CONTINUED FRACTIONS AND HEAVY SEQUENCES

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Abstract. We initiate the study of the sets \( \mathcal{H}(c) \) for real \( x \) for which the sequence \( (kx)_{k \geq 1} \) consistently hits the interval \([0, c)\) at least as often as expected (i.e., with frequency \( \geq c \)). More formally,

\[
\mathcal{H}(c) \overset{\text{def}}{=} \left\{ \alpha \in \mathbb{R} \mid \text{card} \left( \{1 \leq k \leq n \mid \langle k\alpha \rangle < c \} \right) \geq cn, \text{ for all } n \geq 1 \right\},
\]

where \( \langle x \rangle = x - [x] \) stands for the fractional part of \( x \in \mathbb{R} \).

We prove that, for rational \( c \), these sets \( \mathcal{H}(c) \) are of positive Hausdorff dimension and, in particular, are uncountable. For integers \( m \geq 1 \), we obtain a surprising characterization of the numbers \( \alpha \in \mathcal{H}(1/m) \) in terms of their continued fraction expansions: The odd entries (partial quotients) of these expansions are divisible by \( m \). The characterization implies that \( x \in \mathcal{H}(m) \) if and only if \( 1/m \in \mathcal{H}(m) \), for \( x > 0 \). We are unaware of a direct proof of this equivalence without making use of the mentioned characterization of the sets \( \mathcal{H}(m) \).

We also introduce the dual sets \( \hat{\mathcal{H}}(m) \) of reals \( y \) for which the sequence of integers \( ([k/m]y)_{k \geq 1} \) consistently hits the set \( m\mathbb{Z} \) with the at least expected frequency \( 1/m \) and establish the connection with the sets \( \mathcal{H}(m) \):

\[
\text{If } xy = m \text{ for } x, y > 0, \text{ then } x \in \mathcal{H}(m) \iff y \in \hat{\mathcal{H}}(m).
\]

The motivation for the present study comes from Y. Peres’s ergodic lemma.

1. Notation and Results

We write \( \mathbb{R} \supset \mathbb{Q} \supset \mathbb{Z} \supset \mathbb{N} \) for the sets of real numbers, rational numbers, integers and positive integers respectively.

In the paper we initiate the study of the sets \( \mathcal{H}(c) \), \( 0 < c < 1 \), of \( x \in \mathbb{R} \) for which the sequence \( (kx)_{k \geq 1} \) (viewed mod 1) consistently hits the interval \([0, c)\) at least as often as expected. More formally,

\[
\mathcal{H}(c) = \left\{ \alpha \in \mathbb{R} \mid \text{card} \left( \{1 \leq k \leq n \mid \langle k\alpha \rangle < c \} \right) \geq cn, \text{ for all } n \in \mathbb{N} \right\},
\]

where \( \langle x \rangle = x - [x] \) stands for the fractional part of \( x \in \mathbb{R} \). Define

\[
\mathcal{H}(1/m) = \mathcal{H}(\frac{1}{m}), \text{ for } m \in \mathbb{N}.
\]

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The following notation will be used for CF (continued fraction) expansions of finite length $n + 1$:

$$[a_0, a_1, a_2, \ldots, a_n]_↓ = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots + \frac{1}{a_n}}}$$

or of infinite length

$$[a_0, a_1, a_2, \ldots]_↓ = \lim_{n \to \infty} [a_0, a_1, a_2, \ldots, a_n]_↓,$$

where $a_0 \in \mathbb{Z}$ and $a_k \in \mathbb{N}$ for $k \geq 1$.

For some basic facts and standard notation from the theory of CFs we refer to [5] or [4]. (The first few pages in either book should suffice for our purposes.)

Every irrational number has a unique infinite CF expansion, and every rational number has exactly two finite CF expansions

$$[a_0, a_1, a_2, \ldots, a_{n-1} + 1]_↓ = [a_0, a_1, a_2, \ldots, a_{n-1}, 1]_↓,$$

(with the lengths being two consecutive integers, $n$ and $n + 1$).

**Definition 1.** By the odd CF (odd continued fraction) expansion (of $\alpha \in \mathbb{R}$) we mean the CF expansion of length $L \in \{\infty, 1, 3, 5, \ldots\}$. Similarly, in the even CF expansions one assumes $L \in \{\infty, 2, 4, 6, \ldots\}$.

