Abstract

We classify all (-1)-homogeneous axisymmetric no-swirl solutions of incompressible stationary Navier-Stokes equations in three dimension which are smooth on the unit sphere minus the south and north poles, parameterizing them as a four dimensional surface with boundary in appropriate function spaces. Then we establish smoothness properties of the solution surface in the four parameters. The smoothness properties will be used in a subsequent paper where we study the existence of (-1)-homogeneous axisymmetric solutions with non-zero swirl on $S^2 \setminus \{S, N\}$, emanating from the four dimensional solution surface.

1 Introduction

Consider the incompressible stationary Navier-Stokes equations (NSE) in $\mathbb{R}^3$:

$$\begin{cases}
- \Delta u + u \cdot \nabla u + \nabla p = 0, \\
\text{div } u = 0.
\end{cases} \tag{1}$$

The equations are invariant under the scaling $u(x) \to \lambda u(\lambda x)$ and $p(x) \to \lambda^2 p(\lambda x)$, $\lambda > 0$. We study solutions which are invariant under the scaling. For such solutions $u$ is (-1)-homogeneous and $p$ is (-2)-homogeneous. We call them (-1)-homogeneous solutions according to the homogeneity of $u$.

We will write the NSE (1) in spherical coordinates $(r, \theta, \phi)$. A vector field $u$ can be written as

$$u = u_r e_r + u_\theta e_\theta + u_\phi e_\phi,$$
where
\[
e_r = \begin{pmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{pmatrix}, \quad e_\theta = \begin{pmatrix} \cos \theta \cos \phi \\ \cos \theta \sin \phi \\ -\sin \theta \end{pmatrix}, \quad e_\phi = \begin{pmatrix} -\sin \phi \\ \cos \phi \\ 0 \end{pmatrix}.
\]
A vector field \( u \) is called axisymmetric if \( u_r, u_\theta \) and \( u_\phi \) are independent of \( \phi \), and is called no-swirl if \( u_\phi = 0 \).

Landau discovered in [3] a three parameter family of explicit (-1)-homogeneous solutions of the stationary NSE (1), which are axisymmetric and with no swirl. These solutions are now called Landau solutions. The NSE (1) in the axisymmetric no-swirl case was converted earlier to an equation of Riccati type by Slezkin in [11]. The Riccati type equation was later independently derived by Yatseyev using a different method in [17], where various exact solutions were given. The Landau solutions were also independently found by Squire in [13]. Tian and Xin proved in [15] that all (-1)-homogeneous, axisymmetric nonzero solutions of (1) in \( C^2(\mathbb{R}^3 \setminus \{0\}) \) are Landau solutions. A classification of all (-1)-homogeneous solutions was given by Šverák in [14]; all (-1)-homogeneous nonzero solutions of (1) in \( C^2(\mathbb{R}^3 \setminus \{0\}) \) are Landau solutions. He also proved in the same paper that there is no nonzero (-1)-homogeneous solution of the stationary NSE in \( C^2(\mathbb{R}^n \setminus \{0\}) \) for \( n \geq 4 \). In dimension \( n = 2 \), he characterized all such solutions satisfying a zero flux condition.

In [10], Serrin modeled the tornado by (-1)-homogeneous axisymmetric solutions of the three dimensional incompressible stationary Navier-Stokes equations in the half space with zero boundary conditions and one singularity on the unit sphere.

More recently, Karch and Pilarczyk showed in [2] that Landau solutions are asymptotically stable under any \( L^2 \) perturbations. Classifications of homogeneous solutions to the 2-dimensional and 3-dimensional stationary Euler equations are studied respectively in [6] by Luo and Shvydkoy, and in [11] by Shvydkoy. More studies on (-1)-homogeneous axisymmetric solutions of the stationary NSE (1) can be found in [1], [7], [8], [9], [10] and [16].

We are interested in analyzing solutions which are smooth on \( S^2 \) minus finite points. We have classified in [1] all axisymmetric no-swirl solutions with one singularity at the south pole. They form a two dimensional surface with boundary in appropriate function spaces. These solutions are among the solutions found in [17], where the solutions were obtained by a different method. It was proved in [4] that there are no other solutions with precisely one singularity at the south pole. It was also proved there that there exists a curve of axisymmetric solutions with nonzero swirl emanating from every point in the interior and one part of the boundary of the surface of no-swirl solutions, while there is no such curve from any point on the other part of the boundary. Uniqueness results of nonzero swirl solutions near the no-swirl solution surface were also given in [4]. Our main result in this paper is the classification of all (-1)-homogeneous, axisymmetric no-swirl solutions of (1) which are smooth on \( S^2 \setminus \{S, N\} \), where \( S \) is the south pole and \( N \) is the north pole. They are identified as a 4-dimensional surface with boundary in appropriate function spaces. We have established smoothness properties of the solutions surface in the four parameters. These properties are used in a subsequent paper [5] where we study the existence of (-1)-homogeneous axisymmetric solutions.
with non-zero swirl on $S^2 \setminus \{S, N\}$, emanating from the 4-dimensional solution surface.

A $(-1)$-homogeneous axisymmetric vector field $u$ is divergence free if and only if
\begin{equation}
    u_r = -\frac{du_\theta}{d\theta} - u_\theta \cot \theta. \tag{2}
\end{equation}

We work with a new unknown function and a different independent variable:
\begin{equation}
    x := \cos \theta, \quad U_\theta := u_\theta \sin \theta. \tag{3}
\end{equation}

As explained in [4], $(u, p)$ is a $(-1)$-homogeneous axisymmetric no-swirl solution of (1) if and only if $u_\phi = 0$, $u_r$ is given by (2), $p$ is given by
\begin{equation}
    p = -\frac{1}{2} \left( \frac{d^2 u_r}{d\theta^2} + (\cot \theta - u_\theta) \frac{du_r}{d\theta} + u_r^2 + u_\theta^2 \right), \nonumber
\end{equation}
and $U_\theta$ satisfies, for some constants $c_1, c_2, c_3 \in \mathbb{R}$,
\begin{equation}
    (1 - x^2) U'_\theta + 2x U_\theta + \frac{1}{2} U''_\theta = P_c(x) := c_1(1 - x) + c_2(1 + x) + c_3(1 - x^2), \tag{4}
\end{equation}
where "'$'$" denotes differentiation in $x$, and $c := (c_1, c_2, c_3)$.

For each $c_1 \geq -1$ and $c_2 \geq -1$, define
\begin{equation}
    \tilde{c}_3(c_1, c_2) := -\frac{1}{2} \left( \sqrt{1 + c_1} + \sqrt{1 + c_2} \right) \left( \sqrt{1 + c_1} + \sqrt{1 + c_2} + 2 \right). \tag{5}
\end{equation}

Define
\begin{equation}
    J := \{ c \in \mathbb{R}^3 \mid c_1 \geq -1, c_2 \geq -1, c_3 \geq \tilde{c}_3(c_1, c_2) \}. \nonumber
\end{equation}

**Theorem 1.1.** There exist $U^+_\theta(c)(x) \in C^0(J \times [-1, 1])$, such that for every $c \in J$, $U^+_\theta(c) \in C^\infty(-1, 1)$ satisfy (4) in $(-1, 1)$, and $U^+_{\theta}(c) \leq U_\theta \leq U^+_{\theta}(c)$ for any solution $U_\theta$ of (4) in $(-1, 1)$. Moreover, if $c_3 > \tilde{c}_3(c_1, c_2)$, $U^-_{\theta} < U_{\theta}^+$ in $(-1, 1)$, and if $c_3 = \tilde{c}_3(c_1, c_2)$,
\begin{equation}
    U_{\theta}^+(c) = U_{\theta}^-(c) = U_{\theta}^+(c_1, c_2) := (1 + \sqrt{1 + c_1})(1 - x) + (-1 - \sqrt{1 + c_2})(1 + x). \tag{6}
\end{equation}

Next, for $c \in J$, introduce
\begin{equation}
    \gamma^+(c) := U_{\theta}^+(c)(0), \quad \gamma^-(c) := U_{\theta}^-(c)(0). \nonumber
\end{equation}

Define
\begin{equation}
    I := \{ (c, \gamma) \in \mathbb{R}^4 \mid c_1 \geq -1, c_2 \geq -1, c_3 \geq \tilde{c}_3(c_1, c_2), \gamma^- \leq \gamma \leq \gamma^+(c) \}. \nonumber
\end{equation}

**Theorem 1.2.** For each $(c, \gamma)$ in $I$, equation (4) has a unique solution $U^c_{\theta, \gamma}$ in $C^\infty(-1, 1) \cap C^0[-1, 1]$ satisfying $U^c_{\theta, \gamma}(0) = \gamma$. Moreover, these are all $(-1)$-homogeneous axisymmetric no-swirl solutions of the Navier-Stokes equations (1) on $S^2 \setminus \{S, N\}$. 

3
Clearly, $U_{\theta}^{c,-\gamma}(c) = U_{\theta}^{c,+}(c)$ for $c \in J$. Theorem 1.1 and Theorem 1.2 give a classification of all (-1)-homogeneous axisymmetric, no-swirl solutions of Navier-Stokes equations in $C^2(S^2 \setminus \{S, N\})$. There is a 1-1 correspondence between $U_{\theta}^{c,-\gamma}$ and points in the four dimensional surface $I$.

Recall that Landau solutions are

$$U_{\theta}(x) = \frac{2(1 - x^2)}{x + \lambda}, \quad |\lambda| > 1,$$

and they correspond to $U_{\theta}^{c,-\gamma}$ with $c = 0$ and $\gamma \in (-2, 2) \setminus \{0\}$.

The solutions in $C^\infty(S^2 \setminus \{S\})$ correspond to $U_{\theta}^{c,-\gamma}$ with $c_2 = 0$, $c_1 = -2c_3$ and $\gamma^- < \gamma < \gamma^+$.

Define

$$\tau_1(c_1) := 2 - 2\sqrt{1 + c_1}, \quad \tau_2(c_1) := 2 + 2\sqrt{1 + c_1}, \quad \tau'_1(c_2) := -2 - 2\sqrt{1 + c_2}, \quad \tau'_2(c_2) := -2 + 2\sqrt{1 + c_2}. \quad (7)$$

**Theorem 1.3.** Suppose $(c, \gamma) \in I$, then

(i) If $c_3 > c_3(c_1, c_2)$, then $\gamma^- < \gamma^+$, and for any $\gamma^- \leq \gamma < \gamma^+$, $U_{\theta}^{c,-\gamma} < U_{\theta}^{c,+\gamma}$ in $(-1, 1)$. Moreover,

$$\{(x, y) \mid -1 < x < 1, U_{\theta}^{c,-\gamma}(c, \gamma)(x) \leq y \leq U_{\theta}^{c,+\gamma}(c, \gamma)(x)\} = \bigcup_{\gamma \in [\gamma^-, \gamma^+]} \{(x, U_{\theta}^{c,-\gamma}(x)) \mid -1 < x < 1\}.$$

(ii)

$$U_{\theta}^{c,-\gamma}(-1) := \begin{cases} \tau_2(c_1), & \text{when } \gamma = \gamma^+, \\ \tau_1(c_1), & \text{otherwise}, \end{cases} \quad U_{\theta}^{c,+\gamma}(1) := \begin{cases} \tau'_1(c_2), & \text{when } \gamma = \gamma^-, \\ \tau'_2(c_2), & \text{otherwise}. \end{cases}$$

In addition to the continuity of $\gamma^+$ and $\gamma^-$ in $J$, they have further smoothness properties.

**Theorem 1.4.** $\gamma^+$ is in $C^\infty(J \setminus \{c \mid c_1 = -1\})$, and $\gamma^+(-1, c_2, c_3)$ is in $C^\infty(J \cap \{c \mid c_1 = -1\})$ as a function of $(c_2, c_3)$. $\gamma^-$ is in $C^\infty(J \setminus \{c \mid c_2 = -1\})$, and $\gamma^-(c_1, -1, c_3)$ is in $C^\infty(J \cap \{c \mid c_2 = -1\})$ as a function of $(c_1, c_3)$.

