ATTRACTORS OF MULTIVALUED SEMI-FLOWS GENERATED BY SOLUTIONS OF OPTIMAL CONTROL PROBLEMS

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To Professor Valery Melnik, in Memoriam

Abstract. In this paper we study the dynamical system generated by the solutions of optimal control problems. We obtain suitable conditions under which such systems generate multivalued semiprocesses. We prove the existence of uniform attractors for the multivalued semiprocess generated by the solutions of controlled reaction-diffusion equations and study its properties.

1. Introduction. The theory of attractors for infinite-dimensional dynamical systems was initiated in the 70s in the pioneering works [27], [16]. Nowadays it covers wide classes of dissipative evolution systems (see e.g. [2], [34], [36] among many other works), including generalizations to the non-autonomous case (see [8], [9], [13], [17]). The main constructions of the theory of attractors were extended to the case of non-uniqueness of the Cauchy problem in the works of V.S. Melnik (see [30], [31], [32]), A.V. Babin and M.I. Vishik [1] and D.N. Cheban and D.S. Fakhikh (see [14], [15], [12]) using the theory of multivalued operators and by J.M. Ball [3] using generalized semiflows. The application of this theory to various classes of dissipative evolution systems without uniqueness can be found in a great number of papers.

2010 Mathematics Subject Classification. 35B40, 35B41, 35K55, 35Q30, 35Q35, 37B25, 58C06.
Key words and phrases. Multivalued semiflow, multivalued semiprocess, global attractor, optimal control, reaction-diffusion equations.

The first two authors were partially supported by the State Fund for Fundamental Research of Ukraine under grants GP/F66/14921, GP/F78/187 and by the Grant of the National Academy of Sciences of Ukraine 2290/2018. The third author was partially supported by Spanish Ministry of Economy and Competitiveness and FEDER, projects MTM2015-63723-P and MTM2016-74921-P, and by Junta de Andalucía (Spain), project P12-FQM-1492.
in both the autonomous (multivalued semiflows) and nonautonomous (multivalued semiprocesses) cases (see e.g. [3], [4], [10], [11], [18], [21], [22], [23], [24], [26], [29], [32], [33], [35], [37], [38], [39], [40], [41] among many others).

The possibility of applying the theory of global attractors to optimal control problems was mentioned at first in the monograph [42]. In [20] the existence of pullback attractors for the flow generated by controlled three-dimensional Navier–Stokes equations was proved. Moreover, the relationship between the obtained pullback attractor and the global attractor of the three-dimensional Navier-Stokes equations was established as well. As far as we know, there are no more results in the literature concerning existence of attractors for optimal control problems.

It is worth mentioning that optimal control problems are essentially nonautonomous in general, and therefore they generate processes rather than semigroups.

In the present paper we investigate general conditions under which optimal control problems generate dynamical semiprocesses, showing in several examples that this property is not true in general. Also, we prove the existence of the uniform attractor for a reaction-diffusion equation with additive control and quadratic optimality criterion.

2. Semiflows and semiprocesses generated by optimal control problems.

Consider the following abstract optimal control problem

\[
\frac{dy}{dt} = f(t, y(t), u(t)), \quad t > 0,
\]

\[
y|_{t=0} = y_0 \in H,
\]

\[
u(\cdot) \in U_0 \subseteq W(0, +\infty),
\]

\[
J_0(y, u) = \int_0^\infty g(t, y(t), u(t)) dt \rightarrow \inf,
\]

where \(W(0, +\infty)\) is some functional space of measurable functions, the closed set of admissible controls \(U_0\) belongs to \(W(0, +\infty)\), and \(f, g : \mathbb{R}_+ \times H \rightarrow H\) are given functions.

For every \(u(\cdot) \in U_0\) a solution of (1) will be a continuous function \(y(\cdot)\) with values in the phase space \(H\) such that \(\frac{dy}{dt}\) exists and (1) is satisfied for a.a. \(t \in (0, +\infty)\).

A solution \(\{\bar{y}, \bar{u}\}\) of problem (1)-(4) is called an optimal pair. We make the following assumption:

for any \(y_0 \in H\) there exists an optimal pair \(\{\bar{y}, \bar{u}\}\). (5)

Our aim is to investigate the asymptotic behavior of solutions of problem (1)-(4) by using the theory of attractors. For this reason we study the properties of the following map

\[
G : \mathbb{R}_+ \times H \mapsto P(H),
\]

\[
G(t, y_0) = \{\bar{y}(t) \mid \{\bar{y}, \bar{u}\} \text{ is an optimal pair of problem (1)-(4)}\}.
\]

In all the further arguments we will denote by \(P(H) (\beta(H))\) the set of all non-empty (non-empty bounded) subsets of \(H\).

