Adaptive $l_1$-regularization for short-selling control in portfolio selection

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Abstract

We consider the $l_1$-regularized Markowitz model, where a $l_1$-penalty term is added to the objective function of the classical mean-variance one to stabilize the solution process, promoting sparsity in the solution. The $l_1$-penalty term can also be interpreted in terms of short sales, on which several financial markets have posed restrictions. The choice of the regularization parameter plays a key role to obtain optimal portfolios that meet the financial requirements. We propose an updating rule for the regularization parameter in Bregman iteration to control both the sparsity and the number of short positions. We show that the modified scheme preserves the properties of the original one. Numerical tests are reported, which show the effectiveness of the approach.

Keywords: Portfolio selection. Markowitz model. $l_1$-regularization. Bregman iteration.

1 Introduction

In the classical Markowitz mean-variance framework [1], portfolio selection aims at the construction of an investment portfolio that exposes investor to minimum risk providing him a fixed expected return. This approach was proposed by Markowitz in his aforementioned seminal paper, where he stated that portfolio selection strategy should provide an optimal trade-off between expected return and risk (mean-variance approach). In a successive work [2], Markowitz reinforced his theory arguing that, under certain, mild conditions, a portfolio from a mean-variance efficient frontier will approximately maximize the investor’s expected utility.

Markowitz model relies on information about future, since expected returns should actually be computed discounting future flows, that are clearly not available. A common choice is to use historical data as predictive of the future behavior of asset returns. This practice has certain drawbacks; indeed, a limited

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amount of relevant historical data is often available. Moreover, correlation between assets returns can lead to ill-conditioned covariance matrices. It is well known that errors in estimation of expected values affect solutions more severely than errors on variances. For this reason, to overcome this issue some authors focus on minimum-variance portfolios, which do not take into account the return constraint. We recall [3] and references therein. Moreover, different regularization techniques have been suggested; a review of them can be found in [4]. Among these, penalization techniques have been considered, both for the minimum- and the mean-variance approach. In [3] \textit{l}_1 \textit{ and squared-l}_2 \textit{ norm constraints are proposed for the minimum-variance criterion.} In [4] an algorithm for the optimal minimum-variance portfolio selection with a weighted \textit{l}_1 \textit{ and squared-l}_2 \textit{ norm penalty is presented.} In [6] authors regularize the mean-variance objective function with a weighted elastic net penalty.

In this paper, we consider the \textit{l}_1 \textit{ mean-variance regularized model introduced in [7], where a \textit{l}_1 \textit{-penalty term is added to promote sparsity in the solution. Since solutions establish the amount of capital to be invested in each available security, sparsity means that money are invested in a few securities, the active positions. This allows investor to reduce both the number of positions to be monitored and the transaction costs, particularly relevant for small investors, that are not taken into account in the theoretical Markowitz model. Another useful interpretation of \textit{l}_1 \textit{ regularization is related to the amount of shorting in the portfolio; from the financial point of view negative solutions correspond to short sales.} In many markets, among which Italy, Germany and Switzerland, restrictions on short sales have been established in the last years, thus short-controlling is desired as well. Then, the choice of the regularization parameter is crucial in order to provide sparse solutions, with either a limited or null number of negative components, preserving fidelity to data.

In this paper we propose an iterative algorithm based on a modified Bregman iteration. Bregman iteration is a well established method for the solution of \textit{l}_1 \textit{-regularized optimization problems.} It has been successfully applied in different fields, as image restoration [8] matrix rank minimization [9], compressed sensing [10] and finance [6]. Our modification to the original scheme introduces an adaptive updating rule for the regularization parameter in the regularized model. The algorithm selects a value capable to provide solutions satisfying a fixed financial target, formulated in terms of limited number of active and/or short positions.

We show that our modified scheme preserves the properties of the original one and is able to select a good value of the regularization parameter within a negligible computational time. Numerical tests confirm the effectiveness of the proposed algorithm.

The paper is organized as follows. In section 2 we briefly recall Markowitz mean-variance model. In section 3 we introduce Bregman iteration for portfolio selection. Our main results are in section 4, where we introduce our algorithm, based on a modified Bregman iteration, for the \textit{l}_1 \textit{-regularized Markowitz model.} In section 5 we validate our approach by means of several numerical experiments. Finally, in section 6 we give some conclusion and outline future work.
2 Portfolio selection model

We refer to the classical Markowitz mean-variance framework. Given \( n \) traded assets, the core of the problem is to establish the amount of capital to be invested in each available security.

