NEW CLASS OF DISTORTION RISK MEASURES
AND THEIR TAIL ASYMPTOTICS WITH
EMPHASIS ON VAR

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Abstract Distortion risk measures are extensively used in finance and insurance applications because of their appealing properties. We present three methods to construct new class of distortion functions and measures. The approach involves the composting methods, the mixing methods and the approach that based on the theory of copula. Subadditivity is an important property when aggregating risks in order to preserve the benefits of diversification. However, Value at risk (VaR), as the most well-known example of distortion risk measure is not always globally subadditive, except of elliptically distributed risks. In this paper, instead of study subadditivity we investigate the tail subadditivity for VaR and other distortion risk measures. In particular, we demonstrate that VaR is tail subadditive for the case where the support of risk is bounded. Various examples are also presented to illustrate the results.

KEY WORDS: Coherent risk measure; Comonotonicity; Copula; Distortion risk measure; Distortion functions; Extreme value theory; GlueVaR; Maximum domain of attraction; Spectral risk measure; TVaR; Tail sub(super)additivity; Tail distortion risk measure; VaR.
1 INTRODUCTION

A risk measure $\rho$ is a mapping from the set of random variables $\mathcal{X}$, standing for risky portfolios of assets and/or liabilities, to the real line $\mathbb{R}$. In the subsequent discussion, positive values of elements of $\mathcal{X}$ will be considered to represent losses, while negative values will represent gains. Distortion risk measures are a particular and most important family of risk measures that have been extensively used in finance and insurance as capital requirement and principles of premium calculation for the regulator and supervisor. Several popular risk measures belong to the family of distortion risk measures. For example, the value-at-risk (VaR), the tail value-at-risk (TVaR) and the Wang distortion measure. Distortion risk measures satisfy a set of properties including positive homogeneity, translation invariance and monotonicity. When the associated distortion function is concave, the distortion risk measure is also subadditive (Denneberg, 1994; Wang and Dhaene, 1998; Wirch and Hardy, 2001). VaR is one of the most popular risk measures used in risk management and banking supervision due to its computational simplicity and for some regularity reasons, despite has some shortcomings as a risk measure. For example, VaR is not a subadditive risk measure (see, for instance, Artzner et al. (1999), Denuit et al. (2006)), it only concerns about the frequency of risk, but not the size of risk. TVaR, although being coherent, concerns only losses exceeding the VaR and ignores useful information of the loss distribution below VaR. Clearly, it is difficult to believe that a unique risk measure could capture all characteristics of risk, so that an ideal measure does not exist. Moreover, since risk measures associate a single number to a risk, as a matter of fact, they cannot exhaustively all the information of a risk. However, it is reasonable to search for risk measures which are ideal for the particular problem under investigation. As all the proposed risk measures have drawbacks and limited applications, the selection of the appropriate risk measures continues to be a hot topic in risk management.

Zhu and Li (2012) introduced and studied the tail distortion risk measure which was reformulated by Yang (2012) as follows. For a distortion function $g$, the tail distortion risk measure at level $p$ of a loss variable $X$ is defined as the distortion risk measure with distortion function

$$g_p(x) = \begin{cases} 
g\left(\frac{x}{1-p}\right), & \text{if } 0 \leq x \leq 1-p, \\
1, & \text{if } 1-p < x \leq 1. 
\end{cases}$$

Some properties and applications can be found in Mao, Lv and Hu (2012), Mao and Hu (2012) and Lv, Pan and Hu (2013).
As an extension of VaR and TVaR, Belles-Sampera et al. (2014a) proposed a class of new distortion risk measures called GlueVaR risk measures, which can be expressed as a combination of VaR and TVaR measures at different probability levels. They obtain the analytical closed-form expressions for the most frequently used distribution functions in financial and insurance applications, while a subfamily of these risk measures has been shown to satisfy the tail-subadditivity property which means that the benefits of diversification can be preserved, at least they hold in extreme cases. The applications of GlueVaR risk measures in capital allocation can be found in the recent paper Belles-Sampera et al. (2014b).

Cherubini and Mulinacci (2014) propose a class of distortion measures based on contagion from an external “scenario” variable. The dependence between the scenario and the variable whose risk is measured is modeled with a copula function with horizontal concave sections, they give conditions to ensure that coherence requirements be met, and propose examples of measures in this class based on copula functions.

The first purpose of this paper is to construct new risk measures following Zhu and Li (2012), Belles-Sampera et al. (2014a) and Cherubini and Mulinacci (2014). The newly introduced risk measures are included the tail distortion risk measure and the GlueVaR as specials. The second goal of the paper is to investigate the tail asymptotics of distortion risk measures for the sum of possibly dependent risks with emphasis on VaR. The rest of the paper is organized as follows. We review some basic definitions and notations such as distorted functions, distorted expectations and distortion risk measures in Section 2. In Section 3 several new distortion functions and risk measures are introduced. In Section 4 we investigate the tail asymptotics as well as subadditivity/superadditivity of VaR. Finally, in Section 5 we analyze the subadditivity properties of a class of distortion risk measures.

2 DISTORTION RISK MEASURE

2.1 Distorted Functions

A distortion function is a non-decreasing function $g : [0,1] \rightarrow [0,1]$ such that $g(0) = 0, g(1) = 1$. Since Yaari (1987) introduced distortion function in dual theory of choice
under risk, many different distortions \( g \) have been proposed in the literature. Here we list some commonly used distortion functions. A summary of other proposed distortion functions can be found in Denuit et al. (2006).

- \( g(x) = 1_{(x>1-p)}(x) \), where the notation \( 1_A \) to denote the indicator function, which equals 1 when \( A \) holds true and 0 otherwise.

- \( g(x) = \min\{\frac{x}{1-p}, 1\} \).

- Incomplete beta function \( g(x) = \frac{1}{\beta(a,b)} \int_0^x t^{a-1}(1-t)^{b-1}dt \), where \( a > 0 \) and \( b > 0 \) are parameters and \( \beta(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt \). Setting \( b = 1 \) gives the power distortion \( g(x) = x^a \). Setting \( a = 1 \) gives the dual-power distortion \( g(x) = 1 - (1 - x)^b \).

- The Wang distortion \( g(x) = \Phi(\Phi^{-1}(x) + \Phi^{-1}(p)) \), \( 0 < p < 1 \), where \( \Phi \) is the distribution function of the standard normal.

- The lookback distortion \( g(x) = x^p(1 - p \ln x) \), \( p \in (0, 1] \).

Obviously, every concave distortion function is continuous on the interval \((0, 1]\) and can have jumps in 0. In contrast, every convex distortion function is continuous on the interval \([0, 1)\) and can have jumps in 1. For a distortion function \( g \), if there exists a \( t_0 > 0 \) such that \( g(t_0) = 0 \), then \( g \) is not concave; if there exists a \( t_1 < 1 \) such that \( g(t_1) = 1 \), then \( g \) is not convex. The identity function is the smallest concave distortion function and also the largest convex distortion function; \( g_0(x) := 1_{(x>0)} \) is concave on \([0, 1]\) and is the largest distortion function. \( g^0(x) := 1_{(x=1)} \) is convex on \([0, 1]\) and is the smallest distortion function. For \( 0 < p < 1 \), we remark that \( g_1(x) := \min\{\frac{x}{1-p}, 1\} \) is the smallest concave distortion function such that \( g_1(x) \geq 1_{(x>1-p)}(x) \). In fact, we consider a concave distortion function \( g \) such that \( g(x) \geq 1_{(x>1-p)} \), then \( g \equiv 1 \) on \((1-p, 1)\). As \( g \) is concave, it follows that \( g(x) \geq \frac{x}{1-p} \) for \( x \leq 1 - p \), and thus \( g(x) \geq \min\{\frac{x}{1-p}, 1\} \) for \( 0 < x < 1 \). Any concave distortion function \( g \) gives more weight to the tail than the identity function \( g(x) = x \), whereas any convex distortion function \( g \) gives less weight to the tail than the identity function \( g(x) = x \).
2.2 Distorted Risk Measures

Let \((\Omega, \mathcal{F}, P)\) be a probability space on which all random variables involved are defined. Let \(F_X\) be the cumulative distribution function of random variable \(X\) and the decumulative distribution function is denoted by \(\bar{F}_X\), i.e. \(\bar{F}_X(x) = 1 - F_X(x) = P(X > x)\). Let \(g\) be a distortion function. The distorted expectation of the random variable \(X\), notation \(\rho_g[X]\), is defined as

\[
\rho_g[X] = \int_0^{+\infty} g(\bar{F}_X(x)) dx + \int_{-\infty}^{0} [g(\bar{F}_X(x)) - 1] dx,
\]

provided at least one of the two integrals above is finite. If \(X\) a non-negative random variable, then \(\rho_g\) reduces to

\[
\rho_g[X] = \int_0^{+\infty} g(\bar{F}_X(x)) dx.
\]

From a mathematical point of view, a distortion expectation is the Choquet integral (see Denneberg (1994)) with respect to the nonadditive measure \(\mu = g \circ P\). That is \(\rho_g[X] = \int X d\mu\). In view of Dhaene et al. (2012, Theorems 4 and 6) we know that, when the distortion function \(g\) is right continuous on \([0, 1]\), then \(\rho_g[X]\) may be rewritten as

\[
\rho_g[X] = \int_{[0,1]} VaR_{1-q}[X] dg(q),
\]

where \(VaR_{p}[X] = \sup\{x|F_X(x) \leq p\}\), and when the distortion function \(g\) is left continuous on \((0, 1]\), then \(\rho_g[X]\) may be rewritten as

\[
\rho_g[X] = \int_{[0,1]} VaR_{1-q}[X] dg(q) = \int_{[0,1]} VaR_q[X] d\bar{g}(q),
\]

where \(VaR_p[X] = \inf\{x|F_X(x) \geq p\}\) and \(\bar{g}(q) := 1 - g(1 - q)\) is the dual distortion of \(g\). Obviously, \(\bar{g} = g\), \(g\) is left continuous if and only if \(\bar{g}\) is right continuous; \(g\) is concave if and only if \(\bar{g}\) is convex. The distorted expectation \(\rho_g[X]\) is called a distortion risk measure with distortion function \(g\). Distortion risk measures are a particular class of risk measures which as premium principles were introduced by Deneberg (1994) and further developed by Wang (1996, 2000) among others. As it is well known, the mathematical expectation, \(E[X]\), is a distortion risk measure whose distortion function is the identity function. If \(g\) is concave, then

\[
\rho_g[X] \geq \int_0^{+\infty} \bar{F}_X(x) dx + \int_{-\infty}^{0} [\bar{F}_X(x) - 1] dx = E[X],
\]
and if \( g \) is convex, then
\[
\rho_g[X] \leq \int_0^{+\infty} \tilde{F}_X(x) \, dx + \int_{-\infty}^0 [\tilde{F}_X(x) - 1] \, dx = E[X].
\]

Distortion risk measures satisfy a set of properties including positive homogeneity, translation invariance and monotonicity. Hardy and Wirch (2001) have shown that a risk measure based on a distortion function is coherent if and only if the distortion function is concave. A risk measure is said to be coherent if it satisfies the following set of four properties (see, e.g., Artzner et al. 1997 and 1999):

(M) Monotonicity: \( \rho(X) \leq \rho(Y) \) provided that \( P(X \leq Y) = 1. \)

(P) Positive homogeneity: For any positive constant \( c > 0 \) and loss \( X \), \( \rho(cX) = c\rho(X). \)

(S) Subadditivity: For any losses \( X, Y \), then \( \rho(X + Y) \leq \rho(X) + \rho(Y). \)

(T) Translation invariance: If \( c \) is a constant, then \( \rho(X + c) = \rho(X) + c. \)

It is furthermore shown by Artzner et al. (1999) that all mappings satisfying the above properties allow a representation:
\[
\rho(X) = \sup_{p \in \mathcal{P}} E_p[X],
\]
where \( \mathcal{P} \) is a collection of ‘generalised scenarios’. A risk measure \( \rho \) is called a convex risk measure if it satisfies monotonicity, translation invariance and the following convexity (C):
\[
\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y), \quad 0 \leq \lambda \leq 1.
\]
Clearly, under the assumption of positive homogeneity, monotonicity and translation invariance, the convexity of a risk measure is equivalent subadditivity.

