Approximate confidence distribution computing: An effective likelihood-free method with statistical guarantees

Suzanne Thornton* and Min-ge Xie**

Department of Statistics and Biostatistics, Rutgers, The State University of New Jersey, U.S.A.

email: suzanne.thornton@rutgers.edu
email: mxie@stat.rutgers.edu

**SUMMARY:** Approximate Bayesian computing (ABC) is a likelihood-free method that has grown increasingly popular since early applications in population genetics. However, the theoretical justification for Bayesian inference (e.g., construction of credible intervals) based on this method has not yet been fully addressed when using non-sufficient summary statistics. We introduce a more general computational technique, approximate confidence distribution computing (ACC), to overcome a few issues associated with the ABC method, for instance, the lack of theory supporting for constructing credible (or confidence) intervals when the ACC method uses non-sufficient summary statistics, the long computing time, and the necessity of a prior assumption. Specifically, we establish frequentist coverage properties for the outcome of the ACC method by using the theory of confidence distributions, and thus inference based on ACC is justified, even if reliant upon a non-sufficient summary statistic. Furthermore, the ACC method is very broadly applicable; in fact, the ABC algorithm can be viewed as a special case of an ACC method without damaging the integrity of ACC based inference. We supplement the theory with simulation studies and an epidemiological application to illustrate the benefits of the ACC method. It is demonstrated that a well-tended ACC algorithm can greatly increase its computing efficiency over a typical ABC algorithm.

**KEY WORDS:** Confidence distribution; Approximate Bayesian computing; Bernstein von Mises; Large sample theory; Exact inference.
1. Introduction

Approximate Bayesian computing (ABC) is a likelihood-free method that approximates a posterior distribution while avoiding direct calculation of the likelihood. This procedure originated in population genetics where complex demographic histories yield intractable likelihoods. Since then, ABC has been applied to many other areas besides the biological sciences including astronomy and finance. See, e.g., Cameron and Pettitt (2012); Marin et al. (2011); Sisson et al. (2007). Despite its practical popularity, the theoretical justification for inference from ABC is underdeveloped and has only recently been explored in statistical literature; cf., e.g., Robinson et al. (2014); Barber et al. (2015); Frazier et al. (2016); Li and Fearnhead (2016). In this paper, we seek to re-frame the problem within a frequentist setting, proposing a novel likelihood-free method to elegantly fill this gap in the literature.

Rather than deriving a likelihood, ABC uses the assumption that one may treat simulations as artificial experiments and compare observed and simulated summary statistics. These simulations are based on a scientific model $M_\theta$, where $\theta \in \mathcal{P} \subset \mathbb{R}^p$ is an unknown parameter. As explained in Csilléry et al. (2010) (among others), to apply ABC we need not have an analytically tractable expression for the likelihood of the data; instead, we need only assume that data may be generated by simulations of this scientific model.

The goal of ABC is to produce draws from an approximation to a target posterior distribution. The simplest accept-reject version of ABC proceeds as follows:

[Step 1] Generate $\theta^*$ from prior $\pi(\theta)$;

[Step 2] Generate $x^*$ from the model $M_{\theta^*}$ to produce $S_n^*$. 

Steps 1 and 2 are repeated until $\rho(S_n^*, s_{\text{obs}}) \leq \varepsilon$, at which point we set $\theta_i = \theta^*$ and begin the process again. This continues until we obtain $\theta_1, \ldots, \theta_N$. The underlying distribution from which these $N$ copies or draws are simulated is called the ABC posterior. This ABC posterior is sometimes the same or close to the target posterior distribution (but, as is demonstrated later in
Figure 1, it can also be quite different from the target distribution). The user is free to select the
tolerance level, \( \varepsilon \), and the distance metric, \( \rho \). In practice, the tolerance can be as close to zero as
is computationally feasible. Since it is a Bayesian procedure, in addition to the assumption of the
existence of a model \( M_\theta \) that generates artificial data, ABC assumes a prior distribution, \( \pi(\theta) \), on \( \theta \). In the absence of prior information, the user may select a flat prior on \( \theta \).

The approximation of an ABC posterior to its target posterior distribution depends on the close-
ness of the tolerance level to zero and, more crucially for our purposes, on the choice of summary
statistic, \( S_n \). It is known (but often understated in applications of ABC) that if the summary statistic
is not sufficient, the ABC posterior can be a very poor approximation to the target posterior. For
example, consider IID data from a Cauchy(\( \theta, 1 \)) distribution. Only the data itself is sufficient
for \( \theta \) and therefore any reasonable choice of summary statistic will not be sufficient. Figure 1
displays the results of ABC with a “flat” prior, \( \pi(\theta) \propto \exp\{-(\theta - 10)^2/128\} \), for two different
choices of summary statistics, sample mean and sample median. As Figure 1 illustrates, without
sufficiency, ABC will never converge to the targeted posterior distribution and inference from the
ABC posterior will always be difficult to justify within a Bayesian framework. Specifically, the
95\% credible interval from the ABC posterior by the sample mean (Figure 1 left) is (4.972, 15.142),
but the 95\% credible interval from the ABC posterior by the sample median (Figure 1 right) is
(9.991, 10.091). Clearly, both posterior probabilities of \( P(\theta \in (4.972, 15.142)|x_1, \ldots, x_{400}) \) and
\( P(\theta \in (9.991, 10.091)|x_1, \ldots, x_{4000}) \) are not the same and neither of them equal the 95\% credible
intervals of the target posteriors, (9.797, 10.074) for the sample of size \( n = 400 \) (Figure 1 left) and
(10.006, 10.092) for the sample of size \( n = 4000 \) (Figure 1 right).

This aforementioned potential inferential weakness can be overcome by instead considering
a more general frequentist method based on confidence distribution theory. To this end, we in-
troduce a new likelihood-free inference method called approximate confidence-distribution com-
Confidence Distributions

When estimating an unknown parameter, we often desire that our estimators, whether point estimators or interval estimators, have certain properties, such as unbiasedness or correct coverage of the true parameter value in the long run. A confidence distribution is an extension of this tradition in that it is a distribution estimate that satisfies certain properties. Following Xie and Singh (2013), Schweder and Hjort (2016) and references therein, we define a confidence distribution as follows.

**Definition 1:** A sample-dependent function on the parameter space is a confidence distribution (CD) for a parameter \( \theta \) if 1) For each given sample the function is a distribution function on the parameter space; 2) The function can provide confidence intervals/regions of all levels for \( \theta \).

Consider the following example taken from Singh et al. (2007). Suppose \( X_1, \ldots, X_n \) is a sample from \( N(\mu, \sigma^2) \) where both \( \mu \) and \( \sigma^2 \) are unknown. A CD for parameter \( \mu \) is the function \( H_n(y) = F_{t_{n-1}} \left( \frac{y - \bar{X}}{s_n/\sqrt{n}} \right) \) where \( F_{t_{n-1}}(\cdot) \) is the cumulative distribution function of Student’s t-random variable with \( (n - 1) \) degrees of freedom and \( \bar{X} \) and \( s_n^2 \) are the sample mean and variance,
respectively. Note that $H_n(y)$ is a cumulative distribution function in the parameter space of $\mu$ from which we can construct confidence intervals of $\mu$ at all levels. For example, for any $\alpha \in (0, 1)$, one-sided confidence intervals for $\mu$ are $(\infty, H_n^{-1}(\alpha))$ and $[H_n^{-1}(\alpha), \infty)$. Similarly, a CD for parameter $\sigma^2$ is the function $H_n(\sigma^2) = 1 - F_{\chi^2_{n-1}}\left(\frac{(n-1)s^2}{\sigma^2}\right)$, where $F_{\chi^2_{n-1}}(\cdot)$ is the distribution function of a Chi-squared random variable with $(n - 1)$ degrees of freedom. Again, $H_n(\sigma^2)$ is a cumulative distribution function in the parameter space of $\sigma^2$ and, based on it, we can construct confidence intervals of $\sigma$ at all levels.

We emphasize that, by definition, a CD is a sample-dependent distribution function that can represent confidence intervals/regions of all levels for a parameter of interest. Any CD approach uses a sample-dependent distribution function to estimate the unknown parameter; thus a CD is an expression of inference (i.e. an estimator of the target parameter in the form of a distribution function), and not a distribution of the parameter.

A CD estimator has a similar appeal to a Bayesian posterior in that it is a distribution function carrying much information about the parameter. A CD however, is an entirely frequentist notion which treats the parameter as a fixed and unknown quantity and guarantees that any intervals/regions for $\theta$ obtained from a CD can contain the true parameter value, $\theta_0$, at any specified frequency. We will refer to this property as the frequentist coverage property of CDs. A large motivation for conducting inference through a CD, cited by Xie and Singh (2013), is that any statistical approach, whether frequentist, fiducial, or Bayesian, can potentially be unified under the CD framework. We hope to demonstrate that the construction of ACC as a likelihood-free method provides one of many examples in which CD theory provides a useful inferential tool for a problem where a frequentist method with desirable properties was previously unavailable. For more details on confidence distributions see Xie and Singh (2013), Schweder and Hjort (2016), and references therein.
1.2 Approximate CD Computing (ACC)

We now introduce ACC, a wide-reaching frequentist algorithm, as an alternative to existing likelihood-free methods, in particular as an alternative to ABC and the similar method of indirect inference. The theoretical foundation for ACC relies upon the frequentist coverage property of CDs. By drawing upon CD theory, ACC is able to offer theoretical support for ABC but provides a computational method with broader potential applications than ABC. Additionally, as will be discussed later, ACC introduces some useful flexibility to ABC that can greatly decrease computing costs.

As in ABC, the user selects the tolerance level, $\varepsilon$, and the distance metric, $\rho$, and observes the data and the corresponding summary statistic, $s_{\text{obs}}$. Again we assume that given some value of the parameter, $\theta^*$, the model, $M_{\theta^*}$, can generate the observed data. The likelihood for the data, $p(\theta|x_1, \ldots, x_n)$, need not be known. The ACC algorithm proceeds in the same manner as the ABC algorithm, but with Step 1 replaced with

[Step 1'] Generate $\theta^*$ from some $r_n(\theta)$.

