Cotype and summing properties in Banach spaces

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Abstract

It is well known in Banach space theory that for a finite dimensional space $E$ there exists a constant $c_E$, such that for all sequences $(x_k)_k \subset E$ one has

$$\sum_k \|x_k\| \leq c_E \sup_{\varepsilon_k \pm 1} \left\| \sum_k \varepsilon_k x_k \right\|.$$ 

Moreover, if $E$ is of dimension $n$ the constant $c_E$ ranges between $\sqrt{n}$ and $n$. This implies that absolute convergence and unconditional convergence only coincide in finite dimensional spaces. We will characterize Banach spaces $X$, where the constant $c_E \sim \sqrt{n}$ for all finite dimensional subspaces. More generally, we prove that an estimate $c_E \leq cn^{1-\frac{1}{q}}$ holds for all $n \in \mathbb{N}$ and all $n$-dimensional subspaces $E$ of $X$ if and only if the eigenvalues of every operator factoring through $\ell_\infty$ decrease of order $k^{-\frac{1}{q}}$ if and only if $X$ is of weak cotype $q$, introduced by Pisier and Mascioni. We emphasize that in contrast to Talagrand’s equivalence theorem on cotype $q$ and absolutely $(q,1)$-summing spaces this extends to the case $q = 2$. If $q > 2$ and one of the conditions above is satisfied one has

$$\left( \sum_k \|x_k\|^q \right)^{\frac{1}{q}} \leq C^{1+l} \left( 1 + \log_2((1 + \log_2 n)^{\frac{1}{q}}) \right) \mathbb{E} \left\| \sum_k \varepsilon_k x_k \right\|$$

for all $n,l \in \mathbb{N}$ and $(x_k)_k \subset E$, $E$ a $n$ dimensional subspace of $X$. In the case $q = 2$ the same holds if we replace the expected value by the supremum.

Introduction

In Banach spaces unconditional convergence and absolute convergence only coincide for finite dimensional spaces. More precisely, a constant $0 < c < \infty$ such that

$$\sum_k \|x_k\| \leq c \sup_{\varepsilon_k \pm 1} \left\| \sum_k \varepsilon_k x_k \right\|$$

holds for all sequences $(x_k)_k \subset X$ if and only if $X$ is of finite dimension. The best possible constant $c$ is called the absolutely 1-summing norm of the identity of $X$ ($\pi_1(Id_X)$). This notion was originally introduced by Grothendieck under the name ‘semi-integrale à droite’. But Orlicz discovered before that unconditional converging series are at least absolutely 2-summing, provided the underlaying spaces is $L_p$, $1 \leq p \leq 2$. That’s why this property is called Orlicz property. It is best possible, since Dvoretzky’s theorem ensures that for each $\delta > 0$, $n \in \mathbb{N}$ there are elements $(x_k)_{k=1}^n$ in an infinite dimensional Banach space $X$ which satisfies

$$\sum_{1}^{n} \|x_k\| \geq (1-\delta) \sqrt{n} \quad \text{and} \quad \sup_{|\alpha_k| \leq 1} \left\| \sum_{1}^{n} \alpha_k x_k \right\| \leq 1.$$ 

We will study spaces where this estimate is optimal or a certain growth rate occurs. This is contained in the following
Theorem 1 Let $2 \leq q < \infty$. For a complex Banach space $X$ the following properties are equivalent.

i) There exists a constant $c > 0$ such that for all $n \in \mathbb{N}$ and all $n$ dimensional subspace $E \subset X$ one has

$$\sum_k \|x_k\| \leq c n^{1 - \frac{1}{q}} \sup_{\varepsilon_k = \pm 1} \left\| \sum_1^n \varepsilon_k x_k \right\|$$

for all sequences $(x_k)_k \subset E$.

ii) $X$ is of weak cotype $q$, in other words there exists a constant $0 < c_2 < \frac{1}{8e}$ such that for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n$

$$\sum_{k=1}^n |\langle x_k, x^* \rangle|^2 \leq \|x^*\|$$

and $\|x_k\| \geq c_2$ for $k = 1, \ldots, n$

implies $E \left\| \sum_1^n \varepsilon_k x_k \right\| \geq c_2 n^{\frac{1}{q}}$.

iii) There exists a constant $c_3$ such that for all operators $T : X \to X$ which factors through $\ell_\infty$, i.e. $T = SR$, $R : X \to \ell_\infty$ and $S : \ell_\infty \to X$ one has

$$\sup_{n \in \mathbb{N}} n^{\frac{1}{q}} |\lambda_n(T)| \leq c_3 \|S\| \|R\|,$$

where $(\lambda_n(T))_{n \in \mathbb{N}}$ denotes the eigenvalue sequence of $T$ in non-increasing order according to there multiplicity.

If $q > 2$ and one of the conditions above are satisfied there is a constant $C$ such that

$$\left( \sum_k \|x_k\|^q \right)^{\frac{1}{q}} \leq C^{1+l} (\max\{1, \log_2\}(l)(1 + \log_2 n)^{\frac{1}{q}}) E \left\| \sum_1^n \varepsilon_k x_k \right\|$$

holds for all $n, l \in \mathbb{N}$ and $(x_k)_k \subset E$, $E$ a $n$-dimensional subspace of $X$.

This theorem is somehow at end of a fruitful investigation of summing and cotype properties in Banach spaces. Starting point is certainly the pioneering work of Maurey and Pisier [MP]. In their paper they obtained the equivalence in terms of the cotype index. Using deep methods from the theory of stochastic processes, the so-called concentration phenomena, Talagrand improved Maurey/Pisier’s result.

Theorem 2 (Talagrand) Let $2 < q < \infty$ and $X$ a Banach space the following are equivalent

1. The identity of $X$ is absolutely $(q, 1)$-summing, i.e. there is a constant $c_1 > 0$ such that

$$\left( \sum_k \|x_k\|^q \right)^{\frac{1}{q}} \leq c_1 \sup_{\varepsilon_k = \pm 1} \left\| \sum_k \varepsilon_k x_k \right\|,$$

for all $(x_k)_k \subset X$.

2. $X$ is of cotype $q$, i.e. there is a constant $c_2$ such that

$$\left( \sum_k \|x_k\|^q \right)^{\frac{1}{q}} \leq c_2 E \left\| \sum_k \varepsilon_k x_k \right\|,$$

for all $(x_k)_k \subset X$.

In Talagrand’s theorem the case $q = 2$ is not included and for a good reason:
Theorem 3 (Talagrand) There is a symmetric sequence space which has the Orlicz property but is not of cotype 2.

Nevertheless, in the proof of the main theorem, also in the case $q = 2$, we heavily use the probabilistic machinery established by Talagrand. For $q > 2$ the modified cotype condition in ii) can be replaced by the usual cotype condition restricted to vectors of equal norm. This is not possible for $q = 2$, since equal norm cotype 2 is the same as cotype 2. But this modified condition turns out to be a basic tool for the application of the probabilistic method.

By the way, using the main theorem Talagrand’s example yields a symmetric sequence space which is of weak cotype 2 but not of cotype 2. This is impossible in the category of weak Hilbert spaces, since every symmetric weak Hilbert space is actually a Hilbert space.

In this setting the 'weak' theory is more adapted to prove abstract characterization theorems than the 'strong' theory. This is also true for eigenvalue estimates. It happens quite often that eigenvalue estimates for weak $\ell_p$ spaces are easier to prove than eigenvalue estimates for the spaces $\ell_p$ themselves. A useful tool in this context is notion of Weyl numbers. The connection between Weyl numbers and weak cotype was actually discovered by Mascioni [MAS]. We should note that the equivalence between eigenvalue estimates and summing properties can be proved using a generalization of Maurey’s theorem, provided $q > 2$. This approach was pursued in [1], [2].

