Quasi-Clifford algebras, Quadratic forms over $\mathbb{F}_2$, and Lie Algebras

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QUASI-CLIFFORD ALGEBRAS, QUADRATIC FORMS OVER $\mathbb{F}_2$, AND LIE ALGEBRAS

HANS CUYPERS

Abstract. Let $\Gamma = (V, E)$ be a graph, whose vertices $v \in V$ are colored black and white and labeled with invertible elements $\lambda_v$ from a commutative and associative ring $R$ containing $\pm 1$. Then we consider the associative algebra $C_p \Gamma_q$ with identity element $1$ generated by the elements of $V$ such that for all $v, w \in V$ we have

\begin{align*}
v^2 &= \lambda_v 1 \quad \text{if } v \text{ is white}, \\
v^2 &= -\lambda_v 1 \quad \text{if } v \text{ is black}, \\
vw + wv &= 0 \quad \text{if } \{v, w\} \in E, \\
vw - wv &= 0 \quad \text{if } \{v, w\} \notin E.
\end{align*}

If $\Gamma$ is the complete graph, $C_p \Gamma_q$ is a Clifford algebra, otherwise it is a so-called quasi-Clifford algebra.

We describe this algebra as a twisted group algebra with the help of a quadratic space $(V, Q)$ over the field $\mathbb{F}_2$. Using this description, we determine the isomorphism type of $C_p \Gamma_q$ in several interesting examples.

As the algebra $C(\Gamma)$ is associative, we can also consider the corresponding Lie algebra and some of its subalgebras. In case $\lambda_v = 1$ for all $v \in V$, and all vertices are black, we find that the elements $v, w \in V$ satisfy the following relations

\begin{align*}
[v, w] &= 0 \quad \text{if } \{v, w\} \notin E, \\
[v, [v, w]] &= -w \quad \text{if } \{v, w\} \in E.
\end{align*}

In case $R$ is a field of characteristic $0$, we identify these algebras as quotients of the compact subalgebras of Kac-Moody Lie algebras and prove that they admit a so-called generalized spin representation.

1. Introduction

Let $\Gamma = (V, E)$ be a graph, whose vertices $v$ are colored black or white and labeled with invertible elements $\lambda_v$ from a commutative and associative ring $R$ containing $\pm 1$. (By default, an arbitrary graph is considered to have black vertices and all labels equal to $1$.) Then we consider the associative algebra $C(\Gamma)$ with identity element $1$ generated by the elements of $V$ such that for all $v, w \in V$ we have

\begin{align*}
v^2 &= \lambda_v 1 \quad \text{if } v \text{ is white}, \\
v^2 &= -\lambda_v 1 \quad \text{if } v \text{ is black}, \\
vw + wv &= 0 \quad \text{if } v \sim w, \\
vw - wv &= 0 \quad \text{if } v \not\sim w.
\end{align*}

Here $v \sim w$ denotes that $\{v, w\}$ is an edge in $E$.

If $\Gamma$ contains no edges, all vertices are white and $\lambda_v = 1$ for all $v \in V$, then the algebra $C(\Gamma)$ is a Grassmann algebra. On the other hand, if $\Gamma$ is the complete graph on $n$ vertices, $R = \mathbb{R}$ and $\lambda_v = 1$ for all $v \in V$, then the algebra $C(\Gamma)$ is a Clifford algebra $Cl(p, q)$, where $n = p + q$ and $p$ vertices are colored white, while $q$ vertices have the color black.

This construction also appears in [18, 12], where ordinary finite graphs with all vertices black and $R$ the field of complex numbers are considered. For an arbitrary
finite graph $\Gamma$ and field $R$, we obtain a so-called quasi-Clifford algebra as studied by Gastineau-Hills in [10] in connection with orthogonal designs (see also [19, 20, 23] and the recent book [22]).

In this paper we first describe for arbitrary black and white colored graphs $\Gamma$ the algebra $C_p \Gamma$ as a twisted group algebra with the help of an $F_2$-space $V$ and a bilinear form $g$ on $V$. Their isomorphism type turns out to depend only on the quadratic form $Q$ obtained by $Q(v) = g(v, v)$ for $v \in V$. This is shown in the Sections 2 and 3.

Given such a quadratic form we determine the structure of the algebra, focusing on the case where $\lambda_v = 1$ for all $v \in V$. The algebras obtained are called special by Gastineau-Hills [10], and are up to a center isomorphic to (sums of) Clifford algebras.

Using the description as twisted group algebras, we determine the isomorphism type of $C_p \Gamma$ for several interesting graphs $\Gamma$. This is done in Section 5. We apply our results to complete graphs and obtain quickly the classification of Clifford algebras. But we also consider graphs of type $A_n, D_n$ and $E_n$.

As the algebra $C_p \Gamma$ is associative, we can also consider the corresponding Lie algebra and some of its subalgebras. In particular, we determine the isomorphism type of the Lie algebras generated by the generators in $V$. See Section 6.

In case $\lambda_v = 1$ for all $v \in V$, and all vertices are black, we find that the elements $v, w \in V$ satisfy the following relations, where $[\cdot, \cdot]$ denotes the Lie product:

\[
[v, w] = 0 \quad \text{if } v \neq w, \\
[v, [v, w]] = -w \quad \text{if } v \sim w.
\]

In case $R$ is a field of characteristic 0, we identify these Lie algebras with quotients of compact subalgebras of Kac-Moody Lie algebras and prove that they admit a so-called generalized spin representation. In particular, using the computations of Section 5 and 6, we are able to identify various quotients of these compact Lie subalgebras of Kac-Moody algebras and construct spin representations of such algebras extending the results of [7, 8, 11]. This is the topic of Section 8.

2. A CLASS OF ALGEBRAS OBTAINED FROM BILINEAR FORMS OVER $F_2$

In this section we provide a description of a class of algebras as twisted group algebras. The finite dimensional algebras we describe turn out to be quasi-Clifford algebras as introduced by Gastineau-Hills [10]. Our description as twisted group algebra is closely related to the description of Clifford algebras as twisted group algebras, see [2], and relates our algebras to quadratic spaces over the field with two elements as in [9]. (See also the work of Shaw [28, 26, 29, 27].)

Let $V$ be an $F_2$ vector space (with addition $+$) equipped with a bilinear form $g : V \times V \to F_2$.

Let $B$ be a basis for $V$ and $B^*$ a dual basis, where $b^*$ denotes the dual of $b \in B$.

Now assume $R$ is a commutative and associative ring, and $R^*$ its set of invertible elements including the distinct elements 1 and $-1$. Then let $\Lambda : B \to R^*$ be a map which we extend to $V \times V$ by

$$\Lambda(v, w) := \prod_{b \in B} \Lambda(b)^{b^*(v)b^*(w)}$$

for all $v, w \in V$. 

Notice that this is well defined, also for infinite dimensional spaces $V$, since almost all values of $b^*(v)b^*(w)$ are 0, in which case $\Lambda(b^*b^*)$ equals 1.

The algebra $\mathfrak{C}(V, g, \Lambda)$ is then the $R$-algebra with basis $\{e_v \mid v \in V\}$, unit element $e_0 = 1$, and multiplication defined by

$$e_v e_w = (-1)^{g(v, w)} \Lambda(v, w) e_{v+w}$$

for all $v, w \in V$.

If $\Lambda(v) = 1$ for all $v \in V$, we write $\mathfrak{C}(V, g)$ instead of $\mathfrak{C}(V, g, \Lambda)$.

Notice that elements $e_v$ and $e_w$, where $v \neq w \in V$ satisfy the relations

$$e_v e_w - e_w e_v = 0 \text{ if } f(v, w) = 0$$

$$e_v e_w + e_w e_v = 0 \text{ if } f(v, w) = 1.$$

**Proposition 2.1.** The algebra $\mathfrak{C}(V, g, \Lambda)$ is associative.

**Proof.** Let $u, v, w \in V$, then

$$e_u(e_v e_w) = e_u(-1)^{g(v, w)} \Lambda(v, w) e_{v+w}$$

$$= (-1)^{g(v, w)} \Lambda(u, v+w) e_{u+v+w}$$

$$= (-1)^{g(v, w)+g(u,v)+g(w,v)} \Lambda(u, v+w) e_{u+v+w},$$

while

$$(e_u e_v)e_w = (-1)^{g(u,v)} \Lambda(u, v) e_{u+v}$$

$$= (-1)^{g(u,v)+g(u,v)+g(w,v)} \Lambda(u, v+w) e_{u+v+w}$$

$$= (-1)^{g(u,v)+g(u,v)+g(w,v)} \Lambda(u, v+w) e_{u+v+w}.$$

So, we find the algebra to be associative, if and only if the function $\Lambda$ satisfies

$$\Lambda(u, v) \cdot \Lambda(u + v, w) = \Lambda(v, w) \cdot \Lambda(u + v, w).$$

This identity follows from the observation that for all $u, v, w$ and $b^* \in B^*$ we have

$$b^*(u)b^*(v) + b^*(v) b^*(u) + b^*(w) b^*(v) b^*(w) = b^*(v)b^*(w) + b^*(v+b^*(w)) b^*(w)$$

$$= b^*(v)b^*(w) + b^*(v+w) b^*(u).$$

\[
\square
\]

