On the Essential Spectrum of Two-Dimensional Pauli Operators with Repulsive Potentials

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Abstract. We investigate the spectrum of the two-dimensional Pauli operator, describing a spin-$\frac{1}{2}$ particle in a magnetic field $B$, with a negative scalar potential $V$, such that $|V|$ grows at infinity. In particular, we obtain criteria for discrete and dense pure-point spectrum.

1. Introduction

For modelling the kinetic energy of a non-relativistic spin-$\frac{1}{2}$ particle in the plane, under a magnetic field $B$ in the perpendicular direction to the plane, one uses the two-dimensional Pauli operator

$$H_A := \left[\sigma \cdot (-i\nabla - A)\right]^2 = (-i\nabla - A)^2 - \sigma_3 B \quad \text{on } L^2(\mathbb{R}^2, \mathbb{C}^2),$$

where $A$ is a vector potential associated to $B$, i.e. $B = \text{curl} \ A := \partial_1 A_2 - \partial_2 A_1$. Here, $\sigma = (\sigma_1, \sigma_2)$ and $\sigma_3$ are the Pauli-matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

To study the behaviour of such spin-$\frac{1}{2}$ particles (e.g. electrons) in presence of an additional electric potential $V$, we investigate the spectrum of the operator

$$H := H_A + V = \left[\sigma \cdot (-i\nabla - A)\right]^2 + V \quad \text{on } L^2(\mathbb{R}^2, \mathbb{C}^2).$$

If $V$ is non-negative or has a certain decay at infinity (e.g. potentials with Coulomb singularities), spectral properties of the magnetic Schrödinger operator $(-i\nabla - A)^2 + V$, as well as of the Pauli operator $H_A + V$ (in dimension $d = 2$ or $3$), have been widely studied over the last decades (see, e.g. [2,4] or [5] for a latest overview). In this article, instead, we want to point out some interesting features of the spectrum of $H$ for potentials $V$, tending to $-\infty$ as
Since such scalar potentials result in an operator $H$, unbounded from below, it is necessary to discuss questions on self-adjointness related to $H$. We emphasize that, since we also consider unbounded magnetic fields $B$, the self-adjointness of $H$ cannot simply be reduced to that of the magnetic Schrödinger operator.

One motivation for the following considerations is an observation made in [10] for the two-dimensional massless magnetic Dirac operator, coupled to an electric potential $V$: there, an accumulation process of spectral points has been noticed, depending on the ratio $|V^2/B|$ at infinity. This phenomenon can be ascribed to the non-confining effect of $V$ in the case of the Dirac operator. Regarding the Pauli operator, the influence of an additional scalar potential $V$ on the spectrum $\sigma(H)$ depends crucially on the sign of $V$. For simplicity we outline this dependence for a constant magnetic field $B(x) = B_0$: a positive potential $V$, growing at infinity, always leads to discrete spectrum of the operator $H$, independently of $B_0$ (see, e.g. [9]). Such potential wells only enhance a localization effect caused by $B_0$, generating eigenvalues and spectral gaps (proportional to $B_0$). If we instead consider negative potentials $V$, the situation is quite different since the particle lowers its energy by staying in regions where $V$ is small. A potential $V$, converging to $-\infty$ as $|x| \to \infty$, has therefore a delocalizing effect, i.e. the particle tends to escape any compact region of the plane. Our results show that such negative potentials $V$ (describing for example constant radial fields) counteract the localizing effect of “hard” magnetic fields $B$, as they close spectral gaps, induced by $B$:

- If $V$ converges to $-\infty$, but remains small compared to $B$, the spectrum $\sigma(H)$ is discrete, i.e. it consists of eigenvalues of finite multiplicity.
- If $V$ is comparable to $B$, more precisely $|V| \approx 2B$ at infinity, points in the essential spectrum occur.
- If $V$ overtakes $B$, more explicitly $|V/B| \to \infty$ as $|x| \to \infty$ (at least along a path), the spectrum $\sigma(H)$ covers the whole real line.\(^1\)

The precise statements of the claims above are contained in Theorems 1–4 in Sect. 3. We want to remark that the case $|V/B| \to \infty$ as $|x| \to \infty$ is treated by Theorems 3 and 4. Unlike Theorem 4, which is only valid for constant magnetic fields, Theorem 3 covers also non-constant fields $B$, but requires stronger constraints on the growth of $V$. Thus, the important case $B = B_0$ is addressed by two theorems. The ideas of the proofs of Theorems 1–3 originate from those used to prove the results in [10]. However, since we work with a second-order operator, the proofs are technically more laborious. Theorem 4 is based on a further construction of a Weyl sequence, obtained by treating $V$ locally as a potential of a constant electric field. This is a refined approach compared to the method used in the proof of Theorem 3.

The organization of this article is as follows: In the next section some known facts about the Pauli operator are recapitulated. We present our precise results in Sect. 3, provided with some remarks and important applications. In

\(^1\) One may compare this statement with the results about $H_A$ for decaying magnetic fields in [11].
Sect. 4 we give the proof of Theorem 1. The proofs of Theorems 2 and 3 are contained in Sect. 5, while the proof of Theorem 4 can be found in the last section. In the appendix, attached to the main text, we give a proof of the essential self-adjointness of the Pauli operator.

2. Basic Properties of the Pauli Operator

In this section, we point out some basic facts about the Pauli operator and the massless Dirac operator $D_A$, whose square equals $H_A$. For a vector potential $A \in C^1(\mathbb{R}^2, \mathbb{R}^2)$, generating the field $B = \text{curl} A \in C(\mathbb{R}^2, \mathbb{R})$, the Hamiltonian $D_A$ is defined as the closure of the operator

$$\sigma \cdot (-i \nabla - A) = \begin{pmatrix} 0 & d^* \\ d & 0 \end{pmatrix} \quad \text{on } C_0^\infty(\mathbb{R}^2, \mathbb{C}^2),$$

which is essentially self-adjoint on the given core (see [3]). Especially, $d$ and $d^*$ can be seen as closed operators, i.e. we use the notation $d = -i \partial_1 - A_1 + i (-i \partial_2 - A_2)$ on $C_0^\infty(\mathbb{R}^2, \mathbb{C})$ and analogously for $d^*$. One observes that $d, d^*$ satisfy the commutation relation

$$[d, d^*] \varphi := (dd^* - d^*d) \varphi = 2B \varphi \quad \text{for } \varphi \in C_0^\infty(\mathbb{R}^2, \mathbb{C}).$$

We can write $H_A = D_A^2 = \begin{pmatrix} d^*d & 0 \\ 0 & dd^* \end{pmatrix} \quad \text{on } C_0^\infty(\mathbb{R}^2, \mathbb{C}^2)$ and consider $H_A$ as a self-adjoint operator on $\{ \psi \in \mathcal{D}(D_A) \mid D_A \psi \in \mathcal{D}(D_A) \}$, given by the Friedrichs extension. The components $dd^*$ and $d^*d$ of $H_A$ are unitarily equivalent on the orthogonal complement of $\ker(H_A) = \ker(D_A)$. To verify this we note first that due to the matrix structure of $D_A$, we have

$$\text{sgn}(D_A) := \begin{vmatrix} D_A \\ |D_A| \end{vmatrix} = \begin{pmatrix} 0 & s^* \\ s & 0 \end{pmatrix}$$

on $\ker(D_A) = \ker(d) \perp \ker(d^*) \perp$. Since $\text{sgn}(D_A)^2 = \text{Id}$ on $\ker(D_A) \perp$ the maps

$$s: \ker(d) \perp \rightarrow \ker(d^*) \perp, \quad s^*: \ker(d^*) \perp \rightarrow \ker(d) \perp$$

are unitary and conjugated to each other. By the operator identity $H_A = D_A^2 = \text{sgn}(D_A) D_A^2 \text{sgn}(D_A)$ one concludes that

$$\begin{pmatrix} d^*d & 0 \\ 0 & dd^* \end{pmatrix} \varphi = \begin{pmatrix} s^*dd^*s & 0 \\ 0 & sd^*ds^* \end{pmatrix} \varphi,$$

for any $\varphi = (\varphi_1, \varphi_2)^T$ with $\varphi_1 \in \mathcal{D}(d^*d) \cap \ker(d) \perp$ and $\varphi_2 \in \mathcal{D}(dd^*) \cap \ker(d^*) \perp$. Hence, on $\ker(d) \perp$ the operator $d^*d$ is unitarily equivalent to $dd^*$ (considered as an operator on $\ker(d^*) \perp$). In this article we denote the orthogonal projection...
on $\ker(D_A)$ by $P_0$ and the orthogonal projections on $\ker(d)$, $\ker(d^*)$ by $\pi, \pi_*$. In addition, we set

$$P_{0}^\perp := 1 - P_0, \quad \pi^\perp := 1 - \pi, \quad \pi_*^\perp := 1 - \pi_*.$$  

To define our full Hamiltonian, let $A \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ and $B, V \in C(\mathbb{R}^2, \mathbb{R})$ be such that $B = \text{curl} A$, then $H$ is given by

$$H\varphi = \left[ D_A^2 + V \right] \varphi = \left[ ( -i \nabla - A )^2 - \sigma_3 B + V \right] \varphi \quad \text{for} \quad \varphi \in C_0^\infty(\mathbb{R}^2, \mathbb{C}^2).$$

In general, the closure of this densely defined operator is not self-adjoint without any restriction on the growth rate of $V$ at infinity. However, there are conditions, very similar to those for the classical Schrödinger operator, ensuring essential self-adjointness.