Thus, in ACC there is no prior assumption on $\theta$; instead, the user is free to select a data-dependent distribution function, $r_n(\theta)$, from which potential parameter values are generated. Although ACC bears a superficial resemblance to ABC, the flexibility in selecting $r_n(\theta)$ makes ACC a broader concept in practice. If one is willing to assume a prior distribution $\pi(\theta)$ and set $r_n(\theta) = \pi(\theta)$, the ACC is the same as the ABC; thus, ABC can be views as a special case of ACC with $r_n(\theta) = \pi(\theta)$.

From a Bayesian perspective one may view ACC as an extension that permits the use of ABC in the presence of a data-dependent prior. However, there is another natural, frequentist interpretation that views the function $r_n$ as an initial distribution estimate and ACC as an updating algorithm pursuing a better-performing distribution estimate for $\theta$. This is analogous to an application of the EM algorithm, say, which requires an initial point estimate and then updates a search for a better-performing point estimate. In ACC we require the initial guess to be a distribution estimate, $r_n(\theta)$. We are not committing the error of doubly using the data. ACC can guarantee a distribution
estimator with the frequentist coverage property, although it may not necessarily ensure efficiency of this distribution estimator.

Note in the frequentist tradition, ACC treats the parameter of interest as a fixed, unknown quantity. Under general conditions (including some constraints on \( r_n \)) ACC yields a CD. This means that the quantiles of the parameter values kept in the ACC algorithm, \( \{\theta_1, \ldots, \theta_N\} \), can be used to define a confidence interval/region for the parameter of interest at any significance level. This fact (theory) will be further elaborated upon and proven in Section 2.

1.3 Related Work

Though likelihood-free methods such as ABC have existed for more than 20 years, research regarding the theoretical properties of these practical methods is a newly active area (cf. e.g. Li and Fearnhead (2016); Frazier et al. (2016)). Here we do not attempt to give a full review of all likelihood-free methods (e.g. Marin et al. (2011)) but we note that there exist alternatives to ABC such as indirect inference (cf. e.g. Creel and Kristensen (2013); Gourieroux et al. (1993)).

One of our theoretical results specifies conditions under which the ACC produces an asymptotic normal CD. This result mirrors the theory from Li and Fearnhead (2016) and Frazier et al. (2016) regarding the asymptotic normality of the ABC posterior distribution. However, in contrast to these papers, our main focus is on the distribution of the algorithmically retained parameter draws itself and not exclusively on the properties of point estimates derived from this distribution. We are not concerned with viewing this distribution as an approximation to a target posterior, rather we focus on the properties of this distribution inherited through its connection to CDs. More importantly, the properties we develop here allow us to conduct statistical inference (such as construct confidence intervals/regions and \( p \) values) with a guaranteed performance standard.

Moreover, we expect that much of the existing work in this area from the ABC literature can also be applied to ACC to further improve upon its computational performance. For example, work
regarding sequential importance sampling methods (such as Beaumont et al. (2009), Marin et al. (2011)) may also be applied to improve estimates derived from ACC methods.

This paper presents the novel idea that the continued study of asymptotic properties of likelihood-free methods would benefit from the incorporation of CD theory. To this end, and for the ease of presentation, we mainly focus on the basic accept-reject version of ACC though an importance sampling extension is considered at the end of Section 2.

1.4 Notation and Outline
Throughout the paper we will use the following notation. Suppose the observed data is such that \( x_{\text{obs}} \in \mathcal{X} \subset \mathbb{R}^n \). The summary statistics mapping is \( S_n : \mathcal{X} \to \mathbb{R}^m \) and the observed summary statistic is \( s_{\text{obs}} = S_n(x_{\text{obs}}) \). The unknown/intractable likelihood is \( p(\theta|x_1, \ldots, x_n) = p(\theta|x) \). The parameter of interest is \( \theta \in \mathcal{P} \subset \mathbb{R}^p \), where \( p \leq m \). Let \( \theta_0 \) represent the fixed, true value of the parameter. Furthermore, let \( \tilde{f}_n(S_n; \theta) \) denote the (typically unknown) density of the chosen summary statistic, \( S_n \). We will call this distribution an \textit{s-likelihood} to emphasize that it is not a likelihood in any traditional sense. Similarly, \( \tilde{\ell}_n \) represents the log of the \textit{s}-likelihood.

Denote the summary statistic corresponding to simulated data from ACC by \( S^*_n \). Use \( \theta_{\text{ACC}} \) to represent a simulated parameter value draw from the ACC algorithm that produces a \( S^*_n \) sufficiently close to \( s_{\text{obs}} \). The ACC algorithm specifies a distribution estimate for \( \theta \) given \( s_{\text{obs}} \) where the output are \( N \) draws from a CD with density

\[
p(\theta_{\text{ACC}}|s_{\text{obs}}) \propto \int r_n(\theta)\tilde{f}_n(S^*_n; \theta)\mathbb{I}\{p(S^*_n, s_{\text{obs}}) \leq \varepsilon\}dS^*_n.
\]

In Section 2 we elaborate upon the theoretical support for ACC that guarantees the algorithm results in a CD. We study an application of ACC to some genetic data (both real and simulated) in Section 3. We offer some concluding remarks in Section 4. All proofs are deferred until the Appendix.
2. Establishing Frequentist Guarantees for ACC

In this section, we establish some general conditions under which the ACC algorithm produces a CD. To motivate our main theoretical result, we first consider the simple case where we have a scalar parameter and \( \hat{\theta}_S : \mathbb{R}^m \to \mathcal{P} \) is a function, \( \hat{\theta}_S = \hat{\theta}(S_n) \), that maps the summary statistic into the parameter space.

Suppose it is known that \( (\theta_{ACC} - \hat{\theta}_S) \mid S_n = s_{\text{obs}} \) and \( (\hat{\theta}_S - \theta) \mid \theta = \theta_0 \) follow the same (asymmetric) distribution, say \( G \):

\[
(\theta_{ACC} - \hat{\theta}_S) \mid S_n = s_{\text{obs}} \sim (\hat{\theta}_S - \theta) \mid \theta = \theta_0 \sim G(\cdot).
\]

Here, on the left hand side, given \( s_{\text{obs}} \), \( \hat{\theta}_S \) is fixed and the probability measure is with respect to \( \theta_{ACC} \) where the randomness \( \theta_{ACC} \) is due to the simulation conducted in ACC. On the right hand side, \( \hat{\theta}_S \) is a random variable where the randomness is from the sample \( S \) before it is observed to be \( S = s_{\text{obs}} \). This is very similar to that bootstrap central limit theorem establishing \( (\theta_B - \hat{\theta}_S) \mid S_n = s_{\text{obs}} \sim (\hat{\theta}_S - \theta) \mid \theta = \theta_0 \) where appropriate; there, the randomness on the left hand side is from the bootstrap estimator \( \theta_B \) given \( S_n = s_{\text{obs}} \) and the randomness on the right hand side is from random sample \( S_n \); cf, Singh (1981).

If we define the cumulative distribution of \( 2\hat{\theta}_S - \theta_{ACC} \) given the observed summary statistic by \( D_n(t) = P_{ACC}(2\hat{\theta}_S - \theta_{ACC} \leq t \mid s_{\text{obs}}) \) and define the quantile \( \theta_{ACC,\alpha} \) as the value such that

\[
P_{ACC}(2\hat{\theta}_S - \theta_{ACC} \leq 2\hat{\theta}_S - \theta_{ACC,\alpha} \mid s_{\text{obs}}) = 1 - \alpha,
\]

then

\[
D_n(\theta_0) = P_{ACC}(2\hat{\theta}_S - \theta_{ACC} \leq \theta_0 \mid s_{\text{obs}}) = P_{ACC}(\theta_{ACC} - \hat{\theta}_S \geq \hat{\theta}_S - \theta_0 \mid s_{\text{obs}}) = 1 - G(\hat{\theta}_S - \theta_0) \sim U(0, 1).
\]

Thus for any \( \alpha \in (0, 1) \),

\[
P(\theta_0 \leq 2\hat{\theta}_S - \theta_{ACC,\alpha}) = P(\theta_0 \leq D_n^{-1}(1 - \alpha)) = P(D_n(\theta_0) \leq 1 - \alpha) = 1 - \alpha
\]

and we can construct a \((1 - \alpha)\) level confidence interval for \( \theta \) based on the quantiles of the \( N \)
parameter values kept from the ACC algorithm. From this, it immediately follows that

\[ D(\hat{\theta}_S, x) = 1 - G(\hat{\theta}_S - x) = 1 - P_{ACC}(\theta_{ACC} - \hat{\theta}_S \leq \hat{\theta}_S - x|s_{obs}) = P_{ACC}(\theta_{ACC} \geq 2\hat{\theta}_S - x|s_{obs}), \]

is a CD for \( \theta \) since, for \( x \in \mathcal{P} \), \( D(\hat{\theta}_S, x) \) is a continuous CDF on the parameter space given \( s_{obs} \).

Note that in practice, we can only obtain an approximate CD since we cannot obtain \( S_n^* = s_{obs} \) and must settle for values of \( S_n^* \) where \( \rho(S_n^*, s_{obs}) \leq \varepsilon \) for a small, non-zero \( \varepsilon \). The argument for a symmetric \( G \) is much simpler in that it does not require a transformation of the \( \theta_{ACC} \) values but the proof has a similar form.

Now we introduce a key theorem that generalizes the argument above for a multidimensional parameter \( \theta \in \mathcal{P} \subset \mathbb{R}^p \) and for a wider range of relationships between \( S_n \) and \( \theta_{ACC} \). This general theorem assumes a relationship between two mappings \( V, W : \mathcal{P} \times \mathbb{R}^m \rightarrow \mathbb{R} \), where \( V \) is a function that acts on the parameter space \( \mathcal{P} \) given \( s_{obs} \), and \( W \) is a function that acts on \( \mathbb{R}^m \) given \( \theta = \theta_0 \). For example, the argument above required an assumption on the linear mapping \( V(t_1, t_2) = t_1 - \hat{\theta}(t_2) \) where \( \hat{\theta} \) is a function of the summary statistic \( S \); however, we may also wish to consider other mappings such as (say) \( V(t_1, t_2) = t_1/\hat{\theta}(t_2) \).

