DIFFUSION OF COSMIC RAYS IN THE EXPANDING UNIVERSE. II.
ENERGY SPECTRA OF ULTRA–HIGH ENERGY COSMIC RAYS

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ABSTRACT
We consider the astrophysical implications of the diffusion equation solution for ultra–high energy cosmic rays (UHECRs) in the expanding universe, which we obtained in Paper I of this series. The UHECR spectra are calculated in a model with sources located in vertices of a cubic grid with a linear constant (source separation) \( d \). The calculations are performed for various magnetic field configurations \((B_c, l_c)\), where \( l_c \) is the basic scale of the turbulence and \( B_c \) is the coherent magnetic field on this scale. The main purpose of these calculations is to demonstrate the validity of the solution obtained in Paper I and to compare this solution with the Syrovatsky solution used in previous works. The Syrovatsky solution must be necessarily embedded in the static cosmological model. The formal comparison of the two solutions with all parameters being fixed identically reveals the appreciable discrepancies between two spectra. These discrepancies are less if different sets of best-fit parameters are used in these models.

Subject headings: cosmic rays — diffusion

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1. INTRODUCTION
Diffusive propagation of ultra–high energy cosmic rays (UHECRs) in extragalactic space has been recently studied by Aloisio & Berezinsky (2004, 2005), Lemoine (2005), and Aloisio et al. (2007), using the Syrovatsky solution (see Syrovatskii 1959) of the diffusion equation. This solution has been obtained under the rather restrictive assumptions that the diffusion coefficient \( D(E) \) and energy losses \( b(E) = -dE/dt \) of the propagating particles do not depend on the time \( t \). In our recent work (Berezinsky & Gazizov 2006, hereafter Paper I) we found the analytic solution to the diffusion equation in the expanding universe, valid for the time-dependent diffusion coefficient \( D(E, t) \) and energy losses \( b(E, t) \). The aim of this work is to calculate spectra of ultra–high energy (UHE) protons using the solution of Paper I (hereafter the BG solution) and to compare them with the spectra obtained with the help of the Syrovatsky solution in the papers cited above.

The diffusion equation for ultrarelativistic particles propagating in the expanding universe from a single source, as obtained in Paper I, reads

\[
\frac{\partial n}{\partial t} - b(E, t) \frac{\partial n}{\partial E} + 3H(t)n - n \frac{\partial b(E, t)}{\partial E} = - \frac{D(E, t)}{a^2(t)} \nabla_x^2 n
\]

\[= \frac{Q_s(E, t)}{a^3(t)} \delta^3(x - x_g), \tag{1}\]

where the coordinate \( x \) corresponds to the comoving distance and \( a(t) \) is the scaling factor of the expanding universe, \( n = n(t, x, E) \) is the particle number density per unit energy in an expanding volume of the universe, \( dE/dt = -b(E, t) \) describes the total energy losses, which include adiabatic \([H(t)E]\) and interaction \([b_{\text{int}}(E, t)]\) energy losses, and \( Q_s(E, t) \) is the generation function, which gives the number of particles generated by a single source at coordinate \( x_g \) per unit energy and unit time.

According to Paper I, the spherically symmetric solution of equation (1) is

\[
n(x_g, E) = \int_0^z dz \frac{dt}{a(t)} \frac{1}{H_0(1+z)\sqrt{\Omega_m(1+z)^3 + \Lambda}} \frac{\lambda(E, z)}{[4\pi \lambda(E, z)]^{3/2}} E \frac{dE}{dx}, \tag{2}\]

\[
\frac{dt}{dz} = - \frac{1}{H_0(1+z)\sqrt{\Omega_m(1+z)^3 + \Lambda}}, \tag{3}\]

and with cosmological parameters \( \Omega_m = 0.27 \) and \( \Lambda = 0.73 \), \( \lambda(E, z) = \int_0^z dz' \frac{dt'}{dz'} \frac{D(E', z')}{a^2(z')} \), \( \frac{dE}{dz}(E, z) = (1+z) \exp \left[ \int_0^z dz' \frac{dt'}{dz'} \frac{\partial b_{\text{int}}(E', z')}{\partial E'} \right] \).

