Automata and tame expansions of \((\mathbb{Z}, +)\)

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Abstract

The problem of characterizing which automatic sets of integers are stable is here initiated. Given a positive integer \(d\) and a subset \(A \subseteq \mathbb{Z}^m\) whose set of representations base \(d\) is sparse and recognized by a finite automaton, a necessary condition is found for \(x + y \in A\) to be a stable formula in \(\text{Th}(\mathbb{Z}, +, A)\). Combined with a theorem of Moosa and Scanlon this gives a combinatorial characterization of the \(d\)-sparse \(A \subseteq \mathbb{Z}^m\) such that \((\mathbb{Z}, +, A)\) is stable. This characterization is in terms of what were called \(F\)-sets in [14] and elementary \(p\)-nested sets in [9]. For \(A \subseteq \mathbb{Z}\) \(d\)-automatic but not \(d\)-sparse, it is shown that if \(x + y \in A\) is stable then finitely many translates of \(A\) cover \(\mathbb{Z}\). Automata-theoretic methods are also used to produce some NIP expansions of \((\mathbb{Z}, +)\), in particular the expansion by the monoid \((d^\omega, \times)\).

1 Introduction

In [15] Palacin and Sklinos give examples of stable expansions of \(\text{Th}(\mathbb{Z}, +)\), and pose the following general question:

**Question 1.1.** For which \(A \subseteq \mathbb{Z}\) is \(\text{Th}(\mathbb{Z}, +, A)\) stable?

The project of finding sufficient topological or algebraic conditions on \(A\) for stability has been taken up in other recent work; see for example [7] and [12]. The theme also appeared some fifteen years earlier: the results of Moosa and Scanlon in [14] imply that \((\mathbb{Z}, +, A)\) is stable whenever \(A\) is an \(F\)-set (see Definition 3.4). This includes for example the case \(A = d^0\), whose stability was rediscovered in [15]. Of note is that their result applies to \(A \subseteq \mathbb{Z}^m\), not just \(A \subseteq \mathbb{Z}\), and that is the context we will work in.

In this paper, we consider Question 1.1 when \(A \subseteq \mathbb{Z}^m\) is a \(d\)-automatic set for some \(d \geq 2\). Automatic sets are reviewed in Section 2, but let us recall here informally that for \(A \subseteq \mathbb{Z}\) to be \(d\)-automatic means that there is a finite machine that takes strings of digits from \([-d + 1, \ldots, d - 1]\) as input and accepts exactly those strings that are representations base \(d\) of an element of \(A\).

Instead of asking when the first-order theory of \((\mathbb{Z}, +, A)\) is stable, we will focus on a local, and hence combinatorial, notion of stability which we now briefly recall. If \(R \subseteq X \times X\) is a binary relation on a set \(X\) then an \(N\)-ladder for \(R\) is some \(a_0, \ldots, a_N, b_0, \ldots, b_N \in X\) such that \((a_i, b_j) \in R\) if and only if \(i \leq j\). The relation \(R\) is \(N\)-stable if there is no \(N\)-ladder for \(R\), and is stable if it is \(N\)-stable for some \(N\). If \((G, +)\) is a group and \(A \subseteq G\) we say that \(A\) is stable in \(G\) if \(x + y \in A\) is a stable binary relation on \(G\).

The variant of Question 1.1 that we will study is:

**Question 1.2.** Which \(d\)-automatic \(A \subseteq \mathbb{Z}^m\) are stable in \((\mathbb{Z}^m, +)\)?

Automatic sets separate naturally into sparse and non-sparse sets, with “sparse” meaning that the number of accepted strings grows polynomially with length—see Definition 2.9 for a precise definition. Our first result, Theorem 3.1, states that if \(A \subseteq \mathbb{Z}^m\) is \(d\)-sparse and stable in \((\mathbb{Z}^m, +)\) then \(A\) is a finite Boolean combination of translates of finite sums of sets of the form

\[
C(a; \delta) := \{ a + d^0 a + \cdots + d^n \delta a : n < \omega \}
\]

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1It is worth noting that this terminology conflicts somewhat with the terminology used by Conant in [7], in which he calls \(A \subseteq \mathbb{N}\) “stable” if \(\text{Th}(\mathbb{Z}, +, A)\) is stable. His is a stronger notion than ours, which is equivalent to saying that \(\varphi(x; y)\) given by \(x + y \in A\) is a stable formula in \(\text{Th}(G, +, A)\).
where \( a \in \mathbb{Z}^m \) and \( \delta \) is a positive integer. The converse is known to be true: it follows from \([14]\). In fact, these sets are of Diophantine significance in positive characteristic, appearing in both the isotrivial Mordell-Lang theorem of \([14]\) and the Skolem-Mahler-Lech theorem of \([9]\); see \([4, \text{Section 3}]\) for an account of the connection with the latter. In any case, we get a complete answer to Question 1.1 for \( d \)-sparse subsets of \( \mathbb{Z}^m \); this is stated as Corollary 3.8 below.

We then turn our attention to \( d \)-automatic sets that aren’t \( d \)-sparse. We show that if \( A \subseteq \mathbb{Z} \) \( d \)-automatic but not \( d \)-sparse, if \( A \) is stable in \((\mathbb{Z},+)\) then \( A \) is generic (i.e. finitely many translates cover \( \mathbb{Z} \)). This is Theorem 4.2. In particular, every \( d \)-automatic subset of \( \mathbb{N} \) that is stable in \((\mathbb{Z},+)\) must be \( d \)-sparse. Actually, this consequence of our Theorem 4.2 can also be deduced by combining \([7, \text{Theorem 8.8}]\) and \([3, \text{Theorem 1.1}]\), but the general statement requires significantly more work.

Theorems 3.1 and 4.2 suggest the following conjecture, articulated in conversations with Jason Bell, that would offer a complete answer to Question 1.1 for \( d \)-automatic sets.

Conjecture 1.3. The \( d \)-automatic \( A \subseteq \mathbb{Z} \) for which \( \text{Th}(\mathbb{Z},+,A) \) is stable are exactly the finite Boolean combinations of:

- cosets of subgroups of \((\mathbb{Z},+)\), and
- translates of finite sums of sets of the form \( C(a; \delta) \).

In a somewhat different direction, we conclude the paper by using automata-theoretic methods to produce two \( \mathcal{N} \)-interpretations of \((\mathbb{Z},+)\): namely \((\mathbb{Z},+,<,d^n)\) and \((\mathbb{Z},+,d^n,\times [d^n])\), in Theorems 5.1 and 5.9, respectively. The former was recently obtained by Lambotte and Point in \([12]\) using different methods, but the latter is a new example.

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2 Preliminaries on automatic sets

We briefly recall regular languages and finite automata; see \([19]\) for a more detailed presentation.

Definition 2.1. For a finite set \( \Lambda \) viewed as an alphabet we denote by \( \Lambda^* \) the set of words over \( \Lambda \), namely finite strings of letters from \( \Lambda \). The class of regular languages over \( \Lambda \) is the smallest subset of \( \mathcal{P}(\Lambda^*) \) that contains all finite sets and is such that if \( A, B \) are regular then so are \( A \cup B, AB \), and \( A^* \).

A deterministic finite automaton (DFA) over a finite alphabet \( \Lambda \) is a tuple \( \mathcal{A} = (Q, q_0, F, \delta) \) where \( Q \) is a finite set of states, \( q_0 \in Q \) is the start state, \( F \subseteq Q \) is the set of finish states, and \( \delta \colon Q \times \Lambda \to Q \) is the transition function: if \( \mathcal{A} \) is in state \( q \in Q \) and reads the letter \( \ell \in \Lambda \) then it moves into state \( \delta(q,\ell) \). We identify \( \delta \) with its natural extension \( Q \times \Lambda^* \to Q \) inductively by \( \delta(q,\ell_1\cdots\ell_{n+1}) = \delta(\delta(q,\ell_1\cdots\ell_n),\ell_{n+1}) \). The set recognized by \( \mathcal{A} \) is \( \{ \sigma \in \Lambda^* : \delta(q_0,\sigma) \in F \} \). A fundamental fact (see \([19, \text{Lemma 2.2}, \text{Section 3.2}, \text{and Section 3.3}]\) is that the regular languages are precisely the sets recognized by DFAs.

It turns out some behaviour of automata can be captured in Presburger arithmetic:

Proposition 2.2. Suppose \( \Lambda \) is an alphabet; suppose \( L \subseteq \Lambda^* \) is regular and \( \sigma_1, \ldots, \sigma_n \in \Lambda^* \). Then \( \{ (t_1, \ldots, t_n) \in \mathbb{N}^n : \sigma_1^{t_1} \cdots \sigma_n^{t_n} \in L \} \) is definable in \((\mathbb{N},+)\).

Proof. Fix an automaton \((Q, q_0, F, \delta) \) for \( L \). We apply induction on \( n \) to show that \( \delta(q_1, \sigma_1^{t_1} \cdots \sigma_n^{t_n}) = q_2 \) is definable in \((\mathbb{N},+)\) for all \( q_1, q_2 \in Q \). The case \( n = 0 \) is vacuous. For the induction step, note since there are finitely many states that \( \delta(q_1, \sigma_1^t) \) is eventually cyclic in \( t \); so there is \( N \) such that for \( t \geq N \) we have that \( \delta(q_1, \sigma_1^t) \) depends only on the congruence class of \( t \) modulo some \( \mu \). Then our set is defined by

\[
\bigvee_{t < N} \left( (t_1 = t) \land \delta(q_1, \sigma_1^{t_1}) = q_2 \right) \lor \bigvee_{t < \mu} \left( (t_1 \in N + t + \mu\mathbb{N}) \land \delta(q_1, \sigma_1^{N+t}) = q_2 \right)
\]

which is definable in \((\mathbb{N},+)\) by the induction hypothesis.

But then \( \{ (t_1, \ldots, t_n) \in \mathbb{N}^n : \sigma_1^{t_1} \cdots \sigma_n^{t_n} \in L \} \) is the union over \( q \in F \) of \( \{ (t_1, \ldots, t_n) \in \mathbb{N}^n : \delta(q_0, \sigma_1^{t_1} \cdots \sigma_n^{t_n}) = q \} \), which is definable in \((\mathbb{N},+)\). \( \square \) Proposition 2.2
We are primarily interested in the case where the strings in question are representations of integers. Fix a positive integer $d$. Evaluating a string base $d$ gives a natural map $[\cdot] : \mathbb{Z}^\ast \to \mathbb{Z}$ via

$$[k_0k_1 \cdots k_n] = \sum_{i=0}^{n} k_id^i.$$  

Note that unlike usual base $d$ representations the most significant digit occurs last, not first.

**Definition 2.3.** We let $\Sigma = \{0, \ldots, d-1\}$ and $\Sigma_{\pm} = \{-d+1, \ldots, d-1\}$. We say $A \subseteq \mathbb{Z}$ is $d$-automatic if $\{ \sigma \in \Sigma_{\pm}^\ast : [\sigma] \in A \}$ is a regular language over $\Sigma_{\pm}$.

There is a natural extension of this notion to $\mathbb{Z}^m$ for $m \geq 1$. We first extend our map $[\cdot]$ to $(\mathbb{Z}^m)^\ast \to \mathbb{Z}^m$: we set

$$\left[ \begin{pmatrix} k_{10} \\ \vdots \\ k_{m0} \end{pmatrix} \cdots \begin{pmatrix} k_{1n} \\ \vdots \\ k_{mn} \end{pmatrix} \right] = \begin{pmatrix} [k_{10} \cdots k_{m0}] \\ \vdots \\ [k_{1n} \cdots k_{mn}] \end{pmatrix}$$

**Definition 2.4.** We say $A \subseteq \mathbb{Z}^m$ is $d$-automatic if $\{ \sigma \in (\Sigma_{\pm}^m)^\ast : [\sigma] \in A \}$ is a regular language over $\Sigma_{\pm}^m$.

A note on exponential notation: we use $\Lambda^m$ to denote the alphabet $\Lambda \times \cdots \times \Lambda$. This contrasts its usual meaning in formal languages, namely the set of words over $\Lambda$ of length $m$; we use $\Lambda(m)$ to denote this. We will use $\sigma^m$ to denote the $n$-fold concatenation of $\sigma$ with itself; it should be clear from context whether an instance of exponential notation refers to iterated string concatenation or iterated multiplication.

Of course different strings can represent the same integer. It is useful to fix a canonical representation.

**Definition 2.5.** The canonical representation of $0$ is the empty word $\varepsilon$. The canonical representation of a positive integer $a$ is its usual representation base $d$ in $\Sigma^\ast$ (though with the order reversed). The canonical representation of a negative integer $a$ is $(-k_0) \cdots (-k_n)$ where $k_0 \cdots k_n$ is the canonical representation of $-a$.

Finally, the canonical representation of a tuple $(a_1, \ldots, a_m)$ is $\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}$, where $n + 1$ is the maximum of the lengths of the canonical representations of the $a_i$, and $k_{10} \cdots k_{1n}$ is the canonical representation of $a_i$ for each $i$, possibly padded with trailing zeroes to make them of length $n + 1$.

