A Parametrization for the Neutrino Mixing Matrix

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Abstract

We propose a flexible and model independent parametrization of the neutrino mixing matrix, which takes advantage of the fact that there are up to three small quantities in neutrino mixing phenomenology: (i) the deviation from maximal mixing of solar neutrinos, (ii) the mixing matrix element $U_{e3}$ and (iii) the deviation from maximal mixing of atmospheric neutrinos. It is possible to quantify those three observations with a parameter $\lambda \sim 0.2$, which appears at least linearly in all elements of the mixing matrix. The limit $\lambda \to 0$ corresponds to exact bimaximal mixing. Present and future experiments can be used to pin down the power of $\lambda$ required to usefully describe the observed phenomenology. Observing that the ratio of the two measured mass squared differences is roughly $\lambda^2$ allows to further study the structure of the Majorana mass matrix. We comment on the implications of this parametrization for neutrinoless double beta decay and on the oscillation probabilities in long–baseline experiments.
1 Introduction

Neutrino physics has made impressive progress in recent years [1]. In particular, the structure of the neutrino mixing matrix has been identified to a reasonable precision. The final step for the determination of its structure has come from the KamLAND experiment [2], which confirmed the large mixing angle solution for the solar neutrino problem, after a preference for this parameter space was already implied by the data of the various solar neutrino experiments [3]. Very recently, the SNO salt phase data [4] rejected maximal solar mixing by more than 5σ [4, 5, 6]. The (almost) maximal mixing of atmospheric neutrinos has been found by the SuperKamiokande experiment [7] and confirmed by the K2K collaboration [8]. Finally, the presence of a small if not zero angle was implied by reactor experiments [9]. In the present paper we wish to propose a parametrization of the neutrino mixing matrix in terms of a small parameter $\lambda$, whose magnitude is interestingly around 0.2, i.e., close to the Wolfenstein parameter used to parametrize the CKM matrix [10]. In any parametrization of the neutrino mixing matrix (for earlier attempts, see [11, 12, 13]), it is convenient to start from a reference matrix and describe deviations from it. Our reference matrix is the one corresponding to exact bimaximal neutrino mixing. The parameter $\lambda$ describes the up to three small deviations from this mixing scheme, namely the deviation from maximal mixing of solar neutrinos, the deviation from zero $U_{e3}$ and the deviation from maximal mixing of atmospheric neutrinos. The magnitude of $\lambda$ is defined by the observed non–maximality of solar neutrino mixing [4, 5, 6] and future precision experiments can be used to pin down the power of $\lambda$ to usefully describe the other two deviations. In addition, the ratio of the mass squared differences governing solar and atmospheric neutrino oscillations is given by $\lambda^2$, so that it is possible to analyze also the structure of the neutrino mass matrix (provided neutrinos are Majorana particles). We also analyze the oscillation probabilities for long baseline experiments and the effective mass as measured in neutrinoless double beta decay within our parametrization.

The paper is organized as follows: In Section 2 we describe the neutrino mixing parameters as implied by current data and outline the idea of our parametrization. Then, in Section 3 we give the form of the mixing matrix for various special cases of the parametrization and analyze in Section 4 the form of the neutrino mass matrix. In Section 5 we apply our parametrization to the effective mass as measured in neutrinoless double beta decay and to long–baseline oscillation experiments. We conclude in Section 6.

2 Quark versus Lepton Mixing

The Wolfenstein parametrization [10] of the CKM matrix uses the fact that the quark mixing is very small, i.e., the mixing matrix is quasi the unit matrix with only small corrections on the off–diagonal entries. In terms of mixing angles, a hierarchy of the form $\theta_{12} \sim 0.1 > \theta_{23} \sim 0.01 > \theta_{13} \sim 0.001$ is observed. This has been used by Wolfenstein to introduce an expansion parameter $\lambda$ describing the mixing between $u$ and $s$ quarks. The observation that $c = b \ (u \sim b)$
mixing is roughly one (two) orders of magnitude suppressed then leads to
\[
V_{\text{CKM}} \simeq \begin{pmatrix}
1 - \frac{1}{2} \lambda^2 & \lambda & A \lambda^3 (\rho - i\eta) \\
-\lambda & 1 - \frac{1}{2} \lambda^2 & A \lambda^2 \\
A \lambda^3 (1 - \rho + i\eta) & -A \lambda^2 & 1
\end{pmatrix} + O(\lambda^4) .
\] (1)

Of course, \( \lambda \) corresponds to the Cabibbo angle \( \sin \theta_C \simeq 0.22 \), whereas the other parameters are about \( A \simeq 0.83 \), \( \rho \simeq 0.23 \) and \( \eta \simeq 0.36 \). The latter parameter describes the \( CP \) violation in the quark sector; all such effects have to be proportional to \( J_{CP} = \text{Im} \{V_{ud} V_{cb} V_{ub}^* V_{cd}^*\} \simeq -A^2 \lambda^6 \eta \sim -3 \cdot 10^{-5} \). (2)

Therefore, \( CP \) violation in the quark sector is a small effect.

2.1 Neutrino Mixing

The neutrino oscillation data can consistently be described within a 3–neutrino mixing scheme with massive neutrinos, in which the flavor states \( \nu_\alpha \) (\( \alpha = e, \mu, \tau \)) are mixed with the mass states \( \nu_i \) (\( i = 1, 2, 3 \)) via \( U_{\text{PMNS}} \), the unitary Pontecorvo–Maki–Nagakawa–Sakata \( \) lepton mixing matrix. It can be parametrized as
\[
U_{\text{PMNS}} = \begin{pmatrix}
c_{12}c_{13} & s_{12}c_{13} & s_{13} \\
-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13}e^{i\delta} \\
s_{12}s_{23} - c_{12}s_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{23}s_{13}e^{i\delta} & c_{23}s_{13}e^{i\delta}
\end{pmatrix} \text{diag}(1, e^{i\alpha}, e^{i\beta}) ,
\] (3)

where \( \delta \) is a Dirac \( CP \) violating phase, \( \alpha \) and \( \beta \) are possible Majorana \( CP \) violating phases \( \) \( , \) \( c_{ij} = \cos \theta_{ij} \) and \( s_{ij} = \sin \theta_{ij} \). We shall not consider the two Majorana phases in this Section. The angles \( \theta_{12} \) and \( \theta_{23} \) control the oscillations of solar and atmospheric neutrinos, respectively. The angle \( \theta_{13} \) is mainly limited by reactor \( \bar{\nu}_e \) experiments: The Dirac phase \( \delta \) can be measured in long baseline neutrino oscillation experiments (see, e.g., \( \) \( [18] \)).

To obtain information about the PMNS matrix one fits \( \) \( [5, 6, 19] \) the results of neutrino experiments to the hypothesis of neutrino oscillations. The relevant formula for the oscillation probabilities is
\[
P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 2 \text{Re} \sum_{j>i} U_{\alpha i} U_{\beta j}^* U_{\alpha j}^* U_{\beta i}^* \left( 1 - \exp \frac{i\Delta m_{ji}^2 L}{2 E} \right) ,
\] (4)

where \( \Delta m_{ji}^2 = m_j^2 - m_i^2 \). The 1 (3)\( \sigma \) ranges of recent analyzes read in terms of the parametrization Eq. (3):

(0.27) 0.35 \leq \tan^2 \theta_{12} \leq 0.52 (0.72) \ \ [5] ,

(0.45) 0.75 \leq \tan^2 \theta_{23} \leq 1.3 (2.3) \ \ [19] ,

0 (0) \leq \sin^2 \theta_{13} \leq 0.029 (0.074) \ \ [5] .

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The best–fit points are given by

$$\tan^2 \theta_{12} = 0.43 \quad \text{[5]}, \quad \tan^2 \theta_{23} = 1 \quad \text{[19]}.$$  

(6)

For $\theta_{23}$ there is an ambiguity corresponding to $\theta_{23} \leftrightarrow \pi/2 - \theta_{23}$, i.e., the angle lies on the “light” or “dark side”. Matter effects in future long–baseline experiments will resolve this. In terms of the often used parameter $\sin^2 2\theta_{23}$, which is blind to this ambiguity, one has at $1 \ (3)\sigma$: $\sin^2 2\theta_{23} \geq 0.86 \ (0.84)$.

Regarding the mass squared differences, the best–fit values are $(\Delta m^2_\odot)_{\text{BF}} = 7.2 \cdot 10^{-5} \text{ eV}^2$ [5] and $(\Delta m^2_\mu)_{\text{BF}} = 2.6 \cdot 10^{-3} \text{ eV}^2$ [19]. A recent preliminary analysis of the SuperKamiokande collaboration, taking into account e.g. improved identification criteria and neutrino fluxes, yields a value of $(\Delta m^2_\lambda)_{\text{BF}} = 2.0 \cdot 10^{-3} \text{ eV}^2$ [20]. There are two possible mass orderings, the normal and the inverse mass ordering:

normal mass ordering: $\Delta m^2_\odot = \Delta m^2_{31} \ll \Delta m^2_{32} \simeq \Delta m^2_{21} = \Delta m^2_\Lambda$

inverse mass ordering: $\Delta m^2_\odot = \Delta m^2_{31} \ll -\Delta m^2_{21} \simeq -\Delta m^2_{32} = \Delta m^2_\Lambda$

(7)

Extreme cases are the normal (inverse) hierarchy with $m_3 = \Delta m^2_\lambda \gg m_2 = \Delta m^2_\mu \gg m_1 \ (m_2 = \Delta m^2_\lambda \simeq m_1 \gg m_3)$ and the quasi–degenerate mass scheme for which $m^2_3 \simeq m^2_2 \simeq m^2_1 \gg \Delta m^2_\lambda$. The latter is fulfilled for values of the neutrino masses larger than $\sim 0.2 \text{ eV}$.

