HODGE DUALITY OPERATION AND ITS PHYSICAL APPLICATIONS ON SUPERMANIFOLDS

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Abstract: An appropriate definition of the Hodge duality $\star$ operation on any arbitrary dimensional supermanifold has been a long-standing problem. We define a working rule for the Hodge duality $\star$ operation on the $(2 + 2)$-dimensional supermanifold parametrized by a couple of even (bosonic) spacetime variables $x^\mu (\mu = 0, 1)$ and a couple of odd (fermionic) variables $\theta$ and $\bar{\theta}$ of the Grassmann algebra. The Minkowski spacetime manifold, hidden in the supermanifold and parametrized by $x^\mu (\mu = 0, 1)$, is chosen to be a flat manifold on which a two $(1 + 1)$-dimensional (2D) free Abelian gauge theory, taken as a prototype field theoretical model, is defined. We demonstrate the applications of the above definition (and its further generalization) for the discussion of the (anti-)co-BRST symmetries that exist for the field theoretical models of 2D- and 4D free Abelian gauge theories considered on the four $(2 + 2)$- and six $(4 + 2)$-dimensional supermanifolds, respectively.

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1 Introduction

The geometrical superfield formalism is one of the most intuitive approaches to gain an insight into some of the physical and mathematical ideas behind the Becchi-Rouet-Stora-Tyutin (BRST) formalism which plays a very important role in (i) the covariant canonical quantization of the gauge theories that are endowed with the first-class constraints in the language of Dirac’s prescription for the classification of constraints (see, e.g., [1,2]), (ii) the proof of unitarity of the “quantum” gauge theories at any arbitrary order of perturbative computations (see, e.g., [3,4,5]), and (iii) providing a deep connection between the physics of gauge theories with the mathematical ideas behind the cohomology (see, e.g., [6-9]) of the differential geometry. In the usual superfield approach [10-17] to the p-form $(p = 1, 2, 3,...)$ Abelian gauge theories, defined on the $D$-dimensional spacetime manifold, a $(p+1)$-form super curvature $\tilde{F}^{(p+1)} = \tilde{d}\tilde{A}^{(p)}$ is constructed from the super exterior derivative $\tilde{d} = dx^\mu \partial_\mu + d\theta \partial_\theta + d\bar{\theta} \partial_{\bar{\theta}}$ (with $\tilde{d}^2 = 0$) and the super $p$-form connection $\tilde{A}^{(p)}$ on the $(D+2)$-dimensional supermanifold parametrized by $D$-number of even (bosonic) spacetime coordinates $x^\mu (\mu = 0, 1, 2,...,D-1)$ and a couple of odd (fermionic) elements $\theta, \bar{\theta}$ (with $\theta^2 = \bar{\theta}^2 = 0, \theta \bar{\theta} + \bar{\theta} \theta = 0$) of the Grassmann algebra which constitute the superspace variable $Z^M = (x^\mu, \theta, \bar{\theta})$. This $(p+1)$-form super curvature is subsequently equated, due to the so-called horizontality condition $^*$, to the ordinary $(p+1)$-form curvature $F^{(p+1)} = dA^{(p)}$ constructed from the ordinary exterior derivative $d = dx^\mu \partial_\mu$ (with $d^2 = 0$) and the ordinary $p$-form connection $A^{(p)}$ on the ordinary $D$-dimensional Minkowskian spacetime manifold parametrized by the bosonic spacetime variables $x^\mu$ only. This restriction $^\dagger$ provides the geometrical origin and interpretation for (i) the nilpotent (anti-)BRST symmetry transformations (and the corresponding nilpotent and conserved charges) as the translation generators $(\partial/\partial\theta)\partial/\partial\bar{\theta}$ along the Grassmannian directions of the supermanifold, (ii) the nilpotency of the above transformations (and the corresponding nilpotent generators) as a couple of successive translations (i.e. $(\partial/\partial\theta)^2 = (\partial/\partial\bar{\theta})^2 = 0$) along the Grassmannian directions of the supermanifold, and (iii) the anticommutativity of the (anti-)BRST transformations (and the corresponding conserved and nilpotent charges) as the anticommutativity $(\partial/\partial\theta)(\partial/\partial\bar{\theta}) + (\partial/\partial\bar{\theta})(\partial/\partial\theta) = 0$ of the translation generators along the Grassmannian directions of the $(D + 2)$-dimensional supermanifold.

It is obvious from the above discussions that, in the horizontality condition, only one (i.e. $(\tilde{d})d$) of the existing three (super) de Rham cohomological operators $((\tilde{d})d, (\tilde{\delta})\delta, (\tilde{\Delta})\Delta)$ is exploited for the geometrical interpretations of some of the key properties associated with the nilpotent (anti-)BRST transformations and the corresponding conserved charges. To clarify

$^*$This condition is referred to as the soul-flatness condition by Nakanishi and Ojima [18] which amounts to setting equal to zero all the Grassmannian components of the (anti-)symmetric curvature tensor that constitutes the $(p+1)$-form super curvature on the $(D + 2)$-dimensional supermanifold.

$^\dagger$The horizontality condition has also been applied to 1-form 4D non-Abelian gauge theory where the six $(4+2)$-dimensional 2-form super curvature $\tilde{F}^{(2)} = \tilde{d}\tilde{A}^{(1)} + \tilde{A}^{(1)} \wedge \tilde{A}^{(1)}$ is equated with the 4D ordinary 2-form curvature $F^{(2)} = dA^{(1)} + A^{(1)} \wedge A^{(1)}$ leading to the derivation of (anti-)BRST symmetry transformations for the non-Abelian gauge field and the corresponding (anti-)ghost fields (see, e.g., [18]).
the above notations, it is worthwhile to be more specific about the de Rham cohomological operators of the differential geometry defined on an ordinary spacetime manifold without a boundary. On such a manifold, the operators \( d = dx^\mu \partial_\mu \), \( \delta = \pm \ast d \ast \) and \( \Delta = (d + \delta)^2 \) form a set of de Rham cohomological operators where \( (\delta) d \) are the nilpotent (co-)exterior derivatives, \( \Delta \) is the Laplacian operator and \( \ast \) is the Hodge duality operation on the manifold. These operators obey an algebra: \( d^2 = \delta^2 = 0 \), \( \Delta = \{ d, \delta \} \), \( [\Delta, d] = [\Delta, \delta] = 0 \) showing that \( \Delta \) is the Casimir operator for the whole algebra (see, e.g., [6-9] for details). It has been a long-standing problem to exploit the other nilpotent (i.e. \( \tilde{\delta}^2 = 0 \), \( \delta^2 = 0 \)) mathematical entities (\( \tilde{\delta} \)) of the (super) de Rham cohomological operators in the context of the dual-horizontality condition (\( \tilde{\delta} \tilde{A}^{(p)} = \delta A^{(p)} \)) and study its consequences on a \( p \)-form gauge theory in the framework of the geometrical superfield approach to BRST formalism. Here \( \tilde{\delta} = - \ast \tilde{d} \ast \) and \( \delta = - \ast d \ast \) are the super co-exterior derivative and ordinary co-exterior derivative, respectively. The mathematical symbols \( \ast \) and \( \ast \) stand for the Hodge duality operations on the \( (D+2) \)-dimensional supermanifold and \( D \)-dimensional ordinary manifold, respectively, and the super Laplacian operator is defined as \( \tilde{\Delta} = (\tilde{d} + \tilde{\delta})^2 \). To tap the mathematical power of \( \tilde{\delta} \delta \) of the (super) de Rham cohomological operators in the context of the dual-horizontality condition (\( \tilde{\delta} \tilde{A}^{(p)} = \delta A^{(p)} \)) and study its consequences on a \( p \)-form gauge theory in the framework of the geometrical superfield approach to BRST formalism. Here \( \tilde{\delta} = - \ast \tilde{d} \ast \) and \( \delta = - \ast d \ast \) are the super co-exterior derivative and ordinary co-exterior derivative, respectively. The mathematical symbols \( \ast \) and \( \ast \) stand for the Hodge duality operations on the \( (D+2) \)-dimensional supermanifold and \( D \)-dimensional ordinary manifold, respectively, and the super Laplacian operator is defined as \( \tilde{\Delta} = (\tilde{d} + \tilde{\delta})^2 \). To tap the mathematical power of \( \tilde{\delta} = - \ast \tilde{d} \ast \), it is clear that the definition of the Hodge duality \( \ast \) operation on the \( (D+2) \)-dimensional supermanifold is quite important.

A consistent and systematic definition of the Hodge duality \( \ast \) operation on an ordinary spacetime manifold of any arbitrary dimensionality is already quite well-known in the literature (see, e.g., [6-9] for details). In fact, the existence of the totally symmetric metric tensor and the totally antisymmetric Levi-Civita tensor on the spacetime manifold plays a crucial role in such a consistent and systematic definition of the duality operation \( \ast \). However, such a consistent, precise and elaborate definition of the Hodge duality \( \ast \) operation on a supermanifold, to the best of our knowledge, is not well-known in the literature (see, e.g., [18-26] for details). The purpose of our present paper is to provide a working rule for the definition of the Hodge duality \( \ast \) operation on the four \( (2+2) \)- and six \( (4+2) \)-dimensional supermanifolds on which the 2D- and 4D free 1-form \( A^{(1)} = dx^\mu A_\mu \) Abelian gauge theories are defined for the derivation of the nilpotent (anti-)co-BRST symmetry transformations in the framework of superfield approach to BRST formalism. We exploit this working rule for the definition of the \( \ast \) operation in the context of the dual-horizontality condition (\( \tilde{\delta} \tilde{A}^{(1)} = \delta A^{(1)} \)) where the action of the super co-exterior derivative \( \tilde{\delta} = - \ast \tilde{d} \ast \) on the super connection 1-form \( \tilde{A}^{(1)} \) does require, the action of the Hodge duality \( \ast \) operations (in \( \tilde{\delta} \tilde{A}^{(1)} = - \ast \tilde{d} \ast \tilde{A}^{(1)} \)) for the derivations of the (anti-)co-BRST symmetry transformations. To be more precise, for the case of the 4D Abelian gauge theory, defined on the six \( (4+2) \)-dimensional supermanifold, the \( \ast \) operation is defined (i) on the super 1-form \( \tilde{A}^{(1)} \) to produce \( \ast \tilde{A}^{(1)} \) as a super 5-form, and subsequently (ii) on the super 4-form \( \tilde{d} \ast \tilde{A}^{(1)} \) to produce a super 6-form \( \ast \tilde{d} \ast \tilde{A}^{(1)} \) to obtain explicitly \( \tilde{\delta} \tilde{A}^{(1)} = - \ast \tilde{d} \ast \tilde{A}^{(1)} \). In exactly similar fashion, the \( \ast \) operations could be defined for the 2D free Abelian gauge theory, considered on the four \( (2+2) \)-dimensional supermanifold, for the derivation of the nilpotent (anti-)co-BRST symmetries. Towards the above goals in mind, we propose, in
a systematic manner, the Hodge duality \( \star \) operations on all the possible super forms that could be defined on the \((2+2)\)-dimensional supermanifold (cf. Section 2.2) as well as on the \((4+2)\)-dimensional supermanifold (cf. Section 3.2). These definitions are subsequently exploited for the derivation of the nilpotent (anti-)co-BRST symmetries in the framework of superfield formalism (cf. Sections 2.3 and 3.3 below). Our present study is essential on three counts. First and foremost, it has been a long-standing problem to exploit the potential and power of the (super) co-exterior derivatives \( \tilde{\delta} = -\star \tilde{d} \star \) and \( \delta = -\star d \star \) in the context of the derivations of some specific nilpotent symmetries for the BRST formulation of the gauge theories. We find that the above (super) cohomological operators do play a set of decisive roles in the context of the derivations of the nilpotent (anti-)co-BRST symmetry transformations for the 2D- and 4D free Abelian gauge theories. Second, in our recent works [27-32], we have been able to exploit \((\tilde{\delta})\delta\) in the dual-horizontality condition \((\tilde{\delta}\tilde{A}(1) = \delta A(1))\) but the precise expressions for the \(\star\) operations on all the super forms, defined for some suitable supermanifolds, have not yet been obtained. Finally, our present study might turn out to be useful for the discussion of an interacting gauge theory [33,34] which has been shown to provide (i) the field theoretical model for the Hodge theory, and (ii) a model for the interacting topological field theory where topological \(U(1)\) field couples with the matter (Dirac) fields [33,34].

The contents of our present paper are organized as follows.

In Section 2, we very briefly recapitulate the bare essentials of the (anti-)BRST- and (anti-)co-BRST symmetry transformations for the 2D free Abelian gauge theory in the Lagrangian formulation. We also derive the nilpotent (anti-)BRST symmetry transformations in the framework of superfield formalism by exploiting the horizontality condition on the \((2+2)\)-dimensional supermanifold and provide the geometrical interpretation for the nilpotent (anti-)BRST charges \(Q_{(a)b}\) (cf. Section 2.1). For the derivation of the (anti-)co-BRST symmetry transformations in the superfield formalism, we discuss the dual-horizontality condition and define the Hodge duality \(\star\) operation, in a systematic way, for all the (super)forms defined on the four \((2+2)\)-dimensional supermanifold on which a 2D free Abelian gauge theory is considered. The double Hodge duality \(\star\) operations are also defined for all the (super)forms that are supported by the \((2+2)\)-dimensional supermanifold.

Section 3 is devoted to (i) a concise synopsis of the local, covariant, continuous and nilpotent (anti-)BRST- and non-local, non-covariant, continuous and nilpotent (anti-)co-BRST symmetry transformations for the free 4D Abelian theory in the Lagrangian formulation, (ii) a brief discussion for the derivation of the (anti-)BRST symmetry transformations in the usual superfield formalism and its key points of differences with such a derivation for the 2D free Abelian theory, (iii) a systematic definition of the single Hodge duality \(\star\) operation (and the double Hodge duality \(\star\) operations) for all the (super)forms defined on the six \((4+2)\)-dimensional supermanifold, and (iv) the derivation of the nilpotent (anti-)co-BRST symmetries by exploiting the \(\star\) operation in the context of the dual-horizontality condition.

