leq 4 \) the state is entangled and PPT. Then we conclude that it is BE. For \( \alpha > 4 \) Alice and Bob can apply local projectors \( P = |0\rangle\langle 0| + |1\rangle\langle 1| \) obtaining entangled two-qubit state. Hence the initial state is FE in this region of \( \alpha \).

12 Some consequences of existence of bound entanglement

A basic question that arises in the context of bound entanglement is: What is its role in the quantum information theory? We will show in next sections, that despite it is indeed a very poor type of entanglement, it can produce non-classical effect, enhancing quantum communication via a subtle activation-like process [97]. This will lead us to a new paradigm of entanglement processing extending the “LQCC paradigm”. Moreover, the existence of bound entanglement means that there exist stronger limits for distillation rate than it was expected before. These, and other consequences we will report in the next few subsections.

12.1 Bound entanglement and teleportation

By definition, BE states cannot be distilled, hence it is impossible to obtain faithful teleportation via such state. However, it might be the case that the transmission fidelity of imperfect teleportation is still better than the one achievable by purely classical channel i.e. without sharing any entanglement (this was a way of revealing manifestation of quantum features of some mixed state [11]). First searches produced a negative result [34]. Here we present more general results, according to which the most general teleportation scheme cannot produce better than classical fidelity, if Alice and Bob share BE states.

12.1.1 General teleportation scheme

Teleportation, as originally devised [1], is a way of transmission of a quantum state by use of classical channel and bipartite entangled state (pure singlet state) shared by Alice and Bob. The most general scheme of teleportation, would be then of the following form [100]. There are three systems: the one of the input particle, the state of which is to be teleported (ascribe to this system

\[\text{For detailed study of standard teleportation scheme via mixed two-qubit state see [5]. The optimal one-way teleportation via pure states was obtained in [1]}.\]
the Hilbert space $\mathcal{H}_A'$, and two systems, that are in the entangled state $\varrho_{AB}$ (with Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$). For simplicity we assume that $\dim \mathcal{H}_A' = \dim \mathcal{H}_A = \dim \mathcal{H}_B = d$. The initial state is

$$|\psi_A'\rangle\langle\psi_A'| \otimes \varrho_{AB}$$

where $\psi_A'$ is the state to be teleported (unknown to Alice and Bob). Now Alice and Bob perform some trace preserving LQCC operation (trace preserving, because teleportation, is the operation that must be performed with probability 1). The form of the operation depends on the state $\varrho_{AB}$ that is known to Alice and Bob, but are independent of the input state $\psi_A'$ because it is unknown. Now the total system is in new, perhaps very complicated state $\varrho_{A'AB}$. The transmitted state is given by $\text{Tr}_{A'}(\varrho_{A'AB})$. The overall transmission stages are the following

$$\psi_A' \rightarrow |\psi_A'\rangle\langle\psi_A'| \otimes \varrho_{AB} \rightarrow \Lambda(|\psi_A'\rangle\langle\psi_A'| \otimes \varrho_{AB}) \rightarrow \text{Tr}_{A'} \varrho_{A'AB} = \varrho_B.$$ 

Now the transmission fidelity is defined by

$$f = \langle \psi_A'| \varrho_B | \psi_A' \rangle,$$

where the average is taken over uniform distribution of the input states $\psi_A'$.

In the original teleportation scheme (where $\varrho_{AB}$ is a maximally entangled state), the state $\varrho_B$ is exactly equal to the input state, so that $f = 1$. If Alice and Bob share a pair in separable state (or, equivalently, share no pair), then the best one can do is the following: Alice measures the state and sends the results to Bob. Since it is impossible to find the form of the state having only a single system in that state (it would contradict no-cloning theorem) the performance of such process will be very poor. One can check that the best possible fidelity is $f = 2/(d+1)$. If the shared pair is entangled, but it is not a pure maximally entangled state, we will obtain some intermediate values of $f$.

12.1.2 Optimal teleportation

Having defined the general teleportation scheme, one can ask about the maximal fidelity that can be achieved for given state $\varrho_{AB}$ within the scheme. Thus, for given $\varrho_{AB}$ we must maximise $f$ over all possible trace-preserving LQCC operations. The problem is, in general, extremely difficult. However the high symmetry of the chosen fidelity function allows to reduce it in the following way.