### 3. From relations to algebra

Let $R$ be a commutative and associative ring with distinct elements 1, −1. Suppose $V$ is an $F_2$-space equipped with a bilinear form $g$ and for some basis $\mathcal{B}$ of $V$ a map $\Lambda : \mathcal{B} \to R^*$ which we extend to a map $\Lambda : V \times V \to R^*$ defined by $\Lambda(v, w) := \prod_{b \in \mathcal{B}} \Lambda(b^*b^*)$ for all $v, w \in V$. Then we can consider the algebra $\mathfrak{C}(V, g, \Lambda)$ as defined in Section 2. We identify the elements $v \in V$ with the basis vectors $e_v$ of $\mathcal{C}(V, g, \Lambda)$. The algebra $\mathfrak{C}(V, g, \Lambda)$ is defined with the help of the basis $\mathcal{B}$. For any other basis $V$ of $V$ we find that $V$ also generates the algebra. The elements $v \neq w \in V$ then satisfy the following relations:

$$v^2 = (-1)^{Q(v)} \Lambda(v, v) 1$$

$$vw = (-1)^{f(v, w)} vw,$$

where $Q$ is the quadratic form on $V$ defined by $Q(v) = g(v, v)$ and $f$ is the symmetric bilinear form associated to $Q$ and given by $f(v, w) = g(v, w) + g(w, v)$ for all $v, w \in V$. We can capture this information in a black and white colored graph. This graph has vertex set $V$. Two vertices $v \neq w$ are adjacent if and only if $vw = -wv$. A vertex $v \in V$ is labeled by $\Lambda(v, v)$ and is colored black or white. Its color is black if and only if $v^2 = -\Lambda(v, v)v$. 
In this section we reverse this process by showing that each such graph determines the generators and relations of an associative algebra isomorphic to an algebra \( \mathfrak{C}(\mathcal{V}, g, \Lambda) \).

So, let \( \Gamma = (\mathcal{V}, \mathcal{E}) \) be a black and white colored graph with vertex set \( \mathcal{V} \) and edge set \( \mathcal{E} \), and the vertices \( v \in \mathcal{V} \) labeled by nonzero invertible elements \( \lambda_v \) from a commutative and associative ring \( R \) containing the distinct elements 1 and \(-1\).

Then consider \( V_\Gamma \), the vector space of finite subsets of \( \mathcal{V} \), where for two finite subsets \( v, w \) of \( V \) the sum \( v \ast w \) is defined to be the symmetric difference of \( v \) and \( w \).

Put a total ordering \( < \) on the vertex set of \( \Gamma \). Let \( u \) and \( w \) be two finite subsets of \( \mathcal{V} \) and let \( g_v(u, w) \) denote the number of ordered pairs \( (x, y) \in u \times w \), where \( x < y \) and \( \{x, y\} \) is an edge, or \( x = y \) is a black vertex, modulo 2. Then \( g_v(u, v \ast w) = g_v(u, v) + g_v(u, w) \), for any finite subsets \( u, v, w \) of \( \mathcal{V} \), as the ordered edges \( (x, z) \) with \( x < z \) and \( z \in v \cap w \), are counted twice at the right hand side of the equation, just as black vertices in the intersection of \( u \) and \( v \cap w \).

Similarly we find \( g_v(v \ast w, u) = g_v(v, u) + g_v(w, u) \). So, \( g_v : V_\Gamma \times V_\Gamma \to \mathbb{F}_2 \) is bilinear. The map \( Q_\Gamma : V_\Gamma \to \mathbb{F}_2 \) given by \( Q_\Gamma(v) = g_v(v, v) \) for all \( v \in V_\Gamma \) is a quadratic form with associated symmetric (and also alternating) form \( f_\Gamma \) given by \( f_\Gamma(u, w) = g_v(u, w) + g_v(w, u) \).

Now we define an associative algebra \( \mathfrak{C}(\Gamma) \) over \( R \) with basis the set of elements \( v \neq w \in \mathcal{V} \) (after being identified with the subset \( \{v\} \)) satisfy the following relations:

\[
\begin{align*}
v^2 &= \lambda_v 1 \quad \text{if } v \text{ is white}, \\
v^2 &= -\lambda_v 1 \quad \text{if } v \text{ is black}, \\
vw + vw &= 0 \quad \text{if } v \sim w, \\
vw - vw &= 0 \quad \text{if } v \neq w.
\end{align*}
\]

The product is defined as follows.

The element \( \mathcal{G} \) is the unit element of \( \mathfrak{C}(\Gamma) \) and will be denoted by \( 1 \). For \( v, w \) being finite subsets of \( \mathcal{V} \), we define the product of \( v \) and \( w \) by

\[
vw = (-1)^{g_v(v, w)} \left( \prod_{x \in v \cap w} \lambda_x \right) v \ast w.
\]

Clearly this definition of the product is forced upon us by the relations and associativity of the product.

But then it is straightforward to check that with \( \Lambda_\Gamma(v, w) = \prod_{x \in v \cap w} \lambda_x \) we have the following.

**Theorem 3.1.** The algebra \( \mathfrak{C}(\Gamma) \) is isomorphic to \( \mathfrak{C}(V_\Gamma, g_\Gamma, \Lambda_\Gamma) \).

By construction, the algebra \( \mathfrak{C}(\Gamma) \) is the universal associative algebra satisfying the relations prescribed by the graph \( \Gamma \). So, we have:

**Theorem 3.2.** An associative algebra \( \mathfrak{C} \) with unit element \( 1 \) generated by a set of elements \( \mathcal{V} \) satisfying the relations

\[
\begin{align*}
v^2 &= \pm \lambda_v 1, \quad \lambda_v \in R^* \\
vw + vw &= 0 \quad \text{or} \\
vw - vw &= 0
\end{align*}
\]

for \( v \neq w \in \mathcal{V} \), is isomorphic to a quotient of \( \mathfrak{C}(\Gamma) \), where \( \Gamma \) is the black and white colored graph with vertex set \( \mathcal{V} \), two vertices being adjacent if and only if they do not commute and each vertex \( v \) is labeled with \( \lambda_v \) and \( v \) is black if and only if \( v^2 = -\lambda_v 1 \).
Let $V$ be an $F_2$-space equipped with a bilinear form $g$. Let $Q$ be the quadratic form given by $Q(v) = g(v, v)$ for all $v$ and denote by $f$ the associated alternating form given by $f(u, v) = g(u, v) + g(v, u) = Q(u + v) + Q(u) + Q(v)$ for all $v, w \in V$. Then the above results imply that, up to isomorphism, the algebras $Q$ are isomorphic when the two forms are determined by the quadratic form $f$. Suppose $f : C \to \text{Rad}(Q)$ to denote (the isomorphism class of) an algebra $C$. For this reason we will also write $C(V, Q, \Lambda)$ to denote (the isomorphism class of) an algebra $C(V, g, \Lambda)$.

Moreover, two algebras $C(V, Q, \Lambda)$ and $C(V, Q', \Lambda)$, with $Q$ and $Q'$ quadratic forms, are isomorphic when the two forms $Q$ and $Q'$ are equivalent, i.e., when there is a $\gamma \in \text{GL}(V)$ with $Q(v) = Q'(\gamma v)$ for all $v \in V$.

We collect this information in the following theorem.

**Theorem 3.3.** Let $(V, Q)$ be quadratic $F_2$-space with basis $V$ and $\Lambda : V \to R^*$ a map. Suppose $f$ is the symmetric form associated to $Q$. Suppose $g$ is a bilinear form on $V$ with $Q(v) = g(v, v)$ for all $v \in V$.

Then the algebra $C(V, g, \Lambda)$ is isomorphic to $C(\Gamma)$ where $\Gamma$ is the graph with vertex set $V$, in which two vertices $v, w$ are adjacent if and only if $f(v, w) = 1$, a vertex $v$ is labeled by $\Lambda(v)$ and colored black or white, according to $v^2 = -\Lambda(v)1$ or $+\Lambda(v)1$, respectively.

4. **ALGEBRAS AND QUADRATIC FORMS**

As we have seen in the previous section, the algebras $C(\Gamma)$, where $\Gamma$ is a black and white colored graph whose vertices are labeled by invertible elements from an associative ring $R$ are, up to isomorphism, algebras $C(V, Q, \Lambda)$ for some quadratic space $(V, Q)$ over the field $F_2$ and a map $\Lambda : V \to R^*$.

The classification of quadratic forms on vector spaces of finite dimension over the field of 2 elements is well known. We discuss this briefly. The radical of $f$, defined as $\text{Rad}(f) = \{v \in V \mid f(v, w) = 0 \text{ for all } w \in V\}$, is a subspace of $V$. It contains the radical of $Q$, defined as $\text{Rad}(Q) = \{v \in \text{Rad}(f) \mid Q(v) = 0\}$, as a subspace of codimension at most 1. We call the form $Q$ nondegenerate if and only if $\text{Rad}(f) = \{0\}$ and almost nondegenerate if $\text{Rad}(Q) = \{0\}$, but $\text{Rad}(f) \neq \{0\}$.

In dimension one there is, up to isomorphism, a unique nontrivial quadratic form $Q(x) = x^2$, which is almost nondegenerate. It is called of 0-type. In dimension 2 we have, up to isomorphism, exactly two nondegenerate forms, $Q(x_1, x_2) = x_1 x_2$, called of +type, and $Q(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2$, called of −type. In dimension $n > 2$ we can distinguish, up to isomorphism, the following forms:

+type: $V$ is an orthogonal sum $V_1 \perp \cdots \perp V_k \perp \text{Rad}(Q)$, where all $V_i$ are 2-spaces of +type.

−type: $V$ is an orthogonal sum $V_1 \perp \cdots \perp V_k \perp \text{Rad}(Q)$, where all $V_i$ are 2-spaces of +type, except for one, which is of −type.

