Proposal for realizing a multiqubit tunable phase gate of one qubit simultaneously controlling \(n\) target qubits using cavity QED

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We propose a way to realize a multiqubit tunable phase gate of one qubit simultaneously controlling \(n\) target qubits with atoms in cavity QED. In this proposal, classical pulses interact with atoms \emph{outside} a cavity only, thus the experimental challenge of applying a pulse to an intra-cavity single atom without affecting other atoms in the same cavity is avoided. Because of employing a first-order large detuning, the gate can be performed fast when compared with the use of a second-order large detuning. Furthermore, the gate operation time is independent of the number of qubits. This proposal is quite general, which can be applied to various superconducting qubits coupled to a resonator, NV centers coupled to a microsphere cavity or quantum dots in cavity QED.

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\textbf{Introduction}.—Over the past decade, many proposals have been presented for realizing a multiqubit controlled-phase (CP) or controlled-NOT gate with \emph{multiple-control} qubits acting on one target qubit, using various physical systems \cite{1-7}. These proposals are important because they opened new avenues for realizing this type of multiqubit controlled gates, which are of significance in quantum information processing (QIP).

In this work we focus on a multiqubit tunable phase gate of one qubit simultaneously controlling \(n\) target qubits. This gate is useful in QIP (e.g., it has an important application in quantum Fourier transforms). In the following, we will present a way for implementing this gate with atoms in cavity QED.

The proposal has these features: (i) The atoms interact with classical pulses \emph{outside} the cavity, thus the experimental challenge of applying a pulse to an intra-cavity single atom without affecting (many) other atoms in the same cavity is avoided; (ii) The gate operation employs a first-order large detuning, thus the gate can be performed fast when compared with the use of a second-order large detuning; and (iii) The operation time is independent of the number of qubits and thus does not increase with the number of qubits.

\textbf{Multiqubit tunable phase gate}.—The multiqubit phase gate here consists of \(n\) two-qubit CP gates as depicted in Fig. 1(a). Each two-qubit CP gate has a shared control qubit (qubit 1) but a \emph{different} target qubit (2, 3, ..., or \(n+1\)). This multiqubit phase gate has the property: (i) when the control qubit 1 is in the state \(|0\rangle\), phase shifts \(e^{i\theta_2}, e^{i\theta_3}, ..., \) and \(e^{i\theta_{n+1}}\) are simultaneously induced to the state \(|0\rangle\) of the target qubits 2, 3, ..., and \(n+1\) respectively, but nothing happens to the state \(|0\rangle\) of each target qubit; (ii) when the control qubit 1 is in the state \(|1\rangle\), both states \(|0\rangle\) and \(|1\rangle\) of each target qubit remain unchanged. Here, \(\theta_2, \theta_3, ..., \) and \(\theta_{n+1}\) are adjustable as described below, taking values from 0 to \(2\pi\).

\textbf{Atom-cavity dispersive interaction}.—Consider atoms (2, 3, ..., \(n+1\)) with four levels depicted in Fig. 2(a). The cavity mode is coupled to the \(|2\rangle \leftrightarrow |3\rangle\) transition of each atom but highly detuned (decoupled) from the transition between any other two levels [Fig. 2(a)]. In the interaction picture, the interaction Hamiltonian is given by

\[H = \hbar \sum_{k=2}^{n+1} g(e^{-i\Delta_c t} a^+ \sigma_{23,k}^+ + H.c.),\]

where \(\Delta_c = \omega_{32} - \omega_c\) is the detuning of the cavity frequency \(\omega_c\) with the \(|2\rangle \leftrightarrow |3\rangle\) transition frequency \(\omega_{32}\) of the atoms, \(g\) is the coupling constant between the cavity mode and the \(|2\rangle \leftrightarrow |3\rangle\) transition, and \(\sigma_{23,k} = |2\rangle_k |3\rangle_k\).