The characteristic trajectory, \( E' = E'(E, z') \), is a solution of the differential equation

\[
\frac{dE}{dt} = -[H(t)E + b_{\text{int}}(E, t)]. \tag{6}\]

It gives the energy \( E' \) of a particle at epoch \( z' \), if this energy is \( E \) at \( z = 0 \); we will use also the notation \( E_g(E, z) \) for this quantity.

The upper limit \( z_{\text{max}} \) in the integral of equation (2) is provided by the maximum energy of acceleration as \( E_{\text{max}} = E_{g}(E, 0) \).
To calculate the diffuse flux of UHE protons \( J_p(E) \), we sum up the contributions of sources located in the vertices of a three-dimensional (3D) cubic lattice with spacing \( d \). Positions of the sources \( x_{ijk} = (\xi, \eta, \zeta) \) are given by the coordinates \( \xi = d(i + 1/2) \), \( \eta = d(j + 1/2) \), and \( \zeta = d(k + 1/2) \), where \( i, j, k = 0, \pm 1, \pm 2, \ldots \), and the position of the observer is assumed to be at \( \xi = 0, \eta = 0 \), and \( \zeta = 0 \). Thus, we obtain

\[
J_p(E) = \frac{c}{4\pi} \sum_{i,j,k} n(x_{ijk}, E),
\]

(7)

where \( n(x_{ijk}, E) \) is given by equation (2) and

\[
x_{ijk} = d\sqrt{(i + 1/2)^2 + (j + 1/2)^2 + (k + 1/2)^2}.
\]

(8)

For the propagation of UHE protons in magnetic fields, we follow the picture used by Aloisio & Berezinsky (2004, 2005); namely, we assume a turbulent magnetized plasma. The magnetic field produced by turbulence is characterized by the value of the coherent magnetic field \( B_c \) on the basic scale of turbulence \( l_c \), which we will keep in our estimates as 1 Mpc. On the smaller scales \( l < l_c \), the magnetic field is determined by the turbulence spectrum. The critical energy of propagation is determined by the relation \( l_c(E_c) = l_c \), where \( l_c \) is the Larmor radius of a proton. Numerically, \( E_c = 0.93 \times 10^{18} (B_c/1 \text{ nG}) \text{ eV} \). The characteristic propagation length in the magnetic field is the diffusion length \( l_d(E) \). It is defined as the distance on which a particle is deflected by 1 rad. The diffusion coefficient is defined as \( D(E) = c l_d(E)/3 \). For the case in which \( l_c(E) \gg l_c \), i.e., when \( E \gg E_c \), the diffusion length can be straightforwardly found from multiple scattering as

\[
l_d(E) = 1.2 \frac{E_{18}^2}{B_{10}} \text{ Mpc},
\]

(9)

where \( E_{18} = E/(10^{18} \text{ eV}) \) and \( B_{10} = B/(1 \text{ nG}) \). At \( E = E_c \), \( l_d = l_c \).

At \( E \ll E_c \), the diffusion length depends on the spectrum of the turbulence. For the Kolmogorov spectrum, \( l_d(E) = 1.6 \sqrt{E_{18}/B_{10}} \text{ Mpc} \); for the Bohm regime, \( l_d(E) = l_c(E/E_c) \).

The strongest observational upper limit on the magnetic fields in our picture is given by Blasi et al. (1999) as \( B_c \lesssim 10 \text{ nG} \) on the scale of \( l_c = 1 \text{ Mpc} \) and as \( B_c \lesssim 6 \text{ nG} \) on the scale of \( l_c = 50 \text{ Mpc} \). For more general (and less restrictive) observational upper limits, see the reviews by Kronberg (1994) and Vallee (1997).

Another indication of a realistic intergalactic magnetic field is given by magnetohydrodynamic (MHD) simulations combined with large-scale structure (LSS) simulations (Ryu et al. 1998; Sigl et al. 2003, 2004a, 2004b; Dolag et al. 2003, 2004, 2005; Brüggen et al. 2005). Magnetic fields differ by orders of magnitudes in such LSS structures as clusters of galaxies, sheets, filaments, and voids. The magnetic fields in these structures are produced from a very weak primordial field due to adiabatic compression of the structures and the turbulent amplification of the field. In the voids the turbulent amplification is very weak or absent and the field is basically equal to the comoving primordial field. The MHD amplification in more dense structures (clusters of galaxies and filaments) starts relatively late, at redshift \( z \leq 0.3 \), and magnetic fields produced in these structures depend weakly on the assumed primordial field. The clusters of galaxies do not play an important role for the propagation of UHE protons, because these structures occupy too small a fraction of the universe volume.

