Stability of closed characteristics on compact convex hypersurfaces in $\mathbb{R}^{2n}$

Xijun Hu and Yuwei Ou

Dedicate to Professor Paul Rabinowitz.

Abstract. Let $\Sigma \subset \mathbb{R}^{2n}$ with $n \geq 2$ be any $C^2$ compact convex hypersurface which has only finitely many geometrically distinct closed characteristics. Motivated by Long and Zhu’s index jump methods (Ann Math 155:317–368, 2002), we built-up some new index inequalities. Based on this inequalities, we prove that there are at least two geometrically distinct elliptic closed characteristics, and moreover, there exist at least $\varrho_n(\Sigma)$ ($\varrho_n(\Sigma) \geq \lceil \frac{n}{2} \rceil + 1$) geometrically distinct closed characteristics such that for any two elements among them, the ratio of their mean indices is an irrational number.

Mathematics Subject Classification. 58E05, 37J45, 34C25.

Keywords. Compact convex hypersurfaces, closed characteristics, stability, Maslov-type index.

1. Introduction and main results

A very natural and fundamental question is that of existence and stability of characteristics (i.e., periodic orbits) of Hamiltonian flows on a fixed energy hypersurface. The study on closed characteristics on the star-shaped hypersurface in the global sense started in 1978 by Rabinowitz in [22] and Weinstein for the convex hypersurface independently [24]. Since then, and in the past 30 years, these problems have led to some of the most fruitful interactions between analysis, geometry and topology. Up to now, there are many results about the existence and stability of closed characteristics on convex hypersurfaces, but some basic problems are still unsolved, please refer to [4, 7, 8, 10, 12, 14, 16, 18, 21, 23, 25], etc. and references therein. Motivated by the previous results, we study the stability problems of closed characteristics on convex hypersurfaces.

X. Hu: Partially supported by NSFC (Nos. 11425105, 11131004) and NCET.
Y. Ou: Partially supported by NSFC (No. 11131004) and CPSF (No. 2015M580193).
In this paper, let $\Sigma$ be a fixed $C^2$ compact convex hypersurface in $\mathbb{R}^{2n}$, i.e., $\Sigma$ is the boundary of a compact and strictly convex region $U$ in $\mathbb{R}^{2n}$. We denote the set of all such hypersurfaces by $\mathcal{H}(2n)$. Without loss of generality, we suppose $U$ contains the origin. We consider closed characteristics $(\tau, x)$ on $\Sigma$, which are solutions of the following problem

$$
\begin{cases}
\dot{x}(t) = JN_\Sigma(x(t)), & x(t) \in \Sigma, \quad \forall t \in \mathbb{R}, \\
x(\tau) = x(0),
\end{cases}
$$

(1.1)

where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, $I_n$ is the identity matrix in $\mathbb{R}^n$, $\tau > 0$, $N_\Sigma(x)$ is the outward normal vector of $\Sigma$ at $x$ normalized by the condition $N_\Sigma(x) \cdot x = 1$. Here, $a \cdot b$ denotes the standard inner product of $a, b \in \mathbb{R}^{2n}$. Two closed characteristics $(\tau, x)$ and $(\sigma, y)$ are geometrically distinct, if $x(\mathbb{R}) \neq y(\mathbb{R})$. We denote by $T(\Sigma)$, the set of all geometrically distinct closed characteristics on $\Sigma$. $\# A$ denotes the total number of elements in a set $A$.

We follow Ekeland [6], let $j_U(x) = \inf \{ \lambda > 0 | x|_X \in U \}$ be the gauge function of $\Sigma$, then $\Sigma = j_U^{-1}(1)$. Fix a constant $\alpha \in (1, 2)$ and define the Hamiltonian function $H_\alpha : \mathbb{R}^{2n} \to [0, +\infty]$ by $H_\alpha(x) = j_U(x)^\alpha$, $\forall x \in \mathbb{R}^{2n}$. Then the problem (1.1) is equivalent to the following given energy problem of the Hamiltonian system

$$
\begin{cases}
\dot{x}(t) = JH'_\alpha(x(t)), & H_\alpha(x(t)) = 1, \quad \forall t \in \mathbb{R}, \\
x(\tau) = x(0).
\end{cases}
$$

(1.2)

Denoted by $\mathcal{J}(\Sigma, \alpha)$ the set of all geometrically distinct solutions $(\tau, x)$ of (1.2) where $\tau$ is the minimal period of $x$. Note that elements in $T(\Sigma)$ and $\mathcal{J}(\Sigma, \alpha)$ are one to one correspondent to each other.

Let $(\tau, x) \in \mathcal{J}(\Sigma, \alpha)$, the fundamental solution of $x$ is a path of symplectic matrices which satisfied

$$
\dot{\gamma}_x(t) = JH''_\alpha(x(t))\gamma_x(t), \quad \gamma_x(0) = I_{2n}, \quad \text{for all } t \in [0, \tau].
$$

(1.3)

The eigenvalues of $\gamma_x(\tau)$ are called Floquet multipliers of $(\tau, x)$. A closed characteristic $(\tau, x)$ is stable, i.e., elliptic, if all the Floquet multipliers of $x$ are on $U = \{ z \in \mathbb{C} | |z| = 1 \}$, i.e., the unit circle in the complex plane, and is hyperbolic, if 1 is the only double Floquet multipliers on $U$.

In this paper, we will use the Maslov-type index theory for symplectic path to study the stability problem. For $(\tau, x) \in \mathcal{J}(\Sigma, \alpha)$, throughout the paper, we let $(i(x, m), \nu(x, m)) = (i(\gamma_x^m), \nu(\gamma_x^m))$ be the Maslov-type index of the $m$th iteration of $x$, and let $\hat{i}(x) = \hat{i}(\gamma_x, 1)$ be the mean index. For reader’s convenience, we give a brief introduction of Maslov-type index in Appendix.

For the stability problem, Ekeland proved in [5] the existence of at least one elliptic closed characteristic on $\Sigma$ provided $\Sigma \in \mathcal{H}(2n)$ is $\sqrt{2}$-pinched. In [3] of 1992, Dell’Antonio, D’Onofrio and Ekeland proved the existence of at least one elliptic closed characteristic on $\Sigma$ provided $\Sigma \in \mathcal{H}(2n)$ satisfies $\Sigma = -\Sigma$. In [18] of 2000, Long proved that $\Sigma \in \mathcal{H}(4)$ and $\# T(\Sigma) = 2$ imply that both of the closed characteristics must be elliptic. In [21] of 2002,
Long and Zhu further proved when $\# T(\Sigma) < +\infty$, there exists at least one elliptic closed characteristic and there are at least $\left[ \frac{9}{2} \right]$ geometrically distinct closed characteristics on $\Sigma$ possessing irrational mean indices, which are then non-hyperbolic. Moreover, they proved there exist at least two elliptic closed characteristics provided that $\# T(\Sigma) \leq 2\varrho_n(\Sigma) - 2 < \infty$, where $\varrho_n(\Sigma)$ is defined by (2.13). In the recent paper [20], Long and Wang proved that there exist at least two non-hyperbolic closed characteristic on $\Sigma \in \mathcal{H}(6)$ when $\# T(\Sigma) < +\infty$ and in [26], Wang proved that there exist at least two elliptic closed characteristic on $\Sigma \in \mathcal{H}(6)$ when $\# T(\Sigma) = 3$. For other results please refer [13,14,16,18].

