WILD LORENZ LIKE ATTRACTORS

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Abstract. We give the first examples of flows which exhibit robust singular attractors containing a wild hyperbolic set (in the sense of Newhouse). A hyperbolic set is said to be wild, if it has tangencies between its stable and unstable manifolds, in a robust way. The only restriction on the ambient manifold is that its dimension should be at least 5.

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1. Introduction

In [L], motivated by the equations for fluid convection in a two-dimensional layer heated from below, E. Lorenz discovered a flow in $\mathbb{R}^3$ defined by an explicit and simple system of differential equations that exhibited extremely rich and robust dynamical behavior. The combination of the works [ABS, GW, Tu] show that the Lorenz flow has a singular attractor: a transitive compact invariant set that attracts a whole neighborhood of initial conditions and that contains both a singularity (stationary point) and regular orbits. Also, this attractor is robust: every flow that is sufficiently close to the initial one has a singular attractor nearby. The topological, geometrical and ergodic properties of the Lorenz flow have been intensively studied during the last two decades (e.g., see [BS, BDV, LMP, M, S, V] and the references therein). Moreover, the Lorenz flow plays a key role in a global picture for flows in three dimensions. In fact, in [MPP] it is shown that every robust transitive set is...
either hyperbolic or a singular attractor (or repellor) that shares many properties
with the Lorenz attractor.

It is natural to ask whether higher dimensional manifolds support robust singular
attractors with richer dynamical behavior. This question has been addressed in [BPV] and [ShT]. While the singularity of the Lorenz flow has 1 expanding eigen-
value, Bonatti, Pumariño and Viana [BPV] construct robust singular attractors
in an \( n \geq 4 \) dimensional manifold where the singularity has \( k = \max\{n-3, 2\} \geq 2 \)
expanding eigenvalues. They also show that these attractors support a unique
physical (SBR) measure with the no-holes property. Shil’nikov and Turaev [ShT]
construct a robust singular chain transitive set in \( \mathbb{R}^4 \) with the
special feature that it contains a wild hyperbolic set, in the sense of Newhouse [N].
Nevertheless, it is not known if this example has the key properties of being isolated
and transitive. A hyperbolic set is said to be wild, if it has tangencies between its
stable and unstable manifolds, in a robust way. By [MPP], robust singular attrac-
tors in three dimensional manifolds are free of wild hyperbolic sets. Moreover, the
examples in [BPV] can not have a wild hyperbolic set since their dynamics reduces
to a uniformly expanding endomorphism.

The purpose of this article is to give the first examples of robust singular attrac-
tors containing a wild hyperbolic set. The only restriction on the ambient manifold
is that its dimension should be at least 5. In fact, every flow on a manifold of
dimension at least 5 having an attracting periodic orbit, can be modified in an
arbitrarily small neighborhood of this orbit to have such a singular attractor.

**Main Theorem.** Every manifold of dimension at least 5 admits a \( C^1 \) non-empty
open set \( \mathcal{O} \) such that every \( C^2 \) vector field \( X \) in \( \mathcal{O} \) exhibits a singular attractor \( \Lambda_X \)
with the following properties.

1. \( \Lambda_X \) contains a hyperbolic singularity of Morse index 2 of \( X \).
2. \( \Lambda_X \) contains a wild hyperbolic set.
3. There is a residual subset of \( \mathcal{O} \) that is dense in the \( C^\infty \) topology, such that if \( X \) belongs to this residual set, then the set of periodic orbits of \( X \) of Morse
index 1 and the set of periodic orbits of \( X \) of Morse index 2 are both dense
in \( \Lambda_X \).

**Remark 1.1.** In fact we prove the stronger result. that every \( C^1 \) vector field \( X \in \mathcal{O} \)
has an attracting set \( \Lambda_X \) satisfying properties 1, 2, and 3, see Theorem 2.8 in
Section 2. Although it is possible to push the ideas of this paper to establish that
\( \Lambda_X \) is topologically transitive (and hence an attractor) for all \( X \in \mathcal{O} \), it requires a
fairly long and technical proof (see [BKR]).

It is not clear to us what are the ergodic properties of the flows introduced here.
It would be very interesting to show, for example, that these flows admit a physical
or SBR measure.

**1.1. Strategy of the proof.** For the proof of the Main Theorem we proceed as
follows. Fix an integer \( n \geq 5 \). Then, for a given \( \lambda \in (0, 1) \) sufficiently close to
1 we construct a family of vector fields \( \{X_{\lambda,\mu}\}_{\lambda,\mu} \), where \( \mu > 0 \) takes values on
a certain interval to be precised later. These vector fields are defined on a subset
of the closed solid torus \( \mathbb{T}^n \) of dimension \( n \). For some values of \( \mu \) and a suitable
\( C^1 \)-neighborhood \( \mathcal{O} \) of \( X_{\lambda,\mu} \), we show that the conclusions of the Main Theorem
are satisfied. The Main Theorem is obtained by embedding the solid torus \( \mathbb{T}^n \) on
a given manifold of dimension \( n \).
The vector field $X_{\lambda, \mu}$ will have a unique singularity $o$. This singularity will be hyperbolic with eigenvalues $-\eta$, $-\mu$ and $\sigma$ with multiplicities $n - 3$, $1$ and $2$ respectively, where $0 < \mu < \sigma < \eta$. The construction is such that the dynamics of the flow of $X_{\lambda, \mu}$ reduces to the dynamics of the map

$$F_{\lambda, \mu} : \mathbb{C} \setminus \{0\} \to \mathbb{C}$$

$$z \mapsto (1 - \lambda + \lambda|z|^\mu/\sigma) (z/|z|)^2 + 1,$$

in the same way as the dynamics of the geometric Lorenz attractor (see [ABS, GW]) reduces to the dynamics of a map defined on a punctured interval. More precisely, the vector field $X_{\lambda, \mu}$ will induce a first return (or Poincaré) map $\hat{F}_{\lambda, \mu}$ to a certain transversal section of dimension $n - 1$. This first return map $\hat{F}_{\lambda, \mu}$ will have an invariant foliation of dimension $n - 3$ that is uniformly contracted by $\hat{F}_{\lambda, \mu}$.

The map $F_{\lambda, \mu}$ defined above represents the leaf space transformation of $\hat{F}_{\lambda, \mu}$; see Subsection 2.3 below for more details.

The parameter $\mu$ will be taken in an interval of the form $[\mu_0, \sigma]$, where $\mu_0 \in (0, \sigma)$ is sufficiently close to $\sigma$. That is, $F_{\lambda, \mu}$ will be close to the map

$$F_{\lambda} : \mathbb{C} \setminus \{0\} \to \mathbb{C}$$

$$z \mapsto (1 - \lambda + \lambda|z|) (z/|z|)^2 + 1.$$

Note that the endomorphism $z \mapsto (1 - \lambda + \lambda|z|) (z/|z|)^2$ acts as angle doubling on the argument and as an affine contraction of factor $\lambda$ in the radial direction. Thus $F_{\lambda}$ is closely related to the extensively studied quadratic family $Q_c(z) = z^2 + c$, where the $|z| \mapsto |z|^2$ action of $Q_0(z) = z^2$ is replaced by an affine contraction of factor $\lambda$.

The point $0 \in \mathbb{C}$ represents a leaf contained in the stable manifold of the singularity $o$ of the vector field $X_{\lambda, \mu}$. The map $F_{\lambda, \mu}$, which is not defined at $z = 0$, ‘opens’ the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and maps it onto $\{z \in \mathbb{C} \mid |z - 1| > 1 - \lambda\}$, in a 2 to 1 fashion.

The dynamical properties of the maps $F_{\lambda, \mu}$, the corresponding properties of the vector fields $X_{\lambda, \mu}$ and their relation to the Main Theorem are all explained in Section 2.

1.2. Notes and references. To show that the attractor sets introduced here are robustly transitive, we show that the stable and unstable manifolds of a certain saddle fixed point are dense. In our context, the methods used in [BPV] break down, since our leaf space transformation is not uniformly expanding. A key step towards showing that the unstable manifold is dense uses an argument similar to that of [BD1]. In fact, our examples have some kind of “solenoidal blender”. The density of the stable manifold is obtained by ad hoc arguments.

In [ShT], the existence of a wild hyperbolic set is obtained, among other ingredients, from the existence of Newhouse phenomena of persistence of tangencies. Neither in [BD2] nor in our paper, the persistence of tangencies is obtained from the existence of Newhouse phenomena. Although the existence of a wild hyperbolic set implies the co-existence of infinitely many periodic orbits of distinct index [R], we prove part 3 of the Main Theorem directly, independently of part 2.

As mentioned above it is not clear to us what are the ergodic properties of the flows and endomorphisms introduced here. Note that when $\lambda \to 1$ the maps $F_{\lambda}$...
converge to the map \( G(z) = |z|(|z|)^2 + 1 \) in the \( C^\infty \) topology. The map \( G \) extends continuously to \( \mathbb{C} \) and it preserves the Lebesgue measure on \( \mathbb{C} \).

**Question.** Is \( G \) ergodic with respect to the Lebesgue measure?

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2. **Statement of results**

In this section we outline our results and simultaneously describe the structure of the paper. Recall that for the proof of the Main Theorem we construct a family of vector fields \( X_{\lambda,\mu} \) so that the leaf space transformation of a first return map to a cross section is close to

\[ F_\lambda : \mathbb{C} \setminus \{0\} \to \mathbb{C} \]

\[ z \mapsto (1 - \lambda + \lambda |z|) (|z|)^2 + 1. \]

In Subsection 2.1 we introduce a topological space of \( C^1 \) maps \( \mathcal{F}_\lambda \) that contains the endomorphisms induced by \( C^2 \) vector fields which are close to \( X_{\lambda,\mu} \) in the \( C^1 \) topology. The main dynamical properties of maps in \( \mathcal{F}_\lambda \) close to \( F_\lambda \) are summarized in Subsection 2.2. In Subsection 2.3 we describe the vector fields \( X_{\lambda,\mu} \) and in Subsection 2.4 we state a more precise version of the Main Theorem.

2.1. **The space \( \mathcal{F}_\lambda \).** Throughout this paper the punctured complex plane \( \mathbb{C} \setminus \{0\} \) will be denoted by \( \mathbb{C}^* \). Note that the map \( F_\lambda \) can be written as the composition \( F_\lambda = g_\lambda \circ \tau_\lambda \) of the maps

\[ \tau_\lambda : \mathbb{C}^* \to \mathbb{R}/\mathbb{Z} \times (1 - \lambda, +\infty) \]

\[ z \mapsto \left( \frac{1}{2\pi} \arg(z), 1 - \lambda + \lambda |z| \right) \]

and

\[ g_\lambda : \mathbb{R}/\mathbb{Z} \times (0, +\infty) \to \mathbb{C} \]

\[ (\theta, t) \mapsto t \exp(4\pi i\theta) + 1. \]

Moreover note that \( \tau_\lambda \) is a diffeomorphism and that \( g_\lambda \) is a local diffeomorphism. For \( \lambda \in (0,1) \) put

\[ B_\lambda = \{ z \in \mathbb{C} \mid |z| \leq 2(1 - \lambda)^{-1} \} \]

and

\[ B_\lambda^* = B_\lambda \setminus \{0\}. \]

It is easy to check that \( F_\lambda(B_\lambda^*) \) is contained in the interior of \( B_\lambda \).

**Definition 2.1.** For a given \( \lambda \in (0,1) \) define the following spaces of \( C^1 \) maps.

1. The space \( \mathcal{F}_\lambda \) of all homeomorphisms,

\[ \tau : B_\lambda^* \to \mathbb{R}/\mathbb{Z} \times (1 - \lambda, 2(1 - \lambda)^{-1} - 1 - \lambda), \]

that extend to a diffeomorphism onto its image defined on a neighborhood of \( B_\lambda^* \) in \( \mathbb{C}^* \). We endow \( \mathcal{F}_\lambda \) with the weak \( C^1 \) topology.
2. The space $G_{\lambda}$ of all maps

$$g : \mathbb{R}/\mathbb{Z} \times [1 - \lambda, 2(1 - \lambda)^{-1} - 1 - \lambda] \to B_{\lambda},$$

having an extension to a neighborhood of $\mathbb{R}/\mathbb{Z} \times [1 - \lambda, 2(1 - \lambda)^{-1} - 1 - \lambda]$ in $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$ that is a local diffeomorphism. We endow $G_{\lambda}$ with the strong $C^1$ topology.

3. The space $\mathcal{F}_\lambda$ of all maps $F : B_{\lambda}^* \to B_{\lambda}$ that can be written as a composition $F = g \circ \tau$, with $\tau \in \mathcal{T}_\lambda$ and $g \in G_{\lambda}$. We endow $\mathcal{F}_\lambda$ with the largest topology for which the composition map $(\tau, g) \mapsto g \circ \tau$, from $\mathcal{T}_\lambda \times G_{\lambda}$ to $\mathcal{F}_\lambda$, is continuous.

Clearly $F_{\lambda} \in \mathcal{F}_\lambda$. By definition, each map in $\mathcal{F}_\lambda$ is a local diffeomorphism and hence it is an open map.

Notice that two maps in $\mathcal{F}_\lambda$ which are close, need not be close with respect to the uniform topology on $B_{\lambda}^*$. On the other hand, two maps in $\mathcal{F}_\lambda$ that are close in the weak $C^1$ topology need not be close in $\mathcal{F}_\lambda$. In Lemma 2.2 below we show that the composition map from $\mathcal{T}_\lambda \times G_{\lambda}$ to $\mathcal{F}_\lambda$ is open. It follows that for every $\tau \in \mathcal{T}_\lambda$, $g \in G_{\lambda}$ and every $F \in \mathcal{F}_\lambda$ that is close to $g \circ \tau$ in $\mathcal{F}_\lambda$, there is $\tilde{\tau} \in \mathcal{T}_\lambda$ close to $\tau$ and $\tilde{g} \in G_{\lambda}$ close to $g$, so that $F = \tilde{g} \circ \tilde{\tau}$.

The spaces $\mathcal{T}_\lambda$ and $G_{\lambda}$ are Baire spaces as they both admit a complete metric. Since the composition map from $\mathcal{T}_\lambda \times G_{\lambda}$ to $\mathcal{F}_\lambda$ is open (Lemma 2.2), it follows that $\mathcal{F}_\lambda$ is also a Baire space.

**Lemma 2.2.** The composition map from $\mathcal{T}_\lambda \times G_{\lambda}$ to $\mathcal{F}_\lambda$ is open.

**Proof.** Denote the composition map by $\pi$. We need to show that for every open set $\mathcal{U}$ in $\mathcal{T}_\lambda \times G_{\lambda}$ the set $\pi^{-1}(\pi(\mathcal{U}))$ is open. For that, let $(\tilde{\tau}_0, \tilde{g}_0) \in \pi^{-1}(\pi(\mathcal{U}))$ be given and let $(\tau_0, g_0) \in \mathcal{U}$ be such that

$$g_0 \circ \tau_0 = \pi((\tau_0, g_0)) = \pi((\tilde{\tau}_0, \tilde{g}_0)) = \tilde{g}_0 \circ \tilde{\tau}_0. \tag{1}$$

Let $\mathcal{U}'$ (resp. $\mathcal{V}'$) be a neighborhood of $\tau_0$ (resp. $g_0$) in $\mathcal{T}_\lambda$ (resp. $G_{\lambda}$) such that $\mathcal{U}' \times \mathcal{V}' \subset \mathcal{U}$.

From (1) it follows that the map $\tilde{\tau}_0 \circ \tau_0^{-1}$ extends to a diffeomorphism onto its image $h_0$, defined on a neighborhood of $\mathbb{R}/\mathbb{Z} \times [1 - \lambda, 2(1 - \lambda)^{-1} - 1 - \lambda]$. By continuity we have $\tilde{g}_0 \circ h_0 = g_0$. Put $\tilde{\mathcal{U}} = \{h_0 \circ \tau \mid \tau \in \mathcal{U}'\}$ and $\tilde{\mathcal{V}} = \{g \circ h_0^{-1} \mid g \in \mathcal{V}'\}$. By construction the neighborhood $\tilde{\mathcal{U}} \times \tilde{\mathcal{V}}$ of $(\tilde{\tau}_0, \tilde{g}_0)$ in $\mathcal{T}_\lambda \times G_{\lambda}$ is such that $\pi(\tilde{\mathcal{U}} \times \tilde{\mathcal{V}}) = \pi(\mathcal{U} \times \mathcal{V}) \subset \pi(\mathcal{U})$. This shows that $\pi^{-1}(\pi(\mathcal{U}))$ is open in $\mathcal{T}_\lambda \times G_{\lambda}$ and finishes the proof the lemma. \qed

2.2. Dynamics of maps in $\mathcal{F}_\lambda$ near $F_{\lambda}$. For a fixed $\lambda \in (0, 1)$ close to 1, we will be interested on the dynamics of a given map $F$ in $\mathcal{F}_\lambda$ near $F_{\lambda}$. Although a map $F$ in $\mathcal{F}_\lambda$ is not defined at $z = 0$, we will let it act on the subsets of $B_{\lambda}$ by $F(U) = F(U \setminus \{0\})$. For a positive integer $m$ we will denote by $F^m$ the $m$-th iterate of this action. Note that the image of an open subset of $B_{\lambda}$ under this action is again an open subset of $B_{\lambda}$.

Our main object of study will be the maximal invariant set of $F$ in $B_{\lambda}$:

$$\Omega_F = \cap_{m \geq 1} F^m(B_{\lambda}).$$
It is easy to see that $p_\lambda = 1 + (1 - \lambda)^{-1}$ is a saddle fixed point of $F_\lambda$. For a map $F \in \mathcal{F}_\lambda$ close to $F_\lambda$, we denote by $p_F$ the saddle fixed point of $F$ that is the continuation of $p_\lambda$. Of course, $p_F \in \Omega_F$.

**Proposition 2.3** (The topology of $\Omega_F$). For $\lambda \in (0, 1)$ sufficiently close to 1 and for $F$ in $\mathcal{F}_\lambda$ sufficiently close to $F_\lambda$, the set $\Omega_F$ is a topological annulus that contains the origin $z = 0$ in its interior (see Figure 1).

The annulus $\Omega_F$ is neither open nor closed. More precisely, while the external boundary $\gamma_F^+$ is contained in $\Omega_F$, the internal boundary $\gamma_F^-$ is disjoint from $\Omega_F$.

Moreover, $\gamma_F^+$ is a Jordan curve contained in the unstable manifold of $p_F$ and $\gamma_F^+$ is differentiable at every point with the unique exception of a point where the unstable manifold of $p_F$ has a transversal self-intersection. The internal boundary $\gamma_F^-$ is a Jordan curve.

Section 3 contains the proof of this proposition and some technical results which will be needed in the rest of the paper.

In Section 4 we study some dynamical features of maps close to $F_\lambda$ such as the existence of a fundamental annulus with some important covering properties.

When $\lambda \in (2^{-1/2}, 1)$, one of the important features of the map $F_\lambda$ is that its Jacobian,

$$\text{Jac}(F_\lambda)(z) = 2\lambda(\lambda + (1 - \lambda)|z|^{-1}),$$

is everywhere larger than a constant larger than 1. We say that a map $F$ in $\mathcal{F}_\lambda$ is **area expanding**, if there is a constant $\kappa > 1$ such that for every point $z \in B^*_\lambda$ the Jacobian of $F$ at $z$ is at least $\kappa$. A map in $\mathcal{F}_\lambda$ near $F_\lambda$ need not be area expanding, as there is no control on its behavior near $z = 0$.

In Section 5 we prove that for $\lambda \in (0, 1)$ sufficiently close to 1 the map $F_\lambda$ is robustly transitive on the maximal invariant set $\Omega_{F_\lambda}$, in the subspace of $\mathcal{F}_\lambda$ of area expanding maps. More precisely, we prove the following result.

**Theorem 2.4** (Robust transitivity). For $\lambda \in (0, 1)$ sufficiently close to 1 there exists a neighborhood $\mathcal{U}$ of $F_\lambda$ in $\mathcal{F}_\lambda$ such that each area expanding map $F$ in $\mathcal{U}$ is topologically mixing on $\Omega_F$.

To prove the previous theorem we show that the stable manifold of $p_F$ is dense in $B_\lambda$ and that the unstable manifold of $p_F$ is dense in $\Omega_F$. By a standard argument we conclude that $F$ is topologically mixing on $\Omega_F$.  

![Figure 1. The attractor $\Omega_F$](image)
In Section 6 we show the following result about periodic points of maps in $\mathcal{F}_\lambda$. Given an open set $\mathcal{U}$ of $\mathcal{G}_\lambda$ and $\tau \in \mathcal{F}_\lambda$ put

$$\mathcal{U}_\tau = \{ g \in \mathcal{U} \mid F = g \circ \tau \text{ is area expanding} \}.$$  

Note that $\mathcal{U}_\tau$ is an open subset of $\mathcal{G}_\lambda$, that might be empty. For example, if $\text{Jac}(\tau)(z) \to 0$ as $z \to 0$ then the set $\mathcal{U}_\tau$ is empty.

**Proposition 2.5** (Periodic points). There is a neighborhood $\mathcal{U}$ of $\eta_1$ in $\mathcal{G}_\lambda$ such that for every $\tau \in \mathcal{F}_\lambda$ there is a residual subset $\mathcal{R}_\tau$ of $\mathcal{U}_\tau$ with the following property. For every $g \in \mathcal{R}_\tau$ the set of periodic sources and the set of periodic saddles of $F = g \circ \tau$ are both dense in $\Omega_F$.

