HIGHER-ORDER DAEEHEE NUMBERS AND POLYNOMIALS

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Abstract. Recently, Daehee numbers and polynomials are introduced by the authors. In this paper, we consider the Daehee numbers and polynomials of order $k \in \mathbb{N}$ and give some relation between Daehee polynomials of order $k \in \mathbb{N}$ and special polynomials.

1. Introduction

For $\alpha \in \mathbb{N}$, as is well known, the Bernoulli polynomials of order $\alpha$ are defined by the generating function to be

\[(1) \quad \left( \frac{t}{e^t - 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!}, \]

(see [1-14]).

When $x = 0$, $B_n^{(\alpha)} = B_n^{(\alpha)}(0)$ are the Bernoulli numbers of order $\alpha$. In [? , ?, ?], the Daehee polynomials are defined by the generating function to be

\[(2) \quad \left( \frac{\log (1 + t)}{t} \right)(1 + t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!}.

When $x = 0$, $D_n = D_n(0)$ are called the Daehee numbers.

Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of $p$-adic integers, the field of $p$-adic numbers, and the completion of algebraic closure of $\mathbb{Q}_p$. The $p$-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. Let $\text{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in \text{UD}(\mathbb{Z}_p)$, the $p$-adic invariant integral on $\mathbb{Z}_p$ is defined by

\[(3) \quad I(f) = \int_{\mathbb{Z}_p} f(x) \, d\mu(x) = \lim_{n \to \infty} \frac{1}{p^n} \sum_{x=0}^{p^n-1} f(x),

(see [?]).

Let $f_1(x) = f(x+1)$. Then, by (3), we get

\[(4) \quad I(f_1) - I(f) = f'(0), \quad \text{where } f'(0) = \left. \frac{df(x)}{dx} \right|_{x=0}.

The signed Stirling numbers of the first kind $S_1(n,l)$ are defined by

\[(5) \quad (x)_n = x(x-1) \cdots (x-n+1) = \sum_{l=0}^{\infty} S_1(n,l) x^l,

(see [? , ?, ?]).

From (5), we note that

\[
x^{(n)} = x(x+1) \cdots (x+n-1) = (-1)^n (-x)_n = \sum_{l=0}^{n} (-1)^{n-l} S_1(n,l) x^l,
\]
The Stirling numbers of the second kind \(S_2(l, n)\) are defined by the generating function to be

\[
(e^t - 1)^n = n! \sum_{l=0}^{\infty} \frac{S_2(l, n)}{(l+n)!} t^l
\]

In this paper, we study the higher-order Daehee numbers and polynomials and give some relations between Daehee polynomials and special polynomials.

2. Higher-order Daehee polynomials

In this section, we assume that \(t \in \mathbb{C}^p\) with \(|t|^p < p^{\frac{1}{p-1}}\).

For \(k \in \mathbb{N}\), let us consider the Daehee numbers of the first kind of order \(k\):

\[
D_n^{(k)} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + x_2 + \cdots + x_k)_n \, d\mu(x_1) \cdots d\mu(x_k),
\]

where \(n \in \mathbb{Z}_{\geq 0}\).

From (7), we can derive the generating function of \(D_n^{(k)}\) as follows:

\[
\sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \binom{x_1 + \cdots + x_k}{n} t^n \, d\mu(x_1) \cdots d\mu(x_k)
\]

By (4), we easily see that

\[
\int_{\mathbb{Z}_p} (1 + t)^x \, d\mu(x) = \frac{\log (1 + t)}{t}.
\]

Thus, by (8) and (9), we get

\[
\sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!} = \left(\frac{\log (1 + t)}{t}\right)^k.
\]

Now, we observe that

\[
\left(\frac{\log (1 + t)}{t}\right)^k = \frac{k!}{t^k} \sum_{l=k}^{\infty} \frac{S_1(t, k)}{l!} t^l
\]

\[
= \sum_{n=0}^{\infty} S_1(n + k, k) \frac{k!}{(n + k)!} t^n
\]

\[
= \sum_{n=0}^{\infty} S_1(n + k, k) \frac{t^n}{n!}
\]