This way every number $\alpha \in \mathbb{R}$ has unique both odd and even CF expansions; the two coincide if and only if $\alpha$ is irrational. The sequence of (CF) convergents for $\alpha$,

$$\delta_k(\alpha) = [a_0, a_1, \ldots, a_k]_↓,$$

$0 \leq k < L$,

can be alternatively defined as the sequence of rational numbers $\delta_k = \frac{p_k}{q_k}$ with numerators and denominators $p_k = p_k(\alpha)$, $q_k = q_k(\alpha)$ determined by the recurrence relations

$$
\begin{cases}
    p_k = a_k p_{k-1} + p_{k-2}, \\
    q_k = a_k q_{k-1} + q_{k-2},
\end{cases}
$$

for $2 \leq k < L$,

and the initial conditions $p_0 = a_0; q_0 = 1; p_1 = a_0 a_1 + 1; q_1 = a_1$.

The following theorem provides a criterion for the relation $\alpha \in \mathcal{H}_m$ to hold (see [2]).

**Theorem 1.** Let $\alpha \in \mathbb{R}$ and assume that $\alpha = [a_0, a_1, a_2, \ldots]_↓$ is its odd CF expansion (i.e., of the length $L \in \{\infty, 1, 3, 5, \ldots\}$). Let $m \in \mathbb{N}$ be given. Then the following three conditions are equivalent:

(C1) $\alpha \in \mathcal{H}_m$.

(C2) $m \mid a_k$, for all odd $k$, $1 \leq k < L$.

(C3) $m \mid q_k$, for all odd $k$, $1 \leq k < L$, where $q_k = q_k(\alpha)$ are the denominators of the convergents for $\alpha$; see (2).

**Remark 1.** For $m = 1$ the above theorem holds trivially because $\mathcal{H}_1 = \mathbb{R}$. It also holds trivially for $\alpha \in \mathbb{Z}$ (in this case $L = 1$).

**Examples.**

(1) $\alpha = \frac{3}{4}$. The odd CF expansion is $[1, 2, 1]_↓$, $L = 3$, $a_1 = 2$.

Thus $\frac{3}{4} \in \mathcal{H}_m$ if and only if $m = 1$ or 2.
(2) \( \alpha = \frac{\sqrt{5}}{2} \). The odd CF expansion is \([1, 8, 2, 8, 2, 8, \ldots] \), \( L = \infty \), \( a_1 = a_3 = a_5 = \cdots = 8 \).

Thus \( \frac{\sqrt{5}}{2} \in \mathcal{H}_m \) if and only if \( m = 1, 2, 4 \) or 8.

**Corollary 1.** \( \mathcal{H}_m \cap \mathcal{H}_n = \mathcal{H}_{\text{LCM}(m, n)} \), for all \( m, n \in \mathbb{N} \).

**Corollary 2.** For real \( \alpha > 0 \) and \( m \in \mathbb{N} \), we have \( \alpha \in \mathcal{H}_m \) if and only if \( \frac{1}{m\alpha} \in \mathcal{H}_m \).

Both corollaries follow directly from the equivalence of \((C1)\) and \((C2)\) in Theorem 1; the proof of Corollary 2 also uses the identity

\[
(4) \quad m [x_0, mx_1, x_2, mx_3, x_4, mx_5, \ldots] = [mx_0, x_1, mx_2, x_3, mx_4, x_5, \ldots].
\]

In the next three theorems we classify the numbers in the sets \( \mathcal{H}_m \), \( m \in \mathbb{N} \):

\[
(5) \quad \mathcal{H}_m = \{ \alpha \in \mathbb{R} \mid \text{card}(\{1 \leq k \leq n \mid [k\alpha] \in m\mathbb{Z}\}) \geq \frac{n}{m}, \text{ for all } n \in \mathbb{N} \}.
\]

**Theorem 2.** For \( \alpha \in \mathbb{R} \) and \( m \in \mathbb{N} \), we have \( \alpha \in \mathcal{H}_m \iff m\alpha \in \mathcal{H}_m \).

