The Landau–Lifshitz Equation by Semidirect Product Reduction

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March 2000

Abstract
The Landau–Lifshitz equation is derived as the reduction of a geodesic flow on the group of maps into the rotation group. Passing the symmetries of spatial isotropy to the reduced space is an example of semidirect product reduction by stages.

The Landau–Lifshitz equation for the evolution of a field of unit vectors \( n(x, t), x \in \mathbb{R}^p, p = 1, 2, \cdots \), nondimensionalized, is

\[
\frac{\partial n}{\partial t} = -n \times \frac{\delta E}{\delta n},
\]

where \( E \) is some energy functional, typically

\[
E = \frac{1}{2} \int |\nabla n|^2 + \frac{a}{2} \int (n \cdot n - (n \cdot k)^2),
\]

and \( a \) is some coupling constant. This functional encodes the energy due to orientation variance plus, when \( a \neq 0 \), a material anisotropy. With these notations \( E \) is finite if \( n(x) \to -k \) sufficiently fast as \( x \to \infty \). In [1] and [5] one finds, after adjusting dimensions, the conserved quantities

\[
P = 2\pi \int x \times \Omega, \quad \Omega_\alpha = \frac{1}{8\pi} \epsilon_{\alpha\beta\gamma} n \cdot (\nabla_\beta n \times \nabla_\gamma n),
\]

which generate translations and the conserved quantity

\[
N = \int (1 + n \cdot k),
\]

*Supported in part by the Natural Sciences and Engineering Research Council, Canada.
which generates rotations (uniform in space) of \( n \) about \( k \).

In this Letter, I explicitly view the Landau–Lifshitz equation as a right-hand reduction of a geodesic flow on the group of smooth mappings

\[ G = \{ A : \mathbb{R}^p \to SO(3) \mid A(\infty) = \text{Id} \} \]

with pointwise multiplication. The calculations are formal but at the very least require certain asymptotic behaviors the elements of \( G \). For my purpose it suffices to assume that they are the identity outside some compact set.

One reason to view the Landau–Lifshitz equation as a reduction from a cotangent bundle is that, at the level of \( T^*G \), the momenta associated to the group \( SE(p) \) and to right translation by \( G \) itself are easily calculated by well-known formulas. Regardless of whether the energy functional is invariant, one may try to pass these momenta and their symmetries to the reduced space on which, as it turns out, the Landau–Lifshitz equation lives. I show that the conserved quantities \( P \) are obtained in this way. The validity of this rests on whether the subsequent spaces obtained by reductions by those reduced symmetries are isomorphic to the reductions of \( T^*G \) by the whole symmetry. This is formally correct by the reduction by stages theory of [3]. The earlier reduction by stages theory found in [2] is inadequate for the Landau–Lifshitz equation since the group \( G \) is non-Abelian.

When \( p = 2 \) the field \( n \) may be regarded as a map from the 2-sphere \( S^2 \) to itself which maps \( \infty \) to \( -k \). Thus, for \( p = 2 \), the Landau–Lifshitz phase space is decomposed by the degree of \( n \), which is the integer

\[ \deg n = \frac{1}{4\pi} \int n \cdot \frac{\partial n}{\partial x} \times \frac{\partial n}{\partial y} \]

This normalization is such that the degree is 1 on the identity map pulled back to the plane using the stereographic projection which sends \( -k \) to \( \infty \). In [4] it is observed that the \( x \) and \( y \) momenta do not commute in any degree nonzero sector. From a geometric mechanics point of view, this is an apparent contradiction since these momenta do commute on the unreduced phase space. Obtaining the degree nonzero sector from the reduction exercises nontrivial aspects of the reduction by stages theory and that explains fully the apparent contradiction.

When elements of \( G \) are thought of as configurations, I will denote them as \( \psi, \phi \cdots \in G \), while when \( G \) plays its role as a group of symmetries, its elements will be \( A, B \cdots \in G \). Formally, the Lie algebra \( \mathfrak{g} \) of \( G \) is the smooth \( \mathfrak{so}(3) \cong \mathbb{R}^3 \) valued mappings on \( \mathbb{R}^p \), vanishing at \( \infty \), with pointwise Lie bracket. The dual \( \mathfrak{g}^* \) is the same space except its elements do not necessarily vanish at \( \infty \), and the pairing between \( \mathfrak{g} \) and \( \mathfrak{g}^* \) is

\[ \langle \mu, \xi \rangle = \int \mu(x) \cdot \xi(x) \]

