Magnon Bose–Einstein condensation and superconductivity in a frustrated Kondo lattice

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Motivated by recent experiments on magnetically frustrated heavy fermion metals, we theoretically study the phase diagram of the Kondo lattice model with a nonmagnetic valence bond solid ground state on a ladder. A similar physical setting may be naturally occurring in YbAl$_3$C$_2$, CeAgBi$_2$, and TmB$_2$ compounds. In the insulating limit, the application of a magnetic field drives a quantum phase transition to an easy-plane antiferromagnet, which is described by a Bose–Einstein condensation of magnons. Using a combination of field theoretical techniques and density matrix renormalization group calculations we demonstrate that in one dimension this transition is stable in the presence of a metallic Fermi sea, and its universality class in the local magnetic response is unaffected by the itinerant gapless fermions. Moreover, we find that fluctuations about the valence bond solid ground state can mediate an attractive interaction that drives unconventional superconducting correlations. We discuss the extensions of our findings to higher dimensions and argue that depending on the filling of conduction electrons, the magnon Bose–Einstein condensation transition can remain stable in a metal also in dimensions two and three.

Kondo lattice | frustrated magnetism | strongly correlated electrons

Correlated metals with closely competing quantum ground states provide an important platform for studying a range of fascinating phenomena such as strange metallicity (1, 2), unconventional superconductivity (SC) (3), and fractionalized excitations (4, 5). An important example thereof is the Kondo lattice model realized in heavy fermion materials (6), where the competition between magnetic order and screening of local moments induces a non-Fermi liquid in the vicinity of a quantum critical point (4, 7). Moreover, it has been pointed out (8, 9) that Kondo systems with frustrated local moments open a largely unexplored avenue of quantum criticality beyond the Ginzburg–Landau–Wilson paradigm, where screening competes with a quantum disordered spin state, such as a spin liquid (10) or a static-crystalline pattern of local singlets, i.e., a valence bond solid (VBS) (11). The recent discoveries of heavy fermion metals with local moments residing on geometrically frustrated lattices [e.g., the Shastry–Sutherland lattice in Yb$_2$P$_2$T$_2$Pb, Ce$_2$P$_2$T$_2$Pb, and Ce$_2$Ge$_2$Mg (12); a distorted triangular lattice in YbAl$_3$C$_3$ (13); and a distorted Kagome lattice in CeRhSn (14) and CePd$_{1−x}$Ni$_x$Al (15)], provide an excellent platform to study the interplay of magnetic frustration and metallicity.

From a theoretical perspective, it is necessary to establish what properties of frustrated magnetism, including quantum critical phenomena, are stable in the presence of a metallic band. This question is delicate as both the magnetic fluctuations and the electronic excitations are gapless at the critical point. In this work, we consider one of the best understood transitions in insulating quantum magnets, the so-called magnon Bose–Einstein condensation (BEC) (11, 16). It occurs in insulating frustrated VBS magnets, such as TlCuCl$_3$ (17, 18) and SrCu$_2$BO$_3$ (19), where the application of a magnetic field destroys the local singlets to produce an ordered state of spin triplets. The measured critical properties of this transition agree well with the BEC universality class. There are several material candidates to host the magnon BEC phenomena in a metal: YbAl$_3$C$_3$ (13), CeAgBi$_2$ (20), and TmB$_2$ (21). Each of these compounds has either supporting experimental evidence of a VBS ground state in the absence of a field or magnetization plateaus that acquire a finite slope that we expect is due to the Zeeman coupling to the conduction band. Moreover, in YbAl$_3$C$_3$ the field tuned transition to a magnetically ordered phase (22) is accompanied by a logarithmic behavior of specific heat that has been interpreted as a signature of a non-Fermi liquid (13).

Currently, it is unknown if the magnon BEC transition can take place in a metal. To address this question, we study a lattice model (23) exhibiting a magnon BEC transition to an easy-plane (XY) antiferromagnetic (AFM) phase in its insulating limit and solve it in the presence of a metallic conduction band (Fig. 1 A and B). Using a combination of a low-energy field theoretical analysis and density matrix renormalization group (DMRG) calculations we present a comprehensive solution to the problem in one dimension and demonstrate that the magnon BEC transition is stable in a metal.

