Limit Theorems for Generalized Baker’s Transformations

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Abstract

In this paper we study decay of correlations and limit theorems for generalized baker’s transformations [7, 8, 3, 22, 19]. Our examples are piecewise non-uniformly hyperbolic maps on the unit square that posses two spatially separated lines of indifferent fixed points.

We obtain sharp rates of mixing for Lipschitz functions on the unit square and limit theorems for Hölder observables on the unit square. Some of our limit theorems exhibit convergence to non-normal stable distributions for Hölder observables. We observe that stable distributions with any skewness parameter in the allowable range of [−1, 1] can be obtained as a limit and derive an explicit relationship between the skewness parameter and the values of the Hölder observable along the lines of indifferent fixed points.

This paper is the first application of anisotropic Banach space methods [6, 5, 10] and operator renewal theory [21, 12] to generalized baker’s transformations. Our decay of correlations results recover the results of [7]. Our results on limit theorems are new for generalized baker’s transformations.

1 Introduction

Intermittent baker’s transformations (IBTs) are invertible, non-uniformly hyperbolic, and area preserving skew products on the unit square that generalize
the classical baker’s transformation \cite{7, 8, 3, 22, 19}.

If a map $T: X \to X$ preserves a probability measure $\mu$, $\psi: X \to \mathbb{R}$ is in $L^\infty(\mu)$, and $\eta: X \to \mathbb{R}$ is in $L^1(\mu)$, then we define the correlation function by

$$\text{Cor}(k; \psi, \eta, T) = \left| \int \psi \circ T^k \eta \, d\mu - \int \psi \, d\mu \int \eta \, d\mu \right|.$$ 

If the limit of the correlation function as $k$ tends to infinity is zero for all $\psi \in L^\infty$ and $\eta \in L^1$, then the map is strongly mixing. If $\text{Cor}(k; \psi, \eta, T) = O \left( \frac{1}{k^\nu} \right)$ for some $\nu > 0$, then we say that the correlations decay at a polynomial rate. If the rate is independent of the choice of $\psi$ and $\eta$ in some class of functions, then we say that $T$ displays a polynomial rate of decay of correlations for observables in that class. If the class contains functions $\psi$ and $\eta$ such that $\text{Cor}(k; \psi, \eta, T) \asymp \frac{1}{k^\nu}$ as $k \to \infty$, then we say that the rate is sharp. A limit theorem is a statement of the form: If $(H)$ and $\int \psi \, dm = 0$, then

$$\frac{1}{A_n} \sum_{k=0}^{n-1} \psi \circ T^k \xrightarrow{\text{dist}} Z, \quad \text{as } n \to \infty.$$  \hspace{1cm} (1.1)

Where $(H)$ is a dynamical hypothesis, $A_n$ is a sequence of real numbers, and $Z$ is a real valued random variable. It is well known \cite{15} that if a map displays a summable rate of decay of correlations and mild additional hypotheses $(H)$, then \cite{1.1} is satisfied with $A_n = \sqrt{n}$ and $Z = N(0, \sigma)$ a normal distribution with variance determined by the correlation function. When a map displays a rate of decay of correlations that is not summable it is possible \cite{11} to prove that \cite{1.1} is satisfied with a different normalizing sequence and $Z$ a stable law, which may not be normal. In this case more delicate hypotheses are required.

In \cite{7} the authors prove that every IBT displays a sharp polynomial rate of decay of correlations for Hölder observables via the Young tower method \cite{22}. The Young tower method relies on analyzing an expanding factor map of the hyperbolic map in question and obtaining rates of decay of correlations for the factor map. These rates are then lifted to the full hyperbolic map via a posteriori arguments. Operator renewal theory \cite{21, 12, 13} has been used to obtain sharp polynomial rates of decay of correlation estimates and convergence to stable laws when the rate of decay of correlations is not summable. The anisotropic Banach space methods of \cite{6, 5, 10} are used to analyze the transfer operator associated to multidimensional maps directly without the need to pass to one dimensional factors.

In this paper we construct anisotropic Banach spaces adapted to IBTs modeled on the work of \cite{10, 17}. This allows us to analyze the transfer operator of the two dimensional piecewise non-uniformly hyperbolic IBT directly.

\footnote{The notation $f(k) \asymp g(k)$ as $k \to \infty$ indicates that $f$ and $g$ are in bounded ratio for $k$ sufficiently large. See Section 2.2 for a discussion of asymptotics.}
posses lines of indifferent fixed points that obstruct exponential rates of decay of correlations and the Lasota-Yorke type arguments used to obtain such results. In order to treat indifferent fixed points for the full two dimensional map and obtain sharp polynomial rates of decay of correlations we apply operator renewal theory. We also use the operator renewal method to obtain limit theorems for both the summable and non-summable rates of decay of correlations.

Non-normal stable distributions posses a skewness parameter that ranges in $[-1,1]$. In most dynamical applications limit theorems exhibit convergence to a stable distribution with skewness parameter either equal to 1 or $-1$. In this paper we obtain limit theorems that exhibit convergence to stable distributions with any skewness parameter in $[-1,1]$ and derive an explicit relationship between this parameter and properties of the IBT and the observable in question. We also obtain convergence to the normal distribution with both standard and non-standard normalizing sequences.

We will obtain the spectral decomposition required to apply operator renewal theory in Section 5. In Section 5.1 we recover the sharp polynomial rates of decay of correlations for Lipschitz functions. In Section 5.2 we obtain limit theorems for IBTs, which is a new result. See Section 1.1 for statements of the theorems.

1.1 Statement of results

A function $\phi: [0,1] \to [0,1]$ is an intermittent cut function (ICF) if it is smooth, strictly decreasing, and there exist constants $\alpha_0, \alpha_1 > 0$, $c_0, c_1 > 0$, and differentiable functions $h_0$ and $h_1$ defined on a neighborhood of zero with $h_j(0) = 0$ and $Dh_j(x) = o(x^{\alpha_j-1})$, such that

$$1 - \phi(x) = c_0 x^{\alpha_0} + h_0(x), \quad (1.2)$$

$$\phi(1-x) = c_1 x^{\alpha_1} + h_1(x). \quad (1.3)$$

Every IBT is uniquely determined by an ICF. We refer to the constants $c_j$ and $\alpha_j$ above as the contact coefficients and contact exponents of $B$ respectively.

Given an IBT $B$ we will induce on a subset $\Lambda$ of the unit square and apply operator renewal theory to obtain the following.

**Theorem 1.1.** Suppose that $B: [0,1]^2 \to [0,1]^2$ is an Intermittent Baker’s Transformation, as defined in Section 2, with contact exponents $\alpha_j > 0$. Let $\alpha = \max \{\alpha_0, \alpha_1\}$. If $\eta$ and $\psi$ are Lipschitz functions on $\Lambda$, then $\text{Cor}(k; \psi, \eta, B) = O\left(k^{-\frac{1}{\alpha}}\right)$. If additionally $\int \eta d\text{Leb} \neq 0$ and $\int \psi d\text{Leb} \neq 0$, then $\text{Cor}(k; \psi, \eta, B) \asymp k^{-\frac{1}{\alpha}}$.

It is important to note that we obtain a sharp decay rate in Theorem 1.1.
If \( \eta \) and \( \psi \) are supported on \( \Lambda \), \( \int_{\Lambda} \eta \neq 0 \), and \( \int_{\Lambda} \psi \neq 0 \), then Equation (5.8) shows that the rate of decay of correlation is asymptotically in bounded ratio with \( n^{-\frac{1}{\alpha}} \).

The following is a collection of limit theorems for IBTs. See Theorem 5.4 for precise statements.

**Theorem 1.2.** Suppose that \( \psi : [0, 1]^2 \to \mathbb{R} \) is \( \gamma \)-Hölder for some \( \gamma \in (0, 1) \) and \( \int_{[0,1]^2} \psi \, d\text{Leb} = 0 \). Let \( M_0 = \int_0^1 \psi(0, y^{1+\frac{1}{\alpha_0}}) \, dy \) and \( M_1 = \int_0^1 \psi(1, y^{1+\frac{1}{\alpha_1}}) \, dy \).

i. If \( \frac{\alpha_0 + \alpha_1}{\alpha_0} < 1 \), then (1.1) is satisfied with \( A_n = \sqrt{n} \) and \( Z = N(0, \sigma^2) \) where \( \sigma^2 \) depends on \( \text{Cor}(k; \psi, \psi, T) \) for all \( k \geq 0 \).

ii. If \( \alpha_0 > \alpha_1 \), \( \alpha_0 > 1 \), and \( M_0 > 0 \), then (1.1) is satisfied with \( A_n = n^{\frac{\alpha_0}{\alpha_0+1}} \) and \( Z \) a stable law of index \( 1 + \frac{1}{\alpha_0} \) and skewness parameter 1.

iii. If \( \alpha_0 = \alpha_1 =: \alpha \), \( \alpha > 1 \), \( M_0 > 0 \) and \( M_1 < 0 \), then (1.1) is satisfied with \( A_n = n^{\frac{\alpha_0+1}{\alpha_0+2}} \) and \( Z \) a stable law of index \( 1 + \frac{1}{\alpha} \) and skewness parameter determined by \( M_0 \) and \( M_1 \). Any skewness parameter in \([-1, 1]\) is attainable.

iv. If \( \alpha_0 = \alpha_1 = 1 \), \( M_0 \neq 0 \), and \( M_1 \neq 0 \), then (1.1) is satisfied with \( A_n = \sqrt{n \log(n)} \) and \( Z = N(0, \sigma^2) \) where \( \sigma^2 \) is determined by \( M_0 \) and \( M_1 \).

## 2 Maps

Generalized baker’s transformations are area preserving maps of the unit square that generalize the classical baker’s transformation. Roughly speaking a generalized baker’s transformation is a map that realizes the following procedure.\(^2\)

\(^2\)This hypothesis is weakened substantially in Section 5.2.
First, select a function $\phi: [0, 1] \rightarrow [0, 1]$ and let $A = \int \phi$. Second, slice the unit square along the line $x = A$. Third, press the left portion of the square under the graph of $\phi$. Fourth, press the right portion of the square over the graph of $\phi$. If the pressing is done so that area is preserved and every vertical line is mapped affinely to a vertical line, then this procedure determines a map $B: [0, 1]^2 \rightarrow [0, 1]^2$.

![Figure 2: An intermittent baker’s transformation.](image)

We will make the rough description of the last paragraph precise in the case that the function $\phi$ is an ICF as defined in Section 1.1. As before let $A = \int \phi$ denote the area of the region below the graph of $\phi$. The associated IBT $B$ can be defined in terms of an expanding factor map $f: [0, 1] \rightarrow [0, 1]$ and fibre maps $g_x: [0, 1] \rightarrow [0, 1]$, by the formula

$$B(x, y) = (f(x), g_x(y)). \quad (2.1)$$

We define $f$ in Section 2.1 below and note that the fibre maps are defined for each $x \in [0, 1]$ by

$$g_x(y) = \begin{cases} 
\phi(f(x))y, & \text{if } x \in [0, A); \\
[1 - \phi(f(x))]y + \phi(f(x)), & \text{if } x \in [A, 1].
\end{cases} \quad (2.2)$$

For convenience we introduce the following notation for iterates of $B$,

$$g_x^{(0)}(y) = y;$$
$$g_x^{(n+1)}(y) = g_{f^n(x)}(g_x^{(n)}(y)), \quad n \geq 0; \quad (2.3)$$
$$B^n(x, y) = \left(f^n(x), g_x^{(n)}(y)\right). \quad (2.4)$$
2.1 Expanding Factor

We define \( w_0 : [0, 1] \rightarrow [0, A] \) and \( w_1 : [0, 1] \rightarrow [A, 1] \) by

\[
  w_0(x) = \int_0^x \phi(t) \, dt, \quad (2.5)
\]

\[
  w_1(x) = A + \int_0^x 1 - \phi(t) \, dt. \quad (2.6)
\]

Since \( \phi(0) = 1, \phi(1) = 0 \) and \( \phi \) is strictly decreasing we have that \( \phi \) is strictly positive on \([0, 1)\) and hence the functions \( w_0 \) and \( w_1 \) are continuous and strictly increasing and thus are invertible. Define \( f : [0, 1] \rightarrow [0, 1] \) by

\[
  f(x) = \begin{cases} 
    w_0^{-1}(x), & \text{if } x \in [0, A); \\
    w_1^{-1}(x), & \text{if } x \in [A, 1].
  \end{cases} \quad (2.7)
\]

Using Equations (2.5) to (2.7) we compute

\[
  Df(x) = \begin{cases} 
    \left[ \phi(f(x)) \right]^{-1}, & \text{if } x \in [0, A); \\
    \left[ 1 - \phi(f(x)) \right]^{-1}, & \text{if } x \in (A, 1].
  \end{cases} \quad (2.8)
\]

\[
  D^2 f(x) = \begin{cases} 
    -D\phi(f(x)) \left[ Df(x) \right]^3, & \text{if } x \in [0, A); \\
    D\phi(f(x)) \left[ Df(x) \right]^3, & \text{if } x \in (A, 1].
  \end{cases} \quad (2.9)
\]

The alternative representation of \( g_x \) below follows from the displayed equation above and Equation (2.2).

\[
  g_x(y) = \begin{cases} 
    \frac{y}{Df(x)}, & \text{if } x \in [0, A); \\
    \frac{1-y}{1-Df(x)}, & \text{if } x \in (A, 1].
  \end{cases} \quad (2.10)
\]
Taking partial derivatives of the displayed equation above we obtain the displayed equations below. To avoid confusion we write \( g(x, y) \) instead of \( g_y(y) \) to emphasise that \( g : [0, 1]^2 \to [0, 1] \).

\[
\partial_x g(x, y) = \begin{cases} 
    -y\frac{D^2f(x)}{(Df(x))^2}, & \text{if } x \in [0, A); \\
    (1 - y)\frac{D^2f(x)}{(Df(x))^2}, & \text{if } x \in (A, 1].
\end{cases}
\] (2.11)

\[
\partial_y g(x, y) = 1
\] (2.12)

Note that \( Df(x) \) approaches \( \infty \) as \( x \) approaches \( A \) from the left or from the right. From Equation (2.7) we see that \( f(0) = 0 \) and \( f(1) = 1 \). From Equation (2.8) we see that \( Df(0) = Df(1) = 1 \) and therefore \( f \) has neutral fixed points at 0 and 1. It also follows from Equation (2.8) that \( Df(x) \geq 1 \) for all \( x \neq A \), therefore \( f \) is an expanding map.

It should be noted that for \( x \) near 0, the expanding factor \( f \) is approximately \( x \mapsto x(1 + cx^{\alpha_0}) \), with similar behavior near \( x = 1 \). From [18] Theorem 3 we might only expect a finite invariant measure for \( \alpha > 1 \), however \( f \) does not have bounded distortion near \( x = A \) so the main theorem [18] from does not apply. Note that \( f \) is the factor, by projection onto the first coordinate, of \( B \) which preserves two-dimensional Lebesgue measure. It follows that \( f \) must preserve one-dimensional Lebesgue measure. In these examples unbounded distortion near \( x = A \) balances slow escape from the indifferent fixed points at \( x = 0 \) and \( x = 1 \). The map \( f \) associated to an ICF with contact exponent \( \alpha \) preserves one-dimensional Lebesgue measure for any \( \alpha > 0 \).

### 2.2 Exact Rate of Escape from Indifferent Fixed Points

In this section we are concerned with refining asymptotic estimates from [7]. We begin by setting notation.

**Definition 2.1.** Suppose that \( f \) and \( g \) are positive real valued functions.

- We say that \( f(x) \asymp g(x) \) as \( x \to a \) if
  \[
  0 < \liminf_{x \to a} \frac{f(x)}{g(x)} \leq \limsup_{x \to a} \frac{f(x)}{g(x)} < \infty.
  \]

- We say that \( f(x) \in O(g(x)) \) as \( x \to a \) if
  \[
  \limsup_{x \to a} \frac{f(x)}{g(x)} < \infty.
  \]
We will say that $f(x) \sim g(x)$ as $x \to a$ if
\[
\lim_{x \to a} \frac{f(x)}{g(x)} = 1.
\]

We will say that $f(x) \in o(g(x))$ as $x \to a$ if
\[
\limsup_{x \to a} \frac{f(x)}{g(x)} = 0.
\]

We will often abuse notation and let $O(g(x))$ (resp. $o(g(x))$) denote an arbitrary function $h$ such that $h(x) \in O(g(x))$ (resp. $h \in o(g(x))$) as $x \to a$. Note that $f(x) \asymp g(x)$ as $x \to a$ if and only if $f(x) \in O(g(x))$ and $g(x) \in O(f(x))$ as $x \to a$. Similarly $f(x) \sim g(x)$ as $x \to a$ if and only if $f(x) = g(x)(1 + o(1))$.

In this section we refine asymptotic estimates of the form $f(x) \asymp g(x)$ as $x \to a$ from [7] to obtain asymptotic estimates of the form $f(x) \sim g(x)$ as $x \to a$.

Throughout this section $f : [0, 1] \to [0, 1]$ will be the expanding factor map associated to an intermittent cut function with contact exponents $\alpha_0$ and $\alpha_1$, and contact constants $c_0$ and $c_1$. The results of this section are more precise versions of the results contained in [7] Lemma 1. These refinements are needed to prove limit theorems when the rate of decay of correlations is not summable.

