Two-dimensional dispersion of magnetostatic volume spin waves

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Abstract
Owing to the dipolar (magnetostatic) interaction, long-wavelength spin waves in in-plane magnetized films show an unusual dispersion behavior, which can be mathematically described by the model of Damon and Eshbach and refinements thereof. However, solving the two-dimensional dispersion $\omega(k)$ requires the evaluation of a set of coupled transcendental equations and one has to rely on numerics. In this work, we present a systematic perturbative analysis of the spin wave model. An expansion in the in-plane wavevector $k$ allows us to obtain explicit closed-form expressions for the dispersion relation and mode profiles in various asymptotic regimes. Moreover, we derive a very accurate semi-analytical expression for the dispersion relation of the lowest-frequency mode that is straightforward to evaluate.

Keywords: spin wave dispersion, magnetostatic interactions, spin wave profiles, semi-analytical solution

(Some figures may appear in colour only in the online journal)

1. Introduction

The dipolar interaction endows magnetostatic (long-wavelength) spin waves with very peculiar dynamics. In an in-plane magnetized ferromagnetic film, their dispersion shows a strong anisotropy originating from the magnetization vector $M$ [1–3]. Spin waves propagating through the volume of the film appear to move backwards as their group velocity is opposite to their phase velocity (backward volume magnetostatic spin-wave modes, BVMSW) [4]. For $\mathbf{k} \perp M$, surface-localized magnetostatic spin waves (MSSW) or Damon–Eshbach modes also exist, which are forward modes [5]. The surface localization of the DE modes is exponential, with a decay length inversely proportional to the perpendicular component of $k$ [4].

The DE modes have been extensively studied in insulators with microstrip antennas in the Damon–Eshbach geometry [6] and show unusual features such as nonreciprocal propagation [7]. Recently, the surface modes have also been excited in metals using optical methods [8, 9]. The high time resolution of the optical methods are ideal to study the fast magnetization dynamics and have successfully been used to disentangle the difficult interplay between optically induced fields, heating and acoustic wave generation in the metallic magnetic systems [10, 11].

Spin wave propagation in insulators have mainly been studied in magnetic garnets, such as YIG. The intrinsic spin wave damping in these materials is typically orders of magnitudes smaller than in metallic systems, making them ideal to study long living spin waves. Furthermore, since the garnets have a small optical absorption in the infrared and a large magneto-optical coupling, all-optical methods are very suitable [12].

In a film of finite thickness, BVMSWs are not restricted to the case $\mathbf{k} \parallel M$ and can exist with any (in-plane) wavevector $k$ [3, 13]. Their specific dispersion characteristics are an important ingredient in the analysis of all-optical excitations [14] and nonlinear effects [15–17]. In the case $\mathbf{k} \parallel M$, the profiles of the BVMSW modes show an interesting asymmetry in the perpendicular coordinate $z$, reminiscent of the asymmetry of the DE modes but without actual surface localization [3, 18]. The nontrivial two-dimensional dispersion of magnetostatic
modes is of essential importance in opto-magnetic [19, 20] experiments such as [1], where the two-dimensional profile of the uniform excitation (almost homogeneous in the film thickness) can be shaped and the subsequent dynamics observed with spatial and temporal resolution.

For exchange spin waves (wavelength small compared to the exchange length $t_{ex}$), as well as for spin waves propagating in an ultrathin film (thin compared to $t_{ex}$), the dispersion relations are given by fairly simple analytical expressions [21]. On micrometer length scales, however, exchange interactions are negligible and the film thickness $L$ remains as the only parameter. We take $z$ as the direction of magnetization $M$, which is in the plane of the film. In the uniform mode approximation, we assume that $\delta y(z)$ and $\delta z(z)$ are constant functions.

Figure 1. Spin-wave modes in ferromagnetic films with a thickness $L$ larger than the exchange length $t_{ex}$ have a non trivial perpendicular profile $\delta y(z), \delta z(z)$ inside the film ($0 < z < L$). We take $z$ as the film normal and $\mathbf{z}$ as the direction of magnetization $M$, which is in the plane of the film. In the uniform mode approximation, we assume that $\delta y(z)$ and $\delta z(z)$ are constant functions.

In addition to the dipolar interaction [4], we take into account the usual micromagnetic energy functionals for exchange $E_{ex} = AL \int \left( \| \partial_{r} \mathbf{m} \|^2 + \| \partial_{\theta} \mathbf{m} \|^2 \right) d^2r$, intrinsic easy-axis anisotropy $E_{ani} = -KL \int m^2 d^2r$, and Zeeman energy $E_{Hz} = -\mu_{0}M_{S}H_{z} \int m \cdot d^2r$. We define our coordinate system in figure 1. The applied field $H_{z}$ fixes the equilibrium magnetization along $\mathbf{z}$.

Linearization of the Landau–Lifshitz equation [27] without damping

$$\frac{\partial \mathbf{m}}{\partial t} = \left| \frac{\gamma}{M_{S}L} \mathbf{m} \times \frac{\delta E}{\delta \mathbf{m}} \right|$$

around the equilibrium $\mathbf{m}(r) = \hat{\mathbf{z}}$ gives, very generally [28],

$$\left( -\frac{\delta E}{\delta \mathbf{m}} \right) \frac{\partial \mathbf{m}}{\partial t} + \int \left( \frac{\delta^2 E}{\delta \mathbf{m} \delta \mathbf{m}^\top} \right) \left( \frac{\delta \mathbf{y}}{\delta \mathbf{m}} \right) \cdot \frac{\delta \mathbf{z}}{\delta \mathbf{m}} d^2r = 0,$$

where $\gamma$ is the gyromagnetic ratio, and where the functional derivatives of $E$ are to be evaluated for the equilibrium configuration $\mathbf{m}(r) = \hat{\mathbf{z}}$. For brevity, we write $\delta \mathbf{y}$ for $\delta \mathbf{m}_{r}(r)$ and $\delta \mathbf{z}$ for $\delta \mathbf{m}_{\theta}(r)$. The functions $\delta \mathbf{y}, \delta \mathbf{z}$ represent the infinitesimal deviation of magnetization $\mathbf{m}$ from its equilibrium direction. If intrinsic (Gilbert) damping is considered, the angular frequency of the spin waves is modified and acquires an imaginary part. We do not take intrinsic damping into account, as the high-quality films often used in opto-magnetic experiments [1] tend to have a very low damping coefficient.

In the uniform mode approximation [21, 29–31],

$$\omega = |\gamma| \mu_{0} \left( \frac{2A^2 - 2KL}{\mu_{0}M_{S}} + H_{z} + M_{S}N_{k} \right)$$

$$\times \left( \frac{2A^{2}K^{2}}{\mu_{0}M_{S}^{2}} + H_{z} + M_{S}(1 - N_{k}) \sin^{2} \vartheta \right)^{1/2},$$

where $\vartheta$ is the polar angle of wavevector $\mathbf{k}$ and $N_{k} = (1 - e^{-ikL})/(kL)$ is the demagnetizing factor. We provide a full derivation in appendix A. We define $k_{x} = k \cos \vartheta$ and $k_{z} = k \sin \vartheta$. The uniform mode approximation results in a fairly simple expression which is generally only valid for the regime $kL \ll 1$. However, in an experimental setup where the initial excitation of the spin waves matches the uniform behavior of the mode profiles and the dispersion relations, showing numerical examples of the BVMSW behavior for specific cases. In section 4, we present our perturbative closed-form expressions for the dispersion and mode profiles in detail. We successfully compare our results to the numerical solutions. In section 5, we present a semi-analytical expression for the dispersion relation. Section 6 provides a summary of our main conclusions.

2. Formulation

We consider an in-plane magnetized slab of thickness $L$ and saturation magnetization $M_{S}$. In addition to the dipolar interaction [4], we take into account the usual micromagnetic energy functionals for exchange $E_{ex} = AL \int \left( \| \partial_{r} \mathbf{m} \|^2 + \| \partial_{\theta} \mathbf{m} \|^2 \right) d^2r$, intrinsic easy-axis anisotropy $E_{ani} = -KL \int m^2 d^2r$, and Zeeman energy $E_{Hz} = -\mu_{0}M_{S}H_{z} \int m \cdot d^2r$. We define our coordinate system in figure 1. The applied field $H_{z}$ fixes the equilibrium magnetization along $\mathbf{z}$.

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- **Figure 1.** Spin-wave modes in ferromagnetic films with a thickness $L$ larger than the exchange length $t_{ex}$ have a non trivial perpendicular profile $\delta y(z), \delta z(z)$ inside the film ($0 < z < L$). We take $z$ as the film normal and $\mathbf{z}$ as the direction of magnetization $M$, which is in the plane of the film.
profile, for instance an excitation via the inverse Faraday effect [1], a description based on the uniform mode dispersion performs very well [17], even in the case $kL \sim 1$. This makes (3) a good starting point in the analysis of spin wave propagation.

We consider the general case that film thickness $L$ is not small as compared to exchange length $l_{\text{ex}} = \sqrt{2A/\mu_0 M_0^2}$, and the dependence of the modes on the perpendicular coordinate $z$ cannot be neglected. For simplicity, we shall, in fact, assume that both film thickness $L$ and wavelength $2\pi/k$ are much greater than exchange length $l_{\text{ex}}$. This allows us to neglect the exchange energy $E_{\text{ex}}$. In the following, whenever we refer to the short-wavelength limit $k \to \infty$, we mean the regime where the wavelength is much less than film thickness ($kL \gg 1$) but still well above the exchange length $l_{\text{ex}}$ (MSSW, $k l_{\text{ex}} \ll 1$).

