A hyperelliptic Hodge integral

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July 24, 2008

1 Introduction

We work over $\mathbb{C}$. We will use the theories of orbifold stable maps and orbifold Gromov–Witten theory as developed in [2] and [1]. Our notation for the moduli space of degree $\beta$ orbifold stable maps with $n_1$ ordinary marked points and $n_2$ orbifold points will be

$$\mathcal{M}(X; n_1, n_2; \beta),$$

which is an open substack of the corresponding Artin stack of pre-stable maps, $\mathcal{M}(X; n_1, n_2; \beta)$. (Since we will only deal with $\mathbb{Z}/2\mathbb{Z}$ stabilizers in this note, we will not need a more detailed notation.)

Let $C$ be the universal curve over $\mathcal{M} = \mathcal{M}(B(\mathbb{Z}/2\mathbb{Z}); 0, 2g + 2)$. For each $i = 1, \ldots, 2g + 2$, there is a closed substack $D_i$, the $i$-th universal $\mathbb{Z}/2\mathbb{Z}$-gerbe over $\mathcal{M}$. Let $N_{D_i/C}$ be the normal bundle of $D_i$ in $C$ and define $L_i$ to be the line bundle

$$L_i = \pi_\ast \left( N_{D_i/C}^\vee \otimes \rho_1 \right),$$

where $\rho_1$ is the non-trivial representation of $\mathbb{Z}/2\mathbb{Z}$, viewed as a line bundle on $D$ pulled back from $B(\mathbb{Z}/2\mathbb{Z})$.

Remark 1.0.1. Our definition of $L_i$ coincides with the cotangent line bundle on the universal hyperelliptic curve over $\mathcal{M}(B(\mathbb{Z}/2\mathbb{Z}); 0, 2g + 2)$. Indeed, let $p : \tilde{C} \to C$ be the base change of (point) $\to B(\mathbb{Z}/2\mathbb{Z})$ via the universal map $C \to B(\mathbb{Z}/2\mathbb{Z})$ and define $\tilde{D}_i$ analogously. Then $\pi_\ast (N_{\tilde{D}_i/C} \otimes \rho_1)$ can be identified with the pushforward via $\pi p$ of the $-1$-eigenspace of $p_\ast N_{\tilde{D}_i/C}^\vee = N_{\tilde{D}_i/C}^\vee$. Since the hyperelliptic involution acts nontrivially on the fiber of the cotangent bundle at a Weierstrass point, this is just $N_{\tilde{D}_i/C}$, which is the usual cotangent line bundle.

In view of the remark, it is legitimate to say $c_1(L_i) = \psi_i$.

We also have the hyperelliptic Hodge bundle, whose dual is defined to be

$$E^\vee = R^1 \pi_\ast (\rho_1)$$

where $\pi$ is the map from the universal curve $C$ to $\mathcal{M}(B(\mathbb{Z}/2\mathbb{Z}), 2g + 2)$.

Remark 1.0.2. This definition of the Hodge bundle coincides with the usual definition as $R^1 (\pi p)_\ast \mathcal{O}_C$ (where $p : \tilde{C} \to C$ is defined as in the last remark). Indeed, we can identify $R^1 (\pi p)_\ast \mathcal{O}_C$ since $p_\ast \mathcal{O}_C \cong \mathcal{O}_C \otimes (\mathcal{O}_C \otimes \rho_1)$ and $\mathcal{O}_C$ has no higher cohomology (because $C$ has genus 1).

It is therefore justified to write $c_1(E) = \lambda_i$.

Theorem 1.1. We have

$$\int_{\mathcal{M}(B(\mathbb{Z}/2\mathbb{Z}), 2g + 2)} \frac{c(E^\vee)^2}{c(L_1^\vee)} = \int_{\mathcal{M}} \frac{(1 - \lambda_1 + \cdots + (-1)^g \lambda_g)^2}{1 - \psi_1} = \left( -\frac{1}{4} \right)^g .$$

The first equality was proved in the two remarks above. The second equality will be proven by interpreting the integral as a Gromov–Witten invariant on the weighted projective space (Section 2) and evaluating it recursively using the WDVV equations (Section 2).

