A new complemented subspace for the Lorentz sequence spaces, with an application to its lattice of closed ideals

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Abstract. We show that every Lorentz sequence space $d(w, p)$ admits a 1-complemented subspace $Y$ distinct from $\ell_p$ and containing no isomorph of $d(w, p)$. In the general case, this is only the second nontrivial complemented subspace in $d(w, p)$ yet known. We also give an explicit representation of $Y$ in the special case $w = (n^{-\theta})_{n=1}^{\infty}$ ($0 < \theta < 1$) as the $\ell_p$-sum of finite-dimensional copies of $d(w, p)$. As an application, we find a sixth distinct element in the lattice of closed ideals of $\mathcal{L}(d(w, p))$, of which only five were previously known in the general case.

1 Introduction

Little is known about the complemented subspace structure of Lorentz sequence spaces $d(w, p)$. Until recently, the only nontrivial complemented subspace discussed in the literature was $\ell_p$ [ACL73]. Then, in [Wa20], it was shown that for certain weights $w$ (see Theorem 2.2 below), the space $d(w, p)$ contains a 1-complemented subspace isomorphic to $(\bigoplus_{n=1}^{\infty} \ell_n^p)^p$. Up to now, these were the only nontrivial complemented subspaces known to exist.

In this short note, we show that each Lorentz sequence space admits a 1-complemented subspace $Y$ distinct from $\ell_p$ (Section 2). We also give an explicit representation of $Y$ for the case $w = (n^{-\theta})_{n=1}^{\infty}$ ($0 < \theta < 1$), as the $\ell_p$-sum of finite-dimensional copies of $d(w, p)$ (Section 3). Note that this choice of $w = (n^{-\theta})_{n=1}^{\infty}$ corresponds to the classical Lorentz sequence spaces $\ell_{q,p}$ with $p/q = 1 - \theta$. Finally, as an application, we find a sixth distinct element in the lattice of closed ideals in the operator algebra $\mathcal{L}(d(w, p))$, where only five were previously known in the general case (Section 4).

Let us set up the main notation we need to use. We begin by fixing $K \in \{\mathbb{R}, \mathbb{C}\}$. Denote by $\Pi$ the set of all permutations of $\mathbb{N}$, and denote by $\mathbb{W}$ the set of all sequences $w = (w_n)_{n=1}^{\infty} \in c_0 \setminus \ell_1$ satisfying

$$1 = w_1 \geq w_2 \geq w_3 \geq \cdots > 0.$$
Fix $1 \leq p < \infty$ and $w \in \mathbb{W}$. For each $(a_n)_{n=1}^{\infty} \in \mathbb{K}^\mathbb{N}$, we set

$$\| (a_n)_{n=1}^{\infty} \|_{d(w, p)} := \sup_{\pi \in \Pi} \left( \sum_{n=1}^{\infty} |a_{\pi(n)}|^p w_n \right)^{1/p},$$

and let $d(w, p)$ denote the linear space of all $(a_n)_{n=1}^{\infty} \in \mathbb{K}^\mathbb{N}$ with $\| (a_n)_{n=1}^{\infty} \|_{d(w, p)} < \infty$ endowed with the norm $\| \cdot \|_{d(w, p)}$, called a Lorentz sequence space. Recall that if $(a_n)_{n=1}^{\infty} \in c_0$ then there exists a “decreasing rearrangement” $(\hat{a}_n)_{n=1}^{\infty}$ of $(|a_n|)_{n=1}^{\infty}$. In this case, the rearrangement inequality gives us

$$\| (a_n)_{n=1}^{\infty} \|_{d(w, p)} = \left( \sum_{n=1}^{\infty} \hat{a}_n^p w_n \right)^{1/p} \quad \text{for all } (a_n)_{n=1}^{\infty} \in c_0. \tag{1.1}$$

Because $d(w, p) \subset c_0$ as linear spaces (although not as normed spaces), this represents an alternative formulation of the Lorentz sequence space norm.

