SEVERAL PROPERTIES OF α-HARMONIC FUNCTIONS IN THE UNIT DISK

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Abstract. The aim of this paper is to obtain the Schwarz-Pick type inequality for α-harmonic functions $f$ in the unit disk and get estimates on the coefficients of $f$. As an application, a Landau type theorem of α-harmonic functions is established.

1. Introduction and main results

Let $\mathbb{C}$ be the complex plane. For $a \in \mathbb{C}$, let $\mathbb{D}(a, r) = \{z : |z - a| < r\}$ ($r > 0$) and $\mathbb{D}(0, r) = \mathbb{D}_r$. Also, we use the notations $\mathbb{D} = \mathbb{D}_1$ and $\mathbb{T} = \partial \mathbb{D}$, the boundary of $\mathbb{D}$.

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2 \times 2}.$$ 

We will consider the matrix norm

$$|A| = \sup \{|Az| : z \in \mathbb{C}, |z| = 1\}$$

and the matrix function

$$l(A) = \inf \{|Az| : z \in \mathbb{C}, |z| = 1\}.$$ 

Let $D$ and $\Omega$ be domains in $\mathbb{C}$, and let $f = u + iv : D \to \Omega$ be a function that has both partial derivatives at $z = x + iy$ in $D$, where $u$ and $v$ are real functions. The Jacobian matrix of $f$ at $z$ is denoted by

$$Df(z) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$ 

Set

$$\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$ 

Then

(1.1) $|Df(z)| = \sup \{|Df(z)\varsigma| : |\varsigma| = 1\} = |f_z(z)| + |f_{\bar{z}}(z)|,$

(1.2) $l(Df(z)) = \inf \{|Df(z)\varsigma| : |\varsigma| = 1\} = |f_z(z)| - |f_{\bar{z}}(z)|$

and

$$|J_f(z)| = |Df(z)| \cdot l(Df(z)).$$
where $J_f(z)$ stands for the Jacobian of $f$ at $z$.

We denote by $\Delta_\alpha$ the weighted Laplace operator corresponding to the so-called standard weight $w_\alpha = (1 - |z|^2)^\alpha$, that is,

$$\Delta_\alpha, z = \frac{\partial}{\partial z}(w_\alpha)^{-1} \frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial z}(1 - |z|^2)^{-\alpha} \frac{\partial}{\partial \bar{z}}$$

in $\mathbb{D}$, where $\alpha > -1$ (see [20, Proposition 1.5] for the reason for this constraint).

In [20], Olofsson and Wittsten introduced this operator $\Delta_\alpha$ and a counterpart of the classical Poisson integral formula was given.

We remark that in the study of Bergman spaces of $\mathbb{D}$, one often considers the weights $w_\alpha$ in $\mathbb{D}$ ($\alpha > -1$). For an account of recent developments in Bergman space theory, we mention the monograph [13] by Hedenmalm, Korenblum and Zhu. The case $\alpha = 0$ is commonly referred to as the unweighted case, whereas the case $\alpha = 1$ has attracted special attention recently with contributions by Hedenmalm, Shimorin and others (see for instance [14, 15, 16, 22] etc).

Of particular interest to us is the following $\alpha$-harmonic equation in $\mathbb{D}$:

$$\Delta_\alpha(f) = 0.$$  

(1.3)

Denote the associated Dirichlet boundary value problem of functions $f$ satisfying the equation (1.3) by

$$\left\{ \begin{array}{l}
\Delta_\alpha(f) = 0 \quad \text{in } \mathbb{D}, \\
 f = f^* \quad \text{on } \mathbb{T}.
\end{array} \right.$$  

(1.4)

Here the boundary data $f^*$ is a distribution on $\mathbb{T}$, i.e. $f^* \in \mathcal{D}'(\mathbb{T})$, and the boundary condition in the equation (1.4) is to be understood as $f_r \to f^* \in \mathcal{D}'(\mathbb{T})$ as $r \to 1^-$, where

$$f_r(e^{i\theta}) = f(re^{i\theta})$$  

(1.5)

for $\theta \in [0, 2\pi]$ and $r \in [0, 1)$.

For simplicity, we introduce the following definition.