I represent the elements of \( T^*G \) as pairs \((\psi, \mu) \in G \times \mathfrak{g}^*\) using right translation, so the action of right translation of \( G \) on \( T^*G \) is

\[ A \cdot (\psi, \mu) = (\psi(x)A(x)^{-1}, \mu(x)) \]
The momentum map of this action is
\[ J^G(\psi, \mu) = -\psi(x)^{-1}\mu(x), \]
because (see [3])
\[ J^G_\xi(\psi, \mu) = (\psi, \mu) \cdot \frac{d}{dt} \bigg|_{t=0} \psi \exp(-\xi t) \]
\[ = -\int \mu(x) \cdot \psi(x) \xi(x) = \int (-\psi(x)^{-1}\mu(x)) \cdot \xi(x). \]

Fixing the particular momentum value \( \mu \in g^* \) defined by \( \mu(x) = k \), one has
\[ (J^G)^{-1}(\mu) = \{ (\psi, -\psi k) \}, \]
and the quotient \( (J^G)^{-1}(\mu)/G_\mu \) may be realized as
\[ (J^G)^{-1}(\mu) \to \{ n \in g : |n| = 1 \} \quad \text{by} \quad n = -\psi k. \]

It is standard that the evolution equations on \( (J^G)^{-1}(\mu)/G_\mu \) (i.e. the evolution equations for \( n \)) are the Lie–Poisson ('+' because of the right translation) equations
\[ \frac{\partial n}{\partial t} = +\coad{\delta E/\delta n} n \]
which is exactly the Landau–Lifshitz equation. Here, by definition \( \coad{\xi} \mu = -\ad{\xi}^* \mu \), for \( \xi \in g \) and \( \mu \in g^* \).

The question of whether all fields \( n \) are attained through the reduction is the question of whether any \( n : \mathbb{R}^p \to so(3) \) lifts through \( \psi \to -\psi k \) to \( SO(3) \) with \( \psi(\infty) = \text{Id} \). This is true for any \( n \) if \( p \neq 2 \) and if \( \deg n = 0 \) in the case \( p = 2 \). Indeed, suppose \( R > 0 \) is such that \( n = -k \) for \( |x| \geq R \). Choosing any connection on the principle bundle \( SO(3) \to S^2 \) by \( A \to -Ak \) and lifting \( n \) along the radial lines emanating from the origin gives a smooth \( \tilde{\psi}(x) \) such that
\[ \tilde{\psi}(0) = \text{Id}, \quad \tilde{\psi} \text{ is constant along radial lines for } |x| \geq R, \quad \text{and } \tilde{\psi}(x)(-k) = -k \text{ for } |x| \geq R. \]
Thus \( \tilde{\psi} \) restricted to the sphere \( S^{p-1}_R(0) \) of radius \( R \) centered at \( 0 \) is a map into the isotropy group \( (SO(3))_k \cong S^1 \). In the case \( p \neq 2 \) the homotopy group \( \pi_{p-1}(S^1) \) is trivial and there is a homotopy \( \gamma_t, t \in [0, 1], \) such that \( \gamma_0(x) = \text{Id} \) for \( x \in S^{p-1}_R \) and \( \gamma_1 = \tilde{\psi}|S^{p-1}_R \). If \( p = 2 \) and \( \deg n = 0 \), then such a homotopy \( \gamma_t \) can be constructed by using the connection to lift a homotopy between \( n \) and the constant map \(-k\). In any case, choosing a smooth map \( t(r) \) which is 0 for \( r \leq R \) and 1 for \( r \geq R + 1 \) and setting \( \psi(x) = \tilde{\psi}(x)(\gamma_t(|x|))^{-1} \) gives \( \psi \) which lifts \( n \) and is equal to the identity for \( |x| \geq R + 1 \). On the other hand, if \( p = 2 \) and \( \deg n \neq 0 \) then there is no such \( \psi \) since \( \pi_2(SO(3)) = 0 \) and there would be a homotopy of \( \tilde{\psi} \) to a point which would project to a homotopy of \( n \) to a point.

For \( p = 2 \) the degree nonzero sectors can only be obtained by reducing at a nonconstant value for \( \mu \), as opposed to \( \mu = k \). The loss of translation invariance...
of $\mu$ leads to serious complications that require more fully the reduction by stages theory. Towards the end of this Letter, I will attend to this but for now assume $\mu = k$ and that $p \neq 2$, or $p = 2$ and $\deg n = 0$.