Therefore, by (10) and (11), we obtain the following theorem.
Theorem 1. For \( n \in \mathbb{Z}_{\geq 0}, k \in \mathbb{N}, \) we have
\[
D_n^{(k)} = \frac{S_1(n + k, k)}{C_{n+k}}.
\]

It is easy to show that
\[
\left( \frac{\log (1 + t)}{t} \right)^k = \sum_{n=0}^{\infty} B_{n}^{(n+k+1)} (1) \frac{t^n}{n!}.
\]

Therefore, we obtain the following corollary.

Corollary 2. For \( n \in \mathbb{Z}_{\geq 0}, k \in \mathbb{N}, \) we have
\[
D_n^{(k)} = \frac{S_1(n + k, k)}{C_{n+k}} = B_{n}^{(n+k+1)} (1).
\]

From (10), we can derive
\[
\sum_{n=0}^{\infty} D_n^{(k)} \frac{(e^t - 1)^n}{n!} = \left( \frac{t}{e^t - 1} \right)^k = \sum_{n=0}^{\infty} B_{n}^{(k)} \frac{t^n}{n!},
\]
and
\[
\sum_{n=0}^{\infty} D_n^{(k)} \frac{(e^t - 1)^n}{n!} = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} D_n^{(k)} S_2(n, m) \right) \frac{t^n}{m!}.
\]

Therefore, by (14) and (15), we obtain the following theorem.

Theorem 4. For \( m \in \mathbb{Z}_{\geq 0}, k \in \mathbb{N}, \) we have
\[
B_m^{(k)} = \sum_{n=0}^{m} D_n^{(k)} S_2(m, n).
\]

Now, we consider the higher-order Daehee polynomials as follows:
\[
D_n^{(k)} (x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k + x)_n d\mu(x_1) \cdots d\mu(x_k).
\]
Thus, by (16), we get

\[ D_n^{(k)} (x) = \sum_{l=0}^{n} S_1 (n, l) \int_{0}^{\infty} \cdots \int_{0}^{\infty} (x_1 + \cdots + x_k + x)^l d\mu (x_1) \cdots d\mu (x_k) = \sum_{l=0}^{n} S_1 (n, l) B_l^{(k)} (x). \]

Therefore, by (17), we obtain the following theorem.

**Theorem 5.** For \( n \in \mathbb{Z}_{\geq 0}, k \in \mathbb{N}, \) we have

\[ D_n^{(k)} (x) = \sum_{l=0}^{n} S_1 (n, l) B_l^{(k)} (x). \]

From (16), we derive the generating function of \( D_n^{(k)} (x) : \)

\[ \sum_{n=0}^{\infty} D_n^{(k)} (x) \frac{t^n}{n!} = \int_{0}^{\infty} \cdots \int_{0}^{\infty} \sum_{n=0}^{\infty} \frac{\left( x_1 + \cdots + x_k + x \right)^n}{n} d\mu (x_1) \cdots d\mu (x_k) = \int_{0}^{\infty} \cdots \int_{0}^{\infty} (1 + t)^{x_1 + \cdots + x_k + x} d\mu (x_1) \cdots d\mu (x_k) = \left( \frac{\log (1 + t)}{t} \right)^k (1 + t)^x. \]

It is easy to show that

\[ \left( \frac{\log (1 + t)}{t} \right)^k (1 + t)^x = \sum_{n=0}^{\infty} B_n^{(n+k+1)} (x+1) \frac{t^n}{n!}. \]

Therefore, by (18) and (19), we obtain the following theorem.