We also have the smoothness properties of $U_{\theta}^{c,-\gamma}$ in $(c, \gamma)$. Let the subsets $J_k$, $1 \leq k \leq 4$, of $J$ be defined as

$$J_1 := \{c \in J \mid c_1 > -1, c_2 > -1, c_3 > \bar{c}_3\}, \quad J_2 := \{c \in J \mid c_1 = -1, c_2 > -1, c_3 > \bar{c}_3\},$$

$$J_3 := \{c \in J \mid c_1 > -1, c_2 = -1, c_3 > \bar{c}_3\}, \quad J_4 := \{c \in J \mid c_1 = -1, c_2 = -1, c_3 > \bar{c}_3\}.$$

We define the following subsets of $I$: for $1 \leq k \leq 4$, let

$$I_{k,1} := \{(c, \gamma) \in I \mid c \in J_k, \gamma^- < \gamma < \gamma^+(c)\},$$

$$I_{k,2} := \{(c, \gamma) \in I \mid c \in J_k, \gamma = \gamma^+(c)\},$$

$$I_{k,3} := \{(c, \gamma) \in I \mid c \in J_k, \gamma = \gamma^-(c)\}.$$
As mentioned earlier, the following estimates of $U_\theta^{c,\gamma}$ are needed in our next paper on the existence of (-1)-homogeneous axisymmetric solutions of (1) with nonzero swirl on $\mathbb{S}^2 \setminus \{S, N\}$.

**Theorem 1.5.** Let $K$ be a compact set contained in one of $I_{k,l}$, $1 \leq k \leq 4$, $l = 1, 2, 3$. Then $U_\theta^{c,\gamma}$ is in $C^\infty(K \times (-1, 1))$. Moreover,

(i) If $k = 1$ and $l = 1, 2, 3$, or $(k, l) = (2, 2)$ or $(3, 3)$, then for $-1 < x < 1$,

$$|\partial_\gamma^\alpha \partial^j_\gamma U_\theta^{c,\gamma}(x)| \leq C(m, K), \text{ for any } 0 \leq |\alpha| + j \leq m, \tag{9}$$

where $j = 0$ if $l = 2, 3$; $\alpha_1 = 0$ if $k = 2$; and $\alpha_2 = 0$ if $k = 3$.

(ii) If $(k, l) = (2, 1)$ or $(2, 3)$ or $(4, 3)$, then for $-1 < x < 1$,

$$\left(\ln \frac{1 + x}{3}\right)^2 |\partial_\gamma^\alpha \partial^j_\gamma U_\theta^{c,\gamma}(x)| \leq C(m, K), \text{ for any } 1 \leq |\alpha| + j \leq m, \alpha_1 = 0, \tag{10}$$

where $j = 0$ if $l = 3$, and $\alpha_2 = 0$ if $k = 4$.

(iii) If $(k, l) = (3, 1), (3, 2)$ or $(4, 4)$, then for $-1 < x < 1$,

$$\left(\ln \frac{1 - x}{3}\right)^2 |\partial_\gamma^\alpha \partial^j_\gamma U_\theta^{c,\gamma}(x)| \leq C(m, K), \text{ for any } 1 \leq |\alpha| + j \leq m, \alpha_2 = 0, \tag{11}$$

where $j = 0$ if $l = 2$, and $\alpha_1 = 0$ if $k = 4$.

(iv) If $(k, l) = (4, 1)$, then for $-1 < x < 1$, and for any $1 \leq |\alpha| + j \leq m$, $\alpha_1 = \alpha_2 = 0$,

$$\left(\ln \frac{1 + x}{3}\right)^2 \left(\ln \frac{1 - x}{3}\right)^2 |\partial_\gamma^\alpha \partial^j_\gamma U_\theta^{c,\gamma}(x)| \leq C(m, K). \tag{12}$$

To make the above notations clear, we point out that if $(k, l) = (1, 2)$, estimate (9) means that for any compact set $K_1 \subset J_1$, $\left|\partial_\gamma^\alpha \left(U_\theta^{c,\gamma}(x)\right)\right| \leq C(m, K_1)$. For other $I_{k,l}$ with $l = 2$ or $3$, the left hand sides in (9)-(11) are interpreted analogously.

**Remark 1.1.** The estimates in Theorem 1.5 are optimal in each $I_{k,l}$, see examples in Theorem 3.1 in [4].

**Acknowledgment.** The work of the second named author is partially supported by NSF grant DMS-1501004.

# 2 Proof of Theorems

## 2.1 Proof of Theorem 1.1, Theorem 1.2 and Theorem 1.3

As mentioned in Section 1, we work with the function $U_\theta$ and the variable $x$ given in (3). As explained in [4], the stationary NSE (1) of (-1)-homogeneous axisymmetric no-swirl solutions can be reduced to (4) for some constants $c_1, c_2$ and $c_3$. We will show that the existence of solutions of (4) in $C^1(-1, 1)$ depends on the constants $c_1, c_2$ and $c_3$.

Recall the definitions in (7) and (8).
Lemma 2.1. Let \( \delta > 0, U_\theta \in C^1\((-1, -1 + \delta)\) satisfy (4) with \( c_1, c_2, c_3 \in \mathbb{R} \). Then \( c_1 \geq -1 \) and \( U_\theta(-1) := \lim_{x \to -1^+} U_\theta(x) \) exists and is finite. Moreover,

\[ U_\theta(-1) = \tau_1(c_1) \quad \text{or} \quad \tau_2(c_1). \]

Proof. By Proposition 7.1 in [4], \( \lim_{x \to -1^+} U_\theta(x) \) exists and is finite and

\[ \lim_{x \to -1^+} (1 + x)U'_\theta(x) = 0. \]

Sending \( x \) to \(-1\) in (4) leads to

\[ -2U_\theta(-1) + \frac{1}{2}U_\theta(-1)^2 = 2c_1. \]

Thus,

\[ c_1 = \frac{1}{4} [U_\theta(-1) - 2]^2 - 1 \geq -1, \]

and \( U_\theta(-1) = \tau_1(c_1) \) or \( \tau_2(c_1). \)

Lemma 2.1'. Let \( \delta > 0, U_\theta \in C^1(1 - \delta, 1) \) satisfy (4) with \( c_1, c_2, c_3 \in \mathbb{R} \). Then \( c_2 \geq -1 \) and \( U_\theta(1) := \lim_{x \to 1^-} U_\theta(x) \) exists and is finite. Moreover,

\[ U_\theta(1) = \tau'_1(c_2) \quad \text{or} \quad \tau'_2(c_2). \]

Proof. Consider \( \tilde{U}_\theta(x) := -U_\theta(-x) \), and apply Lemma 2.1 to \( \tilde{U}_\theta \).

Lemma 2.2. If \(|c| \leq A\) for some constant \( A > 0\), then there exists some constant \( C \), depending only on \( A \), such that all \( C^1 \) solutions \( U_\theta \) of (4) in \((-1, 1)\) satisfy

\[ |U_\theta(x)| \leq C, \quad -1 < x < 1. \]

Proof. By Lemma 2.1 there is some \( C_1(A) > 0 \), such that \(|U_\theta(\pm 1)| \leq C_1(A)\) for all solutions \( U_\theta \) of (4) in \((-1, 1)\).

If \( \sup_{-1 < x < 1} |U_\theta(x)| \leq 8C_1(A) \), the proof is finished. Otherwise, there exists some \( \bar{x} \in (-1, 1) \) such that \( |U_\theta(\bar{x})| = \max_{-1 \leq x \leq 1} |U_\theta(x)| > 8C_1(A) \). We may assume that \( U_\theta(\bar{x}) > 8C_1(A) \), since the other case can be handled similarly. Then there exists some \(-1 < \tilde{x} < \bar{x} \) such that \( U_\theta(\tilde{x}) = \frac{U_\theta(\tilde{x})}{2} \) and \( U'_\theta(\tilde{x}) \geq 0 \). By equation (4), we have

\[ -U_\theta(\bar{x}) + \frac{1}{8}U_\theta^2(\bar{x}) \leq 2\bar{x}U_\theta(\bar{x}) + \frac{1}{2}U'_\theta(\tilde{x}) \leq P_c(\tilde{x}) \leq C_2(A). \]

It follows that \( U_\theta(\bar{x}) \leq C_3(\bar{x}) \). The proof is finished.

Lemma 2.3. Let \( c_1 \geq -1, \tau = \tau_2(c_1) \) or \( \tau = \tau_1(c_1) \not\in \{0, -2, -4, -6, \ldots \} \). Then for every \( c_2, c_3 \in \mathbb{R} \), there exist \( \delta > 0 \) depending only on an upper bound of \( \sum_{i=1}^3 |c_i| \) and a positive lower bound of \( \inf_{k \in \mathbb{N}} |\tau + 2k| \), and a sequence \( \{a_n\}_{n=1}^\infty \) such that

\[ |a_n| \leq \left( \frac{1}{2\delta} \right)^n, \]
and

\[ U_\theta(x) := \tau + \sum_{n=1}^{\infty} a_n (1 + x)^n \]

is a real analytic solution of \[ f \] in \((-1, -1 + \delta)\). Moreover, \( U_\theta \) is the unique real analytic solution of \[ f \] in \((-1, -1 + \delta')\) satisfying \( U_\theta(-1) = \tau \) for any \( 0 < \delta' \leq \delta \).

**Proof of Lemma 2.3** Let \( s = 1 + x \). Rewrite

\[ P_e(x) = 2c_1 + (-c_1 + c_2 + 2c_3)(1 + x) - c_3(1 + x)^2 =: \tilde{c}_1 + \tilde{c}_2 s + \tilde{c}_3 s^2. \]

Suppose that \( U_\theta = \tau + \sum_{n=1}^{\infty} a_n s^n \), then \( U'_\theta = \sum_{n=1}^{\infty} na_n s^{n-1} \). Plug them into (4),

\[
\begin{align*}
\text{LHS} & = s(2-s) \sum_{n=1}^{\infty} na_n s^{n-1} + 2(s-1)(\tau + \sum_{n=1}^{\infty} a_n s^n) + \frac{1}{2} (\tau + \sum_{n=1}^{\infty} a_n s^n)^2 \\
& = \frac{1}{2} \tau^2 - 2\tau + ((2 + a_1)\tau) s + \sum_{n=2}^{\infty} ((2n - 2 + \tau) a_n + (3 - n) a_{n-1} + \frac{1}{2} \sum_{k+l=n, k,l \geq 1} a_k a_l) s^n \\
& = \tilde{c}_1 + \tilde{c}_2 s + \tilde{c}_3 s^2 = \text{RHS}.
\end{align*}
\]

Compare coefficients,

- \( n = 0 \), \( \frac{1}{2} \tau^2 - 2\tau = \tilde{c}_1 \), so \( \tau = 2 \pm \sqrt{4 + 2c_1} = \tau_1(c_1) \) or \( \tau_2(c_1) \),
- \( n = 1 \), \((a_1 + 2)\tau = \tilde{c}_2\), so \( a_1 = \frac{\tilde{c}_2}{\tau} - 2\),
- \( n = 2 \), \((2 + \tau) a_2 + a_1 + \frac{1}{2} a_1^2 = \tilde{c}_3\), so \( a_2 = \frac{1}{\tau+2}(\tilde{c}_3 - a_1 - \frac{1}{2} a_1^2) \).

For \( n \geq 3 \),

\[(2n - 2 + \tau) a_n + (3 - n) a_{n-1} + \frac{1}{2} \sum_{k+l=n, k,l \geq 1} a_k a_l = 0.\]

Since for any \( n \geq 1 \), \( \tau \neq -2(n-1) \),

\[ a_n = -\frac{1}{2n - 2 + \tau} \left( \frac{1}{2} \sum_{k+l=n, k,l \geq 1} a_k a_l + (3 - n) a_{n-1} \right), \quad (13) \]

it can be seen that \( a_n \) is determined by \( a_1, ..., a_{n-1} \), thus determined by \( c_1, c_2, c_3 \) and \( \tau \).

