SINGULAR SET AND CURVATURE BLOW-UP RATE OF THE LEVEL SET FLOW

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Abstract. Under certain conditions such as the 2-convexity, a singularity of the level set flow is of type I (in the sense that the rate of curvature blow-up is constrained before and after the singular time) if and only if the flow shrinks to either a round point or a $C^1$ curve near that singular point. Analytically speaking, the arrival time is $C^2$ near a critical point if and only if it satisfies a Lojasiewicz inequality near the point.

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1. Introduction

Given an initial hypersurface $\Sigma_0$ in $\mathbb{R}^n$ that is closed, connected, smoothly embedded, and strictly mean-convex, its evolution by the mean curvature vector, i.e., the mean curvature flow, advances monotonically towards the inside until it becomes singular in finite time (cf. E). Nevertheless, there is a unique way to continue the flow through the singularities.

Let $\Omega$ be the domain bounded by $\Sigma_0$. Evans and Spruck in [ES] (see also [CGG]) showed that there is a unique viscosity solution $u \in C^{0,1}(\Omega)$ to the following

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\textsuperscript{2}Here $n \geq 2$ is an integer.

\textsuperscript{3}That is, compact without boundary.

\textsuperscript{4}Namely, the sum of all principal curvatures, with respect to the inward normal, is strictly positive. In fact, one can start with a mean-convex hypersurface since the mean curvature flow will immediately become strictly mean-convex by the strong maximum principle.

\textsuperscript{5}Because the strict mean-convexity is preserved along the flow; specifically, the minimum of the mean curvature of the time-slice is non-decreasing by the maximum principle.

\textsuperscript{6}By the Jordan-Brouwer separation theorem, $\mathbb{R}^n \setminus \Sigma_0$ consists of two open connected sets, the inside $\Omega$ and the outside $\mathbb{R}^n \setminus \Omega$, where $\Omega$ is called the region bounded by $\Sigma_0$. 

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degenerate elliptic equation

\[ - \left( I - \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \cdot \nabla^2 u = 1 \]

using the elliptic regularization and the comparison principle for viscosity solutions. The significance of the solution is that near every regular point of \( u \), that is,

**Definition 1.1.** A point \( x \in \Omega \) is called a regular point of \( u \) provided that \( u \) is smooth near \( x \) with \( \nabla u(x) \neq 0 \). Otherwise, it is called a singular point.

the equation (1.1) is satisfied pointwisely and can be rewritten as

\[ - \nabla \cdot \frac{\nabla u}{|\nabla u|} = \frac{1}{|\nabla u|}, \]

meaning that the flow of the level hypersurfaces\(^6\) is a mean curvature flow\(^7\) with positive mean curvature.\(^8\) Moreover, it is proved in [ES] that \( \Sigma_t = \{ u = t \} \) prior to the first singular time, where \( \Sigma_t \) is the mean curvature flow starting at \( \Sigma_0 \). That is to say, the flow of the level sets of \( u \), which is called a level set flow\(^9\) is indeed an extension of the mean curvature flow starting at \( \Sigma_0 = \partial \Omega \). For each point \( x \in \Omega \), \( u(x) \) is the time when the flow travels through \( x \), so \( u \) is called the arrival time of the level set flow. Note that the level set flow sweeps the domain \( \Omega \) and vanishes right after time \( t = \max_{\Omega} u \), which is called the extinction time.

Regarding the regularity of the level set flow, it is shown in [I1] (see also [W1]) that \( H^{n-1}[\{ u = t \}] \) defines a Brakke flow of integral varifolds. In particular, Huisken’s monotonicity formula (cf. [H2] and [I2]) holds for the level set flow; it follows that the flow has finite entropy (cf. [CM1]). In addition, Ilmanen proved that for almost every \( t \in (0, \max_{\Omega} u) \) the singular set on \( u = t \) is of \( H^{n-1} \)-measure zero. Later, White in [W1] showed that the Hausdorff dimension of the singular set is at most \( n - 2 \); furthermore, he proved in [W2] and [W3] that the tangent flow (see [I2]) at a singularity is, up to a rigid motion, \( \{ \sqrt{-t}C_k \}_{t \in (-\infty, 0)} \), where

\[ C_k = S^{n-k-1} \frac{1}{\sqrt{2(n-k-1)}} \times \mathbb{R}^k, \]

for some \( k \in \{ 0, \cdot \cdot \cdot , n-2 \} \). As such, singular points can be classified according to their associated tangent flows as follows:

**Definition 1.2.** A singular point is called a

\[ \begin{cases} 
\text{round point,} & \text{if } k = 0, \\
\text{k-cylindrical point,} & \text{if } k \in \{ 1, \cdot \cdot \cdot , n-2 \},
\end{cases} \]

where \( k \) is the integer in (1.3)\(^10\)

\(^6\)Note that near a regular point, every level set of \( u \) is a smoothly embedded hypersurface.

\(^7\)The LHS of (1.2) is the mean curvature of the level hypersurface and the RHS is the normal speed of the flow of the level sets.

\(^8\)Because we have \( H = |\nabla u|^{-1} \) by (1.2), where \( H \) denotes the mean curvature of the level hypersurface.

\(^9\)Our definition of a “level set flow” is different from (but closely related to) the convention. For instance, see [H1] to compare the definitions.

\(^10\)Whenever the cylindrical points are mentioned, we always assume that \( n \geq 3 \).
Based on the noncollapsing property of the level set flow, Haslhofer and Kleiner in \cite{HK} gave the smooth estimates (at regular points) in terms of the mean curvature. Colding and Minicozzi in \cite{CM4} established that $u \in C^{1,1}(\Omega)$ and it is twice differentiable everywhere on $\Omega$; moreover, they proved that the singular points are exactly the critical points of $u$. Regarding the further regularity, they proved in \cite{CM5} that $u \in C^2(\Omega)$ if and only if there is exactly one singular time when the level set flow shrinks to either a round point or a closed connected $C^1$ embedded $k$-manifold consisting of $k$-cylindrical points, see Theorem 2.12.

In this paper we give a localized version of Theorem 2.12, namely,

**Theorem 1.3.** The Hessian $\nabla^2 u$ is continuous at a singular point if and only if the point is either a round point or a $C^1$ embedded $k$-manifold.

To distinguish them from other singular points, such points (as stated in Theorem 1.3) are called regular singular points (see Definition 3.1). As is shown in \cite{CM6}, if $p$ is a regular singular point, then $u$ satisfies a Lojasiewicz inequality

$$|u(x) - u(p)|^\frac{1}{2} \leq \beta |\nabla u(x)|$$

in a neighborhood of $p$ (see Proposition 3.2). Since $|\nabla u|^{-1}$ is the mean curvature (see \cite{CM4}), which is comparable with the second fundamental form $\kappa_1$ by the pinching estimate in \cite{HK}, the Lojasiewicz inequality can be interpreted as the rate of blow-up of the second fundamental form does not exceed the rate of $|t - u(p)|^{-\frac{1}{2}}$ near the singularity in space-time (see \cite{CM6}). Such a singularity is then called a type I singularity (see Definition 3.3) of the level set flow. Unlike the traditional definition of a type I singularity (of the mean curvature flow) where the rate of blow-up of curvature is constrained only before the (first) singular time, here the constraint is also imposed to the flow past the singular time. Note that in the case of a regular singular point, the flow locally vanishes after the singular time; thus, the constraint is void after the singular time. However, in the scenario of a neckpinch where the flow is locally split into two pieces after the singular time, the singularity is not of type I due to the rapid clearing-out phenomenon in \cite{CM3} (see Theorem 3.8), even though the rate of blow-up of curvature is possibly bounded by the designated rate prior to the singular time.

It turns out that the type I condition is not only necessary but also sufficient for a singular point to be a regular singular point, so long as the singular point is either a round point or a 1-cylindrical point (which is called a neckpinch singularity of the flow in \cite{SS}). Specifically, we find the following:

**Theorem 1.4.** Suppose that the singular set consists of only round points and/or 1-cylindrical points, then for any singular point $p$ the following three statements are equivalent:

1. $u$ is $C^2$ near $p$.
2. The singular set near $p$ is either a round point or a $C^1$ embedded curve.
3. The Lojasiewicz inequality holds near $p$.

\textsuperscript{11}In other words, the largest principal curvature is comparable with the normal speed of the flow.

\textsuperscript{12}Because we require the Lojasiewicz inequality to hold in a neighborhood of $p$ in $\Omega$.

\textsuperscript{13}That is, the singular point is a saddle point of $u$. 

There are some situations where the hypothesis in Theorem 1.4 is realized. For instance,

**Proposition 1.5.** If either one of the following conditions holds, only round points or 1-cylindrical points can arise as singular points:

- The dimension $n$ is two or three.
- The initial hypersurface $\Sigma_0$ is strictly 2-convex.
- The entropy of $\Sigma_0$ is strictly lower than that of the 2-cylinder $C_2$.

In the second condition, the strictly 2-convexity of $\Sigma_0$ means that $\kappa_1 + \kappa_2 > 0$, where $\kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_{n-1}$ denote the principal curvatures ordered by magnitude. This is a condition stronger than the strict mean-convexity but weaker than the strict convexity.\(^{14}\) In addition, note that by the compactness of $\Sigma_0$ we have

\begin{equation}
\kappa_1 + \kappa_2 \geq \alpha H
\end{equation}

for some $\alpha > 0$. It is shown in CHN (see also HK) that the level set flow is uniformly 2-convex in the sense that (1.4) holds at every regular point. In light of the asymptotic behavior of the level set flow near a singular point (see Remark 2.5) and that $C_k$ is not 2-convex if $k \geq 2$, it is clear that the second condition in Proposition 1.5 yields the result. As for the third condition, the result follows from the observation that the entropy (see (3.12)) of the tangent flow at any singular point is bounded above by $E[\Sigma_0]$ (cf. CM1), where $E[\Sigma_0]$ is the entropy of $\Sigma_0$, and that

\begin{equation}
1 < E[C_0] < E[C_1] < E[C_2] < \cdots < E[C_{n-2}] < 2
\end{equation}

(cf. CM1 and S).

Theorem 1.3 will be proved in Section 2. In Section 2.1 we explain that $u$ is $C^2$ near any round point. It is in Theorem 2.10 that we prove Theorem 1.3 for the case of cylindrical points. In addition, in Corollary 2.11 we show that the continuity of $\nabla^2 u$ at a cylindrical point $p$ yields that $u$ is $C^2$ near $p$. In the end of Section 2 we will see how Colding-Minicozzi’s characterization of the globally $C^2$ regularity, i.e., Theorem 2.12, is related to Theorem 1.3, especially why the flow has only one singular time when $u \in C^2(\bar{\Omega})$.

For Theorem 1.4, the equivalence of the first two statements comes from the discussion in the last paragraph.\(^{16}\) Thus, in Section 3 we focus on associating the regular singular points (see Definition 3.1) with the type I singular points (see Definition 3.3). Given that all regular points must be of type I (see Proposition 3.2), the last ingredient of proving Theorem 1.4 is to show that a type I singular point, especially when it is a 1-cylindrical point\(^{17}\) is a regular singular point. This is the bulk of Section 3.2, where we are devoted to prove Theorem 3.10 with the help of Theorem 3.8 from Section 3.4.

\(^{14}\)More generally, by $m$-convexity we mean that $\kappa_1 + \cdots + \kappa_m \geq 0$. Note that 1-convex is convex and $(n-1)$-convex is mean-convex.

\(^{15}\)That is, 1-convex (i.e. convex) $\Rightarrow$ 2-convex $\Rightarrow$ $\cdots$ $\Rightarrow$ $(n-1)$-convex (i.e. mean-convex).

\(^{16}\)Note that if the singular set near a singular point $p$ is a $C^1$ embedded curve, then $p$ can never be a round point by Section 2.3.

\(^{17}\)By Section 2.1, a round point is already a regular singular point.
2. Continuity of Hessian and Singular Set

The objective of this section is to prove Theorem 1.3, the localized version of Theorem 2.12 in [CM5]. To this end, we will first review Colding-Minicozzi’s regularity theorem in [CM4] and the calculations therein. The analysis will be divided into two cases according to the types of singularities: the round points in Section 2.1 and the cylindrical points in Section 2.2.

For convenience, from now on let us denote the singular set by $S$, the set of round points by $S_0$, and the set of $k$-cylindrical points by $S_k$; thus we have

\[ S = S_0 \cup S_1 \cup \cdots \cup S_{n-2}. \]

Also, as equation (1.1) is translation-invariant, for ease of notation we hereafter assume that 0 is a singular point with $u(0) = 0$.

First of all, let us invoke the following theorem in [CM4] on the regularity of the level set flow and the characterization of singular points.

**Theorem 2.1.** The solution $u$ to (1.1) is globally $C^{1,1}$ and twice differentiable everywhere in the domain. Furthermore, the singular points are exactly the critical points of $u$.

