BIRATIONAL GEOMETRY FOR D-CRITICAL LOCI AND WALL-CROSSING IN CALABI-YAU 3-FOLDS

YUKINOBU TODA

Abstract. The notion of d-critical loci was introduced by Joyce in order to give classical shadows of (−1)-shifted symplectic derived schemes. In this paper, we discuss birational geometry for d-critical loci, by introducing notions such as 'd-critical flips', 'd-critical flops', etc. They are not birational maps of the underlying spaces, but rather should be understood as virtual birational maps.

We show that several wall-crossing phenomena of moduli spaces of stable objects on Calabi-Yau 3-folds are described in terms of d-critical birational geometry. Among them, we show that wall-crossing diagrams of Pandharipande-Thomas (PT) stable pair moduli spaces, which are relevant in showing the rationality of PT generating series, form a d-critical minimal model program.

CONTENTS

1. Introduction
2. Review of birational geometry
3. Birational transformations for d-critical loci
4. Moduli spaces of semistable objects on CY 3-folds
5. Representations of quivers with super-potentials
6. Analytic neighborhood theorem of wall-crossing in CY 3-folds
7. Representations of symmetric and extended quivers
8. D-critical flops of moduli spaces of one dimensional sheaves
9. D-critical flips of moduli spaces of stable pairs
Appendices
Appendix A. Review of Bridgeland stability conditions
Appendix B. Other examples
Appendix C. Wall-crossing formula of DT type invariants
References

1. Introduction

1.1. Background and motivation. Let $X$ be a smooth projective variety over $\mathbb{C}$. For a given numerical class $v \in H^{2*}(X, \mathbb{Q})$ and a stability condition $\sigma$ on the derived category of coherent sheaves on $X$ (e.g. Bridgeland stability condition [Bri07], weak stability condition [Tod10a]), we denote by $M_\sigma(v)$
the coarse moduli space\(^1\) of \(S\)-equivalence classes of \(\sigma\)-semistable objects on \(X\) with Chern character \(v\). The moduli space \(M_\sigma(v)\) depends on a choice of a stability condition \(\sigma\). In general, we have \textit{wall-crossing phenomena}, i.e. there is a wall-chamber structure on the space of stability conditions such that \(M_\sigma(v)\) is constant if \(\sigma\) lies on a chamber but may change if \(\sigma\) crosses a wall. It is an interesting question how the moduli spaces \(M_\sigma(v)\) vary under wall-crossing of \(\sigma\). More precisely, suppose that \(\sigma\) lies on a wall and \(\sigma^\pm\) lie on its adjacent chambers. Then we have the following diagram (called \textit{wall-crossing diagram})

\[
\begin{array}{ccc}
M_{\sigma^+}(v) & \rightarrow & M_{\sigma^-}(v) \\
\downarrow & & \downarrow \\
M_{\sigma}(v) & & 
\end{array}
\]

If \(M_{\sigma^\pm}(v)\) are smooth (or singular with mild singularities) and birational, then it makes sense to ask whether the diagram (1.1) is a flip or flop in birational geometry [KM98]. Note that if this happens, we have the inequality of canonical line bundles of \(M_{\sigma^\pm}(v)\) (see Definition 2.8):

\[
M_{\sigma^+}(v) \geq K_M \cdot M_{\sigma^-}(v).
\]

Indeed this is true in some cases, and birational geometry of the diagram (1.1) has been especially studied when \(X\) is an algebraic surface [EG95, FQ95, MW97, NY11, BM14a, BM14a, ABCH13, Tod14]. In the above articles, birational geometry of the diagram (1.1) has been important in understanding birational geometry of classical moduli spaces (e.g. Hilbert schemes of points), or applications to enumerative geometry (e.g. Donaldson invariants).

However there is a limitation of this research direction. In general, the moduli spaces \(M_{\sigma^\pm}(v)\) can have worse singularities than those which appear in birational geometry [KM98], e.g. terminal singularities, canonical singularities, etc. In fact by Vakil’s Murphy’s law [Vak06], any singularity can appear on such moduli spaces, so they may not be irreducible, may not be reduced, may not be equidimensional, or so on. In such bad cases, it is not even clear what are birational maps between them, nor what are their canonical line bundles. Moreover even if \(M_{\sigma^\pm}(v)\) are smooth, they are not birational in general. So it does not make sense to ask whether the wall-crossing diagram (1.1) is a flip nor a flop, nor satisfy the inequality (1.2).

If we focus on the case that \(X\) is a Calabi-Yau (CY for short) 3-fold, still we have the same issue as above. However in this case, we have additional structures on the moduli spaces \(M_{\sigma^\pm}(v)\) (more precisely on their stable parts) called \textit{d-critical structures}. This notion was introduced by

---

\(^1\)The existence of \(M_\sigma(v)\) is not obvious in general, and we discuss assuming that it exists. It is announced in [AHLH] that \(M_\sigma(v)\) exists if \(\sigma\)-semistable objects with Chern character \(v\) are bounded.
Joyce [Joy15] in order to give classical shadows of (−1)-shifted symplectic structures on derived moduli spaces of stable objects on CY 3-folds [PTVV13]. In particular, we have the notion of virtual canonical line bundles on such moduli spaces.

The purpose of this paper is to introduce the notions of d-critical flips, d-critical flops, etc, for morphisms from d-critical loci to schemes or analytic spaces. They are not birational maps of the underlying spaces, but rather should be understood as ‘virtual’ birational maps. We then show that, despite of the possible bad singularities of $M_{\sigma^\pm}(v)$, several wall-crossing diagrams (1.1) for a CY 3-fold $X$ fit into these notions of d-critical birational transformations. In particular they satisfy an analogue of the inequality (1.2) for virtual canonical line bundles.

1.2. D-critical birational geometry. By definition, a d-critical locus introduced by Joyce [Joy15] consists of data

$$(M, s), \ s \in \Gamma(M, S^0_M)$$

where $M$ is a $\mathbb{C}$-scheme or an analytic space and $S^0_M$ is a certain sheaf of $\mathbb{C}$-vector spaces on $M$ (see Definition 3.1). The section $s$ is called a d-critical structure of $M$. Roughly speaking if $M$ admits a d-critical structure $s$, this means that $M$ is locally written as a critical locus of some function on a smooth space, and the section $s$ remembers how $M$ is locally written as a critical locus.

Let $(M^\pm, s^\pm)$ be two d-critical loci and consider a diagram of morphisms of $\mathbb{C}$-schemes or analytic spaces

$$
\begin{array}{ccc}
M^+ & \rightarrow & M^- \\
\downarrow & & \downarrow \\
A & \rightarrow & A
\end{array}
$$

(1.3)

We introduce the notion of a d-critical flip (resp. d-critical flop) to be a diagram (1.3) satisfying the following: for any $p \in A$, there is a commutative diagram

$$
\begin{array}{ccc}
Y^+ & \rightarrow & Y^- \\
\downarrow & \phi \downarrow & \downarrow \\
Z & \rightarrow & Z \\
\downarrow & w^+ \downarrow & \downarrow w^- \\
\mathbb{C} & \rightarrow & \mathbb{C}
\end{array}
$$

where $\phi: Y^+ \rightarrow Y^-$ is a flip (resp. flop) of smooth varieties (or complex manifolds), such that locally near $p \in A$ there exist isomorphisms between $M^\pm$ and $\{dw^\pm = 0\}$ as d-critical loci (see Definition 3.7 for details). Other notions such as a d-critical divisorial contraction, a d-critical Mori fiber...
We also introduce the inequality of virtual canonical line bundles on d-critical loci, as an analogue of (1.2) (see Definition 3.23):

\[(M^+, s^+) \geq K(M^-, s^-).\]  

(1.4)

For example d-critical flips, d-critical flops satisfy (1.4).

We remark that a diagram (1.3) being a d-critical flip or a d-critical flop does not imply anything on birational geometry of \(M^\pm\) themselves, even when \(M^\pm\) are smooth. Indeed there is an example of a d-critical flip where \(M^\pm\) are smooth but their dimensions are different (see Example 3.8), so in particular they are not birational. Therefore we should interpret these notions as ‘virtual’ birational maps rather than birational maps of the underlying spaces \(M^\pm\).

1.3. Wall-crossing in Calabi-Yau 3-folds. Suppose that \(X\) is a smooth projective CY 3-fold, and consider a wall-crossing diagram (1.1). If \(v\) is primitive, then \(M_{\sigma^\pm}(v)\) consist of \(\sigma^\pm\)-stable objects, and \(M_{\sigma^\pm}(v)\) admit canonical d-critical structures by [BBBBJ15]. Indeed \(M_{\sigma^\pm}(v)\) are classical truncations of derived schemes with \((-1)\)-shifted symplectic structures [PTVV13], and the derived Darboux theorem [BBBBJ15] for \((-1)\)-shifted symplectic derived schemes yield d-critical structures on them.

Therefore we can ask whether the diagram (1.1) is a d-critical flip, a d-critical flop, or so on. We will answer the above question via ‘analytic neighborhood theorem’ given in Theorem 6.1. This theorem describes the diagram (1.1) analytic locally on \(M_\sigma(v)\) in terms of a wall-crossing diagram of moduli spaces of representations of a certain quiver with a (formal but convergent) super-potential. A similar result was already proved in [Todh] for moduli spaces of semistable sheaves, and we will see that the argument can be generalized to our setting, assuming the existence of good moduli spaces of Bridgeland semistable objects. The latter existence problem is recently announced to be settled by Alper-Halpern-Leistner-Heinloth [AHLH]. The analytic neighborhood theorem reduces the above question on the diagram (1.1) to study birational maps of moduli spaces of representations of some quivers without relations.

Using the analytic neighborhood theorem, we study d-critical birational geometry of wall-crossing diagrams (1.1) which appeared in the context of enumerative geometry on CY 3-folds, e.g. Donaldson-Thomas (DT) invariants [Tho00], Pandharipande-Thomas (PT) invariants [PT09] and also Gopakumar-Vafa (GV) invariants [MT]. The results are summarized below:

**Theorem 1.1.** (Theorem 8.3, Theorem 9.13)

(i) In the case of wall-crossing of one dimensional stable sheaves, the diagram (1.1) is a d-critical generalized flop.

(ii) In the case of wall-crossing of PT stable pair moduli spaces, the diagram (1.1) is a d-critical generalized flip at any point in \(\text{Im} \pi^-\), a d-critical MFS at any point in \(M_\sigma(v) \setminus \text{Im} \pi^\pm\).
The wall-crossing diagrams in the above cases have applications to enumerative geometry. The wall-crossing diagrams of one dimensional stable sheaves (i) are used in [Toda] to show that GV invariants defined in [MT] are independent of Bridgeland stability conditions, and also invariant under flops. The wall-crossing diagrams of stable pair moduli spaces (ii) are used in [Bri11, Tod09a, Tod10b, Tod12a] (also in [Dia12] for local curve case) to show the rationality conjecture of the generating series of PT invariants. In this case, we have wall-crossing diagrams which relate PT invariants and L invariants in loc.cit., and Theorem 1.1 (ii) shows that they form a d-critical MMP (see Corollary 9.17).

As a summary, Theorem 1.1 gives an interpretation of wall-crossing diagrams in CY 3-folds relevant in enumerative geometry in terms of d-critical birational geometry. In Appendix B we also discuss some other examples of wall-crossing diagrams in CY 3-folds in terms of d-critical birational geometry, DT/PT correspondence, local K3 surfaces (see Theorem B.3). They also have applications to enumerative geometry [Bri11, Tod10a, Tod12b].

1.4. Speculation toward d-critical D/K equivalence conjecture. Let $Y^+ \rightarrow Y^-$ be a birational map between smooth projective varieties satisfying the relation $Y^+ \geq_K Y^-$. Then by Bondal-Orlov [BO] and Kawamata [Kaw02], it is conjectured that there exists a fully faithful functor of derived categories of coherent sheaves

$$D^b(Y^-) \hookrightarrow D^b(Y^+)$$

which is an equivalence if $Y^+ =_K Y^-$. We call the above conjecture as $D/K$ equivalence conjecture.

We expect a similar conjecture may hold for d-critical loci, or $(-1)$-shifted symplectic derived schemes. Namely for a d-critical locus $(M, s)$ (probably induced by a $(-1)$-shifted symplectic derived scheme equipped with additional data), there may exist a certain triangulated category $\mathcal{D}(M, s)$ such that, if the relation (1.4) holds, we have a fully faithful functor

$$\mathcal{D}(M^-, s^-) \hookrightarrow \mathcal{D}(M^+, s^+)$$

which is an equivalence if (1.4) is an equality. The category $\mathcal{D}(M^-, s^-)$ may be constructed as a gluing of $\mathbb{Z}/2\mathbb{Z}$-periodic triangulated categories of matrix factorizations defined locally on each d-critical chart, though its construction seems to be a hard problem at this moment (see [Joy (J)], [Toe14, Section 6.1]). If it exists, the category $\mathcal{D}(M, s)$ may be interpreted as a kind of ‘Fukaya category’ of d-critical loci, or $(-1)$-shifted symplectic derived schemes (see [JS, Conjecture 1.2]). Moreover we expect that the numerical realization of semi-orthogonal complement of the embedding (1.5) recovers wall-crossing formula of Donaldson-Thomas (DT) invariants on CY 3-folds established in [JS12, KS]. Thus our d-critical birational geometry gives a link of two research subjects developed independently, wall-crossing formula of DT invariants and D/K equivalence conjecture.
If $M^\pm$ are smooth, so in particular $s^\pm = 0$, we can use usual derived categories of coherent sheaves $D^b(M^\pm)$ to ask an analogue of the above question. In [Todc], we address this question in the case of simple wall-crossing diagrams of stable pair moduli spaces.

1.5. Outline of the paper. The outline of this paper is as follows. In Section 2 we review basic terminology of birational geometry. In Section 3 we recall Joyce’s d-critical loci and introduce notions of d-critical birational transformations. In Section 4 we introduce moduli spaces of semistable objects on CY 3-folds and formulate the question on their wall-crossing diagrams. In Section 5 we set notation of moduli spaces of representations of quivers with convergent super-potentials. In Section 6 we state analytic neighborhood theorem for wall-crossing diagrams in CY 3-folds, and give an outline of the proof. In Section 7 we investigate wall-crossing phenomena in symmetric and extended quivers. In Section 8 we describe wall-crossing diagrams of one dimensional stable sheaves on CY 3-folds in terms of d-critical flops. In Section 9 we describe wall-crossing diagrams of stable pair moduli spaces in terms of d-critical flips. In Appendix A we review basics on Bridgeland stability conditions. In Appendix B we give some more examples of wall-crossing in CY 3-folds, and describe them in terms of d-critical birational geometry. In Appendix C we recall how wall-crossing diagrams in this paper have been relevant in the study of Donaldson-Thomas invariants.

1.6. Acknowledgements. The author is grateful to Daniel Halpern-Leistner for explaining the announced work [AHLH] on the existence of good moduli spaces for Bridgeland semistable objects. The author is also grateful to Chen Jiang and Dominic Joyce for valuable discussions. The author is supported by World Premier International Research Center Initiative (WPI initiative), MEXT, Japan, and Grant-in Aid for Scientific Research grant (No. 26287002) from MEXT, Japan.

1.7. Notation and convention. In this paper, all the varieties and schemes are defined over $\mathbb{C}$. For a smooth variety or a complex manifold $M$, we denote by $K_M$ its canonical divisor and $\omega_M = \mathcal{O}_M(K_M)$ its canonical line.
bundle. For a smooth projective variety $X$ and $\beta, \beta' \in H_2(X, \mathbb{Z})$, we write $\beta > \beta'$ if $\beta - \beta'$ is a class of an effective one cycle on $X$. For a projective morphism $f: Y \to Z$ of varieties, we denote by $\rho(Y/Z)$ its relative Picard number. When $f$ is birational, its exceptional locus is denoted by $\text{Ex}(f)$. For a scheme $M$, we denote by $D^b(M) := D^b(Coh(M))$ the bounded derived category of coherent sheaves on $M$.

2. Review of birational geometry

In this section, we review some basics on birational geometry and recall several terminologies. A standard reference is [KM98].

2.1. Terminology from birational geometry. Let $Y$ be a projective variety with at worst terminal singularities (e.g. $Y$ is smooth). A minimal model program (MMP for short) of $Y$ is a sequence of birational maps

$$Y = Y_1 \to Y_2 \to \cdots \to Y_{N-1} \to Y_N$$

satisfying the following:

(i) each $Y_i$ is a projective variety with at worst terminal singularities.

(ii) each birational map $Y_i \to Y_{i+1}$ is either a divisorial contraction or a flip.

(iii) $Y_N$ is either a minimal model, i.e. $K_{Y_N}$ is nef, or has a Mori fiber space structure $Y_N \to Z$. The former occurs if and only if the Kodaira dimension of $Y$ is non-negative.

Here a line bundle $L$ on a variety $Y$ is called nef if for any projective curve $C \subset Y$, we have $\deg(L|_C) \geq 0$. A Cartier divisor $D$ on $Y$ is also called nef if the associated line bundle $O_Y(D)$ is nef. The minimal model $Y_N$ is not necessary unique, but two birational minimal models are known to be connected by a sequence of flops [Kaw08].

The above notions in birational geometry are summarized in the following definitions. In this paper we only treat the case of smooth varieties, but the definitions are the same for varieties with terminal singularities.

Definition 2.1. Let $Y$ be a smooth variety (resp. complex manifold) and $f: Y \to Z$ a projective morphism of varieties (resp. analytic spaces) with $\rho(Y/Z) = 1$ and $f_*O_Y = O_Z$. Then $f$ is called

(i) divisorial contraction if $\dim Y = \dim Z$ (i.e. $f$ is birational or bimeromorphic), $-K_Y$ is $f$-ample and $\text{Ex}(f)$ is a divisor.

(ii) (anti) flipping contraction if $\dim Y = \dim Z$, $-K_Y$ (resp. $K_Y$) is $f$-ample and $f$ is isomorphic in codimension one.

(iii) flopping contraction if $\dim Y = \dim Z$, crepant (i.e. $K_Y \cdot C = 0$ for any curve $C \subset Y$ such that $f(C)$ is a point) and $f$ is isomorphic in codimension one.

(iv) Mori fiber space (MFS for short) if $\dim Z < \dim Y$ and $-K_Y$ is $f$-ample.
We can formulate the relevant birational transformations using the diagram (2.2) below in a unified way:

**Definition 2.2.** Let $Y^+, Y^-$ be smooth varieties (or complex manifolds), and consider a diagram

\[
\begin{array}{c}
  Y^+ \\
  \downarrow f^+ \\
  Z \\
  \downarrow f^- \\
  Y^- 
\end{array}
\]

where $f^+, f^-$ are projective morphisms of varieties (resp. analytic spaces) satisfying $f^\pm_*\mathcal{O}_{Y^\pm} = \mathcal{O}_Z$ if $Y^\pm \neq \emptyset$. Then the diagram (2.2) is called

(i) *divisorial contraction* if $f^+$ is a divisorial contraction and $f^-$ is an isomorphism.

(ii) *flip* if $f^+$ is a flipping contraction, and $f^-$ is an anti-flipping contraction.

(iii) *flop* if $f^+, f^-$ are flopping contraction and the birational map $Y^+ \dashrightarrow Y^-$ is not an isomorphism.

(iv) *MFS* if $f^+$ is a MFS and $Y^- = \emptyset$.

2.2. **Generalized flips, flops and MFS.** We also use the following generalized terminology, without assuming the condition of relative Picard numbers etc:

**Definition 2.3.** In the situation of Definition 2.2, we call a diagram (2.2)

(i) *generalized flip* if $f^\pm$ are birational (or bimeromorphic) morphisms, $-K_{Y^\pm}$ is $f^\pm$-ample and $K_{Y^-}$ is $f^-$-ample.

(ii) *generalized flop* if $f^\pm$ are crepant birational (or bimeromorphic) morphisms, isomorphisms in codimension one, and there exists a $f^+$-ample divisor on $Y^+$ whose strict transform to $Y^-$ is $f^-$-anti-ample.

(iii) *generalized MFS* if $-K_{Y^+}$ is $f^+$-ample and $Y^- = \emptyset$.

**Remark 2.4.** In the definition of generalized flip, we don’t assume that $f^+$ is isomorphic in codimension one. For example, a divisorial contraction is a generalized flip. On the other hand, the morphism $f^-$ for a generalized flip is always isomorphic in codimension one by [KM98, Lemma 3.38].

**Remark 2.5.** The conditions (i) and (ii) in Definition 2.3 are not complementary conditions. Indeed a diagram (2.2) is both of generalized flip and generalized flop if and only if $f^+$ and $f^-$ are isomorphisms.

**Remark 2.6.** In the definition of generalized MFS, we don’t assume that $\dim Z < \dim Y^+$, so $f^+$ can be birational. Indeed such a case may happen in wall-crossing of stable pair moduli spaces in Section 9 (see Remark 9.16).

---

2The reference of this fact was pointed out to the author by Chen Jiang.
Remark 2.7. If the diagram (2.2) is a generalized flip, by applying MMP relative to $Z$ proved in [BCHM10], the birational map $Y^+ \to Y^-$ decomposes into divisorial contractions and flips over $Z$ (though the intermediate varieties may have terminal singularities). Similarly a generalized flop decomposes into flops over $Z$ by [Kaw08].

2.3. Inequalities of canonical divisors. Following Kawamata (for example see [Kawa]), we introduce the inequalities of canonical divisors (with a slight modification allowing empty sets):

Definition 2.8. In the situation of Definition 2.2 we write

(i) $Y^+ >_K Y^-$ if either $Y^- = \emptyset$ (while $Y^+ \neq \emptyset$), or there is a commutative diagram

\[
\begin{array}{ccc}
W & \xrightarrow{g^+} & Y^+ \\
\downarrow{f^+} & & \downarrow{f^-} \\
Y^- & \xleftarrow{g^-} & Z
\end{array}
\]

such that $g^\pm$ are birational and $(g^+)^*K_{Y^+} - (g^-)^*K_{Y^-}$ is linearly equivalent to an effective divisor on $W$.

(ii) $Y^+ =_K Y^-$ if either $Y^\pm = \emptyset$ or there is a commutative diagram (2.3) for birational maps $g^\pm$ such that $(g^+)^*K_{Y^+}$ and $(g^-)^*K_{Y^-}$ are linearly equivalent.

The canonical divisors decrease by divisorial contractions and flips, while flops keep them (see [KM98, Lemma 3.38]). Therefore MMP is interpreted as a process decreasing the canonical divisors, i.e. for a MMP (2.1), we have the inequalities of canonical divisors

\[ Y = Y_1 >_K Y_2 >_K \cdots >_K Y_{N-1} >_K Y_N. \]

Moreover we have $Y_N =_K Y'_N$, if $Y'_N$ is another birational minimal model of $Y$. By Remark 2.7 (or by using [KM98, Lemma 3.38]), if the diagram (2.2) is a generalized flip (resp. generalized flop), we have

\[ Y^+ >_K Y^-, \text{ (resp. } Y^+ =_K Y^-). \]

3. Birational transformations for d-critical loci

The notion of d-critical loci was introduced by Joyce [Joy15], as a classical shadow of $(-1)$-shifted symplectic derived schemes [PTVV13]. In this section, we recall its definition and introduce analogue of birational transformations in the previous section for d-critical loci.
3.1. D-critical locus. Let $M$ be a complex scheme (resp. complex analytic space). In [Joy15], it is proved that there exists a canonical sheaf of $\mathbb{C}$-vector spaces $S_M$ on $M$ satisfying the following property: for any Zariski (resp. analytic) open subset $U \subset M$ and a closed embedding $i: U \hookrightarrow Y$ into a smooth scheme (resp. complex manifold) $Y$, there is an exact sequence

$$0 \rightarrow S_M|_U \rightarrow O_Y/I^2 \xrightarrow{d_{\text{DR}}} \Omega^1_Y/I \cdot \Omega^2_Y.$$  

(3.1)

Here $I \subset O_Y$ is the ideal sheaf which defines $U$ and $d_{\text{DR}}$ is the de-Rham differential. Moreover there is a natural decomposition

$$S_M = S_M^0 \oplus \mathbb{C}_M$$

where $\mathbb{C}_M$ is the constant sheaf on $M$. The sheaf $S_M^0$ restricted to $U$ is the kernel of the composition

$$S_M|_U \hookrightarrow O_Y/I^2 \twoheadrightarrow O_{U_{\text{red}}}.$$  

For example, suppose that $w: Y \rightarrow \mathbb{C}$ is an algebraic (resp. holomorphic) function such that

$$U = \{dw = 0\}, \quad w|_{U_{\text{red}}} = 0.$$  

Then $I = (dw)$ and $w + (dw)^2$ is an element of $\Gamma(U, S_M^0|_U)$.

**Definition 3.1.** ([Joy15]) A pair $(M, s)$ for a complex scheme (resp. analytic space) $M$ and $s \in \Gamma(M, S_M^0)$ is called algebraic (resp. analytic) d-critical locus if for any $x \in M$, there exist a Zariski (resp. analytic) open neighborhood $x \in U \subset M$, a closed embedding $i: U \hookrightarrow Y$ into a smooth scheme (resp. complex manifold) $Y$, an element $w \in \Gamma(O_Y)$ satisfying (3.2) such that $s|_U = w + (dw)^2$ holds. In this case, the data

$$(U, Y, w, i)$$

is called a d-critical chart. The section $s$ is called a d-critical structure of $M$.

**Remark 3.2.** If $M$ is smooth, then $S_M^0 = 0$ so there is a unique (trivial) choice of its d-critical structure, $s = 0$.

Given a d-critical locus $(M, s)$, there exists a line bundle $\omega_{M, s}$ on $M_{\text{red}}$ called virtual canonical line bundle $\mathbb{3}$ (see [Joy15] Section 2.4). It satisfies that, for any d-critical chart (3.3) there is a natural isomorphism

$$\omega_{M, s}|_{U_{\text{red}}} \cong \omega_Y^{\otimes 2}|_{U_{\text{red}}}.$$  

(3.4)

**Remark 3.3.** For a derived scheme $M^{\text{der}}$ with a $(-1)$-shifted symplectic structure [PTVV13], its classical truncation $M$ carries a canonical d-critical structure by [BBBBJ15]. In this case, the virtual canonical line bundle of $M$ is the determinant of the cotangent complex of $M^{\text{der}}$ restricted to $M$. 

$\mathbb{3}$In [Joy15] Section 2.4], it was just called canonical line bundle. We put ‘virtual’ in order to distinguish with the usual canonical line bundle.
We introduce the following relative version of d-critical chart:

**Definition 3.4.** Let \((M, s)\) be an algebraic (resp. analytic) d-critical locus and \(\pi : M \to A\) a morphism of schemes (resp. analytic spaces). For a Zariski (resp. analytic) open subset \(U \subset A\), suppose that there is a following commutative diagram

\[
\begin{array}{c}
\pi^{-1}(U) \\ \downarrow \pi \\
U \\
\end{array} \begin{array}{c}
i \\
Y \\
\downarrow f \\
Z \\
\downarrow g \\
C \\
\end{array}
\]

where \(f : Y \to Z\) is a morphism of schemes (resp. analytic spaces), \(Y\) is smooth, \(i\) and \(j\) are closed immersions, \(g \in \Gamma(O_Z)\) such that the data
\[
(\pi^{-1}(U), Y, w, i)
\]
is a d-critical chart. In this case, we call the diagram (3.5) as a \(\pi\)-relative d-critical chart.

3.2. **D-critical birational transformations.** We formulate the terminology of birational contractions for d-critical loci, using relative d-critical charts in Definition 3.4.

**Definition 3.5.** Let \((M, s)\) be an algebraic (resp. analytic) d-critical locus and
\[
\pi : M \to A
\]
a morphism of schemes (resp. analytic spaces). We call the morphism (3.6) an **algebraic (resp. analytic) d-critical divisorial contraction**, **d-critical (anti) flipping contraction**, **d-critical flopping contraction** at a point \(p \in A\) if there exist a Zariski (resp. analytic) open neighborhood \(p \in U \subset A\), a \(\pi\)-relative d-critical chart (3.5) such that \(f : Y \to Z\) is a divisorial contraction, (anti) flipping contraction, flopping contraction, MFS, as in Definition 2.1 respectively.

We call the morphism (3.6) an **algebraic (resp. analytic) d-critical divisorial contraction**, **d-critical (anti) flipping contraction**, **d-critical flopping contraction**, **d-critical MFS**, if the above corresponding condition holds for any \(p \in A\).