For \(\Delta_c \gg g\) (i.e., the cavity mode is dispersive coupled to the \(|2\rangle \leftrightarrow |3\rangle\) transition of each atom), no energy exchange occurs between the atoms and the cavity mode. In this case, when the level \(|3\rangle\) of each atom is not excited,
H is given by \( \Delta = \omega k \) where \( \Omega \) is the Rabi frequency \( \omega, \phi, t \) for a time \( \pi \Delta / g^2 \). \( \Delta \) is the detuning \( \Delta = \omega_{32} - \omega \) [Fig. 2(b)], which is described by

The time-evolution operator corresponding to \( H_{\text{eff}} \) is given by 
\[
U(t) = \exp \left[ i \left( \frac{g^2}{\Delta} a^+ a \sigma_{22,k} \right) t \right],
\]
where \( U_{\text{kc}}(t) = \exp \left[ i \left( \frac{g^2}{\Delta} a^+ a \sigma_{22,k} \right) t \right] \) acts on the cavity mode and atom \( k \) \( (k = 2, 3, ..., n + 1) \). It is easy to verify that for \( t = \pi \Delta / g^2 \), \( U_{\text{kc}}(t) \) leads to the transformation \( |2\rangle_k |1\rangle_c \rightarrow |2\rangle_k |0\rangle_c \), while leaves the states \( |0\rangle_k |0\rangle_c, |1\rangle_k |0\rangle_c, |2\rangle_k |0\rangle_c, |0\rangle_k |1\rangle_c \) unchanged. Here, \( |1\rangle_c \) is the single-photon state of the cavity.

**Atom-pulse dispersive interaction**—Consider a four-level atom \( k \), which is driven by a pulse with frequency \( \omega \) and initial phase \( \phi = 0 \). The pulse is coupled to the \( |2\rangle \leftrightarrow |3\rangle \) transition of atom \( k \) with a detuning \( \Delta = \omega_{32} - \omega \) [Fig. 2(b)], which is described by

\[
H_{\text{eff}}^{\text{B}} = -\hbar \sum_{k=2}^{n+1} g^2 a^+ a \sigma_{22,k} \Delta_k.
\]  

where \( \Omega_k \) is the pulse Rabi frequency. For \( \Delta \gg \Omega_k \), the Hamiltonian (3) becomes

\[
H_{\text{eff}}^{\text{B}} = \hbar \frac{\Omega_k^2}{\Delta_k} \left( |3\rangle \langle 3| - |2\rangle \langle 2| \right),
\]

where \( \theta_k = \frac{\Omega_k^2 t_k}{\Delta} \). The \( \theta_k \) is adjustable by changing the Rabi frequency \( \Omega_k \) or the duration \( t_k \) of the pulse.

**Gate implementation**—Consider a two-level atom \( 1 \) with two levels \( |0\rangle \) (ground) and \( |1\rangle \) (excited), and \( n \) identical atoms \( (2, 3, ..., n + 1) \) with four levels depicted in Fig. 2. For each atom, the two lowest \( |0\rangle \) and \( |1\rangle \) represent the two logical states of a qubit.

Our gate operation needs to move atoms into or out of the cavity. This can be achieved by employing a translating optical lattice trap [11,12]. Experimentally, using an optical lattice trap to deliver atoms or a single atom into an optical microcavity has been reported [12], and by controlling the motion of the standing wave, a single atom can be transported to a preselected point along the standing wave with a very high precision [11].

FIG. 2: (Color online) (a) Atom-cavity off-resonant interaction. (b) Atom-pulse off-resonant interaction.

The cavity mode is initially in the vacuum state \( |0\rangle_c \). The procedure for realizing the gate is as follows:

**Step (i):** Apply a pulse of \( (\omega_{21}, -\pi/2, \pi/(4\Omega_c)) \) to atoms \( (2, 3, ..., n + 1) \), to transform the state \( |1\rangle_k \) of atom \( k \) to \( |1\rangle_k + |2\rangle_k \) / \( \sqrt{2} \) \( (k = 2, 3, ..., n + 1) \). Meanwhile, move atom 1 into the cavity for a time \( \pi/(2\Omega_c) \) [Fig. 3(a)] to transform the state \( |1\rangle_1 |0\rangle_c \rightarrow |1\rangle_1 |1\rangle_c \), but leave the state \( |0\rangle_1 |0\rangle_c \) unchanged. Then, move atom 1 out of the cavity.

**Step (ii):** Move atoms \( (2, 3, ..., n + 1) \) into the cavity for a time \( \pi \Delta / g^2 \) [Fig. 3(b)]. As a result, the states \( |0\rangle_k |0\rangle_c, |1\rangle_k |0\rangle_c, |2\rangle_k |0\rangle_c, |0\rangle_k |1\rangle_c \) and \( |1\rangle_k |1\rangle_c \) remain unchanged; but the state \( |2\rangle_k |1\rangle_c \) changes to \( -|2\rangle_k |1\rangle_c \) \( (k = 2, 3, ..., n + 2) \) as discussed previously. Then, move atoms \( (2, 3, ..., n + 1) \) out of the cavity.

