ON THE CONTROLLABILITY AND STABILIZATION OF THE BENJAMIN EQUATION

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Abstract. The aim of this paper is to study the controllability and stabilization for the Benjamin equation on a periodic domain $T := \mathbb{R}/(2\pi \mathbb{Z})$. We show that the Benjamin equation is globally exactly controllable and globally exponentially stabilizable in $H_s^p(T)$, with $s \geq 0$. First we prove propagation of compactness, propagation of regularity of solution in Bourgain’s spaces and unique continuation property, and use them to obtain the global exponential stabilizability corresponding to a natural feedback law. Combining the global exponential stability and the local controllability result we prove the global controllability as well. Also, we prove that the closed-loop system with a different feedback control law is locally exponentially stable with an arbitrary decay rate. Finally, a time-varying feedback law is designed to guarantee a global exponential stability with an arbitrary decay rate. The results obtained here extend the ones we proved for the linearized Benjamin equation in [32].

1. Introduction

We consider the Benjamin equation posed on a periodic domain $T := \mathbb{R}/(2\pi \mathbb{Z})$,

$$\partial_t u - \alpha \mathcal{H} \partial_x^2 u - \partial_x^3 u + \partial_x (u^2) = 0, \quad u(x, 0) = u_0 \quad x \in T, \; t \in \mathbb{R}, \quad (1.1)$$

where $u = u(x, t) \in \mathbb{R}$, $\alpha > 0$ is a constant and $\mathcal{H}$ denotes the Hilbert transform defined by $\mathcal{H}(f)(k) = -i \text{sgn}(k) \hat{f}(k)$, $\forall k \in \mathbb{Z}$, with $\hat{f}(k)$ the Fourier transform of $f$ given by

$$\hat{f}(k) := \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} \, dx.$$

The equation (1.1) posed on spatial domain $\mathbb{R}$ was derived by Benjamin [5] to study the gravity-capillarity surface waves of solitary type in deep water and serves as a generic model for unidirectional propagations of long waves in a two-fluid system where the lower fluid with greater density is infinitely deep and the interface is subject to capillarity. The author in [5] also showed that solutions of the Benjamin equation satisfy the conserved quantities,

$$I_1(u) := \frac{1}{2} \int_\mathbb{R} u^2(x, t) \, dx = I_1(u_0) \quad (1.2)$$

and

$$I_2(u) := \int_\mathbb{R} \left[ \frac{1}{2} (\partial_x u)^2(x, t) - \frac{\alpha}{2} u(x, t) \mathcal{H} \partial_x u(x, t) - \frac{1}{3} u^3(x, t) \right] \, dx = I_2(u_0). \quad (1.3)$$

We note that the relations (1.2) and (1.3) hold in the periodic case as well.

The well-posedness of the Cauchy problem (1.1) for given data in $H^s(\mathbb{R})$ and $H_s^p(T)$ has been extensively studied for many years, see [21, 10, 11, 28, 30]. The best known global well-posedness

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result in $L^2(\mathbb{R})$ is due to Linares [28]. The local well-posedness below $L^2(\mathbb{R})$, is studied by Kozono, Ogawa and Tanisaka [21] and Chen, Guo, and Xiao [10] for $s \geq -\frac{3}{4}$. In the periodic case, the best global well-posedness in $L^2(\mathbb{T})$ is due to Linares [28] and local well-posedness in $H^s_p(\mathbb{T})$ for $s \geq -\frac{1}{2}$ is due to Shi and Junfeng [41]. The Benjamin equation also admits solitary waves solutions. Several works have been devoted to study the existence, stability and asymptotic properties of such solutions, see for instance [11, 2, 5, 8].

Our aim here is to study the equation (1.1) in the context of control theory with a forcing term $f = f(x,t)$

$$\partial_t u - \alpha H \partial_x^3 u - \partial_x^3 u + \partial_x u^2 = f(x,t), \quad x \in \mathbb{T}, \ t \in \mathbb{R}. \tag{1.4}$$

In order to keep the mass $I_1(u)$ conserved in the control system (1.4), we demand

$$\int_0^{2\pi} f(x,t) \, dx = 0. \tag{1.5}$$

The control $f$ is allowed to act on only a small subset $\omega$ of the domain $\mathbb{T}$. This situation includes more cases of practical interest and is therefore more relevant in general. For this reason, we consider $g(x)$ as a real non-negative smooth function defined on $\mathbb{T}$, such that,

$$2\pi[g] := \int_0^{2\pi} g(x) \, dx = 1, \tag{1.6}$$

where $[g]$ denotes the mean value $g$ over the interval $(0, 2\pi)$. We assume $\text{supp } g = \omega \subset \mathbb{T}$, where $\omega = \{x \in \mathbb{T} : g(x) > 0\}$ is an open interval. We restrict our attention to controls of the form

$$f = G(h) := g(x) \left[ h(x,t) - \int_0^{2\pi} g(y) h(y,t) \, dy \right], \quad \forall x \in \mathbb{T}, \ t \in [0,T], \tag{1.7}$$

where $h$ is a function defined in $\mathbb{T} \times [0,T]$. Thus, $h \equiv h(x,t)$ can be considered as a new control function. Moreover, for each $t \in [0,T]$ we have that (1.5) is satisfied. Observe that if $s \in \mathbb{R}$, then the operator $G : L^2([0,T]; H^s_p(\mathbb{T})) \to L^2([0,T]; H^s_p(\mathbb{T}))$ is linear and bounded. Furthermore, The operator $G : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is linear, bounded and self-adjoint (see [32]).

In this work, we study the following two important problems in control theory.

**Exact control problem:** Given an initial state $u_0$ and a terminal state $u_1$ in a certain space with $[u_0] = [u_1]$, can one find a control input $f$ such that the solution $u$ of equation (1.4) satisfies $u(x,0) = u_0(x)$ and $u(x,T) = u_1(x) \ \forall x \in \mathbb{T}$, for some final time $T > 0$?

**Stabilization Problem:** Given an initial state $u_0$ in a certain space. Can one find a feedback control law: $f = Ku$ so that the resulting closed loop system

$$\partial_t u - \alpha H \partial_x^3 u - \partial_x^3 u + \partial_x u^2 = Ku, \quad u(x,0) = u_0 \ x \in \mathbb{T}, \ t \in \mathbb{R}^+, \tag{1.8}$$

is asymptotically stable as $t \to \infty$?

Control and stabilization of dispersive equations has been widely studied in the literature, see [13, 20, 24, 27, 34, 35, 38, 37, 38] and references therein. In particular, for the KdV equation, we refer to [24, 40, 45, 39, 34, 14, 31, 36] and for the BO equation we refer to [28, 25, 29] and the references therein. The Benjamin equation (1.1) displays both the third order local term $-\partial_x^3 u$, as in the KdV equation, and the second order nonlocal term $-\alpha H \partial_x^2 u$, as in the BO equation. So, it is natural to analyze the Benjamin equation from the control and stabilization point of view and check whether it behaves in the similar way as the KdV and BO equations. In this regard,
our work is inspired by the works of Laurent, Linares and Rosier [24], Linares and Ortega [26], Rosier and Zhang [24] and Russell and Zhang [40], where the authors studied the controllability and stabilization of the individual BO and the KdV equations posed on periodic domains.

We recall that the authors in [32] considered the controllability and stabilization issues for the linearized Benjamin equation on a periodic domain T, and proved the following results.

**Theorem 1.1** ([32]). Let \( s \geq 0, \alpha > 0, \) and \( T > 0 \) be given. Then for each \( u_0, u_1 \in H^s_p(T) \) with \([u_0] = [u_1]\), there exists a function \( h \in L^2([0,T];H^s_p(T)) \) such that the unique solution \( u \in C([0,T];H^s_p(T)) \) of the non homogeneous linear IVP associated to equation (1.4) (with \( f(x,t) = Gh(x,t) \)) satisfies \( u(x,T) = u_1(x), x \in \mathbb{T} \). Moreover, there exists a positive constant \( \nu = \nu(s,g,T) > 0 \) such that

\[
\|h\|_{L^2([0,T];H^s_p(0,2\pi))} \leq \nu (\|u_0\|_{H^s_p(0,2\pi)} + \|u_1\|_{H^s_p(0,2\pi)}).
\]

Also, employing the feedback control law \( K = -GG^* \), the following result regarding stabilization of the linearized Benjamin equation posed on \( \mathbb{T} \) is proved in [32].

**Theorem 1.2** ([32]). Let \( \alpha > 0, g \) as in (1.6), and \( s \geq 0 \) be given. Then there exist positive constants \( M = M(\alpha,g,s) \) and \( \gamma = \gamma(g) \), such that for any \( u_0 \in H^s_p(T) \), the unique solution \( u \in C([0,+,\infty),H^s_p(T)) \) of the closed-loop system

\[
\partial_t u - \alpha \mathcal{H} \partial_x^2 u - \partial_x^3 u = -GG^* u, \quad u(x,0) = u_0(x), \quad x \in \mathbb{T}, \quad t > 0.
\]  

satisfies

\[
\|u(\cdot,t) - [u_0]\|_{H^s_p(\mathbb{T})} \leq M e^{-\gamma t}\|u_0\|_{H^s_p(\mathbb{T})}, \quad \text{for all } t \geq 0.
\]

Furthermore, the authors in [32] also showed that it is possible to find a linear feedback law such that the resulting closed-loop system (1.9) is stable with arbitrary decay rate.

**Theorem 1.3** ([32]). Let \( s \geq 0, \alpha > 0, \lambda > 0, \) and \( u_0 \in H^s_p(T) \) be given. Then there exists a linear bounded operator \( K_\lambda \) from \( H^s_p(T) \) to \( H^s_p(T) \) such that the unique solution \( u \in C([0,+,\infty),H^s_p(T)) \) of the closed-loop system

\[
\partial_t u - \alpha \mathcal{H} \partial_x^2 u - \partial_x^3 u = -K_\lambda u, \quad u(x,0) = u_0(x), \quad x \in \mathbb{T}, \quad t > 0.
\]  

satisfies

\[
\|u(\cdot,t) - [u_0]\|_{H^s_p(\mathbb{T})} \leq M e^{-\lambda t}\|u_0\|_{H^s_p(\mathbb{T})},
\]

for all \( t \geq 0 \), and some positive constant \( M = M(g,\lambda,\alpha,s) \).

Now, a natural question is, whether one can get similar results for the nonlinear Benjamin equation (1.1). Extending the linear results to the corresponding nonlinear systems is difficult in general. Nevertheless, motivated by the work in Laurent et al. [22, 23, 24] (see also [16, 17]), we use the Bourgain’s spaces and the techniques motivated from the microlocal analysis to get certain propagation of compactness and regularity properties to the solutions of the Benjamin equation posed on a periodic domain \( \mathbb{T} \). We use these properties together with the unique continuation property (see Proposition 1.11 below) to establish the global stabilization and exact controllability for the nonlinear system (1.4).

In what follows, we describe the main results obtained in this work. We start with the following local control result.

\[
\|h\|_{L^2([0,T];H^s_p(0,2\pi))} \leq \nu (\|u_0\|_{H^s_p(0,2\pi)} + \|u_1\|_{H^s_p(0,2\pi)}).
\]
bounded operator 

\[ f \]

Let allows us to get the following local stabilization result for the Benjamin equation. This produces two states which are small enough so that the local controllability result for small data applies. Using this procedure, we obtain the following large data result.

**Theorem 1.5.** Let \( s \geq 0, \alpha > 0, \) and \( \mu \in \mathbb{R} \) be given. There exists a constant \( k' > 0 \) such that for any \( u_0 \in H^s_p(\mathbb{T}) \) with \( [u_0] = \mu \), the corresponding solution \( u \in C([0, +\infty), H^s_p(\mathbb{T})) \) of the closed-loop system (1.8) with \( K = -GG^*u \) satisfies

\[
\|u(\cdot, t) - [u_0]\|_{H^s_p(\mathbb{T})} \leq \alpha_{s,k'}([u_0 - [u_0]\|_{H^s_p(\mathbb{T})}] e^{-k't} \|u_0 - [u_0]\|_{H^s_p(\mathbb{T)}}, \quad \text{for all } t \geq 0,
\]

where \( \alpha_{s,k'} : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is a nondecreasing continuous function depending on \( s \) and \( k' \).

The global controllability result is derived by a combination of the exponential stabilization result presented in Theorem 1.4 and the local control result presented in Theorem 1.5 as is usual in control theory (see for instance [16, 17, 22, 24, 25]). Indeed, given the initial data \( u_0 \) to be controlled, by means of the damping term \( Ku = -GG^*u \) supported in \( \omega \), i.e by solving the IVP (1.8) (with \( Ku = -GG^*u \)), we drive it to a small state in a sufficiently large time. We do the same with the final state \( u_1 \) by solving the system backwards in time, due to the time reversibility of the Benjamin equation. This produces two states which are small enough so that the local controllability result for small data applies. Using this procedure, we obtain the following large data control result as a direct consequence of Theorems 1.4 and 1.5.

**Theorem 1.6** (Large data control). Let \( s \geq 0, \alpha > 0, \mu \in \mathbb{R}, \) and \( R > 0 \) be given. Then there exists a time \( T > 0 \), such that for any \( u_0, u_1 \in H^s_p(\mathbb{T}) \) with \( [u_0] = [u_1] = \mu \) and

\[
\|u_0\|_{H^s_p(\mathbb{T})} \leq R, \quad \|u_1\|_{H^s_p(\mathbb{T})} \leq R,
\]

one can find a control input \( h \in L^2([0, T]; H^s_p(\mathbb{T})) \) such that the IVP associated to (1.4) with \( f = Gh \) admits a unique solution \( u \in C([0, T]; H^s_p(\mathbb{T})) \) satisfying

\[
u(x, 0) = u_0(x), \quad u(x, T) = u_1(x), \quad \text{for all } x \in \mathbb{T}.
\]

Thus, the equation (1.4) is globally exactly controllable.

To get the decay rate \( k' \) in Theorem 1.5, arbitrarily large, a different control law is needed. First, the same feedback control law stabilizing the linearized Benjamin equation given in Theorem 1.3 allows us to get the following local stabilization result for the Benjamin equation.

**Theorem 1.7.** Let \( 0 < \lambda' < \lambda \) and \( s \geq 0 \) be given. Assume \( \alpha > 0 \). There exist \( \delta > 0 \) and a linear bounded operator \( K_\lambda : H^s_p(\mathbb{T}) \rightarrow H^s_p(\mathbb{T}) \) such that for any \( u_0 \in H^s_p(\mathbb{T}) \) with \( \|u_0 - [u_0]\|_{H^s_p(\mathbb{T})} \leq \delta, \)
the corresponding solution \( u \in C([0, +\infty), H_p^s(\mathbb{T})) \) of the closed-loop system (1.8) with \( Ku = K\lambda u \), satisfies
\[
\|u(\cdot, t) - [u_0]\|_{H_p^s(\mathbb{T})} \leq C e^{-\lambda t} \|u_0 - [u_0]\|_{H_p^s(\mathbb{T})}, \quad \text{for all } t \geq 0,
\]
where \( C > 0 \) is a constant independent of \( u_0 \).

Next, the feedback laws involved in Theorems 1.5 and 1.7 can be combined into a time-varying feedback law, as in [15] [24], ensuring a global stabilization result with an arbitrary large decay rate for the Benjamin equation.

**Theorem 1.8.** Let \( s \geq 0, \lambda > 0, \) and \( \alpha > 0 \), be given. For any \( u_0 \in H_p^s(\mathbb{T}) \) with \( \mu = [u_0] \), there exists a continuous map \( Q_\lambda : H_p^s(\mathbb{T}) \times \mathbb{R} \to H_p^s(\mathbb{T}) \) which is periodic in the second variable, and such that the unique solution \( u \in C([0, +\infty), H_p^s(\mathbb{T})) \) of the closed-loop system (1.8) with \( Ku = -GQ_\lambda(u, t) \), satisfies
\[
\|u(\cdot, t) - [u_0]\|_{H_p^s(\mathbb{T})} \leq \gamma_{s, \lambda, \mu}(\|u_0 - [u_0]\|_{H_p^s(\mathbb{T})}) e^{-\lambda t} \|u_0 - [u_0]\|_{H_p^s(\mathbb{T})}, \quad \text{for all } t \geq 0,
\]
where \( \gamma_{s, \lambda, \mu} : \mathbb{R}^+ \to \mathbb{R}^+ \) is a nondecreasing continuous function depending on \( s, \lambda \) and \( \mu \).

Note that, any solution \( u \) of the IVP associated to (1.4) with \( f = Gh \) possesses a constant mean value, say \( \mu := [u(\cdot, t)] = [u_0] \). In order to work on Bourgain’s spaces in the periodic setting we need \( [u(\cdot, t)] = 0 \). To achieve this, it is convenient to define \( \bar{u}(x, t) = u(x, t) - \mu \), so that \( [\bar{u}(\cdot, t)] = 0 \), for all \( t \in [0, T] \). Moreover, \( \bar{u} \) solves
\[
\begin{align*}
\partial_t \bar{u} - \alpha H\partial_x^2 \bar{u} - \partial_t^2 \bar{u} + 2\mu \partial_x \bar{u} + 2\bar{u} H \partial_x \bar{u} = Gh(x, t), & \quad t \in (0, T), \quad x \in \mathbb{T}, \\
\bar{u}(x, 0) = \bar{u}_0(x) := u_0(x) - \mu, & \quad x \in \mathbb{T},
\end{align*}
\]
where \( \bar{u}_0 \in H_p^s(\mathbb{T}) \) with \( s \geq 0 \). Observe that \( \bar{u} \) is a solution of the IVP (1.11), if and only if, \( u(x, t) = \bar{u}(x, t) + \mu \) solves the IVP associated to (1.4) with \( f = Gh \).

From now on, for simplicity, we suppress the notation \( \bar{u} \) by \( u, \mu \) will denote a given real constant, and we work on system (1.11). We define
\[
H_p^s(\mathbb{T}) := \{ u \in H_p^s(\mathbb{T}) : [u(\cdot, t)] = 0, \quad \text{for all } t \geq 0 \}.
\]

If \( s = 0 \), then we denote \( H_p^0(\mathbb{T}) \) by \( L_p^2(\mathbb{T}) \). It is known that \( H_p^0(\mathbb{T}) \) is a closed subspace of \( H_p^s(\mathbb{T}) \) for all \( s \geq 0 \). In particular, \( L_p^2(\mathbb{T}) \) is a closed subspace of \( L^2(\mathbb{T}) \). Furthermore, (\( H_p^0(\mathbb{T}), \| \cdot \|_{H_p^0(\mathbb{T})} \)) is a dense embedding (see Proposition 6.1 and Remark 6.2 in [32]). We establish a local control result in \( H_p^0(\mathbb{T}) \) for the system (1.11) and exponential stability results in \( H_p^0(\mathbb{T}) \) for the system
\[
\begin{align*}
\partial_t u - \alpha H\partial_x^2 u - \partial_t^2 u + 2\mu \partial_x u + 2u H \partial_x u = Ku(x, t), & \quad t \geq 0, \quad x \in \mathbb{T}, \\
u(x, 0) = u_0(x), & \quad x \in \mathbb{T},
\end{align*}
\]
that will imply all the results stated in Theorems 1.3 [1.8].

This paper is organized as follows: In Section 2 we summarize some results on the controllability of the linear system associated to (1.11) and the stabilization of the linear system associated to (1.12). In Section 3 we analyze Bourgain’s spaces properties and derive the propagation of compactness, the propagation of regularity and the unique continuation property for the Benjamin equation. In Section 5 the local controllability is obtained. Section 6 is devoted to study the stabilization of the Benjamin equation by a time-invariant feedback control law. In Section 7 we
investigate the stabilization by constructing a time-varying feedback control law. Finally, in the Appendix we include some results used in this work.

2. Preliminary Results

In this section we recall some results related to the controllability of the linear system associated to (1.11) and the exponential stabilization of the linear system associated to (1.12) (see sections 4 and 5 in [32]).

2.1. Control of the Linear System. We begin by considering the IVP associated to the linear part of (1.11)

\[
\begin{aligned}
\partial_t u - \alpha \Delta_t^2 u - \partial_x^2 u + 2\mu \partial_x u &= Gh(x,t), & t \in (0,T), & x \in \mathbb{T} \\
u(x,0) &= u_0(x), & x \in \mathbb{T}.
\end{aligned}
\]

(2.1)

For \( u_0 \in H_0^s(\mathbb{T}) \), \( s \geq 0 \), the IVP (2.1) possesses a unique global solution and is described by the unitary group \( H_0^s(\mathbb{T}) \)

\[
U_{\mu}(t)u_0 := e^{(\alpha H \partial_x^2 + \partial_t^2 - 2\mu \partial_x) t}u_0 = \left(e^{i(-\kappa^2 - 2\mu k + \alpha |k|^2) t}u_0(k)\right)^\vee.
\]

(2.2)

For details see Remark 4.8 in [32].

As shown in [32] (see Remark 4.9 there), the system (2.1) is exactly controllable in any positive time \( T \) and holds the following property.

Remark 2.1. For \( s \geq 0 \) and any \( T > 0 \) given, there exists a bounded linear operator

\[
\Phi_{\mu} : H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T}) \to L^2([0,T]; H_0^s(\mathbb{T}))
\]

defined by \( h = \Phi_{\mu}(u_0, u_1) \), for all \( u_0, u_1 \in H_0^s(\mathbb{T}) \) such that

\[
u_1 = U_{\mu}(T)u_0 + \int_0^T U_{\mu}(T - s)(G(\Phi(0), u_1))(\cdot, s) \, ds,
\]

for \( (u_0, u_1) \in H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T}) \) and

\[
\|\Phi_{\mu}(u_0, u_1)\|_{L^2([0,T]; H_0^s(\mathbb{T}))} \leq \nu \left( \|u_0\|_{H_0^s(\mathbb{T})} + \|u_1\|_{H_0^s(\mathbb{T})} \right),
\]

where \( \nu \) depends only on \( s \), \( T \), and \( g \). Therefore, for any \( T > 0 \) the following observability inequality holds

\[
\int_0^T \|G U_{\mu}(t)^*(\phi)(x)\|^2_{L_2^2(\mathbb{T})} \, dt \geq \delta^2 \|\phi\|^2_{L_2^2(\mathbb{T})}, \quad \text{for any } \phi \in L_2^0(\mathbb{T}), \text{ some } \delta > 0.
\]

2.2. Stabilization of the Linear Benjamin Equation. In this sub-section we recall the stabilization results (see section 5.1 in [32]) for the linear system

\[
\begin{aligned}
\partial_t u - \alpha \Delta_t^2 u - \partial_x^2 u + 2\mu \partial_x u &= Ku, & t \geq 0, & x \in \mathbb{T} \\
u(x,0) &= u_0(x), & x \in \mathbb{T},
\end{aligned}
\]

(2.3)

associated to (1.12). The authors in [32] proved that, for \( u_0 \in H_0^s(\mathbb{T}) \), \( s \geq 0 \) the IVP (2.3) possesses a unique global solution and is exponentially asymptotically stable when \( t \) goes to infinity with simple feedback control law, \( Ku = -G G^* u \).
Theorem 2.2. Let $\alpha > 0$, $\mu \in \mathbb{R}$, $g$ as in (1.6), and $s \geq 0$ be given. There exist positive constants $M = M(\alpha, \mu, g, s)$ and $\gamma = \gamma(g)$, such that for any $u_0 \in H_0^s(\mathbb{T})$, the unique solution $u$ of (2.3) with $K = -GG^*$ satisfies
\[ \|u(\cdot, t)\|_{H_0^s(\mathbb{T})} \leq Me^{-\gamma t}\|u_0\|_{H_0^s(\mathbb{T})}, \text{ for all } t \geq 0. \]

2.3. Stabilization of the Linear Benjamin Equation in with Arbitrary Decay Rate.
In this subsection, we recall the stabilization result from section 5.2 in [32] where an appropriate linear feedback law is chosen so that the decay rate of the resulting closed-loop system is arbitrarily large.

Let $a > 0$ be any fixed number. For given $\lambda > 0$, and $s \geq 0$, define the operator
\[ L_\lambda \phi = \int_0^a e^{-2\lambda \tau} U_\mu(-\tau)GG^* U_\mu(-\tau)^* \phi d\tau, \text{ for all } \phi \in H_0^s(\mathbb{T}). \] (2.4)
The operator $L_\lambda$ satisfies the following properties.

Lemma 2.3. The operator $L_\lambda : H_0^s(\mathbb{T}) \rightarrow H_0^s(\mathbb{T})$ is linear and bounded. Moreover, $L_\lambda$ is an isomorphism from $H_0^s(\mathbb{T})$ onto $H_0^s(\mathbb{T})$, for all $s \geq 0$.

Remark 2.4. From Lemma 2.3 we infer that there exists a positive constant $C = C(\delta, s, \lambda, a, g)$ such that
\[ \|L_\lambda^{-1}\psi\|_{H_0^s(\mathbb{T})} \leq C\|\psi\|_{H_0^s(\mathbb{T})}, \text{ for all } \psi \in H_0^s(\mathbb{T}). \]

Choosing the feedback control law as
\[ Ku = \begin{cases} -K_\lambda u := -GG^* L_\lambda^{-1} u, & \text{if } \lambda > 0 \\ -K_0 u := -GG^* u, & \text{if } \lambda = 0, \end{cases} \] (2.5)
we can rewrite the resulting closed-loop system (2.3) in the following form
\[ \begin{cases} \partial_t u - \alpha \mathcal{H} \partial_x^2 u - \partial_x^3 u + 2\mu \partial_x u = -K_\lambda u, & t > 0, \ x \in \mathbb{T} \\ u(x, 0) = u_0(x), & x \in \mathbb{T}, \end{cases} \] (2.6)
where $K_\lambda$ is a bounded linear operator on $H_0^s(\mathbb{T})$, with $s \geq 0$. With these considerations, we have the following result.

Theorem 2.5. Let $\alpha > 0$, $\mu \in \mathbb{R}$, $s \geq 0$, and $\lambda > 0$ be given. For any $u_0 \in H_0^s(\mathbb{T})$, the system (2.6) admits a unique solution $u \in C(\mathbb{R}^+, H_0^s(\mathbb{T}))$. Moreover, there exists $M = M(g, \lambda, \delta, \alpha, \mu, s) > 0$ such that
\[ \|u(\cdot, t)\|_{H_0^s(\mathbb{T})} \leq M e^{-\lambda t}\|u_0\|_{H_0^s(\mathbb{T})}, \text{ for all } t \geq 0. \]

3. Bourgain’s Spaces Associated to Benjamin Equation
In this section we introduce the Fourier transform norm spaces, the so called Bourgain’s spaces and derive some preliminary estimates to get a control result for the Benjamin equation.
3.1. Bourgain’s spaces and their properties. In order to simplify the notation, in this subsection, we denote $U(t)$ by $V(t)$, i.e. $V(t)\phi := e^{(\alpha t^2 + \beta t^2 + \gamma \omega t)} \phi$. Given $r \in \mathbb{R}$, we define an operator $D^r : D'(\mathbb{T}) \to \mathbb{C}$ by
\[
D^r v(k) = \begin{cases} 
|k|^r \hat{v}(k), & \text{if } k \neq 0; \\
\hat{v}(0), & \text{if } k = 0,
\end{cases}
\] (3.1)
and $D^r : D'(\mathbb{T}) \to \mathbb{C}$, by $D^r v(k) = (ik)^r \hat{v}(k)$.