A d-critical birational contraction need not to be birational between underlying spaces. Indeed, we have the following example:

**Example 3.6.** Let \(U^\pm\) be the following affine schemes with d-critical structures \(s^\pm\)
\[
U^\pm := \text{Spec} \mathbb{C}[x, y^\pm]/(xy^\pm, y^\pm^2), \quad s^\pm = xy^\pm^2 + (d(xy^\pm)^2)^2.
\]
By gluing $U^+$ and $U^-$ at the smooth open subset $\text{Spec} \mathbb{C}[x, x^{-1}]$, we obtain an algebraic d-critical locus

$$(M, s), \quad M = U^+ \cup U^-, \quad s|_{U^\pm} = s^\pm.$$ 

Note that $M^{\text{red}} = \mathbb{P}^1$, and $M$ is non-reduced at the points $\{0\}$ and $\{\infty\}$. The structure morphism

$$\pi: M \to \text{Spec} \mathbb{C}$$

is an algebraic d-critical divisorial contraction, though they are not birational in the usual sense. Indeed we have the following $\pi$-relative d-critical chart

\[
\begin{array}{ccc}
M & \xrightarrow{i} & \mathbb{C}^2 \\
\pi & \downarrow & \downarrow f \\
\text{Spec} \mathbb{C} & \xrightarrow{j} & \mathbb{C}^2 \\
\end{array}
\]

where $f$ is the blow-up at $0 \in \mathbb{C}^2$ and $g$ is the function $g(u, v) = uv$.

We also formulate a d-critical version of birational transformations below:

**Definition 3.7.** Let $(M^+, s^+), (M^-, s^-)$ be algebraic (resp. analytic) d-critical loci and consider a diagram

\[
\begin{array}{ccc}
M^+ & \xrightarrow{\pi^+} & M^- \\
\downarrow & & \downarrow \\
\text{Spec} \mathbb{C} & \xrightarrow{j} & \mathbb{C} \\
\end{array}
\]

where $\pi^\pm$ are morphisms of schemes (resp. analytic spaces). Then we call the diagram (3.7) an algebraic (resp. analytic) d-critical divisorial contraction, d-critical (generalized) flip, d-critical (generalized) flop, d-critical (generalized) MFS, at $p \in A$ if there exist a Zariski (resp. analytic) open neighborhood $p \in U \subset A$ and $\pi^\pm$-relative d-critical charts

\[
\begin{array}{ccc}
(p^\pm)^{-1}(U) & \xrightarrow{i^\pm} & Y^\pm \\
\downarrow & & \downarrow f^\pm \wedge w^\pm \\
U & \xrightarrow{j} & Z \\
\end{array}
\]

where $g \in \Gamma(O_Z)$ and $j$ are independent of $\pm$, such that the diagram

\[
\begin{array}{ccc}
Y^+ & \xrightarrow{f^+} & Z & \xleftarrow{f^-} & Y^- \\
\end{array}
\]

is a divisorial contraction, (generalized) flip, (generalized) flop, (generalized) MFS, as in Definition 2.2, Definition 2.3 respectively.

We call the diagram (3.7) an algebraic (resp. analytic) d-critical divisorial contraction, d-critical (generalized) flip, d-critical (generalized) flop,
**d-critical MFS**, respectively, if the above corresponding condition holds for any \( p \in A \).

Here we give some examples of d-critical flips, d-critical flops:

**Example 3.8.** Let \( V^+, V^- \) be finite dimensional \( \mathbb{C} \)-vector spaces with dimension \( a, b \) on which \( \mathbb{C}^* \) acts by weight \( 1, -1 \) respectively. We denote by 

\[
\vec{x} = (x_1, \ldots, x_a), \quad \vec{y} = (y_1, \ldots, y_b)
\]

coordinates of \( V^+, V^- \) respectively. For \( c \in \mathbb{Z}_{\geq 0} \), let \( U = \mathbb{C}^c \) (resp. an analytic open neighborhood \( 0 \in U \subset \mathbb{C}^c \)) with a trivial \( \mathbb{C}^* \)-action. By taking GIT quotients of \( V^+ \times V^- \times U \) by the \( \mathbb{C}^* \)-action with respect to the character \( \pm \text{id}: \mathbb{C}^* \to \mathbb{C}^* \), we obtain

\[
Y^+ := \text{Tot}_P (\mathcal{O}_{P(V^+)}(-1) \otimes V^-) \times U
\]

\[
Y^- := \text{Tot}_P (\mathcal{O}_{P(V^-)}(-1) \otimes V^+) \times U.
\]

Then by setting

\[
Z := \text{Spec} \mathbb{C}[x_i y_j : 1 \leq i \leq a, 1 \leq j \leq b] \times U
\]

we obtain the diagram

\[
Y^+ \xrightarrow{f^+} Z \xleftarrow{f^-} Y^-
\]

which is a standard toric flip if \( a > b \geq 2 \), standard toric flop if \( a = b \geq 2 \) (see [Rei92]). Let us consider \( w \in \Gamma(\mathcal{O}_Z) \) of the form 

\[
g = \sum_{i=1}^{a} \sum_{j=1}^{b} w_{ij}(\vec{u}) x_i y_j, \quad w_{ij}(\vec{u}) \in \Gamma(\mathcal{O}_U).
\]

We set \( w^\pm \) by the commutative diagram

\[
Y^+ \xrightarrow{f^+} Z \xleftarrow{f^-} Y^-
\]

\[
\xymatrix{Y^+ \ar[rr]^{f^+} \ar[dr]^w & & Z \ar[dl]^g \ar[rr]_{f^-} \ar[dr] & & Y^- \ar[dl]^w \\
& \mathbb{C} \ar[uu]_a & & & \mathbb{C} \ar[uu]_b & & &}
\]

If we have \( c \gg 0 \) compared to \( a, b \), and \( w_{ij}(\vec{u}) \) are sufficiently general, then the critical loci

\[
M^\pm := \{ dw^\pm = 0 \} \subset Y^\pm
\]

are smooth of dimension \( \pm (a - b) + c - 1 \). Moreover, \( f^\pm(M^\pm) \) are contained in \( \{ 0 \} \times U \). Therefore the diagram

\[
M^+ \xrightarrow{\pi^+} U \xleftarrow{\pi^-} M^-
\]

is an algebraic (resp. analytic) d-critical flip if \( a > b \geq 2 \), an algebraic (resp. analytic) d-critical flop if \( a = b \geq 2 \). Here as \( M^\pm \) are smooth, the d-critical structures \( s^\pm \) on \( M^\pm \) must be zero. Note that in the former case,
the dimensions of $M^\pm$ are different. The fibers of $\pi^\pm$ at $u \in U$ are linear subspaces in $\mathbb{P}(V^\pm)$, whose dimensions depend on $u$.

Here is an example of analytic $d$-critical flips, flops for a diagram of smooth projective varieties.

**Example 3.9.** Let $C$ be a smooth projective curve with genus $g$, and let $S^k(C)$ be the $k$-th symmetric product of $C$:

$$S^k(C) := \underbrace{(C \times \cdots \times C)}_{k} / \mathfrak{S}_k.$$ 

Note that $S^k(C)$ is a smooth projective variety with dimension $k$. Let $\text{Pic}^k(C)$ be the moduli space of line bundles on $C$ with degree $k$, which is a $g$-dimensional complex torus. For each $n > 0$, we consider the classical diagram of Abel-Jacobi maps

$$
\begin{array}{ccc}
S^{n+g-1}(C) & \xrightarrow{\pi^+} & S^{-n+g-1}(C) \\
\text{Pic}^{n+g-1}(C) & \text{Pic}^{n+g-1}(C). \\
\end{array}
$$

Here the morphisms $\pi^\pm$ are given by

$$\pi^+(Z \subset C) = \mathcal{O}_C(Z), \quad \pi^- (Z' \subset C) = \omega_C(-Z').$$

The diagram (3.13) appears as a special case of wall-crossing of stable pair moduli spaces discussed in Theorem 9.22 (see Example 9.24). By loc. cit., at a point $[L] \in \text{Pic}^{n+g-1}(C)$ we see that the diagram (3.13) is an analytic $d$-critical flip if $h^1(L) > 1$, an analytic $d$-critical divisorial contraction if $h^1(L) = 1$ and an analytic $d$-critical MFS if $h^1(L) = 0$. Moreover relative $d$-critical charts are analytic locally on $\text{Pic}^{n+g-1}(C)$ given as in Example 3.8.

Note that $S^{\pm n+g-1}(C)$ are smooth projective varieties, which are not birational for $n > 0$ (as the dimensions are different).

Here is an example of a $d$-critical flop between non-reduced $d$-critical loci:

**Example 3.10.** Let us consider the case $a = b = 2$ and $c = 0$ in Example 3.8. In this case, the diagram (3.10) is the simplest example of a flop, called Atiyah flop. Let us take $g \in \Gamma(\mathcal{O}_Z)$ to be $g = x_1x_2y_1^2$, and define $M^\pm$ as in (3.11). We have the $d$-critical structures on $M^\pm$ by $s^\pm = w^\pm + (dw^\pm)^2$, and an algebraic $d$-critical flop

$$M^+ \xrightarrow{\pi^+} Z \xleftarrow{\pi^-} M^-$$

The schemes $M^\pm$ are described as follows. Let $\mathbb{P}^1 = C^\pm \subset Y^\pm$ be the exceptional loci of $f^\pm$. Then the reduced part of $M^+$ is a smooth divisor on $Y^+$ which contains $C^+$. However $M^+$ is non-reduced along two disjoint curves on $M^+$. On the other hand, the reduced part of $M^-$ is a union of $C^-$ and a smooth divisor on $Y^-$ which intersects $C^-$ at a point $y$. The scheme
$M^-$ is non-reduced along two curves on the above divisor which intersect at $y$. The singularities of schemes $M^\pm$ are not treated in birational geometry.

**Remark 3.11.** As in Remark 2.5 d-critical generalized flips, flops at $p \in A$ in Definition 3.7 include the case that both of $f^+$, $f^-$ in the diagram (3.7) are isomorphisms. In this case, the left vertical arrows in (3.8) are closed immersions. This case also includes the case that $(\pi^\pm)^{-1}(U) = \emptyset$. Indeed one can take a closed embedding $U \subset Z$ for a smooth $Z$ which admits a smooth morphism $g: Z \to \mathbb{C}$. Then by taking $Y^\pm = Z$, $f^\pm = \text{id}$ and $w^\pm = g$, we have $\{dw^\pm\} = \emptyset$.

**Remark 3.12.** In the notation of Definition 3.7 suppose that the function $g$ satisfies $g \in m_0^2$, where $m_0 \subset O_Z$ is the ideal sheaf of $0 := j(p) \in Z$. Then as $g_*: T_{Z,0} \to T_{\mathbb{C},g(0)}$ is a zero map, we see that (set theoretically)

$$(f^\pm)^{-1}(0) \subset \{dw^\pm = 0\}.$$  

In particular if $f^\pm$ contracts a curve in $Y^\pm$ to a point $0 \in Z$, then it lies on $M^\pm$ and $\pi^\pm$ also contracts it to $p$.

**Remark 3.13.** If the condition of Remark 3.12 is not satisfied, it is possible that $\pi^\pm: M^\pm \to A$ do not contract any curve while $f^\pm$ do. For example, let us consider the diagram

\[
\begin{array}{ccc}
\mathbb{C}^2 \xrightarrow{f^+} \mathbb{C}^2 & \xrightarrow{\text{id}} & \mathbb{C}^2 \\
\downarrow{w^+} & & \downarrow{w^-} \\
\mathbb{C} & \xleftarrow{\text{id}} & \mathbb{C}
\end{array}
\]

where $f^+$ is the blow-up at the origin and $g$ is the projection onto one the factors of $\mathbb{C}^2$. By taking the critical locus of $w^\pm$, we obtain the diagram

\[
\{dw^+ = 0\} = \text{Spec} \mathbb{C} \xleftarrow{\text{id}} (0 \in \mathbb{C}^2) \leftarrow \{dw^- = 0\} = \emptyset.
\]

Although $f^+$ contracts a $\mathbb{P}^1$ to a point, the above diagram does not contract curves.

In order to avoid situations as in Remark 3.11 and Remark 3.13, we introduce the following strict notion of birational transformations:

**Definition 3.14.** In the situation of Definition 3.7 we call a diagram (3.7) strict at $p \in A$ if $\dim(\pi^+)^{-1}(p) \geq 1$, i.e. $\pi^+$ is not a finite morphism at $p$. We call a diagram (3.7) is strict if it is strict at some $p \in A$, i.e. $\pi^+$ is not a finite morphism.

**Remark 3.15.** The diagram (3.12) is strict if $a \geq 2$ by Remark 3.12. On the other hand, the diagram (3.14) is not strict.
3.3. D-critical MMP. We define the following d-critical version of MMP as follows:

**Definition 3.16.** Let \((M, s)\) be an algebraic (resp. analytic) d-critical locus. A d-critical MMP of \((M, s)\) is a sequence

\[
\begin{align*}
M_1 & \xrightarrow{\pi_1^+} A_1 & \xrightarrow{\pi_1^-} M_2 \xrightarrow{\pi_2^+} A_2 & \cdots \xrightarrow{\pi_{N-1}^+} A_{N-1} & \xrightarrow{\pi_{N-1}^-} M_N
\end{align*}
\]

where each \((M_i, s_i)\) is an algebraic (resp. analytic) d-critical locus, \((M_1, s_1) = (M, s)\) as d-critical loci, and for each \(i\) the diagram

\[
M_i \xrightarrow{\pi_i^+} A_i \xleftarrow{\pi_i^-} M_{i+1}
\]

is an algebraic (resp. analytic) d-critical generalized flip at any point in \(\text{Im} \pi_i^-\), and d-critical generalized MFS at any point in \(A_i \setminus \text{Im} \pi_i^-\). A d-critical MMP is called **strict** if each diagram (3.16) is strict in the sense of Definition 3.14.

We give an example of a d-critical MMP from a usual MMP:

**Example 3.17.** Let \(\mathcal{X}\) be a complex manifold with a projective morphism \(f: \mathcal{X} \to \Delta\) where \(0 \in \Delta \subset \mathbb{C}\) is a small disc. Suppose that \(f^{-1}(t)\) is a smooth minimal model for any \(t \in \Delta \setminus \{0\}\). Also suppose that we have a \(f\)-relative MMP of \(\mathcal{X}\) over \(\Delta\)

\[
\mathcal{X} = \mathcal{X}_1 \to \mathcal{X}_2 \to \cdots \to \mathcal{X}_{N-1} \to \mathcal{X}_N
\]

where \(\mathcal{X}_N \to \Delta\) is a minimal model over \(\Delta\) such that each \(\mathcal{X}_i\) is smooth. (For example such a MMP always exists when \(\dim \mathcal{X} = 2\).) Then each birational map \(\mathcal{X}_i \to \mathcal{X}_{i+1}\) fits into the diagram

\[
\begin{align*}
\mathcal{X}_i & \xrightarrow{\pi_i^+} \mathcal{Y}_i & \xleftarrow{\pi_i^-} \mathcal{X}_{i+1} \\
\Delta & \xrightarrow{g_i} \mathcal{Y}_i & \xleftarrow{f_{i+1}} \mathcal{X}_{i+1}
\end{align*}
\]

where the top diagram is either a divisorial contraction or a flip. Let \(h: \Delta \to \mathbb{C}\) be defined by \(t \mapsto t^2\) and set

\[
w_i := h \circ f_i : \mathcal{X}_i \to \mathbb{C}, \quad M_i := \{dw_i = 0\}, \quad A_i := \mathcal{Y}_i \times_{\Delta} \text{Spec} \mathbb{C}[t]/t^m
\]

for \(m \gg 0\). Note that \(M_i, A_i\) are projective schemes with \(f_i(M_i) \subset A_i\) for \(m \gg 0\), and \(M_i\) admits a d-critical structure \(s_i = w_i + (dw_i)^2\). Then we obtain a d-critical MMP (3.15), which is strict by Remark 3.12.

**Remark 3.18.** In Example 3.17 note that \((f_i)^{-1}(0) = M_i\) as a set, but their scheme structures may be different. For example if \((f_i)^{-1}(0)\) is a curve with a nodal singularity at \(x \in (f_i)^{-1}(0)\), then the scheme structure of \(M_i\) at
x is given by $\hat{O}_{M,x} = \mathbb{C}[[x,y]]/(x^2y, xy^2)$, which is a critical locus of the function $x^2y^2$, and not isomorphic to the nodal singularity.

As an analogy of minimal model in birational geometry, we introduce the following notion of minimal d-critical loci:

**Definition 3.19.** A d-critical locus $(M, s)$ is called minimal if the virtual canonical line bundle $\omega_{M,s}$ is nef.

**Example 3.20.** (i) For a d-critical locus $(M, s)$, if $M$ is smooth then $\omega_{M,s} = \omega_M^{\otimes 2}$. Therefore $(M, s)$ is minimal if and only if $K_M$ is nef, i.e. $M$ is minimal in the usual sense.

(ii) In the situation of Example 3.17, the d-critical locus $(M_N, s_N)$ is minimal, as $\omega_{M_N,s_N} = \omega_{X_N}^{\otimes 2}|_{M_N}$ is nef. Note that $M_N$ is a projective singular scheme if $(f_N)^{-1}(0)$ is singular.

(iii) There is also an example of a singular projective minimal d-critical locus $(M, s)$, such that $M^\text{red}$ is a smooth non-minimal model. Let $Z$ be the $A_1$ surface singularity

$$Z = \{xy + z^2 = 0\} \subset \mathbb{C}^3$$

and take the blow-up $f: Y \to Z$ at the origin, which is a crepant resolution of the singularity $0 \in Z$. We consider the commutative diagram

$$
\begin{array}{ccc}
Y & \to & Z \\
\downarrow w & & \downarrow (x,y,z) \mapsto x^2+y^2+z^2 \\
\searrow & & \searrow \\
& \mathbb{C}. & \\
\end{array}
$$

We have the following d-critical locus:

$$(M, s), \ M = \{dw = 0\}, \ s = w + (dw)^2.$$

Then $M^\text{red} = \mathbb{P}^1$ and there are 4-points in $\mathbb{P}^1$ at which $M$ is non-reduced. At these points, the scheme structure of $M$ is given by the critical locus of the function $(x, y) \mapsto xy^2$ on $\mathbb{C}^2$. Note that $\omega_{M,s} = \omega_{Y}^{\otimes 2}|_{M_\text{red}} \cong \mathcal{O}_{\mathbb{P}^1}$ as $\omega_Y$ is trivial. Therefore the d-critical locus $(M, s)$ is minimal, while $M^\text{red}$ is not minimal.

Similarly to the usual minimal model in birational geometry, we have the following lemma:

**Lemma 3.21.** Let $(M^+, s^+)$ be a minimal d-critical locus. Then there is no strict diagram

$$(3.17) \quad M^+ \xrightarrow{\pi^+} A \xleftarrow{\pi^-} M^-$$

for a d-critical locus $(M^-, s^-)$, which is a d-critical generalized flip at any point in $\text{Im} \pi^-$, and a d-critical generalized MFS at any point in $A \setminus \text{Im} \pi^-$. 

Proof. Suppose that a diagram (3.17) exists. As $\pi^+$ is not a finite morphism, there is a projective curve $C \subset M^+$ such that $\pi(C) = p$ for some point $p \in A$. Let us take $\pi^+$-relative d-critical charts (3.5) for an open neighborhood $p \in U \subset A$. Then as $-K_{Y^+}$ is $f$-ample, we have
\[
\deg(\omega_{M^+,s^+}|_C) = 2K_{Y^+} \cdot i^+(C) < 0
\]
which contradicts that $(M^+, s^+)$ is minimal. \qed

Remark 3.22. By Lemma 3.21, a strict d-critical MMP (3.15) terminates at $M_N$ if $M_N$ is either minimal or an empty set. In the latter case, the morphism $M_{N-1} \to A_{N-1}$ is a d-critical generalized MFS at any point in $A_{N-1}$.

We generalize the inequality of canonical divisors in Definition 2.8 to virtual canonical line bundles on d-critical loci:

Definition 3.23. In the situation of Definition 3.7, we write
\[
(M^+, s^+) \geq_K (M^-, s^-) \tag{3.18}
\]
if for any $p \in A$ there is a $\pi^+$-relative d-critical chart (3.8) such that $Y^+ \geq_K Y^-$ as in Definition 2.8. The inequality (3.18) is strict if $Y^+ >_K Y^-$ for some $p \in A$.

Remark 3.24. By (3.4), the inequality (3.18) is regarded as an inequality for virtual canonical bundles of d-critical loci.

If the diagram (3.7) is a d-critical divisorial contraction, (generalized) flip, (generalized) MFS, we have the inequality (3.18). In particular for a d-critical MMP (3.15), we have the inequalities
\[
(M_1, s_1) \geq_K (M_2, s_2) \geq_K \cdots \geq_K (M_N, s_N).
\]
Each inequalities are strict if (3.15) is a strict d-critical MMP. Moreover we have the equality of (3.18) if the diagram (3.7) is a d-critical (generalized) flop.

4. Moduli spaces of semistable objects on CY 3-folds

In this section, we discuss moduli spaces of Bridgeland semistable objects on CY 3-folds, and introduce their wall-crossing diagrams as in the introduction. We address a general question whether wall-crossing diagrams in CY 3-folds are described in terms of d-critical birational transformations introduced in the previous section, which is a main topic in this paper.

4.1. Moduli spaces of objects on CY 3-folds. Let $X$ be a smooth projective CY 3-fold, i.e. $\dim X = 3$, $K_X = 0$ and $H^1(O_X) = 0$. Below we fix a trivialization
\[
O_X \xrightarrow{\cong} \omega_X.\tag{4.1}
\]
We denote by $\mathcal{M}$ the 2-functor

$$\mathcal{M}: \text{Sch}/\mathbb{C} \to \text{Groupoid}$$

sending a $\mathbb{C}$-scheme $S$ to the groupoid of relatively perfect object

$$\mathcal{E} \in D^b(X \times S)$$

such that for each $s \in S$, its derived restriction $\mathcal{E}_s$ to $X \times \{s\}$ satisfies $\text{Ext}^{<0}(\mathcal{E}_s, \mathcal{E}_s) = 0$. By the result of Lieblich [Lie06], the 2-functor $\mathcal{M}$ is an Artin stack locally of finite type. We have the open substack $\mathcal{M}^{\text{si}} \subset \mathcal{M}$ consisting of simple objects, i.e. substacks of objects (4.2) which furthermore satisfies $\text{Hom}(\mathcal{E}_s, \mathcal{E}_s) = \mathbb{C}$ for any $s \in S$. Then by [Lie06, Corollary 4.3.3] (also see [Ina02]) there is an algebraic space $M^{\text{si}}$ locally of finite type with a morphism

$$\mathcal{M}^{\text{si}} \to M^{\text{si}}$$

which is a étale locally trivial $B\mathbb{C}^*$-bundle (i.e. $\mathbb{C}^*$-gerbe). By the result of [BBBBJ15], we have the following:

**Theorem 4.1.** ([BBBBJ15]) There is a canonical d-critical structure on the stack $\mathcal{M}$ whose virtual canonical line bundle is given by

$$\omega_{\mathcal{M}}^{\text{vir}} := \text{det}R\mathcal{H}om_{pr,\mathcal{M}}(\mathcal{E}, \mathcal{E}).$$

Here $\mathcal{E}$ is a universal sheaf on $X \times \mathcal{M}$ and $pr,\mathcal{M}: X \times \mathcal{M} \to \mathcal{M}$ is the projection. The restriction of the above d-critical structure to $\mathcal{M}^{\text{si}} \subset \mathcal{M}$ descends to a d-critical structure on $M^{\text{si}}$.

**Remark 4.2.** More precisely, the d-critical structure on $\mathcal{M}$ in Theorem 4.1 is canonically determined once we choose a trivialization (4.1).

### 4.2. Moduli spaces of Bridgeland semistable objects.

We define $\Gamma_X \subset H^{2*}(X, \mathbb{Q})$ to be the image of the Chern character map

$$\text{ch}: K(X) \to H^{2*}(X, \mathbb{Q}).$$

Let $\text{Stab}(X)$ the space of Bridgeland stability conditions on the derived category $D^b(X)$ with respect to the Chern character map $\text{ch}: K(X) \to \Gamma_X$ (see Appendix A.3). By its definition, a point $\sigma \in \text{Stab}(X)$ is written as

$$\sigma = (Z, \mathcal{A}) \in \text{Stab}(X), \quad \mathcal{A} \subset D^b(X), \quad Z: \Gamma_X \to \mathbb{C}$$

where $\mathcal{A}$ is the heart of a bounded t-structure and $Z$ is a group homomorphism, satisfying some conditions.

**Remark 4.3.** So far it is not known whether $\text{Stab}(X) \neq \emptyset$ for a projective CY 3-fold in general. At the present time, this is only known when $X$ is an étale quotient of an abelian 3-fold $\text{MP15, BMS16}$. On the other hand, we can generalize the arguments below in a modified situation where it is easier to construct stability conditions, e.g. $\text{Stab}(X)$ for a non-compact CY 3-fold, $\text{Stab}(\mathcal{D})$ for a triangulated subcategory $\mathcal{D} \subset D^b(X)$, etc.
For a stability condition $\sigma$ as in (4.5) and an element $v \in \Gamma_X$, we have the substacks
\[(4.6) \quad \mathcal{M}_\sigma^s(v) \subset \mathcal{M}_\sigma(v) \subset \mathcal{M}\]
where $\mathcal{M}_\sigma(v)$ consists of $\sigma$-semistable objects $E \in \mathcal{A}$ with $\text{ch}(E) = v$ and $\mathcal{M}_\sigma^s(v)$ is the $\sigma$-stable part of $\mathcal{M}_\sigma(v)$. Below we discuss under the following assumption:

**Assumption 4.4.** The substacks (4.6) are open substacks of $\mathcal{M}$, and they are of finite type.

**Remark 4.5.** The above assumption holds in the cases where $\text{Stab}(X)$ is known to be non-empty (see [PT]). As proven in loc. cit., the Bogomolov-Gieseker type inequality conjecture proposed in [BMT14, BMS] implies both of constructions of stability conditions and Assumption 4.4.

Under Assumption 4.4, we can discuss good moduli spaces for Artin stacks (4.6) in the sense of [Alp13]. A general definition is as follows:

**Definition 4.6.** ([Alp13]) A morphism $p: \mathcal{M} \to M$, where $\mathcal{M}$ is an Artin stack and $M$ an algebraic space, is called a **good moduli space** for $\mathcal{M}$ if the following conditions hold:

(i) $p$ is quasi-compact and $p_*: \text{QCoh}(\mathcal{M}) \to \text{QCoh}(M)$ is exact.

(ii) The natural map $O_M \to p_*O_{\mathcal{M}}$ is an isomorphism.

A good moduli space $p: \mathcal{M} \to M$ is universal for morphisms to algebraic spaces (see [Alp13, Theorem 6.6]). Namely for a morphism $p': \mathcal{M} \to M'$ for another algebraic space $M'$, there is a unique factorization
\[p': \mathcal{M} \xrightarrow{p} M \to M'.\]

Let us return to moduli stacks of (semi)stable objects (4.6). We will use the following result which is announced in [AHLH] (also see [HL, Theorem 4.3], [Sav, Theorem 5.11]):

**Theorem 4.7.** ([AHLH]) The stack $\mathcal{M}_\sigma(v)$ admits a good moduli space
\[p_M: \mathcal{M}_\sigma(v) \to M_\sigma(v)\]
for a separated algebraic space $M_\sigma(v)$ of finite type.

The good moduli space $M_\sigma(v)$ is the coarse moduli space of $S$-equivalence classes of $\sigma$-semistable objects with Chern character $v$. It follows that there is a one to one correspondence between closed points of $M_\sigma(v)$ and $\sigma$-polystable objects, i.e. $\sigma$-semistable objects in $\mathcal{A}$ with Chern character $v$, isomorphic to direct sums of $\sigma$-stable objects.

**Remark 4.8.** The existence of the good moduli space is well-known for moduli stacks given as GIT quotient stacks, say moduli stacks of Gieseker semistable sheaves (see [HL97]). In this case, we can proceed the arguments without relying on [AHLH]. In general it is not known whether $\mathcal{M}_\sigma(v)$
BIRATIONAL GEOMETRY FOR D-CRITICAL LOCI

Let $M^\sigma_\sigma(v) \subset M^\sigma(v)$ be the open subspace consisting of $\sigma$-stable objects. Then we have the morphism
\[ p_M : M^\sigma_\sigma(v) \rightarrow M^\sigma(v) \]  
(4.7)
giving a good moduli space for $M^\sigma_\sigma(v)$. Note that $M^\sigma_\sigma(v)$ is also an open subspace of the moduli space of simple objects $M^{\text{s}i}$ in the previous subsection, and the morphism (4.7) is a pull-back of the morphism (4.3) by the open immersion $M^\sigma_\sigma(v) \subset M^{\text{s}i}$.

