ALMOST SURE WELL-POSEDNESS FOR THE CUBIC NONLINEAR
SCHRÖDINGER EQUATION IN THE SUPER-CRITICAL REGIME ON $\mathbb{T}^d$, $d \geq 3$

HAITIAN YUE

Abstract. In this paper we prove almost sure local well-posedness in both atomic spaces $X^s$ and Fourier restriction spaces $X^{s,b}$ for the cubic nonlinear Schrödinger equation on $\mathbb{T}^d$ ($d \geq 3$) in the super-critical regime.

1. Introduction

We consider the Cauchy initial value problem for the cubic nonlinear Schrödinger equation (NLS) in the $d$-dimensional tori $\mathbb{T}^d$ ($d \geq 3$)

\begin{equation}
\begin{cases}
i u_t + \Delta u = \rho u |u|^2, & \rho = \pm 1, \\
u(0, x) = \phi^\omega(x).
\end{cases}
\end{equation}

The initial data $\phi^\omega(x)$ in (1.1) is defined by randomization.

\begin{equation}
\phi^\omega(x) = \sum_{n \in \mathbb{Z}^d} \frac{g_n(\omega)}{\langle n \rangle^{d-1-\alpha}} e^{in \cdot x}, \text{ where } \langle n \rangle = \sqrt{1 + |n|^2},
\end{equation}

where $(g_n(\omega))_{n \in \mathbb{Z}^d}$ is a sequence of complex i.i.d. mean zero Gaussian random variables on a probability space $(\Omega, A, \mathbb{P})$.

Remark 1.1. Let’s consider a function $\phi \in H^{s_c-\alpha-\epsilon}(\mathbb{T}^d)$ for any $\epsilon > 0$ of the form

\begin{equation}
\phi(x) = \sum_{n \in \mathbb{Z}^d} \frac{1}{\langle n \rangle^{d-1-\alpha}} e^{in \cdot x}.
\end{equation}

If we replace the Fourier coefficients of (1.3) with randomized coefficients $\frac{g_n(\omega)}{\langle n \rangle^{d-1-\alpha}}$, then the randomization of (1.3) becomes the random initial data (1.2) of (1.1). It’s easy to see that $\phi^\omega(x)$ is a.s. in $H^{s_c-\alpha-\epsilon}$, but not in $H^s$, $s \geq s_c - \alpha$. Thus randomization does not regularize the data in the scale of the Sobolev spaces.

In the Euclidean space $\mathbb{R}^d$, the scaling symmetry plays an important role on the well-posedness (existence, uniqueness and continuous dependence of the data to solution map) theory of the Cauchy initial value problem (IVP) for NLS:

\begin{equation}
\begin{cases}
i \partial_t u + \Delta u = |u|^{p-1}u, & p > 1 \\
u(0, x) = u_0(x) \in \dot{H}^s(\mathbb{R}^d).
\end{cases}
\end{equation}

The IVP (1.4) is scaling invariant in the Sobolev norm $\dot{H}^s$, where $s_c := \frac{d}{2} - \frac{2}{p-1}$ is so-called scaling critical regularity. Initial data in $\dot{H}^s$ with $s > s_c$ (sub-critical regime) is the best possible setting for well-posedness. Indeed, local-in-time well-posedness of (1.4) was proven by Cazenave-Weissler in [15].

For $\dot{H}^s$ data with $s = s_c$ (critical regime) the well-posedness problem is more difficult than the one in the sub-critical regime. In fact, the well-posedness in the sub-critical regime can be obtained from the well-posedness in the critical regime by a persistence of regularity argument. Bourgain [4] first proved the large data global-in-time well-posedness and scattering for the
defocusing energy-critical \((s_c = 1)\) NLS in \(\mathbb{R}^3\) with radially symmetric initial data in \(\dot{H}^1\) by introducing an induction method on the size of energy and a refined Morawetz inequality. A different proof of the same result was given by Grillakis in [27]. A breakthrough was made by Colliander-Keel-Staffilani-Takaoka-Tao in [18]. Their work extended the results of Bourgain [4] and Grillakis [27]. They proved global-in-time well-posedness and scattering of the energy-critical problem in \(\mathbb{R}^3\) for general large data in \(\dot{H}^1\). Similar results were then proven by Ryckman-Visan [44] on the higher dimension \(\mathbb{R}^d\) spaces. Furthermore, Dodson proved mass-critical \((s_c = 0)\) global-in-time well-posedness results for \(\mathbb{R}^d\) in his series of papers [22, 23, 24].

Data in \(\dot{H}^s\) with \(s < s_c\) (super-critical regime) is rougher than the critical regularity data. Intuitively, in this case, scaling is ‘against well-posedness’. This intuition was verified for example in [16, 17], where it is shown that super-critical data lead the initial value problem for NLS in \(\mathbb{R}^d\) to ill-posedness. More precisely, they show that the solutions whose \(\dot{H}^s\) norms become arbitrary large in arbitrary small time with arbitrary small initial data can be constructed. These solutions, exhibiting what is called- norm inflation, contradict, in particular, the continuous dependence on the initial data.

However, ill-posedness in some cases can be circumvented by an appropriate probabilistic method in some probability space of initial data, in the other words, one may hope to establish almost sure LWP with respect to certain probability random data space. This random data approach to well-posedness has also been pursued by many authors and applied to several nonlinear evolution equations on different manifolds (\(\mathbb{R}^d\), \(\mathbb{T}^4\) or \(S^d\) etc.) to obtain almost sure local -and in some instances almost sure global- well-posedness results. Some references in the context of NLS include: [38, 19, 21, 7, 8, 9, 40, 11, 31, 33, 40, 1, 31, 33, 41, 37, 26]; in the context of NLW include: [12, 13, 49, 14, 35, 36, 42, 45, 43, 25, 11]; and in the context of Navier-Stokes equations include: [20, 48, 39, 47]. Recently Dodson-Lührmann-Mendelson [25] first established almost sure scattering for cubic NLW in \(\mathbb{R}^4\) with randomized radially symmetric initial data in the super-critical regime. Then Killip-Murphy-Visan [33] and Dodson-Lührmann-Mendelson [26] proved similar almost sure scattering results with randomized radial data for cubic NLS on \(\mathbb{R}^4\).

In this paper, we study the cubic NLS in the super-critical regime on tori \(\mathbb{T}^d\) \((d \geq 3)\) via the probabilistic approach. After Bourgain’s first two papers [5, 6] on \(\mathbb{T}^1\) and \(\mathbb{T}^2\), Nahmod-Staffilani [40] proved an almost sure local-in-time well-posedness result for the periodic 3D quintic NLS with an appropriate gauge transform in the super-critical regime. This paper follows the similar spirit and obtain local-in-time well-posedness in high probability in the adapted atomic spaces \(X^s\) by introducing a new lemma which modifies the "transfer principle" (Prop 3.10) of atomic spaces up and focuses on the estimates in the small time intervals. In this paper, we construct a probability measure for the function space of initial data and show that the solutions exist for high probability of initial data.

Our main result can be stated as following:

**Theorem 1.2** (Main Theorem). Suppose \(d \geq 3\) and

\[
(1.5) \quad s_r(d) = \begin{cases} \frac{1}{4} & d = 3 \\ \frac{1}{19} & d = 4 \\ \frac{1}{4} & d \geq 5. \end{cases}
\]

Let \(0 < \alpha < s_r(d)\), \(s \in [s_c, s_c + s_r(d) - \alpha]\). Then there exists \(\delta_0 > 0\) and \(r = r(s, \alpha) > 0\) such that for any \(0 < \delta < \delta_0\), there exists \(\Omega_\delta \in A\) with

\[
\mathbb{P}(\Omega_\delta^c) < e^{-\frac{1}{r}},
\]
and for each \( \omega \in \Omega_\delta \) there exists a unique solution \( u \) of (1.1) in the space
\[
S(t)\phi^\omega + X^s([0, \delta])_{\text{dist}},
\]
where \( S(t)\phi^\omega \) is the linear evolution of the initial data \( \phi^\omega \) given by (1.2).

Here we denoted by \( X^s([0, \delta])_{\text{dist}} \) the metric space \( (X^s([0, \delta]), \text{dist}) \) where \text{dist} is the metric defined by (1.8) and \( X^s([0, \delta]) \) is the adapted atomic space introduced in the Definition 3.5.

Remark 1.3. We also prove the analog of Main Theorem in \( X^{s,b} \) (Theorem 7.1) instead of the atomic space \( X^s \) in the Section 7, but we hold the theorem in \( X^{s,b} \) only when \( s \in (s_c, s_c + s_r(d) - \alpha) \) (the proof of Theorem 7.1 fails when \( s = s_c \)). If we only consider the statement of theorems, for some \( s > s_c \), the solution space \( S(t)\phi^\omega + X^{s,b}([0, \delta])_{\text{dist}} \) is indeed in the space \( S(t)\phi^\omega + X^{s_c,b}([0, \delta])_{\text{dist}} \). However, the proof of \( s = s_c \) case is still important in the sense that we obtain the nonlinear estimate at the regularity of \( s_c \). Especially in the case of \( s_c = 1 \), the nonlinear estimate at the regularity of \( s_c \) would be necessary if we try to control the energy in a long-time term.

To prove Theorem 1.2 first we consider the initial value problem below,

\[
\begin{align*}
iv_t + \Delta v &= \mathcal{N}(v), & \rho = \pm 1, \quad x \in \mathbb{T}^d \\
v(0, x) &= \phi^\omega(x),
\end{align*}
\]

where
\[
(1.7) \quad \mathcal{N}(v_1, v_2, v_3) := \rho(v_1 v_2 v_3 - 2v_1 \int_{\mathbb{T}^d} v_2 v_3 dx) = \mathcal{N}_1(v_1, v_2, v_3) + \mathcal{N}_2(v_1, v_2, v_3),
\]

and set \( \mathcal{N}(v) := \mathcal{N}(v, \overline{v}, v) \).

Suppose \( \beta_\rho(t) = 2 \int_{\mathbb{T}^d} |v|^2 dx \) and define \( u(t, x) := e^{-i\rho \beta_\rho(s) ds} u(t, x) \). We observe that \( u \) solve IVP (1.1). Now suppose that one obtains well-posedness for the IVP (1.6) in a certain Banach space \( (X, \| \cdot \|) \) then one can transfer those results to the IVP (1.1) by using a metric space \( X_{\text{dist}} := (X, \text{dist}) \) where

\[
(1.8) \quad d(u, v) := \| e^{i\rho \beta_\rho(s) ds} u(t, x) - e^{i\rho \beta_\rho(s) ds} v(t, x) \|.
\]

We define
\[
(1.9) \quad \phi^\omega_0 = S(t)\phi^\omega(x),
\]
and \( w(x, t) \) solves the following the IVP (1.10), then we know that \( v = \phi^\omega_0 + w \) solves the IVP (1.6) which is the gauged NLS we want to solve.

\[
(1.10) \quad \begin{align*}
iw_t + \Delta w &= \mathcal{N}(w + \phi^\omega_0), \quad x \in \mathbb{T}^d \\
w(0, x) &= 0,
\end{align*}
\]

where \( \mathcal{N}(\cdot) \) was defined in (1.7).

We are now ready to state the almost sure well-posedness result for the IVP (1.10) which implies the main theorem (Theorem 1.2).

**Theorem 1.4.** Suppose \( d \geq 3 \) and \( s_r(d) \) is defined as (1.5). Let \( 0 \leq \alpha < s_r(d) \), \( s \in [s_c, s_c + s_r(d) - \alpha] \). Then there exists \( \delta_0 > 0 \) and \( r = r(s, \alpha) > 0 \) such that for any \( 0 < \delta < \delta_0 \), there exists \( \Omega_\delta \in \mathcal{A} \) with

\[
\mathbb{P}(\Omega_\delta) < e^{-\varphi \delta},
\]

and for each \( \omega \in \Omega_\delta \) there exists a unique solution \( w \) of (1.10) in the space \( X^s([0, \delta]) \cap C([0, \delta], H^s(\mathbb{T}^d)) \).
Outline of the following paper. The rest of the paper is organized as follows. In Section 2, we state some basic probabilistic properties the proof depends on. In Section 3, we introduce the adapted atomic spaces $X^s$ and $Y^s$, provide some corresponding embedding properties of the spaces and furthermore obtain a transfer principle proposition (Proposition 3.10) focusing on the small time intervals. Section 4 contains some Strichartz estimates, lattice counting lemmata and other lemmata we rely upon. In Section 5, we estimate the nonlinear terms in the $X^s$-norm case by case. Section 6 contains statements on almost sure local well-posedness for the gauged Cauchy initial value problem (1.10) by using the nonlinear estimate in Section 5. In Section 7, we prove an analog result of almost sure local well-posedness in $X^s,b$ spaces of the main theorem (Theorem 1.2).

Acknowledgments. The author is greatly indebted to his advisor, Andrea R. Nahmod, for suggesting this problem and her patient guidance and warm encouragement over the past years. The author also would like to thank Prof. Gigliola Staffilani for correcting one error in an earlier version of the paper and several helpful discussions in MSRI. The author acknowledges support from the National Science Foundation through his advisor Andrea R. Nahmod’s grants NSF-DMS 1201443 and NSF-DMS 1463714.

2. Probabilistic set up

Lemma 2.1. Let $\{g_n(\omega)\}_{n \in \mathbb{Z}^d}$ be a sequence of complex i.i.d. mean zero Gaussian random variables on a probability space $(\Omega, A, P)$. Then given $\epsilon, \delta > 0$, there exists a subset $\Omega_\delta \subset \Omega$ satisfying $P(\Omega^c_\delta) \leq e^{-\frac{1}{\delta^2}}$, such that

$$|g_n(\omega)| \lesssim \frac{1}{\delta^\epsilon} \log (\langle n \rangle + 1).$$

Proof. For each $n$ and a small $\epsilon > 0$, we have a constant $C$,

$$\mathbb{E} e^{\epsilon |g_n(\omega)|} \leq C.$$ 

Set $M = \frac{1}{\delta^\epsilon}$, and the we have

$$\mathbb{E} \left| \frac{e^{\epsilon |g_n(\omega)|}}{e^M} \right| \leq C e^{-\frac{1}{\delta^2}}.$$

Then we obtain,

$$Ce^{-\frac{1}{\delta^2}} \geq \mathbb{E} \left| \frac{e^{\epsilon |g_n(\omega)|}}{e^M} \right| \geq \sum_{j \in \mathbb{Z}^d} \mathbb{P}(e^{\epsilon |g_j(\omega)|} \geq e^M \langle j \rangle^d) = \sum_{j \in \mathbb{Z}^d} \mathbb{P}(|g_j(\omega)| \geq \frac{1}{\delta^\epsilon} + d \log \langle j \rangle).$$

Exclude $\Omega_\delta^c := \cup_j \{|g_j(\omega)| \leq \frac{1}{\delta^\epsilon} + d \log \langle j \rangle\}$ from $\Omega$, for all $\omega \in \Omega_\delta$, we have

$$|g_n(\omega)| \leq \frac{1}{\delta^\epsilon} + d \log \langle n \rangle \lesssim \frac{1}{\delta^\epsilon} \log (\langle n \rangle + 1),$$

for $n \in \mathbb{Z}^d$.

with $P(\Omega_\delta^c) < C e^{-\frac{1}{\delta^2}}$. \qed

Lemma 2.2 (Lemma 3.1 in [40]). Let $\{g_n(\omega)\}_n$ be a sequence of complex i.i.d. mean zero Gaussian random variables on a probability space $(\Omega, A, P)$ and $(c_n) \in \ell^2$. Define

$$F(\omega) := \sum_n c_n g_n(\omega).$$

Then there exists $C > 0$ such that for every $\lambda > 0$ we have

$$\mathbb{P}(\{\omega : |F(\omega)| > \lambda\}) \leq \exp\left(-\frac{-C\lambda^2}{\|F(\omega)\|_{L^2(\Omega)}^2}\right).$$
As a consequence there exists \( C > 0 \) such that for every \( q \geq 2 \) and every \( (c_n) \in \ell^2 \),
\[
\| \sum_n c_n g_n(\omega) \|_{L^q(\Omega)} \leq C \sqrt{q} \left( \sum_n |c_n|^2 \right)^{1/2}.
\]

**Lemma 2.3** (Lemma 3.5 in [18]). Let \( f^\omega(x, t) = \sum c_n g_n(\omega) e^{i(n \cdot x + |n|^2 t)} \). Then, for \( p, q \geq 2 \), there exists \( \delta_0, C > 0 \) such that
\[
\mathbb{P}(\| f^\omega \|_{L^p L^q(T^4 \times [0, \delta])} > \lambda) < C \exp\left(-\frac{c\lambda^2}{\delta^p \|c_n\|_{l_2}^2}\right)
\]
for \( \delta < \delta_0 \).

**Proof.** By Lemma 2.2 there exists \( C > 0 \) such that
\[
\| \sum_n c_n g_n(\omega) \|_{L^\infty(\Omega)} \leq C \sqrt{\tau} \left( \sum_n |c_n|^2 \right)^{1/2},
\]
for every \( r \geq 2 \). By Minkowski integral inequality, we have
\[
\mathbb{E}(\| f^\omega \|^{r}_{L^p L^q(T^4 \times [0, \delta])})^{1/r} \leq \| f^\omega \|_{L^r(\Omega)} \left( \sum_n |c_n| \right)^{1/r} \leq C \sqrt{r} \left( \sum_n |c_n| \right)^{1/r} \leq C \sqrt{r} \delta^p \|c_n\|_{l_2}^r
\]
for \( r \geq p \). By Chebyshev’s Inequality, we have
\[
\mathbb{P}(\| f^\omega \|_{L^p L^q(T^4 \times [0, \delta])} > \lambda) < C^r \lambda^{-r} \delta^p \|c_n\|_{l_2}^r.
\]
If \( \lambda < \sqrt{pC} \delta^{-\frac{1}{p}} e\|c_n\|_{l_2} \), then (2.3) easily holds.
If \( \lambda \geq \sqrt{pC} \delta^{-\frac{1}{p}} e\|c_n\|_{l_2} \), then we set
\[
r = \left[ \frac{\lambda}{C e \delta^{-\frac{1}{p}} \|c_n\|_{l_2}} \right]^2 \quad (\geq p).
\]
So that (2.4) yields (2.3). \( \square \)

**Corollary 2.4.** Let \( p, q \geq 2 \), and \( P_N R = \sum_{|n| \sim N} g_n(\omega) e^{i(n \cdot x + |n|^2 t)} \), where \( N \) is a dyadic coordinate. There exists \( A \subset \Omega \), \( C \) and \( c > 0 \), with \( \mathbb{P}(A) < C e^{-\frac{1}{N}} \), such that for each \( \omega \in A^c \) and each dyadic coordinate \( N \), we have
\[
\| P_N R \|_{L^p L^q([0, \delta] \times T^d)} \leq \delta^p \log N \frac{\log N}{N^{s_c - \alpha}}.
\]
for \( \delta < \delta_0 \).

**Proof.** By Lemma 2.3 for each dyadic coordinate \( N \), set \( \lambda = \delta^{-\frac{1}{2p}} \log N \| P_N R \|_{l_2} \), there exists \( A_N \subset \Omega \), such that for \( \omega \in A_N^c \), with \( \mathbb{P}(A_N) < C \exp\left(-\frac{\log N}{\delta^p}\right) \) we obtain that
\[
\| P_N R \|_{L^p L^q([0, \delta] \times T^d)} \leq \delta^p \frac{\log N}{N^{s_c - \alpha}},
\]
since \( \| P_N R \|_{l_2} \sim \frac{1}{N^{s_c - \alpha}}. \)
Set \( A = \cup N A_N \), and \( c = \frac{1}{p} \), then we have
\[
\mathbb{P}(A) \leq \sum_{N} \mathbb{P}(A_N) \leq \sum_{N} C \exp\left(-\frac{(\log N)^2}{\delta c}\right)
\leq \sum_{k=1}^{\infty} Ce^{-\frac{k^2}{2\pi}} \leq \sum_{k=1}^{\infty} Ce^{-\frac{k}{\delta'}}
= \frac{Ce^{-\frac{1}{\delta'}}}{1 - e^{-\frac{1}{\delta'}}} < 2Ce^{-\frac{1}{\delta'}}.
\]
when \( \delta \) is small enough. \(\square\)

**Lemma 2.5** (Proposition 3.1 in [30]). For fixed \( n \in \mathbb{Z}^d \), let
\[
D(n) = \{(n_1, n_2, n_3) \in \mathbb{Z}^d \times \mathbb{Z}^d \times \mathbb{Z}^d : n = n_1 - n_2 + n_3, n_2 \neq n_1, n_2 \neq n_3, n_1 \neq n_3\}.
\]
Given \( \{c_{n_1,n_2,n_3}\}_{n} \triangleq D(n) \), define \( F_n \) by
\[
F_n := \sum_{D(n)} c_{n_1,n_2,n_3} g_{n_1}(\omega) g_{n_2}(\omega) g_{n_3}(\omega).
\]
Then there exists \( C > 0 \) such that for every \( \lambda > 0 \) we have
\[
\mathbb{P}(\{\omega : |F_n(\omega)| > \lambda\}) \leq \exp\left(-C\lambda^{2/3}\right) \frac{\|F_n(\omega)\|_{L^2(\Omega)}^{2/3}}{\|F_n(\omega)\|_{L^2(\Omega)}}.
\]

### 3. Function spaces

In this section, we introduce \( X^s \) and \( Y^s \) spaces which are based on the atomic space \( U^p \) and \( V^p \) which were firstly applied to PDEs in [25][29][30]. \( \mathcal{H} \) is a separable Hilbert space on \( \mathbb{C} \), and \( \mathcal{Z} \) denotes the set of finite partitions \(-\infty = t_0 < t_1 < \ldots < t_K = \infty\) of the real line, with the convention that \( v(\infty) := 0 \) for any function \( v : \mathbb{R} \to \mathcal{H} \).

