Detecting nonclassicality and non-Gaussianity by the Wigner function and quantum teleportation in photon-added-and-subtracted two modes pair coherent state

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Abstract

We introduce a new state called photon-added-and-subtracted two modes pair coherent state (PAASTMPCS) by simultaneously adding and subtracting photons to the different modes of a pair coherent state. Its nonclassical and non-Gaussian properties are strengthened by the negative values of its Wigner function as the numbers of photons added or subtracted increases. Based on the linear entropy criterion, we indicate that the PAASTMPCS is an entangled state. When increasing the numbers of photons added or subtracted to a pair coherent state, the degree of entanglement in the PAASTMPCS is enhanced compared with the original pair coherent state. By using a PAASTMPCS as a non-Gaussian entangled resource, the quantum teleportation process is studied in detail. The results show that the number sum and phase difference measurements protocol is more appropriate than the orthogonal quadrature components measurements protocol in the quantum teleportation process of a coherent state.

Keywords  Photon-added and photon-subtracted states · Pair coherent state · Wigner function · Entanglement · Quantum teleportation

1 Introduction

The addition and subtraction of photons to the classical states such as a coherent state [1] or a thermal state [2] are the normal operations that can transform to a nonclassical one [3–5]. Moreover, these effects to the nonclassical states such as pair coherent state [6, 7] or two-mode squeezing state [8] may increase the nonclassicality of the proposed states [9–12]. The schemes of generating the classes of photon-added and photon-subtracted states were proposed [13–15] and the experimental setups to generate these states in the laboratory were performed [16–18]. In continuous variables, the photon-added and photon-subtracted actions to the multi-mode Gaussian and non-Gaussian states can enhance their nonclassicality and nonlocality, especially their entanglement property [19–24]. Besides, the non-Gaussian entangled states can provide the potential applications in quantum information as non-Gaussian entangled resources to perform the quantum tasks such as quantum teleportation [25–27], quantum steering [28], and quantum key distribution [29, 30].

In the two modes of continuous variables systems, the adding and subtracting photons to the Gaussian states can cause these states to become non-Gaussian states. For example, the Gaussian state that is most interested in is the two-mode squeezing state [8]. The photons added and subtracted to a two-mode squeezing state have introduced many non-Gaussian states [31–34], in which the nonclassical and the entanglement properties have been investigated. These non-Gaussian states can be considered to perform many quantum tasks such as improvement of the quantum optical interferometry via the photon-added two-mode squeezed vacuum states [32], enhancement of the quantum entanglement and quantum teleportation by the multiple-photon subtraction and addition to a two-mode squeezing state [27, 33].
Along with continuous variables systems, the pair coherent state is a non-Gaussian state, in which its nonclassical properties were studied [7, 35, 36]. Moreover, the schemes for generating this state were proposed [6, 37–39]. The pair coherent state is an entangled state, and the quantitative measures of entanglement, quantum teleportation, and quantum key distribution by using this state were also studied [40–44]. The photon addition to two modes of pair coherent state as photon-added pair coherent state [45] and generalized photon-added pair coherent state [46] were introduced. Furthermore, the photon-added and photon-subtracted pair coherent state [47], in which the added or subtracted photons only take place in one mode of the pair coherent state was also introduced. In these states, the nonclassical and the entanglement properties of them were studied in detail. Because the pair coherent state is a non-Gaussian state, the proposed states by adding or subtracting photons to this state are non-Gaussian states as well, and the non-Gaussian characteristics of them can be strengthened.

In this paper, so as to extend simultaneously adding and subtracting photons to both modes of a pair coherent state, we introduce a new state called photon-added-and-subtracted two modes pair coherent state (PAASTMPCS) in Sect. 2. We show the nonclassical and non-Gaussian characteristics of the new state by examining the Wigner function in Sect. 3 and the entanglement degree by using the linear entropy of the PAASTMPCS in Sect. 4. We use the PAASTMPCS as a non-Gaussian resource for the quantum teleportation processes by using the orthogonal quadrature components measurements protocol in Sect. 5, and the number sum and phase difference measurements protocol in Sect. 6. Finally, the main results of the paper are summarized in the conclusions.

2 Photon-added-and-subtracted two modes pair coherent state

The pair coherent state (PCS) |ξ, q⟩ab is a state of the two-mode radiation field [7], which is the eigenstate of both the boson annihilation operators ˆa ˆb and the charge operator ˆQ = ˆb† ˆb − ˆa† ˆa as follows:

\[ \hat{a}\hat{b}|\xi, q⟩_{ab} = \xi |\xi, q⟩_{ab}, \]  
\[ \hat{Q}|\xi, q⟩_{ab} = q|\xi, q⟩_{ab}, \]  

where ξ = |ξ|e^iφ is a complex number with φ real, q is an integer referred to the difference in the number of photons between modes a and b. In case q ≥ 0, in the term of Fock states, |ξ, q⟩ab is written as

\[ |\xi, q⟩_{ab} = \left( \sum_{m=0}^{\infty} \frac{|\xi|^{2m}}{m!(m+q)!} \right)^{-1/2} \sum_{n=0}^{\infty} \frac{\xi^n}{n!(n+q)!} |n, n+q⟩_{ab}, \]  

(3)

|n, n+q⟩_{ab} is a two-mode Fock state.

