NORMAL AFFINE SURFACES WITH $\mathbb{C}^*$-ACTIONS

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Abstract. A classification of affine surfaces admitting a $\mathbb{C}^*$-action was given in the work of Bialynicki-Birula, Fieseler and L. Kaup, Orlik and Wagreich, Rynes and others. We provide a simple alternative description of normal quasihomogeneous affine surfaces in terms of their graded rings as well as by defining equations. This is based on a generalization of the Dolgachev-Pinkham-Demazure construction.

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INTRODUCTION

A classification of (normal) affine surfaces admitting a $\mathbb{C}^*$-action was given e.g., in [Bi, BiSo, OrWa, Pi, BasHa, Ry] and [FiKa1]-[FiKa3]. Here we obtain a simple alternative description of normal affine surfaces $V$ with a $\mathbb{C}^*$-action in terms of their graded coordinate rings as well as by defining equations. Our approach is based on a generalization of the Dolgachev-Pinkham-Demazure construction [Do, Pi, De]. Recall (see [FiKa1]-[FiKa3]) that a $\mathbb{C}^*$-action on a normal affine surface $V$ is called elliptic if it has a unique fixed point which belongs to the closure of every 1-dimensional orbit,

parabolic if the set of its fixed points is 1-dimensional, and

hyperbolic if $V$ has only a finite number of fixed points, and these fixed points are of hyperbolic type, that is each one of them belongs to the closure of exactly two 1-dimensional orbits.

In the elliptic case, the complement $V^*$ of the unique fixed point in $V$ is fibered by the 1-dimensional orbits over a projective curve $C$. In the other two cases $V$ is fibered over an affine curve $C$, and this fibration is invariant under the $\mathbb{C}^*$-action.

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Vice versa, given a smooth curve $C$ and a $\mathbb{Q}$-divisor $D$ on $C$, the Dolgachev-Pinkham-Demazure construction provides a normal affine surface $V = V_{C,D}$ with a $\mathbb{C}^*$-action such that $C$ is just the algebraic quotient of $V^*$ or of $V$, respectively. This surface $V$ is of elliptic type if $C$ is projective and of parabolic type if $C$ is affine.

We remind this construction in sections 1 and 2 below. In section 3 we use it to present any normal affine surface $V$ with a parabolic $\mathbb{C}^*$-action as a normalization of the surface $x^d - P(z)y = 0$ in $\mathbb{A}^3_{\mathbb{C}}$ for a certain $d \in \mathbb{N}$ and a certain polynomial $P \in \mathbb{C}[t]$ (see Theorem 3.11).

In section 4 we deal with the hyperbolic case. We generalize the Dolgachev-Pinkham-Demazure construction in order to make it work for any hyperbolic $\mathbb{C}^*$-surface. Instead of one $\mathbb{Q}$-divisor $D$ on a smooth affine curve $C$ as before, it involves now two $\mathbb{Q}$-divisors $D_+$ and $D_-$ on $C$. By our result isomorphism classes of normal affine hyperbolic $\mathbb{C}^*$-surfaces are in 1-1-correspondence to equivalence classes of triples $(C, D_+, D_-)$, where $C$ is a smooth affine curve and $D_+, D_-$ is a pair of $\mathbb{Q}$-divisors on $C$ with $D_+ + D_- \leq 0$; two such triples $(C, D_+, D_-)$ and $(C, D'_+, D'_-)$ are considered to be equivalent if and only if $C \cong C'$ and $D_\pm = D'_\pm \pm D_0$ with a principal divisor $D_0$; cf. Theorem 4.3. We also determine the structure of the singularities, the orbits, the divisor class group and the canonical divisor in terms of the divisors $D_\pm$, see Theorems 4.13, 4.18, 4.22 and Corollary 4.24.

Using our description it is possible to represent any normal hyperbolic $\mathbb{C}^*$-surface fibered over $C = \mathbb{A}^1_{\mathbb{C}}$ as the normalization of a surface in $\mathbb{A}^3_{\mathbb{C}}$ given by

$$x^{dk} - P(t)y = 0, \quad x^{ek}z - Q(t) = 0 \quad \text{and} \quad y^e z^d - R(t) = 0,$$

for certain polynomials $P, Q, R \in \mathbb{C}[t]$ satisfying the relation $P^e R = Q^d$, where $e, d$ are coprime. These polynomials can be easily computed in terms of the data $(D_+, D_-)$ (see Proposition 4.8). For instance, if the divisor $D_-$ is integral then this system reduces to one equation $x^e z - Q(t) = 0$ in $\mathbb{A}^3_{\mathbb{C}}$, and vice versa. When $k = 1$ then it again reduces to one equation $y^e z^d - R(t) = 0$ in $\mathbb{A}^3_{\mathbb{C}}$.

In Proposition 4.11 we show how the pair $(D_+, D_-)$ is transformed when passing to an equivariant cyclic cover of $V$. We deduce, in particular, a characterization of normal hyperbolic $\mathbb{C}^*$-surfaces over $C = \mathbb{A}^1_{\mathbb{C}}$ with the fractional part of $D_-$ supported at one point, as normalized cyclic quotients of the surfaces $x^e z - Q(t) = 0$ in $\mathbb{A}^3_{\mathbb{C}}$.

In the forthcoming paper [FZa], which is actually Part II of the present one, we will apply these results to give a simple description of all normal affine $\mathbb{C}^*$-surfaces equipped in addition by a $\mathbb{C}^+$-action. In fact, this class consists of all normal affine surfaces which admit an algebraic group action with an open orbit.

We note that the results of this paper hold \textit{m.m.} for graded 2-dimensional normal algebras of finite type over a Dedekind domain.

1. Generalities on graded rings

A $\mathbb{Z}$-graded ring $A = \bigoplus_{i \in \mathbb{Z}} A_i$ contains $A_{\geq 0} = \bigoplus_{i \geq 0} A_i$ and $A_{\leq 0} = \bigoplus_{i \leq 0} A_i$ as subrings. The following lemma is “well known”; in lack of a reference we provide a short argument.

Lemma 1.1. If $A = \bigoplus_{i \in \mathbb{Z}} A_i$ is a finitely generated $A_0$-algebra, then so are $A_{\geq 0}$ and $A_{\leq 0}$. Moreover, $A$ is normal if and only if so are both $A_{\geq 0}$ and $A_{\leq 0}$.
Proof. Reversing the grading interchanges the subrings \( A_{\geq 0} \) and \( A_{\leq 0} \). Thus it is sufficient to prove the first part for \( A_{\geq 0} \). If \( a_{ij} \in A_i \) with \(-n \leq i \leq n, j = 1, \ldots, n_i\), is a system of homogeneous generators of \( A_i \), then \( A_{\geq 0} \) is generated (as a module over \( A_0 \)) by the multiplicatively closed system of monomials

\[
a^k := \prod_{i,j} a_{ij}^{k_{ij}},
\]

where \( k := (k_{ij}) \in \mathbb{Z}^N \) satisfies the inequalities

(1) \[ k_{ij} \geq 0, \quad -n \leq i \leq n, \quad j = 1, \ldots, n_i, \quad \sum_{i,j} ik_{ij} \geq 0. \]

By Gordan’s Lemma (see [Od]) the rational polyhedral lattice cone \( K \subseteq \mathbb{Z}^N \) defined by (1) is a finitely generated semigroup. Hence the algebra \( A_{\geq 0} \) is generated by a finite system of monomials \( a^k \in A_{\geq 0} \).

Next we show that the subalgebra \( A_{\geq 0} \) (and then also \( A_{\leq 0} \)) is normal if so is \( A \). Indeed, the integral closure \( (A_{\geq 0})_{\text{norm}} \subseteq A = A_{\text{norm}} \) is graded. Take a homogeneous element \( x \in (A_{\geq 0})_{\text{norm}} \) of degree \( d := \deg x \), and let

(2) \[ x^n + \sum_{i=1}^{n} b_i x^{n-i} = 0, \quad \text{where} \quad b_i \in A_{\geq 0}, \]

be an equation of integral dependence. We may assume that \( b_i \) are also homogeneous, of degree \( \deg b_i = di \geq 0 \). Since \( \deg b_i \geq 0 \) we have \( d \geq 0 \), and so \( x \in A_{\geq 0} \).

Conversely, suppose that both \( A_{\geq 0} \) and \( A_{\leq 0} \) are normal. The ring \( A \otimes_{A_0} \text{Frac}(A_0) \) is normal and so is equal to \( \text{Frac}(A_0)[u, u^{-1}] \) for a homogeneous element \( u \) of minimal degree \( > 0 \) in \( A \otimes_{A_0} \text{Frac}(A_0) \). Hence \( A_{\text{norm}} \) is contained in this subring of \( \text{Frac} A \). If \( f \in A \otimes_{A_0} \text{Frac}(A_0) \) belongs to the normalization \( A_{\text{norm}} \) of \( A \) then so does its top homogeneous component. Thus it is enough to deal with homogeneous elements. Let \( a \) be such an element satisfying an equation of integral dependence \( (\mathbb{2}) \) over \( A \). We may suppose as above that \( b_i \in A_{di} \) \( i = 1, \ldots, n \). Since \( di \) has the same sign as \( d := \deg a \), we have \( a \in (A_{\geq 0})_{\text{norm}} = A_{\geq 0} \) if \( d \geq 0 \) and \( a \in (A_{\leq 0})_{\text{norm}} = A_{\leq 0} \) if \( d \leq 0 \), respectively.

Anyhow, \( a \in A \), whence \( A \) is normal, as stated. \( \square \)

Notation 1.2. Let \( V = \text{Spec} A \) be a normal affine surface over \( \mathbb{C} \) with an effective \( \mathbb{C}^* \)-action. The coordinate ring \( A = \bigoplus_{i \in \mathbb{Z}} A_i \) is then naturally graded so that \( A_i \) is the set of elements of \( A \) on which \( t \in \mathbb{C}^* \) acts via \( t.f = t^i f \). Thus, \( A_0 = A^{\mathbb{C}^*} \) is the subalgebra of invariants, and \( A_i \quad (i \neq 0) \) consists of the quasi-invariants of weight \( i \). Up to reversing the grading we may assume that \( A_+ := \bigoplus_{i > 0} A_i \neq 0 \). The subsets \( A_+ \) and \( A_- := \bigoplus_{i < 0} A_i \) of \( A \) are ideals in \( A_{\geq 0} \) and \( A_{\leq 0} \), respectively.

The following lemma is well known (see e.g., [Dq], [HiKa1], Lemma 1.5]).

Lemma 1.3. (a) If \( A_0 \neq \mathbb{C} \) then the set \( M := \{ i \in \mathbb{Z} \mid A_i \neq 0 \} \) coincides either with \( \mathbb{N} \) or with \( \mathbb{Z} \), and \( A_i \) is a locally free \( A_0 \)-module of rank 1 for all \( i \in M \). Moreover, if \( u \in \text{Frac}(A_0) \cdot A_1 \) is a non-zero element then

\[ A \subseteq \text{Frac}(A_0)[u, u^{-1}], \quad \text{and even} \quad A \subseteq \text{Frac}(A_0)[u] \quad \text{if} \quad M = \mathbb{N}. \]

(b) In particular, if \( A_0 \cong \mathbb{C}[t] \) then \( A_i \) is a free \( A_0 \)-module of rank 1 for all \( i \in M \).
Proof. (a) The $K_0 := \text{Frac}(A_0)$-algebra $A \otimes A_0$ $K_0$ is a 1-dimensional normal graded domain over the field $K_0$. Hence it is isomorphic to the free polynomial ring $K_0[u]$ or the ring of Laurent polynomials $K_0[u, u^{-1}]$, where $u \in K_0 A_d$ and $d > 0$. As the $\mathbb{C}^*$-action is effective $d = 1$, and (a) follows.