The most well-known examples of distortion risk measures are the above-mentioned VaR and TVaR, corresponding to the distortion functions, respectively, are \( g(x) = 1_{(x>1-p)} \) and \( g(x) = \min\left\{\frac{x}{1-p}, 1\right\}. \) Notice that \( \text{TVaR}_p[X] \) can be alternatively expressed as the weighted average of VaR and losses exceeding VaR:
\[
\text{TVaR}_p[X] = \text{VaR}_p[X] + \frac{1 - F_X(\text{VaR}_p[X])}{1 - p} E[X - \text{VaR}_p[X]|X > \text{VaR}_p[X]]. \quad (2.1)
\]
For continuous distributions, TVaR coincide with the expected loss exceeding \( p \)-Value-at-Risk, i.e., the mean of the worst \( (1-p)100\% \) losses in a specified time period which defined by
\[
\text{CTE}_p[X] = E[X|X > \text{VaR}_p[X]].
\]
If $X$ is a real valued random variable and $0 < p < 1$, then we say that $q$ is an $p$-quantile if $P[X < q] \leq p \leq P[X \leq q]$. By definition, $\text{VaR}_p[X]$ is the lower $p$-quantile of the r.v. $X$ and $\text{VaR}_p^+[X]$ is the upper $p$-quantile of the r.v. $X$. $\text{VaR}_p[X]$ is a left-continuous nondecreasing function having $\text{VaR}_0[X]$ as the essential infimum of $X$, possibly $-\infty$, $\text{VaR}_p^+[X]$ is a right-continuous nondecreasing function having $\text{VaR}_1[X]$ as the essential supremum of $X$, possibly $+\infty$. It is easy to see that $\text{VaR}_p[X] \leq \text{VaR}_p^+[X]$, there are at most countably many values of $p \in [0, 1]$ where $\text{VaR}_p[X]$ and $\text{VaR}_p^+[X]$ differ (see, Dhaene et al. (2012)). Moreover, $\text{VaR}_p[X] = \text{VaR}_p^+[X]$ if, and only if $F_X(x) = p$ for at most one $x$, which equivalent to $F_X(\cdot)$ is strictly increasing. The risk measures $\text{VaR}$ and $\text{VaR}^+$ satisfy axioms (M), (P), and (T), but not (S) and (C) (except in some special cases, such as in the multivariate normal distributions or more generally multivariate elliptical distributions), and hence is not coherent in the sense of Artzner et al. (1999). Despite suffers from some serious limitations, $\text{VaR}$ is still the standard of industry and regulatory for the calculation of risk capital in banking and insurance. For example, the Basel Committee on Banking Supervision introduced a 99% Value at Risk requirement, based on a 10-day trading horizon. The TVaR improves the $\text{VaR}$ as a measure of risk by also taking into account the magnitude of loss beyond the $\text{VaR}$. That is TVaR measures average losses in the most adverse cases rather than just the minimum loss, as the $\text{VaR}$ does. Therefore, risk assessment based on the TVaR have to be considerably higher than those based on $\text{VaR}$. The importance of TVaR is also seen from a result of Kusuoka (2001), who proved that $T VaR_p$ is the smallest law invariant coherent risk measure that dominates $VaR_p$. Unlike $\text{VaR}$, the distortion function associated to the TVaR is concave and, then, the TVaR is a coherent risk measure in the sense of Artzner et al. (1999). It means that TVaR is a subadditive risk measure (see, for instance, Denuit et al., 2006). In the literature, the TVaR is sometimes called the expected shortfall. Although TVaR is one of the best coherent risk measures, however, TVaR reflects only the mean size of losses exceeding the $\text{VaR}$. It ignores the useful information in a large part of the loss distribution, and consequently lacks incentive for mitigating losses below the quartile $\text{VaR}$. Moreover, it does not properly adjust for extreme low-frequency and high-severity losses, since it only accounts for the mean value (not higher moments). A recent paper by Frittelli et al. (2014) has proposed a new risk measure, the lambda value at risk $\Lambda\text{VaR})$ as a generalization of the $\text{VaR}$. The novelty of the $\Lambda\text{VaR}$ lies in the fact that the confidence level can change and adjust according to the risk factor profit and loss.

Detailed studies of distortion risk measures and their relation with orderings of risk and the concept of comonotonicity can be found in, for example, Wang (1996), Wang and
Young (1998), H¨ urlimann (1998), Hua and Joe (2012) and the references therein. The following lemma will be used in proofs of later results, which characterizes an ordering of distortion risk measures in terms of their distortion functions.

**Lemma 2.1.** (Belles-Sampera et al. (2014b)). If $g(x) \leq g^*(x)$ for $x \in [0,1]$, then $\rho_g[X] \leq \rho_{g^*}[X]$ for any random variable $X$.

## 3 Generating New Distortion Functions and Measures

Distortion functions can be considered as a starting point for constructing families of distortion risk measures. Thus, constructions of distortion functions play an important role in producing various families of risk measures. Using the technique of mixing, composition and copula allow the construction of new class of distortion functions and measures.

### 3.1 Composting methods

The first approach to construct distortion functions is the composition of distortion functions.

Let $h_1, h_2, \cdots$ be distortion functions, define $f_1(x) = h_1(x)$ and composite functions $f_n(x) = f_{n-1}(h_n(x)), n = 1, 2, \cdots$. It is easy to check that $f_n(x), n = 1, 2, \cdots$ are all distortion functions. If $h_1, h_2, \cdots$ are concave distortion functions, then each $f_n(x)$ is concave and satisfies that

$$f_1 \leq f_2 \leq f_3 \leq \cdots$$

and

$$\lim_{n \to \infty} f_n(x) = 1_{(x > 0)}, \ x \in [0,1].$$

The associated risk measure satisfies (by Lemma 2.1)

$$\rho_{f_1}[X] \leq \rho_{f_2}[X] \leq \rho_{f_3}[X] \leq \cdots$$

and

$$\lim_{n \to \infty} \rho_{f_n}[X] = VaR_1[X] = \text{esssup}(X).$$
If \( h_1, h_2, \cdots \) are convex distortion functions, then each \( f_n(x) \) is convex and satisfies that
\[
f_1 \geq f_2 \geq f_3 \geq \cdots
\]
and
\[
\lim_{n \to \infty} f_n(x) = 1_{(x=1)}, \ x \in [0, 1].
\]
The associated risk measure satisfies (by Lemma 2.1)
\[
\rho f_1[X] \geq \rho f_2[X] \geq \rho f_3[X] \geq \cdots
\]
and
\[
\lim_{n \to \infty} \rho f_n[X] = VaR_0[X] = \text{essinf}(X).
\]

Consider two distortion functions \( g_1 \) and \( g_2 \). If
\[
g_2(x) = \begin{cases} \frac{x}{1-p}, & \text{if } 0 \leq x \leq 1-p, \\ 1, & \text{if } 1-p < x \leq 1, \end{cases}
\]
then we get
\[
g_p(x) := g_1(g_2(x)) = \begin{cases} g_1 \left( \frac{x}{1-p} \right), & \text{if } 0 \leq x \leq 1-p, \\ 1, & \text{if } 1-p < x \leq 1. \end{cases}
\]
The corresponding risk measure \( \rho_{g_p}[X] \) is the tail distortion risk measure which was first introduced by Zhu and Li (2012), and was reformulated by Yang (2012). In particular, on the space of continuous loss random variables \( X \),
\[
\rho_{g_p}[X] = \int_0^\infty g_p \left( 1 - P(X \leq x | X > VaR_p[X]) \right) dx.
\]
If \( g_1(x) = x^r, 0 < r < 1 \) and
\[
g_2(x) = \begin{cases} \frac{x}{1-p}, & \text{if } 0 \leq x \leq 1-p, \\ 1, & \text{if } 1-p < x \leq 1, \end{cases}
\]
then
\[
g_{12}(x) := g_1(g_2(x)) = \begin{cases} \left( \frac{x}{1-p} \right)^r, & \text{if } 0 \leq x \leq 1-p, \\ 1, & \text{if } 1-p < x \leq 1, \end{cases}
\]
and
\[
g_{21}(x) := g_2(g_1(x)) = \begin{cases} \frac{x^r}{1-p}, & \text{if } 0 \leq x \leq (1-p)^\frac{1}{r}, \\ 1, & \text{if } (1-p)^\frac{1}{r} < x \leq 1. \end{cases}
\]
Clearly, \( g_1 < g_{21} \) and \( g_2 < g_{12} \), so that, by Lemma 2.1, \( \rho_{g_1}[X] < \rho_{g_{21}}[X] \) and \( \rho_{g_2}[X] < \rho_{g_{12}}[X] \).

In practice, sometimes one needs to distort the initial distribution more than once.

