INVERSE SPECTRAL THEORY
AS INFLUENCED BY BARRY SIMON

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Dedicated with great pleasure to Barry Simon, mentor and friend,
on the occasion of his 60th birthday.

Abstract. We survey Barry Simon’s principal contributions to the field of
inverse spectral theory in connection with one-dimensional Schrödinger and
Jacobi operators.

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1. Introduction

This Festschrift contribution is devoted to a survey of Barry Simon’s principal
contributions to the area of inverse spectral theory for one-dimensional Schrödinger
and Jacobi operators. We decided to put the emphasis on the following five groups
of topics:

• The Dirichlet spectral deformation method

A general spectral deformation method applicable to Schrödinger, Jacobi, and
Dirac-type operators in one dimension, which can be used to insert eigenvalues into
spectral gaps of arbitrary background operators but is also an ideal technique to
construct isospectral (in fact, unitarily equivalent) sets of operators starting from
a given base operator.

• Renormalized oscillation theory

Renormalized oscillation theory formulated in terms of Wronskians of appropriate
solutions rather than solutions themselves, applies, in particular, to energies

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above the essential spectrum where real-valued solutions exhibit infinitely many zeros and traditional eigenvalue counting methods would naively lead to $\infty - \infty$. While not directly related to inverse spectral methods, we chose to include this topic because of its fundamental importance to the Dirichlet spectral deformation method.

- The xi function and trace formulas for Schrödinger and Jacobi operators

The xi function, that is, essentially, the argument of the diagonal Green’s function, which also takes on the role of a particular spectral shift function, is an ideal tool to derive a hierarchy of (higher-order) trace formulas for one-dimensional Schrödinger and Jacobi operators. The latter are the natural extensions of the well-known trace formulas for periodic and algebro-geometric finite-band potential coefficients to arbitrary coefficients. The xi function provides a tool for direct and inverse spectral theory.

- Uniqueness theorems in inverse spectral theorem

Starting from the Borg–Marchenko uniqueness theorem, the basic uniqueness result for Schrödinger and Jacobi operators in terms of the Weyl–Titchmarsh $m$-coefficient, a number of uniqueness results are discussed. The latter include the Borg-type two-spectra results as well as Hochstadt–Lieberman-type results with mixed prescribed data. In addition to these traditional inverse spectral problems, several new types of inverse spectral problems are addressed.

- Simon’s new approach to inverse spectral theory

In some sense, Simon’s new approach to inverse spectral theory for half-line problems, based on a particular representation of the Weyl–Titchmarsh $m$-function as a finite Laplace transform with control about the error term, can be viewed as a continuum analog of the continued fraction approach (based on the Riccati equation) to the inverse spectral problem for semi-infinite Jacobi matrices (the actual details, however, differ considerably). Among a variety of spectral-theoretic results, this leads to a formulation of the half-line inverse spectral problem alternative to that of Gel’fand and Levitan. In addition, it leads to a fundamental new result, the local Borg–Marchenko uniqueness theorem.

Each individual section focuses on a particular paper or on a group of papers to be surveyed, representing the five items just discussed. Since this survey is fairly long, it was our intention to write each section in such a manner that it can be read independently.

Only self-adjoint Schrödinger and Jacobi operators are considered in this survey. In particular, all potential coefficients $V$ and Jacobi matrix coefficients $a$ and $b$ are assumed to be real-valued throughout this survey (although, we occasionally remind the reader of this assumption).

To be sure, this is not a survey of the state of the art of inverse spectral theory for one-dimensional Schrödinger and Jacobi operators. Rather, it exclusively focuses on Barry Simon’s contributions to and influence exerted on the field. Especially, the bibliography, although quite long, is far from complete and only reflects the particular purpose of this survey. The references included are typically of the following two kinds: First, background references that were used by Barry Simon and his coworkers in writing a particular paper. Such references are distributed throughout the particular survey of one of his papers. Second, at the end of each
such survey we refer to more recent references which complement the results of the particular paper in question.

It was 23 years ago in April of 1983 that Barry and I first met in person at Caltech and started our collaboration. Barry became a mentor and then a friend, and it is fair to say he has had a profound influence on my career since that time. Working with Barry has been exciting and most rewarding for me. Happy Birthday, Barry, and many more such anniversaries!

2. The Dirichlet Spectral Deformation Method

In this section we describe some of the principal results of the paper:

[100] F. Gesztesy, B. Simon, and G. Teschl, Spectral deformations of one-dimensional Schrödinger operators, J. Analyse Math. 70, 267–324 (1996).

Spectral deformations of Schrödinger operators in $L^2(\mathbb{R})$, isospectral and certain classes of non-isospectral ones, have attracted a lot of interest over the past three decades due to their prominent role in connection with a variety of topics, including the Korteweg-de Vries (KdV) hierarchy, inverse spectral problems, supersymmetric quantum mechanical models, level comparison theorems, etc. In fact, the construction of $N$-soliton solutions of the KdV hierarchy (and more generally, the construction of solitons relative to reflectionless backgrounds) is a typical example of a non-isospectral deformation of $H = -\frac{d^2}{dx^2}$ in $L^2(\mathbb{R})$ since the resulting deformation $\tilde{H} = -\frac{d^2}{dx^2} + \tilde{V}$ acquires an additional point spectrum $\{\lambda_1, \ldots, \lambda_N\} \subset (-\infty, 0)$, $N \in \mathbb{N}$, such that

$$\sigma(\tilde{H}) = \sigma(H) \cup \{\lambda_1, \ldots, \lambda_N\}$$

($\sigma(\cdot)$ abbreviating the spectrum). In the $N$-soliton context (ignoring the KdV time parameter for simplicity), $\tilde{V}$ is of the explicit form

$$\tilde{V}(x) = -2\frac{d^2}{dx^2}\ln[W(\Psi_1(x), \ldots, \Psi_N(x))], \quad x \in \mathbb{R},$$

(2.1)

where $W(f_1, \ldots, f_N)$ denotes the Wronskian of $f_1, \ldots, f_N$ and the functions $\Psi_j$, $j = 1, \ldots, N$, are given by

$$\Psi_j(x) = (-1)^{j+1}e^{-\kappa_j x} + \alpha_j e^{\kappa_j x}, \quad x \in \mathbb{R},$$

$$0 < \kappa_1 < \cdots < \kappa_N, \quad \alpha_j > 0, \quad j = 1, \ldots, N.$$

The Wronski-type formula in (2.1) is typical also for general background potentials and typical for the Crum–Darboux-type commutation approach [44], [48] (cf. [90] and the references therein for general backgrounds) which underlies all standard spectral deformation methods for one-dimensional Schrödinger operators such as single and double commutation, and the Dirichlet deformation method presented in this section.

On the other hand, the solution of the inverse periodic problem and the corresponding solution of the algebro-geometric quasi-periodic finite-band inverse problem for the KdV hierarchy (and certain almost-periodic limiting situations thereof) are intimately connected with isospectral (in fact, unitary) deformations of a given base (background) operator $H = -\frac{d^2}{dx^2} + V$. Although not a complete bibliography on applications of spectral deformations in mathematical physics, the interested reader may consult [14], [20], [22], [28], [44], [46], [48], [59, Sect. 4.3], [60], [61], [63], [64], [66], [68], [72], [73], [102], [74], [78], [79, App. G], [90], [100], [103], [105], [112],
The principal result in [100], reviewed in this section (cf. Theorem 2.4 (i)), provides a complete spectral characterization of a new method of constructing isospectral (in fact, unitary) deformations of general Schrödinger operators $H = -\frac{d^2}{dx^2} + V$ in $L^2(\mathbb{R})$. The technique is connected to Dirichlet data, that is, to the spectrum of the operator $H^D$ on $L^2((-\infty, x_0)) \oplus L^2([x_0, \infty))$ with a Dirichlet boundary condition at $x_0$. The transformation moves a single eigenvalue of $H^D$ and perhaps flips the half-line (i.e., $(-\infty, x_0)$ to $(x_0, \infty)$, or vice versa) to which the Dirichlet eigenvalue belongs. On the remainder of the spectrum, the transformation is realized by a unitary operator.

To describe the Dirichlet deformation method (DDM) as developed in [100] in some detail, we suppose that $V \in L^1_{\text{loc}}(\mathbb{R})$ is real valued and introduce the differential expression $\tau = -\frac{d^2}{dx^2} + V(x), x \in \mathbb{R}$. Assuming $\tau$ to be in the limit point case at $\pm \infty$ (for the general case we refer to [100]) one defines the self-adjoint base (i.e., background) operator $H$ in $L^2(\mathbb{R})$ by

$$Hf = \tau f, \quad f \in \text{dom}(H) = \{g \in L^2(\mathbb{R}) | g, g' \in AC_{\text{loc}}(\mathbb{R}); \tau g \in L^2(\mathbb{R})\}. \quad (2.2)$$

Here $W(f, g)(x) = f(x)g'(x) - f'(x)g(x)$ denotes the Wronskian of $f, g \in AC_{\text{loc}}(\mathbb{R})$ (the set of locally absolutely continuous functions on $\mathbb{R}$). Given $H$ and a fixed reference point $x_0 \in \mathbb{R}$, we introduce the associated Dirichlet operator $H^D_{x_0}$ in $L^2(\mathbb{R})$ by

$$H^D_{x_0}f = \tau f, \quad f \in \text{dom}(H^D_{x_0}) = \{g \in L^2(\mathbb{R}) | g, g' \in AC_{\text{loc}}(\mathbb{R}) \setminus \{x_0\}); \lim_{\epsilon \downarrow 0} g(x_0 \pm \epsilon) = 0; \tau g \in L^2(\mathbb{R})\}. \quad (2.3)$$

Clearly, $H^D_{x_0}$ decomposes into $H^D_{x_0} = H^D_{-x_0} \oplus H^D_{x_0}$ with respect to the orthogonal decomposition $L^2(\mathbb{R}) = L^2((-\infty, x_0)) \oplus L^2([x_0, \infty))$. Moreover, for any $\mu \in \sigma_{\text{d}}(H^D_{x_0}) \setminus \sigma(H)$ (\text{\text{d}}(\cdot), the discrete spectrum, $\sigma(\cdot)$ and $\sigma_{\text{ess}}(\cdot)$, the spectrum and essential spectrum, respectively), we introduce the Dirichlet datum

$$\mu, \sigma \in \{\sigma_{\text{d}}(H^D_{x_0}) \setminus \sigma_{\text{d}}(H)\} \times \{-, +\},$$

which identifies $\mu$ as a discrete Dirichlet eigenvalue on the interval $(x_0, \infty)$, that is, $\mu \in \sigma_{\text{d}}(H^D_{x_0})$, $\sigma \in \{-, +\}$ (but excludes it from being simultaneously a Dirichlet eigenvalue on $(x_0, -\infty)$).

Next, we pick a fixed spectral gap $(E_0, E_1)$ of $H$, the endpoints of which (without loss of generality) belong to the spectrum of $H$,

$$(E_0, E_1) \subset \mathbb{R} \setminus \sigma(H), \quad E_0, E_1 \in \sigma(H)$$

and choose a discrete eigenvalue $\mu$ of $H^D_{x_0}$ in the closure of that spectral gap,

$$\mu \in \sigma_{\text{d}}(H^D_{x_0}) \cap [E_0, E_1] \quad (2.4)$$
Given deformation method in the following. For obvious reasons we will allude to (2.7) as the Dirichlet datum
\[(\mu, \sigma) \in (E_0, E_1) \times \{-, +\}, \tag{2.5}\]
or else to a discrete eigenvalue of \(H_{-x_0}^D\) and \(H_{+x_0}^D\), that is,
\[\mu \in \{E_0, E_1\} \cap \sigma_d(H) \cap \sigma_d(H_{-x_0}^D) \cap \sigma_d(H_{+x_0}^D) \tag{2.6}\]
since the eigenfunction of \(H\) associated with \(\mu\) has a zero at \(x_0\). In particular, since \((H_{x_0}^D - z)^{-1}\) is a rank-one perturbation of \((H - z)^{-1}\), one infers
\[\sigma_{\text{ess}}(H_{x_0}^D) = \sigma_{\text{ess}}(H),\]
and thus, \(\mu \in \{E_0, E_1\} \cap \sigma_{\text{ess}}(H)\) is excluded by assumption (2.4). Hence, the case distinctions (2.5) and (2.6) are exhaustive.

In addition to \(\mu\) as in (2.4)--(2.6), we also need to introduce \(\tilde{\mu} \in \{E_0, E_1\}\) and \(\tilde{\sigma} \in \{-, +\}\) as follows: Either
\[(\tilde{\mu}, \sigma) \in (E_0, E_1) \times \{-, +\},\]
or else
\[\tilde{\mu} \in \{E_0, E_1\} \cap \sigma_d(H).\]

Given \(H\), one introduces Weyl–Titchmarsh-type solutions \(\psi_{\pm}(z, x)\) of \((\tau - z)\psi(z) = 0\) by
\[\psi_{\pm}(z, \cdot) \in L^2((R, \pm \infty)), \quad R \in \mathbb{R},\]
\[\lim_{x \to \pm \infty} W(\psi_{\pm}(z), g)(x) = 0 \quad \text{for all } g \in \text{dom}(H).\]

If \(\psi_{\pm}(z, x)\) exist, they are unique up to constant multiples. In particular, \(\psi_{\pm}(z, x)\) exist for \(z \in \mathbb{C}\setminus\sigma_{\text{ess}}(H)\) and we can (and will) assume them to be holomorphic with respect to \(z \in \mathbb{C}\setminus\sigma(H)\) and real-valued for \(z \in \mathbb{R}\) (cf. the discussion in connection with (3.1)).

Given \(\psi_{\sigma}(\mu, x)\) and \(\psi_{-\sigma}(\tilde{\mu}, x)\), one defines
\[W_{(\tilde{\mu}, \sigma)}(x) = \begin{cases} (\tilde{\mu} - \mu)^{-1} W(\psi_{\sigma}(\mu), \psi_{-\sigma}(\tilde{\mu}))(x), & \mu, \tilde{\mu} \in [E_0, E_1], \tilde{\mu} \neq \mu, \\ -\sigma \int_{\sigma_{\text{ess}}} \frac{dx}{d\sigma^2} \psi_{\sigma}(\mu, x^2), & (\tilde{\mu}, \sigma) = (\mu, -\sigma), \mu \in (E_0, E_1), \end{cases}\]
and the associated Dirichlet deformation
\[\tilde{t}_{(\tilde{\mu}, \sigma)} = -\frac{d^2}{dx^2} + \tilde{V}_{(\tilde{\mu}, \sigma)}(x),\]
\[\tilde{V}_{(\tilde{\mu}, \sigma)}(x) = V(x) - 2[\ln|V_{(\tilde{\mu}, \sigma)}(x)|]'', \quad x \in \mathbb{R}, \tag{2.7}\]
\[\mu, \tilde{\mu} \in [E_0, E_1], \mu \neq \tilde{\mu} \text{ or } (\tilde{\mu}, \sigma) = (\mu, -\sigma), \mu \in (E_0, E_1).\]

As discussed in Section 3, \(W_{(\tilde{\mu}, \sigma)}(x) \neq 0, x \in \mathbb{R}\), and hence (2.7) yields a well-defined potential \(\tilde{V}_{(\tilde{\mu}, \sigma)} \in L^1_{\text{loc}}(\mathbb{R})\).

In the remaining cases \((\mu, \tilde{\sigma}) = (\mu, \sigma), \mu \in [E_0, E_1],\) and \(\mu = \tilde{\mu} \in \{E_0, E_1\} \cap \sigma_d(H)\), we define \(\tilde{V}_{(\tilde{\mu}, \sigma)} = V\) which represents the trivial deformation of \(V\) (i.e., none at all), and for notational simplicity these trivial cases are excluded in the remainder of this section. For obvious reasons we will allude to (2.7) as the Dirichlet deformation method in the following.

If \(\tilde{\mu} \in \sigma_d(H)\), then \(\psi_{-\sigma}(\tilde{\mu}) = c \psi_{\sigma}(\tilde{\mu})\) for some \(c \in \mathbb{R}\setminus\{0\}\), showing that \(W_{(\tilde{\mu}, \sigma)}(x)\) and hence, \(\tilde{V}_{(\tilde{\mu}, \sigma)}(x)\) in (2.7) becomes independent of \(\sigma\) or \(\tilde{\sigma}\). In this case we shall
occasionally use a more appropriate notation and write $\hat{V}_\mu$ and $\hat{\tau}_\mu$ (instead of $\hat{V}_{(\hat{\mu}, \hat{\sigma})}$ and $\hat{\tau}_{(\hat{\mu}, \hat{\sigma})}$).

For later reference, we now summarize our basic assumptions on $V$, $\mu$, and $\hat{\mu}$ in the following hypothesis.

**Hypothesis 2.1.** Suppose $V \in L^1_{\text{loc}}(\mathbb{R})$ to be real-valued. In addition, we assume

$$(E_0, E_1) \subset \mathbb{R}\setminus \sigma(H), \quad E_0, E_1 \in \sigma(H),$$

$$\mu \in \sigma_d(H_{x_0}^D), \quad (\mu, \sigma) \in (E_0, E_1) \times \{-, +\} \text{ or } \mu \in \{E_0, E_1\} \cap \sigma_d(H),$$

$$(\hat{\mu}, \hat{\sigma}) \in (E_0, E_1) \times \{-, +\} \text{ or } \hat{\mu} \in \{E_0, E_1\} \cap \sigma_d(H),$$

$$\mu, \hat{\mu} \in [E_0, E_1], \quad \mu \neq \hat{\mu} \text{ or } (\hat{\mu}, \hat{\sigma}) = (\mu, -\sigma), \quad \mu \in (E_0, E_1).$$

Next, introducing the following solutions of $(\hat{\tau}_{(\hat{\mu}, \hat{\sigma})} - z)\hat{\psi}(z) = 0$,

$$\hat{\psi}_{-\sigma}(\mu, x) = \psi_{-\sigma}(\hat{\mu}, x)/W_{(\hat{\mu}, \hat{\sigma})}(x), \quad \hat{\psi}_{\sigma}(\hat{\mu}, x) = \psi_{\sigma}(\mu, x)/W_{(\hat{\mu}, \hat{\sigma})}(x), \quad \hat{\psi}_{\sigma}(\hat{\mu}, x_0) = 0,$$

one infer

$$(\hat{\tau}_{(\hat{\mu}, \hat{\sigma})}\hat{\psi}_{-\sigma}(\mu))(x) = \mu \hat{\psi}_{-\sigma}(\mu, x), \quad (\hat{\tau}_{(\hat{\mu}, \hat{\sigma})}\hat{\psi}_{\sigma}(\hat{\mu})))(x) = \hat{\mu} \hat{\psi}_{\sigma}(\hat{\mu}, x).$$

The Dirichlet deformation operator $\hat{H}_{(\hat{\mu}, \hat{\sigma})}$ associated with $\hat{\tau}_{(\hat{\mu}, \hat{\sigma})}$ in (2.7) is then defined as follows:

$$\hat{H}_{(\hat{\mu}, \hat{\sigma})}f = \hat{\tau}_{(\hat{\mu}, \hat{\sigma})}f, \quad f \in \text{ dom}(\hat{H}_{(\hat{\mu}, \hat{\sigma})}) = \{g \in L^2(\mathbb{R}) \mid g, g' \in AC_{\text{loc}}(\mathbb{R}); g \text{ satisfies the b.c. in } (2.9); \hat{\tau}_{(\hat{\mu}, \hat{\sigma})}g \in L^2(\mathbb{R})\}.$$

The boundary conditions (b.c.'s) alluded to in (2.8) are chosen as follows:

$$\lim_{x \to \hat{\sigma}\infty} W(\hat{\psi}_\sigma(\hat{\mu}), g)(x) = 0 \text{ if } \hat{\tau}_{(\hat{\mu}, \hat{\sigma})} \text{ is l.c. at } \hat{\sigma}\infty, \quad (2.9)$$

$$\lim_{x \to -\hat{\sigma}\infty} W(\hat{\psi}_{-\sigma}(\mu), g)(x) = 0 \text{ if } \hat{\tau}_{(\hat{\mu}, \hat{\sigma})} \text{ is l.c. at } -\hat{\sigma}\infty.$$

Here we abbreviate the limit point and limit circle cases by l.p. and l.c., respectively. As usual, the boundary condition at $\omega\infty$ in (2.8) is omitted if $\hat{\tau}_{(\hat{\mu}, \hat{\sigma})}$ is l.p. at $\omega\infty$, $\omega \in \{-, +\}$.

For future reference we note that in analogy to the Dirichlet operators $H_{x_0}^D$, $H_{x_0}^D$ introduced in connection with the operator $H$, one can also introduce the corresponding Dirichlet operators $\hat{H}_{(\hat{\mu}, \hat{\sigma})}$, $\hat{H}_{(\hat{\mu}, \hat{\sigma})}$ associated with $\hat{H}_{(\hat{\mu}, \hat{\sigma})}$.

Next, we turn to the Weyl–Titchmarsh $m$-functions for the Dirichlet deformation operator $\hat{H}_{(\hat{\mu}, \hat{\sigma})}$ and relate them to those of $H$.

Let $\phi(z, x), \theta(z, x)$ be the standard fundamental system of solutions of $(\tau - z)\psi(z) = 0, \ z \in \mathbb{C}$ defined by $\phi(z, x_0) = \theta'(z, x_0) = 0, \ \phi'(z, x_0) = \theta(z, x_0) = 1, \ z \in \mathbb{C}$, and denote by $\tilde{\theta}_{(\hat{\mu}, \hat{\sigma})}(z, x), \tilde{\phi}_{(\hat{\mu}, \hat{\sigma})}(z, x)$ the analogously normalized fundamental system of solutions of $(\hat{\tau}_{(\hat{\mu}, \hat{\sigma})} - z)\psi(z) = 0, \ z \in \mathbb{C}, \ at \ x_0$. One then has

$$m_\sigma(z, x_0) = \psi_\sigma(z, x_0)/\psi_\sigma(z, x_0), \quad \sigma \in \{-, +\}, \ z \in \mathbb{C}\setminus \mathbb{R},$$

where $m_\sigma(z, x_0)$ denotes the Weyl–Titchmarsh $m$-function of $H$ with respect to the half-line $(x_0, \sigma\infty), \ \sigma \in \{-, +\}$. For the corresponding half-line Weyl–Titchmarsh $m$-functions of $\hat{H}_{(\hat{\mu}, \hat{\sigma})}$ in terms of those of $H$ one then obtains the following result.
Theorem 2.2. Assume Hypothesis 2.1 and \( z \in \mathbb{C}\setminus\mathbb{R} \). Let \( H \) and \( \tilde{H}(\tilde{\mu},\tilde{\sigma}) \) be given by (2.2) and (2.8), respectively, and denote by \( m_\pm \) and \( \tilde{m}(\tilde{\mu},\tilde{\sigma})_\pm \) the corresponding m-functions associated with the half-lines \((x_0,\pm\infty)\). Then,

\[
\tilde{m}(\tilde{\mu},\tilde{\sigma})_\pm(z,x_0) = \frac{z - \mu}{z - \tilde{\mu}} m_\pm(z,x_0) - \frac{\tilde{\mu} - \mu}{z - \tilde{\mu}} m_{-\sigma}(\tilde{\mu},x_0), \quad \tilde{\mu} \neq \mu,
\]

\[
\tilde{m}(\tilde{\mu},\tilde{\sigma})_\pm(z,x_0) = m_\pm(z,x_0) - \left( \int_{\sigma\infty}^x \! dx \phi(\mu,x)^2 \right)^{-1} \frac{1}{z - \mu}, \quad (\tilde{\mu},\tilde{\sigma}) = (\mu, -\sigma).
\]

Given the fundamental relation between \( \tilde{m}(\tilde{\mu},\tilde{\sigma})_\pm \) and \( m_\pm \) in Theorem 2.2, one can now readily derive the ensuing relation between the corresponding spectral functions \( \tilde{\rho}(\tilde{\mu},\tilde{\sigma})_\pm \) and \( \rho_\pm \) associated with the half-line Dirichlet operators \( H_{\tilde{\mu},\tilde{\sigma}}^{D,\pm}(\tilde{\mu},\tilde{\sigma})_\pm,\pm \) and \( H_{\mu,x_0}^{D,\pm}. \) For this and a complete spectral characterization of \( H_{\tilde{\mu},\tilde{\sigma}}^{D,\pm,\pm}(\tilde{\mu},\tilde{\sigma})_\pm,\pm \) in terms of \( H_{\mu,x_0}^{D,\pm,\pm} \) we refer to [100].

Next we turn to the principal results of [100] including explicit computations of the Weyl–Titchmarsh and spectral matrices of \( H_{\tilde{\mu},\tilde{\sigma}} \) in terms of those of \( H \) and a complete spectral characterization of \( H_{\tilde{\mu},\tilde{\sigma}} \) and \( H_{\mu,x_0}^{D,\pm} \) in terms of \( H \) and \( H_{\mu,x_0}^{D}. \)

We start with the Weyl–Titchmarsh matrices for \( H \) and \( \tilde{H}(\tilde{\mu},\tilde{\sigma}) \). To fix notation, we introduce the Weyl–Titchmarsh \( M \)-matrix in \( \mathbb{C}^2 \) associated with \( H \) by

\[
M(z,x_0) = (M_{p,q}(z,x_0))_{1 \leq p,q \leq 2} = [m_-(z,x_0) - m_+(z,x_0)]^{-1}
\times \begin{pmatrix}
(m_-(z,x_0)m_+(z,x_0) & m_-(z,x_0) + m_+(z,x_0))/2 \\
[m_-(z,x_0) + m_+(z,x_0)]/2 & 1
\end{pmatrix}, \quad z \in \mathbb{C}\setminus\mathbb{R},
\]

and similarly \( \tilde{M}(\tilde{\mu},\tilde{\sigma}) \) in connection with \( \tilde{H}(\tilde{\mu},\tilde{\sigma}) \). An application of Theorem 2.2 then yields

Theorem 2.3. Assume Hypothesis 2.1 and \( z \in \mathbb{C}\setminus\mathbb{R} \). Let \( H \) and \( \tilde{H}(\tilde{\mu},\tilde{\sigma}) \) be given by (2.3) and (2.32), respectively. Then the corresponding Weyl–Titchmarsh-matrices \( M \) and \( \tilde{M}(\tilde{\mu},\tilde{\sigma}) \) are related by

\[
\tilde{M}(\tilde{\mu},\tilde{\sigma})_{1,1}(z,x_0) = \frac{z - \mu}{z - \tilde{\mu}} M_{1,1}(z,x_0) - 2\frac{\tilde{\mu} - \mu}{z - \tilde{\mu}} M_{-\sigma}(\tilde{\mu},x_0) M_{1,2}(z,x_0)
+ \frac{(\tilde{\mu} - \mu)^2}{(z - \mu)(z - \tilde{\mu})} M_{-\sigma}(\tilde{\mu},x_0)^2 M_{2,2}(z,x_0),
\]

\[
\tilde{M}(\tilde{\mu},\tilde{\sigma})_{1,2}(z,x_0) = M_{1,2}(z,x_0) - \frac{\tilde{\mu} - \mu}{z - \mu} M_{-\sigma}(\tilde{\mu},x_0) M_{2,2}(z,x_0),
\]

\[
\tilde{M}(\tilde{\mu},\tilde{\sigma})_{2,2}(z,x_0) = \frac{z - \mu}{z - \tilde{\mu}} M_{2,2}(z,x_0), \quad \tilde{\mu} \neq \mu.
\]

Given the basic connection between \( \tilde{M}(\tilde{\mu},\tilde{\sigma}) \) and \( M \) in Theorem 2.3, one can now proceed to derive the analogous relations between the spectral matrices \( \tilde{\rho}(\tilde{\mu},\tilde{\sigma}) \) and \( \rho \) associated with \( H_{\tilde{\mu},\tilde{\sigma}} \) and \( H \), respectively (cf. [100] for details).

The principal spectral deformation result of [100] then reads as follows.

Theorem 2.4. Assume Hypothesis 2.1. Then,

(i) Suppose \( \mu, \tilde{\mu} \in (E_0, E_1) \). Then \( H_{\tilde{\mu}} \) and \( H \) are unitarily equivalent. Moreover, \( H_{\tilde{\mu}}^{D,\pm,\pm}(\tilde{\mu},x_0) \) and \( H_{x_0}^{D,\pm,\pm} \), restricted to the orthogonal complements of the one-dimensional eigenspaces corresponding to \( \tilde{\mu} \) and \( \mu \), are unitarily equivalent.
(ii) Assume \( \mu \in \{E_0, E_1\} \cap \sigma_d(H) \), \( \mu \in (E_0, E_1) \). Then,
\[
\sigma_{\{\mu\}}(\tilde{H}_{\mu, \sigma}) = \sigma_{\{\mu\}}(H) \setminus \{ \mu \}, \quad \sigma_{\{\mu\}}(\tilde{H}_{\mu, \sigma}, x_0) = \sigma_{\{\mu\}}(H_{x_0}) \setminus \{ \mu \} \cup \{ \tilde{\mu} \}.
\]

(iii) Suppose \( \mu \in (E_0, E_1) \), \( \tilde{\mu} \in \{E_0, E_1\} \cap \sigma_d(H) \). Then,
\[
\sigma_{\{\mu\}}(\tilde{H}_{\mu}) = \sigma_{\{\mu\}}(H) \setminus \{ \tilde{\mu} \}, \quad \sigma_{\{\mu\}}(\tilde{H}_{\mu, x_0}) = \sigma_{\{\mu\}}(H_{x_0}) \setminus \{ \mu \}, \quad \tilde{\mu} \notin \sigma(\tilde{H}_{\mu, x_0}).
\]

(iv) Assume \( \mu, \tilde{\mu} \in \{E_0, E_1\} \cap \sigma_d(H) \), \( \mu \neq \tilde{\mu} \). Then,
\[
\sigma_{\{\mu\}}(\tilde{H}_{\mu}) = \sigma_{\{\mu\}}(H) \setminus \{E_0, E_1\}, \quad \sigma_{\{\mu\}}(\tilde{H}_{\mu, x_0}) = \sigma_{\{\mu\}}(H_{x_0}) \setminus \{ \mu \}, \quad \tilde{\mu} \notin \sigma(\tilde{H}_{\mu, x_0}).
\]

In cases (ii)–(iv), the corresponding pairs of operators, restricted to the obvious orthogonal complements of the eigenspaces corresponding to \( \mu \) and/or \( \tilde{\mu} \), are unitarily equivalent. In particular,
\[
\sigma_{\text{ess,ac,sc}}(\tilde{H}_{\mu, \sigma}) = \sigma_{\text{ess,ac,sc}}(\tilde{H}_{\mu, \sigma}, x_0) = \sigma_{\text{ess,ac,sc}}(H_{x_0}) = \sigma_{\text{ess,ac,sc}}(H).
\]

**Remark 2.5.** (i) Perhaps the most important consequence of Theorem 2.4 (i), from an inverse spectral point of view, is the fact that any finite number of deformations of Dirichlet data within spectral gaps of \( V \) yields a potential \( \tilde{V} \) in the isospectral class of \( V \). No further constraints on \( (\mu_j, \sigma_j), (\tilde{\mu}_j, \tilde{\sigma}_j) \), other than \( (\mu_j, \sigma_j), (\tilde{\mu}_j, \tilde{\sigma}_j) \in (E_{j-1}, E_j) \times \{-, +\}, (E_{j-1}, E_j) \in \mathbb{R} \setminus \sigma(H), j = 1, \ldots, N, N \in \mathbb{N}, \) are involved.

(ii) The isospectral property (i) in Theorem 2.4, in the special case of periodic potentials, was first proved by Finkel, Isaacson, and Trubowitz [63]. Further results can be found in Buys and Finkel [28] and Iwaseki [132] (see also [46], [48], [174], [175], [176], [177]). Similar constructions for Schrödinger operators on a compact interval can be found in Pöschel and Trubowitz [192, Chs. 3, 4] andRalston and Trubowitz [194].

(iii) Let \( \mu \in (E_0, E_1) \). Then the (isospectral) Dirichlet deformation \( (\mu, \sigma) \rightarrow (\mu, -\sigma) \) is precisely the isospectral case of the double commutation method (cf. [74], [78], [100, App. B], [103]). It simply flips the Dirichlet eigenvalue \( \mu \) on the half-line \( (x_0, \infty) \) to the other half-line \( (x_0, -\infty) \). In the special case where \( V(x) \) is periodic, this procedure was first used by McKean and van Moerbeke [178].

(iv) The topology of these Dirichlet data strongly depends on the nature of the endpoints \( E_0, E_1 \) of a particular spectral gap. For instance, in cases like the periodic one, different spectral gaps are separated by intervals of absolutely continuous spectrum and the two intervals \( [E_0, E_1] \) together with \( \sigma \in \{-, +\} \) can be identified with a circle (upon identifying the two intervals as two rims of a cut). Globally this then leads to a product of circles, that is, a torus. In particular, the Dirichlet eigenvalues in different spectral gaps can be prescribed independently of each other.

