Cross-Kerr nonlinearity in optomechanical systems

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We consider the response of a nanomechanical resonator interacting with an electromagnetic cavity via a radiation pressure coupling and a cross-Kerr coupling. Using a mean field approach we solve the dynamics of the system, and show the different corrections coming from the radiation pressure and the cross-Kerr effect to the usually considered linearized dynamics.

I. INTRODUCTION

Cavity optomechanics offers a framework to study the coupling between an electromagnetic field and the vibrations of a mechanical resonator. The interaction between these two systems is usually mediated by a radiation-pressure type coupling proportional – through a coupling constant $g$ – to the number of photons $n_c$ in the cavity and the displacement of the mechanical resonator. The radiation pressure coupling offers the possibility of altering the resonant frequency of the mechanical resonator and its damping. The latter can be used for cooling [4–6] or amplification [7]. Moreover, the nonlinearity of the interaction may allow for the observation of macroscopic quantum phenomena such as quantum superposition of states [1, 2] or quantum squeezed states [3]. The requirements for observing these quantum phenomena are the necessity of being close to the ground state and being in the strong coupling regime [8, 9] where $g$ is larger than the cavity and the mechanical resonator decay rate. However, $g$ is usually weak and to bypass this constraint a strong drive to the cavity is applied at the cost of losing the nonlinear property of the interaction.

Our recent proposal [10], in which the cavity and the resonator are coupled to a Josephson junction, shows that the interaction between the cavity and the resonator can be enhanced via the non-linearity of the Josephson effect. Quadratic and higher-order interactions in the displacement have been investigated also in different setups such as membrane in the middle geometries [11, 12].

Analogously to the setup mentioned above, the non-linearity of the Josephson effect leads to an additional nonlinear interaction, namely a cross-Kerr coupling $g_{ck}$ between the cavity and the resonator. The difference resides in the fact that in the Josephson junction setup the relative value of $g_{ck}$ and $g$ depends on the value of the gate charge to a superconducting island, whereas in [12, 13], it generally reflects the position of the resonator within the cavity.

In the context of optomechanical systems, the cross-Kerr coupling between the resonator and the cavity induces a change to the refractive index of the cavity depending on the number of photons in the resonator, whereas the radiation pressure coupling gives rise to an analogous effect, but depending on the displacement of the mechanical resonator.

In this paper we solve the dynamics of the cavity and the mechanical resonator in the presence of the cross-Kerr and the radiation pressure coupling. We determine the effects of the cross-Kerr coupling on the red and blue sidebands within a mean field approach. In particular, we demonstrate that the sideband peak is shifted due to the cross-Kerr coupling. In addition, the cross-Kerr coupling induces a nonmonotonous response of the effective mechanical damping as a function of the number of photons pumped into the cavity.

II. MEAN FIELD APPROACH

We consider an electromagnetic cavity with frequency $\omega_c$ and linewidth $\kappa$ coupled to a mechanical resonator with frequency $\omega_m$ and linewidth $\gamma$. The number of phonons in the cavity $n_c$ is coupled to the vibration am-
The cross-Kerr coupling becomes

\[ H = \omega_c a^\dagger a + \omega_m b^\dagger b - g a^\dagger a(b^\dagger + b) - g_{ck} a^\dagger b^\dagger b, \] (1)

where \( a \) and \( b \) are the annihilation operators of the cavity and the mechanical resonator, respectively. We treat the interactions with a mean-field (MF) approach. Within this frame, the radiation-pressure interaction becomes

\[ g a^\dagger a(b^\dagger + b) = g \left[ \langle a^\dagger + a \rangle - (a^\dagger a) \right](b^\dagger + b) \]

\[ + (a^\dagger a - \langle a^\dagger a \rangle)(b^\dagger + b), \] (2)