It should be noted that most of the ellipticity results above need some additional conditions on $\Sigma$, such as symmetry, pinch or upper bound of the closed characteristics. The symmetry and pinch conditions is not general, and the upper bound is very difficult to verify. It is related to the HZ-conjecture [11], i.e, the number of closed characteristics on convex hypersurfaces is either $n$ or infinite. This conjecture is only solved for the convex hypersurfaces in $\mathbb{R}^4$. In this paper, we remove these conditions and just under a finiteness assumption, we get the ellipticity results on any dimension. This improves the elliptic results ten years ago (see [21]), and it is also an attempt to understand the ellipticity conjecture given in [25], i.e., all the geometrically distinct closed characteristics on $\Sigma$ are irrationally elliptic for $\Sigma \in \mathcal{H}(2n)$ whenever $\# T(\Sigma) < +\infty$.

**Theorem 1.1.** For any $\Sigma \in \mathcal{H}(2n)$ with $n \geq 2$ satisfying $\# T(\Sigma) < +\infty$, there exist at least two elliptic closed characteristics on $\Sigma$.

A typical example is the non-resonant ellipsoid in $\mathbb{R}^{2n}$, that is $\Sigma$ is defined by
\begin{equation}
\sum_{i=1}^{n} \frac{\alpha_i}{2} (p_i^2 + q_i^2) = 1,
\end{equation}
where $\alpha_i/\alpha_j \in \mathbb{R}\setminus\mathbb{Q}$. There just exist $n$ closed characteristics $x_i, i = 1, \ldots, n$ and their mean Maslov-type index satisfy $\hat{i}(x_i)/\hat{i}(x_j) = \alpha_j/\alpha_i \in \mathbb{R}\setminus\mathbb{Q}$. When $\# T(\Sigma)$ is finite, it seems that all the Maslov-type index of the closed characteristics are similar to those in the non-resonant ellipsoid. Another example is in the case $n = 2$, it has been proved in [10] that there are either infinite or 2 closed characteristics. When $n = 2$, in the case $\# T(\Sigma) = 2$, Long, Wang and Hu [25] have proved that both of their mean index are irrational number and all the iterations of their Maslov-type index are the same as a non-resonant ellipsoid, another different proof is given by [1]. As results toward this aspect, we prove that

**Theorem 1.2.** For any $\Sigma \in \mathcal{H}(2n)$ satisfying $\# T(\Sigma) < +\infty$, there exist at least $\varrho_n(\Sigma)$ geometrically distinct closed characteristics on $\Sigma$ such that any two element $(\tau, x), (\tilde{\tau}, \tilde{x})$ satisfy
\begin{equation}
\frac{\hat{i}(x)}{\hat{i}(\tilde{x})} \in \mathbb{R}\setminus\mathbb{Q}.
\end{equation}
The idea of the proof follows [21], let the $m$th iteration interval of $(\tau,x)$ by

$$I_m(\tau,x) = [i(x,m), i(x,m) + \nu(x,m) - 1].$$

Based on the Fadell-Rabinowitz index [9], Ekeland and Hofer [7] proved that for every $k \in \mathbb{N}$, there exist a $(\tau,x) \in J(\Sigma,\alpha)$ and $m \in \mathbb{N}$ such that

$$2k - 2 + n \in I_m(\tau,x).$$

Since we assume $\#T(\Sigma) < +\infty$, all the number with the form $2k - 2 + n$ belongs to the iteration interval of the finite closed characteristics. We get the stability result by the detailed analysis of the Maslov-type index. The main ingredient in our proof of these theorems is the Maslov-type index iteration theory developed by Long and his coworkers, especially based on some new observations on the common index jump theorem of Long and Zhu (Theorem 4.3 of [21], cf. Theorem 11.2.1 of [19]). In Sect. 2, we review briefly the common index jump theorem of Long and Zhu with some further discussion, we prove our main theorems in Sect. 3. For reader’s convenience, we briefly review the Maslov-type index theory in Appendix.

In this paper, let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Q}^+$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of natural integers, integers, rational numbers, positive rational number, real numbers, and complex numbers, respectively. We also define the functions

$$\begin{cases}
[x] = \max\{k \in \mathbb{Z} | k \leq x\}, & E(x) = \min\{k \in \mathbb{Z} | k \geq x\}, \\
\{x\} = x - [x], & \varphi(x) = E(x) - [x].
\end{cases}$$

(1.7)

2. Brief review of Long-Zhu’s index jump Theorem with further discussion

A key theorem of [21] is the following common index jump theorem for a collection of symplectic paths.

**Theorem 2.1.** (cf. P.350 of [21]) Let $\gamma_k \in \mathcal{P}_{\tau_k}(2n)$ for $k = 1, \ldots q$ be a finite collection of symplectic paths. Let $M_k = \gamma_k(\tau_k)$. Suppose that there exists $P_k \in \text{Sp}(2n)$ and $Q_k \in \text{Sp}(2n - 2)$ such that $M_k = P_k^{-1}(N_1(1,1) \circ Q_k)P_k$ and $\hat{i}(\gamma_k,1) > 0$, for all $k = 1, \ldots, q$. Then there exist infinitely many $(N,m_1,\ldots,m_q) \in \mathbb{N}^{q+1}$ such that

$$I(k,m_k) = N + \Delta_k,$$

where

$$I(k,m_k) = m_k(i(\gamma_k,1) + S_{M_k}^+(1) - C(M_k)) + \sum_{\theta \in (0,2\pi)} E\left(\frac{m_k\theta}{\pi}\right)S_{M_k}^-\left(e^{\sqrt{-1}\theta}\right),$$

$$\Delta_k = \sum_{0 < \{m_k\#\} < \delta} S_{M_k}^-\left(e^{\sqrt{-1}\theta}\right)$$

(2.2) (2.3)
for every $k = 1, \ldots, q$. Moreover, we have
\begin{equation}
\min \left\{ \left\{ \frac{m_k \theta}{\pi} \right\}, 1 - \left\{ \frac{m_k \theta}{\pi} \right\} \right\} < \delta,
\end{equation}
where $e^{\sqrt{-1} \theta} \in \sigma(M_k)$, $\frac{\theta}{\pi} \in (0, 2)$ and $\delta$ can be chosen, as small as we want (cf. (4.43) of [21]).