(b) follows from [Alg, Ch. VII, §4, Corollary 2]. □

Lemma 1.3(a) does not hold in general without the assumption that $A_0 \neq \mathbb{C}$ as is seen by the Pham-Brieskorn surfaces $V_{p,q,r} := \{x^p + y^q + z^r = 0\} \subseteq \mathbb{C}^3$.

1.4. Usually (cf. [FiKa]) one distinguishes between the following three cases.

(i) The elliptic case: $A_- = 0, \ A_0 = \mathbb{C}$.

(ii) The parabolic case: $A_- = 0, \ A_0 \neq \mathbb{C}$.

(iii) The hyperbolic case: $A_- \neq 0$.

Below we provide more information in each of these cases.

2. THE ELLIPTIC CASE

In the elliptic case the $\mathbb{C}^*$-action on $V$ is good. In particular, its fixed point set $F := \text{Proj}(\mathbb{C}^*)$ (which is the zero set of the augmentation ideal $A_+$ of $A$) consists of a unique point called the vertex of $V$, and the surface $V$ is smooth outside the vertex. One considers the smooth projective curve $C := \text{Proj}(A) \cong V^*/\mathbb{C}^*$, where $V^* := V \setminus F$, together with the orbit morphism $\pi : V^* \to C$ (the fibers of $\pi$ are the orbits of the $\mathbb{C}^*$-action on $V^*$).

A useful class of examples of normal affine surfaces with a good $\mathbb{C}^*$-action is provided by the affine cones over projective curves. For an ample divisor $D$ on a smooth projective curve $C$ the ring

$$A_{C,D} := \bigoplus_{k \geq 0} H^0(C, \mathcal{O}_C(kD))) \cdot u^k \subseteq \text{Frac}(C)[u],$$

where $u$ is an indeterminate, is the coordinate ring of a normal affine surface $V := \text{Spec} A_{C,L}$ with a good $\mathbb{C}^*$-action. This surface $V$ is a cone over $C$ obtained by blowing down the zero section of the line bundle associated to $\mathcal{O}_C(-D)$.

Let furthermore a finite group $G$ act on $V$ freely off the vertex, and assume that this action commutes with the given good $\mathbb{C}^*$-action on $V$. Then the quotient $V/G$ is again a normal affine surface with a good $\mathbb{C}^*$-action. Conversely, the following result is true.

**Theorem 2.1.** ([Do, Pi, De, Ru]) Every normal affine surface with a good $\mathbb{C}^*$-action appears as the quotient of an affine cone over a smooth projective curve by a finite group acting freely off the vertex of the cone.

Generalizing the construction above, for a smooth projective curve $C$ and a $\mathbb{Q}$-divisor $D$ on $C$ one considers the graded ring

$$A_{C,D} := \bigoplus_{k \geq 0} H^0(C, \mathcal{O}(\lfloor kD \rfloor)) \cdot u^k,$$

where $\lfloor E \rfloor$ denotes the integral part of a $\mathbb{Q}$-divisor $E$. We have the following result.

**Theorem 2.2.** ([P]], [D], Theorem 3.5]) Given a normal affine surface $V = \text{Spec} A$ with a good $\mathbb{C}^*$-action there exists a $\mathbb{Q}$-divisor $D$ on the curve $C = \text{Proj} A$ such that $A \cong A_{C,D}$. 
The affine toric surfaces provide an interesting family of elliptic $\mathbb{C}^*$-surfaces.

**Example 2.3.** ([De], [CQ]) We remind that a normal affine toric surface $V = V_\sigma$ is associated to a strictly convex rational polyhedral cone $\sigma \subseteq \mathbb{R}^2$. If $\dim \sigma = 0$ or $= 1$ then $V_\sigma \approx \mathbb{C}^* \times \mathbb{C}^*$ or $V_\sigma \cong A^1_\mathbb{C} \times \mathbb{C}^*$, respectively, and so $A^\times \neq \mathbb{C}^*$. Consequently, these two cannot be elliptic $\mathbb{C}^*$-surfaces. Otherwise, if $\dim \sigma = 2$ then choosing an appropriate base $e_1, e_2$ of the lattice one may suppose that $\sigma$ is the cone $C(e_2, de_1 - ee_2)$, where $d \geq 1, 0 \leq e < d$ and $\gcd(e, d) = 1$. We denote $V_{d,e} := V_\sigma$; then $V_{d,e} = \text{Spec} \ A_{d,e}$, where

$$A_{d,e} := \bigoplus_{b \geq 0, \ ad-be \geq 0} \mathbb{C} \cdot x^a y^b \subseteq \mathbb{C}[x, x^{-1}, y, y^{-1}]$$

is the semigroup algebra of the dual cone $\sigma^\vee = C(e_1, ee_1 + de_2)$.

The 2-torus $T = (\mathbb{C}^*)^2$ acts on $V_{d,e}$ with an open orbit $V^*_{d,e} := V_{d,e} \setminus \{0\}$. Thus one can introduce on $V_{d,e}$ a number of elliptic, parabolic as well as hyperbolic $\mathbb{C}^*$-actions by choosing appropriate 1-parameter algebraic subgroups of the torus $T$.

In [Ri, BeRi, BeRi2, Co] one can find a description of minimal sets of generators of the algebras $A_{d,e}$ as above, as well as defining equations for the affine varieties $V_{d,e} = \text{Spec} \ A_{d,e} \hookrightarrow \mathbb{C}^N$. An explicit presentation of these algebras as in Theorem 2.2 is given in [De, 5.1].

We would like to emphasize the well known relation between affine toric surfaces and cyclic quotient singularities (see [De, 5.2] or [Od, Proposition 1.24]).

**Lemma 2.4.** If $B$ is the normalization of $A := A_{d,e}$ in the field $L := \text{Frac}(A)[u]$ with $u := \sqrt[ad-be]{v}$, then $B$ is the polynomial ring $B = \mathbb{C}[u, v]$ with $v := u^e y$. The Galois group $\langle \zeta \rangle \cong \mathbb{Z}_d$ of $L : \text{Frac}(A)$ acts on $B$ via the representation, say $G_{d,e}$

$$\zeta \cdot u = \zeta u, \quad \zeta \cdot v = \zeta^e v,$$

and $A = B^{\mathbb{Z}_d}$. Consequently, there is an isomorphism

$$V_{d,e} \cong \mathbb{A}^2_\mathbb{C}/G_{d,e} = \mathbb{A}^2_\mathbb{C}/\mathbb{Z}_d.$$

**Proof.** For the convenience of the reader we give a short argument. By definition, $A$ is generated over $\mathbb{C}$ by the monomials

$$x^a y^b \quad \text{with} \quad b \geq 0, \ ad-be \geq 0.$$

As $x^a y^b = u^{ad-be} v^b$, this shows that $A$ embeds naturally into $\mathbb{C}[u, v]$ and that even $A = \mathbb{C}[x, x^{-1}, y] \cap \mathbb{C}[u, v]$. In particular $A$ is a normal domain. Because of $u^d = x \in A$ and $v^d = x^e y^d \in A$ the ring $B$ is integral over $A$, whence it is the normalization of $A$.

The second part follows from the first one, since $L$ is a cyclic extension of $\text{Frac}(A)$ with Galois group $\mathbb{Z}_d$ acting via $\zeta \cdot u = \zeta u$ and $\zeta \cdot x = x$ for all $x \in A$. \hfill \Box

**Remark 2.5.** Assuming that $e > 0$ and letting $\xi := \zeta^e$ one obtains

$$(\zeta u, \zeta^e v) = (\xi^e u, \xi v),$$

where $0 \leq e' < d$ and $ee' \equiv 1 \mod d$ (note that for $d = 1$ this means $e' = 0$). Hence, with $\tau(u, v) := (u, u)$ the conjugate $\mathbb{Z}_d$-action $G_{d,e}' := \tau^{-1} G_{d,e} \tau$ on $\mathbb{A}^2_\mathbb{C}$

$$\xi \cdot (u, v) = (\xi^e u, \xi v)$$
has the same orbits as $G_{d,e}$ thus providing an isomorphism of toric surfaces

$$V_{d,e} \cong \mathbb{A}^2_{C}/G_{d,e} \cong \mathbb{A}^2_{C}/G'_{d,e} \cong \mathbb{A}^2_{C}/G_{d,e'} \cong V_{d,e'}.$$ 

Moreover, $V_{d,e} \cong V_{d,e'}$ if and only if $d = d'$ and either $e = e'$ or $ee' \equiv 1 \mod d$.

3. The parabolic case

In the parabolic case one considers a normal affine surface $V$ with a $\mathbb{C}^*$-action such that the coordinate ring $A = \bigoplus_{i \geq 0} A_i$ is positively graded and $A_0$ is a 1-dimensional domain. Thus $A_0$ corresponds to a smooth affine curve $C = \text{Spec} A_0$, which can be identified with the algebraic quotient $V/\mathbb{C}^*$ (indeed, $A_0 = A^{\mathbb{C}^*}$ is the ring of invariants of the $\mathbb{C}^*$-action on $A$). The embedding $A_0 \hookrightarrow A$ corresponds to the quotient morphism $\pi : V \to C$, and the projection $A \to A_0$ gives an embedding $\iota : C \hookrightarrow V$ which provides a retraction of $\pi$ and whose image is the fixed point set. Every fiber of $\pi : V \to C$ is the closure of a non-trivial orbit; it contains a unique fixed point (a source of this orbit) \cite[Lemma 1.7]{FiKa}.

A simple example of a parabolic $\mathbb{C}^*$-surface is the cylinder $C \times \mathbb{A}^1_{\mathbb{C}}$ over a smooth affine curve $C$, where $\mathbb{C}^*$ acts on the second factor. More examples can be produced by applying equivariant affine modifications to $C \times \mathbb{A}^1_{\mathbb{C}}$ (see \cite[Theorem 1.1]{KaZa}). Actually, one obtains in this way all normal affine surfaces with a parabolic $\mathbb{C}^*$-action.

3.1. The Dolgachev-Pinkham-Demazure construction (see Theorem \cite[2.2]{D}) is available also in the parabolic case. Let $C = \text{Spec} A_0$ be an affine curve over $\mathbb{C}$ with function field $K_0 := \text{Frac}(A_0)$, and let $D$ be a $\mathbb{Q}$-Cartier divisor on $C$. Similarly as in the elliptic case we can introduce the algebra

$$A_0[D] := A_{C,D} = \bigoplus_{n \geq 0} H^0(C, \mathcal{O}_C([nD])) \cdot u^n \subseteq K_0[u].$$

More explicitly, if $f \in K_0$ then

\begin{equation}
(3) \quad fu^n \in A := A_0[D] \iff \text{div} f + nD \geq 0.
\end{equation}

We note that the algebra $A$ is normal (see Corollary 3.8(b) below) and finitely generated over $A_0$. Notice also that $u \in A_1$ if and only if $D \geq 0$.

The following theorem is well known (cf. \cite[Theorem 3.5]{D}); for the convenience of the reader we include a short proof.

**Theorem 3.2.** Let $C = \text{Spec} A_0$ be a normal affine algebraic curve with function field $K_0 := \text{Frac}(A_0)$. If $A = \bigoplus_{i \geq 0} A_i$ is a normal finitely generated $A_0$-algebra of dimension 2 then the following hold.