Now, we introduce a new state called photon-added-and-subtracted two modes pair coherent state (PAASTMPCS) by acting k times of creation operator ˆa† on mode a and l times with l ≤ q of annihilation operator ˆb on mode b of a PCS |ξ, q⟩ab as

\[ |\xi, q;k,l⟩_{ab} = B_{q;k,l}(\xi)\hat{a}^k\hat{b}^l|\xi, q⟩_{ab}, \]  

where k and l are non-negative integers and the normalized factor B_{q;k,l}(\xi) is determined by

\[ B_{q;k,l}(\xi)^{-2} = \sum_{m=0}^{\infty} \frac{|\xi|^{2m(m+k)!}}{(m!)^2(m+q-l)!}, \]  

(5)

It should be noted that the PAASTMPCS in the form of |ξ, q;k,l⟩_{ab} in Eqs. (3) and (5) does not explicitly depend on q or on l, but it only depends on the difference between q and l. Therefore, we set h = q − l, the PAASTMPCS is written as |ξ, k, h⟩_{ab} in the term of two-mode Fock states in the form

\[ |\xi, q;k,l⟩_{ab} \equiv |\xi, k, h⟩_{ab} = \sum_{n=0}^{\infty} C_{n;k,h}(\xi)|n+k, n+h⟩_{ab}, \]  

(6)

where

\[ C_{n;k,h}(\xi) = \left( \sum_{m=0}^{\infty} \frac{|\xi|^{2m(m+k)!}}{(m!)^2(m+q)!} \right)^{-1/2} \frac{(n+k)!}{(n!)^2(n+h)!}, \]  

(7)

When k = 0 and h = q, or k = l = 0, the PAASTMPCS is reduced to the PCS [6]. In what follows, we will use PAASTMPCS in the form of |ξ, k, h⟩_{ab} in Eqs. (6) and (7) for calculation.

3 Wigner function

In quantum optics, a Wigner function is used to determine the characteristics of physical states in the phase space. Usually, the Wigner function of a given state gets positive values. Based on the positive Wigner function, a family of criteria to detect a quantum state as a quantum non-Gaussian state was introduced [48]. However, in some states, their Wigner function can take on some negative values. Via the negative values of their Wigner function, the states are certainly confirmed as nonclassical and non-Gaussian states [21, 49, 50]. In the PAASTMPCS, the Wigner function can be given in the term of coherent states as
\[
W = \frac{4e^{-2|\alpha_a|^2+|\alpha_b|^2}}{\pi^4} \int d^2\gamma_a d^2\gamma_b e^{2i(\gamma_a^* \gamma_b - \gamma_a \gamma_b^*)} \\
\times h_a(h_a - \hat{\rho}_{ab}| \gamma_a \gamma_b \rangle_{ab},
\]

where \(\alpha_a\) and \(\alpha_b\) are complex numbers in the phase space, \(|\gamma_a\rangle_a\) and \(|\gamma_b\rangle_b\) denote the coherent states, and \(\hat{\rho}_{ab}\) is the density operator of the PAASTMPCS. Using Eq. (6), the density operator \(\hat{\rho}_{ab}\) is written as

\[
\hat{\rho}_{ab} = \sum_{n,m=0}^{\infty} C_{n,k}(\xi) C_{m,k}^*(\xi)|n + k, n + h\rangle_{ab} \langle m + h, m + k|.
\]

Substituting the density operator \(\hat{\rho}_{ab}\) in Eq. (9) into Eq. (8) and then calculating the complex integrals (see Appendix), we obtain the Wigner function of the PAASTMPCS in the form

\[
W = \frac{4e^{-2|\alpha_a|^2+|\alpha_b|^2}}{\pi^2} \left[ \sum_{j=0}^{\infty} \frac{|\xi|^2j(j+k)!}{j!(j+h)!} \right]^{-1} \\
\times \sum_{m,n=0}^{\infty} (4|\alpha_a|e^{i\phi_a})^m (4|\alpha_b|e^{i\phi_b})^n \frac{\cos[(m-n)(\phi_a + \phi_b - \phi)] 2|\alpha_a|^2k |\alpha_b|^2k}{m!n!(m+h)!(n+h)!} \\
\times {}_2F_1[-n-k, -m-k; -1/(2|\alpha_a|)^2] {}_2F_1[-n-h, -m-h; -1/(2|\alpha_b|)^2],
\]

where \(\alpha_a = |\alpha_a|e^{i\phi_a}, \alpha_b = |\alpha_b|e^{i\phi_b}\), and \(\gamma_0\) denotes the hypergeometric function.