For each $i, k \in \mathbb{N}$, we define

$$W_k := \sum_{n=1}^{k} w_n \quad \text{and} \quad w_{i}^{(k)} := \frac{1}{W_k} \sum_{n=(i-1)k+1}^{ik} w_n,$$

and $w^{(k)} := (w_{i}^{(k)})_{i=1}^{\infty}$. It is readily apparent that $w^{(k)} \in \mathbb{W}$. When $p$ is clear from context, we also set

$$d_{i}^{(k)} := \frac{1}{W_k^{1/p}} \sum_{n=(i-1)k+1}^{ik} d_n,$$

where $(d_{n})_{n=1}^{\infty}$ is the canonical unit vector basis for $d(w, p)$. It is routine to verify that $(d_{i}^{(k)})_{i=1}^{\infty}$ is a normalized basic sequence isometrically equivalent to the $d(w^{(k)}, p)$ basis. If necessary, we may sometimes abuse this notation; for instance, if $(j_{k})_{k=1}^{\infty}$ is a sequence in $\mathbb{N}$, then we could write $((d_{i}^{(j_{k})})_{i=1}^{k})_{k=1}^{\infty}$ for appropriately translated successive normalized constant-coefficient blocks of lengths $j_{k}$.

Our main tool for finding complemented subspaces of $d(w, p)$ is the fact that every constant-coefficient block basic sequence of a symmetric basis spans a 1-complemented subspace (cf., e.g., [LT77, Proposition 3.a.4]). We will use this well-known fact freely and without further reference.

2 Lorentz sequence spaces contain at least two nontrivial complemented subspaces

The first discovery of a nontrivial complemented subspace in $d(w, p)$ came almost half a century ago, with the following result.

**Theorem 2.1** ([ACL73, Lemma 1]) \quad Fix $1 \leq p < \infty$ and $w \in \mathbb{W}$, and let

$$x_i = \sum_{n=p_{i-1}+1}^{p_i} a_n d_n, \quad i \in \mathbb{N},$$
form a seminormalized block basic sequence in \( d(w, p) \). If \( a_n \to 0 \), then \( (x_i)_{i=1}^\infty \) admits a subsequence equivalent to \( \ell_p \) and complemented in \( d(w, p) \).

By taking sufficiently long constant-coefficient blocks, it follows that \( d(w, p) \) contains a \( 1 \)-complemented copy of \( \ell_p \). Much later was shown the following.

**Theorem 2.2** ([Wa20, Theorem 4.3]) Let \( 1 \leq p < \infty \) and \( w = (w_n)_{n=1}^\infty \in \mathbb{W} \). If

\[
\inf_{k \in \mathbb{N}} \frac{\sum_{n=1}^{2k} w_n}{\sum_{n=1}^{k} w_n} = 1,
\]

then \( d(w, p) \) admits a \( 1 \)-complemented subspace spanned by constant-coefficient blocks and isomorphic to \( (\bigoplus_{n=1}^\infty \ell_\infty^p) \).

Thanks in large part to the ideas of William B. Johnson, our main result in this section is to generalize this to all Lorentz sequence spaces, as follows.

**Theorem 2.3** Let \( 1 \leq p < \infty \) and \( w \in \mathbb{W} \). Then, there exists an increasing sequence \( (N_k)_{k=1}^\infty \in \mathbb{N}^\mathbb{N} \) such that \( (d^{(N)}_{i=1})_{k=1}^\infty \) spans a \( 1 \)-complemented subspace \( Y \) which contains no isomorph of \( d(w, p) \) and which is not isomorphic to \( \ell_p \).

To prove it, we need a few preliminaries.

**Lemma 2.4** If \( 1 < p < \infty \), then every complemented subspace of \( L_p[0,1] \) with a subsymmetric basis \((x_n)_{n=1}^\infty\) is isomorphic to either \( \ell_p \) or \( \ell_2 \).

**Proof** The case \( p = 2 \) is trivial, because every complemented subspace of \( L_2[0,1] \) is isomorphic to \( \ell_2 \). For the case \( p > 2 \), recall from [KP62, Corollary 6] that every seminormalized basic sequence in \( L_p[0,1], p \in (2, \infty) \), admits a subsequence equivalent to \( \ell_p \) or \( \ell_2 \), and so because \((x_n)_{n=1}^\infty\) is also subsymmetric, then it is in fact equivalent to \( \ell_p \) or \( \ell_2 \). In case \( 1 < p < 2 \), because \((x_n)_{n=1}^\infty\) is complemented in \( L_p[0,1] \), its corresponding sequence of biorthogonal functionals \((x_n^*)_{n=1}^\infty\) is contained in \( L_p'[0,1] \), where \( \frac{1}{p} + \frac{1}{p'} = 1 \). Because \( p' > 2 \), a subsequence of \((x_n^*)_{n=1}^\infty\) is equivalent to \( \ell_{p'} \) or \( \ell_2 \), whence by subsymmetry \((x_n)_{n=1}^\infty\) is equivalent to \( \ell_p \) or \( \ell_2 \).

