Identification of the initial temperature from the given temperature data at the left end of a rod

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Abstract

This study aims to investigate the problem of determining the unknown initial temperature in a variable coefficient heat equation. We obtain the existence and uniqueness of the solution of the optimal control problem considered under some conditions. Using the adjoint problem approach, we get the Frechet differential of the cost functional. We construct a minimizing sequence and give the convergence rate of this sequence. Also, we test the theoretical results in a numerical example by using the MAPLE® program.

Keywords: Optimal Control; Frechet Differential; Adjoint Approach.

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1 Introduction

In this study, we consider the following optimal control problem:

Choose a control $v(x) \in L_2(0,L)$ and a corresponding $u$ such that the pair $(v,u)$ minimizes the functional

$$J_{\alpha}(v) = \int_0^T [u(0,t;v) - y(t)]^2 dt + \alpha \|v(x)\|_{L_2(0,L)}^2$$

subject to the parabolic problem:

$$u_t - (k(x)u_x)_x = f(x,t), \quad (x,t) \in \Omega := (0,L) \times (0,T]$$
$$u(x,0) = v(x), \quad x \in (0,L)$$
$$u_x(0,t) = 0, \quad u(L,t) = 0, \quad t \in (0,T]$$

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where \( y \in L_2(0,T) \) is a given target function and \( f \) is a known function.

\( L_2(0,l) \) is the Banach space consisting of all measurable functions on \((0,l)\) having the norm

\[
\|v\|_{L_2(0,l)} = \left( \int_0^l (v(x))^2 \, dx \right)^{1/2}.
\]

With the choice of the functional in (1), we mention the observation of \( u(0,t;v) \) in \( L_2(0,T) \) for the control \( v(x) \in L_2(0,l) \). In (1), \( \alpha > 0 \) is the parameter of regularization and it can be found by the Tikhonov regularization method [14].

Let admissible set \( V_{ad} \) be closed and convex subset of the space \( L_2(0,l) \). Let’s denote by \( u(x,t;v) \) the solution of the parabolic problem (2), corresponding to the given element \( v \in V_{ad} \).

Many papers have already been published to study the control problems for the parabolic equations. Bushuyev [11] has controlled the function \( f(x) \) in the parabolic problem \( u_t + Au = \sigma(x,t) f(x) \) with Dirichlet boundary conditions. Munch and Periago [2] have studied the optimal distribution of the support of the internal null control of minimal \( L_2 \)-norm for the 1-D heat equation. In [3], the numerical approximation of an optimal control problem for a linear heat equation have been presented by Munch and Periago. Yu [4] has established the equivalence of minimal time and minimal norm control problems for the semi-linear heat equations. Zheng, Guo and Ali [5] have investigated the stability of the optimization problem for a multidimensional heat equation. Zheng and Yin [6] have studied the optimal time for the time optimal control problem governed by an internally controlled semi-linear heat equation. In [7-9], the inverse problems with different controls and cost functions have been investigated.

Lions [10] has examined the optimal control problem of the initial condition for the parabolic systems from the measured temperature at the final time. Hasanov and Mukanova [11] have investigated the problem of determining of the initial condition \( f(x) \) in the heat equation \( u_t = (k(x) u_x)_x \) with boundary conditions \( u(0,t) = 0 \) and \( u_x(l,t) = 0 \) from the measured final data \( u_T(x) = u(x,T) \).

In this study, we formulate the control problem whose solution implies the minimization of the distance measured in a suitable norm between the solution of the problem at left end and a given target. The aim of this work is to determine the optimal function \( \hat{v}^* \) in closed and convex set \( V_{ad} \) such that

\[
J_{\alpha}(\hat{v}^*) < J_{\alpha}(v), \quad \forall v \in V_{ad}.
\]

The plan of the paper is as follows: In section 2, we give the existence of the unique optimal solution of the problem (1)-(2) by getting the weak lower semi-continuity of the cost functional \( J_{\alpha} \). In section 3, we find the gradient of the cost functional via adjoint problem approach and constitute the minimizing sequence for the functional (1). Finally, we obtain the approximate optimal control function for a numerical example.