We begin by setting notation and collecting a few facts. The map $f$ has two smooth onto branches and $Df(x) > 1$ for $x \in (0, A) \cup (A, 1)$, therefore there exist a unique period-2 orbit $\{p, q\}$ such that $0 < p < A < q < 1$, i.e.
\[
f(p) = q, \quad f(q) = p.
\]

For all $n \geq 0$ define,
\[
p_n = w_0^n(p), \quad q_n = w_1^n(q),
\]
\[
p_{n+1}^o = w_1(p_n), \quad q_{n+1}^o = w_0(q_n).
\]

By Equation (2.7) $w_0$ and $w_1$ are inverses of the branches of $f$. For all $n \geq 0$,
\[
f(p_{n+1}) = p_n, \quad f(q_{n+1}) = q_n, \quad (2.16)
\]
\[
f(p_{n+1}^o) = p_n, \quad f(q_{n+1}^o) = q_n. \quad (2.17)
\]

For each $n \geq 0$, intervals are mapped onto one another by $f$ in the following pattern,
\[
[p_{n+2}, p_{n+1}] \mapsto [p_{n+1}, p_n] \mapsto [p_n, p_{n-1}] \mapsto \cdots \mapsto [p_1, p_0] \mapsto [p, q] \quad (2.18)
\]
\[
[q_{n+2}, q_{n+1}] \mapsto [q_{n+1}, q_n] \mapsto [q_n, q_{n-1}] \mapsto \cdots \mapsto [q_0, q_1] \mapsto [p, q]
\]

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Using Equations (2.5) and (2.6) it is easy to check that for all \( n \geq 0, \)
\[
0 < p_{n+1} < p_n, \quad q_n < q_{n+1} < 1.
\] (2.19)

Both of the maps \( w_0 \) and \( w_1 \) are increasing. For all \( n \geq 1, \)
\[
A < p_n^* < p_n^* < q, \quad p < q_n^* < q_{n+1}^* < A.
\] (2.20)

The following is a refinement of Lemma 1 from [7].

**Lemma 2.2.** As \( n \to \infty, \)
\[
p_n \sim \left( \frac{\alpha_0 + 1}{\alpha_0 \alpha_0} \right)^{\frac{1}{\alpha_0}} \left( \frac{1}{n} \right)^{\frac{1}{\alpha_0}}
\] (2.21)
\[
1 - q_n \sim \left( \frac{\alpha_1 + 1}{\alpha_1 \alpha_1} \right)^{\frac{1}{\alpha_1}} \left( \frac{1}{n} \right)^{\frac{1}{\alpha_1}}
\] (2.22)
\[
p_{n+1} - p_n \sim \frac{1}{\alpha_0} \left( \frac{\alpha_0 + 1}{\alpha_0 \alpha_0} \right)^{\frac{1}{\alpha_0}} \left( \frac{1}{n} \right)^{1+\frac{1}{\alpha_0}}
\] (2.23)
\[
q_{n+1} - q_n \sim \frac{1}{\alpha_1} \left( \frac{\alpha_1 + 1}{\alpha_1 \alpha_1} \right)^{\frac{1}{\alpha_1}} \left( \frac{1}{n} \right)^{1+\frac{1}{\alpha_1}}
\] (2.24)
\[
p_n^* - A \sim \frac{1}{\alpha} \left( \frac{\alpha + 1}{\alpha \alpha} \right)^{\frac{1}{\alpha}} \left( \frac{1}{n} \right)^{1+\frac{1}{\alpha}}
\] (2.25)
\[
A - q_n^* \sim \frac{1}{\alpha} \left( \frac{\alpha + 1}{\alpha \alpha} \right)^{\frac{1}{\alpha}} \left( \frac{1}{n} \right)^{1+\frac{1}{\alpha}}
\] (2.26)
\[
p_n^* - p_n^* \sim \frac{\alpha_0}{\alpha_0} \left( \frac{\alpha_0 + 1}{\alpha_0 \alpha_0} \right)^{\frac{1}{\alpha_0}} \left( \frac{1}{n} \right)^{2+\frac{1}{\alpha_0}}
\] (2.27)
\[
q_{n+1}^* - q_n^* \sim \frac{\alpha_0}{\alpha_0} \left( \frac{\alpha_0 + 1}{\alpha_0 \alpha_0} \right)^{\frac{1}{\alpha_0}} \left( \frac{1}{n} \right)^{2+\frac{1}{\alpha_0}}
\] (2.28)

**Proof.** We begin by proving Equation (2.21), the proof of Equation (2.22) is similar. From the definition of \( \phi \) and \( w_0 \) we have, as \( x \to 0, \)
\[
x - w_0(x) = \int_0^x 1 - \phi(t) \, dt = \frac{c_0}{\alpha_0 + 1} x^{\alpha_0 + 1} + o \left( x^{\alpha_0 + 1} \right).
\]

Note that, as \( y \to \infty, \)
\[
\left( \frac{1}{y} \right)^{\frac{1}{\alpha_0}} \left( \frac{1}{y + z} \right)^{\frac{1}{\alpha_0}} = \frac{y^{\frac{1}{\alpha_0}}}{1 + \left( \frac{z}{y} \right)^{\frac{1}{\alpha_0}}} = \frac{\alpha_0 y^{\frac{1}{\alpha_0} + 1}}{z} + o \left( y^{\frac{1}{\alpha_0}} \right).
\]

The second equality above is obtained by computing the MacLaurin series of the middle expression divided by its numerator in terms of the variable \( \frac{1}{y} \). Using the last two displayed equations, we obtain, as \( y \to \infty, \)
\[
\left( \frac{1}{y} \right)^{\frac{1}{\alpha_0}} - w_0 \left( \frac{1}{y} \right)^{\frac{1}{\alpha_0}} = \frac{\alpha_0 c_0}{\alpha_0 + 1} \frac{1}{z} + o(1)
\]
\[
\left( \frac{1}{y} \right)^{\frac{1}{\alpha_0}} - \left( \frac{1}{y + z} \right)^{\frac{1}{\alpha_0}} = \frac{\alpha_0 c_0}{\alpha_0 + 1} \frac{1}{z} + o(1).
\]
Setting \( z = \frac{\alpha_0 c_0}{\alpha_0 + 1} \) and \( \left( \frac{1}{y} \right)^{\frac{1}{\alpha_0}} = p_k \), we obtain, as \( k \to \infty \)

\[
\frac{p_k - p_{k+1}}{p_k - \left( \frac{1}{y + y^2} \right)^{\frac{1}{\alpha_0}}} = 1 + o(1).
\]

An induction argument shows that, as \( k \to \infty \), for all \( j \geq 1 \),

\[
\frac{p_k - p_{k+j}}{p_k - \left( \frac{1}{y + j z} \right)^{\frac{1}{\alpha_0}}} = 1 + o(1).
\]

Rearranging yields,

\[
p_k + j \sim (y + j z)^{-\frac{1}{\alpha_0}} \text{ as } k \to \infty.
\]

Note that \( z (k + j) \sim y + j z \) as \( j \to \infty \), therefore \( p_{k+j} \sim z^{-\frac{1}{\alpha_0}} (k + j)^{-\frac{1}{\alpha_0}} \) as \( j \to \infty \) faster than \( k \to \infty \).

Letting \( n = k + j \) we conclude that

\[
p_n \sim \left( \frac{\alpha_0 + 1}{\alpha_0 c_0} \right)^{\frac{1}{\alpha_0}} \left( \frac{1}{n} \right)^{\frac{1}{\alpha_0}}.
\]

This completes the proof of Equation (2.21).

Equation (2.23) follows from Equation (2.21) since

\[
\left( \frac{1}{n} \right)^{\frac{1}{\alpha_0}} - \left( \frac{1}{n+1} \right)^{\frac{1}{\alpha_0}} \sim \frac{1}{\alpha_0} \left( \frac{1}{n} \right)^{1+\frac{1}{\alpha_0}}.
\]

To prove Equation (2.25) we note that by Equations (1.2), (2.6) and (2.15) we have

\[
p_n - A = w_1(p_{n-1}) - w_1(0)
\]

\[
= \int_0^{p_{n-1}} 1 - \phi(t) \, dt
\]

\[
= \int_0^{p_{n-1}} c_0 t^{\alpha_0} + h(t) \, dt
\]

\[
= \left( \frac{c_0}{\alpha_0 + 1} \right) \left( p_{n-1}^{\alpha_0 + 1} \right) + o \left( \int_0^{p_{n-1}} t^{\alpha_0} \, dt \right)
\]

\[
\sim \frac{1}{\alpha_0} \left( \frac{\alpha_0 + 1}{c_0 \alpha_0} \right)^{\frac{1}{\alpha_0}} \left( \frac{1}{n} \right)^{1+\frac{1}{\alpha_0}}.
\]

Lemma 2.3. Suppose \( n \geq 0 \) and that \((x, y) \in [p, q] \times [0, 1]\). For all \( 1 \leq k \leq n+1 \), let \((x_k, y_k) = B^k(x, y)\).

i. If \( x \in [p_{n+2}^\circ, p_{n+1}^\circ] \), then as \( n - k \to \infty \),

\[
x_k \sim \left( \frac{\alpha_0 + 1}{c_0 \alpha_0} \right)^{\frac{1}{\alpha_0}} \left( \frac{1}{n - k + 2} \right)^{\frac{1}{\alpha_0}}, \quad \text{(2.29)}
\]

\[
y_k \sim \left( 1 - \frac{k+1}{n} \right)^{1+\frac{1}{\alpha_0}}.
\]
ii. If \( x \in [q_{n+1}^0, q_{n+2}^0] \), then as \( n - k \to \infty \),
\[
1 - x_k \sim \left( \frac{\alpha_1 + 1}{e^\alpha} \right)^{\frac{1}{\alpha_1}} \left( \frac{1}{n-k+2} \right)^{\frac{1}{\alpha_1}},
\]
\[
y_k \sim \left( \frac{k+1}{n} \right)^{1+\frac{1}{\alpha}}.
\]

**Proof.** We will only prove the asymptotic for \( x \in [p_{n+2}^0, p_{n+1}^0] \) the case of \( x \in [q_{n+1}^0, q_{n+2}^0] \) being similar. Throughout this proof we will suppress subscripts \((\alpha := \alpha_0 \text{ and } c := c_0)\). By Equation (2.18), \( x_k \in [p_{n-k+2}, p_{n-k+1}] \). By Equation (2.21), as \( n - k \to \infty \),
\[
p_{n-k+2} \sim \left( \frac{\alpha_1 + 1}{e^\alpha} \right)^{\frac{1}{\alpha}} \left( \frac{1}{n-k+2} \right)^{\frac{1}{\alpha}}.
\]
By Equation (2.23), as \( n - k \to \infty \),
\[
x_k - p_{n-k+2} \leq p_{n-k+1} - p_{n-k+2} = o \left( \frac{1}{n-k} \right)^{\frac{1}{\alpha}}.
\]
This verifies the claimed asymptotic behavior of \( x_k \).

Recall Equations (2.2) and (2.3), and note that for \( k \geq 2 \)
\[
y_k = [\phi(x_1) + (1 - \phi(x_1)) y] \prod_{j=2}^{k} \phi(x_j)
\]
and \( y_1 \) can be obtained by omitting the product in the equation above. Applying Equation (1.2) and expanding \( \log(1 - t) \) about \( t = 0 \), we see that as \( t \to 0 \)
\[
\log(\phi(t)) = \log(1 - ct^\alpha + h(t)) \sim -ct^\alpha.
\]
Applying the asymptotic for \( x_k \) from above we obtain, as \( n - k \to \infty \),
\[
\log(\phi(x_j)) \sim -\left( \frac{\alpha_1 + 1}{\alpha} \right) \left( \frac{1}{n-j+2} \right).
\]
It follows that, as \( n - k \to \infty \)
\[
\sum_{j=2}^{k} \log(\phi(x_j)) \sim -\left( \frac{\alpha_1 + 1}{\alpha} \right) \sum_{j=2}^{k} \frac{1}{n-j+2} \sim \frac{\alpha_1 + 1}{\alpha} \log \left( \frac{n-k+1}{n} \right).
\]
Therefore,
\[
\prod_{j=2}^{k} \phi(x_j) \sim \left( 1 - \frac{k+1}{n} \right)^{1+\frac{1}{\alpha}}.
\]
Noting that \( \phi(x_1) = 1 + o \left( \frac{1}{n} \right) \) we see that, as \( n - k \to \infty \),
\[
y_k \sim \left( 1 - \frac{k+1}{n} \right)^{1+\frac{1}{\alpha}},
\]
as desired. \( \Box \)
3 Induced Map

In this section we will construct an induced map that will enjoy uniform hyperbolicity and bounded distortion.

Consider an Intermittent Baker’s Transformation $B : [0, 1]^2 \to [0, 1]^2$ as defined in Section 2. Let $f$ denote the expanding factor that was described in Section 2.1 and let $\{p, q\}$ denote the period-2 orbit described in Equation (2.13). Define the set

$$\Lambda = [p, q] \times [0, 1].$$

(3.1)

We will refer to $\Lambda$ as the base and consider first returns to $\Lambda$.

Define the return time function $r : \Lambda \to \mathbb{N} \cup \{\infty\}$ by

$$r(x, y) = \inf \{n \in \mathbb{N} \cup \{\infty\} : B^n(x, y) \in \Lambda\}.$$  

(3.2)

The induced map $T : \Lambda \to \Lambda$, defined by

$$T(x, y) = B^{r(x,y)}(x, y),$$

(3.3)

maps a point in $\Lambda$ to the first point along its $B$-orbit that lands in $\Lambda$.

Given a point $(x, y)$ the first coordinate of a $B^n(x, y)$ is independent of $y$ for all $n \geq 0$, similarly membership of $(x, y)$ in $\Lambda$ does not depend on $y$. We conclude that $r(x, y)$ does not depend on $y$. It follows that

$$T(x, y) = B^{r(x)}(x, y) = \left(f^{r(x)}(x), g^{r(x)}(y)\right).$$

(3.4)

We see that $T$ is a skew product and define a factor map $u : [p, q] \to [p, q]$ and fibre maps $v_x : [0, 1] \to [0, 1]$ for each $x \in [p, q]$ by,

$$u(x) = f^{r(x)}(x),$$

(3.5)

$$v_x(y) = g^{r(x)}(y).$$

(3.6)

Let $\lambda$ denote the conditional measure on $\Lambda$, defined by

$$\lambda(E) = \frac{\text{Leb}(E \cap \Lambda)}{\text{Leb}(\Lambda)}.$$  

(3.7)

Note that by Equation (2.18) we have, for each $n \geq 0$,

$$[r = n + 2] = \left(\left[q_n + 1, q_{n+2}\right] \cup \left[p_n + 2, p_{n+1}\right]\right) \times [0, 1].$$

(3.8)
It follows from Lemma 2.2 that,
\[ \lambda[r = n] \asymp \left( \frac{1}{n} \right)^{\frac{1}{2} + 2}, \]  
(3.9)
where \( \alpha = \max \{\alpha_0, \alpha_1\} \).

**Lemma 3.1.** If \( x \in (A, q) \), then
\[ Du(x) = [1 - \phi(f(x))]^{-1} \prod_{k=2}^{r(x)} \left[ \phi(f^k(x)) \right]^{-1} \]  
(3.10)
If \( x \in (p, A) \), then
\[ Du(x) = \phi(f(x))^{-1} \prod_{k=2}^{r(x)} \left[ 1 - \phi(f^k(x)) \right]^{-1} \]  
(3.11)

**Proof.** Suppose that \( n \geq 0 \). If \( (x, y) \in \Lambda \) such that \( x \in [p^n_n+2, p^n_{n+1}) \), then \( x \in [A, 1] \), \( \{B^k(x, y) : 1 \leq k \leq n + 1\} \subset [0, p] \times [0, 1] \), and \( B^n+2(x, y) \in \Lambda \). Using the relationship between induced factor map \( u \) and the factor map \( f \) from Equation (3.5) and the derivative formulas from Equation (2.8) we apply the chain rule to verify Equation (3.10). A similar argument verifies Equation (3.11). \( \square \)

**Lemma 3.2.** If \( x \in (A, q) \), then
\[ v(x, y) = g(x, y) \frac{Df}{Du}(x). \]  
(3.12)
If \( x \in (p, A) \), then
\[ 1 - v(x, y) = [1 - g(x, y)] \frac{Df}{Du}(x). \]  
(3.13)

**Proof.** This follows by inspecting Equations (2.8) and (3.6) and Lemma 3.1. Intuitively \( \frac{Df}{Du} \) collects all of the contractions that are applied by the dynamics after the first affine operation on the fiber. \( \square \)

**Lemma 3.3.** If \( x \in (A, q) \), then
\[ \partial x v(x, y) = (1 - y) \frac{D^2u(x)}{[Du(x)]^2} \frac{Df(x)}{Du(x)} - g(x, y) \frac{D^2u(x)}{[Du(x)]^2} \frac{Df(x)}{Du(x)} \]  
(3.14)

\[ + g(x, y) D\phi(f(x)) \frac{[Df(x)]^3}{Du(x)} \]
If \( x \in (p, A) \), then

\[
\partial_x v(x, y) = y \frac{D^2u(x)}{Du(x)^2} \frac{Df(x)}{Du(x)} + [1 - g(x, y)] \frac{D^2u(x)}{Du(x)^2} Df(x) + [1 - g(x, y)] D\phi(f(x)) \frac{Df(x)^3}{Du(x)} Df(x)
\]

(3.15)

Define the projection \( \mu \) of the measure \( \lambda \) onto \([p, q]\), by

\[
\mu(E) = \lambda(E \times [0, 1]).
\]

(3.16)

Recall the usual transfer operator \( T_* \) acting on measures is defined for measurable \( E \) by \( T_* \nu(E) = \nu(T^{-1}E) \). The transfer operator induces the Perron-Frobenius operator \( P \) on \( L^1(\lambda) \). Given \( \eta \in L^1(\lambda) \) and a measurable set \( E \), define \( \nu(E) = \int_E \eta \, d\lambda \), then the Perron-Frobenius operator is defined by \( P\eta = \frac{dT_* \nu}{d\lambda} \) where the right hand side is a Radon-Nikodym derivative. We note that \( T \) is invertible and preserves \( \lambda \), and therefore \( \frac{dT_* \nu}{d\lambda} = \eta \circ T^{-1} \). For this reason we will abuse notation and use \( T_* \) to denote both the transfer operator and the Perron-Frobenius operator associated to \( T \), that is, for all \( \eta \in L^1(\lambda) \),

\[
T_* \eta = \eta \circ T^{-1}.
\]

(3.17)

When we refer to iterates of \( T \) we will use the notation \( v_x^{(k)} \) defined analogously to Equation (2.3) so that we have.

\[
T^k(x, y) = \left( u^k(x), v_x^{(k)}(y) \right)
\]

(3.18)

The map \( u \) preserves \( \mu \).

