THE FIELD THEORY LIMIT
OF INTEGRABLE LATTICE MODELS

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Abstract

The light-cone approach is reviewed. This method allows to find the underlying quantum field theory for any integrable lattice model in its gapless regime. The relativistic spectrum and S-matrix follows straightforwardly in this way through the Bethe Ansatz. We show here how to derive the infinite number of local commuting and non-local and non-commuting conserved charges in integrable QFT, taking the massive Thirring model (sine-Gordon) as an example. They are generated by quantum monodromy operators and provide a representation of $q$–deformed affine Lie algebras $U_q(\hat{g})$. 
1 Yang-Baxter equations and the Light-cone Approach

How to take the continuum limit of integrable lattice models has always been a major problem. In this short review we shall try to convince the reader that the light-cone approach [1, 2] is the best way to perform such continuum limit.

We consider a \(N \times M\) two dimensional square lattice whose links are labeled by an index \(a = 1, \ldots, n\). The statistical weight of each vertex where the four links meet is defined by the \(R\)-matrix \(R_{cd}^{ba}(\theta)\) where \(1 \leq a, b, c, d \leq n\) (see fig.1). Here \(\theta\) is a complex variable called spectral variable. In the present context \(\theta\) can be considered as a sort of coupling constant. It must be noticed that universal magnitudes are \(\theta\)-independent in integrable models [5].

It is convenient to introduce an operator \(T_{ab}(\theta, \tilde{\omega})\) associated to horizontal lines (see fig. 2)

\[
T_{ab}(\theta, \tilde{\omega}) = \sum_{a_1, \ldots, a_{N-1}} t_{a_1 b}(\theta + \omega_1) \otimes t_{a_2 a_1}(\theta + \omega_2) \otimes \cdots \otimes t_{a_{N-1}}(\theta + \omega_N)
\] (1)

For fixed \(a, b\), \(T_{ab}(\theta, \tilde{\omega})\) acts on the vertical space \(\mathcal{V} = \bigotimes_{1 \leq i \leq N} V_i\), \(V_i \equiv \mathbb{C}^n\), and the local vertex operators are defined as \([t_{ab}(\theta)]_{cd} \equiv R_{cd}^{ba}(\theta)\). In eq.(1) we introduced arbitrary inhomogeneity parameters \(\omega_1, \omega_2, \ldots, \omega_N\) associated to each site on a horizontal line.

When periodic boundary conditions (PBC) are considered, it is useful to define the row-to-row transfer matrix as

\[
t(\theta, \tilde{\omega}) = \sum_a T_{aa}(\theta, \tilde{\omega})
\] (2)

For other types of boundary conditions, see refs.[5, 6, 7].

The first relevant physical problem is to compute the partition function \(Z\). It is defined as the sum over all possible configurations of the statistical weights for the whole lattice. For a \(N \times M\) lattice with PBC in both directions, \(Z\) can be written as

\[
Z = Tr \left[ t(\theta, \tilde{\omega})^M \right]
\] (3)

where \(Tr\) stands for the trace on the vertical space \(\mathcal{V}\). The free energy is then given by

\[
f(\theta, \tilde{\omega}) = -\lim_{(N,M) \to \infty} \frac{1}{NM} \log Z
\] (4)

All considerations up to now are valid whether the model is integrable or not. We shall call integrable those models where the \(R\)-matrix \(R(\theta)\) obeys the Yang-Baxter equations (YBE):

\[
\sum_{1 \leq k, l, m \leq n} R_{ba}^{kl}(\theta - \theta') R_{ck}^{lm}(\theta) R_{ml}^{ef}(\theta') = \sum_{1 \leq k, l, m \leq n} R_{ck}^{lk}(\theta') R_{ka}^{mf}(\theta) R_{lm}^{de}(\theta - \theta')
\] (5)

or in tensor product notation

\[
[1 \otimes R(\theta - \theta')] [R(\theta) \otimes 1] [1 \otimes R(\theta')] = [R(\theta') \otimes 1] [1 \otimes R(\theta)] [R(\theta - \theta') \otimes 1]
\] (6)
It must be stressed that the YBE are a heavily overdetermined set of functional algebraic equations. They contain \textit{a priori} \( n^4 \) unknowns [the elements of \( R(\theta) \)] and \( n^6 \) equations. Despite this fact a large set of solutions is known. All of them possess symmetries that reduce the number of independent equations and make possible the existence of solutions. The symmetries may be discrete as cyclic \( Z_n \) symmetries, continuous abelian symmetries as \( U(1)^n \) and non-abelian as \( GL(n) \) (see [5]).

The YBE (5-6) admit the natural graphical representation given in fig. 3. Graphically, the YBE express the freedom to push lines through intersections of pair of lines. This possibility of rigid line shifting can be interpreted as a zero curvature condition on the lattice.

The YBE enjoy a powerful coproduct property. Namely, eqs.(5-6) implies that the operators \( T_{ab}(\theta, \tilde{\omega}) \) fulfill the YB algebra

\[
R(\lambda - \mu) \left[ T(\lambda, \tilde{\omega}) \otimes T(\mu, \tilde{\omega}) \right] = \left[ T(\mu, \tilde{\omega}) \otimes T(\lambda, \tilde{\omega}) \right] R(\lambda - \mu) \tag{7}
\]

For one-site (\( N = 1 \)), \( T_{ab}(\theta, \tilde{\omega}) \) reduces to \( R(\theta) \) and eq.(7) becomes eq.(9). For \( N \)-sites, eq.(7) can be easily proved by repeatedly pushing lines through vertices (see [5]). That is, eq.(7) is the expression of the YBE for \( N \)-sites. The coproduct rule is here defined by eq.(1). Eq.(7) implies the commutativity of transfer matrices

\[
[t(\theta, \tilde{\omega}), t(\theta', \tilde{\omega})] = 0 \tag{8}
\]

That is, the transfer matrices for fixed \( \omega_1, \omega_2, \ldots, \omega_N \) form a commuting family. Hence, one can expect to diagonalize it with \( \theta \)-independent eigenvectors:

\[
t(\theta, \tilde{\omega}) \Psi(\tilde{\omega}) = \lambda(\theta, \tilde{\omega}) \Psi(\tilde{\omega}) \tag{9}
\]

The Bethe Ansatz (BA) actually does this job [3]. Then, the free energy in the thermodynamic limit turns to be given by the largest eigenvalue \( \lambda(\theta, \tilde{\omega})_{\text{max}} \) of \( t(\theta, \tilde{\omega}) \). We find from eqs.(4) and (9)

\[
f(\theta, \tilde{\omega}) = - \lim_{N \to \infty} \frac{1}{N} \log \lambda(\theta, \tilde{\omega})_{\text{max}} \tag{10}
\]

When \( \theta = \theta' \), eq.(7) naturally suggest that \( R(0) \) is a multiple of the unit matrix. This is usually the case. More precisely, a solution of the YBE (3) is called \textit{regular} if

\[
R(0) = c \mathbb{1} \quad \text{that is} \quad R_{ab}(0) = c \delta_a^c \delta_b^d \tag{11}
\]

where \( c \) is a non-zero constant.