The group $H = SE(p) = \{(U, a) \in SO(p) \times \mathbb{R}^p\}$ also acts on $G$ by spatially rotating and translating its elements:

$$(U, a) \cdot \psi = \psi((U, a)^{-1}x) = \psi(U^{-1}(x - a)).$$

This action does not commute with the action of $G$, since

$$(U, a) \cdot (A \cdot \psi) = (U, a) \cdot (A\psi) = A(U^{-1}(x - a))\psi(U^{-1}(x - a))$$

whereas

$$A \cdot ((U, a) \cdot \psi) = A(x)\psi(U^{-1}(x - a)).$$

Since the actions of $G$ and $H$ do not commute they cannot be bound together, using the direct product, into a single action. The noncommutativeness can be written, however, as

$$(U, a) \cdot (A \cdot \psi) = \theta(U, a)(A) \cdot ((U, a) \cdot \psi)$$

as long as one defines $\theta : H \times G \rightarrow G$ by

$$\theta((U, a), A) = A(U^{-1}(x - a)).$$

The assignment $(U, a) \mapsto \theta(U, a)$ is a group morphism from $H$ to $Aut G$.

In general we have two groups $G$ and $H$ acting on $M$, the actions do not commute, but there is a $\theta : H \rightarrow Aut G$ such that for all $m \in M$, $g \in G$, and $h \in H$,

$$h \cdot (g \cdot m) = \theta_h(g) \cdot (h \cdot m).$$

This situation is favorable to quotienting by stages since it is a general situation where the action of $H$ on $M$ passes to the quotient $M/G$. If one wants to define an action of $G \times H$ on $M$ by $(g, h) \cdot m = g \cdot (h \cdot m)$, then

$$(g_1, h_1) \cdot ((g_2, h_2) \cdot m) = g_1 \cdot (h_1 \cdot (g_2 \cdot (h_2 \cdot m))) = g_1g_2(\theta_{h_1}(g_2) \cdot (h_1h_2) \cdot m)$$

and the group product on $G \times H$ must be taken to be the semidirect product

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1g_2, \theta_{h_1}(g_2), h_1h_2).$$

Thus, the two actions may be bound together but using the semidirect product instead of the direct product. The semidirect product of $G$ and $H$ is commonly denoted $G \varphi H$.

Returning to the Landau–Lifshitz equation, the momentum map for the action of $H$ is easily calculated. Let $(\Omega, \dot{a}) \in \mathfrak{h}$, where $\mathfrak{h} = so(p) \times \mathbb{R}^p$ is the Lie
The infinitesimal generator of \((\Omega, \dot{a})\) is, remembering to use right translation,

\[
\frac{d}{d\epsilon} \bigg|_{\epsilon=0} \psi\left(\exp(-\Omega \epsilon)(x - \dot{a} \epsilon)\right) \psi^{-1}(x) = \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \psi(\exp(-\Omega \epsilon)x) \psi^{-1}(x) + \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \psi(x - \dot{a}) \psi^{-1}(x) = \nabla^R_{-\Omega x - \dot{a}} \psi(x),
\]

where \(\nabla^R\), the right-hand gradient, is defined by

\[
(\nabla^R_{b} \psi)(x) = \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \psi(x + \epsilon b) \psi^{-1}(x).
\]

I will use the inner product \(\Omega_1 \cdot \Omega_2 = -\frac{1}{2} \text{trace}(\Omega_1 \Omega_2)\) for \(\Omega_1, \Omega_2 \in \mathfrak{so}(p)\) and the notation \(v \wedge w = v \otimes w - w \otimes v = vw^t - wv^t\) for \(v, w \in \mathbb{R}^p\), so that

\[x \cdot \Omega y = \langle x \wedge y, \Omega \rangle.
\]

As well, I will set

\[(\mu \cdot \nabla^R \psi) = (\mu \cdot \nabla^R_{e_1} \psi, \cdots, \mu \cdot \nabla^R_{e_p} \psi),\]

where \(e_1, \cdots, e_p\) is the standard basis of \(\mathbb{R}^p\), so that

\[v \cdot \nabla^R_{w} \psi = w \cdot (v \cdot \nabla^R \psi),\quad v, w \in \mathbb{R}^p.
\]

Then

\[
J^H_{\Omega, \dot{a}}(\psi, \mu) = \int \mu \cdot \nabla^R_{-\Omega x - \dot{a}} \psi(x) = (\Omega, \dot{a}) \cdot \left( \int x \wedge (\mu \cdot \nabla^R) \psi, - \int (\mu \cdot \nabla^R) \psi \right),
\]

so the \(H\) momentum map is

\[
J^H(\psi, \mu) = \left( \int x \wedge (\mu \cdot \nabla^R) \psi, - \int (\mu \cdot \nabla^R) \psi \right).
\]

The reduction of the symmetry represented by the \(H\) action is possible after the following two observations:

