The third logarithmic coefficient for the class $S$

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Abstract: In this paper we give an upper bound of the third logarithmic coefficient for the class $S$ of univalent functions in the unit disc.

Key words: Univalent, third logarithmic coefficient

1. Introduction

Let $A$ be the class of functions $f$ that are analytic in the open unit disc $D = \{ z : |z| < 1 \}$ of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots,$$

(1.1)

and let $S$ be its subclass consisting of functions that are univalent in the unit disc $D$.

The logarithmic coefficients of the function $f$ given by (1.1) are defined in $D$ by

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n.$$ 

(1.2)

By using (1.1), after differentiation and comparing the coefficients, we can obtain that $\gamma_1 = \frac{1}{2} a_2$, $\gamma_2 = \frac{1}{2} (a_3 - \frac{1}{2} a_2^2)$ and

$$\gamma_3 = \frac{1}{2} \left( a_4 - a_2 a_3 + \frac{1}{3} a_3^2 \right).$$

(1.3)

Very little is known about the estimates of the modulus of the logarithmic coefficients for the whole class $S$ of normalized of univalent functions. The Koebe function $k(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n$ with $\gamma_n = \frac{1}{n}$ being extremal in majority estimates over the class $S$ inspires a conjecture that $|\gamma_n| \leq \frac{1}{n}$ for $n = 1, 2, \ldots$ and $f \in S$. Apparently, this is true only for the class of starlike functions ([8]), but not for the class $S$ in general ([5, Theorem 8.4, p.242]). Sharp estimates for the class $S$ are known only for the first two coefficients, $|\gamma_1| \leq 1$ and $|\gamma_2| \leq \frac{1}{2} + \frac{1}{e}$.

In this paper we give an upper bound of $|\gamma_3|$ for the class $S$.

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It is worth mentioning that the problem of estimating the modulus of the first three logarithmic coefficients is widely studied for the subclasses of $S$ and in some cases sharp bounds are obtained. Namely, sharp estimates for the class of strongly starlike functions of certain order and $\gamma$-starlike functions are given in [8] and [3], respectively, while nonsharp estimates for the class of Bazilevic, close-to-convex and different subclasses of close-to-convex functions are given in [4], [1] and [7], respectively.

2. Main result
As announced before, here is an estimate of the modulus of the third logarithmic coefficient for the whole class of univalent functions.

**Theorem 2.1** For the class $S$ we have

$$|\gamma_3| \leq \frac{\sqrt{133}}{15} = 0.7688\ldots$$

**Proof** In the proof of this theorem we will use mainly the notations and results given in the book of N. A. Lebedev ([6]).

Let $f \in S$ and let

$$\log \frac{f(t) - f(z)}{t - z} = \sum_{p,q=0}^{\infty} \omega_{p,q} t^p z^q,$$

where $\omega_{p,q}$ are called Grunsky’s coefficients with property $\omega_{p,q} = \omega_{q,p}$. For those coefficients we have the next Grunsky’s inequality ([5, 6]):

$$\sum_{q=1}^{\infty} q \left| \sum_{p=1}^{\infty} \omega_{p,q} x_p \right|^2 \leq \sum_{p=1}^{\infty} \frac{|x_p|^2}{p}, \quad (2.1)$$

where $x_p$ are arbitrary complex numbers such that last series converges.

Further, it is well-known that if $f$ given by (1.1) belongs to $S$, then also

$$f_2(z) = \sqrt{f(z^2)} = z + c_3 z^3 + c_5 z^5 + \cdots \quad (2.2)$$

belongs to the class $S$. Then for the function $f_2$ we have the appropriate Grunsky’s coefficients of the form $\omega_{2p-1,2q-1}^{(2)}$ and the inequality (2.1) has the form

$$\sum_{q=1}^{\infty} (2q - 1) \left| \sum_{p=1}^{\infty} \omega_{2p-1,2q-1}^{(2)} x_{2p-1} \right|^2 \leq \sum_{p=1}^{\infty} \frac{|x_{2p-1}|^2}{2p - 1}. \quad (2.3)$$

As it has been shown in [6, p.57], if $f$ is given by (1.1) then the coefficients $a_2, a_3, a_4$ are expressed by Grunsky’s coefficients $\omega_{2p-1,2q-1}^{(2)}$ of the function $f_2$ given by (2.2) in the following way (in the next text we omit upper index 2 in $\omega_{2p-1,2q-1}^{(2)}$):

$$a_2 = 2\omega_{11},$$

$$a_3 = 2\omega_{13} + 3\omega_{11}^2,$$

$$a_4 = 2\omega_{33} + 8\omega_{11}\omega_{13} + \frac{10}{3}\omega_{11}^3. \quad (2.4)$$

