Geometric Motion Planning for Affine Control Systems with Indefinite Boundary Conditions and Free Terminal Time

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Abstract—The problem of motion planning for affine control systems consists of designing control inputs that drive a system from a well-defined initial to final states in a desired amount of time. For control systems with drift, however, understanding which final states are reachable in a given time, or reciprocally the amount of time needed to reach a final state, is often the most difficult part of the problem. We address this issue in this paper and introduce a new method to solve motion planning problems for affine control systems, where the motion desired can have indefinite boundary conditions and the time required to perform the motion is free. The method extends on our earlier work on motion planning for systems without drift. A canonical example of parallel parking of a unicycle with constant linear velocity is provided in this paper to demonstrate our algorithm.

I. INTRODUCTION

Due to its ubiquity in control applications ranging from robotics to autonomous wheeled vehicles, motion planning has been widely studied (see, e.g., [1], [2] and the reference therein) and a host of methods have been developed. From a theoretical point of view, the Chow-Rashevski Theorem provides us with conditions under which a driftless non-holonomic system is controllable [3]. The idea of small time local controllability can be generalized to affine systems with drift as studied in [4], [5]. Beyond that the motion planning is still quite challenging and even the general case of controllability of nonlinear systems is still an open problem. Nevertheless, for some specific nonlinear systems with drift, motion planning or control algorithms are given in [6], [7], [8], [9].

Inspired by the curving shortening flow as studied in [10], we have proposed a geometrical approach for motion planning in [11] for driftless affine systems. The method works by “deforming” an arbitrary path between a given initial state \( x^i \) and a given final state \( x^f \) into an almost admissible trajectory for the system, from which we can extract the controls \( u^* \) that drive the system from \( x^i \) to \( x^f \) approximately. Using a variational approach, we provided a proof of convergence of the method as well. By encoding obstacles into barrier functions, we are able to plan the motion while avoid obstacles. In the later work [12], we modified the algorithm so it is capable of addressing motion planning problems for affine systems with drift and input constraints by solving the so called affine geometric heat flow equation.

Nevertheless, most literature mentioned above including our previous works only focused on motion planning problems with both end points of the motion as well as the time span fixed. In practice, we very often allow indefinite boundary conditions (IBCs) and free terminal time (FTT). It is appreciated that in many motion planning problems there is no need to fix the boundary conditions as a priority, or the optimal/feasible boundary states need to be determined in the motion planning problems. For example, consider the task of somersault performed by a robot made of linkages as studied in [13], [14]. While the final joint angles are prescribed, there are no constraints on the final velocities of the joints if “landing with impact” is allowed. On the other hand, if the final dynamic configuration is fully fixed for the same somersault problem, we have implicit constraints on choosing the initial configuration because of the conservation of angular momentum in mid-air and hence it is better to ask the algorithm to find the feasible initial configuration [15]. In terms of terminal time, it is less important for driftless systems because the time span can always be adjusted by input scaling; however, scaling will not work if there is drift or the control is constrained. In other words, the reachable space depends on the terminal time, which affects the feasibility of our motion planning problem so FTT is indeed crucial in our case and the terminal time cannot be fixed prior to the motion planning problem either.

By deploying some ideas from calculus of variation and state augmentation, in this work we modify our previous algorithm so that it is able to solve motion planning problems with both IBCs and FTT. The paper has 6 sections. In Section II our motion planning problem is formulated with necessary preliminaries. We then state the changes in our algorithm for handling IBCs in Section III followed by the discussion of FTT in Section IV. Our algorithm is then examined in a case study on unicycle in Section V and the paper is concluded in Section VI.

II. PROBLEM FORMULATION AND PRELIMINARIES

Consider an affine control system with drift:

\[
\dot{x} = h(x) + F(x)u
\]  

(1)

Where \( x \in \mathbb{R}^n \) are the states, and for each fixed \( t \) and \( x, h(x) \in \mathbb{R}^n \) is the drift and \( F(x) \in \mathbb{R}^{n \times m} \) consists of columns of control directions and \( u(t) \in \mathbb{R}^m \) is the control input. We further assume \( m < n \) so the system is under-actuated and \( F(x) \) is full rank for all \( x \in \mathbb{R}^n \). Let \( T > 0 \) be the terminal time, either free or fixed. For simplicity we assume that both the functions \( h(\cdot), F(\cdot) \) are smooth and the input function \( u(\cdot) \in L^2([0, T] \rightarrow \mathbb{R}^m) =: \mathcal{U} \). For a curve \( x(\cdot) \in C^1([0, T] \rightarrow \mathbb{R}^n) =: \mathcal{X} \), Let \( L \) be some Lagrangian defined with respect to \( x(t), \dot{x}(t) \). Defined the
action functional $A$ over $X$ as follows:

$$A(x) := \int_0^T L(x(t), \dot{x}(t))dt$$

(2)