Finally we come to the estimate with the iterated logarithm. This will be investigated in chapter 2 and is based on the introduction of optimal cotype spaces. The idea is to measure cotype and summing conditions in terms of maximal, symmetric sequence spaces. It turns out that a certain self concavity is a generalization of the submultiplicativity conditions which occurred in the basic paper of Maurey and Pisier. This broader framework turns out to be more natural to describe cotype conditions of Orlicz spaces, although we will not start this investigation here. In order to find the best possible eigenvalue behavior of operators factoring through $\ell_\infty$ we will also proof the main theorem in a slightly more general setting.

Preliminaries

In what follows $c_0, c_1, \ldots$ always denote universal constants. We use standard Banach space notation. In particular, the classical spaces $\ell_q$ and $\ell_q^n$, $1 \leq q \leq \infty$, $n \in \mathbb{N}$, are defined in the usual way. We will also use the Lorentz spaces $\ell_{pq}$ where $1 \leq p, q \leq \infty$. This space consists of all sequences $\sigma \in \ell_\infty$ such that

$$\|\sigma\|_{pq} := \left(\sum_n \left(\frac{1}{n^p} \sigma_n^*\right)^q \left(\frac{1}{n^q} \sigma_n^*\right)^{-1}\right)^{\frac{1}{q}} < \infty.$$ 

For $q = \infty$ the needed modification is given by

$$\|\sigma\|_{p\infty} := \sup_{n \in \mathbb{N}} \frac{1}{n^p} \sigma_n^* < \infty.$$ 

Here $\sigma^* = (\sigma_n^*)_{n \in \mathbb{N}}$ denotes the non-increasing rearrangement of $\sigma$. More generally, for a non decreasing sequence $(g(n))_{n \in \mathbb{N}}$ with $g(1) = 1$ we denote by $\ell_{g, \infty}$ the space of sequences $\sigma$ such that

$$\|\sigma\|_{g, \infty} := \sup_n g(n) \sigma_n^* < \infty.$$ 

The standard reference on operator ideals is the monograph of Pietsch [P1]. The ideals of linear bounded operators, finite rank operators, integral operators are denoted by $\mathcal{L}$, $\mathcal{F}$.
Let \( 1 \leq q \leq p \leq \infty \) and \( n \in \mathbb{N} \). For an operator \( T \in \mathcal{L}(X, Y) \) the pq-summing norm of \( T \) with respect to \( n \) vectors is defined by

\[
\pi_{pq}^n(T) := \sup \left\{ \left( \frac{1}{n} \sum_{k=1}^{n} \|Tx_k\|^p \right)^{1/p} \sup_{\|x^*\|_X \leq 1} \left( \frac{1}{n} \sum_{k=1}^{n} |\langle x_k, x^* \rangle|^q \right)^{1/q} \leq 1 \right\}.
\]

An operator is said to be absolutely pq-summing, short pq-summing, (\( T \in \Pi_{pq}(X, Y) \)) if

\[
\pi_{pq}(T) := \sup_n \pi_{pq}^n(T) < \infty.
\]

Then \((\Pi_{pq}, \pi_{pq})\) is a maximal and injective Banach ideal (in the sense of Pietsch). As usual we abbreviate \((\Pi_q, \pi_q) := (\Pi_{pq}, \pi_{pq}).\) For further information about absolutely pq-summing operators we refer to the monograph of Tomczak-Jaegermann [TOJ].

In the following \((\varepsilon_k)_{k \in \mathbb{N}}, (g_k)_{k \in \mathbb{N}}\) denotes a sequence of independent normalized Bernoulli, gaussian variables. A Banach space \( X \) is of Rademacher, gaussian cotype \( q \) if there exists a constant \( c > 0 \) such that for all sequences \((x_k)_k \subset X\) one has

\[
\left( \sum_k \|x_k\|^q \right)^{\frac{1}{q}} \leq c \mathbb{E} \left( \sum_k \varepsilon_k x_k \right) \quad \text{or} \quad \left( \sum_k \|x_k\|^q \right)^{\frac{1}{q}} \leq c \mathbb{E} \left( \sum_k g_k x_k \right) \quad \text{resp.}
\]

Here and in the following \(\mathbb{E}\) means expected value. The best possible constant will by denoted by \(Rc_q(X) := Rc_q(id_X), c_q(X) := c_q(Id_X),\) respectively. If this definition is restricted to \(n\) vectors we write \(Rc_q^n, c_q^n,\) respectively. As usual we will use the abbreviation

\[
\ell(u) := \sup_n \left( \mathbb{E} \left( \sum_k \|g_k u(e_k)\|^2 \right) \right)^{\frac{1}{2}}
\]

for all operator \( u \in \mathcal{L}(\ell_2, X). \) Here \((e_k)_k\) is the sequence of unit vectors. By the rotation invariance this norm is invariant by orthogonal transformation of this basis.

Finally some s-numbers are needed. For an operator \( T \in \mathcal{L}(E, F) \) and \( n \in \mathbb{N} \) the \( n \)-th approximation number is defined by

\[
a_n(T) := \inf \{ \|T - S\| \mid \text{rank}(S) < n \},
\]

whereas the \( n \)-th Weyl number is given by

\[
x_n(T) := \sup \{ a_n(Tu) \mid u \in \mathcal{L}(\ell_2, E) \text{ with } \|u\| \leq 1 \}.
\]

Let \( s \in \{a, x\}. \) By \( \mathcal{L}_{pq}^{(s)}, \mathcal{L}_g^{(s)} \) we denote the ideal of operators \( T \) such that \((s_n(T))_n \in \ell_{pq}^s, (s_n(T))_n \in \ell_{g, \infty} \) with the associated quasi-norms

\[
\ell_{pq}^{(s)}(T) := \|(s_n(T))_n\|_{\ell_{pq}^s} \quad \text{and} \quad \ell_{g, \infty}^{(s)}(T) := \|(s_n(T))_n\|_{\ell_{g, \infty}}.
\]

\section{Proof of the main theorem}

We will proof our main theorem in a little bit broader framework. For the eigenvalue estimate we allow a certain growth rate \((g(n))_{n \in \mathbb{N}}.\) Certainly some reasonable conditions are required.
S) i) $g(1) = 1$ and $(g(n))_{n \in \mathbb{N}}$ non decreasing.
   ii) There exists a constant $S_2$ such that for all $1 \leq k \leq n$
   
   $$g(n) \leq S_2 \frac{n}{k} g(k).$$

   iii) The space $\ell_{g,\infty}$ is equivalent to a normed space. The equivalence constant is denoted by $S_3$.
   iv) There is a constant $S_4$ such that for all $n \in \mathbb{N}$
   
   $$\sum_{1}^{n} \frac{1}{g(k)} \leq S_4 \frac{n}{g(n)}.$$

L) There exists a $0 < t < \infty$ such that $n^{\frac{1}{t}} \leq L_t g(n)$.

M) There exists a natural number $r \geq 2$ with $t \leq r$ and a constant $M_r$ such that

$$g(k^{2r}) \leq M_r g(k^r) g(k^r).$$

Condition iii) and iv) are actually equivalent but there is no need to go into further details. The last condition M) is clearly satisfied for supermultiplicative sequences.

