Hardness of Liar’s Domination on Unit Disk Graphs

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Abstract

A unit disk graph is the intersection graph of a set of unit diameter disks in the plane. In this paper we consider liar’s domination problem on unit disk graphs, a variant of dominating set problem. We call this problem as Euclidean liar’s domination problem. In the Euclidean liar’s domination problem, a set \( P = \{p_1, p_2, \ldots, p_n\} \) of \( n \) points (disk centers) are given in the Euclidean plane. For \( p \in P \), \( N[p] \) is a subset of \( P \) such that for any \( q \in N[p] \), the Euclidean distance between \( p \) and \( q \) is less than or equal to 1, i.e., the corresponding unit diameter disks intersect. The objective of the Euclidean liar’s domination problem is to find a subset \( D \subseteq P \) of minimum size having the following properties: (i) \( |N[p_i] \cap D| \geq 2 \) for \( 1 \leq i \leq n \), and (ii) \( |(N[p_i] \cup N[p_j]) \cap D| \geq 3 \) for \( i \neq j, 1 \leq i, j \leq n \). This article aims to prove the Euclidean liar’s domination problem is NP-complete.

1 Introduction

Let \( G = (V, E) \) be a graph. For a vertex \( v \in V \), we define \( N[v] = \{ u \in V \mid (v, u) \in E \} \cup \{ v \} \). A subset \( D \) of \( V \) is a liar’s dominating set if (i) for every \( v \in V \), \( |N[v] \cap D| \geq 2 \), and (ii) for every distinct pair of vertices \( u \) and \( v \), \( |(N[u] \cup N[v]) \cap D| \geq 3 \). Liar’s domination problem in a graph \( G = (V, E) \) asks to find a liar’s dominating set of \( G \) with minimum size.

1.1 Related work

The liar’s domination problem is introduced by Slater in 2009 and showed that the problem is NP-hard for general graphs [7]. Later, Roden and Slater showed that the problem is NP-hard even for bipartite graphs [6]. Panda and Paul [5] proved that the problem is NP-hard for split graphs and chordal graphs. The authors also proposed a linear time algorithm for computing a minimum cardinality liar’s dominating set in a tree.

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1.2 Our work

A unit disk graph (UDG) is an intersection graph of a family of unit diameter disks in the plane. Given a set $C = \{C_1, C_2, \ldots, C_n\}$ of $n$ circular disks in the plane, each having diameter 1, the corresponding UDG $G = (V, E)$ is defined as follows: each vertex $v_i \in V$ corresponds to a disk $C_i \in C$, and there is an edge between two vertices $v_i$ and $v_j$ if and only if $C_i$ and $C_j$ intersect.

In this paper we consider the geometric version of the liar’s domination problem and we call it as Euclidean liar’s domination problem. In the Euclidean liar’s domination problem we are given a UDG and a set $P$ of $n$ disk centers of the given UDG in the plane. For $p \in P$, $N[p]$ is a subset of $P$ such that for any $q \in N[p]$, the Euclidean distance between $p$ and $q$ is less than or equal to 1. We define $\Delta = \max\{|N[p]| : p \in P\}$. The objective of the Euclidean liar’s domination problem is to find a minimum size subset $D$ of $P$ such that (i) for every point in $P$ there exists at least two points in $D$ which are at most distance one, and (ii) for every distinct pair of points $p_i$ and $p_j$ in $P$, $|(N[p_i] \cup N[p_j]) \cap D| \geq 3$, in other words, the number of points in $D$ that are within unit distance with points in the closed neighborhood union of $p_i$ and $p_j$ is at least three.

2 Complexity

In this section we show that the Euclidean liar’s domination problem is NP-complete for UDGs. The decision version of liar’s dominating set of a UDG can be defined as follows.

**UDG LIAR’S DOMINATING SET (UDG-LR-DOM)**

**Instance** : A unit disk graph $G = (V, E)$ and a positive integer $k$.

**Question** : Does there exist a liar’s dominating set $L$ of $G$ such that $|L| \leq k$?

We prove the NP-completeness of UDG-LR-DOM by reducing dominating set problem defined on a planar graph with maximum degree 3 to it, which is known to be NP-complete \([2]\). The decision version of dominating set of a planar graph with maximum degree 3 can be defined as follows.

