A note on sequences that do not have metric Poissonian pair correlations

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Abstract

The purpose of this note is to present a construction of sequences which do not have metric Poissonian pair correlations (MPPC) and whose additive energies grow at rates that come arbitrarily close to a threshold below which it is believed that all sequences have MPPC. A nearly identical result appears already in work of Lachmann and Technau and is proved using a totally different strategy. The main novelty here is the simplicity of the proof, which we arrive at by modifying a construction of Bourgain.

1 Introduction

Let $A \subset \mathbb{N}$ be an infinite subset and denote its smallest $N$ elements $A_N$. For $N \in \mathbb{N}$, $\alpha \in [0, 1]$, and $s > 0$, the quantity

$$F(\alpha, s, N, A) = \frac{1}{N} \sum_{\substack{(a, b) \in A^2 \setminus \{(a, b)\} \ni a \neq b}} 1_{[-s/N, s/N] + \mathbb{Z}}(\alpha(a - b)), \quad (1)$$

measures how often two points in $\alpha A_N(\mod 1)$ lie within a distance $2s/N$ of each other on the circle $\mathbb{T} = [0, 1]/\sim$. If for almost every $\alpha \in [0, 1]$ we have $F(\alpha, s, N, A) \sim 2s$, then $A$ is said to have metric Poissonian pair correlations (MPPC). Since a random point sequence on $\mathbb{T}$ will almost surely have asymptotically Poissonian pair correlations, MPPC is understood as a property connoting random-like behavior for an integer sequence. It is of great interest to understand which integer sequences do and do not have metric Poissonian pair correlations.

In [8], Rudnick and Sarnak showed that the sequence $(n^2)_{n=1}^\infty$ has MPPC whenever $k \geq 2$, whereas it is easy to show that it does not have MPPC if $k = 1$. The intuitive reason that $(n)_{n=1}^\infty$ does not have MPPC is that in this case $A_N = \{1, \ldots, N\}$, and one quickly sees that the quantities $(a - b)$ arising in (2) are too structured to be random-like. Aistleitner, Larcher, and Lewko made this intuition rigorous by connecting MPPC to the behavior of

$$E(A_N) = \#\{(a, b, c, d) \in A_N^4 : a + b = c + d\},$$

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the additive energy of $A_N$. They showed that $A$ has MPPC whenever there exists some $\varepsilon > 0$ for which $E(A_N) \ll N^{3-\varepsilon}$ holds \cite{1}.

In the appendix to the same paper, Bourgain showed that if $E(A_N) \gg N^3$ then $A$ does not have MPPC, and also that there exist sequences for which $E(A_N) = o(N^3)$ which do not have MPPC. This led to a series of papers exploring the connection between additive energy and MPPC. Bloom, Chow, Gafni, and Walker asked the following guiding question.

**Question 1.1** (\cite{2}, Fundamental Question 1.7). Suppose there is a nonincreasing $\psi : \mathbb{N} \to [0, 1]$ such that $E(A_N) \sim N^3\psi(N)$. Is convergence of $\sum \psi(N)/N$ necessary and sufficient for the sequence $A$ to have metric Poissonian pair correlations?

They proved results in support of the answer being “yes,” but as of this writing the overall picture is not complete.

For the sufficiency part, the best result so far is due to Bloom and Walker, and it says that there exists some universal constant $C > 1$ such that if $\psi(N) \ll (\log N)^{-C}$, then $A$ has MPPC \cite{3}. Of course, if one believes the sufficiency part of Question 1.1, then one should believe that any $C > 1$ will do. (Indeed, Hinrichs et al. have established this for a higher dimensional version of the problem \cite{5}.)

The answer to the necessity part of Question 1.1 turns out to be “no.” Aistleitner, Lachmann, and Technau found, for any $\varepsilon > 0$, sequences $A \subset \mathbb{N}$ for which

$$E(A_N) \gg N^3(\log N)^{-\frac{3}{4}-\varepsilon},$$

yet they have metric Poissonian pair correlations. However, the construction is very special. There is still reason to think that perhaps a “randomly chosen” sequence $A \subset \mathbb{N}$ whose additive energy behaves as in the divergence part of Question 1.1 will not have MPPC. Bloom et al. proved a result to this effect, showing that in a certain random model, a sequence $A$ whose additive energy satisfies

$$E(A_N) = N^3(\log N)^{-1}(\log \log N)^{-C}$$

for some $0 \leq C \leq 1$ will almost surely not have MPPC \cite[Theorem 1.6]{2}. In \cite[Theorem 2]{6}, Lachmann and Technau constructed examples where the additive energy is of order $E(A_N) \asymp N^3\psi(N)$ where $\psi$ is any function as in the divergence part of Question 1.1 that satisfies the further condition that $\psi(N) \gg N^{-1/3}(\log N)^{7/3}$. In particular, this yields examples of sets $A \subset \mathbb{N}$ where

$$E(A_N) = N^3(\log N \log \log N \ldots)^{r \text{ iterates}} \log \log \ldots \log N)^{-1}$$

which do not have metric Poissonian pair correlations.