**Definition 1.1.** For $\alpha > -1$, a complex-valued function $f$ is said to be $\alpha$-harmonic if $f$ is twice continuously differentiable in $\mathbb{D}$ and satisfies the condition (1.3).

In [20], Olofsson and Wittsten showed that if an $\alpha$-harmonic function $f$ satisfies

$$\lim_{r \to 1^-} f_r = f^* \in \mathcal{D}'(\mathbb{T}) \quad (\alpha > -1),$$

then it has the form of a Poisson type integral

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{P}_\alpha(ze^{-i\theta})f^*(e^{i\theta})d\theta$$  

(1.6)

in $\mathbb{D}$, where

$$\mathcal{P}_\alpha(z) = \frac{(1 - |z|^2)^{\alpha + 1}}{(1 - z)(1 - \bar{z})^{\alpha + 1}}.$$

In the following, we always assume that any $\alpha$-harmonic function has such a representation which plays a key role in the discussions of this paper.
Obviously, $\alpha$-harmonicity coincides with harmonicity when $\alpha = 0$. See [12] and the references therein for the properties of harmonic mappings. Particularly, Colonna proved the following Schwarz-Pick type inequality.

**Theorem A.** ([11, Theorems 3 and 4]) Let $f$ be a harmonic function of $D$ into $D$. Then for $z \in D$,

$$|Df(z)| \leq \frac{4}{\pi} \cdot \frac{1}{1 - |z|^2}.$$ 

This estimate is sharp and all the extremal functions are

$$f(z) = \frac{2\delta}{\pi} \arg \left( \frac{1 + \psi(z)}{1 - \psi(z)} \right),$$

where $\delta \in \mathbb{C}$, $|\delta| = 1$ and $\psi$ is a conformal automorphism of $D$.

For the related discussions on this topic, see [2, 4, 7, 10, 17, 21] etc.

As the first aim of this paper, we shall generalize Theorem A to the case of $\alpha$-harmonic functions. Our first result is as follows.

**Theorem 1.1.** Suppose that $f$ is an $\alpha$-harmonic function in $D$ with $\alpha > -1$, that $f^* \in C(T)$ and that $\sup_{z \in \mathbb{D}} |f(z)| \leq M$, where $M$ is a constant. Then for $z \in D$,

$$|Df(z)| \leq \frac{M(\alpha + 2)}{c_\alpha} \cdot \frac{1}{1 - |z|^2} \leq \frac{2M(\alpha + 2)}{c_\alpha} \cdot \frac{1}{1 - |z|^2},$$

where $c_\alpha = \frac{\Gamma(\frac{\alpha + 2}{2})}{\Gamma(\frac{\alpha + 1}{2})}$ and $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$ ($s > 0$) is the Gamma function.

In particular, if $f$ maps $D$ into $D$, then

$$|Df(z)| \leq \frac{2(\alpha + 2)}{c_\alpha} \cdot \frac{1}{1 - |z|^2}.$$ 

Let $\lambda_D(z)|dz|$ be the hyperbolic metric of the domain $D$ having constant Gaussian curvature $-1$. The hyperbolic distance $d_{h_D}(z_1, z_2)$ between two points $z_1$ and $z_2$ in $D$ is defined by

$$\inf_\gamma \left\{ \int_\gamma \lambda_D(z)|dz| \right\},$$

where the infimum is taken over all rectifiable curves $\gamma$ in $D$ connecting $z_1$ and $z_2$.

We have known that if $D = \mathbb{D}$, then (cf. [1])

$$\lambda_\mathbb{D}(z) = \frac{2}{1 - |z|^2} \quad \text{and} \quad d_{h_\mathbb{D}}(z_1, z_2) = \log \frac{|1 - z_1 \overline{z_2}| + |z_1 - z_2|}{|1 - z_1 \overline{z_2}| - |z_1 - z_2|}.$$ 

As a consequence of Theorem 1.1, we have

**Corollary 1.1.** Under the assumptions of Theorem 1.1, if $f$ maps $D$ into $D$, then for $z_1$ and $z_2 \in D$,

$$|f(z_1) - f(z_2)| \leq \frac{\alpha + 2}{c_\alpha} d_{h_\mathbb{D}}(z_1, z_2).$$
In [20], the authors got the following homogeneous expansion of \( \alpha \)-harmonic functions (see [20, Theorem 1.2]):