Fixing wavenumber $k = (k_x, k_y)$, we allow the spin-wave mode to have an arbitrary profile $\delta y(z), \delta z(z)$ inside the film. Substituting the energy functionals $E_{\text{ex}}, E_{\text{ani}}, E_{\text{elp}}, E_{\text{dep}}$ into (2), we obtain an eigenvalue equation

$$
\begin{pmatrix}
H_x \hat{S} + M_0 \hat{D}^{\gamma y} \\
M_0 \hat{D}^{yz} - (H_x - \frac{2K}{\mu_0 M_0^2}) \hat{S} + M_0 \hat{D}^{zz} \end{pmatrix}
\begin{pmatrix}
\delta y \\
\delta z
\end{pmatrix}
= \frac{\omega}{\mu_0 |\gamma|} \begin{pmatrix}
0 & iS \\
-iS & 0
\end{pmatrix}
\begin{pmatrix}
\delta y \\
\delta z
\end{pmatrix},
$$

(4)

where $\delta y(z), \delta z(z)$ are now functions of $z$, supported on the interval $0 \leq z \leq L$.

Here $\hat{S}$ represents the identity operator. The operators $\hat{D}^{ab}$ describing the dipolar interaction may be represented in Fourier space as

$$
\hat{D}^{ab}(k_x, k_y) = \frac{k_x k_y}{k_x^2 + k_y^2 + k_z^2},
$$

(5)

where $k_x, k_y$ should be treated as numerical constants (parameters of $\hat{D}^{ab}$) but $k_z$ as an operator $k_z = -i \partial_z$ acting on the functions $\delta y(z), \delta z(z)$.

The functions $\delta y(z), \delta z(z)$ vanish outside the interval $0 < z < L$. The finite film thickness $L$ quantizes the modes that can be excited for any given $k_x, k_y$. We label the modes as $n = 1, 2, \ldots$ in order of increasing $\omega > 0$.

It is convenient to normalize the solutions $\Psi_+$ to satisfy

$$
\Psi_+^* \hat{Q} \Psi_+ = \frac{1}{\mu_0 |\gamma|} \begin{pmatrix}
\delta y \\
\delta z
\end{pmatrix}^* \begin{pmatrix}
0 & iS \\
-iS & 0
\end{pmatrix} \begin{pmatrix}
\delta y \\
\delta z
\end{pmatrix} = \frac{2}{\mu_0 |\gamma|} \text{Im} \int_0^L \delta y(z) \delta z^*(z) \, dz = 1,
$$

(6)

where the asterisk denotes complex conjugation. Because the cross elements $M_0 \hat{D}^{yz}$ in (4) are Hermitian, we may assume without loss of generality that $\delta y(z)$ is purely real and $\delta z(z)$ is purely imaginary. Notice that if $\Psi_+$ is a solution of (4) with eigenvalue $\omega$, its complex conjugate $\Psi_-$ is a solution with eigenvalue $-\omega$ (and norm $\Psi_+^* \hat{Q} \Psi_- = -1$). The fact that solutions occur in conjugate pairs is a result of the Hamiltonianess of the normal-mode problem [28]. The negative-$\omega$ solution $\Psi_-$ is redundant.

2.1. Asymptotic frequencies

Fixing the polar angle $\vartheta$, we now turn to the behavior of (4) in the limits $k \to 0$ and $k \to \infty$ along a radial half-line $(k_x, k_y) = (k \cos \vartheta, k \sin \vartheta)$.

For $k \to \infty$, the operator $\hat{D}^{\gamma y}$ reduces to

$$
\hat{D}^{\gamma y} = \frac{k_x^2}{k_x^2 + k_y^2 + k_z^2} = \frac{k_x^2 \sin^2 \vartheta}{k_x^2 + k_y^2} \to (\sin^2 \vartheta) \hat{S},
$$

(7)

which is a simple scalar operator; analogously, we find

$$
\begin{pmatrix}
\hat{D}^{\gamma y} \\
\hat{D}^{yz} \\
\hat{D}^{zz}
\end{pmatrix} \to \begin{pmatrix}
(\sin^2 \vartheta) \hat{S} & 0 \\
0 & 0
\end{pmatrix},
$$

(8)

We conclude that all modes $n$ are degenerate in the limit $k \to \infty$, since now only the identity operator $\hat{S}$ acts on $\delta y(z), \delta z(z)$ in (4). In particular, it follows that the uniform-mode expression (3) for $\omega$ is exact in this limit, and we have [13]

$$
\omega_{k \to \infty} = |\gamma| \mu_0 \sqrt{(H_x - \frac{2K}{\mu_0 M_0^2}) (H_x + M_0 \sin^2 \vartheta)}.
$$

(9)

In the opposite limit $k \to 0$ (uniform precession), we find

$$
\begin{pmatrix}
\hat{D}^{\gamma y} \\
\hat{D}^{yz} \\
\hat{D}^{zz}
\end{pmatrix} \to \begin{pmatrix}
0 & 0 \\
0 & \hat{S}
\end{pmatrix},
$$

(10)

and again all modes $n$ are degenerate; the precession frequency is given by [13, 32]

$$
\omega_{k=0} = |\gamma| \mu_0 \sqrt{(H_x + M_0 - \frac{2K}{\mu_0 M_0}) H_x},
$$

(11)

in agreement with the uniform-mode expression (3).

2.2. Asymptotic profiles

In the limits $k \to 0$ and $k \to \infty$, the only operator acting on the profiles $\delta y(z), \delta z(z)$ is the identity operator $\hat{S}$. The matrices of operators in (4) reduce to simple $2 \times 2$ scalar matrices. As a result, we can solve (4) analytically and multiply the solution vector by an arbitrary function. We obtain solutions $\Psi_+$ of the form

$$
\Psi_+^{(k=0)} = \begin{pmatrix}
\delta y(z) \\
\delta z(z)
\end{pmatrix} = \sqrt{\frac{\mu_0 |\gamma|}{2ab}} \begin{pmatrix}
a \psi_0(z) \\
-b \psi_0(z)
\end{pmatrix},
$$

(12)

where $\psi_0(z)$ is a real-valued function supported on the interval $0 \leq z \leq L$. Notice that $\delta y(z), \delta z(z)$ differ only by a scalar factor. For the regime $k \to 0$, the values of $a, b$ are given by

$$
a = \sqrt{H_x + M_0 - \frac{2K}{\mu_0 M_0}}, \quad b = \sqrt{H_x},
$$

(13a)

for the regime $k \to \infty$, we have

$$
a = \sqrt{H_x - \frac{2K}{\mu_0 M_0}},
$$

(14a)
In this section we summarize some known results on the behavior of magnetostatic spin-wave modes (see, e.g. [2]). We make a comparison to the properties of the uniform-mode expression (3) derived in appendix A.

Figure 2(a) shows the dispersion relations obtained from a numerical solution of the eigenvalue equation (4). We find a sequence of modes \( n = 1, 2, \ldots \) that can be identified as the BVMSW modes [2, 13]. Their frequencies monotonically decrease in any direction \( \vartheta \) as we move away from the origin \( k = 0 \). In addition, we find, for wavevectors \( k \) pointing predominantly along the \( k_x \) axis (perpendicular to magnetization), a single special branch, which we identify as the DE surface mode [5]. Its frequency increases in \( k \) before leveling off to a constant value.

At \( k = 0 \), all modes have the same frequency \( \omega_{k=0} \) given by (11). The behavior of \( \omega(k) \) in the opposite limit \( k \to \infty \) is somewhat more involved. Any given volume mode \( n \) eventually converges to the same frequency \( \omega_{k=\infty} \), given by (9), as we take \( k \to \infty \). However, for any fixed wavevector \( k \), the frequency in the limit \( n \to \infty \) converges to \( \omega_{k=0} \). As a consequence, there is a quasi-continuum of high-\( n \) volume modes just below the line \( \omega = \omega_{k=0} \).

Figure 3 shows the mode profiles \( \delta y(z), \delta z(z) \) of the two lowest volume modes \( n = 1, 2 \) for a range of wavevectors \( k = (k_x, k_y) = (k \cos \vartheta, k \sin \vartheta) \). While we find that the profiles do not depend in any way on the sign of \( k_z \), notice that the cases \( k_z > 0 \) and \( k_z < 0 \) are inequivalent [2]. In particular, the antinode (amplitude maximum) of the \( n = 1 \) mode tends to move towards one or the other surface of the film (\( z = 0 \) or \( z = L \)) depending on the sign of \( k_z \). A similar nonreciprocity is seen in the DE modes, which exponentially localize near either of the two film surfaces [5].

The explicit perturbative expressions for the mode profiles, which we present in section 4, can be used to quantify the asymmetric behavior.

### 3.1. Relation to uniform-mode analysis

It is interesting to compare the numerical dispersion relation of the \( n = 1 \) volume mode to the dispersion relation (3) obtained in the uniform-mode approximation. As shown in figure 2(b), we find that (3) predicts the correct group velocity \( \frac{d\omega}{dk} \) in the \( k \to 0 \) limit when approaching the point \( k = 0 \) along the \( k_y \) axis (\( \vartheta = 0^\circ \) or \( \vartheta = 180^\circ \)). Along the \( k_x \) axis (\( \vartheta = \pm 90^\circ \)), however, the numerical dispersion relation differs very significantly from the uniform-mode expression. In particular, the slope \( \frac{d\omega}{dk} \) predicted for \( k \to 0 \) is incorrect: we have \( \frac{d\omega}{dk} \to 0 \) for the volume modes, but (3) predicts a positive group velocity. A qualitative explanation for the discrepancy may be found in figure 3. Approaching \( k = 0 \) along the \( k_x \) axis (\( \vartheta = 0^\circ \)), it is found [2] that the limiting profile \( \psi_0(z) \) of the \( n = 1 \) mode is indeed a constant function on the interval \( 0 \leq z \leq L \), as was assumed in the uniform-mode approach.

Along the \( k_y \) axis (\( \vartheta = 90^\circ \)), by contrast, we have that \( \psi_0(z), 0 \leq z \leq L \), is a cosine function with wavenumber...
π/L, and as a result, the uniform-mode analysis is inaccurate even for small k. Regardless of θ, the uniform-mode analysis is also inaccurate in the large-k regime (limiting profile for k → ∞ is a sine function). However, the limiting frequency (9) is reproduced correctly.