**Lemma 2.5** Let \( X \) be a Banach space whose canonical isometric copy in \( X^{**} \) is complemented. Then, for any free ultrafilter \( U \) on \( \mathbb{N} \), the canonical copy of \( X \) in \( X^U \) is complemented in \( X_U \).

**Proof** Let \( q : X \to X^{**} \) denote the canonical embedding, and define the norm-1 linear operator \( V : \ell_\infty(X) \to X^{**} \) by the rule

\[
V(x_n)_{n=1}^\infty = \liminf_{n \to \infty} q x_n,
\]

which exists by the weak\(^*\)-compactness of \( B_{X^{**}} \) together with the fact that if \( K \) is a compact Hausdorff space, then for each \((k_n)_{n=1}^\infty \in K^\mathbb{N}\), the (unique) limit \( \lim_U k_n \)
exists in $K$. Note that if $\lim_{n} x_{n} = 0$, then $V(x_{n})_{n=1}^{\infty} = 0$, and so $V$ induces an operator $\tilde{V} : X^{\ell} \to X^{\star\star}$ which agrees with $V$ along the diagonal. In particular, $\tilde{V}$ sends the canonical copy of $X$ in $X^{\ell}$ isomorphically to the canonical copy of $X$ in $X^{\star\star}$.

**Theorem 2.6** Fix $1 \leq p < \infty$, and let $(x_{n})_{n=1}^{\infty}$ be a basis for a Banach space $X$ whose canonical copy in $X^{\star\star}$ is complemented. If the finite-dimensional spaces $[x_{n}]_{n=1}^{N}$, $N \in \mathbb{N}$, are uniformly complemented in $L_{p}(\mu)$ for some measure $\mu$, then $X$ is complemented in $L_{p}[0, 1]$.

**Proof** Let $X_{N} = [x_{n}]_{n=1}^{N}$ and $P_{N} : X \to X_{N}$ the projection onto $X_{N}$. By uniform complementedness of $X_{N}$, we can find uniformly bounded linear operators $A_{N} : X_{N} \to L_{p}(\mu)$ and $B_{N} : L_{p}(\mu) \to X_{N}$ such that $B_{N}A_{N}$ is the identity on $X_{N}$. Let $\mathcal{U}$ be any free ultrafilter on $\mathbb{N}$. Define the bounded linear operators $A : X \to L_{p}(\mu)^{\ell}$ by the rule $Ax = (A_{N}P_{N}x)_{\mathcal{U}}$, and $B : L_{p}(\mu)^{\ell} \to X^{\ell}$ by $B(y_{N})_{\mathcal{U}} = (B_{N}y_{N})_{\mathcal{U}}$. Let $x \in \operatorname{span}(x_{n})_{n=1}^{\infty}$, so that, for some $k \in \mathbb{N}$,

$$BAx = (P_{1}x, \ldots, P_{k}x, x, x, \ldots)_{\mathcal{U}} = x^{\ell}.$$

By continuity, $BA$ is the canonical injection of $X$ into $X^{\ell}$. Because its range is complemented by Lemma 2.5, we have the identity on $X$ factoring through $L_{p}(\mu)^{\ell}$.

It was proved in [He80, Theorem 3.3] that ultrapowers preserve $L_{p}$ lattice structure, and in particular $L_{p}(\mu)^{\ell}$ is isomorphic to $L_{p}(\nu)$ for some measure $\nu$. Although $L_{p}(\nu)$ itself is nonseparable, we could pass to the closed sublattice generated by $AX$ to find a space isomorphic to a separable $L_{p}$ containing a complemented copy of $X$. Due mostly to a famous result of Lacey and Wojtaszczyk, it is known that separable and infinite-dimensional $L_{p}$ spaces are isomorphic to either $\ell_{p}$ or $L_{p}[0, 1]$ [JL01, Section 4, p. 15]. This means an isomorph of $X$ is complemented in $L_{p}[0, 1]$.

An immediate corollary to Lemma 2.4 and Theorem 2.6 is as follows.

**Corollary 2.7** Let $1 < p < \infty$ and $w \in \mathbb{W}$. Then, no $L_{p}(\mu)$ space contains uniformly complemented copies of $[d_{n}]_{n=1}^{N}$, $N \in \mathbb{N}$.

Now, we are ready to prove the main result of this section, Theorem 2.3.