**Example 3.1** Consider two risks \( X \) and \( Y \) with distributions, respectively, are:

\[
F_X(x) = \begin{cases} 
0, & \text{if } x < 0, \\
0.6, & \text{if } 0 \leq x < 100, \\
0.975, & \text{if } 100 \leq x < 500, \\
1, & \text{if } x \geq 500,
\end{cases}
\]

and

\[
F_Y(x) = \begin{cases} 
0, & \text{if } x < 0, \\
0.6, & \text{if } 0 \leq x < 100, \\
0.99, & \text{if } 100 \leq x < 1100, \\
1, & \text{if } x \geq 1100.
\end{cases}
\]

Then \( EX = EY = 50, \text{VaR}_{0.95}[X]=\text{VaR}_{0.96}[X]=100, \text{VaR}_{0.95}[Y]=\text{VaR}_{0.96}[Y]=100. \)

TVaR can be calculated by formula (2.1):

\( \text{TVaR}_{0.95}[X]=\text{TVaR}_{0.96}[Y]=300, \text{TVaR}_{0.96}[X]=\text{TVaR}_{0.96}[Y]=350. \) So that when \( \alpha = 0.95 \) and \( \beta = 0.96 \), according to the measures of VaR and TVaR, both \( X \) and \( Y \) bear the same risk! However, the maximal loss for \( Y \) (1100) is more than double than for loss \( X \) (500), clearly, risk \( Y \) is more risky than risk \( X \). Now we consider distortion expectation \( \rho_{g_p} \) with

\[
g_1(x) = g_2(x) = \begin{cases} 
\frac{x}{1-p}, & \text{if } 0 \leq x \leq 1 - p, \\
1, & \text{if } 1 - p < x \leq 1.
\end{cases}
\]

One can easily find that, with \( p = 0.95 \), \( \rho_{g_p}[X] = 500 \) and \( \rho_{g_p}[Y] = 1100. \)

### 3.2 Mixing methods

One of the easiest ways to generate distortion functions is to use the method of mixing along with finitely distortion functions or infinitely many distortion functions. Specifically, if \( g_w \ (w \in <a, b>) \) is a one-parameter family of distortion functions, \( \psi \) is an increasing function on \(<a, b>\) such that \( \int_{<a,b>} d\psi(w) = 1 \), then the function \( g = \int_{<a,b>} g_w d\psi(w) \) is a distortion function, the associated risk measure is given by

\[
\rho_g[X] = \int_{<a,b>} \rho_{g_w}[X] d\psi(w). \quad (3.1)
\]
In particular, if \( \psi \) is a discrete distribution, then (3.1) can be written as the form of convex linear combination \( g = \sum_i w_i g_i \) \( (w_i \geq 0, \sum_i w_i = 1) \), the associated risk measure is given by
\[
\rho_g[X] = \sum_i w_i \rho_{g_i}[X].
\] (3.2)

The following lemma is well known (cf. Kriele and Wolf (2014, Theorem 2.1, P. 33)).

**Lemma 3.1.** If all \( \rho_{g_w} \) \( (w \in <a, b>) \) are monotone, positively homogeneous, subadditive and translation invariant, then \( \rho_g[X] \) also has the corresponding properties. That is, if all \( g_w \) \( (w \in <a, b>) \) are coherent, then \( \rho_g[X] \) is also coherent.

Now we list three interesting special cases:

- If \([a, b] = [0, \infty)\), \( g_i(x) = 1 - (1 - x)^i \), \( i \geq 1 \) and \( w_i \geq 0, \sum_i w_i = 1 \), then \( \rho_g[X] \) in (3.2) is coherent since \( g_i(x) \) is concave. As in Tsukahara (2009), if we take \( w_i \) from Bin \((1, \theta) \) \((0 < \theta < 1)\), then \( g_{\theta}(u) = u + u\theta - u^2\theta \). If we take
  \[
  w_i = \frac{\theta^i}{(\theta^i - 1)!}, \quad \theta > 0,
  \]
then
  \[
  g_{\theta}(u) = \frac{e^{\theta}(1 - e^{-\theta u})}{e^{\theta} - 1}.
  \]
Also, if take \( w_i = (1 - \theta)^{i-1}\theta \) \((0 < \theta < 1)\), the geometric distribution, then
  \[
  g_{\theta}(u) = \frac{u}{u + \theta(1 - u)},
  \]
which is the proportional odds distortion; see Example 2.1 in Cherubini and Mulinacci (2014).

- If \([a, b] = [0, 1)\), \( \rho_{g_w} = Var_w[X] \) and \( d\psi(w) = \phi(w)dw \), then \( \rho_g[X] \) in (3.1) reduces to
  \[
  \rho_{\phi}[X] = \int_0^1 Var_w[X]\phi(w)dw,
  \] (3.3)
which is spectral risk measure (see Acerbi 2002, 2004). Here \( \phi \) is called a weighting function satisfies the following properties: \( \phi \geq 0, \int_0^1 \phi(w)dw = 1 \). The following lemma gives a necessity and sufficient condition for \( \rho_{\phi}[X] \) to be a coherent risk measure (cf. Kriele and Wolf (2014)).

**Lemma 3.2.** Spectral risk measure \( \rho_{\phi}[X] \) is coherent if, and only if, \( \phi \) is (almost everywhere) monotone increasing.
Clearly, there exists a one-to-one correspondence between distortion function \( g \) and weighting function \( \phi \), namely, 
\[
  g(1 - t) = 1 - \int_0^t \phi(s)ds.
\]
Obviously, \( g \) is concave if, and only if \( \phi \) is (almost everywhere) monotone increasing. Two well-known members of this class are the VaR and the TVaR. The associated weight functions are \( \phi(w) = \delta_p(w) \) and 
\[
  \frac{1}{1-p}1(w > p),
\]
respectively. Here \( \delta_p(w) \) is a Dirac delta function that gives the outcome \( \alpha = p \) an infinite weight and gives every other outcome a weight of zero. From Lemma 3.2, TVaR is coherent since \( \phi(w) = \frac{1}{1-p}1(w > p) \) is monotone increasing. By contrast, \( \phi(w) = \delta_p(w) \) is not monotone increasing, hence VaR is not coherent. Both of these measures use only the tail of the distribution.

- If \([a, b] = [0, 1], \rho_{gw} = TVaR_w[X] \) and \( \psi = \mu \) is a probability measure on \([0, 1]\), then \( \rho_{g}[X] \) in (3.1) reduces to
\[
  \rho_{\mu}[X] = \int_0^1 TVaR_w[X]d\mu(w),
\]
which is the weighted TVaR (see Cherny 2006). TVaR\(_p\) is a special weighted TVaR with \( \mu(w) = 1(w \geq p) \). According to Lemma 3.1, since each TVaR\(_w\)[X] is coherent risk measure, the weighted TVaR is coherent risk measure. The weighted TVaR can be rewritten as the form of spectral risk measure as following:
\[
  \rho_{\mu}[X] = \int_0^1 TVaR_w[X]d\mu(w) = \int_0^1 \left( \frac{1}{1-w} \int_w^1 VaR_q[X]dq \right) d\mu(w)
\]
\[
  = \int_0^1 \left( VaR_q[X] \int_0^q \frac{1}{1-w}d\mu(w) \right) dq \quad \text{(by the Fubini theorem)}
\]
\[
  = \int_0^1 VaR_q[X]\phi(q)dq
\]
\[
  = \int_0^1 VaR_{1-q}[X]dg(q),
\]
where, \( g \) is a function with \( g(0) = 0 \) and satisfies
\[
  g'(1 - q) = \phi(q) = \int_0^q \frac{1}{1-w}d\mu(w).
\]
Because \( \phi(q) \) is increasing function of \( q \), it follows from Lemma 3.2 that the weighted TVaR \( \rho_{\mu}[X] \) is coherent. Or, equivalently, \( g'(q) \) is decreasing function of \( q \), i.e. \( g \) is a concave function, moreover, \( g \) is increasing and
\[
  g(1) = \int_0^1 g'(1 - w)dw = \int_0^1 dq \int_0^q \frac{1}{1-w}d\mu(w)
\]
\[
  = \int_0^1 \frac{1}{1-w}d\mu(w) \int_w^1 dq
\]
\[
  = \int_0^1 d\mu(w) = 1.
\]
so that $g$ is a concave distortion function, and hence the weighted TVaR $\rho_\mu[X]$ is coherent.

Conversely, the distortion measure with concave distortion function $g$ can be expressed by the weighted TVaR. In fact, note that $\phi(q) = g'(1 - q)$ is monotone increasing, we define a measure $\nu([0, q]) = \phi(q)$. As in the proof of Theorem 2.4 in Kriele and Wolf (2014) we have

$$\rho_\beta g[X] = -\int_0^1 \text{VaR}_w[X]dg(1 - w)$$

$$= \int_0^1 \text{VaR}_w[X]g'(1 - w)dw$$

$$= \int_0^1 \text{VaR}_w[X]\phi(w)dw$$

$$= \int_0^1 \text{TVaR}_w[X](1 - w)d\nu(w)$$

$$= \int_0^1 \text{TVaR}_w[X]d\mu(w),$$

where

$$d\mu(w) = (1 - w)d\nu(w).$$

It can be shown that $\mu$ is a probability measure. In fact,

$$\int_0^1 d\mu(w) = \int_0^1 \nu([0, w])dw$$

$$= \int_0^1 \phi(w)dw = \int_0^1 g'(w)dw = 1.$$

We now give some examples of interesting distortion functions and risk measures.

**Example 3.2** If $w_1, w_2, w_3, w_4 \geq 0, \sum_{i=1}^4 w_i = 1$, then

$$g_{\alpha\beta}(x) = w_1\nu_\beta(x) + w_2\nu_\alpha(x) + w_3\psi_\beta(x) + w_4\psi_\alpha(x),$$

is a distortion function, where $\nu_\beta, \nu_\alpha, \psi_\beta, \psi_\alpha$ are the distortion functions of TVaR and VaR at confidence levels $\beta$ and $\alpha$, respectively. Then the corresponding risk measure

$$\rho_{g_{\alpha\beta}}[X] = w_1\text{TVaR}_\beta[X] + w_2\text{TVaR}_\alpha[X] + w_3\text{VaR}_\beta[X] + w_4\text{VaR}_\alpha[X],$$

is called the GlueVaR risk measure, which were initially defined by Belles-Sampera et al. (2014a) (in the case $w_4 = 0$) and the closed-form expressions of GlueVaR for Normal, Log-normal, Student’s $t$ and Generalized Pareto distributions are provided. Two new
proportional capital allocation principles based on GlueVaR risk measures are studied in Belles-Sampera et al. (2014b).

Although GlueVaR has superior mathematical properties than VaR and TVaR, however, the GlueVaR risk measure may also fail to recognize the differences between two risks. For example, consider two risks \( X \) and \( Y \) in Example 3.1, we have computed that \( \text{VaR}_{0.95}[X] = \text{VaR}_{0.96}[X] = 100 \), \( \text{VaR}_{0.95}[Y] = \text{VaR}_{0.96}[Y] = 300 \), \( \text{TVaR}_{0.95}[X] = \text{TVaR}_{0.96}[Y] = 350 \). So that when \( \alpha = 0.95 \) and \( \beta = 0.96 \), we have \( \rho_{g_{\alpha,\beta}}[X] = \rho_{g_{\alpha,\beta}}[Y] \). Thus according to \( \rho_{g_{\alpha,\beta}} \), both \( X \) and \( Y \) bear the same risk! However, the maximal loss for \( Y \) (1100) is more than double than for loss \( X \) (500), clearly, risk \( Y \) is more risky than risk \( X \).