Moreover, by (4.10), (4.40), and (4.41) in [21], we have
\begin{equation}
m_k = \left( \left\{ \frac{N}{M \hat{i}(\gamma_k, 1)} \right\} + \chi_k \right) M, \quad 1 \leq k \leq q,
\end{equation}
where $M \in \mathbb{N}$, $\chi_k = 0$ or 1 for $1 \leq k \leq q$ will be determined below.

In fact, if we let $\mu_i = \sum_{\theta \in (0, 2\pi)} S_{\hat{i}}(e^{\sqrt{-1} \theta})$ for $1 \leq i \leq q$, $h = q + \sum_{1 \leq i \leq q} \mu_i$ and $\alpha_{i,j} = \frac{\theta}{\pi}$ where $e^{\sqrt{-1} \theta} \in \sigma(M_i)$ for $1 \leq j \leq \mu_i$, then follow (4.22) of [21], the above theorem is reduced to find a vertex
\begin{equation}
\chi = (\chi_1, \ldots, \chi_q, \chi_{1,1}, \chi_{1,2}, \ldots, \chi_{1,\mu_1}, \chi_{2,1}, \ldots, \chi_{q,\mu_q})
\end{equation}
of the cube $[0, 1]^h$ and infinitely many integers $N \in \mathbb{N}$ such that
\begin{equation}
|\{Nv\} - \chi| < \varepsilon
\end{equation}
for any given $\varepsilon$ small enough, where
\begin{equation}
v = \left( \frac{1}{M \hat{i}(\gamma_1, 1)}, \ldots, \frac{1}{M \hat{i}(\gamma_q, 1)}, \frac{\alpha_{1,1}}{i_1(\gamma_1, 1)}, \frac{\alpha_{1,2}}{i_1(\gamma_1, 1)}, \ldots, \frac{\alpha_{1,\mu_1}}{i_1(\gamma_1, 1)} \right) \ldots \left( \frac{\alpha_{q,\mu_q}}{i_q(\gamma_q, 1)} \right).
\end{equation}

This dynamical property can be obtained from the theorem below.

**Theorem 2.2.** (cf. Theorem 4.2 of [21]) Let $H$ be the closure of $\{mv\} | m \in \mathbb{N}$ in $T^h = (\mathbb{R}/\mathbb{Z})^h$ and $V = T_0 \pi^{-1} H$ be the tangent space of $\pi^{-1} H$ at the origin in $\mathbb{R}^h$, where $\pi : \mathbb{R}^h \to T^h$ is the projection map. Define
\begin{equation}
A(v) = V \setminus \cup_{v_k \in \mathbb{R} \setminus \mathbb{Q}} \{x = (x_1, \ldots, x_h) \in V | x_k = 0\}.
\end{equation}

Define $\psi(x) = 0$ when $x \geq 0$ and $\psi(x) = 1$ when $x < 0$. Then for any $a = (a_1, \ldots, a_h) \in A(v)$, the vector
\begin{equation}
\chi(a) = (\psi(a_1), \ldots, \psi(a_h))
\end{equation}
makes (2.8) hold for infinitely many $N \in \mathbb{N}$.

Moreover, this set $A(v)$ possesses the property: If $v \in \mathbb{R}^h \setminus \mathbb{Q}^h$, then $\dim V \geq 1, 0 \not\in A(v) \subset V$, $A(v) = -A(v)$ and $A(v)$ is open in $V$.

Note that when we choose $a \in V$ small enough, then $a + \chi(a) \in [0, 1]^h$, this implies $(V + \chi(a)) \cap [0, 1]^h \neq \emptyset$, and so we can require $N \in \mathbb{N}$ in (2.8) satisfying $\{Nv\} - \chi(a) \in V$. 
Remark 2.3. Given $M_0 \in \mathbb{N}$, by the proof of Theorem 4.1 of [21], we may further require $M_0|N$ (since the closure of the set \{\{Nv\} : N \in \mathbb{N}, M_0|N\}) is still a closed additive subgroup of $T^h$ for some $h \in \mathbb{N}$. Then we can use the step 2 in Theorem 4.1 of [21] to get $N$, hence in our choice of $(N, m_1, \ldots, m_q)$ in Theorem 2.1, we can choose $M_0$ good enough such that $N \in \mathbb{N}$ further satisfies

$$\frac{N}{Mi^*(\gamma_k, 1)} \in \mathbb{Z}, \quad \forall i \in (\gamma_k, 1) \in \mathbb{Q}, \quad k \in \{1, \ldots, q\}. \quad (2.12)$$

Furthermore, from (2.8), we get $\hat{i}(\gamma_k, 1) \in \mathbb{Q}$ implies $\chi_k(a) = \psi(a_k) = 0$.

From the theorems above, we get a useful lemma below, this lemma is very important in our proof of the main Theorem 1.2.

Lemma 2.4. Let $v = (v_1, v_2, \ldots, v_h)$ given by (2.9). If $v_i, v_j \in \mathbb{R} \setminus \mathbb{Q}$ and $\frac{v_i}{v_j} = \frac{p}{q} \in \mathbb{Q}^+(i < j)$, then for $\forall x \in V$, we have $\frac{x_i}{x_j} = \frac{p}{q}$ and for $\forall a \in A(v)$, we have $\chi_i(a) = \chi_j(a)$.

Proof. Since $H = \{(mv) | m \in \mathbb{N}\}$, if $v_i, v_j \in \mathbb{R} \setminus \mathbb{Q}$ and $\frac{v_i}{v_j} = \frac{p}{q} \in \mathbb{Q}^+(i < j)$, that means $v_i, v_j$ are rational dependent, then consider the projection of $H$ onto the coordinate hyperplane

$$D = \{(0, \ldots, 0, x_i, 0, \ldots, 0, x_j, 0, \ldots, 0) | x_i, x_j \in \mathbb{R}\} \subset \mathbb{R}^h,$$

Hence for any $x \in V = T_0 \pi^{-1}H, x_i \neq 0$, we have $\frac{x_i}{x_j} = \frac{p}{q} \in \mathbb{Q}^+$. In particular, if $a \in A(v)$, then $a_i > 0, a_j > 0$ or $a_i < 0, a_j < 0$, hence $\chi_i(a) = \chi_j(a) = 0$ or $\chi_i(a) = \chi_j(a) = 1$, this completes the proof. \hfill \Box

Now we assume that all the proofs below satisfy the finite condition, i.e., 

$\# \mathcal{J}(\Sigma, \alpha) < \infty$. For $(\tau, x) \in \mathcal{J}(\Sigma, \alpha)$, some basic property were proved in [21], that is, $\gamma_x(\tau)$ is sympectic similar to a matrix with the form $N_1(1, 1, 0, 0, \ldots)$ with $M \in \text{Sp}(2n, \mathbb{R})$, and $\hat{i}(x) > 2$. Following the ideas in [21], there is a subset $\mathcal{V}_\infty(\Sigma, \alpha) = \{(\tau_j, x_j) | j = 1, \ldots, q\} \subset \mathcal{J}(\Sigma, \alpha)$ such that Theorem 2.1 can be applied to it. This leads to the following important theorem of Long and Zhu.