We use the analytical expression in Eq. (10) to investigate the nonclassical and non-Gaussian behaviors in the PAASTMPCS. In Fig. 1, we plot the dependence of the Wigner function \(W\) as a function of both real and imaginary parts of \(\alpha_a\) with \(|\xi| = 0.2, \alpha_b = 0.5, \phi = 0, \) and \(h = k = 1\). The result shows that the Wigner function of the PAASTMPCS gets negative values in some regions of the phase space. Therefore, we conclude that the PAASTMPCS is a nonclassical and non-Gaussian state. Besides, in Fig. 2, we have selected a small region in phase space to show that the depth of the Wigner function can be enhanced if the numbers of photon-added \(k\) and photon-subtracted \(l\) are increased.

![Fig. 1 The Wigner function W as a function of real and imaginary parts of \(\alpha_a\) with \(|\xi| = 0.2, \alpha_b = 0.5, \phi = 0, \) and \(h = k = 1\)](image)

The PAASTMPCS are more negative compared with the PCS. This proves that adding and subtracting photons to the different modes are very important in enhancing the nonclassical and non-Gaussian properties of the PAASTMPCS compared with the original PCS.

### 4 Linear entropy

In two modes states, some criteria can be used effectively to detect the entanglement [51–54] and measure the degree of entanglement [40, 55, 56] of them. In order to investigate the degree of entanglement in the PAASTMPCS, we use the linear entropy criterion [40]. Accordingly, the entanglement degree coefficient \(E_{\text{lin}}\) is given in the form

\[
E_{\text{lin}} = 1 - \text{Tr}(\hat{\rho}_{ab}^2),
\]

where \(\text{Tr}\) is denoted as the trace of the matrix. A state is entangled if \(E_{\text{lin}} > 0\). When \(E_{\text{lin}} = 1\), the state becomes maximum entangled. For the PAASTMPCS, from the density operator \(\hat{\rho}_{ab}\) given in Eq. (9), we get

\[
\hat{\rho}_{ab}^2 = [\text{Tr}_b(\hat{\rho}_{ab})]^2 = \sum_{n=0}^{\infty} |C_{n,k}(\xi)|^4 |n + k\rangle_{a} \langle n + k|,
\]
From Eqs. (12) and (13), we have \( \hat{\rho}_b^2 = \text{Tr} \left( \rho_{ab} \right)^2 = \sum_{n=0}^{\infty} |C_{n,k,h}(\xi)|^4 |n+h\rangle_b \langle n+h| \). It is easy to obtain the linear entropy \( E_{\text{lin}} \) of the PAASTMPCS as follows:

\[
E_{\text{lin}} = 1 - \sum_{n=0}^{\infty} |C_{n,k,h}(\xi)|^4 = 1 - \left( \sum_{m=0}^{\infty} \frac{|\xi|^{2m}(m+k)!}{(m!)(m+k)!} \right)^2 - \sum_{n=0}^{\infty} \left( \frac{|\xi|^{2n}(n+h)!}{(n!)(n+h)!} \right)^2.
\]

We examine the entanglement degree of the PAASTMPCS by using Eq. (14). In Fig. 3, we plot the dependence of \( E_{\text{lin}} \) on \( |\xi| \) for several values of \( k \) and \( h = q - l \); therein, the case of \( k = 0 \) and \( h = 6 \) (the solid blue line) corresponds to the PCS, and the others are the PAASTMPCS. At the same values of \( |\xi| \), the curves on the graphs show that the entanglement degree coefficient \( E_{\text{lin}} \) in the PAASTMPCS is always higher than that in the original PCS. Besides, the curves in Fig. 3a and b show the entanglement degree coefficient \( E_{\text{lin}} \) increases if the numbers of photon-added \( k \) and photon-subtracted \( l \) increase (i.e., \( h \) decreases). It means that the degree of entanglement of the PAASTMPCS is enhanced by simultaneously increasing the number of photon-added and photon-subtracted to the original PCS. In addition, in the case of adding and subtracting more photons, the value of the entanglement degree coefficient \( E_{\text{lin}} \) is increasing, especially in the case of the difference of \( k - h \) is getting bigger and bigger (see Fig. 3c).