Setting \( \theta = 0 \) in eqs.(3) yields with the help of eq.(11)

\[
M_{pq}^{ef}(\theta') \delta_c^d = \delta_a^f M_{bc}^{df}(\theta') \quad \text{where} \quad M_{cb}(\theta) = \sum_{1 \leq k, l \leq n} R_{kl}^{ab}(-\theta) R_{cd}(\theta)
\]
We thus see that $M_{ab}(\theta)$ must have the index structure $M_{ab}(\theta) = \delta^a_c \delta^b_d \rho(\theta)$, where $\rho(\theta)$ is a c-number function. This can be written as

$$R(\theta)R(-\theta) = \rho(\theta),$$

that is

$$\sum_{1 \leq c,d \leq n} R^a_{cd}(\theta) R^{cd}_{ef}(-\theta) = \delta^a_e \delta^b_f \rho(\theta).$$

It follows that $\rho(\theta)$ is an even function. This property is usually called ‘unitarity’ although this may not be always the appropriate name.

From eqs.(1)-(2) at zero inhomogeneity $\omega_1 = \omega_2 = \ldots = \omega_N = 0$ and eq.(11), it follows that

$$t(0, \{\omega_k = 0\}) = c^N \Pi_s,$$

where $\Pi_s$ is the unit shift operator in the horizontal direction. The momentum operator is then given by

$$P \equiv -i \log \left[ c^{-N} t(0, \{\omega_k = 0\}) \right].$$

Moreover, it can be shown [8, 5] that the operators

$$C_m \equiv \left. \frac{\partial^m}{\partial \theta^m} \log t(\theta, \{\omega_k = 0\}) \right|_{\theta = 0}$$

couple $m + 1$ neighbor sites on the horizontal line. Usually $C_1$ is a quantum spin chain hamiltonian. The commutativity of the transfer matrices (8) implies

$$[C_k, C_l] = 0 \quad \forall \ k, l.$$

We thus find an infinite number of commuting magnitudes.

We are considering here vertex models where horizontal and vertical lines are of the same type. This is not necessary for integrability. It is possible to choose all vertical lines of a given kind (say with $n_V$ states per vertical link) and all horizontal lines of a different kind (say with $n_H \neq n_V$ states per horizontal link) [3]. Moreover, it is possible to built integrable models mixing vertical (or horizontal) lines of different types [12].

It is also possible to construct integrable face models where the variables $(a, b, c, \ldots)$ are attached to the vertices. That is, to the dual lattice [1, 0, 1].

Up to now, we implicitly consider an euclidean two dimensional lattice. Let us now consider a diagonal-to-diagonal lattice (see fig.4) which represents a discretized Minkowski spacetime in light-cone coordinates. That is, the axis correspond to $x \pm t$, ($x$ and $t$ being the usual space and time variables).

In this approach we start from the discretized Minkowski 2D space–time formed by a regular diagonal lattice of right–oriented and left–oriented straight lines (see fig. 4). These represent true world–lines of “bare” objects (pseudo–particles) which are thus naturally divided in left– and right–movers. The right–movers have all the same positive rapidity $\Theta$, while the left–movers have rapidity $-\Theta$. One can regard $\Theta$ as a cut–off rapidity, which will be appropriately
taken to infinity in the continuum limit. Furthermore, we shall denote by $V$ the Hilbert space of states of a pseudo–particle (we restrict here to the case in which $V$ is the same for both left– and right–movers and has finite dimension $n$, although more general situations can be considered).

The dynamics of the model is fixed by the microscopic transition amplitudes attached to each intersection of a left– and a right–mover, that is to each vertex of the lattice. This amplitudes can be collected into linear operators $R_{ij}$, the local $R$–matrices, acting non–trivially only on the space $V_i \otimes V_j$ of $i$th and $j$th pseudo–particles. $R_{ij}$ thus represent the relativistic scatterings of left–movers on right–movers and depend on the rapidity difference $\Theta – (-\Theta) = 2\Theta$, which is constant throughout the lattice. Moreover, by space–time translation invariance any other parametric dependence of $R_{ij}$ must be the same for all vertices. We see therefore that attached to each vertex there is a matrix $R(2\Theta)^{cd}_{ab}$ where $a,b,c,d$ are labels for the states of the pseudo–particles on the four links stemming out of the vertex, and take therefore $n$ distinct values (see fig. 1). This is the general framework of a vertex model. The difference with the standard statistical interpretation is that the Boltzmann weights are in general complex, since we should require the unitarity of the matrix $R$. In any case, the integrability of the model is guaranteed whenever $R(\lambda)^{cd}_{ab}$ satisfy the Yang–Baxter equations (3).

For periodic boundary conditions, the one–step light–cone evolution operators $U_L(\Theta)$ and $U_R(\Theta)$, which act on the ”bare” space of states $\mathcal{H}_N = (\otimes V)^{2N}$ , ($N$ is the number of sites on a row of the lattice, that is the number of diagonal lines), are built from the local $R$–matrices $R_{ij}$ as [4].

\begin{align*}
U_R(\Theta) &= U(\Theta)V, & U_L(\Theta) &= U(\Theta)V^{-1} \\
U(\Theta) &= R_{12}R_{34}\ldots R_{2N-12N} \tag{12}
\end{align*}

where $V$ is the one-step space translation to the right. $U_R$ ( $U_L$ ) evolves states by one step in right (left) light–cone direction. $U_R$ and $U_L$ commute and their product $U = U_R U_L$ is the unit time evolution operator. The graphical representation of $U$ is given by the section of the diagonal lattice with fat lines in fig. 4. If $a$ stands for the lattice spacing, the lattice hamiltonian $H$ and total momentum $P$ are naturally defined through

\begin{align*}
U &= e^{-iaH}, & U_R U_L^{-1} &= e^{iaP} \tag{13}
\end{align*}

The action of other fundamental operators is naturally defined on the same Hilbert space $\mathcal{H}_N$. These are the $n^2$ Yang-Baxter operators for $2N$ sites, which are conventionally grouped into the $n \times n$ monodromy matrix $T(\lambda) = \{T_{ab}(\lambda), \ a,b = 1,\ldots,n\}$. One usually regards the indices $a,b$ of $T_{ab}$ as horizontal indices fixing the out– and in–states of a reference pseudo–particle. Then $T(\lambda)$ is defined as horizontal coproduct of order $2N$ of the local vertex operators $L_j(\lambda) = R_{0j}(\lambda)P_{0j}$, where $0$ label the reference space and $P_{ij}$ is the transposition in $V_i \otimes V_j$. 

4
Explicitly

\[ T(\lambda) = L_1(\lambda)L_2(\lambda)\ldots L_{2N}(\lambda) \]

The inhomogeneous generalization \( T(\lambda, \vec{\omega}) \) then reads

\[ T(\lambda, \vec{\omega}) = L_1(\lambda + \omega_1)L_2(\lambda + \omega_2)\ldots L_{2N}(\lambda + \omega_{2N}) \]

and has the graphical representation of fig. 2. This expression is identical to eq.(11). \( L_j(\lambda + \omega_j) \) can be regarded as the scattering matrix of the \( j \)th pseudo–particle carrying formal rapidity \( \omega_j \) with the reference pseudo–particle carrying formal rapidity \( -\lambda \).