The situation is entirely different if an endpoint, say \( E_1 \), belongs to the discrete spectrum of \( H \). In this case there are two neighboring spectral gaps \( (E_0, E_1) \) and \( (E_1, E_2) \) and the two Dirichlet eigenvalues \( \mu_j \in (E_{j-1}, E_j), j = 1, 2 \), are not independent of each other. In fact, if one of \( \mu_1 \) or \( \mu_2 \) approaches \( E_1 \), then necessarily so does the other. The topology is then not a product of circles. For instance, a closer analysis of the case of \( N \)-soliton potentials in (2.1) then illustrates that the appropriate coordinates parametrizing the \( N \)-soliton isospectral class are \( \alpha_j \in (0, \infty) \) (compare also with positive norming constants), which results globally in a product of open half-lines.
Remark 2.6. In certain cases where the base (background) potential \( V \) is reflectionless (see, e.g., [106]) and \( H \) is bounded from below and has no singularly continuous spectrum, the isospectral class \( \text{Iso}(V) \) of \( V \) (the set of all reflectionless \( \tilde{V} \)'s such that \( \sigma(\tilde{V}) = \sigma(H) \)) is completely characterized by the distribution of Dirichlet (initial) data \( (\mu_{j+1}(x_0), \sigma_{j+1}(x_0)) \in [E_j, E_{j+1}] \times \{-, +\}, \quad j \in J \), in nontrivial spectral gaps of \( H \). Here \( x_0 \in \mathbb{R} \) is a fixed reference point and \( J = \{0, 1, \ldots, N - 1\}, \quad N \in \mathbb{N}, \) or \( j \in J = \mathbb{N}_0 (= \mathbb{N} \cup \{0\}) \) parametrizes these nontrivial spectral gaps \( (E_j, E_{j+1}) \) of \( H \) (the trivial one being \((-\infty, \inf \sigma(H))\)). Prime examples of this type are periodic potentials, algebro-geometric quasi-periodic finite-band potentials (cf. [15, Ch. 3], [79, Ch. 1], [159, Chs. 8–12], [172, Ch. 4], [183, Ch. II]), and certain limiting cases thereof (e.g., soliton potentials). In these cases, an iteration of the Dirichlet deformation method, in the sense that \( (\mu_{j+1}(x_0), \sigma_{j+1}(x_0)) \to (\tilde{\mu}_{j+1}(x_0), \tilde{\sigma}_{j+1}(x_0)) \) within \( [E_j, E_{j+1}] \times \{-, +\} \) for each \( j \in J \), independently of each other, yields an explicit realization of the underlying isospectral class \( \text{Iso}(V) \) of reflectionless potentials with base \( V \). In the periodic case, this was first proved by Finkel, Isaacson, and Trubowitz [63] (see also [28], [132]). More precisely, the inclusion of limiting cases \( \mu_{j+1}(x_0) \in \{E_j, E_{j+1}\} \cap \sigma_{\text{ess}}(H) \) requires a special argument (since it is excluded by Hypothesis 2.1) but this can be provided in the special cases at hand.

Remark 2.7. Another case of primary interest concerns potentials \( V \) with purely discrete spectra bounded from below, that is,

\[
\sigma(H) = \sigma_d(H) = \{E_j\}_{j \in \mathbb{N}_0}, \quad -\infty < E_0, \quad E_j < E_{j+1}, \quad j \in \mathbb{N}_0, \quad \sigma_{\text{ess}}(H) = \emptyset.
\]

(For simplicity, one may think in terms of the harmonic oscillator \( V(x) = x^2 \), [32]–[36], [99], [160], [180], [193].) In this case, either

\[
(\mu_{j+1}(x_0), \sigma_{j+1}(x_0)) \in (E_j, E_{j+1}) \times \{-, +\} \quad \text{or} \quad \mu_{j+1}(x_0) = E_{j+1} = \mu_{j+2}(x_0),
\]

that is, Dirichlet eigenvalues necessarily meet in pairs whenever they hit an eigenvalue of \( H \). The following trace formula for \( V \) in terms of \( \sigma(H) = \{E_j\}_{j \in \mathbb{N}_0} \) and \( \sigma(H^D_x) = \{\mu_j(x)\}_{j \in \mathbb{N}} \) (with \( H^D_x \) the Dirichlet operator associated with \( \tau = -\frac{\partial^2}{\partial x^2} + V(x) \) and a Dirichlet boundary condition at \( x = y \), proved in [91] (cf. Section 4),

\[
V(x) = E_0 + \lim_{\alpha \to 0} \alpha^{-1} \sum_{j=1}^{\infty} \left( 2e^{-\alpha \mu_j(x)} - e^{-\alpha E_j} - e^{-\alpha E_{j+1}} \right), \quad (2.10)
\]

then shows one crucial difference to the periodic-type cases mentioned previously. Unlike in the periodic case, though, the initial Dirichlet eigenvalues \( \mu_{j+1}(x_0) \) cannot be prescribed arbitrarily in the spectral gaps \( (E_j, E_{j+1}) \) of \( H \). Indeed, the fact that the Abelian regularization in the trace formula (2.10) for \( V(x) \) converges to a limit restricts the asymptotic distribution of \( \mu_{j+1}(x) \in [E_j, E_{j+1}] \) as \( j \to \infty \). For instance, consider \( V(x) = x^2 - 1 \), then \( E_j = 2j, \quad j \in \mathbb{N}_0 \). The choice \( \mu_j(x_0) = 2j - \gamma, \quad \gamma \in (0, 1), \quad \gamma \in (0, 1), \) then yields for the Abelian regularization on the right-hand side of (2.10),

\[
\lim_{\alpha \to 0} \alpha^{-1} \sum_{j=1}^{\infty} \left( 2e^{-\alpha (2j - \gamma)} - e^{-\alpha 2j} - e^{-\alpha (2j - 2)} \right) = \lim_{\alpha \to 0} 2[(\gamma - 1) + O(\alpha)] \frac{1}{1 - e^{-2\alpha}} = \infty.
\]

Put differently, our choice of \( \mu_j(x_0) = 2j - \gamma, \quad \gamma \in (0, 1), \) was not an admissible choice of Dirichlet eigenvalues for the (shifted) harmonic oscillator potential \( V(x) = x^2 - 1 \). However, as stressed in Remark 2.5 (i), one of the fundamental consequences of [100] concerns the fact that there is no such restriction for any finite number of spectral
gaps of $H$. In other words, only the tail end of the Dirichlet eigenvalues $\mu_{j+1}(x_0)$ as $j \to \infty$ is restricted (the precise nature of this restriction being unknown at this point), any finite number of them can be placed arbitrarily in the spectral gaps $(E_j, E_{j+1})$ (with the obvious “crossing” restrictions at the common boundary $E_{j+1}$ of $(E_j, E_{j+1})$ and $(E_{j+1}, E_{j+2})$). The only other known restriction to date on Dirichlet initial data $(\mu_j(x_0), \sigma_j(x_0))$ is that $\sigma_j(x_0) = -\infty$ and $\sigma_j(x_0) = +\infty$ infinitely often, that is, both half-lines $(-\infty, x_0)$ and $(x_0, \infty)$ support (naturally) infinitely many Dirichlet eigenvalues.

For various extensions of the results presented, including a careful discussion of limit point/limit circle properties of the Dirichlet deformation operators, iterations of DDM to insert finitely many eigenvalues in spectral gaps, applications to reflectionless Schrödinger operators, general Sturm–Liouville operators in a weighted $L^2$-space, applications to short-range scattering theory, and a concise summary of single and double commutation methods, we refer to [100].

More recent references: An interesting refinement of Theorem 2.4 (i), in which a unitary operator relating $\tilde{H}(\tilde{\mu}, \tilde{\sigma})$ and $H$ is explicitly characterized, is due to Schmücke [217]. DDM for one-dimensional Jacobi and Dirac-type operators has been worked out by Teschl [238], [243, Ch. 11] (see also [104], [242], [240].

3. Renormalized Oscillation Theory

In this section we summarize some of the principal results of the paper:

[101] F. Gesztesy, B. Simon, and G. Teschl, Zeros of the Wronskian and renormalized oscillation theory, Amer. J. Math. 118, 571–594 (1996).

For over a hundred and fifty years, oscillation theorems for second-order differential equations have fascinated mathematicians. Originating with Sturm’s celebrated memoir [232], extended in a variety of ways by Böcher [21] and others, a large body of material has been accumulated since then (thorough treatments can be found, e.g., in [42], [147], [203], [236], and the references therein). In [101] a new wrinkle to oscillation theory was added by showing that zeros of Wronskians can be used to count eigenvalues in situations where a naive use of oscillation theory would give $\infty - \infty$ (i.e., Wronskians lead to renormalized oscillation theory). In a nutshell, we will show the following result for general Sturm–Liouville operators $H$ in $L^2((a,b);dx)$ with separated boundary conditions at $a$ and $b$ in this section: If $E_{1,2} \in \mathbb{R}$ and if $u_{1,2}$ solve the differential equation $Hu_j = E_j u_j$, $j = 1, 2$ and respectively satisfy the boundary condition on the left/right, then the dimension of the spectral projection $P_{(E_1, E_2)}(H)$ of $H$ equals the number of zeros of the Wronskian of $u_1$ and $u_2$.

The main motivation in writing [101] had its origins in attempts to provide a general construction of isospectral potentials for one-dimensional Schrödinger operators following previous work by Finkel, Isaacson, and Trubowitz [63] (see also [28]) in the special case of periodic potentials. In fact, in the case of periodic Schrödinger operators $H$, the nonvanishing of $W(u_1, u_2)(x)$ for Floquet solutions $u_1 = \psi_{\varepsilon_1}(E_1)$, $u_2 = \psi_{\varepsilon_2}(E_2)$, $\varepsilon_{1,2} \in \{+, -\}$ of $H_p$ for $E_1$ and $E_2$ in the same spectral gap of $H$, is proved in [63]. The extension of these ideas to general one-dimensional Schrödinger operators was done in [100] and is reviewed in Section 2 of this survey. So while [101] is not directly related to the overarching inverse
spectral theory topic of this survey, we decided to include it because of its relevance in connection with Section 2.

To set the stage, we consider Sturm–Liouville differential expressions of the form

\[(\tau u)(x) = r(x)^{-1}[-p(x)u'(x)'] + q(x)u(x)], \quad x \in (a, b), \quad -\infty \leq a < b \leq \infty\]

where

\[r, p^{-1}, q \in L^1_{\text{loc}}((a, b))\] are real-valued and \(r, p > 0\) a.e. on \((a, b)\).

We shall use \(\tau\) to describe the formal differentiation expression and \(H\) for the operator in \(L^2((a, b); r\, dx)\) given by \(\tau\) with separated boundary conditions at \(a\) and/or \(b\).

If \(a\) (resp., \(b\)) is finite and \(q, p^{-1}, r\) are in addition integrable near \(a\) (resp., \(b\)), \(a\) (resp., \(b\)) is called a regular end point. \(\tau\), respectively \(H\), is called regular if both \(a\) and \(b\) are regular. As is usual ([57, Sect. XIII.2], [182, Sect. 17], [249, Ch. 3]), we consider the local domain

\[D_{\text{loc}} = \{u \in AC_{\text{loc}}((a, b)) \mid pu' \in AC_{\text{loc}}((a, b)), \quad \tau u \in L^2_{\text{loc}}((a, b); r\, dx)\},\]

where \(AC_{\text{loc}}((a, b))\) is the set of locally absolutely continuous functions on \((a, b)\). General ODE theory shows that for any \(E \in \mathbb{C}, x_0 \in (a, b)\), and \((\alpha, \beta) \in \mathbb{C}^2\), there is a unique \(u \in D_{\text{loc}}\) such that \(-(pu')' + q - Eu = 0\) for a.e. \(x \in (a, b)\) and \((u(x_0), (pu')(x_0)) = (\alpha, \beta)\).

The maximal and minimal operators are defined by taking

\[\text{dom}(T_{\text{max}}) = \{u \in L^2((a, b); r\, dx) \cap D_{\text{loc}} \mid \tau u \in L^2((a, b); r\, dx)\},\]

with

\[T_{\text{max}}u = \tau u.\]

\(T_{\text{min}}\) is the operator closure of \(T_{\text{max}} \upharpoonright D_{\text{loc}} \cap \{u\ \text{has compact support in} \ (a, b)\}\). Then \(T_{\text{min}}\) is symmetric and \(T_{\text{min}}^* = T_{\text{max}}^*\).

According to Weyl’s theory of self-adjoint extensions ([57, Sect. XIII.6], [182, Sect. 18], [201, App. to X.1], [248, Section 8.4], [249, Chs. 4, 5]), the deficiency indices of \(T_{\text{min}}\) are \((0, 0)\) or \((1, 1)\) or \((2, 2)\) depending on whether it is limit point at both, one, or neither endpoint. Moreover, the self-adjoint extensions can be described in terms of Wronskians ([57, Sect. XIII.2], [182, Sects. 17, 18], [248, Sect. 8.4], [249, Ch. 3]). Define

\[W(u_1, u_2)(x) = u_1(x)(pu_2')(x) - (pu_1')(x)u_2(x).\]

Then if \(T_{\text{min}}\) is limit point at both ends, \(T_{\text{min}} = T_{\text{max}} = H\). If \(T_{\text{min}}\) is limit point at \(b\) but not at \(a\), for \(H\) any self-adjoint extension of \(T_{\text{min}}\), if \(\varphi_\pm\) is any function in \(\text{dom}(H)\) \(\backslash\text{dom}(T_{\text{min}})\), then

\[\text{dom}(H) = \{u \in \text{dom}(T_{\text{max}}) \mid W(u, \varphi_-)(x) \to 0 \text{ as } x \downarrow a\}.\]

Finally, if \(u_1\) is limit circle at both ends, the operators \(H\) with separated boundary conditions are those for which we can find \(\varphi_{\pm} \in \text{dom}(H)\), \(\varphi_\pm = 0\) near \(a\), \(\varphi_- = 0\) near \(b\), and \(\varphi_{\pm} \in \text{dom}(H)\) \(\backslash\text{dom}(T_{\text{min}})\). In that case,

\[\text{dom}(H) = \{u \in D(T_{\text{max}}) \mid W(u, \varphi_-)(x) \to 0 \text{ as } x \downarrow a, W(u, \varphi_+)(x) \to 0 \text{ as } x \uparrow b\}.\]

Of course, if \(H\) is regular, we can just specify the boundary conditions by taking values at \(a, b\) since by regularity any \(u \in \text{dom}(T_{\text{max}})\) has \(u, pu'/u'\) continuous on \([a, b]\).

It follows from this analysis that

if \(u_{1,2} \in \text{dom}(H)\), then \(W(u_1, u_2)(x) \to 0\) as \(x \to a\) or \(b\).
Such operators will be called SL operators (for Sturm–Liouville, but SL includes separated boundary conditions (if necessary)) and denoted by $H$.

It will be convenient to write $\ell_- = a$, $\ell_+ = b$.

Throughout this section we will denote by $\psi_\pm(z, x) \in D_{\text{loc}}$ solutions of $\tau \psi = z \psi$ so that $\psi_\pm(z, \cdot)$ is $L^2$ at $\ell_\pm$ and $\psi_\pm(z, \cdot)$ satisfies the appropriate boundary condition at $\ell_\pm$ in the sense that for any $u \in \text{dom}(H)$, $\lim_{x \to \ell_\pm} W(\psi_\pm(z), u)(x) = 0$. If $\psi_\pm(z, \cdot)$ exist, they are unique up to constant multiples. In particular, $\psi_\pm(z, \cdot)$ exist for $z$ not in the essential spectrum of $H$ and we can assume them to be holomorphic with respect to $z$ in $\mathbb{C}\setminus\sigma(H)$ and real for $z \in \mathbb{R}$. One can choose

$$\psi_\pm(z, x) = ((H - z)^{-1} \chi_{(c,d)})(x) \text{ for } x < c \text{ and } x > d, \quad a < c < d < b \quad (3.1)$$

and uniquely continue $\psi_\pm(z, x)$ for $x > c$ and $x < d$. Here $(H - z)^{-1}$ denotes the resolvent of $H$ and $\chi_{\Omega}$ the characteristic function of the set $\Omega \subseteq \mathbb{R}$. Clearly we can include a finite number of isolated eigenvalues in the domain of holomorphy of $\psi_\pm$ by removing the corresponding poles. Moreover, to simplify notations, all solutions $u$ of $\tau u = Eu$ are understood to be not identically vanishing and solutions associated with real values of the spectral parameter $E$ are assumed to be real-valued in this paper. Thus if $E$ is real and in the resolvent set for $H$ or an isolated eigenvalue, we are guaranteed there are solutions that satisfy the boundary conditions at $a$ or $b$. If $E$ is in the essential spectrum, it can happen that such solutions do not exist or it may happen that they do. In Theorems 3.15, 3.16 below, we shall explicitly assume such solutions exist for the energies of interest. If these energies are not in the essential spectrum, that is automatically fulfilled.

The key idea in [101] is to look at zeros of the Wronskian. That zeros of the Wronskian are related to oscillation theory is indicated by an old paper of Leighton [151], who noted that if $u_j, pu'_j \in AC_{\text{loc}}((a, b))$, $j = 1, 2$ and $u_1$ and $u_2$ have a nonvanishing Wronskian $W(u_1, u_2)$ in $(a, b)$, then their zeros must intertwin each other. (In fact, $pu'_1$ must have opposite signs at consecutive zeros of $u_1$, so by nonvanishing of $W$, $u_2$ must have opposite signs at consecutive zeros of $u_1$ as well. Interchanging the role of $u_1$ and $u_2$ yields strict interlacing of their zeros.) Moreover, let $E_1 < E_2$ and $\tau u_j = E_j u_j$, $j = 1, 2$. If $x_0, x_1$ are two consecutive zeros of $u_1$, then the number of zeros of $u_2$ inside $(x_0, x_1)$ is equal to the number of zeros of the Wronskian $W(u_1, u_2)$ plus one (cf. Theorem 3.20). Hence the Wronskian comes with a built-in renormalization counting the additional zeros of $u_2$ in comparison to $u_1$.

We let $W_0(u_1, u_2)$ be the number of zeros of the Wronskian in the open interval $(a, b)$ not counting multiplicities of zeros. Given $E_1 < E_2$, we let $N_0(E_1, E_2) = \dim(\text{ran}(P(E_1, E_2)(H)))$ be the dimension of the spectral projection $P(E_1, E_2)(H)$ of $H$.

We begin by presenting two aspects of zeros of the Wronskian which are critical for the two halves of our proofs (i.e., for showing $N_0 \geq W_0$ and that $N_0 \leq W_0$). First, the vanishing of the Wronskian lets us patch solutions together:

**Lemma 3.1.** Suppose that $\psi_{+, j}, \psi_- \in D_{\text{loc}}$ and that $\psi_{+, j}$ and $\tau \psi_{+, j}$, $j = 1, 2$ are in $L^2((c, b))$ and that $\psi_-$ and $\tau \psi_-$ are in $L^2((a, c))$ for all $c \in (a, b)$. Suppose, in addition, that $\psi_{+, j}$, $j = 1, 2$ satisfy the boundary condition defining $H$ at $b$ (i.e., $W(u, \psi_{+, j})(c) \to 0$ as $c \uparrow b$ for all $u \in \text{dom}(H)$) and similarly, that $\psi_-$ satisfies the boundary condition at $a$. Then,

1. If $W(\psi_{+, 1}, \psi_{+, 2})(c) = 0$ and $(\psi_{+, 2}(c), (pu'_{+, 2})(c)) \neq (0, 0)$, then there exists a $\gamma$
such that
\[ \eta = \chi_{[c,b]}(\psi_{-1} - \gamma \psi_{+1}) \in \text{dom}(H) \]
and
\[ H\eta = \chi_{[c,b]}(\tau \psi_{+1} - \gamma \tau \psi_{+2}). \]

(ii) If \( W(\psi_{+1}, \psi_{-})(c) = 0 \) and \((\psi_{-1}, (pu')_{(c)})(c) \neq (0, 0), \) then there is a \( \gamma \) such that
\[ \eta = \gamma \chi_{[a,c]} \psi_{+1} + \chi_{(c,b)} \psi_{+1} \in \text{dom}(H) \]
and
\[ H\eta = \gamma \chi_{(a,c)} \tau \psi_{+1} + \chi_{(c,b)} \prescript{\tau}{}{\psi}_{+1}. \]

The second aspect connects zeros of the Wronskian to Prüfer variables \( \rho_u, \theta_u \) (for \( u, pu' \) continuous) defined by
\[ u(x) = \rho_u(x) \sin(\theta_u(x)), \quad (pu')(x) = \rho_u(x) \cos(\theta_u(x)). \]
If \((u(x), (pu')(x))\) is never \((0, 0), \) then \( \rho_u \) can be chosen positive and \( \theta_u \) is uniquely determined once a value of \( \theta_u(x_0) \) is chosen subject to the requirement that \( \theta_u \) be continuous in \( x. \)
Notice that
\[ W(u_1, u_2)(x) = \rho_{u_1}(x) \rho_{u_2}(x) \sin(\theta_{u_1}(x) - \theta_{u_2}(x)). \]
Thus, one obtains the following results.

**Lemma 3.2.** Suppose \((u_j, pu'_j), j = 1, 2\) are never \((0, 0). \) Then \( W(u_1, u_2)(x_0) \) is zero if and only if \( \theta_u(x_0) = \theta_u(x_0)(\mod \pi). \)

In linking Prüfer variables to rotation numbers, an important role is played by the observation that because of
\[ u(x) = \int_{x_0}^x dt \frac{\theta_u(t) \cos(\theta_u(t))}{p(t)}, \]
\( \theta_u(x_0) = 0 (\mod \pi) \) implies \(|\theta_u(x) - \theta_u(x_0)|/(x-x_0) > 0 \) for \( 0 < |x-x_0| \) sufficiently small and hence for all \( 0 < |x-x_0| \) if \((u, pu')(\neq 0, 0). \) (In fact, suppose \( x_1 \neq x_0 \) is the closest \( x \) such that \( \theta_u(x_1) = \theta_u(x_0) \) then apply the local result at \( x_1 \) to obtain a contradiction.) We summarize:

**Lemma 3.3.** (i) If \((u, pu') \neq (0, 0) \) then \( \theta_u(x_0) = 0 (\mod \pi) \) implies
\[ |\theta_u(x) - \theta_u(x_0)|/(x-x_0) > 0 \]
for \( x \neq x_0. \) In particular, if \( \theta_u(c) \in (0, \pi) \) and \( u \) has \( n \) zeros in \((c, d), \) then \( \theta_u(d-\epsilon) \in (n\pi, (n+1)\pi) \) for sufficiently small \( \epsilon > 0. \)
(ii) Let \( E_1 < E_2 \) and assume that \( u_{1,2} \) solve \( \tau u_j = E_j u_j, j = 1, 2. \) Let \( \Delta(x) = \theta_{u_2}(x) - \theta_{u_1}(x). \) Then \( \Delta(x_0) = 0 (\mod \pi) \) implies \( (\Delta(x) - \Delta(x_0))/(x-x_0) > 0 \) for \( 0 < |x-x_0| \).

**Remark 3.4.** (i) Suppose \( r, p \) are continuous on \((a, b). \) If \( \theta_{u_1}(x_0) = 0 (\mod \pi) \) then \( \theta_{u_1}(x) - \theta_{u_1}(x_0) = c_0(x-x_0)+o(x-x_0) \) with \( c_0 > 0. \) If \( \Delta(x_0) = 0 (\mod \pi) \) and \( \theta_{u_1}(x_0) \neq 0 (\mod \pi), \) then \( \Delta(x) - \Delta(x_0) = c_1(x-x_0)+o(x-x_0) \) with \( c_1 > 0. \) If \( \theta_{u_1}(x_0) = 0 = \Delta(x_0) (\mod \pi), \) then \( \Delta(x) - \Delta(x_0) = c_2(x-x_0)^3 + o(x-x_0)^3 \) with \( c_2 > 0. \) Either way, \( \Delta \) increases through \( x_0. \) (In fact, \( c_0 = p(x_0)^{-1}, \) \( c_1 = (E_2 - E_1)r(x_0) \sin^2(\theta_{u_1}(x_0)) \) and \( c_2 = \frac{1}{4}r(x_0)p(x_0)^{-2}(E_2 - E_1). \))
(ii) In other words, Lemma 3.3 implies that the integer parts of \( \theta_u/\pi \) and \( \Delta_{u,v}/\pi \) are increasing with respect to \( x \in (a, b) \) (even though \( \theta_u \) and \( \Delta_{u,v} \) themselves are
not necessarily monotone in $x$).

(iii) Let $E \in [E_1, E_2]$ and assume $[E_1, E_2]$ to be outside the essential spectrum of $H$. Then, for $x \in (a, b)$ fixed,

$$\frac{d \theta_{\psi_{\pm}}(E, x)}{dE} = -\frac{\int_{x}^{x} dt \psi_{\pm}(E, t)^2}{\rho_{\psi_{\pm}}(E, x)}$$

proves that $\mp \theta_{\psi_{\pm}}(E, x)$ is strictly increasing with respect to $E$.

We continue with some preparatory results in the regular case.

**Lemma 3.5.** Assume $H$ to be a regular SL operator.

(i) Let $u_{1, 2}$ be eigenfunctions of $H$ with eigenvalues $E_1 < E_2$ and let $\ell$ be the number of eigenvalues of $H$ in $(E_1, E_2)$. Then $W(u_1, u_2)(x)$ has exactly $\ell$ zeros in $(a, b)$.

(ii) Let $E_1 \leq E_2$ be eigenvalues of $H$ and suppose $[E_1, E_2]$ has $\ell$ eigenvalues. Then for $\epsilon \geq 0$ sufficiently small, $W_0(\psi_-(E_1 - \epsilon), \psi_+(E_2 + \epsilon)) = \ell$.

(iii) Let $E_3 < E_4 < E$ and $u$ be any solution of $\tau u = Eu$. Then,

$$W_0(\psi_-(E_3), u) \geq W_0(\psi_-(E_4), u). \quad (3.2)$$

Similarly, if $E_3 > E_4 > E$ and $u$ is any solution of $\tau u = Eu$, then $(3.2)$ holds.

(iv) Item (iii) remains true if every $\psi_-$ is replaced by a $\psi_+$.

**Remark 3.6.** (i) Since $(E_1, E_2)$ has $\ell - 2$ eigenvalues, Lemma 3.5(i) implies that the Wronskian $W(\psi_-(E_1), \psi_+(E_2))(x)$ has $\ell - 2$ zeros in $(a, b)$ and clearly it has zeros at $a$ and $b$. Essentially, Lemma 3.5(ii) implies that replacing $E_1$ by $E_1 - \epsilon$ and $E_2$ by $E_2 + \epsilon$ moves the zeros at $a, b$ inside $(a, b)$ to give $\ell - 2 + 2 = \ell$ zeros.

(ii) Lemma 3.5(iv) follows from Lemma 3.5(iii) upon reflecting at some point $c \in (a, b)$, implying an interchange of $\psi_+$ and $\psi_-$.

Lemma 3.5 then implies the following result.

**Lemma 3.7.** Let $H$ be a regular SL operator and suppose $E_1 < E_2$. Then,

$$W_0(\psi_-(E_1), \psi_+(E_2)) \geq N_0(E_1, E_2).$$

Using the approach of Weidmann ([249, Ch. 14]) to control some limits, one can remove the assumption that $H$ is regular in Lemma 3.7 as follows.

Fix functions $u_1, u_2 \in D_{loc}$. Pick $c_n \downarrow a$, $d_n \uparrow b$. Define $H_n$ on $L^2((c_n, d_n); r \, dx)$ by imposing the following boundary conditions on $\eta \in \text{dom}(H_n)$

$$W(u_1, \eta)(c_n) = 0 = W(u_2, \eta)(d_n).$$

On $L^2((a, b); r \, dx) = L^2((a, c_n); r \, dx) \oplus L^2((c_n, d_n); r \, dx) \oplus L^2((d_n, b); r \, dx)$ take $H_n = \alpha I \oplus H + \alpha I$ with $\alpha$ a fixed real constant. Then Weidmann proves:

**Lemma 3.8.** Suppose that either $H$ is limit point at $a$ or that $u_1$ is a $\psi_-(E, x)$ for some $E$ and similarly, that either $H$ is limit point at $b$ or $u_2$ is a $\psi_+(E, x)$ for some $E'$. Then $H_n$ converges to $H$ in strong resolvent sense as $n \to \infty$.

The idea of Weidmann’s proof is that it suffices to find a core $D_0$ of $H$ such that for every $\eta \in D_0$ there exists an $n_0 \in \mathbb{N}$ with $\eta \in D_0$ for $n \geq n_0$ and $H_n \eta \to H \eta$ as $n$ tends to infinity (see [248, Theorem 9.16 (i)])]. If $H$ is limit point at both ends, take $\eta \in D_0 = \{ u \in D_{loc} \mid \text{supp}(u) \text{ compact in } (a, b) \}$. Otherwise, pick $\tilde{u}_1, \tilde{u}_2 \in \text{dom}(H)$ with $\tilde{u}_2 = u_2$ near $b$ and $\tilde{u}_2 = 0$ near $a$ and with $\tilde{u}_1 = u_1$ near $a$ and $\tilde{u}_1 = 0$ near $b$. Then pick $\eta \in D_0 + \text{span}(\tilde{u}_1, \tilde{u}_2)$ which one can show is a core for $H$ ([249, Ch. 14]).

Secondly one uses the following fact:
Lemma 3.9. Let \( A_n \to A \) in a strong resolvent sense as \( n \to \infty \). Then,
\[
\dim(\operatorname{ran}(P_{(E_1,E_2)}(A))) \leq \lim_{n \to \infty} \dim(\operatorname{ran}(P_{(E_1,E_2)}(A_n))).
\]

Combining Lemmas 3.7–3.9 then yields the following result.

Theorem 3.10. Let \( E_1 < E_2 \). If \( u_1 = \psi_-(E_1) \) and either \( u_2 = \psi_+(E_2) \) or \( \tau u_2 = E_2u_2 \) and \( H \) is limit point at \( b \). Then,
\[
W_0(u_1,u_2) \geq \dim(\operatorname{ran}((P_{(E_1,E_2)}(H))).
\]

Next, we indicate how the following result can be proved:

Lemma 3.12. \( \langle \eta_j, \eta_k \rangle = \langle \tilde{\eta}_j, \tilde{\eta}_k \rangle \) for all \( j, k \), where \( \langle \cdot, \cdot \rangle \) is the \( L^2((a,b); r \, dx) \) inner product.

Notice that by (3.2),
\[
(H - \frac{E_2 + E_1}{2}) \eta_j = \left( \frac{E_1 - E_2}{2} \right) \tilde{\eta}_j.
\]

This result and Lemma 3.12 imply the following lemma.

Lemma 3.13. If \( \eta \) is in the span of the \( \eta_j \), then
\[
\left\| \left( H - \frac{E_2 + E_1}{2} \right) \eta \right\| = \left| \frac{E_2 - E_1}{2} \right| \| \eta \|.
\]

Thus, \( \dim(\operatorname{ran}(P_{[E_1,E_2]}(H))) \geq \dim(\operatorname{span}(\{\eta_j\})) \). But \( u_1 \) and \( u_2 \) are independent on each interval (since their Wronskian is nonconstant) and so the \( \eta_j \) are linearly independent. This proves Theorem 3.11 in the \( \psi_-(E_1), \psi_+(E_2) \) case. The case \( u_1 = \psi_-(E_1), u_2 = \psi_-(E_2) \) is proved similarly. The cases \( u_1 = \psi_+(E_1), u_2 = \psi_+(E_2) \) can be obtained by reflection.

Next one proves the following result.

Theorem 3.14. Let \( E_1 \neq E_2 \). Let \( \tau u_j = E_j u_j, j = 1, 2, \tau v_2 = E_2v_2 \) with \( u_2 \) linearly independent of \( v_2 \). Then the zeros of \( W(u_1,v_2) \) interlace the zeros of \( W(u_1,v_2) \) and vice versa (in the sense that there is exactly one zero of one function in between two zeros of the other). In particular, \( |W_0(u_1,u_2) - W_0(u_1,v_2)| \leq 1 \).
Theorems 3.10, 3.11, and 3.14 then yield the following two theorems, the principal results of [101]:

**Theorem 3.15.** Suppose $E_1 < E_2$. Let $u_1 = \psi_-(E_1)$ and $u_2 = \psi_+(E_2)$. Then,

$$W_0(u_1, u_2) = N_0(E_1, E_2).$$

**Theorem 3.16.** Suppose $E_1 < E_2$. Let $u_1 = \psi_-(E_1)$ and $u_2 = \psi_-(E_2)$. Then either,

$$W_0(u_1, u_2) = N_0(E_1, E_2),$$

or,

$$W_0(u_1, u_2) = N_0(E_1, E_2) - 1.$$

If either $N_0 = 0$ or $H$ is limit point at $b$, then (3.4) holds.

One infers that if $b$ is a regular point and $E_2 > E_1$ with $\epsilon$ an eigenvalue and $|E_2 - E_1|$ is small, then (3.5) holds rather than (3.4). One also sees that if $u_{1,2}$ are arbitrary solutions of $\tau u_j = E_j u_j$, $j = 1, 2$, then, in general, $|W_0 - N_0| \leq 2$ (this means that if one of the quantities is infinite, the other is as well) and we note that any of $0, \pm 1, \pm 2$ can occur for $W_0 - N_0$. Especially, if either $E_1$ or $E_2$ is in the interior of the essential spectrum of $H$ (or $\dim(ran(P_{(E_1, E_2)}(H))) = \infty$), then $W_0(u_1, u_2) = \infty$ for any $u_1$ and $u_2$ satisfying $\tau u_j = E_j u_j$, $j = 1, 2$ (cf. Theorem 3.19).

**Remark 3.17.** Of course, by reflecting about a point $c \in (a, b)$, Theorems 3.10, 3.15, and 3.16 hold for $u_1 = \psi_+(E_1)$ and $u_2 = \psi_-(E_2)$ (and either $N_0 = 0$ or $H$ is limit point at $a$ in the corresponding analog of Theorem 3.16 yields (3.4)) and similarly, $\tau u_2 = E_2 u_2$ and $H$ is limit point at $a$ yields the conclusion in the corresponding analog of Theorem 3.10.