(a) $A$ is isomorphic to $A_0[D]$ for some $\mathbb{Q}$-divisor $D$ on $C$. More precisely, if $u \in K_0 \cdot A_1$ is a non-zero element and if the divisor $D$ is defined by the equality

$$\pi^* D = \text{div} u - \iota(C),$$

then $A$ and $A_0[D]$ are equal when considered as subrings of $K_0[u]$.

(b) For two $\mathbb{Q}$-divisors $D$ and $D'$ on $C$, the rings $A = A_0[D]$ and $A' = A_0[D']$ are isomorphic as graded $A_0$-algebras if and only if $D$ and $D'$ are linearly equivalent.
Proof. (a) Since \( u \in K_0 \cdot A_1 \) is homogeneous, the divisor \( \text{div} \, u \) on the normal surface \( V = \text{Spec} \, A \) is invariant under the induced \( C^* \)-action on \( V \), and so we have

\[
\text{div} \, u = \sum_{i=1}^{m} p_i F_i + u(C)
\]

with \( p_i \in \mathbb{Z} \), where \( F_i = \pi^{-1}(x_i) \) are the fibers of \( \pi \) over distinct points \( x_i \in \mathbb{C} \), \( i = 1, \ldots, m \). Letting \( \pi^* x_i = q_i F_i \) with \( q_i \in \mathbb{N} \) \((i = 1, \ldots, m)\), the \( \mathbb{Q} \)-divisor \( D := \sum_{i=1}^{m} p_i/q_i x_i \) on \( V \) satisfies

\[
\text{div} \, u = \pi^*(D) + u(C).
\]

Since \( V \) is normal, for a rational function \( \varphi \in K_0 \) on \( C \) the following equivalences hold:

\[
\varphi u^n \in A_n \iff \text{div} (\varphi u^n) \geq 0 \iff \pi^* \text{div} \varphi + n \text{div} \, u \geq 0 \iff \pi^* \text{div} \varphi + n \pi^*(D) + n u(C) \geq 0 \iff \text{div} \varphi + nD \geq 0 \iff \varphi \in H^0(C, \mathcal{O}_C([nD])).
\]

Hence \( A_n = H^0(C, \mathcal{O}_C([nD])) \cdot u^n \) for all \( n \geq 0 \), as desired.

(b) Any isomorphism of graded \( A_0 \)-algebras

\[
\varphi : A_0[D] = \bigoplus_{n \geq 0} H^0(C, \mathcal{O}_C([nD])) \cdot u^n \rightarrow A_0[D'] = \bigoplus_{n \geq 0} H^0(C, \mathcal{O}_C([nD'])) \cdot u^n,
\]

extends to an isomorphism of graded \( K_0 \)-algebras

\[
\varphi_{K_0} : K_0[u] \rightarrow K_0[u']
\]

and so has the form \( u^n \mapsto f^n u^m \), \( n \geq 0 \), for some non-zero \( f \in K_0 \). Conversely, such a morphism \( \varphi_{K_0} \) maps \( A_0[D] \) isomorphically onto \( A_0[D'] \) if and only if

\[
H^0(C, \mathcal{O}_C([nD'])) = f^n \cdot H^0(C, \mathcal{O}_C([nD])) \quad \forall n.
\]

As

\[
f^n \cdot H^0(C, \mathcal{O}_C([nD])) = H^0(C, \mathcal{O}_C([nD - n \text{div} \, f])),
\]

the existence of an isomorphism \( \varphi \) as above is equivalent to the existence of an element \( f \in K_0 \) with \( D' = D - \text{div} \, f \).

3.3. We denote \( \{D\} = D - [D] \) the fractional part of a \( \mathbb{Q} \)-divisor \( D \). Since principal divisors are \( \mathbb{Z} \)-divisors, we have \( \{D\} = \{D'\} \) as soon as \( D \sim D' \).

If \( C = \text{Spec} \, \mathbb{C}[t] = \mathbb{A}^1_{\mathbb{C}} \), then the converse is also true. Indeed, any \( \mathbb{Z} \)-divisor on \( \mathbb{A}^1_{\mathbb{C}} \) is principal, and so the linear equivalence class of a \( \mathbb{Q} \)-divisor \( D \) on \( \mathbb{A}^1_{\mathbb{C}} \) is uniquely determined by the fractional part \( \{D\} \) of \( D \). Thus we obtain the following corollary.

Corollary 3.4. For every normal parabolic \( C^* \)-surface \( V = \text{Spec} \, A \) with \( A = \bigoplus_{n \geq 0} A_n \) and \( A_0 = \mathbb{C}[t] \), there is a unique isomorphism \( A \cong A_0[D] \) of graded \( A_0 \)-algebras, where \( D = 0 \) or \( D = \sum_{i=1}^{n} \frac{p_i}{q_i} x_i \) with \( 0 < p_i < q_i \), \( \gcd(p_i, q_i) = 1 \) \( \forall i = 1, \ldots, n \) and \( x_i \in \mathbb{A}^1_{\mathbb{C}} \), \( x_i \neq x_j \) for \( i \neq j \).

The next lemma is also well known; in lack of a reference we provide a short argument.

Lemma 3.5. Let \( D \) be a \( \mathbb{Q} \)-divisor on a normal affine variety \( S \) and consider the graded ring \( A := \bigoplus_{i \geq 0} A_i \), where \( A_i := H^0(S, \mathcal{O}_S([iD])) \cdot u^i \). For \( d \in \mathbb{N} \) the following conditions are equivalent.

(i) \( dD \) is integral.

(ii) \( A_{d+m} = A_d A_m \) for all \( m \geq 0 \).
(iii) The $d$-th Veronese subring $A^{(d)} := \bigoplus_{m \geq 0} A_{md}$ is isomorphic to the symmetric algebra $S_{A_0}(A_d)$ i.e., $A_{md} = S_m^{d}A_d$.

Proof. Condition (ii) is equivalent to

$$\mathcal{O}_S([m+D]|) \cong \mathcal{O}_S([mD]) \otimes \mathcal{O}_S([D]) \quad \forall m \geq 0,$$

and the latter condition is equivalent to

$$(ii') \quad [(m+d)D] = [mD] + [dD] \quad \forall m \geq 0.$$

Similarly, (iii) is equivalent to

$$(iii') \quad [mdD] = m[dD] \quad \forall m \geq 0.$$ 

The equivalence of (i), (ii') and (iii') now follows from the elementary fact that for a rational number $r = \frac{p}{q}$ and $d \in \mathbb{N}$ the following conditions are equivalent:

1. $dr \in \mathbb{Z}$
2. $[(m+dr)] = [mr] + [d] \quad \forall m \geq 0$
3. $[mdr] = m[dr] \quad \forall m \geq 0.$

□

Notation 3.6. We denote $d(A)$ the smallest positive integer $d$ satisfying the equivalent conditions of Lemma 3.3.

Remark 3.7. In the situation of Theorem 3.2, one can recover $D$ from the graded ring $A = A_0[\mathcal{D}]$ more algebraically as follows. Consider $d \in \mathbb{N}$ with $A_dA_i = A_{d+i}$ for all $i \geq 0$ (or, equivalently, $A_d = S^i(A_d)$, see Lemma 3.3) and let $v$ be a generator of $A_d$ as $A_0$-module; this exists after a suitable localization of $A_0$. If $u^d = f v$ with $f \in \text{Frac } A_0$, then $D = \text{div}(f)/d$. In fact, the ideal $vA$ is equal to $A_{2d}$ and so its zero set has no irreducible components in the fibers of $\pi$. Thus div $v = d \cdot \iota(C)$ on $V$. Since

$$\pi^*(D) = \text{div } u - \iota(C) \quad \text{and} \quad d \cdot \text{div } u = \text{div } v + \text{div } f$$

as divisors on $V$, we obtain $D = \text{div}(f)/d$.

A parabolic $\mathbb{C}^*$-surface $V = \text{Spec } A_0[D]$ has at most cyclic quotient singularities, as follows from Miyanishi’s Theorem (see [MV, Lemma 1.4.4(1)]). In the next result (see [D, Section 5]) we describe their structure in terms of the divisor $D$.

Proposition 3.8. (a) If $A_0 = \mathbb{C}[t]$ and if $D$ is supported on the origin in $\text{Spec } A_0 = \mathbb{A}^1_\mathbb{C}$ so that $D = -\frac{e}{d}[0]$ with $\text{gcd}(e,d) = 1$, then $A := A_0[-\frac{e}{d}[0]]$ is naturally isomorphic to the semigroup algebra

$$A_{d,e} = \bigoplus_{b \geq 0, \; ad - be \geq 0} \mathbb{C} \cdot t^{a}v^{b}$$

graded via $\text{deg } t = 0$, $\text{deg } v = 1$ (cf. Example 2.3). Consequently, $V := \text{Spec } A$ is isomorphic to the toric surface $V_{d,e'} = \text{Spec } A_{d,e'} \cong \mathbb{A}^2_\mathbb{C}/G_{d,e'}$, where $e' \equiv e \mod d$ and $0 \leq e' < d$.

(b) If $C = \text{Spec } A_0$ is any normal affine curve over $\mathbb{C}$ and $D$ is a $\mathbb{Q}$-divisor on $C$, then the surface $V = \text{Spec } A_0[\mathcal{D}]$ is normal with at most cyclic quotient singularities. More precisely, if $D(a) = -e/a$ with $\text{gcd}(e,d) = 1$ then $V$ has a quotient singularity of type $(d,e')$ at $\iota(a)$, where $e'$ is as in (a).
Proof. The first part of (a) follows immediately from (3) in 3.1, whereas the second one is a consequence of Lemma 2.4.

Tensoring the isomorphism in (a) with $- \otimes_{\mathbb{C}[t]} \mathbb{C}[[t]]$ we obtain that (b) holds if $A_0 \cong \mathbb{C}[[t]]$. The general case follows from this by taking completions at the maximal ideals of $A_0$. \qed

The algebra $A_0[D]$ is finitely generated over $A_0$, so there exist $f_1, \ldots, f_n \in K_0$ and $m_1, \ldots, m_n \in \mathbb{N}$ such that

$$A = A_0[f_1 u^{m_1}, \ldots, f_n u^{m_n}] \subseteq K_0[u].$$

In the next result we show how to compute $D$ from such a representation.

**Proposition 3.9.** Let $C = \text{Spec} \ A_0$ be a smooth affine curve and $K_0 := \text{Frac} \ A_0$. If a 2-dimensional subring $B$ of the polynomial ring $K_0[u]$ is represented as

$$B = A_0[f_1 u^{m_1}, \ldots, f_n u^{m_n}] \subseteq K_0[u], \quad m_i > 0 \ \forall i$$

with $f_1, \ldots, f_n \in K_0$, then its normalization $A = B_{\text{norm}}$ coincides as an $A_0$-subalgebra of $K_0[u]$ with $A_0[D]$, where

$$D := -\min_{1 \leq i \leq n} \frac{\text{div} \ f_i}{m_i}.$$

**Proof.** By definition of $D$ we have $\text{div} \ f_i + m_i D \geq 0$ so by (3) $f_i u^{m_i} \in A_0[D]$ and $B$ is a subring of $A_0[D]$. As $A_0[D]$ is normal (see Proposition 3.8(b)), $A$ is also contained in $A_0[D]$. Let us show that these subrings coincide.

