Local Realistic Theories and Quantum Mechanics for the two–neutral–kaon system

R. H. Dalitz\textsuperscript{a} and G. Garbarino\textsuperscript{a,\textsuperscript{b}}
\textsuperscript{a}Department of Theoretical Physics, University of Oxford
1 Keble Rd, Oxford OX1 3NP, UK
\textsuperscript{b}Grup de Física Teòrica, Universitat Autònoma de Barcelona
08193 Bellaterra, Barcelona, Spain
(November 13, 2018)

Abstract

The predictions of local realistic theories for the observables concerning the evolution of a $K^0\bar{K}^0$ quantum entangled pair (created in the decay of the $\phi$–meson) are discussed. It is shown, in agreement with Bell’s theorem, that the most general local hidden–variable model fails in reproducing the whole set of quantum–mechanical joint probabilities. We achieve these conclusion by employing two different approaches. In a first one the local realistic observables are deduced from the most general premises concerning locality and realism, and Bell–like inequalities are not employed. The other approach makes use of Bell’s inequalities. Within the former scheme, under particular conditions for the detection times, the discrepancy between quantum mechanics and local realism for the time–dependent asymmetry turns out to be not less than 20%. The same incompatibility can be made evident by means of a Bell–type test by employing both Wigner’s and (once properly normalized probabilities are used) Clauser–Holt–Shimony–Holt’s inequalities. Because of the relatively low experimental accuracy, the data obtained by the CPLEAR collaboration for the asymmetry parameter do not allow for a decisive test of local realism. Such a test, both with and without the use of Bell’s inequalities, should be feasible in the future at the Frascati $\Phi$–factory.

3.65.Bz
I. INTRODUCTION

In 1935 Einstein Podolsky and Rosen (EPR in the following) advanced a strong criticism concerning the interpretation of quantum theory. They arrived at the conclusion that the description of physical reality given by the quantum wave function is not complete. EPR’s argumentation was based on a condition for a complete theory (every element of physical reality must have a counterpart in the physical theory) and on a criterion which defines physical reality (if, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this quantity). They also assumed the quantum world to be local: this requirement was introduced in order to express relativistic causality, which prevents any action–at–a–distance. Starting from these premises, by considering the behaviour of a correlated and non–interacting system composed by two separated entities, EPR arrived at the following conclusion: contrary to what the indetermination principle states, two non–commuting observables can have simultaneous physical reality, then the description of physical reality given by Copenhagen’s interpretation, which does not permit such a simultaneous reality, is incomplete. At the very heart of their logical conclusion is the following fact: their assumption, according with a quantum system has real and well defined properties also when does not interact with other systems (including a measuring apparatus), is contradicted by quantum mechanics.

This was the point attached by Bohr in his famous replay to EPR’s paper. Here he noticed that EPR’s criterion of reality contained an ambiguity if applied to quantum phenomena. Starting from the complementarity point of view, Bohr stated that *quantum mechanics within its scope* [namely, in its form restricted to human knowledge] *would appear as a completely rational description of the physical phenomena*. In the opinion of Bohr the conclusion of EPR was not justified since they contradicted quantum theory at the beginning, through their criterion of physical reality: following Copenhagen’s interpretation, quantum reality has to be defined by the experimental observation of phenomena.

The probabilistic meaning of the quantum wave function is the main assumption that originated criticisms and debate for a broader interpretation of quantum theory. In fact, the wave function provides a description of the microscopic world in accordance with the laws of chance, namely it is non–deterministic: the actual result of a measurement is selected from the set of possible outcomes at random. It is this interpretation of the quantum state that led Einstein to pronounce the historical sentence: *God does not play dice.*

Within Copenhagen’s interpretation, the measurement process changes the state of the measured system through the reduction of the wave packet. The description of this (non–deterministic and non–local) process given by the hermitian operator associated to the observable one measures is mathematically different from the (deterministic) evolution of the statistical predictions of the wave function, which is accounted for by the Schrödinger equation and its unitary time evolution operator. This matter of fact is also the origin of different paradoxical conclusions of quantum mechanics. It is important to stress that the collapse of
the wave function is a non–local aspect of quantum mechanics. It arises from the fact that the theory does not provide a causal explanation of the anti–correlations which exist between the probabilities of finding a system (say a particle) in two separated regions of space. The EPR–type correlations of two–particle entangled states clearly exhibits a non–locality. To avoid this feature, interpretations of quantum mechanics which do not incorporate the reduction of the wave packet have been introduced (see for instance Bohmian mechanics [3] and Everett’s many–world interpretation [4]). However, we have to stress that the non–local features exhibited by EPR’s states do not contradict the theory of relativity, since they do not allow for faster–than–light communications [5].

Another puzzling question concerns the subdivision of the physical world into quantum system and classical apparatus, the latter being directly controllable and needed to define (through the measurement process) the properties of quantum phenomena. Actually, strictly speaking, real physical properties are possessed only by the combined system of quantum object plus measuring device. This dualistic approach, which leaves the measuring devices out of the world treated by the mathematical formalism of the theory, leads to a description of the physical universe which is not unified, namely to a theoretical framework which is not fully coherent.

The first hypothesis for the solution of the paradoxical conclusion of EPR concerning quantum correlations was proposed by Furry [6] in 1935. He assumed that the quantum–mechanical description of many–body systems could break–down when the particles are sufficiently distant one from another (practically when their wave functions do not overlap any more). This means that in presence of EPR correlations between two quantum subsystems which are very far away one from each other, the state of the global system is no longer given by a superposition of tensorial products of states but it is simply represented by a statistical mixture of products of states (namely it is factorizable). However, Furry’s hypothesis revealed to be incorrect: an old experiment concerning polarization properties of correlated photons [7,9], as well as more recent tests [10–12], excluded a possible separability of the many–body wave function even in the case of space–like separated particles.

In 1952 Bohm [3] suggested an interpretation of quantum theory in terms of hidden–variables, in which the general mathematical formulation and the empirical results of the theory remained unchanged. In Bohm’s interpretation the paradoxical behaviour of correlated and non–interacting systems revealed by EPR find an explanation. However, for such systems Bohm’s theory exhibits a non–local character, which cannot be reconciled with relativity theory.

This result is consistent with what Bell obtained in 1964 [13]. He proved that any deterministic local hidden–variable theory is incompatible with some statistical prediction of quantum mechanics. This is the content of Bell’s theorem in its original form, which has been then generalized [14] to include non–deterministic theories. EPR’s paradox was interpreted as the need for the introduction of additional variables, in order to restore completeness, relativistic causality (namely locality) and realism in the theory (the point of view of realism
asserts that quantum systems have intrinsic and well defined properties even when they are not subject to measurements). In line with this requirement, Bell and other authors \[15–18\] derived different inequalities suitable for testing what has been called local realism.

Once established the particularity of Bell’s local realism in connection with the predictions of quantum mechanics, different experiments have been designed and carried out to test these theories. The oldest ones \[18,19\] measured the linear polarization correlations of photon pairs created in radiative atomic cascade reactions or in electron–positron annihilations, whereas, more recently, parametric down–conversion photon sources have been employed \[11,12,20\]. Essentially all the experiments performed until now (in optics and atomic physics) have proved that the class of theories governed by Bell’s theorem are unphysical: they showed the violation of Bell’s inequalities and were in good agreement with the statistical predictions of quantum mechanics. Actually, to be precise, because of apparata non–idealities and other technical problems, supplementary assumptions are needed in the interpretation of the experiments, and, consequently, no test employed to refute local realism has been completely loophole free \[12,18,21\]. It is then important to continue performing experiments on correlation properties of many particle systems, possibly in new sectors, especially in particle physics, where entangled $K^0\bar{K}^0$ and $B^0\bar{B}^0$ pairs are considerable examples. If future investigations will confirm the violation of Bell’s inequalities, it is clear that, under the philosophy of realism, the locality assumption would be incompatible with experimental evidence. Then, if this were the case, maintaining realism one should consider as a real fact of Nature a non–local behaviour of quantum phenomena. This fact is not in conflict with the theory of relativity. Actually, there is no way to use quantum non–locality for faster–than–light communication: for a correlated system of two separated entities, according to quantum mechanics, the result of a measurement on a subsystem is always independent of the experimental setting used to measure the other subsystem.

In this paper we discuss the predictions of local realistic schemes for a pair of correlated neutral kaons created in the decay of the $\phi$–meson. The two–neutral–kaon system is the most interesting example of massive two–particle system that can be employed to discuss descriptions of microscopic phenomena alternative to quantum mechanics (for a discussion concerning possible violations of quantum mechanics in the $K^0\bar{K}^0$ system see ref. \[22\]). Unlike photons, kaons are detectable with high efficiency (by observing $K_S$ and $K_L$ decays or $K^0$ and $\bar{K}^0$ strong interactions with the nucleons of absorbers). Moreover, for $K^0\bar{K}^0$ pairs, which can be copiously produced at a high luminosity $\Phi$–factory, additional assumptions regarding detection not implicit in local realism (always implemented in the interpretation of experiments with photon pairs \[13\]) are not necessary to derive Bell’s inequalities suitable for experimental tests of local realism \[23\]. Finally, the two–kaon system offers the possibility for tests on unexplored time and energy scales. A correlation experiment discriminating between local realism and quantum mechanics could be performed at the Frascati $\Phi$–factory in the future \[24\]. Indeed, being designed to measure direct $CP$ violation in the $K^0\bar{K}^0$ system, such a factory employs high precision detectors. Unlike the other papers in the literature
which treated the two–kaon correlated system within local realistic models, we shall discuss tests of local realism both with and without the use of Bell’s inequalities.

The work is organized as follows. In section II we introduce, starting from the original EPR’s program, the point of view of local realism for the two–kaon system. The quantum–mechanical expectation values relevant for the evolution of the system are briefly summarized in section III. Section IV is devoted to the presentation of the local realistic scheme we use to describe the observable behaviour of the pair: the philosophy of realism is implemented in our discussion by means of the most general hidden–variable interpretation of the two–kaon evolution. Then, in section V we study the compatibility among the local realistic expectation values and the statistical predictions of quantum mechanics. In agreement with Bell’s theorem, we show how any local hidden–variable theory for the two–kaon entangled state is incompatible with certain predictions of quantum mechanics. In section VI the difficulties of testing local realism for the $K^0 - \bar{K}^0$ state by employing Bell–type inequalities are discussed. We show that, contrary to what is generally believed in the literature, a Bell–type test at a Φ–factory is possible. Our conclusion are given in section VII.

II. FROM EPR’S ARGUMENT TO LOCAL REALISM

The starting point of EPR’s argumentation was the following condition for a complete theory: every element of physical reality must have a counterpart in the physical theory. They defined the physical reality by means of the following sufficient criterion: if, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this quantity. In addition, for a system made of two correlated, spatially separated and non–interacting entities, EPR introduced the following locality assumption: since at the time of measurement the two systems no longer interact, no real change can take place in the second system in consequence of anything that may be done to the first system.

EPR assumed that the physical world is analyzable in terms of distinct and separately existing elements of reality, which are represented, in the supposed complete theory, by well defined mathematical entities. The previous criterion of reality supports the anthropocentric point of view nowadays called realism: it asserts that quantum systems have intrinsic and well defined properties even when they are not subject to measurements. Under this philosophy, the existence of quantum world is (as in classical physics) objective: thus, any measurement performed on a quantum system must produce a result with a definite and predetermined value.

To exemplify EPR’s argumentation, consider the case of a particle with total angular momentum zero which decays, at rest, into two spin $1/2$ particles, 1 and 2, with zero relative orbital angular momentum, which fly apart with opposite momenta. After a certain time (when the particles are separated by a macroscopic distance) suppose they do not interact
any more (this situation corresponds to the EPR–Bohm’s gedanken experiment [8,9]). At this time, the normalized spin wave function of the global system, which does not depend (because of the spherical symmetry of the singlet state) on the quantization direction of the spin, is:

$$|S = 0, S_z = 0\rangle = \frac{1}{\sqrt{2}} [(|+\rangle_1 |−\rangle_2 - |−\rangle_1 |+\rangle_2)$$.

(2.1)

For particles 1 and 2, |+⟩ and |−⟩ represent spin–up and spin–down states, respectively, along a direction chosen as z–axis. Because of the entangled nature of this wave function, the two particles do not have definite values of the spin component along any direction. The superposition of two product states (2.1) produces then non–factorizable joint probabilities. The paradoxical behaviour of correlated and non–interacting systems originates from the fact that the wave function of the global system is not a tensorial product of superpositions of states of the component systems.

When 1 and 2 do not interact any more, a measurement of the spin component of one particle produces a given outcome [which is not predetermined by the quantum state (2.1)] and forces, immediately, the spin of the other particle along the opposite direction; notice that this is independent of whether or not any measurement is then performed on the other particle. For instance, if the result of a measurement along the z–axis finds particle 1 in the spin–up state, we conclude that at the same time particle 2 (which is supposed not to interact with particle 1 nor with the measuring device) has spin–down along z; the wave packet reduction has led to the disentanglement of the superposition (2.1):

$$|S = 0, S_z = 0\rangle \rightarrow |+\rangle_1 |−\rangle_2,$$

(2.2)

and the total angular momentum of the pair is indefinite after the measurement. The instantaneous response (due to the collapse of the wave function) of the particle which is not observed is what Einstein called spooky action–at–a–distance.

The two particles of Bohm’s gedanken experiment are perfectly correlated, and, following EPR, the spin component of particle 2 is an element of physical reality, since it is predicted with certainty and without in any way disturbing particle 2. Moreover, in order to fulfil the locality assumption (no action–at–a–distance), EPR assumed that such an element of reality existed independently of any measurement performed on particle 1. Following EPR’s argumentation, the interpretation of the above experiment by means of quantum mechanics lead to a difficulty. In fact, if we had performed a measurement of the spin component of particle 1 along another direction, say along the x–axis, this would have defined the x component of the spin of particle 2 as another element of reality, again independent of measurement. Obviously, this is also valid for any spin component, then it should be possible, in the supposed complete theory, to assign different spin wave functions to the same physical reality. Therefore, one arrives at the conclusion that two or more physical quantities, which correspond to non–commuting quantum operators, can have simultaneous reality. However, in quantum mechanics two observables corresponding to non-commuting operators cannot
have simultaneous reality. Therefore, there exist elements of physical reality for which quantum mechanics has no counterpart, and, according to EPR’s completeness definition, quantum theory cannot give a complete description of reality.