**Example 3.3** Let \( \lambda \in [0, 1] \), define a distortion function

\[
g_{\lambda}(x) = \lambda g_{0}(x) + (1 - \lambda)g(x),
\]

where \( g_{0}(x) := 1_{(x > 0)} \) and \( g \) is an arbitrary distortion function. Note that \( g_{\lambda} \) can be rewritten as

\[
g_{\lambda}(x) = \begin{cases} 
0, & \text{if } x = 0, \\
\lambda + (1 - \lambda)g(x), & \text{if } 0 < x \leq 1.
\end{cases}
\]

In particular, if \( g(x) = x \), then we get the esssup-expectation convex combination distortion function with weight \( \lambda \) on the essential supremum, which was introduced in Bannör and Scherer (2014). The corresponding risk measure

\[
\rho_{g_{\lambda}}[X] = \lambda \text{esssup}(X) + (1 - \lambda)E(X),
\]

which is a convex combination of the essential supremum of \( X \) and the ordinary expectation of \( X \) w.r.t. \( P \).

If

\[
g(x) = \begin{cases} 
\alpha(1 - \beta) + \frac{1-\alpha}{1-p}x, & \text{if } 0 \leq x \leq 1 - p, \\
\alpha \beta + 1 - \alpha + \alpha(1 - \beta)x, & \text{if } 1 - p < x \leq 1,
\end{cases}
\]

where \( 0 \leq \alpha, \beta \leq 1, 0 < p < 1 \) are constants, then we get

\[
\rho_{g_{\lambda}}[X] = \lambda \text{esssup}(X) + (1 - \lambda)\rho_{g}[X],
\]

where

\[
\rho_{g}[X] = \alpha(1 - \beta)E(X) + \alpha \beta \text{VaR}_{p}[X] + (1 - \alpha)\text{TVaR}_{p}[X].
\]
As illustration, we consider the risks $X$ and $Y$ in Example 3.1, if $p = 0.95$, then
\[ \rho_g[X] = \rho_g[Y] = 50\alpha\beta - 250\alpha + 300. \]
It follows that
\[ \rho_{g\lambda}[X] = 500\lambda + (1 - \lambda)(50\alpha\beta - 250\alpha + 300) \]
and
\[ \rho_{g\lambda}[Y] = 1100\lambda + (1 - \lambda)(50\alpha\beta - 250\alpha + 300). \]
Taking $\lambda = \frac{1}{2}$, $\alpha = 1$, $\beta = 0$, then $\rho_{g\lambda}[X] = 275$ and $\rho_{g\lambda}[Y] = 575$. Taking $\lambda = \alpha = \beta = \frac{1}{2}$, then $\rho_{g\lambda}[X] = 437.5$ and $\rho_{g\lambda}[Y] = 737.5$. Thus the measure $\rho_{g\lambda}$ can measure the differences between two risks $X$ and $Y$.

### 3.3 A copula-based approach

If $F$ is a distribution function on $[0,1]$, then $F$ can be used as a distortion function. The well-known examples are the PH transform and the dual power transform and, more generally, the beta transform; see Wrich and Hardy (1999) for details. Similarly, we use this technique to a distribution function on $[0,1]^2$. We first introduce the notion of copula in the two-dimensional case.

**Definition 3.1.** A two-dimensional copula $C(u,v)$ is a bivariate distribution on the square $[0,1]^2$ having uniform margins. That is a function $C(u,v) : [0,1]^2 \rightarrow [0,1]$ is right-continuous in each variable such that $C(0,v) = C(u,0) = 0$, $C(u,1) = u$, $C(1,v) = v$ and for $u_2 > u_1, v_2 > v_1$,
\[ C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0. \]
For an introduction to copula theory and some of its applications, we refer to Joe (1997), Denuit et al. (2006) and Nelsen (2006).

The well-known examples of copulas are $C^+(u,v) = \min(u,v)$, $C^\perp(u,v) = uv$ and $C^-(u,v) = \max(u + v - 1, 0)$ describing, respectively, comonotone dependence, independence and countermonotone dependence between two random variables $X$ and $Y$. The copula version of the Fréchet-Hoeffding bounds inequality tells us
\[ C^-(u,v) \leq C(u,v) \leq C^+(u,v). \]
Any copula has the following decomposition (cf. Yang et al (2006))
\[ C(u,v) = \alpha C^+(u,v) + \beta C^\perp(u,v) + \gamma C^-(u,v) + lG(u,v), \]
where \( \alpha, \beta, \gamma, l \geq 0 \), \( \alpha + \beta + \gamma + l = 1 \). Here \( G \) is a copula which called the indecomposable part.

For a given two-dimensional copula \( C(\cdot, \cdot) \), define one-parameter family \( \{g_p\}_{p \in (0, 1]} \) by \( g_p(u) = \frac{C(u, p)}{p} \) or \( \frac{C(p, u)}{p} \). Clearly, for each \( p \), \( g_p \) is a right continuous distortion function. For example,

- \( g_p(u) = \frac{C^+(u, p)}{p} = u \) is continuous and both convex and concave, the associated risk measure is \( EX \);
- \( g_{1-p}(u) = \frac{C^+(u, 1-p)}{1-p} = \min \left\{ \frac{u}{1-p}, 1 \right\} \) is continuous and concave, the corresponding risk measure is \( TVaR_p \);
- \( g_p(u) = \frac{C^-(u, p)}{p} = \max \left\{ \frac{u+p-1}{p}, 0 \right\} \) is continuous and convex, the corresponding risk measure is \( \rho_{g_p}[X] = \frac{1}{p} \int_0^p VaR_q[X] dq \).

Conversely, if \( \{g_p\}_{p \in (0, 1]} \) is a family of distortion functions, then, however, \( C(u, p) = pg_p(u) \) is not a copula in general; A sufficient condition can be found in Cherubini and Mulinacci (2014).

A lot of copulas and methods to construct them can be found in the literature, for example, Joe (1997), Denuit et al. (2006) and Nelsen (2006). We give below the most common bivariate copulas and the corresponding distortion functions.

- The Archimedean copulas:

\[
C_\Psi(u, v) = \Psi^{-1}(\Psi(u) + \Psi(v))
\]

for some generator \( \Psi : (0, 1] \rightarrow \mathbb{R}^+ \) with \( \Psi(1) = 0 \) such that \( \Psi \) is convex. The pseudo-inverse of \( \Psi \) is the function \( \Psi^{-1} \) with Dom\(\Psi^{-1} = [0, \infty] \) and Ran\(\Psi^{-1} = [0,1] \) given by

\[
\Psi^{-1}(t) = \begin{cases} 
\Psi^{-1}(t), & \text{if } 0 \leq t \leq \Psi(0), \\
0, & \text{if } \Psi(0) \leq t < \infty.
\end{cases}
\]

If \( \Psi \) is twice differentiable and \( \Psi(0) = \infty \), then \( C_\Psi \) is componentwise concave if, and only if \( \frac{1}{\Psi'} \) is concave, where \( \Psi' \) is the derivative of \( \Psi \) (see Dolati and Nezhad (2014)). As a consequence, we have
Theorem 3.1. For each $v > 0$, the distortion function

$$g_v(u) = \frac{1}{v} \Psi^{-1}(\Psi(u) + \Psi(v))$$

is concave if, and only if $\frac{1}{\Psi}$ is concave.

We list some examples of the Archimedean copulas and the corresponding distortion functions:

(a) The Clayton copula with parameter $\alpha > 0$ is generated by $\Psi(t) = \frac{1}{\alpha}(t^{-\alpha} - 1)$ and takes the form

$$C_\alpha(u, v) = (u^{-\alpha} + v^{-\alpha} - 1)^{-1/\alpha}, \alpha > 0.$$ 

The limit of $C_\alpha(u, v)$ for $\alpha \downarrow 0$ and $\alpha \uparrow \infty$ leads to independence and comonotonicity respectively (Nelsen, 2006). The corresponding distortion functions:

$$g_{\alpha, v}(u) = \frac{1}{v}(u^{-\alpha} + v^{-\alpha} - 1)^{-1/\alpha}, \alpha > 0, v \in (0, 1].$$

In particular, if $\alpha = 1$, we get the proportional odds distortion which is found by Cherubini and Mulinacci (2014):

$$g_{1, v}(u) = \frac{u}{u + v(1 - u)}.$$  

Since $(\frac{1}{\Psi(t)})'' = -\alpha(1 + \alpha)t^{\alpha-1} < 0$, $g_{\alpha, v}(u)$ is concave.

(b) In case $\Psi(t) = -\ln\left(\frac{e^{-at} - 1}{e^{-\alpha t} - 1}\right)$ we get the Frank copulas:

$$C_\alpha(u, v) = -\frac{1}{\alpha} \ln \left(1 + \frac{(e^{-\alpha u} - 1)(e^{-\alpha v} - 1)}{e^{-\alpha} - 1}\right), \alpha \neq 0.$$ 

The corresponding distortion functions:

$$g_{\alpha, v}(u) = -\frac{1}{\alpha v} \ln \left(1 + \frac{(e^{-\alpha u} - 1)(e^{-\alpha v} - 1)}{e^{-\alpha} - 1}\right).$$

Since $(\frac{1}{\Psi(t)})'' = -\alpha e^{at}$, $g_{\alpha, v}(u)$ is convex if $\alpha < 0$ and concave if $\alpha > 0$.

(c) In case $\Psi(t) = t^{-1/\alpha} - 1$ we get the Pareto survival copulas:

$$\hat{C}_\alpha(u, v) = (u^{-1/\alpha} + v^{-1/\alpha} - 1)^{-\alpha}, \alpha > 0.$$ 

The corresponding distortion functions:

$$g_{\alpha, v}(u) = \frac{1}{v}(u^{-1/\alpha} + v^{-1/\alpha} - 1)^{-\alpha}. $$
Since \((\frac{1}{\Psi(t)})'' = -(1 + \frac{1}{\alpha})\frac{\Psi''}{\Psi(t)} < 0\), \(g_{\alpha,v}(u)\) is concave.

(d) In case \(\Psi(t) = \ln \frac{1+\alpha(t-1)}{t}, \alpha \in [-1, 1]\) we get the Ali-Mikhail-Haq copulas:

\[
C_\theta(u, v) = \frac{uv}{1 - \alpha(1 - u)(1 - v)}.
\]

The corresponding distortion functions:

\[
g_{\theta,v}(u) = \frac{u}{1 - \alpha(1 - u)(1 - v)}.
\]

Since \((\frac{1}{\Psi(t)})'' = \frac{2\alpha}{(\alpha-1)^2}\), \(g_{\theta,v}(u)\) is convex if \(\alpha \in [-1, 0]\) and concave if \(\alpha \in [0, 1]\).

