Convex Hulls of Planar Random Walks

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Chang Xu
Department of Mathematics and Statistics
University of Strathclyde
Glasgow, UK

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Abstract

For the perimeter length $L_n$ and the area $A_n$ of the convex hull of the first $n$ steps of a planar random walk, this thesis study $n \to \infty$ mean and variance asymptotics and establish distributional limits. The results apply to random walks both with drift (the mean of random walk increments) and with no drift under mild moments assumptions on the increments.

Assuming increments of the random walk have finite second moment and non-zero mean, Snyder and Steele showed that $n^{-1}L_n$ converges almost surely to a deterministic limit, and proved an upper bound on the variance $\text{Var}[L_n] = O(n)$. We show that $n^{-1}\text{Var}[L_n]$ converges and give a simple expression for the limit, which is non-zero for walks outside a certain degenerate class. This answers a question of Snyder and Steele. Furthermore, we prove a central limit theorem for $L_n$ in the non-degenerate case.

Then we focus on the perimeter length with no drift and area with both drift and zero-drift cases. These results complement and contrast with previous work and establish non-Gaussian distributional limits. We deduce these results from weak convergence statements for the convex hulls of random walks to scaling limits defined in terms of convex hulls of certain Brownian motions. We give bounds that confirm that the limiting variances in our results are non-zero.
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Notations

\( S_n, Z_i \) : random walk with location \( S_n \) and increments \( Z_i \)  
\( \text{hull}(S_0, \ldots, S_n) \) : the convex hull of random walk \( S_n \)  
\( S_n \) : the random walk \( \{S_0, S_1, \ldots, S_n\} \)  
\( L_n \) : the perimeter length of \( \text{hull}(S_0, \ldots, S_n) \)  
\( A_n \) : the area of \( \text{hull}(S_0, \ldots, S_n) \)  
\( \| \cdot \| \) : the Euclidean norm  
\( \mu \) : the mean drift vector  
\( \Sigma \) : the covariance matrix associated with \( Z_i \)  
\( \Sigma^{1/2} \) : the matrix square-root of \( \Sigma \)  
\( \sigma^2 \) : \( \text{tr} \Sigma \)  
\( \hat{\mu} \) : \( \| \mu \|^{-1} \mu \) for \( \mu \neq 0 \)  
\( \sigma^2_\mu \) : \( \mathbb{E} \left[ ((Z_1 - \mu) \cdot \hat{\mu})^2 \right] \)  
\( \sigma^2_{\mu_\perp} \) : \( \sigma^2 - \sigma^2_\mu \)  
\( C([0, T]; \mathbb{R}^d) \) : the class of continuous functions from \([0, T]\) to \( \mathbb{R}^d \)  
\( C^0([0, T]; \mathbb{R}^d) \) : \( \{ f \in C([0, T]; \mathbb{R}^d) : f(0) = 0 \} \)  
\( \rho_\infty(\cdot, \cdot) \) : the supremum metric  
\( \rho(x, A) \) : \( \inf_{y \in A} \rho(x, y) \) for \( A \subseteq \mathbb{R}^d \) and a point \( x \in \mathbb{R}^d \)  
\( C_d \) : \( C([0, 1]; \mathbb{R}^d) \)  
\( C^0_d \) : \( \{ f \in C_d : f(0) = 0 \} \)  
\( S_{d-1} \) : \( \{ u \in \mathbb{R}^d : \| u \| = 1 \} \), the unit sphere in \( \mathbb{R}^d \)  
\( \mathcal{K}_d \) : the collection of convex compact sets in \( \mathbb{R}^d \)  
\( \mathcal{K}_d^0 \) : \( \{ A \in \mathcal{K}_d : 0 \in A \} \)  
\( \rho_H(\cdot, \cdot) \) : the Hausdorff metric  
\( \pi_r(\cdot) \) : the parallel body at distance \( r \)  
\( b \) : \( \{ b(s) \}_{s \in [0, 1]} \), standard Brownian motion in \( \mathbb{R}^d \)  
\( \mathcal{A}(\cdot) \) : the area of convex compact sets in the plane  
\( \mathcal{L}(\cdot) \) : the perimeter length of convex compact sets  
\( h_A(\cdot) \) : the support function of \( A \in \mathcal{K}_d^0 \)  
\( H(f) \) : \( \text{hull}(f[0, 1]) \) for \( f \in C_d \)  
\( h_t \) : the convex hull of the Brownian path up to time \( t \)
\[ \ell_t : = \mathcal{L}(h_t) \]  
\[ a_t : = \mathcal{A}(h_t) \]  
\[ \Rightarrow : \text{ weak convergence} \]  
\[ w : = (w(s))_{s \in [0,1]}, \text{ standard Brownian motion in } \mathbb{R} \]  
\[ \tilde{b}(s) : = (s, w(s)), \text{ for } s \in [0,1] \]  
\[ \tilde{h}_t : = \text{hull } \tilde{b}[0,t] \in \mathcal{K}_2^0 \]  
\[ \tilde{a}_t : = \mathcal{A}(\tilde{h}_t) \]  
\[ \mathbf{1}\{\cdot\} : \text{ the indicator function} \]  
\[ x^+ : = x \mathbf{1}\{x > 0\} \]  
\[ x^+ : = -x \mathbf{1}\{x < 0\} \]  
\[ u_0(\Sigma) : = \text{Var}\mathcal{L}(\Sigma^{1/2}h_1) \]  
\[ v_+, v_0 : \text{ defined in equation (6.1)} \]
Chapter 1

Introduction

1.1 Background on Random Walk

Let $Z_1, Z_2, \ldots$ be independent identically distributed (i.i.d.) random variables taking values in $\mathbb{R}^d$ and let $S_n = \sum_{i=1}^{n} Z_i$. $S_n$ is a random walk [30, p. 88].

Random walk theory is a classical and well-studied topic in probability theory. In 1905, Albert Einstein studied the Brownian motion in his paper “On the Movement of Small Particles Suspended in a Stationary Liquid Demanded by the Molecular-Kinetic Theory of Heat”. Brownian motion is the random motion of particles in a fluid which is found by the botanist Robert Brown in 1827 [32, Sec. 2.1]. He noted that the pollen grains in water kept moved through randomly. Einstein explained in details how the motion that Brown had observed was a result of the pollen being moved by individual water molecules.

Scientists then gave the mathematical formalisation for the Brownian motion and its generalisation: random walk. The term random walk was first used by Karl Pearson in 1905. In a letter to Nature, he gave a simple model to describe a mosquito infestation in a forest. At each time step, a single mosquito moves a fixed length in a randomly chosen direction. Pearson wanted to know the distribution of the mosquitoes after many steps had been taken. The letter was answered by Lord Rayleigh, who had already solved a more general form of this problem in 1880, in the context of sound waves in heterogeneous materials. Modelling a sound wave travelling through the material can be thought of as summing up a sequence of random wave-vectors of constant amplitude but random phase since sound waves
in the material have roughly constant wavelength, but their directions are altered at scattering sites within the material.

There are some classical results we need to bear in mind when we study random walks. First we need to introduce the concepts of recurrence and transience. A random walk \( S_n \) taking values in \( \mathbb{R}^d \) is called point-recurrent if

\[
P(S_n = 0 \text{ infinitely often}) = 1
\]

and point-transient if

\[
P(S_n = 0 \text{ infinitely often}) = 0.
\]

If the random walk is not discrete then these definitions are not very useful. Instead we say that the random walk is neighbourhood-recurrent if for some \( \varepsilon > 0 \),

\[
P(|S_n| < \varepsilon \text{ infinitely often}) = 1
\]

and neighbourhood-transient if

\[
P(|S_n| < \varepsilon \text{ infinitely often}) = 0.
\]

In the discrete case, for a simple random walk we have the Pólya’s theorem \[48\]. A random walk \( S_n = \sum_{i=1}^{n} Z_i \) on \( \mathbb{Z}^d \) is simple if for any \( i \in \mathbb{N} \),

\[
P(Z_i = e) = \begin{cases} (2d)^{-1} & \text{if } e \in \mathbb{Z}^d \text{ and } \|e\| = 1, \\ 0 & \text{otherwise}. \end{cases}
\]

**Theorem 1.1** (Pólya). A simple random walk \( S_n = \sum_{i=1}^{n} Z_i \) in \( \mathbb{Z}^d \) is recurrent for \( d = 1 \) or \( d = 2 \) and transient for \( d \geq 3 \).

This theorem was generalised by Chung and Fuchs \[15\] in 1951.

**Theorem 1.2** (Chung–Fuchs). Let \( S_n \) be a random walk in \( \mathbb{R}^d \). Then,

(i) If \( d = 1 \) and \( n^{-1}S_n \to 0 \) in probability, then \( S_n \) is neighbourhood-recurrent.

(ii) If \( d = 2 \) and \( n^{-1/2}S_n \) converges in distribution to a centred normal distribution, then \( S_n \) is neighbourhood-recurrent.

(iii) If \( d \geq 3 \) and the random walk is not contained in a lower-dimensional subspace, then it is neighbourhood-transient.
1.2 Background on geometric probability

A central theme of classical geometric probability or stochastic geometry concerns the study of the properties of random point sets in Euclidean space and associated structures. For example, a large literature is devoted to study of the lengths of graphs on random vertex sets in Euclidean space $\mathbb{R}^d$, $d \geq 2$. The interests are primarily in the lengths of those graphs representing the solutions to problems in Euclidean combinatorial optimization (see [60] or [67]). In the classical setting, the random point sets are generated by i.i.d. random variables. Some typical problems involve the construction of the shortest possible network of some kind:

Let $X_0, X_1, \ldots, X_n$ be i.i.d. random points with common distribution on $\mathbb{R}^d$ and $V = \{X_i\}_{i=0}^n$.

(i) Travelling salesman problem. Find the length of shortest closed path traversing each vertex in $V$ exactly once.

(ii) Minimal spanning tree. Find the minimal total edge length of a spanning tree through $V$.

(iii) Minimal Euclidean matching. Find the minimal total edge length of a Euclidean matching of points in $V$.

Many of the questions of geometric probability or stochastic geometry are equally valid for point sets generated by random walk trajectories.

1.3 Random convex hulls

We first define the convex hull here. A set $C$ in $\mathbb{R}^d$ is convex if it has the following property [29, p. 42]:

$$(1 - \lambda)x + \lambda y \in C \text{ for any } x, y \in C, \ 0 \leq \lambda \leq 1.$$ 

Given a set $A$ in $\mathbb{R}^d$, its convex hull is the intersection of all convex sets in $\mathbb{R}^d$ which contain $A$. Since the intersection of convex sets is always convex, the convex hull of $A$ is convex and it is the smallest convex set in $\mathbb{R}^d$ with respect to set inclusion, which contains $A$. 
One of the motivations to study the convex hulls is to find the extreme values in the random points. For the 1-dimensional case, the extreme values are just the maximum and minimum values. For higher dimensional cases, the extreme values could be determined by the convex hulls.

However, the extreme values have different meanings in these two different main settings of classical stochastic geometry. For the setting of i.i.d. random points, one important concern is the outlier detection in random sample. For the setting of trajectories of stochastic processes, extremes are important for study of record values. It gives two related but different streams of research, with different underlying probabilistic models and different motivating questions, though generally the motivations are all comes from multidimensional theory of extremes. See for example [4], [5], [6] and [45].

1.3.1 i.i.d. random points

Convex hulls of iid. random points, also known as random polytopes, were first studied by Geffroy [24] (1961), Rényi and Sulanke [50] (1963), and Efron [18] (1965). In the case where the points are normally distributed, the resulting convex hulls are known as Gaussian polytopes. See Reitzner [49, Random polytopes, pp. 45-76] (2010) and Hug [31] (2013) for recent surveys.

Motivation arises in statistics (multivariate extremes) and convex geometry (approximation of convex sets), and there are connection to the isotropic constant in functional analysis: see Reitzner [49]. He also listed some other applications including to the analysis of algorithms and optimization.

For the multivariate extremes, let $X_0, X_1, \ldots, X_n$ be the iid. random points with common distribution on $\mathbb{R}^d$ and $V = \{X_i\}_{i=0}^n$. In the case of $d = 1$, iid. points extremes are used in outlier detection in statistics. In the case of $d \geq 2$, Green [28] describes the peeling algorithm for detection of multivariate outliers via the iterated removal of points on the boundary of the convex hulls.

For the approximation of convex sets, Reitzner [49] insulates the algorithms to efficiently compute convex hull for large point set in $\mathbb{R}^d$. 
1.3.2 Trajectories of stochastic process

Before the study of random polytopes, Lévy [40] had considered the convex hull of planar Brownian motion. The study of convex hull of random walk goes back to Spitzer and Widom [58]. Generally, the convex hull of a stochastic process is an interesting geometrical object, related to extremes of the stochastic processes, giving a multivariate analogue of record values.

