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ON THE CURVATURE ESTIMATES FOR HESSIAN EQUATIONS

By Changyu Ren and Zhizhang Wang

Abstract. The curvature estimates for $k$ curvature equations with general right-hand sides is a longstanding problem. In this paper, we completely solve the problem when $k = n - 1$. We also discuss some applications of our estimates.

1. Introduction. In this paper, we continue to study the longstanding problem about the global curvature estimates for curvature equations with general right-hand sides

$$\sigma_k(\kappa(X)) = \psi(X, \nu(X)), \quad \forall X \in M,$$

where $\sigma_k$ is the $k$-th elementary symmetric function, $\nu(X)$ and $\kappa(X)$ denote the outer normal vector and the principal curvatures of the hypersurface $X : M \to \mathbb{R}^{n+1}$, respectively. This problem was clearly posed by Guan-Li-Li in [22] at first. Moreover, it is very nature to consider the equation (1.1) with the right-hand side containing the normal vector or in other words, the gradient term.

Indeed, equations of the form (1.1) have been studied intensively for a long time since many important geometric problems fall into this type of PDE with special form of $\psi$. The well-known Minkowski problem, namely, the existence of convex hypersurfaces with prescribed Gauss-Kronecker curvature on the outer normal, has been studied by many mathematicians, see for example [12, 32, 33, 34]. More generally, Alexandrov proposed to study the existence of convex hypersurfaces with prescribed Weingarten curvatures on outer normals in [1, 20]. As a counterpart, problems of prescribing curvature measures have also attracted many attentions, see for example [2, 3, 10, 22, 23, 33, 38]. All those problems have been playing important role in the development of both the convex geometry and the theory of the fully nonlinear PDE.

For most of the situations, the main difficulty to study the equations of the form (1.1) is to obtain the $C^2$ estimates. Here, we briefly mention some important results about this estimate. For $k = 1$, equation (1.1) is quasilinear and the $C^2$ estimate follows from the classical theory of the quasilinear PDE. When $k = n$, the equation is of Monge-Ampère type and the $C^2$ estimate in this case with general $\psi(X, \nu)$ was obtained by Caffarelli-Nirenberg-Spruck [8]. However, for $1 < k < n$,
much less is known except some special cases. If \( \psi \) is independent of normal vector \( \nu \), \( C^2 \) estimate was proved by Caffarelli-Nirenberg-Spruck [10]. If \( \psi \) depends only on \( \nu \), \( C^2 \) estimate was proved in [20]. On the other hand, the Dirichlet problem of the equation (1.1) on domains in \( \mathbb{R}^n \) was also considered by Ivochkina [26, 27]. The \( C^2 \) estimate was proved there under some extra conditions on the dependence of \( \psi \) on \( \nu \). In [22, 23], to study the prescribing curvature measure problems, the authors derived the \( C^2 \) estimate for \( \psi(X, \nu) = \langle X, \nu \rangle \tilde{\psi}(X) \). For general right-hand side \( \psi \), the \( C^2 \) estimate of equation (1.1) has been a longstanding problem. Important progress was recently made in the joint work with Guan and the authors. In [24], using a carefully chosen test function and delicate analysis, we are able to get the desired \( C^2 \) estimate when \( k = 2 \) (see [37] for a simplified proof). However, for \( k \geq 3 \), a convexity assumption was required. In this paper, we remove this condition when \( k = n - 1 \) and hence completely solve the problem for this case. We also note the recently important work on the curvature estimates and the \( C^2 \) estimates developed by Guan [19] and Guan-Spruck-Xiao [21].

Before giving the precise statement of the main theorem, we recall the definition of the Gårding’s cone following [9], which is the suitable domain for equation (1.1) to be elliptic.

**Definition 1.** For a domain \( \Omega \subset \mathbb{R}^n \), a function \( v \in C^2(\Omega) \) is called \( k \)-convex if the eigenvalues \( \kappa(x) = (\kappa_1(x), \ldots, \kappa_n(x)) \) of the hessian \( \nabla^2 v(x) \) is in \( \Gamma_k \) for all \( x \in \Omega \), where \( \Gamma_k \) is the Gårding’s cone

\[
\Gamma_k = \{ \kappa \in \mathbb{R}^n | \sigma_m(\kappa) > 0, \ m = 1, \ldots, k \}.
\]

A \( C^2 \) regular hypersurface \( M \subset \mathbb{R}^{n+1} \) is \( k \)-convex if \( \kappa(X) \in \Gamma_k \) for all \( X \in M \).

Now, we state our main theorem.

**Theorem 2.** Suppose \( M \subset \mathbb{R}^{n+1} \) is a closed \( n - 1 \)-convex hypersurface satisfying the curvature equation (1.1) with \( k = n - 1 \) for some positive function \( \psi(X, \nu) \in C^2(\Gamma) \), where \( \Gamma \) is an open neighborhood of the unit normal bundle of \( M \) in \( \mathbb{R}^{n+1} \times S^n \), then there is a constant \( C \) depending only on \( n, k, \|M\|_{C^1}, \inf \psi \) and \( \|\psi\|_{C^2} \), such that

\[
\max_{X \in M, i=1,\ldots,n} \kappa_i(X) \leq C.
\]

Since the calculation is very complicated, we give a rough sketch and the basic ideas of our proof here. The \( C^2 \) estimate is always based on some good choice of test functions. In our case, we use the nonlinear test function

\[
\phi = \log \log \left( \sum_i e^{\kappa_i} \right) - N \log u
\]
proposed in [24], where \( \kappa_1, \kappa_2, \ldots, \kappa_n \) are the principal curvatures of the hypersurface \( M \), \( u \) is the support function of \( M \) and \( N \) is some large positive constant. Differentiating the function \( \phi \) twice and then contracting it with a suitable matrix, we will arrive at the inequality (4.10) after a routine calculation. If the third order terms \( A_i + B_i + C_i + D_i - E_i \), for \( i = 1, 2, \ldots, n \) defined in Section 4 is nonnegative for every index \( i \), we get the non-negativity of (4.10). The proof of the non-negativity of these third order terms is divided into three cases for all indices. The first case deals with those \( \kappa_i \) “growing slowly”, that is, there exists some sufficiently small positive constant \( \delta \) such that \(|\kappa_i| \leq \delta \kappa_1 \). Here, without loss of generality, \( \kappa_1 \) denotes the biggest principal curvature function of \( M \). We adopt some ideas in [24] to derive the non-negativity of the third order terms in Lemma 13 for this case. In fact, the good term \( D_i \) arising from the nonlinearity of the test function is sufficiently large to cancel out the negative terms. The other two cases are for those \( \kappa_i \) “growing quickly”, which means \(|\kappa_i| > \delta \kappa_1 \). To obtain the desired sign, we further divide this case into two sub-cases: \( \kappa_i > \delta \kappa_1 \) and \( -\kappa_i > \delta \kappa_1 \). Since the negative eigenvalue may be very large, the original idea in [24] to control the third order terms cannot work here. For sub case \( \kappa_i > \delta \kappa_1 \), in Lemma 15, we prove that these third order terms are nonnegative if we have a key inequality proved in Theorem 11 in Section 3. Then, the discussion of Theorem 11 shows that the key inequality comes from the non-negativity of two nice quadratic forms in the cone \( \Gamma_{n-1} \). Fortunately, one quadratic form is the Hadamard product of the other one and itself. In Lemma 9, we prove the first quadratic form is nonnegative which complete the proof of this sub case. For the sub case \( -\kappa_i > \delta \kappa_1 \), we also need to know the explicit lower bound of \( \kappa_i \) which has been obtained in Lemma 14. Using some similar tricks as the previous sub case and the lower bound of \( \kappa_i \), we are able to prove the non-negativity of the third order terms in Lemma 16.

We remark that the method to obtain the \( C^2 \) estimate here is completely different from dealing with the scalar curvature equation or convex solutions case in [24].

Next, we consider the Dirichlet problem of the \( n-1 \) Hessian equation with generalized right-hand side in the Euclidean space \( \mathbb{R}^n \).

\[
\begin{align*}
\sigma_{n-1}[D^2 u] &= \psi(x, u, Du), & \text{in } \Omega, \\
u &= \varphi, & \text{on } \partial \Omega,
\end{align*}
\]

where \( \varphi \) is a given function. If \( \psi \) does not depend on \( Du \), the problem has been extensively studied by Caffarelli-Nirenberg-Spruck [9], Krylov [29], Trudinger [40], as well as Li [31] and Guan [17] etc. If the function \( \psi(x, \xi, p) \) defined on \( (x, \xi, p) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \) is convex respect to the vector \( p \), it is well known that the global \( C^2 \) estimate was established by B. Guan [18]. More reference about these type of estimates can be found in [13, 30]. In the following theorem, we remove the condition of convexity in [18].
COROLLARY 3. For the Dirichlet problem (1.3) of the $\sigma_{n-1}$ equation defined on some bounded domain $\Omega \subset \mathbb{R}^n$, the global $C^2$ estimates can be obtained. It means that, we have some constants $C$ depending on $\psi$ and $\nabla u, u$ and the domain $\Omega$, such that

$$\|u\|_{C^2(\bar{\Omega})} \leq C + \max_{\partial \Omega} |\nabla^2 u|.$$  

In the last part of the paper, we give some applications of our estimate. The first one is that we can solve the prescribed $n-1$ curvature equation (1.1) in the cone $\Gamma_{n-1}$. For the sake of the $C^0, C^1$ estimates, we need further barrier conditions on the prescribed function $\psi$ as considered in [3, 10, 38]. We denote $\rho(X) = |X|$.

We assume the following:

*Condition (1).* There are two positive constants $r_1 < 1 < r_2$ such that

$$\begin{cases} 
\psi\left(X, \frac{X}{|X|}\right) \geq \frac{\sigma_k(1, \ldots, 1)}{r_1^k}, & \text{for } |X| = r_1, \\
\psi\left(X, \frac{X}{|X|}\right) \leq \frac{\sigma_k(1, \ldots, 1)}{r_2^k}, & \text{for } |X| = r_2.
\end{cases}$$  

(1.4)

*Condition (2).* For any fixed unit vector $\nu$, we have

$$\frac{\partial}{\partial \rho} (\rho^k \psi(X, \nu)) \leq 0,$$

where $|X| = \rho$.  

(1.5)

Using the above two conditions, we have the following existence theorem.

**Theorem 4.** Suppose $k = n - 1$ and the positive function $\psi \in C^2(\bar{B}_{r_2} \setminus B_{r_1} \times S^n)$ satisfies conditions (1.4) and (1.5), then equation (1.1) has a unique $C^{3, \alpha}$ star-shaped solution $M$ in $\{r_1 \leq |X| \leq r_2\}$.

Here, $M$ is called a star-shaped hypersurface, if it can be viewed as a radial graph of $S^n$, i.e.,

$$R_M : S^n \longrightarrow M, \\
z \mapsto \tilde{\rho}(z)z,$$

where $S^n$ is an $n$ dimensional unit sphere and $\tilde{\rho}(z) > 0$ is a radial function defined on $S^n$.

The second application is to solve the prescribed $n - 1$ curvature problem of the $n$ dimensional spacelike graphic hypersurface $M$ in the Minkowski space $\mathbb{R}^{n,1}$. The Minkowski space $\mathbb{R}^{n,1}$ is the set $\mathbb{R}^{n+1}$ endowed with the following metric

$$ds^2 = dx_1^2 + \cdots + dx_n^2 - dx_{n+1}^2.$$
The terminology “spacelike” means that any tangent vector of $M$ has positive length with respect to the above metric. Since $M$ is a graph over some domain contained in $\mathbb{R}^n$, it can be expressed by a function $u$ which means that $X = (x, u(x))$, $x \in \mathbb{R}^n$ is $M$’s position vector. $u$ is called a spacelike function if its corresponding graph $(x, u(x))$ is spacelike. Again, we suppose $\kappa(X)$ and $\nu(X)$ are the principal curvature and unit normal vector of $M$. Note that $\kappa(X)$ and $\nu(X)$ can be rewritten as the derivatives of the function $u$. Thus, if the prescribed curvature function $\psi$ depends on $M$’s position vector and normal vector, it can be rewritten as another function depending on $x, u, Du$, which we will still denote it by $\psi$. Therefore, solutions of the prescribed $n - 1$ curvature problem can be stated as the following theorem with given boundary functions.

**Theorem 5.** Let $\Omega$ be some bounded domain in $\mathbb{R}^n$ with smooth boundary and $\psi \in C^2(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ be a positive function with $\psi_u \geq 0$. Let $\varphi \in C^4(\bar{\Omega})$ be space like. Consider the following Dirichlet problem

$$
\begin{cases}
\sigma_{n-1}(\kappa_1, \ldots, \kappa_n) = \psi(x, u, Du), & in \Omega \\
u = \varphi, & on \partial \Omega,
\end{cases}
$$

where $\kappa_1, \kappa_2, \ldots, \kappa_n$ are the principal curvatures of a space like graphic hypersurface defined by $(x, u(x))$ in the Minkowski space $\mathbb{R}^{n,1}$. If the problem (1.6) has a sub solution, then it has a unique space like solution $u$ in $\Gamma_{n-1}$ belonging to $C^{3,\alpha}(\bar{\Omega})$ for any $\alpha \in (0, 1)$.

