A theta expression of the Hilbert modular functions for $\sqrt{5}$ via the periods of $K3$ surfaces

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Abstract

In this paper, we give an extension of the classical story of the elliptic modular function to the Hilbert modular case for $\mathbb{Q}(\sqrt{5})$. We construct the period mapping for a family $\mathcal{F} = \{S(X,Y)\}$ of $K3$ surfaces with 2 complex parameters $X$ and $Y$. The inverse correspondence of the period mapping gives a system of generators of Hilbert modular functions for $\mathbb{Q}(\sqrt{5})$. Moreover, we show an explicit expression of this inverse correspondence by theta constants.

Introduction

The symmetric Hilbert modular surface $(\mathbb{H} \times \mathbb{H})/(PSL(2, \mathcal{O}_K), \tau)$, where $\mathcal{O}_K$ is the ring of integers in a real quadratic field $K$ and $\tau$ exchanges the factors of $\mathbb{H} \times \mathbb{H}$, gives the moduli space for the family $\mathcal{F}_K = \{A\}$ of the principally polarized Abelian surfaces with an extra endomorphism structure $K(\subset \text{End}^0(A))$.

In the classical theory, the elliptic modular function $\lambda(z)$ on the moduli space $\mathbb{H}/\Gamma(2)$ is given by the inverse of the multivalued period mapping for a family of elliptic curves. This period mapping gives the Schwarz mapping of the Gauss hypergeometric differential equation $E\left(\frac{1}{2}, \frac{1}{2}, 1\right)$. It is important that the modular function $\lambda(z)$ has an explicit expression given by the Jacobi theta constants.

For the Hilbert modular cases, although there are various studies on the structure of the field of modular functions and the ring of modular forms (for example Gundlach [Gu], Hirzebruch [H] and Müller [M]), still now, to the best of the author’s knowledge, there has not appeared an explicit expression of Hilbert modular functions as an inverse correspondence of the period mapping for a family of algebraic varieties. In this paper, we give an extension of the above classical story to the Hilbert modular functions for $K = \mathbb{Q}(\sqrt{5})$ by using a family of $K3$ surfaces that gives the same variation of Hodge structures of weight 2 with the family $\mathcal{F}_K$ of Abelian surfaces. Namely, we show that the inverse of the period mapping for our family of $K3$ surfaces gives Hilbert modular functions for $\mathbb{Q}(\sqrt{5})$. Moreover, we obtain an explicit theta expression of this inverse correspondence. As our method, we use the fact that our period integrals of $K3$ surfaces satisfy a system of partial differential equations determined in [N1].

Our results is obtained as a combined work with [N1] and based on the results of Hirzebruch [H] and Müller [M] also.

In this paper, we consider the family $\mathcal{F} = \{S(X,Y)\}$ of $K3$ surfaces with 2 complex parameters given by an affine equation in $(x,y,z)$-space:

$$S(X,Y) : z^2 = x^3 - 4y^2(4y - 5)x^2 + 20Xy^3x + Yy^4.$$ 

We show that a system of generators of the field of the Hilbert modular functions for $\mathbb{Q}(\sqrt{5})$ is given by the inverse of the period mapping for $\mathcal{F}$ and obtain an explicit expression of these Hilbert modular functions given by theta constants.

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Running head: Hilbert modular functions via $K3$ surfaces
We use the following results of the Hilbert modular functions for \( \mathbb{Q}(\sqrt{5}) \). Hirzebruch [H] studied the Hilbert modular orbifold \((\mathbb{H} \times \mathbb{H})/(PSL(2, \mathcal{O}), \tau)\), where \( \mathcal{O} = \mathbb{Z} + \frac{1 + \sqrt{5}}{2} \mathbb{Z} \) and \( \tau \) is an involution of \( \mathbb{H} \times \mathbb{H} \), by an algebrogeometric method. He determined the structure of the ring of the symmetric Hilbert modular forms. This ring is isomorphic to the Klein icosahedral ring \( \mathbb{C} \times \mathbb{H} \langle \Delta \rangle \) and \( \mathbb{M} \). Namely, we obtain an expression of the parameters \( X \) and \( Y \) of the symmetric Hilbert modular forms for \( \mathbb{Q}(\sqrt{5}) \) and found the relation \( M(g_2, s_6, s_{10}, s_{15}) = 0 \). These generators are given by the theta constants. Then, they are holomorphic functions on \( \mathbb{H} \times \mathbb{H} \).

We show that the period mapping for \( \mathcal{F} \) gives a biholomorphic correspondence between the monodromy covering of \((X, Y)\)-space and \( \mathbb{H} \times \mathbb{H} \) and the projective monodromy group coincides with the extended Hilbert modular group \((PSL(2, \mathcal{O}), \tau)\). Then, the quotient space \((\mathbb{H} \times \mathbb{H})/(PSL(2, \mathcal{O}), \tau)\) becomes to be the classifying space of the family \( \mathcal{F} \). Consequently, we may regard \( X \) and \( Y \) as Hilbert modular functions for \( \mathbb{Q}(\sqrt{5}) \). This framework enable us to obtain explicit relations between the result of \([H] \) and \([M] \). Namely, we obtain an expression of the parameters \( X \) and \( Y \) as quotients of theta constants by use of the period mapping for our family \( \mathcal{F} \) of \( K3 \) surfaces.

In Section 1, we give a survey of the results of \([M1] \) and the properties of the Hilbert modular orbifold for \( \mathbb{Q}(\sqrt{5}) \). Especially, we recall the family \( \mathcal{F}_0 \) of \( K3 \) surfaces and the period differential equation \((1.14) \) for \( \mathcal{F}_0 \). A generic member of \( \mathcal{F}_0 \) is transformed to \( S(X, Y) \in \mathcal{F} \). The system \((1.14) \) turns out to be the period differential equation for \( \mathcal{F} \), that gives an analogy of the Gauss hypergeometric equation \( 2E_1 \left( \frac{1}{2}, \frac{1}{2}, 1 \right) \).

In Section 2, we study the \( K3 \) surface \( S(X, Y) \). First, we obtain the weighted projective space \( \mathbb{P}(1, 3, 5) \) as a compactification of the \((X, Y)\)-space \( \mathbb{C}^2 \). This remains a parameter space for \( K3 \) surfaces except one point (Theorem \([2.1] \)). We note that, due to \([H] \) together with Klein \([K] \), the orbifold \((\mathbb{H} \times \mathbb{H})/(PSL(2, \mathcal{O}), \tau)\) is isomorphic to \( \mathbb{P}(1, 3, 5) \) as algebraic varieties. Secondly, we define the multi-valued period mapping \( \mathbb{P}(1, 3, 5) - \{ \text{one point} \} \to \mathcal{D} \) for \( \mathcal{F} \), where \( \mathcal{D} \) is a symmetric Hermitian space of type \( IV \). We have a modular isomorphism between \( \mathbb{H} \times \mathbb{H} \) and a connected component \( \mathcal{D}_+ \) of \( \mathcal{D} \). Our period mapping gives an explicit isomorphism between \( \mathbb{P}(1, 3, 5) \) and \((\mathbb{H} \times \mathbb{H})/(PSL(2, \mathcal{O}), \tau)\). Then, we obtain the coordinates of \( \mathbb{H} \times \mathbb{H} \) given by the quotients of period integrals of \((X, Y)\):

\[
(z_1(X, Y), z_2(X, Y)) = \begin{pmatrix}
- \frac{1}{\Gamma_3} \omega + \frac{1 - \sqrt{5}}{2} \frac{1}{\Gamma_1} \omega \\
- \frac{1}{\Gamma_2} \omega - \frac{1 + \sqrt{5}}{2} \frac{1}{\Gamma_1} \omega
\end{pmatrix},
\]

(0.1)

where \( \Gamma_1, \cdots, \Gamma_4 \) are 2-cycles on \( S(X, Y) \in \mathcal{F} \) given in Section 2.2.

Then, the inverse correspondence \((z_1, z_2) \mapsto (X(z_1, z_2), Y(z_1, z_2))\) defines a pair of Hilbert modular functions for \( \mathbb{Q}(\sqrt{5}) \). We obtain an expression of \( X \) and \( Y \) by the following way.

In Section 3, we consider the subfamily \( \mathcal{F}_X = \{ S(X, 0) \} \) of \( K3 \) surfaces. The period mapping for \( \mathcal{F}_X \) gives a correspondence between the \( X \)-space and the diagonal \( \Delta = \{ (z_1, z_2) \in \mathbb{H} \times \mathbb{H} | z_1 = z_2 \} \). We obtain the period differential equation for \( \mathcal{F}_X \). The solutions of this period differential equation are described.
in terms of the solutions of the Gauss hypergeometric equation $\frac{3}{2}E(\frac{1}{12}, \frac{5}{12}, 1)$. Then, we obtain an expression of the parameter $X$ in terms of the elliptic $J$ function (Theorem 3.2).

In Section 4, we obtain an explicit expression of the inverse of the period mapping (1.1) by theta constants:

$$(X, Y) = \left(2^5 \cdot 5^2 \cdot \frac{s_6(z_1, z_2)}{g_2(z_1, z_2)}, 2^{10} \cdot 5^5 \cdot \frac{s_{10}(z_1, z_2)}{g_2(z_1, z_2)}\right)$$

where $g_2, s_6$ and $s_{10}$ are Hilbert modular forms given by Müller (Theorem 1.1).

Our results in this paper are used in the forthcoming paper [N2], in which we shall show a simple and new defining equations of the family of Kummer surfaces for the Humbert surface of invariant 5 and a geometric and intuitive interpretation of period mappings for this family.

1 Preliminaries

1.1 The family $\mathcal{F}_0$

In [N1], we studied the family $\mathcal{F}_0 = \{S_0(\lambda, \mu)\}$ of $K3$ surfaces defined by the equation

$$S_0(\lambda, \mu) : x_0y_0z_0^2(x_0 + y_0 + z_0 + 1) + \lambda x_0y_0z_0 + \mu = 0, \quad (1.1)$$

where $(\lambda, \mu) \in \Lambda = \{(\lambda, \mu)|\lambda\mu(\lambda^2(4\lambda - 1)^3 - 2(2 + 25\lambda(20\lambda - 1))\mu - 3125\mu^2) \neq 0\}$. First, we recall the results of this family.

Set

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}. \quad (1.2)$$

Put

$$\mathcal{D} = \{\xi = (\xi_1 : \xi_2 : \xi_3 : \xi_4) \in \mathbb{P}^3(\mathbb{C})|\xi A\xi = 0, \xi A\xi > 0\}.$$

This is a 2-dimensional symmetric Hermitian space of type $IV$. Note that $\mathcal{D}$ is composed of two connected components: $\mathcal{D} = \mathcal{D}_+ \cup \mathcal{D}_-$. We let $(1 : 1 : -\sqrt{-1} : 0) \in \mathcal{D}_+$. Set $PO(A, \mathbb{Z}) = \{g \in PGL(4, \mathbb{Z})|gAg = A\}$. It acts on $\mathcal{D}$ by $^t\xi \mapsto g^t\xi$. Let $PO^{+}(A, \mathbb{Z}) = \{g \in PO(A, \mathbb{Z})|g(\mathcal{D}_+) = \mathcal{D}_+\}$.

In Section 2 of [N1], we had the multivalued period mapping $\Phi_0 : \Lambda \to \mathcal{D}_+$ for $\mathcal{F}_0$ given by

$$\Phi_0(\lambda, \mu) = \left(\int_{\Gamma_1} \omega : \cdots : \int_{\Gamma_4} \omega\right), \quad (1.3)$$

where $\omega$ is the unique holomorphic 2-form on $S_0(\lambda, \mu)$ up to a constant factor and 2-cycles $\Gamma_1, \cdots, \Gamma_4 \in H_2(S_0(\lambda, \mu), \mathbb{Z})$ are given by this construction.

Let $\text{NS}(S)$ be the Néron-Severi lattice of a $K3$ surface $S$. The orthogonal complement $\text{Tr}(S) = \text{NS}(S)^\perp$ in $H_2(S, \mathbb{Z})$ is called the transcendental lattice of $S$. We proved

**Theorem 1.1.** (1) (N1 Theorem 2.2, 3.1) For a generic point $(\lambda, \mu) \in \Lambda$, the intersection matrix of $\text{NS}(S_0(\lambda, \mu))$ is given by

$$E_8(-1) \oplus E_8(-1) \oplus \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \quad (1.4)$$

and the intersection matrix of $\text{Tr}(S_0(\lambda, \mu))$ is given by

$$U \oplus \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = A. \quad (1.5)$$

(2) (N1 Theorem 5.2) The projective monodromy group of the period mapping $\Phi_0 : \Lambda \to \mathcal{D}_+$ is isomorphic to $PO^{+}(A, \mathbb{Z})$.  