It is well known that Euler-Lagrange equation

$$\frac{\partial L}{\partial x}(x^*(t), \dot{x}^*(t)) - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}}(x^*(t), \dot{x}^*(t)) = 0 \quad \forall t \in [0, T]$$

(3)

is a necessary condition for a curve $x^*$ to be a local optimizer of (2) in $X$. Instead of solving the system of ODEs (3) directly, we can view the solution of (3) as the steady state solution of a system of PDEs by adding one more variable $s$ to the curve $x$ so now it becomes $x(t, s)$, defined on $[0, T] \times [0, \infty)$. Notice that for each fixed $s$, $x(\cdot, s)$ still represents a curve in $X$. As $x(t, s)$ is a multivariable function, we use $x_i$ and $x_{it}$ to represent $\frac{\partial x}{\partial t}$ and $\frac{\partial^2 x}{\partial t^2}$, respectively. We define the affine geometric heat flow (AGHF) equation as follows:

$$x_s = G(x)^{-1} \left( \frac{d}{dt} \frac{\partial L}{\partial x_t}(x, x_t) - \frac{\partial L}{\partial x}(x, x_t) \right),$$

(4)

where $G$ is a positive definite $n \times n$ matrix which will be given later. Unlike the case we have studied in our earlier work that both ends of the curve is fixed in the sense that there exists $x^i, x^f \in \mathbb{R}^d$ such that we have the boundary conditions

$$x(0, s) = x^i, x(T, s) = x^f \quad \forall s \geq 0,$$

we now only fix part of them in the case of IBCs. To be more precise, let $S_{bc} \subset S := \{1, \ldots, d\} \times \{0, T\}$. For some given $x_{bc}(i, t) \in \mathbb{R}^n, (i, t) \in S_{bc}$,

$$x_i(t, s) = x_{bc}^i(i, t) \quad \forall (i, t) \in S_{bc}, s \geq 0$$

(5)

We allow the case that $S_{bc} = \emptyset$, meaning that there are no boundary conditions at all. Meanwhile, in order to solve the parabolic type PDE (4), we still need to feed it with an initial curve

$$x(t, 0) = z(t), \quad t \in [0, T]$$

(6)

where $z(\cdot) \in \mathcal{X}^\prime$ with

$$\mathcal{X}^\prime := \{x \in \mathcal{X} : x_i(t) = x_{bc}^i(i, t) \quad \forall (i, t) \in S_{bc}\}$$

In other words, $\mathcal{X}^\prime$ is the set of all $C^1$ curves with the IBCs satisfied. Note that the curves in $\mathcal{X}^\prime$ do not need to meet the system dynamics (1). We also define the set of admissible paths by

$$\mathcal{X}^* := \{x \in \mathcal{X}^\prime : (1) \text{ is satisfied for some } u \in \mathcal{U}\}$$

We say the motion planning problem is feasible when $\mathcal{X}^* \neq \emptyset$. While finding a curve in $\mathcal{X}^\prime$ is trivial, finding an admissible path in $\mathcal{X}^*$ is difficult. In this work we will show an efficient algorithm that gives an approximation to some $x^* \in \mathcal{X}^*$ when the problem is feasible.

III. Indefinite boundary conditions

We first consider the case when the terminal time $T$ is fixed. Notice that when $S_{bc} = S$, we recover the fully constrained boundary conditions. However, when $S_{bc}$ is a proper subset of $S$, algebraically there are not enough boundary conditions in (3) so the ODEs (3) are under-determined. Nevertheless, For being a minimizer of the functional (2), it should also give 0 variation with respect to perturbations in the free boundary states; In other words, we should also have

$$\frac{\partial L}{\partial x_t}(x(t, s), \dot{x}(t, s)) = 0 \quad \forall (i, t) \in S \setminus S_{bc}, s \geq 0$$

(7)

together with (5) are the new boundary conditions we will use for solving the AGHF equation (4).

A. Decreasing action functional

At this point we provide a lemma showing that the steady state solution of (4) is a solution of (3).