**Claim:** there exists some \( a > 0 \) large, depending only on an upper bound of \( \sum_{i=1}^{3} |c_i| \) and a positive lower bound of \( \inf_{k \in \mathbb{N}} |\tau + 2k| \), such that

\[ |a_n| \leq a^n. \]

**Proof of Claim:** Choose \( a > 1 \) large such that for \( 1 \leq n \leq 100|\tau| + 100 \), \( |a_n| \leq a^n \).

Now for \( n > 100|\tau| + 100 \), suppose that for \( 1 \leq k \leq n - 1 \), \( |a_k| \leq a^k \), then by induction and the recurrence formula (13),

\[ |a_n| \leq \frac{2}{3(n-1)} \left| \frac{1}{2} (n-1)a^n + (n-3)a^{n-1} \right| \leq \left( \frac{1}{3} + \frac{2(n-3)}{3(n-1)a} \right) a^n \leq a^n. \]
The claim is proved.

So for $\delta < \frac{1}{a}$, $U_\theta = \tau + \sum_{n=1}^{\infty} a_n s^n$, with $s = 1 + x$, is a real analytic solution of (1) in $(-1, -1 + \delta)$. The uniqueness of $U_\theta$ is clear from the proof above.

**Lemma 2.3’**. Let $c_2 \geq -1$, $\tau' = \tau'_1(c_2)$ or $\tau' = \tau'_2(c_2) \notin \{0, 2, 4, 6, \ldots \}$. Then for every $c_1, c_3 \in \mathbb{R}$, there exist $\delta > 0$, depending only on an upper bound of $\sum_{i=1}^{3} |c_i|$ and a positive lower bound of $\inf_{k \in \mathbb{N}} |\tau' - 2k|$, and a sequence $\{a_n\}_{n=1}^{\infty}$ such that

$$|a_n| \leq \left(\frac{1}{2\delta}\right)^n,$$

and

$$U_\theta(x) := \tau' + \sum_{n=1}^{\infty} a_n (1 - x)^n$$

is a real analytic solution of (4) in $(1 - \delta, 1)$. Moreover, $U_\theta$ is the unique real analytic solution of (4) in $(1 - \delta, 1)$ satisfying $U_\theta(1) = \tau'$ for any $0 < \delta' \leq \delta$.

The following two lemmas give some local comparison results.

**Lemma 2.4.** Suppose $0 < \delta < 2$, $U_\theta, \tilde{U}_\theta \in C^1(-1, -1 + \delta) \cap C^0[-1, -1 + \delta]$ satisfy

$$(1 - x^2)U'_\theta + 2xU_\theta + \frac{1}{2} U^2_\theta \geq (1 - x^2)\tilde{U}'_\theta + 2x\tilde{U}_\theta + \frac{1}{2} \tilde{U}^2_\theta, \quad -1 < x < -1 + \delta.$$  

Suppose also that one of the following two conditions holds.

(i) $U_\theta(-1) \geq \tilde{U}_\theta(-1) > 2$.

(ii) $U_\theta(-1) = \tilde{U}_\theta(-1) = 2$, and

$$\limsup_{x \to -1+} \int_{-1+\delta}^{x} \frac{-2 + U_\theta(s)}{1 - s^2} ds < +\infty. \quad (14)$$

Then either

$$U_\theta > \tilde{U}_\theta, \quad \text{in } (-1, -1 + \delta),$$

or there exists $\delta' \in (0, \delta)$ such that

$$U_\theta \equiv \tilde{U}_\theta, \quad \text{in } (-1, -1 + \delta').$$

**Proof.** Let $g = U_\theta - \tilde{U}_\theta$, then $g(-1) \geq 0$ and $g$ satisfies

$$g' + b(x)g \geq \frac{1}{2(1 - x^2)} g^2 \geq 0, \quad \text{for all } x \in (-1, -1 + \delta), \quad (15)$$

where $b(x)$ is given by

$$b(x) = (1 - x^2)^{-1}(2x + U_\theta). \quad (16)$$

Let

$$w(x) = e^{\int_{-1+\delta}^{x} b(s) ds} g(x).$$

Then $w$ satisfies, using (15), that

$$w'(x) \geq 0 \quad \text{in } (-1, -1 + \delta).$$  

8
Under condition either (i) or (ii), we have
\[ \limsup_{x \to -1^+} \int_{-1+\delta}^{x} b(s) ds < +\infty. \]
Using this and the fact that \( g(-1) \geq 0 \), we have \( \liminf_{x \to -1^+} w(x) \geq 0 \). Therefore, using (17), we have either \( w > 0 \) in \((-1, -1+\delta)\) or there exists a constant \( \delta' \), \( 0 < \delta' < \delta \) such that \( w \equiv 0 \) in \((-1, -1+\delta')\). The lemma is proved.

**Corollary 2.1.** For \( c_1 > -1, c_2, c_3 \in \mathbb{R} \) and \( 0 < \delta < 2 \), there exists at most one solution \( U_\theta \) of (4) in \( C^1(-1, -1+\delta) \) satisfying
\[ \lim_{x \to -1^+} U_\theta(x) = \tau_2(c_1). \]

**Proof.** Since \( \tau_2(c_1) > 2 \) for \( c_1 > -1 \), the uniqueness follows from (i) of Lemma 2.4. \( \square \)

Similarly, we have

**Lemma 2.4'.** Suppose \( 0 < \delta < 2, \ U_\theta, \tilde{U}_\theta \in C^1[1-\delta, 1] \cap C^0[1-\delta, 1] \) satisfy
\[ (1-x^2)U'_\theta + 2xU_\theta + \frac{1}{2}U^2_\theta \geq (1-x^2)\tilde{U}'_\theta + 2x\tilde{U}_\theta + \frac{1}{2}\tilde{U}^2_\theta, \quad 1-\delta < x < 1. \]
Suppose also that one of the following two conditions holds.

(i) \( U_\theta(1) \leq \tilde{U}_\theta(1) < -2 \),
(ii) \( U_\theta(1) = \tilde{U}_\theta(1) = -2 \), and
\[ \limsup_{x \to 1^-} \int_{1-\delta}^{x} \frac{2 + U_\theta(s)}{1-s^2} ds < +\infty. \]
Then either
\[ U_\theta < \tilde{U}_\theta, \quad \text{in \( (1-\delta, 1) \),} \]
or there exists \( \delta' \in (0, \delta) \) such that
\[ U_\theta \equiv \tilde{U}_\theta, \quad \text{in \( (1-\delta', 1) \).} \]

**Corollary 2.1'.** For \( c_2 > -1, c_1, c_3 \in \mathbb{R} \) and \( 0 < \delta < 2 \), there exists at most one solution \( U_\theta \) of (4) in \( C^1(1-\delta, 1) \) satisfying
\[ \lim_{x \to 1^-} U_\theta(x) = \tau'_1(c_2). \]

Now we are ready to analyze the global behavior of axisymmetric, no-swirl solutions of NSE (4) in \((-1, 1)\). The behavior of solutions depends closely on parameters \( c_1, c_2, c_3 \in \mathbb{R} \).

Recall the definition of \( \tilde{c}_3(c_1, c_2) \) given by (15), we have
Lemma 2.5. Suppose $c_1 \geq -1$, $c_2 \geq -1$, $c_3 = \bar{c}_3(c_1, c_2)$, then $U_{\theta}^*(c_1, c_2)$ given by (4) is the unique $C^1$ solution of (3) in $(-1, 1)$. In particular,

$$U_{\theta}^*(c_1, c_2)(-1) = \tau_2(c_1), \quad U_{\theta}^*(c_1, c_2)(1) = \tau_1'(c_2).$$

Proof. A direct calculation shows that $U_{\theta}^* := U_{\theta}^*(c_1, c_2)$ is a $C^1$ solution of (4) in $(-1, 1)$. It remains to prove the uniqueness.

Let $U_{\theta}$ be a $C^1$ solution of (4) in $(-1, 1)$, $U_{\theta} \neq U_{\theta}^*$. By Lemma 2.1 and Lemma 2.4, $U_{\theta}$ can be extended as a function in $C^0[-1, 1]$, $U_{\theta}(-1) \in \{\tau_1(c_1), \tau_2(c_1)\}$, $U_{\theta}(1) \in \{\tau_1'(c_2), \tau_2'(c_2)\}$. By Corollary 2.1 and (ii) of Lemma 2.4, we know that there exists a constant $0 < \delta_1 < \frac{1}{2}$ such that $U_{\theta} < U_{\theta}^*$ in $(-1, -1 + \delta_1)$. Similarly, by Corollary 2.1 and (ii) of Lemma 2.4, we know that there exists a constant $0 < \delta_2 < \frac{1}{2}$ such that $U_{\theta} > U_{\theta}^*$ in $(1 - \delta_2, 1)$. Therefore, there exists a point $\bar{x} \in (-1 + \delta_1, 1 - \delta_2)$ such that $U_{\theta} (\bar{x}) = U_{\theta}^* (\bar{x})$. Standard uniqueness theory of ODE implies that $U_{\theta} \equiv U_{\theta}^*$ in $(-1, 1)$. This is a contradiction.

Lemma 2.6. Suppose $c_1 \geq -1$, $c_2 \geq -1$, $c_3 < \bar{c}_3(c_1, c_2)$, then (4) has no solution in $C^1(-1, 1)$.

Proof. If $U_{\theta}$ is a $C^1$ solution of (4) in $(-1, 1)$. By Lemma 2.1 and Lemma 2.4, $U_{\theta}$ can be extended as a function in $C^0[-1, 1]$, $U_{\theta}(-1) \in \{\tau_1(c_1), \tau_2(c_1)\}$, $U_{\theta}(1) \in \{\tau_1'(c_2), \tau_2'(c_2)\}$.

By Lemma 2.5, $U_{\theta}^* := U_{\theta}^*(c_1, c_2)$ is the unique solution of (4) with $c_3 = \bar{c}_3(c_1, c_2)$. Since $c_3 < \bar{c}_3(c_1, c_2)$, $U_{\theta} \neq U_{\theta}^*$ in any open interval in $(-1, 1)$. We first assume that $U_{\theta} (\bar{x}) > U_{\theta}^* (\bar{x})$ at some point $\bar{x} \in (-1, 1)$. Since $c_3 < \bar{c}_3(c_1, c_2)$ we have

$$(1 - x^2)U_{\theta}' + 2xU_{\theta} + \frac{1}{2}U_{\theta}^2 < (1 - x^2)U_{\theta}' + 2xU_{\theta} + \frac{1}{2}(U_{\theta}^*)^2, \quad -1 < x < 1. \quad (18)$$

Since $U_{\theta}(-1) \leq U_{\theta}^* (-1)$, we have, in view of Lemma 2.4, there exists $\delta > 0$ such that $U_{\theta} < U_{\theta}^*$ in $(-1, -1 + \delta)$. Now with $U_{\theta} (\bar{x}) > U_{\theta}^* (\bar{x})$ and $U_{\theta} < U_{\theta}^*$ in $(-1, -1 + \delta)$, there exist a point $\xi \in (-1 + \delta, \bar{x})$ such that

$$U_{\theta}(\xi) = U_{\theta}^*(\xi), \quad U_{\theta}'(\xi) \geq U_{\theta}^{'*}(\xi),$$

which contradicts inequality (18) at $\xi$.

Similar arguments lead to a contradiction when $U_{\theta} (\bar{x}) < U_{\theta}^* (\bar{x})$ for some $\bar{x} \in (-1, 1)$ by showing $U_{\theta} > U_{\theta}^*$ near $x = 1$. The lemma is proved.