Actually, one could object, with Bohr [2], that in connection with a correlated system of non–interacting subsystems [described, in quantum mechanics, by eq. (2.1)], EPR’s reality criterion reveals the following weak point: it is not correct to assert that the measurement on subsystem 1 does not disturb system 2; in fact, in quantum mechanics the measurement do separate systems 1 and 2, which are not separated entities before the reduction of the wave packet. It is the measurement on system 1 that fixes (in a way that, however, does not depend only on the experimental setting one uses but contains an element of randomness) the quantum state (before undetermined) of system 2. Any measurement on system 1 is therefore a measurement on the entire system 1+2. Moreover, in quantum mechanics two or more physical quantities can be considered as simultaneous elements of reality only when they can be simultaneously measured. Then, from the point of view of orthodox quantum mechanics, EPR’s argumentation ceases to be a paradox: EPR’s proof of incompleteness is mathematically correct but is founded on premises which are inapplicable to microphenomena.

However, one has to remind that, since in quantum mechanics the elements of reality of quantum systems are our knowings (and not elements concerning the actual behaviour of matter), this interpretation only provides an incomplete description of the dynamics of quantum world, because each knowing originates a collapse of the wave function; this process affects the future behaviour of the system and randomly selects among different and alternative possibilities, whose only known characteristic is the statistical distribution. Thus, in quantum mechanics the reality of two non–commuting observables, which cannot be defined simultaneously, depends on the measurement one performs. In this way, reality is in part created by the observer.

EPR’s paradox was interpreted as the need for the introduction, in quantum mechanics, of additional variables, in order to restore completeness, causality and realism. Then, Bell and other authors developed different inequalities suitable for testing what has been called local realism.

A. Local Realism for the two–neutral–kaon system

Now we come to the entangled system of two neutral kaons. In this paper we neglect the effects of \( CP \) violation. Then, the \( CP \) eigenstates are identified with the short and long living kaons (mass eigenstates): \( |K_+\rangle \equiv |K_S\rangle (CP = +1), |K_-\rangle \equiv |K_L\rangle (CP = -1) \). In this approximation the strong interaction eigenstates \( |K^0\rangle \) and \( |ar{K}^0\rangle \) are given by:

\[
|K^0\rangle = \frac{1}{\sqrt{2}} (|K_S\rangle + |K_L\rangle),
\] 

(2.3)
The time evolution of the mass (weak interaction) eigenstates is:

\[ |K^0⟩ = \frac{1}{\sqrt{2}} [|K_S⟩ - |K_L⟩]. \]

The time evolution of the mass (weak interaction) eigenstates is:

\[ |K_{S,L}(τ)⟩ = e^{-iλ_{S,L}τ}|K_{S,L}⟩, \]  

(2.4)

where \( |K_{S,L}⟩ \equiv |K_{S,L}(0)⟩ \), \( τ = t\sqrt{1 - v^2} \) is the kaon proper time \([t (v) \text{ being the time (kaon velocity) measured in the laboratory frame} \] and:

\[ λ_{S,L} = m_{S,L} + \frac{i}{2} Γ_{S,L}, \]

(2.5)

\( m_{S,L} \) denoting the \( K_S \) and \( K_L \) masses and \( Γ_{S,L} \) the corresponding decay widths: \( Γ_{S,L} = 1/τ_{S,L} \) (we use natural units: \( \hbar = c = 1 \)).

Consider now the strong decay of the \( φ(1020) \)–meson, whose relevant quantum numbers are \( J^{P^C} = 1^{−−} \), into \( K^0\bar{K}^0 \) \([BR(φ → K^0\bar{K}^0) ≃ 34.1%] \). With good approximation the process is non–relativistic: in the center of mass system, the kaons correspond to a Lorentzian factor \( γ ≃ 1.02 \). Just after the decay, at proper time \( τ = 0 \), the quantum–mechanical state is given by the following superposition:

\[ |φ(0)⟩ = \frac{1}{\sqrt{2}} \left[ |K^0⟩_1|\bar{K}^0⟩_2 - |\bar{K}^0⟩_1|K^0⟩_2 \right] \]  

(2.6)

written in both bases we have introduced. Since the kaon is a spinless particle and the \( φ \) has spin 1, angular momentum conservation requires the kaons to be emitted in a spatially antisymmetric state. The state is also antisymmetric under charge conjugation. The second equality in (2.6) is only approximated when one includes the (small) effects of \( CP \) violation. Moreover, in the above equation, 1 and 2 denote the directions of motion of the two kaons. From eqs. (2.3) and (2.4) the time evolution of state (2.6) is obtained in the following form:

\[ |φ(τ_1, τ_2)⟩ = \frac{1}{\sqrt{2}} \left\{ e^{-i(λ_{L_1}τ_1 + λ_{S_1}τ_2)}|K_L⟩_1|\bar{K}_S⟩_2 - e^{-i(λ_{S_1}τ_1 + λ_{L_1}τ_2)}|K_S⟩_1|K_L⟩_2 \right\} \]  

(2.7)

In the following we introduce, within local realism, the elements of physical reality for the two–kaon system. Before doing this, it is important to remind once again that within the philosophy of realism, quantum systems have intrinsic and well defined properties, even when they are not subject to measurements. The existence of quantum world is then (like in classical physics) objective and independent of our observations. As a consequence, any measurement performed on a quantum system produces a result with a definite and pre-determined value. We shall assume locality by requiring that physical phenomena in a
space–time region cannot be affected by what occurs in all space–time regions which are space–like separated from the first one. This means that when the two kaons are space–like separated, the elements of reality belonging to one kaon cannot be created nor influenced by a measurement made on the other kaon. This amounts to express relativistic causality, which prevents any action–at–a–distance. Implicit in our description is also the inexistence, in any reference frame, of influences acting backward in time: a measurement performed on one kaon cannot influence the elements of reality possessed by this kaon for times preceding the measurement.

Quantum mechanics predicts (and we know it is a well tested property) a perfect anti–correlation in strangeness and $CP$ values when both kaons are considered at the same time [see eq. (2.7)]. If an experimenter observes, say along direction 1, a $K^0$ ($K_L$), at the same time $\tau_1$, along direction 2, because of the instantaneous collapse of the two–kaon wave function, one can predict the presence of a $\bar{K}^0$ ($K_S$). Thus, at time $\tau_1$ to the kaon moving along direction 2 we assign an element of reality (since, following EPR’s reality criterion the value of the corresponding physical quantity is predicted with certainty and without in any way disturbing the system), the value $-1$ ($+1$) of strangeness ($CP$). The same discussion is valid when the state observed along direction 1 is $\bar{K}^0$ (or $K_S$) as well as when one exchanges the kaon directions: $1 \leftrightarrow 2$. For times $\tau_2$ successive the observation at time $\tau_1$ along direction 1 of a $K_L$ ($K_S$), a $CP$ measurement on the other kaon will give with certainty the same result $CP = +1$ ($CP = -1$) one expects at time $\tau_1$. This expresses $CP$ conservation.

Obviously, because of the instability of the $K_L$ and $K_S$ components, along direction 2 the experimenter could observe either $CP = +1$ or $CP = -1$ decay products at time $\tau_2$, but what is important in the present discussion is that for any pair of times $(\tau_1, \tau_2)$ these exists perfect anti–correlation on $CP$. In the case in which both kaons are undecayed, when the kaon detected at time $\tau_1$ is $K^0$ ($\bar{K}^0$), at times $\tau_2 > \tau_1$ along direction 2 quantum mechanics predicts the possibility to observe a $\bar{K}^0$ ($K^0$) as well as a $K^0$ ($\bar{K}^0$): since strangeness is not conserved during the evolution of the system (governed by the weak interaction), perfect anti–correlation on strangeness only exists when both particles are considered at the same time.

Following EPR’s argument, in the local realistic approach one then associates to both kaons of the pair, at any time, two elements of reality, which are not created by measurements eventually performed on the partner when the particles are space–like separated (locality): one determines the kaon $CP$ value, the other one supplies the kaon strangeness $S$. They are both well defined also when the meson is not observed (realism) and can take two values, $\pm 1$, which appear at random with the same frequency in a statistical ensemble of kaons. Because of the strangeness non–conservation, a particular value of the element of reality $S$ is defined instantaneously (in fact, instantaneous oscillations between $S = \pm 1$ and $S = \mp 1$ occur), but what is important in the realistic approach is that $S$ has objective and well defined existence at any instant time. For a pair, the instantaneous and simultaneous $|\Delta S| = 2$ oscillations are compatible with locality only if one introduces a hidden–variable interpretation of the
TABLE I. Kaon realistic states.

| State | Strangeness | CP |
|-------|-------------|----|
| $K_1 \equiv K^0_S$ | +1 | +1 |
| $K_2 \equiv \bar{K}^0_S$ | -1 | +1 |
| $K_3 \equiv K^0_L$ | +1 | -1 |
| $K_4 \equiv \bar{K}^0_L$ | -1 | -1 |

pair evolution which predetermines the times of the strangeness jumps.

In conclusion, neglecting $CP$ violation, within local realism a kaon is characterized by two different elements of physical reality, which can both take two values with equal frequency; thus, four different single kaon states can appear just after the $\phi$ decay, with the same frequency (25%). They are quoted in Table I. It is clear that this classification is incompatible with quantum mechanics: in fact, under local realism a kaon has, simultaneously, defined values of strangeness and $CP$, whereas in quantum mechanics these quantities are described by non–commuting operators, then they cannot be measured simultaneously.

### III. QUANTUM–MECHANICAL EXPECTATION VALUES

By introducing the shorthand notation:

$$E_{S,L}(\tau) = e^{-\Gamma_{S,L} \tau},$$

and the mass difference:

$$\Delta m = m_L - m_S,$$

from eq. (2.7) the quantum–mechanical (QM) probability $P_{QM}[K^0(\tau_1), \bar{K}^0(\tau_2)] \equiv |1\langle K^0\mid \bar{K}^0\phi(\tau_1, \tau_2)\rangle|^2$ (eq. (3.3)) that a measurement detects a $K^0$ ($\bar{K}^0$) at time $\tau_1$ along direction 1 and a $\bar{K}^0$ ($K^0$) at time $\tau_2$ along direction 2 is:

$$P_{QM}[K^0(\tau_1), \bar{K}^0(\tau_2)] = P_{QM}[\bar{K}^0(\tau_1), K^0(\tau_2)]$$

$$= \frac{1}{8} \left[ E_L(\tau_1)E_S(\tau_2) + E_S(\tau_1)E_L(\tau_2) + 2\sqrt{E_L(\tau_1 + \tau_2)E_S(\tau_1 + \tau_2)\cos \Delta m(\tau_2 - \tau_1)} \right].$$

The other probabilities relevant for our discussion are the following ones:

$$P_{QM}[K^0(\tau_1), K^0(\tau_2)] = P_{QM}[\bar{K}^0(\tau_1), \bar{K}^0(\tau_2)]$$

$$= \frac{1}{8} \left[ E_L(\tau_1)E_S(\tau_2) + E_S(\tau_1)E_L(\tau_2) - 2\sqrt{E_L(\tau_1 + \tau_2)E_S(\tau_1 + \tau_2)\cos \Delta m(\tau_2 - \tau_1)} \right].$$
A parameter, defined by the following relation for a generic theory:

\[ P_{QM}[K_L(\tau_1), K_S(\tau_2)] = \frac{1}{2} E_L(\tau_1) E_S(\tau_2), \]  

(3.5)

\[ P_{QM}[K_S(\tau_1), K_L(\tau_2)] = \frac{1}{2} E_S(\tau_1) E_L(\tau_2), \]  

(3.6)

\[ P_{QM}[K_S(\tau_1), K_S(\tau_2)] = P_{QM}[K_L(\tau_1), K_L(\tau_2)] = 0, \]  

(3.7)

\[ P_{QM}[K_S(\tau_1), K^0(\tau_2)] = P_{QM}[K_S(\tau_1), \bar{K}^0(\tau_2)] = P_{QM}[K^0(\tau_1), K_L(\tau_2)] \]  

\[ = P_{QM}[\bar{K}^0(\tau_1), K_L(\tau_2)] = \frac{1}{4} E_S(\tau_1) E_L(\tau_2), \]  

(3.8)

\[ P_{QM}[K_L(\tau_1), K^0(\tau_2)] = P_{QM}[K_L(\tau_1), \bar{K}^0(\tau_2)] = P_{QM}[K^0(\tau_1), K_S(\tau_2)] \]  

\[ = P_{QM}[\bar{K}^0(\tau_1), K_S(\tau_2)] = \frac{1}{4} E_L(\tau_1) E_S(\tau_2), \]  

(3.9)

the fourth equation expressing CP conservation.

In the particular case of \( \tau_1 = \tau_2 \equiv \tau \):

\[ P_{QM}[K^0(\tau), K^0(\tau)] = P_{QM}[\bar{K}^0(\tau), \bar{K}^0(\tau)] = \frac{1}{2} E_L(\tau) E_S(\tau), \]  

(3.10)

\[ P_{QM}[K^0(\tau), \bar{K}^0(\tau)] = P_{QM}[\bar{K}^0(\tau), K^0(\tau)] = 0. \]  

(3.11)

These relations, together with eq. (3.7), show the perfect anti–correlation of the quantum–mechanical state (2.7) concerning strangeness and CP.