The prescribing curvature problem of the spacelike graphic hypersurface over some bounded domain in the Minkowski space has been studied by various authors. The prescribe mean curvature problem was proposed and solved by Bartnik-Simon [5]. In [14], Delanö solved the prescribed Gauss curvature curvature problem, which was reduced to the Monge-Ampère type equations. The prescribing $k$-curvature problems with $2 \leq k < n$ were proposed and studied by Bayard [6, 7]. The scalar curvature equation, that is, $k = 2$ case was completely solved by Urbas [41]. The above theorem solves the prescribed $(n - 1)$-curvature problem. The other cases with $2 < k < n - 1$ are still open. We want to remark that, for the prescribing curvature problems, the main difference between in the Euclidean space and in the Minkowski space is that the sectional curvature of the ambient space has opposite sign. Hence, even for the case that the function $\psi$ not depending on gradient terms, these problems can not be successfully solved as in the Euclidean space, comparing [11]. Hypersurfaces with prescribed curvatures in Lorentzian manifolds also have been studied by Gerhardt [15, 16] and Schnürer [35].

The organization of our paper is as follows. In Section 2, we explain more notations and list several needed lemmas. We also will review some algebraic identities of the elementary symmetric functions. In Section 3, we will prove a key inequality. In Section 4, we will divide into three cases to prove our curvature estimates,
namely proving Theorem 2 using the key inequality proved in Section 3. In Section 5, we give some applications. We discuss the prescribed curvature problems in the Euclidean space and in the Minkowski space and prove Theorem 4 and Theorem 5.

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2. Preliminary. The \( k \)-th elementary symmetric function is defined by, for \( \kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n) \in \mathbb{R}^n \) and \( 1 \leq k \leq n \),

\[
\sigma_k(\kappa) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \kappa_{i_1} \cdots \kappa_{i_k}.
\]

We also set \( \sigma_0(\kappa) = 1 \) and \( \sigma_k(\kappa) = 0 \) for \( k > n \). Following [9], a suitable definition domain of \( \sigma_k \) is the Gårding’s cone \( \Gamma_k \), which is an open, convex, symmetric (invariant under the interchange of any two \( \kappa_i \)) cone with vertex at the origin, containing the positive cone: \( \{ \kappa \in \mathbb{R}^n \mid \text{each component } \kappa_i > 0, 1 \leq i \leq n \} \). Korevaar [28] has shown that the cone \( \Gamma_k \) also can be characterized as

\[
\left\{ \kappa \in \mathbb{R}^n; \sigma_k(\kappa) > 0, \frac{\partial \sigma_k(\kappa)}{\partial \kappa_{i_1}} > 0, \ldots, \frac{\partial^k \sigma_k(\kappa)}{\partial \kappa_{i_1} \cdots \partial \kappa_{i_k}} > 0, \right. \\
\left. \text{for all } 1 \leq i_1 < \cdots < i_k \leq n \right\}.
\]

By the definition of \( \Gamma_k \), it is clear that

\[ \Gamma_n \subset \cdots \subset \Gamma_k \subset \cdots \subset \Gamma_1. \]

We explain more notations. We let \( \kappa(A) \) be eigenvalues of a matrix \( A = (a_{ij}) \). Suppose \( F \) is a function defined on a set of symmetric matrices. We let

\[ f(\kappa(A)) = F(A). \]

Thus, we denote

\[ F^{pq} = \frac{\partial F}{\partial a_{pq}}, \quad \text{and} \quad F^{pq,rs} = \frac{\partial^2 F}{\partial a_{pq} \partial a_{rs}}. \]
For a local orthonormal frame, if \( A \) is diagonal at a point, then at this point, we have

\[
F^{pp} = \frac{\partial f}{\partial \kappa_p} = f_p, \quad \text{and} \quad F^{pq\cdot qq} = \frac{\partial^2 f}{\partial \kappa_p \partial \kappa_q} = f_{pq}.
\]

Thus the definition of the \( k \)-th elementary symmetric function can be extended to symmetric matrices. Suppose \( W \) is an \( n \times n \) symmetric matrix and \( \kappa(W) \) are its eigenvalues. We define

\[
\sigma_k(W) = \sigma_k(\kappa(W)),
\]

which is the summation of the \( k \)-th principal minors of the matrix \( W \).

Now we will list some algebraic identities and properties of \( \sigma_k \). For a index \( 1 \leq l \leq n \), the notation \( \sigma_l(\kappa_1 \kappa_2 \ldots) \) means the \( l \)-th elementary symmetric function of \( \kappa_1, \kappa_2, \ldots, \kappa_n \) with \( \kappa_n = 0, \kappa = 0, \ldots \). Thus, we have

\[
\begin{align*}
\text{(i)} & \quad \sigma_k^{pp}(\kappa) := \frac{\partial \sigma_k(\kappa)}{\partial \kappa_p} = \sigma_{k-1}(\kappa | p) \text{ for any } p = 1, \ldots, n; \\
\text{(ii)} & \quad \sigma_k^{pq\cdot qq}(\kappa) := \frac{\partial^2 \sigma_k(\kappa)}{\partial \kappa_p \partial \kappa_q} = \sigma_{k-2}(\kappa | pq) \text{ for any } p, q = 1, \ldots, n \text{ and } \sigma_k^{pp\cdot pp}(\kappa) = 0; \\
\text{(iii)} & \quad \sigma_k(\kappa) = \kappa_i \sigma_{k-1}(\kappa | i) + \sigma_k(\kappa | i) \text{ for any fixed index } i; \\
\text{(iv)} & \quad \sum_{i=1}^n \kappa_i \sigma_{k-1}(\kappa | i) = k \sigma_k(\kappa).
\end{align*}
\]

Thus, for a Codazzi tensor \( W = (w_{ij}) \), we have

\[
\begin{align*}
\text{(v)} & \quad -\sum_{p,q,r,s} \sigma_k^{pp\cdot rs}(w_{pq})w_{rs} = \sum_{p,q} \sigma_k^{pp\cdot qq}w_{pq}^2 - \sum_{p,q} \sigma_k^{pq\cdot qq}w_{pq}^2 w_{pq}w_{qr},
\end{align*}
\]

where \( w_{pq} \) means the covariant derivative of \( w_{pq} \) and \( \sigma_k^{pq\cdot rs} = \frac{\partial^2 \sigma_k(W)}{\partial w_{pq} \partial w_{rs}} \). The meaning of the Codazzi tensor can be found in [24].

For \( \kappa \in \Gamma_k \), if we let \( \kappa_1 \geq \cdots \geq \kappa_n \), then we have

\[
\begin{align*}
\text{(vi)} & \quad \sigma_{k-1}(\kappa | n) \geq \cdots \geq \sigma_{k-1}(\kappa | 1) > 0; \\
\text{(vii)} & \quad \kappa_1 \sigma_{k-1}(\kappa | 1) \geq C_{n,k} \sigma_k(\kappa),
\end{align*}
\]

where \( C_{n,k} \) is a positive constant depending only on \( n, k \). More details of the proof of these formulas can be found in [25, 42].

There are two important concavities for \( \sigma_k \), which is essential to our proof. In [9, 39], it is showed that the function \( \sigma_k^{1/k}(\kappa) \) is a concave function in \( \Gamma_k \) and the function \( \left( \frac{\sigma_k(\kappa)}{\sigma_l(\kappa)} \right)^{1/(k-l)} \) for \( l < k \) is also a concave function in \( \Gamma_l \).

If \( k \geq 1 \), using (2.1), we have

\[
\frac{\partial \sigma_k(\kappa)}{\partial \kappa_i} > 0 \quad \text{in } \Gamma_k
\]

for each \( i \), which implies the ellipticity of the equations (1.1), (1.3) and (1.6) with respective to their admissible solutions, namely the \( k \)-convex solutions.

Now, we give the following two Lemmas, which will be needed in our proof.
**Lemma 6.** Assume that $k > l$, $W = (w_{ij})$ is a Codazzi tensor which is in $\Gamma_k$. Denote $\alpha = \frac{1}{k-l}$. Then, for $h = 1, \ldots, n$, we have the following inequality

\[
- \sum_{p,q} \frac{\sigma_{pp,qq}}{\sigma_k} (W) w_{pph} w_{qqh} + \sum_{p,q} \frac{\sigma_{pp,qq}}{\sigma_l} (W) w_{pph} w_{qqh} \\
\geq \left( \frac{(\sigma_k(W))_h}{\sigma_k(W)} - \frac{(\sigma_l(W))_h}{\sigma_l(W)} \right) \left( (\alpha - 1) \left( \frac{\sigma_k(W)}{\sigma_k(W)} \right)_h - (\alpha + 1) \left( \frac{\sigma_l(W)}{\sigma_l(W)} \right)_h \right).
\]

Furthermore, for any $\delta > 0$,

\[
- \sum_{p,q} \sigma_{pp,qq}^{pp,qq} (W) w_{pph} w_{qqh} + \left( 1 - \alpha + \frac{\alpha}{\delta} \right) \left( \frac{(\sigma_k(W))_h}{\sigma_k(W)} \right)^2 \\
\geq \sigma_k(W) (\alpha + 1 - \delta \alpha) \left( \frac{(\sigma_l(W))_h}{\sigma_l(W)} \right)^2 - \sigma_k(W) \sum_{p,q} \sigma_{pp,qq}^{pp,qq} (W) w_{pph} w_{qqh}.
\]

Here, the meaning of the Codazzi tensor can be found in [24]. A symmetric tensor being in $\Gamma_k$ means that its corresponding eigenvalues are in $\Gamma_k$. The other needful lemma is:

**Lemma 7.** Denote $\text{Sym}(n)$ the set of all $n \times n$ symmetric matrices. Let $F$ be a $C^2$ symmetric function defined in some open subset $\Psi \subset \text{Sym}(n)$. At any diagonal matrix $A \in \Psi$ with distinct eigenvalues, let $\ddot{F}(B,B)$ be the second derivative of $C^2$ symmetric function $F$ in direction $B \in \text{Sym}(n)$, then

\[
\ddot{F}(B,B) = \sum_{j,k=1}^{n} \ddot{f}^{jk} B_{jj} B_{kk} + 2 \sum_{j < k} \ddot{f}^{jk} \dddot{f}^{jk} B_{jj} B_{kk}.
\]

The proof of the first Lemma can be found in [22, 24]. The proof of the second one can be found in [4, 9].

### 3. An inequality.

In this section, we will prove a key inequality for the $n-1$ Hessian equation in the $n$ dimensional Euclidean space. We present some basic algebraic identities before we are going to prove our inequality. We use $h_{ij}$ to denote the second fundamental form of some hypersurface $M$. The eigenvalues of $h_{ij}$ are denoted by $\kappa_1, \kappa_2, \ldots, \kappa_n$. We will always assume that $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n$ from now on.

**Lemma 8.** We have the following identities for $\sigma_{n-1}$ in $\Gamma_{n-1}$ cone. For any indices $1 \leq i, j, p, q \leq n$, we have

(i) \[
\sigma_{n-1}^{jj}( - \sigma_{n-1}^{jj} + 2 \kappa_i \sigma_{n-1}^{ii,jj} + \sigma_{n-1}^{ii} ) = \kappa_i^2 \sigma_{n-3}^{jj}(\kappa|i{j}) + \sigma_{n-1} \sigma_{n-3}(\kappa|i{j});
\]
Then, we have

\[
\sigma_{n-1}^i - \sigma_{n-1}^{ii} - \sigma_{n-1}^{pp} - \sigma_{n-1}^{qq} = \kappa_i^2 \sigma_{n-3}^2 (\kappa_{ipq} - \sigma_{n-1} \sigma_{n-3} (\kappa_{ipq}));
\]

(ii) 

\[
\kappa_i \left( \sigma_{n-1}^{pp} \sigma_{n-1}^{ii,qq} + \sigma_{n-1}^{qq,ii} - \sigma_{n-1}^{ii} \sigma_{n-1}^{pp,qq} \right) - \sigma_{n-1}^{pp} \sigma_{n-1}^{qq} = \kappa_i^2 \sigma_{n-3}^2 (\kappa_{ipq} - \sigma_{n-1} \sigma_{n-3} (\kappa_{ipq}));
\]

(iii) 

\[-\kappa_i^2 \sigma_{n-1}^{pp} \sigma_{n-1}^{ii,qq} + \kappa_i \sigma_{n-1}^{ii} \sigma_{n-1}^{pp,qq} = -\kappa_i^2 \sigma_{n-3}^2 (\kappa_{ipq}) + \kappa_i \sigma_{n-1} \sigma_{n-3} (\kappa_{ipq}).\]

**Proof.** (i) At first, it is easy to calculate that 

\[-\sigma_{n-1}^{jj} + 2 \kappa_i \sigma_{n-1}^{ii,jj} + \sigma_{n-1}^{ii} = (\kappa_i + \kappa_j) \sigma_{n-1} \sigma_{n-3} (\kappa_{ij}).\]

Then, we have

\[
\sigma_{n-1}^{jj} (\kappa_i + \kappa_j) \sigma_{n-1} \sigma_{n-3} (\kappa_{ij})
\]

\[
= \kappa_i \sigma_{n-2} (\kappa_{ij}) + \sigma_{n-1} - \sigma_{n-1} (\kappa_{ij}) \sigma_{n-3} (\kappa_{ij})
\]

\[
= \kappa_i \sigma_{n-3} (\kappa_{ij}) + \sigma_{n-1} - \kappa_i \sigma_{n-2} (\kappa_{ij}) + \sigma_{n-1} \sigma_{n-3} (\kappa_{ij})
\]

\[
= \kappa_i^2 \sigma_{n-3}^2 (\kappa_{ij}) + \sigma_{n-1} \sigma_{n-3} (\kappa_{ij}),
\]

where we have used \(\sigma_{n-1} (\kappa_{ij}) = 0\).