**Step (iii):** Apply a pulse of \( (\omega_{21}, \pi/2, \pi/(4\Omega_c)) \) to atoms \( (2, 3, ..., n + 1) \), to transform the state \( |1\rangle_k + |2\rangle_k \) / \( \sqrt{2} \) of atom \( k \) to \( |1\rangle_k \) while the state \( |1\rangle_k - |2\rangle_k \) / \( \sqrt{2} \) to \( -|2\rangle_k \) \( (k = 2, 3, ..., n + 1) \).

**Step (iv):** Adjust the positions of atoms \( (2, 3, ..., n + 1) \) (e.g., by translating optical lattices) such that the atoms are sufficiently separated in space and then apply a classical pulse to each of them [Fig. 3(d)]. Each pulse has the same frequency and a zero initial phase. The pulse
applied to atom \( k \) \((k = 2, 3, \ldots, n + 1)\) is off-resonant with the \( |2⟩_k \leftrightarrow |3⟩_k \) transition of atom \( k \) with a detuning \( \Delta = \omega_0 - \omega \), which has a Rabi frequency \( \Omega_k \) and a duration \( \tau \). As discussed previously, a phase shift \( \theta_k = \Omega_k^2 \tau \Delta \) on the state \( |2⟩ \) of atom \( k \) is obtained after the pulse is applied to atom \( k \). Note that \( \theta_k \) can be adjusted by changing the pulse Rabi frequency \( \Omega_k \).

**Step (v):** Adjust the positions of the atoms \((2, 3, \ldots, n + 1)\) back to the original positions as depicted in Fig. 3(c) and then apply a pulse of \( \{\omega_{21}, -\pi/2, \pi/(4\Omega_1)\} \) to atoms \((2, 3, \ldots, n + 1)\), to transform the state \(|1⟩_k \) of atom \( k \) to the state \((|1⟩_k + |2⟩_k)/\sqrt{2} \) while the state \(|2⟩_k \) to \((|1⟩_k - |2⟩_k)/\sqrt{2} \) \((k = 2, 3, \ldots, n + 1)\).

**Step (vi):** Move atoms \((2, 3, \ldots, n + 1)\) back into the cavity for a time \( \pi \Delta_c/\gamma^2 \) [Fig. 3(b)]. The results for this step of operation are the same as those given in step (ii). Then, move the atoms \((2, 3, \ldots, n + 1)\) out of the cavity.

**Step (vii):** Apply a pulse of \( \{\omega_{21}, \pi/2, \pi/(4\Omega_1)\} \) to atoms \((2, 3, \ldots, n + 1)\) [Fig. 3(a)], to transform the state \((|1⟩_k + |2⟩_k)/\sqrt{2} \) of atom \( k \) to \(|1⟩_k \) \((k = 2, 3, \ldots, n + 1)\). Meanwhile, move atom 1 back into the cavity for a time \( 3\pi/(2\gamma_c) \) [Fig. 3(a)], such that the state \(|0⟩_1 |1⟩_c \) changes to \(|1⟩_1 |0⟩_c \) while the state \(|0⟩_1 |0⟩_c \) remains unchanged. Then, move atom 1 out of the cavity.

One can check that the \((n + 1)\)-qubit phase gate depicted in Fig. 1(a) was obtained with atoms \((i.e., the control atom 1 and the target atoms 2, 3, \ldots, n + 1)\) above the after manipulation.

Simultaneous interaction of a pulse with all of the target atoms \((2, 3, \ldots, n + 1)\) during steps (i), (iii), (v), and (vii) is unnecessary. Instead, one can apply a pulse to each or part of the atoms \((2, 3, \ldots, n + 1)\) separately.

For step (iv), one can also set \( \Omega_2 = \Omega_1 = \ldots = \Omega_{n+1} = \Omega \). In this case, \( \theta_k = \Omega^2 \tau_k \), which can be tuned by changing the duration \( \tau_k \) of the pulse applied to atom \( k \).