We add a few more results proved in [101].

By applying Theorem 3.14 twice, one concludes

**Theorem 3.18.** Let $E_1 \neq E_2$. Let $u_1, u_2, v_1, v_2$ be the linearly independent functions with $\tau u_j = E_j u_j$ and $\tau v_j = E_j v_j$. Then,

$$|W_0(u_1, u_2) - W_0(v_1, v_2)| \leq 2.$$

**Theorem 3.19.** If $\dim(ran(P_{\{E_1, E_2\}}(H))) = \infty$, then $W_0(u_1, u_2) = \infty$ for any $u_1$ and $u_2$ satisfying $\tau u_j = E_j u_j$, $j = 1, 2$.

**Theorem 3.20.** Let $E_1 < E_2$. Let $\tau u_j = E_j u_j$, $j = 1, 2$. If $a < x_0 < x_1 < b$ are zeros of $u_1$ or of $W(u_1, u_2)(.)$, then the number of zeros of $u_2$ inside $(x_0, x_1)$ equals the number of zeros of $W(u_1, u_2)(.)$ inside $(x_0, x_1)$ plus the number of zeros of $u_1$ inside $(x_0, x_1)$ plus one.

The following result is of special interest in connection with the problem of whether the total number of eigenvalues of $H$ in one of its essential spectral gaps is finite or infinite. In particular, the energies $E_1, E_2$ in Theorem 7.5 below may lie in the essential spectrum of $H$. For this purpose we consider an auxiliary Dirichlet operator $H^D_\beta$, $x_0 \in (a, b)$ associated with $H$. $H^D_\beta$ is obtained by taking the direct sum of the restrictions $H^D_{x_0, x_1}$ of $H$ to $(a, x_0)$, respectively $(x_0, b)$, with a Dirichlet boundary condition at $x_0$. We emphasize that the Dirichlet boundary conditions can be replaced by boundary conditions of the type $\lim_{\epsilon \downarrow 0} [u'(x_0 \pm \epsilon) + \beta u(x_0 \pm \epsilon)] = 0$, $\beta \in \mathbb{R}$.
Theorem 3.21. Let $E_1 < E_2$. Let $\tau u_j = E_j u_j$, $\tau s_j = E_j s_j$, and $s_j(E_j, x_0) = 0$, $j = 1, 2$. Then,

$$\dim(\text{ran}(P_{(E_1, E_2)}(H))) < \infty \text{ if and only if } W_0(u_1, u_2) < \infty,$$

$$\dim(\text{ran}(P_{(E_1, E_2)}(H))) - 1 \leq \dim(\text{ran}(P_{(E_1, E_2)}(H^{D}_{x_0}))) \leq \dim(\text{ran}(P_{(E_1, E_2)}(H))) + 2,$$

$$W_0(s_1, s_2) - 1 \leq \dim(\text{ran}(P_{(E_1, E_2)}(H^{D}_{x_0}))) \leq W_0(s_1, s_2) + 1.$$

For an application of this circle of ideas to the notion of the density of states we refer to [101].

More recent references: Oscillation and renormalized oscillation theory was also put in perspective by Simon’s contribution [229] to the the Festschrift [8] in honor of Sturm and Liouville. Renormalized oscillation theory for one-dimensional Jacobi and Dirac-type operators was developed by Teschl [237] (see also [243, Sect. 4.3]) and [241]. For an interesting application of some of the results in [101] to the stability theory of complete minimal surfaces we refer to a paper by Schmidt [215]. For additional results on oscillation theory, critical coupling constants, and eigenvalue asymptotics we refer to Schmidt [214].

4. Trace Formulas for Schrödinger and Jacobi Operators: The $\xi$ Function

In this section we summarize some of the principal results of the following papers:

[80] F. Gesztesy, H. Holden, and B. Simon, Absolute summability of the trace relation for certain Schrödinger operators, Commun. Math. Phys. 168, 137–161 (1995).

[81] F. Gesztesy, H. Holden, B. Simon, and Z. Zhao, Trace formulae and inverse spectral theory for Schrödinger operators, Bull. Amer. Math. Soc. 29, 250–255 (1993).

[82] F. Gesztesy, H. Holden, B. Simon, and Z. Zhao, Higher order trace relations for Schrödinger operators, Rev. Math. Phys. 7, 893–922 (1995).

[83] F. Gesztesy, H. Holden, B. Simon, and Z. Zhao, A trace formula for multidimensional Schrödinger operators, J. Funct. Anal. 141, 449–465 (1996).

[91] F. Gesztesy and B. Simon, The $\xi$ function, Acta Math. 176, 49–71 (1996).

We start with [91]. One of the principal goals in [91] was to introduce a special function $\xi(\cdot, \cdot)$ on $\mathbb{R} \times \mathbb{R}$ associated with one-dimensional Schrödinger operators $H$ (and Jacobi operators $h$) which led to a generalization of the known trace formula for periodic Schrödinger operators for general potentials $V$ and established $\xi$ as a new tool in the spectral theory of one-dimensional Schrödinger operators and (multi-dimensional) Jacobi operators.

To illustrate this point we recall the well-known trace formula for periodic potentials $V$ of period $\alpha > 0$: Then, by Floquet theory (see, e.g., [58], [163], [202])

$$\sigma(H) = [E_0, E_1] \cup [E_2, E_3] \cup \ldots$$
a set of bands. If $V$ is $C^1(\mathbb{R})$, one can show that the sum of the gap sizes is finite, that is,
\[
\sum_{n=1}^{\infty} |E_{2n} - E_{2n-1}| < \infty. \tag{4.1}
\]

For fixed $y$, let $H_y$ be the operator $-\frac{d^2}{dx^2} + V$ in $L^2([y, y+a])$ with Dirichlet boundary conditions $u(y) = u(y+a) = 0$. Its spectrum is discrete, that is, there are eigenvalues $\{\mu_n(y)\}_{n=1}^{\infty}$ with
\[
E_{2n-1} \leq \mu_n(y) \leq E_{2n}. \tag{4.2}
\]
The trace formula for $V$ then reads
\[
V(y) = E_0 + \sum_{n=1}^{\infty} [E_{2n} + E_{2n-1} - 2\mu_n(y)]. \tag{4.3}
\]
By (4.2), $|E_{2n} + E_{2n-1} - 2\mu_n(y)| \leq |E_{2n} - E_{2n-1}|$ so (4.1) implies the convergence of the sum in (4.3). An elegant direct proof of (4.3) can be found, for instance, in [225, Sect. 26].

The earliest trace formula for Schrödinger operators was found on a finite interval in 1953 by Gel’fand and Levitan [71] with later contributions by Dikii [55], Gel’fand [69], Halberg and Kramer [113], and Gilbert and Kramer [109]. The first trace formula for periodic $V$ was obtained in 1965 by Hochstadt [118], who showed that for finite-band potentials
\[
V(x) - V(0) = 2 \sum_{n=1}^{g} [\mu_n(0) - \mu_n(x)].
\]
Dubrovin [9] then proved (4.3) for finite-band potentials. The general formula (4.3) under the hypothesis that $V$ is periodic and in $C^\infty(\mathbb{R})$ was proved in 1975 by Flaschka [65] and McKean and van Moerbeke [178], and later for general $C^3(\mathbb{R})$ potentials by Trubowitz [246]. Formula (4.3) is a key element of the solution of inverse spectral problems for periodic potentials [56], [65], [118], [159, Ch. 11], [172, Sect. 4.3], [178], [179], [246].

There have been two classes of potentials for which (4.3) has been extended. Certain almost periodic potentials are studied in Craig [43], Levitan [158, [159, Ch. 11], and Kotani-Krishna [141].

In 1979, Deift and Trubowitz [48] proved that if $V(x)$ decays sufficiently rapidly at infinity and $-\frac{d^2}{dx^2} + V$ has no negative eigenvalues, then
\[
V(x) = \frac{2i}{\pi} \int_{-\infty}^{\infty} dk k \ln \left[ 1 + R(k) \frac{f_+(x,k)}{f_-(x,k)} \right] \tag{4.4}
\]
(where $f_{\pm}(x,k)$ are the Jost functions at energy $E = k^2$ and $R(k)$ is a reflection coefficient) which can be shown to be an analog of (4.3). Previously, Venakides [247] studied a trace formula for $V$, a positive smooth potential of compact support, by writing (4.3) for the periodic potential $V_L(x) = \sum_{n=-\infty}^{\infty} V(x+nL)$ and then taking $L$ to $\infty$. He found an integral formula which is precisely (4.4) (although, this was not identified as such in [247]).

The basic definition of $\xi$ depends on the theory of the Lifshits–Krein spectral shift function [146]. If $A$ and $B$ are self-adjoint operators bounded from below, that is, $A \geq \eta, B \geq \eta$ for some real $\eta$, and so that $[(A+i)^{-1} - (B+i)^{-1}]$ is trace
class, then there exists a measurable function $\xi(\lambda)$ associated with the pair $(B, A)$ so that
\[ \text{Tr}[f(A) - f(B)] = -\int_{\mathbb{R}} d\lambda f'(\lambda)\xi(\lambda) \]  
(4.5)
for a class of functions $f$ which are sufficiently smooth and which decay sufficiently rapidly at infinity, and, in particular, for $f(\lambda) = e^{-t\lambda}$ for any $t > 0$; and so that
\[ \xi(\lambda) = 0 \text{ for } \lambda < \eta. \]  
(4.6)
Moreover, (4.5), (4.6) uniquely determine $\xi(\lambda)$ for a.e. $\lambda$. Moreover, if $|(A + i)^{-1} - (B + i)^{-1}|$ is rank $n$, then $|\xi(\lambda)| \leq n$ and if $B \geq A$, then $\xi(\lambda) \geq 0$.

For the rank-one case of importance in this paper, an extensive study of $\xi$ can be found in [227].

Let $V$ be a continuous function on $\mathbb{R}$ which is bounded from below. Let $H = -\frac{d^2}{dx^2} + V$ in $L^2(\mathbb{R})$ which is essentially self-adjoint on $C_0^\infty(\mathbb{R})$ and let $H^D_x$ be the operator on $L^2((-\infty, x)) \oplus L^2((x, \infty))$ with $u(x) = 0$ Dirichlet boundary conditions. Then $[(H_x^D + i)^{-1} - (H + i)^{-1}]$ is rank one, so there results a spectral shift function $\xi(\lambda, x)$ for the pair $(H^D_x, H)$ which, in particular, satisfies,
\[ \text{Tr}[e^{-tH} - e^{-tH^D_x}] = t \int_0^\infty d\lambda e^{-t\lambda}\xi(\lambda, x). \]  
(4.7)
While $\xi$ is defined in terms of $H$ and $H^D_x$, there is a formula that only involves $H$, or more precisely, the Green’s function $G(z, x, y)$ of $H$ defined by
\[ ((H - z)^{-1}f)(x) = \int_\mathbb{R} dy G(z, x, y)f(y), \quad \text{Im}(z) \neq 0. \]
Then by general principles, $\lim_{\epsilon \downarrow 0} G(\lambda + i\epsilon, x, y)$ exists for a.e. $\lambda \in \mathbb{R}$, and
\[ \xi(\lambda, x) = \frac{1}{\pi} \text{Arg}(\lim_{\epsilon \downarrow 0} G(\lambda + i\epsilon, x, x)). \]  
(4.8)
This is formally equivalent to formulas that Krein [146] has for $\xi$ but in a singular setting (i.e., corresponding to an infinite coupling constant). With this definition out of the way, we can state the general trace formula derived in [91]:

**Theorem 4.1.** Suppose $V$ is a continuous function bounded from below on $\mathbb{R}$. Let $\xi(\lambda, x)$ be the spectral shift function for the pair $(H^D_x, H)$ with $H^D_x$ the operator on $L^2((-\infty, x)) \oplus L^2((x, \infty))$ obtained from $H = -\frac{d^2}{dx^2} + V$ in $L^2(\mathbb{R})$ with a Dirichlet boundary condition at $x$. Let $E_0 \leq \inf \sigma(H)$. Then
\[ V(x) = \lim_{\alpha \downarrow 0} \left[ E_0 + \int_{E_0}^\infty d\lambda e^{-\alpha\lambda}[1 - 2\xi(\lambda, x)] \right]. \]  
(4.9)
In particular, if $\int_{E_0}^\infty d\lambda |1 - 2\xi(\lambda, x)| < \infty$, then
\[ V(x) = E_0 + \int_{E_0}^\infty d\lambda [1 - 2\xi(\lambda, x)]. \]
We note that the trace formula extends to real-valued potentials $V \in L^1_{\text{loc}}(\mathbb{R})$ as long as $H$ stays bounded from below (it then is in the limit point case at $\pm\infty$). Equation (4.9) then holds at all Lebesgue points of $V$ and hence for a.e. $x \in \mathbb{R}$.

For certain almost periodic potentials, Craig [43] used a regularization similar to the $\alpha$-regularization in (4.9).
Basically, (4.9) follows from (4.7) and an asymptotic formula,

\[ \text{Tr}[e^{-tH} - e^{-tH^D}] = \frac{1}{2}[1 - tV(x) + o(t)]. \]  

(4.10)

We present a few examples next:

**Example 4.2.** Pick a constant \( C \in \mathbb{R} \) such that \( V(x) = C \) for all \( x \in \mathbb{R} \). Then \( G(\lambda, x, x) = (C - \lambda)^{-1/2}/2 \) and hence one infers that \( \text{Arg}(G(\lambda, x, x)) = 0 \) (resp., \( \pi/2 \)) if \( \lambda < C \) (resp., \( \lambda > C \)). Thus, by (4.8), \( \xi(\lambda, x) = 1/2 \) on \( (C, \infty) \) and \( \xi(\lambda, x) = 0 \) on \( (-\infty, C) \). When \( \xi = 1/2 \) on a subset of \( \sigma(H) \), that set does not contribute to the integral in (4.9) and one verifies for \( E_0 \leq C \),

\[ V(x) = E_0 + \int_{E_0}^{C} d\lambda = E_0 + (C - E_0) = C, \quad x \in \mathbb{R}. \]

**Example 4.3.** Suppose that \( V(x) \to \infty \) as \( |x| \to \infty \). Then \( H \) has eigenvalues \( E_0 < E_1 < E_2 < \cdots \) and \( H_\alpha^D \) has eigenvalues \( \{\mu_j(x)\}_{j=1}^\infty \) with \( E_{j-1} \leq \mu_j(x) \leq E_j \). We have

\[ \xi(\lambda, x) = \begin{cases} 1, & E_{j-1} < \lambda < \mu_j(x), \\ 0, & \lambda < E_0 \text{ or } \mu_j(x) < \lambda < E_j. \end{cases} \]

Thus (4.9) becomes:

\[ V(x) = E_0 + \lim_{\alpha \downarrow 0} \left[ \sum_{j=1}^{\infty} \left( 2e^{-\alpha\mu_j(x)} - e^{-\alpha E_j} - e^{-\alpha E_{j-1}} \right) / \alpha \right]. \]

(4.11)

If we could take \( \alpha \) to zero inside the sum, we would get

\[ V(x) = E_0 + \sum_{j=1}^{\infty} [E_j + E_{j-1} - 2\mu_j(x)] \quad \text{(formal!)} \]

(4.12)

which is just a limit of the periodic formula (4.3) in the limit of vanishing band widths. (4.11) is just a kind of Abelianized summation procedure applied to (4.12).

As a special case of this example, consider \( V(x) = x^2 - 1 \). Then \( E_j = 2j \) and \( \{\mu_j(0)\} \) is the set \( \{2, 2, 6, 6, 10, 10, 14, 14, \ldots \} \) of \( j \) odd eigenvalues, each doubled. Thus (4.12) is the formal sum

\[ -1 = -2 + 2 - 2 + 2 \ldots \quad \text{(formal!)} \]

with (4.11)

\[ -1 = \lim_{\alpha \downarrow 0} \left( (e^{-2\alpha} - 1)/\alpha \right) \left[ 1 - e^{-2\alpha} + e^{-4\alpha} \ldots \right] = \lim_{\alpha \downarrow 0} \left( (e^{-2\alpha} - 1)/\alpha \right) \left[ 1/(1 + e^{-2\alpha}) \right] \]

with a true abelian summation.

**Example 4.4.** Suppose \( V \) is periodic. Let \( E_j, \mu_j(x) \) be the band edges and Dirichlet eigenvalues as in (4.2), (4.3). Then it follows from the fact that the two Floquet solutions are complex conjugates of each other on the spectrum of \( H \), and the Wronskian is antisymmetric in its argument (\( W(f, g) = -W(g, f) \)), that \( g(\lambda, x) \) is purely imaginary on \( \sigma(H) \); that is, \( \xi(\lambda, x) = 1/2 \) there, so

\[ \xi(\lambda, x) = \begin{cases} 1/2, & E_{2n} < \lambda < E_{2n+1}, \\ 1, & E_{2n-1} < \lambda < \mu_n(x), \\ 0, & \mu_n(x) < \lambda < E_{2n}. \end{cases} \]
Then for any \( \mathcal{E} \) terms of the Weyl–Titchmarsh \( \mathcal{m} \)-functions.) Thus,

\[
\int_{E_0}^{\infty} d\lambda |1 - 2\xi(\lambda, x)| = \sum_{n=1}^{\infty} |E_{2n} - E_{2n-1}|
\]

is finite if (4.1) holds. In that case one can take the limit inside the integral in (4.9) and so recover (4.3).

**Example 4.5.** In [80] it is proved that if \( V \) is short-range, that is, \( V \in H^{2,1}(\mathbb{R}) \), then,

\[
\int_{E_0}^{\infty} d\lambda |1 - 2\xi(\lambda, x)| < \infty
\]

and one can take the limit in (4.9) inside the integral. This recovers Venakides’ result [247] with an explicit form for \( H \) arbitrary unit vector in \( \mathbb{R} \). This recovers the associated self-adjoint Schrödinger operator in \( L^2(\mathbb{R}) \) and suppose that \( \sigma(H) = [E_0, \infty) \) for some \( E_0 \in \mathbb{R} \). Then

\[
V(x) = E_0 \quad \text{for a.e. } x \in \mathbb{R}.
\]

Next we mention a striking inverse spectral application of the trace formula (4.9) to a celebrated two-spectra inverse spectral theorem due to Borg [24]:

**Theorem 4.6.** Let \( V \in L^1_{\text{loc}}(\mathbb{R}) \) be real-valued and periodic. Let \( H = -\frac{d^2}{dx^2} + V \) be the associated self-adjoint Schrödinger operator in \( L^2(\mathbb{R}) \) and suppose that \( \sigma(H) = [E_0, \infty) \) for some \( E_0 \in \mathbb{R} \). Then

\[
V(x) = E_0 \quad \text{for a.e. } x \in \mathbb{R}.
\]

Given the trace formula (4.9) (observing the a.e. extension noted after Theorem 4.1) and using the fact that for all \( x \in \mathbb{R} \) and a.e. \( \lambda > E_0 \), \( \lambda(\lambda, x) = 1/2 \) (cf. Example 4.12), the proof of Borg’s Theorem 4.6 is effectively reduced to just a one-line argument (as was observed in [39]). In addition, the new proof permits one to replace periodic by reflectionless potentials and hence applies to algebro-geometric quasiperiodic (KdV) potentials and certain classes of almost periodic potentials.

Now we turn to an analog for Theorem 4.1 for Jacobi operators. This turns out to be a special case of the following result.

**Theorem 4.7.** Let \( A \) be a bounded self-adjoint operator in some complex separable Hilbert space \( \mathcal{H} \) with \( \alpha = \inf \sigma(A), \beta = \sup \sigma(A) \). Let \( \varphi \in \mathcal{H}, \|\varphi\|_\mathcal{H} = 1 \) be an arbitrary unit vector in \( \mathcal{H} \) and let \( \lambda(\lambda) \) be the spectral shift function for the pair \( (A, A), \) where \( A_\infty \) is defined by

\[
(A_\infty - z)^{-1} = (A - z)^{-1} - (\varphi, (A - z)^{-1}\varphi)^{-1}((A - z)^{-1}\varphi, \cdot)(A - z)^{-1}\varphi.
\]

Then for any \( E_- \leq \alpha \) and \( E_+ \geq \beta \):

\[
(\varphi, A\varphi) = E_+ + \int_{E_-}^{E_+} d\lambda |1 - \lambda(\lambda)| = E_+ - \int_{E_-}^{E_+} d\lambda \lambda(\lambda)
\]

\[
= \frac{1}{2} (E_+ + E_-) + \frac{1}{2} \int_{E_-}^{E_+} d\lambda |1 - 2\lambda(\lambda)|.
\]

**Corollary 4.8.** Let \( H \) be a Jacobi matrix on \( \ell^2(\mathbb{Z}^n) \), that is, for a bounded function \( V \) on \( \mathbb{Z}^n \),

\[
(Hu)(n) = \sum_{|n-m|=1} u(m) + V(n)u(n), \quad n \in \mathbb{Z}^n.
\]

(4.13)
For \( r \in \mathbb{Z}^n \), let \( H^D_r \) be the operator on \( L^2(\mathbb{Z}^n \setminus \{ r \}) \) given by (4.13) with \( u(r) = 0 \) boundary conditions. Let \( \xi(\lambda, r) \) be the spectral shift function for the pair \((H^D_r, H)\). Then

\[
V(r) = E_+ + \int_{E_-}^{E_+} d\lambda [1 - \xi(\lambda, r)] = E_+ - \int_{E_-}^{E_+} d\lambda \xi(\lambda, r)
\]

\[
= \frac{1}{2} (E_+ + E_-) + \frac{1}{2} \int_{E_-}^{E_+} d\lambda [1 - 2\xi(\lambda, r)]
\]

(4.14)

for any \( E_- \leq \inf \sigma(H), E_+ \leq \sup \sigma(H) \).

Only when \( n = 1 \) does this have an interpretation as a formula using Dirichlet problems on the half-line.

Next, we look at some applications to absolutely continuous spectra. In particular, we will point out that \( \xi(\lambda, x) \) for a single fixed \( x \in \mathbb{R} \) determines the absolutely continuous spectrum of a one-dimensional Schrödinger or Jacobi operator. We begin with a result that holds for higher-dimensional Jacobi operators as well:

**Lemma 4.9.** (i) For an arbitrary Jacobi matrix, \( H \), on \( \mathbb{Z}^n \), \( \cup_{j \in \mathbb{Z}^n} \{ \lambda \in \mathbb{R} | 0 < \xi(\lambda, j) < 1 \} \) is an essential support for the absolutely continuous spectrum of \( H \).

(ii) For a one-dimensional Schrödinger operator, \( H = -\frac{d^2}{dx^2} + V \) bounded from below, \( \cup_{x \in \mathbb{Q}} \{ \lambda \in \mathbb{R} | 0 < \xi(\lambda, x) < 1 \} \) is an essential support for the absolutely continuous spectrum of \( H \).

**Remark 4.10.** We recall that every absolutely continuous measure, \( d\mu \), has the form \( f(E) dE \). \( S = \{ E \in \mathbb{R} | f(E) \neq 0 \} \) is called an essential support. Any Borel set which differs from \( S \) by sets of zero Lebesgue measure is also called an essential support. If \( A \) is a self-adjoint operator on \( H \) and \( \varphi_n \), an orthonormal basis for \( H \), and \( d\mu_n \), the spectral measure for the pair, \( A, \varphi_n \) (i.e., \( \varphi_n, e^{i\lambda A} \varphi_n \) = \( \int_{E_n} e^{i\lambda E} d\mu_n(E) \)) and if \( d\mu_n^c \) is the absolutely continuous component of \( d\mu_n \) with \( S_n \) its essential support, then \( \cup_n S_n \) is the essential support of the absolutely continuous spectrum for \( A \).

In one dimension though, a single \( x \) suffices:

**Theorem 4.11.** For one-dimensional Schrödinger (resp., Jacobi) operators, \( \{ \lambda \in \mathbb{R} | 0 < \xi(\lambda, x) < 1 \} \) is an essential support for the absolutely continuous measure for any fixed \( x \in \mathbb{R} \) (resp., \( \mathbb{Z} \)).

These results are of particular interest because of their implications for a special kind of semi-continuity of the spectrum.

**Definition 4.12.** Let \( \{ V_n \} \), \( V \) be continuous potentials on \( \mathbb{R} \) (resp., on \( \mathbb{Z} \)). We say that \( V_n \) converges to \( V \) locally as \( n \to \infty \) if and only if

(i) \( \inf_{(n, x) \in \mathbb{N} \times \mathbb{R}} V_n(x) > -\infty \) (\( \mathbb{R} \) case) or \( \sup_{(n, j) \in \mathbb{N} \times \mathbb{Z}} |V_n(j)| < \infty \) (\( \mathbb{Z} \) case).

(ii) For each \( R < \infty \), \( \sup_{|x| \leq R} |V_n(x) - V(x)| \to 0 \) as \( n \to \infty \).

**Lemma 4.13.** If \( V_n \to V \) locally as \( n \to \infty \) and \( H_n, H \) are the corresponding Schrödinger operators (resp., Jacobi matrices), then \( (H_n - z)^{-1} \to (H - z)^{-1} \) strongly for \( \text{Im} \ z \neq 0 \) as \( n \to \infty \).

**Theorem 4.14.** If \( V_n \to V \) locally as \( n \to \infty \) and \( \xi_n(\lambda, x), \xi(\lambda, x) \) are the corresponding \( \xi_i \) functions for fixed \( x \), then \( \xi_n(\lambda, x) d\lambda \) converges to \( \xi(\lambda, x) d\lambda \) weakly in
the sense that for any $f \in L^1(\mathbb{R}; d\lambda)$,
\[
\int_{\mathbb{R}} d\lambda f(\lambda) \xi_n(\lambda, x) \to \int_{\mathbb{R}} d\lambda f(\lambda) \xi(\lambda, x) \quad \text{as } n \to \infty.
\]

**Definition 4.15.** For any $H$, let $|S_{ac}(H)|$ denote the Lebesgue measure of the essential support of the absolutely continuous spectrum of $H$.

**Theorem 4.16.** (For one-dimensional Schrödinger or Jacobi operators) Suppose $V_n \to V$ locally as $n \to \infty$ and each $V_n$ is periodic. Then for any interval $(a, b) \subset \mathbb{R}$,

\[
|(a, b) \cap S_{ac}| \geq \lim_{n \to \infty} |(a, b) \cap S_{ac}(H_n)|.
\]

We note that the periods of $V_n$ need not be fixed; indeed, almost periodic potentials are allowed.

**Example 4.17.** Let $\alpha_n$ be a sequence of rationals and $\alpha = \lim_{n \to \infty} \alpha_n$. Let $H_n$ be the Jacobi matrix with potential $\lambda \cos(2\pi \alpha_n + \theta)$ for $\lambda, \theta$ fixed. Then [2] have shown for $|\lambda| \leq 2$, $|S_n| \geq 4 - 2|\lambda|$. It follows from Theorem 4.16 that $|S| \geq 4 - 2|\lambda|$, providing a new proof (and a strengthening) of a result of Last [148]. At present much more is known about this example and the interested reader may want to consult the survey by Last [149] for additional results.

**Example 4.18.** Let $\{a_m\}_{m \in \mathbb{N}}$ be a sequence with $s = \sum_{m=1}^{\infty} 2^m |a_m| < 2$. Let $V(n) = \sum_{m=1}^{\infty} a_m \cos(2\pi n/2^m)$, a limit periodic potential on $\mathbb{Z}$. Let $h$ be the corresponding Jacobi matrix, then one can show that $|\sigma_{ac}(h)| \geq 2(2 - s)$.

\[
\star \quad \star \quad \star
\]

Next we very briefly turn to higher-order trace formulas derived in [82] obtained by higher-order expansions in (4.10) as $t \downarrow 0$. For simplicity we now assume that $V \in C^\infty(\mathbb{R})$ is bounded from below. Then (4.10) can be extended to

\[
\text{Tr}[e^{itH} - e^{-itH}] \sim \sum_{j=0}^{\infty} s_j(x) t^j, \quad x \in \mathbb{R}.
\]

Similarly, one has,
\[
\text{Tr}[(H - z)^{-1} - (H - z)^{-1}] \sim \sum_{j=0}^{\infty} r_j(x) z^{-j-1},
\]

\[
r_0(x) = 1/2, \quad r_1(x) = V(x)/2, \quad x \in \mathbb{R}
\]

and one can show that
\[
s_j(x) = (-1)^{j+1} (j!)^{-1} r_j(x), \quad j \in \mathbb{N} \cup \{0\}.
\]

In particular, $r_j(x)$ and $s_j(x)$ are the celebrated KdV invariants (up to inessential numerical factors). They can be computed recursively (see, e.g., [82]). The higher-order analogs of (4.9) then read

\[
s_0(x) = \frac{1}{2},
\]

\[
s_j(x) = \frac{(-1)^{j+1}}{j!} \left\{ \frac{E_0^j}{2} + j \lim_{t \downarrow 0} \int_{E_0}^{\infty} d\lambda e^{-t\lambda} \lambda^{-1} \left[ \frac{1}{2} - \xi(\lambda, x) \right] \right\} \quad j \in \mathbb{N}, \quad x \in \mathbb{R}.
\]
and similarly using a resolvent rather than a heat kernel regularization,
\[ r_1(x) = \frac{1}{2} V(x) = \frac{E_0}{2} + \lim_{z \to i\infty} \int_{E_0}^{\infty} d\lambda \frac{z^2}{(\lambda - z)^2} \left[ \frac{1}{2} - \xi(\lambda, x) \right], \]
\[ r_j(x) = \frac{E_j}{2} + \lim_{z \to i\infty} \int_{E_0}^{\infty} d\lambda \frac{z^{j+1}}{(\lambda - z)^{j+1}} j(-\lambda)^{j-1} \left[ \frac{1}{2} - \xi(\lambda, x) \right], \quad j \in \mathbb{N}, \ x \in \mathbb{R}. \]

In the special periodic case, the corresponding extension of (4.3) then reads
\[ 2(-1)^{j+1} j! s_j(x) = 2r_j(x) = E_0^j + \sum_{n=1}^{\infty} [E_{2n-1}^j + E_{2n}^j - 2\mu_n(x)^j], \quad j \in \mathbb{N}, \ x \in \mathbb{R}. \]

The latter formulas were originally found in [65] and [178].

We also note that the use of the Dirichlet boundary condition \( u(x) = 0 \) and hence the choice of the Dirichlet operator \( H^D_x \) in connection with (4.7) can be replaced by any self-adjoint boundary condition of the type \( u'(x) \pm \beta u(x) = 0, \ \beta \in \mathbb{R} \), and the corresponding Schrödinger operator \( H^\beta_x \) in \( L^2((-\infty, x)) \oplus L^2((x, \infty)) \).

This is worked out in detail in [82]. Additional results on trace formulas for Schrödinger operators were presented in [75], [76], [77], [86], [88].

More recent references: Important extensions of the trace formula (4.9), including the case of Schrödinger operators unbounded from below, were discussed by Rybkin [208], [209]. Further discussions of the trace formula (4.14) for Jacobi operators can be found in [94] and Teschl [239], [243, Ch. 6]. An extension of Corollary 4.8 to Schrödinger operators on a countable set was discussed by Shirai [223].

Removal of the resolvent regularization limit in the above trace formula for \( r_1 \) (resp., \( V \)) under optimal conditions on \( V \) has been studied by Rybkin [209], [211] (the latter reference offers necessary and sufficient conditions for absolute summability of the trace formula).

A certain multi-dimensional variant of these trace formulas, inspired by work of Lax [150], was established in [83] (see also [76]).

Matrix-valued extensions of the trace formula for Schrödinger, Dirac-type, and Jacobi matrices, as well as Borg and Hochstadt-type theorems were studied in [16], [37], [38], [39], [40], [77], and [89].

Trace formulas and an ensuing general Borg-type theorem for CMV operators (i.e., in connection with orthogonal polynomials on the unit circle, cf. [229]) appeared in [107].

An application of \( \xi \)-function ideas to obtain Weyl-type asymptotics using \( \zeta \)-function regularizations of determinants of certain operators on complete Riemannian manifolds can be found in Carron [29].

Theorem 4.11 was used in [85] to solve an inverse spectral problem for Jacobi matrices and most recently in [106] in connection with proving purely absolutely continuous spectrum of a class of reflectionless Schrödinger operators with homogeneous spectrum. It has also recently been discussed in [53, Sect. 1.5].

5. VARIOUS UNIQUENESS THEOREMS IN INVERSE SPECTRAL THEORY

In this section we summarize some of the principal results of the following papers:
[50] R. del Rio, F. Gesztesy, and B. Simon, Inverse spectral analysis with partial
One can argue that inverse spectral theory, especially the case of uniqueness theorems in inverse spectral theory, started with the paper by Ambarzumian [7] in 1929 and was turned into a full-fledged discipline by the seminal 1946 paper by Borg [24]. Ambarzumian proved the special uniqueness theorem that if the eigenvalues of a Schrödinger operator \(-d^2/dx^2 + V\) in \(L^2([0, \pi])\) with Neumann boundary conditions at the endpoints \(x = 0\) and \(x = \pi\) coincide with the sequence of numbers \(n^2, n = 0, 1, 2, \ldots\), then \(V = 0\) a.e. on \([0, \pi]\). This result is very special. Indeed, Borg showed that for more general boundary conditions, one set of eigenvalues, in general (i.e., in the absence of symmetries of \(V\)), is insufficient to determine \(V\) uniquely. Moreover, he described in great detail when two spectra guarantee unique determination of the potential \(V\). In this section we will discuss Borg’s celebrated two-spectra uniqueness result and many of its extension due to Gashymov, Hald, Hochstadt, Levitan, Lieberman, Marchenko, and Simon and collaborators.