Moreover, we determined the partial differential equation in 2 variables \(\lambda\) and \(\mu\) of rank 4 that is satisfied by the periods for the family \(F_0\). We call this equation the period differential equation for \(F_0\). This equation has the singular locus \(\Lambda\) (see [N1] Theorem 4.1).

1.2 The Hilbert modular orbifold \((\mathbb{H} \times \mathbb{H})/\langle PSL(2, \mathcal{O}), \tau \rangle\)

Here, we recall the action of the Hilbert modular group on \(\mathbb{H} \times \mathbb{H}\). Let \(\mathcal{O}\) be the ring of integers in the real quadratic field \(\mathbb{Q}(\sqrt{5})\). Set \(\mathbb{H}_\pm = \{z \in \mathbb{C} | \pm \text{Im}(z) > 0\}\). The Hilbert modular group \(PSL(2, \mathcal{O})\) acts on \((\mathbb{H}_+ \times \mathbb{H}_+) \cup (\mathbb{H}_- \times \mathbb{H}_-)\) by

\[
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}: (z_1, z_2) \mapsto \begin{pmatrix}
\alpha z_1 + \beta \\
\gamma z_1 + \delta
\end{pmatrix} \quad \text{for any } \begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \in PSL(2, \mathcal{O}),
\]

where \(\gamma'\) means the conjugate in \(\mathbb{Q}(\sqrt{5})\). We consider the involution

\[
\tau: (z_1, z_2) \mapsto (z_2, z_1)
\]

also.

**Definition 1.1.** If a holomorphic function \(g\) on \(\mathbb{H} \times \mathbb{H}\) satisfies the transformation law

\[
g\left(\frac{az_1 + b}{cz_1 + d}, \frac{a'z_2 + b'}{c'z_2 + d'}\right) = (cz_1 + d)^k(c'z_2 + d')^kg(z_1, z_2)
\]

for any \(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathcal{O})\), we call \(g\) a Hilbert modular form of weight \(k\) for \(\mathbb{Q}(\sqrt{5})\). If \(g(z_2, z_1) = g(z_1, z_2)\), \(g\) is called a symmetric modular form. If \(g(z_2, z_1) = -g(z_1, z_2)\), \(g\) is called an alternating modular form.

If a meromorphic function \(f\) on \(\mathbb{H} \times \mathbb{H}\) satisfies

\[
f\left(\frac{az_1 + b}{cz_1 + d}, \frac{a'z_2 + b'}{c'z_2 + d'}\right) = f(z_1, z_2)
\]

for any \(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathcal{O})\), we call \(f\) a Hilbert modular function for \(\mathbb{Q}(\sqrt{5})\).

Set

\[
W = \begin{pmatrix}
1 & 0 \\
1 - \sqrt{5} & 1 + \sqrt{5}
\end{pmatrix}.
\]

It holds that

\[
A = U \oplus \begin{pmatrix}
2 & 1 \\
1 & -2
\end{pmatrix} = U \oplus WU^tW.
\]

The correspondence

\[
j: (z_1, z_2) \mapsto (z_1, z_2: -1 : z_1 : z_2)(I_2 \oplus W^{-1})
\]

defines a biholomorphic mapping

\[
(\mathbb{H}_+ \times \mathbb{H}_+) \cup (\mathbb{H}_- \times \mathbb{H}_-) \rightarrow D.
\]

The group \(PSL(2, \mathcal{O})\) is generated by three elements

\[
g_1 = \begin{pmatrix}
1 & 1 \\ 0 & 1
\end{pmatrix}, \quad g_2 = \begin{pmatrix}
1 & 1 + \sqrt{5} \\ 0 & 2
\end{pmatrix}, \quad g_3 = \begin{pmatrix}
0 & 1 \\ -1 & 0
\end{pmatrix}.
\]
We have an isomorphism

\[ j : \langle PSL(2, \mathcal{O}), \tau \rangle \rightarrow PO^+(A, \mathbb{Z}) \]

where \( g \Rightarrow j \circ g \circ j^{-1} = \tilde{j}(g) =: \tilde{g} \).

Especially, we see

\[
\begin{align*}
\tilde{g}_1 &= \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
\tilde{g}_2 &= \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \\
\tilde{g}_3 &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
\tilde{\tau} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.
\end{align*}
\] (1.6)

So, the above \( j \) gives a modular isomorphism

\[ j : (\mathbb{H} \times \mathbb{H}, \langle PSL(2, \mathcal{O}), \tau \rangle) \simeq (D_+, PO^+(A, \mathbb{Z})). \] (1.7)

Recall \( D = D_+ \cup D_- \) and the period mapping \( \Phi \) for \( F_0 \). The mapping \( j^{-1} \circ \Phi : \Lambda \rightarrow \mathbb{H} \times \mathbb{H} \) gives an explicit transcendental correspondence between \( \Lambda \) and \( \mathbb{H} \times \mathbb{H} \).

Hirzebruch [H] studied the Hilbert modular orbifold \( \mathbb{H} \times \mathbb{H}/\langle PSL(2, \mathcal{O}), \tau \rangle \). Here, we survey his results.

The Klein icosahedral polynomials are

\[
\begin{align*}
\mathcal{A}(\zeta_0 : \zeta_1 : \zeta_2) &= \zeta_0^2 + \zeta_1 \zeta_2, \\
\mathcal{B}(\zeta_0 : \zeta_1 : \zeta_2) &= 8\zeta_0^3 \zeta_1 \zeta_2 - 2\zeta_0^2 \zeta_1^2 \zeta_2^2 + \zeta_1^3 \zeta_2^3 - \zeta_0 (\zeta_1^5 + \zeta_2^5), \\
\mathcal{C}(\zeta_0 : \zeta_1 : \zeta_2) &= 320 \zeta_0^6 \zeta_1 \zeta_2^2 - 160 \zeta_0^4 \zeta_1^3 \zeta_2^3 + 20 \zeta_0^2 \zeta_1^4 \zeta_2^4 + 6 \zeta_1^5 \zeta_2^5 \\
&\quad - 4\zeta_0 (\zeta_1^5 + \zeta_2^5) (32 \zeta_0^2 - 20 \zeta_0^2 \zeta_1 \zeta_2 + 5 \zeta_1^3 \zeta_2^2) + \zeta_1^{10} + \zeta_2^{10}, \\
12\mathcal{D}(\zeta_0 : \zeta_1 : \zeta_2) &= (\zeta_1^5 - \zeta_2^5) (-1024 \zeta_0^{10} + 3840 \zeta_0^8 \zeta_1 \zeta_2 - 3840 \zeta_0^6 \zeta_1^2 \zeta_2^2 \\
&\quad + 1200 \zeta_0^4 \zeta_1^4 \zeta_2^4 - 100 \zeta_0^2 \zeta_1^6 \zeta_2^6 + \zeta_1^{10} \zeta_2^{10}) \\
&\quad + \zeta_0 (\zeta_1^{10} - \zeta_2^{10}) (352 \zeta_0^4 - 160 \zeta_0^2 \zeta_1 \zeta_2 + 10 \zeta_2^4 \zeta_2^2) + (\zeta_1^{15} - \zeta_2^{15}).
\end{align*}
\] (1.8)

We have the following relation:

\[ R(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) = 144 \mathcal{D}^2 - (-1728 \mathcal{B}^5 + 720 \mathcal{A} \mathcal{C} \mathcal{B}^3 - 80 \mathcal{A}^2 \mathcal{C}^2 \mathcal{B} + 64 \mathcal{A}^3 (5 \mathcal{B}^2 - \mathcal{A} \mathcal{C})^2 + \mathcal{C}^3) = 0. \] (1.9)

Set

\[ X = \frac{\mathcal{B}}{\mathcal{A}^{3/5}}, \quad Y = \frac{\mathcal{C}}{\mathcal{A}}. \] (1.10)

Now, set

\[ \Gamma(\sqrt{5}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mid \alpha \equiv \delta \equiv 1, \beta \equiv \delta \equiv 0 \pmod{\sqrt{5}} \right\}. \]

We note that the group \( PSL(2, \mathcal{O})/\Gamma(\sqrt{5}) \) is isomorphic to the alternating group \( A_5 \). Hirzebruch [H] studied the canonical bundle of the orbifold \( (\mathbb{H} \times \mathbb{H})/\Gamma(\sqrt{5}) \) by an algebrogeometric method. He proved

**Proposition 1.1.** [H] pp.307-310 (1) The non-singular model of \( (\mathbb{H} \times \mathbb{H})/\Gamma(\sqrt{5}), \tau \) is \( \mathbb{P}^2(\mathbb{C}) = \{(\zeta_0 : \zeta_1 : \zeta_2)\} \) by adding six points. A homogeneous polynomial of degree \( k \) in \( \zeta_0, \zeta_1 \) and \( \zeta_2 \) defines a modular form for \( \Gamma(\sqrt{5}) \) of weight \( k \).
(2) The ring of symmetric modular forms for $\text{PSL}(2, \mathcal{O})$ is isomorphic to the ring

$$\mathbb{C}[\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}]/(R(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}) = 0),$$

where $R(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ is the Klein relation (1.9). $\mathfrak{A}$ (resp., $\mathfrak{B}, \mathfrak{C}, \mathfrak{D}$) gives a symmetric modular form for $\text{PSL}(2, \mathcal{O})$ of weight 2 (6, 10, 15, resp.).

(3) There exists an alternating modular form $c$ of weight 5 such that $c^2 = \mathfrak{C}$. The ring of Hilbert modular forms for $\text{PSL}(2, \mathcal{O})$ is isomorphic to the ring

$$\mathbb{C}[\mathfrak{A}, \mathfrak{B}, c, \mathfrak{D}]/(R(\mathfrak{A}, \mathfrak{B}, c^2, \mathfrak{D}) = 0).$$

Let $c' \in \mathbb{C} - \{0\}$. We consider the action $(\zeta_0, \zeta_1, \zeta_2) \mapsto (c' \zeta_0, c' \zeta_1, c' \zeta_2)$. Because $\mathfrak{A}$ (resp., $\mathfrak{B}, \mathfrak{C}, \mathfrak{D}$) is a homogeneous polynomial of degree 2 (6, 10, resp.) in $\zeta_0, \zeta_1$ and $\zeta_2$, we have the action $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}) \mapsto (c^2 \mathfrak{A}, c^6 \mathfrak{B}, c^{10} \mathfrak{C})$. Therefore, we regard $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$-space as the weighted projective space $\mathbb{P}(1, 3, 5)$. Especially, the pair

$$(X, Y) = \left( \frac{\mathfrak{B}}{\mathfrak{A}^3}, \frac{\mathfrak{C}}{\mathfrak{A}^5} \right)$$

(1.11)

gives a system of affine coordinates on $\{\mathfrak{A} \neq 0\}$.

By the arguments of Klein [Kl], Hirzebruch [H] and Kobayashi, Kushibiki and Naruki [KKN], we know the following properties of the action of $A_5$ on $(\mathbb{H} \times \mathbb{H})/\Gamma(\sqrt{5}), \tau) = \mathbb{P}^2(\mathbb{C}) = \{\zeta_0 : \zeta_1 : \zeta_2\}$.

**Proposition 1.2.** (1) The correspondence $(\zeta_0 : \zeta_1 : \zeta_2) \mapsto (\mathfrak{A}(\zeta_0 : \zeta_1 : \zeta_2) : \mathfrak{B}(\zeta_0 : \zeta_1 : \zeta_2) : \mathfrak{C}(\zeta_0 : \zeta_1 : \zeta_2))$ gives an identification between $\mathbb{P}^2(\mathbb{C})/A_5$ and $\mathbb{P}(1, 3, 5)$. Then, the Hilbert modular orbifold $(\mathbb{H} \times \mathbb{H})/\text{PSL}(2, \mathcal{O}), \tau)$ is identified with $\mathbb{P}(1, 3, 5)$. The cusp $(\sqrt{-1} \infty, \sqrt{-1} \infty) \in (\mathbb{H} \times \mathbb{H})/\text{PSL}(2, \mathcal{O}), \tau)$ is given by the point $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}) = (1 : 0 : 0)$. So, the quotient space $(\mathbb{H} \times \mathbb{H})/\text{PSL}(2, \mathcal{O}), \tau)$ corresponds to $\mathbb{P}(1, 3, 5) - \{(1 : 0 : 0)\}$.

(2) The divisor $\{\mathfrak{D} = 0\}$ consists of fifteen lines in $\mathbb{P}^2(\mathbb{C})$. These fifteen lines of $\{\mathfrak{D} = 0\}$ are the reflection lines of fifteen involutions of $A_5$ (note that $A_5$ is generated by three involutions).