In the case of our diagonal lattice of right– and left–moving pseudoparticles, there exists a specific, physically relevant choice of the inhomogeneities, namely

\[ \omega_k = (-1)^k\Theta, \quad k = 1, 2, \ldots 2N \]

leading to the definition of the alternating monodromy matrix

\[ T(\lambda, \Theta) \equiv T(\lambda, \{ \omega_k = (-1)^k\Theta \}) \]

In fact, the evolution operators \( U_L(\Theta) \) and \( U_R(\Theta) \) can be expressed in terms of the alternating transfer matrix \( t(\lambda, \Theta) = t(\lambda, \{ \omega_k = (-1)^k\Theta \}) \) as

\[ U_R(\Theta) = t(\Theta, \Theta), \quad U_L(\Theta) = t(-\Theta, \Theta)^{-1} \quad (14) \]

Notice that \( T(\lambda, \Theta) \) fails to be conserved on the lattice only because of boundary effects. Indeed from fig. 5, which graphically represents the insertion of \( T(\lambda, \Theta) \) in the lattice time evolution, one readily sees that \( U \) and \( T(\lambda, \Theta) \) fail to commute only because of the free ends of the horizontal line. For all vertices in the bulk, the graphical interpretation of the YB equations (3), namely that lines can be freely pulled through vertices, allows to move \( T(\lambda, \Theta) \) up or down, that is to freely commute it with the time evolution. The problem lays at the boundary: if periodic boundary conditions are assumed, then the free horizontal ends of \( T(\lambda, \Theta) \) cannot be dragged along with the bulk, unless they are tied up, to form the transfer matrix \( t(\lambda, \Theta) \). After all, for p.b.c., the boundary is actually equivalent to any point of the bulk and thus \( t(\lambda, \Theta) \) commutes with \( U \), as obvious also from eqs.(14) and the general fact that \( [t(\lambda, \Theta), t(\mu, \Theta)] = 0 \). One might think that the thermodynamic limit \( N \to \infty \), by removing infinitely far away the troublesome free ends of \( T(\lambda, \Theta) \), will allow for its conservation and thus for the existence of an exact YB symmetry with bare \( R \)–matrix. The situation however is not so simple: first of all one must fix the Fock sector of the \( N \to \infty \) non–separable Hilbert space in which to take the thermodynamic limit. Different choices leads to different phases with dramatically different dynamics. Then the non–local structure of \( T(\lambda, \Theta) \) must be taken into account. It is evident, for instance, that in the spin–wave Fock sector above ferromagnetic reference states
$T(\lambda, \Theta)$ can never be conserved. Indeed, the working itself of the Quantum Inverse Scattering Method, where energy eigenstates are built applying non–diagonal elements of $T(\lambda, \Theta)$ on a specific ferromagnetic reference state, of course depends on $T(\lambda, \Theta)$ not commuting with the hamiltonian!

From the field–theoretic point of view, the most interesting phase is the antiferromagnetic one, in which the ground state plays the rôle of densely filled interacting Dirac sea (this holds for all known integrable lattice vertex models \cite{1,2,3}. The corresponding Fock sector is formed by particle–like excitations which become relativistic massive particles within the scaling limit proper of the light–cone approach \cite{2}. This consists in letting $a \to 0$ and $\Theta \to \infty$ in such a way that the physical mass scale

$$\mu = a^{-1} e^{-\kappa \Theta}$$

(15)

stays fixed. Here $\kappa$ is a model–dependent parameter which for the integrable model where the $R$–matrix is a rational function of $\theta$ takes the general form \cite{4}

$$\kappa = \frac{2 \pi t}{h s}$$

(16)

where $h$ is the dual Coxeter number of the underlying Lie algebra, $s$ equals 1, 2 or 3 for simply, doubly and triply laced algebras, respectively, and $t = 1$ ($t = 2$) for non–twisted (twisted) algebras. For the class of model characterized by a trigonometric $R$–matrix (with anisotropy parameter $\gamma$) the expression \cite{10} for $\kappa$ is to be divided by $\gamma$ \cite{4}.

The ground state or (physical vacuum) and the particle–like excitations of this antiferromagnetic phase are extremely more complicated than those of the ferromagnetic phase. It is therefore very hard to control, in the limit $N \to \infty$, the action of the alternating monodromy matrix $T(\lambda, \Theta)$ on the particle–like BA eigenstates of the alternating transfer matrix $t(\lambda, \Theta)$.

2 Bootstrap construction of quantum monodromy operators.

We briefly review in this section the work of refs.\cite{13} where the exact (renormalized) matrix elements of a quantum monodromy matrix $T_{ab}(u)$ ($u$ is the generally complex spectral parameter) were derived using a bootstrap–like approach for a class of integrable local QFT’s. In such theories there is no particle production and the $S$–matrix factorizes. The two–body $S$–matrix then satisfies the Yang–Baxter equations. Moreover, in the models considered in refs.\cite{13} (the $O(N)$ nonlinear sigma model, the $SU(N)$ Thirring model and the $0(2N)$ ($\bar{\psi}\psi)^2$ model), thanks to scale invariance there exist classically conserved monodromy matrices. In general, the quantum $T_{ab}(u)$ can be constructed by fixing its action on the Fock space of physical in and out many–particle states. The starting point are the following three general principles:
1. $\mathcal{T}_{ab}(u)$, $a,b = 1,2,\ldots,n$, exist as quantum operators and are conserved.

2. $\mathcal{T}_{ab}(u)$ fulfill a quantum factorization principle.

3. $\mathcal{T}_{ab}(u)$ is invariant under $P$, $T$ and the internal symmetries of the theory.

The quantum factorization principle referred above under 2. is nowadays called the "coproduct rule". This means that there exists the following relation between the action of $\mathcal{T}_{ab}(u)$ on $k$–particles states and its action on one–particle states

$$\mathcal{T}_{ab}(u) |\theta_1 \alpha_1, \theta_2 \alpha_2, \ldots, \theta_k \alpha_k\rangle_{in} = \sum_{a_1a_2\ldots a_{k-1}} \mathcal{T}_{a a_1}(u) |\theta_1 \alpha_1\rangle \mathcal{T}_{a_1 a_2}(u) |\theta_2 \alpha_2\rangle \ldots \mathcal{T}_{a_{k-1} b}(u) |\theta_k \alpha_k\rangle$$

$$\mathcal{T}_{ab}(u) |\theta_1 \alpha_1, \theta_2 \alpha_2, \ldots, \theta_k \alpha_k\rangle_{out} = \sum_{a_1a_2\ldots a_{k-1}} \mathcal{T}_{ab}(u) |\theta_1 \alpha_1\rangle \mathcal{T}_{a_2 a_1}(u) |\theta_2 \alpha_2\rangle \ldots \mathcal{T}_{a a_{k-1}}(u) |\theta_k \alpha_k\rangle$$

where $\theta_j$ and $\alpha_j$ $(1 \leq j \leq k)$ label the rapidities and the internal quantum numbers of the particles, respectively, in the asymptotic in and out states. Hence it is understood that $\theta_i > \theta_j$ for $i > j$.

Although $\mathcal{T}_{ab}(u)$ acts differently on in and out states, the assumption of conservation is nonetheless consistent. All the eigenvalues of a maximal commuting subset of \{ $\mathcal{T}_{ab}(u)$, $a,b = 1,2,\ldots,n$, $u \in \mathbb{C}$ \} are identical for in and out states with given rapidities. Indeed the two in and out forms of the action on the internal quantum numbers are related by the unitary permutation $|\alpha_1, \alpha_2, \ldots, \alpha_k\rangle \rightarrow |\alpha_k, \alpha_{k-1}, \ldots, \alpha_1\rangle$.

Furthermore, principles 1. and 2. imply that $\mathcal{T}_{ab}(u)$ acts in a trivial way on the physical vacuum state $|0\rangle$:

$$\mathcal{T}_{ab}(u) |0\rangle = \delta_{ab} |0\rangle$$

This also fixes the normalization of $\mathcal{T}_{ab}(u)$ in agreement with the classical limit $[13]$.

An immediate consequence of point 2. is that when $\mathcal{T}_{ab}(u)$ is expanded in powers of the spectral parameter $u$, it generates an infinite set of noncommuting and nonlocal conserved charges. This is the clue to the matching of the quantum monodromy matrix with its classical counterpart which is written nonlocally in terms of the local fields.