In one dimension, a value of a process is a record value if it is either less than all previous values (a lower record) or greater than all previous values (an upper record). In higher dimensions, a natural definition of “record” is then a point that lies outside the convex hull of all previous values.

More recent work on convex hull of Brownian motion includes Burdzy [11] (1985), Cranston, Hsu and March [13] (1989), Eldan [20] (2014), Evans [21] (1985), Pitman and Ross [47] (2012).

For general stochastic processes, convex hulls and related convex minorants or majorants, are studied by Bass [8] (1982) and Sinai [55] (1998).

1.4 Applications for convex hulls of random walks

In recent studies of random walks, attention has focussed on various geometrical aspects of random walk trajectories. Many of the questions of stochastic geometry, traditionally concerned with functionals of independent random points, are also of interest for point sets generated by random walks.

Study of the convex hull of planar random walk goes back to Spitzer and Widom [58] and the continuum analogue, convex hull of planar Brownian motion, to Lévy [40], §52.6, pp. 254–256; both have received renewed interest recently, in part motivated by applications arising for example in modelling the ‘home range’ of animals. Random walks have been extensively used to model the movement of animals; Karl Pearson’s original motivation for the random walk problem originated with modelling the migration of animal species such as mosquitoes, and subsequently random walks have been used to model the locomotion of microbes: see [16,56] for surveys. If the trajectory of the random walker represents the loca-
tions visited by a roaming animal, then the convex hull is a natural estimate of the ‘home range’ of the animal \[65,66\]. Natural properties of interest are the perimeter length and area of the convex hull. See \[42\] for a recent survey of motivation and previous work. The method of Chapter 3 in part relies on an analysis of scaling limits, and thus links the discrete and continuum settings.

### 1.5 Introduction of the model

On each unsteady step, a drunken gardener deposits one of \(n\) seeds. Once the flowers have bloomed, what is the minimum length of fencing required to enclose the garden?

Let \(Z_1, Z_2, \ldots\) be a sequence of independent, identically distributed (i.i.d.) random vectors on \(\mathbb{R}^2\). Write \(0\) for the origin in \(\mathbb{R}^2\). Define the random walk \((S_n; n \in \mathbb{Z}_+)\) by \(S_0 := 0\) and for \(n \geq 1\), \(S_n := \sum_{i=1}^{n} Z_i\). Let \(\text{hull}(S_0, \ldots, S_n)\) be the convex hull of positions of the walk up to and including the \(n\)th step, which is the smallest convex set that contains \(S_0, S_1, \ldots, S_n\). Let \(L_n\) denote the length of the perimeter of \(\text{hull}(S_0, \ldots, S_n)\) and \(A_n\) be the area of the convex hull. (See Figure 1.1.)

We will impose a moments condition of the following form:

\((M_p)\) Suppose that \(\mathbb{E}[\|Z_1\|^p] < \infty\).

For almost everything that follows, we will assume that at least the \(p = 1\) case of \((M_p)\) holds, and frequently we will assume the \(p = 2\) case. For several of our results we assume that \((M_p)\) holds for some \(p > 2\). In any case, we will be explicit about which case we assume at any particular point.

Given that \((M_p)\) holds for some \(p \geq 1\), then \(\mu := \mathbb{E} Z_1 \in \mathbb{R}^2\), the mean drift vector of the walk, is well defined. If \((M_p)\) holds for some \(p \geq 2\), then \(\Sigma := \mathbb{E}[(Z_1 - \mu)(Z_1 - \mu)']\), the covariance matrix associated with \(Z_1\), is well defined; \(\Sigma\) is positive semidefinite and symmetric. We write \(\sigma^2 := \text{tr} \Sigma = \mathbb{E} [\|Z_1 - \mu\|^2]\). Here and elsewhere \(Z_1\) and \(\mu\) are viewed as column vectors, and \(\|\cdot\|\) is the Euclidean norm. We also introduce the decomposition \(\sigma^2 = \sigma^2_\mu + \sigma^2_{\mu^\perp}\) with

\[
\sigma^2_\mu := \mathbb{E} [((Z_1 - \mu) \cdot \hat{\mu})^2] = \mathbb{E} [(Z_1 \cdot \hat{\mu})^2] - \|\mu\|^2 \in \mathbb{R}_+.
\]
Here and elsewhere, ‘·’ denotes the scalar product, $\hat{\mu} := \|\mu\|^{-1}\mu$ for $\mu \neq 0$, and $\mathbb{R}_+ := [0, \infty)$.

Convex hulls of random points have received much attention over the last several decades: see [42] for an extensive survey, including more than 150 bibliographic
references, and sources of motivation more serious than our drunken gardener, such as modelling the ‘home-range’ of animal populations. An important tool in the study of random convex hulls is provided by a result of Cauchy in classical convex geometry. Spitzer and Widom [58], using Cauchy’s formula, and later Baxter [9], using a combinatorial argument, showed that

\[ E[L_n] = 2 \sum_{i=1}^{n} \frac{1}{i} E\|S_i\|. \]  

(1.1)

Note that \( E[L_n] \) thus scales like \( n \) in the case where the one-step mean drift vector \( E[Z_1] \neq 0 \) but like \( n^{1/2} \) in the case where \( E[Z_1] = 0 \) (provided \( E[\|Z_1\|^2] < \infty \)).

The Spitzer–Widdom–Baxter result, in common with much of the literature, is concerned with first-order properties of \( L_n \): see [42] for a summary of results in this direction for various random convex hulls, with a specific focus on (driftless) planar Brownian motion.

Much less is known about higher-order properties of \( L_n \). There is a clear distinction between the zero drift case (\( E[Z_1] = 0 \)) and the non-zero drift case (\( \|E[Z_1]\| > 0 \)). For example, denote \( r_n := \inf_{x \in \partial \text{hull}(S_0,\ldots,S_n)} \|x\| \). Note that \( r_n \) is non decreasing in \( n \), because \( S_0 = 0 \in \text{hull}(S_0,\ldots,S_n) \subseteq \text{hull}(S_0,\ldots,S_{n+1}) \). We investigated the asymptotic behaviour of \( r_n \) in the following two different cases.

**Proposition 1.3.**  
(i) Suppose \( E[\|Z_1\|^2] < \infty \) and \( E[Z_1] = 0 \). Then \( \lim_{n \to \infty} r_n = \infty \) a.s.

(ii) Suppose \( E[\|Z_1\|] < \infty \) and \( E[Z_1] \neq 0 \). Then \( \lim_{n \to \infty} r_n < \infty \) a.s.

**Proof.**  
(i) In the first case, the random walk \( (S_n; n \in \mathbb{Z}_+) \) is recurrent (see e.g. [17]). There exists \( h \in \mathbb{R}_+ \), depending on the distribution of \( Z_1 \), such that \( S_n \) will visit any ball of radius at least \( h \) infinitely often (e.g., in the case of simple symmetric random walk on \( \mathbb{Z}^2 \), it suffices to take \( h = 1 \)). Let \( r > 0 \). Then, \( S_n \) will visit \( B((r + h)y; h) \) infinitely often for each \( y \in \{(1,1),(-1,1),(1,-1),(-1,-1)\} \). Here the notation \( B(x;r) \) is a Euclidean ball (a disk) with centre \( x \in \mathbb{R}^2 \) and radius \( r \in \mathbb{R}_+ \).

Let \( y = (1,1) \). Then \( S_n \) will visit \( B((r + h)(1,1); h) \) infinitely often.

So there exists some random time \( N \) with \( N < \infty \) a.s. such that \( \{S_0,\ldots,S_N\} \) contains a point in each of these four balls, and so \( \text{hull}(S_0,\ldots,S_N) \) contains the square with these points as its corners, which in turn contains \( B(0;r) \). So \( \lim \inf_{n \to \infty} r_n \geq r \) for any \( r \in \mathbb{R}_+ \). So \( \lim_{n \to \infty} r_n = \infty \).
(ii) In the second case, the random walk is transient (see [17]). Let $W_i$ be a wedge with apex $S_i$ with a angle $\theta < \pi$ (say $\theta = \pi/4$) so that $\theta$ is bisected by $E_{Z_1}$. By the Strong Law of Large Numbers, $\|S_n/n - E_{Z_1}\| \to 0$ a.s. and so $S_n/n \cdot E_{Z_1} \to 0$ a.s., where $E_{Z_1}^\perp$ is the normal vector of $E_{Z_1}$. This implies the number of points outside the wedge $W_i$ is finite for any $i \in \mathbb{Z}^+$. We take some $S_k$ inside the wedge $W_0$ and denote the set of finitely many points outside $W_k$ by $\{S_{\sigma_j} : j = 1, 2, \ldots, m\}$. Note that $S_0$ is outside $W_k$ so the set $\{S_{\sigma_j}\}$ is non-empty. Hence, there must be some $S_{\sigma_t} \in \{S_{\sigma_j}\}$ standing on the boundary of the convex hull, $S_{\sigma_t} \in \partial\text{hull}(S_0, \ldots, S_n)$ for all $n \geq \sigma_t$. Then, $\limsup_{n \to \infty} r_n \leq \|S_{\sigma_t}\| < \infty$, which implies $\lim_{n \to \infty} r_n < \infty$ a.s. since $r_n$ is non decreasing.

\[ \square \]

Remark 1.1. The key property for (i) is not (compact set) recurrence, but angular recurrence in the sense that $S_n$ visits any cone with apex at $0$ and non-zero angle infinitely often. Thus the same distinction between (i) and (ii) persists for random walks in $\mathbb{R}^d$, $d \geq 3$, with the notation extended in the natural way.

Because of this distinction, we always separate the arguments of $L_n$ and $A_n$ into the cases of non-zero and zero drift.

To illustrate our model, here we give some pictures of simulation examples (see Figure 1.3).

1.6 Outline of the thesis

Chapter 2 is some necessary mathematical prerequisites for our results. It includes the concepts of the study objects and the essential tools used in the rest chapters.

In Chapter 3 we describe our scaling limit approach, and carry it through after presenting the necessary preliminaries; the main new results of this chapter, Theorems 3.6 and 3.8, give weak convergence statements for convex hulls of random walks in the case of zero and non-zero drift, respectively. Armed with these weak convergence results, we present asymptotics for expectations and variances of the quantities $L_n$ and $A_n$ in Section 5.4, 6.4 and 6.5; the arguments in this section rely in part on the scaling limit apparatus, and in part on direct random walk
Figure 1.3: The number of steps $n = 300$ for all three examples. The top left: Simple random walk on $\mathbb{Z}$. $Z_i$ takes $(\pm 1, 0), (0, \pm 1)$ each with probability $1/4$. The top right: $Z_i$ takes $(\pm 1, 0), (0, \pm 1), (-1, 1), (1, -1)$ each with probability $1/6$. The bottom left: Pearson–Rayleigh random walk. $Z_i$ takes value uniformly on the unit circle.

computations. This section concludes with upper and lower bounds we found for the limiting variances.

Snyder and Steele [57] showed that $n^{-1}L_n$ converges almost surely to a deterministic limit, and proved an upper bound on the variance $\text{Var}[L_n] = O(n)$ [57]. In Chapter 4 we give a different approach to prove their major results, which includes the fact that $n^{-1}\mathbb{E}[L_n]$ converges (Proposition 4.7) and a simple expression for the limit in Proposition 4.5. For the zero drift case, we give a new improved
limit expression in Proposition 4.9.

Chapter 5 gives the convergence of $n^{-1}\text{Var}[L_n]$ in Proposition 5.4, which is first proved by Snyder and Steele [57]. They also gave the law of large numbers for $L_n$ in the non-zero drift case. But we found it also valid for the zero drift case (Proposition 5.5). Apart from that, the following of major results in this chapter are new. For the non-zero drift case, we give a simple expression for the limit of $n^{-1}\text{Var}[L_n]$ in Theorem 5.13 [63, Theorem 1.1], which is non-zero for walks outside a certain degenerate class. This answers a question of Snyder and Steele. It is also the only case where the perimeter length $L_n$ is Gaussian. So we give a central limit theorem for $L_n$ in this case in Theorem 5.14 [63, Theorem 1.2]. For the non-zero drift case, the limit expression of $n^{-1}\text{Var}[L_n]$ is given in Proposition 5.15 [64, Proposition 3.5] and its upper and lower bounds are given by Proposition 5.16 [64, Proposition 3.7].

Chapter 6 is an analogue of Chapter 5 for the area $A_n$. In Theorem 6.8 we give the asymptotic for the expected area $\mathbb{E}A_n$ with zero drift, which is a bit more general than the form given by Barndorff-Nielsen and Baxter [3]. Apart from that, the following of major results in this chapter are new. We give the asymptotic for the expected area $\mathbb{E}A_n$ with drift in Proposition 6.9 [64, Proposition 3.4] and also the asymptotics for their variance $\text{Var}A_n$ in both zero drift (Proposition 6.12 [64, Proposition 3.5]) and non-zero drift cases (Proposition 6.13 [64, Proposition 3.6]). Meanwhile, some upper and lower variance bounds are provided by the last section of this chapter.
Chapter 2

Mathematical prerequisites

2.1 Convergence of random variables

First of all, we define the different modes of convergence we will need in this thesis.