Given a regular point $x$ in the domain, let $N = \frac{\nabla u}{|\nabla u|}$ be the inward unit normal vector and \{e$_1$, · · · , e$_{n−1}$\} be an orthonormal basis for the tangent hyperplane $T_x \{u = u(x)\}$. By the calculations in [CM4] we have

\[ -\nabla^2 u \cdot (N \otimes N) = \left| A \right|^2 + \Delta H \frac{H^3}{H^3}, \]

\[ -\nabla^2 u \cdot (N \otimes e_i) = \frac{\nabla_i H}{H^2}, \]

\[ -\nabla^2 u \cdot (e_i \otimes e_j) = \frac{A_{ij}}{H}, \]

where $A$ denotes the second fundamental form (of the level hypersurface) and $\Delta$ is the Laplace-Beltrami operator (on the level hypersurface). It follows that the Hessian of $u$ can be written as

\[ (2.1) \quad \nabla^2 u(x) = -(N, e_1, \cdots, e_{n−1}) \left( \left| A \right|^2 + \frac{\nabla H}{H} \frac{\nabla H}{H} + \frac{\nabla^2 H}{H^2} \right) (N, e_1, \cdots, e_{n−1})^T, \]

where $(N, e_1, \cdots, e_{n−1})$ denotes the orthogonal $n \times n$ matrix consisting of the column vectors $(N, e_1, \cdots, e_{n−1})$ and the notation $T$ in the upper right corner stands for the transpose of matrices. Note that the Hessian is invariant under the parabolic re-scaling.

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18Namely, $x \mapsto u(x + p) - c$ satisfies (1.1) as well.
19Note that by this setting the point 0 is an interior (singular) point of the domain and the time 0 is no longer the initial time of the flow.
20The function $x \mapsto \lambda^{-2} u(\lambda x)$ is also a solution to (1.1) and has the same Hessian as $u$ at the corresponding points.
2.1. **Round Points.** Let us assume in this subsection that 0 is a round point. By Brakke’s regularity theorem (cf. [B] and [I1]), given $\epsilon > 0$, $M > \sqrt{2(n-1)}$, and $m \geq 2$, there exist $t_0 < 0$ such that for every $t \in [t_0, 0)$, the set
\[
\left( \frac{1}{\sqrt{-t}} \{ u = t \} \right) \cap B_M
\]
is a normal graph over the $(n-1)$-sphere $C_0$ with $C^m$ norm at most $\epsilon$. After undoing the normalization (with respect to the scale $\sqrt{-t}$) and weaving the level sets all together, we see that the contour map of $u$ in $\{ u \geq t_0 \} \cap B_M \sqrt{t_0}$ resembles the contour map of the function
\[
x \mapsto \frac{-1}{2(n-1)} |x|^2,
\]
which is the trajectory of the shrinking sphere $\sqrt{-t}C_0$ as $t \nearrow 0$. By virtue of (2.1) and the asymptotically spherical behavior of the level set flow, Colding and Minicozzi showed in [CM4] that
\[
\nabla^2 u(0) = -\frac{1}{n-1} I_n.
\]
Therefore, $\nabla^2 u$ is continuous at 0 and $u$ is $C^2$ in a neighborhood of 0.

**Remark 2.2.** On the other hand, if the level set flow shrinks to a point $p$ with the property for some $\delta > 0$ the set $u = u(p) - \delta$ is a closed, connected, smoothly embedded, and convex hypersurface in a neighborhood of $p$, then the level set flow in the region $\hat{\Omega}$ bounded by $u = u(p) - \delta$ near $p$ agrees with the mean curvature flow starting at $\partial \hat{\Omega}$ (cf. [ES]), which by [GH] (for $n = 2$) and [H1] (for $n \geq 3$) shrinks to a point (which must be $p$) in an asymptotically spherical manner. Therefore, $p$ is a round point.

2.2. **Cylindrical Points.** Throughout this subsection we assume that 0 is a $k$-cylindrical point for some $k \in \{ 1, \cdots, n-2 \}$. Additionally, by the rotational invariance of (1.1)
we may assume that the tangent flow at 0 is $\sqrt{-t}C_k$.

Also, we will adopt the following notation for coordinates:
\[
x = (y, z) \in \mathbb{R}^{n-k} \times \mathbb{R}^k.
\]
Let us begin the analysis of cylindrical points with the following remarkable theorem in [CM2] (see also [CM3]).

**Theorem 2.3.** Given $\epsilon > 0$ and $M > \sqrt{2(n-k-1)}$, there exist $\delta > 0$ and $L > 0$ so that if for some $t_0 < 0$, the set
\[
\left( \frac{1}{\sqrt{-2t_0}} \{ u = 2t_0 \} \right) \cap B_L
\]
is a normal graph over a cylinder congruent to $C_k$ with $C^m$ norm at most $\delta$ where $m \geq 2$ is an absolute constant, then for every $t \in [t_0, 0)$, the set
$$\left( \frac{1}{\sqrt{-t}} \{ u = t \} \right) \cap B_M$$
is a normal graph over $C_k$ with $C^m$ norm no more than $\epsilon$.

Recovering from the normalization (with respect to the scale $\sqrt{-t}$) and having the level sets woven together, we see that the contour map of $u$ in
$$\{ u \geq t_0 \} \cap B_M \setminus C_\phi,$$
where
$$C_\phi = \{ |y| \leq |z| \tan \phi \}$$
is the solid cone with apex 0, axis $\{0\} \times \mathbb{R}^k$, and angle
$$\phi = \sin^{-1} \frac{\sqrt{2(n-k-1)}}{M},$$
approximates to the contour map of the function
$$\phi(u) \mapsto \frac{-1}{2(n-k-1)} |y|^2,$$which is the trajectory of the shrinking truncated $k$-cylinder $\sqrt{-t} \left( C_k \cap B_M \right)$ as $t \nearrow 0$. This motivates the following definition.

**Definition 2.4.** (Cylindrical Scale) Given positive constants $\phi$ and $\epsilon$, the level set flow is said to be $(\phi, \epsilon)$-asymptotically cylindrical near the $k$-cylindrical point 0 on the cylindrical scale provided that for every $t < 0$, in $B_r \setminus C_\phi$ the two sets $u = t$ and $|y|^2 + 2(n-k-1)t = 0$ are $\epsilon$-close in the $C^m$ topology after rescaling by the factor $\frac{1}{\sqrt{-t}}$.

Regarding the two parameters $\phi$ and $\epsilon$ in Definition 2.4, $\phi$ can be used to determine the extent of the asymptotic regime after the normalization and $\epsilon$ if a measure of the degree to which the normalized level set of $u$ is close to $C_k$.

**Remark 2.5.** By rechoosing $t_0$ in Theorem 2.3 even closer to 0, we may assume (owing to Brakke’s regularity theorem) that
$$\phi(u) \mapsto \frac{-1}{2(n-k-1)} |y|^2,$$
is a normal graph over $C_k$ with $C^m$ norm at most $\frac{\delta}{2}$. Then applying Theorem 2.3 to the nearby $k$-cylindrical points, if any, yields the following. There exists $\rho > 0$ (depending on $n$, $k$, $t_0$, and the Lipschitz constant of $u$) so that every

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26By Brakke’s regularity theorem, such a $t_0$ does exist with the base cylinder chosen to be $C_k$ itself.

27The crux is that the theorem provides a way to determine when does the normalized flow starts to become close to $C_k$ to the specified degree. This is crucial to the work in [CM3].

28In contrast to the circumstance near a round point (see Section 2.1), here the information about $u$ is vague inside the cone $C_\phi$, making it more intricate to analyze.

29By Theorem 2.3, $r = \sqrt{2(n-k-1)(-t_0)}$ is a qualified cylindrical scale, where the constants $\phi$ and $M$ are related by (2.4).

30By (2.4), the smaller the angle $\phi$, the larger the radius $M$. 

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k-cylindrical point in $B^{n-k}_\rho \times B^k_\rho$ has a common $(\phi, \epsilon)$-asymptotically cylindrical scale $r > 0$ (depending on $n$, $k$, and $t_0$, see the footnote in Definition 2.4).

Furthermore, when $\phi$ and $\epsilon$ are chosen sufficiently small (depending on $n$ and $k$), by [CM3] the set $S_k \cap (B^{n-k}_\rho \times B^k_\rho)$ is contained in the graph $y = \psi(z)$, where

$$\psi : B^k_\rho \to B^{n-k}_\rho$$

is a map with small Lipschitz norm. Moreover, in case that $S_k \cap (B^{n-k}_\rho \times B^k_\rho)$ agrees with the graph of $\psi$, then

$$S \cap (B^{n-k}_\rho \times B^k_\rho) = S_k \cap (B^{n-k}_\rho \times B^k_\rho)$$

and it turns out that $\psi \in C^1(B^k_\rho)$; the tangent space of $S \cap (B^{n-k}_\rho \times B^k_\rho)$ at each point is exactly the $k$-dimensional axis of the tangent $k$-cylinder at that point.

**Remark 2.6.** In view of the asymptotic behavior of the level set flow near a singular point (spherical or cylindrical), a critical point of $u$ must be either a local maximum point or a saddle point. That is to say, $u$ has no interior local minimum points.

Due to (1.5) and the upper semi-continuity of the Gaussian density (cf. [E]), in a neighborhood of the $k$-cylindrical point 0, the degree of every singular point is no higher than $k$. The following theorem from [CM3], which is based on the aforementioned property and Remark 2.5, describes the structure of the singular set near 0.

**Theorem 2.7.** There exists $\rho > 0$ so that

$$S \cap (B^{n-k}_\rho \times B^k_\rho) = (S_0 \cup \cdots \cup S_k) \cap (B^{n-k}_\rho \times B^k_\rho).$$

Moreover, we have

1. The set $S_k \cap (B^{n-k}_\rho \times B^k_\rho)$ is contained in a Lipschitz graph $y = \psi(z)$, where $\psi : B^k_\rho \to B^{n-k}_\rho$ is a map with small Lipschitz norm.
2. The Hausdorff dimension of $(S_0 \cup \cdots \cup S_k) \cap (B^{n-k}_\rho \times B^k_\rho)$ is at most $k - 1$.

Analogous to (2.2) for round points, Colding and Minicozzi in [CM4] showed that the Hessian at the $k$-cylindrical point 0 is

$$(2.6) \quad \nabla^2 u(0) = \left( \frac{-1}{n-k-1} I_{n-k} \quad O_k \right).$$

**Remark 2.8.** The Hessian at a different $k$-cylindrical point, say $p$, is similar to the matrix (2.6) in the sense that

$$\nabla^2 u(p) = Q^{-1} \nabla^2 u(0) Q$$

for some orthogonal matrix $Q$ with Ker $\nabla^2 u(p)$ the $k$-dimensional axis of the tangent $k$-cylinder at $p$. Because of the rigid forms of Hessian at singular points (see (2.2) and (2.6)), one can tell the type of a singular point according to the Hessian (such as from the trace) at that point.

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31 The Lipschitz map $\psi$ comes from the McShane-Whitney extension of $S_k \cap (B^{n-k}_\rho \times B^k_\rho)$.

32 In other words, singular points in $B^{n-k}_\rho \times B^k_\rho$ are all $k$-cylindrical points.

33 A more fundamental way to see this is perhaps by the definition of viscosity supersolution to equation (1.1) in [ES].
Regarding the continuity of Hessian near 0, we have the following preliminary remark based on the asymptotically cylindrical behavior of the level set flow outside the cone $C_\phi$ (cf. [CM4]).

**Remark 2.9.** Given positive constants $\phi$ and $\epsilon$, there exists $r > 0$ so that the level set flow is $(\phi, \epsilon)$-asymptotically cylindrical in $B_r$. Thus, for any point in $B_r \setminus C_\phi$, we can choose the tangent vectors in (2.1) in such a way that \{e_{1}, \ldots, e_{n-k-1}\} is close to the round-direction of $C_k$ and \{e_{n-k}, \ldots, e_{n-1}\} is close to the axial-direction of $C_k$. By comparing (2.1) with (2.6) and taking Definition 2.4 into account, we infer that $\nabla^2 u$ can be made as close to $\nabla^2 u(0)$ as pleased in $B_r \setminus C_\phi$, so long as $\epsilon$ is chosen sufficiently small.

Unlike the situation in Section 2.1 where the Hessian is automatically continuous at a round point, a priori we only know that $\nabla^2 u(x) \rightarrow \nabla^2 u(0)$ as $x \rightarrow 0$ from outside of the cone $C_\phi$. In fact, the continuity of Hessian from the inside of the cone depends on the regularity of the singular set $S$ near 0, as will be seen in the following theorem. The proof is based on the ideas in [CM4].

**Theorem 2.10.** $\nabla^2 u$ is continuous at the $k$-cylindrical point 0 if and only if near 0 the singular set $S$ is a $C^1$ embedded $k$-manifold.

In this case, 0 is a local maximum point of $u$ near which we have $S = \{u = 0\}$. Moreover, every singular point $x$ near 0 must be a $k$-cylindrical point with $T_x S$ the $k$-dimensional axis of the tangent $k$-cylinder at $x$.