The main result in refs. $[13]$ was to derive from 1. , 2. and 3. the explicit matrix elements of $\mathcal{T}_{ab}(u)$ on one–particle states. This result can be written as

$$\langle \theta \alpha | \mathcal{T}_{ab}(u) |\theta' \beta\rangle = \delta(\theta - \theta') S^{\alpha \beta}_{b a} (\kappa(u) + \theta)$$

(18)

where $S^{\alpha \beta}_{b a} (\theta - \theta')$ stands for the $S$–matrix of two–body scattering

$$|\theta b, \theta' \beta\rangle_{in} = \sum_{a \alpha} |\theta a, \theta' \alpha\rangle_{out} S^{\alpha \beta}_{b a} (\theta - \theta')$$

and $\kappa(u)$ is an odd function of $u$. Notice that this requires the presence in the model of particles with indices $a,b,\ldots$ as internal state labels. In the simplest situation these new labels
coincide with those of the original particles. The appearance of a nontrivial “renormalization” \(u \to \kappa(u)\) is to be expected when there exist a definition of the spectral parameter outside the bootstrap itself. This is the case of the models of refs.\[13\], which possess Lax pairs and auxiliary problems which fix the definition of \(u\). Here we adopt the purely bootstrap viewpoint and fix the definition of \(u\) so that \(\kappa(u) = u\). In principle, an extra \(u\)– and \(\theta\)–dependent phase factor may appear in the r.h.s. of eq.(18). However, no phase showed up in the specific models of refs.\[13\], when nonperturbative checks were performed using the operator product expansion.

Eq.(18) can be written in a more suggestive way as

\[
T_{ab}(u) |\theta\beta\rangle = \sum_\alpha |\theta\alpha\rangle S^{\alpha \beta}_{\alpha \beta}(u) (19)
\]

This equation, when combined with eqs.(18), completely defines the quantum monodromy operators in the Fock space. From the YB equations satisfied by the \(S\)–matrix it then follows that \(T_{ab}(u)\) fulfills the YB algebra

\[
\hat{R}(u - v) [T(u) \otimes T(v)] = [T(u) \otimes T(v)] \hat{R}(u - v) \tag{20}
\]

where \(\hat{R}_{a\beta}^{\alpha}(u) = S_{a\beta}^{\alpha}(u)\). It should be stressed that the conservation of \(T_{ab}(u)\) implies that this YB algebra is a true non–abelian infinite symmetry algebra of the relativistic local QFT. On the contrary the rôle of the YB algebra in integrable vertex and face models on finite lattices or in nonrelativistic quantum models is that of a dynamical symmetry underlying the Quantum Inverse Scattering Method. In these latter cases, only the transfer matrix, namely

\[
\tau(u) = \sum_\alpha T_{aa}(u)
\]

is conserved. Since \([\tau(u), \tau(v)] = 0\), the transfer matrix just generates an abelian symmetry.

The dynamical symmetry underlying the integrable QFT includes in addition non–conserved operators \(Z_{\alpha}(\theta)\) which create the particle eigenstates out of the vacuum. In the bootstrap framework they can be introduced à la Zamolodchikov–Faddeev, by setting

\[
|\theta_1 \alpha_1, \theta_2 \alpha_2, \ldots, \theta_k \alpha_k\rangle_{in} = Z_{\alpha_k}(\theta_k) Z_{\alpha_{k-1}}(\theta_{k-1}) \ldots Z_{\alpha_1}(\theta_1) |0\rangle
\]

\[
|\theta_1 \alpha_1, \theta_2 \alpha_2, \ldots, \theta_k \alpha_k\rangle_{out} = Z_{\alpha_1}(\theta_1) Z_{\alpha_2}(\theta_2) \ldots Z_{\alpha_k}(\theta_k) |0\rangle
\]

with the fundamental commutation rules

\[
Z_{\alpha_2}(\theta_2) Z_{\alpha_1}(\theta_1) = \sum_{\beta_1, \beta_2} S_{\alpha_2 \beta_2}^{\alpha_1 \beta_1}(\theta_1 - \theta_2) Z_{\beta_1}(\theta_1) Z_{\beta_2}(\theta_2) \tag{21}
\]

Combining now eqs.(18),(19) and (21), we obtain the algebraic relation between monodromy and Zamolodchikov–Faddeev operators:

\[
T_{ab}(u) Z_{\beta}(\theta) = \sum_{c\alpha} Z_{\alpha}(\theta) T_{ac}(u) S_{b\beta}^{\alpha \alpha}(u + \theta)
\]
Together with eqs. (20) and (21), these relations close the complete dynamical algebra of an integrable QFT. For the XXZ spin chain in the regime $|q| < 1$, the ZF operators have been identified in ref. [21] with special vertex operators (or representation intertwiners of the relevant $q$–deformed affine Lie algebra). They are uniquely characterized by being solutions of the $q$–deformed Knizhnik–Zamolodchikov equation and by their normalization [20].

3 Lattice Construction of Quantum Monodromy Operators and Bethe Ansatz

In order to study the infinite volume limit of $T(\lambda, \Theta)$ on the physical Fock space (that is, finite energy excitations around the antiferromagnetic vacuum), one needs to compute scalar products of Bethe Ansatz states to derive relations like (18) or (19) with $T(\lambda, \Theta)$ instead of $T(\lambda, \Theta)$ in the l.h.s. Since this kind of calculations are indeed possible but rather involved, we computed in ref. [3] the eigenvalues of $t(\lambda, \Theta)$ on a generic state of the physical Fock space. Then, we compare these eigenvalues with those of the bootstrap transfer matrix. This tells us whether the bare and the renormalized YB algebras have a common abelian subalgebra. Notice that this fact alone provides a microscopic basis for the TBA, which originally relies solely on the bootstrap.

We consider once more the sG model as example, although the same result would apply to any integrable QFT admitting a light–cone lattice regularization. This class of models contains also the O(N) nonlinear sigma model and the $SU(N)$ Thirring model considered from the bootstrap viewpoint in refs. [13].

The integrable light–cone lattice regularization of the sG–mT model is provided by the six-vertex model [1]. Therefore, the space $V$ is $\mathbb{C}^2$ and the unitarized local $R$–matrices can be written

$$R_{jk}(\lambda) = \frac{1+e^{i\gamma}}{2} + \frac{1-e^{i\gamma}}{2} \sigma_j^x \sigma_k^x + \frac{b}{2}(\sigma_j^y \sigma_k^y + \sigma_j^z \sigma_k^z)$$

$$b(\lambda) = \frac{\sinh \lambda}{\sinh(i\gamma-\lambda)}$$

where $\gamma$ is commonly known as anisotropy parameter and $\sigma_j^x, \sigma_j^y$ and $\sigma_j^z$ are Pauli matrices acting at the site $j$.