Due to non-exact placement of each atom in a pre-selected point of the cavity axis, the real coupling constant \( g_k \) \((g'_k)\) between the target atom \( k \) (the control atom 1) and the cavity may be different from the ideal \( g \) \((g_r)\) above. Hence, it is necessary to investigate how the gate fidelity is affected by the deviation of the real coupling constants from the ideal ones. The fidelity is defined by \( \mathcal{F} = |⟨\psi_{id}| \psi⟩|^2 \), where \( |\psi_{id}⟩ \) and \( |\psi⟩ \) are the states of the system after the above operations for the ideal case and the nonideal case, respectively. For the latter case, the effective Hamiltonian, associated with steps (ii) and (vi), takes the same form as the Hamiltonian (2) with \( g \) replaced by \( g_k \). As an example, consider that each atom is initially in the state \(|0⟩ + |1⟩)/\sqrt{2} \) before the gate. One can easily work out the states \(|\psi_{id}⟩ \) and \(|\psi⟩ \) (not shown to simplify our presentation). We have plotted Fig. 4 to demonstrate how the fidelity changes versus \( \theta \in [0, 2\pi] \) for \( g_k = \theta, g_k = 0.99g \) \((k = 2, 3, \ldots, n + 1)\), and \( g_r = 0.99g_r \). Fig. 4 shows that the gate fidelity decreases as the number \( n \) of target qubits increases but a high fidelity \( \sim 0.96 \) or more can be achieved when \( n \leq 10 \).

**Multiquit phase gate in QFT**—For a two qubit CP gate described by \(|00⟩ \rightarrow |00⟩, |01⟩ \rightarrow |01⟩, |10⟩ \rightarrow |10⟩, \) and \(|11⟩ \rightarrow e^{i\varphi} |11⟩ \), it is obvious that the roles of the two qubits can be interchanged. Thus, the multiquit phase gate in Fig. 1(a) is equivalent to the one with \( n \)-control qubits \((qubits 2, 3, \ldots, n + 1)\) acting on one target qubit \((qubit 1) \) [Fig. 1(b)]. The gate in Fig. 1(b), with \( \theta_k = 2\pi/2^k \) \((k = 2, 3, \ldots, n)\), plays an important role in quantum Fourier transforms (QFT). To implement this gate, the pulse Rabi frequencies \( \Omega_2, \Omega_3, \ldots, \Omega_{n+1} \), involved in step (iv), need to satisfy the relation \( \Omega_{k+1}/\Omega_k = 1/\sqrt{2} \) \((k = 2, 3, \ldots, n)\), and the operation time \( \tau \) for step (iv) needs to be set by \( \tau = (\Delta/\Omega^2)\sqrt{2}(2\pi/2^k) \). In this way, one can obtain \( \theta_k = \Omega^2 \tau/\Delta = 2\pi/2^k \) \((k = 2, 3, \ldots, n + 1)\).

**Fidelity**—Steps (i), (iii), (v), and (vii) can be completed within a very short time because of using resonant interactions only. Thus the dissipation of the atoms \((2, 3, \ldots, n + 1)\) and the cavity for these steps is negligibly small. One can choose atom 1 with sufficiently long spontaneous emission time such that decoherence of this atom is negligible during the entire operation. Thus, the dissipation of the system would appear in steps (ii) and (vi) due to the use of the atom-cavity or atom-pulse dispersive interaction.

During step (ii) or step (vi), the dynamics of the lossy system, composed of the cavity mode and atoms \((2, 3, \ldots, n + 1)\), is determined by

\[
\frac{d\rho}{dt} = -i[H, \rho] + \kappa L[a] + \sum_{i=2,1,0}^{n+1} \gamma_{ai} L[\sigma^-_{ai,k}]
+ \sum_{i=1,0}^{n+1} \gamma_{ai} L[\sigma^+_{ai,k}] + \sum_{i=1}^{n+1} \gamma_{ai} L[\sigma^-_{ai,k}],
\]

where \( H \) is the Hamiltonian (1) with \( g \) replaced by \( g_k \), \( L[a] = (2\rho \alpha^* \alpha + \alpha^* \alpha \rho - \alpha^* \alpha \rho + a^* \alpha \rho + \rho \alpha^* \alpha a) \), \( L[\sigma^-_{ai,k}] = 2\sigma^-_{i,j,k} \rho \sigma^+_{i,j,k} - \sigma^+_{i,j,k} \sigma^-_{i,j,k} \rho - \rho \sigma^+_{i,j,k} \sigma^-_{i,j,k} \) (with \( \sigma^-_{i,j,k} = |i⟩_k |j⟩ \) and \( ij \in \{23, 13, 03, 12, 02, 01\} \)), \( \kappa \) is the decay rate of the cavity mode, \( \gamma_{ji} \) is the decay rate of the level \( |j⟩ \) of atoms \((2, 3, \ldots, n + 1)\) via the decay path
\[ |j \rangle \rightarrow |i \rangle \] (here, \( j, i \in \{32, 31, 30, 21, 20, 10 \} \)). In addition, during step (iv), the dynamics of the lossy system, composed of the cavity mode and atoms (2, 3, ..., \( n + 1 \)), is described by \( \frac{d\rho}{dt} = -i \left[ \sum_{k=0}^{\infty} H^k, \rho \right] + \mathcal{E} \), where \( H^k \) is the Hamiltonian (3) and \( \mathcal{E} \) is the sum of the last four terms of Eq. (4).