The proof of Theorem 2 is derived from the comparison \((1)\) and \((5)\) and taking in account that, for \( x \in \mathbb{R} \), \( \langle x \rangle \in [0, 1/m) \iff [mx] \in m\mathbb{Z} \).

Note that we establish another, deeper connection (than the one indicated in Theorem 2) between the sets \( \mathcal{H}_m \) and \( \mathcal{H}_m \) in Theorem 4 below.

The following result provides an explicit description of the sets \( \mathcal{H}_m \).

**Theorem 3.** Let \( m \in \mathbb{N} \), \( \alpha \in \mathbb{R} \) and assume that \( \alpha = [a_0, a_1, a_2, \ldots] \) is its even CF expansion (of the length \( L \in \{\infty, 2, 4, 6, \ldots\} \)). Let \( m \in \mathbb{N} \) be given. Then the following three conditions are equivalent:

\begin{itemize}
  \item [(C1)] \( \alpha \in \mathcal{H}_m \).
  \item [(C2)] \( m \mid a_k \), for all even \( k \), \( 0 \leq k < L \).
  \item [(C3)] \( m \mid p_k \), for all even \( k \), \( 0 \leq k < L \) where \( p_k = p_k(\alpha) \) are numerators of the convergents for \( \alpha \); see \((3)\).
\end{itemize}

The proof of Theorem 3 easily follows from Theorems 1 and 2 using the identity

\[
(4) \quad m \left[\frac{x_0}{m}, \frac{x_1}{m}, \frac{x_2}{m}, \frac{x_3}{m}, \frac{x_4}{m}, \ldots\right] = \left[m^2x_0, \frac{x_1}{m}, \frac{x_2}{m}, \frac{x_3}{m}, \frac{x_4}{m}, \ldots\right].
\]

Alternatively, Theorem 3 can be derived from the following.

**Theorem 4.** For \( \alpha > 0 \) and \( m \in \mathbb{N} \), we have \( \alpha \in \mathcal{H}_m \iff \frac{1}{m\alpha} \in \mathcal{H}_m \).

Theorem 4 follows from Corollary 2 and identity \((4)\).

The proof of Theorem 4 will be provided in the next section. We also prove (Theorems 5 and 6) that

\[
\mathcal{H}(\frac{n}{m}) \supset \mathcal{H}(\frac{1}{\alpha}) = \mathcal{H}_m,
\]

for arbitrary \( n, m \in \mathbb{N} \), \( n < m \), and conclude that, for rational \( c \), \( 0 < c < 1 \), the sets \( \mathcal{H}(c) \) have a positive Hausdorff dimension (Corollary 3).

Finally, in the last section we discuss briefly the motivation behind our study.

2. **Proof of Theorem 1**

The proof is subdivided into several lemmas, some of which are of independent interest. Let \( \mathbb{I}_{0,1} \) stand for the open unit interval \((0, 1)\). For \( n \in \mathbb{N} \), \( \alpha > 0 \) and \( c \in \mathbb{I}_{0,1} \), consider the following finite subsets of \( \mathbb{N} \):

\[
(6) \quad S(n, \alpha) \overset{\text{def}}{=} \{ k \in \mathbb{N} \mid k\alpha < n \}
\]
and

\[(7) \quad S(n, \alpha, c) \overset{\text{def}}{=} \{ k \in S(n, \alpha) \mid \langle k \alpha \rangle < c \} = \{ k \in \mathbb{N} \mid k \alpha < n \land \langle k \alpha \rangle < c \}. \]

It is easy to see that

\[(8) \quad \text{card}(S(n, \alpha)) = \left\lceil \frac{n}{\alpha} \right\rceil - \right. \]

and

\[(9) \quad \text{card}(S(n, \alpha, c)) = \left\lceil \frac{c}{\alpha} \right\rceil + \sum_{k=1}^{n-1} \left( \left\lceil \frac{k+c}{\alpha} \right\rceil - \left\lceil \frac{k}{\alpha} \right\rceil \right), \]

where \(\left\lceil x \right\rceil^\sim\) stands for the largest integer smaller than \(x \in \mathbb{R}\):

\[(10) \quad \left\lceil x \right\rceil^\sim = \begin{cases} 
\left\lceil x \right\rceil & \text{if } x \notin \mathbb{Z} \\
 x - 1 & \text{if } x \in \mathbb{Z}.
\end{cases} \]

We observe the following.