Along the k_y axis, the uniform-mode dispersion relation (3) coincides, in the small-k regime, with the DE curve. The DE mode, which is exponentially localized to the surface with a decay rate proportional to k_y [5], assumes a uniform profile in the limit k → 0. In other words, the uniform profile, which corresponds to the lowest-frequency mode (n = 1) for θ = 0°, becomes the highest-frequency mode (DE) for θ = 90°. At the same time, the profile of the n = 1 volume mode goes from uniform (θ = 0°) to sinusoidal (θ = 90°). We might interpret the transition as an avoided band crossing, as shown in figure 2(c). The reason for the dependence of the limiting profile ψ_0(z) on θ is given in more formal terms in section 4.

Figure 3. Profiles δy(z) (solid lines) and −iδz(z) (dashed lines), with 0 ≤ z ≤ L, of the lowest-frequency volume modes n = 1, 2, for a range of wavevectors \( \mathbf{k} = (k \cos \vartheta, k \sin \vartheta) \), taking \( H_x = 0.73 M_S \) and \( 2K = 0.46 \mu_0 M_S^2 \). The modes are invariant under a reflection of \( \mathbf{k} \) with respect to the k_y axis (\( \vartheta \leftrightarrow 180° − \vartheta \)). Notice that the limiting profile for k = 0 depends on the direction \( \vartheta \) from which we approach the singularity at \( \mathbf{k} = 0 \) [2]. The critical angle \( \vartheta_{cr} = 49° \) defines the boundary between regions A (uniform limiting profile for n = 1) and B (sinusoidal limiting profile), as shown schematically in figure 4. For k = 1L^{-1} and k = 12L^{-1}, we compare the numerical solutions (black lines) of the normal-mode problem (4) to the first-order approximations (gray lines) given by (16), (18), (19) and table 2. Our first-order expressions provide a good indication of the numerical mode profiles, not only in the \( k \rightarrow 0 \) or \( k \rightarrow \infty \) limits [2] but also for finite k.

Figure 4. Domains of applicability of the three asymptotic regimes A, B, and C. Regions A and B together represent the small-k regime. The behavior of the normal modes and frequencies is qualitatively different depending on whether one approaches the point \( \mathbf{k} = 0 \) from region A or region B. Region C denotes the large-k regime.
Table 1. Asymptotic behavior of the dispersion relations of the volume modes \( n \) in each of the long-wavelength regions A, B, and the A–B boundary line and its the short-wavelength region C (see figure 4). The \( \theta \)-dependent quantities \( H, J, G \) and \( a, b, c \) and \( d \) are defined in (15) and (20). The profiles \( \psi_0, \psi_1, \phi_1 \) (see table 2) are shown for \( n = 1 \) (black lines) and \( n = 2 \) (gray lines).

| Region | Dispersion | \( \psi(z) \) | \( \psi_1(z) \) | \( \phi_1(z) \) |
|--------|------------|---------------|---------------|---------------|
| A \((n = 1)\) | \( \omega = ab - \frac{4kL}{15} kL + \frac{M_0}{10} \left( J - \frac{3k^4}{20} \right) (kL)^2 + O(k^3/H) \) | | | |
| A \((n > 1)\) | \( \omega = ab - \left( \frac{4}{15} kL \right)^2 \frac{M_0 G}{2a b} + O(k^3/H) \) | | | |
| A–B | \( \omega = ab - \left( \frac{4}{15} kL \right)^2 \frac{M_0 G}{2a b} q_0(\eta) + O(\eta[k^3 + k^3]) \) | | | |
| B | \( \omega = ab - \left( \frac{4}{15} kL \right)^2 \frac{M_0 G}{2a b} + O(k^3/(-H)] \) | | | |
| C | \( \omega = ab + \left( \frac{4}{15} kL \right)^2 \frac{M_0 G}{2a b} + O(k^{-3}) \) | | | |

4. Closed-form perturbative expressions

In this section, we present explicit analytical expressions for the frequency \( \omega \) and profiles \( \delta y(z), \delta z(z) \) of the volume modes \( n \) in the small-\( k \) and large-\( k \) regimes. While the limiting profiles for \( k \to 0 \) and \( k \to \infty \) (zeroth order) are well known [2], our expressions, which are accurate up to first order in \( k < 1/k \), give a good impression of the behavior of the modes even for finite \( k \), as shown in figure 3. They can be used to estimate how strongly each volume mode couples to an external field pulse with a given depth profile, or to predict the contribution of the mode to net magnetization \( \delta z(t,x,y) = \int \delta z(t,x,y,z) \, dz \) as measured using Faraday rotation [1]. They also describe quantitatively the asymmetry in the profiles obtained for \( k \neq 0 \). Moreover, we use the perturbation theory derived here to construct an accurate semi-analytical expression for the dispersion relation of the \( n = 1 \) volume mode in section 5.

The limiting behavior for \( k \to 0 \) depends essentially on the polar angle \( \theta \). It is useful to introduce the quantity

\[
H = H_x \cos^2 \theta - \left( M_s - \frac{2k}{\mu_0 M_s} \right) \sin^2 \theta.
\]

The boundary lines \( H = 0 \) separate the small-\( k \) domain into four sectors, as shown in figure 4. We distinguish between region A, where \( H > 0 \), and region B, where \( H < 0 \). Regions A and B meet at the critical angle \( \theta_c = \arctan \sqrt{H_x/(M_s - 2k/\mu_0 M_s)} \) [2]. The large-\( k \) domain is designated as region C.

It is convenient to write the mode profiles as

\[
\Psi_+ = \begin{pmatrix} \delta y \\ \delta z \end{pmatrix} = \sqrt{\frac{\mu_0 |\gamma|}{2a b}} \begin{pmatrix} a [\psi(z) - \phi(z)] \\ -i b [\psi(z) + \phi(z)] \end{pmatrix},
\]

where \( \psi(z), \phi(z) \) are real-valued functions supported on the interval \( 0 \leq z \leq L \). The constants \( a, b > 0 \) are defined by (13) for small \( k \) (regions A and B) and by (14) for large \( k \) (region C). The normalization condition (6) becomes

\[
\Psi_+ \hat{Q} \Psi_+ = \int_0^L [\psi(z)^2 - \phi(z)^2] \, dz = 1.
\]

In the small-\( k \) regime (regions A and B), we expand the wavefunctions and eigenfrequencies as

\[
\omega = \omega_0 + k \omega_1 + k^2 \omega_2 + \ldots,
\]

\[
\psi(z) = \psi_0(z) + k \psi_1(z) + O(k^2),
\]

\[
\phi(z) = k \phi_1(z) + O(k^2);
\]

in the large-\( k \) regime (region C), we define

\[
\omega = \omega_0 + k^{-1} \omega_1 + k^{-2} \omega_2 + \ldots,
\]

\[
\psi(z) = \psi_0(z) + k^{-1} \psi_1(z) + O(k^{-2}),
\]

\[
\phi(z) = k^{-1} \phi_1(z) + O(k^{-2}).
\]

In all three regions, only the \( \psi(z) \) component of the wavefunction contributes at zeroth order \( (k = 0 \text{ or } k = \infty) \); the function \( \phi(z) \) vanishes in those limits (see (12)).

The main results of this section are summarized in tables 1 and 2, which list explicit perturbative expressions for frequency (up to second order) and profiles (up to first order) of the volume modes \( n \), for each of the regions. For brevity, we introduce the quantities

\[
A = \left( H_x + M_s - \frac{2k}{\mu_0 M_s} \right) \sin^2 \theta + H_x,
\]

\[
G = H_x \cos^2 \theta + \frac{2k}{\mu_0 M_s} \sin^2 \theta,
\]

\[
J = H_x \cos^2 \theta + \left( \frac{2k}{\mu_0 M_s} + M_s \right) \sin^2 \theta.
\]

\[
a = \sqrt{H_x + M_s - \frac{2k}{\mu_0 M_s}}
\]

\[
b = \sqrt{H_x}
\]
The asymptotic behavior of the dispersion relations, given by $g(\theta)$, $b$ for $\theta_a$, $\theta_B$, $\theta_c$, $\theta_r$, $\theta_A$, $\theta_B$, $\theta_C$ (section 4.4). The boundary between regions $A$ and $B$, where $|\theta| = \theta_c$, requires special consideration (section 4.3). In the interest of readability, we

\begin{align*}
\psi_0(z) &= \frac{1}{\sqrt{L}} \\
\psi_1(z) &= -\frac{1}{\sqrt{L}} \times \left[ \frac{4M_e \sin \theta}{2M} - \frac{G^2(2\pi/L)^2 - L^2}{L} \right] \\
\phi_1(z) &= \frac{1}{\sqrt{L}} \left( \frac{4M_e}{2M} \sin \theta - \frac{G^2(2\pi/L)^2 - L^2}{L} \right)
\end{align*}

Region A ($n = 1$)

\begin{align*}
\psi_0(z) &= \sqrt{\frac{z}{L}} \\
\psi_1(z) &= -\sqrt{\frac{z}{L}} \times \left( \frac{4M_e \sin \theta}{2M} \sin \left( \frac{\pi}{2} \right) \right) \\
\phi_1(z) &= \sqrt{\frac{z}{L}} \left( \frac{4M_e}{2M} \sin \theta \cos \left( \frac{\pi}{2} \right) \right)
\end{align*}

Region A ($n > 1$)

\begin{align*}
\psi_0(z) &= \sqrt{\frac{2}{L}} \cos \left( \frac{(n-1)\pi}{2L} \right) \\
\psi_1(z) &= -\sqrt{\frac{2}{L}} \sin \left( \frac{(n-1)\pi}{2L} \right) \times \left( \frac{4M_e \sin \theta}{2M} \right) \\
\phi_1(z) &= \sqrt{\frac{2}{L}} \left( \frac{4M_e}{2M} \sin \theta \right)
\end{align*}