Our main results are summarized in Fig. 1 C. We find that the VBS state and the BEC transition survive in the presence of a metallic conduction band. For the case with two partially filled bands we find that superconducting correlations induced by

Significance

Magnetically frustrated Kondo lattices are correlated metals not governed by symmetry-breaking quantum criticality, offering a new perspective on the puzzling phenomenology of strange metal behavior and unconventional superconductivity. While this concept is likely to be realized in certain heavy-fermion compounds, the corresponding models are notoriously difficult to solve, hindering robust theoretical predictions. Here we provide a comprehensive solution for a 1D Kondo ladder with a frustrated valence bond solid ground state in the presence of a magnetic field. We prove the stability of the field-induced magnon Bose–Einstein condensation and demonstrate the emergence of unconventional superconductivity. We also present strong evidence that our results carry over to two and three dimensions. These findings anchor future studies of frustrated heavy fermion systems.

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The conduction electron Hamiltonian reads \( H_c = \sum_{k,p=\pm} E_k(k) \psi_{k,p}^\dagger \psi_{k,p} \), where the dispersion is given by \( E_{k\pm}(k) = -2t_0 \cos k \mp t_L - \mu \), for a chemical potential \( \mu \), the lattice constant is set to unity, and \( \psi_{k,\pm} = (\psi_{k,\uparrow} \pm \psi_{k,\downarrow})/\sqrt{2} \) are two-component spinors in the bonding/antibonding basis. The resulting band structure is presented in Fig. 1B. For the low-energy properties of the system, it is important whether the Fermi energy crosses both bands (as in Fig. 1B) or only one, which we will refer to as two- and one-band cases, respectively. As the localized spins are usually due to \( f \) electrons with a large total angular momentum (27) as compared to the conduction electrons (often from \( s \) or \( d \) states) we have omitted the Zeeman term in \( H_c \). Below we will argue that relaxing this approximation does not fundamentally change any of our main conclusions.

Finally, the conduction electrons interact with the localized spins via an AFM Kondo coupling \( H_K = J_K \sum_{r,\alpha} S_{r,\alpha} \cdot s_{r,\alpha} \) where \( s_{r,\alpha} = \psi_{r,\alpha}^\dagger (\sigma/2) \psi_{r,\alpha} \) and \( J_K > 0 \). To make headway analytically, we project the \( S_{r,\alpha} \) operators in \( H_K \) onto the low-energy sector of Eq. 1 and obtain in the hard-core boson representation

\[
H_K \approx \frac{J_K}{4} \sum_r \left( a_r^\dagger a_r \right) \left( \psi_{r,\uparrow}^\dagger \sigma_z \psi_{r,\uparrow} + \psi_{r,\downarrow}^\dagger \sigma_z \psi_{r,\downarrow} \right)
- \frac{J_K}{2\sqrt{2}} \sum_r \left[ a_r \left( \psi_{r,\uparrow}^\dagger \sigma^+ \psi_{r,\downarrow} + \psi_{r,\downarrow}^\dagger \sigma^+ \psi_{r,\uparrow} \right) + h.c. \right].
\]

One sees that the spin-flip term acts only between the two fermion bands. As is shown below, this has important consequences, namely, stabilizing the BEC transition against Kondo screening.

**Numerical Methods**

For the numerical solution of the model, we use the DMRG algorithm as implemented in the ITensor package (28), targeting the low-lying states and their physical properties. The presence of gapless and near-critical modes requires a careful numerical analysis to avoid a bias toward low-entangled states. To that end, we have monitored the convergence of DMRG results as a function of bond dimension, keeping up to 9,830 states for system sizes up to \( L = 76 \) runs (see SI Appendix, section 6, for details).