**PLANAR DOMINATING SET (PLA-DOM)**

**Instance** : A planar graph $G = (V, E)$ with maximum degree 3 and a positive integer $k$.

**Question** : Does there exist a dominating set $D$ of $G$ such that $|D| \leq k$?

**Lemma 1** (\([8]\)). A planar graph $G = (V, E)$ with maximum degree 4 can be embedded in the plane using $O(|V|)$ area in such a way that its vertices are at integer co-ordinates and its edges are drawn so that they are made up of line segments of the form $x = i$ or $y = j$, for integers $i$ and $j$.

Algorithms to produce such embeddings are discussed in \([3, 4]\). Many standard graph theoretic problems on UDGs are shown to be NP-complete with the aid of Lemma\([1, 1]\).
Figure 1: (a) A planar graph $G$ with maximum degree 3, (b) An embedding of $G$ on a grid in the plane.

**Lemma 2.** Let $G = (V, E)$ be a planar graph with maximum degree 3 and $|E| > 2$. $G$ can be embedded in the plane such that its vertices are at $(4i, 4j)$ and its edges are drawn as a sequence of consecutive line segments drawn on the lines $x = 4i$ or $y = 4j$ for some integers $i$ and $j$.

In summary, we can draw a planar graph $G = (V, E)$ of maximum degree 3 on a grid in the plane, where each grid cell is of size $4 \times 4$, such that:

1. Each vertex $v_i$ in $G$ is replaced by a point $p_i$ in the plane.
2. The co-ordinates of each point $p_i$ (corresponding to a vertex $v_i$) are $(4i, 4j)$ for some integers $i$ and $j$ (see Figure 2).
3. An edge between two points is represented as a sequence of consecutive line segments and is drawn on the lines $x = 4i$ or $y = 4j$ for some integers $i$ or $j$ (these consecutive line segments may bend at some positions of the form $(4i', 4j')$).
4. No two lines representing edges of $G$ intersect each other, i.e., any two set of consecutive line segments correspond to two distinct edges of $G$ can not have a common point unless the edges incident at a vertex in $G$.

**Lemma 3.** A unit disk graph $G' = (V', E')$ can be constructed from the embedding in polynomial time.

**Proof.** Let us first embed the graph $G$ in the plane and divide the set of line segments in the embedding into two categories, namely, proper and improper. We call a line segment is proper if none of its end points corresponds to a vertex in $G$. For each edge $(p_i, p_j)$ of length 4 units we add four points such that two points at distances 1 and 1.5 units from $p_i$ and $p_j$ respectively (see edge $(p_4, p_6)$ in Figure 2(a)). For each edge of length greater than 4 units, we add the following points: for an improper line segment four points at distances 1,
1.5, 2.5, and 3.5 units respectively from the end point corresponds to a vertex in $G$ and four points for a proper line segment at distances 0.5 and 1.5 units from its end points (see Figure 2(a)). If the total number of line segments used in the embedding is $l$, then the sum of the lengths of the line segments is $4l$ as each line segment has length 4 units.

Draw a line segment of length 1.4 units (on the lines $x = 4i$ or $y = 4j$ for some integers $i$ or $j$) for every point $p_i$ (as shown in Figure 2(a)) corresponds a vertex $v_i$ in $G$ without coinciding with the line segments that had already been drawn before. Observe that adding this line segment on the lines $x = 4i$ or $y = 4j$ is possible with out loosing the planarity as the maximum degree of $G$ is 3. Now, add three points (say $x_i$, $y_i$, and $z_i$) at distances 0.2, 1.2, and 1.4 units respectively from $p_i$.

For convenience we name the points added (i) correspond to vertices of $G$ by node points (ii) on the line segments of length greater than or equal to 4 by joint points, and (iii) on the line segments of length 1.4 by support points. Let us denote these three sets of points by $N$, $J$, and $S$ respectively. In Figure 2(a) these sets of points reppresented as set of solid circles, solid squares, and circles respectively. Let $N = \{p_1, p_2, \ldots, p_n\}$, $J = \{q_1, q_2, \ldots, q_m\}$, and $S = \{x_i, y_i, z_i \mid 1 \leq i \leq n\}$. After defining the above sets, remove all the line segments.