Theorem 2.5. (cf. [21]) For given $a \in A(v)$, we define $\chi \equiv \chi(a) = (\psi(a_1), \ldots, \psi(a_h))$ by (2.11). Let $(N, m_1, \ldots, m_q) \in \mathbb{N}^{q+1}$ be given in Remark 2.3 and we define a map $\varrho_n(\Sigma) : \mathcal{H}(2n) \rightarrow \mathbb{N} \cup \{+\infty\}$

$$\varrho_n(\Sigma) = \begin{cases} +\infty, & \text{if } \# \mathcal{V}_\infty(\Sigma, \alpha) = +\infty, \\ \min \left\{ \left[ \left( i(x, 1) + 2s^* - \nu(x, 1), n \right) \right] \in \mathcal{V}_\infty(\Sigma, \alpha) \right\}, & \text{if } \# \mathcal{V}_\infty(\Sigma, \alpha) < +\infty. \end{cases} \quad (2.13)$$

Then for each $s = 1, \ldots, \varrho_n(\Sigma)$, there exists a unique $j(s) \in \{1, \ldots, q\}$ and an injection $p : \mathbb{N} \rightarrow \mathbb{N}$ such that $p(N - s + 1) = ((\tau_j(s), x_j(s)), 2m_j(s))$ and

$$i(x_{j(s)}, 2m_j(s)) \leq 2N - 2s + n \leq i(x_{j(s)}, 2m_j(s)) + \nu(x_{j(s)}, 2m_j(s)) - 1. \quad (2.14)$$

Moreover, for any $s_1, s_2 \in \{1, \ldots, \varrho_n(\Sigma)\}$ with $s_1 < s_2$, we have:

$$\hat{i}(x_{j(s_2)}, 2m_j(s_2)) < \hat{i}(x_{j(s_1)}, 2m_j(s_1)). \quad (2.15)$$
Remark 2.6. It is proved in [21] that for $s = 1, \ldots, \varrho_n(\Sigma)$, $x_{j(s)}$ are geometric different, thus there are at least $\varrho_n(\Sigma)$ closed characteristics, and moreover, at least $\varrho_n(\Sigma) - 1$ among them have irrational mean index and $x_{j(1)}$ is elliptic.

We use the notation above, then from Theorems 2.1 and 2.5, we built up some new index inequalities below. These inequalities are very important in our proof of the main Theorem 1.

**Theorem 2.7.** For any $s_1, s_2 \in \{1, \ldots, \varrho_n(\Sigma)\}$ with $s_1 < s_2$, we have

\[
\left( N + \frac{N}{MD_{j(s_2)}} \right) + \chi_j(s_2)(a) = \left( N + \frac{N}{MD_{j(s_1)}} \right) + \chi_j(s_1)(a) = 2 \left( N + \Delta_j(s) \right) - \left( S_{M_j(s)}^+ (1) + C(M_j(s)) \right),
\]

and

\[
i(x_{j(s)}, 2m_{j(s)}) = 2(N + \Delta_j(s)) - \left( S_{M_j(s)}^+ (1) + C(M_j(s)) \right),
\]

\[
2s \geq n + S_{M_j(s)}^+ (1) + C(M_j(s)) - 2\Delta_j(s) - \nu(x_{j(s)}, 2m_{j(s)}) + 1,
\]

\[
2s \leq n + S_{M_j(s)}^+ (1) + C(M_j(s)) - 2\Delta_j(s),
\]

where $D_j(s) = \hat{i}(x_{j(s)}, 1)$.

**Proof.** Since $s_1 < s_2$, the inequality (2.15) and the property of the mean index $i(x, m) = m \hat{i}(x, 1)$ imply that

\[
2m_{j(s_2)} \hat{i}(x_{j(s_2)}, 1) < 2m_{j(s_1)} \hat{i}(x_{j(s_1)}, 1).
\]

From the definition of $m_{j(s)} = \left( N + \frac{N}{MD_{\gamma_j(s)}} \right) + \chi_{\gamma_j(s)}(a)M$ and $D_j(s) = \hat{i}(x_{j(s)}, 1)$ we get (2.16). To prove formula (2.17), we need some identities (2.1), (2.2) and (4.15) below

\[
I(j(s), m_{j(s)}) = N + \Delta_j(s),
\]

where

\[
I(j(s), m_{j(s)}) = m_{j(s)}(i(\gamma_{j(s)}), 1) + S_{M_j(s)}^+ (1) - C(M_j(s))
\]

\[
+ \sum_{\theta \in (0, 2\pi)} E \left( \frac{m_{j(s)} \theta}{\pi} \right) S_{M_j(s)}^- (e^{\sqrt{-1} \theta}),
\]

\[
i(\gamma_{j(s)}, m_{j(s)}) = m_{j(s)}(i(\gamma_{j(s)}), 1) + S_{M_j(s)}^+ (1) - C(M_j(s))
\]

\[
+ 2 \sum_{\theta \in (0, 2\pi)} E \left( \frac{m_{j(s)} \theta}{2\pi} \right) S_{M_j(s)}^- (e^{\sqrt{-1} \theta}) - (S_{M_j(s)}^+ (1) + C(M_j(s))).
\]

where $C(M_j(s)) = \sum_{0 < \theta < 2\pi} S_{M_j(s)}^- (e^{\sqrt{-1} \theta})$.

Simple calculations show that

\[
i(\gamma_{j(s)}, 2m_{j(s)}) = 2I(j(s), m_{j(s)}) - (S_{M_j(s)}^+ (1) + C(M_j(s))
\]

\[
= 2(N + \Delta_j(s)) - (S_{M_j(s)}^+ (1) + C(M_j(s)).
\]
Therefore, we get the formula (2.17). On the other hand, it is easy to show that formulas (2.14), (2.17) imply (2.18), (2.19).

**Corollary 2.8.** Further properties of inequalities (2.16):

(i) If \( D_j(s_2) \in Q \), then \( \chi_{j(s_2)}(a) = 0 \) and
\[
N = \left( \left\lceil \frac{N}{MD_j(s_2)} \right\rceil + \chi_{j(s_2)}(a) \right) MD_j(s_2) < \left( \left\lceil \frac{N}{MD_j(s_1)} \right\rceil + \chi_{j(s_1)}(a) \right) MD_j(s_1)
\]
with \( D_j(s_1) \in \mathbb{R} \setminus Q \), \( \chi_{j(s_1)}(a) = 1 \).

(ii) If \( \chi_{j(s_2)}(a) = 1 \), then \( D_j(s_2) \in \mathbb{R} \setminus Q \) and
\[
N < \left( \left\lceil \frac{N}{MD_j(s_2)} \right\rceil + \chi_{j(s_2)}(a) \right) MD_j(s_2) < \left( \left\lceil \frac{N}{MD_j(s_1)} \right\rceil + \chi_{j(s_1)}(a) \right) MD_j(s_1)
\]
with \( D_j(s_1) \in \mathbb{R} \setminus Q \), \( \chi_{j(s_1)}(a) = 1 \).