**Magnet BEC Transition**

We first consider the one-band case for fields below the BEC transition \( h_c < h_c \). As the second band is gapped, one can integrate the fermions out (SI Appendix, section 2). Ignoring the hard-core constraint, we find the leading terms (in \( J_K \)) to renormalize the bosonic spectrum. In particular, the critical field is reduced,

\[
h_c(J_K) \approx h_c(0) - \frac{J_K^2}{8t_L} \int_{k_F}^{k_{F}} dk \frac{1}{1 + 2t_L k_F \cos(k)} - \frac{J_K^2}{64\pi^2} \int_{|k| > k_F}^{k_F} dk dk' \frac{1}{1 - \sin^2 \frac{k-k'}{2} - t_0 [\cos(k) - \cos(k')]},
\]

where the expression is for the case of the bottom band being partially filled and \( J_K^2/t_L \ll J_L \) is assumed (see SI Appendix for details). The top line of Eq. 3 derives from the transverse part of the Kondo coupling (Eq. 2, bottom line), whereas the bottom line arises from the longitudinal part (Eq. 2, top line).

Up to second order in \( J_K \) we find that the total magnetization \( M^z = M_1^z + M_2^z \equiv \sum_{r,\alpha} \langle \sigma_{r,\alpha}^z \rangle + \langle s_{r,\alpha}^z \rangle \) vanishes for fields...
$h < h_c(J_K)$. However, $M^z_{\parallel}$ and $M^z_i$ themselves do not vanish. In particular, this results in a Zeeman splitting for the conduction electrons. The nonzero values of $M^z_{\parallel}$ and $M^z_i$ appear in second order in $J_K$ and do not exhibit any singularities (see SI Appendix for details). This effect remains qualitatively similar in the two-band case.

To determine the critical properties at the transition $h_c(J_K)$ we use the fermionic representation of $H_i$. In this representation the BEC transition corresponds to a Lifshitz transition, where the chemical potential touches the bottom of the spinon band. At the critical point, we take the low-energy continuum limit (with $x$ denoting position) to find that the lowest-order interaction term in the gradient expansion $(f^\dagger f)^2$ has to vanish due to the Pauli principle, whereas all of the higher-order terms [such as $(f^\dagger \partial_x f)^2$] are irrelevant at the quantum critical point (QCP) due to the additional derivatives (29). Thus, interactions that arise due to including out the conduction electrons (see SI Appendix for details) do not affect the transition. The critical low-energy field theory is then that of free fermions with a Lagrangian density

$$
\mathcal{L}_{\text{crit}} = f^\dagger(x, \tau) \partial_x f - a \partial_x^2 f - \mu_f(h) f(x, \tau),
$$

where $f(x, \tau)$ is the spinon field in the continuum, $\tau$ is imaginary time, and $\mu_f(h) = h - h_c(J_K)$.

There are several predictions that directly follow from the above discussion that we now confirm with DMRG. First, we examine whether the magnon BEC transition survives at finite Kondo coupling. To that end, we set $J_K = 0.4$ and compute the transverse spin susceptibility at the ordering wave vector $\chi(\pi) \equiv \sum_q e^{i\pi(\pi - q - L/2)} \left(\langle S_{L/2,1}^x S_{L/2,2}^x \rangle - \langle S_{L/2,1}^z S_{L/2,2}^z \rangle\right)$ (Fig. 2A). Indeed, for $h < h_c(J_K)$, the localized spins are short-range correlated, and hence, $\chi(\pi)$ saturates to a value independent of $L$. On the other hand, for $h > h_c(J_K)$, quasi-long-range order leads to a scaling $\chi(\pi) \sim L^\beta$, with $\beta = 1 - 1/(2K)$ for $h > h_c(J_K)$ (23, 30) and a Luttinger parameter $K$ in the AFM phase.

To precisely locate the critical field $h_c$ and determine the critical exponents associated with the transition, we study the finite-size scaling of the spin gap $\Delta$, close to criticality. The observable $\Delta L^\nu$ has a vanishing scaling dimension, and consistent with the magnon BEC we assume a dynamical exponent $z = 2$. Near criticality this implies that $\Delta L^\nu \sim R((h - h_c) L^{1/\nu})$ for an arbitrary scaling function $R$ and that $\Delta L^{\nu}$ is $L$ independent at $h = h_c$, i.e., $\Delta L^\nu$ versus $h$ curves for an increasing set of system sizes precisely cross at a single point identified with the critical field $h_c$. This relation is tested in Fig. 2B, where we observe a clear crossing in $\Delta L^\nu$ versus $h$ for increasing $L$, thus providing an accurate estimate of $h_c$. We use this unbiased method to accurately determine $h_c$ as a function $J_K$ (Fig. 2C) and find a quadratic decrease $h_c(J_K) \sim J_K^2$, in good qualitative agreement with the free-boson result of Eq. 3.

