A universal framework for learning based on the elliptical mixture model (EMM)

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Abstract

An increasing prominence of unbalanced and noisy data highlights the importance of elliptical mixture models (EMMs), which exhibit enhanced robustness, flexibility and stability over the widely applied Gaussian mixture model (GMM). However, existing studies of the EMM are typically of ad hoc nature, without a universal analysis framework or existence and uniqueness considerations. To this end, we propose a general framework for estimating the EMM, which makes use of the Riemannian manifold optimisation to convert the original constrained optimisation paradigms into an un-constrained one. We first revisit the statistics of elliptical distributions, to give a rationale for the use of Riemannian metrics as well as the reformulation of the problem in the Riemannian space. We then derive the EMM learning framework, based on Riemannian gradient descent, which ensures the same optimum as the original problem but accelerates the convergence speed. We also unify the treatment of the existing elliptical distributions to build a universal EMM, providing a simple and intuitive way to deal with the non-convex nature of this optimisation problem. Numerical results demonstrate the ability of the proposed framework to accommodate EMMs with different properties of individual functions, and also verify the robustness and flexibility of the proposed framework over the standard GMM.

1 Introduction

Finite mixture models have a prominent role in statistical machine learning, as these enhanced provide probabilistic awareness in many learning paradigms, including clustering, feature extraction and density estimation. This is achieved in a very intuitive and elegant way, through a linear combination of well understood distributions, which is powerful enough to approximate arbitrary
complex distributions [2]. The Gaussian mixture model (GMM) is the most widely used such model whose popularity stems from the simple formulation and the conjugate property of Gaussian distribution. Despite mathematical elegance, standard Gaussian-based mixture model estimator is subject to robustness issues, and even a slight deviation from the Gaussian assumption or a single outlier can significantly degrade the performance or even break down the estimator [3]. Alternative mixture models are therefore rapidly being sought for robust learning.

Another rapidly emerging issue in modern applications is the requirement for the flexibility in mixture models, this is due to an exponential emergence of multi-faceted data which are almost invariably unbalanced; sources of such imbalance may be vastly different natures of the data channels involved, different powers in the constitutive channels, or temporal misalignment [4]. Another less obvious but equally important obstacle which is prohibitive to the use of current mixture models is that of the different scales of information within multivariate data; for example, in biomedical recordings, respiration and heart beats occupy totally different scales, of < 0.5 Hz and 1-3 Hz respectively, but their harmonics overlap spectrally.

An important class of multivariate analysis techniques are elliptical distributions, which are quite general and flexible and include as special cases a range of standard distributions, such as the Gaussian distribution, the exponential family and the t-distribution [5]. The desirable property of elliptical distributions is their robustness; indeed their use results in robust M-estimators [6], thus making them a natural candidate for robust mixture modelling. In addition to the robustness, it is reported that members of the elliptical mixture model (EMM) class can also effectively mitigate the singular covariance problem experienced in the GMM [7]. Moreover, EMMs are more flexible in capturing intrinsic data structures than the GMM, as the EMM can even use different types of distributions in a single mixture. By virtue of their robustness and flexibility, EMMs are therefore perfectly suited to dealing with data acquired from imperfect sensors, a typical case in modern applications.

Existing mixture models related to elliptical distributions are most frequently based on the t-distribution [7; 8; 9], the Laplace distribution [10], or the hyperbolic distribution [11]. Table 1 summarises the existing results which adopt elliptical distributions that belong to the class of scale mixture of normals [12], where the expectation-maximisation (EM) process, employed in model tuning, is guaranteed to converge. Despite all their desirable properties, a general estimation method for fitting arbitrary elliptical distributions is still lacking.

The development of a general method for estimating the EMM, however, is non-trivial, owing to both theoretical and practical difficulties; for example, different from the GMM, there is no closed-form solution for the maximisation step within the EMM learning. Specifically, the convergence of the iterative re-weighting algorithm, the de facto standard in estimation of elliptical distributions, requires constraints on both the functional formations of elliptical distributions and the data structure. For more detail, readers are referred to [13]. Although these limitations are have been recently somewhat relaxed [14; 15], applications of the EMM are still severely restricted.