(ii) A straightforward calculation shows

\[
\kappa_i \left( \sigma_{n-1}^{pp} \sigma_{n-1}^{ii,qq} + \sigma_{n-1}^{qq,ii} - \sigma_{n-1}^{ii} \sigma_{n-1}^{pp,qq} \right) - \sigma_{n-1}^{pp} \sigma_{n-1}^{qq} = \kappa_i^2 \sigma_{n-3}^2 (\kappa_{ipq}) - \kappa_i \sigma_{n-2} (\kappa_{ipq}) \sigma_{n-3} (\kappa_{ipq}) - \sigma_{n-1} \sigma_{n-3} (\kappa_{ipq})
\]

\[
= \kappa_i \sigma_{n-2} (\kappa_{ipq}) \sigma_{n-3} (\kappa_{ipq}) + \sigma_{n-1} - \sigma_{n-1} (\kappa_{ipq}) \sigma_{n-3} (\kappa_{ipq})
\]

\[
= \kappa_i^2 \sigma_{n-3}^2 (\kappa_{ipq}) + \kappa_i \sigma_{n-2} (\kappa_{ipq}) \sigma_{n-3} (\kappa_{ipq}) + \sigma_{n-1} \sigma_{n-3} (\kappa_{ipq})
\]

\[
= \kappa_i^2 \sigma_{n-3}^2 (\kappa_{ipq}) + \kappa_i \sigma_{n-2} (\kappa_{ipq}) + \kappa_i \sigma_{n-3} (\kappa_{ipq})
\]

\[
= \kappa_i^2 \sigma_{n-3}^2 (\kappa_{ipq}) - \kappa_i \kappa_i \sigma_{n-3}^2 (\kappa_{ipq}) - \kappa_i \sigma_{n-2} (\kappa_{ipq}) \sigma_{n-3} (\kappa_{ipq})
\]

\[
= \kappa_i^2 \sigma_{n-3}^2 (\kappa_{ipq}) - \sigma_{n-1} \sigma_{n-3} (\kappa_{ipq}).
\]

Here we have used \(\sigma_{n-2} (\kappa_{ipq}) = 0\) and

\[
\sigma_{n-1} = \kappa_p \sigma_{n-2} (\kappa_{ip}) + \sigma_{n-1} (\kappa_{ip}) = \kappa_p \sigma_{n-2} (\kappa_{ip}) + \kappa_i \kappa_q \sigma_{n-3} (\kappa_{ipq}).
\]

(iii) We also have

\[
(3.1) \quad \sigma_{n-1}^{pp,qq} = \kappa_i \sigma_{n-4} (\kappa_{ipq}) + \sigma_{n-3} (\kappa_{ipq})
\]
We also need to define another determinant for convenience. That is, for given index \(i\), we define

\[
\sigma_{n-1}^{ii,pp}\sigma_{n-1}^{ii,qq}
\]

\[
\begin{align*}
&= \kappa_q\sigma_{n-4}(\kappa|ipq) + \sigma_{n-3}(\kappa|ipq) \\
&= \kappa_p\sigma_{n-4}(\kappa|ipq) + \sigma_{n-3}(\kappa|ipq) \\
&= \sigma_{n-3}(\kappa|ipq) + \sigma_{n-2}(\kappa|i)\sigma_{n-4}(\kappa|ipq),
\end{align*}
\]

where we have used

\[
\sigma_{n-2}(\kappa|i) = \kappa_p\sigma_{n-3}(\kappa|ip) + \sigma_{n-2}(\kappa|ip)
\]

\[
= \kappa_p\kappa_q\sigma_{n-4}(\kappa|ipq) + (\kappa_p + \kappa_q)\sigma_{n-3}(\kappa|ipq).
\]

Using the above two identities (3.1) and (3.2), we get the identity (iii).

\[
\text{Lemma 9.}
\]

Now we need to consider the following \((n-1) \times (n-1)\) matrix \((a_{pq})\). For some given index \(i\), we define

\[
a_{pq} = \begin{cases} 
\sigma_{n-3}(\kappa|ip), & p = q, \\
-\sigma_{n-3}(\kappa|ipq), & p \neq q,
\end{cases}
\]

where the indices \(p, q \neq i\) and \(1 \leq p, q \leq n\). We would like to prove that the above matrix is semi positive definite. Without loss of generality, we only need to prove the case \(i = 1\) and \(2 \leq p, q \leq n\) which is the following Lemma.

\[
\text{Lemma 9.}
\]

Suppose \(2 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq n\) are \(m\) ordered indices. \(D_m(i_1 \cdots i_m)\) denotes the \(m\)-th principal minor of the matrix \((a_{pq})\). Then we have

\[
D_m(i_1 \cdots i_m) = \det \begin{bmatrix}
a_{i_1i_1} & a_{i_1i_2} & \cdots & a_{i_1i_m} \\
a_{i_2i_1} & a_{i_2i_2} & \cdots & a_{i_2i_m} \\
& \ddots & \ddots & \ddots \\
a_{i_mi_1} & a_{i_mi_2} & \cdots & a_{i_mi_m}
\end{bmatrix}
\]

\[
= \sigma_{n-2}^{m-1}(\kappa|1)\sigma_{n-(m+2)}(\kappa|1i_1 \cdots i_m).
\]

We also need to define another determinant for convenience. That is, for \(k \neq m\),

\[
B_{m-1}(i_1 \cdots i_m; i_k)
\]

\[
\begin{align*}
&= \det \begin{bmatrix}
a_{i_1i_1} & a_{i_1i_2} & \cdots & a_{i_1i_{k-1}} & a_{i_1i_{k+1}} & \cdots & a_{i_1i_m} \\
a_{i_2i_1} & a_{i_2i_2} & \cdots & a_{i_2i_{k-1}} & a_{i_2i_{k+1}} & \cdots & a_{i_2i_m} \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
a_{i_{m-1}i_1} & a_{i_{m-1}i_2} & \cdots & a_{i_{m-1}i_{k-1}} & a_{i_{m-1}i_{k+1}} & \cdots & a_{i_{m-1}i_m}
\end{bmatrix} \\
&= (-1)^{m+k}\sigma_{n-2}^{m-2}(\kappa|1)\sigma_{n-m-1}(\kappa|1i_1 \cdots i_m).
\]
Hence, in $\Gamma_{n-1}$ cone, we have

$$D_{m-1}(2 \cdots m) = \sigma_{n-2}^{m-2}(\kappa|1)\sigma_{n-(m+1)}(\kappa|12 \cdots m) > 0$$

for all $2 \leq m < n$. Especially for $m = n$, we have

$$D_{n-1}(2 \cdots n) = \sigma_{n-2}^{n-2}(\kappa|1)\sigma_{1}(\kappa|12 \cdots n) = 0,$$

which implies the matrix $(a_{pq})$ is a nonnegative definite matrix.

\textbf{Proof.} We prove the above two formulas by induction.

For $m = 2$, we have

$$B_1(i_1i_2;i_1) = a_{i_1i_2} = -\sigma_{n-3}(\kappa|1i_1i_2).$$

Also, we have, by (3.2),

$$D_2(i_1i_2) = \sigma_{n-3}(\kappa|1i_1)\sigma_{n-3}(\kappa|1i_2) - \sigma_{n-3}^2(\kappa|1i_1i_2)$$

$$= \sigma_{n-2}(\kappa|1)\sigma_{n-4}(\kappa|1i_1i_2).$$

Now we assume that (3.4) and (3.5) both hold for the preceding $1, 2, \ldots, m - 1$ cases. Thus, for each $m \geq 3$, we have

$$D_m(i_1 \cdots i_m)$$

$$= \sum_{l=1}^{m-1} (-1)^{m+l}a_{i_mi_l}B_{m-1}(i_1 \cdots i_{m-1};i_l) + a_{i_mi_m}D_{m-1}(i_1 \cdots i_{m-1})$$

$$= \sigma_{n-2}^{m-2}(\kappa|1)$$

$$\left[ - \sum_{l=1}^{m-1} \sigma_{n-3}(\kappa|1i_l)i_l)\sigma_{n-m-1}(\kappa|1i_1 \cdots i_{m-1}) - \sigma_{n-3}(\kappa|1i_m)\sigma_{n-m-1}(\kappa|1i_1 \cdots i_{m-1}) \right].$$

A straightforward computation shows

$$\sigma_{n-3}(\kappa|1i_m)\sigma_{n-m-1}(\kappa|1i_1 \cdots i_{m-1})$$

$$= \sigma_{n-3}(\kappa|1i_m)[\kappa_m\sigma_{n-m-2}(\kappa|1i_1 \cdots i_m) + \sigma_{n-m-1}(\kappa|1i_1 \cdots i_m)]$$

$$= \sigma_{n-2}(\kappa|1)\sigma_{n-m-2}(\kappa|1i_1 \cdots i_m) - \sigma_{n-2}(\kappa|1i_m)\sigma_{n-m-2}(\kappa|1i_1 \cdots i_m)$$

$$+ \sigma_{n-3}(\kappa|1i_m)\sigma_{n-m-1}(\kappa|1i_1 \cdots i_m),$$
\[ \sigma_{n-3}(\kappa|1_i\_m) - \sum_{l=1}^{m-1} \sigma_{n-3}(\kappa|1_i\_m i_l) \]

\[ = \kappa_i \sigma_{n-4}(\kappa|1_i\_1 i_m) + \sigma_{n-3}(\kappa|1_i\_1 i_m) - \sum_{l=1}^{m-1} \sigma_{n-3}(\kappa|1_i\_m i_l) \]

\[ = \kappa_i \sigma_{n-4}(\kappa|1_i\_1 i_m) - \sum_{l=2}^{m-1} \sigma_{n-3}(\kappa|1_i\_m i_l) \]

(3.8)

\[ = \kappa_i \sigma_{n-4}(\kappa|1_i\_1 i_m) - \kappa_i \sum_{l=2}^{m-1} \sigma_{n-4}(\kappa|1_i\_1 i_m i_l) \]

\[ = \kappa_i \kappa_{i2} \sigma_{n-5}(\kappa|1_i\_1 i_2 i_m) + \kappa_i \sigma_{n-4}(\kappa|1_i\_1 i_2 i_m) \]

\[ - \kappa_i \sum_{l=2}^{m-1} \sigma_{n-4}(\kappa|1_i\_1 i_m i_l) \]

\[ = \kappa_i \kappa_{i2} \sigma_{n-5}(\kappa|1_i\_1 i_2 i_m) - \kappa_i \kappa_{i2} \sum_{l=3}^{m-1} \sigma_{n-5}(\kappa|1_i\_1 i_2 i_m i_l) \]

\[ = \ldots \]

\[ = \kappa_i \kappa_{i3} \cdots \kappa_{i_{m-1}} \sigma_{n-2}(\kappa|1_i\cdots i_m). \]

Then using (3.8), we have

\[ \sigma_{n-3}(\kappa|1_i\_m) \sigma_{n-2}(\kappa|1_i\cdots i_m) \]

\[ - \sum_{l=1}^{m-1} \sigma_{n-3}(\kappa|1_i\_m i_l) \sigma_{n-2}(\kappa|1_i\cdots i_m) \]

(3.9)

\[ = \kappa_i \cdots \kappa_{i_{m-1}} \sigma_{n-2}(\kappa|1_i\cdots i_m) \sigma_{n-1}(\kappa|1_i\cdots i_m) \]

\[ = \sigma_{n-2}(\kappa|1_i\cdots i_m) \sigma_{n-1}(\kappa|1_i\cdots i_m) \]

Inserting (3.7) and (3.9) into (3.6), we obtain

\[ D_m(i_1\cdots i_m) = \sigma_{n-2}^{m-1}(\kappa|1) \sigma_{n-2}(\kappa|1_i\cdots i_m). \]

Now, let’s prove the identity (3.5). In what following, \( \hat{B}_{m-1}^l(i_1\cdots i_m;i_k) \) denotes the cofactor of the element \( a_{i_1 i_m} \) in \( B_{m-1}(i_1\cdots i_m;i_k) \) and \( \hat{\cdot} \) means that the index \( i \) does not appear. Expanding the last column of the determinant
Using (3.11) \( B_{m-1}(i_1 \cdots i_m; i_k) \), we have

\[
B_{m-1}(i_1 \cdots i_m; i_k) = \sum_{1 \leq l < k} (-1)^{m-1+l} \left[ -\sigma_{n-3}(\kappa|11i_m) \right] B_{m-2}^l(i_1 \cdots i_{m-1}; i_k) \\
+ \sum_{m-1 \leq l > k} (-1)^{m-1+l} \left[ -\sigma_{n-3}(\kappa|11i_m) \right] B_{m-2}^l(i_1 \cdots i_{m-1}; i_k) \\
+ (-1)^{m-1+k} \left[ -\sigma_{n-3}(\kappa|1i_k i_m) \right] D_{m-2}(i_1 \cdots \hat{i}_k \cdots i_{m-1}).
\]

(3.10)