**Proof.** ($\Rightarrow$) Let $\phi$ and $\epsilon$ be small positive constants (as is required in Remark 2.5), there exists $\delta > 0$ such that for every $k$-cylindrical point $x = (y, z)$ in $B_b^{n-k} \times B_b^k$ the level set flow is $(\phi, \epsilon)$-asymptotically cylindrical about $x$ with cylindrical scale $\delta$; moreover,

$$|\nabla^2 u(x) - \nabla^2 u(0)| < \frac{1}{\sqrt{n}} \min \left\{ \frac{1}{n - k - 1}, \frac{1}{n - l - 1} \right\},$$

for every $x = (y, z)$ in $B_b^{n-k} \times B_b^k$.

Fix $z_0 \in B_b^k$. Let $y_0 \in B_b^{n-k} \cap \{z = z_0\}$ be such that

$$u(y_0, z_0) = \max_{|y| \leq \delta} u(y, z_0).$$

Note that a priori such a point $y_0$ might not be unique. In view of the normal vector field $N = \frac{\nabla u}{|\nabla u|}$ on $\{|y| \geq |z_0| \tan \phi\} \cap \{z = z_0\}$ (with the asymptotically cylindrical behavior of the level set flow taken into account), we infer that $|y_0| < |z_0| \tan \phi$. Then it follows from (2.8) that

$$\nabla u(y_0, z_0) \in \{0\} \times \mathbb{R}^k.$$

We claim that $\nabla u(y_0, z_0) = 0$, i.e., $x_0 = (y_0, z_0)$ is a critical point. To prove the claim, suppose the contrary that $\nabla u(x_0) \neq 0$, so by Theorem 2.1 $x_0$ would be a

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34 That is, parallel to $S^{n-k-1}_{\sqrt{2(n-k-1)}} \times \{0\}$.

35 That is, parallel to $\{0\} \times \mathbb{R}^k$.

36 Geometrically speaking, near the point 0 the level set flow shrinks to $S$ at time 0 and then vanishes.

37 By Remark 2.8 this is Ker $\nabla^2 u(x)$. 

regular point. By (2.6), (2.7), and (2.9) we would have
\[- \left( 1 - \frac{\nabla u}{|\nabla u|} (x_0) \otimes \frac{\nabla u}{|\nabla u|} (x_0) \right) \cdot \nabla^2 u (0) = \frac{n - k}{n - k - 1},\]
\[- \left( 1 - \frac{\nabla u}{|\nabla u|} (x_0) \otimes \frac{\nabla u}{|\nabla u|} (x_0) \right) \cdot \left( \nabla^2 u (x_0) - \nabla^2 u (0) \right) > \frac{-1}{n - k - 1},\]
Thus,
\[- \left( 1 - \frac{\nabla u}{|\nabla u|} (x_0) \otimes \frac{\nabla u}{|\nabla u|} (x_0) \right) \cdot \nabla^2 u (x_0) > 1\]
contradicting (1.1).

Following from the last paragraph, given \( z_0 \in B^k_\delta \) there exists \( y_0 \in \mathcal{B}_\delta \cap \{ z = z_0 \} \) such that \( x_0 = (y_0, z_0) \in \mathcal{S} \). Using Remark 2.8 and (2.10) we deduce that \( x_0 \) is indeed a \( k \)-cylindrical point. It then follows from Remark 2.5 that \( \mathcal{S} \cap (B^{n-k}_\delta \times B^k_\delta) = \mathcal{S}_k \cap (B^{n-k}_\delta \times B^k_\delta) \) is a graph \( y = \psi (z) \), where \( \psi : B^k_\delta \rightarrow B^{n-k}_\delta \) is a \( C^1 \) map with \( \psi (0) = 0 \), and that for every \( x \in \mathcal{S} \cap (B^{n-k}_\delta \times B^k_\delta) \), \( T_x \mathcal{S} \) is the axis of the tangent cylinder of the level set flow at \( x \). Note in particular that the maximum point in (2.8) turns out to be unique and that \( y_0 = \psi (z_0) \). On the other hand, by Theorem 2.1 we have
\[ \frac{\partial}{\partial z} [u (\psi (z), z)] = \nabla u (\psi (z), z) \cdot \left[ \frac{\partial \psi (z)}{\partial z}, 1_k \right] = 0, \]
which, by the mean value theorem, implies that
\[ u (y_0, z_0) = u (\psi (z_0), z_0) = u (0). \]
Therefore, \( u (0) \) is the maximum value of \( u \) in \( B^{n-k}_\delta \times B^k_\delta \) with
\[ \{ u = 0 \} \cap (B^{n-k}_\delta \times B^k_\delta) = \mathcal{S} \cap (B^{n-k}_\delta \times B^k_\delta) \).

(\( \Leftarrow \)) Firstly, note that by Theorem 2.7 the \( k \)-cylindrical points must be densely distributed in the \( C^1 \) embedded \( k \)-manifold \( \mathcal{S} \). It then follows from the upper semicontinuity of the Gaussian density (cf. [12]) that \( \mathcal{S} = \mathcal{S}_k \) near 0.

Given \( \eta > 0 \), choose \( \delta > 0 \) such that \( \mathcal{S} \cap (B^{n-k}_\delta \times B^k_\delta) = \mathcal{S}_k \cap (B^{n-k}_\delta \times B^k_\delta) \) is a graph \( y = \psi (z) \), where \( \psi : B^k_\delta \rightarrow B^{n-k}_\delta \) is \( C^1 \) with small gradient; in particular, in view of Remark 2.9 and Remark 2.9 we may assume that
\[ |\nabla^2 u (\psi (z), z) - \nabla^2 u (0)| \leq \frac{\eta}{2} \]
for every \( z \in B^k_\delta \). Additionally, by Remark 2.5 and Remark 2.9 we may further assume that every \( k \)-cylindrical point in \( B^{n-k}_\delta \times B^k_\delta \) has the uniform cylindrical scale \( \delta \) with the parameters \( \phi \) and \( \epsilon \) (see Definition 2.4) sufficiently small (depending on \( n \), \( k \), and \( \eta \)) that
\[ |\nabla^2 u (y, z) - \nabla^2 u (\psi (z), z)| \leq \frac{\eta}{2} \]
for every \( y \in B^{n-k}_\delta \) and \( z \in B^k_\delta \). Combining (2.10) and (2.11) yield
\[ |\nabla^2 u (x) - \nabla^2 u (0)| \leq \eta \]
for every \( x \in B^{n-k}_\delta \times B^k_\delta \). Thus, \( \nabla^2 u \) is continuous at 0.
\[ \Box \]
If $\nabla^2 u$ is continuous at the $k$-cylindrical point 0, then by (the “only if” part of) Theorem 2.10 $S = S_k$ near 0 and it is a $C^1$ embedded $k$-manifold. On the other hand, applying (the “if” part of) Theorem 2.10 to the nearby singular points yields that $\nabla^2 u$ is continuous at every singular point near 0 and hence it is continuous on a neighborhood of 0. Thus, we have the following corollary.

**Corollary 2.11.** The Hessian $\nabla^2 u$ is continuous at a $k$-cylindrical point 0 if and only if $u \in C^2$ in a neighborhood of 0.

Let us conclude this section with the following proof of the characterization of globally $C^2$ regularity in [CALS] using Theorem 2.10. The focus is on seeing why the flow has only one singular time when $u$ is globally $C^2$.

**Theorem 2.12.** The solution $u$ to (1.1) is globally $C^2$ if and only if the first singular time $T$ of the level set flow is the time of extinction [38] and that the set $u = T$ is either a single round point or a closed connected $C^1$ embedded $k$-manifold consisting of $k$-cylindrical points for some $k \in \{1, \ldots, n-2\}$.

**Proof.** $(\Rightarrow)$ It is clear from Theorem 2.1, Section 2.1, and Theorem 2.10.

$(\Leftarrow)$ Let $T$ be the first singular time and choose a singular point $p \in \{u = T\}$. If $p$ is a round point, then by Section 2.1 there exists $\delta > 0$ so that in a neighborhood of $p$ the set $u = t$ is asymptotic to a sphere for every $t \in (T - \delta, T)$. For every $t \in (T - \delta, T)$, the asymptotically spherical part of $u = t$ near $p$ is clearly compact and relatively open in $\{u = t\}$; therefore, it is indeed the whole of $u = t$ by the connectedness of $\{u = t\}$ [39]. In light of this, the flow shrinks to the point $p$ at time $T$ and so the theorem is proved. Thus, for the rest of our discussion, let us assume that $p$ is a $k$-cylindrical point for some $k \in \{1, \ldots, n-2\}$.

Let $\Gamma$ be the connected component of $p$ in the set of all $k$-cylindrical points in $u = T$. By Theorem 2.10 $\Gamma$ is a closed [39] connected $C^1$ embedded $k$-manifold. Let $\phi$ and $\epsilon$ be sufficiently small positive constants [34] by Remark 2.5 and the compactness of $\Gamma$, there exists $\delta > 0$ so that the $(\phi, \epsilon)$-asymptotically cylindrical surface (see Definition 2.4) of any point on $\Gamma$ is at least $\delta$. It follows that for every $q \in \Gamma$, in $B_\delta(q) \setminus B_\epsilon(q)$ the set $u = t$, whenever it is nonempty, is $\epsilon \sqrt{T - t}$-close in the Hausdorff sense to the $k$-cylinder $c_k^\Gamma, \sqrt{T - t}(q)$, where $c_k^\Gamma(q)$ is the cone obtained by rotating the cone $c_k^\phi$ (see (2.3)) to have the axis $T_\phi^\Gamma$ and then translating by the vector $q$ so that $q$ is the apex of $c_k^\phi(q)$; $C_k^\Gamma, \sqrt{T - t}(q)$ is the $k$-cylinder obtained by firstly rotating the $k$-cylinder $C_k$ (see (1.3)) to have the axis $T_\phi^\Gamma$, then scaling by the factor $\sqrt{T - t}$ (so that the radius becomes $\sqrt{2(n-k-1)(T-t)}$), and lastly translating by the vector $q$. In particular, the distance from $q$ to any point on $\{u = t\} \cap B_\delta(q) \cap \Pi_{n-k}^\Gamma(q)$, where $\Pi_{n-k}^\Gamma(q)$ is the $(n-k)$-dimensional plane passing through $q$ and orthogonal to $T_\phi^\Gamma$, is no more than $\left( \sqrt{2(n-k-1) + \epsilon} \right) \sqrt{T - t}$.

\[38\] Namely, $T = \max u$. In other words, the mean curvature flow starting at the initial hypersurface becomes singular at time $T$ and then vanishes.

\[39\] Note that for every $t \in (0, T)$, the hypersurface $u = t$ is diffeomorphic to $u = 0$ and hence is connected.

\[40\] In proving the compactness of $\Gamma$, we use the continuity of $u$ and the upper semicontinuity of Gaussian density as well.

\[41\] This is for the application of Remark 2.5. In addition, for the latter purpose we require that $\epsilon < 2 - \sqrt{2}$. 
As such, for every \( t \in [0, T) \) with \( T - t \leq \frac{\delta^2}{32(n-k-1)} \), the set
\[
\hat{\Sigma}_t = \left\{ x : u(x) = t \text{ and dist}(x, \Gamma) \leq \frac{\delta}{2} \right\}.
\]
is nonempty and compact; moreover, we will show that \( \hat{\Sigma}_t \) is relatively open in \( u = t \), so by the connectedness\(^4\) of \( u = t \) we then have \( \hat{\Sigma}_t = \{ u = t \} \). To prove the relatively openness of \( \hat{\Sigma}_t \) in \( u = t \), let us fix \( x \in \hat{\Sigma}_t \). By the compactness of \( \Gamma \) there exists \( q \in \Gamma \) so that \( \text{dist}(x, \Gamma) = |x - q| \leq \frac{\delta}{2} \); moreover, \( x - q \) is orthogonal to \( T_q \Gamma \) and hence \( x \in \Pi_{n-k}^q (\Gamma) \). Since \( x \in \{ u = t \} \cap B_{\delta} (q) \cap \Pi_{n-k}^q (\Gamma) \), it follows from the discussion in the last paragraph that
\[
|x - q| < \left( \sqrt{2(n-k-1)} + \epsilon \right) \sqrt{T - t} \leq \frac{\delta}{2}.
\]
where the last inequality is due to \( T - t \leq \frac{\delta^2}{32(n-k-1)} \) and \( \epsilon < 2 - \sqrt{2} \). Then for any \( r \in \left( 0, \frac{\sqrt{2} - 1}{2\sqrt{2}} \delta \right) \) we have \( B_r (x) \cap \{ u = t \} \subset \hat{\Sigma}_t \); namely, \( x \) has an open neighborhood in \( \{ u = t \} \) that is contained in \( \hat{\Sigma}_t \).