- \((J^G)^{-1}(\mu)\) is an invariant subset under the action of \(H\). Indeed,

\[
J^G((U, a) \cdot (\psi, -\psi k)) = -\psi((U, a)^{-1})^{-1} \psi((U, a)^{-1}) k = k.
\]

- \(J^H\) is \(G_\mu\) invariant for \(p \geq 2\). Assume \(A \in G_\mu\). Then using the (easily verified) identity

\[
(\mu \cdot \nabla^R)(\psi \phi) = (\mu \cdot \nabla^R) \psi + ((\psi^{-1} \mu) \cdot \nabla^R) \phi,
\]
one has
\[ J^H(A \cdot (\psi, -\psi k)) = \left( \int x \wedge ((\psi k) \cdot \nabla^R)(\psi A^{-1}), - \int ((\psi k) \cdot \nabla^R)(\psi A^{-1}) \right) \]
\[ = J^H(\psi, -\psi k) + \left( \int x \wedge ((k \cdot \nabla^R)A^{-1}), - \int (k \cdot \nabla^RA^{-1}) \right). \]

Since \( A(x)k = k \), \( A(x) = \exp(\alpha(x)k) \) for some smooth real valued function \( \alpha \) which is an integral multiple of \( 2\pi \) at infinity, one has
\[ (k \cdot \nabla^R)A^{-1} = -\nabla \alpha, \]
so
\[ J^H(A \cdot (\psi, -\psi k)) = J^H(\psi, -\psi k) + \left( - \int x \wedge \nabla \alpha, \int \nabla \alpha \right). \]

In the case \( p \geq 2 \) the space at \( \infty \) is connected so \( \alpha \) is constant there and hence both integrals vanish.

The nontrivial part of the above is the second item about the invariance of \( J^H \); the first item is a specialty of the particular momentum \( \mu \). Yet this kind of thing occurs generally. I will use the notation \( \text{CoAd}_g \mu = \text{Ad}_{g^{-1}}^* \mu \).

**Lemma 1** Let \( J^G : P \to \tilde{g}^* \) be an equivariant momentum map and let \( G \) be a normal subgroup of \( \tilde{G} \). Define \( J^G = i^*_\tilde{G} J^G \), so that \( J^G \) is an equivariant momentum map for the action of \( G \), let \( \mu \in g^* \), define the subgroup \( H_\mu \) of \( \tilde{G} \) by
\[ H_\mu = \{ g \in \tilde{G} : (J^G(g)p) = \mu \text{ for all } p \in (J^G)^{-1}(\mu) \}, \]
and define \( J^{H_\mu} = i^*_{H_\mu} J^G \). Suppose that \( G_\mu \) is connected. Then \( G_\mu = G \cap H_\mu \), \( G_\mu \) is a normal subgroup of \( H_\mu \), and \( J^{H_\mu} \) is \( G_\mu \) invariant on \((J^G)^{-1}(\mu)\).

**Proof** For \( g \in G \) the condition \( J^G(gp) = \mu \) for all \( p \in (J^G)^{-1}(\mu) \) is exactly \( \text{CoAd}_g \mu = \mu \) so \( G_\mu = G \cap H_\mu \). Then \( G_\mu \) normal in \( H_\mu \) follows directly from \( G \) normal in \( \tilde{G} \).

It is required to prove that if \( p \in P \) is such that \( J^G(p) = \mu \), if \( g \in G_\mu \), and if \( \xi \in h_\mu \), then \( J^G_{\xi}(gp) = J^G_{\xi}(p) \). Setting \( \tilde{\mu} = J^G(p) \) and, since \( G_\mu \) is connected, this is equivalent to
\[ \left. d \right|_{t=0} J^G_{\xi}(\exp(\eta t)p) = \langle J^G(p), [\xi, \eta] \rangle = \langle \tilde{\mu}, [\xi, \eta] \rangle = 0 \]
for all \( \eta \in g_\mu \). But if \( \xi \in h_\mu \) and \( \eta \in g_\mu \) then by the definition of \( H_\mu \)
\[ \langle \text{CoAd}_{\exp(\xi t)} \tilde{\mu}, \eta \rangle = \langle \mu, \eta \rangle \]
so differentiation in \( t \) gives \( \langle \tilde{\mu}, [\xi, \eta] \rangle = 0 \). Then \( [\xi, \eta] \in g_\mu \subseteq g \) since \( \eta \in g_\mu \) and \( \xi \in h_\mu \) and \( G_\mu \) is normal in \( H_\mu \), so \( \langle \tilde{\mu}, [\xi, \eta] \rangle = \langle \mu, [\xi, \eta] \rangle = \langle \text{coad}_\eta \mu, \xi \rangle = 0. \)

\[ \square \]
As is easily verified, since the subgroup $G$ is assumed to be normal, the coadjoint action of $\tilde{G}$ restricts to $\mathfrak{g}$ and the subgroup $H_\mu$ is the isotropy group of $\mu$ with respect to this action.