Interchanging the adjacent columns in turn from the \( l \)-th column in \( \tilde{B}_{m-2}^l(i_1 \cdots i_{m-1}; i_k) \), we will get \( B_{m-2}(i_1 \cdots \hat{i}_l \cdots i_{m-1}1i_l; i_k) \). Therefore, the explicit relation between \( \tilde{B}_{m-2}^l(i_1 \cdots i_{m-1}; i_k) \) and \( B_{m-2}(i_1 \cdots \hat{i}_l \cdots i_{m-1}1i_l; i_k) \) is

\[
\tilde{B}_{m-2}^l(i_1 \cdots i_{m-1}; i_k) = (-1)^{m-2-l} B_{m-2}(i_1 \cdots \hat{i}_l \cdots i_{m-1}1i_l; i_k)
\]

for \( 1 \leq l < k \) and

\[
\tilde{B}_{m-2}^l(i_1 \cdots i_{m-1}; i_k) = (-1)^{m-1-l} B_{m-2}(i_1 \cdots \hat{i}_l \cdots i_{m-1}1i_l; i_k)
\]

for \( k < l \leq m-1 \). Then the equality (3.10) becomes

\[
B_{m-1}(i_1 \cdots i_m; i_k) = \sum_{1 \leq l < k} \sigma_{n-3}(\kappa|1i_l i_m) B_{m-2}(i_1 \cdots \hat{i}_l \cdots i_{m-1}1i_l; i_k) \\
- \sum_{m-1 \leq l > k} \sigma_{n-3}(\kappa|1i_l i_m) B_{m-2}(i_1 \cdots \hat{i}_l \cdots i_{m-1}1i_l; i_k) \\
+ (-1)^{m+k} \sigma_{n-3}(\kappa|1i_k i_m) D_{m-2}(i_1 \cdots \hat{i}_k \cdots i_{m-1})
\]

(3.11)

\[
= \sum_{1 \leq l < k} \sigma_{n-3}(\kappa|1i_l i_m)(-1)^{m-1+k-1} \sigma_{n-2}^{m-3}(\kappa|1) \sigma_{n-m}(\kappa|1i_1 \cdots i_{m-1}) \\
- \sum_{m-1 \leq l > k} \sigma_{n-3}(\kappa|1i_l i_m)(-1)^{m-1+k} \sigma_{n-2}^{m-3}(\kappa|1) \sigma_{n-m}(\kappa|1i_1 \cdots i_{m-1}) \\
+ (-1)^{m+k} \sigma_{n-3}(\kappa|1i_k i_m) \sigma_{n-2}^{m-3}(\kappa|1) \sigma_{n-m}(\kappa|1i_1 \cdots \hat{i}_k \cdots i_{m-1}).
\]

Using

\[
\sigma_{n-m}(\kappa|1i_1 \cdots \hat{i}_k \cdots i_{m-1}) = \kappa_k \sigma_{n-m-1}(\kappa|1i_1 \cdots i_{m-1}) + \sigma_{n-m}(\kappa|1i_1 \cdots i_{m-1}),
\]

\( \sigma_{n-m}(\kappa|1i_1 \cdots i_{m-1}) \).
the identity (3.11) becomes

\[ B_{m-1}(i_1 \cdots i_m; i_k) \]
\[ = (-1)^{m+k} \sigma_n^{m-3}(\kappa|1) \left[ \sigma_{n-2}(\kappa|1) \sigma_{n-m-1}(\kappa|1) \cdots \right. \]
\[ + \sum_{l=1}^{m-1} \sigma_{n-3}(\kappa|1) \sigma_{n-m}(\kappa|1) \right] \). 

(3.12)

It is clear that

\[ \sigma_{n-2}(\kappa|1) \sigma_{n-m-1}(\kappa|1) \cdots \]
\[ + \sum_{l=1}^{m-1} \sigma_{n-3}(\kappa|1) \sigma_{n-m}(\kappa|1) \]
\[ = \sigma_{n-3}(\kappa|1) \sigma_{n-m}(\kappa|1) \cdots \]
\[ + \sum_{l=1}^{m-1} \sigma_{n-3}(\kappa|1) \sigma_{n-m}(\kappa|1) \]
\[ = \sigma_{n-3}(\kappa|1) \sigma_{n-m}(\kappa|1) \]
\[ + \sum_{l=1}^{m-1} \sigma_{n-3}(\kappa|1) \sigma_{n-m}(\kappa|1) \]
\[ = \sigma_{n-3}(\kappa|1) \sigma_{n-m}(\kappa|1) \]
\[ + \sum_{l=1}^{m-1} \sigma_{n-3}(\kappa|1) \sigma_{n-m}(\kappa|1) \]
\[ = \sigma_{n-4}(\kappa|1) \sigma_{n-m+1}(\kappa|1) \]
\[ + \sum_{l=2}^{m-1} \sigma_{n-4}(\kappa|1) \sigma_{n-m+1}(\kappa|1) \]
\[ = \sigma_{n-4}(\kappa|1) \sigma_{n-m+1}(\kappa|1) \]
\[ + \sum_{l=3}^{m-1} \sigma_{n-4}(\kappa|1) \sigma_{n-m+1}(\kappa|1) \]
\[ = \cdots \]
\[ = \sigma_{n-m-1}(\kappa|1) \sigma_{n-2}(\kappa|1) \]

(3.13)

Inserting (3.13) into (3.12), we obtain

\[ B_{m-1}(i_1 \cdots i_m; i_k) = (-1)^{m+k} \sigma_n^{m-2}(\kappa|1) \sigma_{n-m-1}(\kappa|1) \sigma_{n-m}(\kappa|1) \sigma_{n-m+1}(\kappa|1) \]

(3.14)

The above lemma implies the non-negativity of the following two quadratic forms.
LEMMA 10. The following two quadratic forms are nonnegative in $\Gamma_{n-1}$

\begin{equation}
\sum_{j \neq i} \sigma_{n-3}^2(\kappa_{ij})h_{jji}^2 + \sum_{p,q \neq i, p \neq q} \sigma_{n-3}^2(\kappa_{ipq})h_{ppi}h_{qqi} \geq 0,
\end{equation}

and,

\begin{equation}
\sum_{j \neq i} \sigma_{n-3}^2(\kappa_{ij})h_{jji}^2 - \sum_{p,q \neq i, p \neq q} \sigma_{n-3}^2(\kappa_{ipq})h_{ppi}h_{qqi} \geq 0,
\end{equation}

where $h_{ijk}$ means the covariant derivatives of the second fundamental form.

Let’s give more explanation of the above theorem. The non-negativity of (3.15) comes from Lemma 9. The Hadamard product of two matrices produces another matrix where each element $ij$ is the product of elements $ij$ of the original two matrices. For example, for matrices $B = (b_{ij}), C = (c_{ij})$, the Hadamard product of $B, C$ is a new matrix $(b_{ij}c_{ij})$. Obviously, the determined matrix by the first quadric form is the Hadamard product of the matrix $(a_{pq})$ and its self defined by (3.3). Thus, the non-negativity of (3.15) yields the non-negativity of (3.14) by the Schur product theorem [36].

Next we will prove the key inequality of this section. We will use the Einstein conversion from now on.

THEOREM 11. Given two positive constants $\delta < 1, \varepsilon$, for any index $i$, if $\kappa_i \geq \delta \kappa_1$ and the positive constant $K$ depending only on $\delta$ and $\varepsilon$ is sufficiently large, then we have

\begin{equation}
\kappa_i \left[ K \left( \sigma_{n-1} \right)_i^2 - \sigma_{n-1}^{pq} h_{ppi}h_{qqi} \right] - \sigma_{n-1}^{ii} h_{iii}^2 + (1 + \varepsilon) \sum_{j \neq i} \sigma_{n-1}^{jj} h_{jji}^2 \geq 0.
\end{equation}

Proof. A straightforward calculation shows

\begin{equation}
\kappa_i \left[ K \left( \sigma_{n-1} \right)_i^2 - \sigma_{n-1}^{pq} h_{ppi}h_{qqi} \right] - \sigma_{n-1}^{ii} h_{iii}^2 + (1 + \varepsilon) \sum_{j \neq i} \sigma_{n-1}^{jj} h_{jji}^2
\end{equation}

\begin{align*}
&= \kappa_i K \left( \sum_{j \neq i} \sigma_{n-1}^{jj} h_{jji} \right)^2 + 2 \kappa_i \sigma_{n-1}^{ii} \left[ \sum_{j \neq i} (K \sigma_{n-1}^{jj} - \sigma_{n-1}^{ii} h_{jji}^2) \right] \\
&+ \left[ \kappa_i K \left( \sigma_{n-1}^{ii} \right)^2 - \sigma_{n-1}^{ii} \right] h_{iii}^2 + (1 + \varepsilon) \sum_{j \neq i} \sigma_{n-1}^{jj} h_{jji}^2 - \kappa_i \sum_{p \neq i; q \neq i} \sigma_{n-1}^{pq} h_{ppi}h_{qqi} \\
&\geq \kappa_i K \left( \sum_{j \neq i} \sigma_{n-1}^{jj} h_{jji} \right)^2 - \frac{\kappa_i^2 \left[ \sum_{j \neq i} (K \sigma_{n-1}^{jj} - \sigma_{n-1}^{ii} h_{jji}^2) \right]^2}{\kappa_i K \left( \sigma_{n-1}^{ii} \right)^2 - \sigma_{n-1}^{ii}} \\
&+ (1 + \varepsilon) \sum_{j \neq i} \sigma_{n-1}^{jj} h_{jji}^2 - \kappa_i \sum_{p \neq i; q \neq i} \sigma_{n-1}^{pq} h_{ppi}h_{qqi}.
\end{align*}
\[
\begin{align*}
\kappa_i K (\sigma_{n-1}^{i j})^2 - \kappa_i^2 \left( \frac{K \sigma_{n-1}^{i i} \sigma_{n-1}^{j j} - \sigma_{n-1}^{i i, j j}}{\kappa_i K (\sigma_{n-1}^{i i})^2 - \sigma_{n-1}^{i i}} \right)^2 + (1 + \varepsilon) \sigma_{n-1}^{j j} \right] h_{j j i}^2 \\
+ \sum_{p, q \neq i; p \neq q} \left[ \kappa_i K \sigma_{n-1}^{p p} \sigma_{n-1}^{q q} - \kappa_i^2 \left( \frac{K \sigma_{n-1}^{i i} \sigma_{n-1}^{p p} - \sigma_{n-1}^{i i, p p}}{\kappa_i K (\sigma_{n-1}^{i i})^2 - \sigma_{n-1}^{i i}} \right) \right] \kappa_i \sigma_{n-1}^{p p, q q} \left[ \sum_{i \neq j} \kappa_i K \sigma_{n-1}^{i i, j j} \right] h_{p p i} h_{q q i},
\end{align*}
\]

where, in the second inequality, we have used

\[
\frac{\kappa_i^2 \left[ \sum_{j \neq i} \left( K \sigma_{n-1}^{i i} \sigma_{n-1}^{j j} - \sigma_{n-1}^{i i, j j} \right) h_{j j i} \right]^2}{\kappa_i K (\sigma_{n-1}^{i i})^2 - \sigma_{n-1}^{i i}} + 2 \kappa_i h_{i i i} \left[ \sum_{j \neq i} \left( K \sigma_{n-1}^{i i} \sigma_{n-1}^{j j} - \sigma_{n-1}^{i i, j j} \right) h_{j j i} \right] + \left[ \kappa_i K (\sigma_{n-1}^{i i})^2 - \sigma_{n-1}^{i i} \right] h_{i i i} \geq 0.
\]

Note that we have

\[
K \kappa_i \sigma_{n-1}^{i i} - 1 \geq K \delta \kappa_i \sigma_{n-1}^{i i} - 1 > 0,
\]

if the positive constant \( K \) is sufficiently large. Thus, we can multiple the term \( \kappa_i K (\sigma_{n-1}^{i i})^2 - \sigma_{n-1}^{i i} \) in (3.17). Then, we get

\[
(3.18)
\]

\[
\begin{align*}
\kappa_i K (\sigma_{n-1}^{i i})^2 - \sigma_{n-1}^{i i} \\
\times \left\{ \kappa_i \left[ K (\sigma_{n-1}^{i i})^2 - \sigma_{n-1}^{p p, q q} h_{p p i} h_{q q i} \right] - \sigma_{n-1}^{i i} h_{i i i}^2 + (1 + \varepsilon) \sum_{j \neq i} \sigma_{n-1}^{j j} h_{j j i}^2 \right\}
\end{align*}
\]

\[
\begin{align*}
\geq \sum_{j \neq i} \left[ \kappa_i K \sigma_{n-1}^{i i, j j} \left( - \sigma_{n-1}^{j j} + 2 \kappa_i \sigma_{n-1}^{i i, j j} + (1 + \varepsilon) \sigma_{n-1}^{i i} \right) \\
- \kappa_i^2 \left( \sigma_{n-1}^{i i, j j} \right)^2 \sigma_{n-1}^{j j} \right] h_{j j i}^2 \\
+ \sum_{p, q \neq i; p \neq q} \left[ \kappa_i K \sigma_{n-1}^{i i} \left( \kappa_i (\sigma_{n-1}^{p p} \sigma_{n-1}^{i i, q q} + \sigma_{n-1}^{q q} \sigma_{n-1}^{i i, p p} - \sigma_{n-1}^{i i} \sigma_{n-1}^{p p, q q}) - \sigma_{n-1}^{p p} \sigma_{n-1}^{q q} \right) \\
- \kappa_i^2 \sigma_{n-1}^{i i, p p} \sigma_{n-1}^{i i, q q} + \kappa_i \sigma_{n-1}^{i i} \sigma_{n-1}^{p p, q q} \right] h_{p p i} h_{q q i}.
\end{align*}
\]
Using the identities in Lemma 8, we have