**Theorem 1.1** Let $g$ be a sequence which satisfies the conditions $Si) - iv), L), and M) and set $D := 2^{2}e^{2}S_2^2$. For a complex Banach space $X$ the following properties are equivalent.

i) There exists a constant $c_1 > 0$ such that for all $n$-dimensional subspaces $E \subset X$ of $X$ one has

$$\pi_1(Id_X) \leq c_1 \frac{n}{g(n)}.$$

i') There exists a constant $c'_1 > 0$ such that for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$ one has

$$\sum_{1}^{n} \|x_i\| \leq c'_1 \frac{n}{g(n)} \sup_{\varepsilon_i = \pm 1} \left\| \sum_{1}^{n} \varepsilon_i x_i \right\|.$$

ii) $X$ is of weak cotype $G$ or there exists a constant $c_2 > 0$ such that for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n$

$$\sum_{1}^{n} |\langle x_k, x^* \rangle|^{2} \leq \|x^*\|$$

and $\|x_k\| \geq \frac{1}{r}$ for $k = 1, \ldots, n$

$$\Rightarrow \mathbb{E} \left\| \sum_{1}^{n} \varepsilon_k x_k \right\| \geq \frac{1}{c_2} g(n).$$

iii) There exists a constant $c_3$ such that for all operators $T : X \to X$ which factors through $\ell_\infty$, i.e. $T = SR$, $R : X \to \ell_\infty$ and $S : \ell_\infty \to X$ one has

$$\sup_{n \in \mathbb{N}} g(n) |\lambda_n(T)| \leq c_3 \|S\| \|R\|,$$

where $(\lambda_n(T))_{n \in \mathbb{N}}$ denotes the eigenvalue sequence of $T$ in non-increasing order according to there multiplicity.
For the proof of the main result we will closely follow Talagrand’s work. The main difference occurs when we establish a situation where the concentration phenomena can be applied. Let us recall that an operator \( T \in \mathcal{L}(X, Y) \) is of weak cotype \( g \) (\( T \in \mathcal{W}_{g}(X, Y) \)), if there is a constant \( c > 0 \) such that
\[
\sup_{k} g(k) \ a_{k}(Tu) \leq c \ell(u) .
\]
The norm \( wc_{g}(T) \) is defined as the infimum over all \( c \) satisfying the inequality above. The following lemma is well known and at origin of the so called weak theory, see [PSW]. Nevertheless, we give a proof in order to check the constants. [B

**Lemma 1.2** An operator \( T \in \mathcal{L}(X, Y) \) is of weak cotype \( g \) if and only if there is a \( 0 < \delta < 1 \) and a constant \( C_{\delta} > 0 \) such that for all \( n \in \mathbb{N} \) and \( u \in \mathcal{L}(\ell_{2}^{n}, X) \) one has
\[
g(n) \ a_{[\delta n]}(Tu) \leq C_{\delta}(T) \ell(u) .
\]
Moreover, we have the following relation for the constants
\[
\frac{\delta}{2S_{2}} C_{\delta}(T) \leq wc_{g}(T) \leq e^{\frac{\delta}{2}} S_{2} (1 - \delta)^{-\frac{1}{2}} C_{\delta}(T) .
\]
**Proof:** The first estimate of \( C_{\delta} \) by \( wc_{g}(T) \) is obvious. For the second let \( u \in \mathcal{L}(\ell_{2}, X) \), \( v \in \Pi_{2}(Y, \ell_{2}) \) and \( n \in \mathbb{N} \). By Schmidt decomposition there is subspace \( H \subset \ell_{2} \) with \( \deg H = n \) such that \( a_{n}(vTu) = a_{n}(vTu_{H}) \). We set \( m := n - [\delta n] \) provided \( \delta n \geq 1 \) and \( m := n \) else. Using the multiplicativity of the Weyl numbers and the Weyl number estimate for the \( 2 \)-summing norm we obtain
\[
a_{n}(vTu) = a_{n}(vTu_{H}) \leq a_{[\delta n]}(Tu_{H}) \ x_{m}(v) \leq C_{\delta} \ g(n)^{-1} \ \ell(u) \ m^{-\frac{1}{2}} \ \pi_{2}(v)
\]
\[
\leq C_{\delta} \ (1 - \delta)^{-\frac{1}{2}} \ (g(n)n^{\frac{1}{2}})^{-1} \ \ell(u) \ \pi_{2}(v)
\]
From [DJ1, ...] we deduce
\[
wc_{g}(T) \leq S_{2} e^{\frac{\delta}{2}} (1 - \delta)^{-\frac{1}{2}} C_{\delta} .
\]
Now we can prove the proposition which initialize Talagrand’s machinery.

**Proposition 1.3** Let \( D := 2e^{\frac{\delta}{2}} S_{2}^{2} \) and \( T \in \mathcal{L}(X, Y) \). If there is a constant \( c > 0 \) such that for all \( n \in \mathbb{N} \), all vectors \( x_{1}, .., x_{n} \) the condition
\[
\sum_{1}^{n} |\langle y^{*}, Tx_{j} \rangle|^{2} \leq \| y^{*} \| \text{ and } \| Tx_{j} \| \geq \frac{1}{D} \text{ for all } j = 1, .., n
\]
implies
\[
\left( \mathbb{E} \left\| \sum_{1}^{n} g_{j} x_{j} \right\|^{2} \right)^{\frac{1}{2}} \geq \frac{1}{c} \ g(n) ,
\]
then \( T \) is of weak cotype \( g \) with
\[
wc_{g}(T) \leq c .
\]
**Proof:** There is no loss of generality to assume $T$ of finite rank. Indeed, if we can prove the assertion for all $T|_E$, $E$ a finite dimensional subspace of $X$ we obtain

$$\text{wc}_q(T) \leq \sup_{E \text{f.d.}} \text{wc}_q(T|_E) \leq c.$$

If $T$ is of finite rank we deduce from the lemma 1.2 above that there is a positive real number $A$ with

$$\frac{\text{wc}_q(T)}{2a} < A < C(T),$$

where $a := \sqrt{2}e^2S_2$. By definition there is an $m \in \mathbb{N}$ and an operator $u \in \mathcal{L}(\ell^m_2, X)$ such that

$$a_{\frac{m}{2}}(Tu) > \frac{1}{16S_2a} \quad \text{and} \quad \ell(u) \leq \frac{g(m)}{16S_2aA}.$$

We define $n := \lceil \frac{m}{4} \rceil$. By definition of the weak cotype $g$ we have

$$g(n) a_n(Tu) \leq \text{wc}_q(T) \ell(u) \leq \frac{\text{wc}_q(T) g(m)}{16S_2aA} < \frac{1}{8S_2} g(m).$$

Let us first assume $m \geq 4$, hence $n \geq m$. Since the approximation numbers coincide with the Gelfand numbers for operators on Hilbert spaces, there exists a subspace $H \subset \ell^m_2$ with $\text{codim}(H) < n$ such that

$$\|Tu|_H\| \leq \frac{g(m)}{8S_2} a_n(Tu) \leq \frac{g(m)}{8S_2} \leq 1.$$

From elementary properties of the approximation numbers we deduce with $\text{codim}(H) < n$

$$\frac{1}{16S_2a} < a_{\frac{m}{4}}(Tu) \leq a_{2n}(Tu) \leq a_n(Tu|_H).$$

By a lemma probably due to Lewis, [P1], there is an orthonormal sequence $(w_j)_1^n \in H$ such that

$$\|Tu(w_j)\| \geq \frac{1}{16S_2a} \quad \text{for all} \quad j = 1, \ldots, n.$$