Lemma 2.7. Suppose $c_1 \geq -1$, $c_2 \geq -1$, $c_3 > \bar{c}_3(c_1, c_2)$. Let $U_{\theta}^+(c)$ be the power series solution, obtained in Lemma 2.3 with $U_{\theta}^+(c)(-1) = \tau_2(c_1)$, of (4) in $(-1, -1 + \delta)$, then $U_{\theta}^+(c)$ can be extended to be a solution of (4) in $(-1, 1)$, and $U_{\theta}^+(c)(1) = \tau_1'(c_2)$. Let $U_{\theta}^-(c)$ be the power series solution, obtained in Lemma 2.3 with $U_{\theta}^-(c)(1) = \tau_1'(c_2)$, of (4) in $(-1, -1 + \delta)$, then $U_{\theta}^-(c)$ can be extended to be a solution of (4) in $(-1, 1)$, and $U_{\theta}^-(c)(-1) = \tau_1(c_1)$. Moreover, $U_{\theta}^{-}(c) < U_{\theta}^+(c)$ in $(-1, 1)$. 

10
Proof. We only need to prove that $U^{+}_\theta := U^{+}_\theta (c)$ can be extended to be a solution of (1) in $(-1,1)$ and $U^{+}_\theta (1) = \tau'_2(c_2)$, since similar arguments work for $U^{-}_\theta (c)$.

Standard existence theory of ODE implies that $U^{+}_\theta$ can be extended to the maximal interval of existence, say $(-1,\xi)$, $\xi \in (-1,\delta,1]$. Since $c_3 > c_3(c_1,c_2)$, we have, with $U^{+}_\theta := U^{+}_\theta (c_1,c_2)$,

$$(1-x^2)U^{+\prime}_\theta + 2xU^{+}_\theta + \frac{1}{2}(U^{+}_\theta)^2 > (1-x^2)U^{\prime}_\theta + 2xU^{*}_\theta + \frac{1}{2}(U^{*}_\theta)^2, \quad -1 < x < \xi.$$  

Since $U^{+}_\theta (-1) = U^{*}_\theta (-1) = \tau_2(c_1)$, by Lemma 2.4 and the fact that $U^{+}_\theta , U^{*}_\theta$ can not coincide in any open interval, we have $U^{+}_\theta > U^{*}_\theta$ in $(-1,\xi)$.

If $\xi < 1$, since $U^{+}_\theta$ is bounded from below by $U^{*}_\theta$, there exists a sequence of points $\{x_i\}$ satisfying

$$x_1 < x_2 < x_3 < \cdots < \xi, \quad \lim_{i \to \infty} x_i = \xi,$$

$$U^{+}_\theta (x_1) < U^{+}_\theta (x_2) < U^{+}_\theta (x_3) < \cdots, \quad \lim_{i \to \infty} U^{+}_\theta (x_i) = +\infty.$$  

Then, in each interval $(x_i, x_{i+1})$, we can find a point $y_i$ such that

$$x_i < y_i < x_{i+1}, \quad U^{+}_\theta (y_i) \geq U^{+}_\theta (x_i), \quad U^{+\prime}_\theta (y_i) \geq 0.$$  

Taking $x = y_i$ in equation (1), and sending $i$ to infinity, we obtain a contradiction. So $\xi = 1$. By Lemma 2.1, $\lim_{r \to 1^+} U^{+}_\theta (x)$ exists and is finite.

We have extended $U^{+}_\theta$ to be a solution of (1) in $C^1(-1,1) \cap C^0[-1,1]$ and $U^{+}_\theta > U^{*}_\theta$ in $(-1,1)$.

Similarly, $U^{-}_\theta$ can be extended to $C^0[-1,1]$, and $U^{-}_\theta < U^{*}_\theta < U^{+}_\theta$ in $(-1,1)$.

By Lemma 2.1, $U^{+}_\theta (1) \in \{\tau'_1(c_2), \tau'_2(c_2)\}$. If $c_2 = -1$, $\tau'_1(c_2) = \tau'_2(c_2)$, so $U^{+}_\theta (1) = \tau'_2(c_2)$. If $c_2 > -1$, since $U^{-}_\theta (1) = \tau'_1(c_2)$ and $U^{+}_\theta > U^{-}_\theta$ in $(-1,1)$, by Corollary 2.1, we have $U^{+}_\theta (1) = \tau'_2(c_2)$. Similarly, $U^{-}_\theta (-1) = \tau_1(c_1)$. Lemma 2.7 is proved.

**Lemma 2.8.** Suppose $c_1 \geq -1$, $c_2 \geq -1$, $c_3 > \bar{c}_3(c_1,c_2)$, then any $C^1$ solution $U^{\prime}_\theta$ of (2) in $(-1,1)$ other than $U^{\pm}_\theta (c)$ satisfies

$$U^{-}_\theta (c) < U^{\prime}_\theta < U^{+}_\theta (c), \quad \text{in} (-1,1),$$

$$U^{\prime}_\theta (1) = \tau_1(c_1), \quad U^{\prime}_\theta (1) = \tau'_2(c_2).$$  

**Proof.** By Lemma 2.1 and Lemma 2.1, $U^{\prime}_\theta$ can be extended to $C^0[-1,1]$ with $U^{\prime}_\theta (-1) = \tau_1(c_1)$ or $\tau_2(c_1)$, and $U^{\prime}_\theta (1) = \tau'_1(c_2)$ or $\tau'_2(c_2)$.

We only need to prove $U^{\prime}_\theta < U^{+}_\theta (c)$ in $(-1,1)$ and $U^{\prime}_\theta (-1) = \tau_1(c_1)$, since similar arguments imply that $U^{\prime}_\theta > U^{-}_\theta (c)$ in $(-1,1)$ and $U^{\prime}_\theta (1) = \tau'_2(c_2)$.

From the standard uniqueness theory of ODE, we know that the graph of $U^{\prime}_\theta$ and $U^{+}_\theta (c)$ can not intersect in $(-1,1)$. So we either have $U^{\prime}_\theta < U^{+}_\theta (c)$ in $(-1,1)$ or $U^{\prime}_\theta > U^{+}_\theta (c)$ in $(-1,1)$.

If $U^{\prime}_\theta > U^{+}_\theta (c)$ in $(-1,1)$, then, by Lemma 2.1, $U^{\prime}_\theta (-1) = U^{+}_\theta (c)(-1) = \tau_2(c_1) \geq 2$. Note that $U^{+}_\theta (c)$ satisfies (1), we can apply Lemma 2.4 to obtain $U^{\prime}_\theta \leq U^{+}_\theta (c)$, a contradiction. So $U^{\prime}_\theta < U^{+}_\theta (c)$ in $(-1,1)$.

If $\tau_1(c_1) < \tau_2(c_1)$, the uniqueness result Corollary 2.1 implies that $U^{\prime}_\theta (1) = \tau_1(c_1)$. If $\tau_1(c_1) = \tau_2(c_1)$, we again have $U^{\prime}_\theta (1) = \tau_1(c_1)$. Lemma 2.8 is proved.

11
Proof of Theorem 1.1 For $c \in J$, if $c_3 = \breve{c}_3$, by Lemma 2.5 $U_\theta^c(c_1, c_2)$ in (1) is the unique solution of (1) in $(-1, 1)$.

If $c_3 > \breve{c}_3$, let $U_\theta^c(c)$ and $U_\theta^c(c)$ be the functions in Lemma 2.7. By Lemma 2.3 Lemma 2.3, Lemma 2.7 and Lemma 2.8, $U_\theta^c(c) \in C^\infty(-1, 1) \cap C^0[-1, 1]$ satisfy (1) in $(-1, 1)$, and $U_\theta^c(c) < U_\theta^c(c)$. Moreover, $U_\theta^c(c) \leq U_\theta^c(c) < U_\theta^c(c)$ for any solution $U_\theta$ of (1) in $(-1, 1)$.

Now we prove the continuity of $U_\theta^c(c)(x)$ in $(c, x)$, the same arguments applies to $U_\theta^c(c)$.

For every $(\hat{c}, \hat{x}) \in J \times [-1, 1]$, we prove the continuity of $U_\theta^c(c)$ at $(\hat{c}, \hat{x})$. By Lemma 2.3 there exists some $\delta > 0$, such that $U_\theta^c(c)(x)$ is continuous in $(B_1(\hat{c}) \cap J) \times [-1, -1 + \delta]$, where $B_1(c)$ is the unit ball in $R^3$ centered at $\hat{c}$.

Consider

\[
\begin{align*}
(1 - x^2)U_\theta' + 2xU_\theta + \frac{1}{2}U_\theta^2 &= P_c(x) = c_1(1 - x) + c_2(1 + x) + c_3(1 - x^2), \\
U_\theta(-1 + \frac{\delta}{2}) &= a,
\end{align*}
\]

(19)

for $a$ close to $a_0 := U_\theta^c(c)(-1 + \frac{\delta}{2})$.

By standard ODE theories, for any $0 < \epsilon < 2 - \delta$, there exists some positive constants $\mu$, such that $U_\theta^c(c) \in C((a_0 - \mu, a_0 + \mu) \times (B_1(\hat{c}) \cap J) \times [-1 + \frac{\delta}{2}, 1 - \epsilon])$.

The continuity of $U_\theta^c(c)(x)$ at $\hat{x} = 1$ follows from Lemma 2.11 which will be given later.

Proof of Theorem 1.2 Let $(c, \gamma) \in I$. If $c_3 = \breve{c}_3$, then $\gamma = \gamma^+ = \gamma^-$ by Theorem 1.1 $U_\theta^{\gamma, \gamma} := U_\theta^{\pm}(c)$ given by (1) is the unique solution of (1) satisfying $U_\theta^{\gamma, \gamma}(0) = \gamma$.

If $c_3 > \breve{c}_3(c_1, c_2)$, and $\gamma = \gamma^\pm(c)$, then $U_\theta^{\gamma, \gamma} := U_\theta^c(c)$ is the unique solution of (1) satisfying $U_\theta^{\gamma, \gamma}(0) = \gamma$.

For $c_3 > \breve{c}_3(c_1, c_2)$, and $\gamma^- < \gamma < \gamma^+(c)$, let $U_\theta^{\gamma, \gamma}$ be the unique local solution of (1) satisfying $U_\theta^{\gamma, \gamma}(0) = \gamma$. By standard ODE theory, $U_\theta^{\gamma, \gamma}$ can be extended to a $C^\infty$ solution in $(-1, 1)$ satisfying $U_\theta^- < U_\theta^c < U_\theta^+(c)$.

By Lemma 2.1 and Lemma 2.1, $U_\theta^{\gamma, \gamma}$ can be extend as a function in $C^0[-1, 1]$.

To complete the proof of Theorem 1.2, it remains to show that $\{U_\theta^{\gamma, \gamma} \mid (c, \gamma) \in J\}$ are all the solutions.

For $c \in R^3$, let $U_\theta^c$ be a solution of (1) in $(-1, 1)$, By Lemma 2.1 and Lemma 2.1, $c_1 \geq -1$ and $c_2 \geq -1$. Then by Lemma 2.6 $c_3 \geq \breve{c}_3$. So $c \in J$. By Theorem 1.1 we have $U_\theta^- < U_\theta < U_\theta^+(c)$. So $\gamma := U_\theta(0)$ satisfies $\gamma^- \leq \gamma \leq \gamma^+(c)$, and $U_\theta = U_\theta^{\gamma, \gamma}$.

Lemma 2.9. Suppose $c_1 \geq -1$, $c_2 \geq -1$, $c_3 > \breve{c}_3(c_1, c_2)$, then $\gamma^- < \gamma^+(c)$, and the graphs

$K_1(\gamma) := \{ (x, U_\theta^{\gamma, \gamma}(x)) \mid -1 < x < 1 \}$, \quad $\gamma^- \leq \gamma \leq \gamma^+(c),$

foliate the set

$K_2 := \{ (x, y) \mid -1 < x < 1, U_\theta^{\gamma, \gamma^+}(x) \leq y \leq U_\theta^{\gamma, \gamma^-}(x) \}$

12
in the sense that for any \( \gamma, \gamma' \in \mathbb{R}, \gamma^-(c) \leq \gamma < \gamma' \leq \gamma^+(c) \), \( U_{\theta}^{c, \gamma} < U_{\theta}^{c, \gamma'} \) in \((-1, 1)\) and \( K_2 = \bigcup_{\gamma^- \leq \gamma \leq \gamma^+(c)} K_1(\gamma) \). Moreover, \( U_{\theta}^{c, \gamma} \) is a continuous function of \((c, \gamma, x)\) in \( J \times [\gamma^-(c), \gamma^+(c)] \times (-1, 1)\).