According to Theorem 3.2, we can represent $A$ as $A = A_0[D']$ with $\pi^*(D') = \text{div} \ u - \nu(C)$. In particular $f_i u^{m_i} \in A = A_0[D']$, so again by (3) $\text{div} f_i + m_i D' \geq 0$ or, equivalently, $D' \geq -\frac{1}{m_i} \text{div} f_i$. Thus $D' \geq D$ and $A_0[D] \subseteq A_0[D'] = A$. As we have already shown the converse inclusion we obtain that $A = A_0[D]$, as desired. \qed

The following examples of parabolic $\mathbb{C}^*$-surfaces ruled over $\mathbb{A}^2_C$ are basic (see Theorem 3.11 below).

**Example 3.10.** For a unitary polynomial $P \in \mathbb{C}[t]$ and for an integer $d \geq 1$ we let

$$B_{d,P}^+ := \mathbb{C}[t, u, v]/(u^d - P(t)v) \cong \mathbb{C}
\begin{bmatrix} t, u, \dfrac{u^d}{P(t)} \end{bmatrix}$$

graded via

$$\deg t = 0, \quad \deg u = 1, \quad \deg v = d.$$  

The normalization

$$A_{d,P}^+ := (B_{d,P}^+)_{\text{norm}}$$

is a positively graded finitely generated $\mathbb{C}$-algebra of dimension 2 with $A_0 = \mathbb{C}[t]$. By Proposition 3.9 and Corollary 3.3 we have

$$A_{d,P}^+ \cong A_0[D] \cong A_0[\{D\}], \quad \text{where} \quad D = D(d, P) := \frac{\text{div}(P)}{d}.$$  

For $P(t) = \prod_{i=1}^n(t-x_i)^{r_i}$ (where $x_i \neq x_j$ if $i \neq j$) we obtain

$$D = \sum_{i=1}^n \frac{r_i}{d} x_i, \quad \text{and} \quad \{D\} = \sum_{i=1}^n \left\{ \frac{r_i}{d} \right\} x_i,$$

whereas $D = 0$ if $P = 1$. Replacing $D$ by $\{D\}$ we may suppose that
algebra

A be a normal 2-dimensional A coincides with the ring of invariants
A:=

Remark 3.12. 1. In the situation of Theorem 3.11 above, the Veronese subring A^{(d)} is equal to A_0[v] = C[t, v]. The cyclic group Z_d acts on A via the C^*-action and A^{(d)} coincides with the ring of invariants A^Z_d, whereas A is the normalization of A^{(d)} in the fraction field Frac(A). Thus the morphism V → A_0^2 = Spec C[t, v] induced by the inclusion C[t, v] ⊆ A represents V as a cyclic covering of the plane branched along the curve u = 0, and V is the normalization of a surface \{u^d - P(t)v = 0\} in C^2.

2. More generally, let C = Spec A_0 be any smooth affine curve and let A = \bigoplus_{i \geq 0} A_i be a normal 2-dimensional A_0-algebra of finite type. If A_1 = u \cdot A_0 and A_d = v \cdot A_0, d := d(A), for suitable elements u \in A_1 and v \in A_d then A is the normalization of an algebra A_0[u, v]/(u^d - P_t v) graded via deg u = 1, deg v = d, for a certain d ∈ N and a certain element P_+ \in A_0.

4. THE HYPERBOLIC CASE

Let A = \bigoplus_{i \in \mathbb{Z}} A_i be the coordinate ring of a normal affine surface V = Spec A with C^*-action such that A_+, A_- are both non-zero. Here again there is a quotient morphism π : V → C = Spec A_0 induced by the inclusion A_0 ⊆ A. Every fiber of π is either a non-trivial orbit or a union of two 1-dimensional orbits and a hyperbolic fixed point, which is a source for one of them and a sink for the other one [FKa]. Thus the fixed point set F is finite and contains Sing V.

By Lemma [11] the proper subalgebras A_{≥0} and A_{≤0} of A are normal and finitely generated, and so V_+ := Spec A_{≥0} and V_- := Spec A_{≤0} are normal affine surfaces with a parabolic C^*-action birationally dominated by V. The natural embeddings A_0 ⊆ A_{≥0} ⊆ A and A_0 ⊆ A_{≤0} ⊆ A yield the commutative diagram

\[
\begin{array}{ccc}
V_+ & \xleftarrow{\sigma_+} & V \\
\pi_+ & \downarrow & \pi_- \\
C & \xrightarrow{\pi} & C
\end{array}
\]

where \sigma_± are equivariant birational morphisms. Hence \sigma_± are equivariant affine modifications [Ka]. Theorem 1.1]. More precisely the following result holds.

Proposition 4.1. V can be obtained from V_± by blowing up a C^*-invariant subscheme and deleting the proper transform of a C^*-invariant divisor D_± on V_±, which contains the fixed point curve \iota_±(C) ⊆ V_±.
Proof. Let us show this for \( V_+ \), the proof for \( V_- \) being similar. Choose a system of homogeneous generators \( a_1, \ldots, a_n \) of the finitely generated \( A_0 \)-subalgebra \( A_{\leq 0} \) and let \( f_0 \in A_+ \) be a non-zero element of degree \( m = -\min_i \deg a_i \). Letting \( f_i := a_i f_0 \) for \( i = 1, \ldots, n \) we obtain

\[
A = A_{\geq 0} \left[ f_1 / f_0, \ldots, f_n / f_0 \right] = A_{\geq 0}[I/f_0] := \left\{ \frac{x_k}{f_0^k} \mid x_k \in I^k, \ k \geq 0 \right\},
\]

where \( I \) is the graded ideal of \( A_{\geq 0} \) generated by \( f_0, \ldots, f_n \). Thus \( V = \text{Spec} A \) is obtained by blowing up \( V_+ = \text{Spec} A_{\geq 0} \) with center \( I \) and deleting the proper transform of the \( \mathbb{C}^* \)-invariant divisor \( \text{div} f_0 \) on \( V_+ \). As this divisor contains \( \iota_+(C) \), the result follows.  \( \square \)

For a more precise description of the affine modifications \( \sigma_\pm \) see Remark 4.20.

4.2. The Dolgachev-Pinkham-Demazure construction is still available in the hyperbolic case. In [De, Theorem 3.5] it is done under the additional assumption that \( A_{-n} \otimes A_{\varepsilon} \to A_0 \) is an isomorphism for all \( n \). Here we generalize the construction in order to make it work for any hyperbolic \( \mathbb{C}^* \)-surface.

Let \( D_+, D_- \) be \( \mathbb{Q} \)-divisors on the smooth affine curve \( C := \text{Spec} A_0 \). For \( n \geq 0 \) we consider the \( A_0 \)-submodules

\[
A_{-n} := H^0(C, \mathcal{O}_C([nD_-])) \cdot u^{-n} \quad \text{and} \quad A_n := H^0(C, \mathcal{O}_C([nD_+])) \cdot u^n
\]

of \( \text{Frac}(A_0)[u, u^{-1}] \), where \( u \) is an indeterminate of degree 1. If \( D_+ + D_- \leq 0 \) then for \( n \geq m \geq 0 \) we have

\[
[nD_+] + [mD_-] \leq [(n-m)D_+],
\]

whence \( A_n \cdot A_{-m} \subseteq A_{n-m} \). Similarly, for \( 0 \leq n \leq m \) we have \( A_n \cdot A_{-m} \subseteq A_{n-m} \). Thus

\[
A := A_0[D_+, D_-] := \bigoplus_{n \in \mathbb{Z}} A_n
\]

is a finitely generated \( A_0 \)-subalgebra of \( \text{Frac}(A_0)[u, u^{-1}] \) with \( A_{\geq 0} = A_0[D_+] \) and \( A_{\leq 0} \cong A_0[D_-] \). The grading on \( A \) defines a natural hyperbolic \( \mathbb{C}^* \)-action on the surface \( \tilde{V} := \text{Spec} A \). The latter surface is normal as so are the algebras \( A_0[D_+] \) and \( A_0[D_-] \) (see Lemma 1.1 and Corollary B.3(b)). Conversely, we have the following theorem.

**Theorem 4.3.** If \( C = \text{Spec} A_0 \) is a smooth affine curve and \( A = \bigoplus_{i \in \mathbb{Z}} A_i \) is a normal graded finitely generated domain of dimension 2 with \( A \neq 0 \), then the following hold.

a) \( A \) is isomorphic to \( A_0[D_+, D_-] \), where \( D_+, D_- \) are \( \mathbb{Q} \)-divisors on \( C \) satisfying \( D_+ + D_- \leq 0 \). More precisely, if \( u \in \text{Frac}(A_0) \cdot A_1 \) and if the divisors \( D_+, D_- \) on \( C \) are defined by

\[
\pi_+(D_+) = \text{div}(u) - \iota_+(C) \quad \text{and} \quad \pi_-(D_-) = \text{div}(u^{-1}) - \iota_-(C),
\]

where \( \pi_\pm \) are as in diagram (4) above and \( \iota_\pm : C \hookrightarrow V_\pm \) are the natural embeddings, then \( D_+ + D_- \leq 0 \) and \( A \cong A_0[D_+, D_-] \).

b) \( A_0[D_+, D_-] \cong A_0[D'_+, D'_-] \) as graded \( A_0 \)-algebras if and only if, for a rational function \( \varphi \in \text{Frac}(A_0) \), one has

\[
D'_+ = D_+ + \text{div} \varphi \quad \text{and} \quad D'_- = D_- - \text{div} \varphi.
\]

**Proof.** (a) By Theorem 3.2 and its proof we have equalities

\[
A_{\geq 0} = A_0[D_+] \quad \text{and} \quad A_{\leq 0} = A_0[D_-]
\]
as subalgebras of Frac($A_0|u, u^{-1}$), whence $A = A_0[D_+, D_-]$. It remains to show that $D_+ + D_- \leq 0$. Applying in (\ref{remark:principal-div}) the functors $\sigma^*_+$ and $\sigma^*_-$ respectively, we obtain

$$
\pi^*(D_+) = \text{div}(u) - \sigma^*_+ t^*_1(C) \quad \text{and} \quad \pi^*(D_-) = \text{div}(u^{-1}) - \sigma^*_- t^*_r(C).
$$

Taking the sum of these equalities yields $\pi^*(D_+ + D_-) = -(\sigma^*_+ t^*_1(C) + \sigma^*_- t^*_r(C))$, whence $D_+ + D_- \leq 0$, as required. Finally (b) follows from Theorem \ref{theorem:principal-div}(b) and its proof.

Consequently, if $A_0 = \mathbb{C}[t]$ then $A$ admits a unique presentation $A = A_0[D_+, D_-]$ with $D_+ = \{D_+\}$ and $D_+ + D_- \leq 0$.

It follows from Theorem \ref{theorem:principal-div} that outside $|D_+| \cup |D_-|$, the map $\pi : V \rightarrow C$ is a locally trivial principal $\mathbb{C}$-bundle. More generally, the Dolgachev-Pinkham-Demazure construction shows the following result (cf. \cite[Proposition 1.11]{BasHa}, \cite[Proposition 1.11]{FtKa}).

**Corollary 4.4.** In all three cases, outside of a finite subset of the curve $C$ the projection $\pi : V^* \rightarrow C$ and $\pi : V \rightarrow C$, respectively, defines a locally trivial fiber bundle. This is a principal $\mathbb{C}$-bundle in the elliptic and hyperbolic cases, and a line bundle in the parabolic case.

Note that if $u \in A_1 \cup A_{-1}$ is a non-zero element then its restriction to a general fiber of $\pi$ gives a fiber coordinate and so a trivialization over a Zariski open subset of $C$.