A function \( f \) in \( \mathbb{D} \) is \( \alpha \)-harmonic if and only if it has the following convergent power series expansion:

\[
(1.7) \quad f(z) = \sum_{k=0}^{\infty} c_k z^k + \sum_{k=1}^{\infty} c_{-k} P_{\alpha,k}(|z|^2) z^k,
\]

where \( P_{\alpha,k}(x) = \int_0^1 t^{k-1} (1-tx)^{\alpha} \, dt \) \((-1 < x < 1)\) and \( \{c_k\}_{k=-\infty}^{\infty} \) denotes a sequence of complex numbers with \( \lim_{|z| \to \infty} \sup |c_k| |z|^k \leq 1 \).

The second aim of this paper is to prove the following estimates on coefficients \( c_k \) and \( c_{-k} \).

\[\textbf{Theorem 1.2.} \] Suppose that \( f \) is an \( \alpha \)-harmonic function in \( \mathbb{D} \) with \( \alpha > -1 \) and that \( \sup_{z \in \mathbb{D}} |f(z)| \leq M \), where \( M \) is a constant. If \( f \) has the series expansion (1.7), then for \( k \in \{0,1,2,...\} \),

\[
(1.8) \quad |c_k| \leq M,
\]

and for \( k \in \{1,2,...\} \),

\[
(1.9) \quad |c_k| + |c_{-k}| B(k, \alpha+1) \leq 4M,\]

where \( B(p,q) \) denotes the Beta function.

By [20, Definition 2.1], we find that

\[
P_{\alpha}(ze^{-i\theta}) = \sum_{k=0}^{\infty} e^{-ik\theta} z^k + \sum_{k=1}^{\infty} \frac{\Gamma(k+\alpha+1)}{\Gamma(k)\Gamma(\alpha+1)} P_{\alpha,k}(|z|^2) e^{ik\theta} z^k.
\]

If \( |f^*(z)| \leq M \), then by (1.6), we get

\[
|c_{-k}| = \left| \frac{\Gamma(k+\alpha+1)}{\Gamma(k)\Gamma(\alpha+1)} \frac{1}{2\pi} \int_{0}^{2\pi} e^{ik\theta} f^*(e^{i\theta}) d\theta \right| \leq M \frac{\Gamma(k+\alpha+1)}{\Gamma(k)\Gamma(\alpha+1)} \to \infty
\]
as \( k \to \infty \).

Moreover, from the proof of [20, Theorem 1.2], we see that

\[
(1-|z|^2)^{-\alpha} \frac{\partial}{\partial z} f(z) = h(z),
\]

where \( h(z) = \sum_{k=0}^{\infty} a_k z^k \), \( z \in \mathbb{D} \) and \( c_{-k} = \overline{a_{k-1}} \) for \( k \geq 1 \). Note that if \( h(z) \) is a normalized (in the sense that \( h(0) = h'(0) - 1 = 0 \)) univalent analytic function in \( \mathbb{D} \), then by Louis de Branges’s theorem it is well-known that \( |a_k| \leq k \) for all \( k \geq 2 \) so that

\[
(1.10) \quad c_{-1} = 0, \quad c_{-2} = 1 \quad \text{and} \quad |c_{-k}| = |a_{k-1}| \leq k-1 \quad \text{for all} \quad k \geq 3.
\]

The classical Landau theorem says that there is a \( \rho = \frac{1}{M+\sqrt{M^2-1}} \) such that every function \( f \), analytic in \( \mathbb{D} \) with \( f(0) = f'(0) - 1 = 0 \) and \( |f(z)| < M \), is univalent in the disk \( \mathbb{D}_\rho \). Moreover, the range \( f(\mathbb{D}_\rho) \) contains a disk of radius \( M\rho^2 \), where \( M \geq 1 \) is a constant (see [18]). Recently, many authors considered Landau type theorem for \( \alpha \)-harmonic functions \( f \) when \( \alpha = 0 \) (see [3, 5, 6, 7, 8, 9] etc).
As an application of Theorems 1.1 and 1.2, we get the following Landau type theorem for $\alpha$-harmonic functions.