Ignoring the phases, the “best–fit PMNS matrix” reads:

$$U^\text{BF}_{\text{PMNS}} = \begin{pmatrix}
0.84 & 0.55 & 0 \\
-0.39 & 0.59 & 0.71 \\
0.39 & -0.59 & 0.71
\end{pmatrix}.$$  

(8)

In the pre–SNO–salt–phase analysis of Ref. [19] there was given the $3\sigma$ range of the PMNS matrix:

$$|U_{\text{PMNS}}| = \begin{pmatrix}
0.73 - 0.88 & 0.47 - 0.67 & 0 - 0.23 \\
0.17 - 0.57 & 0.37 - 0.73 & 0.56 - 0.84 \\
0.20 - 0.58 & 0.40 - 0.75 & 0.54 - 0.82
\end{pmatrix},$$  

(9)

where the phase $\delta$ was allowed to take arbitrary values.

Next generation long–baseline experiments will be able to probe $\Delta m^2_\lambda$ and $\sin 2\theta_{23}$ to the % accuracy [21]. The element $U_{e3}$ can be probed down to the level $10^{-3}$ in future long–baseline or reactor experiments [22]. Neutrino factories [23] can improve these bounds considerably. The solar neutrino mixing angle $\tan^2 \theta_{\odot}$ will see its error reduced below 10 % by experiments investigating the low energy neutrino fluxes from the sun [24].

Currently no information about leptonic $CP$ violation exists. In oscillation experiments one can detect $CP$ violating effects [18], which have to be proportional to [25]

$$J_{CP} = \text{Im} \{U_{e1} U_{\mu 2}^* U_{\mu 1}^* U_{e2}^* \} = \frac{1}{8} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \cos \theta_{13} \sin \delta$$

$$\simeq \frac{\theta_{13}}{4} \left( 1 - \text{corrections from } \theta_{13} \text{ and non–maximal } \theta_{12} \text{ and } \theta_{23} \right),$$  

(10)
where the value $\theta_{13}/4$ is the limit for small $\theta_{13}$, maximal $\theta_{12}$ and $\theta_{23}$.

### 2.2 The Strategy

As a very useful limit, the bimaximal mixing pattern $[26]$, corresponding to $\theta_{12} = \theta_{23} = \pi/4$ and $\theta_{13} = 0$, can be considered. The resulting mixing matrix, ignoring the $CP$ violating phases, reads

$$U_{\text{PMNS}}^{\text{bimax}} = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
-\frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}}
\end{pmatrix}. \tag{11}$$

The non–zero entries therefore take values of $\frac{1}{\sqrt{2}}$ and $\pm \frac{1}{2}$. This form of $U_{\text{PMNS}}$ shall be our reference matrix, whose deviations are to be described by some small parameter $\lambda$. In some sense, the matrix (11) is the analogue to the unit matrix in case of quark mixing. Corrections of order $\lambda$ and higher to the unit matrix lead to the CKM matrix as parametrized in Eq. (1). In the same way, corrections to $U_{\text{PMNS}}^{\text{bimax}}$ of order $\lambda$ and higher can lead to the observed neutrino mixing phenomenology with non–maximal solar neutrino mixing, possible non–maximal atmospheric neutrino mixing and non–zero $U_{e3}$. One might state that the unit matrix in the quark sector and the bimaximal mixing matrix in Eq. (11) are the zeroth order form of the relevant mixing matrix. We shall comment on a possible origin of corrections to bimaximal mixing (along the lines of Ref. [27]) in Section 3.7. Tries to parametrize the PMNS matrix in analogy to the CKM matrix suffer in general from the fact that from the 9 elements of $U_{\text{PMNS}}$ only one is small, namely the element $|U_{e3}| \lesssim 0.27$. The other eight entries take typically values around $1/2$ or $1/\sqrt{2}$. There are to our knowledge two other approaches to do something similar to the PMNS matrix as has been done so successfully with the CKM matrix. The analysis from [11] uses $U_{e2} \simeq \sqrt{2}\lambda$, atmospheric neutrino mixing remains maximal and $U_{e3}$ is proportional to $\lambda^2$. The work [12] chooses the expansion parameter $\lambda = U_{\mu 3} \simeq 1/\sqrt{2}$. Solar neutrino mixing is governed by $U_{e2} = A\lambda^2$ and the small quantity $U_{e3}$ has to be introduced at least the eights power of $\lambda$. Effects due to $CP$ violation in neutrino oscillations are — courtesy of Eq. (10) — proportional to at least $\lambda^{11}$. An Ansatz for $U_{\text{PMNS}}$ corresponding to $\tan^2 \theta_{12} = 0.5$, maximal atmospheric mixing and zero $U_{e3}$ has been used in [13]. To describe deviations from it, it has been multiplied with a Wolfenstein–like matrix.

In this letter we wish to propose a purely phenomenological and model independent parametrization of the PMNS matrix by using a small “expansion” parameter $\lambda$. For a useful analysis in terms of a small parameter one requires small quantities in the mixing matrix. The basic idea is given by the identification of up to three such small numbers in neutrino mixing phenomenology, namely:

(i) the deviation from maximal mixing of solar neutrinos

(ii) the small mixing element $U_{e3}$
(iii) the possible deviation from maximal mixing of atmospheric neutrinos

Those three aspects describe all possible deviations from the bimaximal mixing scheme in Eq. (11). Observation (i) is now a solid experimental evidence, after inclusion of the SNO salt phase data \[4\], it now holds that $\tan^2 \theta_\odot < 1$ at more than 5\(\sigma\) \[4, 5, 6\]. Regarding observation (ii), only the mentioned limit of $|U_{e3}|^2 \leq 0.07$ (at 3\(\sigma\)) exists. Best–fit points of three flavor analyzes of all neutrino data typically yield very small if not vanishing values for this quantity. Finally, atmospheric neutrino mixing insists to be described by solutions with a best–fit corresponding to maximal mixing. This remains true also when the K2K data is included or separately analyzed (e.g., \[19\]). Though exactly maximal mixing and zero $U_{e3}$ would hint to some underlying symmetry in the lepton sector, one can not expect radiative corrections to allow these extreme values to persist down to low energy \[28\]. Thus, one expects non–extreme values for $\theta_{13}$ and $\theta_{23}$. See, e.g., \[27, 29\] for ways to generate deviations from the bimaximal mixing scheme.

All in all, the three observations (i) to (iii) together with the mixing matrices \(8, 9\) lead us to parametrize three elements of the mixing matrix as

\[
\begin{align*}
U_{e2} &= \sqrt{\frac{1}{2}} (1 - \lambda) , \\
U_{e3} &= A \lambda^n , \\
U_{\mu3} &= \sqrt{\frac{1}{2}} (1 - B \lambda^m) e^{i\delta} .
\end{align*}
\]  

(12)

For $\lambda = 0$ one would have the bimaximal scheme from Eq. (11). The two Majorana phases are left out for the moment. Unitarity of $U_{PMNS}$ suffices to calculate the remaining elements. The parameters $A$ and $B$ are numbers of order one. The $\theta_{23} \leftrightarrow \pi/2 - \theta_{23}$ ambiguity reflects in a sign ambiguity of $B$. The power of $\lambda$ in the expressions for $U_{e3}$ ($U_{\mu3}$) can be adjusted when more stringent limits (more precision data) are available.

We can take the best–fit value from Eq. (6) to calculate $\lambda \simeq 0.22$, which is remarkably similar to the Wolfenstein parameter or the sine of the Cabibbo angle. The maximal allowed value of $|U_{e3}|^2 = 0.07$ corresponds to $U_{e3} = A \lambda$ with $A \simeq 1.2$. At 1 (3)\(\sigma\), the range of $\lambda$ lies in $\lambda \simeq (0.08) 0.18 - 0.28 (0.35)$.

(13)

For the best–fit points of the many available analyzes \[6\], which lie in the range between 0.41 and 0.44 for $\tan^2 \theta_{12}$, $\lambda$ is between 0.24 and 0.22.

If indeed $\lambda \simeq 0.22$, then for $m = 1$ it must hold that $B \lesssim 0.91$ in order to fulfill the requirement $\sin^2 2 \theta_{23} \gtrsim 0.85$. In the following we shall work with the “best–fit” value of $\lambda = 0.22$. If the limit on $|U_{e3}|^2$ goes below $\sim 10^{-2}$ one should take the power $n = 2$ in Eq. (12) in order to keep $A$ of order one. If $|U_{e3}|^2 \lesssim 10^{-4}$, then $n = 3$ is advantageous to choose.

Analogously, since typically (see below) $\sin^2 2 \theta_{23} \simeq 1 - 4 B^2 \lambda^{2m}$ one should for values larger than $\sin^2 2 \theta_{23} \simeq 0.95 (0.99)$ use the power $m = 2$ ($m = 3$) in Eq. (12). Values of $m = 4$

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\[^1\]Deviations from maximal solar neutrino mixing have also been analyzed in terms of the parameter $\epsilon = 1 - 2 \sin^2 \theta_{12}$ \[20\], which roughly corresponds to $\lambda$: $\epsilon \simeq \lambda + \mathcal{O}(\lambda^2)$. 
would be required if a precision in $\sin^2 2\theta_{23}$ of order $10^{-4}$ was present, which seems improbable unless a neutrino factory will be operative. In terms of $\tan^2 \theta_{23}$, which in the future will be more appropriate to use, one will find that $\tan^2 \theta_{23} \approx 1 - 4 B \lambda^m$. Thus, for $\tan^2 \theta_{23} \gtrsim 0.7$ (or $\lesssim 1.3$) one should take $m = 2$, while for $\tan^2 \theta_{23} \gtrsim 0.9$ (or $\lesssim 1.1$) the value $m = 3$ is more useful.

We shall now consider several different cases for the powers of $\lambda$ in Eq. (12). The considerations from this Section indicate that current data and the precision of future experiments on $\theta_{23}$ and $\theta_{13}$ limit the realistic values of $m$ and $n$ between 1 and 3.