**Proposition 1.** Let $B, V \in C^1(\mathbb{R}^2, \mathbb{R})$ and $A \in C^2(\mathbb{R}^2, \mathbb{R}^2)$ with $B = \text{curl} A$. In addition, assume that $V$ fulfills the lower bound

$$V(x) \geq -c|x|^2 + d, \quad x \in \mathbb{R}^2, \quad (7)$$

for some constants $c > 0, d \in \mathbb{R}$. Then, $H$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2, \mathbb{C}^2)$.

In Appendix A we recap an argument, already given in [8], to prove this proposition. We added a tightened proof for our relaxed regularity conditions because of completeness.

**Remark 1.** Following the lines of the proof given in Appendix A, we see that the regularity condition on $B, V$ can be relaxed to $B, V \in C^\alpha_{loc}(\mathbb{R}^2, \mathbb{R})$, i.e. both only need to be locally (uniformly) $\alpha$-Hölder continuous. By a perturbation argument one can also see that it suffices to assume that $V, B$ are $C^\alpha_{loc}$ outside some compact set $K \subset \mathbb{R}^2$, while inside $K$ they only need to be continuous.

**Remark 2.** The self-adjoint operator given by Proposition 1 is locally compact, i.e. for any characteristic function $\chi_{B_R(0)}$ on the ball $B_R(0)$ with radius $R$, the operator $\chi_{B_R(0)}(H - i)^{-1}$ is compact.

**Remark 3.** Considering the case $V = 0$, we obtain that $H_A, dd^*$ and $d^*d$ are essentially self-adjoint on $C_0^\infty(\mathbb{R}^2, \mathbb{C}^2)$, respectively, on $C_0^\infty(\mathbb{R}^2, \mathbb{C})$.

Note that (7) is the same lower bound on $V$ as one needs for the (magnetic) Schrödinger operator to ensure the essential self-adjointness, whereas no restriction on the growth of $B$ is necessary. The regularity conditions on $V$ and $A$ are quite strong compared to those of the magnetic Schrödinger operator (see [12]). The reason is that due to the lack of a diamagnetic inequality for $H_A$, one uses a direct argument, which requires more regularity on the potentials $V$ and $A$. The interesting question remains: Could one relax these conditions for the Pauli operator as in the case of the magnetic Schrödinger operator?
3. Main Results

In this section, we assume that $B$, $V$ and $A$ fulfill the conditions of Proposition 1. It is easy to see that in the following results $B, V \in C^1(\mathbb{R}^2, \mathbb{R})$ can be relaxed to hold only outside some compact set $K \subset \mathbb{R}^2$ as in Remark 1.

**Theorem 1.** Assume that

$$V(x) \to -\infty \quad \text{as} \quad |x| \to \infty, \quad (8)$$

$$\left| \frac{\nabla V(x)}{V(x)} \right| \to 0 \quad \text{as} \quad |x| \to \infty, \quad (9)$$

$$\limsup_{|x| \to \infty} \left| \frac{V(x)}{2B(x)} \right| < 1. \quad (10)$$

Then $\sigma_{\text{ess}}(H) = \emptyset$, i.e. $H$ has purely discrete spectrum.

Condition (9) is a restriction of the growth rate of $V$ and rather of technical necessity. The interplay between $B$ and $V$ (as mentioned in the introduction) is described by condition (10). Thus, it is worthwhile to investigate further the dependence of $\sigma(H)$ on this quotient:

One can easily observe that if the quotient of (10) surpasses the constant 1, the spectrum of $H$ changes its character. To see this pick $\Omega \in \ker(dd^*d)$, then

$$(dd^* + V)\Omega = (d^*d + 2B + V)\Omega \approx 0,$$

if $2B \approx -V$. Therefore, if $\ker(d^*d)$ contains enough functions (which is the case for fields $B$ bounded form below by some positive constant), we obtain points in the essential spectrum of $H$, since a Weyl sequence can be constructed out of $\ker(d^*d)$. One can even show that the condition $2B \approx -V$ (at infinity) does not need to hold globally for obtaining $\sigma_{\text{ess}}(H) \neq \emptyset$. We demonstrate this for a certain class of fields $B$ and potentials $V$.

**Definition 1.** A function $f : \mathbb{R}^2 \to \mathbb{R}$ varies with rate $\nu \in [0, 1]$ on a set $S \subset \mathbb{R}^2$, if there is a constant $C > 0$ such that for any $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$, with $\alpha(x) = o(|x|^{\nu})$ as $|x| \to \infty$, $f$ satisfies

$$|f(x + \alpha(x))| \leq C|f(x)| \quad \text{for all} \quad x \in S.$$  

Note that functions of the form $f_1(x) = c|x|^s$, $f_2(x) = c|x_1|^s$, with $s \in \mathbb{R}$, vary with any rate $\nu \in [0, 1]$ on $\mathbb{R}^2 \setminus B_1(0)$, respectively on $\mathbb{R}^2 \setminus [-1, 1] \times \mathbb{R}$.

**Theorem 2.** Assume that there is a sequence $(x_n)_{n \in \mathbb{N}}$, with $|x_n| \to \infty$ as $n \to \infty$, and constants $k \in \mathbb{N}, \varepsilon \in (0, 1)$ such that $|\nabla V|, |\nabla B|$ vary with rate 0 on $(x_n)_{n \in \mathbb{N}}$, as well as

$$V(x_n) \to -\infty, \quad (11)$$

$$\frac{|\nabla B(x_n)|^2}{|B(x_n)|^{1-\varepsilon}}, \frac{|\nabla V(x_n)|^2}{|V(x_n)|^{1-\varepsilon}} \to 0, \quad (12)$$

$$V(x_n) + 2k|B(x_n)| \to 0, \quad (13)$$

as $n \to \infty$. Then $0 \in \sigma_{\text{ess}}(H)$. 

Let us now consider the case $V \gg B$ at infinity. The next two theorems state that the accumulation of eigenvalues intensifies, creating more points in the essential spectrum and closing spectral gaps.

**Theorem 3.** Assume that there is a continuous path $\gamma: \mathbb{R}^+ \to \mathbb{R}^2$, with $|\gamma(t)| \to \infty$ as $t \to \infty$, and constants $\epsilon > 0$, $\nu \in [0, 1]$ such that $|\nabla V|, |\nabla B|$ vary with rate $\nu$ on $\text{Im}(\gamma)$, as well as

$$
\frac{V(\gamma(t))}{2|B(\gamma(t))|} \to -\infty,
$$

(14)

$$
\left( \frac{|\nabla V(\gamma(t))|}{|V(\gamma(t))|} + \frac{|\nabla B(\gamma(t))|}{|B(\gamma(t))|} \right) \left( \frac{|V(\gamma(t))|^3}{B^2(\gamma(t))} \right)^{\frac{1}{2+\epsilon}} \to 0,
$$

(15)

$$
\frac{1}{|\gamma(t)|^{2\nu}} \left( \frac{|V(\gamma(t))|}{B^2(\gamma(t))} \right)^{1+\epsilon} \to 0,
$$

(16)

as $t \to \infty$. In addition, suppose that, for all $t \in (0, \infty)$, the inequality

$$
B_0 \leq |B(\gamma(t))| \leq \alpha \exp \left( \kappa \frac{|V(\gamma(t))|}{B(\gamma(t))} \right)
$$

(17)

holds with constants $\alpha, \kappa, B_0 > 0$. Then $\sigma_{\text{ess}}(H) = \mathbb{R}$.

For our main application, potentials of power-like growth (see discussion after the next theorem), condition (16) imposes unsatisfying restrictions on the growth rate of $V/B$. At least in the case of constant magnetic fields it can be weakened.

**Theorem 4.** Let $B = B_0 > 0$ and $V \in C^2(\mathbb{R}^2, \mathbb{R})$. Assume that there is a continuous path $\gamma: \mathbb{R}^+ \to \mathbb{R}^2$, with $|\gamma(t)| \to \infty$ as $t \to \infty$, and constants $\epsilon > 0$, $\nu \in [0, 1]$ such that the matrix norm of the Hessian matrix $\|\text{Hess}(V)\|_2 : \mathbb{R}^2 \to \mathbb{R}$ varies with rate $\nu$ on $\text{Im}(\gamma)$, as well as

$$
V(\gamma(t)) \to -\infty,
$$

(18)

$$
\|\text{Hess}(V)\|_2(\gamma(t))|V(\gamma(t))|^{1+\epsilon} \to 0,
$$

(19)

$$
\frac{1}{|\gamma(t)|^{2\nu}} |V(\gamma(t))|^{1+\epsilon} \to 0,
$$

(20)

as $t \to \infty$. In addition, assume

$$
\limsup_{t \to \infty} \frac{|\nabla V(\gamma(t))|^2}{|V(\gamma(t))|} < (2B_0)^2.
$$

(21)

Then $\sigma_{\text{ess}}(H) = \mathbb{R}$.

**Remark 4.** Note that a well-known, basic example for this last theorem is a constant electric field $E_0$ in the $x_1$-direction with the corresponding potential $V(x) = E_0 x_1$.