(iii) \( \chi_{j(s_2)}(a) \leq \chi_{j(s_1)}(a) \).

**Proof.** From Remark 2.3, we know that \( D_j(s_2) \in Q \) implies \( \frac{N}{MD_j(s_2)} \in \mathbb{Z} \) and \( \chi_{j(s_2)}(a) = 0 \), hence \( \left\lceil \frac{N}{MD_j(s_2)} \right\rceil + \chi_{j(s_2)}(a) \right) MD_j(s_2) = N \). For this case, it is easy to check that inequality (2.16) holds if and only if \( D_j(s_1) \in \mathbb{R} \setminus Q \), \( \chi_{j(s_1)}(a) = 1 \). This completes the proof of (i). From (i) we know if \( D_j(s_2) \in Q \), \( \chi_{j(s_2)}(a) = 0 \), so \( \chi_{j(s_2)}(a) = 1 \) implies that \( D_j(s_2) \in \mathbb{R} \setminus Q \), easy computation shows that
\[
N = \left( \left\lceil \frac{N}{MD_j(s_2)} \right\rceil + \chi_{j(s_2)}(a) \right) MD_j(s_2) < \left( \left\lceil \frac{N}{MD_j(s_1)} \right\rceil + \chi_{j(s_1)}(a) \right) MD_j(s_1)
\]
Combining (2.16), we get (ii). To prove (iii), note that if \( \chi_{j(s_2)}(a) = 0 \), then (2.21) is obviously right, the case \( \chi_{j(s_2)}(a) = 1 \) is from (ii).

**3. Proofs of the Theorems 1.1 and 1.2**

In this section, we prove Theorems 1.1 and 1.2 based on the index iteration theory developed by Long and his coworkers. Some notations for the Maslov-type index can be found in Appendix. The basic normal forms \( R(\theta_j) \) \((N_2(\omega_j, u_j); N_2(\lambda_j, \nu_j))\) given in Theorem 4.4 is called rational normal form, if \( \frac{\theta_j}{\pi} \in \mathbb{Q} \). For fixed \( a \in A(v) \), the injection map \( p(N - s + 1) = (\tau_{j(s)}(x_{j(s)}), 2m_{j(s)}), s \in \{1, \ldots, g_{n}(\Sigma)\} \) is given in Theorem 2.5.

**Lemma 3.1.** If \( x_{j(2)} \) is not an elliptic closed characteristic, then \( \chi_{j(2)}(a) = 0 \) implies that \( \tilde{i}(x_{j(2)}, 1) \in \mathbb{Q} \).

**Proof.** From Theorem 4.4, we have the symplectic decomposition
\[
\gamma_{j(2)} \simeq N_1(1, 1)^{\circ p_1} \circ I_{2p_0} \circ N_1(1, 1)^{\circ p_2} \circ N_1(-1, 1)^{\circ q_1} \circ -I_{2q_0} \circ N_1(-1, 1)^{\circ q_2}
\circ R(\theta_1) \circ \cdots R(\theta_r) \circ N_2(\omega_1, u_1) \circ \cdots \circ N_2(\omega_r, u_r,)
\circ N_2(\lambda_1, \nu_1) \circ \cdots \circ N_2(\lambda_{n}, \nu_{n}) \circ M_k,
\]
(3.1)
for this decomposition, the number of the rational normal form in \( \{ R(\theta_1), \ldots, R(\theta_r) \} \) is denoted by \( \tilde{r} \). Similarly, for set \( \{ N_2(\omega_1, u_1), \ldots, N_2(\omega_{r^*}, u_{r^*}) \} \) and \( \{ N_2(\lambda_1, \nu_1), \ldots, N_2(\lambda_{r_0}, \nu_{r_0}) \} \), the number of rational normal form is denoted by \( \tilde{r}^* \) and \( \tilde{r}_0 \), respectively, then from (4.22) we have a further estimation of the variable \( \nu(\gamma_{j(2)}, 2m_{j(2)}) \) below

\[
\nu(\gamma_{j(2)}, 2m_{j(2)}) = \nu(\gamma_{j(2)}, 1) + q_- + 2q_0 + q_+ + 2(r + r^* + r_0) \\
- 2(r - \tilde{r} + r^* - \tilde{r}^* + r_0 - \tilde{r}_0) \\
= p_- + 2p_0 + p_+ + q_- + 2q_0 + q_+ + 2(\tilde{r} + \tilde{r}^* + \tilde{r}_0).
\]

(3.2)

Now we prove the lemma by contradiction. Assume that \( \chi_{j(2)}(a) = 0 \) and \( \hat{i}(x_{j(2)}, 1) \in \mathbb{R}\setminus\mathbb{Q} \), then (4.23) implies that at least one of \( \theta_1 \pi, \theta_2 \pi, \ldots, \theta_r \pi \) is irrational number, hence \( r - \tilde{r} \geq 1 \) and

\[
\{ m_{j(2)} D_{j(2)} \} = \left\{ m_{j(2)} \left( i(x_{j(2)}, 1) + p_- + p_0 - r + \sum_{j=1}^{r} \frac{\theta_j}{\pi} \right) \right\} \\
= \left\{ m_{j(2)} \sum_{\frac{\theta_j}{\pi} \in \mathbb{R}\setminus\mathbb{Q}} \frac{\theta_j}{\pi} \right\} \\
\leq \sum_{\frac{\theta_j}{\pi} \in \mathbb{R}\setminus\mathbb{Q}} \left\{ m_{j(2)} \frac{\theta_j}{\pi} \right\},
\]

(3.3)

where \( m_{j(2)} = ([N_{MD_{j(2)}}] + \chi_{j(2)}(a))M = [N_{MD_{j(2)}}]M \). The second equality in (3.3) follows from (2.5) in Theorem 2.1.

On the other hand,

\[
\{ m_{j(2)} D_{j(2)} \} = \left\{ \left[ \frac{N}{MD_{j(2)}} \right] MD_{j(2)} \right\} \\
= \left\{ N - \left[ \frac{N}{MD_{j(2)}} \right] MD_{j(2)} \right\} ,
\]

(3.4)

and from (2.9) and (2.8), we get that \( \left[ \frac{N}{MD_{j(2)}} \right] = \left| \left\{ \frac{N}{MD_{j(2)}} \right\} - \chi_{j(2)}(a) \right| < \varepsilon \) (\( \chi_{j(2)}(a) = 0 \)) for any given \( \varepsilon \) small enough, let \( \varepsilon < \frac{1-\delta}{MD_{j(2)}} \), where \( \delta \) is given by (2.4) of Theorem 2.1, then \( \{ m_{j(2)} D_{j(2)} \} > \delta \), which together with (3.3) gives

\[
\delta < \sum_{\frac{\theta_j}{\pi} \in \mathbb{R}\setminus\mathbb{Q}} \left\{ m_{j(2)} \frac{\theta_j}{\pi} \right\},
\]

