Mathematical simulation of stress-strain state of loaded rods with account of transverse bending

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Abstract. Mathematical simulation of stress-strain state of loaded rods with account of transverse bending is considered in the paper. The urgency, the correctness of mathematical statement of the problem, the mathematical model, the computational algorithm and the computational experiments of the problem set are given in the paper. Results are presented in a graphical form. The oscillations of spatially loaded rods are shown. The analysis is made in linear and non-linear statements.

1. Introduction
All over the world, in order to solve the problems of strain processes of structure elements such as rods in construction, aviation, rocket production, shipbuilding, mechanical engineering and oil industry, the following promising areas are being studied: the development of mathematical models and computational algorithms, improvement and creation of software for linear and nonlinear strain processes in the S.P.Timoshenko beam, simulation of the stress-strain state of rod-like structure elements, that take into account strain processes, geometric and physical nonlinearity, a complete study of intense flexural-torsion, longitudinal-flexural and longitudinal-torsion oscillations propagating in rods[5].

2. Statement of the problem
This article proposes a mathematical simulation of geometrically non-linear rod problems with allowance for transverse bending. In this case, the spatial strain of the rods takes into account two components of the longitudinal displacements $u_1(x,t)$, and one component of the transverse displacements $u_3(x,t)$. Then the motion of the rod points takes the form [7]:

$$u_1(x,y,z,t) = u - z\alpha, \quad u_2 = 0, \quad u_3(x,y,z,t) = w,$$

where $u, w$ are the displacements of the rod midline; $\alpha$ is the angle of rotation of the section at a pure bending; $u_1, u_3$ are the components of the displacement vector. Here, the sought for $u, w, \alpha$ are the functions in the spatial variable $x$ and time $t$, and there are no restrictions on the external load.
3. Mathematical model

Using the Ostrogradsky-Hamilton variation principle [7-11]:

$$\delta \int_{t_1}^{t_2} (K - \Pi + A) dt = 0$$ (2)

where $K, \Pi$ are kinetic and potential energy; $A$ is the work of external volume and surface forces.

Derive a mathematical model of nonlinear problems of rods under static and dynamic spatial loading.

When calculating the variation of kinetic energy, the following ratio is used [7-11]:

$$\int_{t}^{t} \delta K dt = \int_t \int_v (\rho \frac{\partial u_1}{\partial t} \frac{\partial u_1}{\partial t} + \rho \frac{\partial u_2}{\partial t} \frac{\partial u_2}{\partial t} + \rho \frac{\partial u_3}{\partial t} \frac{\partial u_3}{\partial t}) dv dt,$$

To determine the variation of potential energy, the following formula is used [7-11]:

$$\int_{t}^{t} \delta \Pi dt = \int_t \int_v \sum_{i=1}^{3} \sigma_{1i} \delta \varepsilon_{1i} dv dt = \int_t \int_v (\sigma_{11} \delta \varepsilon_{11} + \sigma_{12} \delta \varepsilon_{12} + \sigma_{13} \delta \varepsilon_{13}) dv dt,$$

where $\sigma_{11}, \sigma_{12}, \sigma_{13}$ are the elements of the stress tensor; $\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{13}$ are the elements of the strain tensor.

Take the Cauchy relations [7-11]:

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x} + \frac{1}{2} \left( \frac{\partial u_3}{\partial x} \right)^2, \quad \varepsilon_{13} = \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} \frac{\partial u_3}{\partial z},$$

According to (1), $\frac{\partial u_3}{\partial x} = 0$. So

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x} + \frac{1}{2} \left( \frac{\partial u_3}{\partial x} \right)^2, \quad \varepsilon_{13} = \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}.$$

Stress components based on the Hooke law are written in the form:

$$\sigma_{11} = E \varepsilon_{11} = E \frac{\partial u_1}{\partial x}, \quad \sigma_{13} = G \varepsilon_{13} = G \left( \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} \right).$$

where $E$ is the elasticity modulus, $G = \frac{E}{2(1+\mu)}$ is the shear modulus, $\mu$ is the Poisson ratio.

For the variation of the external forces work the following formula is applied [7-11]:

$$\int_{t}^{t} \delta A dt = \int_v \sum_{i=1}^{3} F_i \delta u_i dv + \int_s \sum_{i=1}^{3} q_i \delta u_i ds + \int_{s_1} \sum_{i=1}^{3} f_i \delta u_i ds_1 =$$

$$= \int_v (F_1 \delta u_1 + F_2 \delta u_2 + F_3 \delta u_3) dv +$$

$$+ \int_s (q_1 \delta u_1 + q_2 \delta u_2 + q_3 \delta u_3) ds + \int_{s_1} (f_1 \delta u_1 + f_2 \delta u_2 + f_3 \delta u_3) ds_1,$$

where $F_i$ are the components of volume forces, referred to the unit volume, $q_i$ is the surface forces, referred to the unit surface area of the rod; $f_i$ is the end face forces, respectively.
Here, \( u, w, \alpha \) are the functions of the \( x \) coordinate and time \( t \), therefore, the derivatives with respect to \( z \) are zero. When considering a symmetric section, the static moments are zero too. Then then from the variation principle (2) we get the following system of equations is obtained with the corresponding initial and boundary conditions[12].