Let $X$ and $X_1, X_2, \ldots$ be random variables in $\mathbb{R}$.

$X_n$ converges almost surely to $X$ ($X_n \xrightarrow{a.s.} X$) as $n \to \infty$ iff

$$\mathbb{P} (\{ \omega : X_n(\omega) \to X(\omega) \text{ as } n \to \infty \}) = 1.$$  

$X_n$ converges in probability to $X$ ($X_n \xrightarrow{p} X$) as $n \to \infty$ iff, for every $\varepsilon > 0$,

$$\mathbb{P} (|X_n - X| > \varepsilon) \to 0 \text{ as } n \to \infty.$$  

The $L^p$ norm of $X$ is defined by

$$\|X\|_p := (\mathbb{E} |X|^p)^{1/p}.$$  

$X_n$ converges in $L^p$ to $X$ ($X_n \xrightarrow{L^p} X$) for $p \geq 1$, as $n \to \infty$ iff

$$\mathbb{E} (|X_n - X|^p) \to 0, \text{ i.e. } \|X_n - X\|_p \to 0, \text{ as } n \to \infty.$$  

Let $F_X(x) = \mathbb{P}(X \leq x), x \in \mathbb{R}$, be the distribution function of $X$ and let $C(F_X) = \{ x : F_X(x) \text{ is continuous at } x \}$ be the continuity set of $F_X$. $X_n$ converges in distribution to $X$ ($X_n \xrightarrow{d} X$) as $n \to \infty$ iff

$$F_{X_n}(x) \to F_X(x) \text{ as } n \to \infty, \text{ for all } x \in C(F_X).$$
The concept of convergence in distribution extends to random variables in \( \mathbb{R}^d \) in terms of the joint distribution functions \( \mathbb{P}[X_n^{(1)} \leq x^{(1)}, \ldots, X_n^{(d)} \leq x^{(d)}] \).

These modes of convergence have the following logical relationships.

\[
X_n \xrightarrow{L^p} X \iff X_n \xrightarrow{p} X \implies X_n \xrightarrow{d} X
\]

\[
X_n \xrightarrow{a.s.} X \iff
\]

Now we collect some basic results on deducing convergence lemmas and theorems.

**Lemma 2.1** (Dominated convergence [30] p.57). Let \( X, Y \) and \( X_1, X_2, \ldots \) be random variables. Suppose that \( |X_n| \leq Y \) for all \( n \), where \( \mathbb{E} Y < \infty \), and that \( X_n \to X \) a.s. as \( n \to \infty \). Then

\[
\mathbb{E} |X_n - X| \to 0 \text{ as } n \to \infty,
\]

In particular,

\[
\mathbb{E} X_n \to \mathbb{E} X \text{ as } n \to \infty.
\]

**Lemma 2.2** (Pratt’s lemma [30] p.221). Let \( X \) and \( X_1, X_2, \ldots \) be random variables. Suppose that \( X_n \to X \) almost surely as \( n \to \infty \), and that

\[
|X_n| \leq Y_n \text{ for all } n, \quad Y_n \to Y \text{ a.s.,} \quad \mathbb{E} Y_n \to \mathbb{E} Y \text{ as } n \to \infty.
\]

Then

\[
X_n \to X \text{ in } L^1 \quad \text{and} \quad \mathbb{E} X_n \to \mathbb{E} X \text{ as } n \to \infty.
\]

**Lemma 2.3** (The Borel–Cantelli lemma [30] p.96, 98). Let \( \{A_n, n \geq 1\} \) be arbitrary events. Then

\[
\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty \implies \mathbb{P}(A_n \text{ i.o.}) = 0.
\]

Moreover, suppose that \( X_1, X_2, \ldots \) are random variables. Then,

\[
\sum_{n=1}^{\infty} \mathbb{P}(|X_n| > \varepsilon) < \infty \text{ for any } \varepsilon > 0 \implies X_n \to 0 \text{ a.s. as } n \to \infty.
\]
Lemma 2.4 (Slutsky’s theorem [30] p.249). Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ be sequences of random variables, Suppose that

$$X_n \overset{d}{\to} X \text{ and } Y_n \overset{p}{\to} a \text{ as } n \to \infty,$$

where $a$ is some constant. Then,

$$X_n + Y_n \overset{d}{\to} X + a \text{ and } X_n \cdot Y_n \overset{d}{\to} X \cdot a.$$

Here we also introduce some useful concepts of uniform integrability.

A collection of random variables $X_i$, $i \in I$, is said to be uniformly integrable if

$$\lim_{M \to \infty} \left( \sup_{i \in I} \mathbb{E} (|X_i| \mathbf{1}(|X_i| > M)) \right) = 0.$$

Lemma 2.5. Let $X$ and $X_1, X_2, \ldots$ be random variables. If $X_n \to X$ in probability then the following are equivalent:

(i) $\{X_n\}_{i=1}^\infty$ is uniformly integrable.

(ii) $X_n \to X$ in $L^1$.

(iii) $\mathbb{E} |X_n| \to \mathbb{E} |X| < \infty$.

Lemma 2.6 (convergence of means [35] p.45). Let $X, X_1, X_2, \ldots$ be $\mathbb{R}_+$-valued random variables with $X_n \overset{d}{\to} X$. If $\{X_i\}_{i=1}^\infty$ is uniformly integrable, then $\mathbb{E} X_n \to \mathbb{E} X$ as $n \to \infty$.

2.2 Martingales

A sequence $\{X_n\}_{i=1}^\infty$ of random variables is $\{\mathcal{F}_n\}$-adapted if $X_n$ is $\mathcal{F}_n$-measurable for all $n$, which means for any $k \in \mathbb{R}$, $\{\omega : X_n(\omega) \leq k\} \in \mathcal{F}_n$.

An integrable $\{\mathcal{F}_n\}$-adapted sequence $X_n$ is called a martingale if

$$\mathbb{E}(X_{n+1} \mid \mathcal{F}_n) = X_n \text{ a.s. for all } n \geq 0.$$ 

It is called a submartingale if

$$\mathbb{E}(X_{n+1} \mid \mathcal{F}_n) \geq X_n \text{ a.s. for all } n \geq 0,$$
and a supermartingale if
\[ E(X_{n+1} \mid \mathcal{F}_n) \leq X_n \text{ a.s. for all } n \geq 0. \]

An integrable, \( \{\mathcal{F}_n\} \)-adapted sequence \( \{D_n\} \) is called a martingale difference sequence if
\[ E(D_{n+1} \mid \mathcal{F}_n) = 0 \text{ for all } n \geq 0. \]

Then, the sequence of \( M_n := \sum_{k=1}^{n} D_k \) is \( \{\mathcal{F}_n\} \)-martingale since
\[ E[M_{n+1} - M_n \mid \mathcal{F}_n] = E[D_{n+1} \mid \mathcal{F}_n] = 0, \]
which indicate
\[ E[M_{n+1} \mid \mathcal{F}_n] = M_n. \]

Lemma 2.7 (Orthogonality of martingale differences [30] p.488). Let \( \{D_n\}_{n=0}^{\infty} \) be a martingale difference sequence. Then \( E[D_m D_n] = 0 \) for \( m \neq n \). Hence,
\[ \text{Var} \left( \sum_{i=0}^{n} D_i \right) = \sum_{i=0}^{n} \text{Var}(D_i). \]

We use a standard martingale difference construction based on resampling. Consider the functional on \( \mathbb{R}^n \), \( f : \mathbb{R}^n \to \mathbb{R} \). Let \( Y_1, Y_2, \ldots, Y_n \) be iid. random variables and \( W_n = f(Y_1, \ldots, Y_n) \). Let \( Y'_1, Y'_2, \ldots, Y'_n \) be independent copies of \( Y_1, Y_2, \ldots, Y_n \) and
\[ W_n^{(i)} = f(Y_1, \ldots, Y_{i-1}, Y'_i, Y_{i+1}, \ldots, Y_n). \]
Let \( D_{n,i} = E[W_n - W_n^{(i)} \mid \mathcal{F}_i] \) where \( \mathcal{F}_i = \sigma(Y_1, \ldots, Y_i) \).

Lemma 2.8. Let \( n \in \mathbb{N} \). Then

(i) \( W_n - E W_n = \sum_{i=1}^{n} D_{n,i}; \)

(ii) \( \text{Var}(W_n) = \sum_{i=1}^{n} E[D_{n,i}^2] \) whenever the latter sum is finite.

Proof. The idea is well known. Since \( W_n^{(i)} \) is independent of \( Y_i \),
\[ E[W_n^{(i)} \mid \mathcal{F}_i] = E[W_n^{(i)} \mid \mathcal{F}_{i-1}] = E[W_n \mid \mathcal{F}_{i-1}]. \]
So,
\[ D_{n,i} = E[W_n \mid \mathcal{F}_i] - E[W_n \mid \mathcal{F}_{i-1}]. \]
Hence $D_{n,i}$ is martingale differences, since
\[ 
\mathbb{E} [D_{n,i} \mid \mathcal{F}_{i-1}] = \mathbb{E} [W_n \mid \mathcal{F}_{i-1}] - \mathbb{E} [W_n \mid \mathcal{F}_{i-1}] = 0 
\]
and
\[ 
\sum_{i=1}^{n} D_{n,i} = \mathbb{E} [W_n \mid \mathcal{F}_n] - \mathbb{E} [W_n \mid \mathcal{F}_0] = W_n - \mathbb{E} W_n. 
\]
So,
\[ 
\mathbb{E} \left( \sum_{i=1}^{n} D_{n,i} \right)^2 = \mathbb{V} \text{ar}(W_n). 
\]
But by orthogonality of martingale differences, (Lemma 2.7),
\[ 
\mathbb{V} \text{ar}(W_n) = \sum_{i=1}^{n} \mathbb{E} [D_{n,i}^2]. 
\]

Note that by the conditional Jensen’s inequality ($\mathbb{E} [(\xi \mid \mathcal{F})]^2 \leq \mathbb{E} [\xi^2 \mid \mathcal{F}]$), we have
\[ 
D_{n,i}^2 \leq \mathbb{E} \left( (W_n - W_n^{(i)})^2 \mid \mathcal{F}_i \right). 
\]
So from part (ii) of Lemma 2.8,
\[ 
\mathbb{V} \text{ar}(W_n) \leq \sum_{i=1}^{n} \mathbb{E} \left( (W_n^{(i)} - W_n)^2 \right). 
\]
This gives a upper bound for the variance of $W_n$, which is a factor of 2 larger than the upper bound obtained from the Efron–Stein inequality (equation (2.3) in [57]):

**Lemma 2.9.**
\[ 
\mathbb{V} \text{ar}(W_n) \leq \frac{1}{2} \sum_{i=1}^{n} \mathbb{E} \left( (W_n^{(i)} - W_n)^2 \right). 
\]

### 2.3 Reflection principle for Brownian motion

**Lemma 2.10** (Reflection principle [44] p.44). If $T$ is a stopping time and \{\(w(t) : t \geq 0\)\} is a standard 1-dimensional Brownian motion, then the process \{\(w^*(t) : t \geq 0\)\} called Brownian motion reflected at $T$ and defined by
\[ 
w^*(t) = w(t) \mathbf{1}\{t \leq T\} + (2w(T) - w(t)) \mathbf{1}\{t > T\} 
\] is also a standard Brownian motion.
Corollary 2.11. Suppose $r > 0$ and \( \{w(t) : t \geq 0\} \) is a standard 1-dimensional Brownian motion. Then,

\[
\mathbb{P}\left(\sup_{0 \leq s \leq t} w(s) > r\right) = 2\mathbb{P}(w(t) > r).
\]

2.4 Useful inequalities

We collect some useful inequalities which is useful in the next chapters.

Lemma 2.12 (Markov’s inequality \[30\] p.120). Let $X$ be a random variable. Suppose that $\mathbb{E}|X|^r < \infty$ for some $r > 0$, and let $x > 0$. Then,

\[
\mathbb{P}(|X| > x) \leq \frac{\mathbb{E}|X|^r}{x^r}.
\]

Lemma 2.13 (Chebyshev’s inequality \[30\] p.121). Let $X$ be a random variable. Suppose that $\text{Var}X < \infty$. Then for $x > 0$,

\[
\mathbb{P}(|X - \mathbb{E}X| > x) \leq \frac{\text{Var}X}{x^2}.
\]

Lemma 2.14 (The Cauchy–Schwarz inequality \[30\] p.130). Suppose that random variables $X$ and $Y$ have finite variances. Then,

\[
|\mathbb{E}XY| \leq \mathbb{E}|XY| \leq \|X\|_2\|Y\|_2 = \sqrt{\mathbb{E}(X^2)\mathbb{E}(Y^2)}.
\]

The next result generalises the Cauchy–Schwarz inequality.

