Eikonal Approximation to 5D Wave Equations as Geodesic Motion in a Curved 4D Spacetime

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Abstract: We first derive the relation between the eikonal approximation to the Maxwell wave equations in an inhomogeneous anisotropic medium and geodesic motion in a three dimensional Riemannian manifold using a method which identifies the symplectic structure of the corresponding mechanics. We then apply an analogous method to the five dimensional generalization of Maxwell theory required by the gauge invariance of Stueckelberg’s covariant classical and quantum dynamics to demonstrate, in the eikonal approximation, the existence of geodesic motion for the flow of mass in a four dimensional pseudo-Riemannian manifold. No motion of the medium is required. These results provide a foundation for the geometrical optics of the five dimensional radiation theory and establish a model in which there is mass flow along geodesics. Finally, we discuss the interesting case of relativistic quantum theory in an anisotropic medium as well. In this case the eikonal approximation to the relativistic quantum mechanical current coincides with the geodesic flow governed by the pseudo-Riemannian metric obtained from the eikonal approximation to solutions of the Stueckelberg-Schrödinger equation. This construction provides a model for an underlying quantum mechanical structure for classical dynamical motion along geodesics on a pseudo-Riemannian manifold. The locally symplectic structure which emerges is that of Stueckelberg’s covariant mechanics on this manifold.

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1. Introduction

It has been known for many years that the Hamilton-Jacobi equation of classical mechanics defines a function which appears to be the eikonal of a wave equation, and therefore that classical mechanics appears to be a ray approximation to some wave theory\(^1\). The propagation of rays of waves in inhomogeneous media appears, from this point of view (as a result of the application of Fermat’s principle), to correspond to geodesic motion in a metric derived from the properties of the medium\(^2\). This geometrical interpretation has been exploited recently by several authors to construct models which exhibit three dimensional analogs of general relativity by studying the wave equations of light in an inhomogeneous medium\(^3\), and, to achieve four dimensional analogs, sound waves and electromagnetic propagation in inhomogeneously moving materials\(^4\). Visser et al\(^5,6,7\) have pointed out that condensed matter systems such as acoustics in flowing fluids, light in moving dielectrics, and quasiparticles in a moving superfluid can be used to mimic kinematical aspects of general relativity. Leonhardt and Piwnicki\(^8\) and Lorenci and Klippert\(^9\), for example, have discussed the case of electromagnetic propagation in moving media. In order to achieve four dimensional geodesic flows, it has been necessary to introduce a motion of the medium. * There is considerable interest in extending these analog models for the kinematical aspects of gravity to include dynamical aspects, i.e., considering gravity as an emergent phenomenon\(^5,6\).

The manifestly covariant classical and quantum mechanics introduced by Stueckelberg\(^12\) in 1941 has the structure of Hamiltonian dynamics with the Euclidean three dimensional space replaced by four-dimensional Minkowski space (since all four of the components of energy-momentum are kinematically independent, the theory is intrinsically “off-shell”). The dynamical evolution of the system is governed by a “world time” \(\tau\). This theory leads to five dimensional wave equations for the associated gauge fields. The self-interaction problem of the relativistic charged particle has been recently studied\(^13\), where it was shown that the radiation field is associated with excursions from the mass shell; highly non-linear terms appear in the resulting generalized Lorentz force. In the limit that the motion maintains a state very close to the mass shell limit, the equation reduces to that of Dirac\(^14\).

The known bound state spectra and corresponding wave functions for the (spinless) two-body problem in potential theories formulated in a manifestly covariant way have been worked out\(^15\). The classical relativistic Kepler orbits have been studied in detail\(^12,16\).

In order for the Stueckelberg-Schrödinger equation of the quantum form of this theory to be gauge invariant, it is necessary to introduce a fifth gauge field, compensating for the derivative with respect to the invariant evolution parameter \(\tau\)\(^17,18\). Generalized gauge invariant field strengths, \(f_{q,p}\), with \(q,p = 0, 1, 2, 3, 5\) occurring in the Lagrangian to second order generate second order field equations analogous to the usual Maxwell equations,

* We remark that Obukhov and Hehl\(^10\) have shown that a conformal class of metrics for spacetime can be derived by imposing constrained linear constitutive relations between the electromagnetic fields \((E,B)\) and the excitations \((D,H)\), using Urbantke’s formulas\(^11\), developed to define locally integrable parallel transport orbits in Yang-Mills theories (on tangent 2-plane elements on which the Yang-Mills curvature vanishes).
with source given by the four-vector matter field current and an additional Lorentz scalar density. The canonical second quantization of this theory was studied in ref. 19. Taking the Fourier transform of these equations over the invariant parameter, as we demonstrate below, one sees that the zero frequency component (zero mode) of the equations coincides with the standard Maxwell theory (the fifth field decouples). The Maxwell theory is therefore properly contained in the five-dimensional generalization as the zero frequency component. The general form of the theory has variations as a function of world time, about the (world time independent) Maxwell theory.

In the quantum case the four-currents are given by bilinears in the wave function containing first derivatives, and the fifth source is the (Lorentz) scalar probability density. The symmetry of the homogeneous equations, which can be $O(3, 2)$ or $O(4, 1)$, depending on the sign chosen for raising and lowering the fifth index; it is not realized in the inhomogeneous equations, since the spacetime current components and the fifth component (the quantum mechanical probability density $|\psi|^2$) cannot be transformed into each other by a linear transformation. Hence, without augmenting the symmetry of the matter fields beyond $O(3, 1)$, the fifth field, whose source in the Maxwell-like equations is the probability density (or, classically, the matter density) in spacetime, can play a special role. There appears to be no kinematic basis for choosing one or the other of these signatures; atomic radiative decay, for example, contains points in phase space (for radiation of off-shell photons) for either type. We note, however, that the homogeneous equations corresponding to the $O(4, 1)$ signature appear, under Fouriers transform of the $\tau$ variable, as Klein-Gordon type wave equations with positive mass-squared (physical particles), but for the $O(3, 2)$ choice of signature, these equations have the “wrong” sign (tachyonic) for interpreting the additive term as a mass-squared. As a physical example of how both metrics may play a role, let us suppose that the off-shell radiation impinging on an atomic nucleus, for example, has the $O(3, 2)$ signature. During the interaction, as the state evolves as a function of $\tau$, the metric may make a transition to the $O(4, 1)$ form. This transition could correspond to the observed phenomenon of photoproduction (the vector dominance model). We therefore leave open the question of a definitive choice of the signature for the five dimensional radiation field at this stage.