We assume that one unit of capital is available and define

\[
\mathbf{w} = (w_1, w_2, \ldots, w_n)^T
\]

the portfolio weight vector, that is, the amount \( w_i \) is invested in the \( i \)-th security.

Asset returns are assumed to be stationary. If we denote with

\[
\mu = (\mu_1, \mu_2, \ldots, \mu_n)^T
\]

the expected asset returns, then the expected portfolio return is their weighted sum:

\[
\sum_{i=1}^{n} w_i \mu_i.
\]

We moreover denote with \( \sigma_{ij} \) is the covariance between returns of securities \( i \) and \( j \). The portfolio risk is measured by means of its variance, given by:

\[
V = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} w_i w_j.
\]

Let \( \rho \) be the fixed expected portfolio return and \( C \) the covariance matrix of returns. Portfolio selection is formulated as the following quadratic constrained optimization problem:

\[
\min_{\mathbf{w}} \mathbf{w}^T C \mathbf{w} \\
\text{s.t.} \\
\mathbf{w}^T \mu = \rho \\
\mathbf{w}^T \mathbf{1}_n = 1,
\]

where \( \mathbf{1} \) is the vector of ones of length \( n \). The first constraint fixes the expected return, according to (1). The second one is a budget constraint which establishes that all the available capital is invested. The non-negativity constraint is often added to avoid short positions. We do not consider it here, since we aim at controlling short positions by tuning the regularization parameter, as it is discussed in the following.

Let us consider a set of \( m \) evenly spaced dates

\[
\mathbf{t} = (t_1, t_2, \ldots, t_m)
\]

at which asset returns are estimated and build the matrix \( \mathbf{R} \in \mathbb{R}^{m \times n} \) that contains observed historical returns of asset \( i \) on its \( i \)-th column. It can be shown that problem (2) can be stated in the following form:

\[
\min_{\mathbf{w}} \frac{1}{m} ||\mathbf{1}_m - \mathbf{Rw}||_2^2 \\
\text{s.t.} \\
\mathbf{w}^T \mu = \rho \\
\mathbf{w}^T \mathbf{1}_n = 1.
\]
As the asset returns are typically correlated, the matrix $R$ could have some singular values close to zero; therefore regularization techniques, that add to objective function some form of a priori knowledge about the solution, must be considered. In this paper we consider the following $l_1$-regularized problem:

$$\begin{align*}
\min_w & \|\rho 1_m - Rw\|_2^2 + \tau \|w\|_1 \\
\text{s.t.} & \\
& w^T \mu = \rho \\
& w^T 1_n = 1,
\end{align*}$$

(4)

where the $1/m$ term has been incorporated into the regularization one. From the second constraint in (4) it follows that the objective function can be equivalently written as:

$$\|\rho 1_m - Rw\|_2^2 + 2\tau \sum_{i: w_i < 0} |w_i| + \tau.$$  

This form points out that $l_1$ penalty is equivalent to a penalty on short positions.

In the limit of very large values of the regularization parameter, we obtain a portfolio with only positive weights, as observed also in [11].

3 \hspace{1em} Bregman iteration for portfolio selection

Portfolio selection can be formulated as the constrained nonlinear optimization problem:

$$\begin{align*}
\min_w & E(w) \\
\text{s.t.} & \\
& Aw = b,
\end{align*}$$

(5)

where

$$E(w) = \|\rho 1 - Rw\|_2^2 + \tau \|w\|_1$$

is strictly convex and non-smooth due to the presence of the $l_1$ penalty term,

$$A = \begin{pmatrix} \mu^T \\ 1_n \end{pmatrix} \in \mathbb{R}^{2 \times n} \quad \text{and} \quad b = (\rho, 1)^T \in \mathbb{R}^2.$$ 

One way to solve (5) is to convert it into an unconstrained problem, for example by using a penalty function/continuation method, which approximates it by a sequence:

$$\min_w E(w) + \frac{\lambda_k}{2} \|Aw - b\|_2^2, \quad \lambda_k \in \mathbb{R}^+.$$ 

(6)