For given $b, s \in \mathbb{R}$ we define the Bourgain’s space $X_{s,b}$ associated to the Benjamin equation on $\mathbb{T}$ as the closure of the space of Schwartz functions $\mathcal{S}(\mathbb{T} \times \mathbb{R})$ under the norm
\[
\|v\|_{X_{s,b}} := \left( \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}} (k)^{2s} \|v(k, \cdot)\|^2 \|v(k, \cdot)\|^2 d\tau \right)^{\frac{1}{2}},
\] (3.2)
where $\phi(k) = -k^3 - 2\mu k + \alpha k |k|$ is the phase function, $\langle \cdot \rangle := (1 + |\cdot|^2)^{\frac{1}{2}}$, and $\hat{v}(k, \tau)$ denotes the Fourier transform of $v$ with respect to the both space and time variables given by
\[
\hat{v}(k, \tau) := \frac{1}{2\pi} \int_{\mathbb{T}} \int_{\mathbb{R}} v(x, t) e^{-i(t\tau + kx)} \, dx \, dt.
\]
Sometimes, we use $\hat{v}(k, t)$ (respectively $\hat{v}(x, t)$) to denote the Fourier transform in space variable $x$ (respectively in time variable $t$). In particular $\|v\|_{X_{s,b}} = \|v\|_{L^2(\mathbb{R}; H^s_p(\mathbb{T}))}$. Note that, $X_{s,b}$ is a Hilbert space and for $b > \frac{1}{2}$
\[
X_{s,b} \subset C(\mathbb{R}; H^s_p(\mathbb{T})),
\] (3.3)
the imbedding being continuous. As noted in [7, 20, 24, 28, 43], while dealing with the bilinear estimates in the periodic case one needs to consider $b = \frac{1}{2}$ for which the imbedding (3.3) fails. To overcome this situation, we introduce the space $Y_{s,b}$ as completion of the space of Schwartz functions $\mathcal{S}(\mathbb{T} \times \mathbb{R})$ under the norm
\[
\|v\|_{Y_{s,b}} := \left( \sum_{k=-\infty}^{\infty} \left( \int_{\mathbb{R}} (k)^{2s} \|\langle \phi(k)\rangle\|^b |\hat{v}(k, \tau)|^2 d\tau \right)^{\frac{1}{2}} \right)^{\frac{1}{2}},
\] (3.4)
and define the space
\[
Z_{s,b} := X_{s,b} \cap Y_{s,b},
\] (3.5)
endowed with the norm $\|v\|_{Z_{s,b}} := \|v\|_{X_{s,b}} + \|v\|_{Y_{s,b}}$. For a given interval $I$, we define $X_{s,b}(I)$ as the restriction space of $X_{s,b}$ to the interval $I$ with the norm
\[
\|f\|_{X_{s,b}(I)} := \inf \left\{ \|f\|_{X_{s,b}} | f = f \text{ on } \mathbb{T} \times I \right\}.
\] (3.6)
If $I = [0, T]$, for simplicity, we denote $X_{s,b}(I)$ by $X_{s,b}^T$. In and analogous manner we define $Y_{s,b}(I)$, $Z_{s,b}(I)$, $Y_{s,b}^T$ and $Z_{s,b}^T$.

In what follows, we record some properties of the spaces $X_{s,b}$ and $Z_{s,b}$.

**Proposition 3.1.** The space $X_{s,b}$ have the following properties:

ii) If $s_1 \leq s_2$ and $b_1 \leq b_2$, then $X_{s_2,b_2}$ is continuously imbedded in $X_{s_1,b_1}$. The same holds for $X_{s,b}(I)$ too.

iii) For a given finite interval $I$, if $s_1 < s_2$ and $b_1 < b_2$, then $X_{s_2,b_2}(I)$ is compactly imbedded in $X_{s_1,b_1}(I)$. 

Proposition 3.2 ([11] [33] [4]). Let I be an interval and \( s \in \mathbb{R} \). The space \( Z_{s,\frac{1}{4}} \) (resp. \( Z_{s,\frac{1}{4}}(I) \)) is continuously imbedded in the space \( C(\mathbb{R}; H^s_p(T)) \) (resp. \( C(I; H^s_p(T)) \)).

Lemma 3.3. Let \( s, b \in \mathbb{R} \). The space \( X_{s,b} \) is reflexive and its dual is given by \( X_{-s,-b} \).

Proof. It follows by the fact that \( \phi(k) \) is an odd function (see Tao [43] page 97). \( \square \)

Lemma 3.4. Let \( s, r \in \mathbb{R} \). Then, for any \( v \in X_{s,b} \) (resp. \( X_{s,b}(I) \)) \( D^r v \in X_{s-r,b} \) (resp. \( X_{s-r,b}(I) \)). The same is valid for the operator \( \partial_x^r \). Moreover, there exists a positive constant \( C \), independent on \( v \), such that

\[
\| D^r v \|_{X_{s-r,b}} \leq C \| v \|_{X_{s,b}}.
\]

The ideas to prove the majority of the results in the following two subsections are similar to those derived in the KdV case (see [7, 20, 11, 12, 19, 18]).

3.2. Linear and integral estimates. To derive some estimates localized in time variable, we introduce a cut-off function \( \eta \in C_c^\infty(\mathbb{R}) \) such that \( \eta \equiv 1 \), if \( t \in [-1,1] \) and \( \eta \equiv 0 \), if \( t \notin (-2,2) \). For \( T > 0 \) given, we define

\[
\eta_T \in C_c^\infty(\mathbb{R}) \quad \text{by} \quad \eta_T(t) := \eta \left( \frac{t}{T} \right).
\]

Proposition 3.5. Let \( s, b \in \mathbb{R} \) and \( T > 0 \) be given. Then for all \( v_0 \in H^s_p(T) \), we have

\[
\| \eta_T(t)V(t)v_0 \|_{X_{s,b}} \leq C_{n,b} T^{\frac{s}{2}} \| v_0 \|_{H^s_p(T)}, \quad \| \eta_T(t)V(t)v_0 \|_{Y_{s,b}} \leq C_{n,b} T^{\frac{s}{2}} \| v_0 \|_{H^s_p(T)},
\] (3.7)

\[
\| V(t)v_0 \|_{X^T_{s,b}} \leq C_{n,b} T^{\frac{s}{2}} \| v_0 \|_{H^s_p(T)}, \quad \| V(t)v_0 \|_{Y^T_{s,b}} \leq C_{n,b} T^{\frac{s}{2}} \| v_0 \|_{H^s_p(T)},
\] (3.8)

where \( C_{n,b} \) is a positive constant that depends only on \( \eta \) and \( b \).

Corollary 3.6. Let \( s, b \in \mathbb{R} \) and \( T > 0 \) be given. Then for all \( v_0 \in H^s_p(T) \), one has

\[
\| \eta_T(t)V(t)v_0 \|_{Z_{s,b}} \leq C_{n,b} T^{\frac{s}{2}} \| v_0 \|_{H^s_p(T)}, \quad \| V(t)v_0 \|_{Z^T_{s,b}} \leq C_{n,b} T^{\frac{s}{2}} \| v_0 \|_{H^s_p(T)},
\] (3.9)

where \( C_{n,b} \) is a positive constant that depends only on \( \eta \) and \( b \).

Proof. It follows from Proposition 3.5 and the definition of the \( Z_{s,b} \) and \( Z^T_{s,b} \)-norms. \( \square \)

Theorem 3.7. Let \( b = \frac{1}{2} \) and \( T > 0 \) be given. Then, one has

\[
\left\| \eta_T(t) \int_0^t V(t - \tau) f(\tau) \ d\tau \right\|_{Z_{s,\frac{1}{2}}} \leq C_{n,T} \| f \|_{Z_{s,\frac{1}{2}}}, \quad \forall f \in Z_{s,\frac{1}{2}},
\] (3.10)

and

\[
\left\| \int_0^t V(t - \tau) f(\tau) \ d\tau \right\|_{Z^T_{s,\frac{1}{2}}} \leq C_{n,T} \| f \|_{Z^T_{s,\frac{1}{2}}}, \quad \forall f \in Z^T_{s,\frac{1}{2}},
\] (3.11)

where \( C_{n,T} \) is a positive constant depending on \( \eta \), and the final time \( T \).

If \( T \leq 1 \), then the positive constant \( C \) involved in (3.10) and (3.11) does not depend on \( T \).

Proof. The proof is similar to the proof of Lemma 3.16 in [18]. \( \square \)

Proposition 3.8. Let \( T > 0 \), \(-\frac{1}{2} < b' < b < \frac{1}{2}\) and \( s \in \mathbb{R} \) be given. Then for all \( v \in X^T_{s,b} \), there exists \( C > 0 \), such that

\[
\| v \|_{X^T_{s,b}} \leq C \ T^{b-b'} \| v \|_{X^T_{s,b}}.
\]
Proof. The proof is similar to the proof of Lemma 2.11 in Tao [43].

**Proposition 3.9** ([43 page 105]). For all \( s, b \in \mathbb{R}, \epsilon > 0 \), and \( v \in X_{s, b} \), we have

\[
\|v\|_{Y_{s, b, \frac{1}{4} - \epsilon}} \leq C_\epsilon \|v\|_{X_{s, b}},
\]

where \( C_\epsilon \) is a positive constant.

**Corollary 3.10.** For all \( T > 0, s, b \in \mathbb{R}, \epsilon > 0 \), and \( v \in X_{s, b}^T \), one has

\[
\|v\|_{Y_{s, b, \frac{1}{4} - \epsilon}} \leq C_\epsilon \|v\|_{X_{s, b}^T},
\]

where \( C_\epsilon \) is a positive constant.

**Proof.** Let \( v \in X_{s, b}^T \) and consider \( \tilde{v} \) an extension in \( X_{s, b} \) such that \( \|\tilde{v}\|_{X_{s, b}} \leq 2 \|v\|_{X_{s, b}^T} \). From Proposition 3.9 we have

\[
\|v\|_{Y_{s, b, \frac{1}{4} - \epsilon}} \leq \|\tilde{v}\|_{Y_{s, b, \frac{1}{4} - \epsilon}} \leq C_\epsilon \|\tilde{v}\|_{X_{s, b}} \leq 2C_\epsilon \|v\|_{X_{s, b}^T}.
\]

\[\square\]

### 3.3. Some Nonlinear Estimates

We start with the following result, which is fundamental to estimate the nonlinear term \( 2u\partial_t u \), in \( Z_{s, -\frac{1}{4}} \) norm.

**Theorem 3.11.** There exists \( C_\alpha > 0 \) depending only on \( \alpha \), such that for all \( v \in X_{0, \frac{1}{4}} \), one has

\[
\|v\|_{L^4(\mathbb{T} \times \mathbb{R})} \leq C_\alpha \|v\|_{X_{0, \frac{1}{4}}}.
\]

**Proof.** Observe that

\[
\|v\|_{X_{0, \frac{1}{4}}}^{\frac{2}{3}} \sim \sum_{k = -\infty}^{\infty} \int_{\mathbb{R}} \left( 1 + |\tau - \phi(k)| \right)^{\frac{2}{3}} |\tilde{v}(k, \tau)|^2 d\tau.
\]

Also note that, we can write \( v(x, t) = \sum_{m = 0}^{\infty} v_{2m}(x, t) \), where

\[
\tilde{v}_{2m}(k, \tau) = \tilde{v}(k, \tau) \cdot \chi_{2^m \leq 1 + |\tau - \phi(k)| < 2^{m+1}}.
\]

In this way, we have

\[
\|v\|_{X_{0, \frac{1}{4}}}^{\frac{2}{3}} \sim \sum_{k = -\infty}^{\infty} \int_{\mathbb{R}} \left| \sum_{m = 0}^{\infty} \tilde{v}(k, \tau) \cdot 2^{\frac{m}{2}} \cdot \chi_{2^m \leq 1 + |\tau - \phi(k)| < 2^{m+1}} \right|^2 d\tau.
\]

(3.14)

Now, using Plancherel’s identity in (3.14), we obtain

\[
\|v\|_{X_{0, \frac{1}{4}}}^{\frac{2}{3}} \sim \sum_{m = 0}^{\infty} 2^{\frac{m}{2}} \|v_{2m}\|_{L^2(\mathbb{T} \times \mathbb{R})}^2.
\]

(3.15)

On the other hand,

\[
\|v\|_{L^4(\mathbb{T} \times \mathbb{R})}^2 = \|v\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 \leq 2 \sum_{m \leq m'} \|v_{2m} \cdot v_{2m'}\|_{L^2(\mathbb{T} \times \mathbb{R})} = 2 \sum_{m \geq 0} \|v_{2m} \cdot v_{2m+n}\|_{L^2(\mathbb{T} \times \mathbb{R})}^2.
\]

(3.16)

Once again, using Plancherel’s identity, we get

\[
\|v_{2m} \cdot v_{2m+n}\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 = \left\| \sum_{k_1 \in \mathbb{Z}} \int_{\mathbb{R}} \tilde{v}_{2m}(k_1, \tau_1) \tilde{v}_{2m+n}(k - k_1, \tau - \tau_1) \, d\tau_1 \right\|_{L^2(\mathbb{T} \times \mathbb{R})}^2.
\]

(3.17)
We estimate the RHS of (3.17) separately in the range $|k| \leq 2^n(|a| + 2)$ and $|k| > 2^n(|a| + 2)$, where the natural number $a$ will be determined later.

$$\|v_{2m}v_{2m+n}\|_{L^2_tL^2_x(\mathbb{R})} \leq \left( \sum_{|k| \leq 2^n(|a| + 2)} \left\| \sum_{k_1, \tau_1} \int_{\mathbb{R}} \hat{v}_{2m}^*(k_1, \tau_1) \hat{v}_{2m+n}^*(k-k_1, \tau-\tau_1) \, dt_1 \right\|_{L^2_t(\mathbb{R})}^2 \right)^{\frac{1}{2}} + \left( \sum_{|k| > 2^n(|a| + 2)} \left\| \sum_{k_1, \tau_1} \int_{\mathbb{R}} \hat{v}_{2m}^*(k_1, \tau_1) \hat{v}_{2m+n}^*(k-k_1, \tau-\tau_1) \, dt_1 \right\|_{L^2_t(\mathbb{R})}^2 \right)^{\frac{1}{2}}$$

(3.18)

To estimate $I$, we use the triangular and Young’s inequalities and obtain

$$\left\| \sum_{k_1, \tau_1} \int_{\mathbb{R}} \hat{v}_{2m}^*(k_1, \tau_1) \hat{v}_{2m+n}^*(k-k_1, \tau-\tau_1) \, dt_1 \right\|_{L^2_t(\mathbb{R})} \leq \sum_{k_1, \tau_1} \| \hat{v}_{2m}^*(k_1, \tau_1) \|_{L^1(\mathbb{R})} \| \hat{v}_{2m+n}^*(k-k_1, \tau-\tau_1) \|_{L^2(\mathbb{R})},$$

(3.19)

Now applying Cauchy-Schwartz inequality, we get

$$\|v_{2m}^*(k, \tau)\|_{L^2(\mathbb{R})} \leq \left( \mu(\tau_1 : 2^m \leq 1 + |\tau_1 - \phi(k_1)| < 2^{m+1}) \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} |v(k_1, \tau_1) \cdot \chi_m \leq 1 + |\tau_1 - \phi(k_1)| < 2^{m+1}|^2 \, dt_1 \right)^{\frac{1}{2}} \leq C_2 2^{\frac{m}{2}} \|v_{2m}^*(k_1, \cdot)\|_{L^2(\mathbb{R})},$$

(3.20)

where $\mu$ is the Lebesgue measure. Therefore, applying Cauchy-Schwartz inequality, Plancherel, and the invariance of the norm under translations, we obtain

$$\left\| \sum_{k_1, \tau_1} \int_{\mathbb{R}} \hat{v}_{2m}^*(k_1, \tau_1) \hat{v}_{2m+n}^*(k-k_1, \tau-\tau_1) \, dt_1 \right\|_{L^2_t(\mathbb{R})} \leq C_2 2^{\frac{m}{2}} \|v_{2m}\|_{L^2(\mathbb{R})} \cdot \|v_{2m+n}\|_{L^2(\mathbb{R})}.$$  

(3.21)

Therefore, from definition of $I$ in (3.18) and inequalities (3.19), (3.20), and (3.21), we get

$$I \leq C_3(\alpha) 2^{\frac{m}{2}} \|v_{2m}\|_{L^2(\mathbb{R})} \cdot \|v_{2m+n}\|_{L^2(\mathbb{R})}.$$  

(3.22)

To estimate $II$, define

$$\theta := \left\| \sum_{k_1, \tau_1} \int_{\mathbb{R}} \hat{v}_{2m}^*(k_1, \tau_1) \hat{v}_{2m+n}^*(k-k_1, \tau-\tau_1) \, dt_1 \right\|_{L^2_t(\mathbb{R})}.$$

First, denote $\chi_m(k, \tau) := \chi_{2^m \leq 1 + |\tau - \phi(k)| < 2^{m+1}}(\tau)$. Applying Cauchy Schwartz inequality in $k_1$ and $\tau_1$, we get

$$\theta \leq \sup_{|k| > 2^n(|a| + 2)} \sup_{\tau \in \mathbb{R}} \left( \chi_m \ast \chi_{m+n}(k, \tau) \right) \left\| \left( \sum_{k_1, \tau_1} \int_{\mathbb{R}} \hat{v}_{2m}^*(k_1, \tau_1)^2 \hat{v}_{2m+n}^*(k-k_1, \tau-\tau_1)^2 \, dt_1 \right)^{\frac{1}{2}} \right\|_{L^2_t(\mathbb{R})}.$$

Therefore, using the translation invariance of the norm, and Plancherel inequality, we obtain

$$II \leq \|\chi_m \ast \chi_{m+n}(k, \tau)\|_{L^\infty_{|k| > 2^n(|a| + 2)}}^2 \cdot \|v_{2m+n}\|_{L^2(\mathbb{R})} \cdot \|v_{2m}\|_{L^2(\mathbb{R})}.$$  

(3.23)

To estimate the convolution term in inequality (3.23), we write for fixed $k$ with $|k| > 2^n(2 + |a|)$ and $\tau$

$$\chi_m \ast \chi_{m+n}(k, \tau) = \sum_{k_1, \tau_1} \int_{\mathbb{R}} \chi_m(k_1, \tau_1) \cdot \chi_{m+n}(k-k_1, \tau-\tau_1) \, dt_1.$$  

(3.24)

From the support condition on $\chi_m$ and $\chi_{m+n}$ we note that for each $k_1$ fixed there exist $C_4 \geq 0$ and $C_5 > 0$ such that

$$C_4 2^m \leq |\tau_1 - \phi(k_1)| < C_2 2^m.$$
Thus, \( \tau_1 = \phi(k_1) + O(2^m) \). In a similar way, we have that \( \tau - \tau_1 = \phi(k - k_1) + O(2^{m+n}) \). In consequence,
\[
\tau = \phi(k_1) + \phi(k - k_1) + O(2^{m+n}), \tag{3.25}
\]
and
\[
\int \chi_m(k_1, \tau_1) \chi_m(n(k - k_1, \tau - \tau_1) \, dt_1 \leq \mu \left( \{ \tau_1 \in \mathbb{R} : 2^n \leq 1 + |\tau_1 - \phi(k_1)| < 2^{m+1} \} \right).
\]

Therefore, for each fixed \( k_1 \), the \( \tau_1 \) integral in (3.24) is \( O(2^m) \). To calculate the numbers of \( k'_1 \)'s for which the integral is non-zero, note that (3.26) implies
\[
\tau = -k^2 + 3kk_1 - 3k_1^2 - 2\mu + \frac{\alpha k_1|k_1|}{k} + \frac{\alpha(k - k_1)|k - k_1|}{k} + O(C_6(\alpha)2^{m+n-a}). \tag{3.26}
\]

Thus, we must study four cases:

**Case 1.** \( k - k_1 \geq 0 \) and \( k_1 \geq 0 \): By identity (3.26), we have
\[
\frac{\alpha k_1^2}{k} + \frac{\alpha(k - k_1)^2}{k} + 3kk_1 - 3k_1^2 - k^2 - 2\mu = \frac{\tau}{k} + O(C_6(\alpha)2^{m+n-a}).
\]

Note that, \( 3kk_1 - 3k_1^2 + \frac{\alpha k_1^2}{k} + \frac{\alpha(k - k_1)^2}{k} = \left( \frac{2\alpha}{k} - 3 \right) (k_1^2 - kk_1) + \alpha k \). Therefore,
\[
\left( \frac{2\alpha}{k} - 3 \right) \left( k_1 - \frac{k}{2} \right)^2 = \frac{\tau}{k} + \left( \frac{2\alpha}{k} - 3 \right) \frac{k^2}{4} - \alpha k + k^2 - 2\mu + O(C_6(\alpha)2^{m+n-a}).
\]

Using that \( |k| > 2^a([\alpha] + 2) \), we observe that \( \left| \frac{2\alpha}{k} - 3 \right| > 1 \). This implies,
\[
\left( k_1 - \frac{k}{2} \right)^2 = \frac{\tau}{2\alpha - 3k} + \frac{k^2}{4} - \frac{\alpha k^2}{2\alpha - 3k} + \frac{k^3}{2\alpha - 3k} - \frac{2\mu k}{2\alpha - 3k} + O(C_6(\alpha)2^{m+n-a}).
\]

**Case 2.** \( k - k_1 \geq 0 \) and \( k_1 \leq 0 \): In this case identity (3.26) implies,
\[
-\frac{\alpha k_1^2}{k} + \frac{\alpha(k - k_1)^2}{k} + 3kk_1 - 3k_1^2 - k^2 - 2\mu = \frac{\tau}{k} + O(C_6(\alpha)2^{m+n-a}).
\]

With the similar calculations as in Case 1, we obtain
\[
\left( k_1 - \frac{-2\alpha + 1}{3k} \frac{k}{2} \right)^2 = \frac{\tau}{3k} + \left( \frac{-2\alpha + 1}{3k} \frac{k}{2} \right)^2 \frac{k^2}{4} + \frac{\frac{k}{3} + \frac{k}{3} - \frac{2\mu}{3}}{k} + O(C_6(\alpha)2^{m+n-a}).
\]

**Case 3.** \( k - k_1 \leq 0 \) and \( k_1 \leq 0 \): In this case identity (3.26) implies,
\[
-\frac{\alpha k_1^2}{k} - \frac{\alpha(k - k_1)^2}{k} + 3kk_1 - 3k_1^2 - k^2 - 2\mu = \frac{\tau}{k} + O(C_6(\alpha)2^{m+n-a}).
\]

Thus,
\[
\left( \frac{-2\alpha}{k} - 3 \right) \left( k_1 - \frac{k}{2} \right)^2 = \frac{\tau}{k} + \left( \frac{-2\alpha}{k} - 3 \right) \frac{k^2}{4} + \alpha k + k^2 + 2\mu + O(C_6(\alpha)2^{m+n-a}).
\]

Using \( |k| > 2^a([\alpha] + 2) \), we observe that \( \left| \frac{-2\alpha}{k} - 3 \right| > 1 \). Therefore,
\[
\left( k_1 - \frac{k}{2} \right)^2 = \frac{\tau}{-2\alpha - 3k} + \frac{k^2}{4} + \frac{\frac{k}{2} + \frac{k}{2} + \frac{2\mu}{2\alpha - 3k}}{2\alpha - 3k} + O(C_6(\alpha)2^{m+n-a}).
\]

**Case 4.** \( k - k_1 \leq 0 \) and \( k_1 \geq 0 \): In this case identity (3.26) implies,
\[
\frac{\alpha k_1^2}{k} - \frac{\alpha(k - k_1)^2}{k} + 3kk_1 - 3k_1^2 - k^2 - 2\mu = \frac{\tau}{k} + O(C_6(\alpha)2^{m+n-a}).
\]
Thus,
\[
\left( k_1 - \frac{2\alpha}{3k} + 1 \right) \frac{k}{2} = \frac{\tau}{3k} + \left( \frac{2\alpha}{3k} + 1 \right) \frac{k^2}{4} - \frac{\alpha k}{3} - \frac{k^2}{3} - \frac{2\mu}{3} + O(C_0(\alpha)2^{m+n-a}).
\]

Therefore, in all cases \( k_1 \) takes at most \( O(C_0(\alpha)2^{m+n-a}) \) values. Thus,
\[
\|\chi_m * \chi_{m+n}(k, \tau)\|_{L^\infty} \leq C(\alpha)2^m \cdot 2^{m+n-a} = C_7(\alpha)2^{3m+n-a}.
\]

We can conclude from inequalities (3.23) and (3.27) that
\[
II \leq C_7(\alpha)2^{3m+n-a} \|v_{2m+n}\|_{L^2} \|v_{2m}\|_{L^2}.
\]

Using estimates \( \|v_{2m}\|_{L^2} \leq C(\alpha) \) and \( \|v_{2m+n}\|_{L^2} \leq C(\alpha) \), we get
\[
\|v_{2m} v_{2m+n}\|_{L^2} \leq C(\alpha) \left( 2^{\frac{m+n-a}{2}} + 2^{\frac{m+n-a}{2}} \right) \|v_{2m+n}\|_{L^2} \|v_{2m}\|_{L^2}.
\]

Taking \( a = \frac{m+n-a}{4} \), we obtain
\[
\|v_{2m} v_{2m+n}\|_{L^2} \leq C_0(\alpha)2^{\frac{3m+n-a}{2}} \|v_{2m+n}\|_{L^2} \|v_{2m}\|_{L^2}.
\]

Therefore, inequalities (3.19) and (3.29) imply
\[
\|v\|^2_{L^4(T\times\mathbb{R})} \leq 2C_0(\alpha) \left( \sum_{m=0}^{\infty} 2^{\frac{m+n-a}{4}} \|v_{2m+n}\|_{L^2}^2 \right) \frac{1}{n} \left( \sum_{m=0}^{\infty} 2^{\frac{m-n}{4}} \|v_{2m}\|_{L^2}^2 \right).
\]

Thus from inequality (3.30) and identity (3.15), we obtain
\[
\|v\|^2_{L^4(T\times\mathbb{R})} \leq C_0(\alpha) \|v\|^2_{X_{0,4}} \left( \sum_{n=0}^{\infty} 2^{-\frac{n}{4}} \right) \leq C(\alpha) \|v\|^2_{X_{0,4}}.
\]

\[ \Box \]

**Corollary 3.12.** Let \( f \in L^4(T\times\mathbb{R}) \). Then, there exists \( C_\alpha > 0 \), such that
\[
\|f\|_{X_{0,-\frac{1}{4}}} \leq C_\alpha \|f\|_{L^4(T\times\mathbb{R})}.
\]

**Proof.** It follows from Theorem 3.11 that \( X_{0,-\frac{1}{4}} \hookrightarrow L^4(T\times\mathbb{R}) \), so that \( (L^4(T\times\mathbb{R}'))' \hookrightarrow \left(X_{0,-\frac{1}{4}}\right)' \), i.e., \( L^4(T\times\mathbb{R}) \hookrightarrow X_{0,-\frac{1}{4}} \). \[ \Box \]

**Lemma 3.13.** For all \( k, k_1 \in \mathbb{Z} \) with \( k \neq 0 \), \( k_1 \neq 0 \), and \( k \neq k_1 \), we have
\[
|3kk_1(k - k_1)| \geq \frac{3}{2} k^2,
\]
\[
|k(k - k_1)| \geq \frac{1}{2} |k|.
\]

**Proof.** The proof of (3.31) follows by simple calculations considering six possible cases:
\[
R^{+++} := \{ k - k_1 > 0, k > 0, k_1 > 0 \}
\]
\[
R^{+++} := \{ k - k_1 > 0, k > 0, k_1 < 0 \}
\]
\[
R^{++-} := \{ k - k_1 > 0, k < 0, k_1 < 0 \}
\]
\[
R^{+-} := \{ k - k_1 < 0, k < 0, k_1 < 0 \}
\]
\[
R^{-+} := \{ k - k_1 < 0, k > 0, k_1 > 0 \}
\]
\[
R^{--} := \{ k - k_1 < 0, k > 0, k_1 > 0 \}.
\]
In the first case $R^{++}$, observe that $3kk_1(k-k_1) \geq \frac{3}{2}k^2 \iff k \geq \frac{k_1^2}{k_1 - \frac{3}{2}}$. The fact $k > k_1$ implies that the right side of the last expression is always true. The other cases are similar. Finally, note that (3.32) is consequence of (3.31) just dividing it by $|3k|$.