where \( \langle A \rangle \) stands for the average of \( A \) over the static nonequilibrium state of the system (mean field). The negative terms in Eq. (2) are included to suppress double counting. The first line of Eq. (2) describes exchange processes between the resonator and the cavity while the second line gives a frequency shift of the cavity which is proportional to the average displacement of the resonator. This decomposition allows us to find the usual results of the weak radiation pressure coupling [24, 25]. In MF, the cross-Kerr coupling becomes

\[ g_{ck} a^\dagger b^\dagger b = g_{ck} \left[ \langle a^\dagger a \rangle b^\dagger b + (b^\dagger b)a^\dagger a \right. \]

\[ \left. + (b^\dagger a)ba^\dagger + (b^\dagger b)a^\dagger + (b^\dagger a^\dagger)ba \right]. \] (3)

The term \( \langle a^\dagger a \rangle b^\dagger b (b^\dagger b)a^\dagger a \) describes a Hartree-like interaction between the resonator and the cavity while the other terms describe exchange processes between the resonator and the cavity. Thus we can rewrite the Hamiltonian as

\[ H = [\omega_c - g_{ck}(b^\dagger b)]a^\dagger a + [\omega_m - g_{ck}(a^\dagger a)]b^\dagger b \]

\[ - G[(a^\dagger b^\dagger b^\dagger) + \langle a^\dagger a \rangle b^\dagger b^\dagger - G^*[a^\dagger a b + (a^\dagger a^\dagger) b^\dagger + b^\dagger (a^\dagger a)]], \] (4)

where the expectation values of the different operators have to be determined self-consistently within the MF picture and \( G = g + g_{ck} b \). We assume the usual experimental situation where \( \omega_c \gg \omega_m \) and where the cavity is driven with a coherent field of strength \( f_p \) oscillating at frequency \( \omega_p = \omega_c + \Delta \). Using the input-output formalism [16] the equations of motion are

\[ \dot{a} = -i[-\Delta - g_{ck}(b^\dagger b)]a - \frac{\kappa}{2} a + \sqrt{\kappa} f_p \]

\[ + iG^*[a]b + iG(a^\dagger b^\dagger - ig(b^\dagger + b)[a - \langle a \rangle] \] (5)

\[ \dot{b} = -i[\omega_m - g_{ck}(a^\dagger a)]b - \frac{\gamma}{2} b + \sqrt{\gamma} b_{in} \]

\[ + iG(a^\dagger a) + iG^*[a^\dagger - ig(a^\dagger a)], \] (6)

Here we have written the cavity operator \( a \) in a frame rotating with frequency \( \omega_p \) neglecting the fast oscillating terms. We define \( b_{in} \) to be the thermal input of the resonator satisfying \( \langle b_{in}(t) \rangle = 0 \) and \( \langle b_{in}(t)b_{in}(t') \rangle = n_{th} \delta(t - t') \), where \( n_{th} \) is the number of phonons in the thermal bath damping the resonator. We split the cavity and the mechanical operators into a sum of coherent and fluctuation parts, i.e., \( a \equiv \delta a + \alpha \) and \( b \equiv \delta b + \beta \) with \( \alpha = \langle a \rangle \), \( \beta = \langle b \rangle \) and \( \langle \delta a \delta b \rangle = \langle \delta b \rangle = 0 \). As usual, we assume that \( \alpha \) and \( \beta \) oscillate at the same frequency as the coherent drive so that \( \dot{\alpha} = \dot{\beta} = 0 \). With these approximations, the solutions of Eqs. (5) (6) are

\[ \alpha = \frac{\sqrt{\kappa} f_p}{\frac{\gamma}{2} - i[\Delta - g_{ck} b^\dagger b] - (G^* \beta + G \alpha \beta)}, \] (7)

\[ \beta = \frac{i(2G - g)|\alpha|^2 - ig\delta a \delta a}{\gamma/2 + i[\omega_m - g_{ck}(a^\dagger a)]}. \] (8)

