Nonlinear cosmological power spectra in Einstein’s gravity

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Is Newton’s gravity sufficient to handle the weakly nonlinear evolution stages of the cosmic large-scale structures? Here we resolve the issue by analytically deriving the density and velocity power spectra to the second order in the context of Einstein’s gravity. The recently found pure general relativistic corrections appearing in the third-order perturbation contribute to power spectra to the second order. In this work the complete density and velocity power spectra to the second order are derived. The power transfers among different scales in the density power spectrum are estimated in the context of Einstein’s gravity. The relativistic corrections in the density power spectrum are estimated to be smaller than the Newtonian one to the second order, but these could be larger than higher-order nonlinear Newtonian terms.

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I. INTRODUCTION

The weakly nonlinear process of gravitating system has fundamental importance in cosmology [1, 2, 3, 4]. The large-scale structure in the observable universe apparently show its nonlinear nature. Considering the success of the Friedmann cosmology with its spatial homogeneity and isotropy assumptions, it is likely that in near horizon scale the structures are in near linear stage. In the small scale, however, the cosmic structure (say, distribution and motion of galaxies) are in fully nonlinear stage. In between the two scales, we have weakly nonlinear stage. Even for the nonlinear structures in the present epoch, as the cosmological structure grows under gravity from linear to nonlinear stages there must be transition era which can be regarded as weakly nonlinear. The density and velocity power spectra provide the main observations of the large-scale structure which can be directly compared with theories of the structure formation. All theoretical studies of the weakly nonlinear evolution of the large-scale structure have been based on Newton’s gravity [1, 2, 3, 4]. Is Newton’s gravity sufficient to handle the situation? In this work we will resolve the issue by analytically deriving the power spectra to the second order in the context of Einstein’s gravity.

We will derive the second-order density and velocity power spectra in the context of relativistic cosmology for the first time. In a zero-pressure medium the pure general relativistic corrections appear in the third order; this was only recently shown by us in [3]. The third-order relativistic corrections, however, contribute to the density and velocity power spectra even to the second order which was unknown previously. We show that compared with the Newtonian contributions to the second-order power spectra the relativistic contributions are generically multiplied by a factor (scale/horizon scale) squared, thus suppressed in the small scale. In the Newtonian theory, in the second-order density power spectrum, the nonlinear power transfer from large-scale nonlinearity is known to exactly cancel to the leading order, thus opening a possibility of relativistic effect becoming important. In this paper we will show that, even in such a situation, the pure general relativistic contribution to the density power spectrum is smaller than the remaining second-order Newtonian ones. We conclude that, even in the context of Einstein’s gravity, up to the second order, the \( k^4 \) long wavelength tail in the density power spectrum previously known in the Newtonian analysis is the only important effect of the nonlinear power transfer among different scales. However, the leading order relativistic contribution in the second-order power spectra could be larger than the higher (third and higher) order pure Newtonian contributions to the power spectra [4], thus demanding a caution in the pure Newtonian study.

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II. BASIC EQUATIONS

We consider an irrotational dust (zero-pressure fluid) without the gravitational waves in a flat Friedmann background. To the third order the basic perturbation equations in Einstein’s gravity are recently derived in [5]. These are

\[\dot{\delta} + \frac{1}{a} \nabla \cdot \mathbf{u} = -\frac{1}{a} \nabla \cdot (\delta \mathbf{u}) + \frac{1}{a} [2 \varphi \mathbf{u} - \nabla (\Delta^{-1} X)] \cdot \nabla \delta,\]  

(1)

\[\frac{1}{a} \nabla \cdot \left( \dot{\mathbf{u}} + \frac{\dot{a}}{a} \mathbf{u} \right) + 4 \pi G \varrho \delta = -\frac{1}{a^2} \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) - \frac{2}{3a^2} \varphi \mathbf{u} \cdot \nabla (\dot{\mathbf{u}} + 3 \Delta^{-1} \nabla \cdot \mathbf{u} + \mathbf{u} \Delta \varphi),\]  

(2)

\[X \equiv 2 \varphi \nabla \cdot \mathbf{u} - \mathbf{u} \cdot \nabla \varphi + \frac{3}{2} \Delta^{-1} \nabla \cdot [\mathbf{u} \cdot \nabla (\nabla \varphi) + \mathbf{u} \Delta \varphi],\]  