In either case, the four dimensional spacetime submanifold coincides with that of Minkowski; on this manifold, the fields are defined as a family of functions parameterized by $\tau$. This parameter controls the evolution generated by the Hamiltonian of the system of particles and radiation. That this parameter occurs in the wave equations with timelike or spacelike metric does not change its physical interpretation. At each value of $\tau$, the four dimensional configuration corresponds to a Maxwell wave; these waves, however, evolve non-trivially in $\tau$, reflecting the dynamics of the system. For example, in classical charged particle scattering, the conserved currents are constructed (as we remark below) in the four-dimensional Maxwell theory, by integration of a local current $\frac{\mu}{M}\delta^4(x - x(s))$ over all proper time $s$ (which we may identify here with the world time). The trajectory $x^\mu(s)$ is not, however, known until the problem is solved; the system is not a priori integrable. In the higher dimensional theory we discuss here, equations of motion determine the trajectory $x(\tau)$, and in the neighborhood of a given $\tau$, the generalized (sometimes called pre-Maxwell) equations determine the fields as functions of $\tau$. This system is explicitly, in principle,
integrable. The integral of the results over all $\tau$ (on both fields and currents) then provides exact solutions of the corresponding Maxwell problem, with currents computed from the resulting trajectories in the standard way. Since, however, the Lorentz force is non-linear, it is clear that the dynamics forming the trajectories may be very different from that of the usual Lorentz force\textsuperscript{13}.

The structure of the gauge theory obtained from the non-relativistic Schrödinger equation is precisely analogous. The four dimensional gauge invariant field strengths obey second order equations, for which the sources are the vector currents and the scalar probability density (or, classically, the matter density in three dimensional space). No linear transformation can connect the Schrödinger probability density with the vector currents, so that the scalar density and the fourth gauge field can play a special role.

Since the eikonal approximation naturally lowers the dimension of the differential equations describing the fields by one, the eikonal approximation to the five-dimensional field equations results in four dimensional differential equations. In the presence of a non-trivial dielectric structure of the medium, the four dimensional field equations resulting from the eikonal approximation can describe geodesic motion in four dimensional spacetime without the necessity of adding motion to the medium. We emphasize that the underlying manifold, on which the fields are defined, is a flat Cartesian space, but that the dynamically induced trajectories are curved, and can be described by the geodesics of a pseudo-Riemannian manifold. This result forms our basic motivation for studying the generalized dynamics of Stueckelberg\textsuperscript{12} in this context.

We start with a review of the simpler case of Maxwell wave propagation in a three dimensional inhomogeneous, anisotropic medium in which we introduce a method which permits us to identify clearly the structure of the effective mechanics which emerges in the eikonal approximation. In Section 3, we study the eikonal structure of waves in a 5D inhomogeneous medium, in which Minkowski spacetime is embedded, and in Section 4, a wave equation of Schrödinger type, and show that the resulting rays have a precise analog with the results of Kline and Kay\textsuperscript{2} (who have studied the three dimensional ray limit of the usual Maxwell equations), but in a 4D spacetime manifold with a pseudo-Riemannian structure. Kline and Kay\textsuperscript{2} show that the rays are geodesic in the metric associated with the anisotropic inhomogeneous medium. As for the case of Kline and Kay\textsuperscript{2}, it follows from the existence of a Hamiltonian that the corresponding Lagrangian obeys an extremum condition which describes the rays as geodesics. We show that there is mass flow along these rays, and that the flow is controlled by generating functions of Hamiltonian type, establishing a relation between geodesic flow and a relativistic particle mechanics of symplectic form.

2. Eikonals of the Maxwell Wave Equation

We first consider the case of Maxwell wave propagation in a three dimensional inhomogeneous, anisotropic dielectric media, as in ref. 2. We find the Fresnel surfaces in terms of quadratic forms which are the solutions of an eigenvalue equation, where each polarization is associated with a Riemannian metric\textsuperscript{2,20} (of Finsler type) giving rise to geodesic flow. The study of the eigenvalue equations, rather than the determinant condition used in ref. 2, permits us to identify more clearly the structure of the effective mechanics which emerges.
We start by writing Maxwell’s equations in an inhomogeneous, anisotropic medium. These are of the usual form

\[
\nabla \cdot D = \rho (1a) \\
\n\nabla \times H - \frac{\partial D}{\partial t} = J (1b) \\
\n\nabla \cdot B = 0 (1c) \\
\n\n\nabla \times E + \frac{\partial B}{\partial t} = 0, (1d)
\]

where \( \rho \) is the charge density, \( J \) the current density of the sources, and, with \( i, j = 1, 2, 3 \) (we assume here that the coordinates labelled by these indices are Cartesian, and we write indices up or down for convenience of expression), where, however, the relations

\[
D_i = \epsilon_{ij}(x)E_j, \quad B_i = \mu_{ij}(x)H_j \quad (2)
\]

reflect the properties of the medium (we assume all medium properties to be independent of time and that, for our present purposes, there is no mixing between electric and magnetic fields through the constitutive tensor relations). We then multiply Eq.(1d) with the matrix \( \mu^{-1} \), act with the \textit{curl} operator and substitute Eq.(1b). We obtain, in the absence of sources,

\[
\nabla \times (\mu^{-1}(\nabla \times E)) + \frac{\partial^2 D}{\partial t^2} = 0 (3)
\]

Substituting for \( D \) using Eq.(2), one obtains

\[
\epsilon^{-1}(\nabla \times (\mu^{-1}(\nabla \times E))) + \frac{\partial^2 E}{\partial t^2} = 0 (4)
\]

We rewrite Eq. (4) in index notation (we use Latin indices for space components 1, 2, 3, and \( t \) for the time index):

\[
(\epsilon^{-1})^{rl}\epsilon^{lmn}\partial^m(\mu^{-1})^{ni}(\epsilon^{ijk}\partial^j E^k) + \partial_t^2 E^r = 0 (5)
\]

Assuming a solution of the form \( E = A \exp(i\omega(t - \varphi(x))) \) and using the eikonal approximation (for large \( \omega \)), one obtains

\[
-(\epsilon^{-1})^{rl}\epsilon^{lmn}(\mu^{-1})^{ni}\epsilon^{ijk}\partial^m\varphi \partial^j \varphi A^k = A^r. (6)
\]

For a given wave front at a given time \( \varphi(x)|_{t=t_0} = c_0 \) one can think of curves, originating from each point on the wave front, and everywhere in the direction of the gradient of \( \varphi(x) \). These lines can be considered to be trajectories on which the different elements of the wave-front propagate. The trajectories of the wave-front normals, in an anisotropic medium, however, are in general not in the same direction at a given point as the direction of propagation of the energy of radiation, i.e., of the Poynting vector. We shall refer to the trajectories which follow the \textit{Poynting vectors} as \textit{rays}. We now show that the equation for
the rays is a geodesic equation. The metric determining the geodesics is fixed by the electric and magnetic properties of the medium and, in general, depends on the polarization of the field.