(e) In case \(\Psi(t) = (-\ln t)^\alpha\) we get the Gumbel-Hougaard copulas:

\[
C_\alpha(u, v) = \exp\left\{\frac{-((\ln u)^\alpha + (-\ln v)^\alpha)}{1/\alpha}\right\}, \alpha \geq 1.
\]

The corresponding distortion functions:

\[
g_{\alpha,v}(u) = \frac{1}{v} \exp\left\{\frac{-((\ln u)^\alpha + (-\ln v)^\alpha)}{1/\alpha}\right\}.
\]

The value \(\alpha = 1\) gives independence and the limit for \(\alpha \uparrow \infty\) leads to comonotonicity. Since

\[
(\frac{1}{\Psi'(t)})'' = \frac{\alpha - 1}{\alpha t}(\ln t)^{\alpha - 2}\left(-1 + \frac{2 - \alpha}{\ln t}\right) \begin{cases} 
  \leq 0, & \text{if } 0 < t \leq 1, 1 \leq \alpha \leq 2, \\
  > 0, & \text{if } e^{2-\alpha} < t \leq 1, \alpha > 2, \\
  < 0, & \text{if } 0 < t < e^{2-\alpha}, \alpha > 2,
\end{cases}
\]

\(g_{\alpha,v}(u)\) is concave if \(1 \leq \alpha \leq 2\) and, if \(\alpha > 2\), \(g_{\alpha,v}(u)\) is convex on \((e^{2-\alpha}, 1]\) and concave on \((0, e^{2-\alpha})\).

Among other copulas, which do not belong to Archimedean family, it is worth to mention the following three copulas, given in the bivariate case as:

- The Farlie-Gumbel-Morgenstern copulas:

\[
C_\alpha(u, v) = uv + u\alpha (1-u)(1-v), \alpha \in [-1, 1],
\]

The corresponding distortion functions:

\[
g_{\alpha,v}(u) = u + u\alpha (1-u)(1-v), \alpha \in [-1, 1], v \in [0, 1],
\]

which is convex if \(\alpha \in [-1, 0]\) and concave if \(\alpha \in [0, 1]\).
• The Marshall-Olkin copulas:

\[ C_{\alpha,\beta}(u, v) = \min\{u^{1-\alpha}v, uv^{1-\beta}\}, \, \alpha, \beta \in [0, 1]. \]

Note that this copula is not symmetric for \( \alpha \neq \beta \). The corresponding distortion functions:

\[ g_{\alpha,\beta,v}(u) = \min\{u^{1-\alpha}, uv^{-\beta}\}, \, \alpha, \beta \in [0, 1], v \in (0, 1], \]

which is concave. In particular, \( g_{\alpha,0,v}(u) = g_{0,\beta,v}(u) = u, \, g_{1,1,v}(u) = \min\{1, \frac{u}{v}\} \).

• The normal copulas:

\[ C_\rho(u, v) = \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v)), \]

where \( \Phi_\rho \) is a bivariate normal distribution with standard normal marginal distributions and the correlation coefficient \(-1 < \rho < 1\), \( \Phi^{-1} \) is the inverse function of the standard normal distribution. The corresponding distortion functions:

\[ g_{\rho,v}(u) = \frac{1}{v}\Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v)). \]

4 TAIL-ASYMPTOTICS FOR VAR

Subadditivity is an appealing property when aggregating risks in order to preserve the benefits of diversification. Subadditivity of two risks is not only dependent on their dependence structure but also on the marginal distributions. Value at risk is one of the most popular risk measures, but this risk measure is not always subadditive, nor convex, exception of elliptically distributed risks. This family consists of many symmetric distributions such as the multivariate normal family, the multivariate Student-\( t \) family, the multivariate logistic family and the multivariate exponential power family, and so on. A recent development in the VaR literature concerns the subadditivity in the tails (see Danielsson et al (2013)) which demonstrate that VaR is subadditive in the tails of all fat tailed distributions, provided the tails are not super fat. However, in most practical models of interest the support of loss is bounded so that the maximum loss is simply finite. We will also show that for this class losses VaR is subadditive in the tail. We can illustrate the ideas here with three simple examples. In Examples 4.1 and 4.3, \( X \) and \( Y \) are independent, while in Example 4.2, \( X \) and \( Y \) are dependent.
Example 4.1 Let $X$ and $Y$ be i.i.d. random variables which are Bernoulli (0.02) distributed, i.e. $P(X = 1) = 1 - P(X = 0) = 0.02$. Then

$$P(X + Y = 0) = P(X = 0)P(Y = 0) = 0.98^2 = 0.9604,$$

$$P(X + Y = 1) = P(X = 1)P(Y = 0) + P(X = 0)P(Y = 1) = 0.0392,$$

$$P(X + Y = 2) = P(X = 1)P(Y = 1) = 0.0004.$$ 

Dhaene et al. (2006) verified that VaR is not subadditive since $VaR_{0.975}[X]=VaR_{0.975}[Y]=0$ and $VaR_{0.975}[X + Y]=1$. However, for $p \geq 0.98$, $VaR_p[X]=VaR_p[Y]=1$ and

$$VaR_p[X+Y] = \begin{cases} 1, & \text{if } 0.98 \leq p < 0.9996, \\ 2, & \text{if } p \geq 0.9996. \end{cases}$$

Thus for $p \geq 0.98$,

$$VaR_p[X+Y] \leq VaR_p[X] + VaR_p[Y].$$

Example 4.2 Suppose we have losses $X$ and $Y$, both dependent on the same underlying Uniform(0,1) random variable $U$ as follows.

$$X = \begin{cases} 1000, & \text{if } U \leq 0.04 \\ 0, & \text{if } U > 0.04 \end{cases}$$

$$Y = \begin{cases} 0, & \text{if } U \leq 0.96 \\ 1000, & \text{if } U > 0.96 \end{cases}$$

Note that

$$P(X + Y = 0) = P(X = 0, Y = 0) = P(U > 0.04, U \leq 0.96) = 0.92,$$

$$P(X + Y = 1000) = P(X = 0, Y = 1000) + P(X = 1000, Y = 0) = 0.08.$$ 

Hardy (2006) found that $VaR_{0.95}[X]=VaR_{0.95}[Y]=0$, $VaR_{0.95}[X + Y]=1000$. Thus

$$VaR_{0.95}[X + Y] \geq VaR_{0.95}[X] + VaR_{0.95}[Y].$$

However, for any $\alpha > 0.96$, $VaR_\alpha[X]=VaR_\alpha[Y]=1000$, $VaR_\alpha[X + Y] = 1000$. Thus

$$VaR_\alpha[X + Y] \leq VaR_\alpha[X] + VaR_\alpha[Y].$$

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**Example 4.3** Let \( X \) and \( Y \) be i.i.d. random variables which are Uniform(0,1) distributed. Then

\[
F_{X+Y}(z) = \begin{cases} 
0, & \text{if } z < 0, \\
\frac{z^2}{2}, & \text{if } 0 \leq z < 1, \\
1 - \frac{1}{2}(2 - z)^2, & \text{if } 1 \leq z < 2, \\
1, & \text{if } z \geq 2,
\end{cases}
\]

and for \( p \in (0,1] \), \( VaR_p[X] = VaR_p[Y] = p \), \( VaR_p[X + Y] = 2 - \sqrt{2(1-p)} \). Thus for \( p \in \left[ \frac{1}{2}, 1 \right] \),

\[
VaR_p[X + Y] \leq VaR_p[X] + VaR_p[Y].
\]

Generally, we have the following result.

**Theorem 4.1.** If the risks \( X_1, X_2, \ldots, X_k \) have finite upper endpoints, then

\[
\limsup_{p \to 1} \frac{VaR_p[\sum_{i=1}^k X_i]}{\sum_{i=1}^k VaR_p[X_i]} \leq 1.
\]

**Proof** The proof is very simple. Denote by \( \text{esssup}(X_i) = \sup \{ x : P(X_i \leq x) < 1 \} \). Then \( \text{esssup}(X_i) < \infty \) and \( P(X_i \leq \text{esssup}(X_i)) = 1, i = 1, 2, \ldots, k \), which lead to

\[
P \left( \sum_{i=1}^k X_i \leq \sum_{i=1}^k \text{esssup}(X_i) \right) = 1.
\]

Hence

\[
\text{esssup} \left( \sum_{i=1}^k X_i \right) \leq \sum_{i=1}^k \text{esssup}(X_i),
\]

and the result follows.

**Remark 4.1.** Many distributions, such as Binomial, Uniform, have finite upper endpoints; Any truncated distribution: whether it is right truncated or doubly truncated all have finite upper endpoints.

Next theorem consider the random variables \( X_1, X_2, \ldots, X_k \) that are not necessarily has finite upper endpoint, we first recall the notion of (extended) regularly varying function:
Definition 4.1. A function \( f \) is called regularly varying at some point \( x^- \) (or \( x^+ \), respectively) with index \( \alpha \in \mathbb{R} \) if for all \( t > 0 \),
\[
\lim_{s \uparrow x} \frac{f(st)}{f(s)} = t^\alpha
\]
(or \( \lim_{s \downarrow x} \frac{f(st)}{f(s)} = t^\alpha \), respectively). We write \( f \in \mathcal{R}_\alpha^- \) (\( f \in \mathcal{R}_\alpha^+ \), respectively). For \( \alpha = 0 \) we say \( f \) is slowly varying; for \( \alpha = -\infty \) rapidly varying.

Definition 4.2. Assume that \( F \) is the distribution function of a nonnegative random. We say \( F \) belongs to the extended regular variation class, if there are some \( 0 \ll \alpha \leq \beta < \infty \) such that
\[
s^{-\beta} \leq \liminf_{x \to \infty} \frac{F(sx)}{F(x)} \leq \limsup_{x \to \infty} \frac{F(sx)}{F(x)} \leq s^{-\alpha}, \quad \text{for all } s \geq 1,
\]
or equivalently
\[
s^{-\alpha} \leq \liminf_{x \to \infty} \frac{F(sx)}{F(x)} \leq \limsup_{x \to \infty} \frac{F(sx)}{F(x)} \leq s^{-\beta}, \quad \text{for all } 0 < s \leq 1.
\]
We write \( F \in \text{ERV}(\alpha, \beta) \).

A standard reference to the topic of (extended) regular variation is Bingham et al. (1987) while main results are summarized by Embrechts et al. (1997).

Theorem 4.2. We assume that \( X_1, X_2, \ldots, X_k \) have the same absolutely continuous marginal distributions \( F \) with infinite upper endpoint.