**Remark 4.5.** The algebra $A = A_0[D_+, D_-]$ contains an invertible element of degree $d > 0$ if and only if $D_- = -D_+$ and $dD_+$ is a principal divisor on $C = \text{Spec} A_0$. In fact, if $v \in A$ is an invertible element of degree $d > 0$ then we can write

$$
v = fu^d \in A_d \quad \text{and} \quad v^{-1} = f^{-1}u^{-d} \in A_{-d},
$$

where $f \in \text{Frac}(A_0)$ satisfies

$$
\text{div}(f) + dD_+ \geq 0 \quad \text{and} \quad -\text{div}(f) + dD_- \geq 0.
$$

Thus $0 \geq D_+ + D_- \geq 0$, whence $D_- = -D_+$. Since $A_d = vA_0$ it also follows that $dD_+$ is principal. Conversely, if $D_+ = -D_-$ and if $dD_+$ is principal, then $vA_0 = A_d$ is free over $A_0$ and $v = fu^d$ with $\text{div} f + dD_+ = 0$ by Remark \ref{remark:principal-div}. Hence also $\text{div} f^{-1} + dD_- = 0$, so $f^{-1}u^{-d} \in A$ and $v = fu^d$ is a unit in $A$.

The following analogue of Proposition \ref{proposition:principal-div} holds with a similar proof.

**Lemma 4.6.** Let $C = \text{Spec} A_0$ be a smooth affine curve with function field $K_0 = \text{Frac}(A_0)$. If a graded 2-dimensional domain $B \subseteq K_0[u, u^{-1}]$ is represented as

$$
B = A_0[h_1u^{-n_1}, \ldots, h_ku^{-n_k}, f_1u^{m_1}, \ldots, f_nu^{m_n}]
$$

(where $n_i, m_j > 0 \forall i, j$)

with $h_1, \ldots, h_k, f_1, \ldots, f_n \in K_0$ and $B_0 = A_0$, then its normalization $A = B_{\text{norm}}$ coincides (as a graded $A_0$-subalgebra of $K_0[u]$) with $A_0[D_+, D_-]$, where

$$
D_- = -\min_{1 \leq i \leq k} \frac{\text{div} h_i}{n_i} \quad \text{and} \quad D_+ = -\min_{1 \leq j \leq n} \frac{\text{div} f_j}{m_j}.
$$

We notice that the assumption $A_0 = B_0$ amounts to the inequalities

$$
\frac{\text{div} h_i}{n_i} + \frac{\text{div} f_j}{m_j} \geq 0 \quad \forall i, j,
$$

which in turn are equivalent to $D_+ + D_- \leq 0$. 

The following lemma provides additional information in the case that $[D_{\pm}]$ and $d_{\pm}(A)D_{\pm}$ are principal divisors.

**Lemma 4.7.** Let $A = \bigoplus_{i \in \mathbb{Z}} A_i = A_0[D_+, D_-] \subseteq \text{Frac}(A_0)[u, u^{-1}]$, and let $d_{\pm} = d_{\pm}(A)$ be the minimal positive integer such that the divisor $d_{\pm}D_{\pm}$ is integral. If $A_{\pm1} = u_{\pm} \cdot A_0$, $A_{\pm d_{\pm}} = v_{\pm} \cdot A_0$ and

$$u_+u_- = Q, \quad u_{\pm}^d = P_{\pm}v_{\pm}$$

for some elements $Q, P_{\pm} \in A_0$, then

$$(6) \quad D_+ = \frac{\text{div} P_+}{d_+} + D_0 \quad \text{and} \quad D_- = \frac{\text{div} P_-}{d_-} - D_0 - \text{div} Q,$$

where $D_0$ is the integral divisor $D_0 = \text{div}(u/u_+)$ on $C = \text{Spec} A_0$. Consequently,

$$(7) \quad \frac{\text{div} P_+}{d_+} + \frac{\text{div} P_-}{d_-} \leq \text{div} Q.$$

Furthermore, $P_+$ and $P_-$ are uniquely determined by $D_+$ and $D_-$ through

$$(8) \quad \{D_+\} = \frac{\text{div} P_+}{d_+} \quad \text{and} \quad \{D_-\} = \frac{\text{div} P_-}{d_-}.$$

**Proof.** We have $u_+^d = P_+ \cdot (u/u_+)^d v_+$ and $u_-^d = P_- \cdot (u/u_+)^{-d} Q^{-d} v_-$ and so by Remark 3.7

$$D_+ = \frac{\text{div}(P_+ \cdot (u/u_+)^d)}{d_+} = \frac{\text{div} P_+}{d_+} + D_0, \quad \text{and}$$

$$D_- = \frac{\text{div}(P_- \cdot (u/u_+)^{-d} Q^{-d})}{d_-} = \frac{\text{div} P_-}{d_-} - D_0 - \text{div} Q.$$

Now (7) follows from the inequality $D_+ + D_- \leq 0$. To show (8), after localizing $A_0$ we can assume that $P_{\pm} = S_{\pm}^d T_{\pm}$, where $S_{\pm}, T_{\pm} \in A_0$ are elements with

$$\text{div} S_{\pm} = \left\lfloor \frac{\text{div} P_{\pm}}{d_{\pm}} \right\rfloor \quad \text{and} \quad \text{div} T_{\pm} = \left\{ \frac{\text{div} P_{\pm}}{d_{\pm}} \right\},$$

respectively. The relation $(u_{\pm}/S_{\pm})^d = T_{\pm} v_{\pm}$ then shows that $u_{\pm}/S_{\pm}$ is integral over $A$ and so by the normality of $A$ is contained in $A_{\pm1}$. As $u_{\pm}$ is a generator of $A_{\pm1}$ this forces that $S_{\pm} \in A_{0}^\times$ are units, proving (8). \qed

In many cases the surfaces $V = \text{Spec} A_0[D_+, D_-]$ can be represented by explicit equations as follows.

**Proposition 4.8.** With the assumptions as in Lemma 4.7 the following hold.

(a) $A = A_0[D_+, D_-]$ is the normalization of the $A_0$-algebra

$$(9) \quad B := A_0[u_-, v_+, v_-]/\left( u_-^d - v_- P_-, v_+^d - v_+ u_-^d - P, v_+ u_-^d - Q_+ \right)$$

graded via $\deg u_- = -1, \deg v_+ = \pm d_+$, where $k := \gcd(d_+, d_-), d'_+ := d_+/k$ and

$$(10) \quad P := \frac{Q^{kd'_+}}{P_+^{d'_+}} \in A_0, \quad Q_+ := \frac{Q^{d_+}}{P_+^{d_+}} \in A_0.$$

1or, equivalently, that $A_{\pm1}$ and $A_{\pm d_{\pm}}$ are free $A_0$-modules of rank 1.
(b) $V = \text{Spec } A$ is a cyclic branched covering of degree $k$ of the normalization of the hypersurface $\{v_+^{d_+} - P = 0\}$ in $C \times \mathbb{A}^2$. \\
(c) If $k = 1$ i.e., if $d_+$ and $d_-$ are coprime and if $v_+$ is not invertible, then $V = \text{Spec } A$ can be represented as the normalization of a hypersurface $X$ in $A^3 = \text{Spec } \mathbb{C}[s, t, u]$ with equation \\

$$q(s, v_+^{d_+} - v_-^{d_-}) = 0,$$

where $q \in \mathbb{C}[s, t]$ is a suitable irreducible polynomial.

**Proof.** (a) First we note that $A$ is integral over the subring $A_0[v_\pm]$. Indeed, if $w \in A_k$ with $k \neq 0$ then $w^{d_+} = aw^{d_+}_+$ if $k > 0$ and $w^{d_-} = av^{d_-}$ if $k < 0$, where $a \in A_0$ (see Lemma 3.3). Since $A$ and its subring $A_0[u_-, v_\pm]$ have the same field of fractions, it follows that $A$ is the normalization of $A_0[u_-, v_\pm]$.

To find the relations between the generators of $A_0[u_-, v_\pm]$, note that $v_\pm = u_\pm^{d_\pm}/P_\pm$ and so

$$v_+^{d_+} v_-^{d_-} = \frac{u_+^{d_+} u_-^{d_-}}{P_+^{d_+} P_-^{d_-}} = Q^{d_+ d_-} P^{d_+ d_-} = P \in A_0,$$

Similarly

$$v_+ u_-^{d_-} = \frac{u_+^{d_+} u_-^{d_-}}{P_+} = Q^{d_-} P_+ = Q_+ \in A_0.$$

The general fibers of the natural map $\text{Spec } B \to C = \text{Spec } A_0$ are irreducible, and every fiber is 1-dimensional and in the closure of the generic fiber. Thus the surface $\text{Spec } B$ is irreducible, and (a) follows.

(b) $A_0[v_\pm]$ is contained in the Veronese subring $A(k)$ of $A$ and the fraction fields of both rings coincide. As $A$ and then also $A(k)$ is integral over $A_0[v_\pm]$ the normalization of $A_0[v_\pm]$ is just $A(k)$. The cyclic group $\mathbb{Z}_k$ acts on $A$ via the $\mathbb{C}^*$-action with invariant ring $A(k)$. Thus $V \to \text{Spec } A(k)$ is a cyclic branched covering of degree $k$, and (b) follows.

(c) In case $k = 1$ the algebra $A = A(k)$ is itself the normalization of the hypersurface $A_0[v_+, v_-]/(v_+^{d_+} - v_-^{d_-} - P)$. Notice that $P$ is non-constant as $A$ is a domain and, by our assumption, the elements $v_\pm$ are not invertible. For a general element $s_0$ of $A_0$ the map $\varphi = (s, t)$ is a finite morphism of $C = \text{Spec } A_0$ onto a plane curve $\tilde{C} \subseteq \mathbb{A}^2_\mathbb{C}$ with an irreducible equation $q(s, t) = 0$, where $t := P = v_+^{d_+} v_-^{d_-} \in A_0$. This implies (c). \qed

**Remarks 4.9.** 1. It is worthwhile mentioning how to get, under the assumptions as in (c), a representation $A \cong A_0[D_+, D_-]$ in terms of $P$ in (10). Choose $p, q \in \mathbb{Z}$ with $|\frac{d_+}{d_-}, q| = 1$ so that $u' := v_+^{d_+} v_-^{d_-}$ has degree 1. By an easy calculation $u'^{d_+} = v_+ P^p$ and $u'^{d_-} = v_- P^q$, whence by Remark 3.7, $A \cong A_0[D_+, D_-]$ with

$$D_+ = \frac{p}{d_+} \text{ div } P, \quad D_- = -\frac{q}{d_-} \text{ div } P, \quad \text{and } D_+ + D_- = -\frac{\text{ div } P}{d_+ d_-}.$$

2. In analogy with (c), any parabolic $\mathbb{C}^*$-surface $V = \text{Spec } A$ with $A = A_0[D]$, where $|D|$ and $d(A)D$ are principal divisors on $C = \text{Spec } A_0$, can be obtained as the normalization of a surface $w^{d_+} - t v = 0 = q(s, t)$ in $\mathbb{A}^3_\mathbb{C} = \text{Spec } \mathbb{C}[s, t, u, v]$ graded via $\deg s = \deg t = 0, \deg u = 1, \deg v = d$, where $q \in \mathbb{C}[s, t]$ is a suitable irreducible polynomial (see also Remark 3.12(2)).

