On contact brackets on the tensor product

Pasha Zusmanovich

Department of Mathematics, University of Ostrava, Ostrava, Czech Republic

ABSTRACT

We study the behaviour of contact brackets on the tensor product of two algebras, in particular, address the question of Martínez and Zelmanov about extension of a contact bracket on the tensor product from the brackets on the factors.

ARTICLE HISTORY

Received 18 November 2020
Accepted 21 February 2021

COMMUNICATED BY

M. Chebotar

KEYWORDS

Tensor product; contact bracket; polynomial algebra

AMS CLASSIFICATIONS

17B60; 17B63; 17B66

1. Introduction

We start by recalling some stuff from a recent interesting survey [1]. The standing assumptions are that the ground field $K$ is of characteristic $\neq 2$, and all commutative associative algebras under consideration contain a unit $1$. Let $A$ be a commutative associative algebra over $K$. A bilinear map $[\cdot, \cdot] : A \times A \to A$ is called a bracket on $A$. A bracket on $A$ is called contact, if $(A, [\cdot, \cdot])$ is a Lie algebra, and the identity

$$[ab, c] = [a, c]b + [b, c]a + [c, 1]ab$$

(1)

holds for any $a, b, c \in A$ (the condition that the linear map $a \mapsto [a, 1]$ is a derivation of $A$, included in the definition of the contact bracket in [1], follows from the identity (1); in some literature such structures appear under the names of Jacobi algebras, or generalized Poisson brackets, or combinations or variations thereof; see, for example, [2, Chapter III, § 5], [3,4], [5, Chapter 5], and references therein). In the particular case $[A, 1] = 0$ this reduces to the classical notion of the Poisson bracket on $A$. The following well-known construction is a paradigmatic example of a contact bracket that is not a Poisson bracket: given a derivation $D$ of an algebra $A$, consider the Lie bracket

$$[a, b] = D(a)b - D(b)a.$$  

(2)

The following question was posed as [1, Question 1], and will be referred to in the sequel as the Martínez–Zelmanov question: given two commutative associative algebras $A, B$, a Poisson bracket $[\cdot, \cdot]_A$ on $A$, and a contact bracket $[\cdot, \cdot]_B$ on $B$, is it always possible to define a contact bracket on the tensor product $A \otimes B$, extending the brackets $[\cdot, \cdot]_A$ and $[\cdot, \cdot]_B$?
It is the purpose of the present note to answer this question. The answer, in a sense, is both ‘yes’ and ‘no’. In general, the answer is negative, the corresponding example is constructed in Section 2. In a sense, this example is quite artificial and ‘degenerate’ – the resulting tensor product is isomorphic to the algebra \( K[x, y, z]/(x^2, y^2, z^2) \); we also indicate how one can produce other examples of a similar sort. On the other hand, for the most ‘interesting’ and ‘natural’ contact brackets appearing in mechanics, differential geometry, and structure theory of Lie algebras – the contact brackets defined on polynomial algebras, or on reduced polynomial algebras in the case of positive characteristic – the answer is affirmative; this is briefly discussed in Section 3.

2. Contact brackets that are not extended to the tensor product

Rewrite the condition (1) in the form

\[
[ab, c] = [a, c]b + [b, c]a - [1, c]ab,
\]

and denote by \( \mathcal{K}^-(A) \), respectively, by \( \mathcal{K}^+(A) \), the vector space of all anticommutative, respectively, commutative, brackets \([·, ·] : A × A → A\) satisfying the condition (3) (without requiring them to satisfy the Jacobi identity or any other additional condition). (Note that while for anticommutative brackets the conditions (1) and (3) are equivalent, for commutative brackets they are not.)

**Proposition 2.1:** Let \( A, B \) be commutative associative algebras, one of them is finite-dimensional. Then there is an embedding of vector spaces

\[
\mathcal{K}^-(A ⊗ B) \hookrightarrow \mathcal{K}^-(A) ⊗ K^+(B) + K^+(A) ⊗ \mathcal{K}^-(B).
\]

More precisely, any bracket from \( \mathcal{K}^-(A ⊗ B) \) can be represented as a (finite) sum \( \sum_{i∈I} f_i ⊗ g_i \), where for each \( i ∈ I \), either \( f_i ∈ \mathcal{K}^-(A) \) and \( g_i ∈ \mathcal{K}^+(A) \), or \( f_i ∈ \mathcal{K}^+(A) \) and \( g_i ∈ \mathcal{K}^-(A) \). Moreover, the equality

\[
\sum_{i∈I} \left( f_i(a, a''')a' - f_i(a', a''')a \right) \otimes \left( g_i(b, b''')b' - g_i(b', b''')b \right) = 0.
\]

holds for any \( a, a', a'' \in A, b, b', b'' \in B \).