Finally, in Section 4, we make some concluding remarks.
2 (Anti-)BRST- and (anti-)co-BRST symmetries for 2D theory: a brief sketch

Let us begin with the BRST- and anti-BRST invariant Lagrangian density \( \mathcal{L}_b \) for the two (1 + 1)-dimensional \(^4\) (2D) free Abelian gauge theory in the Feynman gauge \([3,4,18,35]\)

\[
\mathcal{L}^{(2)}_b = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + B(\partial \cdot A) + \frac{1}{2} B^2 - i \partial_\mu \bar{C} \partial^\mu C,
\]

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the antisymmetric field strength (curvature) tensor derived from the 2-form \( F^{(2)} = dA^{(1)} = \frac{1}{2} (dx^\mu \wedge dx^\nu) F_{\mu\nu} \). The latter is constructed by the application of the exterior derivative \( d = dx^\mu \partial_\mu \) (with \( d^2 = 0 \)) on the 1-form connection \( A^{(1)} = dx^\mu A_\mu \) which defines the vector potential \( A_\mu \) for the Abelian gauge theory. Thus, the operation of \( d \) on 1-form increases the degree by +1. It will be noted that \( F_{\mu\nu} \) has only electric component and the magnetic component of \( F_{\mu\nu} \) is zero in 2D. The Nakanishi-Lautrup auxiliary field \( B \) has been introduced to linearize the gauge-fixing term \(-\frac{1}{2} (\partial \cdot A)^2 \) and the fermionic (\( \bar{C}^2 = C^2 = 0, \bar{C}C + C\bar{C} = 0 \)) (anti-)ghost fields (\( \bar{C}C \)) are required to maintain the unitarity and quantum gauge (i.e. BRST) invariance together for a given physical process allowed by the theory. At this stage, it is worth emphasizing that the gauge-fixing term \((\partial \cdot A)\) owes its origin to the other nilpotent \((\delta^2 = 0)\) cohomological operator \( \delta \) because the operation of the latter \((\delta A^{(1)} = - * d * A^{(1)} = (\partial \cdot A))\) on the 1-form \( A^{(1)} \) produces it. The operator \( \delta = -* d * \), which decreases the degree of a form by 1, is known as the co-exterior derivative and \(* \) is the Hodge duality operation on the 2D spacetime manifold. The action of the Laplacian operator \( \Delta \) on the 1-form \( A^{(1)} \) (i.e. \( \Delta A^{(1)} = dx^\mu \square A_\mu \)) leads to the derivation of the equation of motion \( \square A_\mu = 0 \) for the gauge field \( A_\mu \) if we demand the validity of the Laplace equation \( \Delta A^{(1)} = 0 \). The degree of a form remains intact under the operation of \( \Delta \). Thus, we note that all the three de Rham cohomological operators \((d, \delta, \Delta)\) of differential geometry play very important roles in the description of the gauge theories. One can linearize the kinetic energy term \( \frac{1}{2} E^2 \) of (2.1) by introducing another auxiliary field \( \mathcal{B} \) as

\[
\mathcal{L}^{(2)}_b = \mathcal{B} E - \frac{1}{2} \mathcal{B}^2 + B(\partial \cdot A) + \frac{1}{2} B^2 - i \partial_\mu \bar{C} \partial^\mu C.
\]

For the special case of 2D free Abelian gauge theory, the auxiliary field \( \mathcal{B} \) is analogous to the Nakanishi-Lautrup field \( B \). In fact, the former linearizes of the kinetic energy term \( \frac{1}{2} E^2 \) in exactly the same manner as the latter linearizes the gauge-fixing term \(-\frac{1}{2} (\partial \cdot A)^2 \).

\(^4\)We adopt here the convention and notations such that the flat 2D Minkowskian spacetime manifold is endowed with the flat metric \( \eta_{\mu\nu} = \text{diag} (+1, -1) \) and \( \square = \eta^{\mu\nu} \partial_\mu \partial_\nu = (\partial_0)^2 - (\partial_1)^2 ; (\partial \cdot A) = \partial_0 A_0 - \partial_1 A_1, F_{01} = E = -\varepsilon^{0\mu\nu} \partial_\mu A_\nu = -F^{01} \). Here the 2D antisymmetric Levi-Civita tensor is chosen to satisfy \( \varepsilon_{01} = +1 = -\varepsilon^{01} , \varepsilon_{\mu\nu} \varepsilon_{\nu\lambda} = \delta^\mu_\lambda \), etc., and the Greek indices \( \mu, \nu, \lambda, \ldots = 0, 1 \) correspond to the time- and space directions on the 2D flat Minkowskian spacetime manifold, respectively.
The above Lagrangian density (2.2) is endowed with the following local, off-shell nilpotent $(s^2_{(a)b} = 0)$ and anticommuting $(s_b s_{ab} + s_{ab} s_b = 0)$ (anti-)BRST $s_{(a)b}$ transformations §

\[
\begin{align*}
    s_b A_\mu &= \partial_\mu C, \quad s_b C = 0, \quad s_b \bar{C} = iB, \quad s_b B = 0, \quad s_b E = 0, \\
    s_{ab} A_\mu &= \partial_\mu C, \quad s_{ab} C = 0, \quad s_{ab} \bar{C} = -iB, \quad s_{ab} B = 0, \quad s_{ab} E = 0.
\end{align*}
\]

The key point to be noted, at this stage, is the fact that the kinetic energy term (more precisely the electric field itself), owing its origin to the exterior derivative $\delta$, remains invariant under the (anti-)BRST transformations. In contrast, under the following local, off-shell nilpotent $(s^2_{(a)d} = 0)$ and anticommuting $(s_d s_{ad} + s_{ad} s_d = 0)$ (anti-)co-BRST (or (anti-)dual-BRST) transformations $s_{(a)d}$

\[
\begin{align*}
    s_d A_\mu &= -\varepsilon_{\mu\nu} \partial_\nu \bar{C}, \quad s_d C = 0, \quad s_d \bar{C} = -iB, \\
    s_d B &= 0, \quad s_d B = 0, \quad s_d (\partial \cdot A) = 0, \\
    s_{ad} A_\mu &= -\varepsilon_{\mu\nu} \partial_\nu C, \quad s_{ad} C = 0, \quad s_{ad} \bar{C} = +iB, \\
    s_{ad} B &= 0, \quad s_{ad} B = 0, \quad s_{ad} (\partial \cdot A) = 0,
\end{align*}
\]

it is the gauge-fixing term (more precisely $(\partial \cdot A)$ itself), owing its origin to the co-exterior derivative $\delta$ and $A^{(1)}$, remains invariant. The anticommutator $(s_w = \{s_b, s_d\} = \{s_{ab}, s_{ad}\})$ of the (anti-)BRST- and (anti-)co-BRST transformations leads to the existence of a non-nilpotent $s^2_w \neq 0$ bosonic symmetry transformation in the theory [36-38] under which the (anti-)ghost fields do not transform at all. This bosonic symmetry is the analogue of the Laplacian operator of the differential geometry. There exists a global ghost scale symmetry transformation: $s_g A_\mu = 0, s_g B = 0, s_g \bar{C} = 0, s_g C = -\Lambda \bar{C}, s_g \bar{C} = +\Lambda C$, under which, the Lagrangian density (2.2) remains invariant. Here $\Lambda$ is an infinitesimal spacetime independent (global) parameter. All the above six symmetry transformations can be concisely expressed, in terms of the generic local field $\Sigma(x) = A_\mu(x), C(x), \bar{C}(x), B(x), \bar{B}(x)$, as

\[s_r \Sigma(x) = -i [\Sigma(x), Q_r]_\pm, \quad r = b, ab, d, ad, w, g,\]

where $(+)$ signs on the square brackets stand for the (anti-)commutator for the generic local field $\Sigma$ being (fermionic)bosonic in nature. Here $Q_r$ are the generator of transformations which can be derived from the Noether’s theorem. Their exact form is not required for our present discussion but their explicit and exact form can be found in [36-38].

2.1 Superfield formulation of (anti-)BRST symmetries: a concise review

We begin here with a four $(2+2)$-dimensional supermanifold parametrized by the superspace coordinates $Z^M = (x^\mu, \theta, \bar{\theta})$ where $x^\mu (\mu = 0, 1)$ are the two even (bosonic) spacetime coordinates and $\theta, \bar{\theta}$ are the two odd (Grassmannian) coordinates (with $\theta^2 = \bar{\theta}^2 = 0, \theta \bar{\theta} +$ §We follow here the notations adopted in refs. [3,35]. In its full blaze of glory, the nilpotent $(\delta^2_B = 0)$ BRST transformations $\delta_B$ are the product of an anticommuting $(\eta \bar{C} + C \eta = 0, \eta \bar{C} + \bar{C} \eta = 0)$ spacetime independent parameter $\eta$ and $s_b$ as: $\delta_B = \eta s_b$ where the nilpotency property is carried by $s_b$ (with $s_b^2 = 0$).
\( \tilde{\theta} \theta = 0 \). On this supermanifold, one can define a supervector 1-form superfield \( \tilde{A}^{(1)} = dZ^M \tilde{A}_M(x, \theta, \tilde{\theta}) \) with the following component multiplet superfields (see, e.g., [13,12])

\[
\tilde{A}_M(x, \theta, \tilde{\theta}) = (B_\mu(x, \theta, \tilde{\theta}), \Phi(x, \theta, \tilde{\theta}), \tilde{\Phi}(x, \theta, \tilde{\theta})).
\]  

(2.6)

It will be noted that component superfields \( B_\mu(x, \theta, \tilde{\theta}), \Phi(x, \theta, \tilde{\theta}), \tilde{\Phi}(x, \theta, \tilde{\theta}) \) are the generalization of the basic local fields \( A_\mu(x), C(x), \tilde{C}(x) \), defined on the 2D ordinary spacetime manifold, to the four \((2 + 2)\)-dimensional supermanifold. The most general expansion of these superfields along the Grassmannian directions of the supermanifold, is \([13,27-34]\)

\[
\begin{align*}
B_\mu(x, \theta, \tilde{\theta}) &= A_\mu(x) + \theta \tilde{R}_\mu(x) + \tilde{\theta} R_\mu(x) + i \theta \tilde{\theta} S_\mu(x), \\
\Phi(x, \theta, \tilde{\theta}) &= C(x) + i \theta B(x) - i \tilde{\theta} \tilde{B}(x) + i \theta \tilde{\theta} s(x), \\
\tilde{\Phi}(x, \theta, \tilde{\theta}) &= \tilde{C}(x) - i \theta \tilde{B}(x) + i \theta \tilde{\theta} \tilde{s}(x),
\end{align*}
\]  

(2.7)

where \((+)-\) signs in the above expansion have been chosen for the algebraic convenience. It should be noted that (i) in the limit \( \theta \to 0, \tilde{\theta} \to 0 \), we get back the local basic fields \( A_\mu(x), C(x), \tilde{C}(x) \) of the theory from the superfields \( B_\mu(x, \theta, \tilde{\theta}), \Phi(x, \theta, \tilde{\theta}), \tilde{\Phi}(x, \theta, \tilde{\theta}) \).

(ii) The fermionic degrees of freedom \((C, \tilde{C}, R_\mu, \tilde{R}_\mu, s, \tilde{s})\) match with that of the bosonic \((A_\mu, S_\mu, B, \bar{B}, \bar{B}, \bar{B})\) degrees of freedom so that the expansion can be consistent with the basic tenets of supersymmetry. (iii) All the fields on the r.h.s. of the expansion are the local functions of spacetime \( x^\mu \) alone.