Remark 3.16. For any $k \in \mathbb{Z}$, $\alpha > 0$ and $\mu \in \mathbb{R}$, let $\phi(k) = -k^3 - 2\mu k + \alpha|k|$. For all $k, k_1 \in \mathbb{Z}$ with $k \neq 0$, $k_1 \neq 0$, $\alpha \neq 1$, and $\max\{|k|, |k_1|, |k-k_1|\} \geq \max\{1, \frac{4\alpha}{3} \}$, there exists a constant $C_\alpha > 0$ depending only on $\alpha$ such that, $|E(k, k_1)| \geq 3C_\alpha|kk_1(k-k_1)|$, where

$$E(k, k_1) := (\tau - \phi(k)) - (\tau_1 - \phi(k_1)) - (\tau - \tau_1 - \phi(k-k_1)).$$

Proof. Observe that $E(k, k_1) = -\alpha k|k| + \alpha k_1|k_1| + \alpha(k-k_1)|k-k_1| + 3kk_1(k-k_1)$. Again, the proof follows by straightforward calculations considering the same six cases of Lemma 3.13. We verify three cases, others are similar.

Case 1) In the region $R^{+++}$, one has $k \geq k_1 + 1 \geq 2$ and $\max\{|k|, |k_1|, |k-k_1|\} = k$. Thus $E(k, k_1) = -\alpha k + \alpha k_1 + \alpha(k-k_1)^2 + 3kk_1(k-k_1) = 3k_k(k-k_1)(k-\frac{2\alpha}{3})$, and $|E(k, k_1)| = 3k_k(k-k_1)(k-\frac{2\alpha}{3}) \geq 3k(k-k_1)C_\alpha k = 3|kk_1(k-k_1)|C_\alpha$.

Case 2) In $R^{+-}$, note that $k-k_1 \geq 2$. In this case, max\{|k|, |k_1|, |k-k_1|\} = k-k_1. E(k, k_1) = -\alpha kk + \alpha k_1(k-k_1) + \alpha(k-k_1)^2 + 3kk_1(k-k_1) = 3kk_1((k-k_1) - \frac{2\alpha}{3})$. Thus, $|E(k, k_1)| = 3k(k-k_1)(k-k_1) - \frac{2\alpha}{3} \geq 3k(k-k_1)C_\alpha k = 3|kk_1(k-k_1)|C_\alpha$.

Case 5) In $R^{-+}$, $k \leq -1$, $k_1 \geq 1$, $k-k_1 \leq -2$, and $\max\{|k|, |k_1|, |k-k_1|\} = -(k-k_1)$. $E(k, k_1) = -\alpha kk + \alpha k_1(k-k_1) + \alpha(k-k_1)^2 + 3kk_1(k-k_1) = 3kk_1((k-k_1) + \frac{2\alpha}{3})$. Thus, $|E(k, k_1)| = 3k(k-k_1)(k-k_1) + \frac{2\alpha}{3} \geq 3(k-k_1)k-k_1C_\alpha k = 3|kk_1(k-k_1)|C_\alpha$.

Remark 3.15. Lemmas 3.13 and 3.14 imply the so-called non-resonance property for Benjamin equation, it means, $|E(k, k_1)| \geq \frac{3}{2}C_\alpha k^2$, provided that $\max\{|k|, |k_1|, |k-k_1|\} \geq \max\{1, \frac{4\alpha}{3} \}$.

Remark 3.16. It follows from Lemma 3.14 that if $\max\{|k|, |k_1|, |k-k_1|\} \geq \max\{1, \frac{4\alpha}{3} \}$, then one of the following cases may occur

\begin{enumerate}
  \item $|\tau - \phi(k)| > \frac{3}{2}C_\alpha k^2$,
  \item $|\tau_1 - \phi(k_1)| > \frac{3}{4}C_\alpha k^2$,
  \item $|\tau - \tau_1 - \phi(k-k_1)| > \frac{3}{4}C_\alpha k^2$.
\end{enumerate}

Using similar arguments as in Bourgain [7] (see also [9 42]), we obtain the following key bilinear estimate.

Theorem 3.17. (Bilinear Estimate) Let $u, v : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ be functions in $X_{\alpha, \frac{s}{2}}$, and $X_{\alpha, \frac{s}{2}}$. Assume that the mean $|u(\cdot, t)| = |v(\cdot, t)| = 0$ for each $t \in \mathbb{R}$, $s \geq 0$, $\alpha > 0$. Then

$$\|\partial_x(uy)\|_{Z_{s, \frac{1}{2}}} \leq C_{\alpha, s} \left(\|u\|_{X_{\alpha, \frac{s}{2}}} \|v\|_{X_{\alpha, \frac{s}{2}}} + \|u\|_{X_{\alpha, \frac{s}{2}}} \|v\|_{X_{\alpha, \frac{s}{2}}}\right).$$

Proof. We prove this in two steps.
Step 1. First we estimate the $X_{s,-\frac{1}{2}}$ norm. Using duality and Plancherel, we get

$$
\|\partial_x(uv)\|_{X_{s,-\frac{1}{2}}} = \sup_{w \in X_{s,-\frac{1}{2}}} \left| \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \partial_x(uv)(k,\tau) \hat{w}(k,\tau) \, d\tau \right|
\leq \sup_{w \in X_{s,-\frac{1}{2}}} \left( \sum_{k \in \mathbb{Z}} \sum_{k \neq 0} \int \left| k \right| \left| \hat{u}(k,\tau_1) \right| \left| \hat{v}(k - k_1, \tau - \tau_1) \right| \left| \hat{w}(k,\tau) \right| \, d\tau_1 \, d\tau \right).
$$

(3.33)

Since $[u(\cdot,t)] = [v(\cdot,t)] = 0$, $k = 0$, $k_1 = 0$ and $k - k_1 = 0$ do not contribute to the sum. Now, we move to estimate

$$
I := \sum_{k, k_1 \in \mathbb{Z}} \int \left| k \right| \left| \hat{u}(k,\tau_1) \right| \left| \hat{v}(k - k_1, \tau - \tau_1) \right| \left| \hat{w}(k,\tau) \right| \, d\tau_1 \, d\tau
= \sum_{k, k_1 \in \mathbb{Z}} \int \left| k \right| \left| k \right|^{s} \left| \tau_1 - \phi(k) \right|^{\frac{1}{2}} \left| \hat{u}(k,\tau_1) \right| \left| k - k_1 \right|^{s} \left| \tau - \tau_1 - \phi(k - k_1) \right|^{\frac{1}{2}} \left| \hat{v}(k - k_1, \tau - \tau_1) \right| \left| k \right|^{s} \left| \tau - \phi(k - k_1) \right|^{\frac{1}{2}} \left| \hat{w}(k,\tau) \right| \, d\tau_1 \, d\tau.
$$

(3.34)

Let $u, v, w : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$ with $[u(\cdot,t)] = [v(\cdot,t)] = 0$. We define

$$
c_u(k_1, \tau_1) := (1 + |k_1|)^s \left| \tau_1 - \phi(k_1) \right|^\frac{1}{2} \left| \hat{u}(k_1, \tau_1) \right|,
$$

$$
c_v(k - k_1, \tau - \tau_1) := (1 + |k - k_1|)^s \left| \tau - \tau_1 - \phi(k - k_1) \right|^\frac{1}{2} \left| \hat{v}(k - k_1, \tau - \tau_1) \right|,
$$

$$
c_w(k, \tau) := \langle k \rangle^{-s} \left| \tau - \phi(k) \right|^\frac{1}{2} \left| \hat{w}(k, \tau) \right|,
$$

for all $k, k_1, k - k_1 \in \mathbb{Z} \setminus \{0\}$ and $\tau, \tau_1 \in \mathbb{R}$. Note that $c_u(0, \tau_1) = c_v(0, \tau - \tau_1) = 0$. From inequality (3.33) and definition (3.35), we obtain

$$
I \leq \sum_{k, k_1 \in \mathbb{Z}} \int \frac{\left| k \right| \left| c_u(k_1, \tau_1) c_v(k - k_1, \tau - \tau_1) c_w(k, \tau) \right| \left| \tau_1 - \phi(k_1) \right|^\frac{1}{2} \left| \tau - \tau_1 - \phi(k - k_1) \right|^\frac{1}{2} \left| \tau - \phi(k) \right|^\frac{1}{2}}{\left| k \right|^{s \left| k \right| + (k - k_1)^{s \left| k - k_1 \right|}} \, d\tau_1 \, d\tau.
$$

From (3.32) there exists $C_s > 0$ such that $\frac{(k)^s}{|k|^{s \left| k \right| + (k - k_1)^{s \left| k - k_1 \right|}}} \leq C_s$. Therefore, separating the small frequencies from the large ones, we obtain

$$
I \leq C_s \sum_{k, k_1 \in \mathbb{Z}} \int \frac{\left| k \right| \left| c_u(k_1, \tau_1) c_v(k - k_1, \tau - \tau_1) c_w(k, \tau) \right| \left| \tau_1 - \phi(k_1) \right|^\frac{1}{2} \left| \tau - \tau_1 - \phi(k - k_1) \right|^\frac{1}{2} \left| \tau - \phi(k) \right|^\frac{1}{2}}{\left| k \right|^{s \left| k \right| + (k - k_1)^{s \left| k - k_1 \right|}} \, d\tau_1 \, d\tau
\leq C_{s,\alpha} \sum_{k, k_1 \in \mathbb{Z}} \int \frac{c_u(k_1, \tau_1) c_v(k - k_1, \tau - \tau_1) c_w(k, \tau)}{\left| \tau_1 - \phi(k_1) \right|^\frac{1}{2} \left| \tau - \tau_1 - \phi(k - k_1) \right|^\frac{1}{2} \left| \tau - \phi(k) \right|^\frac{1}{2}} \left| \tau_1 - \tau \right|^\alpha \left| \tau_1 - \tau_1 - \phi(k - k_1) \right|^\alpha \left| \tau - \phi(k) \right|^\alpha \, d\tau_1 \, d\tau
+ \sum_{k, k_1 \in \mathbb{Z}} \int \frac{\left| k \right| \left| c_u(k_1, \tau_1) c_v(k - k_1, \tau - \tau_1) c_w(k, \tau) \right| \left| \tau_1 - \phi(k_1) \right|^\frac{1}{2} \left| \tau - \tau_1 - \phi(k - k_1) \right|^\frac{1}{2} \left| \tau - \phi(k) \right|^\frac{1}{2}}{\left| k \right|^{s \left| k \right| + (k - k_1)^{s \left| k - k_1 \right|}} \, d\tau_1 \, d\tau}
$$

(3.36)
In view of Remark 3.16, we must study three different cases.

**Case 1.** $|\tau - \phi(k)| > \frac{4}{3}C_\alpha k^2$ : In this case, from (3.36), we have

$$I \leq C_{s,\alpha} \sum_{k, k_1 \in \mathbb{Z}} \int_{\mathbb{R}^2} \frac{c_u(k_1, \tau_1) c_u(k - k_1, \tau - \tau_1) c_u(k, \tau)}{(\tau_1 - \phi(k_1))^{\frac{3}{2}} (\tau - \phi(k - k_1))^{\frac{3}{2}}} d\tau_1 d\tau$$

$$+ \sum_{k, k_1 \in \mathbb{Z}} \max\{|k|, |k_1|, |k - k_1|\} \geq \max\left\{1, \frac{4}{3}, \frac{4}{3}ight\} \int_{\mathbb{R}^2} \frac{|k| c_u(k_1, \tau_1) c_u(k - k_1, \tau - \tau_1) c_u(k, \tau)}{(\tau_1 - \phi(k_1))^{\frac{3}{2}} (\tau - \phi(k - k_1))^{\frac{3}{2}} (1 + \frac{4}{3}k^2)^{\frac{3}{2}}} d\tau_1 d\tau$$

(3.37)

We define functions $F, G : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{C}$ by

$$\hat{F}(m, \lambda) = \frac{c_u(m, \lambda)}{(1 + |\lambda - \phi(m)|)^{\frac{3}{2}}}$$

and

$$\hat{G}(m, \lambda) = \frac{c_u(m, \lambda)}{(1 + |\lambda - \phi(m)|)^{\frac{3}{2}}}$$

It means

$$F(x, t) = \sum_{m \in \mathbb{Z}} \int_{\mathbb{R}} e^{i(mx + \lambda t)} \frac{c_u(m, \lambda)}{(1 + |\lambda - \phi(m)|)^{\frac{3}{2}}} d\lambda,$$

and

$$G(x, t) = \sum_{m \in \mathbb{Z}} \int_{\mathbb{R}} e^{i(mx + \lambda t)} \frac{c_u(m, \lambda)}{(1 + |\lambda - \phi(m)|)^{\frac{3}{2}}} d\lambda.$$

From (3.33), (3.37) and Plancherel, we obtain

$$\|\partial_x(uv)\|_{X_{s,-\frac{1}{2}}} \leq C_{s,\alpha} \sup_{w \in X_{s,-\frac{1}{2}}} \left( \int_{k \in \mathbb{Z}} \left\| \hat{F} \ast \hat{G}(k, \tau) \right\|_{L^2_{\mathbb{T}}} \|w\|_{X_{s,-\frac{1}{2}}} \right)$$

$$\leq C_{s,\alpha} \sup_{w \in X_{s,-\frac{1}{2}}} \left( \left\| \hat{F} \hat{G}(k, \tau) \right\|_{L^2_{\mathbb{T}}} \|w\|_{X_{s,-\frac{1}{2}}} \right)$$

where we applied Cauchy-Schwartz, and Theorem 3.11 in the last two inequalities. Note that

$$\|F\|_{X_{0,\frac{1}{2}}} = \left( \sum_{k = -\infty}^{\infty} \int_{\mathbb{R}} (\tau - \phi(k))^{\frac{3}{2}} \frac{|c_u(k, \tau)|^2}{1 + |\tau - \phi(k)|} d\tau \right)^{\frac{1}{2}} \sim \left( \sum_{k = -\infty}^{\infty} \int_{\mathbb{R}} (k)^{\frac{3}{2}} (\tau - \phi(k))^{\frac{3}{2}} |\hat{u}(k, \tau)|^2 d\tau \right)^{\frac{1}{2}} = \|u\|_{X_{s,-\frac{1}{2}}}.$$
Case 2. \(|\tau_1 - \phi(k_1)| > \frac{4}{3}C_3k^2\) : In this case, (3.36) and Cauchy-Schwartz imply

\[
I \leq C_{s,\alpha} \sum_{k, k_1 \in \mathbb{Z}} \max\{|k|, |k_1|, |k-k_1|\} \leq \max\left\{1, \frac{4}{3}\right\} \int_{\mathbb{R}^2} \frac{|c_u(k_1, \tau_1) c_u(k - k_1, \tau - \tau_1) c_u(k, \tau)|}{|\tau - \tau_1 - \phi(k - k_1)|}^{\frac{3}{2}} \int_{\mathbb{R}} |c_u(k_1, \tau_1)| c_u(k - k_1, \tau - \tau_1) c_u(k, \tau_2) d\tau_1 d\tau \\
\leq C_{s,\alpha} \sum_{k \in \mathbb{R}} \int_{\mathbb{R}} \frac{1}{(1 + |\tau - \phi(k)|)^{\frac{3}{2}}} \left( \sum_{k_1 \in \mathbb{Z}} c_u(k_1, \tau_1) \int_{\mathbb{R}} \frac{c_u(k - k_1, \tau - \tau_1)}{|\tau - \tau_1 - \phi(k - k_1)|} d\tau_1 \right) c_u(k, \tau) d\tau \\
\leq C_{s,\alpha} \left( \sum_{k \in \mathbb{R}} \int_{\mathbb{R}} \frac{1}{(1 + |\tau - \phi(k)|)^{\frac{3}{2}}} \left( \int_{\mathbb{R}} |c_u(k, \tau)|^2 d\tau \right)^{\frac{3}{2}} \right),
\]

where \(H_f : \mathbb{T} \times \mathbb{R} \to \mathbb{C}\) is a function defined by \(H_f(m, \lambda) = c_f(m, \lambda)\). It means,

\[
H_f(x, t) = \sum_{m \in \mathbb{Z}} e^{i(mx + \lambda t)} c_f(m, \lambda) d\lambda.
\]

From relations (3.33)-(3.34), (3.38) and \((1 + |\tau - \phi(k)|)^{-1} < (1 + |\tau - \phi(k)|)^{-\frac{3}{2}}\), we have

\[
\|\partial_x(uv)\|_{X_{s,\frac{1}{2}}} \leq C_{s,\alpha} \sup_{w \in X_{s,\frac{1}{2}}} \left[ \left( \sum_{k = -\infty}^{\infty} \int_{\mathbb{R}} (1 + |\tau - \phi(k)|)^{-\frac{3}{2}} \left| \mathcal{H}_u \cdot G(k, \tau) \right|^2 d\tau \right)^{\frac{3}{2}} \right] \|w\|_{X_{s,\frac{1}{2}}} \\
\leq C_{s,\alpha} \left( \sum_{k = -\infty}^{\infty} \int_{\mathbb{R}} (1 + |\tau - \phi(k)|)^{-\frac{3}{2}} \left( \mathcal{H}_u \cdot G(k, \tau) \right)^2 d\tau \right)^{\frac{3}{2}} \\
\leq C_{s,\alpha}\|H_u \cdot G\|_{L^2(\mathbb{T} \times \mathbb{R})} \\
\leq C_{s,\alpha}\|H_u\|_{L^2(\mathbb{T} \times \mathbb{R})} \|G\|_{L^4(\mathbb{T} \times \mathbb{R})},
\]

where we applied Corollary 3.12 and Holder inequality in the last two inequalities. From Theorem 3.11 we have \(\|G\|_{L^4(\mathbb{T} \times \mathbb{R})} \leq \|G\|_{X_{0,\frac{1}{2}}}\|. On the other hand,

\[
\|H_u\|_{L^2(\mathbb{T} \times \mathbb{R})} \sim \left( \sum_{k = -\infty}^{\infty} \int_{\mathbb{R}} |k|^{2s} (\tau - \phi(k))^2 |\mathcal{G}(k, \tau)|^2 d\tau \right)^{\frac{1}{2}} = \|u\|_{X_{s,\frac{1}{2}}}.
\]

Therefore, \(\|\partial_x(uv)\|_{X_{s,\frac{1}{2}}} \leq C_{s,\alpha}\|u\|_{X_{s,\frac{1}{2}}} \|G\|_{X_{0,\frac{1}{2}}} \leq C_{s,\alpha}\|u\|_{X_{s,\frac{1}{2}}} \|v\|_{X_{s,\frac{1}{2}}}\).

**Case 3.** \(|\tau_1 - \phi(k_1)| > \frac{4}{3}C_3k^2\) : Observe that, this case is similar to the second one, just substituting \(H_v\) in the place of \(H_u\) and \(F\) in the place of \(G\). Thus, we obtain

\[
\|\partial_x(uv)\|_{X_{s,\frac{1}{2}}} \leq C_{s,\alpha}\|H_v\|_{L^2(\mathbb{T} \times \mathbb{R})} \|F\|_{X_{0,\frac{1}{2}}} \leq C_{s,\alpha}\|\psi\|_{X_{s,\frac{1}{2}}} \|u\|_{X_{s,\frac{1}{2}}}.
\]
Step 2. Now we estimate the $Y_{s,-1}$ norm. Using duality we have,
\[
\|\partial_x(uv)\|_{Y_{s,-1}} \sim \left\| (1 + |k|)^s(1 + |\tau - \phi(k)|)^{-1} \hat{\partial}_x(uv)(k, \tau) \right\|_{L^1(\mathbb{R})}.
\]

\[
= \sup_{a_k \in \mathbb{Z}, a_k \geq 0} \sum_{k \in \mathbb{Z}} a_k \left( \int_{\mathbb{R}} (1 + |k|)^s(1 + |\tau - \phi(k)|)^{-1} |\hat{\partial}_x(uv)(k, \tau)| \, d\tau \right).
\] (3.39)

We move to estimate
\[
II := (1 + |k|)^s(1 + |\tau - \phi(k)|)^{-1} |\hat{\partial}_x(uv)(k, \tau)|.
\] (3.40)

Note that
\[
II \leq (1 + |k|)^s |k|(1 + |\tau - \phi(k)|)^{-1} \left( \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} |\hat{v}(k_1, \tau_1)||\hat{v}(k-k_1, \tau - \tau_1)| \, d\tau_1 \right)
\]
\[
\leq |k|(1 + |\tau - \phi(k)|)^{-1} \left( \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} (1 + |k|)^s c_u(k_1, \tau_1) c_v(k-k_1, \tau - \tau_1) \, d\tau_1 \right)
\]
\[
\leq C_s \left( \sum_{k_1 \in \mathbb{Z}} \int_{\mathbb{R}} (1 + |\tau - \phi(k_1)|)^{-1} |\tau - \phi(k_1)|^{\frac{1}{2}} \, d\tau_1 \right).
\] (3.41)

It follows from identity (3.39), definition (3.40) and relation (3.41) that
\[
\|\partial_x(uv)\|_{Y_{s,-1}} \leq C_s \sup_{a_k \in \mathbb{Z}, a_k \geq 0} \sum_{k \in \mathbb{Z}} a_k \left( \sum_{k_1 \in \mathbb{Z}} \int_{\mathbb{R}} (1 + |\tau - \phi(k_1)|)^{-1} |\tau - \phi(k_1)|^{\frac{1}{2}} \, d\tau_1 \right).
\]

As in Step 1., in view of Remark 3.16, here we divide the sum into small and large frequencies to consider three different cases and obtain
\[
\|\partial_x(uv)\|_{Y_{s,-1}} \leq C_{s,\alpha} \|u\|_{X_{s,-\frac{1}{2}}} \|v\|_{X_{s,-\frac{1}{2}}};
\]
in the first two cases and
\[
\|\partial_x(uv)\|_{Y_{s,-1}} \leq C_{s,\alpha} \|v\|_{X_{s,-\frac{1}{2}}} \|u\|_{X_{s,-\frac{1}{2}}};
\]
in the third case. We omit the details (see the proof of lemma 7.42 in [7]).

\[\Box\]

**Corollary 3.18.** Let $s \geq 0$, $\alpha > 0$, and $T > 0$ be given. Assume that $u, v : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ are functions in $X_{s,-\frac{1}{2}}^T$ and $X_{s,-\frac{1}{2}}^T$ with mean $[u(\cdot, t)] = [v(\cdot, t)] = 0$ for each $t \in \mathbb{R}$. Then
\[
\|\partial_x(uv)\|_{L_{s,-\frac{1}{2}}^T} \leq C_{\alpha,s} \left( \|u\|_{X_{s,-\frac{1}{2}}^T} \|v\|_{X_{s,-\frac{1}{2}}^T} + \|u\|_{X_{s,-\frac{1}{2}}^T} \|v\|_{X_{s,-\frac{1}{2}}^T} \right).
\]

**Corollary 3.19.** Let $s \geq 0$, $\alpha > 0$, and $T > 0$ be given. Assume that $v : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is a function in $X_{s,-\frac{1}{2}}^T$ with mean $[v(\cdot, t)] = 0$ for each $t \in \mathbb{R}$. Then there exist $0 < \epsilon < \frac{1}{6}$ such that
\[
\|\partial_x(v^2)\|_{L_{s,-\frac{1}{2}}^T} \leq C_{\alpha,s} T^\epsilon \|v\|_{L_{s,-\frac{1}{2}}^T}^2.
\]
Proof. Applying Proposition 3.8 with $b' = \frac{1}{2}$, $0 < \epsilon < \frac{1}{2}$, and $b = \frac{1}{2} + \epsilon$, we obtain
\[
\|u\|_{X^T_{s,b}} \leq C T^\frac{1}{2} + \epsilon \eta(T) \leq C T^\frac{1}{2} \eta(T),
\]
From Corollary 3.11 and 3.14, we have
\[
\|
\]
\[
\leq C \|u\|_{X^T_{s,b}} \leq C T^\frac{1}{2} \|u\|_{X^T_{s,b}} \leq C T^\frac{1}{2} \|u\|_{X^T_{s,b}}.
\]

\[
\]

4. Propagation of Compactness, Regularity and Unique Continuation Property

In this section we establish some results on propagation of compactness, regularity and unique continuation property satisfied by the solution of the Benjamin equation that are essential to prove the global exponential stabilization.

4.1. The Multiplication Property of the Bourgain’s Space. In the following Lemmas we establish the multiplication property of the Bourgain space $X_{s,b}$.

Lemma 4.1. If $\psi = \psi(t) \in C^\infty(\mathbb{R})$, then $\psi v \in X^T_{s,b}$ for all $v \in X^T_{s,b}$. Furthermore, there exists a positive constant $C = C_{\eta,T,b,\psi}$ such that
\[
\|\psi v\|_{X^T_{s,b}} \leq C \|v\|_{X^T_{s,b}}.
\]

If $T \leq 1$, then positive constant $C$ does not depend on the time $T$.

Proof. Let $v = v(x,t) \in X^T_{s,b}$ and consider $w$ an extention to $X_{s,b}$ of $v$ such that
\[
\|w\|_{X_{s,b}} \leq 2 \|v\|_{X_{s,b}}.
\]
Thus, $\eta_T(t) \psi(t)w(x,t)$ is an extention to $X_{s,b}$ of $\psi(t)v(x,t)$. Note that
\[
\|\psi(t)v\|^2_{X^T_{s,b}} = \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}} \langle k \rangle^{2s} \langle \tau - \phi(k) \rangle^{2k} \|\psi(t)\|^2_{L^1(\mathbb{R})} \|\psi(t)\|^2_{L^2(\mathbb{R})} d\tau.
\]

If $b = 0$ in (4.2), we use Young Inequality to obtain
\[
\|\psi(t)v\|^2_{X^T_{s,b}} \leq \sum_{k=-\infty}^{\infty} \langle k \rangle^{2s} \|\psi(t)\|^2_{L^1(\mathbb{R})} \|\psi(t)\|^2_{L^2(\mathbb{R})}.
\]
Since $\eta_T(t) \psi(t) \in C^\infty_c(\mathbb{R})$ we have $\|\psi(t)\|^2_{L^1(\mathbb{R})} < \infty$. Thus, there exists a positive constant $C_{T,\eta,\psi}$ such that
\[
\|\psi(t)v\|^2_{X^T_{s,b}} \leq C_{T,\eta,\psi} \sum_{k=-\infty}^{\infty} \langle k \rangle^{2s} \|\psi(t)\|^2_{L^2(\mathbb{R})} = C_{T,\eta,\psi} \|u\|^2_{X^T_{s,b}} = 2 C_{T,\eta,\psi} \|v\|^2_{X^T_{s,b}}.
\]
On the other hand, if $b > 0$, we have the following inequality
\[
(\tau - \phi(k))^b \leq c_b (\tau - y - \phi(k))^b (y)^b.
\]
From (4.2), (4.3) and Young inequality, we have

\[
\|\psi(t)v\|^2_{X^T_{s,b}} \leq c_b \sum_{k=-\infty}^{\infty} \langle k \rangle^{2s} \|\overline{\langle \cdot \rangle^b (\eta_T(t) \psi(t))^\wedge (\cdot) * (\cdot - \phi(k))^b \hat{\nu}(k,\cdot)} (\tau)\|_{L^2_1(\mathbb{R})}^2 \\
\leq c_b \sum_{k=-\infty}^{\infty} \langle k \rangle^{2s} \|\overline{\langle \cdot \rangle^b (\eta_T(t) \psi(t))^\wedge (\cdot)} (\tau)\|_{L^1_1(\mathbb{R})} \|\langle \tau \phi(k)^b \hat{\nu}(k,\tau)\|_{L^2_1(\mathbb{R})}^2.
\]  

(4.4)

Since \(\eta_T(t)\psi(t) \in C_c^\infty(\mathbb{R})\) we get \(\|\langle \cdot \rangle^b (\eta_T (\cdot) \psi(\cdot))^\wedge (\cdot) \|_{L^1_1(\mathbb{R})}^2 < \infty\). Therefore, there exists a positive constant \(C_{T,\eta,b,\psi}\) such that

\[
\|\psi(t)v\|^2_{X^T_{s,b}} \leq C_{T,\eta,b,\psi} \sum_{k=-\infty}^{\infty} \langle k \rangle^{2s} \|\tau - \phi(k)^b \hat{\nu}(k,\tau)\|_{L^2_1(\mathbb{R})}^2 = 2C_{T,\eta,\psi} \|v\|^2_{X^T_{s,b}}.
\]

Finally, if \(b < 0\) we infer that \(\langle \tau - \phi(k)^b \rangle \leq c_b \langle \tau - y - \phi(k)^b \rangle \langle y \rangle^{-b}\), and similar computations as those yielding (4.3) give the result. Note that, if \(T \leq 1\) then \(\eta(t)\psi(t)v(x,t)\) is an extention to \(X_{s,b}\) of \(\psi(t)v(x,t)\). Consequently, the constant \(C\) in the estimates will only depend on \(\eta, b,\) and \(\psi\).