**Definition 3.1** (Definition 2.1 in [29]). Let \( 1 \leq p < \infty \). For \( \{t_k\}_{k=0}^{K} \in \mathcal{Z} \) and \( \{\phi_k\}_{k=0}^{K-1} \subset \mathcal{H} \) with \( \sum_{k=0}^{K} \|\phi_k\|_{\mathcal{H}}^p = 1 \) and \( \phi_0 = 0 \). A \( U^p \)-atom is a piecewise defined function \( a : \mathbb{R} \to \mathcal{H} \) of the form
\[
a = \sum_{k=1}^{K} 1_{[t_{k-1}, t_k)} \phi_{k-1}.
\]
The atomic Banach space \( U^p(\mathbb{R}, \mathcal{H}) \) is then defined to be the set of all functions \( u : \mathbb{R} \to \mathcal{H} \) such that
\[
u = \sum_{j=1}^{\infty} \lambda_j a_j, \quad \text{for } U^p \text{-atoms } a_j, \quad \{\lambda_j\} \in \ell^1,
\]
with the norm
\[
\|u\|_{U^p} := \inf\left\{\sum_{j=1}^{\infty} |\lambda_j| : u = \sum_{j=1}^{\infty} \lambda_j a_j, \, \lambda_j \in \mathbb{C} \text{ and } a_j \text{ an } U^p \text{ atom}\right\}.
\]
Here \( 1_I \) denotes the indicator function over the time interval \( I \).

**Definition 3.2** (Definition 2.2 in [29]). Let \( 1 \leq p < \infty \). The Banach space \( V^p(\mathbb{R}, \mathcal{H}) \) is defined to be the set of all functions \( v : \mathbb{R} \to \mathcal{H} \) with \( v(\infty) := 0 \) and \( v(-\infty) := \lim_{t \to -\infty} v(t) \) exists, such that
\[
\|v\|_{V^p} := \sup_{\{t_k\}_{k=0}^{K} \in \mathcal{Z}} \left(\sum_{k=1}^{K} \|v(t_k) - v(t_{k-1})\|_{\mathcal{H}}^p\right)^{\frac{1}{p}} \quad \text{is finite}.
\]
Likewise, let $V^p$ denote the closed subspace of all $v \in V^p$ with $\lim_{t \to -\infty} v(t) = 0$. $V^p_{\text{rc}}$ means all right-continuous $V^p$ functions.

Remark 3.3 (Some embedding properties). Note that for $1 \leq p \leq q < \infty$,

\begin{equation}
(3.1) \quad U^p(\mathbb{R}, \mathcal{H}) \hookrightarrow U^q(\mathbb{R}, \mathcal{H}) \hookrightarrow L^\infty(\mathbb{R}, \mathcal{H}),
\end{equation}

equivalently, functions in $U^p(\mathbb{R}, \mathcal{H})$ are right continuous, and $\lim_{t \to -\infty} u(t) = 0$ for each $u \in U^p(\mathbb{R}, \mathcal{H})$. Also note that,

\begin{equation}
(3.2) \quad U^p(\mathbb{R}, \mathcal{H}) \hookrightarrow V^p_{\text{rc}}(\mathbb{R}, \mathcal{H}) \hookrightarrow U^q(\mathbb{R}, \mathcal{H}).
\end{equation}

Definition 3.4 (Definition 2.5 in [29]). For $s \in \mathbb{R}$, we let $U^p_\Delta H^s$, respectively $V^p_\Delta H^s$, be the space of all functions $u : \mathbb{R} \to H^s(\mathbb{T}^d)$ such that $t \mapsto e^{-it\Delta}u(t)$ is in $U^p(\mathbb{R}, H^s)$, respectively in $V^p(\mathbb{R}, H^s)$ with norm

\begin{equation}
\|u\|_{U^p(\mathbb{R}, H^s)} := \|e^{-it\Delta}u(t)\|_{U^p(\mathbb{R}, H^s)}, \quad \|u\|_{V^p(\mathbb{R}, H^s)} := \|e^{-it\Delta}u(t)\|_{V^p(\mathbb{R}, H^s)}.
\end{equation}

Definition 3.5 (Definition 2.6 in [29]). For $s \in \mathbb{R}$, we define $X^s$ as the space of all functions $u : \mathbb{R} \to H^s(\mathbb{T}^d)$ such that for every $n \in \mathbb{Z}^d$, the map $t \mapsto e^{it|n|^2}u(t)(n)$ is in $U^2(\mathbb{R}, \mathcal{C})$, and with the norm

\begin{equation}
(3.3) \quad \|u\|_{X^s} := \left( \sum_{n \in \mathbb{Z}^d} (n)^{2s}\|e^{it|n|^2}u(t)(n)\|_{U^p}^2 \right)^{\frac{1}{2}} \text{ is finite.}
\end{equation}

Definition 3.6 (Definition 2.7 in [29]). For $s \in \mathbb{R}$, we define $Y^s$ as the space of all functions $u : \mathbb{R} \to H^s(\mathbb{T}^d)$ such that for every $n \in \mathbb{Z}^d$, the map $t \mapsto e^{it|n|^2}u(t)(n)$ is in $V^2_{\text{rc}}(\mathbb{R}, \mathcal{C})$, and with the norm

\begin{equation}
(3.4) \quad \|u\|_{Y^s} := \left( \sum_{n \in \mathbb{Z}^d} (n)^{2s}\|e^{it|n|^2}u(t)(n)\|_{V^p_{\text{rc}}}^2 \right)^{\frac{1}{2}} \text{ is finite.}
\end{equation}

Note that

\begin{equation}
(3.5) \quad U^2_\Delta H^s \hookrightarrow X^s \hookrightarrow Y^s \hookrightarrow V^2_\Delta H^s.
\end{equation}

Proposition 3.7 (Proposition 2.10 in [29]). Suppose $u := e^{it\Delta}\phi$ which is a free Schrödinger solution, then we obtain that

\begin{equation}
\|u\|_{X^s([0,\delta])} \leq \|\phi\|_{H^s}.
\end{equation}

Proof. Since $u := e^{it\Delta}\phi$, then $\|u\|_{X^s} = (\sum_{n \in \mathbb{Z}^d} (n)^{2s}\|\hat{\phi}(n)\|_{U^p}^2)^{\frac{1}{2}} \leq \|\phi\|_{H^s}$. \hfill \Box

Definition 3.8 (The corresponding restriction spaces to a time interval $I$). For $p \geq 1$ and a bounded time interval $I$. Define $U^p(I)$, $V^p(I)$, $X^s(I)$ and $Y^s(I)$ with the restriction norms:

\begin{align*}
\|u\|_{U^p(I)} &= \inf\{\|\tilde{u}\|_{U^p} : \tilde{u}(t) = u(t), t \in I\} \quad \text{and} \quad \|u\|_{V^p(I)} = \inf\{\|\tilde{u}\|_{V^p} : \tilde{u}(t) = u(t), t \in I\};
\|u\|_{X^s(I)} &= \inf\{\|\tilde{u}\|_{X^s} : \tilde{u}(t) = u(t), t \in I\} \quad \text{and} \quad \|u\|_{Y^s(I)} = \inf\{\|\tilde{u}\|_{Y^s} : \tilde{u}(t) = u(t), t \in I\}.
\end{align*}

Proposition 3.9 (Proposition 2.19 in [28]). Let $T_0 : L^\infty_2 \times \cdots \times L^\infty_2 \to L^1_{x,\text{loc}}(\mathbb{T}^d)$ be an $m$-linear operator. Assume that for some $1 \leq p \leq \infty$

\begin{equation}
(3.6) \quad \|T_0(e^{it\Delta}\phi_1, \cdots, e^{it\Delta}\phi_m)\|_{L^p(\mathbb{R} \times \mathbb{T}^d)} \leq \prod_{i=1}^{m} \|\phi_i\|_{L^2(\mathbb{T}^d)}.
\end{equation}

Then, there exists an extension $T : U^p_\Delta \times \cdots \times U^p_\Delta \to L^p(\mathbb{R} \times \mathbb{T}^d)$ satisfying

\begin{equation}
(3.7) \quad \|T(u_1, \cdots, u_m)\|_{L^p(\mathbb{R} \times \mathbb{T}^d)} \leq \prod_{i=1}^{m} \|u_i\|_{U^p_\Delta};
\end{equation}

and such that $T(u_1, \cdots, u_m)(t, \cdot) = T_0(u_1(t), \cdots, u_m(t))(\cdot)$, a.e.
**Proposition 3.10.** Let $T_0 : \mathbb{L}^2_x \times \cdots \times \mathbb{L}^2_x \rightarrow \mathbb{L}^1_{x,t,\text{loc}}(\mathbb{T}^d)$ be $m$-linear operator. Assume that for some bounded time interval $I \subset \mathbb{R}$, and $1 < q \leq \infty$

\begin{equation}
(3.8) \quad \left| \int_J \int_{\mathbb{T}^d} T_0(e^{itx} \phi_1, \ldots, e^{itx} \phi_m) \, dx dt \right| \leq |J|^\frac{1}{q} \prod_{i=1}^m \|\phi_i\|_{L^2(\mathbb{T}^d)}, \quad \text{for any } J \subset I.
\end{equation}

Then, for $1 \leq p < \infty$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, there exists an extension $T : U^p_\Delta \times \cdots \times U^p_\Delta \rightarrow \mathbb{L}^1_{x,t,\text{loc}}(I \times \mathbb{T}^d)$ satisfying

\begin{equation}
(3.9) \quad \left| \int_I \int_{\mathbb{T}^d} T(u_1, \ldots, u_m) \, dx dt \right| \leq |I|^\frac{1}{q} \prod_{i=1}^m \|u_i\|_{U^p_\Delta(I)};
\end{equation}

and such that $T(u_1, \ldots, u_m)(t, \cdot) = T_0(u_1(t), \ldots, u_m(t))(\cdot)$, a.e.

**Remark 3.11.** In Hadac-Herr-Koch’s paper [28], they derived a ”transfer principle” as Proposition 3.10 which consider the $L^p$ norm of the multilinear operator $T$ over the whole time space $\mathbb{R}$, while Proposition 3.10 focus on the integral in time (or actually $L^1$ norm is also fine) on a finite time interval $I$. By a stronger assumption (which gives some better estimates on each small intervals $J$), Proposition 3.10 somehow takes advantage of the finite time interval to improve the bounds from $U^1$ norm to $U^p$. In the following proof of Proposition 5.1, the Case B heavily relies on Proposition 3.10.

**Proof.** By multi-linearity of $T_0$ and definition of $U^p$ norm, it will suffice to show that (3.9) is true for all $U^p_\Delta$-atoms $u_i$. Let $a_1, \ldots, a_m$ be $U^p_\Delta$-atoms given as

\begin{equation}
(a_i = \sum_{k_i=1}^{K_i} \mathbb{1}_{I_{k_i,i}} e^{itx} \phi_{k_i-1,i}, \quad \text{for } i = 1, \ldots, m).
\end{equation}

where $I_{k_i,i} = [t_{k_i-1,i}, t_{k_i,i})$, and such that

\begin{equation}
(3.10) \quad \sum_{k_i=1}^{K_i} \|\phi_{k_i-1,i}\|_{L^2_x}^p = 1.
\end{equation}

Then, by (3.8), Cauchy-Schwartz inequality and by induction,

\begin{equation}
(3.11) \quad \left| \int_I \int_{\mathbb{T}^d} T(a_1, \ldots, a_m)(t) \, dx dt \right| \leq \sum_{1 \leq k_1 \leq K_1, \ldots, 1 \leq k_m \leq K_m} \left| \int_{\cap_{i=1}^m I_{k_i,i}} \int_{\mathbb{T}^d} T_0(e^{itx} \phi_{k_1-1,1}, \ldots, e^{itx} \phi_{k_m-1,m}) \, dx dt \right|
\end{equation}

\begin{equation}
(3.12) \quad \leq \sum_{1 \leq k_1 \leq K_1, \ldots, 1 \leq k_m \leq K_m} \prod_{i=2}^m \|\phi_{k_{i-1},i-1}\|_{L^2_x} \left( \sum_{1 \leq k_1 \leq K_1} \left| \cap_{i=1}^m I_{k_i,i} \right| \right)^\frac{1}{q} \left( \sum_{1 \leq k_1 \leq K_1} \|\phi_{k_1-1,1}\|_{L^2_x}^p \right)^\frac{1}{p}
\end{equation}

For fixed $k_2, k_3, \ldots, k_m$, since

\begin{equation}
I_{k_2,2} \cap \cdots \cap I_{k_m,m} = \cup_{1 \leq k_1 \leq K_1} (I_{k_1,1} \cap I_{k_2,2} \cap \cdots \cap I_{k_m,m}),
\end{equation}
we have

\begin{equation}
\left( \sum_{1 \leq k_1 \leq K_1} \mid \bigcap_{i=1}^{m} I_{k_i,i} \right) \frac{1}{q} = \mid \bigcap_{i=1}^{m} I_{k_i,i} \mid \frac{1}{q}.
\end{equation}

Based on (3.10) (3.13) (3.12), we obtain that

\begin{align*}
| \int_{I} \int_{\mathbb{R}^d} T(a_1, \ldots, a_m)(t) \, dx \, dt |
\leq & \sum_{1 \leq k_2 \leq K_2} \prod_{i=2}^{m} \| \phi_{k_{i-1},i} \|_{L^2} \left( \sum_{1 \leq k_1 \leq K_1} \mid \bigcap_{i=1}^{m} I_{k_i,i} \mid \right) \frac{1}{q} \left( \sum_{1 \leq k_1 \leq K_1} \| \phi_{k_{i-1},1} \|_{L^2} \right) \frac{1}{q}.
\end{align*}

If we iterate (3.11) (3.12) on \( k_2, k_3, \ldots, k_m \), finally we obtain that

\begin{align*}
| \int_{I} \int_{\mathbb{R}^d} T(a_1, \ldots, a_m)(t) \, dx \, dt |
\leq & \sum_{1 \leq k_3 \leq K_3} \prod_{i=3}^{m} \| \phi_{k_{i-1},i} \|_{L^2} \left( \sum_{1 \leq k_2 \leq K_2} \mid \bigcap_{i=2}^{m} I_{k_i,i} \mid \right) \frac{1}{q} \left( \sum_{1 \leq k_2 \leq K_2} \| \phi_{k_{i-1},2} \|_{L^2} \right) \frac{1}{q}.
\end{align*}

So we obtain (3.9). \( \square \)

**Proposition 3.12** (Proposition 2.20 in [28]). Let \( q_1, \ldots, q_m > 2 \) (\( m \in \mathbb{N} \)), \( E \) be a Banach space and \( T: U_{\Delta}^{q_1} \times \cdots \times U_{\Delta}^{q_m} \to E \) be a bounded \( m \)-linear operator with

\begin{equation}
\| T(u_1, \ldots, u_m) \|_E \leq C \prod_{i=1}^{m} \| u_i \|_{U_{\Delta}^{q_i}}.
\end{equation}

And also assume there exists \( 0 < C_2 < C \) such that we hold,

\begin{equation}
\| T(u_1, \ldots, u_m) \|_E \leq C_2 \prod_{i=1}^{m} \| u_i \|_{V_{\Delta}^{2}}.
\end{equation}

Then, \( T \) satisfies the estimate

\begin{equation}
\| T(u_1, \ldots, u_m) \|_E \leq C_2 (\log \frac{C}{C_2} + 1) \prod_{i=1}^{m} \| u_i \|_{V_{\Delta}^{2}}, \quad u_i \in V_{\Delta}^{2}, \quad i = 1, \ldots, m.
\end{equation}

To make the proposition 3.12 suitable for the following nonlinear estimates, we also need to introduce a similar interpolation proposition for the integral of \( T \) over a time interval \( I \) as following:
Proposition 3.13. Let $q_1, \ldots, q_m > 2$ ($m \in \mathbb{N}$), and $T : U_{\Delta}^{q_1} \times \cdots \times U_{\Delta}^{q_m} \to L^1_{x,t, loc}(I \times \mathbb{T}^d)$ be a $m$-linear operator with

$$
\int_I \int_{\mathbb{T}^d} |T(u_1, \ldots, u_m)\, dx \, dt| \leq C \prod_{i=1}^m \|u_i\|_{U_{\Delta}^{q_i}}.
$$

And also assume there exists $0 < C_2 < C$ such that we hold,

$$
\int_I \int_{\mathbb{T}^d} |T(u_1, \ldots, u_m)\, dx \, dt| \leq C_2 \prod_{i=1}^m \|u_i\|_{U_{\Delta}^{2}}.
$$

Then, $T$ satisfies the estimate

$$
\int_I \int_{\mathbb{T}^d} |T(u_1, \ldots, u_m)\, dx \, dt| \leq C_2 (\log \frac{C_2}{C_2} + 1) \prod_{i=1}^m \|u_i\|_{V_{\Delta_i}^2}, \quad u_i \in V_{\Delta_i}^2, \; i = 1, \ldots, m.
$$

Proof. The proof is almost the same as that of Proposition 2.20 in [28], since $\int_I \int_{\mathbb{T}^d} |T(u_1, \ldots, u_m)\, dx \, dt|$ is $m$-sublinear for $u_1, \ldots, u_m$ as $\|T(u_1, \ldots, u_m)\|_E$ in Prop 3.12.

Definition 3.14 (Duhamel operator). Let $f \in L^1_{loc}([0, \infty), L^2(\mathbb{T}^d))$, and we define the Duhamel operator $\mathcal{I}$

$$
\mathcal{I}(f)(t) := \int_0^t e^{i(t-t')\Delta} f(t') \, dt',
$$

for $t > 0$ and $\mathcal{I}(f)(t) := 0$ otherwise.

Proposition 3.15 (Proposition 2.11 in [28]). Let $s > 0$, and a time interval $I = [0, \delta]$. For $f \in L^1(I, H^s(\mathbb{T}^d))$ we have $\mathcal{I}(f) \in X^s(I)$ and

$$
\|\mathcal{I}(f)\|_{X^s(I)} \leq \sup_{\|v\|_{Y^{-s}(I)} = 1} \left| \int_{0}^{\delta} \int_{\mathbb{T}^d} f(t, x) \overline{v(t, x)} \, dx \, dt \right|.
$$

4. Auxiliary lemmata and notations

Definition 4.1 (Littlewood-Paley decomposition). For $N > 1$ a dyadic number, we denote by $P_{\leq N}$ the rectangular Fourier projection operator:

$$
P_{\leq N} f = \sum_{n \in \mathbb{Z}^d : |n| \leq N} \hat{f}(n) e^{in \cdot x}.
$$

Then $P_N = P_{\leq N} - P_{\leq N-1}$. Moreover, if $C$ is a subset of $\mathbb{Z}^d$, then the Fourier projection operator onto $C$ is defined by $P_C$

$$
P_C f = \sum_{n \in \mathbb{Z}^d : n \in C} \hat{f}(n) e^{in \cdot x}.
$$

In the Bourgain’s GAF paper [2], he firstly introduced the following Strichartz estimate of Schrödinger operator on tori as a conjecture, and proved parts of the conjecture. And then Bourgain-Demeter [10] proved the following Strichartz estimate.

Proposition 4.2 (Strichartz estimate [2][10]). Let $p > p_c$, where $p_c = \frac{2(d+2)}{d}$. For all $N \geq 1$ we have

$$
\|P_N e^{it\Delta} \phi\|_{L^p_{x,t}(\mathbb{T}^d \times \mathbb{T}^d)} \lesssim N^{\frac{d}{2} - \frac{d+2}{p}},
$$

$$
\|P_C e^{it\Delta} \phi\|_{L^p_{x,t}(\mathbb{T}^d \times \mathbb{T}^d)} \lesssim N^{\frac{d}{2} - \frac{d+2}{p}},
$$

$$
\|P_N u\|_{L^p_{x,t}(\mathbb{T}^d \times \mathbb{T}^d)} \lesssim N^{\frac{d}{2} - \frac{d+2}{p}},
$$

$$
\|P_C u\|_{L^p_{x,t}(\mathbb{T}^d \times \mathbb{T}^d)} \lesssim N^{\frac{d}{2} - \frac{d+2}{p}},
$$

where $C$ is a cube in $\mathbb{Z}^d$ with sides parallel to the axis of side length $N$. 
Note that the last inequality (4.3) follows (4.1) (4.2) and Proposition 3.9.

**Lemma 4.3** (Integer lattice counting estimates [2]). Denote the number of set \(\{(X_1, \cdots, X_d) \in \mathbb{Z}^d : X_1^2 + \cdots + X_d^2 = A\}\) by \(C_{d,A}\). Then \(C_{d,A}\) can be bounded by

\[
\begin{align*}
A^e & \quad (d = 2) \\
A^{d+\epsilon} & \quad (d = 3) \\
A^{1+\epsilon} & \quad (d = 4) \\
A^{d-\epsilon} & \quad (d > 4)
\end{align*}
\]

where \(\epsilon\) is an arbitrary small positive number.

By Lemma 4.3 it’s easy to obtain the following lattice counting lemmas.

**Lemma 4.4.** Let \(S_R\) be a sphere of radius \(R\), \(B_r\) be a ball of radius \(r\), and \(P\) be a plane in \(\mathbb{R}^d\) for \(d \geq 3\). Then

\[
\begin{align*}
|\mathbb{Z}^d \cap S_R| & \leq R^{d-2+\epsilon}, \\
|\mathbb{Z}^d \cap B_r \cap P| & \leq r^{d-1},
\end{align*}
\]

where \(|\cdot|\) denotes cardinality and \(\epsilon\) is an arbitrary small positive number.