We denote the wave front gradient (which can be interpreted, as will be seen later, as the momentum flowing along the ray)

\[ p_m = \partial_m \varphi. \]

Equation (6) then takes the following form:

\[ -(\epsilon^{-1})^r l \varepsilon^{lmn}(\mu^{-1})^{ni} ij k p^m p^j A^k = A^r, \]  
(7)

Multiplying Eq. (7) by \( p^s \epsilon^{sr} \), the left hand side vanishes, and we obtain

\[ p \cdot (\epsilon A) = 0, \]  
(8)

the eikonal form of Eq.(1a). The eikonal solution is therefore consistent with all of Maxwell equations in the non-homogeneous medium. We now define (obviously positive definite for \( \epsilon \) and \( \mu \) scalar; the symmetric in \( mj \) part is symmetric in \( rk \) if \( \epsilon \) and \( \mu \) are symmetric)

\[ M^{rk}_{mj} = -(\epsilon^{-1})^r l \varepsilon^{lmn}(\mu^{-1})^{ni} ij k \]  
(9)

Due to the relation (8), the \( 3 \times 3 \) matrix \( M^{rk}_{ij} p_i p_j \) acts in a two dimensional subspace; one may replace \( A \) by \( v = \epsilon A \) if \( M^{rk}_{mj} \) is replaced by \( \epsilon_{ml} M^{lm}_{ij} \epsilon^{-1}_{mk} \), which takes the two dimensional subspace orthogonal to \( p \) into itself. Eq.(7) may then be written as

\[ \hat{M}^{rk}_{ij} p^i p^j v^k = v^r, \]  
(10)

where \( \hat{M} = \epsilon M \epsilon^{-1} \). Eq.(10) clearly imposes a condition on the magnitude of the vector \( p \). As we shall see, it is restricted to two discrete values.

If we express the tensor \( \hat{M}^{rk}_{mj} \) in coordinates for which the 3 direction is parallel to the momentum \( p \), we obtain the eigenvalue condition

\[ \hat{M}^{rp}_{33} p^i p^j v^k = v^p, \]  
(11)

where the primed quantities are in the momentum oriented coordinate system; since \( v' \) provides support only in the two dimensional subspace orthogonal to \( p \), the matrix \( \hat{M}^{rp}_{33} \) is nonzero only in that subspace. It is not, in general, degenerate. If the eigenvalues are \( \lambda^{(\alpha)} \), \( \alpha = 1, 2 \), it follows from (11) that \( p_3^2 = |p|^2 \) must have the values \( |p^{(\alpha)}|^2 = \lambda^{-1}_{(\alpha)} \). Transforming back to the original frame, we see that (10) can be satisfied only for two polarization modes \( v^{(\alpha)}_k \), and corresponding momenta of the same direction with magnitudes \( |p^{(\alpha)}| = \lambda^{-1}_{(\alpha)} \). Multiplying, for each \( \alpha \), both sides by (normalized) \( v^{(\alpha)}_r \) and summing on \( r \), we obtain the form

\[ H^{(\alpha)} = p_i^{(\alpha)} p_j^{(\alpha)} g_{ij}^{(\alpha)} - 1 = 0, \]  
(12)
where
\[ g_{ij}^{(\alpha)} = v_r^{(\alpha)} \hat{M}_{ij} v^{(\alpha)}_k. \] (13)

We shall identify \( p^{(\alpha)} \) below as a canonical momentum, orthogonal to the surface defined by \( \varphi \); the \( (p^{(\alpha)} \) direction dependent) matrices \( g_{ij}^{(\alpha)} \) therefore act as (Finsler type) metrics for each of the polarizations\(^{20}\).

Since \( g^{(\alpha)} \) depends only on \( x \) and the direction of \( p^{(\alpha)} \) at any given point in space, the condition (12) determines the magnitude of \( p^{(\alpha)} \), and therefore describes a surface. These surfaces, described for \( \alpha = 1, 2 \), coincide with the Fresnel surfaces defined by Kline and Kay\(^2\) by means of the determinant of coefficients of the eigenvalue equation. We have examined here directly the eigenvalue equations since we shall follow this method in the 5D case. It is shown for the three dimensional case in ref. 2 (the proof is given below for the similar four dimensional problem) that \( \partial p^{(\alpha)}_i H^{(\alpha)} \) is parallel to the Poynting vector (clearly the same direction for each \( \alpha \)). If we parametrize the flow along a given ray at \( x \) with some parameter \( s \), this statement can be written as
\[ \dot{x}^i = \frac{dx^i}{ds} = \lambda \partial p^{(\alpha)}_i H^{(\alpha)}, \] (14)
where \( \lambda \) is a scale on \( s \). The total derivative of \( H^{(\alpha)} \) with respect to \( x_i \) is given by
\[ \frac{dH^{(\alpha)}}{dx^i} = \frac{\partial H^{(\alpha)}}{\partial x^i} + \frac{\partial H^{(\alpha)}}{\partial p^{(\alpha)}_k} \frac{\partial p^{(\alpha)}_k}{\partial x^i}. \] (15)
This quantity must vanish, since the derivative relates neighboring Fresnel surfaces, on which (in this mode) \( H^{(\alpha)} \) is zero. Substituting (14) in (15) and using
\[ \frac{\partial p^{(\alpha)}_k}{\partial x^i} = \frac{\partial^2 \varphi}{\partial x^i \partial x^k} = \frac{\partial p^{(\alpha)}_i}{\partial x^k}, \] (16)
we obtain
\[ \lambda \frac{\partial H^{(\alpha)}}{\partial x^i} + \dot{x}^k \frac{\partial p^{(\alpha)}_i}{\partial x^k} = 0 \] (17)
which gives
\[ \dot{p}^{(\alpha)}_i = -\lambda \partial x_i H^{(\alpha)}. \] (18)
Eqs. (14) and (18) correspond to the locally symplectic structure of a Hamiltonian flow generated by the function (12) in each mode. Moreover, one sees that the geodesic equation associated with the metric \( g^{(\alpha)} \) is equivalent to this Hamiltonian flow. This result agrees with application of the Fermat principle.

3. Eikonals of the 5D Wave Equation

In this section we apply a technique similar to that used above to study the structure of wave equations in five dimensions which follow as a consequence of the requirement of gauge invariance of the covariant classical and quantum mechanics of Stueckelberg\(^{12,17−19,21}\). We
show that there is a Hamiltonian form for the generation of the rays in this case as well, and that these rays form spacetime geodesics in a metric space determined by the “dielectric” constitutive tensor relating the tensor fields (analogous to \((E, B)\)) and their corresponding excitation fields (analogous to \((D, H)\)).