Note that the canonical representation of an integer base $d$ is a word over $\Sigma_{\pm}$, and if the integer happens to be non-negative then it is a word over $\Sigma$.

**Example 2.6.** The canonical representation base 10 of $\begin{pmatrix} -23 \\ 432 \end{pmatrix}$ is $\begin{pmatrix} -3 \\ 2 \end{pmatrix} \begin{pmatrix} -2 \\ 3 \end{pmatrix} \begin{pmatrix} 0 \\ 4 \end{pmatrix}$.

**Remark 2.7.** Automaticity is robust under changes in the allowed representations. Indeed, from [10, Proposition 7.14] (together with some basic tools of automata theory—see e.g. [19, Theorems 2.16 and 4.2]) one can show that the following are both equivalent to $A$ being $d$-automatic:

1. The set of canonical representations of elements of $A$ is regular. Note that this is essentially the definition given in [1, Section 5.3]. In particular, our definition generalizes the classical notion of $d$-automatic subsets of $\mathbb{N}$ (see e.g. [2, Chapter 5]).

2. For some (equivalently all) finite $\Lambda \subseteq \mathbb{Z}^m$ containing $\Sigma_{\pm}^m$, $\{ \sigma \in \Lambda^\ast : [\sigma] \in A \}$ is regular over $\Lambda$.

We are particularly interested in sparsity among $d$-automatic sets. If $\Lambda$ is an alphabet we say $L \subseteq \Lambda^\ast$ is sparse if it is regular and the map $k \mapsto |\{ \sigma \in L : |\sigma| \leq k \}|$ grows polynomially in $k$. Several equivalent formulations of sparsity are known; we will in particular make use of the following characterization:

**Fact 2.8** ([4, Proposition 7.1]). If $L \subseteq \Lambda^\ast$ then $L$ is sparse if and only if it is a finite union of sets of the form $u_0w_1^1u_1 \cdots u_{r_n}w_n^nu_n = \{ u_0w_1^1u_1 \cdots w_n^nu_n : r_1, \ldots, r_n \geq 0 \}$ for some $u_0, \ldots, u_n, w_1, \ldots, w_n \in \Lambda^\ast$.

**Definition 2.9.** We say $A \subseteq \mathbb{Z}^m$ is $d$-sparse if the set of canonical representations base $d$ of elements of $A$ is a sparse language over $\Sigma_{\pm}^m$.

Note by **Remark 2.7** that $d$-sparse sets are $d$-automatic. In fact, $d$-sparsity is equivalent to the existence of some finite $\Lambda \supseteq \Sigma_{\pm}^m$ and some sparse $L \subseteq \Lambda^\ast$ such that $A = [L]$, but we will not need this.
3 Characterizing stable sparse sets

In this section we give a complete characterization of the $d$-sparse subsets of $\mathbb{Z}^m$ that are stable in $(\mathbb{Z}^m, +)$. Observe that not every $d$-sparse set is stable: assuming $d > 2$ the set $A = [0^{10}2]^{*}$ is $d$-sparse by Fact 2.8, but it is not hard to verify that $d^i + 2d^{i+1} \in A$ if and only if $i \leq j$.

The main result of this section:

**Theorem 3.1.** Suppose $A \subseteq \mathbb{Z}^m$ is $d$-sparse. If $A$ is stable in $(\mathbb{Z}^m, +)$ then $A$ is a finite Boolean combination of translates of finite sums of sets of the form

$$C(a; \delta) = \{ a + d^i a + \cdots + d^{\delta} a : n < \omega \}$$

where $a \in \mathbb{Z}^m$ and $\delta > 0$.

Sets of this form, namely finite unions of translates of finite sums of $C(a; \delta)$, were studied in [14] in a more general setting, where they were called “groupless $F$-sets”; here $F$ is the multiplication-by-$d$ endomorphism on $(\mathbb{Z}^m, +)$. When $m = 1$, these groupless $F$-sets were rediscovered in a different context by Derksen [9] as “elementary $p$-nested” subsets of $\mathbb{Z}$; see [4, Proposition 3.2] for a proof that they agree up to finite symmetric differences.

Combined with the results of [14] this theorem yields complete answers to both Questions 1.1 and 1.2 for $d$-sparse subsets of $\mathbb{Z}^m$. See Corollary 3.8 below.

Before proving the theorem let us make some observations that may give the reader a better feel for the automata-theoretic nature of the sets $C(a; \delta)$.

**Remark 3.2.**

1. $d^{k+1} = \{ 1 \} \cup (1 + C(d-1;1))$.
2. If $a = [\sigma]$ where $\sigma \in (\mathbb{Z}^m)^*$ is of length $\delta$ then $C(a; \delta) = \{ [\sigma^n] : n > 0 \}$.
3. Every translate of a finite sum of sets of the form $C(a; \delta)$ is $d$-sparse.
4. Let $\mathcal{C}$ denote the collection of subsets of $\mathbb{Z}$ of the form $b + C(a; \delta)$ for some $a, b \in \mathbb{Z}$ and $\delta > 0$. Let $\mathcal{E}$ be the collection of subsets of $\mathbb{Z}$ of the form $[uv^*w]$ for $u, v, w \in \Sigma^*$ or $u, v, w \in (-\Sigma)^*$. Then up to finite symmetric differences, $\mathcal{C}$ and $\mathcal{E}$ agree.

**Proof.** Parts (1) and (2) are easily verified by hand. Part (3) is observed in [4]; see the proof of Theorem 7.4 therein. We prove (4).

$(\supseteq)$ We will see in the proof of Lemma 3.5 below that we can write $[uv^*w]$ as a translate of $[\tau^*]$ for some $\tau \in \Sigma^*$. But then by part (2) this has finite symmetric difference with $C([\tau]; [\tau])$.

$(\subseteq)$ Suppose that we are given $C(a; \delta)$; by negating if necessary we may assume $a \geq 0$. Pick a representation $\sigma \in \Sigma^*$ of $a$ of length $\delta$; then by part (2) we are interested in the canonical representations of $[\sigma^*] \setminus \{ \varepsilon \}$. For $0 < i < \omega$ write $[\sigma^i] = b_i d^i + c_i$ where $0 \leq c_i < d^i$; so $b_i$ is the “carry” when adding up $a + F^i a + \cdots + F^{(i-1)} a$ and cutting off after $i\delta$ digits. Then

$$[\sigma^{i+1}] = d^i [\sigma^i] + [\sigma] = b_i d^{(i+1)} + d^i c_i + [\sigma] \geq b_i d^{(i+1)}$$

so $b_{i+1} \geq b_i$. But $[\sigma^i] = d^{\frac{(i+1)}{\delta}} \leq a d^i$, so each $b_i \leq a$. So the $b_i$ are eventually constant, say $b_N = b_{N+i} = \cdots$. Let $p = b_N + [\sigma]$ mod $d^i$. Then

$$[\sigma^{N+k+1}] = [\sigma^{N+k}] + d^{(N+k)} [\sigma] = b_{N+k} d^{(N+k)} + c_{N+k} + d^{(N+k)} [\sigma] = d^{(N+k)} (b_N + [\sigma]) + c_{N+k}$$

so

$$c_{N+k+1} = [\sigma^{N+k+1}] \mod d^{(N+k+1)} = d^{(N+k)} p + c_{N+k}$$

(since $c_{N+k} < d^{(N+k)}$). So inductively we get $c_{N+k} = c_N + d^{N} p + d^{(N+1)} p + \cdots + d^{N+k-1} p$. So

$$[\sigma^{N+k}] = b_N d^{(N+k)} + c_N + d^{N} p + \cdots + d^{N+k-1} p$$
So if \( u \in \Sigma^{(N, \delta)} \), \( v \in \Sigma^{(\delta)} \), \( w \in \Sigma^* \) represent \( c_N, p, b_N \) respectively then \( [\sigma^{N+k}] = [uv^kw] \). So \( C(a; \delta) = [\sigma^* \setminus \{ a \}] \) has finite symmetric difference with \( [uv^kw] \), as desired.

It remains to show that a translate of a single cycle takes the desired form; by above it suffices to show that a translate of \([uv^kw]\), say by \( \gamma \in \mathbb{Z} \), has finite symmetric difference from some \([xyz^*z]\). (Again we may assume \([uv^kw] \subseteq \mathbb{N} \).) If \( u, v \in (d-1)^* \) then \( \gamma + [uv^kw] = (\gamma - 1) + [0^{\lceil (0^{\lceil |\gamma| \rceil})^* t \rceil}] \) where \( |\gamma| = |w| + 1 \); so we may assume \( uv \notin (d-1)^* \). So for some \( N \) we get that \( 0 \leq \gamma + [uv^N] < d^{uv^N} \); so if \( \sigma \in \Sigma^{(uv^N)} \) has \( [\sigma] = \gamma + [uv^N] \) then \( \gamma + [uv^{N+k}w] = [\sigma v^kw] \). So \( \gamma + [uv^kw] \) has finite symmetric difference from \([\sigma v^kw] \).

\[ \square \] Remark 3.2

We begin working towards a proof of Theorem 3.1. Our approach requires that we first understand the stable formulas in \((\mathbb{N}, 0, S, \delta\mathbb{N}, <)\) where \( S \) is the successor function and \( \delta \) is a fixed positive integer. The following proposition, which is of independent interest, is likely known, but as we could find no reference we include a proof here for completeness.

**Proposition 3.3.** Fix \( \text{Th}(\mathbb{N}, 0, S, \delta\mathbb{N}, <) \) as the ambient theory. Let \( L_\delta = \{ 0, S, P_\delta \} \) and \( L_\delta, < = L_\delta \cup \{ < \} \). Suppose \( \varphi(x_1, \ldots, x_n) \in L_\delta, < \) is quantifier-free and stable with respect to any partition of the variables. Then \( \varphi \) is equivalent to a quantifier-free \( L_\delta \)-formula.\(^2\)

**Proof.** We apply induction on \( n \); the case \( n = 0 \) is vacuous.

With an eye towards constraining the atomic subformulas of \( \varphi \), we rewrite \( \varphi \) as follows:

- Replace any occurrence of \( S^e x_i < K \) by a disjunction of equalities in the obvious way, and of \( S^e x_i < S^f x_j \) for \( e \geq f \) by \( S^{-f} x_i < x_j \). Using this and the fact that \( t_1 \leq t_2 \iff -(t_1 > t_2) \), we may assume all atomic inequalities take the form \( S^e x_i < x_j \).

- Replace any occurrence of \( S^e x_i = K \) by \( x_i = K - e \) and of \( S^e x_i = S^f x_j \) for \( e \geq f \) by \( S^{-f} x_i = x_j \).

So we may assume the atomic subformulas of \( \varphi \) take the following forms:

- \( x_i \equiv K \pmod{\delta} \)
- \( x_i = K \)
- \( S^e x_i < x_j \)
- \( S^e x_i = x_j \)

Let \( M \) be greater than both the largest \( K \) appearing in \( \varphi \) and the largest \( e \) with \( S^e \) appearing in \( \varphi \). Note that the truth value of \( \varphi(\overline{x}) \) is determined by the truth value of the above formulas on \( \overline{x} \). Furthermore since we may assume in said formulas that \( K < M \) and \( e < M \), we get that there are finitely many such formulas; call the set of such formulas \( \Delta \). So we can write \( \varphi \) as a finite disjunction of consistent conjunctions of the form

\[ \psi_f = \bigwedge_{\theta \in \Delta} \theta^{f(\theta)} \]

for some \( f : \Delta \to \{ 0, 1 \} \). (Here \( \psi_f^0 \) denotes \( \neg \psi_f \) and \( \psi_f^1 \) denotes \( \psi_f \).)

Fix one such disjunct \( \psi_f \); we will show that \( \psi_f \) implies some \( L_\delta \)-formula that in turn implies \( \varphi \).

**Case 1.** Suppose \( \psi_f \) contains a conjunct of the form \( S^e x_{j_1} = x_{j_2} \) or \( K = x_{j_2} \). Define a term \( t \) to be \( S^e x_{j_1} \) in the former case and \( K \) in the latter case, and consider \( \varphi'(\overline{x'}) = \varphi'(x_1, \ldots, x_{j_2-1}, x_{j_2+1}, \ldots, x_n) \) obtained by substituting \( x_{j_2} = t \) into \( \varphi \); this is stable because \( \varphi \) is, and because \( t \) involves at most one of the \( x_i \). It also contains one fewer variable, so by the induction hypothesis is equivalent to a quantifier-free \( L_\delta \)-formula \( \theta(\overline{x'}) \). But then \( \theta(\overline{x'}) \land (x_{j_2} = t) \) is our desired formula.