**Proof.** By standard uniqueness theories of ODE,

\[
U_{\theta}^{c, \gamma^-} < U_{\theta}^{c, \gamma} < U_{\theta}^{c, \gamma'} < U_{\theta}^{c, \gamma^+} \quad \text{in} \quad (-1, 1), \quad \text{for} \quad \gamma^- < \gamma < \gamma' < \gamma^+(c).
\]

It is obvious that \( K_1(\gamma) \subseteq K_2 \). On the other hand, let \((x_0, y_0) \in K_2\), so \(-1 < x_0 < 1\) and \( U_{\theta}^{c, \gamma^-}(x_0) < y_0 < U_{\theta}^{c, \gamma^+}(x_0) \). By standard existence and uniqueness theories of ODE, there exists a \( C^1 \) solution \( U_{\theta} \) of (1) in \((-1, 1)\) satisfying \( U_{\theta}(x_0) = y_0 \) and \( U_{\theta}^{c, \gamma^-} < U_{\theta} < U_{\theta}^{c, \gamma^+} \) in \((-1, 1)\). In particular,

\[
\gamma^- = U_{\theta}^{c, \gamma^-}(0) < U_{\theta}(0) < U_{\theta}^{c, \gamma^+}(0) = \gamma^+.
\]

\( U_{\theta} = U_{\theta}^{c, \gamma} \) with \( \gamma = U_{\theta}(0) \) and therefore \( x_0, y_0 \in K_1(\gamma) \). We have proved that \( K_2 = \bigcup_{\gamma^- \leq \gamma \leq \gamma^+} K_1(\gamma) \).

The continuity of \( U_{\theta}^{c, \gamma} \) for \((c, \gamma, x)\) in \( J \times [\gamma^-(c), \gamma^+(c)] \times (-1, 1)\) can be derived from (1), and the continuous dependence of ODE on its boundary conditions.

**Proof of Theorem 1.3** Theorem 1.3 follows from Lemma 2.5 - Lemma 2.9

\[\square\]

### 2.2 Proof of Theorem 1.4 and Theorem 1.5

In the following context, in \( J \cap \{c \in J \mid c_1 = -1\} \), \( U_{\theta}^+(c) = U_{\theta}^+(-1, c_2, c_3) \) is viewed as a function of \((c_2, c_3)\), and \( \partial^\alpha U_{\theta}^+(c)(x) \) means \( \partial^\alpha_{(c_2, c_3)} U_{\theta}^+(c)(x) \). In \( J \cap \{c_1 = -1\} \), \( U_{\theta}^-(c) = U_{\theta}^-(c_1, -1, c_3) \) is viewed as a function of \((c_1, c_3)\), and \( \partial^\alpha U_{\theta}^-(c)(x) \) means \( \partial^\alpha_{(c_1, c_3)} U_{\theta}^-(c)(x) \).

**Lemma 2.10.** For any integer \( m \geq 0 \), and any compact subset \( K \) contained in either \( J \setminus \{c \mid c_1 = -1\} \) or \( J \setminus \{c \in J \mid c_1 = -1\} \), there exist some positive constants \( \delta \) and \( C \), depending only on \( m \) and \( K \), such that \( U_{\theta}^+(c) \in C^m(K \times (-1, -1 + \delta)) \), and

\[
|\partial^\alpha U_{\theta}^+(c)(x)| \leq C, \quad x \in (-1, -1 + \delta), c \in K, |\alpha| \leq m. \tag{20}
\]

**Proof.** Let \( \alpha = (\alpha^1, \alpha^2, \alpha^3) \) denote a multi-index where \( \alpha^i \geq 0, i = 1, 2, 3 \). The partial derivative \( \partial^\alpha = \partial^\alpha_{c_1} \partial^\alpha_{c_2} \partial^\alpha_{c_3} \) and the absolute value \( |\alpha| = \sum_{i=1}^3 \alpha^i \).

By Lemma 2.5 and its proof, there exists \( \delta > 0 \), depending only on \( K \), such that for \( c \in K, U_{\theta}^+(c) \) can be expressed as

\[
U_{\theta}^+(c)(x) = \tau + \sum_{n=1}^{\infty} a_n (1 + x)^n, -1 < x < -1 + \delta,
\]

where

\[
\tau = \tau_2(c_1) = 2 + 2\sqrt{1 + c_1},
\]

13
\[ a_1 = \frac{-c_1 + c_2 + 2c_3}{\tau} - 2, \quad a_2 = \frac{-1}{\tau + 2}(c_3 + a_1 + \frac{1}{2}a_1^2), \]  
(21)

\[ a_n = -\frac{1}{2n - 2 + \tau} \left( \frac{1}{2} \sum_{k+l=n, k,l \geq 1} a_k a_l + (3-n)a_{n-1} \right), \quad n \geq 3, \]  
(22)

and

\[ |a_n| \leq \left( \frac{1}{2\delta} \right)^n. \]  
(23)

Estimate (23) guarantees that the power series expansion of \( U^+_\theta(c)(x) \) is uniformly convergent in \((-1, -1 + \delta)\).

By the above expressions and relations it can be seen that \( \tau(c) \) and \( a_n(c) \) are all \( C^\infty \) functions of \( c \) in \( J \). So to prove the lemma, we just need to show that there exists some \( \delta' > 0 \), depending only on \( m \) and \( K \), such that for any multi-index \( \alpha \) satisfying \( 1 \leq |\alpha| \leq m \), the series

\[ \frac{\partial^{\alpha} \tau}{\partial c^{\alpha}} + \sum_{n=1}^{\infty} \frac{\partial^{\alpha} a_n}{\partial c^{\alpha}} (1 + x)^n \]  
(24)

is absolutely convergent in \((-1, -1 + \delta')\) uniformly for \( c \in K \).

**Case 1:** \( K \subset J \setminus \{ c \mid c_1 = -1 \} \).

Let \( C(m, K) \) be a constant depending only on \( m \) and \( K \) which may vary from line to line. If \( K \) is a compact set in \( J \setminus \{ c \mid c_1 = -1 \} \), there exists some constant \( \delta_1(K) > 0 \), such that \( 4 + 4c_1 \geq \delta_1(K) \). Using this, (21), (22), and the fact that \( \tau > 2 \), we have

\[ \left| \frac{\partial^{\alpha} \tau}{\partial c^{\alpha}} \right| \leq C(m, K), \quad \left| \frac{\partial^{\alpha} a_n}{\partial c^{\alpha}} \right| \leq C(m, K), \quad \forall 1 \leq n \leq 2, 0 \leq |\alpha| \leq m, c \in K. \]  
(25)

Next, let \( g_n(c) := \frac{1}{2n - 2 + \tau} \). By the above estimates and the fact that \( \tau > 2 \), we have

\[ \left| \frac{\partial^{\alpha} g_n}{\partial c^{\alpha}} \right| \leq \frac{C(m, K)}{n}, \quad \text{for all } 1 \leq |\alpha| \leq m, c \in K, \text{ and } n \geq 1. \]  
(26)

To prove the existence of \( \delta' \) such that the series in (24) is convergent for all \( 1 \leq |\alpha| \leq m \) uniformly in \( K \), we will only need to show the following:

**Claim:** there exists some \( a > 0 \), depending only on \( m \) and \( K \), such that

\[ (P_n): \quad |\partial^{\alpha} a_n(c)| \leq a^{n(|\alpha|+1)}, \quad \text{for } 1 \leq |\alpha| \leq m, \text{ and } c \in K \]

holds for all \( n \geq 1 \).

**Proof of Claim:** We prove it by induction on \( n \). Let \( a \) be a constant to be determined in the proof.

By estimate (25), there exists some constant \( \tilde{a} \), depending only on \( m \) and \( K \), such that for all \( a \geq \tilde{a} \), \((P_1)\) and \((P_2)\) hold. We may assume that \( \tilde{a} \geq \frac{1}{2\delta} \) so that we know from (23) that

\[ |a_n(c)| \leq \tilde{a}^n, \]  
(27)
for all \( c \in K \) and \( n \geq 1 \).

Now for \( n \geq 3 \), suppose that for some \( a \geq \bar{a} \), \((P_k)\) holds for all \( 1 \leq k \leq n - 1 \). Let \( Q_n(c) := \sum_{k+l=n, k,l \geq 1} a_k a_l \). Then (22) can be written as

\[
a_n = -\frac{1}{2} g_n Q_n + (n - 3) g_n a_{n-1}.\]

So

\[
\partial^\alpha a_n = -\frac{1}{2} \partial^\alpha (g_n Q_n) + (n - 3) \partial^\alpha (g_n a_{n-1}). \tag{28}
\]

Using (26), by computation we have

\[
|\partial^\alpha (g_n Q_n)| \leq C(m, K) \max_{\alpha_1 \leq \alpha} |\partial^\alpha_1 Q_n|.
\]

Let \( a \geq \bar{a} \), using the definition of \( Q_n(c) \), by induction we have that,

\[
|\partial^\alpha (g_n Q_n)| \leq C(m, K) \max_{\alpha_1 \leq \alpha} \max_{\alpha_2 \leq \alpha_1} \sum_{k+l=n, k,l \geq 1} a_k a_l |\partial^{\alpha_1 - \alpha_2} a_k|,
\]

\[
\leq C(m, K) \max_{\alpha_1 \leq \alpha} \max_{\alpha_2 \leq \alpha_1} a_k^{(|\alpha_2|+1)} a_l^{(|\alpha_1 - \alpha_2|+1)} \leq C(m, K) a^n |\alpha|+1.
\]

Similarly, by (26), (27) and the induction hypothesis, we have

\[
|\partial^\alpha (g_n a_{n-1})| \leq C(m, K) a^{(n-1)|\alpha|+1}, \tag{30}
\]

Plug (29) and (30) in (28), we have that for \(|\alpha| \geq 1\),

\[
|\partial^\alpha a_n| \leq C(m, K) a^{n(|\alpha|+1)}-1.
\]

If from the beginning we use \( a = \max \{\bar{a}, C(m, K)\} \) for the induction hypothesis, we have

\[
|\partial^\alpha a_n| \leq a^{n(|\alpha|+1)}.
\]

So the claim is true for all \( n \). The lemma is proved for \( K \subset J \setminus \{c \mid c_1 = -1\} \).

**Case 2:** \( K \subset J \cap \{c \mid c_1 = -1\} \).

In this case \( \tau = 2 \) and \( g_n(c) \) is a constant in \( K \). By similar arguments as in Case 1, we have the same estimate for \( a_n \) and the proof is finished.

**Corollary 2.2.** For any \( K \subset J \setminus \{c \mid c_1 = -1\} \) or \( J \cap \{c \mid c_1 = -1\} \), \( U_\theta^+(c) \in C^\infty(K \times (-1,1)) \). Moreover, for any \( \epsilon > 0 \), \( m \in \mathbb{N} \), there exists some positive constant \( C \), depending only on \( m, K, \) and \( \epsilon \), such that

\[
||\partial^\alpha U_\theta^+(c)||_{L^\infty(-1,1-\epsilon)} \leq C(m, K, \epsilon), \quad 0 \leq |\alpha| \leq m. \tag{31}
\]
Proof. We know that $U_\theta^+(c)$ satisfies (41) in $(-1,1)$ and $||U_\theta^+||_{L^\infty(-1,1)} \leq C$, where $C$ depends only on $K$. By Lemma 2.10 for any positive integer $m$, there exist some positive constants $\delta$ and $C$, depending only on $m$ and $K$, such that $U_\theta^+(c) \in C^m(K \times (-1,1))$ and $\theta$ holds.