4.3. Wall-crossing diagram in CY 3-folds. By a general theory of Bridge-land stability conditions, there is a collection of locally finite codimension one submanifolds (called walls)
\[ \{W_{\lambda}\}_{\lambda \in \Lambda}, \ W_{\lambda} \subset \text{Stab}(X) \]  
(4.8)
such that the moduli stack $\mathcal{M}_\sigma(v)$ is constant if $\sigma$ lies in a connected component of the complements of walls (called chamber), but may change if $\sigma$ crosses a wall. Each wall $W_{\lambda}$ is defined by the condition
\[ Z(v_1) \in \mathbb{R}_{>0}Z(v_2), \ v_1 + v_2 = v \in \Gamma_X \]
where $(v_1, v_2)$ are not proportional in $(\Gamma_X)_{\mathbb{Q}}$.

Below, we take $v \in \Gamma_X$ to be primitive, i.e. $v$ is not written as a multiple of some element in $\Gamma_X$. Let us take
\[ \sigma = (Z, A) \in \text{Stab}(X), \ \sigma^\pm = (Z^\pm, A^\pm) \in \text{Stab}(X) \]
where $\sigma$ lies on a wall and $\sigma^\pm$ lie on its adjacent chambers. By applying $\mathbb{C}$-action on $\text{Stab}(X)$ if necessary (see Remark A.5), we may assume that $\exists Z(v) > 0$. Then any $\sigma^\pm$-semistable object $E \in A^\pm$ with $\text{ch}(E) = v$ is a $\sigma$-semistable object in $A$. So we have open immersions $\mathcal{M}_{\sigma^\pm}(v) \subset \mathcal{M}_\sigma(v)$, and the commutative diagram
\[ \begin{array}{ccc}
\mathcal{M}_{\sigma^+}(v) & \xrightarrow{p^+_M} & \mathcal{M}_{\sigma}(v) \\
q^+_M \downarrow & & \downarrow q_M \\
M_{\sigma^+}(v) & \rightarrow & M_{\sigma}(v)
\end{array} \]
\[ \begin{array}{ccc}
\mathcal{M}_{\sigma^-}(v) & \xrightarrow{p^-_M} & \mathcal{M}_{\sigma}(v) \\
q^-_M \downarrow & & \downarrow q_M \\
M_{\sigma^-}(v) & \rightarrow & M_{\sigma}(v)
\end{array} \]
(4.9)

Here the top arrows are open immersions, the vertical arrows are morphisms to the good moduli spaces, and the bottom arrows are induced by the universality of good moduli spaces. Moreover we have $M_{\sigma^\pm}(v) = M_{\sigma^\pm}(v)$ as $v$ is primitive and $\sigma^\pm$ lie on chambers. In particular $M_{\sigma^\pm}(v)$ admit d-critical structures by Theorem 4.1, and the following question makes sense:

**Question 4.9.** For a primitive $v \in \Gamma_X$, is the bottom diagram in (4.9) a (generalized) d-critical flip or flop?
In what follows, we study the above question via moduli spaces of representations of Ext-quivers.

5. REPRESENTATIONS OF QUIVERS WITH SUPER-POTENTIALS

In this section, we construct d-critical birational transformations introduced in the previous section via representations of quivers with convergent super-potentials. The descriptions in this section will give analytic local models of d-critical birational transformations for wall-crossing diagrams in CY 3-folds considered in the previous section.

5.1. Representations of quivers. Recall that a quiver $Q$ consists of data

$$Q = (V(Q), E(Q), s, t)$$

where $V(Q), E(Q)$ are finite sets and $s, t$ are maps

$$s, t: E(Q) \to V(Q).$$

The set $V(Q)$ is the set of vertices and $E(Q)$ is the set of edges. For $e \in E(Q)$, $s(e)$ is the source of $e$ and $t(e)$ is the target of $e$. For $i, j \in V(Q)$, we use the following notation

$$(5.1) \quad E_{i,j} := \{e \in E(Q) : s(e) = i, t(e) = j\}, \quad E_{i,j} := \bigoplus_{e \in E_{i,j}} \mathbb{C} \cdot e$$

i.e. $E_{i,j}$ is the set of edges from $i$ to $j$, and $E_{i,j}$ is the $\mathbb{C}$-vector space spanned by $E_{i,j}$. The dual quiver $Q^\vee$ of $Q$ is defined by

$$Q^\vee := (V(Q), E(Q), s^\vee, t^\vee), \quad s^\vee := t, \quad t^\vee := s.$$  

Recall that a path of a quiver $Q$ is a composition of edges in $Q$

$$e_1 e_2 \ldots e_n, \quad e_i \in E(Q), \quad t(e_i) = s(e_{i+1}).$$

The number $n$ above is called the length of the path. The path algebra of a quiver $Q$ is a $\mathbb{C}$-vector space spanned by paths in $Q$:

$$\mathbb{C}[Q] := \bigoplus_{n \geq 0} \bigoplus_{e_1, \ldots, e_n \in E(Q), t(e_i) = s(e_{i+1})} \mathbb{C} \cdot e_1 e_2 \ldots e_n.$$  

Here a path of length zero is a trivial path at each vertex of $Q$, and the product on $\mathbb{C}[Q]$ is defined by the composition of paths.

A $Q$-representation consists of data

$$(5.2) \quad \mathbb{V} = \{(V_i, u_e) : i \in V(Q), \ e \in E(Q), \ u_e : V_{s(e)} \to V_{t(e)}\}$$

where $V_i$ is a finite dimensional $\mathbb{C}$-vector space and $u_e$ is a linear map. It is well-known that a $Q$-representation is nothing but a left $\mathbb{C}[Q]$-module structure on $\bigoplus_{i \in V(Q)} V_i$. Also note that the dual of $\mathbb{V}$

$$(5.3) \quad \mathbb{V}^\vee := \{(V_i^\vee, u_e^\vee) : i \in V(Q), \ e \in E(Q), \ u_e^\vee : V_{t(e)}^\vee \to V_{s(e)}^\vee\}$$

is a $Q^\vee$-representation.
For a \(Q\)-representation \(V\) as in (5.2), the vector
\[
\vec{m} = (m_i)_{i \in V(Q)}, \quad m_i = \dim V_i
\]
is called the \textit{dimension vector} of \(V\). For each \(i \in V(Q)\), let \(S_i\) be the one-dimensional \(Q\)-representation corresponding to the vertex \(i\), whose dimension vector is denoted by \(\vec{i}\). We set
\[
\Gamma_Q := \bigoplus_{i \in V(Q)} \mathbb{Z} \cdot \vec{i}.
\]
Note that the dimension vector (5.4) for a non-zero \(Q\)-representation (5.2) takes its value in the positive cone \(\Gamma_{Q,>0} \subset \Gamma_Q\):
\[
\Gamma_{Q,>0} := \{ \vec{m} = (m_i)_{i \in V(Q)} \in \Gamma_Q : m_i \geq 0 \} \setminus \{0\}.
\]
For a given element \(\vec{m} = (m_i)_{i \in V(Q)} \in \Gamma_{Q,>0}\), let \(V_i\) be \(\mathbb{C}\)-vector spaces with dimension \(m_i\). Let us set
\[
G := \prod_{i \in V(Q)} \text{GL}(V_i), \quad \text{Rep}_Q(\vec{m}) := \prod_{e \in E(Q)} \text{Hom}(V_{s(e)}, V_{t(e)}).
\]
The algebraic group \(G\) acts on \(\text{Rep}_Q(\vec{m})\) by
\[
g \cdot u = \{ g_{t(e)}^{-1} \circ u_e \circ g_{s(e)} \}_{e \in E(Q)}
\]
for \(g = (g_i)_{i \in V(Q)} \in G\) and \(u = (u_e)_{e \in E(Q)}\). A \(Q\)-representation with dimension vector \(\vec{m}\) is determined by a point in \(\text{Rep}_Q(\vec{m})\) up to \(G\)-action. The moduli stack of \(Q\)-representations with dimension vector \(\vec{m}\) is given by the quotient stack
\[
\mathcal{M}_Q(\vec{m}) := [\text{Rep}_Q(\vec{m})/G].
\]
We have the natural morphism to the GIT quotient
\[
p_Q : \mathcal{M}_Q(\vec{m}) \to M_Q(\vec{m}) := \text{Rep}_Q(\vec{m})//G.
\]
Here in general, if a reductive algebraic group \(G\) acts on an affine scheme \(Y = \text{Spec } R\), its GIT quotient is given by
\[
Y//G := \text{Spec } (R^G).
\]
A closed point of \(M_Q(\vec{m})\) corresponds to a semi-simple \(Q\)-representation, i.e. a direct sum of simple \(Q\)-representations, and \(p_Q\) sends a \(Q\)-representation to its semi-simplification. We have the commutative diagram
\[
\begin{array}{ccc}
\text{Rep}_Q(\vec{m}) & \longrightarrow & \mathcal{M}_Q(\vec{m}) \\
\downarrow \pi_Q & & \downarrow p_Q \\
M_Q(\vec{m}) & & \\
\end{array}
\]
The point \( 0 \in \text{Rep}_Q(\vec{m}) \) and its image \( 0 \in M_Q(\vec{m}) \) by the map (5.9) correspond to the semi-simple \( Q \)-representation

\[
\bigoplus_{i \in V(Q)} V_i \otimes S_i.
\]

A \( Q \)-representation (5.2) is called nilpotent if any sufficiently large number of compositions of the linear maps \( u_i \) becomes zero. It is easy to see that a \( Q \)-representation is nilpotent if and only if it is an iterated extensions of simple objects \( \{ S_i \}_{i \in V(Q)} \). In particular, the fiber

\[
p_Q^{-1}(0) \subset M_Q(\vec{m})
\]

for the morphism (5.9) consists of nilpotent \( Q \)-representations with dimension vector \( \vec{m} \). The morphism (5.9) is a good moduli space of the stack \( M_Q(\vec{m}) \) (see Definition 4.6).

### 5.2. Semistable quiver representations

For a quiver \( Q \), let \( K(Q) \) be the Grothendieck group of the abelian category of finite dimensional \( Q \)-representations. Let \( \mathcal{H} \subset \mathbb{C} \) be the upper half plane, and take

\[
(5.11) \quad \xi = (\xi_i)_{i \in V(Q)} \in \mathcal{H}^{\#V(Q)}, \quad \xi_i \in \mathcal{H}.
\]

Then we have the group homomorphism

\[
(5.12) \quad Z_\xi : K(Q) \xrightarrow{\dim} \Gamma_Q \to \mathbb{C}, \quad [S_i] \mapsto \xi_i.
\]

Here \( \dim \) is the map taking the dimension vectors of \( Q \)-representations. Then \( Z_\xi \) defines a Bridgeland stability condition on the abelian category of finite dimensional \( Q \)-representations (see Appendix A). For simplicity, we call \( Z_\xi \)-(semi)stable objects as \( \xi \)-(semi)stable objects.

For a choice of \( \xi \) as in (5.11), let

\[
(5.13) \quad \text{Rep}_Q^\xi(\vec{m}) \subset \text{Rep}_Q(\vec{m})
\]

be the Zariski open locus consisting of \( \xi \)-semistable \( Q \)-representations. We take the associated GIT quotients:

\[
M_Q^\xi(\vec{m}) := [\text{Rep}_Q^\xi(\vec{m})/G], \quad M_Q^\xi(\vec{m}) := \text{Rep}_Q^\xi(\vec{m})/G.
\]

We have the commutative diagram

\[
(5.14)
\begin{array}{ccc}
M_Q^\xi(\vec{m}) & \xrightarrow{j_Q^\xi} & M_Q(\vec{m}) \\
p_Q^\xi \downarrow & & \downarrow p_Q \\
M_Q^\xi(\vec{m}) & \xrightarrow{q_Q^\xi} & M_Q(\vec{m}).
\end{array}
\]

Here \( j_Q^\xi \) is an open immersion, \( p_Q, p_Q^\xi \) are natural morphisms to the good moduli spaces, and \( q_Q^\xi \) is the morphism induced by the open immersion.
By a general theory of GIT, the morphism \( q^\xi_Q \) is a projective morphism of irreducible varieties satisfying \( q^\xi_Q \circ O_{M^\xi_Q(\bar{m})} = O_{M_Q(\bar{m})} \).

Let

\[
M^\xi_Q(\bar{m}) \subset M_Q(\bar{m}), \quad M^\xi_{Q,s}(\bar{m}) \subset M^\xi_Q(\bar{m})
\]

be the open subsets of simple part, \( \xi \)-stable part respectively. It is well-known (for example see [Rei08]) that both of \( M^\xi_Q(\bar{m}) \), \( M^\xi_{Q,s}(\bar{m}) \) are smooth varieties. As any preimage \( (q^\xi_Q)^{-1}(x) \) for \( x \in M^s_Q(\bar{m}) \) is a one point, the morphism \( q^\xi_Q \) is a projective birational morphism if \( M^s_Q(\bar{m}) \neq \emptyset \).

### 5.3. Quivers with convergent super-potentials.

For a quiver \( Q \), by taking the completion of the path algebra \( \mathbb{C}[Q] \) with respect to the length of the path, we obtain the formal path algebra:

\[
\mathbb{C}[Q] := \prod_{n \geq 0} \bigoplus_{e_1, \ldots, e_n \in E(Q), t(e_i) = s(e_{i+1})} \mathbb{C} \cdot e_1 e_2 \ldots e_n.
\]

Note that an element \( f \in \mathbb{C}[Q] \) is written as

\[
f = \sum_{n \geq 0, \{1, \ldots, n+1\} \ni V(Q)} \sum_{e_i \in E_{\psi(i), \psi(i+1)}} a_{\psi, e_*} \cdot e_1 e_2 \ldots e_n.
\]

Here \( a_{\psi, e_*} \in \mathbb{C}, \ e_* = (e_1, \ldots, e_n) \) and \( E_{\psi(i), \psi(i+1)} \) is defined as in (5.1). The above element \( f \) lies in \( \mathbb{C}[Q] \) if and only if \( a_{\psi, e_*} = 0 \) for \( n \gg 0 \).

**Definition 5.1.** The subalgebra

\[
\mathbb{C}\{Q\} \subset \mathbb{C}[Q]
\]

is defined to be elements (5.13) such that \( |a_{\psi, e_*}| < C^n \) for some constant \( C > 0 \) which is independent of \( n \).

Note that \( \mathbb{C}\{Q\} \) contains \( \mathbb{C}[Q] \) as a subalgebra. A *convergent super-potential* of a quiver \( Q \) is an element

\[
W \in \mathbb{C}\{Q\}/[\mathbb{C}\{Q\}, \mathbb{C}\{Q\}].
\]

A convergent super-potential \( W \) of \( Q \) is represented by a formal sum

\[
W = \sum_{n \geq 1} \sum_{\{1, \ldots, n+1\} \ni V(Q)} \sum_{\psi(n+1) = \psi(1)} a_{\psi, e_*} \cdot e_1 e_2 \ldots e_n
\]

with \( |a_{\psi, e_*}| < C^n \) for a constant \( C > 0 \). The above \( W \) is called *minimal* if \( a_{\psi, e_*} = 0 \) for \( e_* = (e_1, \ldots, e_n) \) with \( n \leq 2 \).

For a dimension vector \( \bar{m} \) of \( Q \), let \( \text{tr} W \) be the formal function of \( u = (u_e)_{e \in E(Q)} \in \text{Rep}_Q(\bar{m}) \) defined by

\[
\text{tr} W(u) := \sum_{n \geq 1} \sum_{\{1, \ldots, n+1\} \ni V(Q)} \sum_{\psi(n+1) = \psi(1)} a_{\psi, e_*} \cdot \text{tr}(u_n \circ u_{n-1} \circ \cdots \circ u_1).
\]
The above formal function on $\text{Rep}_Q(\vec{m})$ is $G$-invariant. By the argument of [Todb, Lemma 2.10] and [Toda, Lemma 4.9], there is an analytic open neighborhood $V$ and an analytic function $\text{tr}(W) : V \to \mathbb{C}$ such that the formal function $\text{tr} W$ absolutely converges on $\pi_Q^{-1}(V)$ (here $\pi_Q$ is given in the diagram (5.10)) to give a $G$-invariant analytic function, which factors through $\pi_Q^{-1}(V) \to V$:

\begin{equation}
\text{tr} W : \pi_Q^{-1}(V) \to \pi_Q(\vec{m}) \to \mathbb{C}.
\end{equation}

Then we set

\begin{equation}
\text{Rep}_{(Q,\partial W)}(\vec{m})|_V := \{ d(\text{tr} W) = 0 \} \subset \pi_Q^{-1}(V),
\end{equation}

\begin{equation}
\mathcal{M}_{(Q,\partial W)}(\vec{m})|_V := \{ d(\text{tr} W) = 0 \}/G \subset \pi_Q^{-1}(V)/G,
\end{equation}

\begin{equation}
\text{Rep}_\xi(\vec{m})|_V := \{ d(\text{tr} W) = 0 \}/G \subset V.
\end{equation}

Here $(-)/G$ above is an analytic Hilbert quotient (see [HMP98, Gre15, Todb]).

Remark 5.2. For a $Q$-representation corresponding to a point in $\pi_Q^{-1}(V)$, it satisfies the equation $\{ d(\text{tr} W) = 0 \}$ if and only if it satisfies the relation $\partial W$ of the quiver $Q$ given by derivations of $W$ (see [Todb, Subsection 2.6]).

Let $\xi$ be data as in (5.11) which defines the $\xi$-stability on the category of $Q$-representations, and

\begin{equation}
\text{Rep}_\xi(\vec{m})|_V := \text{Rep}_{(Q,\partial W)}(\vec{m})|_V
\end{equation}

be the open locus consisting of $\xi$-semistable $Q$-representations. Similarly to (5.19), we define

\begin{equation}
\mathcal{M}_\xi(\vec{m})|_V := \text{Rep}_\xi(\vec{m})|_V /G,
\end{equation}

\begin{equation}
\text{Rep}_\xi(\vec{m})|_V := \text{Rep}_\xi(\vec{m})|_V //G.
\end{equation}

Then we have the commutative diagram

\begin{equation}
\begin{array}{ccc}
M_{(Q,\partial W)}(\vec{m})|_V & \xrightarrow{(q_{(Q,\partial W)})^{-1}(V)} & (q_{(Q,\partial W)})^{-1}(V) \\
\downarrow & & \downarrow \quad (q_{(Q,\partial W)})^{-1}(V) \\
M_{(Q,\partial W)}(\vec{m})|_V & \xrightarrow{\text{tr} W} & \mathbb{C}
\end{array}
\end{equation}

Here $\hookrightarrow$ are closed embeddings (see [Todb, Lemma 2.9]), $q_{(Q,\partial W)}$ is given in the diagram (5.14), $q_{(Q,\partial W)}$ is induced by the open immersion (5.20), and $\text{tr} W$ is defined by the commutativity of (5.21).
Lemma 5.3. Suppose that $M_{Q}^{\xi,s}(\vec{m}) = M_{Q}^{\xi}(\vec{m})$ holds. Then there is a $d$-critical structure on $M_{(Q,\partial W)}^{\xi}(\vec{m})|_{V}$ given by
\[ s = \text{tr}^{\xi} W + (d \text{tr}^{\xi} W)^{2} \in \Gamma(S_{M_{(Q,\partial W)}^{\xi}(\vec{m})|_{V}}^{0}) \]
such that the diagram (5.21) is its $q_{\xi}^{q_{(Q,\partial W)}}$-relative $d$-critical chart.

Proof. The assumption implies that $M_{Q}^{\xi,s}(\vec{m})$ and $(q_{\xi}^{q_{(Q,\partial W)}})^{-1}(V)$ (see (5.14) for the definition of the map $q_{\xi}^{q_{(Q,\partial W)}}$) are $C^{\ast}$-gerbes over $M_{Q,\partial W}^{\xi}(\vec{m})|_{V}$ and $(q_{\xi}^{q_{(Q,\partial W)}})^{-1}(V)$ respectively. Since $M_{Q,\partial W}^{\xi}(\vec{m})|_{V}$ is a critical locus of the function $\text{tr}^{\xi} W|_{(q_{\xi}^{q_{(Q,\partial W)}})^{-1}(V)}$, it follows that $M_{Q,\partial W}^{\xi}(\vec{m})|_{V}$ is a critical locus of the function $\text{tr}^{\xi} W$ on the smooth space $(q_{\xi}^{q_{(Q,\partial W)}})^{-1}(V)$. Therefore the lemma holds. \(\square\)

5.4. Wall-crossing of representations of quivers. Suppose that $\vec{m} \in \Gamma_{Q,>0}$ is primitive, i.e. it is not divided by a positive integer greater than one. For each decomposition $\vec{m} = \vec{m}_{1} + \vec{m}_{2}$ for $\vec{m}_{i} \in \Gamma_{Q,>0}$, we set
\[ W_{(\vec{m}_{1},\vec{m}_{2})} := \left\{ \xi \in H^{\text{tr}}(Q) : Z_{\xi}(\vec{m}_{1}) \in \mathbb{R}_{>0}Z_{\xi}(\vec{m}_{2}) \right\}. \]

As $\vec{m}$ is primitive, the vectors $\vec{m}_{1}$, $\vec{m}_{2}$ are not proportional in $(\Gamma_{Q})$. Therefore $W_{(\vec{m}_{1},\vec{m}_{2})}$ is a real codimension one submanifold of $H^{\text{tr}}(Q)$. Since the possible decomposition $\vec{m} = \vec{m}_{1} + \vec{m}_{2}$ is finite, the above real codimension one submanifolds are finite.

The submanifolds $W_{(\vec{m}_{1},\vec{m}_{2})}$ are the set of walls of $\xi$-stability conditions on the category of $Q$-representations, and chambers are connected components of the complement of walls in $H^{\text{tr}}(Q)$. As is well-known, we have the wall-crossing phenomena: the moduli spaces $M_{Q}^{\xi}(\vec{m})$ are constant if $\xi$ lies on a chamber, but may change if $\xi$ crosses a wall. Note that as $\vec{m}$ is primitive, we have $M_{Q}^{\xi,s}(\vec{m}) = M_{Q}^{\xi}(\vec{m})$ if $\xi$ lies on a chamber, hence it is a smooth variety.

Let $W$ be a convergent super-potential of $Q$, and take $\xi^{\pm} \in H^{\text{tr}}(Q)$ which lie on chambers. Then there is an analytic open subset $0 \subset V \subset M_{Q}(\vec{m})$ and the diagram (see the diagram (5.21))
\[ M_{(Q,\partial W)}^{\xi^{+}}(\vec{m})|_{V} \xrightarrow{q_{(Q,\partial W)}^{\xi^{+}}} M_{(Q,\partial W)}^{\xi^{-}}(\vec{m})_{V} \xleftarrow{q_{(Q,\partial W)}^{\xi^{-}}} M_{Q,\partial W}^{\xi}(\vec{m})|_{V}. \]

By Lemma 5.3, the following corollary immediately follows:
Corollary 5.4. The diagram (5.23) is an analytic (generalized) d-critical flip, flop, MFS if the diagram

\[ (5.24) \quad M_Q^+(\vec{m}) \quad M_Q^-(\vec{m}) \]

is a (generalized) flip, flop, MFS, respectively.

Remark 5.5. If \( M_Q^+(\vec{m}) \neq \emptyset \), which follows if the condition in Theorem 7.1 is satisfied, then the diagram (5.24) is a birational map of smooth varieties \( M_Q^+(\vec{m}) \rightarrow M_Q^-(\vec{m}) \). Therefore the diagram (5.24) is a (generalized) MFS only if \( M_Q^+(\vec{m}) = \emptyset \) holds.

6. Analytic Neighborhood Theorem of Wall-Crossing in CY 3-Folds

In this section, we give an analytic neighborhood theorem for wall-crossing diagrams in CY 3-folds. This theorem describes above diagrams in term of wall-crossing diagrams for quivers with convergent super-potentials. A similar result was already proved in [Toda] in the case of moduli spaces of semistable sheaves. We will see that the same argument applies to the case of Bridgeland semistable objects, if we assume the existence of their good moduli spaces (see Theorem 4.7). In this section, we always assume that \( X \) is a smooth projective CY 3-fold.

6.1. Ext-quivers. For a smooth projective CY 3-fold \( X \), let us take a collection of objects in the derived category

\[ E_\bullet = (E_1, E_2, \ldots, E_k), \quad E_i \in D^b(X). \]

Here we recall the notion of Ext-quiver associated with the collection \( E_\bullet \) and the construction of their convergent super-potentials. We will apply the construction here to the collection of objects \( E_\bullet \) associated with polystable objects.

For each \( 1 \leq i, j \leq k \), we fix a finite subset

\[ E_{i,j} \subset \text{Ext}^1(E_i, E_j)^\vee \]

giving a basis of \( \text{Ext}^1(E_i, E_j)^\vee \). Let the quiver \( Q_{E_\bullet} \) be defined as follows. The set of vertices and edges are given by

\[ V(Q_{E_\bullet}) := \{1, 2, \ldots, k\}, \quad E(Q_{E_\bullet}) := \coprod_{1 \leq i, j \leq k} E_{i,j}. \]

The maps \( s, t: E(Q_{E_\bullet}) \rightarrow V(Q_{E_\bullet}) \) are given by

\[ s|_{E_{i,j}} := i, \quad t|_{E_{i,j}} := j. \]
The quiver $Q_{E\bullet}$ is called the Ext quiver associated with the collection $E\bullet$. Note that we have $E_{i,j} = \text{Ext}^1(E_i, E_j)^\vee$, where $E_{i,j}$ is defined as in (5.1).

For a map of sets

$$\psi: \{1, \ldots, n + 1\} \to \{1, \ldots, k\}$$

let $m_n$ be the graded linear maps

$$m_n: \text{Ext}^*(E_{\psi(1)}, E_{\psi(2)}) \otimes \text{Ext}^*(E_{\psi(2)}, E_{\psi(3)}) \otimes \cdots$$

$$\cdots \otimes \text{Ext}^*(E_{\psi(n)}, E_{\psi(n+1)}) \to \text{Ext}^{*+2-n}(E_{\psi(1)}, E_{\psi(n+1)})$$

which give a minimal $A_\infty$-category structure on the dg-category generated by $(E_1, \ldots, E_k)$. Since $X$ is a CY 3-fold, we can take the above $A_\infty$-structure (6.1) to be cyclic (see [Pol01]), i.e. for a map $\psi$ as above with $\psi(1) = \psi(n+1)$, and elements

$$a_i \in \text{Ext}^*(E_{\psi(i)}, E_{\psi(i+1)}), \ 1 \leq i \leq n$$

we have the relation

$$(m_{n-1}(a_1, \ldots, a_{n-1}), a_n) = (m_{n-1}(a_2, \ldots, a_n), a_1).$$

Here $(-,-)$ is the Serre duality pairing

$$(-,-): \text{Ext}^*(E_a, E_b) \times \text{Ext}^{3-*}(E_b, E_a) \to \text{Ext}^{3}(E_a, E_a) \overset{\text{Tr}}{\to} \mathbb{C}.$$

Let $W_{E\bullet} \in \mathbb{C}[Q_{E\bullet}]$ be defined by

$$W_{E\bullet} := \sum_{n \geq 3} \sum_{\substack{1, \ldots, n+1 \subseteq \{1, \ldots, k\} \backslash \psi(1) = \psi(n+1) \psi}} \sum_{e_i \in E_{\psi(i)}, \psi(i+1)} a_{\psi, e\bullet} \cdot e_1 e_2 \cdots e_n.$$ 

Here the coefficient $a_{\psi, e\bullet}$ is given by

$$a_{\psi, e\bullet} := \frac{1}{n}(m_{n-1}(e_1^\vee, e_2^\vee, \ldots, e_{n-1}^\vee, e_n^\vee)).$$

In the RHS, for $e \in E_{i,j}$ the element $e^\vee \in \text{Ext}^1(E_i, E_j)$ is determined by $e^\vee(e) = 1$ and $e^\vee(e') = 0$ for any $e' \in E_{i,j}$ with $e \neq e'$. By taking the Dolbeaut model in defining the $A_\infty$-products (6.1), the result of [Tod03, Lemma 4.1] (based on earlier works [Fuk03, Tu]) shows that

$$W_{E\bullet} \in \mathbb{C}\{Q_{E\bullet}\} \subset \mathbb{C}[Q_{E\bullet}].$$

Here $\mathbb{C}\{Q_{E\bullet}\}$ is given in Definition 5.1. Therefore $W_{E\bullet}$ determines a convergent super-potential of $Q_{E\bullet}$.