It is well known that, if the $k$-th subproblem (6) has solution $w_k$ and $\{\lambda_k\}$ is an increasing sequence tending to $\infty$ as $k \to \infty$, any limit point of $\{w_k\}$ is a solution of (5) [12]. Therefore, in many problems it is necessary to choose very large values of $\lambda_k$ and it makes (6) extremely difficult to solve numerically. Alternatively, Bregman iteration can be used to reduce (5) in a short sequence of unconstrained problems by using the Bregman distance associated with $E$ [12], where, conversely, the value of $\lambda_k$ in (6) remains constant.
The Bregman distance \[13\] associated with a proper convex functional \( E(w) : \mathbb{R}^n \rightarrow \mathbb{R} \) at point \( v \) is defined as:

\[
D^p_E(w, v) = E(w) - E(v) - \langle p, w - v \rangle, \tag{7}
\]

where \( p \in \partial E(v) \) is a subgradient in the subdifferential of \( E \) at point \( v \) and \( \langle . , . \rangle \) denotes the canonical inner product in \( \mathbb{R}^n \). It is not a distance in the usual sense because it is not in general symmetric but it does measure closeness between \( w \) and \( v \) in the sense that if \( u \) lies on the line segment \((w, v)\), then the line segment \((w, u)\) has smaller Bregman distance than \((w, v)\) does. At each Bregman iteration \( E(w) \) is replaced by the Bregman distance so a subproblem in the form of (6) is solved according to the following iterative scheme:

\[
\begin{align*}
\{ w_{k+1} &= \text{argmin}_{w} D^p_k(w, w_k) + \frac{\lambda}{2} \| Aw - b \|_2^2, \\
p_{k+1} &= p_k - \lambda A^T(Aw_{k+1} - b) \in \partial E(w_{k+1}).
\}
\tag{8}
\end{align*}
\]

The updating rule of \( p_{k+1} \) is chosen according to the first-order optimality condition for \( w_{k+1} \) and ensures that \( D^p_{k+1}(w, w_{k+1}) \) is well defined. Under suitable hypotheses the convergence of the sequence \( \{ w_k \} \) to the solution of the constrained problem [5] is guaranteed in a finite number of steps [8]; furthermore, using the equivalence of Bregman iteration with the augmented Lagrangian one [14], convergence is proved also in [14]. Note that the convergence results guarantee the monotonic decrease of \( \| Aw_k - b \|_2^2 \), thus for large \( k \) the constraint conditions are satisfied to an arbitrary high degree of accuracy. This yields a natural stopping criterion according to a discrepancy principle.

Since there is generally no explicit expression for the solution of the sub-minimization problem involved in (8), at each iteration the solution is computed inexactly using an iterative solver. So, in the last years there has been a growing interest about inexact solution of the subproblem involved in Bregman iteration. In recent papers it is proved that, for many applications, Bregman iterations yield very accurate solutions even if subproblems are not solved as accurately [8, 10, 16]. In [17] convergence results are obtained for piece-wise linear convex functionals. In [18] the inexactness in the inner solution is controlled by a criterion that preserves the convergence of the Bregman iteration and its features in image restoration.

4 Modified Bregman iteration

A crucial issue in the solution of (4) is the choice of a suitable value for the regularization parameter \( \tau \), as already pointed out. The aim is to select \( \tau \) so to realize a trade-off between sparsity and short-controlling (requiring sufficiently large values) and fidelity to data (requiring small values). While the literature offers a significative number of methods for Tikhonov regularization [19], \( l_1 \) regularization parameter selection is often based on problem-dependent criteria and related to iterative empirical estimates, that require a high computational cost. In [7] least-angle regression (LARS) algorithm proceeds by decreasing
the value of $\tau$ progressively from very large values, exploiting the fact that the dependence of the optimal weights on $\tau$ is piecewise linear.

In this section, we present a numerical algorithm, based on a modified Bregman iteration with adaptive updating rule for $\tau$. Our basic idea for defining the rule for $\tau$ comes from the well-known properties of the $l_1$ norm and the following proposition [7]:

**Proposition 1** Let $w_{\tau_1}$ and $w_{\tau_2}$ be solution of the $l_1$-regularized problem (4) with $\tau_1$ and $\tau_2$ respectively. If some of $(w_{\tau_2})_i$ are negative and all the entries in $w_{\tau_1}$ are positive or zero, we have $\tau_1 > \tau_2$.

We then propose an updating rule for $\tau$ that generates an increasing sequence of values. Our aim is to modify Bregman iteration, in order to produce solutions satisfying a fixed financial target, defined in terms of sparsity or short-controlling or a combination of them.