**Theorem 6.** For \( n \in \mathbb{Z}_{\geq 0}, k \in \mathbb{N}, \)

\[ D_n^{(k)} (x) = B_n^{(n+k+1)} (x+1) = \sum_{l=0}^{n} \binom{n}{l} B_l^{(n+k+1)} (x+1)^{n-l}. \]

In (18), we note that

\[ \sum_{n=0}^{\infty} D_n^{(k)} (x) \frac{(e^t - 1)^n}{n!} = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} S_2 (n, m) D_n^{(k)} (x) \right) \frac{t^m}{m!} \]

and

\[ \sum_{n=0}^{\infty} D_n^{(k)} (x) \frac{(e^t - 1)^n}{n!} = \left( \frac{t}{e^t - 1} \right)^k e^{xt} = \sum_{m=0}^{\infty} B_m^{(k)} (x) \frac{t^m}{m!}. \]

Therefore, by (20) and (21), we obtain the following theorem.
Theorem 7. For $m \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{N}$, we have

$$B_m^{(k)}(x) = \sum_{n=0}^{m} S_2(m, n) D_n^{(k)}(x).$$

Now, we define Daehee numbers of the second kind of order $k (\in \mathbb{N})$:

$$\hat{D}_n^{(k)} = (-1)^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-x_1 - x_2 - \cdots - x_k)_n \, d\mu(x_1) \cdots d\mu(x_k)$$

$$= (-1)^n \sum_{l=0}^{n} (-1)^{-l} S_1(n, l) B_l^{(k)} = \sum_{l=0}^{n} \left[ \frac{n}{l} \right] B_l^{(k)},$$

where $\left[ \frac{n}{l} \right] = (-1)^{-l} S_1(n, l)$.

Thus, by (22), we get

$$\hat{D}_n^{(k)} = (-1)^n \sum_{l=0}^{n} (-1)^{-l} S_1(n, l) (\sum_{l=0}^{n} (-1)^{-l} S_1(n, l) B_l^{(k)}) = \sum_{l=0}^{n} \left[ \frac{n}{l} \right] B_l^{(k)},$$

Therefore, by (23), we obtain the following theorem.

Theorem 8. For $n \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{N}$, we have

$$\hat{D}_n^{(k)} = \sum_{l=0}^{n} \left[ \frac{n}{l} \right] B_l^{(k)}.$$

From (22), we derive the generating function of $\hat{D}_n^{(k)}$:

$$\sum_{n=0}^{\infty} \hat{D}_n^{(k)} \frac{t^n}{n!} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \frac{1}{n!} \left( x_1 + \cdots + x_k + n - 1 \right)^n \, d\mu(x_1) \cdots d\mu(x_k)$$

$$= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left( \frac{1-t}{1-\frac{t}{e^t-1}} \right)^k \, d\mu(x_1) \cdots d\mu(x_k)$$

By (24), we get

$$\sum_{n=0}^{\infty} \hat{D}_n^{(k)} \frac{(1-e^{-t})^n}{n!} = \left( \frac{e^{-t} - t}{e^t - 1} \right)^k = \left( \frac{t}{e^t - 1} \right)^k = \sum_{m=0}^{\infty} B_m^{(k)} \frac{t^m}{m!}. $$
and
\[
\sum_{n=0}^{\infty} \tilde{D}_n^{(k)} \frac{1 - e^{-t} n}{n!} = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} \tilde{D}_n^{(k)} (-1)^{m-n} S_2(m,n) \right) \frac{t^m}{m!}.
\]

Therefore, by (25) and (26), we obtain the following theorem.

**Theorem 9.** For \( m \in \mathbb{Z}_{\geq 0}, \ k \in \mathbb{N} \), we have
\[
B_m^{(k)} = \sum_{n=0}^{m} \tilde{D}_n^{(k)} (-1)^{n-m} S_2(m,n).
\]

Now, we consider the higher-order Daehee polynomials of the second kind:
\[
\hat{D}_n^{(k)}(x) = \int_{z_p} \cdots \int_{z_p} (x_1 + x_2 + \cdots + x_k - x)^n d\mu(x_1) \cdots d\mu(x_k).
\]

Thus, by (27), we get
\[
\tilde{D}_n^{(k)}(x) = \left( -1 \right)^n n \sum_{l=0}^{n-l} S_1(n,l) B_l^{(k)}(-x). \tag{28}
\]

