Cosmological horizon entropy and generalised second law for flat Friedmann universe

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Abstract
We discuss the generalized second law (GSL) and the constraints imposed by it for two types of Friedmann universes. The first one is the Friedmann universe with radiation and a positive cosmological constant, and the second one consists of non-relativistic matter and a positive cosmological constant. The time evolution of the event horizon entropy and the entropy of the contents within the horizon are analyses in an analytical way by obtaining the Hubble parameter. It is shown that the GSL constraint the temperature of both the radiation and matter of the Friedmann universe. It is also shown that, even though the net entropy of the radiation (or matter) is decreasing at sufficiently large times as the universe expand, it exhibit an increase during the early times when universe is decelerating. That is the entropy of the radiation within the comoving volume is decreasing only when the universe has got an event horizon.

Keywords: Friedmann universe, entropy, generalised second law.

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1 Introduction

Bekenstein and Hawking have showed that the entropy of black holes is proportional to the area of their event horizon [1, 2, 3]. In units of $C = 1, G = 1, k = 1$ and $\hbar = 1$, the black hole entropy is given as

$$S = \frac{A_h}{4}$$  \hspace{1cm} (1)

where $A_h$ is the area of event horizon of the black hole. Hawking have shown that the black hole can evaporate by emitting radiation, consequently it’s event horizon area decreases. He had also shown that the event horizon of the black hole posses temperature, which is inversely proportional to it’s mass or proportional to it’s surface gravity. During the process of evaporation the entropy of the black hole will decrease. But due to the emitted radiation, the entropy of the surrounding universe will increase. Hence the second law of thermodynamics was
modified in such a way that, the entropy of the black hole plus the entropy of the exterior environment of the black hole will never decrease, this is called as the generalized second law (GSL), which can be represented as,

$$\frac{d}{dt}(S_{\text{env}} + S_b) \geq 0 \quad (2)$$

where $S_{\text{env}}$ is the entropy of environment exterior to the black hole and $S_b$ is the entropy of the black hole.

The thermodynamic properties of the event horizon, was shown to exist in a more basic level\[4, 5\], by re-casting the Einstein’s field equation for a spherically symmetric space time as in the form of the first law of thermodynamics. In references \[7, 8\] one can find investigations on the applicability of the first law of thermodynamics to cosmological event horizon. Jacobson \[6\] showed that, Einstein’s field equations are equivalent to the thermodynamical equation of state of the space time.

In cosmology the counterpart of black hole horizon is the cosmological event horizon. Gibbons and Hawking \[9\] proposed that analogous to black hole horizon, the cosmological event horizon also do possess entropy, proportional to their area. They have proved it particularly for de Sitter universe for which an event horizon is existing. For cosmological horizon, GSL implies that, the entropy of the horizon together with the matter enclosed by the event horizon of the universe will never decrease. That is the rate of change of entropy of the cosmological event horizon together with that of material contents within the horizon of the universe, must be greater than or equal to zero,

$$\frac{d}{dt}(S_{\text{CEH}} + S_m) \geq 0 \quad (3)$$

where $S_{\text{CEH}}$ is the entropy of the cosmological event horizon and $S_m$ represents the entropy of the matter or radiation (or both together) of the universe. The validity of GSL for cosmological horizon was confirmed and extended to universe consisting of radiation by numerical analyses by Davies \[10\] and others \[11, 15, 16, 18, 23\]. In reference \[19, 20\], the authors analyzed the GSL with some variable models of $f(T)$ gravity. In reference \[21\] GSL was analyzed with reference to brane scenario. Ujjal Debnath et al. \[22\] have analyzed the GSL for FRW cosmology with power-law entropy correction.

There are investigations connecting the entropy and hidden information. In the case of black hole horizon, the observer is outside the horizon, and the entropy of the black hole is considered as measure of the information hidden within the black hole. While regarding cosmological horizon, the observer is inside the horizon. This will cause problems in explaining the entropy of the cosmological horizon as the measure of hidden information as in the case of black hole. In the case of black hole the hidden region is finite, while in the case of cosmological horizon, there may be infinite region beyond the event horizon of the universe, which causing problems in explaining the cosmological entropy as the hidden information.