**Remark 5.** Results similar to that of Theorems 1–4 can be obtained for the magnetic Schrödinger operator with scalar potentials $V$ using the same techniques as in the proofs of Theorems 1–4.
Finally, we want to discuss some implications of our results, in particular with respect to spherically symmetric fields $B$ and potentials $V$, i.e. $B(x) = b(|x|)$, $V(x) = v(|x|)$ for $x \in \mathbb{R}$. Using the rotational gauge

$$A(x) := \frac{A(r)}{r} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \quad A(r) = \frac{1}{r} \int_0^r b(s)sd,$$

with $r = |x|$, we decompose $H$ in a direct sum of operators on the half-line. More explicitly, there is a unitary map

$$U : L^2(\mathbb{R}^2, \mathbb{C}^2) \to \bigoplus_{j \in \mathbb{Z}} L^2(\mathbb{R}^+, \mathbb{C}^2; dr)$$

such that $UHU^* = \bigoplus_{j \in \mathbb{Z}} h_j$, with

$$h_j := \begin{pmatrix} -\partial_r^2 + \frac{j^2}{r^2} + \frac{1}{2} & 0 \\ 0 & -\partial_r^2 + \frac{(j+1)^2}{r^2} - \frac{1}{2} \end{pmatrix} + \frac{1}{r} A(r) + \frac{m_j}{r} A(r) + \sigma_3 A'(r) + v(r)$$

on $L^2(\mathbb{R}^+, \mathbb{C}^2; dr)$, where $m_j = j + \frac{1}{2}$ (see, e.g. [13]). It is easy to verify that if

$$\liminf_{r \to \infty} b(r) > 0,$$

$$A'(r)/A^2(r) \to 0 \quad \text{as } r \to \infty,$$

$$\limsup_{r \to \infty} |v(r)|/A^2(r) < 1,$$

then $h_j$ has purely discrete spectrum for every $j \in \mathbb{Z}$. As a consequence, one can use the relations

$$\sigma_\#(H) = \bigcup_{j \in \mathbb{Z}} \sigma_\#(h_j), \quad \# \in \{\text{ac, sc, pp}\},$$

to conclude that $\sigma(H) = \sigma_{\text{pp}}(H), \sigma_{\text{ac}}(H) = \sigma_{\text{sc}}(H) = \emptyset$ if (22)–(24) are fulfilled. To get more information about $\sigma(H) = \sigma_{\text{pp}}(H)$ we employ Theorems 1–4 and obtain:

**Corollary 1.** Let $b(r) = b_0 r^s, v(r) = v_0 r^t$ with $v_0 < 0 < b_0$ and exponents $0 \leq s, 0 \leq t \leq 2$. Then

a) $\sigma(H)$ is purely discrete if $0 < t < s$ or $0 < t = s$ and $|v_0| < 2b_0$.

b) $0 \in \sigma_{\text{ess}}(H)$ if $0 < t = s$ and $|v_0| = 2kB_0$ for some $k \in \mathbb{N}$.

c) $\sigma(H) = \mathbb{R}$ is dense pure point if $3t < 3s < 2(s + 1)$.

d) $\sigma(H) = \mathbb{R}$ is dense pure point if $s = 0$ and $0 < t < 1$.

The origins of the strong restrictions on $s, t$ in c), d) can easily be tracked back to conditions (15) and (16) of Theorem 3, and (19) of Theorem 4. Unfortunately, even in the case of a constant magnetic field ($s = 0$), we cannot cover the full range of potentials ($0 < t \leq 2$) for which one might expect $\sigma(H) = \mathbb{R}$.

4. Proof of Theorem 1

Note that the assumptions imply that either $B(x) \to \infty$ or $B(x) \to -\infty$. It suffices to consider the case $B(x) \to \infty$ as $|x| \to \infty$, since otherwise we only have to interchange the roles of $d$ and $d^*$ in the proof. By modifying $B$ and $V$...
on a compact set and comparing the corresponding resolvents, we may assume that $B, V$ satisfy

\[ V(x) \leq -1/\delta, \quad (25) \]
\[ |\nabla V(x)| \leq |\delta V(x)|, \quad (26) \]
\[ |V(x)| \leq 2(1 - \eta)B(x), \quad (27) \]

where $\delta \in (0, \frac{1}{4})$ is fixed, but can be chosen arbitrarily small, and $\eta \in (0, 1)$ is a fixed (\(\delta\)-independent) constant (c.f. [10] Appendix B).

Using the commutator relation (3) we see that

\[ dd^* \geq 2B \geq (1 - \eta)^{-1}|V| \geq (1 - \eta)^{-1} \delta^{-1} \quad (28) \]

on $C_0^\infty(\mathbb{R}^2, \mathbb{C})$ and therefore on $D(dd^*)$. Since $dd^*$ and $d^*d$ are isospectral away from 0, we obtain a spectral gap $(0, \beta) \subset \rho(H_A)$, with $\beta = (1 - \eta)^{-1} \delta^{-1}$. Thus, 0 can be regarded as an isolated point of the spectrum, which is used in the following commutator estimates.

**Lemma 1.** Let $V \in C^1(\mathbb{R}^2, \mathbb{R})$, $B \in C(\mathbb{R}^2, \mathbb{R})$ and $A \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ with $B = \text{curl } A$. Assume further that the conditions (25)–(27) are fulfilled for $\delta \in (0, \frac{1}{4})$ and $\eta \in (0, 1)$. Then:

a) The operators $[P_{0}^\perp, V^{-1}] V$ and $V [P_{0}^\perp, V^{-1}]$ are well-defined on the core $C_0^\infty(\mathbb{R}^2, \mathbb{C}^2)$ and extend to bounded operators on $L^2(\mathbb{R}^2, \mathbb{C}^2)$ with

\[ \|V [P_{0}^\perp, V^{-1}] \|, \| [P_{0}^\perp, V^{-1}] V \| \leq 4\delta^{\frac{3}{2}}. \]

The same holds true if we replace $P_{0}^\perp$ above by $P_0$.

b) $P_0 D(V), P_{0}^\perp D(V) \subset D(V)$.

**Lemma 2.** Let $V \in C^1(\mathbb{R}^2, \mathbb{R})$, $B \in C(\mathbb{R}^2, \mathbb{R})$ and $A \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ with $B = \text{curl } A$. Assume further that the conditions (25)–(27) are fulfilled for $\delta \in (0, \frac{1}{4})$ and $\eta \in (0, 1)$. Then $[\text{sgn}(D_A)P_{0}^\perp, V^{-1}]$ maps $L^2(\mathbb{R}^2, \mathbb{C}^2)$ in $D(V)$ and

\[ \|V [\text{sgn}(D_A)P_{0}^\perp, V^{-1}] \| \leq 4\delta^{\frac{3}{2}}. \quad (29) \]

The proofs of those commutator estimates can be found in [10]. Since $D_A$ is a first-order operator, it is much more convenient to commute $V$ with functions of $D_A$ instead of with functions of $H_A$. For proving Theorem 1, it suffices to find a constant $c > 0$ such that

\[ \|H\varphi\| \geq c\|V\varphi\|, \quad \varphi \in C_0^\infty(\mathbb{R}^2, \mathbb{C}^2) \quad (30) \]