Lemma 2.15 (The H"older inequality \[30\] p.129). Let $X$ and $Y$ be random variables. Suppose that $p^{-1} + q^{-1} = 1$, $\mathbb{E}|X|^p < \infty$ and $\mathbb{E}|Y|^q < \infty$, then

\[
|\mathbb{E}XY| \leq \mathbb{E}|XY| \leq \|X\|_p\|Y\|_q = (\mathbb{E}X^p)^{1/p}(\mathbb{E}Y^q)^{1/q}.
\]

Lemma 2.16 (The Minkowski inequality \[30\] p.129). Let $p \geq 1$. Suppose that $X$ and $Y$ are random variables, such that $\mathbb{E}|X|^p < \infty$ and $\mathbb{E}|Y|^p < \infty$. Then,

\[
\|X + Y\|_p \leq \|X\|_p + \|Y\|_p.
\]

This is the triangle inequality for the $L^p$ norm.

Now we introduce some inequalities on martingales.
Lemma 2.17 (Doob’s inequality \[17\] p.214). If \(X_n\) is a martingale, then for \(1 < p < \infty\),
\[
\mathbb{E} \left( \max_{0 \leq m \leq n} |X_m|^p \right)^{\frac{1}{p}} \leq \left( \frac{p}{p-1} \right)^p \mathbb{E} (|X_n|^p).
\]

Lemma 2.18 (Azuma–Hoeffding inequality \[46\] p.33). Let \(D_{n,i} (i = 1, \ldots, n)\) be a martingale difference sequence adapted to a filtration \(\mathcal{F}_i\), which means \(D_{n,i}\) is \(\mathcal{F}_i\)-measurable and \(\mathbb{E} [D_{n,i}|\mathcal{F}_{i-1}] = 0\). Then, for any \(t > 0\),
\[
\mathbb{P} \left( \left\| \sum_{i=1}^{n} D_{n,i} \right\| > t \right) \leq 2 \exp \left( -\frac{t^2}{2nd^2_\infty} \right),
\]
where \(d_\infty\) is such that \(|D_{n,i}| \leq d_\infty\) a.s. for all \(n, i\).

We also introduce some inequalities for sums of independent random variables.

Lemma 2.19 (Marcinkiewicz–Zygmund inequality \[30\] p.151). Let \(p \geq 1\). Suppose that \(X, X_1, X_2, \ldots, X_n\) are independent, identically distributed random variables with mean \(0\) and \(\mathbb{E} |X|^p < \infty\). Set \(S_n = \sum_{k=1}^{n} X_k\). Then there exists a constant \(B_p\) depending only on \(p\), such that
\[
\mathbb{E} |S_n|^p \leq \begin{cases} B_p n \mathbb{E} |X|, & \text{if } 1 \leq p \leq 2, \\ B_p n^{p/2} \mathbb{E} |X|^{p/2}, & \text{if } p > 2. \end{cases}
\]

Lemma 2.20 (Rosenthal’s inequality \[30\] p.151). Let \(p \geq 1\). Suppose that \(X_1, X_2, \ldots, X_n\) are independent random variables such that \(\mathbb{E} |X_k|^p < \infty\) for all \(k\). Set \(S_n = \sum_{k=1}^{n} X_k\). Then,
\[
\mathbb{E} |S_n|^p \leq \max \left\{ 2^p \sum_{k=1}^{n} \mathbb{E} |X_k|^p, 2^{p^2} \left( \sum_{k=1}^{n} \mathbb{E} |X_k| \right)^p \right\}.
\]

2.5 Useful theorems and lemmas

Lemma 2.21 (Fubini’s theorem \[30\] p.65). Let \((\Omega_1, \mathcal{F}_1, P_1)\) and \((\Omega_2, \mathcal{F}_2, P_2)\) be probability spaces, and consider the product space \((\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, P)\), where \(P = P_1 \times P_2\) is the product measure. Suppose that \(X = (X_1, X_2)\) is a two-dimensional random variable, and that \(g\) is \(\mathcal{F}_1 \times \mathcal{F}_2\)-measurable, and (i) non-negative or (ii) integrable. Then,
\[
\mathbb{E} g(X) = \int_{\Omega} g(X) \, dP = \int_{\Omega_1} \left( \int_{\Omega_2} g(X) \, dP_2 \right) \, dP_1 = \int_{\Omega_2} \left( \int_{\Omega_1} g(X) \, dP_1 \right) \, dP_2.
\]
Lemma 2.22. Let \( \{y_n\}_{n=1}^{\infty} \) be a sequence of real numbers and let \( y \in \mathbb{R} \). If \( y_n \to y \) as \( n \to \infty \), then \( n^{-1} \sum_{i=1}^{n} y_i \to y \) as \( n \to \infty \).

Proof. By assumption, for any \( \varepsilon > 0 \) there exists \( n_0 \in \mathbb{N} \) such that \( |y_n - y| \leq \varepsilon \) for all \( n \geq n_0 \). Then,

\[
\left| \frac{1}{n} \sum_{i=1}^{n} y_i - y \right| = \left| \frac{1}{n} \sum_{i=1}^{n_0} (y_i - y) \right| + \left| \frac{1}{n} \sum_{i=n_0+1}^{n} (y_i - y) \right| \\
\leq \frac{1}{n} \sum_{i=1}^{n_0} |y_i - y| + \frac{1}{n} \sum_{i=n_0+1}^{n} |y_i - y| \\
\leq \frac{1}{n} \sum_{i=1}^{n_0} |y_i - y| + \frac{1}{n} \sum_{i=n_0+1}^{n} |y_i - y| \\
\leq \frac{1}{n} \sum_{i=1}^{n_0} |y_i - y| + \varepsilon \\
\leq 2\varepsilon,
\]

for all \( n \) big enough. Since \( \varepsilon > 0 \) was arbitrary, the result follows.

\[ \square \]

2.6 Multivariate normal distribution

Let \( \Sigma \) be a symmetric positive semi-definite \((d \times d)\) matrix. Then, there exists an unique positive semi-definite symmetric matrix \( \Sigma^{1/2} \) such that \( \Sigma = \Sigma^{1/2} \Sigma^{1/2} \) \cite{41}. The matrix \( \Sigma^{1/2} \) can also be regarded as a linear transform of \( \mathbb{R}^d \) given by \( x \mapsto \Sigma^{1/2} x \).

For a random variable \( Y \), the notation \( Y \sim \mathcal{N}(0, \Sigma) \) means \( Y \) has \( d \) dimensional normal distribution with mean 0 and covariance matrix \( \Sigma \). In the degenerate case, all entries of the covariance matrix is 0, \( \Sigma = 0 \), which means that \( Y = 0 \) almost surely.

Lemma 2.23. Suppose \( X \sim \mathcal{N}(0, I) \) and let \( Y = \Sigma^{1/2} X \). Then \( Y \sim \mathcal{N}(0, \Sigma) \).

Lemma 2.24 (Multidimensional Central Limit Theorem \cite{11} p.62). Suppose \( \{Z_i\}_{i=1}^{\infty} \) is a sequence of i.i.d. random variables on \( \mathbb{R}^d \). \( S_n = \sum_{i=1}^{n} Z_i \) is a random walk on \( \mathbb{R}^d \). If \( \mathbb{E}(\|Z_1\|^2) < \infty \), \( \mathbb{E} Z_1 = 0 \) and \( \mathbb{E}(Z_1 Z_1^\top) = \Sigma \), then

\[
n^{-1/2} S_n \xrightarrow{d} \mathcal{N}(0, \Sigma).
\]
2.7 Analytic and Geometric prerequisites

We recall a few basic facts from real analysis: [53] is an excellent general reference. The *Heine–Borel theorem* states that a set in $\mathbb{R}^d$ is compact if and only if it is closed and bounded [53, p. 40]. Compactness is preserved under continuous mappings: if $(X, \rho_X)$ is a compact metric space and $(Y, \rho_Y)$ is a metric space, and $f : (X, \rho_X) \to (Y, \rho_Y)$ is continuous, then the image $f(X)$ is compact [53, p. 89]; moreover $f$ is uniformly continuous on $X$ [53, p. 91]. For any such uniformly continuous $f$, there is a monotonic *modulus of continuity* $\mu_f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\rho_Y(f(x_1), f(x_2)) \leq \mu_f(\rho_X(x_1, x_2))$ for all $x_1, x_2 \in X$, and for which $\mu_f(\rho) \downarrow 0$ as $\rho \downarrow 0$ (see e.g. [35, p. 57]).

Let $d$ be a positive integer. For $T > 0$, let $\mathcal{C}([0, T]; \mathbb{R}^d)$ denote the class of continuous functions from $[0, T]$ to $\mathbb{R}^d$. Endow $\mathcal{C}([0, T]; \mathbb{R}^d)$ with the supremum metric

$$\rho_\infty(f, g) := \sup_{t \in [0, T]} \rho(f(t), g(t)), \text{ for } f, g \in \mathcal{C}([0, T]; \mathbb{R}^d).$$

Let $\mathcal{C}^0([0, T]; \mathbb{R}^d)$ denote those functions in $\mathcal{C}([0, T]; \mathbb{R}^d)$ that map 0 to the origin in $\mathbb{R}^d$.

Usually, we work with $T = 1$, in which case we write simply

$$\mathcal{C}_d := \mathcal{C}([0, 1]; \mathbb{R}^d), \text{ and } \mathcal{C}_d^0 := \{ f \in \mathcal{C}_d : f(0) = 0 \}.$$ 

For $f \in \mathcal{C}([0, T]; \mathbb{R}^d)$ and $t \in [0, T]$, define $f[0, t] := \{ f(s) : s \in [0, t] \}$, the image of $[0, t]$ under $f$. Note that, since $[0, t]$ is compact and $f$ is continuous, the *interval image* $f[0, t]$ is compact. We view elements $f \in \mathcal{C}([0, T]; \mathbb{R}^d)$ as *paths* indexed by time $[0, T]$, so that $f[0, t]$ is the section of the path up to time $t$.

We need some notation and concepts from convex geometry: we found [29] to be very useful, supplemented by [58] as a convenient reference for a little integral geometry. Let $d$ be a positive integer. Let $\rho(x, y) = \|x - y\|$ denote the Euclidean distance between $x$ and $y$ in $\mathbb{R}^d$. For a set $A \subseteq \mathbb{R}^d$, write $\partial A$ for the boundary of $A$ (the intersection of the closure of $A$ with the closure of $\mathbb{R}^d \setminus A$), and $\text{int}(A) := A \setminus \partial A$ for the interior of $A$. For $A \subseteq \mathbb{R}^d$ and a point $x \in \mathbb{R}^d$, set $\rho(x, A) := \inf_{y \in A} \rho(x, y)$, with the usual convention that $\inf \emptyset = +\infty$. We write $\lambda_d$ for Lebesgue measure on $\mathbb{R}^d$. Write $S_{d-1} := \{ u \in \mathbb{R}^d : \|u\| = 1 \}$ for the unit sphere in $\mathbb{R}^d$. 
Let $\mathcal{K}_d$ denote the collection of convex compact sets in $\mathbb{R}^d$, and write

$$K_0^d := \{ A \in \mathcal{K}_d : 0 \in A \}$$

for those sets in $\mathcal{K}_d$ that include the origin. The Hausdorff metric on $K_0^d$ will be denoted

$$\rho_H(A, B) := \max \left\{ \sup_{x \in B} \rho(x, A), \sup_{y \in A} \rho(y, B) \right\} \text{ for } A, B \in \mathcal{K}_d.$$  

Given $A \in \mathcal{K}_d$, for $r > 0$ set

$$\pi_r(A) := \{ x \in \mathbb{R}^d : \rho(x, A) \leq r \},$$

the parallel body of $A$ at distance $r$. Note that, two equivalent descriptions of $\rho_H$ (see e.g. Proposition 6.3 of [29]) are for $A, B \in K_0^d$,

$$\rho_H(A, B) = \inf \{ r \geq 0 : A \subseteq \pi_r(B) \text{ and } B \subseteq \pi_r(A) \}; \text{ and}$$

$$\rho_H(A, B) = \sup_{e \in \mathbb{S}^{d-1}} |h_A(e) - h_B(e)|, \tag{2.1}$$

where $h_A(x) := \sup_{y \in A} (x \cdot y)$ is the support function of $A$ and $x \cdot y$ is the inner product of $x$ and $y$, i.e. $(x_1, y_1) \cdot (x_2, y_2) = x_1 x_2 + y_1 y_2$.