Let 

$$E_k(w) = \|Rw - \rho\|_2^2 + \tau_k \|w\|_1, \quad k = 0, 1, \ldots$$

We now prove the main result of this paper:

**Theorem 1** Given $(w_{k+1}, p_{k+1})$ provided by (8) applied to $E_k$, it holds

$$\hat{p}_{k+1} = \frac{\tau_{k+1}}{\tau_k} p_{k+1} + 2\left(1 - \frac{\tau_{k+1}}{\tau_k}\right) R^T (Rw_{k+1} - \rho) \in \partial E_{k+1}(w_{k+1}). \quad (9)$$

**Proof.** It holds $p_{k+1} \in \partial E_k(w_{k+1}) = \partial (\tau_k \|w_{k+1}\|_1) + \nabla \left(\|Rw_{k+1} - \rho\|_2^2\right)$, thus a vector $q_{k+1} \in \partial (\tau_k \|w_{k+1}\|_1)$ exists such that

$$p_{k+1} = q_{k+1} + 2 R^T (Rw_{k+1} - \rho).$$

It follows that 

$$q_{k+1} = p_{k+1} - 2R^T (Rw_{k+1} - \rho) \in \partial (\tau_{k+1} \|w_{k+1}\|_1).$$

It is easy to verify that:

$$\frac{\tau_{k+1}}{\tau_k} q_{k+1} \in \partial (\tau_{k+1} \|w_{k+1}\|_1).$$

Then 

$$\frac{\tau_{k+1}}{\tau_k} q_{k+1} + 2 R^T (Rw_{k+1} - \rho) \in \partial E_{k+1}(w_{k+1}),$$

which completes the proof. \[\square\]

We propose the following modified Bregman iteration:

$$\begin{cases}
    p_k = \frac{\tau_k}{\tau_{k-1}} p_k + 2 \left(1 - \frac{\tau_k}{\tau_{k-1}}\right) R^T (Rw_k - \rho), \\
    w_{k+1} = \arg\min_w D_{E_k}^{p_k}(w, w_k) + \frac{\lambda}{2} \|Aw - b\|_2^2, \\
    p_{k+1} = p_k - \lambda A^T (Aw_{k+1} - b), \\
    \tau_{k+1} = h(\tau_k)
\end{cases} \quad (10)$$

where $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is an increasing, bounded function.

Note that relation (9) in Theorem 1 guarantees that the iterative scheme (10) is well defined, thus preserves the properties of the original one.
In this paper we choose a multiplicative form for the function $h$. We set $\tau_{k+1} = \eta_{k+1}\tau_k$, where $\eta_{k+1}$ depends on $w_{k+1}$ according to the financial target, as shown in Algorithm 1. Note that we are not ensured that a finite value of $\tau$ exists that satisfies the financial target, thus we force $h$ to be bounded by setting a maximum value $\tau_{\text{max}}$. If the financial target is met at a certain step, then $\eta_k = 1$ for all successive iterations. Conversely, $\tau$ is set to $\tau_{\text{max}}$. In any case, there exists an iteration $\bar{k}$ such that $\tau_k$ remain fixed at a value $\bar{\tau}$ for $k \geq \bar{k}$.

**Algorithm 1** Modified Bregman Iteration for portfolio selection

Given $\tau_0 > 0$, $\tau_{\text{max}}$, $\lambda$, $\theta > 1$ % Model parameters

Given $n_{\text{short}}$, $n_{\text{act}}$ % Financial target parameters

$k := 0$

$w_0 := 0$, $p_0 := 0$, $\tau_{-1} := \tau_0$, $\lambda$ % Initialization

while “stopping rule not satisfied” do

$p_k = \frac{\tau_{k-1}p_k}{\tau_{k-1}} + \left(1 - \frac{\tau_{k-1}}{\tau_k}\right)R^T(Rw_k - p)$

$w_{k+1} = \text{argmin}_w D_{E_k}(w, w_k) + \frac{\lambda}{2} \|Aw - p\|_2^2$

$p_{k+1} = p_k - \lambda A^T(Aw_{k+1} - b)$

$W_{k+1}^- = \{i : (w_{k+1})_i < 0\}$

$W_{k+1}^+ = \{i : (w_{k+1})_i \neq 0\}$

if $|W_{k+1}^-| > n_{\text{short}}$ or $|W_{k+1}^+| > n_{\text{act}}$ then

$\eta_{k+1} = \theta$

else

$\eta_{k+1} = 1$

end if

$\tau_{k+1} = \min\{\eta_{k+1}\tau_k, \tau_{\text{max}}\}$

$k := k + 1$

end while

**Theorem 2** Let $\bar{\tau}$ be the regularization parameter value produced by the Algorithm 1 at step $\bar{k}$. Suppose that at a certain step $k \geq \bar{k}$ the iterate $w_k$ satisfies $Aw_k = b$. Then $w_k$ is a solution to the constrained problem

$$\min_w E_k(w)$$

s.t.

$$Aw = b.$$ (11)

**Proof.** We note that $\tau_k = \bar{\tau} \forall k \geq \bar{k}$, thus the objective function is fixed for $k \geq \bar{k}$. Therefore, the proof follows the proof of Theorem 2.2 in [14].