To this end, we consider Riemannian manifold optimisation for parameter estimation in this context, which has proven to be extremely effective in problems related to positive definite matrices, as it naturally casts a general ill-posed constrained problem onto that of optimising on a convex half-cone, which can be solved via the vector space of matrices. In contrast, it is always difficult to handle the positive definite constraint in the Euclidean coordinates. Along this direction, Hosseini and Sra successfully applied gradient descent along the Riemannian manifold to the GMM problem, and achieved fast convergence speed without any sacrifice in the accuracy [16][17]. It is therefore natural to ask, whether a general estimation method based on the EMM can be approached from the manifold optimisation perspective?

1.1 Challenges and contributions

The first step towards our aim to introduce a class of feasible and computable EMMs is to define a proper Riemannian metric for elliptical distributions, as the metric completely determines the optimisation procedure of the EMM. A wide variety of works related to positive definite matrices adopt an intrinsic Riemannian metric which comes from the statistics (the Hessian of entropy [18] or the Fisher information [19]) of multivariate normal distributions. Such a metric is also adopted in [16][17] for estimating the GMM. It is therefore natural to first investigate whether such a “Gaussian”-based metric is an appropriate choice for the EMM. To this end, we start from the statistics, and first
assess the rationale of this metric. Then, in addition to the covariance matrices, EMM also needs to estimate the location vector. Location-covariance estimation is typically more complicated but less theoretically supported compared with the covariance problem [20]. A common current strategy is to reformulate the location-estimation problem into that of solely covariance estimation with one more dimension [13]. As reported in [16][17], this reformulation significantly accelerates the convergence speed. For the EMM, the reformulation is not direct due to the non-existence of a closed-form representation. We thus develop the corresponding reformulation for the EMM, and further find that such a reformulation during manifold optimisation attains the same metric of a natural gradient [21] descent for the location vector and the standard covariance estimation. Finally, we propose a general estimation method for the EMM, which overcomes the above limitations [14][15].

Robust EMM estimation is therefore badly needed and is rapidly emerging; for example, a toolbox in [22] which was originally designed for the GMM [17], has already included several types of elliptical distributions. However, the existing toolbox has not been generalised to the EMM. This paper therefore sets out to fill the void in the literature by rigorously establishing a whole new unifying framework for the analysis of EMMs, thus opening a new avenue for practical approaches based on real-world data. Unlike the current inconsistent solutions, the proposed framework is generic and can be considered a natural generalisation of parametrisation from the GMM. Our contributions can be summarised in the following:

- We rigorously unify typical elliptical distributions by means of their intrinsic relationships, which enables simple ways to generate samples and conduct further analysis;
- We introduce a Riemannian metric for the location vector and the covariance matrices within elliptical distributions, which provides us with further understanding of this reformulation. The approach is shown to admit straightforward physical interpretability and to include asymmetric distributions in a seamless and natural way;
- The proposed estimation approach for the EMM is general and generic, and includes the mean-shift algorithm as one of the special cases.

1.2 Related works

The GMM based estimation is well established and its importance has been widely acknowledged in the machine learning community. Since our focus is on the EMM, we omit the review of GMM and the readers are referred to [23] for a comprehensive review. To robustify the GMM model, the mixtures of the $t$-distribution have been thoroughly studied [7][8][9], on the basis of a generalised EM algorithm (expectation-conditional maximization). A more general mixture model has been proposed in [24] based on the Pearson type VII distribution (includes the $t$-distribution as a special case). Moreover, as the transformed coefficients in the wavelet domain tend to be Laplace distributed, a mixture of the Laplace distribution has been proposed in [10] for image denoising. Its more general version, a mixture of hyperbolic distributions, has also been recently introduced in [11]. Typically, these approaches employ generalised EM algorithms because contrary to the GMM, there is no closed-from solution at each maximisation step. Fortunately, the above distributions belong to the scale mixture of normal class, which can be regarded as a convolution of a Gamma distribution and a Gaussian distribution, which ensures the convergence of generalised EM algorithm. However, these approaches lack in generality, as e.g., for other elliptical distributions, the convergence is no longer guaranteed to be generalised. It is important to notice that despite several attempts, current mixture models, including [25][26][27], are of a rather ad hoc nature.