As was pointed out by Laurent et. al [24] for the KdV equation, if \(\phi = \phi(x) \in C_\infty(\mathbb{T})\), then \(\phi\nu\) may not belong to the space \(X^T_{s,b}\) for \(v \in X^T_{s,b}\). For the Benjamin equation too, the same is lost in the index of regularity \(s\) because the structure in space of the harmonics is not kept by the multiplication by a (smooth) function of \(x\) (see Example A.5 in the appendix). In fact, this is reflected in next theorem. We first prove a necessary lemma.

**Lemma 4.2.** Let \(s \in \mathbb{R}\). \(v \in X_{s,1}\) if and only if \(v \in L^2(\mathbb{R}, H^s_p(\mathbb{T}))\), and

\[
\partial_t v - \partial_x^3 v - \alpha H \partial_x^2 v + 2 \mu \partial_x v \in L^2(\mathbb{R}, H^s_p(\mathbb{T})).
\]

In this case we have \(\|v\|_{X_{s,1}}^2 = \|v\|^2_{L^2(\mathbb{R}, H^s_p(\mathbb{T}))} + \|\partial_t v - \partial_x^3 v - \alpha H \partial_x^2 v + 2 \mu \partial_x v\|^2_{L^2(\mathbb{R}, H^s_p(\mathbb{T}))}\).

**Proof.** Let \(s \in \mathbb{R}\) be fixed. Then, applying Plancherel identity in time we obtain

\[
\|v\|_{X_{s,1}}^2 = \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}} \langle k \rangle^{2s} \left( 1 + |\tau + k^3 - \alpha k| + 2\mu k \right) |\hat{\nu}(k,\tau)|^2 d\tau
\]

\[
= \|v\|^2_{L^2(\mathbb{R}, H^s_p(\mathbb{T}))} + \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}} \langle k \rangle^{2s} \left| \hat{\partial_t} v(k,\tau) - \hat{\partial_x}^3 v(k,\tau) - \alpha \hat{H} \hat{\partial_x}^2 v(k,\tau) + 2 \mu \hat{\partial_x} v(k,\tau) \right|^2 dt.
\]

This proves the Lemma.

Using the Fourier transform, \(X_{s,b}\) may be viewed as the weighted \(L^2\) space

\[
L^2(\mathbb{R} \times \mathbb{Z}_k, \langle k \rangle^{2s} (\tau - \phi(k))^b \lambda \otimes \delta),
\]

(4.5)

where \(\lambda\) and \(\delta\) are the Lebesgue measure over \(\mathbb{R}\) and the discrete measure on \(\mathbb{Z}\), respectively.

**Theorem 4.3.** Let \(-1 \leq b \leq 1\), \(s \in \mathbb{R}\), and \(\varphi \in C_\infty(\mathbb{T})\). Then for any \(v \in X_{s,b}\), \(\varphi v \in X_{s-2|b|,b}\). Similarly, the multiplication by \(\varphi\) maps \(X^T_{s,b}\) into \(X^T_{s-2|b|,b}\) i.e. there exists a positive constant \(C = C_{s,\alpha,\varphi,\mu}\) such that

\[
\|\varphi v\|_{X^T_{s-2|b|,b}} \leq C \|v\|_{X^T_{s,b}}.
\]

(4.6)
Case 1. Let $v \in S(\mathbb{T} \times \mathbb{R})$. From definition, we have

$$
\| \varphi v \|_{X,s,0}^2 = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} 2\pi (1 + |k|^2)^s |(\varphi v)^*(k,t)|^2 \, dt.
$$

If $s \geq 0$, one has $$(1 + |k|)^s \leq c_s (1 + |k-j|)^s (1 + |j|)^s.$$ Therefore, the Cauchy-Schwartz inequality for $N > \frac{3}{2}$, yields

$$
\| \varphi v \|_{X,s,0}^2 \leq c_s \sum_{j=-\infty}^{\infty} (1 + |j|)^{2s+2N} \| \hat{\varphi}(j) \|^2 \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} (1 + |k-j|)^{2s} |v(x,t) (k-j,t)|^2 \, dt.
$$

Using the invariance of the $L^2(\mathbb{R}, H^N_p(\mathbb{T}))$ norm by translations, we get

$$
\| \varphi v \|_{X,s,0}^2 \leq c_s \| \varphi \|_{H^N_p(\mathbb{T})}^2 \| v \|_{X,s,0}^2.
$$

If $s < 0$, we apply $(1 + |k|)^s \leq c_s (1 + |k-j|)^s (1 + |j|)^{-s}$ and proceed as above to obtain (4.9).

In the general case, we use density and duality arguments to complete the proof.

Case 2. $b = 1$: From Lemma 4.2, we obtain

$$
\| \varphi v \|_{X,s-2,0}^2 = \| \varphi v \|_{L^2(\mathbb{R}, H^{s-2}_p(\mathbb{T}))}^2 + \| \partial_t (\varphi v) - \partial_x^2 (\varphi v) - \alpha \mathcal{H} \partial_x^2 (\varphi v) + 2\mu \partial_x (\varphi v) - \alpha \varphi \mathcal{H} \partial_x^2 v + \alpha \varphi \mathcal{H} \partial_x^2 v \|_{L^2(\mathbb{R}, H^{s-2}_p(\mathbb{T}))}^2
$$

$$
\leq \| \varphi v \|_{X,s-2,0}^2 + c_\alpha \| \mathcal{H} \partial_x^2 (\varphi v) + \varphi \mathcal{H} \partial_x^2 v \|_{L^2(\mathbb{R}, H^{s-2}_p(\mathbb{T}))}^2
$$

$$
= I + II + III.
$$

From case $b = 0$, we obtain that there exists $A_{s,\varphi} > 0$ such that

$$
I = \| \varphi v \|_{X,s-2,0}^2 \leq A_{s,\varphi} \| v \|_{X,s-2,0}^2 \leq A_{s,\varphi} \| v \|_{X,s,0}^2.
$$

From properties of the operator $\partial_x^2$ on Bourgain’s spaces, and noting that $\mathcal{H}$ is an isometry in $H^{s-2}_p(\mathbb{T})$, we get

$$
II \leq c_\alpha \left( \| \partial_x^2 (\varphi v) \|_{L^2(\mathbb{R}, H^{s-2}_p(\mathbb{T}))}^2 + \| \mathcal{H} \partial_x^2 v \|_{X,s-2,0}^2 \right)
$$

$$
\leq c_\alpha \left( \| \varphi v \|_{X,s,0}^2 + c_{s,\varphi} \| \mathcal{H} \partial_x^2 v \|_{L^2(\mathbb{R}, H^{s-2}_p(\mathbb{T}))}^2 \right)
$$

$$
\leq c_\alpha \left( d_{s,\varphi} \| v \|_{X,s,0}^2 + c_{s,\varphi} \| \partial_x^2 v \|_{X,s-2,0}^2 \right).
$$

Hence, there exists another positive constant $B_{s,\alpha,\varphi}$ such that

$$
II \leq B_{s,\alpha,\varphi} \| v \|_{X,s,0}^2.
$$

We estimate $III$. From the Leibniz’s rule for derivatives one has

$$
\partial_t (\varphi v) - \partial_x^2 (\varphi v) + 2\mu \partial_x (\varphi v) - \alpha \varphi \mathcal{H} \partial_x^2 v = \varphi \left( \partial_t v - \partial_x^2 v + 2\mu \partial_x v - \alpha \mathcal{H} \partial_x^2 v \right)
$$

$$
- 3\partial_x \varphi \partial_x^2 v - 3\partial_x^2 \varphi \partial_x v - \partial_x^2 \varphi v + 2\mu \partial_x \varphi v.
$$

(4.10)
Note that \(-3\partial_x c \partial^2_x v - 3\partial^2_x \varphi \partial_x v - \partial^3_x \varphi v + 2\mu \partial_x \varphi v\) is an operator of second order for \(v\). From identity \((4.10)\), the case \(b = 0\), and the fact that \(\varphi \in C^\infty(\mathbb{T})\), we get

\[
III\leq c\|\varphi(x)\|_{L^2(\mathbb{R},H^s_{\mu}(\mathbb{T}))}^2 + 3|c|\|\partial_x \varphi \partial^2_x v - 3\partial^2_x \varphi \partial_x v - \partial^3_x \varphi v + 2\mu \partial_x \varphi v\|_{L^2(\mathbb{R},H^s_{\mu}(\mathbb{T}))}^2
\]

\[
\leq c_s \|\partial_x v - \partial^3_x v + 2\|\partial_x v - \alpha \mathcal{H}\partial^2_x v\|_{X_{1-2,0}}^2
\]

\[
+ 3\|\partial_x \varphi \partial^2_x v - 3\partial^2_x \varphi \partial_x v - \partial^3_x \varphi v + 2\mu \partial_x \varphi v\|_{X_{1-2,0}}^2
\]

\[
\leq 3\|\partial_x \varphi \partial^2_x v - 3\partial^2_x \varphi \partial_x v - \partial^3_x \varphi v + 2\mu \partial_x \varphi v\|_{X_{1-2,0}}^2.
\]

Using \(X_{s,0} \hookrightarrow X_{s-2,0}\), and \(X_{s,0} \hookrightarrow X_{s-1,0}\), we have that there exists \(D_{s,\mu,\varphi} > 0\) such that

\[
III \leq D_{s,\mu,\varphi}\left(\|\partial_x v - \partial^3_x v + 2\|\partial_x v - \alpha \mathcal{H}\partial^2_x v\|_{X_{s,0}}^2 + \|v\|_{X_{s,0}}^2\right) = D_{s,\mu,\varphi}\|v\|_{X_{s,1}}^2,
\]

where in the last step Lemma \((4.2)\) is used. From \((4.7)-(4.9)\), and \((4.11)\), we obtain that there exists a positive constant \(C_{s,\alpha,\mu,\varphi}\) such that

\[
\|\varphi(x)v\|_{X_{s-2,1}} \leq C_{s,\alpha,\mu,\varphi}\|v\|_{X_{s,1}}^2.
\]

This proves the case \(b = 1\).

**Case 3.** \(0 < b < 1\): In this case we use interpolation. From identification \((4.5)\) and the Complex Interpolation Theorem of Stein-Weiss (see Bergh and Lofstrom [4] page 115), we obtain \((X_{s,0},X_{s,1})_{\theta,2} \approx X_{s(1-\theta)+(\theta,\theta)}\), with \(0 < \theta < 1\). Furthermore, from the cases \(b = 0\) and \(b = 1\) we infer that the operator of multiplication by \(\varphi \in C^\infty(\mathbb{T})\), defined by

\[
T : X_{s,\theta} \approx L^2(\mathbb{R}_r \times \mathbb{Z}_k, \langle k \rangle^{2\theta}\langle \tau - \phi(k)\rangle^{2\theta} \lambda \otimes \delta) \rightarrow X_{s-2\theta,\theta} \approx L^2(\mathbb{R}_r \times \mathbb{Z}_k, \langle k \rangle^{2(s-2\theta)}\langle \tau - \phi(k)\rangle^{2\theta} \lambda \otimes \delta)
\]

satisfies \(\|T\|_{X_{s-2\theta,\theta}} \leq C_{s,\varphi}^\theta C_{s,\varphi,\varphi,\mu}^\theta\|v\|_{s,\theta} \leq C_{\alpha,\varphi,\varphi,\mu}\|v\|_{s,\theta}\), with \(0 < \theta < 1\). Thus, we have a loss of regularity in the spatial variable.

**Case 4.** \(-1 < b < 0\): In this case we use duality,

\[
\|\varphi(x)v\|_{X_{s-2,|b|}} = \sup_{u \in X_{s-2-b,0}} \left| \int_{\mathbb{T}} \int_{\mathbb{R}} u \cdot \varphi(x)v \, dt \, dx \right|.
\]

This completes the proof of the theorem. 

4.2. **Propagation of Compactness and Regularity.** Here we show some properties of propagation of compactness and regularity for the linear operator

\[
L := \partial_t - \alpha \mathcal{H}\partial^2_x - \partial^3_x + 2\mu \partial_x,
\]

associated to the Benjamin equation. These propagation properties are fundamental to study global stabilizability. We begin establishing two technical lemmas.

**Lemma 4.4.** Let \(\alpha > 0\) and \(\mu \in \mathbb{R}\). The operator

\[
L : D(L) \subseteq L^2(\mathbb{T} \times (0,T)) \rightarrow L^2(\mathbb{T} \times (0,T)),
\]

defined by \((4.12)\) is skew-adjoint on \(L^2(\mathbb{T} \times (0,T))\), where

\[
D(L) = \left\{ v \in D'(\mathbb{T} \times (0,T)) : v(x,\cdot) \in H^1(0,T), \text{ and } v(\cdot, t) \in H^3_{\mu}(\mathbb{T}) \right\}.
\]

**Proof.** The proof is similar to that of Proposition 3.3 in [22], so we omit the details. 

\(\square\)
Lemma 4.5. Let \( r \in \mathbb{R} \). The Hilbert transform \( \mathcal{H} \) commutes with the operator \( D^r \) (see (3.11)) in \( L^2(\mathbb{T}) \). Furthermore, \( \mathcal{H} = -D^{-1}\partial_x \) in \( L^2(\mathbb{T}) \). Also, the operator \( \partial_x^r \) commutes with the operators \( D^r \) and \( \mathcal{H} \) in \( L^2(\mathbb{T}) \).

\[
\frac{\partial v_n - \alpha \mathcal{H}\partial_x^2 v_n - \partial_x^3 v_n + 2\mu \partial_x^2 v_n = f_n,}{(4.14)}
\]

for \( n = 1, 2, 3, \ldots \). Suppose that there exists \( C > 0 \) such that

\[
\|v_n\|_{X^r_{b,b}} \leq C,
\]

for all \( n \geq 1 \), and that

\[
\|v_n\|_{X^r_{2+2b,-b}} + \|f_n\|_{X^r_{2+2b,-b}} + \|v_n\|_{X^r_{-1+2b',-b'}} \rightarrow 0,
\]

as \( n \rightarrow \infty \). Additionally, assume that for some nonempty open set \( \omega \subset \mathbb{T} \)

\[
v_n \rightarrow 0, \quad \text{strongly in } L^2((0,T);L^2(\omega)).
\]

Then, there exists a subsequence \( \{v_{n_j}\}_{j \in \mathbb{N}} \) of \( \{v_n\}_{n \in \mathbb{N}} \) such that

\[
v_{n_j} \rightarrow 0, \quad \text{strongly in } L^2_{loc}((0,T);L^2(\mathbb{T})), \quad \text{as } j \rightarrow \infty.
\]

\[
\|v_n\|_{L^2(K, L^2(\mathbb{T})))} \leq \int_0^T \psi(t) \|v_n\|_{L^2(\mathbb{T})}^2 \, dt = \int_0^T \psi(t) \langle v_n, v_n \rangle_{L^2(\mathbb{T})} \, dt.
\]

Since \( \mathbb{T} \) is compact there exists a finite set of points, say \( x_i^j \in \mathbb{T}, i = 1, \ldots, N \), such that we can construct a partition of the unity on \( \mathbb{T} \) involving functions of the form \( \chi_i(\cdot-x_i^0) \) with \( \chi_i(\cdot) \in C^\infty_c(\omega) \). Specifically, there exists \( N \in \mathbb{N} \) such that

\[
\begin{cases}
0 \leq \chi_i(x-x_i^0) \leq 1, & \text{for all } x \in \mathbb{T} \text{ and } i = 1, 2, \ldots, N \\
\chi_i(\cdot) \in C^\infty_c(\omega) & \text{for } i = 1, \ldots, N \\
\sum_{i=1}^N \chi_i(\cdot-x_i^0) = 1 & \text{on } \mathbb{T}.
\end{cases}
\]

Therefore,

\[
\|v_n\|_{L^2(K, L^2(\mathbb{T})))} \leq \int_0^T \left( \psi(t) \left( \sum_{i=1}^N \chi_i(x-x_i^0) \right) v_n, v_n \right)_{L^2(\mathbb{T})} \, dt = \sum_{i=1}^N \left( \psi(t) \chi_i(x-x_i^0) v_n, v_n \right)_{L^2(\mathbb{T})}.
\]

Thus, it is sufficient to show that for any \( \chi(\cdot) \in C^\infty_c(\omega) \) and any \( x_0 \in \mathbb{T} \) there exists a subsequence \( \{v_{n_j}\}_{j \in \mathbb{N}} \) such that

\[
(\psi(t) \chi(x-x_0) v_{n_j}, v_{n_j})_{L^2(\mathbb{T} \times (0,T))} \rightarrow 0, \quad \text{as } j \rightarrow \infty.
\]

For this, consider \( \phi(x) = \chi(x) - \chi(x-x_0) \), where \( \chi \in C^\infty_c(\omega) \) and \( x_0 \in \mathbb{T} \). From Lemma A.1 there exists \( \varphi \in C^\infty(\mathbb{T}) \) such that \( \partial_x \varphi(x) = \chi(x) - \chi(x-x_0) \), for all \( x \in \mathbb{T} \). Consequently,

\[
(\psi(t)\chi(x-x_0) v_n, v_n)_{L^2(\mathbb{T} \times (0,T))} = (\psi(t)\chi(x) v_n, v_n)_{L^2(\mathbb{T} \times (0,T))} - (\psi(t)\partial_x \varphi(x) v_n, v_n)_{L^2(\mathbb{T} \times (0,T))}.
\]
From (4.17), we have that
\[ \left| \psi(t) \chi(x) v_n, v_n \right|_{L^2(\mathbb{T} \times (0, T))} \leq \|\psi\|_{C^\infty} \|\chi\|_{C^\infty} \|v_n\|_{L^2((0, T); L^2(\omega))} \rightarrow 0, \]
as \( n \rightarrow \infty \). So, we only need to show that there exists a subsequence \( \{v_{n_j}\}_{j \in \mathbb{N}} \) such that
\[ \left| \psi(t) \partial_x \varphi(x) v_{n_j}, v_{n_j} \right|_{L^2(\mathbb{T} \times (0, T))} \rightarrow 0, \text{ as } j \rightarrow \infty. \] (4.20)

In what follows, we show (4.20). Taking consideration of definition of \( D^r \) in (3.1) and passing to the frequency space, it is easy to verify that
\[ (\psi(t) \partial_x \varphi(x) v_n, v_n)_{L^2(\mathbb{T} \times (0, T))} = (\psi(t) \partial_x \varphi(x)(-\partial_x^2)D^{-2} v_n, v_n)_{L^2(\mathbb{T} \times (0, T))} \]
\[ + (\psi(t) \partial_x \varphi(x) \overline{v}_n(0, t), v_n)_{L^2(\mathbb{T} \times (0, T))}. \] (4.21)

First, we prove
\[ \lim_{n \rightarrow \infty} \left| (\psi(t) \partial_x \varphi(x)(-\partial_x^2)D^{-2} v_n, v_n)_{L^2(\mathbb{T} \times (0, T))} \right| = 0. \] (4.22)

In fact, from (4.12) and (4.14), we have \( L \overline{v}_n = f_n \), for \( n = 1, 2, 3, \cdots \). Set \( B := \varphi(x)D^{-2} \) and \( A := \psi(t)B \). For \( \epsilon > 0 \), let
\[ A_\epsilon := \psi(t)B_\epsilon, \] (4.23)
be a regularization of \( A \), where
\[ B_\epsilon := B e^{\epsilon \partial_x^2}, \text{ with } e^{\epsilon \partial_x^2} \text{ defined by } e^{\epsilon \partial_x^2}v(\cdot) = \left( e^{-\epsilon k^2} \tilde{v}(k) \right)^\vee (\cdot). \] (4.24)

Define \( \alpha_{n, \epsilon} := ([A_\epsilon, L], v_n, v_n)_{L^2(\mathbb{T} \times (0, T))} \). Using Lemma 4.4 we obtain that
\[ \alpha_{n, \epsilon} := \left( f_n, A_\epsilon^* v_n \right)_{L^2(\mathbb{T} \times (0, T))} + (A_\epsilon v_n, f_n)_{L^2(\mathbb{T} \times (0, T))}. \] (4.25)

We infer from (4.23), (4.24), (4.1) and (4.6) that for any \( r \in \mathbb{R} \), and \( 0 \leq b \leq 1 \), there exists a positive constant \( C \) (independent of \( T \), if \( T \leq 1 \)) such that
\[ \|A_\epsilon^* v\|_{X_{r-2+2b, b}^T} \leq C \|v\|_{X_{r, b}^T}. \] (4.26)

Using \( 0 < b \leq 1 \), we get the immersion \( X_{0, b}^T \hookrightarrow X_{4-4b, b}^T \). From (4.20), (4.13) and (4.16), we obtain
\[ \left| (f_n, A_\epsilon^* v_n)_{L^2(\mathbb{T} \times (0, T))} \right| \leq \|f_n\|_{X_{r-2+2b, -b}^T} \|A_\epsilon^* v_n\|_{X_{r-2b, b}^T}, \]
\[ \leq C \|f_n\|_{X_{r-2+2b, -b}^T} \|v_n\|_{X_{4-4b, b}^T}, \]
\[ \leq C \|f_n\|_{X_{r-2+2b, -b}^T} \|v_n\|_{X_{0, b}^T}, \]
\[ \leq C \|f_n\|_{X_{r-2+2b, -b}^T} \rightarrow 0, \text{ as } n \rightarrow \infty. \] (4.27)

Note that, since the positive constant \( C \) in (4.27) does not depend on \( \epsilon \), one can get
\[ \lim_{n \rightarrow \infty} \sup_{0 < \epsilon \leq 1} \left| (f_n, A_\epsilon^* v_n)_{L^2(\mathbb{T} \times (0, T))} \right| = 0. \] (4.28)

Using a similar procedure, we obtain
\[ \lim_{n \rightarrow \infty} \sup_{0 < \epsilon \leq 1} \left| (A_\epsilon v_n, f_n)_{L^2(\mathbb{T} \times (0, T))} \right| = 0. \] (4.29)

Therefore, (4.26), (4.28) and (4.29) imply that
\[ \lim_{n \rightarrow \infty} \sup_{0 < \epsilon \leq 1} |\alpha_{n, \epsilon}| = 0. \] (4.30)
On the other hand, using that the operator $B_{\epsilon}$ commutes with derivatives in time, we obtain

$$[A_{\epsilon}, L]v_n = -\psi'(t)B_{\epsilon}v_n + [A_{\epsilon}, -\alpha H\partial_x^2]v_n + [A_{\epsilon}, -\partial_x^3 + 2\mu \partial_x]v_n.$$  

Therefore,

$$\alpha_{n, \epsilon} = - (\psi'(t)B_{\epsilon}v_n, v_n)_{L^2(\mathbb{T} \times (0,T))} + (|A_{\epsilon}, -\alpha H\partial_x^2|v_n, v_n)_{L^2(\mathbb{T} \times (0,T))} + (|A_{\epsilon}, -\partial_x^3 + 2\mu \partial_x|v_n, v_n)_{L^2(\mathbb{T} \times (0,T))}.$$  

We infer from (4.24), (4.1) and (4.6) that for any $B$, on the other hand, using that the operator $H$ commutes with derivatives in time, we obtain

$$\left|\psi'(t)B_{\epsilon}v_n, v_n\right|_{L^2(\mathbb{T} \times (0,T))} \leq \|\psi'(t)B_{\epsilon}v_n\|_{X_{\alpha, \epsilon}^T, b} \|v_n\|_{X_{\alpha, \epsilon}^T, b}.$$  

Therefore,

$$\lim_{n \to \infty} \sup_{0 < \epsilon \leq 1} \left|\psi'(t)B_{\epsilon}v_n, v_n\right|_{L^2(\mathbb{T} \times (0,T))} = 0.$$  

Also, observe that

$$[A_{\epsilon}, -\alpha H\partial_x^2]v_n = -\alpha \psi(t) \varphi D^{-2}e^{\epsilon^2}H \partial_x^2 v_n + \alpha \psi(t) H\partial_x^2 \left(\varphi D^{-2}e^{\epsilon^2}v_n\right).$$  

From the Leibniz’s rule for derivatives, we obtain

$$\partial_x^2 \left(\varphi D^{-2}e^{\epsilon^2}v_n\right) = \varphi \partial_x^2 D^{-2}e^{\epsilon^2}v_n + 2\partial_x \varphi \partial_x D^{-2}e^{\epsilon^2}v_n + \partial_x^2 \varphi D^{-2}e^{\epsilon^2}v_n.$$  

Substituting (4.35) into (4.34) and using Lemma 4.5 we obtain

$$[A_{\epsilon}, -\alpha H\partial_x^2]v_n = \alpha \psi(t) \left\{-\varphi H D^{-2}\partial_x^2 e^{\epsilon^2}v_n + H \left(\varphi D^{-2}\partial_x^2 e^{\epsilon^2}v_n\right)\right\} + 2\alpha \psi(t) H \left(\partial_x \varphi \partial_x D^{-2}e^{\epsilon^2}v_n\right) + \alpha \psi(t) H \left(\partial_x^2 \varphi D^{-2}e^{\epsilon^2}v_n\right).$$  

Lemma 4.5 implies that,

$$-\varphi HD^{-2}\partial_x^2 e^{\epsilon^2}v_n + H \left(\varphi D^{-2}\partial_x^2 e^{\epsilon^2}v_n\right) = \varphi D^{-1}\partial_x D^{-2}\partial_x^2 e^{\epsilon^2}v_n - D^{-1}\partial_x \left(\varphi D^{-2}\partial_x^2 e^{\epsilon^2}v_n\right) = \varphi D^{-1}\partial_x D^{-2}\partial_x^2 e^{\epsilon^2}v_n - D^{-1} \left(\varphi \partial_x D^{-2}\partial_x^2 e^{\epsilon^2}v_n\right) = -D^{-1} \left(\partial_x \varphi D^{-2}\partial_x^2 e^{\epsilon^2}v_n\right) = -D^{-1} \left(\varphi \partial_x D^{-2}\partial_x^2 e^{\epsilon^2}v_n\right).$$  

Substituting (4.37) into (4.36), we have

$$[A_{\epsilon}, -\alpha H\partial_x^2]v_n = \alpha \psi(t) \left\{-[D^{-1}, \varphi] D^{-2}\partial_x^2 e^{\epsilon^2}v_n - D^{-1} \left(\partial_x \varphi D^{-2}\partial_x^2 e^{\epsilon^2}v_n\right)\right\} + 2\alpha \psi(t) H \left(\partial_x \varphi \partial_x D^{-2}e^{\epsilon^2}v_n\right) + \alpha \psi(t) H \left(\partial_x^2 \varphi D^{-2}e^{\epsilon^2}v_n\right).$$  

(4.38)
Therefore,

\[
([A_\varepsilon, -\alpha H \partial^2_x] v_n, v_n)_{L^2(T \times (0, T))} = - \left( \alpha \psi(t) [D^{-1}, \varphi] \partial_\varepsilon D^{-2} \partial_x^2 e^{\alpha \partial^2_x} v_n, v_n \right)_{L^2(T \times (0, T))} \\
- \left( \alpha \psi(t) D^{-1} \left( \partial_\varepsilon \varphi \left[ D^{-2} \partial_x^2 e^{\alpha \partial^2_x} v_n \right] \right), v_n \right)_{L^2(T \times (0, T))} \\
+ 2a \left( \psi(t) H \left( \partial_\varepsilon \varphi \left[ D^{-2} \partial_x^2 e^{\alpha \partial^2_x} v_n \right] \right), v_n \right)_{L^2(T \times (0, T))} \\
+ \alpha \left( \psi(t) H \left( \partial_x^2 e^{\alpha \partial^2_x} v_n \right), v_n \right)_{L^2(T \times (0, T))}. \tag{4.39}
\]