The application for this calculation is [3], where it is used to relate the genus zero Gromov–Witten invariants of $[\text{Sym}^2 \mathbb{P}^2]$ and the enumerative geometry of hyperelliptic curves in $\mathbb{P}^2$. 

2 A Gromov–Witten invariant of \( P(1, 1, 2) \)

Let \( \overline{M}(P(1, 1, 2); n_1, n_2; \beta) \) be the moduli space of genus zero orbifold stable maps to \( P(1, 1, 2) \) with \( n_1 \) ordinary marked points and \( n_2 \) orbifold marked points and degree \( \beta \). The degree is evaluated by integrating \( c_1(O(1)) \) over the curve and so is an element of \( \frac{1}{2}\mathbb{Z} \).

The virtual dimension is given by the formula

\[
v \dim \overline{M}(P(1, 1, 2); n_1, n_2; \beta) = \dim P(1, 1, 2) - 3 + \int_{\beta} c_1(TP(1, 1, 2)) + n_1 + n_2 - \sum_{i=1}^{n_2} \text{age}(x_i)
\]

where \( x_i, i = 1, \ldots, n_2 \) is the set of orbifold marked points and \( \text{age}(x_i) \) is the sum of the \( t_j \) such that the eigenvalues of the action of the stabilizer of \( x_i \) acting on \( TP(1, 1, 2) \) are \( e^{2\pi i t_j}, j = 1, \ldots, n_2 \), listed with multiplicity. If \( f : C \to P(1, 1, 2) \) is a representable map then any orbifold point of \( C \) must be carried by \( f \) to the unique stacky point of \( C \), which is represented by \((0, 0, 1)\). The automorphisms act with eigenvalues \(-1, -1\) on the fiber of the tangent bundle at this point, so the age is 1.

The Euler sequence here is

\[
0 \to \mathcal{O} \to O(1) \oplus O(1) \oplus O(2) \to TP(1, 1, 2) \to 0
\]

so \( c_1(TP(1, 1, 2)) = 4c_1(O(1)) \). Thus,

\[
v \dim \overline{M}(P(1, 1, 2); n_1, n_2; \beta) = 4d - 1 + n_1
\]

where \( d = \int_{\beta} c_1(O(1)) \).

The inertia stack of \( P(1, 1, 2) \) is \( P(1, 1, 2) \amalg B(\mathbb{Z}/2\mathbb{Z}) \) and the rigidified inertia stack is \( P(1, 1, 2) \amalg \text{(point)} \). Let \( \gamma \) be the fundamental class of the second component. Let \( p \) be the class of an ordinary point in \( P(1, 1, 2) \).

We’ll compute the invariant,

\[
\langle p, \gamma, \ldots, \gamma \rangle_{\frac{1}{2}}
\]

Let’s put \( h = c_1(O(1)) \). Then \( p = 2h^2 \).

A schematic of the orbifold Chow ring of \( P(1, 1, 2) \) with its structure as a graded vector space is shown below.

\[
\begin{array}{c|c|c|c}
\text{P(1,1,2) (point)} & 0 & 1 & 2 \\
\hline
\text{Q} & \text{Qh} & \text{Qp} \\
\text{Q\gamma} & & & \\
\end{array}
\]

It is easy to see that \( \overline{M}(P(1, 1, 2); 1, 2; 0) \cong B(\mathbb{Z}/2\mathbb{Z}) \) and therefore that \( \gamma^2 = \frac{1}{2}p = h^2 \). Therefore a presentation of the orbifold Chow ring is \( \mathbb{Q}[h, \gamma]/(h^2 - \gamma^2) \). Note in particular that this satisfies Poincaré duality.