\(^2\) In fact both \( \text{Th}(\mathbb{N}, 0, S, \delta\mathbb{N}, <) \) and \( \text{Th}(\mathbb{N}, 0, S, \delta\mathbb{N}) \) admit quantifier elimination; however we don’t make use of this in either the proof or the application of this proposition.
Case 2. Suppose $\psi_f$ contains no such conjuncts; let $\psi_0(\pi)$ be the conjunction of the negations of such. Examining $\Delta$ we see that since $\psi_f$ is consistent it must take the form

$$\psi_0(\pi) \land \left( \bigwedge_{i=1}^{n} x_i \equiv K_i \pmod{\delta} \right) \land (x_{\sigma(1)} < \cdots < x_{\sigma(n)})$$

for some $K_i < \delta$ and some $\sigma \in S_n$. (Note that formulas of the form $S^e x_i < x_j$ for $e < M$ are implied by $x_i < x_j$ and $\psi_0(\pi)$, so we may safely omit them.) I claim that $\chi$ is our desired formula. It is clear that $\models \psi_f \rightarrow \chi$; it remains to show that $\models \chi \rightarrow \varphi$. Fix $1 < j \leq n$, and suppose for contradiction we had $\models \neg \varphi(\pi)$ for some realization $\pi$ of $\chi \land (x_{(j-1)j}; a_{1} < \cdots < x_{(j-1)j}\sigma(n))$; note this last formula takes the form $\psi_{f'}$ for some $f': \Delta \rightarrow \{0, 1\}$. So by definition of $\Delta$ we get that $\models \varphi(\pi)$ for all realizations $\pi$ of $\psi_f$.

I claim this implies that $\varphi(x_1, \ldots, x_{j-1}; x_j, \ldots, x_n)$ has the order property. Indeed, fix $N < \omega$; we construct a ladder of length $N$. For clarity we assume $\sigma = \text{id}$; the argument generalizes with little effort.

Pick $a_1 \geq M$ such that $a_1 \equiv K_1 \pmod{\delta}$, and inductively pick $a_{i+1} \geq a_i + M$ for $1 < i + 1 < j - 1$ such that $a_{i+1} \equiv K_{i+1} \pmod{\delta}$. Now pick $a_{j-1,0} \geq a_j - 2 + M$ with $a_{j-1,0} \equiv K_{j-1} \pmod{\delta}$, and inductively choose $a_{j-1,k}$ and $a_{j,k}$ for $k < N$ to satisfy:

- $a_{j,k} \geq a_{j-1,k} + M$
- $a_{j,k} \equiv K_j \pmod{\delta}$
- $a_{j-1,k+1} \geq a_{j,k} + M$
- $a_{j-1,k+1} \equiv K_{j-1} \pmod{\delta}$

Now pick $a_{j,N-1} + M$ with $a_{j+1} \equiv K_{j+1} \pmod{\delta}$, and proceed inductively to pick $a_{i+1} \geq a_i + M$ with $a_{i+1} \equiv K_{i+1} \pmod{\delta}$ for $j + 1 < i \leq n$. Pictorially:

```
   a_1 → ... → a_j-2 → a_{j-1,0} → a_{j,0} → a_{j+1} → ... → a_n
   \          \          \                \         \      \        \      \     \     \   
   a_{j-1,1} → a_{j,1}   \ \     \          \          \    \          \       \      \      \    
   \        \        \    \        \          \          \      \      \    \      \      \ 
   a_{j-1,N-1} → a_{j,N-1}
```

where an arrow in the diagram indicates that the target is at least the source plus $M$.

For convenience we let $b_k = (a_1, \ldots, a_{j-2}, a_{j-1,k})$ and $c_k = (a_{j,k}, a_{j+1}, \ldots, a_n)$. Note now that for any $k, \ell < N$ we have $\models \chi(b_k, c_\ell)$. Furthermore if $k \leq \ell$ then $(b_k, c_\ell)$ satisfies $x_1 < \cdots < x_n$, so $\models \psi_f(b_k, c_\ell)$ and thus $\models \varphi(b_k, c_\ell)$. Finally if $k > \ell$ then $(b_k, c_\ell)$ satisfies $x_{(j-1)j}; a_{1} < \cdots < x_{(j-1)j}\sigma(n)$, so $\models \psi_f(b_k, c_\ell)$, and thus $\models \neg \varphi(b_k, c_\ell)$.

Thus $\models \varphi(b_k, c_\ell)$ if and only if $k \leq \ell$, and we have constructed a ladder of size $N$ for $\varphi$. So $\varphi$ has the order property and is thus unstable with respect to this partition of the variables, a contradiction. So no such $\pi$ exists, and we can compose $\sigma$ with a transposition of adjacent elements and remain in $\varphi$. But such transpositions generate all of $S_n$; so we may omit the ordering altogether and remain in $\varphi$, and thus $\models \chi \rightarrow \varphi$, as desired.

So we may replace $\psi_f$ in the disjunction with a weaker $L_\delta$-formula. Doing this to all disjuncts, we have written $\varphi$ as an $L_\delta$-formula. □ Proposition 3.3

The remainder of our proof will make use of the $F$-sets of [14]; we briefly recall them here in the context of $\mathbb{Z}^m$ where $F: \mathbb{Z}^m \rightarrow \mathbb{Z}^m$ is multiplication by $d$. 
**Definition 3.4.** A *groupless* $F$-set in $\mathbb{Z}^m$ is a finite union of translates of sums of sets of the form $C(a; \delta)$ as defined above for $a \in \mathbb{Z}^m$ and $\delta > 0$. An $F$-set in $\mathbb{Z}^m$ is a finite union of

$$b + \sum_{i<n} C(a_i; \delta_i) + H$$

for some $b, a_i \in \mathbb{Z}^m$, $\delta_i > 0$, and $H \leq \mathbb{Z}^m$. The $F$-structure on $\mathbb{Z}$, denoted $(\mathbb{Z}, F)$, is the structure with domain $\mathbb{Z}$ and a predicate for every $F$-set in every $\mathbb{Z}^m$. (Note in particular that the graph of addition is a subgroup of $\mathbb{Z}^3$, and hence an $F$-set; so $(\mathbb{Z}, F)$ expands $(\mathbb{Z}, +)$.)

We now describe a simplification of $d$-sparsity that we will use to connect stable sparse sets to $F$-sets.

**Lemma 3.5.** Any $d$-sparse subset of $\mathbb{Z}^m$ can be written as a finite union of translates of sets of the form

$$\{ [\sigma_1^{e_1}] + \cdots + [\sigma_n^{e_n}] : e_1 \leq \cdots \leq e_n \}$$

where $\sigma_i \in (\mathbb{Z}^m)^*$ all have the same length.

To see how this relates to $F$-sets, recall from Remark 3.2 that $|\sigma| = C(\sigma; |\sigma|) \cup \{ 0 \}$; so a set of the above form is (ignoring for the moment the case where some $e_i = 0$) a subset of the groupless $F$-set $C(\sigma_1; |\sigma_1|) + \cdots + C(\sigma_n; |\sigma_n|)$ that is cut out by some kind of order relation.

**Proof.** By Fact 2.8 we can write $A$ as a union of sets of the form $[u_0v_1^* \cdots v_n^* u_n]$ for $u_i, v_i \in (\Sigma^m)^*$. Note first that we may assume all $v_i$ across the union have the same length $N$. Indeed, let $N$ be the least common multiple of the lengths of all the $v_i$ across the union. We can then rewrite any $v_i^*$ as

$$\bigcup_{j<\ell} v_i^j(v_i^j)^*$$

where $\ell = \frac{N}{|v_i|}$ (so $|v_i^j| = N$). Using this to replace each $v_i^*$ in the union and then distributing yields the desired expression for $A$.

It then suffices to show that given $A = [u_0v_1^* \cdots v_n^* u_n]$ with each $|v_i| = N$ we can write $A$ as a translate of a set of the form $\{ [\sigma_1^{e_1}] + \cdots + [\sigma_n^{e_n}] : e_1 \leq \cdots \leq e_n \}$ with each $|\sigma_i| = N$.

**Claim 3.6.** We can write such $[u_0v_1^* \cdots v_n^* u_n]$ in the form $[a\tau_1^* \cdots \tau_n^*]$ where $a \in (\mathbb{Z}^m)^*$ and each $\tau_i \in (\mathbb{Z}^m)^*$ has length $N$.

**Proof.** We apply induction on $n$; the base case is trivial. For the induction step, use the induction hypothesis to write $[u_1v_2^* u_2 \cdots u_{n-1}v_n^* u_n]$ as $[b\tau_2^* \cdots \tau_n^*]$. Let $x = [v_1] + d^{v_1^*}[b]$. Then the following is a telescoping sum:

$$[u_0] + d^{u_0}[b] + d^{u_0|}x + \cdots + d^{u_0|+|v_1|}x = [u_0] + d^{u_0}[v_1] + \cdots + d^{u_0|+|v_1|}[v_1] + d^{u_0|+k|v_1|[b]}$$

(The above equation is taken from a draft of [4]; it was removed from the final paper.) Then if we let

$$a = y \cdot 0^{u_0} + |v_1| - 1$$

$$\tau_1 = x \cdot 0^{v_1} - 1$$

(i.e. strings in $(\mathbb{Z}^m)^*$ whose first entries are $y$ and $x$ and whose later entries are the zero tuple) then

$$[u_0v_1^kb] = \begin{cases} [a\tau_1^{k-1}] & \text{if } k > 0 \\ [u_0b] & \text{else} \end{cases}$$

Hence if $k > 0$ then

$$[u_0v_1^kbw] = [u_0v_1^kb] + d^{u_0v_1^*b}[w] = [a\tau_1^{k-1}] + d^{a\tau_1^{k-1}|}[T_b][w] = [a\tau_1^{k-1}[T_b][w]]$$
where $T_i\sigma$ is the word obtained by replacing each letter $\ell \in \mathbb{Z}^m$ appearing in $\sigma$ with $d^i\ell$. Hence

$$
[u_0v_1^* \cdots v_n^* u_n] = \{[u_0bw] : w \in a_2^* \cdots a_n^* \} \cup \{[u_0b_1^{k-1}w] : k \geq 1, w \in \tau_2^* \cdots \tau_n^* \}
$$

$$
= [u_0b_1^* \tau_2^* \cdots \tau_n^*] \cup \{[a_1^{k-1}(T_{b_1}w)] : k \geq 1, w \in \tau_2^* \cdots \tau_n^* \}
$$

$$
= [u_0b_1^* \tau_2^* \cdots \tau_n^*] \cup [a_1^*(T_{b_1}\tau_2)^* \cdots (T_{b_1}\tau_n)^*]
$$

And $|\tau_i| = |v_i| = |T_{b_1}\tau_i| = N$ for all $i$, as desired. \qed

Claim 3.6

Note that given a set of the form $[\sigma_1^* \cdots \sigma_n^*]$ with each $|\tau_i| = N$ we can rewrite it as $[\sigma_1^*+(T_{\sigma_1}\tau_1)^* \cdots (T_{\sigma_n}\tau_n)^*]$. It then suffices to show that a set of the form $[\tau_1^* \cdots \tau_n^*]$ where each $\tau_i \in (\mathbb{Z}^m)^*$ has length $N$ can be written in the form

$$
\{[\sigma_1^*] + \cdots + [\sigma_n^*] : e_1 \leq \cdots \leq e_n \}
$$

with each $\sigma_i \in (\mathbb{Z}^m)^*$ of length $N$. For $1 \leq i \leq n$ let $\sigma_i \in (\mathbb{Z}^m)^*$ be any string of length $N$ such that $[\sigma_i] = [\tau_i] - \sum_{j=i+1}^n [\sigma_j]$. Then if $e_1 \leq \cdots \leq e_n$ then

$$
[\sigma_1^*] + \cdots + [\sigma_n^*] = [\sigma_1^*] + [\sigma_2^*] + d^{N_1}[\sigma_2^{2-e_1}] + \\
\vdots \\
[\sigma_n^*] + d^{N_{n-1}}[\sigma_n^{e_n-e_{n-1}}] + \cdots + d^{N-1}[\sigma_{n-1}^{e_n-e_{n-1}}] = [\tau_1^*] + d^{N_1}[\tau_2^{2-e_1}] + \cdots + d^{N_{n-1}}[\tau_{n-1}^{e_n-e_{n-1}}] = [\tau_1^* \tau_2^{2-e_1} \cdots \tau_{n-1}^{e_n-e_{n-1}}]
$$

So $[\tau_1^* \cdots \tau_n^*] = \{[\sigma_1^*] + \cdots + [\sigma_n^*] : e_1 \leq \cdots \leq e_n \}$, as desired. \qed

Lemma 3.5

The promised connection between stable sparse sets and $F$-sets:

Lemma 3.7. Suppose $A \subseteq \mathbb{Z}^m$ is $d$-sparse and stable in $(\mathbb{Z}^m, +)$. Then $A$ is definable in $(\mathbb{Z}, F)$.  

Proof. By Lemma 3.5 we can write $A$ as a finite union of sets of the form

$$
\alpha + \{[\sigma_1^*] + \cdots + [\sigma_n^*] : e_1 \leq \cdots \leq e_n \}
$$

with $\alpha \in \mathbb{Z}^m$ and each $\sigma_i \in (\mathbb{Z}^m)^*$ has the same length $N$. Take one such component of the union; we will show that it is contained in a set definable in $(\mathbb{Z}, F)$ that is itself contained in $A$, and hence can be replaced without changing the union.