The standard Algebrized BA can be applied to the diagonalization of the alternating transfer matrix $t(\lambda, \Theta)$ with the following results [1, 2, 14]. The BA states are written

$$\Psi(\vec{\lambda}) = B(\lambda_1)....B(\lambda_M)\Omega$$

where $\vec{\lambda} \equiv (\lambda_1, \lambda_2, \ldots, \lambda_M)$, $B(\lambda_i) = T_{\pm}(\lambda_i + i\gamma/2, \Theta)$ and $\Omega$ is the ferromagnetic ground-state (all spins up). They are eigenvectors of $t(\lambda, \Theta)$

$$t(\lambda, \Theta)\Psi(\vec{\lambda}) = \Lambda(\lambda; \vec{\lambda})\Psi(\vec{\lambda})$$
provided the \( \lambda_i \) are all distinct roots of the “bare” BA equations

\[
\left( \frac{\sinh[i\gamma/2 + \lambda_j - \Theta]}{\sinh[i\gamma/2 - \lambda_j + \Theta]} \frac{\sinh[i\gamma/2 + \lambda_j + \Theta]}{\sinh[i\gamma/2 - \lambda_j - \Theta]} \right)^N = -\prod_{k=1}^M \frac{\sinh[i\gamma + \lambda_j - \lambda_k]}{\sinh[-i\gamma + \lambda_j - \lambda_k]}
\]

(23)

The eigenvalues \( \Lambda(\lambda; \vec{\lambda}) \) are the sum of a contribution coming from \( A(\lambda) = T_{++}(\lambda, \Theta) \) and one coming from \( D(\lambda) = T_{--}(\lambda, \Theta) \),

\[
\Lambda(\lambda; \vec{\lambda}) = \Lambda_A(\lambda; \vec{\lambda}) + \Lambda_D(\lambda; \vec{\lambda})
\]

(24)

Here

\[
\Lambda_A(\lambda; \vec{\lambda}) = \exp \left[ -iG(\lambda, \vec{\lambda}) \right] \\
\Lambda_D(\lambda; \vec{\lambda}) = e^{-iN[\phi(\lambda-i\gamma/2-\Theta, \gamma/2) + \phi(\lambda+i\gamma/2+\Theta, \gamma/2)]} \exp \left[ iG(\lambda-i\gamma, \vec{\lambda}) \right]
\]

and

\[
G(\lambda, \vec{\lambda}) \equiv \sum_{j=1}^M \phi(\lambda - \lambda_j, \gamma/2) , \quad \phi(\lambda, \gamma) \equiv i \log \frac{\sinh(i\gamma + \lambda)}{\sinh(i\gamma - \lambda)}
\]

\( G(\lambda, \vec{\lambda}) \) is manifestly a periodic function of \( \lambda \) with period \( i\pi \). Notice also that \( \Lambda_D(\pm \Theta, \vec{\lambda}) = 0 \). That is, only \( \Lambda_A(\pm \Theta, \vec{\lambda}) \) contributes to the energy and momentum eigenvalues:

\[
E(\Theta) = a^{-1} \sum_{j=1}^M \left[ \phi(\Theta + \lambda_j, \gamma/2) + \phi(\Theta - \lambda_j, \gamma/2) - 2\pi \right]
\]

\[
P(\Theta) = a^{-1} \sum_{j=1}^M \left[ \phi(\Theta + \lambda_j, \gamma/2) - \phi(\Theta - \lambda_j, \gamma/2) \right]
\]

(25)

The ground state and the particle–like excitations of the light–cone six–vertex model are well known [1, 5]: the ground state corresponds to the unique solution of the BAE with \( N/2 \) consecutive real roots (notice that the energy in eq.(25) is negative definite, so that the ground state is obtained by filling the interacting Dirac sea). In the limit \( N \to \infty \) this yields the antiferromagnetic vacuum. Holes in the sea appear as physical particles. A hole located at \( \varphi \) carries energy and momentum, relative to the vacuum,

\[
e(\varphi) = 2a^{-1} \arctan \left( \frac{\cosh(\pi \varphi/\gamma)}{\sinh(\pi \varphi/\gamma)} \right) , \quad p(\varphi) = -2a^{-1} \arctan \left( \frac{\sinh(\pi \varphi/\gamma)}{\cosh(\pi \varphi/\gamma)} \right)
\]

(26)

In the scaling limit \( a \to 0, \Theta \to \infty \) with \( e(0) \) held fixed, we then obtain \((e, p) = m(\cosh \theta, \sinh \theta)\) with

\[
m \equiv 4a^{-1} \exp(-\pi \Theta/\gamma) , \quad \theta \equiv -\pi \varphi/\gamma
\]

(27)

We have thus proved that the continuum limit is \textbf{relativistic} with a finite non-zero mass. It provides a continuum relativistic massive field theory out of \textbf{any} gapless integrable model.
Here, we have only considered the six-vertex model that yields the massive Thirring (mT) model \cite{1}.

We identify \( m \) as the physical mass and \( \theta \) as the physical rapidity of a sG soliton (mT fermion) or antisoliton (antifermion). Complex roots of the BAE are also possible. They correspond to magnons, that is to different polarization states of several sG solitons (mT fermions), or to breather states (in the attractive regime \( \gamma > \pi/2 \)).

Within the light-cone approach one can also perform the continuum limit at the bare level. For the six-vertex model we defined lattice fermion fields \( \psi_n \) and we found their equations of motion on the lattice Minkowski spacetime \cite{1}:

\[
U_R \psi_{2n-2} U_R^+ = U_L \psi_{2n} U_L^+ = \bar{b} \psi_{2n} + \bar{c} \psi_{2n-1} + (c - \bar{c}) \psi_{2n} \psi_{2n-1} - (b + \bar{b}) \psi_{2n-1} \psi_{2n-1} \psi_{2n}
\]

\[
U_L \psi_{2n-1} U_L^+ = U_R \psi_{2n+1} U_R^+ = \bar{b} \psi_{2n-1} + \bar{c} \psi_{2n} + (c - \bar{c}) \psi_{2n-1} \psi_{2n-1} - (b + \bar{b}) \psi_{2n} \psi_{2n-1} \psi_{2n}
\]

where \( b \equiv b(2\Theta) \) and \( c \equiv c(2\Theta) \) (28)

These second quantized field equations are perfectly defined on the lattice. The bare scaling limit is not identical to the renormalized limit defined by eq.(27). The detailed proof in ref.\cite{1} shows that one finds the bare continuum mTm if one takes in eq.(28) \( \Theta \to \infty \), \( a \to 0 \) with \( m_0 \equiv \frac{4}{a} \sin \gamma \ e^{-2\Theta} \) kept fixed.

We see that the bare mass \( m_0 \) scales as \( e^{-2\Theta} \) while the renormalized mass scales as \( e^{-\pi \Theta/\gamma} \) [eq.(27)].

After some calculations \cite{1}, the continuum limit of the momentum and hamiltonian defined by eq.(13) take the form

\[
P = -i \int dx \psi^+ \partial_x \psi \quad \text{and} \quad H = \int dx \left[ -i \psi^+ \left( \gamma^5 \partial_x + im_0 \gamma^0 \right) \psi + \frac{g}{2} \left( \psi \gamma_\mu \psi \right)^2 \right], \quad (29)
\]

where

\[
\psi(x) = \begin{pmatrix} \psi_R(x) \\ \psi_L(x) \end{pmatrix}, \quad \psi_{2n} = \sqrt{a} \psi_R(x + \xi a), \quad \psi_{2n-1} = \sqrt{a} \psi_L(x - \xi a)
\]

with \( 0 < \xi < 1/2 \), \( x = na \), and \( g = -2 \cot \gamma \), \( \gamma_1 = -i \sigma_y \), \( \gamma_0 = \sigma_x \), \( \gamma_5 = \sigma_z \). (30)

Notice that there is an exact and finite relation between the bare continuum \((g)\), the lattice \((\gamma)\) and the renormalized \((\tilde{\gamma} = \frac{\gamma}{1-\gamma/\pi})\) coupling.

For \( R \)-matrices acting on finite dimensional spaces \( V \) one gets fermion or parafermion field theories. In order to describe bosonic field theories one needs infinite dimensional representation spaces \( V \) in the framework of the light-cone approach. [Otherwise, bosons can appear as bound states of fermions as in the mTm-sG model].