3 The Mixing Matrix

3.1 Case $m = n = 1$

In this case we have $U_{e3} = A \lambda$ and $U_{\mu 3} = \sqrt{\frac{1}{2}} (1 - B \lambda) e^{i\delta}$. It corresponds to rather large deviations from the extreme bimaximal values. One “predicts” $U_{e3}$ very close to its current limit and also $\sin^2 2\theta_{23}$ is on the edge of its $3\sigma$ range. We identify

$$\tan^2 \theta_{12} \approx 1 - 4 \lambda + 2 (5 + A^2) \lambda^2 + \mathcal{O}(\lambda^3),$$

$$\sin^2 2\theta_{23} \approx 1 - 4 B^2 \lambda^2 + \mathcal{O}(\lambda^3), \quad \tan^2 \theta_{23} \approx 1 - 4 B \lambda + 2 (A^2 + 5 B^2) \lambda^2 + \mathcal{O}(\lambda^3)$$

(14)

$$\sin^2 \theta_{13} = A^2 \lambda^2.$$ 

The fact that $\sin^2 2\theta_{23}$ is blind to the $\theta_{23} \leftrightarrow \pi/2 - \theta_{23}$ ambiguity is reflected in the fact that $B$ appears quadratically in the last expression. The form of $U_{\text{PMNS}}$ is rather lengthy, and we shall give it therefore only to order $\lambda$:

$$U_{\text{PMNS}} \approx \begin{pmatrix}
\sqrt{\frac{1}{2}} (1 + \lambda) \\
-\frac{1}{2} (1 - (1 - B - A e^{i\delta}) \lambda) \\
\frac{1}{2} (1 - (1 + B + A e^{i\delta}) \lambda)
\end{pmatrix} \begin{pmatrix}
\sqrt{\frac{1}{2}} (1 - \lambda) \\
\frac{1}{2} (1 + (1 + B - A e^{i\delta}) \lambda) \\
\frac{1}{2} (1 + (1 - B + A e^{i\delta}) \lambda)
\end{pmatrix} A \lambda + \mathcal{O}(\lambda^2).$$

(15)

The precise form to a given order of $\lambda$ is easily obtained by using the unitarity of the mixing matrix. The corrections quadratic in $\lambda$ are functions of $A$ and $B$ except for $U_{e1}$, which receives only corrections depending on $A$. Important to note is that the corrections from $\lambda$ are responsible for the deviations from the values $\pm 1/2$ of the entries in the lower left 12 block. Finally, the invariant measure of $CP$ violation in neutrino oscillations is

$$J_{CP} = \frac{A \lambda}{4} \left(1 - (2 + A^2 + 2 B^2) \lambda^2\right) \sin \delta + \mathcal{O}(\lambda^4).$$

(16)

Noting that $A \lambda \approx \theta_{13}$, the corrections stemming from $\theta_{13}$ and non-maximal $\theta_{12,23}$ — as indicated in Eq. (10) — are easily identified. The larger the deviations from maximal $\theta_{12,23}$, i.e., the larger $A$ and $B$, the smaller becomes $J_{CP}$. The “prediction” is that $CP$ violating effects are up to $\sim 5\%$. Note however that actual experiments searching for leptonic $CP$ violation will not just measure $J_{CP}$, see Section 5.2.
3.2 Case \( m = 1 \) and \( n = 2 \)

For these values we have \( U_{e3} = A \lambda^2 \) and \( U_{\mu 3} = \sqrt{\frac{1}{2} (1 - B \lambda)} e^{i\delta} \). We can identify

\[
\tan^2 \theta_{12} \simeq 1 - 4 \lambda + 10 \lambda^2 + O(\lambda^3) ,
\sin^2 2 \theta_{23} \simeq 1 - 4 B^2 \lambda^2 + O(\lambda^3) , \quad \tan^2 \theta_{23} \simeq 1 - 4 B \lambda + 10 B^2 \lambda^2 + O(\lambda^3) , \quad \sin^2 \theta_{13} = A^2 \lambda^4 .
\]

(17)

The “predictions” are \( |U_{e3}|^2 \sim 10^{-3} \) and atmospheric mixing very close to the end of its allowed 3\( \sigma \) range. The mixing matrix \( U_{PMNS} \) reads

\[
U_{PMNS} \simeq \left( \begin{array}{ccc}
\sqrt{\frac{1}{2}} (1 + \lambda) & \sqrt{\frac{1}{2}} (1 - \lambda) & A \lambda^2 \\
- \frac{1}{2} (1 - (1 - B) \lambda) & \frac{1}{2} (1 + (1 + B) \lambda) & \sqrt{\frac{1}{2}} (1 - B \lambda) e^{i\delta} \\
\frac{1}{2} (1 - (1 + B) \lambda) & - \frac{1}{2} (1 + (1 - B) \lambda) & \sqrt{\frac{1}{2}} (1 + B \lambda) e^{i\delta} \\
\end{array} \right) + O(\lambda^2) .
\]

(18)

It is obtained by removing the term \( A e^{i\delta} \) from the PMNS matrix in the case of \( m = n = 1 \) as given in Eq. (15). The corrections of order \( \lambda^2 \) for the lower left 2 by 2 submatrix are functions of \( A \) and \( B \). They are constant for \( U_{e1} \) and only depending on \( B \) for \( U_{e3} \). Effects of \( CP \) violation are proportional to \( \lambda^2 \),

\[
J_{CP} \simeq \frac{A \lambda^2}{4} (1 - 2 (1 + B^2) \lambda^2) \sin \delta + O(\lambda^5) ,
\]

(19)

and not more than a few %.

3.3 Case \( m = 2 \) and \( n = 1 \)

Now our parameters read \( U_{e3} = A \lambda \) and \( U_{\mu 3} = \sqrt{\frac{1}{2} (1 - B \lambda^2)} e^{i\delta} \). The mixing angles are

\[
\tan^2 \theta_{12} \simeq 1 - 4 \lambda + 2 (5 + A^2) \lambda^2 + O(\lambda^3) , \quad \sin^2 2 \theta_{23} \simeq 1 - (A^2 - 2 B)^2 \lambda^4 + O(\lambda^5) , \quad \tan^2 \theta_{23} \simeq 1 + 2 (A^2 - 2 B) \lambda^2 + O(\lambda^4) , \quad \sin^2 \theta_{13} = A^2 \lambda^2 .
\]

(20)

Thus, \( U_{e3} \) is close to its current limit and the deviation from \( \sin^2 2 \theta_{23} = 1 \) is not more than a few %. The mixing matrix \( U_{PMNS} \) is given by

\[
U_{PMNS} \simeq \left( \begin{array}{ccc}
\sqrt{\frac{1}{2}} (1 + \lambda) & \sqrt{\frac{1}{2}} (1 - \lambda) & A \lambda \\
- \frac{1}{2} (1 - (1 - A e^{i\delta}) \lambda) & \frac{1}{2} (1 + (1 - A e^{i\delta}) \lambda) & \sqrt{\frac{1}{2}} (1 - B \lambda^2) e^{i\delta} \\
\frac{1}{2} (1 - (1 + A e^{i\delta}) \lambda) & - \frac{1}{2} (1 + (1 + A e^{i\delta}) \lambda) & \sqrt{\frac{1}{2}} (1 + B \lambda^2) e^{i\delta} \\
\end{array} \right) + O(\lambda^2) ,
\]

(21)

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which is obtained from Eq. (15) by removing $B$ from the lower left 2 by 2 submatrix. The quadratic corrections are functions of $A$ and $B$ except for $U_{e1}$, which only depends on $A$. The rephasing invariant $CP$ violation measure is

$$J_{CP} \simeq \frac{A \lambda}{4} (1 - (2 + A^2) \lambda^2) \sin \delta + O(\lambda^4)$$

being rather sizable but not exceeding 5%.

### 3.4 Case $m = n = 2$

Now it holds $U_{e3} = A \lambda^2$ and $U_{\mu3} = \sqrt{\frac{1}{2}} (1 - B \lambda^2) e^{i \delta}$, yielding the mixing angles

$$\tan^2 \theta_{12} \simeq 1 - 4 \lambda + 10 \lambda^2 + O(\lambda^3),$$

$$\sin^2 2 \theta_{23} \simeq 1 - 4 B^2 \lambda^4 + O(\lambda^5), \quad \tan^2 \theta_{23} \simeq 1 - 4 B \lambda^2 + O(\lambda^3),$$

$$\sin^2 \theta_{13} = A^2 \lambda^4.$$  

The deviation from $\sin^2 2 \theta_{23} = 1$ is not more than a few percent and $|U_{e3}|$ is on the level of $10^{-3}$. The mixing matrix $U_{\text{PMNS}}$ is given by

$$U_{\text{PMNS}} \simeq \begin{pmatrix}
\sqrt{\frac{1}{2}} (1 + \lambda - \lambda^2) & \sqrt{\frac{1}{2}} (1 - \lambda) & A \lambda^2 \\
-\frac{1}{2} (1 - \lambda + (B + A e^{i \delta}) \lambda^2) & \frac{1}{2} (1 + \lambda - (1 - B + A e^{i \delta}) \lambda^2) & \sqrt{\frac{1}{2}} (1 - B \lambda^2) e^{i \delta} \\
\frac{1}{2} (1 - \lambda - (B + A e^{i \delta}) \lambda^2) & -\frac{1}{2} (1 + \lambda - (1 + B - A e^{i \delta}) \lambda^2) & \sqrt{\frac{1}{2}} (1 + B \lambda^2) e^{i \delta}
\end{pmatrix} + O(\lambda^3).$$

It is seen that for the lower left 2 by 2 submatrix the linear corrections to the “bimaximal” values $\pm 1/2$ are constant and the quadratic ones are functions of $A$ and $B$. The rephasing invariant $CP$ violation measure is given by

$$J_{CP} \simeq \frac{A \lambda^2}{4} (1 - 2 \lambda^2) \sin \delta + O(\lambda^5),$$

again on the level of a few percent.