In what follows it will be convenient to define the \( k \)-th return time \( r^{(k)} : \Lambda \to \mathbb{N} \cup \{\infty\} \) by,

\[
r^{(1)}(x, y) = r(x, y)
\]

\[
r^{(k+1)}(x, y) = r^{(k)}(x, y) + r(T^k(x, y)).
\]

(3.19)

Note that if \( n = r^{(k)}(x, y) \), then \( n \) is the smallest positive integer so that the set \( \{B^j(x, y) : j = 1, \ldots, n\} \) contains \( k \) points in \( \Lambda \).
3.1 Dynamical Partitions

Our anisotropic Banach spaces will be built with respect to stable and unstable curves for the IBT. Since $T$ is a skew product, it is easy to check that vertical lines form an equivariant family of stable curves for $T$. For convenience we introduce notation. For every $x \in [p,q]$, define
\[
\ell(x) = \{x\} \times [0,1]. \tag{3.20}
\]
With this notation equivariance takes the form
\[
T(\ell(x)) \subset \ell(u(x)). \tag{3.21}
\]
It is routine to check that for every $x \in [p,q]$ the map $v_x : \ell(x) \rightarrow \ell(u(x))$ is an affine contraction by at least $\beta$.

The next lemma characterizes unstable curves for $T$.

**Lemma 3.4.** There is an equivariant family $\Gamma$ of unstable curves for $T$ such that, each curve is the graph of a function in $C^1([p,q],[0,1])$, the family is bounded in the $C^1$ norm, and the family forms a partition of $\Lambda$.

**Proof.** The proof is a standard but involved application of graph transformations. See [20] Chapter 12 or [9] Lemma 5.4.15. \qed

We define $\gamma : \Lambda \rightarrow \Gamma$ by,
\[
\gamma(x,y) \in \Gamma \text{ such that } (x,y) \in \gamma(x,y). \tag{3.22}
\]
Since $\Gamma$ is a partition $\gamma(x,y)$ is uniquely defined.

Note that by Equation (3.8) the collection $\{[r = n] : n \geq 1\}$ is a partition mod $\lambda$ of $\Lambda$, as is $\{(p,A) \times [0,1], (A,q) \times [0,1]\}$. For all $k \geq 1$ we define,
\[
\Omega_1 = \{[r = n] : n \geq 1\} \cup \{(p,A) \times [0,1], (A,q) \times [0,1]\}, \tag{3.23}
\]
\[
\Omega_{k+1} = \Omega_1 \cup T^{-1}\Omega_k.
\]
All of these collections are partitions mod $\lambda$ since $T$ is measure preserving. Every cell of $\Omega_k$ is a column of the form $[a,b] \times [0,1]$ or $(a,b) \times [0,1]$. We define $\omega_k : \Lambda \rightarrow \Omega_k$ by,
\[
\omega_k(x,y) \in \Omega_k \text{ such that } (x,y) \in \omega_k(x,y). \tag{3.24}
\]
Since $\Omega_k$ is a partition mod $\lambda$, we have that $\omega_k(x,y)$ is uniquely defined for $\lambda$-a.e. $(x,y)$. Note that $r^{(k)}$ is measurable with respect to $\Omega_k$.

Let $\hat{\Omega}_k$ denote the projection of $\Omega_k$ on to the interval $[p,q]$ by the map $(x,y) \mapsto x$. 

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Lastly we define measurable partitions $\Theta_n$ and maps $\theta_n: \Lambda \to \Theta_n$ by

\begin{align}
\Theta_n &= T^n \Omega_n \\
\theta_n(x,y) &\in \Theta_n \text{ such that } (x,y) \in \theta_n(x,y).
\end{align}

The cells of $\Theta_n$ are strips that are bounded above and below by curves in $\Gamma$ and extend across the full width of $\Lambda$.

### 3.2 Derivative Bounds

While an IBT is non-uniformly hyperbolic, the induced map introduced in the last section enjoys uniform hyperbolicity. For our purposes it suffices to show that the factor map $u$ of the induced map $T$ is a well behaved interval map meaning that it enjoys uniform expansion and bounded distortion. The following lemmas from [7] provide the necessary bounds.

**Lemma 3.5** (Lemma 2 from [7]). If

\[
\beta = \sup_{t \in [p,q]} \max \{ \phi(t), 1 - \phi(t) \},
\]

then

\[
\left\| (Du)^{-1} \right\|_\infty \leq \beta.
\]

*Proof.* This follows immediately from Equations (3.10) and (3.11). Note that every term in the product can be bounded above by 1 since $\phi$ takes values in $[0,1]$ and $f^{r(x)}(x) \in [p,q]$.

**Lemma 3.6** (Lemma 3 from [7]). There exists $\kappa < \infty$ such that for all $k \geq 1$, if $w$ and $x$ lie in the same cell of $\hat{\Omega}_k$, then

\[
\left| \frac{Du^k(x)}{Du^k(w)} - 1 \right| \leq \kappa |u^k(x) - u^k(w)|.
\]

*Proof.* See [7].

**Lemma 3.7.** There exists a constant $\tau > 0$ such that

\[
\left\| \frac{\partial_x u^{(k)}}{Du^k} \right\|_\infty \leq \tau.
\]
Proof. Suppose that $k = 1$. By Equation (3.14) we have the identity below for almost every $x \in (A, q)$.

$$
\frac{\partial_x v(x, y)}{D u(x)} = (1 - y) \frac{D^2 u(x)}{[D u(x)]^2} \frac{D f(x)}{[D u(x)]^2} - g(x, y) \frac{D^2 u(x)}{[D u(x)]^2} \frac{D f(x)}{D u(x)}
$$

$$
+ g(x, y) D \phi (f(x)) \frac{[D f(x)]^3}{[D u(x)]^2}
$$

Similarly, by Equation (3.15) we have the identity below for almost every $x \in (p, A)$.

$$
\frac{\partial_x v(x, y)}{D u(x)} = y \frac{D^2 u(x)}{[D u(x)]^2} \frac{D f(x)}{[D u(x)]^2} + [1 - g(x, y)] \frac{D^2 u(x)}{[D u(x)]^2} \frac{D f(x)}{D u(x)}
$$

$$
+ [1 - g(x, y)] D \phi (f(x)) \frac{[D f(x)]^3}{[D u(x)]^2}
$$

Taking norms we obtain the bound below.

$$
\left\| \frac{\partial_x v}{D u} \right\|_{\infty} \leq \left\| \frac{D^2 u}{[D u]^2} \right\|_{\infty} \left\| \frac{D f}{[D u]^2} \right\|_{\infty} + \left\| \frac{D^2 u}{[D u]^2} \right\|_{\infty} \left\| \frac{D f}{D u} \right\|_{\infty} + \left\| \frac{[D \phi \circ f] [D f]^3}{[D u]^2} \right\|_{\infty}
$$

By Lemma 3.6 we have,

$$
\left\| \frac{D^2 u}{[D u]^2} \right\|_{\infty} \leq \kappa.
$$

Suppose that $x \in (p_{n+2}^o, p_{n+1}^o)$. Since $\phi$ is decreasing and $f(x)$ is increasing, $D f(x) = [1 - \phi(f(x))]^{-1}$ is decreasing. Recall from Equation (2.18) that $f$ maps $(p_{n+2}^o, p_{n+1}^o)$ onto $(p_{n+1}, p_n)$. By the Mean Value Theorem there exists $\theta_n \in (p_{n+2}^o, p_{n+1}^o)$ such that

$$
D f(\theta_n) = \frac{p_n - p_{n+1}}{p_{n+1}^o - p_{n+2}^o}.
$$

Since $D f$ is decreasing for all $x \in (p_{n+2}^o, p_{n+1}^o)$,

$$
D f(\theta_{n-1}) \leq D f(x) \leq D f(\theta_{n+1}).
$$

By Equations (2.23) and (2.27) we have

$$
\frac{p_n - p_{n+1}}{p_{n+1}^o - p_{n+2}^o} \approx n
$$

We deduce that for all $x \in (p_{n+2}^o, p_{n+1}^o)$,

$$
D f(x) \approx n.
$$
A similar argument shows that for all \( x \in (p_n^{o_n+2}, p_n^{p_n+1}) \),

\[
Du(x) \approx \frac{q - p}{p_n^{o_n+1} - p_n^{o_n+2}} \approx n^{2 + \frac{1}{o_n}}
\]

By Equation (1.2), \( D\phi(t) = -c_0 o_n t^{a_0 - 1} + o(t^{a_0 - 1}) \) for \( t \) near 0. If \( x \in (p_n^{o_n+2}, p_n^{p_n+1}) \), then \( f(x) \in (p_n^{p_n+1}, p_n^{p_n+1}) \) by Equation (2.18). From Equation (2.29) we see that \( f(x) \approx \left(\frac{1}{n}\right)^{1/o_n} \). Thus, for \( x \in (p_n^{o_n+2}, p_n^{p_n+1}) \),

\[
D\phi(f(x)) \approx n^{1/o_n - 2}.
\]

Similar arguments apply to \( x \in (q_n^{o_n+1}, q_n^{p_n+2}) \).

Combining the last three displayed equations and their analogues for \( x > A \) we see that \( \|\partial_x v^k Du^k\|_\infty \), \( \|\frac{Df}{Du}\|_\infty \), and \( \|\frac{D\phi(f)}{Du}\|_\infty \) are all finite. We conclude that for \( k = 1 \) Equation (3.30) holds with some constant \( \tau_0 > 0 \).

We claim that Equation (3.30) holds with \( \tau = \tau_0 (1 - \beta^2)^{-1} \). To verify this we first prove that

\[
\|\partial_x v^k Du^k\|_\infty \leq \tau_0 \sum_{m=0}^{k-1} \beta^{2m}.
\]

The claim follows by replacing the finite geometric series with the infinite geometric series.

Suppose that the displayed inequality above holds for some \( k \geq 1 \). Let \( DT^k \) denote the Jacobian of \( T^k \). Note that

\[
DT^k = \begin{bmatrix} Du^k & 0 \\
\partial_x v & [Du^k]^{-1} \end{bmatrix}.
\]

Since \( DT^{k+1} = (DT^k \circ T) DT \) we have

\[
\partial_x v^{(k+1)} = \left(\partial_x v^{(k)} \circ T\right) Du + \frac{\partial_x v}{Du^k \circ u} \text{,}
\]

\[
Du^{k+1} = (Du^k \circ u) Du.
\]

Thus,

\[
\frac{\partial_x v^{(k+1)}}{Du^{k+1}} = \frac{\partial_x v^{(k)}}{Du^k} \circ T + \frac{1}{[Du^k \circ u]^2} \frac{\partial_x v}{Du}.
\]

Taking the norm of both sides of the identity above, applying the induction
hypothesis, and applying Equation 3.28 we obtain the bound below.

\[
\| \frac{\partial x^{(k+1)}}{Dx^{k+1}} \|_\infty \leq \| \frac{\partial x^{(k)}}{Dx^{k}} \|_\infty + \left( \frac{1}{Dx^{k}} \right) \| \frac{\partial x}{Du} \|_\infty \\
\leq \tau_0 \sum_{m=0}^{k-1} \beta^{2m} + \tau_0 \sum_{m=0}^{k} \beta^{2m} \\
\leq \frac{\tau_0}{1 - \beta^2}.
\]

\[\square\]

4 Adapted Banach Spaces

In this section we will define Banach spaces \( \mathcal{W} \) and \( \mathcal{S} \) with anisotropic norms that are adapted to the dynamics of the induced map \( T \). These spaces were first introduced in [17] and are a simplified version of the norms defined in [10].

We begin by constructing a space \( \mathcal{L} \) of bounded measurable functions that exhibit regularity along unstable curves, which is one of the key properties that we will need in the space \( \mathcal{S} \). This regularity is necessary in the proof of a Lasota-Yorke inequality (see Proposition 6.6).

**Definition 4.1.** Given a bounded measurable function \( \eta : \Lambda \to \mathbb{R} \), define

\[
\text{Lip}_u(\eta) = \sup_{\gamma \in \Gamma} \sup_{(x,y) \neq (w,z) \in \gamma} \frac{\eta(x,y) - \eta(w,z)}{|x-w|},
\]

\[
\| \eta \|_{\mathcal{L}} = \| \eta \|_{\sup} + \text{Lip}_u(\eta),
\]

\[
\mathcal{L} = \{ \eta : \| \eta \|_{\mathcal{L}} < \infty \}.
\]

Recall that \( \Gamma \) is the partition of \( \Lambda \) by unstable curves.

While elements of the space \( \mathcal{S} \) must exhibit regularity along unstable curves to satisfy a Lasota-Yorke inequality, they will also have a distributional quality along stable lines to facilitate the proof of a compact embedding. By restricting to a stable line we will view elements of \( \mathcal{S} \) and \( \mathcal{W} \) as functionals on spaces of Hölder functions.

**Definition 4.2.** Let \( \mathcal{L} \) denote the space of real valued Lipschitz functions with domain \( [0,1] \). Fix \( \alpha \in (0,1) \) and let \( \mathcal{H} \) denote the space of real values \( \alpha \)-Hölder functions with domain \( [0,1] \). Let \( \| \cdot \|_{\mathcal{L}} \) and \( \| \cdot \|_{\mathcal{H}} \) denote the norms of \( \mathcal{L} \) and \( \mathcal{H} \) respectively.
Next we define norms for the spaces $S$ and $W$, which should be viewed as being related to the strong operator norm on the dual spaces $H^*$ and $L^*$. Specifically, a bounded measurable function $\eta: \Lambda \to \mathbb{R}$ induces a functional for each vertical line $\ell(x)$ through integration. Given $\psi$ in $H$ or $L$

$$\psi \mapsto \int_0^1 \eta(x, y) \psi(y) \, dy$$

defines a bounded linear functional. This motivates the following definition.

**Definition 4.3.** For all bounded measurable functions $\eta: \Lambda \to \mathbb{R}$ define

$$\|\eta\|_W = \sup \left\{ \int_0^1 \eta(x, y) \psi(y) \, dy : x \in [p, q], \|\psi\|_L \leq 1 \right\},$$

$$\|\eta\|_S = \sup \left\{ \int_0^1 \eta(x, y) \psi(y) \, dy : x \in [p, q], \|\psi\|_H \leq 1 \right\},$$

$$\text{Lip}_s (\eta) = \sup \left\{ \int_0^1 \frac{\eta(w, y) - \eta(x, y)}{|w - x|} \psi(y) \, dy : w \neq x \in [p, q], \|\psi\|_L \leq 1 \right\},$$

$$\|\eta\|_S = \|\eta\|_s + \text{Lip}_s (\eta).$$

Since $L \subset H$ we have $\|\cdot\|_W \leq \|\cdot\|_s \leq \|\cdot\|_S$. Both $\|\cdot\|_S$ and $\|\cdot\|_W$ are bounded semi-norms on the space of Lipschitz functions. By taking quotients, $\|\cdot\|_S$ and $\|\cdot\|_W$ induce norms on quotient spaces of $L$. Completing these quotient spaces with respect to their norms produces Banach spaces $S$ and $W$.

## 5 Main Results

In this section we apply operator renewal theory as described in [12][11] to obtain the rate of decay of correlation (Theorem 1.1) and limit theorems (Theorem 5.4) for an IBT $B$.

For each $n \geq 1$ and $k \geq 1$ we define operators by

$$R_n^{(k)} \eta = T_n^{k} \left( \mathbf{1}_{\{\tau^{(k)} = n\}} \eta \right),$$

$$B_n \eta = 1_{\Lambda} B_n \eta = 1_{\Lambda} \mathbf{1}_{\{\tau = n\}} \eta.$$

We will always abbreviate $R_n^{(1)}$ as $R_n$. The operators $R_n$ are a decomposition of $T_n$ by first return time. The operators $B_n$ can be viewed as a restriction of $B_n^{\mathbf{1}_{\Lambda}}$ to an action on functions supported on $\Lambda$.

A key technical observation in operator renewal theory is that the generating functions defined by Equations (5.3) and (5.4) are well defined and related by
Equation (5.5).

\[ B(z) = I + \sum_{n=1}^{\infty} z^n B_n \quad (5.3) \]

\[ R(z) = \sum_{n=1}^{\infty} z^n R_n \quad (5.4) \]

\[ B(z) = [I - R(z)]^{-1} \quad (5.5) \]

In what follows we will make use of the following identities, which are routine to check,

\[ R(1) = T_\ast \quad (5.6) \]

\[ [R(z)]^k = \sum_{n=1}^{\infty} R_n^{(k)} z^n. \quad (5.7) \]

5.1 Decay of Correlations

Heuristically, if \( \eta \) is supported on \( \Lambda \) and \( \int_\Lambda \eta \neq 0 \), then the push forward distributions \( B_n^\ast \eta \) must equilibrate to a multiple \( 1_{[0,1]^2} \), which is the density for the preserved measure. The transfer operator \( B_n \) sends all of the mass represented by \( \eta \) outside of \( \Lambda \). In order for \( B_n^\ast \eta \) to attain its limiting value of \( \int_{[0,1]^2} \eta \text{d}\text{Leb} \) inside of \( \Lambda \), mass must return to \( \Lambda \). The amount of mass that has failed to return after \( n \) steps of the dynamics is \( \text{Leb}[r>n] \), which provides a rough estimate for how quickly the convergence \( B_n^\ast \eta \to 1_{[0,1]^2} \int_{[0,1]^2} \eta \text{d}\text{Leb} \) can occur. Theorem 1.1 shows that this rough estimate is actually sharp.

In this section we will prove Theorem 1.1 by applying [12] Theorem 1.1, which we reproduced below for the convenience of the reader. We have modified notation slightly to match the current setting.

**Theorem 5.1** (Theorem 1.1 from [12]). Let \( B_n \) be bounded operators on \( S \) such that \( B(z) = I + \sum_{n \geq 1} z^n B_n \) converges in \( \text{Hom}(S, S) \) for every \( z \in \mathbb{C} \) with \( |z| < 1 \). Assume that:

1. **Renewal equation:** for every \( z \in \mathbb{C} \) with \( |z| < 1 \), \( B(z) = (I - R(z))^{-1} \) where \( R(z) = \sum_{n \geq 1} z^n R_n \), \( R_n \in \text{Hom}(L, L) \) and \( \sum \| R_n \| < +\infty \).