Starting from probabilities (3.3) and (3.4) it is useful to introduce a time–dependent asymmetry parameter, defined by the following relation for a generic theory:

\[ A(\tau_1, \tau_2) \equiv \frac{P[K^0(\tau_1), K^0(\tau_2)] + P[K^0(\tau_1), K^0(\tau_2)] - P[K^0(\tau_1), K^0(\tau_2)] - P[K^0(\tau_1), K^0(\tau_2)]}{P[K^0(\tau_1), K^0(\tau_2)] + P[K^0(\tau_1), K^0(\tau_2)] + P[K^0(\tau_1), K^0(\tau_2)] + P[K^0(\tau_1), K^0(\tau_2)]}. \]  

(3.12)

The quantum–mechanical expression of this quantity is a function of \( \tau_2 - \tau_1 \) only:

\[ A_{QM}(\tau_1, \tau_2) = 2\sqrt{\frac{E_L(\tau_2 - \tau_1)}{E_L(\tau_2 - \tau_1) + E_S(\tau_2 - \tau_1)}} \cos \Delta m(\tau_2 - \tau_1), \]  

(3.13)

and measures the interference term appearing in like–strangeness \((K^0K^0)\) or \((\bar{K}^0\bar{K}^0)\) and unlike–strangeness \((K^0\bar{K}^0)\) or \((\bar{K}^0K^0)\) events.

IV. LOCAL REALISTIC EXPECTATION VALUES

In this section we discuss the widest class of local hidden–variable models for the two–kaon state and their predictions for the observables provided, in quantum mechanics, by eqs. (3.3)–(3.9). Following the derivation of ref. [31], we start considering how the quantum–mechanical expectation values for the single kaon evolution can be reproduced by a realistic approach. Then, we extend the description to the interesting case of an entangled kaon pair.
A. Evolution of a single kaon

In the realistic approach one introduces the four kaonic states of table I. $K_1$ is a state with defined strangeness (+1) and $CP$ (+1), and the same is true for the other states.

Introduce the notation:

$$p_{ij}(\tau|0) \equiv p[K_j(0) \rightarrow K_i(\tau)],$$

(4.1)

for the conditional probability that a state $K_i$ is present at time $\tau$ if the original state at time $\tau = 0$ was $K_j$. It is immediate to write down the time $\tau = 0$ probabilities; they are:

$$p_{11}(0|0) = p_{22}(0|0) = p_{33}(0|0) = p_{44}(0|0) = 1, \quad p_{ij}(0|0) = 0 \text{ for } i \neq j.$$

(4.2)

When the evolution of the four states is considered, $CP$ conservation requires that, for all times:

$$p_{13}(\tau|0) = p_{14}(\tau|0) = p_{23}(\tau|0) = p_{24}(\tau|0) = p_{31}(\tau|0) = p_{32}(\tau|0) = p_{41}(\tau|0) = p_{42}(\tau|0) \equiv 0.$$  

(4.3)

In quantum mechanics, assuming $CP$ conservation, the mass eigenstates $|K_L\rangle$ and $|K_S\rangle$ are perfectly orthogonal to each other, then:

$$\langle K_L(0)|K_S(\tau)\rangle = \langle K_S(0)|K_L(\tau)\rangle = 0.$$  

(4.4)

During the time evolution, strangeness jumps between $S = +1$ ($-1$) and $S = -1$ ($+1$) states occur. Thus, only transitions $K_1 \leftrightarrow K_2$ and $K_3 \leftrightarrow K_4$ are permitted, and eq. (4.3) is valid.

In order to fix the time evolution of the four states of table I we have to determine 8 probabilities $p_{ij}(\tau|0)$. As we are going to show, it is possible to fix these quantities and reproduce all the quantum–mechanical predictions relevant for the single kaon propagation.

From quantum mechanics [eqs. (2.3), (2.4)] one obtains:

$$|\langle K^0(0)|K_S(\tau)\rangle|^2 = |\langle \bar{K}^0(0)|K_S(\tau)\rangle|^2 = |\langle K_S(0)|K^0(\tau)\rangle|^2 = |\langle K_S(0)|\bar{K}^0(\tau)\rangle|^2 = \frac{1}{2}E_S(\tau),$$

where the different terms have obvious significance. This restrictions correspond to require the following equalities, that we write in the same order as before, among the realistic probabilities:

$$\frac{1}{2}[p_{11}(\tau|0) + p_{12}(\tau|0)] = \frac{1}{2}[p_{21}(\tau|0) + p_{22}(\tau|0)] = \frac{1}{2}[p_{11}(\tau|0) + p_{21}(\tau|0)]$$

(4.6)

$$= \frac{1}{2}[p_{12}(\tau|0) + p_{22}(\tau|0)] = \frac{1}{2}E_S(\tau),$$

which correspond to fix:
\[ p_{21}(\tau|0) = p_{12}(\tau|0), \quad (4.7) \]
\[ p_{22}(\tau|0) = p_{11}(\tau|0), \quad (4.8) \]
\[ p_{11}(\tau|0) + p_{12}(\tau|0) = E_S(\tau), \quad (4.9) \]

the first two equalities being compatible with time–reversal invariance, which follows from \textit{CPT} theorem, having adopted \textit{CP} conservation. In the same way, the equalities:

\[ |\langle K^0(0)|K_L(\tau)\rangle|^2 = |\langle \bar{K}^0(0)|K_L(\tau)\rangle|^2 = |\langle K_L(0)|K^0(\tau)\rangle|^2 \quad (4.10) \]
\[ = |\langle K_L(0)|\bar{K}^0(\tau)\rangle|^2 = \frac{1}{2}E_L(\tau), \]

require:

\[ p_{43}(\tau|0) = p_{34}(\tau|0), \quad (4.11) \]
\[ p_{44}(\tau|0) = p_{33}(\tau|0), \quad (4.12) \]
\[ p_{33}(\tau|0) + p_{34}(\tau|0) = E_L(\tau). \quad (4.13) \]

At this point, two of the 8 \( p_{ij} \)'s are independent. However, other constraints come from quantum mechanics. In fact, one can write:

\[ |\langle K^0(0)|K^0(\tau)\rangle|^2 = \frac{1}{4} \left[ E_L(\tau) + E_S(\tau) + 2\sqrt{E_L(\tau)E_S(\tau)} \cos \Delta m\tau \right] \]
\[ = \frac{1}{2} [p_{11}(\tau|0) + p_{33}(\tau|0)], \quad (4.14) \]

where the first (second) equality follows from quantum mechanics (realism), and, analogously:

\[ |\langle \bar{K}^0(0)|\bar{K}^0(\tau)\rangle|^2 = \frac{1}{4} \left[ E_L(\tau) + E_S(\tau) - 2\sqrt{E_L(\tau)E_S(\tau)} \cos \Delta m\tau \right] \]
\[ = \frac{1}{2} [p_{12}(\tau|0) + p_{34}(\tau|0)], \quad (4.15) \]

where, in the last equality, we have taken into account of eqs. (4.7) and (4.11). The other equations one gets for the quantities \(|\langle K^0(0)|\bar{K}^0(\tau)\rangle|^2\), \(|\langle \bar{K}^0(0)|\bar{K}^0(\tau)\rangle|^2\), \(|\langle K_S(0)|K_S(\tau)\rangle|^2\) and \(|\langle K_L(0)|K_L(\tau)\rangle|^2\) do not supply new constraints but are compatible with the conditions written above. Thus, assuming eqs. (4.7), (4.8), (4.11) and (4.12), among \( p_{11}, p_{12}, p_{33} \) and \( p_{34} \) we have the system of equations:

\[
\begin{cases}
  p_{11}(\tau|0) + p_{12}(\tau|0) = E_S(\tau) \\
  p_{33}(\tau|0) + p_{34}(\tau|0) = E_L(\tau) \\
  p_{11}(\tau|0) + p_{33}(\tau|0) = [E_L(\tau) + E_S(\tau)]Q_+(\tau) \\
  p_{12}(\tau|0) + p_{34}(\tau|0) = [E_L(\tau) + E_S(\tau)]Q_-(\tau),
\end{cases} \quad (4.16)
\]

where the shorthand notation:
TABLE II. Realistic states for the kaon pair at initial time \( \tau = 0 \).

| Direction 1 | Direction 2 |
|-------------|-------------|
| \( K_1 \equiv K_0^S \) \( (S = +1, CP = +1) \) | \( K_4 \equiv K_0^L \) \( (S = -1, CP = -1) \) |
| \( K_2 \equiv \bar{K}_0^S \) \( (S = -1, CP = +1) \) | \( K_3 \equiv K_0^L \) \( (S = +1, CP = -1) \) |
| \( K_3 \equiv \bar{K}_0^L \) \( (S = -1, CP = +1) \) | \( K_2 \equiv K_0^S \) \( (S = +1, CP = +1) \) |
| \( K_4 \equiv \bar{K}_0^L \) \( (S = -1, CP = -1) \) | \( K_1 \equiv K_0^S \) \( (S = +1, CP = +1) \) |

\[
Q_{\pm}(\tau) = \frac{1}{2} \left[ 1 \pm 2 \frac{\sqrt{E_L(\tau)E_S(\tau)}}{E_L(\tau) + E_S(\tau)} \cos \Delta m\tau \right]
\]  

(4.17)

has been employed. Since \( Q_{+}(\tau) + Q_{-}(\tau) = 1 \), in eq. (4.16) only three conditions out of four are independent. A symmetrical choice of \( p_{11}, p_{12}, p_{33} \) and \( p_{34} \) leads to the following realistic probability matrix [31]:

\[
p(\tau|0) = \begin{pmatrix}
E_S(\tau)Q_{+}(\tau) + \delta(\tau) & E_S(\tau)Q_{-}(\tau) - \delta(\tau) & 0 & 0 \\
E_S(\tau)Q_{+}(\tau) - \delta(\tau) & E_S(\tau)Q_{-}(\tau) + \delta(\tau) & 0 & 0 \\
0 & 0 & E_L(\tau)Q_{+}(\tau) - \delta(\tau) & E_L(\tau)Q_{-}(\tau) + \delta(\tau) \\
0 & 0 & E_L(\tau)Q_{+}(\tau) + \delta(\tau) & E_L(\tau)Q_{-}(\tau) - \delta(\tau)
\end{pmatrix}
\]  

(4.18)

where the degree of freedom is given by the function \( \delta(\tau) \). The requirement that all the matrix elements are well defined \( (0 \leq p_{ij}(\tau|0) \leq 1) \) can be satisfied if one chooses properly the function \( \delta(\tau) \). A particular solution correspond to keep \( \delta(\tau) \equiv 0 \). Actually, as we shall see in section IV B 2, an identically vanishing \( \delta(\tau) \) function is the only solution compatible with the local realistic evolution of a correlated pair of kaon.

**B. Evolution of a correlated kaon pair**

Now we come to the time evolution of a correlated and non–interacting \( K^0\bar{K}^0 \) pair emitted in the decay of a \( \phi \)-meson.

From quantum theory [eq. (2.7)] we know that for any time the joint observation of the mesons finds them perfectly correlated. At time \( \tau = 0 \), immediately after the \( \phi \) decay, in the realistic picture there are four possible states for the kaon pair, each appearing with a probability equal to \( 1/4 \): they are listed in table II. Kaon \( K_1 \) is created together with a \( \bar{K}_4 \): we assume, as in quantum mechanics, since it is a well tested property, a perfect anti–correlation in strangeness and \( CP \) when both kaons are considered at equal time. The other three initial states show, obviously, the same correlation property.

When the system evolves, the kaons fly apart from each other, and at two generic times \( \tau_1 \) and \( \tau_2 \) (corresponding to opposite directions of propagation labeled 1 and 2, respectively)
the kaon pair is in one of the states reported in table III. The first row refers to the state with a $K_1$ at time $\tau_1$ along direction 1 (which we define as left direction) and a $K_4$ at time $\tau_2$ along direction 2 (right direction; we have in mind, here, the kaon pair propagation in the center of mass system). Given the classification of the table, in our discussion we consider $\tau_2 \geq \tau_1$: the isotropy of space guarantees the invariance of the two-kaon states by exchanging the directions 1 and 2. In the second row the state corresponds to a left going $K_1$ at time $\tau_1$ and $CP = -1$ decay products (DP) at time $\tau_2$ on the right. These decay products originate from the instability of the $K_3$ and $K_4$ pure states, which are both long living kaons, namely $CP = -1$ states (the corresponding physical processes are: $K_L \to 3\pi, \pi \mu \nu, \pi e \nu\bar{\nu}$). At time $\tau_1$ the state correlated with a left going $K_1$ is necessarily either a $K_4$ or a state containing $CP = -1$ decay products, $K_3^{DP}$ or $K_4^{DP}$. Then, at time $\tau_2 > \tau_1$ on the right we can have: i) a $K_4$ (state in the first row), ii) $CP = -1$ decay products (state in the second row) or iii) a $K_3$ (state in the fourth row). The former case refers to the transition $K_4(\tau_1) \to K_4(\tau_2)$, the latter to $K_4(\tau_1) \to K_3(\tau_2)$, both along direction 2. Occurrence ii) takes contributions from the following transitions: $K_3^{DP}(\tau_1) \to K_3^{DP}(\tau_2)$, $K_4^{DP}(\tau_1) \to K_4^{DP}(\tau_2)$, $K_4(\tau_1) \to K_4^{DP}(\tau_2)$ and $K_4(\tau_1) \to K_3(\tau_1 < \tau < \tau_2) \to K_3^{DP}(\tau_2)$. The other states in table III have similar meaning, the last two rows corresponding to the situation in which both left and right going kaons are decayed at times $\tau_1$ and $\tau_2$, respectively.