A collection $E\bullet$ is called a simple collection if we have

$$\text{Ext}^{\leq 0}(E_i, E_j) = \mathbb{C} \cdot \delta_{ij}.$$ 

Here the RHS is concentrated on degree zero. In this case, the algebra $\mathbb{C}[Q_{E\bullet}] / (\partial W_{E\bullet})$ gives a pro-representable hull of NC deformation functor associated with $E\bullet$, developed in [Lau02, Eri10, Kawb, BB]. By taking
the tensor product with the universal object, we have an equivalence of
categories (see [Todb, Corollary 6.7])

\[ \Phi_{E_\bullet} : \text{mod}_{\text{nil}} \mathbb{C}[Q_{E_\bullet}]/(\partial W_{E_\bullet}) \sim \langle E_1, \ldots, E_k \rangle_{\text{ex}}. \]

Here the LHS is the category of nilpotent \( \mathbb{C}[Q_{E_\bullet}]/(\partial W_{E_\bullet}) \)-modules, and \( \langle - \rangle_{\text{ex}} \) in the RHS is the extension closure. The above equivalence sends simple objects \( S_i \) for \( 1 \leq i \leq k \) corresponding to the vertex \( i \in V(Q_{E_\bullet}) \) to the object \( E_i \).

6.2. Analytic neighborhood theorem. We return to the situation in
Section 4. Namely we take stability conditions

\[ \sigma = (Z, A) \in \text{Stab}(X), \quad \sigma^\pm = (Z^\pm, A^\pm) \in \text{Stab}(X) \]

where \( \sigma^\pm \) lie on adjacent chambers whose closures contain \( \sigma \). We also take
a primitive element \( v \in \Gamma_X \) with \( 3Z(v) > 0 \), the good moduli spaces of
semistable objects \( M_\sigma(v), M_{\sigma \pm}(v) \) and the wall-crossing diagram (6.9). As
we mentioned in Subsection 4.2, a closed point \( p \in M_\sigma(v) \) corresponds to a
\( \sigma \)-polystable object \( E \in \mathcal{A} \). The object \( E \) is of the form

\[ E = \bigoplus_{i=1}^k V_i \otimes E_i \]

where \( V_i \) is a finite dimensional vector space, each \( E_i \in \mathcal{A} \) is a \( \sigma \)-stable
object with \( \arg Z(E_i) = \arg Z(E_j) \) and \( E_i \not\cong E_j \) for \( i \neq j \), satisfying

\[ \sum_{i=1}^k \dim V_i \cdot \text{ch}(E_i) = v. \]

Let \( E_\bullet \) be the collection of \( \sigma \)-stable objects in (6.6)

\[ E_\bullet = (E_1, E_2, \ldots, E_k). \]

Then \( E_\bullet \) is a simple collection, i.e. it satisfies the condition (6.3). Let \( Q_{E_\bullet} \)
be the Ext-quiver associated with the collection \( E_\bullet \). We take data \( \xi^\pm \in H^k \)
as in (5.11) for the Ext-quiver \( Q_{E_\bullet} \) by

\[ \xi^\pm := (\xi^\pm_i)_{1 \leq i \leq k}, \quad \xi^\pm_i = Z^\pm(E_i), \quad 1 \leq i \leq k. \]

Then we have the associated \( \xi^\pm \)-stability condition on the abelian category
of \( Q_{E_\bullet} \)-representations. Let \( \bar{m} = (m_i)_{1 \leq i \leq k} \) be the dimension vector of
\( Q_{E_\bullet} \)-representations given by

\[ m_i = \dim V_i, \quad 1 \leq i \leq k \]

where \( V_i \) is given in (6.6). Note that as \( v \in \Gamma_X \) is primitive, the vector \( \bar{m} \in \Gamma_{Q_{E_\bullet}} \) is also primitive by the identity (6.7).

The following analytic neighborhood theorem describes the wall-crossing
diagram in terms of representations of quivers with convergent super-potentials,
analytic locally on the good moduli spaces:
Theorem 6.1. For a primitive element \( v \in \Gamma_X \) and stability conditions \( \sigma, \sigma^\pm \) as in (6.5), let

\[
M_{\sigma^+}(v) \quad \quad \quad M_{\sigma^-}(v)
\]

be the wall-crossing diagram as in (4.9). For a closed point \( p \in M_{\sigma}(v) \) corresponding to a \( \sigma \)-polystable object (6.6), let \( Q = Q_{E_*} \) be the associated Ext-quiver and \( W = W_{E_*} \) the convergent super-potential as in (6.2). Then there exist analytic open neighborhoods

\[
p \in T \subset M_{\sigma}(v), \quad 0 \in V \subset M_Q(\vec{m})
\]

where \( \vec{m} \) is the dimension vector (6.10), such that we have the commutative diagram of isomorphisms

\[
M_{(Q,\partial W)}^{\xi^\pm}(\vec{m})|_V \xrightarrow{\cong} (q_{M}^{\pm})^{-1}(T)
\]

\[
M_{(Q,\partial W)}^{\xi^\pm}(\vec{m}) \xrightarrow{\cong} (q_{M}^{\pm})^{-1}(T)
\]

Here the left vertical arrow is given in (5.21), the right vertical arrow is given in (6.11) pulled back to \( T \). Moreover the top isomorphism preserves d-critical structures, where the d-critical structure on the LHS is given in Lemma 5.3 and that on the RHS is given in Theorem 4.1.

The above result is proved in [Todb, Theorem 7.7] in the case of one dimensional (twisted) semistable sheaves, and the same argument also applies if we assume the existence of good moduli spaces for Bridgeland semistable objects (see Theorem 4.7). In Subsection 6.3, we will just give an outline of the proof. By Corollary 5.4 and Theorem 6.1, we have the following:

Corollary 6.2. In the situation of Theorem 6.1, the diagram (6.11) is a d-critical (generalized) flip, flop, MFS at \( p \in M_{\sigma}(v) \) (which corresponds to a polystable object (6.6)) if the diagram

\[
M_Q^{\xi^+}(\vec{m}) \quad \quad \quad M_Q^{\xi^-}(\vec{m})
\]

for the Ext-quiver \( Q = Q_{E_*} \), dimension vector \( \vec{m} \) in (6.10) and \( \xi^\pm \) as in (6.7) is a (generalized) flip, flop, MFS, respectively.

Using Corollary 6.2, we give the simplest case of d-critical flips and flops in the following example:
Example 6.3. In the situation of Corollary [6.2] suppose that \( p \in M_\sigma(v) \) corresponds to a \( \sigma \)-polystable object \( E \) of the form

\[
E = E_1 \oplus E_2, \ E_1 \not\cong E_2
\]

where \( E_1, E_2 \) are \( \sigma \)-stable with \( \arg Z(E_1) = \arg Z(E_2) \). Note that we have

\[
\chi(E_1, E_2) := \sum_{i \in \mathbb{Z}} (-1)^i \dim \text{Ext}^i(E_1, E_2)
\]

\[
= \dim \text{Ext}^1(E_2, E_1) - \dim \text{Ext}^1(E_1, E_2)
\]

and the LHS is computed by the Riemann-Roch theorem. The Ext-quiver \( Q \) associated with \( E\bullet_1 = (E_1, E_2) \) has two vertices, \( V(Q) = \{1, 2\} \). Let us set

\[
V^+ = \text{Ext}^1(E_1, E_2), \ V^- = \text{Ext}^1(E_2, E_1),
\]

\[
U = \text{Ext}^1(E_1, E_1) \oplus \text{Ext}^1(E_2, E_2).
\]

The stack of \( Q \)-representations of dimension vector \( \vec{m} = (1, 1) \) is given by

\[
\mathcal{M}_Q(\vec{m}) = [(V^+ \times V^-)/(\mathbb{C}^*)^2] \times U.
\]

Here the action of \((t_1, t_2) \in (\mathbb{C}^*)^2 \) on \((\vec{x}, \vec{y}) \in V^+ \times V^- \) is given by

\[
(t_1, t_2) \cdot (\vec{x}, \vec{y}) = (t_1 t_2^{-1} \cdot \vec{x}, t_1 t_2 \cdot \vec{y}).
\]

We take \( \sigma \) in \([6.5]\) so that

\[
\arg Z^+(E_1) > \arg Z^+(E_2), \ \arg Z^-(E_1) < \arg Z^-(E_2)
\]

hold. It is easy to see that for a point \((\vec{x}, \vec{y}, \vec{u}) \in \mathcal{M}_Q(\vec{m}) \), it is \( \xi^+ \)-(semi)stable if and only if \( \vec{x} \neq 0 \), \( \xi^- \)-(semi)stable if and only if \( \vec{y} \neq 0 \). Therefore the moduli spaces of \( \xi^\pm \)-stable \( Q \)-representations are given by

\[
M^\pm_Q(\vec{m}) = \text{Tot}_{\mathbb{P}(V^+)}(\mathcal{O}_{\mathbb{P}(V^+)}(-1) \otimes V^-) \times U,
\]

\[
M^\pm_Q(\vec{m}) = \text{Tot}_{\mathbb{P}(V^-)}(\mathcal{O}_{\mathbb{P}(V^-)}(-1) \otimes V^+) \times U.
\]

It follows that the diagram \([6.13]\) is a standard toric flip (resp. flop) given in Example \([3.8]\) if \( \chi(E_1, E_2) < 0 \) (resp. \( \chi(E_1, E_2) = 0 \)). Therefore by Corollary [6.2] the the diagram \([6.11]\) is an analytic d-critical flip if \( \chi(E_1, E_2) < 0 \) (resp. flop if \( \chi(E_1, E_2) = 0 \)) at the point \( p = [E_1 \oplus E_2] \in M_\sigma(v) \).

6.3. Outline of the proof of Theorem 6.1.

Proof. Let us take a point \( p \in M_\sigma(v) \) which corresponds to a \( \sigma \)-polystable object \( E \) as in \([6.6]\). Note that for the Ext-quiver \( Q = Q_{E\bullet} \) associated with the collection \([6.8]\), we have

\[
\text{Rep}_Q(\vec{m}) = \text{Ext}^1(E, E), \ G := \prod_{i=1}^k \text{GL}(V_i) = \text{Aut}(E)
\]

where \( \vec{m} \) is the dimension vector \([6.10]\). Under the above identifications, the \( G \)-action on \( \text{Rep}_Q(\vec{m}) \) is compatible with the conjugate \( \text{Aut}(E) \)-action on \( \text{Ext}^1(E, E) \).
Let $E_i \to E_i$ be a resolution of $E_i$ by vector bundles, and set $E^* = \bigoplus_{i=1}^k V_i \otimes E_i^*$. We have linear maps of degree $1 - n$

(6.14) \[ I_n: \text{Ext}^*(E,E)^{\otimes n} \to g_{E_i}^*: = A^{0,*}(\text{Hom}^*(E^*,E^*)) \]

giving an $A_\infty$ quasi-isomorphism between the minimal $A_\infty$-algebra $\text{Ext}^*(E,E)$ and the Dolbeaut dg Lie algebra $g_{E_i}^*$. For $x \in \text{Ext}^1(E,E)$, the formal sum \( I^*(x) := \sum_{n \geq 1} I_n(x,\ldots,x) \) (6.15) has a convergent radius (see [Todb, Lemma 4.1]). Let $\pi_Q: \text{Rep}_Q(\vec{m}) \to M_Q(\vec{m})$ be the natural map to the GIT quotient, and take a sufficiently small analytic open subset $0 \in V \subset M_Q(\vec{m})$. The infinite sum (6.15) determines an $\text{Aut}(E)$-equivariant analytic map \( I^*: \pi_Q^{-1}(V) \to \hat{g}_{E_i}^1 \) (6.16)

Here $\hat{g}_{E_i}^1$ is a certain Sobolev completion of $g_{E_i}^1$ (see [Todb, Lemma 5.1]). On the open subset $\pi_Q^{-1}(V) \subset \text{Ext}^1(E,E)$, the Mauer-Cartan locus of the minimal $A_\infty$-algebra $\text{Ext}^*(E,E)$ is given by the critical locus of the analytic function \( \text{tr} W: \pi_Q^{-1}(V) \to \mathbb{C} \) given in (5.18) for the convergent super-potential $W = W_{E_i}$ defined as in (6.2). Then the restriction of the map (6.16) to the Mauer-Cartan locus determines a smooth morphism of analytic stacks of relative dimension zero (see [Todb, Proposition 4.3])

(6.17) \[ I^*: \mathcal{M}_{(Q,\partial W)}(\vec{m})|_V \to \mathcal{M} \]

Here $\mathcal{M}$ is the moduli stack of complexes given in Subsection 4.1.

By shrinking $V$ if necessary, the image of the above morphism lies in the open substack $\mathcal{M}_\sigma(v) \subset \mathcal{M}$. Let $p_M: \mathcal{M}_\sigma(v) \to M_\sigma(v)$ be the morphism to the good moduli space. Then the argument of [Todb, Proposition 5.4] shows that the map (6.17) induces the commutative diagram of isomorphisms

(6.18) \[
\begin{array}{ccc}
\mathcal{M}_{(Q,\partial W)}(\vec{m})|_V & \xrightarrow{I^*} & p_M^{-1}(T) \\
\downarrow p_Q & & \downarrow p_M \\
M_{(Q,\partial W)}(\vec{m})|_V & \cong & T
\end{array}
\]

for some analytic open neighborhood $p \in T \subset M_\sigma(\beta,n)$. Here in loc. cit., the claim is stated for moduli stacks of semistable sheaves and their good moduli spaces. But the argument can be generalized to the case of Bridgeland semistable objects, as the explicit construction of good moduli spaces are not needed in the proof of loc. cit. The properties we used for the good moduli spaces are their existence and the étale slice theorem. For the
Bridgeland semistable objects, the former is given in [AHLH] and the latter for the map \( p_M : M_\sigma(v) \to M_\tau(v) \) is given in [AHR].

The diagram (6.18) in particular implies the isomorphism

\[
I_* : p_Q^{-1}(0) \cong p_M^{-1}(p).
\]

We show that the above isomorphism restricts to the isomorphisms

\[
I_* : p_Q^{-1}(0) \cap M^{\xi^\pm}_{(Q, \partial W)}(\bar{m})|_V \cong p_M^{-1}(p) \cap M_{\sigma^\pm}(v).
\]

Indeed if the above isomorphism holds, then the argument of [Todh, Theorem 6.8] shows that, after shrinking \( V, T \) if necessary, we have the isomorphism

\[
I_* : M^{\xi^\pm}_{(Q, \partial W)}(\bar{m})|_V \cong p_M^{-1}(T) \cap M_{\sigma^\pm}(v).
\]

By taking the associated isomorphism on good moduli spaces, we obtain the desired diagram (9.18). The comparison of d-critical structures follow from the argument of [Toda, Appendix A].

We show the isomorphism (6.20). Note that \( \mathbb{C} \)-valued points of \( p_Q^{-1}(0) \) consist of nilpotent \( Q \)-representations with dimension vector \( \bar{m} \), and those of \( p_M^{-1}(p) \) consist of objects in the extension closure \( \langle E_1, \ldots, E_k \rangle_{\text{ex}} \) in \( \mathcal{A} \) with Chern character \( v \). Then the isomorphism (6.20) follows from Lemma 6.4 below.

\[
\Phi_{E_*} : \text{mod}_{\text{nil}} \mathbb{C}[\![Q]\!]/(\partial W) \xrightarrow{\sim} \langle E_1, \ldots, E_k \rangle_{\text{ex}}.
\]

Under the above equivalence, a nilpotent \( Q \)-representation \( \mathcal{V} \) is \( \xi^\pm \)-semistable if and only if \( \Phi_{E_*}(\mathcal{V}) \) is \( \sigma^\pm \)-semistable in \( \mathcal{A} \).

**Proof.** The compatibility of \( I_* \) with \( \Phi_{E_*} \) is due to [Todh, Theorem 6.8], and the preservation of the stability follows from the argument of [Todh, Lemma 7.8] without any modification.

### 7. Representations of symmetric and extended quivers

By Corollary 6.2 in order to see whether a given wall-crossing diagram in a CY 3-fold is a d-critical flip or flop, it is enough to see whether a wall-crossing diagram in the Ext-quiver is a flip or flop. In this section, we study the latter problem in detail in the case of symmetric quivers and their extended version. The results in this section will be applied to geometric situations in later sections.
7.1. Some general facts on representations of quivers. Here we give some general facts on moduli spaces of representations of quivers. Below, we use the notation in Section 5. Let \( Q \) be a quiver, \( \vec{m} \in \Gamma_Q \) a dimension vector of \( Q \), and \( V_i \) for each \( i \in V(Q) \) is a vector space with dimension \( m_i \). As in (5.9), we have the moduli stack of \( Q \)-representations with dimension vector \( \vec{m} \)

\[
\mathcal{M}_Q(\vec{m}) = \left[ \prod_{e \in E(Q)} \text{Hom}(V_{s(e)}, V_{t(e)})/G \right], \quad G = \prod_{i \in V(Q)} \text{GL}(V_i)
\]

and its good moduli space \( \mathcal{M}_Q(\vec{m}) \to \mathcal{M}_Q(\vec{m}) \). Let \( \xi \in \mathcal{H}^{\mathcal{V}(Q)} \) be a choice of a stability condition as in (5.11), and \( \mathcal{M}_Q^\xi(\vec{m}) \) the good moduli space of \( \xi \)-semistable representations. As in (5.14), we have the natural morphism

\[
q^{\xi}_Q : \mathcal{M}_Q^\xi(\vec{m}) \to \mathcal{M}_Q(\vec{m}).
\]

Let \( M^s_Q(\vec{m}) \subset M_Q(\vec{m}) \) be the simple part. If \( M^s_Q(\vec{m}) \neq \emptyset \), then the morphism (7.2) is always birational.

A criterion of the condition \( M^s_Q(\vec{m}) \neq \emptyset \) is given in [LBP90]. In order to state this, we prepare some terminology. A full subquiver \( Q' \subset Q \) is said to be strongly connected if each couple from its vertex set belongs to an oriented cycle. For \( \vec{m} \in \Gamma_Q \), let \( \text{supp}(\vec{m}) \) be the set of \( i \in V(Q) \) with \( m_i \neq 0 \). Note that \( \text{supp}(\vec{m}) \) is regarded as a full subquiver of \( Q \). We also use the pairing on \( \Gamma_Q \) defined by

\[
\langle \vec{m}, \vec{m}' \rangle := \sum_{i \in V(Q)} m_i \cdot m'_i - \sum_{e \in E(Q)} m_{s(e)} \cdot m'_{t(e)}.
\]

We denote by \( \tilde{A}_n \) the extended Dynkin \( A_n \)-quiver, i.e. \( Q(\tilde{A}_n) = \{1, 2, \ldots, n\} \) with one arrow from \( i \) to \( i + 1 \) for each \( 1 \leq i \leq n - 1 \) and \( n \) to 1. The result of [LBP90] is stated as follows:

**Theorem 7.1.** ([LBP90, Theorem 4]) For \( \vec{m} \in \Gamma_Q \), we have \( M^s_Q(\vec{m}) \neq \emptyset \) if and only if either one of the following conditions hold:

(i) We have \( m_i = 1 \) for all \( i \in \text{supp}(\vec{m}) \) and \( \text{supp}(\vec{m}) \) is a quiver of type \( A_1 \) or \( \tilde{A}_n \) for \( n \geq 1 \).

(ii) The quiver \( \text{supp}(\vec{m}) \) is not of the above type, is strongly connected and

\[
\langle \vec{m}, \vec{i} \rangle \leq 0, \quad (\vec{i}, \vec{m}) \leq 0, \quad \text{for all } i \in V(Q).
\]

**Remark 7.2.** Suppose that one of the conditions (7.4) fails, e.g.

\[
\langle \vec{m}, \vec{i} \rangle = m_i - \sum_{j \in V(Q), e \in E_{j,i}} m_j > 0
\]
for some \( i \in V(Q) \). Then for a \( Q \)-representation \( V \) as in (5.2), the natural map

\[
\sum_{j \in V(Q), e \in E_{j,i}} u_e : \bigoplus_{e \in E_{j,i}} V_j \to V_i
\]

has a non-trivial cokernel. Therefore we have a surjection \( V \twoheadrightarrow S_i \) and \( M^\xi_s(\vec{m}) = \emptyset \) holds. Similarly if \( \langle \vec{i}, \vec{m} \rangle > 0 \), then there is an injection \( S_i \hookrightarrow V \).

These facts will be used later.

Next, we consider canonical line bundles on moduli spaces of quiver representations. Suppose that \( \vec{m} \) is primitive in \( \Gamma_Q \), and let \( M^\xi_s(\vec{m}) \subset M^\xi_s(\vec{m}) \) be the \( \xi \)-stable part. Then by [Kin94, Proposition 5.3], there exist universal \( Q \)-representations

\[
V_i \to M^\xi_s(\vec{m}), \quad u_e : V_{s(e)} \to V_{t(e)}, \quad i \in V(Q), \quad e \in E(Q).
\]

Here \( V_i \) is a vector bundle on \( M^\xi_s(\vec{m}) \) whose fiber is \( V_i \), \( u_e \) is a map of vector bundles, and for a point \( p \in M^\xi_s(\vec{m}) \) corresponding to a \( Q \)-representation \( (5.2) \) we have \( (u_e)|_p = u_e \). In this case, we have the following lemma on the canonical line bundle of the smooth variety \( M^\xi_s(\vec{m}) \):

**Lemma 7.3.** In the above situation, we have

\[
\omega_{M^\xi_s(\vec{m})} = \bigotimes_{e \in E(Q)} \det V_{s(e)} \otimes \det V_{t(e)}^{-1}.
\]

**Proof.** When \( \vec{m} \) is primitive, the \( \xi \)-stable part of the stack \( M_Q(\vec{m}) \) is a trivial \( \mathbb{C}^* \)-gerbe over \( M^\xi_s(\vec{m}) \). By the description of the stack \( M_Q(\vec{m}) \) in (7.1), the canonical line bundle of the stack \( M_Q(\vec{m}) \) is induced by the one dimensional \( G \)-representation \( G \to \mathbb{C}^* \) given by

\[
g = (g_i)_{i \in V(Q)} \mapsto \prod_{e \in E(Q)} \det g_{s(e)} \cdot (\det g_{t(e)})^{-1}.
\]

Therefore the identity (7.5) holds. \( \square \)

7.2. **Flops via representations of symmetric quivers.** Here we investigate the morphism \( d^\xi_Q \) in (7.2) for a symmetric quiver \( Q \), defined below:

**Definition 7.4.** A quiver \( Q \) is called **symmetric** if \( \sharp E_{i,j} = \sharp E_{j,i} \) for any \( i, j \in V(Q) \). Here \( E_{i,j} \) is defined as in (5.1).

**Remark 7.5.** A symmetric condition of a quiver \( Q \) is equivalent to that the pairing (7.3) is symmetric.

Below for a symmetric quiver \( Q \), we fix identifications \( E_{i,j} = E_{j,i} \), so that \( Q \) and \( Q' \) are identified. In particular for a \( Q \)-representation \( V \), its dual representation \( V^\vee \) given in (5.3) is also a \( Q \)-representation. We have the following lemma on the non-emptiness of the moduli spaces of stable \( Q \)-representations:
Lemma 7.6. For a symmetric quiver $Q$ and $\vec{m} \in \Gamma_Q$, we have $M^\xi_s(\vec{m}) \neq \emptyset$ for some $\xi \in \mathcal{H}^{\Gamma}(Q)$ if and only if $M^\xi_s(\vec{m}) \neq \emptyset$ for any $\xi \in \mathcal{H}^{\Gamma}(Q)$.

Proof. It is enough to show that if $M^\xi_s(\vec{m}) \neq \emptyset$ for some $\xi$, then $M^\xi_s(\vec{m}) \neq \emptyset$. Suppose that $M^\xi_s(\vec{m}) \neq \emptyset$. We apply the criterion in Theorem 7.1 to show that $M^\xi_s(\vec{m}) \neq \emptyset$. Since $\text{supp}(\vec{m})$ is symmetric, it is of type $A_1$ or $\tilde{A}_1$, only if $n = 1$ or $n = 2$. In $A_1$ and $\tilde{A}_1$ case, we have $M^\xi_s(\vec{m}) = M^\xi_s(\vec{m})$ for any dimension vector $\vec{m}$, and the statement is obvious.

Suppose that $\text{supp}(\vec{m})$ is $\tilde{A}_2$, and write its vertices as $\{1, 2\}$. We may assume that $\arg Z_\xi(S_1) > \arg Z_\xi(S_2)$, where $Z_\xi$ is given by (5.12). Let us write a $Q$-representation corresponding to a point in $M^\xi_s(\vec{m})$ as

$$V = \left( V_1 \xrightarrow{e_{12}} V_2 \right)$$

where $V_i$ are vector spaces with dimension $m_i$ and $e_{12}, e_{21}$ are linear maps. The $\xi$-stability implies that $\text{Hom}(S_1, V) = 0$ and $\text{Hom}(V, S_2) = 0$. These conditions imply that $e_{12}$ is an isomorphism, so we can assume $V_1 = V_2 = V$ and $e_{12} = \text{id}$. Let $v \in V$ be an eigen vector of $e_{21}$ with eigen value $\lambda$. Then we have an injection of $Q$-representations

$$\left( \mathbb{C}v \xrightarrow{\text{id}} \mathbb{C}v \right) \hookrightarrow V.$$

As $V$ is $\xi$-stable, the above injection must be an isomorphism. Therefore we have $m_1 = m_2 = 1$ and $M^\xi_s(\vec{m}) \neq \emptyset$ follows from Theorem 7.1.

Suppose that $\text{supp}(\vec{m})$ is not of the above types. If $\text{supp}(\vec{m})$ is not strongly connected, then as it is symmetric it must be disconnected. Then $M^\xi_s(\vec{m}) = \emptyset$ for any $\xi$, which is a contradiction. Therefore $\text{supp}(\vec{m})$ is strongly connected. For $i \in V(Q)$, note that we have $\langle \vec{m}, \vec{i} \rangle = \langle \vec{i}, \vec{m} \rangle$ as $Q$ is symmetric (see Remark 7.5). If $\langle \vec{m}, \vec{i} \rangle = \langle \vec{i}, \vec{m} \rangle > 0$, then by Remark 7.2 for any $Q$-representation $V$ with dimension vector $\vec{m}$, there exist an injection $S_i \hookrightarrow V$ and a surjection $V \twoheadrightarrow S_i$. Such a representation $V$ can never be $\xi$-stable for any choice of $\xi$, which is a contradiction. Therefore the criterion of Theorem 7.1 is applied and $M^\xi_s(\vec{m}) \neq \emptyset$ holds. □

For a symmetric quiver $Q$ and $\vec{m} \in \Gamma_Q$, let us take $\xi^\pm = (\xi^\pm_i)_{i \in V(Q)} \in \mathcal{H}^{\Gamma}(Q)$ satisfying the following:

\begin{equation}
\Re(\xi^+_i) = -\Re(\xi^-_i) \in \mathbb{Z}, \quad \sum_{i \in V(Q)} m_i \cdot \Re(\xi^\pm_i) = 0.
\end{equation}
We have the following diagram

\[
\begin{array}{ccc}
M_\xi^+ (\vec{m}) & \xrightarrow{q_\xi^+} & M_\xi^- (\vec{m}) \\
\downarrow & & \downarrow \\
M_\xi (\vec{m}) & \xrightarrow{q_\xi^-} & \end{array}
\]

(7.7)

By Lemma 7.6 we have \( M_\xi^{\pm, s}(\vec{m}) \neq \emptyset \) if and only if \( M_\xi^{\pm, s}(\vec{m}) \neq \emptyset \).

**Proposition 7.7.** Suppose that \( \vec{m} \) is primitive and \( M_\xi^{\pm, s}(\vec{m}) = M_\xi^{\pm, s}(\vec{m}) \neq \emptyset \) hold. Then the diagram (7.7) is a generalized flop of smooth varieties \( M_\xi^{\pm, s}(\vec{m}) \).