In order to state the results concerning robust homoclinic tangencies, we need to introduce some notation. For $\lambda \in (0, 1)$ close enough to 1 and $F \in \mathcal{F}_\lambda$, we define two subsets of $\Omega_F$: a fundamental annulus $A_F$ (Section 3) and a domain $H_\lambda$ containing $p_F$, where $F$ is uniformly hyperbolic (Subsection 3.3). The set $A_F$ is such that

$$A_F \subset F(A_F) \subset H_\lambda,$$

so the set

$$\Gamma_F = \{ \{ z_j \}_{j \geq 0} \mid z_j \in H_\lambda \text{ and } F(z_{j+1}) = z_j, \text{ for } j \geq 0 \},$$

is non empty. Moreover, every infinite backward orbit $\{ z_j \}_{j \geq 0}$ in $\Gamma_F$ defines a local unstable manifold passing through $z_0$ (Subsection 4.4). In Section 7 we prove the following result.

**Theorem 2.6** (Homoclinic tangencies and wild hyperbolic sets). For $\lambda \in (0, 1)$ close to 1 and every area expanding map $F$ in $\mathcal{F}_\lambda$ close to $F_\lambda$ the following properties hold.

1. The set

$$W_F = \{ z \in H_\lambda \mid F^m(z) \in H_\lambda \text{ for every } m \geq 1 \}$$

is an uniformly hyperbolic and forward invariant set for $F$. Moreover, the local unstable manifold of an infinite backward orbit in $\Gamma_F$ starting at a point in $A_F$, is contained in the unstable manifold of some infinite backward orbit contained in $W_F$.

2. There is an arc $\gamma_\lambda$ of the stable manifold of $p_\lambda$, such that if $\gamma_F$ is an arc of the stable manifold of the fixed point $p_F$ of $F$ that is $C^1$ close to $\gamma_\lambda$, then $\gamma_F$ is tangent to the unstable manifold of an infinite backward orbit in $\Gamma_F$. As the saddle fixed point $p_F$ of $F$ is contained in $W_F$, it follows that the hyperbolic set $W_F$ of $F$ is wild.

Each of the unstable manifolds associated to an infinite backward orbit in $\Gamma_F$ is approximated in the $C^1$ topology by a piece of the unstable manifold of $p_F$ (Subsection 5.2). So by an arbitrarily small perturbation it is possible to create a homoclinic tangency of the fixed point $p_F$.

2.3. The vector fields $X_{\lambda,\mu}$. For an integer $k \geq 0$ denote by $D^k$ the closed unit ball of $\mathbb{R}^k$. Fix an integer $n \geq 5$ and let $T^n = (\mathbb{R}/\mathbb{Z}) \times D^{n-1}$ be the closed solid torus of dimension $n$.

For the proof of the Main Theorem we fix constants $\eta > \sigma > 0$. Given $\lambda \in (0, 1)$ sufficiently close to 1 we construct a one parameter family $\{ X_{\lambda,\mu} \mid \mu \in (\mu_0, \sigma] \}$ of
vector fields, where \( \mu_0 \in (0, \sigma) \) is close to \( \sigma \), defined on an open subset \( U \) of \( \mathbb{T}^n \). For \( \mu \in (\mu_0, \sigma) \) the vector field \( X_{\lambda, \mu} \) will have a hyperbolic singularity \( o = o_{\lambda, \mu} \) with eigenvalues \(-\mu, \sigma\) and \(-\eta\) of multiplicities 1, 2 and \( n - 3 \), respectively.

The vector fields \( X_{\lambda, \mu} \) are closely related to the map \( F_\lambda \) and the space \( \mathcal{F}_\lambda \). More precisely, for \( \mu \) close to \( \sigma \) the map \( F_{\lambda, \mu} : B_\lambda^* \to B_\lambda \)

\[
F_{\lambda, \mu} : B_\lambda^* \to B_\lambda \\
\quad z \mapsto (1 - \lambda|z|^\sigma) (z/|z|)^2 + 1.
\]

is an area expanding endomorphism in \( \mathcal{F}_\lambda \) that is close to \( F_\lambda \). Next we consider the skew product map

\[
\tilde{F}_{\lambda, \mu} : B_\lambda \times \mathbb{T} \times D^{n-5} \to B_\lambda \times \mathbb{T} \times D^{n-5} \\
(z, w, v) \mapsto \left( F_{\lambda, \mu}(z), \frac{z}{2|z|} + \beta|z|^\eta/\sigma \frac{\eta}{\sigma} z, \beta|z|^\eta/\sigma v \right),
\]

where \( \beta > 0 \) is an arbitrarily small constant so that \( \tilde{F}_{\lambda, \mu} \) is well defined and injective. The vector field \( X_{\lambda, \mu} \) will be constructed in such a way that the first return map to a certain cross section \( \Sigma^u \) of the flow of \( X_{\lambda, \mu} \), parametrized by \( B_\lambda \times \mathbb{T} \times D^{n-5} \), is given by \( \tilde{F}_{\lambda, \mu} \).

Note that the base dynamics of \( \tilde{F}_{\lambda, \mu} \) is given by \( F_{\lambda, \mu} \). More precisely, the fibers of the linear projection \( \Pi_\lambda : B_\lambda \times \mathbb{T} \times D^{n-5} \to B_\lambda \) onto the first coordinate form an invariant (strong stable) foliation for the dynamics of \( \tilde{F}_{\lambda, \mu} \) and the action induced by \( \tilde{F}_{\lambda, \mu} \) on the leaf space of this foliation is exactly \( F_{\lambda, \mu} \). The condition \( \mu \in (0, \sigma) \) guarantees that for every vector field \( X \) that is close to \( X_{\lambda, \mu} \), the first return map \( \tilde{F}_X \) to \( \Sigma^u \) for the flow of \( X \), will have an invariant strong stable foliation close to the one formed by the fibers of \( \Pi_\lambda \). When \( X \) is of class \( C^2 \) the corresponding leaf space transformation lies in the space \( \mathcal{F}_\lambda \), is area expanding and is close to \( F_{\lambda, \mu} \), and hence to \( F_\lambda \) (Lemma 7.3). So we will be able to apply the results for singular endomorphisms, described in Subsection 2.2, to study the flow of the vector fields \( X_{\lambda, \mu} \) and its perturbations.

Although \( \tilde{F}_{\lambda, \mu} \) depends on \( \beta > 0 \), for simplicity of the notation we omit it. In theorems 2.7 and 2.8 below we choose \( \beta \) to be conveniently small.

**Theorem 2.7.** Fix an integer \( n \geq 5 \) and \( \eta > \sigma > 0 \). Then for each \( \lambda \in (0, 1) \) sufficiently close to 1, there exists \( \mu_0 \in (0, \sigma) \) and a smooth one parameter family of smooth vector fields \( \{X_{\lambda, \mu} \mid \mu \in (\mu_0, \sigma)\} \) defined on an open set \( U \) of \( \mathbb{T}^n \), such that for all \( \mu \in (\mu_0, \sigma) \) the following hold:

1. The boundary of the open set \( U \subset \mathbb{T}^n \) is a manifold of dimension \( n - 1 \) that is contained in the interior of \( \mathbb{T}^n \). For every \( \mu \in [\mu_0, \sigma] \) the vector field \( X_{\lambda, \mu} \) extends to a smooth vector field defined on a neighborhood of the closure of \( U \), in such a way that on the boundary of \( U \) this vector field points inward. Moreover this extension has a unique singularity \( o = o_{\lambda, \mu} \). The singularity \( o \) is contained in \( U \) and is hyperbolic with eigenvalues \(-\mu, \sigma\) and \(-\eta\) of multiplicities 1, 2 and \( n - 3 \), respectively.
2. There exists a codimension 1 submanifold \( \Sigma^u \cong B_\lambda \times \mathbb{T} \times D^{n-5} \subset U \) transversal to the flow of \( X_{\lambda, \mu} \) so that every forward orbit of the flow of \( X_{\lambda, \mu} \) in \( U \) intersects \( \Sigma^u \) or is contained in a local stable manifold \( W^s_{loc}(o) \) of \( o \). The intersection of this local stable manifold with \( \Sigma^u \) is \( \{0\} \times \mathbb{T} \times D^{n-5} \).
3. The first return map to $\Sigma^u$ is given by
$$\tilde{F}_{\lambda, \mu} : B^*_\lambda \times \mathbb{R} \times D^{n-5} \to B_\lambda \times \mathbb{R} \times D^{n-5}.$$  

2.4. Dynamics of vector fields near $X_{\lambda, \mu}$. Part 1 of Theorem 2.7 implies that for every vector field $X$ that is sufficiently close to $X_{\lambda, \mu}$ and for every $t > 0$ we have $X^t(U) \subset U$. We denote by,
$$\Lambda_X = \bigcap_{t>0} X^t(U),$$
the maximal invariant set of the flow of $X$ in $U$. Moreover we denote by $o_X$ the singularity of $X$ that is the continuation of the hyperbolic singularity $o$ of $X_{\lambda, \mu}$. Clearly we have $o_X \in \Lambda_X$. We will show that for $\lambda \in (0, 1)$ sufficiently close to 1 and $\mu \in (\mu_0, \sigma)$ sufficiently close to $\sigma$, there is a $C^1$ neighborhood of $X_{\lambda, \mu}$ so that statements 1 through 3 of the Main Theorem hold for all vector fields $X$ in $O$. More precisely we prove the following result, which implies the Main Theorem.

**Theorem 2.8.** For each $\lambda \in (0, 1)$ sufficiently close to 1 and each $\mu \in (0, \sigma)$ sufficiently close to $\sigma$ there exist a $C^1$-neighborhood $O$ of $X_{\lambda, \mu}$, such that for each $X \in O$ the maximal invariant set $\Lambda_X$ of the flow of $X$ in $U$ satisfies the following properties.

1. If $X \in O$ is of class $C^2$, then the flow of $X$ restricted to $\Lambda_X$ is topologically mixing. Moreover, $o_X \in \Lambda_X$ and $o_X$ is the unique singularity of $X$ contained in $U$. In particular $\Lambda_X$ is a robustly transitive attractor set in the $C^2$ topology.
2. The attractor set $\Lambda_X$ contains an invariant hyperbolic set $W_X$ having a tangency between its stable and unstable foliations.
3. For vector fields in a residual subset of $O$, the sets of periodic orbits of Morse index 1, and Morse index 2 respectively, are both dense in $\Lambda_X$.

The hypothesis in part 1, that $X \in O$ is of class $C^2$, is unnecessary: the statement still holds for vector fields of class $C^1$. The proof of this fact is more involved and is done in detail in [BKR].

Observe that the singularity $o$ of $X_{\lambda, \mu}$ has a double real unstable eigenvalue $\sigma$. Thus, the singularity $o_X$ of vector fields near $X_{\lambda, \mu}$ may have real or complex unstable eigenvalues.

3. Domain of uniform hyperbolicity and the topology of the attractor

In this section we construct stable and unstable cone fields (subsections 3.1 and 3.2) which are invariant under $F_\lambda$ on certain domain $H_\lambda$, that we call the hyperbolicity domain (Subsection 3.3). After studying a saddle fixed point $p_\lambda$ of $F_\lambda$ in the hyperbolicity domain $H_\lambda$, we prove Proposition 2.3 in Subsection 3.4.

Notice that for every $a, b \in \mathbb{R}$ we have

$$D_{z_0} F_\lambda(z_0(a + ib)) = \left(\frac{z_0}{|z_0|}\right)^2 (a\lambda|z_0| + 2bi(1 - \lambda + \lambda|z_0|))$$

$$= (F_\lambda(z_0) - 1) \left(\frac{a\lambda|z_0|}{1 - \lambda + \lambda|z_0|} + 2bi\right).$$  

(2)
3.1. Stable cone field. Set \( \varepsilon_0 = \frac{1}{2} \sqrt{1 - \lambda} \) and let \( \{ C_z \}_{z \in \mathbb{C}} \) be the cone field defined by

\[
C_z = \{ \rho z (1 + i \varepsilon) \mid \rho \geq 0, |\varepsilon| \leq \varepsilon_0 \}.
\]

**Lemma 3.1.**

1. For every \( z_0 \in \mathbb{C} \) such that \( |z_0| \geq 1 \) and every \( v \in C_{z_0} \) we have

\[
|D_{z_0} F_{\lambda}(v)|^2 \leq \tilde{\lambda} \cdot |v|^2, \text{ where } \tilde{\lambda} = \frac{\lambda^2 + 4 \varepsilon_0^2}{1 + \varepsilon_0^2} \in (0, 1).
\]

2. For \( z_0 \in \mathbb{C} \) satisfying \( 1 - \lambda + \lambda |z_0| \geq 4/\sqrt{1 - \lambda} \) we have

\[
C_{F_{\lambda}(z_0)} \subseteq \text{interior}(D_{z_0} F_{\lambda}(C_{z_0})) \cup \{ 0 \}.
\]

**Proof.**

1. Letting \( v = z_0 (1 + i \varepsilon) \) with \( |\varepsilon| \leq \varepsilon_0 \), we have

\[
|D_{z_0} F_{\lambda}(v)|^2 = |\lambda |z_0| + 2 \varepsilon i (1 - \lambda + \lambda |z_0|)|^2 = (\lambda |z_0|)^2 + (2 \varepsilon (1 - \lambda + \lambda |z_0|))^2.
\]

As by hypothesis \( |z_0| \geq 1 \), we have

\[
|D_{z_0} F_{\lambda}(v)|^2 \leq |z_0|^2 (\lambda^2 + 4 \varepsilon^2) \leq |v|^2 \frac{\lambda^2 + 4 \varepsilon_0^2}{1 + \varepsilon_0^2} \leq |v|^2 \frac{\lambda^2 + 4 \varepsilon_0^2}{1 + \varepsilon_0^2}.
\]

2. Note that

\[
|F_{\lambda}(z_0) - 1| = 1 - \lambda + \lambda |z_0| \geq 4/\sqrt{1 - \lambda} = 2 \varepsilon_0^{-1}.
\]

On the other hand,

\[
D_{z_0} F_{\lambda}(C_{z_0}) = \{(F_{\lambda}(z_0) - 1)(a + ib) \in \mathbb{C} \mid |b/a| \leq 2 \varepsilon_0 (1 - \lambda + \lambda |z_0|)/(\lambda |z_0|)\}.
\]

Let \( \tilde{a}, \tilde{b} \in \mathbb{R} \) be so that \( |\tilde{b}/\tilde{a}| \leq \varepsilon_0 \). Then \( F_{\lambda}(z_0)(\tilde{a} + i \tilde{b}) \in C_{F_{\lambda}(z_0)} \). Define \( a, b \in \mathbb{R} \) by \( a + ib = F_{\lambda}(z_0)(F_{\lambda}(z_0) - 1)^{-1}(\tilde{a} + i \tilde{b}) \). We want to prove that

\[
|b/a| < 2 \varepsilon_0 \frac{1 - \lambda + \lambda |z_0|}{\lambda |z_0|}.
\]

Set \( F_{\lambda}(z_0)(F_{\lambda}(z_0) - 1)^{-1} = 1 + \eta_0 + i \eta_1 \) and notice that we have

\[
|\eta_0|, |\eta_1| \leq |F_{\lambda}(z_0) - 1|^{-1} \leq 2 \varepsilon_0 < \frac{1}{4},
\]

so \( |\eta_1/(1 + \eta_0)| \leq \frac{2}{3} \varepsilon_0 \). On the other hand, \( a = \tilde{a}(1 + \eta_0) - \eta_1 \tilde{b}, b = \tilde{b}(1 + \eta_0) + \eta_1 \tilde{a} \) and

\[
\frac{b}{a} = \frac{\tilde{b}/\tilde{a}(1 + \eta_0) + \eta_1}{1 + \eta_0 - \eta_1 b/\tilde{a}} = \frac{\tilde{b}/\tilde{a} + \eta_1/(1 + \eta_0)}{1 - (\eta_1/(1 + \eta_0))(b/\tilde{a})}.
\]

Therefore,

\[
|b/a| < \varepsilon_0 + \frac{2}{3} \varepsilon_0 < 2 \varepsilon_0 < \frac{2 \varepsilon_0 (1 - \lambda + \lambda |z_0|)}{\lambda |z_0|}.
\]

\( \square \)
3.2. Unstable cone fields. Let \( \{K(z)\}_{z \in C^*}, \{K^-(z)\}_{z \in C^*} \) and \( \{\tilde{K}(z)\}_{z \in C^*} \) be the cone fields defined by
\[
K(z_0) = \{ z_0 \rho (i + \varepsilon) \mid \rho \geq 0, \ |\varepsilon| \leq 1/3 \}, \\
K^-(z_0) = \{ z_0 \rho (i + \varepsilon) \mid \rho \geq 0, \ \varepsilon \in [-1/3, 0] \},
\]
Note that for every \( z \in C^* \) we have \( K(z) \subset \tilde{K}(z) \).

**Lemma 3.2** (Unstable Cone fields).

1. For every \( z_0 \in C^* \) and every \( v \in \tilde{K}(z_0) \) we have
\[
|D_{z_0} F_\lambda(v)| \geq \lambda (5/2)^{1/2} |v|.
\]
2. For every \( z_0 \in C^* \) such that \( 1 - \lambda + \lambda |z_0| \geq 20 \) we have
\[
D_{z_0} F_\lambda(K(z_0)) \subset \text{interior}(K(F_\lambda(z_0))) \cup \{0\} \quad \text{and}
\]
\[
K(F_\lambda(z_0)) \subset \text{interior}(D_{z_0} F_\lambda(\tilde{K}(z_0))) \cup \{0\}. \]
3. For every \( z_0 \in C^* \) such that \( 1 - \lambda + \lambda |z_0| \geq 20 \), \( \Re(z_0) \geq 0 \) and \( \Im(z_0) > 0 \), we have
\[
D_{z_0} F_\lambda(K^-(z_0)) \subset \text{interior}(K^-(F_\lambda(z_0))) \cup \{0\}.
\]

**Proof.**

1. Consider \( v = z_0(i + \varepsilon) \) with \( |\varepsilon| \leq 1 \), so that \( v \in \tilde{K}(z_0) \). Setting \( a = \rho \varepsilon \) and \( b = \rho \) in [2], we have
\[
|D_{z_0} F_\lambda(v)|^2 = (\rho |z_0|)^2 |\lambda \varepsilon + 2i(1 - \lambda) |z_0| + \lambda)|^2 \geq |v|^2 \lambda^2 \varepsilon^2 + 4 \varepsilon^2 + 1.
\]

2. Note that \( |F_\lambda(z_0) - 1| = 1 - \lambda + \lambda |z_0| \geq 20 \), so \( |F_\lambda(z_0)| \geq 19 \). Given \( \rho, \varepsilon \in \mathbb{R} \) such that \( \rho > 0, |\varepsilon| \leq 1/3 \), define \( \rho' \geq 0 \) and \( \varepsilon' \in \mathbb{R} \) by
\[
D_{z_0} F_\lambda(\rho \varepsilon_0(i + \varepsilon)) = F_\lambda(z_0) \rho'(i + \varepsilon').
\]
It is enough to prove that \( |\varepsilon' - \varepsilon|/2 < 1/6 \).

Define \( \rho'', \rho_0 \geq 0 \) and \( \varepsilon'', \varepsilon_0 \in \mathbb{R} \) by
\[
D_{z_0} F_\lambda(\rho \varepsilon_0(i + \varepsilon)) = (F_\lambda(z_0) - 1) \rho''(i + \varepsilon''),
\]
and \( F_\lambda(z_0) - 1 = F_\lambda(z_0) \rho_0(1 + i \varepsilon_0) \).

Then \( \varepsilon' = \varepsilon'' - \varepsilon_{0}^z \). From [2] we have \( \varepsilon'' = (\varepsilon/2) \lambda |z_0|(1 - \lambda + \lambda |z_0|)^{-1} \). So \( |\varepsilon''| < |\varepsilon/2| \leq 1/2 \) and
\[
|\varepsilon'' - \varepsilon/2| < |\varepsilon/2|(1 - \lambda + \lambda |z_0|)^{-1} \leq 1/40.
\]
On the other hand \( |\rho_0(1 + i \varepsilon_0) - 1| = |F_\lambda(z_0)|^{-1} \leq 1/19 \). Hence \( |\rho_0 - 1|, |\rho_0 \varepsilon_0| \leq 1/19 \) and \( |\varepsilon_0| \leq 1/18 \). Since \( |\varepsilon''| < 1/2 \), we have
\[
|\varepsilon' - \varepsilon''| = \left| \varepsilon_0 - \frac{(\varepsilon'')^2}{1 + \varepsilon'' \varepsilon_0} \right| \leq 2|\varepsilon_0| \leq 1/9.
\]
Therefore \( |\varepsilon' - \varepsilon/2| \leq 1/40 + 1/9 < 1/6 \).

3. Let \( z_0 \in C^* \) be such that \( 1 - \lambda + \lambda |z_0| \geq 20 \) and let \( \rho' \geq 0 \) and \( \varepsilon' \in \mathbb{R} \) be such that
\[
D_{z_0} F_\lambda(i z_0) = F_\lambda(z_0) \rho'(i + \varepsilon').
\]
By part 2 is enough to prove that when \( \Re z_0 \geq 0 \) and \( \Im z_0 > 0 \) we have \( \varepsilon' < 0 \). Note that when \( \Re z_0 \geq 0 \) and \( \Im z_0 > 0 \) we have \( \Im F_\lambda(z_0) > 0 \).
denoted by \( \gamma \).

Therefore the image of \(-\) is larger than the original length by a definite factor. Moreover, if the image \( H \) of an arc in \( \mathcal{F} \) is smaller than the original length by a definite factor.

and \( \rho \) of \( F \).

it follows that for every \( \theta \) contained in the domain of hyperbolicity \( p \) of \( \lambda \) manifold of a quasi-angular arc while it remains in \( H \).