For a comprehensive text on the optimisation on the Riemannian manifold, we refer to [28], together with a seminal book on information geometry by Amari [29]. We here mainly focus on manifold optimisation of positive definite matrices. Specifically, pioneering in this direction is the work of Rao, which introduced the Rao distance to define the statistical difference between two multivariate normal distributions [30]. This work was later generalised by [19][31][32]. In the last decade, Wiesel proved the convergence of the iterative reweighting algorithm in [33] via the concept of geodesic convexity, and Zhang et al. further relaxed the convergence conditions in [14]. Sra and Hosseini [15] provided similar results from another perspective of the Riemannian manifold. For more details on the manifold of positive definite matrices, readers are referred to comprehensive works in [34][20]. Recently, Hosseini and Sra directly adopted the gradient descent on the Riemannian manifold for estimating the GMM, and achieved significant improvement over the traditional EM algorithm [16][17].
2 Preliminaries and notations

We first provide a brief introduction and notations of elliptical distributions, focusing especially on their relationships with commonly used distributions in statistical machine learning. Then, several key concepts in manifold optimisation are presented.

2.1 Elliptical distributions

A random variable \( x \in \mathbb{R}^M \) is said to have an elliptical distribution if and only if it admits the following stochastic representation [35].

\[
x = d \mu + R \Lambda u,
\]

where \( R \in \mathbb{R}^+ \) is a non-negative real scalar random variable which models the tail properties of the elliptical distribution, \( u \in \mathbb{R}^M \) is a random vector that is uniformly distributed on a unit spherical surface with the pdf within the class of \( \Gamma(M/2)/(2^{M/2}) \), \( \mu \in \mathbb{R}^M \) is a location (mean) vector, while \( \Lambda \in \mathbb{R}^{M \times M} \) is a matrix that transforms \( u \) from a sphere to an ellipse, and the symbol “=d” designates “the same distribution”. For a comprehensive review of elliptical distributions, we refer to [5][36].

Note that an elliptical distribution does not necessarily possess an explicit pdf, but can always be formulated by its characteristic function. However, when \( M = M \), that is, for a non-singular scatter matrix \( \Sigma = \Lambda \Lambda^T \), the pdf for elliptical distributions does exist and has the following form

\[
p_x(x) = \frac{1}{\sqrt{\det(\Sigma)}} \cdot \frac{\Gamma(M/2)/(2^{M/2})}{\pi^{M/2}} \cdot \left( \int_0^{\infty} t^{M-1} g(t^2) dt \right)^{-1} \cdot g((x - \mu)^T \Sigma^{-1} (x - \mu)),
\]

where \( g(\cdot) \) is called density generator and \( C_M \) is a constant solely related to the dimension \( M \).

Remark. Observe that the term \( C_M \) in (2) serves as a normalisation term, while when \( R \sim \sqrt{\chi^2_M} \), the term \( g(t) = \exp(-t/2) \) formulates the multivariate Gaussian distribution, thus indicating the generality of elliptical distributions.

For simplicity, the elliptical distribution in (2) will be denoted by \( \mathcal{E}(x|\mu, \Sigma, g) \).

2.2 Riemannian manifold

A Riemannian manifold \((M, \rho)\) is a smooth (differential) manifold \( M \) (i.e., locally homeomorphic to the Euclidean space) equipped with a smooth varying inner product \( \rho \) on its tangent space. The inner product also defines the Riemannian metric on the tangent space. So that, the length of a curve and angle of two vectors can be correspondingly defined. Curves on the manifold with the shortest paths are called geodesics, which exhibit constant instantaneous speed and generalise straight lines in the Euclidean space. The distance between two points on \( M \) is defined as the minimum length of all geodesics connecting these two points.

We use the symbol \( T_xM \) to denote the tangent space at the point \( x \), which is the first-order approximation of \( M \) at \( x \). Consequently, vectors on \( T_xM \) generalise the directional derivative, and the Riemannian gradient of a function \( f : M \to \mathbb{R} \) is defined with regard to the equivalence between its inner product with an arbitrary vector \( \xi \) on \( T_xM \) and the Fréchet derivative of \( f \) at \( x \). Moreover, a smooth mapping from \( T_xM \) and \( M \) is called the retraction, whereby an exponential mapping obtains the point on geodesics in the direction. Because the tangent spaces vary across different points on \( M \), parallel transport across different tangent spaces can be introduced on the basis of the Levi-Civita connection, which preserves the inner product and norm. Then, we can convert a complex optimisation problem on \( M \) into a more analysis friendly space, that is, \( T_xM \).