In all three recent simulations by Sigl et al., Dolag et al., and Brüggen et al., as cited above, the magnetic field in voids is found to be \( \sim 10^{-11} \text{ G} \). Magnetic fields in filaments change along the crossing line from 0.1 to 10 nG in simulations by Dolag et al. and from 0.3 to 10 nG in the simulation by Brüggen et al. (2005). The magnetic fields are stronger in the simulations by Sigl et al.: the field averaged over space is \( \langle B^2 \rangle^{1/2} \approx 1 \text{ nG} \), and deflection angles for UHE protons are considerably larger than those in the simulations by Dolag et al. As has been suggested by Sigl et al., it could be a result of different space distribution of the magnetic field in LSS.

Considering diffusion, in our case we should take \( (l_c, B_c) \) to have different values for different structures (Berezinsky et al. 2002b), but since the lifetimes of particles that are of interest in our problem are of the order of the age of the universe, \( t_0 \), a particle crosses many structures in that time, and one can use an average picture, with \( l_c \) and \( B_c \) averaged over the volume of the universe. In the calculations below, \( l_c \) and \( B_c \) imply these averaged values. We assume that the representative values of \( B_c \) are in the range \((0.1\text{–}1) \text{ nG}\) for \( l_c = 1 \text{ Mpc} \), although for some special cases we consider magnetic fields outside this range. For \( B_c \leq 0.01 \text{ nG} \), the spectra are close to those for rectilinear propagation, and \( B_c \leq 2 \times 10^{-3} \text{ nG} \) provides quasi-rectilinear propagation for all protons with \( E \gtrsim 1 \times 10^{17} \text{ eV} \) (see § 2).

We do not put as the aim of this paper the detailed study of diffusion in the time-dependent regime. Such a work (R. Aloisio et al. 2007, in preparation) is at present in progress. Here we want mainly to demonstrate the validity of the solution for the expanding universe obtained in Paper I and to perform the numerical comparison of the UHECR diffuse spectra predicted by the BG and Syrovatsky solutions. The difference is expected to be substantial at energies of \( E \lesssim 3 \times 10^{18} \text{ eV} \), where the effects of the expansion of the universe (in particular, the CMB temperature growth with redshift) are not negligible. We want also to test the new solution obtained in Paper I, namely, to test the compatibility of the BG and Syrovatsky spectra at high energies, where the universe can be considered static, as well as the convergence of BG spectra to the universal spectrum when the source separation \( d \to 0 \).

The paper is organized as follows. Section 2 addresses the question of how reliable is the assumption of the diffusion for the low-energy part of UHECRs. In § 3 we calculate the diffuse spectra using the time-dependent BG solution. In § 4 we consider the static universe, which is the necessary assumption for the Syrovatsky solution, and in § 5 we compare the spectra calculated for the expanding and static universes. The short conclusions are presented in § 6.

2. WHY DIFFUSION?

We argue here that, at least at the low-energy end of extragalactic UHECRs, the diffusion propagation is unavoidable for any reasonable magnetic field. We estimate also the magnetic field \( B_c \) for which protons with energies \( E \gtrsim 10^{17} \text{ eV} \) propagate quasi-rectilinearly, hence producing the same energy spectrum as in the case of rectilinear propagation.

To facilitate the calculations, let us consider the case of the static universe, as in Aloisio & Berezinsky (2005); namely, the universe with “age” \( t_0 = H_0^{-1} \), as follows from WMAP observations, and with a fictitious “adiabatic” energy loss of particles of \( dE/dt = -E H_0 \), where \( H_0 = 72 \text{ km s}^{-1} \text{ Mpc}^{-1} \) is the observed Hubble parameter.
In this picture there is a maximum diffusive distance, the magnetic horizon, that is determined by the distance traversed by a particle during the age of the universe $t_0$:

$$r^2_{\text{hor}}(E) = \int_0^{t_0} dt D(E_0(E,t)),$$

(10)

where $E_0(E,t)$ is the energy that a particle has at time $t$, if it has energy $E$ at $t = t_0$. Putting $dt = -dE_0/b(E_0)$ in equation (10), we obtain

$$r^2_{\text{hor}}(E) = \int E_0^{E} \frac{dE_0}{b(E_0)} D(E_0),$$

(11)

where $E_{\text{max}} = \min(E_0(E, t_0), E_{\text{acc}})$. In $r^2_{\text{hor}}(E)$ one can recognize (see Aloisio & Berezinsky 2005) the Syrovatsky variable $\lambda(E, E_0)$ at $E_0 = E_{\text{max}}$ (for the physical discussion of magnetic horizon, see Parizot 2004).