Remark 1. We do not require uniqueness of solutions for problem (1)-(4). Thus, the map (6) is multivalued in general.
It is known \cite{42} that the map (6) is a multivalued semiflow (m-semiflow) \cite{32}, i.e.
\[ G(t + s, y_0) \subseteq G(t, G(s, y_0)), \forall t, s \geq 0, y_0 \in H, \] (7)
if problem (1)–(4) allows an autonomous optimal control in feedback form, i.e. there exists a map \( \kappa(\cdot) \) (which does not depend on \( y_0 \)) such that \( u(t) = \kappa(y(t)) \) and \( y \) is a solution of the Cauchy problem
\[ \begin{align*}
\frac{dy}{dt} &= f(y(t), \kappa(y(t))), \\
y|_{t=0} &= y_0.
\end{align*} \] (8)

It should be remarked that such optimal stabilization problems are difficult (and unresolvable in general case) even for linear-quadratic distributed systems \cite{19}. Moreover, even in the simplest cases property (7) may fail to be true. We show this situation in the following examples.

Example 1.
\[ \begin{align*}
\dot{y} &= -y + u, \\
y(0) &= y_0, \\
u(\cdot) &\in U_0 = \{ u_1(t) \equiv 1, \ u_2(t) \equiv -1 \} \subset L^2_{loc}(0, +\infty), \\
J_0(y, u) &= +\infty \int_0^t (y^2(t) - 1)dt \to \inf.
\end{align*} \] (9)

There exists a solution of problem (9) for each \( y_0 \). Moreover,
\[ \overline{y}(t) = (y_0 + 1)e^{-t} - 1, \ \overline{u}(t) = -1 \text{ if } y_0 \geq 0, \]
\[ \overline{u}(t) = (y_0 - 1)e^{-t} + 1, \ \overline{u}(t) = 1 \text{ if } y_0 \leq 0. \]

So, we have that the map (6) has the form
\[ G(t, y_0) = \begin{cases}
(y_0 + 1)e^{-t} - 1, & y_0 > 0, \\
(y_0 - 1)e^{-t} + 1, & y_0 < 0, \\
e^{-t} - 1, & y_0 = 0.
\end{cases} \]

We will show that \( G(t, y_0) \) does not satisfy (7). Indeed, for \( y_0 = -1 \) and any \( t \geq 0, s \geq 0 \) we have
\[ G(t + s, y_0) = -2e^{-t-s} + 1. \]

Let \( s = \ln 4 \). Then \( G(s, y_0) = \frac{1}{2} > 0. \) Hence, for arbitrary \( t \geq 0 \) we have that 
\[ G(t, G(s, y_0)) = \frac{3}{2}e^{-t} - 1. \] On the other hand, it is not equal to \( G(t + s, y_0) = -\frac{1}{2}e^{-t} + 1. \)

Example 2.
\[ \begin{align*}
\dot{y} &= u, \\
y(0) &= y_0, \\
u(\cdot) &\in U_0 = L^2_{loc}(0, +\infty), \\
J_0(y, u) &= +\infty \int_0^t e^{-t}(u^2 + y^2 + 2yt)dt \to \inf.
\end{align*} \] (10)
There exists a unique solution of problem (10) for each \( y_0 \) [28]. From Pontryagin’s maximum principle
\[
\max_v \{ pv - e^{-t} v^2 \} = p\bar{u} - e^{-t} \bar{u}^2,
\]
\[
\dot{p} = 2e^{-t}(\bar{y} + t), \quad \lim_{t \to \infty} p(t) = 0,
\]
where \( p(t) \) is the conjugate function. Therefore, we obtain that
\[
\bar{y}(t) = (y_0 - 1)e^{\lambda t} + 1 - t, \quad \bar{u}(t) = \lambda(y_0 - 1)e^{\lambda t} - 1,
\]
where \( \lambda = \frac{1 - \sqrt{5}}{2} \),
and then
\[
G(t, y_0) = (y_0 - 1)e^{\lambda t} + 1 - t, \quad \text{for any } y_0 \in H.
\]
This map does not satisfy (7), because
\[
G(t + s, y_0) = (y_0 - 1)e^{\lambda(t+s)} + 1 - t - s.
\]
On the other hand, we get
\[
G(t, G(s, y_0)) = (y_0 - 1)e^{\lambda(t+s)} - se^{\lambda t} + 1 - t.
\]
These examples show that problem (1)–(4) is essentially non-autonomous in general. So, we need the notion of semiprocess [13], [7] to describe the long-time behavior of solutions of (1)–(4). For this reason we consider the following family of problems
\[
\frac{dy}{dt} = f(t, y(t), u(t)), \quad t > \tau, \quad (11)
\]
\[
y|_{t=\tau} = y_\tau \in H, \quad \tau \geq 0, \quad W(\tau, +\infty), \quad (12)
\]
\[
u(\cdot) \in U_\tau \subseteq W(\tau, +\infty), \quad \tau \geq 0, \quad (13)
\]
\[
J_\tau(y, u) = \int_\tau^{+\infty} g(t, y(t), u(t))dt \to \inf, \quad (14)
\]
where \( \tau \geq 0, \, W(\tau, +\infty) \) is some functional space of measurable functions and the closed set of admissible controls \( U_\tau \) belongs to \( W(\tau, +\infty) \).