0-type: $V$ is an orthogonal sum $V_1 \perp \cdots \perp V_k \perp \text{Rad}(Q)$, where all $V_i$ are 2-spaces of +type, except for one, which is one dimensional and of 0-type.

Notice, in this case we find the radical of $f$ to be larger than the radical of $Q$.

One of the key observations in the proof of this classification is that the type of a direct orthogonal sum of two spaces is determined by the type of the summands. The orthogonal direct sum of spaces of type $x$ and type $y$, where $x, y = \pm 0$ or 0, gives us a space of type $x \cdot y$. We will frequently use these observations in the sequel.

We note that the number of isomorphism classes of quadratic spaces $(V, Q)$ over $F_2$ of infinite dimension is much larger, see [14].
The decomposition of \((V, Q)\) into pairwise orthogonal subspaces provides a decomposition of the algebra \(\mathcal{C}(V, Q, \Lambda)\) into tensor products. Indeed, if we suppose \(R\) is a field, then the following proposition yields this decomposition.

**Proposition 4.1.** Let \(R\) be a field. Suppose \((V, Q)\) is finite dimensional and can be decomposed as a direct orthogonal sum \((V_1, Q_1) \perp (V_2, Q_2)\). Then \(\mathcal{C}(V, Q, \Lambda)\) is isomorphic to \(\mathcal{C}(V_1, Q_1, \Lambda_1) \otimes \mathcal{C}(V_2, Q_2, \Lambda_2)\), where \(\Lambda_i\) is the restriction of \(\Lambda\) to \(V_i \times V_i\).

**Proof.** The map \(\phi\) that sends each tensor \(e_{v_1} \otimes e_{v_2} \in \mathcal{C}(V_1, Q_1, \Lambda_1) \otimes \mathcal{C}(V_2, Q_2, \Lambda_2)\), with \(v_1 \in V_1, v_2 \in V_2\) to \(e_{v_1}, e_{v_2}\) extends uniquely to a linear map

\[
\phi : \mathcal{C}(V_1, Q_1, \Lambda_1) \otimes \mathcal{C}(V_2, Q_2, \Lambda_2) \to \mathcal{C}(V, Q, \Lambda).
\]

Moreover, as the elements \(e_{v_1}\) and \(e_{v_2}\) commute in \(\mathcal{C}(V, Q, \Lambda)\), it is straightforward to check that \(\phi\) is a surjective homomorphism of algebras. As the dimensions of \(\mathcal{C}(V, Q, \Lambda)\) and \(\mathcal{C}(V_1, Q_1, \Lambda_1) \otimes \mathcal{C}(V_2, Q_2, \Lambda_2)\) coincide, we find an isomorphism. \(\square\)

The structure of the algebra \(\mathcal{C}(V, Q, \Lambda)\) not only depends on the quadratic space \((V, Q)\), but also on the ring \(R\) and of course the values \(\Lambda\) takes in \(R\). In case \(R = \mathbb{F}\) is a field, we can use the above Proposition 4.1 and only have to consider small dimensional cases for \(V\) to find the structure of the algebra \(\mathcal{C}(V, Q, \Lambda)\).

These small dimensional cases are worked out in [10]. For later use we describe the situation in the case where \(R = \mathbb{F}\) is a field and \(\Lambda\) is 1. In this situation we consider three types of fields, type I, II and III, defined by:

- type I: There is an element \(i \in \mathbb{F}\) with \(i^2 = -1\).
- type II: There is no \(i \in \mathbb{F}\) with \(i^2 = -1\), but there are \(x, y \in \mathbb{F}\) with \(x^2 + y^2 = -1\).
- type III: There are no \(x, y \in \mathbb{F}\) with \(x^2 + y^2 = -1\).

If \(V\) is 1-dimensional, then \(\mathcal{C}(V, Q)\) is isomorphic to \(\mathbb{F} \times \mathbb{F}\) in case \(Q\) is trivial on \(V\) or \(\mathbb{F}\) is a field of type I. If \(Q\) is non-trivial on \(V\) and \(\mathbb{F}\) is of type II or III, then \(\mathcal{C}(V, Q)\) is isomorphic to \(\mathbb{F}[i]\), where \(i^2 = -1\).

Now assume that \(V = \langle e_1, e_2 \rangle\) is 2-dimensional and suppose \(Q\) is of \(+\)-type, \(Q(e_1) = Q(e_2) = 0\) and \(f(e_1, e_2) = 1\). Then we can identify \(\mathcal{C}(V, Q)\) with \(M(2, \mathbb{F})\), the algebra of \(2 \times 2\) matrices via the map

\[
e_1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

and

\[
e_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

If \(Q\) is of \(-\)-type, then we may assume that \(Q(e_1) = Q(e_2) = f(e_1, e_2) = 1\) and we can identify \(\mathcal{C}(V, Q)\) with \(M(2, \mathbb{F})\) via the map

\[
e_1 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } e_2 \mapsto \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}
\]

if \(\mathbb{F}\) is of type I, and

\[
e_1 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } e_2 \mapsto \begin{pmatrix} x & -y \\ -y & -x \end{pmatrix}
\]

if \(\mathbb{F}\) is of type II and \(x, y \in \mathbb{F}\) with \(x^2 + y^2 = -1\).
If $\mathbb{F}$ is of type III, we can identify $C_{p, V, Q}$ with the matrix algebra
\[
\begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\] This algebra can be identified with the algebra $H$ of quaternions over $\mathbb{F}$.

This implies that for finite dimensional spaces $(V, Q)$ the algebra $C_{p, V, Q}$ is determined, up to isomorphism, by the following parameters:

(a) Dimension $n$ of $V$
(b) Dimension $r$ of Rad$(f)$
(c) Type of $Q$, the form induced by $Q$ on $V$
(d) Type of $\mathbb{F}$

We can now describe the various isomorphism classes of the algebras $C_{p, V, Q}$ in terms of these parameters.

**Proposition 4.2.** Let $(V, Q)$ be a nontrivial, finite dimensional quadratic space over the field $\mathbb{F}_2$. Then the isomorphism type of the algebra $C_{p, V, Q}$ over a field $\mathbb{F}$ of characteristic $\neq 2$ is given in Table 1.

| $\dim(V)$ | Type($Q$) | Type of $\mathbb{F}$ | Algebra |
|-----------|-----------|-----------------------|---------|
| $n = 0 \mod 2$ | $+$ | I | $(M(2, F)^{\otimes \frac{n}{2}})^{2^r}$ |
| $n = 0 \mod 2$ | $-$ | I | $(M(2, F)^{\otimes \frac{n}{2}})^{2^r}$ |
| $n = 1 \mod 2$ | $0$ | I | $(M(2, F)^{\otimes \frac{n-1}{2}})^{2^r}$ |
| $n = 0 \mod 2$ | $+$ | II | $(M(2, F)^{\otimes \frac{n}{2}})^{2^r}$ |
| $n = 0 \mod 2$ | $-$ | II | $(M(2, F)^{\otimes \frac{n}{2}})^{2^r}$ |
| $n = 1 \mod 2$ | $0$ | II | $(M(2, F)^{\otimes \frac{n-1}{2}} \otimes \mathbb{F}[v])^{2^{r-1}}$ |
| $n = 0 \mod 2$ | $+$ | III | $(M(2, F)^{\otimes \frac{n}{2}})^{2^r}$ |
| $n = 0 \mod 2$ | $-$ | III | $(M(2, F)^{\otimes \frac{n}{2}} \otimes \mathbb{F}[v])^{2^r}$ |
| $n = 1 \mod 2$ | $0$ | III | $(M(2, F)^{\otimes \frac{n-1}{2}} \otimes \mathbb{F}[v])^{2^{r-1}}$ |

**Table 1.** The isomorphism types of the algebras $\mathcal{C}(V, Q)$.

We end this section with describing two involutions, related to the grading, reversion and conjugation involutions of Clifford algebras.

Let $H$ be a hyperplane of $V$ and define $\tau_H : \mathcal{C}(V, Q) \to \mathcal{C}(V, Q)$ by linear expansion of
\[
\tau(v) = \begin{cases} 
v & \text{if } v \in H \\
-v & \text{if } v \notin H. \end{cases}
\]

The second involution $\tau_Q : \mathcal{C}(V, Q) \to \mathcal{C}(V, Q)$ is defined as the linear expansion of
\[
\tau(v) = \begin{cases} 
v & \text{if } Q(v) = 0 \\
-v & \text{if } Q(v) = 1. \end{cases}
\]

**Proposition 4.3.** The involution $\tau_H$ is an automorphism of $\mathcal{C}(V, Q)$.

The involution $\tau_Q$ is an anti-automorphism of $\mathcal{C}(V, Q)$.
Proof. First consider $τ_H$, where $H$ is a hyperplane of $V$. It suffices to check for $u, v \in V \setminus \{0\}$ that $τ_H(υv) = τ_H(u)τ_H(v)$. As $H$ is a hyperplane, $τ_H$ fixes either all three vectors $u, v, u \oplus v$ or negates two of them and, indeed, we find $τ_H(υv) = τ_H(u)τ_H(v)$.

To check that $τ_Q$ is an anti-automorphism, we have to check $τ_Q(υv) = τ_Q(v)τ_Q(u)$. If $Q(u) = Q(v) = 0$, then $Q(u \oplus v) = 0$ and $υv = υυv$, or $Q(u \oplus v) = 1$ and $υv = -υυv$. In both cases $τ_Q(υv) = τ_Q(v)τ_Q(u)$.

If $Q(u) = Q(v) = 1$, then $Q(u \oplus v) = 0$ and $υv = υυv$ or $Q(u \oplus v) = 1$ and $υv = -υυv$. Again, in both cases $τ_Q(υv) = τ_Q(v)τ_Q(u)$.