The special case $d_+ = 1$ leads to the following example.
Example 4.10. (Cf. [3, Example 4.11]) For a unitary polynomial $P \in \mathbb{C}[t]$, we let $A = A_{d,P} = B_{\text{norm}}$ be the normalization of the $\mathbb{C}$-algebra

$$B = B_{d,P} := \mathbb{C}[t, u, v]/(u^d v - P(t))$$

graded via $\deg t = 0$, $\deg u = 1$, $\deg v = -d$ so that the normal affine surface $V := \text{Spec} A$ is equipped with a hyperbolic $\mathbb{C}^*$-action. As $A \cong A_0[u, Pu^{-d}]$ we can write

$$A \cong A_0[D_+, D_-], \quad \text{where} \quad D_+ = 0 \quad \text{and} \quad D_- = -\text{div}(P)/d$$

(see Lemma [1.0]). We can recover $P_\pm$ and $Q$ in Lemma [1.1] as follows. By the construction given there $P_+ = 1$ and by (5) $\{D_-\} = \text{div}(P_-)/d_-$ (note that $d = d_-$). This gives

$$\text{div} P_- = d \left\{ -\frac{\text{div} P}{d} \right\} \quad \text{and} \quad \text{div} Q = \frac{\text{div} P + \text{div} P_-}{d}$$

(see (5)). In particular,

$$A_{d_0} \cong A_0[u] \cong \mathbb{C}[t, u] \quad \text{and} \quad A_{d_0} \cong A_{d,P}^+$$

as graded $A_0$-algebras, where for the second isomorphism we have to reverse the grading of one of the rings.

This discussion provides the following characterization of the algebras $A_{d,P}$.

**Proposition 4.11.** If $A = A_0[D_+, D_-]$, where $A_0 \cong \mathbb{C}[t]$ and $D_+$, $D_-$ are $\mathbb{Q}$-divisors on $\mathbb{A}^2$ with $D_+ + D_- \leq 0$, then the following conditions are equivalent.

1. $D_+$ is integral i.e., $\{D_+\} = 0$.
2. $A_{d_0} \cong A_0[u]$ as graded $A_0$-algebras, where $\deg u = 1$.
3. $A \cong A_{d,P}$ as graded $A_0$-algebras, where $D_+ + D_- = \frac{-\text{div} P}{d}$.

Next we study the effect of base change to the Dolgachev-Pinkham-Demazure representation.

**Proposition 4.12.** Let $C = \text{Spec} A_0$ be an affine curve with function field $K_0 = \text{Frac}(A_0)$ and let

$$A := A_0[D_+, D_-] \subseteq K_0[u, u^{-1}],$$

where $D_+$ are $\mathbb{Q}$-divisors on $C$ satisfying $D_+ + D_- \leq 0$. Let $L$ be the field $L := \text{Frac}(A)[\sqrt[t]{uv}]$, where $t \in K_0$ and $b, d \in \mathbb{N}$. If $A'$ is the normalization of $A$ in $L$ then the following hold.

1. $A'_0$ is the normalization of $A_0$ in $K_0[s]$ with $s := \sqrt[t]{t}$, where $k := \text{gcd}(b, d)$.
2. $A' \cong A'_0[D'_+, D'_-]$ with

$$D'_\pm := \frac{k}{d} \left( p^*(D_\pm) \pm \beta \text{div} s \right),$$

where $p : C' := \text{Spec} A'_0 \to C$ is the projection and $\beta$ is defined by $\beta b \equiv k \mod d$.

**Proof.** We let $b = b'k$ and $d = d'k$. The normalization $A'$ admits a natural $\frac{1}{d}$-grading, and the element $u^* := \sqrt[t]{uv}$ is of degree $b/d = b'/d'$. If we write $k = \beta b + \delta d$, then the element $u' := u^\beta u^\delta \in \text{Frac}(A')$ has minimal possible positive degree $1/d'$. Thus

$$A' \subseteq \text{Frac}(A'_0)[u', u'^{-1}].$$

To compute $A'_0$, we note that $u^nu^{-m}$ with $n, m \in \mathbb{N}$ has degree 0 if and only if $nb'/d' = m$. In particular, $n = n'd'$ is an integer multiple of $d'$. Thus $K'_0 := \text{Frac}(A'_0)$ is
generated over $K_0$ by $u'^\ell u'^{-\ell'} = t^{d'/k}$ (i.e., $n' = 1$). As $d'$ and $k$ are coprime, it follows that $s = \sqrt[2]{7}$ also belongs to $K'_0$ and that this field is actually generated by $s$ over $K_0$, proving (1).

After localizing $A_0$ we may assume that there is an element $v_+ \in A$ of degree $d_+ = d(A_{d_+})$ with $A_{d_+} = v_+ A_0$ (see [3.7]). We claim that then $A'_{d_+} = v_+^s A'_0$ for all $s \geq 0$. If not, then for some $s > 0$ and some non-unit $x \in A_0$ the element $v_+^s / x$ belongs to $A'$, so it is integral over $A$ and there is an equation

$$v_+^{sm} / x^m + a_1 v_+^{sm-1} + \cdots + a_m = 0,$$

where $m \geq 0$ and $a_i \in A_{d_+}$. Thus $a_i = v_+^s q_i$ for some elements $q_i \in A_0$, whence dividing the equation above by $v_+^{sm}$ we obtain that

$$1 / x^m + q_1 / x^{m-1} + \cdots + q_m = 0.$$  

As $A'_0$ is integrally closed this is only possible if $x \in A'_0$ contradicting the choice of $x$.

Thus $v = v_+$ is an element satisfying the assumptions of Remark [3.7] and we compute with it the divisor $D'_+$ as follows (the calculation for $D'_-$ is analogous). If we consider the new grading of $A'$ by assigning to $u'$ the degree 1, then $v^k$ becomes an element of degree $dd_+$. Moreover, if $u'^d = P_+ v_+$ with $P_+ \in K_0$ then by Remark [3.7] $D_+ = \text{div}(P_+) / d_+$. Since

$$u'^{dd_+} = (u^s b^d u^\delta)^{dd_+} = (tu^b)^{dd_+} D_+ = t^{\beta d_+} D_+ = t^{\beta d_+} P_+ v_+^k$$

we obtain again by Remark [3.7] that on $C'$

$$D'_+ = \frac{\text{div}(t^{\beta d_+} P_+)}{dd_+} = \beta / d \text{div}(t) + k / d P_+(D_+),$$

and (2) follows. $\Box$

Let us consider the following important example.

**Example 4.13.** With $A_0 := \mathbb{C}[t]$, suppose that $D_+ = -\frac{2}{3}[0]$ and that $D_-$ is any $\mathbb{Q}$-divisor on $\mathbb{A}^1_{\mathbb{C}} = \text{Spec } A_0$. Applying Proposition [4.12] to $s := \sqrt{7} t$ (i.e. $b = 0$) we get that the normalization of $A := A_0[D_+, D_-]$ in the field $L := \text{Frac}(A)[s]$ is given by

$$A' = A_0'[-e[0], D'_-] \subseteq \mathbb{C}(s)[u, u^{-1}],$$

where $A_0' = \mathbb{C}[s]$ and $D'_- = p^*(D_-)$ (as before, $p : \text{Spec } \mathbb{C}[s] \to \text{Spec } \mathbb{C}[t]$ denotes the projection $s \mapsto s^d$). The divisor $D'_+ = -e[0]$ being integral we have

$$A' \cong A_0'[0, D'_+ + D'_-] \subseteq \mathbb{C}(s)[\tilde{u}, \tilde{u}^{-1}],$$

where $\tilde{u} := s^e u$.

More concretely, if $k := d_-(A)$ and if we choose a unitary polynomial $Q \in \mathbb{C}[t]$ with $D_- = -\frac{\text{div}(Q)}{k}$ then $D'_+ + D'_- = -\frac{\text{div}(Q(s^d)^{ske})}{k}$. By Example [4.10] $A' \cong A_{k,P}$ is the normalization of

$$B_{k,P} = \mathbb{C}[s, \tilde{u}, v] / (\tilde{u}^k v - P(s)), \quad \text{where } P(s) := Q(s^d)^{ske}. $$

(12)
The field extension $\text{Frac}(A) \subseteq \text{Frac}(A)[s]$ is Galois with Galois group $\mathbb{Z}_d = \langle \zeta \rangle$, where $\zeta s = \zeta s$. Thus

$$A \cong (A_{k,P})^{\mathbb{Z}_d},$$

and the action of $\zeta$ on $\tilde{u} = s^e u$ is given by $\zeta \tilde{u} = \zeta^e \tilde{u}$. Therefore, the group $\mathbb{Z}_d$ acts on $A_{k,P}$ via

$$\zeta \cdot s = \zeta s, \quad \zeta \cdot \tilde{u} = \zeta^e \tilde{u} \quad \text{and} \quad \zeta \cdot v = v.$$

Thus we obtain the following characterization.

**Proposition 4.14.** For an algebra $A = A_0[D_+, D_-]$ with $A_0 = \mathbb{C}[s]$ the following conditions are equivalent.

(i) $\{-D_+\} = \frac{e}{d}[0]$, where $0 \leq e < d$ and $\gcd(e, d) = 1$.

(ii) $A \cong (A_{k,P})^{\mathbb{Z}_d}$, where $A_{k,P}$ is the normalization of $B_{k,P}$ in (12) and where $\mathbb{Z}_d = \langle \zeta \rangle$ acts via the formulas in (13).

Like in the parabolic case $V$ may possess at most cyclic quotient singularities, as follows from Miyanishi’s Theorem (see [Miy, Lemma 1.4.4(1)]). The type of quotient singularities is determined from the divisors $D_+, D_-$ by the following result. As before, $C = \text{Spec} A_0$ is a smooth affine curve with function field $K_0 = \text{Frac} A_0$ and $A := A_0[D_+, D_-]$ with $\mathbb{Q}$-divisors $D_+$ and $D_-$ on $C$. Denote $\pi : V = \text{Spec} A \to C$ the canonical projection.

**Theorem 4.15.** (a) The set of singular points $\text{Sing} V$ is contained in the fixed point set $F$ which is the zero locus $F = V(I)$ of the ideal $I := A_+ A + A_- A$ of $A$.

(b) The map $\pi|F : F \to C$ is injective, and $\pi(F) = \{a \in C|D_+(a) + D_-(a) < 0\}$.

(c) For a point $a' \in F$ with image $a := \pi(a') \in C$ we write

$$D_+(a) = -\frac{e_+}{m_+} \quad \text{and} \quad D_-(a) = \frac{e_-}{m_-}$$

with the convention that

$$m_+ > 0, \quad m_- < 0, \quad \gcd(e_+, m_+) = \gcd(e_-, m_-) = 1 \quad \text{and} \quad m_+ = 1 \quad \text{if} \quad D_+(a) = 0, \quad m_- = -1 \quad \text{if} \quad D_-(a) = 0.$$

Let $p, q \in \mathbb{Z}$ with $|p \cdot e_+/m_+| = 1$. Then $a' \in F$ is a quotient singularity of type

$$(\Delta(a), e), \quad \text{where} \quad \Delta(a) := -\left| \begin{array}{cc} e_+ & e_- \\ m_+ & m_- \end{array} \right| \quad \text{and} \quad e \equiv \left| \begin{array}{cc} p & e_- \\ q & m_- \end{array} \right| \text{ mod } \Delta(a).$$

In particular, $a' \in \text{Sing} V$ if and only if $\Delta(a) \neq 1$.

**Proof.** As in the proof of Proposition 3.8(b) we can reduce the statement to the case that $A_0 = \mathbb{C}[t]$ and $|D_+| \cup |D_-|$ is contained in the origin, so that $D_\pm = \mp e_\pm / m_\pm[0]$.

(a) The set $\text{Sing} V$ is finite and invariant under the $\mathbb{C}^*$-action. Hence it is contained in the fixed point set $F$.