Applying Cauchy-Schwartz, \((4.1), (4.15), \text{ Lemmas A.2, 3.4 and using that} 0 \leq b' \leq b \leq 1 \) be given (with \( b > 0 \)), we obtain that there exists a positive constant \( C = C_T \) \((C, \text{ if } T \leq 1)\) which does not depend on \( \varepsilon \) such that

\[
\left| \left( \alpha \psi(t) [D^{-1}, \varphi] \partial_\varepsilon D^{-2} \partial_x^2 e^{\alpha \partial^2_x} v_n, v_n \right)_{L^2(T \times (0, T))} \right| \leq C \left\| [D^{-1}, \varphi] \partial_\varepsilon D^{-2} \partial_x^2 e^{\alpha \partial^2_x} v_n \right\|_{L^2(T \times (0, T))} \left\| v_n \right\|_{X^T_{0,b},T} \]

\[
\leq C \left( \sum_{k=-\infty}^{\infty} \int_0^T \left| \left( \partial_\varepsilon D^{-2} \partial_x^2 e^{\alpha \partial^2_x} v_n \right)^\top (k, \tau) \right|^2 d\tau \right)^{1/2} \]

\[
\leq C \left\| \partial_\varepsilon D^{-2} \partial_x^2 e^{\alpha \partial^2_x} v_n \right\|_{X^T_{-2,-b'}} \left\| \partial_\varepsilon D^{-2} \partial_x^2 e^{\alpha \partial^2_x} v_n \right\|_{X^T_{-1,b'}} \]

\[
\leq C \left\| v_n \right\|_{X^T_{-1,b'}} \rightarrow 0 \text{ as } n \rightarrow \infty,
\tag{4.40}
\]

where in the last two inequalities we use \((4.15), (4.16)\) and the immersions \( X^T_{0,b} \hookrightarrow X^T_{-1,b'} \), and \( X^T_{-1+2b' , -b'} \hookrightarrow X^T_{-1,-b'} \). Note that the loss of regularity in \((4.40)\) is too large if one uses the estimates with the same \( b \). Therefore, we have to use the index \( b' \) instead. Consequently,

\[
\lim_{n \to \infty} \sup_{0 < t \leq \infty} \left| \left( \alpha \psi(t) [D^{-1}, \varphi] \partial_\varepsilon D^{-2} \partial_x^2 e^{\alpha \partial^2_x} v_n, v_n \right)_{L^2(T \times (0, T))} \right| = 0. \tag{4.41}
\]

From \((4.1), (4.16), (4.15), (4.16)\), and Lemma \(3.4\) we have that there exists a positive constant \( C = C_T \) \((C, \text{ if } T \leq 1)\) which does not depend on \( \varepsilon \) such that

\[
\left| \left( \alpha \psi(t) D^{-1} \left( \partial_\varepsilon \varphi \left[ D^{-2} \partial_x^2 e^{\alpha \partial^2_x} v_n \right] \right), v_n \right)_{L^2(T \times (0, T))} \right| \leq C \left\| D^{-1} \left( \partial_\varepsilon \varphi \left[ D^{-2} \partial_x^2 e^{\alpha \partial^2_x} v_n \right] \right) \right\|_{X^T_{-2+b',-b'}} \left\| v_n \right\|_{X^T_{-1+b',-b'}} \]

\[
\leq C \left\| D^{-2} \partial_x^2 e^{\alpha \partial^2_x} v_n \right\|_{X^T_{-2+b',-b'}} \left\| v_n \right\|_{X^T_{-1+b',-b'}} \]

\[
\leq C \left\| v_n \right\|_{X^T_{0,b'}} \left\| v_n \right\|_{X^T_{-1+b',-b'}} \]

\[
\leq C \left\| v_n \right\|_{X^T_{-2+b',-b'}} \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

Therefore,

\[
\lim_{n \to \infty} \sup_{0 < t \leq \infty} \left| \left( \alpha \psi(t) D^{-1} \left( \partial_\varepsilon \varphi \left[ D^{-2} \partial_x^2 e^{\alpha \partial^2_x} v_n \right] \right), v_n \right)_{L^2(T \times (0, T))} \right| = 0. \tag{4.42}
\]
Similarly, using that the Hilbert transform $H$ is an isometry in $L^2_p(T)$, we have
\[
\left| \left( 2\alpha \psi(t) H \left( \partial_x \varphi \partial_x D^{-2} e^{t\partial_x^2} v_n \right), v_n \right)_{L^2(T \times (0,T))} \right| \leq 2\alpha \left\| \psi(t) H \left( \partial_x \varphi \partial_x D^{-2} e^{t\partial_x^2} v_n \right) \right\|_{X_{0,\nu}^T} \left\| v_n \right\|_{X_{0,\nu}^T} \\
\leq C \left\| v_n \right\|_{X_{T+2\nu, -\nu}^T} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty,
\]
where the positive constant $C = C_T$ ($C$, if $T \leq 1$) does not depend on $\epsilon$. Hence,
\[
\lim_{n \rightarrow \infty} \sup_{0 < \epsilon \leq 1} \left| \left( 2\alpha \psi(t) H \left( \partial_x \varphi \partial_x D^{-2} e^{t\partial_x^2} v_n \right), v_n \right)_{L^2(T \times (0,T))} \right| = 0. \tag{4.43}
\]
With similar arguments, we get
\[
\lim_{n \rightarrow \infty} \sup_{0 < \epsilon \leq 1} \left| \left( \alpha \psi(t) H \left( \partial_x^2 \varphi D^{-2} e^{t\partial_x^2} v_n \right), v_n \right)_{L^2(T \times (0,T))} \right| = 0. \tag{4.44}
\]
From (4.39) and (4.41)–(4.44), we obtain that
\[
\lim_{n \rightarrow \infty} \sup_{0 < \epsilon \leq 1} \left| \left( [A_{\epsilon}, -\alpha H \partial_x^2] v_n, v_n \right)_{L^2(T \times (0,T))} \right| = 0. \tag{4.45}
\]
Therefore, (4.30), (4.31), (4.33) and (4.45), imply that
\[
\lim_{n \rightarrow \infty} \sup_{0 < \epsilon \leq 1} \left| \left( [A_{\epsilon}, -\partial_x^3 + 2\mu \partial_x] v_n, v_n \right)_{L^2(T \times (0,T))} \right| = 0.
\]
In particular,
\[
\lim_{n \rightarrow \infty} \left( [A, -\partial_x^3 + 2\mu \partial_x] v_n, v_n \right)_{L^2(T \times (0,T))} = 0. \tag{4.46}
\]
Using the Leibniz’s rule for derivatives, we note that
\[
\left( [A, -\partial_x^3 + 2\mu \partial_x] v_n, v_n \right)_{L^2(T \times (0,T))} = \left( 3\psi(t) \partial_x \varphi \partial_x D^{-2} v_n, v_n \right)_{L^2(T \times (0,T))} \tag{4.47}
\]
\[
\quad + \left( 3\psi(t) \partial_x^2 \varphi \partial_x D^{-2} v_n, v_n \right)_{L^2(T \times (0,T))} \\
\quad - \left( \psi(t) \left( -\partial_x^3 + 2\mu \partial_x \varphi \right) D^{-2} v_n, v_n \right)_{L^2(T \times (0,T))}.
\]
Relations (4.46), (4.47) and estimates similar to those in the proof of Proposition 3.5 in [24] shows (4.22).

Second, we prove that there exists a subsequence $\{v_{n_j}\}_{j \in \mathbb{N}}$ of $\{v_n\}_{n \in \mathbb{N}}$ such that
\[
\lim_{j \rightarrow \infty} \left| \left( \psi(t) \partial_x \varphi \widehat{v}_{n_j}(0,t), v_{n_j} \right)_{L^2(T \times (0,T))} \right| = 0. \tag{4.48}
\]
Indeed, observe that
\[
\|\widehat{v}_{n_j}(0,t)\|_{L^2(0,T)} \leq C \left( \int_0^T \int_T \|v_{n_j}(x,t)\|^2 dx \ dt \right)^{\frac{1}{2}} \leq C \|v_{n_j}\|_{X_{0,\nu}^T} \leq C \|v_n\|_{X_{0,\nu}^T} \leq C.
\]
Since $L^2(0,T)$ is a reflexive Banach space, then from weak compactness, there exists a $w \in L^2(0,T)$ and a subsequence $\{v_{n_{j_k}}\}_{j \in \mathbb{N}}$ of $\{v_n\}_{n \in \mathbb{N}}$ such that
\[
\widehat{v}_{n_{j_k}}(0,t) \rightharpoonup w. \tag{4.49}
\]
On the other hand, considering the function
\[
g_{n_j}(t) := \int_T \psi(t) \partial_x \varphi(x) v_{n_j}(x,t) dx, \quad \text{for} \ t \in (0,T),
\]
we note that
\[
\|g_{n_j}(t)\|_{(L^2(0,T))'} := \sup_{\|g\|_{L^2(0,T)} \leq 1} \left| \int_0^T g(t) \int_T \psi(t) \partial_x \varphi(x) v_{n_j}(x,t) \, dx \, dt \right|
\]
\[
\leq C \|g\|_{L^2(0,T)} \|\psi(t) \partial_x \varphi(x)\|_{X^{s-2b,\beta}_{L^2}} \|v_{n_j}(x,t)\|_{X^{s+2b,\beta}_{L^2}} \to 0,
\]
as \( j \to \infty \). Thus, \( g_{n_j} \to 0 \) (strongly) in \((L^2(0,T))'\). Therefore, (4.49) implies
\[
\left(\psi(t) \partial_x \varphi(x) \right. \left. v_{n_j}(0,t), v_{n_j}\right)_{L^2(\mathbb{T} \times (0,T))} = \int_0^T \int \psi(t) g_{n_j}(t) \, dt
\]
\[
= (\hat{v}_{n_j}(0,t), g_{n_j}(t)) \to \langle w, 0 \rangle = 0,
\]
as \( j \to \infty \), which proves (4.48). From (4.21)-(4.22), and (4.48), we obtain (4.20). This completes the proof of the proposition.

\[\square\]

Remark 4.7. If we assume additionally that \( [v_n] = 0 \) for all \( n \in \mathbb{N} \), then the result of in Proposition is valid for the original sequence \( \{v_n\}_{n \in \mathbb{N}} \).

Now, we study the propagation of regularity for the operator \( L \) defined in (3.12).

Proposition 4.8 (Propagation of Regularity). Let \( T > 0 \), \( 0 \leq b \leq 1 \), \( r \geq 0 \) and \( f \in X^{r,-b}_{r,b} \) be given. Let \( v \in X^{r,b}_{r,b} \) solves
\[
L v := \partial_t v - \alpha \partial_x^2 v - \partial_t^2 v + 2\mu \partial_x v = f.
\]

If there exists a nonempty open set \( \omega \) of \( \mathbb{T} \) such that
\[
v \in L^2_{loc}((0,T); H^{r+\rho}(\omega)),
\]
for some \( \rho \) with
\[
0 < \rho \leq \min \left\{ 1 - b, \frac{1}{2} \right\},
\]
then \( v \in L^2_{loc}((0,T); H^{r+\rho}(\mathbb{T})) \).

Proof. Let \( s = r + \rho \). Let \( \Omega \) be a compact subset of the interval \((0,T)\), and \( \psi(t) \in C_c^\infty(0,T) \), such that \( 0 \leq \psi(t) \leq 1 \) and \( \psi(t) = 1 \) in \( \Omega \). Observe that
\[
\|v\|_{L^2(\Omega,H^s(\mathbb{T}))}^2 \leq \int_0^T \psi(t)\|v\|_{H^s(\mathbb{T})}^2 \, dt
\]
\[
\leq c_s \left( \|v\|_{L^2(\mathbb{T} \times (0,T))}^2 + \sum_{k=\infty}^{\infty} \sum_{k \neq 0} |k|^{2s} \psi(t)|\hat{v}(k,t)|^2 \, dt \right)
\]
\[
= c_s \left( \|v\|_{L^2(\mathbb{T} \times (0,T))}^2 + \langle \psi(t)D^{2s-2}\partial_x^2 v, v\rangle_{L^2(\mathbb{T} \times (0,T))} \right).
\]
where the operator \( D \) is defined in (3.1). Thus, we only need to show that there exists a positive constant \( C \) such that
\[
\langle \psi(t)D^{2s-2}\partial_x^2 v, v\rangle_{L^2(\mathbb{T} \times (0,T))} \leq C.
\]
Note that, with a similar argument as in the proof of Theorem 1.6 (see (1.19)), there exist $x_0^i \in \mathbb{T}, i = 1, \ldots, N$, such that we can construct a partition of unity on $\mathbb{T}$ involving functions of the form $\chi_t^{2}(-x_0^i)$ with $\chi_t^{2}(\cdot) \in C_c^\infty(\omega)$. Therefore,

$$\left| \left( \psi(t)D^{2s-2}\partial_x^2, \right)_{L^2(\mathbb{T} \times (0,T))} \right| \leq \sum_{i=1}^{N} \left| \left( \psi(t)D^{2s-2}\chi_t^{2}(x-x_0^i)\partial_x^2, \right)_{L^2(\mathbb{T} \times (0,T))} \right|.$$ 

So, it is sufficient to prove that for any $\chi_t^{2}(\cdot) \in C_c^\infty(\omega)$ and any $x_0 \in \mathbb{T}$, there exists a positive constant $C$ such that

$$\left| \left( \psi(t)D^{2s-2}\chi_t^{2}(x-x_0)\partial_x^2, \right)_{L^2(\mathbb{T} \times (0,T))} \right| \leq C. \quad (4.54)$$

In fact, from Lemma A.1 there exists $\varphi \in C_c^\infty(\mathbb{T})$ such that $\partial_x \varphi(x) = \chi_t^{2}(x) - \chi_t^{2}(x-x_0)$ for all $x \in \mathbb{T}$. Consequently,

$$\left| \left( \psi(t)D^{2s-2}\chi_t^{2}(x-x_0)\partial_x^2, \right)_{L^2(\mathbb{T} \times (0,T))} \right| \leq \left| \left( \psi(t)D^{2s-2}\chi_t^{2}(x)\partial_x^2, \right)_{L^2(\mathbb{T} \times (0,T))} \right| + \left| \left( \psi(t)D^{2s-2}\partial_x \varphi(x)\partial_x^2, \right)_{L^2(\mathbb{T} \times (0,T))} \right|. \quad (4.55)$$

Now, we move to bound the RHS of (4.55). Define $v_n := e^{\pm \partial_x^2}v = E_n v = \left( e^{\pm \frac{i}{2}k^2 \tilde{v}(k,t)} \right)^\vee$ and $f_n := E_n f = E_n L v$, for $n = 1, 2, 3, \ldots$. Passing to the frequency space, it is easy to verify that $E_n$ commutes with $L$, i.e.,

$$f_n := E_n f = E_n L v = LE_n v = Lv_n.$$ 

From hypothesis and the definition of $E_n$, we obtain that there exists $C > 0$ independent on $n$ such that

$$\|v_n\|_{X_r^{r,b}} \leq C, \quad \text{and} \quad \|f_n\|_{X_r^{r,b}} \leq C, \quad \text{for all } n \geq 1. \quad (4.56)$$

Set $B = D^{2s-2} \varphi$, and $A = \psi(t)B$. We infer from (4.1) and (4.6) that for any $r \in \mathbb{R}$, and $0 \leq b \leq 1$, there exists a positive constant $C$ (independent of $T$, if $T \leq 1$) such that

$$\|Av\|_{X_r^{r-2|b|,2r+2,b}} \leq C \|v\|_{X_r^{r,b}}. \quad (4.57)$$

With similar calculations as in the proof of the Proposition 4.6 (see (3.31)), we obtain

$$\langle f_n, A^*v_n \rangle_{L^2(\mathbb{T} \times (0,T))} + \langle Av_n, f_n \rangle_{L^2(\mathbb{T} \times (0,T))} = -\langle \psi'(t)Bv_n, v_n \rangle_{L^2(\mathbb{T} \times (0,T))} + \langle \alpha\mathcal{H}\partial_x^2, A \rangle v_n, v_n \rangle_{L^2(\mathbb{T} \times (0,T))} \quad (4.58)$$

Using that $\rho \leq 1 - b$, (4.56) and (4.57), we get that there exists $C > 0$ independent on $n$ such that

$$\left| \left( \psi(t)D^{2s-2}\partial_x^2, \right)_{L^2(\mathbb{T} \times (0,T))} \right| \leq \left| \psi'(t)\right| B v_n, v_n \rangle_{L^2(\mathbb{T} \times (0,T))} \leq C \|v_n\|_{X_r^{r,b}} \leq C, \quad (4.59)$$

$$\left| \left( \psi(t)D^{2s-2}\partial_x^2, \right)_{L^2(\mathbb{T} \times (0,T))} \right| \leq \left| \psi'(t)\right| B v_n, v_n \rangle_{L^2(\mathbb{T} \times (0,T))} \leq C \|v_n\|_{X_r^{r,b}} \leq C, \quad (4.60)$$

and

$$\left| \left( \psi(t)D^{2s-2}\partial_x^2, \right)_{L^2(\mathbb{T} \times (0,T))} \right| \leq \left| \psi'(t)\right| B v_n, v_n \rangle_{L^2(\mathbb{T} \times (0,T))} \leq C \|v_n\|_{X_r^{r,b}} \leq C. \quad (4.61)$$
Also, using the Leibniz’s rule for derivatives and Lemma \ref{lem:4.3}, a simple calculation yields
\begin{equation}
\mathcal{H}\partial^2_x Av_n - A\mathcal{H}\partial^2_x v_n = -\psi(t)D^{2s-2}([D^{-1}, \varphi]\partial^3_x v_n) - \psi(t)D^{2s-3}\left(\partial_x \varphi \partial^2_x v_n\right) + 2\mathcal{H}\psi(t)D^{2s-2}\left(\partial_x \varphi \partial_x v_n\right) + \mathcal{H}\psi(t)D^{2s-2}\left(\partial^2_x \varphi v_n\right).
\end{equation} \hspace{1cm} (4.62)

Now, using \ref{lem:4.3} we obtain
\begin{equation}
\left(\alpha\mathcal{H}\partial^2_x, A\right)_{v_n, v_n})_{L^2(T(0,T))} = \alpha \left(\mathcal{H}\partial^2_x Av_n - A\mathcal{H}\partial^2_x v_n, v_n\right)_{L^2(T(0,T))} = -\alpha \left(\psi(t)D^{2s-2}([D^{-1}, \varphi]\partial^3_x v_n), v_n\right)_{L^2(T(0,T))} - \alpha \left(\psi(t)D^{2s-3}\left(\partial_x \varphi \partial^2_x v_n\right), v_n\right)_{L^2(T(0,T))} + 2\alpha \left(\mathcal{H}\psi(t)D^{2s-2}\left(\partial_x \varphi \partial_x v_n\right), v_n\right)_{L^2(T(0,T))} + \alpha \left(\mathcal{H}\psi(t)D^{2s-2}\left(\partial^2_x \varphi v_n\right), v_n\right)_{L^2(T(0,T))}.
\end{equation} \hspace{1cm} (4.63)

Using $\rho \leq \frac{1}{2}$, \ref{lem:4.3}, \ref{lem:4.6}, Lemma \ref{lem:A.2} and that $\mathcal{H}$ is an isometry in $H^s_\rho(-\ell, \ell)$, we can obtain $C > 0$ independent on $n$, such that
\begin{equation}
\alpha \left|\left(\psi(t)D^{2s-2}([D^{-1}, \varphi]\partial^3_x v_n), v_n\right)_{L^2(T(0,T))}\right| \leq \alpha \left\|\psi(t)D^{2s-2}([D^{-1}, \varphi]\partial^3_x v_n)\right\|_{X^T_{r,0}} \left\|v_n\right\|_{X^T_{r,0}} \leq \frac{\alpha C \left\|\partial_x \varphi \partial^2_x v_n\right\|_{L^2(0,T):H^{-r+2s-2}(\mathbb{T})} \left\|v_n\right\|_{X^T_{r,b}} \leq \frac{C \left\|v_n\right\|_{X^T_{r+2s-1,0}} \left\|v_n\right\|_{X^T_{r,b}} \leq C, \hspace{1cm} (4.64)
\end{equation}

\begin{equation}
\alpha \left|\left(\psi(t)D^{2s-3}\left(\partial_x \varphi \partial^2_x v_n\right), v_n\right)_{L^2(T(0,T))}\right| \leq \alpha \left\|\psi(t)D^{2s-3}\left(\partial_x \varphi \partial^2_x v_n\right)\right\|_{L^2(0,T):H^{-r}(\mathbb{T})} \left\|v_n\right\|_{L^2(0,T):H^r(\mathbb{T})} \leq \frac{C \left\|\partial_x \varphi \partial^2_x v_n\right\|_{X^T_{r+2s-3,0}} \left\|v_n\right\|_{X^T_{r,0}} \leq \frac{C \left\|v_n\right\|_{X^T_{r+2s-1,0}} \left\|v_n\right\|_{X^T_{r,b}} \leq C \left\|v_n\right\|_{X^T_{r,b}} \leq C, \hspace{1cm} (4.65)
\end{equation}

In similar manner, one can get
\begin{equation}
2\alpha \left|\left(\mathcal{H}\psi(t)D^{2s-2}\left(\partial_x \varphi \partial_x v_n\right), v_n\right)_{L^2(T(0,T))}\right| \leq C \left\|\partial_x \varphi \partial_x v_n\right\|_{X^T_{r+2s-2,0}} \left\|v_n\right\|_{X^T_{r,0}} \leq C, \hspace{1cm} (4.66)
\end{equation}
\begin{equation}
\alpha \left|\left(\mathcal{H}\psi(t)D^{2s-2}\left(\partial^2_x \varphi v_n\right), v_n\right)_{L^2(T(0,T))}\right| \leq C \left\|v_n\right\|_{X^T_{r+2s-2,0}} \left\|v_n\right\|_{X^T_{r,b}} \leq C \left\|v_n\right\|_{X^T_{r,b}} \leq C. \hspace{1cm} (4.67)
\end{equation}

From \ref{4.63}, \ref{4.64}, \ref{4.65}, \ref{4.66}, and \ref{4.67}, we infer that
\begin{equation}
\left|\left(\alpha\mathcal{H}\partial^2_x, A\right)_{v_n, v_n})_{L^2(T(0,T))}\right| \leq C. \hspace{1cm} (4.68)
\end{equation}

It follows from \ref{4.58}, \ref{4.59}, \ref{4.60}, \ref{4.61}, \ref{4.68} that
\begin{equation}
\left|\left(\left[A, -\partial^3_x + 2\partial_x\right] v_n\right)_{L^2(T(0,T))}\right| \leq C, \hspace{1cm} (4.69)
\end{equation}
where $C > 0$ does not depend on $n$. 
Using the Leibniz’s rule, we note that
\[
([A, -\partial^3_x + 2\mu\partial_x]v_n, v_n)_{L^2(\mathbb{T} \times (0, T))} = (3\psi(t)D^{2s-2}\partial_x^2 \partial_x^2 v_n, v_n)_{L^2(\mathbb{T} \times (0, T))} + (3\psi(t)D^{2s-2}\partial_x^3 \partial_x^2 v_n, v_n)_{L^2(\mathbb{T} \times (0, T))} - (\psi(t)D^{2s-2}(-\partial_x^3 \partial_x^2 v_n, v_n)_{L^2(\mathbb{T} \times (0, T))}.
\]

It follows from (4.69), (4.70) and similar estimates as those in the proof of Proposition 3.6 in [24] that there exists \( C > 0 \) independent of \( n \) such that
\[
\left| (\psi(t)D^{2s-2}\partial_x^2 \partial_x^2 v_n, v_n)_{L^2(\mathbb{T} \times (0, T))} \right| \leq C, \quad \text{for any } n \geq 1.
\]

Therefore, letting \( n \to \infty \) we get that the second term on the right side of (4.55) is bounded.

Finally, estimates similar to those in the proof of Proposition 3.6 in [24] shows that there exists \( C > 0 \) independent of \( n \) such that
\[
\left| (\psi(t)D^{2s-2}\partial_x^2 \partial_x^2 v_n, v_n)_{L^2(\mathbb{T} \times (0, T))} \right| \leq C.
\]

Letting \( n \to \infty \) we get that the first term on the right side of inequality (4.55) is bounded. Thus, (4.55), (4.71), and (4.72) imply (4.54) and completes the proof. \( \square \)

**Corollary 4.9.** Let \( \mu \in \mathbb{R}, \alpha > 0 \). Let \( v \in X^T_{0, \omega} \) be a solution of
\[
\partial_t v - \partial_x^3 v - \alpha \mathcal{H}\partial_x^2 v + 2\mu \partial_x v + 2v\partial_x v = 0, \quad \text{on } (0, T),
\]
with \( [u] = 0 \). Assume that \( v \in C^\infty(\omega \times (0, T)), \) where \( \omega \) is a nonempty open set in \( \mathbb{T} \). Then \( v \in C^\infty(\mathbb{T} \times (0, T)) \).

**Proof.** This result is a direct consequence of Corollary 3.10 and an iterated application of Proposition 1.8 with \( f = -2v\partial_x v \) (see Corollary 3.7 in [24]). \( \square \)

### 4.3. Unique Continuation Property

In this subsection we prove the unique continuation property for the Benjamin equation. We start with a result proved in [29].

**Lemma 4.10 ([29] Lemma 2.9).** Let \( s \in \mathbb{R} \) and let \( h(x) = \sum_{k \geq 0} \hat{h}(k)e^{ikx} \) be such that \( h \in H^s(\mathbb{T}) \) and \( h = 0 \) in \((a, b) \subset \mathbb{T} \). Then \( \hat{h} \equiv 0 \).

The following is the main result of this subsection.

**Proposition 4.11.** Let \( \mu \in \mathbb{R}, \alpha > 0, c(t) \in L^2(0, T) \) and \( v \in L^2((0, T); L^2_0(\mathbb{T})) \) be a solution of
\[
\begin{cases}
\partial_t v - \partial_x^3 v - \alpha \mathcal{H}\partial_x^2 v + 2\mu \partial_x v + 2v\partial_x v = 0, & t > 0, \quad \text{on } \mathbb{T} \times (0, T) \\
v(x, t) = c(t), & \text{for almost every } (x, t) \in (a, b) \times (0, T),
\end{cases}
\]
for some \( T > 0 \) and \( 0 \leq a < b \leq 2\pi \). Then \( v(x, t) = 0 \) for almost every \( (x, t) \in \mathbb{T} \times (0, T) \).