5 Quantum teleportation uses the orthogonal quadrature components measurements protocol

The scheme of teleportation using continuous variables as known as orthogonal quadrature components measurements protocol was first proposed by Furusawa et al [57] and later improved by several authors [41, 50, 58, 59]. Using this protocol, both sender (Alice) and receiver (Bob) must be shared by a two-mode entangled state. In this section, we use the PAASTMPCS as an entanglement resource for teleportation by exploiting the orthogonal quadrature components measurements protocol [60]. In this first protocol, we assume that Alice possesses the mode \( a \) and Bob holds the mode \( b \). Alice needs to teleport to Bob a coherent state \( |\alpha\rangle_c \) of mode \( c \), in which this state is expanded in the term of the Fock states as

\[
\hat{\rho}_b^2 = \text{Tr} \left( \rho_{ab} \right)^2 = \sum_{n=0}^{\infty} |C_{n,k,h}(\xi)|^4 |n+h\rangle_b \langle n+h|.
\]

Fig. 3 The linear entropy \( E_{\text{lin}} \) as a function of \( |\xi| \) for in a \((k, h) = (0, 6)\) (the blue solid line), \((2, 6)\) (the purple dotted curve), \((6, 6)\) (the green dot-dashed curve), and \((12, 6)\) (the red dashed curve), in b \((k, h) = (0, 6)\) (the blue solid line), \((4, 6)\) (the purple dotted curve), \((4, 4)\) (the green dot-dashed curve), and \((4, 1)\) (the red dashed curve), and in c \((k, h) = (0, 6)\) (the blue solid line), \((5, 5)\) (the purple dotted curve), \((14, 2)\) (the green dot-dashed curve), and \((64, 1)\) (the red dashed curve) (Color figure online)
The input state of the system is written as

$$|\alpha\rangle_c = \sum_{m=0}^{\infty} d_m |m\rangle_c,$$

with $d_m = e^{-|\alpha|^2/2} a^m / \sqrt{m!}$. The input state of the system is written as

$$|\Phi_{in}\rangle_{abc} = |\xi, k, h\rangle_{ab} \otimes |\alpha\rangle_c$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{n,k,h}(\xi) d_m |n + k, n + h\rangle_{ab} |m\rangle_c,$$

(16)

where $C_{n,k,h}(\xi)$ is given by Eq. (7). Next, Alice measures the orthogonal quadrature components on two modes $a$ and $c$. The measurement corresponds to projecting the state $|\Phi_{in}\rangle_{abc}$ onto the eigenstate of the orthogonal quadrature components operators of the modes $a$ and $c$. After this measurement, the state of Bob (non-normalized) becomes

$$|\Phi_b\rangle_{ac} = |\beta\rangle_c |\Phi_{in}\rangle_{abc} = |\beta\rangle_c |\xi, k, h\rangle_{ab} \otimes |\alpha\rangle_c,$$

(17)

where $|\beta\rangle_c$ is the eigenstate of two commutative operators $\hat{x}_c = \hat{x}_c - \hat{\beta}_c$ and $\hat{y}_c = \hat{y}_c + \hat{\beta}_c$ [60]. Such state is expressed in the term of the Fock states as

$$|\beta\rangle_{ac} = \frac{1}{\sqrt{P(\beta)}} \sum_{m=0}^{\infty} \hat{D}(\beta) |m, m\rangle_{ac},$$

(18)

with $\hat{x}_c |\beta\rangle_{ac} = \text{Re}(\beta) |\beta\rangle_{ac}$, $\hat{y}_c |\beta\rangle_{ac} = \text{Im}(\beta) |\beta\rangle_{ac}$, where $\hat{D}(\beta)$ is denoted as the displacement operator acting on the state of mode $c$. The state in Eq. (17) is expanded in normalized form as follows:

$$|\Phi_{nor}\rangle_b = \frac{1}{\sqrt{P(\beta)}} |\Phi_b\rangle_b = \frac{e^{-|\beta|^2/2}}{\sqrt{P(\beta)}} \sum_{n=0}^{\infty} C_{n,k,h}(\xi) \frac{(\alpha - \beta)^n}{\sqrt{(n + k)!}} |n + h\rangle_b,$$

(19)

where $P(\beta)$ is the probability of the measurement given by

$$P(\beta) = b \langle \Phi | \Phi \rangle_b = \frac{e^{-|\alpha|^2/2}}{\pi} \sum_{n=0}^{\infty} |C_{n,k,h}(\xi)|^2 \frac{1}{(n + k)!} (n + k)!.$$

(20)

The results of the measurement are sent to Bob by a classical channel. Finally, Bob restores the teleported state by using the operator $\hat{D}(\beta)$ acting on his state. The quantum teleportation process is completed. The output state (normalized) reads

$$|\Psi_{nor}\rangle_{out} = \hat{D}(\beta) |\Phi_{nor}\rangle_b = \frac{e^{-|\beta|^2/2}}{\sqrt{P(\beta)}} \sum_{n=0}^{\infty} C_{n,k,h}(\xi) \frac{(\alpha - \beta)^n}{\sqrt{(n + k)!}} |n + h\rangle_b,$$