1. If
\[
\lim_{z \to \infty} \frac{P(\sum_{i=1}^k X_i > z)}{P(X_1 > \frac{z}{k})} < 1,
\]
then
\[
\lim_{p \to 1} \frac{\text{VaR}_p[\sum_{i=1}^k X_i]}{\sum_{i=1}^k \text{VaR}_p[X_i]} < 1;
\]

2. If
\[
\lim_{z \to \infty} \frac{P(\sum_{i=1}^k X_i > z)}{P(X_1 > \frac{z}{k})} = 1,
\]

then
\[
\lim_{p \to 1} \frac{\text{VaR}_p[\sum_{i=1}^k X_i]}{\sum_{i=1}^k \text{VaR}_p[X_i]} = 1.
\]
(3) If
\[ \lim_{z \to \infty} \frac{P(\sum_{i=1}^{k} X_i > z)}{P(X_1 > \frac{z}{k})} > 1, \]
then
\[ \lim_{p \to 1} \frac{VaR_p[\sum_{i=1}^{k} X_i]}{\sum_{i=1}^{k} VaR_p[X_i]} > 1. \]

Proof We prove (1) only since the other cases follow immediately in the same way. Because all the marginal distributions are absolutely continuous, so we have for any \( p \in (0, 1), \)
\[ P(X_1 > VaR_p[X_1]) = P \left( \sum_{i=1}^{k} X_i > VaR_p \left[ \sum_{i=1}^{k} X_i \right] \right) = 1 - p. \]
This, together with (4.1), implies that
\[ \lim_{p \to 1} P(X_1 > VaR_p[X_1]) \left( X_1 > \frac{1}{k} VaR_p[\sum_{i=1}^{k} X_i] \right) < 1. \] (4.3)
The absolute continuity of \( F \) implies that \( \overline{F} \) is continuous and strictly monotone decreasing. Then from (4.3) we have
\[ \lim_{p \to 1} \frac{VaR_p[X_1]}{\frac{1}{k} VaR_p[\sum_{i=1}^{k} X_i]} > 1, \]
which is (4.2). This completes the proof.

Example 4.4 Suppose that each \( X_i \) is regularly varying with index \(-\alpha < 0\). When the \( X_i \) are mutually independent, it follows from (Feller 1971, p. 279) that
\[ \lim_{s \to \infty} \frac{P(\sum_{i=1}^{k} X_i > s)}{P(X_1 > \frac{s}{k})} = \frac{k}{k^\alpha}. \]
Thus we get
\[ \lim_{p \to 1} \frac{VaR_p[\sum_{i=1}^{k} X_i]}{\sum_{i=1}^{k} VaR_p[X_i]} \begin{cases} < 1, & \text{if } \alpha > 1, \\ = 1, & \text{if } \alpha = 1, \\ > 1, & \text{if } \alpha < 1. \end{cases} \]
Suppose that the \( X_i \) are comonotonic, i.e. \( P(X_1 = \cdots = X_k) = 1 \), then
\[ \lim_{s \to \infty} \frac{P(\sum_{i=1}^{k} X_i > s)}{P(X_1 > \frac{s}{k})} = 1. \]
So that in the case \( \alpha = 1 \) the result for the independent and the comonotonic case are the same.
Theorem 4.3. Suppose $Y_1, \cdots, Y_k$ are nonnegative random variables (but not necessarily independent or identically distributed.) If $Y_1$ has distribution $F$ satisfying $1 - F(x) = x^{-\alpha}L(x), \alpha > 0, x > 0$, where $L \in R_0^\infty$ is slowly varying at infinity. If $\frac{P(Y_i > x)}{F(x)} \to c_i$ and $\frac{P(Y_i > x, Y_j > x)}{F(x)} \to 0, i \neq j$, as $x \to \infty$, $i, j = 1, 2, \cdots, k$, then

$$\lim_{p \to 1} \frac{VaR_p[\sum_{i=1}^k Y_i]}{\sum_{i=1}^k VaR_p[Y_i]} \begin{cases} < 1, & \text{if } \alpha > 1, \\ = 1, & \text{if } \alpha = 1, \\ > 1, & \text{if } \alpha < 1. \end{cases}$$

Proof It follows from Lemma 2.1 in Davis and Resnick (1996) that

$$\lim_{p \to 1} \frac{P(\sum_{i=1}^k Y_i > x)}{1 - F(x)} \to \sum_{i=1}^k c_i, \text{ as } x \to \infty.$$  

This leads to

$$\lim_{p \to 1} \frac{P(\sum_{i=1}^k Y_i > VaR_p[\sum_{i=1}^k Y_i])}{P(Y_1 > VaR_p[\sum_{i=1}^k Y_i])} = \sum_{i=1}^k c_i. \quad (4.4)$$

Because

$$P(Y_1 > VaR_p[Y_1]) = P\left(\sum_{i=1}^k Y_i > VaR_p\left[\sum_{i=1}^k Y_i\right]\right) = 1 - p.$$  

Thus from (4.4) that

$$\lim_{p \to 1} \frac{P(Y_1 > VaR_p[Y_1])}{P(Y_1 > VaR_p[\sum_{i=1}^k Y_i])} = \sum_{i=1}^k c_i,$$

which is equivalent to

$$\lim_{p \to 1} \frac{P(Y_1 > VaR_p[Y_1])}{P\left(Y_1 > \frac{1}{\sum_{i=1}^k c_i}VaR_p[\sum_{i=1}^k Y_i]\right)} = 1.$$  

This implies that

$$\lim_{p \to 1} \frac{VaR_p[Y_1]}{\left(\sum_{i=1}^k c_i\right)^{-\frac{1}{\alpha}}VaR_p[\sum_{i=1}^k Y_i]} = 1,$$

since $F$ is continuous and strictly monotone decreasing. Note that $c_1 = 1, c_i^{1/\alpha}VaR_p[Y_1] \sim VaR_p[Y_i]$ (as $p \to 1$) and

$$\left(\sum_{i=1}^k c_i\right)^{1/\alpha} \begin{cases} < \sum_{i=1}^k c_i^{1/\alpha}, & \text{if } \alpha > 1, \\ = \sum_{i=1}^k c_i, & \text{if } \alpha = 1, \\ > \sum_{i=1}^k c_i^{1/\alpha}, & \text{if } \alpha < 1, \end{cases}$$

completing the proof.
Remark 4.2. The above result is obtained by Embrechts et al. (2009) for identically distributed and Archimedean copula dependent $Y_i$'s. However, our result can not obtained from theirs due to the following fact: The famous Farlie-Gumbel-Morgenstern family, does not belong to Archimedean family, which has the form

$$F(x, y) = F_1(x)F_2(y)(1 + \alpha F_1(x)F_2(y))$$

where $F_1, F_2$ are two distributions and $\alpha \in [-1, 1]$ is a constant. When $F_1 = F_2$, it satisfying $\frac{F(x, x)}{F_1(x)} \to 0$ as $x \to \infty$.

In the next theorem we consider the extended regularly varying instead of regularly varying.

**Theorem 4.4.** Suppose $Y_1, \ldots, Y_k$ are nonnegative random variables with the common identical distribution function $F$. If $F \in ERV(-\alpha, -\beta)$ and $\frac{P(Y_i > x, Y_j > x)}{F(x)} \to 0, i \neq j$, as $x \to \infty$, $i, j = 1, 2, \cdots, k$, then

1. If $\beta < 1$, 
   $$\limsup_{p \to 1} \frac{VaR_p[\sum_{i=1}^{k} Y_i]}{\sum_{i=1}^{k} VaR_p[Y_i]} > 1;$$

2. If $\alpha > 1$, 
   $$\liminf_{p \to 1} \frac{VaR_p[\sum_{i=1}^{k} Y_i]}{\sum_{i=1}^{k} VaR_p[Y_i]} < 1;$$

3. If $\alpha = \beta = 1$, 
   $$\lim_{p \to 1} \frac{VaR_p[\sum_{i=1}^{k} Y_i]}{\sum_{i=1}^{k} VaR_p[Y_i]} = 1.$$

**Proof** It follows from Lemma 2.2 in Zhang et al. (2009) that

$$\frac{P(\sum_{i=1}^{k} Y_i > x)}{1 - F(x)} \to k, \text{ as } x \to \infty.$$ 

This leads to

$$\lim_{p \to 1} \frac{P(\sum_{i=1}^{k} Y_i > VaR_p[\sum_{i=1}^{k} Y_i])}{P(Y_1 > VaR_p[\sum_{i=1}^{k} Y_i])} = k,$$

from which and using the same argument as that in the proof of Theorem 4.3 leads to

$$\lim_{p \to 1} \frac{P(Y_1 > VaR_p[Y_i])}{P(Y_1 > VaR_p[\sum_{i=1}^{k} Y_i])} = k.$$ \hfill (4.5)
If $\beta < 1$, then
\[
\limsup_{p \to 1} \frac{\bar{F}(k^{-\beta} \text{VaR}_p[\sum_{i=1}^{k} Y_i])}{(k^{-\beta})^{-\alpha} \bar{F}(\text{VaR}_p[\sum_{i=1}^{k} Y_i])} \leq 1.
\]

This and (4.5) imply that
\[
\limsup_{p \to 1} \frac{\bar{F} (\text{VaR}_p[Y_1])}{\bar{F}(k^{-\beta} \text{VaR}_p[\sum_{i=1}^{k} Y_i])} \geq 1.
\]

It follows that
\[
\limsup_{p \to 1} \frac{k^{\frac{-\beta}{\alpha}} \text{VaR}_p[Y_1]}{\text{VaR}_p[\sum_{i=1}^{k} Y_i]} \leq 1. \tag{4.6}
\]

Thus
\[
\limsup_{p \to 1} \frac{k \text{VaR}_p[Y_1]}{\text{VaR}_p[\sum_{i=1}^{k} Y_i]} < 1.
\]

Similarly, if $\alpha > 1$,
\[
\liminf_{p \to 1} \frac{k^{\frac{1}{\alpha}} \text{VaR}_p[Y_1]}{\text{VaR}_p[\sum_{i=1}^{k} Y_i]} \geq 1. \tag{4.7}
\]

and hence
\[
\liminf_{p \to 1} \frac{k \text{VaR}_p[Y_1]}{\text{VaR}_p[\sum_{i=1}^{k} Y_i]} > 1.
\]

If $\alpha = \beta = 1$, then by (4.6) and (4.7) one has
\[
\lim_{p \to 1} \frac{k \text{VaR}_p[Y_1]}{\text{VaR}_p[\sum_{i=1}^{k} Y_i]} = 1. \tag{4.8}
\]

This ends the proof of Theorem 4.4.

To give applications of our results we employ extreme value theory techniques. A distribution function $F$ (or the rv $X$) is said to belong to the Maximum Domain of Attraction (MDA) of the extreme value distribution $H$ if there exist constants $c_n > 0, d_n \in \mathbb{R}$ such that $c_n^{-1}(\max(X_1, \ldots, X_n) - d_n) \overset{d}{\to} H$. We write $X \in \text{MDA}(H)$ or $F \in \text{MDA}(H)$. According to the Fisher-Tippett theorem (see Theorem 3.2.3 in Embrechts et al. (1997)) $H$ belongs to one of the three standard extreme value distributions:

**Frechet type**:
\[
\Phi_\alpha(x) = \begin{cases} 
0, & \text{if } x \leq 0, \\
\exp\{-x^{-\alpha}\}, & \text{if } x > 0, \quad \alpha > 0.
\end{cases}
\]

**Weibull type**:
\[
\Psi_\alpha(x) = \begin{cases} 
\exp\{-(x)^\alpha\}, & \text{if } x \leq 0, \\
1, & \text{if } x > 0, \quad \alpha > 0.
\end{cases}
\]

**Gumbel type**:
\[
\Lambda(x) = \exp\{-e^{-x}\}, \quad x \in \mathbb{R}.
\]
Let \( x_F \) denote the right-endpoint of the support of \( F \): \( x_F = \inf \{ x : F(x) = 1 \} \). Then we have the following results (see Embrechts et al. (1997), PP. 132-157).