Another important fact is regarding the existence of dominant energy condition for the non-decreasing horizon area. In the case of black hole, Hawking proved an area theorem, that the area of the black hole will never decrease if it is not radiating \[25\]. Davies \[12\] proved an analogous theorem for cosmological event horizon that the area of the cosmological event horizon will never decrease, provided it satisfies the dominant energy condition,

$$\rho + p \geq 0 \quad (4)$$

where $\rho$ is the density of the cosmic fluid and $p$ is its pressure.

Regarding the applicability of the generalized second law to the Friedmann universe, analysis were done by considering the Friedmann universe as a small deviation form the de Sitter phase\[11, 12\]. In these works the authors calculated the horizon entropy through a numerical computation of the comoving distance to the event horizon. In the present work we obtained an analytical equation for the Hubble parameter and proceeded.
to the calculation of the entropy of the cosmological event horizon in an analytical way. We also checked the validity of dominant energy condition by using the derived Hubble parameter. Our analysis is for a flat universe which consists of (i) radiation and positive cosmological constant and (ii) non-relativistic matter and positive cosmological constant. We have considered the flat universe because of the fact that, the inflationary cosmological models predicts flat universe and more over the flatness of the space is confirmed by observations, for example, the current value of the curvature parameter is $\Omega_k \sim 10^{-3}$ \cite{26}.

The paper is arranged as follows. In section two, we consider the flat Friedmann universe with radiation and a positive cosmological constant. We are presenting the calculation of the entropy of radiation, event horizon and the total entropy of universe and the respective time evolutions. We have also checked the validity of the generalized second law in this section. In section three we present the analogous calculations for the flat Friedmann universe with non-relativistic matter and a positive cosmological constant. In section four we present the particular behaviour of the radiation entropy in the Friedmann universe with reference to the development of the event horizon. In section five we present the discussion followed by conclusions.

2 Friedmann universe with radiation and cosmological constant.

For a flat Friedmann universe with FRW metric, the dynamics are governed by the Friedmann equations (by choosing $8\pi G = 1$),

$$H^2 = \frac{\rho_r}{3} + \frac{\Lambda}{3} \quad (5)$$

and

$$\dot{\rho}_r + 3H (\rho_r + P_r) = 0 \quad (6)$$

where $\rho_r$ is the radiation density, $p_r$ is the radiation pressure, $\Lambda$ is the constant cosmological constant, $H$ is the Hubble parameter and the dot over the density represents derivative with respect to time. For radiation, the pressure is, $p_r = \rho_r/3$. From equations (5) and (6) the scale factor of this universe can be obtained as,

$$a(t) = \left(\frac{\Omega_{r0}}{\Omega_{\Lambda}}\right)^{1/4} \sinh^{1/2}(2\sqrt{\Omega_{\Lambda}H_0(t - t_0)}) \quad (7)$$

where $\Omega_{r0} = \rho_{r0}/3H_0^2$, $\Omega_{\Lambda} = \Lambda/3H_0^2$ and $H_0$ is the present value of the Hubble parameter. This equation shows that as $t \rightarrow t_0$ the scale factor $a(t) \rightarrow t^{1/2}$ the radiation dominated phase of the Friedmann universe and as $t \rightarrow \infty$ the scale factor $a(t) \rightarrow e^{\sqrt{2}\lambda}$, the de Sitter phase. The behaviour of the scale factor with time is shown in figure 1 in comparison with the scale factor of the de Sitter phase. From the plot it is evident that the scale factor of the Friedmann universe tends to the de Sitter phase at large times. So at smaller times the universe is in the radiation dominated phase and it is decelerating, consequently it doesn’t have event horizon. At larger times the universe enters the accelerated expansion phase, where it posses an event horizon.