**Lemma 4.5.** Consider the set

\(S = \{(n_1, n_2, n_3) \in \mathbb{Z}^d \times \mathbb{Z}^d \times \mathbb{Z}^d : n_2 \neq n_1, n_3, |n_i| \sim N_i \text{ for } i = 1, 2, 3, \text{ and } (n_2 - n_1, n_2 - n_3) = \mu\}\).

For a fixed \(n_2\), \(|S(n_2)| \leq N_1^d N_2^{d-1} \min\{N_1, N_3\}^\epsilon\), where \(|\cdot|\) denotes cardinality and \(\epsilon\) is an arbitrary small positive number.

**Lemma 4.6** (Bounds of Fourier coefficients of Characteristic function). Consider \(1_{[a,b]}(t)\) as a function in \(L^2([0,2\pi])\) where \(a, b \in [0,2\pi]\), then the Fourier coefficients \(\|F(1_{[a,b]})(k)\| \leq \frac{2}{|k|}\) for all \(k \in \mathbb{Z}\).

**Proof.** \(|F(1_{[a,b]})(k)| = |\frac{e^{ika} - e^{ika}}{ik}| \leq \frac{2}{|k|}|. \)

**Lemma 4.7** (Lemma 6.3 in [4]). Let \(A = (A_{ik})_{1 \leq i \leq N \atop 1 \leq k \leq M}\) be an \(N \times M\) matrix. Then

\[
\|AA^\ast\| \leq \max_{1 \leq j \leq N} \sum_{k=1}^M |A_{jk}|^2 + \left(\sum_{i \neq j} \sum_{k=1}^M |A_{ik}A_{jk}|^2\right)^{\frac{1}{2}}
\]

where \(\|\cdot\|\) is the 2-norm.

5. Estimate for nonlinear term

To estimate \(\|I(\mathcal{N}(w + v_0^\omega))\|_{X^s([0,\delta])}\), by Prop 4.15 the we just need to bound the integral \(I_\delta \int_{\mathbb{T}^d} \mathcal{N}(w + v_0^\omega)u(0) dxdt\), where \(\delta < 1\). This section will focus on estimating this integral.

**Proposition 5.1.** Suppose \(d \geq 3\) and \(s_r(d)\) is given in (1.2). Let \(0 \leq \alpha < s_r(d), s \in [s_c, s_c + s_r(d) - \alpha], r > 0, 0 \leq \delta < 1, \text{ and } I = [0, \delta]. \) There exist \(\Omega_\delta \subset \Omega\) with \(P(\Omega_\delta) < e^{-1/\delta^c}\), and \(c > 0\), such that we obtain that

\[
\left| \int_0^\delta \int_{\mathbb{T}^d} \mathcal{N}(w(1) + v_0^\omega, w(2) + v_0^\omega, w(3) + v_0^\omega)u(0) dxdt \right|
\]

\[
\leq \|u(0)\|_{Y^{-s}(I)} \left(\delta^{c \min\{1, s-s_c\}} \|w(1)\|_{X^s(I)} \|w(2)\|_{X^s(I)} \|w(3)\|_{X^s(I)} + \delta^c \sum_{J \subset \{1,2,3\}} \prod_{j \notin J} \|w(j)\|_{X^s(I)} \right),
\]

where \(v_0^\omega\) is defined (1.2), \(u(0) \in Y^{-s}(I)\) and \(w(i) \in X^s(I)\) for \(i = 1, 2, 3\). (when the subset \(S_J = 0, \prod_{j \in S_J} \|w(j)\|_{X^s(I)} = 1.\))
To show Proposition 5.3 it is clear that $\mathcal{N}(w + v_0^i)$ can be expressed as

$$\sum_{u^{(i)} \in \{w, v_0^i\}, i=1,2,3} \mathcal{N}(u^{(1)}, u^{(2)}, u^{(3)}).$$

We dyadic decompose

$$u_i = P_N u^{(i)},$$ where $i \in \{0, 1, 2, 3\}.$

By the symmetry, in the following paper we suppose that $N_1 \geq N_2 \geq N_3,$ and we need to estimate the following integral case by case,

$$\int_{0}^{\delta} \int_{\mathbb{T}^d} \mathcal{N}(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) \overline{\xi_0} \, dx \, dt,$$

where $\tilde{u}_i = u_i$ or $\overline{u_i}$ and only one of $\tilde{u}_i$ can be $\overline{u_i}$.

Remark 5.2. To make the integral $\int_{0}^{\delta} \int_{\mathbb{T}^d} \tilde{u}_1 \tilde{u}_2 \tilde{u}_3 \overline{\xi_0} \, dx \, dt$ (which is the main term of (5.2)) nontrivial, the two highest frequencies must be comparable, which means $N_1 \sim \max \{N_0, N_2 \}$ ($\frac{1}{4}N_1 \leq \max \{N_0, N_2 \} \leq 4N_1 \). It is easy to show if $\frac{3}{4}N_1 \geq \max \{N_0, N_2 \}$ or $\max \{N_0, N_2 \} \geq 4N_1$, then the integral $\int_{0}^{\delta} \int_{\mathbb{T}^d} \tilde{u}_1 \tilde{u}_2 \tilde{u}_3 \overline{\xi_0} \, dx \, dt$ is zero. Then the following two cases need to be considered:

- $N_0 \sim N_1 \geq N_2$;
- $N_0 < N_2 \sim N_1$.

Now let’s summarize all cases of $(u_1, u_2, u_3)$ we should consider. Denote

$$R_i = P_N v_0^i \text{ and } D_i = P_N w$$ for $i \in \{1, 2, 3\}.$

The list of all cases of $(u_1, u_2, u_3)$ is below:

A. $u_1 = D_1 :$
- (a) $(D_1, D_2, D_3);$  
- (b) $(D_1, D_2, R_3);$  
- (c) $(D_1, R_2, D_3);$  
- (d) $(D_1, R_2, R_3);$  

B. $u_1 = R_1 :$
- (a) $(R_1, R_2, R_3);$  
- (b) $(R_1, R_2, D_3);$  
- (c) $(R_1, D_2, R_3);$  
- (d) $(R_1, D_2, D_3).$

5.1. Case A (a). We consider the all deterministic case $u_i = D_i$ for all $i \in \{1, 2, 3\}.$ It’s directly the local well-posed result for the critical data following the strichartz estimates Proposition 4.2 (the case $d = 4$ is in [34]).

Proposition 5.3. Assume $N_i$, $i = 0, 1, 2, 3$, are dyadic numbers and $N_1 \geq N_2 \geq N_3,$ and $0 \leq \delta \leq 1.$ For $s \geq s_c$, there exists $c > 0,$ so that we can bound the integral:

$$\left| \int_{0}^{\delta} \int_{\mathbb{T}^d} \mathcal{N}(\tilde{D}_1, \tilde{D}_2, \tilde{D}_3) \overline{\xi_0} \, dx \, dt \right| \lesssim c^{\min \{1, s-s_c \}} \frac{N_3 \min \{N_0, N_2 \}}{N_2^2} \frac{1}{N_2^{s-s_c}} \|u_0\|_{Y^{-s}} \|D_1\|_{X^s} \|D_2\|_{X^s} \|D_3\|_{X^s},$$

where $u_0, \tilde{D}_1, \tilde{D}_2,$ and $\tilde{D}_3$ is defined as (5.3).

Proof. We decompose $\mathbb{R}^d = \cup_j C_j,$ where each $C_j$ is a cube of side-length $N_2$. Let $P_{C_j}$ denote the family of Fourier projections onto the cube $C_j$. We write $C_j \sim C_k$ if the sum set \{ $c_1 + c_2 : c_1 \in C_j, c_2 \in C_k$ \} overlaps the Fourier support of $P_{\leq 2N_2}$. Observe that given $C_k$ there are a bounded number of $C_j \sim C_k$. If $N_0 \sim N_1 \geq N_2,$ and we decompose $u_0$ and $D_1$ with Fourier projections onto the small cubes of size $N_2$. If $N_0 \leq N_2 \sim N_1$, and we also decompose $u_0$ and $D_1$ with Fourier projections onto the cubes of size $N_2$, however the frequency of $u_0$ has Fourier support of $P_{\leq N_0}$ which is only in one cube of size $N_2$. For the case of $N_0 < N_2 \sim N_1,$
the cube decomposition doesn’t help, but for simplicity of notations, we use the same cube decomposition.

(1) Case: $d = 3$

First, let’s consider $N_1(D_1, D_2, D_3) = \pm \tilde{D}_1 \tilde{D}_2 \tilde{D}_3$. Set $\frac{1}{2^+}$ satisfying

$$
\frac{1}{2^+} = \frac{2}{11} - c_1,
$$

where $c_1 = \min \{\frac{2}{11} - \epsilon, \frac{2}{5}(s - s_c)\}$. (In this paper, we always use $\epsilon$ as a small positive number which can be chosen arbitrarily small, and $\epsilon$ may be different in the different positions.)

By the cube decomposition, and Hölder inequality, we obtain that

$$
\int_0^\delta \int_{T^3} \tilde{\pi}_0 \tilde{D}_1 \tilde{D}_2 \tilde{D}_3 \, dx \, dt \\
\leq \sum_{C_j \sim C_k} \left| \int_0^\delta \int_{T^3} (P_{C_j} \tilde{\pi}_0)(P_{C_k} \tilde{D}_1) \tilde{D}_2 \tilde{D}_3 \, dx \, dt \right| \\
\leq \sum_{C_j \sim C_k} \| P_{C_j} u_0 \|_{L_t^\infty L_x^2} \| P_{C_k} D_1 \|_{L_t^\infty L_x^2} \| D_2 \|_{L_t^\infty L_x^2} \| D_3 \|_{L_t^\infty L_x^2} \\
\lesssim \delta^{c_1} \sum_{C_j \sim C_k} \| P_{C_j} u_0 \|_{L_t^\infty L_x^2} \| P_{C_k} D_1 \|_{L_t^\infty L_x^2} \| D_2 \|_{L_t^\infty L_x^2} \| D_3 \|_{L_t^\infty L_x^2}.
$$

By Strichartz estimates (Lemma 1.2) and (5.6), we obtain that

$$
\lesssim \delta^{c_1} \sum_{C_j \sim C_k} \min \{N_0, N_2\} \left(\frac{\tilde{N}_2}{N_2}\right)^{\frac{3}{11}} \tilde{N}_3 \frac{N_2^{d-5} + 5 c_1}{N_3^2} \sum_{C_j \sim C_k} \| P_{C_j} u_0 \|_{Y^{-s}} \| P_{C_k} D_1 \|_{X^s} \| D_2 \|_{X^{s_c}} \| D_3 \|_{X^{s_c}} \\
\lesssim \delta^{c_1} \min \{N_0, N_2\} \frac{1}{N_2^{d-5}} \| u_0 \|_{Y^{-s}} \| D_1 \|_{X^s} \| D_2 \|_{X^{s_c}} \| D_3 \|_{X^{s_c}},
$$

where $c = \frac{1}{11}$.

Second, let’s consider $N_2(D_1, D_2, D_3) = \pm \tilde{D}_1 \int_{T^3} \tilde{D}_2 \tilde{D}_3 \, dx$.

$$
\left| \int_0^\delta \int_{T^3} \tilde{\pi}_0 \tilde{D}_1 \, dx \int_{T^3} \tilde{D}_2 \tilde{D}_3 \, dx \, dt \right| \\
\leq \sum_{C_j \sim C_k} \left| \int_0^\delta \int_{T^3} (P_{C_j} \tilde{\pi}_0)(P_{C_k} \tilde{D}_1) \, dx \int_{T^3} \tilde{D}_2 \tilde{D}_3 \, dx \, dt \right| \\
\leq \sum_{C_j \sim C_k} \| P_{C_j} u_0 \|_{L_t^{\frac{2}{3}} L_x^2} \| P_{C_k} D_1 \|_{L_t^{\frac{2}{3}} L_x^2} \| D_2 \|_{L_t^{\frac{2}{3}} L_x^2} \| D_3 \|_{L_t^{\frac{2}{3}} L_x^2} \\
\lesssim \sum_{C_j \sim C_k} \| P_{C_j} u_0 \|_{L_t^{\frac{2}{3}} L_x^2} \| P_{C_k} D_1 \|_{L_t^{\frac{2}{3}} L_x^2} \| D_2 \|_{L_t^{\frac{2}{3}} L_x^2} \| D_3 \|_{L_t^{\frac{2}{3}} L_x^2}.
$$

Then we follow the same approach for $N_1$ term, we can hold the same bound of $N_2$.

(2) Case: $d \geq 4$

First, let’s consider $N_1(D_1, D_2, D_3) = \pm \tilde{D}_1 \tilde{D}_2 \tilde{D}_3$. Set $3^+$ and $\infty^-$ satisfying the following conditions:

$$
\frac{1}{\infty^-} = c_2, \quad \frac{1}{3^+} = \frac{1}{3} - \frac{c_2}{3},
$$

where $c_2 = \frac{2}{d+7} \min \{\frac{1}{7}, s - s_c\} + c_3$, $c_3 = \frac{1}{d+2} \min \{s - s_c, \frac{1}{7}\}$.
By the cube decomposition, Hölder inequality, and Lemma 4.2, we obtain that

\[
\left| \int_0^\delta \int_{T^d} \overline{u}_0 \overline{D}_1 \overline{D}_2 \overline{D}_3 \, dx \, dt \right|
\leq \sum_{C_j \sim C_k} \left| \int_0^\delta \int_{T^d} (P_{C_j} \overline{u}_0)(P_{C_k} \overline{D}_1) \overline{D}_2 \overline{D}_3 \, dx \, dt \right|
\lesssim \sum_{C_j \sim C_k} \|P_{C_j} u_0\|_{L_t^\infty} \|P_{C_k} D_1\|_{L_t^\infty} \|D_2\|_{L_t^\infty} \|D_3\|_{L_t^\infty}
\lesssim \delta^{\frac{1}{3}} \min\{N_0, N_2\} \left( \frac{\delta}{r^2} \right) ^{\frac{2}{3}} \left( \frac{N_3}{N_2} \right) \left( \frac{D_1}{D_3} \right) ^{\frac{2}{3}} \|u_0\|_{L_t^\infty} \|D_1\|_{L_t^\infty} \|D_2\|_{L_t^\infty} \|D_3\|_{L_t^\infty},
\]

where \( c = \frac{1}{3(d+2)} \) (it’s easy to check that \( \min \left\{ \frac{1}{2}, 1 - 2(s - s_c) \right\} - \min\{s - s_c, \frac{1}{3}\} \geq \frac{1}{6} > c \).

Second, let’s consider \( N_2(\overline{D}_1, \overline{D}_2, \overline{D}_3) = \pm \overline{D}_1 \int_{T^d} \overline{D}_2 \overline{D}_3 \, dx. \)

\[
\left| \int_0^\delta \int_{T^d} \overline{u}_0 \overline{D}_1 \, dx \int_{T^d} \overline{D}_2 \overline{D}_3 \, dx \right|
\leq \sum_{C_j \sim C_k} \left| \int_0^\delta \int_{T^d} (P_{C_j} \overline{u}_0)(P_{C_k} \overline{D}_1) \, dx \int_{T^d} \overline{D}_2 \overline{D}_3 \, dx \right|
\lesssim \sum_{C_j \sim C_k} \|P_{C_j} u_0\|_{L_t^\infty} \|P_{C_k} D_1\|_{L_t^\infty} \|D_2\|_{L_t^\infty} \|D_3\|_{L_t^\infty},
\]

Then following the same approach for \( N_1 \) term, we can hold the bound of \( N_2 \).

Case A (b). We consider the case \((u_1, u_2, u_3) = (D_1, D_2, R_3)\).

**Proposition 5.4.** Assume \( N_i, i = 0, 1, 2, 3, \) are dyadic numbers and \( N_1 \geq N_2 \geq N_3, \) and \( 0 \leq \delta \leq 1. \) For \( s \geq s_c \) and \( 0 \leq \alpha < s_c, \) there exist \( c, r > 0 \) and subset \( \Omega_{\delta} \subset \Omega \) with \( P(\Omega_{\delta}) \leq e^{-1/\delta^2} \), so that for all \( \omega \in \Omega_{\delta} \) and all \( N_1 \geq N_2 \geq N_3, \) we can bound the integral:

\[
\left| \int_0^\delta \int_{T^d} N(\overline{D}_1, \overline{D}_2, \overline{R}_3) \overline{u}_0 \, dx \right| \lesssim \delta^c \left( \frac{1}{N_2 N_3} \right) \|u_0\|_{L_t^\infty} \|D_1\|_{L_t^\infty} \|D_2\|_{L_t^\infty},
\]

where \( u_0, \overline{D}_1, \overline{D}_2, \) and \( \overline{R}_3 \) is defined as \((5.3)\).

**Proof.** Let \( P_{C_j} \) denote the family of Fourier projections onto the cube \( C_j \) of size \( N_2. \) We write \( C_j \sim C_k \) if the sum set overlaps the Fourier support of \( P_{2N_2}. \)

(1) Case: \( d = 3 \)

First, let’s consider \( N_0(\overline{D}_1, \overline{D}_2, \overline{R}_3) = \pm \overline{D}_1 \overline{D}_2 \overline{R}_3. \)
By Corollary 2.4 there exists $\Omega_\delta$ with $\mathbb{P}(\Omega^c_\delta) < e^{-1/\delta^r}$ and $c' > 0$, such that for all $N_3$ and $\omega \in \Omega_\delta$, we obtain that

$$\tag{5.8} \|R_3\|_{L_{t,x}^{\frac{1}{2}}([0,\delta] \times \mathbb{T}^3)} \leq \delta' \frac{\log N_3}{N_3^{s_c - \alpha}}.$$  

By Lemma 4.2 Cauchy-Schwartz inequality and (5.8),

$$\left| \int_0^\delta \int_{\mathbb{T}^3} \overline{p}_0 \tilde{D}_1 \tilde{D}_2 \tilde{R}_3 dx \right| dx dt \leq \sum_{C_j \sim C_k} \left| \int_0^\delta \int_{\mathbb{T}^3} (PC_j \overline{p}_0)(PC_k \tilde{D}_1) \tilde{D}_2 \tilde{R}_3 dx \right|.$$  

where $c = \min(c', s_c - \alpha - \epsilon, \frac{1}{10})$.

Second, $N_2(\tilde{D}_1, \tilde{D}_2, \tilde{R}_3) = \pm \tilde{D}_1 \tilde{D}_2 \tilde{R}_3$, we can bound $| \int_0^\delta \int_{\mathbb{T}^3} N_2(\tilde{D}_1, \tilde{D}_2, \tilde{R}_3) dx dt |$ by $\sum_{C_j \sim C_k} \|PC_j u_0\|_{L_{t,x}^{\frac{1}{2}}} \|PC_k D_1\|_{L_{t,x}^{\frac{1}{2}}} \|D_2\|_{L_{t,x}^{\frac{1}{2}}} \|R_3\|_{L_{t,x}^{\frac{1}{2}}}$, using H"older inequality. Then we can bound the second part via the same way.

(2) Case: $d \geq 4$

First, let’s consider $N_1(\tilde{D}_1, \tilde{D}_2, \tilde{R}_3) = \pm \tilde{D}_1 \tilde{D}_2 \tilde{R}_3$.

Set $3^{++}$ and $\infty^{--}$ as following:

$$\tag{5.9} \frac{1}{3^{++}} = \frac{1}{3} - \frac{1}{6(d + 2)}, \quad \frac{1}{\infty^{--}} = \frac{1}{2(d + 2)}.$$  

By Corollary 2.4 there exists $\Omega_\delta$ with $\mathbb{P}(\Omega^c_\delta) < e^{-1/\delta^r}$ and $c' > 0$, such that for all $N_3$ and $\omega \in \Omega_\delta$, we obtain that

$$\tag{5.10} \|R_3\|_{L_{t,x}^{\infty^{--}}([0,\delta] \times \mathbb{T}^3)} \leq \delta' \frac{\log N_3}{N_3^{s_c - \alpha}}.$$
By Lemma 4.2, Cauchy-Schwartz inequality and \((5.10)\),

\[
\left| \int_0^\delta \int_{T^d} \tilde{u}_0 \tilde{D}_1 \tilde{D}_2 \tilde{R}_3 dx dt \right|
\leq \sum_{C_j \sim C_k} \left| \int_0^\delta \int_{T^d} (P_{C_j} \tilde{u}_0)(P_{C_k} \tilde{D}_1) \tilde{D}_2 \tilde{R}_3 dx dt \right|
\leq \sum_{C_j \sim C_k} \|P_{C_j} \tilde{u}_0\|_{L^{3+}} \|P_{C_k} \tilde{D}_1\|_{L^{3+}} \|D_2\|_{L^{3+}} \|R_3\|_{L^{\infty}}
\leq \sum_{C_j \sim C_k} \min\{N_0, N_2\}^{\frac{4}{3}} \|P_{C_j} u_0\|_{L^\infty} \|P_{C_k} D_1\|_{L^\infty} \|D_2\|_{L^\infty} \|R_3\|_{L^{\infty}}
\leq \delta'^c \left( \frac{1}{N_2 N_3} \right)^c \|P_{C_j} u_0\|_{L^\infty} \|P_{C_k} D_1\|_{L^\infty} \|D_2\|_{L^\infty},
\]

where \(c = \min(c', s_c - \alpha - \epsilon, \frac{d}{2})\).