**Proof.** Under the assumption, the morphisms \( q_\xi^{\pm} \) are projective birational morphisms from smooth varieties \( M_\xi^{\pm, s}(\vec{m}) \). Moreover the canonical divisors of \( M_\xi^{\pm, s}(\vec{m}) \) are trivial by Lemma 7.3 and the symmetric condition of \( Q \). We show that \( q_\xi^{\pm} \) are isomorphisms in codimension one. By [Toda Lemma 4.4], the maps \( q_\xi^{\pm} \) are semismall, i.e. there is a stratification \( \{ S_\lambda \}_\lambda \) of \( M_\xi (\vec{m}) \) such that for any \( x \in S_\lambda \) we have the inequality

\[
\dim(q_\xi^{\pm})^{-1}(x) \leq \frac{1}{2} \text{codim } S_\lambda.
\]

(7.8)

Moreover from the proof of loc.cit. and [MR Theorem 1.4] (which is used in loc.cit.), under the assumption \( M_\xi^{s}(\vec{m}) \neq \emptyset \), the equality holds in (7.8) only for the dense strata \( S_\lambda = M_\xi(\vec{m}) \), i.e. \( q_\xi^{\pm} \) are small maps. In particular, \( q_\xi^{\pm} \) are isomorphisms in codimension one.

It remains to show that there exists a \( q_\xi^{\pm} \)-ample divisor on \( M_\xi^{\pm, s}(\vec{m}) \) whose strict transform to \( M_\xi^{\pm, s}(\vec{m}) \) is \( q_\xi^{\pm} \)-anti ample. Let us consider the following characters of \( G \):

\[
g = (g_i)_{i \in V(\Omega)} \mapsto (\det g_i)^{\Re(\xi^\pm)}.
\]

(7.9)

They define \( G \)-equivariant line bundles on \( \text{Rep}_Q(\vec{m}) \), hence on the stack \( M_\xi(\vec{m}) \), which we write as \( L_\pm \). Note that by the condition (7.3), the characters (7.9) are trivial on the diagonal torus \( \mathbb{C}^* \subset G \). Therefore the restrictions of \( L_\pm \) to \( M_\xi^{\pm, s}(\vec{m}) \), \( M_\xi^{\pm, s}(\vec{m}) \) descend to line bundles \( (L_\pm)^+, (L_\pm)^- \) on \( M_\xi^{\pm, s}(\vec{m}) \), \( M_\xi^{\pm, s}(\vec{m}) \) respectively. By the GIT construction of \( M_\xi^{\pm, s}(\vec{m}) \) (see [Kin94]), the line bundle \( (L_\pm)^+ \) is \( q_\xi^{\pm} \)-ample, and \( (L_-)^- \) is \( q_\xi^- \)-ample. By the construction of the above line bundles, the strict transform of \( (L_\pm)^+ \) on \( M_\xi^{\pm, s}(\vec{m}) \) to \( M_\xi^{\pm, s}(\vec{m}) \) is \( (L_\pm)^- = ((L_-)^-) \), which is \( q_\xi^- \)-anti-ample. Therefore the diagram (7.7) is a generalized flop. \( \square \)
For a convergent super-potential $W$ of $Q$, let us take an analytic open neighborhood $0 \in V \subset M_Q(\vec{m})$ as in (5.17). For two data $\xi^\pm$ as above, as in (5.23) we have the diagram

\[
\begin{array}{c}
M_{(Q,\partial W)}(\vec{m})|_V \xrightarrow{\eta_{(Q,\partial W)}^+} M_{(Q,\partial W)}^+(\vec{m})|_V \\
\downarrow q_{(Q,\partial W)}^+ \\
M_{(Q,\partial W)}(\vec{m})|_V \xrightarrow{\eta_{(Q,\partial W)}^-} M_{(Q,\partial W)}^-(\vec{m})|_V \\
\downarrow q_{(Q,\partial W)}^-
\end{array}
\]

By Lemma 5.3 and Proposition 7.7, we obtain the following corollary:

**Corollary 7.8.** Let $(Q,W)$ be a symmetric quiver $Q$ with a convergent super-potential $W$. Suppose that $\vec{m} \in \Gamma_Q$ is primitive and take $\xi^\pm$ as above, as in (7.6). If $M_{(Q,\partial W)}(\vec{m})|_V \neq \emptyset$ holds, then the diagram (7.10) is an analytic d-critical generalized flop.

### 7.3. Flips via representations of extended quivers.

Let $Q$ be a symmetric quiver, and choose non-negative integers $a_i, b_i$ for each $i \in V(Q)$, and another non-negative integer $c$. We construct an extended quiver $Q^*$ in the following way: the set of vertices is

$$V(Q^*) = \{0\} \cup V(Q).$$

For $i, j \in V(Q) \subset V(Q^*)$, the set of edges from $i$ to $j$ in $Q^*$ is the same as that in $Q$. The numbers of other edges are given by

$$\sharp E_{0,i} = a_i, \quad \sharp E_{i,0} = b_i, \quad i \in V(Q), \quad \sharp E_{0,0} = c.$$  

For example, see Figure 1. Note that $Q^*$ contains $Q$ as a subquiver. In particular, any $Q$-representation is regarded as a $Q^*$-representation in a natural way.

For a dimension vector $\vec{m} \in \Gamma_Q$ of $Q$, we define the extended dimension vector $\vec{m}^* \in \Gamma_{Q^*}$ by

$$\vec{m}^* := \vec{0} + \vec{m}$$

i.e. $(\vec{m}^*)_0 = 1$ and $(\vec{m}^*)_i = m_i$ for $i \in V(Q)$. The following lemma is obvious from the construction of $Q^*$:
Lemma 7.9. Giving a $Q^*$-representation $\mathbb{V}^*$ with dimension vector $\vec{m}^*$ is equivalent to giving a $Q$-representation

$$\mathbb{V} = \{ (V_i, u_e) \}_{i \in V(Q), e \in E(Q)}$$

of dimension vector $\vec{m}$, together with linear maps

$$E_{0,i} \rightarrow V_i, \ E_{i,0} \otimes V_i \rightarrow \mathbb{C}$$

for each $i \in V(Q)$. Here $E_{i,j}$ is the $\mathbb{C}$-vector space defined as in (5.1).

Let us take data $\xi^\pm = (\xi_i^\pm) \in \mathcal{H}^V(Q^*)$ for $Q^*$ satisfying

$$\xi_i^\pm = \sqrt{-1}, \ i \in V(Q), \ \Re(\xi_i^+) < 0, \ \Re(\xi_i^-) > 0.$$

The following lemma characterizes $\xi^\pm$-semistable $Q^*$-representations:

Lemma 7.10. Let $\mathbb{V}^*$ be a $Q^*$-representation of dimension vector $\vec{m}^*$, given by a $Q$-representation $\mathbb{V}$ as in (7.11) together with linear maps (7.12).

(i) The object $\mathbb{V}^*$ is $\xi^+$-semistable if and only if it is $\xi^+$-stable if and only if the images of linear maps $E_{0,i} \rightarrow V_i$ in (7.12) generate $\bigoplus_{i \in V(Q)} V_i$ as a $\mathbb{C}[Q]$-module.

(ii) The object $\mathbb{V}^*$ is $\xi^-$-semistable if and only if it is $\xi^-$-stable if and only if the images of linear maps $E_{i,0} \rightarrow V_i^\vee$ induced by right maps in (7.12) generate $\bigoplus_{i \in V(Q)} V_i^\vee$ as a $\mathbb{C}[Q]$-module. Here the $\mathbb{C}[Q]$-module structure on $\bigoplus_{i \in V(Q)} V_i^\vee$ is given by the dual $Q^\vee (= Q)$ representation $\mathbb{V}^\vee$ of $\mathbb{V}$.

Proof. (i) By a choice of $\xi^+$, a $Q^*$-representation $\mathbb{V}^*$ of dimension vector $\vec{m}^*$ is $\xi^+$-(semi)stable if and only if there is no surjection $\mathbb{V}^* \twoheadrightarrow \mathbb{V}'$ as $Q^*$-representations where $\mathbb{V}'$ is a non-zero $Q$-representation. The last condition is equivalent to that the images of linear maps $E_{0,i} \rightarrow V_i$ generate $\bigoplus_{i \in V(Q)} V_i$ as a $\mathbb{C}[Q]$-module.

(ii) By a choice of $\xi^-$, a $Q^*$-representation $\mathbb{V}^*$ of dimension vector $\vec{m}^*$ is $\xi^-$-(semi)stable if and only if there is no injection $\mathbb{V}' \hookrightarrow \mathbb{V}^*$ as $Q^*$-representations where $\mathbb{V}'$ is a non-zero $Q$-representation. This is equivalent to that the dual representation $(\mathbb{V}^*)^\vee$ does not admit a surjection to $\mathbb{V}''$ in the category of $(Q^*)^\vee$-representations, where $\mathbb{V}''$ is a non-zero $Q^\vee(= Q)$-representation. Therefore similarly to (i), we conclude (ii).

By Lemma 7.10 we have $M_{Q^*}(\vec{m}^*) = M_{Q^*}(\vec{m}^*)$, so they are smooth quasi-projective varieties. Let us consider the following diagram

$$M_{Q^*}^{\xi^+}(\vec{m}^*) \xrightarrow{\xi^+_{Q^*}} M_{Q^*}(\vec{m}^*) \xleftarrow{\xi^-_{Q^*}} M_{Q^*}^{\xi^-}(\vec{m}^*)$$

Lemma 7.11. Suppose that $a_i > b_i$ for all $i \in V(Q)$. Then in the diagram (7.14), we have the following:
(i) The anti-canonical divisor of \( M_{\xi}^{\pm}(\vec{m}^*) \) is ample.

(ii) The canonical divisor of \( M_{\xi}^{\pm}(\vec{m}^*) \) is ample.

Proof. For \( i \in V(Q^*) \) and \( e \in E(Q^*) \), let

\[
\mathcal{V}_i^\pm \to M_{\xi}^{\pm}(\vec{m}^*), \quad \mathbf{u}_e : \mathcal{V}_{s(e)}^\pm \to \mathcal{V}_{t(e)}^\pm
\]

be a universal \( Q^* \)-representations. Note that such a universal representations exists as \( \vec{m}^* \) is primitive and \( M_{\xi}^{\pm}, s \cdot Q^* \cdot (\vec{m}^*) = M_{\xi}^{\pm}(\vec{m}^*) \) hold. For \( i = 0 \), the vector bundles \( \mathcal{V}_0^\pm \) are line bundles by our choice of \( \vec{m}^* \), so by replacing \( \mathcal{V}_i^\pm \) by \( \mathcal{V}_i^\pm \otimes (\mathcal{V}_0^\pm)^{-1} \) we may assume that \( \mathcal{V}_0^\pm \) are trivial line bundles. Then by (7.15), we have

\[
\omega_{M_{\xi}^{\pm}(\vec{m}^*)} = \bigotimes_{i \in V(Q)} \det(\mathcal{V}_i^\pm)^{h_i-a_i}.
\]

(7.15)

Let \( E \subset C[Q] \) be the vector subspace generated by paths of the form \( e_1 e_2 \cdots e_n \) for \( n \geq 1 \) such that \( s(e_1) = 0, t(e_1) \in V(Q) \) and \( e_2, \ldots, e_n \in E(Q) \). Then the compositions \( \mathbf{u}_{e_n} \circ \cdots \circ \mathbf{u}_{e_1} \) determine the morphism of sheaves

\[
E \otimes \mathcal{O}_{M_{\xi}^{\pm}(\vec{m}^*)} \to \bigoplus_{i \in V(Q)} \mathcal{V}_i^+.
\]

Then Lemma 7.10 (i) implies that the above morphism is surjective. Therefore each \( \mathcal{V}_i^+ \) is generated by its global sections. By (7.15) and the assumption \( a_i > b_i \) for all \( i \in V(Q) \), the line bundle \( (\omega_{M_{\xi}^{\pm}(\vec{m}^*)})^{-1} \) is generated by its global sections. In order to show that it is ample, it is enough to show that it has positive degree on any projective curve on \( M_{\xi}^{\pm}(\vec{m}^*) \). Let \( C \) be a smooth projective curve and take a non-constant map

\[
h : C \to M_{\xi}^{\pm}(\vec{m}^*).
\]

Note that each degree of \( h^* \mathcal{V}_i^+ \) is non-negative, as it is globally generated. Therefore if the degree of \( h^* (\omega_{M_{\xi}^{\pm}(\vec{m}^*)})^{-1} \) is non-positive, then each degree of \( h^* \mathcal{V}_i^+ \) must be zero. By Sublemma 7.12 below, in this case the vector bundle \( h^* \mathcal{V}_i^+ \) must be a direct sum of \( \mathcal{O}_C \). Then the pull-back of the universal map \( \mathbf{u}_e \) to \( C \) by the map \( h \) has to be constant. But this implies that the map \( h \) is constant, which is a contradiction. Therefore (i) of the lemma holds. The result of (ii) follows from Lemma 7.10 (ii) and the dual argument of (i). \( \square \)

We used the following sublemma:

Sublemma 7.12. Let \( C \) be a smooth projective curve and \( \mathcal{V} \) a vector bundle on it. Suppose that \( \mathcal{V} \) is generated by its global sections, and \( \deg \mathcal{V} = 0 \). Then \( \mathcal{V} \) is isomorphic to a direct sum of \( \mathcal{O}_C \).
Proof. We set $W = H^0(C, V)$, and $r = \text{rank } V$. The natural surjection $W \otimes \mathcal{O}_C \to V$ induces a morphism
$$\gamma : C \to \text{Gr}(W, r) \hookrightarrow \mathbb{P}^N.$$ Here $N = \dim \mathcal{O}_C^\ast W$, and the right arrow is the Plücker embedding. Since $\text{deg } V = \text{deg } \gamma^* \mathcal{O}(1)$, the assumption $\text{deg } V = 0$ implies that $\gamma$ is a constant map. Also the vector bundle $V$ is a pull-back of the universal quotient bundle on $\text{Gr}(W, r)$. Therefore $V$ is isomorphic to a direct sum of $\mathcal{O}_C$. \hfill $\square$

Using the above lemma, we show the following proposition:

**Proposition 7.13.** Suppose that $a_i > b_i$ for all $i \in V(Q)$. Then in the diagram (7.14), either one of the followings hold:

(i) We have $M^\varepsilon_Q(\vec{m}^*) = \emptyset$ and the diagram (7.14) is a generalized MFS.

(ii) We have $M^\varepsilon_Q(\vec{m}^*) \neq \emptyset$ and the diagram (7.14) is a generalized flip.

**Proof.** If $M^\varepsilon_Q(\vec{m}^*) \neq \emptyset$, then the maps in (7.14) are birational so (ii) holds by Lemma 7.11. Below we show that $M^\varepsilon_Q(\vec{m}^*) = \emptyset$ implies that $M^\varepsilon_Q(\vec{m}^*) = \emptyset$. Note that for $i \in V(Q)$ we have
$$\langle \vec{m}^*, \vec{i} \rangle = \langle \vec{m}, \vec{i} \rangle - a_i, \quad \langle \vec{i}, \vec{m}^* \rangle = \langle \vec{i}, \vec{m} \rangle - b_i.$$ We also have the identities
$$\langle \vec{m}^*, 0 \rangle = 1 - \sum_{i \in V(Q)} b_i m_i, \quad \langle 0, \vec{m}^* \rangle = 1 - \sum_{i \in V(Q)} a_i m_i.$$ By our assumption $a_i > b_i$, we have the inequalities
$$\langle \vec{m}^*, \vec{i} \rangle < \langle \vec{i}, \vec{m}^* \rangle, \quad \langle 0, \vec{m}^* \rangle < \langle \vec{m}^*, 0 \rangle.$$ If $\langle \vec{i}, \vec{m}^* \rangle > 0$ for $i \in V(Q)$, then by Remark 7.2 any $Q^*\text{-representation } V^*$ of dimension vector $\vec{m}^*$ admits an injection $S_i \hookrightarrow V^*$. Therefore we have $M^\varepsilon_Q(\vec{m}^*) = \emptyset$. Similarly if $\langle \vec{m}^*, 0 \rangle > 0$, then any such $V^*$ admits a surjection $V^* \twoheadrightarrow S_0$, which implies $M^\varepsilon_Q(\vec{m}^*) = \emptyset$. Therefore by the inequalities (7.16), we may assume that $\langle \vec{m}^*, \vec{j} \rangle \leq 0$ and $\langle \vec{j}, \vec{m}^* \rangle \leq 0$ for any $j \in V(Q^*)$.

By Theorem 7.1 the condition $M^\varepsilon_Q(\vec{m}^*) = \emptyset$ implies that $\text{supp}(\vec{m}^*)$ is not strongly connected. Let $Q_1, \cdots, Q_l$ be the connected components of $\text{supp}(\vec{m})$, which are subquivers of $Q$. As $\text{supp}(\vec{m}^*)$ is not strongly connected and $Q$ is symmetric, there is $1 \leq k \leq l$ such that we have $b_i = 0$ for any $i \in V(Q_k)$. This implies that any $Q^*\text{-representation } V^*$ of dimension vector $\vec{m}^*$ admits an injection $V^' \hookrightarrow V^*$ for a non-zero $Q_k\text{-representation } V^'$. Therefore we have $M^\varepsilon_Q(\vec{m}^*) = \emptyset$ and (i) holds. \hfill $\square$

**Example 7.14.** Let $Q$ be the symmetric quiver with one vertex and no loops, and write $V(Q) = \{1\}$. Then $Q^*$ has two vertices $V(Q^*) = \{0, 1\}$, with $a_1\text{-arrows from } 0 \text{ to } 1, b_1\text{-arrows from } 1 \text{ to } 0, \text{ and } c\text{-loops at } 0$. The dimension vector $\vec{m}^*$ is written as $\vec{0} + m \cdot \vec{1}$ for $m \in \mathbb{Z}_{>0}$. Let $V$ be a vector
space with dimension $m$. Then by Lemma 7.10, the stack $M_{Q^*}(\vec{m}^*)$ is written as

$$M_{Q^*}(\vec{m}^*) = [(\text{Hom}(E_{0,1}, V) \times \text{Hom}(E_{1,0}, V^*)) / (\mathbb{C}^* \times \text{GL}(V))] \times \mathbb{E}_{0,0}^\vee.$$  

By Lemma 7.10 we see that

$$M_{Q^*}^{\xi^+}(\vec{m}^*) = \text{Tot}_{\text{Gr}(E_{0,1}, m)}(Q_0^\vee \otimes \mathbb{E}_{1,0}^\vee) \times \mathbb{E}_{0,0}^\vee,$$

$$M_{Q^*}^{\xi^-}(\vec{m}^*) = \text{Tot}_{\text{Gr}(E_{1,0}, m)}(Q_1^\vee \otimes \mathbb{E}_{0,1}^\vee) \times \mathbb{E}_{0,0}^\vee.$$  

Here $Q_{i,j}$ is the universal quotient bundle on $\text{Gr}(E_{i,j}, m)$. In this case, the birational map $M_{Q^*}^{\xi^+}(\vec{m}^*) \to M_{Q^*}^{\xi^-}(\vec{m}^*)$ is a Grassmannian flip.

Let $W^*$ be a convergent super-potential of $Q^*$, and take an analytic neighborhood $0 \in V \subset M_{Q^*}(\vec{m}^*)$ as in (7.17). We have the diagram

$$M_{Q^*,\partial W^*}(\vec{m}^*)|_V \quad \xrightarrow{q_{Q^*,\partial W^*}} \quad M_{Q^*,\partial W^*}(\vec{m}^*)|_V.$$  

By Lemma 5.3 and Proposition 7.13, we have the following corollary:

**Corollary 7.15.** For the diagram (7.17), either one of the followings holds:

(i) We have $M_{Q^*}^{\xi^-}(\vec{m}^*) = M_{Q^*,\partial W^*}(\vec{m}^*)|_V = \emptyset$ and the diagram (7.17) is a d-critical generalized MFS.

(ii) The diagram (7.17) is a d-critical generalized flip.

We also have the strictness (see Definition 3.14) of the diagram (7.17) under some conditions:

**Lemma 7.16.** Suppose that $W^*$ is minimal (see (5.16)) and $a_i > m_i$ for any $i \in V(Q)$. Then the diagram (7.17) is strict at $0 \in M_{Q^*,\partial W^*}(\vec{m}^*)|_V$.

**Proof.** We need to show that the map $q_{Q^*,\partial W^*}^{\xi^+}$ in the diagram (7.17) is not a finite morphism at $0 \in M_{Q^*,\partial W^*}(\vec{m}^*)|_V$. We consider nilpotent $Q^*$-representations $V^*$ given by $Q$-representations (7.11) where $u_e = 0$ for all $e \in E(Q)$, together with surjective linear maps $E_{0,i} \to V_i$, and zero maps $E_{1,0} \otimes V_i \to 0$ in (7.12). The isomorphism classes of such $Q^*$-representations form the product of Grassmannians $\text{Gr}(E_{0,i}, m_i)$ for all $i \in V(Q)$. By Lemma 7.10, such $Q^*$-representations are $\xi^+$-stable. They also satisfy the relation $\partial W^*$ (see Remark 5.2) by the minimality of $W^*$, so we have

$$\prod_{i \in V(Q)} \text{Gr}(E_{0,i}, m_i) \subset \left(q_{Q^*,\partial W^*}^{\xi^+}\right)^{-1}(0).$$

Since the LHS is not zero dimensional by the assumption $a_i > m_i$, the lemma holds. \qed
8. D-critical flops of moduli spaces of one dimensional sheaves

In this section, we show that wall-crossing phenomena of one dimensional stable sheaves on CY 3-folds are described in terms of d-critical (generalized) flops. The proof of this result is related to the proof of the wall-crossing formula of Gopakumar-Vafa invariants given in [Toda].

8.1. Twisted semistable sheaves. For a smooth projective CY 3-fold $X$, let $\text{Coh}_{\leq 1}(X) \subset \text{Coh}(X)$ be the abelian subcategory of coherent sheaves $E$ on $X$ whose supports have dimensions less than or equal to one, and $D^b_{\leq 1}(X) := D^b(\text{Coh}_{\leq 1}(X)) \subset D^b(X)$ its bounded derived category. Let $\Gamma_{\leq 1}$ be defined by

$$\Gamma_{\leq 1} := H^2(X, \mathbb{Z}) \oplus \mathbb{Z}. \quad (8.1)$$

The Chen character of an object in $D^b_{\leq 1}(X)$ takes its value in $\Gamma_{\leq 1}$, and given by

$$\text{ch}(E) = (\text{ch}_2(E), \text{ch}_3(E)) = ([E], \chi(E)). \quad (8.2)$$

Here $[E]$ is the fundamental one cycle associated with $E$.

We denote by $\text{Stab}_{\leq 1}(X)$ the space of Bridgeland stability conditions on $D^b_{\leq 1}(X)$ with respect to the Chern character map $(8.2)$. Namely a point $\sigma \in \text{Stab}_{\leq 1}(X)$ is a pair

$$\sigma = (Z, A), \ A \subset D^b_{\leq 1}(X), \ Z : \Gamma_{\leq 1} \to \mathbb{C}$$

where $A$ is the heart of a bounded t-structure on $D^b_{\leq 1}(X)$ and $Z$ is a group homomorphism, satisfying some conditions (see Appendix A). By Theorem A.4 the forgetting map $(Z, A) \mapsto Z$ gives a local homeomorphism

$$\text{Stab}_{\leq 1}(X) \to (\Gamma_{\leq 1})^\vee.$$

Let $A(X)_\mathbb{C}$ be the complexified ample cone of $X$ defined by

$$A(X)_\mathbb{C} := \{ B + i \omega \in H^2(X, \mathbb{C}) : \omega \text{ is ample } \}.$$  

For a given element $B + i \omega \in A(X)_\mathbb{C}$, let $Z_{B, \omega}$ be the group homomorphism defined by

$$Z_{B, \omega} : \Gamma_{\leq 1} \to \mathbb{C}, \ (\beta, n) \mapsto -n + (B + i \omega) \beta. \quad (8.3)$$

Then the pair

$$\sigma_{B, \omega} := (Z_{B, \omega}, \text{Coh}_{\leq 1}(X)) \quad (8.4)$$

determines a point in $\text{Stab}_{\leq 1}(X)$. The map

$$A(X)_\mathbb{C} \to \text{Stab}_{\leq 1}(X), \ (B, \omega) \mapsto \sigma_{B, \omega}$$

is a continuous injective map, whose image is denoted by

$$U(X) \subset \text{Stab}_{\leq 1}(X).$$
Remark 8.1. For an object $E \in \text{Coh}_{\leq 1}(X)$, it is $\sigma_{B,\omega}$-(semi)stable if and only if for any subsheaf $0 \neq F \subseteq E$, we have the inequality
\[ \mu_{B,\omega}(F) < (\leq) \mu_{B,\omega}(E). \]
Here $\mu_{B,\omega}(E) \in \mathbb{R} \cup \{\infty\}$ is defined by
\[ \mu_{B,\omega}(E) := \frac{\chi(E) - B \cdot [E]}{\omega \cdot [E]} = \frac{\Re Z_{B,\omega}(E)}{\Im Z_{B,\omega}(E)}, \tag{8.5} \]
when $\omega \cdot [E] \neq 0$, and $\mu_{B,\omega}(E) = \infty$ when $\omega \cdot [E] = 0$.

8.2. Moduli spaces of one dimensional stable sheaves. Let us take $\beta \in H^2(X, \mathbb{Z})$, $\sigma = (Z, \text{Coh}_{\leq 1}(X)) \in U(X)$ where $\beta$ is an effective curve class. We denote by $M_{\sigma}(\beta)$ the moduli stack of $\sigma$-semistable $E \in \text{Coh}_{\leq 1}(X)$ satisfying $\text{ch}(E) = (\beta, 1)$. The stack $M_{\sigma}(\beta)$ is an Artin stack locally of finite type, with a good moduli space $p_M : M_{\sigma}(\beta) \to M_{\sigma}(\beta)$ for a projective scheme $M_{\sigma}(\beta)$ (see [Todb, Lemma 7.4]). A closed point of $M_{\sigma}(\beta)$ corresponds to a $\sigma$-polystable sheaf, i.e. a direct sum
\[ E = \bigoplus_{i=1}^k V_i \otimes E_i \tag{8.6} \]
where each $V_i$ is a finite dimensional vector space, $E_i \in \text{Coh}_{\leq 1}(X)$ is a $\sigma$-stable sheaf with $\text{arg} Z(E_i) = \text{arg} Z(E)$ and $E_i \not\sim E_j$ for $i \neq j$.

Remark 8.2. The stack $M_{\sigma}(\beta)$ is a GIT quotient stack (see [Todb, Lemma 7.4]), so the good moduli space $M_{\sigma}(\beta)$ exists without relying on [AHLH] (see Remark 4.8).

As we mentioned in Subsection 4.3 there is a wall-chamber structure on $\text{Stab}_{\leq 1}(X)$. On the subset $U(X) \subset \text{Stab}_{\leq 1}(X)$, each wall is given by
\[ \{(Z, \text{Coh}_{\leq 1}(X)) \in U(X) : Z(v_1) \in \mathbb{R}_{>0}Z(v_2)\} \]
for each decomposition
\[ (\beta, 1) = v_1 + v_2, \quad v_i = (\beta_i, n_i) \in \Gamma_{\leq 1}. \]
Here $\beta_i$ is an effective curve class. Suppose that $\sigma \in U(X)$ lies in one of the above walls, and write $\sigma = \sigma_{B,\omega}$ as in (8.2) for $B + i\omega \in A(X)_\mathbb{C}$. Let us take another stability conditions $\sigma^\pm \in U(X)$ written as
\[ \sigma^\pm = \sigma_{B \pm \varepsilon B, \omega \pm i\varepsilon_\omega} \in U(X) \tag{8.7} \]
for $\varepsilon B + i\varepsilon_\omega \in H^2(X, \mathbb{C})$. We assume that $\varepsilon B + i\varepsilon_\omega$ is sufficiently small and general so that both of $\sigma^\pm$ lie on chambers. Similarly to the diagram (6.11),
we have the diagram

\[ (8.8) \quad M_{\sigma^+}(\beta) \xrightarrow{q_M^+} M_{\sigma}(\beta) \xleftarrow{q_M} M_{\sigma^-}(\beta). \]

Note that as \((\beta, 1)\) is primitive in \(\Gamma \leq 1\) and \(\sigma^\pm\) lie on chambers, both of \(M_{\sigma^\pm}(\beta)\) consist of \(\sigma^\pm\)-stable sheaves. Therefore by Theorem 4.1, they admit d-critical structures. Applying Corollary 6.2, we have the following:

**Theorem 8.3.** The diagram \((8.8)\) is an analytic d-critical generalized flop.

**Proof.** For a point \(p \in M_{\sigma}(\beta)\), suppose that it corresponds to a polystable sheaf \(E\) of the form \((8.6)\). Since each \(E_i\) has at most one dimensional support, the Ext-quiver \(Q = Q_{E_*}\) associated with the collection \(E_* = (E_1, E_2, \ldots, E_k)\) is symmetric (see [Toda, Lemma 5.1]). We take data \(\xi^\pm \in H^k\) as in \((5.11)\) for the quiver \(Q\), given by

\[ \xi^\pm_i = Z_{B,\omega}(E_i) \pm (\varepsilon_B + i\varepsilon_\omega) \cdot [E_i]. \]

As \(\text{arg} Z_{B,\omega}(E_i) = \text{arg} Z_B(E)\), by taking rotations and scaling of \(\xi^\pm\), and also perturbing \(\varepsilon_B + i\varepsilon_\omega\) if necessary, we may assume that \(\xi^\pm\) satisfy the condition \((7.6)\). Then the result follows from Corollary 6.2 and Corollary 7.8.