The co-moving distance to the event horizon, can be obtained by using the relation,

$$\chi = \int_t^{\infty} \frac{dt}{a(t)} < \infty \quad (8)$$

Thus the proper distance to the event horizon is $r_h = a(t)\chi$. For the existence of the event horizon, the integral has to converge. With the scale factor in equation (7), the integral in the equation for comoving distance to the event horizon cannot be solved analytically. So as a first step we made a numerical computation of the comoving
Figure 1: Evolution of scale factor $a(t)$ with time $t$ in Gyrs, the blue line (lower line) corresponds to the Friedmann universe and red line (upper line) for de Sitter universe.

Figure 2: Variation of the comoving distance $\chi$ to the event horizon with time in Gyrs for Friedmann universe with radiation and cosmological constant.
distance to the event horizon, as it is necessary to understand the time evolution of the comoving distance and the result is shown in figure 2. The plot shows that the comoving distance to the event horizon is decreasing with time. Since the comoving horizon distance is decreasing, the comoving volume of the universe within the horizon also decreases. The radiation density behaves as $\rho_\gamma \sim a^{-4}$, therefore the radiation content within the horizon is decreasing with time. Which nevertheless implies that the radiation is crossing the horizon, hence the radiation entropy within the horizon is decreasing. This method and conclusion is in line with the result of T M Davies et. al. [11]. One can also find investigations of the same spirit regarding the heat flow through the cosmic horizon in references [17, 24]. In fact this result is true for any model of the universe having an event horizon.

The horizon entropy can be obtained as per the Bekenstein equation [1]. For that the area of the event horizon can be taken as $A_h = 4\pi \chi^2$. In the work of Davies et, al. the entropy was calculated in a numerical way, but we are obtaining the entropy of the horizon using the Hubble parameter obtained form the scale factor. We are substituting $\chi$ in terms of the Hubble parameter. The scale factor in equation (7) shows that, at large time the scale factor is approaching to that of de Sitter phase. For de Sitter phase, it can be shown that,

$$\chi = \frac{1}{H}$$  \hspace{1cm}  (9)

Since the Friedmann universe considered here is approaching the de Sitter phase at large times, it will not be unfair in taking, the comoving distance $\chi \sim 1/H$ for the Friedmann universe in consideration. For the scale factor in equation (7), the Hubble parameter is,

$$H = H_0 \sqrt{\Omega_\Lambda} \coth(2\sqrt{\Omega_\Lambda} H_0 t)$$  \hspace{1cm}  (10)

Before going for a calculation of the entropy of the event horizon, we will check here the validity of the area theorem proposed by Davies, with the obtained Hubble parameter. From equation (5) and (10), the condition for non-decreasing horizon area, equation (4), leads to

$$\frac{H^2}{H_0^2 \Omega_\Lambda} \geq 1$$  \hspace{1cm}  (11)

Using equation (10) we have plotted $\frac{H^2}{H_0^2 \Omega_\Lambda}$ versus time in figure 3. We have used the parameter values, $H_0 = 73 \text{ km s}^{-1} \text{ Mpc}^{-1}$ [13] and a standard value $\Omega_\Lambda = 0.7$ through out for our calculations. The plot shows that the area of the event horizon of the Friedmann universe with radiation and a positive cosmological constant will never decrease, hence the entropy of horizon will never decrease. On the other hand, the entropy of the radiation is decreasing with time as we have argued earlier.

In order to satisfy the GSL, the decrease in the entropy of the radiation is to be balanced by the increase in the horizon entropy. The horizon area is always increasing, implies that there exist some kind of trading of the entropy between the horizon and the radiation content of the universe. The entropy of the event horizon is $S_{CEH} = \pi \chi^2$. As we have argued earlier, taking $\chi = 1/H$, the horizon entropy become,

$$S_{CEH} = \frac{\pi}{H^2} \equiv \frac{\pi}{H_0^2 \Omega_\Lambda \coth^2(2\sqrt{\Omega_\Lambda} H_0 t)}$$  \hspace{1cm}  (12)