To finish the proof, it suffices to show that \( \Gamma = \{ u = T \} \). Were \( \Gamma \neq \{ u = T \} \), then by Theorem 2.10 and the compactness of both \( \{ u = T \} \) and \( \Gamma \), we would have
\[
\text{dist}(\{ u = T \} \setminus \Gamma, \Gamma) = \rho > 0.
\]
On the other hand, from the result\(^1\) and argument in the last paragraph, we infer that the hypersurface \( u = t \) is indeed contained in the \( \left( \sqrt{2(n-k-1)} + \epsilon \right) \sqrt{T - t} \)-tubular neighborhood of \( \Gamma \) for every \( t \in [0, T) \) with \( T - t \leq \frac{\delta^2}{32(n-k-1)} \). Particularly, for every \( t \in [0, T) \) sufficiently close to \( T \) the set \( u = t \) is contained in the \( \frac{\delta}{2} \)-tubular neighborhood of \( \Gamma \), contradicting (2.12) because the level set flow is non-fattening (cf. [11] and [W1]). \( \square \)

### 3. Łojasiewicz Inequality and Type I Singularity

In this section we aim to prove Theorem 1.4 which, in view of Theorem 1.3 and the footnote in Definition 3.1, amounts to characterizing the regular singular points (see Definition 3.1) by means of the type I condition (see Definition 3.3) under the hypothesis that \( S = S_0 \cup S_1 \). This is done in three steps. Firstly, it is seen in Proposition 3.2 that the type I condition is a necessary condition for a singular point to be a regular singular point. Next, in Section 3.1 we prove that a type I singular point must be a local maximum point (see Theorem 3.3). Lastly, the proof is completed in Section 3.4 by establishing Theorem 3.10.

To start with, let us introduce the following definition.

**Definition 3.1.** (Regular Singular Points) A singular point of \( u \) is called a regular singular point if near which \( u \) is \( C^2 \)\(^2\).

---

\(^1\)Note that the hypersurface \( u = t \) is diffeomorphic to the hypersurface \( u = 0 \) when \( t < T \).

\(^2\)That is, \( \Sigma_1 = \{ u = t \} \).

\(^3\)A round point is automatically a regular singular point (see Section 2.1). For a \( k \)-cylindrical point \( p \), \( p \) is a regular singular point if \( \nabla^2 u \) is continuous at \( p \) (see Corollary 2.11) if there exists \( \delta > 0 \) so that \( S \cap B_{\delta} (p) \) is a \( C^1 \) embedded \( k \)-manifold (see Theorem 2.10).
As pointed out in [CM6], near a regular singular point the solution $u$ must satisfy a Lojasiewicz inequality. For the sake of completeness, we write the complete statement in the following proposition and provide a proof.

**Proposition 3.2.** If $0$ is a regular singular point, then for every $\epsilon > 0$ there exists $\delta > 0$ so that

$$|u|^2 \leq C_{n,k,\epsilon} |\nabla u|$$

in $B_\delta$, where $k = \text{nullity}(\nabla^2 u(0)) \in \{0, \cdots , n - 2\}$ and $C_{n,k,\epsilon} \to \sqrt{\frac{n-k-1}{2}}$ as $\epsilon \searrow 0$.

**Proof.** Let us assume that $k > 0$ so $0$ is a $k$-cylindrical point; the proof for $k = 0$ (i.e., round point) is similar.

Given $\epsilon > 0$ (sufficiently small depending on $n$ and $k$), choose $\delta > 0$ such that $u$ is $C^2$ in $B_\delta$ and that the singular set

$$(3.1) \quad S \cap B_\delta = S_k \cap B_\delta = \{u = 0\} \cap B_\delta$$

is a $C^1$ embedded $k$-manifold (see Theorem 2.10) whose normal bundle $NS$ parametrizes $B_\delta$. Particularly, for every $x \in B_\delta$ there exists a unique $x' \in S \cap B_\delta$ such that $x - x' \in N_{x'}S$.

Furthermore, by the continuity of Hessian and Remark 2.5, we may also assume that

$$|\nabla^2 u(x) - \nabla^2 u(x')| \leq \epsilon,$$

$$(3.3) \quad \left| \frac{\nabla u(x)}{|\nabla u(x)|} + \frac{x - x'}{|x - x'|} \right| \leq \epsilon$$

in case when $x \in B_\delta \setminus S$.

To prove the proposition, let us fix $x \in B_\delta$ and assume that $x \notin S$; otherwise $u(x) = 0$ and the Lojasiewicz inequality would hold trivially. Let $x' \in S \cap B_\delta$ be as defined in the last paragraph. Since $u(x') = 0$ by (3.1) and $\nabla u(x') = 0$ (cf. Theorem 2.10), the Taylor’s theorem says that there exist $x_0$ on the segment $x'x$ such that

$$|u(x)| = |u(x) - u(x')| \leq \frac{1}{2} \nabla^2 u(x_0) \cdot [(x - x') \otimes (x - x')]$$

$$(3.4) \quad \leq \frac{1}{2} \left( \frac{1}{n-k-1} + \epsilon \right) |x - x'|^2$$

Note that the last inequality is due to (3.2).

Likewise, by the mean value theorem, (3.2), (3.3), and (3.4) (see also Remark 2.5), we get

$$|\nabla u(x)| = \nabla u(x) \cdot \frac{\nabla u(x)}{|\nabla u(x)|} = \left( \nabla u(x) - \nabla u(x') \right) \cdot \frac{\nabla u(x)}{|\nabla u(x)|}$$

$$= \nabla^2 u(x_1) \cdot \left[ (x - x') \otimes \frac{\nabla u(x)}{|\nabla u(x)|} \right],$$

where $x_1 \in \overline{x'x}$ is a point arising from the mean value theorem,

$$\geq \nabla^2 u(x') \cdot \left[ (x - x') \otimes \frac{\nabla u(x)}{|\nabla u(x)|} \right] - \epsilon |x - x'|$$

$45$Since $(N_{x'}S)^\perp = T_{x'}S$ is the axis of the tangent cylinder of the level set flow at $x'$ (cf. Theorem 2.10), $x$ is on the $n - k$ dimensional plane passing through $x'$ and orthogonal to the axis of the tangent cylinder at $x'$.
\[
\geq \nabla^2 u(x') \cdot \left[(x-x') \otimes -\frac{x-x'}{|x-x'|}\right] - \frac{\epsilon}{n-k-1} |x-x'| - \epsilon |x-x'|
\]
(3.5)

Note that in the last equality we use the fact that \(x-x'\) is orthogonal to \(T_{x'}S = \text{Ker} \nabla^2 u(x')\) (cf. Theorem 2.10). Lastly, combining (3.4) with (3.5) yields
\[
|u(x)|_{1/2} \leq C_{n,k,\epsilon} |\nabla u(x)|,
\]
where \(C_{n,k,\epsilon} = \sqrt{\frac{1}{2} (\frac{1}{n-k-1} + \epsilon \frac{n-k}{n-k-1})} \).

On the set of regular points, the level set flow is a strictly mean-convex mean curvature flow with \(H = |\nabla u|^{-1}\) (see (1.2)). It can be seen that the mean curvature increases without bound as tending to critical points of \(u\), which by Theorem 2.1 are singular points. In the case where the point 0 is a regular singular point, the level set flow is a mean curvature flow near the point 0 prior to time 0; then it shrinks to a lower dimensional set \(S\) near the point 0 at time 0 and vanishes afterwards (see Theorem 1.3). So the solution \(u\) in Proposition 3.2 is actually non-positive near the point 0 and the Łojasiewicz inequality can be reformulated as
\[
\sqrt{-t} \leq C_{n,k,\epsilon} H^{-1} \Leftrightarrow H \leq \frac{C_{n,k,\epsilon}}{\sqrt{-t}}
\]
as \(t \searrow 0\). Taking into account the fact that the mean curvature is comparable with the norm of the second fundamental form (cf. [HK]), the singular point 0 is the so-called “type I singularity” of the flow at the singular time 0 (cf. [H2]).

As is seen in the last paragraph, the information of type I singularities is incorporated in the Łojasiewicz inequality; moreover, compared with (3.6), the Łojasiewicz inequality holds even at singularities of the flow. Thus, it seems reasonable to define the notion of type I singularities of the level set flow using the Łojasiewicz inequality. This is the following definition.

**Definition 3.3.** The singular point 0 is called a type I singular point if there exists \(\beta > 0\) so that the Łojasiewicz inequality
\[
|u|^{1/2} \leq \beta |\nabla u|
\]
holds in a neighborhood of 0\(^{46}\). Otherwise, it is called a type II singularity.

**Remark 3.4.** If 0 is a type I singular point, then by Theorem 2.1 the singular set \(S\) near 0 is contained in \(\{u = 0\}\) (cf. [CM6]). In other words, in a neighborhood of 0, a point \(x\) is a regular point provided \(u(x) \neq 0\).

**3.1. Saddle Points.** The highlight of this subsection is Theorem 3.8 which says that all saddle points are type II singular points (see Definition 3.3). As there are no interior local minimum points (see Remark 2.6), a type I singular point must be a local maximum point. This is essential to establishing Theorem 3.10 which is the last piece of puzzle of proving Theorem 1.4.

The proof of Theorem 3.8 is based on Proposition 3.5 and Proposition 3.7; the latter is the so-called rapid clearing-out phenomenon of the level set flow (cf. [CM3]). In this subsection we assume that 0 is a saddle point.

\(^{46}\)The corresponding definition for a different singular point, say \(p\), should be modified as \(|u - u(p)|^{1/2} \leq C |\nabla u|\) in some neighborhood of \(p\).
Proposition 3.5. Suppose that the saddle point 0 is a type I singular point. Then there exist $\delta > 0$ and an integral curve

$$x(s) \in C^1_{\text{loc}}(0, \delta] \cap C[0, \delta]$$

of the normal vector field $N = \frac{\nabla u}{|\nabla u|}$ that starts at 0 at time 0; that is, $x(0) = 0$ and

$$\frac{dx}{ds} = \frac{\nabla u(x)}{|\nabla u(x)|} \quad \text{for } 0 < s \leq \delta.$$

Moreover, for every $s \in [0, \delta]$ we have $u(x(s)) > 0$ and

$$|x(s)| \leq 2\beta \sqrt{u(x(s))},$$

where $\beta$ is the positive constant in Definition 3.3.

Proof. Since the level set flow and the Lojasiewicz inequality are invariant under the parabolic scaling\textsuperscript{47} for simplicity let us assume that the Lojasiewicz inequality in Definition 3.3 holds in $B_1$.

As 0 is a saddle point of $u$, there exists a sequence of points $\{p_i\}_{i \in \mathbb{N}}$ in $B_1$ that converges to 0 with $u(p_i) > u(0) = 0$ for every $i$. Because $\nabla u$ is Lipschitz continuous (cf. Theorem 2.1), the existence and uniqueness theorem of ODE gives that for each $i$ there is a unique flow line of the vector field $\nabla u$ passing through the point $p_i$.\textsuperscript{48} Moreover, these flow lines are indeed smooth curves due to Theorem 2.1\textsuperscript{49} and the regularity theory in ODE. It then follows from a reparametrization by the arc length that for every $i$ there is a smooth curve $x_i(s)$ in $B_1$ satisfying

$$\frac{d}{ds} x_i = N(x_i) = \frac{\nabla u(x_i)}{|\nabla u(x_i)|}$$

with $x_i(0) = p_i$. It is clear that $x_i$ is defined for $0 \leq s \leq s_i$ with $s_i \to 1$ as $i \to \infty$.

By the chain rule we have

$$\frac{d}{ds} [u(x_i)] = \nabla u(x_i) \cdot N(x_i) = |\nabla u(x_i)|,$$

which particularly implies that $u(x_i(s)) > 0$ for $0 \leq s \leq s_i$; moreover, invoking the Lojasiewicz inequality gives

$$\frac{d}{ds} [u(x_i)] = |\nabla u(x_i)| \geq \frac{1}{\beta} \sqrt{u(x_i)},$$

that is, $\frac{d}{ds} \sqrt{u(x_i)} \geq \frac{1}{\beta s}$. Thus, we obtain

$$\beta |\nabla u(x_i(s))| \geq \sqrt{u(x_i(s))} \geq \sqrt{u(x_i(s))} - \sqrt{u(x_i(0))} \geq \frac{s}{2\beta}$$

for $0 \leq s \leq s_i$. In addition, since

$$\frac{d^2}{ds^2} x_i = \frac{d}{ds} \left[ \frac{\nabla u(x_i)}{|\nabla u(x_i)|} \right] = \left( \frac{\nabla^2 u}{|\nabla u|} \right) \frac{\nabla (N \cdot (N \otimes N))}{\nabla u} (x_i),$$

by (3.8) we have\textsuperscript{50}

$$\frac{d^2}{ds^2} x_i \leq 2 \left\| \frac{\nabla^2 u}{|\nabla u(x_i)|} \right\|_{L^\infty(B_1)} \leq \frac{4\beta^2 \left\| \nabla^2 u \right\|_{L^\infty(B_1)}}{s}.$$

\textsuperscript{47}Namely, $x \mapsto \lambda^{-2} u(\lambda x)$ satisfies (1.1) and the Lojasiewicz inequality for every $\lambda > 0$.