The point of Lemma 1 is that when reducing first by the subgroup $G$ at the momentum value $\mu$ to obtain $(J^G)^{-1}(\mu)/G_\mu$, the residual symmetry after the reduction would intuitively be those elements $\tilde{g} \in \tilde{G}$ which map $(J^G)^{-1}(\mu)$ to itself; this is exactly the group $H_\mu$. Then, in order to pass the ‘residual momentum’ to the quotient it is required that $J^H_{\mu}$ be $G_\mu$-invariant, and the Lemma gives sufficient conditions for this. As for the relevance to the Landau–Lifshitz equation, the Euclidean invariance of momentum $\mu = k$ gives $H_\mu = G_\mu \theta \times H$ (that is rotation fields fixing $k$ paired with all Euclidean motions), $G$ is normal in the $G_\theta \times H$, and so the Lemma explains the invariance of $J^H$ under $G_\mu$ for $m \geq 2$. Also, for dimension $q = 1$, the group $G_\mu$ is not connected, having components in one-to-one correspondence with $\mathbb{Z}$, and so the invariance of $J^H_{\mu}$ fails when $q = 1$, just when a hypothesis of the Lemma fails.

Obtaining a formula for the momentum for the Landau–Lifshitz equation means actually passing the momentum map $J^H$ to the quotient space $\{n\}$. One way to do this is, given some $n$, to find $\psi_n$ such that $\psi_n k = -n$ and then calculate $J^H(\psi_n)$, and one such $\psi_n$ is

$$\psi_n = \exp\left(\frac{-\arccos(-k \cdot n)}{|k \times n|}k \times n\right).$$

A long calculation gives

$$(n \cdot \nabla^R)\psi_n = \frac{-1}{1 - k \cdot n}((k \times n) \cdot \nabla)n,$$

so the momentum map on the reduced space is

$$J^H(n) = \left(\int x \wedge \frac{1}{1 - k \cdot n}((k \times n) \cdot \nabla)n, \int \frac{-1}{1 - k \cdot n}((k \times n) \cdot \nabla)n\right).$$

This is somewhat unsatisfactory due to the singularity at $n = k$, which is due to the singularity of $\psi_n$ at $n = k$. A concern might be that the singularities in the choice of $\psi_n$ might give a value for the momentum that differs from nonsingular choices but the singularities are tame enough not to contribute to the integrals making up the momenta.

There remains the problem of the degree nonzero sectors in the case $p = 2$. Hitting the degree $m$ sector is not a problem: it is the reduced space $(J^G)^{-1}(\mu)/G_\mu$ for some $\mu \in \mathfrak{g}$ such that $\deg \mu = m$ and $|\mu(x)| = 1$ for all $x \in \mathbb{R}^2$. The difficulty is that the subgroup $H$, which ought to represent Euclidean symmetries, does not act on $(J^G)^{-1}(\mu)$, since

$$J^G((U,a)(\psi,-\psi \mu)) = J^G(\psi(U^{-1}(x-a)), -\psi(U^{-1}(x-a))\mu(U^{-1}(x-a)))$$

$$= \mu(U^{-1}(x-a)) \neq \mu(x).$$
This is a fundamental obstruction and it is best to have recourse to the general theory. An appropriate general context is that of Lemma 1. Since $G_\mu$ is normal in $H_\mu$, the quotient group $\tilde{H}_\mu = H_\mu/G_\mu$ acts on $P_\mu = (J^G)^{-1}(\mu)/G_\mu$ and the obvious adjustment is to use the group $\tilde{H}_\mu$ to represent the ‘residual’ symmetries on $P_\mu$. Since $J^{H_\mu}$ is $G_\mu$ invariant it passes to the quotient, but it is not clearly a momentum map for the action of the group $\tilde{H}_\mu$ since it has values in $\tilde{h}_\mu^*$ rather than $\tilde{h}_\mu$. The values of $J^{H_\mu}$ are in $\tilde{h}_\mu^* = (\tilde{h}_\mu/\tilde{g}_\mu)^*$ if and only if they annihilate $\tilde{g}_\mu$. This cannot be expected since if $p \in (J^G)^{-1}(\mu)$ and $\xi \in \tilde{g}_\mu$ then