As before, for every \( u(\cdot) \in U_\tau \) we will say that \( y(\cdot) \) is a solution of (11) if it is a continuous function with values in the phase space \( H \) such that \( \frac{dy}{dt} \) exists and satisfies (11) a.e. on \( (\tau, +\infty) \).

A solution \( \{\bar{y}, \bar{u}\} \) of problem (11)-(14) is called an optimal pair. We make the assumption that
\[
\text{for any } \tau \geq 0, \, y_\tau \in H \text{ there exists an optimal pair } \{\bar{y}, \bar{u}\}. \quad (15)
\]

We consider the following map associated to (11)–(14):
\[
U : \mathbb{R}_{+\tau} \times H \mapsto P(H),
\]
\[
U(t, \tau, y_\tau) = \{\bar{y}(t) \mid \{\bar{y}, \bar{u}\} \text{ is an optimal pair of (11)-(14)}\}, \quad (16)
\]
where \( \mathbb{R}_{+\tau} = \{(t, \tau) \mid t \geq \tau \geq 0\} \).

Does the map (16) generates a semiprocess, that is, is the relation
\[
U(t, \tau, y_\tau) \subseteq U(t, s, U(s, \tau, y_\tau)), \quad \forall t \geq s \geq \tau, \quad y_\tau \in H, \quad \text{true?} \quad \text{Example 3 shows that the answer to the above question may be “not”}
\]
Example 3. Let us consider the problem:

\[
\begin{aligned}
\dot{y} &= -y + u, \quad t > \tau, \\
y(\tau) &= y_\tau, \\
u(\cdot) &\in U_\tau \equiv U_0 = \{u_1(t) \equiv 1, \ u_2(t) \equiv -1\} \subset L^2_{loc}(\tau, +\infty), \\
J_\tau(y, u) &= \int_{\tau}^{+\infty} (y^2(t) - 1) dt \rightarrow \inf,
\end{aligned}
\]  

(18)

whose solutions generate the following map:

\[
U(t, \tau, y_\tau) = \begin{cases} 
(y_\tau + 1)e^{-(t-\tau)} - 1, & \text{if } y_\tau \geq 0, \\
(y_\tau - 1)e^{-(t-\tau)} + 1, & \text{if } y_\tau \leq 0, \\
\{e^{-(t-\tau)} - 1, -e^{-(t-\tau)} + 1\}, & \text{if } y_\tau = 0.
\end{cases}
\]

Then for \(\tau = 0, y_\tau = -1\) we have

\[U(t, 0, -1) = -2e^{-t} + 1, \ \forall t \geq 0,\]

and for \(s = \ln 4, t \geq s\) we obtain

\[U(t, s, U(s, 0, -1)) = U(t, s, \frac{1}{2}) = \frac{3}{2}e^{-(t-s)} - 1 = 6e^{-t} - 1,\]

i.e. (17) is not fulfilled.

Nevertheless, for wide classes of optimal control problems relation (17) is true. In particular, for Example 2. Indeed, relation (17) for Example 2 can be easily obtained from the fact that \(\bar{y}\) is an optimal solution if and only if it is a solution of the differential equation

\[\dot{y} = \lambda(y + t - 1) - 1.\]

The Principle of Optimality [6] (the final part of the optimal process is optimal) plays a crucial role for (17). This principle holds for Example 2 but not for Example 1.

In terms of problem (11)-(14) it means that if \(\{\bar{y}, \bar{u}\}\) is an optimal pair of (11)-(14), then for any \(s > \tau\) its restriction to \([s, +\infty)\), i.e. the pair \(\{\bar{y}, \bar{u}\}|_{[s, +\infty)}\), will be optimal for the problem (11)-(14) with initial time \(s\) and initial condition \(y_s = \bar{y}(s)\).

If this principle is satisfied for \(\bar{y}(t) \in U(t, \tau, y_\tau)\), we have \(\bar{y}(s) \in U(s, \tau, y_\tau)\) and \(\bar{y}(t) \in U(t, s, \bar{y}(s)),\)

so (17) holds.

The following lemma gives us sufficient conditions for (17) to be fulfilled in terms of the coefficients of problem (11)-(14).