Let us consider a transition from diffusive to rectilinear propagation, allowing a considerable deflection angle of $\theta \gtrsim 1$, when the spectrum is the same as in the rectilinear propagation. Two conditions must be fulfilled:

$$r_1(E) > l_c,$$

(12)

$$l_p(E) > r_{\text{hor}}(E).$$

(13)

Equation (12) gives a necessary (but not sufficient!) condition: the scattering angle on a basic scale must be small enough, $\theta \leq l_c/r_1(E) < 1$. It implies propagation in the regime with $E > E_c$, where $l_p(E) = l_c(E/E_c)^2$. Considering the low-energy case $E \leq 1 \times 10^{18}$ eV, when the adiabatic energy loss dominates, so that $b(E) = E H_0$ and

$$E_{\text{max}} = E_0(E, t_0) = E e^{H_0 t_0} = e E,$$

(14)

one readily obtains from equation (11) the expression

$$r^2_{\text{hor}}(E) = \frac{c l_c}{6 H_0} \left( \frac{E}{E_{\text{acc}}} \right)^2 (e^2 - 1).$$

(15)

Using equation (13), we get

$$E_c \leq \left\{ \frac{c H_0^{-1}}{6 l_c} (e^2 - 1) \right\}^{-1/2}.$$  

(16)

From $r_1(E_c) = l_c$, we obtain $E_c = q B_c l_c$, where $q$ is an electric charge, which equals 300 for a proton, if $B$ is measured in gauss and $E$ in eV.

Finally, we have

$$B_c \leq \frac{E_{\text{acc}}}{q l_c \left[ \frac{c H_0^{-1}}{6 l_c} (e^2 - 1) \right]^{1/2}},$$

(17)

or, numerically,

$$B_c \leq 1.6 \times 10^{-3} \frac{E}{10^{18} \text{ eV}} \left( \frac{l_c}{\text{1 Mpc}} \right)^{-1/2} \text{ nG.}$$

(18)

Therefore, $B_c \leq 1.6 \times 10^{-3} \text{ nG}$ provides quasi-rectilinear propagation for all protons with energies $E \geq 1 \times 10^{17}$ eV, while at lower energies the diffusion description is applicable. For a reasonably low field value of $B_c \approx 0.01 \text{ nG}$, diffusion becomes valid at $E \leq 1 \times 10^{18}$ eV.

3. DIFFUSIVE ENERGY SPECTRA OF UHECRs IN THE EXPANDING UNIVERSE

In the following calculations we will use the simplified illustrative description of magnetic field evolution with redshift; namely, we parameterize the evolution of the magnetic configuration $(l_c, B_c)$ as

$$I_c(z) = l_c/(1+z), \quad B_c(z) = B_c(1+z)^{2-m},$$

where the factor $(1+z)^2$ describes the diminishing of the magnetic field with time due to magnetic flux conservation and the factor $(1+z)^{-m}$ is that due to MHD amplification of the field. The critical energy $E_c(z)$ found from $r_1(E) = l_c(z)$ is given by

$$E_c(z) = 0.93 \times 10^{18}(1+z)^{1-m} \frac{B_c}{1 \text{ nG}},$$

for $l_c = 1 \text{ Mpc}$. The maximum redshift used in the calculations is $z_{\text{max}} = 4$.

The diffuse flux is calculated for the lattice distribution of the sources (in the coordinate space $x$) with lattice parameter (the source separation) $d$ and a power-law generation function for a single source,

$$Q_s(E) = \frac{q_0(\gamma_0 - 2)}{E_0^\gamma} \left( \frac{E}{E_0} \right)^{-\gamma_0},$$

(19)

where $E_0$ is the normalizing energy, for which we will use $1 \times 10^{18}$ eV, and $q_0$ has a physical meaning of a source luminosity in protons with energies $E \geq E_0, L_0(z > z_0)$. The corresponding emissivity $\varepsilon_0 = q_0/d^2$, i.e., the energy production rate in particles with $E \geq E_0$ per unit comoving volume, will be used to fit the observed spectrum to the calculated one.