\textsuperscript{48}Note that $p_i$ is not a stationary point of $\nabla u$ by the Lojasiewicz inequality.

\textsuperscript{49}Each of these flow lines consists of regular points as $\nabla u$ vanishes at no points along the curve.

\textsuperscript{50}The Hessian of $u$ is bounded according Theorem 2.1.
for 0 ≤ s ≤ s_i.

In view of \(3.7\) and \(3.9\), the Arzelà-Ascoli compactness theorem implies that \(\{x_i(s)\}_{i \in \mathbb{N}}\) subconverges to \(x(s)\) in \(C[0, 1 - \epsilon] \cap C^1[\epsilon, 1 - \epsilon]\) for every \(\epsilon \in (0, 1)\). Passing \(3.8\) to the limit (and noting that \(\nabla u\) is continuous) gives

\[
\beta |\nabla u(x(s))| \geq \sqrt{u(x(s))} \geq \frac{s}{2\beta} > 0
\]

for 0 < s ≤ 1. Likewise, with the help of \(3.10\), taking the limit of \(3.7\) gives

\[
\frac{dx}{ds} = \frac{\nabla u(x)}{|\nabla u(x)|}
\]

for 0 < s ≤ 1 with \(x(0) = 0\). Lastly, noting that

\[
|\mathbf{x}(s)| \leq \int_0^s |\mathbf{x}'(\xi)| d\xi = s,
\]

\(3.10\) implies

\[
\sqrt{u(x(s))} \geq \frac{s}{2\beta} \geq \frac{|\mathbf{x}(s)|}{2\beta}
\]

for 0 ≤ s ≤ 1.

Proposition \(3.7\) is the rapid clearing-out lemma in [CM3]. For reader’s convenience, a proof will be provided. To streamline the proof, we need Lemma \(3.6\) concerning the Gaussian area. Recall that given \(p \in \mathbb{R}^n\) and \(\Lambda > 0\), the Gaussian area with center \(p\) and scale \(\sqrt{\Lambda}\) of a \((n - 1)\)-rectifiable set \(\Sigma\) in \(\mathbb{R}^n\) is defined as

\[
F_{p,\Lambda}(\Sigma) = \int_{\Sigma} e^{-\frac{|y-p|^2}{4\Lambda}} d\mathcal{H}^{n-1}(y).
\]

The entropy of \(\Sigma\) (cf. [CM1]) is defined as

\[
E[\Sigma] = \sup\{F_{p,\Lambda}(\Sigma) : p \in \mathbb{R}^n, \Lambda > 0\}.
\]

**Lemma 3.6.** Given \(M > 0\), \(k \in \{1, \cdots, n-2\}\), and \(\lambda \geq 1\), there exist \(\Lambda > 2\) (depending on \(n, k\)) and \(\phi, \epsilon > 0\) (depending on \(n, k, M, \lambda\)) with the following property: If \(\Sigma\) is a \((n-1)\)-rectifiable set in \(\mathbb{R}^n\) with entropy no higher than \(\lambda\); additionally, \(\Sigma\) is \(\epsilon\)-close in the \(C^1\) topology to the \(k\)-cylinder \(C_k\) in the ball \(B_{\sqrt{2(n-k-1)/\csc\phi}}\). Then

\[
F_{p,\Lambda}(\Sigma) \leq \frac{1}{2}
\]

for every \(p \in B_{M\sqrt{\Lambda-1}}\).

**Proof.** Firstly, we claim that there exists \(\Lambda > 2\) depending on \(n\) and \(k\) such that

\[
F_{p,\Lambda}(C_k) \leq \frac{1}{8}
\]

for every \(p \in \mathbb{R}^n\). To see that, let us adopt the notation \(x = (y, z) \in \mathbb{R}^{n-k} \times \mathbb{R}^k\) for the following calculation. Given \(p = (p_1, p_2) \in \mathbb{R}^{n-k} \times \mathbb{R}^k\) and \(\Lambda > 1\), Tonelli’s theorem gives

\[
F_{p,\Lambda}(C_k) = \left(\int_{S_n^{n-k-1} / \sqrt{2(n-k-1)/\csc\phi}} \frac{e^{-\frac{|y-p|^2}{4\Lambda}}}{(4\pi\Lambda)^{n-k-1}} d\mathcal{H}^{n-k-1}(y)\right) \left(\int_{\mathbb{R}^k} \frac{e^{-\frac{|z|^2}{4\Lambda}}}{(4\pi\Lambda)^{\frac{k}{2}}} d\mathcal{H}^k(x)\right)
\]
\[
\leq \frac{1}{(4\pi) \frac{n-k}{2}} \mathcal{H}^{n-k-1} \left( S^{n-k-1} \frac{\sqrt{2(n-k-1)}}{} \right),
\]
from which the claim can be verified easily. This is how the constant \( \Lambda = \Lambda(n,k) \) is determined. Below we show how to choose the constants \( \phi \) and \( \epsilon \).

Let \( \Sigma \) be as stated and fix \( p \in B_M \sqrt{\lambda} \). Using the technique in \([\text{CIM}]\) we have

\[
F_{p,\Lambda} \left( \Sigma \setminus B \sqrt{2(n-k-1) \csc \phi} \right) = \int_{\Sigma \setminus B \sqrt{2(n-k-1) \csc \phi}} \left( \sqrt{2}^{n-1} e^{-\frac{|x-p|^2}{4\lambda}} \right) e^{-\frac{|x-p|^2}{8\lambda}} d\mathcal{H}^{n-1}(x)
\]

\[
\leq \sqrt{2}^{n-1} e^{-\left( \sqrt{2(n-k-1) \csc \phi} - M \sqrt{\lambda} \right)^2} F_{p,2\Lambda}(\Sigma) \leq \sqrt{2}^{n-1} e^{-\left( \sqrt{2(n-k-1) \csc \phi} - M \sqrt{\lambda} \right)^2} \lambda.
\]

Thus, by choosing \( \phi = \phi(n,k,M,\lambda) \) sufficiently small we have

\[(3.14)\]

\[
F_{p,\Lambda} \left( \Sigma \setminus B \sqrt{2(n-k-1) \csc \phi} \right) \leq \frac{1}{4}.
\]

To proceed, let us parametrize \( \Sigma \) and \( C_k \) in \( B \sqrt{2(n-k-1) \csc \phi} \) as

\[
x_{\Sigma} : M \rightarrow B \sqrt{2(n-k-1) \csc \phi} \quad \text{and} \quad x_{\Sigma_k} : M \rightarrow B \sqrt{2(n-k-1) \csc \phi},
\]

respectively, where \( M \) is some compact \((n-1)\)-manifold diffeomorphic to \( C_k \cap B \sqrt{2(n-k-1) \csc \phi} \), in such a way that the two immersions are \( \epsilon \)-close in the \( C^1 \) norm. It follows that

\[
F_{p,\Lambda} \left( \Sigma \cap B \sqrt{2(n-k-1) \csc \phi} \right) - F_{p,\Lambda} \left( C_k \cap B \sqrt{2(n-k-1) \csc \phi} \right)
\]

\[
= \frac{1}{(4\pi \lambda)^{\frac{n-k}{2}}} \int_{M} \left( e^{-\frac{|x-p|^2}{4\lambda}} \sqrt{\deg (g_{\Sigma})} = e^{-\frac{|x-p|^2}{4\lambda}} \sqrt{\deg (g_{\Sigma_k})} \right) d\mathcal{H}^{n-1},
\]

where \( g_{\Sigma} \) and \( g_{\Sigma_k} \) are the respective induced metric on \( M \). Consequently, if \( \epsilon = \epsilon(n,k,M,\lambda) \) is sufficiently small we have

\[
\left| F_{p,\Lambda} \left( \Sigma \cap B \sqrt{2(n-k-1) \csc \phi} \right) - F_{p,\Lambda} \left( C_k \cap B \sqrt{2(n-k-1) \csc \phi} \right) \right| \leq \frac{1}{8},
\]

which together with \((3.13)\) yield

\[(3.15)\]

\[
F_{p,\Lambda} \left( \Sigma \cap B \sqrt{2(n-k-1) \csc \phi} \right) \leq \frac{1}{4}.
\]

Lastly, the conclusion follows from \((3.14)\) and \((3.15)\). \( \square \)

**Proposition 3.7.** Given that 0 is a saddle point, for any \( M > 0 \) there exists \( \tau > 0 \) so that

\[
\{ u = t \} \cap B_M \sqrt{\tau} = \emptyset
\]

for every \( t \in (0, \tau] \).

**Proof.** As 0 is a saddle point, it must be a \( k \)-cylindrical point for some \( k \in \{1, \cdots, n-2\} \) (see Section \([2.1]\) ). Now set \( \lambda \) as the entropy of the level set flow\(^{51}\) and let \( \Lambda, \phi, \) and \( \epsilon \) be the constants from Lemma \([3.6]\). By Remark \([2.3]\) there exists \( \tau > 0 \) such that \( u \) is \((\phi, \epsilon)\)-asymptotically cylindrical near the \( k \)-cylindrical point 0 on the cylindrical scale \( \sqrt{-\tau} \); particularly, \( \sqrt{-\tau} \Sigma \) is \( \epsilon \)-close in the \( C^1 \) topology.

\(^{51}\) More accurately, \( \lambda \) is chosen to be the entropy of the initial time-slice; the entropy of the latter time-slice does not exceed \( \lambda \) owing to Huisken's monotonicity formula (cf. \([12]\) and \([\text{CM1}]\)).
to the \( k \)-cylinder \( C_k \) in the ball \( B_{\sqrt{2(n-k-1)\csc\phi}} \) for every \( t \in [-\tau,0) \) where \( \Sigma_t = \{ u = t \} \).

Given \( t \in [-\tau,0) \) and \( p \in B_M \sqrt{\Lambda(-1)(-t)} \), the change of variable in Gaussian integrals (see (3.11)) and Lemma 3.6 yield

\[
F_{p,\Lambda(-t)}(\Sigma_t) = F_{p,\sqrt{-t},\Lambda}(\frac{1}{\sqrt{-t}} \Sigma_t) \leq \frac{1}{2}.
\]

Then it follows from Huisken’s monotonicity formula (cf. [H2]) that the Gaussian density of the flow at the point \( p \) and time \( (\Lambda - 1)(-t) \) satisfies

\[
\Theta[p, (\Lambda-1)(-t)] = \lim_{s \to \Lambda(-t)} F_{p,\Lambda(-t)-s}(\Sigma_{t+s}) \leq F_{p,\Lambda(-t)}(\Sigma_t) \leq \frac{1}{2}.
\]

So we must have \( u(p) \neq (\Lambda - 1)(-t) \), i.e.,

\[ p \notin \Sigma(\Lambda-1)(-t). \]

To see this, suppose the contrary that \( u(p) = (\Lambda - 1)(-t) \). As the singular set has dimension at most \( n-2 \), we can choose a sequence of regular points \( p_i \to p \), and we automatically obtain \( u(p_i) \to u(p) \) by the continuity of \( u \). The upper semicontinuity of Gaussian density yields

\[ \Theta[p, (\Lambda - 1)(-t)] = \Theta[p, u(p)] \geq \limsup_{i \to \infty} \Theta[p_i, u(p_i)] \geq 1 \]

(cf. [E]), which is a contradiction.

Finally, we get

\[ \Sigma(\Lambda-1)(-t) \cap B_{\sqrt{\Lambda(-1)(-t)}} = \emptyset \quad \forall \ t \in [-\tau,0), \]

which, by setting \( t' = (\Lambda - 1)(-t) \), can be rewritten as

\[ \Sigma_t \cap B_{\sqrt{\Lambda(-1)}} = \emptyset \quad \forall \ t' \in (0,(\Lambda - 1)\tau]. \]

□

We are now in the position to prove the main theorem of this subsection.

**Theorem 3.8.** The saddle point 0 is a type II singular point.

In other words, a type I point cannot be a saddle point and hence (by Remark 2.6) must be a local maximum point.

**Proof.** By Proposition 3.7 with \( M = 3\beta \), where \( \beta \) is the positive constant in Definition 3.3 there exists \( \tau > 0 \) such that

\[ \Sigma_t \cap B_{3\beta\sqrt{\tau}} = \emptyset \]

for every \( t \in (0,\tau] \), where \( \Sigma_t = \{ u = t \} \). On the other hand, by Proposition 3.5 there exists a continuous curve \( x(s) \) satisfying \( u(x(0)) = u(0) = 0, u(x(s)) > 0 \), and

\[ x(s) \in \Sigma_{u(x(s))} \cap \bar{B}_{2\beta\sqrt{u(x(s))}} \]

for every \( s \in [0,\delta] \). A contradiction then follows. □

**Remark 3.9.** Remark 3.4 can be improved by Theorem 3.8 as follows: When 0 is a type I singular point, there exists a neighborhood of 0 in which a point \( x \) is a regular point if and only if \( u(x) \neq 0 \) (\( \Leftrightarrow u(x) < 0 \)).

\[^{52}\text{See Definition 2.3.}\]
3.2. Neckpinch Singularity. In this subsection 0 is assumed to be a 1-cylindrical point. Our goal is to prove

**Theorem 3.10.** If the 1-cylindrical point 0 is a type I singular point (see Definition 3.3), then it must be a regular singular point (see Definition 3.1).

by the method of contradiction, so let us assume throughout this subsection that

**Assumption 3.11.** The 1-cylindrical point 0 is a type I singular point but fails to be a regular singular point.

and we manage to get a contradiction.