Boundary A-B (for $\eta = 0$ and $k_r > 0$)\(^a\)

\begin{align*}
\psi_0(z) &= \sqrt{\frac{z}{L}} \cos \left( \frac{\pi}{2L} \right) \\
\psi_1(z) &= -\sqrt{\frac{z}{L}} \times \left( \frac{4M_e \sin \theta}{2M} \sin \left( \frac{\pi}{2L} \right) \right) \\
\phi_1(z) &= \sqrt{\frac{z}{L}} \left( \frac{4M_e}{2M} \sin \theta \cos \left( \frac{\pi}{2L} \right) \right)
\end{align*}

Region B\(^b\)

\begin{align*}
\psi_0(z) &= \pm \sqrt{\frac{z}{L}} \sin \left( \frac{\pi}{2L} \right) \\
\psi_1(z) &= \mp \sqrt{\frac{z}{L}} \times \left( \frac{4M_e \sin \theta}{2M} \sin \left( \frac{\pi}{2L} \right) \right) \\
\phi_1(z) &= \sqrt{\frac{z}{L}} \left( \frac{4M_e}{2M} \sin \theta \right)
\end{align*}

Region C

\begin{align*}
\psi_0(z) &= \sqrt{\frac{z}{L}} \sin \left( \frac{\pi}{2L} \right) \\
\psi_1(z) &= -\sqrt{\frac{z}{L}} \times \left( \frac{4M_e \sin \theta}{2M} \cos \left( \frac{\pi}{2L} \right) \right) \\
\phi_1(z) &= \sqrt{\frac{z}{L}} \left( \frac{4M_e}{2M} \sin \theta \right)
\end{align*}

\(^a\) For $k_r < 0$, take $\psi_0(z) \leftarrow (-1)^{n-1} \psi_0(L - z)$ for $\psi_0, \psi_1, \phi_1$.

\(^b\) For $k_r > 0$ (upper signs) and $k_r < 0$ (lower signs).

In figure 3, we successfully compare our first-order mode profiles, given by the expressions in table 2, to the numerical results.

In the remainder of this section, we present in more detail the derivations for each of the regions A (section 4.1), B (section 4.2), and C (section 4.4). The boundary between regions $A$ and $B$, where $|\theta| = \theta_c$, requires special consideration (section 4.3). In the interest of readability, we

\begin{align*}
c &= \sqrt{H_x - \frac{2K}{\mu_0 M_S}} & (20f) \\
d &= \sqrt{H_x + M_S \sin^2 \theta} & (20g)
\end{align*}

The asymptotic behavior of the dispersion relations, given by the expressions in table 1, is shown for $n = 1$ in figure 2(b).
focus on the limiting profiles \(\psi_0(z)\) of the \(n = 1\) mode and on the asymptotic behavior of its dispersion relation. A more mathematical derivation of the perturbation theory used to obtain all results in tables 1 and 2 is given in appendix C.

4.1. Region A

The operators \(\hat{D}^{ab}(k_x, k_y)\), defined in (5), are the only non-trivial operators appearing in the eigenvalue equation (4). We expand \(\hat{D}^{ab}(k \cos \vartheta, k \sin \vartheta)\) in the parameter \(k\) (for fixed \(\vartheta\)). Using appendix B, we obtain

\[
\begin{pmatrix}
\frac{D_x}{D^x} \\
\frac{D_z}{D^z}
\end{pmatrix} = D_0 + kD_1 + k^2D_2 + k^3D_3 + \ldots
\]

\[
= \begin{pmatrix} 0 & 0 \\ 0 & \hat{k} \end{pmatrix} + k \begin{pmatrix} \frac{\pi(\sin^2 \vartheta)\delta(\hat{k}_z) - \pi(\delta(\hat{k}_z))}{\sin \vartheta} \\ \frac{(\sin \vartheta)\delta'(\hat{k}_z) - \pi(\sin \vartheta)\delta''(\hat{k}_z)}{\sin \vartheta} \end{pmatrix} + k^2 \begin{pmatrix} \frac{(\sin^2 \vartheta)\delta''(\hat{k}_z) - \pi(\sin \vartheta)\delta''(\hat{k}_z)}{\sin \vartheta} \\ \frac{2\pi(\sin \vartheta)\delta'(\hat{k}_z)}{\sin \vartheta} - \pi(\sin \vartheta)\delta''(\hat{k}_z) \end{pmatrix} + k^3 \left[ \begin{pmatrix} -\frac{1}{2}\pi(\sin^2 \vartheta)\delta''(\hat{k}_z) - \pi(\sin \vartheta)\delta''(\hat{k}_z) \\ -\pi(\sin \vartheta)\delta''(\hat{k}_z) \end{pmatrix} \right] + O(k^4) \tag{21}
\]

where \(\delta\) represents the Dirac delta distribution. The expressions containing \(\hat{k}\) represent (in real space) convolution operators acting on the profiles \(\delta y(z), \delta y(z)\); for example, the action of \(D_1\) may be expressed as

\[
\hat{D}_1 \begin{pmatrix} \delta y(z) \\ \delta z(z) \end{pmatrix} = \frac{1}{2} \left[ \begin{pmatrix} \int [\sin^2 \vartheta \delta y(z') + i \sin \vartheta \delta \text{sgn}(z - z') \delta z(z')] \, dz' \\ \int [i \sin \vartheta \delta \text{sgn}(z - z') \delta y(z') - \delta z(z')] \, dz' \end{pmatrix} \right].
\]

4.2. Region B

In region B, where \(H < 0\), minimization of (23) is equivalent to minimization of \(\left(\int_0^L \psi_0(z) \, dz\right)^2\). The minimum value

\[
\omega_1 = \lim_{k \to 0} \frac{d\omega}{dk} = 0
\]

is obtained for any profile \(\psi_0(z)\) for which \(\int_0^L \psi_0(z) \, dz = 0\). In other words, the zeroth-order profile \(\psi_0(z)\) is indeterminate even in first-order perturbation theory. The degeneracy is lifted by the second-order term of (21). By (C.17), the profile \(\psi_0(z)\) minimizes

\[
\omega_2 = -\frac{\gamma |\mu_0 M_S|}{2ab} \left[ H_x \cos^2 \vartheta + \frac{2K}{\mu_0 M_S} \sin^2 \vartheta \right] \times \int_0^L \psi_0 k_z^2 \psi_0 \, dz. \tag{27}
\]

The operator \(k_z^2\) represents, as usual, a convolution in real space. Using the Fourier transform \(f(k) = k^{-n} \leftrightarrow f(x) = i(x)^n \sin(x)/[2(n - 1)!]\), we have

\[
\int_0^L \psi_0 k_z^2 \psi_0 \, dz = -\frac{1}{2} \int_0^L \psi_0(z) \int_0^L |z - z'| \psi'(z') \, dz'. \tag{28}
\]

Minimization of \(\omega_2\) gives

\[
\psi_0(z) = \frac{1}{\sqrt{L}} \cos \left( \frac{\pi x}{L} \right) \Pi^* \left( \frac{z}{L} \right). \tag{29}
\]

and we evaluate

\[
\omega_2 = \lim_{k \to 0} \frac{d\omega}{dk} = \lim_{k \to k_0} \frac{1}{2k} \frac{d\omega}{dk} = -\frac{\mu_0 |\gamma M_S L^2}{2n\pi ab} \left[ H_x \cos^2 \vartheta + \frac{2K}{\mu_0 M_S} \sin^2 \vartheta \right]. \tag{30}
\]

As in region A, minimization of \(\omega_2\) immediately fixes \(\phi_1(z)\) according to (C.16). However, the third-order expression (C.19) for \(\psi_1(z)\) is indeterminate in region B. We need to minimize the fourth-order functional \(\omega_4\), given by (C.22), to determine \(\psi_1(z)\). The resulting expressions are listed in table 2.
Figure 5. In regions A (blue) and B (red), we carry out the expansion in $k$ along lines of constant $\vartheta$. In the A–B boundary region (green), we expand instead along curves of constant $\eta$, as defined by (36). Together, the three perturbative expressions for $\omega$ (see table 1) provide, up to the uncertainty in $q_n(\eta)$, a description of the dispersion relation that is accurate to second order in $\vartheta$ uniformly in $k$.

4.3. Boundary line A–B

The boundary line between regions A and B requires special consideration, as neither the perturbation theory of region A nor of region B is valid on this line. We find that the boundary line in some sense interpolates between the profiles in the interior of regions A and B.

On the boundary, where $H = 0$, we have that $\omega_1$ as given by (23) is identically zero. This means that, as in region B, the profile $\psi_0(z)$ is fixed by minimization of $\omega_2$. In contrast to region B, however, there is no constraint $\int_0^L \psi_0(z) \, dz = 0$ from minimization of $\omega_1$. By (C.17), $\psi_0(z)$ minimizes

$$\omega_2 = -\frac{1}{2\omega_0} \int_0^L |(\bar{Y}_1^2 \psi_0)'|^2 \, dz,$$

where

$$\bar{Y}_1^2 \psi_0(z) = -C \int_0^z \psi_0(z') \, dz' \quad \text{for } k_2 > 0,$$

$$\bar{Y}_1^2 \psi_0(z) = C \int_z^L \psi_0(z') \, dz' \quad \text{for } k_2 < 0,$$

with $C = \mu_0|\mu_0| |M_S| \sin \vartheta$. Minimization gives

$$\psi_0(z) = \frac{\sqrt{2}}{L} \cos \left( \frac{\pi z}{2L} \right) \Pi^* \left( \frac{\vartheta}{L} \right) \quad \text{for } k_2 > 0,$$

$$\psi_0(z) = \frac{\sqrt{2}}{L} \sin \left( \frac{\pi z}{2L} \right) \Pi^* \left( \frac{\vartheta}{L} \right) \quad \text{for } k_2 < 0,$$

and we evaluate

$$\omega_2 = \lim_{k \to 0} \frac{1}{2} \left( \frac{\partial^2 \omega}{\partial k^2} \right)_{H=0} = -|\mu_0| \frac{4L^2 M_5 G_0}{\pi^2} 2ab q_1(\eta).$$

Minimization of $\omega_2$ also fixes $\phi_1(z)$ by (C.16). The other first-order component $\psi_1(z)$ is again determined only at fourth order of perturbation theory.