Using equations (2)–(8), one obtains the diffuse spectrum as

$$J_p(E) = \frac{c}{4\pi H_0} \frac{q_0(\gamma_0 - 2)}{E_0^\gamma} \left( \frac{E}{E_0} \right)^{-\gamma_0}$$

$$\times \sum_i \int_0^{z_i} dz \frac{[E_p(E, z)/E_0]^{-\gamma_0}}{(1+z)\sqrt{\Omega_m(1+z)^3 + \Lambda}} \times \exp \left\{ -\frac{x_i}{x_0} (4\pi \xi(E, z) \right\} dE (E, z) / \left[ 4\pi H_0 (E, z) \right]^{3/2} dE,$$

(20)

where the summation is performed over the sources as in equations (7) and (8); the upper limit $z_i$ is provided by the maximum energy of acceleration as $E_p(E, z_0) = E_{\text{acc}}$ or by $z_{\text{max}}$, whichever is smaller; $\gamma_0$ is the generation index; and a formula for $dE_p/dE$ can be found in Berezinsky & Grigorieva (1988) and Berezinsky et al. (2002a). The analog of the Syrovatsky variable, $\xi(E, z)$, is given by

$$\xi(E, z) = \frac{1}{H_0} \int_0^z dz \sqrt{(1+z)/(1+z)^3 + \Lambda} D(E_p(E, z, z).$$

(21)

First of all we test the correctness of the obtained solution with the help of the propagation theorem (Aloisio & Berezinsky 2004), which states that the diffusive solution given in equation (7) converges to the universal spectrum, i.e., one calculated for homogeneous source distribution (see Berezinsky et al. 2002a), when the distances between sources $d \to 0$. Figure 1 demonstrates
in the case of interpolation there is only one (unknown) interpolation $n(E, r)$ that conserves the number of particles. This problem will be studied in detail in R. Aloisio et al. (2007, in preparation), but for the purposes of this paper we can accept the rough recipe of transition from the diffusive to rectilinear regimes as described above. The appearance of the artificial peculiarity connected with the accepted propagation transition will be useful as a mark for the position of the transition in the spectrum.

For rectilinear propagation for the lattice distribution of the sources, the diffuse spectrum is calculated as (Berezinsky et al. 2006)

$$J_p(E) = \frac{(\gamma_0 - 2)C_0}{(4\pi)^2E_0^2} \times \sum \frac{E_{ijk}(E, z_{ijk})/E_0}{(i + 1/2)^2 + (j + 1/2)^2 + (k + 1/2)^2}(1 + z_{ijk})$$

where $C_0 = q_0/d = 2.4 \times 10^{45}$ erg (Mpc$^3$ yr)$^{-1}$ is the emissivity, $z_{ijk}$ is the redshift for a source with coordinates $i, j, k$, and the factor of $(1 + z_{ijk})$ takes into account the time dilation.

The calculated spectra in the expanding universe for $B_e = 0.1$ nG and $B_e = 1$ nG, both for $m = 1$ and $E_{\text{max}} = 1 \times 10^{22}$ eV, are shown in Figure 2 in comparison with the Yakutsk data (Egorova et al. 2004; left) and with all data (right). The all-data spectrum is obtained using the energy calibration of all detectors with the help of the dip (Berezinsky et al. 2006). One can observe the peculiarity in the predicted spectrum in the right panel ($B_e = 1$ nG) at energy $E \approx 2 \times 10^{19}$ eV. This is the energy of the transition to rectilinear propagation. This peculiarity is unphysical and is connected with the simplified description of the transition as described above. When the magnetic field diminishes, the peculiarity shifts toward lower energies (left), as it should. At small values of $d$ the calculated spectra converge to the universal spectrum, as they must.

The spectra for a very low magnetic field with $B_e = 0.01$ nG is shown in Figure 3. Note that the peculiarity is shifted here to the energy $E \approx 2 \times 10^{17}$ eV; above this energy, the spectrum is rectilinear. This behavior corresponds to that described in the discussion in § 2.