The entropy of the radiation can be obtained using the relation,

$$S_\gamma = \left( \frac{\rho_\gamma + p_\gamma}{T_\gamma} \right) V_{CEH}$$  \hspace{1cm}  (13)
where $V_{CEH}$ is the volume of the event horizon and $T_{\gamma}$ is the temperature of the radiation. Taking $V_{CEH} = 4\pi \chi^3/3$, substituting temperature form radiation energy density, $\rho_{\gamma} = \sigma T_{\gamma}^4$,

$$S_{\gamma} = \frac{16\pi \sigma^{1/4}}{9} \frac{\rho_{\gamma}^{3/4}}{H^3}$$

which after substituting $H$ parameter form equation (10) and $\rho_{\gamma}$ in terms of $H$, using the Friedmann equations, become

$$S_{\gamma} = \frac{16\pi \sigma^{1/4}}{9} \frac{\left(3H_0^2 \Omega_{\Lambda} \coth^2(2\sqrt{\Omega_{\Lambda}}H_0t) - 3H_0^2 \Omega_{\Lambda}\right)^{3/4}}{\left(H_0\sqrt{\Omega_{\Lambda}} \coth(2\sqrt{\Omega_{\Lambda}}H_0t)\right)^3}$$

where $\sigma = \pi^2/15$, the radiation constant.

We have plotted the time variation of $S_{CEH}$, $S_{\gamma}$ and $S_{CEH} + S_{\gamma}$ in figure 4. The figure shows that, at sufficiently large times the radiation entropy is decreasing, while the horizon entropy is increasing. The increase
in the horizon entropy is more than required to compensate for the decrease in the radiation entropy because of that, total entropy comprising the entropy of the radiation and horizon will increase. This is confirming the validity of the generalized second law for the cosmological horizon, that the entropy of the horizon plus the entropy of the fluid within the horizon will never decrease. This result is in confirmation with the earlier works of Davies and others, but they have arrived at the conclusion through straight numerical work, on the other hand our work is more of an analytical way.

The conditions for satisfying the generalized second law can be obtained by analysing the validity of the exact statement of the law as given equation (3). The time rate of the horizon entropy is,

\[
\frac{dS_{CEH}}{dt} = -2\pi \left( \frac{\dot{H}}{H^3} \right)
\]

where the dot over \( H \) represents the derivative with time given as \( \dot{H} = -\frac{2\rho_\gamma}{3} \), leads to

\[
\frac{dS_{CEH}}{dt} = \frac{4\pi \rho_\gamma}{3H^3}. \tag{17}
\]

The time rate of radiation entropy can be given from (14) as,

\[
\frac{dS_\gamma}{dt} = -\frac{16\pi \sigma^{1/4} \rho_\gamma^{3/4}}{3} \left( \frac{1}{H^2} + \frac{\dot{H}}{H^4} \right). \tag{18}
\]

The above two equations reveal that the time rate of horizon entropy is positive hence the horizon entropy is at the increase, while the time rate of radiation entropy is negative hence the radiation entropy is at the decrease. The generalized second law, can then be represented as,

\[
-2\pi \left( \frac{\dot{H}}{H^3} \right) - \frac{16\pi \sigma^{1/4} \rho_\gamma^{3/4}}{3} \left( \frac{1}{H^2} + \frac{\dot{H}}{H^4} \right) \geq 0 \tag{19}
\]

Replacing \( H \) and \( \dot{H} \) using the equation (10), the above condition become,

\[
\frac{H_0 \sqrt{\Omega_\Lambda} \coth^2(2\sqrt{\Omega_\Lambda} H_0 t)}{\cosh(2\sqrt{\Omega_\Lambda} H_0 t)} - \frac{4\sigma^{1/4} \rho_\gamma^{3/4}}{3} \left( 1 - 2\coth^2(2\sqrt{\Omega_\Lambda} H_0 t) \right) \geq 0 \tag{20}
\]

Expressing \( \rho_\gamma \), in terms of \( H \), using the Friedmann equation, we have evaluated the time evolution of the left hand side of the above inequality condition and the result is shown in figure [5]. The figure shows that the condition for the GSL is always satisfied.