**Proof** Fix $k \in \mathbb{N}$, and note that $(d_{i}^{(k)})_{i=1}^{N_{k}}$ is isometric to the $d(w^{(k)}, p)$ basis. Consider the case where $p = 1$. Then, we can choose the $N_{k}$’s large enough that each $(d_{i}^{(k)})_{i=1}^{N_{k}}$ fails to be $k$-equivalent to $\ell_{1}^{N_{k}}$, and hence $((d_{i}^{(k)})_{i=1}^{N_{k}})_{k=1}^{\infty}$ fails to be equivalent to $\ell_{1}$. As $\ell_{1}$ has a unique unconditional basis by a result of Lindenstrauss and Pełczyński, it follows that $Y$ is not isomorphic to $\ell_{1}$.

Next, consider the case where $1 < p < \infty$. By Corollary 2.7, we can select $N_{k}$’s large enough that $[d_{i}^{(k)}]_{i=1}^{N_{k}}$ fails to be $k$-complemented in $\ell_{p}$. As $[d_{i}^{(k)}]_{i=1}^{N_{k}}$’s are all 1-complemented in $Y$, that means $Y$ is not isomorphic to $\ell_{p}$.

It remains to show that $Y$ contains no isomorph of $d(w, p)$. Suppose toward a contradiction that it does. As $(d_{n})_{n=1}^{\infty}$ is weakly null (cf., e.g., [ACL73, Proposition 1]), we can use the gliding hump method together with symmetry to find a normalized
A new complemented subspace for the Lorentz sequence spaces block sequence of \(( (d_i^{(k)})_{i=1}^{N_k})_{k=1}^{\infty}\) equivalent to \((d_n)_{n=1}^{\infty}\). However, every such block sequence is also a block sequence w.r.t. \((d_n)_{n=1}^{\infty}\) with coefficients tending to zero. By Theorem 2.1, it follows that \((d_n)_{n=1}^{\infty}\) admits a subsequence equivalent to \(\ell_p\), which is impossible.

3 A special case

In this section, we show that when \(w = (n^{-\theta})_{n=1}^{\infty}\) for some fixed \(0 < \theta < 1\), the space \(Y\) described in Theorem 2.3 can be chosen to be isomorphic to the space

\[
Y_{w,p} := \left( \bigoplus_{N=1}^{\infty} D_N \right)_p,
\]

where \(D_N := [d_n]_{n=1}^{N}\), for each \(N \in \mathbb{N}\). As usual, we require some preliminaries.

Lemma 3.1 Let \(0 < \theta < 1\) and \(j, k \in \mathbb{N}\). Then,

\[
\left( \frac{j+1}{k+1} \right)^{1-\theta} - \left( \frac{j+1}{k} \right)^{1-\theta} \leq \frac{\sum_{n=j+1}^{j+k} n^{-\theta}}{\sum_{n=1}^{k} n^{-\theta}} \leq \frac{(j/k+1)^{1-\theta} - (j/k)^{1-\theta}}{2^{1-\theta} - 1}.
\]

Proof Observe that the map

\[
f(t) = (1 + 1/t)^{1-\theta} - (1/t)^{1-\theta}
\]

is increasing on \([1, \infty)\) and hence has a minimum \(f(1) = 2^{1-\theta} - 1\). Hence,

\[
\left( \frac{j+1}{k+1} \right)^{1-\theta} - \left( \frac{j+1}{k} \right)^{1-\theta} \leq \frac{(j+k+1)^{1-\theta} - (j+1)^{1-\theta}}{k^{1-\theta} - \theta}
\]

\[
= \int_{j+1}^{j+k+1} t^{-\theta} dt
\]

\[
\leq \int_{1}^{k} t^{-\theta} dt
\]

\[
= \frac{\sum_{n=j+1}^{j+k} n^{-\theta}}{\sum_{n=1}^{k} n^{-\theta}}
\]

\[
\leq \frac{\int_{j}^{j+k} t^{-\theta} dt}{\int_{1}^{k+1} t^{-\theta} dt}
\]

\[
= \frac{(j+k)^{1-\theta} - j^{1-\theta}}{(k+1)^{1-\theta} - 1}
\]

\[
= \frac{(j/k+1)^{1-\theta} - (j/k)^{1-\theta}}{(1 + 1/k)^{1-\theta} - (1/k)^{1-\theta}}
\]

\[
\leq \frac{(j/k+1)^{1-\theta} - (j/k)^{1-\theta}}{2^{1-\theta} - 1}.
\]
Lemma 3.2  Let \(0 < \theta < 1\) and \(w = (w_n)_{n=1}^\infty = (n^{-\theta})_{n=1}^\infty \in \mathbb{W}\). Then,
\[
\frac{1 - \theta}{2} \cdot w_i \leq w_i^{(k)} \leq \frac{2 - 2^\theta}{2^{1-\theta} - 1} \cdot w_i \quad \text{for all } i, k \in \mathbb{N}.
\]
In particular, if \(1 \leq p < \infty\), then there is a constant \(C \in [1, \infty)\), depending only on \(\theta\), such that
\[
(d_n)_{n=1}^\infty \preceq C (d_i^{(k)})_{i=1}^\infty \quad \text{for all } k \in \mathbb{N}.
\]