Second, \(N_2(\tilde{D}_1, \tilde{D}_2, \tilde{R}_3) = \pm \tilde{D}_1 F \tilde{D}_2 \tilde{R}_3 dx\). We can bound \(\left| \int_0^\delta \int_{T^d} N_2(\tilde{D}_1, \tilde{D}_2, \tilde{R}_3) \tilde{u}_0 dx dt \right|\) by \(\sum_{C_j \sim C_k} \|P_{C_j} u_0\|_{L^{3+}} \|P_{C_k} D_1\|_{L^{3+}} \|D_2\|_{L^{3+}} \|R_3\|_{L^{\infty}}\), using Hölder inequality. Then we can bound the second part via the same way.

\[\square\]

Case A (c). We consider the case \((u_1, u_2, u_3) = (D_1, R_2, D_3)\).

**Proposition 5.5.** Assume \(N_i, i = 0, 1, 2, 3,\) are dyadic numbers and \(N_1 \geq N_2 \geq N_3,\) and \(0 \leq \delta \leq s_c\). For \(s \geq s_c\) and \(0 \leq \alpha < \frac{d}{6}\), there exists \(c, r > 0\) and subset \(\Omega_\delta \subset \Omega\) with \(\mathbb{P}(\Omega_\delta) \leq e^{-1/\delta^c}\), so that for all \(\omega \in \Omega_\delta\) and all \(N_1 \geq N_2 \geq N_3,\) we can bound the integral:

\[
\left| \int_0^\delta \int_{T^d} \mathcal{N}(\tilde{D}_1, \tilde{R}_2, \tilde{D}_3) \tilde{u}_0 dx dt \right| \leq \delta'^c \left( \frac{1}{N_2 N_3} \right)^c \|u_0\|_{L^{3+}} \|D_1\|_{L^{3+}} \|D_3\|_{L^{3+}},
\]

where \(u_0, \tilde{D}_1, \tilde{R}_2,\) and \(\tilde{D}_3\) is defined as \((5.3)\).

**Proof.** Let \(P_{C_j}\) denote the family of Fourier projections onto the cube \(C_j\) of size \(N_2\). We write \(C_j \sim C_k\) if the sum set overlaps the Fourier support of \(P_{2N_2}\).

(1) Case: \(d = 3\)

First, let’s consider \(N_1(\tilde{D}_1, \tilde{R}_2, \tilde{D}_3) = \pm \tilde{D}_1 R_2 \tilde{D}_3\).

By Corollary 2.4, there exists \(\Omega_\delta\) with \(\mathbb{P}(\Omega_\delta) < e^{-1/\delta^c}\) and \(c' > 0\), such that for all \(N_2\) and \(\omega \in \Omega_\delta\), we obtain that

\[\|R_2\|_{L^{3+}}^{\frac{4}{3}} \left| [0, \delta] \times T^3 \right| \leq \delta'^c \log \frac{N_2}{N_2^{s_c - \alpha}}.\]

\[\square\]
By Lemma 4.2, Hölder inequality and \((5.11)\),

\[
\left| \int_0^\delta \int_{\mathbb{T}^d} \overline{u}_0 \overline{D}_1 \overline{R}_2 \overline{D}_3 dx dt \right|
\leq \sum_{C_j \sim C_k} \left| \int_0^\delta \int_{\mathbb{T}^d} (P_{C_j} \overline{u}_0)(P_{C_k} \overline{D}_1) \overline{R}_2 \overline{D}_3 dx dt \right|
\leq \sum_{C_j \sim C_k} \| P_{C_j} u_0 \|_{L_t^\infty \times L_x^\infty} \| P_{C_k} D_1 \|_{L_t^1 \times L_x^1} \| R_2 \|_{L_t^1 \times L_x^1} \| D_3 \|_{L_t^1 \times L_x^1}
\leq \min \{ N_0, N_2 \} \frac{3}{2} \frac{3}{2} \frac{3}{2} \sum_{C_j \sim C_k} N_2^{3/2} \| P_{C_j} u_0 \|_{H^s} \| P_{C_k} D_1 \|_{H^s} \| D_3 \|_{H^s}
\leq \delta^{c'} \frac{\log(N_2)}{N_2^{\frac{s}{2} - \frac{\alpha}{2}} N_3^{\frac{s}{2} - \frac{\alpha}{2}}} \sum_{C_j \sim C_k} \| P_{C_j} u_0 \|_{H^s} \| P_{C_k} D_1 \|_{H^s} \| D_3 \|_{H^s},
\]

where \( c = \min(\epsilon, s_c - \alpha - \frac{d}{2} - \epsilon, s_c - \frac{d}{2}) \).

Second, \( N_2(\overline{D}_1, \overline{R}_2, \overline{D}_3) = \pm \overline{D}_1 \int_{\mathbb{T}^d} \overline{R}_2 \overline{D}_3 dx \). We can bound \( \int_0^\delta \int_{\mathbb{T}^d} N_2(\overline{D}_1, \overline{R}_2, \overline{D}_3) \overline{u}_0 dx dt \) by \( \sum_{C_j \sim C_k} \| P_{C_j} u_0 \|_{L_t^\infty \times L_x^\infty} \| P_{C_k} D_1 \|_{L_t^1 \times L_x^1} \| D_3 \|_{L_t^1 \times L_x^1} \| R_2 \|_{L_t^1 \times L_x^1} \), using Hölder inequality. Then we can bound the second part via the same way.

(2) Case: \( d \geq 4 \)

First, let’s consider \( N_1(\overline{D}_1, \overline{R}_2, \overline{D}_3) = \pm \overline{D}_1 \overline{R}_2 \overline{D}_3 \).

Set \( 3^{++} \) and \( \infty^{--} \) as \((5.9)\). By Corollary 2.3, there exists \( \Omega_\delta \) with \( \mathbb{P}(\Omega_\delta^c) < e^{-1/\delta'} \) and \( \epsilon' > 0 \), such that for all \( N_2 \) and \( \omega \in \Omega_\delta \), we obtain that

\[
\| R_2 \|_{L_t^{3^{++}} \times L_x^{\infty^{--}}(\mathbb{T}^d)} \leq \delta^{c'} \frac{\log(N_2)}{N_2^{\frac{s}{2} - \frac{\alpha}{2}} N_3^{\frac{s}{2} - \frac{\alpha}{2}}},
\]

By Lemma 4.2, Cauchy-Schwartz inequality and \((5.12)\),

\[
\left| \int_0^\delta \int_{\mathbb{T}^d} \overline{u}_0 \overline{D}_1 \overline{R}_2 \overline{D}_3 dx dt \right|
\leq \sum_{C_j \sim C_k} \left| \int_0^\delta \int_{\mathbb{T}^d} (P_{C_j} \overline{u}_0)(P_{C_k} \overline{D}_1) \overline{R}_2 \overline{D}_3 dx dt \right|
\leq \sum_{C_j \sim C_k} \| P_{C_j} u_0 \|_{L_t^{3^{++}} \times L_x^{\infty^{--}}} \| P_{C_k} D_1 \|_{L_t^{1+} \times L_x^{1-}} \| D_3 \|_{L_t^{1+} \times L_x^{1-}} \| R_2 \|_{L_t^{\infty^{--}}}
\leq \sum_{C_j \sim C_k} \min \{ N_0, N_2 \} \frac{d-1}{2} \frac{d}{2} \frac{d}{2} \frac{d-1}{2} \frac{d}{2} \frac{d}{2} \| P_{C_j} u_0 \|_{H^s} \| P_{C_k} D_1 \|_{H^s} \| D_3 \|_{H^s}
\leq \delta^{c'} \frac{\log N_2}{N_2^{\frac{s}{2} - \frac{\alpha}{2}} N_3^{\frac{s}{2} - \frac{\alpha}{2}}} \sum_{C_j \sim C_k} \| P_{C_j} u_0 \|_{H^s} \| P_{C_k} D_1 \|_{H^s} \| D_3 \|_{H^s},
\]

where \( c = \min(\epsilon, \frac{d}{2} - \alpha - \epsilon, \frac{d}{2} - \frac{1}{2}) \).

Second, \( N_2(\overline{D}_1, \overline{D}_2, \overline{D}_3) = \pm \overline{D}_1 \int_{\mathbb{T}^d} \overline{D}_2 \overline{D}_3 dx \). We can bound \( \int_0^\delta \int_{\mathbb{T}^d} N_2(\overline{D}_1, \overline{D}_2, \overline{D}_3) \overline{u}_0 dx dt \) by \( \sum_{C_j \sim C_k} \| P_{C_j} u_0 \|_{L_t^{3^{++}} \times L_x^{\infty^{--}}} \| P_{C_k} D_1 \|_{L_t^{1+} \times L_x^{1-}} \| D_3 \|_{L_t^{1+} \times L_x^{1-}} \| R_3 \|_{L_t^{\infty^{--}}} \), using Hölder inequality. Then we can bound the second part via the same way.

□
**Case A (d).** We consider the case \((u_1, u_2, u_3) = (D_1, R_2, R_3)\).

**Proposition 5.6.** Assume \(N_i, i = 0, 1, 2, 3\), are dyadic numbers and \(N_1 \geq N_2 \geq N_3\), and \(0 \leq \delta \leq 1\). For \(s \geq s_c\) and \(0 \leq \alpha < s_c\), there exist \(c, r > 0\) and subset \(\Omega_\delta \subset \Omega\) with \(\mathbb{P}(\Omega_\delta^c) \leq e^{-1/\delta^r}\), so that for all \(\omega \in \Omega_\delta\) and all \(N_1 \geq N_2 \geq N_3\), we can bound the integral:

\[
\left| \int_0^\delta \int_{\mathbb{T}^4} N(\tilde{D}_1, \tilde{R}_2, \tilde{R}_3) \mu_0 dx dt \right| \lesssim \delta^c \left( \frac{1}{N_2 N_3} \right)^c \| u_0 \|_{Y^s} \| D_1 \|_{X^s},
\]

where \(u_0, \tilde{D}_1, \tilde{R}_2, \text{ and } \tilde{R}_3\) is defined as \((5.3)\).

**Proof.** Let \(P_{C_j}\) denote the family of Fourier projections onto the cube \(C_j\) of size \(N_2\). We write \(C_j \sim C_k\) if the sum set overlaps the Fourier support of \(P_{2N_2}\).

First, let’s consider \(N_1(\tilde{D}_1, \tilde{R}_2, \tilde{R}_3) = \pm \tilde{D}_1 \tilde{R}_2 \tilde{R}_3\).

Set \(p^+_c, q\) as

\[
\frac{1}{p^+_c} = \frac{d}{2(d+2)} - \epsilon \text{ and } \frac{1}{q} = \frac{1}{2} - \frac{1}{p^+_c}.
\]

By Corollary 2.4, there exists \(\Omega_\delta\) with \(\mathbb{P}(\Omega_\delta^c) < e^{-1/\delta^r}\) and \(c' > 0\), such that for all \(N\) and \(\omega \in \Omega_\delta\), we obtain that

\[
\left\| P_{N} v_0^{\omega} \right\|_{L^2_{t,x}([0, \delta] \times \mathbb{T}^4)} \leq \delta \frac{c' \log N}{N^{s_c - \alpha}}.
\]

By Lemma 4.2, Cauchy-Schwartz inequality and \((5.13)\),

\[
\left| \int_0^\delta \int_{\mathbb{T}^4} \mu_0 \tilde{D}_1 \tilde{R}_2 \tilde{R}_3 dx dt \right|
\leq \sum_{C_j \sim C_k} \left| \int_0^\delta \int_{\mathbb{T}^4} (P_{C_j} \mu_0)(P_{C_k} \tilde{D}_1) \tilde{R}_2 \tilde{R}_3 dx dt \right|
\leq \sum_{C_j \sim C_k} \left\| P_{C_j} u_0 \right\|_{L^p_{t,x}} \left\| P_{C_k} D_1 \right\|_{L^q_{t,x}} \left\| R_2 \right\|_{L^q_{t,x}} \left\| R_3 \right\|_{L^q_{t,x}}
\leq \sum_{C_j \sim C_k} N_j^2 \left\| P_{C_j} u_0 \right\|_{Y^0} \left\| P_{C_k} D_1 \right\|_{X^0} \left\| R_2 \right\|_{L^q_{t,x}} \left\| R_3 \right\|_{L^q_{t,x}}
\leq \delta^c \frac{\log N \log N_3}{N_2^{s_c - \alpha}} \frac{N_2}{N_3^{s_c - \alpha}} \sum_{C_j \sim C_k} \left\| P_{C_j} u_0 \right\|_{Y^s} \left\| P_{C_k} D_1 \right\|_{X^s}
\leq \delta^c \left( \frac{1}{N_2 N_3} \right)^c \left\| P_{C_j} u_0 \right\|_{Y^s} \left\| u_1 \right\|_{X^s},
\]

where \(c = \min(2c', s_c - \alpha - \epsilon)\).

Second, \(N_j(\tilde{D}_1, \tilde{R}_2, \tilde{R}_3) = \pm \tilde{D}_1 \int_{\mathbb{T}^4} \tilde{R}_2 \tilde{R}_3 dx\). We can bound \(\int_0^\delta \int_{\mathbb{T}^4} N_j(\tilde{D}_1, \tilde{R}_2, \tilde{R}_3) \mu_0 dx dt\) by \(\sum_{C_j \sim C_k} \left\| P_{C_j} u_0 \right\|_{L^p_{t,x}} \left\| P_{C_k} D_1 \right\|_{L^q_{t,x}} \left\| R_2 \right\|_{L^q_{t,x}} \left\| R_3 \right\|_{L^q_{t,x}}\), using Hölder inequality. Then we can bound the second part via the same way. \(
\)

In **Case B**, the top frequency is random term, so that the approach in **Case A** fails. In the following proofs of subcases of **Case B**, it will suffice to focus on the frequencies satisfying \(N_0 \sim N_1 \geq N_2\), since if \(N_0 < N_2 \sim N_1\), then **Case B** can be treated as **Case A** which the top frequency is deterministic term.

**Case B (a).** We consider the all random case \((u_1, u_2, u_3) = (R_1, R_2, R_3)\).

**Proposition 5.7.** Assume \(N_i, i = 0, 1, 2, 3\), are dyadic numbers and for any \(N_1, N_2, N_3\), satisfying \(N_1 \geq N_2 \geq N_3\), and \(0 \leq \delta \leq 1\). For \(\alpha < \frac{1}{4}\) and \(s_c \leq s < s_c + \frac{1}{4} - \alpha\), there exist
c, r > 0 and subset $\Omega_\delta \subset \Omega$ with $P(\Omega_\delta^c) \leq e^{-1/\delta^r}$, so that for all $\omega \in \Omega_\delta$ and all $N_1 \geq N_2 \geq N_3$, we can bound the integral:

$$\left| \int_0^\delta \int_{\mathbb{R}^d} \mathcal{N}(\tilde{R}_1, \tilde{R}_2, \tilde{R}_3) \varpi_0 \, dx dt \right| \leq \delta^c \left( \frac{1}{N_1} \right)^c \| u_0 \|_{Y^{0,s}},$$

where $\tilde{R}_1$, $\tilde{R}_2$, and $\tilde{R}_3$ is defined as (5.3) and only one of $\tilde{R}_i$ can be $\bar{R}_i$.

**Proof.** Let’s suppose that $\tilde{R}_1 = \bar{R}_1$, $\tilde{R}_2 = R_2$ and $\tilde{R}_3 = R_3$, and the other cases are similar (we will also explain how to prove in the others in the following proof).

Define $S(n, m) := \{(n_1, n_2, n_3) \in \mathbb{Z}^d \times \mathbb{Z}^d \times \mathbb{Z}^d : -n_1 + n_2 + n_3 = n, -|n_1|^2 + |n_2|^2 + |n_3|^2 = m, n_1 \neq n_2, n_3, and n_i \sim N_i\}$ (For example, if we consider $\mathcal{N}(R_1, R_2, R_3)$ case, then the corresponding $S(n, m) := \{(n_1, n_2, n_3) \in \mathbb{Z}^d \times \mathbb{Z}^d \times \mathbb{Z}^d : n_1 - n_2 + n_3 = n, |n_1|^2 - |n_2|^2 + |n_3|^2 = m, n_2 \neq n_1, n_3, and n_i \sim N_i\}$, where $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^d$. Then we have

\[
\mathcal{N}(\tilde{R}_1, \tilde{R}_2, R_3) := J_1 + J_2
\]

\[
= \sum_{n \in \mathbb{Z}^d, m \in \mathbb{Z}} e^{in \cdot x + itm} \sum_{S(n, m)} \frac{g_{n_1}(\omega)}{\langle n_1 \rangle^{d-1-\alpha}} \frac{g_{n_2}(\omega)}{\langle n_2 \rangle^{d-1-\alpha}} \frac{g_{n_3}(\omega)}{\langle n_3 \rangle^{d-1-\alpha}}
\]

\[
+ \sum_{n \in \mathbb{Z}^d, n \sim N_i, i=1,2,3} |g_n(\omega)|^2 g_n(\omega) \langle n \rangle^{3d-3-3\alpha} e^{in \cdot x + it|n|^2}
\]

**Step 1 a)** First, let’s consider $J_1$ term. By Prop[4.10] to estimate $| \int_0^\delta \int_{\mathbb{R}^d} \varpi_0 J_1(\tilde{R}_1, R_2, R_3) \, dx dt |$, we can first consider $u_0$ as a linear solution $1_J e^{it\Delta} \phi$ in any small interval $J \subset [0, \delta]$ and get the bound of $| \int_{J \times \mathbb{R}^d} J_1(\tilde{R}_1, R_2, R_3) \, dx dt |$. Suppose $\phi(x) = \sum_{n \in \mathbb{Z}^d} a_n e^{in \cdot x}$ and $1_J(t) = \sum_{k \in \mathbb{Z}} b_k e^{ikt}$.

\[
\left| \int_{J \times \mathbb{R}^d} J_1(\tilde{R}_1, R_2, R_3) \, dx dt \right|
\]

\[
= \left| \sum_{n \in \mathbb{Z}^d, |n| \sim N_0} \sum_{k \in \mathbb{Z}, |k| \leq N_1^2} b_k a_n \frac{g_{n_1}(\omega)}{\langle n_1 \rangle^{d-1-\alpha}} \frac{g_{n_2}(\omega)}{\langle n_2 \rangle^{d-1-\alpha}} \frac{g_{n_3}(\omega)}{\langle n_3 \rangle^{d-1-\alpha}} \right|
\]

then by Lemma[4.6] we have that $\sum_{|k| \leq N_1^2} |b_k| \lesssim \log N_1$. So

\[
\sum_{n \in \mathbb{Z}^d, |n| \sim N_0} \sum_{k \in \mathbb{Z}, |k| \leq N_1^2} b_k a_n \frac{g_{n_1}(\omega)}{\langle n_1 \rangle^{d-1-\alpha}} \frac{g_{n_2}(\omega)}{\langle n_2 \rangle^{d-1-\alpha}} \frac{g_{n_3}(\omega)}{\langle n_3 \rangle^{d-1-\alpha}}
\]

\[
\leq \| P_{N_0} \phi \|_{L^2_x} \sum_{|k| \leq N_1^2} |b_k| \left( \sum_{n \in \mathbb{Z}^d, |n| \sim N_0} \sum_{k \in \mathbb{Z}, |k| \leq N_1^2} \left| \frac{g_{n_1}(\omega)}{\langle n_1 \rangle^{d-1-\alpha}} \frac{g_{n_2}(\omega)}{\langle n_2 \rangle^{d-1-\alpha}} \frac{g_{n_3}(\omega)}{\langle n_3 \rangle^{d-1-\alpha}} \right|^2 \right)^{\frac{1}{2}}
\]

\[
\left( \sum_{n \in \mathbb{Z}^d, |n| \sim N_0} \sum_{k \in \mathbb{Z}, |k| \leq N_1^2} \left| \frac{g_{n_1}(\omega)}{\langle n_1 \rangle^{d-1-\alpha}} \frac{g_{n_2}(\omega)}{\langle n_2 \rangle^{d-1-\alpha}} \frac{g_{n_3}(\omega)}{\langle n_3 \rangle^{d-1-\alpha}} \right|^2 \right)^{\frac{1}{2}}
\]
By Lemma 2.20, after choosing a subset $\Omega_3^1$ with $\mathbb{P}(\Omega_3^1) \lesssim e^{-\frac{1}{2\epsilon}}$, and by Lattice counting lemma (Lemma 4.4), we obtain that

$$\|P_N \phi\|_{L^2_t} \sum_{|k| \leq N_1^3} \left| b_k \right| \left( \sum_{n \in \mathbb{Z}^d, |n| \sim N_1} \left| \sum_{|s| < k} \frac{g_n(\omega)}{n_1^2-1-\alpha} \frac{g_n(\omega)}{n_2^d-1-\alpha} \frac{g_n(\omega)}{n_3^d-1-\alpha} \right)^{\frac{1}{2}} \right)$$

$$\lesssim \|P_N \phi\|_{L^2_t} \sum_{|k| \leq N_1^3} \left| b_k \right| N_1^c \frac{N_1^c}{N_1^{s_\cdot} + N_2^{s_\cdot} - \alpha N_3^{s_\cdot} - \alpha} \|P_N \phi\|_{L^2_t}.$$ 

**Step 1 b)** Second, let’s consider $J_2$. By Lemma 2.21, there exists a set $\Omega_3^2$ with $\mathbb{P}(\Omega_3^2) < e^{-1/\delta^2}$, for all $\omega \in \Omega_3^2$, we have $|g_n(\omega)| \lesssim \frac{\log((n+1)^{d-3})}{\delta}$. 

$$\|J_2\|_{L^2_{t,x}} \sum_{n \in \mathbb{Z}^d, n \sim N_1, i = 1,2,3} \left| \frac{g_n(\omega)^2}{n} \frac{g_n(\omega)}{n} \right|^2 \lesssim \delta^{-\epsilon} \frac{1}{N_1^{5d-6\alpha-6\epsilon}}.$$ 

If we choose $\Omega_3 = \Omega_3^1 \cap \Omega_3^2$, then we obtain that

$$\left| \int_{J \times \mathbb{T}^d} P_N e^{it\Delta} \phi N_1(R_1, R_2, R_3) dx dt \right| \lesssim \delta^{-\epsilon} \frac{N_1^c}{N_1^{s_\cdot} + N_2^{s_\cdot} - \alpha N_3^{s_\cdot} - \alpha} \|P_N \phi\|_{L^2_t}.$$ 

(For simplicity, we use $\epsilon$ vaguely as a constant which we can choose arbitrary small and $\epsilon$’s in the different inequalities don’t have to be the exactly same.)