## 2 Existence and uniqueness of the optimal solution

It is assumed that

\[
f \in L_2(\Omega), \quad k(x) \in L_{\infty}(0,l), \quad 0 < k_1 \leq k(x) \leq k_2, \quad \forall x \in (0,l).
\]

We can define the generalized solution \( u(x,t) \) of the problem (2) as a element of \( \hat{V}^{1.0}(\Omega) \) satisfying the following identity

\[
\int_0^T \int_0^l \left( -u_t \eta_t + ku_x \eta_x \right) \, dx \, dt = \int_0^T \int_0^l f(x,t) \eta(x,t) \, dx \, dt + \int_0^l v(x) \eta(x,0) \, dx
\]

for all \( \eta \in \hat{H}^{1.1}(\Omega) \) and \( \eta(x,T) = 0 \) (see [12]). Here \( H^{1.1}(\Omega) \) is the Sobolev space of functions with the norm [12]

\[
\|u\|_{H^{1.1}(\Omega)} = \left( \int_{\Omega} \left[ u^2 + u_x^2 + u_t^2 \right] \, dx \right)^{1/2}
\]
and \( \hat{H}^{1,1} (\Omega) := \{ \eta \in H^{1,1} (\Omega) : \eta (l, t) = 0, \forall t \in (0, T) \} \).

Here \( V^{1,0} (\Omega) = C ([0, T] ; L_2 (0, l)) \cap L_2 ((0, T) ; H^1 (0, l)) \) and \( \hat{V}^{1,0} (\Omega) \) is a subspace of \( V^{1,0} (\Omega) \) whose elements equal to zero for \( x = l \).

It is known that for every \( v (x) \in L_2 (0, l) \), the boundary value problem (2) admits a unique generalized solution \( u \in V^{1,0} (\Omega) \) (see [7], [12]).

We give the solvability of the optimal control problem (1)-(2). Let’s give the increment \( \Delta v \) to \( v \) such that \( v + \Delta v \in V_{ad} \) and show the solution of (2) corresponding \( v + \Delta v \) by \( u_\Delta = u(x, t; v + \Delta v) \). Then the function \( \Delta u = u_\Delta - u \) will be the solution of the following difference problem:

\[
\begin{align*}
\Delta u_t - (k(x) \Delta u)_x &= 0, \quad (x, t) \in \Omega := (0, l) \times (0, T] \\
\Delta u(x, 0) &= \Delta v(x), \quad x \in (0, l) \\
\Delta u_x (0, t) &= 0, \quad \Delta u(l, t) = 0, \quad t \in (0, T].
\end{align*}
\]

(5)

**Lemma 1.** Let \( \Delta u \) be the solution of the problem (5). Then the following estimate is valid:

\[
\| \Delta u(0, \cdot) \|_{L_2 (0, T)} \leq c_1 \| \Delta v \|_{L_2 (0, l)}
\]

(6)

**Proof.** The proof Lemma 1 can be obtained same as in [7].

We can rewrite the cost functional \( J_\alpha (v) \) as

\[
J_\alpha (v) = \pi (v, v) - 2Lv + b
\]

(7)

for

\[
\pi (v, v) = \int_0^T [u(0, t; v) - u(0, t; 0)]^2 dt + \alpha \int_0^l v^2 dx,
\]

(8)

\[
Lv = \int_0^T [u(0, t; v) - u(0, t; 0)] [y(t) - u(0, t; 0)] dt
\]

(9)

and

\[
b = \int_0^T [y(t) - u(0, t; 0)]^2 dt
\]

(10)

The functional \( \pi (v, v) \) defined by (8) is bilinear and symmetric. Further, we can write

\[
| \pi (v, v) | \geq \alpha \| v \|^2_{L_2 (0, l)}
\]

and this implies the coercivity of \( \pi (v, v) \). Using Lemma 1, we can prove that the continuities of the functional \( \pi (v, v) \) in (8) and the functional \( Lv \) in (9).

**Theorem 2.** Let \( \pi (v, v) \) be a continuous symmetric bilinear coercive form and \( Lv \) be a continuous linear form. Then there exists a unique element \( v^* \in V_{ad} \) such that

\[
J_\alpha (v^*) = \inf_{v \in V_{ad}} J_\alpha (v).
\]

**Proof.** Proof of this theorem can easily be obtained by showing the weak lower semi-continuity of \( J_\alpha \) same as in [10].

**Theorem 3.** Let the assumptions of Theorem 2 remain valid. The minimizing element \( v^* \) of \( V_{ad} \) is characterized by

\[
\pi (v^*, v - v^*) \geq L(v - v^*)
\]

(11)

for \( \forall \alpha \in V_{ad} \) [10].