(3) The involution $\tau$ induces an involution on the orbifold $(\mathbb{H} \times \mathbb{H})/\text{PSL}(2, \mathcal{O})$. The branch locus of the canonical projection $(\mathbb{H} \times \mathbb{H})/\text{PSL}(2, \mathcal{O}) \rightarrow \mathbb{P}(1, 3, 5)$ is given by $\{\mathfrak{C} = 0\}$.

Set

$$\mathfrak{X} = \{(X, Y) \in \mathbb{C}^2 | (1728X^5 - 720X^3Y + 80XY^2 - 64(5X^2 - Y)^2 - Y^3) \neq 0\}.$$  

(1.12)

In Section 6 of [N1], we obtained the birational mapping $\Lambda \rightarrow \mathfrak{X}$ given by

$$(\lambda, \mu) \mapsto (X, Y) = \left( \frac{25\mu}{2(\lambda - 1/4)^3}, \frac{-3125\mu^2}{(\lambda - 1/4)^9} \right).$$

(1.13)

**Theorem 1.2.** ([N1] Theorem 6.3) By the correspondence (1.13), the period differential equation for the family $\mathcal{F}_0 = \{S_0(\lambda, \mu)\}$ is transformed to the system of differential equations

$$\begin{cases}
    u_{XX} = L_1 u_{XY} + A_1 u_X + B_1 u_Y + P_1 u, \\
    u_{YY} = M_1 u_{XY} + C_1 u_X + D_1 u_Y + Q_1 u
\end{cases}$$

(1.14)

with

$$\begin{align*}
L_1 &= \frac{-20(4X^2 + 3XY - 4Y)}{36X^2 - 32X - Y}, \\
A_1 &= \frac{-2(20X^3 - 8XY + 9X^2Y + Y^2)}{XY(36X^2 - 32X - Y)}, \\
C_1 &= \frac{-2(-25X^2 + 27X^3 + 2Y - 3XY)}{5Y(36X^2 - 32X - Y)}, \\
P_1 &= \frac{-2(8X - Y)}{X^2(36X^2 - 32X - Y)}, \\
M_1 &= \frac{-2(54X^3 - 50X^2 - 3XY + 2Y)}{5Y(36X^2 - 32X - Y)}, \\
B_1 &= \frac{10Y(-8 + 3X)}{X(36X^2 - 32X - Y)}, \\
D_1 &= \frac{-2(-120X^2 + 135X^3 - 2Y - 3XY)}{5Y(36X^2 - 32X - Y)}, \\
Q_1 &= \frac{-2(-10 + 9X)}{25Y(36X^2 - 32X - Y)}.
\end{align*}$$
Remark 1.1. In [14], we saw that (1.14) is an uniformizing differential equation of the Hilbert modular orbifold \( \mathbb{H} \times \mathbb{H} / \langle \text{PSL}(2, \mathbb{O}), \tau \rangle \). In other words, the solutions of (1.14) define the developing map of the canonical projection \( \mathbb{H} \times \mathbb{H} \to (\mathbb{H} \times \mathbb{H}) / \langle \text{PSL}(2, \mathbb{O}), \tau \rangle \). This gives an alternative proof of Theorem 1.1 (2).

2 The period of the family \( \mathcal{F} \)

2.1 The family \( \mathcal{F} \) of K3 surfaces

We obtain a new family \( \mathcal{F} \) of K3 surfaces with explicit defining equations from the family \( \mathcal{F}_0 = \{ S_0(\lambda, \mu) \} \).

**Proposition 2.1.** The family of K3 surfaces \( \mathcal{F}_0 = \{ S_0(\lambda, \mu) \} \) for \( (\lambda, \mu) \in \Lambda \) is transformed to the family \( \mathcal{F} = \{ S(X, Y) \} \) for \( (X, Y) \in \mathfrak{X} \):

\[
S(X, Y) : z^2 = x^3 - 4y^2(4y - 5)x^2 + 20Xy^3x + Yy^4. \tag{2.1}
\]

**Proof.** By the transformation (1.13) and the birational transformation given by

\[
\begin{align*}
x_0 &= \frac{Yy}{10Xx_1}, \\
y_0 &= \frac{4Y^2x_1y_1^2}{-50X^2Yx_1y_1 - 5XY^2y_1^2 + 5XYz_1}, \\
z_0 &= \frac{-10XYx_1y_1 + Y^2y_1^2 - Yz_1}{20XYx_1y_1},
\end{align*}
\]

the family \( \mathcal{F}_0 = \{ S_0(\lambda, \mu) \} \) is transformed to the family \( \mathcal{F}_1 = \{ S_1(X, Y) \} \) given by

\[
S_1(X, Y) : z_1^2 = Y(x_1^3 - 4y_1^2(4y_1 - 5)x_1^2 + 20Y^3x_1 + Yy_1^4)
\]

over \( \mathfrak{X} \). Then, by the correspondence \( (x_1, y_1, z_1) \mapsto (x, y, z) = (x_1, y_1, \frac{1}{\sqrt{Y}}z_1) \), we have the family \( \mathcal{F} = \{ S(X, Y) \} \) given by (2.1). \( \square \)

From (1.3), we obtain the multivalued analytic period mapping

\[
\Phi_1 : \mathfrak{X} \to \mathcal{D}_+ ; (X, Y) \mapsto \left( \int_{\Gamma_1} \omega : \int_{\Gamma_2} \omega : \int_{\Gamma_3} \omega ; \int_{\Gamma_4} \omega \right), \tag{2.2}
\]

where \( \omega = \frac{dx \wedge dy}{z} \) is the unique holomorphic 2-form on \( S(X, Y) \) up to a constant factor and \( \Gamma_1, \cdots , \Gamma_4 \) are certain 2-cycles on \( S(X, Y) \) (this period mapping is stated in detail at the beginning of Section 2.2).

**Remark 2.1.** The correspondence \( (x_1, y_1, z_1) \mapsto (x, y, z) = (x_1, y_1, \frac{1}{\sqrt{Y}}z_1) \) in the proof of Proposition 2.1 induces the double covering \( \mathfrak{X}' \to \mathfrak{X} \) given by \( (X, Y') \mapsto (X, Y) = (X, Y'^2) \). However, \( (X, Y') \) and \( (X, -Y') \in \mathfrak{X}' \) define mutually isomorphic \( P \)-marked K3 surfaces (see Definition 2.1). So, we obtain the above period mapping \( \Phi_1 \) on \( \mathfrak{X} \).

Hence, from Theorem 1.1 for a generic point \( (X, Y) \in \mathfrak{X} \), the intersection matrix of the Néron-Severi lattice \( \text{NS}(S(X, Y)) \) is given by (1.4) and that of the transcendental lattice \( \text{Tr}(S(X, Y)) \) is given by \( A \) in (1.3). The projective monodromy group of \( \Phi_1 \) is isomorphic to \( PO^+(A, \mathbb{Z}) \). From Theorem 1.2 the period differential equation for the family \( \mathcal{F} = \{ S(X, Y) \} \) is given by (1.14).

**Proposition 2.2.** Under the correspondence (1.11), the surface \( S(X, Y) \) is birationally equivalent to

\[
S(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}) : z^2 = x^3 - 4(4y^3 - 5\mathfrak{A}y^2)x^2 + 20\mathfrak{B}y^3x + \mathfrak{C}y^4. \tag{2.3}
\]
Proof. Putting $X = \frac{B}{A^3}, Y = \frac{C}{A^5}$ to \textbf{(2.1)}, we have

$$\mathcal{A}^5 z^2 = \mathcal{A}^5 x^3 + (20g^2 - 16g^3) \mathcal{A}^2 y^3 + 20 \mathcal{A}^2 \mathcal{B} y^3 x + \mathcal{C} y^4.$$ 

Then, by the correspondence

$$x \mapsto \frac{x}{\mathcal{A}^3}, \quad y \mapsto \frac{y}{\mathcal{A}}, \quad z \mapsto \frac{z}{\sqrt{\mathcal{A}}^3},$$

we obtain \textbf{(2.4)}. \hfill \Box

Remark 2.2. For two surfaces

$$\begin{cases}
S(\mathcal{A} : \mathcal{B} : \mathcal{C}) : z^2 = x^3 - 4(4y^3 - 5\mathcal{A}y^2)x^2 + 20\mathcal{B}y^3x + \mathcal{C}y^4, \\
S(k^2\mathcal{A} : k^6\mathcal{B} : k^{10}\mathcal{C}) : z^2 = x^3 - 4(4y^3 - 5k^2\mathcal{A}y^2)x^2 + 20k^6\mathcal{B}y^3x + k^{10}\mathcal{C}y^4,
\end{cases}$$

we have an isomorphism $S(\mathcal{A} : \mathcal{B} : \mathcal{C}) \to S(k^2\mathcal{A} : k^6\mathcal{B} : k^{10}\mathcal{C})$ given by $(x, y, z) \mapsto (k^6x, k^2y, k^3z)$ as elliptic surfaces. Therefore, $(\mathcal{A} : \mathcal{B} : \mathcal{C}) \in \mathbb{P}(1 : 3 : 5)$ gives an isomorphism class of these elliptic K3 surfaces.

We set $K_1 = \{Y = 0\}$ and $K_2 = \{1728X^5 - 720X^3Y + 80XY^2 - 64(5X^2 - Y)^2 - Y^3 = 0\}$.

Theorem 2.1. The $(\mathcal{A} : \mathcal{B} : \mathcal{C})$-space $\mathbb{P}(1,3,5)$ gives a compactification of the parameter space $\mathcal{X}$ of the family $\mathcal{F} = \{S(X,Y)\}$ of K3 surfaces given by \textbf{(2.1)}. Namely, if $(1 : 0 : 0) \neq (\mathcal{A} : \mathcal{B} : \mathcal{C}) \in \mathbb{P}(1,3,5)$, then the corresponding surface $S(\mathcal{A} : \mathcal{B} : \mathcal{C})$ is a K3 surface. On the other hand, $S(1 : 0 : 0)$ is a rational surface.

Proof. First, we prove the case $\mathcal{A} \neq 0$. In this case, we consider $S(X,Y)$ in \textbf{(2.1)}. We have the Kodaira normal form of \textbf{(2.1)}:

$$z_i^2 = x_i^3 - g_2(y)x - g_3(y) \quad (y \neq \infty), \quad (2.4)$$

with

$$\begin{cases}
g_2(y) = -\left(20Xy^3 - \frac{16}{3}y^4(4y - 5)^2\right), \\
g_3(y) = -\left(Yy^4 + \frac{80}{3}y^5(4y - 5)X - \frac{128}{27}y^6(4y - 5)^3\right),
\end{cases}$$

and

$$z_2^2 = x_2^3 - h_2(y_1)x_2 - h_3(y_1) \quad (y \neq 0), \quad (2.5)$$

with

$$\begin{cases}
h_2(y_1) = -\left(20Xy_1^5 - \frac{256}{3}y_1^2 + \frac{640}{3}y_1^3 - \frac{400}{3}y_1^4\right), \\
h_3(y_1) = -\left(20Xy_1^7 - \frac{320}{3}Xy_1^5 - \frac{400}{3}Xy_1^7 - \frac{8192}{27}y_1^3 + \frac{10240}{9}y_1^4 - \frac{12800}{9}y_1^5 + \frac{16000}{27}y_1^6\right),
\end{cases}$$

where $y_1 = \frac{1}{y}$. The discriminant $D_0, D_\infty, \text{ resp.}$ of the right hand side of \textbf{(2.4), (2.5), resp.} is given by

$$\begin{cases}
D_0 = y^8(27Y^2 + 32000X^3y - 72000XYy - 160000X^2y^2 + 320000Yy^2 + 57600XY^2y^2 + 256000X^2y^3 - 768000Y^3y^3 + 102400X^2y^4 + 61440Yy^4 - 16384Yy^5), \\
D_\infty = y_1^9(-16384Y - 102400X^2y^2 + 61440Yy_1 + 256000X^2y_1^2 - 768000Yy_1^2 - 160000X^2y_1^3 + 320000Yy_1^3 + 57600XYy_1^3 + 32000X^3y_1^4 - 72000XYy_1^4 + 27Y^2y_1^5).
\end{cases}$$

If $(X,Y) \in \mathcal{X}$, then we have

$$\text{ord}_y(D_0) = 8, \quad \text{ord}_y(g_2) = 3, \quad \text{ord}_y(g_3) = 4,$$
so \(\pi^{-1}(0)\) is the singular fibre of type \(IV^*\) (for detail, see [Ko] or [Sh]). Similarly, we have

\[
\text{ord}_y(D_\infty) = 11, \quad \text{ord}_y(h_2) = 2, \quad \text{ord}_y(h_3) = 3,
\]

so \(\pi^{-1}(\infty) = I_5^3\). We have other 5 singular fibres of type \(I_1\). Therefore, for \((X,Y) \in \mathfrak{X}\), \(S(X,Y)\) is an elliptic \(K3\) surface whose singular fibres are of type \(IV^* + 5I_1 + I_3^2\).