Lemma 1. Let for problem (11)-(14) condition (15) hold. Moreover,

if \(\tau_2 \geq \tau_1\), then for \(u \in U_{\tau_1}\) we have \(u|_{[\tau_2, +\infty)} \in U_{\tau_2};\)

(19)

and if \(\tau_2 \geq \tau_1, u_2 \in U_{\tau_2}, u_1 \in U_{\tau_1}\), then

\[
u(t) = \begin{cases} 
u_1(t), & t \in [\tau_1, \tau_2) \\
u_2(t), & t \geq \tau_2
\end{cases} \quad \in U_{\tau_1}.
\]

(20)

Then formula (16) defines a strict multivalued semiprocess, i.e.

\[U(t, \tau, y_\tau) = U(t, s, U(s, \tau, y_\tau)), \ \forall t \geq s \geq \tau, \ y_\tau \in H.\]

(21)
Proof. Let $\xi \in U(t, \tau, y_r)$. Then $\xi = \bar{y}(t), \{\bar{u}, \bar{\pi}\}$ is an optimal pair in problem \((11)-(14)\) with initial conditions $(\tau, y_r)$, so $\bar{y}(s) \in U(s, \tau, y_r)$. Let us prove that $\{\bar{y}, \bar{\pi}\}|_{s, +\infty}$ is an optimal pair of problem \((11)-(14)\) with initial conditions $(s, \bar{y}(s))$. Suppose it is not true. Then there exists a solution $\{\bar{y}, \hat{u}\}$ of problem \((11)-(14)\) with initial conditions $(s, \bar{y}(s))$ such that

$$J_s(\bar{y}, \hat{u}) < J_s(\bar{y}, \bar{\pi}).$$

Consider the control

$$u(t) = \begin{cases} \bar{u}(t), & t \geq s \\ \bar{\pi}(t), & t \in [\tau, s) \end{cases} \in U,$$

and the corresponding phase variable

$$y(t) = \begin{cases} \bar{y}(t), & t \in [\tau, s), \\ \hat{u}(t), & t \geq s. \end{cases}$$

Then $J_\tau(y, u) \geq J_\tau(\bar{y}, \bar{\pi})$. On the other hand,

$$J_\tau(y, u) = \int_0^\tau g(t, \bar{y}(t), \bar{\pi}(t))dt + \int_0^{+\infty} g(t, \bar{y}(t), \hat{u}(t))dt$$

$$< \int_0^\tau g(t, \bar{y}(t), \bar{\pi}(t))dt + \int_0^{+\infty} g(t, \bar{y}(t), \bar{\pi}(t))dt = J_\tau(\bar{y}, \bar{\pi}).$$

This contradiction proves that

$$\xi = \bar{y}(t) \in U(t, s, \bar{y}(s)) \subset U(t, s, U(s, \tau, y_r)).$$

Let $\xi \in U(t, s, U(s, \tau, y_r))$. Then $\xi = \bar{y}_2(t), \{\bar{u}_2, \bar{\pi}_2\}$ is an optimal pair of \((11)-(14)\) with initial datum $(s, \bar{y}_2(s)), \bar{y}_2(s) = \bar{y}_1(s), \{\bar{\pi}_1, \bar{\pi}_2\}$ is an optimal pair of \((11)-(14)\) with initial datum $(\tau, y_r)$. Let us put

$$u(t) = \begin{cases} \bar{u}_1(t), & t \in [\tau, s), \\ \bar{u}_2(t), & t \geq s, \end{cases}$$

$$y(t) = \begin{cases} \bar{y}_1(t), & t \in [\tau, s), \\ \bar{y}_2(t), & t \geq s. \end{cases}$$

Then

$$J_\tau(y, u) = \int_\tau^\infty g(t, \bar{y}_1(t), \bar{u}_1(t))dt + \int_\infty^{+\infty} g(t, \bar{y}_2(t), \bar{u}_2(t))dt \leq$$

$$\leq \int_\tau^\infty g(t, \bar{y}_1(t), \bar{u}_1(t))dt + \int_\infty^{+\infty} g(t, \bar{y}_1(t), \bar{u}_1(t))dt = J_\tau(\bar{y}_1, \bar{u}_1).$$

Thus, $\{y, u\}$ is an optimal pair of \((11)-(14)\) with initial datum $(\tau, y_r)$, i.e. $\xi = y(t) \in U(t, \tau, y_r)$.

As usual, denote by $\text{dist}(B, C) = \sup_{b \in B} \inf_{c \in C} ||b - c||$ the Hausdorff semi-distance from the set $B$ to the set $C$. The following definition seems to be natural for semiprocesses.

**Definition 1.** \cite{13} The compact set $A \subset H$ is called a uniform (w.r.t. $\tau$) attractor for the multivalued semiprocess $U$ if

$$\lim_{\tau \to 0} \sup_{\tau \geq 0} \text{dist}(U(t + \tau, \tau, B), A) = 0 \text{ for every } B \in \beta(H),$$

and $A$ is the minimal closed set satisfying such property.
In order to formulate the conditions for the existence of a uniform attractor of the m-semiprocess $U$, it is natural to consider the following map:

$$G : \mathbb{R}_+ \times H \mapsto P(H),$$

$$G(t, x) = \bigcup_{\tau \geq 0} U(t + \tau, \tau, x).$$

(22)

**Lemma 2.** [20] $G$ is a multivalued semiflow, i.e., for any $t, s \geq 0, x \in H$ we have

$$G(0, x) = x,$$

$$G(t + s, x) \subseteq G(t, G(s, x)).$$

**Definition 2.** The compact set $A \subset H$ is called a uniform attractor for m-semiflow $G$ if

$$\lim_{t \to \infty} \text{dist}(G(t, B), A) = 0, \ \forall B \in \beta(H),$$

and $A$ is the minimal closed set satisfying such property.