### 3.5 The “Wolfenstein Case” $m = 2$ and $n = 3$ 

How could one not be tempted to put the third power of the expansion parameter in the $U_{e3}$ and the second power in the $U_{\mu3}$ element. This would resemble the Wolfenstein parametrization Eq. (11). In this case, i.e., $U_{e3} = A \lambda^3$ and $U_{\mu3} = \sqrt{\frac{1}{2}} (1 - B \lambda^2) e^{i \delta}$, we have

$$\tan^2 \theta_{12} \simeq 1 - 4 \lambda + 10 \lambda^2 + O(\lambda^3),$$

$$\sin^2 2 \theta_{23} \simeq 1 - 4 B^2 \lambda^4 + O(\lambda^5), \quad \tan^2 \theta_{23} \simeq 1 - 4 B \lambda^2 + O(\lambda^3),$$

$$\sin^2 \theta_{13} = A^2 \lambda^6.$$
The “prediction” for $\sin^2 2\theta_{23} - 1$ is not more than a few percent and $|U_{e3}|^2$ is on the level of $10^{-4}$. For the PMNS matrix one has

$$U_{\text{PMNS}} \simeq$$

$$\begin{pmatrix}
\sqrt{1 \over 2} (1 + \lambda - \lambda^2 + \lambda^3) & \sqrt{1 \over 2} (1 - \lambda) & A \lambda^3 \\
-1 \over 2 (1 - \lambda + B \lambda^2 - (B - A e^{i\delta}) \lambda^3) & 1 \over 2 (1 + \lambda - (1 - B) \lambda^2 + (1 + B - A e^{i\delta}) \lambda^3) & \sqrt{1 \over 2} (1 - B \lambda^2) e^{i\delta} \\
1 \over 2 (1 - \lambda - B \lambda^2 + (B - A e^{i\delta}) \lambda^3) & -1 \over 2 (1 + \lambda - (1 + B) \lambda^2 + (1 - B + A e^{i\delta}) \lambda^3) & \sqrt{1 \over 2} (1 + B \lambda^2) e^{i\delta} \\
\end{pmatrix} + \mathcal{O}(\lambda^4).$$

In contrast to the Wolfenstein parametrization, however, $\lambda$ appears linearly in e.g. the $U_{\mu 2}$ element. Also, $CP$ violation is proportional to the third power of $\lambda$,

$$J_{CP} \simeq {A \lambda^3 \over 4} (1 - 2 \lambda^2) \sin \delta + \mathcal{O}(\lambda^5),$$

not exceeding one percent.

### 3.6 The remaining cases

The remaining interesting cases are $m = 3$ with $n = 2$ and also $m = n = 3$. The former case — that is $U_{\mu 3} = \sqrt{1 \over 2} (1 - B \lambda^3) e^{i\delta}$ and $U_{e 3} = A \lambda^2$ — yields

$$\tan^2 \theta_{12} \simeq 1 - 4 \lambda + 10 \lambda^2 + \mathcal{O}(\lambda^3),$$

$$\sin^2 2\theta_{23} \simeq 1 - 4 B^2 \lambda^6 + \mathcal{O}(\lambda^7),$$

$$\tan^2 \theta_{23} \simeq 1 - 4 B \lambda^3 + \mathcal{O}(\lambda^4),$$

$$\sin^2 \theta_{13} = A^2 \lambda^4,$$

with $|U_{e 3}|^2 \sim 10^{-3}$ and $\sin^2 2\theta_{23} \neq 1$ with a precision of less than 1%. For the PMNS matrix holds:

$$U_{\text{PMNS}} \simeq$$

$$\begin{pmatrix}
\sqrt{1 \over 2} (1 + \lambda - \lambda^2 + \lambda^3) & \sqrt{1 \over 2} (1 - \lambda) & A \lambda^2 \\
-1 \over 2 (1 - \lambda + A e^{i\delta} \lambda^2 + (B + A e^{i\delta}) \lambda^3) & 1 \over 2 (1 + \lambda - (1 + A e^{i\delta}) \lambda^2 + (1 + B + A e^{i\delta}) \lambda^3) & \sqrt{1 \over 2} (1 - B \lambda^3) e^{i\delta} \\
1 \over 2 (1 - \lambda - A e^{i\delta} \lambda^2 - (B + A e^{i\delta}) \lambda^3) & -1 \over 2 (1 + \lambda - (1 - A e^{i\delta}) \lambda^2 + (1 - B - A e^{i\delta}) \lambda^3) & \sqrt{1 \over 2} (1 + B \lambda^3) e^{i\delta} \\
\end{pmatrix} + \mathcal{O}(\lambda^4).$$

As for the case $m = n = 2$ one obtains:

$$J_{CP} \simeq {A \lambda^2 \over 4} (1 - 2 \lambda^2) \sin \delta + \mathcal{O}(\lambda^5),$$

10
again at most a few percent.

If \( m = n = 3 \), i.e., \( U_{\mu 3} = \sqrt{\frac{1}{2}} (1 - B \lambda^3) e^{i \delta} \) and \( U_{e 3} = A \lambda^3 \), then it holds for the mixing parameters:

\[
\begin{align*}
\tan^2 \theta_{12} & \simeq 1 - 4 \lambda + 10 \lambda^2 + \mathcal{O}(\lambda^3), \\
\sin^2 2\theta_{23} & \simeq 1 - 4 B^2 \lambda^6 + \mathcal{O}(\lambda^7), \\
\tan^2 \theta_{23} & \simeq 1 - 4 B \lambda^3 + \mathcal{O}(\lambda^4), \\
\sin^2 \theta_{13} & = A^2 \lambda^6 ,
\end{align*}
\]

(32)
i.e., except for \( \sin^2 \theta_{13} \) to the given order identical for the \( m = 3 \) and \( n = 2 \) case above. The parameters \( A \) and \( B \) appear in the mixing matrix only a third order in \( \lambda \):

\[
U_{PMNS} \simeq \begin{pmatrix}
\sqrt{\frac{1}{2}} (1 + \lambda - \lambda^2 + \lambda^3) & \frac{1}{2} (1 + \lambda - \lambda^2 + (1 + B - A e^{i \delta}) \lambda^3) & A \lambda^3 \\
-\frac{1}{2} (1 - \lambda + (B + A e^{i \delta}) \lambda^3) & \frac{1}{2} (1 + \lambda - \lambda^2 + (1 + B - A e^{i \delta}) \lambda^3) & \sqrt{\frac{1}{2}} (1 - B \lambda^3) e^{i \delta} \\
\frac{1}{2} (1 - \lambda - (B + A e^{i \delta}) \lambda^3) & -\frac{1}{2} (1 + \lambda - \lambda^2 + (1 - B + A e^{i \delta}) \lambda^3) & \sqrt{\frac{1}{2}} (1 + B \lambda^3) e^{i \delta} \\
\end{pmatrix} + \mathcal{O}(\lambda^4).
\]

(33)

There are no quadratic terms in the elements \( U_{\mu 1} \) and \( U_{\tau 1} \). Finally, \( CP \) violation is governed by

\[
J_{CP} \simeq \frac{A \lambda^3}{4} (1 - 2 \lambda^2) \sin \delta + \mathcal{O}(\lambda^6)
\]

(34)
being below one percent.

The remaining cases \( m = 3 \) and \( n = 1 \) \((m = 1 \text{ and } n = 3)\) are obtained from the cases \( m = 2 \) and \( n = 1 \) \((m = 1 \text{ and } n = 2)\) by setting \( B = 0 \) \((A = 0)\) in the relevant expressions for the mixing parameters and \( J_{CP} \).

### 3.7 Speculations

One can speculate about the origin of the corrections induced by the \( \lambda \) terms. It is possible to imagine, e.g., that the bimaximal mixing scheme from Eq. (11) stems from the diagonalization of the neutrino mass matrix (this is possible, e.g., when a \( L_e - L_\mu - L_\tau \) symmetry is present) and any corrections are implied by the unitary matrix \( U_\ell \) that diagonalizes the charged lepton mass matrix. Recall that in a basis in which the charged lepton mass matrix is not diagonal the PMNS matrix is given by \( U_\ell^\dagger U \), where \( U \) diagonalizes the neutrino mass matrix in that basis. If we define the matrix \( U_\lambda \), which induces the correction to the bimaximal scheme, we may write \( U_{PMNS} \equiv U_\lambda U_{PMNS}^{\text{bimax}} \), where \( U_{PMNS}^{\text{bimax}} \) is given in Eq. (11). Then one can simply

\footnote{This is similar to the strategy in [13], where however a different mixing matrix to start with was used.}
solve for $U_{\lambda}$. Taking for definiteness the example $m = 3$ and $n = 2$, one finds

$$U_{\lambda} \simeq \begin{pmatrix}
1 - \lambda^2/2 & -\lambda/\sqrt{2} + \frac{1 + 2A}{2\sqrt{2}} \lambda^2 & \lambda/\sqrt{2} + \frac{2A - 1}{2\sqrt{2}} \lambda^2 \\
\lambda/\sqrt{2} - \frac{1 + 2A e^{i\delta}}{2\sqrt{2}} \lambda^2 & \frac{1}{2} + e^{i\delta} - \frac{\lambda^2}{4} & \frac{1}{2} - e^{i\delta} + \frac{\lambda^2}{4} \\
-\lambda/\sqrt{2} + \frac{1 - 2A}{2\sqrt{2}} \lambda^2 & \frac{1}{2} - e^{i\delta} + \frac{\lambda^2}{4} & \frac{1}{2} + e^{i\delta} - \frac{\lambda^2}{4}
\end{pmatrix} + O(\lambda^3). \quad (35)$$

It is seen that 23 and 32 entries are in general of order one but reduce to order $\lambda^2$ for CP conservation. Those entries can also be of order $\lambda$, however only for the cases $m = n = 1$ and $m = 1$ with $n = 2$. Thus, if CP is conserved and atmospheric neutrino mixing is very close to maximal, the matrix $U_{\lambda}$ takes the unit matrix as the dominant form with corrections of order $\lambda$. The typical “CKM–structure” with very small $\lambda^3$ terms is however not necessary.