It has been shown that the best Alice and Bob can do is the following. They first perform some LQCC action that aims at increasing $F(\varrho_{AB})$ as much as possible. Then they perform the standard teleportation scheme, via the new state $\varrho'_{AB}$ (just as if it were the state $P_+$). The obtained fidelity is given by

$$f_{\text{max}} = \frac{F_{\text{max}}d + 1}{d + 1},$$

where $F_{\text{max}} = F(\varrho'_{AB})$ is the maximal $F$ that can be obtained by trace-preserving LQCC actions if the initial state is $\varrho_{AB}$.

12.1.3 Teleportation via bound entangled states

According to (67), to check the performance of teleportation via BE states of $d \otimes d$ system, we should find maximal $F$ attainable from BE states via trace-preserving LQCC actions. As it was noted that so defined fidelity is not a unique criterion of performance of teleportation. For example, one can consider restricted input: Alice receives one of two nonorthogonal vectors with some probabilities. Then the formula for fidelity would be different. In general, fidelity is determined by a chosen distribution over input states.
argued in Sect. 8, a BE state subjected to any LQCC operation remains BE. Moreover, singlet fraction $F$ of a BE states of $d \otimes d$ system satisfies $F \leq 1/d$ (because states with $F > 1/d$ are distillable, as shown in Sect. 8). We conclude that if the initial state is BE then the highest $F$ achievable by any (not only trace preserving) LQCC actions is $F = 1/d$. However, as we have argued, this gives fidelity $f = 2/(d+1)$, that can be achieved without entanglement, too. Thus the BE states behave here like separable states – their entanglement does not manifest itself.

12.2 Activation of bound entanglement

Here we will show that bound entanglement can produce a non-classical effect, even though the effect is very subtle one. It is the so-called activation of bound entanglement [97]. The underlying concept originates from formal entanglement-energy analogy developed in [104, 96, 14, 79, 14]. One can imagine that the bound entanglement is like energy of the system confined in a shallow potential well. Then, as in the process of chemical activation, if we add a small amount of extra energy to the system, its energy can be liberated.

In our case, the role of the system will be played by a huge amount of bound entangled pairs, while the extra energy – by a single pair that is free entangled. More specifically, we will show that a process called conclusive teleportation [105] can be performed with arbitrarily high fidelity if Alice and Bob can perform joint operations over the BE pairs and the FE pair. We will argue that it is impossible if either of the two elements is lacking.

12.2.1 Conclusive teleportation

Suppose that Alice and Bob have a pair in a state for which the optimal teleportation fidelity is $f_0$. Suppose further, that the fidelity is too poor for some Alice and Bob purposes. What they can do to change the situation is to perform the so-called conclusive teleportation. Namely, they can perform some LQCC operation with two final outcomes 0 and 1. Obtained the outcome 0 they fail and decide to discard the pair. If the outcome is 1 they perform teleportation, and the fidelity is now better than the initial $f_0$. Of course, the price they must pay is that the probability of the success (outcome 1) may be small. The scheme is illustrated on Fig. 6.

A simple example is the following. Suppose that Alice and Bob share a pair in pure state $\psi = a|00\rangle + b|11\rangle$ which is nearly product (e.g. $a$ is close to 1). Then the standard teleportation scheme provides a rather poor fidelity $f = 2(1 + ab)/3$ [106, 98]. However, Alice can subject her particle to filtering procedure [83, 84] described by the operation

\[ \Lambda = W(\cdot)W^\dagger + V(\cdot)V^\dagger \]

with $W = \text{diag}(b, a)$, $V = \text{diag}(a, b)$. Here the outcome 1 (success) corresponds to operator $W$. Indeed, if this outcome was obtained, the state collapses to the singlet one

\[ \tilde{\psi} = \frac{W \otimes I \psi}{||W \otimes I \psi||} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \]

Then, in this case perfect teleportation can be performed. Thus, if Alice and Bob teleported directly via the initial state, they would obtain a very poor performance. Now, they have a small, but nonzero chance of performing perfect teleportation.

Similarly as in the usual teleportation, the conclusive teleportation can be reduced to conclusive increasing $F$ (illustrated on Fig. 6) followed by the original teleportation protocol. If in the first stage Alice and Bob obtain a state with some $F$, then the second stage will produce the corresponding fidelity $f = (Fd + 1)/(d + 1)$. Thus we can restrict our consideration to conclusive
Figure 6: Conclusive teleportation. Starting with a weakly entangled pair Alice and Bob prepare with probability $p$ a strongly entangled pair and then perform teleportation.