**Lemma 1.** Given \(\alpha > 0\) and \(c \in \mathbb{I}_{0,1} = (0, 1)\), the following two conditions are equivalent:

1. \(\alpha \in \mathcal{H}(c)\);
2. \(\text{card}(S(n, \alpha, c)) \geq c \times \text{card}(S(n, \alpha))\), for all \(n \in \mathbb{N}\).

**Proof.** The claim of Lemma 1 follows directly from the definitions of the sets \(\mathcal{H}(c), S(n, \alpha)\) and \(S(n, \alpha, c)\) (see (1), (6), (7)). \(\square\)

**Lemma 2.** Let \(\alpha, \beta > 0\) and \(c \in \mathbb{I}_{0,1}\). Assume that the following two conditions are met:

\[(11) \quad (1) \quad \frac{1}{\alpha} - \frac{1}{\beta} \in \mathbb{Z} \quad \text{and} \quad (2) \quad \frac{1}{\alpha} - \frac{1}{\beta} \in \mathbb{Z}. \]

Then:

(A) \(\text{card}(S(n, \alpha)) - \text{card}(S(n, \beta)) = \frac{n(\beta - \alpha)}{\alpha \beta}\);

(B) \(\text{card}(S(n, \alpha, c)) - \text{card}(S(n, \beta, c)) = \frac{c n(\beta - \alpha)}{\alpha \beta}\);

(C) \(\alpha \in \mathcal{H}(c) \iff \beta \in \mathcal{H}(c)\).

**Proof.** The claims (A) and (B) of the lemma follow from formulae (8), (9) and the obvious implications

\(x - y \in \mathbb{Z} \iff \langle x \rangle = \langle y \rangle \iff \left\lceil x \right\rceil^\sim - \left\lceil y \right\rceil^\sim = x - y.\)

Finally, (C) follows from (A), (B) and Lemma 1. \(\square\)

The next lemma is just a specification of Lemma 2.

**Lemma 3.** Let \(\alpha, \beta > 0, c \in \mathbb{I}_{\alpha, \beta}, m \in \mathbb{N}\) be given and assume that the following two conditions are met:

(a) \(\frac{1}{\alpha} - \frac{1}{\beta} \in m \mathbb{Z}\), \quad (b) \(mc \in \mathbb{N}\).

Then \(\alpha \in \mathcal{H}(c)\) if and only if \(\beta \in \mathcal{H}(c)\).

**Proof.** We need only validate condition (2) of Lemma 2. It follows from (a) that \(\frac{1}{\alpha} - \frac{1}{\beta} = km\), for some \(k \in \mathbb{Z}\). But then \(\frac{1}{\alpha} - \frac{1}{\beta} = ckm = k(mc) \in \mathbb{Z}\), in view of (b). The proof is complete. \(\square\)
For $\alpha \in \mathbb{R}$, denote by $L(\alpha)$ the length of the odd CF expansion of $\alpha = [a_0(\alpha), a_1(\alpha), \ldots]$. Observe that $L(\alpha) = 1$ if and only if $\alpha \in \mathbb{Z}$; otherwise $L(\alpha) \geq 3$. Define the map $\phi : \mathbb{R} \to \mathbb{R}$ by the rule

$$
\phi(\alpha) = \begin{cases} 
\frac{1}{(\alpha)} = [a_1, a_2, \ldots] & \text{if } \alpha \notin \mathbb{Z}, \\
0 & \text{if } \alpha \in \mathbb{Z}.
\end{cases}
$$

One easily verifies that for $\alpha = [a_0, a_1, a_2, \ldots] \notin \mathbb{Z}$, one has $\phi^2(\alpha) = [a_2, \ldots];$ i.e., $\phi^2$ removes the first two entries (partial quotients) in the odd CF expansion of any noninteger.