(b) The map $A_0 \to A/I$ is obviously surjective. Thus

$$\pi|F : F = \text{Spec}(A/I) \to C$$

is a closed embedding. Moreover, $F = \emptyset$ if and only if $1 \in I$ if and only if $1 = a_+ a_-$ for some homogeneous elements of $A$ of opposite degrees, and the latter happens if and only if $D_+ + D_- = 0$ by Remark 4.13.
(c) Notice first that the elements
\[ v_+ := t^{e_+}u^{m_+}, \quad v_- := t^{e_-}u^{m_-} \in K_0[u, u^{-1}] \]
belong to A. Indeed, by definition, the ideal \( I + tA \) of \( A \) (this is just the maximal ideal of the point \( a' \in F' \)) is generated by the monomials \( t^eu^m \) with \( (e, m) \in \mathbb{Z} \times \mathbb{Z} \), where
\[ (e, m) \neq (0, 0) \]
and
\[ e + mD_+(0) \geq 0 \quad \text{if} \quad m \geq 0, \quad e - mD_-(0) \geq 0 \quad \text{if} \quad m \leq 0. \]
In other words, \((e, m)\) is an element of the cone \( C := C((e_+, m_+), (e_-, m_-)) \) generated by the vectors \((e_+, m_+)\) in the plane. Hence \( A \) is a toric algebra generated by the semigroup \( C \cap \mathbb{Z}^2 \), and so is a quotient \( A_{d,e} \) for some \( d, e \geq 0 \) (see Lemma 2.4). To determine \( d, e \), we must find a basis of \( \mathbb{Z}^2 \) such that \((e_+, m_+)\) is one of the basis vectors. This is done as follows.

If we choose \( p, q \in \mathbb{Z} \) with \(|p/m_+| = 1\), then the vectors \( \tilde{e}_1 := (e_+, m_+) \) and \( \tilde{e}_2 := (p, q) \) form a basis of \( \mathbb{Z}^2 \), and
\[(e_-, m_-) = \Delta \tilde{e}_1 + \Delta \tilde{e}_2, \quad \text{where} \quad \Delta := \begin{vmatrix} p & e_- \\ q & m_- \end{vmatrix} \]
and \( \Delta := \Delta(0) \).

As \( \tilde{e}_1 \) and \( (e_-, m_-) \) form a basis of the cone \( C \), it follows from Lemma 2.4 that \( A \) has a quotient singularity of type \((\Delta, e)\), where \( 0 \leq e < d \) and \( e \equiv \begin{vmatrix} p & e_- \\ q & m_- \end{vmatrix} \mod \Delta \). Note that \( \Delta \) and \( \Delta' \) are coprime since so are \( e_- \) and \( m_- \).

The determinant \( \Delta \) has always positive sign as
\[(14) \quad D_+(0) + D_-(0) = \frac{\Delta}{m_+m_-} \leq 0 \quad \text{and} \quad m_+ > 0, \quad m_- < 0, \]
and so (c) follows. \( \square \)

Corollary 4.16. If \( A_{d,P} \) is the normalization of the algebra
\[ B_{d,P} = \mathbb{C}[t, u, v]/(u^dv - P(t)), \]
where \( P(t) = \prod_{i=1}^k (t - a_i)^{r_i} \) with \( a_i \neq a_j \) for \( i \neq j \) (see Example 4.10), then the singular points of the surface \( V_{d,P} = \text{Spec} A_{d,P} \) are the points \( a_i' \in V_{d,P} \) (1 \( \leq i \leq k \)), where \( t = a_i, u = v = 0 \) and \( r_i \mid d \).

Proof. It was shown in Example 4.10 that \( D_+ = 0 \) and \( D_-(a_i) = -\frac{r_i}{d} \). Therefore, \( \Delta(a_i) = e_+ > 1 \) if and only if \( r_i \nmid d \), which implies our assertion. \( \square \)

In the sequel we use the following notation.

Definition 4.17. Let \( O = \mathbb{C}^*z \) be the orbit through a point \( z \in V \setminus F \). Following [FiKa3] we say that \( O \) is of type \((d, q)\) if \( d \) is the order of the stabilizer
\[ \text{Stab}_z = \ker(\mathbb{C}^* \to \text{Aut } O) \subseteq \mathbb{C}^*, \quad \text{so that} \quad \text{Stab}_z = \langle \zeta \rangle \cong \mathbb{Z}_d, \]
and \( q \) (0 \( \leq q < d \)) is determined from the tangent representation of \( \text{Stab}_z \) on the tangent plane \( T_zV \) via pseudo-reflections
\[ \text{Stab}_z \ni \zeta \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \zeta^q \end{pmatrix}. \]
The orbit \( O \) is called principal if \( d = 1 \) and exceptional otherwise (see [FiKa1, FiKa2] for a detailed description of the structure of \( V \) near the exceptional orbits).
In the next result we will characterize the orbit types of the surface $V = \text{Spec} A$ with $A := A_0[D_+, D_-]$, where $D_+$ and $D_-$ are $\mathbb{Q}$-divisors on the smooth affine curve $C = \text{Spec} A_0$. Let $\pi : V \to C$ denote the projection. To examine the orbits over a point $a \in C$, we write

$$D_+(a) = -e_+/m_+ \quad \text{and} \quad D_-(a) = e_-/m_-$$

with the conventions as in Theorem 4.15(c). Let $q_+$ be defined by $0 \leq q_+ < m_+$ and $q_+e_+ \equiv -1 \mod m_+$, and similarly $q_-$ by $0 \leq q_- < -m_-$ and $q_-e_- \equiv 1 \mod m_-$. With this notation the following result holds.

**Theorem 4.18.** The exceptional orbits of $V$ are located over $|D_+| \cup |D_-|$. The orbits over a given point $a \in |D_+| \cup |D_-|$ are as follows.

(a) If $D_+(a) + D_-(a) = 0$ then $\pi^* (a) = m_+O$ consists of one orbit $O$ of type $(m_+, q_+)$ with multiplicity $m_+$. Moreover, $O$ appears with coefficient $-e_+$ in $\text{div} u$.

(b) If $D_+(a) + D_-(a) < 0$ then $\pi^{-1}(a)$ contains two orbits $O^+$ and $O^-$ of types $(m_+, q_+)$ and $(-m_-, q_-)$, respectively. Their closures $\bar{O}^\pm$ intersect in the unique fixed point of the fiber, and $\pi^* (a) = m_+O^+ - m_-O^-$. Moreover, $\bar{O}^\pm$ appears with multiplicity $\mp e_\pm$ in $\text{div} u$.

**Proof.** With the same reasoning as in the proof of Proposition 3.5(b) it is sufficient to treat the case where $A_0 = \mathbb{C}[t]$ and $D_{\pm}$ are supported on $a = 0 \in A_0^1$, i.e. $D_{\pm} = \mp e_+/m_{\pm}[0]$. Note that in this case $m_+ = d(A_{\geq 0})$ and $m_- = -d(A_0)$.

(a) If $D_+ + D_- = 0$, so that $e_+ = -e_- = e$ and $m_+ = -m_- =: m$ then $A$ is the semigroup algebra $\mathbb{C}[C \cap \mathbb{Z}^2]$, where $C$ is the cone generated over $\mathbb{R}$ by the vectors $\pm(e, m)$ and $(1, 0)$. If we choose $p, q \in \mathbb{Z}$ with $|\frac{p}{q} | = 1$ then

$$C \cap \mathbb{Z}^2 = \{(a, b)| (a, b) \in \mathbb{Z}(e, m) + \mathbb{N}(p, q)\}.$$ 

Hence $A$ is the algebra of Laurent polynomials

$$A = \mathbb{C}[x, x^{-1}, y], \quad \text{where} \quad x := t^eu^m \in A_m \quad \text{and} \quad y := t^pu^q \in A_q.$$ 

Clearly then

$$t = x^{-q}y^m \quad \text{and} \quad u = x^py^{-e}.$$ 

The action of $\mathbb{C}^*$ is given by $\lambda.x = \lambda^m x$ and $\lambda.y = \lambda^q y$, whence there is only one orbit $O$ over $t = 0$, and it is given by the equation $y = 0$. By (10) we have

$$\pi^*(0) = \text{div} t = m \cdot O \quad \text{and} \quad \text{div} u = -e \cdot O.$$ 

The stabilizer of any point of $O$ is the group $E_m \subseteq \mathbb{C}^*$ of $m$-th roots of unity, and the type of the orbit is $(m, q) = (m_+, q_+)$, as required in (a).

(b) Let now $D_+ + D_- < 0$. Consider a generator $v_\pm = t^{e_\pm}u^{m_\pm}$ of $A_{m_\pm}$ as $A_0$-module (cf. the proof of Theorem 4.15(c)). The localization $A_{v_\pm} = A[t^{e_\pm}u^{-m_\pm}]$ is the subring $A_0[D_+, -D'_-]$ of $\text{Frac}(A_0)[u, u^{-1}]$ with $D'_- := \max(D_-, -D_-)$ (see Lemma 4.1). As $D_+ + D_- \leq 0$ we have $D'_- = -D_-$, so by (a) the open subset $\text{Spec} A_{v_\pm}$ of $V$ contains an orbit $O^+$ of type $(m_+, q_+)$, and it has multiplicities $m_+$ and $-e_+$ in $\pi^*(0)$ and $\text{div} u$, respectively. Similarly, $\text{Spec} A_{v_-}$ contains an orbit $O^-$ of type $(-m_-, q_-)$, which has multiplicities $-m_-$ and $e_-$ in $\pi^*(0)$ and $\text{div} u$, respectively. We have $\text{div}(v_+v_-) = \Delta \cdot (\bar{O}^+ + \bar{O}^-)$, where by our assumption $\Delta = m_+m_- (D_+(0) + D_-(0)) > 0$ (see (14)). Thus the fiber of $\pi$ over $t = 0$ can be given by $v_+ \cdot v_- = 0$, where the functions $v_+, v_-$
vanish on $\mathcal{O}^-$ and $\mathcal{O}^+$, respectively. The intersection $\mathcal{O}^+ \cap \mathcal{O}^-$ is given by $v_+ = v_- = 0$, and so is the unique fixed point of the fiber.

**Example 4.19.** In the example of the algebra $A = A_{d,P}$ treated in Corollary 4.16 we have $D_+ = 0$ and $D_- = -\text{div}(P)/d = \sum_{i} -\frac{a_i}{d}[a_i]$ (see Example 4.10). The exceptional orbits are located over the points $a_i \in \mathbb{A}^1_C$, and $\pi^{-1}(a_i) = O_i^+ \cup \{a_i\} \cup O_i^-$, where $a_i$ is the unique fixed point of the fiber (located over the point $(0,0,a_i)$ of Spec $B_{d,P} \subseteq \mathbb{C}^3$). Applying Theorem 4.18, the orbit $O_i^+$ is principal, and if we write $r_i/d = e_i/m_i$ with $\gcd(e_i,m_i) = 1$ then $O_i^+$ is of type $(m_i,q_i)$, where

$q_i e_i \equiv -1 \mod m_i$ with $0 \leq q_i < m_i$.