holds (see Lemma 4 in the appendix).
Proof of Theorem 1. Let \( \varphi \in C_0^\infty(\mathbb{R}^2, \mathbb{C}) \). By Lemma 1 we can split \( \|H\varphi\| \) as
\[
\|H\varphi\|^2 = \|(H_A + V)(P_0 + P_0^\perp)\varphi\|^2
= \|(VP_0 + (H_A + V)P_0^\perp)\varphi\|^2
= \|(H_A + V)P_0^\perp\varphi\|^2 + 2\text{Re}\langle(H_A + V)P_0^\perp\varphi, VP_0\varphi\rangle + \|VP_0\varphi\|^2
= \|(H_A + V)P_0^\perp\varphi\|^2 - \delta \|VP_0^\perp\varphi\|^2
+ 2\text{Re}\langle VP_0\varphi, H_A P_0^\perp\varphi\rangle + \|V\varphi\|^2 - (1 - \delta) \|VP_0^\perp\varphi\|^2.
\]
For the cross-term, condition (26) yields
\[
\|\langle VP_0\varphi, H_A P_0^\perp\varphi\rangle\| = \|\langle D_A VP_0\varphi, D_A P_0^\perp\varphi\rangle\|
\leq \frac{1}{2}\delta^{3/2}\|(-i\sigma \nabla V)VP_0\varphi\|^2 + \frac{1}{2}\delta^{3/2}\|D_A P_0^\perp\varphi\|^2
\leq \frac{1}{2}\delta^{3/2}\|VP_0\varphi\|^2 + \frac{1}{2}\delta^{3/2}\|H_A P_0^\perp\varphi\|^2
\leq \frac{1}{4}\delta^{3/2}\|H_A P_0^\perp\varphi\|^2 + \frac{1}{4}\delta^{3/2}\|V\varphi\|^2 + 2\|VP_0\varphi\|^2.
\]
By Lemma 1 a) we have
\[
\|VP_0^\perp\varphi\|, \|VP_0\varphi\| \leq (1 + 4\delta^{3/2})\|V\varphi\|,
\]
and therefore
\[
\|V\varphi\|^2 - (1 - \delta) \|VP_0^\perp\varphi\|^2 - \frac{1}{4}\delta^{3/2}\|V\varphi\|^2 - \frac{1}{2}\delta^{3/2}\|VP_0\varphi\|^2
\geq (\delta - 14\delta^{3/2})\|V\varphi\|^2. \tag{32}
\]
Because
\[
\|(H_A + V)P_0^\perp\varphi\|^2 - \delta^{3/2}\|H_A P_0^\perp\varphi\|^2 - \delta\|VP_0\varphi\|^2
\geq (1 - \varepsilon - \delta^{3/2})\|H_A P_0^\perp\varphi\|^2 + (1 - \varepsilon^{-1} - \delta)\|VP_0^\perp\varphi\|^2
\]
for any \( \varepsilon \in (0, 1) \), it suffices to show, in view of (31) and (32), that
\[
\langle H_A P_0^\perp\varphi, H_A P_0^\perp\varphi\rangle + \frac{1 - \varepsilon^{-1} - \delta}{1 - \varepsilon - \delta^{3/2}}\langle VP_0^\perp\varphi, VP_0^\perp\varphi\rangle \geq 0 \tag{33}
\]
for \( \delta > 0 \) small enough and some \( \varepsilon \in (0, 1) \). We choose \( \varepsilon = 1 - \delta^{1/2} \), then
\[
-\frac{1 - \varepsilon^{-1} - \delta}{1 - \varepsilon - \delta^{3/2}} = \frac{1}{1 - \delta} \left( \frac{1}{1 - \delta^{1/2}} + \delta^{1/2} \right) =: c_\delta > 0.
\]
Since \( dd^* \geq 2B \), and therefore \( \text{ker}(d^*) = \{0\} \), we have
\[
P_0^\perp = \begin{pmatrix} \pi^\perp & 0 \\ 0 & \pi^\perp \end{pmatrix} = \begin{pmatrix} \pi^\perp & 0 \\ 0 & 1 \end{pmatrix}.
\]
Setting \( \varphi = (\varphi_1, \varphi_2)^T \), one can rewrite (33) as
\[
\|H_A P_0^\perp\varphi\|^2 - c_\delta \|VP_0^\perp\varphi\|^2
= \|d^*d\pi^\perp\varphi_1\|^2 - c_\delta \|V\pi^\perp\varphi_1\|^2 + \|dd^*\varphi_2\|^2 - c_\delta \|V\varphi_2\|^2.
\]
Using the isometries $s, s^*$ given in (5), relation (6), and the estimate (28), one obtains
\[
\|dd^*\varphi_2\|^2 - c_\delta \|V\varphi_2\|^2
= \langle d^*\varphi_2, d^*dd^*\varphi_2 \rangle - c_\delta \langle \sqrt{-V}\varphi_2, |V|\sqrt{-V}\varphi_2 \rangle
= \langle sd^*\varphi_2, dd^*sd^*\varphi_2 \rangle - c_\delta \langle \sqrt{-V}\varphi_2, |V|\sqrt{-V}\varphi_2 \rangle
\geq \langle sd^*\varphi_2, 2Bsd^*\varphi_2 \rangle - c_\delta \langle d^*\sqrt{-V}\varphi_2, d^*\sqrt{-V}\varphi_2 \rangle
\geq \langle sd^*\varphi_2, 2Bsd^*\varphi_2 \rangle - c_\delta \langle \sqrt{-V}d^*\varphi_2, \sqrt{-V}d^*\varphi_2 \rangle
- c_\delta \langle [d^*, \sqrt{-V}]\varphi_2, \sqrt{-V}d^*\varphi_2 \rangle
- c_\delta \langle \sqrt{-V}d^*\varphi_2, [d^*, \sqrt{-V}]\varphi_2 \rangle
- c_\delta \langle [d^*, \sqrt{-V}]\varphi_2, [d^*, \sqrt{-V}]\varphi_2 \rangle
\geq \|\sqrt{2Bsd^*}\varphi_2\|^2 - c_\delta \left(\|d^*\varphi_2\| + \|\sqrt{-V}d^*\varphi_2\|\right)^2
\geq \|\sqrt{2Bsd^*}\varphi_2\|^2 - c_\delta \left(\delta \|sd^*\varphi_2\| + (1 + 4\delta^2)\|\sqrt{-V}sd^*\varphi_2\|\right)^2
\geq \left[1 - c_\delta (1 - \eta) (1 + 15\delta^2)\right] \|\sqrt{2Bsd^*}\varphi_2\|^2,
\] (35)
where we applied the bound $\|\sqrt{-V}[s\pi^+, \sqrt{-V}^{-1}]\| \leq 4\delta^2$. For the latter write
\[
G[\text{sgn}(DA)P_0^+, G^{-1}] = \begin{pmatrix} 0 & G[s^+, G^{-1}] \\ G[s\pi^+, G^{-1}] & 0 \end{pmatrix},
\] (36)
where $G = \sqrt{-V}$ and therefore, by Lemma 2 with $\sqrt{-V}$ instead of $V$, we get
\[
\|\sqrt{-V}[s\pi^+, \sqrt{-V}^{-1}]\| \leq \|\sqrt{-V}[\text{sgn}(DA)P_0^+, \sqrt{-V}^{-1}]\| \leq 4\delta^2.
\]
Similarly, we obtain a lower bound for $\|d^*d\pi^+\varphi_1\|^2 - c_\delta \|V\pi^+\varphi_1\|^2$ by applying again the upper relation of Eq. (6). More precisely,
\[
\|d^*d\pi^+\varphi_1\|^2 - c_\delta \|V\pi^+\varphi_1\|^2
= \|d^*ds^*s\pi^+\varphi_1\|^2 - c_\delta \|Vs^*s\pi^+\varphi_1\|^2
= \|dd^*s\pi^+\varphi_1\|^2 - c_\delta \|Vs^*V^{-1}Vs\pi^+\varphi_1\|^2
= \|dd^*s\pi^+\varphi_1\|^2 - c_\delta \|(s^* + T)Vs\pi^+\varphi_1\|^2,
\]
where $T = V[s^*, V^{-1}]$. As above, Lemma 2 together with relation (36) for $G = V$ yields $\|T\| \leq 4\delta^2$. Thus,
\[
\|d^*d\pi^+\varphi_1\|^2 - c_\delta \|V\pi^+\varphi_1\|^2 \geq \|dd^*s\pi^+\varphi_1\|^2 - c_\delta (1 + 10\delta^2)\|Vs\pi^+\varphi_1\|^2.
\]
We note that $s\pi^+\varphi_1 \in \mathcal{D}(dd^*) \subset \mathcal{D}(V)$, hence we can use (35) (by approximating $s\pi^+\varphi_1$ through $C_0^\infty$-functions in the graph norm of $dd^*$) to conclude that
\[
\|d^*d\pi^+\varphi_1\|^2 - c_\delta \|V\pi^+\varphi_1\|^2
\geq \left[1 - c_\delta (1 - \eta) (1 + 15\delta^2)\right] \|\sqrt{2Bsd^*s\pi^+}\varphi_1\|^2.
\]
Combining this inequality with (35) leads to
\[
\|H_A P_0^\perp \varphi\|^2 - c_\delta \|VP_0^\perp \varphi\|^2 \\
\geq [1 - c_\delta (1 - \eta)(1 + 50\delta^2)] \left( \|\sqrt{2B}sd^* \varphi_2\|^2 + \|\sqrt{2Bd^\perp \varphi_1}\|^2 \right),
\]
where the r.h.s is non-negative for \(\delta\) small enough. □

5. Proof of Theorems 2 and 3

The basic strategy of the proofs is to represent \(B\) and \(V\) locally through constant values \(V_n := V(x_n)\) and \(B_n := B(x_n)\) along a sequence \((x_n)_{n \in \mathbb{N}} \subset \mathbb{R}^2\). Since one also needs to compare vector potentials associated to \(B_n\) and \(B\), we use the gauges
\[
A_n(x) := \int_0^1 B_n \wedge (x - x_n) \text{sd}s = \frac{1}{2} B_n \wedge (x - x_n),
\]
\[
\tilde{A}_n(x) := \int_0^1 B(x_n + s(x - x_n)) \wedge (x - x_n) \text{sd}s,
\]
where \(a \wedge v := a(-v_2, v_1)\) for \(a \in \mathbb{R}\) and \(v = (v_1, v_2) \in \mathbb{R}^2\). The two given vector potentials fulfill \(\text{curl} A_n = \text{curl} \tilde{A}_n = B\); hence, for every \(n \in \mathbb{N}\) there exists a function \(g_n \in C^2(\mathbb{R}^2, \mathbb{R})\) such that \(\nabla g_n = A_n - \tilde{A}_n\). In addition, for every vector potential \(A_n\), representing the constant magnetic fields \(B_n\), we obtain operators \(d_n\) and \(d_n^*\), \(n \in \mathbb{N}\), defined as in (2). For a sequence of natural numbers \((k_n)_{n \in \mathbb{N}}\) we set
\[
\psi_n(x) := \begin{pmatrix} (d_n^*)^{k_n} e^{-\frac{B_n}{4}|x - x_n|^2} \\ 0 \end{pmatrix}.
\]
Iterating the commutator relation (3) for \(d_n, d_n^*\) yields
\[
d_n^*d_n \left[ (d_n^*)^{k_n} e^{-\frac{B_n}{4}|x - x_n|^2} \right] = 2k_n B_n \left[ (d_n^*)^{k_n} e^{-\frac{B_n}{4}|x - x_n|^2} \right], \quad n \in \mathbb{N}, \tag{37}
\]
i.e. \(\psi_n\) is an eigenfunction of \(H_{A_n}\) for the corresponding eigenvalue \(2k_n B_n\). For the localization let \(\chi \in C_0^\infty(\mathbb{R}^2, [0, 1])\) be such that \(\chi(x) = 1\) for \(|x| \leq 1\) and \(\chi(x) = 0\) for \(|x| \geq 2\). We set
\[
\chi_n(x) := \chi \left( \frac{x - x_n}{r_n} \right),
\]
where \(r_n > 0\) will be chosen in the proofs. For the Weyl sequence, we define functions \(\varphi_n\) through
\[
\varphi_n(x) := e^{i g_n(x)} \chi_n(x) \psi_n(x), \quad x \in \mathbb{R}^2, \tag{38}
\]
with \(n \in \mathbb{N}\). Bounds on the norm of \(\varphi_n\) can be obtained in [10]. They are given through:
Lemma 3. For all $n \in \mathbb{N}$ large enough we have
\[
\|\varphi_n\|^2 = 2\pi \int_0^\infty (B_n r)^{2k_n} e^{-\frac{B_n}{2} r^2} dr = 2^{k_n+1} \pi B_n^{k_n-1} k_n! \tag{39}
\]
\[
\|\varphi_n\|^2 \geq \|\psi_n\|^2 \left( 1 - \frac{1}{k_n!} \int_{\frac{1}{2} B_n r_n^2}^{\infty} s^{k_n} e^{-s} ds \right). \tag{40}
\]

Now $H\varphi_n$ can be written as
\[
e^{-i g_n} (H_A + V)\varphi_n = (H_{\tilde{A}} + V)\chi_n \psi_n
\]
\[
= (H_A + V)\chi_n \psi_n + 2(\tilde{A}_n - A_n)(-i \nabla - A_n)\chi_n \psi_n
\]
\[
+ (\tilde{A}_n - A_n)^2 \chi_n \psi_n + \text{div}(\tilde{A}_n - A_n)\chi_n \psi_n
\]
\[
+ (B - B_n)\chi_n \psi_n, \tag{41}
\]
with the localization error
\[
(H_{\tilde{A}_n} + V)\chi_n \psi_n - \chi_n (H_{A_n} + V)\psi_n
\]
\[
= -(\Delta \chi_n) \psi_n + 2(-i \nabla \chi_n)(-i \nabla - A_n)\psi_n. \tag{42}
\]

To prove Theorems 2 and 3 we estimate each term of (41) separately. For the proofs we use the notation $K_n := \{ x \in \mathbb{R}^2 \mid r_n \leq |x - x_n| \leq 2r_n \}$ with $n \in \mathbb{N}$.