We shall use Greek letters for space time indices \((\mu = 0, 1, 2, 3)\) and Latin letters to include a fifth index representing the Poincaré invariant \(\tau\) parameter in addition to the usual 4 spacetime coordinates (e.g., \(q = 0, 1, 2, 3, 5\)). The analysis proceeds by a generalization of the method discussed in Section 2 above. The generalized electromagnetic field tensor is written

\[
f_{q_1 q_2} \equiv \partial_{q_1} a_{q_2} - \partial_{q_2} a_{q_1},
\]

where \(a_q\) are the so-called pre-Maxwell electromagnetic potentials (the fifth gauge potential \(a_5\) is required for gauge compensation of \(i\partial_5\), generating the evolution of the Stueckelberg wave function\(^{12,17}\)).

We introduce the dual (third rank) tensor

\[
k_{l_1 l_2 l_3} = \varepsilon_{l_1 l_2 l_3 q_1 q_2} f_{q_1 q_2},
\]

where \(\varepsilon_{l_1 l_2 l_3 q_1 q_2}\) is the antisymmetric fifth rank Levi-Civita tensor density. The homogeneous pre-Maxwell equations are then given by

\[
\partial_{l_3} k_{l_1 l_2} = 0,
\]

or, more explicitly (the 5 index is raised with signature \(\pm\), according to, as discussed above, \(O(4, 1)\) or \(O(3, 2)\) symmetry of the homogeneous field equations),

\[
\partial_5 \varepsilon_{l_1 l_2 5 q_1 q_2} f_{q_1 q_2} + \partial_\sigma \varepsilon_{l_1 l_2 \sigma q_1 q_2} f_{q_1 q_2} = 0.
\]

We now divide Eq.(20) into two cases. In the first, we take the indices \(l_1, l_2\) to correspond only to space-time indices:

\[
\partial_5 \varepsilon^{\mu \nu \lambda \sigma} f_{\lambda \sigma} + 2 \partial_\sigma \varepsilon^{\mu \nu \sigma \lambda} f_{\lambda 5} = 0,
\]

or

\[
\partial_5 \varepsilon^{\mu \nu \lambda \sigma} f_{\lambda \sigma} + 2 \partial_\sigma \varepsilon^{\mu \nu \sigma \lambda} f_{\lambda 5} = 0,
\]

where \(\varepsilon^{\mu \nu \sigma \lambda}\) is the four dimensional Levi-Civita tensor density. This equation, on the 0-mode (\(\tau\) independent Fourier components) does not involve any of the usual Maxwell fields but only the fifth (Lorentz scalar) electromagnetic field. The second set from Eq.(20) corresponds to \(l_1\) or \(l_2 = 5\). It is clear then that all the other 4-remaining indices must be space-time indices and we obtain

\[
\varepsilon^{5 \mu \sigma \delta \nu} \partial_\sigma f_{\delta \nu} = 0 \rightarrow \varepsilon^{\mu \sigma \delta \nu} \partial_\sigma f_{\delta \nu} = 0.
\]

It is this equation that reduces on integration, over all \(\tau\), to the two usual homogeneous Maxwell equations. On the zero mode the \(\tau\) derivative disappears and therefore Eq.(21) reads for \(\mu = 0\), \(\nu = i\), \(\nabla \times f = 0\) and for the \(\mu = i\), \(\nu = j\) components, \(\partial_i f + \nabla f_0 = 0\) where
we have called \( f^{\mu 5} = (f_0, f) \). Equation (22) then reduces to the standard homogeneous Maxwell equations; therefore we see that the homogeneous pre-Maxwell equations (i.e., Eq.(19)) for the 5 component fields and the spacetime component fields decouple on the 0-mode. This has the effect of reducing the pre-Maxwell system of equations, as we discuss below as well for the inhomogeneous equations, to the usual Maxwell equations. With appropriate identification of the integrated quantities, the zero mode of the pre-Maxwell equations coincides with the Maxwell theory (it is for this reason that the five dimensional gauge fields associated with the Stueckelberg theory are called “pre-Maxwell” fields).

We now turn to the current-dependent pre-Maxwell equations. These can be written:

\[
\partial_l n^{l_1 l_2} = -j^{l_2},
\]

where \( n^{l_1 l_2} \) are the matter induced (excitation) fields (corresponding to \( H, D \) in the 4D theory). We remark that, restricting our attention to the spacetime components of Eq. (23), which then reads

\[
\partial_5 n^{\mu 5} + \partial_\nu n^{\mu \nu} = -j^\mu, \tag{23'}
\]

we may extract the 0-mode by integrating over all \( \tau \). Since \( j^k \) satisfies the five dimensional conservation law \( \partial_k j^k = 0 \), its integral over \( \tau \) (assuming \( j^5 \rightarrow 0 \) for \( \tau \rightarrow \pm \infty \)) reduces to the four dimensional conservation law \( \partial_\mu J^\mu = 0 \), where \( J^\mu = \int j^\mu (x, \tau) d\tau \) is the 0-mode part of \( j^\mu \) (this formula for the conserved \( J^\mu \) is given in Jackson\(^22\)). The first term of the left side of (23') vanishes under integration (assuming that \( n^{\mu 5} \rightarrow 0 \) for \( \tau \rightarrow \pm \infty \)), and one obtains the form

\[
\partial^\nu F^{\mu \nu} = J^\mu,
\]

where we may identify the zero mode fields \( F^{\mu \nu} \), as above, with the Maxwell fields \( H, D \). With the zero mode of (22), we see that the Maxwell theory is properly contained in the five dimensional generalization we are studying here. We see that in Eq. (23) the \( l_2 = \mu \) component reduces on the 0-mode to the standard Maxwell equations and for \( l_2 = 5 \) we get \( \partial_1 n^0 + \nabla \cdot n = 0 \) (using similar notation as above, \( n^{\mu 5} = (n_0, n) \)); thus for the pre-Maxwell equation involving the currents, on the 0-mode, the fields associated with the fifth component and the spacetime fields decouple.

We now assume the existence of linear constitutive equations in the dynamical structure of the 5D fields in a medium which connects the \( n \) tensor-field to the \( k \) tensor-fields using a fifth rank tensor \( \mathcal{E} \) which is a generalization of the fourth rank covariant permeability-dielectric tensor\(^23\) which relates the \( E, B \) fields to the excitation fields \( D, H \) in the usual Maxwell electrodynamics. The constitutive equations have the form

\[
n^{l_1 l_2} = \mathcal{E}^{l_1 l_2 q_1 q_2 q_3} k_{q_1 q_2 q_3}, \tag{24}
\]

antisymmetric in \( l_1 l_2 \) as well as \( q_1 q_2 q_3 \) (the indices of \( k \) have been lowered with the generalized Minkowski metric tensor; we shall treat other tensors in the same way in the following). It is useful at this point to distinguish between the space-time elements \( f^{\mu \nu} \) and the elements \( f^{\mu 5} \). Let us assume that the tensor introduced in Eq. (24) does not mix these fields (for \( n^{\mu 5} \), if \( \varepsilon_5^{q_1 q_2 q_3} \) has \( q_1 q_2 q_3 = \alpha \beta \gamma \), then the components of \( k \) that enter are of the form \( k_{\alpha \beta \gamma} = \mathcal{E}_{\alpha \beta \gamma \mu 5} f^{\mu 5} \) only; similarly, for \( n^{\mu \nu} \), only the components \( \mathcal{E}_{\mu \nu \alpha \beta 5} \)
can occur, and \( k^{\alpha \beta 5} \) connects only to the components \( f_{\lambda \sigma} \) of the field tensor). As we have pointed out above, the vector \( f_{\mu 5} \) is physically distinguished from the antisymmetric tensor \( f_{\mu \nu} \) in the inhomogeneous field equations, since the source terms break the higher symmetry of the homogeneous field equations. The assumption that the constitutive equations do not couple these components results in a simpler system to analyze, although (as for Hall type effects in the non-relativistic theory) it is conceivable that the more general case could occur.