In this note, we present a modified version of Bourgain’s construction \cite[Appendix]{1} which gives examples of sequences which do not have MPPC and whose additive energies meet the threshold proposed in Question 1.1. That is, we prove the following.

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1 We use $f \ll g$ to mean that there exists some universal constant $c > 0$ for which $f \leq cg$ holds for all large arguments of the functions $f$ and $g$. We use $f \asymp g$ to mean $f \ll g \ll f$. 

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Theorem 1.2. Suppose $\psi : \mathbb{N} \to [0,1]$ is a nonincreasing function such that $N^{3-\delta} \psi(N)$ is non-decreasing for some fixed $\delta > 0$, and such that $\sum \psi(N)/N$ diverges. Then there exists an infinite set $\mathcal{A} \subset \mathbb{N}$ such that $E(A_N) \asymp N^3 \psi(N)$ and such that $\mathcal{A}$ does not have metric Poissonian pair correlations.

Remark. As in [6, Theorem 2], Theorem 1.2 has a condition on $\psi$ besides just divergence of the series. Since $E(A_N)$ must increase to infinity it is unavoidable that such a theorem should have extra conditions on $\psi$. Indeed, the extra condition in Theorem 1.2 is only used in the proof that the constructed sequence $\mathcal{A}$ actually satisfies $E(A_N) \ll N^3 \psi(N)$. It is not used in proving the assertion that $\mathcal{A}$ does not have MPPC.

Given that there has to be some extra condition on $\psi$, perhaps it would be most natural to only require that $N^{3-\delta} \psi(N)$ increase to infinity. Instead, we make the slightly stronger assumption that there is some $\delta > 0$ for which $N^{3-\delta} \psi(N)$ is unbounded whenever $0 < \delta < 3$. In particular, Theorem 1.2 applies when

$$\psi(N) = (\log N \log \log N \ldots \log \log \ldots \log N)^{-1}$$

as in [6].

The rest of this note consists of the proof. For a more detailed discussion of pair correlations and additive energy, we recommend the surveys [9, 7].

2 Proof of Theorem 1.2

Notice that we lose no generality in assuming that $\psi(N) = o(1)$, for otherwise we would have $E(A_N) = \Omega(N^3)$, and in this case it is known that $\mathcal{A}$ cannot have MPPC. We may also assume that $\psi(N)^{-1}$ takes only integer values.

Let $\iota(N) : \mathbb{N} \to \mathbb{R}$ decrease to 0 slowly enough that $\sum \frac{\psi(N)\iota(N)}{N}$ still diverges. Let $(\Delta_N)_N$ be a positive integer sequence that increases fast enough that the sets

$$S_N := \left\{ \alpha \in [0,1] : \|\Delta_N \alpha\| \leq \frac{\psi(N)\sqrt{\iota(N)}}{N} \text{ for some } 0 < d \leq N \sqrt{\iota(N)} \right\},$$

where $\|\cdot\|$ denotes distance to $\mathbb{Z}$, are pairwise quasi-independent. To see that it is possible to do this, note that $S_N = \Delta_N^{-1} S$ where $S$ is a union of finitely many intervals in $\mathbb{T}$. In particular, $S$ is measurable. Recall that for any $m \geq 2$, the “times $m$ modulo 1” map $T_m : \mathbb{T} \to \mathbb{T}$ is measure-preserving, meaning that for any measurable set $S$ we have $\text{meas}(T_m^{-1} S) = \text{meas}(S)$, and mixing, meaning that for any two measurable sets $S, T \subset \mathbb{T}$ we have

$$\lim_{k \to \infty} \text{meas} \left( T_m^{-k} (S) \cap T \right) = \text{meas}(S) \text{meas}(T).$$

We may therefore take $\Delta_1 = 1$ and inductively set $\Delta_N$ to be a large enough power of $m$ that

$$\text{meas}(S_N \cap S_M) \leq 2 \text{meas}(S_N) \text{meas}(S_M)$$
for all $M < N$.

Notice that $\text{meas}(S_N) \gg \psi(N)t(N)$. Since $\sum_N \frac{\psi(N)t(N)}{N}$ is a divergent sum of nonincreasing terms, by Cauchy’s condensation test we have that $\sum_t \psi(2^t)t(2^t)$ diverges, hence $\sum_t \text{meas}(S_{2^t})$ diverges. Since the sets $(S_{2^t})$ are pairwise quasi-independent, the version of the second Borel–Cantelli lemma proved by Erdős–Renyi [4] guarantees that the limsup set $S_\infty := \limsup_{t \to \infty} S_{2^t}$ has full measure.