## 2.8 Continuous mapping theorem and Donsker’s Theorem

We consider random walks in $\mathbb{R}^d$ in this section. First we need to define the weak convergence in $\mathbb{R}^d$.

Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $(M, \rho)$ is a metric space. For $n \geq 1$, suppose that

$$X_n, X : \Omega \rightarrow M$$

are random variables taking values in $M$. If

$$\mathbb{E} f(X_n) \rightarrow \mathbb{E} f(X) \text{ as } n \rightarrow \infty,$$

for all bounded, continuous functional $f : M \rightarrow \mathbb{R}$, then we say that $X_n$ converges weakly to $X$ and write $X_n \Rightarrow X$. The weak convergence generalises the concept of convergence in distribution for random variables on $\mathbb{R}^d$. 
Lemma 2.25 (continuous mapping theorem \footnote{35} p.41). Fix two metric spaces \((M_1, \rho_1)\) and \((M_2, \rho_2)\). Let \(X, X_1, X_2, \ldots\) be random variables taking values in \(M_1\) with \(X_n \Rightarrow X\). Suppose \(f\) is a mapping on \((M_1, \rho_1) \to (M_2, \rho_2)\), which is continuous everywhere in \(M_1\) apart from possible on a set \(A \subseteq M_1\) with \(\mathbb{P}(X \in A) = 0\). Then, \(f(X_n) \Rightarrow f(X)\).

We generalise the definition of \(Z_i\) and \(S_n\) a little in this section. Let \(\{Z_i\}_{i=1}^{\infty}\) be a i.i.d. random vectors on \(\mathbb{R}^d\) and \(S_n = \sum_{i=1}^{n} Z_i\). For each \(n \in \mathbb{N}\) and all \(t \in [0, 1]\), define
\[
X_n(t) := S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor) (S_{\lfloor nt \rfloor + 1} - S_{\lfloor nt \rfloor}) = S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor) Z_{\lfloor nt \rfloor + 1}.
\]
Let \(b := (b(s))_{s \in [0,1]}\) denote standard Brownian motion in \(\mathbb{R}^d\), started at \(b(0) = 0\).

Lemma 2.26 (Donsker’s Theorem). Let \(d \in \mathbb{N}\). Suppose that \(\mathbb{E}(\|Z_1\|^2) < \infty\), \(\|\mathbb{E} Z_1\| = 0\), and \(\mathbb{E} [Z_1 Z_1^{\top}] = \Sigma\). Then, as \(n \to \infty\),
\[
n^{-1/2} X_n \Rightarrow \Sigma^{1/2} b,
\]
in the sense of weak convergence on \((C_0^d, \rho_\infty)\).

Remark 2.1. Donsker’s theorem generalizes the multidimensional central limit theorem (Lemma 2.24) to a \textit{functional} central limit theorem, because weak convergence of paths implies convergence in distribution of the endpoints. Indeed, taking \(t = 1\) in Donsker’s Theorem, the marginal convergence gives
\[
n^{-1/2} X_n(1) = n^{-1/2} S_n \overset{d}{\to} \Sigma^{1/2} b(1).
\]
Here by Lemma 2.23 \(\Sigma^{1/2} b(1) \sim \mathcal{N}(0, \Sigma)\) since \(b(1) \sim \mathcal{N}(0, I)\). Then we have \(n^{-1/2} S_n \overset{d}{\to} \mathcal{N}(0, \Sigma)\), which is Lemma 2.24.

2.9 Cauchy formula

For this section we take \(d = 2\). We consider the \(\mathcal{A} : \mathcal{K}_2 \to \mathbb{R}_+\) and \(\mathcal{L} : \mathcal{K}_2 \to \mathbb{R}_+\) given by the area and the perimeter length of convex compact sets in the plane. Formally, we may define
\[
\mathcal{A}(A) := \lambda_2(A), \quad \text{and} \quad \mathcal{L}(A) := \lim_{r \downarrow 0} \left( \frac{\lambda_2(\pi_r(A)) - \lambda_2(A)}{r} \right), \quad \text{for} \ A \in \mathcal{K}_2. \quad (2.3)
\]
The limit in (2.3) exists by the Steiner formula of integral geometry (see e.g. [54]), which expresses $\lambda_2(\pi_r(A))$ as a quadratic polynomial in $r$ whose coefficients are given in terms of the intrinsic volumes of $A$:

$$\lambda_2(\pi_r(A)) = \lambda_2(A) + r\mathcal{L}(A) + \pi r^2 1\{A \neq \emptyset\}. \quad (2.4)$$

In particular,

$$\mathcal{L}(A) = \begin{cases} \mathcal{H}_1(\partial A) & \text{if } \text{int}(A) \neq \emptyset, \\ 2\mathcal{H}_1(\partial A) & \text{if } \text{int}(A) = \emptyset, \end{cases}$$

where $\mathcal{H}_d$ is $d$-dimensional Hausdorff measure on Borel sets. We observe the translation-invariance and scaling properties

$$\mathcal{L}(x + \alpha A) = \alpha \mathcal{L}(A), \quad \text{and} \quad \mathcal{A}(x + \alpha A) = \alpha^2 \mathcal{A}(A),$$

where for $A \in \mathcal{K}_2$, $x + \alpha A = \{x + \alpha y : y \in A\} \in \mathcal{K}_2$.

For $A \in \mathcal{K}_2$, Cauchy obtained the following formula:

$$\mathcal{L}(A) = \int_0^\pi \left( \sup_{y \in A} (y \cdot e_\theta) - \inf_{y \in A} (y \cdot e_\theta) \right) d\theta. \quad (2.5)$$

We will need the following consequence of (2.5).

**Proposition 2.27.** Let $K = \{z_0, \ldots, z_n\}$ be a finite point set in $\mathbb{R}^2$, and let $\mathcal{C} = \text{hull}(K)$. Then

$$\mathcal{L}(\mathcal{C}) = \int_0^\pi \left( \max_{0 \leq i \leq n} (z_i \cdot e_\theta) - \min_{0 \leq i \leq n} (z_i \cdot e_\theta) \right) d\theta. \quad (2.6)$$

In particular, for the case of our random walk, (2.6) says

$$L_n = \mathcal{L}(\text{hull}(S_0, \ldots, S_n)) = \int_0^\pi \left( \max_{0 \leq i \leq n} (S_i \cdot e_\theta) - \min_{0 \leq i \leq n} (S_i \cdot e_\theta) \right) d\theta. \quad (2.7)$$

An immediate but useful consequence of (2.7) is that

$$L_{n+1} \geq L_n, \ a.s. \quad (2.8)$$

In the case where $K$ is a finite point set, $\text{hull}(K)$ is a convex polygon, the boundary of which contains vertices $V \subseteq K$ (extreme points of the convex hull) and the line-segment edges connecting them; note that $\text{hull}(K) = \text{hull}(V)$. 

Now, by convexity,

\[ \sup_{y \in C} (y \cdot e_\theta) = \max_{0 \leq i \leq n} (z_i \cdot e_\theta) = \sup_{y \in V} (y \cdot e_\theta), \]

and similarly for the infimum. So (2.5) does indeed imply (2.6). However, to keep this presentation as self-contained as possible, we give a direct proof of (2.6) without appealing to the more general result (2.5).

**Proof of Proposition 2.27.** The above discussion shows that it suffices to consider the case where \( V = K \) in which all of the \( z_i \) are on the boundary of the convex hull. Without loss of generality, suppose that \( 0 \in C \). Then we may rewrite (2.6) as

\[ \mathcal{L}(C) = \int_0^{2\pi} \max_{0 \leq i \leq n} (z_i \cdot e_\theta) \, d\theta. \]

Suppose also that \( z_i = \|z_i\|e_{\theta_i} \) in polar coordinates, labelled so that \( 0 \leq \theta_0 < \theta_1 < \cdots < \theta_n < 2\pi \). Thus starting from the rightmost point of \( \partial C \) on the horizontal axis and traversing the boundary anticlockwise, one visits the vertices \( z_0, z_1, \ldots, z_n \) in order.

Let \( z_{n+1} := z_0 \). Draw the perpendicular line of \( z_k - z_{k-1} \) passing through point
0 and denote the foot as \( y_k \). For \( 1 \leq k \leq n + 1 \), let

\[
\hat{z}_k := \begin{cases} 
  y_k, & \text{if } y_k \in \text{line segment } z_{k-1}z_k \\
  z_k, & \text{if } y_k \in \text{extended line of } z_{k-1}z_k \\
  z_{k-1}, & \text{if } y_k \in \text{extended line of } z_kz_{k-1}
\end{cases}
\]

and let \( z_0 := z_{n+1} \). Notice that \( \hat{z}_1, \ldots, \hat{z}_{n+1} \) are ordered in the same way as \( z_0, \ldots, z_n \) (see Figure 2.1). Therefore,

\[
\partial C = \bigcup_{k=0}^{n} \left[ (\hat{z}_{k+1} - z_k) \cup (z_k - \hat{z}_k) \right].
\]

Write \( \hat{z}_i = \|\hat{z}_i\|e_{\hat{\theta}_i} \) for \( 0 \leq i \leq n + 1 \) in the polar coordinates, we have

\[
\int_0^{2\pi} \max_{0 \leq i \leq n} (z_i \cdot e_\theta) \, d\theta = \sum_{k=0}^{n} \int_{\hat{\theta}_k}^{\hat{\theta}_{k+1}} z_k \cdot e_\theta \, d\theta.
\]

Consider \( \int_{\hat{\theta}_k}^{\hat{\theta}_{k+1}} z_k \cdot e_\theta \, d\theta \). Let \( z_k := (\alpha_1, \beta_1), z_{k+1} := (\alpha_2, \beta_2) \) and \( z_{k-1} := (\alpha_0, \beta_0) \). Without loss of generality, we can set \( \beta_1 = 0 \) and \( \alpha_1 > 0 \). Then we have \( \beta_2 \geq 0, \beta_0 \leq 0, 0 \leq \hat{\theta}_{k+1} \leq \pi/2 \) and \( -\pi/2 \leq \hat{\theta}_k \leq 0 \). So,

\[
\int_{\hat{\theta}_k}^{\hat{\theta}_{k+1}} z_k \cdot e_\theta \, d\theta = \int_{\hat{\theta}_k}^{\hat{\theta}_{k+1}} (\alpha_1, 0) \cdot (\cos \theta, \sin \theta) \, d\theta
\]

\[
= \alpha_1 (\sin \hat{\theta}_{k+1} - \sin \hat{\theta}_k)
\]

\[
= \alpha_1 \left( \frac{\|\hat{z}_{k+1} - z_k\|}{\alpha_1} - \frac{-\|z_k - \hat{z}_k\|}{\alpha_1} \right)
\]

\[
= \|\hat{z}_{k+1} - z_k\| + \|z_k - \hat{z}_k\|.
\]

Hence,

\[
\int_0^{2\pi} \max_{0 \leq i \leq n} (z_i \cdot e_\theta) \, d\theta = \sum_{k=0}^{n} \int_{\hat{\theta}_k}^{\hat{\theta}_{k+1}} z_k \cdot e_\theta \, d\theta = \sum_{k=0}^{n} (\|\hat{z}_{k+1} - z_k\| + \|z_k - \hat{z}_k\|) = L(C). \quad \square
\]
Chapter 3

Scaling limits for convex hulls

3.1 Overview

For some of the results that follow, scaling limit ideas are useful. Recall that
\( S_n = \sum_{k=1}^{n} Z_k \) is the location of our random walk in \( \mathbb{R}^2 \) after \( n \) steps. Write \( S_n := \{S_0, S_1, \ldots, S_n\} \). Our strategy to study properties of the random convex set hull \( S_n \) (such as \( L_n \) or \( A_n \)) is to seek a weak limit for a suitable scaling of hull \( S_n \), which we must hope to be the convex hull of some scaling limit representing the walk \( S_n \).

In the case of zero drift (\( \mu = 0 \)) a candidate scaling limit for the walk is readily identified in terms of planar Brownian motion. For the case \( \mu \neq 0 \), the ‘usual’ approach of centering and then scaling the walk (to again obtain planar Brownian motion) is not useful in our context, as this transformation does not act on the convex hull in any sensible way. A better idea is to scale space differently in the direction of \( \mu \) and in the orthogonal direction.

In other words, in either case we consider \( \phi_n(S_n) \) for some affine continuous scaling function \( \phi_n : \mathbb{R}^2 \to \mathbb{R}^2 \). The convex hull is preserved under affine transformations, so
\[
\phi_n(\text{hull } S_n) = \text{hull } \phi_n(S_n),
\]
the convex hull of a random set which will have a weak limit. We will then be able to deduce scaling limits for quantities \( L_n \) and \( A_n \) provided, first, that we work in suitable spaces on which our functionals of interest enjoy continuity, so that we can appeal to the continuous mapping theorem for weak limits, and, second, that

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\( \phi_n \) acts on length and area by simple scaling. The usual \( n^{-1/2} \) scaling when \( \mu = 0 \) is fine; for \( \mu \neq 0 \) we scale space in one coordinate by \( n^{-1} \) and in the other by \( n^{-1/2} \), which acts nicely on area, but not length. Thus these methods work exactly in the three cases corresponding to \( (6.8) \).