We define $x_j := u(w_j)$ and $z := \sum_1^n e_j \otimes x_j : \ell^m_2 \to X$. Since the system $w_j$ is an orthonormal sequence we get $\|Tz\| \leq \|Tu|_H\| \leq 1$. By the properties of the $\ell$ norm we deduce

$$\left(\mathbb{E} \sum_1^n g_j x_j^2\right)^{\frac{1}{2}} = \ell(z) \leq \ell(u) \leq \frac{g(m)}{16S_2aA} \leq \frac{g(n)}{2aA}.$$

By assumption this implies $2aA \leq c$ and therefore

$$\text{wc}_q(T) \leq 2aA \leq c.$$

If $m \leq 4$ we see that

$$\|Tu\| > \frac{1}{16S_2a} \quad \text{and} \quad \ell(u) \leq \frac{1}{4aA}.$$
From the weak cotype $g$ definition we have
\[ \|Tu\| \leq wc_g(T) \ell(u) < \frac{g(m)}{S_2} \leq \frac{1}{2}. \]

Let $h$ be a norm 1 vector were the norm is attained and let $x_1 := 2u(h)$. From $\|Tx_1\| \leq 1$, $\|Tx_1\| \geq \frac{1}{8S_2a}$ and
\[ \left(\mathbb{E}\|g_1x_1\|^2\right)^{\frac{1}{2}} = 2 \ell(u|_{\text{span}(h)}) \leq \frac{1}{2aA} \]
we infer $2aA \leq c$ which implies the assertion. \hspace{8em} \square

A precise calculation in the case $g(k) = k^{\frac{1}{q}}$, $2 \leq q < \infty$ shows that we can take $D = 8e$.

Although the condition in the proposition above is a little bit technical it is nonetheless equivalent to the usual definition of weak cotype $g$.\[B\]

**Proposition 1.4** If $T \in \mathcal{L}(X, Y)$ is of weak cotype $g$, $0 < \rho < 1$ and $x_1, ..., x_n \in X$ satisfying
\[ \sup_{y^* \in B_{Y^*}} \sum_{1}^{n} |\langle y^*, Tx_j \rangle|^2 \leq 1 \quad \text{and} \quad \|Tx_j\| \geq \rho \quad \text{for} \quad j = 1, ..., n \]
then one has
\[ \rho^4 g(n) \leq S_2 2048 wc_g(T) \left( \mathbb{E}\left\| \sum_{1}^{k} g_jx_j \right\| \right)^{\frac{1}{2}}. \]

**Proof:** We choose functionals $(y_j^*)^n \subset B_{Y^*}$ with
\[ \rho \leq \|Tx_j\| = \langle y_j^*, Tx_j \rangle \]
for $j = 1, ..., n$. We define the operators
\[ u := \sum_{1}^{n} e_j \otimes x_j : \ell_2^n \rightarrow X \quad \text{and} \quad v := \sum_{1}^{n} y_j^* \otimes e_j : Y \rightarrow \ell_\infty^n. \]

By definition of $\|v\| \leq 1$ and by assumption $\|Tu\| \leq 1$. On the other hand fix $1 \leq k \leq n$. Using an a well known estimate of the 2-summing norm by approximation numbers, \[P\] we get
\[ \rho \sqrt{n} \leq \pi_2(vTu) \leq 2 \sum_{1}^{n} \frac{a_j(vTu)}{\sqrt{j}} \leq 2 \sum_{1}^{k-1} \frac{1}{\sqrt{j}} + 2 \sum_{1}^{n} \frac{a_k(vTu)}{\sqrt{k}} \leq 4\sqrt{k-1} + 2 \frac{n}{\sqrt{k}} a_k(Tu). \]

If we choose $k-1 \leq \frac{\rho^2}{64} n \leq k$ we deduce
\[ \rho = \frac{2}{\sqrt{n}} \frac{\rho \sqrt{n}}{2} \leq \frac{2}{\sqrt{n}B} \frac{n}{\sqrt{k}} a_k(Tu) \leq \frac{32}{\rho} a_k(Tu). \]

Finally we get
\[ g(n) \leq \frac{S_2 64}{\rho^2} g(k) \leq \frac{S_2 2048}{\rho^4} g(k) \frac{\rho^2}{32} \leq \frac{S_2 2048}{\rho^4} g(k) a_k(Tu) \leq S_2 \frac{2048}{\rho^4} wc_g(T) \ell(u). \]
Now we will briefly proof the easy implications of our theorem.

\textbf{ii) $\Rightarrow$ iii)} Let $T = RS$, where $R : X \to \ell_\infty$ and $S : \ell_\infty \to X$. By proposition [1.4], $X$ is of weak cotype $g$ and in particular of finite cotype by condition $L)$. From Maurey’s theorem, see [TOJ], every operator $S : \ell_\infty \to X$ is $p$-summing for some $p < \infty$. Using the gaussian version of Kintchine’s inequality we get for all $u : t_2 \to \ell_\infty$

\[ \sup_k g(k) a_k(Su) \leq wc_g(X) \ell(Su) \leq wc_g(X) \sqrt{p} \pi_p(S) \|u\| \]

\[ \leq wc_g(X) \sqrt{p} c(p, X) \|S\| \|u\|. \]

This means $\ell_{g, \infty}(S) \leq \sqrt{p} c(p, X) wc_g(X) \|S\|$ for some constant $c(p, X)$. (Actually this part of the proof is due to Mascioni.) To conclude we only have to note that $\mathcal{L}_{g, \infty}$ is of eigenvalue type $\ell_{g, \infty}$ by the generalized Weyl’s inequality, see [PII], together with condition $S_4$. Hence we get

\[ \sup_k g(k) |\lambda_k(T)| \leq c_1 \ell_{g, \infty}(T) \leq c_1 \|R\| \ell_{g, \infty}(S) \leq c_1 \sqrt{p} c(p, X) wc_g(X) \|R\| \|S\|. \quad \square \]

\textbf{iii) $\Rightarrow$ i)} Let $E$ be a $n$-dimensional subspace of $X$ and $(x_k)_1^N \subset E$. We choose functionals $x_1^*, \ldots, x_N^*$ in $B_{X^*}$ such that

\[ \|x_i\| = \langle x_i, x_i^* \rangle. \]

We define the operators

\[ v := \sum_{i=1}^N x_i^* \otimes e_i : X \to \ell_\infty^N \quad \text{and} \quad u := \sum_{i=1}^N e_i \otimes x_i : \ell_\infty^N \to X. \]

By definition of $v$ we have $\|v\| \leq 1$ and

\[ \|u\| = \sup_{\|x_i\| \leq 1} \left\| \sum_{i=1}^N \alpha_i x_i \right\|. \]

Therefore we conclude

\[ \sum_{i=1}^N \|Tx_i\| = tr(uv) \leq \sum_{i=1}^n |\lambda_i(uv)| \]

\[ \leq \sum_{i=1}^n g(k)^{-1} \sup_k g(k) |\lambda_i(uv)| \]

\[ \leq S_4 \frac{n}{g(n)} c_3 \|v\| \|u\| \]

\[ \leq 4 S_4 c_3 \frac{n}{g(n)} \sup_{\|\alpha_i\| \leq 1} \left\| \sum_{i=1}^n \alpha_i x_i \right\|. \quad \square \]

The implication $i) \Rightarrow i')$ follows obviously from the contraction principle, see [LTII]

\[ \sup_{\|\alpha_k\| \leq 1} \left\| \sum_{i=1}^n \alpha_k x_k \right\| \leq 4 \sup_{\|\varepsilon_k\| \leq 1} \left\| \sum_{i=1}^n \varepsilon_k x_k \right\| \]

and the trivial observation that $n$ elements are contained in a $n$ dimensional subspace of $X$. \quad \square
Till the end of this chapter we are concerned with the proof of the implication \( i' \Rightarrow ii \).
Assuming \( i' \) we first observe

\[
\|\|Tx_k\||\|_{g, \infty} \leq c_1 \sup_{x^* \in B_{X^*}} \sum_k |\langle x^*, x_k \rangle|.
\]