**Step 2** If we set $1^+$ and $\infty^-$ satisfying $\frac{1}{1^+} = 1 - \epsilon$ and $\frac{1}{\infty^-} = \frac{\epsilon}{3}$, by Hölder inequality and Lemma 2.4 after excluding a subset of probability $e^{-\frac{1}{2\epsilon}}$, we have

$$\left| \int_{J \times \mathbb{T}^d} P_N e^{it\Delta} \phi N_2(R_1, R_2, R_3) dx dt \right|$$

$$= \left| \int_{J \times \mathbb{T}^d} P_N e^{it\Delta} \phi R_1 R_2 R_3 dx dt \right|$$

$$\lesssim \|P_N e^{it\Delta} \phi\|_{L^1_t L^2_x(J \times \mathbb{T}^d)} \|R_1\|_{L^{\infty^-} L^2_x} \|R_2\|_{L^{\infty^-} L^2_x} \|R_3\|_{L^{\infty^-} L^2_x}$$

$$\lesssim |J|^{1-\epsilon} \frac{N_1^c}{N_1^{s_\cdot} - \alpha N_2^{s_\cdot} - \alpha N_3^{s_\cdot} - \alpha} \|P_N \phi\|_{L^2_t}.$$ 

And also we have

$$\left| \int_{J \times \mathbb{T}^d} P_N e^{it\Delta} \phi N_3(R_1, R_2, R_3) dx dt \right|$$

$$\lesssim \left| \int_{J} \left( \int_{\mathbb{T}^d} P_N e^{it\Delta} \phi R_1 dx \right) \left( \int_{\mathbb{T}^d} R_2 R_3 dx \right) dt \right|$$

$$\lesssim \|P_N e^{it\Delta} \phi\|_{L^{1^+}_t L^2_x(J \times \mathbb{T}^d)} \|R_1\|_{L^{\infty^-} L^2_x} \|R_2\|_{L^{\infty^-} L^2_x} \|R_3\|_{L^{\infty^-} L^2_x}$$

$$\lesssim |J|^{1-\epsilon} \|P_N e^{it\Delta} \phi\|_{L^{\infty^-} L^2_x(J \times \mathbb{T}^d)} \|R_1\|_{L^{\infty^-} L^2_x} \|R_2\|_{L^{\infty^-} L^2_x} \|R_3\|_{L^{\infty^-} L^2_x}$$

$$\lesssim |J|^{1-\epsilon} \frac{N_1^c}{N_1^{s_\cdot} - \alpha N_2^{s_\cdot} - \alpha N_3^{s_\cdot} - \alpha} \|P_N \phi\|_{L^2_t}.$$
So we obtain that
\[ \left| \int_{J \times \mathbb{T}^d} P_{N_0} e^{it\Delta} \phi N(R_1, R_2, R_3) \, dx dt \right| \lesssim |J|^{1-\epsilon} \frac{N_1^\epsilon}{N_1^{s_\epsilon} N_2^{s_\epsilon} N_3^{s_\epsilon}} \| P_{N_0} \phi \|_{L_x^2}. \]

**Step 3** Average the estimates in Step 1 and Step 2, we obtain that
\[ (5.15) \quad \left| \int_{J \times \mathbb{T}^d} P_{N_0} e^{it\Delta} \phi N(R_1, R_2, R_3) \, dx dt \right| \lesssim |J|^{\frac{1}{2}} \delta^{-\epsilon} \frac{N_1^\epsilon}{N_1^{s_\epsilon+\frac{1}{4}-\alpha} N_2^{s_\epsilon} N_3^{s_\epsilon}} \| P_{N_0} \phi \|_{L_x^2}. \]

By the estimate (5.10) in Step 3 and Lemma 3.10, we hold that
\[ \left| \int_{0}^{\delta} \int_{\mathbb{T}^d} N(R_1, R_2, R_3) \varpi_0 \, dx dt \right| \lesssim \delta^{\frac{1}{2}} \frac{N_1^\epsilon}{N_1^{s_\epsilon+\frac{1}{4}-\alpha} N_2^{s_\epsilon} N_3^{s_\epsilon}} \| u_0 \|_{U_x^2}. \]

**Step 4** Set $p_1^+, q$ as (5.13) and $\frac{1}{\infty^*} = \epsilon$. Using Strichartz estimate (4.4), we have
\[ \left| \int_{[0, \delta] \times \mathbb{T}^d} u_0 N(R_1, R_2, R_3) \, dx dt \right| \lesssim \| u_0 \|_{L_{t,x}^{p_1^+}} \| R_1 \|_{L_{t,x}^q} \| R_2 \|_{L_{t,x}^q} \| R_3 \|_{L_{t,x}^q} \lesssim \delta^{\frac{d+4}{2(d+2)}} \| u_0 \|_{L_{t,x}^{p_1^+}} \| R_1 \|_{L_{t,x}^{q}} \| R_2 \|_{L_{t,x}^{q}} \| R_3 \|_{L_{t,x}^{q}} \lesssim \delta^{\frac{d+4}{2(d+2)}} \frac{1}{N_1^{s_\epsilon} N_2^{s_\epsilon} N_3^{s_\epsilon}} \| u_0 \|_{U_x^{p_1^+}}. \]

By the interpolation lemma (Lemma 3.13) and the embedding property (3.5), we obtain that
\[ \left| \int_{0}^{\delta} \int_{\mathbb{T}^d} N(R_1, R_2, R_3) \varpi_0 \, dx dt \right| \lesssim \delta^{\frac{1}{2}} \frac{N_1^\epsilon}{N_1^{s_\epsilon+\frac{1}{4}-\alpha} N_2^{s_\epsilon} N_3^{s_\epsilon}} \| u_0 \|_{Y^{s}} \| D_3 \|_{X^{s}}, \]

Since $s < s_\epsilon + \frac{1}{4} - \alpha$, we hold the proposition. \(\square\)

**Case B (b).** We consider the all case $(u_1, u_2, u_3) = (R_1, R_2, D_3)$.

**Proposition 5.8.** Assume $N_1, i = 0, 1, 2, 3$, are dyadic numbers and $N_1 \geq N_2 \geq N_3$, and $0 \leq \delta \leq 1$. For $s_\epsilon \leq s < s_\epsilon + \frac{1}{4} - \alpha$ and $0 \leq \alpha < \frac{1}{4}$, there exist $c, r > 0$ and subset $\Omega_\delta \subset \Omega$ with $\mathbb{P}(\Omega_\delta^c) \leq e^{-1/\delta^r}$, so that for all $\omega \in \Omega_\delta$ and all $N_1 \geq N_2 \geq N_3$, we can bound the integral:
\[ \left| \int_{0}^{\delta} \int_{\mathbb{T}^d} N(\widetilde{R}_1, \widetilde{R}_2, \widetilde{D}_3) \varpi_0 \, dx dt \right| \lesssim \delta^{\frac{1}{2}} \frac{N_1^\epsilon}{N_1^{s_\epsilon+\frac{1}{4}-\alpha} N_2^{s_\epsilon} N_3^{s_\epsilon}} \| u_0 \|_{Y^{s}} \| D_3 \|_{X^{s}}, \]

where $\widetilde{R}_1, \widetilde{R}_2,$ and $\widetilde{D}_3$ is defined as (5.3) and only one of $\{ \widetilde{R}_1, \widetilde{R}_2, \widetilde{D}_3 \}$ can be the conjugate.

**Proof.** Let’s suppose that $\widetilde{R}_1 = \overline{R}_1, \widetilde{R}_2 = R_2$ and $\widetilde{D}_3 = D_3$, and the other cases are similar.

Define $S_3(n, n_2, m) := \left\{ (n_1, n_2) \in \mathbb{Z}^d \times \mathbb{Z}^d : -n_1 + n_2 + n_3 = n, -|n_1|^2 + |n_2|^2 + |n_3|^2 = m, n_1 \neq n_2, n_3, \text{ and } n_1 \sim N_1 \right\}$. Then we have
\[ N(\widetilde{R}_1, \widetilde{R}_2, D_3) := J_1 + J_2 \]
\[ = \sum_{n \in \mathbb{Z}^d, m \in \mathbb{Z}} e^{in \cdot x + itm} \sum_{S(n,m)} \frac{g_{n_1}(\omega)}{(n_1)^{d-1-\alpha}} \frac{g_{n_2}(\omega)}{(n_2)^{d-1-\alpha}} \tilde{D}_3(n_3) \]
\[ + \sum_{n \in \mathbb{Z}^d, n_2 \sim N_2, i=1,2,3} \frac{|g_\omega|}{(n_2)^{2d-2-2\alpha}} \tilde{D}_3(n_3) e^{in \cdot x + it|n|^2} \]

where $\tilde{D}_3(n_3) = \sum_{n \sim N_3} e^{in \cdot x + it|n|^2}$. The case for $\tilde{R}_1 = R_1, \tilde{R}_2 = \overline{R}_2, \tilde{D}_3 = D_3$ is similar.
Step 1 a) First, let’s consider $J_1$ term. By Prop[3.10] to estimate $|\int_{\delta}^{\delta} \int_{\mathbb{T}^d} \overline{u}_0(f_1) d\tau_1 d\tau_2 d\tau_3|$, we can first consider $u_0$ as a linear solution $1_{J} e^{it\Delta} \phi$ in any small interval $J \subset [0, \delta]$ and get the bound of $|\int_{\mathbb{T}^d} P_{N_0} e^{it\Delta} \phi f_1(\mathbb{T}_1, R_2, R_3) e^{it\Delta} \phi^{(3)} d\tau|$. Suppose $\phi(x) = \sum_{n \in \mathbb{Z}^d} a_n e^{inx}$, $\phi^{(3)}(x) = \sum_{n \in \mathbb{Z}^d} (3) e^{inx}$ and $\mathbb{I}_{J(t)} = \sum_{k \in \mathbb{Z}} b_k e^{ikt}$.

$$\left| \int_{\mathbb{T}^d} P_{N_0} e^{it\Delta} \phi f_1(\mathbb{T}_1, R_2, R_3) e^{it\Delta} \phi^{(3)} d\tau \right|$$

$$= \left| \sum_{n \in \mathbb{Z}^d, |n| \sim N_0} \sum_{k \in \mathbb{Z}, |k| \leq N_1^2} b_k a_n \frac{g_{n_1}(\omega)}{\langle n_1 \rangle^{d-1-\alpha}} \frac{g_{n_2}(\omega)}{\langle n_2 \rangle^{d-1-\alpha}} a^{(3)}_n \right|$$

then by Lemma[4.6] we have that $\sum_{|k| \leq N_1^2} |b_k| \lesssim \log N_1$. So

$$\leq \left\| P_{N_0} \phi \right\|_{L^2} \left\| P_{N_3} \phi^{(3)} \right\|_{L^2} \sum_{|k| \leq N_1^2} |b_k| \left( \sum_{n \in \mathbb{Z}^d, |n| \sim N_0} \sum_{n_3 \in \mathbb{Z}^d, |n_3| \sim N_3} \frac{g_{n_1}(\omega)}{\langle n_1 \rangle^{d-1-\alpha}} \frac{g_{n_2}(\omega)}{\langle n_2 \rangle^{d-1-\alpha}} \right)^2$$

By Lemma[2.5], after choosing a subset $\Omega_1^\delta$ with $\mathbb{P}(\Omega_1^\delta) \lesssim e^{-\frac{1}{\delta^2}}$, and by Lattice counting lemma (Lemma[4.1]), we obtain that

$$\left\| P_{N_0} \phi \right\|_{L^2} \left\| P_{N_3} \phi^{(3)} \right\|_{L^2} \sum_{|k| \leq N_1^2} |b_k| \left( \sum_{n \in \mathbb{Z}^d, |n| \sim N_0} \sum_{n_3 \in \mathbb{Z}^d, |n_3| \sim N_3} \frac{1}{\langle n \rangle^{d-1-\alpha} \langle n_2 \rangle^{d-1-\alpha}} \right)^2$$

$$\lesssim \left\| P_{N_0} \phi \right\|_{L^2} \left\| P_{N_3} \phi^{(3)} \right\|_{L^2} \sum_{|k| \leq N_1^2} |b_k| \frac{1}{N_1^{d-1-\alpha} N_2^{d-1-\alpha}}$$

$$\times \left\{ \left( \{ n, n_1, n_2, n_3 \} \in \mathbb{Z}^d \times \mathbb{Z}^d \times \mathbb{Z}^d : (n_1, n_2, n_3) \in S(n, |n|^2 + k) \right) \right\}^{1/2}$$

$$\leq \frac{N_1^d N_3^d}{N_1^{d+\frac{1}{2}-\alpha} N_2^{d-\alpha}} \left\| P_{N_0} \phi \right\|_{L^2} \left\| P_{N_3} \phi^{(3)} \right\|_{L^2}.$$
By Lemma 2.1, there exists a set $\Omega_3^2$ with $\mathbb{P}(\Omega_3^2) < e^{-1/\delta}$, for all $\omega \in \Omega_3^2$, we have $|g_n(\omega)| \leq N^2 \log(N_1)N^2/|\lambda^2 - 2x|$. 

(5.19) 

$\|P_{N_0} \phi\|_L^2 \|P_{N_0} \phi(3)\|_L^2 \leq N^2 \log(N_1)N^2/|\lambda^2 - 2x|$. 

(5.20) 

$\|P_{N_0} \phi\|_L^2 \|P_{N_0} \phi(3)\|_L^2 \leq N^2 \log(N_1)N^2/|\lambda^2 - 2x|$. 

If we choose $\Omega_3 = \Omega_1^2 \cap \Omega_2^2$, then we obtain that 

$\int_{J \times T^d} P_{N_0} e^{it\Delta} \phi N(\overline{R}_1, \overline{R}_2, P_{N_3} e^{it\Delta} \phi(3)) dx dt \leq N^2 \log(N_1)N^2/|\lambda^2 - 2x|$. 

Step 2 If we set $2^+, \infty^-, 1^+$ and $q$ satisfying $1/2 = 1/2 - \epsilon$, $2^+ = 1/2 + \epsilon$, and $q = 2^+$. By Hölder inequality and Lemma 2.3 after excluding a subset of probability $e^{-\delta}$, we have 

$\int_{J \times T^d} P_{N_0} e^{it\Delta} \phi N_{1}(\overline{R}_1, \overline{R}_2, P_{N_3} e^{it\Delta} \phi(3)) dx dt \leq N^2 \log(N_1)N^2/|\lambda^2 - 2x|$. 

And also we have 

$\int_{J \times T^d} P_{N_0} e^{it\Delta} \phi N_{2}(\overline{R}_1, \overline{R}_2, P_{N_3} e^{it\Delta} \phi(3)) dx dt \leq N^2 \log(N_1)N^2/|\lambda^2 - 2x|$. 

So we obtain that 

$\int_{J \times T^d} P_{N_0} e^{it\Delta} \phi N(\overline{R}_1, \overline{R}_2, P_{N_3} e^{it\Delta} \phi(3)) dx dt \leq N^2 \log(N_1)N^2/|\lambda^2 - 2x|$. 

Step 3 Average the estimates in Step 1 and Step 2, we obtain that 

(5.21) 

$\int_{J \times T^d} P_{N_0} e^{it\Delta} \phi N(\overline{R}_1, \overline{R}_2, P_{N_3} e^{it\Delta} \phi(3)) dx dt \leq N^2 \log(N_1)N^2/|\lambda^2 - 2x|$. 

(5.22) 

$\int_{J \times T^d} P_{N_0} e^{it\Delta} \phi N(\overline{R}_1, \overline{R}_2, P_{N_3} e^{it\Delta} \phi(3)) dx dt \leq N^2 \log(N_1)N^2/|\lambda^2 - 2x|$.
By the estimate (5.24) in Step 3 and Lemma 3.10 we hold that
\[
\left| \int_0^\delta \int_{\mathbb{T}^d} N(R_1, R_2, D_3) \overline{u}_0 dx dt \right| \lesssim \delta^{\frac{1}{2}} \frac{N_1 N_3^{\frac{d}{2}}}{N_1^{s_c + \frac{1}{2} - s} N_2^{s_c - s}} \| u_0 \|_{Y^s} \| D_3 \|_{X^s}
\]

**Step 4** Set \( p^+_c \), \( q \) as (5.13) and \( \frac{1}{\infty} = \epsilon \). Replacing \( P_{Q_0} e^{it\Delta} \phi \) by \( \overline{u}_0 \), following the similar idea, and using Strichartz estimate (4.4), we have
\[
\left| \int_{[0, \delta] \times \mathbb{T}^d} \overline{u}_0 N(R_1, R_2, D_3) \, dx \right| \lesssim \| u_0 \|_{L^p_{t,x}} \| R_1 \|_{L^q_{t,x}} \| R_2 \|_{L^q_{t,x}} \| D_3 \|_{L^p_{t,x}}
\]
\[
\lesssim \delta^{\frac{1}{2(d+2)} - \epsilon} \frac{1}{N_1^{s_c - s - \epsilon} N_2^{s_c - s}} \| u_0 \|_{L^p_{t,x}} \| D_3 \|_{L^p_{t,x}}.
\]

By Lemma 3.13 and the embedding property (3.5), we obtain that
\[
\left| \int_0^\delta \int_{\mathbb{T}^d} N(R_1, R_2, D_3) \overline{u}_0 dx dt \right| \lesssim \delta^{\frac{1}{2}} \frac{N_1 N_3^{\frac{d}{2}}}{N_1^{s_c + \frac{1}{2} - s} N_2^{s_c - s}} \| u_0 \|_{Y^s} \| D_3 \|_{X^s}.
\]
Since \( s < s_c + \frac{1}{4} - \alpha \), we hold the proposition. \qed

**Case B (c).** We consider the all case \((u_1, u_2, u_3) = (R_1, D_2, R_3)\). By the similar approach with **Case B (b)**, we can hold following property:

**Proposition 5.9.** Assume \( N_i \), \( i = 0, 1, 2, 3 \), are dyadic numbers and \( N_1 \geq N_2 \geq N_3 \), and \( 0 \leq \delta \leq 1 \). For \( s_c \leq s < s_c + \frac{1}{6} - \alpha \) and \( 0 \leq \alpha < \frac{1}{6} \), there exist \( c, r > 0 \) and subset \( \Omega_\delta \subset \Omega \) with \( \mathbb{P}(\Omega_\delta) \leq e^{-1/s^r} \), so that for all \( \omega \in \Omega_\delta \) and all \( N_1 \geq N_2 \geq N_3 \), we can bound the integral:
\[
\left| \int_0^\delta \int_{\mathbb{T}^d} N(R_1, D_2, R_3) \overline{u}_0 dx dt \right| \lesssim \delta^{\prime} \frac{1}{N_1} \| u_0 \|_{Y^{-s}} \| D_2 \|_{X^s},
\]

where \( R_1, D_2, \) and \( R_3 \) is defined as (5.5) and only one of \( \{ R_1, D_2, R_3 \} \) can be the conjugate.

**Proof.** For \( d \geq 4 \).

Following the **Case B (b)**, by choosing a subset \( \Omega_\delta \subset \Omega \) with \( \mathbb{P}(\Omega_\delta) < e^{-1/s^r} \), for \( \omega \in \Omega_\delta \), we have similar estimate:
\[
\left| \int_0^\delta \int_{\mathbb{T}^d} N(R_1, D_2, R_3) \overline{u}_0 dx dt \right| \lesssim \delta^{\frac{1}{2}} \frac{N_1 N_3^{\frac{d}{2}}}{N_1^{s_c + \frac{1}{2} - s} N_2^{s_c - s}} \| u_0 \|_{Y^{-s}} \| D_2 \|_{X^s},
\]

since \( s < s_c + \frac{1}{6} - \alpha \), the proposition holds.

For \( d = 3 \). Following the **Case B (b)**, in Step 3, we average the estimates in Step 1 and Step 2 with different weights, we have that
\[
\left| \int_0^\delta \int_{\mathbb{T}^3} N(R_1, D_2, R_3) \overline{u}_0 dx dt \right| \lesssim \delta^{\frac{1}{2}} \frac{N_1 N_3^{\frac{1}{2}}}{N_1^{s_c + \frac{1}{2} - s} N_2^{s_c - s}} \| u_0 \|_{Y^{-s}} \| D_2 \|_{X^s},
\]

since \( s_c \leq s < s_c + \frac{1}{6} - \alpha \), and \( s_c = \frac{1}{2} \), the proposition holds. \qed
Case B (d). We consider the case \((u_1, u_2, u_3) = (R_1, D_2, D_3)\).