Lemma 1 (Gradient decent rule under IBCs) Let $x(t, s)$ be a solution to the AGHF (4) with an initial condition (6) and boundary conditions (5). Then $\frac{\partial L(x(t, s), \dot{x}(t, s))}{\partial x_t} \leq 0$ and it is 0 if and only if (3) is satisfied on the curve $x(\cdot, s)$.

With some modification on the boundary conditions, its proof is essentially the same as the proof of Lemma 1 in [12] and provided in the appendix.

As a remark, similar arguments also hold if we require the indefinite boundary states not fully free but in some subsets of $\mathbb{R}^n$. In other words, if the motion planning problem needs to satisfy the boundary condition that

$$x(0) \in \Omega^i := \{x \in \mathbb{R}^n : \phi_j^i(x) = 0, j = 1, \ldots, n^i\},$$

$$x(T) \in \Omega^f := \{x \in \mathbb{R}^n : \phi_k^f(x) = 0, k = 1, \ldots, n^f\},$$

then all we need are some transversality conditions such that $\frac{\partial^2 L}{\partial x_t \partial x}^T(x(0, s), \dot{x}(0, s))$ is a linear combination of $\nabla \phi_j^i$‘s and $\frac{\partial^2 L}{\partial x_t \partial x}(x(T, s), \dot{x}(T, s))$ is a linear combination of $\nabla \phi_k^f$‘s in the non-degenerate case. The proof is similar and hence omitted. For more information on transversality conditions for optimization, see, e.g., [16].

B. Algorithm

Our algorithm for motion planning with IBCs and fixed terminal time is very similar to our old algorithm for fixed boundary states, consisting of the following steps:

S1: Find a bounded $n \times (n-m)$ $x$-dependent matrix $F_c(x)$, differentiable in $x$, such that

$$F(x) := (F_c(x)|F(x)) \in \mathbb{R}^{n \times n}$$

(8)

is invertible for all $x \in \mathbb{R}^n$. The matrix $F_c(x)$ can be obtained using, e.g., the Gram-Schmidt procedure.

S2: Evaluate

$$G(x) := (F(x)^{-1})^T D F(x)^{-1}$$

(9)
where \( D := \text{diag}(\lambda, \ldots, \lambda, 1, \ldots, 1) \) for some sufficiently large \( \lambda > 0 \) and set
\[
L(x, \dot{x}) = (\dot{x} - h(x))'G(x)(\dot{x} - h(x)). \tag{10}
\]

S3: Solve the AGHF (4) with boundary conditions (5), (7) and initial condition (6). Denote the solution by \( x(t, s) \);
S4: For some sufficiently large \( \bar{s} \), Evaluate the extracted control by
\[
u(t) := (0 \ I_{m \times m}) \bar{F}(x(t, \bar{s}))^{-1} (x_1(t, \bar{s}) - h(x(t, \bar{s}))) \tag{11}
\]

Output: The control \( u(\cdot) \) obtained in (11) yields a trajectory \( \tilde{x}(\cdot) \) when integrating (1) from the initial value \( \tilde{x}(0) = x(0, \bar{s}) \), which is our solution to the motion planning problem. We call it integrated path.

Again, as we stated in [12], the integrated path from this algorithm will only be an approximated admissible path and how close the end states to the desired boundary conditions depends on how large are the two parameters \( \lambda \) and \( \bar{s} \). In addition, this algorithm will work only if the motion planning problem is feasible and the initial curve \( v \) is chosen in some neighborhood of an admissible path, which is not pre-known. The theoretical result is almost the same as our previous one so we only provide the statements but omit the proof here.

**Theorem 1** Consider the system (1) and assume the motion planning problem with IBCs (5) and fixed terminal time \( T \) is feasible. Then there exists \( C > 0 \) such that for any \( \lambda \) such that \( \lambda > 0 \), there exists an open set \( \Omega_\lambda \subseteq X^0 \) so that as long as the initial curve \( z \in \Omega_\lambda \). The integrated path \( \tilde{x}(\cdot) \) from our algorithm with sufficiently large \( \bar{s} \) has the properties that for all \( (i, 0) \in S_{bc} \)
\[
\tilde{x}_i(0) = x_{bc}(i, 0) \tag{12}
\]
and for all \( (i, T) \in S_{bc} \),
\[
|\tilde{x}_i(T) - x_{bc}(i, T)| \leq \sqrt{3TMC \over \lambda} \exp \left( \frac{3T}{2} (L_1^2T + L_2^2C) \right) \tag{13}
\]
where \( L_1, L_2 \) are the Lipschitz constants of \( h, F \) and \( M \) is the upper bound of \( \|F_c\| \), which is constructed in the algorithm S1.