Lemma 2.1. The Gromov–Witten invariants of \( P(1, 1, 2) \) have the following properties.

(a) If \( \langle \gamma^\alpha \alpha, \alpha \rangle_0 \neq 0 \) then \( n = 2 \) and \( \alpha = 1 \).

(b) The invariant \( \langle \gamma^\alpha h, \ast \rangle_0 \) is zero for all \( n \).

Proof. For (a), the invariant is computed on the moduli space \( \overline{M}(P(1, 1, 2); a, n; 0) \) which has virtual dimension \( a - 1 \). Hence the invariant will be zero unless \( a = 1 \) (so \( \alpha \) comes from the untwisted sector) and \( \alpha = 1 \). But then the invariant will be zero by the unit axiom unless \( n = 2 \).

For (b), it is sufficient by linearity to show that \( \langle \gamma^\alpha, 2h, \ast \rangle_0 = 0 \). But the Chow class \( 2h \) can be represented by a line that doesn’t pass through the unique orbifold point \((0, 0, 1)\). Since this is a degree zero invariant, this means it is computed on an empty moduli space, i.e., it is zero. \( \square \)
The WDVV equations give

$$\sum_{a+b=2g-1 \atop d_1+d_2=\frac{1}{2}} \left\langle h, h, \gamma \otimes a, * \right\rangle _{d_1}, \gamma, \gamma \otimes b \right\rangle _{d_2} = \sum_{a+b=2g-1 \atop d_1+d_2=\frac{1}{2}} \left\langle h, \gamma, \gamma \otimes a, * \right\rangle _{d_1}, h, \gamma, \gamma \otimes b \right\rangle _{d_2},$$

where $d_1$ and $d_2$ can take the values 0 and $\frac{1}{2}$ in the sums.

Consider first the right side of the equality. One of the $d_i$ must be zero, so consider the invariant $\left\langle h, \gamma, \ldots, \gamma, * \right\rangle _0$. This is zero by Lemma 2.13. On the left side, note that if $d_i = 0$ then the corresponding term of the sum will be zero by the divisor axiom unless $a = 0$ also. Thus we get

$$\left\langle h^2, \gamma \otimes (2g+1) \right\rangle _\frac{1}{2} + \frac{1}{4} \sum_{a+b=2g-1} \left\langle \gamma \otimes a, *, \gamma \otimes (b+2) \right\rangle _\frac{1}{4} = 0.$$

But $\left\langle \gamma \otimes n, * \right\rangle _0 = 0$ for $n > 2$ by Lemma 2.1 (a). Thus we are left with

$$\left\langle h^2, \gamma \otimes (2g+1) \right\rangle _\frac{1}{2} = -\frac{1}{4} \left\langle \gamma^2, \gamma \otimes (2g-1) \right\rangle _\frac{1}{2}.$$

Since $h^2 = \gamma^2 = \frac{1}{2}P$ we get

$$\left\langle p, \gamma \otimes (2g+1) \right\rangle _\frac{1}{2} = \left( -\frac{1}{4} \right) ^g \left\langle p, \gamma \right\rangle _\frac{1}{2}$$

by induction. The invariant on the right side of this equality is easily seen to be 1. Indeed, $\overline{M}(P(1,1,2); 1, 1; \frac{1}{2})$ may be identified with $P(\Gamma(P(1,1,2), O(1))) \cong P^1$. The virtual dimension of $\overline{M}(P(1,1,2); 0, 1; \frac{1}{2})$ is also 1, so we only need to solve the enumerative problem to compute $\left\langle p, \gamma \right\rangle _\frac{1}{2}$. If $(u,v) \in P^1$ is a point, then the condition that the corresponding curve interpolates the point $(x,y,z) \in P(1,1,2)$ is $ux + vy = 1$. This has exactly one solution if $(x,y) \neq (0,0)$ so we conclude that $\left\langle P, \gamma \right\rangle _\frac{1}{2} = 1$.