For covariance matrices, or more generally, positive definite matrices, although there are various metrics designed for measuring the distance between matrices [37][38][39][40], not all of them arise from the smooth varying inner product (i.e., Riemannian manifold), which would consequently give a “true” geodesic distance. The most popular such metric comes from the statistical manifold in which each point defined as a probability distribution. The inner product in such a manifold was adopted by Skovgaard [19] to measure dissimilarities through covariance matrices of two multivariate Gaussian distributions, in the form of \( \rho_\Sigma(\eta, \xi) \triangleq <\eta, \xi> \triangleq \text{tr}(\eta \Sigma^{-1} \xi \Sigma^{-1}) \), and its effectiveness has been comprehensively verified [16][17][32][37]. It is also possible to obtain a closed-form solution for the geodesic.
between two positive definite matrices $\Sigma_0$ and $\Sigma_1$, $\gamma(t) = \Sigma_0^{1/2}(\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2})^t\Sigma_0^{1/2}$, to yield its geodesic distance $d(\Sigma_0, \Sigma_1) = ||\ln(\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2})||_F$ [21]. A geodesic convex function $f$ can be defined as $f(\gamma(t)) \leq (1-t)f(\Sigma_0) + tf(\Sigma_1)$ with $t \in [0,1]$. We should point out that a function with a geodesic convex form ensures that the global optimum can be found, although such function may not be Euclidean convex.

3 Statistical Riemannian metrics for elliptical distributions

Typically, in the manifold $\mathbb{R}^M$, the metric has been widely defined through the Fisher information, which results in the natural gradient in manifold gradient descent [21] and represents an information geodesic measure, as outlined in a seminal book by Amari [29]. However, for positive definite matrices, natural gradient is not guaranteed to be explicitly obtained [21]. Alternatively, Burbea and Rao introduced the “entropy differential metric” [42] on the basis of entropy, which was later used by Hiai and Petz to define the Riemannian metric for positive definite matrices [18]. It needs to be pointed out that the intrinsic metric, that is, $tr(d\Sigma\Sigma^{-1}d\Sigma\Sigma^{-1})$, is obtained via the Hessian of Boltzmann entropy of multivariate normal distributions.

This allows us here to calculate the corresponding Riemannian metrics for the elliptical distributions.

**Proposition 1.** Consider the class of elliptical distributions, $\mathcal{E}(x|\mu, \Sigma, g)$. Then, the Riemannian metric for the location vector is given by

$$ds^2 = \left[\frac{4}{M} \int_0^\infty t^{1/2}(\dot{g}(t))^2 p_R(\sqrt{t}) dt \right] d\mu^T \Sigma^{-1} d\mu, \quad (3)$$

and the Riemannian metric for the covariance by

$$ds^2 = tr(d\Sigma\Sigma^{-1}d\Sigma\Sigma^{-1}) \quad (4)$$

**Proof.** The Riemannian metric for the mean vectors is directly obtained from the Fisher information matrix [43]. To obtain the Riemannian metric for the covariance matrix, we first calculate the Hessian of Boltzmann entropy, as follows,

$$H(x|\Sigma) = \int_{\mathbb{R}^M} p(x) \ln p(x) dx = \int_{\mathbb{R}^M} p(x)[-\frac{1}{2} \ln |\Sigma| + \ln C_M + \ln g(t)] dx$$

$$= -\frac{1}{2} \ln |\Sigma| + \ln C_M + \int_{\mathbb{R}^M} p_R(t) \ln g(t) dt. \quad (5)$$

Because $(\ln C_M + \int_{\mathbb{R}^M} p_R(t) \ln g(t) dt)$ is irrelevant to $\Sigma$, the Hessian of $H(x)$ can be calculated

$$\frac{\partial H(x|\Sigma + t\Sigma + h\Sigma)}{\partial \theta \partial h}|_{t=0, h=0} = \text{tr}(\Sigma \Sigma^{-1} \dot{\Sigma} \Sigma^{-1}). \quad (6)$$

The Riemannian metric thus can thus be obtained as $ds^2 = tr(d\Sigma\Sigma^{-1}d\Sigma\Sigma^{-1})$, which is the same as the case for multivariate normal distributions and is the mostly widely used metric.

Finally, on the basis of [43], we can now provide the following treatment for the elliptical distributions.

**Theorem 1.** Consider the class of elliptical distributions, $\mathcal{E}(x|\mu, \Sigma, g)$. Then, upon reformulating $x$ and $\Sigma$ as

$$y = [x^T, 1]^T, \quad \hat{\Sigma} = \begin{pmatrix} \Sigma + \lambda \mu \mu^T & \lambda \mu \\ \lambda \mu^T & \lambda \end{pmatrix}$$

(7)

gives the following Riemannian metric for the reformulated covariance,

$$ds^2 = tr(d\hat{\Sigma}\hat{\Sigma}^{-1}d\hat{\Sigma}\hat{\Sigma}^{-1}) = \lambda d\mu^T \Sigma^{-1} d\mu + tr(d\Sigma\Sigma^{-1}d\Sigma\Sigma^{-1}), \quad (8)$$

where $\lambda = 4/M \int_0^\infty t^{1/2}(g'(t))^2 p_R(t) dt$.