| Probabilities | Direction 1 (Left) Time $\tau_1$ | Direction 2 (Right) Time $\tau_2$ |
|---------------|----------------------------------|----------------------------------|
| $P_1(\tau_1, \tau_2; \lambda)$ | $K_1 \equiv K_0^S$ | $K_1 \equiv K_0^S$ |
| $P_2(\tau_1, \tau_2; \lambda)$ | $K_1 \equiv K_0^S$ | $CP = -1$ DP |
| $P_3(\tau_1, \tau_2; \lambda)$ | $CP = +1$ DP | $K_4 \equiv K_0^L$ |
| $P_4(\tau_1, \tau_2; \lambda)$ | $K_1 \equiv K_0^S$ | $K_3 \equiv K_0^S$ |
| $P_5(\tau_1, \tau_2; \lambda)$ | $K_2 \equiv K_0^S$ | $K_3 \equiv K_0^S$ |
| $P_6(\tau_1, \tau_2; \lambda)$ | $K_2 \equiv K_0^S$ | $CP = -1$ DP |
| $P_7(\tau_1, \tau_2; \lambda)$ | $CP = +1$ DP | $K_3 \equiv K_0^S$ |
| $P_8(\tau_1, \tau_2; \lambda)$ | $K_2 \equiv K_0^S$ | $K_4 \equiv K_0^L$ |
| $P_9(\tau_1, \tau_2; \lambda)$ | $K_3 \equiv K_0^S$ | $K_2 \equiv K_0^S$ |
| $P_{10}(\tau_1, \tau_2; \lambda)$ | $K_3 \equiv K_0^S$ | $CP = +1$ DP |
| $P_{11}(\tau_1, \tau_2; \lambda)$ | $CP = -1$ DP | $K_2 \equiv K_0^S$ |
| $P_{12}(\tau_1, \tau_2; \lambda)$ | $K_3 \equiv K_0^S$ | $K_1 \equiv K_0^S$ |
| $P_{13}(\tau_1, \tau_2; \lambda)$ | $K_4 \equiv K_0^S$ | $K_1 \equiv K_0^S$ |
| $P_{14}(\tau_1, \tau_2; \lambda)$ | $K_4 \equiv K_0^S$ | $K_2 \equiv K_0^S$ |
| $P_{15}(\tau_1, \tau_2; \lambda)$ | $CP = -1$ DP | $K_1 \equiv K_0^S$ |
| $P_{16}(\tau_1, \tau_2; \lambda)$ | $K_4 \equiv K_0^S$ | $K_2 \equiv K_0^S$ |
| $P_{17}(\tau_1, \tau_2; \lambda)$ | $CP = +1$ DP | $CP = -1$ DP |
| $P_{18}(\tau_1, \tau_2; \lambda)$ | $CP = -1$ DP | $CP = +1$ DP |
1. Interpretation of the states with local hidden–variables

At this point it is important to stress that the states listed in table II are assumed to be well defined for all times \( \tau_1 \) and \( \tau_2 \) with \( \tau_1 \leq \tau_2 \): this is the main requirement of the realistic approach (analogous discussion is valid for states in tables I and II). For a given kaon pair we assume that only one of the 18 possibilities of table II really occurs for fixed \( \tau_1 \) and \( \tau_2 \). This means that we are making the hypothesis (realism) that there exist additional variables, usually called hidden–variables (with respect to orthodox quantum mechanics these variables are hidden in the sense that they are uncontrollable), that provide a complete description of the pair, which is viewed as really existing and with well defined properties independently of any observation. The state representing the meson pair for given times \( (\tau_1, \tau_2) \) is completely defined by these hidden–variables: they are supposed to determine in advance (say when the two kaons are created) the future behaviour of the pair. Thus, the times in correspondence of which the instantaneous \( |\Delta S| = 2 \) jumps and the decay occur for a given kaon are predetermined by its hidden–variables. Under this hypotheses there is no problem concerning a possible causal influence acting among the different entities of entangled systems when a measurement takes place on one subsystem. However, the new variables, which we denote with the compact symbol \( \lambda \), are unobservable because they are averaged out in the measuring processes, and unobservable are the states of table III. In principle, also the measuring apparatus could be described by means of hidden–variables, which influence the results of measurement. Besides, hidden–variables associated to the kaon pair could show a non–deterministic behaviour. It is important to stress that in the approach with hidden–variables, the probabilistic character of quantum mechanics is viewed as a practical necessity for treating problems at the observation level, but (and this is a strong difference compared to the orthodox interpretation) does not originate from the intrinsic behaviour of microphenomena: the indetermination principle is supposed to act only during the observation process.

The realistic probabilities listed in table III:

\[
P_i(\tau_1, \tau_2; \lambda) \equiv P_i(\tau_1, \tau_2|\lambda)\rho(\lambda),
\]

(4.19)
correspond to the situation in which a single meson pair, described by the value \( \lambda \) of the hidden–variables, is considered. Once \( \tau_1 \) and \( \tau_2 \) are fixed, the state of the kaon pair \( \lambda \) (we can think it is fixed at the time of the pair creation) can take values in the set \( \{\lambda^{[\tau_1, \tau_2]}_i; i = 1, .., 18\} \) (however, we stress again, \( \lambda \) is fixed when a single pair is considered), and for a deterministic theory we have:

\[
P_i(\tau_1, \tau_2; \lambda^{[\tau_1, \tau_2]}_j) = \delta_{ij}\rho(\lambda^{[\tau_1, \tau_2]}_i).
\]

(4.20)
In the previous relations, \( \rho \) is the probability distribution of the kaon pair hidden–variables and \( P_i(\tau_1, \tau_2|\lambda^{[\tau_1, \tau_2]}_j) \) \( (= \delta_{ij}) \) is the probability of the \( i \)–th state of table III conditional on the presence of a pair in the state \( \lambda^{[\tau_1, \tau_2]}_j \). For a single meson pair, only one of the probabilities
of table (11) is different from zero at instants \((\tau_1, \tau_2)\) in a deterministic model: if \(\lambda \equiv \lambda_{k}^{[\tau_1, \tau_2]}\), the non-vanishing probability is the \(k\)-th of the table. As far as different times \(\tau_1'\) and \(\tau_2'\) are considered, eq. (4.20) is valid for the same set of hidden-variables, but in general with the variables appearing in a different permutation, \(\{\lambda_{i}^{[\tau_1', \tau_2']}\} \equiv \mathcal{P}\{\lambda_{j}^{[\tau_1, \tau_2]}\}\) (indexes \(i\) and \(j\) always refer to the classification of table (11)). Therefore, in the model we are describing the pair can be created in 18 different realistic states, namely with 18 different values of the hidden-variables. We stress again: the hidden-variable sets at different pair of times can be created in 18 different realistic states, namely with 18 different values of the hidden-variables. We stress again: the hidden-variable sets at different pair of times contain the same objects, but in one of the 18! different orderings, and notation used in eq. (4.20) does not mean that the hidden-variables are time-dependent. From eq. (4.20) it follows that in a model with deterministic kaon pair hidden-variables, the normalization of the local realistic probabilities corresponds to that of the hidden-variables:

\[
\sum_{i=1}^{18} P_{i}(\tau_1, \tau_2; \lambda_{i}^{[\tau_1, \tau_2]}) = \sum_{i=1}^{18} \rho(\lambda_{i}^{[\tau_1, \tau_2]}) = 1. \tag{4.21}
\]

In the realistic interpretation, the quantum state (27) corresponds to a statistical ensemble of meson pairs, which are further specified by different values of the hidden-variables. To exemplify, let us consider such a (large) ensemble of identical kaon pairs specified by different \(\lambda\)'s (which we suppose now to be continuous variables), whose distribution \(\rho\) we assume to be independent of the apparatus parameters \(\tau_1\) and \(\tau_2\), the kaons being emitted in a way which does not depend on the adjustable times \(\tau_1\) and \(\tau_2\) (we assume, here, no retroactive causality). We can give a statistical characterization of this ensemble by means of the set of observables (3.3)–(3.9). Considering, as an example, \(P[K^0(\tau_1), \bar{K}^0(\tau_2)]\), within a general deterministic local hidden-variable interpretation:

\[
P_{LR}[K^0(\tau_1), \bar{K}^0(\tau_2)] \equiv \int d\lambda \rho(\lambda) P(K^0, \tau_1; \bar{K}^0, \tau_2; \lambda) \tag{4.22}
\]

\[
= \int d\lambda \rho(\lambda) P_{\text{Left}}(K^0, \tau_1; \lambda) P_{\text{Right}}(\bar{K}^0, \tau_2; \lambda),
\]

where \(LR\) stands for local realism. The joint probability \(P(K^0, \tau_1; \bar{K}^0, \tau_2; \lambda)\), which is conditional on the presence of the particular value \(\lambda\) of the hidden-variables, has been assumed to be locally explicable, then it appears in the factorized form in the last equality. The function \(P_{\text{Left}}(K^0, \tau_1; \lambda) [P_{\text{Right}}(\bar{K}^0, \tau_2; \lambda)]\) is the conditional probability that, once fixed \(\lambda\), the left (right) going kaon at time \(\tau_1\) (\(\tau_2\)), which is fully specified by \(\lambda\), is \(K^0\) (\(\bar{K}^0\)). As required by locality, given \(\lambda\) and the apparatus parameters \(\tau_1\) and \(\tau_2\), \(P_{\text{Left}}\) and \(P_{\text{Right}}\) are independent. They only take two values:

\[
P_{\text{Left}}(K^0, \tau_1; \lambda) = \begin{cases} 
1, & \text{when the state at time } \tau_1 \text{ is } K^0 \\
0, & \text{when the state at time } \tau_1 \text{ is not } K^0
\end{cases}. \tag{4.23}
\]

The knowledge (impossible, we emphasize again) of the hidden-variables associated to an individual kaon pair emission would permit to determine the precise instants the \(K_S\) and \(K_L\) components decay, then eq. (4.23) follows. The locality condition is motivated by the
requirement of relativistic causality, which prevents faster-than-light influences between space-like separated events. In the present case, assuming there is no delay among the times at which the experimenters choose to perform their observations and the real kaon measurement times \( \tau_1 \) and \( \tau_2 \), the locality requirement is fulfilled when the two observation events are separated by a space-like interval [see eq. (5.4)]. However, to be precise, as we shall explain in section \( \text{V} \), a loophole that is impossible to block exists and could permit, in principle and without requiring the existence of action-at-a-distance, an information to reach both the measuring devices for any choice of the detection times \( \tau_1 \) and \( \tau_2 \).

In the above we have restricted our argumentation to deterministic theories only, but it is possible to extend the same description given by eq. \( \text{(I.22)} \) to non-deterministic (namely stochastic) theories as well as to deterministic theories in which additional hidden-variables correspond to the measurement devices \( \text{[18]} \). Let us make the hypothesis that also the experimental apparata are described in terms of hidden-variables, which influence the measurement outcomes. In this case, denoting with \( \lambda' (\lambda'') \) the hidden-variables specifying the behaviour of the apparatus measuring on the left (right), in the new local hidden-variable theory the expectation value of eq. \( \text{(4.22)} \) is given by:

\[
P_{\text{LR}}[K^0(\tau_1), \bar{K}^0(\tau_2)] \equiv \int d\lambda d\lambda' d\lambda'' \rho(\lambda, \lambda', \lambda'') P(K^0, \tau_1; \bar{K}^0, \tau_2 | \lambda, \lambda', \lambda'')
\]

\[
= \int d\lambda d\lambda' d\lambda'' \rho(\lambda) p(\lambda'|\lambda) p(\lambda''|\lambda) P_{\text{Left}}(K^0, \tau_1 | \lambda, \lambda') P_{\text{Right}}(\bar{K}^0, \tau_2 | \lambda, \lambda''),
\]

where:

\[
\rho(\lambda, \lambda', \lambda'') \equiv p(\lambda', \lambda''|\lambda)\rho(\lambda) = p(\lambda'|\lambda) p(\lambda''|\lambda)\rho(\lambda).
\]

In the second equality of eq. \( \text{(4.24)} \), locality has been assumed for both kaon pair and apparata hidden-variables; \( p(\lambda'|\lambda) \) \[ p(\lambda''|\lambda) \] is the conditional probability that, when the kaon pair is specified by the variables \( \lambda \), the device measuring the left (right) going kaon is described by the variables \( \lambda' (\lambda'') \). Since the distribution \( \rho(\lambda) \) of the kaon pair hidden-variables is normalized to unity, the same occurs for \( p(\lambda'|\lambda) \) and \( p(\lambda''|\lambda) \) for any \( \lambda \). It is then clear, by comparing eqs. \( \text{(4.22)} \) and \( \text{(4.24)} \), that:

\[
P(K^0, \tau_1; \bar{K}^0, \tau_2 | \lambda) = \int d\lambda' d\lambda'' p(\lambda', \lambda''|\lambda) P(K^0, \tau_1; \bar{K}^0, \tau_2 | \lambda, \lambda', \lambda''),
\]

and when one implements locality for the kaon pair and apparata hidden-variables, equality \( \text{(4.23)} \) is replaced by:

\[
0 \leq P_{\text{Left}}(K^0, \tau_1 | \lambda) \equiv \int d\lambda' p(\lambda'|\lambda) P_{\text{Left}}(K^0, \tau_1 | \lambda, \lambda') \leq 1.
\]

The difference compared to the deterministic case without apparata hidden-variables is now clear, and in the new picture an equality like \( \text{(4.23)} \) is valid for \( P_{\text{Left}}(K^0, \tau_1 | \lambda, \lambda') \). It is important to stress here that for the most general non-deterministic local hidden-variable theory, elements of randomness entering the probabilities \( P_{\text{Left}}(K^0, \tau_1 | \lambda) \) and \( P_{\text{Right}}(\bar{K}^0, \tau_2 | \lambda) \) could
be related not only to apparata hidden–variables, but to other unknown mechanisms. Nevertheless, the above discussion that have led to eqs. (4.24)–(4.27) also applies for the most general non–deterministic local hidden–variable theory.

In our local realistic theory (table III) the set of hidden–variables describing the kaon pair forms a discrete set, and eq. (4.22) reduces to:

\[
\begin{align*}
P_{LR}[K^0(\tau_1), \bar{K}^0(\tau_2)] &= \rho \left( \lambda_1^{[\tau_1,\tau_2]} \right) P \left( K^0_S, \tau_1; \bar{K}^0_L, \tau_2 | \lambda_1^{[\tau_1,\tau_2]} \right) \\
&\quad + \rho \left( \lambda_9^{[\tau_1,\tau_2]} \right) P \left( K^0_L, \tau_1; \bar{K}^0_S, \tau_2 | \lambda_9^{[\tau_1,\tau_2]} \right) \\
&= P_1 \left( \tau_1, \tau_2; \lambda_1^{[\tau_1,\tau_2]} \right) + P_9 \left( \tau_1, \tau_2; \lambda_9^{[\tau_1,\tau_2]} \right),
\end{align*}
\]

where in the last line the notation of table III has been introduced. It is important to stress that, contrary to what occurs for quantum–mechanical probabilities, in the description with hidden–variables of eqs. (4.22) and (4.28) the transitions that lead to two–kaon states which contribute to \( P_{LR}[K^0(\tau_1), \bar{K}^0(\tau_2)] \) do not interfere with one another.