From the GSL condition in equation \( [19] \), we can obtain a condition regarding the temperature of the radiation within the horizon. With \( \dot{H} = -2\rho_\gamma/3 \), the condition \( [19] \), become,

\[
\frac{\rho_\gamma^{1/4}}{H} + 8\sigma^{1/4} \frac{\rho_\gamma}{3H} \geq 4\sigma^{1/4} \tag{21}
\]

Taking \( \rho_\gamma^{1/4} = \sigma^{1/4}T_\gamma \), then an inequality condition constraints the present value of the temperature of the radiation can be obtained as,

\[
T_{\gamma 0} \geq (4H_0 - 8H_0 \Omega_\gamma 0) \tag{22}
\]
the above condition leads to a numerical value, $T_{\gamma 0} \geq 10^{-29}$K. Compared to the present temperature of the radiation $T_{\gamma 0} \sim 2.73$K, this is very much in favour of the validity of second law in the Friedmann universe. This result is agreeing with the result obtained by Davies et. al, that $T_{\gamma 0} \geq H$, where $H$ is now taken as the temperature of the horizon. By using the fundamental constants, the temperature of the horizon, is

$$T_{CEH} = \frac{h H}{k 2\pi}$$

(23)

where $k$ is the Boltzmann constant, which implies a present value, $T_{CEH0} \sim 10^{-30}$K. So the temperature of the horizon is less than that of the event horizon, which indicates that, the radiation can approach the horizon.

### 3 Freidmann universe with matter and cosmological constant.

In this section we are analysing the Friedmann universe with matter and a positive cosmological constant, regarding the horizon entropy and the generalized second law. The Friedmann equation, in this case is,

$$H^2 = \frac{\rho_m}{3} + \frac{\Lambda}{3}$$

(24)

The scale factor can then be obtained as,

$$a(t) = \left( \frac{\Omega_{m0}}{\Omega_{\Lambda}} \right)^{1/3} \sinh^{2/3} \left( \frac{3\sqrt{\Omega_{\Lambda}} H_0 t}{2} \right)$$

(25)

As in the previous case, here also the solution will tend to the de Sitter phase, $a(t) \to e^{\sqrt{\Lambda/3}}$ as $t \to \infty$. Which means that the model poses an event horizon. The Hubble parameter corresponds to the scale factor is,

$$H = H_0 \sqrt{\Omega_{\Lambda}} \coth \left( \frac{3}{2} \sqrt{\Omega_{\Lambda}} H_0 t \right)$$

(26)

It can be seen that the dominant energy condition is being satisfied, as in the case of Friedmann universe with radiation, such that $H^2/(H_0^2 \Omega_{\Lambda}) \geq 1$ at all time.
The comoving distance to the horizon is evaluated using the scale factor in the equation (25), and is decreasing with time as shown in figure 6. So the matter entropy within the horizon will decrease and hence the matter will cross the event horizon.

Entropy of the event horizon is calculated as,

$$S_{CEH} = \frac{\pi}{H_0^2 \Omega_\Lambda \coth^2\left(\frac{3}{2} \sqrt{\Omega_\Lambda} H_0 t\right)}$$  \hspace{1cm} (27)

Entropy of matter can be calculated using an analogous relation corresponding to equation (13), and taking temperature of matter approximately as $T_m \sim \rho_m^{1/4}$,

$$S_m = \frac{4\pi}{3} \frac{\rho_m^{3/4}}{H^3}$$  \hspace{1cm} (28)

which after substitution of $H$ parameter becomes,

$$S_m = \frac{4\pi}{3} \frac{(3H_0^2 \Omega_\Lambda \coth^2\left(\frac{3}{2} \sqrt{\Omega_\Lambda} H_0 t\right) - 3H_0^2 \Omega_\Lambda)^{3/4}}{H_0^2 \Omega_\Lambda^{1/2} \coth^3\left(\frac{3}{2} \sqrt{\Omega_\Lambda} H_0 t\right)}.$$  \hspace{1cm} (29)

The behaviour of $S_{CEH}$, $S_m$ and $S_{CEH} + S_m$ with time is shown in figure 7. The figure shows that the total entropy of the universe is increasing and the increase in the entropy of the horizon is more than that required for compensating the decrease in the matter entropy. The general behaviour is the same as that of Friedmann universe with radiation, that universe with matter also will satisfy the generalized second law.