4. SPECTRA IN THE STATIC UNIVERSE

In this section we study the Syrovatsky solution of the UHECR diffusion equation with the aim of comparing it to the BG solution for the expanding universe. The Syrovatsky solution is valid in the case of infinite space with a time-independent diffusion coefficient $D(E)$ and energy losses $b(E)$. This implies the static universe.

We define the static universe as one in which the stationary diffusion equation with the Syrovatsky solution is embedded in the following way. There is no expansion. The Hubble constant is assumed as a formal parameter, $H_0 = 72$ km s$^{-1}$ Mpc$^{-1}$, that defines the “age” $t_0$ of the universe according to the WMAP relation $H_0t_0 = 0.993$, such that $c_0$ is the size of the universe: the space density of the UHECR sources outside the sphere of radius $c_0$ is $n_s = 0$. The temperature of CMB photons is constant, $T_0 = 2.728$ K, and thus the interaction energy losses $dE/dt = -b(E)$ are time-independent.

Apart from interaction energy losses, we assume the fictitious adiabatic energy losses described by $dE/dt = -H_0E$. In this approach we follow the work by Aloisio & Berezinsky (2005).
The universal spectrum in the static universe is different from that in the expanding universe. It is defined in the same way as in the expanding universe (Berezinsky et al. 2006), namely, from the conservation of the number of particles:

$$n_{\text{univ}}(E) = \int_0^\infty dt \frac{Q_{\text{gen}}(E_g(E, t))}{dE_g} \frac{dE_g}{dE},$$

(23)

where $Q_{\text{gen}}(E_g)$ is the generation rate per unit volume and $E_g(E, t)$ is determined by the evolution equation $dE/dt = -b(E)$. Note that in the expanding universe $Q_{\text{gen}}(E, t)$ and $n(E, t)$ are related to a comoving volume. In contrast to the case in the expanding universe, the ratio $dE_g/dE$ in equation (23) is given by a simple expression (Berezinsky & Grigorieva 1988), $dE_g/dE = b(E_g)/b(E)$. Using

$$Q_{\text{gen}}(E_g) = \left(\frac{\gamma_g - 2}{E_0}\right) L_0 \left(\frac{E_g}{E_0}\right)^{-\gamma_g},$$

(24)

where $L_0$ is the emissivity in particles with energies $E \geq E_0$ for spectral index $\gamma_g > 2$, one easily obtains the universal spectrum in analytic form:

$$n_{\text{univ}}(E) = \frac{\gamma_g - 2}{\gamma_g - 1} L_0 \left(\frac{E}{E_0}\right)^{-\gamma_g} \left[1 - \left(\frac{E}{E_{\text{max}}}\right)^{-\gamma_g - 1}\right],$$

(25)

where $E_{\text{max}} = \min[E_0, (E_{\max}^\nu, E_{\max}^{\nu_{\text{acc}}})]$. For the diffuse spectra calculations we use the lattice distribution of the sources, including the procedure of transition from diffusive to rectilinear propagation, but using the Syrovatsky solution for the BG solution (see also Aloisio & Berezinsky 2005) instead of the BG solution.

In Figure 4 we present the diffuse UHECR spectra calculated, as in § 3, in the grid model with spacing $d$. UHECR sources are located at vertices of the grid. The spectra are calculated as a combination of the diffuse flux, described by the Syrovatsky solution, combined with the rectilinear flux. They are presented for two magnetic field configurations $(B, \hat{L})$ equal to $(0.1 \, \text{nG}, 1 \, \text{Mpc})$ and $(0.1 \, \text{nG}, 1 \, \text{Mpc})$ and for different spacings $d$, indicated by the numbers in units of Mpc beside the corresponding curves. As in § 3, the spectra have irregularities at the energy of the transition from diffusive to rectilinear propagation, caused by the rough method through which the two propagation regimes are sewn together.

The calculated diffuse fluxes converge to the universal spectrum given in equation (25) when $d \to 0$, as it should, according to the propagation theorem. In contrast to calculations by Aloisio & Berezinsky (2005), we have obtained the best fit of the data using the value $\gamma_g = 2.65$, which is insignificantly different from the value $\gamma_g = 2.7$ of Aloisio & Berezinsky (2005). One can see the basic agreement of these spectra with those in the expanding universe, including spectrum peculiarities caused by the transition from diffusive to rectilinear propagation.