Within the scheme of table III it is easy to obtain the predictions of local realism for measurements concerning an individual kaon. For instance, the probability to observe a left going \( K^0 \) at time \( \tau_1 \) is given by:

\[
P_{LR}^{\text{Left}}[K^0(\tau_1)] = \sum_{i=1,2,4,9,10,12} P_i \left( \tau_1, \tau_2; \lambda_i^{[\tau_1,\tau_2]} \right).
\]

The hidden–variable interpretation of the observables we have discussed in this section has also been assumed, even if not explicitly declared, when, in section IV A, we treated the propagation of a single kaon from the point of view of realism.

2. Evaluation of the observables

Now we proceed discussing the range of variability of the meson pair observables compatible with the most general local realistic model. We shall use the rules of classical probability theory.

Start considering the probabilities of table III. In particular, we concentrate on the state in the fourth row. At time \( \tau_1 \) the left going kaon is \( K_1 \), then, requiring \( CP \) conservation, at time \( \tau = 0 \) the initial state was either a \( K_1 \) or a \( K_2 \). Since either a \( K_1 \) or a \( K_2 \) must be present as initial state for the left going kaon (both with equal frequency \( 1/4 \)), from matrix (4.18) the probability that at time \( \tau_1 \) the state is \( K_1 \) equals to \([p_{11}(\tau_1|0) + p_{12}(\tau_1|0)]/4 = E_S(\tau_1)/4\). Correlated with this \( K_1 \), at the same time \( \tau_1 \) on the right there is either a \( K_4 \) or \( CP = -1 \) decay products. Since the two–kaon state we are considering corresponds to a \( K_3 \) at time \( \tau_2 \), at time \( \tau_1 \) we must require the presence of a \( K_4 \): the probability that at this time a \( K_4 \) is not decayed is \( p_{43}(\tau_1|0) + p_{44}(\tau_1|0) = E_L(\tau_1) \). Finally, from \( \tau_1 \) the \( K_4 \) must evolve into \( K_3 \) at time \( \tau_2 \). This transition occurs with (conditional) probability that we denote:

\[
p_{34}(\tau_2|\tau_1) \equiv p[K_4(\tau_1) \to K_3(\tau_2)].
\]
Then, probability \( P_4 \) of table [II] is given by:

\[
P_4 \left( \tau_1, \tau_2; \lambda_4^{[\tau_1, \tau_2]} \right) = \frac{1}{4} E_S(\tau_1) E_L(\tau_1) p_{34}(\tau_2 | \tau_1). \tag{4.31}
\]

By using the same line of reasoning one obtains the other probabilities. They have the following expressions:

\[
P_1 \left( \tau_1, \tau_2; \lambda_1^{[\tau_1, \tau_2]} \right) = \frac{1}{4} E_S(\tau_1) E_L(\tau_1) p_{44}(\tau_2 | \tau_1), \tag{4.32}
\]

\[
P_2 \left( \tau_1, \tau_2; \lambda_2^{[\tau_1, \tau_2]} \right) = P_6 \left( \tau_1, \tau_2; \lambda_6^{[\tau_1, \tau_2]} \right) = \frac{1}{4} E_S(\tau_1) [1 - E_L(\tau_2)],
\]

\[
P_3 \left( \tau_1, \tau_2; \lambda_3^{[\tau_1, \tau_2]} \right) = \frac{1}{4} \left[ 1 - E_S(\tau_1) \right] E_L(\tau_1) [p_{43}(\tau_2 | \tau_1) + p_{44}(\tau_2 | \tau_1)],
\]

\[
P_5 \left( \tau_1, \tau_2; \lambda_5^{[\tau_1, \tau_2]} \right) = \frac{1}{4} E_S(\tau_1) E_L(\tau_1) p_{33}(\tau_2 | \tau_1),
\]

\[
P_7 \left( \tau_1, \tau_2; \lambda_7^{[\tau_1, \tau_2]} \right) = \frac{1}{4} \left[ 1 - E_S(\tau_1) \right] E_L(\tau_1) [p_{33}(\tau_2 | \tau_1) + p_{34}(\tau_2 | \tau_1)],
\]

\[
P_8 \left( \tau_1, \tau_2; \lambda_8^{[\tau_1, \tau_2]} \right) = \frac{1}{4} E_S(\tau_1) E_L(\tau_1) p_{43}(\tau_2 | \tau_1),
\]

\[
P_9 \left( \tau_1, \tau_2; \lambda_9^{[\tau_1, \tau_2]} \right) = \frac{1}{4} E_S(\tau_1) E_L(\tau_1) p_{22}(\tau_2 | \tau_1),
\]

\[
P_{10} \left( \tau_1, \tau_2; \lambda_{10}^{[\tau_1, \tau_2]} \right) = P_{14} \left( \tau_1, \tau_2; \lambda_{14}^{[\tau_1, \tau_2]} \right) = \frac{1}{4} E_L(\tau_1) [1 - E_S(\tau_2)],
\]

\[
P_{11} \left( \tau_1, \tau_2; \lambda_{11}^{[\tau_1, \tau_2]} \right) = \frac{1}{4} E_S(\tau_1) [1 - E_L(\tau_1)] [p_{21}(\tau_2 | \tau_1) + p_{22}(\tau_2 | \tau_1)],
\]

\[
P_{12} \left( \tau_1, \tau_2; \lambda_{12}^{[\tau_1, \tau_2]} \right) = \frac{1}{4} E_S(\tau_1) E_L(\tau_1) p_{12}(\tau_2 | \tau_1),
\]

\[
P_{13} \left( \tau_1, \tau_2; \lambda_{13}^{[\tau_1, \tau_2]} \right) = \frac{1}{4} E_S(\tau_1) E_L(\tau_1) p_{11}(\tau_2 | \tau_1),
\]

\[
P_{15} \left( \tau_1, \tau_2; \lambda_{15}^{[\tau_1, \tau_2]} \right) = \frac{1}{4} E_S(\tau_1) [1 - E_L(\tau_1)] [p_{11}(\tau_2 | \tau_1) + p_{12}(\tau_2 | \tau_1)],
\]

\[
P_{16} \left( \tau_1, \tau_2; \lambda_{16}^{[\tau_1, \tau_2]} \right) = \frac{1}{4} E_S(\tau_1) E_L(\tau_1) p_{21}(\tau_2 | \tau_1),
\]

\[
P_{17} \left( \tau_1, \tau_2; \lambda_{17}^{[\tau_1, \tau_2]} \right) = \frac{1}{2} [1 - E_S(\tau_1)] [1 - E_L(\tau_2)],
\]

\[
P_{18} \left( \tau_1, \tau_2; \lambda_{18}^{[\tau_1, \tau_2]} \right) = \frac{1}{2} [1 - E_L(\tau_1)] [1 - E_S(\tau_2)].
\]

The description of eqs. (4.28) and (4.29) and table [II] corresponds to the most general hidden–variable theory. Actually, the local realistic probabilities of eqs. (4.31), (4.32) must be interpreted by means of equations like (4.26), namely they contain elements of randomness related both to apparata hidden–variables and, in general, to other unknown mechanisms:

\[
P_i(\tau_1, \tau_2; \lambda_i^{[\tau_1, \tau_2]}) = P_i(\tau_1, \tau_2 | \lambda_i^{[\tau_1, \tau_2]} \rho(\lambda_i^{[\tau_1, \tau_2]}), \text{ where } 0 \leq P_i(\tau_1, \tau_2 | \lambda_i^{[\tau_1, \tau_2]} \leq 1.
\]

When \( \tau_1 = \tau_2 = 0 \), only the probabilities for the four states of table [II]:

\[
P_1 \left( 0, 0; \lambda_1^{[0,0]} \right) = P_5 \left( 0, 0; \lambda_5^{[0,0]} \right) = P_9 \left( 0, 0; \lambda_9^{[0,0]} \right) = P_{13} \left( 0, 0; \lambda_{13}^{[0,0]} \right) = \frac{1}{4}, \tag{4.33}
\]
are non–vanishing. Moreover, for \( \tau_1 = \tau_2 \equiv \tau \neq 0 \), four probabilities of our set are still zero:

\[
P_4(\tau, \tau; \lambda_4^{[\tau, \tau]}) = P_8(\tau, \tau; \lambda_8^{[\tau, \tau]}) = P_{12}(\tau, \tau; \lambda_{12}^{[\tau, \tau]}) = P_{16}(\tau, \tau; \lambda_{16}^{[\tau, \tau]}) = 0, \tag{4.34}
\]

because of the requirement of perfect anti–correlation on strangeness at equal times.

Consider now the contribution to \( P_1(\tau, \tau; \lambda_1^{[\tau, \tau]} \rangle \) coming from the transitions \( K_1(0) \rightarrow K_1(\tau) \) on the left and \( K_4(0) \rightarrow K_4(\tau) \) on the right. It can be written in the following two equivalent ways: 1) the probability that the left going kaon is created in the state \( K \) and is then subject to the transition \( K_1(0) \rightarrow K_1(\tau) \) is \( p_{11}(\tau | 0) / 4 \); in order to obtain the required probability we have to multiply this quantity by the probability \( E_L(\tau) \) that the right going kaon at time \( \tau \), that is correlated with the left going \( K_1 \), is an undecayed \( K_4 \); 2) the probability that the right going kaon is created in the state \( K_4 \) and is then subject to the transition \( K_4(0) \rightarrow K_4(\tau) \) is \( p_{44}(\tau | 0) / 4 \); to obtain the required probability we have to multiply this quantity by the probability \( E_S(\tau) \) that the left going kaon at time \( \tau \) is an undecayed \( K_1 \). Therefore, the following equality is valid:

\[
p_{11}(\tau | 0) E_L(\tau) = p_{44}(\tau | 0) E_S(\tau), \tag{4.35}
\]

and from eq. (4.18) we obtain that it is verified only when \( \delta(\tau) \equiv 0 \). This property can also be proved starting from the other probabilities of eqs. (4.31), (4.32) which are non–vanishing when \( \tau_1 = \tau_2 \).

The independent observables relevant for the problem [given, in quantum mechanics, by eqs. (3.3)–(3.6)] can be written, within local realism, as follows:

\[
P_{LR}[K^0(\tau_1), \bar{K}^0(\tau_2)] \equiv P_1(\tau_1, \tau_2; \lambda_1^{[\tau_1, \tau_2]}) + P_9(\tau_1, \tau_2; \lambda_9^{[\tau_1, \tau_2]})
\]

\[
= \frac{1}{4} E_S(\tau_1) E_L(\tau_1) [p_{22}(\tau_2 | \tau_1) + p_{44}(\tau_2 | \tau_1)],
\]

\[
P_{LR}[K^0(\tau_1), K^0(\tau_2)] \equiv P_5(\tau_1, \tau_2; \lambda_5^{[\tau_1, \tau_2]}) + P_{13}(\tau_1, \tau_2; \lambda_{13}^{[\tau_1, \tau_2]})
\]

\[
= \frac{1}{4} E_S(\tau_1) E_L(\tau_1) [p_{11}(\tau_2 | \tau_1) + p_{33}(\tau_2 | \tau_1)],
\]

\[
P_{LR}[K^0(\tau_1), \bar{K}^0(\tau_2)] \equiv P_4(\tau_1, \tau_2; \lambda_4^{[\tau_1, \tau_2]}) + P_{12}(\tau_1, \tau_2; \lambda_{12}^{[\tau_1, \tau_2]})
\]

\[
= \frac{1}{4} E_S(\tau_1) E_L(\tau_1) [p_{12}(\tau_2 | \tau_1) + p_{34}(\tau_2 | \tau_1)],
\]

\[
P_{LR}[\bar{K}^0(\tau_1), \bar{K}^0(\tau_2)] \equiv P_8(\tau_1, \tau_2; \lambda_8^{[\tau_1, \tau_2]}) + P_{16}(\tau_1, \tau_2; \lambda_{16}^{[\tau_1, \tau_2]})
\]

\[
= \frac{1}{4} E_S(\tau_1) E_L(\tau_1) [p_{21}(\tau_2 | \tau_1) + p_{43}(\tau_2 | \tau_1)],
\]

\[
P_{LR}[K_L(\tau_1), K_9(\tau_2)] \equiv P_9(\tau_1, \tau_2; \lambda_9^{[\tau_1, \tau_2]}) + P_{12}(\tau_1, \tau_2; \lambda_{12}^{[\tau_1, \tau_2]})
\]

\[
+ P_{13}(\tau_1, \tau_2; \lambda_{13}^{[\tau_1, \tau_2]}) + P_{16}(\tau_1, \tau_2; \lambda_{16}^{[\tau_1, \tau_2]})
\]

\[
= \frac{1}{4} E_S(\tau_1) E_L(\tau_1) [p_{11}(\tau_2 | \tau_1) + p_{12}(\tau_2 | \tau_1) + p_{21}(\tau_2 | \tau_1) + p_{22}(\tau_2 | \tau_1)],
\]

\[
P_{LR}[K_S(\tau_1), K_9(\tau_2)] \equiv P_1(\tau_1, \tau_2; \lambda_1^{[\tau_1, \tau_2]}) + P_{14}(\tau_1, \tau_2; \lambda_{14}^{[\tau_1, \tau_2]})
\]
\[ + P_5 (\tau_1, \tau_2; \lambda_5^{[\tau_1, \tau_2]}) + P_8 (\tau_1, \tau_2; \lambda_8^{[\tau_1, \tau_2]}) \]
\[ = \frac{1}{4} E_S (\tau_1) E_L (\tau_1) [p_{32} (\tau_2 | \tau_1) + p_{34} (\tau_2 | \tau_1) + p_{43} (\tau_2 | \tau_1) + p_{41} (\tau_2 | \tau_1)]. \]

The probabilities of eqs. (3.8) and (3.9) do not supply new information since they are not independent of the other ones just considered, whereas eq. (3.7), which ensures CP conservation, was assumed, in section II A and then in table III, when we introduced local realism for the two–kaon system.