**Proposition 5.10.** Assume \(N_i, i = 0, 1, 2, 3\), are dyadic numbers and \(N_1 \geq N_2 \geq N_3\), and \(0 \leq \delta \leq 1\). For \(d \geq 3\),

\[
s_r(d) = \begin{cases} 
\frac{3}{4} & d = 3 \\
\frac{4}{5} & d = 4 \\
\frac{3}{4} & d \geq 5 
\end{cases}
\]

For \(s_c \leq s < s_c + s_r(d) - \alpha\) and \(0 \leq \alpha < s_r(d)\), there exist \(c, r > 0\) and subset \(\Omega_\delta \subset \Omega\) with \(\mathbb{P}(\Omega_\delta) \leq e^{-1/\delta^r}\), so that for all \(\omega \in \Omega_\delta\) and all \(N_1 \geq N_2 \geq N_3\), we can bound the integral:

\[
\left| \int_0^\delta \int_{\mathbb{T}^3} \mathcal{N}(\tilde{R}_1, \tilde{D}_2, \tilde{D}_3) \bar{\eta}_0 dx dt \right| \lesssim \delta^r \left( \frac{1}{N_1} \right)^c \| u_0 \| \| Y^{-\alpha} \| \| D_2 \| \| X \| \| D_3 \|_{X^*},
\]

where \(\tilde{R}_1, \tilde{D}_2, \text{ and } \tilde{D}_3\) is defined as [5.3] and only one of \(\{\tilde{R}_1, \tilde{D}_2, \tilde{D}_3\}\) can be the conjugate.

**Proof.** Let’s suppose that \(\tilde{R}_1 = R_1, \tilde{D}_2 = D_2\) and \(\tilde{D}_3 = D_3\), and the other cases are similar.

Define \(S_{2,3}(n, n_2, n_3, m) := \{ n_1 \in \mathbb{Z}^d : n_1 - n_2 + n_3 = n, \ |n_1|^2 - |n_2|^2 + |n_3|^2 = m, \ n_2 \neq n_1, \ n_3, \text{ and } n_1 \sim N_1 \}\). Then we have

\[
\mathcal{N}(R_1, D_2, D_3) := J_1 + J_2
\]

\[
= \sum_{n \in \mathbb{Z}^d, m \in \mathbb{Z}} e^{in \cdot x + itm} \sum_{S(n, m)} \frac{g_{n_1}(\omega)}{\langle n_1 \rangle^{d-1-\alpha}} \overline{D_2(n_2)} \overline{D_3(n_3)}
\]

\[
+ \sum_{n \in \mathbb{Z}^d, n \sim N_1, i = 1, 2, 3} g_{n}(\omega) \overline{D_2(n_2)} \overline{D_3(n_3)} e^{in \cdot x + it|n|^2}
\]

**Step 1 a)** First, let’s consider \(J_1\) term. By Proposition 3.11 to estimate \(\left| \int_0^\delta \int_{\mathbb{T}^d} \mathbb{1}_J(u_0) J_1(R_1, D_2, D_3) dx dt \right|\),

we can first consider \(u_0\) as a linear solution \(1 \cdot e^{it \Delta} \phi\) in any small interval \(J \subset [0, \delta]\) and get the bound of \(\left| \int_{J \times \mathbb{T}^d} P_{N_0} e^{it \Delta} \phi J_1(R_1, D_2, D_3) dt \right|\). Suppose \(\phi(x) = \sum_{n \in \mathbb{Z}^d} a_n e^{in \cdot x}, \phi_i(x) = \sum_{n \in \mathbb{Z}^d} a_n^{(i)} e^{in \cdot x} \) for \(i = 2, 3\) and \(1_J(t) = \sum_{k \in \mathbb{Z}} b_k e^{ikt}\).

\[
= \sum_{n \in \mathbb{Z}^d, |n| \sim N_0} \sum_{k \in \mathbb{Z}, |k| \leq N^2_1} \sum_{S(n, |n|^{2+k})} a_n^{(2)} a_n^{(3)} g_{n_1}(\omega) \langle n_1 \rangle^{d-1-\alpha} 
\]

\[
\leq \| P_{N_0} \phi \|^2_{L^2_x} \| P_{N_3} \phi^{(3)} \|^2_{L^2_x} \sum_{|k| \leq N^2_1} |b_k| \left( \sum_{n \in \mathbb{Z}^d, |n| \sim N_0} \sum_{n_3 \in \mathbb{Z}^d, n_3 |n|^{2+k}} \frac{g_{n_1}(\omega) \langle n_1 \rangle^{d-1-\alpha}}{n_1^{d-1-\alpha}} \right)^{\frac{1}{2}}
\]

Next, fix \(k\). Let’s focus on

\[
(5.23) \sum_{n \in \mathbb{Z}^d, |n| \sim N_0} \sum_{n_3 \in \mathbb{Z}^d, n_3 |n|^{2+k}} \frac{g_{n_1}(\omega) \langle n_1 \rangle^{d-1-\alpha}}{n_1^{d-1-\alpha}}
\]

To bound \(5.23\), we use the matrix \(\mathcal{G}^* \mathcal{G}\) argument in Bourgain’s paper [6] as follows. Fix \(n_3 \) and \(|n_3| \sim N_3\). Define

\[
\mathcal{G} = \mathcal{G}_0 = (\sigma_{n, n_2})_{|n| < N_0, |n_2| < N_2, n \neq n_3}.
\]
where

\[
\sigma_{n,n_2} = \begin{cases} 
\frac{1}{N_1^{2d-1-\alpha}} g_{n+n_2-n_3}(\omega) & \text{if } 2\langle n-n_3, n_2-n_3 \rangle = k \\
0 & \text{otherwise}
\end{cases}
\]

Then (5.23) is bounded by \( N_1^d \| G_\omega \|^{\frac{d}{2}} \) and by Lemma 4.7 we obtain that

\[
\| G^* G \| \leq \max_n \left( \sum_n |\sigma_{n,n_2}|^2 + \left( \sum_{n \neq n'} \sum_n |\sigma_{n,n_2} \sigma_{n',n_2'}|^2 \right) \right)^{\frac{1}{2}}.
\]

Using Lemma 2.1 the first term in (5.24) is bounded as follows,

\[
\sum_{n_2} \left| \frac{1}{N_1^{d-1-\alpha}} g_{n+n_2-n_3}(\omega) \right|^2 \leq \frac{N_2^d}{N_1^{2(d-1-\alpha)-\varepsilon}} \leq \frac{N_2}{N_1^{2\varepsilon+1-2\alpha-\varepsilon}}.
\]

Then we will show that the second term in (5.24) is bounded as follows

\[
\left( \sum_{n \neq n'} \sum_{n_2} |\sigma_{n,n_2} \sigma_{n',n_2'}|^2 \right)^{\frac{1}{2}} \leq N_1^{-2\varepsilon-1+2\alpha+\varepsilon} N_2^{d+1}.
\]

Indeed, write

\[
\sum_{n \neq n'} \sum_{n_2} |\sigma_{n,n_2} \sigma_{n',n_2'}|^2 = \frac{1}{N_1^{4(d-1-\alpha)}} \sum_{n \neq n'} \sum_{n_2} g_{n+n_2-n_3}(\omega) g_{n'+n_2-n_3}(\omega).
\]

Then we use Lemma 2.5 there exists a set \( \Omega_1^{\varepsilon} \) with \( \mathbb{P}(\Omega_1^{\varepsilon}) < e^{-1/\delta_r} \), for all \( \omega \in \Omega_1^{\varepsilon} \), (5.27) can be bounded by

\[
\frac{1}{N_1^{4(d-1-\alpha)}} |\{(n,n',n_2) : n \neq n', 2\langle n-n_3, n_2-n_3 \rangle = k, 2\langle n'-n_3, n_2-n_3 \rangle = k\}|.
\]

To bound the number of the elements in \( \{(n,n',n_2) : n \neq n', 2\langle n-n_3, n_2-n_3 \rangle = k, 2\langle n'-n_3, n_2-n_3 \rangle = k\} \), first we can count the number of pair \((n,n_2)\) and by Lemma 4.5 it is bounded by \( N_1^{d-1+\varepsilon} N_2^{d-1} \). And then the number of possible \( n' \) is bounded \( N_1^{d-1+\varepsilon} N_2^{d-1} \). So the size of the upper lattice set is bounded by \( N_1^{2(d-1)+\varepsilon} N_2^{d-1} \), and hence we hold (5.28).

By the estimates (5.25) and (5.26), we obtain that

\[
\left| \int_{\mathbb{R}^d} |\overline{P_N e^{i t \Delta}} \phi_j(\mathbb{R}_1, P_{N_2} e^{i t \Delta} \phi_j(2), P_{N_3} e^{i t \Delta} \phi_j(3)) dx dt \right|
\]

\[
\lesssim N_1^{d-1+\frac{d}{2}} N_3^{d} \frac{N_2^{d}}{N_1^{\frac{\varepsilon}{2}+\frac{d}{2}-\alpha}} \| P_N \phi \|_{L^2} \| P_{N_2} \phi(2) \|_{L^2} \| P_{N_3} \phi(3) \|_{L^2}.
\]
Step 1 b) Second, let’s consider \( \mathcal{J}_2 \). By Lemma 2.4, there exists a set \( \Omega_2^\delta \) with \( \mathbb{P}(\Omega_2^c) < e^{-1/\delta^{2/3}} \), for all \( \omega \in \Omega_2^\delta \), we have \( |g_n(\omega)| \lesssim \frac{(n_1')^2}{\delta} \).

\[
\int_{J \times \mathbb{T}^d} |P_{N_0} e^{it\Delta} \mathcal{J}_2(\overline{R}_1, P_{N_2} e^{it\Delta} \phi^{(2)}, P_{N_3} e^{it\Delta} \phi^{(3)})| \, dx \, dt \\
\leq \sum_{n \in \mathbb{Z}^d, |n|_0 \sim N_0} b_n \mathcal{N} \frac{|g_n(\omega)|}{(n_1')^{d-1-\alpha}} a_n \phi^{(2)} a_n \phi^{(3)} \\
\leq \frac{N_1^{d-1-\alpha}}{N_1^{d-1-\alpha}} \sum_{n \in \mathbb{Z}^d, |n|_0 \sim N_0} b_n \mathcal{N} a_n \phi^{(2)} a_n \phi^{(3)} \\
\lesssim \frac{N_2^d \log(N_1)}{N_1^{d-1-\alpha}} \|P_{N_0} \phi\|_{L_t^2} \|P_{N_2} \phi^{(2)}\|_{L_t^2} \|P_{N_3} \phi^{(3)}\|_{L_t^1(\mathbb{T}^d)} \\
\lesssim \frac{N_2^d \log(N_1)}{N_1^{d-1-\alpha}} \|P_{N_0} \phi\|_{L_t^2} \|P_{N_2} \phi^{(2)}\|_{L_t^2} \|P_{N_3} \phi^{(3)}\|_{L_t^1(\mathbb{T}^d)}.
\]

If we choose \( \Omega_\delta = \Omega_1^\delta \cap \Omega_2^\delta \), then we obtain that

\[
\int_{J \times \mathbb{T}^d} |P_{N_0} e^{it\Delta} \phi N(\overline{R}_1, P_{N_2} e^{it\Delta} \phi^{(2)}, P_{N_3} e^{it\Delta} \phi^{(3)})| \, dx \, dt \\
\lesssim \frac{N_1^{d-1-\alpha} N^{d_2} N^{d_3}}{N_1^{s+c/2}} \|P_{N_0} \phi\|_{L_t^2} \|P_{N_2} \phi^{(2)}\|_{L_t^2} \|P_{N_3} \phi^{(3)}\|_{L_t^2}.
\tag{5.30}
\]

Step 2 If we set \( 2^+, \infty^- \), \( 1^+ \) and \( q \) satisfying \( \frac{1}{2^+} = \frac{1}{2} - \epsilon, \frac{2^+}{\infty^-} + \frac{2^+}{1^+} = \frac{1}{2}, \frac{1}{1^+} + \frac{2^-}{\infty^-} = 1 \), and \( \frac{2}{\alpha} + \frac{d}{\alpha} = \frac{d}{\alpha} \). By Hölder inequality and Lemma 2.4 after excluding a subset of probability \( e^{-1/\delta} \), we have

\[
\int_{J \times \mathbb{T}^d} \left| \frac{P_{N_0} e^{it\Delta} \phi N_1(\overline{R}_1, P_{N_2} e^{it\Delta} \phi^{(2)}, P_{N_3} e^{it\Delta} \phi^{(3)})} \right| \, dx \, dt \\
= \int_{J \times \mathbb{T}^d} \left| \frac{P_{N_0} e^{it\Delta} \phi}{\overline{R}_1} P_{N_2} e^{it\Delta} \phi^{(2)} P_{N_3} e^{it\Delta} \phi^{(3)} \right| \, dx \, dt \\
\leq \|P_{N_0} e^{it\Delta} \phi\|_{L_t^2} \overline{R}_1 \|P_{N_2} e^{it\Delta} \phi^{(2)}\|_{L_t^2} \|P_{N_3} e^{it\Delta} \phi^{(3)}\|_{L_t^2} \\
\lesssim |J|^{1-\epsilon} \frac{N^{d_2-d_-\epsilon}}{N^{s+c\alpha}} \|P_{N_0} \phi\|_{L_t^2} \|P_{N_2} \phi^{(2)}\|_{L_t^2} \|P_{N_3} \phi^{(3)}\|_{L_t^2}.
\]

And also we have

\[
\int_{J \times \mathbb{T}^d} \left| \frac{P_{N_0} e^{it\Delta} \phi N_2(\overline{R}_1, P_{N_2} e^{it\Delta} \phi^{(2)}, P_{N_3} e^{it\Delta} \phi^{(3)})} \right| \, dx \, dt \\
\lesssim \int_{J} \left( \int_{\mathbb{T}^d} \frac{P_{N_0} e^{it\Delta} \phi}{\overline{R}_1} \, dx \right) \left( \int_{\mathbb{T}^d} P_{N_2} e^{it\Delta} \phi^{(2)} P_{N_3} e^{it\Delta} \phi^{(3)} \, dx \right) \, dt \\
\leq \|P_{N_0} e^{it\Delta} \phi\|_{L_t^\infty} \overline{R}_1 \|P_{N_2} e^{it\Delta} \phi^{(2)}\|_{L_t^\infty} \|P_{N_3} e^{it\Delta} \phi^{(3)}\|_{L_t^\infty} \\
\lesssim |J|^{-1-\epsilon} \frac{1}{N^{s+c\alpha}} \|P_{N_0} \phi\|_{L_t^2} \|P_{N_2} \phi^{(2)}\|_{L_t^2} \|P_{N_3} \phi^{(3)}\|_{L_t^2}.
\]

So we obtain that

\[
\int_{J \times \mathbb{T}^d} |P_{N_0} e^{it\Delta} \phi N(\overline{R}_1, P_{N_2} e^{it\Delta} \phi^{(2)}, P_{N_3} e^{it\Delta} \phi^{(3)})| \, dx \, dt \\
\lesssim |J|^{-1-\epsilon} \frac{N^{d_2-d_-\epsilon}}{N^{s+c\alpha}} \|P_{N_0} \phi\|_{L_t^2} \|P_{N_2} \phi^{(2)}\|_{L_t^2} \|P_{N_3} \phi^{(3)}\|_{L_t^2}. \tag{5.31}
\]
Step 3 Average the estimates \[ [5.30] \text{ and } [5.31] \] in Step 1 and Step 2, we obtain that
\[
\left| \int_{J \times \mathbb{T}^d} P_N e^{it \Delta} \phi N(R_1, P_N e^{it \Delta} \phi^{(2)}, P_N e^{it \Delta} \phi^{(3)}) \, dx \, dt \right| \\
\lesssim \left| J \right| \frac{d-1}{2} \frac{N_2^{s(s-2)} N_3^{s(s-2)}}{N_1^{\frac{d}{2}} + N_1^{-\alpha - \epsilon}} \| P_N \phi \|_{L^2_x} \| P_N \phi^{(2)} \|_{L^2_x} \| P_N \phi^{(3)} \|_{L^2_x}.
\]
(5.32)

By (5.32) and Lemma 3.10, we hold that
\[
\left| \int_0^\delta \int_{\mathbb{T}^d} N(R_1, D_2, D_3) \mu_0 \, dx \, dt \right| \lesssim \delta^{\frac{s}{d}} \frac{N_1^d N_2^{s(s-2)} N_3^{s(s-2)}}{N_1^{\frac{d}{2}} + N_1^{-\alpha - \epsilon}} \| u_0 \|_{U^2_{\alpha, d}} \| D_2 \|_{U^2_{\alpha, d}} \| D_3 \|_{U^2_{\alpha, d}}.
\]
(5.33)

Step 4 (only for \( d = 3, 4 \)) If we set \( \frac{1}{p_3} = \frac{d}{2(d+2)} - \epsilon \) and \( \frac{1}{p_4} = 1 - \frac{3}{p_3} \). By Hölder inequality and Lemma 2.4, after excluding a subset of probability \( e^{-\frac{1}{T}} \), we have
\[
\left| \int_{J \times \mathbb{T}^d} \overline{\mu}_0 N_1(R_1, D_2, D_3) \, dx \, dt \right| = \left| \int_{J \times \mathbb{T}^d} \overline{\mu}_0 R_1 D_2 D_3 \, dx \, dt \right|
\leq \| u_0 \|_{L^p_{t,x}(J \times \mathbb{T}^d)} \| R_1 \|_{L^q_{t,x}} \| D_2 \|_{L^{p_2}_{t,x}} \| D_3 \|_{L^{p_4}_{t,x}}
\lesssim \frac{N_1^d}{N_1^{\frac{d}{2}} + N_1^{-\alpha - \epsilon}} \| u_0 \|_{U^p_{\alpha, d}} \| D_2 \|_{U^p_{\alpha, d}} \| D_3 \|_{U^p_{\alpha, d}}.
\]
And also we have
\[
\left| \int_{J \times \mathbb{T}^d} \overline{\mu}_0 N_2(R_1, D_2, D_3) \, dx \, dt \right| \lesssim \left| \int_J \left( \int_{J \times \mathbb{T}^d} \overline{\mu}_0 R_1 \, dx \right) \left( \int_{J \times \mathbb{T}^d} D_2 D_3 \, dx \right) \, dt \right|
\leq \| u_0 \|_{L^{p_3}_{t,x}(J \times \mathbb{T}^d)} \| R_1 \|_{L^{q_3}_{t,x}} \| D_2 \|_{L^{p_3}_{t,x}} \| D_3 \|_{L^{p_4}_{t,x}}
\lesssim \frac{N_1^d}{N_1^{\frac{d}{2}} + N_1^{-\alpha - \epsilon}} \| u_0 \|_{U^p_{\alpha, d}} \| D_2 \|_{U^p_{\alpha, d}} \| D_3 \|_{U^p_{\alpha, d}}.
\]
So we obtain that
\[
\left| \int_{J \times \mathbb{T}^d} \overline{\mu}_0 N(R_1, D_2, D_3) \, dx \, dt \right| \lesssim \frac{N_1^d}{N_1^{\frac{d}{2}} + N_1^{-\alpha - \epsilon}} \| u_0 \|_{U^p_{\alpha, d}} \| D_2 \|_{U^p_{\alpha, d}} \| D_3 \|_{U^p_{\alpha, d}}.
\]
(5.34)

Step 5 Finally, when \( d = 3, 4 \), by the embedding properties Remark 3.3 and (3.5), we average (5.33) and (5.34) with weights: \( \left( \frac{d-1}{5d-3}, \frac{5d-3d}{5d-1} \right) \); when \( d \geq 5 \), we directly use (5.33). Summarizing these two cases, we obtain that
\[
\left| \int_0^\delta \int_{\mathbb{T}^d} N(R_1, D_2, D_3) \mu_0 \, dx \, dt \right| \lesssim \delta^{\frac{s}{d}} \frac{N_2^{s(s-2)} N_3^{s(s-2)}}{N_1^{\frac{d}{2}} + N_1^{-\alpha - \epsilon}} \| u_0 \|_{U^s_{\alpha, d}} \| D_2 \|_{X^s} \| D_3 \|_{X^s}
\]
(5.35)

where
\[
s_\tau(d) = \begin{cases} \frac{1}{7} & d = 3 \\ \frac{4}{19} & d = 4 \\ \frac{1}{4} & d \geq 5. \end{cases}
\]

Since \( s < s_\epsilon + s_\tau(d) - \alpha \), we have the proposition. \( \square \)

Remark 5.11. The proofs when the conjugate is on the different position are similar, for example, \( N(R_1, D_2, D_3) \) in the case \( B(d) \). The only difference between \( N(R_1, D_2, D_3) \) and \( N(R_1, D_2, D_3) \) in \( B(d) \) is (5.28). In the \( N(R_1, D_2, D_3) \), the set in (5.28) should be \( \{(n, n', n_2) : n \neq n', 2(n_3 - n, n + n_2) = k, 2(n_3 - n', n_2 + n_2) = k \} \). First, we can count the number of pair \( (n, n_2) \) and by Lemma 4.4 it is bounded by \( N_1^{d-2+\epsilon} N_2^d \). And then the number of possible
$n'$ is bounded $N_1^{d-2+\epsilon}$ by Lemma 4.5. So the size of the upper lattice set is bounded by $N_1^{2(d-2)+\epsilon} N_2^d$, which is better than the corresponding bound in $\mathcal{N}(R_1, D_2, D_3)$.