This result shows that if the sequence provided by Algorithm 1 converges in the sense of $\lim_{k \to \infty} |Aw_k - b|_2 = 0$, then the iterates $w_k$ will get arbitrarily close to a solution to the original constrained problem with $\tau = \bar{\tau}$.  

7
5 Experimental results

In this section, we discuss some computational issues and show the effectiveness of Algorithm 1 for solving the regularized portfolio optimization problem (4). In Algorithm 1 we set $\lambda = 1$, $\tau_0 = 2^{-5}$, $\tau_{\text{max}} = 1$ and $\theta = 2$. Iterations are stopped as soon as $\|A w_k - b\|_2 \leq T ol$ with $T ol = 10^{-4}$ that, from the financial point of the view, guarantees constraints at a sufficient accuracy. We implement the Fast Proximal Gradient method with backtracking stepsize rule (FISTA) [20] to solve the unconstrained subproblem at each modified Bregman iteration in Algorithm 1. FISTA is an accelerated variant of Forward Backward (FB) algorithm, built upon the ideas of Güler [21] and Nesterov [22]. Note that FB is a first-order method for minimizing objective functions $F(x) \equiv f(x) + g(x)$, where $g : \mathbb{R}^n \to \mathbb{R}$ is a proper, convex, lower semicontinuous function with $\text{dom}(g)$ closed, $f : \mathbb{R}^n \to \mathbb{R}$ is convex and $\nabla f$ is $L$-Lipschitz continuous. It generates a sequence $(x_n)_{n \in \mathbb{N}}$ in two separate stages; the former performs a forward (explicit) step which involves only $f$, while the latter performs a backward (implicit) step involving a proximal map associated to $g$ [23]. In our case we set $f = \|\rho_1 - R w\|_2^2 - < p, w > + \frac{1}{2} \|A w - b\|_2^2$ and $g = \tau \|w\|_1$, then the proximal map of $g$ is the simple and explicit Soft threshold operator:

$$\text{Prox}_g(w_i) = \text{sgn}(w_i)\left(|w_i| - \min\{|w_i|, \tau\}\right).$$

Inner iterations are stopped when the relative difference in Euclidean norm between two successive iterates is less than $T ol_{\text{inn}} = 10^{-4}$. All our experiments, some of which are reported in the following, show that it is not worth to require a great accuracy to the inner solver.

The tests have been performed in Matlab R2015a (v. 8.5, 64-bit) environment, on a six-core Xeon processor with 24 GB of RAM and 12 MB of cache memory, running Ubuntu/Linux 12.04.5. We compare our optimal portfolios with the evenly weighted one (the naive portfolio), usually taken as benchmark in literature [24]. This essentially for three reasons: it is easy to implement, many investors still use such simple rule to allocate their wealth across assets and it allows one to diversify the risk.

We evaluate our approach observing the out-of-sample performances of optimal portfolios as in [3]. This means that for each T-years period of asset returns, we use historical series to solve (4); the target return $\rho$ is fixed to the average return provided by the naive portfolio in those years. The optimal solution obtained in this way is used to build a portfolio that is retained for one year. We continue this process by moving one year ahead until we reach the end of the period, ending with a series of out-of-sample portfolios. We then compare the so obtained average return $\hat{\rho}$ and standard deviation values $\hat{\sigma}$ with the corresponding ones of the naive portfolio. We moreover compute the Sharpe ratio $SR = \hat{\rho}/\hat{\sigma}$; since one would desire great return and small variance values, the Sharpe ratio can be taken as reference value for the comparison. We present the results on three test problems; the first and the second one come from Fama and French database.

\footnote{data available at http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html#BookEquity}
used in [7]. We obtain comparable results in terms of optimal portfolio Sharpe ratio. The last test problem is built on data from Italian market.

5.1 Test 1: FF48

We consider the first database - FF48 - which contains monthly returns of 48 industry sector portfolios from July 1926 to December 2015. Using data from 1970 to 2015, we construct optimal portfolios and analyze their out-of-sample performance. Starting from July 1970, we use the \( T = 5 \)-years so 40 optimal portfolios are built, until June 2015. Portfolios in FF48 exhibit moderate correlation, indeed the condition number of \( C \) is \( O(10^4) \) for all simulations. We tested difference values of \( Tol_{inn} \); all our experiments, show that lower values of \( Tol_{inn} \) do not improve results, thus we show results obtained for \( Tol_{inn} = 10^{-4} \).