**Remark.** From Theorem 1, we can see that after reformulation manifold optimisation is actually performed under the same Riemannian metric as a simultaneous estimation of the location and the covariance in their respective Riemannian manifolds. This provides another perspective in understanding the proposed reformulation and enhanced physical interpretability.
Table 1: Typical members of the elliptical family of distributions

| Types                | \( g(t) & C_M \)                              | Typical Multivariate Distributions |
|----------------------|-----------------------------------------------|-----------------------------------|
| Kotz Type [5]        | \( \mathbb{R}^2 = d, G^{1/2}, G \sim \text{Ga}(2a + M, \frac{2}{|d|} b) \), \( a > 1 - \frac{2}{|d|}, b, s > 0 \) |                               |
| Pearson Type VII [5] | \( \tau^{-1} \sim \text{Ga}(s - \frac{d}{|d|}, \frac{1}{s}) \), \( v > 0, s > \frac{d}{|d|} \) |                               |
| Scale Mixture Type   | \( \mathbb{R}^2 = d, G : V, G \sim \text{Ga}( \frac{1}{|d|}, 2) \) |                               |
| Hyperbolic Type [5]  | \( \tau \sim \text{GIG}(v, a, \lambda), v > 0, a > 0, \lambda \in \mathbb{R} \) |                               |
| Other Types [2][3]   | \( \tau \sim \partial \text{Kov}(\frac{X}{\sqrt{\tau}}) / \partial \tau \) \( \tau \sim \text{Sos}(\frac{a}{d}), a \in (0, 2) \) |                               |
| Pearson Type II [5]  | \( \mathbb{R}^2 \sim \text{Beta}(M/2, s), s > 0 \) |                               |

4 Manifold optimisation for the elliptical mixture model (EMM)

We next introduce a concise summary on the elliptical distributions, in order to provide clarify in handling different types of elliptical distribution in later sections. Then, we lay out the EMM optimisation problem, following by the reformulation and manifold optimisation.

4.1 Elliptical family of distributions

The elliptical family of distributions is quite general, and includes many widely used standard distributions as special cases, e.g., the Gaussian distribution. A comprehensive summary can be found in Chapter 3 in [5], but involves complicated closed-form formulations for each type of elliptical distribution. In addition, the open literature employs different notations and formulations to categorise these distributions, which may lead to confusion. To this end, we here list provide a unifying summary of typical elliptical distributions which is achieved through stochastic representations of (1). This makes it possible to avoid complicated formulations, and instead classify different categories simply through several typical distributions of \( \mathcal{R} \). Uniquely, this makes it possible to generate high-dimensional samples from the one-dimensional \( \mathcal{R} \) for a range of elliptical distributions, and also further clarifies the commonalities between the members of elliptical family of distributions.

In general, according to [1], an arbitrary elliptical distribution can be represented by \( \mathcal{R} \) and \( \mathbf{u} \). As the uniformly distributed \( \mathbf{u} \) only relates to the dimension \( M \), we focus on the parameter \( \mathcal{R} \), or equivalently, \( \mathbb{R}^2 \), in order to provide a unifying summary of typical elliptical distributions which are listed[1] in Table 1. The proof of this is obvious and can be achieved by direct validation. We omit it here due to the space limitation.

4.2 The elliptical mixture model (EMM)

Generally, we assume the EMM consists of \( K \) mixtures, each elliptically distributed. To make the proposed EMM flexible enough to capture inherent structures in data, in our framework it is not necessary for every elliptical distribution to have the same density generator (denoted by \( E_k(\mathbf{x} | \mu_k, \Sigma_k, g_k) \)).

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[1] The term \( \mathbb{R}^2 \) is frequently used in practice because it has the same distribution as the quadratic form \( (\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu) \) (i.e., the Mahalanobis distance).