System of oscillation equations:

\[
-\rho F \frac{\partial^2 u}{\partial t^2} + EF \frac{\partial^2 u}{\partial x^2} + (f_1 + \ddot{q}_1) = 0, \\
-\rho F \frac{\partial^2 w}{\partial t^2} + GF \frac{\partial^2 w}{\partial x^2} + EF \frac{\partial u}{\partial x} \left( \frac{\partial^2 u}{\partial x^2} \right) + EF \frac{\partial^2 w}{\partial x^2} \left( \frac{\partial u}{\partial x} \right) + (f_3 + \ddot{q}_3) = 0, \\
-\rho I_y \frac{\partial^2 \alpha}{\partial t^2} + EI_y \frac{\partial^2 \alpha}{\partial x^2} - (M(f_1) + (M(q_1)) = 0.
\]

Boundary conditions:

\[
[-E \frac{\partial u}{\partial x} + \phi_1] \frac{\partial u}{\partial x} = 0, \\
[-GF \frac{\partial w}{\partial x} + E \frac{\partial u}{\partial x} \left( \frac{\partial u}{\partial x} \right) + \phi_3] \frac{\partial w}{\partial x} = 0, \\
[-EI_y \frac{\partial \alpha}{\partial x} + M(\phi_1)] \frac{\partial \alpha}{\partial x} = 0.
\]

Initial conditions:

\[
\left( \rho F \frac{\partial u}{\partial t} \right) \frac{\partial u}{\partial t} = 0, \quad \left( \rho I_y \frac{\partial \alpha}{\partial t} \right) \frac{\partial \alpha}{\partial t} = 0, \quad \left( \rho \frac{\partial w}{\partial t} \right) \frac{\partial w}{\partial t} = 0.
\]

Here, performing some mathematical calculations, we have:

\[
-\frac{\partial^2 \ddot{u}^k}{\partial t^2} + \frac{\partial^2 \ddot{u}^k}{\partial x^2} + \frac{l^2}{EFa} \left( f_1 + \ddot{q}_1 \right) = 0, \\
-\frac{\partial^2 \ddot{w}^k}{\partial t^2} + \frac{1}{2(1 + \mu)} \frac{\partial^2 \ddot{w}^k}{\partial x^2} + \Phi^{-1}_{2} \frac{l^2}{EFa} \left( f_3 + \ddot{q}_3 \right) = 0, \\
-\frac{\partial^2 \ddot{\alpha}^k}{\partial t^2} + \frac{I_y}{Fa^2} \frac{\partial^2 \ddot{\alpha}^k}{\partial x^2} - \frac{l^2}{EFa^2} \left( M(f_1) + (M(q_1)) = 0. \right.
\]

Here

\[
\Phi^{-1}_2 = \frac{\partial^2 \ddot{w}^k - 1}{\partial x^2} \left( l \frac{\partial \ddot{u}^k - 1}{a \partial x} \right) + \partial \ddot{w}^k - 1 \left( l \frac{\partial^2 \ddot{u}^k - 1}{a \partial x^2} \right).
\]

Boundary conditions:

\[
\left[ -\frac{l \dot{\ddot{u}}^k}{\partial x} + \frac{l^2}{EFa} \phi_1 \right] \frac{\partial \ddot{u}}{\partial x} = 0, \\
\left[ -\frac{1}{2(1 + \mu)} \left( \frac{\partial \ddot{w}^k}{\partial x} - \Phi^{-1}_2 \phi_3 \right) \right] \frac{\partial \ddot{w}}{\partial x} = 0, \\
\left[ \frac{I_y}{Fa^2} \frac{\partial \ddot{\alpha}^k}{\partial x} + \frac{l^2}{EFa^2} \left( M(\phi_1) \right) \right] \frac{\partial \ddot{\alpha}}{\partial x} = 0.
\]

Here

\[
\Phi^{-1}_2 = \frac{\partial \ddot{w}^k - 1}{\partial x} \left( l \frac{\partial \ddot{u}^k - 1}{a \partial x} \right).
\]
Initial conditions:
\[
\left[ \frac{\partial \bar{u}_k}{\partial t} \right]_{t_0} \delta \bar{u} \bigg|_{t} = 0, \quad \left[ \frac{\partial \bar{w}_k}{\partial t} \right]_{t_0} \delta \bar{w} \bigg|_{t} = 0, \quad \left[ \frac{I_y}{F a^2} \frac{\partial \alpha_k}{\partial t} \right]_{t_0} \delta \alpha \bigg|_{t} = 0. \tag{5}
\]

Thus, using the hypothesis (1), a system of equations for problems of geometrically non-linear rods under spatial loading with allowance for transverse bending is obtained. By specifying the boundary conditions at \( x = 0 \) and \( x = l \), one can solve a number of practical problems.