The condition for satisfying the generalized second law for this universe can be obtained by incorporating the time derivatives of the corresponding entropies into the second law, as

$$-\frac{2\pi H}{H} - \pi \rho_m^{3/4} \left(\frac{3}{H^2} + \frac{4H}{H^4}\right) \geq 0$$  \hspace{1cm} (30)
Using the Hubble parameter equation (26), the above condition become,

\[
\frac{H_0 \sqrt{\Omega_\Lambda \text{Cosech}^2 \left( \frac{3}{2} \sqrt{\Omega_\Lambda} H_0 t \right)}}{\text{Coth} \left( \frac{3}{2} \sqrt{\Omega_\Lambda} H_0 t \right)} - \rho_m^{\frac{3}{4}} \\
\left( 1 - 2 \text{Sech}^2 \left( \frac{3}{2} \sqrt{\Omega_\Lambda} H_0 t \right) \right) \geq 0
\]  

(31)

Substituting \( \rho_m \) in terms of the Hubble parameter using the Friedmann equation, we have plotted the time evolution of the left hand side of the above equation and is shown in figure 8. As in the case of the Friedmann universe with radiation, here also the plot shows that the inequality condition corresponds to the generalized second law is satisfied.

Figure 7: Variation \( S_{CEH} \cdot S_m \) and \( S_{CEH} + S_m \) with time in Gyrs for Friedman universe with matter and cosmological constant. The continuous line representing entropy of horizon plus that of matter, dashed line representing the entropy of horizon and dash-dot line is for entropy of matter.

Figure 8: Plot showing the validity of the generalized second law as per the condition in equation (29)
Figure 9: The plot on the left represents the time evolution of the entropy of the radiation in the Friedmann universe with radiation and $\Lambda$, and the plot on the right represents the time evolution of the $q$–factor of the same Friedmann universe with time in giga years.

As in the previous section, the generalized second law can leads to constraint on the temperature of matter. Equation (30) can be recast, by taking $\dot{H} = -\rho_m/2$, as

$$\rho_m H - \rho^{3/4} \left( 3 - \frac{2\rho_m}{H^2} \right) \geq 0. \tag{32}$$

From this it can be shown that, the present temperature of matter in the universe satisfies,

$$T_{m0} \geq (3H_0 - 6\Omega_{m0}H_0) \tag{33}$$

For the standard value $\Omega_{m0} = 0.3$ the above condition also gives, $T_{m0} \geq 10^{-29} K$. The temperature of the horizon is $T_h \sim H \tag{9}$, and with proper parameters, $T_h \sim 10^{-30} K$. So the present temperature of the matter is greater than the temperature of the horizon, which supports the conclusion that the matter can cross the event horizon.

4 Growth of event horizon and evolution of the entropy

In this section we will restrict our analysis to Friedmann universe with radiation and cosmological constant only. However one can easily see that the conclusions made are in general true for Friedmann universe with matter also, but with different numerical values. Our aim here is to show that the entropy of the contend of the universe does have a small increase before the development of event horizon. In the last two sections we have discussed the behaviour of horizon entropy and entropy of the material within the horizon. We have concentrated on checking the validity of the generalized second law. We have shown that the total entropy of the universe is always increasing, and the cosmological event horizon is satisfying the generalized second law. However it is to be noted from the figure[4](from figure[7]) that the entropy of the radiation (matter) is increasing first, attaining a maximum, then after it is decreasing. For clarity regarding this we will show in figure[9] the time evolution of the radiation entropy for a Friedmann universe having radiation and a positive cosmological constant. The figure shows that the radiation entropy first increases and then decreases to zero at very large times.

In the previous section we have concluded that the decrease in the entropy of radiation (or matter) is due to the escape of the radiation (or matter) from within the horizon. The horizon will exist only when the universe is accelerating. From the behaviour of the scale factor we have noted that, the universe will be in the radiation
dominated (or matter dominated) phase as time $t \to 0$. In the radiation dominated or matter dominated phase the expansion of the universe is decelerating, hence no horizon. The horizon will develop only when the universe enters the $\Lambda$ dominated phase. A clear demarcation between the deceleration and acceleration phases during the evolution of the universe can be obtained by calculating the deceleration parameter $q$, which is defined as

$$ q = -1 - \frac{\dot{H}}{H^2} $$

(34)

we have calculated the $q$–factor using the Hubble parameter in equation (10) for the Friedmann universe with radiation and $\Lambda$ and the time evolution of which is shown in figure 9 along with the time evolution of the radiation entropy for an easy comparison. The universe enters the accelerating phase, corresponding to the time at which the $q$–factor starts to have negative values and as per the figure that is around a time $t \sim 7\text{Gyrs}$.