\[
(k_i K (\sigma_{n-1}^{ii})^2 - \sigma_{n-1}^{ii})
\]

\[
\times \left\{ k_i [K(\sigma_{n-1})^2 - \sigma_{n-1}^{pq} h_{ppi} h_{qqi}] - \sigma_{n-1}^{ii} h_{iij}^2 + (1 + \varepsilon) \sum_{j \neq i} \sigma_{n-1}^{jj} h_{jjj}^2 \right\}
\]

\[\geq \sum_{j \neq i} \left[ k_i K \sigma_{n-1}^{ii} (k_i \sigma_{n-3}^2 (k_i i j) + \sigma_{n-3} (k_i i j) \sigma_{n-1} + \varepsilon \sigma_{n-3}^{jj} \sigma_{n-1}^{ii})
\]

\[\quad - k_i^2 (\sigma_{n-1}^{ii} j j)^2 - (1 + \varepsilon) \sigma_{n-1}^{ii} \sigma_{n-1}^{jj} h_{jjj}^2
\]

\[+ \sum_{p, q \neq i, p \neq q} \left[ k_i K \sigma_{n-1}^{ii} (k_i \sigma_{n-3}^2 (k_i i p q) - \sigma_{n-1} \sigma_{n-3} (k_i i p q))
\]

\[\quad - k_i^2 \sigma_{n-3}^2 \sigma_{n-3} (k_i i p q) + k_i \sigma_{n-1} \sigma_{n-3} (k_i i p q) h_{ppi} h_{qqi}
\]

\[= \sum_{j \neq i} \left[ (k_i K \sigma_{n-1}^{ii} - 1) k_i \sigma_{n-3}^2 (k_i i j) + k_i K \sigma_{n-1}^{ii} \sigma_{n-3} (k_i i j) \sigma_{n-1}
\]

\[\quad + (k_i K \sigma_{n-1}^{ii} \varepsilon - (1 + \varepsilon)) \sigma_{n-1}^{ii} \sigma_{n-1}^{jj} h_{jjj}^2
\]

\[+ \sum_{p, q \neq i, p \neq q} \left[ (k_i K \sigma_{n-1}^{ii} - 1) k_i \sigma_{n-3}^2 (k_i i p q)
\]

\[\quad - k_i K \sigma_{n-1}^{ii} \left( \sigma_{n-1} - \frac{1}{K} \right) \sigma_{n-3} (k_i i p q) h_{ppi} h_{qqi}
\]

\[\geq \sum_{j \neq i} \left[ (k_i K \sigma_{n-1}^{ii} - 1) k_i \sigma_{n-3}^2 (k_i i j) + k_i K \sigma_{n-1}^{ii} \sigma_{n-3} (k_i i j) \left( \sigma_{n-1} - \frac{1}{K} \right) h_{jjj}^2
\]

\[+ \sum_{p, q \neq i, p \neq q} \left[ (k_i K \sigma_{n-1}^{ii} - 1) k_i \sigma_{n-3}^2 (k_i i p q)
\]

\[\quad - k_i K \sigma_{n-1}^{ii} \left( \sigma_{n-1} - \frac{1}{K} \right) \sigma_{n-3} (k_i i p q) h_{ppi} h_{qqi}.
\]

Here, the last inequality holds if the positive constant \( K \) is sufficiently large. By Lemma 10, the above quadratic form is nonnegative. We have completed our proof. \( \square \)

As the end of this section, we give a counterexample which says that our inequality (3.16) does not hold in general. Let’s consider the second elementary symmetric function \( \sigma_2 \) in a 4 dimensional space as an example. For the parameter \( t > 0 \), we suppose the eigenvalues \( \kappa_1, \kappa_2, \kappa_3, \kappa_4 \) are

\[ \kappa_1 = 2 t + \frac{1}{t}, \kappa_2 = 2 t, \kappa_3 = 0, \text{ and } \kappa_4 = - t. \]
Then, a straightforward calculation shows
\[\sigma_{11}^2 = t, \quad \sigma_{22}^2 = t + \frac{1}{t}, \quad \sigma_{33}^2 = 3t + \frac{1}{t}, \quad \sigma_{44}^2 = 4t + \frac{1}{t}, \quad \sigma_2 = 1.\]

Thus, \((\kappa_1, \kappa_2, \kappa_3, \kappa_4)\) is in the cone \(\Gamma_2\) for \(t > 0\). If the quadratic form (3.16) is nonnegative for any \(\epsilon\), the corresponding coefficient matrix is semi positive definite. Therefore, let's calculate the determinant of the coefficient matrix for \(i = 1, \varepsilon = 1/5\) in (3.16), which is equal to
\[
\frac{46260K - 48528}{125} + \frac{363K - 363}{25} + \frac{15136K - 15356}{125t^2} + \frac{(60360K - 67704)t^2}{125} + \frac{(25760K - 34672)t^4}{125} - \frac{3936Kt^6}{125}.
\]
Obviously, it is negative if \(t\) is sufficiently large, which yields that (3.16) is not always nonnegative for the above example.

4. The global curvature estimates. In this section, we will derive the global \(C^2\)-estimates for the curvature equation with \(k = n - 1\).

Denote \(X, \nu\) to be the position vector and outer normal vector of \(M\). Set \(u(X) = \langle X, \nu(X) \rangle\), where \(\langle \cdot, \cdot \rangle\) denotes the inner product of the ambient space. By the assumption that \(M\) is a starshaped hypersurface with a \(C^1\) bound, \(u\) is bounded from below and above by two positive constants. At every point in the hypersurface \(M\), choose a local coordinate frame \(\{\partial/\partial x_1, \ldots, \partial/\partial x_{n+1}\}\) in \(\mathbb{R}^{n+1}\) such that the first \(n\) vectors are the local coordinates of the hypersurface and the last one is the unit outer normal vector. We let \(h_{ij}\) be the second fundamental form of the hypersurface \(M\). The following geometric formulas are well known (e.g., [22]),
\[
(4.1) \quad h_{ij} = \langle \partial_i X, \partial_j \nu \rangle,
\]
and
\[
(4.2) \begin{align*}
X_{ij} &= -h_{ij} \nu & \text{(Gauss formula)} \\
(\nu)_i &= h_{ij} \partial_j & \text{(Weigarten equation)} \\
h_{ijk} &= h_{ikj} & \text{(Codazzi formula)} \\
R_{ijkl} &= h_{ik}h_{jl} - h_{il}h_{jk} & \text{(Gauss equation)},
\end{align*}
\]
where \(R_{ijkl}\) is the \((4,0)\)-Riemannian curvature tensor. We also have
\[
(4.3) \quad h_{ijkl} = h_{ijk} + h_{mj}R_{imlk} + h_{im}R_{jmlk} = h_{klij} + (h_{mj}h_{il} - h_{ml}h_{ij})h_{mk} + (h_{mj}h_{kl} - h_{ml}h_{kj})h_{mi}.
\]
For the function \(u\), we consider the following test function which appears first in [24],
\[
\phi = \log \log P - N \ln u.
\]
Here the function $P$ is defined by

$$P = \sum_{i} e^{\kappa_i}.$$  

We may assume that the maximum of $\phi$ is achieved at some point $X_0 \in M$. By a proper rotation of the coordinates, we may assume the matrix $(h_{ij})$ is diagonal at that point, and we can further assume that $h_{11} \geq h_{22} \cdots \geq h_{nn}$. Since $\kappa_1, \kappa_2, \ldots, \kappa_n$ denote the principal curvatures of $M$, then we have $\kappa_i = h_{ii}$.

Covariant differentiating the function $\phi$ twice at $X_0$, we have

*(4.4)* \begin{equation}
\phi_i = \frac{P_i}{\log P} - N \frac{h_{ii}(X, \partial_i)}{u} = 0,
\end{equation}

and

\begin{align*}
\phi_{ii} &= \frac{P_{ii}}{\log P} - \frac{P_i^2}{P^2 \log P} - \frac{P^2_i}{(P \log P)^2} - \frac{N}{u} \sum_{l} h_{il,i}(\partial_l, X) \\
&\quad - \frac{N h_{ii}}{u} + N h_{ii}^2 + N \frac{h_{ii}(X, \partial_i)^2}{u^2} \\
&= \frac{1}{P \log P} \left[ \sum_l e^{\kappa_l} h_{ll,ii} + \sum_l e^{\kappa_l} h_{l,l}^2 + \sum_{\alpha \neq \beta} \frac{e^{\kappa_{\alpha}} - e^{\kappa_{\beta}}}{\kappa_{\alpha} - \kappa_{\beta}} h_{\alpha,\beta i}^2 - \left( \frac{1}{P} + \frac{1}{P \log P} \right) P_i^2 \right] \\
&\quad - \frac{N \sum_l h_{il,i}(\partial_l, X)}{u} - \frac{N h_{ii}}{u} + N h_{ii}^2 + N \frac{h_{ii}(X, \partial_i)^2}{u^2} \\
&= \frac{1}{P \log P} \left[ \sum_l e^{\kappa_l} h_{ii,il} + \sum_l e^{\kappa_l} (h_{il}^2 - h_{ii}h_{il}) h_{ii} + \sum_l e^{\kappa_l} (h_{ii}h_{ll} - h_{il}^2) h_{il} \\
&\quad + \sum_l e^{\kappa_l} h_{ill}^2 + \sum_{\alpha \neq \beta} \frac{e^{\kappa_{\alpha}} - e^{\kappa_{\beta}}}{\kappa_{\alpha} - \kappa_{\beta}} h_{\alpha,\beta i}^2 - \left( \frac{1}{P} + \frac{1}{P \log P} \right) P_i^2 \right] \\
&\quad - \frac{N \sum_l h_{il,i}(\partial_l, X)}{u} - \frac{N h_{ii}}{u} + N h_{ii}^2 + N \frac{h_{ii}(X, \partial_i)^2}{u^2}.
\end{align*}

Contracting with $\sigma_{n-1}^i$, we have

*(4.5)* \begin{equation}
\sigma_{n-1}^i \phi_{ii} \\
= \frac{1}{P \log P} \left[ \sum_l e^{\kappa_l} \sigma_{n-1}^i h_{ii,il} + (n-1) \psi \sum_l e^{\kappa_l} h_{il}^2 - \sigma_{n-1}^i h_{il}^2 \sum_l e^{\kappa_l} h_{il} \\
&\quad + \sum_l \sigma_{n-1} h_{il}^2 + \sum_{\alpha \neq \beta} \frac{e^{\kappa_{\alpha}} - e^{\kappa_{\beta}}}{\kappa_{\alpha} - \kappa_{\beta}} h_{\alpha,\beta i}^2 - \left( \frac{1}{P} + \frac{1}{P \log P} \right) \sigma_{n-1}^i P_i^2 \right] \\
&\quad - \frac{N \sum_l \sigma_{n-1}^i h_{il,i}(\partial_l, X)}{u} - \frac{N(n-1) \psi}{u} + N \sigma_{n-1}^i h_{ii}^2 + N \frac{\sigma_{n-1}^i (X, \partial_i)^2}{u^2}.
\end{equation}
At $X_0$, differentiating the equation (1.1) twice, we have

\begin{equation}
\sigma_{n-1}^{ii} h_{iik} = d_X \psi(\partial_k) + h_{kk} d_{\nu} \psi(\partial_k),
\end{equation}

and

\begin{equation}
\sigma_{n-1}^{ji} h_{iikk} + \sigma_{n-1}^{pq,rs} h_{pqkl} h_{rsnk} \geq -C - Ch_{11}^2 + \sum_l h_{lkk} d_{\nu} \psi(\partial_l),
\end{equation}

where $C$ is some uniform constant.