4. Computational algorithm

The solution of differential equations of motion (3) with the corresponding boundary (4) and initial (5) conditions obtained from the variation principle (2) in a scalar form is rather difficult. Therefore, the system of differential equations, the boundary and initial conditions will be represented in a vector form. Introduce the vectors in the following form \([13-16]\):

\[
\vec{U} = [\bar{u}, \bar{w}, \alpha]^T, \quad \vec{F} = [(f_1 + \bar{q}_1), (f_3 + \bar{q}_3), (M(f_1) + M(q_1))]^T, \\
\vec{F}(\varphi) = [\bar{\varphi}_1, \bar{\varphi}_3, M(\varphi_1)]^T, \quad \Phi^{k-1} = [0, \Phi_2^{k-1}, 0]^T, \quad \Phi^{-1} = [0, \Phi_2^{k-1}, 0]^T. \tag{6}
\]

The system of equations (3), boundary conditions (4) and initial conditions (5), with introduced matrix elements, are written in the vector form:

\[
M \frac{\partial^2 \vec{U}_k}{\partial t^2} + A \frac{\partial^2 \vec{U}_k}{\partial x^2} + E \Phi^{k-1} + D \vec{F} = 0, \tag{7}
\]

\[
\left[- M \frac{\partial \vec{U}_k}{\partial t}\right]_{t_0} E \delta \vec{U} \bigg|_{t} = 0, \tag{8}
\]

\[
\left[A \frac{\partial \vec{U}_k}{\partial x} + E \Phi^{k-1} + D \vec{F}(\varphi)\right] E \delta \vec{U} \bigg|_{x} = 0. \tag{9}
\]

The linear problem is solved, which is the first approximation of the solution.

\[
M \frac{\partial^2 \vec{U}_k}{\partial t^2} + A \frac{\partial^2 \vec{U}_k}{\partial x^2} + D \vec{F} = 0, \tag{10}
\]

\[
\left[- M \frac{\partial \vec{U}_k}{\partial t}\right]_{t_0} E \delta \vec{U} \bigg|_{t} = 0, \tag{11}
\]

\[
\left[A \frac{\partial \vec{U}_k}{\partial x} + D \vec{F}(\varphi)\right] E \delta \vec{U} \bigg|_{x} = 0. \tag{12}
\]

Further, an iteration process is organized until the following condition is satisfied:

\[
MAX |\vec{U}^k_{ij} - U_{ij}^{k-1}| \leq \varepsilon, \quad \text{Here } k- \text{ is the number of iterations. When constructing a computational algorithm for a system of differential equations (10) with initial (11) and boundary (12) conditions, the central finite-difference relations of second-order accuracy are applied [17-18].}
\]

After some mathematical calculations, a system of algebraic equations is built in the form:

\[
\vec{U}_{i,j+1} = -A \vec{U}_{i-1,j} + B \vec{U}_{i,j} - C \vec{U}_{i+1,j} - \vec{U}_{i,j-1} - \vec{F}_{i,j}.
\]
Coefficients of all resolving equations have the form:
\[ \tilde{A} = \frac{\tau^2 AM^{-1}}{h^2}, \tilde{B} = 2 + \frac{\tau^2 AM^{-1}}{h^2}, \tilde{C} = \frac{\tau^2 AM^{-1}}{h^2}, \tilde{F}_{i,j} = D\tau^2 M^{-1}\tilde{F}_{i,j}, \]

In a static statement equation (7) and boundary conditions (9) are rewritten in the form
\[
A \frac{\partial^2 \tilde{U}_k}{\partial x^2} + E\tilde{\Phi}_{k-1} + D\tilde{F} = 0, \tag{13}
\]

\[
\left[ \tilde{A} \frac{\partial \tilde{U}_k}{\partial x} + B\tilde{U}_k + E\tilde{\Phi} + D\tilde{F} (\varphi) \right] E l \delta \tilde{U} \bigg|_{x} = 0. \tag{14}
\]