Since $A$ is stable in $(\mathbb{Z}^m, +)$ and addition is associative we get that $x_0 + x_1 + \cdots + x_n \in A$ is stable under any partition of the variables; thus so too is

$$
(x_1 + \cdots + x_n \in A - \alpha) \land \bigwedge_{i=1}^n (x_i \in [\sigma_i^*])
$$

Thus $X := \{(e_1, \ldots, e_n) \in \mathbb{N}^n : [\sigma_1^*] + \cdots + [\sigma_n^*] \in A - \alpha \}$ is a stable relation on $\mathbb{N}^n$ under any partition of the variables. Furthermore if $f \in S_n$ and we let $\tau_i \in (\mathbb{Z}^m)^*$ be of length $N$ such that $[\tau_i] = \sum_{j=i}^n [\sigma_{f(j)}]$, then as in the proof of Lemma 3.5 we get for $e_{f(1)} \leq \cdots \leq e_{f(n)}$ we have

$$
(e_1, \ldots, e_n) \in X \iff [\tau_1^{e_{f(1)}} \tau_2^{e_{f(2)}-e_{f(1)}} \cdots \tau_{n-1}^{e_{f(n)-e_{f(n-1)}}} \in A - \alpha
$$

Let $A \supseteq \Sigma^n_{\mathbb{Z}}$ be any alphabet containing all the entries of the $\tau_i$; so by Remark 2.7 we get that $\{ \mu \in A^* : [\mu] \in A - \alpha \}$ is regular. So, by the proof of Proposition 2.2 we get that $[\tau_1^{e_{f(1)}} \tau_2^{e_{f(2)}-e_{f(1)}} \cdots \tau_{n-1}^{e_{f(n)-e_{f(n-1)}}}] \in A - \alpha$ can be expressed by a Boolean combination of congruences and equalities between a $t_i$ and a constant. But a congruence or equality between $e_{f(i)+1} - e_{f(i)}$ and a constant $k$ can be expressed as a congruence or equality between $e_{f(i)+1}$ and $k$ (where equality is $S^k(e_{f(i)})$, and is thus expressible by an $L_0$-formula for some $\delta$; furthermore by taking disjunctions and LCMs we may assume all congruences that occur have the same modulus $\delta$. So

$$
[\tau_1^{e_{f(1)}} \tau_2^{e_{f(2)}-e_{f(1)}} \cdots \tau_{n-1}^{e_{f(n)-e_{f(n-1)}}}] \in A - \alpha
$$

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Lemma 3.7

Let $\mathbb{Z}^m$ be the tuple all of whose entries are 1. I claim that the map $\mathbb{N} \to \mathbb{Z}^m$ given by $e \mapsto [1^{Ne}]$ defines an interpretation of $(\mathbb{N},0,S,P_3)$ in $(Z,F)$. Indeed, the image is definable: it’s simply $C(\mathbb{N})\cup\{0\}$. Furthermore the unnested atomic $L_\delta$-formulas all map to definable sets in $(Z,F)$:

- $P_\delta(x)$ can be expressed as a condition of $[\mathbb{N}^x]$ by demanding that it lie in $C(\mathbb{N})\cup\{0\}$.
- $y = Sx$ can be expressed by requiring that $\left[\frac{\mathbb{N}^x}{\mathbb{N}^y}\right] \in C\left(\left[\frac{\mathbb{N}}{d\mathbb{N}}\right] ; N\cup\{0\}\right) + \left[\frac{0}{\mathbb{N}}\right]$.

Furthermore the map $[\mathbb{N}^e] \mapsto [\sigma_i^e]$ is definable in $(Z,F)$ for each $i$: its graph is simply $C\left(\left[\frac{\mathbb{N}}{[\sigma_i]}\right] ; N\cup\{0\}\right)$. (Recall that $|\sigma_i| = N$.) Then since $X$ is definable in $(\mathbb{N},0,S,P_3)$ we get that

$$Y := \left\{ \sum_{i=1}^n [\sigma_i^e] : (e_1,\ldots,e_n) \in X \right\} = (A - \alpha) \cap ([\sigma_1^e] + \cdots + [\sigma_n^e])$$

is definable in $(Z,F)$. (Recall that addition is definable in $(Z,F)$.) But $[\sigma_1^e] + \cdots + [\sigma_n^e] \in A - \alpha$ if $e_1 \leq \cdots \leq e_n$; so $\alpha + \{ [\sigma_1^{e_1}] + \cdots + [\sigma_n^{e_n}] : e_1 \leq \cdots \leq e_n \} \subseteq A + Y \subseteq A$.

So we can replace $\alpha + \{ [\sigma_1^{e_1}] + \cdots + [\sigma_n^{e_n}] : e_1 \leq \cdots \leq e_n \}$ in the union defining $A$ with $\alpha + Y$. Doing this for all such terms in the union, we can write $A$ as a union of sets definable in $(Z,F)$; so $A$ is definable in $(Z,F)$.

Our theorem now follows:

**Proof of Theorem 3.1.** By previous lemma $A$ is definable in $(Z,F)$. But by [14, Theorem A] $(Z,F)$ admits quantifier elimination. So $A$ is definable by a Boolean combination of $F$-sets, say in disjunctive normal form; we must show the $F$-sets can be taken to be groupless. Take one disjunct

$$\bigcap_{i<k} B_i \setminus \bigcup_{j<\ell} C_j$$

where the $B_i,C_j$ are $F$-sets. By Lemma 3.5 $A$ is contained in a finite union of translates of finite sums of $C(\sigma_i;\delta_i)$, and hence in a groupless $F$-set $\hat{A}$. So if $k > 0$ we may replace every $B_i$ and $C_j$ in our disjunct with $B_i \cap \hat{A}$ and $C_j \cap \hat{A}$, respectively, and the result of the disjunction will still be $A$. If $k = 0$ we instead replace our disjunct with $\hat{A} \setminus \bigcup_{j<\ell} (C_j \cap \hat{A})$, and again the result of the disjunction is still $A$. But $B_i \cap \hat{A}, C_j \cap \hat{A}$ are intersections of $F$-sets, and hence themselves $F$-sets by [14, Proposition 3.9]. Furthermore $\hat{A}$ is $d$-sparse by Remark 3.2; so $B_i \cap \hat{A}, C_j \cap \hat{A}$ cannot contain a translate of a subgroup, and hence are groupless $F$-sets. Applying the above replacement to every disjunct, we get that $A$ is a Boolean combination of groupless $F$-sets, i.e. translates of sums of $C(\alpha;\delta)$, as desired. \[ \blacksquare \] Theorem 3.1

We conclude by pointing out that combined with [14] we obtain the following characterization of the stable $d$-sparse sets:

**Corollary 3.8.** Suppose $A \subseteq \mathbb{Z}^m$ is $d$-sparse. The following are equivalent:

1. $\text{Th}(Z,+,A)$ is stable.
2. $A$ is stable in $(\mathbb{Z}^m,+)$.
3. $A$ is a finite Boolean combination of translates of sums of sets of the form $C(\alpha;\delta)$.
4. $A$ is definable in $(Z,F)$.

**Proof.** (1) $\implies$ (2) is clear, (2) $\implies$ (3) is Theorem 3.1, (3) $\implies$ (4) is clear, and (4) $\implies$ (1) is by the fact (Theorem A of [14]) that $\text{Th}(Z,F)$ is stable. \[ \blacksquare \] Corollary 3.8
4 Beyond sparsity: the non-generic case

In the previous section we characterized the $d$-sparse sets that are stable in $(\mathbb{Z}^m, +)$. So the question of which automatic sets are stable in $(\mathbb{Z}^m, +)$ reduces to the non-sparse case, which remains open. We begin to study this problem in this section, restricting our attention to subsets of $\mathbb{Z}$.

As an example of a non-sparse automatic set that is stable in $(\mathbb{Z}, +)$, consider a coset of a subgroup, say $A = r + s\mathbb{Z}$ where $s > 0$. Then $A$ is stable in $(\mathbb{Z}, +)$ since it's definable in $(\mathbb{Z}, +)$. It isn't $d$-sparse: the number of $a \in A$ with $d^{-k} a < d^k$ grows exponentially with $k$, so the set of canonical representations of $A$ isn't sparse. It is $d$-automatic: see [2, Theorem 5.4.2] (though recall as mentioned in Remark 2.7 that they use a different convention for representing integers, so the automaton will be slightly different).

One can also take Boolean combinations of cosets and the stable sparse sets of the previous section to get further examples, as long as the result isn't $d$-sparse. But all examples produced in this way will be "generic":

**Definition 4.1.** We say $A \subseteq \mathbb{Z}$ is generic if some finite union of additive translates of $A$ covers $\mathbb{Z}$.

We show that in the non-sparse setting all stable automatic sets are generic.

**Theorem 4.2.** Suppose $A \subseteq \mathbb{Z}$ is $d$-automatic and not $d$-sparse. If $A$ is stable in $(\mathbb{Z}, +)$ then $A$ is generic.

It will be easier to work first in $\mathbb{N}$, and in particular to use $\Sigma = \{0, \ldots, d - 1\}$ for our representations rather than $\Sigma_{\pm} = \{-d + 1, \ldots, d - 1\}$; the main advantage to doing so is that whenever $\sigma, \tau \in \Sigma^*$ have the same length we have $\sigma = \tau$ if and only if there are arbitrarily large gaps in $(\mathbb{Z}, +)$ since it's definable in $(\mathbb{Z}, +)$. Recall from Remark 2.7 that $A \subseteq \mathbb{N}$ is a $d$-automatic subset of $\mathbb{Z}$ if and only if it is a $d$-automatic subset of $\mathbb{N}$ in the classical sense; i.e. $\{\sigma \in \Sigma^* : [\sigma] \in A\}$ is regular. Note also that if $A \subseteq \mathbb{N}$ then the canonical representations of the elements of $A$ all lie in $\Sigma^*$, and up to trailing zeroes these are the only representations over $\Sigma$ of elements of $A$. So $A \subseteq \mathbb{N}$ is $d$-sparse as a subset of $\mathbb{Z}$ if and only if $\{\sigma \in \Sigma^* : [\sigma] \in A, \sigma$ has no trailing zeroes $\}$ is sparse. On the other hand stability and genericity when relativized to $\mathbb{N}$ give something new:

**Definition 4.3.** We say $A \subseteq \mathbb{N}$ is stable in $\mathbb{N}$ if $x + y \in A$ is a stable relation on $\mathbb{N}$. We say $A$ is generic in $\mathbb{N}$ if some finite union of (possibly negative) translates of $A$ covers $\mathbb{N}$.

We will first focus on proving:

**Proposition 4.4.** Suppose $A \subseteq \mathbb{N}$ is $d$-automatic and not $d$-sparse. If $A$ is stable in $\mathbb{N}$ then $A$ is generic in $\mathbb{N}$.

We begin with a characterization of the generic $d$-automatic sets.

**Lemma 4.5.** Suppose $A \subseteq \mathbb{N}$ is $d$-automatic; let $L \subseteq \Sigma^*$ be the set of representations of elements of $A$. Then the following are equivalent:

1. $A$ is generic in $\mathbb{N}$.

2. For any $r, s \in \mathbb{N}$, every $\tau \in \Sigma^*$ occurs as a suffix of a word in $L$ of length $r + sk$ for some $k \geq 0$.

In other words, $A$ is not generic in $\mathbb{N}$ if and only if there are $r, s \in \mathbb{N}$ such that $L \cap \Sigma^{(r + sk\mathbb{N})}$ has a forbidden suffix.

**Proof.** Note first that $A$ is not generic in $\mathbb{N}$ if and only if there are arbitrarily large gaps in $A$ (i.e. runs of naturals not in $A$).

(1) $\implies$ (2) Suppose we are given $\tau, r, s$ such that $\tau$ is a forbidden suffix for $L \cap \Sigma^{(r + sk\mathbb{N})}$. Then $r + sk > |\tau|$ then $A$ is disjoint from $[\Sigma^{(r + sk - |\tau|)\tau}] = \{b \in \mathbb{N} : d^{r+sk-|\tau|}[\tau] \leq b < d^{r+sk-|\tau|}([\tau] + 1)\}$

So $A$ has a gap of size $d^{r+sk-|\tau|}$. So as $k \to \infty$ we get arbitrarily large gaps in $A$; so $A$ isn't generic in $\mathbb{N}$. 