Inserting (4.7) into (4.5), we have

\begin{equation}
\sigma_{n-1}^{ii} \phi_{ii} \geq \frac{1}{P \log P} \left[ \sum_l e^{\kappa_l} \left( -C - Ch_{11}^2 - \sigma_{n-1}^{pq,rs} h_{pqkl} h_{rsnk} \right) + \sum_l e^{\kappa_l} h_{lkk} d_{\nu} \psi(\partial_l) 
\right.
\end{equation}

\begin{equation}
\left. + (n-1) \psi \sum_l e^{\kappa_l} h_{ll}^2 - \sigma_{n-1}^{ii} h_{ll}^2 \right] + \sum_{\alpha \neq \beta} \sigma_{n-1}^{ii} e^{\kappa_\alpha - \kappa_\beta} h_{\alpha \beta i}^2
\end{equation}

\begin{equation}
+ \frac{1}{P} \left( \frac{1}{P \log P} \right) \frac{\sigma_{n-1}^{ii} P^2}{P \log P}
\end{equation}

\begin{equation}
- N \sum_l \sigma_{n-1}^{ii} h_{il} \langle \partial_l, X \rangle \frac{1}{u} - N(n-1) \psi \frac{1}{u} + N \sigma_{n-1}^{ii} h_{ii}^2 + N \frac{\sigma_{n-1}^{ii} h_{ll}^2(\partial_i) X, \partial_i}{u^2}.
\end{equation}

By (4.4) and (4.6), we have

\begin{equation}
\sum_k d_{\nu} \psi(\partial_k) \frac{e^{\kappa_{lkk}}}{P \log P} - \frac{N}{u} \sum_k \sigma_{n-1}^{ii} h_{iik} \langle \partial_k, X \rangle = - \frac{N}{u} \sum_k d_X \psi(\partial_k) \langle X, \partial_k \rangle.
\end{equation}

Denote

\begin{equation}
A_i = e^{\kappa_i} \left[ K(\sigma_{n-1})^2 \right] - \sum_{p \neq q} \sigma_{n-1}^{pp,qq} h_{ppi} h_{qqi}, \quad B_i = 2 \sum_{l \neq i} \sigma_{n-1}^{ii,ll} e^{\kappa_i} h_{ll}^2,
\end{equation}

\begin{equation}
C_i = \sigma_{n-1}^{ii} \sum_l e^{\kappa_l} h_{ll}^2; \quad D_i = 2 \sum_{l \neq i} \sigma_{n-1}^{ll} e^{\kappa_i - \kappa_l} h_{ll}^2; \quad E_i = \frac{1 + \log P}{P \log P} \frac{\sigma_{n-1}^{ii} P^2}{P \log P}.
\end{equation}

Using

\begin{equation}
- \sum_l \sigma_{n-1}^{pq,rs} h_{pqkl} h_{rsnk} = \sum_{p \neq q} \sigma_{n-1}^{pp,qq} h_{pql}^2 - \sum_{p \neq q} \sigma_{n-1}^{pp,qq} h_{ppl} h_{qql},
\end{equation}
and (4.8), for any $K > 1$, we have

\[
\sigma_{n-1}^{ii} \phi_{ii} \geq \frac{1}{P \log P} \left[ \sum_l e^{\kappa l} \left( K (\sigma_{n-1})^2_l - \sum_{p \neq q} \sigma_{n-1}^{pp,qq} h_{ppl} h_{qql} + \sum_{p \neq q} \sigma_{n-1}^{pp,qq} h_{pql}^2 \right) \right.
\]
\[
+ \sum_l \sigma_{n-1}^{ii} e^{\kappa_l} h_{ii}^2 + \sum_{\alpha \neq \beta} \sigma_{n-1}^{ii} e^{\kappa_\alpha} - e^{\kappa_\beta} h_{\alpha \beta i}^2
\]
\[
- \frac{1 + \log P}{P \log P} \sigma_{n-1}^{ii} P_i^2 - CP - CK P h_{11}^2 \right]
\]
\[
+ (N - 1) \sigma_{n-1}^{ii} h_{ii}^2 + N \sigma_{n-1}^{ii} h_{ii}^2 (X, \partial_i)^2 \]
\[
\geq \frac{1}{P \log P} \left[ A_i + B_i + C_i + D_i - E_i \right]
\]
\[
+ (N - 1) \sigma_{n-1}^{ii} h_{ii}^2 + N \sigma_{n-1}^{ii} h_{ii}^2 (X, \partial_i)^2 - \frac{C + CK h_{11}^2}{\log P}
\]

We claim that

\[
0 \geq \sigma_{n-1}^{ii} \phi_{ii}
\]

\[
\geq \frac{1}{P \log P} \sum_i (A_i + B_i + C_i + D_i - E_i)
\]
\[
+ (N - 1) \sigma_{n-1}^{ii} h_{ii}^2 + N \sigma_{n-1}^{ii} h_{ii}^2 (X, \partial_i)^2 - \frac{C + CK h_{11}^2}{\log P}
\]
\[
\geq (N - 1) c_0 h_{11} - \frac{C + CK h_{11}^2}{\log P}
\]

Here we have used

\[
\sigma_{n-1}^{11} h_{11} \geq c_0,
\]

where $c_0$ is some positive constant only depending on $n, k$. Choosing a sufficiently large positive constant $N$, we get an upper bound of $h_{11}$.

For $n = 2$, the equation is a quasi linear elliptic equation. The $C^2$ estimate is well known. In a word, we complete the proof of Theorem 2.

Next, we will divide into three cases to prove our claim (4.11). At first, we need to prove the following lemma.
Thus we divide the proof of (4.12) into four cases to discuss.

(4.13)  

$$e^{\kappa_l} \sigma_{k-2}(\kappa|i) + (1 + \varepsilon_T) \frac{e^{\kappa_l} - e^{\kappa_i}}{\kappa_l - \kappa_i} \sigma_{k-1}(\kappa|i) \geq e^{\kappa_i} \sigma_{k-1}(\kappa|i).$$

**Proof.** It is clear that we have the following identity

$$\sigma_{k-1}(\kappa|i) = \sigma_{k-1}(\kappa|i) + (\kappa_i - \kappa_l) \sigma_{k-2}(\kappa|i).$$

Multiplying $\frac{e^{\kappa_l} - e^{\kappa_i}}{\kappa_l - \kappa_i}$ in the both sides of the above identity, we have

(4.13)  

$$e^{\kappa_l} \sigma_{k-2}(\kappa|i) + \frac{e^{\kappa_l} - e^{\kappa_i}}{\kappa_l - \kappa_i} \sigma_{k-1}(\kappa|i) = e^{\kappa_l} e^{\kappa_i-\kappa_l} - 1 \frac{e^{\kappa_i} e^{\kappa_l} - 1}{\kappa_l - \kappa_i} \sigma_{k-1}(\kappa|i) \geq e^{\kappa_l} \sigma_{k-1}(\kappa|i).$$

Thus we divide the proof of (4.12) into four cases to discuss.

**Case (i).** $\kappa_l \leq \kappa_i$.

In this case, we have

$$e^{\kappa_l} e^{\kappa_i-\kappa_l} - 1 \frac{e^{\kappa_i} e^{\kappa_l} - 1}{\kappa_l - \kappa_i} \sigma_{k-1}(\kappa|i) \geq e^{\kappa_i} \sigma_{k-1}(\kappa|i).$$

Hence, by (4.13), we get (4.12) if we assume that $\kappa_1$ is sufficiently large.

**Case (ii).** $0 < \kappa_l - \kappa_i \leq 1$.

In this case, obviously, we have $\kappa_i \geq \kappa_l - 1$. By the mean value theorem, there exists some constant $\kappa_i < \xi < \kappa_l$, such that

$$e^{\kappa_l} e^{\kappa_i-\kappa_l} - 1 \frac{e^{\kappa_i} e^{\kappa_l} - 1}{\kappa_l - \kappa_i} \sigma_{k-1}(\kappa|i) \geq e^{\kappa_i} \sigma_{k-1}(\kappa|i),$$

if $\kappa_1$ is sufficiently large. By (4.13), we get (4.12).

**Case (iii).** $\kappa_l - \kappa_i > 1$ and $\frac{\kappa_l}{\kappa_i} \leq \frac{1}{3}$.

Using the assumption $|\kappa_i| < \delta \kappa_1$, we have

$$\kappa_l - \kappa_i \leq \left(\delta + \frac{1}{3}\right) \kappa_1.$$  

In turn, we have

$$e^{\kappa_l} e^{\kappa_i-\kappa_l} - 1 \frac{e^{\kappa_i} e^{\kappa_l} - 1}{\kappa_l - \kappa_i} \sigma_{k-1}(\kappa|i) \geq e^{\kappa_i} \sigma_{k-1}(\kappa|i),$$

$$\frac{1 - e^{-1}}{\kappa_l - \kappa_i} \sigma_{k-1}(\kappa|i) \geq \frac{1 - e^{-1}}{\frac{1}{3} + \delta} \kappa_1 \sigma_{k-1}(\kappa|i).$$
Since the constant $\delta < \frac{1}{3}$, we get
\[
\frac{1 - e^{-1}}{\frac{1}{2} + \delta} \geq 1.
\]
Thus inserting the above two inequalities into (4.13), we get (4.12).

**Case (iv).** $\kappa_i - \kappa > 1$ and $\frac{\kappa_i}{\kappa_i} > \frac{1}{3}$.

In this case, (4.12) can be rewritten as
\[
(1 + \varepsilon_T) e^{K_i} \sigma_{k-2}(\kappa|il) + (1 + \varepsilon_T) \frac{e^{K_i} - e^{K_i}}{\kappa_i - \kappa_i} \left[ \kappa_i \sigma_{k-2}(\kappa|il) + \sigma_{k-1}(\kappa|il) \right]
\geq \frac{e^{K_i}}{\kappa_i} \left[ \kappa_i \sigma_{k-2}(\kappa|il) + \sigma_{k-1}(\kappa|il) \right].
\]

(4.14)

If $\sigma_{k-1}(\kappa|il) \leq 0$, obviously, we have
\[
e^{K_i} \sigma_{k-2}(\kappa|il) \geq \frac{e^{K_i}}{\kappa_i} \left[ \kappa_i \sigma_{k-2}(\kappa|il) + \sigma_{k-1}(\kappa|il) \right].
\]

Therefore (4.14) obviously holds. Thus, we can assume $\sigma_{k-1}(\kappa|il) > 0$. In order to get (4.14), we only need to prove the following two inequalities
\[
(1 + \varepsilon_T) e^{K_i} \sigma_{k-2}(\kappa|il) + (1 + \varepsilon_T) \frac{e^{K_i} - e^{K_i}}{\kappa_i - \kappa_i} \kappa_i \sigma_{k-2}(\kappa|il) \geq 0,
\]
(4.15)

and
\[
\varepsilon_T e^{K_i} \sigma_{k-2}(\kappa|il) + (1 + \varepsilon_T) \frac{e^{K_i} - e^{K_i}}{\kappa_i - \kappa_i} \kappa_i \sigma_{k-2}(\kappa|il) \geq 0.
\]
(4.16)

For (4.15), since $\sigma_{k-1}(\kappa|il) > 0$, if we have
\[
\varepsilon_T \kappa_i e^{K_i} + \kappa_i e^{K_i} - (1 + \varepsilon_T) \kappa_i e^{K_i} \geq 0,
\]
then (4.15) will hold. Using $|\kappa_i| < \delta \kappa_1$, we in fact need
\[
(\varepsilon_T - \delta) \kappa_1 e^{K_i} - (1 + \varepsilon_T) \kappa_1 e^{K_i} \geq 0,
\]
which implies the following requirement
\[
\kappa_i - \kappa_i \geq \log \left( \frac{1 + \varepsilon_T}{\varepsilon_T - \delta} \right).
\]

Since $|\kappa_i| < \delta \kappa_1$, the above assumption can be satisfied, if we require
\[
\left( \frac{1}{3} - \delta \right) \kappa_1 \geq \log \left( \frac{1 + \varepsilon_T}{\varepsilon_T - \delta} \right).
\]
Hence, if we assume $\kappa_1$ is sufficiently large, the above inequality obviously will hold.
For (4.16), we only need to prove
\[ \varepsilon_T + (1 + \varepsilon_T) \frac{1 - e^{\kappa_i - \kappa_i}}{\kappa_i - \kappa_i} \kappa_i \geq 0. \]
If \( \kappa_i \geq 0 \), it is clearly right. Thus, we only consider the case \( \kappa_i < 0 \), in which we need to require
\[ (\kappa_i - \kappa_i) \varepsilon_T \geq -(1 + \varepsilon_T) \kappa_i, \]
which implies
\[ \kappa_i \varepsilon_T \geq -\kappa_i. \]
By our assumption, \( \kappa_i \geq \frac{1}{2} \kappa_1 \), \( |\kappa_i| \leq \delta \kappa_1 \), the constants \( \delta \) and \( \varepsilon_T \) only need to satisfy
\[ \frac{\varepsilon_T}{3} \geq \delta. \]
We complete our proof. \( \square \)

Now, we are in the position to prove our claim. We choose a positive constant \( \delta \) which will be determined in the proof of the Lemmas 13-16. Then, we divide our proof of the claim into three cases to deal with every \( \kappa_i \), for \( i = 1, 2 \ldots, n \),
(a) \( |\kappa_i| \leq \delta \kappa_1 \);
(b) \( \kappa_i \geq \delta \kappa_1 \);
(c) \( -\kappa_i \geq \delta \kappa_1 \).
The first two cases will be discussed in Lemma 13 and Lemma 15. The last lemma will concern the last case.

**Lemma 13.** There exists a positive constant \( \delta < \frac{1}{2} \) such that, for any \( |\kappa_i| \leq \delta \kappa_1 \), \( 1 \leq i \leq n \), we have
\[ A_i + B_i + C_i + D_i - E_i \geq 0, \]
if the constant \( K \) and the biggest eigenvalue \( \kappa_1 \) both are sufficiently large.