In view of the scaling limits that we expect, it is natural to work not with point sets like \( S_n \), but with continuous paths; instead of \( S_n \) we consider the interpolating path constructed as follows. For each \( n \in \mathbb{N} \) and all \( t \in [0,1] \), define

\[
X_n(t) := S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor) (S_{\lfloor nt \rfloor + 1} - S_{\lfloor nt \rfloor}) = S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor) Z_{\lfloor nt \rfloor + 1}.
\]

Note that \( X_n(0) = S_0 \) and \( X_n(1) = S_n \). Given \( n \), we are interested in the convex hull of the image in \( \mathbb{R}^2 \) of the interval \([0,1]\) under the continuous function \( X_n \). Our scaling limits will be of the same form.

### 3.2 Convex hulls of paths

In this section we study some basic properties of the map from a continuous path to its convex hull. Let \( f \in C([0,T],\mathbb{R}^d) \). For any \( t \in [0,T] \), \( f[0,t] \) is compact, and so Carathéodory’s theorem for convex hulls (see Corollary 3.1 of [29, p. 44]) shows that \( \text{hull}(f[0,t]) \) is compact. So \( \text{hull}(f[0,t]) \in \mathcal{K}_d \) is convex, bounded, and closed; in particular, it is a Borel set.

For reasons that we shall see, it mostly suffices to work with paths parametrized over the interval \([0,1]\). For \( f \in \mathcal{C}_d \), define

\[
H(f) := \text{hull} (f[0,1]) .
\]

First we prove continuity of the map \( f \mapsto H(f) \).

**Lemma 3.1.** For any \( f, g \in \mathcal{C}^0_d \), we have

\[
\rho_H(H(f), H(g)) \leq \rho_{\infty}(f, g).
\]

Hence the function \( H : (\mathcal{C}^0_d, \rho_{\infty}) \to (\mathcal{K}^0_d, \rho_H) \) is continuous.

**Proof.** Let \( f, g \in \mathcal{C}^0_d \). Then \( H(f) \) and \( H(g) \) are non-empty, as they both contain \( f(0) = g(0) = 0 \). Consider \( x \in H(f) \). Since the convex hull of a set is the set of
all convex combinations of points of the set (see Lemma 3.1 of [29, p. 42]), there exist a finite positive integer \( n \), weights \( \lambda_1, \ldots, \lambda_n \geq 0 \) with \( \sum_{i=1}^{n} \lambda_i = 1 \), and \( t_1, \ldots, t_n \in [0, 1] \) for which \( x = \sum_{i=1}^{n} \lambda_i f(t_i) \). Then, taking \( y = \sum_{i=1}^{n} \lambda_i g(t_i) \), we have that \( y \in H(g) \) and, by the triangle inequality,

\[
\rho(x, y) = \sum_{i=1}^{n} \lambda_i \rho(f(t_i), g(t_i)) \leq \rho_\infty(f, g).
\]

Thus, writing \( r = \rho_\infty(f, g) \), every \( x \in H(f) \) has \( x \in \pi_r(H(g)) \), so \( H(f) \subseteq \pi_r(H(g)) \). The symmetric argument gives \( H(g) \subseteq \pi_r(H(f)) \). Thus, by (2.1), we obtain (3.1).

Given \( f \in C_d \), let \( E(f) := \text{ext}(H(f)) \), the extreme points of the convex hull (see [29, p. 75]). The set \( E(f) \) is the smallest set (by inclusion) that generates \( H(f) \) as its convex hull, i.e., for any \( A \) for which \( \text{hull}(A) = H(f) \), we have \( E(f) \subseteq A \); see Theorem 5.5 of [29, p. 75]. In particular, \( E(f) \subseteq f[0, 1] \).

**Lemma 3.2.** Let \( f \in C_d \). Let \( q : \mathbb{R}^d \to \mathbb{R} \) be continuous and convex. Then \( q \) attains its supremum over \( H(f) \) at a point of \( f \), i.e.,

\[
\sup_{x \in H(f)} q(x) = \max_{t \in [0,1]} q(f(t)).
\]

**Proof.** Theorem 5.6 of [29, p. 76] shows that any continuous convex function on \( H(f) \) attains its maximum at a point of \( E(f) \). Hence, since \( E(f) \subseteq f[0,1] \),

\[
\sup_{x \in H(f)} q(x) = \sup_{x \in E(f)} q(x) \leq \sup_{x \in f[0,1]} q(x).
\]

On the other hand, \( f[0,1] \subseteq H(f) \), so \( \sup_{x \in [0,1]} q(x) \). Hence

\[
\sup_{x \in H(f)} q(x) = \sup_{x \in f[0,1]} q(x) = \sup_{t \in [0,1]} q(f(t)).
\]

Since \( q \circ f \) is the composition of two continuous functions, it is itself continuous, and so the supremum is attained in the compact set \([0,1] \).

For \( A \in \mathcal{K}_d^0 \), the **support function** of \( A \) is \( h_A : \mathbb{R}^d \to \mathbb{R}_+ \) defined by

\[
h_A(x) := \sup_{y \in A} (x \cdot y).
\]
For $A \in \mathcal{K}^0_2$, Cauchy’s formula (2.5) states
\[ L(A) = \int_{S_1} h_A(u) du = \int_0^{2\pi} h_A(e^{i\theta}) d\theta. \]

We end this section by showing that the map $t \mapsto \text{hull}(f[0,t])$ on $[0,T]$ is continuous if $f$ is continuous on $[0,T]$, so that the continuous trajectory $t \mapsto f(t)$ is accompanied by a continuous ‘trajectory’ of its convex hulls. This observation was made by El Bachir [19, pp. 16–17]; we take a different route based on the path space result Lemma 3.1. First we need a lemma.

**Lemma 3.3.** Let $T > 0$ and $f \in C([0,T]; \mathbb{R}^d)$. Then the map defined for $t \in [0,T]$ by $t \mapsto g_t$, where $g_t : [0,1] \to \mathbb{R}^d$ is given by $g_t(s) = f(ts)$, $s \in [0,1]$, is a continuous function from $([0,T], \rho)$ to $(\mathcal{C}_d, \rho_\infty)$.

**Proof.** First we fix $t \in [0,T]$ and show that $s \mapsto g_t(s)$ is continuous, so that $g_t \in \mathcal{C}_d$ as claimed. Since $f$ is continuous on the compact interval $[0,T]$, it is uniformly continuous, and admits a monotone modulus of continuity $\mu_f$. Hence
\[
\rho(g_t(s_1), g_t(s_2)) = \rho(f(ts_1), f(ts_2)) \leq \mu_f(\rho(ts_1, ts_2)) = \mu_f(t \rho(s_1, s_2)),
\]
which tends to 0 as $\rho(s_1, s_2) \to 0$. Hence $g_t \in \mathcal{C}_d$.

It remains to show that $t \mapsto g_t$ is continuous. But on $\mathcal{C}_d$,
\[
\rho_\infty(g_{t_1}, g_{t_2}) = \sup_{s \in [0,1]} \rho(f(t_1s), f(t_2s)) \leq \sup_{s \in [0,1]} \mu_f(\rho(t_1s, t_2s)) \leq \mu_f(\rho(t_1, t_2)),
\]
which tends to 0 as $\rho(t_1, t_2) \to 0$, again using the uniform continuity of $f$.

Here is the path continuity result for convex hulls of continuous paths; cf [19, p. 16–17].

**Corollary 3.4.** Let $T > 0$ and $f \in C^0([0,T]; \mathbb{R}^d)$ with $f(0) = 0$. Then the map defined for $t \in [0,T]$ by $t \mapsto \text{hull}(f[0,t])$ is a continuous function from $([0,T], \rho)$ to $(\mathcal{K}_d^0, \rho_H)$. 
Proof. By Lemma 3.3, \( t \mapsto g_t \) is continuous, where \( g_t(s) = f(ts), \ s \in [0,1] \). Note that, since \( f(0) = 0, \ g_t \in C_0^0 \). But the sets \( f[0,t] \) and \( g_t[0,1] \) coincide, so \( \text{hull}(f[0,t]) = H(g_t) \), and, by Lemma 3.1, \( g_t \mapsto H(g_t) \) is continuous. Thus \( t \mapsto H(g_t) \) is the composition of two continuous functions, hence itself a continuous function:

\[
[0,T] \to C_0^0 \to K_0^0 \to H(g_t)
\]

Recall definitions of the functionals for perimeter length \( \mathcal{L} \) and area \( \mathcal{A} \) in (2.3). We give the following inequalities in the metric spaces.

**Lemma 3.5.** Suppose that \( A,B \in K_2^0 \). Then

\[
\rho(\mathcal{L}(A), \mathcal{L}(B)) \leq 2\pi \rho_H(A,B); \tag{3.2}
\]

\[
\rho(\mathcal{A}(A), \mathcal{A}(B)) \leq \pi \rho_H(A,B)^2 + (\mathcal{L}(A) \vee \mathcal{L}(B)) \rho_H(A,B). \tag{3.3}
\]

Hence, the functions \( \mathcal{L} \) and \( \mathcal{A} \) are both continuous from \( (K_2^0, \rho_H) \) to \( (\mathbb{R}_+, \rho) \).

**Proof.** First consider \( \mathcal{L} \). By Cauchy’s formula,

\[
|\mathcal{L}(A) - \mathcal{L}(B)| = \left| \int_{S_1} (h_A(u) - h_B(u)) \, du \right| \\
\leq \int_{S_1} \sup_{u \in \mathbb{R}_1} |h_A(u) - h_B(u)| \, du = 2\pi \rho_H(A,B),
\]

by the triangle inequality and then (2.2). This gives (3.2).

Now consider \( \mathcal{A} \). Set \( r = \rho_H(A,B) \). Then, by (2.1), \( A \subseteq \pi_r(B) \). Hence

\[
\mathcal{A}(A) \leq \mathcal{A}(\pi_r(B)) \leq \mathcal{A}(B) + r \mathcal{L}(B) + \pi r^2,
\]

by (2.4). With the analogous argument starting from \( B \subseteq \pi_r(A) \), we get (3.3). \( \square \)

### 3.3 Brownian convex hulls as scaling limits

Now we return to considering the random walk \( S_n = \sum_{k=1}^n Z_k \) in \( \mathbb{R}^2 \). The two different scalings outlined in Section 3.1 for the cases \( \mu = 0 \) and \( \mu \neq 0 \), lead to different scaling limits for the random walk. Both are associated with Brownian motion.
In the case $\mu = 0$, the scaling limit is the usual planar Brownian motion, at least when $\Sigma = I$, the identity matrix. Let $b := (b(s))_{s \in [0,1]}$ denote standard Brownian motion in $\mathbb{R}^2$, started at $b(0) = 0$. For convenience we may assume $b \in C^0_2$ (we can work on a probability space for which continuity holds for all sample points, rather than merely almost all). For $t \in [0,1]$, let

$$h_t := \text{hull } b[0,t] \in K^0_2$$

(3.4)
de note the convex hull of the Brownian path up to time $t$. By Corollary 3.4, $t \mapsto h_t$ is continuous. Much is known about the properties of $h_t$: see e.g. [13, 19, 21, 36].

We also set

$$\ell_t := L(h_t), \quad a_t := A(h_t),$$

(3.5)
the perimeter length and area of the standard Brownian convex hull. By Lemma 3.5, the processes $t \mapsto \ell_t$ and $t \mapsto a_t$ also have continuous sample paths.

We also need to work with the case of general covariances $\Sigma$; to do so we introduce more notation and recall some facts about multivariate Gaussian random vectors. For definiteness, we view vectors as Cartesian column vectors when required. Since $\Sigma$ is positive semidefinite and symmetric, there is a (unique) positive semidefinite symmetric matrix square-root $\Sigma^{1/2}$ for which $\Sigma = (\Sigma^{1/2})^2$. The map $x \mapsto \Sigma^{1/2}x$ associated with $\Sigma^{1/2}$ is a linear transformation on $\mathbb{R}^2$ with Jacobian $\det \Sigma^{1/2} = \sqrt{\det \Sigma}$; hence $A(\Sigma^{1/2}A) = A(A)\sqrt{\det \Sigma}$ for any measurable $A \subseteq \mathbb{R}^2$.