By the same way, we know the structure of the elliptic surface \(S(X,Y)\) for \((X,Y) \notin \mathfrak{X}\). If \(X \neq 0\) and \(Y = 0\) (namely, \((X,Y) \in K_1 - \{(0,0)\}\)) then \(S(X, 0)\) is an elliptic \(K3\) surface with the singular fibres of type \(III^* + 3I_1 + I_6^3\). If \((X,Y) \in K_2 - \{(0,0)\}\), \(S(X,Y)\) is an elliptic \(K3\) surface with the singular fibres of type \(IV^* + 3I_1 + I_2 + I_3^2\). However, we see easily that \(S(0, 0)\) is not a \(K3\) surface, but a rational surface.

Next, we consider the case \(\mathfrak{A} = 0\). In this case, note that \((\mathfrak{B}, \mathfrak{C}) \neq (0, 0)\). We have the equation of  

\[
S(0: \mathfrak{B}: \mathfrak{C}): z^2 = x^3 - 16y^3x^2 + 20\mathfrak{B}y^3x + \mathfrak{C}y^4. \tag{2.45}
\]

On \(\{\mathfrak{A} = 0\} \subset \mathbb{P}(1,3,5)\), we use the parameter \(l = \frac{c^3}{\mathfrak{B}^5}\).

By the correspondence \(x = \frac{c^3}{\mathfrak{B}^4}x', y = \frac{c^2}{\mathfrak{B}^3}y'\) and \(z = \sqrt{\frac{c^3}{\mathfrak{B}^5}}z'\), we have

\[
S(l) : z'^2 = x'^3 - 16y'^3x'^2 + 20y'^3x' + y'^4.
\]

The discriminant of the right hand side is given by 

\[
y^8(27 + 32000y' + 5760ly'^2 - 102400l^2y'^4 - 16384l^3y'^5). \tag{2.46}
\]

From this, we can see that \(S(l)\) is an elliptic \(K3\) surface with the singular fibres of type \(IV^* + 5I_1 + I_3^2\) \(\Box\).

Hence, we obtain the extended family \(\{S(\mathfrak{A}: \mathfrak{B}: \mathfrak{C})| (\mathfrak{A}: \mathfrak{B}: \mathfrak{C}) \in \mathbb{P}(1,3,5) - \{(1:0:0)\}\}\) of \(K3\) surfaces. For simplicity, let \(F\) denotes this extended family.

\section{The extension \(\Phi\) of the period mapping \(\Phi_1\)}

Set \(c_0 = (1:0:0) \in \mathbb{P}(1,3,5)\). In this subsection, we extend the period mapping \(\Phi_1: \mathfrak{X} \to \mathcal{D}_+\) in (2.29) to \(\Phi: \mathbb{P}(1,3,5) - \{c_0\} \to \mathcal{D}_+\).

First, we recall the \(S\)-marking on \(\mathfrak{X}\). According to Theorem 2.1 and its proof, we have the elliptic \(K3\) surface

\[
\pi(\mathfrak{X}: \mathfrak{B}: \mathfrak{C}) : S(\mathfrak{A}: \mathfrak{B}: \mathfrak{C}) \to \mathbb{P}^1(\mathbb{C}) = (y\text{-sphere})
\]

for any \((\mathfrak{A}: \mathfrak{B}: \mathfrak{C}) \in \mathbb{P}(1,3,5) - \{c_0\}\).

Take a generic point \((X_0, Y_0) \in \mathfrak{X}\). The elliptic \(K3\) surface \(\hat{S} = S(X_0, Y_0)\) given by (2.24) and (2.25) has the singular fibres of type \(IV^* + 5I_1 + I_3^2\). Let \(F\) be a general fibre of this elliptic fibration and \(O\) be the zero of the Mordell-Weil group of sections. We have two irreducible components of the divisor \(C\) given by \(\{x = 0, z^2 = Yy^4\}\). We take the section \(R\) given by \(y \mapsto (x, y, z) = (0, y, \sqrt{Y}y^2)\). This gives a component of the divisor \(C\). Let us consider the irreducible decomposition \(\bigcup_{j=0}^6 a_j \bigcup_{j=0}^9 b_j\) resp.) of the singular fibre \(\pi^{-1}_{(X,Y)}(0) \pi^{-1}_{(X,Y)}(\infty)\) resp.) of type \(IV^* (I_5^3)\) resp.). These curves are illustrated in Figure 1. Note that \(a_0 \cap O \neq \emptyset, b_0 \cap O \neq \emptyset, a_6 \cap R \neq \emptyset\) and \(b_9 \cap R \neq \emptyset\).

We set \(\Gamma_5 = F, \Gamma_6 = O, \Gamma_7 = R, \Gamma_{8+k} = a_k+1 (0 \leq k \leq 5), \Gamma_{14+l} = b_{l+1} (0 \leq l \leq 8)\). We have the lattice \(\hat{L} = (\Gamma_5, \cdots, \Gamma_{22})_Z \subset H_2(\hat{S}, Z)\). We can check that \(|\det(\hat{L})| = 5\). Hence, we have \(\hat{L} = \text{NS}(\hat{S})\).

Since \(\hat{L}\) is a primitive lattice, there exists \(\Gamma_1, \cdots, \Gamma_4 \in H_2(\hat{S}, Z)\) such that \(\langle \Gamma_1, \cdots, \Gamma_4, \Gamma_5, \cdots, \Gamma_{22}\rangle_Z = H_2(\hat{S}, Z)\). Let \(\{\hat{\Gamma}_1^*, \cdots, \hat{\Gamma}_{22}^*\}\) be the dual basis of \(\{\Gamma_1, \cdots, \Gamma_{22}\}\) in \(H_2(\hat{S}, Z)\). Then, we see that \(\langle \hat{\Gamma}_1^*, \cdots, \hat{\Gamma}_4^*\rangle_Z\) is the transcendental lattice. We may assume that its intersection matrix is

\[
(\hat{\Gamma}_j^* \cdot \hat{\Gamma}_k^*)_{1 \leq j, k \leq 4} = A \quad (2.6)
\]
where $A$ is given by (1.2). We define the period of $\tilde{S}$ by

$$\Phi_1(X_0, Y_0) = \left( \int_{\Gamma_1} \omega : \cdots : \int_{\Gamma_4} \omega \right).$$

Take a small connected neighborhood $V_0$ of $(X_0, Y_0)$ in $X$ so that we have a local topological trivialization $\tau : \{ S(p) \mid p \in V_0 \} \rightarrow \tilde{S} \times V_0$. (2.7)

Let $\varpi : \tilde{S} \times V_0 \rightarrow \tilde{S}$ be the canonical projection. Set $r = \varpi \circ \tau$. Then,

$$r'_p = r_{\mid S(p)} \quad \forall p \in V_0$$

gives a $C^\infty$-isomorphism of surfaces. For any $p \in V_0$, we have an isometry $\psi_p : H_2(S(p), \mathbb{Z}) \rightarrow H_2(\tilde{S}, \mathbb{Z})$ given by

$$\psi_p = r'_p.$$

We call this isometry the S-marking on $V_0$. By an analytic continuation along an arc $\alpha \subset X$, we define the S-marking on $X$. This depends on the choice of $\alpha$. The S-marking preserves the Néron-Severi lattice. We define the period mapping $\Phi_1 : X \rightarrow \mathcal{D}_+$ by

$$p \mapsto \left( \int_{\psi_p^{-1}(\Gamma_1)} \omega : \cdots : \int_{\psi_p^{-1}(\Gamma_4)} \omega \right).$$

This is equal to the period mapping in (2.2).

Here, we recall the P-marking for $K3$ surfaces, which is defined in \[N1\] Section 5.

**Definition 2.1.** Let $S$ be an algebraic $K3$ surface. An isometry

$$\psi : H_2(S, \mathbb{Z}) \rightarrow H_2(\tilde{S}, \mathbb{Z})$$

is called the P-marking if

(i) $\psi^{-1} (\text{NS}(\tilde{S})) \subset \text{NS}(S)$,

(ii) $\psi^{-1} (F), \psi^{-1} (O), \psi^{-1} (R), \psi^{-1} (a_j) \ (1 \leq j \leq 6)$ and $\psi^{-1} (b_j) \ (1 \leq j \leq 9)$ are all effective divisors,

(iii) $(\psi^{-1}(F) \cdot C) \geq 0$ for any effective class $C$. Namely, $\psi^{-1}(F)$ is nef.

A pair $(S, \psi)$ is called a P-marked $K3$ surface.
**Definition 2.2.** Two P-marked K3 surfaces \((S_1, \psi_1)\) and \((S_2, \psi_2)\) are said to be isomorphic if there is a biholomorphic mapping \(f : S_1 \to S_2\) with
\[
\psi_2 \circ f_* \circ \psi_1^{-1} = \text{id}_{H_2(\tilde{S}, \mathbb{Z})}.
\]

Two P-marked K3 surfaces \((S_1, \psi_1)\) and \((S_2, \psi_2)\) are said to be equivalent if there is a biholomorphic mapping \(f : S_1 \to S_2\) with
\[
(\psi_2 \circ f_* \circ \psi_1^{-1})|_{NS(\tilde{S})} = \text{id}_{NS(\tilde{S})}.
\]

**Remark 2.3.** The other connected component \(R'\) of the divisor \(C\) given by the section \(y \to (x, y, -\sqrt{y^2})\) intersects \(a_4\) \((b_9, \text{resp.})\) at \(y = 0\) \((y = \infty, \text{resp.})\). Letting \(q\) be the involution of \(S(X, Y)\) given by \((x, y, z) \to (x, -y, -z)\), we have \(q_*(R') = R\), \(q_*(a_4) = a_6\), \(q_*(a_3) = a_5\) and \(q_*(b_9) = b_9\). Then, we can see that P-marked K3 surfaces \((\tilde{S}, \text{id})\) and \((\tilde{S}, q)\) are isomorphic by \(q\). This shows that our argument does not depend on the choice of the curves \(R\) or \(R'\).

The period of a P-marked K3 surface \((S, \psi)\) is given by
\[
\tilde{\Phi}'(S, \psi) = \left( \int_{\psi^{-1}(\Gamma_1)} \omega : \cdots : \int_{\psi^{-1}(\Gamma_n)} \omega \right).
\]
(2.8)

It is a point in \(\mathcal{D}\). Let \(\mathcal{X}\) be the isomorphism classes of P-marked K3 surfaces and let
\[
\mathcal{X} = \mathcal{X}/(P\text{-marked equivalence}).
\]

By the Torelli theorem for K3 surfaces, the period mapping \(\tilde{\Phi}' : \mathcal{X} \to \mathcal{D}\) for P-marked K3 surfaces defined by (2.8) gives an identification between \(\mathcal{X}\) and \(\mathcal{D}\). Moreover, a P-marked K3 surface \((S_1, \psi_1)\) is equivalent to a P-marked K3 surface \((S_2, \psi_2)\) if and only if
\[
\tilde{\Phi}'(S_1, \psi_1) = g \circ \tilde{\Phi}'(S_2, \psi_2)
\]
for some \(g \in PO(A, \mathbb{Z})\) (see [N1] Lemma 5.1). Therefore, we identify \([\mathcal{X}]\) with
\[
\mathcal{D}/PO(A, \mathbb{Z}) = \mathcal{D}_+ / PO^+(A, \mathbb{Z}) \simeq (\mathbb{H} \times \mathbb{H})/(PSL(2, \mathbb{O}), \tau).
\]
(2.9)

Recall that the above isomorphism is given by the modular isomorphism \(j\) in (1.7).

We note that \(\mathcal{X}\) is embedded in \([\mathcal{X}]\) (see [N1] Remark 5.3). Then, an S-marked K3 surface is a P-marked K3 surface and the period mapping for P-marked K3 surfaces is an extension of the period mapping for S-marked K3 surfaces. From \(\tilde{\Phi}' : \mathcal{X} \to \mathcal{D}\), we obtain a multivalued mapping \(\Phi' : [\mathcal{X}] \to \mathcal{D}_+\).

We have
\[
\Phi'|_{\mathcal{X}} = \Phi_1,
\]
(2.10)
where \(\Phi_1\) is the period mapping in (2.2) for S-marked K3 surfaces.