**Theorem 1.3.** Suppose that $f$ is an $\alpha$-harmonic function in $D$ with $\alpha \geq 0$, that $f^* \in C(T)$, that $\sup_{z \in D} |f(z)| \leq M$, where $M$ is a constant, and that $f(0) = |J_f(0)| - \beta = 0$. If $f$ satisfies (1.10), then we have the following:

1. $f$ is univalent in $D_{\rho_0}$, where $\rho_0$ satisfies the following equation
   \[
   \frac{\beta c_\alpha}{M(\alpha + 2)} - (M + 5) \frac{\rho_0(2 - \rho_0)}{(1 - \rho_0)^2} = 0;
   \]
2. $f(D_{\rho_0})$ contains a univalent disk $D_{R_0}$ with
   \[
   R_0 \geq (M + 5) \left( \frac{\rho_0}{1 - \rho_0} \right)^2.
   \]

The arrangement of the rest of this paper is as follows. In Section 2, we shall prove Theorem 1.1 and Corollary 1.1. Section 3 will be devoted to the proof of Theorem 1.2. In Section 4, Theorem 1.3 will be demonstrated.

### 2. Schwarz-Pick Type Inequality

The aim of this section is to prove Theorem 1.1 and Corollary 1.1. The proofs need a result from [19]. Before the statement of this result, we do some preparation.

In [19], the author considered the following integral means:

\[
\mathcal{M}_\alpha(r) = \frac{1}{2\pi} \int_0^{2\pi} K_\alpha(re^{i\theta})d\theta,
\]

where $r \in [0, 1)$ and

\[ K_\alpha(z) = c_\alpha |P_\alpha(z)| = c_\alpha \frac{(1 - |z|^2)^{\alpha+1}}{|1 - z|^{\alpha+2}} \]

in $D$.

Let us recall the following result from [19].

**Theorem B.** [19, Theorem 3.1] Let $\alpha > -1$. The integral means function $\mathcal{M}_\alpha(r)$ given by (2.1) satisfies the following assertions.

1. $\lim_{r \to 1-} \mathcal{M}_\alpha(r) = 1$;
2. $\mathcal{M}_\alpha^{(n)}(r) \geq 0$ for $r \in [0, 1)$ and $n \geq 0$.

The following result also plays a key role in the proof of Theorem 1.1.

**Lemma 2.1.** If $\alpha > -1$ and $f^* \in C(T)$, then

\[
\frac{\partial}{\partial z} \int_0^{2\pi} P_\alpha(ze^{-i\theta})f^*(e^{i\theta})d\theta = \int_0^{2\pi} \frac{\partial}{\partial z} P_\alpha(ze^{-i\theta})f^*(e^{i\theta})d\theta
\]

and

\[
\frac{\partial}{\partial \bar{z}} \int_0^{2\pi} P_\alpha(ze^{-i\theta})f^*(e^{i\theta})d\theta = \int_0^{2\pi} \frac{\partial}{\partial \bar{z}} P_\alpha(ze^{-i\theta})f^*(e^{i\theta})d\theta.
\]
Proof. By elementary calculations we see that the following equalities hold:

\begin{equation}
(2.2) \quad \frac{\partial}{\partial z} P_a(ze^{-i\theta}) = \frac{(1 - |z|^2)^\alpha}{(1 - ze^{-i\theta})^{\alpha+1}} \left[ e^{-i\theta}(1 - |z|^2) - (\alpha + 1)\bar{z}(1 - ze^{-i\theta}) \right] /
\end{equation}

and

\begin{equation}
(2.3) \quad \frac{\partial}{\partial \bar{z}} P_a(ze^{-i\theta}) = \frac{(\alpha + 1)(1 - |z|^2)^\alpha e^{i\theta}}{(1 - ze^{-i\theta})^{\alpha+2}}.
\end{equation}

Then we know that functions

\[ \frac{\partial}{\partial z} P_a(ze^{-i\theta}) f^*(e^{i\theta}) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} P_a(ze^{-i\theta}) f^*(e^{i\theta}) \]

are continuous on $\mathbb{D}_r \times [0, 2\pi]$, where $r \in [0, 1)$.