As per the above analysis the universe enters the accelerating phase at around $t \sim 7\text{Gyrs}$, and at around this time the event horizon starts developing. At this transition time the horizon was tiny. Even at this time the difference in the entropy of the event horizon and radiation was very high. Entropy of the radiation given in the equation (15), gives a value for $t = 7\text{Gyrs}$, $S_\gamma \sim 10^{27}$. While the entropy of the event horizon, as in equation (12), leads to value of $S_{CEH} \sim 10^{35}$, for the same time. These shows that, even at the formation of the event horizon, the entropy of it is eight orders of magnitude greater than the radiation entropy. So even at the tiny stage of the event horizon the entropy of radiation is not so significant.

A comparison of the plots in figure 9 shows that the entropy of radiation is increasing at first and is start decreasing at the same time when $q$–factor become negative, as the universe entering the accelerating phase. In the decelerating phase, corresponds to positive values of $q$–factor the radiation entropy is increasing as there is no horizon for the radiation to escape. Since the radiation entropy is increasing during the initial stages, the generalized second law is still valid such that

$$ \frac{dS_\gamma}{dt} \geq 0, $$

(35)

will become the GSL as there in no event horizon. When the universe enters the accelerated expanding phase, where it has event horizon, the radiation entropy is decreasing, because now the radiation is crossing the event horizon. But nevertheless, in the accelerating phase, the horizon entropy is increasing at a faster rate compensated to the decrease in the radiation entropy, which in turn leads to the increase in the total entropy of the universe, guaranteeing the validity of the generalized second law.

The time rate of radiation entropy is given in equation (18). Substituting for $\dot{H} = -2\rho_\gamma/3$ for Friedmann universe with radiation, the equation can be reduced to,

$$ \frac{dS_\gamma}{dt} = \frac{16\pi\sigma^{1/4}}{3} \rho_\gamma^{1/4} \left( \frac{2\rho_\gamma}{3H^4} - \frac{1}{H^2} \right) $$

(36)

When the radiation entropy is maximum, the time rate is zero, then the above equation leads to the condition,

$$ H = \sqrt{\frac{2\Lambda}{3}} $$

(37)

where we have used the Friedmann equation to substitute for the $\rho_\gamma$. From which the corresponding time can be obtained as

$$ t = \frac{1}{2\sqrt{\Omega_\Lambda H_0}} \coth^{-1}(\sqrt{2}) $$

(38)
For the standard parameters, the value of the above time, corresponds to the decreasing of radiation entropy, is around $t \sim 7\text{Gyrs}$, which is in confirmation with the figure 9. In the case Friedmann universe with matter and a positive cosmological constant also, it is evident form the figure 7 that the entropy of matter too have an increase before the formation of the event horizon. So one can easily see that in the case matter also, the above conclusions are true in general.

5 Conclusions

Gibbons and Hawking have conjectured that cosmological event horizon of the de Sitter universe have entropy like black hole event horizon, and the total entropy of such a universe will never decrease, that is it satisfies the GSL. Later Davies and others have extended this conjecture to Friedmann universe with radiation and dust such that the Friedmann universe satisfies the GSL. However their work is mainly based on the numerical computation. In this paper we have presented an analytical analysis of the entropy of the event horizon and fluid within the horizon and also the constraints followed from the validity of the GSL. We have considered two types of Friedmann universes. Type one is the Friedmann universe with radiation and a positive cosmological constant. The other type is the Friedmann universe with non-relativistic matter and a positive cosmological constant.