**Proof:** The proof utilizes a simple linear-algebraic technique employed by us earlier to compute various structures on the tensor product of algebras in terms of the tensor factors, see [6,7].

Due to the finite-dimensionality assumption, we have an isomorphism of vector spaces

\[
\text{Hom}_K(A ⊗ B ⊗ A ⊗ B, A ⊗ B) \cong \text{Hom}_K(A ⊗ A, A) ⊗ \text{Hom}_K(B ⊗ B, B),
\]

thus any bracket \([·, ·] \) on \( A ⊗ B \) can be represented in the form

\[
[a ⊗ b, a' ⊗ b'] = \sum_{i∈I} f_i(a, a') ⊗ g_i(b, b')
\]
for any \(a, a' \in A, b, b' \in B\), where \(f_i\) are some brackets on \(A\), \(g_i\) are some brackets on \(B\). The condition \((1)\) is then equivalent to

\[
\sum_{i \in \mathbb{I}} f_i(aa', a'') \otimes g_i(bb', b'') - f_i(a, a')a' \otimes g_i(b, b'')b' - f_i(a', a'')a \otimes g_i(b', b'')b

- f_i(a''', 1)aa' \otimes g_i(b'', 1)bb' = 0
\]

(6)

for any \(a, a', a'' \in A, b, b', b'' \in B\).

Substituting in \((6)\) \(b = b' = 1\) yields:

\[
\sum_{i \in \mathbb{I}} \left( f_i(aa', a'') - f_i(a, a'')a' - f_i(a', a'')a \right) \otimes g_i(1, b'') - f_i(a'', 1)aa' \otimes g_i(b'', 1) = 0.
\]

(7)

The condition of anticommutativity of \([\cdot, \cdot]\) is equivalent to

\[
\sum_{i \in \mathbb{I}} f_i(a, a') \otimes g_i(b, b') + f_i(a', a) \otimes g_i(b', b) = 0.
\]

Symmetrizing this equality with respect to \(a, a'\) yields

\[
\sum_{i \in \mathbb{I}} (f_i(a, a') - f_i(a', a)) \otimes (g_i(b, b') - g_i(b', b)) = 0
\]

(8)

and

\[
\sum_{i \in \mathbb{I}} (f_i(a, a') + f_i(a', a)) \otimes (g_i(b', b) + g_i(b, b')) = 0.
\]

(9)

Applying to the last two equalities [6, Lemma 1.1], and observing that the condition of commutativity and anticommutativity of the same bilinear map entails zero map (characteristic is not 2), we may partition the index set \(\mathbb{I} = \mathbb{I}_1 \cup \mathbb{I}_2\) such that \(f_i\) is anticommutative and \(g_i\) is commutative for \(i \in \mathbb{I}_1\), and \(f_i\) is commutative and \(g_i\) is anticommutative for \(i \in \mathbb{I}_2\).\(^1\)

Using this partition, the equality \((7)\) can be rewritten as

\[
\sum_{i \in \mathbb{I}} \left( f_i(aa', a'') - f_i(a, a'')a' - f_i(a', a'')a + f_i(1, a'')a' \right) \otimes g_i(1, b'') = 0.
\]

(10)

Symmetrizing the equality \((6)\) with respect to \(a, a'\), we get the equality \((4)\).

Applying [6, Lemma 1.1] to the equalities \((10)\) and \((4)\), we get a partition \(\mathbb{I} = \mathbb{I}_{11} \cup \mathbb{I}_{12} \cup \mathbb{I}_{21} \cup \mathbb{I}_{22}\) such that

\[
\begin{align*}
  f_i(aa', a'') - f_i(a, a'')a' - f_i(a', a'')a & = f_i(1, a'')a' \quad \text{if } i \in \mathbb{I}_{11}, \\
  f_i(1, a'')a' = f_i(a', a'')a & \quad \text{if } i \in \mathbb{I}_{12}, \\
  f_i(aa', a'') - f_i(a, a'')a' - f_i(a', a'')a & + f_i(1, a'')a' = 0, \\
  g_i(b, b')b' = g_i(b', b'')b & \quad \text{if } i \in \mathbb{I}_{21}, \\
  f_i(a, a'')a' = f_i(a', a'')a & \quad \text{if } i \in \mathbb{I}_{22}, \\
  g_i(1, b'') = 0, & \quad g_i(b, b')b' = g_i(b', b'')b & \quad \text{if } i \in \mathbb{I}_{22}.
\end{align*}
\]

Observe that the identity \(\varphi(x, z)y = \varphi(y, z)x\) for a bracket \(\varphi\) on an algebra with unit is equivalent to the identity \(\varphi(x, y) = \varphi(1, y)x\). Consequently, the second condition for the
brackets $f_i$ with $i \in I_{11}$ implies the first one, $g_i$ with $i \in I_{22}$ vanish, and thus the identity
\[ f_i(aa', a'') - f_i(a, a'')a' - f_i(a', a'')a + f_i(1, a'')aa' = 0 \] (11)
holds for any $i \in I$.