We will denote by \( D \) of \( \gamma \) will be called the domain of hyperbolicity.

For every map \( F \) contained in \( \mathcal{F} \), the cone field \( \mathcal{C} \) (resp. \( \mathcal{K} \)) is an stable (resp. unstable) and invariant cone field of \( F : H \rightarrow F(H) \). For such \( F \), the image of a quasi-angular arc contained in \( H \) is a quasi-angular arc whose length is larger than the original length by a definite factor. Moreover, if the image of an arc in \( H \) is quasi-radial, then the original arc is quasi-radial and the length of the image is smaller than the original length by a definite factor.

3.4. The saddle fixed point \( p \) and the proof of Proposition 2.3. For \( \lambda \in (0,1) \) sufficiently close to 1, the saddle fixed point \( p_\lambda = (1 - \lambda)^{-1} + 1 \) of \( F_\lambda \) is contained in the domain of hyperbolicity \( H_\lambda \). So the unstable manifold of \( p_\lambda \) is a quasi-angular arc while it remains in \( H_\lambda \). As the unstable cone field is given by \( \mathcal{K} = \{ z \rho(i + \varepsilon) : \rho \geq 0, |\varepsilon| \leq 1/3 \} \), we see that a piece of the unstable manifold of \( p_\lambda \) is a quasi-angular arc parameterized by \( \theta \mapsto \rho_\lambda(\theta) \exp(i\theta) \), with \( \rho_\lambda(0) = p_\lambda \) and \( \rho_\lambda(\theta) \geq p_\lambda \exp(-\theta/3) \). This remains valid while

\[
p_\lambda \exp(-\theta/3) \geq \lambda^{-1} \left( \lambda - 1 + 4/\sqrt{1-\lambda} \right).
\]

From now on we assume that \( \lambda \in (0,1) \) is sufficiently close to 1, so that the function \( \rho_\lambda \) is defined on the interval \([-\pi, \pi]\).

By part 3 of Lemma 3.2, for every \( \theta \in (0,\pi] \) we have \( \rho_\lambda'(\theta) < 0 \). By symmetry it follows that for every \( \theta \in [-\pi, 0) \) we have that \( \rho_\lambda'(\theta) > 0 \) and that \( \rho_\lambda'(0) = 0 \).

Therefore the image of \([-\pi, \pi] \) by the map \( \theta \mapsto \rho_\lambda(\theta) \exp(i\theta) \) forms a Jordan curve, denoted by \( \gamma_\lambda^+ \), that is contained in the unstable manifold of \( p_\lambda \). So \( \gamma_\lambda^+ \) is smooth, except at \( \theta = \pm\pi \), which is a point of transversal self-intersection of the unstable manifold of \( p_\lambda \).

Note that every map \( F \) in \( \mathcal{F} \) that is sufficiently close to \( F_\lambda \), has a saddle fixed point \( p_F \) that is the continuation of \( p_\lambda \). Moreover, a piece of the unstable manifold of \( p_F \) forms a Jordan curve \( \gamma_\lambda^+ \) that is smooth, except at a point of transversal self-intersection. We will denote by \( D_F \) the closed disk in \( \mathbb{C} \) bounded by the Jordan curve \( \gamma_\lambda^+ \).

Recall that the maximal invariant set \( \Omega_F \) of \( F \) is defined by \( \Omega_F = \cap_{m \geq 1} F^m(B_\lambda) \). In the proof of Proposition 2.3 below, we will show that \( \Omega_F = D_F \cap F(B_\lambda^+) \). We say that \( \gamma_\lambda^+ \) is the external boundary of \( \Omega_F \).
Proof of Proposition 2.3. In part 1 we prove that the the conclusions of the proposition hold for the set $D_F \cap F(B_\lambda)$ and in part 2 we show that $\Omega_F = D_F \cap F(B_\lambda)$.

1. For $F \in \mathcal{F}_\lambda$ sufficiently close to $F_\lambda$ the set $D_F \cap F(B_\lambda)$ clearly contains $z = 0$ and $\gamma_F^+$. So we just need to prove that the set $D_F \cap F(B_\lambda)$ is homeomorphic to an annulus that is bounded by $\gamma_F^+$ and by a Jordan curve $\gamma_F^-$, and is disjoint from $\gamma_F^+$.

Fix $\epsilon \in (0,1)$, put $\gamma^{-} = \{|z-1| = 1 - \lambda + \lambda \epsilon\}$ and let $C_F \subset \subset \mathbb{C}$ be the closed annulus bounded by $\gamma_F^+$ and $\gamma^{-}$. When $F = F_\lambda$, the set $C_F$ is contained in the interior of $F_\lambda(B_\lambda)$. As $F_\lambda$ is a 2 to 1 covering between $B_\lambda^1$ and $F_\lambda(B_\lambda)$, it follows that if $F \in \mathcal{F}_\lambda$ is sufficiently close to $F_\lambda$, then $F$ is a 2 to 1 covering map between $\tilde{C}_F = F^{-1}(C_F)$ and $C_F$. In particular the sets $\gamma_F^+ = F^{-1}(\gamma_F^+)$ and $\gamma_F^- = F^{-1}(\gamma^-)$ are Jordan curves.

Denote by $\tilde{D}_F$ the disc bounded by $\gamma_F^-$. Let $F \in \mathcal{F}_\lambda$ be close to $F_\lambda$ and put $F = g \circ \tau$, with $g \in \mathcal{G}_\lambda$ close to $g_1$ and $\tau \in \mathcal{F}_\lambda$ close to $\tau_\lambda$. Note that $F(B_\lambda) = g(R/Z \times (1 - \lambda, 2(1 - \lambda)^{-1} - 1 - \lambda))$. As the space $\mathcal{G}_\lambda$ is endowed with the strong $C^1$ topology, it follows that for every $F \in \mathcal{F}_\lambda$ sufficiently close to $F_\lambda$ there is a $C^1$ map $\rho_F : \mathbb{R}/Z \to \mathbb{R}$ that is close to the constant map $\theta \mapsto 1 - \lambda$ and such that

$$g(R/Z \times \{1 - \lambda\}) = \{\theta \in \mathbb{R}/Z \mid 1 + \rho_F(\theta) \exp(4\pi i \theta)\}.$$ 

Taking $F$ closer to $F_\lambda$ if necessary we have

$$g(\tilde{D}_F) = \{1 + t \exp(4\pi i \theta) \mid \theta \in \mathbb{R}/Z, \rho_F(\theta) \leq t \leq 1 - \lambda + \lambda \epsilon\} = \{1 + t \exp(2\pi i \psi) \mid \psi \in \mathbb{R}/Z, \min\{|\rho_F(\psi/2)|, |\rho_F((\psi + 1)/2)|\} \leq t \leq 1 - \lambda + \lambda \epsilon\}.$$

As $C_F \subset F(B_\lambda)$, it follows that $D_F \cap F(B_\lambda)$ is the annulus bounded by $\gamma_F^+$ and by the Jordan curve

$$\gamma_F^- = \{1 + \min\{|\rho_F(\psi/2)|, |\rho_F((\psi + 1)/2)|\} \exp(2\pi i \psi) \mid \psi \in \mathbb{R}/Z\}.$$ 

Clearly $D_F \cap F(B_\lambda)$ is disjoint form $\gamma_F^-$. 

2. To prove that $\Omega_F = D_F \cap F(B_\lambda)$ we first prove that the set $D_F \cap F(B_\lambda)$ is invariant by $F$ by showing in part 2.1 that $F(D_F) = D_F \cap F(B_\lambda)$ and in part 2.2 that $F(D_F \cap F(B_\lambda)) = F(D_F)$. This implies that $D_F \cap F(B_\lambda) \subset \Omega_F$. Then we complete the proof by showing in part 2.3 that $\Omega_F \subset D_F \cap F(B_\lambda)$.

2.1. We keep the notation of part 1. Denote by $\tilde{D}_F$ the disc bounded by $\gamma_F^+$. As $\gamma_F^+ = F^{-1}(\gamma_F^+)$ and $F : \gamma_F^+ \to \gamma_F^+$ is a 2 to 1 covering map, we have $F(\tilde{D}_F) = D_F \cap F(B_\lambda)$.

We will prove now that $D_F \subset \tilde{D}_F$. Note that $\ell_F^+ = \gamma_F^+ \cap \gamma_F^+$ is a $C^1$ arc whose image by $F$ is equal to $\gamma_F^-$. Moreover, the closure of $\tilde{\gamma}_F^- \setminus \ell_F^+$ is a $C^1$ arc $\gamma_F^-$ having the same end points as $\ell_F^+$, where these arcs intersect transversally. When $F = F_\lambda$ we have $\ell_{F_\lambda}^- = -\ell_F^+$, and since $\gamma_F^+$ is tangent to the cone field $K^-$, it follows that $\gamma_F^+ \setminus \ell_F^+$ is contained in the interior of $\tilde{C}_F$. Now, for $F \in \mathcal{F}_\lambda$ close to $F_\lambda$ the arc $\ell_F^+$ intersects $\gamma_F^-$ at its 2 extreme points in a transversal way. Thus, taking $F$ closer to $F_\lambda$ if necessary, we have that the arc $\gamma_F^+ \setminus \ell_F^+$ is contained in the interior of $\tilde{D}_F$. Thus $D_F \subset \tilde{D}_F$ and $F(D_F) \subset D_F \cap F(B_\lambda)$.

To prove that $F(D_F) = D_F \cap F(B_\lambda)$ we just need to prove that $F(D_F) = F(\tilde{D}_F)$. Put $E_F = \tilde{D}_F \setminus D_F$ and note that for every $F$ sufficiently close to $F_\lambda$, we have $E_F \subset \tilde{C}_F$. As $F$ is a 2 to 1 covering map between $\tilde{C}_F$ and $C_F$, to prove that $F(D_F) = F(\tilde{D}_F)$ is enough to prove that $F$ is injective on $E_F$. Let $\bar{q}_F$ and $\bar{g}_F$ be
the end points of ℓ^+_F. Note that these points are mapped to the unique point q_F in γ^+_F where the unstable manifold of p_F has a transversal self-intersection. When F = F_λ the set \( \overline{E_F} \setminus \{ q_F, \bar{q}_F \} \) is contained in the left half plane, where F_λ is injective. So, if F is sufficiently close to F_λ it follows that F is locally injective on E_F. To prove that F is injective on E_F is enough to prove that a point in E_F near q_F and a point in E_F near \( \bar{q}_F \) cannot have the same image. A piece of the unstable manifold of p_F slightly larger than \( \gamma^+_F \), locally separates the plane at q_F into four regions. The points in E_F near q_F and \( \bar{q}_F \) are mapped to opposite regions and are therefore disjoint. Thus it follows that F is injective on E_F and this completes the proof that \( F(D_F) = D_F \cap F(B_\lambda) \).

2.2. We prove now that \( F(D_F \cap F(B_\lambda)) = F(D_F) \). As for every,

\[
D_\lambda \setminus F_\lambda(B_\lambda^+) = \{ z \in \mathbb{C} \mid |z - 1| \leq 1 - \lambda \},
\]

have \( -z \in F_\lambda(B_\lambda^+) \), it follows that for every \( z \in D_F \setminus F(B_\lambda^+) \) there is \( z' \in D_F \cap F(B_\lambda^+) \) close to \( -z \) such that \( F(z') = F(z) \). Thus \( F(D_F) = F(D_F \cap F(B_\lambda)) \).

2.3. We now prove that \( \Omega_F \subset D_F \cap F(B_\lambda) \). As \( \Omega_F \subset F(B_\lambda) \) by definition, we just need to prove that \( \Omega_F \subset D_F \). Since \( \gamma^+_F \) is a quasi-angular Jordan curve, it follows that there exists a constant \( C > 0 \) such that for every \( z \in B_\lambda \setminus D_F \) the length \( L \) of each quasi-radial arc joining \( z \) to \( \gamma^+_F \) satisfies \( \text{dist}(z, D_F) \leq L \leq C \text{dist}(z, D_F) \).

For a given integer \( n \geq 1 \), let \( z_0, \ldots, z_n \in B_\lambda \setminus D_F \) be such that for every \( j = 0, \ldots, n - 1 \) we have \( F(z_j) = z_{j+1} \). Let \( \ell_j \) be the radial arc joining \( z_j \) to \( \gamma^+_F \) and such that \( \ell_j \cap F(\ell_j) \). It follows that the length of \( \ell_j \) is exponentially small with \( j \). This proves that \( F^n(B_\lambda) \) is contained in an exponentially small neighborhood of \( D_F \) and that \( \Omega_F \subset D_F \).

4. Fundamental annulus and local unstable manifolds

In this section we construct, for every \( \lambda \in (0,1) \) sufficiently close to 1 and every \( F \) in \( \mathcal{F}_\lambda \) sufficiently close to \( F_\lambda \), an annulus \( A_F \) such that \( A_F \subset F(A_F) \) (Subsection 4.2) and such that exists a positive integer \( N \) for which \( F^N(A_F) = \Omega_F \) (Subsection 4.3). The annulus \( A_F \) is defined in Subsection 4.2. From the property \( A_F \subset F(A_F) \) we deduce that for every point \( z_0 \in A_F \) there is an infinite backward orbit of \( F \) in \( A_F \) that starts at \( z_0 \) and that each one of these backward orbits has associated a local unstable manifold of definite size (Subsection 4.3).

4.1. Fundamental annulus. Fix \( r \in (0, \exp(-\pi/3)) \) and \( \lambda \in (0,1) \) close to 1. Then put

\[
U_+ = \{ z \in \mathbb{C} \mid \Re z \geq 0, \ |z| \geq r(1 - \lambda)^{-1} \}.
\]

\[
U_- = \{ z \in \mathbb{C} \mid \Re z \leq 0, \ |z - 1| \geq r(1 - \lambda)^{-1} \}
\]

and \( U_0 = U_+ \cup U_- \). Notice that the boundary \( \gamma_0 \) of \( U_0 \) is a Jordan curve. As the Jordan curve \( \gamma_0^+ \) is contained in \( \{ z \in \mathbb{C} \mid |z| \geq \exp(-\pi/3) \ (1 - \lambda)^{-1} + 1 \} \) (Subsection 5.1), it follows that for every \( F \) in \( \mathcal{F}_\lambda \) sufficiently close to \( F_\lambda \), the Jordan curve \( \gamma_0 \) is contained in the disc bounded by \( \gamma_0^+ \). Let \( A_F \) be the annulus bounded by \( \gamma_0^+ \) and \( \gamma_0^+ \), that contains \( \gamma_0^+ \) and is disjoint from \( \gamma_0 \). The annulus \( A_F \) will be called the fundamental annulus of \( F \).
As $F_\lambda(\gamma^+_{\lambda})$ is contained in $\{z \in \mathbb{C} \mid |z| \geq \lambda \exp(-\pi/3) \left((1-\lambda)^{-1} + 1 \right) - \lambda \}$, it follows that if $\lambda \in (0,1)$ is close enough to 1 and $F \in \mathcal{F}_\lambda$ is close enough to $F_\lambda$, then

$$F(\gamma^+_{\lambda}) \subset \{z \in \mathbb{C} \mid |z| > r(1-\lambda)^{-1}\},$$

and that $A_F$ and $F(A_F)$ are both contained in the domain of hyperbolicity $H_\lambda$.

4.2. Self-covering property of the fundamental annulus. The purpose of this subsection is to prove the following proposition.

**Proposition 4.1.** For $\lambda \in (0,1)$ sufficiently close to 1 and for $F \in \mathcal{F}_\lambda$ sufficiently close to $F_\lambda$, the annulus $A_F$ satisfies $A_F \subset F(A_F)$.

As $\gamma^+_{\lambda} \subset F(\gamma^+_{\lambda}) \subset D_F$ and by the set $F_\lambda(\gamma^+_{\lambda})$ is outside the disc bounded by $\gamma_0$, the proposition is an immediate consequence of the following lemma.

**Lemma 4.2.** The closed set $U_0$ is contained in the interior of $F_\lambda(U_0)$.

**Proof.** As

$$F_\lambda(U_+) = \{w \in \mathbb{C} \mid |w-1| \geq 1 - \lambda + \lambda r(1-\lambda)^{-1}\},$$

and $1 - \lambda + \lambda r(1-\lambda)^{-1} < r(1-\lambda)^{-1}$, it follows that $U_-$ is contained in the interior of $F_\lambda(U_+)$. So it is enough to show that $U_+$ is contained in the interior of $F_\lambda(U_-)$.

Note that the image by $F_\lambda$ of the quasi-angular arc $\{z \in \mathbb{C} \mid \Re z \leq 0, |z - 1| = r(1-\lambda)^{-1}\}$ is a quasi-angular Jordan curve. So it is enough to prove that for every $z_0 \in \mathbb{C}$ such that $|z_0 - 1| = r(1-\lambda)^{-1}$ and $\arg(z_0) \in [\pi/2, 3\pi/2]$, we have $|F_\lambda(z_0)| < r(1-\lambda)^{-1}$. For such $z_0$ put $\rho = |z_0|$ and $\alpha = \arg(z_0)$. Thus,

$$\tilde{\rho} = |F_\lambda(z_0) - 1| = 1 - \lambda + \lambda \rho \quad \text{and} \quad \arg(F_\lambda(z_0) - 1) = 2\alpha \mod 2\pi.$$

We have,

$$(r(1-\lambda)^{-1})^2 = 1 + \rho^2 - 2\rho \cos \alpha,$$

$$|F_\lambda(z_0)|^2 = 1 + \tilde{\rho}^2 - 2\tilde{\rho} \cos(\pi - 2\alpha) = 1 + \rho^2 + 2\rho \cos 2\alpha.$$

As $\rho \geq r(1-\lambda)^{-1} - 1 > 1$, we have $\tilde{\rho} < \rho$ and $|F_\lambda(z_0)|^2 < 1 + \rho^2 + 2\rho \cos 2\alpha$. Hence

$$|F_\lambda(z_0)|^2 < (r(1-\lambda)^{-1})^2 + 2\rho(\cos 2\alpha + \cos \alpha).$$

Since $\alpha \in [\pi/2, 3\pi/2]$ we have $\cos \alpha \leq 0$, so

$$\cos 2\alpha + \cos \alpha = (2 \cos \alpha - 1)(\cos \alpha + 1) \leq 0.$$

It follows that $|F_\lambda(z_0)| < r(1 - \lambda)^{-1}$, as wanted. $\square$

4.3. Covering property of the fundamental annulus. The purpose of this subsection is to prove the following proposition.

**Proposition 4.3.** For every $\lambda \in (0,1)$ sufficiently close to 1 and every $F$ in $\mathcal{F}_\lambda$ sufficiently close to $F_\lambda$ there is a positive integer $N$ such that $F^N(A_F) = \Omega_F$.

The proof of this proposition depends on some lemmas.

**Lemma 4.4.** Assume that $\lambda \in (0,1)$ is sufficiently close to 1 and that $F$ in $\mathcal{F}_\lambda$ is sufficiently close to $F_\lambda$. Then there is $N \geq 1$ such that for every $z_0 \in F(B^*_\lambda)$ there exists a positive integer $m_0 \leq N$ and a backward orbit $z_0, z_1, \ldots, z_{m_0}$ (i.e. for $j = 1, \ldots, m_0$ we have $F(z_j) = z_{j-1}$), such that $z_{m_0} \in B_\lambda \setminus F(B^*_\lambda)$. 
We will prove this lemma for $F = F_\lambda$. The case when $F$ is sufficiently close to $F_\lambda$ is an easy consequence.

Let $Q^+ = \{ z \in \mathbb{C} | \Re z \leq 0 \text{ and } 3z \geq 0 \}$ and $Q^- = \{ z \in \mathbb{C} | \Re z \leq 0 \text{ and } 3z \leq 0 \}$. As $F_\lambda((Q^+ \cup Q^-) \setminus \{ 0 \}) = F_\lambda(\mathbb{C} \setminus \{ 0 \})$, it follows that for every $z_0 \in F_\lambda(\mathbb{C} \setminus \{ 0 \})$ there is a pre-image $z_1$ of $z_0$ by $F_\lambda$ in $Q^+ \setminus \{ 0 \}$ or in $Q^- \setminus \{ 0 \}$. Moreover, note that $Q^+ \subseteq F_\lambda(Q^+ \setminus \{ 0 \})$ and $Q^- \subseteq F_\lambda(Q^- \setminus \{ 0 \})$. Thus the previous lemma is an immediate consequence of the following one.

Lemma 4.5. Let $z' \in Q^+ \setminus \{ 0 \}$ be such that $z = F_\lambda(z') \in Q^-$. Then the following properties hold.
1. $|z' - 1| \geq \sqrt{2}$.
2. $|z' - 1| - 1 > \lambda^{-1}(|z - 1| - 1)$.

Proof. 1. If $|z'| \in (0, 1)$, then $\Re(F_\lambda(z')) > 0$ so $F_\lambda(z') \notin Q^-$. But $|z' - 1| < \sqrt{2}$ implies $|z'| < 1$ (since $\Re z' \leq 0$), which is not possible by the previous observation.
2. Set $|z - 1| = 1 + \mu$, where $\mu \geq 0$. Since $1 + \mu = |z - 1| = 1 - \lambda + |z'|$, we have $|z'| = 1 + \lambda^{-1}\mu$. Then observe that, since $\Re z' \leq 0$, we have $|z' - 1| > |z'|$.

Proof of Proposition 4.3. Let $N$ be the integer given by Lemma 4.4 and for a given $z_0 \in \Omega \cap F(B_\lambda)$ let $m_0 \leq N$ and $z_1, \ldots, z_m \in B_\lambda$ be as in this lemma. Thus there is $m \in \{ 1, \ldots, m_0 \}$ such that $z_m \notin D_F$ and $F(z_m) = z_{m-1} \in D_F$. Then 3 implies that $z_{m-1} \in A_F$, and therefore we have $z_0 \in F^{m-1}(\Omega_F)$.