To estimate the correlation length exponent $\nu$, we study the finite-size scaling of $R$ curves at criticality (see SI Appendix, section 6, for details). We find $\nu = 0.49(1)$, in compliance with the predicted magnon-BEC universality class value, $\nu_{\text{AB}} = 1/2$ (5). Last, close to criticality, the free fermion result implies a Luttinger parameter $\nu = 1$ (23), which gives $\beta = 1/2$. Again, we find excellent agreement with the DMRG calculation at a field close to $h_c(J_K)$, as shown in Fig. 2D, thus further confirming the universality class of the magnon BEC transition in a metal.

We now argue that the BEC transition also remains stable in the two-band case. While the boson self-energy in this regime could have divergences at $q = \pm (K_L^z \pm K_F)$, generally, these momenta are not equal to the ordering wave vector $Q = \pi$. Assuming $J_K \ll J_L$, the bosons in the vicinity of $q = \pi$ are then expected to be described by the same theory Eq. 4 in the vicinity of $h_c(J_K)$ as in the one-band case. Hence, the universality

![Fig. 2 results of DMRG calculations for the one-band case, for $J_L = 1, J_i = 0.3, t_\perp = 1.5, t_i = 1.0, J_K = 0.4$, and 1/8 filling. (A) $\chi(\pi)$ as a function of magnetic field $h$ for various system sizes. (B) Curve crossing analysis of the universal amplitude $\Delta L^\nu$ as a function of $h$. (C) Values of the critical field extracted from the location of the curve crossing in $\chi$ for various $J_K$. The dashed line depicts the perturbative result of Eq. 3 computed neglecting the hard-core constraint (D) $\chi(\pi)$ versus system size on a log-log scale for a field close to $h_c(J_K) = 0.4$. For comparison, solid line depicts the scaling prediction of Eq. 4.](image-url)
class of the transition is unchanged, and corrections to Eq. 3 are subleading at weak coupling.

**SC in the Two-Band Case**

We now consider the properties of conduction electrons for the two-band case (Fig. 1B). We determine the emergent phases using bosonization (26), with the low-energy excitations of the two bands being described with the real bosonic fields \( \varphi_\pm^\alpha(x) \) and \( \varphi_\mp^\alpha(x) \) for the spin and charge sectors, respectively [each field also has a canonically conjugate one, \( \theta_\pm^\alpha(x) \)]. In the absence of a Kondo coupling, the low-energy excitations of each of these fields are described as a Luttinger liquid with \( K = 1 \), \( u = v_p \equiv 2k_\parallel \sin k_F \).

Ignoring for the moment the aforementioned Zeeman splitting, integrating out the gapped hard-core bosons leads to two interaction terms

\[
H_{i,\pm} = \frac{J_K^2}{8k_F^4} \int dx \cos \left( 2\theta^\alpha_\parallel(x) - \theta^\alpha_\parallel \right) \cos \sqrt{2} (\varphi_\pm^\alpha(x) + \varphi_\mp^\alpha(x)),
\]

where the index \( \pm \) refers to individual bands (see Fig. 1B), \( 1/\alpha \) is a high-energy cutoff, and \( \theta_{\pm,\pm}^\alpha + \theta_{\mp,\mp}^\alpha \) is the boson dispersion at \( q = k_F^\alpha \pm k_F^\alpha \). Semiclassically, the terms in Eq. 5 create a pinning potential for the fields making their excitations gapped. To assess their possible impact in the quantum regime, we performed one-loop renormalization group (RG) analysis. The corresponding equations have been derived in ref. 31; we take the perturbatively generated interactions, including Eq. 5, as the initial conditions and solve the equations numerically (see SI Appendix, section 3, for details). We find that terms proportional to \( \cos(2\sqrt{2}\varphi_\pm^\alpha) \) are generated and flow to strong coupling together with the ones in Eq. 5, allowing for a semiclassical analysis. The remaining terms lead to a renormalization of the Luttinger parameters. Minimizing the action including the cosine terms, we find that out of four gapless fermion modes, only the excitations of the total charge mode \( \varphi_\uparrow + \varphi_\downarrow \) remain gapless, corresponding to a state with power law correlations of the superconducting order parameter \( O_{SC}(x) = \psi_\uparrow(x) \psi_\downarrow(x) \) and the conduction electron density at \( 2k_F^\alpha \). Furthermore, the equilibrium values of the gapped fields are such that the superconducting correlations are sign-changing between the bands, i.e., there is a \( \pi \) phase shift between the SC order parameters of the \( + \) and \( - \) bands, which is an analogue of \( d^-\)wave pairing on the ladder. Additionally, the dominant velocity renormalization is such that the SC correlations are stronger, i.e., decay slower, than the \( 2k_F^\alpha \) density ones. These results resemble the case of two-leg Hubbard ladders (32) that have \( \delta^-\)wave superconducting and charge density wave correlations. Similar results have also been obtained for the Kondo–Heisenberg model away from the VBS limit (with \( J_1 = J_0 \)) in zero magnetic field (33).