Let $z = \rho e^{i\varphi} \in \mathbb{D}_r$. It follows from

\[ \frac{\partial}{\partial \rho} P_a(ze^{-i\theta}) = \frac{\partial}{\partial z} P_a(ze^{-i\theta}) e^{i\varphi} + \frac{\partial}{\partial \bar{z}} P_a(ze^{-i\theta}) e^{-i\varphi} \]

and

\[ \frac{\partial}{\partial \varphi} P_a(ze^{-i\theta}) = \frac{\partial}{\partial z} P_a(ze^{-i\theta}) iz - \frac{\partial}{\partial \bar{z}} P_a(ze^{-i\theta}) i\bar{z} \]

that both

\[ \frac{\partial}{\partial \rho} P_a(ze^{-i\theta}) f(e^{i\theta}) \quad \text{and} \quad \frac{\partial}{\partial \varphi} P_a(ze^{-i\theta}) f(e^{i\theta}) \]

are continuous in $\mathbb{D}_r \times [0, 2\pi]$. Hence

\[ \int_0^\rho \int_0^{2\pi} \frac{\partial}{\partial \rho} P_a(ze^{-i\theta}) f^*(e^{i\theta}) d\theta d\rho = \int_0^{2\pi} \int_0^\rho \frac{\partial}{\partial \rho} P_a(ze^{-i\theta}) f^*(e^{i\theta}) d\rho d\theta = \int_0^{2\pi} \left( P_a(ze^{-i\theta}) - P_a(0) \right) f^*(e^{i\theta}) d\theta \]

and

\[ \int_0^{2\pi} \int_0^\rho \frac{\partial}{\partial \varphi} P_a(ze^{-i\theta}) f^*(e^{i\theta}) d\theta d\varphi = \int_0^{2\pi} \int_0^\rho \frac{\partial}{\partial \varphi} P_a(ze^{-i\theta}) f^*(e^{i\theta}) d\varphi d\theta = \int_0^{2\pi} \left( P_a(ze^{-i\theta}) - P_a(\rho e^{-i\theta}) \right) f^*(e^{i\theta}) d\theta. \]

By differentiating with respect to $\rho$ and $\varphi$, respectively, we get

\begin{equation}
(2.4) \quad \int_0^{2\pi} \frac{\partial}{\partial \rho} P_a(ze^{-i\theta}) f^*(e^{i\theta}) d\theta = \frac{\partial}{\partial \rho} \int_0^{2\pi} P_a(ze^{-i\theta}) f^*(e^{i\theta}) d\theta
\end{equation}

and

\begin{equation}
(2.5) \quad \int_0^{2\pi} \frac{\partial}{\partial \varphi} P_a(ze^{-i\theta}) f^*(e^{i\theta}) d\theta = \frac{\partial}{\partial \varphi} \int_0^{2\pi} P_a(ze^{-i\theta}) f^*(e^{i\theta}) d\theta.
\end{equation}

Since

\[ \frac{\partial}{\partial z} P_a(ze^{-i\theta}) = e^{-i\varphi} \left( \frac{\partial}{\partial \rho} P_a(ze^{-i\theta}) - \frac{i}{\rho} \frac{\partial}{\partial \varphi} P_a(ze^{-i\theta}) \right) \]
Several properties of $\alpha$-harmonic functions in the unit disk

and

\[ \frac{\partial}{\partial z} P_\alpha(ze^{-i\theta}) = \frac{e^{i\theta}}{2} \left( \frac{\partial}{\partial \rho} P_\alpha(ze^{-i\theta}) + \frac{i}{\rho} \frac{\partial}{\partial \varphi} P_\alpha(ze^{-i\theta}) \right), \]

it follows from (2.4) and (2.5) that the proof of the lemma is complete. \qed

Now, we are ready to present the proofs of Theorem 1.1 and Corollary 1.1.