**Remark 2.** In this definition the semi-invariance property ($A \subseteq G(t, A)$ for all $t \geq 0$) in the classical definition of global attractor for m-semiflows (see [30], [32]) is replaced by the minimality condition.

It is easy to see that $A$ is a uniform attractor of the m-semiprocess $U$ if and only if $A$ is a uniform attractor of the m-semiflow $G$ which is defined in (22).

If $\tau \in (-\infty, +\infty)$, i.e. if $U$ is a multivalued process, then there exists another way to describe the dynamics of $U$, which comes from the theory of pullback attractors (see [20], [7]).

In the present paper we consider the multivalued semiflow (22) as the basic dynamical object that is associated with the optimal control problem (11)–(14).

**Lemma 3.** [41] Let the following dissipative condition be fulfilled: there exists $B_0 \in \beta(H)$ for which for any $B \in \beta(H)$ there exists $T = T(B)$ such that

$$G(t, B) \subset B_0, \ \forall t \geq T.$$ (23)

Then $G$ has a uniform attractor $A$ if and only if $G$ is asymptotically compact, i.e., for any $B \in \beta(H)$ and all $\xi_n \in G(t_n, B)$ one has

$$\{\xi_n\}_{n \geq 1} \text{ is relatively compact in } H.$$ (24)

Moreover,

$$A = \omega(B_0) = \bigcap_{\tau > 0} \bigcup_{t \geq \tau} G(t, B_0).$$ (25)

If for any $t > 0$ $G(t, \cdot)$ has closed graph, then

$$A \subset G(t, A), \ \forall t \geq 0,$$ (26)

that is, $A$ is a global attractor of $G$.

**Remark 3.** In terms of the m-semiprocess $U$ the dissipativity condition (23) means that there is a set $B_0 \in \beta(H)$ for which for any $B \in \beta(H)$ there exists $T = T(B)$ such that

$$U(t + \tau, \tau, B) \subset B_0, \ \forall t \geq T, \ \tau \geq 0.$$ (27)
3. Application to parabolic inclusions with additive control and quadratic quality criterion. Let \( V \subset H \subset V^* \) be an evolution triple of Hilbert spaces with compact and dense embeddings, \( (\cdot,\cdot) \) be the duality between \( V \) and \( V^* \), \( \|\cdot\| \), \( (\cdot,\cdot) \) be the norm and the inner product in \( H \), respectively, \( \|\cdot\|_V \) be the norm in \( V \), and suppose that there exists \( \alpha > 0 \) such that
\[
\alpha \|y\|^2 \leq \|y\|^2_V, \quad \forall y \in V. \tag{28}
\]
Let \( A : V \to V^* \) be a linear continuous self-adjoint operator such that for some \( \beta > 0 \) we have
\[
(Ay,y) \geq \beta \|y\|^2_V, \quad \forall y \in V. \tag{29}
\]
Finally, let \( F : H \to P(H) \) be such that
\[
F(y) \text{ is closed and convex in } H, \quad \forall y \in H, \tag{30}
\]
\[
\exists c > 0 \text{ such that } \|F(y)\|_+ := \sup_{z \in F(y)} \|z\| \leq c(1 + \|y\|), \quad \forall y \in H, \tag{31}
\]
\[
\text{dist}(F(y), F(y_0)) \to 0, \quad y \to y_0, \quad \forall y_0 \in H. \tag{32}
\]
Let us consider the optimal control problem:
\[
\begin{align*}
\frac{dy}{dt} + Ay &\in F(y) + u(t), \quad \text{for a.a. } t > \tau, \\
y|_{t=\tau} &\in H, \\
u(\cdot) &\in U_\tau \subseteq L^2_\text{loc}(\tau, +\infty; H),
\end{align*} \tag{33}
\]
\[
J_\tau(y,u) = \int_{\tau}^{+\infty} e^{-2\kappa t} \|y(t)\|^2 dt + \gamma \int_{\tau}^{+\infty} \|u(t)\|^2 dt \to \inf,
\tag{36}
\]
where \( \tau \geq 0 \) is the initial moment of time, \( \kappa > 0, \gamma > 0 \), and
\[
U_\tau \text{ is convex, closed in } L^2_\text{loc}(\tau, +\infty; H),
\tag{37}
\]
\[
J_\tau \text{ satisfies (19), (20), } 0 \in U_\tau.
\]
For example, \( U_\tau = L^2_\text{loc}(\tau, +\infty; H) \) or
\[
U_\tau = \{ u \in L^2_\text{loc}(\tau, +\infty; H) \mid \|u(t)\| \leq \xi \text{ for a.a. } t > \tau \}.
\]
In all the subsequent considerations the constants \( c_i > 0, \delta_i > 0 \) will depend only on constants \( \alpha, \beta, c, \kappa, \gamma \) of the problem (33)–(36).