We have obtained the expansion scale factor and the Hubble parameter for the Friedmann universe with radiation (and matter) and cosmological constant. The time evolution of the scale factor is plotted and have found that at sufficiently small times the Friedmann universe is radiation(or matter) dominated and is in the decelerating phase. But at large times, the universe become dominated by the cosmological constant, hence in the accelerated expansion and will approach de Sitter phase at very large times. During the accelerated expansion phase, the universe has got an event horizon. We have numerically evaluated the time evolution of the comoving distance to the event horizon and verified that the comoving distance is decreasing with time in both types of the universes. As a result the comoving volume of the event horizon decreases, subsequently the radiation (matter) can cross the event horizon. This implies that the entropy of the radiation (or matter) is decreasing consequent to the escaping of radiation (or matter) through the horizon.

Analogous to the area theorem in black hole, Davies proposed a corresponding theorem for the cosmological event horizon which implies a dominant energy condition as given in equation (4). In the present case of the Friedmann universe, the dominant energy condition implies that, $H^2/(H_0^2\Omega) \geq 1$. The plot in figure 3 conclusively proves this. So once the event horizon is formed it’s area will never decrease. So unlike in the case of the black holes, where the area of the event horizon decreases when it is radiating, the area of the cosmological event horizon increases when radiation (matter) crosses the horizon.

We have obtained the analytical relations for the entropy of the event horizon and radiation (matter) for the Friedman universe. The entropy of the event horizon is given in equation (12), according to which the present value of the event horizon entropy will be around, $S_{CEH} \sim 10^{35}$. We have plotted the time evolution of these entropies and found that the net entropy of the radiation (or matter) is decreasing but the entropy of the event horizon is increasing at faster rate as the universe expands. This implies that the total entropy of the Friedmann universe, that is the sum of the entropy of the radiation (or matter) and event horizon, is increasing. This indicate the validity of the GSL for both types of the Friedmann universes. The constraints imposed by the GSL is obtained. For the Friedmann universe with radiation and cosmological constant, the GSL constraint the present temperature of the radiation as, $T_{r0} \geq 10^{-29}\text{K}$ in standard units. Compared to the latest value of the radiation temperature form COBE, $2.725 \pm 0.002 \text{K}$ [27], the above constraint implies the Friedmann universe in consideration is very well in the purview of the GSL. The temperature of the horizon, is $T_{CEH} = H$ in the
standard units. For the present case, this temperature is around \( T_{CEH} \sim 10^{-30} \text{K} \). The comparison of the above temperatures shows that there is a radiation drain form within the horizon. At this point one should note the result of Davies et. al.\(^{11}\) that, the temperature of the radiation is higher than that of the horizon. So there is natural flow direction towards the horizon. Therefore in the present context, we can conclude that the temperature of the horizon of the Friedmann universe with radiation and cosmological constant at present is less than \( 10^{-29} \text{K} \). For the universe with non-relativistic matter and cosmological constant, the GSL constraint the matter temperature as \( T_{m0} \geq 10^{-29} \text{K} \), which in turn implies that the horizon temperature of the Friedmann universe with matter and cosmological constant is less than \( 10^{-29} \text{K} \), so that there is matter flow towards the event horizon.

The time evolution of the radiation (matter) entropy shows that, it increases first, attains a maximum and then decreases as shown in the figure.\(^{9}\) It is seen that the increase in the radiation (matter) entropy is during the deceleration phase of the universe, when the radiation (matter) is dominating the cosmological constant. It is to be noted that there is no event horizon when the universe is decelerating. So the corresponding increase in the entropy of the radiation is due to the non-existence of the of the event horizon. If the event horizon is absent, there is no crossing of the radiation over the horizon, and it retained within causal region of our universe, which facilitate the small increase in the radiation entropy. We made this point clear by comparing the time evolution of the radiation entropy and \( q \)–factor, such that the entropy of the radiation is start decreasing when the \( q \)–factor become negative, consequently the expansion is accelerating at which condition the universe posses an event horizon. We have computed the the time corresponding to the maximum of the radiation entropy at which the \( q \)–factor is critically become negative, as \( t \sim 7\text{Gyrs} \), and is evident form figure.\(^{9}\)

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