Now we construct a UDG $G' = (V', E')$, where $V' = N \cup J \cup S$ and there is an edge between two points in $V'$ if and only if the Euclidean distance between the points is at most 1 (see Figure 2(b)). Observe that, $|N| = n$, $|J| = 4l(= m)$, where $l$ is the total length of the segments having length greater than or equal to 4, and $|S| = 3n$. Hence, $|V'| = 4(n + l)$ and $l$ is bounded by a polynomial of $n$. Therefore $G'$ can be constructed in polynomial time.

**Theorem 1.** UDG-LR-DOM is NP-complete.

**Proof.** For any given subset $L$ of $V'$ and a positive integer $k'$, it is easy to verify that the subset $L$ is a liar’s dominating set of size at most $k'$ or not. Hence UDG-LR-DOM belongs to the class NP.
We prove the hardness of UDG-LR-DOM by reducing PLA-DOM to it. Let an instance, \( G = (V, E) \), of PLA-DOM has been given. Construct an instance, a UDG \( G' = (V', E') \), of UDG-LR-DOM as discussed in Lemma 3. We now prove the following claim: \( G \) has a dominating set of size at most \( k \) if and only if \( G' \) has a liar’s dominating set of size at most \( k' = k + 4l + 3n \).

**Necessity**: Let \( D \subseteq V \) be the given dominating set of \( G \) with \( |D| \leq k \). Let \( L = D \cup J \cup S \). We prove that \( L \) is a liar’s dominating set of \( G' \).

(i) Every point \( p_i \) in \( N \) is dominated once by a point \( x_i \) in \( S \) and by at least one point in \( J \). Since \( J \subseteq L \), every point in \( J \) is dominated by itself and by its neighbor and maybe by one point in \( D \). Similarly, \( S \subseteq L \), every point in \( S \) is double dominated by points in \( L \). Thus, every point in \( V' \) satisfies the first condition of liar’s dominating set.

(ii) Now consider every distinct pair of points in \( V' \). Every point \( p_i \) in \( N \) is dominated by \( x_i \) and some \( q_i \) in \( J \). Therefore, \( |(N[p_i] \cup N[q_j]) \cap L| \geq |\{x_i, q_i, x_j, q_j\}| = 4 \). Similarly, \( |(N[p_i] \cup N[x_j]) \cap L| \geq |\{x_i, q_i, x_j, y_j\}| = 4 \) and \( |(N[q_j] \cup N[p_i]) \cap L| \geq |\{q_j, x_j, q_i\}| = 3 \). Also \( |(N[x_i] \cup N[z_i]) \cap L| \geq |\{x_i, y_i, z_i\}| = 4 \). In the same way we can prove that the rest of the pair combinations have at least three points of \( L \) in their closed neighborhood union. Thus every distinct pair of points in \( V' \) satisfies the second condition of liar’s dominating set.

So \( L \) is a liar’s dominating set of \( G' \) and \( |L| = |D| + |J| + |S| \leq k + 4l + 3n = k' \). Thus the necessity follows.

**Sufficiency**: Let \( L \subseteq V' \) be a liar’s dominating set of size at most \( k' = k + 4l + 3n \) We prove that \( G \) has a dominating set of size at most \( k \).