**Proof.** Since \( v(x, t) = c(t) \) for a.e \( (x, t) \in (a, b) \times (0, T), \) we have that
\[
\partial_x v(x, t) = \partial_x^2 v(x, t) = (2v\partial_x v)(x, t) = 0 \quad \text{for a.e } (x, t) \in (a, b) \times (0, T).
\]
Thus, from the first relation in (4.74), we infer that \( v \) satisfies
\[
\partial_t v - \alpha \mathcal{H}\partial_x^2 v = 0 \quad \text{in } (a, b) \times (0, T).
\]
Thus, the second relation in (4.74) implies that
\[ \alpha \mathcal{H} \partial_2^3 v = \partial_t v = c'(t) \text{ in } (a, b) \times (0, T). \] (4.76)

Using (4.75) and (4.76) we deduce that, for almost every \( t \in (0, T) \), it holds that
\[
\begin{align*}
\partial_2^3 v(\cdot, t) &\in H^{-3}(\mathbb{T}) \\
\partial_3^3 v(\cdot, t) &\equiv 0, \text{ in } (a, b).
\end{align*}
\] (4.77)

Using (4.75) and (4.76) we deduce that, for almost every \( t \in (0, T) \),
\[ \mathcal{H} \partial_2^3 v(\cdot, t) = \partial_x \mathcal{H} \partial_2^2 v(\cdot, t) = 0, \text{ in } (a, b). \]

Thus, for a.e. \( t \in (0, T) \),
\[ \mathcal{H} \partial_2^2 v(\cdot, t) = 0, \text{ in } (a, b). \]

Pick a time \( t \) as above, and set \( h(x) = \partial_3^3 v(x, t) \), for \( x \in \mathbb{T} \). Decompose \( h \) as
\[ h(x) = \sum_{k \in \mathbb{Z}} \hat{h}(k)e^{ikx}, \]
where the convergence of the series being in \( H^{-3}(\mathbb{T}) \). Observe that
\[
(ih - \mathcal{H} h)(x) = \sum_{k \in \mathbb{Z}} (ih - \mathcal{H} h)^{(k)} e^{ikx} = i \sum_{k \in \mathbb{Z}} (1 + \text{sgn}(k)) \hat{h}(k)e^{ikx} = 2i \sum_{k > 0} \hat{h}(k)e^{ikx}.
\] (4.78)

From (4.77), we have
\[ 0 = ih(x) - \mathcal{H} h(x) = 2i \sum_{k > 0} \hat{h}(k)e^{ikx}, \text{ for all } x \in (a, b). \]

Therefore, \( \sum_{k > 0} \hat{h}(k)e^{ikx} = 0 \), in \( (a, b) \). From Lemma 4.10, we obtain that \( \sum_{k > 0} \hat{h}(k)e^{ikx} = 0 \), in \( \mathbb{T} \). Since \( h \) is real-valued, we also have that \( \hat{h}(-k) = \overline{\hat{h}(k)} \), for all \( k \in \mathbb{Z} \). Thus,
\[ \sum_{k > 0} \hat{h}(-k)e^{-ikx} = 0, \text{ in } \mathbb{T}. \]

Consequently, for a.e. \( t \in (0, T) \), \( \partial_3^3 v(\cdot, t) = 0 \) in \( \mathbb{T} \). Then, for a.e. \( t \in (0, T) \), \( \partial_2^2 v(\cdot, t) = c_1(t) \) in \( \mathbb{T} \). But from (4.75) we obtain that \( c_1(t) = 0 \), for a.e. \( t \in (0, T) \). Thus, for a.e. \( t \in (0, T) \), \( \partial_2^2 v(\cdot, t) = 0 \) in \( \mathbb{T} \). Arguing in a similar way we obtain that for a.e. \( t \in (0, T) \), \( \partial_x v(\cdot, t) = 0 \) in \( \mathbb{T} \). Thus, for a.e. \( t \in (0, T) \),
\[ v(x, t) = c(t) \text{ in } \mathbb{T}. \] (4.79)

Substituting (4.79) in the first relation of (4.74), we obtain that \( c'(t) = 0 \) for a.e. \( t \in (0, T) \).

Therefore, \( v(x, t) = c(t) = cte =: \beta \) a.e. in \( \mathbb{T} \times (0, T) \).

Finally, using \( v \in L^2_0(\mathbb{T}) \), we have \([v] = 0\). Consequently, we obtain \( v(x, t) = \beta = 0 \) a.e. in \( \mathbb{T} \times (0, T) \).

**Corollary 4.12.** Let \( T > 0, \mu \in \mathbb{R}, \) and \( \alpha > 0 \) be given. Assume that \( \omega \) is a nonempty open set in \( \mathbb{T} \) and let \( v \in X^T_0 \) be a solution of
\[
\begin{align*}
\begin{cases}
\partial_t v - \partial_3^3 v - \alpha \mathcal{H} \partial_2^2 v + 2\mu \partial_x v + 2v \partial_x v = 0, & t > 0, \text{ on } \mathbb{T} \times (0, T) \\
v(x, t) = c, & \text{on } \omega \times (0, T),
\end{cases}
\end{align*}
\] (4.80)

where \( c \in \mathbb{R} \) and \([v] = 0\). Then \( v(x, t) = c = 0 \) on \( \mathbb{T} \times (0, T) \).

**Proof.** Using Corollary 4.11, we infer that \( v \in C^{\alpha}(\mathbb{T} \times (0, T)) \). It follows that \( v(x, t) = c \) on \( \mathbb{T} \times (0, T) \) by Proposition 4.11 and from the fact that \([v] = 0\), we obtain \( c = 0 \). \[]
5. **Local control for the Benjamin equation**

In this section we obtain the main results regarding controllability of nonlinear Benjamin equation (1.11) in $H_0^s(\mathbb{T})$ with $s \geq 0$. Observe that the result obtained in this section will imply Theorem 1.3. We proceed as in [40], by rewriting the system (1.11) in its equivalent integral equation form,

$$ u(t) = U_\mu(t)u_0 + \int_0^t U_\mu(t-\tau)G(h)(\tau) \, d\tau - \int_0^t U_\mu(t-\tau)(2u\partial_x u)(\tau) \, d\tau, $$

where $U_\mu(t)$ is the semigroup defined in (2.2). Let us choose $h = \Phi_\mu(u_0, u_1 + \omega(T, u))$, where $\Phi_\mu$ is the bounded linear operator given in the Remark 2.1 and define

$$ w(T, u) := \int_0^T U_\mu(T - \tau)(2u\partial_x u)(\tau) \, d\tau. $$

According to Remark 2.1 the linear system (2.1) is exactly controllable in any positive time $T$. Therefore, for given $u_0$, $u_1 \in H_0^s(\mathbb{T})$, we have

$$ u(t) = U_\mu(t)u_0 + \int_0^t U_\mu(t-\tau)(G(\Phi_\mu(u_0, u_1 + w(T, u))))(\tau) \, d\tau - \int_0^t U_\mu(t-\tau)(2u\partial_x u)(\tau) \, d\tau, $$

and $u(0) = u_0$, $u(T) = u_1$. This suggests that we should consider the map

$$ \Gamma(u) := U_\mu(t)u_0 + \int_0^t U_\mu(t-\tau)(G(\Phi_\mu(u_0, u_1 + \omega(T, u))))(\tau) \, d\tau - \int_0^t U_\mu(t-\tau)(2u\partial_x u)(\tau) \, d\tau, $$

and show that $\Gamma$ is a contraction in an appropriate space. The fixed point $u$ of $\Gamma$ is a mild solution of IVP (1.11) with $h = \Phi_\mu(u_0, u_1 + \omega(T, u))$ and satisfies $u(x, T) = u_1(x)$. To complete this argument, we use the Bourgain’s space associated to the Benjamin equation and show that $\Gamma$ is a contraction mapping. This is the content of the following result.

**Theorem 5.1 (Small Data Control).** Let $T > 0$, $s \geq 0$, $\mu \in \mathbb{R}$, and $\alpha > 0$ be given. Then there exists a $\delta > 0$ such that for any $u_0, u_1 \in H_0^s(\mathbb{T})$ with $[u_0] = [u_1] = 0$ and

$$ \|u_0\|_{H_0^s(\mathbb{T})} \leq \delta, \quad \|u_1\|_{H_0^s(\mathbb{T})} \leq \delta, $$

one can find a control $h \in L^2([0, T]; H_0^s(\mathbb{T}))$ such that the IVP (1.11) has a unique solution $u \in C([0, T]; H_0^s(0, 2\pi))$ satisfying

$$ u(x, 0) = u_0(x), \quad u_1(x, T) = u_1(x), \quad \text{for all } x \in \mathbb{T}. $$

**Proof.** Let $T > 0$ be given. For $s \geq 0$ we will show that there exists $M > 0$ such that $\Gamma$ defined by (5.1) is a contraction on the ball $B_M(0) := \left\{ u \in Z_T^\infty_{s,\frac{1}{2}} : [u] = 0, \|u\|_{Z_T^\infty_{s,\frac{1}{2}}} \leq M \right\}$. In fact, using Corollary 3.6, Theorem 3.7 and Corollary 3.19, we obtain

$$ \|\Gamma(u)\|_{Z_T^\infty_{s,\frac{1}{2}}} \leq c_1 \left( \|u_0\|_{H_0^s(\mathbb{T})} + \|G(\Phi_\mu(u_0, u_1 + w(T, u)))\|_{Z_T^\infty_{s,\frac{1}{2}}} + \|u\|_{Z_T^\infty_{s,\frac{1}{2}}}^2 \right), $$

(5.2)
where \( c_1 \) is a positive constant depending on \( s, T, \) and \( \alpha \). Using that \( G \) (see \((1.7)\)) is a bounded operator, the immersion \( X^T_{s, \alpha} \hookrightarrow X^T_{s, \alpha} \) and Remark 2.1, we get

\[
\|G(\Phi_\mu(u_0, u_1 + w(T, u)))\|_{X^T_{s, \alpha}^{-\frac{1}{2}}} \leq c\|G(\Phi_\mu(u_0, u_1 + \omega(T, u)))\|_{X^T_{s, \alpha}} \leq c\|G(\Phi_\mu(u_0, u_1 + \omega(T, u)))\|_{L^2(0,T), H^\alpha_0(T)} \leq c_2 \left( \|u_0\|_{H^\alpha_0(T)} + \|u_1\|_{H^\alpha_0(T)} + \|w(T, u)\|_{H^\alpha_0(T)} \right),
\]

(5.3)

where \( c_2 > 0 \) depends on \( s, T, \) and \( g \). Using Proposition 5.2, Theorem 3.7, and Corollary 5.19, we obtain

\[
\|w(T, u)\|_{H^\alpha_0(T)} \leq \sup_{0 \leq \tau \leq T} \left\| \int_0^\tau U_\mu(t - \tau)(2u\partial_x u)(\tau)d\tau \right\|_{H^\alpha_0(T)} \leq c \left\| \int_0^\tau U_\mu(t - \tau)(\partial_x (u^2))(\tau)d\tau \right\|_{Z^T_{1, \frac{1}{2}}} \leq c_3 \|u\|_{Z^T_{1, \frac{1}{2}}}^2,
\]

(5.4)

where \( c_3 > 0 \) depends on \( s, \alpha, T \). From (5.3) and (5.4), we have

\[
\|G(\Phi_\mu(u_0, u_1 + w(T, u)))\|_{X^T_{s, \alpha}^{-\frac{1}{2}}} \leq c_2 \left( \|u_0\|_{H^\alpha_0(T)} + \|u_1\|_{H^\alpha_0(T)} + c_3 \|u\|_{Z^T_{1, \frac{1}{2}}}^2 \right),
\]

(5.5)

From Corollary 5.19 with \( b = -1 \) and \( 0 < \epsilon \leq \frac{1}{2} \), we get

\[
\|G(\Phi_\mu(u_0, u_1 + w(T, u)))\|_{Y^T_{s, \epsilon}^{-\frac{1}{2}}} \leq c(\epsilon) \|G(\Phi_\mu(u_0, u_1 + w(T, u)))\|_{X^T_{s, \epsilon}^{-\frac{1}{2}}} \leq c(\epsilon) \|G(\Phi_\mu(u_0, u_1 + w(T, u)))\|_{X^T_{s, 0}},
\]

(5.6)

From inequality (5.6) and the same calculations as above, we obtain

\[
\|G(\Phi_\mu(u_0, u_1 + w(T, u)))\|_{Y^T_{s, \epsilon}^{-\frac{1}{2}}} \leq c_4(\epsilon) \left( \|u_0\|_{H^\alpha_0(T)} + \|u_1\|_{H^\alpha_0(T)} + c_3 \|u\|_{Z^T_{1, \frac{1}{2}}}^2 \right),
\]

(5.7)

where \( c_4(\epsilon) > 0 \) depends on \( s, \alpha, T \). From (5.5) and (5.7), we infer that

\[
\|G(\Phi_\mu(u_0, u_1 + w(T, u)))\|_{Z^T_{s, \frac{1}{2}}} \leq (c_2 + c_4) \left( \|u_0\|_{H^\alpha_0(T)} + \|u_1\|_{H^\alpha_0(T)} + (c_2c_3 + c_4c_3) \|u\|_{Z^T_{1, \frac{1}{2}}}^2 \right).
\]

(5.8)

Combining (5.2) and (5.8), we obtain that there exists \( C = C_{s, \epsilon, \alpha, g, T} > 0 \) such that

\[
\|\Gamma(u)\|_{Z^T_{s, \frac{1}{2}}} \leq C(\|u_0\|_{H^\alpha_0(T)} + \|u_1\|_{H^\alpha_0(T)} + \|u\|_{Z^T_{1, \frac{1}{2}}}^2).
\]

(5.9)

Choosing \( \delta > 0 \) and \( M > 0 \) such that

\[
CM < \frac{1}{4} \quad \text{and} \quad 2C\delta + CM^2 \leq M,
\]

(5.10)

we obtain from (5.9) that \( \|\Gamma(u)\|_{Z^T_{s, \frac{1}{2}}} \leq M \), for each \( u \in B_M(0) \), provided that \( \|u_0\|_{H^\alpha_0(T)} \leq \delta \) and \( \|u_1\|_{H^\alpha_0(T)} \leq \delta \).

Furthermore, for all \( u, v \in B_M(0) \), with a similar computations as above, we can obtain

\[
\|\Gamma(u) - \Gamma(v)\|_{Z^T_{s, \frac{1}{2}}} \leq C\|u - v\|_{Z^T_{s, \frac{1}{2}}} \leq \frac{1}{2}\|u - v\|_{Z^T_{s, \frac{1}{2}}}.
\]

Thus the map \( \Gamma \) is a contraction on \( B_M(0) \) provided that \( \delta \) and \( M \) are chosen according to (5.10) with \( \|u_0\|_{H^\alpha_0(T)} \leq \delta \) and \( \|u_1\|_{H^\alpha_0(T)} \leq \delta \). \qed
Note that Theorem 14 is a direct consequence of Theorem 5.1.

6. Stabilization of the Benjamin Equation

In this section we study the stabilization problem for the Benjamin equation in $H_0^2(T)$, with $s \geq 0$. Consider the IVP,

$$
\begin{align*}
\begin{cases}
\partial_t u - \partial_x^2 u - \alpha \partial_x u + 2 \mu \partial_x u + 2 u \partial_x u = -K \lambda u, & t > 0, \ x \in T \\
u(x, 0) = u_0(x), & x \in T,
\end{cases}
\end{align*}
$$

(6.1)

with $[u] = 0, \lambda \geq 0$. The feedback control law $K \lambda$ is as defined in (2.5). We first check that the system (6.1) is globally well-posed in $H_0^2(T)$ for any $s \geq 0$. Let $U_\mu(t)$ be the group defined in (2.2) that describes solution $u$ of the linear IVP associated to (6.1). The following estimate is needed.

**Lemma 6.1.** For any $0 < \epsilon < 1$ there exists a positive constant $C(\epsilon)$ such that

$$
\left\| \int_0^t U_\mu(t - \tau)(K \lambda v)(\tau) \, d\tau \right\|_{Z_{s, \frac{1}{2}}} \leq C(\epsilon) T^{1-\epsilon} \|v\|_{Z_{s, \frac{1}{2}}}.
$$

**Proof.** The proof follows from Theorem 3.7, Propositions 3.8, 2.4, Corollary 3.10 with similar arguments as in the proof of Lemma 4.2 in [24]. □

**Theorem 6.2.** Let $s \geq 0$, $\lambda \geq 0$, $\alpha > 0$ and $\mu \in \mathbb{R}$, be given. For any $u_0 \in H_0^2(T)$ there exists a maximal time of existence $T^* > 0$ and a unique solution $u \in C([0, T^*]; H_0^2(T))$ to the IVP (6.1) such that $u$ satisfies the following properties:

i) For every interval $[0, T] \subset [0, T^*)$, $u \in Z_{s, \frac{1}{2}} \cap C([0, T]; L_1^2(T))$.

ii) (Blow-up Alternative) If $T^* < +\infty$, then $\lim_{t \to T^*} \|u(t)\|_{H_0^2(T)} = +\infty$.

iii) $u$ depends continuously on the initial data in the following sense: If $\lim_{n \to \infty} u_{n, 0} = u_0$ in $H_0^2(T)$ and if $u_n$ is the corresponding maximal solution of the IVP (6.1) with initial data $u_{n, 0}$, then $\lim_{n \to \infty} u_n = u$ in $Z_{s, \frac{1}{2}}$, for every interval $[0, T] \subset [0, T^*)$. In particular, $\lim_{n \to \infty} u_n = u$ in $C([0, T]; H_0^2(T))$, for every interval $[0, T] \subset [0, T^*)$.

Furthermore, denoting $S(t)u_0$ the unique solution $u$ of the IVP (6.1) corresponding to the initial data $u_0$, the operator $S(t) : H_0^2(T) \to Z_{s, \frac{1}{2}}$, defined by

$$
S(t)u_0 = u
$$

(6.2)

is continuous on every interval $[0, T] \subset [0, T^*)$.

**Proof.** We rewrite the IVP (6.1) in its integral form and for given $u_0 \in H_0^2(T)$, $0 < T < 1$, we define the map

$$
\Gamma(v) = U_\mu(t)u_0 - \int_0^t U_\mu(t - \tau)(2v \partial_x v)(\tau) \, d\tau - \int_0^t U_\mu(t - \tau)(K \lambda v)(\tau) \, d\tau.
$$

Observe that,

$$
\Gamma(v_1) - \Gamma(v_2) = \int_0^t U_\mu(t - \tau)[\partial_x((v_2 - v_1)(v_2 + v_1))](\tau) \, d\tau + \int_0^t U_\mu(t - \tau)[K \lambda(v_2 - v_1)](\tau) \, d\tau.
$$

It follows then from Corollary 3.6, Theorem 3.7, Corollary 3.10 and Lemma 6.1 that there exists some positive constants $C_1, C_2, C_3$, $0 < \theta < \frac{\epsilon}{2}$, and $0 < \epsilon < 1$ such that

$$
\|\Gamma(v)\|_{Z_{s, \frac{1}{2}}} \leq C_1 \|u_0\|_{H_0^2(T)} + C_2 T^\theta \|v\|_{Z_{s, \frac{1}{2}}}^2 + C_3 T^{1-\epsilon} \|v\|_{Z_{s, \frac{1}{2}}}.
$$

(6.3)
\[ \| \Gamma (v_1) - \Gamma (v_2) \|_{Z^T_{s, \frac{1}{2}}} \leq C_2 T^\theta \| v_2 - v_1 \|_{Z^T_{s, \frac{1}{2}}} + \| v_2 + v_1 \|_{Z^T_{s, \frac{1}{2}}} + C_3 T^{1-\epsilon} \| v_2 - v_1 \|_{Z^T_{s, \frac{1}{2}}}, \]  

for any \( v, v_1, v_2 \in Z^T_{s, \frac{1}{2}} \cap L^2([0, T]; L^0_2(T)) \). Pick \( M = 2 C_1 \| u_0 \|_{H^0_2(T)} \), and \( T > 0 \) such that,
\[
2 C_2 M T^\theta + C_3 T^{1-\epsilon} \leq \frac{1}{2}.
\]

Note that, if we choose \( 0 < \epsilon < 1 \) such that \( 0 < \theta < 1 - \epsilon \), then \( T^{1-\epsilon} \leq T^\theta \) and the time \( T > 0 \) can be taken as
\[
T = T(\| u_0 \|_{H^0_2(T)}) = \left( \frac{1}{8 C_1 C_2 \| u_0 \|_{H^0_2(T)} + 2 C_3} \right)^{\frac{1}{\theta}}.
\]

Therefore, from (6.6), we infer that for any \( v, v_1, v_2 \in B_M(0) \), \( \| \Gamma (v) \|_{Z^T_{s, \frac{1}{2}}} \leq M \), and
\[
\| \Gamma (v_1) - \Gamma (v_2) \|_{Z^T_{s, \frac{1}{2}}} \leq \frac{1}{2} \| v_2 - v_1 \|_{Z^T_{s, \frac{1}{2}}}.
\]

Therefore, \( \Gamma \) is a contraction map in the closed ball \( B_M(0) \) and its unique fixed point \( u \) is the desired solution of (6.1) in the space \( Z^T_{s, \frac{1}{2}} \). It follows from the Proposition 3.2 that \( u \in C([0, T]; H^0_2(T)) \) with
\[
\| u \|_{L^\infty([0, T]; H^0_2(T))} \leq C_4 \| u \|_{Z^T_{s, \frac{1}{2}}} \leq 2 C_1 C_4 \| u_0 \|_{H^0_2(T)},
\]
for some \( C_4 > 0 \).

Now we turn our attention to prove the blow-up alternative. We use the ideas given in [33]. Define
\[
T^* := \sup \{ T > 0 : \exists ! u \in Z^T_{s, \frac{1}{2}} \text{ solution of (6.1) on } [0, T] \}.
\]

Assume \( T^* < +\infty \) and \( \lim_{t \to T^*} \| u(t) \|_{H^0_2(T)} = +\infty \). Then there exist a sequence \( t_j \to T^* \) and a positive constant \( R \) such that \( \| u(t_j) \|_{H^0_2(T)} \leq R \). In particular, for \( k \in \mathbb{N} \) such that \( t_k \) is close to \( T^* \) we have that \( \| u(t_k) \|_{H^0_2(T)} \leq R \). Now we solve the IVP (6.1) with initial data \( u(t_k) \). Thus from (5.6) we obtain that \( T(\| u(t_k) \|_{H^0_2(T)}) \geq T(R) \). Therefore, we can extend our solution to the interval \([t_k, t_k + T(R)]\). If we pick \( k \) such that \( t_k + T(R) > T^* \), then it contradicts the definition of \( T^* \) in (6.7).

Finally, we will prove the continuous dependence on the initial data. Let \( u_0 \in H^0_2(T) \) and consider a sequence \( u_{n, 0} \in H^0_2(T) \) such that \( \lim_{n \to \infty} \| u_{n, 0} \|_{H^0_2(T)} = u_0 \), where the limit is taken in the \( L^2(T) \) norm. Let \( u \) and \( u_n \) be the maximal solutions of the IVP (6.1) in the spaces \( C\left([0, T^*]; H^0_2(T) \right) \) and \( C\left([0, T_n^*]; H^0_2(T) \right) \) with initial data \( u_0 \) and \( u_{n, 0} \), respectively. For \( n \) sufficiently large we have that \( \| u_{n, 0} \|_{H^0_2(T)} \leq 2 \| u_0 \|_{H^0_2(T)} \). So, by the local theory there exists \( T_0 = T_0(\| u_0 \|_{H^0_2(T)}) \) such that \( u \) and \( u_n \) are defined in \([0, T_0]\) for any \( n > N_0 \). Observe that
\[
\Gamma(u) - \Gamma(u_n) = \Gamma(u) - \Gamma(u_0) + \int_0^t U_\mu(t - \tau) [\partial_x (u_n^2 - u^2)](\tau) d\tau + \int_0^t U_\mu(t - \tau) [K_\lambda(u_n - u)](\tau) d\tau.
\]

With a similar procedure that led to (6.4), we get
\[
\| u - u_n \|_{Z^T_{s, \frac{1}{2}}} \leq C_1 \| u_0 - u_{n, 0} \|_{H^0_2(T)} + (2 C_2 T_0^\theta \| u \|_{H^0_2(T)} + C_3 T_0^{1-\epsilon} \| u \|_{Z^T_{s, \frac{1}{2}}}) \| u_0 - u_n \|_{Z^T_{s, \frac{1}{2}}} \]
\[
\leq C_1 \| u_0 - u_{n, 0} \|_{H^0_2(T)} + \frac{1}{2} \| u - u_n \|_{Z^T_{s, \frac{1}{2}}}.
\]
Thus
\[
\| u - u_n \|_{L^2_{T_0}} \leq 2C_1 \| u_0 - u_{n,0} \|_{H^s_0(T)}.
\] (6.8)

From (6.8) we infer
\[
\lim_{n \to \infty} \| u - u_n \|_{L^\infty([0,T_0];H^s(T))} \leq C_4 \lim_{n \to \infty} \| u - u_n \|_{L^2_{T_0}} = 0.
\]

Iterating this property to cover the compact subset \([0,T]\) of \([0,T^*]\) we finish the proof. Also, the continuous dependence shows that the operator \(S(t)\) defined in (6.2) is continuous. This completes the proof of the theorem.

Next we study the global existence of solutions to the IVP (6.1).

**Theorem 6.3.** Let \(s \geq 0, \lambda \geq 0, \alpha > 0, \) and \(\mu \in \mathbb{R}\) be given. For any \(u_0 \in H^s_0(\mathbb{T})\) and for any \(T > 0\) there exists a unique solution \(u \in Z^{T,1}_{s,1} \cap C([0,T];L^2_0(T))\) to the IVP (6.1). Furthermore, the following estimates hold
\[
\| u \|_{Z^{T}_{s,1}} \leq \beta_{T,s}(\| u_0 \|_{L^2_0(T)}) \| u_0 \|_{H^s_0(T)},
\] (6.9)
where \(\beta_{T,s} : \mathbb{R}^+ \to \mathbb{R}^+\) is a nondecreasing continuous function depending only on \(T\) and \(s\). In particular, \(u \in C([0,T];H^s_0(\mathbb{T}))\) and \(\| u \|_{L^\infty([0,T];H^s_0(\mathbb{T}))} \leq C_4 \beta_{T,s}(\| u_0 \|_{L^2_0(T)}) \| u_0 \|_{H^s_0(T)}\), where \(C_4\) is a positive constant. Moreover, denoting \(S(t)u_0\) the unique solution \(u\) of the IVP (6.1) corresponding to the initial data \(u_0\), the operator \(S(t) : H^s_0(\mathbb{T}) \to Z^{T,1}_{s,1}\) defined by (6.2) is continuous in the interval \([0,T]\).

**Proof.** First, we assume that \(s = 0\). Multiplying the equation (6.1) by \(u\) and integrating in space we obtain
\[
\frac{1}{2} \frac{d}{dt} \left( \| u(\cdot,t') \|_{L^2_0(T)}^2 \right) = - (GG^*L^{-1}_\lambda u(\cdot,t'),u(\cdot,t'))_{L^2_0(T)}, \quad \text{for all } t' \geq 0.
\] (6.10)

Now integrating in the time variable in \((0,t)\), and using the properties of operators \(G\) and \(L^{-1}_\lambda\), we infer that
\[
\frac{1}{2} \| u(\cdot,t) \|_{L^2_0(T)}^2 - \frac{1}{2} \| u_0 \|_{L^2_0(T)}^2 = - \int_0^t (GL^{-1}_\lambda u(\cdot,t'),Gu(\cdot,t'))_{L^2_0(T)} dt' \leq \int_0^t \| G \|_{L^2_0(T)} \| u(\cdot,t') \|_{L^2_0(T)} dt'
\]
(6.11)
for all \(t \geq 0\). Gronwall’s inequality implies that
\[
\| u(\cdot,t) \|_{L^2_0(T)} \leq \| u_0 \|_{L^2_0(T)} e^{C t}, \quad \text{for all } t \geq 0, \text{ where } C = \| G \|_{L^2_0(T)}^2 \| L^{-1}_\lambda \|.
\] (6.12)
From the first line of (6.11), we note that
\[
\| u(\cdot,t) \|_{L^2_0(T)} \leq \| u_0 \|_{L^2_0(T)}, \quad \text{when } \lambda = 0 \text{ and } t \geq 0.
\] (6.13)

It follows that equation (6.1) is globally well-posed in \(L^2_0(\mathbb{T})\) by the blow-up alternative. An standard continuation argument shows the estimate (6.3) with \(s = 0\).