In the derivation of Eqs. (7) (8), we have assumed, in agreement with what is usually done in the optomechanical literature (see e.g. [15]), \( \Delta + g(b^\dagger + b) \approx \Delta \). The equations of motion for the fluctuations in the Fourier space are given by

\[ \left[ \frac{\kappa}{2} - i(\omega + \Delta) \right] \delta a = iG \delta b + iG^* \alpha \delta b \]

\[ \left[ \frac{\gamma}{2} - i(\omega - \omega_m) \right] \delta b = iG \delta a + iG \delta \alpha + \sqrt{\gamma} b_{in}, \] (10)

where \( \Delta = \Delta + g_{ck} b^\dagger b \) and \( \omega_m = \omega_m - g_{ck}(a^\dagger a) \). The equation of the thermal drive \( b_{in} \) on the response of the cavity is mediated by the coupling \( G \). Through this coupling the oscillations of the mechanical resonator produce sideband peaks at \( \omega_d \pm \omega_m \) in the cavity response. They allow for the exchange of energy between the cavity and the resonator when \( \Delta \approx \pm \omega_m \) [15,17]. These processes are depicted in Fig. 2. For \( \Delta \approx -\omega_m \) the system is in the red sideband regime and one can transfer energy from the resonator to the cavity, thus the mechanical resonator is damped and cooled. For \( \Delta \approx \omega_m \), the system is in the blue sideband regime and one can transfer energy from the cavity to the resonator, thus the mechanical resonator is excited and heated. In order to find the correction to the damping, we solve the response function of \( \delta a \) for the thermal input \( \delta b_{in} \). We find that it is a Lorentzian function peaked at \( \omega_m + \omega_{shift} \) with

\[ \omega_{shift} = -|G|^2 |\alpha|^2 \left( \frac{\Delta^2 - \omega_m^2 + \frac{\kappa^2}{4}}{\omega_m^2} \right) \]

\[ \left( \frac{\omega_m^2 + (\omega_m + \Delta)^2}{\omega_m^2 + (\omega_m - \Delta)^2} \right), \] (11)

and whose linewidth is \( \gamma + \Gamma_{opt} \) with

\[ \Gamma_{opt} = |G|^2 |\alpha|^2 \kappa \]

\[ \left( \frac{\omega_m^2 + (\omega_m + \Delta)^2}{\omega_m^2 + (\omega_m - \Delta)^2} \right). \] (12)
FIG. 2. Cooling (heating) process. The cavity is driven with a frequency \( \omega_d = \omega_c - \omega_m \) (\( \omega_d = \omega_c + \omega_m \)). The drive does not allow a transition from \( |n_m, n_c\rangle \to |n_m, n_c + 1\rangle \) but allow the transition from \( |n_m, n_c\rangle \to |n_m - 1, n_c + 1\rangle \) \( \langle n_m, n_c\rangle \to |n_m + 1, n_c + 1\rangle \). The cavity relaxes then to the state \( |n_m - 1, n_c\rangle \langle n_m + 1, n_c\rangle \) resulting into cooling (heating) of the mechanical resonator.

Integrating the Lorentzian function obtained above, we obtain the number of phonons and photons coming from the thermal vibrations of the resonator. We get

\[
\langle \delta b \delta b \rangle = \frac{\gamma n^b + \Gamma_{opt} n_{m0}}{\gamma + \Gamma_{opt}},
\]

\[
\langle \delta a \delta a \rangle = G^2 |\alpha|^2 \langle \delta b \delta b \rangle + \frac{1}{\frac{\kappa}{4} + (\omega_m + \Delta)^2} + \frac{1}{\frac{\kappa}{4} + (\omega_m - \Delta)^2}
\]

with

\[n_{m0} = \frac{(\omega_m + \Delta)^2 + \frac{\kappa}{4}}{4\Delta^2}.\]

Eqs. (7), (8), (13) and (14) form a set of self-consistency equations and are solved in the next sections in order to find the number of phonons in the resonator and photons in the cavity. We now focus on the sidebands.