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Suppose Lemma 4.5

Proposition 4.4

Case 1. \( \tau \) of the same length as \( |uv| \) for \( \tau \in \Sigma^* \) with the property that \( \Sigma^{|(m)} \tau \cap L = \emptyset \); in other words, if we replace each zero with any letter and delete the separator, the result is never in \( L \). So \( 0^m \$ \tau \in S \) if and only if \( \tau \) is a forbidden suffix for \( L \cap \Sigma^{|(m+|\tau|)} \). Then \( S \) is regular: it’s not too hard to construct a non-deterministic finite automaton (NFA) for the complement, which suffices (see e.g. [19, Section 2.2]). Since there are arbitrarily large gaps in \( A \) we get that there are elements \( 0^m \$ \tau \in S \) with \( m \) arbitrarily large. Indeed, suppose we are given \( m \). Find a gap of size \( 2d^m \); then this gap will contain two multiples of \( d^m \), say \( a, a + d^m \). Then if \( \tau \in \Sigma^* \) is such that \( |\tau| = \frac{d^m}{2} \) then \( \tau \) is a forbidden suffix for \( L \cap \Sigma^{|(m+|\tau|)} \); so \( 0^m \$ \tau \in S \).

Recall the pumping lemma for regular languages (see [19, Lemma 4.1]): if \( R \) is regular then there is a pumping length \( p > 0 \) such that if \( \mu \in R \) has length \( \geq p \) then we can write \( \mu = uvw \) such that

- \( v \neq \varepsilon \)
- \( |uv| \leq p \)
- \( uv^n w \subseteq R \).

Pick \( 0^m \$ \tau \in S \) with \( m \) bigger than the pumping length of \( S \). Then by the pumping lemma we can write \( m = r + s \) so that \( 0^r (0^s)^r \$ \tau \subseteq S \); so \( \tau \) is a forbidden suffix for \( L \cap \Sigma^{|(r+|\tau|)} \). \( \square \) Lemma 4.5

The following technical lemma is the source of instability in Proposition 4.4. For \( K < \omega \) we define a partial binary operation \( +_K \) on \( \Sigma^* \) by setting \( \sigma +_K \tau \) to be the unique representation of \( |\sigma| + |\tau| \) of length \( K \), if one exists.

Lemma 4.6. Suppose \( L \subseteq \Sigma^* \) is regular but not sparse, and satisfies \( L = L^* \) and

\[
\text{there are } r, s \in \mathbb{N} \text{ such that } L \cap \Sigma^{(r+s|n|)} \text{ is infinite and has a forbidden suffix } \sigma.
\]

Then for all \( N < \omega \) there is \( K < \omega \) such that the binary relation \( x +_K y \in L \) on \( \Sigma^* \) has an \( N \)-ladder.

Proof. Pick \( \sigma, r, s \) as in \((\dagger)\). Since \( L \cap \Sigma^{(r+s|n|)} \) is infinite, there is \( a \in \Sigma^{(|\sigma|)} \) that occurs as a suffix of some element of \( L \cap \Sigma^{(r+s|n|)} \). Suppose \( |a| \leq |\sigma| \); we will see at the end how to modify the argument in the case \( |a| > |\sigma| \).

Pick such a maximal under \( \leq_N \) (the preorder induced from the ordering on \( \mathbb{N} \)); so if \( a' \in \Sigma^{(|\sigma|)} \) has \( |a'| = |a| + 1 \) then \( a' \) does not occur as a suffix of some element of \( L \cap \Sigma^{(r+s|n|)} \). (Note \( \text{such } a' \text{ exists since } |a| < |\sigma| < d^{s|\sigma|} \), and hence \( |a| + 1 < d^{s|\sigma|} \) can be represented by a string of length \( |\sigma| \).) Consider the set \( S \) of \( \tau \in L \cap \Sigma^{(r+s|n|)} \) with \( a \) as a suffix such that \( \tau \) is \( \leq_{s|\sigma|} \text{-maximal among the elements of } L \text{ ending in } a \) that are of the same length as \( \tau \). Then \( S \) is infinite: since \( L = L^* \) and \( a \) occurs as a suffix of some \( \mu \in L \cap \Sigma^{(r+s|n|)} \), we get that \( \mu_{1+s|\sigma|} \subseteq L \cap \Sigma^{(r+s|n|)} \) also has \( a \) as a suffix, and hence that \( S \) contains a word of length \((1+sk)|\mu|\) for \( k < \omega \). Furthermore \( S \) is regular: using the fact that \( \{ (\mu, \nu) \in (\Sigma^*)^2 : |\mu| \leq |\nu| \} \) and \( \Sigma^2 \)-a are regular, one can construct an NFA for the complement of \( S \). So by the pumping lemma \( S \) contains a set of the form \( uv^s w \) with \( v \neq \varepsilon \). By prepending a power of \( v \) to \( w \) we may assume \( |w| \geq |a| \), and in particular that \( w \) has \( a \) as a suffix (and is non-empty).

Since \( L = L^* \) and \( uv^s w \subseteq S \subseteq L \) we get that \( L \supseteq (uv^s w)^* \supseteq u \{ wu, v \}^* w \). This, together with the maximality of elements of \( S \), the fact that \( a' \) is a forbidden suffix for \( L \cap \Sigma^{(r+s|n|)} \), and the fact that \( |uv^s w| \in r + s|N| \), will be enough to construct our ladder.

Pick \( n, m \) such that \( n|wu| = m|v| \); then \( u(wu)^n w \in L \) and ends in \( a \), so since \( uv^m w \in S \) and \( |uv^m w| = |uw| v^n w \) we get that \( |uw| v^n w \subseteq [uv^m w] \), and hence that \( [(uw)^n] \leq [v^n] \).

Case 1. Suppose \( [(uw)^n] < [v^n] \); then since \( [v^n] - [(uw)^n] \leq [v^n] \) there is \( \alpha \in \Sigma^{|(m)||v|} \) such that \( |\alpha| = [v^n] - [(uw)^n] > 0 \). We let

\[
K = |u| + N|m|v| + |v| \in r + s|N|
\]
\[
d_i = u(wu)^{n(N-i)} v^{mi} w
\]
\[
e_i = 0^{|u| \alpha^{N-i}}
\]
for \( i \leq N \). Then \( d_i + K e_j \) is defined for all \( i, j \); i.e., \([d_i] + [e_j]\) has a representation of length \( K \). Indeed, 
\( |e_j| \leq K - |w| \leq K - |a| \); so \([e_j] < d_K - |a|\). So if we write \( d_i = \tau a\) for some \( \tau \) (possible since \( d_i \) has \( w \), and hence \( a \), as a suffix) then 
\[
[d_i] + [e_j] < [\tau a] + d_K - |a| = [\tau a] + d^{|\tau|} = [\tau a'] < d^K
\]
since \( |\tau a'| = |d_i| = K \). So \([d_i] + [e_j]\) has a representation of length \( K \), and \( d_i + K e_j \) is defined. In fact the above proof shows that \( d_i + K e_j \) has either \( a \) or \( a' \) as a suffix.

Since \( [\alpha] > 0 \) it is clear that the \( e_j \) are strictly decreasing. Suppose \( i > j \); then \([d_i + K e_j] = [d_i] + [e_j] > |d_i| + |e_j| = uw^m N w\). So if \( d_i + K e_j \) has \( a \) as a suffix then since \( uw^m N w \in S \) and \( d_i + K e_j \) has the same length, has \( a \) as a suffix, and represents a strictly larger number, we get that \( d_i + K e_j \notin L \).

Otherwise as noted above we get that \( d_i + K e_j \) has \( a' \) as a suffix, in which case \( d_i + K e_j \notin L \) since \( a' \) is a forbidden suffix for \( L \cap \Sigma^{(r+sN)} \) and \( |d_i + K e_j| = K \in r + sN \). Conversely suppose \( i \leq j \); then \( d_i + K e_j = uw^m (N - j)(uw)^{j - i} v^m w \in u\{uw, v\}^* w \subseteq L \). So the \( d_i, e_i \) form an \( \Lambda \)-ladder for \( x + K y \in L \).

**Case 2.** Suppose \([wu]^n = [v]^m\); so \((wu)^n = v^m\). Then \( uw^* w \supseteq u((wu)^n)^* w = ((uw)^n)^* uw \); so if we let \( u' = \varepsilon \), \( v' = (uw)^n \), and \( w' = uw^* w \subseteq uw^* w \subseteq S \). Furthermore \( v' \neq \varepsilon \) since \( |v'| = n|m| = m|v| > 0 \), and \( v' \in L \) since \( L = L^* \) and \( uw \in uw^* w \subseteq L \). So we may replace \( u, v, w \) with \( u', v', w \) respectively, and we thus assume that \( u = \varepsilon \) and \( v, w \in L \). (Recall that the only requirement we had of \( u, v, w \) was that \( uw^* w \subseteq S \) and \( v \neq \varepsilon \).

By [4, Proposition 7.1] since \( L \) isn’t sparse there are \( x, y_1, y_2, z \in \Sigma^* \) with \( y_1, y_2 \) distinct, non-trivial, and of the same length such that \( x \{y_1, y_2\}^* z \subseteq L \). Let \( b = xy_1 z \) and \( c = xy_2 z \); so \([b] = |c| \) with \( b, c \in L \) and \( b \neq c \). By replacing \( b, c, v \) with powers thereof we may assume \([b] = |c| = |v| \). Then since \( b \neq c \) we get that one of \( b, c \), without loss of generality say \( b \), has \( b \neq v \), and thus \([b] \neq [v] \). Note since \( L = L^* \) that \( L \supseteq \{b, v\}^* w \).

Since \( vw \in S \) and since \( bw \) has the same length as \( vw \), has \( a \) as a suffix, and lies in \( L \), we get that \([bw] \leq [vw] \). So \([b] \leq [v] \), and since \( b \neq v \) we get \([b] < [v] \). Then since \([v] - [b] < [v] \) there is \( \alpha \in \Sigma^{(|v|)} \) such that \([\alpha] = [v] - [b] \). We then let
\[
K = N|v| + |w| \in r + sN \\
d_i = b^{N - i} v^i w \\
e_i = \alpha^{N - i}
\]
for \( i \leq N \). Then by an argument identical to the previous case the \( d_i, e_i \) form an \( \Lambda \)-ladder for \( x + K y \in L \).

The case \([\alpha] > [\tau] \) is similar; we outline it here. We take minimal such \( a \) under \( \leq N \), and define \( S \) to be the set of \( \tau \in L \cap \Sigma^{(r+sN)} \) ending in \( a \) that are \( \leq N \)-minimal among the elements of \( L \) ending in \( a \) that are of the same length as \( \tau \). Then \( S \) is again infinite and regular, and thus contains a set of the form \( uw^* w \); we again assume \( w \) has \( a \) as a suffix. If \( n|m| = m|v| \) then dually to before we get \([uw]^m \geq [v]^n \). If \([uw]^m > [v]^m \), say with \( \alpha \in \Sigma^{(m|v|)} \) with \([\alpha] = [((uw)^n - [v]^m] > 0 \), then we’d like to let
\[
K = |u| + N|m| |v| + |w| \\
d_i = u(wu)^n m^{-1} v^i w \\
e_i = 0^{|v|(-\alpha)^{N-i}}
\]
and claim this as our ladder. Unfortunately we’re working over \( \Sigma \), not \( \Sigma \), so we can’t allow the \( e_i \) to use negative digits. This is easily fixed, however: note for all \( i, j \) that \([d_i] \geq d^{|u| + N|w|m|v|} - |e_j| \) (since \([\alpha] \neq 0 \) and \( w \), and hence \( d_i \), has \( a \) as a suffix). So we can take \( d_i', e_i' \in \Sigma^* \) such that \([d_i'] = [d_i] - d^{|u| + N|w|m|v|} \) and \([e_i'] = e_i + d^{|u| + N|w|m|v|} \). Then \([d_i'] + [e_i'] = [d_i] + [e_j] \), and now as before one can show that \( d_i' + K e_j' \) is always defined and is \( L \) if and only if \( i < j \).

If \([uw]^n = [v]^m \) we do a similar trick. As before we may assume \( u = \varepsilon \) and \( v, w \in L \), and get some \( b \in L \) with \([b] = |v| \) and \([b] \neq [v] \); dually to before we get \([b] > [v] \), say with \( \alpha \in \Sigma^{(|v|)} \) such that \([\alpha] = [b] - [v] \).
Our initial attempt at a ladder will now be:

\[ K = N |v| + |w| \]
\[ d_i = b^{N-i} \alpha^i w \]
\[ e_i = (-\alpha)^{N-i} \]

Now we have \(d_i \geq d^N |v| \geq -|e_j|\); so we can pull the same trick to turn the \(d_i, e_i\) into a ladder.

**Lemma 4.6**

Suppose \(M = (Q, q_0, F, \delta)\) is a DFA over \(\Sigma\). For \(q \in Q\) we let \(L_q = \{ \sigma \in \Sigma^* : \delta(q, \sigma) = q \}\); that is, \(L_q\) is the set of words which take state \(q\) back to state \(q\) in \(M\). Note that \(L_q\) is regular: it is recognized by the automaton \((Q, q, \{ q \}, \delta)\).