We have therefore proved

$$\left\langle p, \gamma \otimes (2g+1) \right\rangle _\frac{1}{2} = \left( -\frac{1}{4} \right) ^g \tag{1}$$

3 The virtual fundamental class

**Lemma 3.1.** Let $C$ be a smooth orbifold curve. Suppose there is a representable map $f : C \rightarrow P(1,1,2)$ of degree $\frac{1}{2}$. Then $C$ has at most 1 orbifold point.

**Proof.** In this case, $f^* O(1)$ is a line bundle of degree $\frac{1}{2}$ on $C$. The only such line bundles on $C$ are the $O(P)$ where $P$ is an orbifold point of $C$. Suppose $f^* O(1) = O(P)$ for a particular orbifold point $P$ and that $C$ has another orbifold point $Q \neq P$. Let $\sigma$ be any nonzero section of $O(1)$ over $P(1,1,2)$. Then $f^* \sigma$ is a section of $O_C(P)$, hence vanishes only $P$. But this means $f(Q) \neq (0,0,1)$, which contradicts the representability of $f$. $\square$

**Proposition 3.2.** There are isomorphisms

$$\overline{M}(P(1,1,2); 0, 2g+1; \frac{1}{2}) \cong \overline{M}(P(1,1,2); 0, 1; \frac{1}{2}) \times \overline{M}(B(Z/2Z); 2g + 2)$$

**Proof.** If $(f, C) \in \overline{M}(P(1,1,2); 0, 2g+1; \frac{1}{2})$, then $C$ has a unique irreducible component $C_0$ with $\deg f \mid _{C_0} = \frac{1}{2}$; all other components have degree 0. By the lemma, $C_0$ has exactly 1 orbifold point. The remaining orbifold points must lie on a component that is attached at the unique orbifold point of $C$. Thus every point of $\overline{M}(P(1,1,2); 0, 2g+1; \frac{1}{2})$ lies in the image of the gluing map

$$\iota : \overline{M}(P(1,1,2); 0, 1; \frac{1}{2}) \times \overline{M}(B(Z/2Z); 2g + 2) \rightarrow \overline{M}(P(1,1,2); 0, 2g + 1)$$

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that attaches the marked point from the first component to the first marked point from the second component.

This is a closed embedding, so to complete the proof, we must show that the image of this map is in open in \( \mathfrak{M}(B(\mathbb{Z}/2\mathbb{Z}); 2g + 2) \). Consider a first-order deformation \((C', f')\) of \((C, f)\). Let \(C_1\) be the contracted component of \(C\) and let \(C_0\) be the component of positive degree. If \((C', f')\) were not in the image of \(\iota\), then \(C'\) would be a first-order smoothing of \(C\). But then consider the map \(N_{C_1/C'} \to (f_{|C_1})^* T_{\mathbb{P}(0,1)}(1, 1, 2)\). If \(P\) is the point of attachment between \(C_0\) and \(C_1\), then \(N_{C_1/C'}|_P\) is spanned by \(T_P C_0\). Moreover, \(C_0\) meets \((0, 0, 1)\) transversally (since \(f_{|C_0}\) has degree \(\frac{1}{2}\)), which implies that the map \(N_{C_1/C'} \to (f_{|C_1})^* T_{\mathbb{P}(0,1)}(1, 1, 2)\) is nonzero at \(P\).

On the other hand, \(N_{C_1/C} \cong \mathcal{O}_{C_1}(-P)\), and \((f_{|C_1})^* T_{\mathbb{P}(0,1)}(1, 1, 2) \cong (f_{|C_1})^* (\rho_1 \oplus \rho_1)\) because \(f\) contracts \(C_1\) onto the point \((0,0,1)\) and \(T_{\mathbb{P}(0,1)}(1,1,2) \cong \rho_1 \oplus \rho_1\). Thus we obtain a pair of sections of \(\rho_1 \oplus \mathcal{O}_{C_1}(P)\), at least one of which does not vanish at \(P\).