- Fréchet case: For some \( \alpha > 0 \), \( F \in MDA(\Phi_\alpha) \Leftrightarrow F \in R_\alpha^\infty. \) Examples are Pareto, Cauchy, Burr, Loggamma and Stable with index \( \beta < 2 \).

- Weibull case: For some \( \alpha > 0 \), \( F \in MDA(\Psi_\alpha) \Leftrightarrow x_F < \infty, \overline{F}(x_F - 1/x) \in R_\alpha^\infty. \) Examples are Uniform and Beta distribution.

- Gumbel case: \( F \in MDA(\Lambda_\alpha) \Leftrightarrow x_F \leq \infty \) and there exists a positive measurable function \( a \) such that for \( t \in \mathbb{R} \)

\[
\lim_{x \uparrow x_F} \frac{\overline{F}(x + ta(x))}{\overline{F}(x)} = e^{-t}.
\] (4.9)

Examples are Exponential-like, Weibull-like, Gamma, Normal, Lognormal, Benktander-type-I and Benktander-type-II.

**Remark 4.3.** (1) For \( \alpha > 0 \), if \( X_1, X_2, \cdots, X_k \in MDA(\Psi_\alpha) \), in view of Weibull case above they are all have finite supports. It follows from Theorem 4.1, \( \text{VaR}_p \) is subadditive for \( p \) is sufficiently close to 1.

(2) For \( \alpha > 0 \), if \( X_1, X_2, \cdots, X_k \in MDA(\Phi_\alpha) \) and are identically distributed, \((-X_1, -X_2, \cdots, -X_k)\) has an Archimedean copula with generator \( \psi \), which is regularly varying at 0 with index \(-\beta < 0 \). We apply (2.2) in Alink et al. (2004) and Definition 4.1 to obtain

\[
\lim_{z \to \infty} \frac{P(\sum_{i=1}^k X_i > z)}{P(X_1 > \frac{z}{k})} = \lim_{z \to \infty} \frac{P(\sum_{i=1}^k X_i > z)}{P(X_1 > z)} \frac{P(X_1 > z)}{P(X_1 > \frac{z}{k})}
= q_k(\beta, \alpha) \lim_{z \to \infty} \frac{P(X_1 > \frac{z}{k})}{P(X_1 > \frac{z}{k})}
= \lim_{z \to \infty} \frac{P(X_1 > z q_k(\beta, \alpha)^{-1/\alpha})}{P(X_1 > \frac{z}{k})}
= k^{-\alpha} q_k(\beta, \alpha) \begin{cases} < 1, & \text{if } \alpha > 1, \\ = 1, & \text{if } \alpha = 1, \\ > 1, & \text{if } \alpha < 1, \end{cases}
\]

where in the last step we have used Lemma 3.1(d) in Embrechts et al. (2009) which states that

\[ \min \{ k^\alpha, k \} \leq q_k(\beta, \alpha) \leq \max \{ k^\alpha, k \}. \]
This, together with Theorem 4.2 we recover the result Theorem 2.5 in Embrechts et al. (2009).

(3). If \( X_1, X_2, \cdots, X_k \in MDA(\Lambda_\alpha) \) have common distribution \( F, (-X_1, -X_2, \cdots, -X_k) \) has an Archimedean copula with generator \( \psi \), which is regularly varying at 0 with index \(-\beta < 0\). We apply (2.6) in Alink et al. (2004) to obtain

\[
\lim_{z \to \infty} \frac{P(\sum_{i=1}^{k} X_i > z)}{P(X_1 > \frac{z}{k})} = e^{-\frac{1}{\beta}} q_k^G(\beta),
\]

where

\[
q_k^G(\beta) = \int_{\sum_{i=1}^{k} x_i \leq 1} \frac{d^{k}x_1 \cdots dx_k}{dx_1 \cdots dx_k} \left( \sum_{i=1}^{k} e^{-\beta x_i} \right)^{-1/\beta} dx_1 \cdots dx_k.
\]

The constant \( q_k^G(\beta) \leq e^{\frac{1}{k}} \) is strictly increasing in \( \beta \) with

\[
\lim_{\beta \to 0} q_k^G(\beta) = 0, \quad \lim_{\beta \to \infty} q_k^G(\beta) = e^{\frac{1}{k}}.
\]

For more details, see Alink et al. (2004) for the case \( k = 2 \) and Chen et al. (2012) for general case. Thus by Theorem 4.2,

\[
\lim_{p \to 1} \frac{VaR_p[\sum_{i=1}^{k} X_i]}{\sum_{i=1}^{k} VaR_p[X_i]} \leq 1.
\]

In particular, when \( \alpha \to \infty \),

\[
\lim_{p \to 1} \frac{VaR_p[\sum_{i=1}^{k} X_i]}{\sum_{i=1}^{k} VaR_p[X_i]} = 1
\]

**Remark 4.4.** Note that \( VaR_p[X] \) is a left-continuous nondecreasing function having \( VaR_0[X] \) as the essential infimum of \( X \) and \( VaR_1[X] \) as the essential supremum of \( X \). Thus under the assumptions of Theorem 4.1 or Theorem 4.2, if \( p \) close to 1, we have

\[
VaR_p[X_1 + X_2] \leq VaR_p[X_1] + VaR_p[X_2],
\]

which, together with the positive homogeneity of \( VaR_p[X] \), implies that, if \( p \) close to 1, the convexity is holds:

\[
VaR_p[\lambda X + (1 - \lambda)Y] \leq \lambda VaR_p[X] + (1 - \lambda)VaR_p[Y], \quad 0 \leq \lambda \leq 1.
\]

From above analysis we see that, although, in general the VaR risk measure lack of subadditivity and convexity. However, one should not too worries about violations of subadditivity for risk management applications relying on VaR, since in most practical circumstances it is subadditive, at least is subadditive in the tail, and the failure to be subadditive in a few situations is not sufficiently important to reject the VaR risk measure.
5 TAIL-SUBADDITIVITY FOR DISTORTION RISK MEASURES

The tail-subadditivity property for GlueVaR risk measures were initially defined by Belles-Sampera et al. (2014a) and the milder condition of subadditivity in the tail region is investigated. Furthermore, they verified that a GlueVaR risk measure is tail-subadditive if its associated distortion function $k_{\beta,\alpha}^{h_1,h_2}(u)$ is concave in $[0, 1 - \alpha)$, where parameters $\alpha$ is confidence level and $\beta$ is an extra confidence level such that $0 \leq \alpha \leq \beta \leq 1$ and,

$$k_{\beta,\alpha}^{h_1,h_2}(u) = \begin{cases} 
\frac{h_1}{1-\beta}u, & \text{if } 0 \leq u < 1 - \beta, \\
h_1 \frac{h_2 - h_1}{\beta - \alpha} (u - 1 + \beta), & \text{if } 1 - \beta \leq u < 1 - \alpha, \\
1, & \text{if } 1 - \alpha \leq u \leq 1,
\end{cases}$$

where $h_1$ and $h_2$ are two distorted survival probabilities at levels $1 - \beta$ and $1 - \alpha$, respectively. Here $0 \leq h_1 \leq h_2 \leq 1$. We note, however, from their proof to Theorem 6.1 that the result will hold for any distortion function that is concave in $[0, 1 - \alpha)$, not restricted to $k_{\beta,\alpha}^{h_1,h_2}(u)$. In this section we state the result and give an alternative proof. As in Belles-Sampera et al. (2014a), for a given confidence level $\alpha$, the tail region of a random variable $Z$ is defined as $Q_{\alpha,Z} = \{w | Z(w) > s_\alpha\} \subseteq \Omega$, where $s_\alpha = \inf \{z | F_Z(z) \leq 1 - \alpha\}$ is the $\alpha$-quantile. For simplicity, we use the notation $S_Z(z) := F_Z(z)$.

**Theorem 5.1.** For a confidence level $\alpha \in [0, 1)$ and two risks $X, Y$ defined on the same probability space. If $Q_{\alpha,X} \cap Q_{\alpha,Y} \cap Q_{\alpha,X+Y} \neq \emptyset$ and $g$ is a concave distortion function in $[0, 1 - \alpha)$, then the distortion risk measure $\rho_g$ is tail-subadditive. That is

$$\int_{0 \wedge m_\alpha}^{0} [g(S_{X+Y}(z)) - 1]dz + \int_{0 \vee m_\alpha}^{\infty} g(S_{X+Y}(z))dz$$

$$\leq \int_{0 \wedge m_\alpha}^{0} [g(S_X(z)) - 1]dz + \int_{0 \vee m_\alpha}^{\infty} g(S_X(z))dz$$

$$+ \int_{0 \wedge m_\alpha}^{0} [g(S_Y(z)) - 1]dz + \int_{0 \vee m_\alpha}^{\infty} g(S_Y(z))dz,$$

where $m_\alpha = \sup \{s_\alpha(X), s_\alpha(Y), s_\alpha(X + Y)\}$. 

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Proof Without loss of the generality, we assume that the risks $X$ and $Y$ are nonnegative, so that $m_\alpha = S_{X+Y}^{-1}(1-\alpha) \geq 0$. It follows that

$$\int_{m_\alpha}^{\infty} g(S_{X+Y}(z))dz = \int_{S_{X+Y}^{-1}(1-\alpha)}^{\infty} g(S_{X+Y}(z))dz$$

$$= \int_{S_{X+Y}^{-1}(1-\alpha)}^{\infty} dx \int_{[0,S_{X+Y}(x)]} dg(q)$$

$$= \int_{[0,S_{X+Y}(S_{X+Y}^{-1}(1-\alpha))]} dg(q) \int_{S_{X+Y}^{-1}(1-\alpha)} F_{X+Y}^{-1}(1-q) dx$$

$$= \int_{[0,S_{X+Y}(S_{X+Y}^{-1}(1-\alpha))]} F_{X+Y}^{-1}(1-q)dg(q) - S_{X+Y}^{-1}(1-\alpha)g(1-\alpha),$$

where in the third step we have used the Fubini’s theorem to change the order of integration. As above, we have

$$\int_{[0,S_{X+Y}(S_{X+Y}^{-1}(1-\alpha))]} F_{X+Y}^{-1}(1-q)dg(q) = \int_{1-S_{X+Y}(S_{X+Y}^{-1}(1-\alpha))}^{1} F_{X+Y}^{-1}(q)g'(1-q) dq$$

$$= \int_{0}^{1} TVaR_{X+Y}(w)d\mu_{X+Y}(w),$$

where

$$d\mu_{X+Y}(w) = (1-w)d\nu_{X+Y}(w), \nu_{X+Y}([0,q]) = 1_{(1-S_{X+Y}(S_{X+Y}^{-1}(1-\alpha)),1]}(q)g'(1-q).$$