2. **Spectral Gap:** \( 1 \) is a simple isolated eigenvalue of \( R(1) \).

3. **Aperiodicity:** for every \( z \neq 1 \) with \( |z| \leq 1 \), \( I - R(z) \) is invertible.

\(^3\)With the strong operator topology
Let $P$ be the eigenprojection of $R(1)$ at 1. If $\sum_{k>n} \|R_k\| = O\left(1/n^\beta\right)$ for some $\beta > 1$ and $PR'(1)P \neq 0$, then for all $n$

$$B_n = \frac{1}{\mu} P + \frac{1}{\mu^2} \sum_{k=n+1}^{\infty} P_k + E_n$$

where $\mu$ is given by $PR'(1)P = \mu P$, $P_n = \sum_{l>n} PR_l P$ and $E_n \in \text{Hom}(\mathcal{L}, \mathcal{L})$ satisfy

$$\|E_n\| = \begin{cases} O\left(1/n^\beta\right), & \text{if } \beta > 2; \\ O\left(\log(n)/n^2\right), & \text{if } \beta = 2; \\ O\left(1/n^{2\beta-2}\right), & \text{if } 2 > \beta > 1. \end{cases}$$

The following two propositions verify that the hypotheses of Theorem 5.1 are satisfied and will be proved later.

**Proposition 5.2** (Convergence and Renewal Equation).

- For all $n \geq 1$, the operators $B_n$ and $R_n$ are bounded on $\mathcal{S}$.
- For all $z$ in the open unit disk of $\mathbb{C}$, the operators $B(z)$ and $R(z)$ converge in $\text{Hom}(\mathcal{S}, \mathcal{S})$ and satisfy $B(z) = (I - R(z))^{-1}$.
- The operator $R(z)$ converges in $\text{Hom}(\mathcal{S}, \mathcal{S})$ for $z$ in the closed unit disk of $\mathbb{C}$, that is $\sum_{n \geq 1} \|R_n\|_S < \infty$.
- Let $\alpha = \max\{\alpha_0, \alpha_1\}$. As $n \to \infty$, $\sum_{k>n} \|R_k\|_S = O\left(n^{-(1+\frac{1}{\alpha})}\right)$.

**Proof.** See Section 6.4.

**Proposition 5.3** (Spectral Gap and Aperiodicity).

- $1$ is a simple isolated eigenvalue of $R(1)$.
- For $z \neq 1$ with $|z| \leq 1$, $I - R(z)$ is invertible.
- For $\eta \in \mathcal{L}$, the spectral projector $P$ can be computed by the formula $P\eta = 1_\Lambda \int_\Lambda \eta d\lambda$.

**Proof.** See Section 6.6.

**Proof of Theorem 1.1** Suppose that $\eta$ and $\psi$ are Lipschitz functions on $\Lambda$. Let $\alpha = \max\{\alpha_0, \alpha_1\}$.

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4Here $R'(1)$ denotes the operator $\frac{d}{dz} R|_{z=1}$.  

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We start by identifying the parameter $\beta$ from Theorem 5.1. By Proposition 5.2, we have
\[
\sum_{k>n} \|R_k\|_S = O \left( \left( \frac{1}{n} \right)^{1+\frac{1}{\alpha}} \right).
\]
Therefore, $\beta = 1 + \frac{1}{\alpha}$.

Next we identify the parameter $\mu$ from Theorem 5.1. Note that since $\eta$ is Lipschitz on $\Lambda$ we have $\eta \in L^1$. Applying the spectral projector formula from Proposition 5.3, we obtain,
\[
Pd \eta = P_{\infty} \sum_{n=1}^{\infty} nR_n \eta = P_{\infty} \sum_{n=1}^{\infty} nR_n \frac{1}{\lambda} \int \eta d\lambda\]
By Kac’s Lemma $\sum_{n=1}^{\infty} n\lambda[r=n] = \frac{1}{\text{Leb}(\Lambda)}$. Therefore $\mu = \frac{1}{\text{Leb}(\Lambda)}$.

Next we identify the operators $P_k$ from Theorem 5.1. By a calculation similar to the one above,
\[
P_k \eta = \sum_{l>k} PR_l P \eta = P \sum_{l>k} \lambda \eta[l] = \lambda \eta[k] P \eta.
\]
Therefore,
\[
P_k = \lambda \eta[k] P.
\]

From Theorem 5.1, we obtain the expansion
\[
B_n = \text{Leb}(\Lambda) P + \text{Leb}(\Lambda)^2 \sum_{k>n} P_k + E_n
\]
where
\[
\|E_n\| = \begin{cases} 
O \left( \left( \frac{1}{n} \right)^{1+\frac{1}{\alpha}} \right), & \text{if } \alpha > 1; \\
O \left( \frac{\log(n)}{n^\alpha} \right), & \text{if } \alpha = 1; \\
O \left( \left( \frac{1}{n} \right)^{2/\alpha} \right), & \text{if } \alpha < 1.
\end{cases}
\]
Since $\lambda$ is the conditional measure obtained by restricting $\text{Leb}$ to $\Lambda$, see Equation (3.7), we have for any $\eta \in L^1(\Lambda, \lambda)$, $\text{Leb}(\Lambda) \int \eta d\lambda = \int \eta d\text{Leb}$. Applying
the expansion of $B_n$ that we have just obtained we see that

$$B_n\eta = \text{Leb}(\Lambda) P\eta + \text{Leb}(\Lambda)^2 \sum_{k>n} P_k\eta + E_n\eta$$

$$= 1_\Lambda \text{Leb}(\Lambda) \int_\Lambda \eta \, d\lambda + \sum_{k>n} 1_{r>k} \text{Leb}(\Lambda)^2 \lambda[r > k] \int_\Lambda \eta \, d\lambda + E_n\eta$$

$$= 1_\Lambda \int_\Lambda \eta \, d\text{Leb} + 1_\Lambda \sum_{k>n} \text{Leb}[r > k] \int_\Lambda \eta \, d\text{Leb} + E_n\eta.$$ 

Since $\eta$ and $\psi$ are Lipschitz on the square, $1_\Lambda \eta \in L$ and we obtain

$$\int B_n \eta \psi \, d\text{Leb} = \int 1_\Lambda B_n^*(1_\Lambda \eta) \psi \, d\text{Leb} = \int 1_\Lambda \eta (1_\Lambda \psi) \circ B_n \, d\text{Leb}$$

If $\eta$ and $\psi$ are the restrictions to $\Lambda$ of Lipschitz functions on the square, then

$$\int_\Lambda \eta \psi \circ B_n \, d\text{Leb} = \int_\Lambda \eta \, d\text{Leb} \int_\Lambda \psi \, d\text{Leb} + \sum_{k>n} \text{Leb}[r > k] \int_\Lambda \eta \, d\text{Leb} \int_\Lambda \psi \, d\text{Leb}$$

$$+ \int_\Lambda E_n \eta \psi \, d\text{Leb}.$$ 

Note that $\sum_{k>n} \text{Leb}[r > k] \simeq (\frac{1}{n})^{\frac{1}{\alpha}}$ and that regardless of the value of $\alpha$ this decays slower than $\|E_n\|$. If $\int \eta \neq 0$ and $\int \psi \neq 0$, then

$$\int_\Lambda \eta \psi \circ B_n \, d\text{Leb} - \int_\Lambda \eta \, d\text{Leb} \int_\Lambda \psi \, d\text{Leb} = \sum_{k>n} \text{Leb}[r > k] \int_\Lambda \eta \, d\text{Leb} \int_\Lambda \psi \, d\text{Leb}$$

$$+ \int_\Lambda E_n \eta \psi \, d\text{Leb}$$

$$\simeq \left( \frac{1}{n} \right)^{\frac{1}{\alpha}}.$$ 

For functions with integral zero the rate of decay may be faster than $\left( \frac{1}{n} \right)^{\frac{1}{\alpha}}$. □

**Corollary 5.3.1.** If the hypotheses of Theorem 1.1 are satisfied and additionally either $\int_\Lambda \psi = 0$ or $\int_\Lambda \eta = 0$, then Cor $k, \psi, \eta, B$ is a summable sequence.

### 5.2 Limit Theorems

In this section we will select an observable $X: [0, 1]^2 \to \mathbb{R}$ with mean zero and deduce distributional limit behavior of the form

$$\frac{1}{A_n} \sum_{k=0}^{n-1} X \circ B^k \xrightarrow{\text{dist}} Z, \quad \text{as } n \to \infty,$$  

(5.9)
where $A_n$ is a sequence of real numbers, and $Z$ is a real valued random variable and $B$ is an IBT with contact coefficients $c_j$ and contact exponents $\alpha_j$.

The random variables that can arise as the limits in Equation (5.9) are stable distributions. Stable distributions with mean zero can be parameterized as follows. Let $p \in (1, 2]$, $a > 0$ and $b \in [-1, 1]$. Let $St(p, a, b)$ be the distribution such that if $Z \sim St(p, a, b)$, then

$$E[e^{itZ}] = e^{-a|t|^p(1 - b \text{sgn}(t) \tan(\frac{p\pi}{2}))}.$$  

Note that if $p = 2$, then $Z$ is normally distributed with mean zero and standard deviation $\sigma = \sqrt{2a}$.

Below we collect a precise technical version of Theorem 1.2. In order to state the theorem we need to define several constants.

$$M_0 = \int_0^1 X(0, y^{1 + \frac{1}{\alpha_0}}) dy,$$

$$M_1 = \int_0^1 X(1, y^{1 + \frac{1}{\alpha_1}}) dy,$$

$$C_0 = \frac{|M_0|}{\alpha_0 \text{Leb}(\Lambda)} \left( \frac{|M_0|}{c_0 \alpha_0} \right)^{\frac{1}{p_0}},$$

$$C_1 = \frac{|M_1|}{\alpha_1 \text{Leb}(\Lambda)} \left( \frac{|M_1|}{c_1 \alpha_1} \right)^{\frac{1}{p_1}}.$$

**Theorem 5.4.** Suppose that $X: [0, 1]^2 \to \mathbb{R}$ is $\gamma$-Hölder for some $\gamma \in (0, 1]$ and $\int_{[0,1]^2} X d\text{Leb} = 0$.

i. Suppose that $\xi \in L^2$ and that $\xi$ is not a coboundary. Then $\sigma^2 = \int_\Lambda |\xi|^2 d\lambda + 2 \sum_{k=1}^\infty \int_\Lambda \xi \circ T^k \xi d\lambda$ converges, $\sigma^2 > 0$, and as $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} X \circ B_k \xrightarrow{\text{dist}} N(0, \sigma^2).$$

ii. Suppose that $\alpha_0 > \alpha_1$, $\alpha_0 > 1$, and $M_0 > 0$. Let $p = 1 + \frac{1}{\alpha_0}$, $a = C_0 \Gamma(1 - p) \cos \left( \frac{p\pi}{2} \right)$, and $b = 1$. As $n \to \infty$,

$$\frac{1}{n^{\alpha_0 + 1}} \sum_{k=0}^{n-1} X \circ B_k \xrightarrow{\text{dist}} St(p, a, b).$$

iii. Suppose that $\alpha_0 = \alpha_1 =: \alpha$, $\alpha > 1$, $M_0 > 0$ and $M_1 < 0$. Let $p = 1 + \frac{1}{\alpha}$, 

$a = (C_0 + C_1) \Gamma(1 - p) \cos \left( \frac{p\pi}{2} \right)$, and $b = \frac{C_0 - C_1}{C_0 + C_2}$. As $n \to \infty$,

$$\frac{1}{n^{\alpha + 1}} \sum_{k=0}^{n-1} X \circ B_k \xrightarrow{\text{dist}} St(p, a, b).$$

\footnote{In particular if the hypotheses of Lemma 5.5 are satisfied}
iv. Suppose that \( \alpha_0 = \alpha_1 = 1, M_0 \neq 0, \) and \( M_1 \neq 0, \) then as \( n \to \infty, \)

\[
\frac{1}{\sqrt{n \log(n)}} \sum_{k=0}^{n-1} X \circ B^k \xrightarrow{\text{dist}} N(0, C_0 + C_1).
\]

Note that by manipulating the values of \( M_0 \) and \( M_1 \) in the third limit theorem above on can obtain stable distributions with any skewness parameter \( b \in [-1, 1]. \)

The choices of parameter ranges in the last three limit theorems above are motivated by the following lemma.

**Lemma 5.5 (Finite Variance Conditions).** Suppose that \( X : [0, 1]^2 \to \mathbb{R} \) is \( \gamma \)-Hölder for some \( \gamma \in (0, 1]. \) If for \( j = 0 \) one of the conditions below is satisfied, and similarly for \( j = 1 \) one of the conditions below is satisfied, then \( \xi \in L^2. \)

i. \( \alpha_j < 1, \)

ii. \( M_j = 0 \) and \( \alpha_j = 1, \)

iii. \( M_j = 0, \) \( 1 < \alpha < 3, \) and \( \gamma > \frac{\alpha - 1}{2}, \)

**Proof.** See Section 6.7

The proof of this Theorem 5.4 is an application of [11] Theorem 2.1. For the convenience of the reader we reproduce the theorem here. We have modified the notation slightly to fit our setting.

**Theorem 5.6 (Theorem 2.1 from [11]).** Let \( S \) be a Banach space and \( R_n \in \text{Hom}(S, S) \) be operators on \( S \) with \( \|R_n\| \leq r_n \) for a sequence \( r_n \) such that \( a_n = \sum_{k>n} r_k \) is summable. Write \( R(z) = \sum R_n z^n \) for \( z \in \mathbb{D}. \) Assume that 1 is a simple isolated eigenvalue of \( R(1) \) and that \( 1 - R(z) \) is invertible for \( z \in \mathbb{D} - \{1\}. \) Let \( P \) denote the spectral projection of \( R(1) \) for the eigenvalue 1, and assume that \( PR(1)P = \mu P \) for some \( \mu > 0. \) Let \( R_n(t) \) be an operator depending on \( t \in [-\delta_0, \delta_0], \) continuous at \( t = 0 \) with \( R_n(0) = R_n \) and \( \|R_n(t)\| \leq C r_n \) for all \( t \in [-\delta_0, \delta_0], \) for some constant \( C > 0. \) For \( z \in \mathbb{D} \) and \( t \in [-\delta_0, \delta_0] \) write

\[
R(z, t) = \sum_{n=1}^{\infty} z^n R_n(t).
\]

This is a continuous perturbation of \( R(z). \) For \( t \) small and \( z \) close to 1, \( R(z, t) \) is close to \( R(1), \) whence it admits an eigenvalue \( \chi(z, t) \) close to 1. Assume that \( \chi(1, t) = 1 - (c + o(1))M(|t|) \) for \( c \in \mathbb{C} \) with \( \text{Re}(c) > 0, \) and some continuous function \( M : \mathbb{R}_+ \to \mathbb{R}_+ \) vanishing only at 0. Then
1. There exists $\epsilon_0 > 0$ such that for all $|t| < \epsilon_0$, $I - R(z,t)$ is invertible for all $z \in \mathbb{D}$. We can write $(I - R(z,t))^{-1} = \sum T_{n,t}z^n$.

2. Furthermore, there exist functions $\epsilon(t)$ and $\delta(n)$ tending to 0 when $t \to \infty$ and $n \to \infty$ such that for all $|t| < \epsilon_0$, for all $n \in \mathbb{N}^*$, we have

$$\|T_{n,t} = \frac{1}{\mu} \left(1 - \frac{\epsilon}{\mu} M(|t|)^n P \right) \leq \epsilon(t) + \delta(n).$$

Before we can apply Theorem 5.6 we must define the operators $R_n(t)$. First, we define an observable $\xi: \Lambda \to \mathbb{R}$ derived from the observable $X$ as follows,

$$\xi(x,y) = \sum_{k=0}^{r(x)-1} (X \circ B^k)(x,y). \quad (5.10)$$

For all $t \in \mathbb{R}$ and $\eta \in \mathbb{L}$, let

$$R_n(t)\eta = R_n \left[ \exp (it\xi) \eta \right]. \quad (5.11)$$

Note that the hypotheses of Theorem 5.6 that pertain to the unperturbed operators have already been verified in Propositions 5.2 and 5.3. Recall from the proof of Theorem 1.1 that $\mu = \frac{1}{\text{Leb}(\Lambda)}$. The following two propositions verify the remaining hypotheses of Theorem 5.6 that pertain to the perturbed operator.

**Proposition 5.7** (Convergence and Continuity of Perturbations).

- The operators $R_n(t)$ are continuous at $t = 0$.
- As $t \to 0$, $\|R(z,t) - R(z,0)\| = O(|t|)$.
- There exists $\delta_0 > 0$ and $C > 0$ such that for all $t \in [-\delta_0, \delta_0]$ and $n \in \mathbb{N}^*$, $\|R_n\| \leq Cr_n$.

**Proof.** See Section 6.8.

**Proposition 5.8** (Expansion of Dominant Eigenvalue). Let $\chi(t)$ denote the eigenvalue near 1 of the operator $R(1,t)$ for small $t$. Suppose that $X: [0,1]^2 \to \mathbb{R}$ is $\gamma$-Hölder for some $\gamma \in (0,1]$ and $\int_{[0,1]^2} X \, d\text{Leb} = 0$.