We introduce the new set of fields:

\[
b^{\mu \nu} = \frac{1}{2} \varepsilon^{\mu \nu \lambda \sigma} f_{\lambda \sigma}, \tag{25}\]

so that

\[
n^{\lambda \sigma} = 2 \mathcal{E}^{\lambda \sigma \alpha \beta 5} b_{\alpha \beta}. \tag{26}\]

On the zero mode, the fields \( b^{\mu \nu} \) correspond to the dual Maxwell fields; in this theory they play a role analogous to the \( B \) fields in the Maxwell theory. In a similar way, the \( f_{\mu 5} \) fields are analogous to \( E \). The part of the tensor \( \mathcal{E}_{l_1 l_2 q_1 q_2 q_3} \) connecting the \( \mu 5 \) fields is discussed below.

Working with these fields enables us to construct the equations in a form which, as we shall show, generalizes the Maxwell theory to a form where the invariant time \( \tau \) plays the role of \( t \) and spacetime plays the role of space. This analogy helps to interpret the physics and it distinguishes between the familiar physical quantities \( f_{\mu \nu} \) and the new fields \( f_{\mu 5} \). Substituting these fields in (21) and (22), we find

\[
\partial_5 b^{\mu \nu} + \partial_\sigma \varepsilon^{\mu \nu \sigma \lambda} f_{\lambda 5} = 0, \tag{27}\]

analogous to (1d), and

\[
\partial_\sigma b^{\mu \sigma} = 0, \tag{28}\]

analogous to (1c). For the spacetime excitation fields we define

\[
h_{\mu \nu} = \frac{1}{2} \varepsilon_{\mu \nu \lambda \sigma} n^{\lambda \sigma}
\]

and we get from (23), for \( l_2 = \mu \),

\[
\partial_5 n^{\mu 5} - \frac{1}{2} \varepsilon^{\mu \nu \lambda \sigma} \partial_\sigma h_{\lambda \nu} = - j^{\mu}, \tag{29}\]

analogous to (1b), where we have used

\[
\varepsilon_{\alpha \beta \eta \delta} \varepsilon^{\eta \delta \gamma \mu} = -2(\delta^\gamma_\alpha \delta^\mu_\beta - \delta^\mu_\alpha \delta^\gamma_\beta) \tag{30}\]

To complete the set of equations, we note that for \( l_2 = 5 \), we get from (23)

\[
\partial_\sigma n^{5 \sigma} = - j^{5}, \tag{31}\]
analogous to (1a).

To obtain a mass-energy conservation law for the fields, we multiply (29) by $f_{\mu 5}$ and (27) by $h_{\mu \nu}$, and then combine them, obtaining

$$[f_{\mu 5} \partial_5 n^{\mu 5} + \frac{1}{2} h_{\mu \nu} \partial_\tau b^{\mu \nu}] + \frac{1}{2} \epsilon^{\sigma \mu \lambda \nu} \partial_\sigma (f_{\mu 5} h_{\lambda \nu}) = -j^{\mu} f_{\mu 5}. \quad (32)$$

Assuming the dielectric tensor reduced into the $\mu 5$ and $\mu \nu$ subspaces is symmetric (the relations of $n^{\mu 5}$ to $f_{\mu 5}$ and $n^{\mu \nu}$ to $f_{\mu \nu}$ go by the contraction $\mathcal{E} \epsilon$; the exclusive property of indices of $\epsilon$ then imply simple conditions on $\mathcal{E}$ for the symmetry of these forms) we can write (32) as

$$\frac{1}{2} \partial_5 [f_{\mu 5} n^{\mu 5} + \frac{1}{2} h_{\mu \nu} b^{\mu \nu}] + \frac{1}{2} \epsilon^{\sigma \mu \lambda \nu} \partial_\sigma (f_{\mu 5} h_{\lambda \nu}) = -j^{\mu} f_{\mu 5}. \quad (33)$$

From the Stueckelberg Hamiltonian $^{17}$

$$K = \frac{1}{2M} (p^{\mu} - e_0 a^{\mu}) (p_\mu - e_0 a_\mu) - e_0 a_5, \quad (34)$$

where $e_0$ differs from the usual electric charge by a dimensional constant, one can derive the relativistic Lorentz force $^{13,17,21,24}$

$$M \ddot{x}^{\mu} = e_0 f^{\mu \nu} \dot{x}^{\nu} + e_0 f^{\mu 5}, \quad (35)$$

from which it follows that

$$M \dot{x}^{\mu} \ddot{x}^{\mu} = M \frac{d}{d\tau} (\dot{x}^{\mu} \dot{x}^{\mu}) = \dot{x}^{\mu} f^{\mu 5} \quad (36)$$

Since, in the Stueckelberg theory, the Hamilton equations imply that

$$\dot{x}^{\mu} = \frac{1}{M} (p^{\mu} - e_0 a^{\mu}),$$

and hence the $\tau$ derivative in the central equality of (36) corresponds to a change in the mass-squared $(p^{\mu} - e_0 a^{\mu}) (p_\mu - e_0 a_\mu)$ of the particle. It therefore follows that $j^{\mu} f_{\mu 5}$, is the rate of change of the mass of the particle. We therefore identify $s^{\alpha} = \frac{1}{2} \epsilon^{\alpha \sigma \mu \lambda \nu} f_{\mu 5} h_{\lambda \nu}$ as the analogue of the Maxwell Poynting vector. This Poynting 4-vector is the mass radiation of the field. We see, furthermore, that $\frac{1}{2} \left[ f_{\mu 5} n^{\mu 5} + \frac{1}{2} h_{\mu \nu} b^{\mu \nu} \right]$ is the scalar mass density of the field (its four integral is the dynamical generator of evolution of the non-interacting field $^{17,19}$).