In table 1 for both optimal and naive portfolio, expected return, standard deviation and Sharpe ratio are reported, all expressed on annual basis. The optimal portfolios are no-short ones, \( (n_{short} = 0, n_{act} = 48) \), that is, the target is to obtain positive solutions. Values refer to average values computed over 8 years, grouped as described in the first column of the table. The first row contains average values computed over the all 40-years period of simulation. In all cases, optimal portfolio exhibits greater values of Sharpe ratio than the naive one.

| Period          | Optimal portfolio | Naive portfolio |
|----------------|------------------|----------------|
|                | \( \hat{\rho} \) | \( \hat{\sigma} \) | SR | \( \hat{\rho} \) | \( \hat{\sigma} \) | SR |
| 1975/07 - 2015/06 | 14%             | 38%            | 37% | 15%             | 60%            | 26% |
| 1975/07 - 1983/06 | 22%             | 44%            | 51% | 29%             | 63%            | 47% |
| 1983/07 - 1991/06 | 14%             | 39%            | 37% | 7%              | 59%            | 12% |
| 1991/07 - 1999/06 | 14%             | 29%            | 50% | 15%             | 50%            | 30% |
| 1999/07 - 2007/06 | 13%             | 34%            | 38% | 17%             | 58%            | 29% |
| 2007/07 - 2015/06 | 7%              | 44%            | 15% | 10%             | 69%            | 14% |

Table 1: Comparison between optimal no-short \( (n_{short} = 0, n_{act} = 48) \) and naive portfolio for FF48. Reported return and standard deviation are average values over 40 years (first line) and over groups of 8 years (lines 2 - 6).

In table 2 for the same financial target, we report results on optimal portfolios containing at most ten active positions \( (n_{short} = 48, n_{act} = 10) \). Values are interpreted as in table 1. In all cases, optimal portfolio exhibits again greater values of Sharpe ratio than the naive one. In figure 2 we report the number of active and short positions in optimal portfolios (top) and the number of modified...
Figure 1: Optimal portfolio for FF48, with $n_{\text{short}} = 0, n_{\text{act}} = 48$. Top: active positions. Bottom: number of modified Bregman iterations.
Table 2: Comparison between optimal and naive portfolio for FF48. Optimal portfolios contain at most ten active positions ($n_{short} = 48$, $n_{act} = 10$). Reported return and standard deviation are average values over 40 years (first line) and over groups of 8 years (lines 2 − 6).

| Period          | Optimal portfolio | Naive portfolio |
|-----------------|-------------------|-----------------|
|                 | $\hat{\rho}$ | $\hat{\sigma}$ | SR | $\hat{\rho}$ | $\hat{\sigma}$ | SR |
| 1975/07 − 2015/06 | 14% | 37% | 38% | 15% | 60% | 26% |
| 1975/07 − 1983/06 | 21% | 41% | 49% | 29% | 63% | 47% |
| 1983/07 − 1991/06 | 15% | 38% | 40% | 7% | 59% | 12% |
| 1991/07 − 1999/06 | 14% | 29% | 49% | 15% | 50% | 30% |
| 1999/07 − 2007/06 | 12% | 33% | 37% | 17% | 58% | 29% |
| 2007/07 − 2015/06 | 8% | 43% | 18% | 10% | 69% | 14% |

Table 3: Comparison between no-short optimal portfolios for FF48 produced by Algorithm 1 and LARS in [7]. Table 1. Reported return and standard deviation are average values over 30 years (first line) and over groups of 5 years (lines 2 − 7).

| Period          | Algorithm 1 | LARS |
|-----------------|-------------|------|
|                 | $\hat{\rho}$ | $\hat{\sigma}$ | SR | $\hat{\rho}$ | $\hat{\sigma}$ | SR |
| 1976/07 − 2006/06 | 17% | 37% | 46% | 41% | 41% |
| 1976/07 − 1981/06 | 23% | 43% | 53% | 48% | 49% |
| 1981/07 − 1986/06 | 23% | 36% | 64% | 41% | 57% |
| 1986/07 − 1991/06 | 9% | 45% | 20% | 45% | 20% |
| 1991/07 − 1996/06 | 16% | 21% | 76% | 26% | 62% |
| 1996/07 − 2001/06 | 16% | 38% | 42% | 40% | 40% |
| 2001/07 − 2006/06 | 13% | 39% | 33% | 43% | 30% |

Bregman iterations (bottom) for each year of simulation. In this case the average number of Bregman iterations is equal to 6. The values of $\tau$ range between $2^{-5}$ and $2^{-3}$. Finally in table 3 we report a comparison with results exhibited in Table 1 of paper [7]. We denote with Algorithm 1 the results produced by our optimization procedure and with LARS the results provided in [7]. We refer to the same 30-years simulation period, with the average taken over 5-years for each break-out period. We note that our procedure of regularization parameter selection produces higher values of Sharpe ratio; since the expected return is fixed by the constraint, this means that we obtain less risky portfolios.