**Theorem 1**: If there exists mappings \( V, W : \mathcal{P} \times \mathbb{R}^m \rightarrow \mathbb{R} \) such that

\[ V(\theta_{ACC}, S_n)|s_{obs} \sim W(\theta, S_n)|\theta = \theta_0, \quad (1) \]

then as \( \varepsilon \rightarrow 0 \), ACC can be utilized to derive a CD for \( \theta \). If (1) holds asymptotically as \( n \rightarrow \infty \), then as \( \varepsilon \rightarrow 0 \) ACC can derive an asymptotic CD (aCD).

Theorem 1 states that if (1) holds (asymptotically or exactly) for some choice of summary statistic, then ACC is guaranteed to yield inferential results with valid frequentist coverage. Specifically, if we let \( G \) represent the distribution of \( V(\theta_{ACC}, S_n)|s_{obs} \) and define \( q_\alpha = G^{-1}(1 - \alpha) \) for any \( \alpha \in (0, 1) \), then a \( (1 - \alpha)100\% \) confidence region for \( \theta_0 \) is found to be \( \{\theta|W(\theta, s_{obs}) \leq q_\alpha\} \).
2.1 Estimation by Maximizing the $s$-likelihood

The classical Bernstein von Mises theorem (BvM) is well known for establishing a link between Bayesian and frequentist methods by guaranteeing the posterior is asymptotically Gaussian. In this section, following the example of Li and Fearnhead (2016), we establish a theorem similar to BvM to justify ACC-based inference. However, in contrast to any existing results in the ABC literature, in conjunction with Theorem 1, our theorem guarantees that ACC methods produce a CD for the parameter of interest.

We begin by assuming some standard regularity conditions on the parameter space and on the $s$-likelihood. These assumptions establish two lemmas from which Theorem 2 immediately follows. Define a neighborhood around the true $\theta_0$ by $P_0 = \{\theta : ||\theta - \theta_0|| < \delta\}$ for some $\delta > 0$. Define the estimator $\tilde{\theta}_S = \text{ArgMin}_\theta \left\{ -\left[ \frac{\partial}{\partial \theta} \tilde{f}_n(s_{\text{obs}}; \theta) \right] \right\}$, so that $\tilde{\theta}_S$, is a function of $s_{\text{obs}}$ and assume the following:

1. $P$ is an open interval;
2. $\{S_n : \tilde{f}_n(S_n; \theta) > 0\}$ is the same for all $\theta \in P$;
3. For any $S_n$ in the sample space of the summary statistic and for any $\theta$ in $P_0$, $\left[ \frac{\partial^2}{\partial \theta_i \partial \theta_j} \tilde{f}_n(\theta) \right]_{p \times 1}$ exists and is finite and $\left[ \frac{\partial^2}{\partial \theta_i \partial \theta_j} \tilde{f}_n(\theta) \right]_{p \times p}$ exists and is negative definite;
4. Both the first and second derivatives with respect to $\theta$ of $\tilde{f}_n(S_n; \theta)$ can be obtained by differentiating under the integral sign on the left hand side of $\int \tilde{f}_n(S_n; \theta) dS_n = 1$.

It follows that the information matrix based on the distribution of the summary statistic is

$$I_S(\theta_0) = E_{\theta_0} \left[ \left( \frac{\partial}{\partial \theta_i} \tilde{f}_n(\theta) \right)^T \left( \frac{\partial}{\partial \theta_i} \tilde{f}_n(\theta) \right) | S_n \right] = E_{\theta_0} \left[ \left( -\frac{\partial^2}{\partial \theta_i \partial \theta_j} \tilde{f}_n(\theta) \right) | S_n \right].$$

The observed “Fisher” information corresponding to $\tilde{f}_n(\tilde{\theta}_S)$ is written as $\tilde{I}_n = \frac{1}{n} \frac{\partial^2}{\partial \theta_i \partial \theta_j} \tilde{f}_n(\tilde{\theta}_S)$. Lemma 1 below derives the asymptotic distribution of $\tilde{\theta}_S$ given $\theta_0$. Lemma 2 derives the distribution of the parameter values obtained from the ACC algorithm $\theta_{\text{ACC}}$, conditional on $s_{\text{obs}}$ and as $\varepsilon \to 0$. Similar results are stated in Li and Fearnhead (2016, Theorem 3.1) and Frazier et al. (2016, Theorem 2), but with $r_n(\theta)$ replaced by a data-free prior function $\pi(\theta)$. 
LEMMA 1: If in addition to the four regularity conditions above we assume

(5) $\left[ \frac{\partial}{\partial \theta_i} \tilde{\ell}_n(\theta) \right]$ is monotone in $\theta$ and is continuous in $\mathcal{P}_0$;

(6) $E_{\theta} \left[ \frac{\partial}{\partial \theta_i} \tilde{\ell}_n(\theta) \right]$ changes sign uniquely in $\mathcal{P}_0$ and is differentiable at $\theta_0$ with non-zero derivative;

(7) $E_{\theta} \left( \left[ \frac{\partial}{\partial \theta_i} \tilde{\ell}_n(\theta) \right]^T \left[ \frac{\partial}{\partial \theta_i} \tilde{\ell}_n(\theta) \right] \right) < \infty$ in $\mathcal{P}_0$ and is continuous at $\theta_0$;

then $\sqrt{n}(\tilde{\theta}_S - \theta_0) \xrightarrow{d} N\left(0, I_{\tilde{\sigma}}^{-1}(\theta_0)\right)$.

LEMMA 2: If in addition to the previous seven conditions on the distribution of the summary statistic and on the parameter space we also assume

(8) $\tilde{\ell}_n(\theta) \in C^3(\mathcal{P}_0)$ and $\frac{1}{n} \sup_{\theta \in \mathcal{P}} | \frac{\partial^3}{\partial \theta_i \partial \theta_j \partial \theta_k} \tilde{\ell}_n(\theta) | \leq M(s_{\text{obs}})$ for any $i, j, k$ and $M(s_{\text{obs}}) = O_p(1)$;

and we assume the following two conditions on the data-dependent generating function in the ACC algorithm

(9) $\lim_{n \to \infty} r_n(\theta_0) \neq 0$ and $r_n(\theta)$ is continuous in some $\mathcal{P}_0$;

(10) $\lim_{n \to \infty} \frac{\exp\{\tilde{\ell}_n(\theta)\}}{r_n(\theta)} = \infty$ and $\lim_{n \to \infty} \frac{\exp\{\tilde{\ell}_n(\theta)\}}{r_n(\theta_0)} = 0$ for any $\theta \in \mathcal{P}_0^C$ and some $\delta > 0$ with $P_{\theta_0}$-probability one;

then $\sqrt{n} \left( \theta_{\text{ACC}} - \tilde{\theta}_S \right) | S_n = s_{\text{obs}} \xrightarrow{d} N(0, I_{\tilde{\sigma}}^{-1}(\theta_0))$.

Lemma 2 establishes a pointwise convergence given any $S_n = s_{\text{obs}}$. Note that for $\theta \in \mathcal{P}_0$, $\lim_{n \to \infty} \frac{r_n(\theta)}{r_n(\theta_0)} = 1$ follows from Assumption 9 and note that Assumption 10 simply states that we must choose an initial distribution estimate, $r_n$, that grows at a slower rate than the s-likelihood.

The next result is an immediate consequence of these two lemmas.

THEOREM 2: If assumptions 1 – 10 hold, then

$$\left( \theta_{\text{ACC}} - \tilde{\theta}_S \right) | S_n = s_{\text{obs}} \sim (\tilde{\theta}_S - \theta_0) | \theta_0$$

with $P_{\theta_0}$-probability one; therefore, by Theorem 7 as $\varepsilon \to 0$, a CD for $\theta$ can be derived from ACC.
Note that if $S_n$ is sufficient, then the likelihood for the data is proportional to the $s$-likelihood; therefore,

$$
\frac{p(\theta|x_1, \ldots, x_n)}{\int p(\theta|x_1, \ldots, x_n)d\theta} \propto \frac{\tilde{f}_n(S_n; \theta)}{\int \tilde{f}_n(S_n; \theta)d\theta}.
$$

Thus, assuming $\tilde{f}_n$ is continuous, we have that

$$
p_{\varepsilon}(\theta|s_{obs}) = \int p_{\varepsilon}(\theta, S_n|s_{obs})dS_n = \frac{\int r_n(\theta)\tilde{f}_n(S_n; \theta)\mathbb{I}\{\rho(S_n, s_{obs}) < \varepsilon\}dS_n}{\int \int r_n(\theta)\tilde{f}_n(S_n; \theta)\mathbb{I}\{\rho(S_n, s_{obs}) < \varepsilon\}dS_n d\theta}
$$

where $p_{\varepsilon}(\theta|s_{obs})$ is the distribution of $\theta_{ACC}$. Therefore, if in addition to Assumptions 1-10 above we also have that $S_n$ is sufficient and $r_n(\theta) \propto 1$, then not only does ACC produce confidence regions with correct frequentist coverage, but ACC-based inference is also efficient. Hence the CD produced by ACC is actually the normalized likelihood rather than the normalized $s$-likelihood.

**Example 1:** In both applications of ABC of Figure 1 we observed that, since there was no sufficient summary statistic, the ABC posterior did not match the target posterior. However, Theorem 2 guarantees that confidence intervals based on draws from the ACC algorithm can have the correct frequentist coverage, regardless of the sufficiency of the summary statistic. Thus the inaccuracy of ABC in approximating a targeted posterior distribution is of no matter provided one can apply the assumptions of Theorem 2.