Proof of Proposition 5.1

Proof. Suppose dyadic coordinates $N_1 \geq N_2 \geq N_3$, consider arbitrary $w_i^{(i)}$ and $u_i^{(0)}$ satisfying $w_i^{(i)}(t) = w^{(i)}(t)$ and $u_i^{(0)}(t) = u^{(0)}(t)$, for $t \in I$ and $i = 1, 2, 3$. Then we have

$$
\left| \int_0^\delta \int_{T^d} \mathcal{N}(w^{(1)} + v_0^\omega, w^{(2)} + v_0^\omega, w^{(3)} + v_0^\omega) u^{(0)} dx dt \right|
= \left| \int_0^\delta \int_{T^d} \mathcal{N}(w_i^{(1)} + v_0^\omega, w_i^{(2)} + v_0^\omega, w_i^{(3)} + v_0^\omega) u_i^{(0)} dx dt \right|
= \sum_{N_0 \leq N_1 \geq N_2 \geq N_3} \left| \int_0^\delta \int_{T^d} \mathcal{N}(P_{N_1}(w_i^{(1)} + v_0^\omega), P_{N_2}(w_i^{(2)} + v_0^\omega), P_{N_3}(w_i^{(3)} + v_0^\omega)) u_i^{(0)} dx dt \right|

$$

There are only two cases: $N_0 \sim N_1 \geq N_2 \geq N_3$ and $N_0 \lesssim N_1 \sim N_2 \geq N_3$.

By Proposition 5.3 - 5.11, we can always sum up for these two cases to obtain the following estimate:

$$
\left| \int_0^\delta \int_{T^d} \mathcal{N}(w^{(1)} + v_0^\omega, w^{(2)} + v_0^\omega, w^{(3)} + v_0^\omega) u^{(0)} dx dt \right|
\lesssim \| u_i^{(0)} \|_{Y^s} \left( \delta^{\min \{1, s-s_c\}} \| w_i^{(1)} \|_{X^s} \| w_i^{(2)} \|_{X^s} \| w_i^{(3)} \|_{X^s} + \delta^c \sum_{J \in J} \prod_{j \in J} \| w_i^{(j)} \|_{X^s} \right),
$$

by the definition of $X^s(I)$ and $Y^{-s}(I)$ in Definition 3.8 we obtain Proposition 5.1. \qed

6. Proof of the Theorem 1.1

To prove Theorem 1.1 (especially the case $s = s_c$), we should introduce two weaker norms $Z^s(I)$ and $Z^s(I)$-norm than $X^s(I)$-norm.

Definition 6.1.

$$
\| v \|_{Z^s(I)} := \sup_{J \subset I} \left( \sum_{N \in 2^\mathbb{Z}} N^{4s+2-d} \| P_N v \|_{L^4(T^d \times J)}^4 \right)^{\frac{1}{4}} \text{ and } \| v \|_{Z^s(I)} := \| v \|_{V^s(I)} \| v \|_{X^s(I)}.
$$

The following property show us that $Z^s(I)$ is a weaker norm than $X^s(I)$.

Proposition 6.2.

$$
\| v \|_{Z^s(I)} \lesssim \| v \|_{X^s(I)}.
$$

Proof. By the definition of $Z^s(I)$ and the following Strichartz type estimates (Proposition 1.2), we obtain that

$$
\sup_{J \subset I} \left( \sum_{N \text{ dyadic number}} N^{4s+2-d} \| P_N v \|_{L^4(T^d \times J)}^4 \right)^{\frac{1}{4}} \lesssim \left( \sum_{N \text{ dyadic number}} N^{4s} \| P_N v \|_{U^4_2}^4 \right)^{\frac{1}{4}} \lesssim \| v \|_{X^s(I)}.
$$

\qed
Lemma 6.3 (Bilinear estimates in \[30\]). Assuming \(|I| \leq 1\) and \(N_1 \geq N_2\), for any \(v_1 \in Y^0(I)\) and \(v_2 \in Y^{s_c}(I)\), where \(s_c = \frac{d}{2} - 1\), there holds that

\[
\begin{aligned}
\|P_{N_1}v_1P_{N_2}v_2\|_{L^2_{x,t}(\mathbb{T}^d \times I)} & \lesssim \left( \frac{N_2}{N_1} + \frac{1}{N_2} \right)^\kappa \|P_{N_1}v_1\|_{Y^0(I)}\|P_{N_2}v_2\|_{Y^{s_c}(I)}
\end{aligned}
\]

for some \(\kappa > 0\).

Remark 6.4. This Bilinear estimate is a simple d-dimension generalization of Proposition 2.8 in \[30\]. The proof of Lemma 6.3 is almost the same as the \(d = 4\) case in \[30\] and heavily rely on \(L^p\) estimates in Proposition 4.2 (for some \(p < 4\)). In the proof not only we need the decoupling properties for spatial frequency, but also we need further trip partitions to apply the decoupling properties for time frequency.

Let’s introduce an refined nonlinear estimate for \(s = s_c\) case, which is a d-dimension generalization of Lemma 3.2 in \[32\].

Proposition 6.5 (Refined nonlinear estimate). For \(v_k \in X^{s_c}(I)\), \(k = 1, 2, 3\), \(|I| \leq 1\), we hold the estimate

\[
\|I(N(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3))\|_{X^{s_c}(I)} \lesssim \sum_{\{i,j,k\} = \{1,2,3\}} \|v_i\|_{X^{s_c}(I)}\|v_j\|_{Z^{s_c}(I)}\|v_k\|_{Z^{s_c}(I)}
\]

where \(\tilde{v}_k = v_k\) or \(\tilde{v}_k = \overline{v}_k\) for \(k = 1, 2, 3\).

Proof. By Proposition 5.15 we suppose \(N_0, N_1, N_2, N_3\) are dyadic, and by the symmetry, we assume \(N_1 \geq N_2 \geq N_3\). Since it’s easy to check that \(N_2\) is simple to bound, we just need to show the case \(N_1\).

\[
\|I(N(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3))\|_{X^{s_c}(I)} \lesssim \sup_{\|u_0\|_{Y^{s_c}}} \left| \int_{\mathbb{T}^d \times I} \prod_{k=1}^3 \tilde{v}_k \, dxdt \right|
\]

\[
\leq \sup_{\|u_0\|_{Y^{s_c}}} \sum_{N_0, N_1 \geq N_2 \geq N_3} \left| \int_{\mathbb{T}^d \times I} \prod_{k=1}^3 P_{N_0}u_0 P_{N_k} \tilde{v}_k \, dxdt \right|
\]

Then we know \(N_1 \sim \max\{N_2, N_0\}\) by the spatial frequency orthogonality. There are two cases:

1. \(N_0 \sim N_1 \geq N_2 \geq N_3\);
2. \(N_0 \leq N_2 \sim N_1 \geq N_3\).

Case 1: \(N_0 \sim N_1 \geq N_2 \geq N_3\)

By Cauchy-Schwartz inequality and Lemma 6.3, we have that

\[
\left| \int_{\mathbb{T}^d} \prod_{k=1}^3 P_{N_0}u_0 P_{N_1} \tilde{v}_1 P_{N_2} \tilde{v}_2 P_{N_3} \tilde{v}_3 \, dxdt \right| \leq \|P_{N_0}u_0\|_{L^2_{x,t}}\|P_{N_1}v_1\|_{L^2_{x,t}}\|P_{N_2}v_2\|_{L^2_{x,t}}\|P_{N_3}v_3\|_{L^2_{x,t}}
\]

\[
\lesssim \left( \frac{N_0}{N_1} + \frac{1}{N_3} \right)^\kappa \left( \frac{N_2}{N_0} + \frac{1}{N_2} \right)^\kappa \|P_{N_0}u_0\|_{Y^0(I)}\|P_{N_1}v_1\|_{Y^0(I)}\|P_{N_2}v_2\|_{Y^{s_c}(I)}\|P_{N_3}v_3\|_{Y^{s_c}(I)}
\]

Assume \(\{C_j\}\) is a cube partition of size \(N_2\), and \(\{C_k\}\) is a cube partition of size \(N_3\). By \(\{\overline{P_{C_j}P_{N_0}u_0P_{N_2}v_2}\}_{j}\) and \(\{\overline{P_{C_k}P_{N_1}v_1P_{N_3}v_3}\}_{k}\) are both almost orthogonality, Proposition 4.2
and definition of $Z^{sc}$ norm, we obtain that

(6.4) \[
\left| \int P_{N_0}^* u_0 P_{N_1} \tilde{v}_1 P_{N_2} \tilde{v}_2 P_{N_3} \tilde{v}_3 dx dt \right| \leq \|P_{N_0} u_0 P_{N_2} v_2\|_{L^{s,c}_{x,t}} \|P_{N_1} v_1 P_{N_3} v_3\|_{L^{s,c}_{x,t}}
\]
\[
\lesssim \left( \sum_{C_j} \|P_{C_j} P_{N_0} u_0 P_{N_2} v_2\|_{L^{s,c}_{x,t}}^2 \right)^{1/2} \left( \sum_{C_k} \|P_{C_k} P_{N_1} v_1 P_{N_3} v_3\|_{L^{s,c}_{x,t}}^2 \right)^{1/2}
\]
\[
\lesssim \left( \sum_{C_j} \|P_{C_j} P_{N_0} u_0\|_{L^4_{x,t}}^2 \|P_{N_2} v_2\|_{L^{s,c}_{x,t}}^2 \right)^{1/2} \left( \sum_{C_k} \|P_{C_k} P_{N_1} v_1\|_{L^4_{x,t}}^2 \|P_{N_3} v_3\|_{L^{s,c}_{x,t}}^2 \right)^{1/2}
\]
\[
\lesssim \|P_{N_0} u_0\|_{Y^0(I)} \|P_{N_1} v_1\|_{Y^0(I)} \|P_{N_2} v_2\|_{Z^{sc}(I)} \|P_{N_3} v_3\|_{Z^{sc}(I)}.
\]

Interpolate (6.3) and (6.4), and $N_0 \sim N_1$, we have

(6.5) \[
\left| \int P_{N_0}^* u_0 P_{N_1} \tilde{v}_1 P_{N_2} \tilde{v}_2 P_{N_3} \tilde{v}_3 dx dt \right| \leq \left( \sum_{N_0 \sim N_1 \geq N_2 \geq N_3} \left( \frac{N_3}{N_1} + \frac{1}{N_3} \right)^{\kappa} \right) \|P_{N_0} u_0\|_{Y^0(I)} \|P_{N_1} v_1\|_{Y^0(I)} \|P_{N_2} v_2\|_{Z^{sc}(I)} \|P_{N_3} v_3\|_{Z^{sc}(I)}.
\]

Sum (6.5) over all $N_0 \sim N_1 \geq N_2 \geq N_3$,

\[
\sum_{N_0 \sim N_1 \geq N_2 \geq N_3} \left( \frac{N_3}{N_1} + \frac{1}{N_3} \right)^{\kappa} \|P_{N_0} u_0\|_{Y^0(I)} \|P_{N_1} v_1\|_{Y^0(I)} \|P_{N_2} v_2\|_{Z^{sc}(I)} \|P_{N_3} v_3\|_{Z^{sc}(I)} \lesssim \|u_0\|_{Y^0(I)} \|v_1\|_{Y^0(I)} \|v_2\|_{Z^{sc}(I)} \|v_3\|_{Z^{sc}(I)}.
\]

**Case 2:** $N_0 \leq N_2 \sim N_1 \geq N_3

Similar, we have

(6.6) \[
\left| \int P_{N_0}^* u_0 P_{N_1} \tilde{v}_1 P_{N_2} \tilde{v}_2 P_{N_3} \tilde{v}_3 dx dt \right| \leq \left( \sum_{N_0 \sim N_2 \sim N_1 \geq N_3} \left( \frac{N_3}{N_1} + \frac{1}{N_3} \right)^{\kappa} \right) \|P_{N_0} u_0\|_{Y^0(I)} \|P_{N_1} v_1\|_{Y^0(I)} \|P_{N_2} v_2\|_{X^{sc}(I)} \|P_{N_3} u_3\|_{X^{sc}(I)}.
\]

Similar with (6.3), we obtain that:

(6.7) \[
\left| \int P_{N_0}^* u_0 P_{N_1} \tilde{v}_1 P_{N_2} \tilde{v}_2 P_{N_3} \tilde{v}_3 dx dt \right| \lesssim \|P_{N_0} u_0\|_{Y^0(I)} \|P_{N_1} v_1\|_{Y^0(I)} \|P_{N_2} v_2\|_{Z^{sc}(I)} \|P_{N_3} v_3\|_{Z^{sc}(I)}.
\]

Interpolating (6.6) and (6.7), and summing over $N_0 \leq N_2 \sim N_1 \geq N_3$, we have

\[
\sum_{N_0 \leq N_2 \sim N_1 \geq N_3} \left| \int P_{N_0}^* u_0 P_{N_1} \tilde{v}_1 P_{N_2} \tilde{v}_2 P_{N_3} \tilde{v}_3 dx dt \right| \lesssim \|P_{N_0} u_0\|_{Y^0(I)} \|P_{N_1} v_1\|_{X^{sc}(I)} \|P_{N_2} v_2\|_{Z^{sc}(I)} \|P_{N_3} v_3\|_{Z^{sc}(I)}.
\]

Summarize these two cases, and similarly consider $N_1 \geq N_3 \geq N_2$, $N_2 \geq N_1 \geq N_3$, $N_2 \geq N_3 \geq N_1$, $N_3 \geq N_1 \geq N_2$, and $N_3 \geq N_2 \geq N_1$, then we can get the desired estimate (6.2).

\[\square\]

**Proof of Theorem 1.4** Suppose $d \geq 3$, $s_r(d)$ is defined as (1.5) and $0 \leq \alpha < s_r(d)$.

Consider the mapping

\[\Phi(w) = \mathcal{I}(\mathcal{N}(w + w_0^c)),\]

where \(\mathcal{N}(w + w_0^c)\) is the cubic NLS on tori.
Let $\Theta_k$ be given by Tao (Section 2.6 in [46]).

Reduced by Bourgain [2][3] in the context of Schrödinger and KdV equations. A nice summary

$I < k < s$ small enough, there exists $0 \leq a < s_c$

Consider the set

$$S = \{w \in X^{s_c}(I) : \|w\|_{X^{s_c}(I)} \leq 1\}.$$

where $I = [0, \delta]$ and $\delta$ is to be determined.

To show $\Phi$ is a contraction mapping in $S$. Using Proposition 3.15 Proposition 5.1 and choosing $\delta$ small enough, we obtain that

$$\|\Phi(w)\|_{X^{s_c}(I)} \lesssim \delta^c \min\{1, s_c, s\} (1 + \|w\|_{X^{s_c}(I)} + \|w\|_{X^{s_c}(I)}^2 + \|w\|_{X^{s_c}(I)}^3) \leq 1.$$

For any $w, v \in S$, using Proposition 3.15 Proposition 5.1 and choosing $\delta$ small enough, there exists $0 < k < 1$ such that

$$\|\Phi(w) - \Phi(v)\|_{X^{s_c}(I)} \lesssim \delta^c \min\{1, s_c, s\} (1 + \|w\|_{X^{s_c}(I)} + \|v\|_{X^{s_c}(I)} + \|w\|_{X^{s_c}(I)}^2 + \|v\|_{X^{s_c}(I)}^3) \|w - v\|_{X^{s_c}(I)} \leq k \|w - v\|_{X^{s_c}(I)}.$$

So $\Phi$ is a contraction mapping.

**Case 2:** $s = s_c$

Consider the set

$$S = \{w \in X^{s_c}(I) : \|w\|_{X^{s_c}(I)} \leq 1, \|w\|_{Z^{s_c}(I)} \leq a\}.$$

where $I = [0, \delta]$, $a$ and $\delta$ is to be determined.

To show $\Phi$ is a contraction mapping in $S$. By Proposition 3.15 Proposition 5.1 Proposition 6.5 choosing $\delta$ small enough, we obtain that

$$\|\Phi(w)\|_{X^{s_c}(I)} \lesssim \delta^c (1 + \|w\|_{X^{s_c}(I)} + \|w\|_{Z^{s_c}(I)}^2 + \|w\|_{X^{s_c}(I)}^2) \lesssim \delta^c + a^2.$$

and also

$$\|\Phi(w)\|_{Z^{s_c}(I)} \lesssim \delta^c (1 + \|w\|_{X^{s_c}(I)} + \|w\|_{Z^{s_c}(I)}^2 + \|w\|_{X^{s_c}(I)}^2 \lesssim \delta^c + a^2.$$

For any $w, v \in S$, by Proposition 3.15 Proposition 5.1 and Proposition 6.5 choosing $\delta$ small enough, there exists $0 < k < 1$ such that

$$\|\Phi(w) - \Phi(v)\|_{X^{s_c}(I)} \lesssim (\|w\|_{Z^{s_c}(I)} + \|v\|_{Z^{s_c}(I)})(\|w\|_{X^{s_c}(I)} + \|v\|_{X^{s_c}(I)})(\|w - v\|_{X^{s_c}(I)} + \delta^c (\|w\|_{X^{s_c}(I)} + \|v\|_{X^{s_c}(I)} + 1) \|w - v\|_{X^{s_c}(I)} \lesssim (a + \delta^c) \|w - v\|_{X^{s_c}(I)}.$$

Set $a = \delta$ and let $\delta$ small enough, then we obtain that $\Phi$ is a contraction mapping.

\[ \square \]

7. **The analog result in $X^{s,b}$ space**

$X^{s,b}$ spaces (also known as *Fourier restriction spaces* or *Bourgain spaces*) were firstly introduced by Bourgain [2][3] in the context of Schrödinger and KdV equations. A nice summary is give by Tao (Section 2.6 in [40]).

**Theorem 7.1** (Analog of Theorem 1.2 in $X^{s,b}$). Suppose $d \geq 3$ and $s_r(d)$ is defined by (1.10)

Let $0 \leq \alpha < s_r(d)$, $s \in (s_c, s_c + s_r(d) - \alpha)$. Then there exist some $b > \frac{1}{2}, \delta_0 > 0$ and $r = r(s, \alpha) > 0$ such that for any $0 < \delta < \delta_0$, there exists $\Omega_{\delta} \in A$ with

$$\mathbb{P}(\Omega_{\delta}) < e^{-\frac{1}{2}},$$

and for each $\omega \in \Omega_{\delta}$ there exists a unique solution $u$ of (1.1) in the space

$$S(t) \phi^\omega + X^{s,b}(0, \delta_{\text{dist}}),$$

where $S(t) \phi^\omega$ is the linear evolution of the initial data $\phi^\omega$ given by (1.2).
Definition 7.3. Suppose $d \in \mathbb{Z}^+$ and $s, b \in \mathbb{R}$, for any $u : \mathbb{R} \times \mathbb{T}^d \to \mathbb{C}$, $u \in X^{s, d} (\mathbb{R} \times \mathbb{T}^d)$ (short for $\hat{X}^{s, b}$ in Proposition 7.3) if
\[
\|u\|_{X^{s, b}} := \|\langle n \rangle^s \lambda + |n|^b \hat{u}(n, \lambda)\|_{L^2(\mathbb{Z}^d)} < +\infty,
\]
where $\hat{u}(n, \lambda)$ is the space-time Fourier transformation of $u$. Note that $\|u\|_{X^{s, b}}$ can be also defined as $\|e^{-it\Delta} u\|_{H^b_{L^2} (\mathbb{R}^d)}$.

Definition 7.4 (The corresponding restriction spaces to a time interval $I$). Suppose $d \in \mathbb{Z}^+$, $s, b \in \mathbb{R}$ and $I$ is a time interval in $\mathbb{R}$, for any $u : I \times \mathbb{T}^d \to \mathbb{C}$, then $u \in X^{s, b} (I \times \mathbb{T}^d)$ (short for $X^{s, b} (I)$) if
\[
\|u\|_{X^{s, b} (I)} := \inf_{v \in X^{s, b} \cap C^\infty (I \times \mathbb{T}^d)} \{\|v\|_{X^{s, b} : v(t) = u(t) \text{ for all } t \in I\} < +\infty.
\]

Remark 7.5 (Some embedding properties). For $s \leq s'$ and $b \leq b'$, we obtain that
\[
X^{s, b} \hookrightarrow X^{s', b'}.
\]
Furthermore, if $b > \frac{1}{2}$, then
\[
X^{s, b} \hookrightarrow L^\infty_t (\mathbb{R}, H^s (\mathbb{T}^d)).
\]

Proposition 7.6 (Analog of Proposition 3.9 in $X^{s, b}$). Let $T_0 : L^2_x \times \cdots \times L^2_x \to L^1_{x, loc} (\mathbb{T}^d)$ be an $m$-linear operator. Assume that for some $1 \leq p \leq \infty$
\[
(7.1)
\|T_0 (e^{it\phi_1} \cdots, e^{it\phi_m})\|_{L^p (\mathbb{R} \times \mathbb{T}^d)} \leq \prod_{i=1}^m \|\phi_i\|_{L^2 (\mathbb{T}^d)}.
\]

Then, for any $b > \frac{1}{2}$, there exists an extension $T : X^{0, b} \times \cdots \times X^{0, b} \to L^p (\mathbb{R} \times \mathbb{T}^d)$ satisfying
\[
(7.2)
\|T (u_1, \cdots, u_m)\|_{L^p (\mathbb{R} \times \mathbb{T}^d)} \leq \prod_{i=1}^m \|u_i\|_{X^{0, b}};
\]
and such that $T(u_1, \cdots, u_m)(t, \cdot) = T_0 (u_1(t), \cdots, u_m(t)) (\cdot)$, a.e.

Proof. Suppose that for all $i = 1, \cdots, m,$
\[
(7.3)
\int d\lambda_i \sum_{n_i \in \mathbb{Z}^d} \hat{u}_i(n_i, \lambda_i) e^{i n_i \cdot x + \lambda_i t} = \int \phi^{(i)}_{\mu_i} d\mu_i,
\]
where $\phi^{(i)}_{\mu_i} := \sum_{n_i \in \mathbb{Z}^d} \hat{u}_i(n_i, \mu_i - |n_i|^2) e^{\mu_i t} e^{i n_i \cdot x}$ and $\mu_i = \lambda_i + |n_i|^2$.