$$\langle J^{H_\mu}(p), \xi \rangle = \langle \mu, \xi \rangle,$$

which vanishes for all $\xi \in \tilde{g}_\mu$ if and only if $\mu$ annihilates $\tilde{g}_\mu$, an unusual circumstance. For example, if $\mu \neq 0$, and in the presence of an $\text{Ad}$ invariant metric with the usual identification of $\tilde{g}_\mu^*$ and $\tilde{g}_\mu$, one would have $\mu \in \tilde{g}_\mu$ and $\mu$ would not annihilate itself. Also if $\tilde{g}$ is Abelian then $\mu$ annihilates $\tilde{g}_\mu$ if and only if $\mu$ is zero. The obvious adjustment here is to subtract from $J^{H_\mu}$ any extension, say $\hat{\mu}$, of $\mu|\tilde{g}_\mu$ to $\tilde{h}_\mu$. Then $J^{H_\mu} - \hat{\mu}$ has values in $\tilde{h}_\mu$ and is $G_\mu$ invariant on $(J^G)^{-1}(\mu)$, and so defines a map $J^{\tilde{H}_\mu} : P_\mu \to \tilde{h}_\mu^*$. This is a momentum map for the action of $\tilde{H}_\mu$ of $P_\mu$. Indeed, given $\xi \in \tilde{h}_\mu = \tilde{h}_\mu/\tilde{g}_\mu$ there is a $\tilde{\xi} \in \tilde{h}_\mu$ projecting to $\xi$, and the flow of $J^{\tilde{H}_\mu} - \hat{\mu}$, which is multiplication by $\exp(\tilde{\xi}(t))$, projects to the flow of $J^{\tilde{H}_\mu}_\tilde{\xi}$. By construction the projection of multiplication by $\exp(\tilde{\xi}(t))$ is multiplication by $\exp(\xi(t))$, so $J^{\tilde{H}_\mu}$ generates the action of $\tilde{H}_\mu$, which is what is required for $J^{\tilde{H}_\mu}$ to be a momentum map. This reduction procedure is identical to that found in [3].

What is interesting about all this is that the momentum map $J^{\tilde{H}_\mu}$ need not be equivariant, even though it has been reduced from an equivariant momentum map. This effect is also observed in [3] and [3]. The lack of equivariance is characterized (see [3]) by the cocycle $\sigma^{\tilde{H}_\mu} = J^{\tilde{H}_\mu}(\hat{h}\hat{p}) - \text{CoAd}_{\hat{h}} J^{\tilde{H}_\mu}(\hat{p}), h \in \tilde{H}_\mu$. The value of $p$ is irrelevant as long as $P_\mu$ is connected. To calculate this cocycle, given $p \in P_\mu$, choose $\hat{p} \in (J^G)^{-1}(\mu)$ projecting to $p$, and given $h \in \tilde{H}_\mu$ choose $\hat{h} \in \tilde{H}_\mu$ projecting to $h$. Then

$$\sigma^{\tilde{H}_\mu}(h) = J^{\tilde{H}_\mu}(\hat{h}\hat{p}) - \hat{\mu} - \text{CoAd}_{\hat{h}}(J^{\tilde{H}_\mu}(\hat{p}) - \hat{\mu}) = \text{CoAd}_{\hat{h}} \hat{\mu} - \hat{\mu}.$$

The values of $\sigma^{\tilde{H}_\mu}$ annihilate $\tilde{g}_\mu$ and are to be regarded as lying in $\tilde{h}_\mu^*$. The infinitesimal version of this cocycle is the antisymmetric, bilinear cocycle $\Sigma^{\tilde{H}_\mu}$ on $\tilde{h}_\mu$ defined by

$$\Sigma^{\tilde{H}_\mu}(\xi, \eta) = \frac{d}{de} \bigg|_{e=0} \langle \sigma^{\tilde{H}_\mu}(\exp(e\eta)), \xi \rangle.$$

Given $\xi, \eta \in \tilde{h}_\mu$, and choosing $\tilde{\xi}, \tilde{\eta} \in \tilde{h}_\mu$ projecting to $\xi, \eta$ respectively, it is immediate that

$$\Sigma^{\tilde{H}_\mu}(\xi, \eta) = \langle \hat{\mu}, [\tilde{\xi}, \tilde{\eta}] \rangle.$$
Nonequivalence of the momentum map is reflected in the commutation relations
\[
\{ J^\xi_{\mu}, J^\eta_{\mu} \} = J^\mu_{[\xi,\eta]} - \Sigma^\mu (\xi, \eta)
\]
which differ from the usual ones where no cocycle appears.