Indeed this is a classical argument. We can assume \( \|Tx\| \) non increasing and fix \( k \in \mathbb{N} \). Then we have

\[
k \|Tx_k\| \leq \sum_1^k \|Tx_i\| \leq c'_1 \frac{k}{g(k)} \sum_1^k \|\varepsilon_i x_i\|
\]

\[
\leq c'_1 \frac{k}{g(k)} \sup_{x^* \in B_{X^*}} \sup_{|\alpha_i| \leq 1} |\langle x^*, \sum_1^k \alpha_i x_i \rangle|.
\]

The best possible constant in (*) will be denoted by \( H \). Let us note that for vectors \( (x_i)_1^n \) with \( \|x_i\| \geq 1 \) we certainly have

\[
g(n) \leq H \sup_{x^* \in B_{X^*}} \sum_1^n |\langle x^*, x_i \rangle|.
\]

With this observation the following two lemmata from Talagrand can be formulated in our setting. The first one is a lemma which allows to regroup a certain collection of disjoint blocs.

**Lemma 1.5 (Talagrand: Lemma 4.2.)** There exists a constant \( K > 0 \) with the following property. Consider disjoint subsets \( I_1, \ldots, I_k \) of \( \{1, \ldots, n\} \) with union \( I \). Let \( \alpha > 0 \) such that for all \( 1 \leq j \leq k \)

\[
\mathbb{E} \left\| \sum_{i \in I_j} g_i x_i \right\| \geq \alpha \quad \text{and} \quad \sqrt{k} \leq \frac{\alpha}{2S_3KH} g(k).
\]

Then one has

\[
\mathbb{E} \left\| \sum_{i \in I} g_i x_i \right\| \geq \frac{\alpha}{2S_3H} g(k).
\]

In the following we want to prove that (*) implies weak cotype \( g \). By proposition 1.3 and Kahane’s inequality we are left to verify that for all vectors \( x_1, \ldots, x_n \) with \( \|x_i\| \geq \frac{1}{n} \) and \( \sum_1^n |\langle x^*, x_i \rangle|^2 \leq \|x^*\|^2 \) one has

\[
\mathbb{E} \left\| \sum_1^n g_i x_i \right\| \geq \frac{g(n)}{c(g, H)}.
\]

Therefore we will fix in the following this sequence of vectors. The second lemma proved with the concentration of measure phenomena reads as follows [B]

**Lemma 1.6 (Talagrand: Lemma 4.3.)** Let \( 8 \leq s \leq n \) such that

\[
\frac{s}{\sqrt{n}} \leq \frac{1}{16HD} g(s).
\]

10
and $J$ a subset of $\{1, ..., n\}$ with $\text{card} J \geq \frac{n}{2}$. Then there exists a subset $I \subset J$ with $\text{card} I = s$ and

$$E \left\| \sum_{i \in I} \varepsilon_i x_i \right\| \geq \frac{g(s)}{64HD}.$$ 

**Remark 1.7** The most interesting application of the theorem is certainly given by the sequence $g(k) = \sqrt{k}$. In this case the conclusion of the theorem is very easy. Indeed, we can choose $s \leq \frac{n}{64HD^2} \leq 2s$ and get a sequence $I \subset \{1, ..., n\}$ with $\text{card} I = s$ such that

$$E \left\| \sum_{i=1}^{n} g_i x_i \right\| \geq \frac{\sqrt{\pi}}{2} \frac{s}{64HD} \geq \frac{\sqrt{2}}{\pi} \frac{1}{(64HD)^2} \sqrt{n}.$$ 

**Proposition 1.3** implies

$$\text{wc}_2(X) \leq c_0 H^2.$$ 

Now we start with the main proof. In the sequel we will assume some conditions to be verified. At the end we will discuss the influence of this conditions on the constants. We choose an even number $M \in \mathbb{N}$ such that

$$2^{rM+1} \leq n \leq 2^{rM+1} 2^{2r}.$$ 

Since $g(2^{2r} 2^{rM+1}) \leq S_2 2^{2r} g(2^{rM+1})$ we can even assume $n = 2^{rM+1}$. Furthermore, we set

$$N := \frac{M}{2}, \quad p := s := 2^N.$$ 

The condition

$$\sqrt{2} 16 HD \leq g(s) \quad \text{and} \quad 8 \leq s \quad (1)$$ 

implies $\frac{s}{\sqrt{n}} \leq \frac{g(s)}{64HD^2}$. From a successive application of lemma 1.6 we can find $p$ disjoint subsets $I_1, ..., I_p$ each of cardinality $s$ such that

$$E \left\| \sum_{i \in I_j} g_i x_i \right\| \geq \frac{\sqrt{\pi}}{2} \frac{s}{100HD} \geq \frac{g(s)}{100HD}.$$ 

Now we will apply the iteration procedure to regroup disjoint blocs. This lemma is also essentially contained in [TAL].

**Lemma 1.8** Let $k \in \mathbb{N}$ satisfy

$$g(k) \geq 2S_3(K+1)H \quad \text{and} \quad g(s) \geq 100DH \sqrt{k}. \quad (2)$$ 

Given a subset $T \subset \{1, ..., p\}$ with $\text{card}(T) = k^l$, $k^l \leq p$ and $I_T = \bigcup_{j \in T} J_t$ one has

$$E \left\| \sum_{i \in I_T} g_i x_i \right\| \geq \frac{g(s)}{100DH} \left( \frac{g(k)}{2S_3 H} \right)^l.$$ 

11
Proof: The case \( l = 0 \) is (c). Proceeding by induction we can assume that the statement is valid for \( l \). A set \( T \) of cardinality \( k^{l+1} \) can be split up into \( k \) sets \( T_j \) with cardinality \( k^l \). By induction hypothesis we have for all \( j = 1, \ldots, k \)

\[
\mathbb{E} \left| \sum_{i \in T_j} g_i x_i \right| \geq \frac{g(s)}{100DH} \left( \frac{g(k)}{2S_3H} \right)^l =: \alpha .
\]

The assertion follows from lemma 1.5 provided we have

\[
\sqrt{k} \leq \frac{\alpha}{2S_3KH} \frac{g(k)}{g(k)} = \frac{g(s)}{100HD} \frac{g(k)}{2S_3KH} \left( \frac{g(k)}{2S_3H} \right)^l .
\]

which is obvious by our assumption. \( \square \)

Now we set \( k := 2^N \) and assume (1) and (2) to be satisfied. Then we have can apply lemma 1.8 to find

\[
\mathbb{E} \left| \sum_{i \in \bigcup_{j=1}^{p} J_j} x_i g_i \right| \geq \frac{1}{100DH} \left( \frac{1}{2S_3H} \right)^r g(s) g(k)^r \geq \frac{1}{100DM_{r}H} \left( \frac{1}{2S_3H} \right)^r g(2^{2r + 1}) .
\]

In this case we set

\[
c_1(g, H) := S_2 2^{2r + 1} 100DM_{r} (2S_3)^r H^{r+1} .
\]

In order to guarantee the conditions (1) and (2) we define

\[
B := \max \{ 2S_3L_t(K + 1), 100L_tD \}^{\max \{2, t\}}
\]

and assume first \( k \geq BH^{\max \{2, t\}} \). Since \( H \geq 1, L_t \geq 1 \), we trivially have \( s \geq 8 \). Furthermore, we get

\[
2S_3(K + 1)H \leq \frac{k^l}{L_t} \leq g(k) \quad \text{and} \quad 100DH \sqrt{k} \leq \frac{\sqrt{k}}{L_t} \sqrt{k} \leq g(k^r) = g(s) .
\]

If the remaining case \( k \leq BH^{\max \{2, t\}} \) we define

\[
c_2(g, H) := S_2 2^{2r + 3} B^r g([H^{\max \{2r, tr\}}]) .
\]

For \( c(g, H) = \max \{ c_1(g, H), c_2(g, H) \} \) we have \( \mathbb{E} \left| \sum_{i} g_i x_i \right| \geq \frac{g(n)}{c(g, H)} \) in any case and the proof is finished.