**Proof.** Using Lemma 6, we first have \( A_i > 0 \), if the constant \( K \) is sufficiently large. By the Cauchy-Schwarz inequality, we have
\[
P_i^2 = e^{2\kappa_i} h_{iii}^2 + 2 \sum_{l \neq i} e^{\kappa_i + \kappa_l} h_{iii} h_{lli} + \left( \sum_{l \neq i} e^{\kappa_i} h_{lli} \right)^2 \\
\leq e^{2\kappa_i} h_{iii}^2 + 2 \sum_{l \neq i} e^{\kappa_i + \kappa_l} h_{iii} h_{lli} + (P - e^{\kappa_i}) \sum_{l \neq i} e^{\kappa_i} h_{lli}^2. \]
(4.17)
Using (4.17), we have
\[
B_i + C_i + D_i - E_i \\
\geq 2 \sum_{l \neq i} e^{\kappa_l} \sigma_{n-1}^{l,ii} h_{lli}^2 + 2 \sum_{l \neq i} \frac{e^{\kappa_l} - e^{\kappa_i}}{\kappa_l - \kappa_i} \sigma_{n-1}^{ii} h_{lli}^2 - \frac{1}{\log P} \sum_{l \neq i} e^{\kappa_i} \sigma_{n-1}^{ii} h_{lli}^2 \\
+ \frac{1 + \log P}{P \log P} \sum_{l \neq i} e^{\kappa_l + \kappa_i} \sigma_{n-1}^{ii} h_{lli}^2 + e^{\kappa_i} \sigma_{n-1}^{ii} h_{ili}^2 - \frac{1 + \log P}{P \log P} e^{2 \kappa_i} \sigma_{n-1}^{ii} h_{ili}^2 \\
- 2 \frac{1 + \log P}{P \log P} \sum_{l \neq i} e^{\kappa_l + \kappa_i} \sigma_{n-1}^{ii} h_{iii} h_{ili}.
\]

(4.18)

Using Lemma 12, there exists a positive constant \( \delta < \frac{1}{2} \), such that

\[
\frac{3}{2} \sum_{l \neq i} e^{\kappa_i} \sigma_{n-1}^{l,ii} h_{lli}^2 + \frac{3}{2} \sum_{l \neq i} \frac{e^{\kappa_l} - e^{\kappa_i}}{\kappa_l - \kappa_i} \sigma_{n-1}^{ii} h_{lli}^2 - \frac{1}{\log P} \sum_{l \neq i} e^{\kappa_i} \sigma_{n-1}^{ii} h_{lli}^2 \geq 0.
\]

On the other hand, it is clear that

\[
\sum_{l \neq i, 1} e^{\kappa_l + \kappa_i} \sigma_{n-1}^{ii} h_{lli}^2 - 2 \sum_{l \neq i, 1} e^{\kappa_l + \kappa_i} h_{iii} h_{lli} \geq - \sum_{l \neq i, 1} e^{\kappa_l + \kappa_i} \sigma_{n-1}^{ii} h_{lli}^2.
\]

Then, using the above two inequalities, (4.18) becomes

\[
B_i + C_i + D_i - E_i \\
\geq \frac{1 + \log P}{P \log P} e^{\kappa_l + \kappa_i} \sigma_{n-1}^{ii} h_{lli}^2 + e^{\kappa_i} \sigma_{n-1}^{ii} h_{ili}^2 - \frac{1 + \log P}{P \log P} \sum_{l \neq 1} e^{\kappa_l + \kappa_i} \sigma_{n-1}^{ii} h_{ili}^2 \\
- 2 \frac{1 + \log P}{P \log P} e^{\kappa_l + \kappa_i} \sigma_{n-1}^{ii} h_{iii} h_{ili} + \frac{1}{2} e^{\kappa_1} \sigma_{n-1}^{1,ii} h_{ili}^2 + \frac{1}{\kappa_1 - \kappa_i} e^{\kappa_l} - e^{\kappa_i} \sigma_{n-1}^{1,ii} h_{ili}^2.
\]

(4.21)

A straightforward calculation shows that

\[
e^{\kappa_i} \sigma_{n-1}^{ii} h_{ili}^2 - \frac{1 + \log P}{P \log P} \sum_{l \neq 1} e^{\kappa_l + \kappa_i} \sigma_{n-1}^{ii} h_{ili}^2 \geq \left( \frac{e^{\kappa_1}}{P} - \frac{1}{\log P} \right) e^{\kappa_i} \sigma_{n-1}^{ii} h_{ili}^2 \\
\geq \frac{1}{n + 1} e^{\kappa_i} \sigma_{n-1}^{ii} h_{ili}^2
\]

and

\[-2 \frac{1 + \log P}{P \log P} e^{\kappa_l + \kappa_i} \sigma_{n-1}^{ii} h_{ili} h_{ili} \geq -3 \frac{e^{\kappa_l + \kappa_i} \sigma_{n-1}^{ii} h_{ili} h_{ili}}{P} \geq -3 e^{\kappa_i} \sigma_{n-1}^{ii} |h_{ili} h_{ili}|,\]
hold at the same time if \( \kappa_1 \) is sufficiently large. We let \( l = 1, k = n - 1 \) in (4.13) and we have
\[
(4.22) \quad e^{\kappa_1} \sigma_{n-1}^{11} h_{11}^2 + \frac{e^{\kappa_1} - e^{\kappa_i}}{\kappa_1 - \kappa_i} \sigma_{n-1}^{11} h_{11}^2 = e^{\kappa_i} \sigma_{n-1}^{11} h_{11}^2 + \frac{e^{\kappa_1} - e^{\kappa_i}}{\kappa_1 - \kappa_i} \sigma_{n-1}^{ii} h_{11}^2. 
\]
By Taylor’s Theorem, we also have
\[
(4.23) \quad \frac{e^{\kappa_1} - e^{\kappa_i}}{\kappa_1 - \kappa_i} \sigma_{n-1}^{ii} h_{11}^2 = e^{\kappa_i} \sum_{m \geq 1} \frac{(\kappa_1 - \kappa_i)^{m-1}}{m!} \sigma_{n-1}^{ii} h_{11}^2. 
\]
Combining the previous four formulas and using (4.21), we obtain
\[
(4.24) \quad B_i + C_i + D_i - E_i \geq e^{\kappa_i} \sigma_{n-1}^{ii} \left[ \frac{1}{n + 1} h_{11}^2 - 3|h_{iii} h_{11i}| + \frac{1}{2} \sum_{m \geq 1} \frac{(\kappa_1 - \kappa_i)^{m-1}}{m!} h_{11i}^2 \right] \geq 0, 
\]
if \( \kappa_1 \) is sufficiently large. \( \square \)

In \( \Gamma_{n-1} \) cone, it is well known that the only possible negative eigenvalue is the smallest one. Since we have assumed that \( \kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n \), the possible non positive eigenvalue is \( \kappa_n \). Hence, we have the following lower bound of \( \kappa_n \).

**Lemma 14.** In the cone \( \Gamma_{n-1} \), if \( \kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n \) and \( \kappa_n \leq 0 \), we have
\[
-\kappa_n < \frac{\kappa_1}{n - 1}. 
\]

**Proof.** It is easy to see that
\[
\sigma_{n-1}(\kappa|n) = \kappa_1 \cdots \kappa_{n-1}, \quad \text{and} \quad \sigma_{n-2}(\kappa|1n) = \kappa_2 \cdots \kappa_{n-1}. 
\]
Assume that \( \lambda = -\kappa_n / \kappa_1 \). Thus we have
\[
(4.24) \quad \kappa_1 \cdots \kappa_{n-1} = \sigma_{n-1} - \kappa_n \sigma_{n-2}(\kappa|n) > -\kappa_n \sigma_{n-2}(\kappa|n) = \lambda \kappa_1 \sigma_{n-2}(\kappa|n) = \lambda \kappa_1^2 \sigma_{n-3}(\kappa|1n) + \lambda \kappa_1 \sigma_{n-2}(\kappa|1n). 
\]
Hence, we get
\[
(1 - \lambda) \kappa_2 \cdots \kappa_{n-1} > \lambda \kappa_1 \sigma_{n-3}(\kappa|1n) \geq (n - 2) \lambda \kappa_2 \cdots \kappa_{n-1}, 
\]
which implies \( \lambda < \frac{1}{n - 1} \). \( \square \)

**Lemma 15.** For the chosen positive constant \( \delta \) in Lemma 13, for any \( \kappa_i \geq \delta \kappa_1 \), \( 1 \leq i \leq n \) and \( n \geq 3 \), we have
\[
A_i + B_i + C_i + D_i - E_i \geq 0, 
\]
if the positive constant \( K \) and the biggest eigenvalue \( \kappa_1 \) both are sufficiently large.
\textbf{Proof.} Using (4.18), we have
\begin{align*}
A_i + B_i + C_i + D_i - E_i \\
\geq e^{\kappa_i} \left[ K \left( \sigma_{n-1} \right)^2 - \sigma_{n-1}^{pp,qq} h_{pp} h_{qq} \right] + 2 \sum_{l \neq i} e^{\kappa_l} \sigma_{n-1}^{ll,ii} h_{li}^2
- \frac{1}{\log P} \sum_{l \neq i} e^{\kappa_i} \sigma_{n-1}^{ii} h_{li}^2 + \frac{1 + \log P}{\log P} \sum_{l \neq i} e^{\kappa_l + \kappa_i} \sigma_{n-1}^{ii} h_{li}^2
+ 2 \sum_{l \neq i} e^{\kappa_l - \kappa_i} \sigma_{n-1}^{ll} h_{li}^2
+ e^{\kappa_i} \sigma_{n-1}^{ii} h_{ii}^2 - \frac{1 + \log P}{P \log P} e^{2\kappa_i} \sigma_{n-1}^{ii} h_{ii}^2
- 2 \frac{1 + \log P}{P \log P} \sum_{l \neq i} e^{\kappa_i + \kappa_l} \sigma_{n-1}^{ii} h_{li} h_{ii}.
\end{align*}
(4.25)

We claim that the following inequality holds if \( \kappa_1 \) is sufficiently large,
\begin{align*}
e^{\kappa_i} \left[ K \left( \sigma_{n-1} \right)^2 - \sigma_{n-1}^{pp,qq} h_{pp} h_{qq} \right] + 2 \sum_{l \neq i} e^{\kappa_l} \sigma_{n-1}^{ll,ii} h_{li}^2
\geq \frac{1}{\log P} e^{\kappa_i} \sigma_{n-1}^{ii} h_{ii}^2.
\end{align*}
(4.26)

In view of Theorem 11 and \( \log P > \kappa_1 \), we need to prove that, there exists a small positive constant \( \epsilon_0 = 1/6 \), such that if \( \kappa_1 \) is sufficiently large, we have
\begin{align*}
2 \frac{1 - e^{\kappa_l - \kappa_i}}{\kappa_i - \kappa_l} \kappa_1 \geq 1 + \epsilon_0,
\end{align*}
(4.27)
for all \( l \neq i \). We further need to divide the proof into three cases.

\textbf{Case (i).} \( \kappa_l \geq \kappa_i \). In this case, we obviously have
\begin{align*}
\frac{1 - e^{\kappa_l - \kappa_i}}{\kappa_i - \kappa_l} = \frac{e^{\kappa_l - \kappa_i} - 1}{\kappa_l - \kappa_i} \geq 1.
\end{align*}

It is easy to get (4.27) if \( \kappa_1 \) is sufficiently large.

\textbf{Case (ii).} \( \kappa_l - \kappa_i \geq 3 \). Using our assumption, we have
\begin{align*}
2 \frac{1 - e^{\kappa_l - \kappa_i}}{\kappa_i - \kappa_l} \kappa_1 \geq \frac{2 \kappa_1}{\kappa_i - \kappa_l} (1 - e^{-3}).
\end{align*}

Since \( 0 < \kappa_i \leq \kappa_1 \), if \( \kappa_l \geq 0 \), it is easy to see that
\begin{align*}
\frac{2 \kappa_1}{\kappa_i - \kappa_l} \geq 2.
\end{align*}
If $\kappa_l < 0$, in $\Gamma_{n-1}$, we only have one negative eigenvalue. Thus, by Lemma 14, we have

$$\frac{2\kappa_1}{\kappa_i - \kappa_l} \geq 2 \frac{n-1}{n}.$$  

Combining the previous four inequalities and $n \geq 3$, we have (4.27).

**Case (iii).** $0 < \kappa_i - \kappa_l \leq 3$. In this case, using the mean value theorem, we have

$$\frac{1 - e^{\kappa_i - \kappa_l}}{\kappa_i - \kappa_l} = \frac{e^{\kappa_i} - e^{\kappa_l}}{\kappa_i - \kappa_l} = e^{\xi} \frac{e^{\kappa_l}}{e^{\kappa_i}} \geq e^{-3},$$

where $\kappa_l \leq \xi \leq \kappa_i$. Since $\kappa_1$ can be assumed to be sufficiently large, the above inequality yields (4.27). In a word, (4.26) holds for all case.

Note that, in $\Gamma_{n-1}$, the following formulas

$$2\kappa_1\sigma_{n-3}(\kappa|i) - \sigma_{n-2}(\kappa|i) = 2\kappa_1\sigma_{n-3}(\kappa|il) - \kappa_1\sigma_{n-3}(\kappa|il) - \sigma_{n-2}(\kappa|il)$$

$$\geq \kappa_1\sigma_{n-3}(\kappa|il) - \sigma_{n-2}(\kappa|il)$$

$$= \kappa_1^2\sigma_{n-4}(\kappa|il1) + \kappa_1\sigma_{n-3}(\kappa|il1) - \sigma_{n-2}(\kappa|il)$$

$$= \kappa_1^2\sigma_{n-4}(\kappa|il1)$$

$$> 0$$

are true for all $i, l$. It implies that

$$2\kappa_1\sigma_{n-1}^{ll,ii} \geq \sigma_{n-1}^{ii}.$$  

Using the above inequality, we have

$$2 \sum_{l \neq i} e^{\kappa_l}\sigma_{n-1}^{ll,ii}h_{li}^2 - \frac{1}{\log P} \sum_{l \neq i} e^{\kappa_l}\sigma_{n-1}^{ii}h_{li}^2 \geq 0.$$  

(4.28)

On the other hand, we have

$$\frac{1 + \log P}{P\log P} \sum_{l \neq i} e^{\kappa_l+\kappa_i}\sigma_{n-1}^{ii}h_{li}^2 - 2 \frac{1 + \log P}{P\log P} \sum_{l \neq i} e^{\kappa_i+\kappa_l}\sigma_{n-1}^{ii}h_{lii}h_{lli}$$

$$\geq - \frac{1 + \log P}{P\log P} \sum_{l \neq i} e^{\kappa_l+\kappa_i}\sigma_{n-1}^{ii}h_{lli}^2.$$  

(4.29)
Inserting (4.26), (4.28) and (4.29) into (4.25), we obtain

\[
A_i + B_i + C_i + D_i - E_i \geq \frac{1}{\log P} e^{\kappa_l} \sigma_{n-1}^i h_{i i i}^2 + e^{\kappa_1} \sigma_{n-1}^2 h_{i i i}^2 - \frac{1}{P \log P} e^{2\kappa_1} \sigma_{n-1}^i h_{i i i}^2 \\
- \frac{1}{P \log P} \sum_{l \neq i} e^{\kappa_{l+\kappa_1}} \sigma_{n-1}^i h_{i i i}^2
\]

\[= 0. \]

For the last case (c), in which the smallest eigenvalue \( \kappa_n \) is negative, we have the following estimate.