Observe that we added points \( x_i, y_i, z_i \) in such a way that \( p_i \) is adjacent to \( x_i \), \( x_i \) is adjacent to \( y_i \) and \( y_i \) is adjacent to \( z_i \) i.e., \( \{(p_i, x_i), (x_i, y_i), (y_i, z_i)\} \in E' \) for each \( i \). Hence, \( z_i \) and \( y_i \) must be in \( L \) due to the first condition of liar’s domination. Also, every component of \( L \) must contain at least three vertices due to the second condition of liar’s domination. Hence, \( x_i \in L \). Therefore, any liar’s dominating set of \( G' \) must contain \( \{x_i, y_i, z_i\}, 1 \leq i \leq n \) i.e., \( S \subseteq L \). These account for \( 3n \) vertices of \( L \). Let \( L' = L \setminus S \). Now we shall show that, by removing or replacing some points in \( L' \), \( k \) node points can be chosen such that the corresponding vertices in \( G \) is a dominating set of \( G \). Note that \( L' \) is a dominating set of the UDGG \( G'' = (V'', E'') \), where \( V'' = V' \setminus S \), \( E'' = E' \setminus \{(p_i, x_i), (x_i, y_i), (y_i, z_i) | 1 \leq i \leq n\} \) and \( |L'| = k + 4l \). In order to ensure the liar’s domination, every segment of length greater than or equal to 4 in \( G'' \) should have at least two joint points in \( L' \). If there are more than two joint points corresponding to a segment in \( L' \), then we remove and/or replace the joint points so that each segment will have only two joint points while ensuring the domination. Now, \( L' \) has been updated. Let \( L'' \) be the set obtained after updating \( L' \) and \( L'' \) is also a dominating set of \( G'' \) with cardinality at most \( k + 2l \).

We obtain the required dominating set \( D \) of \( G \) from \( L'' \) as follows: consider a series of line segments, say \( I = [p_i, p_j] \), corresponding an edge \( (p_i, p_j) \) of \( G'' \), where \( |I| = 4l' \) i.e., \( I \) has \( l' \) segments. If none of \( p_i \) and \( p_j \) are in \( L'' \), then replace a point in \( L'' \) by \( p_i \) with out loosing the domination property (existence of such a point is guaranteed as \( L'' \) is a dominating set). We apply this to all \( I \)'s. After applying the above process to all \( I \)'s, if there is an edge \( (p_i, p_j) \) such that none of \( p_i \) and \( p_j \) are in \( L' \), then there must exist \( I_1 = [p_s, p_i] \) and \( I_2 = [p_s, p_j] \) with lengths \( 4l_1 \) and \( 4l_2 \) corresponding to some edges in \( G'' \) such that \( p_s \) and \( p_t \) are
in $L''$. From the above preprocessing it is clear that $I_1$ and $I_2$ have at least $2l_1$ and $2l_2$ joint points in $L''$.

From the above argument, there are at least $2l$ joint points in $L''$, where $l$ is the total number of line segments used in $G''$. This means that there are at most $|L''| - 2l(=k)$ node points in $L''$.

Let $D = \{v_i \in V \mid v_i$ corresponds to a node point in $L''\}$. So, $D$ is a dominating set of $G$ and $|D| \leq k$. Thus the sufficiency follows.

Hence, UDG-LR-DOM is NP-complete.

\[ \square \]

3 Conclusion

In this article we considered the liar’s domination problem on unit disk graphs and proved that the problem belongs to NP-complete class.

References

[1] Brent N Clark, Charles J Colbourn, and David S Johnson. Unit disk graphs. *Discrete mathematics*, 86(1-3):165–177, 1990.

[2] Michael R Gary and David S Johnson. Computers and intractability: A guide to the theory of NP-completeness, 1979.

[3] John Hopcroft and Robert Tarjan. Efficient planarity testing. *Journal of the ACM (JACM)*, 21(4):549–568, 1974.

[4] Alon Itai, Christos H Papadimitriou, and Jayme Luiz Szwarcfiter. Hamilton paths in grid graphs. *SIAM Journal on Computing*, 11(4):676–686, 1982.

[5] BS Panda and S Paul. Liar’s domination in graphs: Complexity and algorithm. *Discrete Applied Mathematics*, 161(7):1085–1092, 2013.

[6] Miranda L Roden and Peter J Slater. Liar’s domination in graphs. *Discrete mathematics*, 309(19):5884–5890, 2009.

[7] Peter J Slater. Liar’s domination. *Networks*, 54(2):70–74, 2009.

[8] Leslie G Valiant. Universality considerations in VLSI circuits. *IEEE Transactions on Computers*, 100(2):135–140, 1981.