**Proof of Theorem 1.1** From (2.2) and (2.3), we can easily get

\[ \left| \frac{\partial}{\partial z} P_\alpha(ze^{-i\theta}) \right| \leq \frac{1}{c_\alpha} \cdot \frac{(\alpha + 2)|z| + 1}{1 - |z|^2} K_\alpha(ze^{-i\theta}) \]

and

\[ \left| \frac{\partial}{\partial z} P_\alpha(ze^{-i\theta}) \right| = \frac{\alpha + 1}{c_\alpha} \cdot \frac{1}{1 - |z|^2} K_\alpha(ze^{-i\theta}). \]

In the first inequality above, the fact “\(1 - |z| \leq |1 - ze^{-i\theta}|\)” is applied. By (1.1), (1.6) and Lemma 2.1 yield

\[ |Df(z)| = \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial z} P_\alpha(ze^{-i\theta}) f^*(e^{i\theta}) d\theta \right| + \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial z} P_\alpha(ze^{-i\theta}) f^*(e^{i\theta}) d\theta \right|, \]

we see from (2.1) and Theorem B that

\[ |Df(z)| \leq \frac{M(\alpha + 2)}{c_\alpha} \cdot \frac{1}{1 - |z|^2} M_\alpha(|z|) \leq \frac{M(\alpha + 2)}{c_\alpha} \cdot \frac{1}{1 - |z|}, \]

and so the proof of Theorem 1.1 is complete. \qed

**Proof of Corollary 1.1** For any \(z_1\) and \(z_2\) \(\in \mathbb{D}\), let \(\gamma\) be the hyperbolic geodesic connecting \(z_1\) and \(z_2\). It follows from Theorem 1.1 that

\[ |f(z_1) - f(z_2)| \leq \int_{\gamma} |Df(z)| \cdot |dz| \leq \frac{\alpha + 2}{c_\alpha} \int_{\gamma} \frac{2}{1 - |z|^2} |dz| = \frac{\alpha + 2}{c_\alpha} d_{h_\alpha}(z_1, z_2), \]

as required. \qed

### 3. Estimates on coefficients

The aim of this paper is to prove Theorem 1.2. We start with a lemma.

**Lemma 3.1.** Under the assumptions of Theorem 1.2, if \(f\) has the series expansion (1.7), then

1. \(|c_k| \leq M\) for \(k \geq 0\);
2. \((|c_k| + |c_{-k}| P_{\alpha,k}(r^2)) r^k \leq \frac{4}{\pi} M\) for \(k > 0\) and \(r \in (0, 1)\).

**Proof.** If \(k \neq 0\), let \(z = re^{i\theta} \in \mathbb{D}\). Then by (1.7), we have

\[ c_k r^k = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) e^{-ik\theta} d\theta \text{ and } c_{-k} P_{\alpha,k}(r^2) r^k = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) e^{ik\theta} d\theta. \]
Letting $c_k = |c_k| e^{i\mu_k}$ and $c_{-k} = |c_{-k}| e^{i\nu_k}$ leads to
\[
(|c_k| + |c_{-k}| P_{\alpha,k}(r^2)) r^k = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) (e^{-i(k\theta + \mu_k)} + e^{i(k\theta - \nu_k)}) d\theta
\]
\[
\leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})| \cdot |e^{-i(k\theta + \mu_k)} + e^{i(k\theta - \nu_k)}| d\theta
\]
\[
\leq \frac{M}{\pi} \int_0^{2\pi} \left| \cos \left( k\theta + \frac{\mu_k - \nu_k}{2} \right) \right| d\theta,
\]
and so [4, Lemma 1] gives
\[
(|c_k| + |c_{-k}| P_{\alpha,k}(r^2)) r^k \leq \frac{4M}{\pi}.
\]
Thus the assertion (2) in the lemma is true.