Proof of Theorem 2. We set $k_n = k$ and choose the radii to be $r_n^{-4} = B_n^{(2-\epsilon)}$. Then, for any $p \geq 0$,
\[
\frac{(B_n)^p}{k_n!} \int_{\frac{1}{2} B_n r_n^2}^{\infty} s^{k_n} e^{-s} ds = \frac{(B_n)^p}{k!} \int_{\frac{1}{2} B_n^{1/2}}^{\infty} s^{k} e^{-s} ds \to 0 \quad \text{as } n \to \infty.
\]
In addition, we have $\|\psi_n\|^2 \leq 2 \|\varphi_n\|^2$ for $n \in \mathbb{N}$ large enough. For treating the terms on the r.h.s. of (41), we estimate:
\[
\| (\tilde{A}_n - A_n)(-i \nabla - A_n)\chi_n \psi_n \|^2
\]
\[
\leq C_1 r_n^4 \| \nabla B(x_n) \|^2 \| (\tilde{A}_n - A_n)\chi_n \psi_n \|^2
\]
\[
\leq 2C_1 r_n^4 \| \nabla B(x_n) \|^2 \left[ (2k + 1)B_n \| \psi_n \|^2 + r_n^{-2} \| \nabla \chi \|^2 \int_{K_n} |\psi_n(x)|^2 dx \right]
\]
\[
\leq 16kC_1 \| \nabla B(x_n) \|^2 \| \psi_n \|^2 + 4C_1 \| \nabla \chi \|^2 \| \psi_n \|^2 B_n^{\epsilon/2} \frac{1}{k!} \int_{\frac{1}{2} B_n^{1/2}}^{\infty} s^{k} e^{-s} ds.
\]
In addition,
\[
\| (\tilde{A}_n - A_n)^2 \chi_n \psi_n \|^2 \leq C_2 r_n^4 \| \nabla B(x_n) \|^4 \| \psi_n \|^2 \leq C_2 \| \nabla B(x_n) \|^4 B_n^{2(1-\epsilon)} \| \psi_n \|^2,
\]
\[
\| \text{div} (\tilde{A}_n - A_n)\chi_n \psi_n \|^2 = \| \text{div} \tilde{A}_n \chi_n \psi_n \|^2 \leq C_3 r_n^2 \| \nabla B(x_n) \|^2 \| \psi_n \|^2
\]
\[
\leq C_3 \| \nabla B(x_n) \|^2 B_n^{1-\epsilon} \| \psi_n \|^2,
\]
\[
\| (B - B_n)\chi_n \psi_n \|^2 \leq C_4 r_n^2 \| \nabla B(x_n) \|^2 \| \psi_n \|^2 \leq C_4 \| \nabla B(x_n) \|^2 B_n^{-\epsilon/2} \| \psi_n \|^2.
\]
For the first term of the r.h.s of (41) we get, due to (42), that

\[ \| (H_{A_n} + V) \chi_n \psi_n \| \leq \| \chi_n (H_{A_n} + V) \psi_n \| + \| (\Delta \chi_n) \psi_n \| + 2\| (-i \nabla \chi_n) (-i \nabla - A_n) \psi_n \|, \]

with

\[ \| (\Delta \chi_n) \psi_n \| ^2 \leq r_n^{-1}\| \Delta \chi \| _\infty ^2 \int _{K_n} |\psi_n(x)| ^2 dx \]

\[ \leq 2\| \Delta \chi \| _\infty ^2 \| \psi_n \| ^2 \frac{1}{k!} B_n ^{2-\epsilon} \int _{\frac{1}{2} B_n ^{1/2}} s^k e^{-s} ds, \]

and

\[ \| (-i \nabla \chi_n) (-i \nabla - A_n) \psi_n \| ^2 \leq r_n^{-2} \| \nabla \chi \| _\infty ^2 \int _{K_n} |(-i \nabla - A_n) \psi_n(x)| ^2 dx \]

\[ \leq \| \nabla \chi \| _\infty ^2 \| \psi_n \| ^2 (2k + 1) B_n ^{2-\epsilon/2} \int _{\frac{1}{2} B_n ^{1/2}} s^k e^{-s} ds. \]

Because of (37) and since \( |\nabla V| \) vary with rate 0, we conclude by the mean value theorem

\[ \| \chi_n (H_{A_n} + V) \psi_n \| ^2 \leq \| \chi_n (V + 2kB_n) \psi_n \| ^2 \]

\[ \leq C_5 \| \nabla V(x_n) \| ^2 r_n ^2 \| \chi_n \psi_n \| ^2 + (2kB_n + V_n) ^2 \| \chi_n \psi_n \| ^2 \]

\[ \leq C_5 (4k)^{1-\epsilon} \frac{\| \nabla V(x_n) \| ^2}{|V_n| ^{1-\epsilon}} \| \psi_n \| ^2 + (2kB_n + V_n) ^2 \| \varphi_n \| ^2. \]

Hence, by (41) and conditions (11)–(13) we see that \( \| (H_{A_n} + V) \varphi_n \| / \| \varphi_n \| \to 0 \) as \( n \to \infty \). In addition, note that \( r_n \to 0 \) as \( n \to \infty \), so we can assume that the \( \varphi_n \) have mutually disjoint support, i.e. \( (\varphi_n) _{n \in \mathbb{N}} \) is a Weyl sequence for 0.

**Proof of Theorem 3.** We first note that it suffices to prove \( 0 \in \sigma_{\text{ess}}(H) \), since for \( E \in \mathbb{R} \) we consider \( V_E := V - E \) instead of \( V \), which also fulfills (14)–(17) along \( \gamma \). Because \( \mathbb{R}^+ \ni t \mapsto V(\gamma(t))/B(\gamma(t)) \) is continuous and (14) holds we find points \( (x_n) _{n \in \mathbb{N}} \subset \text{Ran}(\gamma) \), with \( |x_n| \to \infty \) as \( n \to \infty \), such that \( 2nB(x_n) = -V(x_n) \). We choose \( k_n = n \) and set

\[ r_n := \sqrt{2n ^{1+\epsilon} / B_n}. \tag{43} \]

Note that \( r_n / |x_n| ^\nu \to 0 \) as \( n \to \infty \) by (16). In particular, we might assume the \( \varphi_n \) to have mutually disjoint support. Further, for any \( \lambda \geq 0 \),

\[ \frac{e^{\lambda n}}{n!} \int _{n ^{1+\epsilon}} ^{\infty} s ^n e ^{-s} ds \leq e^{\lambda n} \exp \left( n \ln(2n) - n ^{1+\epsilon} / 2 + n \right) \to 0 \quad \text{as} \quad n \to \infty. \]

Hence, we can choose \( N \in \mathbb{N} \) so large that \( \| \varphi_n \| ^2 \leq \| \psi_n \| ^2 \leq 2\| \varphi_n \| ^2 \) for \( n \geq N \). Proceeding as in the proof of Theorem 2, we obtain for the terms on the r.h.s. of (41):
\[(\tilde{A}_n - A_n)(-i \nabla - A_n) \chi_n \psi_n \|^2 \]
\[
\leq C_6 r_n^4 |\nabla B(x_n)|^2 (2n + 1) B_n \left[ |\psi_n|^2 + r_n^{-2} |\nabla \chi|^2 \right] \int_{K_n} |\psi_n(x)|^2 \, dx
\]
\[
\leq 16C_6 n^{3+2\epsilon} B_n \left| \frac{\nabla B(x_n)}{B_n^2} \right|^2 |\psi_n|^2 + C_6 \left| \nabla \chi \right|^2 |\psi_n|^2 B_n \frac{1}{n!} \int_{n^{1+\epsilon}}^{\infty} s^n e^{-s} \, ds
\]
\[
\leq \tilde{C}_6 \left( \frac{|V_n|^3}{B_n^2} \right)^{1+\epsilon} \frac{|\nabla B(x_n)|^2}{B_n^2} |\psi_n|^2 + \tilde{C}_6 \left| \nabla \chi \right|^2 |\psi_n|^2 \frac{2\epsilon + 1}{n!} \int_{n^{1+\epsilon}}^{\infty} s^n e^{-s} \, ds.
\]