**Lemma 4.7.** Suppose \(A \subseteq \mathbb{N}\) is \(d\)-automatic but not \(d\)-sparse; suppose \(A\) is not generic in \(\mathbb{N}\). Fix an automaton \(M = (Q, q_0, F, \delta)\) that recognizes the set of representations over \(\Sigma\) of elements of \(A\). Then there is a non-dead \(q \in Q\) such that \(L_q\) satisfies the hypotheses of Lemma 4.6: namely \(L_q\) is regular but not sparse, \(L_q = L_q^*\), and \(L_q\) satisfies (†).

(We say \(q \in Q\) is a dead state if there is no \(\sigma\) such that \(\delta(q, \sigma) \in F\).

**Proof.** Note we always have that \(L_q\) is regular and \(L_q = L_q^*\); so we only need non-sparsity and (†). We first note some facts about how non-sparsity and (†) interact with the \(L_q\).

**Claim 4.8.**

1. If \(q\) is a finish state of \(M\) and \(L_q\) is infinite then \(L_q\) satisfies (†).

2. There is a non-dead \(q\) such that \(L_q\) isn't sparse.

3. If \(q, q'\) are states in \(M\) with a path from \(q\) to \(q'\) and vice-versa then \(L_q\) is sparse if and only if \(L_{q'}\) is.

**Proof.**

1. Let \(L\) be the set of representations of elements of \(A\), and fix \(\mu \in \Sigma^*\) such that \(\delta(q_0, \mu) = q\). (We may assume such \(\mu\) exists: otherwise we can remove \(q\) from \(M\) without changing the set recognized by \(M\).) By non-genericity of \(A\) in \(\mathbb{N}\) and Lemma 4.5 there is some forbidden suffix for \(L \cap \Sigma^{(\tau + d\mathbb{N})}\). Note that if \(\tau\) is a forbidden suffix for \(L \cap \Sigma^{(\tau + d\mathbb{N})}\) then \(\tau^0\) is a forbidden suffix for \(L \cap \Sigma^{(\tau + d\mathbb{N})}\) (since \(L\) is closed under removing trailing zeroes). So there is a forbidden suffix for \(L \cap \Sigma^{(\tau + d\mathbb{N})}\) for any \(\tau' \geq \tau\); pick \(\tau'\) such that \(L_q \cap \Sigma^{(\tau' + d\mathbb{N})}\) is infinite. Then since \(q\) is a finish state the forbidden suffix for \(L \cap \Sigma^{(\tau' + d\mathbb{N})}\) is also a forbidden suffix for \(L_q \cap \Sigma^{(\tau' + d\mathbb{N})}\). So \(L_q\) satisfies (†).

2. By [4, Proposition 7.1] there is a non-\(M\)-dead state \(q\) and distinct non-empty \(u, v \in \Sigma^*\) such that 
\(\delta(q, u) = \delta(q, v) = q\) and \(\delta(q, x) \neq q\) for \(x\) any proper non-empty prefix of \(u\) or \(v\). Then taking \(b, c\) to be powers of \(u, v\) respectively such that \(|b| = |c|\), we get that \(b \neq c\) (otherwise \(u\) or \(v\) would be a prefix of the other); also \(\delta(q, a) = \delta(q, b) = q\), so \(b, c \in L_q\). So \(L_q \supseteq \{ b, c \}^*\), and a quick computation shows that \(L_q\) isn’t sparse.

3. Let \(\delta(q, \mu) = q'\) and \(\delta(q', \nu) = q\). Suppose \(L_q\) isn’t sparse. Then \(L_{q'} \supseteq \nu L_q \mu\) also isn’t sparse.

**Claim 4.8**

By Claim 4.8 (2) there is \(q\) such that \(L_q\) isn’t sparse and \(a\) has a path to a finish state \(q'\). If \(L_q\) isn’t sparse then by Claim 4.8 (1) we’re done; suppose then that it is sparse. We show in this case that there is a forbidden infix for \(L_q\) (i.e. some \(\sigma\) that does not appear as a substring of any element of \(L_q\)), and hence in particular that \(L_q\) satisfies (†) with \(r = 0\) and \(s = 1\).

Note that there is no path from \(q'\) to \(q\), else by Claim 4.8 (3) \(L_{q'}\) wouldn’t be sparse. Enumerate the states of \(M\) with a path to \(q\) (and hence to \(q'\)) as \((q_i : i < n)\). Inductively pick \(\sigma_i \in \Sigma^*\) as follows: if \(\delta(q_i, \sigma_0 \cdots \sigma_{i-1})\) has no path to \(q\) we let \(\sigma_i = \varepsilon\), and otherwise we pick \(\sigma_i\) such that \(\delta(q_i, \sigma_0 \cdots \sigma_i) = q'\). Note then that \(\delta(q_i, \sigma_0 \cdots \sigma_j)\) has no path to \(q\); hence neither does \(\delta(q_i, \sigma_0 \cdots \sigma_{n-1})\). Let \(\tau = \sigma_0 \cdots \sigma_{n-1}\). We have shown that if \(r\) is a state with a path to \(q\) (so one of the \(q_i\)) then \(\delta(r, \tau)\) has no path to \(q\). Clearly if \(r\) has no path to \(q\) then neither does \(\delta(r, \tau)\). Hence for all \(r \in Q\) we get that \(\delta(r, \tau)\) has no path to \(q\); that is, \(\tau\) is a forbidden infix.

**Lemma 4.7**
Proof of Proposition 4.4. Suppose \( A \subseteq \mathbb{N} \) is \( d \)-automatic and neither \( d \)-sparse nor generic in \( \mathbb{N} \). Fix a minimal automaton \( M = (Q, q_0, F, \delta) \) for the set of representations over \( \Sigma \) of elements of \( A \). (The minimal automaton of a regular language \( L \) is an automaton recognizing \( L \) where all states are reachable from the start state and such that given distinct \( q, q' \in Q \) there is \( \nu \) such that \( \delta(q, \nu) \in F \) if and only if \( \delta(q', \nu) \notin F \). Such automata exist and are unique: see the proof of the right-to-left direction of [19, Theorem 4.7].)

By Lemma 4.7 there is a non-dead \( q \) such that \( L_q \) satisfies the hypotheses of Lemma 4.6. Using minimality, for each \( q' \neq q \) let \( \sigma_{q'} \in \Sigma^* \) and \( \varepsilon_{q'} \in \{0, 1\} \) be such that \((\delta(q, \sigma_{q'}) \in F)^{\varepsilon_{q'}} \land (\delta(q', \sigma_{q'}) \in F)^{1 - \varepsilon_{q'}} \) holds (where as before \( \varphi^0 \) denotes \( \neg \varphi \) and \( \varphi^1 \) denotes \( \varphi \)). If \( \theta \in Q \), then \( \theta = q \) if and only if

\[
\bigwedge_{q' \neq q} (\delta(\theta, \sigma_{q'}) \in F)^{\varepsilon_{q'}}
\]

holds. Consider then the following formula in the variables \( \overline{x} = (x_{q'} : q' \neq q) \) and \( y \):

\[
\varphi(\overline{x}; y) = \bigwedge_{q' \neq q} (x_{q'} + y \in A)^{\varepsilon_{q'}}
\]

We show that \( \varphi \) is unstable in \( \mathbb{N} \), and hence since \( \varphi \) is a Boolean combination of instances of \( x + y \in A \) that \( A \) is unstable in \( \mathbb{N} \).

Recall that \( L_q \) satisfies the hypotheses of Lemma 4.6; so for some \( K < \omega \) there is an \( N \)-ladder \((d_i, e_i : i \leq N)\) for \( x + K y \in L_q \). We may assume each \( |d_i| = K \). Take any \( \mu \in \Sigma^* \) such that \( \delta(q_0, \mu) = q_i \), and let

\[
\begin{align*}
    b_{i,q'} &= [\mu d_i \sigma_{q'}] \\
    c_i &= [0^{\mu} e_i]
\end{align*}
\]

These \( \overline{b}_i := (b_{i,q'} : q' \neq q) \), \( c_i \) will be our ladder for \( \varphi \). Note that \( b_{i,q'} + c_j = [\mu(d_i + K e_j) \sigma_{q'}] \). Then

\[
\varphi(\overline{b}_i; c_j) \iff \bigwedge_{q' \neq q} (b_{i,q'} + c_j \in A)^{\varepsilon_{q'}}
\]

\[
\iff \bigwedge_{q' \neq q} (\delta(q_0, \mu(d_i + K e_j) \sigma_{q'}) \in F)^{\varepsilon_{q'}}
\]

\[
\iff \bigwedge_{q' \neq q} (\delta(q, d_i + K e_j), \sigma_{q'}) \in F)^{\varepsilon_{q'}}
\]

\[
\iff \delta(q, d_i + K e_j) = q
\]

\[
\iff d_i + K e_j \in L_q
\]

\[
\iff i \leq j.
\]

So \( \varphi \) is unstable in \( \mathbb{N} \), and thus \( A \) is unstable in \( \mathbb{N} \). \( \Box \) Proposition 4.4

We can now do the case \( A \subseteq \mathbb{Z} \):

Proof of Theorem 4.2. Suppose \( A \subseteq \mathbb{Z} \) is \( d \)-automatic but neither \( d \)-sparse nor generic in \( \mathbb{Z} \).

Case 1. Suppose one of \( A \cap \mathbb{N} \) and \( -A \cap \mathbb{N} \) is generic in \( \mathbb{N} \) and the other is \( d \)-sparse. Then taking finitely many translates and unioning we get a set \( B \) where (say) \( B \cap \mathbb{N} \) is \( d \)-sparse and \( B \supseteq -\mathbb{N} \). (Note that \( d \)-sparsity is closed under translation and finite union.) Recall by [19, Theorem 3.8] that the set of prefixes of a sparse set is also sparse, and in particular that every sparse set has a forbidden prefix. So there is \( \sigma \in \Sigma^* \) such that \( \sigma \) is not a prefix of any canonical representative of an element of \( B \cap \mathbb{N} \); by possibly appending a 1, we may assume that \( \sigma \) has no trailing zeroes. So if \( r = [\sigma] \) and \( s = d^{[\sigma]} \) then \((r + s\mathbb{N}) \cap B = \emptyset \); so \((r + s\mathbb{Z}) \cap B = r + s\mathbb{Z}_{<0} \), and thus \((r + s\mathbb{Z}) \cap B \) is unstable in \( (\mathbb{Z}, +) \) since if \( x, y \in s\mathbb{Z} \), then

\[
x \leq y \iff r + x - y - s \in r + s\mathbb{Z}_{<0}
\]

So \((x + y \in r + s\mathbb{Z}) \land (x + y \in B) \) is unstable in \( (\mathbb{Z}, +) \). But \( x + y \in r + s\mathbb{Z} \) is stable in \((\mathbb{Z}, +)\), since it’s definable in \((\mathbb{Z}, +)\); so \( B \) is unstable in \((\mathbb{Z}, +) \). So since \( B \) is a finite union of translates of \( A \) we get that \( A \) is unstable in \((\mathbb{Z}, +) \).
Case 2. Suppose otherwise. Since $A$ isn’t generic in $\mathbb{Z}$, at most one of $A \cap \mathbb{N}$ or $\neg A \cap \mathbb{N}$ is generic in $\mathbb{N}$; likewise with $d$-sparse. Since we precluded the previous case we know there can’t be one of each, and generic in $\mathbb{N}$ and $d$-sparse are contradictory. So one of $A \cap \mathbb{N}$ or $\neg A \cap \mathbb{N}$ is neither generic in $\mathbb{N}$ nor $d$-sparse. Note that $A$ is stable in $(\mathbb{Z}, +)$ if and only if $-A$ is. Hence replacing $A$ by $-A$ if necessary we may assume $A \cap \mathbb{N}$ is neither generic in $\mathbb{N}$ nor $d$-sparse. Then by Proposition 4.4 there are arbitrarily large ladders in $\mathbb{N}$ for $x + y \in A \cap \mathbb{N}$; since $\mathbb{N}$ is closed under addition, we get that these are also ladders in $\mathbb{Z}$ for $x + y \in A$. Hence $A$ is unstable in $(\mathbb{Z}, +)$.

As an illustration of our theorem we note that the following automatic sets are not stable in $(\mathbb{Z}, +)$. Indeed, it is easily checked that they are all neither sparse nor generic.

**Corollary 4.9.** The following automatic sets are unstable in $(\mathbb{Z}, +)$:

- The set of $a \in \mathbb{Z}$ such that the canonical base-$d$ representation of $a$ ends in $\pm 1$ (assuming $d > 2$).
- The set of $a \in \mathbb{Z}$ such that the canonical base-$d$ representation of $a$ doesn’t contain a 0 (assuming $d > 2$).
- The set of $a \in \mathbb{Z}$ such that the canonical base-$d$ representation of $a$ is of even length.
- The set of $a \in \mathbb{Z}$ such that in the canonical binary representation of $a$ takes the form $0^{k_0}10^{k_1}1 \cdots 0^{k_m}1$ or $0^{k_0}(-1)0^{k_1}(-1) \cdots (-1)0^{k_m}(-1)$ for some even $k_0, \ldots, k_m$ (possibly zero); i.e. does not contain a block of zeroes of odd length. These are precisely the $a \in \mathbb{Z}$ such that the Baum-Sweet sequence has a 1 in the $|a|^{th}$ position. See [2, Section 5.1] for more details on the Baum-Sweet sequence.