Our goal now is to construct a sequence $\mathcal{A} \subset \mathbb{N}$ such that $E(A_N) \asymp N^3\psi(N)$ and such that for every $\alpha \in S_\infty$, we have $\limsup_{N \to \infty} F(\alpha, 1, N, \mathcal{A}) = \infty$. We will construct $\mathcal{A}$ block by block. For each $N$, let

$$B_N = \left\{ \Delta_N \left( \frac{N}{\psi(N)} + n \right) : 1 \leq n \leq \frac{N}{\psi(N)} \text{ and } \xi^{(N)}_n(\omega) = 1 \right\},$$

with $\xi^{(N)}_1, \ldots, \xi^{(N)}_{N/\psi(N)}$ independent Bernoulli random variables with $\mathbb{P}(\xi^{(N)}_n = 1) = \psi(N)$. For comparison, these blocks $B_N$ are dilates of the blocks in [1] by the factor $\Delta_N$.

In light of [1, Lemma 6], the following three properties hold with positive probability, and so we may henceforth assume that $B_N$ is an instantiation of $B_N(\omega)$ where:

1. For all $d \in \mathbb{Z} \setminus \{0\}$ we have $|B_N \cap (B_N + \Delta_N d)| \leq 2N\psi(N)$.
2. For all $d \in \mathbb{Z} \setminus \{0\}$ with $|d| < \frac{N}{10\psi(N)}$ we have $|B_N \cap (B_N + \Delta_N d)| \geq \frac{1}{2}N\psi(N)$.
3. We have $N/2 \leq |B_N| \leq 2N$.

Since any two elements of $B_N$ differ by a multiple of $\Delta_N$, we have

$$E(B_N) = \sum_{d \in \mathbb{Z}} |B_N \cap (B_N + \Delta_N d)|^2.$$

With this, the first two properties above show us that $E(B_N) \asymp N^3\psi(N)$.

Let $\mathcal{A} := \{B_1, B_2, B_4, \ldots\}$ be the concatenation of the blocks $B_{2^t}$, $t \geq 0$. Suppose that $A_N$ is a truncation of $\mathcal{A}$ in the block $B_{2^t}$. It is obvious then that

$$E(A_N) \gg E(B_{2^{t-1}}) \gg (2^{t-1})^3\psi(2^{t-1}) \gg N^3\psi(N).$$

Also, by possibly making $\Delta_N$ if needed, we have

$$E(A_N) \leq \sum_{k=0}^t E(B_{2^k})$$

$$\ll \sum_{k=0}^t 2^{3k}\psi(2^k)$$

which, by our assumption that $N^{3-\delta}\psi(N)$ is increasing,

$$\ll 2^{3t}\psi(2^t)$$

$$\ll N^3\psi(N).$$

This shows that $\mathcal{A}$ has the desired behavior in additive energy, namely, $E(A_N) \asymp N^3\psi(N)$. 

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As for pair correlations, note that
\[
F(\alpha, s, N, \mathcal{A}) = \frac{1}{N} \sum_{d \in \mathbb{Z} \setminus \{0\}} |A_N \cap (A_N + d)| \mathbf{1}_{d,s/N}(\alpha),
\]
where \(\mathbf{1}_{d,\varepsilon}\) denotes the indicator function of the set \(\{\alpha \in [0,1] : \|d\alpha\| \leq \varepsilon\}\). In particular, for \(N = 2^t\) we have
\[
F(1, |B_1| + |B_2| + |B_4| + \cdots + |B_N|, \mathcal{A}) \geq \frac{1}{4N} \sum_{d \neq 0} |B_N \cap (B_N + \Delta_N d)| \mathbf{1}_{\Delta_N d,1/(4N)}
\]
\[
\geq \frac{\psi(N)}{8} \sum_{0<d \leq \frac{N}{10\psi(N)}} \mathbf{1}_{\Delta_N d,1/(4N)}
\]
\[
\geq \frac{\psi(N)}{4} \sum_{0<d \leq \frac{N}{10\psi(N)}} \mathbf{1}_{\Delta_N d,1/(4N)}.
\]
Notice that for any \(\alpha \in S_N\), we will have
\[
F(\alpha, 1, |B_1| + \cdots + |B_N|, \mathcal{A}) \geq \frac{1}{40\sqrt{\psi(N)}}.
\]
Since almost every \(\alpha \in [0,1]\) is contained in infinitely many \(S_{2^t}\)'s, this implies that
\[
\limsup_{N \to \infty} F(\alpha, 1, N, \mathcal{A}) = \infty
\]
for almost every \(\alpha \in [0,1]\). Therefore \(\mathcal{A}\) does not have metric Poissonian pair correlations.

\[\square\]

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