We now introduce the eikonal approximation, i.e., set

$$f(t_1, t_2, x, \tau) = f(t_1, t_2, x) \exp i\kappa (\tau - \Psi (x))$$

for large $\kappa$. In the absence of sources the 5D-Maxwell equations (27), (28), (29), (31) take the form (for large $\kappa$)

$$b^{\mu \nu} - \epsilon^{\mu \nu \sigma \lambda} p_\sigma f_{\lambda 5} = 0, \quad (37)$$
\[ p_\sigma b^{\mu \sigma} = 0, \quad (38) \]
\[ n^{\mu 5} + \frac{1}{2} \varepsilon^{\mu \sigma \lambda \nu} p_\sigma h_{\lambda \nu} = 0, \quad (39) \]
\[ p_\sigma n^{5 \sigma} = 0, \quad (40) \]

where \( p_\sigma = \partial_\sigma \Psi \).

We now relate the direction of \( p_\mu \) to the polarization of the fields. We write the “cross product” of \( n \) and \( b \) (analogous to the cross product of \( D \) and \( B \) in Maxwell’s theory):

\[ \varepsilon_{\mu \nu \sigma \lambda} n^{\nu 5} b^{\sigma \lambda} = 2 n^{\nu 5} f_{\nu 5} p_\mu, \quad (41) \]

or

\[ p_\mu = \frac{1}{2 n^{\alpha 5} f_{\alpha 5}} \varepsilon_{\mu \nu \sigma \lambda} n^{\nu 5} b^{\sigma \lambda}, \]

where we have used (37) and (40). It is clear that since \( p_\mu \) and the Poynting four-vector are cross products of tensors which are not necessarily aligned in the same four-directions, they are in general not parallel to each other (in space-time) due to the anisotropy of the medium, i.e., the wave normal and radiation flow directions are not, in general, the same.

The relations (37) – (40), along with the constitutive relations relating \( n^{\mu \nu}, f_{\sigma \lambda}, \) and \( n^{\mu 5}, f_{\sigma 5} \), provide relations analogous to (6) characterizing the possible field strengths of the eikonal approximation in terms of properties of the medium. We shall not treat these relations here, but discuss the mass-radiation flows, along the rays, on spacetime geodesics in the interesting special case where \( h^{\mu \nu} = b^{\mu \nu} \), which is analogous to the case of materials with \( \mu = 1 \) in Maxwell’s electromagnetism. This case is interesting since, although the space is empty in the usual sense (in analogy to \( E = D, B = H \)), the dielectric effect involving the \( f_{\mu 5} \) components can drive the radiation on curved trajectories, i.e., the corresponding spacetime can have a non-trivial metric structure.

We multiply (37) by \( \frac{1}{2} \varepsilon^{\alpha \beta \mu \nu} p_\beta \). We then use (39) and (30) to obtain

\[ n_\alpha^{5} - p_\alpha p_\beta f_{\beta 5} + p_\beta p_\alpha f_{\alpha 5} = 0 \quad (42) \]

Defining the reduced dielectric tensor \( \mathcal{E}_{\alpha}^{\beta} \) as the part of the general dielectric tensor which connects only the \( \alpha 5 \) components of the fields, i.e.,

\[ n_{\alpha 5} = \mathcal{E}_{\alpha}^{\beta} f_{\beta 5}, \quad (43) \]

the condition (40) then implies that \( \mathcal{E}_{\alpha}^{\beta} f_{\beta 5} \) cannot be in the direction of \( p_\alpha \) (unless it is lightlike). We obtain from Eq. (42)

\[ (\mathcal{E}_{\alpha}^{\beta} - p_\sigma p_\rho \delta_\alpha^{\rho} + p_\rho p_\beta) f_{\beta 5} = 0, \quad (44) \]

where we have chosen the negative sign for the signature of the fifth index, \( n_{\alpha 5} = -n_\alpha^{5} \) [with this choice the flat space limit, for which \( \mathcal{E}_{\alpha}^{\beta} = \delta_\alpha^{\beta} \), Eq. (44), with (40), admits only spacelike \( p_\alpha \); for positive signature of the fifth index, in this limit, \( p_\alpha \) would be timelike].
Eq. (41) has a solution only if the determinant of the coefficients vanishes (a similar calculation in which the field strengths \( f_{\mu5} \) enter in place of \( f_{\mu5} \) results in the same condition on these coefficients, as it must). It is somewhat simpler to work with the eigenvalue equation (44). Assuming as before that this dielectric tensor is symmetric, we can work in a Lorentz frame in which it is diagonal. In this frame we have (for the transformed fields)

\[ f_{\alpha5} = -\frac{p_\alpha}{(\mathcal{E}^\alpha - p^2)}(p^\beta f_{\beta5}). \]  

(45)

Note that in the isotropic case for which all of the \( E^\alpha \) are equal, one obtains \( p_\beta f_{\beta5} = 0 \), and the metric becomes conformal, i.e., one obtains the condition

\[ \mathcal{E}^{-1}\eta^{\mu\nu}p_\mu p_\nu = -1, \]

where \( \eta^{\mu\nu} \) is the flat space Minkowski metric \((-1, 1, 1, 1)\).

Multiplying the equation (45) on both sides by \( p_\alpha \), and summing over \( \alpha \), one obtains the condition \( (p^2 \equiv p_\mu p^\mu) \),

\[ 0 = K = \frac{p_1^2}{\mathcal{E}_1 - p^2} + \frac{p_2^2}{\mathcal{E}_2 - p^2} + \frac{p_3^2}{\mathcal{E}_3 - p^2} - \frac{p_0^2}{\mathcal{E}_0 - p^2} + 1. \]

(46)

This condition determines, in this case, the Fresnel surface of the wave fronts.

It then follows that

\[ \frac{\partial K}{\partial p_\mu} = \frac{2p^\mu}{\mathcal{E}^\mu - p^2} + 2p^\mu \frac{\partial K}{\partial p^2}. \]

(47)

Calculating the scalar product of (45) and (47) one then obtains

\[ f_{\mu5} \frac{\partial K}{\partial p_\mu} =
\]

\[ = -2(p_\nu f^{\nu5}) \left\{ \sum_{i=1,2,3} \frac{(p^i)^2}{(\mathcal{E}^i - p^2)^2} - \frac{p_0^2}{(\mathcal{E}^0 - p^2)^2} - \frac{\partial K}{\partial p^2} \right\} = 0. \]

(48)

Multiplying the expression (37) for \( b_{\mu\nu}(h_{\mu\nu}) \) by (47), the contribution of the second term of (47) vanishes since the Levi-Civita tensor is antisymmetric; the first term, according to (45), is proportional to \( f^{\alpha5} \), and vanishes for the same reason. It therefore follows that

\[ \frac{\partial K}{\partial p^\mu} h^{\mu\nu} = 0. \]

(49)