To prove the assertions (1), we first recall from [20, Definition 2.1] that
\[
P_{\alpha}(z e^{-i\theta}) = \sum_{k=0}^{\infty} e^{-i k\theta} z^k + \sum_{k=1}^{\infty} \frac{\Gamma(k + \alpha + 1)}{\Gamma(k) \Gamma(\alpha + 1)} P_{\alpha,k}(|z|^2) e^{i k\theta} z^k.
\]
Then by (1.6), we get
\[
f(z) = \sum_{k=0}^{\infty} z^k \frac{1}{2\pi} \int_0^{2\pi} e^{-i k\theta} f^*(e^{i\theta}) d\theta
\]
\[
+ \sum_{k=1}^{\infty} \frac{\Gamma(k + \alpha + 1)}{\Gamma(k) \Gamma(\alpha + 1)} P_{\alpha,k}(|z|^2) z^k \frac{1}{2\pi} \int_0^{2\pi} e^{i k\theta} f^*(e^{i\theta}) d\theta,
\]
which implies
\[
|c_k| = \left| \frac{1}{2\pi} \int_0^{2\pi} e^{-i k\theta} f^*(e^{i\theta}) d\theta \right| \leq M,
\]
as required. \qed

**Proof of Theorem 1.2** To prove this theorem, by Lemma 3.1, we only need to check (1.9) in the theorem. By letting $r \to 1^-$ in Lemma 3.1(2), we see that the inequalities (1.9) easily follows. \qed

## 4. Landau Type Theorem

This section consists of two subsections. In the first subsection, we shall prove an auxiliary result. In the second subsection, Theorem 1.3 will be checked.

### 4.1. A lemma.

**Lemma 4.1.** For constants $\alpha > -2$, $\beta > 0$ and $M > 0$, let
\[
\varphi(x) = \frac{\beta c_0}{M(\alpha + 2)} + (M + 5) \frac{x(x - 2)}{(1 - x)^2}
\]
in $[0,1)$. Then

1. $\varphi$ is continuous in $[0,1)$ and strictly decreasing in $(0,1)$;
2. there is a unique $x_0 \in (0,1)$ such that $\varphi(x_0) = 0$. 


Proof. For \( x \in [0, 1) \), obviously,
\[
\varphi'(x) = -\frac{2(M + 5)}{(1 - x)^3} < 0.
\]
Hence \( \varphi(x) \) is continuous and strictly decreasing in \([0, 1)\). It follows from
\[
\varphi(0) = \frac{\beta c_\alpha}{M(\alpha + 2)} > 0 \quad \text{and} \quad \lim_{x \to 1^-} \varphi(x) = -\infty < 0
\]
that there is a unique \( x_0 \in (0, 1) \) such that \( \varphi(x_0) = 0 \). The proof of this lemma is complete. \( \square \)

4.2. Proof of Theorem 1.3. To prove this theorem, we need estimates on two quantities \(|f_z(z) - f_z(0)| + |f_{\bar{z}}(z) - f_{\bar{z}}(0)|\) and \(l(Df(0))\). First, we estimate \(|f_z(z) - f_z(0)| + |f_{\bar{z}}(z) - f_{\bar{z}}(0)|\). Obviously, by (1.7), we see that
\[
f_z(z) - f_z(0) = \sum_{k=2}^{\infty} kc_k z^{k-1} + \sum_{k=2}^{\infty} c_{-k} \frac{d}{dw} P_{\alpha,k}(w) z^{k+1}
\]
and
\[
f_{\bar{z}}(z) - f_{\bar{z}}(0) = \sum_{k=2}^{\infty} k c_{-k} P_{\alpha,k}(w) z^{k-1} + \sum_{k=2}^{\infty} c_{-k} \frac{d}{dw} P_{\alpha,k}(w) z^{k+1},
\]
where \( w = |z|^2 \).
Since
\[
\frac{d}{dw} P_{\alpha,k}(w) = -\int_0^1 t^k \alpha(1 - tw)^{\alpha-1} dt \leq 0,
\]
we get that
\[
(4.1) \quad P_{\alpha,k}(w) \leq P_{\alpha,k}(0) = \frac{1}{k}.
\]
Moreover, since
\[
P_{\alpha,k}(w) = \frac{1}{w^k} \int_0^w x^{k-1}(1 - x)^\alpha dx,
\]
we easily get
\[
(4.2) \quad \frac{d}{dw} P_{\alpha,k}(w) = -\frac{k}{w} P_{\alpha,k}(w) + \frac{(1 - w)^\alpha}{w}.
\]
Then (1.9), (1.10), (4.1) and (4.2) guarantee that

\[
|f(z) - f(0)| + |\bar{f}(z) - \bar{f}(0)| \leq \sum_{k=2}^{\infty} k(|c_k| + |c_{-k}| |P_{\alpha,k}(w)|) |z|^{k-1} \\
+ 2 \sum_{k=2}^{\infty} |c_{-k}| (kP_{\alpha,k}(w) + 1) |z|^{k-1} \\
\leq (M + 5) \sum_{k=2}^{\infty} k|z|^{k-1} \\
= (M + 5) \frac{|z|(2 - |z|)}{(1 - |z|)^2},
\]

which is what we want.