8.3. **Example: elliptic CY 3-fold.** Here we discuss an example of wall-crossing of one dimensional stable sheaves on an elliptic CY 3-fold. Let \(S = \mathbb{P}^2\) and take general elements

\[ u \in H^0(S, O_S(-4K_S)), \quad v \in H^0(S, O_S(-6K_S)). \]

Then as in [Tod12a, Section 6.4], we have a simply connected CY 3-fold \(X\) with a flat elliptic fibration

\[ (8.9) \quad \pi_X : X \to S \]

defined by the equation \(zy^2 = uxz^2 + vz^3\) in the projective bundle

\[ \mathbb{P}_S(O_S(-2K_S) \oplus O_S(-3K_S) \oplus O_S) \to S. \]

Here \([x : y : z]\) is the homogeneous coordinate of the above projective bundle. Note that \(\pi_X\) admits a section \(\iota : S \to X\) whose image \(D := \iota(S)\) correspond to the fiber point \([0 : 1 : 0]\). Let \(H \subset X\) be the pull-back of a hyperplane in \(\mathbb{P}^2\) to \(X\) by \(\pi_X\). We have

\[ H^2(X, \mathbb{R}) = \mathbb{R}[D] + \mathbb{R}[H]. \]
Let $F$ be a fiber of $\pi_X$ and $l \subset D$ a line. Then $[F]$ and $[l]$ span the Mori cone $\overline{NE}(X)$ of $X$. The intersection matrix is given by

$$
\begin{pmatrix}
H \cdot l & D \cdot l \\
H \cdot F & D \cdot F
\end{pmatrix} = 
\begin{pmatrix}
1 & -3 \\
0 & 1
\end{pmatrix}.
$$

We fix an ample divisor $\omega_0$ on $X$ and write $d_1 = \omega_0 \cdot F > 0$, $d_2 = \omega_0 \cdot l > 0$. Let us take an effective curve class $\beta$ and write it as $\beta = r[F] + k[l]$, $r, k \in \mathbb{Z}_{\geq 0}$.

We consider wall-chamber structure on the subset of $U(X)$, given by the image of the map

$$H^2(X, \mathbb{R}) \to U(X), \; B \mapsto \sigma_B,\omega_0.$$  

(8.10)

We identify the image of (8.10) with $H^2(X, \mathbb{R})$ by the above map. For each decomposition in $\Gamma_{\leq 1}(X)$

$$\beta, 1 = (\beta_1, n_1) + (\beta_2, n_2), \; \beta_i = r_i[F] + k_i[l]$$

the wall is given by the equation $\mu_{B,\omega_0}(\beta_1, n_1) = \mu_{B,\omega_0}(\beta, 1)$. By writing $B = x[D] + y[H]$, a direct computation shows that the above condition is equivalent to

$$\begin{equation}
(3d_1 + d_2)x - d_1y = \frac{r_1d_1 + k_1d_2 - n_1(rd_1 + kd_2)}{rk_1 - kr_1}.
\end{equation}$$

(8.12)

It follows that every wall is proportional to the line $y = (3 + d_2/d_1)x$, so any two wall are disjoint if they do not coincide.

In the case of $k = 1$, i.e. $\beta = r[F] + [l]$, we have the following decomposition

$$\begin{equation}
(r[F] + [l], 1) = (r[F], 1) + ([l], 0).
\end{equation}$$

(8.13)

We set

$$B_0 = \frac{d_2}{r(3d_1 + d_2)}[D], \; B_\pm = B_0 \pm \varepsilon[D], \; 0 < \varepsilon \ll 1.$$

The above $B_0$ satisfies the equation (8.12) determined by the decomposition (8.13). Therefore $\sigma_0 := \sigma_{B_0,\omega_0}$ lies on a wall and $\sigma_\pm := \sigma_{B_\pm,\omega_0}$ lie on its adjacent chambers.

In the above case $k = 1$ case, we can describe the wall-crossing diagram (8.8) for $\sigma = \sigma_0$ in terms of d-critical simple flop. It is easy to see that $\sigma_0$ does not lie on a wall determined by a decomposition of the form (8.11), other than (8.13). Therefore any point $p \in M_{\sigma_0}(\beta)$, which do not correspond to $\sigma_0$-stable sheaf, corresponds to a a $\sigma_0$-polystable sheaf $E$ of the form

$$E = E_1 \oplus E_2, \; \text{ch}(E_1) = (r[F], 1), \; \text{ch}(E_2) = ([l], 0).$$

Here $E_1, E_2$ are $\sigma$-stable sheaves. Then $E_1$ is a stable vector bundle on a fiber $F$ with rank $r$ and degree 1, and $E_2 = \mathcal{O}_l(-1)$ for some line $l \subset D$. If $F \cap l = \emptyset$, then

$$\text{Ext}^1(E_1, E_2) = \text{Ext}^1(E_2, E_1) = 0.$$
and we have \((q_M^+)^{-1}(U) = \emptyset\) for a small open subset \(p \in U \subset M_\sigma(\beta)\). This is an obvious case of \(d\)-critical generalized flop (see Remark 3.11). Otherwise \(F \cap l\) is a one point, and it is easy to check that
\[
\text{Ext}^1(E_1, E_1) = \mathbb{C}^3, \quad \text{Ext}^1(E_2, E_2) = \mathbb{C}^2, \\
\text{Ext}^1(E_1, E_2) = \text{Ext}^1(E_2, E_1) = \mathbb{C}^r.
\]

The associated Ext-quiver \(Q\) has two vertices \(\{1, 2\}\), \(r\)-arrows from 1 to 2 and 2 to 1, and the number of loops at 1, 2 are 3, 2 respectively. In this case, the diagram (8.8) is a \(d\)-critical simple flop at \(p \in M_\sigma(\beta)\) as we mentioned in Example 6.3.

8.4. D-critical flops under flops. For a CY 3-fold \(X\) and \(\beta \in H_2(X, \mathbb{Z})\), we define
\[
M_X(\beta) := M_{\sigma = \sigma(\omega)(\beta)}.
\]

Here \(\omega\) is an ample divisor on \(X\). Note that \(M_X(\beta)\) is independent of a choice of \(\omega\) (see [MT, Remark 3.2]). Moreover \(M_X(\beta)\) consists of stable sheaves, so it admits a \(d\)-critical structure by Theorem 4.1. The moduli space (8.14) was used in [MT] in the definition of Gopakumar-Vafa invariants on CY 3-folds.

Suppose that we have a flop diagram of CY 3-folds
\[
\begin{xy}
(0,-10) *+{X} = "X", (0,0) *+{Y}, (0,10) *+{X^\dagger}
\end{xy}
\]

Let \(\phi_* \beta \in H_2(X^\dagger, \mathbb{Z})\) be defined by \(\phi_* \beta \cdot D = \beta \cdot \phi^{-1}_* D\) for any divisor \(D\) on \(X^\dagger\). Here \(\phi^{-1}_* D\) is a strict transform of \(D\) to \(X\). In the above situation, we have the following:

**Theorem 8.4.** The \(d\)-critical loci \(M_X(\beta)\) and \(M_{X^\dagger}(\phi_* \beta)\) are connected by a sequence of analytic \(d\)-critical generalized flops.

**Proof.** Under a 3-fold flop (8.15), we have the commutative diagram (see [Bri02, Tod08]):
\[
\begin{array}{ccc}
D_{\leq 1}^b(X) & \xrightarrow{\Phi} & D_{\leq 1}^b(X^\dagger) \\
\downarrow \text{ch} & & \downarrow \text{ch} \\
\Gamma_{\leq 1} & \xrightarrow{\Phi^\dagger} & \Gamma_{\leq 1}^\dagger.
\end{array}
\]

Here \(\Phi\) is an equivalence of derived categories, \(\Gamma_{\leq 1}^\dagger = H_2(X^\dagger, \mathbb{Z}) \oplus \mathbb{Z}\) and \(\Phi^\dagger\) is an isomorphism which takes \((\beta, n)\) to \((\phi_* \beta, n)\). The above equivalence induces the isomorphism
\[
\Phi_* : \text{Stab}_{\leq 1}(X) \xrightarrow{\cong} \text{Stab}_{\leq 1}(X^\dagger).
\]
Under the above isomorphism, the closures of $\Phi^* U(X)$ and $U(X^\dagger)$ intersect (see [Toda, Lemma 6.4]). Let $\omega^\dagger$ be an ample divisor on $X^\dagger$. We take a path

$$\gamma: [0, 1] \to \Phi^* U(X) \cup U(X^\dagger)$$

such that $\gamma(0) = \Phi^* \sigma_{0, \omega}$ and $\gamma(1) = \sigma(0, \omega^\dagger)$. By perturbing $\gamma$ if necessary, we can assume that each wall-crossing of the path is given as in (8.7). Moreover the intersection of $\Phi^* U(X)$ and $U(X^\dagger)$ is not a wall by [Toda, Lemma 6.7]. Therefore applying Theorem 8.3 at each wall, we obtain the result. □

9. D-critical flips of moduli spaces of stable pairs

In this section, we show that wall-crossing phenomena of Pandharipande-Thomas stable pair moduli spaces [PT09], studied in [Bri11, Tod09a, Tod10b, Tod12a, Dia12], are described in terms of d-critical birational geometry. The wall-crossing phenomena here were used in loc.cit. to show the rationality of the generating series of stable pair invariants, which we will review in Appendix C.

9.1. Stable pairs. Let $X$ be a smooth projective CY 3-fold over $\mathbb{C}$. The notion of stable pairs by Pandharipande-Thomas is given below:

**Definition 9.1.** ([PT09]) A stable pair consists of data

$$(F, s), F \in \text{Coh}_{\leq 1}(X), s: O_X \to F$$

where $F$ is a pure one dimensional sheaf and $s$ is a morphism of coherent sheaves which is surjective in dimension one.

As in (8.1), we set $\Gamma_{\leq 1} = H_2(X, \mathbb{Z}) \oplus \mathbb{Z}$. For $(\beta, n) \in \Gamma_{\leq 1}$, let

$$(9.1) \quad P_n(X, \beta)$$

be the moduli space of stable pairs $(F, s)$ such that $\text{ch}(F) = (\beta, n)$. It is proved in [PT09, HT10] that the moduli space (9.1) is a projective scheme with a symmetric perfect obstruction theory. Indeed the moduli space (9.1) is identified with the moduli space of two term complexes in the derived category (here $O_X$ is located in degree zero)

$$(9.2) \quad I^* = (\cdots \to 0 \to O_X \xrightarrow{s} F \to 0 \to \cdots) \in D^b(X)$$

satisfying a certain stability condition on it (see [PT09, Tod10a]). It follows that by Theorem 4.1 there is a canonical d-critical structure on the moduli space of stable pairs (9.1).

9.2. Weak semistable objects. Below we study wall-crossing in some abelian subcategory in $D^b(X)$ defined below:

**Definition 9.2.** We define the subcategory $A_X$ in $D^b(X)$ by

$$(9.3) \quad A_X := \langle O_X, \text{Coh}_{\leq 1}(X)[-1] \rangle_{\text{ex}} \subset D^b(X).$$

Here $\langle \ast \rangle_{\text{ex}}$ is the smallest extension closed subcategory containing $\ast$. 
The category \([9.3]\) is an abelian subcategory of \(D^b(X)\) (see [Tod10a, Lemma 6.2]). We have the Chern character map
\[
\text{cl}: K(A_X) \to \Gamma_{\leq 1}^* := \mathbb{Z} \oplus \Gamma_{\leq 1}
\]
sending \(O_X\) to \((1, 0)\) and \(F \in \text{Coh}_{\leq 1}(X)\) to \((0, \text{ch}(F))\). We will be interested in certain rank one objects in \(A_X\). We have the following lemma describing rank one objects in \(A_X\), whose proof is obvious:

**Lemma 9.3.** An object \(E \in D^b(X)\) with \(\text{rank}(E) = 1\) is an object in \(A_X\) if and only if there exist distinguished triangles in \(D^b(X)\)
\[
0 = E_0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 = E
\]
such that
\[
F_1 \in \text{Coh}_{\leq 1}(X)[-1], \quad F_2 = O_X, \quad F_3 \in \text{Coh}_{\leq 1}(X)[-1].
\]
In this case, the top sequence of \((9.5)\) is a filtration in \(A_X\).

**Remark 9.4.** For a two term complex \(I^*\) in \((9.2)\) associated with a stable pair, it is an object in \(A_X\) as it fits into the exact sequence in \(A_X\)
\[
0 \rightarrow F[-1] \rightarrow I^* \rightarrow O_X \rightarrow 0.
\]
Similarly the derived dual \(\mathbb{D}(I^*)\) for \(\mathbb{D}(*) = R\text{Hom}(*, O_X)\) is an object in \(A_X\) as it fits into the exact sequence in \(A_X\)
\[
0 \rightarrow O_X \rightarrow \mathbb{D}(I^*) \rightarrow F^\vee[-1] \rightarrow 0
\]
where \(F^\vee := \text{Ext}^2_{O_X}(F, O_X)\).

In what follows, we fix an ample divisor \(\omega\) on \(X\). For \(t \in \mathbb{R}\), let \(\mu_t^*\) be the slope function on the abelian category \(A_X\) defined by, for \(E \in A_X\)
\[
\mu_t^*(E) := \begin{cases} 
  t & \text{if } E \notin \text{Coh}_{\leq 1}(X)[-1] \\
  \mu_\omega(E) = \chi(E)/\omega \cdot [E] & \text{if } E \in \text{Coh}_{\leq 1}(X)[-1].
\end{cases}
\]
Here \(\mu_\omega := \mu_{0, \omega}\) is the slope function on one dimensional sheaves defined as in \((8.5)\) for \(B = 0\). The above slope function \(\mu_t^*\) on \(A_X\) satisfies the weak seesaw property, and defines the weak stability condition on \(A_X\) (see [Tod10b, Tod12a]):

**Definition 9.5.** An object \(E \in A_X\) is \(\mu_t^*\)-(semi)stable if for any exact sequence \(0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0\) in \(A_X\), we have the inequality
\[
\mu_t^*(F) < (\leq) \mu_t^*(G).
\]

**Remark 9.6.** The above \(\mu_t^*\)-stability condition on \(A_X\) can also be formulated in terms of a Bridgeland-type weak stability condition introduced in [Tod10a], by taking the filtration \(\{0\} \oplus \Gamma_{\leq 1} \subset \Gamma_{\leq 1}^*\). See [Tod12a] for details.
9.3. Moduli spaces of weak semistable objects. Let $\mathcal{M}$ be the moduli stack of objects in $D^b(X)$ considered in Subsection 4.1. For $(\beta, n) \in \Gamma_{\leq 1}$ and $t \in \mathbb{R}$, we have the substack

$$M_t^* (\beta, n) \subset \mathcal{M}$$

consisting of $\mu_t^*$-semistable objects $E \in \mathcal{A}_X$ satisfying

$$\text{cl}(E) = (1, -\beta, -n) \in \Gamma_{\leq 1}^*.$$ 

The substack (9.6) is an open substack of $\mathcal{M}$, which is an Artin stack of finite type over $\mathbb{C}$ (see [Tod10b, Proposition 3.17], [Tod12a, Proposition 5.4]). Similarly to Theorem 4.7, the result of [AHLH] is applied to the stack $M_t^* (\beta, n)$. So it admits a good moduli space $p_t: M_t^* (\beta, n) \rightarrow M_t^* (\beta, n)$ where $M_t^* (\beta, n)$ is a separated algebraic space of finite type. A closed point of $M_t^* (\beta, n)$ corresponds to a $\mu_t^*$-polystable object $E \in \mathcal{A}_X$ written as

$$E = \bigoplus_{i=0}^k V_i \otimes E_i, \ E_i \in \mathcal{A}_X. \ (9.8)$$

Here each $V_i$ is a finite dimensional vector space with $V_0 = \mathbb{C}$, the object $E_0 \in \mathcal{A}_X$ is a rank one $\mu_t^*$-stable object, and each $E_i$ for $1 \leq i \leq k$ is isomorphic to $F_i[-1]$ for a $\mu_*\omega$-semistable sheaf $F_i \in \text{Coh}_{\leq 1}(X)$ with $\mu_*\omega(F_i) = t$, such that $F_i \not\cong F_j$ for $i \neq j$. The moduli space $M_t^* (\beta, n)$ is related to stable pair moduli spaces as follows:

**Proposition 9.7.** ([Tod10b, Theorem 3.21], [Tod12a, Proposition 5.4]) For $|t| \gg 0$, the moduli spaces $M_t^* (\beta, n)$ consists of $\mu_t^*$-stable objects. Moreover we have the isomorphisms

$$P_n(X, \beta) \cong M_t^* (\beta, n), \ t \gg 0, \ (9.9)$$

$$P_{-n}(X, \beta) \cong M_t^* (\beta, n), \ t \ll 0. \ (9.10)$$

The isomorphism (9.9) sends a stable pair $(F, s)$ to the two term complex (9.2), and the isomorphism (9.10) sends $(F, s)$ to the derived dual of (9.2) (see Remark 9.4).

**Proof.** For $|t| \gg 0$, it is proved in [Tod10b, Theorem 3.21], [Tod12a, Proposition 5.4] that the stack $M_t^* (\beta, n)$ consists of $\mu_t^*$-stable objects, and we have the isomorphism

$$[P_{\pm n}(X, \beta)/\mathbb{C}^*] \cong M_t^* (\beta, n), \ \pm t \gg 0.$$ 

Here $\mathbb{C}^*$ acts on $P_{\pm n}(X, \beta)$ trivially, and the above isomorphisms are defined as in the statement of the proposition. By taking the good moduli spaces of both sides, we obtain the proposition. \qed
Remark 9.8. By the isomorphism (9.10), an object \( E \in A_X \) is isomorphic to \( D(I^\bullet) \) for a stable pair (9.2) as in Remark 9.4 if and only if \( E \) fits into an exact sequence in \( A_X \)

\[
0 \to \mathcal{O}_X \to E \to F'[\geq -1] \to 0
\]

for some \( F' \in \text{Coh}_{\leq 1}(X) \) such that \( \text{Hom}(F'[\geq -1], E) = 0 \) for any \( F'' \in \text{Coh}_{\leq 1}(X) \). This fact will not be used later in this paper.

9.4. Wall-crossing of weak semistable objects. For \( t \in \mathbb{R} \), we set \( t^\pm := t \pm \varepsilon \), \( 0 < \varepsilon \ll 1 \).

Then we have open immersions of stacks

\[
\mathcal{M}_{t^+}^*(\beta, n) \subset \mathcal{M}_t^*(\beta, n) \supset \mathcal{M}_{t^-}^*(\beta, n).
\]

We define \( W \subset \mathbb{R} \) to be the subset of \( t \in \mathbb{R} \) where at least one of the open immersions in (9.11) is not an isomorphism, i.e. \( W \) is the set of walls for the \( \mu_t^\ast \)-stability for \( t \in \mathbb{R} \).

Lemma 9.9. The subset \( W \subset \mathbb{R} \) is a finite set.

Proof. An element \( t \in W \) is of the form

\[
t = \frac{n'}{\omega \cdot \beta'}
\]

for \( 0 < \beta' < \beta \) and \( n' \in \mathbb{Z} \). In particular, \( W \) is locally finite. It is also bounded by Proposition 9.7, hence \( W \) is a finite set. \( \square \)

We write \( W \) as

\[
W = \{ \infty = t_0 > t_1 > t_2 > \cdots > t_l > t_{l+1} = -\infty : t_1, \ldots, t_l \in \mathbb{R} \}.
\]

Note that each \( \mathcal{M}_t^*(\beta, n) \) is constant if \( t \) lies in a connected component of \( \mathbb{R} \setminus W \), but may change if \( t \) crosses one of \( t_i \). For \( t \in W \), the open immersions (9.11) induce the diagram of good moduli spaces

\[
\begin{array}{ccc}
\mathcal{M}_{t^+}^*(\beta, n) & \xrightarrow{q_{M^*}} & \mathcal{M}_t^*(\beta, n) \\
\downarrow q_{M^*} & & \downarrow q_{M^*} \\
\mathcal{M}_t^*(\beta, n) & & \mathcal{M}_t^*(\beta, n)
\end{array}
\]

Note that \( \mathcal{M}_{t^\pm}^*(\beta, n) \) consists of \( \mu_{t^\pm}^\ast \)-stable objects, hence they admit \( d \)-critical structures by Theorem 4.1. Below, we investigate \( d \)-critical birational geometry of the diagram (9.13).

Let us take a point \( p \in \mathcal{M}_t^*(\beta, n) \) in the diagram (9.13), and suppose that it corresponds to a \( \mu_t^* \)-polystable object (9.8). Let \( Q^* = Q_{E^*} \) be the Ext-quiver associated with the collection

\[
E^*_\bullet = (E_0, E_1, \ldots, E_k).
\]
On the other hand, let $Q = Q_{E^*}$ be the Ext-quiver associated with the collection of shift of one dimensional sheaves

\[(9.15) \quad E^* = (E_1, E_2, \ldots, E_k).\]

As we observed in the proof of Theorem 8.3, the quiver $Q$ is symmetric. Then we have

\[V(Q^*) = \{0\} \sqcup V(Q)\]

and the numbers of arrows from 0 to $i \in V(Q)$, from $i$ to 0, and the loops at 0 are

\[(9.16) \quad a_i := \text{ext}^1(E_0, E_i), \quad b_i := \text{ext}^1(E_i, E_0), \quad c := \text{ext}^1(E_0, E_0)\]

respectively. Therefore $Q^*$ is obtained from $Q$ as in the construction of Subsection 7.3. We also have the convergent super-potential

\[W^* := W_{E^*} \in \mathbb{C}\{Q^*\}/[\mathbb{C}\{Q^*\}, \mathbb{C}\{Q^*\}]\]

of $Q^*$ associated with the collection \((9.14)\) as in the construction of \((6.2)\). The following lemma (which will be used in Lemma 9.23) is obvious from the constructions, and we leave readers for details:

**Lemma 9.10.** Let $\mathbb{C}\{Q^*\} \to \mathbb{C}\{Q\}$ be the linear map sending a path $e_1 e_2 \ldots e_n$ to itself if each $e_i \in E(Q)$ and to zero otherwise. The above map induces the linear map

\[\mathbb{C}\{Q^*\}/[\mathbb{C}\{Q^*\}, \mathbb{C}\{Q^*\}] \to \mathbb{C}\{Q\}/[\mathbb{C}\{Q\}, \mathbb{C}\{Q\}], \quad f \mapsto f|_Q.\]

Under the above map, we have $W^*|_Q = W$, where $W = W_{E^*}$ is the convergent super-potential of $Q$ associated with the collection \((9.15)\).

Let $\bar{m}^*$ be the dimension vector of $Q^*$ given by

\[(9.17) \quad (\bar{m}^*)_i = \dim V_i, \quad 0 \leq i \leq k\]

where $V_i$ is given in \((9.8)\). Note that we have $\bar{m}^* = \bar{0} + \bar{m}$, where $\bar{m}$ is the dimension vector of $Q$ given by $m_i = \dim V_i$, $1 \leq i \leq k$. Let us also take

\[\xi^\pm = (\xi^\pm_i)_{0 \leq i \leq k} \in \mathcal{H}^{\mathcal{W}(Q^*)}\]

as in \((7.13)\). Similarly to Theorem 6.1, we have the local description of the morphisms in \((9.13)\) in terms of $Q^*$:

**Theorem 9.11.** For a closed point $p \in M^*_t(\beta, n)$ corresponding to a $\mu^*_t$-polystable object \((9.8)\), let $Q^* = Q_{E^*}$ be the Ext-quiver associated with $E^*$ and $W^* = W_{E^*}$ the convergent super-potential of $Q^*$ constructed in \((6.2)\). Then there exist analytic open neighborhoods

\[p \in T \subset M^*_t(\beta, n), \quad 0 \in V \subset M_{Q^*}(\bar{m}^*)\]
where \( \vec{m}^* \) is the dimension vector \((9.17)\), such that we have the commutative diagram of isomorphisms

\[
\begin{array}{ccc}
M_{(Q^*, \partial W^*)}(\vec{m}^*)|_V & \xrightarrow{\cong} & (q_{M^*})^{-1}(T) \\
\downarrow q_{(Q^*, \partial W^*)} & & \downarrow q_{M^*} \\
M_{(Q^*, \partial W^*)}(\vec{m}^*)|_V & \cong & T.
\end{array}
\]

Here the left vertical arrow is given in \((7.17)\), the right vertical arrow is given in \((9.13)\) pulled back to \(T\). Moreover the top isomorphism preserves the equivalence of categories given in \((6.4)\). Under the above equivalence, a nilpotent \(Q^*\)-representation \(V\) is \(\xi^\pm\)-semistable if and only if there is no non-trivial surjection \(\xi^\pm\)-semistable objects in \(A_X\).

Proof. The proof is almost identical to Theorem \(6.1\). The only required modification is the proof of the preservation of stability in Lemma \(6.4\), as we need to compare Bridgeland stability on \(Q^*\)-representations given by \(\xi^\pm\) with weak stability \(\mu^*\) on \(A_X\). In Lemma \(9.12\) below, we give an ad-hoc proof for this using characterizations of semistable objects in both sides.

\[\Phi_{E^*} : \text{mod}_{nil} C[[Q^*]]/\langle \partial W^* \rangle \xrightarrow{\cong} \langle E_0, E_1, \ldots, E_k \rangle_{\text{ex}}\]

be the equivalence of categories given in \((6.4)\). Under the above equivalence, a nilpotent \(Q^*\)-representation \(V\) is \(\xi^\pm\)-semistable if and only if \(\Phi_{E^*}(V)\) is \(\mu^{*\pm}\)-semistable in \(A_X\).

Proof. Let \(S_i\) for \(0 \leq i \leq k\) be the simple \(Q^*\)-representation corresponding to the vertex \(i \in V(Q^*)\). As in the proof of Lemma \(7.10\) a \(Q^*\)-representation \(V\) is \(\xi^\pm\)-semistable if and only if there is no non-trivial surjection \(V \rightarrow V'\) of \(Q^*\)-representations such that \(V'\) is an object in \(\langle S_1, \ldots, S_k \rangle_{\text{ex}}\).

For an object \(F \in \langle E_0, E_1, \ldots, E_k \rangle_{\text{ex}}\), we claim that it is \(\mu^{*\pm}\)-semistable if and only if there is no non-trivial surjection \(F \rightarrow F'\) in \(\langle E_0, E_1, \ldots, E_k \rangle_{\text{ex}}\) such that \(F' \in \langle E_1, \ldots, E_k \rangle_{\text{ex}}\). The only if direction is obvious from the definition of \(\mu^{*\pm}\)-stability. In order to show the if direction, suppose that \(F\) is not \(\mu^{*\pm}\)-semistable in \(A_X\). Then there exists an exact sequence

\[
0 \rightarrow A \rightarrow F \rightarrow B \rightarrow 0
\]

in \(A_X\) such that \(\mu^{*\pm}(A) > \mu^{*\pm}(B)\). By taking the limit \(t_+ \rightarrow t\), we have \(\mu^{*\pm}(A) \geq \mu^{*\pm}(B)\). On the other hand, since \(F\) is \(\mu^{*\pm}\)-semistable, we have \(\mu^{*\pm}(A) \leq \mu^{*\pm}(B)\). It follows that \(\mu^{*\pm}(A) = \mu^{*\pm}(B)\), therefore both \(A, B\) are \(\mu^{*\pm}\)-semistable. By the uniqueness of Jordan-H"older factors, the exact sequence \((9.19)\) is an exact sequence in \(\langle E_0, E_1, \ldots, E_k \rangle_{\text{ex}}\). Then by the inequality \(\mu^{*\pm}(A) > \mu^{*\pm}(B)\), we have \(B \in \langle E_1, \ldots, E_k \rangle_{\text{ex}}\). Therefore the if direction is also proved.