In the above expressions, we assume that we approach the point $k = 0$ along a line $H = 0$; in other words, we fix $\vartheta = \pm \vartheta_0$. We can generalize (34) by carrying out the expansion along a curve of constant $\eta$, as shown in figure 5, where we define

$$\eta = \left( \frac{a^2}{2b^2 M_5 L} \right) \frac{H}{k}.$$

We obtain, for the lowest mode $n = 1$,

$$\omega_2 = \lim_{k \to 0} \frac{1}{2} \left( \frac{\partial^2 \omega}{\partial k^2} \right)_{H=0} = -|\mu_0| \frac{4L^2 M_5 G_0}{\pi^2} 2ab q_1(\eta).$$

Notice that (34) corresponds to $\eta = 0$. The generalization $\eta \neq 0$ interpolates between regions A ($\eta \to \infty$) and B ($\eta \to -\infty$) and allows one, in principle, to construct a second-order approximation of $\omega$ around $k = 0$ uniform in $\vartheta$.

Table 1 gives $\omega_2$ for arbitrary mode index $n$. While we are unaware of a closed-form expression for the functions $q_n(\eta)$, we have a small-$k$ expansion

$$q_n(\eta) = 1 + 2\eta + \eta^2 - \frac{1}{6} (2n - 1) \pi^2 \eta^3 (1 - \eta) + O(\eta^5)$$

and large-$k$ expansions

$$q_n(\eta) = \frac{n^2}{12} (1 + 3\eta) + O(\eta^{-1}) \quad \text{for } \eta > 0, n = 1,$$

$$q_n(\eta) = \left( \frac{n - \frac{1}{n}}{n - 1} \right)^2 + O(\eta^{-1}) \quad \text{for } \eta > 0, n > 1,$$

$$q_n(\eta) = \left( \frac{n - \frac{1}{n}}{n - 1} \right)^2 + O(\eta^{-1}) \quad \text{for } \eta < 0.$$

A very good approximation for $q_1(\eta)$, with a maximal absolute error of 0.00363, is given by

$$q_1(\eta) \approx \frac{1}{24} \left[ 3 + \pi^2 (1 + 3\eta) + \sqrt{3 + \pi^2 (1 + 3\eta)^2 - 12\pi^2 (1 + 3\eta) + 432} \right].$$

4.4. Region C

For large $k$, we have the expansion (see (21))

$$\begin{pmatrix} \tilde{D}_{1\varphi} & \tilde{D}_{1z} \\ \tilde{D}_{2\varphi} & \tilde{D}_{2z} \end{pmatrix} = \tilde{D}_{-0} + k^{-1} \tilde{D}_{-1} + k^{-2} \tilde{D}_{-2} + \cdots,$$

where

$$\tilde{D}_{-0} = \sin^2 \vartheta \begin{pmatrix} 0 & k_0^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{k^2} \begin{pmatrix} -\sin^2 \vartheta & k_0^2 & 0 \\ 0 & -\sin^2 \vartheta & k_0^2 \\ 0 & 0 & \sin^2 \vartheta \end{pmatrix} + O(k^{-5}).$$


Figure 6. Mode profiles \( \delta y(z) \) and \( \delta z(z) \) of the lowest-frequency mode \( n = 1 \) and their approximate representations as linear combinations (48) of the basis functions \( u_0, u_1 \) (blue lines), \( u_2, u_3 \) (red lines), for three wavevectors \( k = (k_x, k_y) \), taking \( H_z = 0.73 M_S \) and \( 2K = 0.46 \mu_0 M_S^2 \). There is no visible difference between the full profiles and the approximations.

All modes are degenerate at zero order. In the \( k \to \infty \) limit, the mode profiles are of the form (12) with \( a, b \) given by (14). Notice that, for region C, the value of \( b \) depends on \( \phi \). The second-order term \( D_{-2} \) in (41) lifts the degeneracy and \( \psi(z) \).

Regardless of \( k_y \), we have

\[
\lim_{k \to \infty} -k^2 \frac{d \omega}{dk} = \omega_1 = \Psi_0^1(M_S D_{-1}) \Psi_0 = 0. \tag{42}
\]

For \( k_y = 0 \), the first-order term \( D_{-1} \) in (41) even vanishes identically. In this case, we may somewhat simplify our calculations by treating \( \omega_2 \) as the first-order term of a perturbation series in \( k^2 \). We find the limiting profile \( \psi_0(z) \) of the \( n = 1 \) mode by minimization of

\[
\omega_2 = \Psi_0^1(M_S D_{-2}) \Psi_0 = \frac{\mu_0 |\gamma| M_S}{2ab} H_z \int_0^L \left( \frac{d\psi_0}{dz} \right)^2 dz, \tag{43}
\]

under the constraint (C.14a). We obtain

\[
\psi_0(z) = \sqrt{\frac{2}{L}} \sin \left( \frac{\pi z}{L} \right) \tag{44}
\]

and

\[
\lim_{k \to \infty} -k^2 \frac{d \omega}{dk} = \omega_2 = \mu_0 |\gamma| M_S \pi^2 H_z L^2 / 2ab. \tag{45}
\]

Notice that \( \psi_0(z) \) satisfies Dirichlet boundary conditions \( \psi_0(0) = \psi_0(L) = 0 \). Such conditions are necessary to give (43) a finite value, since \( \psi_0(z) \) must vanish outside the interval \( 0 \leq z \leq L \). However, higher-order terms \( \phi_1(z), \psi_1(z) \) of the expansion can have finite values for \( z = 0 \) or \( z = L \).

In the general case \( k_y \neq 0 \), the profile \( \psi_0(z) \) is still a sine function (44). Using (C.17), we evaluate

\[
\omega_2 = \frac{\mu_0 |\gamma| M_S G}{2ab} \int_0^L \left( \frac{d\psi_0}{dz} \right)^2 dz = \mu_0 |\gamma| M_S \pi^2 G / L^2 / 2ab. \tag{46}
\]

The profile \( \phi_1(z) \) is fixed by (C.16). The other first-order profile \( \psi_1(z) \) is determined, again, only by minimization of the fourth-order functional \( \omega_4 \) (C.22). Table 2 lists the resulting expressions.

When performing the derivation of the profile \( \psi_1(z) \), we take into account the following. Writing out (C.18) and (C.22) for region C, we find that the functionals \( \omega_3 \) and \( \omega_4 \) contain terms such as \( \int \psi_1 k_0 \partial_z \psi_1 dz = \int \psi_1 \partial_z \psi_1 dz \), and higher-order terms \( \omega_3 \). Indeed, the profiles \( \psi_0(z), \psi_1(z) \) have discontinuities at \( z = 0 \) and \( z = L \), while \( \psi_0(z) \) has discontinuities in its first derivative. The functionals \( \omega_3, \omega_4 \) can each be written as a sum of regular integral terms plus boundary terms of the forms (a) \( \lim_{\Delta \to 0^+} \int_{-\Delta}^{\Delta} S(z) \delta(z) dz \) and (b) \( \lim_{\Delta \to 0^+} \int_{-\Delta}^{\Delta} S(z) \delta'(z) dz \), where \( S(z) \) is the Heaviside step function. It is natural to assign the value \( \frac{1}{2} \) to (a). As for terms (b), which diverge, we must require that the sum of their prefactors vanishes, yielding a boundary condition that acts as a constraint in the minimization of \( \omega_4 \).

5. Semi-analytical solution

In this section, we present a semi-analytical expression for the dispersion relation of the \( n = 1 \) BVMSW mode that can be evaluated in constant time using standard numerical routines. The expression is accurate up to an error that is negligible for any practical purpose (well below 0.01% in the example of figure 7). It takes a given wavevector \( (k_x, k_y) \) as input; evaluation does not require an initial guess for \( \omega \).

Figure 6 shows that typical profiles \( \delta y(z), \delta z(z) \) of the \( n = 1 \) mode can be written, to a very reasonable approximation, as a linear combination of only four basis functions

\[
u_0(z) = \Pi^* \left( \frac{z}{L} \right), \tag{47a}\]

\[
u_1(z) = \left( \frac{z}{L} - \frac{1}{2} \right) \Pi^* \left( \frac{z}{L} \right), \tag{47b}\]

\[
u_2(z) = \sin \left( \frac{\pi z}{L} \right) \Pi^* \left( \frac{z}{L} \right), \tag{47c}\]

\[
u_3(z) = - \cos \left( \frac{\pi z}{L} \right) \Pi^* \left( \frac{z}{L} \right). \tag{47d}\]

For our semi-analytical approximation, we restrict the profiles to such linear combinations

\[
\delta y(z) = c_{10} u_0(z) + c_{11} u_1(z) + c_{12} u_2(z) + c_{13} u_3(z), \tag{48a}\]

\[
\delta z(z) = d_{10} u_0(z) + d_{11} u_1(z) + d_{12} u_2(z) + d_{13} u_3(z). \tag{48b}\]

On this basis set, the operators \( S, D^\|, D^\perp, D^{\text{xx}} \) reduce to simple \( 4 \times 4 \) matrix blocks, given below, and (4) becomes an \( 8 \times 8 \) eigenvalue problem

\[
\begin{bmatrix}
H_z S + M_S D^\| & M_S D^\perp \\
M_S D^\perp & M_S D^{\text{xx}}
\end{bmatrix}
\begin{pmatrix}
u \\
\omega / \mu_0 |\gamma|
\end{pmatrix}
= 0. \tag{49}
\]
where square brackets indicate block matrices and $v = (c_0, c_1, c_2, d_0, d_1, d_2, d_3)^T$ represents the eigenvector. The approximate frequency of the $n = 1$ mode is given by the lowest positive eigenvalue $\omega$.