Consider (19) for $a$ close to $a_0 := U_\theta^+(c)(-1 + \frac{\delta}{2})$. By standard ODE theories, for any $0 < \epsilon < 2 - \delta$, there exist some positive constants $\mu$ and $C$, depending on $m$, $K$ and $\epsilon$, such that if $|a - a_0| < \mu$, then there exists a solution $U_\theta > C^m((a_0 - \mu, a_0 + \mu) \times K \times [-1 + \frac{\delta}{4}, 1 - \epsilon])$ of (19), and

$$|\partial_\theta^\alpha \partial_\epsilon^\beta U_\theta| \leq C, \quad |\beta|, |\alpha| \leq m, c \in K, -1 + \frac{\delta}{4} < x < 1 - \epsilon.$$  

It follows, also in view of (20), that $U_\theta^+(c) = U_\theta|_{a=a_0}$ satisfies (31). $\Box$

Similarly to Lemma 2.10 and Corollary 2.2 we have

**Lemma 2.10'.** For any integer $m \geq 0$, and any compact set $K$ contained in either $J \setminus \{c \in J \mid c_2 = -1\}$ or $J \cap \{c \in J \mid c_2 = -1\}$, there exist some positive constants $\delta$ and $C$, depending only on $m$ and $K$, such that $U_\theta^-(c) \in C^m(K \times (1 - \delta, 1))$, and

$$|\partial_\theta^\alpha \partial_\epsilon^\beta U_\theta| \leq C, \quad x \in (1 - \delta, 1), c \in K, |\alpha| \leq m.$$  

**Corollary 2.2'.** For any $K \subset J \setminus \{c \in J \mid c_2 = -1\}$ or $J \cap \{c \in J \mid c_2 = -1\}$, $U_\theta^- \in C^\infty(K \times (-1,1))$. Moreover, for any $\epsilon > 0$, $m \in \mathbb{N}$, there exists some positive constant $C$, depending only on $m$, $K$, and $\epsilon$, such that

$$||\partial_\theta^\alpha \partial_\epsilon^\beta U_\theta^-||_{L^\infty(-1+\epsilon,1)} \leq C, \quad 0 \leq |\alpha| \leq m.$$  

Theorem 1.3 can be obtained from Corollary 2.2 and Corollary 2.2.

To prove Theorem 1.3, we make the following observations.

By Corollary 2.2 and Corollary 2.2, we know that for $1 \leq k \leq 4$ and $l = 2,3$, $U_\theta^+(c)$ and $U_\theta^-(c)$ are smooth in $I_{k,l}$. Here the smoothness means $U_\theta^+(c)$ and $U_\theta^-(c)$ are smooth restricted to each $I_{k,l}$.

By standard ODE theory, since $U_\theta$ satisfies (41), it is smooth in $I_{k,1}$ for each $1 \leq k \leq 4$. So a solution $U_\theta$ of the initial value problem

$$\begin{cases}
(1 - x_2)U_\theta' + 2xU_\theta + \frac{1}{2}U_\theta'' = P_c(x), 
-1 < x < 1,
U_\theta(0) = \gamma,
\end{cases}$$  

(32)

is smooth with respect to $(c, \gamma)$ in each $I_{k,l}$, $1 \leq k \leq 4, 1 \leq l \leq 3$. It remains to prove the estimates (i)-(iv) in Theorem 1.3.

We first make some estimates about the solutions $U_\theta$ of (32).

Recall that for each $(c, \gamma) \in I$, there is a solution $U_\theta = U_\theta^{c,\gamma}$ satisfying (32).

**Lemma 2.11.** Let $K$ be a compact subset of $I \setminus \{(c, \gamma) \mid \gamma = \gamma^+(c)\}$. Then for any $\epsilon > 0$, there exists some $\delta > 0$, depending only on $\epsilon$ and $K$, such that for any $(c, \gamma) \in K$,

$$|U_\theta^{c,\gamma}(x) - U_\theta^{c,\gamma}(-1)| < \epsilon, \quad -1 < x < -1 + \delta.$$  

16
Proof. We prove it by contradiction. Suppose the contrary, there exist some $\epsilon > 0$ and a sequence $(c^i, \gamma^i) \in K$ and $-1 < x_i < -1 + \frac{1}{j}$, such that

$$|U^c_{\theta} \gamma^i(x_i) - U^c_{\theta} \gamma^i(-1)| \geq \epsilon.$$  

Since $K$ is compact, there exist a subsequence, still denoted as $(c^i, \gamma^i)$, and some $(c, \gamma) \in K$, such that $(c^i, \gamma^i) \to (c, \gamma)$ as $i \to \infty$.

Denote $U^c_{\theta} = U^c_{\theta} \gamma^i$. By standard ODE theory, we have that $U^c_{\theta} \to U^c_{\theta} := U^c_{\theta} \gamma$ in $C^1_{loc}(-1, 1)$. We first assume that

$$U^c_{\theta}(x_i) \geq U^c_{\theta}(-1) + \epsilon. \quad (33)$$

Since $(c, \gamma), (c^i, \gamma^i) \in K$, we have $\gamma < \gamma^+(c)$ and $\gamma^i < \gamma^+(c^i)$. Then, by Theorem 1.3 (ii), $U^c_{\theta}(-1) = 2 - 2\sqrt{1 + c_1}$ and $U^c_{\theta}(-1) = 2 - 2\sqrt{1 + c_1}$.

Since $c^i \to c$, we have $U^c_{\theta}(-1) \to U^c_{\theta}(-1)$, and therefore for sufficiently large $i$,

$$U^c_{\theta}(x_i) > U^c_{\theta}(-1) + \frac{\epsilon}{2}. \quad (34)$$

Case 1: $U^c_{\theta}(-1) < 2$.

There exists some $\epsilon_1 > 0$, such that $U^c_{\theta}(-1) + 3\epsilon_1 < \min\{2, U^c_{\theta}(-1) + \frac{\epsilon}{2}\}$. For sufficiently large $i$ we have $U^c_{\theta}(-1) < 2 - \epsilon_1$. Since $U^c_{\theta} \to U^c_{\theta}$ in $C^1_{loc}(-1, 1)$, we have

$$\lim_{i \to \infty} U^c_{\theta}(-1 + \frac{1}{j}) = U^c_{\theta}(-1 + \frac{1}{j}).$$

By the continuity of $U^c_{\theta}$,

$$\lim_{j \to \infty} U^c_{\theta}(-1 + \frac{1}{j}) = U^c_{\theta}(-1).$$

Thus for large $j$, there exists $i_j \geq j$, such that $-1 < x_i < -1 + \frac{1}{j}$ and

$$U^c_{\theta}(-1 + \frac{1}{j}) \leq U^c_{\theta}(-1 + \frac{1}{j}) + \frac{\epsilon_1}{10} \leq U^c_{\theta}(-1) + 2\epsilon_1.$$  

By $(33)$, $U^c_{\theta}(x_{i_j}) > U^c_{\theta}(-1) + 2\epsilon_1$.

Choose $\tilde{x}_{i_j} \in (x_{i_j}, -1 + \frac{1}{j})$, satisfying

$$U^c_{\theta}(\tilde{x}_{i_j}) = U^c_{\theta}(-1) + 2\epsilon_1 \leq 2 - \frac{\epsilon_1}{2} \quad \text{and} \quad (U^c_{\theta})'(\tilde{x}_{i_j}) \leq 0.$$  

Plugging $U^c_{\theta}$ and $\tilde{x}_{i_j}$ in $(\ref{eq:energy})$, using the above, we have

$$2\tilde{x}_{i_j} U^c_{\theta}(\tilde{x}_{i_j}) + \frac{1}{2} (U^c_{\theta})^2(\tilde{x}_{i_j}) \geq P_{c^i}(\tilde{x}_{i_j}). \quad (35)$$

Sending $j \to \infty$ in $(35)$ leads to

$$h(\xi) := -2\xi + \frac{1}{2} \xi^2 \geq P_c(-1),$$

17
where $\xi := U_\theta(-1) + 2\epsilon_1 \in (U_\theta(-1), 2)$.

Since $h(s)$ is a decreasing function when $s \leq 2$, we have

$$h(\xi) < h(U_\theta(-1)) = P_c(-1),$$

a contradiction.

**Case 2: $U_\theta(-1) = 2$.**

By (34) and the convergence of $U_i^\theta(-1)$ to $U_\theta(-1)$, we may choose $\tilde{x}_i \in (-1, x_i)$ satisfying

$$U_\theta^i(\tilde{x}_i) = U_\theta(-1) + \frac{\epsilon}{4} = 2 + \frac{\epsilon}{4}, \text{ and } (U_\theta^i)'(\tilde{x}_i) \geq 0.$$ 

Plugging $U_\theta^i$ and $\tilde{x}_i$ in (4), using the above, we have

$$2\tilde{x}_i U_\theta^i(\tilde{x}_i) + \frac{1}{2}(U_\theta^i(\tilde{x}_i))^2 \leq P_c(\tilde{x}_i).$$

Sending $i \to \infty$, the above leads to

$$h(2) < h(2 + \frac{\epsilon}{4}) \leq P_c(-1) = h(U_\theta(-1)) = h(2),$$

a contradiction.

Now, if instead of (33),

$$U_\theta^i(x_i) \leq U_\theta^i(-1) - \epsilon,$$

then for sufficiently large $i$, we have

$$U_\theta^i(x_i) < U_\theta(-1) - \frac{\epsilon}{2}.$$ 

As in the proof of Case 1, there exists $\tilde{x}_{ij} \to -1$, such that

$$U_\theta^{ij}(\tilde{x}_{ij}) = U_\theta(-1) - \frac{\epsilon}{2} =: \xi, \text{ and } (U_\theta^{ij})'(\tilde{x}_{ij}) \geq 0.$$ 

Plugging $U_\theta^{ij}$ and $\tilde{x}_{ij}$ in (4), using the above, we have

$$2\tilde{x}_{ij} U_\theta^{ij}(\tilde{x}_{ij}) + \frac{1}{2}(U_\theta^{ij}(\tilde{x}_{ij}))^2 \leq P_c^{ij}(\tilde{x}_{ij}).$$

Sending $j \to \infty$ in the above leads to

$$h(\xi) := -2\xi + \frac{1}{2}\xi^2 \leq P_c(-1).$$

Since $h(s)$ is a decreasing function when $s \leq 2$, we have

$$h(\xi) > h(U_\theta(-1)) = P_c(-1),$$

a contradiction. 

Similarly we have
Lemma 2.11’. Let \( K \) be a compact subset of \( I \setminus \{(c, \gamma) \mid \gamma = \gamma^-(c)\} \). Then for any \( \epsilon > 0 \), there exists some \( \delta > 0 \), depending only on \( \epsilon \) and \( K \), such that for any \( (c, \gamma) \in K \),

\[
|U^c_{\theta} - U^c_{\theta}(1)| < \epsilon, \quad 1 - \delta < x < 1.
\]

Proof. For convenience, let us denote \( U_{\theta} = U^c_{\theta} \), \( \alpha_0 = \sqrt{1 + \alpha_1} \), \( \tau_1 = \tau_1(c_1) = 2 - 2\sqrt{1 + \alpha_1} \), and \( \tau_2 = \tau_2(c_1) = 2 + 2\sqrt{1 + \alpha_1} \). Since \( \gamma < \gamma^+(c) \), \( U^c_{\theta}(1) = \tau_1 \).

By Lemma 2.11, there exists some \( \delta \), such that for any \( (c, \gamma) \in K \),

\[
|U^c_{\theta}(x) - U^c_{\theta}(1)| \leq C|1 + x|^{\min\{\sqrt{1 + \alpha_1}, 1\} - \epsilon}, \quad -1 < x < -1 + \delta.
\]

Lemma 2.12. Let \( K \) be a compact subset of \( I \setminus \{(c, \gamma) \mid c_1 = -1 \text{ or } \gamma = \gamma^+(c)\} \). Then for any \( \epsilon > 0 \), there exist some positive constants \( \delta \) and \( C \), depending only on \( \epsilon \) and \( K \), such that for any \( (c, \gamma) \in K \),

\[
|U^c_{\theta}(x) - U^c_{\theta}(1)| \leq C(1 - x)^{\min\{1 + \alpha_2, 1\} - \epsilon}, \quad 1 - \delta < x < 1.
\]

Proof. For convenience, let us denote \( U_{\theta} = U^c_{\theta} \), \( \alpha_0 = \sqrt{1 + \alpha_1} \), \( \tau_1 = \tau_1(c_1) = 2 - 2\sqrt{1 + \alpha_1} \), and \( \tau_2 = \tau_2(c_1) = 2 + 2\sqrt{1 + \alpha_1} \). Since \( \gamma < \gamma^+(c) \), \( U^c_{\theta}(1) = \tau_2 \).