(3)

where \(\delta\) and \(\mathbf{u}\) are the density perturbation \((\delta \equiv \delta \varrho / \varrho)\) and the perturbed velocity, respectively; \(a\) is the cosmic scale factor and \(\varrho\) is the background density. A perturbed order variable \(\varphi\) is a metric perturbation variable related to the perturbed three-space curvature. To the linear order \(\varphi\) can be related to \(\delta\) and \(\mathbf{u}\) as [5]

\[\varphi = \frac{1}{c^2} (-\delta \Phi + \dot{\Delta}^{-1} \nabla \cdot \mathbf{u}),\]  

(4)

where \(\delta \Phi\) is the perturbed order Newtonian gravitational potential; \(\delta \Phi\) is related to \(\delta\) by Poisson’s equation

\[\Delta \Phi = 4 \pi G \varrho a^2 \delta.\]  

(5)

Up to the second order, remarkably, Eqs. (1) and (2) coincide exactly with the ones in Newtonian theory; we call it a relativistic/Newtonian correspondence to the second order [6]. This is why we simply call \(\delta\) and \(\mathbf{u}\) as the (Newtonian) density and velocity perturbations even in the present relativistic situation; in the relativistic context \(\delta\) and \(\varphi\) are related to the certain gauge-invariant combinations of variables, see [5, 6]. It is also remarkable to notice that all the pure relativistic third-order correction terms have \(\varphi\) factor compared with the relativistic/Newtonian second-order terms. Contribution of the gravitational waves to the third order can be found in [5]; for equations in the multi-component case, see the third reference in [5]. We note that as the above equations are derived in the relativistic perturbation theory, these are valid in the fully general relativistic situation and in all scales (including the super-horizon scale) as long as the perturbative assumption is met.

III. SOLUTIONS IN THE PHASE SPACE

As we are considering a flat background, we may take the Fourier transformation defined as \(F(k) = \int d^3x F(x) e^{ik \cdot x}\). By introducing \(\mathbf{u} \equiv \nabla \mathbf{u}\) and \(\theta \equiv \nabla \cdot \mathbf{u} = \Delta \mathbf{u}\), we have \(\mathbf{u}(k, t) = -ik \mathbf{u}(k, t)\) and \(\theta(k, t) = -k^2 u(k, t)\). In the phase
space, Eqs. (11) - (3) become

$$
\dot{\delta}(k, t) + \frac{1}{a} \theta(k, t) = -\frac{1}{a} \int d^3k' \frac{1}{2} \left[ \delta(k', t) \theta(k - k', t) \frac{k \cdot (k - k')}{|k - k'|^2} + C^+ \right] \\
+ \frac{2}{a} \int d^3k' \int d^3k'' \frac{1}{3} \left[ \delta(k' - k'', t) \theta(k'', t) \frac{(k - k' - k'') \cdot k''}{|k''|^2} + C^{++} \right] \\
- \frac{1}{a} \int d^3k' \frac{1}{2} \left[ \frac{(k - k') \cdot k'}{|k'|^2} X(k', t) \delta(k - k', t) + C^+ \right],
$$

(6)

$$
\frac{1}{a} \left[ \dot{\theta}(k, t) + \frac{\dot{a}}{a} \theta(k, t) \right] + 4\pi G \delta(k, t) = -\frac{1}{a^2} \int d^3k' \theta(k', t) \theta(k - k', t) \frac{k^2}{|k - k'|^2} e^{-i (k - k') \cdot t} \\
- \frac{1}{a^2} \int d^3k' \int d^3k'' \frac{1}{3} \left[ \varphi(k', t) \theta(k'', t) \theta(k - k' - k'', t) \\
\times \left( \frac{2}{3} \frac{(3k - k' - k'') \cdot k''}{|k''|^2} - 4 \frac{(k - k' - k'') \cdot k'' k \cdot (k - k' - k'')}{|k - k' - k''|^2} \right) + C^{++} \right] \\
+ \frac{1}{a^2} \int d^3k' \frac{1}{2} \left[ \frac{(k - k') \cdot k'}{|k'|^2} \left( 1 - \frac{k^2}{|k - k'|^2} \right) \theta(k', t) X(k - k', t) \right. \\
\left. + \frac{2}{3} \frac{\dot{\theta}(k - k', t) X(k', t) + C^+}{k^2} \right],
$$

(7)

$$
X(k, t) = \frac{1}{a^2} \int d^3k' \left[ \theta(k', t) \varphi(k - k', t) \left[ 1 - \frac{1}{2} \frac{1}{k^2} \left( 1 - \frac{3}{2} \frac{(k - k') \cdot k'}{k^2} \right) \frac{3}{4} \frac{k' \cdot (k - k')}{|k - k'|^2} \right] + C^+ \right],
$$

(8)

where $C^+$ indicates terms replacing $k'$ to $k - k'$; $C^{++}$ indicates two sets of terms, one replacing $k'$ to $k - k' - k''$, and the other replacing $k''$ to $k - k' - k''$.