Now the lemma for the plus sign holds by the above descriptions of \(\xi^+\)-semistable \(Q^*\)-representations and \(\mu^{*\pm}\)-semistable objects in \(A_X\), together
with the fact that the equivalence (6.21) sends $S_i$ to $E_i$. The result for the minus sign holds in a similar way.  

Using Theorem 9.11 we can describe the diagram (9.13) in terms of d-critical birational transformations:

**Theorem 9.13.** For $t \in \mathcal{W}$ with $t > 0$, the diagram (9.13) is an analytic d-critical generalized flip at any $p \in \text{Im} \, q_{M^*}$, and an analytic d-critical generalized MFS at any $p \in M^*_t(\beta, n) \setminus \text{Im} \, q_{M^*}$.

Moreover for each effective curve class $\beta$, there is $t(\beta) > 0$ (which is independent of $n$) such that if $t > t(\beta)$ then the diagram (9.13) is strict. There is also $n(\beta) > 0$ such that if $n > n(\beta)$, then any $t \in \mathcal{W}$ satisfies $t > t(\beta)$. Therefore in this case, the diagram (9.13) is always strict.

**Proof.** Suppose that $p \in M^*_t(\beta, n)$ corresponds to a $\mu^*_t$-polystable object (9.8), and take $1 \leq i \leq k$. Let $a_i, b_i$ be as in (9.16). Since $\mu_\omega(E_i) = t > 0$ and $E_i \in \text{Coh}_{\leq 1}(X)[-1]$, we have $\chi(E_i) < 0$. By the Riemann-Roch theorem and $\text{hom}(E_0, E_i) = \text{hom}(E_i, E_0) = 0$, we have

$$0 > \chi(E_i) = \chi(E_0, E_i) = -a_i + b_i.$$  

Therefore we have $a_i > b_i$, and the first statement follows from Corollary 7.15 and Theorem 9.11.

We show the second strictness statement. Again suppose that $p$ corresponds to a $\mu^*_t$-polystable object (9.8), but now assume that (9.8) is not $\mu^*_t$-stable, i.e. $k \geq 1$. We write

$$\text{cl}(E_0) = (1, -\beta_0, -n_0), \quad \text{ch}(E_i) = -\beta_i, n_i$$

for $1 \leq i \leq k$. Then we have $\beta_0 \geq 0$, $\beta_i > 0$ for $1 \leq i \leq k$, and

$$\beta_0 + m_1 \beta_1 + \cdots + m_k \beta_k = \beta, \tag{9.20}$$

$$n_0 + m_1 n_1 + \cdots + m_k n_k = n \tag{9.21}$$

where $m_i = \dim V_i$ for $1 \leq i \leq k$. By the identity (9.20), for a fixed $\beta \geq 0$ there is only a finite number of possibilities for $k, \beta_i$ and $m_i$. If we have $n_i \leq m_i$ for some $1 \leq i \leq k$, then we have

$$t = \frac{n_i}{\omega \cdot \beta_i} \leq \frac{m_i}{\omega \cdot \beta_i}$$

and the RHS is bounded above. Therefore there is $t(\beta) > 0$ such that if $t > t(\beta)$ then $n_i > m_i$ for any $1 \leq i \leq k$. But then

$$-n_i = \chi(E_i) = -a_i + b_i$$

which implies $a_i \geq n_i > m_i$. Therefore the diagram (9.13) is strict at $p$ by Lemma 7.16.

Suppose that $t \in \mathcal{W}$ satisfies $t \leq t(\beta)$. Applying Lemma 9.14 below for $\beta_0$ and all $0 < t \leq t(\beta)$, we can find $n'(\beta_0) \in \mathbb{Z}$ such that $n_0 \leq n'(\beta_0)$ holds.
By (9.21) and using $t = n_i / \omega \cdot \beta_i$, we obtain
\[ t \geq \frac{n - n'(\beta_0)}{\sum_{i=1}^k m_i(\omega \cdot \beta_i)}. \]
There is $n(\beta) > 0$ such that for $n > n(\beta)$, the RHS is bigger than $t(\beta)$. Therefore for such $n(\beta)$, the desired statement holds. \qed

We have used the following lemma:

**Lemma 9.14.** For each effective curve class $\beta$ and $t \in \mathbb{R}$, there is $n_t(\beta) > 0$ such that for $n > n_t(\beta)$, the RHS is bigger than $t(\beta)$.

**Proof.** The lemma is proved in the proof of \cite{toda2010ample} Lemma 4.4. \qed

In the following example, we see that Grassmannian flips appear as relative d-critical charts:

**Example 9.15.** In Theorem 9.13, suppose that $p \in M_t^\ast(\beta, n)$ corresponds to a $\mu_t^\ast$-polystable object $E \in A_X$ of the form
\[ E = E_0 \oplus (V \otimes F[-1]) \]
where $E_0 \in A_X$ is a rank one $\mu_t^\ast$-stable object, $V$ is a finite dimensional vector space, $F \in \text{Coh}_{\leq 1}(X)$ satisfying $\text{Ext}^1(F, F) = 0$, e.g. $F = \mathcal{O}_C(k)$ for a rational curve $\mathbb{P}^1 = C \subset X$ with $N_{C/X} = \mathcal{O}_C(-1) \oplus 2$. In this case the Ext-quiver $Q^\ast$ for $\{E_0, F[-1]\}$ is the same one considered in Example 7.14, and the birational map
\[ M_{Q^\ast}[\bar{m}^\ast] \to M_{Q^\ast}[\bar{m}^\ast] \]
is a Grassmannian flip as in Example 7.14. For example, the wall-crossing diagrams in local $\mathbb{P}^1$ studied in \cite{nakajima2011wall} are described as d-critical Grassmannian flips as above.

**Remark 9.16.** In Theorem 9.13, it is possible that the diagram (9.13) is a strict analytic d-critical generalized MFS at $p \in M_t^\ast(\beta, n) \setminus \text{Im} q_{M^\ast}$, but in the notation of Theorem 9.11 we have $M_{Q^\ast}[\bar{m}^\ast] = \emptyset$ and the morphism
\[ q_{Q^\ast}^\ast : M_{Q^\ast}[\bar{m}^\ast] \to M_{Q^\ast}[\bar{m}^\ast] \]
is birational.

Indeed suppose that there exist disjoint smooth curves $C_1, C_2 \subset X$ such that $C_1 \cong \mathbb{P}^1$, and $\omega \cdot C_2 = (\omega \cdot C_1) \cdot m$ for some integer $m \geq 2$. Then for $t = 1/(\omega \cdot C_1)$, $\beta = [C_1] + [C_2]$ and $n = m + 1$, we have the point $p \in M_t^\ast(\beta, n)$ corresponding to the $\mu_t^\ast$-polystable object
\[ \mathcal{O}_X \oplus \mathcal{O}_{C_1}[-1] \oplus \mathcal{O}_{C_2}(D)[1] \]
where $D$ is a divisor on $C_2$ with $\deg D = m + g(C_2) - 1$. Suppose that $h^1(\mathcal{O}_{C_2}(D)) \neq 0$. Then at the point $p$, it is easy to see that we have the situation mentioned above.
9.5. PT/L correspondence via d-critical MMP. Let \( l \in \mathbb{Z} \) be the number of elements in \( W \) given in (9.12). For each \( 1 \leq i \leq l + 1 \), we set

\[
M_i := M_i^\star (\beta, n) = M_{i-1}^\star (\beta, n), \quad A_i := M_i^\star (\beta, n)
\]

with d-critical structure \( s_i \) on \( M_i \) given in Theorem 9.14. We also define \( L_n^\pm(X, \beta) \) to be

\[
L_n^\pm(X, \beta) := M^\star_{l, \pm \varepsilon} (\beta, n), \quad 0 < \varepsilon \ll 1.
\]

Let us take unique \( 1 \leq l' \leq l + 1 \) satisfying \( t_{l' - 1} > 0 \geq t_{l'} \). We have the following zigzag diagram which connects \( M_1 \cong P_n(X, \beta) \) and \( M_{l'} = L_n^+(X, \beta) \):

\[
(9.23) \quad M_1 \xrightarrow{\pi^+_1} M_2 \xrightarrow{\pi^+_2} \cdots \xrightarrow{\pi^+_t} M_{l'}
\]

\[
A_1 \xrightarrow{\pi^-_1} A_2 \xrightarrow{\pi^-_2} \cdots \xrightarrow{\pi^-_{l'}} A_{l'}
\]

The wall-crossing diagrams at \( t \leq 0 \) are given by the following zigzag diagram, which connects \( M_{l+1} \cong P_n(X, \beta) \) and \( M_{l'} = L_n^+(X, \beta) \):

\[
(9.24) \quad M_{l+1} \xrightarrow{\pi^+_l} M_l \xrightarrow{\pi^+_{l-1}} \cdots \xrightarrow{\pi^+_{l'}} M_{l'}
\]

\[
A_{l+1} \xrightarrow{\pi^-_l} A_l \xrightarrow{\pi^-_{l-1}} \cdots \xrightarrow{\pi^-_{l'}} A_{l'}
\]

**Corollary 9.17.** The diagrams (9.23), (9.24) are d-critical MMP. In particular, we have the inequalities of virtual canonical line bundles:

\[
(M_1, s_1) \geq_K (M_2, s_2) \geq_K \cdots \geq_K (M_{l'}, s_{l'}),
\]

\[
(M_{l+1}, s_{l+1}) \geq_K (M_l, s_l) \geq_K \cdots \geq_K (M_{l'}, s_{l'}).
\]

Moreover for each effective curve class \( \beta \), there is \( n(\beta) > 0 \) such that the diagrams (9.23), (9.24) are strict and \( M_{l'} = \emptyset \) if \( |n| > n(\beta) \) holds. In this case, the morphisms

\[
(9.25) \quad \pi^+_{l'-1}: M_{l'-1} \to A_{l'-1}, \quad \pi^-_{l'}: M_{l'+1} \to A_{l'}
\]

are d-critical generalized MFS which are strict at any point in \( A_{l'-1}, A_{l'} \) respectively.

**Proof.** The inequalities of virtual canonical line bundles for the diagram (9.23) is immediate from Theorem 9.13. The same argument also applies to the diagram (9.24).

If we take \( n(\beta) > 0 \) enough large, then the diagrams (9.23), (9.24) are strict as in the argument of Theorem 9.13. Moreover we have \( M_{l'} = \emptyset \) by Lemma 9.14. Then any point in \( A_{l'-1}, A_{l'} \) does not correspond to \( \mu^*_{l'-1} \)-stable object, \( \mu^*_{l'} \)-stable object, respectively (as otherwise \( M_{l'} \neq \emptyset \)). Then as in the proof of Theorem 9.13, the morphisms in (9.25) are strict at any point in \( A_{l'-1}, A_{l'} \) respectively. \( \square \)
If $0 \in \mathcal{W}$, i.e. $t_\nu = 0$, the wall-crossing at $t = 0$ is given by the diagram

\begin{equation}
(9.26) \quad M_\nu = L_n^+(X, \beta) \quad \text{and} \quad L_n^-(X, \beta) = M_{\nu+1}
\end{equation}

\[ M_{t=0}^*(\beta, n). \]

**Corollary 9.18.** The diagram (9.26) is an analytic d-critical generalized flop. In particular, we have

\[ L_n^+(X, \beta) =_K L_n^-(X, \beta). \]

**Proof.** For a point $p \in M_{t=0}^*(\beta, n)$, let $a_i, b_i$ be as in (9.16). Then similarly to the proof of Theorem 9.13, we have $a_i = b_i$. This implies that the Ext-quiver $Q^*$ associated with the collection (9.14) is symmetric. Therefore similarly to Theorem 8.3, the diagram (9.26) is an analytic d-critical generalized flop. \qed

In the next subsection, we discuss the case that $\beta$ is irreducible in detail. Here we give some explicit examples for non-irreducible curve classes, discussed in [Tod09a, Section 5].

**Example 9.19.** Let $X \to Y$ be a birational contraction with exceptional locus $C = C_1 \cup C_2$, where each $C_i$ is isomorphic to $\mathbb{P}^1$, $C_1 \cap C_2 = \{p\}$ and $N_{C_i/X} = \mathcal{O}_{C_i}(-1)^{\oplus 2}$. We set $d_i := C_i \cdot \omega$ and assume $d_1 > d_2 > 0$. Let us consider the diagram (9.23) in the case $(\beta, n) = ([C], 2)$. In this case, we have two walls

\[ \mathcal{W} = \left\{ \infty > t_1 = \frac{1}{d_1} > t_2 = \frac{2}{d_1 + d_2} > 0 \right\}. \]

The reduced part of the diagram (9.23) becomes (see [Tod09a, Section 5.2])

\begin{equation}
(9.27) \quad \begin{array}{ccc}
C & \rightarrow & C_1 \\
\pi_1^+ & \downarrow & \pi_1^- \\
C_1 & \rightarrow & \emptyset \\
\pi_2^- & \downarrow & \pi_2^+ \\
& \text{pt} & \\
\end{array}
\end{equation}

The map $\pi_1^+$ contracts to $C_2 \subset C$ to the point $p \in C_1$ and $\pi_1^- = \text{id}$. The point $p$ corresponds to the $\mu_1^+$-polystable object $I_{C_1} \oplus \mathcal{O}_{C_2}[-1]$, where $I_{C_1}$ is the ideal sheaf of $C_1$. The Ext-quiver $Q^*$ of $\{I_{C_1}, \mathcal{O}_{C_2}[-1]\}$ is of the following form

\[ Q^* = \left( \begin{array}{c} 0 \end{array} \right) \] \hspace{1cm} \text{1} \] .

Therefore locally at $p$, the $\pi_1^+$-relative d-critical charts are given by a diagram of the form

\begin{equation}
\begin{array}{ccc}
\hat{C}^2 & \xrightarrow{f^+} & C^2 \\
\downarrow g & & \downarrow \text{id} \\
C & \xrightarrow{w^+} & C^2 \\
\downarrow w^- & & \end{array}
\end{equation}
where \( f^\dagger \) is the blow-up at the origin. It seems likely that \( g(u, v) = u^2 \), and the scheme structure of \( P_2(X, \beta) \) at \( p \in C = P^\text{red}_2(X, \beta) \) is given by the critical locus of \( \mathbb{C}^2 \to \mathbb{C}, (x, y) \mapsto x^2y^2 \). In particular, the left diagram of (9.27) is an analytic \( d \)-critical divisorial contraction. Similarly the right diagram of (9.27) is a \( d \)-critical MFS.

**Example 9.20.** Let \( X \to Y \) be a birational contraction with \( C \cong \mathbb{P}^1, N_{C/X} = \mathcal{O}_C(-1)^{\oplus 2} \). We set \( d := C \cdot \omega \). Let us consider the diagram (9.23) in the case \( (\beta, n) = (2[\mathcal{C}], 4) \). In this case, we have two walls

\[
W = \left\{ \infty > t_1 = \frac{3}{d} > t_2 = \frac{2}{d} > 0 \right\}.
\]

The reduced part of the diagram (9.23) becomes (see [Tod09a, Section 5.3] and [PT09, Section 4.1] for \( P^\text{red}_4(X, \beta) = \mathbb{P}^3 \))

\[
\begin{array}{ccc}
\mathbb{P}^3 & \xrightarrow{\pi_1^+} & \mathbb{P}^3 \\
\downarrow \pi_1^- & & \downarrow \pi_2^- \\
\text{pt} & & \text{pt} \\
\end{array}
\]

The map \( \pi_1^+ \) contracts \( \mathbb{P}^3 \) to a point, corresponding to the \( \mu_1^\omega \)-polystable object \( I_C \oplus \mathcal{O}_C(2)[-1] \). The Ext-quiver \( Q^* \) of \( \{ I_C, \mathcal{O}_C(2)[-1] \} \) is of the following form

\[
Q^* = \begin{pmatrix}
0 & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & 1
\end{pmatrix}.
\]

Therefore the \( \pi_1^+ \)-relative \( d \)-critical charts are given by a diagram of the form

\[
\begin{array}{ccc}
\mathbb{C}^4 & \xrightarrow{f^+} & \mathbb{C}^4 \\
\downarrow w^+ & & \downarrow w^- \\
\mathbb{C} & \xrightarrow{g} & \mathbb{C}^4 \\
\end{array}
\]

where \( f^\dagger \) is the blow-up at the origin. It seems likely that \( g(u, v, t, s) = uv + ts \) and the scheme structure on \( P^4_4(X, \beta) \) is non-reduced along the quadric in \( \mathbb{P}^3 = P^\text{red}_4(X, \beta) \). In particular, the left diagram of (9.28) is an analytic \( d \)-critical divisorial contraction. Similarly the right diagram of (9.28) is a \( d \)-critical MFS.

**9.6. The case of irreducible curve class.** Suppose that \( \beta \) is an irreducible curve class, i.e. it is not written as \( \beta_1 + \beta_2 \) for effective curve classes \( \beta_i > 0 \). Let \( M_n(X, \beta) \) be the moduli stack of \( \mu_\omega \)-semistable one dimensional sheaves \( F \) on \( X \) with \( \text{ch}(F) = (\beta, n) \), and

\[
M_n(X, \beta) \to M_n(X, \beta)
\]

its good moduli space for a projective scheme \( M_n(X, \beta) \).
Remark 9.21. When $\beta$ is irreducible, a one dimensional sheaf $F$ with $[F] = \beta$ is $\mu_\omega$-semistable if and only if it is a pure one dimensional sheaf. In particular any point in $\mathcal{M}_n(X, \beta)$ corresponds to a $\mu_\omega$-stable sheaf, and the morphism (9.29) is a $\mathbb{C}^*$-gerbe.

As $\beta$ is irreducible, by Remark 9.21 we have the following diagram:

\[
\begin{array}{ccc}
P_\beta(X, \beta) & \xrightarrow{q_1^\pm} & P_{-\beta}(X, \beta) \\
\downarrow q_1^+ & & \downarrow q_1^- \\
M_n(X, \beta) & & M_n(X, \beta)
\end{array}
\]

Here the morphisms $q_1^\pm$ are given by

\[
qu_1^+(F, s) = F, \quad q_1^-(F', s') = \mathcal{E}xt^2_{\mathcal{O}_X}(F', \mathcal{O}_X).
\]

The diagram (9.30) appeared in [PT10] to show the BPS rationality of the generating series of stable pair invariants with irreducible curve classes. The above diagram is indeed a special case of the wall-crossing in the previous subsection, and we have the following:

**Theorem 9.22.** (i) Suppose that $n > 0$. Then at a point $p = [F] \in M_n(X, \beta)$, the diagram (9.30) is

\[
\begin{array}{l}
analytic \ d\text{-}critical \ flip \quad \text{if } h^1(F) > 1 \\
analytic \ d\text{-}critical \ divisorial \ contraction \quad \text{if } h^1(F) = 1 \\
analytic \ d\text{-}critical \ MFS \quad \text{if } h^1(F) = 0
\end{array}
\]

(ii) Suppose that $n = 0$. Then at a point $p = [F] \in M_n(X, \beta)$, the diagram (9.30) is

\[
\begin{array}{l}
analytic \ d\text{-}critical \ flop \quad \text{if } h^1(F) > 1 \\
isomorphisms \quad \text{if } h^1(F) = 1 \\
empty \ sets \quad \text{if } h^1(F) = 0
\end{array}
\]

**Proof.** If $\beta$ is irreducible, there is only one wall (see [Tod09a Subsection 5.1]):

\[
\mathcal{W} = \left\{ t_1 = \frac{n}{\omega \cdot \beta} \right\}.
\]

As in the previous subsection, for $(\beta, n) \in \Gamma_{\leq 1}$ we have the wall-crossing diagram

\[
\begin{array}{ccc}
M^*_{t_1}(\beta, n) & \xrightarrow{q_{M^*}} & M^*\left(t_1, (\beta, n)\right) \\
\downarrow q^*_{M^*} & & \downarrow q^*_{M^*} \\
M^*_{t_1}(\beta, n) & & M^*\left(t_1, (\beta, n)\right)
\end{array}
\]

The algebraic space $M^*_{t_1}(\beta, n)$ parametrizes $\mu^*_{t_1}$-polystable objects $E$ of the form

\[
E = E_0 \oplus E_1, \quad E_0 = \mathcal{O}_X, \quad E_1 = F[-1]
\]
where $F$ is a pure one dimensional sheaf satisfying $\text{ch}(F) = (\beta, n)$ (see Remark 9.21). Let us take $p \in M^*_t(\beta, n)$ which corresponds to a $\mu^*_t$-polystable object (9.33). The Ext-quiver $Q^*$ associated with the collection $\{E_0, E_1\}$ has two vertices $\{0, 1\}$, the number of arrows from 0 to 1, 1 to 0 and the loops at 1, 0 are

$$a := \text{h}^0(F), \ b := \text{h}^1(F), \ c := \text{ext}^1(F, F), \ 0 = \text{ext}^1(\mathcal{O}_X, \mathcal{O}_X)$$

respectively. Note that $a - b = n$ by the Riemann-Roch theorem. Let us set

$$V^+ = \text{H}^0(F), \ V^- = \text{H}^1(F)^\vee, \ U = \text{Ext}^1(F, F).$$

Let $\vec{m}^* = (1, 1)$ be the dimension vector of $Q^*$. Similarly to the argument in Example 6.3, we have

$$M^+_Q(\vec{m}^*) = \text{Tot}_{\mathcal{F}(V^+)}(\mathcal{O}_{\mathcal{F}(V^+)}(-1) \otimes V^-) \times U,$$

$$M^-_Q(\vec{m}^*) = \text{Tot}_{\mathcal{F}(V^-)}(\mathcal{O}_{\mathcal{F}(V^-)}(-1) \otimes V^+) \times U.$$ 

Therefore for $n > 0$, the diagram

$$\begin{array}{ccc}
M^+_Q(\vec{m}^*) & \xrightarrow{q^+_Q} & M^+_Q(\vec{m}^*) \\
\downarrow{q^+_Q} & & \downarrow{q^+_Q} \\
M^-_Q(\vec{m}^*) & \xrightarrow{q^-_Q} & M^-_Q(\vec{m}^*)
\end{array}$$

is a standard toric flip if $b > 1$, a divisorial contraction (indeed a blow-up of a smooth variety at a smooth center) if $b = 1$, and MFS if $b = 0$ (see Example 3.8). By Theorem 9.11, it follows that the diagram (9.32) satisfies the condition (9.31).

On the other hand, we have the isomorphisms by Proposition 9.7

$$\mathcal{P}_{\pm n}(X, \beta) \xrightarrow{\gamma} M^*_t(\beta, n).$$

We also have the morphism of stacks

$$\mathcal{M}_n(X, \beta) \to M^*_t(\beta, n)$$

sending a flat family of one dimensional sheaves $\mathcal{F}$ on $X \times S$ over a $\mathbb{C}$-scheme $S$ to the family of $\mu^*_t$-semistable objects $\mathcal{O}_{X \times S} \oplus \mathcal{F}[-1]$ over $S$. By the universality of good moduli spaces, we have the induced morphism

$$\gamma: \mathcal{M}_n(X, \beta) \to M^*_t(\beta, n).$$

The above morphism $\gamma$ is bijective on closed points, by the description of $\mu^*_t$-polystable objects in (9.33). We can also show that $\gamma$ is a closed immersion (see Lemma 9.23 below), so $M^*_t(\beta, n)$ is a nilpotent thickening of $\mathcal{M}_n(X, \beta)$. Therefore by comparing the diagram (9.30) with (9.32), the result for $n > 0$ follows. The result for $n = 0$ also holds by a similar argument.

We have used the following lemma:

**Lemma 9.23.** The morphism (9.34) is a closed immersion.
Example 9.24. Suppose that products of curves appear as special cases of the diagram (9.30): It follows that the analytic closed subspace in $E$ contains the analytic closed subspace defined by the ideal (9.36)

$$W^* \in \mathbb{C}\{Q^*\}, \ W \in \mathbb{C}\{Q\}$$

be the convergent super-potentials defined as in (6.2) for the collection \{E_0, E_1\}, \{E_1\} respectively. Let us take an analytic open neighborhood $0 \in V \subset M_{Q^*}(\vec{m}^*)$ for $\vec{m}^* = (1, 1)$. As in (5.18), we have the $G = (\mathbb{C}^*)^2$-invariant analytic function

$$\text{tr}W^* : \pi_{Q^*}^{-1}(V) \to \mathbb{C}$$

where $\pi_{Q^*} : \text{Rep}_{Q^*}(\vec{m}^*) \to M_{Q^*}(\vec{m}^*)$ is the quotient morphism. By Theorem 9.11, the local analytic structure of $M_{\beta}^*(\beta, n)$ at $p$ is isomorphic to the analytic closed subspace in $\mathbb{C}[x_1, \ldots, x_n]$ defined by the ideal

$$\text{tr}W^* \subset (\mathbb{C}\{\pi_{Q^*}^{-1}(V)\})^G = O_V.$$

Let

$$\vec{x} = (x_1, \ldots, x_a), \ \vec{y} = (y_1, \ldots, y_b), \ \vec{z} = (z_1, \ldots, z_c)$$

be the coordinates of $E_{0,1}, E_{1,0}, E_{1,1}$ corresponding to the basis $E_{i,j} \subset E_{i,j}$ (see the notation 5.1) respectively. Since $W^*|Q = W$ by Lemma 9.10, the function (9.35) is of the following form

$$\text{tr}W^*(\vec{x}, \vec{y}, \vec{z}) = \text{tr}W(\vec{z}) + \sum_{i,j} f_{i,j}(\vec{z})x_iy_j + \sum_{i',j',j''} f_{i',j',j''}(\vec{z})x_{i'}y_jy_{j'} + \cdots .$$

Here $f_*(\vec{z})$ is an analytic function on an open neighborhood of $0 \in E_{1,1}$. Since each $x_iy_j$ is $G$-invariant, the above description of $\text{tr}W^*$ implies that

$$(\text{tr}W^*)^G \subset (\text{tr}W(\vec{z}), x_iy_j : 1 \leq i \leq a, 1 \leq j \leq b).$$

Since we have

$$M_{Q^*}(\vec{m}^*) = \text{Spec} \mathbb{C}[x_iy_j : 1 \leq i \leq a, 1 \leq j \leq b] \times E_{1,1}$$

it follows that the analytic closed subspace in $V$ defined by the ideal (9.36) contains the analytic closed subspace defined by $\text{tr}W(\vec{z}) = 0$ on $V \cap (\{0\} \times E_{1,1})$. Since the latter is analytic locally isomorphic to $M_{\beta}(X, \beta)$ at $[F] \in M_n(X, \beta)$ (see Theorem 6.1), the lemma holds.

In the following example, we see that the classical diagrams on symmetric products of curves appear as special cases of the diagram (9.30):

**Example 9.24.** Suppose that $X$ contains a smooth projective curve $C$ of genus $g$, which is super-rigid in $X$, i.e. $H^0(N_C/X) = 0$, and set $\beta = [C]$.
Suppose that $C \subset X$ is the unique curve on $X$ with $[C] = \beta$. For example, we can take a local model (see Remark 9.25)

\[(9.38) \quad X = \text{Tot}_C(L_1 \oplus L_2)\]

for general line bundles $L_1, L_2$ on $C$ satisfying $\deg L_i = g - 1$ and $L_1 \otimes L_2 \cong \omega_C$. Here $X$ contains $C$ as a zero-section. By setting $\beta = [C]$, we have the isomorphism

\[S^{n+g-1}(C) \cong P_n(X, \beta), \quad Z \mapsto (O_C(Z), s)\]

where $s$ is the section of $O_C(Z)$ which vanishes at $Z$. Therefore in this case, the diagram (9.30) coincides with (3.13) and the statement in Example 3.9 follows from Theorem 9.22.

Remark 9.25. A subtlety of using the local model (9.38) is that it is non-compact. Let us compactify $X$ to the $\mathbb{P}^2$-bundle $X \rightarrow C$. Still $X$ is not CY3, and also $H^1(O_X) \neq 0$, so we need to modify the argument. Let $E_0 = O_X$, $E_1 = F[-1]$ where $F$ is a line bundle on $C$ pushed forward to $X$ by the zero section of $X \rightarrow C$. Then for $E = E_0 \oplus E_1$, we replace the $A_\infty$-structure on $\text{Ext}^*(E, E)$ with the $L_\infty$-structure on its traceless part $\text{Ext}^*(E, E)_0$. Then the latter is cyclic (though the former is not), and the argument similar to Theorem 9.11 shows that the diagram (3.13) satisfies the desired property in Example 3.9.