It is known [25], [23] that under conditions (30)–(32) for every \( \tau \geq 0, y_\tau \in H, u_\tau \in U_\tau \) problem (33), (34) has at least one strong solution \( y(\cdot) \), i.e., \( y \in C([\tau, +\infty); H) \), \( y(t) \in D(A) \) for a.a. \( t \), \( y \) is absolutely continuous on compact subsets of \( (\tau, +\infty) \) and there is \( f \in L^2_\text{loc}(\tau, +\infty; H) \) such that
\[
\begin{cases}
\frac{dy(t)}{dt} + Ay(t) = f(t) + u(t) \text{ for a.a. } t > \tau, \\
y|_{t=\tau} = y_\tau,
\end{cases}
\tag{38}
\]
\[
f(t) \in F(y(t)) \text{ for a.a. } t > \tau. \tag{39}
\]
Since \( y \in L^2(\tau,T;V) \) [5, p.188], it follows that for all \( T > \tau \) every strong solution belongs to the space
\[
W(\tau,T) = \{ y \in L^2(\tau,T;V) \mid \frac{dy}{dt} \in L^2(\tau,T;V^*) \}
\]
and for a.a. $t > \tau$
\[
\frac{1}{2} \frac{d}{dt} \|y(t)\|^2 + \langle Ay, y \rangle = \langle f(t), y(t) \rangle + \langle u(t), y(t) \rangle.
\] (40)

**Lemma 4.** Let conditions (30)–(32) be fulfilled and
\[c - \alpha \beta < \kappa.\] (41)
Then the optimal control problem (33)-(36) has at least one solution \(\{\overline{y}, \overline{u}\}\).

**Proof.** From (40) any solution of (33)-(34) satisfies
\[
\frac{d}{dt} \|y\|^2 \leq 2(c - \alpha \beta + \varepsilon) \|y\|^2 + \frac{c^2}{\varepsilon} \|u(t)\|^2,
\] (42)
where \(\varepsilon > 0\) is such that
\[
\delta_1 = c - \alpha \beta + \varepsilon < \kappa.
\] (43)
Then
\[
\|y(t)\|^2 \leq \|y_\tau\|^2 e^{2\delta_1 (t - \tau)} + c_1 e^{2\delta_1 (t - \tau)} + c_1 \int_\tau^t e^{2\delta_1 (t - s)} \|u(s)\|^2 ds.
\] (44)
So, when \(u(t) \equiv 0 \in U_\tau\) we have
\[
\|y(t)\|^2 \leq (\|y_\tau\|^2 + c_1) e^{2\delta_1 (t - \tau)}.
\] (45)
This implies by virtue of (43) that the set of admissible pairs in (33)-(36) is non-empty for any \(\tau \geq 0\), \(y_\tau \in H\).

Let \(d \geq 0\) is the infimum value of the cost functional \(J_\tau\). Let us choose a sequence of admissible pairs \(\{y_n, u_n\}\) such that
\[
J_\tau(y_n, u_n) \leq d + \frac{1}{n}.
\] (46)
Then for all \(T > \tau\), \(\{u_n\}\) is bounded in \(L^2(\tau, T; H)\),
\[
\text{which by virtue of (44), (40), (29) leads to}
\]
\[
\{y_n\}\text{ is bounded in } W(\tau, T)\text{ and } C([\tau, T]; H).
\] (48)
Then from the Compactness Lemma [34] we have that for some pair of functions \(\{\overline{y}, \overline{u}\}\) and for every \(T > \tau\) up to a subsequence
\[
u_n \rightharpoonup \overline{u} \text{ in } L^2(\tau, T; H),
\]
\[
y_n \rightarrow \overline{y} \text{ in } L^2(\tau, T; H)\text{ and a.e. in } H,
\]
\[
y_n \rightharpoonup \overline{y} \text{ in } L^2(\tau, T; V).
\] (49)
Then, passing to the limit in (38), (39) we obtain that \(\{\overline{y}, \overline{u}\}\) is a strong solution of problem (33), (34) and \(\overline{u} \in U_\tau\) (see [25], [23]). Thus from (46) we have
\[
\int_\tau^T e^{-2\kappa t} \|y(t)\|^2 dt + \gamma \int_\tau^T \|\overline{u}(t)\|^2 dt \leq d, \forall T > \tau.
\] (50)
From (50) it follows that \(\{\overline{y}, \overline{u}\}\) is a solution of problem (33)-(36).
Hence, the multivalued semiflow
\[ G(t, y) = \bigcup_{\tau \geq 0} U(t + \tau, \tau, y), \]  
where
\[ U(t, \tau, y_{\tau}) = \{ y(t) \mid y \text{ is an optimal trajectory of (33)-(36)} \}, \]
is correctly defined.