Let us now discuss the interplay of the above effects and the Zeeman splitting \( E_Z \) due to a finite \( J_0 \). The Zeeman splitting \( \Delta_Z \) can thwart SC (34, 35) unless the SC gap \( \Delta_{SC} \) is sufficiently larger then \( E_Z \) (for an alternative discussion, see SI Appendix). As the interactions are marginal, the gap is expected to be exponentially small \( \Delta_{SC} \sim v_F/\sqrt{\alpha} \exp[-1/g] \), where \( g \sim v_F J_K^2/\sqrt{\alpha \varepsilon_{\mp,\mp}} \), whereas \( E_Z \propto J_0^2 \). It follows that for infinitesimal \( J_0 \), \( E_Z \) dominates, while at larger \( J_0, \Delta_{SC} \) takes over, which represents a Doniach-like competition between Zeeman splitting and SC.

This is similar to the competition between the Kondo coupling and the Ruderman–Kittel–Kasuya–Yosida (RKKY) interaction.

We note that the superconducting pairing due to VBS fluctuations differs from the conventional scenario of a spin density wave QCP (3), as well as the one in the 1D Hubbard (32, 36) or \( t - J \) model. Unlike the SDW QCP (1, 3), the magnon BEC critical mode has \( z = 2 \) even without an interaction with fermions, while the 1D Hubbard model does not possess the VBS subsystem. In the case of the \( t - J \) model, the binding of holes is achieved due to the energy cost \( J_1 \) of breaking a singlet bond (37, 38). The difference from the results for the \( t - J \) model is that the VBS fluctuations are not completely local, especially for \( h \) close to \( h_c \) and no requirements on the magnitude of \( t_z \) (37).

The expectations above are confirmed by the DMRG results in Fig. 3. First, to estimate the SC gap, we obtain the spin gap \( \Delta_{\uparrow} \), in Fig. 3A with a finite size scaling analysis of the energy difference \( E(M^\uparrow = 1, L) - E(M^\uparrow = 0, L) \), between the ground states in the \( M^\uparrow = 1 \) and \( M^\uparrow = 0 \) sectors (see SI Appendix for details). Indeed, we find a nearly vanishing spin gap \( \Delta_{\uparrow} \approx 0 \) for weak Kondo coupling, \( J_k = 0.2 \), and a finite gap, \( \Delta_{\uparrow} \approx 0.17 \), at larger Kondo coupling, \( J_k = 2.4 \). To further characterize the above phases, in Fig. 3B, we investigate the \( e^- \) intraband spin-density wave (SDW), \( O_{SDW}(x) = \psi_{\uparrow,\uparrow}(x) \psi_{\downarrow,\downarrow}(x) \), and SC (defined above), \( O_{SC} \), order parameters, through their respective correlation functions \( \chi_{+/-}(r) = \langle O_{SC}^{+/-}(r) \rangle (L/2) O_{SC}^{+/-}(r) \). At \( J_k = 0.2 \), we find that both order parameters fall off like a power law, akin to the decoupled free-electron limit. By contrast, in the spin gapped phase, \( J_k = 2.4 \), the SDW correlation decays exponentially, while the SC correlation remains quasi-long range.