Inequalities of the type given by (11) are termed variational inequalities.
3 Frechet differential of the cost functional and numerical example

Let us introduce the Lagrangian \( L(u, v, \psi) \) given by

\[
L(u, v, \psi) = J_\alpha(v) + \langle \psi, u_t - (k(x)u_x)_x - f(x,t) \rangle_{L_2(\Omega)}
\]

(12)

where the functional \( J_\alpha(v) \) is defined by (1) and the function \( \psi(x,t) \) is the Lagrange multiplier.

Using the \( \delta L = 0 \) stationarity condition, we have the following adjoint problem:

\[
\begin{align*}
\psi_t + (k(x)\psi_x)_x &= 0, \quad (x,t) \in \Omega := (0,1) \times (0,T) \\
\psi(x,T) &= 0, \quad x \in (0,1) \\
k(0)\psi_x(0,t) &= 2[u(0,t;v) - y(t)], \quad \psi(I,t) = 0, \quad t \in (0,T)
\end{align*}
\]

(13)

Now, we investigate the variation of the functional \( J_\alpha(v) \). The difference functional

\[
\Delta J_\alpha(v) = J_\alpha(v + \Delta v) - J_\alpha(v)
\]

is such as

\[
\Delta J_\alpha(v) = \int_0^T \left[ 2u(0,t;v) - 2y(t) \right] \Delta u(0,t) dt \\
+ \int_0^T \left[ \Delta u(0,t) \right]^2 dt + \alpha \int_0^T (2v + \Delta v) \Delta v dx
\]

(14)

Using the identity between the difference problem and the adjoint problem the equation (14) can be rewritten as follows:

\[
\Delta J_\alpha(v) = \int_0^T \left\{ -\psi(x,0) + 2\alpha v \right\} \Delta v dx + \int_0^T \left[ \Delta u(0,t) \right]^2 dt + \alpha \int_0^T (\Delta v)^2 dx.
\]

(15)

The Lemma 1 implies that the second integral in (15) is bounded by term \( o(||v||^2_{L_2(0,T)}) \). So Frechet differential at \( v \in V_{ad} \) of the cost functional \( J_\alpha(v) \) can be defined as follows:

\[
J_\alpha' (v) = -\psi(x,0) + 2\alpha v.
\]

We use the conjugate gradient method that is known to be very successful in linear optimization problems in order to compute a numerical approximation of the optimal control. According to this method the minimizing sequence is set by

\[
v_{k+1} = v_k - \beta_k J_\alpha' (v_k), \quad k = 0, 1, 2, \ldots
\]

(16)

where \( v_0 \in V_{ad} \) is a given initial iteration and \( \beta_k \) is a small enough relaxation parameter and assures that

\[
J_\alpha(v_{k+1}) < J_\alpha(v_k).
\]

Concerning the choice of the parameter \( \beta_k \), there are several possibilities and these can be found in any optimization books.

**Lemma 4.** The cost functional (1) is strongly convex with the strong convexity constant \( \alpha \).

From the following strongly convex functional definition:

\[
J_\alpha(\lambda v_1 + (1-\lambda)v_2) \leq \lambda J_\alpha(v_1) + (1-\lambda) J_\alpha(v_2) - \lambda(1-\lambda)||v_1 - v_2||^2_{L_2(0,T)}
\]

we can see that the cost functional (1) is strongly convex with the constant \( \chi = \alpha \).

Using the strongly convexity of the cost functional, we can write the following inequality

\[
||v_k - v^*||^2 \leq (J_\alpha(v_k) - J_\alpha(v^*)), \quad k = 0, 1, 2, \ldots
\]

This inequality show that the minimizing sequence (16) converges to optimal solution \( v^* \).
Example 5. Let us assume in the problem (2) that $k(x) = 1$, $l = 1$, $T = 1$ and $f(x,t) = e^t (3-x^2)$. We use the cost functional $J_\alpha(v) = \int_0^1 [u(0,t;v) - e^t]^2 \, dt + \alpha \|v(x)\|_{L_2(0,l)}^2$ and want to solve the minimizing problem $J_\alpha(v^*) = \inf J_\alpha(v)$. We solve the direct problem (2) by the Fourier method. Starting with the initial element $v_0 = 0$ and choosing $\alpha = 0.1$ and using the minimizing sequence (16) for $\beta_k = 0.01$, then after 100 iterations, we obtain the following approximate solution

$$v_{100} = +0.88459 - 0.88459x - 0.000017\cos(32.98672x) - 0.000020\cos(29.84153x) - 0.000025\cos(26.70353x) - 0.000033\cos(23.56194x) - 0.000044\cos(20.42035x) - 0.000062\cos(17.27875x) - 0.000092\cos(14.13716x) - 0.000153\cos(10.99557x) - 0.000301\cos(7.85398x) - 0.000834\cos(4.71238x) - 0.07512\cos(1.57079x)$$