(21)

where

$$\sum_{n=0}^{\infty} C_{n,k,h}(\xi) |\alpha - \beta|^n \hat{D}(\beta) |n + h\rangle_b \sqrt{(n + k)!}.$$

The quality of the quantum teleportation process is shown by the average fidelity. The fidelity is an overlap of the output and input states

$$F = \frac{\langle \Phi | \Phi_{nor}\rangle_{out}^2}{P(\beta)} = \frac{e^{-2|\alpha|^2}}{P(\beta)} \sum_{n=0}^{\infty} \frac{C_{n,k,h}(\xi) C^*_{m,k,h}(\xi) |\alpha - \beta|^{2(n + m + k + h)}}{(n + k)! (n + h)! (m + k)! (m + h)!}.$$

(22)

Therefore, the average fidelity is determined as

$$F_{av} = \int P(\beta) F d^2 \beta.$$

(23)

The quantum teleportation process is successful when $F_{av} > 0.5$ and perfect at $F_{av} = 1$. By replacing the fidelity in Eq. (22) into Eq. (23), then calculating the integral, the our calculated result is

$$F_{av} = \sum_{m,n=0}^{\infty} \frac{C_{n,k,h}(\xi) C^*_{m,k,h}(\xi) (n + m + k + h)!}{2(n+m+k+h+1) \sqrt{(n+k)!(n+h)!(m+k)!(m+h)!}}.$$n

(24)

In case $\xi = |\xi|$, the average fidelity in Eq. (24) is explicitly written as

$$F_{av} = \left( \sum_{j=0}^{\infty} \frac{|\xi|^2 (j + k)!}{(j + h)!} \right)^{-1} \sum_{m,n=0}^{\infty} \frac{2^{-2|\alpha|^2}}{\pi} \sum_{n=0}^{\infty} |C_{n,k,h}(\xi)|^2 \frac{1}{(n + k)!} (n + k)! |n + h\rangle_b.$$

(25)

From Eq. (25), it is easy to see that the average fidelity $F_{av}$ does not depend on the amplitude $|\alpha|$ of a coherent state to be teleported. In Fig. 4, the curves show the dependence of the average fidelity $F_{av}$ as a function of $|\xi|$ with the different values of parameters $k$ and $h$, in which the case $k = 0$ and $h = q = 6$ (the solid blue line) corresponds to the PCS, and the others are the PAAMPCPS. It is clear that the average fidelity in the PAAMPCPS is always bigger than that in the PCS. More importantly, when the parameter $|\xi|$ is increasing, the average fidelity in the PAAMPCPS is improved by increasing the number of photon added and photon subtracted to the PCS. That is shown in Fig. 4a and b. Besides, the average fidelity $F_{av}$ gets the biggest values in case the number of photons in mode $a$ and mode $b$ are equal, i.e., $h = k = q = l$ (see Fig. 4c). Thus, the role of adding...
and subtracting photons is very important in enhancing the degree of average fidelity of the quantum teleportation process by using the first protocol.

6 Quantum teleportation uses the number sum and phase difference measurements protocol

In the second protocol as the number sum and phase difference measurements [61], the input state of the system is also given by Eq. (16). Now, Alice measures the photon number sum and phase difference on two modes \(a\) and \(c\). The measurement corresponds to projecting the state in Eq. (16) on the eigenstate of the number sum and phase difference operators of the two modes \(a\) and \(c\). After the measurement of Alice, the state of the system (non-normalized) becomes

\[ |\Phi_{ac}\rangle = \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{N} e^{i\phi(j)} |j\rangle_a |N-j\rangle_c, \]

(27)

in which \(\phi^c\) is restricted in the windows \(\phi^c_0 \leq \phi^c < \phi^c_0 + 2\pi\). \(\phi^c_0\) is an arbitrary real number. We clearly write Eq. (26) after inserting the normalization factor as

\[ |\Phi_{ac}\rangle = \frac{1}{\sqrt{2\pi P(N)}} \sum_{j=0}^{N} e^{-i(n+k)\phi^c} C_{n,k,(j)}(\xi) |N-(n+k)\rangle_n |n+h\rangle_b, \]

(28)

where \(P(N) = \frac{1}{2\pi} \sum_{\xi=0}^{N-k} \left| C_{n,k,\xi}(\xi) \right|^2 \left| d_{N-(n+k)} \right|^2 \) is the probability to obtain the photon number sum \(N\) and the phase difference \(\phi^c\). After the measurement, Alice sends to Bob the numbers of \(N\) and \(\phi^c\) by a classical channel. Based on these data, Bob rotates his phase by using the unitary operator \(U = e^{i(N_k+k-h)\phi^c}\), with \(N_k\) being the photon number operator of mode \(b\), acting on his state. The state of Bob becomes

\[ |\Phi_{abc}\rangle = \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{N} e^{-i(n+k)\phi^c} C_{n,k,(j)}(\xi) |N-(n+k)\rangle_n |n+h\rangle_b, \]