The linear problem is solved and thus the zero approximation of the solution is made:
\[
A \frac{\partial^2 \tilde{U}_k}{\partial x^2} + D\tilde{F} = 0, \tag{15}
\]

\[
\left[ \tilde{A} \frac{\partial \tilde{U}_k}{\partial x} + B\tilde{U}_k + D\tilde{F} (\varphi) \right] E l \delta \tilde{U} \bigg|_{x} = 0. \tag{16}
\]

When constructing a computational algorithm for the system of differential equations (15) with boundary conditions (16), the central finite-difference relations of the second-order accuracy are applied [16-19].

\[ A\tilde{U}_{i-1} - B\tilde{U}_i + C\tilde{U}_{i+1} + \tilde{F}_i = 0, \text{ here } \tilde{A} = \frac{\tilde{A}}{h^2}, \ B = \frac{2A}{h^2}, \ C = \frac{A}{h^2}, \ \tilde{F}_i = D\tilde{F}. \]

Consider the boundary conditions of the case when the two ends of the rod are rigidly fixed. In this case, the boundary conditions are
\[ \tilde{U}_1 \bigg|_{x=0} = 0, \quad \frac{\partial \tilde{U}}{\partial x} \bigg|_{x=0} = 0, \quad \tilde{U}_N \bigg|_{x=l} = 0, \quad \frac{\partial \tilde{U}}{\partial x} \bigg|_{x=l} = 0. \tag{17} \]

Solution order of the problem is:
1. At \( i = 0 \) the following equation is solved
\[ \tilde{U}_1 = -\left( 1 + \tilde{C}^{-1}\tilde{A} \right)^{-1}\tilde{C}^{-1}\tilde{F}_0. \]
2. At \( i = 1 \) the following equation is solved
\[ \tilde{U}_2 = \tilde{C}^{-1} \left( B\tilde{U}_1 - \tilde{F}_0 \right). \]
3. At \( i \geq 2 \) the following equation is solved
\[ \tilde{U}_{i+1} = \tilde{C}^{-1} \left( -A\tilde{U}_{i-1} + B\tilde{U}_i - \tilde{F}_i \right). \]
4. At \( i = N-2 \) the following equation is solved
\[ \tilde{U}_{N-1} = \tilde{C}^{-1} \left( -A\tilde{U}_{N-3} + B\tilde{U}_{N-2} - \tilde{F}_{N-2} \right). \]

As a result the first approximation of the problem is obtained:
\[ A \frac{\partial^2 \tilde{U}_k}{\partial x^2} = -D\tilde{F} - E\tilde{\Phi}^{k-1}. \]
Further, an iteration process is going on until the following condition is met:

$$MAX \left| U_i^k - U_i^{k-1} \right| \leq \varepsilon.$$ 

Applying the central finite-difference relations of the second-order accuracy [13-16] for $\Phi_{k-1}^{k-1}$ and $\bar{\Phi}_{k-1}^{k-1}$, we have

$$\Phi_{k-1}^{k-1} = \left[ \frac{1}{h^2} (w_{i+1} - 2w_i + w_{i-1}) \right] \left[ \frac{1}{2h} (u_{i+1} - u_{i-1}) \right] + \left[ \frac{1}{2h} (w_{i+1} - w_{i-1}) \right] \left[ \frac{1}{ah} (u_{i+1} - 2u_i + u_{i-1}) \right]$$

and

$$\bar{\Phi}_{k-1}^{k-1} = \frac{1}{2h} (w_{i+1} - w_{i-1}) \frac{a}{2h} (u_{i+1} - u_{i-1}).$$

5. Computational experiment

Consider the numerical solution of the problem. The following parameters are used to calculate the rod: the Young’s modulus $E = 2 \times 10^5$ Pa, the Poisson’s ratio $\nu = 0.3$ (for steel), the length $l = 10$ m, cross-sections $a = 0.02$ m, $b = 0.02$ m, surface loads $q_1 = 0.015H$, $q_3 = 0.02H$, $M = 0.012Hm$.

The results obtained correspond to the specified boundary value problems (17).

![Figure 1. Comparison of linear and non-linear results of longitudinal displacement $u$.](image)

![Figure 2. Comparison of linear and non-linear results of transverse displacement $w$.](image)
Figure 3. Comparison of linear and non-linear results in the angle of inclination $\alpha$.

6. Conclusion

The results obtained are consistent with the results of studies on the nonlinear dynamics of rods; a comparison of elastic elements displacements in the linear statement gives differences with the nonlinear statement of the problem. This suggests that the rod structure of geometric and physical parameters studied in this paper is considered as essentially non-linear one. Therefore, the linear simulation of this problem will lead to the first (overestimated) approximation of the solution. Besides, it will reflect the variety of complex oscillatory processes that occur at multiple frequencies, which will remain outside the framework of linear dynamic models.

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