5. COMPARISON OF SPECTRA IN EXPANDING AND STATIC UNIVERSES

The direct comparison of the BG and Syrovatsky solutions of the diffusion equations is not possible because they are embedded in different cosmological environments. While the BG solution is...
valid for the expanding universe, the Syrovatsky solution is valid only for the static universe. Since we have used two different cosmological models for these solutions, there are two ways of comparison.

The first one is given by using equal values of parameters in both solutions. In this method, for the BG solution we use the standard cosmological parameters for the expanding universe, $H_0$, $\Omega_m$, and $\Lambda$, the maximum redshift $z_{\text{max}}$ up to which UHECR sources are still active, the magnetic field configuration $(B_c, L)$, the separation $d$, and UHECR parameters $\gamma_g$ and $\mathcal{L}_0$, determined by the best fit of the observed spectrum. For the static universe with the Syrovatsky solution we use the same parameters: $H_0$, $d$, $(B_c, L)$, $\gamma_g$, and $\mathcal{L}_0$. The maximum redshift in the BG solution we chose to be that providing the age of the universe that equals $t_0 = H_0^{-1}$ in the static universe ($z_{\text{max}} = 1.465$). We refer to this formal method of comparison as the "equal-parameter method."

A physically better justified comparison is given by the best-fit method, in which $\gamma_g$ and $\mathcal{L}_0$ are chosen as the best-fit parameters for the BG and Syrovatsky solutions, respectively. As a matter of fact, we would use the best-fit parameters $\gamma_g$ and $\mathcal{L}_0$ for each solution, if we considered them independently. The weakness of this method is a "forced" agreement at the energy range of the measured UHECR spectrum, provided by the best-fit parameters to the same spectrum. The difference between the two solutions becomes most appreciable at $E \approx 1 \times 10^{18}$ eV.

The equal-parameter comparison of the expanding and static universe solutions is shown in Figure 5 for $\gamma_g = 2.7$, $\mathcal{L}_0 = 2.4 \times 10^{45}$ erg Mpc$^{-1}$ yr$^{-1}$, and $d = 30$ Mpc. In the left panel, $B_c = 0.1$ nG, and in the right panel, $B_c = 1$ nG. The universal spectra for the expanding and static universes are shown by solid lines. The Kolmogorov diffusion is presented by dashed lines, and the Bohm diffusion is shown with dotted lines. Note that the universal spectra are different at $E \approx 1 \times 10^{19}$ eV, as they must be, due to excessive energy losses in the expanding universe caused by the increasing of the CMB temperature $T(z)$ with redshift. At the highest energy end, $E \approx 6 \times 10^{19}$ eV, the two solutions coincide exactly, because at these energies the energy attenuation time is short, the CMB temperature does not change during the time
of flight, and the expanding universe case becomes static. As one can see in Figure 5, at these energies both the universal spectra are the same, and all spectra for $d = 30$ Mpc merge into one curve. The difference of fluxes at lower energies is naturally explained by the increase of energy losses in the expanding universe because of the $T_{\text{CMB}}(z)$ dependence and cosmological evolution of the magnetic field.

The best-fit comparison of the expanding and static universe solutions is shown in Figure 6. The best-fit parameters are $\gamma_\text{diff} = 2.7$ and $\ell_0 = 2.4 \times 10^{16}$ erg Mpc$^{-3}$ yr$^{-1}$ and $\gamma_\text{diff} = 2.65$ and $\ell_0 = 5.7 \times 10^{14}$ erg Mpc$^{-3}$ yr$^{-1}$ for the expanding and static universes, respectively. The left panel shows the case in which $B_c = 0.1$ nG, and the right panel shows the case in which $B_c = 1$ nG. One can see a good (although “forced”) agreement between the expanding universe and static universal solutions in the energy range of UHECR observations, which is improved in comparison with the equal-parameter method because of the choice of the best-fit parameters ($\gamma_\text{diff}$, $\ell_0$), which are different for each case.

However, we emphasize again that this is a natural method of selecting the solution through independent analysis. At energies below $1 \times 10^{18}$ eV, the larger discrepancies can be seen, being induced by larger energy losses and the evolution of magnetic field in the case of the expanding universe.

In conclusion, one can see a reasonably good agreement between the Syrovatsky solution, embedded in the static universe model, and the BG solution for the expanding universe at energies of $E > 1 \times 10^{18}$ eV, with noticeable discrepancies at smaller energies, which are natural and understandable.