Let \(\pi : C_1 \to \overline{C}_1\) be the coarse moduli space. Then we get a section of \(\pi_* (\rho_1 \oplus \mathcal{O}_{C_1}(P))\) that is not everywhere zero. But \(\pi_* (\rho_1 \oplus \mathcal{O}_{C_1}(P)) = \mathcal{O}_{\overline{C}_1}(-g)\) where \(2g + 2\) is the number of orbifold points on \(C_1\). By stability of \((C, f)\), \(g > 0\), so all sections of \(\pi_* (\rho_1 \oplus \mathcal{O}_{C_1}(P))\) vanish. This contradicts the nonvanishing of the section at \(P\).

Now that we know how the moduli space looks, we must determine the virtual fundamental class. We use the deformation–obstruction sequence,

\[
\text{Def}(C) \to \text{Obs}(f) \to \text{Obs}(C, f) \to \text{Obs}(C) = 0.
\]

We know that \(\text{Obs}(C, f)\) is a vector bundle because \(\overline{\mathfrak{M}(\mathbb{P}(1, 1, 2); 0, 2g + 1; \frac{1}{2})}\) is smooth. The virtual fundamental class is the top Chern class of this vector bundle.

\[
\text{Obs}(f) \text{ is the relative obstruction space for the map } \overline{\mathfrak{M}(\mathbb{P}(1, 1, 2); 0, 2g + 1; \frac{1}{2})} \to \mathfrak{M}(B(\mathbb{Z}/2\mathbb{Z}); 2g + 1).
\]

If \((C, f)\) is a curve in \(\overline{\mathfrak{M}(\mathbb{P}(1, 1, 2); 0, 2g + 1; \frac{1}{2})}\) then we have just seen that \(C\) is the union of two curves, \(C_0\) and \(C_1\), along an orbifold point, with \(\deg(f_{|C_0}) = \frac{1}{2}\) and \(\deg(f_{|C_1}) = 0\). It is clear that any deformation of \(C\) that is trivial near the node will extend to a deformation of \((C, f)\) — indeed, \(C_0\) is rigid and \(C_1\) is contracted by \(f\). Thus, the image of \(\text{Def}(C) \to \text{Obs}(f)\) is the space of deformations of the node. If we name the nodal point \(P\), then the deformations of the node are parameterized by \(\pi_* (T_P C_0 \otimes T_P C_1)\), so we have an exact sequence on \(\overline{\mathfrak{M}(\mathbb{P}(1, 1, 2); 0, 2g + 1; \frac{1}{2})}\).

\[
0 \to \pi_* (T_P C_0 \otimes T_P C_1) \to \text{Obs}(f) \to \text{Obs}(C, f) \to 0.
\]

Explicitly, \(\text{Obs}(f) = \mathcal{R}^1 \pi_* f^* T \mathbb{P}(1, 1, 2)\), where \(f : C \to \mathbb{P}(1, 1, 2)\) is the universal map. Tensoring the normalization sequence for the node \(P\) with \(f^* T \mathbb{P}(1, 1, 2)\) and taking cohomology, we obtain

\[
H^0(T|_P) \to H^1(T) \to H^1(T|_{C_0}) \oplus H^1(T|_{C_1}) \to H^1(T|_P) = 0,
\]

writing \(T = f^* T \mathbb{P}(1, 1, 2)\). Note that \(H^0(T|_P) = 0\) since \(P\) is an orbifold point and \(T|_P \cong \rho_1 \oplus \rho_1\) has no invariant sections.