Equation (49) takes the form of an $8 \times 8$ generalized Hermitian eigenvalue problem, the solutions $\omega$ of which can be found numerically using standard routines. While not all linear-algebra computer packages may support the generalized format $Hv = \omega Qv$, it can always be rewritten as $Q^{-1}Hv = \omega v$ and solved as an ordinary non-Hermitian eigenvalue problem.

Explicit analytical expressions exist for all matrix elements in (49). The identity operator $I$ becomes the overlap matrix

$$ S = L \left( \begin{array}{cccc}
1 & 0 & 2 \pi & 0 \\
0 & \frac{\pi}{2} & 0 & \frac{\pi}{2} \\
\frac{\pi}{2} & 0 & 1 & 0 \\
0 & \frac{\pi}{2} & 0 & \frac{1}{2}
\end{array} \right), \quad (50) $$

where the matrix elements are defined by $S_{ij} = \int u_i(z) u_j(z) dz$. The elements of the $D_{i\alpha}$ matrix blocks can be evaluated in Fourier space as

$$ D_{i\alpha} = \frac{1}{2\pi} \int u_i^*(k_c) k_{\alpha} k_{\epsilon} u_i(k_c) dk_c, \quad (51) $$

and analogously for $D_{\alpha j}$ and $D_{\epsilon j}$, where $u_i(k_c)$ is the Fourier transform of the basis function $u_i(z)$. We obtain

$$ D_{i\epsilon} = \frac{L^2}{k^2} \times \left( \begin{array}{ccc}
1 - N_{00} & 0 & \frac{2(1 - N_{01})}{\pi} \\
0 & 1 - N_{11} & 0 \\
\frac{2(1 - N_{10})}{\pi} & 0 & 1 - N_{10}
\end{array} \right) \quad (52) $$(52)

and

$$ D_{\alpha j} = L \left( \begin{array}{ccc}
N_{00} & 0 & \frac{2}{\pi} N_{02} \\
0 & \frac{2}{\pi} N_{11} & 0 \\
\frac{2}{\pi} N_{02} & 0 & \frac{2}{\pi} N_{13}
\end{array} \right), \quad (53) $$

where

$$ N_{00} = \frac{1 - e^{-\frac{4kL}{kL}}}{kL}, \quad (54a) $$

$$ N_{11} = 12 \left( \frac{1 + e^{-\frac{4kL}{kL}}}{4kL} + (1 + kL)e^{-\frac{4kL}{kL}} - 1 \right), \quad (54b) $$

$$ N_{22} = \frac{\pi^2}{k^2L^2 + \pi^2} - \frac{2\pi^2 kL (1 + e^{-4kL})}{(k^2L^2 + \pi^2)^2}, \quad (54c) $$

$$ N_{33} = \frac{\pi^2}{k^2L^2 + \pi^2} + \frac{2k^2 L^3 (1 + e^{-4kL})}{(k^2L^2 + \pi^2)^2}, \quad (54d) $$

$$ N_{02} = \frac{\pi^2}{k^2L^2 + \pi^2} \frac{1 + e^{-\frac{4kL}{kL}}}{2}, \quad (54e) $$

$$ N_{13} = \frac{\pi^2}{k^2L^2 + \pi^2} \frac{(1 + e^{-\frac{4kL}{kL}})(2 + kL)}{4} \quad (54f) $$(54f)

are the so-called demagnetizing factors; and we obtain

$$ D_{\alpha j} = \frac{k_j L_2}{2\pi} \left( \begin{array}{ccc}
0 & \frac{\pi}{2} Z_{j1} & 0 \\
-\frac{\pi}{2} Z_{j1} & 0 & 0 \\
0 & 0 & -Z_{j3}
\end{array} \right) \quad (55) $$

and

$$ Z_{j3} = N_{02}, \quad (56b) $$

$$ Z_{21} = \frac{\pi^2}{k^2L^2 + \pi^2} \left( 1 - \frac{4}{\pi} Z_{j1} kL \right), \quad (56c) $$

$$ Z_{23} = N_{22}, \quad (56d) $$

where

Notice that the basis set (47) has been chosen in such a way that it can represent exactly the profiles $\psi_i(z), \phi_i(z)$ of the $n = 1$ mode in each of the regions A, B, and C (see table 2). As a result, we have at least second-order accuracy of $\omega$ in $k$ or $1/k$ in those regions, as shown in figure 7. Since the mode profile $\psi_1(z)$ is fixed only at third (or, in regions B and C, fourth) order of perturbation theory (see section 4), it does not need to be included in the basis set to obtain second-order accuracy.

On the boundary line between regions A and B ($\theta = \phi_{cut}$), we have only first-order accuracy, because the corresponding
profiles $\psi_0(z), \phi_1(z)$ are not represented in the basis set. Figure 7 shows that the error $\Delta \omega$ nonetheless remains very small. The exact small-$k$ behavior of $\omega$ on the A–B boundary is given in section 4.3.

We comment on our claim that the approximate expression for $\omega(k_x, k_y)$ can be evaluated in constant time. In general, the solution of an eigenvalue problem for matrices of size $5 \times 5$ or larger requires the use of iterative methods, the convergence rate of which may depend on system parameters. In our case, the characteristic equation $\text{Det}(H - \omega Q) = 0$ of the eigenvalue problem (49) contains only even powers of $\omega$, since all eigenvalues appear in conjugate pairs (Hamiltonian problem (28)). We could therefore write the characteristic equation, a polynomial of eighth degree in $\omega$, as a quartic polynomial in $\omega^2$, which can be explicitly solved by radicals. This guarantees the existence of an analytical expression for $\omega$ in principle.

Note that the choice for the basis functions (47) matches possible excitation profiles achievable in experiments. As we have seen before, the uniform profile (47a) can be excited via optical methods [1] and, tuning the frequency to increase absorption, the linear profile is accessible as well. The sinusoidal profiles are important matching the DE modes. This choice for the basis functions makes our semi-analytical expression, besides accurate, very versatile in describing the spin wave dispersion in various experimental setups.

6. Summary

MSSW show a rich spectrum of interesting physical phenomena such as strong anisotropic behavior, nonreciprocal propagation and nonlinear effects. A comprehensive understanding of the dispersion of these spin waves is an important factor in the analysis of experimental results and can aid in the design of new experiments. For an in-plane magnetized film, two distinct modes can be classified. The DE mode has an has an evanescent character and is thus located at the surface whereas the BVMSW has a nonvanishing profile throughout the material. In the regime where exchange interactions can be neglected ($\lambda \gg l_{ex}, L \gg l_{ex}$) the perpendicular profile of the spin waves is essential to an effective description of the propagation. As is shown in figure 2(c), the DE modes exists for spin waves that propagate with an angle $\theta > \theta_{cr}$ with the external field. In contrast, the BVMSW modes exist for any direction of $k$.

Since the defining equations of the BVMSW modes can be solved only numerically, we believe that it is useful to have some approximate analytical results describing their essential features. In table 1, we summarize the simple analytical expressions that we have derived for the mode frequencies in the short-wavelength and long-wavelength regimes, including the behavior for wavevectors $k$ pointing in a direction close to the critical angle $\theta_{cr}$. We have also obtained explicit first-order expressions for the depth profiles $\delta y(z), \delta z(z)$ of the modes, given by (16), (18), (19) and table 2. These expressions highlight and quantify the asymmetry in $z$ found for $k_y \neq 0$ (nonreciprocal behavior). These mode profiles are also important when one wants to assess the coupling of an excitation to the various spin wave modes in a system. For instance, the lowest order BVMSW has a homogeneous profile and couples strongly to an excitation that acts uniform throughout the system, as is the case for optical excitations relying on the inverse Faraday effect at a frequency where the sample is transparent.

In addition to the perturbative results, we provide a semi-analytical expression for the dispersion relation of the lowest mode $n = 1$ valid for arbitrary wavevector $(k_x, k_y)$. While this expression is, strictly speaking, an approximation, we find that the error is so small as to be negligible for practical purposes. The semi-analytical expression is straightforward to implement using standard numerical routines. This makes it very useful as a quick way to interpret experimental results or to be used in a numerical scheme.

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Appendix A. Uniform-mode analysis

In this section, we review the derivation of the dispersion relation of spin waves in a film in the uniform-mode approximation, where we assume that the precession amplitudes $\delta y(z), \delta z(z)$ of the magnetization inside the film do not depend on the perpendicular coordinate $z$. Formally, this approximation is valid only in the limit of ultrathin films ($L \ll l_{ex}$). While there are some important qualitative differences between the uniform-mode expression and the dispersion relation for large film thickness $L$, it provides a useful first indication of the dispersion behavior of the BVMSW modes.

Specifically, taking $\hat{z}$ as the film normal, we assume

$$M(t, x, y, z) = M_s \mathbf{m}(t, x, y) \Pi^*(z/L),$$

(A.1)

where $M_s$ is saturation magnetization, unit vector $\mathbf{m}(t, x, y)$ is the magnetization direction, and $\Pi^*(z/L)$ is the rectangular function

$$\Pi^*(z/L) = \begin{cases} 1 & \text{for } 0 < z/L < 1 \\ 0 & \text{for } z/L < 0 \text{ or } z/L > 1 \end{cases} \quad (A.2)$$

In view of section 4.4, it is convenient to define $\Pi^*(0) = \Pi^*(1) = \frac{1}{2}$.