**Theorem 1.** Let conditions (30)–(32) be fulfilled and
\[ c < \alpha \beta. \]  
Then the multivalued semiflow \( G \) has the uniform attractor \( A \).

**Proof.** Let us prove the dissipativity condition (23). In virtue of (52) there exists \( \delta_2 > 0 \) such that for every solution of (33), (34) we have
\[ \|y(t)\|^2 \leq \|y(s)\|^2 e^{-2\delta_2(t-s)} + c_2 + c_2 \int_{s}^{t} e^{-2\delta_2(t-p)} \|u(p)\|^2 dp, \ \forall s \geq \tau. \]  
In particular,
\[ \|\mathcal{y}(t + \tau)\|^2 \leq \|\mathcal{y}(\tau)\|^2 e^{-2\delta_2 t} + c_2 + c_2 \int_{\tau}^{t+\tau} e^{-2\delta_2(t+\tau-p)} \|\mathcal{u}(p)\|^2 dp, \ \forall t \geq 0, \ \forall \tau \geq 0. \]  
Let the initial conditions \( y_{\tau} = \mathcal{y}(\tau) \) satisfies the inequality
\[ \|y_{\tau}\| \leq R. \]
Since \( \mathcal{u} \) is an optimal control, we have
\[ J_{\tau}(\mathcal{y}, \mathcal{u}) \leq J_{\tau}(y, 0), \]  
where \( y \) is a strong solution of (33), (34) with control \( u = 0 \). From (55) we have
\[ \int_{\tau}^{+\infty} e^{-2\kappa t} \|\mathcal{y}(t)\|^2 dt + \gamma \int_{\tau}^{+\infty} \|\mathcal{u}(t)\|^2 dt \leq \int_{\tau}^{+\infty} e^{-2\kappa t}(R^2 e^{-2\delta_2(t-\tau)} + c_2) dt \leq \frac{1}{2\kappa}(R^2 + c_2)e^{-2\kappa \tau}. \]  
Then
\[ \|\mathcal{y}(t + \tau)\|^2 \leq c_3(R^2 + 1). \]  
In virtue of the Principle of Optimality (see the proof of Lemma 1) we get that
\[ J_t(\mathcal{y}, \mathcal{u}) \leq J_t(y, 0), \ \forall t \geq \tau, \]  
where \( y \) is a strong solution of (33)-(34) on \( (t, +\infty) \) with control \( u \equiv 0 \) and initial datum \( y(t) = \mathcal{y}(t) \).

From (58) we obtain
\[ \int_{t}^{+\infty} \|\mathcal{u}(p)\|^2 dp \leq \frac{1}{\gamma} \int_{t}^{+\infty} e^{-2\kappa p} \|y(p)\|^2 dp \leq \frac{1}{\gamma} \int_{t}^{+\infty} e^{-2\kappa p} \|\mathcal{y}(p)\|^2 dp \leq c_4(R^2 + 1)e^{-2\kappa t}. \]  
Since in virtue of (56)
\[ \int_{\tau}^{t+\tau} e^{-2\delta_2(t+\tau-p)} \|u(p)\|^2 dp \leq e^{-2\delta_2 t} c_5(R^2 + 1), \]  
Therefore, the multivalued semiflow
\[ G(t, y) = \bigcup_{\tau \geq 0} U(t + \tau, \tau, y), \]  
where
\[ U(t, \tau, y_{\tau}) = \{ y(t) \mid y \text{ is an optimal trajectory of (33)-(36)} \}, \]
is correctly defined.
and in virtue of (59)
\[
\int_{\tau+t}^{\tau+2t} e^{-2\kappa(t+\tau)}\|u(p)\|^2 dp \leq \int_{\tau+t}^{\tau+t+1} \|u(p)\|^2 dp + e^{-2\kappa_2} \int_{\tau+t+1}^{\tau+2t-1} \|u(p)\|^2 dp + \ldots \\
\leq c_4(R^2 + 1) e^{-2\kappa(\tau+\frac{t}{2})} \frac{1}{1-e^{-2\kappa_2}},
\]
from (54), (60), (61) we have the existence of \( T = T(R) \) such that
\[
\|y(t+\tau)\| \leq 1 + 2c_2, \quad \forall t \geq T(R), \quad \tau \geq 0,
\]
that implies the fulfillment of condition (23).