**C. Comments on implementation in MATLAB**

Our algorithm is implemented in MATLAB. The first and second step of the algorithm is processed symbolically. Step 3 is implemented via the command pdepe, where the type of hyperbolic PDEs it is capable of solving is of the form
\[
c(t, s, x, x_1)x_s = t^{-m} \frac{\partial}{\partial t} (t^m f(t, s, x, x_1)) + s(t, s, x, x_1) \tag{14}
\]
By comparison with (4), we see that in our case we simply need \( m = 0, c = C, f = {h}t^m \), and \( s = -\frac{\partial}{\partial x} \), which can also be done symbolically. In addition, the boundary conditions are formulated in the form
\[
p(t, s, x) + q(t, s)f(t, s, x, x_1) = 0
\]
where the \( f \) is the same as in (14), and thus is \( \frac{\partial}{\partial x} \) in our case. Therefore for the boundary conditions (5) and (7), the implementation is simply
\[
p_i(t, s, x) = x_i(t, s) - x_{bc}(i, t), \quad q_i(t, s) = 0,
\]
\[
\forall (t, s) \in S_{bc}, s \geq 0;
\]
\[
p_i(t, s, x) = 0, \quad q_i(t, s) = 1,
\]
\[
\forall (t, s) \in S \setminus S_{bc}, s \geq 0.
\]

**IV. Free terminal time**

For a driftless affine system, If \( u(\cdot) : [0, 1] \to \mathbb{R}^m \) gives an admissible path satisfying the boundary condition \( x(0) = x^l, x(1) = x^f \), then \( \frac{1}{\tau}u(\frac{t}{\tau}) \) gives a time scaled admissible path with \( x'(0) = x^l, x'(T) = x^f \). In other words, there is no need to consider the terminal time \( T \) other than 1 since it can always be done by scaling the inputs. However, when there are input constraints or in the presence of the drift term, such scaling is no longer true and hence the minimization of \( A \) in (12) as studied in Lemma 1 should be with respect to \( T \) as well. In other words, unlike the driftless case where the reachable space of a driftless system is independent of \( T \), the reachable space of an affine system with drift or constrained inputs is somehow related to the terminal time \( T \). For example, the unicycle with unit linear velocity can only reach planar positions within the ball of radius \( T \) at time \( T \). Thus we cannot simply just fix a random \( T \) prior to minimizing \( A \), in which case solutions may even not exist.

**A. Augmenting the true time**

Compared with the case of fixed terminal time, FTT in the view of maximum principle means that the Hamiltonian is identically 0 along the optimal trajectory. That information is not helpful in our case, as we do not rely on our analysis on the costates nor Hamiltonian. Instead, while we still consider functions defined over a fixed domain \([0, 1]\), the way to tackle FTT is to augment a new state \( \tau \in \mathbb{R} \) to the system, which is the true time variable that starts from \( \tau(0) = 0 \) and \( \tau(1) = T \) yet to be determined. There is also an additional constraint on the function \( \tau(\cdot) \) that it needs to be strictly increasing, in which case the inverse function \( \tau^{-1}(\cdot) \) exists and we can recover the control as a function of the true time from \( u(\cdot) \) by \( u(\tau^{-1}(t)) \). For smooth \( \tau(\cdot) \), this monotonicity constraint can be resolved by deploying our earlier technique on constrained inputs by treating the derivative of \( \tau \) as another extra state, or simply we define \( \tilde{\tau}(t) = a(t)^2, \tilde{a}(t) = u_0(t) \) where \( u_0 \) is the additional input to the twice-augmented system. Notice that since \( \tau \) is the true time, \( \frac{d\tau}{dt} \) should obey the true system dynamics (1) instead of \( \frac{d\tau^*}{dt} \). Thus using chain rule, we have
\[
\dot{x} := \frac{dx}{dt} = \frac{dx}{d\tau} \frac{d\tau}{dt} = h(x)a^2 + F(x)a^2u.
\]
In summary, denote the augmented state
\[
x' = \begin{pmatrix} x \\ \tau \\ a \end{pmatrix}, \tag{15}
\]
we have
\[
\dot{x}' = \begin{pmatrix} \dot{x} \\ \dot{\tau} \end{pmatrix} = \begin{pmatrix} h(x) a^2 \\ a^2 \\ a^2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} F(x) a \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} u \\ u_0 \end{pmatrix} \quad \text{(16)}
\]