1. Suppose that $\xi \in L^2$ and that $\xi$ is not a coboundary. Then $\sigma^2 = \int_{\Lambda} |\xi|^2 \, d\lambda + 2 \sum_{k=1}^{\infty} \int_{\Lambda} \xi \odot T^k \xi \, d\lambda$ converges, $\sigma^2 > 0$, and as $t \to 0$,

$$\chi(t) \sim 1 - \left( \frac{1}{2}\sigma^2 + o(1) \right) t^2.$$

6In particular if the hypotheses of Lemma 5.5 are satisfied
ii. Suppose that $\alpha_0 > \alpha_1$, $\alpha_0 > 1$, and $M_0 > 0$. Let $p = 1 + \frac{1}{\alpha_0}$, $a = C_0 \Gamma(1 - p) \cos \left( \frac{p\pi}{2} \right)$, and $b = 1$. As $t \to 0$,

$$\chi(t) = 1 - \left( a \left( 1 - ib \operatorname{sgn}(t) \tan \left( \frac{p\pi}{2} \right) \right) + o(1) \right) |t|^p$$

iii. Suppose that $\alpha_0 = \alpha_1 =: \alpha > 1$, $M_0 > 0$ and $M_1 < 0$. Let $p = 1 + \frac{1}{\alpha}$, $a = (C_0 + C_1) \Gamma(1 - p) \cos \left( \frac{p\pi}{2} \right)$, and $b = \frac{C_0 - C_1}{\alpha_0 + \alpha_2}$. As $t \to 0$,

$$\chi(t) = 1 - \left( a \left( 1 - ib \operatorname{sgn}(t) \tan \left( \frac{p\pi}{2} \right) \right) + o(1) \right) |t|^p$$

iv. Suppose that $\alpha_0 = \alpha_1 = 1$, $M_0 \neq 0$, and $M_1 \neq 0$. As $t \to 0$,

$$\chi(t) = 1 + \left( \frac{1}{2} (C_0 + C_1) + o(1) \right) |t|^2 \log |t|.$$

Proof. See Section 6.9.

Proof of Theorem 5.4. The results follow from arguments similar to those presented in [11] Sections 4.3 and 4.4. For the proof of (iv) it is worth noting that

$$\chi \left( \frac{t}{\sqrt{n \log(n)}} \right) = 1 + (C_0 + C_1)t^2 \frac{1}{n} \left[ \frac{\log(t)}{\log(n)} - \frac{\log(\log(n))}{2 \log(n)} - \frac{1}{2} \right]$$

$$= 1 - \frac{1}{2} (C_0 + C_1) t^2 \frac{1}{n} \left[ 1 - o(1) \right]$$

$$\sim 1 - \frac{1}{2} (C_0 + C_1) t^2 \frac{1}{n}.$$

Therefore,

$$\lim_{n \to \infty} \left[ \chi \left( t \sqrt{\frac{\log(n)}{n}} \right) \right]^n = \exp \left( - \frac{1}{2} (C_0 + C_1) t^2 \right).$$

\[\Box\]

6 Technical Results

In this section we verify Propositions 5.2, 5.3, 5.7 and 5.8.

6.1 Compact Embedding

In this section we will show that $\mathcal{S}$ is compactly embedded into $\mathcal{W}$. This is necessary for us to apply Hennion’s theorem in Section 6.2 to show that the operators $R(z)$ acting on $\mathcal{S}$ are quasi-compact.
Proposition 6.1. The inclusion of $S$ into $W$ is a compact embedding.

Lemma 6.2. Let $U$ be a linear subspace of a Banach space with norm $||\cdot||$. Suppose that for all $\epsilon > 0$ there exist a finite set of linear functionals $\{\alpha_1, \ldots, \alpha_k\}$ defined on $U$ such that for all $\eta \in U$,

$$||\eta|| \leq \max_{1 \leq i \leq k} |\alpha_i(\eta)| + \epsilon.$$ 

Then $U$ is a compactly embedded subspace.

Proof. Let $\alpha: U \to \mathbb{R}^k$, be the linear mapping with coordinate functions $\alpha_1, \ldots, \alpha_k$. We will view $\mathbb{R}^k$ as a normed linear space equipped with the max-norm. By the supposed bound, $\alpha$ has operator norm 1. Let $U_1$ denote the unit ball of $U$ and note that $\alpha(U_1)$ is a subset of the unit ball of $\mathbb{R}^k$. Fix $\epsilon > 0$ and let $\{V_1, \ldots, V_j\}$ be a finite cover of the unit ball of $\mathbb{R}^k$ by balls of radius $\epsilon$.

The collection $\{\alpha^{-1}V_1, \ldots, \alpha^{-1}V_k\}$ is a cover of $U_1$. In fact, this collection is a cover by sets of diameter at most $3\epsilon$. To verify this, fix $j \in \{1, \ldots, k\}$ and suppose that $\eta$ and $\nu$ are in $\alpha^{-1}V_j$. Since $V_j$ is a max-norm ball of radius $\epsilon$ we have

$$\|\alpha(\eta - \nu)\|_{\text{max}} = \|\alpha(\eta) - \alpha(\nu)\|_{\text{max}} \leq \text{diam}(V_j) \leq 2\epsilon.$$ 

By the supposed bound, we obtain

$$\|\eta - \nu\| \leq \max_{1 \leq i \leq k} |\alpha_i(\eta - \nu)| + \epsilon = \|\alpha(\eta - \nu)\|_{\text{max}} + \epsilon \leq 3\epsilon.$$ 

For each $j \in \{1, \ldots, k\}$, select $\eta_j \in \alpha^{-1}V_j$ and let $B_j$ be the open $\|\cdot\|$-ball of radius $4\epsilon$ centered at $\eta_j$. By the choice of radius, we see that $\alpha^{-1}V_j \subset B_j$. Therefore, $\{B_1, \ldots, B_k\}$ is an open cover of $U_1$ by balls of radius $4\epsilon$. Since $\epsilon > 0$ was arbitrary, we conclude that $U_1$ is totally bounded with respect to the metric induced by $\|\cdot\|$. Therefore, $U$ is a compactly embedded subspace of the Banach space. \qed

Lemma 6.3. The space $L$ is compactly embedded into $H$.

Proof. Recall that $L$ and $H$ are respectively Lipschitz and H"older functions on $[0,1]$. The result is classical. \qed

Proof of Proposition 6.1. Fix $\epsilon > 0$. Chose a set $\{w_1, \ldots, w_m\} \subset [p,q]$ that is $\frac{\epsilon}{2}$-dense. Chose $\{\xi_1, \ldots, \xi_n\} \subset L$ that is $\frac{\epsilon}{2}$-dense in the unit ball of $L$ with respect to $||\cdot||_H$. Note that for all $j \in \{1, \ldots, n\}$, $||\xi_j||_H \leq ||\xi_j||_L \leq 1$.

For $\eta \in L$ with $||\eta||_S < \infty$, $i \in \{1, \ldots, m\}$, and $j \in \{1, \ldots, n\}$, define $\alpha_{ij}(\eta) = \int_0^1 \eta(w_i, y) \xi_j \, dy$. These linear functionals are in $W^*$ with norm at most 1. Therefore, the functionals $\alpha_{ij}$ are defined on the linear subspace $S \subset W$. 

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For all $\eta \in L$ with $\|\eta\|_S < \infty$, $x \in [p,q]$, $\psi \in \mathcal{L}$, $i \in \{1, \ldots, m\}$, and $j \in \{1, \ldots, n\}$,
\[
\int_0^1 \eta(x,y) \psi(y) \, dy = \int_0^1 \eta(x,y) [\psi - \xi_j] (y) \, dy \\
+ \int_0^1 [\eta(x,y) - \eta(w_i, y)] \xi_j(y) \, dy \\
+ \int_0^1 \eta(x_i, y) \xi_j(y) \, dy \\
\leq \|\eta\|_S \|\psi - \xi_j\|_{\mathcal{H}} + \text{Lip}_S (\eta) |x - w_i| + |\alpha_{ij}(\eta)|
\]
By selecting $i$ so that $w_i$ and $x$ are close, and selecting $j$ so that $\psi$ and $\xi_j$ are close we see that
\[
\int_0^1 \eta(x,y) \psi(y) \, dy \leq |\alpha_{ij}(\eta)| + \epsilon.
\]
Take a maximum over $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ on the right hand side of the inequality above, and a supremum over $x \in [p,q]$ and $\psi \in \mathcal{L}$ with $\|\psi\|_{\mathcal{L}} \leq 1$ on the left hand side to obtain
\[
\|\eta\|_{\mathcal{W}} \leq \max_{i,j} |\alpha_{ij}(\eta)| + \epsilon. \tag{6.1}
\]
Since the set of $\eta \in L$ with $\|\eta\|_S < \infty$ is dense in $\mathcal{S}$, the bound above extends to all $\eta \in \mathcal{S}$. We have shown that for all $\epsilon > 0$ there exists a finite collection of bounded linear functionals $\alpha_{ij} \in \mathcal{W}^*$, such that for all $\eta \in \mathcal{S}$, we have Equation (6.1). By Lemma 6.2, the inclusion of $\mathcal{S}$ into $\mathcal{W}$ is compact. □

### 6.2 Essential Spectrum

In this section we show that the operators $R(z)$ are quasi-compact for $|z| \leq 1$ (see Proposition 6.4). This is the first step toward proving the finer spectral properties of $R(z)$ obtained in Proposition 5.3. In the process of proving quasi-compactness we will produce bounds on the operators $R_k$ (see Lemma 6.7) that will be sufficient to prove Proposition 5.2.

Recall that $\beta$ is defined in Equation (3.27) and is related to the unstable expansion of the map $T$ and that $a \in (0,1]$ is a parameter related to the norm on $\mathcal{H}$ and is fixed in Section 4.

**Proposition 6.4 (Quasi-Compactness).** For each $|z| \leq 1$ the operator $R(z): \mathcal{S} \to \mathcal{S}$ is quasi-compact with spectral radius less than or equal to $|z|$ and essential spectral radius less than or equal to $\beta^a |z|$.

The proof of Proposition 6.4 is an application of the following theorem of Hennion.
Theorem 6.5 (Hennion [14] via Liverani [16]). If $S \subseteq W$ are Banach spaces with norms $\|\cdot\|_S$ and $\|\cdot\|_W$ respectively, such that $\|\cdot\|_W \leq \|\cdot\|_S$, and $L : S \to S$ is a bounded linear operator such that:

1. $L : S \to W$ is a compact operator;
2. There exists $\theta, A, B, C > 0$ such that for all $k \in \mathbb{N}$ there exists $M_k > 0$ such that for all $f \in S$, we have
   
   \begin{align*}
   (a) \quad & \|L^k f\|_W \leq CM_k \|f\|_W, \\
   (b) \quad & \|L^k f\|_S \leq A\theta^k \|f\|_S + BM_k \|f\|_W.
   \end{align*}

Then $L : S \to S$ is quasi compact with essential spectral radius less than or equal to $\theta$. We will refer to the second inequality above as the Lasota-Yorke inequality.

In order to apply Theorem 6.5 to obtain Proposition 6.4 we need the following inequalities.

Proposition 6.6 (Uniform Lasota-Yorke Inequality). For all $\eta \in L$ and $k \geq 1$

\begin{align*}
\|R(z)^k \eta\|_S & \leq [\kappa + 1] |z|^k \|\eta\|_S, \\
\|R(z)^k \eta\|_W & \leq [\kappa + 1] : |z|^k \|\eta\|_W, \\
\|R(z)^k \eta\|_S & \leq [\kappa + 1] |z|^k \left[(\beta^a)^k \|\eta\|_S + (\kappa + \tau + 1) \|\eta\|_W\right].
\end{align*}

Proof. This proposition is proved in Section 6.5 \qed

We are now in a position to prove Proposition 6.4.

Proof of Proposition 6.4. As is discussed in Section 4, $\|\cdot\|_W \leq \|\cdot\|_S$ and as a result $S$ can be viewed as a subset of $W$. The operator $R(z) : S \to S$ is bounded by the first inequality from Proposition 6.6.

The compactness of $R(z) : S \to W$ follows from Proposition 6.1 which states that the inclusion of $S$ into $W$ is compact. The transformation $R(z) : S \to W$ is the composition of the bounded operator $R(z) : S \to S$ followed by the compact inclusion from $S$ into $W$, and thus is compact. Therefore, the first hypothesis of Theorem 6.5 is satisfied.

The second hypothesis of Theorem 6.5 is verified by the second and third inequalities obtained in Proposition 6.6. By inspecting the third inequality from Proposition 6.6 we see that $\theta = |z| \beta^a$ and we have verified the claimed bound on the essential spectral radius of $R(z)$.
The claimed bound on the spectral radius of $R(z)$ follows from the first inequality in Proposition 6.6 and the Gelfand spectral radius formula.

### 6.3 Basic Norm Bounds

The following lemma provides the key estimates required to prove Propositions 5.2 and 6.6.

**Lemma 6.7 (Basic Norm Bounds).** For all $k \geq 1$, $n \geq 1$, and $\eta \in \mathbb{L}$,

\[
\left\| R_k^n \eta \right\|_W \leq [\kappa + 1] \lambda \left\{ r(k) = n \right\} \| \eta \|_W,
\]

\[
\left\| R_k^n \eta \right\|_s \leq [\kappa + 1] \lambda \left\{ r(k) = n \right\} \| \eta \|_s,
\]

\[
\text{Lip}_s \left( R_k^n \eta \right) \leq [\kappa + 1] \lambda \left\{ r(k) = n \right\} \left[ 2 (\beta^n)^k \| \eta \|_s + \| \eta \|_W \right],
\]

\[
\left\| R_k^n \eta \right\|_s \leq [\kappa + 1] \lambda \left\{ r(k) = n \right\} \left[ (\beta^a)^k \| \eta \|_s + (\kappa + \tau + 1) \| \eta \|_W \right].
\]

**Proof.** Recall the definitions of norms from Section 4 and the definition of the operators $R_k^n$ from Equation (5.1). All of the quantities that we wish to bound are defined through integrals of the form

\[
\int_0^1 R_k^n \eta(x, y) \psi(y) \, dy = \int_0^1 T_{k^n} \left( \eta 1_{\{ r(k) = n \}} \right)(x, y) \psi(y) \, dy \tag{6.10}
\]

\[
= \int_0^1 T_{k^n} \eta(x, y) \psi(y) 1_{T_{k^n} \left\{ r(k) = n \right\}} \, dy.
\]

The set $T_{k^n} \left\{ r(k) = n \right\}$ is a countable collection of horizontal strips. Similarly the set $\left\{ r(k) = n \right\}$ is a countable collection of vertical strips. That is, there exists a countable collection of intervals $\{ I_j \subseteq [p, q] : j \in \mathbb{Z}^+ \}$ such that

\[
\left\{ r(k) = n \right\} = \bigcup_{j \in \mathbb{Z}^+} I_j \times [0, 1].
\]

It will be convenient to define $V_j = I_j \times [0, 1]$. Note that for all $j \in \mathbb{Z}^+$, $T_{k^n} V_j$ is a horizontal strip and $T_{k^n} V_j$ is $C^1$. In the horizontal coordinate, $u^k|_{I_j} : I_j \rightarrow [p, q]$ is a $C^2$ bijection.

The integration in Equation (6.10) is over the set $T_{k^n} \left\{ r(k) = n \right\} \cap (\{ x \} \times [0, 1])$. By the comments of the previous paragraph, for each $j \in \mathbb{Z}^+$, there exists $s_j \in I_j$
so that \( u^k(s_j) = x \). Therefore, the preimage of the region of integration under the map \( T^k \) is the countable union of full vertical lines as follows,

\[
T^{-k} \left( T^k \{ r^{(k)} = n \} \cap \{ x \times [0, 1] \} \right) = \left\{ r^{(k)} = n \right\} \cap T^{-k} \left( \{ x \times [0, 1] \} \right) = \bigcup_{j \in \mathbb{Z}^+} \{ s_j \} \times [0, 1].
\]

Suppose that \((s_j, t) \in V_j\) and that \((x, y) = T^k(s_j, t) = \left( u^k(s_j), v^k(s_j)(t) \right)\). Let \( \eta \in L \) and \( \psi \in L \) or \( H \). Suppose that \( n \in \mathbb{Z}^+ \) and \( k \in \mathbb{Z}^+ \) are numbers such that the set \( \{ r^{(k)} = n \} \) is non-empty. An elementary change of variables on each line segment \( \{ s_j \} \times [0, 1] \) shows that

\[
\int_0^1 R_n^{(k)} \eta(x, y) \psi(y) \, dy = \sum_{j=1}^{\infty} \int_0^1 \eta(s_j, t) \psi \left( v^k(s_j, t) \right) \partial_y v^k(s_j, t) \, dt,
\]

where we have written \( v^k(s, t) \) instead of \( v^k(s_j, t) \), this helps to avoid nested subscripts and reduces confusion when taking partial derivatives of the map \( v^k : \Lambda \rightarrow [0, 1] \).