By Lemma 2.11, there exists some \( \delta = \delta(\epsilon, K) > 0 \), such that \( |U_{\theta}(x) - \tau_1| < \epsilon \) for all \( x \in (-1, -1 + \delta) \). By Lemma 2.2, \( |U_{\theta}(x)| \leq C(K) \) and therefore \( |h(x)| \leq C(K)(1 + x) \) for \( x \in (-1, 1) \). So for all \( x \in (-1, -1 + \delta) \),

\[
w < \int_{-1+\delta}^{x} \frac{\tau_1 - \tau_2 - \epsilon}{2(1 - s^2)} ds + C(K) \leq (-\alpha_0 - \frac{\epsilon}{4}) \ln(1 + x) + C(\epsilon, K),
\]

and

\[
w > \int_{-1+\delta}^{x} \frac{\tau_1 - \tau_2 + \epsilon}{2(1 - s^2)} ds - C(K) \geq (-\alpha_0 + \frac{\epsilon}{4}) \ln(1 + x) - C(\epsilon, K).
\]

Thus

\[
e^w \leq C(\epsilon, K)(1 + x)^{-\alpha_0 - \frac{\epsilon}{4}}, \quad e^{-w} \leq C(\epsilon, K)(1 + x)^{\alpha_0 - \frac{\epsilon}{4}}.
\]

Plugging this into (36), we have

\[
|U_{\theta} - \tau_1| \leq C(\epsilon, K)(1 + x)^{\alpha_0 - \frac{\epsilon}{4}} + C(\epsilon, K)(1 + x), \quad -1 < x < -1 + \delta.
\]

Lemma 2.12’. Let \( K \) be a compact subset of \( I \setminus \{(c, \gamma) \mid c_2 = -1 \text{ or } \gamma = \gamma^+(c)\} \). Then for any \( \epsilon > 0 \), there exists some positive constants \( \delta \) and \( C \), depending only on \( \epsilon \) and \( K \), such that for any \( (c, \gamma) \in K \),

\[
|U^c_{\theta}(x) - U^c_{\theta}(1)| \leq C(1 - x)^{\min\{1 + \alpha_2, 1\} - \epsilon}, \quad 1 - \delta < x < 1.
\]
Lemma 2.13. Let $K$ be a compact subset of $I \cap \{(c, \gamma) \mid c_1 = -1, \gamma < \gamma^+(c)\}$. Then for any $\epsilon > 0$, there exists some $\delta > 0$, depending only on $\epsilon$ and $K$, such that for any $(c, \gamma) \in K$,

$$|(U_{\theta}^{c, \gamma} - 2) \ln \left(\frac{1 + x}{3}\right) - 4| < \epsilon, \quad -1 < x < -1 + \delta.$$ 

Proof. If $U_{\theta} := U_{\theta}^{c, \gamma}$ is a solution of (32) with $(c, \gamma) \in I$, $c_1 = -1$, and $\gamma < \gamma^+(c)$, we have $U_{\theta}(-1) = 2$. Denote

$$g := g^{c, \gamma} = (U_{\theta} - 2) \ln \left(\frac{1 + x}{3}\right), \quad -1 < x < 0.$$ 

Then by Theorem 1.3 in [4], $g(-1) = 4$, $g(x)$ satisfies

$$(1 - x^2) \ln \left(\frac{1 + x}{3}\right) g' - (1 - x)g + \frac{1}{2}g^2 = H_{c, \gamma}(x) := (P_c(x) - 2(1 + x)U_{\theta} + 2) \left(\ln \frac{1 + x}{3}\right)^2. \quad (37)$$

We prove the lemma by contradiction. Assume there exist some $\epsilon > 0$ and a sequence $(c^i, \gamma^i) \in K$ and $-1 < x_i < -1 + \frac{1}{i}$, such that

$$|g^{c^i, \gamma^i}(x_i) - g^{c^i, \gamma^i}(-1)| = |g^{c^i, \gamma^i}(x_i) - 4| \geq \epsilon.$$ 

Since $K$ is compact, there exist a subsequence, still denoted as $(c^i, \gamma^i)$, and some $(c, \gamma) \in K$, such that $(c^i, \gamma^i) \to (c, \gamma)$ as $i \to \infty$.

Denote $g_i := g^{c^i, \gamma^i}$. By standard ODE theory, we have that $g_i \to g := g^{c, \gamma}$ in $C^1_{loc}(-1, 1)$. As explained earlier, $g(-1) = 4$.

We first assume that

$$g_i(x_i) \geq 4 + \epsilon. \quad (38)$$

Using this and the fact that $g_i \to g$ in $C^1_{loc}(-1, 1)$, by similar arguments as in the proof of Lemma 2.11, we have that there exist $x_i \leq \tilde{x}_i \to -1$, such that

$$\xi_i := g_i(\tilde{x}_i) = 4 + \frac{\epsilon}{\sqrt{i}} =: \xi, \text{ and } g'_i(\tilde{x}_i) \leq 0.$$ 

Let $h(s) := -2s + \frac{1}{2}s^2$. By (37) we have that

$$-(1 - \tilde{x}_i)g_i(\tilde{x}_i) + \frac{1}{2}g_i^2(\tilde{x}_i) \leq H_{c^i, \gamma^i}(\tilde{x}_i).$$

Sending $i \to \infty$, we have

$$h(\xi) \leq H_{c, \gamma}(-1) = 0.$$ 

On the other hand, since $\xi > 0$, so $h(\xi) > 0$. A contradiction.

Now if instead of (38), we have

$$g_i(x_i) \leq 4 - \epsilon.$$ 

Without loss of generality, we assume that $0 < \epsilon < 1$. As in Case 1 of the proof of Lemma 2.11, there exists $\tilde{x}_{ij} \to -1$, such that

$$g_i(\tilde{x}_{ij}) = 4 - \frac{\epsilon}{\sqrt{i}} =: \xi, \text{ and } g'_i(\tilde{x}_{ij}) \geq 0.$$ 

20
By (37) we have that

\[-(1 - \tilde{x}_{ij})g_{ij}(\tilde{x}_{ij}) + \frac{1}{2}g_{ij}^2(\tilde{x}_{ij}) \geq H_{c_{ij},\gamma_{ij}}(\tilde{x}_{ij}).\]

Sending \(i \to \infty\), we have

\[h(\xi) \geq H_{c,\gamma}(-1) = 0.\]

On the other hand, since \(3 < \xi < 4\), so \(h(\xi) < h(4) = 0\). A contradiction. \(\square\)

Similarly, we have

**Lemma 2.13'**. Let \(K\) be a compact subset of \(I \cap \{(c, \gamma) \mid c_2 = -1, \gamma > \gamma^-(c)\}\). Then for any \(\varepsilon > 0\), there exists some \(\delta > 0\), depending only on \(\varepsilon\) and \(K\), such that for any \((c, \gamma) \in K\),

\[|(U_0^{c,\gamma} + 2) \ln \left(\frac{1 - x}{3}\right) + 4| < \varepsilon, \quad 1 - \delta < x < 1.\]

The next lemma strengthens Lemma 2.13.

**Lemma 2.14**. Let \(K\) be a compact subset of \(I \cap \{(c, \gamma) \mid c_1 = -1, \gamma < \gamma^+(c)\}\). Then for any \(\varepsilon > 0\), there exists some positive constants \(\delta\) and \(C\), depending only on \(\varepsilon\) and \(K\), such that for any \((c, \gamma) \in K\),

\[|U_0^{c,\gamma}(x) - 2 - \frac{4}{\ln \frac{1 + x}{3}}| \leq C \left| \ln \frac{1 + x}{3} \right|^{-2+\varepsilon}, \quad -1 < x < -1 + \delta.\]

**Proof.** For convenience let us denote \(U_0 = U_0^{c,\gamma}\). Let \(V := U_0 - 2 - \frac{4}{\ln \frac{1 + x}{3}}\). Then \(V\) satisfies the equation

\[(1 - x^2)V' + \frac{4}{\ln \frac{1 + x}{3}}V + \frac{1}{2}V^2 = h(x),\]

where \(h := P_c(x) - P_c(-1) - \frac{4(1+x)}{(\ln \frac{1 + x}{3})^2} - 2(1 + x)V - 4(1 + x) - \frac{8(1+x)}{\ln \frac{1 + x}{3}}.\) We have, using Lemma 2.11 that there exists some \(\delta = \delta(c, K)\), such that \(|h| \leq C(c, K)(1 + x)\) for all \(x \in (-1, -1 + \delta)\).

Let \(w := \int_{-1}^{x} \frac{1}{2}V' + \frac{4}{\ln \frac{1 + x}{3}} ds\). We have

\[V = V \left(-\frac{1}{2}\right) e^{-w} + e^{-w} \int_{-1}^{x} e^{w} \frac{h}{1 - s^2} ds. \tag{39}\]

Since \(V \left(-\frac{1}{2}\right) = U_0 \left(-\frac{1}{2}\right) - 2 + \frac{4}{\ln 2}\), we have \(|V \left(-\frac{1}{2}\right)| \leq C(K)\). By Lemma 2.13 making \(\delta = \delta(c, K) > 0\) smaller if necessary, we have \(|(U_0 - 2) \ln \frac{1 + x}{3} - 4| < \varepsilon\), i.e. \(|V| \leq \frac{\varepsilon}{\ln \frac{1 + x}{3}}\), for all \(-1 < x < -1 + \delta\). We also have \(|h| \leq C(c, K)(1 + x)\) for all \(x \in (-1, -1 + \delta)\). Thus for all \(-1 < x < -1 + \delta < -\frac{1}{2}\), we have

\[w \leq C(c, K) + \int_{-1+\delta}^{x} \frac{4 + \frac{5}{2}}{(1 - s^2) \ln \frac{1 + x}{3}} \leq C(c, K) + (2 + \varepsilon) \ln \left(-\frac{1 + x}{3}\right),\]

21
and

\[ w \geq -C(\epsilon, K) + \int_{-1+\delta}^{x} \frac{4 - \epsilon}{(1 - s^2) \ln \frac{1 + s}{3}} \geq -C(\epsilon, K) + (2 - \epsilon) \ln(- \ln \frac{1 + x}{3}). \]

So

\[ e^w \leq C(\epsilon, K) \left| \ln \frac{1 + x}{3} \right|^{2+\epsilon}, \quad e^{-w} \leq C(\epsilon, K) \left| \ln \frac{1 + x}{3} \right|^{-2+\epsilon}. \]

Plugging this into (39), we have

\[ |V| \leq C(\epsilon, K) \left| \ln \frac{1 + x}{3} \right|^{-2+\epsilon}. \]

The proof is finished. \(\square\)

Similarly we have the following strengthening of Lemma 2.13:

**Lemma 2.14’.** Let \( K \) be a compact subset of \( I \cap \{(c, \gamma) \mid c_1 = -1, \gamma > \gamma^+(c)\} \). Then for any \( \epsilon > 0 \), there exists some positive constants \( \delta \) and \( C \), depending only on \( \epsilon \) and \( K \), such that for any \( (c, \gamma) \in K \),

\[ |U_{c,\gamma}^{\theta}(x) + 2 + \frac{4}{\ln \frac{1 + x}{3}}| < C \left| \ln \frac{1 - x}{3} \right|^{-2+\epsilon}, \quad 1 - \delta < x < 1. \]

Now using Lemma 2.11–Lemma 2.14, we prove the following estimates of partial derivatives of \( U_{\theta} := U_{c,\gamma}^{\theta} \) with respect to \( (c, \gamma) \) on each \( I_{k,l} \).