Finally, if $Q(u) = 0$ and $Q(v) = 1$ (or $Q(u) = 1$ and $Q(v) = 0$), then $Q(u \oplus v) = 0$ and $υv = -υυv$ or $Q(u \oplus v) = 1$ and $υv = υυv$. Also now we can check $τ_Q(υv) = τ_Q(v)τ_Q(u)$.

\[\square\]

Proposition 4.4. Let $τ$ be a nontrivial linear map of $C(V, Q)$ mapping any $v \in V$ to $±v$.

If $τ$ is an automorphism of $C(V, Q)$, then $τ = τ_H$ for some hyperplane $H$ of $V$.

If $τ$ is an anti-automorphism of $C(V, Q)$, then $τ = τ_Q$ or $τ_Qτ_H$ for some hyperplane $H$ of $V$.

Proof. First assume that $τ$ is an automorphism. If $τ$ negates two vectors $v, w \in V \setminus \{0\}$, then $v \oplus w$, should be fixed. So, the vectors in $V$ fixed by $τ$ form a hyperplane $H$ of $V$ and $τ = τ_H$.

Next, assume that $τ$ is an anti-automorphism. The $τ_Qτ$ is an automorphism, and by the above, we either have $τ = τ_Q$ or $τ_Qτ_H$ for some hyperplane $H$ of $V$.

\[\square\]

Remark 4.5. The anti-automorphism $τ_Q$ acts on the matrix algebras of Table 1 by transposition followed by complex or quaternion conjugation (if applicable) on $F[i]$ or $H$, respectively. This can easily be checked in small dimensional cases, as described above, and hence on the tensor products. See also [1].

5. Examples

In this section we consider a few examples of algebras given by some black and white colored graph $Γ$. We only consider cases where the ring $R = F$ is a field and where the values of the vertices are $±1$. Up to changing the colors of the vertices, we can assume the map $Λ$ to be the constant map 1. When drawing a graph $Γ$ we use the color gray for a vertex to indicate that we have not yet determined whether its color should be black or white.

Example 5.1 (Clifford algebras and graphs of type $A$). Let $Γ$ be the complete graph on $n$ vertices with $p$ white vertices and $q$ black vertices. Then of course $C(Γ)$ is isomorphic to the Clifford algebra $Cl(p, q)$. Consider the corresponding quadratic space $(V, Q) = (VΓ, QΓ)$ obtained from $Γ$. Suppose the vectors $e_1, \ldots, e_n \in V$ correspond to the vertices of $Γ$, where $Q(e_i) = 1$ for all $i$ with $1 \leq i \leq q$. Then with $f_1 = e_1, f_2 = e_1 \oplus e_2, f_3 = e_2 \oplus e_3, \ldots, f_n = e_{n-1} \oplus e_n$ we find a spanning set for $V$ with corresponding graph of type $A_n$ as in Figure 1.

All vertices are black, except for $f_q + 1$, which is white. (If $q = n$, then all vertices are black, if $q = 0$, only $f_1$ is white.)

This implies that for $q \geq 1$ we find $Cl(p, q)$ to be isomorphic to $Cl(q − 1, p + 1)$. Just read the diagram from right to left.
QUASI-CLIFFORD ALGEBRAS, QUADRATIC FORMS OVER $\mathbb{F}_2$, AND LIE ALGEBRAS

Figure 1. Graph of type $A_n$ obtained by changing the generators.

| $p\cdot q$ | 0 | 1 | 2 | 3 | 4 |
|-----------|---|---|---|---|---|
| 0         | + | 0 | - | - | - |
| 1         | + | + | 0 | - | - |
| 2         | + | + | + | 0 | - |
| 3         | 0 | + | + | + | 0 |
| 4         | - | 0 | + | + | + |

Table 2. Type of $Q$ for small values of $p+q$.

Now let $g_1 = f_1$, and for $i$ with $2 \leq 2i \leq n$ let $g_i = f_{2i}$ and $g_{i-1} = f_1 \circ f_3 \circ \cdots \circ f_{2i-1}$.

If $n$ is odd, then let $g_n = f_1 \circ f_n$. Then the graph on these vertices is given in Figure 2.

Figure 2. The graphs for $n$ even (left) or odd (right).

First assume $n$ is even. Notice that $Q(g_{2i}) = 1$ for all $i$, except when $q \leq n$ is odd. Then $Q(g_{q+1}) = 0$. Moreover, $Q(g_1) = 1$ and $Q(g_{2i+1})$ is $i$ (mod 2) if $q$ is odd. For even $q$ we find that $Q(g_{2i+1})$ is $i + 1$ (mod 2) if $2i + 1 \leq q$ and $i$ (mod 2) for $2i + 1 > q$. For odd $n$, we find $Q(g_n) = Q(g_1) + Q(g_{n-1})$.

From this information we can deduce the type of $Q$. In particular, we see that the type of $Q$ is multiplied with $-1$ if we raise $p$ or $q$ with 4, and hence stays the same if we add 8 to $q$ or $p$ (Bott-periodicity). Indeed, adding 4 to $p$ or $q$ adds

to the graph and multiplies the type of $Q$ with $-1$.

For small values of $p$ and $q$ we have collected this information in Table 2.

Using the results of Table 2 and the above information, we find in Table 3 the isomorphism type of the Clifford algebras over fields $F$ of type III.

We notice that the above also classifies the algebras $\mathcal{C}(\Gamma)$ where $\Gamma$ is a graph of type $A_n$ as in Figure 3, since we can replace the vertices $f_i$ by $e_i$, i.e., by reversing the above described process, and end up with a complete graph. In particular, we find that we only have to consider those graphs of type $A_n$ in which at most one vertex is white.

Example 5.2 (Graphs of type $D$). Next we consider graphs of type $D_n$, where $n \geq 4$. See Figure 4.
To classify the corresponding algebras we only have to consider the cases where at most one of the vertices 2, . . . , n is white. Moreover, we notice that \( e_1 + e_2 \) is an element which is in the radical of the form \( f \) induced on \( V \). If both the vertices 1 and 2 are black or both are white, we find \( e_1 + e_2 \) to be in the radical of \( Q_\Gamma \) and \( \mathfrak{C}(\Gamma) \) is the direct product \( \mathfrak{C}(\Gamma_1) \times \mathfrak{C}(\Gamma_1) \), where \( \Gamma_1 \) is obtained from \( \Gamma \) by deleting vertex 1. If only one of the two vertices 1 and 2 is black, then \( Q_\Gamma(e_1 + e_2) = 1 \) and we find \( \mathfrak{C}(\Gamma) \) to be isomorphic to \( \mathfrak{C}(\Gamma_1) \otimes \mathbb{F}[i] \).

**Example 5.3** (Graphs of type \( E_n \)). Let \( \Gamma \) be a graph of type \( E_n \), where \( n \geq 1 \) as in Figure 5.

Assume that all vertices are colored black. Consider the quadratic form on \( V = \mathbb{F}^n \) given by

\[
Q(x_1, \ldots, x_n) = \sum_{i=1}^n x_i^2 + x_2x_4 + x_1x_3 + x_3x_4 + \cdots + x_{n-1}x_n.
\]

Then \( Q(e_i) = 1 \) and \( f(e_i, e_j) = 1 \) if and only if \( i \) is adjacent to \( j \). So, \( \mathfrak{C}(\Gamma) \) is isomorphic to \( \mathfrak{C}(V, Q) \).
For $n \geq 4$ even, we find that we can split $V$ into the orthogonal sum of the spaces
\[ \langle e_1, e_3 \rangle \perp \langle e_2, e_4 + \cdots + e_n \rangle \perp \langle e_8, e_5 + e_7 \rangle \perp \cdots \perp \langle e_n, e_5 + e_7 + \cdots + e_{n-1} \rangle.\]
 Such a 2-dimensional space is of $+\text{-type}$ if the second generator is of even weight, and of $-\text{-type}$ if the second generator is of odd weight. So we find $Q$ to be of $+\text{-type}$ if $n = 0, 2 \pmod 8$ and of $-\text{-type}$ for $n = 4, 6 \pmod 8$.

For $n \geq 5$ odd we find the vector $e_2 + e_5$ (for $n = 5$) or $e_2 + e_5 + e_7 + e_9 + \cdots + e_n$ (for $n \geq 9$) to span the radical of $f$, the bilinear form associated to $Q$. This vector is isotropic if and only if $n = 1 \pmod 4$. It remains to find the type of the form induced on $V/\text{Rad}(f)$ in case $n = 1 \pmod 4$. As modulo $e_2 + e_5 + e_7 + e_9 + \cdots + e_n$, we find that $e_2$ is in the subspace spanned by $e_1, e_3, e_4, \ldots, e_n$, the type of $Q$ is determined by the type of $Q$ restricted to this subspace. As above we find that this is of $+\text{-type}$ if $n - 1 = 0, 2 \pmod 8$ and of $-\text{-type}$ if $n - 1 = 4, 6 \pmod 8$. So, also for graphs $E_n$ we find Bott-periodicity. The information is summarized in Table 4.

**Algorithm 5.4.** In this section we have seen three examples on how to identify the algebra $\mathfrak{C}(\Gamma)$ from the graph $\Gamma$. The described method can be turned into an algorithm, which consists of the following steps:

(a) Apply a (modified) Gram-Schmidt procedure to decompose $V_\Gamma$ into an orthogonal sum of nondegenerate 2-dimensional spaces and 1-dimensional spaces.