Proof  We can assume \(i, k \geq 2\). Observe that
\[
t \mapsto t - (t - 1)^{1-\theta} \cdot t^\theta
\]
is decreasing on \([2, \infty)\), and hence has the maximum \(2 - 2^\theta\). Furthermore, the function
\[
t \mapsto t - (t - 1/2)^{1-\theta} \cdot t^\theta
\]
is decreasing on \([2, \infty)\) and hence has infimum
\[
\lim_{t \to \infty} (t - (t - 1/2)^{1-\theta} \cdot t^\theta) = \frac{1 - \theta}{2}.
\]
Thus, by the above, and applying Lemma 3.1 with \(j = k(i - 1)\),
\[
\frac{1 - \theta}{2} \cdot i^{-\theta} \leq \left( i - (i - 1/2)^{1-\theta} \cdot i^\theta \right) i^{-\theta}
\]
\[
= i^{1-\theta} - (i - 1/2)^{1/\theta}
\]
\[
\leq (i + 1/k)^{1-\theta} - (i - 1 + 1/k)^{1/\theta}
\]
\[
\leq \frac{\sum_{n=(i-1)k+1}^{ik} n^{-\theta}}{\sum_{n=1}^{k} n^{-\theta}}
\]
\[
\leq \frac{i^{1-\theta} - (i - 1)^{1-\theta}}{2^{1-\theta} - 1}
\]
\[
= \frac{i - (i - 1)^{1-\theta} \cdot i^\theta}{2^{1-\theta} - 1} \cdot i^{-\theta}
\]
\[
\leq \frac{2 - 2^\theta}{2^{1-\theta} - 1} \cdot i^{-\theta}. \quad \blacksquare
\]

Remark 3.3  Suppose \(x = \sum_{n \in A} a_n d_n\) and \(y = \sum_{n \in B} b_n d_n\) for finite and disjoint sets \(A, B \subset \mathbb{N}\), where \((a_n)_{n \in A}\) and \((b_n)_{n \in B}\) are sequences of scalars. Then,
\[
\|x + y\|^p \leq \|x\|^p + \|y\|^p.
\]

Lemma 3.4  Let \((j_k)_{k=1}^\infty\) be a sequence of positive integers, and, for each \(k\), set
\[
J_k = j_1 + 2j_2 + 3j_3 + \cdots + k j_k.
\]
Suppose that there are constants $A, B \in (0, \infty)$ such that
\begin{equation}
\tag{3.1}
w_i^{(j_k)} \leq Aw_i
\end{equation}
and
\begin{equation}
\tag{3.2}
Bw_i \leq \frac{1}{W_{j_k}} \sum_{n=J_{k-1}+j_k}^{J_{k-1}+j_k} w_n,
\end{equation}
for all $i = 1, \ldots, k$ and all $k \in \mathbb{N}$. Then, $((d_i^{(j_k)})_{i=1}^\infty)_{k=1}^\infty$ is equivalent to the canonical $Y_{w,p}$ basis.

**Proof** Due to (3.1), we have $((d_i^{(j_k)})_{i=1}^k) \leq_A d(w, p)^k$. Now, using Remark 3.3, for any finitely supported scalar sequence $((a_i^{(k)})_{i=1}^k)_{k=1}^\infty$,
\[
\left\| \sum_{k=1}^\infty \sum_{i=1}^k a_i^{(k)} d_i^{(j_k)} \right\|_p \leq A \left\| \sum_{k=1}^\infty \sum_{i=1}^k a_i^{(k)} d_i^{(j_k)} \right\|_p \leq A^p \sum_{k=1}^\infty \left\| (a_i^{(k)})_{i=1}^k \right\|_{d(w,p)}^p = A^p \left\| ((a_i^{(k)})_{i=1}^k)_{k=1}^\infty \right\|_{Y_{w,p}}^p.
\]