Consider the case where we observe IID data from a Cauchy $(\theta, \tau)$ distribution where $\tau = 1$ is known. Choosing $S_n = \text{Median}(x)$, note that $\sqrt{n}(S_n - \theta) \overset{\mathcal{L}}{\to} \text{N}(0, \tau^2/4)$ and therefore, for large $n$, $\tilde{f}_n(\theta) \propto \ln(\sqrt{n}) - \frac{2\mu}{\pi}(s_{obs} - \theta)^2$.

Figure 2 compares the results of ABC and ACC for a sample of size $n = 100$, showing that both methods yield the same result. However, a well-chosen $r_n$ greatly reduces the computing time necessary for ACC. In fact, ACC with $r_n(\theta) \propto 1/[(1 + (\theta - \bar{x})^2]$ preformed about five times faster than ABC with a “flat” prior $\pi(\theta) \propto \exp\{-(\theta - 10)^2/128\}$ and the acceptance rate for ACC was 0.520%, much greater than 0.098%, the acceptance rate for ABC. We can apply the results of
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Theorem 2 to ACC in this example since, for any \( \theta, r_n(\theta) \neq 0, r_n \) is continuous, and

\[
\frac{\exp\{\tilde{\ell}_n(\theta)\}}{r_n(\theta)} \propto \frac{\sqrt{n} \exp\left\{-\frac{2n}{\pi^2} (s_{\text{obs}} - \theta)^2\right\}}{1 + (\theta - \bar{x}_{\text{obs}})^2}.
\]

Since \( \sqrt{n}|S_n - \theta_0| = o_p(\log \log n) \), we have

\[
\frac{\exp\{\tilde{\ell}_n(\theta_0)\}}{r_n(\theta_0)} \propto \sqrt{n} \exp(n)[1 + (\theta' - \bar{x}_{\text{obs}})^2] \xrightarrow{n \to \infty} \infty.
\]

Furthermore, \([1 + (\theta_0 - \bar{x}_{\text{obs}})^2]^{-1} \leq 1 \) and \(|\theta' - s_{\text{obs}}| \geq |\theta' - \theta_0| - |s_{\text{obs}} - \theta_0| \geq \delta - O(n^{-\frac{1}{2}}) \geq \delta/2, \)

for \( \theta' \notin \mathcal{P}_0 \) and \( n \) large enough and thus we have

\[
\frac{\exp\{\tilde{\ell}_n(\theta')\}}{r_n(\theta_0)} \propto \sqrt{n} \exp\left\{-\frac{n}{2\pi^2\delta^2}\right\} \xrightarrow{n \to \infty} 0.
\]

Therefore, by Theorem 2, for large enough \( n \),

\[
(\theta_{\text{ACC}} - \text{Median}(x))|S = s_{\text{obs}} \sim (\text{Median}(x) - \theta) | \theta = \theta_0 \sim \mathcal{N}(0, \frac{\pi^2}{4n})
\]

and ACC produces a CD for \( \theta \).

**Figure 2:** Left: Matching results for the distribution of \( \theta_{\text{ACC}} \) (solid red) and the ABC-posterior for \( \theta \) (solid yellow). The target distribution of \( \theta_{\text{ACC}}, \mathcal{N}(\text{Median}(x), \frac{\pi^2}{4n}) \), is also shown (dashed black). Right: The dashed lines show the data dependent \( r_n \) for ACC (red) and the flat prior for ABC (yellow). The true parameter value here is \( \theta_0 = 10 \).

Furthermore, additional simulation studies counted the proportion of times out of 100 runs that 95% confidence intervals based on observations of the ACC method covered the true value of \( \theta \). For a sample of size \( n = 100 \), 96% of the resulting ACC-based confidence intervals contained the correct parameter value thereby verifying the claim that ACC produced a CD.
2.2 Guidelines for Selecting $r_n$

The key to the generality of ACC is that it can produce justifiable inferential results regardless of how one generates potential values of $\theta$. Theorem 2 asserts conditions under which ACC-based confidence regions cover the true parameter values at a given level even when the generating function on the parameter space is data dependent. As a general rule, one should be careful in choosing $r_n$ to ensure the growth of the distribution with respect to the sample size is controlled according to Assumption 10 of Theorem 2. In a Bayesian context, ABC automatically satisfies this condition whenever $r_n$ is defined to be a flat prior. However, the desirable coverage properties of ACC-based confidence regions remains valid even if $r_n$ is data-driven.

For example, suppose one observes data $x_1, \ldots, x_n$, where $k = n^\delta$, for some $\delta \in (0, 1)$ and $k \in \mathbb{Z}_+$, and let $\hat{\theta}_S$ be an estimate for $\theta$ that is a function of the selected summary statistic. If we partition the original data set into $k$ subsets of size $n^{1-\delta}$ computing the summary statistic over each of these $k$ subsets to obtain $k$ estimates $\hat{\theta}_1, \ldots, \hat{\theta}_k$, then by defining $r_n(\theta)$ as any usual distribution approximation over these $k$ estimates we ensure Assumption 10 of Theorem 2 is satisfied.

This data-driven $r_n$ is always more varied than the $s$-likelihood, meaning, the observed summary statistic carries more information about $\theta$ than the generating function $r_n$ and this is true regardless of whether or not the $s$-likelihood is known. For practical purposes, if $n$ is large one may simply choose $\delta = 1/2$; however, for smaller $n$, say $n < 100$, it is better to select a larger $\delta$ to produce more subsets of smaller size upon which to define $r_n$. Note that the above is one example of how to choose $r_n$ but there are many more possible choices outlined by the theory supporting ACC.

2.3 Estimation with a Pivotal Summary Statistic

Although Theorem 2 applies in many circumstances, ACC-based inference still remains valid (in terms of proper frequentist coverage) beyond the assumptions of the Bernstien von Mises Theorem. Theorem 3 explores a special case of ACC when the limiting distribution of the summary statistic is not necessarily Gaussian.
**Theorem 3:** Suppose \( T = \Psi(\theta, S_n)|\theta = \theta_0 \sim G \) where \( G \) is free of \( \theta \) with density \( g \). Let \( u_{t,\theta} \) be the solution of \( t = \Psi(u, \theta) \) for any given values of \( t \) and \( \theta \). If \( r_n \) and \( \rho \) are chosen so that

\[
\int r_n(\theta) \mathbb{I}\{\rho(u_{t,\theta}, s_{obs}) \leq \varepsilon\} d\theta = h_\varepsilon(s_{obs}), \tag{2}
\]

where \( h_\varepsilon(s_{obs}) \) is some function that is independent of \( t \), then

\[
\Psi(\theta_{ACC}, S_n)|s_{obs} \sim \Psi(\theta, S_n)|\theta = \theta_0 \tag{3}
\]

and therefore, by Theorem 1 as \( \varepsilon \to 0 \), ACC can be used to derive a CD for \( \theta \). (If \( T \overset{d}{\to} G \) or if also \( h_\varepsilon(s_{obs}) \) is replaced with \( h_\varepsilon(s_{obs})(1 + o(1)) \), then by Theorem 1 as \( \varepsilon \to 0 \), ACC can produce an aCD for \( \theta \).)

The existing literature on likelihood-free methods typically relies upon obtaining a “nearly sufficient” summary statistic to justify inferential results; see e.g., Joyce and Marjoram (2008). Theorems 2 and 3 however, guarantee good frequentist properties of ACC-based inference while bypassing the struggle to choose a “sufficient enough” summary statistic. However, as mentioned earlier, if the summary statistic happens to be sufficient for \( \theta \), then for an appropriate choice of \( r_n \), ACC-based inference is also efficient.

As a special case of Theorem 3 if we let \( \rho \) to be Euclidean distance, condition (2) means that the area under \( r_n \) corresponding to values of \( \theta \) that produce a simulated summary statistic that lies within an \( \varepsilon \)-neighborhood of \( s_{obs} \) is the same over any range of \( T \) values. The next corollary identifies a specific case of Theorem 3 where we assume the summary statistic is from a location, scale, or location-scale family.

**Corollary 4:** Let \( \rho \) be the Euclidean distance metric and let \( \hat{\mu}_S = \hat{\mu}(S_1) \) and \( \hat{\sigma}_S = \hat{\sigma}(S_2) \) for \( S_1, S_2 \in \mathbb{R} \).

**Part 1** Suppose \( \hat{\mu}_S \sim g_1(\hat{\mu}_S - \mu) \). If \( r_n(\mu) \propto 1 \) then

\[
(\mu_{ACC} - \hat{\mu}_S)|s_{obs} \sim (\hat{\mu}_S - \mu)|\mu = \mu_0.
\]
**Part 2** Suppose \( \hat{\sigma}_S \sim \frac{1}{\sigma}g_2 \left( \frac{\hat{\sigma}_S}{\sigma} \right) \). If \( r_n(\sigma) \propto 1/\sigma^2 \) then
\[
\frac{\sigma_{\text{ACC}}}{\hat{\sigma}_S} \bigg| S_{\text{obs}} \sim \frac{\hat{\sigma}_S}{\sigma} \bigg| \sigma = \sigma_0.
\]

**Part 3** Let \( \theta = (\mu, \sigma)^T \), \( S_n = (S_1, S_2)^T \), and \( \hat{\theta}_S = (\hat{\mu}_S, \hat{\sigma}_S)^T \). Suppose \( \hat{\mu}_S \sim \frac{1}{\sigma}g_1 \left( \frac{\hat{\mu}_S - \mu}{\sigma} \right) \), and \( \hat{\sigma}_S \sim \frac{1}{\sigma}g_2 \left( \frac{\hat{\sigma}_S}{\sigma} \right) \). If \( r_n(\mu, \sigma) \propto 1/\sigma^2 \) then
\[
\begin{pmatrix}
\frac{\mu_{\text{ACC}} - \hat{\mu}_S}{\sigma_{\text{ACC}}} \\
\frac{\sigma_{\text{ACC}}}{\hat{\sigma}_S}
\end{pmatrix}
\bigg| S_{\text{obs}} \sim \begin{pmatrix}
\frac{\hat{\mu}_S - \mu}{\sigma} \\
\frac{\hat{\sigma}_S}{\sigma}
\end{pmatrix}
\bigg| \theta = \theta_0.
\]

Furthermore, we may derive \( H_1(\hat{\mu}_S, x) = 1 - \int_{-\infty}^{\hat{\mu}_S - x} g_1(w)dw \), a CD for \( \mu \) induced by \( (\hat{\mu}_S - \mu)|\mu = \mu_0 \) or \( H_2(\hat{\sigma}_S, x) = 1 - \int_{0}^{\hat{\sigma}_S} g(w)dw \), a CD for \( \sigma \) induced by \( \frac{\hat{\sigma}_S}{\sigma}|\sigma = \sigma_0 \).