Now, we extend the period mapping \(\Phi_1 : \mathcal{X} \to \mathcal{D}_+\) in (2.2) to \(\Phi : \mathbb{P}(1, 3, 5) - \{c_0\} \to \mathcal{D}_+\). We recall that \((\mathbb{P}(1, 3, 5) - \{c_0\}) - \mathcal{X} = (K_1 \cup K_2 \cup \{0\}) - \{c_0\}\).

First, since the local topological trivialization on \(\mathcal{X}\) in (2.7) is naturally extended to \(\{0\}\), there exist \(\mathcal{X}\)-markings on \(\{0\}\) and the period mapping (2.2) on \(\mathcal{X}\) is extended to \(\mathbb{P}(1, 3, 5) - (K_1 \cup K_2 \cup \{c_0\}) \to \mathcal{D}_+\).

Let us recall that the projective monodromy group of \(\Phi_1\) is isomorphic to \(PO^+(A, \mathbb{Z})\). According to (2.8) and Proposition [126] (3) (Proposition [125] (2), resp.), the local monodromy of the period mapping \(\Phi_1\) in (2.2) around \(K_1\) \((K_2, \text{resp.})\) is locally finite. Hence, the period mapping \(\mathbb{P}(1, 3, 5) - (K_1 \cup K_2 \cup \{c_0\}) \to \mathcal{D}_+\) can be extended to \(\mathbb{P}(1, 3, 5) - \{c_0\} \to \mathcal{D}_+\). We note that this extension is assured by Theorem (9.5) in Griffiths [G1].
Therefore, we have the extended period mapping
\[ \Phi : \mathbb{P}(1,3,5) - \{ c_0 \} \to D_+ \]
with
\[ \Phi \big|_X = \Phi_1. \] (2.12)

Since we have \([2.9]\) and Proposition \([1.2](1)\), the P-marked equivalence classes \([X]\) is identified with \(\mathbb{P}(1,3,5) - \{ c_0 \}\). Because we have \([2.10], [2.12]\) and \(X\) is a Zariski open set in \(\mathbb{P}(1,3,5) - \{ c_0 \}\), \(\Phi\) in \([2.11]\) is equal to the period mapping \(\Phi'\) on \([X]\).

Let \([\Phi(p)] \in D_+/PO^+(A,\mathbb{Z})\) be the equivalence class of \(\Phi(p) \in D_+\). From the above argument, we have the following proposition.

**Proposition 2.3.** The period mapping \(\Phi' : [X] \to D_+\) for P-marked K3 surfaces is given by the period mapping \(\Phi\) in \([2.11]\) for the family \(\mathcal{F} = \{ S(p) | p \in \mathbb{P}(1,3,5) - \{ c_0 \} \}\) of K3 surfaces. This is an extension of the period mapping in \([2.2]\) for S-marked K3 surfaces. Especially, if \([\Phi(p_1)] = [\Phi(p_2)]\) in \(D_+/PO^+(A,\mathbb{Z})\), then \(p_1 = p_2\).

For \(p \in \mathbb{P}(1,3,5) - \{ c_0 \}\), let
\[ \psi_p : H_2(S(p), \mathbb{Z}) \to H_2(\hat{S}, \mathbb{Z}) \]
be a P-marking naturally induced by the above proposition. The period of \(S(p)\) is given by
\[ \Phi(p) = \left( \int_{\psi_p^{-1}(\Gamma_1)} \omega : \int_{\psi_p^{-1}(\Gamma_2)} \omega : \int_{\psi_p^{-1}(\Gamma_3)} \omega : \int_{\psi_p^{-1}(\Gamma_4)} \omega \right). \] (2.13)

According to Remark \([1.1]\) the multivalued analytic mapping \((j^{-1} \circ \Phi)|_X : X \to \mathbb{H} \times \mathbb{H}\) gives a developing map of the canonical projection \(\Pi : \mathbb{H} \times \mathbb{H} \to (\mathbb{H} \times \mathbb{H})/(PSL(2,\mathcal{O}),\tau)\). Hence, by Proposition \([2.3]\) \((j^{-1} \circ \Phi)|_X\) is extended to the analytic mapping
\[ j^{-1} \circ \Phi : \mathbb{P}(1,3,5) - \{ c_0 \} \to \mathbb{H} \times \mathbb{H}. \]

This gives a developing map of \(\Pi\).

**Remark 2.4.** Sato \([5a]\) showed that the system of differential equations on \(X\)
\[
\begin{cases}
  u_{XX} = Lu_{XY} + Au_X + Bu_Y + Pu, \\
  u_{YY} = Mu_{XY} + Cu_X + Du_Y + Qu
\end{cases}
\]
with \(L = \frac{-20(4X^2 + 3XY - 4Y)}{36X^2 - 32X - Y}, M = \frac{-2(54X^3 - 50X^2 - 3XY + 2Y)}{5(36X^2 - 32X - Y)}\) is an uniformizing differential equation of \((\mathbb{H} \times \mathbb{H})/(PSL(2,\mathcal{O}),\tau)\). Namely, taking linearly independent solutions \(y_0, y_1, y_2\) and \(y_3\), the mapping \(p \mapsto (y_0(p) : \cdots : y_3(p))\) gives a developing map \(X \to D_+\). Of course, our equation \([1.14]\) is also an uniformizing differential equation in this sense. But, note that we do not know whether we can extend it to the singular locus applying the theory of the uniformizing differential equations. Since we regard \(\mathbb{P}(1,3,5) - \{ c_0 \}\) as the parameter space of \(\mathcal{F}\) and \(p \mapsto (y_0(p) : \cdots : y_3(p))\) is the period mapping for \(\mathcal{F}\), we obtain the extension of the solutions of \([1.14]\) to the singular locus.

Hence, we obtain the following theorem.

**Theorem 2.2.** The multivalued mapping \(j^{-1} \circ \Phi : \mathbb{P}(1,3,5) - \{ c_0 \} \to \mathbb{H} \times \mathbb{H}\) gives the developing map of \(\Pi\). Namely, the inverse mapping of \(\Pi : \mathbb{H} \times \mathbb{H} \to (\mathbb{H} \times \mathbb{H})/(PSL(2,\mathcal{O}),\tau)\) is given by \(j^{-1} \circ \Phi\) through the identification \((\mathbb{H} \times \mathbb{H})/(PSL(2,\mathcal{O}),\tau) \simeq \mathbb{P}(1,3,5) - \{ c_0 \}\) given by Proposition \([1.2](1)\).
Let $\Delta$ be the diagonal:
\[
\Delta = \{(z_1, z_2) \in \mathbb{H} \times \mathbb{H} | z_1 = z_2\}.
\]
From the above theorem and Proposition 1.2 (3), we have

**Corollary 2.1.** It holds that
\[
\Pi(\Delta) = \{(\mathfrak{A} : \mathfrak{B} : 0)\} - \{c_0\}
\]
through the identification $(\mathbb{H} \times \mathbb{H})/(PSL(2, \mathcal{O}), \tau) \simeq \mathbb{P}(1, 3, 5) - \{c_0\}$ given by Proposition 1.2 (1).

Due to Theorem 2.2, we obtain the system of coordinates $(z_1, z_2)$ of $\mathbb{H} \times \mathbb{H}$ coming from the multivalued period mapping (2.13) for the family of $K3$ surfaces $\{S(p)\}$:
\[
(z_1(p), z_2(p)) = \left(-\int_{\Gamma_3} \omega + \frac{1 - \sqrt{5}}{2} \int_{\Gamma_4} \omega, -\int_{\Gamma_3} \omega + \frac{1 + \sqrt{5}}{2} \int_{\Gamma_4} \omega\right).
\]
(2.14)

Here, for simplicity, let $\Gamma_j$ denotes the 2-cycle $\psi_p^{-1}(\Gamma_j)$ on $S(p)$ for $j \in \{1, 2, 3, 4\}$.

According to Proposition 1.2 (1), by adding one cusp, we have the compactification $(\mathbb{H} \times \mathbb{H})/(PSL(2, \mathcal{O}), \tau)$. Then, putting $\Pi \circ j^{-1} \circ \Phi(c_0) = (\sqrt{-1\infty}, \sqrt{-1\infty})$, we obtain an extended mapping
\[
\Pi \circ j^{-1} \circ \Phi : (1, 3, 5) \rightarrow (\mathbb{H} \times \mathbb{H})/(PSL(2, \mathcal{O}), \tau),
\]
(2.15)
where $(\sqrt{-1\infty}, \sqrt{-1\infty})$ stands for the $(PSL(2, \mathcal{O}), \tau)$ orbit of $(\sqrt{-1\infty}, \sqrt{-1\infty})$.

# 3 The family $\mathcal{F}_X$ and the period differential equation

In this section, we consider the family $\mathcal{F}_X = \{S(X, 0)\}$ and the diagonal $\Delta = \{(z_1, z_2) \in \mathbb{H} \times \mathbb{H} | z_1 = z_2\}$.

## 3.1 The family $\mathcal{F}_X$

In Section 2, we had the $K3$ surfaces $S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})$ for $(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \in \mathbb{P}(1, 3, 5) - \{c_0\}$ and the period mapping (2.13). Restricting them to $\{\mathfrak{C} = 0\}$, we obtain the family $\{S(\mathfrak{A} : \mathfrak{B} : 0) | (\mathfrak{A} : \mathfrak{B} : 0) \neq c_0\}$ of $K3$ surfaces with $S(\mathfrak{A} : \mathfrak{B} : 0) : z^2 = x^3 - 4y^2(4y - 5\mathfrak{A})x^2 + 20\mathfrak{B}y^3x$. Then, we have the family $\mathcal{F}_X = \{S(X, 0)\}$ of $K3$ surfaces with
\[
S(X, 0) : z^2 = x^3 - 4y^2(4y - 5\mathfrak{A})x^2 + 20\mathfrak{B}y^3x,
\]
where $X \left( = \frac{\mathfrak{B}}{\mathfrak{A}^2} \right) \in \mathbb{P}^1(\mathbb{C}) - \{0\}$. In this section, we consider the family $\mathcal{F}_X$ and the period mapping for $\mathcal{F}_X$.

Set $\Sigma = (X -$ sphere $\mathbb{P}^1(\mathbb{C})) - \{(0, \frac{25}{27}, \infty)\}$. Because we have Propostion 2.2, we can prove the following theorem for the subfamily $\mathcal{F}'_X = \{S(X, 0) | X \in \Sigma\}$ as in [N1].

**Theorem 3.1.** (1) For a generic point $X \in \Sigma$, the intersection matrix of the Néron-Severi lattice $NS(S(X, 0))$ is given by
\[
E_8(-1) \oplus E_8(-1) \oplus U \oplus (-2)
\]
and that of transcendental lattice $Tr(S(X, 0))$ is given by
\[
U \oplus (2) =: A_X.
\]
(2) The projective monodromy group of the multivalued period mapping for $\mathcal{F}'_X$ is isomorphic to $PO^+(A_X, \mathbb{Z})$. 

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From the period mapping $\Phi$ in (2.13), the system of coordinates $(z_1, z_2)$ in (2.14), Corollary 2.1 and the above theorem, we obtain a multivalued period mapping $\Phi_X$ for $\mathcal{F}_X$ such that

$$j^{-1} \circ \Phi_X : \{X | X \in \mathbb{P}^1(\mathbb{C}) - \{0\}\} \to \Delta,$$

(3.1)

where $\Phi_X$ is given by $X \mapsto (\xi_1 : \xi_2 : \xi_3 : \xi_4) = \left(\int_{\Gamma_1} \omega : \int_{\Gamma_2} \omega : \int_{\Gamma_3} \omega : 0\right) \in \mathcal{D}_+$. These are induced by the monodromy matrices in (1.6) generated by two elements:

Recalling (2.15), we obtain an extended mapping $\Pi \circ j^{-1} \circ \Phi_X : \mathbb{P}^1(\mathbb{C}) \to \overline{\Delta/M_X}$. We note $\Pi \circ j^{-1} \circ \Phi_X(0)$ is the $M_X$ orbit of $(\sqrt{-1} \infty, \sqrt{-1} \infty)$. The action of $M_X$ on $\Delta(\mathbb{C} \times \mathbb{H})$ induces the action of $PSL(2, \mathbb{Z})$ on $\mathbb{H}$, for we have the coordinate $z$ in (3.2). Namely, there exist $\gamma_1, \gamma_2 \in \pi_1(\Sigma, *)$ such that

$$\gamma_1(z) = z + 1, \quad \gamma_2(z) = -\frac{1}{z}.$$

So, $\overline{\Delta/M_X}$ is identified with the orbifold $\mathbb{H}/PSL(2, \mathbb{Z}) \simeq \mathbb{P}^1(\mathbb{C})$.