The converse of Theorem 4.2 is certainly false. For example, let $A \subseteq \mathbb{Z}$ be as in the example at the beginning of Section 3; so $A$ is $d$-sparse and unstable in $(\mathbb{Z}, +)$. Then the complement of $A$ remains unstable, and is generic since $A$ doesn’t contain a pair of adjacent integers.

Theorems 3.1 and 4.2 together reduce the question of which automatic subsets of $\mathbb{Z}$ are stable to the case of generic subsets of $\mathbb{Z}$.

## 5 Two NIP expansions of $(\mathbb{Z}, +)$

In this final section we show how to apply automata-theoretic methods to produce some NIP expansions of $(\mathbb{Z}, +)$; see [18] for background on NIP.

### 5.1 $(\mathbb{Z}, +, <, d^{\mathbb{N}})$ is NIP

Fix $d > 0$. That $\text{Th}(\mathbb{N}, +, d^{\mathbb{N}})$ is NIP was shown recently by Lambotte and Point (it is an instance of [12, Corollary 2.33]), but our proof is novel and short. It will be convenient to work in $(\mathbb{N}, +)$ rather than $(\mathbb{Z}, +, <)$. Since $(\mathbb{Z}, +, <, d^{\mathbb{Z}})$ is interpretable in $(\mathbb{N}, +, d^{\mathbb{N}})$, it will suffice to prove:

**Theorem 5.1.** $\text{Th}(\mathbb{N}, +, d^{\mathbb{N}})$ is NIP.

Before proving the theorem, let us observe that since all $d$-sparse subsets of $\mathbb{N}$ are definable in $(\mathbb{N}, +, d^{\mathbb{N}})$—see [17, Theorem 5]—and as $A \subseteq \mathbb{Z}$ is $d$-sparse if and only if both $A \cap \mathbb{N}$ and $\neg A \cap \mathbb{N}$ are, we get:

**Corollary 5.2.** The expansion of $(\mathbb{Z}, +)$ by all $d$-sparse subsets is NIP.

Our proof of Theorem 5.1 will make use of a result of Chernikov and Simon on NIP pairs of structures; we briefly recall their setup and result. We let $L = \{ + \}$ and $\mathcal{N} = (\mathbb{N}, +)$; we fix $\text{Th}(\mathcal{N})$ as our ambient theory.

**Definition 5.3.** Let $L_P$ be $L$ expanded by a unary predicate $P$. A bounded $L_P$-formula is one of the form $(Q_1 x_1 \in P) \cdots (Q_n x_n \in P) \varphi$ for some quantifiers $Q_i$ and some $\varphi \in L$. If $M$ is an $L$-structure and $A \subseteq M$ we say $A$ is bounded in $M$ if every $L_P$-formula is $\text{Th}(M, A)$-equivalent to a bounded one.

**Definition 5.4.** Suppose $M$ is a structure and $A \subseteq M$. The induced structure $A_M$ of $M$ on $A$ has domain $A$ and atomic relations $D \cap A^n$ for each $0$-definable $D \subseteq M^n$. 


Fact 5.5 ([6, Corollary 2.5]). Suppose $M$ is a structure and $A \subseteq M$ is bounded in $M$. If $\text{Th}(M)$ and $\text{Th}(A_M)$ are NIP then so is $\text{Th}(M, A)$.

We wish to apply this to $(M, A) = (N, d^N)$. Boundedness follows from earlier work of Point:

**Proposition 5.6.** $d^N$ is bounded in $N$.

**Proof.** [16, Propositions 9 and 11] say that $\text{Th}(N, +, <, 0, 1, \frac{1}{n}, \lambda_d, S, S^{-1})_{n \geq 1}$ admits quantifier elimination, where

- $S(d^n) = d^{n+1}$ and $S(a) = a$ for other $a$
- $S^{-1}(d^{n+1}) = d^n$ and $S^{-1}(a) = a$ for other $a$
- $\lambda_d(x) = d^{\log_d(x)}$ for $x > 0$ and $\lambda_d(0) = 0$.

It then remains to show that any quantifier-free formula in this signature is equivalent to a bounded $L_P$-formula. But a quantifier-free formula $\varphi(\ldots, \lambda_d(t), \ldots)$ involving $\lambda_d$ is equivalent to

$$(\exists x \in d^N)((x \leq t) \land (\forall y \in d^N)\neg(x < y \leq t) \land \varphi(\ldots, x, \ldots)) \lor ((t = 0) \land \varphi(\ldots, 0, \ldots))$$

So at the cost of quantifying over $d^N$ we can eliminate occurrences of $\lambda_d$; we can similarly dispense with occurrences of $S$ and $S^{-1}$. Repeatedly applying this yields that any quantifier-free formula in the given signature is equivalent to one of the form $(Q_1 x_1 \in d^N) \cdots (Q_n x_n \in d^N) \psi$ where $\psi$ is a formula in $\{0, 1, +, <, \frac{1}{n}\}_{n \geq 1}$. But since $(N, 0, 1, +, <, \frac{1}{n})_{n \geq 1}$ is a definitional expansion of $(N, +)$, we get that $\varphi$ is equivalent to a bounded $L_P$-formula. \qed

It is well-known that $N$ is NIP; it is definable in $(Z, +, <)$, which is NIP as all ordered abelian groups are (see [11]). It remains to show that the induced structure $(d^N)_N$ is NIP.

The following is well-known; see e.g. [5, Theorem 6.1], of which it is a weakening.

**Fact 5.7.** All definable subsets of $N$ are $d$-automatic.

We therefore wish for a description of how $d$-automatic sets can intersect $d^N$.

**Proposition 5.8.** If $X \subseteq N^n$ is $d$-automatic then the relation

$$(\{(k_1, \ldots, k_n) \in N^n : (d^{k_1}, \ldots, d^{k_n}) \in X\}$$

is definable in $(N, +)$.

**Proof.** By symmetry and disjunction it suffices to check the case $k_1 \leq \cdots \leq k_n$.

It will be more convenient to work with $(X \cap N^n_{>0}) - 1$, which is also $d$-automatic. Then taking

$$\sigma_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ d - 1 \\ \vdots \\ d - 1 \end{pmatrix}$$

with $i - 1$ zeroes, we get for $k_1 \leq \cdots \leq k_n$ that

$$(d^{k_1}, \ldots, d^{k_n}) \in X \iff (d^{k_1 - 1}, \ldots, d^{k_n - 1}) \in (X \cap N^n_{>0}) - 1 \iff [\sigma_1^{k_1}, \sigma_2^{k_2 - k_1}, \ldots, \sigma_n^{k_n - k_{n-1}}] \in (X \cap N^n_{>0}) - 1$$

(since the base-$d$ representation of $d^{k_i} - 1$ consists of $d - 1$ repeated $k_i$ times). But by Proposition 2.2 the last condition is definable in $(N, +)$, as desired. \qed

Our theorem now follows easily:

**Proof of Theorem 5.1.** Proposition 5.8 and Fact 5.7 imply that the map $k \mapsto d^k$ induces an interpretation of $(d^N)_N$ in $(N, +)$. But $\text{Th}(N, +)$ is NIP; so $\text{Th}(d^N)_N$ is NIP. But $d^N$ is bounded in $N$ by Proposition 5.6, and $N$ is NIP. So $\text{Th}(N, d^N)$ is NIP by Fact 5.5. \qed
5.2 \((\mathbb{Z}, +, d^\mathbb{N}, \times | d^\mathbb{N})\) is NIP

Next we consider the expansion of \((\mathbb{Z}, +)\) by the monoid \((d^\mathbb{N}, \times)\). Note that as the ordering on \(d^\mathbb{N}\) is definable here, \((\mathbb{Z}, +, d^\mathbb{N}, \times | d^\mathbb{N})\) is not stable. However:

**Theorem 5.9.** \(\text{Th}(\mathbb{Z}, +, d^\mathbb{N}, \times | d^\mathbb{N})\) is NIP.

Surprisingly, our methods apply even though \(\times | d^\mathbb{N}\) itself isn’t \(d\)-automatic: since

\[
\begin{pmatrix}
0^i1 & 0^{i+1} \\
0^i1 & 0^{i+1} \\
0^i1 & 0^{i+1}
\end{pmatrix} \in \times | d^\mathbb{N} \iff i = j + 1
\]

it follows from the Myhill-Nerode theorem (see e.g. [19, Theorem 4.7]) that the set of canonical representations of elements of \(\times | d^\mathbb{N}\) isn’t regular. The reason automatic methods still apply is **Fact 5.7**, together with the following generalization of **Proposition 5.8**, which tells us that the interaction between iterated concatenation and membership in automatic sets can be described using Presburger arithmetic.

**Lemma 5.10.** Suppose \(X \subseteq \mathbb{Z}^m\) is \(d\)-automatic and \((\ell_{11}, \ldots, \ell_{1n_1}), \ldots, (\ell_{m1}, \ldots, \ell_{mn_m})\) are tuples from \(\Sigma_+\). Then the relation

\[
\left\{(k_{ij}) : \begin{pmatrix} k_{11} \cdots k_{1n_1} \\ \vdots \\ k_{m1} \cdots k_{mn_m} \end{pmatrix} \in X \right\} \subseteq \mathbb{N}^{n_1} \times \cdots \times \mathbb{N}^{n_m}
\]

is definable in \((\mathbb{N}, +)\).

**Proof.** We show that for any \(\ell_{ij}\), any automaton \((Q, q_0, \delta, F)\), and any \(q_1, q_2 \in Q\) the relation

\[
\left\{(k_{ij}) : \delta \left(q_1, P \begin{pmatrix} k_{11} \cdots k_{1n_1} \\ \vdots \\ k_{m1} \cdots k_{mn_m} \end{pmatrix} = q_2 \right) \right\}
\]

is definable in \((\mathbb{N}, +)\), where \(P : (\Sigma_+^*)^m \rightarrow (\Sigma_+^m)^*\) takes in \(m\) strings and pads them on the right with zeroes so they all have the same length as the longest one. This claim, applied to an automaton for the set of representations over \(\Sigma_+^m\) of elements of \(X\), yields the desired result.

We apply induction on \((m, n_1, \ldots, n_m)\). The base case \(m = 0\) is vacuous. For the induction step, suppose first that some \(n_i = 0\); say for ease of notation that \(i = 1\). Then we can construct an automaton \((Q, q_0, \delta', F)\) over \(\Sigma_+^{m-1}\) such that \(\delta'(q, \sigma) = \delta(q, \begin{pmatrix} 0 | \sigma \end{pmatrix})\) for any \(\sigma \in \Sigma_+^{m-1}\); that is, it behaves like the original automaton would if the input had an extra string of zeroes attached. In particular we have

\[
\delta \left(q_1, P \begin{pmatrix} \varepsilon \\ \ell_{21} \cdots \ell_{2n_2} \\ \vdots \\ \ell_{m1} \cdots \ell_{mn_m} \end{pmatrix} = q_2 \right) \iff \delta' \left(q_1, P \begin{pmatrix} \ell_{21} \cdots \ell_{2n_2} \\ \vdots \\ \ell_{m1} \cdots \ell_{mn_m} \end{pmatrix} = q_2 \right)
\]

and by the induction hypothesis the latter is definable in \((\mathbb{N}, +)\).

Suppose then that no \(n_i = 0\). Suppose \(k_{11}\) is minimum among the \(k_{ij}\). Then if

\[
q = \delta \left(q_1, P \begin{pmatrix} \ell_{11} \\ \vdots \\ \ell_{1n_1} \end{pmatrix} \right)
\]

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then our relation is equivalent to
\[
\delta \left( q, P \left( f_{k12}^{n1}, \ldots, f_{k1n1}^{n1} \right) \right) = q_2
\]

which by the induction hypothesis is definable in \((\mathbb{N}, +, \cdot)\). Similarly we get definability in the case \(k_1i\) is minimum for some \(i > 1\). So taking disjunctions we get that our relation is definable in \((\mathbb{N}, +, \cdot)\). □

Lemma 5.10

For our proof of Theorem 5.9 it will be convenient to assume \(d \geq 8\). In fact this suffices: consider for example the case \(d = 4\). Assuming the theorem holds when \(d = 4^2 = 16\), we get that \((\mathbb{Z}, +, 16\mathbb{N}, \times|16\mathbb{N})\) is NIP. But \(\times|4\mathbb{N}\) is definable in \((\mathbb{Z}, +, 16\mathbb{N}, \times|16\mathbb{N})\): we have \((a, b, c) \in \times|4\mathbb{N}\) if and only if \((4^ia, 4^ib, 4^{i+j}c) \in \times|16\mathbb{N}\) for some \(i, j \in \{0, 1\}\). This is because \(x\) is a power of 4 if and only if one of \(x, 4x\) is a power of 16. So \((\mathbb{Z}, +, 4\mathbb{N}, \times|4\mathbb{N})\) is definable in \((\mathbb{Z}, +, 16\mathbb{N}, \times|16\mathbb{N})\), and is thus NIP. Similar arguments work for all \(2 \leq d < 8\).