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Now, from (1.3) and (2.3) we have

\[
\gamma_3 = \omega_{33} + 2\omega_{11}\omega_{13}
\]

On the other hand, from (2.4) for \( x_{2p-1} = 0, p = 3, 4, \ldots \) we have

\[
|\omega_{11}x_1 + \omega_{31}x_3|^2 + 3|\omega_{13}x_1 + \omega_{33}x_3|^2 \leq |x_1|^2 + \frac{|x_3|^2}{3}.
\]  

(2.5)

From (2.5) for \( x_1 = 2\omega_{11}, x_3 = 1 \) and since \( \omega_{31} = \omega_{13} \), we have

\[
|2\omega_{11}^2 + |\omega_{13}|^2 + 3|\gamma_3|^2 \leq 4|\omega_{11}|^2 + \frac{1}{3},
\]

and from here

\[
|\gamma_3|^2 \leq \frac{1}{9} + \frac{4}{3}|\omega_{11}|^2 - \frac{1}{3}|2\omega_{11}^2 + \omega_{13}|^2
\]

\[
= \frac{1}{9} + \frac{4}{3}|\omega_{11}|^2 - \frac{1}{3}\left(4|\omega_{11}|^4 + |\omega_{13}|^2 + 4\text{Re} \{\omega_{13}\overline{\omega_{11}}^2\}\right)
\]

\[
= \frac{1}{9} + \frac{4}{3}|\omega_{11}|^2 - \frac{4}{3}|\omega_{11}|^4 - \frac{1}{3}|\omega_{13}|^2 - \frac{4}{3}\text{Re} \{\omega_{13}\overline{\omega_{11}}^2\}.
\]

Using the fact that

\[-|\omega_{13}|^2 \leq -|\text{Re} \{\omega_{13}\}|^2 = -(\text{Re} \{\omega_{13}\})^2,
\]

we obtain

\[
|\gamma_3|^2 \leq \frac{1}{9} + \frac{4}{3}|\omega_{11}|^2 - \frac{4}{3}|\omega_{11}|^4 - \frac{1}{3}(\text{Re} \{\omega_{13}\})^2 - \frac{4}{3}\text{Re} \{\omega_{13}\overline{\omega_{11}}^2\}.
\]

Next, without loss of generality using suitable rotation of \( f \) we can assume that \( 0 \leq a_2 \leq 2 \) and \( a_2 = 2\omega_{11} \) receive that \( 0 \leq \omega_{11} \leq 1 \). So, let put \( \omega_{11} = a, 0 \leq a \leq 1 \), and continue analysing

\[
|\gamma_3|^2 \leq \frac{1}{9} + \frac{4}{3}a^2 - \frac{4}{3}a^4 - \frac{1}{3}(\text{Re} \{\omega_{13}\})^2 - \frac{4}{3}a^2\text{Re} \{\omega_{13}\}.
\]  

(2.6)

It is a classical result that for the class \( S \) we have \(|a_3 - a_2|^2 \leq 1 \) (see [9, p.5]), which is by (2.4) equivalent with

\[
|2\omega_{13} - \omega_{11}|^2 \leq 1.
\]

From here,

\[-1 \leq \text{Re} \{2\omega_{13} - \omega_{11}^2\} \leq 1,
\]

i.e.

\[-\frac{1}{2}(1 - a^2) \leq \text{Re} \{\omega_{13}\} \leq \frac{1}{2}(1 + a^2).
\]

(2.7)

If we put \( x_1 = 1 \) and \( x_3 = 0 \) in (2.5), then we get

\[
|\omega_{11}|^2 + 3|\omega_{13}|^2 \leq 1,
\]

which implies

\[
|\omega_{13}| \leq \frac{1}{\sqrt{3}}\sqrt{1 - |\omega_{11}|^2} = \frac{1}{\sqrt{3}}\sqrt{1 - a^2}.
\]

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Combining this with (2.7), we receive

\[-\frac{1}{2}(1 - a^2) \leq \text{Re} \{\omega_{13}\} \leq \frac{1}{\sqrt{3}} \sqrt{1 - a^2}\]

(because \(-\frac{1}{2}(1 - a^2) \geq -\frac{1}{\sqrt{3}} \sqrt{1 - a^2}\)).