The secondary fields (i.e. \( R_\mu, \tilde{R}_\mu, S_\mu, s, \tilde{s} \)) can be expressed in terms of the basic fields (i.e. \( A_\mu, C, \tilde{C}, B, \bar{B} \)) of the Lagrangian density (2.2) by exploiting the horizontality condition \((\tilde{F}^{(2)} = F^{(2)})\) where the super curvature 2-form \( \tilde{F}^{(2)} = \tilde{d}\tilde{A}^{(1)} \), defined on the \((2 + 2)\)-dimensional supermanifold, is equated with the ordinary 2-form curvature \( F^{(2)} = dA^{(1)} \), defined on the 2D ordinary flat Minkowskian spacetime manifold. The explicit expressions for these forms are

\[
\begin{align*}
\tilde{F}^{(2)} &= \frac{1}{2} (dZ^M \wedge dZ^N) \tilde{F}_{MN} = \tilde{d}\tilde{A}^{(1)}, \\
F^{(2)} &= \frac{1}{2} (dx^\mu \wedge dx^\nu) F_{\mu\nu} = dA^{(1)},
\end{align*}
\]  

(2.8)

where the exact expressions for \( \tilde{d}, \tilde{A}^{(1)} \) and \( \tilde{d}\tilde{A}^{(1)} = \tilde{F}^{(2)} \) (constructed by \( \tilde{d}, \tilde{A}^{(1)} \)), are

\[
\begin{align*}
\tilde{d} &= dZ^M \partial_M = dx^\mu \partial_\mu + d\theta \partial_\theta + d\tilde{\theta} \partial_{\tilde{\theta}}, \\
\tilde{A}^{(1)} &= dZ^M \tilde{A}_M = dx^\mu B_\mu(x, \theta, \tilde{\theta}) + d\theta \Phi(x, \theta, \tilde{\theta}) + d\tilde{\theta} \tilde{\Phi}(x, \theta, \tilde{\theta}),
\end{align*}
\]  

(2.9)

\[
\begin{align*}
\tilde{F}^{(2)} &= \frac{1}{2} \left( dx^\mu \wedge dx^\nu \right) \left( \partial_\mu B_\nu - (d\theta \wedge d\tilde{\theta}) (\partial_\theta \Phi - \partial_{\tilde{\theta}} B_\nu) \\
&\quad + (dx^\mu \wedge d\theta)(\partial_\mu \Phi - \partial_{\tilde{\theta}} B_\nu) - (d\theta \wedge d\tilde{\theta})(\partial_\theta \Phi - \partial_{\tilde{\theta}} B_\nu)ight).
\end{align*}
\]  

(2.10)

Ultimately, the application of soul-flatness (horizontality) condition \((\tilde{d}\tilde{A}^{(1)} = dA^{(1)})\), leads to the following restrictions (cf. (2.11)) and thereby the ensuing relationships (cf. (2.12))

\[
\begin{align*}
\partial_\mu \Phi &= \partial_\theta B_\mu, \\
\partial_\mu R_\nu &= \partial_\nu R_\mu, \\
\partial_\mu \tilde{R}_\nu &= \partial_\nu \tilde{R}_\mu, \\
\partial_\theta \Phi &= \partial_{\tilde{\theta}} \tilde{\Phi} = 0, \\
\partial_{\tilde{\theta}} \Phi &= \partial_\theta \Phi = 0,
\end{align*}
\]  

(2.11)

\[
\begin{align*}
R_\mu (x) &= \partial_\mu C(x), \\
\tilde{R}_\mu (x) &= \partial_\mu \tilde{C}(x), \\
S_\mu (x) &= \partial_\mu B (x), \\
\mathcal{B} (x) &= \mathcal{B} (x) = 0, \\
\tilde{S} (x) &= \tilde{s} (x) = 0, \\
B(x) + \tilde{B}(x) &= 0.
\end{align*}
\]  

(2.12)
The insertion of all the above values into the most general expansion (2.7) on the (2 + 2)-dimensional supermanifold leads to the derivation of the off-shell nilpotent (anti-)BRST transformations for the most basic fields $A_{\mu}, C, \bar{C}$ as expressed below

$$
B_{\mu} (x, \theta, \bar{\theta}) = A_{\mu} (x) + \theta (s_{ab} A_{\mu} (x)) + \bar{\theta} (s_b A_{\mu} (x)) + \theta \bar{\theta} (s_b s_{ab} A_{\mu} (x)),
$$

$$
\Phi (x, \theta, \bar{\theta}) = C (x) + \theta (s_{ab} C (x)) + \bar{\theta} (s_b C (x)) + \theta \bar{\theta} (s_b s_{ab} C (x)),
$$

$$
\bar{\Phi} (x, \theta, \bar{\theta}) = \bar{C} (x) + \theta (s_{ab} \bar{C} (x)) + \bar{\theta} (s_b \bar{C} (x)) + \theta \bar{\theta} (s_b s_{ab} \bar{C} (x)).
$$

(2.13)

It should be noted that (i) the third- and the fourth terms in the above expansion of $\Phi (x, \theta, \bar{\theta})$ and the second- and the fourth terms of the above expansion of $\bar{\Phi} (x, \theta, \bar{\theta})$ are exactly equal to zero because $s_b C = 0, s_{ab} C = 0, s_{(a)b} B = 0$. (ii) A comparison with (2.5) establishes the geometrical interpretation for the nilpotent $(Q^2_{(a)b} = 0)$ (anti-)BRST charges $Q_{(a)b}$ as the translation generators along $(\theta)\bar{\theta}$-directions of the supermanifold. In fact, there exists a mapping

$$
\begin{align*}
& s_b \leftrightarrow \text{Lim}_{\theta \to 0} \frac{\partial}{\partial \theta} \leftrightarrow Q_b, \quad s_{ab} \leftrightarrow \text{Lim}_{\bar{\theta} \to 0} \frac{\partial}{\partial \bar{\theta}} \leftrightarrow Q_{ab}, \\
& (2.14)
\end{align*}
$$

among the (anti-)BRST transformations $s_{(a)b}$, the translation generators along $(\theta)\bar{\theta}$-direction of the supermanifold and the nilpotent (anti-)BRST charges $Q_{(a)b}$. (iii) The geometrical interpretation of the nilpotency property is contained in the translations generators which satisfy $(\partial / \partial \theta)^2 = (\partial / \partial \bar{\theta})^2 = 0$. (iv) The anticommutativity properties of the transformations $s_b s_{ab} + s_{ab} s_b = 0$ and their corresponding generators $Q_b Q_{ab} + Q_{ab} Q_b = 0$ are reflected in the specific property of the translation generators $(\partial / \partial \theta)(\partial / \partial \bar{\theta}) + (\partial / \partial \bar{\theta})(\partial / \partial \theta) = 0$. (v) Under the (anti-)BRST transformations, the superfields $(\Phi)\bar{\Phi}$ convert themselves from the general superfields (cf. (2.7)) to the (anti-)chiral superfields (i.e. $\Phi (x, \theta, \bar{\theta}) = C (x) - i \theta B (x), \bar{\Phi} (x, \theta, \bar{\theta}) = \bar{C} (x) + i \bar{\theta} B (x)$) because these satisfy $\partial_{\theta} \Phi = 0, \partial_{\bar{\theta}} \bar{\Phi} = 0$.

2.2 Hodge duality operation on (2 + 2)-dimensional supermanifold

It is evident from the previous Section that we have been able to derive the local, covariant, nilpotent $(s^2_{(a)b} = 0)$ and anticommuting $(s_b s_{ab} + s_{ab} s_b = 0)$ (anti-)BRST symmetry transformations $s_{(a)b}$ without any recourse to the definition of the Hodge duality operation. This is primarily due to the fact that we have exploited only the (super) exterior derivatives ($\bar{d}$)$d$ and the (super) 1-form connections ($\bar{A}^{(1)} A^{(1)}$) in the horizontality condition $\bar{d} A^{(1)} = d A^{(1)}$ where the Hodge duality operation plays no role at all. For the derivation of the local, covariant, nilpotent $(s^2_{(a)d} = 0)$ and anticommuting $(s_d s_{ad} + s_{ad} s_d = 0)$ (anti-)co-BRST symmetry transformations $s_{(a)d}$, we have to tap the potential and power of the super co-exterior derivative $\delta = - \star d \star$ and the ordinary co-exterior derivative $\delta = - \star d \star$ in the dual-horizontality condition $\tilde{\delta} \bar{A}^{(1)} = \delta A^{(1)}$, where (i) $\star$ and $\ast$ are the Hodge duality operations, and (ii) $A^{(1)}$ and $\bar{A}^{(1)}$ are the (super) 1-form connections on the supermanifold and ordinary manifold, respectively. On the four (2 + 2)-dimensional supermanifold, there exist three independent 4-forms (and their linear combinations are also allowed). These
independent 4-forms are
\[ \phi_1 = \frac{1}{2!}(dx^\mu \wedge dx^\nu \wedge d\theta \wedge d\bar{\theta}) F_{\mu\nu\theta\bar{\theta}}, \quad \phi_2 = \frac{1}{2!}(dx^\mu \wedge dx^\nu \wedge d\bar{\theta} \wedge d\theta) F_{\mu\nu\theta\bar{\theta}}, \]
\[ \phi_3 = \frac{1}{2!}(dx^\mu \wedge dx^\nu \wedge d\theta \wedge d\bar{\theta}) F_{\mu\nu\theta\bar{\theta}}. \]  
(2.15)

It will be noted that (i) the wedge product between the pure Grassmannian differentials is symmetric (i.e. \( d\theta \wedge d\bar{\theta} = d\bar{\theta} \wedge d\theta, d\theta \wedge d\bar{\theta} = d\bar{\theta} \wedge d\theta, d\theta \wedge d\bar{\theta} = d\bar{\theta} \wedge d\theta \)), the wedge product between the pure spacetime differentials is antisymmetric (i.e. \( dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu \)), and the wedge product between the mixed differentials is also antisymmetric (i.e \( dx^\mu \wedge d\theta = -d\theta \wedge dx^\mu, dx^\mu \wedge d\bar{\theta} = -d\bar{\theta} \wedge dx^\mu \)). Accordingly, the covariant indices of \( F \)'s will also be symmetric as well as antisymmetric corresponding to our specific choice of these indices. (ii) On the (2 + 2)-dimensional supermanifold, more than two spacetime- as well as two Grassmannian differentials (e.g. \( dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge d\theta, dx^\mu \wedge d\theta \wedge d\theta \wedge d\bar{\theta} \)) are not allowed. (iii) For the present supermanifold, the overall numerical factors (e.g. \( \frac{1}{2!} \)), present in the definition of the superforms (e.g. (2.15)), correspond to such numerical factors present in the definition of ordinary forms on the ordinary spacetime manifold. (iv) The Hodge duality \( \star \) operation for some selected superforms on a six (4 + 2)-dimensional supermanifold have been defined in our earlier work [42]. However, some ad-hoc assumptions have been made in [42]. No such assumptions have been made in our present Hodge duality \( \star \) definitions. (v) The operation of the Hodge duality on a given form does not affect \( F \)'s per se. However, the wedge products, present in the above forms, are affected by the Hodge duality operation. For instance, a single Hodge duality \( \star \) operation on the wedge product of the above cited differentials of the 4-forms, on the (2 + 2)-dimensional supermanifold, is
\[ \star (dx^\mu \wedge dx^\nu \wedge d\theta \wedge d\bar{\theta}) = \varepsilon^{\mu\nu} s^{\theta\bar{\theta}}, \]
\[ \star (dx^\mu \wedge dx^\nu \wedge d\bar{\theta} \wedge d\theta) = \varepsilon^{\mu\nu} s^{\theta\bar{\theta}}, \]
which, ultimately, imply the following zero-forms:
\[ \star \phi_1 = \frac{1}{2!} \varepsilon^{\mu\nu} F_{\mu\nu\theta\bar{\theta}}, \quad \star \phi_2 = \frac{1}{2!} \varepsilon^{\mu\nu} s^{\theta\bar{\theta}} F_{\mu\nu\theta\bar{\theta}}, \quad \star \phi_3 = \frac{1}{2!} \varepsilon^{\mu\nu} s^{\theta\bar{\theta}} F_{\mu\nu\theta\bar{\theta}}. \]
(2.17)

At this juncture, a few comments are in order. First, in contrast to the ordinary spacetime differentials where \( (dx^\mu \wedge dx^\mu) = 0 \), the Grassmann differentials of the form \((d\theta \wedge d\theta)\) and \((d\bar{\theta} \wedge d\bar{\theta})\) are non-zero on the supermanifold. Second, the coordinates \( x^0, x^1, \theta, \bar{\theta} \) correspond to the four linearly independent directions on the (2 + 2)-dimensional supermanifold. This is why, a single \( \star \) operation on \((dx^\mu \wedge dx^\nu \wedge d\theta \wedge d\bar{\theta})\) yields only \( \varepsilon^{\mu\nu} \) on the r.h.s. The same does not happen when we take a single \( \star \) operation on \((dx^\mu \wedge dx^\nu \wedge d\theta \wedge d\theta)\) and \((dx^\mu \wedge dx^\nu \wedge d\bar{\theta} \wedge d\bar{\theta})\) because \((d\theta \wedge d\theta)\) and \((d\bar{\theta} \wedge d\bar{\theta})\) do not incorporate the linearly independent differentials \( d\theta \) and \( d\bar{\theta} \) together. Third, the symmetric quantities \( s^{\theta\bar{\theta}} \) and \( s^{\theta\bar{\theta}} \) have been introduced so that one can keep track of the Grassmannian wedge products when a second Hodge duality operation is applied on a given form. For instance, two successive \( \star \) operations on the wedge products corresponding to the independent 4-forms, yield the following
\[ \star [\star (dx^\mu \wedge dx^\nu \wedge d\theta \wedge d\bar{\theta})] = - (dx^\mu \wedge dx^\nu \wedge d\theta \wedge d\theta), \]
\[ \star [\star (dx^\mu \wedge dx^\nu \wedge d\bar{\theta} \wedge d\theta)] = - (dx^\mu \wedge dx^\nu \wedge d\bar{\theta} \wedge d\theta), \]
\[ \star [\star (dx^\mu \wedge dx^\nu \wedge d\theta \wedge d\bar{\theta})] = - (dx^\mu \wedge dx^\nu \wedge d\theta \wedge d\theta), \]
(2.18)
where we have used the following inputs while taking the second \( \star \) operation
\[
\star [\varepsilon^\mu_\nu] = \frac{1}{2^}\varepsilon_{\sigma\rho} (dx^\sigma \wedge dx^\rho \wedge d\theta \wedge d\bar{\theta}) \varepsilon^{\mu_\nu},
\star [\varepsilon^\mu_\nu s^{\theta\bar{\theta}}] = \frac{1}{2^}\varepsilon_{\sigma\rho} (dx^\sigma \wedge dx^\rho \wedge d\theta \wedge d\bar{\theta}) \varepsilon^{\mu_\nu},
\star [\varepsilon^\mu_\nu \bar{s}^{\theta\bar{\theta}}] = \frac{1}{2^}\varepsilon_{\sigma\rho} (dx^\sigma \wedge dx^\rho \wedge d\theta \wedge d\bar{\theta}) \varepsilon^{\mu_\nu}.
\]

Thus, it is clear that the presence of the constant symmetric factors \( s^{\theta\bar{\theta}}, \bar{s}^{\theta\bar{\theta}} \) in (2.16) do provide a kind of guidance for the operation of a couple of Hodge duality \( \star \) operations on a given wedge product (see, e.g., (2.18) and (2.19)). The double \( \star \) operations are essential because our \( \star \) definition should comply with the general requirements of a duality invariant theory where \( \star (\star G) = \pm G \) is true for any arbitrary form \( G \) (see, e.g., [39]).