(b) Determine the type of $Q_\Gamma$ by taking the product of the types of the nondegenerate 2-dimensional spaces and 1-dimensional spaces from step (a) on which $Q_\Gamma$ is nontrivial.

(c) Determine the isomorphism type of $\mathfrak{C}(\Gamma)$ using the type of $Q_\Gamma$ as computed in step (b) and Table 1.

If $n$ denotes the number of vertices of $\Gamma$, then this algorithm has complexity of order $n^3$, as the Gram-Schmidt procedure has complexity of order $n^3$.

6. Lie algebras

We continue with the notation of the previous sections. Consider the algebra $\mathfrak{C}(V, Q, \Lambda)$ as in Section 2, where $(V, Q)$ is a quadratic space over the field $\mathbb{F}_2$ and $\Lambda : V \times V \to R^3$ is defined as in Section 2. Then $\mathfrak{C}(V, Q, \Lambda)$ is an associative algebra...
and we can consider the associated Lie algebra, where the Lie bracket is defined by

\[
[u, v] = \frac{1}{2}(uv - vu) = \frac{1}{2}((-1)^{g(u,v)} - (-1)^{g(v,u)})\Lambda(u, v) \cdot u \cdot v
\]

for all \( u, v \in V \). Here \( g \) is a bilinear form with \( Q(v) = g(v, v) \) for \( v \in V \), and \( f(u, v) = g(u, v) + g(v, u) \) the corresponding alternating form defined by \( Q \). Notice that we identify the values of \( f(u, v) \in F_2 \) with 0 and 1 in \( R \).

This Lie algebra does depend only on the symplectic space \((V, f)\) and the map \( \Lambda \), and can actually be defined for any symplectic space \((V, f)\), even if 2 is not invertible in \( R^* \). We denote this Lie algebra by \( g(V, f, \Lambda) \).

As the elements of \( V \) form a basis for \( \mathfrak{C}(V, Q, \Lambda) \), they also form a basis for \( g(V, f, \Lambda) \). Elements \( u, w \in V \) satisfy the following relations in \( g(V, f, \Lambda) \):

\[
[u, [u, w]] = -f(u, w)\Lambda(u, w)\Lambda(u, u \cdot w)w = -f(u, w)\Lambda(u, u)w.
\]

Clearly, the element 1 is in the center of this Lie algebra, but so are all elements \( u \in V \) that are in the radical of \( f \).

We now concentrate on the case where \( R \) is a field \( F \) of characteristic \( \neq 2 \), and \( \Lambda(u, v) = 1 \) for all \( u, v \in V \). In this case we write \( g(V, f) \) for \( g(V, f, \Lambda) \).

If \( u, v \in V \) with \( r_0 := u \cdot v \) in the radical of \( f \), we find

\[
[u + v, w] = [u, w] + [v, w] = -f(u, w)(u \cdot w) - f(v, w)(v \cdot w)
\]

As \( (u \cdot w) \cdot (v \cdot w) = u \cdot v = r_0 \), we find that the linear span of the elements \( u + v, \) where \( v = u \cdot r_0 \), is an ideal of \( g(V, f, \Lambda) \), which we denote by \( \mathfrak{J}^+_r \). Similarly we find

\[
[u - v, w] = -f(u, w)(u \cdot w - v \cdot w)
\]

so that \( \mathfrak{J}^-_r \), the linear span of the elements \( u - v, \) where \( v = u \cdot r_0 \) is also an ideal. This implies the following.

**Proposition 6.1.** Let \( 0 \neq r_0 \in \text{Rad}(f) \), then \( g(V, f) = \mathfrak{J}^+_r \oplus \mathfrak{J}^-_r \).

Moreover, \( g/\mathfrak{J}^+_r \) is isomorphic to \( g(\overline{V}, \overline{f}) \), where \( (\overline{V}, \overline{f}) \) is the quotient space of \((V, f)\) modulo \( \langle r_0 \rangle \).

Using the above proposition and the information in Table 1, we can deduce the isomorphism types of the Lie algebras \( g(V, f) \) obtained from the various algebras \( \mathfrak{C}(V, Q) \). This information can be found in Table 5. Here \( r \) denotes the dimension of the radical and \((\overline{V}, Q)\) is obtained from \((V, Q)\) by taking the quotient modulo the radical of \( Q \).

Although the Lie algebra \( g \) of the algebra \( \mathfrak{C}(V, Q) \) only depends on the symplectic form \( f \) but not on \( Q \), it does contain a Lie subalgebra that is related to \( Q \), and in fact is the centralizer of \( -\tau_Q \).

**Proposition 6.2.** Let \( H \) be a hyperplane of \( V \). Then \( \tau_H \) and \( -\tau_Q \) are automorphisms of \( g \).

**Proof.** By 4.3 we find \( \tau_H \) to be an automorphism. So, we consider \( -\tau_Q \).
Let $u$, $v \in V$, then

$$-\tau_Q([u, v]) = -\tau_Q(uv - vu)$$

$$= -(\tau_Q(v)\tau_Q(u) - \tau_Q(u)\tau_Q(v))$$

$$= -[\tau_Q(v), \tau_Q(u)]$$

$$= [\tau_Q(u), \tau_Q(v)]$$

$$= [-\tau_Q(u), -\tau_Q(v)].$$

The centralizer in $\mathfrak{g}(V, f)$ of an automorphism $\sigma$ is a Lie subalgebra, which we denote by $\mathfrak{g}_{\sigma}(V, f)$.

Clearly $\mathfrak{g}_{-\tau_Q}(V, f)$ is isomorphic to $\mathfrak{g}(H, f_H)$. The subalgebra $\mathfrak{g}_{-\tau_Q}(V, f)$ depends on $Q$ and therefore is also denoted by $\mathfrak{g}(V, Q)$. It is the linear span of the set $\{v \in V \mid Q(v) = 1\}$ of non-isotropic vectors in $V$ inside $\mathfrak{g}(V, f)$.

The isomorphism types of these subalgebras can also be found in Table 5. They can be deduced using the description of the matrix algebras as given in Section 4 and Remark 4.5.

**Remark 6.3.** If we fix a hyperplane $H$ of $V$, then the group $\langle -\tau_Q, \tau_H \rangle$ is elementary abelian of order $2^2$. The Lie algebra $\mathfrak{g}(V, f)$ can be decomposed as

$$\mathfrak{g}(V, f) = \mathfrak{g}_{1,1} \oplus \mathfrak{g}_{1,-1} \oplus \mathfrak{g}_{-1,1} \oplus \mathfrak{g}_{-1,-1},$$

where $\mathfrak{g}_{i,j}$ for $i, j = \pm 1$ denotes the intersection of the $i$-eigenspace of $-\tau_Q$ and $j$-eigenspace of $\tau_H$.

Notice that for $i, j, k, l = \pm 1$ we have

$$[\mathfrak{g}_{i,j}, \mathfrak{g}_{k,l}] \subseteq \mathfrak{g}_{ik,jl}.$$
When $F$ is a field of type III, one can also consider the $\mathbb{F}$-Lie subalgebras
\[ g_{1,1} \oplus i g_{k,l} \]
(where $k, l = \pm 1$) of the Lie algebra $\mathfrak{g}(V, Q)$ defined over $\mathbb{F}[i]$ with $i^2 = -1$. See also [30, 32, 31].

7. LIE ALGEBRAS OBTAINED FROM GRAPHS

Let $\Gamma = (V, \mathcal{E})$ be a black and white colored graph with all labels equal to 1. Then let $\mathfrak{g}(\Gamma)$ be the Lie algebra of $\mathfrak{C}(\Gamma)$. The vertices in $V$ do generate $\mathfrak{C}(\Gamma)$, but need not generate the Lie algebra $\mathfrak{g}(\Gamma)$.

In this section we provide a characterization of the Lie algebras $\mathfrak{g}(\Gamma)$ and its subalgebra generated by the vertices of $\Gamma$.

So, consider a connected black and white colored graph $\Gamma = (V, \mathcal{E})$ and consider the Lie algebra $\mathfrak{g}(\Gamma)$ over a field $\mathbb{F}$, with characteristic different from 2. As in the previous sections we identify $\mathfrak{C}(\Gamma)$ with the algebra $\mathfrak{C}(V, Q_\Gamma)$. By $f_\Gamma$ we denote the bilinear form associated to $Q_\Gamma$.

The Lie subalgebra of $\mathfrak{g}(\Gamma)$ generated by the vertices of $\Gamma$ will be studied with the help of the geometry of $(V_\Gamma, Q_\Gamma)$ and $(V, f_\Gamma)$. We denote this subalgebra by $\mathfrak{H}(\Gamma)$. Notice that the coloring of the vertices of $\Gamma$ has no effect on the isomorphism type of this Lie algebra. So, from now on we assume that all vertices are black.

Let $(V, Q)$ be a quadratic space over $\mathbb{F}_2$ with addition $\phi$. If $v \neq w \in V$ are nonzero vectors with $Q(v) = Q(w) = f(v, w) = 1$, then we call the 2-dimensional subspace $\langle v, w \rangle$ an elliptic line of $(V, Q)$. We identify this 2-space with the set of three nonzero vectors $\{v, w, v \mp w\}$ contained in it. By $\Pi(V, Q)$ we denote the partial linear space $(P, L)$ where $P$ consists of all the vectors $v$ of $V \setminus \text{Rad}(f)$ with $Q(v) = 1$ and whose lines in $L$ are the elliptic lines. (Notice that a vector $v$ with $Q(v) = 1$ but $v \in \text{Rad}(f)$ is not in $P$.) It is a so-called cotriangular space, having the property that for each point $p$ and line $\ell$ not on $p$, the point $p$ is collinear to 0 or all but one of the points of $\ell$.