Using (as in the first inequality above) that $|\nabla B|$ vary with rate $\nu$, we conclude that

\[
\|(\tilde{A}_n - A_n)^2 \chi_n \psi_n\|^2 \leq C_7 r_n^4 \left| \frac{\nabla B(x_n)}{B_n^4} \right|^4 |\psi_n|^2,
\]
\[
\|\text{div}(\tilde{A}_n - A_n) \chi_n \psi_n\|^2 \leq C_8 r_n^2 \left| \frac{\nabla B(x_n)}{B_n^2} \right|^2 |\psi_n|^2,
\]
\[
\|(B - B_n) \chi_n \psi_n\|^2 \leq C_9 r_n^2 \left| \frac{\nabla B(x_n)}{B_n^2} \right|^2 |\psi_n|^2.
\]

We estimate, using equality (42), the first term on the r.h.s. of (41) by

\[
\|(H_{A_n} + V) \chi_n \psi_n\| \leq \|(V - V_n) \chi_n \psi_n\| + \|(\Delta \chi_n) \psi_n\|
\]
\[+ 2 \left\| (-i \nabla \chi_n)(-i \nabla - A_n) \psi_n \right\|,
\]

with

\[
\|(\Delta \chi_n) \psi_n\|^2 \leq \|\Delta \chi\|^2_{\infty} \|\psi_n\|^2 \frac{B_n^2}{n^{2+2\epsilon}} \frac{1}{n!} \int_{n^{1+\epsilon}}^{\infty} s^n e^{-s} \, ds
\]
\[
\leq \alpha^2 \|\Delta \chi\|^2_{\infty} \|\psi_n\|^2 \frac{e^{4\kappa n}}{n^{2+2\epsilon}} \frac{1}{n!} \int_{n^{1+\epsilon}}^{\infty} s^n e^{-s} \, ds,
\]

and

\[
\left\| (-i \nabla \chi_n)(-i \nabla - A_n) \psi_n \right\|^2 \leq \|\nabla \chi\|^2_{\infty} \|\psi_n\|^2 \frac{B_n}{n^{1+\epsilon}} \int_{K_n} (2n + 1) B_n \left| \psi_n(x) \right|^2 \, dx
\]
\[
\leq \alpha \left\| \nabla \chi \right\|^2_{\infty} \|\psi_n\|^2 \frac{e^{2\kappa n}}{n^{1+\epsilon}} \frac{1}{n!} \int_{n^{1+\epsilon}}^{\infty} s^n e^{-s} \, ds.
\]

Since $|\nabla V|$ vary with rate $\nu$, we have

\[
\|\chi_n (H_{A_n} + V) \psi_n\|^2 \leq \|\chi_n (V - V_n) \psi_n\|^2 \leq C_{10} \|\nabla V(x_n)\|^2 r_n^2 |\psi_n|^2
\]
\[
\leq \tilde{C}_{10} \left( \frac{|V_n|^3}{B_n^2} \right)^{1+\epsilon} \frac{|\nabla V(x_n)|^2}{|V_n|^2} |\psi_n|^2.
\]

We see that $\|(H_{A_n} + V) \varphi_n\|/\|\varphi_n\| \to 0$ as $n \to \infty$, and therefore, by (41) and the above estimates, that $\|(H_{A} + V) \varphi_n\|/\|\varphi_n\| \to 0$ as $n \to \infty$. \qed
6. Proof of Theorem 4

Throughout this section we consider the case of a constant magnetic field \( B(x) = B_0 \). In addition, we assume that \( A \) is in the rotational gauge, i.e.

\[
A(x) = \frac{B_0}{2} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}.
\]

Note that \( H_A \) is invariant under rotations. More precisely, for a matrix \( R \in SO(2, \mathbb{R}) \) define the unitary map

\[
U_R : L^2(\mathbb{R}^2, \mathbb{C}^2) \to L^2(\mathbb{R}^2, \mathbb{C}^2), \quad \psi(\cdot) \mapsto \psi(R^{-1} \cdot),
\]

then \( U_R^{-1} H_A U_R = H_A \) and therefore

\[
U_R^{-1} (H_A + V) U_R = H_A + V_R \quad \text{with} \quad V_R(\cdot) = V(R \cdot).
\]

To construct a Weyl sequence, consider a second gauge \( \tilde{A} \), the Landau gauge \( \tilde{A}(x) = B_0 x_1 \hat{e}_2 \). Then, our Hamiltonian reads

\[
H_{\tilde{A}} + V = -\partial_1^2 + (-i \partial_2 - B_0 x_1)^2 - \sigma_3 B_0 + V = \tilde{d}^* \tilde{d} + B_0 - \sigma_3 B_0 + V,
\]

with

\[
\tilde{d} = -i \partial_1 + i (-i \partial_2 - B_0 x_1), \quad \tilde{d}^* = -i \partial_1 - i (-i \partial_2 - B_0 x_1).
\]

For electric fields of the form \( V(x) = V_0 + \mathcal{E}_0 (x_1 - \zeta) \), with constants \( V_0, \mathcal{E}_0, \zeta \in \mathbb{R} \), we can write

\[
H_{\tilde{A}} + V = -\partial_1^2 + (-i \partial_2 - B_0 x_1)^2 - \sigma_3 B_0 + V_0 + \mathcal{E}_0 (x_1 - \zeta) = -\partial_1^2 + B_0^2 (x_1 - \frac{1}{B_0} (-i \partial_2 - \frac{\mathcal{E}_0}{2B_0}))^2 + \frac{\mathcal{E}_0}{B_0} (-i \partial_2 - \frac{\mathcal{E}_0}{2B_0}) - \mathcal{E}_0 \zeta - \sigma_3 B_0 + V_0 + \left( \frac{\mathcal{E}_0}{2B_0} \right)^2. \tag{45}
\]

Performing a Fourier transform in \( x_2 \), we obtain the direct integral representation

\[
H_{\tilde{A}} + V \approx \int_{\mathbb{R}} h(\xi) d\xi
\]

on \( L^2(\mathbb{R}_\xi, L^2(\mathbb{R}, \mathbb{C}^2)) \), with

\[
h(\xi) = -\partial_1^2 + B_0^2 (x_1 - \frac{1}{B_0} (\xi - \frac{\mathcal{E}_0}{2B_0}))^2 + \frac{\mathcal{E}_0}{B_0} (\xi - \frac{\mathcal{E}_0}{2B_0}) - \mathcal{E}_0 \zeta - \sigma_3 B_0 + V_0 + \left( \frac{\mathcal{E}_0}{2B_0} \right)^2
\]

\[
= -\partial_1^2 + B_0^2 (x_1 - \hat{\zeta})^2 + \mathcal{E}_0 \hat{\zeta} - \mathcal{E}_0 \zeta - \sigma_3 B_0 + V_0 + \left( \frac{\mathcal{E}_0}{2B_0} \right)^2.
\]

Here we set \( \hat{\zeta} = \frac{1}{B_0} (\xi - \frac{\mathcal{E}_0}{2B_0}) \). Note that \( h(\xi) \) is the Hamiltonian of a shifted harmonic oscillator. Thus, we define for \( n \in \mathbb{N}_0 \)

\[
\phi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x) e^{-\frac{1}{2} x^2}, \quad x \in \mathbb{R},
\]

\[
\]
where $H_n$ denotes the $n$th Hermite polynomial. The normalized functions
\[
\hat{\psi}_{E_0,n,\xi}(x_1) := \sqrt{B_0} \left( \phi_n \left( \sqrt{B_0} (x_1 - \frac{1}{B_0} (\xi - \frac{\xi_0}{2B_0})) \right) \right)
\]
fulfill the equation
\[
h(\xi) \hat{\psi}_{E_0,n,\xi} = \left( 2nB_0 + \mathcal{E}_0 \left( \frac{1}{B_0} (\xi - \frac{\xi_0}{2B_0}) - \zeta \right) + V_0 + \left( \frac{\xi_0}{2B_0} \right)^2 \right) \hat{\psi}_{E_0,n,\xi}.
\]
Hence,
\[
\psi_{E_0,n,\xi}(x_1, x_2) := e^{i \xi x_2} \hat{\psi}_{E_0,n,\xi}(x_1)
\]
satisfies
\[
[H_A + V] \psi_{E_0,n,\xi}
= \left( 2nB_0 + \mathcal{E}_0 \left( \frac{1}{B_0} (\xi - \frac{\xi_0}{2B_0}) - \zeta \right) + V_0 + \left( \frac{\xi_0}{2B_0} \right)^2 \right) \psi_{E_0,n,\xi},
\]
for $\xi \in \mathbb{R}$ and $n \in \mathbb{N}_0$, seen as a differential equation. In addition, we have
\[
\begin{align*}
d\psi_{E_0,n,\xi} &= -i \sqrt{2nB_0} \psi_{E_0,n-1,\xi} + i \frac{\xi_0}{2B_0} \psi_{E_0,n,\xi}, \quad (48) \\
\hat{d}^* \psi_{E_0,n,\xi} &= i \sqrt{2(n+1)B_0} \psi_{E_0,n+1,\xi} - i \frac{\xi_0}{2B_0} \psi_{E_0,n,\xi}. \quad (49)
\end{align*}
\]
**Proof of Theorem 4.** As discussed in the proof of Theorem 3, it is sufficient to find a Weyl sequence for $E = 0$. Because of (18) and (21) there exists a sequence $\{y_n\}_{n \in \mathbb{N}} \subset \text{Ran}(\gamma)$ such that
\[
V(y_n) = -2nB_0 - \left( \frac{\nabla V(y_n)}{2B_0} \right)^2. \quad (50)
\]
Further, one can find rotations $R_n \in SO(2, \mathbb{R})$ such that $\nabla V_{R_n}(x_n) = \nabla V_{R_n}(x_n)|\dot{e}_1$ with $x_n = R_n^{-1} y_n = (x_{n,1}, x_{n,2})^T$ for $n \in \mathbb{N}$. We set
\[
\begin{align*}
V_n := V(y_n) &= V_{R_n}(x_n), \quad (51) \\
\mathcal{E}_n := |\nabla V(y_n)| &= |\nabla V_{R_n}(x_n)|, \quad (52) \\
\xi_n := B_0 x_{n,1} + \frac{\xi_0}{2B_0}. \quad (53)
\end{align*}
\]
For the Weyl functions let $\chi \in C_c^\infty(\mathbb{R}, [0, 1])$ with $\chi(x) = 1$ for $|x| \leq 1$, $\chi(x) = 0$ for $|x| \geq 2$. Define
\[
\chi_{n,j}(x) := \chi \left( \frac{x-x_{n,j}}{r_n} \right), \quad j = 1, 2,
\]
and
\[
\varphi_n(x) := \chi_{n,1}(x_1) \chi_{n,2}(x_2) \psi_{E_0,n,\xi_0}(x_1, x_2)
= \chi \left( \frac{x_2-x_{n,2}}{r_n} \right) e^{-i \xi_n x_2} \chi \left( \frac{x_1-x_{n,1}}{r_n} \right) \left( \sqrt{B_0} \phi_n \left( \sqrt{B_0} (x_1 - x_{n,1}) \right) \right),
\]
where the localization radii $r_n$ are chosen to be $r_n := \sqrt{n^{1+\epsilon}/B_0}$. Note that
\[
r_n \leq 2r_n \int_{-\sqrt{n}^{1+\epsilon}}^{\sqrt{n}^{1+\epsilon}} |\phi_n(x)|^2 dx \leq \|\varphi_n\|^2
\leq 4r_n \int_{-2\sqrt{n}^{1+\epsilon}}^{2\sqrt{n}^{1+\epsilon}} |\phi_n(x)|^2 dx \leq 4r_n,
\]
(54)
for $n \in \mathbb{N}$ large enough (see Lemma 5 in the appendix). By denoting $g(x) = \frac{B_0}{2} x_1 x_2$ for $x \in \mathbb{R}^2$, we get, due to (44), (47), (50) and (53), that