As $A_F \subset F(A_F)$ and $m \geq N$, it follows that $\Omega_F \subset F^N(A_F)$.

4.4. Local unstable manifolds. Assume that $\lambda \in (0, 1)$ is close enough to 1, so that the conclusions of Lemma 5.2 hold for every point $z_0$ in $H_\lambda$. It follows that for $F$ sufficiently close to $F_\lambda$, the set

$$
\Gamma_F = \{ \{ z_i \}_{i \geq 0} | z_i \in H_\lambda \text{ and } F(z_{i+1}) = z_i, \text{ for } i \geq 0 \}
$$

is non empty and that for every $z_0 \in A_F$ there is an infinite backward orbit in $\Gamma_F$ starting from $z_0$. From the theory of hyperbolic sets we know that there exists $\alpha > 0$ such that for every infinite backward orbit $\underline{z}$ in $\Gamma_F$ starting from a point $z_0 \in A_F$, the set

$$
W^u_\alpha(\underline{z}) = \{ w_0 \in H_\lambda | w_0 \text{ has a backward orbit } w \in \Gamma_F \}
$$

with $|w_m - z_m| < \alpha$ for all $m \geq 0$

is a quasi-angular arc. The set $W^u_\alpha(\underline{z})$ will be called the local unstable manifold of $\underline{z}$. Moreover, for every $\varepsilon > 0$ there exists $\delta > 0$ and a positive integer $N$ such that, if $z'$ is an infinite backward orbit in $\Gamma_F$ such that for every $0 \leq m \leq N$ we have $|z'_m - z_m| < \delta$, then the $C^1$ distance between the local unstable manifolds $W^u_\alpha(\underline{z})$ and $W^u_\alpha(\underline{z}')$ is at most $\varepsilon$.

5. Topological dynamics on the attractor

In this section we prove Theorem 2.4 (Subsection 5.3), by showing that for every $\lambda \in (0, 1)$ sufficiently close to 1 and every area expanding map $F$ in $\mathcal{F}_\lambda$ sufficiently close to $F_\lambda$, the stable manifold of the saddle fixed point $p_F$ is dense in $B_\lambda$ (Subsection 5.1) and that the unstable manifold of $p_F$ is dense in $\Omega_F$ (Subsection 5.2).
Throughout this section we fix $\lambda \in (0, 1)$ close to 1 and an area expanding map $F$ in $\mathcal{F}_\lambda$ close to $F_\lambda$.

5.1. Density of the stable manifold of $p_F$. Consider a connected open subset $V$ of $B_\lambda$. Suppose for a moment that there is a positive integer $m$ such that for every $k = 1, \ldots, m$ the map $F$ is injective on $F^k(V)$. As $F$ is area expanding, it follows that the area of $F^k(V)$ grows exponentially fast with $k$, as long as $k \leq m$. We conclude that there is an integer $m$ for which $F$ is not injective on $F^m(V)$. The following lemma implies that in this case $F^m(V)$, and hence $V$, intersects the stable manifold of $p_F$. From this it follows that the stable manifold of $p_F$ is dense in $B_\lambda$.

**Lemma 5.1.** Every connected open subset of $B_\lambda$ where $F$ is not injective intersects the stable manifold of $p_F$.

**Proof.** Recall that $F_\lambda = g_1 \circ \tau_\lambda$, where $\tau_\lambda \in \mathcal{T}_\lambda$ and $g_1 \in \mathcal{G}_\lambda$ are defined in Subsection 2.1. Note that $F_\lambda$ maps the interval $(0, 2(1 - \lambda)^{-1} + 1) \subset \mathbb{R} \subset \mathbb{C}$ onto $(2 - \lambda, 2(1 - \lambda)^{-1})$. It follows that $(0, 2(1 - \lambda)^{-1} + 1)$ is contained in the stable manifold of $p_F$. On the other hand, note that the preimage of the arc $(1, 2(1 - \lambda)^{-1})$ by $g_1$ consists of the arcs $\{0\} \times I$ and $\{1/2\} \times I$, where $I \subset \mathbb{R}$ is the interval $I = (0, 2(1 - \lambda)^{-1} - 1)$, and that each one of these arcs intersects transversally $\mathbb{R}/\mathbb{Z} \times \{1 - \lambda\}$ and $\mathbb{R}/\mathbb{Z} \times \{2(1 - \lambda)^{-1} - 1 - \lambda\}$. Moreover, these arcs separate $\mathbb{R}/\mathbb{Z} \times \{1 - \lambda, 2(1 - \lambda) - 1 - \lambda\}$ into two sets where $g_1$ is injective.

Given $F \in \mathcal{F}_\lambda$ close to $F_\lambda$, let $\tau \in \mathcal{T}_\lambda$ be close to $\tau_\lambda$ and $g \in \mathcal{G}_\lambda$ be close to $g_1$, such that $F = g \circ \tau$. Denote by $\gamma^s$ an arc of the stable manifold of $p_F$ that is $C^1$ close to $(1, 2(1 - \lambda)^{-1})$. Since $g$ is $C^1$ close to $g_1$ in the $C^1$ topology, the preimage of $\gamma^s$ by $g$ contains 2 arcs $\gamma$ and $\gamma'$, that intersect transversally the circles $\mathbb{R}/\mathbb{Z} \times \{1 - \lambda\}$ and $\mathbb{R}/\mathbb{Z} \times \{2(1 - \lambda)^{-1} - 1 - \lambda\}$. Moreover, these arcs separate the cylinder $\mathbb{R}/\mathbb{Z} \times \{1 - \lambda, 2(1 - \lambda)^{-1} - 1 - \lambda\}$ into two parts, on each of which $g$ is injective.

It follows that $\tau^{-1}(\gamma \cup \gamma') = F^{-1}(\gamma^s)$ divides $B_\lambda^s$ into two parts, on each of which $F$ is injective. So every open and connected subset of $B_\lambda$ where $F_\lambda$ is not injective must intersect $F^{-1}(\gamma^s)$. As this last set is contained in the stable manifold of $p_F$, the statement follows.

5.2. Density of the unstable manifold of $p_F$. Together with Proposition 1.8 the following lemma implies that the unstable manifold of $p_F$ is dense in $\Omega_F$. In fact it is easy to see that the local unstable manifold of each infinite backward orbit in $\Gamma_F$ is approximated in the $C^1$ topology by arcs of the unstable manifold of $p_F$.

**Lemma 5.2.** Every quasi-radial arc contained in $A_F$ intersects transversally the unstable manifold of $p_F$ at some point.

**Proof.** Given a quasi-radial arc $\gamma_0$ contained in $A_F$, choose $z_0 \in \gamma_0$ and let $\underline{z} = \{z_m\}_{m \geq 0} \in \Gamma_F$ be an infinite backward orbit of points in $A_F$, starting at $z_0$. For $j \geq 1$ define inductively a quasi-radial arc $\gamma_j$ joining $z_j$ to some point in $\gamma_0^+$ and such that $F(\ell_j)$ is contained in $\ell_{j-1}$. As for every $j \geq 0$ the point $z_j$ belongs to $A_F$, the length of $\ell_j$ is bounded independently of $j$. It follows that the length of the arc $F^j(\ell_j)$ is exponentially small with $j$. Thus $F^j(\gamma_j \cap \gamma_0^+)$ is a point of the unstable manifold of $p_F$ that is exponentially close to $z_0$. \[\square\]
5.3. Proof of Theorem 2.4. Let $U, V$ be open subsets of $B_\lambda$ that intersect the attractor set $\Omega_F$ of $F$. We now prove that there exists a positive integer $m_0$ such that for every $m \geq m_0$ we have $F^m(U) \cap V \neq \emptyset$.

As the unstable manifold of $p_F$ is dense in $\Omega_F$, there exists a small open set $V_0$ and a positive integer $j_0$ such that $V_0$ intersects the local unstable manifold of $p_F$ and such that $F^{j_0}(V_0) \subset V$. As the stable manifold of $p_F$ is dense in $\Omega_F$, there exists a positive integer $k_0$ such that $F^{k_0}(U) \cap V_0 \neq \emptyset$.

Taking $m_0 = j_0 + k_0$ we see that for every $m \geq m_0$ we have $F^m(U) \cap V \neq \emptyset$. This proves Theorem 2.4. \qed

6. Eventually onto property and periodic points

This section is devoted to the proof that the set of periodic sources and the set of periodic saddles of a generic area expanding map $F$ in $\mathcal{F}_\Lambda$ that is close to $F_\lambda$, are both dense in $\Omega_F$ (Proposition 2.5 in Subsection 2.2). The proof is based on the following intermediate property.

**Definition 6.1.** We say that $F \in \mathcal{F}_\Lambda$ has the eventually onto property if for every open subset $V$ of $\Omega_F$

\[
\text{interior}(\Omega_F) \subset \bigcup_{m \geq 0} F^m(V).
\]

We first prove that the conclusions of Proposition 2.5 hold for every map $F$ close to $F_\lambda$ with the eventually onto property and satisfying a certain genericity condition (Proposition 6.2 in Subsection 6.2). This genericity condition is expressed in terms of the eigenvalues of the derivative of $F$ at a fixed source $p_F^+$ of $F$ (see Subsection 6.3 below).

We then show that for every $F$ in $\mathcal{F}_\Lambda$ near $F_\lambda$, the eventually onto property holds for all open sets $V$ containing $p_F^+$ (Proposition 6.4 in Subsection 6.3) and we complete the proof of Proposition 2.5 in Subsection 6.3 by showing that the eventually onto property holds for a generic area expanding map near $F_\lambda$.

6.1. Limit behavior and the fixed source $p_F^+$. When $\lambda \to 1$ the map $F_\lambda$ converges in the $C^\infty$ topology to the map $G_\uparrow : \mathbb{C}^* \to \mathbb{C}$ defined by,

\[
G_\uparrow(z) = |z|^2 + 1.
\]

The map $G_\uparrow$ extends continuously to $z = 0$ by setting $G_\uparrow(0) = 1$. Moreover, note that $G_\uparrow$ has constant Jacobian equal to $2$, and that $G_\uparrow$ is injective on the upper half plane $\mathbb{H} = \{3z > 0\}$.

It is easy to see that the point $p^+ = \exp(\pi i/3)$ is the unique fixed point of $G_\uparrow$ in the upper half plane. Moreover, the derivative of $G_\uparrow$ at $p^+$ has complex eigenvalues (not in $\mathbb{R}$) of modulus larger than $1$, so that $p^+$ is a hyperbolic source.

It follows that for $\lambda \in (0, 1)$ sufficiently close to $1$ and $F$ sufficiently close to $F_\lambda$, the map $F$ has a (unique) fixed source $p_F^+$ near $p^+$. Moreover the derivative of $F$ at $p_F^+$ has complex eigenvalues (not in $\mathbb{R}$).
6.2. Dynamics of maps with the eventually onto property. This subsection is devoted to the proof of the following proposition. Recall that the homoclinic class of a saddle periodic point is, by definition, the closure of the set formed by the transversal intersections between the stable and unstable manifolds of the saddle periodic point. Every point in a homoclinic class is accumulated by saddle periodic points, see Remark 6.3.

**Proposition 6.2.** There exists a neighborhood $\mathcal{U}$ of $F_\lambda$ in $\mathcal{F}_\lambda$ such that for every $F \in \mathcal{U}$ with the eventually onto property the following hold:

1. The periodic sources of $F$ are dense in $\Omega_F$.
2. If the arguments of the eigenvalues of $p^+_F$ are irrational multiples of $\pi$, then the homoclinic class of $p_F$ is equal to $\Omega_F$. In particular, the periodic saddles of $F$ are dense in $\Omega_F$.

**Proof.** Let $\lambda \in (0,1)$ be close to 1 and let $F$ be a map in $\mathcal{F}_\lambda$ close to $F_\lambda$ and that satisfies the eventually onto property. In particular, the iterated preimages of $p^+_F$ are dense in $\Omega_F$.

Let $V$ be a small open neighborhood of $p^+_F$ such that $F$ has a local inverse $f$ defined on $V$, such that $f(V) \subset V$ and such that $f$ is uniformly contracting, so that $\bigcap_{m \geq 0} f^m(V) = \{p^+_F\}$.

1. Let us prove that the set of periodic sources of $F$ is dense in $\Omega_F$. Let $q \in \Omega_F$ and $k \geq 1$ be such that $F^k(q) = p^+_F$. It is enough to prove that close to $q$ there is a periodic source of $F$.

Let $U$ be a small open neighborhood of $q$. Let $\{q_m\}_{m \geq 1}$ be such that for every $m$ we have that $F^m(q_m) = q$ and such that $q_m \to p_F^+$ as $m \to \infty$. Let $m_0$ be such that for every $m \geq m_0$ we have $q_m \in V$. Let $U_0$ be a small open neighborhood of $q_{m_0}$ contained in $V$, such that $F^{m_0}(U_0) \subset U$, $F^{m_0+k}(U_0) \subset V$ and such that for every $j = 1, \ldots, m_0 + k$ we have that $F^j : U_0 \to f^j(U_0)$ is a diffeomorphism.

Notice that $F^{m_0+k}(U_0)$ is an open neighborhood of $p^+_F$. Finally, take $\ell \geq 1$ large enough so that $f^\ell(U_0) \subset F^{m_0+k}(U_0)$ and so that $F^{\ell+m_0+k}$ restricted to $W = f^\ell(U_0)$ is uniformly expanding. It follows that $W \subset F^{\ell+m_0+k}(W)$ and that $W$ contains a periodic source $\tilde{q}$ of $F$ of period $\ell + m_0 + k$. Thus, $F^{\ell+m_0}(\tilde{q}) \in U$ is a periodic source for $F$ close to $q$.

2. Assume that the argument of the eigenvalues of $D_{p^+_F}F$ are irrational multiples of $\pi$.

As the stable manifold of the saddle fixed point $p_F$ of $F$ is dense in $\Omega_F$ it follows that there is a point $z_0$ in $V$ contained in the stable manifold of $p_F$. Let $v_0 \in T_{z_0}B_\lambda$ be a vector that is tangent to the stable manifold of $p_F$ at $z_0$. For $m \geq 1$ put $z_m = f^m(z_0)$ and $v_m = D_{z_m}f^m v_0 \in T_{z_m}B_\lambda$. As the argument of the eigenvalues of $D_{p_F}F$ are irrational multiples of $\pi$, it follows that the arguments of the $v_m$ are dense in $[-\pi, \pi]$.

So, each iterated preimage $q$ of $p^+_F$ in $A_F$ is accumulated by arcs of the stable manifold that are quasi-radial. Hence Lemma 5.2 implies that $q$ is accumulated by points of transversal intersection between the stable and unstable manifold of $p_F$. From Proposition 4.3 it follows that the transversal homoclinic intersections of $p_F$ are dense in $\Omega_F$. □

**Remark 6.3.** We now prove that arbitrarily close to a transversal homoclinic point of $p_F$ there is a saddle periodic point of $F$. Although this fact is well-known in the
case of diffeomorphisms, some care should be taken for the singular endomorphisms considered here.

Let $U$ be an open subset of $\Omega_F$ and let $q \in U$ be a point of transversal intersection between the stable and unstable manifolds of $p_F$. Then, for every positive integer $m$ we have $q_m = F^m(q) \in C^*$ and $q_m \to p_F$ as $m \to \infty$. Consider an infinite backward orbit $\{\eta_m\}_{m \geq 0} \subset C^*$ of $q$ (that is $\eta_0 = q$ and $F(\eta_{m+1}) = \eta_m$ for every $m$) such that $\eta_m \to p_F$ as $m \to \infty$.

Let $N$ be a large integer, such that $q_N$ (resp. $\eta_N$) is in the local stable (resp. unstable) manifold of $p_F$. There is a small open neighborhood $\overline{V}_N$ of $\eta_N$ whose boundary are two small quasi-radial arcs and two quasi-angular arcs, such that the following properties hold.

1. For every $m$, $0 \leq m \leq 2N$, $F^m(\overline{V}_N) \subset C^*$ and $V_0 = F^N(\overline{V}_N)$ is a small neighborhood of $q$ that has the shape of a rectangle with two sides ‘parallel’ to the piece of stable manifold of $p_F$ that contains $q$, and two sides ‘parallel’ to the piece of the unstable manifold of $p_F$ that contains $q$.
2. $V_N = F^{2N}(\overline{V}_N) = F^N(V_0)$ is a small neighborhood of $q_N$ that has the shape of a rectangle with two sides ‘parallel’ to the local stable manifold of $p_F$, and two sides that are transversal to the local stable manifold of $p_F$.

Clearly, the two sides of $\overline{V}_N$ that are quasi-radial arcs map onto the two sides of $V_N$ that are parallel to the local stable manifold of $p_F$. We now take a sufficiently large integer $M$ so that $F^M(V_N) \cap \overline{V}_N$ is a thin strip that crosses from one quasi-radial arc in the boundary of $\overline{V}_N$ to the other.

It follows that $F^M(V_N) \cap \overline{V}_N \neq \emptyset$ and therefore $V_0$ contains a saddle periodic point of $F$.

### 6.3. Eventually onto property at $p^+_F$.

This subsection is devoted to prove the following proposition.

**Proposition 6.4.** For all $\lambda \in (0,1)$ sufficiently close to 1 and every $F$ sufficiently close to $F_\lambda \in \mathcal{F}_\lambda$, the following property holds. For every neighborhood $U$ of the fixed point $p^+_F$,

$$\text{interior}(\Omega_F) \subset \cup_{m \geq 0} F^m(U).$$

The following lemma reduces this proposition to show that a certain arc $I$ in $\mathbb{C}$ is contained in the basin of the fixed source $p^+_F$. This last fact is proven in lemmas 6.6 and 6.7.

**Lemma 6.5.** Let $V$ be a neighborhood of the arc $I = \{t \in \mathbb{C} | t \in \mathbb{R}, t \in [0,1]\}$. If $\lambda \in (0,1)$ is sufficiently close to 1 and $F$ is sufficiently close to $F_\lambda$, then $\text{interior}(\Omega_F) \subset \cup_{m \geq 0} F^m(V)$.

**Proof.**

It is enough to prove that

$$\text{interior}(A_F) = A_F \setminus \gamma^+_F \subset \cup_{m \geq 0} F^m(V).$$

In fact, by Proposition 6.3 it follows that there exists an integer $N \geq 1$ such that $F^N(\text{interior}(A_F)) = \text{interior}(\Omega_F)$.

After some preliminary considerations in part 1 this is proven in part 2.

1. First notice that if $\lambda \in (0,1)$ is close enough to 1, then $F^\lambda(V) \cup F^\lambda_2(V)$ contains the fundamental domain $[2,2+\lambda) \times \{0\}$ of the stable manifold of $p_\lambda$. Therefore, $\cup_{m=0}^{m=2\lambda} F^m(V)$ contains a neighborhood $V_0$ of a fundamental domain $D_0 \subset \mathbb{R}$ of the
stable manifold of \( p_\lambda \) which is contained in \( \{ z \in \mathbb{C} \mid |z| \geq 20 \} \). For every \( F \) close to \( F_\lambda \) the open set \( V_0 \) is also a neighborhood of a fundamental domain \( D_0(F) \) of the stable manifold of \( p_F \).

Recall that if \( \lambda \) is sufficiently close to 1, then the unstable cone fields of \( F_\lambda \) (see Subsection 3.2) are defined and invariant in \( \{ z \in \mathbb{C} \mid |z| \geq 20 \} \) (Lemma 3.2). The same happens for every \( F \) close to \( F_\lambda \). In particular, if \( x \) is a real positive number satisfying \( x \geq 20 \exp(\pi/3) \), then a quasi-angular arc through \( x \) touches the negative real axis before intersecting the circle \( \{ z \in \mathbb{C} \mid |z| = 20 \} \). We will assume that if \( x \in D_0 \) then \( 20 \exp(\pi/3) \leq x \leq 180 \).

2. At a point \( x_0 \in D_0 \) let \( \eta \) be a quasi-angular arc of length \( \ell \) that contains \( x_0 \). Then, since for every positive real number \( x > 0 \) we have, \( F_\lambda(x) < x + 2 \), it follows by induction that for every positive integer \( m \) we have \( F_\lambda^m(x_0) < x_0 + 2m < 180 + 2m \) and \( F_\lambda^m(\eta) \) is quasi-angular arc of length \( 2m\ell \) or contains a quasi-angular Jordan curve around the origin.

For \( m_0 \) so that \( 2m_0\ell > 180 + 2m_0 \) we have that \( F_\lambda^{m_0}(\eta) \) contains a quasi-angular Jordan curve around the origin.

Now take a small rectangle shaped region \( W_0 \) contained in \( V_0 \) foliated by quasi-angular arcs through points in \( D_0 \) (so that \( D_0 \) crosses \( W_0 \) from one side to the opposite one) and such that the image of the arc through the endpoint of \( D_0 \) closer to the origin contains the arc through the other endpoint of \( D_0 \).

From the above argument, there exists a positive integer \( m_0 \) such that \( F_\lambda^{m_0}(W_0) \) contains an annulus \( A_{m_0} \) around the origin. Notice that the image by \( F_\lambda \) of the portion of the internal boundary of \( A_{m_0} \) with positive real part is exactly the external boundary of \( A_{m_0} \). This \( m_0 \) is the same for every \( \lambda \) close enough to 1. And by continuity a similar construction is valid for \( F \) close enough to \( F_\lambda \).