6. CONCLUSIONS

In this paper we study the application of the solution of the diffusion equation in the expanding universe that was obtained in Paper I to the propagation of UHE protons. However, we do not consider here the detailed picture with realistic evolution of the magnetic field in the expanding universe and a realistic transition from diffusive to quasi-rectilinear propagation. This will be considered in our next work (R. Aloisio et al. 2007, in preparation).

In this paper we demonstrate that the solution of the diffusion equation found in Paper I looks quite reasonable when applied to realistic models. Numerically, these solutions are similar to the Syrovatsky solutions, which are valid for the static universe, with understandable distinctions at low energies.

The diffusion spectra in the expanding universe are presented in Figures 1, 2 (right and left panels), and 3 for the following magnetic configurations: $(B_c, \ell_c) = (100 \text{ nG}, 1 \text{ Mpc})$, $(1 \text{ nG}, 1 \text{ Mpc})$, $(0.1 \text{ nG}, 1 \text{ Mpc})$, and $(0.01 \text{ nG}, 1 \text{ Mpc})$, respectively. In the latter case, the energy spectra at $E > 1 \times 10^{17}$ eV are practically the same as those in the case of rectilinear propagation. The evolution of magnetic field in the expanding universe is given by one illustrative example. The spectra are shown for different separations of the sources. The transition from diffusive to rectilinear propagation is described by a simplified recipe, which results in the artificial spectral features described in \S 3. The best fit of the observed spectrum needs the same index of generation spectrum $\gamma_0 = 2.7$ as in the case of the universal spectrum. The diffusive spectrum converges to the universal spectrum when $d \to 0$, as it should, according to the propagation theorem.

The Syrovatsky solution of the diffusion equation, i.e., one for which the diffusion coefficient $D(E)$ and energy losses $b(E)$ do not depend on time $t$, is valid only for the static universe. In the static universe we make several additional assumptions. We introduce the Hubble constant $H_0$ as a formal parameter that determines the fictitious “adiabatic” energy losses $dE/dt = -EH_{\text{d}}$. The “age” of the universe, $t_0 = H^{-1}_0$, determines the sphere of radius $r_0 = ct_0$ occupied by the sources. For this universe we calculate the universal spectrum given by equation (25) and the diffusive spectra for a given magnetic configuration $(B_c, \ell_c)$ and different separations of the sources $d$. The convergence to the universal spectrum occurs when $d \to 0$, as it should. The best fit of the observational data is obtained when $\gamma_0 = 2.65$, which can be considered as a good agreement with the work by Aloisio & Berezinsky (2004), where $\gamma_0 = 2.7$ was used.

For the comparison of the BG and Syrovatsky solutions we use two schemes. In the formal one we compare the BG and Syrovatsky solutions for the same magnetic configurations $(B_c, \ell_c)$ and the same emissivities $\ell_0$. One can see from Figure 5 that both solutions coincide exactly at $E \geq 6 \times 10^{19}$ eV, when the effect of universe expansion (most notably variation of CMB temperature) can be neglected. At lower energies the difference in the spectra naturally emerges due to the $T_{\text{CMB}}(t)$ dependence and thus due to different energy losses in the two solutions. Since the
energy losses in the expanding universe are larger, the BG spectra occur below the Syrovatsky spectra.

For practical applications, the discrepancy in the spectra at the energy range $1 \times 10^{18} - 5 \times 10^{19}$ eV is not essential, because it is eliminated by renormalization of the calculated flux, i.e., by changing the emissivity $L_0$ for the static universe solution. In fact, this procedure is necessary for fitting of the observed spectra.

The comparison of the two solutions as given above is formal. As was emphasized above, the Syrovatsky solution, which is valid for infinite space and time, with time-independent physical quantities $D(E)$ and $b(E)$, needs the specific definition of the static universe. Only in this case can one obtain the physically viable solution. This solution needs the best-fit parameters $\gamma_0$ and $L_0$, which are different from those in the expanding universe. It is physically more meaningful to compare the spectra using, for the static universe, its own best-fit parameters, $\gamma_0$ and $L_0$, which are different from those in the expanding universe. The comparison shown in Figure 6 reveals fewer discrepancies than in the formal scheme of comparison.

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