The fidelity of the gate operations is given by \( F = \langle \psi_{id} | \tilde{\rho} | \psi_{id} \rangle \), where \( |\psi_{id} \rangle \) is the state of the whole system after the gate operations, in the ideal case without considering the dissipation of the system during the entire operation and non-exact placement of atoms inside the cavity; and \( \tilde{\rho} \) is the final density operator of the whole system after the gate operations are performed in a real situation.

As an example, let us consider realizing a four-qubit CP gate in QFT, using a two-level atom 1 and three identical four-level atoms (2, 3, 4). The three identical atoms (2, 3, 4) are chosen as Rydberg atoms with the principle quantum numbers 49, 50, and 51, which correspond to the three levels [1], [2], and [3] as depicted in Fig. 2, respectively. We label the energy relaxation times for the three levels [1], [2], and [3] by \( \gamma_1^{-1}, \gamma_2^{-1}, \) and \( \gamma_3^{-1} \), which are on the order of \( \sim 3 \times 10^{-2} \) s (e.g., see [13-15]). Without loss of generality, assume that each of the four atoms is initially in the state \((|0 \rangle + |1 \rangle)/\sqrt{2}\) and the cavity mode is in the vacuum state before the gate. The expression for the ideal state \(|\psi_{id} \rangle\) of the system after the entire operation is straightforward (not shown here to simply our presentation). As a conservative estimation, consider \( \gamma_{21}^{-1} = \gamma_{31}^{-1} = \gamma_{30}^{-1} = \gamma_{3}^{-1} \), \( \gamma_{21}^{-1} = \gamma_{20}^{-1} = \gamma_{2}^{-1} \), and \( \gamma_{1}^{-1} = \gamma_{1}^{-1} \). In addition, choose \( g_r = 0.99g_r, g_2 = g_3 = g_4 = 0.99g, g_r = g = 2\pi \times 50 \) KHz [13,14], \( \Omega_c = g_r, \Omega_2 = g, \Omega_4/\Omega_3 = \Omega_3/\Omega_2 = 1/\sqrt{2}, \tau = (\Delta/\Omega_2^2) (2\pi/2^2), \tau_m = 1 \mu\text{s} \) (a typical time for loading atoms into or out of the cavity), and \( \kappa^{-1} = 3.0 \times 10^{-2} \) s. Our numerical calculation shows that a high fidelity \( \sim 97\% \) can be achieved when the ratio \( b = \Delta_c/g = \Delta/\Omega_2 \) is about 10 (Fig. 5).

For Rydberg atoms chosen here, the \( |2 \rangle \leftrightarrow |3 \rangle \) transition frequency is \( \sim 51 \) GHz [15]. The cavity mode frequency is then \( \sim 50.9995 \) GHz for \( \Delta_c/g = 10 \). For the cavity-photon lifetime \( \kappa^{-1} \) used in our calculation, the required quality factor \( Q \) of the cavity is \( \sim 9.6 \times 10^9 \). Note that cavities with a high \( Q \) \( \sim 3 \times 10^{10} \) was previously reported [16].

**Discussion**—The idea of coupling qubits to a cavity to implement the proposed gate was previously presented [17]. However, our present proposal differs from the one in [17]. First, in our present proposal, application of pulses to the qubits is carried out outside a cavity, while the one in [17] requires that each pulse was applied to a different target qubit inside a cavity. Second, the present proposal is based on a first-order large detuning, i.e., \( \Delta_c \gg g \) or \( \Delta \gg \Omega_k \), while the proposal in [17] was based on a second-order large detuning \( \delta = \Delta_c - \Delta \gg g^2/\Delta_c, \Omega_k^2/\Delta \) between the cavity mode and the pulse (for the detailed discussion, see [17]).

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