Choose \( \lambda \) sufficiently close to 1 in such a way that the fundamental annulus \( A_F \) is disjoint from \( A_{m_0} \) and contained in the unbounded component of the complement of \( A_{m_0} \). For every \( m \geq m_0 \) inductively define an annulus \( A_m \) by

\[
A_{m+1} = F_\lambda(A_m \cap \{ z \in \mathbb{C} \mid \Re z \geq 0 \}).
\]

Similarly define the annulus \( A_m(F), m \geq m_0 \), for every \( F \) close enough to \( F_\lambda \).

By the Inclination Lemma it follows that

\[
A_F \setminus \gamma_\delta^+ \subset \bigcup_{m \geq m_0} F(A_m(F)) \subset \bigcup_{m \geq 0} F^m(V_0).
\]

\[\square\]

**Lemma 6.6.** There is an open subset \( D \) of the upper half plane \( \mathbb{H} \), that is bounded by a Jordan curve and such that \( G_\Gamma(D) \) contains both the closure of \( D \) and the arc \( \Gamma \).

**Proof.** Given \( \varepsilon > 0 \) small, set

\[
D' = \{ z \in \mathbb{C}^* \mid \arg(z) \in (\varepsilon, \pi/2), |z - p^+| < |2i - p^+| \}.
\]

1. We will prove that if \( \varepsilon > 0 \) is sufficiently small, then \( \overline{D'} \cap G_\Gamma(\partial D') = \emptyset \). In part 2 below we conclude the proof of the lemma from this fact.

The boundary of \( D' \) consists of the arc \( J_1 = \{ it \mid t \in \mathbb{R}, 0 \leq t \leq 2 \} \), an arc of the form \( J_2 = \{ p \exp(i\varepsilon) \mid 0 \leq p \leq \rho_0 \} \), for some \( \rho_0 > 0 \), and an arc \( J_3 \) of the circle \( \{ z \in \mathbb{C} \mid |z - p^+| = |2i - p^+| \} \).
Clearly, $G_1(J_3) = \{t \in \mathbb{C} \mid t \in \mathbb{R}, -1 \leq t \leq 1\}$ intersects $D'$ only at 0. Let us show that $G_1(J_2) = \{\rho \exp(2i\epsilon) + 1 \mid 0 \leq \rho \leq \rho_0\}$ is disjoint from $D'$. As $\arg(1 + \exp(2i\epsilon)) = \epsilon$, it follows that for $0 \leq \rho < 1$ the point $1 + \rho \exp(2i\epsilon)$ is disjoint from $D'$. As $|2i - p^+| < |2 - p^+|$, it follows that if $\epsilon$ is sufficiently small, then for all $\rho \geq 1$

$$|1 + \rho \exp(2i\epsilon) - p^+| \geq |1 + \exp(2i\epsilon) - p^+| > |2i - p^+|,$$

so that $\rho \exp(2i\epsilon) + 1 \not\in D'$.

To see that $G_1(J_3)$ is disjoint from $D'$, observe that for each pair of points $z, z' \in \mathbb{C}^*$ such that $\arg(z), \arg(z') \in [0, \pi/2]$, we have $|G(z) - G(z')| \geq |z - z'|$, with equality if and only if $z/z' \in \mathbb{R}$. So, for every $z \in J_3$ we have $|G(z) - p^+| \geq |z - p^+|$, with equality if and only if $z = (1 + |2i - p^+|) \exp(\pi i/3)$. It is easy to check that the image of this point has negative real part. We conclude that $G_1(J_3)$ is disjoint from $D'$.

2. For $\delta > 0$ small set $D'' = \{z \in D' \mid |z| > \delta\}$. Note that $D''$ is bounded by a Jordan curve and $D'' \subset \mathbb{H}$. We will show that, if $\delta > 0$ is sufficiently small, then $D'' \cup I \subset G_1(D'')$. Then any sufficiently small neighborhood $D$ of $D'' \subset G_1(D'')$. When $\delta < 1$, the arc $I$ is clearly contained in the closure of $G_1(D'')$. □

By the previous lemma it follows that, if we take $\lambda$ sufficiently close to 1, then every map $F$ sufficiently close to $F_\lambda$ satisfies the following properties.

1. $F$ is injective on $\overline{D}$ and $\overline{D} \subset F(D)$.
2. The Jacobian of $F$ on $D$ is larger than a constant larger than one.
3. The fixed source $p_F^+$ is contained in $D$ and it is the unique fixed point of $F$ in $\overline{D}$. Moreover, the derivative of $F$ at $p_F^+$ has complex eigenvalues (not in $\mathbb{R}$) of norm larger than 1.

**Lemma 6.7.** Let $\lambda \in (0, 1)$ be close to 1 and let $F$ be an endomorphism in $\mathcal{F}_\lambda$ close to $F_\lambda$ satisfying the properties above. Denote by $f : F(D) \to D$ the inverse of $F$ restricted to $F(D)$. Then

$$\cap_{m \geq 1} f^m(D) = \{p_F^+\}.$$

In particular, for every neighborhood $U$ of $p_F^+$ there is an integer $m$ such that $D \subset f^m(U)$.

**Proof.** Set $K = \cap_{m \geq 1} f^m(D)$. As the Jacobian of $f = F^{-1}|_{F(D)}$ is smaller than a constant smaller than 1, it follows that $K$ has 0 Lebesgue measure, and hence empty interior. Assume by contradiction that $K$ is not equal to $\{p_F^+\}$. We will prove then that $f$ has a fixed point in $D$ distinct from $p_F^+$. This contradicts property 3 above and proves the lemma.

Let $z_0 \in K$ be different from $p_F^+$ and denote by $K_0$ the set of accumulation points of the forward orbit of $z_0$ by $F$. So $K_0 \subset K$ and $F(K_0) = K_0$. Moreover, as $p_F^+$ is a source, $p_F^+ \not\in K_0$.

Let $h : \mathbb{D} \to D$ be a linearizing coordinate of $F$ near $p_F^+$, so that $h(0) = p_F^+$ and so that for every $w \in \mathbb{D}$ satisfying $|w| < 1/2$ we have $F(h(w)) = h(2w)$. The forward
orbit by \( f \) of each point in \( h(\mathbb{D}) \) converges to \( p^+_F \). As \( f(K_0) = K_0 \) and \( p^+_F \notin K_0 \), it follows that \( K_0 \cap h(\mathbb{D}) = \emptyset \).

Since \( K \) has empty interior, there is \( w_0 \in \mathbb{D} \) such that \( h(w_0) \notin K \). Let \( m \geq 1 \) be such that \( h(w_0) \notin f^m(D) \) and set \( t_0 = \inf \{ t \mid h(tw_0) \notin f^m(D) \} \) and \( \gamma = h(\{ tw_0 \mid t \in [0, t_0] \}) \).

As \( D \), and hence \( f^m(D) \), are bounded by a Jordan curve, it follows that \( D_0 = f^m(D) \setminus \gamma \) is homeomorphic to a disk. Moreover, as \( f(\gamma) \subset \gamma \), we have \( f(D_0) \subset D_0 \), and since \( K_0 \) is disjoint from \( \gamma \subset h(\mathbb{D}) \), we have \( K_0 \subset D_0 \). Let \( \tilde{f} : D_0 \to D_0 \) be a homeomorphism that coincides with \( f \) on \( f(D_0) \), such that \( \tilde{f}(\gamma) = \gamma \) and such that every point in \( D_0 \) enters \( f(D_0) \) under forward iteration by \( \tilde{f} \). In particular \( \tilde{f} \) does not have fixed points. But, since \( K_0 \) is a compact subset of \( D_0 \) that is invariant by \( \tilde{f} \), Brouwer’s translation theorem applied to \( \tilde{f} \) implies that \( \tilde{f} \) has a fixed point in \( f(D_0) \). So we get a contradiction that proves the lemma. \( \square \)

### 6.4. Proof of Proposition 2.5

In view of Proposition 6.3, a map \( F \) in \( \mathcal{F}_\lambda \) close to \( F_\lambda \) satisfies the eventually onto property if and only if the iterated preimages of \( p^+_F \) are dense in \( \Omega_F \). So Proposition 2.5 is a direct consequence of Proposition 6.2 and the following proposition.

Recall that for an open set \( \mathcal{U} \) of \( \mathcal{G}_\lambda \) and \( \tau \in \mathcal{T}_\lambda \) we denote
\[
\mathcal{U}_\tau = \{ g \in \mathcal{U} \mid F = g \circ \tau \text{ is area expanding} \}.
\]

**Proposition 6.8.** There is a neighborhood \( \mathcal{U} \) of \( g_1 \) in \( \mathcal{G}_\lambda \) such that for every \( \tau \in \mathcal{T}_\lambda \), there is a residual subset \( \mathcal{R}_{\tau} \) of \( \mathcal{U}_\tau \) with the following property. For every \( g \in \mathcal{U}_\tau \) the map \( F = g \circ \tau \) is such that the iterated preimages of the fixed source \( p^+_F \) are dense in \( B_\lambda \). In particular, \( F = g \circ \tau \) satisfies the eventually onto property.

**Proof.** Let \( \mathcal{U} \) be a sufficiently small neighborhood of \( g_1 \) in \( \mathcal{G}_\lambda \) such that for every \( \tau \in \mathcal{T}_\lambda \) sufficiently close to \( \tau_1 \), all the results of Section 4 hold for \( F = g \circ \tau \).

Given an open subset \( U \) of \( B_\lambda \), put
\[
\mathcal{A}(U) = \{ g \in \mathcal{U} \mid p^+_F \in \bigcup_{m \geq 0} F^m(U) \text{ where } F = g \circ \tau \}.
\]

Clearly \( \mathcal{A}(U) \) is an open subset of \( \mathcal{U}_\tau \). We will show that \( \mathcal{A}(U) \) is dense in \( \mathcal{U}_\tau \). The residual set \( \mathcal{R}_{\tau} \) will be the intersection of the sets \( \mathcal{A}(U) \), where \( U \) runs through a countable basis for the topology of \( B_\lambda \).

Given \( g \in \mathcal{U} \), put \( F = g \circ \tau \) and consider an iterated preimage \( q_0 \) of \( p^+_F \) in the fundamental annulus \( A_F \). Consider a \( C^1 \) family of functions \( \{ h_\varepsilon \} \) in \( \mathcal{U}_\tau \), such that \( h_0 = g \), such that \( h_\varepsilon \) coincides with \( g \) on \( \{ |z| \geq 15 \} \) and such that, if we denote by \( q_\varepsilon \) the iterated preimage of \( p^+_F \) by \( F_\varepsilon = h_\varepsilon \circ \tau \) that is the continuation of \( q \), then \( q_\varepsilon \) moves following a radial arc.

Note that for every small \( \varepsilon > 0 \) the maps \( F_\varepsilon \) and \( F_0 \) coincide on \( H_\lambda \) and we have \( p_{F_\varepsilon} = p_{F_0} \). As the stable manifold of the saddle fixed point \( p_{F_0} \) is dense in \( B_\lambda \), there exists \( m \) such that \( F^m_0(U) \) intersects the local stable manifold of \( p_{F_0} \).

Let \( \ell \subset A_{F_0} \) be a smooth arc contained in \( F^m_0(U) \) that intersects transversally the local stable manifold of \( p_{F_0} \). Shrinking \( \ell \) if necessary, we assume that for every sufficiently small \( \varepsilon > 0 \) we have \( \ell \subset F^m(U) \). From the Inclination Lemma and from Lemma 6.7 it follows that there exists a sub-arc \( \ell' \) of \( \ell \) and a positive integer \( N \), such that for every \( j = 0, \ldots, N \) the arc \( F^N_j(\ell') \) is a quasi-angular arc contained in \( H_\lambda \) and such that \( F^N_0(\ell') \) intersects transversally, and in a non empty way, the
radial arc defined by \( q_\varepsilon \). So, for some \( \varepsilon \in (\varepsilon_0, \varepsilon_0) \), we have \( q_\varepsilon \in F_0^N(\ell ') \). As for every \( j = 0, \ldots, N \) the arc \( F_0^j(\ell ') \) is contained in \( H_\lambda \), where \( F_0 \) coincides with \( F \), we have \( F_\varepsilon^j(\ell ') = F_0^j(\ell ') \). Thus \( q_\varepsilon \in F_\varepsilon^N(\ell ') \subset F_\varepsilon^{N+m}(U) \).

\[ \square \]

7. Robust tangencies and wild hyperbolic sets

In this section we prove Theorem 2.6. Part 1 of this theorem is a direct consequence of Lemma 7.1 in Subsection 7.1. The proof of part 2 is based on a strong property: every “curved” arc near \( A_F \) is tangent to the unstable manifold of some infinite backward orbit in \( \Gamma_F \) (Proposition 7.3). We complete the proof of part 2 of Theorem 7.1 in Subsection 7.1. The proof of part 2 is based on a strong property: every “curved” arc near \( A_F \) is tangent to the unstable manifold of the saddle fixed point \( p_F \) of \( F \) that is “curved”. Throughout this section we assume that \( \lambda \in (0,1) \) is close enough to 1, so that all the properties of Section 3 are satisfied.

7.1. Wild hyperbolic set. Fix \( F \) in \( \mathcal{F}_\lambda \) close to \( F_\lambda \). Then the set

\[ W_F = \{ z \in H_\lambda \mid F^m(z) \in H_\lambda \text{ for every } m \geq 1 \}, \]

is an uniformly hyperbolic forward invariant set for \( F \).

The saddle fixed point \( p_\lambda \) of \( F_\lambda \) is contained in \( W_{F_\lambda} \) and the part of the real axis contained in \( H_\lambda \) belongs to \( W_{F_\lambda} \). More generally, if \( F \) is sufficiently close to \( F_\lambda \), then the saddle fixed point \( p_F \) of \( F \) is contained in \( W_F \) and the stable manifold \( W_s(p_F) \) contains an arc close to the real axis connecting \( p_F \) to the boundary of \( H_\lambda \). This arc is contained in \( W_F \). Hence, every quasi-angular arc contained in \( H_\lambda \) that turns once around the origin intersects \( W_F \).

To every infinite backward orbit \( z = \{ z_m \}_{m \geq 0} \in \Gamma_F \) we associate a global unstable manifold \( W_u(z) = \cup_{m \geq 0} F^m(W_u(z_m)) \), where \( z_m \) is the infinite backward orbit \( \{ z_{m+j} \}_{j \geq 0} \). Below, in Lemma 7.1 we will prove that there is \( M \geq 1 \) such that for every \( z \in \Gamma_F \) starting at a point in \( A_F \) and every \( m = 1, \ldots, M \); we have that \( F^m(W_u(z)) \subset H_\lambda \) and that \( F^M(W_u(z)) \) is a quasi-angular arc which turns once around the origin. It follows that the local unstable manifold \( W_{u_F}(z) \) is contained in the global unstable manifold of an infinite backward orbit contained in \( W_F \). So this proves part 1 of Theorem 7.1.

Recall that for \( F \in \mathcal{F}_\lambda \) close to \( F_\lambda \), the distance from the fundamental domain \( A_F \) to the origin has the order of \( (1 - \lambda)^{-1} \), and the distance from the boundary of the fundamental domain \( H_\lambda \) to the origin has the order of \( (1 - \lambda)^{-1} \). So, the distance from the fundamental domain \( A_F \) of \( F \) to the boundary of the hyperbolic set \( H_\lambda \) tends to infinity as \( \lambda \) tends to 1.

**Lemma 7.1.** For every \( \lambda \) sufficiently close to 1 there exists an integer \( M \geq 0 \) such that for every \( F \in \mathcal{F}_\lambda \) sufficiently close to \( F_\lambda \), and every infinite backward orbit \( z = \{ z_k \}_{k \geq 0} \in \Gamma_F \) starting at \( z_0 \in A_F \), the following properties hold.

1. For every \( j = 0, \ldots, M \) the set \( F^j(W_u(z)) \) is contained in the domain of hyperbolicity \( H_\lambda \).
2. \( F^M(W_u(z)) \) is a quasi-angular arc that turns at least once around the origin.

**Proof.** Let \( L > 0 \) be a constant such that every quasi-angular of length at least \( L(1 - \lambda)^{-1} \) that is contained in \( B_\lambda \), turns at least once around the origin. On the
other hand, choose \( \eta \in (1, (5/2)^{1/2}) \) and let \( \lambda_1 \in (0, 1) \) be sufficiently close to 1 so that for every \( \lambda \in (\lambda_1, 1) \) we have \( \eta^4 < \lambda (5/2)^{1/2} \).

Notice that for every \( w \in \mathbb{C} \) such that \( |w| > R > 1 \) we have \( |F_\lambda(w)| > \lambda R - 1 \). Inductively, we have that \( |F_\lambda^m(w)| > \lambda^m R - m \) whenever \( \lambda^{m-1} R - (m-1) > 1 \). Take \( R = \frac{1}{3} r(1 - \lambda)^{-1} < \text{dist}(\{0\}, A_\lambda) \) and consider an infinite backward orbit \( z \in \Gamma_{\hat{w}_\lambda} \) starting from a point \( z_0 \) in \( A_\lambda \). Since for \( w \in W_\alpha^u(z) \) we have that \( |w| > R \), then \( F_\lambda^m(W_\alpha^u(z)) \subset H_\lambda \) whenever \( \lambda \) is sufficiently close to 1 and \( \lambda^{m-1} \frac{1}{3} r(1 - \lambda)^{-1} (m-1) > 5(1 - \lambda)^{-\frac{1}{2}} \). Let \( C > 0 \) be such that for every \( m \geq 0 \) we have \( C \eta^m (1 - \lambda)^{-\frac{1}{2}} \geq m - 1 + 5(1 - \lambda)^{-\frac{1}{2}} \). Then, whenever

\[
m < \frac{\ln((2C\lambda)^{-1} r) - \frac{1}{2} \ln(1 - \lambda)}{\ln(\eta/\lambda)},
\]

we have \( F_\lambda^m(W_\alpha^u(z)) \subset H_\lambda \).

On other hand, while the iterates of \( W_\alpha^u(z) \) remain in \( H_\lambda \) their length is increased by a factor \( \lambda (5/2)^{1/2} > \eta^3 > 1 \) (part 1 of Lemma 3.2). So, the length of \( F_\lambda^m(W_\alpha^u(z)) \) is greater than \( 2\eta^m \alpha \). Moreover, we have \( 2\eta^{3m} \alpha > L(1 - \lambda)^{-1} \) if and only if

\[
m > \frac{\ln(L/(2\alpha)) - \ln(1 - \lambda)}{3 \ln \eta}.
\]

Let \( \lambda \in (\lambda_1, 1) \) be sufficiently close to 1 so that,

\[
\frac{\ln(L/(2\alpha)) - \ln(1 - \lambda)}{3 \ln \eta} < \frac{\ln((2C\lambda)^{-1} r) - \frac{1}{2} \ln(1 - \lambda)}{\ln(\eta/\lambda)},
\]

and so that there is an integer \( M \) satisfying

\[
\frac{\ln(L/(2\alpha)) - \ln(1 - \lambda)}{3 \ln \eta} < M < \frac{\ln((2C\lambda)^{-1} r) - \frac{1}{2} \ln(1 - \lambda)}{\ln(\eta/\lambda)}.
\]

Then the desired properties hold for \( F = F_\lambda \) and for maps \( F \in \mathcal{F}_\lambda \) close enough to \( F_\lambda \). 

\[\square\]

### 7.2. A repelling annulus.

**Lemma 7.2.** There is a closed annulus \( A^0_\lambda \subset A_\lambda \) that is contained in the interior of its image under \( F_\lambda \). Moreover, \( A^0_\lambda \) contains at least one iterated preimage of the fixed source \( p^+_\lambda \) in its interior.

**Proof.** Recall that the outer boundary \( \gamma^+_F \) of \( A_\lambda \) is a Jordan curve formed by a piece of the unstable manifold of the saddle fixed point \( p_\lambda \) that intersects itself transversely at some point. Let \( \tilde{\gamma}^+_\lambda \) be a Jordan curve in the interior of \( A_\lambda \) formed by a \( C^1 \) arc close to \( \gamma^+_F \), that intersects itself transversely at some point, and such that its image \( F(\tilde{\gamma}^+_\lambda) \) is closer to the unstable manifold of \( p_\lambda \), than \( \gamma^+_F \). Then the closed annulus \( A^0_\lambda \) obtained from \( A_\lambda \) by replacing its outer boundary \( \gamma^+_F \) by \( \tilde{\gamma}^+_\lambda \), is contained in \( A_\lambda \) and in the interior \( F(A^0_\lambda) \). We can choose \( \tilde{\gamma}^+_\lambda \) close enough to \( \gamma^+_F \) in such a way that \( A^0_\lambda \) contains an iterated preimage of \( p^+_\lambda \) in its interior. 

\[\square\]
7.3. Curved arcs and tangencies. Let $A^0_\lambda$ be the annulus given by Lemma 7.1. Denote by $C_\lambda$ the collection of class $C^1$, such that the following properties hold.

1. The image of $\gamma$ intersects $A^0_\lambda$ and $\text{length}(\gamma) \leq d_\lambda = \text{dist}(A^0_\lambda, \partial F_\lambda(A^0_\lambda))/2$.
2. For all $t \in [0,1]$ the vector $\gamma'(t)$ is non zero and it belongs to $K(\gamma(t))$.
3. There is $\rho \in \mathbb{R}$ (resp. $\rho' \in \mathbb{R}$) such that vector $\gamma'(0)$ (resp. $\gamma'(1)$) is of the form

$$\gamma'(0) = \gamma(0)\rho(i+1/3) \quad \text{(resp. } \gamma'(1) = \gamma(1)\rho'(i-1/3)\text{)}.$$ 

Note that property 1 implies that $\gamma$ is contained in $F_\lambda(A^0_\lambda)$ and that property 3 implies that $\gamma'(0)$ and $\gamma'(1)$ are in the boundary of $K(\gamma(0))$ and $K(\gamma(1))$, respectively.