$$H U_{R_n} e^{-ig} \varphi_n = U_{R_n} e^{-ig} [H + V_{R_n}] \varphi_n = U_{R_n} e^{-ig} (\tilde{d}^* \tilde{d} \varphi_n - \chi_{n,1} \chi_{n,2} \tilde{d}^* \tilde{d} \psi_{\eta, n, \xi_n}) + \frac{\| \tilde{d}^* \tilde{d} \varphi_n - \chi_{n,1} \chi_{n,2} \tilde{d}^* \tilde{d} \psi_{\eta, n, \xi_n} \|}{\varphi_n},$$

with, using (48) and (49),

$$\| - i \chi_{n,2} \partial_1 \chi_{n,1} + \chi_{n,1} \partial_2 \chi_{n,2} \| \tilde{d}^* \psi_{\eta, n, \xi_n} \| \leq \sqrt{2(n + 1)} B_0 \left\| - i \chi_{n,2} \partial_1 \chi_{n,1} + \chi_{n,1} \partial_2 \chi_{n,2} \psi_{\eta, n+1, \xi_n} \right\| + \frac{E}{2B_0} \left\| - i \chi_{n,2} \partial_1 \chi_{n,1} + \chi_{n,1} \partial_2 \chi_{n,2} \psi_{\eta, n, \xi_n} \right\| \leq 2 \sqrt{2(n + 1)} B_0 \| \chi' \|_\infty \sqrt{2r_n} \| \varphi_{n+1} \| + \frac{E}{2B_0} r_n \| \chi' \|_\infty \sqrt{2r_n} \| \varphi_n \| \leq 2 \sqrt{2} \| \chi' \|_\infty \left( B_0 \sqrt{\frac{2n+2}{n+1}} + \frac{E}{2} \sqrt{\frac{B_0}{n+1}} \right) \sqrt{r_n},$$

and

$$\| - i \chi_{n,2} \partial_1 \chi_{n,2} - \chi_{n,1} \partial_2 \chi_{n,2} \| \tilde{d} \psi_{\eta, n, \xi_n} \| \leq \sqrt{2nB_0} \left\| - i \chi_{n,2} \partial_1 \chi_{n,1} - \chi_{n,1} \partial_2 \chi_{n,2} \psi_{\eta, n-1, \xi_n} \right\| + \frac{E}{2B_0} \left\| - i \chi_{n,2} \partial_1 \chi_{n,1} - \chi_{n,1} \partial_2 \chi_{n,2} \psi_{\eta, n, \xi_n} \right\| \leq 2 \sqrt{2} \| \chi'' \|_\infty r_n^{-2} \sqrt{r_n},$$

Thus, in view of condition (21) and estimate (54), we get

$$\| \tilde{d}^* \tilde{d} \varphi_n - \chi_{n,1} \chi_{n,2} \tilde{d}^* \tilde{d} \psi_{\eta, n, \xi_n} \| \leq 0 \text{ as } n \to \infty.$$ 

To estimate the remaining term on the r.h.s of (55), we expand $V_{R_n}$ up to second order and obtain, by (51) and (52), that

$$\| [V_{R_n} (x) - V_n - \mathcal{E}_n (x_1 - x_1, n)] \varphi_n (x) | \leq \| \text{Hess}(V_{R_n}) \|_2 (\eta_{x,x_n}) |x - x_n|^2 | \varphi_n (x)|,$$

with $\eta_{x,x_n} \in [x, x_n]$. Because $R_n$ are rotations, we have that $\| \text{Hess}(V_{R_n}) \|_2 (\cdot) = \| \text{Hess}(V) \|_2 (R_n \cdot)$ for $n \in \mathbb{N}$. Since $\| \text{Hess}(V) \|_2$ varies with rate $\nu$ along $\text{Im}(\gamma)$ and since (by (20) and (50)) $r_n/|x_n|^\nu \to 0$ as $n \to \infty$, we find a constant $C_{11} > 0$ such that, for $n \in \mathbb{N}$ large enough,

$$\| \text{Hess}(V_{R_n}) \|_2 (\eta) \leq C_{11} \| \text{Hess}(V_{R_n}) \|_2 (x_n), \quad \eta \in B_{2r_n} (x_n)$$
holds. As a consequence,

\[ \| U_{R_n} e^{-ig} [V_{R_n} - V_n - \varepsilon_n(x_1 - x_{1,n})] \varphi_n \| \]
\[ \leq 4C_{11} \| \text{Hess}(V) \|_2 (R_n x_n) r_n^2 \| \varphi_n \| \]
\[ \leq 4C_{11} \| \text{Hess}(V) \|_2 (y_n) \left( \frac{|V_n|}{B_0} \right)^{1+\epsilon} \| \varphi_n \| \]

for \( n \in \mathbb{N} \) large enough. With (19) we conclude that \( (U_{R_n} e^{-ig} \varphi_n)_{n \in \mathbb{N}} \) is a Weyl sequence for 0.

\[ \square \]

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Appendix A. Essential Self-Adjointness of the Pauli Operator

For the proof of Proposition 1 we first note that for \( \varphi \in C_0^\infty(\mathbb{R}^2, \mathbb{C}) \) we can write

\[ \begin{align*}
\left[ (-i \nabla - A)^2 + B \right] \varphi &= \sum_{k,l=1}^{2} (-i \partial_k - A_k) C_{k,l} (-i \partial_l - A_l) \varphi, \\
\left[ (-i \nabla - A)^2 - B \right] \varphi &= \sum_{k,l=1}^{2} (-i \partial_k - A_k) C_{k,l} (-i \partial_l - A_l) \varphi,
\end{align*} \]

where \( C_{k,l} \) denote the coefficients of the symmetric non-negative definite matrix

\[ C = 1 - \sigma_2 = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = C^*. \]

Furthermore, along the proof we use the notation \( B_R := \{ x \in \mathbb{R}^2 | ||x|| \leq R \} \) and \( S_R := \{ x \in \mathbb{R}^2 | ||x|| = R \} \)

**Proof of Proposition 1.** Since \( H \) is a diagonal matrix operator, it suffices to show that both operators on the diagonal,

\[ Q_\pm := [(-i \nabla - A)^2 \pm B + V], \]

are essentially self-adjoint on \( C_0^\infty(\mathbb{R}^2, \mathbb{C}) \). Because \( Q_\pm \) are symmetric on \( C_0^\infty(\mathbb{R}^2, \mathbb{C}) \), we have to show that \( Q_+^* \varphi = \pm i \varphi \) implies \( \varphi \equiv 0 \) for \( \varphi \in D(Q_+^*), \) respectively, that \( Q_-^* \varphi = \pm i \varphi \) implies \( \varphi \equiv 0 \) for \( \varphi \in D(Q_-^*). \) We only treat the case \( Q_-^* \varphi = i \varphi, \) since the others are completely analogous. Let \( \varphi \in D(Q_-^*) \) be such that \( Q_-^* \varphi = i \varphi, \) then

\[ [(-i \nabla - A)^2 - B + V] \varphi = i \varphi \] (57)
holds in distributional sense. By elliptic regularity theory (see, e.g. [6, 7]) we obtain \( \varphi \in C^2(\mathbb{R}^2, \mathbb{C}) \), and that (57) holds strongly. Using integration by parts gives

\[
\int_{B_R} \left[ \sum_{k,l=1}^{2} (-i \partial_k - A_k)C_{k,l}(-i \partial_l - A_l)\varphi \right] \bar{\varphi} \, dx \\
= \int_{B_R} \left[ \sum_{k,l=1}^{2} (-i \partial_k - A_k)\varphi C_{k,l}(-i \partial_l - A_l)\varphi \right] \, dx \quad (58)
\]

with \( R > 0 \), where \( \nu_k(x) = x_k/|x| \) for \( k = 1, 2 \). By taking the imaginary part of (58), we conclude with (57) that

\[
\int_{B_R} |\varphi|^2 \, dx = \int_{S_R} \left[ \sum_{k,l=1}^{2} \nu_k C_{k,l}(-i \partial_l - A_l)\varphi \right] \varphi \, dS
\]

for any \( R > 0 \). Applying Cauchy–Schwarz yields

\[
\int_{B_R} |\varphi|^2 \, dx \leq \left( \int_{S_R} \sum_{k,l=1}^{2} (-i \partial_k - A_k)\varphi C_{k,l}(-i \partial_l - A_l)\varphi \, dS \right)^{1/2} \left( \int_{S_R} |\varphi|^2 \, dS \right)^{1/2}.
\]