[2] The symbol \( \text{Ga}(x, y) \) represents the Gamma distribution, \( \text{Ga}(x, y) = \frac{y^x e^{-x}}{\Gamma(x)} \); \( \text{GIG}(x, y) \) is the generalised inverse Gaussian distribution, \( \text{GIG}(x, y) = \frac{(s/y)^{y/2} \Gamma(x+y)}{\Gamma(x) \sqrt{\pi y}} e^{-xs/(y+y)} \); \( \text{Kov}(x) \) denotes the Kolmogorov-Smirnov distribution, \( \text{Kov}(x) = 1 - 2 \sum_{n=1}^{\infty} (-1)^{n+1} \exp(-2n^2 x^2) \); \( \text{Beta}(x, y) \) is the Beta distribution, \( \frac{\Gamma(x+y)}{\Gamma(x) \Gamma(y)} y^{x-1} (1 - t)^{y-1} \) and \( \text{Sos}(\alpha) \) is the symmetric \( \alpha \)-stable distribution with index \( \alpha \). In addition, \( B_s(y) \) is the Bessel function of the second kind, whilst \( \Gamma(x) \) is the Gamma function.
In finite mixture models, latent variables \( z_k \in \{0, 1 \} \) are binary, to represent membership to the \( k \)-th mixture. The probability of choosing the \( k \)-th mixture is denoted by \( p(z_k = 1) = \pi_k \), so that \( \sum_{k=1}^{K} \pi_k = 1 \) and \( \sum_{k=1}^{K} \pi_k = 1 \). Upon rearranging the scalars \( z_k \) into a vector \( z \), we can further simply write \( p(z) = \prod_{k=1}^{K} \pi_k \). For a set of observed i.i.d samples \( x_n, n = 1, 2, 3, \cdots, N \), the negative log-likelihood can be obtained as

\[
J = -\ln p(x_1, x_2, \cdots, x_N) = -\sum_{n=1}^{N} \sum_{k=1}^{K} \pi_k \cdot C_M \cdot g_k \left( (x_n - \mu_k)^T \Sigma_k^{-1} (x_n - \mu_k) \right). \tag{9}
\]

4.3 Manifold optimisation and reformulation

The estimation of \( \pi_k, \mu_k \) and \( \Sigma_k \) requires the minimisation of \( J \) in (9), which is not possible to achieve in closed-form. We therefore proceed to introduce the manifold optimisation for the EMM framework, by first reformulating the terms \( \mu_k \) and \( \Sigma_k \) to \( \tilde{\Sigma}_k \) on the basis of Theorem 7.

**Theorem 2.** Given the \( \tilde{\Sigma}_k \) reformulated in (7), optimisation on the following function

\[
\tilde{J} = -\sum_{n=1}^{N} \sum_{k=1}^{K} \frac{\pi_k \cdot C_M}{\det(\Sigma_k)} \cdot g_k \left( y_n^T \tilde{\Sigma}_k^{-1} y_n - c_k \right), \tag{10}
\]

has the same global optimum as the original non-convex function in (9) when \( c_k = \frac{\sum_{n=1}^{N} E[z_{nk} | \theta]}{\sum_{n=1}^{N} E[z_{nk} | \psi_k(t_{nk})]}; \quad \tilde{\Sigma}_k^* = \left( \begin{array}{cc} \Sigma_k^* + \lambda_k^* \mu_k^T \mu_k & \lambda_k^* \mu_k^T \\ \lambda_k^* \mu_k & \lambda_k^* \end{array} \right), \)

where \( \lambda^* = 1/c_k; \quad \tilde{\Sigma}_k^* \) and \( \mu_k^* \) achieve the minimum of (9).

**Proof.** The proof of the first property rests upon a generalisation of the result in the GMM [17], and its proof is analogous to that of Theorem in [17]. In fact, for Gaussian distribution, \( \psi_k(t_{nk}) \equiv 1 \) leads to \( \lambda_k^* \equiv 1 \), which is the reformulation adopted in the GMM in [17].

The second property can be proved via the relationship \( y_n^T \tilde{\Sigma}_k^{-1} y_n = (x_n - \mu_k)^T \Sigma_k^{-1} (x_n - \mu_k) + 1/\lambda_k \). This relationship can be easily verified through a decomposition of \( \tilde{\Sigma}_k \) to \( \begin{bmatrix} 1 & \mu_k \\ \mu_k^T & \lambda_k \end{bmatrix} \) [48].