The following proposition is the key step to produce robust tangencies.

**Proposition 7.3.** If $\lambda$ is close to 1 and $F$ in $\mathcal{F}_\lambda$ is close to $F_\lambda$, then every arc in $C_\lambda$ is tangent to the local unstable manifold of an infinite backward in $\Gamma_F$.

**Proof.** Consider $F \in \mathcal{F}_\lambda$ close to $F_\lambda$ in such a way that $\text{dist}(A^0_\lambda, \partial F(A^0_\lambda)) > d_\lambda$. Property 1 of the definition of $C_\lambda$ implies that the image of every arc in $C_\lambda$ is contained in $F(A^0_\lambda)$.

1. We will show that for any arc $\gamma$ in $C_\lambda$ there is a lift $\tilde{\gamma}$ by $F$, such that a sub-arc of $\tilde{\gamma}$, re-parameterized to be defined in $[0,1]$, belongs to $C_\lambda$. In fact, as $A^0_\lambda \subset F(A^0_\lambda)$, there is a lift $\tilde{\gamma}$ of $\gamma$ by $F$, such that for some $t \in [0,1]$ we have $\tilde{\gamma}(t) \in A^0_\lambda$. Moreover, for each $t \in [0,1]$ the vector $\tilde{\gamma}'(t)$ belongs to $\tilde{K}(\gamma(t))$, so length$(\tilde{\gamma}) \leq 2/3 \cdot \text{length}(\gamma) < d_\lambda$.

On the other hand, since the set $F^{-1}(F(A^0_\lambda))$ is contained in $H_\lambda$, it follows that for every $z_0$ in this set we have $Dz_0 F(K(z_0)) \subset K(F(z_0))$. Hence, there are $\tau, \tau' \in [0,1]$ and $\rho, \rho' \in \mathbb{R}$, such that $\tilde{\gamma}'(\tau)$ and $\tilde{\gamma}'(\tau')$ are of the form $\gamma(\tau)\rho(i+1/3)$ and $\gamma(\tau')\rho'(i-1/3)$, respectively. By taking $\tau'$ closer to $\tau$ if necessary, we assume that for all $t$ between $\tau$ and $\tau'$, we have $\tilde{\gamma}'(t) \in K(\gamma(t))$. So the arc $t \mapsto \tilde{\gamma}(\frac{t-\tau}{\tau'-\tau})$, defined on $[0,1]$, belongs to $C_\lambda$.

2. Let $\gamma_0$ be an arc in $C_\lambda$. For $m \geq 1$ define inductively an arc $\gamma_m \in C_\lambda$ in such a way that $\gamma_m$ is constructed from $\gamma_{m-1}$, as described in part 1. It follows that there is a sequence $\{t_m\}_{m \geq 1} \subset [0,1]$ such that for $m \geq 1$ we have $F(\gamma_m(t_m)) = \gamma_{m-1}(t_{m-1})$, so that $w = \{\gamma_m(t_m)\}_{m \leq 0}$ is an infinite backward orbit of points in $F_\lambda(A^0_\lambda) \subset H_\lambda$. We have then $w \in \Gamma_F$. Moreover, the vector $\gamma'_m(t_m)$ is non-zero and belongs $K(\gamma_m(t_m))$ and for $m \geq 1$, the vector $DF(\gamma'_m(t_m))$ is parallel to $\gamma'_{m-1}(t_{m-1})$. It follows that $\gamma'_0(t_0)$ is parallel to the unstable direction associated to the backward orbit $w$ and that the arc $\gamma$ is tangent to the corresponding local unstable manifold at $t = t_0$.

7.4. The stable manifold of $p_F$ contains a curved arc. Note that every arc of length at most $d_\lambda$ which intersects $A^0_\lambda$ and that turns around some point at least 2 times contains an arc in $C_\lambda$.

1. Define $R_0 = \{ t \in \mathbb{C} \mid t \in \mathbb{R}, t \leq 0 \}$ and for $m \geq 1$ define inductively $R_m$ as the preimage of $R_{m-1}$ contained in $\mathbb{H}$, by the limit map $G_1$. The corresponding backward orbit of 0 converges to the fixed point $p^+$ by Lemma 7.1.
2. As the derivative of $p^+$ has complex multipliers, it follows that there is a positive integer $N$ and a piece $\gamma$ of $R_N$ contained in $D$, such that $\gamma$ and all its successive preimages under $G_{\gamma} | H$ 'turn around $p^+$ at least 5 times'.

3. As $F_\lambda$ converges in the $C^2$ topology to $G_1$ on $D$, when $\lambda \to 1$, it follows that if $\lambda$ is sufficiently close to 1, then there is a curve $\gamma_\lambda$ contained in $D$ such that $F_N(\gamma_\lambda)$ is a piece of $R_0$, and such that $\gamma_\lambda$ and all its successive preimages under $F_\lambda | H$ 'turn around $p^+_{F_\lambda}$ at least 4 times'.

4. Let $z_0 \in A^0_\lambda$ be an iterated preimage of $p^+_{F_\lambda}$ by $F_\lambda$. Then there is a preimage $\tilde{\gamma}_\lambda$ of $\gamma_\lambda$ by some iterate of $F_\lambda$ such that $\tilde{\gamma}_\lambda$ is contained in a $d_\lambda/2$ neighborhood of $z_0$, it has length at most $d_\lambda/2$, and it 'turns around $z_0$ at least 3 times'.

5. So for $F \in \mathcal{F}_\lambda$ sufficiently close to $F_\lambda$ there is a piece of the stable manifold $\tilde{\gamma}_F$ that is contained in a $d_\lambda$ neighborhood of $z_0$, it has length at most $d_\lambda$ and it 'turns around $z_0$ at least 2 times'. It follows that a piece of $\tilde{\gamma}_F$ is contained in $C_\lambda$ and it is therefore tangent to an unstable manifold of an infinite backward orbit in $\Gamma_F$.

8. The vector fields $X_{\lambda,\mu}$

This section contains the proof of Theorem 2.7, which consists of the construction of a family of vector fields $\{X_{\lambda,\mu}\}$. In Subsection 8.1, we state a refined version of Theorem 2.7, make some remarks and introduce some notation. In Subsection 8.2, we describe the strategy of the construction of $X_{\lambda,\mu}$ and give the initial steps. The heart of the construction is contained in subsections 8.3 and 8.4. We end the construction in Subsection 8.5.

8.1. Remarks and notation. Recall that for an integer $k \geq 0$ we denote by $D^k$ the closed unit ball of $\mathbb{R}^k$. Also $\mathbb{D}$ is the closed unit disc in $\mathbb{C}$ and $\mathbb{T}^n = \mathbb{R}/\mathbb{Z} \times D^{n-1}$.

Fix an integer $n \geq 5$, $\lambda \in (0,1)$ and $\eta > \sigma > 0$. Given $\mu \in (0,\sigma]$, let $F_{\lambda,\mu} : \mathbb{C}^* \to \mathbb{C}$ be defined by

$$F_{\lambda,\mu}(z) = (1 - \lambda + \lambda |z|^{\mu/\sigma})(z/|z|)^2 + 1.$$ 

Note that $F_{\lambda,\mu}$ depends smoothly on $\lambda$ and $\mu$ and that $F_{\lambda,\sigma} = F_\lambda$. For $\lambda$ close to 1 and $\mu \in (0,\sigma]$ close to $\sigma$, the map $F_{\lambda,\mu}$ is area expanding.

Observe that $F_{\lambda,\mu}$ can be written as the composition $F_{\lambda,\mu} = G_t \circ T_{\lambda,\mu}$ of the maps

$$T_{\lambda,\mu} : \mathbb{C}^* \to \mathbb{C}^*$$

$$z \mapsto (1 - \lambda + \lambda |z|^{\mu/\sigma})(z/|z|),$$

$$G_t : \mathbb{C}^* \to \mathbb{C}$$

$$z \mapsto (z^2/|z|) + 1.$$ 

Recall that $B_\lambda = \{ z \in \mathbb{C} \mid |z| \leq 2(1-\lambda)^{-1}\}$, $B_\lambda^* = B_\lambda \setminus \{0\}$ and put

$$\tilde{B}_\lambda = \{ z \in \mathbb{C} \mid (1 - \lambda)/2 \leq |z| \leq 2(1-\lambda)^{-1} - 1 - \lambda/2 \} \subset B_\lambda.$$ 

Then for all $\mu \in (0,\sigma]$ we have $T_{\lambda,\mu}(B_\lambda^*) \subset \tilde{B}_\lambda$, $G_t(\tilde{B}_\lambda) \subset B_\lambda$ and $F_\lambda(B_\lambda^*) \subset B_\lambda$.

Similarly, for $\beta > 0$ sufficiently small, the map

$$\tilde{F}_{\lambda,\mu} : B_\lambda^* \times \mathbb{D} \times D^{n-5} \to B_\lambda^* \times \mathbb{D} \times D^{n-5}$$

$$(z, w, v) \mapsto \left( F_{\lambda,\mu}(z), \frac{z}{2|z|} + \beta |z|^{\eta/\sigma} \frac{|z|}{z} w, \beta |z|^{\eta/\sigma} v \right).$$
is well defined and injective. This map may be written as the composition \( \tilde{T}_{\lambda,\mu} = \tilde{G}_1 \circ \tilde{T}_{\lambda,\mu} \) of the maps

\[
\tilde{T}_{\lambda,\mu} : B_3^5 \times \overline{D} \times D^{n-5} \rightarrow \tilde{B}_{\lambda,\mu} \times \overline{D} \times D^{n-5} \quad (z, w, v) \mapsto (T_{\lambda,\mu}(z), \beta_0|z|^{\eta/\sigma} w, \beta_0|z|^{\eta/\sigma} v),
\]

where \( \beta_0 \in (0, 1) \) is sufficiently small, and

\[
\tilde{G}_1 : \tilde{B}_{\lambda} \times \overline{D} \times D^{n-5} \rightarrow B_3^5 \times \overline{D} \times D^{n-5} \quad (z, w, v) \mapsto \left( G_1(z), \frac{z}{2|z|} + \beta_1|z| w, \beta_1 v \right),
\]

where \( \beta_1 \in (0, 1/2) \) is such that \( \beta_0 \beta_1 = \beta \).

From the considerations above, Theorem 8.1 is a direct consequence of the following theorem.

**Theorem 8.1.** Fix an integer \( n \geq 5 \) and \( \eta > \sigma > 0 \). Then for each \( \beta \in (0, 1) \) sufficiently small and for each \( \lambda \in (0, 1) \) sufficiently close to 1, there exists \( \mu_0 \in (0, \sigma) \) and a smooth one parameter family of smooth vector fields \( \{X_{\lambda,\mu} | \mu \in (\mu_0, \sigma)\} \) defined on an open set \( U \subset T^n \) such that for all \( \mu \in (\mu_0, \sigma) \) the following hold:

1. The boundary of the open set \( U \subset T^n \) is a manifold of dimension \( n - 1 \) that is contained in the interior of \( T^n \). For every \( \mu \in (\mu_0, \sigma) \) the vector field \( X_{\lambda,\mu} \) extends to a smooth vector field defined on a neighborhood of the closure of \( U \), in such a way that on the boundary of \( U \) this vector field points inward. Moreover this extension has a unique singularity \( o = o_{\lambda,\mu} \). The singularity \( o \) is contained in \( U \) and is hyperbolic with eigenvalues \( -\mu \), \( \sigma \) and \( -\eta \) of multiplicities 1, 2 and \( n - 3 \), respectively.
2. There exist codimension 1 submanifolds

\[
\Sigma^u \approx B_3^5 \times \overline{D} \times D^{n-5} \quad \text{and} \quad \Sigma^s \approx \tilde{B}_{\lambda} \times \overline{D} \times D^{n-5},
\]

which are transversal to the flow of \( X_{\lambda,\mu} \) and so that every forward orbit of the flow of \( X_{\lambda,\mu} \) in \( U \) intersects \( \Sigma^u \) or is contained in a local stable manifold \( W^s_{\text{loc}}(o) \) of \( o \). The intersection of this local stable manifold \( W^s_{\text{loc}}(o) \) with \( \Sigma^u \) is \( \{0\} \times \overline{D} \times D^{n-5} \).
3. The Poincaré maps from \( \Sigma^u \) to \( \Sigma^s \) and from \( \Sigma^s \) to \( \Sigma^u \) induced by \( X_{\lambda,\mu} \) are given by \( \tilde{T}_{\lambda,\mu} \) and \( \tilde{G}_1 \), respectively.

### 8.2. Strategy of the construction.

In the rest of this section, we identify \( \mathbb{R}^n \) with \( \mathbb{R} \times \mathbb{C} \times \mathbb{C} \times \mathbb{R}^{n-5} \) where we use coordinates \((s, z, w, v)\). Also, we view the solid torus \( T^n = \mathbb{R} / \mathbb{Z} \times D^{n-1} \) as a subset of \( \mathbb{R} / \mathbb{Z} \times \mathbb{C} \times \mathbb{C} \times \mathbb{R}^{n-5} \).

We will define a vector field \( \tilde{X}_{\lambda,\mu} \) on a closed subset \( \tilde{V} \) of \([-2, 3] \times 2B_3 \times 2D \times D^{n-5} \) so that \( \tilde{X}_{\lambda,\mu} \) is constant equal to \((1, 0)\) for \( s \) close to \(-2\) and 3. Hence, if we let \( a = (1 - \lambda)/16 \), then the image of \( X_{\lambda,\mu} \) under the map

\[
Q : [-2, 3] \times 2B_3 \times 2D \times D^{n-5} \rightarrow \mathbb{R} / \mathbb{Z} \times D^{n-1} \quad (s, z, w, v) \mapsto ((s + 2)/5 \mod 1, az, aw, v/4)
\]

will be a smooth vector field \( DQ(\tilde{X}_{\lambda,\mu}) \) defined on the subset \( V = Q(\tilde{V}) \) of \( \mathbb{R} / \mathbb{Z} \times \frac{1}{4}D \times \frac{1}{4}D \times D^{n-5} \). Our desired vector field \( X_{\lambda,\mu} \) will be obtained extending \( DQ(\tilde{X}_{\lambda,\mu}) \) to a small neighborhood \( U \) of \( V \).
The construction of $\tilde{X}_{\lambda,\mu}$ will be such that the codimension 1 submanifolds $\Sigma_u = \{-2\} \times B_\lambda \times \mathbb{D} \times D^{n-5}$, $\Sigma^s = \{2\} \times B_\lambda \times \mathbb{D} \times D^{n-5}$ and $\Sigma_+^u = \{3\} \times B_\lambda \times \mathbb{D} \times D^{n-5}$ are transversal to $\tilde{X}_{\lambda,\mu}$ (see Figure 2). The Poincaré map from $\Sigma_u^+$ into $\Sigma^s$ will be given by the map $(-2, z, w, v) \mapsto (2, \tilde{T}_{\lambda,\mu}(z, w, v))$. This Poincaré map corresponds to a transition through a singularity with eigenvalues $-\mu$ in the $s$-direction, $\sigma$ in $z$-directions, and $-\eta$ in the $w$ and $v$ directions. The Poincaré map from $\Sigma^s$ into $\Sigma_+^u$ will be given by $(2, z, w, v) \mapsto (3, \tilde{G}_1(z, w, v))$ and corresponds to an isotopy between the identity and $\tilde{G}_1$.

The heart of the proof of Theorem 8.1 is to construct the vector field $\tilde{X}_{\lambda,\mu}$. In Subsection 8.3 we construct the part corresponding to the passage through a singularity and in Subsection 8.4 we construct the part corresponding to the isotopy. Then, in Subsection 8.5 we finish the construction of $X_{\lambda,\mu}$.

8.3. Through the singularity. Recall that we identify $\mathbb{R}^n$ with $\mathbb{R} \times C \times C \times \mathbb{R}^{n-5}$ and use coordinates $(s, z, w, v)$. In $\mathbb{R}^n$ we consider the linear diagonal vector field $L_\mu$ which has eigenvalue $-\mu$ in the $s$-direction, $\sigma$ in the two $z$-directions, and $-\eta$ in all the $w$ and $v$ directions. Although one should think of $L_\mu$ as a vector field defined on a copy of $\mathbb{R}^n$ distinct from the one where we construct $\tilde{X}_{\lambda,\mu}$, at one step of the construction will be convenient to think of $L_\mu$ as defined in a neighborhood of $s = -1$ in the copy of $\mathbb{R}^n$ which corresponds to the vector field $\tilde{X}_{\lambda,\mu}$.

Definition of $\tilde{X}_{\lambda,\mu}$ for $s \in [-2, -1]$. Let $\tilde{X}_{\lambda,\mu}$ be constant equal to $(1, 0)$ on a neighborhood of $s = -2$ and let $\tilde{X}_{\lambda,\mu}$ coincide with $L_\mu$ on a neighborhood of $s = -1$. Then, letting $b = 4(1 - \lambda)^{-1}$, extend $\tilde{X}_{\lambda,\mu}$ to $s \in (-2, -1)$ so that the Poincaré map from $s = -2$ to $s = -1$ is

$$(2, z, w, v) \mapsto (-1, z/b, w, v).$$

Thus the Poincaré map shrinks $B_\lambda$ to the closed disc $\frac{1}{2} D$ of radius 1/2 in the $z$-coordinate.

Let $V_{[-2, -1]}$ be the set of points $(s, z, w, v)$ with $s \in [-2, -1]$ which belong to an orbit of the flow of $\tilde{X}_{\lambda,\mu}$ that starts at $\Sigma_+^u = \{-2\} \times B_\lambda \times \mathbb{D} \times D^{n-5}$

Definition of $\tilde{X}_{\lambda,\mu}$ for $s \in [-1, 1]$. Now observe that, under the linear flow $L_\mu$ we have the following transition map through the singularity:

$$\begin{align*}
\{1\} \times \frac{1}{4} D^* \times \mathbb{D} \times D^{n-5} & \rightarrow \{-2^{-\mu/\sigma}, 0\} \times \{\|z\| = 1\} \times \mathbb{D} \times D^{n-5} \\
(-1, z, w, v) & \mapsto (-\|z\|^{-\mu/\sigma}, z/\|z\|, |z|^{\eta/\sigma} w, |z|^{\eta/\sigma} v).
\end{align*}$$

For $\epsilon > 0$ small, let $W$ be the closed region in $\mathbb{R}^n$ bounded by $s = -1$, $s = \epsilon$, $|z| = 1$ and the orbits of $\{-1, \epsilon\} \times \partial \left(\frac{1}{2} D\right) \times \mathbb{D} \times D^{n-5}$ under the linear flow $L_\mu$ (see Figure 2).

Choose $\beta_0 > 0$ small and consider a smooth family of maps

$$\psi_\mu : W \rightarrow [-1, 1] \times 2B_\lambda \times 2\mathbb{D} \times D^{n-5}$$

such that the following hold:

(a) $\psi_\mu$ extends to a smooth family of diffeomorphisms between a neighborhood of $W$ and a neighborhood $\psi_\mu(W)$.

(b) For all $(s, z, w, v)$ in a neighborhood of $\{-1\} \times \frac{1}{4} D^* \times \mathbb{D} \times D^{n-5}$,

$$\psi_\mu(s, z, w, v) = (s, z, w, v).$$
Figure 2. Illustrates the construction of $\tilde{X}_{\lambda,\mu}$

(c) For all $(s, z, w, v) \in [-2^{-\mu/\sigma}, 2^{-\mu/\sigma}] \times \{|z| = 1\} \times \mathbb{D} \times D^{n-5}$,

$$
\psi_\mu(s, z, w, v) = (1, (1 - \lambda + \lambda b^\mu/\sigma (-s)) z, \beta_0 b^n/\sigma w, \beta_0 b^n/\sigma v)
$$

For example we may consider a non-increasing $C^\infty$ function $h_0 : [0, 1] \to [0, 1]$ such that $h_0(r) = 0$ for $r \in [0, 1/2]$, $h_0$ is strictly increasing on $[1/2, 1]$ and $h_0(1) = 1$. Then we may take $\psi_\mu(s, z, w, v) = (s', z', w', v')$ where,

\[
\begin{align*}
  s' &= h_0(|z|)(1 - s) + s \\
  z' &= \left(h_0(|z|)(-\lambda + \lambda b^\mu/\sigma (-s)) + 1\right) z \\
  w' &= \left(h_0(|z|)(\beta_0 b^n/\sigma - 1) + 1\right) w \\
  v' &= \left(h_0(|z|)(\beta_0 b^n/\sigma - 1) + 1\right) v.
\end{align*}
\]

A standard exercise shows that such $\psi_\mu$ is a diffeomorphism onto its image.

Now define $\tilde{X}_{\lambda,\mu}$ on $\tilde{V}_{[-1,1]} = \psi_\mu(W)$ as $D\psi_\mu(L_\mu)$. Note that the Poincaré map between $\tilde{\Sigma}^\circ$ and $s = 1$ is given by

$$
(-2, z, w, v) \mapsto (1, \tilde{T}_{\lambda,\mu}(z, w, v)).
$$

Denote by $\tilde{o}$ the unique singularity $\psi_\mu(0)$ of $\tilde{X}_{\lambda,\mu}$ in $\tilde{V}_{[-1,1]}$.

**Definition of $\tilde{X}_{\lambda,\mu}$ for $s \in [1, 2]$.** We let $\tilde{X}_{\lambda,\mu}$ be constant equal to $(1, 0)$ on a neighborhood of $s = 2$ and extend $\tilde{X}_{\lambda,\mu}$ to $s \in (1, 2)$ so that the Poincaré map from $s = 1$ to $s = 2$ is the identity in the $(z, w, v)$ coordinates.