Let us concentrate now on the 3-forms. These independent forms are five in number on the \((2 + 2)\)-dimensional supermanifold. These are as given below
\[
\tau_1 = \frac{1}{2^} (dx^\mu \wedge dx^\nu \wedge d\theta) T_{\mu\nu\theta}, \quad \tau_2 = \frac{1}{2^} (dx^\mu \wedge dx^\nu \wedge d\bar{\theta}) T_{\mu\nu\bar{\theta}},
\tau_3 = (dx^\mu \wedge d\theta \wedge d\bar{\theta}) T_{\mu\theta\bar{\theta}}, \quad \tau_4 = (dx^\mu \wedge d\bar{\theta} \wedge d\bar{\theta}) T_{\mu\bar{\theta}\bar{\theta}},
\tau_5 = (dx^\mu \wedge d\theta \wedge d\bar{\theta}) T_{\mu\theta\bar{\theta}}.
\]

As discussed earlier, the operation of the Hodge duality would affect the wedge products. This is why, we shall obtain a set of 1-forms as the dual to the above 3-forms. The explicit expressions for a single \( \star \) operation on the wedge products corresponding to 3-forms, are
\[
\star (dx^\mu \wedge dx^\nu \wedge d\theta) = \varepsilon^{\mu_\nu} (d\bar{\theta}), \quad \star (dx^\mu \wedge dx^\nu \wedge d\bar{\theta}) = \varepsilon^{\mu_\nu} (d\theta),
\star (dx^\mu \wedge d\theta \wedge d\bar{\theta}) = \varepsilon^{\mu_\nu} (dx^\nu), \quad \star (dx^\mu \wedge d\bar{\theta} \wedge d\bar{\theta}) = \varepsilon^{\mu_\nu} (dx^\nu) s^{\theta\bar{\theta}},
\star (dx^\mu \wedge d\theta \wedge d\bar{\theta}) = \varepsilon^{\mu_\nu} (dx^\nu) \bar{s}^{\theta\bar{\theta}}.
\]

Application of (2.21) to (2.20) (with inputs as the analogue of (2.19)) imply
\[
\star \tau_1 = \frac{1}{2^} \varepsilon^{\mu_\nu} (d\bar{\theta}) T_{\mu\nu\theta}, \quad \star \tau_2 = \frac{1}{2^} \varepsilon^{\mu_\nu} (d\theta) T_{\mu\nu\bar{\theta}},
\star \tau_3 = \varepsilon^{\mu_\nu} s^{\theta\bar{\theta}} (dx^\nu) T_{\mu\theta\bar{\theta}}, \quad \star \tau_4 = \varepsilon^{\mu_\nu} \bar{s}^{\theta\bar{\theta}} (dx^\nu) T_{\mu\bar{\theta}\bar{\theta}},
\star \tau_5 = \varepsilon^{\mu_\nu} (dx^\nu) T_{\mu\theta\bar{\theta}},
\]
which are dual to the 3-forms given in (2.20). The double \( \star \) operation on the wedge products corresponding to 3-forms, are
\[
\star [\star (dx^\mu \wedge dx^\nu \wedge d\theta)] = -(dx^\mu \wedge dx^\nu \wedge d\theta),
\star [\star (dx^\mu \wedge dx^\nu \wedge d\bar{\theta})] = -(dx^\mu \wedge dx^\nu \wedge d\bar{\theta}),
\star [\star (dx^\mu \wedge d\theta \wedge d\bar{\theta})] = +(dx^\mu \wedge d\theta \wedge d\bar{\theta}),
\star [\star (dx^\mu \wedge d\bar{\theta} \wedge d\bar{\theta})] = +(dx^\mu \wedge d\bar{\theta} \wedge d\bar{\theta}),
\star [\star (dx^\mu \wedge d\theta \wedge d\bar{\theta})] = +(dx^\mu \wedge d\theta \wedge d\bar{\theta}).
\]

There exist six independent 2-forms on the four \((2 + 2)\)-dimensional supermanifold as
\[
\chi_1 = \frac{1}{2^} (dx^\mu \wedge dx^\nu) S_{\mu\nu}, \quad \chi_2 = (d\theta \wedge d\bar{\theta}) S_{\theta\bar{\theta}},
\chi_3 = (d\theta \wedge d\bar{\theta}) S_{\theta\bar{\theta}}, \quad \chi_4 = (d\theta \wedge d\bar{\theta}) S_{\theta\bar{\theta}},
\chi_5 = (dx^\mu \wedge d\theta) S_{\mu\theta}, \quad \chi_6 = (dx^\mu \wedge d\bar{\theta}) S_{\mu\bar{\theta}}.
\]
A single $\ast$ operation on the wedge products corresponding to the above 2-forms are as
\begin{align*}
\ast (dx^\mu \wedge dx^\nu) &= \varepsilon^{\mu\nu} (d\theta \wedge d\bar{\theta}), \\
\ast (d\theta \wedge d\theta) &= \frac{1}{2} \sigma^a \varepsilon^{\mu\nu} (dx_\mu \wedge dx_\nu), \\
\ast (d\theta \wedge d\bar{\theta}) &= \frac{1}{2} \sigma^a \varepsilon^{\mu\nu} (dx_\mu \wedge dx_\nu), \\
\ast (d\mu \wedge d\bar{\theta}) &= \varepsilon^{\mu\nu} (dx_\nu \wedge d\bar{\theta}),
\end{align*}
(2.25)
which clearly establish the fact that the dual of 2-forms (cf. 2.24) are 2-forms on a four $(2 + 2)$-dimensional supermanifold as listed below
\begin{align*}
\ast \chi_1 &= \frac{1}{2} \varepsilon^{\mu\nu} (d\theta \wedge d\bar{\theta}) S_{\mu\nu}, \\
\ast \chi_3 &= \frac{1}{2} \sigma^a \varepsilon^{\mu\nu} (dx_\mu \wedge dx_\nu) S_\theta, \\
\ast \chi_5 &= \varepsilon^{\mu\nu} (dx_\nu \wedge d\bar{\theta}) S_{\mu\theta}, \\
\ast \chi_2 &= \frac{1}{2} \varepsilon_{\sigma\rho} (dx^\sigma \wedge dx^\rho) S_{\theta\bar{\theta}}, \\
\ast \chi_4 &= \frac{1}{2} \sigma^a \varepsilon^{\mu\nu} (dx_\mu \wedge dx_\nu) S_{\bar{\theta}\bar{\theta}}, \\
\ast \chi_6 &= \varepsilon^{\mu\nu} (dx_\nu \wedge d\theta) S_{\mu\bar{\theta}}.
\end{align*}
(2.26)
The double $\ast$ operation on the wedge products corresponding to the six independent 2-forms on the $(2 + 2)$-dimensional supermanifold is
\begin{align*}
\ast \ast (dx^\mu \wedge dx^\nu) &= -(dx^\mu \wedge dx^\nu), \\
\ast \ast (d\theta \wedge d\theta) &= -(d\theta \wedge d\bar{\theta}), \\
\ast \ast (d\theta \wedge d\bar{\theta}) &= + (dx^\mu \wedge d\theta), \\
\ast \ast (dx^\mu \wedge d\theta) &= + (dx^\mu \wedge d\theta),
\end{align*}
(2.27)
It is straightforward to guess that there exist only three independent 1-forms on the four $(2 + 2)$-dimensional supermanifold as $^\dagger$
\begin{align*}
\alpha_1 &= (dx^\mu) A_\mu, \\
\alpha_2 &= (d\theta) A_\theta, \\
\alpha_3 &= (d\bar{\theta}) A_{\bar{\theta}}.
\end{align*}
(2.28)
A single $\ast$ operation on the above independent 1-forms would lead to the 3-forms on the four $(2 + 2)$-dimensional supermanifold. The operation of the single Hodge duality on the independent 1-form differentials are
\begin{align*}
\ast (dx^\mu) &= \varepsilon^{\mu\nu} (dx_\nu \wedge d\theta \wedge d\bar{\theta}), \\
\ast (d\theta) &= \frac{1}{2} \sigma^a \varepsilon^{\mu\nu} (dx_\mu \wedge dx_\nu \wedge d\bar{\theta}), \\
\ast (d\bar{\theta}) &= \frac{1}{2} \sigma^a \varepsilon^{\mu\nu} (dx_\mu \wedge dx_\nu \wedge d\theta),
\end{align*}
(2.29)
which finally imply the following independent 3-forms corresponding to the independent 1-forms of equation (2.28), defined on the $(2 + 2)$-dimensional supermanifold, namely;
\begin{align*}
\ast \alpha_1 &= \varepsilon^{\mu\nu} (dx_\nu \wedge d\theta \wedge d\bar{\theta}) A_\mu, \\
\ast \alpha_3 &= \frac{1}{2} \sigma^a \varepsilon^{\mu\nu} (dx_\mu \wedge dx_\nu \wedge d\theta) A_{\bar{\theta}}.
\end{align*}
(2.30)
The result of a couple of successive $\ast$ operations on the differentials, corresponding to the 1-forms on the $(2 + 2)$-dimensional supermanifold, is given by
\begin{align*}
\ast \ast (dx^\mu) &= + (dx^\mu), \\
\ast \ast (d\theta) &= - (d\theta), \\
\ast \ast (d\bar{\theta}) &= - (d\bar{\theta}).
\end{align*}
(2.31)

$^\dagger$In general, a set of three 1-forms can be constructed from the spacetime differential $(dx_\mu)$. These are $\alpha_1^{(1)} = dx^\mu A_\mu^{(1)}, \alpha_1^{(2)} = dx^\mu \sigma^a A_\mu^{(2)}, \alpha_1^{(3)} = dx^\mu \sigma^a A_\mu^{(3)}$. A single $\ast$ operation yields $\ast (dx^\mu) = \varepsilon^{\mu\nu} (dx_\nu \wedge d\theta \wedge d\bar{\theta}), \ast (dx^\mu \sigma^a) = \varepsilon^{\mu\nu} (dx_\nu \wedge d\theta \wedge d\bar{\theta}), \ast (dx^\mu \sigma^a) = \varepsilon^{\mu\nu} (dx_\nu \wedge d\theta \wedge d\bar{\theta})$. For obvious reasons, such kind of a triplet of 1-forms can not be constructed from the 1-forms $\alpha_2$ as well as $\alpha_3$ because their Hodge dual forms are not well defined on a $(2 + 2)$-dimensional supermanifold.
We shall be exploiting the above Hodge duality operations on the wedge products of the differentials of a given form in the forthcoming Section 2.3 in the context of the derivation of the (anti-)co-BRST symmetries for the 2D free 1-form Abelian gauge theory considered on a four \((2 + 2)\)-dimensional supermanifold.

### 2.3 Superfield formulation of (anti-)co-BRST symmetries for 2D theory

It is clear from the symmetry transformations (2.4) that the local, covariant, continuous, nilpotent \(s_{(a)d}^2 = 0\) and anticommuting \(s_{d} s_{a} + s_{a} s_{d} = 0\) (anti-)co-BRST symmetries \(s_{(a)d}\) exist for the Lagrangian density (2.2) describing the free (non-interacting) Abelian gauge theory on the flat 2D Minkowskian spacetime manifold. Exploiting the dual-horizontality condition \(\delta \tilde{A}^{(1)} = \delta A^{(1)}\) with the following inputs

\[
\delta \tilde{A}^{(1)} = - \star \tilde{d} \star \tilde{A}^{(1)}, \quad \delta A^{(1)} = - \star d \star A^{(1)} = (\partial \cdot A), \tag{2.32}
\]

we expect to obtain all the secondary fields of the super expansion (2.7) in terms of the basic fields of the Lagrangian density (2.2) of the theory. Towards this goal in mind, we first explicitly compute \(\delta \tilde{A}^{(1)} = - \star \tilde{d} \star \tilde{A}^{(1)}\) taking the help of the definitions (2.9) and the Hodge duality operations discussed earlier. First, the dual \((\star A^{(1)})\) of the super 1-form connection \(\tilde{A}^{(1)} = dZ^M \tilde{A}_M\) is a 3-form on the \((2 + 2)\)-dimensional supermanifold. The explicit expression for this 3-form (i.e. dual to the 1-form super connection \(\tilde{A}^{(1)}\)) is

\[
\star \tilde{A}^{(1)} = \epsilon^{\mu \nu} (dx_\nu \wedge d\theta \wedge d\bar{\theta}) B_\mu + \frac{1}{2!} \epsilon_{\sigma \rho} (dx_\sigma \wedge dx_\rho \wedge d\theta \wedge d\bar{\theta}) \Phi, \tag{2.33}
\]

where we have used the definition of the 1-form super connection \(\tilde{A}^{(1)}\) from (2.9) and the Hodge duality operations on the 1-forms from (2.29). We apply now the super exterior derivative \(\tilde{d} = dZ^M \partial_M\) from (2.9) on the 3-form dual super connection (2.33), the outcome is

\[
\tilde{d} (\star \tilde{A}^{(1)}) = \epsilon^{\mu \nu} (dx_\xi \wedge dx_\nu \wedge d\theta \wedge d\bar{\theta}) (\partial_\xi B_\mu) - \frac{1}{2!} \epsilon_{\sigma \rho} (dx_\sigma \wedge dx_\rho \wedge d\theta \wedge d\bar{\theta}) (\partial_\sigma \Phi) - \frac{1}{2} \epsilon_{\sigma \rho} (dx_\sigma \wedge dx_\rho \wedge d\theta \wedge d\bar{\theta}) (\partial_\sigma \Phi) - \frac{1}{2} \epsilon_{\sigma \rho} (dx_\sigma \wedge dx_\rho \wedge d\theta \wedge d\bar{\theta}) (\partial_\sigma \Phi), \tag{2.34}
\]