11
Let us discuss briefly here the rational limit of the six-vertex $R$-matrix in its spin $S$ representation \cite{15}. That is, for $V = C^{2S+1}$.

\[
R_{jk}(\theta) = \frac{\Gamma(2S + 1 + i\theta)\Gamma(J + 1 - i\theta)}{\Gamma(2S + 1 - i\theta)\Gamma(J + 1 + i\theta)}
\]  

(31)

where the operator $J$ is defined by

\[
J(J + 1) = 2S(S + 1) + 2S_j \otimes S_k
\]

where $S_j$ and $S_k$ are spin $S$ operators acting on the spaces $V_j$ and $V_k$ respectively. [ $(S_j)^2 = (S_k)^2 = S(S + 1)$]. The hamiltonian and momentum (13) describe in the $S = \infty$ limit the principal chiral model (PCM) \cite{13, 14}. However, this is not the full hamiltonian. One finds in this way states which are left (or right) $SU(2)$-singlets. The lattice current construction (35)-(38) [see below] holds for the PCM. Notice that for large $\theta$ the $R$-matrix (11) possess an expansion like eq.(33). Then, the whole procedure works yielding a conserved a curvatureless current that we can identify either with the $SU(2)_L$ or with the $SU(2)_R$ current. This whole construction generalizes to the $SU(N)$ PCM. It also generalizes to PCM with one anisotropy axis (trigonometric $R$-matrices)\cite{18}.

The light-cone approach to the sine-Gordon-mTm model using bosonic fields is worked out in ref.\cite{19}.

4 Vertex Models and Field Theories associated to q-deformed Lie Algebras

The $R$-matrices solutions of the YB eq.(5) can be classified according to

1. The Lie algebra (or q-deformed Lie algebra) to whom they are associated.

2. The couple of Lie algebra representations where they act: $V \otimes V'$, ( $V$ may coincide or not with $V'$).

The six-vertex $R$-matrix is associated to the $q - A_1$ Lie algebra in its fundamental (spin 1/2) representation. The $R$-matrix \cite{3}

\[
R_{ab}^{\theta}(\theta) = \frac{\sin \gamma}{\sin(\gamma - \theta)} e^{i\theta \text{sign}(a-b)}, \quad a \neq b; \\
R_{ba}^{\theta}(\theta) = \frac{\sin \gamma}{\sin(\gamma - \theta)}, \quad a \neq b; \\
R_{aa}^{\theta}(\theta) = 1
\]

(32)

\[1 \leq a, b \leq n\]
corresponds to the $q - A_{n-1}$ Lie algebra in its $n$-dimensional (quark) representation. When $q = e^{i\gamma}$ is a root of unity representations which are unknown for $q = 1$ appear and hence new models can be constructed. They are better defined in face language. For the $q - A_1$ case they are called RSOS models[9]. $R$-matrices associated to other $q$-Lie algebras can be found in [22].

The eigenvectors of the transfer matrix can be obtained via the Bethe Ansatz (BA) in its various generalizations. (The BA can also be formulated in face language [11, 10]). When the $R$-matrix corresponds to a Lie algebra of rank larger than one, the nested Bethe Ansatz (NBA) must be used.

The nested Bethe Ansatz (NBA) is probably the most sophisticated algebraic construction of eigenvectors for integrable lattice models. It consists of several levels of BA each one inside the previous one. This is the reason of its name. The NBA has been worked out for the $A_{n-1}$ trigonometric and hyperbolic vertex model [4], for the $Sp(2n)$ symmetric vertex model [23] and for $O(2n)$ symmetric vertex model [24] (always in the fundamental representation). The structure of the NBA is closely related to the respective Dynkin diagram. For $D_n$, one starts by one end of the ‘fork’, goes till the end of the diagram and then back till the other end of the ‘fork’ [24].

The NBA equations (NBAE) for a class of vertex models associated to simple Lie Algebras has been proposed in ref.[25] and solved (in a large extent) in ref.[4]. Let us summarize the more relevant results for the field theory limit.

The structure of the NBAE for a given $R$-matrix is dictated by the associated Dynkin diagram. There are sets of NBA roots associated to each spot of the Dynkin diagram. The NBAE couple the roots associated to the same spot and to the roots associated to spots connected to it in the Dynkin diagram. The structure of the ground state may be antiferromagnetic (AF) or ferromagnetic (F) depending on the chosen regime (values of $\theta$ and $\gamma$).

The AF ground state yields the more interesting field theories. It is formed by filling all ‘Dirac’ seas with BA roots. There is a Dirac sea for each NBA level. The ground state roots are real for simply laced Lie algebras apart of a constant imaginary part that depends on the level in a simple way [3, 4]. For non-simply laced cases [4] and for non-fundamental representations of all algebras, complex (‘string’ type) roots form the ground state.

On the top of the AF ground state (renormalizes or physical vacuum) there are excitations. One finds as many branches of excitations as the rank of the Lie algebra. They follow making holes and adding complex roots to each Dirac sea. The mass spectrum is usually $q$-independent except for non-compact $q$-Lie algebras [26]. The mass spectrum coincides (for simply laced cases) up to a general factor with the components of the Perron-Frobenius eigenvector of the Cartan matrix.

The field theories obtained for $R$-matrices in the fundamental representations are basically fermions or para-fermion models. In ref.[4] we computed the mass spectrum and the S-matrix.
in these models for most of the q-deformed simple Lie algebras.

Let us now briefly discuss the scaling limit of vertex models with rational $R$-matrices associated to a Lie algebra $G$. These $R$-matrices have the asymptotic behaviour

$$R(\theta) \overset{\theta \to \infty}{\to} P \left[ 1 + \frac{\Pi + \mu}{i\theta} + O\left(\frac{1}{\theta^2}\right) \right]$$  \hspace{1cm} (33)

where $\mu$ is a numerical constant, $P_{cd} = \delta_a^d \delta_c^b$ is the exchange operator and

$$\Pi = \sum_{\alpha=1}^{\dim G} T_\alpha \otimes T_\alpha$$  \hspace{1cm} (34)

We then introduce the lattice operator

$$T_\alpha^n \equiv 1 \otimes \ldots \otimes T_\alpha$$  \hspace{1cm} (35)

Using eqs.(12),(33) and (34) and the Lie algebra commutators

$$[T_\alpha, T_\beta] = i f_{\gamma}^{\alpha \beta} T_\gamma$$

we can show that the operators $T_\alpha^n$ obey local equations of motion on the lattice

$$U_R T_{2n-2}^\alpha U_L^+ = U_L T_{2n}^\alpha U_L^+ = \left( T_{2n}^\alpha + \frac{2i}{\theta} f_{\gamma}^{\alpha \beta} T_{2n-1}^\gamma + O\left(\frac{1}{\theta^2}\right) \right)$$
$$U_R T_{2n-1}^\alpha U_L^+ = U_L T_{2n+1}^\alpha U_L^+ = \left( T_{2n}^\alpha - \frac{2i}{\theta} f_{\gamma}^{\alpha \beta} T_{2n-1}^\gamma + O\left(\frac{1}{\theta^2}\right) \right).$$  \hspace{1cm} (36)

The bare scale limit is now defined as $a \to 0$, $\theta \to \infty$, $x = na$ fixed. We find

$$\partial^\mu J_\mu^\alpha (x) = 0 \quad , \quad \partial_0 J_1^\alpha - \partial_1 J_0^\alpha + ig f_{\gamma}^{\alpha \beta} [J_0^\beta, J_1^\gamma] = 0.$$  \hspace{1cm} (37)

where $\mu = 0, 1$ and in light-cone coordinates

$$J_R^\alpha \equiv \frac{1}{g a \theta} T_{2n}^\alpha \quad , \quad J_L^\alpha \equiv \frac{1}{g a \theta} T_{2n-1}^\alpha$$  \hspace{1cm} (38)