For ease of notations, the orientation is chosen in such a way that the tangent flow at 0 is the self-shrinking of the 1-cylinder

\[ C_1 = S^{n-2} \sqrt{2(n-2)} \times \mathbb{R}^1 \]

and we will adopt the notation \( x = (y, z) \in \mathbb{R}^{n-1} \times \mathbb{R} \) for coordinates.

Let \( \phi, \varepsilon \in (0, 1) \) be small constants to be determined. By Remark 2.5 and the parabolic rescaling, we may assume that the level set flow is \((\phi, \varepsilon)-\)asymptotically cylindrical in \( B_1^{n-1} \times (-1, 1) \). Likewise, by the type I condition and the parabolic rescaling, we may also assume that the Lojasiewicz inequality in Definition 3.3 holds in \( B_1^{n-1} \times (-1, 1) \). Moreover, by Theorem 2.1 and the parabolic rescaling, we may even assume that that \( u(0) = 0 \) is the maximum value of \( u \) in \( B_1^{n-1} \times (-1, 1) \).

As is considered in (2.8) in Theorem 2.10 let us define the function

\[ u_{\text{max}}(z) = \max_{|y| \leq 1} u(y, z) = \max_{|y| < |z| \tan \phi} u(y, z). \]

Note that the last equality in (3.16) comes from the asymptotically cylindrical behavior of the level set flow in \( B_1^{n-1} \times (-1, 1) \) outside the cone \( C_0 = \{|y| < |z| \tan \phi\} \).

**Lemma 3.12.** There exists a sequence \( \{z_i\}_{i \in \mathbb{N}} \) in \((-1, 1)\) that converges to 0 with \( u_{\text{max}}(z_i) < 0 \) for every \( i \).

Upon passing to a subsequence and changing the orientation of the \( z \)-axis if necessary, we may assume that \( z_i \in (0, 1) \) for every \( i \).

**Proof.** Suppose the contrary that there exists \( \delta > 0 \) such that

\[ u_{\text{max}}(z) = 0 \quad \forall -\delta \leq z \leq \delta. \]

As 0 is the maximum value of \( u \) near the point 0, the singular set \( S \), which by Theorem 2.1 is the set of critical points of \( u \), intersects the hyperplane \( z = z_0 \) for every \( z_0 \in [-\delta, \delta] \). Note that the points of intersection are inside the cone \( C_0 \).

On the other hand, near the 1-cylindrical point 0 Theorem 2.7 says that \( S = S_1 \cup S_0 \); the set \( S_1 \) is contained in a Lipschitz curve \( y = \psi(z) \) and the set \( S_0 \) is countable. Consequently, the set \( S_1 \) intersects \( z = z_0 \) (inside the cone \( C_0 \)) for all

---

53 The constant \( \phi \) will be determined in 3.23, where \( \beta \) is the constant in Definition 3.3 and then \( \epsilon \) will be chosen sufficiently small depending on \( n \) and the choice of \( \phi \).

54 If \( u \) satisfies (1.1) and is \((\phi, \varepsilon)-\)asymptotically cylindrical in \( B_1 \), then \( x \mapsto r^{-2} u(rx) \) satisfies (1.1) and is \((\phi, \varepsilon)-\)asymptotically cylindrical in \( B_1 \).

55 If \( u \) satisfies the Lojasiewicz inequality in Definition 3.3 in \( B_1 \), then \( x \mapsto r^{-2} u(rx) \) satisfies the same Lojasiewicz inequality in \( B_1 \).

56 In view of Section 2.1 every round point is isolated and hence the set \( S_0 \) is discrete.
Remark 2.5. Note that the contour map of $u$ turns our attention to the region $B^{n-1} \times [0, z_*]$. By Remark 2.3, Definition 2.4, and Remark 2.5, the error of the approximation in (3.18) can be made tiny. Moreover, we have

$$\lim_{z \to 0} \frac{u(z \omega \tan \phi, z)}{z^2} = \frac{\tan^2 \phi}{2(n-2)}$$

for $0 < z \leq z_*$ and $\omega \in \mathbb{S}^{n-2}$.

Remark 3.13. By Remark 3.4 every $t \in (t_*, 0)$ is a regular value of $u$ in $B^{n-1}_1 \times [0, z_*]$. Specifically, the set $u = t$ in $B^{n-1}_1 \times [0, z_*]$ is a non-empty compact, smoothly embedded, and strictly mean-convex hypersurface with boundary on the hyperplane $z = 0$. Moreover, it is asymptotic to a cylinder outside the cone $\mathcal{C}_\phi^*$. Let $\hat{\Sigma}_t$ be the path component of $u = t$ in $B^{n-1}_1 \times [0, z_*]$ that includes the asymptotically cylindrical part of $u = t$ outside the cone $\mathcal{C}_\phi^*$. It is not hard to see that $\hat{\Sigma}_t$ is a compact hypersurface in $B^{n-1}_1 \times [0, z_*]$ with boundary

$$\partial \hat{\Sigma}_t = \{ u = t \} \cap (B^{n-1}_1 \times \{0\}) = \partial (\{ u = t \} \cap (B^{n-1}_1 \times [0, z_*])).$$

Every other path component of $u = t$ in $B^{n-1}_1 \times [0, z_*]$, if any, is a closed hypersurface in $B^{n-1}_1 \times [0, z_*]$ in view of (3.19). Additionally, since all the path components are compact and mutually disjoint, they are away from each other.

Definition 3.14. For every $t \in (t_*, 0)$, let $\hat{\Omega}_t$ be the path component of $u \geq t$ in $B^{n-1}_1 \times [0, z_*]$ that contains the asymptotically cylindrical part of $u \geq t$ outside the cone $\mathcal{C}_\phi^*$. Note that $\hat{\Omega}_t$ is away from the hyperplane $z = z_*$.
In the following lemma we investigate the structure of $\hat{\Omega}_t$.

**Lemma 3.15.** For every $t \in (t_*, 0)$, $\hat{\Omega}_t$ is the region bounded by $\hat{\Sigma}_t$ (see Remark 3.13) and the hyperplane $z = 0$; moreover, $\hat{\Sigma}_t \subset \hat{\Omega}_t$.

**Proof.** Fix $t \in (t_*, 0)$. By the path-connectedness of $\hat{\Sigma}_t$, it is clear that $\hat{\Sigma}_t \subset \hat{\Omega}_t$. Moreover, as $t$ is a regular value of $u$ in $B_1^{n-1} \times [0, z_\star]$ (see Remark 3.4), we have $\hat{\Sigma}_t \subset \partial \hat{\Omega}_t$. In what follows we would like to show that

$$\partial \hat{\Omega}_t \cap \{z > 0\} \subset \hat{\Sigma}_t$$

by contradiction.

Suppose the contrary that $(\partial \hat{\Omega}_t \setminus \hat{\Sigma}_t) \cap \{z > 0\} \neq \emptyset$. Since

$$\partial \hat{\Omega}_t \cap \{z > 0\} \subset \{u = t\} \cap (B_1^{n-1} \times (0, z_\star)),$$

it follows from Remark 3.13 that there exists a closed, path-connected, smoothly embedded, and strictly mean-convex hypersurface $\Gamma_t$ that is contained in $\{u = t\} \cap (B_1^{n-1} \times (0, z_\star))$, is away from $\hat{\Sigma}_t$, and satisfies $\Gamma_t \cap \partial \hat{\Omega}_t \neq \emptyset$. By Remark 3.4 and the path-connectedness of $\Gamma_t$ we then have $\Gamma_t \subset \hat{\Omega}_t$.

By the Jordan-Brouwer separation theorem (cf. [GP]), $\Gamma_t$ bounds an open connected set $D_t$ in $B_1^{n-1} \times (0, z_\star)$ that is, $\Gamma_t = \partial D_t$. In view of Remark 2.6 we have $D_t \subset \{u \geq t\}$. It follows that the level set flow near $\partial D_t$, which by Remark 3.4 is a strictly mean-convex mean curvature flow of closed and path-connected hypersurfaces, is outside $D_t$ prior to time $t$ and inside $D_t$ past to time $t$. Namely, in a tubular neighborhood of $\partial D_t$, $u < t$ outside $D_t$ and $u > t$ inside $D_t$.

Fix $p \in \hat{\Sigma}_t \cap \{z = 0\}$. As $\Gamma_t$ and $p$ are included in the path-connected set $\hat{\Omega}_t$, there exists a continuous curve

$$x : [0, 1] \to \mathbb{R}^n$$

such that $x(0) \in \Gamma_t$, $x(1) = p$, and $x(s) \in \hat{\Omega}_t$ for every $s \in (0, 1)$. In addition, since $\Gamma_t$ and $\hat{\Sigma}_t$ are away from each other (see Remark 3.13), there is $\delta > 0$ so that $\hat{\Sigma}_t$ is out of the $\delta$-tubular neighborhood of $\Gamma_t = \partial D_t$ and that in the $\delta$-tubular neighborhood of $\partial D_t$ we have $u > t$ inside $D_t$ and $u < t$ outside $D_t$. Recall that the signed distance function to $\partial D_t$, i.e.,

$$d(x) = \begin{cases} \text{dist}(x, \partial D_t), & x \in D_t \\ -\text{dist}(x, \partial D_t), & x \notin D_t \end{cases},$$

is smooth in a tubular neighborhood of $\partial D_t$ and continuous in $\mathbb{R}^n$. So $d(x(s))$ is a continuous function on $[0, 1]$ with $d(x(0)) = 0$ and $|d(x(1))| > \delta$. In view of $x(1) = p \in \{z = 0\}$ and $D_t \subset B_1^{n-1} \times (0, z_\star)$, we get that $d(x(1)) < -\delta$. It then

---

62 As $\Gamma_t \subset B_1^{n-1} \times (0, z_\star)$, the “outside” of $\mathbb{R}^n \setminus \Gamma_t$ includes the complement of $B_1^{n-1} \times (0, z_\star)$. Therefore, the “inside” should be contained in $B_1^{n-1} \times (0, z_\star)$.

63 Actually, more can be said. Firstly, there exists $\delta > 0$ so that in a tubular neighborhood of $\Gamma_t$, the set $\Gamma_t = \{u = \tau\}$, where $\tau \in [t - \delta, t + \delta]$, is a strictly mean-convex hypersurface in $B_1^{n-1} \times (0, z_\star)$ that is diffeomorphic to $\Gamma_t$; moreover, $\Gamma_t$ bounds an open connected set $D_\tau$ contained in $\{u \geq \tau\} \cap B_1^{n-1} \times (0, z_\star)$. As the flow $\{\Gamma_t\}$ moves inwardly, we have $D_{\tau_1} \supset D_{\tau_2}$ for $t - \delta \leq \tau_1 < \tau_2 < t + \delta$. Thus, the domain $D_t$ can be written as $\bigcup_{\tau \in [t, t + \delta]} \Gamma_t \cup D_{\tau + \delta}$, from which we deduce that $D_t \subset \{u > t\}$. Indeed, the domain $D_\tau$ keeps contracting by the mean curvature vector until it becomes $S \cap D_t$ when $\tau = 0$. 

---
follows from the intermediate value theorem that there exists \( \hat{s} \in (0, 1) \) such that 
\[
d(x(\hat{s})) = -\delta, \text{ yielding } u(x(\hat{s})) < t.
\]
However, we have
\[
x(\hat{s}) \in \hat{\Omega}_t \subset \{ u \geq t \},
\]
which is a contradiction. \( \square \)

**Remark 3.16.** For each \( t \in (t_*, 0) \), by the continuity of \( u \) at 0 there exists \( r \in (0, z_*) \) so that \( u \geq t \) in \( B_r^{n-1} \times [-r, r] \). By Definition 3.13 \( \text{we have } \mathcal{C}_t^{*} \cap \{ 0 \leq z \leq r \} \subset \hat{\Omega}_t \). Thus, it is not hard to see that
\[
\hat{\zeta}_t = \sup \left\{ \zeta \in (0, z_*) : \mathcal{C}_t^{*} \cap \{ 0 \leq z \leq \zeta \} \subset \hat{\Omega}_t \right\}
\]
belongs to \( (0, z_*) \) (by virtue of the continuity of \( u \)) and that
\[
\mathcal{C}_t^{*} \cap \{ 0 \leq z < \hat{\zeta}_t \} \subset \hat{\Omega}_t.
\]
Note that \( \hat{\zeta}_t \searrow 0 \) as \( t \nearrow 0 \).