A subspace of $\Pi(V, Q)$ is a subset $S$ of the point set of $\Pi$ such that each line meeting $S$ in two points is contained in $S$. A subspace $S$ is often identified with the partial linear space $(S, \{\ell \in L \mid \ell \subseteq S\})$. As the intersection of subspaces is again a subspace, we can define the subspace generated by a subset $X$ of $P$ to be the intersection of all subspaces containing $X$.

Cotriangular spaces (and their subspaces) have been studied by several authors, see for example [15, 33, 24]. Their connection with Lie algebras has been considered in [6, 5].

Notice that $V$ is a basis for $V_\Gamma$ and $\Gamma$ is connected. Then the subspace of $\Pi(V_\Gamma, Q_\Gamma)$ generated by $V$ is denoted by $\Pi(\Gamma)$.

**Proposition 7.1.** Let $\Gamma = (V, \mathcal{E})$ be a connected graph. The subspace $\Pi(\Gamma)$ of $\Pi(V_\Gamma, Q_\Gamma)$ is a basis for $\mathfrak{H}(\Gamma)$.

**Proof.** This follows immediately from the following observation: if $v, w \in V_\Gamma$ are collinear points in $\Pi(\Gamma)$, then $Q_\Gamma(v) = Q_\Gamma(w) = 1 = f_\Gamma(v, w)$. So $Q(v \mp w) = 1$ and $v \mp w$ is a point of $\Pi(\Gamma)$ and $[v, w] = \pm v \mp w$. If $v, w$ are not collinear, then $[v, w] = 0$. \(\square\)

Let $\Pi = (P, L)$ be an arbitrary cotriangular space with point set $P$ and set of lines $L$. Then on $P$ we can define an equivalence relation $\sim$, where two points
\( p, q \in P \) are equivalent if and only if the set of points collinear with but different from \( p \) coincides with the set of points collinear with, but different from \( q \). Notice that two points that are collinear, are never equivalent. Now for each line \( \ell \in L \) we can consider \( \mathcal{T} \) to be the set of three equivalence classes of the points on \( \ell \). If \( P \) denotes the equivalence classes of \( P \) and \( \mathcal{E} \) the set \( \{ \ell \mid \ell \in L \} \), then \( \Pi = (P, \mathcal{E}) \) is also a cotriangular space. Moreover, it is reduced, meaning that no two distinct points are \(~\)-equivalent.

If \( \mathcal{V} \) is a subset of \( P \) and \( \Gamma = (\mathcal{V}, \mathcal{E}) \) the graph with vertex set \( \mathcal{V} \) and two vertices \( v, w \in \mathcal{V} \) adjacent if and only if \( f(v, w) = 1 \), then \( \Gamma \) denotes the graph with vertices the \(~\)-equivalence classes of the vertices in \( \mathcal{V} \) and two classes adjacent if and only if there are vertices adjacent vertices in these classes.

Besides the cotriangular spaces obtained from the elliptic lines of a quadratic space over the field \( \mathbb{F}_2 \), there is a second class of examples. Let \( \Omega \) be a finite set and \( P \) be the set of unordered pairs of elements from \( \Omega \). As lines we take the triples of points contained in any subset of \( \Omega \) of size 3. This space will be denoted as \( \mathcal{T}(\Omega) \).

As follows from the work of Hall [15], cotriangular spaces come only in these two types:

**Theorem 7.2.** [15] Let \( \Pi \) be a connected and reduced cotriangular space. Then up to isomorphism \( \Pi \) is one of the following.

(a) The geometry \( \Pi(V, Q) \) of elliptic lines in an orthogonal space \( (V, Q) \) over \( \mathbb{F}_2 \), where the radical of \( Q \) is \( \{0\} \).

(b) The geometry \( \mathcal{T}(\Omega) \) for some set \( \Omega \).

Hall also determined how these spaces can embed in each other. In particular, he has proven the following result.

**Theorem 7.3.** [15] Let \( (V, Q) \) be an orthogonal space over \( \mathbb{F}_2 \), where \( \text{Rad}(Q) = \{0\} \).

Let \( \Pi \) be a proper connected subspace of \( \Pi(V, Q) \), where \( \text{Rad}(Q) = \{0\} \). Then either there is a proper subspace \( U \) of \( V \) such that the points of \( \Pi \) are in \( P \setminus U \), or \( \Pi \) is isomorphic to \( \mathcal{T}(\Omega) \) for some set \( \Omega \).

Moreover, in the latter case, \( V \) can be identified with the vector subspace of \( \mathbb{F}_2\Omega \) of even weight vectors, and \( Q \) takes the value 1 on all weight 2 vectors.

**Corollary 7.4.** Let \( \Gamma = (\mathcal{V}, \mathcal{E}) \) be a connected graph. Then either \( \Pi(\Gamma) = \Pi(V_{\Gamma}, Q_{\Gamma}) \), or \( \Gamma \) is a line graph and \( \Pi(\Gamma) \) isomorphic to \( \mathcal{T}(\Omega) \) for some set \( \Omega \).

**Proof.** Suppose \( \Gamma = (\mathcal{V}, \mathcal{E}) \) is a connected graph. As we can identify \( \Pi(V_{\Gamma}, Q_{\Gamma}) \) with \( \Pi(V, Q) \), we can assume \( \Gamma = \overline{\Gamma} \).

If \( \Pi(\Gamma) \neq \Pi(V_{\Gamma}, Q_{\Gamma}) \), then, as the vertices in \( \Gamma \) linearly span \( V_{\Gamma} \), the above Theorem 7.3 can be applied to find \( \Pi(\Gamma) \) to be isomorphic to \( \mathcal{T}(\Omega) \) for some set \( \Omega \). But then \( \Gamma \) is a line graph of a graph with vertex set \( \Omega \).

We use the above theorem and its corollary to determine when \( \mathcal{S}(\Gamma) \) and \( g(\Gamma) \) do or do not coincide. In order to describe the Lie algebras thus obtained we need to introduce one more class of Lie algebras connected to the cotriangular spaces \( \mathcal{T}(\Omega) \). So, let \( \Omega \) be a set and \( \mathcal{T}(\Omega) \) the corresponding cotriangular space. Then the points of \( \mathcal{T}(\Omega) \) can be identified with the vectors of weight 2 in the \( \mathbb{F}_2 \) vector space \( \mathbb{F}_2\Omega \) with the elements of \( \Omega \) as basis and addition \( + \).

On the space \( \mathbb{F}_2\Omega \) we can define a quadratic form \( Q \) by \( Q(\omega) = 0 \) and \( Q(\omega + \omega') = 1 \) for all distinct \( \omega, \omega' \in \Omega \). Then, consider \( \mathcal{E}(\mathbb{F}_2\Omega, Q) \) and in its Lie algebra \( g(\mathbb{F}_2\Omega, Q) \) the subalgebra \( g(\Omega) \) spanned by the weight two vectors.
For two weight two vectors \( \omega_1 + \omega_2 \) and \( \omega_3 + \omega_4 \) we have
\[
[\omega_1 + \omega_2, \omega_3 + \omega_4] = -f(\omega_1 + \omega_2, \omega_3 + \omega_4)\omega_1 + \omega_2 + \omega_3 + \omega_4,
\]
where \( f \) is the bilinear form associated to \( Q \).

This is equal to 0 if \( \omega_1 + \omega_2 = \omega_3 + \omega_4 \) or \( \omega_1, \ldots, \omega_4 \) are all distinct, and \(-\omega_2 + \omega_3\) if \( \omega_1, \omega_2, \omega_3 \) are distinct, and \( \omega_4 = \omega_2 \).

So indeed, \( g(\Omega) \) is a Lie subalgebra.

We can identify the Lie algebra \( g(\Omega) \) with a Lie subalgebra of \( gl(F\Omega) \).

Indeed, an element \( \omega_1 + \omega_2 \), where \( \omega_1, \omega_2 \) are distinct element from \( \Omega \) acts linearly on \( F\Omega \) as \( \epsilon_{\omega_1 + \omega_2} \), which is defined by
\[
\epsilon_{\omega_1 + \omega_2}(\omega) = f(\omega_1 + \omega_2, \omega)(-1)^g(\omega_1 + \omega_2, \omega)\omega_1 + \omega_2 + \omega_3
\]
for all \( \omega_3 \in \Omega \). Here \( g \) is a bilinear form with \( g(v, v) = Q(v) \) for all \( v \in F_2\Omega \).

So, \( \epsilon_{\omega_1 + \omega_2}(\omega_1) = \pm \omega_2 \) and \( \epsilon_{\omega_1 + \omega_2}(\omega_2) = \mp \omega_1 \), while \( \epsilon_{\omega_1 + \omega_2}(\omega) = 0 \) for \( \omega \in \Omega \) different from \( \omega_1, \omega_2 \).

One easily checks that \( \epsilon \) maps \( g(\Omega) \) to the Lie algebra of finitary anti-symmetric linear maps in \( gl(F\Omega) \). In particular, if \(|\Omega| = n \) is finite, then \( g(\Omega) \) is isomorphic to \( so(n, F) \).

**Theorem 7.5.** Suppose \( \Gamma \) is a connected graph and all its vertices are black.

If \( \Gamma \) is not a line graph, then \( R(\Gamma) \) admits a quotient isomorphic to \( g(\Gamma) \).

If \( \Gamma \) is a line graph, then \( \Pi(\Gamma) \simeq T(\Omega) \) for some set \( \Omega \) and \( R(\Gamma) \) admits a quotient isomorphic to \( g(\Omega) \).