**Example 2:** Consider the case of IID data from a Cauchy distribution where neither the location parameter, \( \theta \), nor the scale parameter, \( \tau \), is known. According to Theorem\(^3\) we can still choose \( r_n \) such that the result of Corollary\(^4\) holds for ACC with \( \hat{\theta}_S = \bar{X} \) and \( \hat{\tau}_S = \exp\{0.5(\text{Median}(\ln |(X_i - m)/(X_j - m)|))\} \), for \( m = \text{Median}(X_1, \ldots, X_n) \) and \( 1 \leq i, j \leq n, 1 \leq j \). Note \( \hat{\tau}_S \) is the median-adjusted Hodges-Lehmann estimator for the Cauchy scale parameter from Kravchuk and Pollett (2012); therefore,
\[
f(\hat{\theta}_S; \theta) \propto [1 + (\hat{\theta}_S - \theta)^2]^{-1} \quad \text{and} \quad f(\hat{\tau}_S; \tau) \propto [a\hat{\tau}_S]^{-1} \exp \left\{ -[2a^2(\ln \hat{\tau}_S - \ln \tau)^2]^{-1} \right\},
\]
where \( a = 2/(n(\pi^2/4\sqrt{6})^2) \).

For a sample of size \( n = 400 \), applying ABC with a “flat” prior on \( \theta \), \( \pi(\theta) \propto \exp\{- (\theta - 10)^2/128\} \), and a “flat” prior on \( \tau \), \( \pi(\tau) = \tau^{-2} \exp\{1/2\tau\} \), Figure\(^3\) compares the ABC-posterior distributions of \( \hat{\theta}_S/\tau \) and \( \hat{\tau}_S \) to the distributions of \( \hat{\theta}_S/\hat{\tau}_S \) and \( \hat{\tau}_S/\tau \). With \( \rho((\hat{\theta}_S, \hat{\tau}_S), (\hat{\theta}_{\text{obs}}, \hat{\tau}_{\text{obs}})) = \sqrt{(\hat{\theta}_S - \hat{\theta}_{\text{obs}})^2 + (\hat{\tau}_S - \hat{\tau}_{\text{obs}})^2} \) and \( \varepsilon = 0.1 \), the acceptance rate for \((\theta, \tau)\) was 0.042%.

Figure\(^4\) shows the results of applying ACC using the method of Section 2.2 to derive a data-dependent \( r_n \). Therefore, to define \( r_n \), we created a set of \( \hat{\theta}^{(i)} \) values for \( i = 1, \ldots, n^{1/2} \) and drew \( \theta \) values from a density estimate over these values. As in the ABC application, we used \( \hat{\tau}_S \) to estimate
Figure 3: ABC-posterior distributions of $\frac{\theta - \hat{\theta}_S}{\tau}$ (left) and $\frac{\tau}{\hat{\tau}_S}$ (right) are shown by solid (black) lines and their asymptotic distributions shown by the (black) dashed lines. The “flat” priors are shown in blue (dashed). The acceptance rate here was 0.042%.

$\tau$ which is drawn from a flat prior. The resulting distributions of $\frac{\theta_{ACC} - \hat{\theta}_S}{\tau_{ACC}}$ and $\frac{\tau_{ACC}}{\hat{\tau}_S}$ are shown in Figure 4. The (improved) acceptance rate for $(\theta_{ACC}, \tau_{ACC})$ was 0.1230%.

Figure 4: The distributions of $\frac{\theta_{ACC} - \hat{\theta}_S}{\tau_{ACC}}$ (left) and $\frac{\tau_{ACC}}{\hat{\tau}_S}$ (right) are shown by solid (black) lines and their asymptotic distributions (i.e. the distributions of $(\hat{\theta}_S - \theta) | \theta = 10$ and $\frac{\tau}{\hat{\tau}} | \tau = 1$) are shown by the dashed (black) lines. The choice of $r_\alpha$ is shown in blue (dashed). The acceptance rate here is 0.1230%.

For both ABC and ACC, Figures 3 and 4 verify the conclusions of Corollary 4 parts 1 and 2 in that we observe (approximately)

$$\frac{\theta_{ACC} - \hat{\theta}_S}{\tau_{ACC}} \sim (\hat{\theta}_S - \theta) | \theta = 10$$

and

$$\frac{\tau_{ACC}}{\hat{\tau}_S} \sim \frac{\hat{\tau}_S}{\tau} | \tau = 1.$$
Furthermore, additional simulation studies investigating the marginal coverage of the true $\theta$ and $\tau$ values for this example of ACC counted the number of times a 95% confidence interval for $\theta$ ($\tau$) contained the true value $\theta_0 = 10$ ($\tau_0 = 1$). The coverage out of 100 runs for $\theta$ was 94% and the coverage out of 100 runs for $\tau$ was 97%.

2.4 Importance Sampling

Practically speaking, the accept-reject version of ABC is too computationally costly to be useful. Li and Fearnhead (2016) examine the properties of an importance and rejection sampling ABC method; Bonassi and West (2015) introduce the ABC-SMC algorithm which, in addition to tolerance updating, takes advantage of importance sampling to improve upon the performance of ABC. Such extensions make ABC more useful by decreasing the computational cost of the method via an arbitrary data-dependent distribution function weighting the $N$ selected parameter values. We now extend the ACC algorithm by incorporating importance sampling.

[Step 1] Generate $\theta^*$ from $q_n(\theta)$

[Step 2] Generate $x^*$ from the model $M_{\theta^*}$ to produce $S_n^{*}$

Repeat steps 1 and 2 until $\rho(S_n^{*}, s_{obs}) \leq \varepsilon$. Set $\theta_i = \theta^*$ and assign the corresponding weight $w_i = r_n(\theta_i)/q_n(\theta_i)$. Start again at step 1 and continue until $N$ parameter and weight pairs are obtained.

IS-ACC produces samples from the distribution

$$p(\theta|s_{obs}) \propto q_n(\theta) \int \tilde{f}_n(S_n^{*}; \theta) I\{\rho(S_n^{*}, s_{obs}) \leq \varepsilon\} dS_n^{*}$$

and therefore the weighted accepted parameter values have density

$$p(\theta_{\text{ACC}}|s_{obs}, w_i, \ldots, w_N) \propto \frac{r_n(\theta)}{q_n(\theta)} p(\theta|s_{obs}) = r_n(\theta) \int \tilde{f}_n(S_n^{*}; \theta) I\{\rho(S_n^{*}, s_{obs}) \leq \varepsilon\} dS_n^{*}.$$ 

Thus we may modify ACC to include importance sampling while still maintaining the integrity of ACC-based inference. We summarize this fact in the following theorem.
THEOREM 5: Under the same assumptions of Theorem 1 as \( \varepsilon \to 0 \), IS-ACC can be used to derive a CD and if holds asymptotically as \( n \to \infty \), then IS-ACC can be used to derive an aCD.

To define their IS-ABC algorithm, Li and Fearnhead (2016) replace \( r_n(\theta) \) in IS-ACC with a prior distribution, \( \pi(\theta) \); however, in such circumstances, we suggest that rather than use IS-ABC, one instead sets \( q_n(\theta) = r_n(\theta) \) in ACC to generate an estimate based on a CD. Alternatively, one could follow the method of Section 2.2 to define \( r_n \) based on the observed data and use any other data dependent \( q_n \) to implement IS-ACC. Either of these adjustments could greatly reduce the necessary computing time.

3. Empirical Examples

3.1 A Birth-Death-Mutation Process with an Intractable Likelihood

In genetics it is common for inference to be hindered by the difficulty of constructing analytical likelihood functions. A model of disease transmission and genetic marker mutation as investigated in Tanaka et al. (2006) is such an example of a model with an intractable likelihood. Here the transmission of tuberculosis is modeled with a birth-death-mutation process, a continuous-time stochastic model that describes the growth in the number of infectious cases over time where the mutation of the disease marker is modeled to reflect instantaneous genetic fixation.

The pseudo-code to simulate a population of genetic marker mutations is described in Tanaka et al. (2006). Begin with one infected case, then randomly generate through cycles of birth, death, or mutation. If the event that occurs at a particular time point is birth/death, then the cluster corresponding to the current genotype increases/decreases by one. If the event is a mutation, the cluster corresponding to that genotype decreases by one while simultaneously a new cluster of size one is created.

Let \( \alpha \) be the birth rate per case per year, \( \delta \) be the death rate, and \( \theta \) be the mutation rate. Functions of the parameters \( \alpha \) and \( \delta \) are of biological interest. Let \( X_i(t) = \text{Number of cases of genotype } i \)
at time $t$, $G(t) =$ Number of distinct genotypes at time $t$, and $N(t) = \sum_{i=1}^{G(t)} X_i(t)$. Define the following probabilities: $P_{i,x}(t) = P(X_i(t) = x)$, $\bar{P}_n(t) = P(N(t) = n)$, $\bar{P}_g(t) = P(G(t) = g)$. Mathematically one may write the BDM process as a system of three stochastic differential equations

$$\frac{dP_{i,x}(t)}{dt} = -(\alpha + \delta + \theta) x P_{i,x}(t) + \alpha (x-1) P_{i,x-1}(t) + (\delta + \theta) (x+1) P_{i,x+1}(t)$$

$$\frac{d\bar{P}_g(t)}{dt} = -\theta N(t) \bar{P}_g(t) + \theta N(t) \bar{P}_{g-1}(t)$$

$$\frac{d\bar{P}_n(t)}{dt} = -(\alpha - \delta) n \bar{P}_n(t) + \alpha \bar{P}_{n-1}(t) + \delta (n+1) \bar{P}_{n+1}(t)$$

subject to the boundary conditions: $\frac{dP_{i,0}(t)}{dt} = (\delta + \theta) P_{i,1}(t)$; $\frac{d\bar{P}_1(t)}{dt} = -\theta N(t) \bar{P}_1(t)$; and $\frac{d\bar{P}_g(t)}{dt} = \delta \bar{P}_1(t)$ and subject to the initial conditions: $G(0) = 1$; $P_{g,1}(t_g) = 1$ and $P_{g,x}(t_g) = 0$ for $x \neq 1$ where $t_g = \{ \text{time at which genotype } g \text{ first appears} \}$; and $\bar{P}_1(0) = 1$ and $\bar{P}_n(0) = 0$ for $n \neq 1$.