Now let us fix \( t_0 \in (t_*, 0) \). The following proposition is based on Lemma 3.15 and is crucial to Lemma 3.18.

**Proposition 3.17.** For every \( t \in [t_0, 0) \) we have
\[
\hat{\Omega}_t = \{ u \geq t \} \cap \hat{\Omega}_{t_0};
\]
in particular, \( \{ u \geq t \} \cap \hat{\Omega}_{t_0} \) is path-connected.

**Proof.** By Definition 3.14 we have \( \hat{\Omega}_t \subset \{ u \geq t \} \cap \hat{\Omega}_{t_0} \) for every \( t \in [t_0, 0) \). Suppose the contrary that there exists \( t_1 \in (t_0, 0) \) such that
\[
\hat{\Omega}_{t_1} \subset \{ u \geq t_1 \} \cap \hat{\Omega}_{t_0}.
\]
Then the first thing we would like to show is that
\[
\{ u = t_1 \} \cap \hat{\Omega}_{t_0} \setminus \hat{\Omega}_{t_1} \neq \emptyset.
\]
To see that, choose \( p \in \{ u \geq t_1 \} \cap \hat{\Omega}_{t_0} \setminus \hat{\Omega}_{t_1} \). If \( u(p) = t_1 \), then we are done. So let us assume that \( u(p) > t_1 \). By the path-connectedness of \( \hat{\Omega}_{t_0} \), there exists a continuous curve \( x : [0, 1] \to B_1^{n-1} \times [0, z_*] \) such that \( x(0) = p, x(1) \in \hat{\Omega}_{t_0} \), and \( x(s) \in \hat{\Omega}_{t_0} \) for every \( s \in [0, 1] \). Since \( u(x(0)) > t_1 \) and \( u(x(1)) = t_0 \), we have
\[
\hat{s} = \sup \{ \sigma \in [0, 1] : u(x(\hat{s})) \geq t_1 \forall s \in [0, \sigma] \} \in (0, 1)
\]
and \( u(x(\hat{s})) = t_1 \); in addition, as \( p = x(0) \notin \hat{\Omega}_{t_1} \) and \( x(s) \in \{ u \geq t_1 \} \) for every \( s \in [0, \hat{s}] \), we have \( x(\hat{s}) \notin \hat{\Omega}_{t_1} \). Thus, we find \( q = x(\hat{s}) \in \{ u = t_1 \} \cap \hat{\Omega}_{t_0} \setminus \hat{\Omega}_{t_1} \).

Next, by Remark 3.13 \( \text{and Lemma 3.15} \) there exists a closed, path-connected, smoothly embedded, and strictly mean-convex hypersurface \( \Gamma_{t_1} \) contained in \( \{ u = t_1 \} \cap (B_1^{n-1} \times (0, z_*)) \) such that \( \Gamma_{t_1} \cap \hat{\Omega}_{t_0} \setminus \hat{\Omega}_{t_1} \neq \emptyset. \) Due to the path-connectedness \( \text{of } \Gamma_{t_1} \), and Definition 3.14 \( \text{it is clear that } \Gamma_{t_1} \subset \hat{\Omega}_{t_0} \setminus \hat{\Omega}_{t_1}. \) By the argument in proving Lemma 3.15, \( \Gamma_{t_1} \) bounds an open connected set \( \mathcal{D}_{t_1} \) contained in \( \{ u \geq t_1 \} \cap (B_1^{n-1} \times (0, z_*)), \) i.e., \( \Gamma_{t_1} = \partial \mathcal{D}_{t_1}. \) It then follows from the path-connectedness\(^6\) of \( \mathcal{D}_{t_1} = \mathcal{D}_{t_1} \cup \Gamma_{t_1} \text{ that } \mathcal{D}_{t_1} \subset \hat{\Omega}_{t_0} \setminus \hat{\Omega}_{t_1}. \)

\(^6\)The interior \( \mathcal{D}_{t_1} \) is open connected and hence path-connected. In a tubular neighborhood of the smoothly embedded hypersurface \( \partial \mathcal{D}_{t_1} \), we can find a continuous path joining a given point on \( \partial \mathcal{D}_{t_1} \) with a nearby point in \( \mathcal{D}_{t_1}. \) Thus, the closure \( \overline{\mathcal{D}_{t_1}} \) is path-connected.
Due to the type I condition (see Definition 3.3), on the set \( \{ t_0 \leq u \leq t_1 \} \cap (B_1^{n-1} \times [0, z_\ast]) \) we have 
\[
|\nabla u| \geq \beta^{-1} \sqrt{-t_1} = \rho.
\]

Then by the interior curvature estimate in [HK], there exists \( \delta > 0 \) (depending on the dimension \( n \), the scaling-invariant constant of noncollapsing, the entropy of the flow, \( \rho \), and \( t_0 - t_\ast \)) so that in a tubular neighborhood of \( \Gamma_t \), the set \( \Gamma_t = \{ u = t \} \) for \( t \in [t_1 - \delta, t_1] \) forms a strictly mean-convex mean curvature flow of hypersurfaces in \( B_1^{n-1} \times (-1, 1) \) that is diffeomorphic to \( \Gamma_t \). In view of the asymptotically cylindrical behavior of \( u \) outside the cone \( \mathcal{C}_0 \) and the choice of \( z_\ast \), it is clear that \( \Gamma_t \subset \{ u = t \} \) is away from \( \partial B_1^{n-1} \times (-1, 1) \) and stays strictly below the hyperplane \( z = z_\ast \) for every \( t \in [t_1 - \delta, t_1] \). We would like to show that \( \Gamma_t \) actually stays in \( B_1^{n-1} \times (0, z_\ast) \) for \( t \in [t_2, t_1] \), where \( t_2 = \max \{ t_1 - \delta, t_0 \} \). To this end, let 
\[
\hat{t} = \inf \{ \tau \in [t_2, t_1] : \Gamma_{\hat{t}} \text{ is contained in } B_1^{n-1} \times (0, z_\ast) \text{ for all } t \in [\tau, t_1] \}
\]
As \( \Gamma_{t_1} \) is contained in \( B_1^{n-1} \times (0, z_\ast) \), it is clear that \( \hat{t} < t_1 \). Were \( \Gamma_{\hat{t}} \) not contained in \( B_1^{n-1} \times (0, z_\ast) \), it would be that \( \Gamma_{\hat{t}} \subset B_1^{n-1} \times \{ 0, z_\ast \} \) with \( \Gamma_{\hat{t}} \cap \{ z = 0 \} \neq \emptyset \). Choose \( \hat{x} \in \Gamma_{\hat{t}} \cap \{ z = 0 \} \). Note that \( \hat{x} \neq 0 \) because \( u(x) = \hat{t} \). Since 
\[
\hat{z}(\hat{x}) = 0 = \min_{\Gamma_{\hat{t}}} z,
\]
we obtain that \( \nabla z(\hat{x}) = (0, 1) \) is a normal vector of \( \Gamma_{\hat{t}} \subset \{ u = \hat{t} \} \) at \( \hat{x} \) and hence it must be parallel to \( \nabla u(\hat{x}) \); a contradiction follows immediately from the asymptotically cylindrical behavior of \( u \) on the hyperplane \( z = 0 \). Therefore, \( \Gamma_{\hat{t}} \) is contained in \( B_1^{n-1} \times (0, z_\ast) \) as well; it follows that \( t = t_2 \).

Furthermore, arguing as in the proof of Lemma 3.13, for every \( t \in [t_2, t_1] \), \( \Gamma_t \) bounds an open connected set \( \mathcal{D}_t \) contained in \( \{ u \geq \hat{t} \} \cap (B_1^{n-1} \times (0, z_\ast)) \) with \( \mathcal{D}_t \supset \overline{\mathcal{D}}_t \) whenever \( t_2 \leq \hat{t} \leq t \leq t_1 \). As \( \mathcal{D}_{t_1} \subset \hat{\Omega}_{t_0} \) and that 
\[
\overline{\mathcal{D}}_{t_2} = \bigcup_{t \in [t_2, t_1]} \Gamma_t \cup \mathcal{D}_t,
\]
is path-connected, we have \( \overline{\mathcal{D}}_{t_2} \subset \hat{\Omega}_{t_0} \).

If \( t_2 = t_0 \), then we have found an open connected set \( \mathcal{D}_{t_0} = \mathcal{D}_{t_2} \) that is contained in \( \hat{\Omega}_{t_0} \) whose boundary \( \Gamma_{t_0} \) is a closed, path-connected, and smoothly embedded hypersurface contained \( \{ u = t_0 \} \cap \hat{\Omega}_{t_0} \cap (B_1^{n-1} \times (0, z_\ast)) \). Else, \( t_2 > t_0 \), then we will continue the aforementioned process. As the constant \( \delta \) is chosen so that it works for every following step, so we can set \( t_{i+1} = t_i - \delta \) for \( i = 2, 3, \ldots \) until \( t_{m-1} \in (t_0, t_0 + \delta) \) for some \( m \in \mathbb{N} \) and then we set \( t_m = t_0 \). In that case we will have 
\[
\hat{\Omega}_{t_0} \supset \overline{\mathcal{D}}_{t_0} = \overline{\mathcal{D}}_{t_m} \supset \cdots \supset \overline{\mathcal{D}}_{t_2} = \overline{\mathcal{D}}_{t_1},
\]
where each \( \mathcal{D}_{t_i} \) is an open connected set in \( \{ u \geq t_i \} \cap (B_1^{n-1} \times (0, z_\ast)) \) whose boundary \( \Gamma_{t_i} \) is a closed, path-connected, and smoothly embedded hypersurface contained \( \{ u = t_i \} \cap (B_1^{n-1} \times (0, z_\ast)) \). In particular, we obtain an open connected set \( \mathcal{D}_{t_0} = \mathcal{D}_{t_m} \) contained in \( \hat{\Omega}_{t_0} \) whose boundary \( \Gamma_{t_0} \) is a path-connected smooth closed hypersurface contained \( \{ u = t_0 \} \cap \hat{\Omega}_{t_0} \cap (B_1^{n-1} \times (0, z_\ast)) \).

---

\(^{65}\)Considering the asymptotically cylindrical behavior of \( u \) outside the cone \( \mathcal{C}_0 \) and the choice of \( z_\ast \), the set \( \{ t_0 \leq u \leq t_1 \} \cap (B_1^{n-1} \times [0, z_\ast]) \) is roughly contained in \( \hat{B}_1^{n-1} \times [0, \frac{1}{2}] \).
By Remark 3.13 we have $\Gamma_{t_0} \subset \partial \hat{\Omega}_{t_0}$. It follows from Lemma 3.15 that $\Gamma_{t_0} \subset \hat{\Sigma}_{t_0}$.

We claim that $\Gamma_{t_0}$ is relatively clopen in $\hat{\Sigma}_{t_0}$; therefore, by the connectedness of $\hat{\Sigma}_{t_0}$ we have $\Gamma_{t_0} = \hat{\Sigma}_{t_0}$, giving a contradiction since $\Gamma_{t_0} \subset B^{n-1}_1 \times (0, z_*)$ while $\hat{\Sigma}_{t_0} \cap \{ z = 0 \} \neq \emptyset$. To prove the claim, note first that $\Gamma_{t_0}$ is compact and hence it must be relatively closed in $\hat{\Sigma}_{t_0}$. To verify the relatively openness, let us fix $x_0 \in \Gamma_{t_0}$. By Remark 3.13 there exists $r > 0$ so that $B_r(x_0) \subset B^{n-1}_1 \times (0, z_*)$ and that $B_r(x_0) \setminus \Gamma_{t_0}$ is separated into two open connected sets: one is contained in $\{ u > t_0 \}$ and the other is contained in $\{ u < t_0 \}$. As a result, we have

$$\hat{\Sigma}_{t_0} \cap B_r(x_0) \subset \{ u = t_0 \} \cap B_r(x_0) \subset \Gamma_{t_0} \cap B_r(x_0);$$

that is, there is a neighborhood of $x_0$ in $\hat{\Sigma}_{t_0}$ that is contained in $\Gamma_{t_0}$. \qed

We are about to prove Proposition 3.19, which says that $0$ is the only singular point in $B^{n-1}_1 \times [0, z_0]$ for some $z_0 > 0$. The proof relies on the following lemma:

**Lemma 3.18.** For every $z \in (0, \hat{z}_{t_0})$\(^{66}\) such that

$$u(y, z) = u_{\max}(z) \in [t_0, 0),$$

where $y \in B^{n-1}_{z \tan \phi}$ is any maximum point of $u$ on level $z$ (see 3.10)\(^{67}\) there holds

$$\nabla u(y, z) \cdot (0, 1) < 0.$$

*Proof.* Suppose the contrary that there exists $\hat{z} \in (0, \hat{z}_{t_0})$ and $y \in B^{n-1}_{z \tan \phi}$ such that $u(\hat{y}, \hat{z}) = u_{\max}(\hat{z}) \in [t_0, 0)$ and $\nabla u(\hat{y}, \hat{z}) = (0, \lambda)$ for some $\lambda > 0$. Choose $\delta \in (0, \hat{z}_{t_0} - \hat{z})$ such that

$$t_0 \leq u(\hat{y}, \hat{z}) < u(\hat{y}, \hat{z} + \delta) < 0.$$

Since $\hat{z} + \delta < \hat{z}_{t_0}$ and $\hat{y} < (\hat{z} + \delta) \tan \phi$, by Remark 3.16 ($\hat{y}, \hat{z} + \delta$) belongs to $\{ u \geq u(\hat{y}, \hat{z} + \delta) \} \cap \hat{\Omega}_{t_0}$. Then it follows from Proposition 3.17 that there exists a continuous curve $x : [0, 1] \to B^{n-1}_1 \times [0, z_*]$ so that $x(0) = (\hat{y}, \hat{z} + \delta), x(1) = 0$, and $x(s) \in \{ u \geq u(\hat{y}, \hat{z} + \delta) \} \cap \hat{\Omega}_{t_0}$ for $s \in [0, 1]$.