III. OPTIMAL COOLING/HEATING

In order to minimize/maximize the optical damping \( \Gamma_{opt} \), we set \( \Delta = \mp \Delta_m \). The upper sign refers to the red sideband \( (\Gamma_{opt} > 0) \) and the lower sign to the blue sideband \( (\Gamma_{opt} < 0) \). In the resolved sideband limit, \( \omega_m \gg \kappa \gg \gamma \), the frequency shift and the optical damping become

\[
\omega_{\text{shift}} = \pm \frac{|G|\alpha|^2}{\omega_m} = \pm \frac{|G|^2 |\alpha|^2}{\omega_m - g_{ck} \langle a^\dagger a \rangle},
\]

\[
\Gamma_{opt} = \pm \frac{4|G|^2 |\alpha|^2}{\kappa} \frac{\omega_m - g_{ck} \langle a^\dagger a \rangle}{4(\omega_m - g_{ck} \langle a^\dagger a \rangle)^2}.
\]

Now both the frequency shift and the optical damping depend on the cross-Kerr coupling. In Figs. 4-5 we plot the number of phonons and the optical damping as a function of \( \omega_m/\kappa \) for the red sideband in the Doppler limit. Since the cross-Kerr coupling shifts the mechanical frequency, the value of \( \Gamma_{opt} \) is shifted as well. The sign of the shift is given by the sign of \( g_{ck} \). Otherwise we recover the cooling of the resonator for the red sideband (Fig. 3) and the parametric instability when \( \Gamma_{opt} = -\gamma \) for the blue sideband (Fig. 5).

IV. CASE WITH \( \Delta = \omega_m \).

In experiments the parameter one can tune directly is the detuning \( \Delta \) and not \( \Delta \) as it can be difficult to set \( \Delta = \mp \Delta_m \) for each value of \( |\alpha| \) as the pump strength is varied. Therefore, another regime we consider is the case where \( \Delta = \mp \omega_m \), i.e., setting \( \Delta = \mp \omega_m + g_{ck} \langle b^\dagger b \rangle \). In this case the frequency shift and optical damping in the red (upper sign) and in the blue (lower sign) sideband
The number of photons pumped into the cavity is fixed to $|\alpha|^2 = 100$ and the bath temperature corresponds to $n^{th} = 100$.

When $g_{ck} = 0$, $g_{ck} = 0.1g$, and $g_{ck} = -0.1g$, the frequency shift increases (decreases) as $\alpha$ increases until $|\alpha|^2 \lesssim \kappa^2/4$. For the blue sideband when $g_{ck} > 0$ the frequency shift increases (decreases) as $\alpha$ increases until $|\alpha|^2 \lesssim \kappa^2/4$. This thus competes with the usual limitation coming from the intrinsic (Duffing) nonlinearity of the resonator.

In Figs. 6 and 7 the optical damping becomes inversely proportional to the number of photons pumped into the cavity, consequently, the cooling deteriorates when pumping more phonons in the cavity.

In the blue sideband (Fig. 7) the main effect of a small cross-Kerr coupling is to limit the instability to a finite number of phonons, $\langle b^\dagger b \rangle \approx \sqrt{\kappa/\gamma}|G||\alpha|/|g_{ck}| + |\alpha|^2$. This thus competes with the usual limitation coming from the intrinsic (Duffing) nonlinearity of the resonator.
V. CONCLUSION

In conclusion, we have solved the dynamics of a mechanical resonator coupled to an electromagnetic cavity via a radiation pressure coupling and a cross-Kerr coupling using a mean field approach. We have shown that the cross-Kerr coupling shifts the frequency of the mechanical resonator and of the optical cavity, the shift depending on the number of photons in the cavity and phonons in the resonator. In addition, we have shown that when the detuning of the pump is equal to the frequency of the mechanical resonator the variation of the optical damping saturates instead of being linearly dependent on the number of phonons pumped into the cavity.

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