The reduction just described can be worked out in detail for the Landau–Lifshitz equation. Fix some \( \mu \in \mathfrak{g} \) which is not necessarily constant but \( |\mu(x)| = 1 \); when \( p = 2 \) this \( \mu \) could have nonzero degree. The subgroup \( H_\mu \) is found by finding the subgroup under which \( (J^G)^{-1}(\mu) \) is mapped into itself. Since

\[
J^G \left( (A, (U, a)) (\psi, -\psi \mu) \right)
= J^G \left( \psi(U(x)^{-1}(x - a))A(x)^{-1}, -\psi(U(x)^{-1}(x - a))\mu(U(x)^{-1}(x - a)) \right)
= -A(x)\psi(U(x)^{-1}(x - a))^{-1} \times -\psi(U(x)^{-1}(x - a))\mu(U(x)^{-1}(x - a))
= A(x)\mu(U(x)^{-1}(x - a))
\]

it follows that

\[
H_\mu = \{ (A, (U, a)) : A(x)\mu(U^{-1}(x - a)) = \mu(x) \}.
\]
The projection onto the second factor is a quotient map for the left action of \( G_\mu \) on \( H_\mu \), so \( H_\mu = H_\mu / G_\mu \cong H \) and the action of \( H_\mu \) on the space of Landau–Lifshitz fields \( P_\mu = \{ n \} \) is

\[
(U, a)n = n(U(x)^{-1}(x - a)),
\]
as expected.

To calculate the infinitesimal cocycle first extend \( \mu \) to \( \mathfrak{h}_\mu \). The extension that comes to mind is

\[
\langle \dot{\mu}, (\xi, (\Omega, \dot{a})) \rangle = \int \mu(x) \cdot \xi(x),
\]
or just \( \iota^*_1 \mu \) where \( \iota_1 : G \rightarrow G \times H \) is the inclusion. The subgroup \( H_\mu \) is defined by the condition \( A(x)\mu(U^{-1}(x - a)) = \mu(x) \), so its Lie algebra \( \mathfrak{h}_\mu \) is obtained by differentiating this condition at the identity. Given \( (\xi, (\Omega, \dot{a})) \in \mathfrak{se}(p) \), this gives

\[
\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \exp(\xi(x))\mu(\exp(\Omega \epsilon)(x - \dot{a} \epsilon)) = \xi(x) \times \mu(x) + \nabla_{\Omega x - \dot{a} \mu(x)} = 0,
\]
so given an \( (\Omega, \dot{a}) \in \tilde{H}_\mu \cong \mathfrak{se}(p) \) one finds a \( (\xi, (\Omega, \dot{a})) \in \mathfrak{se}(p) \in \mathfrak{h}_\mu \) projecting to this by solving

\[
\xi(x) \times \mu(x) = \nabla_{\Omega x + \dot{a}} \mu(x).
\]
One solution is
\[ \xi(x) = \mu(x) \times \nabla_{\Omega + \dot{a}} \mu(x) \]
so that one can set
\[ (\Omega, \dot{a})^\wedge = (\mu(x) \times \nabla_{\Omega + \dot{a}} \mu(x), (\Omega, \dot{a})). \]

The Lie bracket of \( G_{\theta} \times H \) is
\[
[\langle (\xi_1, (\Omega_1, \dot{a}_1)), (\xi_2, (\Omega_2, \dot{a}_2))\rangle] \\
= (\xi_1(x) \times \xi_2(x) - \nabla_{\Omega_{x + \dot{a}} \xi_2(x)} + \nabla_{\Omega_{x + \dot{a}} \xi_1(x)}, [(\Omega_1, a_1), (\Omega_2, a_2)])
\]
so the cocycle is (many of the terms vanish because they are obviously perpendicular to \( \mu \) and so vanish when paired with \( \mu \) in the outermost dot product)
\[
\Sigma^H_{\mu}((\Omega_1, \dot{a}_1), (\Omega_2, \dot{a}_2)) = - \int \mu(x) \cdot \nabla_{\Omega_{x + \dot{a}}} \mu(x) \times \nabla_{\Omega_{x + \dot{a}}} \mu(x).
\]
When \( \mu = k \) this cocycle is zero. If \( \mu \) is spherically symmetric, then \( \Sigma^H_{\mu} \) simplifies to
\[
\Sigma^H_{\mu}((\Omega_1, \dot{a}_1), (\Omega_2, \dot{a}_2)) = - 4 \pi \omega_0(\dot{a}_1, \dot{a}_2) \deg \mu,
\]

where
\[
\omega_0(a_1, a_2) = a_1^T J a_2, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\]

A similar \( \mathfrak{se}(2) \) cocycle appears in the planar point vortex system. In order for the further reduction of the Landau–Lifshitz system by the symmetry \( \bar{H}_\mu \) to be symplectomorphic to the original system on \( T^*G \) reduced by \( G_{\theta} \times H \), an additional ‘stages hypothesis’ ([3]) must be satisfied. It is sufficient that \( \mathfrak{g} + \mathfrak{h}_\mu = \mathfrak{g} \times \mathfrak{h} \), a fact which is obvious from the calculation of \( \mathfrak{h}_\mu \) above.