Here $\frac{d\bar{P}_g(t)}{dt}$ describes the creation of new genotypes and $\frac{d\bar{P}_n(t)}{dt}$ describes the changes in the total number of cases.

As specified in [Tanaka et al. (2006)](#), this model assumes the time until the next event is exponentially distributed with parameter $N(t) \times (\alpha + \delta + \theta)$ but these times are not simulated since the total time experienced by the infectious population is not of interest. Instead the simulations run until the population reaches a particular size determined to reflect appropriate genetic diversity. Once the population reaches this size a sample is taken from which we calculate summary statistics.

The observed data $x_1, \ldots, x_g$ consists of a random sample of size $n$ drawn without replacement from the population of size $G(t)$. Each point $x_i$ expresses the number of occurrences of genotype $i$ in the sample for $i = 1, \ldots, g$. Thus $\sum_{i=1}^{g} x_i = n$. [Tanaka et al. (2006)](#) justifies selecting the biologically relevant (though somewhat mathematically arbitrary) summary statistics $S_1 = g$, and $S_2 = 1 - \sum_{i=1}^{g} (x_i/n)^2$.

To define $r_n$ and apply ACC, we consider a new parameter $\tau = (\alpha - \delta) - \theta$. This parameter is related to the clustering of the genotypes. One may (naively) estimate $\tau$ with $\hat{\tau} = \frac{1}{n} \sum_{i=1}^{g} \{ n_i : n_i > 1 \}$. Note that $\hat{\tau}$ is not necessarily a good estimate for $\tau$, nevertheless it does contain some
additional information about the data. Since \( \hat{\tau} = \frac{1}{n} \sum_{i=1}^{g} \{ n_i : n_i > 1 \} \) averages the number of clusters that have had enough time to mutate, \( \hat{\tau} \) reflects some additional information about the clusteredness of the data. Therefore the function \( r_n \) contains more information about \((\alpha, \delta)\) from the data than the flat prior in Tanaka et al. (2006).

Tanaka et al. (2006) uses ABC to approximate the posterior of \((\alpha - \delta)\) with prior

\[
p(\alpha, \delta, \theta) \propto \exp \left\{ -\frac{(\theta - 0.198)^2}{2 \cdot 0.06735^2} \right\} \mathbb{I}\{0 < \delta < \alpha\}.
\]  

(4)

We consider the basic accept-reject version of ABC with this prior setting \( \varepsilon = 0.1 \) and compare the performance to the accept-reject version of ACC with \( r_n(\alpha, \delta, \theta) \) defined to be

\[
p(\tau, \theta) \propto \exp \left\{ -\frac{(\tau - \hat{\tau})^2}{2c_1^2} \right\} \exp \left\{ -\frac{(\theta - 0.198)^2}{2 \cdot 0.06735^2} \right\} \mathbb{I}\{\theta, \tau > 0\}
\]

(5)

\[
p(\alpha, \delta | \tau, \theta) \propto \exp \left\{ -\frac{(\alpha - \tau)^2}{2c_2^2} \right\} \mathbb{I}\{0 < \delta < \tau + \theta < \alpha\},
\]

where \( c_1 \) and \( c_2 \) are constants chosen to be large enough to satisfy assumptions 9 and 10 from Theorem 2. For a comparison of the regions specified by \( \pi(\alpha, \delta, \theta) \) to those specified by \( r_n(\alpha, \delta, \theta) \) for various choices of \( c = c_1 = c_2 \) see Figure 5. Note that as \( c \) increases, the searched region for ACC becomes more similar to the uninformative region of ABC (shown in black). The observed data used in each trial produced the summary statistics \( s_{\text{obs}_1} = 48 \), \( s_{\text{obs}_2} = 0.9591 \), and \( \hat{\tau} = 0.5625 \).

The true values of the parameters used to simulate the data are \( \alpha_0 = 0.90 \), \( \delta_0 = 0.30 \), and \( \theta_0 = 0.15 \).

**Figure 5:** Drawing \( \alpha \) (horizontal axis) and \( \delta \) (vertical axis) from a flat prior (black) and from \( r_n \) in equation (5) with \( c_1 = c_2 = \{0.25, 0.1, 0.01\} \) (red ordered left to right).
Figure 6 shows the results of applying ACC (with \(c_1 = c_2 = 0.1\)) and ABC to this simulated example to collect \(N = 1000\) samples from a CD for \((\alpha - \delta)\). The distance function from Tanaka et al. (2006), \(\rho((S_1, S_2), (s_{obs_1}, s_{obs_2})) = \frac{1}{n}|S_1 - s_{obs_1}| + |S_2 - s_{obs_2}|\), is used for both ABC and ACC. The computations were executed in parallel across four cores. In each trial we allow the population of genotypes to grow until reaching size 2000 (not biologically accurate but suitable for the purpose of illustration) and we randomly select a sample of size 80 from this simulated population. The true value of \((\alpha - \delta)\) in this example is 0.60. The ABC method produces an ABC posterior with a mode that is lower than the truth whereas the ACC method produces a distribution estimate that is more concentrated around the true value \((\alpha - \delta = 0.6)\). The first and third quartiles of the computing time to generate and keep a value for \((\alpha, \delta, \theta)\) in ABC were (9.61s, 24s), respectively. In contrast, the first and third quartiles of computing time for ACC with \(c_1 = c_2 = 0.10\) were (9.61s, 16.47s). Simulations based on the ACC method verify that the resulting confidence intervals have (at least) the nominal frequentist coverage.

Figure 6: Left: Computing time (excluding outliers) to generate and keep a single parameter value in ABC and ACC with \(N = 1000\). Middle: ABC posterior for \((\alpha - \delta)\) \((\varepsilon = 0.1)\). Right: Distribution of \((\alpha_{\text{ACC}} - \delta_{\text{ACC}})\) \((\varepsilon = 0.1)\). The true value of \((\alpha - \delta)\) is 0.6 and is marked with a line.
3.2 Real Data Application

In this section, we apply ACC to some real data found in Tanaka et al. (2006) investigating the transmission of tuberculosis using the birth-death-mutation process model specified in the previous section. The observed data is

\[30^123^115^110^18^15^24^43^13^22^01^282\]

where \(m^k\) indicates an observation of \(k\) genotype clusters of size \(m\). The sample size is \(n = 473\).

The summary statistics \(S_1 = g\) and \(S_2 = 1 - \sum_{i=1}^{g}(x_i/n)^2\) are the same as in the previous section and for this data set are \(s_{\text{obs1}} = 326\) and \(s_{\text{obs2}} = 0.9892236\). However, we follow the example set by Li and Fearnhead (2016) and select a third summary statistic in order to match the number of summary statistics to the number of unknown parameters \((\alpha, \delta, \theta)\). With \(S_3 = \hat{\tau}\) and \(s_{\text{obs3}} = 0.4038055\) we modify \(\rho\) so that \(\rho((S_1, S_2, S_3), (s_{\text{obs1}}, s_{\text{obs2}}, s_{\text{obs3}})) = \frac{1}{n}|S_1 - s_{\text{obs1}}| + |S_2 - s_{\text{obs2}}| + |S_3 - s_{\text{obs3}}|\) and as Tanaka et al. (2006) we set \(\varepsilon = 0.025\). The BDM simulations must run long enough to generate a population of size \(O(10^4)\) in order to yield biologically relevant results.

Figure 7 compares the ABC-posterior from Tanaka et al. (2006) with a distribution estimate for \((\alpha - \delta)\) based on ACC with \(r_n\) chosen according to (5) for \(c_1 = c_2 = 0.1\). The median acceptance rate of ACC was 16.25% compared to the acceptance rate in Tanaka et al. (2006) of 10.3% for the same value of \(\varepsilon\).

4. Discussion

We have introduced ACC, a new likelihood-free method that does not depend on any Bayesian assumptions such as prior information. Rather than compare the output to a target posterior distribution, ACC quantifies the uncertainty in estimation from drawing upon a direct connection to a CD. This connection to CD theory guarantees that confidence intervals/regions based on ACC methods capture the truth about the parameters of interest at least at the nominal level and thus we provide theoretical support for ACC-based inference including but not limited to, the special case
Figure 7: The distribution of $\left(\alpha_{ACC} - \delta_{ACC}\right)$ (left) with $\varepsilon = 0.025$ compared to a digitally reproduced version of the ABC-posterior for $\alpha - \delta$ from Figure 1 of Tanaka et al. 2006 with $\varepsilon = 0.025$ (right, solid gray).

where we do have prior information (ABC). Furthermore, in the special case where the selected summary statistic is sufficient, the ACC method is also efficient.

In addition to providing sound theoretical results for inference, the framework of ACC (which can encompass likelihood-free methods from Bayesian, frequentist, or even fiducial paradigms; see, e.g., Hannig et al. (2016)) sets the user up for better computational performance by allowing the data to drive the algorithm through the choice of $r_n$. The potential computational advantage of ACC has been illustrated through an epidemiological example, first with oracle knowledge and then, with a real dataset drawn from the ABC literature.