We start with paper [92]. It contains a variety of new uniqueness theorems for potentials \(V\) in one-dimensional Schrödinger operators \(-d^2/dx^2 + V\) on \(\mathbb{R}\) and on the half-line \(\mathbb{R}_+ = [0, \infty)\) in terms of appropriate spectral shift functions introduced in a series of papers describing new trace formulas for \(V\) on \(\mathbb{R}\) [80], [81], [82], [91] and on \(\mathbb{R}_+\) [76]. In particular, it contains a generalization of a well-known uniqueness theorem of Borg and Marchenko for Schrödinger operators on the half-line with purely discrete spectra to arbitrary spectral types and a new uniqueness result for Schrödinger operators with confining potentials on the entire real line.

Turning to the half-line case first, we recall one of the principal uniqueness results proved in [92], which extends a well-known theorem of Borg [25] and Marchenko [171] in the special case of purely discrete spectra to arbitrary spectral types. We suppose

\[
V \in L^1([0, R]) \text{ for all } R > 0, \quad V \text{ real-valued,} \tag{5.1}
\]
and introduce the differential expression
\[
\tau_+ = -\frac{d^2}{dx^2} + V(x), \quad x \geq 0,
\]
for simplicity assuming that \(\tau\) is in the limit point case at \(\infty\). (We refer to [92] for a general treatment that includes the limit circle case.) Associated with \(\tau_+\) one introduces the following self-adjoint operator \(H_{+\alpha}\) in \(L^2(\mathbb{R}_+)\).

\[
H_{+\alpha}f = \tau_+f, \quad \alpha \in [0, \pi),
\]
\[
f \in \text{dom}(H_{+\alpha}) = \{g \in L^2(\mathbb{R}_+) \mid g, g' \in AC([0, R]) \text{ for all } R > 0; \sin(\alpha)g'(0_+) + \cos(\alpha)g(0_+) = 0; \tau_+g \in L^2(\mathbb{R}_+)\}.
\]

Then \(H_{+\alpha}\) has uniform spectral multiplicity one.

Next we introduce the fundamental system \(\phi_{\alpha}(z, x), \theta_{\alpha}(z, x), z \in \mathbb{C}\), of solutions of \(\tau_+ \psi(z, x) = z\psi(z, x), x \geq 0\), satisfying
\[
\phi_{\alpha}(z, 0) = -\theta_{\alpha}'(z, 0) = -\sin(\alpha), \quad \phi_{\alpha}'(z, 0) = \theta_{\alpha}(z, 0) = \cos(\alpha)
\]
such that \(W(\theta_{\alpha}(z), \phi_{\alpha}(z)) = 1\). Furthermore, let \(\psi_{+,\alpha}(z, x), z \in \mathbb{C}\setminus\mathbb{R}\) be the unique solution of \(\tau_+ \psi(z) = \psi(z)\) which satisfies
\[
\psi_{+,\alpha}(z, \cdot) \in L^2(\mathbb{R}_+), \quad \sin(\alpha)\psi_{+,\alpha}'(z, 0_+) + \cos(\alpha)\psi_{+,\alpha}(z, 0_+) = 1.
\]

\(\psi_{+,\alpha}\) is of the form
\[
\psi_{+,\alpha}(z, x) = \theta_{\alpha}(z, x) + m_{+,\alpha}(z)\phi_{\alpha}(z, x)
\]
with \(m_{+,\alpha}(z)\) the half-line Weyl–Titchmarsh \(m\)-function. Being a Herglotz function (i.e., an analytic function in the open upper-half plane that maps the latter to itself), \(m_{+,\alpha}(z)\) has the following representation in terms of a positive measure \(d\rho_{+,\alpha}\),

\[
m_{+,\alpha} = \begin{cases}
  a_{+,\alpha} + \int_{\mathbb{R}} \left[ \frac{1}{\lambda - z} - \frac{1}{\lambda + z} \right] d\rho_{+,\alpha}(\lambda), & \alpha \in [0, \pi), \\
  \cot(\alpha) + \int_{\mathbb{R}} (\lambda - z)^{-1} d\rho_{+,\alpha}(\lambda), & \alpha \in (0, \pi).
\end{cases}
\]

The basic uniqueness criterion for Schrödinger operators on the half-line \([0, \infty)\), due to Marchenko [171], that we shall rely on repeatedly in the following, can be stated as follows.

**Theorem 5.1.** Suppose \(\alpha_1, \alpha_2 \in [0, \pi), \alpha_1 \neq \alpha_2\) and define \(H_{+j,\alpha_j}, m_{+,\alpha_j}\), \(\rho_{+,\alpha_j}\), associated with the differential expressions \(\tau_j = -\frac{d^2}{dx^2} + V_j(x), x \geq 0\), where \(V_j, j = 1, 2\) satisfy assumption (5.1). Then the following three assertions are equivalent:

(i) \(m_{+1,\alpha_1}(z) = m_{+2,\alpha_2}(z), z \in \mathbb{C}_+\).

(ii) \(\rho_{+1,\alpha_1}((-\infty, \lambda]) = \rho_{+2,\alpha_2}((-\infty, \lambda]), \lambda \in \mathbb{R}\).

(iii) \(\alpha_1 = \alpha_2\) and \(V_1(x) = V_2(x)\) for a.e. \(x \geq 0\).

Next we relate Green’s functions for different boundary conditions at \(x = 0\).

**Lemma 5.2.** Let \(\alpha_j \in [0, \pi), j = 1, 2, x, x' \in \mathbb{R}_+,\) and \(z \in \mathbb{C}\setminus\{\sigma(H_{+,\alpha_1}) \cup \sigma(H_{+,\alpha_2})\}\). Then,

\[
G_{+,\alpha_2}(z, x, x') - G_{+,\alpha_1}(z, x, x') = -\frac{\psi_{+,\alpha_1}(z, x)\psi_{+,\alpha_1}(z, x')}{\cot(\alpha_2 - \alpha_1) + m_{+,\alpha_1}(z)}.
\]

\[
\frac{G_{+,\alpha_2}(z, 0, 0)}{G_{+,\alpha_1}(z, 0, 0)} = \frac{1}{(\beta_1 - \beta_2) \sin^2(\alpha_1) \left| \cot(\alpha_2 - \alpha_1) + m_{+,\alpha_1}(z) \right|}
\]
Lemma 5.3. \((\beta_1 - \beta_2)\sin^2(\alpha_2)[\cot(\alpha_2 - \alpha_1) - m_{+\alpha_2}(z)], \quad \beta_j = \cot(\alpha_j), j = 1, 2\)

\[
\text{Tr}[(H_{+\alpha_2} - z)^{-1} - (H_{+\alpha_1} - z)^{-1}] = -\frac{d}{dz}\ln[\cot(\alpha_2 - \alpha_1) + m_{+\alpha_1}(z)]
\]

\[
= \frac{d}{dz}\ln[\cot(\alpha_2 - \alpha_1) - m_{+\alpha_2}(z)].
\]

Since \(m_{+\alpha}(z)\) is a Herglotz function, we may now introduce spectral shift function [27] \(\xi_{\alpha_1,\alpha_2}(\lambda)\) for the pair \((H_{+\alpha_2}, H_{+\alpha_1})\) via the exponential Herglotz representation of \(m_{+\alpha}(z)\) (cf. [12])

\[
\cot(\alpha_2 - \alpha_1) + m_{+\alpha_1}(z) = \exp\left\{\Re[\ln(\cot(\alpha_2 - \alpha_1) + m_{+\alpha_1}(i))]\right\}
\]

\[
+ \int_{\mathbb{R}} \left[\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2}\right] \xi_{\alpha_1,\alpha_2}(\lambda) d\lambda, \quad 0 \leq \alpha_1 < \alpha_2 < \pi, z \in \mathbb{C}\setminus\mathbb{R}.
\]

This is extended to all \(\alpha_1, \alpha_2 \in [0, \pi)\) by

\[
\xi_{\alpha_1,\alpha_2}(\lambda) = 0, \quad \xi_{\alpha_2,\alpha_1}(\lambda) = -\xi_{\alpha_1,\alpha_2}(\lambda) \text{ for a.e. } \lambda \in \mathbb{R}.
\]

Next we summarize a few properties of \(\xi_{\alpha_1,\alpha_2}(\lambda)\).

**Lemma 5.3.** (i) Suppose \(0 \leq \alpha_1 < \alpha_2 < \pi\). Then for a.e. \(\lambda \in \mathbb{R},\)

\[
\xi_{\alpha_1,\alpha_2}(\lambda) = \begin{cases}
\lim_{\epsilon \downarrow 0} \pi^{-1}\text{Im}\{\ln[\cot(\alpha_2 - \alpha_1) + m_{+\alpha_1}(\lambda + i\epsilon)]\}, \\
\lim_{\epsilon \downarrow 0} \pi^{-1}\text{Im}\{\ln[\cot(\alpha_2 - \alpha_1) - m_{+\alpha_2}(\lambda + i\epsilon)]\}, \\
\lim_{\epsilon \downarrow 0} \pi^{-1}\text{Im}\{\ln[\frac{G_{+\alpha_1}(\lambda + i\epsilon, 0, 0)}{G_{+\alpha_2}(\lambda + i\epsilon, 0, 0)}]\}.
\end{cases}
\]

(For \(\alpha_1 = 0, G_{+\alpha_1}(\lambda + i\epsilon, 0, 0)/\sin(\alpha_1)\) has to be replaced by \(-1\) in the last expression.) Moreover, \(0 \leq \xi_{\alpha_1,\alpha_2}(\lambda) \leq 1\) a.e.

(ii) Let \(\alpha_j \in [0, \pi), 1 \leq j \leq 3\). Then the “chain rule”

\[
\xi_{\alpha_1,\alpha_3}(\lambda) = \xi_{\alpha_1,\alpha_2}(\lambda) + \xi_{\alpha_2,\alpha_3}(\lambda)
\]

holds for a.e. \(\lambda \in \mathbb{R},\)

(iii) For all \(\alpha_1, \alpha_2 \in [0, \pi),\)

\[
\xi_{\alpha_1,\alpha_2} \in L^1(\mathbb{R}; (1 + \lambda^2)^{-1} d\lambda).
\]

(iv) Assume \(\alpha_1, \alpha_2 \in [0, \pi), \alpha_1 \neq \alpha_2\). Then,

\[
\xi_{\alpha_1,\alpha_2} \in L^1(\mathbb{R}; (1 + |\lambda|)^{-1} d\lambda) \text{ if and only if } \alpha_1, \alpha_2 \in (0, \pi).
\]

(v) For all \(\alpha_1, \alpha_2 \in [0, \pi),\)

\[
\text{Tr}[(H_{+\alpha_2} - z)^{-1} - (H_{+\alpha_1} - z)^{-1}] = -\int_{\mathbb{R}} \frac{d\lambda \xi_{\alpha_1,\alpha_2}(\lambda)}{(\lambda - z)^{2}}.
\]

We note that \(\xi_{\alpha_1,\alpha_2}(\lambda)\) (for \(\alpha_1, \alpha_2 \in (0, \pi)\)) has been introduced by Javrjan [133], [134]. In particular, he proved Lemma 5.2 (iii) and Lemma 5.3 (v) in the non-Dirichlet cases where \(0 < \alpha_1, \alpha_2 < \pi\).

Given these preliminaries, we are now able to state the main uniqueness result for half-line Schrödinger operators of [92].
Theorem 5.4. Suppose $V_j$ satisfy assumption (5.1) and define $H_{+,j,\alpha,j,\ell}$, $j, \ell = 1, 2$, associated with the differential expressions $\tau_j = -\frac{d^2}{dx^2} + V_j(x)$, $x \geq 0$, $j = 1, 2$, where $\alpha_{j,\ell} \in [0, \pi)$, $\ell = 1, 2$, and we suppose $0 \leq \alpha_{1,1} < \alpha_{1,2} < \pi$, $0 \leq \alpha_{2,1} < \alpha_{2,2} < \pi$. In addition, let $\xi_{j,\alpha_{1,j},\alpha_{2,j}}$, $j = 1, 2$ be the spectral shift function for the pair $(H_{+,j,\alpha_{1,1}}, H_{+,j,\alpha_{2,2}})$. Then the following two assertions are equivalent:
(i) $\xi_{1,\alpha_{1,1},\alpha_{1,2}}(\lambda) = \xi_{2,\alpha_{2,1},\alpha_{2,2}}(\lambda)$ for a.e. $\lambda \in \mathbb{R}$.
(ii) $\alpha_{1,1} = \alpha_{2,1}$, $\alpha_{1,2} = \alpha_{2,2}$, and $V_1(x) = V_2(x)$ for a.e. $x \geq 0$.

As a corollary, one obtains a well-known uniqueness result originally due to Borg [25, Theorem 1] and Marchenko [171, Theorem 2.3.2] (see also [161]).

Corollary 5.5. Define $\tau_j$, $H_{+,j,\alpha}$, $\alpha \in [0, \pi)$ as in Theorem 5.4. Assume in addition that $H_{+,1,\alpha_1}$ and $H_{+,2,\alpha_2}$ have purely discrete spectra for some (and hence for all) $\alpha_j \in [0, \pi)$, that is,
\[ \sigma_{\text{ess}}(H_{+,j,\alpha_j}) = \emptyset \] for some $\alpha_j \in [0, \pi), j = 1, 2$.

Then the following two assertions are equivalent:
(i) $\sigma(H_{+,1,\alpha_1}) = \sigma(H_{+,2,\alpha_2})$, $\sigma(H_{+,1,\alpha_1}) = \sigma(H_{+,2,\alpha_2})$, $\alpha_j,\ell \in [0, \pi)$, $j, \ell = 1, 2$, $\sin(\alpha_{1,1} - \alpha_{2,2}) \neq 0$.
(ii) $\alpha_{1,1} = \alpha_{2,1}$, $\alpha_{1,2} = \alpha_{2,2}$, and $V_1(x) = V_2(x)$ for a.e. $x \geq 0$.

Roughly speaking, Corollary 5.5 implies that two sets of purely discrete spectra $\sigma(H_{+,\alpha_1}), \sigma(H_{+,\alpha_2})$ associated with distinct boundary conditions at $x = 0$ (but a fixed boundary condition (if any) at $+\infty)$, that is, $\sin(\alpha_2 - \alpha_1) \neq 0$, uniquely determine $V$ a.e. The first main result in [92], Theorem 5.4, removes all a priori spectral hypotheses and shows that the spectral shift function $\xi_{\alpha_1,\alpha_2}(\lambda)$ for the pair $(H_{+,\alpha_2}, H_{+,\alpha_1})$ with distinct boundary conditions at $x = 0$, $\sin(\alpha_2 - \alpha_1) \neq 0$, uniquely determines $V$ a.e. This illustrates that Theorem 5.4 is the natural generalization of Borg’s and Marchenko’s theorems from the discrete spectrum case to arbitrary spectral types.

Now we turn to uniqueness results for Schrödinger operators on the whole real line. We shall rely on the notation $\tau$, $\phi_{\alpha}$, $\theta_{\alpha}$, $\psi_{\pm,\alpha}$, $m_{\pm,\alpha}$, $d\varphi_{\pm,\alpha}$, which are defined in complete analogy to the half-line case (with $x \in \mathbb{R}$), and we shall assume
\[ V \in L^1_{\text{loc}}(\mathbb{R}), \quad V \text{ real-valued.} \] (5.3)

Following [82], we introduce, in addition, the following family of self-adjoint operators $H^\beta_y$ in $L^2(\mathbb{R})$,
\[ H^\beta_y f = \tau f, \quad \beta \in \mathbb{R} \cup \{\infty\}, \quad y \in \mathbb{R}, \]
\[ \text{dom}(H^\beta_y) = \{ g \in L^2(\mathbb{R}) \mid g, g' \in AC([y, \pm R]) \text{ for all } R > 0; g'(y_{\pm}) + \beta g(y_{\pm}) = 0; \] \[ \lim_{R \rightarrow \pm \infty} W(f_{\pm}(z_{\pm}), g)(R) = 0; \tau g \in L^2(\mathbb{R}) \}. \]

Thus $H_y^\infty := H^\infty_y$ (resp., $H_y^N := H^N_y$) corresponds to the Schrödinger operator with an additional Dirichlet (resp., Neumann) boundary condition at $y$. In obvious notation, $H^\beta_y$ decomposes into the direct sum of half-line operators
\[ H^\beta_y = H^\beta_{-y} \oplus H^\beta_{+,y} \]
with respect to $L^2(\mathbb{R}) = L^2((-\infty, y)) \oplus L^2([y, \infty))$. In particular, $H^\beta_{+,y}$ equals $H_{+,\alpha}$ for $\beta = \cot(\alpha)$ (and $y = 0$) in our notation (5.2) and, as done in our previous
Sections 2 and 4, the reference point $y$ will be added as a subscript to obtain
$\theta_{\alpha,y}(z,x), \psi_{\sigma_{\alpha,y}}(z,x), m_{\pm,\alpha,y}(z), M_{\alpha,y}(z)$, etc.

Next, we recall a few results from [82]. With $G(z, x, x')$ and $G^\beta_y(z, x, x')$ the
Green's functions of $H$ and $H^\beta_y$, one obtains (for $z \in \mathbb{C}\setminus\{\sigma(H_y^\infty) \cup \sigma(H)\}$)

$$G^\beta_y(z, x, x') = G(z, x, x') - \frac{(\beta + \partial_2)G(z, x, y)(\beta + \partial_1)G(z, y, x')}{(\beta + \partial_1)(\beta + \partial_2)G(z, y, y)},$$

$$G^\infty_y(z, x, x') = G(z, x, x') - G(z, y, y)^{-1}G(z, x, y)G(z, y, x').$$

Here we abbreviated

$$\partial_1 G(z, y, x') = \partial_x G(z, x, x')|_{x=y}, \quad \partial_2 G(z, z, x') = \partial_x G(z, x, x')|_{x'=y},$$

$$\partial_1 \partial_2 G(z, y, y) = \partial_x \partial_x G(z, x, x')|_{x=y=x'},$$

$$\partial_1 G(z, y, x) = \partial_2 G(z, x, y), \quad x \neq y.$$ 

As a consequence,

$$\text{Tr}[(H^\beta_y - z)^{-1} - (H - z)^{-1}] = -\frac{d}{dz} \ln[(\beta + \partial_1)(\beta + \partial_2)G(z, y, y)], \quad \beta \in \mathbb{R},$$

$$\text{Tr}[(H^\infty_y - z)^{-1} - (H - z)^{-1}] = -\frac{d}{dz} \ln[G(z, y, y)].$$

In analogy to the Herglotz property of $G(z, y, y)$, $(\beta + \partial_1)(\beta + \partial_2)G(z, y, y)$ is
also Herglotz for each $y \in \mathbb{R}$. Hence, both admit exponential representations of the form

$$G(z, y, y) = \exp\left\{c_\infty + \int_\mathbb{R} d\lambda \left[ \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right] \xi^{\infty}(\lambda, y) \right\},$$

$$c_\infty \in \mathbb{R}, \quad 0 \leq \xi^{\infty}(\lambda, y) \leq 1 \text{ a.e.,}$$

$$\xi^{\infty}(\lambda, y) = \lim_{\epsilon \downarrow 0} \pi^{-1} \text{Im}\left\{\text{ln}[G(\lambda + i\epsilon, y, y)]\right\} \text{ for a.e. } \lambda \in \mathbb{R},$$

$$(\beta + \partial_1)(\beta + \partial_2)G(z, y, y) = \exp\left\{c_\beta + \int_\mathbb{R} d\lambda \left[ \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right] \xi^{\beta}(\lambda, y) + 1 \right\},$$

$$c_\beta \in \mathbb{R}, \quad -1 \leq \xi^{\beta}(\lambda, y) \leq 0 \text{ a.e.,} \quad \beta \in \mathbb{R},$$

$$\xi^{\beta}(\lambda, y) = \lim_{\epsilon \downarrow 0} \pi^{-1} \text{Im}\left\{\text{ln}[(\beta + \partial_1)(\beta + \partial_2)G(\lambda + i\epsilon, y, y)]\right\} - 1, \quad \beta \in \mathbb{R}$$

for each $y \in \mathbb{R}$. Moreover,

$$\text{Tr}[(H^\beta_y - z)^{-1} - (H - z)^{-1}] = -\int_\mathbb{R} d\lambda (\lambda - z)^{-2} \xi^{\beta}(\lambda, y), \quad \beta \in \mathbb{R} \cup \{\infty\}.$$

Applying the basic uniqueness criterion for Schrödinger operators to both half-
lines $(-\infty, y)$ and $[y, \infty)$ then yields the following principal characterization result
for Schrödinger operators on $\mathbb{R}$ first proved in [92].

**Theorem 5.6.** Let $\beta_1, \beta_2 \in \mathbb{R} \cup \{\infty\}$, $\beta_1 \neq \beta_2$, and $x_0 \in \mathbb{R}$. Then the following assertions hold:

(i) $\xi^{\beta_1}(\lambda, x_0)$ and $\xi^{\beta_2}(\lambda, x_0)$ for a.e. $\lambda \in \mathbb{R}$ uniquely determine $V(x)$ for a.e. $x \in \mathbb{R}$
if the pair $(\beta_1, \beta_2)$ differs from $(0, \infty)$, $(\infty, 0)$.

(ii) If $(\beta_1, \beta_2) = (0, \infty)$ or $(\infty, 0)$, assume in addition that $\tau$ is in the limit point case at $+\infty$ and $-\infty$. Then $\xi^{\infty}(\lambda, x_0)$ and $\xi^{0}(\lambda, x_0)$ for a.e. $\lambda \in \mathbb{R}$ uniquely determine
V a.e. up to reflection symmetry with respect to \( x_0 \); that is, both \( V(x) \), \( V(2x_0 - x) \) for a.e. \( x \in \mathbb{R} \) correspond to \( \xi^\infty(\lambda, x_0) \) and \( \xi^0(\lambda, x_0) \) for a.e. \( \lambda \in \mathbb{R} \).

**Corollary 5.7.** Suppose \( \tau \) is in the limit point case at \( +\infty \) and \( -\infty \) and let \( \beta \in \mathbb{R} \cup \{ \infty \} \) and \( x_0 \in \mathbb{R} \). Then \( \xi^\infty(\lambda, x_0) \) for a.e. \( \lambda \in \mathbb{R} \) uniquely determines \( V(x) \) for a.e. \( x \in \mathbb{R} \) if and only if \( V \) is reflection symmetric with respect to \( x_0 \), that is, \( V(2x_0 - x) = V(x) \) a.e.

In view of Corollary 5.5, it seems appropriate to formulate Theorem 5.6 in the special case of purely discrete spectra.

**Corollary 5.8.** Suppose \( H \) (and hence \( H_y^\infty \) for all \( y \in \mathbb{R} \), \( \beta \in \mathbb{R} \cup \{ \infty \} \)) has purely discrete spectrum, that is, \( \sigma_{\text{ess}}(H) = \emptyset \) and let \( \beta_1, \beta_2 \in \mathbb{R} \cup \{ \infty \} \), \( \beta_1 \neq \beta_2 \), and \( x_0 \in \mathbb{R} \).

(i) \( \sigma(H), \sigma(H_{x_0}^\beta), j = 1, 2 \) uniquely determine \( V \) a.e. if the pair \( (\beta_1, \beta_2) \) differs from \((0, \infty)\) and \((\infty, 0)\).

(ii) If \( (\beta_1, \beta_2) = (0, \infty) \) or \((\infty, 0)\), assume in addition that \( \tau \) is in the limit point case at \( +\infty \) and \( -\infty \). Then \( \sigma(H), \sigma(H_{x_0}^\infty), \text{ and } \sigma(H_{x_0}^0) \) uniquely determine \( V \) a.e. up to reflection symmetry with respect to \( x_0 \), that is, both \( V(x) \) and \( \tilde{V}(x) = V(2x_0 - x) \) for a.e. \( x \in \mathbb{R} \) correspond to \( \sigma(H) = \sigma(\tilde{H}), \sigma(H_{x_0}^\infty) = \sigma(\tilde{H}_{x_0}^\infty), \text{ and } \sigma(H_{x_0}^0) = \sigma(\tilde{H}_{x_0}^0) \).

Here, in obvious notation, \( \tilde{H}, \tilde{H}_{x_0}^\infty, \tilde{H}_{x_0}^0 \) correspond to \( \tilde{\tau} = -\frac{d^2}{dx^2} + \tilde{V}(x), x \in \mathbb{R} \).

(iii) Suppose \( \tau \) is in the limit point case at \( +\infty \) and \( -\infty \) and let \( \beta \in \mathbb{R} \cup \{ \infty \} \). Then \( \sigma(H) \) and \( \sigma(H_{x_0}^\beta) \) uniquely determine \( V \) a.e. if and only if \( V \) is reflection symmetric with respect to \( x_0 \).

(iv) Suppose that \( V \) is reflection symmetric with respect to \( x_0 \) and that \( \tau \) is nonoscillatory at \( +\infty \) and \( -\infty \). Then \( V \) is uniquely determined a.e. by \( \sigma(H) \) in the sense that \( V \) is the only potential symmetric with respect to \( x_0 \) with spectrum \( \sigma(H) \).

Of course, Corollary 5.8 (iii) is implied by the result of Borg [5] and Marchenko [32] (see Corollary 5.5 with \( \alpha_1 = 0, \alpha_2 = \pi/2 \)).

Thus far, we exclusively dealt with \( \xi \)-functions and spectra in connection with uniqueness theorems. A variety of further uniqueness results can be obtained by invoking alternative information such as the left/right distribution of \( \lambda_{\beta}(x_0) \) (i.e., whether \( \lambda_{\beta}(x_0) \) is an eigenvalue of \( H_{x_0}^\infty \) in \( L^2((-\infty, x_0]) \) or of \( H_{x_0}^0 \) in \( L^2((x_0, \infty]) \)) and/or associated norming constants. For details we refer to the discussion in [92].

**More recent references:** Uniqueness theorems related to Theorem 5.4 in the short-range case with spectral shift data replaced by scattering data were studied by Akhtosun and Weder [5], [6]. Analogs of Corollaries 5.5 and 5.8 (i) for Jacobi operators were derived by Teschl [239].

\[ \star \star \star \]

Next we focus on [96], which discussed results where the discrete spectrum (or partial information on the discrete spectrum) and partial information on the potential \( V \) of a one-dimensional Schrödinger operator \( H = -\frac{d^2}{dx^2} + q \) determines the potential completely. Included are theorems for finite intervals and for the whole line. In particular, a new type of inverse spectral problem involving fractions of the eigenvalues of \( H \) on a finite interval and knowledge of \( V \) over a corresponding fraction of the interval was posed and solved in [96]. The methods employed in
let \( h_0 \in \mathbb{R}, h_1 \in \mathbb{R} \cup \{ \infty \} \) and assume \( V_1, V_2 \in L^1((0, 1)) \) to be real-valued. Consider the Schrödinger operators \( H_j, H_2 \) in \( L^2([0, 1]) \) given by

\[
H_j = -\frac{d^2}{dx^2} + V_j, \quad j = 1, 2,
\]
with the boundary conditions

\[
u'(0) + h_0 u(0) = 0, \]
\[u'(1) + h_1 u(1) = 0. \tag{5.4}
\]

Let \( \sigma(H_j) = \{ \lambda_{j,n} \} \) be the (necessarily simple) spectra of \( H_j, j = 1, 2 \). Suppose that \( V_1 = V_2 \) a.e. on \([0, 1/2] \) and that \( \lambda_{1,n} = \lambda_{2,n} \) for all \( n \). Then \( V_1 = V_2 \) a.e. on \([0, 1] \).

Here, in obvious notation, \( h_1 = \infty \) in (5.4) singles out the Dirichlet boundary condition \( u(1) = 0 \).

For each \( \varepsilon > 0 \), there are simple examples where \( V_1 = V_2 \) on \([0, (1/2) - \varepsilon] \) and \( \sigma(H_1) = \sigma(H_2) \) but \( V_1 \neq V_2 \). (Choose \( h_0 = -h_1 \), \( V_1(x) = 0 \) for \( x \in (0, (1/2) - \varepsilon) \cup [1/2, 1] \) and nonzero on \((1/2) - \varepsilon, 1/2)\), and \( V_2(x) = V_1(1 - x) \). See also Theorem \( \Gamma \) in the appendix of [234].)

Later refinements of Theorem 5.9 in [115], [234] (see also the summary in [233]) showed that the boundary condition for \( H_1 \) and \( H_2 \) at \( x = 1 \) need not be assumed a priori to be the same, and that if \( V \) is continuous, then one only needs \( \lambda_{1,n} = \lambda_{2,m(n)} \) for all values of \( n \) but one. The same boundary condition for \( H_1 \) and \( H_2 \) at \( x = 0 \), however, is crucial for Theorem 1.1 to hold (see [115], [49]).

Moreover, analogs of Theorem 5.9 for certain Schrödinger operators are considered in [136] (see also [192, Ch. 4]). Reconstruction techniques for \( V \) in this context are discussed in [206].

Our purpose in [96] was to provide a new approach to Theorem 5.9 that we felt was more transparent and, moreover, capable of vast generalizations. To state our generalizations, we will introduce a shorthand notation to paraphrase Theorem 5.9 by saying “\( V \) on \([0, 1/2] \) and the eigenvalues of \( H \) uniquely determine \( V \).” This is just a shorthand notation for saying \( V_1 = V_2 \) a.e. if the obvious conditions hold.

Unless explicitly stated otherwise, all potentials \( V, V_1, \) and \( V_2 \) will be real-valued and in \( L^1((0, 1)) \) for the remainder of this paper. Moreover, to avoid too many case distinctions we shall assume \( h_0, h_1 \in \mathbb{R} \) in (5.4). In particular, for \( h_0, h_1 \in \mathbb{R} \) we index the corresponding eigenvalues \( \lambda_n \) of \( H \) by \( n \in \mathbb{N}_0 = \mathbb{N} \cup \{ 0 \} \). The case of Dirichlet boundary conditions, where \( h_0 = \infty \) and/or \( h_1 = \infty \) has been dealt with in detail in [96, Appendix A].

Here is a summary of the generalizations proved for Schrödinger operators on \([0, 1] \) in [96]:

**Theorem 5.10.** Let \( H = -\frac{d^2}{dx^2} + V \) in \( L^2([0, 1]) \) with boundary conditions (5.4) and \( h_0, h_1 \in \mathbb{R} \). Suppose \( V \) is \( C^2k \) \((((1/2) - \varepsilon, (1/2) + \varepsilon)) \) for some \( k = 0, 1, \ldots \) and for some \( \varepsilon > 0 \). Then \( V \) on \([0, 1/2] \), \( h_0 \), and all the eigenvalues of \( H \) except for \((k + 1)\) uniquely determine \( h_1 \) and \( V \) on all of \([0, 1] \).
Remark 5.11. (i) The case \( k = 0 \) in Theorem 5.10 is due to Hald [115].

(ii) In the non-shorthand form of this theorem (cf. the paragraphs preceding Theorem 5.10), we mean that both \( V_1 \) and \( V_2 \) are \( C^{2k} \) near \( x = 1/2 \).

(iii) One need not know which eigenvalues are missing. Since the eigenvalues asymptotically satisfy

\[
\lambda_n = (\pi n)^2 + 2(h_1 - h_0) + \int_0^1 dx \, V(x) + o(1) \quad \text{as} \quad n \to \infty,
\]
given a set of candidates for the spectrum, one can tell how many are missing.

(iv) For the sake of completeness we mention the precise definition of \( H \) in \( L^2([0,1]) \) for real-valued \( V \in L^1([0,1]) \) and boundary condition parameters \( h_0, h_1 \in \mathbb{R} \cup \{\infty\} \) in (1.1):

\[
H = -\frac{d^2}{dx^2} + V,
\]

\[
\text{dom}(H) = \{ g \in L^2([0,1]) \mid g, g' \in AC([0,1]);\ (-g'' + Vg) \in L^2([0,1]);\ g'(0) + h_0 g(0) = 0,\ g'(1) + h_1 g(1) = 0 \},
\]

where \( AC([0,1]) \) denotes the set of absolutely continuous functions on \([0,1]\) and \( h_{x_0} = \infty \) represents the Dirichlet boundary condition \( g(x_0) = 0 \) for \( x_0 \in \{0,1\} \) in (5.5).

By means of explicit examples, it has been shown in Section 3 of [96], that Theorem 5.10 is optimal in the sense that if \( V \) is only assumed to be \( C^{2k-1} \) near \( x = 1/2 \) for some \( k \geq 1 \), then it is not uniquely determined by \( V|_{[0,1/2]} \) and all the eigenvalues but \((k + 1)\).

Theorem 5.10 works because the condition that \( V \) is \( C^{2k} \) near \( x = 1/2 \) gives us partial information about \( V \) on \([1/2,1] \); indeed, we know the values of \( V(1/2), V'(1/2), \ldots, V^{(2k)}(1/2) \) computed on \([1/2,1]\) since one can compute them on \([0,1/2]\). This suggests that knowing \( V \) on more than \([0,1/2]\) should let one dispense with a finite density of eigenvalues. That this is indeed the case is the content of the following theorem.

We denote by \( \#\{\ldots\} \) the cardinality of the set \( \{\ldots\} \).