Next we introduce the sets

$$\mathbb{R}(m) \overset{\text{def}}{=} \{ \alpha \in \mathbb{R} \setminus \mathbb{Z} \mid a_1(\alpha) \in m\mathbb{N} \} \cup \mathbb{Z}, \quad m \in \mathbb{N}.$$

**Lemma 4.** Let $m \in \mathbb{N}$, $c \in \mathbb{I}_{0,1}$ and assume that $mc \in \mathbb{N}$. Then, for every $\alpha \in \mathbb{R}(m)$,

$$\alpha \in \mathcal{H}(c) \iff \phi^2(\alpha) \in \mathcal{H}(c).$$

**Proof.** Denote $\beta = \phi^2(\alpha)$ and $u = \frac{1}{c} - \frac{1}{(\beta)} = -a_1$. Since $\alpha \in \mathbb{R}(m)$, we conclude that $m|u$ and use Lemma 3 to complete the proof of Lemma 4 $\alpha \in \mathcal{H}(c) \iff \langle \beta \rangle \in \mathcal{H}(c) \iff \beta \in \mathcal{H}(c).$ \hfill $\square$

It turns out that Lemma 4 can be used to explicitly exhibit uncountable subsets of $\mathcal{H}(c)$, for $c \in \mathbb{I}_{0,1} \cap \mathbb{Q}$. ($\mathbb{Q}$ stands for the set of rational numbers.) Those are the sets

$$\mathbb{R}_m \overset{\text{def}}{=} \{ \alpha \in \mathbb{R} \mid \phi^{2k}(\alpha) \in \mathbb{R}(m), \text{ for all } k \geq 0 \}, \quad m \in \mathbb{N}$$

(see Theorem 5 below).

The following lemma provides an alternative, more explicit description of the classes $\mathbb{R}_m$.

**Lemma 5.** Let $m \in \mathbb{N}$ and assume that $\alpha \in \mathbb{R}$ is given in terms of its odd CF expansion $\alpha = [a_0, a_1, \ldots]_\uparrow$ of length $L = L(\alpha) \in \{ \infty, 1, 3, \ldots \}$. Then $\alpha \in \mathbb{R}_m$ if and only if $a_k \in m\mathbb{Z}$, for all odd $k < L$.

**Proof.** The proof follows directly from the nature of the map $\phi^2$ and the trivial fact that $\mathbb{Z} \subset \mathbb{R}(m)$. \hfill $\square$

**Examples.**

1. $\alpha = \frac{3}{4}$. The odd CF expansion is $[1, 2, 1]_\uparrow$, $L = 3$, $a_1 = 2$. Thus $\frac{3}{4} \in \mathbb{R}_m \iff m = 1$ or 2.

2. $\alpha = \frac{\sqrt{5} - 1}{2}$. The odd CF expansion is $[1, 8, 2, 8, 2, \ldots]_\uparrow$, $L = \infty$, $a_1 = a_3 = a_5 = \cdots = 8$. Thus $\frac{\sqrt{5} - 1}{2} \in \mathbb{R}_m \iff m = 1, 2, 4$ or 8.

**Theorem 5.** Let $m \in \mathbb{N}$, $c \in \mathbb{I}_{0,1} = (0, 1)$ and assume that $cm \in \mathbb{N}$. Then $\mathbb{R}_m \subset \mathcal{H}(c)$.

**Proof of Theorem 5.** Recall that $\mathbb{Q} \subset \mathbb{R}$ stands for the set of rational numbers. We prove that

$$\alpha \in \mathbb{R}_m \implies \alpha \in \mathcal{H}(c).$$
Case 1. $L < \infty$, i.e. $\alpha \in \mathbb{Q}$. The proof goes by induction in $L = L(\alpha) \in \{1, 3, 5, \ldots \}$.
If $L = 1$, then $\alpha \in \mathbb{Z}$ and one has both $\alpha \in \mathbb{R}_m$ and $\alpha \in \mathcal{H}(c)$. For the
inductional step, we use Lemma 4 and the obvious fact that $\phi^2(\mathbb{R}_m) \subset \mathbb{R}_m$ (see
(13) and Lemma 5).