Figure 7: Conclusive increasing of singlet fraction. Alice and Bob with probability $p$ of success obtain a state with higher singlet fraction than the one of the initial state.
increasing singlet fraction. The latter was developed in [85, 100]. An interesting peculiarity of conclusive increasing singlet fraction is that sometimes it is impossible to obtain $F = 1$, but still $F$ arbitrarily close to 1 can be obtained. However, if $F \to 1$ then the probability of success tends to 0, so that, indeed, it is impossible to reach $F = 1$ [100].

12.2.2 Activation protocol

Suppose that Alice and Bob share a single pair of spin-1 particles in the following free entangled mixed state

$$\varrho_{\text{free}} = \varrho(F) \equiv F|\psi_+\rangle\langle\psi_+| + (1 - F)\sigma_+, \quad 0 < F < 1,$$

(70)

where $\sigma_{\pm}$ are separable states given by (41). It is easy to see that the state (70) is free entangled. Namely after action of the local projections ($|0\rangle\langle0| + |1\rangle\langle1| \otimes (|0\rangle\langle0| + |1\rangle\langle1|)$ we get an entangled $2 \otimes 2$ state (its entanglement can be revealed by calculating partial transposition). Thus, according to the Theorem 4, the state (70) is FE. By complicated considerations one can show [100] that there is a threshold $F_0 < 1$ that cannot be exceeded in the process of conclusive increasing singlet fraction. In other words Alice and Bob have no chance to obtain a state $\varrho'$ with $F(\varrho') > F_0$ (we do not know the value $F_0$, we only know that such a number exists).

Suppose now that Alice and Bob share in addition a very large number of pairs in the following BE state (the one considered in Sect. (11))

$$\sigma_\alpha = 2^7|\psi_+\rangle\langle\psi_+| + \frac{\alpha}{7}\sigma_+ + \frac{5 - \alpha}{7}\sigma_-.$$

(71)

As stated in Sect. 11 for $3 < \alpha \leq 4$ the state is BE. As one knows, from the BE pairs of $3 \otimes 3$ system there is no chance to obtain even a pair with $F > 1/3$. Now it turns out that if Alice and Bob have both FE pair and the BE pairs, they can apply a simple protocol to obtain $F$ arbitrarily close to 1. Thus, due to the connection between conclusive increasing singlet fraction and conclusive teleportation, the fidelity of the latter can be arbitrarily close to unity only if both FE pair and BE pairs are shared.

The protocol [97] is similar to the recurrence distillation protocol described in Sect. 7.1. It is an iteration of the following two steps

(i) Alice and Bob take the free entangled pair in the state $\varrho_{\text{free}}(F)$ and one of the pairs being in the state $\sigma_\alpha$. They perform the bilateral XOR operation $U_{\text{BXOR}} \equiv U_{\text{XOR}} \otimes U_{\text{XOR}}$, each of them treating the member of free (bound) entangled pair as a source (target)

(ii) Alice and Bob measure the members of source pair in basis $|0\rangle, |1\rangle, |2\rangle$. Then they compare their results via classical communication. If the compared results differ from each other they have to discard both pairs and then the trial of improvement of $F$ fails. If the results agree then the trial succeeds and they discard only the target pair, coming back with (as we shall see) improved source pair to the first step (i).

After some algebra one can see that the success in the step (ii) occurs with nonzero probability

$$P_{F \to F'} = \frac{2F + (1 - F)(5 - \alpha)}{7}$$

(73)

22Here we need the quantum XOR gate not for two qubits as in Sect. 7.1 but for two qutrits (three-level systems). A general XOR operation for $d \otimes d$ system that was used in in [33, 107] is defined as

$$U_{\text{XOR}}|a\rangle|b\rangle = |a\rangle(|b + a\rangle \mod d),$$

(72)

where initial state $|a\rangle$ ($|b\rangle$) corresponds to source (target) state.
Figure 8: Liberating bound entanglement. The singlet fraction of the FE state is plotted versus the number of successful iterations (i-ii) and the parameter $\alpha$ of the state $\rho_{\alpha}$ of the used BE pairs. The initial singlet fraction of the FE pair is taken $F_{in} = 0.3$ (This figure is reproduced from Phys. Rev. Lett. 82, 1056 (1999) by permission of authors.)

leading then to the transformation $\rho(F) \rightarrow \rho(F')$ with the improved fidelity

$$F'(F) = \frac{2F}{2F + (1 - F)(5 - \alpha)}.$$  (74)