If $W \sim \mathcal{N}(0, I)$, then by Lemma 2.23, $\Sigma^{1/2}W \sim \mathcal{N}(0, \Sigma)$, a bivariate normal distribution with mean 0 and covariance $\Sigma$; the notation permits $\Sigma = 0$, in which case $\mathcal{N}(0,0)$ stands for the degenerate normal distribution with point mass at 0. Similarly, given $b$ a standard Brownian motion on $\mathbb{R}^2$, the diffusion $\Sigma^{1/2}b$ is correlated planar Brownian motion with covariance matrix $\Sigma$. Recall that ‘$\Rightarrow$’ (see Section 2.8) indicates weak convergence.

**Theorem 3.6.** Suppose that $\mathbb{E}(\|Z_1\|^2) < \infty$ and $\mu = 0$. Then, as $n \to \infty$,

$$n^{-1/2} \text{hull } \{S_0, S_1, \ldots, S_n\} \Rightarrow \Sigma^{1/2}h_1,$$

in the sense of weak convergence on $(K^0_2, \rho_H)$.  

**Proof.** Donsker’s theorem (see Lemma 2.26) implies that $n^{-1/2}X_n \Rightarrow \Sigma^{1/2}b$ on $(C^0_2, \rho_\infty)$. Now, the point set $X_n[0,1]$ is the union of the line segments $\{S_k +
\[ \theta(S_{k+1} - S_k) : \theta \in [0, 1] \text{ over } k = 0, 1, \ldots, n - 1. \] Since the convex hull is preserved under affine transformations,

\[ H(n^{-1/2}X_n) = n^{-1/2}H(X_n) = n^{-1/2} \text{hull}\{S_0, S_1, \ldots, S_n\}. \]

By Lemma 3.1, \( H \) is continuous, and so the continuous mapping theorem (see Lemma 2.25) implies that

\[ n^{-1/2} \text{hull}\{S_0, S_1, \ldots, S_n\} \Rightarrow H(\Sigma^{1/2}b) \text{ on } (\mathbb{K}_0^2, \rho_H). \]

Finally, invariance of the convex hull under affine transformations shows

\[ H(\Sigma^{1/2}b) = \Sigma^{1/2}H(b) = \Sigma^{1/2}h_1. \]

Theorem 3.6 together with the continuous mapping theorem and Lemma 3.5 implies the following distributional limit results in the case \( \mu = 0 \). Recall that \( \xrightarrow{i.d.} \) (see Section 2.1) denotes convergence in distribution for \( \mathbb{R} \)-valued random variables.

**Corollary 3.7.** Suppose that \( \mathbb{E}(\|Z_1\|^2) < \infty \) and \( \mu = 0 \). Then, as \( n \to \infty \),

\[ n^{-1/2}L_n \xrightarrow{d} \mathcal{L}(\Sigma^{1/2}h_1), \quad \text{and} \quad n^{-1}A_n \xrightarrow{d} \mathcal{A}(\Sigma^{1/2}h_1) = a_1\sqrt{\det \Sigma}. \]

**Remark 3.1.** Recall that \( a_1 = \mathcal{A}(h_1) \) is the area of the standard 2-dimensional Brownian convex hull run for unit time. The distributional limits for \( n^{-1/2}L_n \) and \( n^{-1}A_n \) in Corollary 3.7 are supported on \( \mathbb{R}_+ \) and, as we will show in Proposition 5.16 and Proposition 6.14 below, are non-degenerate if \( \Sigma \) is positive definite; hence they are non-Gaussian excluding trivial cases.

In the case \( \mu \neq 0 \), the scaling limit can be viewed as a space-time trajectory of one-dimensional Brownian motion. Let \( w := (w(s))_{s \in [0, 1]} \) denote standard Brownian motion in \( \mathbb{R} \), started at \( w(0) = 0 \); similarly to above, we may take \( w \in C^0_1 \). Define \( \tilde{b} \in C^0_2 \) in Cartesian coordinates via

\[ \tilde{b}(s) = (s, w(s)), \quad \text{for } s \in [0, 1]; \]

thus \( \tilde{b}[0, 1] \) is the space-time diagram of one-dimensional Brownian motion run for unit time. For \( t \in [0, 1] \), let \( \tilde{h}_t := \text{hull}\tilde{b}[0, t] \in \mathbb{K}^0_2 \), and define \( \tilde{a}_t := \mathcal{A}(\tilde{h}_t) \). (Closely related to \( \tilde{h}_t \) is the greatest convex minorant of \( w \) over \( [0, t] \), which is of interest in its own right, see e.g. 47 and references therein.)
Figure 3.1: Simulated path of $n = 1000$ steps a random walk with drift $\mu = (\frac{1}{2}, \frac{1}{4})$ and its convex hull (top left) and (not to the same scale) the image under $\psi_n^\mu$ (bottom right).

Suppose $\mu \neq 0$ and $\sigma_\mu^2 \in (0, \infty)$. Given $\mu \in \mathbb{R}^2 \setminus \{0\}$, let $\hat{\mu}_\perp$ be the unit vector perpendicular to $\mu$ obtained by rotating $\hat{\mu}$ by $\pi/2$ anticlockwise. For $n \in \mathbb{N}$, define $\psi_n^\mu : \mathbb{R}^2 \to \mathbb{R}^2$ by the image of $x \in \mathbb{R}^2$ in Cartesian components:

$$\psi_n^\mu(x) = \left( \frac{x \cdot \hat{\mu}}{n \|\mu\|}, \frac{x \cdot \hat{\mu}_\perp}{\sqrt{n \sigma_\mu^2}} \right).$$

In words, $\psi_n^\mu$ rotates $\mathbb{R}^2$, mapping $\hat{\mu}$ to the unit vector in the horizontal direction, and then scales space with a horizontal shrinking factor $\|\mu\|/n$ and a vertical factor $\sqrt{n \sigma_\mu^2}$; see Figure 3.1 for an illustration.

**Theorem 3.8.** Suppose that $\mathbb{E}(\|Z_1\|^2) < \infty$, $\mu \neq 0$, and $\sigma_\mu^2 > 0$. Then, as $n \to \infty$,

$$\psi_n^\mu(\text{hull}\{S_0, S_1, \ldots, S_n\}) \Rightarrow \tilde{h}_1,$$

in the sense of weak convergence on $(\mathcal{K}_2^0, \rho_H)$.

**Proof.** Observe that $\hat{\mu} \cdot S_n$ is a random walk on $\mathbb{R}$ with one-step mean drift $\hat{\mu} \cdot \mu = \|\mu\| \in (0, \infty)$, while $\hat{\mu}_\perp \cdot S_n$ is a walk with mean drift $\hat{\mu}_\perp \cdot \mu = 0$ and increment variance

$$\mathbb{E}[(\hat{\mu}_\perp \cdot Z)^2] = \mathbb{E}[(\hat{\mu}_\perp \cdot (Z - \mu))^2]$$
Chapter 3

\[ = \mathbb{E}[\|Z - \mu\|^2] - \mathbb{E}[(\hat{\mu} \cdot (Z - \mu))^2] = \sigma^2 - \sigma^2_{\mu} \]
\[ = \sigma^2_{\mu}.\]

According to the strong law of large numbers, for any \( \varepsilon > 0 \) there exists \( N_\varepsilon \in \mathbb{N} \) a.s. such that \( |m^{-1}\hat{\mu} \cdot S_m - \|\mu\|| < \varepsilon \) for \( m \geq N_\varepsilon \). Now we have that
\[
\sup_{N_\varepsilon/n \leq t \leq 1} \left| \frac{\hat{\mu} \cdot S_{[nt]}}{n} - t\|\mu\| \right| \leq \sup_{N_\varepsilon/n \leq t \leq 1} \left( \frac{|nt|}{n} \right) \left| \frac{\hat{\mu} \cdot S_{[nt]}}{[nt]} - \|\mu\| \right|
\[ + \|\mu\| \sup_{0 \leq t \leq 1} \left| \frac{|nt|}{n} - t \right|
\]
\[ \leq \sup_{N_\varepsilon/n \leq t \leq 1} \left| \frac{\hat{\mu} \cdot S_{[nt]}}{[nt]} - \|\mu\| \right| + \frac{\|\mu\|}{n}
\]
\[ \leq \varepsilon + \frac{\|\mu\|}{n}.\]

On the other hand,
\[
\sup_{0 \leq t \leq N_\varepsilon/n} \left| \frac{n^{-1}\hat{\mu} \cdot S_{[nt]} - t\|\mu\|}{n} \right| \leq \frac{1}{n} \max \{\hat{\mu} \cdot S_0, \ldots, \hat{\mu} \cdot S_{N_\varepsilon} \} + \frac{N_\varepsilon\|\mu\|}{n} \to 0, \text{ a.s.}
\]

since \( N_\varepsilon < \infty \) a.s. Combining these last two displays and using the fact that \( \varepsilon > 0 \) was arbitrary, we see that
\[
\sup_{0 \leq t \leq 1} \left| n^{-1}\hat{\mu} \cdot S_{[nt]} - t\|\mu\| \right| \to 0, \text{ a.s. (the functional version of the strong law).}
\]

Similarly,
\[
\sup_{0 \leq t \leq 1} \left| n^{-1}\hat{\mu} \cdot S_{[nt]+1} - t\|\mu\| \right| \to 0, \text{ a.s. as well.}
\]

Since \( X_n(t) \) interpolates \( S_{[nt]} \) and \( S_{[nt]+1} \), it follows that
\[
\sup_{0 \leq t \leq 1} \left| n^{-1}\hat{\mu} \cdot X_n(t) - t\|\mu\| \right| \to 0, \text{ a.s.}
\]

In other words, \( (n\|\mu\|)^{-1}X_n \cdot \hat{\mu} \) converges a.s. to the identity function \( t \mapsto t \) on \([0,1]\).

For the other component, Donsker’s theorem (Lemma 2.26) gives \((n\sigma^2_{\mu},) \to X_n \cdot \hat{\mu}_\perp \Rightarrow w \) on \( (C^0_1, \rho_\infty) \). It follows that, as \( n \to \infty \), \( \psi_n^\mu(X_n) \Rightarrow \tilde{b} \), on \( (C_2^0, \rho_\infty) \). Hence by Lemma 3.1 and since \( \psi_n^\mu \) acts as an affine transformation on \( \mathbb{R}^2 \),
\[
\psi_n^\mu(H(X_n)) = H(\psi_n^\mu(X_n)) \Rightarrow H(\tilde{b}),
\]
on \( (K_2^0, \rho_H) \), and the result follows. \( \square \)
Theorem 3.8 with the continuous mapping theorem (Lemma 2.25), Lemma 3.5, and the fact that \( A(\psi_n^\mu(A)) = n^{-3/2}\|\mu\|^{-1}(\sigma_{\mu_\perp}^2)^{-1/2}A(A) \) for measurable \( A \subseteq \mathbb{R}^2 \), implies the following distributional limit for \( A_n \) in the case \( \mu \neq 0 \).

**Corollary 3.9.** Suppose that \( \mathbb{E}(\|Z_1\|^2) < \infty \), \( \mu \neq 0 \), and \( \sigma_{\mu_\perp}^2 > 0 \). Then
\[
n^{-3/2} A_n \xrightarrow{d} \|\mu\|(\sigma_{\mu_\perp}^2)^{1/2}\tilde{a}_1, \text{ as } n \to \infty.
\]

**Remarks 3.2.** (i) Only the \( \sigma_{\mu_\perp}^2 > 0 \) case is non-trivial, since \( \sigma_{\mu_\perp}^2 = 0 \) if and only if \( Z \) is parallel to \( \pm \mu \) a.s., in which case all the points \( S_0, \ldots, S_n \) are collinear and \( A_n = 0 \) a.s. for all \( n \).
(ii) The limit in Corollary 3.9 is non-negative and non-degenerate (see Proposition 6.14 below) and hence non-Gaussian.

The framework of this chapter shows that whenever a discrete-time process in \( \mathbb{R}^d \) converges weakly to a limit on the space of continuous paths, the corresponding convex hulls converge. It would be of interest to extend the framework to admit discontinuous limit processes, such as Lévy processes with jumps [36] that arise as scaling limits of random walks whose increments have infinite variance.
Chapter 4

Spitzer–Widom formula for the expected perimeter length and its consequences

4.1 Overview

Our contribution in this Chapter is giving a new proof of the Spitzer–Widom formula in Section 4.2 and giving the asymptotics for the expected perimeter length in Section 4.3 by using that formula. Firstly, we show how to deduce the Spitzer–Widom formula from the Cauchy formula.

The following theorem is Theorem 2 in [58].