Similarly, the identity
\[ g_i(bb', b'') - g_i(b, b'')b' - g_i(b', b'')b + g_i(1, b'')bb' = 0 \] (12)
also holds for any $i \in I$.

Applying again [6, Lemma 1.1] to the equalities (8) and (9), we get again the partition $I = I_1 \cup I_2$ such that $f_i$ is anticommutative and $g_i$ is commutative for $i \in I_1$, and $f_i$ is commutative and $g_i$ is anticommutative for $i \in I_2$, what, together with (11) and (12), gives decomposition of the bracket (5) exactly as in the statement of the theorem.

One might be tempted to try to pursue these reasonings further in an attempt to establish a formula expressing $K^-(A \otimes B)$ in terms of certain invariants of $A$ and $B$, similarly how it is done for various structures on the tensor product of two algebras in [6,7]. However, this does not seem to be possible. Indeed, for this approach to succeed, all the brackets from $K^-(A \otimes B)$ should be, at the end, representable as the sum of decomposable ones, i.e. the brackets of the form $f \otimes g$, where $f$ is a bracket on $A$, and $g$ is a bracket on $B$; but a glance, for example, at the brackets of type (2), or the more complicated brackets defining the simple Lie algebras of contact type (both infinite dimensional, and finite dimensional in positive characteristic; for the most general brackets of this type, see Section 3) defies such a possibility. In some particular cases, however, it is possible to get such exact formulas – for example, in Proposition 2.2. Note also that if we assume in (3) $[1, A] = 0$ (thus getting the condition defining Poisson brackets), the situation becomes much more tractable, even without assuming commutativity of algebras and anticommutativity of the bracket; see, for example, [8] for a sample of possible results in this direction.

Now we start to construct an example providing a negative answer to the Martínez–Zelmanov question. At the end, our example turns out to be $K[x, y, z]/(x^2, y^2, z^2)$, a quite trivial 8-dimensional algebra; the fact that it provides a negative answer to the question could be established by trivial, if a bit tedious, calculations. However, we choose to establish it as a corollary of intermediate statements formulated in a greater generality; this will allow us to understand better the structure of contact brackets on the tensor product in terms of the tensor factors, and opens possibilities to construct further examples and counter-examples of contact brackets with desired properties.

By generalized derivation of a commutative associative algebra $A$, we will understand a linear map $D : A \to A$ such that
\[ D(ab) = D(a)b + D(b)a - D(1)ab \] (13)
for any $a, b \in A$. (Generalized derivations of associative rings not necessarily commutative, and not necessarily having a unit – in the latter case $D(1)$ in the formula (13) is replaced by an arbitrary fixed element of the ring – were studied in a number of papers, see, for example, [9].)

The vector space of all generalized derivations of $A$ will be denoted by GDer($A$). Particular cases of generalized derivations are the usual derivations (with $D(1) = 0$), and
multiplications $R_u(a) = au$ on a fixed element $u \in A$. Thus, we always have an inclusion of vector spaces

$$\text{Der}(A) \oplus A \subseteq \text{GDer}(A),$$

where $\text{Der}(A)$ denotes the vector space (actually, a Lie algebra) of (ordinary) derivations of $A$, and the second direct summand at the left-hand side, $A$, is identified with the vector space of all multiplications by elements of $A$, via $u \leftrightarrow R_u$.

A commutative associative algebra $A$ will be called univariate-like, if it satisfies the following conditions:

(a) The inclusion (14) is an equality, i.e. each generalized derivation of $A$ is a sum of a derivation and a multiplication by an element of $A$;
(b) $\text{Der}(A)$ is an one-generated free $I_A$-module for some ideal $I_A$ of $A$, i.e. $\text{Der}(A) = I_A \partial_A$ for some linear map $\partial_A : A \rightarrow A$ (note that $\partial_A$ does not have to be a derivation of $A$);
(c) The preimage of 1 under the free generator $\partial_A$ is not empty.