In order to derive the perturbative solutions we expand

$$
\delta(k, t) = \delta_1(k, t) + \delta_2(k, t) + \delta_3(k, t) + \ldots,
$$

(9)

and similarly expand $\theta$ and $\varphi$. To the linear order, Eqs. (11) and (7) give $\ddot{\delta}_1 + 2(\dot{a}/a)\dot{\delta}_1 - 4\pi G \delta_1 = 0$. Equations up to this point are valid in the presence of the cosmological constant. In the following, in order to derive analytic solutions we assume an absence of the cosmological constant. In a flat background without the cosmological constant, we have $a \propto t^{2/3}$, $6\pi G \delta = t^{-2}$, thus we have two solutions $\delta_1(k, t) \propto t^{2/3}$ and $t^{-1}$. We ignore the decaying solution in an expanding phase and set

$$
\delta_1(k, t) \equiv A(k)e^{i\phi(k)}t^{2/3}, \quad \theta_1(k, t) = -\frac{2}{3} A(k)e^{i\phi(k)}at^{-1/3}, \quad \varphi_1(k, t) = \frac{5}{2} \frac{\ell}{\ell_H} t^{2/3} A(k)e^{i\phi(k)}.
$$

(10)

In a flat background without the cosmological constant, to the linear order, from Eqs. (11) and (7) we have $\varphi = -(5/3)\dot{\phi}/c^2$. We introduced a scale $\ell \equiv a/k$ and the Hubble horizon scale $\ell_H \equiv c/(\dot{a}/a)$; thus, $\ell/\ell_H \equiv \dot{a}/kc$. Notice that $\varphi_1$ is time independent. To the second order from Eqs. (9) and (7) we have the solutions

$$
\delta_2(k, t) = \frac{1}{14} \frac{t^{4/3}}{a^{1/3}} \int d^3k' A(k') A(k - k')e^{i\phi(k')}J(k, k', k - k'),
$$

(11)

$$
\theta_2(k, t) = -\frac{1}{21} \frac{\ell}{\ell_H} \frac{1}{a^{1/3}} \int d^3k' A(k') A(k - k')e^{i\phi(k')}L(k, k', k - k'),
$$

(12)

where

$$
J(k, k', k - k') \equiv \left[ 4F(k, k', k - k') + 5H(k, k') + 5H(k, k - k') \right],
$$

$$
L(k, k', k - k') \equiv \left[ 8F(k, k', k - k') + 3H(k, k') + 3H(k, k - k') \right],
$$

$$
H(k, k') \equiv \frac{k \cdot k'}{k^2}, \quad F(k, k', k - k') \equiv \frac{1}{2} \frac{k'}{k^2} \frac{k \cdot (k - k')}{|k - k'|^2}.
$$

(13)

In order to derive third-order solutions we need $X$ to the second order. Using Eq. (11), Eq. (8) becomes

$$
X_2(k, t) = -\frac{20}{27} \frac{1}{c^2 k^2} \frac{1}{(2\pi)^3} \int d^3k' A(k') A(k - k')e^{i\phi(k')}M(k, k', k - k') \alpha^3 t^{-5/3},
$$

(14)

$$
M(k, k', k - k') \equiv \frac{k^2}{k^2} + \frac{1}{4} \frac{k \cdot k'}{k^2} + \frac{3}{4} \frac{k \cdot (k - k')}{|k - k'|^2} - \frac{1}{4} \frac{k^2}{k^2} \frac{k' \cdot (k - k')}{|k - k'|^2}.
$$