In order to determine the observables of eq. (4.36), we now ask whether it is possible to derive useful relations among the \( p_{ij} (\tau_2 | \tau_1) \)'s and the probabilities \( p_{ij} (\tau | 0) \) of matrix (4.18). By introducing three–time probabilities:

\[ p_{ijk} (\tau_2, \tau_1, 0) = p_{ijk} (\tau_2, \tau_1 | 0) p_k (0) = p_{ijk} (\tau_2 | \tau_1, 0) p_{j} (\tau_1 | 0) p_k (0), \]

and using the multiplication theorem, \( p_{11} (\tau_2 | \tau_1), p_{12} (\tau_2 | \tau_1), p_{21} (\tau_2 | \tau_1) \) and \( p_{22} (\tau_2 | \tau_1) \) can be written as follows:

\[ p_{11} (\tau_2, \tau_1) \equiv p_{11} (\tau_2 | \tau_1) p_1 (\tau_1) = \frac{1}{4} \left[ p_{111} (\tau_2, \tau_1 | 0) + p_{112} (\tau_2, \tau_1 | 0) \right], \]

\[ p_{12} (\tau_2, \tau_1) \equiv p_{12} (\tau_2 | \tau_1) p_2 (\tau_1) = \frac{1}{4} \left[ p_{121} (\tau_2, \tau_1 | 0) + p_{122} (\tau_2, \tau_1 | 0) \right], \]

\[ p_{21} (\tau_2, \tau_1) \equiv p_{21} (\tau_2 | \tau_1) p_1 (\tau_1) = \frac{1}{4} \left[ p_{211} (\tau_2, \tau_1 | 0) + p_{212} (\tau_2, \tau_1 | 0) \right], \]

\[ p_{22} (\tau_2, \tau_1) \equiv p_{22} (\tau_2 | \tau_1) p_2 (\tau_1) = \frac{1}{4} \left[ p_{221} (\tau_2, \tau_1 | 0) + p_{222} (\tau_2, \tau_1 | 0) \right]. \]

In the previous relations, \( p_1 (\tau_1) = [p_{11} (\tau_1) + p_{12} (\tau_1)] / 4 = E_S (\tau_1) / 4 \) and \( p_2 (\tau_1) = [p_{21} (\tau_1) + p_{22} (\tau_1)] / 4 = E_L (\tau_1) / 4 \) is the probability to observe a \( K_1 \) \( (K_2) \) along direction 1 at time \( \tau_1 \) (analogous relations are valid for \( CP = -1 \) states, and \( p_3 (\tau) = p_4 (\tau) = E_L (\tau) / 4 \) the \( p_{ijk} (\tau | 0) \)'s are given in eq. (4.18) with \( \delta (\tau) \equiv 0 \), whereas \( p_{ij} (\tau_2, \tau_1) \) denote standard (namely non–conditional) two–times probabilities. Moreover, \( p_{ijk} (\tau_2, \tau_1, 0) \) is the probability to have states \( K_k, K_j \) and \( K_i \) at times 0, \( \tau_1 \) and \( \tau_2 \), respectively, \( p_{ijk} (\tau_2, \tau_1 | 0) \) is the probability that at times \( \tau_1 \) and \( \tau_2 \) the states are \( K_j \) and \( K_i \), respectively, if the state at time 0 was \( K_k \), and, finally, \( p_{ijk} (\tau_2 | \tau_1, 0) \) is the probability of a \( K_i \) at time \( \tau_2 \) conditional on the presence of a \( K_k \) at time 0 and a \( K_j \) at \( \tau_1 \). It is then clear that, in eqs. (4.38)–(4.41), the two–times probabilities \( p_{ij} (\tau_2, \tau_1) \) are obtained by summing over the possible states appearing at time \( \tau = 0 \).

Let us now consider probability \( p_{11} (\tau_2 | 0) \). Introduce a time \( \tau_1 \) in the interval \([0, \tau_2]\): at instant \( \tau_1 \) the state can be either a \( K_1 \) or a \( K_2 \), then the contributions to \( p_{11} (\tau_2 | 0) \) come from two transitions with different intermediate state. They are \( K_1 (0) \rightarrow K_1 (\tau_1) \rightarrow K_1 (\tau_2) \) and \( K_1 (0) \rightarrow K_2 (\tau_1) \rightarrow K_1 (\tau_2) \), thus:

\[ p_{11} (\tau_2 | 0) = p_{111} (\tau_2, \tau_1 | 0) + p_{121} (\tau_2, \tau_1 | 0), \]

for any \( \tau_1 \in [0, \tau_2] \). Limiting again the discussion to probabilities relevant for the evolution of \( CP = +1 \) states, one obtains the remaining relations:
\begin{align}
p_{12}(\tau_2|0) &= p_{112}(\tau_2, \tau_1|0) + p_{122}(\tau_2, \tau_1|0), \quad (4.43) \\
p_{21}(\tau_2|0) &= p_{211}(\tau_2, \tau_1|0) + p_{221}(\tau_2, \tau_1|0), \quad (4.44) \\
p_{22}(\tau_2|0) &= p_{212}(\tau_2, \tau_1|0) + p_{222}(\tau_2, \tau_1|0). \quad (4.45)
\end{align}

Now, the sum of two three–times probabilities corresponding to the same states at times 0 and \(\tau_1\) but with different states at \(\tau_2\) provides a known result; in fact:

\begin{align}
p_{111}(\tau_2|\tau_1, 0) + p_{211}(\tau_2|\tau_1, 0) &= E_S(\tau_2 - \tau_1), \quad (4.46) \\
p_{112}(\tau_2|\tau_1, 0) + p_{212}(\tau_2|\tau_1, 0) &= E_S(\tau_2 - \tau_1), \quad (4.47) \\
p_{121}(\tau_2|\tau_1, 0) + p_{221}(\tau_2|\tau_1, 0) &= E_S(\tau_2 - \tau_1), \quad (4.48) \\
p_{122}(\tau_2|\tau_1, 0) + p_{222}(\tau_2|\tau_1, 0) &= E_S(\tau_2 - \tau_1). \quad (4.49)
\end{align}

Each of these equalities accounts for the contributions to transitions into final states \(K_1\) and \(K_2\) once the kaonic states at times 0 and \(\tau_1\) are fixed: these probabilities equal the probability \(E_S(\tau_2 - \tau_1)\) that a \(CP = +1\) kaon does not decay during the time interval between \(\tau_1\) and \(\tau_2\).

By using the shorthand notation \(p_{ijk} \equiv p_{ijk}(\tau_2, \tau_1|0)\), it follows from (4.18) that the above equations (4.42)–(4.49) can be written in the equivalent form:

\begin{align}
p_{111} + p_{121} &= p_{222} + p_{212} = E_S(\tau_2)Q_+(\tau_2) \quad (4.50) \\
p_{112} + p_{122} &= p_{221} + p_{211} = E_S(\tau_2)Q_-(\tau_2) \quad (4.51) \\
p_{111} + p_{211} &= p_{222} + p_{122} = E_S(\tau_2)Q_+(\tau_1) \quad (4.52) \\
p_{112} + p_{212} &= p_{221} + p_{121} = E_S(\tau_2)Q_-(\tau_1). \quad (4.53)
\end{align}

These conditions on the 8 \(CP = +1\) three–times probabilities supplies two system of equations:

\begin{align}
\begin{cases}
p_{111} + p_{121} &= E_S(\tau_2)Q_+(\tau_2) \\
p_{221} + p_{211} &= E_S(\tau_2)Q_-(\tau_2) \\
p_{111} + p_{211} &= E_S(\tau_2)Q_+(\tau_1) 
\end{cases} \quad (4.54)
\end{align}

\begin{align}
\begin{cases}
p_{222} + p_{212} &= E_S(\tau_2)Q_+(\tau_2) \\
p_{112} + p_{122} &= E_S(\tau_2)Q_-(\tau_2) \\
p_{222} + p_{122} &= E_S(\tau_2)Q_+(\tau_1)
\end{cases} \quad (4.55)
\end{align}

each containing three independent conditions and four unknown probabilities.

From previous results one obtains:

\begin{align}
p_{11}(\tau_2|\tau_1) + p_{12}(\tau_2|\tau_1) &= \frac{1}{4} \left[ \frac{p_{111} + p_{112}}{p_{1}(\tau_1)} + \frac{p_{121} + p_{122}}{p_{2}(\tau_1)} \right] = E_S(\tau_2 - \tau_1). \quad (4.56)
\end{align}

Analogously:
\[ p_{21}(\tau_2 | \tau_1) + p_{22}(\tau_2 | \tau_1) = p_{11}(\tau_2 | \tau_1) + p_{21}(\tau_2 | \tau_1) = p_{12}(\tau_2 | \tau_1) + p_{22}(\tau_2 | \tau_1) = E_S(\tau_2 - \tau_1), \]

thus:

\[ p_{21}(\tau_2 | \tau_1) = p_{12}(\tau_2 | \tau_1), \]  \hspace{1cm} (4.58)

\[ p_{22}(\tau_2 | \tau_1) = p_{11}(\tau_2 | \tau_1). \]  \hspace{1cm} (4.59)

Exactly the same derivation can be repeated for the \( CP = -1 \) probabilities: the relations valid in this case are obtained from (4.38)–(4.59) simply by replacing \( E \) with \( E_L \) and \( 1 \rightarrow 3, 2 \rightarrow 4 \) for the state indexes. Obviously, the normalization of the local realistic probabilities (4.31) and (4.32), \( \sum_{i=1}^{18} P_i(\tau_1, \tau_2; \lambda^{[\tau_1, \tau_2]}) = 1 \), is automatically ensured by the above results.

From eqs. (4.36) and previous analysis one thus obtains the following expression for the observables within the local realistic approach:

\[
P_{LR}[K^0(\tau_1), K^0(\tau_2)] = P_{LR}[\bar{K}^0(\tau_1), K^0(\tau_2)] = \frac{1}{8}[E_S(\tau_1)E_L(\tau_2) + E_L(\tau_1)E_S(\tau_2)]
\times[1 + A_{LR}(\tau_1, \tau_2)],
\]

\[
P_{LR}[K^0(\tau_1), K^0(\tau_2)] = P_{LR}[\bar{K}^0(\tau_1), K^0(\tau_2)] = \frac{1}{8}[E_S(\tau_1)E_L(\tau_2) + E_L(\tau_1)E_S(\tau_2)]
\times[1 - A_{LR}(\tau_1, \tau_2)],
\]

\[
P_{LR}[K_L(\tau_1), K_S(\tau_2)] = \frac{1}{4}E_L(\tau_1)E_S(\tau_2),
\]

\[
P_{LR}[K_S(\tau_1), K_L(\tau_2)] = \frac{1}{4}E_S(\tau_1)E_L(\tau_2),
\]

written directly in terms of the asymmetry parameter [see definition (3.12)]:

\[
A_{LR}(\tau_1, \tau_2) = 2 \frac{[p_{11}(\tau_2 | \tau_1) + p_{33}(\tau_2 | \tau_1)]}{E_S(\tau_2 - \tau_1) + E_L(\tau_2 - \tau_1)} - 1
\]

\[= 2 \left( \frac{p_{11} + p_{12}}{E_S(\tau_1)} + \frac{(p_{333} + p_{334})}{E_L(\tau_1)} \right) / \frac{E_S(\tau_2 - \tau_1) + E_L(\tau_2 - \tau_1)} - 1, \]

where, compatibly with constraints (1.54) and (1.55), the three–times probabilities can vary in the following intervals:

\[
\text{Max}\{0; Q_+(\tau_2) - Q_-(\tau_1)\} \leq \frac{p_{11}}{E_S(\tau_2)}; \quad \frac{p_{333}}{E_L(\tau_2)} \leq \text{Min}\{Q_+(\tau_1); Q_+(\tau_2)\}, \qquad (4.65)
\]

\[
\text{Max}\{0; Q_-(\tau_1) - Q_+(\tau_2)\} \leq \frac{p_{112}}{E_S(\tau_2)}; \quad \frac{p_{334}}{E_L(\tau_2)} \leq \text{Min}\{Q_-(\tau_1); Q_-(\tau_2)\}.
\]

**V. COMPATIBILITY BETWEEN LOCAL REALISM AND QUANTUM MECHANICS**

From eqs. (4.29), (4.31), (4.32), (4.56), and (4.57) we obtain that local realism reproduces the single kaon quantum–mechanical expectation values:
This result is of general validity: for all EPR–like particle pairs it is always possible to take into account of the single particle observables by employing a local hidden–variable model.

Results (4.62) and (4.63) reproduce the quantum–mechanical predictions (3.5) and (3.6); it is easy to see that expectation values (3.8) and (3.9) are obtained too. The same conclusion would be true for the joint observables (4.60) and (4.61) involving $K_S–K_L$ mixing if the time–dependent local realistic asymmetry parameter had the same expression it has in quantum mechanics. Thus:

Local Realism equivalent to Quantum Mechanics $\iff A_{LR}(\tau_1, \tau_2) \equiv A_{QM}(\tau_1, \tau_2).

From eqs (4.64) and (4.65) it follows that the asymmetry corresponding to the most general local realistic theory satisfies the following inequality:

$$2|Q_+(\tau_2) – Q_-(\tau_1)| – 1 \leq A_{LR}(\tau_1, \tau_2) \leq 1 – 2|Q_+(\tau_2) – Q_+(\tau_1)|.$$  (5.3)

If one considers the special case in which $\tau_2 = \tau_1 \equiv \tau$, local realism is compatible with quantum mechanics: in fact, $A_{LR}^{\text{Min}}(\tau, \tau) \leq A_{QM}(\tau, \tau) = A_{LR}^{\text{Max}}(\tau, \tau) \equiv 1$ for all times. This is obvious, since within the class of local realistic theories supplying asymmetry parameters in the interval (5.3), a model that reproduces the perfect anti–correlation properties of the two–kaon state at equal times must exist. When $\tau_1 = 0$, both descriptions supplies the same asymmetry: $A_{LR}(0, \tau) = A_{QM}(0, \tau) \equiv Q_+(\tau) – Q_-(\tau)$. Another special case is when, for instance, $\tau_2 = 1.5\tau_1$: in this situation, the local realistic asymmetry does not satisfy the compatibility requirement (5.2). This is depicted in figure 1. The maximum values of the local realistic asymmetry stands below the quantum–mechanical ones for $0 < \tau_1 \lesssim 2.3\tau_S$. The largest incompatibility corresponds to $\tau_1 \simeq 1.5\tau_S$, where $[A_{QM} – A_{LR}^{\text{Max}}]/A_{QM} \simeq 20\%$. In general, local realism and quantum mechanics are incompatible when $\tau_2 = \alpha\tau_1$ with $\alpha > 1$. The degree of incompatibility increases for increasing $\alpha$. For instance, when $\tau_2 = 2\tau_1 \simeq 2.4\tau_S$, $A_{QM}$ is 27 \% larger than $A_{LR}^{\text{Max}}$. The large differences among quantum–mechanical and local realistic predictions justify our approach, which neglected CP violation.