Next, in Fig. 3C, we examine the spin resolved momentum distribution of the bonding band, \( n_{\uparrow,\uparrow}(k_x,k_y) - \langle \psi_{\uparrow,\uparrow}(x) \psi_{\uparrow,\uparrow}(x) \rangle \). The distribution evolves from a sharp Fermi edge, at small \( J_k \), to an incoherent distribution, characteristic of a Luttinger liquid, upon approach to the spin gapped phase.

The Fermi wave vector is unchanged throughout this transition, unlike the usual Kondo lattice model (39, 40), due to the number of spins per unit cell being even (i.e., two) (9, 41).

In the superconducting spin-gapped phase, there is only a single gapless channel \( (e = 1) \) corresponding to the total charge mode. These results are consistent with the expectations from weak-coupling RG.

**Interactions in the AFM Phase**

For \( h > h_c(J_K) \), there is a finite density of spinons in the fermionic representation of the Hamiltonian Eq. 1. The low-energy excitations around the spinon Fermi points at \( k_F^\pm \) are of the Luttinger liquid type (23) that can be described using bosonization. The Luttinger parameters \( K \) and \( u \) are known functions of \( h \) (23). One can now rewrite the low-energy part of the Kondo coupling in terms of the bosonic fields resulting in three contributions \( H_S = H_{K}\uparrow + H_{K}\downarrow + H_{SI} \) where we use \( \varphi(x) \) for the spinon fields. First, a velocity renormalization appears due to

\[
H_V = \begin{pmatrix} J_K \end{pmatrix} \int dx \cos(2k_F^\alpha x - 2\varphi),
\]

where \( k_F^\alpha \) is the Fermi wave vector of the spinons. As \( \varphi, \varphi_{\rho_{\rho}} \) are slowly varying functions of \( x \) the integral averages to zero except for two special cases, \( k_F^\alpha = k_F, \pi - k_F \). If that is so, however, this term is relevant throughout the AFM phase,* and it

\*The scaling dimension of this term is \( 1 - K \times O_{KH}/\nu \) with \( K < 1 \) throughout the AFM phase (23).
pins the values of two bosonic fields resulting in a spectral gap of the order $v_F J_K$ (with $v_F/\alpha$ to be averaged to zero), similar to the opening of the SDW gap in itinerant magnets.

The presence of $H_S = -J_K \sum_{k,\sigma} \int dx M_F \partial_x \phi_\sigma$ term leads additionally to a Zeeman splitting for the conduction electrons. For $h > h_c$, $M_F$ is nonzero even in the absence of Kondo coupling, and thus, this term is linear in $J_K$ and parametrically larger then both the AFM gap and the perturbatively induced Zeeman splitting. Thus, Zeeman splitting is the dominant effect of the Kondo coupling, and we need to reconsider the effect of the interaction in Eq. 6 starting with the Zeeman split Fermi points. The condition for Eq. 6 not to be averaged to zero is then $k_F^z = k_F \pm \frac{J_K (5+\sqrt{5})}{4v_F}$, $\pi - k_F \pm \frac{J_K (5+\sqrt{5})}{4v_F}$. In each case, one of the three gapless modes is gapped out. At finite $J_K$ one expects each of these special values to broaden into an interval, as is shown by the shaded green regions in Fig. 1C.

Conduction Electron $g$ Factor

While throughout we have neglected the $g$ factor, $g_0$ of the conduction electrons, it is a reasonable assumption for a range of fields $g_0 \ll J_K$; thus, we consider our results to carry over to the $g_0 \neq 0$ case, albeit in finite fields, we expect the superconducting correlations to be additionally suppressed.

Extensions to 2D and 3D

We now show that the magnon BEC transition is also stable at finite $J_K$ in the one-band case for 2D and 3D extensions of the model considered here. The 2D extension of the model in Fig. 1A consists of the ladders arranged in a columnar pattern with a weak interladder coupling, and for 3D, we stack the resulting layers on top of one another (Fig. 4).