(15)
To the third order from Eqs. (16) and (7), using Eqs. (10)–(15), we have the solutions

\[
\delta_3(k, t) = \frac{1}{4} \frac{t^2}{(2\pi)^6} \int d^3 k' \int d^3 k'' A(k'')e^{i\phi(k'')} \left\{ A(k') A(k - k' - k'')e^{i\phi(k') + i\phi(k - k' - k'')} \right. \\
\times \left[ \frac{2}{5} F(k', k - k') L(k - k', k'; k - k') \right. \\
+ \frac{1}{18} H(k, k') J(k - k', k''; k - k' - k'') + \frac{1}{18} H(k, k - k') L(k - k', k''; k - k' - k'') \left. \right] + C^+ \right\} \\
+ \frac{5}{21} \left( \ell / \ell_H \right)^2 t^2 \frac{1}{(2\pi)^6} \int d^3 k' \int d^3 k''' A(k''')e^{i\phi(k''')} \left\{ A(k') A(k - k' - k'''e^{i\phi(k') + i\phi(k - k' - k'''}) \right. \\
\times \left[ \frac{k^2}{k'^2} \left( -3 \frac{1}{k'^2} - 3 \frac{1}{k''^2} + \frac{3}{k'^2} \right) + \frac{15 k' \cdot (k - k')}{4 |k - k'|^2} + \frac{3 k'^2 \cdot (k - k')}{2 |k - k'|^2} \left. \right] + C^+ \right\},
\]

\[
\theta_3(k, t) = -\frac{1}{63} \frac{at}{(2\pi)^6} \int d^3 k' \int d^3 k''' A(k''')e^{i\phi(k''')} \left\{ A(k') A(k - k' - k'''e^{i\phi(k') + i\phi(k - k' - k'''}) \right. \\
\times \left[ F(k', k - k') L(k - k', k''; k - k') \right. \\
+ \frac{1}{4} H(k, k') J(k - k', k''; k - k' - k'') + \frac{1}{4} H(k, k - k') L(k - k', k''; k - k' - k'') \left. \right] + C^+ \right\} \\
+ \frac{5}{21} \left( \ell / \ell_H \right)^2 \frac{a t}{(2\pi)^6} \int d^3 k' \int d^3 k''' A(k''')e^{i\phi(k''')} \left\{ A(k') A(k - k' - k'''e^{i\phi(k') + i\phi(k - k' - k'''}) \right. \\
\times \left[ \frac{k^2}{k'^2} \left( -\frac{2}{3} \frac{1}{k'^2} - \frac{16}{3} \frac{1}{k''^2} + \frac{16}{3} \frac{1}{k'^2} \right) + \frac{15 k' \cdot (k - k')}{4 |k - k'|^2} + \frac{3 k'^2 \cdot (k - k')}{2 |k - k'|^2} \left. \right] + C^+ \right\}.
\]

We note that the pure general relativistic contributions first appearing in the third order are generally multiplied by a \((\ell / \ell_H)^2\) factor which came from \(\phi\) terms in Eqs. (11)–(13), see \(\phi\) in Eq. (10); \((\ell / \ell_H)^2\) is small in the small scale but becomes of order unity in near horizon scale. The Newtonian part of \(\delta_3\) is proportional to \(a^2\), and the relativistic part is proportional to \(a^2\). The Newtonian part of \(\theta_3\) is proportional to \(at\), and the relativistic part is proportional to \(at^{1/3}\).

### IV. POWER SPECTRA

The density power spectrum is

\[
|\delta(k, t)|^2 = |\delta_1(k, t)|^2 + 2Re [\delta_1^*(k, t)\delta_2(k, t)] + |\delta_2(k, t)|^2 + 2Re [\delta_1^*(k, t)\delta_3(k, t)] + \ldots,
\]

where \(\mathcal{R}\) indicates the real part. The velocity power spectrum \(|\theta(k, t)|^2\) can be similarly expanded. Assuming the random phase, the second term in the right-hand-side vanishes. The second-order power spectra of density and
velocity follow from Eqs. (10)-(17). These are

\[ |\delta(k, t)|^2 = |\delta_1(k, t)|^2 + \frac{1}{(2\pi)^3} \int d^3k' \left\{ \frac{2}{144} |\delta_1(k', t)|^2 |\delta_1(k-k', t)|^2 J^2(k, k', k-k') \right. \\
+ |\delta_1(k, t)|^2 \left[ |\delta_1(k', t)|^2 \left( \frac{2}{63} F(k, k', k-k') L(k-k', k, -k') + \frac{1}{18} H(k, k-k') L(k-k', k, -k') \right) \right. \\
+ \frac{1}{18} H(k, k') J(k-k', k, -k') + \frac{1}{18} H(k, k-k') L(k-k', k, -k') \right] \right\} \\
+ \frac{10}{21} \left( \frac{\ell}{\ell_H} \right)^2 |\delta_1(k', t)|^2 \left( \frac{1}{(2\pi)^3} \int d^3k' \left\{ \frac{2}{144} |\delta_1(k', t)|^2 |\delta_1(k-k', t)|^2 J^2(k, k', k-k') \right. \\
+ |\delta_1(k, t)|^2 \left[ |\delta_1(k', t)|^2 \left( \frac{2}{63} F(k, k', k-k') L(k-k', k, -k') + \frac{1}{18} H(k, k-k') L(k-k', k, -k') \right) \right. \\
+ \frac{1}{18} H(k, k') J(k-k', k, -k') + \frac{1}{18} H(k, k-k') L(k-k', k, -k') \right] \right\} \\
+ \left[ |\delta_1(k', t)|^2 M(k-k', k, -k') \right] \frac{k^2}{|k-k'|^2} \left( 1 - \frac{9 k^2}{k^2} + \frac{15 k' \cdot (k-k')}{|k-k'|^2} + \frac{3 k^2}{k^2} \frac{k' \cdot (k-k')}{|k-k'|^2} + \frac{C^+}{C+} \right) \right\}. \tag{19} \]