For the reverse inequality, let $(\hat{a}_i^{(k)})_{i=1}^k$ denote the decreasing rearrangement of $(|a_i^{(k)}|)_{i=1}^k$. Then, applying (3.2),
\[
\left\| \sum_{k=1}^\infty \sum_{i=1}^k a_i^{(k)} d_i^{(j_k)} \right\|_p \geq \left\| \sum_{k=1}^\infty \sum_{i=1}^k \hat{a}_i^{(k)} d_i^{(j_k)} \right\|_p \geq B \sum_{k=1}^\infty \sum_{i=1}^k \hat{a}_i^{(k)} w_i \geq B \left\| ((a_i^{(k)})_{i=1}^k)_{k=1}^\infty \right\|_{Y_{w,p}}^p.
\]

**Theorem 3.5** Let $(j_k)_{k=1}^\infty$ and $(J_k)_{k=1}^\infty$ be as in Lemma 3.4. Suppose there is $M \in [1, \infty)$ such that
\[
\frac{J_{k-1}}{j_k} \leq M, \quad \text{for all } k = 2, 3, 4, \ldots.
\]
Then, $((d_i^{(j_k)})_{i=1}^k)_{k=1}^\infty$ is equivalent to the canonical $Y_{w,p}$ basis.

**Proof** Due to Lemma 3.4, it suffices to show that (3.2) and (3.1) both hold. To do this, fix an arbitrary $k \in \mathbb{N}$. We may assume, without loss of generality, that $j_k \geq 2$. 
Now, by Lemma 3.1,
\[
\frac{1}{W_{jk}} \sum_{n=J_{k-1} + (i-1)j_k + 1}^{J_{k-1} + (i-1)j_k + 1} w_n \geq \left( \frac{J_{k-1} + (i-1)j_k + 1}{j_k} - \frac{J_{k-1} + (i-1)j_k + 1}{j_k} \right)^{1-\theta}.
\]
\[
= \left( \frac{J_{k-1} + i + 1}{j_k} - \frac{J_{k-1} + i - 1 + 1}{j_k} \right)^{1-\theta}.
\]
\[
\geq \left( \frac{J_{k-1} + i}{j_k} - \frac{J_{k-1} + i - 1 + 1}{j_k} \right)^{1-\theta}.
\]
\[
= i^{\theta} \left[ \left( \frac{J_{k-1} + i}{j_k} \right)^{1-\theta} - \left( \frac{J_{k-1} + i - 1}{j_k} \right)^{1-\theta} \right].
\]
Applying the Mean Value Theorem to the function \( x \mapsto (\phi + x)^{1-\theta}, \phi \in [1, \infty), \) we can find \( x_\phi \in (-1/2, 0) \) such that
\[
\phi^{1-\theta} - (\phi + x_\phi)^{1-\theta} = \frac{(1-\theta)(\phi + x_\phi)^{-\theta}}{2} = \frac{(1-\theta)x^{-\theta}}{2}.
\]
Hence, letting \( \phi = J_{k-1}/j_k + i, \) we have
\[
i^{\theta} \left[ \left( \frac{J_{k-1} + i}{j_k} \right)^{1-\theta} - \left( \frac{J_{k-1} + i - 1}{j_k} \right)^{1-\theta} \right] \geq i^{\theta} \left[ \frac{(1-\theta)(J_{k-1}/j_k + i)^{-\theta}}{2} \right].
\]
\[
1 - \theta \left( \frac{i}{J_{k-1}/j_k + i} \right)^{\theta} \geq \frac{1-\theta}{2} \left( \frac{1}{M+1} \right)^{\theta}.
\]
This proves (3.2), and (3.1) follows immediately from Lemma 3.2.

Taking inductively \( j_1 = 1 \) and \( j_{k+1} = J_k, \) the following is now immediate.

**Corollary 3.6** Let \( 1 \leq p < \infty, \ 0 < \theta < 1, \) and \( w = (w_n)_{n=1}^{\infty} = (n^{-\theta})_{n=1}^{\infty} \in \mathbb{W}. \) Then, \( d(w, p) \) admits a 1-complemented subspace isomorphic to \( Y_{w, p}. \)

### 4 Application to the lattice of closed ideals

In [KPSTT12], it was shown (among other results) that the lattice of closed ideals for the operator algebra \( \mathcal{L}(d(w, p)) \) can be put into a chain:

\[
\{0\} \not\subseteq \mathcal{K}(d(w, p)) \not\subseteq SS(d(w, p)) \not\subseteq S_{d(w, p)}(d(w, p)) \not\subseteq \mathcal{L}(d(w, p)).
\]