A few remarks are in order. First, all the wedge products with more than two spacetime differentials- as well as Grassmannian differentials are dropped out because a \((2 + 2)\)-dimensional supermanifold cannot support such forms. Second, the negative signs, in the above, have cropped up because \((d\theta \partial_\theta)(dx_\mu \wedge dx_\nu \wedge d\theta)\Phi = -(dx_\mu \wedge dx_\nu \wedge d\theta \wedge d\theta) \partial_\theta \Phi\), etc. The stage is now set for the application of the \((-\star\cdotp)\) on the above super 4-forms which will lead to the derivation of a 0-form (superscalar) on the supermanifold. Exploiting the Hodge duality operation, defined in (2.16), we obtain the following expression

\[
\delta \tilde{A}^{(1)} = - \star \tilde{d} \star \tilde{A}^{(1)} = (\partial \cdot B) - (\partial_\theta \Phi) - (\partial_\bar{\theta} \Phi) - s^{\theta \bar{\theta}} (\partial_\theta \Phi) - s^{\theta \bar{\theta}} (\partial_\bar{\theta} \Phi), \tag{2.35}
\]
where we have used $\varepsilon_{\rho\sigma} = -2!, \varepsilon^\mu_\nu \xi^\nu = -\delta^\mu_\nu$, etc. When the above superscalar is equated with the ordinary scalar (i.e. $\delta A^{(1)} = -*d*A^{(1)} = (\partial \cdot A)$) due to the requirement of the dual-horizontality condition (i.e. $\tilde{A}^{(1)} = \bar{A}^{(1)}$), we obtain the following restrictions

\begin{equation}
(\partial \cdot B) - (\partial_y \Phi + \partial_y \Phi) = (\partial \cdot A), \quad \partial_y \Phi = 0, \quad \partial_y \Phi \bar{=} 0.
\end{equation}

The insertion of the most general super expansions (cf. (2.7)) on the $(2 + 2)$-dimensional supermanifold for the superfields $B_\mu(x, \theta, \bar{\theta}), \Phi(x, \theta, \bar{\theta}), \bar{\Phi}(x, \theta, \bar{\theta})$ into the above restrictions leads to

\begin{equation}
(\partial \cdot R)(x) = 0, \quad (\partial \cdot \bar{R})(x) = 0, \quad (\partial \cdot S)(x) = 0, \quad s(x) = \bar{s}(x) = B(x) = \bar{B}(x) = 0, \quad B(x) + \bar{B}(x) = 0.
\end{equation}

It is worthwhile to mention that, unlike the horizontality condition where the secondary fields are expressed explicitly and exactly in terms of the basic fields of the Lagrangian density (2.2), the dual-horizontality condition provides only the restrictions that are quoted in (2.37). For the 2D free Abelian gauge theory, the local, covariant and continuous solutions for the above restrictions exist as given below

\begin{equation}
R_\mu = -\varepsilon_{\mu\nu}\partial^\nu C, \quad \bar{R}_\mu = -\varepsilon_{\mu\nu}\partial^\nu \bar{C}, \quad S_\mu = +\varepsilon_{\mu\nu}\partial^\nu B.
\end{equation}

Substitution of the above values into the most general super expansion in (2.7) leads to the following expression for the expansion vis-à-vis the off-shell nilpotent (anti-)co-BRST transformations of equation (2.4):

\begin{align*}
B_\mu(x, \theta, \bar{\theta}) &= A_\mu(x) + \theta (s_{a\sigma} A_\mu(x)) + \bar{\theta} (s_{d\sigma} A_\mu(x)) + \theta \bar{\theta} (s_d s_{a\sigma} A_\mu(x)), \\
\Phi(x, \theta, \bar{\theta}) &= C(x) + \theta (s_{a\sigma} C(x)) + \bar{\theta} (s_{d\sigma} C(x)) + \theta \bar{\theta} (s_d s_{a\sigma} C(x)), \\
\bar{\Phi}(x, \theta, \bar{\theta}) &= \bar{C}(x) + \theta (s_{a\sigma} \bar{C}(x)) + \bar{\theta} (s_{d\sigma} \bar{C}(x)) + \theta \bar{\theta} (s_d s_{a\sigma} \bar{C}(x)).
\end{align*}

This equation is the analogue of the expansion in (2.13) where (anti-)BRST symmetry transformations have been derived. It is clear from (2.39) (which produces the (anti-)co-BRST transformations for the basic fields $A_\mu, C, \bar{C}$) that (anti-)co-BRST nilpotent charges $Q_{(a)d}$, similar to the (anti-)BRST charges $Q_{(a)b}$, correspond to the translation generators $(\partial/\partial \theta, \partial/\partial \bar{\theta})$ along the Grassmannian directions $(\theta, \bar{\theta})$ of the supermanifold. However, there is a clear-cut distinction between these two sets of charges when it comes to the discussion of the nilpotent transformations for the (anti-)ghost fields corresponding to the fermionic superfields $\Phi$ and $\bar{\Phi}$. For instance, under the nilpotent anti-BRST transformations, the superfield $\Phi$ becomes anti-chiral (i.e. $\Phi = C + \theta (s_{a\sigma} C(x))$) but the same superfield becomes chiral (i.e. $\Phi = C(x) + \bar{\theta} (s_d C(x))$) due to the co-BRST transformations. Similar arguments and interpretations can be provided for the nature of the superfield $\Phi$ as far as the off-shell nilpotent BRST and anti-co-BRST transformations are concerned.

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\[\text{It will be noted that the non-local and non-covariant solutions to the restrictions (2.37) also exist. For the 2D Abelian case, we have } R_0 = i\bar{C}, R_1 = i(\partial_0 \partial_1/\nabla^2)\bar{C}, R_2 = i\bar{C}, R_3 = i(\partial_0 \partial_1/\nabla^2)\bar{C}, \text{ etc. However, for our present discussions, we avoid such kind of pathological choices. In fact, for the 4D Abelian theory, this kind of symmetries exist, too (see, e.g. [40,41], for details).}\]
3 (Anti-)BRST and (anti-)co-BRST symmetries for 4D theory: a brief synopsis

Let us start off with the analogue of the Lagrangian density (2.1) for the 4D free Abelian gauge theory defined on the four dimensional ** ordinary flat Minkowski spacetime manifold

\[
L_b^{(4)} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + B(\partial \cdot A) + \frac{1}{2} B^2 - i\partial_\mu \bar{C} \partial^\mu C,
\]

where the field strength tensor \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \) constructed from \( d = dx^\mu \partial_\mu \) and the 1-form \( A^{(1)} = dx^\mu A_\mu \) through \( F^{(2)} = dA^{(1)} = \frac{1}{2} (dx^\mu \wedge dx^\nu) F_{\mu\nu}, \) has the electric \( (F_{0i} = E_i = E) \) and the magnetic \( (F_{ij} = \epsilon_{ijk} B_k, B_i = B = \frac{1}{2} \epsilon_{ijk} F_{jk}) \) components and the gauge-fixing term \( (\partial \cdot A) = \partial_0 A_0 - \partial_\alpha A_\alpha \) is constructed by the application of the nilpotent \((\delta^2 = 0)\) co-exterior derivative \( \delta = - * d * \) on the 1-form \( A^{(1)} = dx^\mu A_\mu \) (i.e. \( \delta A^{(1)} = (\partial \cdot A) \)). Here the Hodge duality * operation is defined on the 4D Minkowskian flat spacetime manifold. All the other symbols carry the same meaning as discussed in Section 2. The above Lagrangian density can be linearized by introducing a couple of vector auxiliary fields \( b^{(1)}, b^{(2)} \) as [42]

\[
L_b^{(4)} = b_i^{(1)} E_i - \frac{1}{2}(b^{(1)})^2 - b_i^{(2)} B_i + \frac{1}{2}(b^{(2)})^2 + B(\partial \cdot A) + \frac{1}{2} B^2 - i\partial_\mu \bar{C} \partial^\mu C.
\]

The above Lagrangian density respects the following local, covariant, continuous, off-shell nilpotent \((s_{(a)b}^2 = 0)\) and anticommuting \((s_b s_{ab} + s_{ab} s_b = 0)\) (anti-)BRST \((s_{(a)b})\) symmetry transformations [42]

\[
\begin{align*}
s_b A_\mu &= \partial_\mu C, & s_b C &= 0, & s_b \bar{C} &= i B, & s_b B &= 0, \\
s_b B &= 0, & s_b b^{(1)} &= 0, & s_b b^{(2)} &= 0, & s_b E &= 0, \\
s_{ab} A_\mu &= \partial_\mu C, & s_{ab} C &= 0, & s_{ab} \bar{C} &= -i B, & s_{ab} B &= 0, \\
s_{ab} B &= 0, & s_{ab} b^{(1)} &= 0, & s_{ab} b^{(2)} &= 0, & s_{ab} E &= 0,
\end{align*}
\]

because (3.2) transforms to a total derivative under the above transformations. Furthermore, the same Lagrangian density is endowed with the following non-local, non-covariant, continuous, off-shell nilpotent \((s_{(a)d}^2 = 0)\) and anticommuting \((s_d s_{ad} + s_{ad} s_d = 0)\) (anti-)co-BRST symmetry transformations \((s_{(a)d})\) (see, e.g., [40–42] for details)

\[
\begin{align*}
s_d A_0 &= i C, & s_d A_i &= \frac{\partial_0 \partial_i}{\sqrt{2}} C, & s_d C &= 0, & s_d \bar{C} &= 0, & s_d B &= 0, & s_d b^{(1)} &= 0, \\
s_d C &= \frac{\partial_0 b^{(1)}}{\sqrt{2}}, & s_d B &= 0, & s_d b^{(2)} &= 0, & s_d (\partial \cdot A) &= 0, \\
s_{ad} A_0 &= i C, & s_{ad} A_i &= \frac{\partial_0 \partial_i}{\sqrt{2}} C, & s_{ad} C &= 0, & s_{ad} \bar{C} &= 0, & s_{ad} B &= 0, & s_{ad} b^{(1)} &= 0, \\
s_{ad} \bar{C} &= \frac{\partial_0 b^{(1)}}{\sqrt{2}}, & s_{ad} B &= 0, & s_{ad} b^{(2)} &= 0, & s_{ad} (\partial \cdot A) &= 0,
\end{align*}
\]

**We follow here the notations and conventions such that 4D Minkowskian manifold is endowed with a flat metric \( \eta_{\mu\nu} = \text{diag} (+1,-1,-1,-1) \) and the totally antisymmetric 4D Levi-Civita tensor \( \epsilon_{\mu\nu\lambda\kappa} \) is chosen to satisfy \( \epsilon_{0123} = +1 = -\epsilon_{0231}, \epsilon_{0ijk} = \epsilon_{ijk} = -\epsilon_{0ijk}, \epsilon_{\mu\nu\lambda\kappa} \epsilon^{\mu\nu\lambda\rho} = -4!, \epsilon_{\mu\nu\lambda\kappa} \epsilon^{\mu\nu\lambda\rho} = -3! \delta^\rho_\omega, \) etc. Here the Greek indices \( \mu, \nu, \lambda, \ldots = 0, 1, 2, 3 \) correspond to the spacetime directions on the 4D ordinary manifold and the Latin indices \( i, j, k, \ldots = 1, 2, 3 \) stand for the space directions only. The 3-vectors are occasionally represented by the bold faced letters (i.e. \( \mathbf{B} = B_i, \mathbf{E} = E_i, \mathbf{b}^{(4)} = b_i^{(1)}, \mathbf{b}^{(2)} = b_i^{(2)}, \) etc.)
where $\nabla^2 = \partial_i \partial_i = (\partial_1)^2 + (\partial_2)^2 + (\partial_3)^2$. At this stage, a few comments are in order. (i) It is clear that the (anti-)BRST symmetry transformations are local, covariant, continuous, nilpotent and anticommuting. In contrast, the (ant-)co-BRST symmetry transformations are non-local, non-covariant, continuous, nilpotent and anticommuting. (ii) The nilpotent (anti-)BRST as well as (anti-)co-BRST transformations keep the magnetic field $B$ invariant. (iii) Under the (anti-)BRST and (anti-)co-BRST transformations, the 2-forms $\{s_d, s_b\} = \{s_{ab}, s_{ad}\} = s_w$ lead to the definition of a non-nilpotent bosonic symmetry $s_w$. However, the exact expressions for these transformations are not essential for our present discussions. (v) The global scale transformations on the (anti-)ghost fields define the ghost symmetry in the theory. The corresponding conserved charge is the ghost charge $Q_{g}$. (vi) The above conserved Noether charges generate the transformations (2.5).