4 The Majorana Mass matrix

4.1 Basics

Up to now our analysis assumed only the neutrino oscillation explanation of the experimental data. Now we assume in addition that neutrinos are Majorana particles, which is, e.g., a prediction of the see–saw mechanism \[32\]. Thus, the neutrino mass matrix $m_{\nu}$ in the basis in which the charged lepton mass matrix is diagonal is given by:

$$m_{\nu} = U_{\text{PMNS}} m_{\nu}^{\text{diag}} U_{\text{PMNS}}^T. \quad (36)$$

Here $m_{\nu}^{\text{diag}}$ is a diagonal matrix containing the masses $m_{1,2,3}$ of the three massive Majorana neutrinos. Immediate consequence of the Majorana nature of the neutrinos is the presence of two Majorana phases $\alpha$ and $\beta$ to which neutrino oscillations are insensitive \[33\]. Information about these phases can be obtained by studying processes in which the total lepton charge $L$ changes by two units, e.g., neutrinoless double beta decay, $K^+ \rightarrow \pi^- + \mu^+ + \mu^+$, etc. Realistically, only neutrinoless double beta decay can expected to be measured \[34\]. The decay width of this process is sensitive to the $ee$ element of $m_{\nu}$.

An interesting observation is that the ratio of typical best–fit values of the mass squared differences corresponds roughly to the expansion parameter $\lambda$:

$$R \equiv \sqrt{\frac{(\Delta m^2_{\odot})_{\text{BF}}}{(\Delta m^2_{A})_{\text{BF}}}} \simeq \sqrt{\frac{7.2 \cdot 10^{-5}}{2.0 \cdot 10^{-3}}} \simeq 0.19 \sim \lambda. \quad (37)$$

We took for $\Delta m^2_{\odot}$ the best–fit point of the preliminary new analysis of the SuperKamiokande collaboration \[20\]. Using for $\Delta m^2_{A}$ the 90 % C.L. analysis from \[20\], which is $(1.3 \ldots 3.1) \cdot 10^{-3}$ eV$^2$, with the 90 % C.L. range of $\Delta m^2_{\odot}$ from \[5\], we find that $R$ lies between 0.13 and 0.28. This corresponds to a good precision to the $3\sigma$ range of $\lambda$ as given in Eq. (13). In the following we shall assume that $R \simeq \lambda$ and study the resulting structure of the neutrino mass matrix.
The results do not change much unless $\Delta m^2_A$, $\Delta m^2_{\odot}$ and $\tan^2 \theta_{12}$ are on the very edges of their allowed ranges. Before we perform this analysis, it is useful to study the mass matrix again in the limit of exact bimaximal mixing. In the following, we will neglect the $CP$ violating phases, see, e.g., [35] for an analysis of the structure of $m_\nu$ in case of complex entries. Using Eqs. (3) and (36), the mass matrix reads

$$m_\nu = \begin{pmatrix} A & B & -B \\ \cdot & D + \frac{A}{2} & D - \frac{A}{2} \\ \cdot & \cdot & D + \frac{A}{2} \end{pmatrix},$$  \tag{38}$$

where

$$A = \frac{m_1 + m_2}{2}, \quad B = \frac{m_2 - m_1}{2\sqrt{2}}, \quad D = \frac{m_3}{2}. \tag{39}$$

As mentioned, there are three extreme cases for the mass hierarchies, the normal hierarchy (NH) with $\sqrt{\Delta m^2_A} = m_3 \gg m_2 \simeq \sqrt{\Delta m^2_{\odot}} \gg m_1 \simeq 0$, the inverse hierarchy (IH) with $\sqrt{\Delta m^2_A} = m_2 \simeq m_1 \gg m_3 \simeq 0$ and quasi–degenerate neutrinos (QD) with $m_0 \equiv m_3 \simeq m_2 \simeq m_1$. Depending on the relative signs of the mass states, several extreme forms of the mass matrix result. In case of NH, one finds for $m_{1,2} = 0$:

$$m_\nu = \frac{\sqrt{\Delta m^2_A}}{2} \begin{pmatrix} 0 & 0 & 0 \\ \cdot & 1 & 1 \\ \cdot & \cdot & 1 \end{pmatrix},$$  \tag{40}$$
i.e., a mass matrix with a leading $\mu\tau$ block. Regarding IH, the third mass $m_3$ can safely be neglected. The form of $m_\nu$ then depends on the relative sign of the two mass states $m_1$ and $m_2$:

$$m_\nu = \sqrt{\Delta m^2_A} \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ \cdot & \frac{1}{2} & -\frac{1}{2} \\ \cdot & \cdot & \frac{1}{2} \end{pmatrix} & \text{same sign} \\ \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \cdot & 0 & 0 \\ \cdot & \cdot & 0 \end{pmatrix} & \text{opposite sign} \end{cases} \tag{41}$$
For the QD spectrum one finds

\[
\begin{pmatrix}
1 & 0 & 0 \\
\cdot & 1 & 0 \\
\cdot & \cdot & 1 \\
0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\cdot & \frac{1}{2} & \frac{1}{2} \\
\cdot & \cdot & \frac{1}{2} \\
0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\cdot & \frac{1}{2} & \frac{1}{2} \\
\cdot & \cdot & \frac{1}{2} \\
1 & 0 & 0 \\
\cdot & 0 & -1 \\
\cdot & \cdot & 0
\end{pmatrix}
\]  

\text{sign}(m_1) = \text{sign}(m_2) = \text{sign}(m_3)

(42)

We can expect that in our parametrization the parameter \( \lambda \) will appear in the neutrino mass matrix at least linearly in order to correct the extreme values 0, \( \pm \frac{1}{\sqrt{2}}, \pm 1 \) and \( \pm 1/2 \).

### 4.2 Normal hierarchy

In case of the normal hierarchy we have

\[
m_3 = \sqrt{\Delta m^2_A}, \ m_2 = \sqrt{\Delta m^2_{3\alpha}} \lambda \text{ and } m_1 = D \sqrt{\Delta m^2_A} \lambda^{2+l}, \text{ where } l \geq 0 .
\]

(43)

The expression for \( m_1 \) with \( D = O(1) \) expresses our lacking knowledge of it. A similar Ansatz for the structure of \( m_\nu \) in case of a normal hierarchical mass scheme has been made in [11]. For \( m = n = l = 1 \) and all mass states positive the mass matrix looks like

\[
m_\nu = \sqrt{\Delta m^2_A} \begin{pmatrix}
\lambda \\
1 + (\frac{1}{2} - 2B) \lambda \\
1 - \frac{1}{2}
\end{pmatrix},
\]

(44)

neglecting terms of order \( O(\lambda^2) \). The characteristic “leading \( \mu \tau \) block” structure of \( m_\nu \) from Eq. (40) is identified. Corrections at order \( \lambda \) depend on \( A \) in the \( e \)-row of \( m_\nu \) and on \( B \) for the \( \mu \mu \) and \( \tau \tau \) entries. Higher powers of \( \lambda \) in \( m_1 \) will — to order \( \lambda^2 \) — lead to the disappearance of \( D \) in the formula for \( m_\nu \). Setting \( U_{e3} = A \lambda^2 \) leads to a mass matrix in which to order \( \lambda^2 \) the parameter \( A \) does not appear in the \( \mu \tau \) submatrix as well as in the \( ee \) entry. It is obtained by removing \( A \) from the last equation in the indicated entries. The matrix for \( U_{\mu 3} = \sqrt{\frac{1}{2}} (1 - B \lambda^2) \)
but $U_{e3} = A \lambda$ is given by Eq. (44) by removing $B$ from the first row and from the linear terms of the $\mu\mu$ and $\tau\tau$ entries. The “Wolfenstein like” parametrization with $U_{e3} = A \lambda^3$ and $U_{\mu 3} = \sqrt{\frac{1}{2}} (1 - B \lambda^2)$ together with $m_1 = D \sqrt{\Delta m^2_A} \lambda^3$ leads to the particularly simple form

$$m_\nu = \frac{\sqrt{\Delta m^2_A}}{2} \left( \begin{array}{ccc} \lambda - 2 \lambda^2 & \frac{\lambda}{\sqrt{2}} & -\frac{\lambda}{\sqrt{2}} \\ \cdot & 1 + \frac{\lambda}{2} + (1 - 2B) \lambda^2 & 1 - \frac{\lambda}{2} - \lambda^2 \\ \cdot & \cdot & 1 + \frac{\lambda}{2} + (1 + 2B) \lambda^2 \end{array} \right) + O(\lambda^3). \quad (45)$$

The $\theta_{23} \leftrightarrow \pi/2 - \theta_{23}$ ambiguity, which translates into a sign ambiguity of $B$, is seen to have origin in the size of the $\mu\mu$ and $\tau\tau$ entries. E.g., for all masses positive and $\theta_{23} > \pi/4$ the $\tau\tau$ entry is larger. The structure of the mass matrix does not depend on the exactness of the relation $R = \lambda$ or the relative signs of the mass states. When $\text{sign}(m_1) = -\text{sign}(m_2) = -\text{sign}(m_3)$, then the mass matrix looks as above. For $\text{sign}(m_1) = -\text{sign}(m_2) = \text{sign}(m_3)$ and $\text{sign}(m_1) = \text{sign}(m_2) = -\text{sign}(m_3)$ one has to replace $A$ with $-A$, $B$ with $-B$ and the 1 in the $\mu\tau$ block with $-1$. If we further choose $U_{\mu 3} = \sqrt{\frac{1}{2}} (1 - B \lambda^3)$ than we obtain a mass matrix in which up to order $\lambda^2$ the parameters $A, B$ and $D$ do not appear at all. It is obtained by setting in the last equation $B = 0$.