Paradigmatic examples of univariate-like algebras are provided by the following:

**Lemma 2.1:** The algebras $K[x]$ and $K[x]/(x^n)$ for any $n \in \mathbb{N}$, are univariate-like.

**Proof:** This is well known (and could be established by straightforward calculations). Note that the properties of the algebra $K[x]/(x^n)$ are quite different depending on whether $n$ is divided by the characteristic of the ground field or not (in the latter case each derivation is of the form $f \frac{d}{dx}$, where $f \in xK[x]/(x^n)$), but the conclusion is valid in the both cases. ■

**Lemma 2.2:** Let $A$ be an univariate-like commutative associative algebra. Then there are the following isomorphisms of vector spaces:

(i) $K^-(A) \simeq \text{Der}(A) \simeq I_A$; each bracket from $K^-(A)$ is of the form (2), i.e.

$$[a, b] = u(a \partial_A(b) - b \partial_A(a))$$

for some $u \in I_A$.

(ii) $K^+(A) \simeq I_A \oplus \text{GDer}(A) \simeq I_A \oplus I_A \oplus A$; each bracket from $K^+(A)$ is of the form

$$[a, b] = u \partial_A(a) \partial_A(b) + v \partial_A(ab) + abw,$$

for some $u, v \in I_A, w \in A$.

**Proof:** Let $[-, -]$ be a bracket on $A$ satisfying the identity (3). Then for each $c \in A$, the linear map $a \mapsto [a, c]$ is a generalized derivation of $A$. Hence

$$[a, c] = \varphi(c) \partial(a) + a \psi(c)$$

for some maps $\varphi : A \rightarrow I_A, \psi : A \rightarrow A$, and any $a, c \in A$. Obviously, $\varphi, \psi$ can be chosen to be linear.

Also, according to the condition (c), fix $x \in A$ such that $\partial_A(x) = 1$. 

(i) If $[\cdot,\cdot]$ is anticommutative, then

$$\varphi(c)\partial_A(a) + a\psi(c) + \varphi(a)\partial_A(c) + c\psi(a) = 0.$$  

Substituting here 1’s yields $\psi(a) = -\varphi(1)\partial_A(a)$, and hence

$$\left(\varphi(c) - c\varphi(1)\right)\partial_A(a) + \left(\varphi(a) - a\varphi(1)\right)\partial_A(c) = 0$$

for any $a, c \in A$. Substituting here $x$ instead of $a$ and $c$, we get $\varphi(x) = x\varphi(1)$, $\varphi(1) = a\varphi(1)$, and, finally,

$$[a, c] = \varphi(1)(c\partial_A(a) - a\partial_A(c)).$$

(ii) If $[\cdot,\cdot]$ is commutative, then

$$\varphi(c)\partial_A(a) + a\psi(c) = \varphi(a)\partial_A(c) + c\psi(a).$$

Substituting here $c = 1$ yields $\psi(a) = \varphi(1)\partial_A(a) + a\varphi(1)$, and hence

$$\left(\varphi(c) - c\varphi(1)\right)\partial_A(a) = \left(\varphi(a) - a\varphi(1)\right)\partial_A(c)$$

for any $a, c \in A$. Substituting here $c = x$ yields

$$\varphi(a) = \left(\varphi(x) - x\varphi(1)\right)\partial_A(a) + a\varphi(1),$$

and hence

$$[a, c] = (\varphi(x) - x\varphi(1))\partial_A(a)\partial_A(c) + \varphi(1)\partial_A(ac) + ac\psi(1).$$

\[\blacksquare\]

**Proposition 2.2:** For any commutative associative algebra $A$,

$$\mathcal{K}^-(A \otimes K[x]/(x^2)) \simeq \mathcal{K}^-(A) \oplus \mathcal{K}^-(A) \oplus \text{Der}(A) \oplus A.$$  

Each bracket from $\mathcal{K}^-(A \otimes K[x]/(x^2))$ is of the form

$$[a \otimes 1, b \otimes 1] = \alpha(a, b) \otimes 1 + \beta(a, b) \otimes x,$$

$$[a \otimes 1, b \otimes x] = (\alpha(a, b) + bD(a) + abu) \otimes x,$$

$$[a \otimes x, b \otimes x] = 0$$

for any $a, b \in A$, where $\alpha, \beta \in \mathcal{K}^-(A)$, $D \in \text{Der}(A)$, $u \in A$.  

(15)
Proof: By Lemmas 2.1 and 2.2, $\mathcal{K}^{-}(K[x]/(x^2))$ is 1-dimensional, linearly spanned by the bracket

$$\varphi(f,g) = x \left( \frac{d}{dx}(g) - g \frac{d}{dx}(f) \right),$$

and $\mathcal{K}^{+}(K[x]/(x^2))$ is 4-dimensional, with the basis

$$\varphi_1(f,g) = x \frac{d}{dx}(f) \frac{d}{dx}(g), \quad \varphi_2(f,g) = x \frac{d}{dx}(fg),$$

$$\varphi_3(f,g) = fg - x \frac{d}{dx}(fg), \quad \varphi_4(f,g) = xfg,$$

where $f, g \in K[x]/(x^2)$. We have:

$$\varphi(1,x) = -\varphi(x,1) = x, \quad \varphi_1(x,x) = x, \quad \varphi_2(1,x) = \varphi_2(x,1) = x,$$

$$\varphi_3(1,1) = 1, \quad \varphi_4(1,1) = x,$$

and the values on all other pairs of the monomials $1, x$, are zero.