(27)

Fig. 4 The average fidelity \(F_{av}\) as a function of \(|\xi|\) for in a \((k,h) = (0,6)\) (the blue solid line), \((3,6)\) (the purple dotted curve), \((4,6)\) (the green dot-dashed curve), and \((6,6)\) (the red dashed curve), in b \((k,h) = (0,6)\) (the blue solid line), \((2,6)\) (the purple dotted curve), \((2,5)\) (the green dot-dashed curve), and \((2,2)\) (the red dashed curve), and in c \((k,h) = (0,6)\) (the blue solid line), \((1,5)\) (the purple dotted curve), \((2,4)\) (the green dot-dashed curve), and \((3,3)\) (the red dashed curve) (Color figure online)
Then, Bob transforms the photon number \( n + h \) to become \( N - (n + k) \). The teleportation process is completed. The output state is

\[
|\Psi_{\text{nor}}\rangle_{\text{out}} = \frac{1}{\sqrt{2\pi P(N)}} \sum_{n=0}^{N-k} C_{n,k,h}(\xi) d_{N-(n+k)}|N-(n+k)\rangle.
\]  

In order to estimate the efficacy of this teleportation process, we also use the average fidelity. The fidelity of the quantum teleportation process is determined by

\[
F = \langle \alpha | |\Psi_{\text{nor}}\rangle_{\text{out}}^\dagger \rangle^2 = \frac{1}{2\pi P(N)} \left| \sum_{n=0}^{N-k} C_{n,k,h}(\xi) d_{N-(n+k)} \right|^2.
\]  

From that, the average fidelity is represented by

\[
F_{av} = \sum_{N=k}^{\infty} \int_{\phi_0}^{\phi_0 + 2\pi} P(N) F d\phi^-
\]

\[
= \sum_{N=k}^{\infty} \sum_{n=0}^{N-k} \left| \sum_{n=0}^{N-k} C_{n,k,h}(\xi) d_{N-(n+k)} \right|^2.
\]  

Using Eq. (32) with the assumption that \( \xi = |\xi| \), we obtain the result as follows:

\[
F_{av} = \left( \sum_{n=0}^{\infty} |\xi|^{2m} |\xi|! 2^{m} \right)^{-1} \sum_{m=0}^{\infty} \sum_{n=0}^{m} |\xi|^{m-n} |\alpha|^{2m-n} |\xi|! 2^{m-n} \sqrt{n!} \sqrt{(n+k)!}.
\]  

It is clear in Eq. (33) that the average fidelity \( F_{av} \) not only depends on the parameters in the PAAST-MPCS, but also depends on the amplitude \( |\alpha| \) of a teleported state. In Fig. 5, we plot the dependence of the average fidelity \( F_{av} \) on \( |\xi| \) with several values of \( |\alpha| \) when \( k = 4 \) and \( h = 6 \). We see that the average fidelity \( F_{av} \) increases with the decreasing of \( |\alpha| \), and it can approach to zero if \( |\alpha| = 0.25 \) and \( |\xi| \) is large enough. In order to compare the degree of average fidelity \( F_{av} \) between PAAST-MPCS and PCS, we plot the dependence of the average fidelity \( F_{av} \) on \( |\xi| \), \( k \) and \( h \) for \( |\alpha| = 1.00 \) in Fig. 6. The case \( k = 0, h = q = 6 \) (the blue solid line) corresponds to the PCS, and the others are the PAAST-MPCS. When increasing the numbers of photon-added and photon-subtracted to both modes \( a \) and \( b \) of PCS, the average fidelity \( F_{av} \) in the PAAST-MPCS is always higher than that in the PCS (see Fig. 6a and b). It proves that the average fidelity of the quantum teleportation process is enhanced by adding and subtracting photons. These indicate the important role of the photon addition and photon subtraction operation to enhance

\[
|\Psi_{\text{nor}}\rangle_b = \hat{U} |\Phi_{\text{nor}}\rangle_b = \frac{1}{\sqrt{2\pi P(N)}} \sum_{n=0}^{N-k} C_{n,k,h}(\xi) d_{N-(n+k)}|n+h\rangle_b.
\]  