Then, by (7.3), Minkowski integral inequality and (7.1), we obtain that
\[
(7.4)
\|T (u_1, \cdots, u_m)\|_{L^p (\mathbb{R} \times \mathbb{T}^d)} = \|T (\int \phi^{(1)}_{\mu_1} d\mu_1, \cdots, \int \phi^{(m)}_{\mu_m} d\mu_m)\|_{L^p (\mathbb{R} \times \mathbb{T}^d)}
\]
For a fixed $i$ and any $b > \frac{1}{2}$, by Hölder inequality and $\int_{\mathbb{R}} \frac{1}{\|\mu_i\|^2} \, d\mu < +\infty$, we have that

$$\int_{\mathbb{R}} \|\phi(x)\|_{L^2_x(T^d)} \, d\mu_i = \int_{\mathbb{R}} \frac{1}{\|\mu_i\|^2} \|\mu_i\|^b \|\phi(x)\|_{L^2_x(T^d)} \, d\mu_i \leq \left( \int_{\mathbb{R}} \frac{1}{\|\mu_i\|^2b} \, d\mu_i \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} \|\mu_i\|^b \|\phi(x)\|_{L^2_x(T^d)} \, d\mu_i \right)^{\frac{1}{2}} = \left( \int_{\mathbb{R}} \sum_{n_i} \|\lambda_i + |n_i|^2b \|\tilde{u}_i(n_i, \lambda_i)\|^2 \, d\lambda \right)^{\frac{1}{2}} \leq \|u_i\|_{X^{0,b}}. \quad (7.5)$$

By (7.4) and (7.5), we obtain the proposition. \hfill \Box

Following Proposition 7.6 and Proposition 7.7, we obtain the following corollary.

**Corollary 7.7** (Analog of Proposition 4.2). Let $b > \frac{1}{2}$ and $p > p_c$, where $p_c = \frac{2(d+2)}{d}$. For all $N \geq 1$ we have

$$\|P_\omega u\|_{L^p_{x,t}(T^d)} \leq N^{\frac{d+2-2p}{2p}} \|P_C u\|_{X^{0,b}}, \quad (7.6)$$

$$\|P_C u\|_{L^p_{x,t}(T^d)} \leq N^{\frac{d+2-2p}{2p}} \|P_C u\|_{X^{0,b}}, \quad (7.7)$$

where $C$ is a cube in $\mathbb{Z}^d$ with sides parallel to the axis of side length $N$.

**Proposition 7.8** ($X^{s,b}$ norm of Duhamel’s formula). For any $s > \frac{1}{2}$, $s \in \mathbb{R}$, and $I$ is a time interval $[0, \delta]$, we obtain that

$$\|I(f)\|_{X^{s,b}(I)} \leq \sup_{v \in \tilde{X}^{s,1-b}(I)} \left| \int_0^\delta \int_{T^d} f(t, x) \overline{v(t, x)} \, dx \, dt \right|. \quad (7.8)$$

where $\tilde{X}^{s,1-b}(I) := \{e^{it\Delta} v \mid H^{1-b}_x H^{s}_x(T^d)\}$

### 7.2 Nonlinear estimate in $X^{s,b}$

**Proposition 7.9** (Analog of Proposition 5.1 in $X^{s,b}$). Suppose $d \geq 3$ and $s_\omega(d)$ is given in (1.3). Let $0 \leq \alpha < s_\omega(d)$, $s \in (s_\omega, s_\omega + s_\omega'(d) - \alpha)$, $r > 0$, $0 \leq \delta < 1$, and $I = [0, \delta]$. There exist some $b > \frac{1}{2}$, $\Omega_\delta \subset \Omega$ with $P(\Omega_\delta^c) < e^{-1/\delta^c}$ and $c > 0$, such that we obtain that

$$\left| \int_0^\delta \int_{T^d} \mathcal{N}(w^{(1)} + v^{(\omega)}_0, w^{(2)} + v^{(\omega)}_0, w^{(3)} + v^{(\omega)}_0) \, dx \, dt \right| \leq \|u^{(0)}\|_{\tilde{X}^{s,1-b}(I)} \left( \delta^{c \min \{1, s - s_c\}} \|w^{(1)}\|_{X^{s,b}(I)} \|w^{(2)}\|_{X^{s,b}(I)} \|w^{(3)}\|_{X^{s,b}(I)} + \delta^c \sum_{S_j \subset \{1, 2, 3\}} \prod_{j \in S_j} \|w^{(j)}\|_{X^{s,b}(I)} \right), \quad (7.9)$$

where $v^{(\omega)}_0$ is defined in (1.9), $w^{(0)} \in \tilde{X}^{s,1-b}(I)$ and $w^{(i)} \in X^{s,b}(I)$ for $i = 1, 2, 3$. (when the subset $S_j = \emptyset$, $\prod_{j \in S_j} \|w^{(j)}\|_{X^{s,b}(I)} = 1$.)

**Proof.** The proof of Proposition 7.9 is similar with the proof of Proposition 5.1, we first dyadically decompose the terms in each position of the nonlinear term $\mathcal{N}(u_1, u_2, u_3)$, and then we have the same cases: Denote

$$R_i = P_{N_i} v^{(\omega)}_0 \quad \text{and} \quad D_i = P_{N_i} w \quad \text{for } i \in \{1, 2, 3\}.$$ 

The list of all cases of $(u_1, u_2, u_3)$ is below:

A. $u_1 = D_1$:

(a) $(D_1, D_2, D_3)$;

(b) $(D_1, D_2, R_3)$;
(c) \((D_1, R_2, D_3)\);
(d) \((D_1, R_2, R_3)\);

B. \(u_1 = R_1\):
(a) \((R_1, R_2, R_3)\);
(b) \((R_1, R_2, D_3)\);
(c) \((R_1, D_2, R_3)\);
(d) \((R_1, D_2, D_3)\).

For the Case A, we can control the nonlinear terms as Proposition 5.3 - 5.6 in a similar approach. For simplicity, let me just show the Case A (a) as an example.

Choose \(b = \frac{1}{2} + \) which is close to \(\frac{1}{2} \) enough. We follow the proof of Proposition 5.3 almost identically, but Proposition \(7.7\) take place of Proposition \(4.2\) Then we obtain that

\[
\left| \int_0^\delta \int_T^d \mathcal{N}(\vec{D}_1, \vec{D}_2, \vec{D}_3)\vec{p}_0 \, dxdt \right| \\
\lesssim \delta^{\min\{1, s-s_c\}} (N_3 \min \{N_0, N_2\})^\epsilon \frac{1}{N_2^s-s_c} \|u_0\|_{X^{s,b}} \|D_1\|_{X^{s,b}} \|D_2\|_{X^{s,b}} \|D_3\|_{X^{s,b}}.
\]

To get the \(\|u_0\|_{X^{s,1-b}}\) instead of \(\|u_0\|_{X^{s,b}}\), we need another nonlinear estimate. By the same cube decomposition, Hölder inequality, and Proposition 7.7 we obtain that (here we only consider the main part of \(\mathcal{N}(\vec{D}_1, \vec{D}_2, \vec{D}_3)\), and the remaining part is easily bounded)

\[
\left| \int_0^\delta \int_T^d \vec{p}_0 \vec{D}_1 \vec{D}_2 \vec{D}_3 \, dxdt \right| \\
\leq \sum_{C_j} \int_0^\delta \int_T^d \left| (P_{C_j} \vec{p}_0)(P_{C_k} \vec{D}_1) \vec{D}_2 \vec{D}_3 \right| \, dxdt \\
\lesssim \sum_{C_j} \|P_{C_j} u_0\|_{L_{t,x}^s} \|P_{C_k} D_1\|_{L_{t,x}^s} \|D_2\|_{L_{t,x}^s} \|D_3\|_{L_{t,x}^s} \\
\lesssim \sum_{C_j} N_2^{s_c} N_3^{s_c+1} \|P_{C_j} u_0\|_{L_{t,x}} \|P_{C_k} D_1\|_{L_{t,x}^s} \|D_2\|_{L_{t,x}^s} \|D_3\|_{L_{t,x}^s} \\
\lesssim \frac{N_3^{1-(s-s_c)}}{N_2^{s-s_c}} \|u_0\|_{X^{s,0}} \|D_1\|_{X^{s,b}} \|D_2\|_{X^{s,b}} \|D_3\|_{X^{s,b}}.
\]

Using complex interpolation method from \(X^{s,b}\) and \(X^{s,0}\) to \(X^{s,1-b}\) and interpolating \(7.10\) and \(7.11\) (actually we don’t interpolate \(7.10\) and \(7.11\) directly but interpolate two estimates in the process of \(7.10\) and \(7.11\)), we obtain that

\[
\left| \int_0^\delta \int_T^d \mathcal{N}(\vec{D}_1, \vec{D}_2, \vec{D}_3)\vec{p}_0 \, dxdt \right| \\
\lesssim \delta^{\min\{1, s-s_c\}} (N_3 \min \{N_0, N_2\})^\epsilon \frac{N_3^5}{N_2^{s-s_c}} \|u_0\|_{X^{s,1-b}} \|D_1\|_{X^{s,b}} \|D_2\|_{X^{s,b}} \|D_3\|_{X^{s,b}}.
\]

Observe the bound in \(7.12\), to sum up over \(N_2\) and \(N_3\), we need \(s > s_c + \epsilon\), which is the reason why we can’t obtain \(s = s_c\) case in \(X^{s,b}\) space.

For the Case B, we could obtain analogs of Proposition 5.7-5.10 by modifying the proofs a little bit. Let me show the Case B (d) as an example. Similar with the proof of Proposition 5.10, we only focus \(N_0 \sim N_1 \geq N_2 \geq N_3\). Then

\[
\left| \int_0^\delta \int_T^d \mathcal{N}(\vec{R}_1, \vec{D}_2, \vec{D}_3)\vec{p}_0 \, dxdt \right| \leq \|R_1\|_{L_{t,x}^{s}} \|D_2\|_{L_{t,x}^s} \|D_3\|_{L_{t,x}^s} \|u_0\|_{L_{t,x}^s}.
\]
By Hausdorff-Young inequality w.r.p.t the time $t$ and Hölder inequality, we obtain that for a general function $u$ and dyadic number $N,$

$$
\|P_N u\|_{L^3_{t,x}} \lesssim \sum_{|n| \sim N} \|e^{ix\cdot n} \int \tilde{u}(n, \lambda) e^{\lambda t} d\lambda\|_{L^3_{t,x}}
$$

(7.14)

$$\lesssim \sum_{|n| \sim N} \left( \int |\tilde{u}(n, \lambda)|^2 d\lambda \right)^{\frac{3}{4}} \lesssim \|P_N u\|_{X^{\frac{1}{3}, \frac{3}{1} + \epsilon}}.
$$

By Corollary 2.3 and (7.14), we obtain that

$$
\text{LHS of (7.13)} \lesssim \log N_1 \left( \frac{N^s_0}{N_1^{1 - \alpha}} \right) \|D_2\|_{X^{\frac{1}{3}, \frac{3}{1} + \epsilon}} \|D_3\|_{X^{\frac{1}{3}, \frac{3}{1} + \epsilon}} \|u_0\|_{X^{\frac{1}{3}, \frac{3}{1} + \epsilon}}
$$

(7.15)

$$\lesssim \frac{N^s_0}{N_1 N_3^{\frac{3}{8}}} \|D_2\|_{X^{\frac{1}{3}, \frac{3}{1} + \epsilon}} \|D_3\|_{X^{\frac{1}{3}, \frac{3}{1} + \epsilon}} \|u_0\|_{X^{-\frac{1}{2}, \frac{1}{2} + \epsilon}}
$$

Thus the estimate (7.15) is conclusive provided for some deterministic term, we consider the contribution $\tilde{u}_0 |(\lambda + |n|^2)| > N_1^{1-\epsilon}$. Thus in this case, LHS of (7.13) may be estimated assuming

$$
(\lambda + |n|^2) \leq N_1^{4(s - s_c + \alpha)}.
$$

For $s - s_c + \alpha < \frac{1}{3}$, replacing Proposition 5.9 and 5.10 by Proposition 7.5, we could recover the proof of Proposition 5.10 and obtain an analog of (5.35),

$$
\left| \int_0^\delta \int T^{d} N(\mathbf{T}, D_1, D_2, D_3) \pi_0 dx dt \right|
$$

(7.16)

$$\lesssim \left( d_e(d) - \epsilon \right) \frac{N^s_0 - s}{N_1^{s - s_c + \epsilon} (d) + \frac{1}{4} - \alpha - \epsilon} \|u_0\|_{X^{-\frac{1}{2}, \frac{1}{2} + \epsilon}} \|D_2\|_{X^{1, \frac{1}{2} + \epsilon}} \|D_3\|_{X^{1, \frac{1}{2} + \epsilon}}
$$

$$\lesssim \left( d_e(d) - \epsilon \right) \frac{N^s_0 - s}{N_1^{s - s_c + \epsilon} (d) + \frac{1}{4} - \alpha - \epsilon} \|u_0\|_{X^{-\frac{1}{2}, \frac{1}{2} + \epsilon}} \|D_2\|_{X^{1, \frac{1}{2} + \epsilon}} \|D_3\|_{X^{1, \frac{1}{2} + \epsilon}},
$$

where $s_e(d)$ is defined by (1.5), since $s < s_c + s_e(d) - \alpha,$ we have the proposition.

In a similar idea, we can also recover the other cases in $X^{s,b}.$

□

References

1. Árpád Bényi, Tadahiro Oh, and Oana Pocovnicu, On the probabilistic Cauchy theory of the cubic nonlinear Schrödinger equation on $\mathbb{R}^d,$ $d > 3,$ Trans. Amer. Math. Soc. Ser. B 2 (2015), 1–50. MR 3350022

2. J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations, Geom. Funct. Anal. 3 (1993), no. 2, 107–156. MR 1209299

3. , Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. II. The KdV-equation, Geom. Funct. Anal. 3 (1993), no. 3, 209–262. MR 1215780

4. , Global wellposedness of defocusing critical nonlinear Schrödinger equation in the radial case, J. Amer. Math. Soc. 12 (1999), no. 1, 145–171. MR 1626257

5. Jean Bourgain, Periodic nonlinear Schrödinger equation and invariant measures, Comm. Math. Phys. 166 (1994), no. 1, 1–26.

6. , Invariance of the blowup profile for the 2D-defocusing nonlinear Schrödinger equation, Comm. Math. Phys. 176 (1996), no. 2, 421–445. MR 1374429

7. Jean Bourgain and Ayman B calcul, Almost sure global well posedness for the radial nonlinear Schrödinger equation on the unit ball I: the 2D case, Ann. Inst. H. Poincaré Anal. Non Linéaire 31 (2014), no. 6, 1267–1288. MR 3280067

8. , Almost sure global well posedness for the radial nonlinear Schrödinger equation on the unit ball II: the 3d case, J. Eur. Math. Soc. (JEMS) 16 (2014), no. 6, 1289–1325. MR 3226743

9. , Invariant Gibbs measure evolution for the radial nonlinear wave equation on the 3d ball, J. Funct. Anal. 266 (2014), no. 4, 2319–2340. MR 3150162
The wave data remarks the of LA with cubic scattering the Random critical well-posedness and energy-critical with the for critical, equation for I. Schrödinger nonlinear low nonlinear sure sure nonlinearity, Discrete Contin. Dyn. Syst. Global Ill-posedness the, canonical data, arXiv preprint arXiv:1703.09655 (2017).

Global Strichartz in nonlinear Global mass-subcritical energy Well-posedness 4D scattering defocusing, H On Schrödinger and Random Scale radial Schrödinger energy-critical of scattering scattering proof random with wave nonlinear in partially equations, Amer. J. Math. for scattering Almost to conjecture, Ann. of Math. (2) KP-II for space, Ann. Inst. H. Poincaré Anal. Non Linéaire Local theory equation, J. Eur. Math. Soc. (JEMS) Almost and NLS the well-posedness Asymptotics, l data, Anal. PDE in data R global nonlinear for nonlinear Cauchy with estimates Probabilistic energy-critical for of radial case, New methods and results in nonlinear field equations (Bielefeld, 1987), Lecture Notes in energy the nonlinear frequency nonlinearity, the energy-critical decoupling theory a for sure Global data and scattering well-posedness result, Invent. data nls the Schrödinger Random-data applications, Math. Res.

30. Sebastian Herr, Daniel Tataru, and Nikolay Tzvetkov, Almost sure well-posedness of the cubic NLS on tori 37
31. Hiroyuki Hirayama and Mamoru Okamoto, Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in $\mathbb{R}^3$, Ann. of Math. (2) 167 (2008), no. 3, 767–865. MR 2415387
32. Alexandru D. Ionescu and Benoît Pausader, The proof of the $l^2$ decoupling conjecture, Ann. of Math. (2) 182 (2015), no. 1, 351–389. MR 3374964
33. Rowan Killip, Jason Murphy, and Monica Vișan, Ill-posedness for nonlinear schrödinger and wave equations, arXiv preprint math/0311048 (2003).
34. Rowan Killip and Monica Vișan,
35. Jonas Lührmann and Dana Mendelson, Almost sure well-posedness of the cubic nonlinear Schrödinger equation below $L^2(\mathbb{T})$, Duke Math. J. 161 (2012), no. 3, 367–414. MR 2881226
36. Martin Hadac, Sebastian Herr, and Herbert Koch, Well-posedness and scattering for the KP-II equation in a critical space, Ann. Inst. H. Poincaré Anal. Non Linéaire 26 (2009), no. 3, 917–941. MR 2526409
37. Jonas Lührmann and Dana Mendelson, Random data Cauchy theory for nonlinear wave equations of power-type on $\mathbb{R}^3$, Comm. Partial Differential Equations 39 (2014), no. 12, 2262–2283. MR 3259556
38. Rowan Killip and Monica Vișan, Scale invariant Strichartz estimates on tori and applications, Math. Res. Lett. 23 (2016), no. 2, 445–472. MR 3512894
39. Jonas Lührmann and Dana Mendelson, Random data Cauchy theory for nonlinear wave equations of power-type on $\mathbb{R}^3$, Comm. Partial Differential Equations 39 (2014), no. 12, 2262–2283. MR 3259556
40. On the almost sure global well-posedness of energy sub-critical nonlinear wave equations on $\mathbb{R}^3$, New York J. Math. 22 (2016), 209–227. MR 3484682
41. Jason Murphy, Random data final-state problem for the mass-subcritical nls in $L^2$. arXiv preprint arXiv:1703.09849 (2017).
38. Andrea R. Nahmod, Tadahiro Oh, Luc Rey-Bellet, and Gigliola Staffilani, Invariant weighted Wiener measures and almost sure global well-posedness for the periodic derivative NLS, J. Eur. Math. Soc. (JEMS) 14 (2012), no. 4, 1275–1330. MR 2928851

39. Andrea R. Nahmod, Nataša Pavlović, and Gigliola Staffilani, Almost sure existence of global weak solutions for supercritical Navier-Stokes equations, SIAM J. Math. Anal. 45 (2013), no. 6, 3431–3452. MR 3131480

40. Andrea R. Nahmod and Gigliola Staffilani, Almost sure well-posedness for the periodic 3D quintic nonlinear Schrödinger equation below the energy space, J. Eur. Math. Soc. (JEMS) 17 (2015), no. 7, 1687–1759. MR 3361727

41. Tadahiro Oh, Mamoru Okamoto, and Oana Pocovnicu, On the probabilistic well-posedness of the nonlinear Schrödinger equations with non-algebraic nonlinearities, arXiv preprint arXiv:1708.01568 (2017).

42. Tadahiro Oh and Oana Pocovnicu, Probabilistic global well-posedness of the energy-critical defocusing quintic nonlinear wave equation on $\mathbb{R}^3$, J. Math. Pures Appl. (9) 105 (2016), no. 3, 342–366. MR 3465807

43. Oana Pocovnicu, Almost sure global well-posedness for the energy-critical defocusing nonlinear wave equation on $\mathbb{R}^d$, $d = 4$ and 5, J. Eur. Math. Soc. (JEMS) 19 (2017), no. 8, 2521–2575. MR 3668066

44. E. Ryckman and M. Vişan, Global well-posedness and scattering for the defocusing energy-critical nonlinear Schrödinger equation in $\mathbb{R}^{1+4}$, Amer. J. Math. 129 (2007), no. 1, 1–60. MR 2288737

45. Chenmin Sun and Bo Xia, Probabilistic well-posedness for supercritical wave equations with periodic boundary condition on dimension three, Illinois J. Math. 60 (2016), no. 2, 481–503. MR 3680544

46. Terence Tao, Nonlinear dispersive equations, CBMS Regional Conference Series in Mathematics, vol. 106, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006, Local and global analysis. MR 2233925

47. Jingrui Wang and Keyan Wang, Almost sure existence of global weak solutions to the 3d incompressible navier-stokes equation, Discrete and Continuous Dynamical Systems - A 37 (2017), 5003.

48. Ting Zhang and Daoyuan Fang, Random data Cauchy theory for the generalized incompressible Navier-Stokes equations, J. Math. Fluid Mech. 14 (2012), no. 2, 311–324. MR 2925111

49. Sijia Zhong, The Cauchy problem of null form wave equation on $\mathbb{T}^d$ with random initial data, Funkcial. Ekvac. 55 (2012), no. 3, 367–403. MR 3052716

Department of Mathematics and Statistics, University of Massachusetts Amherst

E-mail address: hyue@math.umass.edu