**Proof.** The elements of \( \Gamma \) generate a subalgebra \( R(\Gamma) \) of \( g(\Gamma) \). Clearly if, \( u, v \in V_\Gamma \) are in \( R(\Gamma) \), then so is \( [u, v] \). This implies that the elements of \( V_\Gamma \) that are contained in \( R(\Gamma) \) form a subspace \( S \) of the the geometry \( \Pi := \Pi(V_\Gamma, Q_\Gamma) \).

Now let \( R \) be the radical of \( Q_\Gamma \) on \( V_\Gamma \). For points \( p, q \) of \( \Pi \) we have \( p = q \) if and only if \( p + q \in R \).

As factoring out the radical of \( Q_\Gamma \) also implies taking a quotient of \( R(\Gamma) \), we find that \( R(\Gamma) \) admits a quotient isomorphic to \( R(\Gamma) \).

Moreover, \( S \) is mapped to a subspace \( \overline{S} \) of \( \Pi \).

If \( \Gamma \) is not a line graph, then, by Corollary 7.4, this subspace \( \overline{S} \) is the full cotriangular space \( \Pi \), and \( R(\Gamma) = g(\Gamma) \).

If \( \Gamma \) is a line graph of a graph with vertex set \( \Omega \), then its vertices can be identified with pairs from \( \Omega \), and we find \( \overline{S} \) to be isomorphic to \( T(\Omega) \). But then \( R(\Gamma) \) admits a quotient isomorphic to \( g(\Omega) \). \( \square \)

We notice that due to Beineke’s characterization of line graphs, see [3], we can conclude that \( \Gamma \) is not a line graph if it contains an induced subgraph \( \Delta \) which is one of the nine graphs from Figure 6. The three graphs on the first row of Figure 6 are not reduced, while the others are. So, if \( \Delta \) is one of these three graphs contained as an induced subgraph in some reduced graph \( \Gamma \), then \( \Gamma \) contains a vertex distinguishing the vertices that have in \( \Delta \) the same set of neighbors. So, in \( \Gamma \) we find two vertices if \( \Delta \) is the first graph and one vertex in case \( \Delta \) is the second or third graph, such that adding these vertices to \( \Delta \) we obtain a reduced graph.

This implies that \( \Gamma \) contains a reduced connected subgraph \( \Gamma_0 \) on 6 vertices which is not a line graph. In particular, if we determine the quadratic space \( (V_{\Gamma_0}, Q_{\Gamma_0}) \) for this subgraph, then this is a nondegenerate orthogonal \( F_2 \)-space of +or --type.
Figure 6. The nine forbidden subgraphs for a line graph.

But, if \((V_{\Gamma_0}, Q_{\Gamma_0})\) is of \(+-\) type, then its cotriangular space \(\Pi(V_{\Gamma_0}, Q_{\Gamma_0})\) is isomorphic to \(T(\Omega)\), where \(\Omega\) is of size 8, contradicting that \(\Gamma_0\) is not a line graph.

We have proven the following.

**Proposition 7.6.** Suppose \(\Gamma\) is a connected graph such that \(\overline{\Gamma}\) is not a line graph. Then \(\Gamma\) contains a subgraph \(\Gamma_0\) on 6 vertices spanning a nondegenerate 6-dimensional orthogonal \(F_2\) space \((V_{\Gamma_0}, Q_{\Gamma_0})\) of \(+-\) type.

**Corollary 7.7.** Suppose \(\Gamma\) is a connected graph such that \(\overline{\Gamma}\) is not a line graph. Then \(F(\Gamma_0)\), defined over a field \(F\) of odd characteristic, contains a subalgebra isomorphic to \(\text{sp}(4, \mathbb{H})\), where \(\mathbb{H}\) is a quaternion algebra over \(F\).

**Proof.** Let \(\Gamma_0\) be the subgraph on 6 vertices guaranteed by Proposition 7.6. Then \(F(\Gamma_0)\) is the subalgebra we are looking for. \(\square\)

**Remark 7.8.** The Lie algebra \(\text{sp}(4, \mathbb{H})\), where \(\mathbb{H}\) are the real quaternions, is the maximal compact Lie subalgebra of a split real Lie algebra of type \(\mathfrak{e}_6\). See Example 8.7.

**Remark 7.9.** Proposition 7.6 and Corollary 7.7 are closely related to some results of Seven [25]. See in particular [25, Theorem 2.7]. Seven shows, among other things, the following:

Let \((V, Q)\) be an orthogonal space over the field with two elements corresponding bilinear form \(f\). To each vector \(v\) with \(Q(v) = 1\) we can assign a transvection \(\tau_v : V \to V\) in the orthogonal group of \(O(V, Q)\), such that for all \(w \in V\) we have

\[
\tau_v(w) = w + f(v, w)v.
\]

Let \(V\) be a basis of anisotropic vectors of \(V\), and denote by \(\Gamma\) the graph where two elements \(v, w \in V\) are adjacent if and only if \(f(v, w) = 1\). If \(\Gamma\) is connected, but \(\overline{\Gamma}\) is not a line graph, then \(\Gamma\) contains an induced subgraph \(\Gamma_0\) on six points that generate a nondegenerate 6-dimensional orthogonal \(F_2\) space \((V_{\Gamma_0}, Q_{\Gamma_0})\) of \(+-\) type on which the corresponding transvections induce the orthogonal group \(O(V_{\Gamma_0}, Q_{\Gamma_0})\) which is isomorphic to the Weyl group of type \(E_6\).
Remark 7.10. We notice that we can consider the various algebras of this and the previous section over a ring $R$. In particular, we can consider the Lie algebras $\mathfrak{g}(V, f)$ and $\mathfrak{g}(V, Q)$, as well as $\mathfrak{g}(\Omega)$ for some quadratic $F_2$-space $(V, Q)$ with associated bilinear form $f$ and set $\Omega$ over the integers $\mathbb{Z}$.

If we reduce scalars modulo an odd prime $p$, we obtain the Kaplansky Lie algebras as considered in [5], and if we reduce scalars modulo 2 we find the Lie algebras considered by Kaplansky in [16]. See also [6].

Algorithm 7.11. Let $\Gamma$ be a finite connected black colored graph. The above considerations also provide an algorithm to determine $\mathfrak{K}(\Gamma)$ as in Theorem 7.5 from the input $\Gamma$.

(a) Find the decomposition into the $\sim$-equivalence of $V$. This can be done using a standard partition algorithm. See for example Algorithm 2 in [13].

(b) Take a single vertex from each $\sim$-class and determine the induced subgraph of $\Gamma$. This graph is isomorphic to $\mathfrak{K}(\Gamma)$.

(c) Determine, if possible, a graph $\Delta$ such that $\Gamma$ is the line graph of $\Delta$ (several algorithms exist, see for example [21]). In case $\Gamma$ is $K_3$, the complete graph on 3 points, there are two graphs $\Delta$, namely $K_3$ and $K_1, 3 = D_4$ having $\Gamma$ as line graph. In this case, take $\Delta$ to be $K_3$.

(d) If $\Gamma$ is the line graph of the graph $\Delta$, then $\mathfrak{K}(\Gamma)$ is isomorphic to $\mathfrak{g}(\Omega)$, where $\Omega$ is the vertex set of $\Delta$.

(e) If $\Gamma$ is not a line graph, then $\mathfrak{K}(\Gamma)$ equals $\mathfrak{g}(\Gamma)$, the Lie algebra of $C(\Gamma)$. The isomorphism type of the latter can be determined using Table 5 and Algorithm 5.4.

If the input of the algorithm is a graph on $n$ vertices, then the complexity of the algorithm is of order at most $n^3$, as for each step there exist algorithms of order at most $n^3$.

8. Spin representations and compact subalgebras of Kac-Moody algebras

Suppose $\Gamma = (V, \mathcal{E})$ is a graph with all vertices colored black. Then the generators $x \neq y \in V$ of the Lie algebra $\mathfrak{K}(\Gamma)$ do satisfy the relations

$$[x, y] = 0 \quad \text{if } (x, y) \notin \mathcal{E},$$
$$[x, [x, y]] = -y \quad \text{if } (x, y) \in \mathcal{E}.$$

So, the free Lie algebra $\mathfrak{g}_\Gamma$ with generators in $V$ subject to the above relations has then $\mathfrak{K}(\Gamma)$ as a quotient.

The next result is motivated by, and a generalisation of the results of [11]. We consider linear representations $\phi$ of the free Lie algebra $\mathfrak{g}_\Gamma$ into $\mathfrak{g}(W)$, the general linear Lie algebra on a vector space $W$ over a field of characteristic not 2. If $x, y$ are two linear maps on $W$, then by $xy$ we denote the composition, and we consider the Lie product of $\mathfrak{g}(W)$ to be defined as

$$[x, y] = \frac{1}{2}(xy - yx).$$

Such a representation $\phi$ is called a generalized spin representation of $\mathfrak{g}_\Gamma$, if and only if

$$\phi(x)^2 = -1_W$$

for all generators $x \in V$. Our first observation is that $\mathfrak{g}_\Gamma$ always admits such a representation.
Proposition 8.1. The Lie algebras $\mathfrak{g}_\Gamma$ and $\mathfrak{g}(\Gamma)$ admit a generalized spin representation.

Proof. As $\mathfrak{g}(\Gamma)$ is a quotient of $\mathfrak{g}_\Gamma$, we only have to show that $\mathfrak{g}(\Gamma)$ admits a generalized spin representation.