Recall from Section 3 that \( T \) preserves the measure \( \lambda \) on \( \Lambda \), which is a probability measure obtained by restricting Lebesgue measure to \( \Lambda \). From this we deduce that, \( \lambda \) almost everywhere the Jacobian determinant of \( T \) is 1. Calculating the Jacobian determinant of \( T^k \) using Equation (3.18) we obtain the identity below.

\[
\partial_y v^k(x, y) Du^k(x) = 1 \quad (6.11)
\]

It follows that for \( \mu \) almost every \( x \),

\[
\int_0^1 R_n^{(k)} \eta(x, y) \psi(y) \, dy = \sum_{j=1}^{\infty} \int_0^1 \eta(s_j, t) \psi \left( v^k(s_j, t) \right) \left[ Du^k(s_j) \right]^{-1} \, dt. \quad (6.12)
\]

We apply the definitions of \( \| \cdot \|_s \) and \( \| \cdot \|_W \) (see Equations (4.1) and (4.2)) to provide the following preliminary bounds. If \( \psi \in L \), then

\[
\int_0^1 R_n^{(k)} \eta(x, y) \psi(y) \, dy \leq \sum_{j \in \mathbb{Z}^+} \left[ Du^k(s_j) \right]^{-1} \| \eta \|_W \| \psi \circ v^k(s_j) \|_L. \quad (6.13)
\]

If \( \psi \in H \), then

\[
\int_0^1 R_n^{(k)} \eta(x, y) \psi(y) \, dy \leq \sum_{j=1}^{\infty} \left[ Du^k(s_j) \right]^{-1} \| \eta \|_s \| \psi \circ v^k(s_j) \|_H. \quad (6.14)
\]
In order to bound \( \text{Lip}_s \left( R_n^{(k)} \eta \right) \) we must consider integrals of the form

\[
\int_0^1 \left[ R_n^{(k)} \eta(x, y) - R_n^{(k)} \eta(w, y) \right] \psi(y) \, dy,
\]

where \( x, w \in [p, q] \) are fixed horizontal coordinates and \( \psi \in \mathcal{L} \) is a test function. If we separate the integral above by linearity and apply Equation (6.12) to each term we obtain

\[
\int_0^1 \left[ R_n^{(k)} \eta(x, y) - R_n^{(k)} \eta(w, y) \right] \psi(y) \, dy = \sum_{j=1}^{\infty} \left[ Du^k(s_j(x)) \right]^{-1} \int_0^1 \eta(s_j(x), t) \psi \left( v^{(k)}(s_j(x), t) \right) \, dt
\]

\[
- \sum_{j=1}^{\infty} \left[ Du^k(s_j(w)) \right]^{-1} \int_0^1 \eta(s_j(w), t) \psi \left( v^{(k)}(s_j(w), t) \right) \, dt
\]

where \( s_j(x) \) and \( s_j(w) \) are points in \( I_j \) such that \( u^k(s_j(x)) = x \) and \( u^k(s_j(w)) = w \). We will fix \( j \) and expand \( j \)-th term of right hand side of the identity above following the pattern

\[
A_x B_x C_x - A_w B_w C_w = (A_x - A_w) B_x C_x + A_w (B_x - B_w) C_x + A_w B_w (C_x - C_w)
\]

and apply the definitions of \( \| \cdot \|_s \) and \( \text{Lip}_s (\cdot) \) (see Equations (4.3) and (4.4)) to obtain the preliminary bound below.

\[
\int_0^1 \left[ R_n^{(k)} \eta(x, y) - R_n^{(k)} \eta(w, y) \right] \psi(y) \, dy \leq \sum_{j=1}^{\infty} \left[ Du^k(s_j(w)) \right]^{-1} \left[ 1 - \frac{Du^k(s_j(w))}{Du^k(s_j(x))} \right] \| \eta \|_W \text{Lip}_s \left( v^{(k)} \right) \| \psi \|_s \| v^{(k)} \|_{s_j(x)} \|_\mathcal{L}
\]

\[
+ \sum_{j=1}^{\infty} \left[ Du^k(s_j(w)) \right]^{-1} \text{Lip}_s (\eta) \| s_j(x) - s_j(w) \| \| \psi \|_{s_j(x)} \| v^{(k)} \|_{s_j(x)} \|_\mathcal{L}
\]

\[
+ \sum_{j=1}^{\infty} \left[ Du^k(s_j(w)) \right]^{-1} \| \eta \|_W \text{Lip}(\psi) \sup_{0 \leq t \leq 1} \left| v^{(k)}(s_j(x)) (t) - v^{(k)}(s_j(w)) (t) \right|
\]

(6.15)

Having collected the preliminary bounds Equations (6.13) to (6.15) we wish to improve them to bounds that are uniform in the horizontal coordinates \( x \) and \( w \). To this end we collect the following uniform bounds.

Fix \( j \in \mathbb{Z}^+ \), suppose that \( s \in I_j \) and \( u^k(s) = x \). Since \( u^k \) maps \( I_j \) onto \([p, q] \),

\[
\frac{1}{\mu(I_j)} \int_{I_j} Du^k(s) \, d\mu(s) = \frac{1}{\mu(I_j)} \int_p^q 1 \, d\mu(x) = \frac{1}{\mu(I_j)}
\]

Since \( u^k|_{I_j} \) is \( C^2 \), we can apply the Integral Mean Value Theorem to conclude that there exists \( \theta \in I_j \) such that

\[
Du^k(\theta) = \frac{1}{\mu(I_j)}
\]

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Since $s, \theta \in I_j \subset [r^{(k)} = n]$ we may apply Equation (3.29) to obtain
\[
\left| \frac{D u^k(\theta)}{D u^k(s)} - 1 \right| \leq \kappa \left| u^k(s) - u^k(\theta) \right| \leq \kappa
\]
Since $u^k|_{I_j}$ is order preserving, we have $D u^k(s) \geq 0$. In combination with the last two displayed equations this yields,
\[
0 \leq \left[ D u^k(s) \right]^{-1} \leq (\kappa + 1) \mu(I_j).
\] (6.16)

By another application of Equation (3.29), for all $w, x \in [p, q]$,
\[
\left| 1 - \frac{D u^k(s_j(w))}{D u^k(s_j(x))} \right| \leq \kappa \left| u(s_j(x)) - u(s_j(w)) \right| = \kappa |x - w|.
\] (6.17)

Since $v_{s(z)}^{(k)}$ viewed as a map of the interval $[0, 1]$ into itself is a contraction for all $z \in [p, q]$, we have
\[
\left\| \psi \circ v_{s(z)}^{(k)} \right\|_\mathcal{L} \leq \| \psi \|_\mathcal{L}.
\] (6.18)

Let us view $x \mapsto s_j(x)$ as a local inverse of $u^k$ on $I_j$ and note that,
\[
\partial_x \left[ v_{s(z)}^{(k)}(s_j(x), t) \right] = \frac{\partial_x v_{s(z)}^{(k)}(s_j(x), t)}{D u^k(s_j(x))}.
\]

By Lemma 3.7 we have
\[
\left\| \frac{\partial_x v_{s(z)}^{(k)}}{D u^k} \right\|_\infty \leq \tau.
\]

From the previous two lines it follows that
\[
\sup_{0 \leq t \leq 1} \left| v_{s(z)}^{(k)}(t) - v_{s(w)}^{(k)}(t) \right| \leq \left\| \frac{\partial_x v_{s(z)}^{(k)}}{D u^k} \right\|_\infty |x - w|.
\]
\[
\leq \tau |x - w|.
\] (6.19)

Finally, note that $|s_j(x) - s_j(w)| \leq \beta |x - w|$ by Equation (3.28).

Having collected the uniform bounds above we refine Equations (6.13) to (6.15) as follows. If $\psi \in \mathcal{L}$ and $x \in [p, q]$, then
\[
\int_0^1 R_{n}^{(k)} \eta(x, y) \psi(y) \, dy \leq \sum_{j \in \mathbb{Z}^+} (\kappa + 1) \mu(I_j) \| \eta \|_\mathcal{W} \| \psi \|_\mathcal{L}
\]
\[
= (\kappa + 1) \lambda \left\{ r^{(k)} = n \right\} \| \eta \|_\mathcal{W} \| \psi \|_\mathcal{L}.
\]

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By taking a supremum over $\psi$ such that $\|\psi\|_L \leq 1$ and $x \in [p,q]$ we obtain Equation (6.5). Similarly, if $\psi \in H$ and $x \in [p,q]$, then
\[
\int_0^1 R_n^{(k)} \eta(x,y) \psi(y) dy \leq (\kappa + 1) \lambda \left\{ r^{(k)} = n \right\} \|\eta\|_s \|\psi\|_H.
\]

By taking a supremum over $\psi$ such that $\|\psi\|_H \leq 1$ and $x \in [p,q]$ we obtain Equation (6.6).

If $\psi \in L$ and $w, x \in [p,q]$, then
\[
\int_0^1 \left[ R_n^{(k)} \eta(x,y) - R_n^{(k)} \eta(w,y) \right] \psi(y) dy \leq (\kappa + 1) \lambda \left\{ r^{(k)} = n \right\} \kappa |w - x| \|\eta\|_W \|\psi\|_L + (\kappa + 1) \lambda \left\{ r^{(k)} = n \right\} \|\eta\|_W Lip(\psi) \beta \kappa |w - x| \|\psi\|_L + (\kappa + 1) \lambda \left\{ r^{(k)} = n \right\} \|\eta\|_W Lip(\psi) \tau |w - x| \|\psi\|_L.
\]

By taking a supremum over $\psi$ such that $\|\psi\|_L \leq 1$ and $w, x \in [p,q]$ we obtain Equation (6.7).

Finally, we must verify Equation (6.8). Suppose that $\psi \in H$ and $x \in [p,q]$. By Equation (6.12),
\[
\int_0^1 R_n^{(k)} \eta(x,y) \psi(y) dy = \sum_{j=1}^\infty \int_0^1 \eta(s_j,t) \psi \left( v^{(k)}(s_j,t) \right) \left[ Du^k(s_j) \right]^{-1} dt
\]
\[
= \sum_{j=1}^\infty \int_0^1 \eta(s_j,t) \left[ \psi \left( v^{(k)}(s_j,t) \right) - \psi \left( v^{(k)}(s_j,0) \right) \right] \left[ Du^k(s_j) \right]^{-1} dt
\]
\[
+ \sum_{j=1}^\infty \int_0^1 \eta(s_j,t) \psi \left( v^{(k)}(s_j,0) \right) \left[ Du^k(s_j) \right]^{-1} dt
\]
\[
\leq (\kappa + 1) \lambda \left\{ r^{(k)} = n \right\} \|\eta\|_s \|\Psi_1\|_H
\]
\[
+ (\kappa + 1) \lambda \left\{ r^{(k)} = n \right\} \|\eta\|_W \|\Psi_0\|_L
\]
\[
\Psi_0 := \psi \left( v^{(k)}(s_j,0) \right)
\]
\[
\Psi_1 := \psi \left( v^{(k)}(s_j,t) \right) - \psi \left( v^{(k)}(s_j,0) \right)
\]

In the third line we have use the fact that $\Psi_0$ is constant and hence Lipschitz. An elementary calculation shows that
\[
\|\Psi_0\|_L \leq \|\psi\|_L \leq \|\psi\|_H
\]
\[
\|\Psi_1\|_H \leq 2(\beta \sigma)^k \|\psi\|_H
\]

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We conclude that,
\[
\int_0^1 R_n^{(k)}(x, y) \psi(y) dy \leq (\kappa + 1) \lambda \left\{ r^{(k)} = n \right\} \|\psi\|_{\mathcal{H}} \left[ 2 (\beta^n)^k \|\eta\|_s + \|\eta\|_W \right].
\]
By taking a supremum over \(\psi\) such that \(\|\psi\|_{\mathcal{H}} \leq 1\) and \(x \in [p, q]\) we obtain Equation (6.8).

Lastly Equation (6.9) follows by adding Equation (6.7) and Equation (6.8) and noting that \(\beta^k < \beta^\alpha k\). \(\square\)

### 6.4 Proof of Convergence and Renewal Equation

In this section we use Lemma 6.7 to prove Proposition 5.2.

**Claim 1** For all \(n \geq 1\), the operators \(B_n\) and \(R_n\) are bounded on \(\mathcal{S}\).

**Proof.** Boundedness of the \(R_n\) is given by Equation (6.6) from Lemma 6.7 with \(k = 1\). Note that \(B_n = \sum_{k=1}^n R_n^{(k)}\), and that the collection of sets \(\{ r^{(k)} = n \} : k = 1, \ldots, n \) are disjoint. It follows from Equation (6.6) that
\[
\|B_n \eta\|_{\mathcal{S}} \leq \sum_{k=1}^n \left\|R_n^{(k)} \eta\right\|_{\mathcal{S}} \leq \sum_{k=1}^n [\kappa + 1] \lambda \left[ r^{(k)} = n \right] \|\eta\|_{\mathcal{S}} \leq [\kappa + 1] \|\eta\|_{\mathcal{S}}.
\]
Therefore, the operators \(B_n\) are bounded on \(\mathcal{S}\). \(\square\)

**Claim 2** For all \(z\) in the open unit disk of \(\mathbb{C}\), the operators \(B(z)\) and \(R(z)\) converge in \(Hom(\mathcal{S}, \mathcal{S})\) and satisfy \(B(z) = (I - R(z))^{-1}\).

**Proof.** Since the operators \(R_n\) and \(B_n\) are uniformly bounded in \(n\), \(R(z)\) and \(B(z)\) converge for \(|z| < 1\) as desired.

The IBT \(B\) is clearly conservative and non-singular with respect to Lebesgue measure and \(\Lambda\) has positive measure. That \(B(z)\) and \(R(z)\) satisfy \(B(z) = (I - R(z))^{-1}\) can be verified by applying [21] Proposition 1. \(\square\)

**Claim 3** The operator \(R(z)\) converges in \(Hom(\mathcal{S}, \mathcal{S})\) for \(z\) in the closed unit disk of \(\mathbb{C}\), that is \(\sum_{n \geq 1} ||R_n||_{\mathcal{S}} < \infty\).
Proof. Note that the sets \([ r = n ]\) partition \(\Lambda\). By Equation (6.6) with \(k = 1\), for any \(\eta \in L\),
\[
\| R(1)\eta \|_S \leq \sum_{n=1}^{\infty} \| R_n \eta \|_S \leq [\kappa + 1] \| \eta \|_S \sum_{n=1}^{\infty} \lambda \{ r = n \}
= [\kappa + 1] \| \eta \|_S
\]
So \(R(z)\) is bounded on \(S\) for all \(|z| \leq 1\).

Claim 4  Let \(\alpha = \max \{ \alpha_0, \alpha_1 \}\). As \(n \to \infty\), \(\sum_{k>n} \| R_k \|_S = O \left( n^{-(1+\frac{1}{\alpha})} \right) \).

Proof. It follows directly from the definition of \(\lambda\) in Equation (3.7) and the relationship between \(p_n^0\), \(q_n^0\) and return time outlined in Equation (2.18) that for \(n \geq 2\),
\[
\lambda \{ r = n \} = \frac{p_{n-1}^0 - p_n^0 + q_n^0 - q_{n-1}^0}{\text{Leb}(\Lambda)}
\]
By the asymptotic behavior of \(p_{n-1}^0 - p_n^0\) and \(q_{n-1}^0 - q_n^0\) described in Equations (2.27) and (2.28) we have
\[
\lambda \{ r = n \} \asymp \left( \frac{1}{n} \right)^{2+\frac{1}{\alpha}}.
\]
By the norm bound obtained in Equation (6.6) with \(k = 1\), \(\| R_n \|_S = O \left( \lambda \{ r = n \} \right)\), therefore
\[
\sum_{k>n} \| R_k \|_S = O \left( \left( \frac{1}{n} \right)^{\frac{1}{\alpha}+1} \right).
\]

6.5 Proof of Lasota-Yorke inequality

In this section we will use Lemma 6.7 to prove Proposition 6.6

Proof of Proposition 6.6  We will prove Equation (6.4). The proofs of the other inequalities are similar.

Note that \(\min r^{(k)} \geq 2k\) and apply Lemma 6.7, so that,
\[
\| R(z)^k \eta \|_S \leq \sum_{n=2k}^{\infty} |z^n| \left\| R_n^{(k)} \eta \right\|_S
\leq |z|^k \sum_{n=2k}^{\infty} \{ r^k = n \} \left[ 2 (\beta^n)^k \| \eta \|_S + (\kappa + \tau + 1) \| \eta \|_W \right]
= [\kappa + 1] |z|^k \left[ 2 (\beta^n)^k \| \eta \|_S + (\kappa + \tau + 1) \| \eta \|_W \right].
\]
Obviously we could have obtained $|z|^{2k}$ as a multiplier in the inequalities above. We opt for the weaker bound as it makes no difference in what follows and is slightly less cumbersome.

### 6.6 Proof of Spectral Gap and Aperiodicity

In this section we will prove Proposition 5.3. The next two lemmas will be useful in the proof.

**Lemma 6.8.** If $|z| \leq 1$ and $\eta \in L$, then

$$\| R(z)\eta \|_L \leq \| \eta \|_L. \quad (6.20)$$

**Proof.** We begin by bounding the sup-norm term in $\| \cdot \|_L$,

$$\| R(z)\eta \|_{sup} = \sup_{(x,y) \in A} \left| \sum_{n=1}^{\infty} z^n R_n \eta(x,y) \right|$$

$$= \sup_{(x,y) \in A} \left| \sum_{n=1}^{\infty} z^n T_n \left(1_{\{r=n\}} \eta \right)(x,y) \right|$$

$$= \sup_{(x,y) \in A} \left| \sum_{n=1}^{\infty} z^n \left[1_{\{r=n\}} \circ T^{-1}\right](x,y) \left[\eta \circ T^{-1}\right](x,y) \right|$$

$$= \sup_{(x,y) \in A} \left| z^{r\left(T^{-1}(x,y)\right)} \left[\eta \circ T^{-1}\right](x,y) \right|$$

$$\leq \sup_{(x,y) \in A} \left| \left[\eta \circ T^{-1}\right](x,y) \right|$$

$$\leq \| \eta \|_{sup}$$

For the $\text{Lip}_u (\cdot)$-term, fix $\gamma \in \Gamma$ and $(x,y), (w,z) \in \gamma$. Note that $T^{-1}\gamma$ is a segment of some unstable curve $\gamma' \in \Gamma$. Note that $r \circ T^{-1}$ is constant on unstable curves, that is $T_*r$ is $B^u$ measurable. Let $N(\gamma) = r \left(T^{-1}(x,y)\right) = r \left(T^{-1}(w,z)\right)$. Let $x'$ and $w'$ denote the first coordinates of $T^{-1}(x,y)$ and $T^{-1}(w,z)$ respectively. Note that $|x' - w'| \leq \beta |x - w|$, since $T$ is uniformly expanding along unstable curves. Computing as above we obtain,

$$R(z)\eta(x,y) - R(z)\eta(w,z) = z^{N(\gamma)} \eta \left(T^{-1}(x,y)\right) - z^{N(\gamma)} \eta \left(T^{-1}(w,z)\right)$$

$$\leq \left| z^{N(\gamma)} \text{Lip}_u (\eta) |x' - w'| \right|$$

$$\leq \beta \text{Lip}_u (\eta) |x - w|.$$
Since \((x, y)\) and \((w, z)\) were arbitrary, \(\text{Lip}_u (R(z)\eta) \leq \beta \text{Lip}_u (\eta)\). We conclude that,
\[
\| R(z)\eta \|_L \leq \| \eta \|_{\text{sup}} + \beta \text{Lip}_u (\eta) \leq \| \eta \|_L.
\]

\[\square\]

**Lemma 6.9.** For each \(z\) with \(|z| = 1\),

1. The peripheral spectrum of \(R(z)\) consists of semi-simple eigenvalues.
2. Every peripheral eigenvector of \(R(z)\) is in \(L\).

**Proof.** This follows from Lemma 6.8 by a standard argument, and can be found in a slightly different setting in [4] Proposition 3.5. We will outline the proof for the convenience of the reader.