Here, \( \mathcal{K} \) denotes the compact operators, \( SS \) the strictly singular operators, and \( S_{d(w, p)} \) the ideal of operators which fail to be bounded below on any isomorph of \( d(w, p). \) While, in [Wa20, Corollary 2.7], for the special case where \( 1 < p < 2 \) and \( w \in \mathbb{W} \cap \ell_2/(\ell_2), \) a chain of distinct closed ideals with cardinality of the continuum were identified lying between \( \mathcal{K}(d(w, p)) \) and \( SS(d(w, p)), \) for the general case, the only distinct elements known were those of the above chain.
For an operator $T$, let $\mathcal{J}_T$ denote the class of operators factoring through $T$. If $Z$ is any Banach space, we then set $\mathcal{J}_Z = \mathcal{J}_{Id_Z}$. By Theorem 4.3 below, we can extend the chain above as follows:

$$\{0\} \not\subseteq \mathcal{K}(d(w, p)) \not\subseteq SS(d(w, p)) \not\subseteq (\overline{\mathcal{J}_p} \vee SS)(d(w, p))$$

$$\not\subseteq S_{d(w, p)}(d(w, p)) \not\subseteq \mathcal{L}(d(w, p)).$$

Furthermore, by [KPSTT12, Corollary 3.2 and Theorem 5.3] together with the fact that $d(w, p)$ has the approximation property, any additional distinct closed ideals in the above chain must lie between $\mathcal{K}(d(w, p))$ and $SS(d(w, p))$, or else between $(\overline{\mathcal{J}_p} \vee SS)(d(w, p))$ and $S_{d(w, p)}(d(w, p))$, although there may be other ideals in the lattice which are not a part of the chain.

To prove Theorem 4.3, we need a couple of preliminary results.

**Proposition 4.1** Let $X$ and $Z$ be an infinite-dimensional Banach spaces such that $Z^2 \cong Z$, and $X$ fails to be isomorphic to a complemented subspace of $Z$. Then, $\mathcal{J}_Z(X)$ is a proper ideal in $\mathcal{L}(X)$. Furthermore, if $P \in \mathcal{L}(X)$ is a projection with image isomorphic to $Z$, then

$$\mathcal{J}_P(X) = \mathcal{J}_Z(X).$$

**Proof** Because $Z^2 \cong Z$, [KPSTT12, Lemma 2.2] guarantees that $\mathcal{J}_Z(X)$ is an ideal in $\mathcal{L}(X)$. Suppose toward a contradiction that $Id_X \in \mathcal{J}_Z(X)$. Then, $Id_X = AB$ for operators $A \in \mathcal{L}(Z, X)$ and $B \in \mathcal{L}(X, Z)$. By [KPSTT12, Lemma 2.1], $BX$ is complemented in $Z$ and isomorphic to $X$, which contradicts our hypotheses. It follows that $\mathcal{J}_Z(X)$ is a proper ideal in $\mathcal{L}(X)$. Recall that the closure of a proper ideal in a unital Banach algebra is again proper; in particular, $\overline{\mathcal{J}_Z}(X)$ is a proper ideal in $\mathcal{L}(X)$.

To prove the “furthermore” part, assume $A \in \mathcal{L}(Z, X)$ and $B \in \mathcal{L}(X, Z)$. Let $Q : Z \to X$ be the canonical embedding, so that $PQ = Id_Z$ and hence $AB = APQB \in \mathcal{J}_P(X)$. It follows that $\mathcal{J}_Z(X) \subseteq \mathcal{J}_P(X)$, and the reverse inclusion is even more obvious. \[\square\]

For the next result, $\mathcal{F}$ denotes the class of finite-rank operators and $\mathcal{E}$ the class of inessential operators. Recall also that a basis $\mathcal{B}$ is called semисpreading whenever every subsequence of $\mathcal{B}$ is dominated by $\mathcal{B}$ itself. In particular, the unit vector basis of $\ell_p$ is semisprading.

**Proposition 4.2** ([LLR04, Corollary 3.8]) Let $Z$ be a Banach space with a semispreading basis $(z_n)$, and let $X$ be a Banach space with basis $(x_n)$ such that any seminormalized block sequence of $(x_n)$ contains a subsequence equivalent to $(z_n)$ and spanning a complemented subspace of $X$. Then,

$$\{0\} \not\subseteq \mathcal{F}(X) = \mathcal{K}(X) = \mathcal{E}(X) \subseteq \overline{\mathcal{J}_Z}(X),$$

and any additional distinct closed ideals must lie between $\overline{\mathcal{J}_Z}(X)$ and $\mathcal{L}(X)$.