### 4.3 Inverse hierarchy

In this case we have

$$m_2 = \sqrt{\Delta m^2_A}, \quad m_1 \simeq \sqrt{\Delta m^2_A (1 - \lambda^2/2)} \quad \text{and} \quad m_3 = D \sqrt{\Delta m^2_A} \lambda^{2+l}, \quad \text{where} \quad l \geq 0. \quad (46)$$

The dependence on the power of $\lambda$ in $m_3$ is almost vanishing. The form of $m_\nu$ depends strongly on the signs of the masses $m_1$ and $m_2$. For identical relative signs between $m_1$ and $m_2$, the $e\mu$ and $e\tau$ entries are suppressed by $\lambda$ or $\lambda^2$, depending on the powers of $\lambda$ in $U_{\mu 3}$ or $U_{e3}$. If e.g., $m = 2$ and $n = 1$ or $m = n = 1$, then the entries are of order $\lambda$. For all other cases under consideration, these terms are of order $\lambda^2$. The remaining independent entries of $m_\nu$ are order one. If $m_1$ and $m_2$ have opposite relative signs, then $m_{e\mu}$ and $m_{e\tau}$ are order one and the remaining entries of $m_\nu$ are linear in $\lambda$, independent of $m$ and $n$. One finds for $m = n = 1$ that for same signs of $m_1$ and $m_2$

$$m_\nu \simeq \sqrt{\Delta m^2_A} \left( \begin{array}{ccc} 1 & -\frac{A}{\sqrt{2}} \lambda & -\frac{A}{\sqrt{2}} \lambda \\ \cdot & \frac{1}{2} + B \lambda & -\frac{1}{2} + B \lambda \\ \cdot & \cdot & \frac{1}{2} - B \lambda \end{array} \right) + O(\lambda^2), \quad (47)$$

while for opposite signs

$$m_\nu \simeq \sqrt{\Delta m^2_A} \left( \begin{array}{ccc} 2 \lambda & -(\frac{1}{\sqrt{2}} - \frac{B}{\sqrt{2}}) \lambda & \frac{1}{\sqrt{2}} - \frac{B}{\sqrt{2}} \lambda \\ \cdot & (A - 1) \lambda & \lambda \\ \cdot & \cdot & -(A + 1) \lambda \end{array} \right) + O(\lambda^2). \quad (48)$$

---

*The convention here and in the following will be such that the sign of the $ee$ entry is positive.*
The parameter $\lambda$ appears at least linearly to correct the extreme “bimaximal” mass matrices from Eq. (41). For same signs, $A$ appears in the $e$–row and $B$ in the $\mu\tau$ sector, whereas for opposite signs it is vice versa. Taking as another example again the “Wolfenstein–like” case $m = 2$ and $n = 3$ one finds to order $\lambda^2$:

$$m_\nu \simeq \sqrt{\Delta m^2} \begin{pmatrix} 1 - \frac{1}{4} \lambda^2 & \frac{1}{4\sqrt{2}} \lambda^2 & -\frac{1}{4\sqrt{2}} \lambda^2 \\ \cdot & \frac{1}{2} + (B - \frac{1}{8}) \lambda^2 & -\frac{1}{2} + \frac{1}{8} \lambda^2 \\ 2\lambda - \frac{5}{4} \lambda^2 & -\frac{1}{\sqrt{2}} + \frac{a - 4B - 1}{4\sqrt{2}} \lambda^2 & \frac{1}{\sqrt{2}} - \frac{a + 4B - 1}{4\sqrt{2}} \lambda^2 \end{pmatrix}$$

same signs

$$m_\nu \simeq \sqrt{\Delta m^2} \begin{pmatrix} 1 - \eta & \frac{A}{\sqrt{2}} \eta \lambda & \frac{A}{\sqrt{2}} \eta \lambda \\ \cdot & 1 - \frac{\eta}{2} - B \eta \lambda & \frac{\eta}{2} \\ \cdot & \cdot & 1 - \frac{\eta}{2} + B \eta \lambda \end{pmatrix} + O(\eta \lambda^3).$$

(51)

As usual, $B$ will not show up for higher orders of $\lambda$ in $U_{\mu3}$, leading to a mass matrix that is at order $\lambda^2$ only a function of $\lambda$. It is again obtained by setting in the last equation $B = 0$.

\section{4.4 Quasi–degenerate Neutrinos}

For quasi–degenerate neutrinos, i.e., $m_3^2 \simeq m_2^2 \simeq m_1^2 \gg \Delta m^2$, there is another small quantity introduced, namely the ratio of the common mass scale $m_0$ with $\Delta m^2$. For simplicity we work with the normal mass ordering. In this case we can express the three mass states as

$$m_3 \equiv m_0, \quad m_2 = a m_0 \text{ and } m_1 = b m_0,$$

where $a = 1 - \eta$, $b = 1 - \eta (1 + \lambda^2)$

and $\eta = \frac{\Delta m^2}{2 m_0^2}$.

The common mass scale is denoted by $m_0$. These expressions for the masses are valid to order $\eta$. Since the spectrum is quasi–degenerate for $m_0 \simeq 0.2$ eV, we can estimate $\eta \simeq 0.04$ eV, therefore $\lambda^2 > \eta$.

First, we take the case that all mass states have the same relative sign. For $m = n = 1$ we find:

$$\begin{pmatrix} 1 - \eta & \frac{A}{\sqrt{2}} \eta \lambda & \frac{A}{\sqrt{2}} \eta \lambda \\ \cdot & 1 - \frac{\eta}{2} - B \eta \lambda & \frac{\eta}{2} \\ \cdot & \cdot & 1 - \frac{\eta}{2} + B \eta \lambda \end{pmatrix} + O(\eta \lambda^2).$$

(51)

Taking the case $m = 2$ and $n = 3$ we have

$$\begin{pmatrix} 1 - \eta - \frac{\eta \lambda^2}{2} & \frac{\eta \lambda^2}{2\sqrt{2}} & -\frac{\eta \lambda^2}{2\sqrt{2}} \\ \cdot & 1 - \frac{\eta}{2} - \frac{1 + 4B - 1}{4} \eta \lambda^2 & \frac{\eta}{2} + \frac{\eta \lambda^2}{4} \\ \cdot & \cdot & 1 - \frac{\eta}{2} + \frac{4B - 1}{4} \eta \lambda^2 \end{pmatrix} + O(\eta \lambda^3).$$

(52)
It is seen that for $\eta = 0$ the mass matrix is proportional to the unit matrix, irrespective of $m$ and $n$. The corrections to the zero values of the extreme bimaximal form from Eq. (42) are very small.

If $\text{sign}(m_1) = -\text{sign}(m_2) = \text{sign}(m_3)$, then the dependence on $\eta$ is not so important. Neglecting $\eta$ with respect to terms of order 1, we find for $m = n = 1$:

$$
(m_{\nu})_{+-} = m_0 \begin{pmatrix}
2\lambda & -\frac{1}{\sqrt{2}} + \frac{A-B}{\sqrt{2}} \lambda & \frac{1}{\sqrt{2}} + \frac{A-B}{\sqrt{2}} \lambda \\
\cdot & \frac{1}{2} + (A - B - 1) \lambda & \frac{1}{2} + \lambda \\
\cdot & \cdot & \frac{1}{2} - (A + 1 - B) \lambda
\end{pmatrix} + O(\eta, \lambda^2) \tag{53}
$$

and for $m = 2, n = 3$:

$$
(m_{\nu})_{++} = m_0 \begin{pmatrix}
2\lambda - \lambda^2 & -\frac{1}{\sqrt{2}} - \frac{A+B}{\sqrt{2}} \lambda & \frac{1}{\sqrt{2}} - \frac{A+B}{\sqrt{2}} \lambda \\
\cdot & \frac{1}{2} - \lambda + \left(\frac{1}{2} - B\right) \lambda^2 & \frac{1}{2} + \lambda - \frac{1}{2} \lambda^2 \\
\cdot & \cdot & \frac{1}{2} - \lambda + \left(\frac{1}{2} + B\right) \lambda^2
\end{pmatrix} + O(\eta \lambda, \lambda^3). \tag{54}
$$

For the case $\text{sign}(m_1) = -\text{sign}(m_2) = -\text{sign}(m_3)$ one finds for $m = n = 1$

$$
(m_{\nu})_{--} = m_0 \begin{pmatrix}
2\lambda & -\frac{1}{\sqrt{2}} - \frac{A+B}{\sqrt{2}} \lambda & \frac{1}{\sqrt{2}} - \frac{A+B}{\sqrt{2}} \lambda \\
\cdot & -\frac{1}{2} + (A + B - 1) \lambda & -\frac{1}{2} + \lambda \\
\cdot & \cdot & -\frac{1}{2} - (A + B + 1) \lambda
\end{pmatrix} + O(\eta, \lambda^2) \tag{55}
$$

and for $m = 2, n = 3$:

$$
(m_{\nu})_{--} = m_0 \begin{pmatrix}
2\lambda - \lambda^2 & -\frac{1}{\sqrt{2}} + \frac{A+B}{\sqrt{2}} \lambda & \frac{1}{\sqrt{2}} - \frac{A+B}{\sqrt{2}} \lambda \\
\cdot & -\frac{1}{2} - \lambda + \left(\frac{1}{2} + B\right) \lambda^2 & -\frac{1}{2} + \lambda - \frac{1}{2} \lambda^2 \\
\cdot & \cdot & -\frac{1}{2} - \lambda + \left(\frac{1}{2} - B\right) \lambda^2
\end{pmatrix} + O(\eta \lambda, \lambda^3) \tag{56}
$$