Theorem 5.12. Let \( H = -\frac{d^2}{dx^2} + V \) in \( L^2([0,1]) \) with boundary conditions (5.4) and \( h_0, h_1 \in \mathbb{R} \). Then \( V \) on \([0,(1+\alpha)/2]\) for some \( \alpha \in (0,1) \), \( h_0 \), and a subset \( S \subseteq \sigma(H) \) of all the eigenvalues \( \sigma(H) \) of \( H \) satisfying

\[
\#\{\lambda \in S \mid \lambda \leq \lambda_0\} \geq (1 - \alpha)\#\{\lambda \in \sigma(H) \mid \lambda \leq \lambda_0\} + (\alpha/2) \quad (5.6)
\]

for all sufficiently large \( \lambda_0 \in \mathbb{R} \), uniquely determine \( h_1 \) and \( V \) on all of \([0,1]\).

Remark 5.13. (i) As a typical example, knowing slightly more than half the eigenvalues and knowing \( V \) on \([0,2]\) determines \( V \) uniquely on all of \([0,1]\). To the best of our knowledge, Theorem 5.12 introduced and solved a new type of inverse spectral problem.

(ii) As in the case \( \alpha = 0 \), one has an extension of the same type as Theorem 5.10. Explicitly, if \( V \) is assumed to be \( C^{2k} \) near \( x = (1+\alpha)/2 \), we only need

\[
\#\{\lambda \in S \mid \lambda \leq \lambda_0\} \geq (1 - \alpha)\#\{\lambda \in \sigma(H) \mid \lambda \leq \lambda_0\} + (\alpha/2) - (k + 1) \]

instead of (5.6).
One can also derive results about problems on all of \( \mathbb{R} \).

**Theorem 5.14.** Suppose that \( V \in L^1_{\text{loc}}(\mathbb{R}) \) satisfies the following two conditions:

(i) \( V(x) \geq C|x|^{2+\varepsilon} - D \) for some \( C, \varepsilon, D > 0 \).

(ii) \( V(-x) \geq V(x) \quad x \geq 0 \).

Then \( V \) on \( [0, \infty) \) and the spectrum of \( H = -\frac{d^2}{dx^2} + V \) in \( L^2(\mathbb{R}) \) uniquely determine \( V \) on all of \( \mathbb{R} \).

Hochstadt-Lieberman [123] used the details of the inverse spectral theory in their proof. In a sense, we only used in [96] the main uniqueness theorem of that theory due to Marchenko [171], which we now describe. For \( V \in L^1([a, b]) \) real-valued, \(-\infty < a < b < \infty\), consider \(-u'' + Vu = zu\) with the boundary condition

\[
u'(b) + h_b u(b) = 0
\]

at \( x = b \). Let \( u_+(z, x) \) denote the solution of this equation, normalized, say, by \( u_+(z, b) = 1 \). The \( m_+\)-function is then defined by

\[
m_+(z, a) = \frac{u'(z, a)}{u_+(z, a)}.
\]

Similarly, given a boundary condition at \( x = a \),

\[
u'(a) + h_au(a) = 0,
\]

we define the solution \( u_-(z, x) \) of \(-u'' + Vu = zu\) normalized by \( u_-(z, a) = 1 \) and then define

\[
m_-(z, b) = \frac{u'_-(z, b)}{u_-(z, b)}.
\]

In our present context where \(-\infty < a < b < \infty, m_\pm \) are even meromorphic on \( \mathbb{C} \). Moreover,

\[
\text{Im}(z) > 0 \implies \text{Im}(m_-(z, b)) < 0, \text{Im}(m_+(z, a)) > 0.
\]

Marchenko’s [171] fundamental uniqueness theorem of inverse spectral theory then reads as follows:

**Theorem 5.15.** \( m_+(z, a) \) uniquely determines \( h_b \) as well as \( V \) a.e. on \([a, b]\).

If \( V \in L^1_{\text{loc}}([a, \infty)) \) is real-valued (with \(|a| < \infty\) and \(-\frac{d^2}{dx^2} + V\) is in the limit point case at infinity, one can still define a unique \( m_+(z, a) \) function but now for \( \text{Im}(z) \neq 0 \) rather than all \( z \in \mathbb{C} \). For such \( z \), there is a unique function \( u_+(z, \cdot) \) which is \( L^2 \) at infinity (unique up to an overall scale factor which drops out of \( m_+(z, a) \) defined by (5.8)). Again, one has the following uniqueness result independently proved by Borg [25] and Marchenko [171].

**Theorem 5.16.** \( m_+(z, a) \) uniquely determines \( V \) a.e. on \([a, \infty)\).

It is useful to have \( m_-(z, b) \) because of the following basic fact:

**Theorem 5.17.** Let \( H = -\frac{d^2}{dx^2} + V \) be a Schrödinger operator in \( L^2([a, b]) \) with boundary conditions (5.7), (5.9) and let \( G(z, x, y) \) be the integral kernel of \((H-z)^{-1}\). Suppose \( c \in (a, b) \) and let \( m_+(z, c) \) be the corresponding \( m_+\)-function for \([c, b]\) and \( m_-(z, c) \) the \( m_-\)-function for \([a, c]\). Then

\[
G(z, c, c) = \frac{1}{m_-(z, c) - m_+(z, c)}.
\]
Theorems 5.15 and 5.16 are deep facts; Theorem 5.17 is an elementary calculation following from the explicit formula for the integral kernel of \((H - z)^{-1}\):

\[
G(z, x, y) = \frac{u_-(z, \min(x, y))u_+(z, \max(x, y))}{W(u_-(z), u_+(z))(x)},
\]

where as usual \(W(f, g)(x) = f'(x)g(x) - f(x)g'(x)\) denotes the Wronskian of \(f\) and \(g\). An analog of Theorem 5.17 holds in case \([a, b]\) is replaced by \((-\infty, \infty)\).

We can now describe the strategy of our proofs of Theorems 5.9–5.14. \(G(z, c, c)\) has poles at the eigenvalues of \(H\) (this is not quite true; see below), so by (5.11), at eigenvalues \(\lambda_n\) of \(H\):

\[
m_+(\lambda_n, c) = m_-(\lambda_n, c).
\]

(5.12)

If we know \(V\) on a left partial interval \([a, c]\) and we know some eigenvalue \(\lambda_n\), then we know \(m_-(z, c)\) exactly; so by (5.12), we know the value of \(m_+(\lambda_n, c)\) at the point \(\lambda_n\). Below we indicate when knowing the values of \(f(\lambda_n)\) of an analytic function of the type of the \(m\)-functions uniquely determines \(f(z)\). If \(m_+(z, c)\) is determined, then by Theorem 5.15, \(V\) is determined on \([a, b]\) and so is \(h_0\).

So the logic of the argument for a theorem like Theorem 5.9 is the following:

(i) \(V\) on \([0, 1/2]\) and \(h_0\) determine \(m_-(z, 1/2)\) by direct spectral theory.

(ii) The \(\lambda_n\) and (5.12) determine \(m_+(\lambda_n, 1/2)\), and then by suitable theorems in complex analysis, \(m_+(z, 1/2)\) is uniquely determined for all \(z\).

(iii) \(m_+(z, 1/2)\) uniquely determines \(V\) (a.e.) on \([1/2, 1]\) and \(h_1\) by inverse spectral theory.

It is clear from this approach why \(h_0\) is required and \(h_1\) is free in the context of Theorem 5.9 (see [49] for examples where \(h_1\) and \(V\) on \([0, 1/2]\) do not determine \(V\)); without \(h_0\) we cannot compute \(m_-(z, 1/2)\) and so start the process.

As indicated before (5.12), \(G(z, c, c)\) may not have a pole at an eigenvalue \(\lambda_n\) of \(H\). It will if \(u_n(c) \neq 0\), but if \(u_n(c) = 0\), then \(G(z, c, c) = 0\) rather than \(\infty\). Here \(u_n\) denotes the eigenfunction of \(H\) associated with the (necessarily simple) eigenvalue \(\lambda_n\). Nevertheless, (5.12) holds at points where \(u_n(c) = 0\) since then \(u_-(c) = u_+(c) = 0\), and so both sides of (5.12) are infinite. (In spite of (5.12), \(m_- - m_+\) is also infinite at \(z = \lambda_n\) and so \(G(\lambda_n, c, c) = 0\).) We summarize this discussion next:

**Theorem 5.18.** For any \(c \in (a, b)\), (5.12) holds at any eigenvalue \(\lambda_n\) of \(H_{[a, b]}\) (with the possibility of both sides of (5.12) being infinite).

**More recent references:** A new inverse nodal problem was reduced to Theorem 5.12 by Yang [251]. A substantial generalization of Theorem 5.14, replacing condition (i) by \(H\) being bounded from below with purely discrete spectrum, was proved by Khodakovsky [137], [138]. He also found other variants of Theorem 5.14.

We end our survey of [96] by briefly indicating the uniqueness theorems for entire functions needed in the proofs of Theorems 5.9–5.14. In discussing extensions of Hochstadt’s discrete (finite matrix) version [122] of the Hochstadt–Lieberman theorem in [94], we made use of the following simple lemma which is an elementary consequence of the fact that any polynomial of degree \(d\) with \(d + 1\) zeros must be the zero polynomial:
Lemma 5.19. Suppose $f_1 = \frac{P_1}{Q_1}$ and $f_2 = \frac{P_2}{Q_2}$ are two rational fractions where the polynomials satisfy $\deg(P_1) = \deg(P_2)$ and $\deg(Q_1) = \deg(Q_2)$. Suppose that $d = \deg(P_1) + \deg(Q_1)$ and that $f_1(z_n) = f_2(z_n)$ for $d + 1$ distinct points $\{z_n\}_{n=1}^{d+1} \in \mathbb{C}$. Then $f_1 = f_2$.

In the context of [96], one is interested in entire functions of the form

$$f(z) = C \prod_{n=0}^{\infty} \left(1 - \frac{z}{x_n}\right), \quad (5.13)$$

where $0 < x_0 < x_1 < \cdots$ is a suitable sequence of positive numbers which are the zeros of $f$ and $C$ is some complex constant.

Given a sequence $\{x_n\}_{n=0}^{\infty}$ of positive reals, we define

$$N(t) = \#\{n \in \mathbb{N} \cup \{0\} \mid x_n < t\}.$$  

We recall the following basic theorem (see, e.g., [152, Ch. I], [173, Sects. II.48, II.49]):

**Theorem 5.20.** Fix $0 < \rho_0 < 1$. Then:

(i) If $\{x_n\}_{n=0}^{\infty}$ is a sequence of positive reals with

$$\sum_{n=0}^{\infty} x_n^{-\rho} < \infty \text{ for all } \rho > \rho_0, \quad (5.14)$$

then the product in (5.13) defines an entire function $f$ with

$$|f(z)| \leq C_1 \exp(C_2|z|^\rho) \text{ for all } \rho > \rho_0. \quad (5.15)$$

(ii) Conversely, if $f$ is an entire function satisfying (5.15) with all its (complex) zeros on $(0, \infty)$, then its zeros $\{x_n\}_{n=0}^{\infty}$ satisfy (5.14), and $f$ has the canonical product expansion (5.13).

Moreover, (5.14) holds if and only if

$$N(t) \leq C|t|^\rho \text{ for all } \rho > \rho_0. \quad (5.16)$$

Given this theorem, we single out the following definition.

**Definition 5.21.** A function $f$ is called of $m$-type if and only if $f$ is an entire function satisfying (5.15) (of order $0 < \rho < 1$ in the usual definition) with all the zeros of $f$ on $(0, \infty)$.

Our choice of “$m$-type” in Definition 5.21 comes from the fact that in many cases we discuss in this paper, the $m$-function is a ratio of functions of $m$-type. By Theorem 5.20, $f$ in Definition 5.21 has the form (5.13) and $N(t)$, which we will denote as $N_f(t)$, satisfies (5.16).

**Lemma 5.22.** Let $f$ be a function of $m$-type. Then there exists a $0 < \rho < 1$ and a sequence $\{R_k\}_{k=1}^{\infty}$, $R_k \to \infty$ as $k \to \infty$, so that

$$\inf \{|f(z)| \mid |z| = R_k\} \geq C_1 \exp(-C_2 R_k^\rho).$$

**Lemma 5.23.** Let $F$ be an entire function that satisfies the following two conditions:

(i) $\sup_{|z|=R_k} |F(z)| \leq C_1 \exp(C_2 R_k^\rho)$ for some $0 \leq \rho < 1$, $C_1, C_2 > 0$, and some sequence $R_k \to \infty$ as $k \to \infty$. 

(ii) \( \lim_{|x| \to \infty, x \in \mathbb{R}} |F(ix)| = 0. \)

Then \( F \equiv 0. \)

Lemmas 5.22 and 5.23 finally yield the following result.

**Theorem 5.24.** Let \( f_1, f_2, g \) be three functions of \( m \)-type so that the following two conditions hold:

(i) \( f_1(z) = f_2(z) \) at any point \( z \) with \( g(z) = 0. \) (ii) For all sufficiently large \( t, \)

\[
\max(N_{f_1}(t), N_{f_2}(t)) \leq N_g(t) - 1.
\]

Then, \( f_1 = f_2. \)

\[\star\ \ \ \ \star\ \ \ \ \star\]

Refinements of the results of [96] can be found in [50], [51]. Here we just mention the following facts.

**Theorem 5.25.** Let \( H_1(h_0), H_2(h_0) \) be associated with two potentials \( V_1, V_2 \) on \([0, 1]\) and two potentially distinct boundary conditions \( h_1^{(1)}, h_1^{(2)} \in \mathbb{R} \) at \( x = 1. \) Suppose that \( \{(\lambda_n, h_0^{(n)})\}_{n \in \mathbb{N}} \) is a sequence of pairs with \( \lambda_0 < \lambda_1 < \cdots \to \infty \) and \( h_0^{(n)} \in \mathbb{R} \cup \{\infty\} \) so that both \( H_1(h_0^{(n)}) \) and \( H_2(h_0^{(n)}) \) have eigenvalues at \( \lambda_n. \) Suppose that

\[
\sum_{n=0}^{\infty} \frac{(\lambda_n - \frac{1}{4} \sigma^2 n^2)_{+}}{n^2} < \infty
\]

holds. Then \( V_1 = V_2 \) a.e. on \([0, 1]\) and \( h_1^{(1)} = h_1^{(2)}. \)

This implies Borg’s celebrated two-spectra uniqueness result [24] (see also, [154], [161], [159, Ch. 3], [171]):

**Corollary 5.26.** Fix \( h_0^{(1)}, h_0^{(2)} \in \mathbb{R}. \) Then all the eigenvalues of \( H(h_0^{(1)}) \) and all the eigenvalues of \( H(h_0^{(2)}) \), save one, uniquely determine \( V \) a.e. on \([0, 1]\).

It also implies the following amusing result:

**Corollary 5.27.** Let \( h_0^{(1)}, h_0^{(2)}, h_0^{(3)} \in \mathbb{R} \) and denote by \( \sigma_j = \sigma(H(h_0^{(j)}) \) the spectra of \( H(h_0^{(j)}), \ j = 1, 2, 3. \) Assume \( S_j \subseteq \sigma_j, \ j = 1, 2, 3 \) and suppose that for all sufficiently large \( \lambda \) one has

\[
\#\{\lambda \in \{S_1 \cup S_2 \cup S_3\} \text{ with } \lambda \leq \lambda_0\} \geq 2 \#\{\lambda \in \{\sigma_1 \cup \sigma_2 \cup \sigma_3\} \text{ with } \lambda \leq \lambda_0\} - 1.
\]

Then \( V \) is uniquely determined a.e. on \([0, 1]\).

In particular, two-thirds of three spectra determine \( V. \)

**More recent references:** Further refinements of Corollary 5.27, involving \( N \) spectra, were proved by Horváth [124] (he also studies the corresponding analog for a Dirac-type operator). Optimal and nearly optimal conditions for a set of eigenvalues to determine the potential in terms of closedness properties of the exponential system corresponding to the known eigenvalues (implying Theorem 5.25 and a generalization thereof) were also derived by Horváth [125]. For an interesting half-line problem related to this circle of ideas we also refer to Horváth [126]. A variant of Theorem 5.25 was discussed by Ramm [197], [198]. Hochstadt–Lieberman-type problems for Schrödinger operators including a reconstruction algorithm has been presented by L. Sakhnovich. The analog of the two-spectra result, Corollary 5.26,
including a reconstruction algorithm, for a class of singular potentials has been discussed by Hryniv and Mykytyuk [127], [129] (see also [131]). They also studied Hochstadt–Lieberman-type results for such a class of singular potentials in [130]. Hochstadt–Lieberman-type results for a class of Dirac-type operators relevant to the AKNS system were published by del Rio and Grébert [52]. Borg- and Hochstadt–Lieberman-type inverse problems for systems including matrix-valued Schrödinger and Dirac-type equations, were studied in depth by M. Malamud [164], [165], [166] [167]. He also studied Borg-type theorems for nth-order scalar equations [168]. Borg- and Hochstadt–Lieberman-type inverse problems for matrix-valued Schrödinger operators were also studied by Shen [220]. He also considered Borg-type inverse problems for Schrödinger operators with weights [221].

Additional results on determining the potential uniquely from spectra associated to three intervals of the type $[0, 1]$, $[0, a]$, and $[a, 1]$ for some $a \in (0, 1)$ (and similarly for whole-line problems with purely discrete spectra) can be found in [95]. This has been inspired by work of Pivovrachik [184], who also addressed the reconstruction algorithm from three spectra in the symmetric case $a = 1/2$ (see also [185], [188], [189]). He also considered the analogous Sturm–Liouville problem applicable to a smooth inhomogeneous partially damped string in [186] and extended some of these results to Sturm–Liouville equations on graphs in [187], [190]. Uniqueness and characterization problems for a class of singular Sturm–Liouville problems associated with three spectra were studied by Hryniv and Mykytyuk [128]. The reconstruction of a finite Jacobi matrix from three of its spectra was presented by Michor and Teschl [181].

These results are related to two other papers: In [94], we considered, among other topics, analogs of Theorems 5.9 and 5.12 for finite tri-diagonal (Jacobi) matrices, extending a result in [122]. The approach there is very similar to the current one except that the somewhat subtle theorems on zeros of entire functions in this paper are replaced by the elementary fact that a polynomial of degree at most $N$ with $N + 1$ zeros must be identically zero. In [93], we consider results related to Theorem 5.14 in that for Schrödinger operators on $(-\infty, \infty)$, “spectral” information plus the potential on one of the half-lines determine the potential on all of $(-\infty, \infty)$. In that paper, we considered situations where there are scattering states for some set of energies and the “spectral” data are given by a reflection coefficient on a set of positive Lebesgue measure in the a.c. spectrum of $H$. The approach is not as close to this paper as is [94], but $m$-function techniques (see also [92]) are critical in all three papers.

More recent references: For additional results on inverse scattering with partial information on the potential we refer to Aktosun and Papanicolaou [2], Aktosun and Sacks [3], Aktosun and Weder [4], and the references therein.

We conclude this section by briefly describing some of the results in [94], where inverse spectral analysis for finite and semi-infinite Jacobi operators $H$ was studied. While discussing a variety of topics (including trace formulas), we also provided a new proof of a result of Hochstadt [122] and its extension, which can be viewed as the discrete analog of the Hochstadt and Lieberman result in [123]. Moreover, we solved the inverse spectral problem for $(\delta_n, (H - z)^{-1}\delta_n)$ in the case of finite Jacobi
matrices. As mentioned earlier, the tools we apply are grounded in \(m\)-function techniques.

Explicitly, [94] studied finite \(N \times N\) matrices of the form:

\[
H = \begin{pmatrix}
    b_1 & a_1 & 0 & 0 & \cdots \\
    a_1 & b_2 & a_2 & 0 & \cdots \\
    0 & a_2 & b_3 & a_3 & \cdots \\
    \vdots & \vdots & \vdots & \vdots & \ddots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & 0 & a_{N-1} & b_N
\end{pmatrix}
\]

and the semi-infinite analog \(H\) defined on

\[
\ell^2(\mathbb{N}) = \left\{ u = (u(1), u(2), \ldots) \left| \sum_{n=1}^{\infty} |u(n)|^2 < \infty \right. \right\}
\]

given by

\[
(Hu)(n) = \begin{cases}
    a_n u(n+1) + b_n u(n) + a_{n-1} u(n-1), & n \geq 2, \\
    a_1 u(2) + b_1 u(1), & n = 1.
\end{cases}
\]

In both cases, we assume \(a_n, b_n \in \mathbb{R}\) with \(a_n > 0\). To avoid inessential technical complications, we will only consider the case where \(\sup_n |a_n| + |b_n| < \infty\) in which case \(H\) is a map from \(\ell^2\) to \(\ell^2\) and defines a bounded self-adjoint operator. In the semi-infinite case, we will set \(N = \infty\). It will also be useful to consider the \(b\)'s and \(a\)'s as a single sequence \(b_1, a_1, b_2, a_2, \ldots = c_1, c_2, \ldots\), that is,

\[
c_{2n-1} = b_n, \quad c_{2n} = a_n, \quad n \in \mathbb{N}.
\]

Concerning the recovery of a finite Jacobi matrix from parts of the matrix and additional spectral information (i.e., mixed data), Hochstadt [122] proved the following remarkable theorem.

**Theorem 5.28.** Let \(N \in \mathbb{N}\). Suppose that \(c_{N+1}, \ldots, c_{2N-1}\) are known, as well as the eigenvalues \(\lambda_1, \ldots, \lambda_N\) of \(H\). Then \(c_1, \ldots, c_N\) are uniquely determined.

The discrete Hochstadt–Lieberman-type theorem proved in [94] reads as follows.

**Theorem 5.29.** Suppose that \(1 \leq j \leq N\) and \(c_{j+1}, \ldots, c_{2N-1}\) are known, as well as \(j\) of the eigenvalues. Then \(c_1, \ldots, c_j\) are uniquely determined.

We emphasize that one need not know which of the \(j\) eigenvalues one has.

Borg [24] proved the celebrated theorem that the spectra for two boundary conditions of a bounded interval regular Schrödinger operator uniquely determine the potential. Later refinements (see, e.g., [25], [120], [153], [154], [161], [171]) imply that they even determine the two boundary conditions.

Next, we consider analogs of this result for a finite Jacobi matrix. Such analogs were first considered by Hochstadt [119], [121] (see also [23], [67], [110], [111], [114], [122]). The results below are adaptations of known results for the continuum or the semi-infinite case, but the ability to determine parameters by counting sheds light on facts like the one that the lowest eigenvalue in the Borg result is not needed under certain circumstances.

Given \(H\), an \(N \times N\) Jacobi matrix, one defines \(H(b)\) to be the Jacobi matrix where all \(a\)'s and \(b\)'s are the same as \(H\), except \(b_1\) is replaced by \(b_1 + b\), that is,

\[
H(b) = H + b(\delta_1, \cdot)\delta_1.
\]
Theorem 5.30. The eigenvalues $\lambda_1, \ldots, \lambda_N$ of $H$, together with $b$ and $N-1$ eigenvalues $\lambda(b)_1, \ldots, \lambda(b)_{N-1}$ of $H(b)$, determine $H$ uniquely.

Again it is irrelevant which $N-1$ eigenvalues of the $N$ eigenvalues of $H(b)$ are known.

Theorem 5.31. The eigenvalues $\lambda_1, \ldots, \lambda_N$ of $H$, together with the $N$ eigenvalues $\lambda(b)_1, \ldots, \lambda(b)_N$ of some $H(b)$ (with $b$ unknown), determine $H$ and $b$.

Remark 5.32. Since

$$b = \text{Tr}(H(b) - H) = \sum_{j=1}^{N} (\lambda(b)_j - \lambda_j),$$

we can a priori deduce $b$ from the $\lambda(b)$’s and $\lambda$’s and so deduce Theorem 5.31 from Theorem 5.30. We note that the parameter counting works out. In Theorem 5.30, $2n - 1$ eigenvalues determine $2n - 1$ parameters; and in Theorem 5.31, $2n$ eigenvalues determine $2n$ parameters.

The basic inverse spectral theorem for finite Jacobi matrices shows that $(\delta_1, (H-z)^{-1}\delta_1)$ determines $H$ uniquely. In [94] we considered $N \in \mathbb{N}, 1 \leq n \leq N$, and asked whether $(\delta_n, (H-z)^{-1}\delta_n)$ determines $H$ uniquely. For notational convenience, we occasionally allude to $G(z, n, n)$ as the $n, n$ Green’s function in the remainder of this section. The $n = 1$ result can be summarized via:

Theorem 5.33. $(\delta_1, (H-z)^{-1}\delta_1)$ has the form $\sum_{j=1}^{N} \alpha_j (\lambda_j - z)^{-1}$ with $\lambda_1 < \cdots < \lambda_N, \sum_{j=1}^{N} \alpha_j = 1$ and each $\alpha_j > 0$. Every such sum arises as the $1, 1$ Green’s function of an $H$ and of exactly one such $H$.

For general $n$, define $\tilde{n} = \min(n, N + 1 - n)$. Then the following theorems were proved in [94]:

Theorem 5.34. $(\delta_n, (H-z)^{-1}\delta_n)$ has the form $\sum_{j=1}^{k} \alpha_j (\lambda_j - z)^{-1}$ with $k$ one of $N, N-1, \ldots, N - \tilde{n} + 1$ and $\lambda_1 < \cdots < \lambda_k, \sum_{j=1}^{k} \alpha_j = 1$ and each $\alpha_j > 0$. Every such sum arises as the $n, n$ Green’s function of at least one $H$.

Theorem 5.35. If $k = N$, then precisely $\binom{N-1}{n-1}$ operators $H$ yield the given $n, n$ Green’s function.

Theorem 5.36. If $k < N$, then infinitely many Jacobi matrices $H$ yield the given $n, n$ Green’s function. Indeed, the inverse spectral family is then a collection of $(k-1)\binom{N-1}{n-1-N+k}$ disjoint manifolds, each of dimension $N - k$ and diffeomorphic to an $(N - k)$-dimensional open ball.

More recent references: Additional geometric information in connection with Theorem 5.36 and a version for off-diagonal Green’s functions were studied by Gibson [108]. Borg- and discrete Hochstadt–Lieberman-type results for generalized (i.e., certain tri-diagonal block) Jacobi matrices were studied by Derevyagin [54] (see also Shieh [222]). The case of non-self-adjoint Jacobi matrices with a rank-one imaginary part, and an extension of Hochstadt–Lieberman-type results to this situation was recently discussed by Arlinskiǐ and Tsedanovskii [11]. An extension of results of Hochstadt [121] to the case of normal matrices was found by S. Malamud [169]. A detailed treatment of two-spectra inverse problems of semi-infinite Jacobi operators, including reconstruction, has recently been presented by Silva and Weder [224].
6. The Crown Jewel: Simon’s New Approach to Inverse Spectral Theory

In this section we summarize some of the principal results of the following papers:

[228] B. Simon, *A new approach to inverse spectral theory, I. Fundamental formalism*, Ann. Math. 150, 1029–1057 (1999).

[97] F. Gesztesy and B. Simon, *A new approach to inverse spectral theory, II. General real potentials and the connection to the spectral measure*, Ann. Math. 152, 593–643 (2000).

[200] A. Ramm and B. Simon, *A new approach to inverse spectral theory, III. Short range potentials*, J. Analyse Math. 80, 319–334 (2000).

[98] F. Gesztesy and B. Simon, *On local Borg–Marchenko uniqueness results*, Commun. Math. Phys. 211, 273–287 (2000).

As the heading of this section suggests, we are approaching the pinnacle of Barry Simon’s contributions to inverse scattering theory thus far: In his spectacular paper [228], he single-handedly developed a new approach to inverse spectral theory for Schrödinger operators on a half-line, by starting from a particular representation of the Weyl–Titchmarsh $m$-function as a finite Laplace-type transform with control over the error term. In addition to establishing this feat, it also led to a completely unexpected uniqueness result for Weyl–Titchmarsh functions, what is now called the local Borg–Marchenko uniqueness theorem, but which really should have been named Simon’s local uniqueness theorem. The inverse spectral approach for Schrödinger operators on a half-line (including a reconstruction algorithm for the potential) originated with the celebrated paper [70] by Gelfand and Levitan in 1951 and an independent approach by Krein [142] in the same year, followed by a seminal contribution [171] by Marchenko in 1952. The Borg–Marchenko uniqueness result was first published by Marchenko [170] in 1950 but Borg apparently had it in 1949 and it was independently published by Borg [25] and again by Marchenko [171] in 1952. Both results, the uniqueness theorem and the Gel’fand–Levitan (reconstruction) formalism remained pillars of the inverse spectral theory that withstood any reformulation or improvement for nearly fifty years. Hence it was an incredible achievement by Barry Simon to have changed the inverse spectral landscape by offering such a reformulation of inverse spectral theory and in the very same paper [228] to have been able to substantially improve the Borg–Marchenko uniqueness theorem from a global to a local version.

We start by highlighting the approach in Simon’s paper [228] and then switch to a more detailed treatment of some aspects of the theory by borrowing from [97].

Inverse spectral methods have been actively studied in the past years both via their relevance in a variety of applications and due to their connection with integrable evolution equations such as the KdV equation. In this section, however, we will not deal with the full-line inverse spectral approach relevant to integrable equations but exclusively focus on inverse spectral theory for half-line Schrödinger operators. In this particular context, a major role is played by the Gel’fand–Levitan equations [70] (see also, [30, Chs. 3, 4], [31], [143], [144], [145], [159, Ch. 2], [171], [172, Ch. 2], [182, Ch. VIII], [204], [235], [244]). The goal in Barry Simon’s paper [228] was to present a new approach to their basic results. In particular, he introduced a new basic object, the $A$-function (see (6.20) below), the remarkable
equation (6.23) it satisfies, and illustrated its fundamental importance with several new results including improved asymptotic expansions of the Weyl–Titchmarsh $m$-function in the high-energy regime and the local uniqueness result.

To present some of these new results, we will first describe the major players in this game. One is concerned with self-adjoint differential operators on either $L^2([0,b])$ with $b < \infty$, or $L^2([0,\infty))$ associated with differential expressions of the form

$$-\frac{d^2}{dx^2} + V(x), \quad x \in (0,b).$$

(6.1)

If $b$ is finite, we suppose

$$\int_0^b dx |V(x)| < \infty$$

and place a boundary condition

$$u'(b) + hu(b) = 0$$

(6.2)

at $b$, where $h \in \mathbb{R} \cup \{\infty\}$ with $h = \infty$ shorthand for the Dirichlet boundary condition $u(b) = 0$. If $b = \infty$, we suppose

$$\int_y^{y+1} dx |V(x)| < \infty \text{ for all } y \geq 0$$

(6.3)

Under condition (6.3), it is known that (6.1) is limit point at infinity [201, App. to Sect. X.1]. In addition, a fixed self-adjoint boundary condition at $x = 0$ is assumed when talking about the self-adjoint operator associated with (6.1).

In either case, for each $z \in \mathbb{C} \setminus [-\beta, \infty)$ with $-\beta$ sufficiently large, there is a unique solution (up to an overall constant), $u(z,x)$, of $-u'' + Vu = zu$ which satisfies (6.2) at $b$ if $b < \infty$ or which is $L^2$ at $\infty$ if $b = \infty$. The principal $m$-function $m(z)$ is defined by

$$m(z) = \frac{u'(z,0)}{u(z,0)}.\quad (6.4)$$

If we replace $b$ by $b_1 = b - x_0$ with $x_0 \in (0,b)$ and let $V(s) = V(x_0 + s)$ for $s \in (0,b_1)$, we get a new $m$-function we will denote by $m(z,x_0)$. It is given by

$$m(z,x) = \frac{u'(z,x)}{u(z,x)}.$$

$m(z,x)$ satisfies the Riccati-type equation

$$\frac{d}{dx} m(z,x) = V(x) - z - m^2(z,x).$$

(6.5)

Obviously, $m(z,x)$ depends only on $V$ on $(x,b)$ (and on $h$ if $b < \infty$). A basic result of the inverse spectral theory says that the converse is true as was shown independently by Borg [25] and Marchenko [171] in 1952:

**Theorem 6.1.** $m$ uniquely determines $V$. Explicitly, if $V_j$ are potentials with corresponding $m$-functions $m_j$, $j = 1,2$, and $m_1 = m_2$, then $V_1 = V_2$ a.e. (including $h_1 = h_2$).

In 1999, Simon [228] spectacularly improved this to obtain a local version of the Borg–Marchenko uniqueness result as follows:
Theorem 6.2. If \((V_1, b_1, h_1), (V_2, b_2, h_2)\) are two potentials and \(a < \min(b_1, b_2)\) and if
\[ V_1(x) = V_2(x) \text{ on } (0, a), \] (6.5)
then as \(\kappa \to \infty\),
\[ m_1(-\kappa^2) - m_2(-\kappa^2) = \tilde{O}(e^{-2\kappa \alpha}). \] (6.6)

Conversely, if (6.6) holds, then (6.5) holds.

In (6.6), we use the symbol \(\tilde{O}\) defined by
\[ f = \tilde{O}(g) \text{ as } x \to x_0 \text{ (where } \lim_{x \to x_0} g(x) = 0) \]
if and only if \(\lim_{x \to x_0} \frac{|f(x)|}{|g(x)|^{1-\varepsilon}} = 0 \) for all \(\varepsilon > 0\).

From a result's point of view, this local version of the Borg–Marchenko uniqueness theorem was the most significant new result in Simon's paper [228], but a major thrust of this paper was the new set of methods introduced which led to a new approach of the inverse spectral problem. Theorem 6.2 implies that \(V\) is determined by the asymptotics of \(m(-\kappa^2)\) as \(\kappa \to \infty\). One can also read off differences of the boundary condition from these asymptotics. Moreover, the following result is proved in [228]:

Theorem 6.3. Let \((V_1, b_1, h_1), (V_2, b_2, h_2)\) be two potentials and suppose that
\[ b_1 = b_2 \equiv b < \infty, \quad |h_1| + |h_2| < \infty, \quad V_1(x) = V_2(x) \text{ on } (0, b). \] (6.7)
Then
\[ \lim_{\kappa \to \infty} e^{2\kappa \alpha} |m_1(-\kappa^2) - m_2(-\kappa^2)| = 4(h_1 - h_2). \] (6.8)

Conversely, if (6.8) holds for some \(b < \infty\) with a limit in \((0, \infty)\), then (6.7) holds.