We can also calculate \(H^1(T|_{C_0}) = 0\) using the Euler sequence, which pulls back to

\[
0 \to \mathcal{O} \to \mathcal{O}(P) \oplus \mathcal{O}(P) \oplus \mathcal{O}(2P) \to T|_{C_0} \to 0
\]

since \(f_{|C_0}\) has degree \(\frac{1}{2}\) and \(f^* \mathcal{O}(1) = \mathcal{O}(P)\). Pushing this sequence forward to the coarse moduli space via \(q : C_0 \to \overline{C}_0\) (note \(q_*\) is exact) gives

\[
0 \to \mathcal{O}_{C_0} \to \mathcal{O}_{\overline{C}_0} \oplus \mathcal{O}_{\overline{C}_0} \oplus \mathcal{O}_{\overline{C}_0}(q(P)) \to \pi_* T \to 0.
\]
Now taking cohomology and noting that \( H^1(\mathcal{O}_{\mathcal{C}_0}) = H^1(\mathcal{O}_{\mathcal{C}_0}(\pi(P))) = H^2(\mathcal{O}_{\mathcal{C}_0}) = 0 \), we deduce that
\[
H^1(T|_{\mathcal{C}_0}) = 0
\]
from the long exact sequence.

It now follows that \( \mathrm{Obs}(f) = H^1(T|_{\mathcal{C}_1}) \). But, as already remarked, \( f|_{\mathcal{C}_1} \) factors through the orbifold point of \( \mathbf{P}(1,1,2) \), so \( T|_{\mathcal{C}_1} \) is the pullback of the tangent bundle at this point, which is \( \rho_1 \oplus \rho_1 \). Thus,
\[
\mathrm{Obs}(f) = R^1 \pi_* (\rho_1 \oplus \rho_1) \cong E^\vee \oplus E^\vee
\]
where \( E \) is the Hodge bundle defined in the introduction.

We therefore have an exact sequence,
\[
0 \to \pi_*(T_{PC_0} \otimes T_{PC_1}) \to E^\vee \oplus E^\vee \to \mathrm{Obs}(C,f) \to 0.
\]

Now, consider the cartesian diagram
\[
\begin{array}{ccc}
e^{-1}(p) & \to & \overline{M}(\mathbf{P}(1,1,2);1,2g+1;\frac{1}{2}) \\
\downarrow & & \downarrow e \\
p & \to & \mathbf{P}(1,1,2).
\end{array}
\]

Under the identification of Proposition 3.2, \( e \) factors through the evaluation map on \( \overline{M}(\mathbf{P}(1,1,2);1,1;\frac{1}{2}) \). Thus, \( e^{-1}(p) \) may be identified with \( \overline{M}(\mathbf{B}(\mathbb{Z}/2\mathbb{Z});2g+2) \). We have \( i^{-1}(T_{PC_0}) \cong \rho_1 \), we get the exact sequence,
\[
0 \to L^\vee_1 \to E^\vee \oplus E^\vee \to i^* \mathrm{Obs}(C,f) \to 0.
\]

Now,
\[
\langle p, \gamma \otimes (2g+1) \rangle \frac{1}{2} = \int i^! \overline{M}(\mathbf{P}(1,1,2);1,2g+1;\frac{1}{2}) \overline{\mathrm{vir}} = \int \overline{M}(\mathbf{B}(\mathbb{Z}/2\mathbb{Z});2g+2) \frac{c(E^\vee)^2}{c(L^\vee_1)}.
\]

But we have also seen in Section 2 that
\[
\langle p, \gamma \otimes (2g+1) \rangle \frac{1}{2} = \left( -\frac{1}{4} \right)^g
\]
and this completes the proof of the theorem.

References

[1] Dan Abramovich, Tom Graber, and Angelo Vistoli. Gromov–Witten theory of Deligne–Mumford stacks. math.AG/0603151, 2006.

[2] Dan Abramovich and Angelo Vistoli. Compactifying the space of stable maps. J. Amer. Math. Soc., 15(1):27–75 (electronic), 2002.

[3] Jonathan Wise. The genus zero Gromov–Witten invariants of \([\text{Sym}^2 P^2]\), 2007.