\[ |\theta(k, t)|^2 = \frac{4}{9} \left( a^2/t^2 \right) |\delta_1(k, t)|^2 + \frac{1}{(2\pi)^3} \int d^3k' \left\{ \frac{2}{21} |\delta_1(k', t)|^2 |\delta_1(k-k', t)|^2 L^2(k, k', k-k') \right. \\
+ \frac{2}{189} |\delta_1(k, t)|^2 \left[ |\delta_1(k', t)|^2 \left( 4 F(k, k', k-k') L(k-k', k, -k') \right) \right. \\
+ H(k, k') J(k-k', k, -k') + H(k, k-k') L(k-k', k, -k') \right] \right\} \\
+ \frac{20}{63} \left( \frac{\ell}{\ell_H} \right)^2 \left( a^2/t^2 \right) |\delta_1(k, t)|^2 \left( \frac{2}{(2\pi)^3} \int d^3k' \left\{ \frac{2}{3} |\delta_1(k', t)|^2 \left( \frac{5}{3} + \frac{8 k^2}{k^2} \right) \right. \\
+ \frac{7 k \cdot k'}{k^2} - \frac{24 (k \cdot k')^2}{k^4} + \frac{14 k^2}{k^2} \right) \\
+ \left[ |\delta_1(k', t)|^2 M(k-k', k, -k') \right] \frac{k^2}{|k-k'|^2} \left( \frac{4}{3} - 4 k^2 \frac{k'}{k^2} + 3 k' \cdot (k-k') \frac{|k-k'|^2}{|k-k'|^2} + k^2 \frac{k' \cdot (k-k')}{|k-k'|^2} + \frac{C^+}{C+} \right) \right\}. \tag{20} \]

Notice that, in the second-order power spectrum, compared with the Newtonian contributions the pure general relativistic effects are simply multiplied by a \((\ell/\ell_H)^2\) factor. Newtonian part of the density spectrum is proportional to \(a^4\) and the relativistic part is proportional to \(a^3\). In the case of the velocity power spectrum, Newtonian part is proportional to \(a^3\) and the relativistic part is proportional to \(a^2\). Newtonian parts of the power spectra are known in the literature \[2, 4\]. The pure relativistic contributions to the second-order power spectra are our new contribution in this work.

(i) For the power transfer from the small-scale, thus \(k' \to \infty\), we have (in the following we assume isotropic power spectrum, thus \(|\delta(k, t)| = |\delta_1(k, t)|\), etc)

\[ |\delta(k, t)|^2 \approx |\delta_1(k, t)|^2 + \frac{k^4}{(2\pi)^2} \int k^4 dk' |\delta_1(k', t)|^2 - \frac{1}{21 \pi^2} k^2 |\delta_1(k, t)|^2 \int dk' |\delta_1(k', t)|^2. \tag{21} \]

Using \(\delta_k^2 \approx |\delta_1(k, t)|^2 k^3\), with \(\delta_k\) a density contrast at a given wavenumber \(k\), and introducing a small-scale cut-off frequency \(k_c\) we have

\[ |\delta(k, t)|^2 \approx |\delta_1(k, t)|^2 + c_1 (k/k_c)^4 \delta_{k_c}^4 + c_2 |\delta_1(k, t)|^2(k/k_c)^2 \delta_{k_c}^2 + c_3 |\delta_1(k, t)|^2 (\ell/\ell_H)^2 \delta_{k_c}^2, \tag{22} \]

where \(c_i\) are constants of order unity. The \(c_1\) term is the well known \(k^4\) long wavelength tail generated by a power transfer from the small-scale nonlinearity; this was first shown by Zel’dovich \[1, 2, 3\]. The Newtonian and relativistic contributions in \(2 R (\delta_1^2 \delta_3)\) are smaller than the linear term by factors \((k/k_c)^2 \delta_{k_c}^2\) and \((\ell/\ell_H)^2 \delta_{k_c}^2\), respectively; we have \(k/k_c = k_c/\ell \ll 1\) with \(\ell_c \equiv a/k_c\).