By this augmentation we have new drift term \( h' \) and new admissible control direction matrix \( F' \). The reason why we take one \( \alpha \) out from \( F' \) and multiply it to the control \( u \) will be explained later when we discuss the total energy consumption of the planned path. In addition, by observation we see that if the inadmissible control direction matrix is constructed by
\[
F_c'(x') := \begin{pmatrix} F_c(x) \\ 0 \\ 0 \\ 0 \end{pmatrix}
\quad \text{(17)}
\]

then \( \bar{F}^T = (F_c^T F') \) is full rank if \( \bar{F} = (F_c F) \) is full rank as we needed earlier. Because the dimension of the system (16) is now \( n + 2 \), \( D', G', L' \) should all be defined accordingly. There are also some small tweaks on the boundary conditions. We still have the old boundary conditions (5), (7); we also have a new boundary condition defined in (8) is invertible for all \( x \). Let
\[
\dot{x}(1, s) = x^b(1, s) = x^b(0, s) \quad \text{(18)}
\]

We do not have any constraints on \( \tau(1, s), a(0, s), a(1, s) \); nevertheless, according to the previous discussion on IBCs we should have
\[
\frac{\partial L}{\partial \tau_1}(\bar{x}(1, s), \bar{z}(1, s), 1) = 0,
\frac{\partial L}{\partial a_2}(\bar{x}(0, s), \bar{z}(0, s), 0) = 0,
\frac{\partial L}{\partial a_4}(\bar{x}(1, s), \bar{z}(1, s), 1) = 0
\quad \text{(19)}
\]

for all \( s \geq 0 \) as a complement in order to solve our AGHF. The rest can be proceed similarly to the algorithm in Section III-B and we summarize it in the following subsection:

### B. Algorithm for FTT problem

**S1:** Augment the states as in (15). Find a bounded \( n \times (n - m) \) \( x \)-dependent matrix \( F_c(x) \), differentiable in \( x \), such that \( \bar{F} \) defined in (8) is invertible for all \( x \in \mathbb{R}^n \). Denote \( \bar{F}^T(x') = (F_c^T(x) F'(x')) \) where \( F', F_c' \) come from (16), (17).

**S2:** Evaluate
\[
G'(x') := (\bar{F}^T(x')^{-1})^T D' \bar{F}'(x')^{-1}
\quad \text{(20)}
\]

where \( D' = \text{diag}(\lambda, \lambda, \lambda, \lambda, \lambda, \lambda, \lambda, \lambda) \) and set
\[
L'(x', \dot{x}') = (\dot{x}' - h'(x'))^T G'(x') (\dot{x}' - h'(x')).
\quad \text{(21)}
\]

**S3:** Pick some \( T_0 > 0, a_0, a_T \in \mathbb{R} \) as the initial guess for \( T, a(0), a(1) \). Let \( \dot{z}' \in C^0([0, 1] \rightarrow \mathbb{R}^{n+2}) \) be an initial curve such that \( \dot{z}'(t) = x^b(i, t) \) for all \( (i, t) \in S_{bc} \),
\[
z_{n+1}'(0) = 0, z_{n+2}'(0) = a_g^i, z_{n+2}'(1) = a_T^i.
\]

Solve the AGHF (4) with boundary conditions (5), (7), (18), (19) and initial curve \( \dot{z}' \) described above.

Denote the solution by \( \dot{x}'(t, \bar{s}) \).

**S4:** Fix \( \bar{s} \) sufficiently large. Define
\[
w(t) := \bar{F}'(x(t, \bar{s}))^T (\dot{x}'(t, \bar{s}) - h'(x(t, \bar{s}))).
\]

Split \( w \) so \( w^T = (v^T, v_0, u^T, u_0) \) for some \( v \in C^0([0, 1] \rightarrow \mathbb{R}^m), u \in C^0([0, 1] \rightarrow \mathbb{R}^m), v_0, u_0 \in C^0([0, 1] \rightarrow \mathbb{R}) \). Define
\[
\bar{r}(s) = \int_0^s a(t)^2 dt
\quad \text{(22)}
\]

where
\[
a(t) = \int_0^t u_0(r) dr + x_{n+2}(0, \bar{s}) = x_{n+2}(t, \bar{s}),
\quad \text{(23)}
\]

then \( T = \bar{r}(1) \) is our resultant terminal time and
\[
u^T(t) = \frac{u^T(\bar{r}^{-1}(t))}{a(\bar{r}^{-1}(t))}
\]

is our extracted control.