Next, we suppose \(s = 3\). In fact, we will prove that for any \(T > 0\) and any \(u_0 \in H^3_0(\mathbb{T}) \subset L^2_0(\mathbb{T})\) the solution of the IVP (6.1) belongs to the space \(u \in Z^{T,1}_{4,1} \cap C([0,T];H^3_0(\mathbb{T}))\).
For this, let $T > 0$ and $u_0 \in H^3_0(T) \subset L^2_0(T)$. Then, the local solution $u$ of the IVP (6.1) belongs to the space $u \in Z_{0,1}^{T_1} \cap C([0, T_1]; L^2_0(T))$, where $T_1$ is the time of local existence given by relation (6.6) in Theorem 6.2 with $s = 0$. Then $u$ satisfies
\[
\|u\|_{L^\infty([0,T_1])} \leq C_4 \|u\|_{Z_{0,1}^{T_1}} \leq 2C_4C_1 \|u_0\|_{L^2_0(T)}.
\] (6.14)

Define $v = \partial_t u$, so that $[v] = 0$ and $v$ satisfies
\[
\begin{cases}
\partial_t v - \partial_{xx} v - \alpha H \partial_x^2 v + 2\mu \partial_x v + 2 \partial_x(uv) = -K_\lambda v, & 0 < t \leq T_1, \ x \in \mathbb{T} \\
v(x, 0) = v_0 = u_0'' + \alpha Hu_0' - 2\mu u_0' - 2u_0u_0' - K_\lambda u_0, & x \in \mathbb{T}.
\end{cases}
\] (6.15)

Note that, applying the Gagliardo-Niremberg’s inequality (see the Theorem 3.70 in [3]), we obtain that there exists $c_1 > 0$ such that
\[
2\|u_0\|_{L^2_0(T)} \leq 2\|u_0\|_{L^2_0(T)}\|u'_0\|_{L^\infty(T)} \leq 2c_1\|u_0\|_{L^2_0(T)}\|u''_0\|_{L^2_0(T)}^{1/2}\|u_0\|_{L^2_0(T)}^{1/2} \leq 2c_1\|u_0\|_{L^2_0(T)}\|u_0\|_{H^3_0(T)}.
\] (6.16)

Therefore, $v_0 \in L^2_0(T)$, with
\[
\|v_0\|_{L^2_0(T)} \leq \left(c_2 + 2c_1\|u_0\|_{L^2_0(T)}\right)\|u_0\|_{H^3_0(T)},
\] (6.17)
where $c_2 > 0$ depends on $\alpha, \mu, \lambda, g$ and $\delta$. On the other hand, considering the map
\[
\Gamma(w) = U_\mu(t)v_0 - 2 \int_0^t U_\mu(t - \tau)\partial_x(u_0(t, .)w)(\tau) \, d\tau - \int_0^t U_\mu(t - \tau)\partial_x(u_0(t, .)w)(\tau) \, d\tau,
\]
using Corollary 3.18 and doing the same calculations as those conducing to (6.3), yield
\[
\|\Gamma(w)\|_{Z_{0,1}^{T_2}} \leq C_1\|v_0\|_{L^2_0(T)} + 2C_2T_2 \|u\|_{Z_{0,1}^{T_2}} \|w\|_{Z_{0,1}^{T_2}} + C_3T_2^{1-\epsilon} \|w\|_{Z_{0,1}^{T_2}} \leq C_1\|v_0\|_{L^2_0(T)} + \left(4C_1C_2T_2 \|u_0\|_{L^2_0(T)} + C_3T_2^{1-\epsilon}\right)\|w\|_{Z_{0,1}^{T_2}}.
\]

Note that,
\[
\Gamma(w_1) - \Gamma(w_2) = -2 \int_0^t U_\mu(t - \tau)\partial_x(u_0(t, w_1 - w_2))(\tau) \, d\tau - \int_0^t U_\mu(t - \tau)\partial_x(u_0(t, w_1 - w_2))(\tau) \, d\tau.
\]
Thus,
\[
\|\Gamma(w_1 - w_2)\|_{Z_{0,1}^{T_2}} \leq \left(4C_1C_2T_2 \|u_0\|_{L^2_0(T)} + C_3T_2^{1-\epsilon}\right)\|w_1 - w_2\|_{Z_{0,1}^{T_2}},
\]
for any $w, w_1, w_2 \in Z_{0,1}^{T_2} \cap L^2([0, T_2]; L^2_0(T))$. Therefore, taking $T_2 = T_1(\|u_0\|_{L^2_0(T)})$ (note that $T_2$ can be taken bigger that $T_1$, but we take $T_2 = T_1$ in order to guarantee the existence of solutions for systems (6.1) and (6.15) simultaneously), we obtain that the map $\Gamma$ is a contraction in a closed ball $\bar{B}_M(0) = \left\{ w \in Z_{0,1}^{T_1}; [w] = 0, \|w\|_{Z_{0,1}^{T_1}} \leq M \right\}$ with $M = 2C_1\|v_0\|_{L^2_0(T)}$. Its unique fixed point $v$ is the desired solution of (6.13) in the space $Z_{0,1}^{T_1} \cap L^2([0, T_1]; L^2_0(T))$. Thus, $\|v\|_{Z_{0,1}^{T_1}} \leq 2C_1\|v_0\|_{L^2_0(T)}$.

From Proposition 3.2 we infer that $v \in C([0, T_1]; L^2_0(T))$ with
\[
\|v\|_{L^\infty([0,T_1])} \leq C_4\|v\|_{Z_{0,1}^{T_1}} \leq 2C_4C_1\|v_0\|_{L^2_0(T)}.
\] (6.18)

From equation (6.1), we have $\partial_{xx}^2 u = v - \alpha H \partial_x^2 u + 2\mu \partial_x u + 2u \partial_x u + K_\lambda u$. 
Consequently,
\[
\|\partial_t^2 u\|_{L^2_0(\mathbb{T})} \leq \|v\|_{L^2_0(\mathbb{T})} + \alpha\|H\partial_t^2 u\|_{L^2_0(\mathbb{T})} + 2\|u\|\|\partial_x u\|_{L^2_0(\mathbb{T})} + 2\|u\|\partial_x u\|_{L^2_0(\mathbb{T})} + \|K\alpha u\|_{L^2_0(\mathbb{T})},
\]  
(6.19)

The analogous computations as those leading to (5.27) and (5.30) in [32], yield for any \(\epsilon > 0\),
\[
2\|\partial_t u(\cdot, t)\|_{L^2_0(\mathbb{T})} \leq c_4\epsilon\|u(\cdot, t)\|_{L^2_0(\mathbb{T})} + \frac{c_4}{4\epsilon}\|\partial_x^2 u(\cdot, t)\|_{L^2_0(\mathbb{T})},
\]  
(6.20)
\[
\alpha\|H\partial_t^2 u(\cdot, t)\|_{L^2_0(\mathbb{T})} \leq c_4\epsilon^2\|u(\cdot, t)\|_{L^2_0(\mathbb{T})} + \frac{c_4}{4\epsilon}\|\partial_x^2 u(\cdot, t)\|_{L^2_0(\mathbb{T})}.
\]  
(6.21)

The similar computations as those leading to (6.10), but using Cauchy’s inequality with \(\epsilon > 0\), yield
\[
2\|u(\cdot, t)\partial_x u(\cdot, t)\|_{L^2_0(\mathbb{T})} \leq c_3\epsilon\|u(\cdot, t)\|_{L^2_0(\mathbb{T})} + \frac{c_3}{4\epsilon}\|\partial_x^2 u(\cdot, t)\|_{L^2_0(\mathbb{T})},
\]  
(6.22)

We already know that
\[
\|K\alpha u(\cdot, t)\|_{L^2_0(\mathbb{T})} \leq c_4\|u(\cdot, t)\|_{L^2_0(\mathbb{T})}.
\]  
(6.23)

From (6.14), (6.17) and (6.18)-(6.23), we get that for \(0 < t \leq T_1\)
\[
\left(1 - \frac{c_4}{\epsilon^2} + \frac{c_4}{4\epsilon}\right)\|\partial_t^2 u\|_{L^2_0(\mathbb{T})} \leq 2C_4C_1\left(c_2 + 2c_1\|u_0\|_{L^2_0(\mathbb{T})}\right)\|u_0\|_{H^3_0(\mathbb{T})} + \left(c_4\epsilon^2 + c_4\epsilon + c_4\right)2C_4C_1\|u_0\|_{H^3_0(\mathbb{T})} + c_3\epsilon2C_4C_1\|u_0\|_{L^2_0(\mathbb{T})}\|u_0\|_{H^3_0(\mathbb{T})}.
\]

Taking \(\epsilon\) large enough, we can conclude that there exists \(C > 0\) such that
\[
\|\partial_t^2 u\|_{L^2_0(\mathbb{T})} \leq C\left(1 + \|u_0\|_{L^2_0(\mathbb{T})} + \|u_0\|_{L^2_0(\mathbb{T})}^2\right)\|u_0\|_{H^3_0(\mathbb{T})},
\]
Consequently,
\[
\|u\|_{L^\infty([0,T_1];H^3_0(\mathbb{T}))} \leq \beta_{T_1,3}(\|u_0\|_{L^2_0(\mathbb{T})})\|u_0\|_{H^3_0(\mathbb{T})},
\]  
(6.24)
where \(\beta_{T_1,3}\) is a nondecreasing continuous function depending only on \(T_1\).

Next, if we assume that the maximal time of existence \(T^* > 0\) of the solution \(u\) of the IVP (6.1) with initial data in \(H^3_0(\mathbb{T})\) is finite, then from (6.12) we have that \(\lim_{t \to T^*} \|u(t)\|_{L^2_0(\mathbb{T})} < +\infty\). Then there exist a sequence \(t_j \to T^*\) and a positive constant \(R\) such that \(\|u(t_j)\|_{L^2_0(\mathbb{T})} \leq R\). In particular for \(k \in \mathbb{N}\) such that \(t_k\) is close to \(T^*\) we have that \(\|u(t_k)\|_{H^3_0(\mathbb{T})} \leq R\). Now we solve the equation (6.1) with initial data \(u(t_k)\) in \(H^3_0(\mathbb{T})\). Then from (6.10) we obtain that \(T(\|u(t_k)\|_{L^2_0(\mathbb{T})}) \geq T(R)\). Therefore, applying similar arguments as those leading to (6.24) we can extend our solution to the interval \([t_k, t_k + T(R)]\). If we pick \(k\) such that \(t_k + T(R) > T^*\), then it contradicts the definition of \(T^*\) in (6.7).

Consequently, we can iterate the procedure leading to (6.24) in order to cover the compact interval \([0, T]\), thus we obtain that \(u \in C([0, T]; H^3_0(\mathbb{T}))\) and satisfies (6.9) with \(s = 3\). This completes the proof of case \(s = 3\).

Finally, observe that a similar result can be obtained for \(s \in 3\mathbb{N^*}\). The global well-posedness for other values of \(s\), follows by nonlinear interpolation (see [44, 6]). This completes the proof. \(\Box\)
Next, we prove a local exponential stability result when applying the feedback law \( f = -K\lambda u \). For this, we need to observe that the system (6.11) can be rewritten as

\[
\partial_t u = A\mu u - 2u \partial_x u - K\lambda u, \quad t > 0, \quad x \in \mathbb{T},
\]

where \( A\mu = \alphaH\partial_x^2 + \partial_x^3 - 2\mu \partial_x \). Let \( T_\lambda(t) = e^{(\alphaH\partial_x^2 + \partial_x^3 - 2\mu \partial_x - K\lambda)t} \) be the \( C_0 \)-semigroup on \( H^1_0(\mathbb{T}) \) with infinitesimal generator \( A\mu - K\lambda \). The system (6.1) can be written in an equivalent integral form

\[
u(t) = T_\lambda(t)u_0 - \int_0^t T_\lambda(t - \tau)(2u \partial_x u)(\tau) \, d\tau. \tag{6.25}
\]

Now, we need to extend some estimates for the \( C_0 \)-semigroup \( \{T_\lambda(t)\} \).

**Lemma 6.4.** Let \( s \geq 0, \lambda \geq 0, \) and \( T > 0 \) be given. Assume that \( \mu \in \mathbb{R} \), and \( \alpha > 0 \). Then, there exists a constant \( C > 0 \), such that

\[
\|T_\lambda(t)\phi\|_{Z^T_{s, \frac{1}{2}}} \leq C \|\phi\|_{H^1_0(\mathbb{T})}, \text{ for any } \phi \in H^1_0(\mathbb{T}), \tag{6.26}
\]

\[
\left\| \int_0^t T_\lambda(t - \tau)\partial_x(u \cdot v)(\tau) \, d\tau \right\|_{Z^T_{s, \frac{1}{2}}} \leq C \|u\|_{Z^T_{s, \frac{1}{2}}} \cdot \|v\|_{Z^T_{s, \frac{1}{2}}}, \text{ for any } u, v \in Z^T_{s, \frac{1}{2}}, \tag{6.27}
\]

where the constant \( C \) does not depend on \( T \) if \( T \in [0, 1] \).

**Proof.** By definition of \( T_\lambda \),

\[
u(t) = T_\lambda(t)\phi \tag{6.28}
\]

is a solution to

\[
\begin{cases}
\partial_t u - \partial_x^3 u - \alphaH\partial_x^2 u + 2\mu \partial_x u + K\lambda u = 0, & t > 0, \quad x \in \mathbb{T} \\
u(x, 0) = \phi(x), & x \in \mathbb{T}. \tag{6.29}
\end{cases}
\]

On the other hand, using the Duhamel’s formula, we can write (6.29) as

\[
u(t) = U_\mu(t)\phi - \int_0^t U_\mu(t - \tau)(K\lambda\phi)(\tau) \, d\tau. \tag{6.30}
\]

Then, from (6.28) and (6.30), we infer that

\[
T_\lambda(t)\phi = U_\mu(t)\phi - \int_0^t U_\mu(t - \tau)(K\lambda\phi)(\tau) \, d\tau. \tag{6.31}
\]

Therefore, using the Lemma 6.1, we obtain

\[
\|T_\lambda(t)\phi\|_{Z^T_{s, \frac{1}{2}}} \leq C_1 \|\phi\|_{H^1_0(\mathbb{T})} + C(\epsilon) \|K\lambda\phi\|_{Z^T_{s, \frac{1}{2}}},
\]

for some \( 0 < \epsilon < 1 \). Thus, for \( T_0 \) sufficiently small such that \( 1 - C(\epsilon)T_0^{-1-\epsilon} > 0 \) we get that there exists a positive constant \( C = C(T_0) \) such that

\[
\|T_\lambda(t)\phi\|_{Z^T_{s, \frac{1}{2}}} \leq C(T_0) \|\phi\|_{H^1_0(\mathbb{T})}.
\]

For \( T \geq T_0 \), the result follows from an easy induction and the fact that

\[
\|T_\lambda(t)\phi\|_{Z^T_{s, \frac{1}{2}}} \leq \|T_\lambda(t)\phi\|_{Z^T_{s, \frac{1}{2}}} + \|T_\lambda(t)\phi\|_{Z^T_{s, \frac{1}{2}}} + \cdots + \|T_\lambda(t)\phi\|_{Z^T_{s, \frac{1}{2}}}, \tag{6.32}
\]

for some \( k \in \mathbb{Z} \).
Now, we move to prove (6.27). Note that, from (6.31)
\[
\int_0^t T_\lambda(t-\tau)f(\tau) \, d\tau = \int_0^t U_\mu(t-\tau)f(\tau) \, d\tau - \int_0^t \int_0^{t-\tau} U_\mu(t-\tau-s)(K_\lambda T_\lambda(s)f(\tau)) \, ds \, d\tau.
\]
Performing a change of variable \(s = -\tau + \theta\) and changing the order of integration, we obtain
\[
\int_0^t T_\lambda(t-\tau)f(\tau) \, d\tau = \int_0^t U_\mu(t-\tau)f(\tau) \, d\tau - \int_0^t U_\mu(t-\theta) \int_0^\theta [K_\lambda T_\lambda(\theta-\tau)f(\tau)] \, d\tau \, d\theta. \tag{6.33}
\]
From Fubini’s theorem, we infer
\[
\int_0^\theta [K_\lambda T_\lambda(\theta-\tau)f(\tau)] \, d\tau = K_\lambda \left( \int_0^\theta [T_\lambda(\theta-\tau)f(\tau)] \, d\tau \right). \tag{6.34}
\]
It follows from (6.33) and (6.34) that
\[
\int_0^t T_\lambda(t-\tau)f(\tau) \, d\tau = \int_0^t U_\mu(t-\tau)f(\tau) \, d\tau - \int_0^t U_\mu(t-\theta) K_\lambda \left( \int_0^\theta [T_\lambda(\theta-\tau)f(\tau)] \, d\tau \right) \, d\theta. \tag{6.35}
\]
We conclude the proof by using (6.35) and similar arguments as those in Lemma 4.4 [24]. □

**Theorem 6.5.** Let \(0 < \lambda' < \lambda\) and \(s \geq 0\) be given. Assume \(\mu \in \mathbb{R}\) and \(\alpha > 0\). Then there exists \(\delta > 0\) such that for any \(u_0 \in H_0^s(\mathbb{T})\) with \(\|u_0\|_{H_0^s(\mathbb{T})} \leq \delta\), the corresponding solution \(u\) of the IVP (6.1) satisfies
\[
\|u(\cdot, t)\|_{H_0^s(\mathbb{T})} \leq C e^{-\lambda't}\|u_0\|_{H_0^s(\mathbb{T})}, \quad \text{for all } t \geq 0,
\]
where \(C > 0\) is a constant that does not depend on \(u_0\).

**Proof.** Using Theorem 4.5 and Lemma 6.3 we can complete the proof as in [24 Theorem 4.3], so we omit the details. □

The stability result presented in Theorem 6.5 is local. We will extend it to a global stability result. In order to do that, the following observability inequality is needed.

**Proposition 6.6.** Let \(s \geq 0, \lambda \geq 0, \mu \in \mathbb{R}, \alpha > 0, T > 0, \) and \(R_0\) be given. Then, there exists a constant \(\beta > 1\) such that for any \(u_0 \in L^2(\mathbb{T})\) satisfying
\[
\|u_0\|_{L^2_\lambda(\mathbb{T})} \leq R_0, \tag{6.36}
\]
the corresponding solution \(u\) of the IVP (6.1) satisfies
\[
\|u_0\|_{L^2_\lambda(\mathbb{T})}^2 \leq \beta \int_0^T \|Gu\|_{L^2_\lambda(\mathbb{T})}^2(t) \, dt. \tag{6.37}
\]

**Proof.** We argue by contradiction, assuming that (6.37) is not true, then for any \(n \geq 1\), equation (6.1) admits a solution \(u_n\) satisfying
\[
u_n \in Z_{\lambda, \frac{1}{2}}^T \cap C([0, T]; L^2_\lambda(\mathbb{T})), \tag{6.38}
\]
\[
\|u_n(0)\|_{L^2_\lambda(\mathbb{T})} \leq R_0, \tag{6.39}
\]
and
\[
\int_0^T \|Gu_n\|_{L^2_\lambda(\mathbb{T})}^2(t) \, dt < \frac{1}{n\lambda} \|u_{0,n}\|_{L^2_\lambda(\mathbb{T})}^2, \tag{6.40}
\]
where \( u_{0,n} = u_n(0) \). As \( \alpha_n := \|u_{0,n}\|_{L^2(\mathbb{T})} \leq R_0 \), we can extract a subsequence of \( \{\alpha_n\} \), still denoted by \( \{\alpha_n\} \) such that \( \lim_{n \to \infty} \alpha_n = \alpha \). In what follows, we consider two cases \( \alpha > 0 \), and \( \alpha = 0 \) separately.

**Case 1.** \( \alpha > 0 \): From (6.38) and (6.39) we obtain that the sequence \( \{\alpha_n\} \) is bounded in both spaces \( L^\infty([0,T];L^2(\mathbb{T})) \) and \( X_{0 \frac{T}{2}}^T \). Corollary 3.18 implies that the sequence \( \{\partial_x u_n^2\} \) is bounded in \( X_{0 \frac{T}{2}}^T \). On the other hand, from Proposition 3.1 we infer that \( X_{0 \frac{T}{2}}^T \) is compact. After, extracting a subsequence of \( \{u_n\} \), still denoted by \( \{u_n\} \), we may assume that

\[
\begin{align*}
\lim_{n \to \infty} u_n & \to u \text{ in } X_{0 \frac{T}{2}}^T, \\
\lim_{n \to \infty} u_n & \to u \text{ in } X_{-T^{-1},0}^T,
\end{align*}
\]  

(6.41)  

(6.42)

and

\[
\begin{align*}
-\partial_x (u_n^2) & \to f \text{ in } X_{0 - \frac{T}{2}}^T,
\end{align*}
\]  

(6.43)

where \( u \in X_{0 \frac{T}{2}}^T \) and \( f \in X_{0 \frac{T}{2}}^{-\alpha} \). Also, from Theorem 3.11, \( X_{0 \frac{T}{2}}^T \) is continuously imbedded in \( L^4(\mathbb{T} \times [0,T]) \) and

\[
\|u_n^2\|_{L^2(\mathbb{T} \times [0,T])} = \|u_n\|_{L^4(\mathbb{T} \times [0,T])}^2 \leq C \|u_n\|_{X_{0 \frac{T}{2}}^T}^2 \leq C \|u_n\|_{X_{0 \frac{T}{2}}^T}.
\]

Thus, \( u_n^2 \) is bounded in \( L^2(\mathbb{T} \times [0,T]) \) and it follows that

\[
\|\partial_x (u_n^2)\|_{L^2([0,T];H^{-1}(\mathbb{T}))} = \|\partial_x (u_n^2)\|_{X_{-1,0}^T} \leq \|u_n^2\|_{L^2(\mathbb{T} \times [0,T])}.
\]

Therefore, \( \partial_x (u_n^2) \) is bounded in \( L^2([0,T];H^{-1}(\mathbb{T})) = X_{0 \frac{T}{2}}^T \). Applying interpolation between \( X_{0 \frac{T}{2}}^T \) and \( X_{-1,0}^T \) (see proof of Theorem 3.13) we conclude that \( \partial_x (u_n^2) \) is bounded in \( X_{-\theta, \frac{T}{2} + \frac{T}{4}}^T \), for \( 0 < \theta < 1 \). Since \( X_{-\theta, \frac{T}{2} + \frac{T}{4}}^T \) is compactly imbedded in \( X_{-1,\frac{T}{2}}^T \), for \( 0 < \theta < 1 \), one can extract a subsequence of \( \{u_n\} \), still denoted by \( \{u_n\} \), such that

\[
\begin{align*}
-\partial_x (u_n^2) & \to f \text{ in } X_{-1,\frac{T}{2}}^T.
\end{align*}
\]  

(6.44)

It follows from (6.40) that

\[
\int_0^T \|G |u_n|\|_{L^2(\mathbb{T})}^2(t) \, dt \to \int_0^T \|G |u|\|_{L^2(\mathbb{T})}^2(t) \, dt = 0,
\]  

(6.45)

which implies \( u(x,t) = c(t) = \int_0^T g(y)u(y,t) \, dy \), on \( \omega \times (0,T) \) (see (1.7)). Thus, passing to the limit in equation (6.1) verified by \( u_n \), we obtain

\[
\begin{align*}
\begin{cases}
\partial_t u - \partial_x^2 u - \alpha \mathcal{H} \partial_x^2 u + 2\mu \partial_x u = f, & \text{on } \mathbb{T} \times (0,T) \\
u(x,t) = c(t), & \text{on } \omega \times (0,T). 
\end{cases}
\end{align*}
\]  

(6.46)

Let

\[
u_n := u_n - u \quad \text{and} \quad f_n := -\partial_x (u_n^2) - f - K_0 u_n.
\]

(6.47)

Note first that, (6.45) implies

\[
\int_0^T \|G |u_n|\|_{L^2(\mathbb{T})}^2(t) \, dt = \int_0^T \|G |u|\|_{L^2(\mathbb{T})}^2(t) \, dt + \int_0^T \|G |u|\|_{L^2(\mathbb{T})}^2(t) \, dt \\
- 2 \int_0^T (G u_n, Gu)_{L^2(\mathbb{T})} \, dt \to 0, \quad \text{as } n \to \infty.
\]

(6.48)
From \((6.41)\), we obtain that \(w_n \to 0\) in \(X^T_{0, \frac{T}{2}}\). Furthermore, \((6.41)\), and \((6.44)-(6.47)\) imply that \(w_n\) satisfies
\[
\partial_t w_n - \partial_x^2 w_n - \alpha \mathcal{H} \partial_y^2 w_n + 2 \mu \partial_x w_n = f_n, \quad \text{on } T \times (0,T).
\]

(6.49)

Observe that
\[
\int_0^T \int_T |G w_n|^2 \, dx \, dt = \int_0^T \int_T g^2(x) w_n^2(x,t) \, dx \, dt
\]
\[
- 2 \int_0^T \left( \int_T g(y) w_n(y,t) \, dy \right) \left( \int_T g^2(x) w_n(x,t) \, dx \right) \, dt
\]
\[
+ \int_0^T \left( \int_T g(y) w_n(y,t) \, dy \right)^2 \left( \int_T g^2(x) \, dx \right) \, dt.
\]

(6.50)

At this point we need the following Lemma.

**Lemma 6.7.** Let \(\{w_n\}_{n \geq 1}\) be a sequence of solutions of equation \((6.49)\) and \(y\) defined in \((1.6)\). If \(w_n \to 0\) in \(X^T_{0, \frac{T}{2}}\), then there exists a subsequence of \(c_n(t) := \int_T g(y) w_n(y,t) \, dy, \, t \in (0,T)\), still denoted by \(\{c_n\}_{n \geq 1}\), such that \(c_n \to 0\) in \(L^2(0,T)\) as \(n \to \infty\).

Proof. From hypotheses we infer that \(w_n \to 0\) in \(X^T_{0,0}\). So, \(\{w_n\}_{n \geq 1}\) is bounded in \(X^T_{0,0}\). From \((6.49), (6.47)\) and integration by parts, we have
\[
\frac{d}{dt} c_n(t) = \int_T g(y) \left( \partial_y^3 w_n + \alpha \mathcal{H} \partial_y^2 w_n - 2 \mu \partial_y w_n + f_n \right)(y) \, dy
\]
\[
= \int_T w_n(y) \left( - \partial_y^3 \alpha - \alpha \mathcal{H} \partial_y^2 g + 2 \mu \partial_y g \right)(y) + g(y) \left( - \partial_y (u_n^2) - \alpha \right)(y) - GGg(y) u_n(y) \, dy.
\]

(6.51)

Integrating \((6.51)\) in \((0,T)\) and using Cauchy-Schwarz inequality on space variable, we obtain
\[
\left\| \frac{d}{dt} c_n(t) \right\|_{L^2(0,T)} \leq C \left( \int_0^T \left( \int_T |w_n|^2 \, dy \right) \left( \int_T |f_n + \alpha \mathcal{H} \partial_y^2 g(y) + 2 \mu \partial_y g(y) - f(y)\right|^2 \, dy \right)^{\frac{1}{2}}
\]
\[
+ \left| \int_0^T \int_T \partial_y g(y) u_n^2(y,t) \, dy \, dt \right| + \left| \left( g(y), f(y,t) \right)_{L^2(T \times (0,T))} \right|
\]
\[
+ C \left( \int_0^T \left( \int_T |GGg(y)|^2 \, dy \right) \left( \int_T |u_n|^2 \, dy \right) \right)^{\frac{1}{2}}.
\]

Further simplifying and using \((6.41)\) and \((6.43)\), we obtain
\[
\left\| \frac{d}{dt} c_n(t) \right\|_{L^2(0,T)} \leq C_{\alpha, \mu} \|g\|_{H^3(T)} \|w_n\|_{X^T_{0,0}} + \|\partial_y g\|_{L^\infty(T)} \int_0^T \int_T |u_n(y,t)|^2 \, dy \, dt
\]
\[
+ C_{T, \alpha, \mu} \|g\|_{H^2(T)} \|f\|_{X^T_{0,0}} + \|GGg\|_{L^2(T)} \|u_n\|_{X^T_{0,0}}
\]
\[
< +\infty.
\]

(6.52)

On the other hand,
\[
\|c_n(t)\|_{L^2(0,T)} \leq \|g\|_{L^2(T)} \|w_n\|_{X^T_{0,0}} < +\infty.
\]

(6.53)

From \((6.52)\) and \((6.53)\), we have that \(c_n(t) \in H^1(0,T)\). The Rellich’s Theorem, and the fact that \(w_n \to 0\) in \(X^T_{0,0}\) imply the desired conclusion. \(\square\)
We continue with the proof of Proposition 6.6. From (6.48), (6.50) and Lemma 6.7, we deduce that
\[ \int_0^T \int_{\mathbb{T}} g^2(x) w_n^2(x,t) \, dx \, dt \to 0. \]
Hence,
\[ \|w_n\|_{L^2((0,T);L^2(\bar{\omega}))} \leq \frac{4}{\|g\|_{L^\infty(T)}} \int_0^T \int_{\omega} g^2(y) w_n^2(x,t) \, dx \, dt \to 0, \text{ as } n \to +\infty, \]
where \( \omega := \{ x \in \mathbb{T} : g(x) > \|g\|_{L^\infty(T)} \} \). It follows from (6.44) and (6.45) that
\[ \|f_n\|_{X^{T-1,0}} \leq \| - \partial_x (u_n^2) - f\|_{X^{T-1,0}} + C\|G u_n\|_{X^{T^0,0}} \to 0, \text{ as } n \to +\infty. \]
Applying the propagation of compactness property (see Proposition 4.9 with \( b = \frac{1}{2} \), and \( b' = 0 \)), we obtain that
\[ \|w_n\|_{L^2_{loc}((0,T);L^2(\mathbb{T}))} \to 0, \text{ as } n \to +\infty. \] (6.54)

Hence, \( u_n^2 \to u^2 \) in \( L^1_{loc}((0,T);L^1(\mathbb{T})) \). Consequently, \( \partial_x(u_n^2) \to \partial_x(u^2) \) in the distributional sense. Thus, \( f = -\partial_x(u^2) \) and \( u \in X^{T,0,\frac{1}{2}} \) satisfy
\[ \begin{aligned}
\partial_t u - \partial_x^2 u - \alpha H \partial_x^2 u + 2 \mu \partial_x u + \partial_x(u^2) &= 0, & \text{on } \mathbb{T} \times (0,T) \\
u(x,t) &= c(t), & \text{on } \omega \times (0,T).
\end{aligned} \] (6.55)

From the unique continuation property (see Proposition 4.11) we get that \( u = 0 \). Now (6.54) implies that \( u_n \to 0 \) in \( L^2_{loc}((0,T);L^2(\mathbb{T})) \). Hence, there exists a time \( t_0 \in [0,T] \) such that \( u_n(t_0) \to 0 \) in \( L^2(\mathbb{T}) \). From (6.10) with \( \lambda = 0 \), we get
\[ \|u_n(0)\|_{L^2(\mathbb{T})}^2 = \|u_n(t_0)\|_{L^2(\mathbb{T})}^2 + \int_0^{t_0} \|Gu_n\|_{L^2(\mathbb{T})} \, dt' \to 0 \text{ as } n \to +\infty, \]
which contradicts the assumption that \( \alpha > 0 \).