Next, we estimate \( l(Df(0)) \). Applying Theorem 1.1 leads to

\[
\beta = |J_f(0)| = |Df(0)| l(Df(0)) \leq \frac{M(\alpha + 2)}{c_\alpha} l(Df(0)),
\]

which gives

\[
(4.4) \quad l(Df(0)) \geq \frac{\beta c_\alpha}{M(\alpha + 2)}.
\]

Now, we are ready to finish the proof of the theorem. First, we demonstrate the univalence of \( f \) in \( \mathbb{D}_{\rho_0} \), where \( \rho_0 \) is determined by the equation (1.11). For this, let \( z_1, z_2 \) be two points in \( \mathbb{D}_{\rho_0} \) with \( z_1 \neq z_2 \), and denote the segment from \( z_1 \) to \( z_2 \) with the endpoints \( z_1 \) and \( z_2 \) by \([z_1, z_2]\). Since

\[
|f(z_2) - f(z_1)| = \left| \int_{[z_1, z_2]} f_z(z) dz + f(z) d\bar{z} \right| \\
\geq \left| \int_{[z_1, z_2]} f_z(z) dz + f(z) d\bar{z} \right| \\
- \left| \int_{[z_1, z_2]} [f_z(z) - f_z(0)] dz + [f(z) - f(0)] d\bar{z} \right|,
\]

we see from (4.3), (4.4) and Lemma 4.1 that

\[
|f(z_2) - f(z_1)| \geq l(Df(0)) \cdot |z_2 - z_1| - (M + 5) \int_{0}^{\rho_0-\rho_0} \frac{|z|(2 - |z|)}{(1 - |z|)^2} |dz| \\
> \left[ \frac{\beta c_\alpha}{M(\alpha + 2)} - (M + 5) \frac{\rho_0(2 - \rho_0)}{(1 - \rho_0)^2} \right] |z_2 - z_1| \\
= 0.
\]

Thus, for arbitrary \( z_1 \) and \( z_2 \in \mathbb{D}_{\rho_0} \) with \( z_1 \neq z_2 \), we have

\[
f(z_1) \neq f(z_2),
\]

which implies the univalence of \( f \) in \( \mathbb{D}_{\rho_0} \).

Next, we prove Theorem 1.3(2). For any \( \zeta = \rho_0 e^{i\theta} \in \partial\mathbb{D}_{\rho_0} \), we obtain that
Several properties of $\alpha$-harmonic functions in the unit disk

$$|f(\zeta) - f(0)| = \left| \int_{[0, \zeta]} f_z(z)dz + f_{\overline{z}}(z)d\overline{z} \right|$$

$$\geq \left| \int_{[0, \zeta]} f_z(0)dz + f_{\overline{z}}(0)d\overline{z} \right|$$

$$- \left| \int_{[0, \zeta]} [f_z(z) - f_z(0)]dz + [f_{\overline{z}}(z) - f_{\overline{z}}(0)]d\overline{z} \right|$$

$$\geq l(Df(0))\rho_0 - (M + 5) \int_0^{\rho_0} \frac{|z|(2 - |z|)}{(1 - |z|)^2} |dz| \quad \text{(by (4.3))}$$

$$= \frac{\beta c_\alpha \rho_0}{M(\alpha + 2)} - (M + 5) \frac{\rho_0}{1 - \rho_0}.$$  

$$= (M + 5) \frac{\rho_0}{1 - \rho_0}.$$  

(by (1.11))

Hence $f(\mathbb{D}_{\rho_0})$ contains a univalent disk $\mathbb{D}_{R_0}$, where

$$R_0 \geq (M + 5) \frac{\rho_0}{1 - \rho_0}.$$  

The proof of this theorem is complete.  

\[ \square \]

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