**Output:** The integrated path \( \hat{x}^1 \) is obtained by integrating \( \bar{x}^1(\cdot) \) with the extracted control \( u^T(\cdot) \) and initial value \( \hat{x}^1(0) = \bar{x}^1(0, \bar{s}) \bar{x}^2(0, \bar{s}) \cdots \bar{x}^n(0, \bar{s}) \).

**C. Cost minimizing**

We provide a theorem in the FTT case here, which is similar to Theorem 4. Its proof is contained in the appendix.

**Theorem 2** Consider the system (11) and assume the motion planning problem with IBCs (5) and FTT is solvable. The integrated path \( \hat{x}^1(\cdot) \) from our algorithm with properly chosen initial curve \( \bar{z}' \) and sufficiently large \( \bar{s} \) has the properties that for all \( (i, 0) \in S_{bc} \),
\[
\hat{x}^1(0) = x^{bc}(i, 0)
\quad \text{(24)}
\]

and there exists \( K > 0 \) such that for all \( (i, T) \in S_{bc} \),
\[
|\hat{x}^1(T) - x^{bc}(i, T)| \leq \frac{K}{\sqrt{\lambda}}.
\quad \text{(25)}
\]

Next we provide a heuristic argument that our extracted control is “economical”. Plug \( x'(t, \bar{s}) \) which is derived from Step 3 into \( L' \) defined in Step 2, we have
\[
L'(x'(t, \bar{s}), x'_1(t, \bar{s})) = w(t)^T D' w(t)
= \lambda |v(t)|^2 + \lambda v_0(t)^2 + |u(t)|^2 + u_0(t)^2 \geq |u(t)|^2.
\]

From our analysis of Lemma 11 we see that the action functional \( A = \int_0^T L'dt \) is minimized when solving the AGHF; in other words, the \( L^2 \)-norm of \( u(\cdot) \) is relatively small from our algorithm. On the other hand, again by the change of variable via \( \bar{r}^{-1}(r) = s \), this \( L^2 \)-norm is exactly the energy of the actual input:
\[
E = \int_0^T |u^T(r)|^2 dr = \int_0^T |u(\bar{r}^{-1}(r))|^2 \bar{r} dr = \int_0^1 |u(s)|^2 ds
\]
This in fact is not a coincidence; it is only achieved when we design the $F'(x')$ in that particular form as in [15] where we have shifted one $a$ to the input. As a summary, our algorithm not only finds an approximation to an admissible path which satisfying the IBCs with FTT, the energy consumption of the planned path is also relatively small.

V. CASE STUDY: UNICYCLE WITH UNIT LINEAR VELOCITY

A planar unicycle has three state variables, $x, y$ which represent its planar position and $\theta$ which represents its orientation. The kinematics of a unicycle with unit linear velocity is given by

$$
\begin{pmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta}
\end{pmatrix} = 
\begin{pmatrix}
\cos \theta \\
\sin \theta \\
0
\end{pmatrix} +
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix} u,
$$

(26)

from which we have $h = (\cos \theta \ sin \theta \ 0)\top$ and $F = (0 \ 0 \ 1)\top$. It is not hard to see that $F_x = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 0
\end{pmatrix}$ is the orthogonal complement to $F$. Thus according to our algorithm we have

$$
h'(x') = \begin{pmatrix}
\frac{a^2 \cos \theta}{a^2 \sin \theta} \\
0 \\
\frac{a^2}{0}
\end{pmatrix}, \quad \tilde{F}'(x') = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$

Consider the canonical parallel parking problem that $(x, y, \theta) |_{\tau=0} = (0, 0, 0)$ and $(x, y, \theta) |_{\tau=T} = (0, 1, 0)$ where $T$ is to be determined. We would also like to ensure that the energy cost $E = \int_0^T u(t)^2 dt$ is small. We pick $\lambda = 1000, T_0 = 10, a^i_g = a^f_g = 1$ and Let $z'$ be the line segment connecting the boundary conditions for our algorithm.