A.1. Magnetostatic energy: general case

It is well known that the interaction between two magnetic point dipoles $\mathbf{v}_r, \mathbf{v}_t$ located at $\mathbf{r}_r, \mathbf{r}_t$ is given by
\[ E_{\text{dip}} = \frac{\mu_0}{4\pi} \frac{3(\mathbf{v}_i \cdot \mathbf{e}_j)(\mathbf{v}_j \cdot \mathbf{e}_i) - \mathbf{v}_i \cdot \mathbf{v}_j}{r_{ij}} , \]  
(A.3)

where \( r_{ij} = r_j - r_i, \) \( r_{ij} = ||r_{ij}||, \) and \( \mathbf{e}_i = r_{ij}/r_{ij}. \) For a continuous magnetization distribution \( \mathbf{M}(\mathbf{r}) = M_\text{S} \mathbf{m}(\mathbf{r}), \) total energy becomes, in tensor notation,

\[ E_{\text{dip}} = \frac{1}{2} \mu_0 M_\text{S}^2 \int \int m_a(r') \bar{f}_{ab}(r' - r) m_b(r) \, d^3r' \, d^3r \]

\[ = \frac{1}{2} \mu_0 M_\text{S}^2 \int \tilde{m}_a(k) \tilde{f}_{ab}(k) \tilde{m}_b(k) \frac{d^3k}{(2\pi)^3} , \]  
(A.4)

where \( a, b \) represent the spatial directions \( x, y, z; \) \( \tilde{m}_a(\mathbf{r}) \) is the Fourier transform of \( m_a(\mathbf{r}); \) and where we define

\[ \tilde{f}_{ab}(\mathbf{k}) = -\frac{A_{ab}(\mathbf{k})}{4\pi r^2} . \]  
(A.5)

The factor \( 1/2 \) is a double-counting correction. The functions \( A_{ab}^{(2)}(\mathbf{r}) \) are the second-order spherical polynomials

\[ A_{ab}^{(2)}(\mathbf{r}) = 3r_\alpha r_\beta - \delta_{ab} r_\alpha r_\beta \]  
(A.6)

(e.g. \( A_{ab}^{(2)}(\mathbf{r}) = 3x^2 - r^2). \) The Fourier transform of (A.5) is given by\(^1\)

\[ \tilde{f}_{ab}(\mathbf{k}) = \frac{A_{ab}^{(2)}(\mathbf{k})}{3k^2} . \]  
(A.7)

A.2. Magnetostatic energy: uniform mode

For a magnetization profile (A.1) that is homogeneous in \( z \) inside the film, we have

\[ E_{\text{dip}} = \frac{1}{2} \mu_0 M_\text{S}^2 \int \int m_a(r') \tilde{g}_{ab}(r' - r) m_b(r) \, d^3r' \, d^3r \]

\[ = \frac{1}{2} \mu_0 M_\text{S}^2 L \int \tilde{m}_a(\mathbf{k}) \tilde{g}_{ab}(\mathbf{k}) \tilde{m}_b(\mathbf{k}) \frac{d^2k}{2\pi} , \]  
(A.8)

where

\[ \tilde{g}_{ab}(x, y) = \frac{1}{L} \int \Pi^+(\frac{z}{L}) \tilde{f}_{ab}(x, y, z - \tilde{z}) \Pi^+(\frac{\tilde{z}}{L}) \, dz \, d\tilde{z} . \]  
(A.9)

By the convolution theorem,

\[ \tilde{g}_{ab}(k_x, k_y) = L \int_{-\infty}^{\infty} \tilde{f}_{ab}(k_x, k_y, k_z) \sin^2 k_z L \frac{dk_z}{2\pi} , \]  
(A.10)

where we have used the Fourier transform \( \Pi^+(kL) = e^{-kL/2} \sin(kL/2) \) with \( \sin(\phi)/\phi. \) We evaluate

\[ \tilde{g}_{uv}(k_x, k_y) = (1 - N_k) \frac{k_y k_v}{k_x} - \frac{1}{3} \delta_{uv} , \]  
(A.11a)

\[ \tilde{g}_{uc}(k_x, k_y) = 0 , \]  
(A.11b)

\(^1\) We use the nonunitary definition of the Fourier transform

\[ f(\mathbf{k}) = \int f(\mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}} \, d^nx, \]  
where \( n \) is the dimension of space. The inverse transform is given by \( f(\mathbf{x}) = (2\pi)^{-n} \int f(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}} \, d^nk. \) We use the result that, for a spherical polynomial \( A^{(n)}(\mathbf{r}) \) of order \( n, \) the Fourier transform of a function \( f(\mathbf{r}) = f_0(r) A^{(n)}(\mathbf{r}) \) is given by \( \tilde{f}(\mathbf{k}) = f_0(k) \tilde{A}^{(n)}(\mathbf{k}), \) where \( f_0(k) = (2\pi)^{n/2} \frac{n!}{(n-2)!} \frac{1}{k_n} f_0(r) r^{(n-2)/2} J_{n-2}(kr) \, dr \) with \( J_n(z) \) a Bessel function of the first kind.

---

**Figure A1.** Dispersion relation of BVMSWs in the uniform-mode approximation, for \( H_\text{f} = 80 \, \text{kA/m}, K = 3.5 \, \text{kJ/m}^3, L = 100 \, \mu \text{m}, \) and \( M_\text{S} = 110 \, \text{kA/m}. \) We neglect exchange \( A, \) assuming that \( k \) is much smaller than the inverse exchange length \( 1/k_x = \sqrt{\mu_0 M_\text{S}^2/(24)}. \) Notice that, in the magnetostatic regime, \( \omega \) (mostly) decreases in \( k, \) giving the spin waves a backward-propagating character. We define \( \vartheta \) as the polar angle of the wavevector \( \mathbf{k} = (k_x, k_y) = (k \cos \theta, k \sin \theta). \)

\[ \tilde{g}_{zz}(k_x, k_y) = N_k = \frac{1}{3}, \]  
(A.11c)

where \( u, \nu \) represent the in-plane coordinates \( x, y. \) The demagnetizing factor \( N_k \) is given by

\[ N_k = \frac{1 - e^{-kL}}{kL} . \]  
(A.12)

We define \( N_{k=0} = 1 \) (continuity); notice that \( N_k \rightarrow 0. \)

If we assume that the magnetization of the film is completely homogeneous \((\mathbf{k} = 0),\) we get

\[ E_{\text{dip}} = \frac{1}{2} \mu_0 M_\text{S}^2 V \left( -\frac{1}{3} m_1^2 + \frac{2}{3} m_2^2 \right) , \]

where \( V \) is the total film volume. Due to the constraint \( ||\mathbf{m}|| = 1, \) this effectively is a hard-axis anisotropy of strength \( \frac{1}{2} \mu_0 M_\text{S}^2 \) where \( \mathbf{k} \) is the easy axis. This confirms that the dipolar interaction favors in-plane magnetization, and gives the well-known condition \( K > \frac{1}{2} \mu_0 M_\text{S}^2 \) for perpendicular (out-of-plane) magnetization due to an intrinsic perpendicular anisotropy \( K \) in the absence of an applied field. As a second limiting case, let us consider a system where \( \mathbf{m}(x, y) \) depends only on \( x (k_y = 0) \) and where \( L \) is very large (thick film, \( L \gg |k_x|^{-1} \)). For a fixed \( k_x \neq 0, \) we get, in the limit \( L \rightarrow \infty, \) an effective hard-axis anisotropy of strength \( \frac{1}{2} \mu_0 M_\text{S}^2 \) where the hard axis is \( \mathbf{k}. \)

A.3. Linearization

Substituting \( E = E_{xx} + E_{anis} + E_H + E_{\text{dip}} \) and passing to Fourier space, (2) becomes

\[
\begin{pmatrix}
\frac{H_s}{M_\text{S}} + 2 \frac{H_s^2}{M_\text{S}^2} + (1 - N_k) \frac{k^2}{r^2} & -\frac{1}{m_0(\gamma) \omega} \\
\frac{H_s}{M_\text{S}} + 2 \frac{H_s^2}{M_\text{S}^2} + N_k & \frac{2 \alpha^2}{m_0(\gamma) \omega} + N_k
\end{pmatrix}
\begin{pmatrix}
\delta r \\
\delta z
\end{pmatrix} = 0.
\]
The positive solution for $\omega$ in the characteristic equation gives the dispersion relation of (3).

Figure A1 shows an example of the dispersion relation (3) for typical parameters. Notice that the dispersion relation has a cusp at the origin $k = 0$. With the exception of a small area right above and below the point $k = 0$, the frequency decreases with increasing $k$. This implies that the spin waves have a group velocity that is opposite to their wavevector $k$ (backward modes).

In our example, the backward dispersion behavior is seen even for the strictly perpendicular wavevectors $k \perp M$ (i.e. along the $y$-axis, $k_y = 0$). This behavior is related to the perpendicular crystalline anisotropy $K > 0$ considered. The positive bump in the dispersion relation seen for very small values of $k_y$ is an artifact of the uniform mode approximation (see figure 2(b)).

Appendix B. Distributional limits

This appendix provides some elementary results needed to carry out the small-$k$ expansion (21). If we set $(k_x, k_y) = (k \cos \vartheta, k \sin \vartheta)$, the operators $\hat{D}^x, \hat{D}^y, \hat{D}^z$, defined by (5), become

$$\hat{D}^x = k^2 + k_z^2 \sin^2 \vartheta,$$
 $$\hat{D}^y = k_z k_x \sin \vartheta,$$
 $$\hat{D}^z = k^2 + k_z^2 = \hat{s} - k^2 + k_z^2. \quad (B.1c)$$

Equation (21) is obtained by expanding the operators $k^2/(k^2 + k_z^2)$ and $k_z k_x/(k^2 + k_z^2)$ as a Taylor series in the parameter $k > 0$. In the following, $k_z$ may be substituted for $x$ and $k$ for $\varepsilon$.