To understand Simon’s new approach, it is useful to recall briefly the two approaches to the inverse problem for Jacobi matrices on \(\ell^2(\mathbb{N}_0)\) [19, Ch. VII], [94], [231]:
\[ A = \begin{pmatrix} b_0 & a_0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \]
with \(a_j > 0, b_j \in \mathbb{R}\). Here the \(m\)-function is just \((\delta_0, (A - z)^{-1}\delta_0) = m(z)\) and, more generally, \(m_n(z) = (\delta_n, (A^{(n)} - z)^{-1}\delta_n)\) with \(A^{(n)}\) on \(\ell^2(\{n, n+1, \ldots\})\) obtained by truncating the first \(n\) rows and \(n\) columns of \(A\). Here \(\delta_n\) is the Kronecker vector, that is, the vector with 1 in slot \(n\) and 0 in other slots. The fundamental theorem in this case is that \(m(z) \equiv m_0(z)\) determines the \(b_n\)'s and \(a_n\)'s.

\(m_n(z)\) satisfies an analog of the Riccati equation (6.4):
\[ a_n^2 m_{n+1}(z) = b_n - z - \frac{1}{m_n(z)}. \] (6.9)

One solution of the inverse problem is to turn (6.9) around to see that
\[ m_n(z)^{-1} = -z + b_n - a_n^2 m_{n+1}(z) \] (6.10)
which, first of all, implies that as \(z \to \infty\), \(m_n(z) = -z^{-1} + O(z^{-2})\), so (6.10) implies
\[ m_n(z)^{-1} = -z + b_n + a_n^2 z^{-1} + O(z^{-2}). \] (6.11)
Thus, (6.11) for \( n = 0 \) yields \( b_0 \) and \( a_2^0 \) and so \( m_1(z) \) by (6.9), and then an obvious induction yields successive \( b_k \), \( a_2^k \), and \( m_{k+1}(z) \).

A second solution involves orthogonal polynomials. Let \( P_n(z) \) be the eigensolutions of the formal \( (A - z)P_n = 0 \) with boundary conditions \( P_{-1}(z) = 0, P_0(z) = 1 \). Explicitly,

\[
P_{n+1}(z) = a_n^{-1}[(z - b_n)P_n(z)] - a_{n-1}P_{n-1}.
\]

Let \( d\rho \) be the spectral measure for \( A \) and vector \( \delta_0 \) so that \( m(z) = \int \frac{d\rho(\lambda)}{\lambda - z} \).

Then one can show that

\[
\int d\mu(\lambda) P_n(\lambda)P_m(\lambda) = \delta_{n,m}, \quad n, m \in \mathbb{N}_0.
\]

Thus, \( P_n(z) \) is a polynomial of degree \( n \) with positive leading coefficients determined by (6.14). These orthonormal polynomials are determined via Gram–Schmidt from \( \rho \) and by (6.13) from \( m \). Once one has the polynomials \( P_n \), one can determine the \( a \)'s and \( b \)'s from equation (6.12).

Of course, these approaches via Riccati equation and orthogonal polynomials are not completely disjoint. The Riccati solution gives the \( a_n \)'s and \( b_n \)'s as continued fractions and the connection between continued fractions and orthogonal polynomials played a fundamental role in Stieltjes' work [231] on the moment problem in 1895.

The Gel’fand–Levitan approach to the continuum case (cf. [70], [159, Ch. 2], [171], [172, Ch. 2]) is a direct analog of this orthogonal polynomial case. One looks at solutions \( U(k, x) \) of

\[
- U''(k, x) + V(x)U(k, x) = k^2 U(k, x)
\]
satisfying \( U(k, 0) = 1, U'(k, 0) = ik \), and proves that they satisfy a representation

\[
U(k, x) = e^{ikx} + \int_{-x}^x dy K(x, y)e^{iky},
\]

the analog of \( P_n(z) = cz^n + \) lower order. One defines \( s(k, x) = (2ik)^{-1}[U(k, x) - U(-k, x)] \) which satisfies (6.15) with \( s(k, 0) = 0, s'(k, 0) = 1 \).

The spectral measure \( d\rho \) associated to \( m(z) \) by

\[
d\rho(\lambda) = (2\pi)^{-1}\lim_{\varepsilon, \delta \to 0} \text{Im}(m(\lambda + i\varepsilon))\ d\lambda
\]
satisfies

\[
\int d\rho(k^2) s(k, x)s(k, y) = \delta(x - y),
\]

at least formally. (6.16) and (6.17) yield an integral equation for \( K \) depending only on \( d\rho \) and then once one has \( K \), one can find \( U \) and hence \( V \) via (6.15) (or via another relation between \( K \) and \( V \)).

The principal goal in [228] was to present a new approach to the continuum case, that is, an analog of the Riccati equation approach to the discrete inverse problem. The simple idea for this is attractive but has a difficulty to overcome. \( m(z, x) \) determines \( V(x) \), at least if \( V \) is continuous by the known asymptotics ([45], [210]):

\[
m(-\kappa^2, x) = -\kappa - \frac{V(x)}{2\kappa} + o(\kappa^{-1}).
\]
We can therefore think of (6.4) with $V$ defined by (6.18) as an evolution equation for $m$. The idea is that using a suitable underlying space and uniqueness theorem for solutions of differential equations, (6.4) should uniquely determine $m$ for all positive $x$, and hence $V(x)$ by (6.18).

To understand the difficulty, consider a potential $V(x)$ on the whole real line. There are then functions $u_{\pm}(z,x)$ defined for $z \in \mathbb{C}\setminus[\beta, \infty)$ which are $L^2$ at $\pm \infty$ and two $m$-functions $m_{\pm}(z,x) = u'_{\pm}(z,x)/u_{\pm}(z,x)$. Both satisfy (6.4), yet $m_+(z,0)$ determines and is determined by $V$ on $(0, \infty)$ while $m_-(z,0)$ has the same relation to $V$ on $(-\infty, 0)$. Put differently, $m_+(z,0)$ determines $m_+(z,x)$ for $x > 0$ but not at all for $x < 0$. $m_-$ is the reverse. So uniqueness for (6.4) is one-sided and either side is possible! That this does not make the scheme hopeless is connected with the fact that $m_-$ does not satisfy (6.18), but rather

$$m_-(\kappa^2, x) = \kappa + \frac{V(x)}{2\kappa} + o(\kappa^{-1}).$$

(6.19)

We will see the one-sidedness of the solvability is intimately connected with the sign of the leading $\pm \kappa$ term in (6.18), (6.19).

The key object in this new approach is a function $A(\alpha)$ defined for $\alpha \in (0,b)$ related to $m$ by

$$m(-\kappa^2) = -\kappa - \int_0^{\alpha} d\alpha A(\alpha)e^{-2\alpha \kappa} + O(e^{-2\alpha \kappa})$$

(6.20)

as $\kappa \to \infty$. We have written $A(\alpha)$ as a function of a single variable but we will allow similar dependence on other variables. Since $m(-\kappa^2, x)$ is also an $m$-function, (6.20) has an analog with a function $A(\alpha, x)$.

By uniqueness of inverse Laplace transforms (see [228, Appendix 2, Theorem A.2.2]), (6.20) and $m$ near $-\infty$ uniquely determine $A(\alpha)$.

Not only will (6.20) hold but, in a sense, $A(\alpha)$ is close to $V(\alpha)$. Explicitly, one can prove the following result:

**Theorem 6.4.** Let $m$ be the $m$-function of the potential $V$. Then there is a function $A \in L^1((0,b])$ if $b < \infty$ and $A \in L^1([0,a])$ for all $a < \infty$ if $b = \infty$ so that (6.20) holds for any $a \leq b$ with $a < \infty$. $A(\alpha)$ only depends on $V(y)$ for $y \in [0,\alpha]$. Moreover, $A(\alpha) = V(\alpha) + E(\alpha)$ where $E(\alpha)$ is continuous and satisfies

$$|E(\alpha)| \leq \left(\int_0^\alpha dy |V(y)|^2 \right)^{1/2} \exp\left(\alpha \int_0^\alpha dy |V(y)|\right).$$

Restoring the $x$-dependence, we see that $A(\alpha, x) = V(\alpha + x) + E(\alpha, x)$ where

$$\lim_{\alpha \downarrow 0} \sup_{0 \leq x \leq a} |E(\alpha, x)| = 0$$

for any $a > 0$, so

$$\lim_{\alpha \downarrow 0} A(\alpha, x) = V(x),$$

(6.21)

where this holds in general in the $L^1$-sense. If $V$ is continuous, (6.21) holds point-wise. In general, (6.21) will hold at any point of right Lebesgue continuity of $V$.

Because $E$ is continuous, $A$ determines any discontinuities or singularities of $V$. More is true. If $V$ is $C^k$, then $E$ is $C^{k+2}$ in $\alpha$, and so $A$ determines $k$th order kinks in $V$. Much more is true and one can also prove the following result:
\textbf{Theorem 6.5.} \(V\) on \([0,a]\) is only a function of \(A\) on \([0,a]\). Explicitly, if \(V_1, V_2\) are two potentials, let \(A_1, A_2\) be their \(A\)-functions. If \(a < b_1, a < b_2, \text{ and } A_1(\alpha) = A_2(\alpha) \text{ for } \alpha \in [0,a], \text{ then } V_1(x) = V_2(x) \text{ for } x \in [0,a].\)

Theorems 6.4 and 6.5 imply Theorem 6.2.

As noted, the singularities of \(V\) come from singularities of \(A\). A boundary condition is a kind of singularity, so one might hope that boundary conditions correspond to very singular \(A\). In essence, we will see that this is the case – there are delta-function and delta-prime singularities at \(\alpha = b\). Explicitly, one can prove the following result:

\textbf{Theorem 6.6.} Let \(m\) be the \(m\)-function for a potential \(V\) with \(b < \infty\). Then for \(a < 2b\),

\[
m(-\kappa^2) = -\kappa - \int_0^a d\alpha A(\alpha)e^{-2\alpha\kappa} - A_1 \kappa e^{-2\kappa b} - B_1 e^{-2\kappa b} + O(e^{-2\alpha\kappa}),
\]

where the following facts hold:

(a) If \(h = \infty\), then \(A_1 = 2\), \(B_1 = -2 \int_0^b V(y) dy\).

(b) If \(|h| < \infty\), then \(A_1 = -2\), \(B_1 = 2[2h + \int_0^b V(y) dy]\).

This implies Theorem 6.3.

The reconstruction theorem, Theorem 6.5, depends on the differential equation that \(A(\alpha, x)\) satisfies. Remarkably, \(V\) drops out of the translation of (6.4) to the equation for \(A\):

\[
\frac{\partial A(\alpha, x)}{\partial x} = \frac{\partial A(\alpha, x)}{\partial \alpha} + \int_0^\alpha d\beta A(\beta, x)A(\alpha - \beta, x).
\]

(6.23)

If \(V\) is \(C^1\), the equation holds in classical sense. For general \(V\), it holds in a variety of weaker senses. Either way, \(A(\alpha, 0)\) for \(\alpha \in [0,a]\) determines \(A(\alpha, x)\) for all \(x, \alpha\) with \(\alpha > 0\) and \(0 < x + \alpha < a\). (6.21) then determines \(V(x)\) for \(x \in [0,a]\).

That is the essence from which uniqueness comes. We will return to this circle of ideas later on when discussing Simon’s approach to the inverse spectral problem in detail.

\[\star \quad \star \quad \star \]

Now we switch to [97] and take a closer look at some of the concepts introduced in [228]. In particular, we continue the study of the \(A\)-amplitude associated to half-line Schrödinger operators, \(-\frac{d^2}{dx^2} + V\) in \(L^2([0,b]), b \leq \infty\). \(A\) is related to the Weyl–Titchmarsh \(m\)-function via \(m(-\kappa^2) = -\kappa - \int_0^a d\alpha A(\alpha)e^{-2\alpha\kappa} + O(e^{-2a-\varepsilon\kappa})\) for all \(\varepsilon > 0\). Three main issues will be discussed:

- First, we describe how to extend the theory to general \(V\) in \(L^1([0,a])\) for all \(a > 0\), including \(V\)’s which are limit cycle at infinity.
- Second, the following relation between the \(A\)-amplitude and the spectral measure \(\rho\):

\[
A(\alpha) = -2 \int_{-\infty}^{\infty} d\rho(\lambda) \lambda^{-\frac{1}{2}} \sin(2\alpha \sqrt{\lambda}),
\]

will be discussed. (Since the integral is divergent, this formula has to be properly interpreted.)

- Third, a Laplace transform representation for \(m\) without error term in the case \(b < \infty\) will be presented.
We consider Schrödinger operators
\[ -\frac{d^2}{dx^2} + V \] (6.24)
in \( L^2([0, b]) \) for \( 0 < b < \infty \) or \( b = \infty \) and real-valued locally integrable \( V \). There are essentially four distinct cases.

**Case 1.** \( b < \infty \). We suppose \( V \in L^1([0, b]) \). We then pick \( h \in \mathbb{R} \cup \{\infty\} \) and add the boundary condition at \( b \)
\[ u'(b-) + hu(b-) = 0, \] (6.25)
where \( h = \infty \) is shorthand for the Dirichlet boundary condition \( u(b-) = 0 \).

For Cases 2–4, \( b = \infty \) and
\[ \int_{0}^{a} dx \left| V(x) \right| < \infty \quad \text{for all} \quad a < \infty. \] (6.26)

**Case 2.** \( V \) is “essentially” bounded from below in the sense that
\[ \sup_{a>0} \left( \int_{a}^{a+1} dx \max(-V(x), 0) \right) < \infty. \] (6.27)
Examples include \( V(x) = c(x+1)^\beta \) for \( c > 0 \) and all \( \beta \in \mathbb{R} \) or \( V(x) = -c(x+1)^\beta \) for all \( c > 0 \) and \( \beta \leq 0 \).

**Case 3.** (6.27) fails but (6.24) is limit point at \( \infty \) (see, e.g., [41, Ch. 9], [201, Sect. X.1] for a discussion of limit point/limit circle), that is, for each \( z \in \mathbb{C}_+ = \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \} \),
\[ -u'' + Vu = zu \] (6.28)
has a unique solution, up to a multiplicative constant, which is \( L^2 \) at \( \infty \). An example is \( V(x) = -c(x+1)^\beta \) for \( c > 0 \) and \( 0 < \beta \leq 2 \).

**Case 4.** (6.24) is limit circle at infinity, that is, every solution of (6.28) is \( L^2([0, \infty)) \) at infinity if \( z \in \mathbb{C}_+ \). We then pick a boundary condition by choosing a nonzero solution \( u_0 \) of (6.28) for \( z = i \). Other functions \( u \) satisfying the associated boundary condition at infinity then are supposed to satisfy
\[ \lim_{x \to \infty} W(u_0, u)(x) = \lim_{x \to \infty} [u_0(x)u'(x) - u'_0(x)u(x)] = 0. \] (6.29)
Examples include \( V(x) = -c(x+1)^\beta \) for \( c > 0 \) and \( \beta > 2 \).

The Weyl–Titchmarsh \( m \)-function, \( m(z) \), is defined for \( z \in \mathbb{C}_+ \) as follows. Fix \( z \in \mathbb{C}_+ \). Let \( u(x, z) \) be a nonzero solution of (6.28) which satisfies the boundary condition at \( b \). In Case 1, that means \( u \) satisfies (6.25); in Case 4, it satisfies (6.29); and in Cases 2–3, it satisfies \( \int_{R}^\infty |u(z, x)|^2 \, dx < \infty \) for some (and hence for all) \( R \geq 0 \). Then,
\[ m(z) = \frac{u'(z, 0_+)}{u(z, 0_+)} \] (6.30)
and, more generally,
\[ m(z, x) = \frac{u'(z, x)}{u(z, x)}. \] (6.31)
\( m(z, x) \) satisfies the Riccati equation (with \( m' = \partial m/\partial x \)),
\[ m'(z, x) = V(x) - z - m(z, x)^2. \] (6.32)
\( m \) is an analytic function of \( z \) for \( z \in \mathbb{C}_+ \), and the following properties hold:
Case 1. $m$ is meromorphic in $\mathbb{C}$ with a discrete set $\lambda_1 < \lambda_2 < \cdots$ of poles on $\mathbb{R}$ (and none on $(-\infty, \lambda_1)$).

Case 2. For some $\beta \in \mathbb{R}$, $m$ has an analytic continuation to $\mathbb{C} \setminus [\beta, \infty)$ with $m$ real on $(-\infty, \beta)$.

Case 3. In general, $m$ cannot be continued beyond $\mathbb{C}_+$ (there exist $V$’s where $m$ has a dense set of polar singularities on $\mathbb{R}$).

Case 4. $m$ is meromorphic in $\mathbb{C}$ with a discrete set of poles (and zeros) on $\mathbb{R}$ with limit points at both $+\infty$ and $-\infty$.

Moreover, if $z \in \mathbb{C}_+$ then $m(z, x) \in \mathbb{C}_+$, so $m$ admits the Herglotz representation,

$$m(z) = \text{Re}(m(i)) + \int_{\mathbb{R}} d\rho(\lambda) \left[ \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right], \quad z \in \mathbb{C} \setminus \mathbb{R},$$

where $\rho$ is a positive measure called the spectral measure, which satisfies

$$\int_{\mathbb{R}} \frac{d\rho(\lambda)}{1 + |\lambda|^2} < \infty,$$

$$d\rho(\lambda) = \text{w-lim}_{\varepsilon \to 0} \frac{1}{\pi} \text{Im}(m(\lambda + i\varepsilon)) d\lambda,$$

where w-lim is meant in distributional sense.

All these properties of $m$ are well known (see, e.g. [162, Ch. 2]).

In (6.33), the constant $\text{Re}(m(i))$ is determined by the result of Everitt [62] that for each $\varepsilon > 0$, $m(-\kappa^2) = -\kappa + o(1)$ as $|\kappa| \to \infty$ with $-\frac{\pi}{2} + \varepsilon < \text{arg}(\kappa) < -\varepsilon < 0$. (6.36)

Atkinson [13] improved (6.36) to read,

$$m(-\kappa^2) = -\kappa + \int_{0}^{a_0} d\alpha V(\alpha)e^{-2\alpha\kappa} + o(\kappa^{-1})$$

(6.37) again as $|\kappa| \to \infty$ with $-\frac{\pi}{2} + \varepsilon < \text{arg}(\kappa) < -\varepsilon < 0$ (actually, he allows $\text{arg}(\kappa) \to 0$ as $|\kappa| \to \infty$) as long as $\text{Re}(\kappa) > 0$ and $\text{Im}(\kappa) > -\exp(-D|\kappa|)$ for suitable $D$). In (6.37), $a_0$ is any fixed $a_0 > 0$.

One of the main results in [97] was to go way beyond the two leading orders in (6.37).

Theorem 6.7. There exists a function $A(\alpha)$ for $\alpha \in [0, b)$ so that $A \in L^1([0, a])$ for all $a < b$ and

$$m(-\kappa^2) = -\kappa - \int_{0}^{a} d\alpha A(\alpha)e^{-2\alpha\kappa} + \tilde{O}(e^{-2\alpha\kappa})$$

(6.38) as $|\kappa| \to \infty$ with $-\frac{\pi}{2} + \varepsilon < \text{arg}(\kappa) < -\varepsilon < 0$. Here we say $f = \tilde{O}(g)$ if $g \to 0$ and for all $\varepsilon > 0$, $\left(\frac{1}{\varepsilon}\right)|g|^\varepsilon \to 0$ as $|\kappa| \to \infty$. Moreover, $A - q$ is continuous and

$$|(A - q)(\alpha)| \leq \left[\int_{0}^{a} dx |V(x)|^2\right]^2 \exp\left(\alpha \int_{0}^{a} dx |V(x)|\right).$$

(6.39)
This result was proved in Cases 1 and 2 in [228]. The proof of this result if one only assumes (6.26) (i.e., in Cases 3 and 4) has been provided in [97].

Actually, in [228], (6.38) was proved in Cases 1 and 2 for $\kappa$ real with $|\kappa| \to \infty$. The proof in [97], assuming only condition (6.26), includes Case 2 in the general $\kappa$-region $\arg(\kappa) \in (-\pi/2 + \varepsilon, -\varepsilon)$ and, as can be shown, the proof also holds in this region for Case 1.

**Remark 6.8.** At first sight, it may appear that Theorem 6.7, as stated, does not imply the $\kappa$ real result of [228], but if the spectral measure $\rho$ of (6.33) has $\text{supp}(\rho) \subseteq [a, \infty)$ for some $a \in \mathbb{R}$, (6.38) extends to all $\kappa$ in $|\arg(\kappa)| < \frac{\pi}{2} - \varepsilon$, $|\kappa| \geq a + 1$. To see this, one notes by (6.33) that $m'(z)$ is bounded away from $[a, \infty)$ so one has the a priori bound $|m(z)| \leq C|z|$ in the region $\text{Re}(z) < a - 1$. This bound and a Phragmén–Lindelöf argument let one extend (6.38) to the real $\kappa$ axis.

The following is a basic result from [228]:

**Theorem 6.9.** (Theorem 2.1 of [228]) Let $V \in L^1([0, \infty))$. Then there exists a function $A$ on $(0, \infty)$ so that $A - V$ is continuous and satisfies (6.39) such that for $\text{Re}(\kappa) > \|V\|_{1/2}$,

$$m(-\kappa^2) = -\kappa - \int_0^\infty d\alpha A(\alpha)e^{-2\alpha \kappa}. \tag{6.40}$$

**Remark 6.10.** In [228], this is only stated for $\kappa$ real with $\kappa > \|V\|_{1/2}$, but (6.39) implies that $|A(\alpha) - V(\alpha)| \leq \|V\|^2_1 \exp(\alpha \|V\|_1)$ so the right-hand side of (6.40) converges to an analytic function in $\text{Re}(\kappa) > \|V\|_{1/2}$. Since $m(z)$ is analytic in $\mathbb{C}\setminus[a, \infty)$ for suitable $\alpha$, we have equality in $\{\kappa \in \mathbb{C} | \text{Re}(\kappa) > \|V\|_{1/2}\}$ by analyticity.

Theorem 6.7 in all cases follows from Theorem 6.9 and the following result which was proved in [97]:

**Theorem 6.11.** Let $V_1, V_2$ be potentials defined on $(0, b_j)$ with $b_j > a$ for $j = 1, 2$. Suppose that $V_1 = V_2$ on $[0, a]$. Then in the region $\arg(\kappa) \in (-\pi/2 + \varepsilon, -\varepsilon)$, $|\kappa| \geq K_0$, we have that

$$|m_1(-\kappa^2) - m_2(-\kappa^2)| \leq C_{\varepsilon, \delta} \exp(-2a\text{Re}(\kappa)), \tag{6.41}$$

where $C_{\varepsilon, \delta}$ depends only on $\varepsilon, \delta$, and $\sup_{0 \leq \alpha \leq a} \left( \int_0^{x+\delta} dy|V_j(y)| \right)$, where $\delta > 0$ is any number so that $a + \delta \leq b_j$, $j = 1, 2$.

**Remark 6.12.** (i) An important consequence of Theorem 6.11 is that if $V_1(x) = V_2(x)$ for $x \in [0, a]$, then $A_1(\alpha) = A_2(\alpha)$ for $\alpha \in [0, a]$. Thus, $A(\alpha)$ is only a function of $V$ on $[0, x]$.

(ii) This implies Theorem 6.7 by taking $V_1 = V$ and $V_2 = V\chi_{[0,a]}$ and using Theorem 6.9 on $V_2$.

(iii) The actual proof implies (6.41) on a larger region than $\arg(\kappa) \in (-\pi/2 + \varepsilon, -\varepsilon)$. Basically, one needs $\text{Im}(\kappa) \geq -C_1 \exp(-C_2|\kappa|)$ as $\text{Re}(\kappa) \to \infty$.

The basic connection between the spectral measure $d\rho$ and the $A$-amplitude established in [97] says

$$A(\alpha) = -2 \int_{-\infty}^{\infty} d\rho(\lambda) \lambda^{-\frac{1}{2}} \sin(2\alpha \sqrt{\lambda}). \tag{6.42}$$
However, the integral in (6.42) is not convergent. Indeed, the asymptotics (6.36) imply that $\int_0^R d\rho(\lambda) \sim \frac{2}{3\pi} R^{\frac{3}{2}}$ so (6.42) is never absolutely convergent. Thus, (6.42) has to be suitably interpreted.

We will indicate how to demonstrate (6.42) as a distributional relation, smeared in $\alpha$ on both sides by a function $f \in C^\infty_0((0, \infty))$. This holds for all $V$’s in Cases 1–4. Finally, we will discuss an Abelianized version of (6.42), namely,

$$A(\alpha) = -2 \lim_{\varepsilon \downarrow 0} \int_{-\infty}^{\infty} d\rho(\lambda) e^{-\varepsilon \lambda} \lambda^{-\frac{1}{2}} \sin(2\alpha \sqrt{\lambda}) \tag{6.43}$$

at any point, $\alpha$, of Lebesgue continuity for $V$. (6.43) is proved only for a restricted class of $V$’s including Case 1, 2 and those $V$’s satisfying

$$V(x) \geq -Cx^2, \quad x \geq R$$

for some $R > 0$, $C > 0$, which are always in the limit point case at infinity. Subsequently, we will use (6.43) as the point of departure for relating $A(\alpha)$ to scattering data.

In order to prove (6.42) for finite $b$, one needs to analyze the finite $b$ case extending (6.38) to all $a$ including $a = \infty$ (by allowing $A$ to have $\delta$ and $\delta'$ singularities at multiples of $b$). This was originally done in [228] for $\kappa$ real and positive and $a < \infty$.

We now need results in the entire region $\text{Re}(\kappa) \geq K_0$. Explicitly, the following was proved in [97]:

**Theorem 6.13.** In Case 1, there are $A_n, B_n$ for $n = 1, 2, \ldots$, and a function $A(\alpha)$ on $(0, \infty)$ with

$$|A_n| \leq C, \quad |B_n| \leq Cn,$$

$$\int_0^a d\alpha |A(\alpha)| \leq C \exp(K_0|a|) \quad \text{so that for } \text{Re}(\kappa) > \frac{1}{2} K_0:$$

$$m(-\kappa^2) = -\kappa - \sum_{n=1}^{\infty} A_n \kappa e^{-2\kappa bn} - \sum_{n=1}^{\infty} B_n e^{-2\kappa bn} - \int_0^a d\alpha A(\alpha) e^{-2\kappa \alpha}.$$ 

(6.42) can be used to obtain a priori bounds on $\int_{-R}^0 d\rho(\lambda)$ as $R \to \infty$.

Now we turn to more details and start by illustrating how to use the Riccati equation and a priori control on $m_j$ to obtain exponentially small estimates on $m_1 - m_2$.

**Lemma 6.14.** Let $m_1, m_2$ be two absolutely continuous functions on $[a, b]$ so that for some $Q \in L^1([a, b])$,

$$m_j'(x) = Q(x) - m_j(x)^2, \quad j = 1, 2, \quad x \in (a, b). \tag{6.44}$$

Then

$$|m_1(a) - m_2(a)| = |m_1(b) - m_2(b)| \exp \left( \int_a^b dy [m_1(y) + m_2(y)] \right).$$

As an immediate corollary, one obtains the following result (which implies Theorem 6.11):
Theorem 6.15. Let \( m_j(\kappa^2, x) \) be functions defined for \( x \in [a, b] \) and \( \kappa \in K \) some region of \( \mathbb{C} \). Suppose that for each \( \kappa \) in \( K \), \( m_j \) is absolutely continuous in \( x \) and satisfies (note that \( V \) is the same for \( m_1 \) and \( m_2 \)),

\[
m_j'(-\kappa^2, x) = V(x) + \kappa^2 - m_j(-\kappa^2, x)^2, \quad j = 1, 2.
\]

Suppose \( C \) is such that for each \( x \in [a, b] \) and \( \kappa \in K \),

\[
|m_j(-\kappa^2, x) + \kappa| \leq C, \quad j = 1, 2, \tag{6.45}
\]

then

\[
|m_1(-\kappa^2, a) - m_2(-\kappa^2, a)| \leq 2C \exp[-2(b-a)|\text{Re}(\kappa) - C|]. \tag{6.46}
\]

Theorem 6.15 places importance on a priori bounds of the form (6.45). Fortunately, by modifying ideas of Atkinson [13], we can obtain estimates of this form as long as \( \text{Im}(\kappa) \) is bounded away from zero.

Atkinson’s method allows one to estimate \( |m(-\kappa^2) + \kappa| \) in two steps. We will fix some \( a < b \) finite and define \( m_0(-\kappa^2) \) by solving

\[
m_0'(-\kappa^2, x) = V(x) + \kappa^2 - m_0(-\kappa^2, x)^2, \tag{6.47a}
\]

\[
m_0(-\kappa^2, a) = -\kappa \tag{6.47b}
\]

and then setting

\[
m_0(-\kappa^2) := m_0(-\kappa^2, 0_+). \tag{6.47c}
\]

One then proves the following result.

Lemma 6.16. There is a \( C > 0 \) depending only on \( V \) and a universal constant \( E > 0 \) so that if \( \text{Re}(\kappa) \geq C \) and \( \text{Im}(\kappa) \neq 0 \), then

\[
|m(-\kappa^2) - m_0(-\kappa^2)| \leq E \frac{|\kappa|^2}{|\text{Im}(\kappa)|} e^{-2a \text{Re}(\kappa)}. \tag{6.48}
\]

In fact, one can take

\[
C = \max \left( a^{-1} \ln(6), \, 4 \int_0^a dx \, |V(x)| \right), \quad E = \frac{3 \cdot 2 \cdot 12^2}{5}.
\]

Lemma 6.17. There exist constants \( D_1 \) and \( D_2 \) (depending only on \( a \) and \( V \)), so that for \( \text{Re}(\kappa) > D_1 \),

\[
|m_0(-\kappa^2) + \kappa| \leq D_2.
\]

Indeed, one can take

\[
D_1 = D_2 = 2 \int_0^a dx \, |V(x)|.
\]

These Lemmas together with Theorem 6.9 yield the following explicit form of Theorem 6.11.

Theorem 6.18. Let \( V_1, V_2 \) be defined on \((0, b_j)\) with \( b_j > a \) for \( j = 1, 2 \). Suppose that \( V_1 = V_2 \) on \([0, a]\). Pick \( \delta \) so that \( a + \delta \leq \min(b_1, b_2) \) and let \( \eta = \sup_{0 \leq x \leq a; j = 1, 2} (\int_x^{x+\delta} dy \, |V_j(y)|) \). Then if \( \text{Re}(\kappa) \geq \max(4\eta, \delta^{-1} \ln(6)) \) and \( \text{Im}(\kappa) \neq 0 \), one obtains

\[
|m_1(-\kappa^2) - m_2(-\kappa^2)| \leq 2g(\kappa) \exp(-2a|\text{Re}(\kappa) - g(\kappa)|),
\]

where

\[
g(\kappa) = 2\eta + \frac{864}{5} \frac{|\kappa|^2}{|\text{Im}(\kappa)|} e^{-2\delta \text{Re}(\kappa)}.
\]
Remark 6.19. (i) To obtain Theorem 6.11, we need only note that in the region 
\[ \arg(\kappa) \in (-\frac{\pi}{2} + \varepsilon, -\varepsilon), \; |\kappa| > K_0, \; g \text{ is bounded.} \]
(ii) We need not require that \( \arg(\kappa) < -\varepsilon \) to obtain \( g \) bounded. It suffices, for 
e.g., that \( \text{Re}(\kappa) \geq |\text{Im}(\kappa)| \geq e^{-\alpha \text{Re}(\kappa)} \) for some \( \alpha < 2\delta \).
(iii) For \( g \) to be bounded, we need not require that \( \arg(\kappa) > -\frac{\pi}{2} + \varepsilon \). It suffices that 
\[ |\text{Im}(\kappa)| \geq \text{Re}(\kappa) \geq \alpha \ln|\text{Im}(\kappa)| \] 
for some \( \alpha > (2\delta)^{-1} \). Unfortunately, this does not include the region \( \text{Im}(-\kappa^2) = c, \; \text{Re}(-\kappa^2) \to \infty \), where \( \text{Re}(\kappa) \) goes to zero as \( |\kappa|^{-1} \). However, as \( \text{Re}(-\kappa^2) \to \infty \), we only need that \( |\text{Im}(-\kappa^2)| \geq 2\alpha|\kappa|\ln(|\kappa|) \).

Next, we turn to finite \( b \) representations with no errors: Theorem 6.9 implies that 
if \( b = \infty \) and \( V \in L^1([0, \infty)) \), then (6.40) holds, a Laplace transform representation
for \( m \) without errors. It is, of course, of direct interest that such a form ula holds, but we are especially interested in a particular consequence of it – namely, that
it implies that the formula (6.38) with error holds in the region \( \text{Re}(\kappa) > K_0 \) with
error uniformly bounded in \( \text{Im}(\kappa) \); that is, one proves the following result:

Theorem 6.20. If \( V \in L^1([0, \infty)) \) and \( \text{Re}(\kappa) > |V|_{1/2} \), then for all \( a > 0 \):

\[
\left| m(-\kappa^2) + \kappa + \int_0^a d\alpha A(\alpha)e^{-2\alpha\kappa} \right| \leq \left[ |V|_1 + \frac{|V|^2_{1/2}|V|_1}{2\text{Re}(\kappa) - |V|_1} \right] e^{-2a\text{Re}(\kappa)}. \tag{6.49}
\]

The principal goal is to prove an analog of this result in the case \( b < \infty \). To do
so, we will need to first prove an analog of (6.40) in case \( b < \infty \) – something of
interest in its own right. The idea will be to mimic the proof of Theorem 2 from
[228] but use the finite \( b \), \( V^{(0)}(x) = 0, \; x \geq 0 \) Green’s function where [228] used
the infinite \( b \) Green’s function. The basic idea is simple, but the arithmetic is a bit
involved.