Case 2. $L = \infty$, i.e. $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ ($\alpha$ is irrational). The proof uses an approximation
argument. For a subset $S \subset \mathbb{R}$, denote by $\overline{S}$ the closure of $S$ in $\mathbb{R}$. Next we validate
the following inclusion:

$$\mathcal{H}(c) \cap (\mathbb{R} \setminus \mathbb{Q}) \subset \mathcal{H}(c)$$ (for $c \in \mathbb{Q} \cap (0, 1)$).

Indeed, it follows from (14) that $\mathcal{H}(c) = \bigcap_{n \in \mathbb{N}} \mathcal{H}(c, n)$, where
$$\mathcal{H}(c, n) = \{ \alpha \in \mathbb{R} \mid \text{card}(\{1 \leq m \leq n \mid \langle m\alpha \rangle < c\}) \geq cn\}$$
is a finite union of intervals of the form $[u, v)$ with rational endpoints $u, v$:
$$u, v \in \bigcup_{0 \leq r < s \leq n} \left\{ \frac{r}{s}, \frac{r+\epsilon}{s} \right\} \subset \mathbb{Q}$$
because $c \in \mathbb{Q}$. In particular, for all $n \geq 1$,
$$\mathcal{H}(c, n) \cap (\mathbb{R} \setminus \mathbb{Q}) \subset \mathcal{H}(c, n).$$

The proof of (14) is completed as follows:

$$\mathcal{H}(c) \cap (\mathbb{R} \setminus \mathbb{Q}) = \bigcap_{n \in \mathbb{N}} (\mathcal{H}(c, n) \cap (\mathbb{R} \setminus \mathbb{Q})) \subset \left( \bigcap_{n \in \mathbb{N}} (\mathcal{H}(c, n)) \right) \cap (\mathbb{R} \setminus \mathbb{Q})$$

$$= \bigcap_{n \in \mathbb{N}} (\mathcal{H}(c, n) \cap (\mathbb{R} \setminus \mathbb{Q})) \subset \bigcap_{n \in \mathbb{N}} \mathcal{H}(c, n) = \mathcal{H}(c).$$

Next one considers the sequence $\delta_{2k} = \delta_{2k}(\alpha), k \geq 0$, of even CF convergents
of $\alpha$ (with $L(\delta_{2k}) = 2k + 1$, an odd number). By what has been proven in Case 1,
$\delta_{2k} \in \mathbb{R}_m \cap \mathbb{Q} \subset \mathcal{H}(c)$, for all $k \geq 0$. Since
$$\lim_{k \to \infty} \delta_{2k} = \alpha \in \mathbb{R} \setminus \mathbb{Q},$$
the proof is complete in view of (14). □

Corollary 3. Let $C \subset \mathbb{Q} \cap \mathbb{N}$ be a finite subset of rational numbers. Then the
intersection $\bigcap_{c \in C} \mathcal{H}(c)$ is an uncountable set of positive Hausdorff dimension.

Proof. Let $m \in \mathbb{N}$ be a common denominator for all the numbers $c \in C$. Then
$$\mathbb{R}_m \subset \bigcap_{c \in C} \mathcal{H}(c),$$
in view of Theorem 5.

The set $\mathbb{R}_m$ is clearly uncountable, and it is easily seen to have a positive Haus-
dorff dimension. (One way to see it is to observe that it contains the set $\mathbb{R}_m$ of
all numbers of the form $[0, m, a_1, m, a_2, m, \ldots]_1$, where $a_i \in \{1, 2\}$. This set is the
disjoint union of its images under the two contractions:
$$f_i(x) = \frac{1}{m + \frac{1}{x}}, \quad i = 1, 2,$$
and therefore must be of positive Hausdorff dimension (see Chapter 9 in [2]). □

Corollary 4. $\mathbb{R}_m \subset \mathcal{H}_m$, for all $m \in \mathbb{N}$.

Proof of Corollary 4. Take $c = \frac{1}{m}$ in Theorem 5 (Recall that $\mathcal{H}_m = \mathcal{H}(\frac{1}{m})$.) □

As is pointed out in Section 1, the inclusion in Corollary 4 can be reversed.