7 Conclusions

In this paper, we have introduced a new state called photon-added-and-subtracted two modes pair coherent state (PAAST-MPCS) and studied their nonclassical and non-Gaussian properties based on the Wigner function. It is shown that the Wigner function of the PAAST-MPCS gets negative values in some regions of the phase space and depends on the adding and subtracting photons to the PCS. It proves that the PAAST-MPCS is a nonclassical and non-Gaussian state. The nonclassical and non-Gaussian properties of this state are enhanced by adding and subtracting photons to the original PCS. We have quantified the entanglement degree of PAAST-MPCS based on the linear entropy criterion. It is shown that the PAAST-MPCS is an entangled state, and the entanglement degree of this state is greater than that of the original PCS. The more photon-added and photon-subtracted are, the more increasing the entanglement degree is. The entanglement degree can approach to the unit when the amplitude of the PAAST-MPCS \( |\xi| \) as well as the numbers of adding and subtracting photons are very large. It proves that the PAAST-MPCS is a non-Gaussian resource for performing quantum teleportation processes. For the first protocol by using the orthogonal quadrature components measurements, the average fidelity of the quantum teleportation process is always bigger than that of the original PCS. Moreover, the average fidelity gets the maximum values in case the numbers of photons in mode \( a \) and mode \( b \) are equal, i.e., the photon-added \( k \) and photon-subtracted \( l \) satisfy the condition \( k + l = q \). For the second protocol by using the number sum and phase difference measurements,
it is shown that the average fidelity can approach to the unit when the amplitude $|\alpha|$ of the input state $|\alpha\rangle$, is small and the amplitude $|\xi|$ of the PAAASTMPCS is not large. For example, when $|\alpha| = 0.25$, and $|\xi| = 4$, the $F_{av}$ reaches 0.998. Besides, if the numbers of photon-added and photon-subtracted to the PCS increase, the average fidelity $F_{av}$ also increases and the value of $F_{av}$ is higher than that compared with the case of using the PCS. Since the average fidelity can reach to unit, the second protocol is more suitable than the first one for quantum teleportation of a coherent state using the non-Gaussian entangled resource is the PAAASTMPCS. In addition, the schemes for generating PCS have been proposed [37, 38]. From that, the PAAASTMPCS can be experimentally generated by adding $k$ photons to mode $a$ and subtracting $l$ photons from mode $b$ of a PCS by using the ways as exploiting a downconverter with the weak parameter [16] and realizing with a beam splitter of high transmittance [63].

**Appendix**

**A derivation of Eq. (10)**

Substituting the density operator $\hat{\rho}$ in Eq. (9) to Eq. (8), then expanding a coherent state $|\gamma\rangle_z$ in terms of a Fock state as $|\gamma\rangle_z = e^{-|\gamma|^2/2} \sum_k (\xi^k/\sqrt{k!})|k\rangle$, with $x = \{a, b\}$, we get

$$W = \frac{4e^{2(|\alpha|^2+|\xi|^2)}}{\pi^4} \sum_{m,n=0}^{\infty} \frac{C_{n,k} C^*_{m,k} (\xi)(-1)^{k+h}}{\sqrt{(n+h)!(m+h)!(n+k)!(m+k)!}}$$

$$\times \int d^2\gamma_a d^2\gamma_b e^{[2\gamma_a^*\gamma_b^* \gamma_a - 2\xi_a \gamma_a^* + 2\gamma_b^* \gamma_b]}$$

$$\times e^{-2|\xi|^2/2} (\gamma_a^*+\gamma_b^*)^{n+k} (\gamma_a^{*+h}+\gamma_b^{*+h})^{m+k}.$$

(34)

We rewrite Eq. (34) as follows:

$$W = \frac{4e^{2(|\alpha|^2+|\xi|^2)}}{\pi^4} \sum_{m,n=0}^{\infty} \frac{C_{n,k} C^*_{m,k} (\xi)(-1)^{k+h}}{\sqrt{(n+h)!(m+h)!(n+k)!(m+k)!}}$$

$$\times \frac{1}{\pi^4} \int d^2\gamma_a e^{-|\xi|^2/2} (\gamma_a^*+\gamma_b^*)^{n+k} (\gamma_a^{*+h}+\gamma_b^{*+h})^{m+k}$$

$$\times \frac{1}{\pi^4} \int d^2\gamma_b e^{-|\xi|^2/2} (\gamma_a^*+\gamma_b^*)^{n+k} (\gamma_a^{*+h}+\gamma_b^{*+h})^{m+k}$$

$$= \frac{4e^{2(|\alpha|^2+|\xi|^2)}}{\pi^4} \sum_{m,n=0}^{\infty} \frac{C_{n,k} C^*_{m,k} (\xi)(-1)^{k+h}}{\sqrt{(n+h)!(m+h)!(n+k)!(m+k)!}} J_1 J_2,$$

(35)

where

$$J_1 = \frac{1}{\pi^4} \int d^2\gamma_a e^{-|\xi|^2/2} (\gamma_a^*+\gamma_b^*)^{n+k} (\gamma_a^{*+h}+\gamma_b^{*+h})^{m+k},$$