Let us consider the generating function of \( D_n^{(k)}(x) \) as follows:
\[
\sum_{n=0}^{\infty} \tilde{D}_n^{(k)}(x) \frac{t^n}{n!} = \int_{z_p} \cdots \int_{z_p} \left( \sum_{n=0}^{\infty} \binom{x_1 + \cdots + x_k - x + n - 1}{n} t^n d\mu(x_1) \cdots d\mu(x_k) \right) \frac{1}{(1-t)^x}.
\]

From (30), we have
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\[
\sum_{n=0}^{\infty} \hat{D}^{(k)}_n (x) \frac{(-1)^n}{n!} = \left( \frac{\log (1 + t)}{t} \right)^k (1 + t)^{x+k}
\]

\[
= \sum_{n=0}^{\infty} B^{(n+k+1)}_n (x + k + 1) \frac{t^n}{n!}
\]

Therefore, by (31), we obtain the following theorem.

**Theorem 10.** For \( n \in \mathbb{Z}_{\geq 0}, k \in \mathbb{N} \), we have

\[
(-1)^n \hat{D}^{(k)}_n (x) = B^{(n+k+1)}_n (x + k + 1).
\]

By (30), we get

\[
\sum_{n=0}^{\infty} \hat{D}^{(k)}_n (x) \frac{1-e^{-t}}{n!} = e^{-tx} \left( \frac{t}{e^t-1} \right)^k = \sum_{m=0}^{\infty} B^{(k)}_m (-x) \frac{t^m}{m!},
\]

and

\[
\sum_{n=0}^{\infty} \hat{D}^{(k)}_n (x) \frac{1}{n!} (1-e^{-t})^n
\]

\[
= \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} \hat{D}^{(k)}_n (x) (-1)^{m-n} S_2 (m, n) \right) \frac{t^m}{m!}.
\]

Therefore, by (32) and (12), we obtain the following theorem.

**Theorem 11.** For \( m \in \mathbb{Z}_{\geq 0}, k \in \mathbb{N} \), we have

\[
B^{(k)}_m (-x) = \sum_{n=0}^{m} \hat{D}^{(k)}_n (x) (-1)^{m-n} S_2 (m, n).
\]

Now, we observe that

\[
(-1)^n \frac{D^{(k)}_n (x)}{n!} = \left( (-1)^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \frac{(x_1 + \cdots + x_k + x)}{n} d\mu (x_1) \cdots d\mu (x_k) \right)
\]

\[
= \left( \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-x_1 + \cdots + x_k) d\mu (x_1) \cdots d\mu (x_k) \right)
\]

\[
= \sum_{m=0}^{n} \frac{(n-1)}{m!} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \frac{(-x_1 + \cdots + x_k - x)}{m} d\mu (x_1) \cdots d\mu (x_k)
\]

\[
= \sum_{m=0}^{n} \frac{(n-1)}{m!} (-1)^{m} \hat{D}^{(k)}_m (-x).
\]

Therefore, by (34), we obtain the following theorem.
Theorem 12. For $n \in \mathbb{Z}_{\geq 0}, k \in \mathbb{N}$, we have

$$(-1)^n \frac{D_n^{(k)}(x)}{n!} = \sum_{m=1}^{n} \left(\frac{n-1}{m!}\right) (-1)^m \tilde{D}_m^{(k)}(-x).$$

By the same method as Theorem 12, we get

$$\tilde{D}_n^{(k)}(x) = \left(\frac{n}{n!}\right) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(\frac{x_1 + \cdots + x_k - x + n - 1}{n}\right) d\mu(x_1) \cdots d\mu(x_k)$$

$$= \sum_{m=0}^{n} \frac{n-1}{m!} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(\frac{x_1 + \cdots + x_k - x}{m}\right) d\mu(x_1) \cdots d\mu(x_k)$$

$$= \sum_{m=1}^{n} \frac{n-1}{m!} D_m^{(k)}(-x).$$

Thus, by (35), we get

$$\tilde{D}_n^{(k)}(x) = \sum_{m=1}^{n} \frac{n-1}{m!} D_m^{(k)}(-x).$$

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