### 3.1 Superfield formulation of (anti-)BRST symmetries for 4D theory

We consider the free four $(3+1)$-dimensional (4D) Abelian gauge theory on a six $(4+2)$-dimensional supermanifold parametrized by the four spacetime $x^\mu (\mu = 0, 1, 2, 3)$ bosonic variables and a couple of odd $\theta^2 = \bar{\theta}^2 = 0, \theta \bar{\theta} + \bar{\theta} \theta = 0$ Grassmannian variables $\theta$ and $\bar{\theta}$. The local basic fields $(A_\mu (x), C(x), \bar{C}(x))$ of the Lagrangian density (3.1) are now generalized to the superfields $(B_\mu (x, \theta, \bar{\theta}), \Phi(x, \theta, \bar{\theta}), \bar{\Phi}(x, \theta, \bar{\theta}))$ on the six dimensional supermanifold. These latter superfields can be expanded in terms of the basic fields as given in (2.7). However, there is a subtle difference between the expansion on the four $(2+2)$-dimensional (cf. Section 2) and the six $(4+2)$-dimensional supermanifold. For instance, in the following

\begin{align}
B_\mu (x, \theta, \bar{\theta}) &= A_\mu (x) + \theta \bar{R}_\mu (x) + \bar{\theta} R_\mu (x) + i \theta \bar{\theta} S_\mu (x), \\
\Phi (x, \theta, \bar{\theta}) &= C(x) + i \theta \bar{B}(x) - i \bar{\theta} B(x) + i \theta \bar{\theta} s(x), \\
\bar{\Phi} (x, \theta, \bar{\theta}) &= \bar{C}(x) - i \theta \bar{B}(x) + i \bar{\theta} B(x) + i \theta \bar{\theta} \bar{s}(x),
\end{align}

the auxiliary scalar fields $B$ and $\bar{B}$ are not the ones that have been written for the 2D free Abelian gauge theory. In particular, the auxiliary scalar field $B$ appears explicitly in the Lagrangian density (2.2) for the 2D theory. However, it does not appear explicitly in the Lagrangian density of the 4D theory. All the rest of the steps are exactly the same (see, e.g., equations (2.8)–(2.12)) as discussed in the sub-section 2.1 for the discussion of the 2D Abelian theory on a $(2+2)$-dimensional supermanifold. Finally, the horizontality condition $\delta A^{(1)} = dA^{(1)}$ leads to the derivation of the nilpotent (anti-)BRST symmetry transformations (3.3) for the 4D free Abelian gauge theory as expressed below in the language of the superfield expansion on the six $(4+2)$-dimensional supermanifold (see, e.g., [42] for details)

\begin{align}
B_\mu (x, \theta, \bar{\theta}) &= A_\mu (x) + \theta (s_{ab} A_\mu (x)) + \bar{\theta} (s_b A_\mu (x)) + \theta \bar{\theta} (s_b s_{ab} A_\mu (x)), \\
\Phi (x, \theta, \bar{\theta}) &= C(x) + \theta (s_{ab} C(x)) + \bar{\theta} (s_b C(x)) + \theta \bar{\theta} (s_b s_{ab} C(x)), \\
\bar{\Phi} (x, \theta, \bar{\theta}) &= \bar{C}(x) + \theta (s_{ab} \bar{C}(x)) + \bar{\theta} (s_b \bar{C}(x)) + \theta \bar{\theta} (s_b s_{ab} \bar{C}(x)).
\end{align}

The above equation establishes the geometrical interpretation for the off-shell nilpotent (anti-)BRST charges $Q_{(a)b}$ as the translation generators $(\partial/\partial \theta) \partial/\partial \bar{\theta}$ along the $\theta\bar{\theta}$-
directions of the six \((4 + 2)\)-dimensional supermanifold. In fact, the process of translations of the superfields \((B_\mu, \Phi, \bar{\Phi})\) along \((\theta)\bar{\theta}\)-directions of the supermanifold produces the internal (anti-)BRST symmetry transformations \(s_{(a)b}\) (cf. (3.3)) for the local fields \((A_\mu, C, \bar{C})\).

### 3.2 Hodge duality on \((4 + 2)\)-dimensional supermanifold

To obtain the nilpotent \((s^2 \mu_{(a)d} = 0)\) and anticommuting \((s_d s_{ad} + s_{ad} s_d = 0)\) (anti-)co-BRST transformations \(s_{(a)d}\) for the basic fields \((A_\mu, C, \bar{C})\) of the 4D free Abelian gauge theory, we have to exploit the dual-horizontality condition \(\delta \bar{A}(1) = \delta A(1)\) where \(\delta = - \star d\star\) and \(\delta = - \star d\star\) are the super co-exterior derivative and the ordinary co-exterior derivative, respectively. These derivatives are defined on the six \((4 + 2)\)-dimensional supermanifold and the ordinary 4D Minkowskian spacetime manifold. As discussed earlier, \(\delta\) is the super exterior derivative (see, e.g., for the definition, equation (2.9)) and \(\star\) and \(\star\) are the Hodge duality operations on the supermanifold and the ordinary manifold, respectively. The \((4 + 2)\)-dimensional supermanifold can support only three (super) 1-forms as given below

\[
\mathcal{O}_1 = dx^\mu P_\mu, \quad \mathcal{O}_2 = d\theta P_\theta, \quad \mathcal{O}_3 = d\bar{\theta} P_{\bar{\theta}}, \quad (3.7)
\]

However, as will become clear later, a triplet of (super) 1-forms can be constructed from the differential \(dx^\mu\) that is present in the definition of \(\mathcal{O}_1\). The Hodge duality \(\star\) operation on the above 1-forms produces the following 5-forms

\[
\star \mathcal{O}_1 = \frac{1}{4!} \varepsilon^{\mu \nu \lambda \zeta} (dx_\nu \wedge dx_\lambda \wedge dx_\zeta \wedge d\theta \wedge d\bar{\theta}) P_\mu, \\
\star \mathcal{O}_2 = \frac{1}{4!} \varepsilon^{\mu \nu \lambda \zeta} (dx_\mu \wedge dx_\nu \wedge dx_\lambda \wedge dx_\zeta \wedge d\bar{\theta}) P_\theta, \\
\star \mathcal{O}_3 = \frac{1}{4!} \varepsilon^{\mu \nu \lambda \zeta} (dx_\mu \wedge dx_\nu \wedge dx_\lambda \wedge dx_\zeta \wedge d\theta) P_{\bar{\theta}}. \quad (3.8)
\]

It will be noted, in the above, that (i) the \(\star\) operation acts basically on the differentials and it does not act on \(P\)'s. (ii) A linear combination of the 1-forms of (3.7) can also be considered as 1-form. (iii) The double \(\star\) operation on the above 1-forms yields

\[
\star [\star \mathcal{O}_1] = + \mathcal{O}_1, \quad \star [\star \mathcal{O}_2] = - \mathcal{O}_2, \quad \star [\star \mathcal{O}_3] = - \mathcal{O}_3, \quad (3.9)
\]

where we have used \(\varepsilon^{\mu \nu \lambda \zeta} \varepsilon_{\nu \lambda \zeta \rho} = +3! \delta^{\mu}_{\rho}, \varepsilon^{\mu \nu \lambda \zeta} \varepsilon_{\mu \nu \lambda \zeta} = -4!\). Furthermore, we have used the \(\star\) operation on the 5-forms which are found to be dual to 1-forms. In fact, the six \((4 + 2)\)-dimensional supermanifold can support five independent 5-forms:

\[
\tilde{\phi}_1 = \frac{1}{3!} (dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge d\theta \wedge d\bar{\theta}) \bar{F}_{\mu \nu \lambda \theta \bar{\theta}}, \\
\tilde{\phi}_2 = \frac{1}{3!} (dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge d\theta \wedge d\bar{\theta}) \bar{F}_{\mu \nu \lambda \bar{\theta}}, \\
\tilde{\phi}_3 = \frac{1}{3!} (dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge d\bar{\theta} \wedge d\bar{\theta}) \bar{F}_{\mu \nu \lambda \bar{\theta}}, \\
\tilde{\phi}_4 = \frac{1}{3!} (dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\xi \wedge d\theta) \bar{F}_{\mu \nu \lambda \xi \theta}, \\
\tilde{\phi}_5 = \frac{1}{3!} (dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\xi \wedge d\bar{\theta}) \bar{F}_{\mu \nu \lambda \xi \bar{\theta}}. \quad (3.10)
\]

The Hodge duality \(\star\) operation on the wedge products of the differentials, present in the
above 5-forms, yields the following 1-form differentials

\[ \star (dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge d\theta \wedge d\bar{\theta}) = \varepsilon^{\mu \nu \lambda \zeta} (dx_\zeta), \]
\[ \star (dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge d\theta \wedge d\bar{\theta}) = s^{\theta \bar{\theta}} \varepsilon^{\mu \nu \lambda \zeta} (dx_\zeta), \]
\[ \star (dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge d\theta \wedge d\bar{\theta}) = s^{\bar{\theta} \theta} \varepsilon^{\mu \nu \lambda \zeta} (dx_\zeta), \]
\[ \star (dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\zeta \wedge d\theta) = \varepsilon^{\mu \nu \lambda \zeta} (d\theta), \]
\[ \star (dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\zeta \wedge d\bar{\theta}) = \varepsilon^{\mu \nu \lambda \zeta} (d\bar{\theta}). \]

(3.11)

The presence of the symmetric constants, \( s^{\theta \bar{\theta}} \) and \( s^{\bar{\theta} \theta} \) on the r.h.s. of (3.11), enforces the following Hodge duality \( \star \) operation

\[ \star [ s^{\theta \bar{\theta}} (dx^\mu) ] = \frac{1}{3!} \varepsilon^{\mu \nu \lambda \zeta} (dx_\nu \wedge dx_\lambda \wedge dx_\zeta \wedge d\theta \wedge d\bar{\theta}), \]
\[ \star [ s^{\bar{\theta} \theta} (dx^\mu) ] = \frac{1}{3!} \varepsilon^{\mu \nu \lambda \zeta} (dx_\nu \wedge dx_\lambda \wedge dx_\zeta \wedge d\theta \wedge d\bar{\theta}). \]

(3.12)

Taking into account (3.8), (3.11) and (3.12), it is clear that the double \( \star \) operation on the 5-forms in (3.10) leads to

\[ \star [ \star \phi_1 ] = + \phi_1, \quad \star [ \star \phi_2 ] = + \phi_2, \quad \star [ \star \phi_3 ] = + \phi_3, \]
\[ \star [ \star \phi_4 ] = - \phi_4, \quad \star [ \star \phi_5 ] = - \phi_5. \]

(3.13)

The six \((4+2)\)-dimensional supermanifold can support six 2-forms analogous to (2.24). Their explicit expressions are as under

\[ \tilde{\chi}_1 = \frac{1}{3!} (dx^\mu \wedge dx^\nu) \tilde{S}_{\mu \nu}, \quad \tilde{\chi}_2 = (d\theta \wedge d\bar{\theta}) \tilde{S}_{\theta \bar{\theta}}, \]
\[ \tilde{\chi}_3 = (d\theta \wedge d\bar{\theta}) \tilde{S}_{\bar{\theta} \theta}, \quad \tilde{\chi}_4 = (d\theta \wedge d\bar{\theta}) \tilde{S}_{\theta \bar{\theta}}, \]
\[ \tilde{\chi}_5 = (dx^\mu \wedge d\theta) \tilde{S}_{\mu \theta}, \quad \tilde{\chi}_6 = (dx^\mu \wedge d\bar{\theta}) \tilde{S}_{\mu \bar{\theta}}. \]

(3.14)

It is clear that the components \( \tilde{S}_{\mu \nu}, \tilde{S}_{\mu \theta}, \tilde{S}_{\mu \bar{\theta}} \) are antisymmetric. However, the components with the Grassmannian indices \( \tilde{S}_{\theta \bar{\theta}}, \tilde{S}_{\bar{\theta} \theta}, \tilde{S}_{\theta \bar{\theta}} \) are symmetric. On the above supermanifold, the operation of a single Hodge duality \( \star \) operation leads to the definition of 4-forms which are dual to the above 2-forms. In fact, a single \( \star \) operation on the wedge products of the differentials of the above 2-forms, are

\[ \star (dx^\mu \wedge dx^\nu) = \frac{1}{2!} \varepsilon^{\mu \nu \lambda \zeta} (dx_\lambda \wedge dx_\zeta \wedge d\theta \wedge d\bar{\theta}), \]
\[ \star (d\theta \wedge d\bar{\theta}) = \frac{1}{3!} \varepsilon^{\mu \nu \lambda \zeta} (dx_\mu \wedge dx_\nu \wedge dx_\lambda \wedge dx_\zeta), \]
\[ \star (d\theta \wedge d\bar{\theta}) = \frac{1}{3!} s^{\theta \bar{\theta}} \varepsilon^{\mu \nu \lambda \zeta} (dx_\mu \wedge dx_\nu \wedge dx_\lambda \wedge dx_\zeta), \]
\[ \star (d\theta \wedge d\bar{\theta}) = \frac{1}{3!} s^{\bar{\theta} \theta} \varepsilon^{\mu \nu \lambda \zeta} (dx_\mu \wedge dx_\nu \wedge dx_\lambda \wedge dx_\zeta), \]
\[ \star (dx^\mu \wedge d\theta) = \frac{1}{3!} \varepsilon^{\mu \nu \lambda \zeta} (dx_\nu \wedge dx_\lambda \wedge dx_\zeta \wedge d\theta), \]
\[ \star (dx^\mu \wedge d\bar{\theta}) = \frac{1}{3!} \varepsilon^{\mu \nu \lambda \zeta} (dx_\nu \wedge dx_\lambda \wedge dx_\zeta \wedge d\bar{\theta}). \]