On the basis of the reformulated \( \tilde{J} \), we have the Euclidean gradient for the reformulated \( \tilde{\Sigma}_k \):

\[
\nabla_E \tilde{J}(\tilde{\Sigma}_k) = \sum_{n=1}^{N} E[z_{nk} | \theta] \psi_k(y_n^T \tilde{\Sigma}_k^{-1} y_n) \tilde{\Sigma}_k^{-1} y_n y_n^T \tilde{\Sigma}_k^{-1} + \frac{\sum_{n=1}^{N} E[z_{nk} | \theta]}{2} \tilde{\Sigma}_k^{-1}. \tag{11}
\]

Before moving to the Riemannian manifold optimisation, we shall first inspect the properties of the introduced gradient. The optimum value occurs when \( \nabla_E \tilde{J}(\tilde{\Sigma}_k) = 0 \), and we arrive at \( \tilde{\Sigma}_k = -2 \cdot \frac{\sum_{n=1}^{N} E[z_{nk} | \theta] \psi_k(y_n^T \tilde{\Sigma}_k^{-1} y_n) y_n y_n^T \} \cdot \psi_k(y_n^T \tilde{\Sigma}_k^{-1} y_n) \), that is, the M-estimators, decrease to 0 when the Mahalanobis distance \( y_n^T \tilde{\Sigma}_k^{-1} y_n \) increases. It is well-known that the Mahalanobis distance is a scale-free metric, which is particularly suited to measure outliers. Thus, \( y_n \) with large Mahalanobis distance result in small values of \( \psi_k(y_n^T \tilde{\Sigma}_k^{-1} y_n) \), and would have little impact on the \( \tilde{\Sigma}_k \), which therefore generates the robustness of the EMM. Furthermore, the existence of \( \psi_k \) also mitigate the problem of the singular distributions during estimation.

Next, we obtain the Riemannian gradient \( \nabla_R \tilde{J}(\tilde{\Sigma}_k) = \tilde{\Sigma}_k \nabla_E \tilde{J}(\tilde{\Sigma}_k) \tilde{\Sigma}_k \) for the manifold optimisation [16], through the retraction of

\[
\Re_{\tilde{\Sigma}_k} \left( \nabla_R \tilde{J}(\tilde{\Sigma}_k) \right) = \tilde{\Sigma}_k + \nabla_R \tilde{J}(\tilde{\Sigma}_k) + \frac{1}{2} \nabla_R \tilde{J}(\tilde{\Sigma}_k) \tilde{\Sigma}_k^{-1} \nabla_R \tilde{J}(\tilde{\Sigma}_k),
\]

which is an approximation of the exponential mapping whilst ensuring of the computational feasibility. This makes it possible to implement the steepest manifold gradient descent algorithm to minimise \( \tilde{J} \).
We verified the proposed framework on both synthetic and real-world data. The synthetic dataset was generated without (in orange) and with the noise (in red) in Figure 5. As can be seen from this figure, the outliers at (10,0) have dramatically biased the location estimation in the GMM away from the ground-truth, while estimations based on the EMM model remain almost unchanged. This demonstrates the robustness of the EMM, compared with the GMM.

4.4 Regularisation

We have shown that the EMM can relieve the problem of singular distributions, however, it cannot completely alleviate the problem for all the elliptical distributions. We therefore resort to the regularisation of the covariance matrix, which is basically imposing the sparsity of the precision matrix. While those regulators cannot ensure the geodesic convex property, we here follow the approach by Ollila and Tyler [49] to impose the $\text{tr}(\Sigma_k^{-1})$ as the regulator, where $\alpha$ controls the weight of the regulator. In fact, similar regularisation forms can be obtained when adding an inverse-Wishart prior distribution of $\Sigma_k$ followed by the maximum a posterior instead of the maximum likelihood process [50]. Specific to the EMM, the advantages of this regulator are that it is strictly geodesic convex in $\Sigma_k$ and the solutions are ensured to exist for any data configuration [49]. In this case, the reformulation turns to

$$\tilde{\Sigma}_k = \left(\Sigma_k - \alpha I_M + \lambda_k \mu_k \mu_k^T \begin{array}{l} \lambda_k \mu_k \\ \lambda_k \end{array} \right).$$

Therefore, the $\tilde{\Sigma}_k$ are always full-rank, and thus completely avoid the singular distributions during estimation as desired.