**Remark 3.1.** The projective monodromy group $M_X \simeq PO^+(A_X, \mathbb{Z})$ of the period mapping $\Phi_X$ is generated by two elements:

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(3.4)

These are induced by the monodromy matrices in (1.6).

## 3.2 The Gauss hypergeometric equation $2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; t\right)$

We recall the Gauss hypergeometric equation

$$2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; t\right) : t(1-t)\frac{d^2}{dt^2}u + (1 - \frac{3}{2}t)\frac{du}{dt} - \frac{5}{144}u = 0.$$

(3.5)

The Riemann scheme of $2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; t\right)$ is given by

$$\left\{ \begin{array}{lll} t = 0 & t = 1 & t = \infty \\ 0 & 0 & 1/12 \\ 0 & 1/2 & 5/12 \end{array} \right\}.$$

We can take the solutions $y_1(t)$ and $y_2(t)$ of $2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; t\right)$ such that the inverse mapping of the Schwarz mapping

$$\sigma : \mathbb{C} - \{0, 1\} \to \mathbb{H} : t \mapsto \sigma(t) = \frac{y_2(t)}{y_1(t)} = z_0$$

(3.6)

is given by

$$z_0 \mapsto \frac{1}{J(z_0)}.$$

(3.7)

where $J(z)$ is the elliptic $J$ function with $J\left(\frac{1+\sqrt{-3}}{2}\right) = 0$, $J(\sqrt{-1}) = 1$ and $J(\sqrt{-1}\infty) = \infty$.  

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Remark 3.2. The above \( J \) function is given by

\[
J(z) = \frac{1}{1728} \left( \frac{1}{q} + 744 + 196884q + \cdots \right), \tag{3.8}
\]

where \( q = e^{2\pi \sqrt{-1} z} \).

Note that the Schwarz mapping \( \sigma \) is a multivalued analytic mapping. We can choose the single-valued branch of the Schwarz mapping \( \sigma \) on \((0, 1) \subset \mathbb{R}\) such that \( \sigma(t) \in \sqrt{-1} \mathbb{R} \) and

\[
\lim_{t \to +0} \sigma(t) = \sqrt{-1} \infty, \quad \lim_{t \to -1} \sigma(t) = \sqrt{-1}. \tag{3.9}
\]

Then, the single-valued branch of the solutions \( y_1(t) \) and \( y_2(t) \) near \((0, 1) \subset \mathbb{R}\) is in the form

\[
\begin{aligned}
    y_1(t) &= u_{11}(t), \\
    y_2(t) &= \log(t) \cdot u_{21}(t) + u_{22}(t),
\end{aligned} \tag{3.10}
\]

where \( u_{jk}(t) \) are unit holomorphic functions around \( t = 0 \) and \( \log \) stands for the principal value.

The projective monodromy group of \( 2E_1 \left( \frac{1}{12}, \frac{5}{12}; t \right) \) is isomorphic to \( PSL(2, \mathbb{Z}) \). In other words, the action of the fundamental group \( \pi_1(\mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty, *\}) \) on \( \mathbb{H} = \left\{ z_0 = \frac{y_2}{y_1} \right\} \) is generated by the two actions

\[
\begin{aligned}
    z_0 &\mapsto z_0 + 1, \\
    z_0 &\mapsto -\frac{1}{z_0},
\end{aligned} \tag{3.11}
\]

if we normalize a basis \( y_1, y_2 \) of the solutions of \( 2E_1 \left( \frac{1}{12}, \frac{5}{12}; t \right) \) around a base point.

Remark 3.3. The projective monodromy group for the system \((y_2^2(t); -y_1^2(t); y_1(t)y_2(t))\) is generated by the two matrices in (3.4).

3.3 The period differential equation

In this subsection, we determine the period differential equation for the family \( \mathcal{F}_X \). Then, considering the solutions of this period differential equation, we shall obtain the expression of \( X \) using the coordinate \( z \) in (3.2).

Proposition 3.1. On the locus \( \{ Y = 0 \} \), the period differential equation \((1.14)\) is restricted to the following ordinary differential equation of rank 4:

\[
\begin{aligned}
    \frac{d^4}{dX^4} u &= \frac{1}{2} \frac{3(243X^2 - 4060X + 2000)}{2X(81X^2 - 1155X + 1000)} \frac{d^3}{dX^3} u + \frac{2034X^2 - 40680X + 8000}{8X^2(81X^2 - 1155X + 1000)} \frac{d^2}{dX^2} u + \frac{15(3X - 80)}{8X^2(81X^2 - 1155X + 1000)} \frac{d}{dX} u = 0. \tag{3.12}
\end{aligned}
\]

Proof. Recalling the period differential equation \((1.14)\), set

\[
\begin{aligned}
    E_1 u &= L_1 u_{XY} + A_1 u_X + B_1 u_Y + P_1 u, \\
    E_2 u &= M_1 u_{XY} + C_1 u_X + D_1 u_Y + Q_1 u.
\end{aligned}
\]

Deriving these equations, we have the system of equations

\[
\begin{aligned}
    u_{XX} &= E_1 u, \quad u_{XXX} = \frac{\partial}{\partial X} E_1 u, \quad u_{XXY} = \frac{\partial}{\partial Y} E_1 u, \quad u_{XXX} = \frac{\partial^2}{\partial X^2} E_1 u, \quad u_{XXXY} = \frac{\partial^2}{\partial X \partial Y} E_1 u, \\
    u_{YY} &= E_2 u, \quad u_{XY} = \frac{\partial}{\partial X} E_2 u, \quad u_{Y} = \frac{\partial}{\partial Y} E_2 u, \quad u_{XXY} = \frac{\partial^2}{\partial Y^2} E_1 u = \frac{\partial^2}{\partial X^2} E_2 u.
\end{aligned}
\]
Our periods satisfy this system. From this system, canceling the terms \( u_Y, u_{XY}, u_{YY}, u_{XYY}, u_{YYY}, \) we can obtain the differential equation

\[
a_4(X, Y)u_{XXX} + a_3(X, Y)u_{XX} + a_2(X, Y)u_X + a_1(X, Y)u + a_0(X, Y)u = 0,
\]

where \( a_j(X, Y) \) \((j = 1, 2, 3, 4)\) is a polynomial in \( X \) and \( Y \). Putting \( Y = 0 \), we have (3.12).

Set

\[
\bar{\eta}_j(X) = \int_{\gamma_j} \omega \quad (j \in \{1, 2, 3\}).
\]

The equation (3.12) has the 4-dimensional space of solutions generated by \( \bar{\eta}_1(X), \bar{\eta}_2(X), \bar{\eta}_3(X) \) and 1. The Riemann scheme of (3.12) is given by

\[
\begin{align*}
X = 0 & \quad X = 25/27 & \quad X = 40/3 & \quad X = \infty \\
0 & \quad 0 & \quad 0 & \quad 0 \\
1 & \quad 1/2 & \quad 1 & \quad -5/6 \\
1 & \quad 1 & \quad 2 & \quad -1/2 \\
1 & \quad 2 & \quad 4 & \quad -1/6
\end{align*}
\]

Setting \( X = \frac{25}{27}t \), the equation (3.12) is transposed to

\[
W_4u = 0,
\]

where

\[
W_4 = \frac{d^4}{dt^4} + \frac{1620t^3 - 29232t^2 + 15552t}{72t^2(t - 1)(5t - 72)} \frac{d^3}{dt^3} + \frac{565t^2 - 12204t + 2592}{36t^2(t - 1)(5t - 72)} \frac{d^2}{dt^2} + \frac{25t - 720}{72t^2(t - 1)(5t - 72)} \frac{d}{dt}.
\]

Straightforward calculation shows the following.

**Proposition 3.2.** Set

\[
W_4 = \frac{d^4}{dt^4} + \frac{1620t^3 - 29232t^2 + 15552t}{72t^2(t - 1)(5t - 72)} \frac{d^3}{dt^3} + \frac{565t^2 - 12204t + 2592}{36t^2(t - 1)(5t - 72)} \frac{d^2}{dt^2} + \frac{25t - 720}{72t^2(t - 1)(5t - 72)} \frac{d}{dt}.
\]

It holds that

\[
W_4 = W_1 \circ W_3.
\]

Set \( \eta_j(t) = \bar{\eta}_j\left(\frac{25}{27}t\right) \) for \( j \in \{1, 2, 3\} \).

**Proposition 3.3.** The periods \( \eta_1(t), \eta_2(t) \) and \( \eta_3(t) \) are the solutions of

\[
W_3u = 0
\]

satisfying

\[
\eta_1\eta_2 + \eta_3^2 = 0.
\]

**Proof.** Let \( V = \langle \eta_1, \eta_2, \eta_3 \rangle_\mathbb{C} \) and \( V' = \langle W_3\eta_1, W_3\eta_2, W_3\eta_3 \rangle_\mathbb{C} \). Since the linear mapping \( W_3 : V \to V' \) given by \( f \mapsto W_3f \) is monodromy-equivalent and \( V \) is an irreducible representation, according to Schur’s lemma, we have \( V \cong V' \) or \( V' = \{0\} \). It follows from (3.13) that \( V' \subseteq \text{Ker}(W_1) \). Because \( \dim(\text{Ker}(W_1)) = 1 \), we have \( V' = \{0\} \).

For \( t \mapsto (\eta_1(t) : \eta_2(t) : \eta_3(t)) \) is the period mapping \( \Phi_X \), the relation (3.14) is clear.

**Proposition 3.4.** If \( u_1 \) and \( u_2 \) are solutions of \( 2E_1\left( \frac{1}{12}, \frac{5}{12}; 1; t \right) \), then \( tu_1^2(t), tu_2^2(t) \) and \( tu_1(t)u_2(t) \) are solutions of the period differential equation \( W_3u = 0 \).
Proof. Take any solutions of \( _2F_1 \left( \frac{1}{12}, \frac{5}{12}; 1; t \right) \) \( u_1(t) \) and \( u_2(t) \). For \( j \in \{1, 2\} \),

\[
u_j'' = \frac{1 - 3t/2}{t(t - 1)} u_j' - \frac{5}{144(t - 1)} u_j,
\]

then

\[
u_j^{(3)} = \frac{535t^2 - 715t + 288}{144t^2(t - 1)^2} u_j' + \frac{5(7t - 4)}{288t^2(t - 1)^2} u_j.
\]

Here, by a straightforward calculation, we have

\[
W_3(tu_1u_2) = \frac{5}{72(t - 1)} u_1u_2 + \frac{113t - 36}{36(t - 1)} (u_1'u_2 + u_1u_2') + \frac{3(3t - 2)}{t - 1} u_1'u_2
\]

\[
+ \frac{3(3t - 2)}{2(t - 1)} (u_1''u_2 + u_1u_2'') + 3t(u_1''u_2' + u_1'u_2'') + t(u_1^{(3)}u_2 + u_1u_2^{(3)}). \tag{3.17}
\]

Substituting (3.15) and (3.16) for (3.17), we have \( W_3(tu_1u_2) = 0 \). \( \square \)

Remark 3.4. According to (3.12), the derivation \( \frac{d}{dt} \eta_j \ (j = 1, 2, 3) \) of the period is a solution of the equation

\[
\frac{d^3}{dt^3} + \frac{1620t^3 - 29232t^2 + 15552t}{72t^2(t - 1)(5t - 72)} \frac{d^2}{dt^2} + \frac{1130t^2 - 24408t + 5184}{72t^2(t - 1)(5t - 72)} \frac{d}{dt} + \frac{25t - 72}{72t^2(t - 1)(5t - 72)} v = 0. \tag{3.18}
\]

Then, set

\[
S(t) = \left( \begin{array}{c} \frac{7}{6} \\ \frac{1}{2} \\ \frac{5}{6} \\ \frac{1}{6} \\ \frac{1}{2} \end{array} \right) + \frac{1}{5} \left( \begin{array}{c} 3 \\ 6 \\ 36 \\ 6 \\ 36 \end{array} \right),
\]

where \( _3F_2 \) is the generalized hypergeometric series:

\[
_3F_2(a_1, a_2, a_3; b_1, b_2; t) = \sum_{n=0}^{\infty} \frac{(a_1, n)(a_2, n)(a_3, n)}{(b_1, n)(b_2, n)n!} t^n.
\]

We see that \( S(t) \) is a holomorphic solution of (3.18) around \( t = 0 \). The indefinite integral of \( S(t) \) with the integral constant \( 0 \) is given by

\[
t \cdot _3F_2 \left( \frac{1}{6}, \frac{1}{2}, \frac{5}{6}; 1, 1; t \right) + \frac{1}{5} t \cdot _3F_2 \left( \frac{7}{6}, \frac{1}{2}, \frac{5}{6}; 1, 1; t \right)
\]

\[
= \frac{6}{5} t \cdot _3F_2 \left( \frac{1}{6}, \frac{1}{2}, \frac{5}{6}; 1, 1; t \right) = \frac{6}{5} \cdot \left( _2F_1 \left( \frac{1}{12}, \frac{5}{12}; 1 \right) \right)^2.
\]

Here, we applied Clausen's formula. From the above proposition, this gives a holomorphic solution of \( W_3u = 0 \) around \( t = 0 \).