The evolution of the $(x, y, \theta)$ coordinates in the AGHF solution $x'(s, t)$ with respect to $s$ is shown in Figure 1. Notice that in Figure 1a we have $s = 0$ and hence it is indeed the first three coordinates of the initial curve $z'$. In Figure 1d the curve barely changes any more with respect to $s$ so the PDE solver is stopped and we use $\bar{s} = 1$ to extract the control. The extracted control is shown as the black curve in Figure 2 and integrated path is shown as the black curve in Figure 3. It turns out that by our algorithm $T = 1.4072$ and $E = 21.1022$. As a comparison, a heuristic admissible path for the unicycle system (26) which consists of two semicircles is considered. Such a path has a total length of $\frac{\pi}{2}$, and hence the total traveling time is also $\frac{\pi}{2}$ because of unit velocity. By observation we see that $u$, or the turning rate, is equal to the curvature of the path and thus $u(t) = 4$ for the first half and $u(t) = -4$ for the second half. As a result, the total energy cost in this case is $4^2 \times \frac{\pi}{2} \approx 25.13$. Both the total time and energy cost of this heuristic path is larger than what we derived from our proposed algorithm. In addition, we also applied our motion planning algorithm studied in [12] for fixed terminal time with varies $T$. Their extracted

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Fig1.png}
\caption{The $(x, y, \theta)$ plots of the solution of (4) for different values of $s$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Fig2.png}
\caption{The extracted control for the FTT algorithm, the heuristic control and the extracted controls from our previous algorithm with fixed $T$.}
\end{figure}
controls and integrated paths are also illustrated in Figure 2 and Figure 3.

The energy costs of these different results are shown in Figure 4. Notice that this motion planning problem has no solution with global minimal energy cost. This is because the parallel parking of the constant linear velocity unicycle can always be accomplished by an S-shaped curve with arbitrarily large turning radius \( r \), as seen by the trend of solutions with larger \( T \) in Figure 3. By doing that the magnitude to the control is approximately \( O(1/r) \) while the total time is approximately \( O(r) \) and hence \( E = \int_0^T |u|^2 \, dt = O(1/r) \to 0 \) as \( r \) increases to infinity. This asymptotic behavior is also reflected on the plot of energy in Figure 4. However finding globally optimal or sub-optimal solution for the motion planning is out of our interest because it requires infinite or extremely long travel time of the unicycle and hence impractical. On the other hand, our FTT algorithm is able to produce a value of \( \Theta \) that is quite systematic and usually gives a good approximating solution within reasonable amount of computation time.

As a final remark, we want to comment on finding the true optimal solution to this motion planning problem via maximum principle. We start by formulating the Hamiltonian as

\[
H = p^\top f - L = p_1 \cos(\theta) + p_2 \sin(\theta) + p_3 u - u^2 \tag{27}
\]

Now by maximum principle we have \( u = \frac{p_3}{2}, \) \( p_1, p_2 \) are constants and \( \dot{p}_3 = p_1 \sin(\theta) - p_2 \cos(\theta) \). In addition, since the problem has a free terminal time, we have \( H \equiv 0 \). While we still have boundary conditions on the states \( x, y, \theta \), we do not know the value of \( p_1, p_2 \) and there are neither any boundary conditions for \( p_3 \). On the other hand, if we take the second derivative of \( \theta \), we see that

\[
\ddot{\theta} = \frac{d}{d\theta} \left( \frac{1}{2} \frac{dp_3}{dt} \right) = \frac{p_1}{2} \sin(\theta) - \frac{p_2}{2} \cos(\theta), \tag{28}
\]

which is a second order nonlinear ODE. While there are no general solutions for nonlinear ODEs with IBCs and FTT, the general solution of (28) can be expressed in terms of Jacobi elliptic functions [17], which requires quite a lot of work. It can also be seen that finding the exact optimal path via maximum principle is difficult to be generalized for more complicated systems or in higher dimensions. On the contrary, in trade of the true minimal cost and the accuracy of the exact expression of the optimal solution, our algorithm is quite systematic and usually gives a good approximating solution within reasonable amount of computation time.

VI. CONCLUSION

In this paper we have extended our earlier method for motion planning for affine control systems with fixed boundary conditions and fixed terminal time to indefinite boundary conditions and free terminal time. We have first shown that the deficiency of boundary conditions can be completed by constraints on the Lagrangian with respect to the derivative of the corresponding states and it was verified via a perturbation argument. On the other hand, a time scaling has been applied to the motion planning problem, which resulted in an augmented affine system with drift and indefinite boundary conditions and hence the motion with indefinite terminal time can be planned by appealing to the techniques we developed in the previous work and the analysis on indefinite boundary conditions. In the end of the paper we have also studied a canonical example of unicycle with constant linear velocity and used our algorithm to show how parallel parking can be accomplished in the most economical way.