We will start with the \( h = \infty \) case. Three functions for \( V^{(0)}(x) = 0, \; x \geq 0 \) are
significant. First, the kernel of the resolvent \((-\frac{d^2}{dx^2} + \kappa^2)^{-1} \) with \( u(0_+) = u(b_-) = 0 \)
boundary conditions. By an elementary calculation (see, e.g., [228, Sect. 5]), it has the form

\[
G^{(0)}_{h=\infty}(-\kappa^2, x, y) = \frac{\sinh(\kappa x)}{\kappa} \left[ e^{-\kappa x} - e^{-\kappa(2b-x)} \right] \frac{1}{1 - e^{-2\kappa b}},
\tag{6.50}
\]

with \( x_\leq = \min(x, y), \; x_\geq = \max(x, y) \).

The second function is

\[
\psi^{(0)}_{h=\infty}(-\kappa^2, x) = \lim_{y \to 0} \left. \frac{\partial G^{(0)}_{h=\infty}(-\kappa^2, x, y)}{\partial y} \right|_{y=0} = \frac{e^{-\kappa x} - e^{-\kappa(2b-x)}}{1 - e^{-2\kappa b}}, \tag{6.51}
\]

and finally (notice that \( \psi^{(0)}_{h=\infty}(-\kappa^2, 0_+) = 1 \) and \( \psi^{(0)}_{h=\infty} \) satisfies the equations
\(-\psi'' = -\kappa^2 \psi \) and \( \psi(-\kappa^2, b_-) = 0 \):

\[
m^{(0)}_{h=\infty}(-\kappa^2) = \psi^{(0)}_{h=\infty}(-\kappa^2, 0_+) = \frac{\kappa + \kappa e^{-2\kappa b}}{1 - e^{-2\kappa b}}. \tag{6.52}
\]

In (6.52), prime means \( d/dx \).

Now fix \( V \in C_0^\infty((0, b)) \). The pair of formulas

\[
\left( -\frac{d^2}{dx^2} + V + \kappa^2 \right)^{-1}
= \sum_{n=0}^{\infty} (-1)^n \left( -\frac{d^2}{dx^2} + \kappa^2 \right)^{-1} \left[ V \left( -\frac{d^2}{dx^2} + \kappa^2 \right)^{-1} \right]^n
\]
\[ m(-\kappa^2) = \lim_{x<y; y \downarrow 0} \frac{\partial^2 G(-\kappa^2, x, y)}{\partial x \partial y} \]
yields the following expansion for the \( m \)-function of \( -\frac{d^2}{dx^2} + V \) with \( u(b_-) = 0 \) boundary conditions.

**Lemma 6.21.** Let \( V \in C_0^\infty((0, b)) \), \( b < \infty \). Then

\[ m(-\kappa^2) = \sum_{n=0}^{\infty} M_n(-\kappa^2; V), \]  
(6.53)

where

\[ M_0(-\kappa^2; V) = m_{h=\infty}^{(0)}(-\kappa^2), \]  
(6.54)

\[ M_1(-\kappa^2; V) = -\int_0^b V(x)\psi_{h=\infty}^{(0)}(-\kappa^2, x)^2 \, dx, \]  
(6.55)

and for \( n \geq 2 \),

\[ M_n(-\kappa^2; V) = (-1)^n \int_0^b dx_1 \ldots \int_0^b dx_n V(x_1) \ldots V(x_n) \]
\[ \times \psi_{h=\infty}^{(0)}(-\kappa^2, x_1)\psi_{h=\infty}^{(0)}(-\kappa^2, x_n) \prod_{j=1}^{n-1} G_{h=\infty}^{(0)}(-\kappa^2, x_j, x_{j+1}). \]  
(6.56)

The precise region of convergence is unimportant since one can expand regions by analytic continuation. For now, we note it certainly converges in the region \( \kappa \) real with \( \kappa^2 > \|V\|_\infty \).

Writing each term in (6.53) as a Laplace transform then yields the following result:

**Theorem 6.22.** (Theorem 6.13 for \( h = \infty \)) Let \( b < \infty \), \( h = \infty \), and \( V \in L^1([0, b]) \). Then for \( \text{Re}(\kappa) > \|V\|_1/2 \), we have that

\[ m(-\kappa^2) = -\kappa - \sum_{j=1}^{\infty} A_j e^{-2\kappa b j} - \sum_{j=1}^{\infty} B_j e^{-2\kappa b j} - \int_0^{\alpha} d\alpha A(\alpha) e^{-2\alpha \kappa}, \]  
(6.57)

where

\[ A_j = 2, \quad B_j = -2j \int_0^b dx V(x), \quad j \in \mathbb{N}, \]

\[ |A(\alpha) - A_1(\alpha)| \leq \frac{(2\alpha + b)(2\alpha + 2b)}{2b^2} \|V\|_1^2 \exp(\alpha \|V\|_1) \text{ with } A_1 \text{ given by} \]

\[ A_1(\alpha) = \begin{cases} V(\alpha), & 0 \leq \alpha < b, \\ (n+1)V(\alpha - nb) + nq((n+1)b - \alpha), & nb \leq \alpha < (n+1)b, \quad n \in \mathbb{N}. \end{cases} \]

In particular, for all \( a \in (0, b) \),

\[ \int_0^a d\alpha |A(\alpha)| \leq C(b, \|V\|_1)(1 + a^2) \exp(a \|V\|_1). \]

This implies the following estimate:
Corollary 6.23. If \( V \in L^1([0, \infty)) \) and \( \text{Re}(\kappa) \geq \frac{1}{2}\|V\|_1 + \varepsilon \), then for all \( a \in (0, b) \), \( b < \infty \), we have that
\[
|m(-\kappa^2) + \kappa + \int_0^a d\alpha A(\alpha)e^{-2\alpha \kappa}| \leq C(a, \varepsilon)e^{-2\alpha \text{Re}(\kappa)},
\]
where \( C(a, \varepsilon) \) depends only on \( a \) and \( \varepsilon \) (and \( \|V\|_1 \)) but not on \( \text{Im}(\kappa) \).

Next, we turn to the case \( h \in \mathbb{R} \). Then (6.50)–(6.52) become
\[
G_h^{(0)}(-\kappa^2, x, y) = \frac{\sinh(\kappa x \varepsilon)}{\kappa} \psi_h^{(0)}(-\kappa^2, x_>,)
\]
(6.58)
\[
\psi_h^{(0)}(-\kappa^2, x) = \left[ e^{-\kappa x} + \zeta(h, \kappa)e^{-\kappa(b-x)} \right] \frac{1}{1 + \zeta(h, \kappa)e^{-2bx}},
\]
(6.59)
\[
m_h^{(0)}(-\kappa^2) = -\kappa + 2\kappa \frac{\zeta(h, \kappa)e^{-2\kappa b}}{1 + \zeta(h, \kappa)e^{-2\kappa b}},
\]
(6.60)
where
\[
\zeta(h, \kappa) = \frac{\kappa - h}{\kappa + h}.
\]
(6.61)

This then leads to the following result:

Theorem 6.24. (Theorem 6.13 for general \( h \in \mathbb{R} \)) Let \( b < \infty \), \( |h| < \infty \), and \( V \in L^1([0, b]) \). Then for \( \text{Re}(\kappa) > \frac{1}{2}D_1[\|V\|_1 + |h| + b^{-1} + 1] \) for a suitable universal constant \( D_1 \), (6.57) holds, where
\[
A_j = 2(-1)^j, \quad B_j = 2(-1)^{j+1} \left[ 2h + \int_0^b dx V(x) \right],
\]
(6.62)
\[
|A(\alpha) - V(\alpha)| \leq \|V\|_1 \exp(\alpha\|V\|_1) \quad \text{if } |\alpha| < b, \text{ and for any } a > 0,
\]
(6.63)
\[
\int_0^a d\alpha |A(\alpha)| \leq D_1(b, \|V\|_1, h) \exp(D_1a(\|V\|_1 + |h| + b^{-1} + 1)).
\]
(6.64)

Hence, one obtains the following estimate:

Corollary 6.25. Fix \( b < \infty \), \( V \in L^1([0, b]) \), and \( |h| < \infty \). Fix \( a < b \). Then there exist positive constants \( C \) and \( K_0 \) so that for all complex \( \kappa \) with \( \text{Re}(\kappa) > K_0 \),
\[
|m(-\kappa^2) + \kappa + \int_0^a d\alpha A(\alpha)e^{-2\alpha \kappa}| \leq Ce^{-2\alpha \kappa}.
\]

Next we return to the relation between \( A \) and \( \rho \) and discuss a first distributional form of this relation: Our primary goal in the following is to discuss a formula which formally says that
\[
A(\alpha) = -2\int_{-\infty}^{\infty} d\rho(\lambda) \lambda^{-\frac{1}{2}} \sin(2\alpha \sqrt{\lambda}),
\]
(6.65)
where for \( \lambda \leq 0 \), we define
\[
\lambda^{-\frac{1}{2}} \sin(2\alpha \sqrt{\lambda}) = \begin{cases} 2\alpha & \text{if } \lambda = 0, \\ (-\lambda)^{-\frac{1}{2}} \sinh(2\alpha \sqrt{-\lambda}) & \text{if } \lambda < 0. \end{cases}
\]

In a certain sense which will become clear, the left-hand side of (6.65) should be \( A(\alpha) - A(-\alpha) + \delta'(\alpha) \).
To understand (6.65) at a formal level, note the basic formulas,

\[ m(-\kappa^2) = -\kappa - \int_0^\infty d\alpha A(\alpha)e^{-2\alpha\kappa}, \]  

(6.66)

\[ m(-\kappa^2) = \Re(m(i)) + \int_{-\infty}^\infty d\rho(\lambda) \left[ \frac{1}{\lambda + \kappa^2} - \frac{\lambda}{1 + \lambda^2} \right], \]  

(6.67)

and

\[ (\lambda + \kappa^2)^{-1} = 2 \int_0^\infty d\alpha \lambda^{-\frac{1}{2}} \sin(2\alpha \sqrt{\lambda})e^{-2\alpha\kappa}, \]  

(6.68)

which is an elementary integral if \( \kappa > 0 \) and \( \lambda > 0 \). Plug (6.68) into (6.67), formally interchange the order of integrations, and (6.66) should only hold if (6.65) does. However, a closer examination of this procedure reveals that the interchange of the order of integrations is not justified and indeed (6.65) is not true as a simple integral since,\n
\[ \int_0^\infty d\rho(\lambda) \sim R \to \infty \frac{2}{3\pi^2} \pi R^2, \]  

which implies that (6.65) is not absolutely convergent. We will even see later that the integral sometimes fails to be conditionally convergent.

Our primary method for understanding (6.65) is as a distributional statement, that is, it will hold when smeared in \( \alpha \) for \( \alpha \in (0, b) \). We discuss this next if \( V \in L^1([0, \infty)) \) or if \( b < \infty \). Later it will be extended to all \( V \) (i.e., all Cases 1–4) by a limiting argument. Subsequently, we will study (6.65) as a pointwise statement, where the integral is defined as an Abelian limit.

Suppose \( b < \infty \) or \( b = \infty \) and \( V \in L^1([0, b)) \). Fix \( a < b \) and \( f \in C_0^\infty((0, a)) \). Define

\[ m_a(-\kappa^2) := -\kappa - \int_0^a d\alpha A(\alpha)e^{-2\alpha\kappa}, \]  

(6.69)

for \( \Re(\kappa) \geq 0 \). Fix \( \kappa_0 \) real and let

\[ g(y, \kappa_0, a) := m_a(-(\kappa_0 + iy)^2), \]

with \( \kappa_0, a \) as real parameters and \( y \in \mathbb{R} \) a variable. As usual, define the Fourier transform by (initially for smooth functions and then by duality for tempered distributions [201, Ch. IX])

\[ \hat{F}(k) = \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} dy e^{-iky} F(y), \quad \hat{F}(k) = \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} dy e^{iky} F(y). \]  

(6.70)

Then by (6.69),

\[ \tilde{g}(k, \kappa_0, a) = -\sqrt{2\pi} \kappa_0 \delta(k) - \sqrt{2\pi} \delta'(k) - \frac{\sqrt{2\pi}}{2} e^{-k\kappa_0} A \left( \frac{k}{2} \right) \chi_{(0, 2a)}(k). \]  

(6.71)

Thus, since \( f(0) = f'(0) = 0 \), in fact, \( f \) has support away from 0 and \( a \),

\[ \int_0^a d\alpha A(\alpha) f(\alpha) = -\frac{2}{\sqrt{2\pi}} \int_0^a d\alpha \tilde{g}(2\alpha, \kappa_0, a) e^{2\alpha\kappa_0} f(\alpha) = -\frac{1}{\sqrt{2\pi}} \int_0^{2a} d\alpha \tilde{g}(\alpha, \kappa_0, a) e^{\alpha\kappa_0} f \left( \frac{\alpha}{2} \right) = -\frac{1}{\sqrt{2\pi}} \int_\mathbb{R} dy g(y, \kappa_0, a) \hat{F}(y, \kappa_0), \]  

(6.72)
where we have used the unitarity of $\hat{\gamma}$ and
\[
\hat{F}(y, \kappa_0) = \frac{1}{\sqrt{2\pi}} \int_0^{2\alpha} d\alpha \, e^{\alpha(y_0 + iy)} f \left( \frac{\alpha}{2} \right) = \frac{2}{\sqrt{2\pi}} \int_0^{\alpha} d\alpha \, e^{2\alpha(y_0 + iy)} f(\alpha).
\] (6.73)

Notice that
\[
|\hat{F}(y, \kappa_0)| \leq Ce^{2(a-\varepsilon)(1 + |y|^2)^{-1}}
\] (6.74)
since $f$ is smooth and supported in $(0, a - \varepsilon)$ for some $\varepsilon > 0$.

By Theorem 6.20 and Corollary 6.25,
\[
|m_a(-(\kappa_0 + iy)^2) - m(-(\kappa_0 + iy)^2)| \leq Ce^{-2\alpha\kappa_0}
\] (6.75)
for large $\kappa_0$, uniformly in $y$. From (6.72), (6.74), and (6.75), one concludes the following fact:

Lemma 6.26. Let $f \in C^\infty_0((0, a))$ with $0 < a < b$ and $V \in L^1([0, b])$. Then
\[
\int_0^a d\alpha \, A(\alpha) f(\alpha) = \lim_{\kappa_0 \to \infty} \left[ -\frac{1}{\pi} \int_R dy \, m(-(\kappa_0 + iy)^2) \int_0^a d\alpha \, e^{2\alpha(y_0 + iy)} f(\alpha) \right].
\] (6.76)

As a function of $y$, for $\kappa_0$ fixed, the alpha integral is $O((1 + y^2)^{-N})$ for all $N$ because $f$ is $C^\infty$. Now define
\[
\tilde{m}_R(-\kappa^2) = \left[ c_R + \int_{\lambda \leq R} \frac{d\rho(\lambda)}{\lambda + \kappa^2} \right],
\] (6.77)
where $c_R$ is chosen so that $\tilde{m}_R \to m$. Because $\int_R \frac{d\rho(\lambda)}{1 + \lambda^2} < \infty$, the convergence is uniform in $y$ for $\kappa_0$ fixed and sufficiently large. Thus in (6.76) we can replace $m$ by $m_R$ and take a limit (first $R \to \infty$ and then $\kappa_0 \to \infty$). Since $f(0^+) = 0$, the $\int dy \, c_R \, d\alpha$-integrand is zero. Moreover, we can now interchange the $dy \, d\alpha$ and $d\rho(\lambda)$ integrals. The result is that
\[
\int_0^a d\alpha \, A(\alpha) f(\alpha) = \lim_{\kappa_0 \to \infty} \lim_{R \to \infty} \int_{\lambda \leq R} d\rho(\lambda)
\times \left[ \int_0^a d\alpha \, e^{2\alpha\kappa_0} f(\alpha) \left[ -\frac{1}{\pi} \int_R \frac{dy}{(\kappa_0 + iy)^2 + \lambda} \right] e^{2\alpha\kappa_0} f(\alpha) \right].
\] (6.78)

In the case at hand, $d\rho$ is bounded below, say $\lambda \geq -K_0$. As long as we take $\kappa_0 > K_0$, the poles of $(\kappa_0 + iy)^2 + \lambda$ occur in the upper half-plane
\[
y_\pm = i\kappa_0 \pm \sqrt{\lambda}.
\]
Closing the contour in the upper plane, we find that if $\lambda \geq -K_0$,
\[
-\frac{1}{\pi} \int_R \frac{d\rho e^{2\alpha i y}}{(\kappa_0 + iy)^2 + \lambda} = -2e^{-2\alpha\kappa_0} \frac{\sin(2\alpha\sqrt{\lambda})}{\sqrt{\lambda}}.
\]
Thus (6.78) becomes
\[
\int_0^a d\alpha \, A(\alpha) f(\alpha) = -2 \lim_{\kappa_0 \to \infty} \lim_{R \to \infty} \int_{\lambda \leq R} d\rho(\lambda) \left[ \int_0^a d\alpha \, f(\alpha) \frac{\sin(2\alpha\sqrt{\lambda})}{\sqrt{\lambda}} \right].
\]
$\kappa_0$ has dropped out and the $\alpha$ integral is bounded by $C(1 + \lambda^2)^{-1}$, so one can take the limit as $R \to \infty$ since $\int_R \frac{d\rho(\lambda)}{1 + \lambda^2} < \infty$. One is therefore led to the following result.
Theorem 6.27. Let \( f \in C^\infty_0((0, a)) \) with \( a < b \) and either \( b < \infty \) or \( V \in L^1((0, \infty)) \) with \( b = \infty \). Then
\[
\int_0^a d\alpha A(\alpha)f(\alpha) = -2\int_{\mathbb{R}} d\rho(\lambda) \left[ \int_0^a d\alpha f(\alpha) \frac{\sin(2\alpha \sqrt{\lambda})}{\sqrt{\lambda}} \right].
\] (6.79)

One can strengthen this in two ways. First, one wants to allow \( a > b \) if \( b < \infty \). As long as \( A \) is interpreted as a distribution with \( \delta \) and \( \delta' \) functions at \( \alpha = nb \), this is easy. One also wants to allow \( f \) to have a nonzero derivative at \( \alpha = 0 \). The net result is described in the next theorem:

Theorem 6.28. Let \( f \in C^\infty_0(\mathbb{R}) \) with \( f(-\alpha) = -f(\alpha) \), \( \alpha \in \mathbb{R} \) and either \( b < \infty \) or \( V \in L^1((0, \infty)) \) with \( b = \infty \). Then
\[
-2\int_{\mathbb{R}} d\rho(\lambda) \left[ \int_{-\infty}^\infty d\alpha f(\alpha) \frac{\sin(2\alpha \sqrt{\lambda})}{\sqrt{\lambda}} \right] = \int_{-\infty}^\infty d\alpha \tilde{A}(\alpha)f(\alpha),
\] (6.80)

where \( \tilde{A} \) is the distribution
\[
\tilde{A}(\alpha) = \chi_{(0, \infty)}(\alpha)A(\alpha) - \chi_{(-\infty, 0)}(\alpha)A(-\alpha) + \delta'(\alpha)
\] (6.81a)

if \( b = \infty \) and
\[
\tilde{A}(\alpha) = \chi_{(0, \infty)}(\alpha)A(\alpha) - \chi_{(-\infty, 0)}(\alpha)A(-\alpha) + \delta'(\alpha)
\]
\[
+ \sum_{j=1}^\infty B_j[\delta(\alpha - 2bj) - \delta(\alpha + 2bj)]
\]
\[
+ \sum_{j=1}^\infty \frac{1}{2} A_j[\delta'(\alpha - 2bj) + \delta'(\alpha + 2bj)]
\] (6.81b)

if \( b < \infty \), where \( A_j, B_j \) are \( h \) dependent and given in Theorems 6.22 and 6.24.

Next we change the subject temporarily and turn to bounds on \( \int_0^{\pm R} d\rho(\lambda) \) which are of independent interest: As we will see, (6.36) implies asymptotic results on \( \int_{-R}^R d\rho(\lambda) \), and (6.65) will show that \( \int_{-\infty}^0 e^{ib\sqrt{\lambda}} d\rho(\lambda) < \infty \) for all \( b > 0 \) and more. It follows from (6.67) that
\[
\text{Im}(m(ia)) = a \int_{\mathbb{R}} \frac{d\rho(\lambda)}{\lambda^2 + a^2}, \quad a > 0.
\]

Thus, Everitt’s result (6.36) implies that
\[
\lim_{a \to \infty} a^\frac{1}{2} \int_{\mathbb{R}} \frac{d\rho(\lambda)}{\lambda^2 + a^2} = 2^{-\frac{1}{2}}.
\]

Standard Tauberian arguments (see, e.g., in [225, Sect. III.10], which in this case shows that on even functions \( R^\frac{1}{2} d\rho(\lambda/R) \to (2\pi)^{-1} |\lambda|^\frac{1}{2} d\lambda \)) then imply the following result:

Theorem 6.29.
\[
\lim_{R \to \infty} R^{-\frac{1}{2}} \int_{-R}^R d\rho(\lambda) = \frac{2}{3\pi}.
\] (6.82)
Remark 6.30. (i) This holds in all cases (1–4) we consider here, including some with \( \text{supp}(d\rho) \) unbounded below.

(ii) Since one can show that \( \int_{-\infty}^{0} d\rho \) is bounded, one can replace \( \int_{-R}^{R} \) by \( \int_{0}^{R} \) in (6.82).

Next, we recall the following a priori bound that follows from Lemmas 6.16 and 6.17:

Lemma 6.31. Let \( d\rho \) be the spectral measure for a Schrödinger operator in Cases 1–4. Fix \( a < b \).

Then there is a constant \( C_a \) depending only on \( a \) and \( \int_{0}^{a} dy |V(y)| \) so that

\[
\int_{-\infty}^{R} d\rho(\lambda) \left( 1 + \frac{\lambda}{1 + \lambda^2} \right) \leq C_a. \tag{6.83}
\]

The goal is to bound \( \int_{-\infty}^{0} e^{2\alpha \sqrt{-\lambda}} d\rho(\lambda) \) for any \( \alpha < b \) and to find an explicit bound in terms of \( \sup_{0 \leq x \leq \alpha + 1} [-V(y)] \) when that sup is finite. As a preliminary, we need the following result from the standard limit circle theory [41, Sect. 9.4].

Lemma 6.32. Let \( b = \infty \) and let \( d\rho \) be the spectral measure for some Schrödinger operator in Cases 2–4. Let \( d\rho_{R,h} \) be the spectral measure for the problem with \( b = R < \infty, h \) and potential equal to \( V(x) \) for \( x \leq R \). Then there exists \( h(R) \) so that

\[
\int_{-\infty}^{R} d\rho_{R,h}(\lambda) \rightarrow R \rightarrow \infty d\rho, \text{ when smeared with any function } f \text{ of compact support}. \tag{6.84}
\]

This result implies that we need only obtain bounds for \( b < \infty \) (where we have already proved (6.79)).

Lemma 6.33. If \( \rho_1 \) has support in \([-E_0, \infty), E_0 > 0\), then

\[
\int_{-\infty}^{0} e^{\gamma \sqrt{-\lambda}} d\rho_1(\lambda) \leq e^{\gamma \sqrt{E_0}} (1 + E_0^2) \int_{-\infty}^{0} d\rho_1(\lambda) \left( 1 + \frac{\lambda}{1 + \lambda^2} \right). \tag{6.84}
\]

Lemmas 6.31, 6.32 and Lemma 6.33 imply the following result.

Theorem 6.34. Let \( \rho \) be the spectral measure for some Schrödinger operator in Cases 2–4. Let

\[
E(\alpha_0) := -\inf \left\{ \int_{0}^{\alpha_0+1} dx \left( |\varphi'_n(x)|^2 + V(x)|\varphi(x)|^2 \right) \Big| \varphi \in C_0^\infty((0, \alpha_0 + 1)), \right. \\
\left. \int_{0}^{\alpha_0+1} dx |\varphi(x)|^2 \leq 1 \right\}.
\]

Then for all \( \delta > 0 \) and \( \alpha_0 > 0 \),

\[
\alpha_0 \delta \int_{-\infty}^{0} e^{2(1-\delta)\alpha_0 \sqrt{-\lambda}} d\rho(\lambda) \leq \left[ C_1 (1 + \alpha_0) + C_2 (1 + E(\alpha_0)^2) e^{2(\alpha_0+1)\sqrt{E(\alpha_0)}} \right], \tag{6.85}
\]

where \( C_1, C_2 \) only depend on \( \int_{0}^{1} dx |V(x)| \). In particular,

\[
\int_{-\infty}^{0} e^{B \sqrt{-\lambda}} d\rho(\lambda) < \infty \tag{6.86}
\]

for all \( B < \infty \).
As a special case, suppose $V(x) \geq -C(x+1)^2$. Then $E(\alpha_0) \geq -C(\alpha_0 + 2)^2$ and we see that
\[
\int_{-\infty}^{0} e^{B\sqrt{-\lambda}} d\rho(\lambda) \leq D_1e^{D_2B^2}. \tag{6.87}
\]
This implies the next result.

**Theorem 6.35.** If $d\rho$ is the spectral measure for a potential which satisfies
\[
V(x) \geq -Cx^2, \quad x \geq R \tag{6.88}
\]
for some $R > 0$, $C > 0$, then for $\varepsilon > 0$ sufficiently small,
\[
\int_{-\infty}^{0} e^{-\varepsilon \lambda} d\rho(\lambda) < \infty. \tag{6.89}
\]

If in addition $V \in L^1([0, \infty))$, then the corresponding Schrödinger operator is bounded from below and hence $d\rho$ has compact support on $(-\infty, 0]$. This fact will be useful later in the scattering-theoretic context.

The estimate (6.86), in the case of non-Dirichlet boundary conditions at $x = 0_+$, appears to be due to Marchenko [171]. Since it is a fundamental ingredient in the inverse spectral problem, it generated considerable attention; see, for instance, [70], [155], [156], [157], [161], [171], [172, Sect. 2.4]. The case of a Dirichlet boundary at $x = 0_+$ was studied in detail by Levitan [157]. These authors, in addition to studying the spectral asymptotics of $\rho(\lambda)$ as $\lambda \downarrow -\infty$, were also particularly interested in the asymptotics of $\rho(\lambda)$ as $\lambda \uparrow \infty$ and established Theorem 6.29. In the latter context, we also refer to Bennewitz [17], Harris [116], and the literature cited therein. In contrast to these activities, we were not able to find estimates of the type (6.85) (which implies (6.86)) and (6.89) in the literature.

At this point one can return to the relation between $A$ and $\rho$ and discuss a second distributional form of this relation which extends Theorem 6.27 to all four cases.

**Theorem 6.36.** Let $f \in C_0^\infty((0, \infty))$ and suppose $b = \infty$. Assume $V$ satisfies (6.26) and let $d\rho$ be the associated spectral measure and $A$ the associated $A$-function. Then (6.80) and (6.81) hold.

Next we establish a third relation between $A$ and $\rho$ and turn to Abelian limits: For $f \in C_0^\infty(\mathbb{R})$, define for $\lambda \in \mathbb{R}$,
\[
Q(f)(\lambda) = \int_{-\infty}^{\infty} d\alpha \, f(\alpha) \frac{\sin(2\alpha \sqrt{\lambda})}{\sqrt{\lambda}} \tag{6.90}
\]
and then
\[
T(f) = -2 \int_{\mathbb{R}} d\rho(\lambda) Q(f)(\lambda) \tag{6.91}
\]
\[= \int_{-\infty}^{\infty} d\alpha \, \tilde{A}(\alpha) f(\alpha). \tag{6.92}
\]
Relations (6.80), (6.81) show that for $f \in C_0^\infty(\mathbb{R})$, the two expressions (6.91), (6.92) define the same $T(f)$. This was proved for odd $f$’s but both integrals vanish for even $f$’s. Now one wants to use (6.91) to extend to a large class of $f$, but needs to exercise some care not to use (6.92), except for $f \in C_0^\infty(\mathbb{R})$.

$Q(f)$ can be defined as long as $f$ satisfies
\[
|f(\alpha)| \leq C_k e^{-k|\alpha|}, \quad \alpha \in \mathbb{R} \tag{6.93}
\]
for all \( k > 0 \). In particular, a simple calculation shows that
\[
    f(\alpha) = (\pi \varepsilon)^{-\frac{3}{2}} e^{-(\alpha - \alpha_0)^2/\varepsilon} \quad \text{implies} \quad Q(f)(\lambda) = \frac{\sin(2\alpha_0 \sqrt{\lambda})}{\sqrt{\lambda}} e^{-\varepsilon \lambda}. \tag{6.94}
\]

We use \( f(\alpha, \alpha_0, \varepsilon) \) for the function \( f \) in (6.94).

For \( \lambda \geq 0 \), repeated integrations by parts show that
\[
    |Q(f)(\lambda)| \leq C(1 + \lambda^2)^{-1} \left[ \|f\|_1 + \left\| \frac{d^3 f}{d\alpha^3} \right\|_1 \right], \tag{6.95}
\]
where \( \| \cdot \|_1 \) represents the \( L^1(\mathbb{R}) \)-norm. Moreover, essentially by repeating the calculation that led to (6.94), one sees that for \( \lambda \leq 0 \),
\[
    |Q(f)(\lambda)| \leq C |\varepsilon| \| e^{\alpha^2/\varepsilon} f \|_\infty. \tag{6.96}
\]

One then concludes the following result.

**Lemma 6.37.** If \( \int_{\mathbb{R}} (1 + \lambda^2)^{-1} \, d\rho(\lambda) < \infty \) (always true!) and \( \int_{-\infty}^0 e^{-\varepsilon_0 \lambda} \, d\rho(\lambda) < \infty \) (see Theorem 6.35 and the remark following its proof), then using (6.91), \( T(\cdot) \) can be extended to functions \( f \in C^3(\mathbb{R}) \) that satisfy \( e^{\alpha^2/\varepsilon_0} f \in L^\infty(\mathbb{R}) \) for some \( \varepsilon_0 > 0 \) and \( \frac{d^2 f}{d\alpha^2} \in L^1(\mathbb{R}) \), and moreover,
\[
    |T(f)| \leq C \left[ \left\| \frac{d^3 f}{d\alpha^3} \right\|_1 + \| e^{\alpha^2/\varepsilon_0} f \|_\infty \right] := C \|f\|_{\varepsilon_0}. \tag{6.97}
\]

Next, fix \( \alpha_0 \) and \( \varepsilon_0 > 0 \) so that \( \int_{0}^{\infty} e^{-\varepsilon_0 \lambda} \, d\rho(\lambda) < \infty \). If \( \varepsilon < \varepsilon_0 \), \( f(\alpha, \alpha_0, \varepsilon) \) satisfies \( \|f\|_{\varepsilon_0} < \infty \) so we can define \( T(f) \). Fix \( g \in C^\infty_0(\mathbb{R}) \) with \( g := 1 \) on \((-2\alpha_0, 2\alpha_0)\). Then \( \|f(\cdot, \alpha_0, \varepsilon)(1 - g)|_{\varepsilon_0} \to 0 \) as \( \varepsilon \downarrow 0 \). So
\[
    \lim_{\varepsilon \downarrow 0} T(f(\cdot, \alpha_0, \varepsilon)) = \lim_{\varepsilon \downarrow 0} T(gf(\cdot, \alpha_0, \varepsilon)).
\]

For \( gf \), we can use the expression (6.92). \( f \) is approximately \( \delta(\alpha - \alpha_0) \) so standard estimates show if \( \alpha_0 \) is a point of Lebesgue continuity of \( \tilde{A}(\alpha) \), then
\[
    \int_{-\infty}^{\infty} d\alpha f(\alpha, \alpha_0, \varepsilon) g(\alpha) \tilde{A}(\alpha) \to \tilde{A}(\alpha_0).
\]

Since \( A - q \) is continuous, points of Lebesgue continuity of \( A \) exactly are points of Lebesgue continuity of \( V \). Thus, one obtains the following theorem.