Let us prove the asymptotic compactness for \( G \). Due to dissipativity for \( t \to \infty \)
\[
\xi_n \in G(t_n, B) \subset G(1, B_0).
\]
Thus,
\[
\xi_n \in U(1+\tau_n, \tau_n, \eta_n), \quad \text{where } \tau_n \geq 0, \quad \eta_n \overset{w}{\to} \eta \text{ in } H, \quad \|\eta_n\| \leq R_0,
\]
being \( R_0 \) the radius if the absorbing ball \( B_0 \). Then \( \xi_n = \xi_n(1+\tau_n), \) \( \xi_n(\tau_n) = \eta_n, \) where \( \{\xi_n, \eta_n\} \) is an optimal pair of (33)-(36) with initial datum \( (\tau_n, \eta_n) \).

Let us put
\[
y_n(t) := \xi_n(t+\tau_n), \quad u_n(t) := \eta_n(t+\tau_n).
\]
Then \( y_n(\cdot) \) is a strong solution of (33), (34) with \( y_n(0) = \eta_n \) and control function \( u_n(t) \). From inequality (56) we deduce that
\[
\{u_n\}_{n \geq 1} \text{ is bounded in } L^2(0,2; H).
\]
Then the properties of problem (33), (34) (see [25] or [23]) guarantee that up to subsequence
\[
y_n \to y \text{ in } C([\delta,2]; H) \quad \forall 0 < \delta < 2.
\]
In particular, \( y_n(1) = \xi_n \to y(1) \) in \( H \) and the theorem is proved.

Unfortunately, whether the uniform attractor \( A \) is invariant with respect to the semiflow \( G \) or not is still an open question. However, we can say something about the properties of the set \( A \) using the semiflow generated by (33) with \( u \equiv 0 \).

More precisely, let us consider the problem
\[
\frac{dy}{dt} + Ay \in F(y), \\
y|_{t=0} = y_0 \in H.
\]
It is known (see [25] or [23]) that under conditions (52) problem (63) generates the dissipative m-semiflow \( V : \mathbb{R}_+ \times H \to P(H) \) given by
\[
V(t, y_0) = \{y(t) \mid y(\cdot) \text{ is solution of (63)}\}, \quad \forall t \geq 0, \quad y_0 \in H,
\]
which has the invariant global attractor \( \Theta \).

**Theorem 2.** Let the conditions (30)-(32), (52) be fulfilled, \( A \) be a uniform attractor of the optimal control problem (33)-(36) and let \( \Theta \) be the global attractor of problem (63). Then the following properties hold:
\[
A \subset V(t, A), \quad \forall t \geq 0; \quad (64)
\]
\[
A \subset \Theta. \quad (65)
\]
Proof. Let us prove (64). If \( \xi \in \mathcal{A} \), then by (25) \( \xi = \lim_{n} \xi_n \), where \( t_n \to +\infty \), \( \xi_n \in G(t_n, B_0) \). Since

\[
\xi_n \in U(t_n + \tau_n, \tau_n, B_0) = U(t + t_n + \tau_n - t, t_n + \tau_n - t, U(t_n + \tau_n - t, \tau_n, B_0)),
\]

for \( s_n = t_n + \tau_n - t \to +\infty \) we have

\[
\xi_n = \overline{y}_n(t + s_n), \quad \overline{y}_n(s_n) = \eta_n \to \eta \in \mathcal{A},
\]

where \( \{\overline{y}_n, \overline{u}_n\} \) is an optimal pair of (33)-(36) with initial datum \((s_n, \eta_n)\).

Let us put

\[
\hat{y}_n(p) := \overline{y}_n(p + s_n), \quad \hat{u}_n(p) := \overline{u}_n(p + s_n).
\]

Then \( \hat{y}_n \) is a solution of the following problem

\[
\frac{dy}{dt} + Ay \in F(y) + \hat{u}_n,
\]

\[
y|_{t=0} = \eta_n.
\]

From (56)

\[
u_n \to 0 \text{ in } L^2(0, T; H), \quad \forall \ T > t.
\]

Therefore from the strong convergence \( \eta_n \to \eta \) we deduce

\[
\hat{y}_n \to \hat{y} \text{ in } C([0, T]; H),
\]

where \( \hat{y} \) is a solution of the problem (63) with \( y(0) = \eta \). Therefore,

\[
\xi_n = \hat{y}_n(t) \to \xi = \hat{y}(t) \in V(t, A)
\]

and (64) is proved.

Since

\[
\text{dist}(V(t, A), \Theta) \to 0, \quad t \to \infty,
\]

from (64) we obtain (65). The theorem is proved.

\[ \square \]

**Remark 4.** It would be interesting to study the robustness of the global attractor when we vary the target function \( J_\tau(y, u) \), that is, if we consider functions \( J_\varepsilon(y, u) \) converging to \( J_\tau \) in some sense as \( \varepsilon \to 0 \).

**Acknowledgments.** This paper is dedicated to the memory of Professor Valery S. Melnik, on the tenth anniversary of his passing away. As his former students, we would like to express our sincere gratitude to him for his invaluable guidance and his always altruistic support. We will keep him forever in our hearts.

We would like to thank the anonymous referees for their useful remarks.

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Received February 2018; revised June 2018.

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