Exact Duality of The Dissipative Hofstadter Model on a Triangular Lattice: T-Duality and Non-commutative Algebra

Taejin Lee

Kangwon National University
Contents

- Introduction: Dissipative Hofstadter model on a triangular lattice
- Dissipative Hofstadter model and Quantum Wires
- Target Space Dual Transformation: $O(2, 2; \mathbb{R})$
- Boundary State and Magic Circles
- Conclusions
- Ref.: T. Lee, (2016) arXiv:1601.07757
I. Introduction

- Dissipative Hofstadter model
  = Open string theory in the background of the NS-B field and the tachyon potential

\[ S = \frac{1}{4\pi\alpha'} \int d\tau d\sigma E_{ab} (\partial_\tau + \partial_\sigma) X^a (\partial_\tau - \partial_\sigma) X^b \]
\[ + \frac{g\pi}{2} \int_{\partial M} \frac{d\sigma}{2\pi} \sum_a \left( e^{iX^a} + e^{-iX^a} \right) \]

where \( E_{ab} = (g + 2\pi\alpha' B)_{ab} \) and \( \alpha = 1/\alpha', \quad \beta = 2\pi B \)
Dissipative Hofstadter model

- Dissipative Hofstadter model on a Triangular Lattice

\[
S = \frac{\eta}{4\pi\hbar} \int_{-\beta_T/2}^{\beta_T/2} dt dt' \frac{(X(t) - X(t'))^2}{(t - t')^2} \\
+ \frac{ieB_H}{2\hbar c} \int_{-\beta_T/2}^{\beta_T/2} dt \sum_{a,b=1}^{2} \epsilon^{ab} \partial_t X^a X^b \\
+ \frac{V_0}{\hbar} \int_{-\beta_T/2}^{\beta_T/2} dt \sum_{a=1}^{2} \cos \left( \frac{2\pi k^a \cdot X}{l} \right),
\]

where \( \beta_T = 1/T \) and

\[
k_1 = \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right), \quad k_2 = \left( \frac{1}{2}, -\frac{\sqrt{3}}{2} \right), \quad k_3 = (-1, 0).
\]
Dissipative Hofstadter model

- Dissipative Hofstadter model and Quantum Wires

\[ \alpha = \frac{1}{\alpha'} = \eta / 2\pi. \]

- Luttinger liquid and String theory: TL parameter, Regge slope and friction coefficient
Dissipative Hofstadter model

- DHM on triangular lattice and Y-Junction of quantum wires
- Hopping interaction between wires

\[
\sum_{a=1}^{3} \left( \psi_{L}^{a} \psi_{L}^{a+1} - \psi_{R}^{a} \psi_{R}^{a+1} \right)
\]

where \( \psi_{L/R}^{4} = \psi_{L/R}^{1} \)

- Bosonization of hopping interaction

\[
\sum_{a=1}^{3} \left( e^{i \frac{\phi_{a} - \phi_{a+1}}{\sqrt{2}}} + e^{-i \frac{\phi_{a} - \phi_{a+1}}{\sqrt{2}}} \right) = \sum_{a=1}^{3} \left( e^{i k_{a} \cdot X} + e^{-i k_{a} \cdot X} \right)
\]
Non-Abelian Statistics and Quantum Computation

- Non-commutativity and Q-Deformation
- Triangular Lattice:

\[ \text{Elementary excitations of the system} \rightarrow SU(3) \text{ Non-Abelian Statistics} \rightarrow \text{Quantum Computation} \]
Boundary condition in commutative basis

\[ X^a = X^a_L + X^a_R, \]
\[ X^a_L = \frac{1}{\sqrt{2}} x^a_L + \frac{1}{\sqrt{2}} p^a_L \sigma + \frac{i}{\sqrt{2}} \sum_{n \neq 0} \frac{\alpha^a_n}{n} e^{-ni\sigma}, \]
\[ X^a_R = \frac{1}{\sqrt{2}} x^a_R - \frac{1}{\sqrt{2}} p^a_R \sigma + \frac{i}{\sqrt{2}} \sum_{n \neq 0} \frac{\tilde{\alpha}^a_n}{n} e^{ni\sigma}, \]

\[ \left( E_{ab} \alpha^b_{-n} + E^t_{ab} \tilde{\alpha}^b_n \right) |B_E\rangle = 0, \quad p^b |B_E\rangle = 0, \quad a, b = 1, 2. \]
Target Space Dual Transformation: $O(2,2; \mathbb{R})$

- $O(2,2; \mathbb{R})$ T-dual transformation

\[ T = \begin{pmatrix} 1 & 0 \\ \theta/(2\pi) & 1 \end{pmatrix}, \quad T^t J T = J, \]

\[ \theta/(2\pi) = \frac{1}{E} (2\pi B) \frac{1}{E^t} = \frac{\beta}{\alpha^2 + \beta^2 \epsilon}, \]

\[ \alpha_n^a = \left( G(E)^{-1} \right)^a_b \beta_n^b, \]

\[ \tilde{\alpha}_n^a = \left( G(E^t)^{-1} \right)^a_b \tilde{\beta}_n^b \]

\[ G = E^t g^{-1} E = \left( \frac{\alpha^2 + \beta^2}{\alpha} \right) I. \]