Theorem 6. $\mathbb{R}_m = \mathcal{H}_m$, for all $m \in \mathbb{N}$. 
We first need to prove the following:

**Lemma 6.** \( \mathcal{H}_m \subset \mathbb{R}(m), \) for all \( m \in \mathbb{N}. \)

**Proof of Lemma 6.** Since \( \mathbb{Z} \subset \mathbb{R}(m) \) (see (12)), it suffices to prove that if \( \alpha \in (\mathbb{R} \setminus \mathbb{Z}) \cap \mathcal{H}_m, \) then \( \alpha \in \mathbb{R}(m). \) We assume without loss of generality that \( \lfloor \alpha \rfloor = 0 \) (otherwise replacing \( \alpha \) by \( \langle \alpha \rangle \)). Let \( [0, a_1, a_2, \ldots] \) be the odd CF expansion of \( \alpha. \)

We have to show that \( m \mid a_1. \) If not, then \( a_1 \equiv r \pmod{m}, \) for some integer \( r, \) \( 1 \leq r \leq m - 1. \) Define \( \beta \in \mathbb{R} \) by its CF expansion \( [0, r, a_2, a_3, \ldots] \), with all the entries \( a_k, \) for \( k \neq 1, \) being the same as in the CF expansion of \( \alpha. \)

Then, with \( c = \frac{1}{m}, \) we easily validate the conditions of Lemma 3. Indeed, \( \alpha \in \mathcal{H}_m = \mathcal{H}(c), mc = 1 \in \mathbb{N} \) and \( \frac{1}{m} - \frac{r}{m} = a_1 - r \in m\mathbb{Z}. \) By Lemma 3, \( \beta \in \mathcal{H}(c) = \mathcal{H}_m, \) which is impossible because \( \beta = \langle \beta \rangle > \frac{1}{r+1} \geq \frac{1}{m} = c, \) so that the relation \( \beta \in \mathcal{H}(c) \) contradicts (11) for \( n = 1. \)

**Proof of Theorem 6.** In view of Corollary 4, we only have to establish the inclusion \( \mathcal{H}_m \subset \mathbb{R}_m. \) Let \( \alpha \in \mathcal{H}_m \) be given. We claim that then

\[
\phi^{2k}(\alpha) \in \mathbb{R}(m) \cap \mathcal{H}_m, \quad \text{for all } k \geq 0.
\]

The proof goes by induction in \( k. \) For \( k = 0, \) (15) holds in view of Lemma 6. For the inductive step, we use Lemmas 4 and 6. This completes the proof of (15).

Now we are ready to complete the proof of Theorem 6 in the introduction. The main part of the work has already been done: Theorem 6 is a rephrasing of the equivalence \( (C1) \iff (C2). \)

It remains to prove the equivalence of the following two conditions:

\[
(C2) \quad m \mid a_k, \quad \text{for all odd } k, 1 \leq k < L.
\]

\[
(C3) \quad m \mid q_k, \quad \text{for all odd } k, 1 \leq k < L.
\]

The proof goes by induction in \( k. \) For \( k = 1 \) the equivalence is immediate because \( q_1 = a_1. \)

Now assume that both \( (C2) \) and \( (C3) \) hold for some odd \( k = n < L - 2. \) It suffices to show that

\[
m \mid a_{n+2} \iff m \mid q_{n+2}.
\]

From the identity \( q_{n+2} = a_{n+2}q_{n+1} + q_n \) we derive the congruence \( q_{n+2} \equiv a_{n+2}q_{n+1} \pmod{m}, \) so that the implication \( \implies \) is immediate. The opposite implication is also valid because \( q_{n+2}, q_{n+1} \) are relatively prime.

### 3. Motivation: Heavy sequences

The following result by Y. Peres is closely related to the Maximal Ergodic Theorem:

**Lemma 7** (Peres). Let \( T : X \to X \) be a continuous transformation of a compact space, and let \( \mu \) be a probability measure preserved by \( T. \) For every continuous \( g : X \to \mathbb{R} \) there exists some \( x \in X \) such that

\[
\forall N \in \mathbb{N} \quad \frac{1}{N} \sum_{n=0}^{N-1} g(T^n x) \geq \int_X g d\mu.
\]
Example 2. Fix parameter 

\[
\mathcal{H}^f_T = \{ x \in X : S_n(x) - n \int_X f \, d\mu \geq 0, \quad \forall n \in \mathbb{N} \}. 
\]

Then Lemma 4 tells us that in this situation, for any \( f \in L^1(X, \mu) \), \( \mathcal{H}^f_T \neq \emptyset \).