We consider the expressions $\varepsilon^2/(\varepsilon^2 + \varepsilon^2)$ and $\varepsilon x/(\varepsilon^2 + \varepsilon^2)$ as functions of $\varepsilon$ and calculate derivatives with respect to the parameter $\varepsilon$ in the limit $\varepsilon \to 0^+$. Most of these limits can only be defined if we turn to generalized functions (distributions) of $x$. We obtain

$$\lim_{\varepsilon \to +0^+} \frac{\varepsilon^2}{x^2 + \varepsilon^2} = 0, \quad (B.2a)$$
$$\lim_{\varepsilon \to +0^+} \frac{\varepsilon x}{x^2 + \varepsilon^2} = 0; \quad (B.2b)$$
$$\lim_{\varepsilon \to +0^+} \frac{\partial}{\partial \varepsilon^0} \frac{\varepsilon^2}{x^2 + \varepsilon^2} = \pi \delta(x), \quad (B.3a)$$
$$\lim_{\varepsilon \to +0^+} \frac{\partial}{\partial \varepsilon^0} \frac{\varepsilon x}{x^2 + \varepsilon^2} = \frac{1}{x}, \quad (B.3b)$$

where $\delta(x)$ is the Dirac delta distribution;

$$\lim_{\varepsilon \to +0^+} \frac{\partial^2}{\partial \varepsilon^2} \frac{\varepsilon^2}{x^2 + \varepsilon^2} = \frac{2}{x^2}, \quad (B.4a)$$
$$\lim_{\varepsilon \to +0^+} \frac{\partial^2}{\partial \varepsilon^2} \frac{\varepsilon x}{x^2 + \varepsilon^2} = 2 \pi \delta'(x); \quad (B.4b)$$

and

$$\lim_{\varepsilon \to +0^+} \frac{\varepsilon^2}{x^2 + \varepsilon^2} = -3 \pi \delta''(x), \quad (B.5a)$$
$$\lim_{\varepsilon \to +0^+} \frac{\varepsilon x}{x^2 + \varepsilon^2} = -6 \frac{x}{x^2}. \quad (B.5b)$$

Expressions of the form $1/x^2$ are formally defined as distributional derivatives $\xi^{-\nu}$.

Appendix C. Perturbation theory

The generalized Hermitian eigenvalue problem

$$H\Psi = \omega Q \Psi, \quad (C.1)$$

where $H$ and $Q$ are Hermitian operators one of which is positive definite, can be cast as a problem of minimization of the functional

$$\omega = \Psi^\dagger H \Psi \quad (C.2)$$

under the constraint

$$\Psi^\dagger Q \Psi = 1. \quad (C.3)$$

We suppose that $H$ depends on a parameter $k$ and expand the solution $\Psi$ around $k = 0$. Equations (C.2) and (C.3) become

$$(\omega_0 + k \omega_1 + \ldots) = (\Psi_0 + k \Psi_1 + \ldots)^\dagger \cdot (H_0 + k H_1 + \ldots) (\Psi_0 + k \Psi_1 + \ldots) \quad (C.4)$$

and

$$(\Psi_0 + k \Psi_1 + \ldots)^\dagger Q (\Psi_0 + k \Psi_1 + \ldots) = 1. \quad (C.5)$$

Collecting like powers of $k$, we obtain

$$\omega_0 = \Psi_0^\dagger H_0 \Psi_0, \quad (C.6a)$$
$$\omega_1 = 2 \Psi_0^\dagger H_0 \Psi_1 + \Psi_0^\dagger H_1 \Psi_0, \quad (C.6b)$$
$$\omega_2 = 2 \Psi_0^\dagger H_0 \Psi_2 + \Psi_1^\dagger H_0 \Psi_1 + 2 \Psi_0^\dagger H_1 \Psi_1 + \Psi_0^\dagger H_2 \Psi_0, \quad (C.6c)$$
$$\omega_3 = 2 \Psi_0^\dagger H_0 \Psi_3 + 2 \Psi_1^\dagger H_0 \Psi_2 + 2 \Psi_2^\dagger H_1 \Psi_1 + 2 \Psi_0^\dagger H_3 \Psi_0 + \Psi_0^\dagger H_2 \Psi_2 + \Psi_0^\dagger H_3 \Psi_0, \quad (C.6d)$$
$$\omega_4 = 2 \Psi_0^\dagger H_0 \Psi_4 + 2 \Psi_1^\dagger H_0 \Psi_3 + 2 \Psi_2^\dagger H_0 \Psi_2 + 2 \Psi_3^\dagger H_1 \Psi_1 + 2 \Psi_4^\dagger H_1 \Psi_1 + 2 \Psi_0^\dagger H_4 \Psi_0 + \Psi_0^\dagger H_3 \Psi_2 + \Psi_0^\dagger H_4 \Psi_0 \quad (C.6e)$$

and

$$1 = \Psi_0^\dagger Q \Psi_0, \quad (C.7a)$$
$$0 = 2 \Psi_0^\dagger Q \Psi_1, \quad (C.7b)$$
$$0 = 2 \Psi_0^\dagger Q \Psi_2 + \Psi_1^\dagger Q \Psi_1, \quad (C.7c)$$
where we assume that all terms $\Psi_q^H H \Psi_B$ and $\Psi_A H \Psi_B$ are real. To obtain the expansion $\Psi_0 + k \Psi_1 + k^2 \Psi_2 + \ldots$ of the eigenfunction $n = 1$ with the lowest eigenvalue $\omega$, we sequentially minimize the functionals $\omega_0, \omega_1, \omega_2, \ldots$ under the constraints (C.7a).

In the following, we assume that the eigenvectors $\Psi$ are written in the form (16), and we assume that $Q$ and $H_0$ are given by

$$\Psi_0^A Q \Psi_B = \psi_A \psi_B - \phi_A \phi_B,$$

$$\Psi_0^A H_0 \Psi_B = \mu_0 |\gamma| \langle a | \psi_A \psi_B + \phi_A \phi_B \rangle,$$

The positive-$\omega$ solutions of the zeroth-order eigenvalue equation $H_0 \Psi_0 = \omega_0 Q \Psi_0$ have $\phi_0 = 0$ and $\psi_0$ arbitrary (provided $\Psi_0 = 1$). We further assume that the $H_i$ for $i \geq 1$ are of the form

$$H_i = \left( \hat{A}_i, \hat{C}_i, \hat{B}_i \right),$$

where $\hat{C}$ is Hermitian. We have

$$\Psi_i^A H_i \Psi_B = \psi_i \hat{X}_i \psi_B + \phi_i \hat{X}_i \phi_B + \psi_i \hat{Y}_i \phi_B + \phi_i \hat{Y}_i \psi_B$$

and

$$\hat{X}_i = \frac{\mu_0 |\gamma|}{2ab} (a^2 \hat{A}_i + b^2 \hat{B}_i),$$

$$\hat{Y}_i = \frac{\mu_0 |\gamma|}{2ab} (-a^2 \hat{A}_i + b^2 \hat{B}_i - 2iab \hat{C}_i).$$

The functional $w_0$ becomes

$$w_0 = \mu_0 |\gamma| \langle ab, \rangle,$$

$$w_1 = \psi_0 \hat{X}_1 \psi_0,$$

$$w_2 = 2 \omega_0 \phi_0^2 + 2 \psi_0 \hat{X}_1 \psi_0 + 2 \psi_0 \hat{Y}_1 \phi_0 + \psi_0 \hat{X}_2 \psi_0,$$

$$w_3 = 4 \omega_0 \phi_1 \phi_2 + 2 \psi_0 \hat{Y}_1 \phi_2$$

$$+ \psi_1 \hat{X}_1 \psi_1 + \phi_1 \hat{X}_1 \phi_1 + 2 \psi_1 \hat{Y}_1 \phi_1 + 2 \psi_0 \hat{X}_2 \psi_2 + 2 \psi_0 \hat{Y}_2 \phi_1$$

$$+ \psi_0 \hat{X}_3 \psi_0 + \psi_0 \hat{X}_3 \phi_0,$$

$$w_4 = 4 \omega_0 \phi_1 \phi_2 + 2 \omega_0 \phi_0^2$$

$$+ 2 \psi_0 \hat{X}_1 \psi_1 + 2 \psi_0 \hat{Y}_1 \phi_1 + 2 \psi_1 \hat{X}_2 \psi_2 + 2 \phi_1 \hat{X}_1 \psi_0$$

$$+ 2 \psi_1 \hat{Y}_1 \phi_2 + 2 \phi_1 \hat{Y}_1 \psi_2$$

$$+ 2 \psi_0 \hat{X}_2 \psi_2 + 2 \psi_0 \hat{Y}_2 \phi_2$$

$$+ \psi_1 \hat{X}_2 \psi_2 + \psi_1 \hat{X}_2 \phi_1 + 2 \phi_1 \hat{Y}_2 \psi_1$$

$$+ 2 \psi_0 \hat{X}_3 \psi_1 + 2 \psi_0 \hat{Y}_3 \phi_1 + \psi_0 \hat{X}_3 \psi_0,$$

where we have substituted constraint (C.7b) into (C.6b) and so on. The constraints (C.7a) become

$$1 = \psi_0^2,$$

$$0 = 2 \psi_0 \psi_1,$$

$$0 = 2 \psi_0 \psi_2 + \psi_1^2 - \phi_1^2,$$

$$0 = 2 \psi_0 \psi_3 + 2 \psi_1 \psi_2 - 2 \phi_1 \phi_2.$$
Eliminating \( \phi_2 \), we obtain

\[
\omega_4 = - \frac{1}{2 \alpha_0} \left( (\hat{X}_1 + \omega_1) \phi_1 + \hat{Y}_1^T \psi_1 + \hat{Y}_2^T \psi_0 \right) \left( (X_2 - \lambda) \psi_1 + \phi_1 (X_2 + \lambda) \phi_1 + 2 \psi_1 \hat{Y}_2 \phi_1 \right) + 2 \psi_0 \hat{X}_1 \psi_1 + 2 \psi_0 \hat{Y}_2 \phi_1 + \psi_0 \hat{X}_2 \psi_0,
\]

which functional should be minimized treating (C.19) as an additional constraint.

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