The momentum map \( J_{\Omega, \dot{a}}^H(n) \) is to be calculated by finding \( J_{(\Omega, \dot{a})^\wedge}(\psi, -\psi \mu) \) where \( \psi \) such that \( \psi \mu = n \). With the chosen extension \( \hat{\mu} \) of \( \mu \) the difference \( J^H_{\mu} - \hat{\mu} \) annihilates all of the first factor of \( \mathfrak{g} \times \mathfrak{h} \), so this is equivalent to calculating \( J_{(\Omega, \dot{a})^\wedge}(\psi, -\psi \mu) \). The result is that the reduced momentum is calculated in the same way as the case \( \mu = k \), which is just to find \( \psi \) such that \( \psi \mu = n \) and then calculate \( J^H(\psi, -\psi \mu) \). If one then takes \( n = \mu \), the momentum is zero since one can take \( \psi(x) = \text{Id} \). Thus, the reduced momentum calculated this way is the unique momentum that vanishes at \( \mu \).
As has already been mentioned, from $[1]$ and $[5]$ the conserved quantities

$$P = 2\pi \int x \times \Omega, \quad \Omega_{\alpha} = \frac{1}{8\pi} \epsilon_{\alpha\beta\gamma} n \cdot (\nabla_{\beta} n \times \nabla_{\gamma} n),$$

generate translations in the case $p = 3$. Actually, working out the $x$-component of this where, $\mathbb{R}^3 = \{(x, y, z)\}$, easily gives

$$P_x = \frac{1}{2} \int n \cdot \left( \frac{\partial n}{\partial x} \times \left( y \frac{\partial n}{\partial y} + z \frac{\partial n}{\partial z} \right) \right),$$

and leads directly to the generalization (summation over repeated indices)

$$P_i = \frac{1}{p-1} \int n \cdot \left( \frac{\partial n}{\partial x^i} \times x^j \frac{\partial n}{\partial x^j} \right).$$

It is easily verified that this generates translations for any value of $p \neq 1$. If a vector density $\mathcal{P}$ is defined by

$$\mathcal{P}_i = \frac{1}{p-1} \int n \cdot \left( \frac{\partial n}{\partial x^i} \times x^j \frac{\partial n}{\partial x^j} \right),$$

then, in view of the expression for $J^H$, one might guess that $-\Omega \cdot (x \wedge \mathcal{P}) = \Omega x \cdot \mathcal{P}$ is a density generating the rotations corresponding to $\Omega$. This is also easily verified, and so leads to the momentum map

$$\bar{J}^H = \left( -\int x \wedge \mathcal{P}, \int \mathcal{P} \right).$$

When $n$ is spherically symmetric, the density $\mathcal{P}$ is zero since

$$(p-1)P_i = n \cdot \left( \frac{\partial n}{\partial x^i} \times x^j \frac{\partial n}{\partial x^j} \right) = n \cdot \left( \frac{\partial n}{\partial r} \frac{\partial r}{\partial x^i} \times x^j \frac{\partial n}{\partial r} \frac{\partial r}{\partial x^j} \right) = n \cdot \left( \frac{\partial n}{\partial r} \frac{x^i}{r} \times \frac{\partial n}{\partial r} \frac{x^j}{r} \right) = 0.$$

Thus, $J^H = \bar{J}^H$, and the momentum of Papanicolaou and Tomaras $[5]$ is seen to be the result of semidirect product reduction. In particular, their observation that in the case $p = 2$ one has \{\(P_x, P_y\)\}(n) = 4\pi \deg n follows from the commutation relations and the cocycle, as follows:

$$\{P_x, P_y\}(n) = \{J_{\bar{i}}^H, J_{\bar{j}}^H\}(n) = J_{[\bar{i},\bar{j}]}^H(n) - \Sigma_{\bar{\mu}}^H(\bar{i},\bar{j}) = 4\pi \omega_0(\bar{i},\bar{j}) \deg \mu = 4\pi \deg n.$$

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