We also say that there is some point \( x \) whose orbit is heavy for \( f \). If \( f \) is the characteristic function of a set \( A \), we will generally simply refer to “the heavy set of \( A \)”, or call a sequence “heavy for \( A \)”. Restricting ourselves only to the reals modulo one, \( \mathbb{R}/\mathbb{Z} = S^1 \), we derive the following results:

**Example 1.** Fix \( \alpha \in S^1 \). Then for any closed subset \( A \subset S^1 \), there exists some point \( x \) whose orbit is heavy for \( A \).

The previous example can be viewed as the following: for any choice of a closed set \( A \subset S^1 \) and leading coefficient \( \alpha \), there exists some choice of \( \beta \) such that the polynomial \( \alpha n + \beta \), considered modulo one, is heavy for \( A \). This example may be generalized as follows:

**Example 2.** Fix \( \alpha \in \mathbb{R} \), a closed set \( A \subset S^1 \), and a choice of \( k \in \mathbb{N} \). Then there exists a choice of coefficients \( a_0, a_1, \ldots, a_{k-1} \) such that the sequence

\[
\{ \alpha n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0 \}_{n=0}^{\infty} 
\]

is heavy for \( A \) (when taken modulo one). For details on how to derive this sequence as the orbit of a measure preserving system, we refer the reader to pp. 35–37 in \cite{3} or, for a more detailed derivation of heaviness properties, to \cite{7}.

Finally, the reader may be tempted to try to generalise the results of Theorem 3 and Example 1 to claim that the set

\[
\mathcal{H}[A] \overset{\text{def}}{=} \{ x \in S^1 \mid \text{the sequence } (kx)_{k\geq 1} \text{ is heavy for } A \}
\]

is always nonempty. This cannot be done.

**Example 3.** There exists a closed set \( A \subset S^1 \), a finite union of closed intervals, whose measure is larger than \( 1/3 \), such that \((x \in A) \Rightarrow (2x \notin A, 3x \notin A)\). In particular, for such an \( A \) one has \( \mathcal{H}[A] = \emptyset \).

The measure of the set \( A \) in the above example can be made arbitrarily close to \( \frac{1}{3} \). For details, see \cite{7}, where techniques from ergodic theory are used (in a nonconstructive way) to establish the existence of such a set \( A \).

We have examples of closed subintervals \( J \subset (0, 1) \subset S^1 \) for which the set \( \mathcal{H}[J] \) is countable or finite \((J = \left[ \frac{1}{3}, \frac{2}{3} \right] \) and \( J = \left[ \frac{2}{3}, \frac{3}{3} \right] \), respectively). We don’t know whether it can be made empty.

Note that the subject of our paper is somehow related to that in \cite{1}, where some sufficient conditions for the one-sided boundedness of the sequence

\[
\text{card}(\{1 \leq k \leq n \mid \langle k\alpha \rangle < c \}) - cn
\]
have been established (cf. equation (1)). All our results are new and imply some of the results in [1].

REFERENCES

[1] Y. Dupain, T. Vera Sós, On the one-sided boundedness of discrepancy-function of the sequence \{na\}, Acta Arith. 37 (1980), 363–374. MR598889 (82c:10058)

[2] K. Falconer, Fractal Geometry: Mathematical Foundations and Applications, John Wiley & Sons, Ltd., 1990. MR1062577 (92k:28008)

[3] H. Furstenberg, Recurrence in ergodic theorem and combinatorial number theory, Princeton University Press, 1981. MR603625 (82c:28010)

[4] A. Y. Khinchin, Continued Fractions, The University of Chicago Press, 1964. MR0161833 (28:5037)

[5] S. Lang, Introduction to Diophantine Approximations, Springer-Verlag, 1995. MR1348400 (96h:11067)

[6] Y. Peres, A combinatorial application of the maximal ergodic theorem, Bull. London Math. Soc. 20 (1988), 248–252. MR931186 (89e:28033)

[7] D. Ralston, Heaviness—An Extension of a Lemma of Y. Peres, Houston Journal of Mathematics. To appear.

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