(36)

and

$$J_2 = \frac{1}{\pi^4} \int d^2\gamma_b e^{-|\xi|^2/2} (\gamma_a^*+\gamma_b^*)^{n+k} (\gamma_a^{*+h}+\gamma_b^{*+h})^{m+k}.$$

(37)

To calculate $J_1$ and $J_2$ in Eqs. (36) and (37), we consider an integral

$$J = \frac{1}{\pi^4} \int d^2\beta e^{-|\beta|^2} f(\beta^*) f(\beta)$$

(38)

By using a complex integral

$$\frac{1}{\pi^4} \int d^2\beta e^{-|\beta|^2} f(\beta^*) f(\beta) = (\partial/\partial a) f(a),$$

the integral in Eq. (38) is given as

$$J = (\partial/\partial a) f(e^{-a'a'}. (a'))^n.$$

(39)

From the definition of the Laguerre function $L_n^{(a)}(z) = \frac{e^{-z}}{\pi^{n/2} (\pi z)^{a+0.5}}$, and setting $|\alpha|^2 = y$, we have $\alpha = y/\alpha^*$ or $(\partial/\partial a)^y = (a)^y (\partial/\partial y)^y$. Thus the integral in Eq. (38) is given in form

$$J(a) = e^{-a'a'} (a')^y.$$
\[ J = q!(-|a|^2)^{-q}L^{-|b|-q}_{-q}(|a|^2)(-1)^q a^*(a^*)^q e^{-|a|^2}. \] (40)

Using the correlation between the Laguerre function and the hypergeometric function

\[ _2F_0(-n, b;; -1/z) = n!(z)^{-n}L_n^{b-n}(z), \]

we have

\[ J = (-1)^q a^*(a^*)^q e^{-|a|^2} _2F_0(-q, -l;; -1/|a|^2). \] (42)

From Eq. (42), the integrals \( J_1 \) and \( J_2 \) in Eqs. (36) and (37) are given by

\[ J_1 = (-1)^{n+k}(2a_n)^{m+k}(2a_n^a)^{m+k}e^{-|2a_n|^2} _2F_0(-n-k, -m-k;; -1/|2a_n|^2), \] (43)

and

\[ J_2 = (-1)^{n+k}(2a_n)^{m+k}(2a_n^a)^{m+k}e^{-|2a_n|^2} _2F_0(-n-h, -m-h;; -1/|2a_n|^2). \] (44)

After putting Eqs. (43) and (44) to Eq. (35), the Wigner function is determined as

\[ W = \frac{4e^{-2(|a|^2)+|a^*_b|^2}}{\pi^2} \sum_{m,n=0}^{\infty} \frac{C_{n,k,h}(\xi)C_{m,k,h}(\xi)}{(n+h)!(m+h)!(n+k)!(m+k)!} \times (2a_n^a2a_n)^{m+n}2a_n^a2a_n^{b^*}2a_n^b2a_n^{b^*} _2F_0(-n-k, -m-k;; -1/|2a_n|^2) \times _2F_0(-n-h, -m-h;; -1/|2a_n|^2). \] (45)

Note that \( \xi = |\xi|^2 e^{i\phi}, a_z = |a_z|^2 e^{i\phi} \) with \( x = \{a, b\} \), the Wigner function in Eq. (45) becomes

\[ W = \frac{4e^{-2(|a|^2)+|a^*_b|^2}}{\pi^2} \sum_{j=0}^{\infty} \frac{|\xi|^2(j+k)!}{(j!)^2(j+h)!} \sum_{m,n=0}^{\infty} \frac{C_{n,k,h}(\xi)C_{m,k,h}(\xi)}{(n+m)!n!(m+h)!} \times (2a_n^a2a_n)^{m+n}2a_n^a2a_n^{b^*}2a_n^b2a_n^{b^*} _2F_0(-n-k, -m-k;; -1/|2a_n|^2) \times _2F_0(-n-h, -m-h;; -1/|2a_n|^2). \] (46)

Because the imaginary parts of the Wigner function in Eq. (46) are vanished, the Wigner function is reduced to

\[ W = \frac{4e^{-2(|a|^2)+|a^*_b|^2}}{\pi^2} \sum_{j=0}^{\infty} \frac{|\xi|^2(j+k)!}{(j!)^2(j+h)!} \sum_{m,n=0}^{\infty} \frac{|\xi|^2(m+n)!n!(m+h)!\cos[(m-n)(\varphi_a + \varphi_b - \phi)]}{(m+h)!} \times (2a_n^a2a_n)^{m+n}2a_n^a2a_n^{b^*}2a_n^b2a_n^{b^*} _2F_0(-n-k, -m-k;; -1/|2a_n|^2) \times _2F_0(-n-h, -m-h;; -1/|2a_n|^2). \] (47)

Obviously, the Wigner function in Eq. (47) coincides with the function in Eq. (10).

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