5.2 Test 2: FF100

We here show results on the second database by Fama and French - FF100 - containing data of 100 portfolios which are the intersections of 10 portfolios formed on size and 10 portfolios formed on the ratio of book equity to market
Figure 2: Optimal portfolio for FF48, with $n_{\text{short}} = 48, n_{\text{act}} = 10$. Top: active and short positions in optimal portfolios. Bottom: number of modified Bregman iterations.
Table 4: Comparison between no-short ($n_{short} = 0, n_{act} = 100$) optimal and naive portfolio for FF100. Reported return and standard deviation are average values over 40 years (first line) and over groups of 8 years (lines 2–6).

| Period         | Optimal portfolio | Naive portfolio |
|----------------|-------------------|-----------------|
|                | $\hat{\rho}$     | $\hat{\sigma}$ | $\hat{\rho}$ | $\hat{\sigma}$ | $\hat{\rho}$ | $\hat{\sigma}$ | $\hat{\rho}$ | $\hat{\sigma}$ | $\hat{\rho}$ | $\hat{\sigma}$ |
|                | 14%               | 50%             | 29%           | 15%             | 57%             | 27% |
| 07/1975 – 06/1983 | 18%               | 54%             | 33%           | 24%             | 58%             | 41% |
| 07/1983 – 06/1991 | 15%               | 54%             | 28%           | 12%             | 60%             | 20% |
| 07/1991 – 06/1999 | 19%               | 38%             | 49%           | 18%             | 45%             | 39% |
| 07/1999 – 06/2007 | 14%               | 47%             | 29%           | 14%             | 56%             | 25% |
| 07/2007 – 06/2015 | 7%                | 56%             | 13%           | 10%             | 67%             | 14% |

equity. Also FF100 contains monthly returns from from July 1926 to December 2015.

We apply the same strategy as in FF48 ($T = 5$—years, 40 optimal portfolios constructed). Correlation values observed in FF100 are higher than in the previous test, the conditioning of $C$ is $O(10^{18})$.

In table 4 we show optimal no-short portfolios. We report the expected return, the standard deviation and the Sharpe ratio expressed on annual basis. On the overall period, optimal portfolio outperforms the naive one. The values of $\tau$ range between $2^{-4}$ and $2^{-2}$, the percentage of sparsity varies from 4% to 17% (Fig. 3). Note that, looking at details on each year of simulation, we observe negative returns for both optimal and naive portfolio. For instance, in the 8th year of simulation, optimal portfolio produces a loss of 4%, the naive one of 12%. In the 10th year the losses are of 1% and 10% respectively. This happens because almost all components in portfolios show decreased returns. Finally, we note that in the period 07/1975 – 06/1983 naive portfolio outperforms the optimal one. For instance, in the 3rd year of simulation the optimal portfolio, which contains 5 assets (56, 90, 91, 93, 95), produces a gain of 8%, versus a gain of 18% of the naive one. This behavior is essentially due to a drastic change in asset returns with respect to historical data. This situation could be controlled by a dynamic asset allocation strategies, for which at the beginning of each period during the investment horizon, the investor can freely rearrange the portfolio, but it isn’t the aim of this paper. We finally report also in this case a comparison with results exhibited in [7], Table 3. We note that we obtain higher values of the Sharpe ratio.

### 5.3 Test 3: IT72

We here consider a portfolio constructed on real data from Italian market. It considers the monthly returns of 72 equities, from September 2009 to August 2016. Assets are reported in table 6. 25 assets are included in the FTSE MIB index computation. The FTSE MIB is the primary benchmark Index for the Italian equity markets. The Index is comprised of highly liquid, leading compa-
Figure 3: Optimal portfolio for FF100, with $n_{\text{short}} = 0, n_{\text{act}} = 100$. Top: active positions. Bottom: number of modified Bregman iterations.
### Optimal portfolio

| Period         | Algorithm 1 | LARS |
|----------------|-------------|------|
|                | $\hat{\rho}$ | $\hat{\sigma}$ | $\hat{\rho}$ | $\hat{\sigma}$ |
| 1976/07 – 2006/06 | 16%         | 48%           | 33%         | 53%         |
| 1976/07 – 1981/06 | 12%         | 54%           | 22%         | 59%         |
| 1981/07 – 1986/06 | 24%         | 44%           | 55%         | 49%         |
| 1986/07 – 1991/06 | 10%         | 61%           | 16%         | 65%         |
| 1991/07 – 1996/06 | 19%         | 29%           | 66%         | 31%         |
| 1996/07 – 2001/06 | 18%         | 52%           | 35%         | 52%         |
| 2001/07 – 2006/06 | 11%         | 49%           | 22%         | 55%         |