**Theorem 6.38.** Suppose either \( b < \infty \) and \( V \in L^1([0, b]) \) or \( b = \infty \), and then either \( V \in L^1((0, \infty)) \) or \( V \in L^1([0, a]) \) for all \( a > 0 \) and
\[
    V(x) \geq -Ca^2, \quad x \geq R
\]
for some \( R > 0, C > 0 \). Let \( \alpha_0 \in (0, b) \) and be a point of Lebesgue continuity of \( V \). Then
\[
    A(\alpha_0) = -2 \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}} d\rho(\lambda) e^{-\varepsilon \lambda} \frac{\sin(2\alpha_0 \sqrt{\lambda})}{\sqrt{\lambda}}. \tag{6.98}
\]

Finally, we specialize (6.98) to the scattering-theoretic setting. Assuming \( V \in L^1((0, \infty); (1 + x) \, dx) \), the corresponding Jost solution \( f(z, x) \) is defined by
\[
    f(z, x) = e^{i\sqrt{z}x} - \int_{-\infty}^{\infty} dx' \frac{\sin(\sqrt{z}(x - x'))}{\sqrt{z}} V(x')f(z, x'), \quad \text{Im}(\sqrt{z}) \geq 0, \tag{6.99}
\]
and the corresponding Jost function, \( F(\sqrt{z}) \), and scattering matrix, \( S(\lambda) \), \( \lambda \geq 0 \), then read

\[
F(\sqrt{z}) = f(z,0_+), \\
S(\lambda) = \frac{F(\sqrt{\lambda})}{F(\sqrt{\lambda})}, \quad \lambda \geq 0.
\]

(6.100) (6.101)

The spectrum of the Schrödinger operator \( H \) in \( L^2([0,\infty)) \) associated with the differential expression \( -\frac{d^2}{dx^2} + V(x) \) and a Dirichlet boundary condition at \( x = 0_+ \) (cf. (6.128) for precise details) is simple and of the type

\[
\sigma(H) = \{-\kappa_j^2 < 0\}_{j \in J} \cup [0, \infty).
\]

Here \( J \) is a finite (possibly empty) index set, \( \kappa_j > 0 \), \( j \in J \), and the essential spectrum is purely absolutely continuous. The corresponding spectral measure explicitly reads

\[
d\rho(\lambda) = \begin{cases} 
\pi^{-1}|F(\sqrt{\lambda})|^{-2}\sqrt{\lambda}d\lambda, & \lambda \geq 0, \\
\sum_{j \in J} c_j \delta(\lambda + \kappa_j^2) d\lambda, & \lambda < 0,
\end{cases}
\]

(6.102)

where

\[
c_j = \|\varphi(-\kappa_j^2, \cdot)\|^2_2, \quad j \in J
\]

(6.103)

are the norming constants associated with the eigenvalues \( \lambda_j = -\kappa_j^2 < 0 \). Here the regular solution \( \varphi(z, x) \) of \(-\psi''(z, x) + [V(x) - z] \psi(z, x) = 0 \) (defined by \( \varphi(z, 0_+) = 0 \), \( \varphi'(z, 0_+) = 1 \)) and \( f(z, x) \) in (6.99) are linearly dependent precisely for \( z = -\kappa_j^2 \), \( j \in J \).

Since

\[
|F(\sqrt{\lambda})| = \prod_{j \in J} \left(1 + \frac{\kappa_j^2}{\lambda}\right) \exp \left( -\frac{1}{\pi} P \int_0^\infty \frac{d\lambda'}{\lambda - \lambda'} \right), \quad \lambda \geq 0,
\]

where \( P \int_0^\infty \) denotes the principal value symbol and \( \delta(\lambda) \) the corresponding scattering phase shift, that is, \( S(\lambda) = \exp(2i\delta(\lambda)) \), \( \delta(\lambda) \to 0 \), the scattering data

\[
\{-\kappa_j^2, c_j\}_{j \in J} \cup \{S(\lambda)\}_{\lambda \geq 0}
\]

uniquely determine the spectral measure (6.102) and hence \( A(\alpha) \). Inserting (6.102) into (6.98) then yields the following expression for \( A(\alpha) \) in terms of scattering data.

**Theorem 6.39.** Suppose that \( V \in L^1([0,\infty); (1 + x)dx) \). Then

\[
A(\alpha) = -2 \sum_{j \in J} c_j \kappa_j^{-1} \sinh(2\alpha \kappa_j)
\]

\[
- 2\pi^{-1} \lim_{\varepsilon \downarrow 0} \int_0^\infty d\lambda \, e^{-\varepsilon^2 |F(\sqrt{\lambda})|^{-2} \sin(2\alpha \sqrt{\lambda})}
\]

(6.104)

at points \( \alpha \geq 0 \) of Lebesgue continuity of \( V \).

**Remark 6.40.** In great generality \( |F(k)| \to 1 \) as \( k \to \infty \), so one cannot take the limit in \( \varepsilon \) inside the integral in (6.104). In general, though, one can can replace \( |F(\sqrt{\lambda})|^{-2} \) by \( (|F(\sqrt{\lambda})|^{-2} - 1) \equiv X(\lambda) \) and ask if one can take a limit there. As long as \( V \) is \( C^2((0,\infty)) \) with \( V'' \in L^1([0,\infty)) \), it is not hard to see that as \( \lambda \to \infty \)

\[
X(\lambda) = -\frac{V(0)}{2\lambda} + O(\lambda^{-2}).
\]
Thus, if \( V(0) = 0 \), then

\[
A(\alpha) = -2 \sum_{j \in J} c_j k_j^{-1} \sinh(2\alpha k_j) - 2\pi^{-1} \int_0^\infty d\lambda (|F(\sqrt{\lambda})|^{-2} - 1) \sin(2\alpha \sqrt{\lambda}).
\]  

(6.105)

The integral in (6.105) is only conditionally convergent if \( V(0) \neq 0 \).

We note that in the present case, where \( V \in L^1([0, \infty); (1 + x) \ dx) \), the representation (6.40) of the \( m \)-function in terms of the \( A(\alpha) \)-amplitude was considered in a paper by Ramm [195] (see also [196, p. 288–291]).

We add a few more remarks in the scattering-theoretic setting. Assuming \( V \in L^1([0, \infty); (1 + x) \ dx) \), one sees that

\[
|F(k)| = 1 + o(k^{-1})
\]  

(6.106)

(cf. [31, eq. II.4.13] and apply the Riemann-Lebesgue lemma; actually, one only needs \( V \in L^1([0, \infty)) \) for the asymptotic results on \( F(k) \) as \( k \uparrow \infty \) but we will ignore this refinement in the following). A comparison of (6.106) and (6.104) then clearly demonstrates the necessity of an Abelian limit in (6.104). Even replacing \( d\rho \) in (6.98) by \( d\sigma = d\rho - d\rho(0) \), that is, effectively replacing \( |F(\sqrt{\lambda})|^{-2} \) by \([|F(\sqrt{\lambda})|^{-2} - 1]\) in (6.104), still does not necessarily produce an absolutely convergent integral in (6.104).

The latter situation changes upon increasing the smoothness properties of \( V \) since, for example, assuming \( V \in L^1([0, \infty); (1 + x) \ dx) \), \( V' \in L^1([0, \infty)) \), yields

\[
|F(k)|^{-2} - 1 = O(k^{-2})
\]  

as detailed high-energy considerations (cf. [87]) reveal. Indeed, if \( V'' \in L^1([0, \infty)) \), then the integral one gets is absolutely convergent if and only if \( V(0) = 0 \).

As a final issue related to the representation (6.65), we discuss the issue of bounds on \( A \) when \( |V(x)| \leq Cx^2 \). One has two general bounds on \( A \): the estimate of [228] (see (6.39)),

\[
|A(\alpha) - V(\alpha)| \leq \left[ \int_0^\alpha dy |V(y)| \right]^2 \exp \left[ \alpha \int_0^\alpha dy |V(y)| \right],
\]  

(6.107)

and the estimate in Theorem 6.42,

\[
|A(\alpha)| \leq \frac{\gamma(\alpha)}{\alpha} I_1(2\alpha \gamma(\alpha)),
\]  

(6.108)

where \( |\gamma(\alpha)| = \sup_{0 \leq x \leq \alpha} |V(x)|^{1/2} \) and \( I_1(\cdot) \) is the modified Bessel function of order one (cf., e.g., [1], Ch. 9). Since ([1], p. 375)

\[
0 \leq I_1(x) \leq e^x, \quad x \geq 0,
\]  

(6.109)

one concludes that

\[
|A(\alpha)| \leq \sqrt{C} e^{2\sqrt{C} \alpha^2}
\]  

(6.110)

if \( |V(x)| \leq Cx^2 \).

We continue with a discussion of the case of constant \( V \):
Example 6.41. If \( b = \infty \) and \( V(x) = V_0, \ x \geq 0 \), then if \( V_0 > 0 \),
\[
A(\alpha) = \frac{V_0^{1/2}}{\alpha} J_1(2\alpha V_0^{1/2}), \tag{6.111}
\]
where \( J_1(\cdot) \) is the Bessel function of order one (cf., e.g., [1], Ch. 9); and if \( V_0 < 0 \),
\[
A(\alpha) = \frac{(-V_0)^{1/2}}{\alpha} I_1(2\alpha (-V_0)^{1/2}), \tag{6.112}
\]
with \( I_1(\cdot) \) the corresponding modified Bessel function.

This example is important because of the following monotonicity property:

**Theorem 6.42.** Let \( |V_1(x)| \leq -V_2(x) \) on \([0,a]\) with \( a \leq \min(b_1,b_2) \). Then,
\[
|A_1(\alpha)| \leq A_2(\alpha) \text{ on } [0,a].
\]

In particular, for any \( V \) satisfying \( \sup_{0 \leq x \leq a} |V(x)| < \infty \), one obtains
\[
|A(\alpha)| \leq \frac{\gamma(\alpha)}{\alpha} I_1(2\alpha\gamma(\alpha)), \tag{6.113}
\]
where
\[
\gamma(\alpha) = \sup_{0 \leq x \leq a} (|V(x)|^{1/2}). \tag{6.114}
\]

In particular, (6.109) implies
\[
|A(\alpha)| \leq \alpha^{-1}\gamma(\alpha)e^{2\alpha\gamma(\alpha)}, \tag{6.115}
\]
and if \( V \) is bounded,
\[
|A(\alpha)| \leq \alpha^{-1}\|V\|_{\infty}^{1/2} \exp(2\alpha\|V\|_{\infty}^{1/2}). \tag{6.116}
\]

For \( \alpha \) small, (6.115) is a poor estimate and one should use (6.107) which implies that \( |A(\alpha)| \leq |V|_{\infty} + \alpha^2\|V\|_{\infty}^2 e^{\alpha^2\|V\|_{\infty}} \).

This lets one prove the following result:

**Theorem 6.43.** Let \( h = \infty \) and \( V \in L^\infty([0,\infty)) \). Suppose \( \kappa^2 > \|V\|_{\infty} \). Then
\[
m(-\kappa^2) = -\kappa - \int_0^\infty d\alpha A(\alpha)e^{-2\alpha} \tag{6.117}
\]
(with an absolutely convergent integral and no error term).

**Remark 6.44.** We recall (cf. (6.40)) that the representation (6.117) also holds with \( A \in L^1([0,a]) \) for all \( a > 0 \) and as an absolutely convergent integral for \( \Re(\kappa) > \|V\|_{1/2} \) if \( V \in L^1([0,\infty)) \). This fact will be used below.

The case of Bargmann potentials has been worked out in [97, Sect. 1] and explicit formulas for the \( A \)-function have been obtained.

We end this survey of [97] and [228] by recalling the major thrust of [228] – the connection between \( A \) and the inverse spectral theory. Namely, there is an \( A(\alpha,x) \) function associated to \( m(z,x) \) by
\[
m(-\kappa^2,x) = -\kappa - \int_0^a d\alpha A(\alpha,x)e^{-2\alpha\kappa} \tag{6.118}
\]
for \( a < b - x \). This, of course, follows from Theorem 6.7 by translating the origin. The point is that \( A \) satisfies the simple differential equation in distributional sense
\[
\frac{\partial A}{\partial x}(\alpha,x) = \frac{\partial A}{\partial \alpha}(\alpha,x) + \int_0^a d\beta A(\alpha - \beta, x)A(\beta, x). \tag{6.119}
\]
This is proved in [228] for $V \in L^1([0, a])$ (and some other $V$’s) and so holds in the generality of [97] since Theorem 6.11 implies $A(\alpha, x)$ for $\alpha + x \leq a$ is only a function of $V(y)$ for $y \in [0, a]$.

Moreover, by (6.39), one has

$$\lim_{\alpha \to 0} |A(\alpha, x) - V(\alpha + x)| = 0 \quad (6.120)$$

uniformly in $x$ on compact subsets of the real line, so by the uniqueness theorem for solutions of (6.119) (proved in [228]), $A$ on $[0, a]$ determines $V$ on $[0, a]$.

In the limit circle case, there is an additional issue to discuss. Namely, that $m(z, x = 0)$ determines the boundary condition at $\infty$. This is because, as we just discussed, $m$ determines $A$ which determines $V$ on $[0, \infty)$. $m(z, 0_+)$ and $V$ determine $m(z, x)$ by the Riccati equation. Once we know $m$, we can recover $u(i, x) = \exp \left( \int_0^x m(i, y) \, dy \right)$, and so the particular solution that defined the boundary condition at $\infty$.

Thus, the inverse spectral theory aspects of the framework easily extend to the general case of potentials considered in [97].

To turn this into an inverse spectral approach alternative to and fully equivalent to that of Gel’fand and Levitan, one needs to settle necessary and sufficient conditions for solvability of the differential equation (6.119) in terms of an initial condition $A(\alpha, 0_+) = A_0(\alpha)$, that is, in terms of properties of $A_0$. This final step was accomplished by Remling [205] and we briefly describe its major elements next.

Remling’s first result is of local nature and determines a necessary and sufficient condition on $A$ to be the $A$-function of a potential $V$. Assuming $V \in L^1([0, b])$ for all $b > 0$, he introduces the set

$$A_b = \{ A \in L^1([0, b]) \mid A \text{ real-valued, } I + K_A > 0 \}, \quad (6.121)$$

where

$$(K_A f)(\alpha) = \int_0^b d\beta K(\alpha, \beta) f(\beta), \quad \alpha \in [0, b], \quad f \in L^2([0, b]),$$

$$K(\alpha, \beta) = [\phi(\alpha - \beta) - \phi(\alpha + \beta)]/2, \quad \phi(\alpha) = \int_0^{\alpha/2} d\gamma A(\gamma), \quad \alpha, \beta \in [0, b].$$

Based on his reformulation of the Gel’fand–Levitan approach in terms of de Branges spaces in [204], Remling obtained the following characterization of $A$-functions:

**Theorem 6.45.** $A_b$ is precisely the set of $A$-functions in

$$m(-\kappa^2) = -\kappa - \int_0^a d\alpha A(\alpha) e^{-2\alpha \kappa} + O(e^{-2\alpha \kappa}) \text{ for all } a < b.$$  

Equivalently, given $A_0 \in L^1([0, b])$, there exists a potential $V \in L^1([0, b])$ such that $A_0$ is the $A$-function of $V$ if and only if $A_0 \in A_b$.

(We recall that all potentials $V$ in this survey are assumed to be real-valued.)

As a second result, Remling also proved in [205] that the positivity condition in (6.121) is necessary and sufficient to solve (6.119) on $\Delta_b = \{ (\alpha, x) \in \mathbb{R}^2 \mid \alpha \in [0, b - x], x \in [0, b] \}$ given an initial condition $A(\cdot, 0_+) = A_0 \in L^1([0, b])$. The potential $V$ can then be read off from

$$V(x) = A(0_+, x) \text{ for } x \in [0, b]. \quad (6.122)$$
Necessity of this positivity condition had been established independently by Keel and Simon (unpublished). To make this precise, it pays to slightly rewrite (6.119) as follows: Let

$$B(\alpha, x) = A(\alpha - x, x) - A_0(\alpha), \quad (\alpha, x) \in \tilde{\Delta}_b,$$

(6.123)

where

$$\tilde{\Delta}_b = \{(\alpha, x) \in \mathbb{R}^2 | 0 \leq x \leq \alpha \leq b\}.$$

Then (6.119) together with the initial condition

$$A(\cdot, 0_+) = A_0 \in L^1([0, b]),$$

becomes

$$B(\alpha, x) = \int_0^x dy \int_0^{\alpha-y} d\beta [B(y + \beta, y) + A_0(y + \beta)]$$

$$B(\alpha, 0_+) = 0, \quad (\alpha, x) \in \tilde{\Delta}_b.$$

(6.124)

If \(A\) is actually the \(A\)-function of a potential, then \(B \in C(\tilde{\Delta}_b)\) by [228, Theorem 2.1]. Remling [205] then proves the following result:

**Theorem 6.46.** Suppose \(A_0 \in L^1([0, b])\). Then (6.124) has a solution \(B \in C(\tilde{\Delta}_b)\) if and only if \(A_0 \in \mathcal{A}_b\).

This brings Simon’s inverse approach to full circle and one can envision the following two scenarios. First, Simon’s inverse \(A\)-function approach, as complemented by Remling [205]:

$$A_0 \in \mathcal{A}_b \xrightarrow{\text{by (6.124)}} B(\alpha, x), \quad (\alpha, x) \in \tilde{\Delta}_b \xrightarrow{\text{by (6.123)}} A(\alpha, x), \quad (\alpha, x) \in \Delta_b$$

$$\xrightarrow{\text{by (6.122)}} V = A(0_+, \cdot) \in L^1([0, b]).$$

(6.125)

Second, denote by \(\mathcal{R}\) the set of spectral functions \(\rho\) associated with self-adjoint half-line Schrödinger operators with a Dirichlet boundary condition at \(x = 0\) and a self-adjoint boundary condition (6.29) at infinity (if any, i.e., if (6.24) is in the limit circle case at \(\infty\)). For characterizations of \(\mathcal{R}\) we refer, for instance, to [161], [159, Ch. 2], [172, Ch. 2]. Then combining (6.125) with (6.80) yields Simon’s inverse spectral approach as an alternative to that by Gel’fand and Levitan:

$$\rho \in \mathcal{R} \xrightarrow{\text{by (6.80)}} A_0 \in \mathcal{A}_b \text{ for all } b > 0$$

$$\xrightarrow{\text{by (6.124)}} B(\alpha, x), \quad (\alpha, x) \in \tilde{\Delta}_b \text{ for all } b > 0$$

$$\xrightarrow{\text{by (6.123)}} A(\alpha, x), \quad (\alpha, x) \in \Delta_b \text{ for all } b > 0$$

$$\xrightarrow{\text{by (6.122)}} V = A(0_+, \cdot) \in L^1([0, b]) \text{ for all } b > 0.$$

(6.126)

**More recent references:** Local solvability and a necessary condition for global solvability of the \(A\)-equation (6.119) were recently discussed by Zhang [252], [253]. Connections between the \(A\)-amplitude and the scattering transform for Schrödinger operators on the real line have been discussed by Hitrik [117].

⋆ ⋆ ⋆

Next we briefly quote the main results by Ramm and Simon [200]. The primary goal in this paper was to study \(A\) as an interesting object in its own right and, in
particular, using ideas implicit in Ramm [195] to obtain detailed information on the behavior of $A(\alpha)$ as $\alpha \to \infty$ when $V$ decays sufficiently fast as $x \to \infty$. Indeed, for potentials decaying rapidly enough, Ramm [195] stated the representation (6.117) (actually, (6.40)), but no proof was given (nor was there any connection of the function $A$ to the inverse problem for $V$). In [195] the inverse problem of finding the potential from the knowledge of the $m$-function has been solved for short-range potentials. A more detailed discussion of the result in [195] can be found in [198], [199].

Throughout [200] it is assumed that

$$
\int_0^\infty (1 + x) \, dx \, |V(x)| < \infty \tag{6.127}
$$

and the Dirichlet-type Schrödinger operator $H$ in $L^2([0, \infty))$ defined by

$$
Hf = -f'' + Vf, \quad f \in \text{dom}(H) = \{ u \in L^2([0, \infty)) \mid u, u' \in AC_{\text{loc}}([0, b]) \}
$$

for all $b > 0$; $u(0+) = 0$; $(-g'' + Vg) \in L^2([0, \infty))$ (6.128)

is considered.

More generally, for $n \in \mathbb{N}_0$, $B \leq 0$ and $\ell \geq 0$, the space $C_B^{\ell}$ of all functions $q$ with $n - 1$ classical derivatives and $q^{(n)} \in L^1([0, \infty))$ so that

$$
\int_0^\infty (1 + x)^\ell \, e^{-Bx} \, dx \, |q(j)(x)| < \infty
$$

for $j = 0, 1, \ldots, n$. Thus, (6.127) says $V \in C_0^{B=0, \ell=1}$.

Under condition (6.127), general principles (see, e.g., [172, Ch. 3]) imply that for all $\kappa \in \mathbb{C}$ with $\text{Re}(\kappa) \geq 0$, there is a unique solution $F(\kappa, x)$ of $-f'' + Vf = -\kappa^2 f$ normalized so that $F(\kappa, x) = e^{-\kappa x}(1 + o(1))$ as $x \to \infty$. We set $F(\kappa) := F(\kappa, 0+)$. Except for the change of variables $\kappa = -ik$, $F(\kappa, x)$ and $F(\kappa)$ are the standard Jost solution and Jost function. Both $F(\kappa, x)$ and $F(\kappa)$ are analytic with respect to $\kappa$ in $\{ \kappa \in \mathbb{C} \mid \text{Re}(\kappa) > 0 \}$. If $V \in C_B^{\ell}$ for any $n, \ell$ and $B < 0$, then $F(\kappa, x)$ and $F(\kappa)$ have analytic continuations into the region $\text{Re}(\kappa) > B/2$.

The following is easy to see and well known (cf. [172, Ch. 3]):

1. The zeros of $F$ in $\{ \kappa \in \mathbb{C} \mid \text{Re}(\kappa) > 0 \}$ occur precisely at those points $\kappa_j$ with $-\kappa_j^2$ an eigenvalue of the operator $H$ and each such zero is simple.
2. $F$ has no zeros in $\{ \kappa \in \mathbb{C} \mid \text{Re}(\kappa) = 0, \kappa \neq 0 \}$.
3. If $F(0) = 0$ and $V \in C_{n=0, \ell=2}$, then $F$ is $C^1$ and $F'(0) \neq 0$. If $F(0) = 0$, we say that $H$ has a zero energy resonance.

If $F$ can be analytically continued to $\{ \kappa \in \mathbb{C} \mid \text{Re}(\kappa) > B/2 \}$ for $B < 0$, then zeros of $F$ in $\{ \kappa \in \mathbb{C} \mid \text{Re}(\kappa) < 0 \}$ are called resonances of $H$. They occur in complex conjugate pairs (since $F$ is real on the real axis). If $F'(\kappa_0) \neq 0$ at a zero $\kappa_0$, we say that $\kappa_0$ is a simple resonance. Resonances need not be simple if $\text{Re}(\kappa_0) < 0$ although they are generically simple.

The result stated in [195] can be phrased as follows:

**Theorem 6.47.** Suppose that $V$ satisfies (6.127) (i.e., it lies in $C_{n=0, \ell=1}^{B=0}$) and that $H$ does not have a zero energy resonance. Let $\{-\kappa_j^2\}_{j=1}^J$ be the negative eigenvalues of $H$ with $\kappa_j > 0$. Then

$$
A(\alpha) = \sum_{j=1}^J B_j e^{2\kappa_j^\alpha} + g(\alpha), \tag{6.129}
$$
where $g \in L^1([0, \infty))$. In particular, if $H$ has no eigenvalues and no zero energy resonance (e.g., if $V \geq 0$), then $A \in L^1([0, \infty))$.

**Remark 6.48.** (i) The result stated in [195] assumes implicitly that there is no zero energy resonance. Details can be found in [198].

(ii) If $A \in L^1([0, \infty))$, then the representation (6.117) (resp., (6.40)) can be analytically continued to the entire region $\text{Re}(\kappa) \geq 0$.

(iii) If $u_j$ is the eigenfunction of $H$ corresponding to the eigenvalue $-\kappa_j^2$, normalized by $\|u_j\|_2 = 1$, then

$$B_j = -\frac{|u_j'(0_+)|^2}{\kappa_j}.$$ 

This follows from (6.104) and the fact that

$$d\rho(\lambda) \mid (-\infty, 0) = \sum_{j=1}^{J} |u_j'(0_+)|^2 \delta(\lambda + \kappa_j^2) \, d\lambda.$$

To handle zero energy resonances of $H$, one needs an extra two powers of decay (just as (6.28) says more or less that $|V(x)|$ is bounded by $O(x^{-2-e})$, the condition in the next theorem says that $|V(x)|$ is more or less $O(x^{-4-e})$):

**Theorem 6.49.** Let $V \in C_{n=0, \ell=3}^{B}$. Suppose that $H$ has a zero energy resonance and negative eigenvalues at $\{ -\kappa_j^2 \}_{j=1}^{J}$ with $\kappa_j > 0$. Then

$$A(\alpha) = B_0 + \sum_{j=1}^{J} B_j e^{2\alpha \kappa_j} + g(\alpha), \quad (6.130)$$

where $g \in L^1([0, \infty))$.

These results are special cases of the following theorem:

**Theorem 6.50.** Let $V \in C_{n=0, \ell=3}^{B}$ where $\ell \geq 1$ and, if $H$ has a zero energy resonance, then $\ell \geq 3$. Then (6.129) (resp., (6.130)) if there is a zero energy resonance) holds, where $g \in C_{n=0, \ell=1}^{B}$ (resp., $C_{n=0, \ell=3}^{B}$).

Finally, for $B < 0$, the following result was proved in [200]:

**Theorem 6.51.** Let $V \in C_{n=0, \ell=0}^{B}$ with $B < 0$. Let $\hat{B} \in (B, 0)$ and let $\{-\kappa_j^2\}_{j=1}^{J}$ with $\kappa_j > 0$ be the negative eigenvalues of $H$, $\{\lambda_j\}_{j=1}^{M}$ with $\lambda_j \leq 0$ the real resonances (a.k.a. anti-bound states) of $H$, and $\{\mu_j \pm i\nu_j\}_{j=1}^{N}$ the complex resonances of $H$ with $\hat{B} \leq \mu_j < 0$ and $\nu_j > 0$. Suppose each resonance is simple. Then for suitable $\{B_j\}_{j=1}^{J}$, $\{C_j\}_{j=1}^{M}$, $\{D_j\}_{j=1}^{N}$, $\{\theta_j\}_{j=1}^{N}$, one obtains

$$A(\alpha) = \sum_{j=1}^{J} B_j e^{2\alpha \kappa_j} + \sum_{j=1}^{M} C_j e^{2\alpha \lambda_j} + \sum_{j=1}^{N} D_j e^{2\mu_j \alpha} \cos(2\nu_j \alpha + \theta_j) + \tilde{g}(\alpha),$$

where $\tilde{g} \in C_{n=0, \ell=0}^{B}$. In particular, if $H$ has no negative eigenvalues, the rate of decay of $A$ is determined by the resonance with the least negative value of $\lambda$ or $\mu$. 

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We conclude this section with a brief look at the principal results in [98].

Let \( H_j = -\frac{d^2}{dx^2} + V_j \), \( V_j \in L^1([0,b]) \) for all \( b > 0 \), \( V_j \) real-valued, \( j = 1, 2 \), be two self-adjoint operators in \( L^2([0, \infty)) \) with a Dirichlet boundary condition at \( x = 0_+ \). Let \( m_j(z), z \in \mathbb{C} \setminus \mathbb{R} \) be the Weyl-Titchmarsh \( m \)-functions associated with \( H_j, j = 1, 2 \). The main purpose of [98] was to provide a short proof of the following local uniqueness theorem in the spectral theory of one-dimensional Schrödinger operators, originally obtained by Simon [228], but under slightly more general assumptions than in [228].

We summarize the principal results of [98] as follows:

**Theorem 6.52.** (i) Let \( a > 0, 0 < \varepsilon < \pi/2 \) and suppose that
\[
|m_1(z) - m_2(z)| \sim O(e^{-2\text{Im}(z^{1/2})a})
\]
along the ray \( \arg(z) = \pi - \varepsilon \). Then
\[
V_1(x) = V_2(x) \quad \text{for a.e. } x \in [0, a].
\]

(ii) Conversely, let \( \arg(z) \in (\varepsilon, \pi - \varepsilon) \) for some \( 0 < \varepsilon < \pi \) and suppose \( a > 0 \). If
\[
V_1(x) = V_2(x) \quad \text{for a.e. } x \in [0, a],
\]
then
\[
|m_1(z) - m_2(z)| \sim O(e^{-2\text{Im}(z^{1/2})a}). \tag{6.131}
\]

(iii) In addition, suppose that \( H_j, j = 1, 2 \), are bounded from below. Then (6.131) extends to all \( \arg(z) \in (\varepsilon, \pi] \).

**Corollary 6.53.** Let \( 0 < \varepsilon < \pi/2 \) and suppose that for all \( a > 0 \),
\[
|m_1(z) - m_2(z)| \sim O(e^{-2\text{Im}(z^{1/2})a})
\]
along the ray \( \arg(z) = \pi - \varepsilon \). Then
\[
V_1(x) = V_2(x) \quad \text{for a.e. } x \in [0, \infty).
\]

Theorem 6.52 and Corollary 6.53 follow by combining some of the Riccati equation methods in [97] with properties of transformation operators (cf. [172, Sect. 3.1]) and a uniqueness theorem for finite Laplace transforms [228, Lemma A.2.1].

In particular, Corollary 6.53 represents a considerable strengthening of the original Borg–Marchenko uniqueness result [25], [170], [171]:

**Theorem 6.54.** Suppose
\[
m_1(z) = m_2(z), \quad z \in \mathbb{C} \setminus \mathbb{R},
\]
then
\[
V_1(x) = V_2(x) \quad \text{for a.e. } x \in [0, \infty).
\]

**Remark 6.55.** (i) Marchenko [170] first published Theorem 6.54 in 1950. His extensive treatise on spectral theory of one-dimensional Schrödinger operators [171], repeating the proof of his uniqueness theorem, then appeared in 1952, which also marked the appearance of Borg’s proof of the uniqueness theorem [25] (apparently, based on his lecture at the 11th Scandinavian Congress of Mathematicians held at Trondheim, Norway, in 1949).

We emphasize that Borg and Marchenko also treat the general case of non-Dirichlet boundary conditions at \( x = 0_+ \) (see also item (vi) below). Moreover, Marchenko
simultaneously discussed the half-line and finite interval case (cf. item \(\text{(viii)}\) below).

\(\text{(ii)}\) As pointed out by Levitan [159] in his Notes to Chapter 2, Borg and Marchenko were actually preceded by Tikhonov [245] in 1949, who proved a special case of Theorem 6.54 in connection with the string equation (and hence under certain additional hypotheses on \(V_j\)).

\(\text{(iii)}\) Since Weyl–Titchmarsh functions \(m\) are uniquely related to the spectral measure \(d\rho\) of \(H\) by the standard Herglotz representation theorem, (6.33), Theorem 6.54 is equivalent to the following statement: Denote by \(d\rho\) the spectral measures of \(H_j\), \(j = 1, 2\). Then

\[ d\rho_1 = d\rho_2 \text{ implies } V_1 = V_2 \text{ a.e. on } [0, \infty). \]

In fact, Marchenko took the spectral measures \(d\rho_j\) as his point of departure while Borg focused on the Weyl–Titchmarsh functions \(m_j\).

\(\text{(iv)}\) The Borg–Marchenko uniqueness result, Theorem 6.54 (but not the strengthened version, Corollary 6.53), under the additional condition of short-range potentials \(V_j\) satisfying \(V_j \in L^1([0, \infty); (1 + x)\, dx)\), \(j = 1, 2\), can also be proved using Property C, a device used by Ramm [197], [198] in a variety of uniqueness results.

\(\text{(v)}\) The ray \(\arg(z) = \pi - \varepsilon, 0 < \varepsilon < \pi/2\) chosen in Theorem 6.52 \((\text{i})\) and Corollary 6.53 is of no particular importance. A limit taken along any non-self-intersecting curve \(C\) going to infinity in the sector \(\arg(z) \in ((\pi/2) + \varepsilon, \pi - \varepsilon)\) will do since we can apply the Phragmén–Lindelöf principle ([191, Part III, Sect. 6.5]) to the region enclosed by \(C\) and its complex conjugate \(\overline{C}\).

\(\text{(vi)}\) For simplicity of exposition, we only discussed the Dirichlet boundary condition

\[ u(0+) = 0 \]

in the Schrödinger operator \(H\). Everything extends to the the general boundary condition

\[ u'(0+) + hu(0+) = 0, \quad h \in \mathbb{R}, \]

and we refer to [98, Remark 2.9] for details.

\(\text{(vii)}\) Similarly, the case of a finite interval problem on \([0, b]\), \(b \in (0, \infty)\), instead of the half-line \([0, \infty)\) in Theorem 6.52 \((\text{i})\), with \(0 < a < b\), and a self-adjoint boundary condition at \(x = b_-\) of the type

\[ u'(b_-) + h_b u(b_-) = 0, \quad h_b \in \mathbb{R}, \]

can be discussed (cf. [98, Remark 2.10]).

While we have separately described a few extensions in Remarks 6.55 \((\text{v})-(\text{viii})\), it is clear that they can all be combined at once.

Without going into further details, we also mention that [98] contains the analog of the local Borg–Marchenko uniqueness result, Theorem 6.52 \((\text{i})\) for Schrödinger operators on the real line. In addition, the case of half-line Jacobi operators and half-line matrix-valued Schrödinger operators was dealt with in [98].

More recent references: An even shorter proof of Theorem 6.52 \((\text{i})\), close in spirit to Borg’s original paper [25], was found by Bennewitz [18]. Still other proofs were presented by Horváth [124] and Knudsen [139]. Various local and global uniqueness results for matrix-valued Schrödinger, Dirac-type, and Jacobi operators were considered in [84]. The analog of the local Borg–Marchenko theorem for certain Dirac-type systems was also studied by A. Sakhnovich [212]. The matrix-valued weighted Sturm–Liouville case has further been studied by Andersson [9]. He also
studied uniqueness questions for certain scalar higher-order differential operators in [10]. A local Borg–Marchenko theorem for complex-valued potentials has been proved by Brown, Peacock, and Weikard [26]. The case of semi-infinite Jacobi operators with complex-valued coefficients was studied by Weikard [250]. A (global) uniqueness result for trees in terms of the (generalized) Dirichlet-to-Neumann map was found by Brown and Weikard [27].

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