Let \( y_1(t) \) and \( y_2(t) \) are the single-valued branches of the solutions of \( _2F_1 \left( \frac{1}{12}, \frac{5}{12}; 1; t \right) \) near \( (0, 1) \subset \mathbb{R} \) given in (3.9). Let

\[
s_1(t) = ty_1^2(t), \quad s_2(t) = ty_1(t)y_2(t), \quad s_3(t) = ty_2^2(t).
\]

Note that, if \( t \in (0, 1) \subset \mathbb{R} \), we have

\[
\begin{align*}
s_1(t) &= t \cdot v_1(t), \\
s_2(t) &= t \cdot (\log(t)v_2(t) + v_3(t)), \\
s_3(t) &= t \cdot (\log^2(t)v_3(t) + \log(t)v_3(t) + v_3(t)),
\end{align*}
\]

where \( v_j(t) \) are unit holomorphic functions around \( t = 0 \). Moreover, they satisfy

\[
-s_1(t)s_3(t) + s_2^2(t) = 0. \tag{3.20}
\]
Lemma 3.1. A branch of the multivalued analytic mapping \( t \mapsto (\eta_1(t) : \eta_2(t) : \eta_3(t)) \) satisfies

\[
(\eta_1(t) : \eta_2(t) : \eta_3(t)) = (s_3(t) : -s_1(t) : s_2(t)) \in \mathbb{P}^2(\mathbb{C}).
\]

Proof. Because we have Proposition \ref{prop:1.2} (1) and the coordinate \( z \) in \ref{eq:3.2}, we take the single-valued branch of the multivalued period mapping \( t \mapsto (\eta_1(t) : \eta_2(t) : \eta_3(t)) \) on \( t \in (0, 1) \subset \mathbb{R} \) such that

\[
\lim_{t \to +0} \frac{\eta_3(t)}{\eta_2(t)} = \sqrt{-1}\infty.
\]

(3.21)

In this proof, we consider \( \eta_1(t), \eta_2(t) \) and \( \eta_3(t) \) near \((0, 1) (\subset \mathbb{R})\).

According to Proposition \ref{prop:3.1}, we have

\[
\eta_j(t) = \sum_{k=1}^{3} a_{jk} s_k(t) \quad (j = 1, 2, 3),
\]

where \( a_{jk} \ (j, k = 1, 2, 3) \) are constants. Since we have \ref{eq:3.21}, we obtain \( a_{23} = 0 \). So, it follows that \( \eta_2(t) = a_{21} s_1(t) + a_{22} s_2(t) \). From \ref{eq:3.19}, we see that \( \eta_1(t) \eta_2(t) \) does not contain \( \log^4(t) \). Then, from \ref{eq:3.14}, we have \( a_{33} = 0 \). Recalling \ref{eq:3.21} again, we obtain \( a_{22} = 0 \). Because we consider \( y \mapsto (\eta_1(t) : \eta_2(t) : \eta_3(t)) \in \mathbb{P}^2(\mathbb{C}) \), we assume that \( a_{21} = -1 \). Then, the single-valued branches \( \eta_j(t) \ (j = 1, 2, 3) \) are in the form

\[
\left\{ \begin{array}{l}
\eta_1(t) = a_{11} s_1(t) + a_{12} s_2(t) + a_{13} s_3(t), \\
\eta_2(t) = -s_1(t), \\
\eta_3(t) = a_{31} s_1(t) + a_{32} s_2(t).
\end{array} \right.
\]

Hence, using \ref{eq:3.6}, the coordinate \( z \) in \ref{eq:3.2} is given by

\[
z = a_{32} \frac{s_2(z)}{s_1(z)} + a_{31} = a_{32} z_0 + a_{31}.
\]

Considering the actions of \( \pi_1(\mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}) \) on \( z = \frac{\eta_3}{\eta_2} \) -space in \ref{eq:3.3} and \( z_0 = \frac{y_2}{y_1} \) -space in \ref{eq:3.11}, we have \( a_{31} = 0 \) and \( a_{32} = 1 \).

Therefore, using \ref{eq:3.14} again, we obtain

\[
\eta_1(t) = s_3(t), \quad \eta_2(t) = -s_1(t), \quad \eta_3(t) = s_2(t).
\]

Corollary 3.1. A coordinate \( z \) in \ref{eq:3.2} of the diagonal \( \Delta \ (\simeq \mathbb{H}) \) is equal to

\[
z = \frac{y_2(t)}{y_1(t)}.
\]

Proof. From the above lemma, this is clear.

Theorem 3.2. The inverse of the multivalued period mapping \( j^{-1} \circ \Phi_X : X \mapsto (z, z) \) in \ref{eq:3.1} is given by

\[
X(z, z) = \frac{25}{27} \cdot \frac{1}{J(z)}.
\]

Proof. From the above Corollary and the inverse Schwarz mapping \ref{eq:3.7}, we have \( t(z) = \frac{1}{J(z)} \). Therefore, we obtain

\[
X(z, z) = \frac{25}{27} \cdot t(z) = \frac{25}{27} \cdot \frac{1}{J(z)}.
\]
4 The theta expressions of $X$ and $Y$

First, we recall the classical elliptic functions. Let $z \in \mathbb{H}$.

The classical Eisenstein series are given by

$$G_2(z) = 60 \sum_{(0,0) \neq (m,n) \in \mathbb{Z}^2} \frac{1}{(mz+n)^4}, \quad G_3(z) = 140 \sum_{(0,0) \neq (m,n) \in \mathbb{Z}^2} \frac{1}{(mz+n)^6}.$$

$G_2(z)$ ($G_3(z)$, resp.) is a modular form of weight 4 (6, resp.) for $PSL(2, \mathbb{Z})$. The ring of modular forms for $PSL(2, \mathbb{Z})$ is $\mathbb{C}[G_2, G_3]$. We have $G_2(\sqrt{-1}\infty) = \frac{4\pi^4}{3}$ and $G_3(\sqrt{-1}\infty) = \frac{8\pi^6}{27}$. Let $E_4(z) = \frac{3}{4\pi^4} G_2(z)$ and $E_6(z) = \frac{27}{8\pi^6} G_3(z)$ be the normalized Eisenstein series. The discriminant form is

$$\Delta(z) = G_2^3(z) - 27G_3^2(z).$$

We have $\Delta(\sqrt{-1}\infty) = 0$. This is a cusp form of weight 12. The cusp form of weight 12 is $\Delta$ up to a constant factor. The $J$ function in $(3.8)$ is given by

$$J(z) = \frac{G_3(z)}{G_2(z) - 27G_3^2(z)} = \frac{G_3(z)}{\Delta(z)}.$$ (4.1)

The field of modular functions for the modular group $PSL(2, \mathbb{Z})$ is $\mathbb{C}(J(z))$.

For $a, b \in \{0, 1\}$, the Jacobi theta constants are defined by

$$\vartheta_{ab}(z) = \sum_{n \in \mathbb{Z}} \exp \left( \sqrt{-1}\pi \left( n + \frac{a}{2} \right)^2 z + 2\sqrt{-1}\pi \left( n + \frac{a}{2} \right) \frac{b}{2} \right)$$

for $(a, b) = (0, 0), (0, 1)$ and $(1, 0)$. The functions $\vartheta_{00}(z), \vartheta_{01}(z)$ and $\vartheta_{10}(z)$ are the modular forms of weight 2 for the principal congruence subgroup $\Gamma(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \, | \, \alpha \equiv \delta \equiv 1, \beta \equiv \gamma \equiv 0 \pmod{2} \right\}$.

The ring of modular forms for $\Gamma(2)$ is

$$\mathbb{C}[\vartheta_{00}^4, \vartheta_{01}^4, \vartheta_{10}^4]/(\vartheta_{01}^4 + \vartheta_{10}^4 = \vartheta_{00}^4) = \mathbb{C}[\vartheta_{00}^4, \vartheta_{01}^4].$$

We note that

$$\frac{1}{1728} \left( \frac{3}{4\pi^4} \right)^3 \Delta(z) = \frac{1}{2\pi^8} \vartheta_{00}^8(z) \vartheta_{01}^8(z) \vartheta_{10}^8(z).$$

Next, we survey the theta constants for Hilbert modular forms for $\mathbb{Q}(\sqrt{5})$. They are introduced by Müller [M].

Set

$$\mathcal{S}_2 = \{ Z \in \text{Mat}(2, 2) \mid \text{tr}Z = Z, \text{Im}(Z) > 0 \}.$$

This is the Siegel upper half plane consisting of $2 \times 2$ complex matrices. For $a, b \in \{0, 1\}^2$ with $^t ab \equiv 0 \pmod{2}$, set

$$\vartheta(Z; a, b) = \sum_{g \in \mathbb{Z}^2} \exp \left( \pi \sqrt{-1} \left( g + \frac{1}{2} a \right) Z \left( g + \frac{1}{2} a \right)^t + \frac{1}{2} gb \right).$$

We use the mapping $\psi : \mathbb{H} \times \mathbb{H} \to \mathcal{S}_2$ given by

$$(z_1, z_2) = \zeta \mapsto \begin{pmatrix} \text{Tr} \left( \frac{\zeta}{\sqrt{5}} \right) \\ \text{Tr} \left( \frac{\zeta}{\sqrt{5}} \right) \\ \text{Tr} \left( \frac{-\zeta}{\sqrt{5}} \right) \end{pmatrix} = \frac{1}{2\sqrt{5}} \begin{pmatrix} (1 + \sqrt{5})z_1 - (1 - \sqrt{5})z_2 \\ 2(z_1 - z_2) \\ (1 + \sqrt{5})z_1 + (1 + \sqrt{5})z_2 \end{pmatrix},$$

where $\zeta = \frac{1 + \sqrt{5}}{2}$. 

\[19\]
Remark 4.1. Set
\[ N_5 = \left\{ \left( \begin{array}{cc} \sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_3 \end{array} \right) \in \mathfrak{S}_2 \mid -\sigma_1 + \sigma_2 + \sigma_3 = 0 \right\}. \]
Let \( p \) be the canonical projection \( \mathfrak{S}_2 \to \mathfrak{S}_2/Sp(4, \mathbb{Z}). \) Then, the Humbert surface \( \mathcal{H}_5 = p(N_5) \) of invariant 5 gives the moduli space of principally polarized Abelian surfaces \( A \) such that \( \mathbb{Q}(\sqrt{5}) \subset \text{End}(A) \otimes \mathbb{Q}. \) We note that the above \( \psi \) is a mapping \( \mathbb{H} \times \mathbb{H} \to N_5. \)

For \( j \in \{0, 1, \ldots, 9\}, \) we set
\[ \theta_j(z_1, z_2) = \psi(z_1, z_2); a, b, \]
where the correspondence between \( j \) and \( (a, b) \) is given by Table 1. These theta constants are holomorphic functions on \( \mathbb{H} \times \mathbb{H} \).