(3.5)

hence at least one of the elements in \( \{ \frac{\theta_j}{\pi} | \frac{\theta_j}{\pi} \in \mathbb{R}\setminus\mathbb{Q}, j = 1, \ldots, r \} \) satisfies \( \{ m_{j(2)} \frac{\theta_j}{\pi} \} \notin (0, \delta) \). We have the estimation of the variable \( \Delta_{j(2)} \) below
\[
\Delta_{j(2)} = \sum_{0 < \{m_{j(2)} \theta \} < \delta} S_{M_{j(2)}}^{-}(e^{\sqrt{-1} \theta}) \\
\leq \sum_{\frac{\pi}{\delta} \in \mathbb{R} \setminus \mathbb{Q}} S_{M_{j(2)}}^{-}(e^{\sqrt{-1} \theta}) - 1 \\
= r - \bar{r} - 1 + 2(r_* - \bar{r}_*), \tag{3.6}
\]
where the last equality comes from the calculation of the splitting number of the basic normal given in Definition 4.1. To prove the Lemma, we rewrite the useful inequality (2.18) and equalities (4.20), (4.24), (4.25), (3.2) below.

\[
2s \geq n + S_{M_{j(s)}}^{+}(1) + C(M_{j(s)}) - 2\Delta_{j(s)} \\
- \nu(x_{j(s)}, 2m_{j(s)}) + 1, \quad (s = 2) \\
n = p_- + p_0 + p_+ + q_- + q_0 + q_+ + r + 2r_* + 2r_0 + k \\
S_{M_{j(2)}}^{+}(1) = p_- + p_0 \\
C(M_{j(2)}) = \sum_{0 < \theta < 2\pi} S_{M_{j(2)}}^{-}(e^{\sqrt{-1} \theta}) = q_0 + q_+ + r + 2r_* \\
\nu(\gamma_{j(2)}, 2m_{j(2)}) = p_- + 2p_0 + p_+ + q_- + 2q_0 + q_+ + 2(\bar{r} + \bar{r}_* + \bar{r}_0)
\]
This combines with inequality (3.6), easy computation shows that for \( s = 2 \), we have

\[
4 \geq p_- + q_+ + 2(r_0 - \bar{r}_0) + 2\bar{r}_* + k + 3 \tag{3.7}
\]
From Lemma 1.3 of [21], for any \((\tau, x) \in \mathcal{J}(\Sigma, \alpha)\), there exist \( P \in \text{Sp}(2n) \) and \( Q \in \text{Sp}(2n - 2) \) such that \( \gamma_x(\tau) = P^{-1}(N_1(1, 1) \circ Q)P \), hence we always have \( p_- \geq 1 \). On the other hand, from the condition of the lemma, we know \( x_{j(2)} \) is not an elliptic closed characteristic, this implies that \( k \geq 1 \), hence we get \( 4 \geq 5 \). This contradiction completes the proof.

**Corollary 3.2.** If \( x_{j(2)} \) is not elliptic, then \( \chi_{j(1)}(a) = 1 \) and \( \hat{i}(x_{j(1)}, 1) \in \mathbb{R} \setminus \mathbb{Q} \).

**Proof.** If \( \chi_{j(2)}(a) = 1 \), from ii) of Corollary 2.8 we have \( \chi_{j(1)}(a) = 1 \) and \( \hat{i}(x_{j(1)}, 1) \in \mathbb{R} \setminus \mathbb{Q} \). If \( \chi_{j(2)}(a) = 0 \), then from Lemma 3.1, we have \( \hat{i}(x_{j(2)}, 1) \in \mathbb{Q} \). This combines with i) of Corollary 2.8 we get \( \hat{i}(x_{j(1)}, 1) \in \mathbb{R} \setminus \mathbb{Q} \) and \( \chi_{j(1)}(a) = 1 \).

Now we start to prove Theorems 1.1 and 1.2.

**Proof of Theorem 1.1.** From Remark 2.6, we know \( x_{j(1)} \) is elliptic. If \( x_{j(2)} \) is also elliptic, then the proof is complete. If \( x_{j(2)} \) is not an elliptic closed characteristic, from Corollary 3.2, we get \( \chi_{j(1)}(a) = 1 \) and \( \hat{i}(x_{j(1)}, 1) \in \mathbb{R} \setminus \mathbb{Q} \). Now from the property of set \( A(v) \) in Theorem 2.2, we can choose \(-a \in A(v)\), since Theorem 2.5 still holds for \(-a \in A(v)\), so we have \((\bar{N}, \bar{m}_1, \ldots, \bar{m}_q), \bar{j}(s) \) and injection map \( p(\bar{N} - s + 1) = ((\tau_j(s), x_j(s), 2m_j(s)), s \in \{1, \ldots, g_n(\Sigma)\} \).

If \( \bar{j}(1) \neq j(1) \), then \( x_{j(1)}, x_{j(1)} \) are two different elliptic closed characteristics, then we complete the proof. If \( \bar{j}(1) = j(1) \), from the definition of \( \chi(a) \), we know that \( \chi_{j(1)}(a) = 1 \) implies \( \chi_{j(1)}(-a) = 0 \), hence \( \chi_{j(1)}(-a) = \chi_{j(1)}(a) = 0 \), this combines with (i), (iii) of Corollary 2.8, we have \( \chi_{j(2)}(-a) = 0 \) and
\(\hat{i}(x,1) \in R \setminus Q\), but Lemma 3.1 still holds in the case \(-a \in A(v)\), that means if \(x_{j(2)}\) is not elliptic, we should have \(\chi_{j(2)}(-a) = 0\) implies \(\hat{i}(x_{j(2)}, 1) \in Q\), this contradiction completes the proof. \(\square\)

**Proof of Theorem 1.2.** For the \(q_n(\Sigma)\) geometrically distinct closed characteristics in Remark 2.6, for any two closed characteristics \((\tau, x)\) and \((\tilde{\tau}, \tilde{x})\), we know that if one of them has rational mean index, then another must have irrational mean index, hence for this case, the theorem is true. Now we can assume that \(\hat{i}(x, 1), \hat{i}(\tilde{x}, 1) \in R \setminus Q\). For this case, we get the proof by contradiction.