Remark 1.9 1. In the case \( g(k) = k^{\frac{1}{t}} \) the proof above gives a constant of order \( c_q H^{2r + 2} \) which is certainly not optimal.

2. If the condition M) is not satisfied we can define the new sequence

\[
\tilde{g}(n) := \max \{ g(k^r)g(k)^r \mid k^{2r} \leq n \}
\]

It is easy to check that the conditions Si) – Siii) as well as L) are still satisfied (for probably different constants). The proof above shows that a summing condition of order \( \ell_{g, \infty} \) for a Banach space \( X \) implies weak cotype \( \tilde{g} \).
2 Optimal summing and cotype spaces

In the following we will define sequence spaces which are associated with the cotype and summing properties of Banach spaces. In this setting it is more convenient to study the Rademacher Cotype. We will use the following definition of a maximal symmetric sequence space $Y$ which is a sequence space with the following properties

i) $\|\tau\|_{\infty} \leq \|\tau\|_{Y} \leq \|\tau\|_{1}$ for all sequences with finite support.

ii) $\|\tau^*\| = \|\tau\|$, where $\tau^*$ denotes the non increasing rearrangement of $|\tau|$.

iii) $\|\tau\| = \sup_{n} \|P_{n}(\tau)\|$, where $P_{n}$ denotes the projection onto the first $n$ coordinates.

An operator $T \in L(X, Y)$ is said to be $(Y, 1)$-summing, of cotype $Y$, if there is a constant $c > 0$ such that for all $n \in \mathbb{N}$, $x_{1}, \ldots, x_{n}$ one has

$$\left\| \sum_{1}^{n} \|Tx_{k}\|_{e_{k}} \right\|_{Y} \leq c \sup_{x* \in B_{X^*}} \sum_{1}^{n} \|\langle x*, x_{k} \rangle\|, \quad \left\| \sum_{1}^{n} \|Tx_{k}\|_{e_{k}} \right\|_{Y} \leq c \mathbb{E} \left\| \sum_{1}^{n} \varepsilon_{k} x_{k} \right\|, \text{ resp.}$$

The corresponding norm is denoted by $\pi_{Y, 1}(T) := \inf \{c\}$, $c_{Y}(T) := \inf \{c\}$, where the infimum is taken over all $c$ satisfying the inequality above.

**Remark 2.1** This definition can in particular be applied for $Y = \ell_{g, \infty}$. We want to compare cotype $\ell_{g, \infty}$ with the notion of weak (gaussian) cotype $g$ defined in the chapter before. Using Lewis’ Lemma as in the proof of proposition [L3] it is quite easy to see that Rademacher cotype $\ell_{g, \infty}$ implies weak cotype $g$. The converse is not always true. If we consider $g(n) = \sqrt{n}$ we see that every Banach spaces with weak cotype 2, not having cotype 2, yields an example of a Banach space having cotype $g$, but not cotype $\ell_{g, \infty}$. A further example is given by $g(n) = \sqrt{1 + \ln n}$, since every Banach space has cotype $g$ but cotype $\ell_{g, \infty}$ only holds for Banach spaces with finite cotype by Maurey/Pisier’s theorem. Therefore, it is natural to require two additional conditions, namely $g(n) \geq cn^{\frac{1}{2}}$ for some $Bp < \infty$ and

$$g(n) \leq c_{q} \left( \frac{n}{k} \right)^{\frac{1}{q}} g(k)$$

for some $q > 2$. In this case a Banach space with cotype $g$ is of finite cotype and using the inequality

$$\pi_{2}(T) \leq C_{q} c_{q} \sqrt{n} \sup_{k} g(k) x_{k}(T)$$

valid for all operators of rank at most $n$, we easily see that every Banach space of cotype $g$ is also of cotype $\ell_{g, \infty}$.

The main tool of this chapter are the properties of the optimal summing and cotype space associated to a Banach space $X$. Given $\tau = (\tau_{k})_{k}$ we define

$$\|\alpha\|_{S} := \inf \left\{ \sup_{\|\alpha\|_{S} \leq 1} \left\| \sum_{1}^{n} \tau_{k} \alpha_{k} x_{k} \right\| \mid (x_{k})_{1}^{n} \subset X, \|x_{k}\| = 1 \right\}$$
and
\[ \|\alpha\|_C := \inf \left\{ \mathbb{E} \left| \sum_{k} \varepsilon_k \alpha_k x_k \right| : (x_k) \subset X, \|x_k\| = 1 \right\}. \]

Clearly this is homogeneous expression which are invariant under permutations and change of signs. In order to guarantee the triangle inequality we define for \( T \in \{C, S\} \)
\[ \|\tau\|_T^0 := \inf \left\{ \sum_{i=1}^{m} \|\tau^i\|_T : n \in \mathbb{N}, \tau^i \text{ with finite support and } |\tau^i| \leq \sum_{i=1}^{m} |\tau^i| \right\} \]
and
\[ \|\tau\|_T := \sup_{n} \|P_n(\tau)\|_T^0. \]

The space \( Y_S, Y_C \) defined by this norm will be called \textit{optimal summing space}, \textit{optimal cotype space}, respectively. We summarize the properties of this spaces in the following

\textbf{Lemma 2.2} Let \( X \) be a Banach space and \( Y_S, Y_C \) it’s optimal summing, optimal cotype space, respectively, and let \( Z \) be a maximal sequence space. Then one has

1. The identity of \( X \) is \((Y_S, 1)\) summing and of cotype \( Y_C \) with constant 1.

2. The identity of \( X \) is \((Z, 1)\)-summing (of cotype \( Z \)) if and only if
\[ Y_S \subset Z \quad (Y_C \subset Z, \text{ resp.}) \]

The norm of the inclusion is \( \pi_{(Z, 1)}(id_X) \) \((C_Z(id_X), \text{ respectively})\).