We find the philosophical interpretation admitted through ACC to be more natural than the Bayesian interpretation of ABC. Within a frequentist setting, it makes sense to view the many different (due to choice of summary statistic) potential CDs produced by ACC as various choices of estimators. However, within the Bayesian framework, there is no clear way to choose from among the different ABC posteriors due to various choices of summary statistics. In particular, there is an ambiguity in defining the probability measure on the joint space $(\mathcal{P}, \mathcal{X})$ when choosing among different ABC posteriors. Rather than engaging in a pursuit to define a moving target such as this, ACC maintains a consistently clear frequentist interpretation (under the same random sample
setting with a fixed set of parameters) and thereby offers a consistently cohesive interpretation of likelihood-free methods.

Appendix

Proof of Theorem[1] Suppose there exist mappings \( V, W : \mathcal{P} \times \mathbb{R}^m \rightarrow \mathbb{R} \) such that equation (1) holds and let \( G^*(t) = P(V(\theta_{ACC}, \hat{\theta}_S) \leq t | x_{obs}) \) and \( G(t) = P(W(\hat{\theta}_S, \theta_0) \leq t) \).

It follows that \( G(W(\hat{\theta}_S, \theta_0)) \sim \text{Unif}(0,1) \) and since \( G^* \) is monotone we can define \( q^* = (G^*)^{-1}(1-\alpha) \). Thus for \( S_\alpha = \{ \theta | W(\hat{\theta}_S, \theta) \leq q^* \} \),

\[
P(\theta_0 \in S_\alpha) = P(\theta_0 \in \{ \theta | G^*(W(\hat{\theta}_S, \theta)) \leq 1-\alpha \}) = P(\theta_0 \in \{ \theta | G(W(\hat{\theta}_S, \theta)) \leq 1-\alpha \}) = 1-\alpha,
\]

where the second to last equality holds by equation (1). Therefore we have that \( S_\alpha \) is a \((1-\alpha)100\%\) confidence region for \( \theta \) from which it immediately follows that \( \theta_{ACC} \) is a CD random variable.

Proof of Lemma[1]: The proof of this lemma is simply a multivariate extension of Theorem A in Serfling (1980) pages 250—251. Note by the assumptions in Lemma 1 we have that \( \tilde{T}_S \) is consistent for \( \theta_0 \) and therefore \( \tilde{T}_n \) is consistent for \( I_S(\theta_0) \).

Proof of Lemma[2]: Following the format of the proof of Theorem 4.2 in Ghosh et al. (2007), we prove here that

\[
\int \left| p(t|S_n) - \sqrt{\frac{\text{Det}(I_S(\theta_0))}{(2\pi)^p}} e^{-\frac{1}{2} t^T I_S(\theta_0) t} \right| dt \rightarrow 0 \text{ with } P_{\theta_0}-\text{probability one where}
\]

\[
t = \sqrt{n}(\theta - \hat{\theta}_S) \quad \text{and} \quad p(t|S_n) = \frac{r_n(\hat{\theta}_S + t/\sqrt{n})}{\text{Det}(I_S(\theta_0))} \exp\left\{ \frac{\tilde{\ell}_n(\hat{\theta}_S + t/\sqrt{n}) - \tilde{\ell}_n(\hat{\theta}_S)}{\text{Det}(I_S(\theta_0))} \right\}.
\]

For simplicity define \( h_n(t) = \frac{r_n(\hat{\theta}_S + t/\sqrt{n})}{r_n(\theta_0)} \exp\left\{ \frac{\tilde{\ell}_n(\hat{\theta}_S + t/\sqrt{n}) - \tilde{\ell}_n(\hat{\theta}_S)}{\text{Det}(I_S(\theta_0))} \right\} e^{-\frac{1}{2} t^T I_S(\theta_0) t} \). First we show that it is sufficient to prove \( \int |h_n(t)| dt \rightarrow 0 \) with \( P_{\theta_0}-\text{probability one.} \)

If \( \int |h_n(t)| dt \rightarrow 0 \), then \( \int \left| \frac{r_n(\hat{\theta}_S + t/\sqrt{n})}{r_n(\theta_0)} \exp\left\{ \frac{\tilde{\ell}_n(\hat{\theta}_S + t/\sqrt{n}) - \tilde{\ell}_n(\hat{\theta}_S)}{\text{Det}(I_S(\theta_0))} \right\} e^{-\frac{1}{2} t^T I_S(\theta_0) t} \right| dt \rightarrow 0 \) that is,

\[
\int \frac{r_n(\hat{\theta}_S + t/\sqrt{n})}{r_n(\theta_0)} \exp\left\{ \frac{\tilde{\ell}_n(\hat{\theta}_S + t/\sqrt{n}) - \tilde{\ell}_n(\hat{\theta}_S)}{\text{Det}(I_S(\theta_0))} \right\} dt \rightarrow 0 \quad \text{and} \quad \int e^{-\frac{1}{2} t^T I_S(\theta_0) t} dt = \sqrt{\frac{(2\pi)^p}{\text{Det}(I_S(\theta_0))}}.
\]

For ease of notation let \( k_n(t|S_n) = \frac{r_n(\hat{\theta}_S + t/\sqrt{n})}{r_n(\theta_0)} \exp\left\{ \frac{\tilde{\ell}_n(\hat{\theta}_S + t/\sqrt{n}) - \tilde{\ell}_n(\hat{\theta}_S)}{\text{Det}(I_S(\theta_0))} \right\} \) so that \( \int k_n(t|S_n) dt \rightarrow 0 \).
\[
\sqrt{\frac{(2\pi)^p}{\text{Det}(I_S(\theta_0))}}. \text{ Now since } \int k_n(t|S_n) - e^{-\frac{1}{2}t^T I_S(\theta_0)t} \, dt \to 0 \text{ we can write }
\]

\[
\int \left| p(t|S_n) - \sqrt{\frac{\text{Det}(I_S(\theta_0))}{(2\pi)^p}} e^{-\frac{1}{2}t^T I_S(\theta_0)t} \right| \, dt 
= \left( \int k_n(t|S_n) \, dt \right)^{-1} \int \left| k_n(t|S_n) - \left( \int k_n(t|S_n) \, dt \right) \sqrt{\frac{\text{Det}(I_S(\theta_0))}{(2\pi)^p}} e^{-\frac{1}{2}t^T I_S(\theta_0)t} \right| \, dt 
\leq \left( \int k_n(t|S_n) \, dt \right)^{-1} \left[ \int \left| k_n(t|S_n) - e^{-\frac{1}{2}t^T I_S(\theta_0)t} \right| \, dt 
+ \int \left| e^{-\frac{1}{2}t^T I_S(\theta_0)t} - \left( \int k_n(t|S_n) \, dt \right) \sqrt{\frac{\text{Det}(I_S(\theta_0))}{(2\pi)^p}} e^{-\frac{1}{2}t^T I_S(\theta_0)t} \right| \, dt \right].
\]

Note the first term in the bracketed sum tends toward zero. Since we assume \( I_S(\theta_0) \) is positive definite, we can apply the dominated convergence theorem to the second term inside the bracketed sum as follows,

\[
\lim_{n \to \infty} \int \left| e^{-\frac{1}{2}t^T I_S(\theta_0)t} - \left( \int k_n(t|S_n) \, dt \right) \sqrt{\frac{\text{Det}(I_S(\theta_0))}{(2\pi)^p}} e^{-\frac{1}{2}t^T I_S(\theta_0)t} \right| \, dt 
= \int e^{-\frac{1}{2}t^T I_S(\theta_0)t} \lim_{n \to \infty} \left| 1 - \left( \int k_n(t|S_n) \, dt \right) \sqrt{\frac{\text{Det}(I_S(\theta_0))}{(2\pi)^p}} \right| \, dt 
= \int e^{-\frac{1}{2}t^T I_S(\theta_0)t} \lim_{n \to \infty} \left| 1 - \sqrt{\frac{(2\pi)^p}{\text{Det}(I_S(\theta_0))}} \right| \, dt = 0.
\]

Hence \( \int |h_n(t)| \, dt \to 0 \) implies \( \int |p(t|S_n) - \sqrt{\frac{\text{Det}(I_S(\theta_0))}{(2\pi)^p}} e^{-\frac{1}{2}t^T I_S(\theta_0)t}| \to 0 \). We complete the proof by considering two regions, \( A_1 = \{t : ||t|| > \delta \sqrt{n}\} \) and \( A_2 = \{t : ||t|| < \delta \sqrt{n}\} \). First,

\[
\int_{A_1} |h_n(t)| \, dt \leq \int_{A_1} \frac{r_n(\tilde{\theta}_S + \frac{t}{\sqrt{n}})}{r_n(\theta_0)} \exp \left\{ \tilde{\ell}_n \left( \tilde{\theta}_S + \frac{t}{\sqrt{n}} \right) - \tilde{\ell}_n(\tilde{\theta}_S) \right\} \, dt + \int_{A_1} e^{-\frac{1}{2}t^T I_S(\theta_0)t} \, dt.
\]

By Assumption 9 the first term of the inequality tends toward zero. The second term tends to zero since it represents the tails of a multivariate Gaussian distribution.
Now to prove convergence to zero in $A_2$ we derive the series expansion

$$
\bar{\ell}_n \left( \dot{\theta}_S + \frac{t}{\sqrt{n}} \right) - \bar{\ell}_n(\dot{\theta}_S) = \frac{1}{\sqrt{n}} \left[ \frac{\partial}{\partial \dot{\theta}_i} \bar{\ell}_n(\dot{\theta}_S) \right] t + \frac{1}{2} \left( \frac{1}{\sqrt{n}} \right)^2 t^T \left[ \frac{\partial^2}{\partial \dot{\theta}_i \partial \dot{\theta}_j} \bar{\ell}_n(\dot{\theta}_S) \right] t
$$

$$
+ \frac{1}{6} \left( \frac{1}{\sqrt{n}} \right)^3 t^T \left[ \sum_{k=1}^p \left( \frac{\partial^3}{\partial \dot{\theta}_i \partial \dot{\theta}_j \partial \dot{\theta}_k} \bar{\ell}_n(\dot{\theta}') \right) t_k \right] t
$$

$$
= -\frac{1}{2} t^T \tilde{I}_n t + \frac{1}{6} t^T \sum_{k=1}^p \left( \frac{\partial^3}{\partial \dot{\theta}_i \partial \dot{\theta}_j \partial \dot{\theta}_k} \bar{\ell}_n(\dot{\theta}') \right) \frac{t_k}{n} t
$$

$$
+ \frac{1}{4} t^T I_S(\theta_0) t - \frac{1}{4} t^T I_S(\theta_0) t
$$

where $\theta' = \dot{\theta}_S + \epsilon_1(t)$ for some $|\epsilon_1(t)| \leq \delta$. Here $\frac{1}{\sqrt{n}} \left( \frac{\partial}{\partial \dot{\theta}_i} \bar{\ell}_n(\dot{\theta}_S) \right) t = 0$ by the definition of $\dot{\theta}_S$.