In the proof of what follows, we use the fact that if $\mathcal{I}$ and $\mathcal{J}$ are ideals in $\mathcal{L}(X)$, then $\overline{\mathcal{I} \vee \mathcal{J}} = \overline{\mathcal{I}} + \overline{\mathcal{J}}$. 

Theorem 4.3  Fix $1 \leq p < \infty$ and $w \in \mathbb{W}$. Let $Y$ be as in Theorem 2.3, and $P_Y \in \mathcal{L}(d(w, p))$ any continuous linear projection onto $Y$. Then,

$$P_Y \in S_{d(w, p)}(d(w, p)) \cap (\overline{\mathcal{J}_Y} \vee SS)(d(w, p)).$$

Proof  Let $P_{\ell_p} \in \mathcal{L}(d(w, p))$ be any projection onto an isomorphic copy of $\ell_p$ spanned by basis vectors of $Y$. (Such a copy exists by Theorem 2.1.) By Theorem 2.3, $Y$ contains no isomorph of $d(w, p)$ and hence $P_Y \in S_{d(w, p)}(d(w, p))$. Because $S_{d(w, p)}(d(w, p))$ is the unique maximal ideal in $\mathcal{L}(d(w, p))$, and $\mathcal{J}_{P_{\ell_p}}(d(w, p)) = \mathcal{J}_{\ell_p}(d(w, p))$ by Proposition 4.1, it is sufficient to prove that $P_Y \notin (\overline{\mathcal{J}_{P_{\ell_p}}} \vee SS)(d(w, p))$.

Next, we claim that $P_Y \notin (\overline{\mathcal{J}_{P_{\ell_p}}} \vee SS)(d(w, p))$ only if $Id_Y \notin (\overline{\mathcal{J}_{\ell_p}} \vee SS)(Y)$. To prove it, fix $\varepsilon > 0$, and suppose there are $A, B \in \mathcal{L}(d(w, p))$ and $S \in SS(d(w, p))$ such that

$$\|AP_{\ell_p}B + S - P_Y\| < \varepsilon.$$

Let $J_Y : Y \to d(w, p)$ be an embedding satisfying $P_Y J_Y = J_Y$, or $P_Y J_Y = Id_Y$ when viewed as an operator in $\mathcal{L}(Y)$. Composing $P_Y$ on the left and $J_Y$ on the right, we have

$$\|P_Y AP_{\ell_p}BJ_Y + P_Y SJ_Y - Id_Y\|_{\mathcal{L}(Y)} < \|P_Y\| \cdot \varepsilon \cdot \|J_Y\|.$$

On the other hand, because $AP_{\ell_p} = A|_{\ell_p} P_{\ell_p}$ and $P_{\ell_p} = P_{\ell_p} P_Y$, we have

$$P_Y AP_{\ell_p}BJ_Y = (P_Y A|_{\ell_p}) P_{\ell_p} (P_Y B|_{\ell_p}),$$

and hence

$$\|(P_Y A|_{\ell_p}) P_{\ell_p} (P_Y B|_{\ell_p}) + P_Y SJ_Y - Id_Y\|_{\mathcal{L}(Y)} < \|P_Y\| \cdot \varepsilon \cdot \|J_Y\|.$$

Because $\mathcal{J}_{\ell_p}(Y) = \mathcal{J}_{P_{\ell_p}}(Y)$ by Proposition 4.1, where $P_{\ell_p}$ is likewise viewed as an operator in $\mathcal{L}(Y)$, from the above together with the ideal property of $SS$, the claim follows.

Let $B_Y = \{(d_i^{(k)})_{i=1}^{N_k}\}_{k=1}^{\infty}$ denote the canonical basis of $Y$ from Theorem 2.3. Note that because $B_Y$ is made up of constant coefficient blocks of $(d_n)$ of increasing length, any seminormalized blocks of $B_Y$ will contain a subsequence equivalent to $\ell_p$ by Theorem 2.1. In fact, in [CL74, Lemma 15], this result was refined to show that we can choose that subsequence to span a complemented subspace of $d(w, p)$, and hence of $Y$ itself. We can therefore apply Theorem 4.2 to conclude that $SS(Y) \subset \overline{\mathcal{J}_{\ell_p}}(Y)$. Meanwhile, again by Proposition 4.1, $\overline{\mathcal{J}_{\ell_p}}(Y)$ is a proper ideal in $\mathcal{L}(Y)$, which means $Id_Y \notin \overline{\mathcal{J}_{\ell_p}}(Y)$. Hence, $P_Y \notin (\overline{\mathcal{J}_{P_{\ell_p}}} \vee SS)(d(w, p))$ as desired. \hfill \blacksquare

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