Therefore we have a lattice version of the $G$-algebra currents $J_\mu^\alpha(x)$ associated to an exactly integrable discretization of the field theory model. Eqs.(37) characterize the currents in the non-abelian Thirring model associated to the Lie algebra $G$. This model has as Lagrangian

$$\mathcal{L} = i \bar{\psi} \gamma_\mu T^\alpha \psi - \frac{g}{4} \left( \bar{\psi} \gamma_\mu T^\alpha \psi \right) \left( \bar{\psi} \gamma^\beta T^\beta \psi \right) K_{\alpha \beta}$$  \hspace{1cm} (39)

Here $\psi$ transforms under an irreducible representation $\rho$ of $G$, $T^\alpha$ are the $G$-generators in that representation and $K_{\alpha \beta}$ is proportional to the inverse of the Killing form. Actually the hamiltonian and momentum $[H and P defined by eq.(13)]$ describe the zero-chirality (massive) sector of the model (39) (see ref.[2]) and we can identify

$$J_\mu^\alpha(x) = \bar{\psi} \gamma_\mu T^\alpha \psi$$  \hspace{1cm} (40)

The renormalized scaling limit is discussed in [2, 3] and through eqs.(13)-(16).
5 Thermodynamic limit of the transfer matrix from the Bethe Ansatz

In ref.[3] it is shown that the bootstrap construction (discussed in sec.2) of conserved $T_{ab}(u)$ generalizes to integrable models with trigonometric $R-$matrices such as the sine-Gordon or massive Thirring model. In such cases the classical limit is abelian, as shown explicitly in ref.[3].

The main aim of ref.[3] was to investigate and clarify, from a microscopic point of view, the problem of unveiling the existence of the infinite YB symmetry of the sG–mT model. In other words, since lattice models provide regularized version of QFT, we seek an explicit connection between the lattice and the bootstrap YB algebras.

In order to investigate the operators present in such QFT, it is important to learn how the monodromy operators $T_{ab}(\lambda, \Theta)$ act on physical states. In ref.[3] we explicitly compute the eigenvalues of the alternating six–vertex transfer matrix $t(\lambda, \Theta)$, on a generic $n$–particle state, in the thermodynamic limit.

The eigenvalues of $t(\lambda, \Theta)$ turned out to be $i\pi$–periodic and multi–valued functions of $\lambda$, each determination of $t(\lambda, \Theta)$ being a meromorphic function of $\lambda$. We call $t^{II}(\lambda, \Theta)$ and $t^{I}(\lambda, \Theta)$ the determinations associated with the periodicity strips closer to the real axis. The ground–state contribution $\exp[-iG(\lambda)V]$ is exponential on the lattice size, as expected, whereas the excited states contributions are finite and express always in terms of hyperbolic functions.

In strip I $|\text{Im}\,\lambda| < \gamma/2$, we define the renormalized type I transfer matrix

$$t^I(\lambda) = \lim_{N \to \infty} t(\lambda, \Theta) \exp[iG^I(\lambda)V](-)^{J_z-N/2}$$

where $J_z = N/2 - M$ is to be identified with the soliton (or fermion) charge of the continuum sG–mT model. The last sign factor in eq.(41) corresponds to square–root branch choice suitable to obtain the relation

$$t^I(\pm \Theta) = \exp\{-ia[P_\pm - (P_\pm)_V]\}$$

where $P_\pm \equiv (H \pm P)/2$ [see eqs.(13), (14), (25)] and $(P_\pm)_V$ stands for the vacuum contribution. Notice that the $\Theta$–dependence of $t^I(\lambda)$ has been completely canceled out, since it is present only in the vacuum contribution. In fact, from the Bethe Ansatz calculations in ref.[3], we read the eigenvalue $\Lambda^I(\lambda)$ of $t^I(\lambda)$ on a generic particle state:

$$\Lambda^I(\lambda) = \exp \left[ -2i \sum_{n=1}^{k} \arctan \left( e^{\pi \lambda/\gamma + \theta_n} \right) \right] = \prod_{n=1}^{k} \coth \left( \frac{\pi \lambda}{2\gamma} + \frac{\theta_n}{2} + \frac{i\pi}{4} \right)$$

where $\theta_n \equiv -\pi \varphi_n/\gamma$ are the physical particle rapidities. Suppose now we expand log $\Lambda^I(\lambda)$ in powers of $z = e^{-\pi|\lambda|/\gamma}$ around $\lambda = \pm \infty$,

$$\pm i \log \Lambda^I(\lambda) = \sum_{j=0}^{\infty} z^{2j+1} \frac{(-1)^j}{j + 1/2} \sum_{n=1}^{k} e^{\pm(2j+1)\theta_n}$$
One has to regard the coefficients of the expansion parameter $z$ as the eigenvalues of the conserved abelian charges generated by the transfer matrix. The additivity of the eigenvalues implies the locality of the charges. In terms of operators we can write, around $\lambda = \pm \infty$,

$$\pm i \log t^I(\lambda) = \sum_{j=0}^{\infty} \left( \frac{4z}{m} \right)^{2j+1} I_j^\pm$$

(44)

where $I_0^\pm = p_\pm$ is the continuum light–cone energy–momentum and the $I_j^\pm$, $j \geq 1$, are local conserved charges with dimension $2j + 1$ and Lorentz spin $\pm(2j + 1)$. Their eigenvalues

$$\frac{(-1)^j}{j + 1/2} \sum_{n=1}^{k} \left( \frac{m e^{\pm \theta_n}}{4} \right)^{2j+1}$$

coincide with the values on multisoliton solutions of the higher integrals of motion of the sG equation [27]. It is remarkable that these eigenvalues are free of quantum corrections although the corresponding operators in terms of local fields certainly need renormalization. Let us stress that explicit expressions for these conserved charges can be obtained by writing the local $R$–matrices in terms of fermi operators, as in ref.[1]. Notice also that, combining eqs.(42) with (44), and recalling the scaling law (27), we can write

$$P_\pm - (P_\pm)_V = p_\pm + \frac{m}{4} \sum_{j=1}^{\infty} \left( \frac{ma}{4} \right)^{2j} I_j^\pm$$

(45)

That is, the light-cone lattice hamiltonian and momentum can be expressed in a precise way as the continuum hamiltonian and momentum plus an infinite series of continuum higher conserved charges, playing the rôle of irrelevant operators.

The explicit Bethe Ansatz calculation in ref.[3] showed that in the strip II, the lattice transfer matrix eigenvalues match with the bootstrap transfer matrix eigenvalues.