By Proposition 2.1, each bracket from $\mathcal{K}^{-}(A \otimes K[x]/(x^2))$ can be represented as

$$[a \otimes f, b \otimes g] = \chi(a, b) \otimes \varphi(f,g) + \sum_{i=1}^{4} \chi_i(a, b) \otimes \varphi_i(f,g),$$

where $\chi \in \mathcal{K}^{+}(A)$, and $\chi_i \in \mathcal{K}^{-}(A)$, $i = 1, 2, 3, 4$.

Writing the equality (3) for triple $a \otimes 1, b \otimes x, c \otimes 1$, and utilizing the fact that $\chi \in \mathcal{K}^{+}(A)$, $\chi_2 \in \mathcal{K}^{-}(A)$, we get:

$$-b\chi(a,c) + ab\chi_1(1,c) + b\chi_2(a,c) - ab\chi_2(1,c) = b\chi_3(a,c) - ab\chi_3(1,c)$$

for any $a, b, c \in A$.

Substitute here $b = 1$:

$$-\chi(a,c) + a\chi(1,c) + \chi_2(a,c) - a\chi_2(1,c) = \chi_3(a,c) - a\chi_3(1,c). \quad (16)$$

Substituting in (16), in its turn, $c = 1$, using skew-symmetry of $\chi_2$ and $\chi_3$, and substituting back the obtained equality into (16), we get

$$\chi_2(a,c) = \chi_3(a,c) + \chi(a,c) - 2a\chi(1,c) + ac\chi(1,1). \quad (17)$$

Symmetrizing this equality with respect to $a, c$, we get

$$\chi(a,c) = a\chi(1,c) + c\chi(1,a) - ac\chi(1,1). \quad (18)$$

Substitute (18) back into (17):

$$\chi_2(a,c) = \chi_3(a,c) + c\chi(1,a) - a\chi(1,c).$$

Taking into account (18), the condition $\chi \in \mathcal{K}^{+}(A)$ is equivalent to

$$\chi(1,ab) = b\chi(1,a) + a\chi(1,b) - ab\chi(1,1),$$

i.e. $\chi(1,\cdot) \in \text{GDer}(A)$. 

Denoting $D(a) = 2(\chi(1, a) - \chi(1, 1)a)$ and $u = \chi(1, 1)$, the obtained formulas can be rewritten as: $D \in \text{Der}(A)$, and

$$\chi(a, b) = \frac{1}{2}(aD(b) + bD(a)) + abu,$$

$$\chi_2(a, b) = \chi_3(a, b) + \frac{1}{2}(bD(a) - aD(b)).$$

Writing the equality (3) for triple $1 \otimes 1, b \otimes x, c \otimes x$, we get $\chi_1(b, c) = 0$.

Summarizing, we get that the bracket $[\cdot, \cdot]$ is represented in the form (15) (where $\alpha = \chi_3$ and $\beta = \chi_4$). Conversely, it is trivial to verify that each such skew-symmetric bracket lies in $\mathcal{K}^{-}(A \otimes K[x]/(x^2))$. \hfill \blacksquare

**Corollary 2.1**: For any commutative associative algebra $A$, each contact bracket on $A \otimes K[x]/(x^2)$ is of the form (15), subject to additional conditions

$$\beta(\alpha(a, b), c) + \beta(\alpha(b, c), a) + \beta(\alpha(c, a), b) + \alpha(\beta(b, c), a) + \alpha(\beta(c, a), b)$$

$$- \beta(a, b)D(c) - \beta(b, c)D(a) - \beta(c, a)D(b)$$

$$- (c\beta(a, b) + a\beta(b, c) + b\beta(c, a))u = 0$$

and

$$D(\alpha(a, b)) - \alpha(D(a), b) + \alpha(D(b), a) + \alpha(1, b)D(a) - \alpha(1, a)D(b)$$

$$+ \alpha(u, a)b - \alpha(u, b)a - \alpha(a, b)u + 2\alpha(1, b)au - 2\alpha(1, a)bu = 0$$

for any $a, b, c \in A$.