These two last cases look very similar. Finally, the situation for $\text{sign}(m_1) = \text{sign}(m_2) = -\text{sign}(m_3)$ looks simpler: e.g., when $m = n = 1$:

$$
(m_{\nu})_{++} = m_0 \begin{pmatrix}
1 - 2 A^2 \lambda^2 & -\sqrt{2} A \lambda + \sqrt{2} A B \lambda^2 & -\sqrt{2} A \lambda - \sqrt{2} A B \lambda^2 \\
\cdot & 2 B \lambda - B^2 \lambda^2 & -1 + (A^2 + 2 B^2) \lambda^2 \\
\cdot & \cdot & -2 B \lambda + (2 A^2 + B^2) \lambda^2
\end{pmatrix} + O(\eta, \lambda^3). \tag{57}
$$

The corrections to the entry 1 (0) that is present in the extreme bimaximal form from Eq. (42)
are at least quadratic (linear). For \( m = 2 \) and \( n = 3 \) it holds:

\[
(m_{\nu})_{++-} = m_0 \begin{pmatrix}
1 - \eta - \frac{1}{2} \eta \lambda^2 & \frac{1}{2 \sqrt{2}} \eta \lambda^2 & -\frac{1}{2 \sqrt{2}} \eta \lambda^2 \\
\cdot & -\frac{\eta}{2} + (B (2 - \eta) - \frac{\eta}{4}) \lambda^2 & -1 + \frac{\eta}{2} + \frac{1}{4} \eta \lambda^2 \\
\cdot & \cdot & -\frac{\eta}{2} - (B (2 - \eta) + \frac{\eta}{4}) \lambda^2
\end{pmatrix} + O(\lambda^3)
\]  

(58)

4.5 Summary for the Mass Matrix

Looking at the cases considered in the last Subsections, the following summarizing statements can be made:

- Roughly, for \(|U_{e3}| \sim 0.01 \) and \( \sin^2 2\theta_{23} \lesssim 0.9 \), corrections to the extreme forms of the mass matrices in Eqs. (40, 41, 42) are linear in \( \lambda \). When \(|U_{e3}| \lesssim 10^{-3} \) and \( \sin^2 2\theta_{23} \gtrsim 0.95 \), the corrections become quadratic.

- For the normal hierarchy, corrections to the exact bimaximal form are at least order \( \lambda \). To lowest order, the parameter \( A \) appears in the \( e \) row of \( m_{\nu} \) and \( B \) in the \( \mu \mu \) and \( \tau \tau \) elements. The number \( D \), parameterizing the unknown lightest mass state, appears in all entries. There is basically no dependence on the relative signs of the mass states.

- For the inverse hierarchy, the dependence on \( D \) is highly suppressed. For identical signs of the two heaviest mass states, the correction to the \( ee \) entry, whose extreme value in case of bimaximal mixing is (in units of \( \Delta m^2_{31} \)) 1, is at least order \( \lambda^2 \). The remaining elements receive at least linear corrections. \( A \) appears at leading order in the \( e \)-row and \( B \) in the \( \mu \tau \) sector. Opposite signs of the two leading mass states lead to linear corrections to the entries and the appearance of \( A \) in the \( \mu \tau \) sector and \( B \) in the \( e \)-row.

- In case of a quasi–degenerate spectrum and identical signs of the masses, the corrections to the unit matrix are at least quadratical. The cases \( \text{sign}(m_1) = -\text{sign}(m_2) = \text{sign}(m_3) \) and \( \text{sign}(m_1) = -\text{sign}(m_2) = -\text{sign}(m_3) \) look very similar. For \( \text{sign}(m_1) = \text{sign}(m_2) = -\text{sign}(m_3) \) there are only quadratic corrections to the \( ee \) and \( \mu \tau \) entries.

- The \( \mu \tau \) entry is special in our parametrization since the parameters \( A \) and \( B \) do typically only appear there for rather large deviations from zero \( U_{e3} \) and from maximal atmospheric neutrino mixing.

- If we would consider the inverse ordering in the QD mass spectrum, one has to change Eq. (50) to \( m_2 = m_0, m_1 = am_0 \) and \( m_3 = bm_0 \). Only simple sign changes for \( A \) and/or \( B \) in some elements of \( m_{\nu} \) would be the result. The main difference would be for the case \( \text{sign}(m_1) = \text{sign}(m_2) = \text{sign}(m_3) \), where the corrections to the \( e \mu \) and \( e \tau \) elements are now order \( \eta \) and not just \( \eta \lambda \) or \( \eta \lambda^2 \).
5 Applications

It is surely useful to study formulae which are obtained by expansions in small parameters or by certain simplifications within our parametrization. We shall perform this analysis now for the effective mass as measurable in neutrinoless double beta decay and the oscillation probabilities for long–baseline neutrino oscillations.

5.1 Neutrinoless Double Beta Decay

We shall analyze now within our parametrization the form of the $ee$ element of $m_\nu$, which is denoted by $\langle m \rangle$. In a given mass scheme or hierarchy one can considerably simplify the expression for $\langle m \rangle$ [36]. We first note that since

$$\langle m \rangle = \sum m_i U_{ei}^2$$

(59)

the results are independent on the power of $\lambda$ in $U_{\mu 3}$.

5.1.1 Normal Hierarchy

With the help of Eqs. (43, 59) we can evaluate the effective mass in case of the normal hierarchy. We find for $U_{e3} = A \lambda$ and $m_3 = D \lambda^2$:

$$\langle m \rangle \simeq \frac{\sqrt{\Delta m_A^2}}{2} \left[ \lambda - 2 \lambda^2 \left( 1 - D^2 c_2 \right) \right] + O(\lambda^3),$$

(60)

where $c_2 = \cos 2\alpha$ and so on. Terms of order $\lambda^3$ depend on $A, D$ and the two Majorana phases. Choosing $m_3 = D \lambda^3$ or higher powers of $\lambda$ leads to the disappearance of $D$ in the formula. For $m_3 = D \lambda^2$ and higher orders of $\lambda$ in $U_{e3}$, i.e., $n \geq 2$, it holds

$$\langle m \rangle \simeq \frac{\sqrt{\Delta m_A^2}}{2} \left[ \lambda - 2 \lambda^2 \left( 1 - \frac{D}{2} c_2 \right) + \frac{\lambda^3}{4} \left( 4 (1 + 2 D c_2) + D^2 (1 - c_4) \right) \right] + O(\lambda^4).$$

(61)

The formulae for $m_3 = D \lambda^3$ correspond to setting $D^2 = 0$ in this last equation. Roughly, we can estimate the effective mass in the normal hierarchy as

$$\langle m \rangle \sim \frac{\lambda}{2} \sqrt{\Delta m_A^2} \lesssim 0.005 \text{ eV}.$$  

(62)

5.1.2 Inverse Hierarchy

From Eqs. (46, 59) one sees that in the expression for $\langle m \rangle$ the dependence on $\beta$ practically vanishes. The result for $U_{e3} = A \lambda$ is

$$\langle m \rangle \simeq \frac{\sqrt{\Delta m_A^2}}{2} \sqrt{c_2^2 + \frac{1}{4} \left( 7 - 4 A^2 - (9 + 4 A^2) c_2 \right) \lambda^2} + O(\lambda^3).$$

(63)
Higher powers of $\lambda$ in $U_{e3}$ lead to the disappearance of $A$ in this equation. The maximal and minimal values are obtained when $\alpha$ takes the values 0 and $\pi/2$, respectively. Thus,

$$2 \sqrt{\Delta m^2} \lambda \lesssim \langle m \rangle \lesssim \sqrt{\Delta m^2} \left(1 - \left(A^2 + \frac{1}{4}\right) \lambda^2\right),$$

(64)

up to corrections of $O(\lambda^3)$. For no extreme values of $\alpha$, the scale of the effective mass is

$$\langle m \rangle \sim \sqrt{\Delta m^2} \gtrsim 0.05 \text{ eV}.$$

(65)

Comparing this with the value of $\langle m \rangle$ in the normal hierarchy in Eq. (62) one sees that the expansion parameter $\lambda$ shows up as the ratio of the typical values of $\langle m \rangle$ in the inverted and normal hierarchy.

It is known that extraction of information from a measurement of $0\nu\beta\beta$ suffers from a large uncertainty stemming from the calculation of the nuclear matrix elements. This uncertainty is a number of order one [37]. It is therefore an important question to ask and an even more important one to answer whether future $0\nu\beta\beta$ experiments can distinguish [36, 38] between the normal and inverted mass hierarchy. Let us parametrize the nuclear matrix element uncertainty with a factor $\xi$ as done in [39]. In order to distinguish the normal from the inverted hierarchy it must hold that the maximal value of $\langle m \rangle$ in the normal hierarchy times the uncertainty $\xi$ has to be smaller than the minimal value of $\langle m \rangle$ in the inverted hierarchy. Therefore, choosing $U_{e3} = A \lambda$ and small $m_3$ we find from Eqs. (60, 64)

$$\xi \lesssim 4 \left(1 + 2 \lambda(1 \pm A^2)\right) + O(\lambda^3).$$

(66)

Needless to say, $A$ vanishes for smaller values of $U_{e3}$. If that is the case, then $\xi \lesssim 6$, which is a very realistic number. Thus, with our expansion parameter $\lambda \simeq 0.2$ and $|U_{e3}|^2 \lesssim 0.01$ it is easily possible to distinguish between the normal and inverted mass hierarchy.