Table 5: Comparison between no-short optimal portfolios for FF100 produced by Algorithm 1 and LARS in [7]. Table 3. Reported return and standard deviation are average values over 30 years (first line) and over groups of 5 years (lines 2 – 7).

| Sector                          | Company Name                   |
|---------------------------------|--------------------------------|
| ACCA SPA                        | EL TOWERS SPA                  |
| ACEA SPA                        | PRYSMIAN SPA                   |
| AUTOGRILL SPA                   | RECORDATI SPA                  |
| AMPLIFON SPA                    | REPLY SPA                      |
| ATLANTIA SPA                    | SABAF SPA                      |
| AZIMUT HOLDING SPA              | SAGILE SPA                     |
| BASICNET SPA                    | SALINI IMPREGILO SPA           |
| BIALETTI INDUSTRIE SPA          | SAFILO GROUP SPA               |
| BANCA MEDOLANUM SPA             | SARES GETTERS SPA              |
| BANCA MONTE DEI PASCHI SIENA   | SIA SPA                        |
| BANCO POPOLARE SC               | SPEG SPA                       |
| BANCA POPOL EMILIA ROMAGNA      | SMA SPA                        |
| BREMBO SPA                      | SMN SPA                        |
| BUIZZI UNICEM SPA               | SABAS SPA                      |
| BUIZZI UNICEM SPA-RSP           | ANSALDO STS SPA                |
| CAIRO COMMUNICATIONS SPA        | SAI SPA                        |
| CEMENTIR HOLDING SPA            | TELECOM ITALIA SPA             |
| DAVIDE CAMPARI-MILANO SPA       | TELECOM ITALIA-RSP             |
| CREDITO VALTALLESE SCARL        | TDS SPA                        |
| DATALOGIC SPA                   | TERN SPAA                      |
| DANIIELI & CO                   | UBI BANCA SPA                   |
| DIARIOVIN SPA                   | UNICREDIT SPA                   |
| D'AMICO INTERNATIONAL SHIPPI    | UNIPOL GRUPPO FINANZIARIO SPA   |
| DE LONGHI SPA                   | UNIPOLSAI SPA                   |
|                                | ZIGNAGO VETRO SPA               |

Table 6: IT72 assets.

nies across different sectors, indeed it captures about the 80% of the domestic market capitalization. The FTSE MIB is computed on 40 Italian equities and seeks to replicate the broad sector weights of the Italian stock market.

Starting from September 2009, we use the $T = 6$-years data to build the optimal portfolio from September 2015 until August 2016. The conditioning of $R^T R$ is $O(10^9)$. In figure [4] we graphically show the composition of the optimal portfolio we constructed. The optimization strategy allocates the investor wealth on 14 equities, with weights represented as percentage in the figure, among which 5 belong to the FTSE MIB set. The result is obtained in 10 Bregman iterations, with $\tau = 2^{-3}$. We note that the optimal portfolio has return and standard deviation, on annual basis, given by 11% and 34% respectively. The same values for the naive portfolio are $-14\%$ and 60%, thus the latter provides a loss to the investor.
Figure 4: Optimal portfolio on Italian market equities. Built on monthly historical returns of 72 equities from September 2009 to August 2016.
6 Conclusions

We have proposed an algorithm, which exploits the Bregman iteration method, for the portfolio selection problem formulated as an $l_1$-regularized mean-variance model. The choice of the regularization parameter is the key point in order to provide solutions with either a limited or null number of negative components and/or a limited number of active positions. Our main contribution is the modification of the Bregman iteration, which adaptively sets the value of the regularization parameter depending on the financial target. It is observed that both sparsity and short-controlling are obtained for sufficiently large values of the regularization parameter. The basic idea is then to generate an increasing sequence of values and fix it when requirements are met. We show that our modification to the Bregman iteration preserves the convergence of the original scheme. Numerical experiments confirm the effectiveness of the proposed algorithm.

We saw in our experiments that sometimes the effectiveness of the optimization strategy can be affected by changes in market conditions. Future work could consider dynamic asset allocation, which involves frequent portfolio adjustments.

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