- Boundary condition in non-commutative basis:

\[ \left( \beta_n^a + \tilde{\beta}_n^a \right) |B_E\rangle = 0, \quad a = 1, 2. \]
Boundary State and Magic Circles

- **Non-Commutativity**

\[
\left[ X^a(\sigma_1), X^b(\sigma_2) \right] = i \theta^{ab}.
\]

\[
X^a(\sigma, 0) = Z^a(\sigma, 0) + \frac{i}{\sqrt{2}} \frac{\beta}{\alpha} \sum_{n \neq 0} \frac{1}{n} \epsilon^{ab} \left( \beta^b_n + \tilde{\beta}^b_{-n} \right) e^{in\sigma},
\]

\[
Z^a(\sigma, 0) = x^a + \omega^a \sigma + i \frac{1}{\sqrt{2}} \sum_{n \neq 0} \frac{1}{n} \left[ \beta^a_n e^{in\sigma} + \tilde{\beta}^a_{-n} e^{-in\sigma} \right],
\]

- **Boundary State**

\[
Z = \langle 0 | B \rangle,
\]

\[
\langle TO_1 \ldots O_n \rangle = \langle 0 : O_1 \ldots O_n : | B \rangle.
\]
Boundary State of DHM on Triangular Lattice

\[ |B\rangle = T \exp \left[ \frac{V_0}{2} \int d\sigma \sum_{a=1}^{3} \left( e^{ik^a \cdot X} + e^{-ik^a \cdot X} \right) \right] |B_E\rangle. \]

\[
\sum_{a=1}^{3} \left( e^{ik^a \cdot X} + e^{-ik^a \cdot X} \right) = \sum_{a=1}^{3} \left\{ \exp \left( i \sum_{b=1}^{2} \sqrt{\frac{3}{2}} R_{ab} X^b \right) \right\}
+ \exp \left( -i \sum_{b=1}^{2} \sqrt{\frac{3}{2}} R_{ab} X^b \right) \}
\]

where \( R_{ab} \), for \( a = 1, 2, 3 \) and \( b = 1, 2 \), are the components of \( 3 \times 2 \) submatrix of an \( SO(3) \) rotation matrix \( (R) \)

\[
(R) = \begin{pmatrix}
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\
-\frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}}
\end{pmatrix}, \quad (R)^t(R) = (R)(R)^t = I.
\]
Boundary state in the non-commutative basis

\[ |B⟩ = \sum_{n^1, n^2, n^3} \frac{1}{n^1!n^2!n^3!} \left( \frac{V}{2} \right)^{n^1+n^2+n^3} \int \prod_{i=1}^{n^1} dσ^1_i \prod_{i=1}^{n^2} dσ^2_i \prod_{i=1}^{n^3} dσ^3_i \exp \left\{ -i \frac{3}{2} θ \sum_{a,d=1}^{3} \sum_{b,c=1}^{2} \sum_{i=1}^{n^a} \sum_{σ^a_i > σ^b_j} e^{a_i} R_{ab} ε_{bc} (R^t)_{cd} e^{d_j} \right\} \]

\[ T \exp \left\{ i \sqrt{3} \frac{2}{2} \sum_{a=1}^{3} \sum_{b=1}^{2} \sum_{i=1}^{n^a} R_{ab} Z^b(σ^a_i) \right\} |B_E⟩ \]

\[ e^{a_i} = ±1 \text{ for } a = 1, 2, 3 \text{ and } i = 1, \ldots, n^a. \]
Non-commutative Phase

\[ \frac{3}{2} \theta \sum_{a,b=1}^{3} \sum_{i=1}^{n^a} \sum_{\sigma^a_i > \sigma^b_j} \frac{1}{\sqrt{3}} e^{a_i(N)}_{ab} e^{b_j}, \]

where

\[ (N) = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}. \]

Equivalence Relation

\[ \frac{\sqrt{3}}{2} \theta = \frac{\sqrt{3}}{2} \hat{\theta} + 2\pi n, \quad n \in \mathbb{Z}. \]
Equivalence Relations

- Critical Circle: All the points on the circle
  \[
  \left( \alpha - \frac{\sqrt{\det G}}{2} \right)^2 + \beta^2 = \left( \frac{\sqrt{\det G}}{2} \right)^2,
  \]
  have the same closed string metric \( G_{ab} = \sqrt{\det G} \delta_{ab} \).

- Magic Circles: All the points on the circles
  \[
  \alpha^2 + \left( \beta - \frac{1}{2 \left( \theta + 2n/\sqrt{3} \right)} \right)^2 = \left( \frac{1}{2 \left( \theta + 2n/\sqrt{3} \right)} \right)^2, \quad n \in \mathbb{Z}
  \]
  share the same non-commutativity parameter \( \theta \). When \( \theta = 0 \), magic circles.
Critical Circles and Magic Circles
Conclusions

- The exact $O(2, 2; R)$ duality of the DHM on a triangular lattice: Phase Diagram
- The perturbation analysis of the DHM on a triangular lattice (to be done): RG exponent
- Particle-Kink Duality (to be done)
- Q-deformed Algebra (to be done)
- Realization of non-Abelian Statistics (to be done)