**Appendix A. Review of Bridgeland stability conditions**

Here we recall basic definitions on Bridgeland stability conditions on triangulated categories [Bri07].

A.1. **Stability conditions on abelian categories.** Let $\mathcal{A}$ be an abelian category, and $K(\mathcal{A})$ its Grothendieck group.

**Definition A.1.** A stability condition on an abelian category $\mathcal{A}$ is a group homomorphism

\[Z: K(\mathcal{A}) \rightarrow \mathbb{C}\]

satisfying the followings:

(i) (Positivity property): For any non-zero $0 \neq E \in \mathcal{A}$, we have

\[Z(E) \in \{ z \in \mathbb{C} : \text{Im } z > 0 \} \cup \mathbb{R}_{<0}.\]

A non-zero object $E \in \mathcal{A}$ is called $Z$-(semi)stable if for any non-zero subobject $0 \neq F \subset E$ in $\mathcal{A}$, we have the inequality in $(0, \pi]$

\[\arg Z(F) < (\leq) \arg Z(E).\]

(ii) (Harder-Narasimhan property): For any $E \in \mathcal{A}$, there exists a filtration (called Harder-Narasimhan filtration)

\[0 = E_0 \subset E_1 \subset \cdots \subset E_n = E\]
such that each $F_i := E_i/E_{i-1}$ is $Z$-semistable with $\arg Z(F_i) > \arg Z(F_{i+1})$ for all $1 \leq i \leq n - 1$.

A.2. Stability conditions on triangulated categories. Let $\mathcal{D}$ be a $\mathbb{C}$-linear triangulated category, and $K(\mathcal{D})$ its Grothendieck group. We fix a finitely generated abelian group $\Gamma$ together with a group homomorphism

$\text{cl}: K(\mathcal{D}) \to \Gamma$.

Remark A.2. A choice of $(\Gamma, \text{cl})$ corresponds to a choice of a Chern character map. For example if $\mathcal{D} = D^b(X)$ for a smooth projective variety $X$, we can take $\Gamma = \Gamma_X$ where $\Gamma_X$ is the image of the Chern character map $\text{ch}: K(X) \to H^{2*}(X, \mathbb{Q})$ and $\text{cl} = \text{ch}$.

Definition A.3. ([Bri07]) A Bridgeland stability condition on $\mathcal{D}$ consists of data

$\sigma = (Z, A), \ A \subset \mathcal{D}, \ Z : \Gamma \to \mathbb{C}$.

Here $A$ is the heart of a bounded $t$-structure on $\mathcal{D}$, $Z$ is a group homomorphism (called central charge) such that $Z \circ \text{cl}$ is a stability condition on $A$ as in Definition A.1. An object $E \in \mathcal{D}$ is called $\sigma$-(semi)stable if $E[k] \in A$ for some $k \in \mathbb{Z}$ and it is $Z$-(semi)stable in $A$.

A.3. The space of Bridgeland stability conditions. Let $\text{Stab}_\Gamma(\mathcal{D})$ be the set of Bridgeland stability conditions on $\mathcal{D}$ with respect to the group homomorphism $\text{(A.1)}$ satisfying the following condition (called support property):

$$\sup \left\{ \frac{\|\text{cl}(E)\|}{\|Z(E)\|} : E \text{ is } \sigma\text{-semistable} \right\} < \infty.$$ 

Here $\|*\|$ is a fixed norm on $\Gamma_\mathbb{R}$. The following is the main result in [Bri07]:

Theorem A.4. ([Bri07, Theorem 1.2]) The set $\text{Stab}_\Gamma(\mathcal{D})$ has a structure of a complex manifold such that the map

$$\text{Stab}_\Gamma(\mathcal{D}) \to \text{Hom}(\Gamma, \mathbb{C})$$

sending $(Z, A)$ to $Z$ is a local isomorphism.

Let $X$ be a smooth projective variety and take $\mathcal{D} = D^b(X)$. By setting $\Gamma$ to be the image of the Chern character map as in Remark A.2, we have the complex manifold

$$\text{Stab}(X) := \text{Stab}_{\Gamma_X}(D^b(X)).$$

Remark A.5. Note that we have the natural $\mathbb{C}^*$-action on $\text{Hom}(\Gamma, \mathbb{C})$ by the multiplication. This action lifts to a $\mathbb{C}$-action on $\text{Stab}_\Gamma(\mathcal{D})$ via the universal cover $\mathbb{C} \to \mathbb{C}^*$, which does not change semistable objects. See [Bri09, Section 3.3].
Appendix B. Other examples

In this section, we discuss some other examples of wall-crossing in CY 3-folds in terms of d-critical birational geometry. We just describe the results without details, since the arguments are similar to Theorem 1.1.

B.1. DT/PT correspondence. Let $X$ be a smooth projective CY 3-fold. For $(\beta, n) \in \Gamma_{\leq 1}$, let

\[ I_n(X, \beta) \]

be the moduli space of subschemes $C \subset X$ such that $[C] = \beta$ and $\chi(O_C) = n$. The moduli space (B.1) is identified with the moduli space of rank one torsion free sheaves $I$ with Chern character $(1, 0, -\beta, -n)$. In particular, it has a canonical d-critical structure.

The moduli spaces (B.1) and (9.1) are related by wall-crossing phenomena with respect to certain Bridgeland-type weak stability conditions in the derived category (see [Tod10a]). The above wall-crossing is relevant in showing the DT/PT correspondence conjecture [PT09] (see Subsection C.3). As in Section 9, we have the diagram

\[ \begin{array}{ccc}
I_n(X, \beta) & \xrightarrow{q} & P_n(X, \beta) \\
\downarrow q & & \downarrow q_P \\
T_n(X, \beta)
\end{array} \]

where $T_n(X, \beta)$ is an algebraic space which parametrizes objects of the form

\[ I_C \oplus \left( \bigoplus_{i=1}^{k} V_i \otimes O_{x_i}[-1] \right) \]

where $I_C$ is an ideal sheaf of a pure one dimensional subscheme $C \subset X$, and $x_i \neq x_j$ for $i \neq j$, satisfying

\[ \chi(O_C) + \sum_{i=1}^{k} \dim V_i = n. \]

We have the following:

**Theorem B.1.** The diagram (B.2) is an analytic d-critical generalized flip at any point in $\text{Im } q_P$, an analytic d-critical generalized MFS at any point in $T_n(X, \beta) \setminus \text{Im } q_P$. In particular, we have

\[ I_n(X, \beta) \geq_K P_n(X, \beta). \]

**Proof.** The proof is the same as in Theorem 9.13. \qed

Here is an example of DT/PT correspondence for a fixed smooth curve:

**Example B.2.** In the situation of Example 9.24, we have

\[ \text{Quot}(I_C, n + g - 1) \xrightarrow{\approx} I_n(X, \beta). \]
Here the LHS is the Quot scheme parameterizing quotients $I_C \to Q$ such that $Q$ is a zero dimensional sheaf with length $n + g - 1$, and the above isomorphism is given by taking the kernel of $I_C \to Q$. By setting $m = n + g - 1$, the diagram (B.2) is

$$
\begin{array}{ccc}
\text{Quot}(I_C, m) & \xrightarrow{q_I} & \text{Sm}(C) \\
\text{Sm}(X) & \xrightarrow{q_P} & \end{array}
$$

Here $q_I$ sends $I_C \to Q$ to the support of $Q$, and $q_P$ is induced by $C \subset X$. Note that $\text{Sm}(C)$ is always smooth while $\text{Quot}(I_C, m)$ can be singular. By Theorem [B.1], the above diagram is an analytic d-critical generalized flip at any point in $\text{Im} q_P$, an analytic d-critical generalized MFS at any point in $\text{Sm}(X) \setminus \text{Im} q_P$.

**B.2. Wall-crossing in local K3 surfaces.** Let $S$ be a smooth projective K3 surface over $\mathbb{C}$, and $\Gamma_S$ its Mukai lattice:

$$
\Gamma_S := \mathbb{Z} \oplus \text{NS}(S) \oplus \mathbb{Z}.
$$

Let $v(-)$ be the Mukai vector

(B.3) \quad v: K(S) \to \Gamma_S, \quad E \mapsto \text{ch}(E) \cdot \sqrt{\text{td}X}.

Then we have the space of Bridgeland stability conditions on $D^b(S)$ with respect to the group homomorphism (B.3), denoted by $\text{Stab}(S)$. The structure of the space $\text{Stab}(S)$ is studied in [Bri08], and moduli spaces of Bridgeland stable objects on $S$ are studied in [BM14b].

Let $X$ be the non-compact CY 3-fold defined by

$$
X := \text{Tot}_S(K_S) = S \times \mathbb{C}.
$$

We consider moduli spaces of semistable objects on the triangulated category

$$
D_c^b(X) := \{ E \in D^b(X) : \text{Supp}(E) \text{ is compact } \}.
$$

Let $p_S: X \to S$ be the projection. We have the group homomorphism

(B.4) \quad \text{cl}: K(D_c^b(X)) \to \Gamma_S, \quad E \mapsto v(p_S^*E)

Let $\text{Stab}_c(X)$ be the space of Bridgeland stability conditions on $D_c^b(X)$ with respect to the group homomorphism (B.4). Then we have the isomorphism (see [Tod09b Theorem 6.5, Lemma 5.3])

$$
p_S*: \text{Stab}_c(X) \xrightarrow{\cong} \text{Stab}(S)
$$

which is identity on central charges. Under the above isomorphism, an object $E \in D_c^b(X)$ is $\sigma$-(semi)stable for $\sigma \in \text{Stab}_c(X)$ if and only if $p_S^*E \in D^b(S)$ is $p_S*\sigma$-(semi)stable.
For $\sigma = (Z, A) \in \text{Stab}_c(X)$ and $v \in \Gamma_S$ with $\Im Z(v) > 0$, let $\mathcal{M}_\sigma(v)$ be the moduli stack of $\sigma$-semistable objects $E \in A$ such that $\text{cl}(E) = v$, and $M_\sigma(v) \to M_\sigma(v)$ be its good moduli space. Note that if $v$ is primitive and $\sigma$ is general, then $M_\sigma(v)$ consists of $\sigma$-stable objects of the form $i_c \ast F$ for $c \in \mathbb{C}$, where $F \in D^b(S)$ is $p_S, \sigma$-stable and $i_c : S \times \{c\} \hookrightarrow X$ is the inclusion. Therefore we have $M_\sigma(v) = M_{p_S, \sigma}(v) \times \mathbb{C}$ where $M_{p_S, \sigma}(v)$ is the moduli space of $p_S, \sigma$-stable objects in $D^b(S)$ with Mukai vector $v$. By [BM14b], the moduli space $M_{p_S, \sigma}(v)$ is a projective holomorphic symplectic manifold of dimension $2 + (v, v)$, where $(\cdot, \cdot)$ is the Mukai product on $\Gamma_S$.

Suppose that $v \in \Gamma_S$ is primitive and $\sigma \in \text{Stab}_c(X)$ lies on a wall with respect to $v$. If $\sigma^\pm \in \text{Stab}_c(X)$ lie on its adjacent chambers, we obtain the diagram

$$
\begin{array}{ccc}
M_{\sigma^+}(v) & \xrightarrow{q_M} & M_{\sigma^-}(v) \\
\downarrow & & \downarrow \\
M_\sigma(v) & \xleftarrow{q_M} & M_\sigma(v)
\end{array}
$$

(B.5)

By noting the Ext-quiver associated with any collection in $D^b(S)$ is symmetric, we have the following:

**Theorem B.3.** The diagram (B.5) is an analytic $d$-critical generalized flop.

**Appendix C. Wall-crossing formula of DT type invariants**

Here we review the previous works [Brill, Tod10a, Tod09a, Tod10b, Tod12a, Dia12] where wall-crossing phenomena in this paper were applied to show several properties on DT invariants.

**C.1. Product expansion formula of PT invariants.** For a CY 3-fold $X$ and an element $(\beta, n) \in \Gamma_{\leq 1}$, the moduli space of stable pairs $P_n(X, \beta)$ admits a zero dimensional virtual class $[\text{PT09}]$. The PT invariant $P_{n, \beta} \in \mathbb{Z}$ is defined by the integration of the virtual class. It also coincides with weighted Euler characteristics

$$
P_{n, \beta} = \int_{[P_n(X, \beta)]} \chi_B \; de := \sum_{m \in \mathbb{Z}} m \cdot e(\chi_B^{-1}(m)).
$$

Here $\chi_B$ is the Behrend constructible function [Beh09] on $P_n(X, \beta)$. The generating series of PT invariants satisfies the following product expansion
formula
\begin{equation}
1 + \sum_{\beta>0, n \in \mathbb{Z}} P_{n, \beta} q^n t^\beta
= \exp \left( \sum_{\beta>0, n>0} (-1)^{n-1} n N_{n, \beta} q^n t^\beta \right) \cdot \left( \sum_{\beta>0, n \in \mathbb{Z}} L_{n, \beta} q^n t^\beta \right).
\end{equation}

Here $N_{n, \beta}$ and $L_{n, \beta}$ are as follows:

1. The invariant $N_{n, \beta} \in \mathbb{Q}$ is the generalized DT invariant [JS12] counting one dimensional semistable sheaves on $X$ with Chern character $(\beta, n) \in \Gamma_{\leq 1}$. It satisfies the symmetric property $N_{n, \beta} = N_{-n, \beta}$ and the periodicity property $N_{n, \beta} = N_{n, \beta + \omega \cdot \beta}$ for any ample divisor $\omega$ on $X$.

2. The invariant $L_{n, \beta} \in \mathbb{Z}$ is a DT type invariant counting certain stable objects $E \in D^b(X)$ satisfying $\text{ch}(E) = (1, 0, -\beta, -n)$ (see (C.3) below). It satisfies the symmetric property $L_{n, \beta} = L_{-n, \beta}$ and the vanishing $L_{n, \beta} = 0$ for $|n| \gg 0$.

The formula (C.1) is proved using wall-crossing phenomena in Section 9, which led to the proof of the rationality conjecture of the generating series of PT invariants [MNOP06, PT09].

C.2. Wall-crossing formula of PT invariants. Below we recall how to derive the formula (C.1) from wall-crossing in Section 9. For $t \in \mathbb{R}$, let us consider the moduli stack $M^*_t(\beta, n)$ and its good moduli space $M^*_t(\beta, n)$ as in Subsection 9.3. Let $W \subset \mathbb{R}$ be the set of walls (9.12), and take $t \notin W$. By integrating the Behrend function on $M^*_t(\beta, n)$, we have the invariant
\begin{equation}
L_{n, \beta}^t := \int_{M^*_t(\beta, n)} \chi_B \, de \in \mathbb{Z}. \tag{C.2}
\end{equation}

By Proposition 9.7, we have the identities
\begin{equation}
L_{n, \beta}^t = \begin{cases} P_{n, \beta}, & t \gg 0, \\ P_{-n, \beta}, & t \ll 0. \end{cases} \tag{C.3}
\end{equation}

For $t_i \in W$, the wall-crossing formula of the invariants (C.2) is given by (see [Tod12a, Theorem 5.7])
\begin{equation}
\lim_{\varepsilon \to +0} \sum_{\beta>0, n \in \mathbb{Z}} L_{n, \beta}^{t_i + \varepsilon} q^n t^\beta
= \exp \left( \sum_{n/\omega \cdot \beta = t_i} (-1)^{n-1} n N_{n, \beta} q^n t^\beta \right) \cdot \left( \lim_{\varepsilon \to +0} \sum_{\beta>0, n \in \mathbb{Z}} L_{n, \beta}^{t_i - \varepsilon} q^n t^\beta \right).
\end{equation}

Applying the above formula from $t \to \infty$ to $t \to +0$, using the identity (9.9) and setting
\begin{equation}
L_{n, \beta}^t := L_{n, \beta}^t, \quad 0 < t \ll 1
\end{equation}
we obtain the formula (C.1).

### C.3. DT/PT Correspondence

Let $I_n(X, \beta)$ the moduli space defined in (B.1). The rank one DT invariants

$$I_{n, \beta} := \int I_n(X, \beta) \chi_B \, de$$

are related to PT invariants by the identity

$$\sum_{n \in \mathbb{Z}} I_{n, \beta} q^n = \prod_{n \geq 1} (1 - q^n)^{-n \cdot e(X)} \sum_{n \in \mathbb{Z}} P_{n, \beta} q^n.$$

The above formula, called DT/PT correspondence, was conjectured in [PT09] and proved in [Bri11, Tod10a], via wall-crossing phenomena in the derived category discussed in Subsection B.1.

### References

[ABCH13] Daniele Arcara, Aaron Bertram, Izzet Coskun, and Jack Huizenga, The minimal model program for the Hilbert scheme of points on $\mathbb{P}^2$ and Bridgeland stability, Adv. Math. 235 (2013), 580–626. MR 3010070

[AHLH] J. Alper, D. Halpern-Leistner, and J. Heinloth, to appear.

[AHR] J. Alper, J. Hall, and D. Rydh, A Luna étale slice theorem for algebraic stacks, preprint, arXiv:1504.06467.

[Alp13] Jarod Alper, Good moduli spaces for Artin stacks, Ann. Inst. Fourier (Grenoble) 63 (2013), no. 6, 2349–2402. MR 3237451

[BB] A. Bodzenta and A. Bondal, Flops and spherical functors, preprint, arXiv:1511.00665.

[BBBBBJ15] Oren Ben-Bassat, Christopher Brav, Vittoria Bussi, and Dominic Joyce, A ‘Darboux theorem’ for shifted symplectic structures on derived Artin stacks, with applications, Geom. Topol. 19 (2015), no. 3, 1287–1359. MR 3352237

[BCHM10] C. Birkar, P. Cascini, C. Hacon, and J. McKernan, Existence of minimal models for varieties of log general type, J. Amer. Math. Soc. 23 (2010), 405–468.

[Beh09] K. Behrend, Donaldson-Thomas type invariants via microlocal geometry, Ann. of Math 170 (2009), 1307–1338.

[BM14a] Arend Bayer and Emanuele Macrì, MMP for moduli of sheaves on $K3$s via wall-crossing: nef and movable cones, Lagrangian fibrations, Invent. Math. 198 (2014), no. 3, 505–590. MR 3279532

[BM14b] Arend Bayer and Emanuele Macrì, Projectivity and birational geometry of Bridgeland moduli spaces, J. Amer. Math. Soc. 27 (2014), no. 3, 707–752. MR 3194493

[BMS] A. Bayer, E. Macrì, and P. Stellari, The space of stability conditions on abelian threefolds, and on some Calabi-Yau threefolds, preprint, arXiv:1410.1585.

[BMS16] Arend Bayer, Emanuele Macrì, and Paolo Stellari, The space of stability conditions on abelian threefolds, and on some Calabi-Yau threefolds, Invent. Math. 206 (2016), no. 3, 869–933. MR 3573975

[BMT14] A. Bayer, E. Macrì, and Y. Toda, Bridgeland stability conditions on 3-folds I: Bogomolov-Gieseker type inequalities, J. Algebraic Geom. 23 (2014), 117–163.

[BO] A. Bondal and D. Orlov, Semiorthogonal decomposition for algebraic varieties, preprint, arXiv:9506012.

[Bri02] T. Bridgeland, Flops and derived categories, Invent. Math 147 (2002), 613–632.
[Bri07] YUKINOBU TODA, Stability conditions on triangulated categories, Ann. of Math 166 (2007), 317–345.
[Bri08] YUKINOBU TODA, Stability conditions on $K3$ surfaces, Duke Math. J. 141 (2008), 241–291.
[Bri09] YUKINOBU TODA, Spaces of stability conditions, Proc. Sympos. Pure Math. 80 (2009), 1–21, Algebraic Geometry-Seattle 2005.
[Bri11] YUKINOBU TODA, Hall algebras and curve-counting invariants, J. Amer. Math. Soc. 24 (2011), 969–998.

Dia12] D.-E. Diaconescu, Moduli of ADHM sheaves and the local Donaldson-Thomas theory, J. Geom. Phys. 62 (2012), no. 4, 763–799. MR 2888981

EG95] Geir Ellingsrud and Lothar Göttsche, Variation of moduli spaces and Donaldson invariants under change of polarization, J. Reine Angew. Math. 467 (1995), 1–49. MR 1355920

Eri10] E. Eriksen, Computing noncommutative deformations of presheaves and sheaves of modules, Canad. J. Math. 62 (2010), 520–542.

FQ95] Robert Friedman and Zhenbo Qin, Flips of moduli spaces and transition formulas for Donaldson polynomial invariants of rational surfaces, Comm. Anal. Geom. 3 (1995), no. 1-2, 11–83. MR 1362648

Fuk03] Kenji Fukaya, Deformation theory, homological algebra and mirror symmetry, Geometry and physics of branes (Como, 2001), Ser. High Energy Phys. Cosmol. Gravit., IOP, Bristol, 2003, pp. 121–209. MR 1950958

Gre15] Daniel Greb, Complex-analytic quotients of algebraic G-varieties, Math. Ann. 363 (2015), no. 1-2, 77–100. MR 3394374

[HL] D. Halpern-Leistner, The D-equivalence conjecture for moduli spaces of sheaves.

[HL97] D. Huybrechts and M. Lehn, Geometry of moduli spaces of sheaves, Aspects in Mathematics, vol. E31, Vieweg, 1997.

[HMP98] Peter Heinzner, Luca Migliorini, and Marzia Polito, Semistable quotients, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 26 (1998), no. 2, 233–248. MR 1631577

[HT10] D. Huybrechts and R. P. Thomas, Deformation-obstruction theory for complexes via Atiyah-Kodaira-Spencer classes, Math. Ann. (2010), 545–569.

Ina02] M. Inaba, Toward a definition of moduli of complexes of coherent sheaves on a projective scheme, J. Math. Kyoto Univ. 42-2 (2002), 317–329.

[Joy] D. Joyce, Shifted symplectic geometry, Calabi-Yau moduli spaces, and generalizations of Donaldson-Thomas theory: our current and future research, Talks given Oxford, October 2013, at a workshop for EPSRC Programme Grant research group, https://people.maths.ox.ac.uk/joyce/PHandout.pdf.

[Joy15] D. Joyce, A classical model for derived critical loci, J. Differential Geom. 101 (2015), 289–367.

[JS] Dominic Joyce and Pavel Safronov, A Lagrangian Neighbourhood Theorem for shifted symplectic derived schemes, preprint, arXiv:1506.04024.

[JS12] D. Joyce and Y. Song, A theory of generalized Donaldson-Thomas invariants, Mem. Amer. Math. Soc. 217 (2012).

[Kawa] Y. Kawamata, Birational geometry and derived categories, arXiv:1710.07370.

[Kawb] Y. Kawamata, On multi-pointed non-commutative deformations and Calabi-Yau threefolds, preprint, arXiv:1512.06170.

[Kaw02] Y. Kawamata, D-equivalence and K-equivalence, J. Differential Geom. 61 (2002), 147–171.

[Kaw08] Y. Kawamata, Flops connect minimal models, Publ. Res. Inst. Math. Sci. 44 (2008), 419–423.

[Kin94] A. King, Moduli of representations of finite-dimensional algebras, Quart. J. Math. Oxford Ser.(2) 45 (1994), 515–530.
[KM98] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, 1998.

[KS] M. Kontsevich and Y. Soibelman, *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations*, preprint, arXiv:0811.2435.

[Lau02] O. Laudal, *Noncommutative deformations of modules*, Homology Homotopy Appl 4 (2002), 357–396.

[LBP90] Lieven Le Bruyn and Claudio Procesi, *Semisimple representations of quivers*, Trans. Amer. Math. Soc. 317 (1990), no. 2, 585–598. MR 958897

[Lie06] M. Lieblich, *Moduli of complexes on a proper morphism*, J. Algebraic Geom. 15 (2006), 175–206.

[MP15] Antony Maciocia and Dulip Piyaratne, *Fourier-Mukai transforms and Bridgeland stability conditions on abelian threefolds*, Algebr. Geom. 2 (2015), no. 3, 270–297. MR 3370123

[MR] Sven Meinhardt and Markus Reineke, *Donaldson-Thomas invariants versus intersection cohomology of quiver moduli*, arXiv:1411.4062.

[MT] D. Maulik and Y. Toda, *Gopakumar-Vafa invariants via vanishing cycles*, to appear in Inventiones, arXiv:1610.07303.

[NN11] K. Nagao and H. Nakajima, *Counting invariant of perverse coherent sheaves and its wall-crossing*, Int. Math. Res. Not. (2011), 3855–3938.

[NY11] Hiraku Nakajima and Kôta Yoshioka, *Perverse coherent sheaves on blow-up. II. Wall-crossing and Betti numbers formula*, J. Algebraic Geom. 20 (2011), no. 1, 47–100. MR 2729275

[Pol01] A. Polishchuk, *Homological mirror symmetry with higher products*, Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds (Cambridge, MA, 1999), AMS/IP Stud. Adv. Math., vol. 23, Amer. Math. Soc., Providence, RI, 2001, pp. 247–259. MR 1876072

[PT] D. Piyaratne and Y. Toda, *Moduli of Bridgeland semistable objects on 3-folds and Donaldson-Thomas invariants*, to appear in Crelle, arXiv:1504.01177.

[PT09] R. Pandharipande and R. P. Thomas, *Curve counting via stable pairs in the derived category*, Invent. Math. 178 (2009), 407–447.

[PT10] , *Stable pairs and BPS invariants*, J. Amer. Math. Soc. 23 (2010), 267–297.

[PTVV13] T. Pantev, B. Toën, M. Vaquie, and G. Vezzosi, *Shifted symplectic structures*, Publ. Math. IHES 117 (2013), 271–328.

[Rei92] M. Reid, *What is a flip*, Colloquium talk, University of Utah.

[Rei08] Markus Reineke, *Moduli of representations of quivers*, Trends in representation theory of algebras and related topics, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2008, pp. 589–637. MR 2484736

[Sav] M. Savvas, *Generalized Donaldson-Thomas invariants via Kirwan blowup II*, preprint.

[Tho00] R. P. Thomas, *A holomorphic Casson invariant for Calabi-Yau 3-folds and bundles on K3-fibrations*, J. Differential. Geom. 54 (2000), 367–438.

[Toda] Y. Toda, *Gopakumar-Vafa invariants and wall-crossing*, arXiv:1710.01843.

[Todc] , *Semiorthogonal decompositions of stable pair moduli spaces via d-critical flips*, preprint.
[Tod08] YUKINOBU TODA, Birational Calabi-Yau 3-folds and BPS state counting, Communications in Number Theory and Physics 2 (2008), 63–112.
[Tod09a] YUKINOBU TODA, Limit stable objects on Calabi-Yau 3-folds, Duke Math. J. 149 (2009), 157–208.
[Tod09b] YUKINOBU TODA, Stability conditions and Calabi-Yau fibrations, J. Algebraic Geom. 18 (2009), 101–133.
[Tod10a] YUKINOBU TODA, Curve counting theories via stable objects I: DT/PT correspondence, J. Amer. Math. Soc. 23 (2010), 1119–1157.
[Tod10b] YUKINOBU TODA, Generating functions of stable pair invariants via wall-crossings in derived categories, Adv. Stud. Pure Math. 59 (2010), 389–434. New developments in algebraic geometry, integrable systems and mirror symmetry (RIMS, Kyoto, 2008).
[Tod12a] YUKINOBU TODA, Stability conditions and curve counting invariants on Calabi-Yau 3-folds, Kyoto Journal of Mathematics 52 (2012), 1–50.
[Tod12b] YUKINOBU TODA, Stable pairs on local K3 surfaces, J. Differential. Geom. 92 (2012), 285–370.
[Tod14] YUKINOBU TODA, Stability conditions and birational geometry of projective surfaces, Compos. Math. 150 (2014), 1755–1788.
[Toe14] Bertrand Toen, Derived algebraic geometry and deformation quantization, Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II, Kyung Moon Sa, Seoul, 2014, pp. 769–792. MR 3728637
[Tu] J. Tu, Homotopy L-infinity spaces, arXiv:1411.5115.
[Vak06] Ravi Vakil, Murphy’s law in algebraic geometry: badly-behaved deformation spaces, Invent. Math. 164 (2006), no. 3, 569–590. MR 2227692

Kavli Institute for the Physics and Mathematics of the Universe, University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa, 277-8583, Japan.
E-mail address: yukinobu.toda@ipmu.jp