### 5.1.3 Quasi–degenerate Neutrinos

The formulae for the mass states are given in Eq. (43). Ignoring $\eta$ and taking $U_{e3} = A \lambda$ one finds

$$\frac{\langle m \rangle}{m_0} \simeq \sqrt{c^2_\alpha + (2 - A^2 - (2 + A^2) c_{2\alpha} + 2 A^2 c_\alpha c_{\alpha - 2\beta}) \lambda^2} + O(\lambda^3).$$

(67)

Interesting are the cases which correspond to $CP$ conservation, and which are obtained by setting $\alpha, \beta$ to $\pi/2, \pi$. They read up to $O(\eta \lambda^2, \lambda^3)$:

$$\frac{\langle m \rangle}{m_0} \simeq \begin{cases} 
1 - \eta & \alpha = \beta = 0 \quad \leftrightarrow \quad \text{sign}(m_1) = \text{sign}(m_2) = \text{sign}(m_3) \\
2 \lambda & 2 \alpha = \beta = \pi \quad \leftrightarrow \quad \text{sign}(m_1) = -\text{sign}(m_2) = \text{sign}(m_3) \\
2 \lambda & \alpha = \beta = \pi/2 \quad \leftrightarrow \quad \text{sign}(m_1) = -\text{sign}(m_2) = -\text{sign}(m_3) \\
1 - \eta - 2 A^2 \lambda^2 & \alpha = 2 \beta = \pi \quad \leftrightarrow \quad \text{sign}(m_1) = \text{sign}(m_2) = -\text{sign}(m_3) 
\end{cases}$$

(68)
As usual, for $U_{e3} = A \lambda^2$ and above, the dependence on $A$ (and thus also on $\beta$) drops and appears only to order $\lambda^4$. Noting that the minimal value of $\langle m \rangle$ is $2\lambda m_0$, we can investigate if future experiments can distinguish between the normal and quasi-degenerate mass hierarchy \[39\]. In analogy to the discussion leading to Eq. (66), it follows from Eqs. (60,68)

$$
\xi \lesssim 2 \sqrt{\frac{\eta}{\lambda^2}} \left(1 + 2\lambda (1 \pm A^2)\right),
$$

which will be easily possible. It is a bit more tricky to distinguish between the quasi-degenerate and the inverted hierarchies. The requirement for $\xi$ is

$$
\xi \lesssim \sqrt{\frac{2}{\eta}} \lambda \left(1 + \left(A^2 + \frac{1}{4}\right) \lambda^2\right),
$$

which is suppressed roughly by a factor $\lambda$ with respect to the limit on $\xi$ in order to distinguish the normal and quasi-degenerate mass hierarchy. Also in this aspect the parameter $\lambda$ shows up as a scaling factor.

### 5.2 Long Baseline Oscillation Experiments

There is another field of neutrino physics in which expansion in small parameters gives insight in the physics involved and which is therefore useful to study within our parameterization. These are the oscillation probabilities for long-baseline experiments \[18\]. The determination of some currently unknown neutrino parameters, namely $U_{e3}$, the sign of $\Delta m^2_{31}$ and the Dirac-like $CP$ violating phase are the purpose of such experiments. There are helpful expansions of the relevant oscillation probabilities in vacuum \[10\]. Here, we do not consider matter effects since they will not change our conclusions. Let us first comment on $CP$ violation. Using Eq. (1), one finds for the difference of the oscillation probabilities:

$$
\Delta P \equiv P(\nu_e \rightarrow \nu_\mu) - P(\bar{\nu}_e \rightarrow \bar{\nu}_\mu) =
8 J_{CP} \left( \sin \frac{\Delta m^2_{31} L}{2 E} - \sin \frac{\Delta m^2_{32} L}{2 E} - \sin \frac{\Delta m^2_{21} L}{2 E} \right),
$$

which, using $\Delta m^2_{32} = \Delta m^2_{31} - \Delta m^2_{21}$, can easily be shown to vanish for two masses being equal. The invariant $J_{CP}$ was defined in Eq. (10). Since $\Delta m^2_{21}/\Delta m^2_{31} = \pm \lambda^2$, where the “+” in case of normal ordering and the “−” for inverse ordering, we can expand the last equation:

$$
\Delta P \simeq \pm 4 J_{CP} \lambda^2 \frac{\Delta m^2_{31} L}{2 E} \sin^2 \frac{\Delta m^2_{21} L}{4 E} + O(\lambda^3).
$$

Thus, the $CP$ violating effects in realistic experiments are suppressed by another two orders of $\lambda$ in addition to the suppression present in $J_{CP}$. If $n$ is the power of $\lambda$ in $U_{e3}$, then the total suppression is $\lambda^{2+n}$.

One can also consider the bare oscillation probability for the “golden channel”, which is given by $\nu_e \rightarrow \nu_\mu$ oscillations. Using the form of $P(\nu_e \rightarrow \nu_\mu)$ as given, e.g., in \[22\], one finds for the oscillation probability in case of $m = n = 1$:

$$
P(\nu_e \rightarrow \nu_\mu) \simeq 2 A^2 \sin^2 \Delta_{32} \lambda^2 - 2 A \sin^2 \Delta_{32} \left(2 A B + \cos(\delta - \Delta_{32})\right) \lambda^3 + O(\lambda^4).
$$

21
Here, the “−” sign is for neutrinos and the + for antineutrinos. We defined $\Delta_{32} = (m_3^2 - m_2^2) L/4E$. The first term proportional to $\lambda^2$ is the term that probes $U_{e3}$ whereas the second term proportional to $\lambda^3$ is the one probing the $CP$ phase $\delta$.

As another example, assume $n = m = 2$. Then, the terms probing $U_{e3}$ and $CP$ violation will both be proportional to $\lambda^4$:

$$P(\nu_e \rightarrow \nu_\mu) \simeq \frac{\sin^2 \Delta_{32}}{2} \left( 1 + 4 A^2 - 4 A \cos(\delta \mp \Delta_{32}) \right) \lambda^4 + O(\lambda^6).$$  \hspace{1cm} (74)

The parameter $B$ only appears at order $\lambda^6$, since there are no terms of order $\lambda^5$. For $m = 2$ and $n = 3$ it holds:

$$P(\nu_e \rightarrow \nu_\mu) \simeq \frac{\sin^2 \Delta_{32}}{2} \lambda^4 - 2 A \sin^2 \Delta_{32} \cos(\delta \mp \Delta_{32}) \lambda^5 + O(\lambda^6).$$  \hspace{1cm} (75)

A characteristic combination of the oscillation parameters that appears in the relevant probabilities is $\Delta m_{21}^2 / \Delta m_{31}^2 \sin 2\theta_{12} [40]$. Neglecting terms of order $\lambda^6$, we find for this parameter in our parametrization that

$$\frac{\Delta m_{21}^2}{\Delta m_{31}^2} \sin 2\theta_{12} \simeq \pm \lambda^2 \left\{ \begin{array}{ll}
(1 - 2 \lambda^2 + 2 (1 + A^2) \lambda^3) & \text{for } n = 1 \\
(1 - 2 \lambda^2 + 2 \lambda^3) & \text{for } n = 2 \\
(1 - 2 \lambda^2 + 2 \lambda^3) & \text{for } n = 3
\end{array} \right., \hspace{1cm} (76)$$

where again the “+” in case of normal ordering and the “−” for inverse ordering. The difference between the cases $n = 2$ and $n = 3$ appears only at seventh order in $\lambda$. The characteristic parameter is therefore to order $\lambda^4$ independent on the precise form of the parametrization.

### 6 Conclusions

The zeroth order approximation for neutrino mixing can be the bimaximal scheme with two maximal and one zero angle in the mixing matrix. It can be used as a reference matrix, whose corrections can be described in a similar manner as the Wolfenstein parametrization describes corrections to the unit matrix. Indeed, at least one of the angles in neutrino mixing is different from the extreme value corresponding to bimaximal mixing, namely the angle describing solar neutrino oscillations. To take this into account, a flexible parametrization of the neutrino mixing matrix was proposed in which the expansion parameter $\lambda \simeq 0.2$ is introduced to quantify this deviation from maximal mixing of solar neutrinos. It can also be used to quantify the possible deviation from zero $U_{e3}$ and maximal mixing of atmospheric neutrinos. The power of $\lambda$ to usefully describe these two latter aspects can be adjusted to future data. Depending on the power of $\lambda$, rather simple forms of the PMNS matrix are obtained, where the deviations from the “bimaximal” values $0, \pm 1/2$ and $\pm 1/\sqrt{2}$ are implied by $\lambda$. If $U_{\mu3}$ and $U_{e3}$ are close to their maximally allowed values, $\lambda$ appears at first order in all elements of $U_{\text{PMNS}}$. For values of $|U_{e3}| \lesssim 10^{-3}$ and $\sin^2 2\theta_{23} \gtrsim 0.95$, the corrections become quadratic. The invariant measure for
leptonic $CP$ violation is proportional to $\lambda^n$, where $n$ is the power of $\lambda$ in $U_{e3}$. One can interpret these corrections to the exact bimaximal mixing scheme in the same way as corrections to the unit matrix lead to the CKM matrix for the quark sector. Observing further that the ratio of the mass squared differences as measured in experiments is roughly $\lambda^2$, allows to study the form of the Majorana neutrino mass matrix $m_\nu$. Also here, the corrections to the extreme forms of $m_\nu$ in case of bimaximal mixing and extreme hierarchies are linear or quadratic in $\lambda$, depending on the precise values of $U_{e3}$, $U_{\mu 3}$ or the value of the smallest mass state. The $ee$ element of $m_\nu$ can be measured in experiments probing neutrinoless double beta decay. Here, $\lambda$ appears as the scale factor of the typical values of $\langle m \rangle$ in the normal and inverted hierarchy. It also influences the maximal value of the uncertainty in the calculations of the nuclear matrix elements allowed to distinguish the normal, inverted or quasi–degenerated mass hierarchies. We furthermore commented on how our parametrization applies to realistic long–baseline oscillation experiments. Simple forms of the relevant oscillation probabilities are obtained. In particular, due to the small ratio of the two independent mass squared differences, effects of $CP$ violation are suppressed by another two orders of $\lambda$.

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