APPENDIX

A. Proof of Lemma 7

Firstly, by first order approximation we have

\[
x(t, s + \delta) = x(t, s) + \delta x_s(t, s) + o(\delta)
\]
Plugging it into the first order variation of $L$, we have
\[
\mathcal{A}(x, s + \delta)) = \int_0^T L(x(t, s + \delta), x(t, s + \delta)) dt
\]
\[
= \int_0^T L(x(t, s), x(t, s)) + (\delta_x x(t, s) + o(\delta))^T \frac{\partial L}{\partial x} + \left( \frac{d}{dt}(\delta_x x(t, s) + o(\delta)) \right)^T \frac{\partial L}{\partial x_t} + o(\delta) dt
\]
\[
= \mathcal{A}(x, s) + \delta \int_0^T x(t, s)^T \frac{\partial L}{\partial x} + x(t, s)^T \frac{\partial L}{\partial x_t} dt + o(\delta),
\]
where all $o(\delta)$ terms are collected together. Using integration by parts for the $x(t, s)^T \frac{\partial L}{\partial x_t}$ term, we have
\[
\mathcal{A}(x, s + \delta)) = \mathcal{A}(x, s) + \delta \left( x(t, s)^T \frac{\partial L}{\partial x} \right)_0^T + \int_0^T x(t, s)^T \frac{\partial L}{\partial x} - x(t, s)^T \frac{d}{dt} \frac{\partial L}{\partial x_t} dt + o(\delta).
\]

Our new boundary conditions (5), (7) imply that $x(t, s)^T \frac{\partial L}{\partial x_t} = 0$ for all $t \in [0, T]$ and all $s \geq 0$. Hence the integrated term $x(t, s)^T \frac{\partial L}{\partial x_t}$ vanishes. Thus plug in the AGHF (4) here,
\[
\Delta A = \delta \int_0^T x(t, s)^T \frac{\partial L}{\partial x} dt + o(\delta)
\]
\[
= -\delta \int_0^T G(x)|x(t, s)|^2 dt + o(\delta)
\]
where $\Delta A = \mathcal{A}(x, s + \delta)) - \mathcal{A}(x, s))$ and hence
\[
\frac{\partial A}{\partial s} = \lim_{\delta \to 0} \Delta A = -\int_0^T G(x)|x(t, s)|^2 dt \leq 0
\]
and equality is achieved if and only if $x(t, s) = 0$ almost everywhere for $t \in [0, T]$. Because of (4) and the fact that $x(., s) \in C^1$, (3) is satisfied on $x(., s)$. □

**B. Proof of Theorem 2**

Note the property (24) is directly given by the construction. The major difference between the two algorithms is in Step 4, where a time scaling is involved in the second algorithm on FTT. Nevertheless, if we directly feed $(u, u_0)$ to the system (16) as required by the Step 4 in the first algorithm, by the results of Theorem 1 we again conclude the bounds of (13) for all $(i, T) \in S_{bc}$, where
\[
\tilde{x}(t) = \tilde{x}(0) + \int_0^t h(\tilde{x}(s)) a(s)^2 + F(\tilde{x}(s)) \alpha(s) u(s) ds
\]
and $a$ as defined in (25). On the other hand, the integrated path is given by
\[
\hat{x}(t) = \hat{x}(0) + \int_0^t h(\hat{x}(r)) + F(\hat{x}(r)) u(r) dr
\]
\[
= \hat{x}(0) + \int_0^t h(\hat{x}(r)) + \frac{F(\hat{x}(r))}{\alpha(\hat{r}^{-1}(r))} u(\hat{r}^{-1}(r)) dr
\]
Set $\hat{r}^{-1}(r) = s$ and notice that $\frac{d}{ds} \hat{x}(s) = \alpha(s)^2$, we see that
\[
\hat{x}(t) = \hat{x}(0) + \int_0^t h(\hat{x}(\hat{r}(s))) + \frac{F(\hat{x}(\hat{r}(s)))}{\alpha(s)} a(s)^2 ds
\]
which implies that $\hat{x}(t) = \tilde{x}(\hat{r}^{-1}(t))$. In particular, $\hat{x}(T) = \hat{x}(1)$ and we conclude (25) from (13). □

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