Since the scalar product of \( \frac{\partial K}{\partial p^\mu} \) with both \( h^{\mu\nu} \) and \( f^{\mu5} \) is zero, it is proportional to their “cross product” i.e., it is parallel to the Poynting vector. To make the proof explicit, it is convenient to define \( V^\mu = \frac{\partial K}{\partial p_\mu} \), \( H_i = -h_{0i} \), \( F_i = f_{i5} \), and \( D^i = \varepsilon^{ijk} h_{jk} \) (the space index may be raised or lowered without changing sign in our Minkowski metric). In this case, the conditions \( V^\mu h_{\mu\nu} = 0 \) and \( V^\mu f_{\mu5} \) become

\[ V^0 H - V \times D = 0 \]

\[ -V^0 f^{05} + V \cdot f = 0 \]

\[ V \cdot H = 0, \]

(50)

13
where we have used boldface to represent the space components of the vector. In these terms, the Poynting vector is given by

\[
S^0 = D \cdot \mathbf{f} \\
S = f^{05}D + \mathbf{f} \times \mathbf{H}
\]  

(51)

Taking the cross product of \( \mathbf{f} \) with the first of (50), one obtains

\[
V^0(\mathbf{f} \times \mathbf{H}) = V(\mathbf{f} \cdot D) - D(\mathbf{f} \cdot V).
\]

For \( V^0 \neq 0 \), one may substitute this into the second of (51). The \( f^{05}D \) term, with the help of the second of (50), cancels, and we are left with

\[
S = \frac{S^0}{V^0}V.
\]

It then follows that \( S^\mu = \frac{S^0}{V^\mu}V^\mu \), i.e., \( V^\mu \) is proportional to the Poynting vector. For the case \( V^0 = 0 \), the second and third of (50) imply that

\[
V \cdot \mathbf{f} = V \cdot \mathbf{H} = 0,
\]

i.e., if \( V \neq 0 \) (the case \( V^\mu = 0 \) is exceptional in the eikonal approximation), it must be proportional to \( \mathbf{f} \times \mathbf{H} \). From the first of (50), we see that \( \mathbf{V} \times \mathbf{D} = 0 \), and if \( \mathbf{D} \neq 0 \), it must be proportional to \( \mathbf{V} \). The space part of \( S^\mu \), from the second of (51) is then proportional to \( \mathbf{V} \). Under these conditions, the time part of \( S^\mu \) vanishes, and therefore we again obtain the result that \( V^\mu \) is proportional to \( S^\mu \). If \( \mathbf{D} = 0 \), then \( S^0 = 0 \) and, since \( \mathbf{V} \) is proportional to \( \mathbf{f} \times \mathbf{H} \), it again follows that \( V^\mu \) is proportional to \( S^\mu \).

From this point, one may follow the same procedure used in the case of Maxwell’s electromagnetism (Eqs.(14) to (18)) to obtain the Hamiltonian flow corresponding to the admissible modes. The Lagrangian associated with the Hamiltonian (46) satisfies a minimal principle, from which it follows that the Hamiltonian flow is geodesic on this manifold. Replacing \( f^{05} \) in (44) by \((\mathcal{E}^{-1})^{05}n^5\), one obtains

\[
\{\delta^\alpha_{\gamma} - M^\alpha_{\gamma\mu\nu}p^\mu p^\nu\}n^5 = 0,
\]

(52)

where

\[
M^\alpha_{\gamma\mu\nu} = \delta^\alpha_{\mu}(\mathcal{E}^{-1})^\nu_{\gamma} - \delta^\nu_{\gamma}(\mathcal{E}^{-1})^\alpha_{\mu}.
\]

(53)

The condition (40) implies that the solutions can lie only in the hyperplane orthogonal to \( p^\sigma \). The projection of the matrix \( M^\alpha_{\gamma\mu\nu}p^\mu p^\nu \) (in the indices \( \alpha, \gamma \)) into the three dimensional subspace orthogonal to \( p^\sigma \) is symmetric, and can therefore be diagonalized by an orthogonal (or pseudo-orthogonal) transformation in three dimensions. In fact, the Gauss law and a gauge condition restrict the polarization degrees of freedom to three\(^19\) (the eikonal approximation is far from the zero mode, which corresponds to the Maxwell limit, for which only two polarizations survive) and hence one finds three geodesics.
For $p^\sigma$ timelike, one can choose a Lorentz frame in which the eigenvalue condition (52) has the form
\begin{equation}
\{\delta^{\alpha\gamma} - M^{\alpha\gamma}_{\mu\nu}(p^\mu p^\nu)^2\} n^\gamma n^5 = 0,
\end{equation}
and for $p^\sigma$ spacelike,
\begin{equation}
\{\delta^{\alpha\gamma} - M^{\alpha\gamma}_{\mu\nu}(p^\mu p^\nu)^2\} n^\gamma n^5 = 0.
\end{equation}
In each of these cases, the matrix can be diagonalized under the little group acting in the space orthogonal to $p^\sigma$ (leaving it invariant). For the lightlike case, up to a rotation, $p^\sigma$ has the form $(p, 0, 0, 0)$. The remaining matrix may then be diagonalized under $\text{SO}(2)$ rotations, to obtain just two geodesics, corresponding to the polarization states of a massless Maxwell-like theory. This special limiting case will be investigated in detail elsewhere.

With the same procedure as applied to the Maxwell case treated above, with $H$ replaced by $K$, and the space indices replaced by spacetime indices, one finds the symplectic structure of the flow of matter in space time.

It has been shown by Kline and Kay 2, as discussed above, that for the three dimensional Maxwell case, the Hamilton equations resulting from the eikonal coincide with the geodesic flow generated by the resulting metric (recall that the direction of momentum associated with both eigenstates is the same); a similar proof can be applied to the 4D cases we have studied here.

4. Eikonal of the Relativistic Stueckelberg-Schrödinger Equation

It is interesting to apply the eikonal method to a relativistic Schrödinger equation in a medium which is not isotropic, for example, in a crystal with shear forces25, with locally varying band structure (as in a crystal under nonuniform stress, or near the boundaries or impurities). In this case, the rays are directly associated with the (probability) flow of particles. The eikonal eigenvalue condition is one dimensional in this case, since the field is scalar. For an analog of this structure (corresponding, for example, to a distribution of events in a potential periodic in both space and time) in four dimensions described by a relativistically covariant equation of Stueckelberg-Schrödinger type, the metric one obtains is a spacetime metric, and the geodesic flow is that of the quantum probability for the spacetime events (matter) described by the Stueckelberg wave function. We show elsewhere26, moreover, that such an equation may be derived from a relativistic generalization of a stochastic procedure analogous to that of Nelson27, with non-trivial correlations between the stochastic motions in different directions of spacetime. This simplest analog of the nonrelativistic problem is given by
\begin{equation}
i \frac{\partial}{\partial \tau} \psi_\tau(x) = \partial^\mu E_{\mu\nu} \partial^\nu \psi_\tau(x),
\end{equation}
where $E_{\mu\nu}$ corresponds to the effect of the medium, and is assumed to be symmetric. The Schrödinger current is then
\begin{equation}
j_\tau(x)_\nu = -i (\psi_\tau^* E_{\mu\nu} \partial^\mu \psi_\tau - \psi_\tau E_{\mu\nu} \partial^\mu \psi_\tau^*).
\end{equation}
In the eikonal approximation, for which the frequency associated with $\tau$ (essentially the total mass of the particle15) is large, one obtains the condition
\begin{equation}
K = E_{\mu\nu} p^\mu p^\nu - 1 = 0,
\end{equation}
analogous to the Fresnel surface condition (46) for the optical case. It is clear that $\partial K/\partial p_\mu$ is in the direction of $j^\mu$. This implies that $K$ is the operator of evolution for the dynamical flow of particles, corresponding to the rays. It follows from the Hamilton equations that the flow is geodesic, where $E_{\mu\nu}$ is the metric.