If only $\alpha > 3$, then the above continuous function of $F$ exceeds the value of $F$ on the whole region $(0, 1)$. Thus the successful repeating of the steps (i-ii) produces the sequence of source fidelities $F_n \rightarrow 1$. On Fig. 8 we plotted the obtained $F$ versus number of iteration of the protocol and the parameter $\alpha$. For $\alpha \leq 3$ the singlet fraction goes down: separable states cannot help to increase it. We can see the dramatic qualitative change at the “critical” point that occurs at the borderline between separable states and bound entangled ones ($\alpha = 3$). On the other hand it is surprising that there is no qualitative difference between the behaviour of BE states ($3 < \alpha \leq 4$) and FE states ($4 < \alpha \leq 5$). Here the change is only quantitative while the shape of the corresponding curves is basically the same. To authors knowledge this is the only effect we know about, where the bound entanglement manifests its quantumness. Since the effect is very subtle, one must conclude that bound entanglement is essentially different from the free entanglement, and it is enormously weak.

12.3 Entanglement enhanced LQCC operations

The activation effect suggests to extend the paradigm of LQCC operations by including quantum communication (under suitable control). Then we obtain entanglement enhanced LQCC (LQCC+EE) operations (see [108] in this context). For example, if we allow LQCC operation and arbitrary amount of shared bound entanglement, we obtain LQCC+BE paradigm. One can now ask about entanglement of formation and distillation in this regime. Since the BE states contain entanglement, even though very weak, then infinite amount of it could make $D_{LQCC+BE}$ much larger than usual $D_{LQCC}$: one might expect $D_{LQCC+BE}$ to be maximal possible, independently of the input state $\rho$ ([11] for two-qubit pairs we would have $D_{LQCC+BE} = 1$ for any state). In [57, 112] it was shown that it is impossible. The argument of [57] is as follows. First, the authors recall that $D_{LQCC} \leq F_{LQCC}^F$ ([7]). Otherwise it would be possible to increase entanglement by means of LQCC actions. Indeed, suppose, that for some state $\rho$ we have $D_{LQCC}(\rho) > F_{LQCC}^F(\rho)$. Then Alice and Bob could take $n$ two-qubit pairs in singlet state, produce $n/F_{LQCC}^F$ pairs of the state $\rho$. Then they could distill $n(D_{LQCC}/F_{LQCC}^F)$ singlets, that would be greater number than $n$. A similar argument is applied to $LQCC+BE$ action: the authors show that it is impossible to increase number of singlet pairs by LQCC+BE actions and conclude that $D_{LQCC+BE} \leq F_{LQCC+BE}^F$. On the other hand, obviously we have $F_{LQCC+BE}^F \leq F_{LQCC}^F$. Combining the inequalities we obtain that $D_{LQCC+BE}$ is bounded by the usual entanglement of formation $F_{LQCC}^F$, that is maximal.

The term “critical” we used here reflects the rapid character of the change (see [53] for a similar “phase transition” between separable and FE states). On they other hand, the present development of thermodynamical analogies in entanglement processing [104, 106, 18, 79, 14] allows to hope that in future one will be able to build a synthetic theory of entanglement based on thermodynamical analogies: then the “critical” point would become truly critical.

In multipartite case two other effects have been recently found [110].

Entanglement of formation $E_{LQCC}^F(\rho)$ of a state $\rho$ is the amount of input singlet pairs per output pair needed to produce the state $\rho$ by LQCC operations [17].
only for singlet type states. A different argument in [112] bases on Rains results on bounds for distillation of entanglement [89] (see Sect. 12.4). Thus, even though employing infinite amount of BE pairs the LQCC+BE operations are not enormously powerful. However, it is still possible, that they are better than LQCC themselves, i.e. one conjectures that $D_{LQCC+BE}(\varrho) > D_{LQCC}(\varrho)$ for some states $\varrho$.

12.4 Bounds for entanglement of distillation

Bound entanglement is an achievement in qualitative description, however, as we could see in previous section it has an impact on quantitative approach. Here we will see, that it helped to obtain a strong upper bound for entanglement of distillation $D$ (recall that the latter has the meaning of the capacity of the noisy teleportation, channel constituted by bipartite mixed states, hence is a central parameter of quantum communication theory).