Let $\tilde{V}_{[1,2]}$ be the set of points $(s, z, w, v)$ with $s \in [1, 2]$ which belong to an orbit starting at $\psi_\mu(W) \cap \{s = 1\}$.

**8.4. The isotopy.** Our aim now is to define $\tilde{X}_{\lambda,\mu}$ for $s \in [2, 3]$ so that $\tilde{X}_{\lambda,\mu}$ is the horizontal vector field $(1, 0)$ in a neighborhood of $s = 2$ and $s = 3$ and so that the Poincaré map from $s = 2$ to $s = 3$ is given by $(2, z, w, v) \mapsto (3, \tilde{G}_1(z, w, v))$. The idea is to construct an isotopy between the identity and $\tilde{G}_1$.

We need the following Lemmas.

**Lemma 8.2.** Consider the map $\gamma : [0, 1] \times \mathbb{R}/\mathbb{Z} \to \mathbb{C}^2$ defined by

$$
\gamma_s(\theta) = ((1 - s) \exp(2\pi i \theta) + s \exp(4\pi i \theta), s \exp(2\pi i \theta)).
$$

Then the following properties hold.

---

[Diagram and text content as described in the original document]
1. For every \( s \in [0, 1] \) the map \( \gamma_s \) is injective, and for every \( \theta \) we have \( \gamma'_s(\theta) \neq 0 \). In particular \( \gamma \) defines an isotopy between \( \gamma_0 \) and \( \gamma_1 \).

2. There is a smooth map \( u : [0, 1] \times \mathbb{R}/\mathbb{Z} \to \mathbb{C}^2 \) such that \( u_0(\theta) \equiv (0, 1) \), \( u_1(\theta) = (0, \exp(-2\pi i \theta)) \) for every \( \theta \), and for every \( s \) and \( \theta \), the vectors \( u_s(\theta) \) and \( \gamma'_s(\theta) \) are linearly independent over \( \mathbb{C} \).

**Proof.** The first assertion is easily proved. To prove the second let \( \chi : [0, 1] \to [0, 1] \) be a smooth function such that \( \chi \equiv 0 \) on \([0, 1/3]\) and \( \chi \equiv 1 \) on \([2/3, 1]\). For \( 0 < \varepsilon < 1/2 \) small define,

\[
u_s(\theta) = \begin{cases} (-4\chi(\varepsilon^{-1}s), 1 - \chi(\varepsilon^{-1}s)) & \text{if } s \in [0, \varepsilon] \\ (-4, 0) & \text{if } x \in [\varepsilon, 1 - \varepsilon] \\ (-4(1 - \chi(1 - \varepsilon^{-1}(1 - s))), \chi(1 - \varepsilon^{-1}(1 - s)) \exp(-2\pi i \theta)) & \text{if } s \in [1 - \varepsilon, 1] \end{cases}
\]

The lemma is now proved by straightforward calculations. \(\square\)

**Lemma 8.3.** For \( s \in [0, 1] \) and \( \beta_1 \in (0, 1) \), let \( \hat{G}_s : \hat{B}_\lambda \times \hat{D} \times D^{n-5} \to 2B_\lambda \times 2D \times D^{n-5} \) be the map defined by

\[
\hat{G}_s(z, w, v) = (\gamma_s(\theta) + \varepsilon(-it\gamma'_s(\theta) + \varepsilon u_s(\theta), \beta_1 v)
\]

where \( t = |z| \) and \( \theta = \frac{\arg(z)}{2\pi} \in \mathbb{R}/\mathbb{Z} \). If \( \beta_1 \in (0, 1) \) is sufficiently small, then for every \( s \in [0, 1] \) the map \( \hat{G}_s : \hat{B}_\lambda \times \hat{D} \times D^{n-5} \to 2B_\lambda \times 2D \times D^{n-5} \) is injective.

**Proof.** Observe that \( \|\partial_\theta \hat{G}_s - (\gamma'_s(\theta), 0)\| \leq \varepsilon(M|\gamma'_s(\theta)| + \|u'_s(\theta)\|) \) and \( \partial_\theta \hat{G}_s = (-\varepsilon \gamma'_s(\theta), 0) \). Moreover, \( \hat{G}_s \) is holomorphic in \( w \) and \( \partial_w \hat{G}_s = (\varepsilon u_s(\theta), 0) \). So, if \( \varepsilon \) is sufficiently small, the vectors \( \partial_\theta \hat{G}_s \) and \( \partial_w \hat{G}_s \) are linearly independent over \( \mathbb{R} \), and the plane generated by real combinations of \( \partial_\theta \hat{G}_s \) and \( \partial_w \hat{G}_s \) is close to the complex plane \( \{(\lambda \gamma'_s(\theta), 0) | \lambda \in \mathbb{C}\} \).

By construction \( \gamma'_s(\theta) \) and \( u_s(\theta) \) are linearly independent over \( \mathbb{C} \). It follows that, if \( \varepsilon \) is sufficiently small, \( \hat{G}_s \) is a local diffeomorphism, and that there is \( \delta > 0 \) such that for every \( \theta_0 \) the map \( \hat{G}_s \) restricted to \( \{t \exp(2\pi i \theta), w, v | |\theta - \theta_0| < \delta\} \) is injective. As \( \gamma_s \) is injective for all \( s \in [0, 1] \), it follows that, if \( \varepsilon \) is sufficiently small, \( \hat{G}_s \) is injective for all \( s \). \(\square\)

Clearly, \( \hat{G}_0(z, w, v) = ((z/|z|) + 2\varepsilon z, \varepsilon w, \beta_1 v) \) is isotopic to the identity, and

\[
\hat{G}_1(z, w, v) = \left( \frac{z^2}{|z|^2} + 4\varepsilon \frac{z^2}{|z|^2} \frac{z}{|z|} + 2\varepsilon |z| \frac{|z|}{z}, \beta_1 v \right)
\]

is isotopic to \( \hat{G}_1 \).

Let \( \hat{H} : \hat{B}_\lambda \times \hat{D} \times D^{n-5} \to 2B_\lambda \times 2D \times D^{n-5} \) be an isotopy between the identity and \( \hat{G}_1 \). Let \( h_1 : [2, 3] \to [0, 1] \) be a \( C^\infty \) function that is constant equal to 0 (resp. 1) on a neighborhood of 2 (resp. 3). The vector field \( \tilde{X}_{\lambda, \mu} \) will be defined on the set \( \tilde{V}_{[2,3]} = \{ (s, \hat{H}_{h_1(s)}(x)) | s \in [2, 3] \text{ and } x \in \hat{B}_\lambda \times \hat{D} \times D^{n-5} \} \), by

\[
\tilde{X}_{\lambda, \mu}(s, z, w, v) = \left( 1, \frac{\partial}{\partial s} \left( \hat{H}_{h_1(s)} \left( \hat{H}_{h_1(s)}^{-1}(z, w, v) \right) \right) \right).
\]
By construction \( \widetilde{X}_{\lambda,\mu} \) induces the Poincaré map \((2, x) \mapsto (3, \hat{G}_1(x))\). Since the function \( h_1 \) is constant on a neighborhood of \( s = 2 \) and of \( s = 3 \), it follows that \( \widetilde{X}_{\lambda,\mu} \) is constant equal to \((1, 0)\) on a neighborhood of \( s = 2 \) and of \( s = 3 \).

8.5. End of the construction. We let the domain of definition of the vector field \( \widetilde{X}_{\lambda,\mu} \) be

\[
\widetilde{V} = \widetilde{V}_{[-2, -1]} \cup \widetilde{V}_{[-1, 1]} \cup \widetilde{V}_{[1, 2]} \cup \widetilde{V}_{[2, 3]}
\]

and observe that the unique singularity \( \bar{o} \) of \( \widetilde{X}_{\lambda,\mu} \) is \( \psi_\mu(0) \in \widetilde{V}_{[-1, 1]} \). The stable manifold of this singularity \( W^s(\bar{o}) \) consists of \( \psi_\mu(W \cap \{ z = 0 \}) \subset \widetilde{V}_{[-1, 1]} \) together with the points in \( \widetilde{V}_{[-2, -1]} \) that are in the backward orbit of points in \( \psi_\mu(W \cap \{ z = 0 \}) \). In particular, \( W^s(\bar{o}) \cap \Sigma^u_\pm = \{-1\} \times \{0\} \times \mathbb{B}^2 \times D^{n-5} \). Moreover, for every neighborhood \( N \) of \( W^s(\bar{o}) \), there exists \( t_0 > 0 \) such that the orbit of every point \((s, x) \in V \setminus N \) hits \( \Sigma^u_\pm \) before time \( t_0 \). Note that \( W^s(\bar{o}) \cap \Sigma^u_\pm = \emptyset \) and \( W^s(\bar{o}) \cap \Sigma^s_\pm = \emptyset \).

Now we pass to the solid torus \( T^n \) by the map \( Q : (s, z, w, v) \mapsto ((s+2)/5, az, aw, v/4) \), and obtain the vector field \( X_{\lambda,\mu} = DQ(\widetilde{X}_{\lambda,\mu}) \) defined on \( V = Q(\widetilde{V}) \) whose flow is transversal to the cross sections \( \Sigma^u = Q(\Sigma^u_\pm) \) and \( \Sigma^s = Q(\Sigma^s_\pm) \). Moreover, \( X_{\lambda,\mu} \) has a unique singularity \( o = Q(\bar{o}) \) in \( V \) with local stable manifold \( W^s_\text{loc}(o) = Q(W^s(\bar{o})) \).

Take \( \mu = \sigma \) and observe that for every \( t_0 > 0 \) we have \( X^\mu_{\lambda,\sigma}(V) \subseteq \text{interior}(V) \). Therefore, we may extend the definition of \( X_{\lambda,\sigma} \) to a neighborhood \( V' \) of \( V \) and find a neighborhood \( U \) of \( V \) that is bounded by a smooth manifold of dimension \( n - 1 \) contained in \( V' \), in such a way that \( X_{\lambda,\sigma} \) points inward on the boundary of \( U \). Since \( U \) contains \( V \) for \( \mu \) close to \( \sigma \) we may extend the definition of \( X_{\lambda,\mu} \) from \( V \) to a neighborhood of the closure of \( U \), so that we have a smooth family of vector fields defined in \( U \) for \( \mu \) close to \( \sigma \). It follows that there exists \( \mu_0 \) so that for all \( \mu \in (\mu_0, \sigma) \) the vector field \( X_{\lambda,\mu} \) points inward on the boundary of \( U \). The rest of the properties required in Theorem 8.1 for \( X_{\lambda,\mu} \) easily follow.

9. First return map and leaf space transformation

In this section we show that for vector fields \( X \) close to \( X_{\lambda,\mu} \), the first return map \( \hat{F}_X \) to \( \Sigma^u \) of the flow of \( X \) admits a strong stable foliation of codimension 2 (Subsection 9.1) and we study the corresponding leaf space transformation (Subsection 9.2). In particular, we show that when \( X \) is of class \( C^2 \), the corresponding leaf space transformation belongs to \( \mathcal{F}_\lambda \) (Lemma 9.3).

9.1. First return map. Fix an integer \( n \geq 5 \). For a given \( \lambda \in (0, 1) \) close to 1, let \( X_\lambda = X_{\lambda,\sigma} \) be the vector field defined on \( T^n \) and let \( \Sigma^u \approx B_\lambda \times \mathbb{B}^2 \times D^{n-5} \) be the transversal section to \( X_\lambda \), given by Theorem 8.1.

Note that the intersection between a local stable manifold of the singularity \( o = o_{\lambda,\sigma} \) and \( \Sigma^u \) is equal to \( \{0\} \times \mathbb{B}^2 \times D^{n-5} \). For \( X \) close to \( X_\lambda \) in the \( C^1 \) topology we denote by \( o_X \) the singularity of \( X \) that is the continuation of the hyperbolic singularity \( o \) of \( X_\lambda \). After a smooth coordinate change, we assume that for every vector field \( X \) close to \( X_\lambda \) in the \( C^1 \) topology, the intersection of a local stable manifold of the singularity \( o_X \) of \( X \) with \( \Sigma^u \) is equal to \( \{0\} \times \mathbb{B}^2 \times D^{n-5} \). Then we set \( \Sigma^{u*} = \Sigma^u \setminus \{0\} \times \mathbb{B}^2 \times D^{n-5} \) and observe that there is a well defined Poincaré map \( \hat{F}_X : \Sigma^{u*} \to \Sigma^u \) of the flow of \( X \).
The following lemma is a consequence of [HPS]. Recall that \( \Pi_\lambda : \Sigma^u \approx B_\lambda \times \mathbb{D} \times D^{n-5} \rightarrow B_\lambda \) is the projection to the first coordinate.

**Lemma 9.1 (Strong stable foliation).** Let \( \lambda \in (0,1) \) be close to 1, let \( \mu \in (0,\sigma) \) be close to \( \sigma \) and let \( X_{\lambda,\mu} \) be the vector field given by Theorem 8.1. Then there is a neighborhood \( \mathcal{O} \) of \( X_{\lambda,\mu} \) in the \( C^1 \) topology, such that for every \( X \) in \( \mathcal{O} \) we have the following properties.

1. There is a strong stable foliation \( \mathcal{F}^s_X \) of \( \hat{F}_X \) in \( \Sigma^u \), having \( \{0\} \times \mathbb{D} \times D^{n-5} \) as a leaf. The leaves of \( \mathcal{F}^s_X \) are submanifolds of codimension 2 in \( \Sigma^u \) that are of class \( C^1 \). Moreover the foliation \( \mathcal{F}^s_X \) is \( C^{0} \) close to the foliation formed by the fibers of the projection \( \Pi_\lambda \).
2. Every leaf of the strong stable foliation intersects \( B_\lambda \times \{0\} \times \{0\} \) in at most one point. For every point \( x \in \Sigma^u \) in a leaf intersecting this set, we denote by \( \Pi_X(x) \) the point in \( B_\lambda \) such that \( (\Pi_X(x),0,0) \) is in the same leaf as \( x \). Then the map \( \Pi_X \) is continuous and \( C^{0} \) close to the projection \( \Pi_\lambda \). In the particular case when \( X \) is of class \( C^2 \), the map \( \Pi_X \) is of class \( C^1 \) and it depends on a \( C^1 \) way on \( X \).

**Proof.** As \( \mu \in (0,\sigma) \), it follows that \( \hat{F}_{\lambda,\mu} \) contracts the fibers of the projection \( \Pi_\lambda \) in a stronger way than any contraction of \( F_{\lambda,\mu} \).

Moreover, the stronger contraction factor of \( F_{\lambda,\mu} \) in \( B_\lambda^s \),
\[
\inf\{\|D(F_{\lambda,\mu})z(v)\| \mid z \in B_\lambda^s, \|v\| = 1\},
\]
can be made arbitrarily close to \( \lambda \) by choosing \( \mu \in (0,\sigma) \) sufficiently close to \( \sigma \). On the other hand, the fiber contraction can be made arbitrarily strong by choosing \( \beta > 0 \) small enough. Then the results follow from [HPS] using a graph transforming method. For a direct exposition of these methods which applies to our case see [BLMP].

### 9.2. Leaf space transformation.

Let \( \lambda \in (0,1) \), \( \mu \in (0,\sigma) \) and \( \mathcal{O} \) be as in Lemma 9.1. As the closure of \( \hat{F}_{\lambda,\mu}(\Sigma^{u^+}) \) is in the interior of \( \Sigma^u \), reducing \( \mathcal{O} \) if necessary we assume that every leaf through a point in \( \hat{F}_X(\Sigma^{u^+}) \) intersects \( B_\lambda \times \{0\} \times \{0\} \) in a unique point. Then the leaf space transformation \( F_X : B_\lambda^s \rightarrow B_\lambda \) is defined by
\[
F_X(z) = \Pi_X(\hat{F}_X(z,0,0)).
\]
The map \( F_X \) is continuous, but in general not differentiable.

**Lemma 9.2.** Reducing \( \mathcal{O} \) if necessary we have that for every \( X \in \mathcal{O} \) the map \( F_X \) is a local homeomorphism.

**Proof.** Note that two points \( z, z' \in \Sigma^+ \) have the same image under \( F_{\lambda,\mu} \) if and only if \( z + z' = 0 \). On the other hand, the images under \( \hat{F}_{\lambda,\mu} \) of the leaves \( \{z\} \times \mathbb{D} \times D^{n-5} \) and \( \{-z\} \times \mathbb{D} \times D^{n-5} \) of \( F_{\lambda,\mu} \) lie in the same leaf of \( \mathcal{F}^s_X \) and their distance is at least \( 2(1 - \beta) \).

By continuity, for every vector field \( X \) that is sufficiently close to \( X_{\lambda,\mu} \) in the \( C^1 \) topology, the distance between 2 points in the same leaf of \( \mathcal{F}^s_X \) whose preimages by \( \hat{F}_X \) are in distinct leaves, is at least \( 1 - \beta \). This implies that arbitrarily close leaves of \( \mathcal{F}^s_X \) can not have images contained in the same leaf. It follows that \( F_X \) is
locally injective and therefore a local homeomorphism.

For $C^2$ vector fields we can say even more.

**Lemma 9.3.** Denote by $O'$ the subset of $O$ of vector fields of class $C^2$. Then, reducing $O$ if necessary, the following properties hold.

1. For each vector field $X$ in $O'$, there are $\tau_X \in \mathcal{T}_\lambda$ and $g_X \in \mathcal{G}_\lambda$ such that $F_X = g_X \circ \tau_X$. In particular $F_X \in \mathcal{F}_\lambda$.
2. The maps $X \mapsto \tau_X$, $X \mapsto g_X$ and $X \mapsto F_X$ from $O'$ to $\mathcal{T}_\lambda$, $\mathcal{G}_\lambda$ and $\mathcal{F}_\lambda$, respectively, are all continuous. Note that here $O' \subset O$ is provided with the $C^1$ topology.
3. For each $X \in O'$ there is a neighborhood $\mathcal{U}$ of $g_X$ in $\mathcal{G}_\lambda$, such that for every $g \in \mathcal{U}$ there is $Y \in \mathcal{O}$ satisfying $F_Y = g \circ \tau_X$.
4. For each $X \in O'$ the map $F_X$ is area expanding.

**Proof.**

1. For $X \in O'$ the first return map $\tilde{F}_X$ is of class $C^2$ and $\Pi_X$ is of class $C^1$. So $F_X$ is of class $C^1$.

For each vector field $X$ that is close to $X_\lambda$ in the $C^1$ topology, let $\tilde{T}_X$ (resp. $\hat{G}_X$) be the Poincaré map from $\Sigma^u$ to $\Sigma^s$ (resp. from $\Sigma^s$ to $\Sigma^u$). Clearly $\tilde{F}_X = \hat{G}_X \circ \tilde{T}_X$. The pull-back of the strong stable foliation $\mathcal{F}_{\mathfrak{s}_{\lambda}}$ from $\Sigma^u$ to $\Sigma^s$ by $\hat{G}_X$ defines a foliation $\mathcal{F}_{\tilde{F}_X}^{\mathfrak{s}_{\lambda}}$ in $\Sigma^s \approx \overline{B}_\lambda \times \mathbb{B} \times D^{n-5}$. The map $\tilde{T}_X$ carries leaves of $\mathcal{F}_{\mathfrak{s}_{\lambda}}^{\mathfrak{s}_{\lambda}}$ into leaves of $\mathcal{F}_{\tilde{F}_X}^{\mathfrak{s}_{\lambda}}$. When $X = X_\lambda$, the foliation $\mathcal{F}_{\tilde{F}_X}^{\mathfrak{s}_{\lambda}}$ is the one formed by the fibers of the projection $\tilde{\Pi}_X : \overline{B}_\lambda \times \mathbb{B} \times D^{n-5} \to \overline{B}_\lambda$ to the first coordinate. Thus the foliation $\mathcal{F}_{\tilde{F}_X}^{\mathfrak{s}_{\lambda}}$ is close to $\mathcal{F}_{\mathfrak{s}_{\lambda}}$. Let $\tilde{\Pi}_X$ be the projection along leaves of $\mathcal{F}_{\mathfrak{s}_{\lambda}}$, in such a way that $x$ and $(\tilde{\Pi}_X(x), 0, 0)$ are in the same leaf of $\mathcal{F}_{\tilde{F}_X}^{\mathfrak{s}_{\lambda}}$. The map $\tilde{\Pi}_X$ is defined on a large subset of $\overline{B}_\lambda \times \mathbb{B} \times D^{n-5}$ and takes images in $\overline{B}_\lambda$. Note that $\tilde{\Pi}_X = \Pi_X$ and that when $X$ of class $C^2$ varies continuously in the $C^1$ topology, $\tilde{\Pi}_X$ varies in a $C^1$ way.

Let $\iota : B_\lambda \to \overline{B}_\lambda \times \mathbb{B} \times D^{n-5}$ (resp. $\tilde{\iota} : \overline{B}_\lambda \to \overline{B}_\lambda \times \mathbb{B} \times D^{n-5}$) be defined by $z \mapsto (z, 0, 0)$ and put

$$T_X = \tilde{\Pi}_X \circ \tilde{T}_X \circ \iota : B_\lambda^* \to \overline{B}_\lambda,$$

$$G_X = \Pi_X \circ \hat{G}_X \circ \tilde{\iota} : \overline{B}_\lambda \to B_\lambda.$$

Note that $F_X = G_X \circ T_X$, $\tilde{T}_X = \tilde{T}_X$, $\hat{G}_X = \hat{G}_X$, $\tilde{T}_X = \tilde{T}_X$ and $G_X = G_X$. So if we denote by $\psi : \mathbb{C}^* \to \mathbb{R}/\mathbb{Z} \times \mathbb{R}$ the map $\psi(z) = (\frac{1}{2\pi} \arg z, |z|)$, with inverse $\psi^{-1}(\theta, t) = t \exp(2\pi i \theta)$, then $\tau_X = \psi \circ T_X$, and $g_1 = G_X \circ \psi^{-1}$.