(3.15)

\[ ^{\dagger}\text{It will be noted that there are three 1-form differentials } (dx_\mu), s^{\theta \bar{\theta}}(dx_\mu), s^{\bar{\theta} \theta}(dx_\mu), \text{ constructed from } (dx_\mu), \text{ because the dual 5-form differentials on supermanifold are different for each individual of them. For instance, } \star (dx_\mu) = \frac{1}{3!} \varepsilon^{\mu \nu \lambda \zeta} (dx^\mu \wedge dx^\lambda \wedge dx^\zeta \wedge d\theta \wedge d\bar{\theta}), \star [ s^{\theta \bar{\theta}}(dx_\mu) ] = \frac{1}{3!} \varepsilon^{\mu \nu \lambda \zeta} (dx^\mu \wedge dx^\lambda \wedge dx^\zeta \wedge d\theta \wedge d\bar{\theta}), \star [ s^{\bar{\theta} \theta}(dx_\mu) ] = \frac{1}{3!} \varepsilon^{\mu \nu \lambda \zeta} (dx^\mu \wedge dx^\lambda \wedge dx^\zeta \wedge d\theta \wedge d\bar{\theta}). \text{ Only for the sake of brevity, a single 1-form } dx^\mu P_\mu \text{ (constructed from } dx^\mu) \text{ is given in (3.7). It will be noted that such kind of a triplet of superforms cannot be associated with } \mathcal{O}_2 \text{ and } \mathcal{O}_3 \text{ because their Hodge duals are not well-defined on the six } (4+2)\text{-dimensional supermanifold of our present discussion.} \]
This shows that the wedge products of the differentials corresponding to the 4-forms in the above equations are Hodge dual to the wedge products of the differentials corresponding to 2-forms considered (cf. (3.14)) on the six \((4 + 2)\)-dimensional supermanifold. The total number of the independent 4-forms on the above supermanifold are

\[
\begin{align*}
\bar{\tau}_1 &= \frac{1}{4}(dx^\mu \wedge dx^\nu \wedge d\theta \wedge d\bar{\theta})\bar{T}_{\mu\nu\theta\bar{\theta}}, \\
\bar{\tau}_2 &= \frac{1}{24}(dx^\mu \wedge dx^\nu \wedge \bar{d}\theta \wedge d\theta)\bar{T}_{\mu\nu\bar{\theta}}, \\
\bar{\tau}_3 &= \frac{1}{24}(dx^\mu \wedge dx^\nu \wedge \bar{d}\theta \wedge \bar{d}\theta)\bar{T}_{\mu\nu\bar{\theta}}, \\
\bar{\tau}_4 &= \frac{1}{24}(dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge \bar{d}\theta)\bar{T}_{\mu\nu\lambda\bar{\theta}}, \\
\bar{\tau}_5 &= \frac{1}{24}(dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge d\theta)\bar{T}_{\mu\nu\lambda\theta}, \\
\bar{\tau}_6 &= \frac{1}{24}(dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\zeta)\bar{T}_{\mu\nu\lambda\zeta}.
\end{align*}
\]

(3.16)

It will be noted that the 4-forms with the wedge products \((dx_\mu \wedge dx_\nu \wedge dx_\lambda \wedge dx_\zeta), (dx_\mu \wedge dx_\nu \wedge dx_\lambda \wedge dx_\zeta) s^{\theta\bar{\theta}}, (dx_\mu \wedge dx_\nu \wedge dx_\lambda \wedge dx_\zeta) s^{\theta\bar{\theta}}\) are different because their dual 2-forms are different as can be seen from (3.15). However, for the sake of brevity, we have chosen only one \(^{14}\) of these in (3.16). A single \(*\) operation on the wedge products of the differentials corresponding to 4-forms are

\[
\begin{align*}
* (dx^\mu \wedge dx^\nu \wedge \bar{d}\theta \wedge \bar{d}\bar{\theta}) &= \frac{1}{2} \varepsilon^{\mu\nu\lambda\zeta} (dx_\lambda \wedge dx_\zeta) , \\
* (dx^\mu \wedge dx^\nu \wedge d\theta \wedge d\bar{\theta}) &= \varepsilon^{\mu\nu\lambda\zeta} (dx_\lambda \wedge dx_\zeta) , \\
* (dx^\mu \wedge dx^\nu \wedge d\theta \wedge d\theta) &= \varepsilon^{\mu\nu\lambda\zeta} (dx_\lambda \wedge dx_\zeta) , \\
* (dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge d\bar{\theta}) &= \varepsilon^{\mu\nu\lambda\zeta} (dx_\lambda \wedge dx_\zeta) , \\
* (dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge d\theta) &= \varepsilon^{\mu\nu\lambda\zeta} (dx_\lambda \wedge dx_\zeta) .
\end{align*}
\]

It is clear from (3.15)--(3.17) that one can compute now the double \(*\) operations on the 2-forms as well as 4-forms. Finally, we focus on the independent 3-forms that can be supported on the \((4 + 2)\)-dimensional supermanifold. There exist six such forms:

\[
\begin{align*}
\sigma_1 &= \frac{1}{4}(dx^\mu \wedge dx^\nu \wedge d\theta)\ R_{\mu\nu\theta}, & \sigma_2 &= \frac{1}{4}(dx^\mu \wedge dx^\nu \wedge d\bar{\theta})\ R_{\mu\nu\bar{\theta}}, \\
\sigma_3 &= (dx^\mu \wedge d\theta \wedge d\bar{\theta})\ R_{\mu\theta\bar{\theta}}, & \sigma_4 &= (dx^\mu \wedge \bar{d}\theta \wedge d\theta)\ R_{\mu\bar{\theta}\theta}, \\
\sigma_5 &= (dx^\mu \wedge \bar{d}\theta \wedge d\bar{\theta})\ R_{\mu\bar{\theta}\bar{\theta}}, & \sigma_6 &= \frac{1}{4}(dx^\mu \wedge dx^\nu \wedge dx^\lambda)\ R_{\mu\nu\lambda}.
\end{align*}
\]

(3.18)

A single Hodge duality \(*\) operation on the above 3-forms will lead to the derivation of the dual 3-forms on the six \((4 + 2)\)-dimensional supermanifold. Such an operation will affect the wedge products of the differentials as given below

\[
\begin{align*}
* (dx^\mu \wedge dx^\nu \wedge d\theta) &= \frac{1}{2} \varepsilon^{\mu\nu\lambda\zeta} (dx_\lambda \wedge dx_\zeta \wedge d\bar{\theta}) , \\
* (dx^\mu \wedge dx^\nu \wedge d\bar{\theta}) &= \frac{1}{2} \varepsilon^{\mu\nu\lambda\zeta} (dx_\lambda \wedge dx_\zeta \wedge d\theta) , \\
* (dx^\mu \wedge d\theta \wedge d\bar{\theta}) &= \frac{1}{2} \varepsilon^{\mu\nu\lambda\zeta} (dx_\lambda \wedge dx_\zeta \wedge d\theta) , \\
* (dx^\mu \wedge d\theta \wedge d\theta) &= \frac{1}{2} \varepsilon^{\mu\nu\lambda\zeta} (dx_\lambda \wedge dx_\zeta \wedge d\bar{\theta}) , \\
* (dx^\mu \wedge \bar{d}\theta \wedge d\bar{\theta}) &= \frac{1}{2} \varepsilon^{\mu\nu\lambda\zeta} (dx_\lambda \wedge dx_\zeta \wedge d\bar{\theta}) , \\
* (dx^\mu \wedge \bar{d}\theta \wedge d\theta) &= \frac{1}{2} \varepsilon^{\mu\nu\lambda\zeta} (dx_\lambda \wedge dx_\zeta \wedge d\theta) .
\end{align*}
\]

\(^{14}\)In principle, one can define a triplet of \(\tau_6\) form in (3.16). These are \(\tau_6^{(1)} = \frac{1}{4}(dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\zeta)\bar{T}_{\mu\nu\lambda\zeta}, \tau_6^{(2)} = \frac{1}{4}(dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\zeta) s^{\theta\bar{\theta}}\bar{T}_{\mu\nu\lambda\zeta}, \tau_6^{(3)} = \frac{1}{4}(dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\zeta) s^{\theta\bar{\theta}}\bar{T}_{\mu\nu\lambda\zeta}\). It is obvious that the Hodge dual of these forms are distinct and different. For obvious reasons, no other forms in (3.16) support such kind of a triplet of superforms (as their Hodge dual forms are not well-defined on the supermanifold in the sense that they will contain more than two Grassmannian wedge products).
As expected, there are three 3-forms constructed by the wedge products of the spacetime differentials \((dx^\mu \wedge dx^\nu \wedge dx^\lambda), (dx^\mu \wedge dx^\nu \wedge dx^\lambda)s^{\theta \bar{\theta}}, (dx^\mu \wedge dx^\nu \wedge dx^\lambda)s^{\bar{\theta} \theta}\) whose Hodge duals are different 3-forms as given below

\[
\begin{align*}
\ast (dx^\mu \wedge dx^\nu \wedge dx^\lambda) &= \varepsilon^{\mu \nu \lambda \zeta} (dx^\zeta \wedge d\theta \wedge d\bar{\theta}), \\
\ast \left[ (dx^\mu \wedge dx^\nu \wedge dx^\lambda)s^{\theta \bar{\theta}} \right] &= \varepsilon^{\mu \nu \lambda \zeta} (dx^\zeta \wedge d\theta \wedge d\bar{\theta}), \\
\ast \left[ (dx^\mu \wedge dx^\nu \wedge dx^\lambda)s^{\bar{\theta} \theta} \right] &= \varepsilon^{\mu \nu \lambda \zeta} (dx^\zeta \wedge d\theta \wedge d\bar{\theta}).
\end{align*}
\]

(3.20)

The above considerations allow us to define the following triplet of \(\sigma_6\) of (3.18)

\[
\begin{align*}
\sigma_6^{(1)} &= \frac{1}{3!}(dx^\mu \wedge dx^\nu \wedge dx^\lambda) R^{(1)}_{\mu \nu \lambda}, \\
\sigma_6^{(2)} &= \frac{1}{3!}(dx^\mu \wedge dx^\nu \wedge dx^\lambda) s^{\theta \bar{\theta}} R^{(2)}_{\mu \nu \lambda}, \\
\sigma_6^{(3)} &= \frac{1}{3!}(dx^\mu \wedge dx^\nu \wedge dx^\lambda) s^{\bar{\theta} \theta} R^{(3)}_{\mu \nu \lambda}.
\end{align*}
\]

(3.21)

However, for the sake of brevity, we have taken only one of the above triplets in equation (3.18). It is now straightforward to check that the double Hodge duality \(\ast\) operations on the 3-forms of (3.18) yield the following

\[
\begin{align*}
\ast [ \ast \sigma_1 ] &= - \sigma_1, & \ast [ \ast \sigma_2 ] &= - \sigma_2, \\
\ast [ \ast \sigma_3 ] &= + \sigma_3, & \ast [ \ast \sigma_4 ] &= + \sigma_4, \\
\ast [ \ast \sigma_5 ] &= + \sigma_5, & \ast [ \ast \sigma_6 ] &= + \sigma_6.
\end{align*}
\]

(3.22)

Finally, we do know that the six \((4 + 2)\)-dimensional supermanifold can support three 6-forms, modulo some constant factors, as given below

\[
\begin{align*}
\Psi_1 &= \frac{1}{4!} (dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\zeta \wedge d\theta \wedge d\bar{\theta}) \tilde{G}_{\mu \nu \lambda \zeta \theta \bar{\theta}}, \\
\Psi_2 &= \frac{1}{4!} (dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\zeta \wedge d\theta \wedge d\bar{\theta}) \tilde{G}_{\mu \nu \lambda \zeta \theta \bar{\theta}}, \\
\Psi_3 &= \frac{1}{4!} (dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\zeta \wedge d\theta \wedge d\bar{\theta}) \tilde{G}_{\mu \nu \lambda \zeta \theta \bar{\theta}}.
\end{align*}
\]

(3.23)

A single Hodge duality \(\ast\) operation on the above 6-forms produces 0-form scalars on the six dimensional supermanifold. Such an operation on the wedge products of the differentials are

\[
\begin{align*}
\ast (dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\zeta \wedge d\theta \wedge d\bar{\theta}) &= \varepsilon^{\mu \nu \lambda \zeta}, \\
\ast (dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\zeta \wedge d\theta \wedge d\bar{\theta}) &= \varepsilon^{\mu \nu \lambda \zeta} s^{\theta \bar{\theta}}, \\
\ast (dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\zeta \wedge d\theta \wedge d\bar{\theta}) &= \varepsilon^{\mu \nu \lambda \zeta} s^{\bar{\theta} \theta}.
\end{align*}
\]

(3.24)

Two consecutive \(\ast\) operation on the 6-forms of (3.23) leads to

\[
\begin{align*}
\ast [ \ast \Psi_1 ] &= - \Psi_1, & \ast [ \ast \Psi_2 ] &= - \Psi_2, & \ast [ \ast \Psi_3 ] &= - \Psi_3.
\end{align*}
\]

(3.25)

It is evident that we have collected, in the present section, all the possible super-forms, their single Hodge dual- as well as their double Hodge dual superforms, etc., that could be defined on the \((4 + 2)\)-dimensional supermanifold.