**Proof**: Amounts to writing the Jacobi identity for the bracket (15), which is equivalent to the specified equalities. \hfill \blacksquare

**Corollary 2.2**: Let $A$ be a commutative associative algebra, and $[\cdot, \cdot]_A \in \mathcal{K}^{-}(A)$. Then any bracket from $\mathcal{K}^{-}(A \otimes K[x]/(x^2))$ extending the bracket $[\cdot, \cdot]_A$ on $A$, and the bracket $\varphi$ on $K[x]/(x^2)$, is of the form (15) such that $\alpha = [\cdot, \cdot]_A$, $\beta = 0$, and $u = 1$.

**Proof**: Amounts to substituting the equalities $[a \otimes 1, b \otimes 1] = [a, b]_A \otimes 1$ and $[1 \otimes f, 1 \otimes g] = 1 \otimes \varphi(f, g)$ in (15). \hfill \blacksquare

Now we are ready to prove the main result of this note, answering in negative the Martinez–Zelmanov question.

**Theorem 2.1**: There exist two commutative associative algebras $A, B$, a Poisson bracket $[\cdot, \cdot]_A$ on $A$, and a contact bracket $[\cdot, \cdot]_B$ on $B$, such that there is no contact bracket on $A \otimes B$ extending brackets $[\cdot, \cdot]_A$ and $[\cdot, \cdot]_B$.

**Proof**: Let $B = K[x]/(x^2)$ and $[\cdot, \cdot]_B = \varphi$. Assume $[\cdot, \cdot]$ is a contact bracket on $A \otimes K[x]/(x^2)$, extending the brackets $[\cdot, \cdot]_A$ on $A$, and $\varphi$ on $K[x]/(x^2)$. By Corollaries 2.1
and 2.2,
\[ [a \otimes 1, b \otimes 1] = [a, b]_A \otimes 1 \]
\[ [a \otimes 1, b \otimes x] = ([a, b]_A + bD(a) + ab) \otimes x \]
\[ [a \otimes x, b \otimes x] = 0, \]

where \( D \in \text{Der}(A) \) is such that
\[
D([a, b]_A) = [D(a), b]_A + [D(b), a]_A + [1, b]_A D(a) - [1, a]_A D(b)
+ [1, a]_A b - [1, b]_A a - [a, b]_A + 2[1, b]_A a - 2[1, a]_A b = 0
\]
for any \( a, b \in A \). If \([\cdot, \cdot]_A\) is a Poisson bracket, the last equality is reduced to
\[ D([a, b]_A) = [D(a), b]_A + [D(b), a]_A = [a, b]_A. \] (19)

The left-hand side of this equality coincides with the standard action of \( D \) on the space of bilinear maps on \( A \), so our task reduces to finding an algebra \( A \) with a Poisson bracket \([\cdot, \cdot]_A\) not invariant with respect to this action for any derivation \( D \) of \( A \). There seems to be a plethora of such examples, one of the easiest is provided by the algebra \( A = K[x, y]/(x^2, y^2) \) and the Poisson bracket \([x, y]_A = xy\).

Indeed, any derivation \( D \) of \( K[x, y]/(x^2, y^2) \) is of the form \( f \frac{d}{dx} + g \frac{d}{dy} \), where \( f \in Kx \oplus Kxy \) and \( g \in Ky \oplus Kxy \), so substituting \( x = a \) and \( y = b \) in (19) yields
\[ fy + gx - [f, y] + [g, x] \] (20)
at the left-hand side and \( xy \) at the right-hand side. But a quick check for all possibilities for \( f, g \) shows that (20) always vanishes.

Our scheme allows to construct further examples providing a negative answer to the Martínez–Zelmanov question, and to its modifications, for example, when both brackets on the tensor factors are contact. This is left as an exercise to the reader.

Further, more elaborated, examples quite possibly could be obtained by considering quotient of polynomial algebras by non-homogeneous ideals. Such quotients may possess quite involved Poisson brackets, see, for example, [10] and references therein.

Finally, let us mention another series of examples. These examples are brackets of the form
\[ [a \otimes b, a' \otimes b'] = [a, a']_A \otimes bb' + aa' \otimes [b, b']_B, \] (21)
where \( a, a' \in A, b, b' \in B \). If \([\cdot, \cdot]_A\) and \([\cdot, \cdot]_B\) are Poisson brackets on \( A \) and \( B \), respectively, then this is again a Poisson bracket on \( A \otimes B \), a classical construction known from the literature as the tensor product of two Poisson brackets.