**Remark 4.20.** We can now precise the character of the affine modifications $\sigma_{\pm} : V \to V_{\pm}$ as in Proposition 4.1. Doing this locally we assume first that $A_0 = \mathbb{C}[t]$ and $D_{\pm}$ is supported on $a = 0 \in \mathbb{A}^1_C$. If $D_+ + D_- = 0$ then $A = A_{\geq 0}[v_+] = (A_{\geq 0})_{v_+}$, whence $\sigma_{+} : V \to V_{+}$ is an open embedding and $V = V_{+} \setminus V_{-}$ is the divisor $\text{div} v_+ = m_+ \cdot \mathcal{I}_{+}(C)$. In case $D_+ + D_- < 0$, letting in the proof of Proposition 4.1 $f_0 := v_+^{m_-}$, we obtain that $\sigma_{+} : V \to V_{+}$ consists in blowing up a graded ideal $I \subseteq (t,v_+)$ of the algebra $A_{\geq 0}$ supported at a fixed point and deleting the proper transform of the divisor $\text{div} v_+ = m_+ \cdot \mathcal{I}_{+}(C)$. The exceptional curve in $V$ is just the orbit $O^- = \{v_+ = 0\}$.

Globalizing we see that $\sigma_{\pm} : V \to V_{\pm}$ blows up a graded ideal with support at the fixed points $b_1', \ldots, b_l' \in \mathcal{I}_{\pm}(C)$ over the points $b_i := \pi(\mathcal{I}_{\pm}(C)) \in C$ with $D_{+}(b_i) + D_{-}(b_i) < 0$, and deleting the proper transform of the fixed point curve $\mathcal{I}_{\pm}(C) \subseteq V_{\pm}$. Moreover the exceptional set of $\sigma_{\pm}$ is $O_{\pm}^+ \cup \cdots \cup O_{\pm}^i$.

**4.21.** We let as before $C = \text{Spec} A_0$ be a smooth affine curve with function field $K_0 = \text{Frac} A_0$, and we let $D_+, D_-$ be $\mathbb{Q}$-divisors on $C$. In what follows we compute the Picard group and the divisor class group of $A := A_0[D_+, D_-]$ (see also [Mor, Thm. 5.1] and [Wa, Cor. 1.7] for the elliptic case). We denote by $a_1, \ldots, a_k$ the points in $C$ for which $D_+(a) = -D_-(a) \neq 0$, and we let $b_1, \ldots, b_l \in C$ be the points with $D_+(b) + D_-(b) < 0$. Let us write

$$D_+(a_i) = \pm \frac{e_i}{m_i}, \quad D_+(b_j) = \pm \frac{e_j^+}{m_j^+} \quad \text{and} \quad D_-(b_j) = \pm \frac{e_j^-}{m_j^-}$$

with the conventions as in Theorem 4.15. If $\pi : V := \text{Spec} A \to C$ denotes the canonical map then the preimage $\pi^{-1}(a_i)$ consists of only one orbit $O_i$, and $\pi^{-1}(b_j)$ consists of two orbit closures $\bar{O}_j^+ \cup \bar{O}_j^-$, so that

$$\pi^*(V_i) = m_i O_i \quad \text{and} \quad \pi^*(V_j) = m_j^+ \bar{O}_j^+ - m_j^- \bar{O}_j^-$$

as divisors on $V$, see Theorem 4.18.

**Theorem 4.22.** The divisor class group $\text{Cl} A$ of $A$ is the group

$$\pi^*(\text{Cl} A_0) \oplus \bigoplus_{i=1}^k \mathbb{Z}[O_i] \oplus \bigoplus_{j=1}^l (\mathbb{Z}[\bar{O}_j^+] \oplus \mathbb{Z}[\bar{O}_j^-])$$

modulo the relations

$$\pi^*(a_i) = m_i(O_i), \quad i = 1, \ldots, k,$$

$$\pi^*(b_j) = m_j^+ [\bar{O}_j^+] - m_j^- [\bar{O}_j^-], \quad j = 1, \ldots, l,$$

$$0 = \sum_{j=1}^l e_i(O_i) + \sum_{j=1}^l (e_j^+ [\bar{O}_j^+] - e_j^- [\bar{O}_j^-]).$$
Proof. Let $\text{Div}_h A \subseteq \text{Div} A$ be the subgroup of all Weil divisors on $V$ that are homogeneous, i.e. finite sums of irreducible divisors given by homogeneous prime ideals. The homogeneous principal divisors $\text{Prin}_h A$ form a subgroup of $\text{Div}_h A$, which consists of all divisors $\text{div} f$, where $f = g/h \in \text{Frac} A$ is a quotient of homogeneous elements. By [AC §1, Ex. 16]

$$\text{Cl} A \cong \text{Cl} h A := \text{Div}_h A/\text{Prin}_h A.$$ The group $\text{Div}_h A$ is freely generated by all $\mathbb{C}^*$-invariant subvarieties of codimension 1 in $V$, that is by all irreducible components of the fibers of $\pi : V \to C$. If $D_+(a) = D_-(a) = 0$ then the fiber over $a$ is the prime divisor $\pi^*(a)$. If $a = a_i$ for some $i$ then the fiber over $a$ consists of just one orbit $O_i$ of type $(m_i, q_i)$, and by (17) $\pi^*(a_i) = m_i O_i$ as divisors on $V$. If $a = b_j$ for some $j$ then by (17) $\pi^*(b_j) = m_j O_j^+ - m_j O_j^-$. Thus the natural map $\pi^* : \text{Div} A_0 \to \text{Div}_h A$ is injective, and

$$\text{(18)} \quad \text{Div}_h A \cong \frac{\pi^*(\text{Div} A_0) \oplus \bigoplus_{i=1}^k \mathbb{Z}[O_i] \oplus \bigoplus_{j=1}^l (\mathbb{Z}[O_j^+] \oplus \mathbb{Z}[O_j^-])}{(\pi^*(a_i) - m_i O_i), \pi^*(b_j) - m_j O_j^+ + m_j O_j^-}. \quad (18)$$

The group $\text{Prin}_h A$ is generated by all divisors $\text{div}(fu^k) = \text{div} f + k \text{div} u$, where $f \in K_0^*$ is non-zero. Dividing out $\pi^*(\text{Prin} A_0) = \pi^* \text{div}(K_0^*)$ in (18) gives the group

$$\frac{\pi^*(\text{Cl} A_0) \oplus \bigoplus_{i=1}^k \mathbb{Z}[O_i] \oplus \bigoplus_{j=1}^l (\mathbb{Z}[O_j^+] \oplus \mathbb{Z}[O_j^-])}{(\pi^*(a_i) - m_i [O_i], \pi^*(b_j) - m_j^+ [O_j^+] + m_j^- [O_j^-])}. \quad (19)$$

By Theorem 1.18 the divisor of $u$ is given by

$$\text{div} u = - \sum_{j=1}^k e_i [O_i] + \sum_{j=1}^l (-e_j^+ [O_j^+] + e_j^- [O_j^-]).$$

Hence, taking (19) modulo this relation leads to the divisor class group, as required. $\square$

Corollary 4.23. A is factorial if and only if $C \subseteq \mathbb{A}^1_C$ (i.e. $A_0$ is a localization of $\mathbb{C}[t]$) and one of the following two conditions is satisfied.

(i) $l = 0$ and $\gcd(m_i, m_j) = 1$ for $1 \leq i < j \leq k$.

(ii) $l = 1$, $m_i = 1$ for all $i$ and $|e^+ - e^-| = \pm 1$, where $e^\pm := e^\pm_1$ and $m^\pm := m^\pm_1$.

Proof. If $C$ is a curve of genus $g \geq 1$ then the group $\text{Cl} A$ is not finitely generated. Thus assuming that $A$ is factorial, $C$ is isomorphic to an open subset of $\mathbb{A}^1_C$. By Theorem 1.22 the group $\text{Cl} A$ has then $k + 2l$ generators and $k + l + 1$ independent relations, whence necessarily $l \leq 1$. In the case $l = 1$ the number of generators and the number of relations are equal, and so the order of $\text{Cl} A$ is the absolute value of the determinant

$$\begin{vmatrix} e^+ & e^- & e_1 & e_2 & \cdots & e_k \\ m^+ & m^- & 0 & 0 & \cdots & 0 \\ 0 & 0 & m_1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & m_k \\ \end{vmatrix} = \begin{vmatrix} e^+ & e^- \\ m^+ & m^- \\ \end{vmatrix} \cdot m_1 \cdot m_2 \cdots m_k.$$ 

Thus, if $\text{Cl} A = 0$ then all the factors of this product are equal to 1, and we are in case (ii). If $l = 0$ then $\text{Cl} A$ is the group $\bigoplus_{i=1}^k \mathbb{Z} m_i \cdot [O_i]$ modulo the relation $\sum_i e_i [O_i] = 0$. As $e_i$ and $m_i$ are coprime, this group is trivial if and only if (i) holds. Conversely, if
(i) or (ii) is satisfied then the discussion above shows that $\text{Cl}_A$ is trivial, finishing the proof. 

Finally, we determine the Picard group and the canonical divisor of $A$. The local divisor class group at the point $b_j$ is generated by $\hat{O}_j^\pm$ modulo the relations $e_j^+\hat{O}_j^+ - e_j^-\hat{O}_j^- = 0$ and $m_j^+\hat{O}_j^+ - m_j^-\hat{O}_j^- = 0$. Since the Picard group $\text{Pic}_A$ is the kernel of the map of $\text{Cl}_A$ into the direct product of all local divisor class groups, we obtain the following result.

**Corollary 4.24.** Pic $A$ is the group

$$
\pi^*(\text{Cl}_A) \oplus \bigoplus_{i=1}^k \mathbb{Z}[O_i] \oplus \bigoplus_{j=1}^l \mathbb{Z}(e_j^+\hat{O}_j^+ - e_j^-\hat{O}_j^-)
$$

modulo the relations

$$
\pi^*(a_i) = m_i[O_i], \quad i = 1, \ldots, k,
$$

$$
0 = \sum_{j=1}^k e_i[O_i] + \sum_{j=1}^l \left(e_j^+\hat{O}_j^+ - e_j^-\hat{O}_j^-\right).
$$

In particular, Pic $A$ vanishes if and only if $C \subseteq A^1$ and case (i) in Corollary 4.23 is satisfied or $l = 1$ and $m_i = 1$ for all $1 \leq i \leq k$.

**Corollary 4.25.** The canonical divisor of the surface $V = \text{Spec} A$ is given by

$$
K_V = \pi^*(K_C) + \sum_{j=1}^k (m_i - 1)[O_i] + \sum_{j=1}^l \left((m_j^+ - 1)[\hat{O}_j^+] + (-m_j^- - 1)[\hat{O}_j^-]\right).
$$

**Proof.** We claim that multiplication by the meromorphic differential form $du/u$ on $V$ gives an isomorphism

$$
\frac{du}{u} \wedge : \pi^*(\omega_C)\left(\sum_{j=1}^k (m_i - 1)[O_i] + \sum_{j=1}^l \left((m_j^+ - 1)[\hat{O}_j^+] + (-m_j^- - 1)[\hat{O}_j^-]\right)\right) \cong \omega_V.
$$

This is a local problem, so with the same arguments as in the proof of Theorem 4.18 we can reduce to the case that $A_0 \cong \mathbb{C}[t]$ and $D_+ = -D_- = -\frac{1}{2}[0]$, where $e$, $m$ are coprime. In this case (13) in the proof of Theorem 4.18 shows that $A = \mathbb{C}[x, x^{-1}, y]$ with $x := t^eu^m$ and $y := t^pu^q$, where $p, q$ are integers with $p^q | m = 1$. Moreover by (16) $t = x^{-q}y^m$ and $u = x^py^{-e}$. By an elementary calculation $\frac{du}{u} \wedge dt = x^{-q-1}y^{m-1}dx \wedge dy$, whence the result follows. 

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