The first upper bound for $D$ was entanglement of formation [17] calculated explicitly for two-qubit states [53]. However, a stronger bound has been provided in [19] (see also [91]). It is given by the following measure of entanglement [18, 19] based on the relative entropy

$$E_{VP}(\varrho) = \inf_{\sigma} S(\varrho|\sigma),$$

(75)

where infimum is taken over all separable states $\sigma$. The relative entropy is defined by

$$S(\varrho|\sigma) = \text{Tr}_\varrho \log \varrho - \text{Tr}_\varrho \log \sigma.$$

Vedral and Plenio provided a tricky argumentation [19] showing that $E_{VP}$ is upper bound for $D(\varrho)$, at additional assumption, that it is additive. Even though we still do not know if it is indeed additive, Rains showed [89] that it is a bound for $D$ even without this assumption. He also obtained a stronger bound by use of BE states (more precisely: PPT states). It appears, that if the infimum in (75) is taken over PPT states (that are bound entangled), then the new measure $E_R$ is a bound for distillable entanglement, too. However, since the set of PPT states is strictly greater than the set of separable states, the bound is stronger. For example, the entangled PPT states have zero distillable entanglement. Since they are not separable, $E_{VP}$ does not vanish for them, hence the evaluation of $D$ by means of $E_{VP}$ is too rough. The Rains measure vanishes for these states.

We will not provide here the original proof of the Rains result. Instead we demonstrate a general theorem on bounds for distillable entanglement obtained in [113], that allow essential simplification of the proof of the result.

**Theorem 6** Any function $B$ satisfying the conditions a)-c) below is an upper bound for entanglement of distillation:

a) Weak monotonicity: $B(\varrho) \geq B(\Lambda(\varrho))$ where $\Lambda$ is superoperator realizable by means of LQCC operations.

b) Partial subadditivity: $B(\varrho^\otimes n) \leq nB(\varrho)$

c) Continuity for isotropic state $\varrho(F,d)$: suppose that we have a sequence of isotropic states $\rho(F_d,d)$, (see Sect. 4, formula (37)) such that $F_d \to 1$ if $d \to \infty$. Then we require

$$\lim_{d \to \infty} \frac{1}{\log d} B(\varrho(F_d,d)) \to 1.$$

(76)

**Remarks.** If, instead of LQCC operations we take other class $C$ of operations including classical communication at least in one direction (e.g. the mentioned LQCC+BE operations), the proof *mutatis mutandis* also applies. (then the condition a) would involve the class $C$)
Proof. The main idea of the proof is to exploit the monotonicity condition: We will show that if $D$ were greater than $B$ then during distillation protocol the function $B$ would have to increase. But it cannot be so, because distillation is LQCC action, hence $B$ would violate the assumption a). By subadditivity we have

$$B(\rho) \geq \frac{1}{n} B(\rho^\otimes n).$$

(77)

Distillation of $n$ pairs aims at obtaining $k$ pairs each in nearly singlet state. Then the asymptotic rate is $\lim k/n$. It was shown [89] that equally well one can think of final $d \otimes d$ system in the state close to $P_d^{+}$. The asymptotic rate is now $\lim (\log d)/n$. Then the only relevant parameters of the final state $\rho_{\text{out}}$ is dimension $d$ and fidelity $F(\rho_{\text{out}})$. Thus distillation protocol can be followed by $U \otimes U^*$ twirling, producing isotropic final state $\rho(d, F)$ (see Sect. 4). By condition a), distillation does not increase $B$, hence

$$\frac{1}{n} B(\rho^\otimes n) \geq \frac{1}{n} B(\rho(F_{dn}, d_n)).$$

(78)

Now, in distillation process $F \to 1$, and if we consider optimal protocol, then $(\log d)/n \to D$. Hence, by condition c) the right hand side of the inequality tends to $D(\rho)$. Thus we obtain that $B(\rho) \geq D(\rho)$.}

Part IV

Concluding remarks

In contrast to pure states case the problem of mixed-state entanglement is “non-degenerate” in the sense that various scalar and structural separability criteria are not equivalent. There is a fundamental connection between entanglement and positive maps represented by Theorem 1. However, still there is a problem of turning it into an operational criterion for higher-dimensional systems. Recently [114, 115] the question was reduced to problem of investigation of the so-called “edge”
PPT entangled states as well as the positive maps and entanglement witnesses detecting their entanglement. Some operational criteria for low rank density matrices (also for multiparticle case) have been worked out in [116].