By using (2.6), (2.7) and the notation \(t = \text{Re} \{\omega_{13}\}\) we obtain

\[|\gamma_3| \leq \frac{1}{9} + \frac{4}{3} a^2 - \frac{4}{3} a^4 - \frac{1}{3} t^2 - \frac{4}{3} a^2 t : \equiv \psi(a, t) = \frac{1}{9} + \frac{1}{3} \varphi(a, t),\]

where \(0 \leq a \leq 1, -\frac{1}{2}(1 - a^2) \leq t \leq \frac{1}{\sqrt{3}} \sqrt{1 - a^2}\) and \(\varphi(a, t) = 4a^2 - 4a^4 - t^2 - 4a^2 t\).

It remains to show that the maximal value of the function \(\psi(a, t)\) over the region \(\Omega = [0, 1] \times [-\frac{1}{2}(1 - a^2), \frac{1}{\sqrt{3}} \sqrt{1 - a^2}]\) equals \(\left(\frac{\sqrt{133}}{15}\right)^2 = \frac{133}{225}\), or equivalently that \(\varphi(a, t)\) has maximal value \(\frac{36}{25}\) on the same region.

Indeed, the system of equations

\[
\begin{cases}
\varphi'_a(a, t) = 8a - 16a^3 - 8at = 0 \\
\varphi'_t(a, t) = -4a^2 - 2t = 0
\end{cases}
\]

has unique real solution \(a = t = 0\) with \(\varphi(0, 0) = 0\), while on the edges of the region \(\Omega\) we have the following:

- for \(a = 0\) we have that the function \(\varphi(0, t) = -t^2\) on the interval \(-\frac{1}{2} \leq t \leq \frac{1}{\sqrt{3}}\) attains maximal value \(\varphi(0, 0) = 0\);

- when \(a = 1\), \(t\) can take single value, \(t = 0\), and in that case \(\varphi(1, 0) = 0\);

- for \(t = -\frac{1}{2}(1 - a^2)\), the function \(\varphi_a \left( -\frac{1}{2}(1 - a^2) \right) = -\frac{1}{4} (a^2 - 1) (a^2 - \frac{1}{25})\) is with maximal value \(\frac{36}{25}\) on the interval \(0 \leq a \leq 1\) attained for \(a = \frac{\sqrt{133}}{5}\);

- for \(t = \frac{1}{\sqrt{3}} \sqrt{1 - a^2}\), the values of the function

\[
\varphi \left( a, \frac{1}{\sqrt{3}} \sqrt{1 - a^2} \right) = \frac{1}{3} (-12a^4 + 13a^2 - 1) - \frac{4a^2}{\sqrt{3}} \sqrt{1 - a^2}
\]

\[\leq \frac{1}{3} (-12a^4 + 13a^2 - 1) < \frac{36}{25}.
\]

on the interval \(0 \leq a \leq 1\) are smaller than \(\frac{36}{25}\).

This completes the proof.

\[\square\]

References

[1] Ali MF, Vasudevarao A. On logarithmic coefficients of some close-to-convex functions. Proceedings of the American Mathematical Society 2018; 146 (3): 1131-1142. doi: 10.1090/proc/13817
[2] Cho NE, Kowalczyk B, Kwon OS, Lecko A, Sim YJ. On the third logarithmic coefficient in some subclasses of close-to-convex functions. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas 2020; 114. doi: 10.1007/s13398-020-00786-7

[3] Darus M, Thomas DK. $\alpha$-logarithmically convex functions. Indian Journal of Pure and Applied Mathematics 1998; 29 (10): 1049-1059.

[4] Deng Q. On the logarithmic coefficients of Bazilevič functions. Applied Mathematics and Computation 2011; 217 (12): 5889-5894. doi: 10.1016/j.amc.2010.12.075

[5] Duren PL. Univalent function. New York, NY, USA: Springer-Verlag, 1983.

[6] Lebedev NA. Area principle in the theory of univalent functions. Moscow, Russia: Nauka, 1975 (in Russian).

[7] Thomas DK. The logarithmic coefficients of close-to convex functions. Proceedings of the American Mathematical Society 2016; 144 (2): 1681-1687. doi: 10.1090/proc/12921

[8] Thomas DK. On the coefficients of strongly starlike functions. Indian Journal of Mathematics 2016; 58 (2): 135-146.

[9] Thomas DK, Tuneski N, Vasudevarao A. Univalent Functions: A Primer. De Gruyter Studies in Mathematics, 69. Berlin, Germany: De Gruyter, 2018.