As the elements of $\mathfrak{g}(\Gamma)$ act by left multiplication on $\mathfrak{C}(\Gamma)$, and $x^2 = -1$ for all $x \in \mathcal{V}$, we have found a generalized spin representation. $\square$

For finitely generated $\mathfrak{g}_\Gamma$ generalized spin representations have been constructed by [11], generalizing [8, 7] in which such representations have been constructed for graphs of type $E_6$ and $E_{10}$.

The following characterization of the generalized spin representation is also obtained in [11].

Theorem 8.2. Suppose $\phi : \mathfrak{g}_\Gamma \to \mathfrak{g}(W)$ for some vector space $W$ over a field of characteristic $\neq 2$ is a linear representation of $\mathfrak{g}_\Gamma$.

If $\phi$ is a generalized spin representation, then $\phi(\mathfrak{g}_\Gamma)$ is isomorphic to a quotient of $\mathfrak{g}(\Gamma)$.

Proof. We identify the elements $x \in \mathfrak{x}$ with their images under $\phi$ and compute in $\text{End}(W)$. Then, as $x^2 = -1_W$, we find $x$ to be invertible with inverse $\phi(x)^{-1} = -\phi(x)$. Now for $x, y \in \mathfrak{x}$ we have $xy - yx = 0$ or $\phi(x)(xy - yx) - (xy - yx)x = -y$.

Suppose we are in the latter case. Then $x^2y - 2xyx + yx^2 = -4y$ and hence $2y - 2xyx = 0$. Now multiplying with $x$ yields $2(xy + yx) = 0$ from which we deduce $xy + yx = 0$.

So, the (images under $\phi$ of the) elements $x \in \mathcal{V}$ satisfy, as linear maps from $W$ to itself, the relations of the generators of $\mathfrak{C}(\Gamma)$, where all vertices of $\Gamma$ are considered to be black. But then the subalgebra of $\text{End}(W)$ generated by $\phi(\mathcal{V})$ is isomorphic to a quotient of $\mathfrak{C}(\Gamma)$. In particular, $\mathfrak{g}(\Gamma)$ maps onto a subalgebra of $\mathfrak{g}(W)$ containing $\phi(\mathfrak{g}_\Gamma)$ as the subalgebra generated by the elements of $\phi(x)$ with $x \in \mathcal{V}$. This implies that $\phi(\mathfrak{g}_\Gamma)$ is isomorphic to a quotient of $\mathfrak{g}(\Gamma)$. $\square$

A result of Berman [4] relates the free Lie algebra $\mathfrak{g}_\Gamma$ to the so-called compact subalgebras of Kac-Moody algebras over fields $F$ of characteristic 0. Let us explain this connection, restricting ourselves to the simply laced case.

Let $A = (a_{ij})$ be a generalized Cartan matrix indexed by the set $\mathcal{V}$, which is simply laced. That means

$$a_{ii} = 2$$

and

$$a_{ij} = a_{ji} = 0 \text{ or } -1$$

for $i \neq j \in \mathcal{V}$.

Then the Kac-Moody Lie algebra $\mathfrak{m}(A)$ is the free Lie algebra over $F$ with generators

$$e_i, f_i, h_i, \text{ where } i \in \mathcal{V}$$

subject to the relations

$$[h_i, h_j] = 0, \ [h_i, e_j] = a_{ij}e_j, \ [h_i, f_j] = -a_{ij}f_j \text{ for all } i, j \in \mathcal{V}$$

$$[e_i, f_j] = 0, \ [e_i, f_i] = h_i, \ [e_i, [e_i, e_j]] = 0 = [f_i, [f_i, f_j]] \text{ for all } i \neq j \in \mathcal{V}.$$

The so-called compact subalgebra $\mathfrak{A}(A)$ of $\mathfrak{m}(A)$ is the Lie subalgebra generated by the elements
If for each $x \in \mathcal{V}$ we denote by $x$ the element $e_x + f_x$, and consider the associated graph $\Gamma = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V}$ and two distinct vertices $x, y$ adjacent if and only if $a_{xy} \neq 0$, then we obtain the following.

**Lemma 8.3.** Let $x \neq y \in \mathcal{V}$. Then

$$[x, y] = 0 \quad \text{if } (x, y) \notin \mathcal{E}$$

$$[x, [x, y]] = -y \quad \text{if } (x, y) \in \mathcal{E}.$$

**Proof.** If $x$ and $y$ are non-adjacent then clearly $[x, y] = 0$. So, assume $x$ and $y$ are adjacent. Then

$$[x, [x, y]] = [e_x + f_x, [e_x + f_x, e_y + f_y]]$$

$$= [e_x + f_x, [e_x, e_y] + [f_x, f_y]]$$

$$= [e_x, [e_x, e_y] + [f_x, f_y]] + [f_x, [e_x, e_y] + [f_x, f_y]]$$

$$= [e_x, [f_x, f_y]] + [f_x, [e_x, e_y]]$$

$$= -[f_y, [e_x, f_x]] - [e_y, [f_x, e_x]]$$

$$= -[f_y, h_x] + [e_y, h_x]$$

$$= -f_y - e_y$$

$$= -y.$$

\[ \square \]

**Theorem 8.4.** (Berman [4]) Let $\mathbb{F}$ be a field of characteristic 0 and $A = (a_{ij})$ a simply laced generalized Cartan matrix with associated graph $\Gamma = (\mathcal{V}, \mathcal{E})$. Let $\mathfrak{R}(\mathcal{A})$ the Kac-Moody Lie algebra over $\mathbb{F}$. Then the compact Lie subalgebra $\mathfrak{R}(\mathcal{A})$ of $\mathfrak{R}(\mathcal{A})$ is isomorphic to the free Lie algebra $\mathfrak{g}_\mathbb{F}$ over $\mathbb{F}$ generated by $\mathcal{V}$ subject to the relations

$$[x, y] = 0 \quad \text{if } (x, y) \notin \mathcal{E}$$

$$[x, [x, y]] = -y \quad \text{if } (x, y) \in \mathcal{E}$$

for $x \neq y \in \mathcal{V}$.

Combining the above Theorem 8.4 with Proposition 8.1, we obtain the following.

**Corollary 8.5.** Let $\mathbb{F}$ be a field of characteristic 0 and $A = (a_{ij})$ a simply laced generalized Cartan matrix with associated graph $\Gamma = (\mathcal{V}, \mathcal{E})$. Let $\mathfrak{R}(\mathcal{A})$ the Kac-Moody Lie algebra over $\mathbb{F}$. The compact Lie subalgebra $\mathfrak{R}(\mathcal{A})$ of $\mathfrak{R}(\mathcal{A})$ admits a quotient isomorphic to $\mathfrak{R}(\Gamma)$, and in particular, admits a spin representation.

**Example 8.6.** If $\Gamma$ is the graph $E_{10}$, with all vertices black, and $\mathbb{F}$ is a field of type III, for example $\mathbb{R}$, then $\mathfrak{C}(\Gamma)$ is isomorphic to $\mathfrak{C}(V, Q)$, where $(V, Q)$ is a nondegenerate form of $+\text{-type}$. But then $\mathfrak{g}(\Gamma) = \mathfrak{R}(\Gamma)$ is isomorphic to $\mathfrak{so}(32, \mathbb{F})$. So, if $\mathbb{F} = \mathbb{R}$, we find that the compact Lie subalgebra $\mathfrak{R}(E_{10})$ of $\mathfrak{R}(E_{10})$ admits a quotient isomorphic to $\mathfrak{g}(\Gamma)$. Using Table 4, we obtain similar results for graphs of type $E_n$ for all $n$. See also [8, 7, 11].

**Example 8.7.** As in [11] we can use the above result to determine the maximal compact Lie subalgebra $\mathfrak{K}$ of the semi-simple split real Lie algebras of type $A_n, D_n$, where $n \geq 1$ and $E_n$, where $6 \leq n \leq 8$.

Indeed, using 8.5, we find that the maximal compact Lie subalgebra $\mathfrak{K}$ of a semi-simple split real Lie algebras $\mathfrak{g}$ of type $A_n, D_n$, where $n \geq 1$ and $E_n$, where $6 \leq n \leq 8$, admits a quotient isomorphic to $\mathfrak{R}(\Gamma)$, where $\Gamma$ is the corresponding graph of type $A_n, D_n$, or $E_n$. 
QUASI-CLIFFORD ALGEBRAS, QUADRATIC FORMS OVER $\mathbb{F}_2$, AND LIE ALGEBRAS

| Type of $\mathfrak{g}$ | Maximal compact subalgebra $\mathfrak{H}$ | Dimension $n$ |
|----------------------|----------------------------------------|--------------|
| $A_n$ ($n > 3$)      | $\mathfrak{so}(n + 1, \mathbb{R})$      | $n(n - 1)$   |
| $D_n$ ($n > 3$)      | $\mathfrak{so}(n, \mathbb{R}) \oplus \mathfrak{so}(n, \mathbb{R})$ | $n(n - 1)$   |
| $E_6$                | $\mathfrak{sp}(4, \mathbb{H})$          | 36           |
| $E_7$                | $\mathfrak{su}(8, \mathbb{C})$          | 63           |
| $E_8$                | $\mathfrak{so}(16, \mathbb{R})$         | 120          |

Table 6. Maximal compact subalgebras of the split real Lie algebras.

Using the results of the previous sections, we find these quotients to be as in Table 6. This provides a lower bound for the dimension of $\mathfrak{H}$ which coincides with the upper bound of the dimension of $\mathfrak{H}$ that one can obtain from the Iwasawa decomposition of $\mathfrak{g}$.

These results can also be found in [17], where $\mathfrak{H}$ is embedded in the Lie algebra of a (generalized) Clifford algebra.

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