3. For \( Y \in \{Y_S, Y_C\} \) and each finitely supported sequence \((\tau^k)^n_i\) one has
\[ \left\| \sum_{i=1}^{n} \tau^k \right\|_Y e_k \leq \left\| \sum_{i=1}^{n} \tau^k \right\|_Y. \]

\textbf{Proof:} 1., 2. are obvious. We will only consider the cotype case in 3.. We denote by \( \tau := \sum_{k} |\tau^k| \).
Given \( \delta > 0 \) we can find a finite sequence \((x_i) \subset X \) with \( \|x_i\| = 1 \) such that
\[ \mathbb{E} \left| \sum_{i} \varepsilon_i \tau^k x_i \right| \leq (1 + \delta) \|\tau\|_C. \]

For any sequence of signs \( \rho_k \) we can find a sequence \((\gamma_i)\), \( \gamma \in [-1, 1] \) such that
\[ \sum_{k} \rho_k |\tau^k| = \gamma \tau. \]

By the sign invariance of \((\varepsilon_i)\) and the fact that extreme points in the unit ball \( \ell_\infty^n \) over \( \mathbb{R} \) are sequences of signs, see e.g. \([P] \), we get
\[ \mathbb{E}_\varepsilon \left| \sum_{i} \varepsilon_i \left( \sum_{k} \rho_k |\tau^k_i| \right) x_i \right| = \mathbb{E}_\varepsilon \left| \sum_{i} \varepsilon_i \gamma_i \tau^k x_i \right| \leq \mathbb{E}_\varepsilon \left| \sum_{i} \varepsilon_i \tau^k x_i \right| \leq (1 + \delta) \|\tau\|_C. \]
Taking expectations we deduce from 1. and the triangle inequality in \( Y \)
\[
(1 + \delta) \left\| \tau \right\|_{C} \geq \mathbb{E}_x \mathbb{E}_{\rho} \left\| \sum_{i} \varepsilon_i \left( \sum_{k} \rho_k \tau_i^k \right) x_i \right\| \geq \mathbb{E}_x \left\| \sum_{i} \sum_{k} \varepsilon_i \tau_i^k x_i \right\|_X e_k \left\| \tau^k \right\|_{C}^{Y_C} \]
\[
\geq \left\| \sum_{k} \left( \mathbb{E}_x \left\| \sum_{i} \varepsilon_i \tau_i^k x_i \right\| \right) e_k \right\|_{Y_C} \geq \left\| \sum_{k} \left\| \tau^k \right\|_{C} e_k \right\|_{Y_C} .
\]
Letting \( \delta \) to zero we have proved
\[
\left\| \sum_{k} \left\| \tau^k \right\|_{C} e_k \right\| \leq \left\| \sum_{k} \left| \tau^k \right| \right\|_{C} . \tag{\ast}
\]
Now let \( \tau \leq \sum_{j} |\sigma^j| \). We define \( \beta^k := \frac{1}{|\sigma^k|} \tau \) by pointwise multiplication and using the convention \( \frac{0}{0} = 0 \). For the sequences \( \sigma^{kj} := \beta^k \sigma^j \) we clearly have
\[
\sum_{k} |\sigma^{kj}| \leq |\sigma^j| \quad \text{and} \quad |\tau^k| \leq \sum_{j} |\sigma^{kj}| .
\]
From (\ast) applied for each sequence \( (|\sigma^{kj}|)_k \) we deduce
\[
\left\| \sum_{k} \left\| \tau^k \right\|_{Y} e_k \right\|_{Y_C} \leq \left\| \sum_{k} \left( \sum_{j} \left\| \sigma^{kj} \right\|_{C} \right) e_k \right\|_{Y_C} \leq \sum_{j} \left\| \sum_{k} \left\| \sigma^{kj} \right\|_{C} e_k \right\|_{Y_C}
\]
\[
\leq \sum_{j} \left\| \sum_{k} \left| \sigma^{kj} \right| \right\|_{C} \leq \sum_{j} \left\| \sigma^j \right\|_{C} .
\]
Taking the infimum over all \( \tau \leq \sum_{j} \sigma^j \) yields the assertion. \( \square \)

We will study in more detail the spaces which satisfy the last condition. For a maximal symmetric sequence space \( Y \) we define
\[
f_Y(n) := \left\| \sum_{1}^{n} e_k \right\|_{Y} \quad \text{and} \quad q_Y := \inf \left\{ 0 < q < \infty \mid \exists C : \frac{1}{n^{\frac{1}{q}}} \leq C f_Y(n) \right\} .
\]
Obviously we have the following inclusions for all \( q > q_Y \)
\[
Y \subset \ell_{f_Y, \infty} \subset \ell_q .
\]
If \( Y \) satisfies the concavity condition 3, we have the following alternative which is somehow an improvement of the classical Maurey/Pisier argument in the context of finite cotype.

**Proposition 2.3** Let \( Y \) be a maximal symmetric sequence space which satisfies
\[
\left\| \sum_{1}^{n} \left\| \tau^k \right\|_{Y} e_k \right\|_{Y} \leq \left\| \sum_{1}^{n} \left| \tau^k \right| \right\|_{Y} .
\]
For all \( 1 \leq p < \infty \) we have either \( \ell_p \subset Y \) with inclusion norm 1 or there exists a \( q < p \) with \( Y \subset \ell_q \not\subset \ell_p \). In particular, we have
\[
\ell_{q_Y} \subset Y .
\]
Proof: Let $\tau$ be a sequence of finite support, $\tau_k = 0$ for $k \geq n$ say. For $i = 1, \ldots, n$ we define

$$\sigma_i := \sum_{j=1}^{n} \tau_j e_{(i-1)n+j}$$

and the product $\tau \otimes \tau := \sum_{i=1}^{n} \tau_i \sigma_i$.

Usually, $\tau \otimes \tau$ is defined in $\ell_p(\mathbb{N}^2)$ which is isometric isomorphic to $\ell_p$ by a renumbering of $\mathbb{N}^2$. That’s what we try to imitate with the definition above. Clearly, we have $\|\tau \otimes \tau\|_p = \|\tau\|^2_p$.

Our assumption on $Y$ implies

$$\|\tau\|^2_Y = \sum_{i=1}^{n} \tau_i \|\sigma_i\|_Y e_i \leq \|\tau \otimes \tau\|_Y. \quad (*)$$

In particular, $f_Y$ is submultiplicative, i.e. $f_Y(n) f_Y(k) \leq f_Y(nk)$ . Now we consider the following alternative.

1. There exists an $n_0 \in \mathbb{N}$ such that $f_Y(n_0) > \frac{1}{n^p}$.

2. For all $n \in \mathbb{N}$ one has $f_Y(n) \leq n^\frac{1}{p}$.

In the first case we choose $q < p$ such that $f_Y(n_0) = n_0^\frac{1}{q}$. For $n \in \mathbb{N}$ we choose $m \in \mathbb{N}$ with $n_0^{m-1} \leq n \leq n_0^m$. By the submultiplicativity and the triangle inequality we deduce

$$n^\frac{1}{q} \leq n_0^\frac{m}{q} = f_Y(n_0)^m \leq f_Y(n_0) \leq n_0 f_Y(n).$$

This means $Y \subset \ell_{q,\infty}$ and therefore for all $q < r < p$ the inclusion $Y \subset \ell_r \subset \ell_p$. Now we consider the second case. We will first show $\ell_{p,1} \subset Y$. Indeed, let $\tau$ ba a non increasing positive sequence with finite support. Then we have

$$\|\tau\|_Y \leq \sum_{k=0}^{\infty} \left\| \sum_{j=2^k}^{2^{k+1}} \tau_j e_j \right\|_Y \leq \sum_{k=0}^{\infty} \tau_2^k f_Y(2^k) \leq \sum_{k=0}^{\infty} \tau_2^k (2^k)^\frac{1}{p} \leq 5 \|\tau\|_{p,1}.$$

Defining $C_n := \|P_n : \ell_p \to Y\|$, this means $C_n \leq 5(1 + \ln n)$. Now we will use a tensor trick to finish the proof. For this we prove $C_n^2 \leq C_n^2$. Indeed, let $\tau$ a sequence with support contained in $\{1, \ldots, n\}$. From $(*)$ we deduce

$$\|\tau\|^2_Y \leq \|\tau \otimes \tau\|_Y \leq C_n^2 \|\tau \otimes \tau\|_p \leq C_n^2 \|\tau\|^2_p.$$