Hence it suffices to show that

\[
\int_{\mathbb{R}^2} \sum_{k,l=1}^{2} \frac{(-i \partial_k - A_k)\varphi C_{k,l}(-i \partial_l - A_l)\varphi}{|x|^2 + 1} \, dx < \infty, \quad (59)
\]

since this implies that \((1, \infty) \ni r \mapsto r^{-1} \int_{B_r} |\varphi|^2 \, dx\) is an \( L^1\)-function, i.e. \( \varphi \equiv 0 \). For (59) we consider the function

\[
f(R) := \int_{B_R} \sum_{k,l=1}^{2} \frac{(-i \partial_k - A_k)\varphi C_{k,l}(-i \partial_l - A_l)\varphi}{|x|^2 + 1} \, dx,
\]

with \( R > 0 \). Using Eq. (57) and integration by parts, we obtain, with \( \zeta(x) = (|x|^2 + 1)^{-1} \), \( M \geq c + |d| \), that

\[
f(R) - M\|\varphi\|^2 \\
\leq f(R) + \int_{B_R} \zeta V |\varphi|^2 dx
\]
\[\int_{B_R} \zeta(Q^* \varphi) \phi \, dx - i \int_{B_R} \left[ \sum_{k,l=1}^2 (\partial_l \zeta) C_{k,l} (-i \partial_k - A_k) \varphi \right] \phi \, dx + i \int_{S_R} \zeta \left[ \sum_{k,l=1}^2 \nu_i C_{k,l} (-i \partial_k - A_k) \varphi \right] \phi \, dS.\]

By the estimates
\[
\left| \int_{B_R} \left[ \sum_{k,l=1}^2 (\partial_l \zeta) C_{k,l} (-i \partial_k - A_k) \varphi \right] \phi \, dx \right| \\
\leq \int_{B_R} 2 \zeta^{1/2} \left| \sum_{k,l=1}^2 \nu_i C_{k,l} (-i \partial_k - A_k) \varphi \right| |\phi| \, dx \\
\leq 2 \int_{B_R} \left[ \sum_{k,l=1}^2 \zeta (-i \partial_k - A_k) \varphi C_{k,l} (-i \partial_k - A_k) \varphi \right]^{1/2} |\phi| \, dx \\
\leq 2 [f(R)]^{1/2} \|\varphi\| \leq \frac{1}{2} f(R) + 2 \|\varphi\|^2,
\]

and
\[
\left| \int_{S_R} \zeta \left[ \sum_{k,l=1}^2 \nu_k C_{k,l} (-i \partial_k - A_k) \varphi \right] \phi \, dS \right| \\
\leq \int_{S_R} \zeta \left[ \sum_{k,l=1}^2 (-i \partial_k - A_k) \varphi C_{k,l} (-i \partial_k - A_k) \varphi \right]^{1/2} |\phi| \, dS \\
\leq \left( \int_{S_R} \sum_{k,l=1}^2 \zeta (-i \partial_k - A_k) \varphi C_{k,l} (-i \partial_k - A_k) \varphi \, dS \right)^{1/2} \left( \int_{S_R} |\varphi|^2 \, dS \right)^{1/2} \\
= \left( f'(R) \int_{S_R} |\varphi|^2 \, dS \right)^{1/2},
\]

we conclude that
\[f(R) \leq 2(3 + c) \|\varphi\|^2 + 2 \left( f'(R) \int_{S_R} |\varphi|^2 \, dS \right)^{1/2}.\]

If \(f(R) = 0\) for all \(R > 0\), then clearly (59) holds. If \(f(R_0) > 0\) for some \(R_0 > 0\), then \(f(R) > 0\) for all \(R > 0\) and \(f'(R)/f^2(R) \in L^1(R_0, \infty)\), which implies that
\[
\left( \frac{f'(R)}{f^2(R)} \int_{S_R} |\varphi|^2 \, dS \right)^{1/2} \in L^1(R_0, \infty).
\]

Hence, there exists a sequence \((R_n)_{n \in \mathbb{N}} \subset (0, \infty)\) such that \(R_n \to \infty\) as \(n \to \infty\) and
\[
\left( f'(R_n) \int_{S_{R_n}} |\varphi|^2 dS \right)^{1/2} \leq \frac{1}{4} f(R_n).
\]

Therefore, \( f(R_n) \leq 4(3 + c)\|\varphi\|^2 \) for all \( n \in \mathbb{N} \), which implies (59) since \( f(R) \) is a monotonically increasing function. \( \square \)

**Appendix B. Remarks on Local Compact Operators**

**Lemma 4.** Let \( A \) be a locally compact, self-adjoint operator on \( L^2(\mathbb{R}^n, \mathbb{C}^m) \) with \( n, m \geq 1 \). Assume there is a function \( G \in L^\infty_{loc}(\mathbb{R}^n, [0, \infty)) \) with \( G(x) \to \infty \) as \( |x| \to \infty \) such that

\[
\|A\varphi\| \geq \|G\varphi\| \quad \text{for } \varphi \in \mathcal{D}(A).
\]

Then \( \sigma_{\text{ess}}(A) = \emptyset \), i.e. \( A \) has purely discrete spectrum.

**Proof.** Assume \( \lambda \in \sigma_{\text{ess}}(A) \subseteq \mathbb{R} \). Then one can find a normalized sequence \((\varphi_n)_{n \in \mathbb{N}} \subseteq \mathcal{D}(A)\) such that \( \varphi_n \rightharpoonup 0 \) and \( \|(A - \lambda)\varphi_n\| \to 0 \) as \( n \to \infty \). Let \( R > 0 \) be a fixed constant. We have

\[
\|\chi_R \varphi_n\| = \|\chi_R (A - i - \lambda)^{-1}(A - i - \lambda)\varphi_n\| \leq \|\chi_R (A - i - \lambda)^{-1}||\|(A - \lambda)\varphi_n\| + \|\chi_R (A - i - \lambda)^{-1}\varphi_n\|
\]

using the notation \( \chi_R := \chi_{B_R(0)} \). Since \( \chi_R (A - i - \lambda)^{-1} \) is compact, this inequality implies that \( \|\chi_R \varphi_n\| \to 0 \) as \( n \to \infty \). Let \( R > 0 \) be so large that \( G(x) \geq 5|\lambda| + 1 \) for \( |x| \geq R \). Choosing \( N \in \mathbb{N} \) large enough, we can estimate, for \( n \geq N \),

\[
\|(A - \lambda)\varphi_n\| \geq \|G\varphi_n\| - |\lambda| \\
\geq \|G(1 - \chi_R)\varphi_n\| - \|G\chi_R\| \|\chi_R \varphi_n\| - |\lambda| \\
\geq (5|\lambda| + 1)(\|1 - \chi_R\| \varphi_n\| - \|G\chi_R\| \|\chi_R \varphi_n\| - |\lambda| \\
\geq (|\lambda| + 1/2) - \|G\chi_R\| \|\chi_R \varphi_n\|.
\]

Hence, \( \|(A - \lambda)\varphi_n\| \to 0 \) as \( n \to \infty \), which is a contradiction. \( \square \)

**Appendix C. Integral Estimates**

**Lemma 5.** Let \( H_n \) be the \( n \)th Hermite polynomial. Then, for any \( \epsilon > 0 \),

\[
\frac{1}{2^n n!} \int_{\sqrt{n}^{1+\epsilon}}^{\infty} |H_n(x)|^2 e^{-x^2} dx \to 0 \quad \text{as } n \to \infty,
\]

\[
\frac{1}{2^n n!} \int_{-\infty}^{-\sqrt{n}^{1+\epsilon}} |H_n(x)|^2 e^{-x^2} dx \to 0 \quad \text{as } n \to \infty.
\]

**Proof.** We only treat the first case, since the second claim can be deduced from the first one by a symmetry argument. Due to the identity

\[
H_n(x) = (-1)^n \sum_{k_1 + 2k_2 = n} \frac{n!}{k_1! k_2!} (-1)^{k_1 + k_2} (2x)^{k_1}
\]
for the $n$th Hermite polynomial (see, e.g. [1]), we obtain for $|x| \geq 1$ the estimate
\[
|H_n(x)| \leq \sum_{k_1+2k_2=n} \frac{n!}{k_1!k_2!} (2|x|)^{k_1} \leq \frac{n+1}{2} 2^n n! |x|^n.
\]

Thus, for $n \in \mathbb{N}$ large enough we have
\[
\int_{\sqrt{n^{1+\epsilon}}}^{\infty} |\phi_n(x)|^2 dx \leq \frac{(n+1)^2}{4} 2^n n! \int_{\sqrt{n^{1+\epsilon}}}^{\infty} x^{2n} e^{-x^2} dx
\leq \frac{(n+1)^2}{4} 2^n n^{1+\epsilon} n! \exp\left(-n^{1+\epsilon}/2\right) \int_{\sqrt{n^{1+\epsilon}}}^{\infty} e^{-x^2/2} \, dx \rightarrow 0
\]
as $n \to \infty$, using Stirling’s Formula. \qed

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