What happens if \([\cdot, \cdot]_A, [\cdot, \cdot]_B\) are contact brackets? It is straightforward to check that in this case the bracket (21) satisfies the condition (1), so the question reduces to whether (21) satisfies the Jacobi identity or not. It turns out that this is no longer necessarily true if at least one of \([\cdot, \cdot]_A, [\cdot, \cdot]_B\) is a contact bracket.² The corresponding examples were considered in an old interesting survey [11].
In the examples considered in that survey, $A$ and $B$ are reduced polynomial algebras of the form $K[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$, defined over a field of characteristic $p > 0$, with corresponding brackets yielding simple Lie algebras of the Cartan type of the series $W_n$, $H_{2n}$, $K_{2n+1}$ (or algebras close to them). For example, it was observed in [11, § 3] that the bracket (21) defined on $W_1 \otimes W_1$ satisfies the Jacobi identity, while on $W_1 \otimes H_2$, $H_2 \otimes K_3$, and $K_3 \otimes K_3$, it does not. This still does not give a full answer to the Martínez–Zelmanov question, as it could happen that the brackets on $A$ and $B$ can be extended to $A \otimes B$ in a way different from (21). In fact, as briefly explained in the next section, this is always the case: any contact bracket on two (reduced) polynomial algebras could be extended to a contact bracket on their tensor product.

3. Contact brackets that are extended to the tensor product

Most of the content of this section is hardly new: it is either contained in the literature, implicitly or explicitly, or can be obtained by immediate analogy with the known (Poisson) case; we omit almost all of the proofs. Still, the final conclusion – that any contact bracket on two polynomial algebras could be extended to a contact bracket on their tensor product – seems to be not explicitly recorded in the literature, and provides a nice contrast with Theorem 2.1.

When discussing brackets on polynomial and close to them algebras, it is convenient to adopt the following shorthand notation. For two linear operators $D, F : A \to A$ on an algebra $A$, its exterior product $D \wedge F : A \times A \to A$ is defined as

$$(D \wedge F)(a, b) = D(a)F(b) - D(b)F(a),$$

where $a, b \in A$. For example, using this notation, the bracket (2) can be written as $D \wedge \text{id}_A$, where $\text{id}_A$ is the identity map on $A$. More generally, consider the bracket of the form

$$[\cdot, \cdot] = \sum_{i=1}^n (D_i \wedge F_i) + D \wedge \text{id}_A,$$

(22)

where $D, D_1, \ldots, D_n, F_1, \ldots, F_n \in \text{Der}(A)$. It is easy to check that each such a bracket belongs to $\mathcal{K}^-(A)$. Let us call an associative commutative algebra $A$ standard, if, conversely, each bracket from $\mathcal{K}^-(A)$ is of the form (22). In particular, Lemma 2.2(i) implies that an univariate-like algebra is standard. We also have

**Proposition 3.1:** The polynomial algebra $K[x_1, \ldots, x_n]$, and the reduced polynomial algebra $K[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$ are standard.

**Proof:** More precisely, any bracket from $\mathcal{K}^-(K[x_1, \ldots, x_n])$ is of the form

$$[\cdot, \cdot] = \sum_{1 \leq i < j \leq n} \left( f_{ij} \frac{d}{dx_i} \wedge \frac{d}{dx_j} \right) + \left( \sum_{i=1}^n f_i \frac{d}{dx_i} \right) \wedge \text{id}_{K[x_1, \ldots, x_n]}$$

for some elements $f_{ij}, f_i \in K[x_1, \ldots, x_n]$. An elementary proof goes similarly to the classical Poisson case (where all $f_i$’s vanish); see, for example, [12, Proposition 1.6]. We set $f_i = [1, x_i]$
and \( f_{ij} = [x_i, x_j] + f_i x_j - f_j x_i \), and then proceed by induction on the sum of degrees of the monomials \( a, b \) in the expression \([a, b]\).

The reasoning in the case of reduced polynomial algebra is the same.

Necessary and sufficient conditions for the bracket \((22)\) to be a contact bracket are

\[
\begin{align*}
[[D, D]] &= 0, \\
[[D, D]] &= 2 D \wedge D,
\end{align*}
\]

where \( D = \sum_{i=1}^{n} D_i \wedge F_i \), and \([\cdot, \cdot]\) is the Schouten bracket ([13, Erratum], see also [3, Lemma 3.7]; note in passing that the conditions of Corollary 2.1 are reminiscent of the conditions \((23)\), as the left-hand sides of equalities there are reminiscent of the respective Schouten brackets, but since the algebra \( A \) there is, generally, not standard, the conditions of Corollary 2.1 look a bit more involved).

Now we can easily establish

**Theorem 3.1:** Let \( A, B \) be two standard commutative associative algebras, \([\cdot, \cdot]_A\) a contact bracket on \( A \), and \([\cdot, \cdot]_B\) a contact bracket on \( B \). Then there exists a contact bracket on \( A \otimes B \) extending brackets \([\cdot, \cdot]_A\) and \([\cdot, \cdot]_B\).