When reaching the optimum, i.e., $\tilde{\Sigma}_k = -2 \cdot \sum_{n=1}^{N} \frac{\mathbb{E}[z_{nk}|\theta] \psi_k(t_{nk})x_n}{\mathbb{E}[z_{nk}|\theta]}$, we can obtain the following equations for $\mu_k$ and $\Sigma_k$:

$$\mu_k = \frac{\sum_{n=1}^{N} \mathbb{E}[z_{nk}|\theta] \psi_k(t_{nk})x_n}{\sum_{n=1}^{N} \mathbb{E}[z_{nk}|\theta]} ,$$

$$\Sigma_k = -2 \cdot \frac{\sum_{n=1}^{N} \mathbb{E}[z_{nk}|\theta] \psi_k(t_{nk})(x_n - \mu_k)(x_n - \mu_k)^T}{\sum_{n=1}^{N} \mathbb{E}[z_{nk}|\theta]} + \alpha I_M .$$

We can find from (11) that when $\alpha$ increases, $\Sigma_k$ is more likely to be an identity matrix, in which the estimation on $\mu_k$ is actually the mean-shift algorithm.

5 Numerical results

We verified the proposed framework on both synthetic and real-world data. The synthetic dataset was generated according to [23] and consisted of two Gaussian-distributed clusters, with their location vectors satisfying

$$||\mu_1 - \mu_2||^2 \geq c \cdot \max\{\text{tr}(\Sigma_1), \text{tr}(\Sigma_2)\},$$

where $c$ is a constant that controls the separation. We set $c = 5$ here for clear illustration. Each Gaussian-distributed cluster contained 10,000 samples, and the small set of points centred at (10,0) was treated as noise, which contained 100 samples from a spherical Gaussian distribution. The elliptical distribution used was the $t$-distribution with $v = 1$, as in Table 1. The estimation results were generated without (in orange) and with the noise (in red) in Figure 5. As can be seen from this figure, the outliers at (10,0) have dramatically biased the location estimation in the GMM away from the ground-truth, while estimations based on the EMM model remain almost unchanged. This demonstrates the robustness of the EMM, compared with the GMM.

Then, in order to assess the quality of features extracted by the EMM, we considered the probability of the location vector in image for reconstruction. We adopted five types of EMMs for comprehensive comparison: the $t$-distribution with $v = 1$ (denoted by $Tdist.1$), the $t$-distribution with $v = 100$ (denoted by $Tdist.100$), the Kotz type with $\alpha = 1.5, b = 1, s = 1.5$ (denoted by Kotz1), the Kotz type with $\alpha = 1.5, b = 1, s = 0.5$ (denoted by Kotz2), and the logistic distribution (denoted by Logi). We
randomly chose 50 images from the BSDS500 database \cite{51} for mixture modelling. We also used the k-means in vl-feat toolbox \cite{52} as initialisations. After estimations of each EMM together with the GMM, the mean value was assigned to each pixel with regard to the posterior distribution of $\pi_k$, followed by the reconstruction. We used the peak signal-to-noise ratio (PSNR) as the quality assessment metric, whereby a higher PSNR value indicates lower reconstruction error, that is, the more accurate and effective features were captured by the elliptical mixture model. The results are presented in Table 2.

For some representative results are visualised in Fig. 5. By comparing Fig. 5(b) and (c), we can see that because $t$-distributions with $v = 1$ is more heavily-tailed than that with $v = 100$, its reconstruction is more clear and it also well reconstructs the details of the ship. Furthermore, both Kotz1 and Kotz2 are not geodesic convex, which means they cannot be estimated via the iterative re-weighting algorithm. However, the proposed manifold optimisation ensures that an optimum for these distributions is found. Moreover, the Kotz1 model in Figure 5(d) is the only one which is inferior to the GMM. This may be due to its lighter tails ($s > 1$), which indicates it is sensitive to the details. In contrast, other elliptical distributions given here are all heavy-tailed than the Gaussian distribution, and are thus more robust to outliers. This all demonstrates the flexibility of the EMM framework, and the ability of the proposed manifold optimisation algorithm to provide a general solution to the EMM.

6 Conclusions

We have proposed a universal framework for estimating the EMM for the general case of unbalanced data and mixtures of different members of the class of elliptical distributions. We have revisited the
statistics of elliptical distributions to justify the effectiveness of the Riemannian metrics adopted the EMM learning process. We have also analysed the rationale for the problem reformulation under the framework of Riemannian manifold, and have introduced its EMM version. The existing elliptical distributions have also been unified in this paper, to provide much needed flexibility in choosing the EMM. Numerical results have not only demonstrated the robustness and flexibility of the proposed EMM framework, but also further highlighted the physical interpretability and the effects of individual distributions on the EMM.

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