Note that by Proposition 6.4 the operator \(R(z)\) is quasi-compact and therefore admits a decomposition \(R(z) = Q + F\) such that the spectral radius of \(R(z)\), \(F\) is supported on a finite dimensional subspace of \(S\), and \(QF = FQ = 0\). It follows directly that \(R(z)^k = Q^k + F^k\) for \(k \geq 1\).

It follows from Equation (6.2) that for all \(k \geq 1\) and \(\eta \in L\), we have the uniform bound \(\| R(z)^k\eta \|_S \leq |z| (\kappa + 1) \| \eta \|_S\). From the decomposition in the last paragraph we have \(\| F^k\eta \|_S \leq |z| (\kappa + 1) \| \eta \|_S\).

The peripheral spectrum of \(R(z)\) coincides with the peripheral spectrum of \(F\) since the spectral radius of \(Q\) is strictly less than that of \(R(z)\). It follows that the peripheral spectrum of \(R(z)\) consists of eigenvalues.

Recall that the spectral radius of \(R(z)\) and hence \(F\) is \(|z|\). If \(F\) had a generalized eigenvector associated to a peripheral eigenvalue, then \(|z|^{-k} \| F^k \|_{\text{op}}\) would grow linearly in \(k\). This cannot be the case since \(|z|^{-k} \| F^k \|_{\text{op}} \leq \kappa + 1\). Therefore, the peripheral spectrum of \(R(z)\) consists of semi-simple eigenvalues.

Note that \(F(S)\) is finite dimensional and therefore closed in \(S\). By definition \(L\) is dense in \(S\), thus \(F(L)\) is dense in \(F(S)\). Since \(F(S)\) is closed \(F(L) = F(S)\).

By Lemma 6.8, \(R(z)\) is bounded on \(L\) and thus \(F\) is also. Therefore, \(F(S) = F(L) \subset L\). Every eigenvector of \(F\) is contained in \(F(S)\). We conclude that every eigenvector associated to a peripheral eigenvalue of \(R(z)\) is in \(L\).

We are now in a position to prove Proposition 5.3.

**Proof of Proposition 5.3** The proof of this lemma will be divided into several distinct parts.

\[\text{An eigenvalue is semi-simple if its algebraic and geometric multiplicities match.}\]
Claim 1: For all $|z| \leq 1$ the operator $R(z) - I$ is invertible if and only if 1 is not an eigenvalue of $R(z)$.

Proof of Claim 1. If 1 is an eigenvalue of $R(z)$, then $R(z) - I$ is not invertible by the definition of an eigenvalue. Suppose that $R(z) - I$ is not invertible. Then 1 is a point in the spectrum of $R(z)$. By Proposition 6.4 the operator $R(z)$ is quasi-compact with essential spectral radius less than $\beta^2 |z|$, which is strictly less than 1, therefore 1 is a point in the spectrum of $R(z)$ that is outside the essential spectrum. It follows that 1 is an eigenvalue of $R(z)$ and that any eigenvector associated to the eigenvalue 1 lies in a finite dimensional $R(z)$ invariant subspace of $S$.

Claim 2: If $|z| < 1$, then $R(z) - I$ is invertible.

Proof of Claim 2. Fix $z$ such that $|z| < 1$. It follows from Proposition 6.4 that the spectral radius of $R(z)$ is at most $|z|$. By assumption $|z| < 1$, so 1 is not an eigenvalue of $R(z)$. By the previous claim $R(z) - I$ is invertible.

Claim 3: If $|z| = 1$ and $z \neq 1$, then $I - R(z)$ is invertible. The operator $R(1)$ has a simple eigenvalue at 1 and the associated eigenspace is $span \{ \Lambda \}$.

Proof of Claim 3. We will verify both parts of the claim simultaneously. Let $z$ be a complex number such that $|z| = 1$ and let $\eta \in S$ be an eigenvector of $R(z)$ with eigenvalue 1, that is

$$R(z)\eta = \eta.$$  \hspace{1cm} (6.21)

The proof relies on two observations about $\eta$:

Observation 1: If $|z| = 1$ and $\eta \in S$ such that $R(z)\eta = \eta$, then for almost every $(x, y) \in \Lambda$,

$$[\eta \circ T](x, y) z^r = \eta(x, y).$$

Observation 2: If $|z| = 1$ and $\eta \in S$ so that $R(z)\eta = \eta$, then $\eta$ is a constant multiple of $\Lambda$.

We will verify both observations after completing the proof of Claim 3.
We will show that, if $\eta \neq 0$, then $z = 1$. By Observation 2, $\eta$ is constant. Since $T$ preserves Lebesgue measure $\eta \circ T = \eta$. It follows that Equation (6.21) reduces to

$$(z^r(x) - 1)\eta = 0.$$ 

The equation above is satisfied if $\eta = 0$ or if $z^r(x) = 1$.

The equation $z^r(x) = 1$ is satisfied if and only if for all $a \in \text{image}(r) \subseteq \mathbb{Z}$,

$$a \frac{\text{arg}(z)}{2\pi} \in \mathbb{Z}.$$ 

The inclusion above can hold if and only if there exists a rational number $b/c$ such that $\frac{\text{arg}(z)}{2\pi} = b/c$. Assuming that $b/c$ is reduced we see that $ab/c \in \mathbb{Z}$ and if and only if $c$ divides $a$. Therefore, $\frac{\text{arg}(z)}{2\pi} = b/c$ and $c$ divides $a$ for all $a \in \text{image}(r)$. From Section 2.2 it follows that image $(r) = \{n \in \mathbb{N} : n \geq 2\}$ and hence the greatest common divisor of image $(r)$ is 1 so that $c = 1$ and hence $\frac{\text{arg}(z)}{2\pi} \in \mathbb{Z}$. Therefore the principal value of the argument of $z$ is 0 and hence $z = 1$.

$T$ preserves Lebesgue measure on $\Lambda$. By Equation (5.6) we have that $R(1)$ is the Frobenius-Perron operator of $T$. It follows that $R(1)1_\Lambda = 1_\Lambda$. By Observation 2 any $\eta$ that satisfies the eigenvector equation $R(1)\eta = \eta$ is a multiple of $1_\Lambda$. By Lemma 6.9 the eigenvalue 1 is semi-simple. We have verified that $1_\Lambda$ is a basis for the eigenspace associated to the eigenvalue 1. We conclude that 1 is a simple eigenvalue of $R(1)$.

\[\blacksquare\]

To complete the proof of the lemma it remains to verify Observation 1 and Observation 2 from the proof of the last claim.

**Proof of Observation 1.** By Lemma 6.9 we have $\eta \in \mathbf{L}$. Since $\|\eta\|_\infty \leq \|\eta\|_{\text{sup}} \leq \|\eta\|_\Lambda$ we have $\eta \in L^\infty (\Lambda, \lambda)$. For all $\psi$ and $\eta$ in $\mathbf{L}$ we have

$$\int R(z)\eta \psi \, d\lambda = \int \sum_{n=1}^\infty z^n R_n \eta \psi \, d\lambda = \sum_{n=1}^\infty \int z^n T_n(\eta \mathbf{1}_{\{r=n\}}) \psi \, d\lambda$$

$$= \sum_{n=1}^\infty \int \eta z^n \mathbf{1}_{\{r=n\}} \psi \circ T \, d\lambda = \int \sum_{n=1}^\infty \eta z^n \mathbf{1}_{\{r=n\}} \psi \circ T \, d\lambda$$

$$= \int \eta z^r \psi \circ T \, d\lambda.$$ 

Since $\eta \in L^\infty (\lambda)$ we have $\eta \in L^2 (\lambda)$. Define $W(z)$ on $L^\infty (\lambda)$ by $W(z)\psi = \int \eta z^r \psi \, d\lambda$.
by the Mean Value Theorem there exists \( s \).

Further application of the Mean Value Theorem yields

\[
|W(z)|^2 = |W(z)|^2 - 2\text{Re}(W(z)\eta, \eta) + |\eta|^2
\]

\[
= |W(z)|^2 - 2\text{Re}(\eta, R(z)\eta) + |\eta|^2
\]

\[
= |W(z)|^2 - 2\text{Re}(\eta, \eta) + |\eta|^2
\]

\[
= |W(z)|^2 - |\eta|^2,
\]

and note that

\[
|W(z)|^2 = \int |\eta|^2 \, T \, d\lambda = \int |\eta|^2 \, d\lambda = |\eta|^2,
\]

from which we conclude that \( W(z)\eta = [\eta \circ T] \, z' = \eta \) except possibly on a \( \lambda \) null set. We have verified Equation \( \text{(6.21)}. \)

**Proof of Observation 2.** We begin by showing that \( \eta \) is essentially constant along stable fibres. Recall that by Lemma \( \text{6.9} \) we have \( \eta \in \mathbf{L} \). For each \( j \geq 1 \) select \( \tau_j \in C^{\infty}(\Lambda) \) such that \( |\tau_j - \eta|_1 < 2^{-j} \). Note that \( |W(\tau_j - \eta)|_1 = |z'(\tau_j - \eta) \circ T|_1 = |\tau_j - \eta|_1 < 2^{-j} \). Let \( \tau_j(x,y) = \int \tau_j(x,y) \, dy \) and note that by the Mean Value Theorem there exists \( s \in (0,1) \) and \( t \in (y,s) \) such that

\[
|\tau_j(x,y) - \bar{\tau}_j(x,y)| = |\tau_j(x,y) - \tau_j(x,s)| = |\partial_y \tau_j(x,t)||y-s| \leq \|\partial_y \tau_j\|_{\infty} |y-s|.
\]

Further application of the Mean Value Theorem yields

\[
|W^n \tau_j(x,y) - W^n \bar{\tau}_j(x,y)| \leq \|\partial_y \tau_j\|_{\infty} \|\partial_y v^{(n)}_x\|_{\infty} \leq \|\partial_y \tau_j\|_{\infty} \beta^n.
\]

For each \( j \geq 1 \) select \( n = n(j) \) such that \( \|\partial_y \tau_j\|_{\infty} \beta^n + 2^{-j} < 10 \cdot 2^{-j} \) and note that

\[
|\eta - \bar{\tau}_j|_1 \leq |W^n \eta - W^n \bar{\tau}_j|_1 + |W^n \tau_j - W^n \bar{\tau}_j|_1 \leq 10 \cdot 2^{-j}.
\]

We see that \( \eta \) is the \( L^1 \)-limit of functions that are constant along stable fibres. It follows that for \( \mu \text{-a.e. } x \in [p,q] \),

\[
\text{for } \text{Leb-a.e. } y, \eta(x,y) = \int_{\ell(x)} \eta(x,z) \, d\text{Leb}(z), \quad (\text{6.22})
\]

Next we will use the unstable regularity of \( \eta \) to show that Property \( \text{6.22} \) holds for every \( x \in [p,q] \). To verify this suppose that \( x \) failed to satisfy Property \( \text{6.22} \). This can happen if and only if there exist sets \( A_x, B_x \subset \ell(x) \) and \( \epsilon > 0 \), such that \( \text{Leb}(A_x) > 0 \), \( \text{Leb}(B_x) > 0 \), and for all \( y \) in \( A_x \) and \( z \) in \( B_x \)

\[
\eta(x,y) - \eta(x,z) \geq \epsilon. \quad (\text{6.23})
\]

For \( w \neq x \) let \( A_w \subset \ell(w) \) be the set obtained by sliding along unstable curves into \( \ell(w) \) and let \( B_w \) be defined similarly. Note that \( \text{Leb}(A_w) > 0 \) if and only if \( \text{Leb}(A_w) > 0 \). Since \( \eta \) is in \( \mathbf{L} \) we have that

\[
|\eta(x,y) - \eta(\ell(w) \cap \gamma(x,y))| \leq \text{Lip}_n(\eta) |x-w|.
\]

\(^8\text{By sliding along unstable curves we mean } (x,y) \mapsto \gamma(x,y) \cap \ell(w)\)
Choose $\delta > 0$ so that $\text{Lip}_u(\eta) \delta < \epsilon/3$. Fix $w \in [p,q]$ such that $|w - x| < \delta$. Select $(w, y) \in A_w$ and $(w, z) \in B_w$ and let $(x, y') \in A_x$ and $(x, z') \in B_x$ denote the points obtained by sliding along unstable disks back to $\ell(x)$. We compute,

$$
\eta(w, y) - \eta(w, z) \geq \eta(x, y') - \eta(x, z') - 2\text{Lip}_u(\eta) |x - w| \geq \epsilon - 2\text{Lip}_u(\eta) \delta \geq \frac{\epsilon}{3}.
$$

We have just shown that for every $w \in [p,q]$ with $|w - x| < \delta$ Property 6.23 holds at $w$, thus Property 6.22 fails at $w$. This contradicts our observation that Equation (6.22) fails and let $\eta$ be defined similarly. By the previous paragraph both $A_x$ and $A_w$ are null sets. Let $B \subset \ell(x)$ be the set obtained by sliding $A_w$ along unstable disks into $\ell(x)$. The set $B$ is a null set, therefore the set $G = \ell(x) - (A_x \cup B)$ consisting of points in $\ell(x)$ where $\eta(x, y) = h(x)$ and $\eta(\gamma(x, y) \cap \ell(w)) = h(w)$ has full measure. Choose $(x, y) \in G$ and note that

$$
|h(x) - h(w)| = |\eta(x, y) - \eta(\gamma(x, y) \cap \ell(x))| \leq \text{Lip}_u(\eta) |x - w|,
$$

so $h$ is Lipschitz with Lipschitz constant at most $\text{Lip}_u(\eta)$.

Next we would like to verify $\int [W(z)\eta](x, y) \, dy = z^r [h \circ u](x)$. Note that $T$ maps $\ell(x)$ into $\ell(u(x))$ affinely. We will apply the change of variable $y' = v_x(y)$ noting that $dy' = \partial_{y'} v_x(y) dy$ and that $\partial_{y'} v_x(y)$ is constant and exactly equal to the length of the interval $T\ell(x) \subset \ell(u(x))$

$$
\int_0^1 z^{r(x)}(\eta \circ T)(x, y) \, dy = z^{r(x)} \frac{1}{|T\ell(x)|} \int_{T\ell(x)} \eta(u(x), y') \, dy' = z^{r(x)} h(u(x))
$$

Applying Observation 1 we obtain

$$
z^r [h \circ u](x) = h(x) \quad (6.24)
$$

Next we deduce that $h$ is an essentially constant function. We will apply Corollary 3.2 from [2]. We reformulate the Corollary in our notation for the convenience of the reader.

Suppose that:

- $u: [p,q] \to [p,q]$ is a probability preserving, almost onto Gibbs-Markov map with respect to the partition $\alpha = \{ I_j, I_j' : j = 2, \cdots, \infty \}$

- $\varphi: [p,q] \to \{ z \in \mathbb{C} : |z| = 1 \}$ is $\alpha$-measurable.

---

9 see Section 2.2
• \( h: [p, q] \to \{ z \in \mathbb{C} : |z| = 1 \} \) is Borel measurable and \( \varphi(x) = h \cdot \bar{h} \circ u \)

Then \( h \) is essentially constant.

Let us verify that \( u \) satisfies the first hypothesis of the Corollary. For each \( a \in \alpha \) the map \( u|_a \) is a homeomorphism onto \( [p, q] \) with \( C^2 \) inverse \( v_a: [p, q] \to a \). The map \( u \) is uniformly expanding by Lemma 3.5 and satisfies Adler’s bounded distortion property by Lemma 3.6. By Example 2 of [2] it follows that \( u \) is a mixing Gibbs-Markov map. Since every branch of \( u \) is onto, \( u \) is almost onto as defined immediately after Theorem 3.1 of [2].

Since \( u \) is a Gibbs-Markov map, \( u \) is ergodic. Taking the complex modulus of Equation (6.24) yields \( |h| = |h \circ u| = |\bar{h} \circ u| \), thus \( |h| \) is an essentially constant function. Since \( h \) is Lipschitz, we have that \( |h| \) is Lipschitz and therefore pointwise constant. Without loss of generality assume that \( |h| = 1 \).

Since \( h \) is a circle valued function we have \( \bar{h} = 1/h \). Let \( \varphi(x) = h \cdot \bar{h} \circ u \). By Equation (6.24) we have

\[
\varphi(x) = h \cdot \bar{h} \circ u = \frac{h}{h \circ u} = z^r(x).
\]

Since \( r(x) \) is measurable with respect to the partition \( \alpha \) we have that \( \varphi \) is circle valued and \( \alpha \)-measurable. We have just verified that \( \varphi \) satisfies the second hypothesis above and that \( h \) and \( \varphi \) are related as required in the third hypothesis by definition.

Applying the Corollary we see that \( h \) is essentially constant. Since \( h \) is Lipschitz we conclude that \( h \) is pointwise constant. Let \( h_0 \) denote the constant value of \( h \).

Define \( H(x, y) = h_0 \), this function is clearly in \( L \). On each vertical line the function \( H \) agrees with \( \eta \) except possibly on a set of one dimensional Lebesgue measure zero. It follows that for all \( t \in [p, q] \) there exists a \( \lambda \)-null set \( N_t \) such that for all \( (x, y) \in \Lambda - N_t \) we have \( \eta(\ell(t) \cap \gamma(x, y)) - H(x, y) = 0 \). With this fact it follows directly from Equations (4.2) and (4.3) that \( \|\eta - H\|_s = 0 \) and \( \text{Lip}_s(\eta - H) = 0 \), thus \( \|\eta - H\|_s = 0 \). We conclude that \( \eta \) and \( H \) are in the same \( S \)-equivalence class.

Having verified Observation 1 and Observation 2 from the proof of Claim 3 we see that the lemma follows by combining Claim 2 and Claim 3.