It is remarkable that structure of entanglement reveals discontinuity. There are two qualitatively different types of entanglement: distillable – “free” entanglement and the “bound” one, that cannot be distilled. All the two qubit entangled states are free entangled. Moreover, free entangled state in any dimension must have some features of two-qubit entanglement. The bound entanglement is practically useless for quantum communication. However it is not a marginal phenomenon, as the volume of the set BE states in the set of all states for finite dimension is nonzero.

Activation of bipartite bound entanglement suggested [97] nonadditivity of corresponding quantum communication channels in a sense, that distillable entanglement $D(\varrho_{BE} \otimes \varrho_{EF})$ could exceed $D(\varrho_{FE})$ for some free entangled $\varrho_{FE}$ and bound entangled $\varrho_{BE}$ state. Quite recently it has been shown [26] that in the multipartite case two different bound entangled states, if tensored together, can make a distillable state: $D(\varrho_{BE}^1 \otimes \varrho_{BE}^2) > D(\varrho_{BE}^1) + D(\varrho_{BE}^2) = 0$. This new nonclassical effect was called superactivation. On the other hand in [109] it was shown that the four-party “unlockable” bound entangled states [120] can be used for remote concentration of quantum information. It is intriguing that for bipartite systems, with the exception of the activation effect, the bound entanglement is permanently passive. In general, there may be a qualitative difference between bipartite bound entanglement and multipartite one. Still, in the light of the recent results [117] it is quite possible that also bipartite bound entanglement is nonadditive. The very recent investigations of bound entanglement for continuous variables [118, 119] rise analogous questions also in this latter domain.

As we have seen there is a basic connection between bound entanglement and irreversibility. Then it would be interesting to investigate some dynamical features of BE. It cannot be excluded that some systems involving BE states may reveal nonstandard (non-exponential) decay of entanglement. In general, it seems that the role of bound entanglement in quantum communication will be negative: in fact, existence of BE constitutes a fundamental restriction for entanglement processing. One can speculate that it is ultimate restriction in the context of distillation, i.e. that it may allow to determine the value distillable entanglement. Then it seems important to develop the approach combining BE and the entanglement measures involving relative entropy. It also seems reasonable to conjecture that in the case of the general distillation processes involving mixed states conversion $\varrho \rightarrow \varrho'$ [17] bound entanglement $E_B$ never decreases [27] (i.e. $\Delta E_B \geq 0$) in optimal processes.

The irreversibility inherently connected with distillation encourages to develop some natural formal analogies between mixed-state entanglement processing and phenomenological thermodynamics. The construction of “thermodynamics of entanglement” (cf. [104, 79, 96, 122]) would be essential for a synthetic understanding of entanglement processing. Of course, the progress in the above direction would require to develop various techniques of search for bound entangled states.

One of the challenges of mixed-state entanglement theory is to determine which states are useful for quantum communication at given additional resources. In particular, one still does not know: (i) which states are distillable under LOCC (i.e. which states are free entangled) (ii) which states are distillable under one-way classical communication and local operations.

A promising direction of mixed-state entanglement theory is application to the theory of quantum channel capacity, pioneered in [17]. In particular, the methods leading to upper bounds for distillable entanglement in Sect. [12, 4] allow to obtain upper bounds for quantum channel capacities 26It could be then reformulated in terms of the so called binding entanglement channels [111, 57].

27Bound entanglement can be quantified [4] as the difference between entanglement of formation and entanglement of distillation (defined within the original distillation scheme): $E_B = E_F - D$.
It has been shown that the following hypothetical inequality

\[ D_1(\varrho) \geq S(\varrho_B) - S(\varrho) \] (79)

where \( D_1(\varrho) \) is one-way distillable entanglement\(^{28}\), would imply equality between capacity of quantum channel and the maximal rate of coherent information \(^{124}\). The latter equality would be nothing but quantum Shannon theorem, with coherent information being the counterpart of mutual information. All the results obtained so far in the domain of quantifying entanglement indicate that the inequality is true. However, the proof of the inequality has not been found so far.

Finally, one would like to have a clear connection between entanglement and its basic manifestation – nonlocality. One can assume that free entangled states exhibit nonlocality via distillation process \(^{12,13}\). However, the question concerning possible nonlocality of BE states remains open (see \(^{126,127,128}\) in this context).

To answer the above and many other questions, one must develop the mathematical description of the structure of mixed-state entanglement. In this context, it would be especially important to push forward the mathematics of positive maps. One hopes that the exciting physics connected with by mixed-state entanglement we presented in this contribution will stimulate the progress in this domain.

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BE states = 3
(border-line between separable and BE states)