Case 2. \( \alpha = 0 \) : Using the unique continuation property for the linearized Benjamin equation which can be proved in a similar way as Proposition 4.11 and a similar argument as those in Proposition 4.6 [24] we arrive at a contradiction. This completes the proof. \( \square \)

Theorem 6.8. Let \( \lambda = 0 \) in (6.11). Assume \( \mu \in \mathbb{R} \) and \( \alpha > 0 \). Then there exists \( k > 0 \) such that for any \( R_0 > 0 \), there exists a constant \( C > 0 \) independent of \( u_0 \), such that for any \( u_0 \in L^2_0(\mathbb{T}) \) with \( \|u_0\|_{L^2_0(\mathbb{T})} \leq R_0 \), the corresponding solution \( u \) of the IVP (6.11) (with \( \lambda = 0 \)) satisfies
\[ \|u(\cdot, t)\|_{L^2(\mathbb{T})} \leq C e^{-kt} \|u_0\|_{L^2_0(\mathbb{T})}, \text{ for all } t \geq 0. \] (6.56)

Proof. This theorem is a direct consequence of the observability inequality (6.37) (see [24] Theorem 4.5). Observe that the constant \( k \) is independent of \( R_0 \). \( \square \)

Now, we prove that the solution \( u \) of (6.11) (with \( \lambda = 0 \)) decays exponentially in any space \( H^0_0(\mathbb{T}) \). For this, we need an exponential stability result for the linearized system
\[ \begin{aligned}
\partial_t w - \partial_x^2 w - \alpha H \partial_x^2 w + 2 \mu \partial_x w + 2 \partial_x(aw) &= -K_0 w, & x \in \mathbb{T}, \ t > 0, \\
w(x,0) &= w_0(x), & x \in \mathbb{T},
\end{aligned} \] (6.57)
where \( a \in Z^T_{\frac{1}{2}} \cap L^2([0,T]; L^2(\mathbb{T})) \) is a given function. This is done in the following two Lemmas.
Lemma 6.9. Let $s \geq 0$, $\alpha > 0$, and $\mu \in \mathbb{R}$ be given. Assume $a \in Z_{s, \frac{T}{2}} T \cap L^2([0, T]; L^3(\mathbb{T}))$ for all $T > 0$, and that there exists $T' > 0$ such that
\[
\sup_{n \geq 1} \|a\|_{Z_s^n [T, (n+1)T]} \leq \beta.
\] (6.58)

Then for any $w_0 \in H_0^s(\mathbb{T})$ and any $T > 0$ there exists a unique solution $w \in Z_{s, \frac{T}{2}} T \cap C([0, T]; H_0^s(\mathbb{T}))$ of the IVP (6.57). Furthermore, the following estimate holds
\[
\|w\|_{Z_{s, \frac{T}{2}} T} \leq v(\|a\|_{Z_{s, \frac{T}{2}} T}) \|w_0\|_{H_0^s(\mathbb{T})},
\] (6.59)

where $v : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function.

Moreover, denote by $S(t)w_0$ the unique solution $u$ of equation (6.57) corresponding to the initial data $w_0$. Then the operator $S(t) : H_0^s(\mathbb{T}) \to Z_{s, \frac{T}{2}} T$, defined by $S(t)w_0 = w$ is continuous in the interval $[0, T]$.

Proof. We first establish the existence and uniqueness of a solution $w \in Z_{s, \frac{T}{2}} T \cap L^2([0, T]; L^3(\mathbb{T}))$ of (6.57) for $0 < T \leq 1$ small enough and then show that $T$ can be taken arbitrarily large. Let us rewrite system (6.57) in its integral form and for given initial datas $w_0, w_1 \in H_0^s(\mathbb{T})$ we define the map
\[
\Gamma(w_j) = U_\mu(t)w_j - \int_0^t U_\mu(t-\tau)(2\partial_z(a \cdot w_j))(\tau) \, d\tau - \int_0^t U_\mu(t-\tau)(K_0v_j)(\tau) \, d\tau,
\]
where $j = 0, 1$ and $U_\mu(t) = e^{(a^2 + \alpha H\partial_z^2 - \mu \partial_z)t}$. Assume $0 < T \leq T'$. Then, calculations similar to those in Theorem 6.2 yield
\[
\|\Gamma(w_1) - \Gamma(w_2)\|_{Z_{s, \frac{T}{2}} T} \leq C_1 \|w_0 - w_1\|_{H_0^s} + 2C_2 T^\beta \|a\|_{Z_{s, \frac{T}{2}} T} \|v_1 - v_2\|_{Z_{s, \frac{T}{2}} T}^\gamma + C_3 T^{1-\epsilon} \|v_2 - v_1\|_{Z_{s, \frac{T}{2}} T},
\] (6.60)

for any $a, v_1, v_2 \in Z_{s, \frac{T}{2}} T \cap L^2([0, T]; L^3(\mathbb{T}))$. Choosing $w_1 = 0$, $M = 2C_1\|w_0\|_{H_0^s(\mathbb{T})}$, and $T > 0$ such that $2C_2 T^\beta + C_3 T^{1-\epsilon} \leq \frac{1}{2}$, we obtain that the map $\Gamma$ is a contraction in a closed ball $B_M(0)$ with $M = 2C_1\|w_0\|_{H_0^s(\mathbb{T})}$. Its unique fixed point $w$ is the desired solution of (6.57) in $Z_{s, \frac{T}{2}} T \cap L^2([0, T]; L^3(\mathbb{T}))$. Note that, the time of existence, can be taken as
\[
T = \min \left\{ \frac{1}{2}, T', \left( \frac{1}{2C_2 \beta + C_3} \right)^{\frac{1}{\epsilon}} \right\}.
\] (6.61)

Furthermore, (6.60) shows that the solution depends continuously on the initial data and satisfies (6.59).

Now, we prove the global existence of the solution. Let $T^*$ be the maximal time of existence of the solution $w$ of the IVP (6.57) satisfying (6.59) with initial data $w_0 \in H_0^s(\mathbb{T})$. If $T^* < \infty$, then from (6.58), we have
\[
\lim_{r \to T^*} \|w(r)\|_{H_0^s(\mathbb{T})} \leq \lim_{r \to T^*} v(\|a\|_{Z_s^r [0, T']}) \|w_0\|_{H_0^s(\mathbb{T})} \leq v((n_0 + 1)\beta) \|w_0\|_{H_0^s(\mathbb{T})} < +\infty,
\]
for some $n_0 \in \mathbb{Z}$. Following a similar argument as in the proof of the blow-up alternative in Theorem 6.2 we finish the proof.

\[\square\]

Lemma 6.10. Let $s \geq 0$, $\alpha > 0$, and $\mu \in \mathbb{R}$ be given. Assume $a \in Z_{s, \frac{T}{2}} T \cap L^2([0, T]; L^3(\mathbb{T}))$ for all $T > 0$. Then for any $k' \in (0, k)$ there exists $T > 0$, and $\beta > 0$ such that if
\[
\sup_{n \geq 1} \|a\|_{Z_s^r [0, T]} \leq \beta,
\] (6.62)
the solution of the IVP (6.57) satisfies
\[ \|w(\cdot, t)\|_{H^s(\mathbb{T})} \leq C e^{-k't} \|w_0\|_{H^s(\mathbb{T})}, \quad \text{for all } t \geq 0, \] (6.63)
where \( C > 0 \) is a constant that does not depend on \( w_0 \).

**Proof.** From Lemma 6.9 we have that for any \( T > 0 \) the IVP (6.57) admits a unique solution \( w \in Z^{T}_{\frac{1}{2}} \cap C([0, T]; H^s(\mathbb{T})) \) and
\[ \|w\|_{Z^{T}_{\frac{1}{2}}} \leq C \|w_0\|_{H^s(\mathbb{T})}, \] (6.64)
where \( v : \mathbb{R}^+ \to \mathbb{R}^+ \) is a nondecreasing continuous function. Rewrite (6.57) in its integral form
\[ w(t) = T_0(t)w_0 - \int_0^t T_0(t - \tau)(2 \partial_x(a \cdot w))(\tau) d\tau, \]
where \( T_0(t) = e^{(\alpha H^{2} + 2\mu \partial_x - K_0)t} \) is the \( C_0 \)-semigroup on \( H^s(\mathbb{T}) \) with infinitesimal generator \( A \mu - K_0 \). For any \( T > 0 \), we infer from Corollary 2.2, Lemma 6.4, and (6.64) that
\[ \|w(\cdot, T)\|_{H^s(\mathbb{T})} \leq C_1 e^{-k'T} \|w_0\|_{H^s(\mathbb{T})} + C_2 \|a\|_{Z^{T}_{\frac{1}{2}}} v(\|a\|_{Z^{T}_{\frac{1}{2}}}) \|w_0\|_{H^s(\mathbb{T})}, \] (6.65)
where \( C_1 > 0 \) is independent of \( T \) and \( C_2 > 0 \) may depend on \( T \). Let
\[ y_n := w(\cdot, nT), \quad \text{for } n = 1, 2, 3, \ldots \]
Using the semigroup property, we have
\[ y_{n+1} = w(\cdot, nT + T) \]
\[ = T_0(T) \left[ T_0(nT)w_0 - \int_0^{nT} T_0(nT - \tau)(2 \partial_x(a \cdot w))(\tau) d\tau \right] \]
\[ - T_0(nT) \int_0^T T_0(T - (\theta + nT))(2 \partial_x(a \cdot w))(\theta + nT) d\theta. \]
Defining \( I_2 := -T_0(nT) \int_0^T T_0(T - (\theta + nT))(2 \partial_x(a \cdot w))(\theta + nT) d\theta \), we observe that
\[ \|I_2\|_{H^s(\mathbb{T})} \leq C \left\| \int_0^t T_0(t - (\theta + nT))(2 \partial_x(a \cdot w))(\theta + nT) d\theta \right\|_{Z^{T}_{\frac{1}{2}}} \]
\[ \leq C_2 \left\| a(\theta + nT) \right\|_{Z^{T}_{\frac{1}{2}}} \left\| w(\theta + nT) \right\|_{Z^{T}_{\frac{1}{2}}} \]
\[ \leq C_2 \left\| a \right\|_{Z^{[nT,(n+1)T]}_{\frac{1}{2}}} \left\| w \right\|_{Z^{[nT,(n+1)T]}_{\frac{1}{2}}} \]
\[ \leq C_2 \left\| a \right\|_{Z^{[nT,(n+1)T]}_{\frac{1}{2}}} v(\left\| a \right\|_{Z^{[nT,(n+1)T]}_{\frac{1}{2}}}) \left\| w(\cdot, nT) \right\|_{H^s(\mathbb{T})} \]
\[ \leq C_2 \beta \left\| y_n \right\|_{H^s(\mathbb{T})}. \]
Therefore,
\[ \|y_{n+1}\|_{H^s(\mathbb{T})} \leq \|T_0(T)y_n\|_{H^s(\mathbb{T})} + C_2 \beta \left\| y_n \right\|_{H^s(\mathbb{T})} \]
\[ \leq \left( C_1 e^{-k'T} + C_2 \beta \right) \|y_n\|_{H^s(\mathbb{T})}, \quad \text{for } n \geq 1. \] (6.66)
Choosing \( T > 0 \) sufficiently large and \( \beta \) small enough so that
\[ C_1 e^{-k'T} + C_2 \beta = e^{-k'T}, \] (6.67)
we get from \((6.66)\) that \(\|y_{n+1}\|_{H_0^s(\mathbb{T})} \leq e^{-k'T}\|y_n\|_{H_0^s(\mathbb{T})}\), for \(n \geq 1\), as long as \((6.62)\) holds. Thus, \(w\) satisfies \((6.63)\) and the proof is complete. \(\square\)

**Theorem 6.11.** Let \(\lambda = 0\) in \((6.1)\). Assume \(\mu \in \mathbb{R}, \alpha > 0\) and \(k_0 > 0\) be the infimum of the numbers \(\gamma, k\) given respectively in Theorem 2.2 and Theorem 6.8. Let \(s \geq 0\) and let \(k' \in (0,k_0)\) be given. Then there exists a nondecreasing continuous function \(\alpha_{s,k'} : \mathbb{R}^+ \rightarrow \mathbb{R}^+\) such that for any \(u_0 \in H_0^s(\mathbb{T})\), the corresponding solution \(u\) of the IVP \((6.1)\) (with \(\lambda = 0\)) satisfies

\[
\|u(\cdot,t)\|_{H_0^s(\mathbb{T})} \leq \alpha_{s,k'}(\|u_0\|_{L_2^s(\mathbb{T})}) e^{-k't}\|u_0\|_{H_0^s(\mathbb{T})}, \quad \text{for all } t \geq 0.
\]

**Proof.** Note that, in Theorem 6.8 we already established \((6.68)\) for \(s = 0\) (with \(k' = k\)). Now, we consider the case \(s = 3\). Let \(R_0 > 0\) be any number and \(u_0 \in H_0^3(\mathbb{T})\) with \(\|u_0\|_{L_2^3(\mathbb{T})} \leq R_0\). Let \(u\) be the solution of \((6.1)\) with \(\lambda = 0\) and initial data \(u_0\), and define \(v = \partial_t u\). Then \(v\) solves

\[
\begin{align*}
\partial_t v - \partial_x^3 v - \alpha \partial_x^2 w + 2\mu \partial_x w + 2 \partial_x(wv) &= -K_0v, \quad x \in \mathbb{T} \quad t > 0, \\
v(x,0) = v_0 &= w_0 + \alpha \partial_x u_0' - 2\mu v_0' - 2\mu w_0' - K_0 w_0 \in L_2^3(\mathbb{T}), \quad x \in \mathbb{T}.
\end{align*}
\]

From \((6.69)\) and \((6.70)\) we infer that for any \(T > 0\) there exists a constant \(C > 0\) that depends only on \(R_0\) and \(T\) such that

\[
\|u\|_{Z(0,T)} \leq C(T) e^{-k'T}\|u_0\|_{L_2^3(\mathbb{T})}, \quad \text{for all } t \geq 0.
\]

Therefore, for any \(\epsilon > 0\), there exists \(t^* > 0\) such that if \(t \geq t^*\), we get

\[
\|u\|_{Z(0,t^*+T)} \leq \epsilon.
\]

One can choose \(\epsilon < \beta\) in \((6.70)\), where \(\beta\) is given by \((6.67)\), and use the exponential stability result (Lemma 6.10) for the linearized system

\[
\begin{align*}
\partial_t w - \partial_x^2 w - \alpha \partial_x^2 w + 2\mu \partial_x w + 2 \partial_x(wv) &= -K_0w, \quad x \in \mathbb{T} \quad t > t^*, \\
\partial_x w(0) &= v(t^*), \quad x \in \mathbb{T},
\end{align*}
\]

where \(u \in Z^{T}_{s,2} \cap L_2^s(0,T; L_2^3(\mathbb{T}))\) is a given function and \(v = v(t-t^*)\), to infer that

\[
\|v(\cdot,t-t^*)\|_{L_2^3(\mathbb{T})} \leq C e^{-k'(t-t^*)}\|v(\cdot,t^*)\|_{L_2^3(\mathbb{T})}, \quad \text{for all } t \geq t^*.
\]

This means

\[
\|v(\cdot,t)\|_{L_2^3(\mathbb{T})} \leq C e^{-k't}\|v_0\|_{L_2^3(\mathbb{T})}, \quad \text{for any } t \geq 0,
\]

where \(C > 0\) depends only on \(R_0\). It follows from Theorem 6.8 and the equation

\[
\partial_x^3 u = v - \alpha \partial_x^2 u + 2\mu \partial_x u + 2 \partial_x u + K_0 u
\]

that

\[
\|u(\cdot,t)\|_{H_0^3(\mathbb{T})} \leq C e^{-k't}\|u_0\|_{H_0^3(\mathbb{T})}, \quad \text{for any } t \geq 0,
\]

where \(C > 0\) depends only on \(R_0\).

Now, we move to prove theorem for \(0 < s < 3\). Applying a similar argument as above to \(u_1 - u_2\) and \(a = u_1 + u_2\), where \(u_1\) and \(u_2\) are two different solutions, we obtain the following Lipschitz stability estimate which is useful in the interpolation argument

\[
\|(u_1 - u_2)(\cdot,t)\|_{L_2^s(\mathbb{T})} \leq C e^{-k't}\|(u_1 - u_2)(\cdot,0)\|_{L_2^s(\mathbb{T})}, \quad \text{for any } t \geq 0.
\]
The case $0 < s < 3$ follows by an interpolation argument similar to the one applied in Theorem 6.3. One case use similar argument for other values of $s$. □

Note that Theorem 6.11 is a direct consequence of Theorem 6.11.

7. Time-varying feedback law

In this section we construct a smooth time-varying feedback law such that a semiglobal stabilization holds with an arbitrary large decay rate.

Let $\lambda > 0$, $\mu \in \mathbb{R}$, $\alpha > 0$ and $s \geq 0$ be given. Theorem 6.11 implies that there exists $\kappa > 0$ and a nondecreasing continuous function $\alpha_s : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that for any $u_0 \in H^s_0(\mathbb{T})$, the corresponding solution $u$ of the IVP

$$
\begin{aligned}
\begin{cases}
\partial_t u - \partial_x^2 u - \alpha \partial_t^2 u + 2u\partial_x u + 2u_\alpha u = -GG^* u, & t > t_0, \ x \in \mathbb{T}, \\
u(x, t_0) = u_0(x), & x \in \mathbb{T},
\end{cases}
\end{aligned}
$$

(7.1)

satisfies

$$
\|u(\cdot, t)\|_{H^s_0(\mathbb{T})} \leq \alpha_s(\|u_0\|_{L^2_0(\mathbb{T})}) e^{-\kappa(t-t_0)} \|u_0\|_{H^s_0(\mathbb{T})}, \text{ for all } t \geq t_0.
$$

(7.2)

Also, for any fixed $\lambda' \in (0, \lambda)$ and any $u_0 \in H^s_0(\mathbb{T})$, the Theorem 6.5 asserts that the solution of the IVP

$$
\begin{aligned}
\begin{cases}
\partial_t u - \partial_x^2 u - \alpha \partial_t^2 u + 2u\partial_x u + 2u_\alpha u = -K_{\lambda} u, & t > t_0, \ x \in \mathbb{T}, \\
u(x, t_0) = u_0(x), & x \in \mathbb{T},
\end{cases}
\end{aligned}
$$

(7.3)

satisfies

$$
\|u(\cdot, t)\|_{H^s_0(\mathbb{T})} \leq C_s e^{-\lambda'(t-t_0)} \|u_0\|_{H^s_0(\mathbb{T})}, \text{ for all } t \geq t_0,
$$

(7.4)

for some $C_s > 0$, provided that $\|u_0\|_s \leq r_0$ for some $r_0 \in (0, 1)$. Define $\rho \in C^\infty(\mathbb{R}^+; [0, 1])$ a function such that

$$
\rho(r) = 1, \text{ for } r \leq r_0, \quad \rho(r) = 0, \text{ for } r \geq 1.
$$

(7.5)

Also, consider any function $\theta \in C^\infty(\mathbb{R}; [0, 1])$ with the following properties:

$$
\begin{aligned}
\begin{cases}
\theta(t + 2) = \theta(t) & \text{for all } t \in \mathbb{R}, \\
\theta(t) = 1 & \text{for } 0 \leq 1 - 1 - \delta, \\
\theta(t) = 0 & \text{for } 1 \leq t \leq 2.
\end{cases}
\end{aligned}
$$

(7.6)

for some $\delta \in (0, \frac{1}{10})$. Let $T > 0$ be given. We define the following time-varying feedback law

$$
K(u, t) := \rho(\|u\|_{H^s_0(\mathbb{T})}^2) \left[\theta\left(\frac{t}{T}\right)K_{\lambda} u + \theta\left(\frac{t}{T} - T\right)GG^* u \right] + (1 - \rho(\|u\|_{H^s_0(\mathbb{T})}^2))GG^* u
$$

$$
= GG^* \left\{ \rho(\|u\|_{H^s_0(\mathbb{T})}^2) \left[\theta\left(\frac{t}{T}\right)L_\lambda^{-1} u + \theta\left(\frac{t}{T} - T\right) u \right] + (1 - \rho(\|u\|_{H^s_0(\mathbb{T})}^2)) u \right\}.
$$

(7.7)

Observe that $K$ has the following behaviour of the trajectories. In a first time, when $\|u\|_{H^s_0(\mathbb{T})}$ is large, we choose $K = GG^*$ to guarantee the decay of the solution. Then, after a transient period, we have $\|u\|_{H^s_0(\mathbb{T})} \leq r_0$ and we get into an oscillatory regime. During each period of length $2T$, we have three steps:

- A period of time for which the damping $K_\lambda$ is active, leading to a decay like $e^{-\lambda' (t-t_0)}$;
- A short transition time of order $\delta$ where a deviation from the origin may occur;
A period of time for which the damping $GG^*$ is active, leading to a decay like $e^{-\kappa(t-t_0)}$. The expected decay is a “mean value” of the two decays above. We consider the system

$$
\begin{align*}
\frac{\partial_t u - \partial_x^2 u - \alpha \partial_t^2 u + 2\mu \partial_x u + 2u \partial_x u = -K(u,t)}{u(x,t_0) = u_0(x)},
\end{align*}
$$

Lemma A.3. We set $\mu = \Delta^2$ and using Lemma A.3 (see Corollary A.2 in [22]).

Finally, we establish the following semiglobal stabilization result with an arbitrary decay rate.

**Theorem 7.1.** Let $s \geq 0$, $\lambda > 0$, $\alpha > 0$, and $\mu \in \mathbb{R}$ be given. Consider any $\lambda' \in (0, \lambda)$ and any $\lambda'' \in \left(\frac{\lambda'}{2}, \frac{\lambda'}{2}\right)$ where $\kappa$ is given in (7.2). Then there exists a time $T > 0$ such that for any $T > T_0$, $t_0 \in \mathbb{R}$ and $u_0 \in H^s_0(\mathbb{T})$, the unique solution of the closed-loop system (7.8) satisfies

$$
\|u(\cdot,t)\|_{H^s(\mathbb{T})} \leq \gamma_s(\|u_0\|_{H^s_0(\mathbb{T})}) e^{-\lambda''(t-t_0)} \|u_0\|_{H^s_0(\mathbb{T})}, \quad \text{for all } t \geq t_0,
$$

where $\gamma_s$ is a nondecreasing continuous function.

**Proof.** The proof follows as in [24, Theorem 5.1].

**Appendix A.**

**Lemma A.1.** A function $\phi \in C^\infty(\mathbb{T})$ can be written in the form $\partial_x \varphi$ for some function $\varphi \in C^\infty(\mathbb{T})$, if and only if,

$$
\int_\mathbb{T} \phi(x) \, dx = 0.
$$

**Lemma A.2.** Let $s,r \in \mathbb{R}$, and $f$ denotes the operator of multiplication by $f \in C^\infty(\mathbb{T})$. Then, $[D^r, f] := D^r f - f D^r$ maps any $H^s(\mathbb{T})$ into $H^{s-r+1}(\mathbb{T})$, i.e., there exists a constant $c = c_f$ depending on $f$ such that

$$
\|[D^r, f]\phi\|_{H^{s-r+1}(\mathbb{T})} \leq c_f \|\phi\|_{H^s(\mathbb{T})}.
$$

**Proof.** This result is proved in [22] (see Lemma A.1).

**Lemma A.3.** If $f \in C^\infty(\mathbb{T})$, then for every $s \in \mathbb{R}$ there exists a positive constants $C$ and $C_s$ such that the following estimate holds

$$
\|f v\|_{H^s(\mathbb{T})} \leq C \|v\|_{H^s(\mathbb{T})} + C_s \|v\|_{H^{s-1}(\mathbb{T})}.
$$

**Proof.** This result follows by writing

$$
D^s(fv) = f D^s v + [D^s, f] v,
$$

and using Lemma A.3 (see Corollary A.2 in [22]).

**Lemma A.4.** Let $f \in C^\infty(\mathbb{T})$ and $\rho_e = e^2 \partial_x^2$ with $0 \leq \epsilon \leq 1$. Then $[\rho_e, f]$ is uniformly bounded as an operator from $H^s$ into $H^{s+1}$ and

$$
\|\rho_e f\|_{H^{s+1}(\mathbb{T})} \leq c_s \|\phi\|_{H^s(\mathbb{T})}, \quad \text{for all } \phi \in H^s(\mathbb{T}).
$$

**Proof.** This result is proved in [22] (see Lemma A.3).
Example A.5. For $j \geq 1$, consider the function $v_j(x,t) := \psi(t) e^{ijx} e^{i\phi(j)t}$, where $\psi \in C^\infty_c(\mathbb{R})$ takes the value 1 on $[-1,1]$. Note that,

$$\hat{v}_j(k,t) = \psi(t) e^{i\phi(j)t} \hat{e}^{ijx}(k) = \psi(t) e^{i\phi(j)t} \delta_{kj},$$

where $\delta_{kj}$ is the Kronecker delta function. Then,

$$\hat{v}_j(k,\tau) = \delta_{kj} \left( \psi(t) e^{i\phi(j)t} \right)^\wedge (\tau) = \delta_{kj} \hat{\psi}(\tau - \phi(j)).$$

Therefore,

$$\|v_j\|_{X_{0,b}}^2 = \int_{\mathbb{R}} (\tau - \phi(j))^2 |\hat{\psi}(\tau - \phi(j))|^2 \, d\tau \leq c_b \|\psi\|_{H^b_t(\mathbb{R})}^2.$$

Thus, the sequence $\{v_j\}$ is uniformly bounded in the space $X_{0,b}$, for every $b \geq 0$.

However, multiplying $v_j$ by $\varphi(x) = e^{ix}$, we observe

$$\|e^{ix}v_j\|_{X_{0,b}}^2 = \int_{\mathbb{R}} (\tau - \phi(1 + j))^2 |\hat{\psi}(\tau - \phi(j))|^2 \, d\tau.$$ 

Using that $\tau - \phi(1 + j) = \tau - \phi(j) + P(j)$ with $P(j) = 3j^2 + (3 - 2\alpha)j + 1 + 2\mu - \alpha$, we have

$$\|e^{ix}v_j\|_{X_{0,b}}^2 \sim \int_{\mathbb{R}} (1 + |\tau + P(j)|)^{2b} |\hat{\psi}(\tau)|^2 \, d\tau \approx j^{2b},$$

for $j$ large enough.

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