We obtained as general form of the $A$ and $D$ contributions to the eigenvalue of $t(\lambda, \Theta)$ [see eq.(24)] on the $N \to \infty$ limit of the BA states for $\lambda$ in strip II [3]:

$$\Lambda_A(\lambda) = -e^{-iG^{II}(\lambda)} \prod_{n=1}^{k} S(x_n) \coth \frac{x_n}{2} \prod_{j=1}^{m} \frac{\sinh \tilde{\gamma}[i/2 + (\xi(\lambda + i\gamma/2) + u_j)/\pi]}{\sinh \tilde{\gamma}[i/2 - (\xi(\lambda + i\gamma/2) + u_j)/\pi]}$$

and

$$\Lambda_D(\lambda) = -e^{-iG^{II}(\lambda)} \prod_{n=1}^{k} S(x_n) \hat{b}(x_n) \coth \frac{x_n}{2} \prod_{j=1}^{m} \frac{\sinh \tilde{\gamma}[3i/2 + (\xi(\lambda + i\gamma/2) - u_j)/\pi]}{\sinh \tilde{\gamma}[-i/2 + (\xi(\lambda + i\gamma/2) - u_j)/\pi]}$$

where for definiteness we chose the strip II , $-\pi + \gamma/2 < \text{Im} \lambda < -\gamma/2$ and set $x_n = \xi(\lambda + i\gamma/2) + \theta_n$. The distinct numbers $u_1, u_2, \ldots, u_m$ must satisfy the BA equations

$$\prod_{n=1}^{k} \frac{\sinh \tilde{\gamma}[i/2 + (u_j + \theta_n)/\pi]}{\sinh \tilde{\gamma}[i/2 - (u_j + \theta_n)/\pi]} = -\prod_{r=1}^{m} \frac{\sinh \tilde{\gamma}[-i + (u_j - u_r)/\pi]}{\sinh \tilde{\gamma}[-i + (u_j - u_r)/\pi]}$$
These last two expressions can be connected with that for the eigenvalues of the bootstrap transfer matrix $\tau(u)$ [3], provided we identify $u$ with $\frac{\pi}{\gamma} (\lambda + i\gamma/2)$. We find indeed [3]:

$$\Lambda(\lambda) = -e^{-iG^{II}(\lambda)V} \xi(u) \prod_{n=1}^{k} \coth \left( \frac{u + \theta_n}{2} \right)$$

(46)

where $\xi(u)$ is the eigenvalue of the bootstrap transfer matrix $\tau(u)$ and $\lambda$ is in strip II. In analogy with eq.(41), we now define the type II renormalized transfer matrix

$$t^{II}(\lambda) = \lim_{N \to \infty} t(\lambda, \Theta) \exp[iG^{II}(\lambda)V](-)^{J_{z} - N/2}$$

Then, taking into account eq.(43), eq.(46) can be rewritten

$$\xi(u) = \frac{\Lambda^{II}\left(\frac{\pi}{\gamma} u - i\frac{\gamma}{2}\right)}{\Lambda^{I}\left(\frac{\pi}{\gamma} u - i\frac{\gamma}{2}\right)}$$

(47)

Notice that the dependence on the cutoff rapidity $\Theta$ has completely disappeared from the r.h.s. of eq.(47). This holds true both for the explicit dependence in the vacuum function $G(\lambda)V$ and for the implicit dependence through the bare BAE, which are now replaced by the $\Theta$–independent higher–level ones. In other words, the eigenvalues of the bootstrap transfer matrix can be recovered from the light–cone regularization already on the infinite diagonal lattice, with no need to take the continuum limit. This should cause no surprise, since after all a factorized scattering can be defined also on the infinite lattice, with physical rapidities replaced by lattice rapidities [see eq.(26)]. The bootstrap construction of the quantum monodromy operators $T_{ab}(u)$ then proceeds just like on the continuum. In this case, some $q_0$–deformation of the two dimensional Lorentz algebra should act as a symmetry on the physical states. This $q_0$ becomes unit when $\Theta \to \infty$.

We then compare these Bethe Ansatz eigenvalues with the eigenvalues of the bootstrap transfer matrix $\tau(u)$. Remarkably enough, we find the following simple relation between the two results, for $0 < \gamma < \pi/2$ (repulsive regime),

$$\tau(u) = t^{II}\left(\frac{\gamma}{\pi} u - i\frac{\gamma}{2}, \Theta\right) t^{I}\left(\frac{\gamma}{\pi} u - i\frac{\gamma}{2}, \Theta\right)^{-1}$$

(48)

where $t^{II}(\lambda, \Theta)$ and $t^{I}(\lambda, \Theta)$ have been normalized to one on the ground state. Thus, we succeed in connecting the bootstrap transfer matrix $\tau(u)$ of the sG-mT model with the alternating transfer matrix $t(\lambda, \Theta)$ of the six vertex model. In the thermodynamic limit $\tau(u)$ coincide with the jump between the two main determinations of $t(\lambda, \Theta)$. Notice the renormalization of the rapidity by $\gamma/\pi$ and the precise overall shift by $i\gamma/2$ in the argument in order the equality to hold.

We find in addition that $t(\lambda, \Theta)$, for $0 < \text{Im} \lambda < \gamma/2$, generates the hamiltonian and momentum together with an infinite number of higher–dimension and higher–spin conserved
abelian charges, through expansion in powers of $e^{\pm \pi \lambda / \gamma}$. We see therefore that the same bare operator generates two kinds of conserved quantities. Energy and momentum as well the higher-spin abelian charges are local in the basic fields which interpolate physical particles, whereas the infinite set of charges obtained from the jump from $t^{II}(\lambda, \Theta)$ to $t^{I}(\lambda, \Theta)$ are nonlocal in the same fields. The fact that local and nonlocal charges come from different sides of a natural boundary, clearly shows that they carry independent information. That is, one cannot produce the nonlocal charges from the sole knowledge of the local charges. We also recall that the monodromy matrix $T(\lambda, \Theta)$ can be written in terms of the lattice Fermi fields of the mT model \[\text{(1)}\], so that local and nonlocal charges do admit explicit expressions in terms of local field operators.

We expect eqs. (48) and (45), and the discussion below eq. (48), to be valid for many other integrable models provided the appropriate rapidity renormalization and imaginary shift are introduced.

The quantum monodromy operators $\mathcal{T}_{ab}(u)$ generate a Fock representation of the $q$–deformed affine Lie algebra $U_q(\hat{G})$ corresponding to the given $R$–matrix. More precisely, by expanding $\mathcal{T}_{ab}(u)$ in powers of $z = e^u$ around $z = 0$ and $z = \infty$, one obtains non–abelian non-local conserved charges representing the algebra $U_q(\hat{G})$ on the Fock space of in– and out–particles. This connects our approach based on the YB symmetry, to the $q$–deformed algebraic approach of ref. [28]. $U_q(\hat{G})$ is a Hopf algebra endowed with an universal $R$–matrix, which reduces to the $R$– explicitly entering the YB algebra, upon projection to the finite–dimensional vector space spanned by the indexes of $\mathcal{T}_{ab}(u)$ [24]. In particular, the two expansions around $z = 0$ and $z = \infty$ generate the two Borel subalgebras of $U_q(\hat{G})$. A single monodromy matrix $T(u)$ is sufficient for this purpose, since this field–theoretic representation has level zero. This fact receives a new explanation in the light–cone approach, since $U_q(\hat{G})$ emerges as true symmetry only in the infinite–volume limit above the antiferromagnetic ground state (with no need to take the continuum limit), but its action is uniquely defined already on finite lattices, and all finite–dimensional representations have level zero.

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**Figure Captions**

Fig.1. The $R$-matrix elements $R_{cd}^{ab} (\theta)$ define the statistical weight of the depicted vertex configuration.

Fig.2. Graphical representation of the inhomogeneous monodromy matrix. The angles between the horizontal and the vertical lines are site-dependent in an arbitrary way.

Fig.3. The Yang-Baxter equation.

Fig.4. Light–cone lattice representing a discretized portion of Minkowski space–time. A $R$–matrix of probability amplitudes is attached to each vertex. The bold lines correspond to the action, at a given time, of the one–step evolution operator $U$.

Fig.5. Insertion of the alternating monodromy matrix in the light–cone lattice.

Fig.6. The two main determinations, $G^I (\lambda)$ and $G^{II} (\lambda)$ are defined by $G (\lambda)$ with $\lambda$ in strips I and II, respectively.
This figure "fig1-1.png" is available in "png" format from:

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