However, it is important to stress the following restriction concerning the choice (which must be at free will) of the detection times $\tau_1$ and $\tau_2$. In order to satisfy the locality condition, namely to make sure that the measurement event on the right is causally disconnected from that on the left, these events must be space–like separated. For a two–kaon system in which
the kaons fly back–to–back in the laboratory frame system, this requirement corresponds to choose detection times which satisfy the inequality [27]:

\[ 1 \leq \frac{\tau_2}{\tau_1} < \frac{1 + v}{1 - v} = 1.55, \]  

(5.4)

where \( v \approx 0.22 \) is the kaon velocity (in units of \( c \)) in the laboratory frame system. Nevertheless, concerning the locality assumption, a loophole that is impossible to avoid could allow, in principle (it is completely unknown, however, in which way), an information to reach both devices at the instants of measurement, whatever the choice of these times is. In fact, events in the overlap region of the two backward light–cones corresponding to the measurements at \( \tau_1 \) and \( \tau_2 \) might be responsible for the choice of the times \( \tau_1 \) and \( \tau_2 \) as well as for the experimental outcomes. If this were the actual case, even for causally disconnected measurement events one could not infer that the non–occurrence of action–at–a–distance implies locality. Thus, a non–local behaviour of microscopic phenomena could be still compatible with relativistic causality.

An experiment that measured the asymmetry parameter was performed by the CPLEAR collaboration at CERN [10]. The \( K^0 \bar{K}^0 \) pairs were produced by proton–antiproton annihilation at rest, while the kaon strangeness was detected through kaon strong interactions with bound nucleons of absorber materials. The data, corrected for a comparison with pure
quantum–mechanical predictions [eq. (3.13)], are reported in table IV. The temporal uncertainty of data is not considered here. Asymmetry values compatible with local realism depend on the detection times $\tau_1$ and $\tau_2$ separately: the CPLEAR set–up corresponds to the following corrected times: $\tau_1 = \tau_2 = 0.55\tau_S$ when $\tau_2 - \tau_1 = 0$ and $\tau_1 = 0.55\tau_S$, $\tau_2 = 1.92\tau_S$ when $\tau_2 - \tau_1 = 1.37\tau_S$. We notice that also in the second case the two observation events were space–like separated: in fact, $\tau_2/\tau_1 = 3.5$ (also when uncorrected times are considered), and, since the kaon velocity in the center of mass system for $p\bar{p} \rightarrow K^0\bar{K}^0$ is $v \simeq 0.85$, condition (5.4) gives $1 \leq \tau_2/\tau_1 < 12.2$. It is evident from table IV that the data are in agreement, within one standard deviation, with both quantum mechanics and local realism. For a decisive test of local realistic theories more precise data are needed.

In agreement with Bell’s theorem, in this section we have seen that local realism contradicts some statistical predictions of quantum mechanics concerning the evolution of the two–neutral–kaon system. Local realism has already been tested against quantum mechanics (by employing Bell–type inequalities) in optics and atomic physics: neglecting existing loopholes, apart form some irrelevant exception, all the experimental results revealed incompatible with the local realistic viewpoint and were in good agreement with quantum mechanics. For the two–kaon correlated system one avoids the detection loophole and, in particular situations, the differences among the predictions of local realism and quantum mechanics are so evident that a future measurement at the Frascati $\Phi$–factory (say for $\tau_2 = 1.5\tau_1$, with $\tau_1$ around $1.5\tau_S$) should be able to confirm one of the two pictures.

### VI. ON THE POSSIBILITY TO TEST LOCAL REALISM WITH BELL’S INEQUALITIES FOR THE TWO–NEUTRAL–KAON SYSTEM

When $CP$ non–conservation is taken into account, a Bell’s inequality violated by quantum mechanics has been derived in the special case of a gedanken experiment [28]. Unfortunately, the magnitude of violation of this inequality is very small, of the order of the $K^0–\bar{K}^0$ $CP$ violating parameter, $\epsilon$, thus representing a problem from the experimental point of view. A similar inequality, which is violated by a non–vanishing value of the direct $CP$ and $CPT$ violating parameter, $\epsilon'$, is discussed in ref. [29]. Moreover, experimental set–up exploiting $K_S–K_L$ regeneration processes have also been proposed in order to formulate Bell’s inequalities that show incompatibilities with some statistical predictions of quantum theory [26,27,30]. Unfortunately, in order to avoid a tiny violation of the inequalities that one ob-

| Time difference: $\tau_2 - \tau_1$ | Experiment | Quantum Mechanics | Local Realism |
|----------------------------------|------------|------------------|--------------|
| 0                                | 0.88 ± 0.17 | 1                | 0.86 ± 1     |
| $1.37\tau_S$                     | 0.56 ± 0.12 | 0.64             | 0.34 ± 0.48  |

TABLE IV. Asymmetry parameter measured by CPLEAR collaboration [10].
tains for thin regenerators, this kind of Bell–type test requires large amount of regenerator materials. Moreover, the test proposed in ref. [26] can be performed only at asymmetric Φ–factories.

In ref. [25] the authors concluded that, under the hypothesis of CP conservation, because of the specific properties of the kaon, it is impossible to test local realism by using Bell’s inequalities, since whatever inequality one considers, a violation by quantum–mechanical expectation values cannot be found. In this section we consider again this question in order to prove how such a test is actually feasible with Wigner’s inequalities [16]. Moreover, in agreement with the discussion of ref. [32], we shall also show that a Bell–test is possible when properly normalized observables and Clauser–Horne–Shimony–Holt’s (CHSH’s) inequalities [15,17] are employed.

The two–kaon system presents some analogies but also a significant difference compared to the case of the singlet state of two spin–1/2 particles (2.1). The (free) choice of the times τ_1 and τ_2 at which a strangeness measurements on the kaon pair (2.7) is performed is analogous to the (free) choice of the orientation along which the spin is observed in the case of the singlet state (2.1). If we consider the ideal limit in which the weak interaction eigenstates K_S and K_L are stable (Γ_L = Γ_S = 0), quantum–mechanical kaon probabilities (3.3) and (3.4) have exactly the same expressions (proportional to 1 ± cos θ_{12}) of the spin–singlet case, provided one replaces the angle between the two spin analyzers with θ_{12} = ∆m(τ_2 − τ_1). Then, a strangeness measurement on the two–kaon system is perfectly equivalent to a spin measurement on the singlet state (2.1). It is then obvious that if the above hypothesis were realized in Nature, one could find violations of Bell’s inequalities of the same magnitude of the ones that characterize the spin system.

However, this hypothesis is far from being realistic, and the kaon joint probabilities decreases with time because of the K_S and K_L weak decays. This leads to an important difference with respect to the spin case. Because of the particular values of the kaon lifetimes (Γ_S and Γ_L) and of the quantity ∆m ≡ m_L − m_S, which controls the quantum–mechanical interference term (i.e., the K_S–K_L mixing), ref. [25] concluded that no choice of the detection times is able to show a violation, by quantum mechanics, of Bell’s inequalities. The authors of ref. [25] reached this conclusion on the basis of CHSH’s inequalities.

Actually, the interplay between kaon exponential damping and strangeness oscillations only makes it more difficult (but not impossible) a Bell–type test. The reason of this behaviour lies in the very short K_S lifetime (τ_S) compared with the typical time (2π/∆m ≃ 13τ_S) of the strangeness oscillations. The situation would be different (namely the discrimination between quantum mechanics and local realistic theories would be easier) if one treated the B^0–B^0 system (this is due to the fact that the states analogous to K_S and K_L for the B^0–meson have the same lifetime).

We start discussing Wigner’s inequalities. We must recall that these inequalities are derivable for deterministic theories only, therefore they are less general than CHSH’s. Let us consider the following Wigner’s inequality involving K_S–K_L mixing:
\( P_{LR}[\bar{K}^0(\tau_1), \bar{K}^0(\tau_2)] \leq P_{LR}[\bar{K}^0(\tau_1), \bar{K}^0(\tau_3)] + P_{LR}[\bar{K}^0(\tau_3), \bar{K}^0(\tau_2)], \) \hspace{1cm} (6.1)

where \( \tau_1 \leq \tau_3 \leq \tau_2 \). It has been written for \( \bar{K}^0 \bar{K}^0 \) joint detection since \( \bar{K}^0 \bar{K}^0 \) states are easier to detect than \( K^0 \) states. Obviously, the same conclusions that we shall obtain in the following are valid for the inequality corresponding to \( K^0 \bar{K}^0 \) detection. Inequalities that contain \( K^0 \bar{K}^0 \) joint probabilities turn out to be useless for Bell-type tests.

In the limit \( \Gamma_S = \Gamma_L = 0 \), eq. (6.1) reduces to the analogous inequality for the spin–singlet case:

\[
P_{LR}(s_a = -, s_b = -) \leq P_{LR}(s_a = -, s_c = -) + P_{LR}(s_c = -, s_b = -). \hspace{1cm} (6.2)
\]

Since:

\[
P_{QM}(s_a = -, s_b = -) = \frac{1}{4}(1 - \cos \theta_{\alpha\beta}), \hspace{1cm} (6.3)
\]

\( \theta_{\alpha\beta} \) being the angle between the spin measurement directions characterized by the unitary vectors \( \vec{\alpha} \) and \( \vec{\beta} \), inequality (6.2) is violated by quantum mechanics when one chooses \( \theta_{ab} = 2\theta_{ac} = 2\theta_{cb} \equiv 2\theta \) with \( \theta \) in the interval \([0, \pi/2]\) (see figure 2). The greatest violation of eq. (6.2) (0.375 > 0.250) is for \( \theta = \pi/3 \) and is significant, since it corresponds to \( P_{QM}(s_a = -, s_b = -) = 0.375 \) and \( P_{QM}(s_a = -, s_c = -) = P_{QM}(s_c = -, s_b = -) = 0.125 \).

![Image of a graph titled WQM(θ)](image)

FIG. 2. Violation of Wigner’s inequality (6.2) for the spin–singlet state (2.1). The function \( W_{QM}(\theta) \equiv P_{QM}(s_a = -, s_b = -) - P_{QM}(s_a = -, s_c = -) - P_{QM}(s_c = -, s_b = -) \) is plotted versus \( \theta \) \( (\theta \equiv \theta_{ab}/2 = \theta_{ac} = \theta_{cb}). \) The inequality is violated by quantum mechanics when \( \theta \) is in the interval \([0, \pi/2]\).

Coming back to the two–kaon system in the real case with \( \Gamma_S/\Gamma_L \approx 579 \), we must require the three detection times of inequality (6.1) to satisfy restriction (5.4), dictated by the necessity to avoid any causal connection between the measurements that could be present if the two observation events would not be space–like separated. By introducing the relation:
\[ \tau_2 - \tau_1 = 2(\tau_3 - \tau_1) = 2(\tau_2 - \tau_3) \equiv (p - 1)\tau \]  

(6.4) among the observation times and choosing \( \tau_1 = \tau \), one obtains \( \tau_2 = p\tau \) and \( \tau_3 = (p + 1)\tau/2 \), and the locality requirement \((5.4)\) is fulfilled when \( 1 \leq p < 1.55 \). Apart from the case with \( p = 1 \), for all values of \( p \) in this range, inequality \((6.1)\) is incompatible with quantum mechanics at small \( \tau \): in figure 3 this is shown for \( p = 1.5, 1.3, \) and 1.1. In the case with \( p = 1.5 \), the largest violation of eq. \((6.1)\) \((0.0051 > 0.0026)\) corresponds to \( \tau \simeq 1.6\tau_S \), \( P_{QM}[\bar{K}^0(\tau_1), \bar{K}^0(\tau_2)] = 0.0051, P_{QM}[\bar{K}^0(\tau_1), \bar{K}^0(\tau_3)] = 0.0016 \) and \( P_{QM}[\bar{K}^0(\tau_3), \bar{K}^0(\tau_2)] = 0.0010 \). An experimental test of a tiny violation like this one is difficult to perform since, relatively to the typical experimental accuracy, the function \( W_{QM}(\tau) \) of figure 3 is too close to 0 in the region of maximum violation. In particular, a precision like that of the CPLEAR experiment (see table IV) is insufficient to achieve this purpose, since \( W(\tau) \) would be measured with an error larger than the maximum violation shown in figure 3.