5. Summary and Discussion

We have reviewed the eikonal treatment of Maxwell waves in an anisotropic medium in a form which exhibits the structure of a canonical mechanics on a three dimensional pseudo-Riemannian manifold, where there are two metrics, one for each admissible polarization mode. The canonical momenta for each of the corresponding geodesics have the same direction, but differ in magnitude.

We then studied the eikonal approximation for the 5D wave equation for the generalized Maxwell fields implied by the requirements of gauge invariance of the Stueckelberg manifestly covariant quantum theory. One finds in this case the structure of a canonical mechanics on a four dimensional pseudo-Riemannian manifold, where there are in general three metrics for each of the timelike or spacelike possibilities for the canonical momenta, and two for the limiting lightlike case. No motion of the medium is required. This canonical mechanics has the form of Stueckelberg’s covariant mechanics, where the form of the Hamiltonian describes the Fresnel surfaces, and the evolution generated by the Hamilton equations carries mass flow along the geodesics.

We finally studied briefly the eikonal approximation to an non-isotropic Stueckelberg-Schrödinger equation written in a form analogous to the non-relativistic theory in a crystal as it appears in a Brillouin zone, where the local energy surface carries curvature (an example of a model providing an analogy to a crystal in spacetime is that of an electromagnetic field in a resonant cavity), or, as derived from a relativistic Nelson type stochastic procedure with non-trivial correlations. The eikonal approximation corresponds to (total system) mass large compared to typical Hamilton-Jacobi momenta. In this case the Hamiltonian flow is in the direction of the eikonal approximation to the quantum mechanical current, so that the analog of the Fresnel surface condition is the generator of evolution for the flow of particles, corresponding to the rays. The Hamiltonian flow is geodesic with respect to the pseudo-Riemannian metric identified with the effective mass matrix. This construction provides a model for an underlying quantum mechanical structure for classical dynamical motion along geodesics in a pseudo-Riemannian manifold, an analog to general relativity.

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References

1. H. Goldstein, *Classical Mechanics*, Addison-Wesley, N.Y. (1950)
2. M. Kline and I.W. Kay, *Electromagnetic Theory and Geometrical Optics*, John Wiley and Sons, N.Y. (1965).
3. P. Piwnicki, Geometrical approach to light in inhomogeneous media, [gr-qc/0201007](gr-qc/0201007).
4. M. Visser, Class. Quant. Grav. 15, 1767 (1998); L.J. Garay, J.R. Anglin, J.I. Cirac and P. Zoller, Phys. Rev. A63, 026311 (2001); Phys. Rev Lett. 85, 4643 (2000).
5. M. Visser, Carlos Barceló, and Stefan Liberati, Analogue models of and for gravity gr-qc/0111111 (2001).
6. Carlos Barceló and Matt Visser, Int. Jour. Mod. Phys. D 10, 799 (2001).
7. Carlos Barceló, Stefano Liberati and Matt Visser, Class. and Quantum Grav. 18, 3595 (2001).
8. U. Leonhardt and P. Piwnicki, Phys. Rev A 60, 4301 (1999).
9. V.A. De Lorenci and R. Klippert, Phys. Rev. D 65, 064027 (2002).
10. Y.N. Obukhov and F.W. Hehl, Phys. Lett. B 458(1999) 466. See also, M. Schönberg, Rivista Brasileira de Fisica 1, 91 (1971); A Peres, Ann. Phys.(NY) 19,279 (1962).
11. H. Urbantke, J. Math. Phys. 25 3231(1984); Acta Phys. Austriaca Suppl. XIX,875 (1978).
12. E.C.G. Stueckelberg, Helv. Phys. Acta 14, 322, 588 (1941); J.S. Schwinger, Phys. Rev. 82, 664 (1951); R.P. Feynman, Rev. Mod. Phys. 20, 367 (1948); R.P. Feynman, Phys. Rev. 80, 440 (1950). C. Piron and L.P. Horwitz, Helv. Phys. Acta 46, 316 (1973), extended this theory to the many-body case throught he postulate of a universal invariant evolution parameter.
13. O. Oron and L.P. Horwitz, Phys. Lett. A 280, 265 (2001).
14. P.A.M. Dirac, Proc. Roy. Soc. London, Ser. A 167, 148 (1938).
15. R.I. Arshansky and L.P. Horwitz, Jour. Math. Phys. 30, 66 and 380 (1989).
16. M.A. Trump and W.C. Schieve, Classical Relativistic Many-Body Dynamics, Fund. Theories of Physics, Kluwer, Dordrecht (1999).
17. D. Saad, L.P. Horwitz and R.I. Arshansky, Found. Phys. 19, 1126 (1989).
18. M.C. Land, N. Shnerb and L.P. Horwitz, Jour. Math. Phys. 36, 3263 (1995).
19. N. Shnerb and L.P. Horwitz, Phys. Rev A48, 4058 (1993).
20. M. Visser, Birefringence versus bi-metricity, contribution to Festschrift in honor of Mario Novello (2002).
21. M.C. Land and L.P. Horwitz, Found. Phys. Lett. 4, 61 (1991).
22. J.D. Jackson, Classical Electrodynamics, 2nd edition, Wiley, N.Y. (1975).
23. A.O. Barut, Electrodynamics and Classical Theory of Fields and Particles, Dover, N.Y. (1964).
24. C. Piron (personal communication) has obtained this result independently.
25. H. Brooks, Adv. in Electronics 7, 85 (1955); C. Kittel, Quantum Theory of Solids, p. 131, John Wiley and Sons, N.Y. (1963).
26. O. Oron and L.P. Horwitz, in preparation.
27. Edward Nelson, Dynamical Theories of Brownian Motion, Princeton University Press, Princeton (1967); Edward Nelson, Quantum Fluctuations, Princeton University Press Princeton (1985). See also, Ph. Blanchard, Ph. Combe and W. Zheng, Mathematical and Physical Aspects of Stochastic Mechanics, Springer-Verlag, Heidelberg (1987), for further helpful discussion.