Note that for each $k$, $\left[ \frac{\partial^3}{\partial \dot{\theta}_i \partial \dot{\theta}_j \partial \dot{\theta}_k} \bar{\ell}_n(\dot{\theta}') \right]$ is a $p \times p$ matrix and $t_k$ is a scalar. By Assumption 10, for any pair of $i, j \in \{1, \ldots, p\}$, $\left[ \sum_{k=1}^p \left( \frac{\partial^3}{\partial \dot{\theta}_i \partial \dot{\theta}_j \partial \dot{\theta}_k} \bar{\ell}_n(\dot{\theta}') \right) \frac{t_k}{n} \right]_{i,j} \leq p M(s_{obs}) \sum_{k} \frac{t_k}{n} \leq p^2 M(s_{obs}) \delta$. By Assumption 3, provided $\delta$ is small enough, $1/4 I_S(\theta_0) - \frac{1}{6} \sum_{k=1}^p \left( \frac{\partial^3}{\partial \dot{\theta}_i \partial \dot{\theta}_j \partial \dot{\theta}_k} \bar{\ell}_n(\dot{\theta}') \right) \frac{t_k}{n}$ is positive definite. Therefore

$$
\exp \left\{ \bar{\ell}_n \left( \dot{\theta}_S + \frac{t}{\sqrt{n}} \right) - \bar{\ell}_n(\dot{\theta}_S) \right\}
$$

$$
= \exp \left\{ -\frac{1}{2} t^T \tilde{I}_n t + \frac{1}{4} t^T I_S(\theta_0) t - \frac{1}{6} t^T \sum_{k=1}^p \left( \frac{\partial^3}{\partial \dot{\theta}_i \partial \dot{\theta}_j \partial \dot{\theta}_k} \bar{\ell}_n(\dot{\theta}') \right) \frac{t_k}{n} t \right\}
$$

$$
\leq \exp \left\{ -\frac{1}{2} t^T \tilde{I}_n t + \frac{1}{4} t^T I_S(\theta_0) t \right\}
$$

Now we can apply the dominated convergence theorem for $t \in A_2$ and thus

$$
\lim_{n \to \infty} \int_{A_2} |h_n(t)| dt = \int_{A_2} \lim_{n \to \infty} \left| \frac{r_n \left( \dot{\theta}_S + \frac{t}{\sqrt{n}} \right)}{r_n(\theta_0)} \right| \exp \left\{ \bar{\ell}_n \left( \dot{\theta}_S + \frac{t}{\sqrt{n}} \right) - \bar{\ell}_n(\dot{\theta}_S) \right\} - e^{-\frac{1}{2} t^T I_S(\theta_0) t} \right| dt
$$

$$
= \int_{A_2} \lim_{n \to \infty} \left| \frac{r_n \left( \dot{\theta}_S + \frac{t}{\sqrt{n}} \right)}{r_n(\theta_0)} \right| \exp \left\{ -\frac{1}{2} t^T \tilde{I}_n t + \frac{1}{6} \left( \frac{1}{\sqrt{n}} \right)^3 t^T \times \right.
$$

$$
\left[ \sum_{k=1}^p \left( \frac{\partial^3}{\partial \dot{\theta}_i \partial \dot{\theta}_j \partial \dot{\theta}_k} \bar{\ell}_n(\dot{\theta}') \right) \right] t_k \right\} - e^{-\frac{1}{2} t^T I_S(\theta_0) t} \right| dt
$$

$$
\xrightarrow{n \to \infty} \int_{A_2} e^{-\frac{1}{2} t^T I_S(\theta_0) t} - e^{-\frac{1}{2} t^T I_S(\theta_0) t} dt = 0 \text{ with } P_{\theta_0} \text{-probability one.}
$$

**Proof of Theorem 3** In an ACC algorithm, we simulate $\theta^* \sim r_n(\theta)$ and $S_n^* \sim M_{\theta^*}$. Since $\Psi$ is a pivot function, we have $\Psi(S_n^*, \theta^*) \mid \theta^* \sim g(t)$. Denote by $T^* = \Psi(S_n^*, \theta^*)$, it follows that unconditionally, $(T^*, \theta^*) \sim r_n(\theta) g(t)$. In an ACC algorithm, we only keep those $\theta^*$ which
generate $S^*_n$ such that $\rho(S^*_n, s_{obs}) \leq \varepsilon$. So conditional on $s_{obs}$, $T_{ACC} = \Psi(S_n, \theta_{ACC})$ follows the distribution with density
\[
p_{\varepsilon}(t|s_{obs}) \propto \int r_n(\theta) g(t) \mathbb{I}\{\rho(u_{t, \theta}, s_{obs}) \leq \varepsilon\} d\theta \propto g(t) \int r_n(\theta) \mathbb{I}\{\rho(u_{t, \theta}, s_{obs}) \leq \varepsilon\} d\theta
\]
In the above formula, $u_{t, \theta}$ is the solution of $t = \Psi(u, \theta)$ for any given values of $t$ and $\theta$.

If $\int r_n(\theta) \mathbb{I}\{\rho(u_{t, \theta}, s_{obs}) \leq \varepsilon\} d\theta$ is free of $t$, then $\Psi(S_n, \theta_{ACC})|s_{obs} \sim G(t)$ and we may apply the results of Theorem 1.

References

Barber, S., Voss, J., and Webster, M. (2015). The rate of convergence for approximate Bayesian computation. *Electronic Journal of Statistics* **9**, 80–105.

Beaumont, M. A., Cornuet, J.-M., Marin, J.-M., and Robert, C. P. (2009). Adaptive approximate Bayesian computation. *Biometrika* **20**, 1–9.

Bonassi, F. V. and West, M. (2015). Sequential Monte Carlo with adaptive weights for approximate Bayesian computation. *Bayesian Analysis* **10**, 171–187.

Cameron, E. and Pettitt, A. N. (2012). Approximate bayesian computation for astronomical model analysis: A case study in galaxy demographics and morphological transformation at high redshift. *Monthly Notices of the Royal Astronomical Society* **425**, 44–65.

Creel, M. and Kristensen, D. (2013). Indirect likelihood inference. *Manuscript, Department of Economics. Columbia University*.

Csilléry, K., Blum, M. G. B., Gaggiotti, O. E., and François, O. (2010). Approximate Bayesian computation (ABC) in practice. *Trends in Ecology and Evolution* **25**, 410–418.

Frazier, D. T., Martin, G. M., Robert, C. P., and Rousseau, J. (2016). Asymptotic properties of approximate Bayesian computation. *arXiv: 1607.06903*.

Ghosh, J. K., Delampady, M., and Samanta, T. (2007). *An Introduction to Bayesian Analysis Theory and Methods*. Springer Science+Business Media, LLC.
Gourieroux, C., Monfort, A., and Renault, E. (1993). Indirect inference. *Journal of Applied Econometrics* **8**, S85–S118.

Hannig, J., Iyer, H., Lai, C., and Lee, T. (2016). Generalized fiducial inference: A review and new results. *Journal of American Statistical Association*.

Joyce, P. and Marjoram, P. (2008). Approximately sufficient statistics and Bayesian computation. *Statistical Applications in Genetics and Molecular Biology* **7**, 26.

Kravchuk, O. Y. and Pollett, P. K. (2012). Hodges-lehmann scale estimator for Cauchy distribution. *Communications in Statistics - Theory and Methods* **41**, 3621–3632.

Li, W. and Fearnhead, P. (2016). On the asymptotic efficiency of ABC estimators. *arXiv: 1506.03481*.

Marin, J.-M., Pudlo, P., Robert, C. P., and Ryder, R. J. (2011). Approximate Bayesian computational methods. *Statistics and Computing* **22**, 1167–1180.

Robinson, J. D., Bunnefeld, L., Hearn, J., Stone, G. N., and Hickerson, M. J. (2014). ABC inference of multi-population divergence with admixture from unphased population genomic data. *Molecular Ecology* **23**, 4458–4471.

Schweder, T. and Hjort, N. L. (2016). *Confidence, Likelihood, Prabability*. Cambridge University Press.

Serfling, R. J. (1980). *Approximation Theorems of Mathematical Statistics*. John Wiley and Sons.

Singh, K. (1981). On the asymptotic accuracy of Efron's bootstrap. *The Annals of Statistics* **9**, 1187–1195.

Singh, K., Xie, M., and Strawderman, W. E. (2007). Confidence distribution (CD) - distribution estimator of a parameter. *IMS Lecture Notes* **54**, 132–150.

Sisson, S., Fan, Y., and Tanaka, M. (2007). Seuqential Monte Carlo without likelihoods. *Proceedings of the National Academie of Science USA* **104**, 1760–1765.

Tanaka, M. M., Francis, A. R., Luciani, F., and Sisson, S. A. (2006). Using approximate Bayesian
computation to estimate tuberculosis transmission parameters from genotype data. *Genetics* **173**, 1511–1520.

Xie, M. and Singh, K. (2013). Confidence distribution, the frequentist distribution estimator of a parameter: A review. *International Statistical Review* **81**, 3–39.