Finally, we get

$$\int_{m_\alpha}^{\infty} g(S_{X+Y}(z))dz = \int_{0}^{1} TVaR_{X+Y}(w)d\mu_{X+Y}(w) - S_{X+Y}^{-1}(1-\alpha)g(1-\alpha). \quad (5.1)$$

Similarly,

$$\int_{m_\alpha}^{\infty} g(S_{X}(z))dz = \int_{0}^{1} TVaR_{X}(w)d\mu_{X}(w) - S_{X+Y}^{-1}(1-\alpha)g(1-\alpha), \quad (5.2)$$

and

$$\int_{m_\alpha}^{\infty} g(S_{Y}(z))dz = \int_{0}^{1} TVaR_{Y}(w)d\mu_{Y}(w) - S_{X+Y}^{-1}(1-\alpha)g(1-\alpha), \quad (5.3)$$

where

$$d\mu_{X}(w) = (1-w)d\nu_{X}(w), \nu_{X}([0,q]) = 1_{(1-S_{X}(S_{X}^{-1}(1-\alpha)),1]}(q)g'(1-q),$$

and

$$d\mu_{Y}(w) = (1-w)d\nu_{Y}(w), \nu_{Y}([0,q]) = 1_{(1-S_{Y}(S_{X+Y}^{-1}(1-\alpha)),1]}(q)g'(1-q).$$
By the subadditivity of TVaR and note that $\nu_{X+Y}([0, q]) \leq \nu_X([0, q]), \nu_Y([0, q])$, we obtain

$$
\int_0^1 TVaR_{X+Y}(w) d\mu_{X+Y}(w) \leq \int_0^1 TVaR_w(X) d\mu_{X+Y}(w)
+ \int_0^1 TVaR_w(X) d\mu_{X+Y}(w)
\leq \int_0^1 TVaR_X(w) d\mu_X(w)
+ \int_0^1 TVaR_Y(w) d\mu_Y(w),
$$

this, together with (5.1)-(5.3), implies that

$$
\int_{m_\alpha}^{\infty} g(S_{X+Y}(z)) dz \leq \int_{m_\alpha}^{\infty} g(S_X(z)) dz + \int_{m_\alpha}^{\infty} g(S_Y(z)) dz,
$$
as desired.

**Remark 5.1.** Consider the distortion functions associated with the Gumbel-Hougaard copulas (cf. Section 3.3):

$$
g_{\alpha,v}(u) = \frac{1}{v} \exp \left\{ - \left( -\ln u \right)^\alpha + \left( -\ln v \right)^\alpha \right\}^{1/\alpha}.
$$

If $\alpha > 2$, then $g_{\alpha,v}(u)$ is concave on $(0, e^{2-\alpha})$ and convex on $(e^{2-\alpha}, 1]$. Thus the distortion risk measure $\rho_{g_{\alpha,v}(u)}$ is tail-subadditive.

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**References**

[1] Acerbi, C., (2002): Spectral measures of risk: a coherent representation of subjective risk aversion. Journal of Banking Finance 26, 1505-1518.

[2] Alink, S., Löwe, M., Wüthrich, M. V., (2004): Diversification of aggregate dependent risks. Insurance Mathematics & Economics 35, 77-95.

[3] Artzner, P., Delbaen, F., Eber, J.-M., Heath, D., (1997): Thinking coherently. Risk 10(11), 68-71.
[4] Artzner, P., Delbaen, F., Eber, J.-M., Heath, D., (1999): Coherent measures of risk. Mathematical Finance 9, 203-228.

[5] Balbás, A., Garrido, J., Mayoral, S., (2009): Properties of distortion risk measures. Methodology and Computing in Applied Probability 11, 385-399.

[6] Bannöhr, Karl F., Scherer, M., (2014): On the calibration of distortion risk measures to bid-ask prices. Quantitative Finance 14(7), 1217-1228.

[7] Belles-Sampera, J., Guillén, M., Santolino, M., (2014a): Beyond Value-at-Risk: Glue-VaR distortion risk measures. Risk Analysis 34 (1), 121-134.

[8] Belles-Sampera, J., Guillén, M., Santolino, M., (2014b): GlueVaR risk measures in capital allocation applications. Insurance: Mathematics and Economics 58, 132-137.

[9] Bingham, N. H., Goldie, C. M., Teugels, J. L., (1987): Regular Variation. Cambridge University Press, Cambridge.

[10] Chen, D., Mao, T., Pan, X., Hu, T. Z., (2012): Extreme value behavior of aggregate dependent risks. Insurance: Mathematics and Economics 50, 99-108.

[11] Cherny, A. S., (2006): Weighted VaR and its properties. Finance and Stochastics 10, 367-393.

[12] Cherubini, U., Mulinacci, S., (2014): Contagion-based distortion risk measures. Applied Mathematics Letters 27, 85-89.

[13] Danielsson, J., Jorgensen, B. N., Samorodnitsky, G., Sarmad, M., Vries, C. G., (2013): Fat tails, VaR and subadditivity. Journal of Econometrics 172, 283-291.

[14] Davis, R. A., Resnick, Sidney I., (1996): Limit theory for bilinear processes with heavy-tailed noise. The Annals of Applied Probability 6(4), 1191-1210.

[15] Denneberg, D., (1994): Non-additive measure and integral. Theory and Decision Library 27, Kluwer Academic Publishers.

[16] Denuit, M., Dhaene, J., Goovaerts, M. and Kaas, R., (2006): Actuarial Theory for Dependent Risks : Measures, Orders and Models. John Wiley & Sons, Ltd.

[17] Dhaene, J., Kukush, A., Linders, D., Tang, Q., (2012): Remarks on quantiles and distortion risk measures. European Actuarial Journal 2 (2), 319-328.
[18] Dhaene, J., Vanduffel, S., Tang, Q., Goovaerts, M., Kaas, R., Vyncke, D., (2006): Risk measures and comonotonicity: a review. Stochastic Models 22, 573-606.

[19] Dolati, A., Nezhad, A. D., (2014): Some Results on Convexity and Concavity of Multivariate Copulas. Iranian Journal of Mathematical Sciences and Informatics 9(2), 87-100.

[20] Dowd, K., Blake, D., (2006): After var: the theory, estimation, and insurance applications of quantile-based risk measures. The Journal of Risk and Insurance 73(2), 193-229.

[21] Embrechts, P., Klüppelberg, C., Mikosch, T., (1997): Modelling Extremal Events for Insurance and Finance. Springer-Verlag, Berlin.

[22] Embrechts, P., Nešehová, J., Wüthrich, M. V., (2009): Additivity properties for Value-at-Risk under Archimedean dependence and heavy-tailedness. Insurance: Mathematics and Economics 44, 164-169.

[23] Feller, W., (1971): An introduction to probability theory and its applications, V.II, Second Edition, Wiley, New York.

[24] Frittelli, M., Maggis, M., Peri, I., (2014): Risk measures on $\mathbb{P}(\mathbb{R})$ and Value at Risk with Probability/Loss function. Mathematical Finance 24, 442-463.

[25] Hardy, M. R., (2001): A regime-switching model of long-term stock returns. North American Actuarial Journal 5(2), 41-53.

[26] Hardy, M. R., (2006): An introduction to risk measures for actuarial applications. Construction and Evaluation of Actuarial Models Study Note, Education and Examination Committee of the Society of Actuaries.

[27] Hua, L., Joe, H., (2012): Tail comonotonicity: Properties, constructions, and asymptotic additivity of risk measures. Insurance: Mathematics and Economics 51 (2), 492-503.

[28] Hürlimann, W., (1998): Inequalities for Look Back Option Strategies and Exchange Risk Modelling. In Proceedings of the 1st Euro-Japanese Workshop on Stochastic Modelling for Finance, Insurance, Production and Reliability, Brussels.

[29] Joe, H., (1997): Multivariate Models and Dependence Concepts. Chapman & Hall, London.
[30] Kusuoka, S., (2001): On law invariant coherent risk measures. Adv Math Econ 3, 83-95.

[31] Kriele, M., Wolf, J., (2014): Value-Oriented Risk Management of Insurance Companies, Springer-Verlag, London.

[32] Lv, W., Pan, X., Hu, T., (2013): Asymptotics of the risk concentration based on the tail distortion risk measure. Statistics & Probability Letters 83(12), 2703-2710.

[33] Mao, T., Hu, T., (2013): Second-order properties of risk concentrations without the condition of asymptotic smoothness. Extremes 16(4), 383-405.

[34] Mao, T., Lv, W., Hu, T., (2012): Second-order expansions of the risk concentration based on CTE. Insurance: Mathematics & Economics 51, 449-456.

[35] Nelsen, R. B., (1999): An Introduction to Copulas, Springer-Verlag, New York.

[36] Rockafellar, R. T., Uryasev, S. P., (2000): Optimization of conditional value-at-risk. Journal of Risk 2, 21-42.

[37] Rockafellar, R. T., Uryasev, S. P., (2002): Conditional value-at-risk for general loss distributions. Journal of Banking & Finance 26, 1443-1471.

[38] Tsukahara, H., (2009): One-parameter families of distortion risk measures. Mathematical Finance 19, 691-705.

[39] Wang, S. S., (1996): Premium calculation by transforming the layer premium density. ASTIN Bulletin 26, 71-92.

[40] Wang, S. S., (2000): A class of distortion operators for pricing financial and insurance risks. Journal of Risk and Insurance 67(1), 15-36.

[41] Wang, S., Dhaene, J., (1998): Comonotonicity, correlation order and premium principles. Insurance: Mathematics and Economics 22, 235-242.

[42] Wang, S., Young, V. R., (1998): Ordering risks. Expected utility theory versus Yaari dual theory of risk. Insurance: Mathematics Economics 22, 145-161.

[43] Wirch, J. L., Hardy, M. R., (1999): A synthesis of risk measures for capital adequacy. Insurance: Math. Econom. 25, 337-347.

[44] Wirch, J. L., Hardy, M. R., (2001): Distortion risk measures: coherence and stochastic dominance. Working paper, [http://pascal.iseg.utl.pt/cemapre/ime2002/](http://pascal.iseg.utl.pt/cemapre/ime2002/)
[45] Yang, J. P., Cheng, S., Zhang, L. H., (2006): Bivariate copula decomposition in terms of comonotonicity, countermonotonicity and independence. Insurance: Mathematics and Economics 39, 267-284.

[46] Yaari, M. E., (1987): The dual theory of choice under risk. Econometrica 55, 95-115.

[47] Yang, F., (2015): First- and second-order asymptotics for the tail distortion risk measure of extreme risks. Communications in Statistics - Theory and Methods 44(3), 520-532.

[48] Zhang, Y., Shen, X., Weng, C., (2009): Approximation of the tail probability of randomly weighted sums and applications. Stochastic Processes and their Applications 119, 655-675.

[49] Zhu, L., Li, H., (2012): Tail distortion risk and its asymptotic analysis. Insurance: Mathematics & Economics 51, 115-121.