**Lemma 16.** For the chosen positive constant \( \delta \) in Lemma 13, assume that \( -\kappa_n \geq \delta \kappa_1 \), and \( n \geq 3 \). Thus we also have

\[ A_n + B_n + C_n + D_n - E_n \geq 0, \]

if the positive constant \( K \) and the biggest eigenvalue \( \kappa_1 \) both are sufficiently large.

**Proof.** If the positive constant \( K \) is sufficiently large, by Lemma 6, we first have \( A_n > 0 \). Now we need to show

\[
\sum_{l \neq n} \frac{5}{3} \kappa_1 \sigma_{n-3}(\kappa|l) + \frac{5}{3} \kappa_1 \frac{1 - e^{\kappa_1 n - \kappa_l}}{\kappa_l - \kappa_n} \sigma_{n-2}(\kappa|l) - \sigma_{n-2}(\kappa|n) \geq 0
\]

for all \( l \neq n \). Since \( n \geq 3 \), by Lemma 14, we know that \( -\kappa_n < \frac{1}{3} \kappa_1 \). Therefore we have

\[
\frac{5}{3} \frac{1 - e^{\kappa_1 n - \kappa_l}}{\kappa_l - \kappa_n} > \frac{5}{3} \frac{1 - e^{-\delta \kappa_1}}{3 \kappa_1} > 1
\]

if \( \kappa_1 \) is sufficiently large. Since

\[ \sigma_{n-2}(\kappa|n) = \kappa l \sigma_{n-3}(\kappa|l) + \sigma_{n-2}(\kappa|l), \]

we have

\[
\frac{5}{3} \kappa_1 \sigma_{n-3}(\kappa|l) + \frac{5}{3} \frac{1 - e^{\kappa_1 n - \kappa_l}}{\kappa_l - \kappa_n} \sigma_{n-2}(\kappa|l) - \sigma_{n-2}(\kappa|n)
\]

\[\geq \frac{2}{3} \kappa_1 \sigma_{n-3}(\kappa|l) + \sigma_{n-2}(\kappa|l) - \sigma_{n-2}(\kappa|l)\]

\[= \frac{2}{3} \kappa_1 \sigma_{n-3}(\kappa|l) + \kappa n \sigma_{n-3}(\kappa|n) > 0.\]
Here we have used \(-\kappa_n < \frac{1}{2}\kappa_1\) and \(\sigma_{n-3}(\kappa|ln) > 0\). Thus, the inequality (4.30) has been proved.

Using (4.20), (4.30) and (4.18), we obtain

\[
B_n + C_n + D_n - E_n \\
\geq \frac{1 + \log P}{P \log P} e^{\kappa_1 + \kappa_n} \sigma_{n-1}^{nn} h_{11n}^2 + e^{\kappa_n} \sigma_{n-1}^{nn} h_{nnn}^2 \\
- \frac{1 + \log P}{P \log P} \sum_{l \neq 1} e^{\kappa_l + \kappa_n} \sigma_{n-1}^{nn} h_{nnn}^2 - 2 \frac{1 + \log P}{P \log P} e^{\kappa_n + \kappa_1} \sigma_{n-1}^{nn} h_{nnn} h_{11n} \\
+ \frac{1}{3} e^{\kappa_1} \sigma_{n-1}^{11, nn} h_{11n}^2 + \frac{1}{3} e^{\kappa_1 - \kappa_n} \sigma_{n-1}^{11} h_{11n}^2.
\]

The proof of the non-negativity of the last expression is similar to (4.21) in Lemma 13. We complete our proof. □

5. Some applications. In this section, let’s give two main applications of our curvature estimates. The first one is to prove an existence result of the prescribed \(n - 1\) curvature problem for star shaped hypersurfaces in the Euclidean space, namely Theorem 4.

**Proof of Theorem 4.** We use the continuity method to solve the existence result. For \(0 \leq t \leq 1\), according to [10], we consider a family of functions,

\[
\psi^t(X, \nu) = t\psi(X, \nu) + (1 - t)C_n^2 \left( \frac{1}{|X|^k} + \varepsilon \left( \frac{1}{|X|^k} - 1 \right) \right),
\]

where \(\varepsilon\) is a sufficiently small constant satisfying

\[
0 < \psi_0 \leq \min_{r_1 \leq \rho \leq r_2} \left( \frac{1}{\rho^k} + \varepsilon \left( \frac{1}{\rho^k} - 1 \right) \right),
\]

and \(\psi_0\) is another positive constant. The \(C^0\) and \(C^1\) estimates is same as the proof in [24]. For \(n \geq 3\), the \(C^2\) estimate derives from Theorem 2. The openness comes from [10]. By the continuity method and the Evans-Krylov theory, we obtain Theorem 4. We complete our proof. □

The proof of the Corollary 3 is similar to Theorem 2. Using the \(C^2\) estimates proved in Corollary 3 and the boundary estimates obtained in [18], we have the following existence result of the Dirichlet problem of the \(n - 1\) Hessian equations.

**Theorem 17.** Suppose \(\Omega \subset \mathbb{R}^n\) is a bounded domain with smooth boundary and \(\psi(x, u, p) \in C^2(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)\) is a positive function with \(\psi_u \geq 0\). Suppose there
is a subsolution $u \in C^3(\bar{\Omega})$ satisfying

\begin{equation}
\sigma_{n-1}[D^2u] \geq \psi(x, u, Du),
\end{equation}

where $\varphi$ is a given boundary function, then the Dirichlet problem (1.3) has a unique $C^{3,\alpha}$ solution $u$ for any $0 < \alpha < 1$.

In the following, we consider the prescribed curvature problem for the space-like graphic hypersurface in the Minkowski space which is the other application of our estimates.

We first present some setting of our problem. As mentioned in the introduction, the Minkowski space $\mathbb{R}^{n,1}$ is the set $\mathbb{R}^{n+1}$ endowed with the following metric

$$ds^2 = dx_1^2 + \cdots + dx_n^2 - dx_{n+1}^2.$$ 

Let $u$ be a function defined in some bounded domain $\Omega \subset \mathbb{R}^n$. Suppose $M$ is a space like graphic hypersurface defined by the function $u$, which means

$$M = \{ X = (x, u(x)); x \in \mathbb{R}^n \},$$

where $X = (x, u(x))$ is $M$’s position vector. Since $M$ is space like, in [7], the uniform $C^1$ bound has been obtained for the equation (1.6). Namely, there is a positive constant $\theta$, such that

$$\sup_{\Omega} |Du| \leq \theta < 1.$$ 

The induced metric on $M$ is

$$g_{ij} = \delta_{ij} - D_i u D_j u, \quad 1 \leq i, j \leq n,$$

where $D_i u = \frac{\partial u}{\partial x_i}$. Suppose $\nu$ is the unit exterior normal vector of $M$. We define the second fundamental form of $M$

\begin{equation}
\langle \nu \rangle = \langle \partial_i X, \partial_j \nu \rangle,
\end{equation}

where $\langle \nu \rangle$ is the Minkowski inner product defined by metric $ds^2$ in the above paragraph. Using the function $u$, $h_{ij}$ can be rewritten as

$$h_{ij} = \frac{D_{ij} u}{\sqrt{1 - |Du|^2}},$$

where $D_{ij} u = \frac{\partial^2 u}{\partial x_i \partial x_j}$. The principal curvatures of $M$ are still denoted by $\kappa_1, \ldots, \kappa_n$. Then for space like hypersurfaces, the Gauss formula and the Gauss
The equation are different

\[ X_{ij} = h_{ij} \nu \quad \text{(Gauss formula)} \]

\[ R_{ijkl} = -(h_{ik}h_{jl} - h_{il}h_{jk}) \quad \text{(Gauss equation)} , \]

where \( R_{ijkl} \) is a \((4,0)\)-Riemannian curvature tensor. The Weingarten formula and the Codazzi equations are same. Hence, the formula (4.3) also changes a little

\[ h_{ijkl} = h_{ijlk} + h_{mj} R_{imlk} + h_{im} R_{jmlk} \]

\[ = h_{klij} - (h_{mj}h_{il} - h_{ml}h_{ij}) h_{mk} - (h_{mj}h_{kl} - h_{ml}h_{kj}) h_{mi} . \]

Now let’s prove Theorem 5.

**Proof of Theorem 5.** Let’s first discuss the a priori estimates of the solutions of the Dirichlet problem (1.3). The \( C^0 \) estimate comes from the comparison principle. The \( C^1 \) estimate has been obtained in [7]. The \( C^2 \) estimate on the boundary has been derived in [18]. For the \( C^2 \) estimate in the interior, we use a similar procedure as in Section 4 to obtain it. Let’s give a short proof. For the function \( u \), we consider the following test function

\[ \phi = \log \log P + \frac{N}{2} |Du|^2 , \]

where the function \( P \) is defined by

\[ P = \sum_l e^{\kappa_l} . \]

Suppose that \( \phi \) achieves its maximum value in \( \Omega \) at some point \( x_0 \). We can assume that the matrix \( (u_{ij}) \) is diagonal at \( x_0 \) by a proper rotation of the coordinate, and further assume \( \kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n \). Hence, at \( x_0 \), covariant differentiating the test function \( \phi \) twice, we have

\[ \phi_i = \frac{P_i}{P \log P} + Nu_i u_{ii} = 0 , \]

and

\[ \phi_{ii} = \frac{P_{ii}}{P \log P} - \frac{(1 + \log P)P_i^2}{(P \log P)^2} + \sum_s Nu_s u_{sii} + Nu_{ii}^2 . \]
By a similar calculation as the equalities (4.4) and (4.5), we have

$$
\sigma_{n-1}^{ii} \phi_{ii} = \frac{1}{P \log P} \left[ \sum_l e^{\kappa_l} \sigma_{n-1}^{ii} h_{ii,l} - (n-1)\psi \sum_l e^{\kappa_l} h_{ii,l}^2 + \sigma_{n-1}^{ii} h_{ii,l}^2 \sum_l e^{\kappa_l} h_{ll} \right. \\
+ \sum_l \sigma_{n-1}^{ii} e^{\kappa_l} h_{li}^2 + \sum_{\alpha \neq \beta} \sigma_{n-1}^{ii} e^{\kappa_\alpha - e^{\kappa_\beta}} h_{\alpha\beta i}^2 - \left( \frac{1}{P} + \frac{1}{P \log P} \right) \sigma_{n-1}^{ii} P_i^2 \\
+ \sum_s N u_s \sigma_{n-1}^{ii} u_{sii} + \sigma_{n-1}^{ii} N u_{ii}^2.
$$

At $x_0$, differentiating equation (1.6) twice, we have

$$
\sigma_{n-1}^{ii} h_{ii,j} = \psi_j + \psi_u u_j + \psi_{p_j} u_{jj},
$$

and

$$
\sigma_{n-1}^{ii} h_{ii,jj} + \sigma_{n-1}^{pq,rs} h_{pqj} h_{rsj} \geq -C u_{jj}^2 + \sum_s \psi_{p_s} u_{sjj}.
$$

Inserting (5.9) into (5.7), we have

$$
\sigma_{n-1}^{ii} \phi_{ii} \geq \frac{1}{P \log P} \left[ \sum_l e^{\kappa_l} \left( K (\sigma_{n-1})_l^2 - \sum_{p \neq q} \sigma_{n-1}^{pp,qq} h_{ppl} h_{ql} + \sum_{p \neq q} \sigma_{n-1}^{pp,qq} h_{pql}^2 \right) \\
+ \sum_l \sigma_{n-1}^{ii} e^{\kappa_l} h_{li}^2 + \sum_{\alpha \neq \beta} \sigma_{n-1}^{ii} e^{\kappa_\alpha - e^{\kappa_\beta}} h_{\alpha\beta i}^2 \\
- \frac{1 + \log P}{P \log P} \sigma_{n-1}^{ii} P_i^2 - CP - CK P h_{11}^2 \right] + N \sigma_{n-1}^{ii} u_{ii}^2
$$

Here, in the last two lines, the definition of $A_i, B_i, C_i, D_i, E_i$ is same as the previous section. Then, according to the exactly same procedure of the previous two sections, we can derive the $C^2$ estimate in the interior which implies the global $C^2$ estimate, using the $C^1$ bound. Now we use the continuity method to solve the Dirichlet problem (1.3). The openness is standard and the closeness has been proved. Using Evans-Krylov theory, we obtain our theorem. \qed
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