For a vector field $X$ of class $C^2$ that is close to $X_\lambda$ in the $C^1$ topology, the image of $B_\lambda^*$ by $T_X$ is an annulus that is bounded by the Jordan curve $T_X(\{z \in \mathbb{C} \mid |z| = 2(1 - \lambda)^{-1}\})$ and the Jordan curve equal to the image by $\tilde{\Pi}_X$ of the intersection of the local unstable manifold of $o_X$ with $\Sigma^s$. These Jordan curves are of class $C^1$ and they vary continuously in the $C^2$ topology, when $X$, of class $C^2$, varies continuously in the $C^1$ topology. So for such $X$ we can find a homeomorphism

$$\psi_X : T_X(B_\lambda^*) \to \mathbb{R}/\mathbb{Z} \times [1 - \lambda, 2(1 - \lambda) - 1 - \lambda],$$

that extends to a diffeomorphism onto its image, defined on a neighborhood of $T_X(B_\lambda^*)$, in such a way that $\psi_{X_\lambda}$ coincides with $\psi$ and that $\psi_X$ varies continuously in the (strong) $C^1$ topology when $X$, of class $C^2$, varies continuously in the $C^1$ topology.
topology. Then we have \( \tau_X = \psi_X \circ T_X \in \mathcal{F}_\lambda \), \( g_X = G_X \circ \psi_X^{-1} \in \mathcal{F}_\lambda \) and \( F_X = G_X \circ T_X = g_X \circ \tau_X \in \mathcal{F}_\lambda \).

2. Clearly when \( X \) of class \( C^2 \) varies continuously in the \( C^1 \) topology, \( G_X \), \( \psi_X \) and \( g_X = G_X \circ \psi_X^{-1} \) vary continuously in the (strong) \( C^1 \) topology. It remains to show that \( T_X \) varies continuously in the (weak) \( C^1 \) topology. For that, just observe that for each neighborhood \( \hat{U} \) of \( \{0\} \times \mathbb{D} \times D^{n-5} \) in \( \Sigma^u \) (the intersection of the stable manifold of \( o_{\lambda,\mu} \) with \( \Sigma^u \)) the map \( X \mapsto \widehat{T_X\mid_{\Sigma^u \setminus \hat{U}}} \) is continuous in the \( C^1 \) topology.

3. Keep the notation of part 1. Note first that for every map \( \hat{G} \) of class \( C^1 \) that is close to \( \hat{G}_X \) there is a vector field \( Y \in \mathcal{O} \) that coincides with \( X \) between \( \Sigma^u \) and \( \Sigma^s \) (so that \( \hat{T}_Y = \hat{T}_X \)) and such that \( \hat{G}_Y = \hat{G}_X \). On the other hand, note that for every \( g \) close to \( g_X \) we can find \( \hat{G} \) close to \( \hat{G}_X \) mapping leaves of \( \hat{F}_X \mathcal{F}^s \) into leaves of \( \hat{F}_X \mathcal{F}^s \) and such that \( \Pi_X \circ \hat{G} \circ \tilde{\iota} \) coincides with \( g \circ \psi_X \) on \( \tau_X(B_\lambda) \).

For a given \( g \in \mathcal{F}_\lambda \) near \( g_X \) let \( \hat{G} \) and \( Y \) be as above, so that \( \hat{T}_Y = \hat{T}_X \), \( \hat{G}_Y = \hat{G}_X \) and \( \hat{G}_Y \) maps leaves of \( \hat{F}_X \mathcal{F}^s \) into leaves of \( \hat{F}_X \mathcal{F}^s \). It follows that the strong stable foliation \( \hat{F}_Y \mathcal{F}^s \) of \( \hat{F}_Y \) is equal to \( \hat{F}_X \mathcal{F}^s \) and that \( \Pi_Y = \Pi_X \). So \( G_Y = \Pi_X \circ \hat{G} \circ \tilde{\iota} \) coincides with \( g \circ \psi_X \) on \( \tau_X(B_\lambda) \) and we have,

\[
F_Y = G_Y \circ T_Y = g \circ (\psi_X \circ T_X) = g \circ \tau_X.
\]

4. To prove that \( F_X \) is area expanding, we first notice that the map \( F_{\lambda,\mu} \) is area expanding. For each vector field \( X \in \mathcal{O} \) the map \( F_X \) is \( C^1 \)-close to \( F_{\lambda,\mu} \) outside a small neighborhood of the origin. So \( F_X \) is area expanding on this set. That \( F_X \) is area expanding close to 0 is clear from the eigenvalues of the linear part of \( X \) at the singularity \( o_X \), as these are close to those of the linear part of \( X_{\lambda,\mu} \) at \( o_{\lambda,\mu} \), \( \square \)

10. Robust transitivity

This section is dedicated to the proof of Theorem 2.8

10.1. Proof of part 1 of Theorem 2.8 Let \( X_{\lambda,\mu}, U, o, \ldots \) be as in the statement of Theorem 2.8. For a vector field \( X \) close to \( X_{\lambda,\mu} \), let \( \Lambda_X \) be the maximal invariant set in \( U \) of the flow of \( X \) and let \( o_X \) be the singularity of \( X \) that is the continuation of the hyperbolic singularity \( o \) of \( X_{\lambda,\mu} \). Clearly we have \( o_X \in \Lambda_X \). By construction of \( X_{\lambda,\mu} \), it follows that for every vector field \( X \) that is sufficiently close to \( X_{\lambda,\mu} \), the point \( o_X \) is the unique singularity of \( X \) in \( U \). So, to prove part 1 of Theorem 2.8 it remains to show that for \( \lambda \in (0, 1) \) sufficiently close to 1 and \( \mu \in (0, \sigma) \) sufficiently close to \( \sigma \), there is a neighborhood \( \mathcal{O} \) of \( X_{\lambda,\mu} \) in the \( C^1 \) topology such that for every \( X \in \mathcal{O} \) the restriction of the flow of \( X \) to \( \Lambda_X \) is topologically mixing.

Recall that \( \widehat{F}_X : \Sigma^{uu} \to \Sigma^u \) is the first return map to \( \Sigma^u \) of the flow of \( X \). Although \( \widehat{F}_X \) is not defined on \( \{0\} \times \mathbb{D} \times D^{n-5} \) we let it act on subsets of \( \Sigma^u \) by \( \widehat{F}_X(\hat{U}) := \widehat{F}_X(\hat{U} \setminus \{0\} \times \mathbb{D} \times D^{n-5}) \). We denote by

\[
\widehat{\Omega}_X = \cap_{m \geq 1} \widehat{F}_X^m(\Sigma^u),
\]

the maximal invariant set of \( \widehat{F}_X \) in \( \Sigma^u \). The maximal invariant set \( \Lambda_X \) of the flow of \( X \) is just the closure of the suspension of \( \widehat{\Omega}_X \) by the flow of \( X \). So to prove part 1 of Theorem 2.8 is enough to prove that the restriction of \( \widehat{F}_X \) to \( \widehat{\Omega}_X \) is topologically mixing.
Proposition 10.1. Let $\lambda \in (0, 1), \mu \in (0, \sigma)$ and $\Omega$ be as in Lemma 9.3. Then for every vector field $X \in \mathcal{O}$ such that the restriction of $F_X$ to $\Omega_X$ is topologically mixing, the map $\hat{F}_X$ restricted to $\hat{\Omega}_X$ is topologically mixing.

For the proof of Proposition 10.1 we need the following lemma.

Lemma 10.2. Given an open set $\hat{U}$ in $\Sigma^u$ intersecting $\hat{\Omega}_X$, there exists an open set $U_0$ in $B_\lambda$ intersecting $\Omega_X$ and an integer $j \geq 0$, such that $\hat{F}_X^j(\Pi_X^{-1}(U_0)) \subset \hat{U}$.

Proof. Let $\hat{U}$ be an open set as in the hypothesis and take $x \in \hat{U} \cap \hat{\Omega}_X$. Then for every integer $j \geq 0$ there exists a point $x_j \in \hat{\Omega}_X$ such that $\hat{F}_X^j(x_j) = x$. As $\hat{F}_X$ contracts the leaves of the foliation $F_X^u$ uniformly, for $j$ sufficiently large the leaf of $F_X^u$ through the point $x_j$ is mapped well inside $\hat{U}$ by $\hat{F}_X^j$. The same happens for a neighborhood of leaves of $F_X^u$.

Proof of Proposition 10.1. Let $\hat{U}, \hat{V}$ be two open subsets of $\Sigma^u$ that intersect $\hat{\Omega}_X$. Take an open set $U_0$ in $B_\lambda$ and a positive integer $j$ such that $\hat{F}_X^j(\Pi_X^{-1}(U_0)) \subset \hat{U}$, given by the previous lemma. Since the restriction of $F_X$ to $\Omega_X$ is topologically mixing and $U_0$ intersects $\Omega_X$, there exists a positive integer $k_0$ such that for every $k \geq k_0$ we have $F_X^k(\Pi_X^{-1}(\hat{U})) \cap U_0 \neq \emptyset$. Hence for every $k \geq k_0$ we have $\hat{F}_X^k(\hat{V}) \cap \Pi_X^{-1}(U_0) \neq \emptyset$. But this implies that for every $\ell \geq j + k_0$ we have $\hat{F}_X^\ell(\hat{V}) \cap \hat{U} \neq \emptyset$. So, the restriction of $\hat{F}_X$ to $\hat{\Omega}_X$ is topologically mixing.

10.2. Proof of part 2 of Theorem 2.8 Let $\lambda \in (0, 1), \mu \in (0, \sigma)$ and $\mathcal{O}$ be as in Lemma 9.3. Let $\hat{\rho}_\lambda$ be the unique fixed point of $\hat{F}_{\lambda, \sigma}$ in $\Pi_X^{-1}(p_\lambda)$. Reducing $\mathcal{O}$ if necessary, we assume that for each $X \in \mathcal{O}$ there is a fixed point $\hat{p}_X$ of $\hat{F}_X$ that is the continuation of $\hat{\rho}_\lambda$ and such that $\Pi_X(\hat{p}_X) \in H_x$. Thus, for every $X \in \mathcal{O}$ of class $C^2$ we have $\Pi_X(\hat{p}_X) = p_{F_X}$.

For $X \in \mathcal{O}$ let

$$W_X = \{x \in \hat{\Omega}_X \mid \hat{F}_X^m(x) \in \Pi_X^{-1}(H_\lambda) \text{ for all } m \geq 0\},$$

be the maximal invariant set of $\hat{F}_X$ in $\Pi_X^{-1}(H_\lambda)$. Clearly $W_X$ is a compact set invariant by $\hat{F}_X$.

1. Identify the tangent space at each point of $\Sigma^u \approx B_\lambda \times \overline{D} \times D^n$ with $\mathbb{C} \times \mathbb{C} \times \mathbb{R}^{n-5}$ and denote by $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^{n-5}$. Put $\varepsilon_0 = \frac{1}{2}(1 - \lambda)^{1/2}$, as in Subsection 3.1 and let $\hat{C}$ and $\hat{K}$ be the cone fields defined on $B_\lambda \times \overline{D} \times D^n$ by

$$\hat{C}(z_0, v_0, w_0) = \{(z_0 \rho(1 + i\varepsilon), w, v) \mid \rho \geq 0, |\varepsilon| \leq \varepsilon_0 \text{ and } \rho \geq (1 - \lambda) \max\{|w|, |v|\}\},$$

$$\hat{K}(z_0, v_0, w_0) = \{(z_0 \rho(i + \varepsilon), w, v) \mid \rho \geq 0, |\varepsilon| \leq 1/3 \text{ and } \rho \geq (1 - \lambda) \max\{|w|, |v|\}\}.$$
$\mu \in (0, \sigma)$ sufficiently close to $\sigma$, the cone field $\mathcal{C}$ (resp. $\mathcal{K}$) is an stable (resp. unstable) and invariant cone field for $F_{\lambda, \mu} : \Pi_{\lambda, \mu}^{-1}(H_{\lambda}) \to \Pi_{\lambda, \mu}^{-1}(H_{\lambda}))$. Then, reducing $\mathcal{O}$ if necessary, for every $X \in \mathcal{O}$ the cone field $\mathcal{C}$ (resp. $\mathcal{K}$) is an stable (resp. unstable) and invariant cone field for $F_X : \Pi_X^{-1}(H_{\lambda}) \to F_X(\Pi_X^{-1}(H_{\lambda}))$. It follows that $\tilde{W}_X$ is a uniformly hyperbolic set for $F_X$.

2. For $X \in \mathcal{O}$ put,

$$\tilde{\Gamma}_X = \{\{x_j\}_{j \geq 0} \mid \Pi_X(x_j) \in H_{\lambda}, \, \tilde{F}_X(x_{j+1}) = x_j\}.$$ 

Note that for every infinite backward orbit $\{x_j\}_{j \geq 0}$ of $\tilde{F}_X$ in $\tilde{\Gamma}_X$, the sequence $\{\Pi_X(x_j)\}_{j \geq 0}$ is an infinite backward orbit of $F_X$ in $\Gamma_X := \Gamma_{F_X}$. Conversely, for every infinite backward orbit $\{z_j\}_{j \geq 0}$ of $F_X$, the sequence $\{x_j\}_{j \geq 0}$ defined by the property

$$\{x_j\} = \cap_{j \geq 0} \tilde{F}_X^{-1}(\Pi_X^{-1}(z_{j+1})),$$

is an infinite backward orbit of $\tilde{F}_X$ in $\tilde{\Gamma}_X$. From the theory of uniformly hyperbolic sets we know that for every infinite backward orbit $\tilde{z} \in \tilde{\Gamma}_X$ there is a one dimensional local unstable manifold $W^u_{\tilde{z}}(x_0)$ of $\tilde{F}_X$ through $x_0$.

Reducing $\mathcal{O}$ if necessary, part 1 of Theorem 2.6 implies that, if $X \in \mathcal{O}$ is of class $C^2$, then the local unstable manifold of each infinite backward orbit in $\tilde{\Gamma}_X$ is contained in the unstable manifold of some point of $\tilde{W}_X$.

3. Let $\lambda \in (0, 1)$ be sufficiently close to 1 and let $\mathcal{U}$ be a sufficiently small neighborhood of $F_{\lambda}$ in $\mathcal{F}_{\lambda}$, so that for every $F \in \mathcal{U}$ there is an arc $\mathcal{C}_F$ of the stable manifold of the fixed point $p_F$ of $F$ that is tangent to the local unstable manifold of an infinite backward orbit $\tilde{z}_F$ of $F$ in $\Gamma_F$ (part 2 of Theorem 2.6).

Reducing $\mathcal{O}$ if necessary we assume that for every $X \in \mathcal{O}$ of class $C^2$ we have $F_X \in \mathcal{U}$ (Lemma 2.3). As all the elements of $\tilde{W}_{F_X}$ belong to $A_{F_X}$, which is well inside $H_{\lambda}$, it follows that there is an infinite backward orbit $\tilde{z}_X$ of $\tilde{F}_X$ in $\tilde{\Gamma}_X$ that projects to $\tilde{z}_F$ by $\Pi_X$. Therefore the local unstable manifold $W^u_{\tilde{z}_X}(x_0)$ is tangent to the submanifold $\Pi_X^{-1}(\tilde{\gamma}_{F_X})$ of the stable manifold of $\tilde{p}_X$.

4. Let $X$ be an arbitrary vector field in $\mathcal{O}$ and let $\{X_j\}_{j \geq 0}$ be a sequence of vector fields of class $C^2$ in $\mathcal{O}$, that converge to $X$ in the $C^1$ topology. Let $\mathcal{S} = \tilde{\mathcal{S}}_{X_j} \in \tilde{\Gamma}_{X_j}$ be given by part 3, so that the local unstable manifold of $\mathcal{S}$ is tangent to $\Pi_{X_j}^{-1}(\tilde{\gamma}_{F_{X_j}})$. Setting $\mathcal{S}^j = \{x_{kj}\}_{k \geq 0}$ and taking a subsequence if necessary, we assume that for every $k \geq 0$ the $x_{kj}$ converge to some point $x_k$ as $j \to \infty$. It follows that $\Pi_X(x_k) \in H_{\lambda}$ and that $\mathcal{S} = \{x_k\}_{k \geq 0}$ is an infinite backward orbit of $\tilde{F}_X$ in $\tilde{\Gamma}_X$. Moreover is easy to check that the local unstable manifolds $W^u_{\alpha}(\mathcal{S}^j)$ converge to $W^u_{\alpha}(\mathcal{S})$ in the $C^1$ topology and that the manifolds $\Pi_{X_j}^{-1}(\tilde{\gamma}_{F_{X_j}})$ converge in the $C^1$ topology to a...
submanifold of the stable manifold of \( \hat{p}_X \), as \( j \to \infty \). So \( W^u(x) \) is tangent to the stable manifold of \( \hat{p}_X \). As \( \hat{p}_X \) is contained in \( \hat{W}_X \) and by part 2 the set \( W^u(x) \) is contained in the unstable foliation of \( \hat{W}_X \), it follows that \( \hat{W}_X \) is a wild hyperbolic set for \( \hat{F}_X \).

10.3. **Proof of part 3 of Theorem** Let \( U \) be the neighborhood of \( F_\lambda \) in \( \mathcal{F}_\lambda \) given by Proposition 2.6. Let \( \lambda \in (0,1) \) sufficiently close to 1, \( \mu \in (0,\sigma) \) sufficiently close to \( \sigma \) and \( \mathcal{O} \) a sufficiently small neighborhood of \( X_{\lambda,\mu} \) such that for every \( X \in \mathcal{O} \) of class \( C^2 \) we have \( F_X \in U \) (Lemma 9.3).

1. We will show that there is a dense subset \( D \) of \( \mathcal{O} \) of vector fields \( X \) such that the set of periodic sources and the set of periodic saddles of \( F_X \) are both dense in \( \Omega_X \). To prove this assertion, let \( Y \in \mathcal{O} \) be a given vector field of class \( C^2 \) and let \( \tau_Y \in \tau \) and \( g_Y \in \mathcal{G}_\lambda \) be given by Lemma 9.3. Part 4 of the same lemma implies that \( g_Y \in U_{\lambda,\mu} \) and Proposition 2.6 implies that there is \( g \) close to \( g_Y \) in \( \mathcal{G}_\lambda \) such that the set of periodic sources and the set of periodic saddles of \( F = g \circ \tau_Y \) are both dense in \( \Omega_F \). By part 3 of Lemma 9.3 there is a vector field \( X \in \mathcal{O} \) near \( Y \) such that \( F_X = g \circ \tau_Y \). This shows the assertion.

2. We will now prove that for \( X \in D \) the set of periodic points of Morse index 1 (resp. 2) of \( \hat{F}_X \) is dense in \( \Omega_X \).

Given a point \( x \) in \( \hat{\Omega}_X \) and an integer \( j \geq 0 \) let \( x_j \) be the unique point in \( \Sigma^u \) such that \( \hat{F}_X^j(x_j) = x \). Let \( \{p_j\}_{j \geq 0} \) be a sequence of saddle (resp. repelling) periodic points of \( F_X \) close enough to \( \Pi_X(x_j) \), so that \( F_X^j(p_j) \) converges to \( \Pi_X(x) \) as \( j \to \infty \). Since \( \hat{F}_X \) contracts uniformly the fibers of the map \( \Pi_X \), it follows that the sequence of periodic point of Morse index 1 (resp. 2) \( \{\hat{F}_X^j(p_j)\}_{j \geq 0} \) of \( \hat{F}_X \) converges to \( x \).

3. Let \( \hat{T} \) be a trapping region for \( \hat{F}_{X_{\lambda,\mu}} \) containing \( \hat{\Omega}_{X_{\lambda,\mu}} \). Reducing \( \mathcal{O} \) if necessary we assume that for every \( X \in \mathcal{O} \), the set \( \hat{T} \) contains \( \Omega_X \) and is a trapping region for \( \hat{F}_X \). Let \( \hat{B} \) be a countable base of the topology of \( \hat{T} \).

For \( \hat{U} \in \hat{B} \) and \( X \in \mathcal{O} \cap D \) we have either \( \Omega_X \cap \hat{U} = \emptyset \) or \( \Omega_X \cap \hat{U} \neq \emptyset \). In the former case there exists a \( C^1 \) open neighborhood \( \mathcal{O}_X^U \) of \( X \) such that for every \( Y \in \mathcal{O}_X^U \) we have \( \Omega_Y \cap \hat{U} = \emptyset \). In the later case there exists a \( C^1 \) open neighborhood \( \mathcal{O}_X^U \) of \( X \) such that for every \( Y \in \mathcal{O}_X^U \) the set \( \hat{U} \) contains a periodic saddle (resp. source) of \( \hat{F}_Y \).

Then, \( D_{\hat{U}} = \bigcup_{X \in \mathcal{O} \cap D} \mathcal{O}_X^U \) is an \( C^1 \) open and dense subset of \( \mathcal{O} \) and \( \mathcal{R} = \bigcap_{\hat{U} \in \hat{B}} D_{\hat{U}} \) is a \( C^1 \) residual subset of \( \mathcal{O} \) such that for every \( Y \in \mathcal{R} \), the set of periodic points of Morse index 1 (resp. 2) of \( \hat{F}_Y \) is dense in \( \hat{\Omega}_X \). \( \square \)

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