When one considers joint probabilities normalized to undecayed kaon pairs:

\[
P[\bar{K}^0(\tau), \bar{K}^0(\tau')] \rightarrow P^{\text{ren}}[\bar{K}^0(\tau), \bar{K}^0(\tau')] \equiv \frac{P[\bar{K}^0(\tau), \bar{K}^0(\tau')]}{P[-(\tau), -(\tau')]} = \frac{1}{4} [1 - A(\tau, \tau')],
\]

(6.5) since these quantities are less damped than the original ones, a Bell–type test can be performed also with CHSH’s inequalities. In the previous equation, the probability that at times \( \tau \) (on the left) and \( \tau' \) (on the right) both kaons are undecayed is:
\[ P[-(\tau), -(\tau')] = P[\bar{K}^0(\tau), \bar{K}^0(\tau')] + P[K^0(\tau), \bar{K}^0(\tau')] + P[K^0(\tau), K^0(\tau')] \]
\[ + P[K^0(\tau), K^0(\tau')] = \frac{1}{2} [E_S(\tau)E_L(\tau') + E_L(\tau)E_S(\tau')] \]  

(6.6)

the last equality being valid both in the local realistic description [eqs. (5.10), (5.11)] and in quantum mechanics [eqs. (3.3), eqs. (3.4)], since it is independent of the \(K^0-\bar{K}^0\) oscillations. The same derivation that supplies the CHSH’s inequality in the unrenormalized case can be applied to the renormalized observables of eq. (5.3). By introducing four detection times \((\tau_1\text{ and } \tau_2\text{ for the left going meson, } \tau_3\text{ and } \tau_4\text{ for the right going meson})\), the CHSH’s inequality for strangeness \(-1\text{ detection is then:} \]
\[ -1 \leq S_{LR}(\tau_1, \tau_2, \tau_3, \tau_4) \leq 0, \]  

(6.7)

with:
\[ S_{LR}(\tau_1, \tau_2, \tau_3, \tau_4) \equiv P_{LR}^{\text{ren}}[\bar{K}^0(\tau_1), \bar{K}^0(\tau_3)] - P_{LR}^{\text{ren}}[\bar{K}^0(\tau_1), K^0(\tau_4)] + P_{LR}^{\text{ren}}[K^0(\tau_2), \bar{K}^0(\tau_3)] \]
\[ + P_{LR}^{\text{ren}}[K^0(\tau_2), K^0(\tau_4)] - P_{LR}^{\text{ren}}[\bar{K}^0(\tau_2), \bar{K}^0(\tau_3)] - P_{LR}^{\text{ren}}[K^0(\tau_2), K^0(\tau_3)], \]  

(6.8)

where the single meson observables are given by [see eq. (5.1)]:
\[ P_{LR}^{\text{ren}}[\bar{K}^0(\tau)] = \frac{P_{LR}[\bar{K}^0(\tau)]}{P_{LR}[-(\tau)\text{]} = \frac{1}{2}. \]  

(6.9)

Consider the special case in which the four times are related by:
\[ \tau_3 - \tau_1 = \tau_2 - \tau_3 = \tau_4 - \tau_2 = \frac{1}{3}(\tau_4 - \tau_1) \equiv \tau. \]  

(6.10)

Thus, in quantum mechanics quantity (6.8) reduces to [see eq. (5.3)]:
\[ S_{QM}(\tau) = \frac{1}{4} [2 - 3A_{QM}(\tau) + A_{QM}(3\tau)] - 1. \]  

(6.11)

If we choose \(\tau_1 \equiv \tau\), the other times become: \(\tau_2 = 3\tau, \tau_3 = 2\tau\text{ and } \tau_4 = 4\tau\), and, in the limit of stable kaons, both side of inequality (6.7) are violated by quantum mechanics in periodical intervals of \(\tau\) (see curve marked spin in figure 4): the largest violations are: 
\(-1.21 < -1, 0.21 > 0\).

As far as the real case for kaons is considered, quantum mechanics does not violate inequality (6.7) when unrenormalized expectation values are used (see curve unren in figure 4). The conclusion is different once one employs probabilities normalized to undecayed kaon pairs: as it is shown in figure 4 (curve ren), for \(0 < \tau \lesssim 1.4\tau_S\text{ quantum–mechanical expectation values are incompatible with the left hand side of inequality (6.7). The largest violation of the inequality \((-1.087 < -1\) corresponds to } \tau \simeq 0.81\tau_S\text{ and } P_{QM}^{\text{ren}}[\bar{K}^0(\tau_1), \bar{K}^0(\tau_3)] \simeq 0.036, P_{QM}^{\text{ren}}[\bar{K}^0(\tau_1), \bar{K}^0(\tau_4)] \simeq 0.195.

With the previous choice of the four detection times the locality condition (5.4) is not satisfied, since: \(\tau_4/\tau_1 = 4 > 1.55\). In order to fulfil this requirement when relation (6.10) is
used, one can introduce times $\tau_1 = p\tau$, $\tau_2 = (p+2)\tau$, $\tau_3 = (p+1)\tau$ and $\tau_4 = (p+3)\tau$ ($p \geq 0$) and require $\tau_4/\tau_1 = (p+3)/p < 1.55$, thus $p > 5.45$. However, since the renormalized quantum–mechanical probabilities only depend on the difference between the observation times [see eqs. (6.5), (3.13)], the result ren of figure 4 is independent of $p$, and the locality condition is satisfied. Thus, experimentally one could choose to use, for instance, $p = 6$, namely $\tau_1 = 6\tau$, $\tau_2 = 8\tau$, $\tau_3 = 7\tau$, $\tau_4 = 9\tau$, and the largest violation of the inequality would be again for $\tau \simeq 0.81\tau_S$. However, as $p$ increases, even if the renormalized probabilities are unchanged, the strangeness detection becomes more and more difficult, because of the kaon decays, thus small $p$ are preferable. Also the curve corresponding to the limit $\Gamma_S = \Gamma_L = 0$ is the same for any $p$. Of course, the curve corresponding to the inequality that makes use of unrenormalized probabilities depends on $p$, but this case is not interesting since it cannot be used for a discriminating test whatever the choice of $p$ is.
Assuming the same relative error in the measurement of all joint probabilities appearing in eq. (6.8) and disregarding the uncertainties on the single kaon detection, the need for an error on $S_{\text{Exp}}$ much smaller than the maximum violation (0.087) of figure 4 is satisfied if the joint probabilities of (6.8) can be determined with an accuracy $\delta P_{\text{Exp}}^{\text{ren}}/P_{\text{Exp}}^{\text{ren}} << 40\%$.

The CPLEAR data did not fulfil this condition. The experimental accuracy required to perform a conclusive test of local realism with CHSH’s inequality (6.7) is of the same order of magnitude of that needed in the use of Wigner’s inequality (6.1). To give an idea of the comparison between the potentialities of a test with Bell’s inequalities and a test through the measurement of the asymmetry parameter, let us consider the following hypothetical case in the situation of figure 1 with $\tau_2 = 1.5\tau_1 \simeq 2.3\tau_s$. By assuming a relative precision on the measurement of $\bar{K}_0K_0$ pairs five times better than the one for $K_0\bar{K}_0$ detection, the requirement $\delta P_{\text{Exp}}^{\text{ren}}/P_{\text{Exp}}^{\text{ren}} << 40\%$ for the joint observables of (6.8) corresponds to an accuracy on the asymmetry, $\delta A_{\text{Exp}}/A_{\text{Exp}} << 20\%$, that would allow a clear test of local realism.

### VII. CONCLUSIONS

In agreement with Bell’s theorem, in this paper we have shown that quantum mechanics for the two–neutral–kaon system cannot be completed by a theory which is both local and realistic: the separability assumed in Bell’s local realistic theories for the joint probabilities contradicts the non–separability of quantum entangled states. Although both the locality condition and the realistic viewpoint seem reasonable, they are not prescribed by any first principle. Any local realistic approach is only able to reproduce the non–paradoxical predictions of quantum mechanics like the perfect anti–correlations in strangeness and $CP$ and the single meson observables. On this point it is important to recall that the authors of ref. [33] showed how for entangled systems of three or more particles the incompatibility between local realism and quantum mechanics is even deeper: in fact, for these systems, a contradiction already arises at the level of perfectly correlated quantum states, the premises of local realism being in conflict with the non–statistical predictions of quantum mechanics. For EPR’s pairs, maintaining the realistic viewpoint, in order to reproduce the prediction of quantum mechanics (which, up to now, have been strongly supported by experimental evidence), one is forced to consider as a real fact of Nature a non–local behaviour of microscopic phenomena.

In the present paper, the incompatibility proof among quantum mechanics and local realistic models has been carried out by employing two different approaches. We started discussing the variability of the expectation values deduced from the general premises concerning locality and realism. The realistic states have been interpreted within the widest class of hidden–variable models. As far as the process $e^+e^- \rightarrow \phi \rightarrow K^0\bar{K}^0$ is considered, under particular conditions for the experimental parameters (the detection times), the discrepancies among quantum mechanics and local realistic models for the time–dependent
asymmetry are not less than 20%. The data collected by the CPLEAR collaboration (which used the reaction \( p\bar{p} \rightarrow K^0\bar{K}^0 \)) do not allow for conclusive answers concerning a refutation of local realism: these data are compatible not only with quantum–mechanical asymmetries, but with the range of variability of local realistic predictions. Therefore, a decisive test of local realism needs for more precise data.

The other approach we followed in this paper makes use of Bell–like inequalities involving \( K_S – K_L \) mixing. Contrary to what is generally believed in the literature, we have shown that a Bell–type test is feasible at a \( \Phi \)-factory, both with Wigner’s and (once probabilities normalized to undecayed kaons are used) CHSH’s inequalities. For an experiment at a \( \Phi \)-factory, the degree of inconsistency between quantum mechanics and local realism shown by a Bell test is of the same order of magnitude of that obtained in the first part of the paper through the comparison of the asymmetry parameters.

Concluding, by employing an experimental accuracy for joint kaon detection considerably higher than that corresponding to the CPLEAR measurement, a decisive test of local realism vs quantum mechanics both with and without the use of Bell’s inequalities will be feasible in the future at the Frascati \( \Phi \)-factory.

ACKNOWLEDGMENTS

We are grateful to Albert Bramon for many valuable discussions. One of us (G.G.) acknowledge financial support by the EEC through TMR Contract CEE–0169.
REFERENCES

[1] A. Einstein, B. Podolsky and N. Rosen, *Phys. Rev.* **47** (1935) 777.
[2] N. Bohr, *Phys. Rev.* **48** (1935) 696.
[3] D. Bohm, *Phys. Rev.* **85** (1952) 166; ibid. 180.
[4] H. Everett, *Rev. Mod. Phys.* **29** (1957) 454.
[5] A. Shimony, in *Proceedings of the International Symposium on Foundation of Quantum Mechanics* (Physical Society of Japan, Tokyo, 1984), p.25.
[6] W. H. Furry, *Phys. Rev.* **49** (1936) 393; ibid. 476.
[7] C. S. Wu and I. Shaknov, *Phys. Rev.* **77** (1950) 136.
[8] D. Bohm, *Quantum Theory*, (Prentice Hall, Englewood Cliffs, N.J., 1951) p.614.
[9] D. Bohm and Y. Aharonov, *Phys. Rev.* **108** (1957) 1070.
[10] A. Apostolakis et al., *Phys. Lett.* **B 422** (1998) 339.
[11] D. Bouwmeester, J.-W. Pan, M. Daniell, H. Weinfurther and A. Zeilinger, *Phys. Rev. Lett.* **82** (1999) 1345.
[12] W. Tittel, J. Brendel, H. Zbinden and N. Gisin, *Phys. Rev. Lett.* **81** (1998) 3563; *Phys. Rev. A* **59** (1999) 4150.
[13] J. S. Bell, *Physics* **1** (1964) 195.
[14] J. S. Bell, Proceedings of the *International School of Physics ‘Enrico Fermi’,* Course XLIX, ed B. d’Espagnat (Academic, New York, 1971) p.171; reprinted in J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics,* (Cambridge U. P., Cambridge, 1987) p.29.
[15] J. F. Clauser, M. A. Horne, A. Shimony and R. A. Holt, *Phys. Rev. Lett.* **23** (1969) 880.
[16] E. P. Wigner, *Am. J. Phys.* **38** (1970) 1005.
[17] J. F. Clauser and M. A. Horne, *Phys. Rev. D* **10** (1974) 526.
[18] J. F. Clauser and A. Shimony, *Rep. Prog. Phys.* **41** (1978) 1881.
[19] A. Aspect, P. Grangier and G. Roger, *Phys. Rev. Lett.* **47** (1981) 460; ibid. **49** (1982) 91; A. Aspect, J. Dalibard and G. Roger, *Phys. Rev. Lett.* **49** (1982) 1804.
[20] G. Weihs, T. Jennewein, C. Simon, H. Weinfurter, and A. Zeilinger, *Phys. Rev. Lett.* **81** (1998) 5039.
[21] E. Santos, *Phys. Rev. A* **46** (1992) 3646; *Phys. Lett. A* **212** (1996) 10; L. De Caro and A. Garuccio, *Phys. Rev. A* **46** (1994) R2803.
[22] P. Huet and M. E. Peskin, *Nucl. Phys. B* **434** (1995) 3.
[23] V. L. Lepore and F. Selleri, *Found. Phys. Lett.* **3** (1990) 203.
[24] Proceedings of the *Workshop on Physics and Detectors for DaΦne,* edited by G. Pancheri (INFN, LNF, 1991); *The DaΦne Handbook,* edited by L. Maiani, G. Pancheri and N. Paver (INFN, LNF, 1992); *The Second DaΦne Physics Handbook,* edited by L. Maiani, G. Pancheri and N. Paver (INFN, LNF, 1995).
[25] G. C. Ghirardi, R. Grassi and T. Webern, Proceedings of the *Workshop on Physics and Detectors for DaΦne,* edited by G. Pancheri (INFN, LNF, 1991) p.261; G. C. Ghirardi,
R. Grassi and R. Regazzon, *The DaΦne Handbook*, edited by L. Maiani, G. Pancheri and N. Paver (INFN, LNF, 1992) p.283.

[26] P. H. Eberhard, *Nucl. Phys. B* **398** (1993) 155; *The Second DaΦne Physics Handbook*, edited by L. Maiani, G. Pancheri and N. Paver (INFN, LNF, 1995) p.99.

[27] A. Di Domenico, *Nucl. Phys. B* **450** (1995) 293.

[28] F. Uchiyama, *Phys. Lett. A* **231** (1997) 295.

[29] F. Benatti and R. Floreanini, *Phys. Rev. D* **57** (1998) R1332.

[30] A. Bramon and M. Nowakowski, *Phys. Rev. Lett. B* **83** (1999) 1; B. Ancochea, A. Bramon and M. Nowakowski, *Phys. Rev. D* **60** (1999) 094008.

[31] F. Selleri, *Phys. Rev. A* **56** (1997) 3493; R. Foadi and F. Selleri, *Phys. Lett. B* **461** (1999) 123; *Phys. Rev. A* **61** (2000) 012106.

[32] N. Gisin and A. Go, arXiv:quant-ph/0004063.

[33] D. M. Greenberger, M. A. Horne, A. Shimony and A. Zeilinger, *Am. J. Phys.* **58** (1990) 1131.