### 3.3 Superfield formulation of (anti-)co-BRST symmetries for 4D theory

As evident from (3.4) that the non-local, non-covariant, continuous, off-shell nilpotent and anticommuting \((s_d s_{ad} + s_{ad} s_d = 0)\) (anti-)co-BRST symmetries \(s_{(a)d}\) do exist for the 4D
free Abelian gauge theory. To obtain these symmetries in the framework of superfield formulation, we have to exploit the dual-horizontality condition \( \delta \tilde{A}^{(1)} = \delta A^{(1)} \) on the six \((4+2)\)-dimensional supermanifold. It is clear that the r.h.s. of the above condition (i.e \( \delta A^{(1)} = - \ast d \ast A^{(1)} = (\partial \cdot A) \)) is the usual gauge-fixing term on the ordinary 4D spacetime manifold. For the computation of the l.h.s. \( \delta \tilde{A}^{(1)} = - \ast \delta \ast \tilde{A}^{(1)} \), we first concentrate on the dual \((\ast \tilde{A}^{(1)} = \ast dZ^M \tilde{A}_M)\) of the super 1-form connection \( \tilde{A}^{(1)} \). The ensuing expression for \((\ast \tilde{A}^{(1)})\), due to the Hodge duality operation given in (3.8) and definition (2.9), is

\[
\ast \tilde{A}^{(1)} = \frac{1}{3} \varepsilon^{\mu \nu \lambda \zeta} (dx^\mu \wedge dx^\nu \wedge dx^\chi \wedge dx^\zeta \wedge d\theta \wedge d\bar{\theta}) B_\mu(x, \theta, \bar{\theta}) + \frac{1}{3} \varepsilon^{\mu \nu \lambda \zeta} (dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\zeta \wedge d\theta \wedge d\bar{\theta}) \Phi(x, \theta, \bar{\theta}) \tag{3.26}
\]

which is nothing but the 5-form defined on the six \((4+2)\)-dimensional supermanifold. Applying now the super exterior derivative \( \tilde{d} = dZ^M \partial_M \) on the above 5-form, we obtain the following 6-form

\[
\tilde{d} (\ast \tilde{A}^{(1)}) = \frac{1}{3} \varepsilon^{\mu \nu \lambda \zeta} (dx^\mu \wedge dx^\nu \wedge dx^\chi \wedge dx^\zeta \wedge d\theta \wedge d\bar{\theta}) (\partial^\rho B_\mu)(x, \theta, \bar{\theta}) - \frac{1}{3} \varepsilon^{\mu \nu \lambda \zeta} (dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\zeta \wedge d\theta \wedge d\bar{\theta}) (\partial_\rho \Phi)(x, \theta, \bar{\theta}) - \frac{1}{3} \varepsilon^{\mu \nu \lambda \zeta} (dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\zeta \wedge d\theta \wedge d\bar{\theta}) (\partial_\rho \Phi)(x, \theta, \bar{\theta}) \tag{3.27}
\]

It should be noted that all the wedge products with more than four spacetime differentials and two Grassmannian differentials have been dropped out because on a \((4+2)\)-dimensional supermanifold one cannot define such kind of differential forms. On (3.27), we now apply another \((-\ast)\) to obtain a super 0-form (superscalar) by exploiting the Hodge duality operation defined in (3.24). Such a superscalar is

\[
\delta \tilde{A}^{(1)} = - \ast \tilde{d} \ast \tilde{A}^{(1)} = (\partial \cdot B) - (\partial_\theta \Phi + \partial_{\bar{\theta}} \Phi) - s^{\theta \theta} (\partial_\theta \Phi) - \delta \tilde{d} \ast \tilde{A}^{(1)} \tag{3.28}
\]

Equating the above superscalar with the ordinary scalar \( \delta A^{(1)} \), due to the dual-horizontality condition \( (\delta \tilde{A}^{(1)} = \delta A^{(1)}) \), we obtain the following relationships

\[
(\partial \cdot B) - (\partial_\theta \Phi + \partial_{\bar{\theta}} \Phi) = (\partial \cdot A), \quad \partial_\theta \Phi = 0, \quad \partial_{\bar{\theta}} \Phi = 0. \tag{3.29}
\]

The insertion of the most general super expansions (cf. (3.5)) leads to the following restrictions on the secondary fields of expansion (3.5):

\[
(\partial \cdot R)(x) = 0, \quad \langle \partial \cdot \hat{R} \rangle(x) = 0, \quad (\partial \cdot S)(x) = 0, \quad s(x) = \tilde{s}(x) = \hat{B}(x) = \hat{\tilde{B}}(x) = 0, \quad \mathcal{B}(x) + \hat{\mathcal{B}}(x) = 0. \tag{3.30}
\]

Consistent with the statements made after (3.5), the following choices of the secondary fields in terms of the basic fields (see, e.g., [42] for details)

\[
R_0 = i \hat{C}, \quad R_i = i \frac{\partial_0 \partial_i}{\nabla^2} \hat{C}, \quad \hat{R}_0 = i \hat{C}, \quad \hat{R}_i = i \frac{\partial_0 \partial_i}{\nabla^2} \hat{C}, \quad \mathcal{B} = +i \frac{\partial b_i^{(1)}}{\nabla^2}, \quad \hat{\mathcal{B}} = -i \frac{\partial b_i^{(1)}}{\nabla^2}, \quad S_0 = \frac{\partial b_i^{(1)}}{\nabla^2}, \quad S_i = \frac{\partial_0 \partial_i}{\nabla^2} \left( \frac{\partial b_i^{(1)}}{\nabla^2} \right), \tag{3.31}
\]
do satisfy all the above conditions (3.30), emerging from the application of the dual-horizontality condition. It is worth emphasizing, at this juncture, that the auxiliary field $b^{(2)}$ has not been taken into account in the expansion (3.5) as well as in the choices (3.31) because this field (and its equivalent magnetic field $B$) do not appear in any transformations listed in (3.3) and (3.4). Furthermore, this field, on its own, does not transform under (co-)BRST transformations. In terms of the transformations in (3.4) and expressions (3.31), we obtain the following expansions

\[
\begin{align*}
B_0 (x, \theta, \bar{\theta}) &= A_0 (x) + \theta (s_{ad} A_0 (x)) + \bar{\theta} (s_d A_0 (x)) + \theta \bar{\theta} (s_d s_{ad} A_0 (x)), \\
B_i (x, \theta, \bar{\theta}) &= A_i (x) + \theta (s_{ad} A_i (x)) + \bar{\theta} (s_d A_i (x)) + \theta \bar{\theta} (s_d s_{ad} A_i (x)), \\
\Phi (x, \theta, \bar{\theta}) &= C (x) + \theta (s_{ad} C (x)) + \bar{\theta} (s_d C (x)) + \theta \bar{\theta} (s_d s_{ad} C (x)), \\
\bar{\Phi} (x, \theta, \bar{\theta}) &= \bar{C} (x) + \theta (s_{ad} \bar{C} (x)) + \bar{\theta} (s_d \bar{C} (x)) + \theta \bar{\theta} (s_d s_{ad} \bar{C} (x)).
\end{align*}
\]

The above expansion does establish the geometrical interpretation for the conserved and nilpotent (anti-)co-BRST charges $Q_{(a)d}$ as the translation generators along the Grassmannian directions of the six $(4 + 2)$-dimensional supermanifold. In fact, there exists some inter-connections among the nilpotent transformations $s_{(a)d}$, the translations generators along the Grassmannian directions of the supermanifold and the nilpotent charges $Q_{(a)d}$, as

\[
s_d \leftrightarrow \lim_{\theta \to 0} \frac{\partial}{\partial \bar{\theta}} \leftrightarrow Q_d, \quad s_{ad} \leftrightarrow \lim_{\bar{\theta} \to 0} \frac{\partial}{\partial \theta} \leftrightarrow Q_{ad}.
\]

The above relationship is the analogue of exactly the same kind of relation existing in the context of the nilpotent (anti-)BRST symmetries (cf. (2.14)).

4 Conclusions

In our present investigation, we have been able to define a consistent Hodge duality $\star$ operation on (i) the four $(2 + 2)$-dimensional supermanifold, and (ii) the six $(4 + 2)$-dimensional supermanifold. These definitions are essential for the derivation of the nilpotent $(s_{(a)d}^2 = 0)$ (anti-)co-BRST symmetries $s_{(a)d}$ for (i) the two $(1+1)$-dimensional (2D) free Abelian gauge theory, and (ii) the four $(3 + 1)$-dimensional (4D) free Abelian gauge theory in the framework of superfield formulation. In fact, the above 2D- and 4D free Abelian gauge theories (described by the local fields that take values on the 2D and 4D flat Minkowskian space-time manifold) are considered on the four $(2 + 2)$-dimensional- and six $(4 + 2)$-dimensional supermanifolds, respectively. Our study on these supermanifolds (described by the superfields that take values on the supermanifold parametrized by the superspace variables $Z^M = (x^\mu, \theta, \bar{\theta})$) does provide the geometrical origin and interpretation for the nilpotent (anti-)BRST- and (anti-)co-BRST symmetries (and the corresponding nilpotent generators). The physical application of a consistent definition of the Hodge duality $\star$ operation turns up in the context of the dual-horizontality condition $\tilde{\delta} \tilde{\Lambda}^{(1)} = \delta \Lambda^{(1)}$ where the use of the super co-exterior derivative $\tilde{\delta} = - \star \tilde{d} \star$ (on the l.h.s.) does require a consistent definition of the $\star$ operation. In fact, the existence of the nilpotent (anti-)co-BRST symmetry
transformations owes its origin to the (super) co-exterior derivatives where the definition
of $\star$ plays a very decisive role. In the language of physics, it is the gauge-fixing term of the
(anti-)BRST invariant Lagrangian density of a gauge theory that remains invariant under
the (anti-)co-BRST transformations (cf. Sections 2 and 3). This statement has been shown
to be true for both the 2D- and 4D free Abelian gauge theories where there is no interaction
between the $U(1)$ gauge field and the matter fields.

One of the novel and the most decisive ingredients in our whole discussion is the intro-
duction of the constant symmetric parameters $s^\theta$ and $s^{\bar{\theta}}$ in the definition of the Hodge
duality $\star$ operation on the wedge products of the differentials of some given (super)forms on
the $(D+2)$-dimensional supermanifold. The usefulness of these parameters, in our whole
discussion, are primarily four folds. First, these allow us, for instance, to take into account
the fact that there are three (super) differentials corresponding to the 1-forms defined on
the four $(2+2)$-dimensional supermanifold. These are, for the sake of emphasis, once again
written as
\[(dx^\mu), \quad (dx^\mu) s^\theta, \quad (dx^\mu) s^{\bar{\theta}}, \quad (\text{4.1})\]
whose Hodge duals correspond to wedge products of the differentials corresponding to the
3-forms on the supermanifold as given below
\[\star (dx^\mu) = \varepsilon^{\mu\nu}(dx_\nu \wedge d\theta \wedge d\bar{\theta}), \quad \star [(dx^\mu s^\theta)] = \varepsilon^{\mu\nu}(dx_\nu \wedge d\theta \wedge d\bar{\theta}), \quad \star [(dx^\mu s^{\bar{\theta}})] = \varepsilon^{\mu\nu}(dx_\nu \wedge d\bar{\theta} \wedge d\bar{\theta}). \quad (\text{4.2})\]
In fact, the above prescription can be generalized to any $(D+2)$-dimensional supermanifold.
Second, the presence of these parameters facilitate the action of the double $\star$ operations on
any arbitrary form $f$ that is supposed to obey $\star(\star f) = \pm f$ [39]. For instance, in the above
equation, the following results turn out automatically
\[\star [\star (dx^\mu)] = dx^\mu, \quad \star [\star (dx^\mu) s^\theta] = dx^\mu s^\theta, \quad \star [\star (dx^\mu) s^{\bar{\theta}}] = dx^\mu s^{\bar{\theta}}. \quad (\text{4.3})\]
Third, it is evident that the Hodge dual of a 2-superform (e.g. $d\theta \wedge d\bar{\theta}$) will be a 2-superform
on a $(2+2)$-dimensional supermanifold. The existence of $s^\theta$ does allow such a definition
because $\star (d\theta \wedge d\bar{\theta}) = \frac{1}{2}\varepsilon^{\mu\nu}(dx_\mu \wedge dx_\nu)s^\theta$. Fourth, the existence of the above parameters is
at the heart of the accurate derivation of the nilpotent (anti-)co-BRST symmetry transfor-
mations for the gauge field as well as the (anti-)ghost fields of the 2D- and 4D free Abelian
gauge theories as is evident from the key equations (2.35), (2.36), (3.28) and (3.29).

It is clear from our present discussion that the geometrical superfield formalism pro-
vides an exact and unique way of deriving the local, covariant, continuous, nilpotent and
anticommuting (anti-)BRST symmetry transformations. However, this is not the case with
the derivation of the (anti-)co-BRST symmetries which are not found to be unique. In
fact, for both the 2D- and 4D free 1-form Abelian gauge theories, the dual-horizontality
condition ($\delta \tilde{A}^{(1)} = \delta A^{(1)}$) leads to the conditions $(\partial \cdot \tilde{R} = 0, (\partial \cdot \tilde{R}) = 0, (\partial \cdot S) = 0$ on the
secondary fields of the expansions in (2.7) as well as (3.5). For the 2D theory, there exist
local, covariant, continuous and nilpotent solutions for $R_\mu, \tilde{R}_\mu, S_\mu$ so that one obtains the
(anti-)co-BRST transformations of (2.4). However, for the 4D free Abelian gauge theory, only non-local, non-covariant, continuous and nilpotent solutions exist for $R_\mu, \bar{R}_\mu, S_\mu$. Furthermore, these solutions for the latter case are not unique. In fact, there has been a whole lot of discussion on the various possibilities of the existence of the dual-BRST symmetry transformations for the Abelian gauge theory in [41]. All these possibilities of symmetries are captured by different choices of $R_\mu$ and $\bar{R}_\mu$ (see, e.g., [42] for details). Thus, in some sense, the superfield formalism with the super co-exterior derivative $\tilde{\delta}$ does provide the reasons behind the non-uniqueness of the nilpotent (anti-)co-BRST symmetry transformations where the dual-horizontality condition plays a very decisive role.

It would be an interesting endeavour to generalize our present work to the case of the interacting gauge theories where the gauge fields couple to the matter fields. In fact, one such example, where the $U(1)$ gauge field $A_\mu$ couples with the Dirac fields in 2D, has been shown to present the field theoretical model for the Hodge theory. In this model, the nilpotent (anti-)BRST and (anti-)co-BRST symmetries co-exist together [33,34]. In a recent set of papers [43-47], the nilpotent symmetries for all the basic fields of (i) the interacting 2D- as well as 4D (non-)Abelian gauge theories, and (ii) a reparametrization invariant theory, have been derived by exploiting the augmented superfield formulation. In this formalism, in addition to the (dual-)horizontality conditions, the invariance of the (super)matter conserved currents on the supermanifold has also been exploited. In fact, the latter restriction yields the nilpotent symmetries for the matter fields of an interacting gauge theory. Furthermore, it would be an interesting venture to generalize our present work to the discussion of the free 4D 2-form Abelian gauge theory where the existence of the local, covariant, continuous and nilpotent (anti-)co-BRST symmetries has been shown [48,49]. Yet another direction that could be pursued is to generalize the superfield formalism with only two Grassmann variables (i.e. $\theta$ and $\bar{\theta}$) to the superfield approach depending upon multiple Grassmann variables (e.g. $\theta_\alpha$ and $\bar{\theta}_\dot{\alpha}$ with $\alpha, \dot{\alpha} = 1, 2, 3...$). These are some of the issues that are under investigation and our results would be reported elsewhere [50].

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