6.7 Proof of Finite Variance Conditions

In this section we prove Lemma 5.5.
Proof of Lemma 5.5. Recall that $\Omega_1$ is a partition mod $\lambda$ of $\Lambda$ that is the common refinement of the return time partition and the partition that splits $\Lambda$ along the vertical line $\ell_A$ (see Equation (3.23)). Let $\omega_1(n,0)$ be the cell of $\Omega_1$ with return time $n$ that lies to the right of $\ell_A$ and $\omega_1(n,1)$ be the cell of $\Omega_1$ that lies to the left of $\ell_A$. Note that $\omega_1(n,j)$ must pass near the fixed line at $\ell_j$ before returning to $\Lambda$. By the monotone convergence theorem and Hölder’s inequality

$$
\int_{\Lambda} |\xi|^2 \, d\lambda = \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \int_{\Lambda} 1_{\omega_1(n,j)} |\xi|^2 \, d\lambda
\leq \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \left\| 1_{\omega_1(n,j)} |\xi|^2 \right\|_{\sup} \lambda(\omega_1(n,j)).
$$

By Equation (2.18) we have $\lambda(\omega_1(n,0)) = p_{n+1}^o - p_{n+2}^o$ and $\lambda(\omega_1(n,0)) = q_{n+2}^o - q_{n+1}^o$. By Equations (2.27) and (2.28), as $n \to \infty$,

$$
\lambda(\omega_1(n,j)) = O \left( \left( \frac{1}{n} \right)^{2+\frac{1}{\alpha_j}} \right).
$$

We will show that

$$
\sum_{n=0}^{\infty} \left\| 1_{\omega_1(n,0)} |\xi|^2 \right\|_{\sup} \lambda(\omega_1(n,0)),
$$

converges for each of the stated conditions. The sum with $j = 1$ converges by analogous arguments that are independent of the $j = 0$ case.

i. By Lemma 6.10 as $n \to \infty$ Suppose that $\alpha_0 < 1$

$$
\left\| 1_{\omega_1(n,0)} |\xi|^2 \right\|_{\sup} = O(n^2).
$$

Therefore,

$$
\left\| 1_{\omega_1(n,0)} |\xi|^2 \right\|_{\sup} \lambda(\omega_1(n,0)) = O \left( \left( \frac{1}{n} \right)^{2+\frac{1}{\alpha_0}} \right).
$$

and the terms in (⋆) are summable.

ii. Suppose that $M_0 = 0$ and $\alpha_0 = 1$. By Lemma 6.10 as $n \to \infty$,

$$
\left\| 1_{\omega_1(n,0)} |\xi|^2 \right\|_{\sup} = O(n^{2-2\gamma}).
$$

Therefore,

$$
\left\| 1_{\omega_1(n,0)} |\xi|^2 \right\|_{\sup} \lambda(\omega_1(n,0)) = O \left( \left( \frac{1}{n} \right)^{2\gamma+1} \right).
$$

Since $2\gamma + 1 > 1$, the terms in (⋆) are summable.
iii. Suppose that $M_0 = 0$, $1 < \alpha_0 < 3$, and $\gamma > \frac{2\alpha_0 - 1}{2}$. By Lemma 6.10 as $n \to \infty$, 
\[ \left\| 1_{\omega_1(n,0)} |\xi|^2 \right\|_{\text{sup}} = O \left( n^{2-2\gamma/\alpha_0} \right). \]
Therefore, 
\[ \left\| 1_{\omega_1(n,0)} |\xi|^2 \right\|_{\text{sup}} \lambda(\omega_1(n,0)) = O \left( \left( \frac{1}{n} \right)^{2\gamma/\alpha_0 + 1} \right). \]
Since $\frac{2\gamma+1}{\alpha_0} > 1$, the terms in $(\star)$ are summable.

\[ \]

6.8 Proof of Convergence and Continuity of Perturbations

In this section we prove Proposition 5.7

Proof of Proposition 5.7 We will show that for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $t \in (-\delta, \delta)$ and for all $\eta \in L$, 
\[ \| [R_n(t) - R_n(0)] \eta \|_S \leq \epsilon \| \eta \|_S. \]
First note that, 
\[ [R_n(t) - R_n(0)] \eta = T_* \left( (\exp(it\xi) - 1) 1_{[r=n]} \eta \right) \]
Second note that, 
\[ \| [R_n(t) - R_n(0)] \eta \|_S = \| [R_n(t) - R_n(0)] \eta \|_s + \operatorname{Lip}_s ([R_n(t) - R_n(0)] \eta). \]
Fix $\epsilon > 0$. We will estimate the two terms on the right. Let $\psi \in \mathcal{H}$ be a test function with $\| \psi \|_H \leq 1$, $x$ be a point in $[p, q]$, and consider a typical integral from the definition of $\| [R_n(t) - R_n(0)] \eta \|_s$, 
\[ I = \int_0^1 T_* \left( (\exp(it\xi) - 1) 1_{[r=n]} \eta \right) (x, y) \psi(y) dy \]
We will use the following facts to bound $I$.

A. For all $x \in [p, q]$, $\frac{d\nu}{dy}$ is a constant and for all $x \in [r = n]$, $0 < \frac{d\nu}{dy} \leq [\kappa + 1] \lambda [r = n]$. 
B. The value of $1_{[r=n]}(x, y)$ is independent of $y$. 

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Similarly, we consider a typical integral from the definition of $x, y$
Applying the change of variables $(x, y) = T(w, z) = (u(x), v_w(z))$, we compute as follows.

$$I = \int_0^1 T_s ((\exp (it\xi) - 1) 1_{[r=n]} \eta) (x, y) \psi(y) dy$$
$$= \int_0^1 ((\exp (it\xi) - 1) 1_{[r=n]} \eta) (w, z) \psi(v_w(z)) \frac{dv_w}{dz} dz$$
$$\leq [\kappa + 1] \lambda \int_0^1 (\exp (it\xi) - 1) 1_{[r=n]} \eta(w, z) dz$$
(by A)
$$= [\kappa + 1] \lambda \int_0^1 \eta(w, z) (\exp (it\xi) - 1) 1_{[r=n]} \psi(v_w(z)) dz$$
(by B)
$$\leq 2[\kappa + 1] \lambda \| \eta \|_S \| \psi \|_C \| (\exp (it\xi) - 1)(w, \cdot) \|_C$$
(by C)

Similarly, we consider a typical integral from the definition of $\text{Lip}_s ((R_n(t) - R_n(0)) [\eta])$.
Let $x_1, x_2 \in [p, q]$ such that $x_1 \neq x_2$ and $\psi_1$ and $\psi_2$ be test functions. We will apply changes of variable $(x_1, y) = T(w_1, z_1)$ and $(x_2, y) = T(w_2, z_2)$. Define $\Psi_j = \psi_j \circ v_{w_j} (\exp(it\xi) - 1)$ and $\Phi_j = (\max_j \{ \| \Psi_j \|_C \})^{-1} \Psi_j$. We compute as follows.

$$II = \int_0^1 \frac{R_n(t) \eta(x_1, y) \psi_1(y) - R_n(t) \eta(x_2, y) \psi_2(y)}{|x_1 - x_2|} dy$$
$$= \frac{1}{|x_1 - x_2|} \int_0^1 \eta(w_1, z_1) \Psi_1(z_1) \frac{dv_{w_1}}{dz_1} dz_1$$
$$- \frac{1}{|x_1 - x_2|} \int_0^1 \eta(w_2, z_2) \Psi_2(z_2) \frac{dv_{w_2}}{dz_2} dz_2$$
$$\leq [\kappa + 1] \lambda [r=n] \frac{|x_1 - x_2|}{|x_1 - x_2|} \int_0^1 \eta(w_1, z_1) \Psi_1(z_1) dz_1$$
$$- \frac{1}{|x_1 - x_2|} \int_0^1 \eta(w_2, z_2) \Psi_2(z_2) dz_2$$
$$\leq 2 \int_0^1 \eta(w_1, z) \Phi_1(z) - \eta(w_2, z) \Phi_2(z) dz$$
$$\leq [\kappa + 1] \lambda [r=n] \max_j \{ \| \Psi_j \|_C \} L_{\text{Lip}_s}(\eta)$$
$$\leq 2 \int_0^1 \eta(x, \cdot) \Phi_j(\cdot) dz$$

It is necessarily the case that $w, w_1, w_2 \in [r = n]$ in the calculations above.
By Lemma 6.10, $\| \xi(x, \cdot) \|_C$ is uniformly bounded on $[r = n]$ by some constant
Choose $\delta > 0$ such that $\delta(\kappa + 1)M(n)\|\eta\|_s < \epsilon/3$. It follows from the calculations above that for $|t| < \delta$ we have $I < \epsilon/3$ and $II < 2\epsilon/3$, which completes the proof of continuity.

By the estimates above,

$$\|R_n(t) - R_n(0)\|_S = O\left(|t| \lambda[r = n] \|\xi\|^{1/\alpha}_1\right).$$

As $n \to \infty$, $\|\xi(x, \cdot)\|_{L^\infty} = O(n)$ and $\lambda[r = n] = O\left(n^{-2-\frac{1}{\alpha}}\right)$, where $\alpha = \max\{\alpha_0, \alpha_1\}$, thus

$$\|R_n(t) - R_n(0)\|_{op} = O\left(|t| n^{-1-\frac{1}{\alpha}}\right).$$

The estimate above is summable in $n$. The result for $R(z, t)$ follows by an easy application of the triangle inequality and monotone convergence.

By unpacking the definition of $\|R_n(t)\eta\|_S$ as was done above and choosing to use the bound $|e^{-it\xi}| \leq 1$ yields

$$\|R_n(t)\|_{op} = O\left(\lambda[r = n]\right).$$

Note that we could have obtained a similar bound in the argument above by taking $2 \left|\sin\left(\frac{t\xi}{2}\right)\right| < 2$ instead of $2 \left|\sin\left(\frac{t\xi}{2}\right)\right| < |t| |\xi|$, however we would not have obtained the desired dependence on $|t|$. Since $\lambda([r = n]) = O\left(n^{-2-\frac{1}{\alpha}}\right)$, where $\alpha = \max\{\alpha_0, \alpha_1\}$, the sequence $r_n = \lambda[r = n]$ verifies the third claim.

\[\square\]

### 6.9 Proof of Expansion of the Dominant Eigenvalue

In this section we will prove Proposition 5.8. Before we can complete the proof we will need the following two lemmas on the asymptotic behavior of $\xi$.

**Lemma 6.10.** Suppose that $X: [0, 1]^2 \to \mathbb{R}$ is $\gamma$-Hölder for some $\gamma \in (0, 1]$ and that $(x, y) \in \Lambda$ is a point such that $x \in [A, q]$ and $r(x, y) = n + 2$ for some $n \geq 0$. As $n \to \infty$,

$$\xi(x, y) = n \int_0^1 X(0, y^{1 + \frac{1}{\alpha_0}}) dy + O\left(n^{1-\gamma}\right) + O\left(n^{1-\frac{1}{\alpha_0}}\right).$$

If $x \in [p, A]$, then as $n \to \infty$,

$$\xi(x, y) = n \int_0^1 X(1, y^{1 + \frac{1}{\alpha_1}}) dy + O\left(n^{1-\gamma}\right) + O\left(n^{1-\frac{1}{\alpha_1}}\right).$$
Proof. We will prove the first asymptotic expansion, the proof of the second is similar. Throughout this proof we will suppress the subscript on the contact parameters ($\alpha = \alpha_0$ and $c = c_0$). By Equation (5.10)

$$\xi(x, y) = \sum_{k=0}^{n+1} X(x_k, y_k).$$

Since $X$ is $\gamma$-Hölder,

$$|X(x_k, y_k) - X(0, y_k)| = O(x_k^\gamma) = O\left(n^{-\frac{2\gamma}{\alpha}}\right).$$

$$|X(0, y_k) - X\left(0, \left(1 - \frac{k+1}{n}\right)^{1+\frac{1}{\gamma}}\right)| = O(n^{-\gamma}).$$

An end point approximation to the Riemann sum shows that

$$\left|\int_0^1 X(0, y^{1+\frac{1}{\gamma}}) dy - \frac{1}{n} \sum_{k=1}^{n-1} X\left(0, \left(1 - \frac{k+1}{n}\right)^{1+\frac{1}{\gamma}}\right)\right| = O(n^{-\gamma}).$$

A standard triangle inequality argument shows that

$$\xi(x, y) = n \int_0^1 X(0, y^{1+\frac{1}{\gamma}}) dy + O\left(n^{1-\gamma}\right) + O\left(n^{1-\frac{2}{\gamma}}\right)$$

and therefore the claimed asymptotic holds.

Next we investigate the cumulative distribution function of $\xi$.

Lemma 6.11. Suppose that $X : [0, 1]^2 \to \mathbb{R}$ is $\gamma$-Hölder for some $\gamma \in (0, 1]$.

- If $M_0 > 0$, then for $t$ sufficiently large,
  $$\lambda([\xi > t] \cap [A, q]) \sim \frac{M_0}{\alpha_0 \text{Leb}(A)} \left(\frac{M_0(a_0+1)}{\epsilon_0 a_0}\right)^\frac{1}{\alpha_0} \left(\frac{1}{t}\right)^{1+\frac{1}{\alpha_0}}.$$
  $$\lambda([\xi < -t] \cap [A, q]) = 0.$$

- If $M_0 < 0$, then for $t$ sufficiently large,
  $$\lambda([\xi > t] \cap [A, q]) = 0,$$
  $$\lambda([\xi < -t] \cap [A, q]) \sim \frac{|M_0|}{\alpha_0 \text{Leb}(A)} \left(\frac{|M_0(a_0+1)|}{\epsilon_0 a_0}\right)^\frac{1}{\alpha_0} \left(\frac{1}{t}\right)^{1+\frac{1}{\alpha_0}}.$$

- If $M_1 > 0$, then for $t$ sufficiently large,
  $$\lambda([\xi > t] \cap [p, A]) \sim \frac{M_1}{\alpha_1 \text{Leb}(A)} \left(\frac{M_1(a_1+1)}{\epsilon_1 a_1}\right)^\frac{1}{\alpha_1} \left(\frac{1}{t}\right)^{1+\frac{1}{\alpha_1}},$$
  $$\lambda([\xi < -t] \cap [p, A]) = 0.$$
• If $M_1 < 0$, then for $t$ sufficiently large,
\[
\lambda ([\xi > t] \cap [p, A]) = 0,
\]
\[
\lambda ([\xi < -t] \cap [p, A]) \sim \frac{|M_1|}{\alpha_{Leb}(\Lambda)} \left( \frac{|M_1| (\alpha_1 + 1)}{c_1} \right)^{\frac{1}{\alpha_1}} \left( \frac{1}{t} \right)^{1 + \frac{1}{\alpha_1}}.
\]

Proof. We will prove the first asymptotic, the proofs of the others are similar. We will suppress subscripts throughout this proof ($M = M_0$, $\alpha = \alpha_0$, and $c = c_0$). For convenience define for any function $f$ on $\Lambda$ and real number $t$, $U(f, t) = [f > t] \cap [A, q]$. Note that by Equation (2.18), $\lambda(U(r, n)) = p \circ n - A = p \circ n - A \text{Leb}(\Lambda)$, thus by Equation (2.25)
\[
\lambda(U(r, t)) \sim \frac{1}{\alpha_{Leb}(\Lambda)} \left( \frac{(\alpha + 1)}{\alpha} \right)^{\frac{1}{\alpha}} \left( \frac{t}{t'} \right)^{1 + \frac{1}{\alpha}} \left( \frac{1}{t} \right)^{1 + \frac{1}{\alpha}}.
\]
Let $g(x, y) = \xi(x, y) - Mr(x, y)$, then fix $\epsilon > 0$ and note that,
\[
\lambda(U(\xi, t)) \geq \lambda(U(Mr, t(1 + \epsilon))) - \lambda(U(|g|, \epsilon t)), \quad
\lambda(U(\xi, t)) \leq \lambda(U(Mr, t(1 - \epsilon))) + \lambda(U(|g|, \epsilon t)).
\]
Note that $|g| > \epsilon t$ iff $r > \frac{\epsilon}{|g|} t$. By Lemma 6.10 $|g| = o(r(x, y))$, thus the quantity $\frac{r}{|g|} \epsilon$ is unbounded as $r \to \infty$. We conclude that as $t \to \infty$,
\[
\lambda(U(|g|, \epsilon t)) = o(\lambda(U(r, t))).
\]
Therefore as $t \to \infty$,
\[
|\lambda(U(\xi, t)) - \lambda(U(r, t))| = o(\lambda(r > t)).
\]
The claimed asymptotic for $\lambda(U(\xi, t))$ follows, since
\[
\left( \frac{t}{M} \left[ \frac{M}{t} \right] \right)^{1 + \frac{1}{\alpha}} = 1 + o(1)
\]
as $t \to \infty$.

It is not hard to check that $\xi$ is continuous on each set $[r = n + 2] \cap [A, q]$ for $n \geq 0$. By Lemma 6.10 for $(x, y) \in [r = n + 2] \cap [A, q]$,
\[
\xi(x, y) = Mn + O \left( n^{1-\gamma} \right) + O \left( n^{1-\frac{3}{4}} \right).
\]
For $n$ sufficiently large the first term dominates the last two and $\xi$ is strictly positive on $[r = n + 2] \cap [A, q]$. This leaves finitely many sets where $\xi$ may be negative, on each $\xi$ is continuous, therefore $\xi$ is bounded below. We conclude that, for $t$ sufficiently large,
\[
\lambda([\xi < -t] \cap [A, q]) = 0.
\]
\qed
We are now in a position to prove Proposition 5.8.

Proof of Proposition 5.8. By Proposition 5.7 we have
\[ \|R(z,t) - R(z,0)\|_S = O(|t|). \]
If \( e(t) \) is the eigenfunction of \( R(1,t) \) associated to the eigenvalue \( \chi(t) \) with integral 1, then because eigenvectors depend holomorphically on operators
\[ \|e(t) - 1\|_S = O(\|R(z,t) - R(z,0)\|_S) = O(|t|). \]
With this estimate in place the claimed expansions follow directly from Lemma 6.11 and the following theorems.

i. By arguments similar to [11] Theorem 3.7 we obtain the claimed expansion.

ii. The estimate above is sufficient to apply [2] Theorem 5.1, which yields the desired expansion of the eigenvalue \( \chi(t) \) for \( t \) near 0.

iii. The estimate above is sufficient to apply [2] Theorem 5.1, which yields the desired expansion of the eigenvalue \( \chi(t) \) for \( t \) near 0.

iv. Similarly we apply [11] Theorem 3.1 to obtain the claimed expansion.

\[ \Box \]

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