**Lemma 2.15.** For any \( \epsilon > 0 \), \( m \in \mathbb{N} \), and compact subset \( K \) of \( I \setminus \{(c, \gamma) \mid c_1 = -1 \text{ or } \gamma = \gamma^+(c)\} \), there exists some positive constant \( C \), depending only on \( m, K, \) and \( \epsilon \), such that

\[ \sum_{1 \leq |\alpha| + |j| \leq m} |\partial^\alpha_{\gamma} \partial_j^c U_{\theta}| \leq C, \quad -1 < x < 1 - \epsilon. \]

**Proof.** We prove the lemma by induction. We use \( C(m, K, \epsilon) \) and \( C \) to denote constants which may be different from line to line, and their dependence is clear from the context.

We know by (32) that

\[ (1 - x^2) \left( \frac{\partial U_{\theta}}{\partial \gamma} \right)' + (2x + U_{\theta}) \left( \frac{\partial U_{\theta}}{\partial \gamma} \right) = 0, \]

\[ (1 - x^2) \left( \frac{\partial U_{\theta}}{\partial c_i} \right)' + (2x + U_{\theta}) \left( \frac{\partial U_{\theta}}{\partial c_i} \right) = \partial_{c_i} P_c(x), \]

and \( \frac{\partial U_{\theta}(0)}{\partial \gamma} = 1, \frac{\partial U_{\theta}(0)}{\partial c_i} = 0, i = 1, 2, 3. \) Denote

\[ a(x) = a_{c,\gamma}(x) = \int_0^x \frac{2s + U_{\theta}}{1 - s^2} ds. \quad (40) \]
Then
\[ \frac{\partial U_\theta}{\partial \gamma} = e^{-a(x)}, \]  

(41)

and for \( i = 1, 2, 3, \)
\[ \frac{\partial U_\theta}{\partial c_i} = e^{-a(x)} \int_0^x e^{a(s)} \frac{\partial c_i}{\partial c} P_c(s) \frac{ds}{1 - s^2}. \]  

(42)

By the definition of \( a(x), \) Lemma 2.2 and Lemma 2.12 we have that there exists some constant \( C = C(\epsilon, K) \) such that
\[ e^{-a(x)} \leq C(1 + x)^{-\frac{v_\epsilon + 1}{2}}, \quad e^a(x) \leq C(1 + x)^{-\frac{v_\epsilon + 1}{2}}, \quad -1 < x < 1 - \epsilon. \]

Since when \((c, \gamma) \in K, U_\theta(1) < 2,\) there exists some \( C(K, \epsilon), \) such that \( e^{-a(x)} \leq C(K, \epsilon).\) Thus by (41) and (42) we have that for \(-1 < x < 1 - \epsilon,\)
\[ \sum_{|\alpha|+j=1} |\partial_\alpha \partial_j U_\theta| \leq C(K, \epsilon). \]

Now for \( m \geq 2,\) suppose that \( C(m_1, K, \epsilon) \) exist for all \( 1 \leq m_1 \leq m - 1,\) then for any \(|\alpha| + j = m,\)
\[ (1 - x^2)(\partial_\alpha \partial_j U_\theta)' + 2x \partial_\alpha \partial_j U_\theta + \frac{1}{2} \partial_\alpha \partial_j (U_\theta^2) = \partial_\alpha \partial_j P_c(x). \]

This leads to
\[ (1 - x^2)(\partial_\alpha \partial_j U_\theta)' + (2x + U_\theta) \partial_\alpha \partial_j U_\theta = h, \]
where
\[ h := -\frac{1}{2} \sum_{0 \leq (\alpha, j), 0 < |\alpha| + j < m} \begin{pmatrix} \alpha \\ \alpha_1 \end{pmatrix} \begin{pmatrix} j \\ j_1 \end{pmatrix} \partial^\alpha \partial^{j_1} \partial_\gamma U_\theta \partial^{\alpha_1-j_1} U_\gamma. \]

Notice that \( \partial_\alpha \partial_j U_\theta(0) = 0 \) for all \(|\alpha| + j \geq 2,\) we have
\[ \partial_\alpha \partial_j U_\theta = e^{-a(x)} \int_0^x e^{a(s)} h(s) \frac{ds}{1 - s^2}. \]

By the induction assumption, \( h \in L^\infty(-1, 1 - \epsilon) \) and there exists some positive constant \( C, \) depending only on \( m, K, \) and \( \epsilon \) such that \(|h|_{L^\infty(-1, 1 - \epsilon)} \leq C.\) So we have
\[ |\partial_\alpha \partial_j U_\theta|_{L^\infty(-1, 1 - \epsilon)} \leq C. \]

The proof is finished. \( \Box \)

Similarly, using Lemma 2.2 and Lemma 2.12 we have

**Lemma 2.15'.** For any \( \epsilon > 0, m \in \mathbb{N}, \) and compact subset \( K \) of \( I \setminus \{(c, \gamma) | c_2 = -1 \text{ or } \gamma = \gamma^-(c)\}, \) there exists some positive constant \( C, \) depending only on \( m, K, \) and \( \epsilon, \) such that
\[ \sum_{1 \leq |\alpha| + j \leq m} |\partial_\alpha \partial_j U_\theta| \leq C, \quad -1 + \epsilon < x < 1. \]
Lemma 2.16. For any $\epsilon > 0$, $m \in \mathbb{N}$, and compact subset $K$ of $I \cap \{(c,\gamma) \mid c_1 = -1, \gamma < \gamma^+(c)\}$, there exists some positive constant $C$, depending only on $m$, $K$, and $\epsilon$, such that

$$\sum_{1 \leq |\alpha| + j \leq m, \alpha_1 = 0} \left( \ln \frac{1 + x}{3} \right)^2 |\partial_c^\alpha \partial_j^i U_\theta| \leq C, \quad -1 < x < 1 - \epsilon.$$  

Proof. We prove the lemma by induction. Denote $C(m, K, \epsilon)$ and $C$ to be constants which may vary from line to line, and their dependence is clear from the context. Similar as the proof of Lemma 2.15, we have (41) and (42) where $a(x)$ is defined by (40). By the definition of $a(x)$, Lemma 2.2 and Lemma 2.14, there exists some constant $C = C(m, K, \epsilon)$, such that

$$e^{-a(x)} \leq C \left( \ln \frac{1 + x}{3} \right)^{-2}, \quad e^{a(x)} \leq C \left( \ln \frac{1 + x}{3} \right)^2.$$  

Notice in this case, $i = 2$ or $3$ in (42), and $|\partial_c P_c| \leq C(1 + x)$ for some constant $C$ depending only on $K$, so we have that for $-1 < x < 1 - \epsilon$,

$$\sum_{|\alpha| + j = 1} \left( \ln \frac{1 + x}{3} \right)^2 |\partial_c^\alpha \partial_j^i U_\theta| \leq C(K, \epsilon).$$  

Now suppose that $C(m_1, K, \epsilon)$ exists for all $1 \leq m_1 \leq m - 1$. As in the proof of the previous lemma we have, for all $|\alpha| + j = m$ and $\alpha_1 = 0$, that

$$\partial_c^\alpha \partial_j^i U_\theta = C e^{-a(x)} + e^{a(x)} \int_{-1 + \frac{1}{2}}^x e^{a(s)} \frac{h(s)}{1 - s^2} ds,$$

where

$$h := \frac{1}{2} \sum_{0 \leq (\alpha_1, j_1) \leq (\alpha, j), 0 < |\alpha_1| + j_1 < m} \left( \begin{array}{c} \alpha \\ \alpha_1 \end{array} \right) \left( \begin{array}{c} j \\ j_1 \end{array} \right) \partial_c^{\alpha_1} \partial_j^{j_1} U_\theta \partial_c^{\alpha - \alpha_1} \partial_j^{j - j_1} U_\theta.$$  

Then, by the induction assumption, $h \in L^\infty(-1, 1 - \epsilon)$ and there is some positive constant $C$, depending only on $m$, $K$, and $\epsilon$, such that $(\ln \frac{1 + x}{3})^4 |h(x)| \leq C$ for all $-1 < x < 1 - \epsilon$. Using this estimate we then have

$$\left( \ln \frac{1 + x}{3} \right)^2 |\partial_c^\alpha \partial_j^i U_\theta| \leq C.$$  

The lemma is proved. \hfill \Box

Similarly, using Lemma 2.14, we have

Lemma 2.16'. For any $\epsilon > 0$, $m \in \mathbb{N}$, and compact subset $K$ of $I \cap \{(c,\gamma) \mid c_2 = -1, \gamma > \gamma^-(c)\}$, there exists some positive constant $C$, depending only on $m$, $K$, and $\epsilon$, such that

$$\sum_{1 \leq |\alpha| + j \leq m, \alpha_2 = 0} \left( \ln \frac{1 - x}{3} \right)^2 |\partial_c^\alpha \partial_j^i U_\theta| \leq C(m, K, \epsilon), \quad -1 + \epsilon < x < 1.$$  

Theorem 1.3 follows from Corollary 2.2, Corollary 2.2', Lemma 2.15, 2.15', 2.16 and 2.16'.
References

[1] M. A. Goldshtik, A paradoxical solution of the Navier-Stokes equations. Prikl. Mat. Mekh. 24 (1960), 610-621. Transl., J. Appl. Math. Mech. (USSR) 24 (1960), 913-929.

[2] G. Karch and D. Pilarczyk, Asymptotic stability of Landau solutions to Navier-Stokes system, Arch. Ration. Mech. Anal. 202 (2011), 115-131.

[3] L. Landau, A new exact solution of Navier-Stokes Equations, Dokl. Akad. Nauk SSSR 43 (1944), 299-301.

[4] L. Li, Y.Y. Li and X. Yan, Homogeneous solutions of stationary Navier-Stokes equations with isolated singularities on the unit sphere. I. One singularity, arXiv: 1609.08197 v1[math. AP] 26 Sep 2016. To appear in Arch. Ration. Mech. Anal.

[5] L. Li, Y.Y. Li and X. Yan, Homogeneous solutions of stationary Navier-Stokes equations with isolated singularities on the unit sphere. III, in preparation.

[6] X. Luo and R. Shvydkoy, 2D homogeneous solutions to the Euler equation, Communications in Partial Differential Equations 40 (2015), 1666-1687.

[7] A. F. Pillow and R. Paull, Conically similar viscous flows. Part 1. Basic conservation principles and characterization of axial causes in swirl-free flow, Journal of Fluid Mechanics 155 (1985), 327-341.

[8] A. F. Pillow and R. Paull, Conically similar viscous flows. Part 2. One-parameter swirl-free flows, Journal of Fluid Mechanics 155 (1985), 343-358.

[9] A. F. Pillow and R. Paull, Conically similar viscous flows. Part 3. Characterization of axial causes in swirling flow and the one-parameter flow generated by a uniform half-line source of kinematic swirl angular momentum, Journal of Fluid Mechanics 155 (1985), 359-379.

[10] J. Serrin, The Swirling Vortex, Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences 271 (1972), 325-360.

[11] R. Shvydkoy, Homogeneous Solutions to the 3D Euler System, arXiv:1510.03378 v1[math.AP] 12 Oct 2015.

[12] N. A. Slezkin, On an exact solution of the equations of viscous flow, Uch. zap. MGU, no. 2, 89-90, 1934.

[13] H. B. Squire, The round laminar jet, Quart. J. Mech. Appl. Math. 4 (1951), 321-329.

[14] V. Šverák, On Landau’s solutions of the Navier-Stokes equations, Problems in mathematical analysis. No. 61. J. Math. Sci. (N. Y.) 179 (2011), 208-228. arXiv: math/0604550 (2006).

[15] G. Tian and Z. P. Xin, One-point singular solutions to the Navier-Stokes equations, Topol. Methods Nonlinear Anal. 11 (1998), 135-145.

[16] C. Y. Wang, Exact solutions of the steady state Navier-Stokes equation, Annu. Rev. Fluid Mech. 23 (1991), 159-177.
[17] V. I. Yatseyev, On a class of exact solutions of the equations of motion of a viscous fluid, National Advisory Committee for Aeronautics, Technical Memorandum, no 1349, 1950.