Let \( a \in \mathbb{Z} \) and \( j_1, \ldots, j_r \in \{0, \ldots, 9\}. \) We set \( \theta_{j_1}^a \cdots j_r = \theta_{j_1}^a \cdots \theta_{j_r}^a. \)

Set \( s_5 = 2^{-6} \theta_{0123456789}. \) This is an alternating modular form of weight 5. The following \( g_2 \) \( (s_6, s_{10}, s_{15}, \text{resp.}) \) is a symmetric Hilbert modular form of weight 2 \( (6, 10, 15, \text{resp.}) \) for \( \mathbb{Q}(\sqrt{5}):\)

\[
\begin{align*}
  g_2 &= \theta_{0145} - \theta_{1279} - \theta_{3478} + \theta_{0268} + \theta_{3569}, \\
  s_6 &= 2^{-3}(\theta_{012478}^2 + \theta_{012569}^2 + \theta_{034568}^2 + \theta_{236789}^2 + \theta_{134579}^2), \\
  s_{10} &= s_5^2 = 2^{-12} \theta_{0123456789}, \\
  s_{15} &= -2^{-18}(\theta_{07}^3 \theta_{18} \theta_{24} - \theta_{25}^6 \theta_{16} \theta_{09} + \theta_{03}^5 \theta_{32} - \theta_{16}^2 \theta_{07}^3 \theta_{16} + \theta_{09}^2 \theta_{16}^3 \theta_{25} - \theta_{07}^2 \theta_{23}^3 \theta_{89} + \theta_{18}^2 \theta_{25} \theta_{07} - \theta_{24}^2 \theta_{15} \theta_{07} - \theta_{26}^2 \theta_{03}^5 \theta_{58} - \theta_{24}^2 \theta_{18}^3 \theta_{18} - \theta_{89}^2 \theta_{23}^3 \theta_{23} - \theta_{07}^2 \theta_{24}^3 \theta_{18} \\
&\quad + \theta_{89}^2 \theta_{25}^3 \theta_{27} - \theta_{16}^2 \theta_{09}^3 \theta_{25} - \theta_{03}^5 \theta_{16}^3 \theta_{58} + \theta_{16}^2 \theta_{25}^3 \theta_{09} - \theta_{16}^2 \theta_{25}^3 \theta_{03} \\
&\quad - \theta_{25}^2 \theta_{09}^3 \theta_{16} - \theta_{57}^2 \theta_{49}^3 \theta_{13} + \theta_{07}^2 \theta_{89}^3 \theta_{23} + \theta_{58}^2 \theta_{26}^3 \theta_{03} + \theta_{57}^2 \theta_{13}^3 \theta_{49} - \theta_{23}^2 \theta_{89}^3 \theta_{07} + \theta_{18}^2 \theta_{07}^3 \theta_{24} + \theta_{03}^5 \theta_{58}^3 \theta_{46} + \theta_{25}^2 \theta_{07}^3 \theta_{89} + \theta_{16}^2 \theta_{25}^3 \theta_{13} - \theta_{13}^2 \theta_{57}^3 \theta_{49} + \theta_{13}^2 \theta_{49}^3 \theta_{57}). \quad (4.2)
\end{align*}
\]

Proposition 4.1. (M Satz 1) (1) The ring of the symmetric Hilbert modular forms for \( \mathbb{Q}(\sqrt{5}) \) is given by
\[ \mathbb{C}[g_2, s_6, s_{10}, s_{15}]/(M(g_2, s_6, s_{10}, s_{15}) = 0), \]
where
\[
M(g_2, s_6, s_{10}, s_{15}) = s_{15}^2 - \left( \frac{5^5 3^3}{2} s_{10}^2 + \frac{5^3}{2} g_2 s_{10}^2 + \frac{3^2 \cdot 5^2}{2} g_2^2 s_{10} - \frac{1}{23} g_2^4 s_{10} - 2 \cdot 3^3 s_{6}^2 + \frac{1}{23} g_2^3 s_{6}^2 \right). \quad (4.3)
\]

(2) The ring of the Hilbert modular forms for \( \mathbb{Q}(\sqrt{5}) \) is given by
\[ \mathbb{C}[g_2, s_5, s_{6}, s_{15}]/(M(g_2, s_5^2, s_6, s_{15}) = 0). \]

Proposition 4.2. (M pp.244-245) Müller’s modular forms satisfy
\[
\begin{align*}
g_2(i\infty, i\infty) &= 1, \\
s_6(z, z) &= \frac{2}{1728} \left( \frac{3}{4\pi i} \right)^3 \Delta(z) = \frac{1}{2^7} \theta_{00}^8(z) \theta_{01}^8(z) \theta_{10}^8(z), \\
s_{10}(z, z) &= 0.
\end{align*}
\]
Especially, the relations
\[
\begin{align*}
\frac{4\pi^4}{3} g_2(z, z) &= \frac{4\pi^4}{3} E_4(z) = G_2(z), \\
2^{11} \pi^{12} s_6(z, z) &= G_2^3(z) - 27G_3^2(z) = \Delta(z)
\end{align*}
\]
hold.

Now, we obtain the theta expressions of the parameters \( X \) and \( Y \) for the family \( F \). According to Proposition 1.1 \( \{X = \frac{a_3}{3a_4}, Y = \frac{c}{3a_5}\} \) gives a system of generators of symmetric Hilbert modular functions for \( \mathbb{Q}(\sqrt{5}) \). From Theorem 2.2 the inverse correspondence \((z_1, z_2) \mapsto (X(z_1, z_2), Y(z_1, z_2))\) of the multivalued period mapping for \( F \) defines the pair of Hilbert modular functions of variables \( z_1 \) and \( z_2 \) in (2.14). In the following argument, we shall obtain the expression of \( X(z_1, z_2) \) and \( Y(z_1, z_2) \) as the quotients of Müller’s modular forms.

For our argument, we set \( Z = \frac{D^2}{3A^15} \). This defines a symmetric Hilbert modular function for \( \mathbb{Q}(\sqrt{5}) \) also.

**Lemma 4.1.** The modular functions \( X(z_1, z_2), Y(z_1, z_2) \) and \( Z(z_1, z_1) \) have the expressions
\[
\begin{align*}
X(z_1, z_2) &= k_1 \frac{s_6(z_1, z_2)}{g_2^6(z_1, z_2)}, \\
Y(z_1, z_2) &= k_2 \frac{s_{10}(z_1, z_2)}{g_2^6(z_1, z_2)}, \\
Z(z_1, z_2) &= k_3 \frac{s_{15}(z_1, z_2)}{g_2^{15}(z_1, z_2)},
\end{align*}
\]
for some \( k_1, k_2 \) and \( k_3 \in \mathbb{C} \).

**Proof.** Since \( X = \frac{a_3}{3a_4} \), the modular function \( X \) is given by the quotient of Hilbert modular forms of weight 6 and its denominator is the cube of a Hilbert modular form of weight 2. Note that, a Hilbert modular form of weight 2 is equal to \( g_2 \) up to a constant factor. Then, we have
\[
X(z_1, z_2) = \frac{k_{11} s_6(z_1, z_2) + k_{12} g_2^3(z_1, z_2)}{k_{13} g_2^3(z_1, z_2)},
\]
where \( k_{11}, k_{12} \) and \( k_{13} \) are constants. Recalling Proposition 1.2 (1), we have \( X(\sqrt{-1\infty}, \sqrt{-1\infty}) = 0 \). Then, from Proposition 4.2 we obtain \( k_{12} = 0 \) and
\[
X(z_1, z_2) = k_1 \frac{s_6(z_1, z_2)}{g_2^6(z_1, z_2)}.
\]

Since \( Y = \frac{c}{3a_5} \), the modular function \( Y \) is given by the quotient of Hilbert modular forms of weight 10. Its denominator is the 5-th power of a modular form of weight 2. Then, we have
\[
Y(z_1, z_2) = \frac{k_{21} s_{10}(z_1, z_2) + k_{22} g_2^5(z_1, z_2) + k_{23} g_2^2(z_1, z_2) s_6(z_1, z_2)}{k_{24} g_2^5(z_1, z_2)},
\]
where \( k_{21}, k_{22}, k_{23} \) and \( k_{24} \) are constants. By Proposition 1.2 (3), we have \( Y(z, z) = 0 \). According to (1.2) and Proposition 1.2 if a modular form \( g \) of weight 10 vanishes on the diagonal \( \Delta \), then we have \( g = \text{const} \cdot s_{10} \). So, it holds that \( k_{22} = k_{23} = 0 \). Therefore, we obtain
\[
Y(z_1, z_2) = k_2 \frac{s_{10}(z_1, z_2)}{g_2^6(z_1, z_2)}.
\]
Recalling Proposition 1.1 (2), we note that \( \mathcal{D} \) defines a symmetric Hilbert modular form of weight 15. Since \( Z = \frac{\mathcal{D}^2}{15} \), the modular function \( Z \) is given by the quotient of modular forms of weight 30. Its denominator is the 15-th power of a modular form of weight 2 and its numerator is given by the square of a symmetric modular form of weight 15. According to Proposition 4.1 (2), a symmetric modular form of weight 15 is given by \( \text{const} \cdot s_{15} \). Then, we have
\[
Z(z_1, z_2) = k_3 \frac{s_{15}^2(z_1, z_2)}{g_2(z_1, z_2)}.
\]

**Theorem 4.1.** The inverse correspondence of the multivalued period mapping \( j^{-1} \circ \Phi : (X,Y) \mapsto (z_1, z_2) \) in (2.14) for the family \( \mathcal{F} \) is given by the quotient of Müller's modular forms:
\[
\begin{cases}
X(z_1, z_2) = 2^5 \cdot 5^2 \cdot \frac{s_6(z_1, z_2)}{g_2(z_1, z_2)};
Y(z_1, z_2) = 2^{10} \cdot 5^5 \cdot \frac{s_{10}(z_1, z_2)}{g_2(z_1, z_2)}.
\end{cases}
\]

**Proof.** First, we obtain the expression of \( X \). To obtain it, we determine the constant \( k_1 \) in (4.3). Due to Theorem 3.2 (11) and Proposition 4.2 we have
\[
X(z, z) = \frac{25}{27} \cdot \frac{1}{J(z)} = 25 \cdot \frac{2^{11} \pi^{12} s_6(z, z)}{\left( \frac{4 \pi^4}{3} \right)^3} g_2^3(z, z) = 2^5 \cdot 5^2 \cdot \frac{s_6(z, z)}{g_2^2(z, z)}.
\]
So, we obtain \( k_1 = 2^5 \cdot 5^2 \).

Next, we determine the constant \( k_3 \) in (4.4). By (3.9), we have
\[
144Z(z_1, z_2) = -1728X^5(z_1, z_2) + 720X^3(z_1, z_2)Y(z_1, z_2)
- 80X(z_1, z_2)Y^2(z_1, z_2) + 64(5X^2(z_1, z_2) - Y(z_1, z_2))^2 + Y^3(z_1, z_2).
\]
(4.5)

Recalling that \( Y(z, z) = 0 \), we have
\[
144Z(z, z) = -1728X^5(z, z) + 64 \cdot 25 \cdot X^4(z, z)
= -2^{26} \cdot 5^{10} \cdot \left( 2^5 \cdot 3^3 \cdot \frac{s_6(z, z)}{g_2^2(z, z)} - 1 \right) \left( \frac{s_6(z, z)}{g_2^2(z, z)} \right)^4.
\]
(4.6)

On the other hand, from (4.3), we have
\[
\frac{s_{15}^2(z_1, z_2)}{g_2^{15}(z_1, z_2)} = 5^5 \left( \frac{s_{10}(z_1, z_2)}{g_2^4(z_1, z_2)} \right)^3 - \frac{5^3}{2} \frac{s_6(z_1, z_2)}{g_2^2(z_1, z_2)} \left( \frac{s_{10}(z_1, z_2)}{g_2^4(z_1, z_2)} \right)^2 + \frac{3^2 \cdot 5^2}{2} \frac{s_6(z_1, z_2)}{g_2^2(z_1, z_2)} \left( \frac{s_{10}(z_1, z_2)}{g_2^4(z_1, z_2)} \right)^2 \left( \frac{s_6(z_1, z_2)}{g_2^2(z_1, z_2)} \right)
+ \frac{1}{2^4} \left( \frac{s_{10}(z_1, z_2)}{g_2^4(z_1, z_2)} \right)^2 - \frac{1}{2^3} \left( \frac{s_6(z_1, z_2)}{g_2^2(z_1, z_2)} \right)^2 \left( \frac{s_{10}(z_1, z_2)}{g_2^4(z_1, z_2)} \right) - 2 \cdot 3^3 \left( \frac{s_6(z_1, z_2)}{g_2^2(z_1, z_2)} \right)^5 + \frac{1}{2^4} \left( \frac{s_6(z_1, z_2)}{g_2^2(z_1, z_2)} \right)^4.
\]
(4.7)

So, because \( s_{10}(z, z) = 0 \), we have
\[
\left( \frac{s_{15}^2(z, z)}{g_2^{15}(z, z)} \right) = \frac{1}{2^4} \left( -2^{26} \cdot 5^{10} \cdot 3^{-2} \cdot \frac{s_6(z, z)}{g_2^2(z, z)} + 1 \right) \left( \frac{s_6(z, z)}{g_2^2(z, z)} \right)^4.
\]
(4.8)

Since
\[
Z(z, z) = k_3 \frac{s_{15}^2(z, z)}{g_2^{15}(z, z)},
\]
comparing (4.10), (4.8), we have \( k_3 = 2^{26} \cdot 5^{10} \cdot 3^{-2} \).

Finally, from (4.3), (4.4), \( k_1 = 2^5 \cdot 5^2 \) and \( k_3 = 2^{26} \cdot 5^{10} \cdot 3^{-2} \), we have
\[
k_2 = 2^{10} \cdot 5^5.
\]

Thus, we obtain the explicit theta expression of the inverse correspondence \((z_1, z_2) \mapsto (X(z_1, z_2), Y(z_1, z_2))\) of the period mapping for our family \( \mathcal{F} \) of K3 surfaces.
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