Assume \(\frac{\hat{i}(x, 1)}{i(x, 1)} = \frac{p}{q} \in Q^+\), then from Theorem 2.5, we know there exist \(s_1, s_2 \in \{1, 2, \ldots, q_n(\Sigma)\}\) such that \(x = x_{j(s_1)}\) and \(\tilde{x} = x_{j(s_2)}\). Without loss of generality, we can assume that \(s_1 < s_2\), then we have (2.16)

\[
\left(\frac{N}{MD_j(s_2)} + \chi_{j(s_2)}(a)\right) MD_j(s_2) < \left(\frac{N}{MD_j(s_1)} + \chi_{j(s_1)}(a)\right) MD_j(s_1),
\]

simple calculations shows that

\[
\left(\frac{N}{MD_j(s_2)} - \chi_{j(s_2)}(a)\right) MD_j(s_2) > \left(\frac{N}{MD_j(s_1)} - \chi_{j(s_1)}(a)\right) MD_j(s_1),
\]

(3.8)

where \(D_j(s) = \hat{i}(x_j(s), 1)\). Let

\[
v = \left(\frac{1}{M \hat{i}(\gamma_1, 1)}, \ldots, \frac{1}{M \hat{i}(\gamma_q, 1)}, \frac{1}{\hat{i}_1(\gamma_1, 1)}, \frac{1}{\hat{i}_1(\gamma_1, 1)}, \ldots, \frac{1}{\hat{i}_1(\gamma_1, 1)}\right)
\]

given by (2.9), where \(\gamma_k\) is the associated symplectic path of \((\tau_k, x_k) \in V_\infty(\Sigma, \alpha)\), then we have

\[
\frac{v_j(s_2)}{v_j(s_1)} = \frac{1}{M_i(\gamma_j(s_2), 1)} = \frac{\hat{i}(x, 1)}{\hat{i}(\tilde{x}, 1)} = \frac{q}{p},
\]

(3.9)

On the other hand, from Theorem 2.2, for fixed \(a \in V\), we choose \(N \in \mathbb{N}\) such that \(\{Nv\} - \chi(a)\) small enough, recall that we also have \(\{Nv\} - \chi(a) \in V\). From Lemma 2.4, we get

\[
\frac{\{Nv_j(s_2)\} - \chi_{j(s_2)}(a)}{\{Nv_j(s_1)\} - \chi_{j(s_1)}(a)} = \frac{N}{MD_j(s_2)} - \chi_{j(s_2)}(a) = \frac{N}{MD_j(s_1)} - \chi_{j(s_1)}(a) = \frac{q}{p},
\]

(3.10)

(3.9), (3.10) imply

\[
\left(\frac{N}{MD_j(s_1)} - \chi_{j(s_1)}(a)\right) MD_j(s_1) = \left(\frac{N}{MD_j(s_2)} - \chi_{j(s_2)}(a)\right) MD_j(s_2)
\]

(3.11)

This contradicts to (3.8), then the proof is complete. \(\square\)
Acknowledgements
The first author sincerely thanks Y. Long and C. Zhu for the explanation of their methods and helpful discussion of this problem. The authors thank the referee for helpful suggestions which make the paper more readable.

Appendix: Index iteration theory for Maslov-type index
In this section, we briefly recall the index theory for symplectic paths. The index theory is introduced by Conley and Zehnder [2] and developed by Long and others (see [15,18,19] for details).

As usual, we defined the set of symplectic path by
\[ P_\tau(2n) = \{ \gamma \in C([0, \tau], \text{Sp}(2n)) \mid \gamma(0) = I_{2n} \} . \]

For any \( \gamma \in P_\tau(2n) \) and \( \omega \in U \), let \( i_\omega(\gamma) \) be the Maslov-type index (see [19]) and \( \nu_\omega(\gamma) = \dim \ker C(\gamma(\tau) - \omega I_{2n}) \), they are called the index function of \( \gamma \). We also define its \( m \)th iteration \( \gamma^m : [0, m\tau] \to \text{Sp}(2n) \) by \( \gamma^m(t) = \gamma(t - j\tau)\gamma(\tau)^j \), for \( j\tau \leq t \leq (j+1)\tau, j = 0,1, \ldots, m-1 \).

Definition 4.1. (cf. [18,19]) For any \( \gamma \in P_\tau(2n) \), we define
\[ (i(\gamma, m), \nu(\gamma, m)) = (i(\gamma^m), \nu(\gamma^m)) \], \( \forall m \in \mathbb{N} \). (4.1)

The mean index \( \hat{i}(\gamma, m) \) per \( m\tau \) for \( m \in \mathbb{N} \) is defined by
\[ \hat{i}(\gamma, m) = \lim_{k \to +\infty} \frac{i(\gamma, mk)}{k} . \] (4.2)

For any \( M \in \text{Sp}(2n) \) and \( \omega \in U \), the splitting numbers \( S^\pm_M(\omega) \) of \( M \) at \( \omega \) are defined by
\[ S^\pm_M(\omega) = \lim_{\epsilon \to 0^+} i_\omega \exp(\pm \sqrt{-1}\epsilon)(\gamma) - i_\omega(\gamma) \], \( \forall \gamma \in P_\tau(2n) \) satisfying \( \gamma(\tau) = M \). (4.3)

For \( \Sigma \in \mathcal{H}(2n) \) and \( \alpha \in (1, 2) \), let \( (\tau, x) \in J(\Sigma, \alpha) \). we define
\[ S^+(x) = S^+_{\gamma_x(\tau)}(1) \], (4.4)
\[ (i(x, m), \nu(x, m)) = (i(\gamma_x, m), \nu(\gamma_x, m)) \], (4.5)
\[ \hat{i}(x, m) = \hat{i}(\gamma_x, m) \], (4.6)

For all \( m \in \mathbb{N} \), where \( \gamma_x \) is the associated symplectic path of \((\tau, x)\).

In [16–19], the following symplectic matrices were introduced as basic normal forms:
\[ D(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} , \quad \lambda = \pm 2, \] (4.7)
\[ N_1(\lambda, b) = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix} , \quad \lambda = \pm 1, b = \pm 1, 0, \] (4.8)
\[ R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad \theta \in (0, \pi) \cup (\pi, 2\pi), \]  
(4.9)

\[ N_2(\omega, b) = \begin{pmatrix} R(\theta) \\ 0 \end{pmatrix}, \quad \theta \in (0, \pi) \cup (\pi, 2\pi), \]  
(4.10)

where \( b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \) with \( b_i \in \mathbb{R} \) and \( b_2 \neq b_3 \), \( \omega = e^{\sqrt{-1}\theta} \). We call \( N_2(\omega, b) \) is nontrivial if \( (b_2 - b_3) \sin \theta < 0 \) and \( N_2(\omega, b) \) is trivial if \( (b_2 - b_3) \sin \theta > 0 \).