(ii) For the power transfer from the large-scale, thus \(k' \to 0\), we have

\[ |\delta(k, t)|^2 \approx |\delta_1(k, t)|^2 + \left( \frac{\ell}{\ell_H} \right)^2 \frac{5}{21 \pi^2} k^2 |\delta_1(k, t)|^2 \int dk' |\delta_1(k', t)|^2. \tag{23} \]
To the lowest order in \( k' \rightarrow 0 \), the Newtonian contributions, which have order \( k^2|\delta_1(k,t)|^2 \int dk'|\delta_1(k',t)|^2 \), exactly cancel out to the second order; this was shown by Vishniac \[3,4\]. The next order terms in that limit have order \( |\delta_1(k,t)|^2 \int k'^2dk'|\delta_1(k',t)|^2 \). Thus, using a large-scale cut-off frequency \( k_c \) we have

\[
|\delta(k,t)|^2 \sim |\delta_1(k,t)|^2 + c_4|\delta_1(k,t)|^2\delta^2_{k_c} + c_5|\delta_1(k,t)|^2 (\ell/\ell_H)^2(k/k_c)^2\delta^2_{k_c}.
\]  

(24)

Thus, the second-order terms, both Newtonian and relativistic, are smaller than the linear ones by factors \( \delta^2_{k_c} \) and \( (\ell/\ell_H)^2(k/k_c)^2\delta^2_{k_c} \), respectively. Compared with the Newtonian second-order term the relativistic one has

\[
\sim (\ell/\ell_H)^2(k/k_c)^2 \sim (\ell_c/\ell_H)^2
\]

which is smaller than unity but has no scale dependence.

V. DISCUSSION

Our results show that even to the second order the power spectra receive corrections from pure relativistic third-order perturbations. Compared with the Newtonian terms, the relativistic contributions are multiplied by a factor \( (\ell/\ell_H)^2 \), thus generally suppressed in the small-scale limit, but comparable in near horizon scale. In near horizon scale, however, as the perturbations are supposed to be in near linear stage, the second-order contributions are negligible compared with the linear-order terms. In the case of the power transfer from the large-scale nonlinearity it was previously known that the leading order Newtonian nonlinear contribution in the density power spectrum cancels exactly. Even in that situation our investigation shows that the relativistic effect is subdominant compared with the remaining Newtonian nonlinear effect. Our analysis of the asymptotic cases shows that the \( k^4 \) long wavelength tail of density power spectrum generated by the small-scale nonlinearity in the Newtonian theory is the only important effect of the nonlinear power transfer even in Einstein’s gravity.

Resolution of the issue of whether the relativistic contributions are always smaller than the Newtonian nonlinear effects requires quantitative estimation of the general power spectra presented in Eqs. (19) and (20). This may depend on the specific form of linear density power spectrum, and may require numerical integration of Eqs. (19) and (20). As the higher perturbational order contribution to the power spectrum is in general suppressed by power of \( \delta^2_{k} \) term \[4\], it is likely that such Newtonian contributions are smaller than the relativistic contribution to the second order; quantitative estimation may depend on the linear spectrum and the scale. Although the Newtonian perturbation theory has a recursion formulae to all orders \[4\], the relativistic situation should be handled at each order separately. Estimation of non-Gaussian contribution of the pure relativistic corrections is also an important and interesting issue to be addressed: our Eqs. (6)-(8) or Eqs. (10)-(17) provide the starting point for evaluating the bispectrum or the higher order correlation functions. In the presence of the cosmological constant we have to go back to Eqs. (6)-(8) which are valid in the presence of the cosmological constant; for a general expression of \( \varphi \) in terms of \( \delta \) and \( \theta \) to the linear order, see Eq. (4). Our perturbation equations are fully relativistic while assuming perturbations to be weakly nonlinear. In the small scale where structures are in fully nonlinear stage while the relativistic effects are small, the cosmological post-Newtonian approximation provides a complementary theoretical framework to handle the structure formation \[7\]. Density power spectrum based on the cosmological post-Newtonian equations has not been studied in the literature. Investigation of these issues are left for future studies.

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