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$$H_{K, \pm}^{\text{intra}} = \frac{J_K}{4v_F} \sum_{k,q} f(\mathbf{k}, \mathbf{q}) a(\mathbf{q}) \psi_{\pm, \mathbf{k}+\mathbf{q}} \psi_{\pm, \mathbf{k}} + h.c.,$$

where for $q_o$ close to $\pi$, $f(\mathbf{k}, \mathbf{q}) \sim f(\mathbf{k}, \pi)(q_o - \pi)$ (see SI Appendix for details). Assuming that the BEC occurs for bosons with $q_o = Q_0$ [e.g., $Q_0 = (\pi, \pi, \pi)$ in 3D], only fermions around the Fermi surface points connected by $Q_0$ (hot spots) are coupled to the critical mode. To assess the influence of the interactions, we calculate their scaling dimension for two cases with the Fermi velocities at the hot spots being antiparallel or not. We define the scaling dimension of energy and the momentum parallel to the Fermi velocities to be 1, $[\varepsilon] = [k_0] = 1$, while the remaining momenta have scaling dimension 2 to keep $z = 2$ intact (29, 43). Under these assumptions, $H_{K, \pm}^{\text{intra}}$ (Eq. 7) as well as the first term

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3}
\caption{DMRG results for the two-band case, for $J_\perp = 1$, $J_\parallel = 0.3$, $t_\perp = 0.1$, $t_\parallel = 1.0$, $h = 0.5$, and 1/4 filling, for various $J_K$ in the metallic VBS and superconducting spin gapped phase. (A) The spin gap, extracted from a finite size scaling of the energy level spectroscopy, $E(M = 1) - E(M = 0)$. Curves are smoothed using a moving average. Dashed line is fit to a linear function in $1/L$. (B) SDW ($\chi_{\perp}$) and SC ($\chi_{\parallel}$) correlation functions on a log–log scale, for $L = 44$. (C) The spin-resolved momentum occupation number $n_{\sigma, k}$ of the bonding band, for $L = 44$. (D) The extraction of the central charge from the scaling of the bipartite von Neumann entanglement entropy versus the logarithm of system size $\log(L)$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4}
\caption{Schematic depiction of the (A) 2D and (B) 3D extensions of the model in Fig. 1A.}
\end{figure}
of $H_K$ (Eq. 2) are irrelevant in $d > 1$. For the case of antiparallel [noncollinear] Fermi velocities at the hot spots, we find that the former has a scaling dimension $(1 - d)/2 - [d - 2]/2$ and the latter $1 - d$. This provides a strong indication that the BEC transition should retain its universality class in the one-band case.

The above result implies that the quantum critical behavior is governed by the same theory as in the undoped case, i.e., that of a dilute $z = 2$ Bose gas. Interestingly, this prediction may be verified in $\text{YbAl}_2C_2$, where a VBS ground state of the Yb moments (44) is formed on a deformed triangular lattice (45), while the conductivity suggests metallic behavior (44).

Application of a magnetic field results in a quantum phase transition (46), with the specific heat having been found to exhibit a $C_V / T \sim \log(1 / T)$ behavior close to the QCP. Initially, this behavior has been attributed to a possible non-Fermi liquid state formed due to Kondo coupling (46). However, our results suggest that the field-induced transition should not be affected by Kondo coupling; indeed, for $z = 2$, $d = 2$ BEC a $\log(1 / T)$ divergence is expected (47, 48). Thus, our results allow one to interpret the observed anomalies as a signature of the stability of the magnon BEC transition in metallic systems.

**Discussion and Conclusions**

In this work we have studied quantum critical properties and phases of a 1D frustrated Kondo lattice with a nonmagnetic VBS state in magnetic field. We show that the field-induced magnon BEC transition that occurs in the insulating limit is stable in the presence of a metallic conduction band and retains its universality class, with the critical field of the value being lowered. We have demonstrated that VBS fluctuations lead to unconventional SC in the case of two partially filled bands. Finally, we have shown that the stability of the magnon BEC transition extends to higher-dimensional versions of the model. Our results allow us to draw conclusions regarding field-induced transitions in heavy-fermion materials with spins residing on frustrated lattices. In particular, our results have lead us to a clear interpretation of the observed criticality in $\text{YbAl}_2C_2$ and anchor future studies of the phase diagrams and quantum criticality of frustrated Kondo lattices. It will be also interesting to extend our theory to VBS states that may break a crystalline symmetry as well as field-tuned VBS to AFM transitions that occur at fractional magnetization plateaus (19).

**Data Availability.** All data needed to evaluate the conclusions in the paper are available in the manuscript and SI Appendix.

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