Splitting numbers possess the following properties:

**Lemma 4.2.** (cf. [16], Lemma 9.1.5 and List 9.1.12 of [19]) For \( M \in \text{Sp}(2n) \) and \( \omega \in \mathbb{U} \), there hold

\[ S_M^\pm(\omega) = 0, \quad \text{if} \quad \omega \not\in \sigma(M). \]  
(4.11)

\[ (S_{N_1(1,a)}^+(1), S_{N_1(1,a)}^-(1)) = \begin{cases} (1,1), & \text{if} \quad a \geq 0, \\ (0,0), & \text{if} \quad a < 0. \end{cases} \]  
(4.12)

\[ (S_{N_1(-1,a)}^+(1), S_{N_1(-1,a)}^-(1)) = \begin{cases} (1,1), & \text{if} \quad a \leq 0, \\ (0,0), & \text{if} \quad a > 0. \end{cases} \]  
(4.13)

\[ (S_{R(\theta)}(e^{\sqrt{-1}\theta}), S_{R(\theta)}(e^{-\sqrt{-1}\theta})) = (0,1) \quad \text{if} \quad e^{\sqrt{-1}\theta} \in \mathbb{U}\setminus\mathbb{R}. \]  
(4.14)

If \( e^{\sqrt{-1}\theta} \in \mathbb{U}\setminus\mathbb{R} \) and \( N_2(\omega, b) \) is nontrivial, then

\[ (S_{N_2(\omega,b)}^+(e^{\sqrt{-1}\theta}), S_{N_2(\omega,b)}^-(e^{-\sqrt{-1}\theta})) = (1,1). \]  
(4.15)

If \( e^{\sqrt{-1}\theta} \in \mathbb{U}\setminus\mathbb{R} \) and \( N_2(\omega, b) \) is trivial, then

\[ (S_{N_2(\omega,b)}^+(e^{\sqrt{-1}\theta}), S_{N_2(\omega,b)}^-(e^{-\sqrt{-1}\theta})) = (0,0). \]  
(4.16)

For any \( M_i \in \text{Sp}(2n_i) \) with \( i = 0 \) and 1, there holds

\[ S_{M_0 \circ M_1}^\pm(\omega) = S_{M_0}^\pm(\omega) + S_{M_1}^\pm(\omega) \quad \text{and} \quad S_M^\pm(\omega) = S_{\bar{M}}^\pm(\bar{\omega}), \quad \forall \omega \in \mathbb{U}. \]  
(4.17)

where \( \bar{\omega} \) is the conjugate of \( \omega \) and the \( \circ \)-product of \( M_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix} \) with \( k = 0,1 \) is defined by the following

\[ M_0 \circ M_1 = \begin{pmatrix} A_0 & 0 & B_0 & 0 \\ 0 & A_1 & 0 & B_1 \\ C_0 & 0 & D_0 & 0 \\ 0 & C_1 & 0 & D_1 \end{pmatrix}. \]

Also denote by \( M^{\circ k} \) the \( k \)-fold \( \circ \)-product \( M \circ \cdots \circ M \).

The following is the precise index iteration formulae for symplectic paths, which is due to Y.Long (cf. Chapter 8 [19] or Theorem 2.1, 6.5 and 6.7 of [21]).
Theorem 4.3. For \( n \in \mathbb{N}, \tau > 0 \), and any path \( \gamma \in \mathcal{P}_\tau(2n) \), set \( M = \gamma(\tau) \). Extend \( \gamma \) to the whole \([0, +\infty)\). Then for any \( m \in \mathbb{N} \),

\[
i(\gamma, m) = m(i(\gamma, 1) + S^+_M(1) - C(M)) + 2 \sum_{\theta \in (0, 2\pi)} E\left(\frac{m\theta}{2\pi}\right) S^-_M(e^{\sqrt{-1}e^{\theta}}) - (S^+_M(1) + C(M)).
\] (4.18)

where \( C(M) = \sum_{0<\theta<2\pi} S^-_M(e^{\sqrt{-1}e^{\theta}}) \).

Theorem 4.4. Let \( \gamma \in \mathcal{P}_\tau(2n) \), then there exists a homotopy such that the index function of \( \gamma \) is unchanged and

\[
\gamma(\tau) \simeq N_1(1, 1)^{op} \circ I_{2p_0} \circ N_1(1, -1)^{op} \circ N_1(-1, 1)^{op} \circ -I_{2q_0} \circ N_1(-1, -1)^{op} \circ R(\theta_1) \circ \cdots \circ R(\theta_r) \circ N_2(\omega_r, u_r) \circ \cdots \circ N_2(\omega_r, u_r)
\] (4.19)

where \( N_2(\omega_j, u_j) \) are nontrivial form with some \( \omega_j = e^{\sqrt{-1}\alpha_j}, \alpha_j \in (0, \pi) \cup (\pi, 2\pi) \) and \( u_j = \left(\begin{array}{cc}
u_{j1} & \nu_{j2} \\
u_{j3} & \nu_{j4}\end{array}\right) \in \mathbb{R}^{2 \times 2} \), \( N_2(\omega_j, u_j) \) are trivial form with some \( \lambda_j = e^{\sqrt{-1}\beta_j}, \beta_j \in (0, \pi) \cup (\pi, 2\pi) \) and \( \nu_j = \left(\begin{array}{cc}\nu_{j1} & \nu_{j2} \\
u_{j3} & \nu_{j4}\end{array}\right) \in \mathbb{R}^{2 \times 2} \), \( M_k = D(2)^k \) or \( D(-2) \circ D(2)^{(k-1)} \); \( p_-, p_0, p_+, q_-, q_0, q_+, r, r_+ \) and \( r_0 \) are non-negative integers; these integer and real number are uniquely determined by \( \gamma(\tau) \). It holds that

\[
n = p_- + p_0 + p_+ + q_- + q_0 + q_+ + r + 2r_+ + 2r_0 + k.
\] (4.20)

Then using the functions defined in (1.2), we have

\[
i(\gamma, m) = m(i(\gamma, 1) + p_- + p_0 - r) + 2 \sum_{j=1}^{r} E\left(\frac{m\theta_j}{2\pi}\right) - r - p_- - p_0
\]

\[\quad - \frac{1 + (-1)^m}{2} (q_0 + q_+) + 2 \left(\sum_{j=1}^{r} \varphi\left(\frac{m\alpha_j}{2\pi}\right) - r_+\right)\] (4.21)

\[
\nu(\gamma, m) = \nu(\gamma, 1) + \frac{1 + (-1)^m}{2} (q_- + 2q_0 + q_+) + 2(r + r_* + r_0)
\]

\[\quad - 2 \left(\sum_{j=1}^{r} \varphi\left(\frac{m\theta_j}{2\pi}\right) + \sum_{j=1}^{r} \varphi\left(\frac{m\alpha_j}{2\pi}\right) + \sum_{j=1}^{r_0} \varphi\left(\frac{m\beta_j}{2\pi}\right)\right)\] (4.22)

\[
i(\gamma, 1) = i(\gamma, 1) + p_- + p_0 - r + \sum_{j=1}^{r} \frac{\theta_j}{\pi}\] (4.23)
\[ S_M^+(1) = p_- + p_0 \quad (4.24) \]

\[ C(M) = \sum_{0 < \theta < 2\pi} S^{-}_M(e^{\sqrt{-1}\theta}) = q_0 + q_+ + r + 2r^* \quad (4.25) \]

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Xijun Hu
Department of Mathematics
Shandong University
Jinan 250100 Shandong
People's Republic of China
e-mail: xjhu@sdu.edu.cn

Yuwei Ou
Chern Institute of Mathematics
Nankai University
Tianjin 300071
People's Republic of China
e-mail: yuweiou@163.com