The values of the cost functional for this element is $J_{0.1}(v_{100}) = 0.04794$ and the norm of the difference between the approximate solution and the exact solution is $\|v_{100} - v\|_{L_2(0,1)} = 0.082440$.

We give the values of the $\|u(0,t;v_{100}) - e^t\|_{L_2(0,1)}^2$ and $\|v_{100}\|_{L_2(0,1)}^2$ for some $\alpha$’s in Table 1.

Table 1. Some $\alpha$, $\|u(0,t;v_{100}) - e^t\|_{L_2(0,1)}^2$ and $\|v_{100}\|_{L_2(0,1)}^2$ values

| $\alpha$ | $\|u(0,t;v_{100}) - e^t\|_{L_2(0,1)}^2$ | $\|v_{100}\|_{L_2(0,1)}^2$ |
|-----|-----------------|-----------------|
| 0.1 | 0.02703 | 0.20904 |
| 0.2 | 0.03356 | 0.18194 |
| 0.3 | 0.04015 | 0.16205 |
| 0.4 | 0.04673 | 0.14153 |
| 0.5 | 0.05348 | 0.12598 |
| 1.0 | 0.08095 | 0.07398 |
| 2.0 | 0.11747 | 0.03282 |

4 Conclusions

In this paper, we prove that the initial temperature can be controlled from the target $u(0,t)$ in the heat problem. The Frechet differential of the cost functional considered can be obtained by using the adjoint problem. The minimizing sequence (16) converges to the optimal solution of the optimal control problem considered. We give a numerical example which shows that theoretical results are verified.

References

[1] I. Bushuyev, Global uniqueness for inverse parabolic problems with final observation, Inverse Problems. (1995) 11-16.
[2] A. Munch and F. Periago, Optimal distribution of the internal null control for the one-dimensional heat equation, Journal of Differential Equations, 250 (2011) 95-111.
[3] A. Munch and F. Periago, Numerical approximation of bang-bang controls for the heat equation: an optimal design approach, Systems and Control Letters, 62 (2013) 643-655.
[4] H. Yu, Equivalence of minimal time and minimal norm control problems for semi linear heat equation, Systems and Control Letters, 73 (2014) 17-24.
[5] G. Zheng, B. Guo and M. M. Ali, Stability of optimal control of heat equation with singular potential, Systems and Control Letters, 74 (2014) 18-23.
[6] G. Zheng, J. Yin, Numerical approximation for a time optimal control problems governed by semi-linear heat equations, Advances in Difference Equations, (2014), 1-7.
[7] A. Hasanov and B. Pektas, Identification of an unknown time-dependent heat source term from overspecified Dirichlet boundary data by conjugate gradient method, Computers and Mathematics with Applications, 65 (2013), 42-57.
[8] A. Hasanov, M. Otelbaev and B. Akpayev, Inverse heat conduction problems with boundary and final time measured output data, Inverse Problems in Science and Engineering, 19 (7), (2011), 985-1006.
[9] A. Hasanov, Identification of spacewise and time dependent source terms in 1D heat conduction equation from temperature measurement at a final time, International Journal of Heat and Mass Transfer, 55, (2012), 2069-2080.
[10] Lions, J. L., 1971, Optimal Control of Systems Governed by Partial Differential Equations, Springer-Verlag, New York.
[11] A. Hasanov and B. Mukanova, Relationship between representation formulas for unique regularized solutions of inverse source problems with final overdetermination and singular value decomposition of input-output operators, IMA Journal of Applied Mathematics, (2014), 1-21.
[12] Ladyzhenskaya, O. A., Boundary Value Problems in Mathematical Physics, Springer, New York, 1985.
[13] Vasilyev, F.P., Ekstremal problemlerin cozum metotlari, Nauka, Moskova, 1981.
[14] Tikhonov, A. N. and Samarskii, A. A., 1963. Equations of Mathematical Physics. Dover Publications, 765 p, New York.