Proof of Theorem 5.9. We assume \(d \geq 8\). We will apply an extension due to Conant and Laskowski of the result of Chernikov and Simon we used previously (Fact 5.5). Since these results only apply to subsets of the domain, our first task is to encode \(d^\mathbb{N}\) and \(\times|d^\mathbb{N}\) as such. Let
\[
B = d^\mathbb{N} \cup \{ [7^i6^{i+j}] : i, j \in \mathbb{N} \}
\]
The point is that from \(B\) we will be able to extract both \(d^\mathbb{N}\) and
\[
\left\{ \frac{a - 1}{d - 1} + \frac{b}{d - 1} + \frac{c - 1}{d - 1} : (a, b, c) \in \times|d^\mathbb{N}, a \leq b \right\}
\]
These together will be enough to recover \(\times|d^\mathbb{N}\).

Claim 5.11. \(d^\mathbb{N}\) and \(\times|d^\mathbb{N}\) are definable in \((\mathbb{Z}, +, B)\).

Proof. Note first that \(d^\mathbb{N}\) is definable in \((\mathbb{Z}, +, B)\): we have \(a \in d^\mathbb{N}\) if and only if \(a = 1 \text{ or } 0 \neq a \in B\) and \(a \equiv 0 \pmod{d}\). I now claim that \((a, b, c) \in \times|d^\mathbb{N}\) with \(a \leq b\) if and only if \(a, b, c \in d^\mathbb{N}\) and
\[
\frac{a - 1}{d - 1} + \frac{b}{d - 1} + \frac{c - 1}{d - 1} \in B
\]
For the left-to-right direction, note that if \((d^i, d^j, d^{i+j}) \in \times|d^\mathbb{N}\) with \(i \leq j\) then
\[
\frac{d^i - 1}{d - 1} + \frac{d^j - 1}{d - 1} + \frac{d^{i+j} - 1}{d - 1} = [1^i] + [2^j] + [4^{i+j}] = [7^{i+j}] \in B
\]
For the right-to-left direction, suppose \(d^i, d^j, d^k\) satisfy
\[
[1^i] + [2^j] + [4^k] = \frac{d^i - 1}{d - 1} + \frac{d^j - 1}{d - 1} + \frac{d^k - 1}{d - 1} \in B
\]
If \(i = j = k = 0\) then \((d^i, d^j, d^k) \in \times|d^\mathbb{N}\) and \(d^i \leq d^j\), as desired; suppose then that at least one is non-zero. Then \([1^i] + [2^j] + [4^k] \neq 0 \pmod{d}\), so \([1^i] + [2^j] + [4^k] \in B \setminus d^\mathbb{N}\), and is thus equal to \([7^{i+j}] \in B\) for some \(i', j', k'\). But the map \((x, y, z) \mapsto [1^x] + [2^y] + [4^z]\) is injective. Indeed, we can represent \([1^x] + [2^y] + [4^z]\) by element of \([1, \ldots, 7]^3\); note that each element of \([1, \ldots, 7]\) can be represented uniquely as a sum of a subset of \([1, 2, 4]\). We can then recover \(x, y, z\) from the canonical representation of \([1^x] + [2^y] + [4^z]\) as the number of occurrences of \(\ell \in \{1, \ldots, 7\}\) that use a 1 in this sum representation; we can likewise recover \(y, z\).

So since \([1^i] + [2^j] + [4^k] = [1^{i'}] + [2^{j'}] + [4^{k'}]\) we get by injectivity that \(j = j' + i' \geq i = i\) and \(k = 2i' + j' = i + j;\) so \((d^i, d^j, d^k) \in \times|d^\mathbb{N}\) and \(d^i \leq d^j\), as desired.

But \((a, b, c) \in \times|d^\mathbb{N}\) if and only if \((b, a, c) \in \times|d^\mathbb{N}\) so
\[
(x \leq y \land (x, y, z) \in \times|d^\mathbb{N}\) \lor (y \leq x \land (y, x, z) \in \times|d^\mathbb{N})
\]
defines \(\times|d^\mathbb{N}\) in \((\mathbb{Z}, +, B)\). □

Claim 5.11
So it suffices to show that \((\mathbb{Z}, +, B)\) is NIP. We again check that the induced structure on \(B\) is NIP. When using Fact 5.5, we only concerned ourselves with the structure induced from the \(0\)-definable sets; however, to use the result of Conant and Laskowski, we will need that the structure induced by all sets definable with parameters from \(Z\) is NIP.

**Claim 5.12.** Let \(Z\) be \((\mathbb{Z}, +)\) expanded by names for all the constants. Then the induced structure \(B_Z\) is NIP.

**Proof.** Let \(D = \{(e_1, 1, 0, 0) : e_1 \in \mathbb{N}\} \cup \{(0, 0, e_3, e_4) : e_3, e_4 \in \mathbb{N}\} \subseteq \mathbb{N}^4\); note that \(D\) is definable in \((\mathbb{N}, +)\).

Consider \(\Phi : \mathbb{N}^4 \to \mathbb{Z}\) given by \((e_1, e_2, e_3, e_4) \mapsto [0^{e_1 + 1}1^{e_2}7^{e_3}6^{e_4}4^{e_5}]\); note that \(\Phi(D) \subseteq B\), and in fact \(\Phi : D \to B\) is bijective. I claim that \(\Phi\) defines an interpretation of \(B_Z\) in \((\mathbb{N}, +)\). Recall that \((\mathbb{Z}, +, 0, 1, \delta\mathbb{N})_{\delta > 0}\) admits quantifier elimination (see e.g. [13, Exercise 3.4.6]). So if \(X \subseteq Z\) is definable in \(Z\) then \(X\) is a Boolean combination of congruences and equalities, and hence \(X \cap \mathbb{N}\) is definable in \((\mathbb{N}, +)\); likewise with \(-X \cap \mathbb{N}\). So since \(N\) is a \(d\)-automatic subset of \(Z\) and \(d\)-automatic sets are closed under Boolean combinations we get that \(X\) is \(d\)-automatic. One argues similarly that if \(X \subseteq Z^m\) is definable in \(Z\) then \(X\) is \(d\)-automatic. So to show that \(\Phi\) defines an interpretation it suffices to show that whenever \(X \subseteq Z^m\) is \(d\)-automatic we have that

\[
\left\{(e_{ij}) \in D^m : \begin{bmatrix}
0^{e_1 + 1}1^{e_2}7^{e_3}6^{e_4}4^{e_5} \\
0^{e_6}1^{e_7}7^{e_8}6^{e_9}4^{e_{10}} \\
0^{e_{11}}1^{e_{12}}7^{e_{13}}6^{e_{14}}4^{e_{15}}
\end{bmatrix} \in X
\right\}
\]

is definable in \((\mathbb{N}, +)\). But this follows from Lemma 5.10 (and definability of \(D\)). So \(\Phi\) defines an interpretation of \(B_Z\) in \((\mathbb{N}, +)\); so \(B_Z\) is NIP. \(\square\) Claim 5.12

Now by [8, Theorem 2.9] we get since \(\text{Th}(\mathbb{Z}, +)\) is weakly minimal (see e.g. [8, Proposition 3.1]) and \(B_Z\) is NIP that \((\mathbb{Z}, +, B)\) is NIP. So \((\mathbb{Z}, +, d^\mathbb{N}, \times |d^\mathbb{N})\) is NIP. \(\square\) Theorem 5.9

Despite the similarity of methods in Theorems 5.1 and 5.9, we don’t know whether \(\text{Th}(\mathbb{Z}, +, <, d^\mathbb{N}, \times |d^\mathbb{N})\) is NIP. One might hope to apply Fact 5.5 with \((\mathbb{Z}, +, <)\) as the base NIP structure and \(B\) as the new predicate. Indeed, as in the proof of Claim 5.12 one can show that the induced structure on \(B\) is NIP by observing that the definable subsets of \((\mathbb{Z}, +, <)\) are \(d\)-automatic. Checking boundedness, however, isn’t simply a matter of adapting the arguments of Theorem 5.1 as the quantifier elimination result of Point that applied to \(d^\mathbb{N}\) doesn’t seem to apply to \(B\). Nor does the result of Conant and Laskowski yield boundedness as \((\mathbb{Z}, +, <)\) is not weakly minimal. So if one wishes to use our approach to show that \(\text{Th}(\mathbb{Z}, +, <, d^\mathbb{N}, \times |d^\mathbb{N})\) is NIP one needs a new way to check boundedness.

One can restate Theorem 5.9 as saying that expanding \((\mathbb{Z}, +)\) by a singly generated submonoid of \((\mathbb{Z} \setminus \{0\}, \times)\) yields an NIP structure. It would be natural to ask about finitely generated submonoids in general, but it seems unlikely that our automata-theoretic methods will apply as there is no obvious choice of \(d\) in general.

**References**

[1] Boris Adamczewski and Jason P Bell. “On vanishing coefficients of algebraic power series over fields of positive characteristic”. In: Inventiones mathematicae 187.2 (2012), pp. 343–393 (cit. on p. 3).

[2] Jean-Paul Allouche and Jeffrey Shallit. *Automatic Sequences: Theory, Applications, Generalizations*. Cambridge University Press, 2003. DOI: 10.1017/CBO9780511546563 (cit. on pp. 3, 10, 15).

[3] Jason Bell, Kathryn Hare, and Jeffrey Shallit. “When is an automatic set an additive basis?” In: Proceedings of the American Mathematical Society, Series B 5.6 (2018), pp. 50–63 (cit. on p. 2).

[4] Jason Bell and Rahim Moosa. “\(F\)-sets and finite automata”. In: Journal de théorie des nombres de Bordeaux 31.1 (2019), pp. 101–130 (cit. on pp. 2–4, 7, 12, 13).

[5] Véronique Bruyère et al. “Logic and \(p\)-recognizable sets of integers.” In: Bulletin of the Belgian Mathematical Society Simon Stevin (1994) (cit. on p. 16).

[6] Artem Chernikov and Pierre Simon. “Externally definable sets and dependent pairs”. In: Israel Journal of Mathematics (Mar. 1, 2013) (cit. on p. 16).
[7] Gabriel Conant. “Stability and sparsity in sets of natural numbers”. In: *Israel J. Math.* 230.1 (2019), pp. 471–508. ISSN: 0021-2172. DOI: 10.1007/s11856-019-1835-0 (cit. on pp. 1, 2).

[8] Gabriel Conant and Michael C. Laskowski. “Weakly minimal groups with a new predicate”. In: *Journal of Mathematical Logic* (2020). ISSN: 0219-0613 (cit. on p. 19).

[9] Harm Derksen. “A Skolem–Mahler–Lech theorem in positive characteristic and finite automata”. In: *Inventiones mathematicae* 168.1 (2007), pp. 175–224 (cit. on pp. 1, 2, 4).

[10] Christiane Frougny. “Numeration systems”. In: *Algebraic Combinatorics on words*. Vol. 90. Cambridge University Press, 2002, pp. 230–268 (cit. on p. 3).

[11] Yuri Gurevich and Peter H Schmitt. “The theory of ordered abelian groups does not have the independence property”. In: *Transactions of the American Mathematical Society* (1984) (cit. on p. 16).

[12] Quentin Lambotte and Françoise Point. “On expansions of (Z, +, 0)”. In: *Annals of Pure and Applied Logic* (2020), p. 102809 (cit. on pp. 1, 2, 15).

[13] David Marker. *Model theory: an introduction*. Vol. 217. Springer Science & Business Media, 2006 (cit. on p. 19).

[14] Rahim Moosa and Thomas Scanlon. “F-structures and integral points on semiabelian varieties over finite fields”. In: *American Journal of Mathematics* 126.3 (2004), pp. 473–522 (cit. on pp. 1, 2, 4, 6, 9).

[15] Daniel Palacín and Rizos Sklinos. “On Superstable Expansions of Free Abelian Groups”. In: *Notre Dame J. Formal Logic* 59.2 (2018), pp. 157–169. DOI: 10.1215/00294527-2017-0023 (cit. on p. 1).

[16] Françoise Point. “On decidable extensions of Presburger arithmetic: from A. Bertrand numeration systems to Pisot numbers”. In: *The Journal of Symbolic Logic* (2000) (cit. on p. 16).

[18] Pierre Simon. *A guide to NIP theories*. Cambridge University Press, 2015 (cit. on p. 15).

[19] Sheng Yu. “Regular Languages”. In: *Handbook of Formal Languages: Volume 1 Word, Language, Grammar*. Ed. by Grzegorz Rozenberg and Arto Salomaa. Berlin, Heidelberg: Springer Berlin Heidelberg, 1997, pp. 41–110. ISBN: 978-3-642-59136-5. DOI: 10.1007/978-3-642-59136-5_2 (cit. on pp. 2, 3, 11, 14, 17).