Hence we get

$$C_n \leq \inf_k \left( C_{n^2}^k \right)^\frac{1}{2k} \leq \inf_k \left( 5(1 + 2^k \ln n) \right)^\frac{1}{2k} = 1.$$

If we apply the alternative for the space $\ell_{q_Y}$ we only have to observe that an inequality $Y \subset \ell_q$ implies $n^\frac{1}{q} \leq C f_Y(n)$. By definition this is impossible for all $q < q_Y$. \hfill \Box

As an application we will investigate cotype properties with respect to the Lorentz space $\ell_{q,w}$.
Proposition 2.4 Let $2 \leq q < \infty$, $1 \leq w \leq \infty$. A Banach space $X$ is of cotype $\ell_{q,w}$ if and only if

$$X \text{ is of cotype } \begin{cases} p \text{ for some } p < q & \text{if } w < q \\ \ell_{p,\infty} & \text{if } q < w. \end{cases}$$

If $X$ is of cotype $\ell_{q,\infty}$ there exists a constant $C$ such that

$$c_q(id_E) \leq \sqrt{\pi} C^{k+1} (\max\{1, \log_2\})^{(k)}((1 + \log_2 n)^{\frac{1}{2}})$$

holds for all $k \in \mathbb{N}$ and $n$-dimensional subspaces $E \subset X$. In particular,

$$c_q(id_E) \leq \sqrt{\pi} 2 C^{1+k_n},$$

where $k_n$ is the smallest integer $k$ with $n \leq \frac{2^k}{k}$.

**Proof:** If $w < q$ and $X$ is of cotype $\ell_{q,w}$ we have $Y_C \subset \ell_{q,w}$ by lemma 2.2, but certainly not $\ell_q \subset Y_C$. By lemma 2.3 there must be a $p < q$ such that $Y_C \subset \ell_p$. Since $X$ is of cotype $Y_C$ it is also of cotype $\ell_p$.

Now let us assume that $X$ is of cotype $\ell_{q,\infty}$ with constant $D$, say. This implies in particular $n^\frac{1}{q} \leq C f_{Y_C}(n)$. Let $q \leq w < \infty$ and $\tau$ a positive non increasing sequence of finite support.

For $k \in \mathbb{N}$ we define the disjoint elements $x_k := \tau_{2^k} \sum_{j=2^{k-1}+1}^{2^k} e_j$. Then we get

$$\|\tau\|_{q,w} \leq \left(\sum_n (\tau_{n,\frac{1}{q}})^w \frac{1}{n}\right)^{\frac{1}{w}} \leq \left(\sum_k (\tau_{2^k, \frac{1}{q}})^w \frac{1}{2^k}\right)^{\frac{1}{w}} \leq \|\tau\|_{\infty} + 2^\frac{1}{2} D \sum_{k \in \mathbb{N}} \|x_k\|_{Y_C} e_k \|_{w}.$$ 

If $w > q$ we have $\ell_{q,\infty} \subset \ell_w$ with inclusion norm $c_{qw}$. Therefore we deduce from $\sum_k x_k \leq \tau$ and condition 3. in lemma 2.2,

$$\|\tau\|_{q,w} \leq 2^\frac{1}{2} D \left(\|\tau\|_{\infty} + c_{qw} \sum_k \|x_k\|_{Y_C} e_k \|_{q,\infty}\right) \leq 2^\frac{1}{2} D \left(\|\tau\|_{Y_C} + c_{qw} \sum_k \|x_k\|_{Y_C} e_k \|_{Y_C}\right) \leq 2^{1+\frac{1}{2}} c_{qw} D^2 \|\tau\|_{Y_C}.$$ 

Since $X$ is of cotype $Y_C$ and $Y_C \subset \ell_{q,w}$ we obtain the assertion in this case. Now we come to the case $q = w$. We denote by $\alpha_n := \|P_n : Y_C \to \ell_q\|$, with the convention $\alpha_0 = 1$. We will prove

$$\alpha_n \leq 2^{1+\frac{1}{2}} D \alpha_{\log_2 n}.$$ 

Indeed, if the support of the given sequence $\tau$ above is contained in $\{1, \ldots, n\}$ then we will have $x_k = 0$ whenever $2^k > n$. Therefore we obtain again by lemma 2.2

$$\|\tau\|_q \leq \left(\|\tau\|_{\infty} + 2^\frac{1}{2} D \sum_{k=1}^{\log_2 n} \|x_k\|_{Y_C} e_k \right) \leq \left(\|\tau\|_{\infty} + 2^\frac{1}{2} D \sum_{k=1}^{\log_2 n} \|x_k\|_{Y_C} e_k \right) \leq \left(1 + 2^\frac{1}{2} D \alpha_{\log_2 n}\right) \|\tau\|_{Y_C}.$$ 

17
Together with the trivial estimate \( \| id : \ell^q_{n,\infty} \to \ell^q_{n} \| \leq (1 + \log_2 n)^{\frac{1}{q}} \) and induction this implies
\[
Rc^q_{\ell^q_{n}}(id_X) \leq D^{k+1} (\max\{1,\log_2\}(1 + \log_2 n)^{\frac{1}{q}}).
\]
Since the gaussian cotype constant of a \( n \) dimensional space \( E \) can be well estimated by the gaussian cotype constant, see \([TOJ, DJ2]\), and as a consequence of the comparison principle of gaussian and Rademacher variables, see e.g. \([TOJ]\), we deduce
\[
c_{q}(id_E) \leq \sqrt{2} c^{n}_{q}(Id_E) \leq \sqrt{\pi Rc^{q}_{\ell^q_{n}}(id_X)} \leq \sqrt{\pi D^{k+1} (\max\{1,\log_2\}(1 + \log_2 n)^{\frac{1}{q}})}.
\]

**Final remark 2.5**  
1. The same contraction argument can also be applied in the space \( Y_S \), provided we have \( Y_S \subset \ell_{q,\infty} \). This is only interesting in the case \( q = 2 \). Hence in a weak cotype 2 space we have
\[
\pi_{21}^{n}(id_X) \leq C^{k+1} (\max\{1,\log_2\}(1 + \log_2 n)^{\frac{1}{2}}).
\]
It is still open whether such an estimate is valid for the cotype 2 constant.

2. Let \( X \) be a Banach space with non-trivial cotype. Then \( q_{YS} \) is finite and we can fix a natural number \( q_{YS} < r \). If we define
\[
g_{S}(n) := f_{YS}([n^{\frac{1}{q}}]) f_{YS}([n^{\frac{1}{q}}])^{r},
\]
we can apply the proof of \( i) \Rightarrow ii) \) of the main theorem to deduce that \( X \) is of weak cotype \( g_{S} \). Given a number \( q > 2 \) and a sequence \( g(n)_{n \in \mathbb{N}} \) we define
\[
g_{q}(n) := n^{-\frac{1}{q}} \inf\{k^{-\frac{1}{q}} g(k) \mid 1 \leq k \leq n\}.
\]
From remark \([2,3]\) we have clearly
\[
\frac{1}{cr_{r,q}} (g_S)_q(n) \leq f_{YC}(n) \leq f_{YS}(n).
\]
Nevertheless, the eigenvalue estimate for the sequence \( (f_{YS})_q \) can be directly derived from a generalization of Maurey theorem’s theorem to \( p \)-convex sequences, see \([J1, J2]\). (The space \( \ell_{f_{YS},\infty} \) is \( p \)-convex for each \( 2 < p < q \).) This is of particular interest if the cotype properties of Orlicz spaces associated to the function \( M(t) = \left( \frac{t^r}{1+|\ln t|} \right)^{\frac{1}{q}} \) are studied in more detail.

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