Since $z(x(s))$ is a continuous function with $z(x(0)) = \hat{z} + \delta$ and $z(x(1)) = 0$, by the intermediate value theorem there exists $\hat{s} \in (0, 1)$ so that

$$z(x(\hat{s})) = \hat{z}.$$

By the choice of the continuous path, we obtain

$$u(x(\hat{s})) \geq u(\hat{y}, \hat{z} + \delta) > u(\hat{y}, \hat{z}),$$

contradicting the assumption that $u_{\max}(\hat{z}) = u(\hat{y}, \hat{z})$. \qed

**Proposition 3.19.** There exists $z_0 \in (0, \hat{z}_{t_0})$, where $\hat{z}_{t_0}$ is the constant in Lemma 3.18, so that the function $u_{\max}$ is nonincreasing on $[0, z_0]$ with $u_{\max}(z) < 0$ for every $z \in [0, z_0]$.

It follows from Remark 3.9 and the asymptotically cylindrical behavior of the level set flow on the hyperplane $z = 0$ that the point $0$ is the only singular point in $B^{n-1}_1 \times [0, z_0]$.

---

\(^{66}\)See Remark 3.10 for the definition of $\hat{z}_{t_0}$.

\(^{67}\)Note that $\nabla u(y, z) \neq 0$ by the type I condition (see Definition 3.3) and that it must be parallel to $(0, 1)$ as $(y, z)$ is a maximum point of $u$ on level $z$. 

Proof. Firstly we would like to show that $u_{\text{max}}$ is a Lipschitz continuous on $[0, z_0]$; in particular, it is absolutely continuous. To see that, given $z$ and $\tilde{z}$ in $[0, z_0]$, choose $y$ and $\tilde{y}$ in $B_1^{n-1}$ so that $u_{\text{max}}(z) = u(y, z)$ and $u_{\text{max}}(\tilde{z}) = u(\tilde{y}, \tilde{z})$ (see (3.16)). Then we have

$$u_{\text{max}}(z) - u_{\text{max}}(\tilde{z}) \leq u(y, z) - u(y, \tilde{z}) \leq \|\nabla u\|_{L^\infty(B_1^{n-1} \times [0, z_0])} |z - \tilde{z}|,$$

$$u_{\text{max}}(z) - u_{\text{max}}(\tilde{z}) \geq u(\tilde{y}, z) - u(\tilde{y}, \tilde{z}) \geq -\|\nabla u\|_{L^\infty(B_1^{n-1} \times [0, z_0])} |z - \tilde{z}|.$$

Particularly, since $u_{\text{max}}(0) = 0$ and that 0 is the maximum value of $u_{\text{max}}$ on $[0, z_0]$, there exists $z_0 \in (0, \tilde{z}_0)$ such that

$$(3.20) \quad t_0 \leq u_{\text{max}}(z) \leq 0$$

for every $z \in [0, z_0]$.

Note that an absolutely continuous function is differentiable almost everywhere. Assume that $u_{\text{max}}$ is differentiable at $z \in (0, z_0)$. Let us choose $y \in B_1^{n-1} \tan \phi$ so that $u_{\text{max}}(z) = u(y, z)$. Then either $u_{\text{max}}(z) = 0$, in which case we have $u'(z) = 0$ as 0 is the maximum value of $u_{\text{max}}$ on $[0, z_1]$; or $u_{\text{max}}(z) \in [t_0, 0)$, in which case Lemma 3.18 gives $\nabla u(y, z) \cdot (0, 1) < 0$, yielding

$$u_{\text{max}}'(z) = \lim_{h \to 0} \frac{u_{\text{max}}(z) - u_{\text{max}}(z - h)}{h} \leq \lim_{h \to 0} \frac{u(y, z) - u(y, z - h)}{h} = \nabla u(y, z) \cdot (0, 1) < 0.$$

Thus, $u_{\text{max}}' \leq 0$ almost everywhere.

It follows from the fundamental theorem of calculus (for absolutely continuous functions) that for every $0 \leq z < \tilde{z} \leq z_0$ we have

$$\int_z^{\tilde{z}} u_{\text{max}}'(\zeta) d\zeta \leq 0.$$

Thus, $u_{\text{max}}$ is nonincreasing on $[0, z_0]$. Moreover, by Lemma 3.12 there exists a sequence $z_i \in [0, z_0]$ tending to 0 with $u_{\text{max}}(z_i) < 0$. Therefore, we have

$$u_{\text{max}}(z) \leq u_{\text{max}}(z_i) < 0 \quad \forall z \in [z_i, z_0].$$

As $z_i \to 0$, we obtain $u_{\text{max}}(z) < 0$ for every $z \in (0, z_0]$.

For every $p \in B_1^{n-1} \times [0, z_0]$ with $p \neq 0$ [6] by the argument in Proposition 3.5 [7] there exists a smooth curve $x(s) = \Psi(p, s)$ in $B_1^{n-1} \times [0, z_0]$ satisfying

$$\frac{dx}{ds} = N(x) = \frac{\nabla u(x)}{\|\nabla u(x)\|}$$

with $x(0) = \Psi(p, 0) = p$. In fact, $s \mapsto \Psi(p, s)$ is the unique integral curve of the unit normal vector field $N = \frac{\nabla u}{|\nabla u|}$ that starting at $p$. Let

$$\mathcal{C}_\phi^0 = \mathcal{C}_\phi \cap \{0 \leq z \leq \tilde{z}_0\} = \{y| |y| \leq z \tan \phi \cap \{0 \leq z \leq \tilde{z}_0\}.$$

---

[6] The positive constant $z_0$ is chosen in Proposition 3.19.

[7] So that $p$ is not a critical point by Proposition 3.19.

[7] That is, $\nabla u$ is Lipschitz continuous by Theorem 2.3 [8] the existence and uniqueness theorem of ODE yields a unique flow line of the vector field $\nabla u$ passing through the non-stationary point $p$. The flow line is smooth owing to Theorem 2.3 and the regularity theory in ODE. After a reparametrization by the arc length, we then get the desired curve.
What follows is showing that if the initial point sits in the cone \( \mathcal{C}_\phi^0 \) with its \( u \) value at least \( u_{\text{max}}(z_0) \), then the associated integral curve stays in the cone \( \mathcal{C}_\phi^0 \) and converges to 0.

**Proposition 3.20.** For every \( p \in \mathcal{C}_\phi^0 \) such that
\[
\phi(p) \cdot \frac{\nabla u}{|\nabla u|} = \cos \phi,
\]
the integral curve \( s \to \Psi(p,s) \) of \( N = \frac{\nabla u}{|\nabla u|} \) starting at \( p \) stays in \( \mathcal{C}_\phi^0 \) and tends to 0 in finite distance. Moreover, the arc length is bounded above by \( 2\beta \sqrt{-u(p)} \).

**Proof.** Fix \( p \in \mathcal{C}_\phi^0 \cap \{u_{\text{max}}(z_0) \leq u < 0\} \) and let \( x(s) = \Psi(p,s) \). Set
\[
\dot{s} = \{\sigma \geq 0 : x(s) \text{ is defined and stays in the cone } \mathcal{C}_\phi^0 \text{ for every } s \in [0,\sigma]\}.
\]
Firstly we would like to show that \( \dot{s} > 0 \). This is clear when \( p \) is in the interior of the cone \( \mathcal{C}_\phi^0 \), so our discussion will focus on the case where \( p \) is located on the boundary of the cone \( \mathcal{C}_\phi^0 \). If \( p \) is on the top of the cone \( \mathcal{C}_\phi^0 \), i.e., \( z(p) = z_0 \), then by the conditions \( u(p) \geq u_{\text{max}}(z_0) \) and \( \phi(p) \cdot \frac{\nabla u}{|\nabla u|} = \cos \phi \), we infer that \( u(p) = u_{\text{max}}(z_0) \) and \( p \in B_{z_0} \tan \phi \). It follows from Lemma 3.18 that \( \nabla u(p) = (0,-\lambda) \) for some \( \lambda > 0 \).

So the curve immediately enters the interior of the cone \( \mathcal{C}_\phi^0 \) after \( s = 0 \), yielding that \( \dot{s} > 0 \). The other possibility is that when \( p \) is on the lateral boundary of the cone with \( z(p) \in (0,z_0) \). In view of the asymptotically cylindrical behavior of \( u \) on the boundary of the cone \( \mathcal{C}_\phi^0 \), we have
\[
N(p) \cdot N_{\partial \mathcal{C}_\phi^0} = \cos \phi,
\]
where \( N_{\partial \mathcal{C}_\phi^0} \) is the inward unit normal vector of \( \partial \mathcal{C}_\phi^0 \). Thus, in this case the curve also enters the interior of the cone \( \mathcal{C}_\phi^0 \) after \( s = 0 \) and hence we have \( \dot{s} > 0 \).

Next, we show that if \( \dot{s} < \infty \), then the curve must converge to 0 as \( s \to \dot{s} \). Note first that as both \( x(s) \) and \( \frac{dx}{ds} = N(x) \) are bounded on \([0,\dot{s}]\), we have
\[
x(s) \to q \in \partial \mathcal{C}_\phi^0
\]
as \( s \to \dot{s} \). Suppose that \( q \neq 0 \), then by the argument in the last paragraph, the integral curve \( s \to \Psi(q,s) \) is defined on \([0,\delta]\) for some \( \delta > 0 \) and it moves to the
interior of the cone $\mathcal{C}_\phi^0$ for $s \in (0, \delta)$. Using the semigroup property of the flow induced by a vector field, we have

$$\Psi (p, \hat{s} + s) = \Psi (q, s),$$

for $s \in [0, \delta]$, contradicting the choice of $\hat{s}$. Thus, $q = 0$.

Lastly, let us see why $\hat{s}$ is finite and how to estimate it. By the type I condition, on $[0, \hat{s})$ we have

$$\frac{d}{ds} [u (x)] = \nabla u (x) \cdot N (x) = |\nabla u (x)| \geq \frac{1}{\beta} \sqrt{-u (x)},$$

yielding

$$\sqrt{-u (p)} = \sqrt{-u (x (0))} \geq \sqrt{-u (x (s))} + \frac{s}{2\beta} \geq \frac{s}{2\beta}$$

for every $s \in [0, \hat{s})$. Therefore, $\hat{s} \leq 2\beta \sqrt{-u (p)}$. □

Now we are ready to get a desired contradiction to finish this subsection. By Remark 3.16 and Proposition 3.19 we can choose $\check{z} \in (0, z_0)$ and $\check{p} \in \partial \mathcal{C}_\phi^0$ with $z (\check{p}) = \check{z}$, i.e., $\check{p} = (\check{z} \check{\omega} \tan \phi, \check{z})$ for some $\check{\omega} \in S^{n-2}$, so that

$$\check{t} = u (\check{p}) \in (u_{\max} (z_0), 0).$$

By (3.18) we have

$$\frac{-\check{t}}{\check{z}^2} = \frac{-u (\check{z} \check{\omega} \tan \phi, \check{z})}{\check{z}^2} \approx \frac{\tan^2 \phi}{2 (n-2)} \leq \frac{\tan^2 \phi}{n-2},$$

namely,

$$\check{z} \geq \sqrt{n-2} \cot \phi \sqrt{-\check{t}}. \quad (3.21)$$

On the other hand, by Proposition 3.20 the integral curve $s \mapsto \Psi (\check{p}, s)$ of $N = \frac{\nabla u}{|\nabla u|}$ starting at $\check{p}$ tends to 0 as $s \nearrow \check{s}$, where $\check{s} \in \left(0, 2\beta \sqrt{-u (\check{p})}\right)$ is the arc length of the curve. In particular, we obtain

$$\check{z} \leq |\check{p}| \leq \check{s} \leq 2\beta \sqrt{-u (\check{p})} = 2\beta \sqrt{-\check{t}}. \quad (3.22)$$

In view of (3.21) and (3.22), a contradiction follows if

$$\phi < \tan^{-1} \left( \frac{\sqrt{n-2}}{2\beta} \right). \quad (3.23)$$

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