**Proof:** We have

\[
\begin{align*}
[\cdot, \cdot]_A &= \sum_{i=1}^{n} (D_i \wedge D'_i) + D \wedge \text{id}_A, \\
[\cdot, \cdot]_B &= \sum_{i=1}^{n} (F_i \wedge F'_i) + F \wedge \text{id}_B
\end{align*}
\]

for some \( D, D_i, D'_i \in \text{Der}(A) \) and \( F, F_i, F'_i \in \text{Der}(B) \).

Let us define the bracket \([\cdot, \cdot]\) on \( A \otimes B \) as

\[
[\cdot, \cdot] = \sum_{i=1}^{n} \left( (D_i \otimes \text{id}_B) \wedge (D'_i \otimes \text{id}_B) + (\text{id}_A \otimes F_i) \wedge (\text{id}_A \otimes F'_i) \right) \\
+ (D \wedge \text{id}_B) \otimes m_B + m_A \otimes (\text{id}_A \wedge F),
\]

where \( m_A : A \times A \to A \) and \( m_B : B \times B \to B \) are multiplications in algebras \( A \) and \( B \), respectively. It is a matter of routine verification, using the equalities \((23)\) for the brackets \([\cdot, \cdot]_A\) and \([\cdot, \cdot]_B\), to establish the same equalities for the bracket \([\cdot, \cdot]\). ■

Proposition 3.1 implies that the conclusion of Theorem 3.1 is applicable to the tensor product of two (reduced) polynomial algebras.

**Notes**

1. Of course, this is just a manifestation of the vector space isomorphism \( C^2(A \otimes B) \cong C^2(A) \otimes S^2(B) + S^2(A) \otimes C^2(B) \), where \( C^2 \) and \( S^2 \) denote the vector space of anticommutative and
commutative brackets on the corresponding algebra, respectively. However, we use a similar reasoning in more complicated situations, where the argument based on the above decomposition of $C^2(A \otimes B)$ is not enough.

2. Note that Remark 5.1.1(2) in [5], which claims the contrary, is incorrect.

Acknowledgments

Thanks are due to Ivan Kaygorodov for pointing out some relevant literature. GAP [14] was used to check some of the computations performed in this note.

Disclosure statement

No potential conflict of interest was reported by the author(s).

References

[1] Martínez C, Zelmanov E. Brackets, superalgebras and spectral gap. São Paulo J Math Sci. 2019;13(1):112–132.
[2] Hermann R. Yang-Mills, Kaluza-Klein, and the Einstein program. Interdisciplinary mathematics. Vol. XIX. Brookline (MA): Math Sci Press; 1978.
[3] Cantarini N, Kac V. Classification of linearly compact simple Jordan and generalized Poisson superalgebras. J Algebra. 2007;313(1):100–124.
[4] Kaygorodov I. Algebras of Jordan brackets and generalized poisson algebras. Lin Multilin Algebra. 2017;65(6):1142–1157.
[5] Agore AL, Militaru G. Extending structures. fundamentals and applications. Boca Raton (FL): CRC Press; 2020.
[6] Zusmanovich P. Low-dimensional cohomology of current lie algebras and analogs of the Riemann tensor for loop manifolds. Lin Algebra Appl. 2005;407:71–104. arXiv:math/0302334.
[7] Zusmanovich P. A compendium of Lie structures on tensor products. Zapiski Nauchnykh Seminarov POMI 2013;414:40–81. (N.A. Vavilov Festschrift), reprinted in J Math Sci. 2014;199(3):266–88. arXiv:1303.3231
[8] Eremita D. Biderivations on tensor products of algebras. Comm Algebra. 2018;46(4):1722–1726.
[9] Nakajima A. On categorical properties of generalized derivations. Sci Math. 1999;2(3):345–352.
[10] Kubo F. Lie structures on $\xi x_1, \ldots, x_n, y/(y^3 - 3py - q)$. Bull Kyushu Inst Tech Math Natur Sci. 1988;35:1–6.
[11] Kostrikin AI, Dzhumadil’daev AS. Modular Lie algebras: new trends. In: Bahturin Yu, editor. Algebra. Proceeding of the International Algebraic Conference on the Occasion of the 90th Birthday of A.G. Kurosh; 1998 May 25–30; Moscow, Russia. Berlin: De Gruyter; 2000. p. 181–203.
[12] Laurent-Gengoux C, Pichereau A, Vanhaecke P. Poisson structures. Berlin: Springer; 2013.
[13] Kirillov AA. Local lie algebras. Uspekhi Mat Nauk. 1976;31(4):57–76. Erratum: 1977;32(1)267. (in Russian); Russ Math Surv. 1976;31(4):55–75 (English translation).
[14] The GAP Group. GAP – Groups, Algorithms, and Programming. Version 4.10.2, 2019. Available from: https://www.gap-system.org/