**Theorem 4.1** (Spitzer–Widom formula). Suppose that \( E \|Z_1\| < \infty \). Then

\[
E L_n = 2 \sum_{k=1}^{n} \frac{1}{k} E \|S_k\|.
\]

The basis for our derivation of the Spitzer–Widom formula is an analogous result for one-dimensional random walk, stated in Lemma 4.3 below, which is itself a consequence of the combinatorial result given in Lemma 4.2. Lemma 4.2 was stated by Kac [34, pp. 502–503 and Theorem 4.2 on p. 508] and attributed to Hunt; the proof given is due to Dyson. Lemma 4.3 is variously attributed to Chung, Hunt, Dyson and Kac; it is also related to results of Sparre Andersen [1] and is a special case of what has become known as the Spitzer or Spitzer–Baxter...
identity [35, Ch. 9] for random walks, which is a more sophisticated result usually deduced from Wiener–Hopf Theory.

4.2 Derivation of Spitzer–Widom formula

Let $X_1, X_2, \ldots$ be i.i.d. random variables. Let $T_n = \sum_{i=1}^{n} X_i$ and $M_n = \max\{0, T_1, \ldots, T_n\}$. Let $\sigma : (1, 2, \ldots, n) \mapsto (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathbb{Z}_n^+$ be a permutation on $\{1, \ldots, n\}$. Then $(\pi_n; \circ)$ is a group consisting of $\sigma$ under the composition operation. For $\sigma \in \pi_n$, let $T_\sigma^n = \sum_{i=1}^{n} X_{\sigma_i}$ and $M_\sigma^n = \max\{0, T_1^\sigma, \ldots, T_n^\sigma\}$.

**Lemma 4.2.**

$$\sum_{\sigma \in \pi_n} M_\sigma^n = \sum_{\sigma \in \pi_n} X_{\sigma_1} \sum_{k=1}^{n} 1\{T_k^\sigma > 0\}.$$ 

**Proof.** Note that if $T_k^\sigma \leq 0$, then $M_k^\sigma - M_{k-1}^\sigma = 0$. If $T_k^\sigma > 0$, then

$$M_k^\sigma = \max(T_1^\sigma, T_2^\sigma, \ldots, T_k^\sigma) = X_{\sigma_1} + \max(0, X_{\sigma_2}, X_{\sigma_2} + X_{\sigma_3}, \ldots, \sum_{l=2}^{k} X_{\sigma_l}).$$

Combining these two cases, we get

$$M_k^\sigma - M_{k-1}^\sigma = 1\{T_k^\sigma > 0\} \left[ X_{\sigma_1} + \max\left(0, X_{\sigma_2}, X_{\sigma_2} + X_{\sigma_3}, \ldots, \sum_{l=2}^{k} X_{\sigma_l}\right) - \max\left(0, X_{\sigma_1} + X_{\sigma_2}, \ldots, \sum_{j=1}^{k-1} X_{\sigma_j}\right) \right].$$

Fix $k \in \{1, \ldots, n\}$. Let $G(\omega_{k+1}, \ldots, \omega_n)$ be the subset of $\pi_n$ consisting of permutations whose last $(n-k)$ indices are $\omega_{k+1}, \ldots, \omega_n$, where $1 \leq \omega_i \leq n$. Then $\pi_n$ is decomposed into $\frac{n!}{k!}$ disjoint subsets $G(\omega_{k+1}, \ldots, \omega_n)$ of size $k$.

Denote

$$f(\sigma_1, \ldots, \sigma_{k-1}, \sigma_k) := \max\left(0, X_{\sigma_1}, X_{\sigma_1} + X_{\sigma_2}, \ldots, \sum_{j=1}^{k-1} X_{\sigma_j}\right).$$

Then,

$$M_k^\sigma - M_{k-1}^\sigma = 1\{T_k^\sigma > 0\} \left[ X_{\sigma_1} + f(\sigma_2, \ldots, \sigma_k, \sigma_1) - f(\sigma_1, \ldots, \sigma_{k-1}, \sigma_k) \right].$$
Summing both sides of the equation over \( \{\sigma \in \pi_n\} \), since
\[
\sum_{\sigma \in \pi_n} = \sum_{1 \leq \sigma_{k+1}, \ldots, \sigma_n \leq n} \sum_{\sigma \in G(\sigma_{k+1}, \ldots, \sigma_n)} ,
\]
and
\[
\sum_{\sigma \in G(\sigma_{k+1}, \ldots, \sigma_n)} f(\sigma_2, \ldots, \sigma_k, \sigma_1) = \sum_{\sigma \in G(\sigma_{k+1}, \ldots, \sigma_n)} f(\sigma_1, \ldots, \sigma_{k-1}, \sigma_k),
\]
we get
\[
\sum_{\sigma \in \pi_n} (M^\sigma_n - M^\sigma_{k-1}) = \sum_{\sigma \in \pi_n} X_{\sigma_1} 1 \{T^\sigma_k > 0\}. \tag{4.1}
\]

The result is implied by summing both sides of the equation \(4.1\) from \( k = 1 \) to \( n \). Note that \( M^0_0 = \max(0) = 0 \).

Here we use the notation \( x^+ := x 1 \{x > 0\} \) and \( x^- := -x 1 \{x < 0\} \) for \( x \in \mathbb{R} \). So \( x = x^+ - x^- \) and \( |x| = x^+ + x^- \).

The following result on the expected maximum of 1-dimensional random walk is variously attributed to Chung, Hunt, Dyson and Kac. A combinatorial proof similar to the one given here can be found on page 301-302 of [14].

**Lemma 4.3.** Suppose that \( \mathbb{E}|X_k| < \infty \). Then,
\[
\mathbb{E} M_n = \sum_{k=1}^{n} \frac{\mathbb{E}(T^+_k)}{k}.
\]

**Proof.** By Lemma 4.2 we have
\[
\mathbb{E} M_n = \mathbb{E} M^\sigma_n = \frac{1}{n!} \sum_{\sigma \in \pi_n} \mathbb{E} M^\sigma_n
\]
\[
= \frac{1}{n!} \sum_{\sigma \in \pi_n} \mathbb{E} \left[ X_{\sigma_1} \sum_{k=1}^{n} 1 \{T^\sigma_k > 0\} \right]
\]
\[
= \mathbb{E} \left[ X_1 \sum_{k=1}^{n} 1 \{T_k > 0\} \right],
\]
since the \( X_i \) are i.i.d., \( \mathbb{E} (X_1 1 \{T_k > 0\}) = \mathbb{E} (X_i 1 \{T_k > 0\}) \) for any \( 1 \leq i \leq k \). Also, \( \mathbb{E} (X_1 1 \{T_k > 0\}) = k^{-1} \mathbb{E} (T_k 1 \{T_k > 0\}) \). Then,
\[
\mathbb{E} \left[ X_1 \sum_{k=1}^{n} 1 \{T_k > 0\} \right] = \sum_{k=1}^{n} \mathbb{E} \left[ X_1 1 \{T_k > 0\} \right]
\]
Chapter 4

\[= \sum_{k=1}^{n} \mathbb{E} \left[ \frac{T_k}{k} \mathbf{1}\{T_k > 0\} \right] \]

\[= \sum_{k=1}^{n} \frac{\mathbb{E}(T_k^+)}{k}. \]

**Remark 4.1.** Fluctuation theory for one-dimensional random walks concerns a series of important identities involving the distributions of \(M_n, T_n\), and other quantities associated with the random walk path. A cornerstone of the theory is the celebrated double generating-function identity of Spitzer which states that

\[\sum_{n=0}^{\infty} t^n \mathbb{E}[e^{iuM_n}] = \exp \left\{ \sum_{k=1}^{\infty} \frac{t^k}{k} \mathbb{E}[e^{iuT_k^+}] \right\}\]

for \(|t| < 1\). Lemma \[3.3\] is a corollary to Spitzer’s identity, obtained on differentiating with respect to \(u\) and setting \(u = 0\). The proof of Spitzer’s identity may be approached from an analytic perspective, using the Wiener–Hopf factorization (see e.g. Resnick \[51\] Ch. 7), or from a combinatorial one (see e.g. Karlin and Taylor \[37\] Ch. 17). These references discuss many other aspects of fluctuation theory, as do Chung \[14\] §§8.4 & 8.5, Feller \[23\], Asmussen \[2\] Ch. VIII], and Takács \[62\]. In particular, Chung \[14\] pp. 301–302] gives a direct proof of Lemma 4.3 closely related to the one presented here; essentially the same proof is in \[2\] p. 232.

**Proof of the Spitzer–Widom formula.**

Denote \(M_n(\theta) := \max_{0 \leq i \leq n}(S_i \cdot \mathbf{e}_\theta)\) and \(m_n(\theta) := \min_{0 \leq i \leq n}(S_i \cdot \mathbf{e}_\theta)\). Note that \(M_n(\theta) \geq 0\) and \(m_n(\theta) \leq 0\) since \(\mathbf{0} \in \mathcal{H}_n\).

Applying Fubini’s theorem (see Lemma \[2.21\] in Cauchy formula \[2.7\], we get

\[\mathbb{E} L_n = \int_0^\pi (\mathbb{E} M_n(\theta) - \mathbb{E} m_n(\theta)) \, d\theta. \]

Observe that \(S_n \cdot \mathbf{e}_\theta\) is a one-dimensional random walk on \(\mathbb{R}\). Take \(T_k = S_k \cdot \mathbf{e}_\theta\) in Lemma \[4.3\]. Then,

\[\mathbb{E} M_n(\theta) = \sum_{k=1}^{n} \frac{\mathbb{E} [(S_k \cdot \mathbf{e}_\theta)^+]}{k} \quad \text{and} \quad \mathbb{E} m_n(\theta) = -\sum_{k=1}^{n} \frac{\mathbb{E} [(-S_k \cdot \mathbf{e}_\theta)^+]}{k},\]

since \(m_n(\theta) = -\max_{0 \leq i \leq n}(-S_i \cdot \mathbf{e}_\theta)\). So, since \(x^- = (-x)^+\),

\[\mathbb{E} L_n = \int_0^\pi \sum_{k=1}^{n} \frac{1}{k} \mathbb{E} \left[ (S_k \cdot \mathbf{e}_\theta)^+ + (S_k \cdot \mathbf{e}_\theta)^- \right] \, d\theta \]
\[
\int_0^\pi \sum_{k=1}^n \frac{\mathbb{E} |S_k \cdot e_\theta|}{k} d\theta.
\]

Then, by Fubini’s theorem,
\[
\mathbb{E} L_n = \sum_{k=1}^n \frac{1}{k} \int_0^\pi \mathbb{E} |S_k \cdot e_\theta| d\theta
= \sum_{k=1}^n \frac{1}{k} \mathbb{E} \int_0^\pi |S_k \cdot e_\theta| d\theta
= 2 \sum_{k=1}^n \frac{\mathbb{E} \|S_k\|}{k}.
\]

### 4.3 Asymptotics for the expected perimeter length

To investigate the first-order properties of \(\mathbb{E} L_n\), we suggested by the Spitzer-Widom formula (1.1) that the first-order properties of \(\mathbb{E} \|S_n\|\) need to be studied first.

**Lemma 4.4.** If \(\mathbb{E} \|Z_1\| < \infty\), then \(n^{-1}\mathbb{E} \|S_n\| \to \|\mu\|\) as \(n \to \infty\).

**Proof.** The strong law of large numbers for \(S_n\) says \(\|S_n/n - \mathbb{E} Z_1\| \to 0\) a.s. as \(n \to \infty\). Then by the triangle inequality,
\[
\|S_n/n\| = \|S_n/n - \mathbb{E} Z_1 + \mathbb{E} Z_1\| \leq \|S_n/n - \mathbb{E} Z_1\| + \|\mathbb{E} Z_1\|
\]
and
\[
\|\mathbb{E} Z_1\| \leq \|\mathbb{E} Z_1 - S_n/n\| + \|S_n/n\|.
\]
So, \(\|S_n\|/n \to \|\mathbb{E} Z_1\|\) a.s. as \(n \to \infty\).

Similarly, let \(Y_n = \sum_{i=1}^n \|Z_i\|\), then \(Y_n/n \to \mathbb{E} \|Z_1\|\) a.s. as \(n \to \infty\). Also we simply have \(\mathbb{E} [Y_n/n] = \mathbb{E} \|Z_1\|\) and \(0 \leq \|S_n\|/n \leq Y_n/n\). Hence, the result is proved by Pratt’s Lemma (see Lemma 2.2). \(\square\)

The following asymptotic result for \(\mathbb{E} L_n\) was obtained as equation (2.16) by Snyder & Steele [57] under the stronger condition \(\mathbb{E} (\|Z_1\|^2) < \infty\); as Lemma 4.4 shows, a finite first moment is sufficient.
**Proposition 4.5.** Suppose $E \|Z_1\| < \infty$, then $n^{-1}E L_n \to 2\|\mu\|$, as $n \to \infty$.

**Proof.** The result is implied by the Spitzer–Widom formula (1.1) and Lemma 2.22 with $y_n = n^{-1}E \|S_n\|$, since $y_n \to \|\mu\|$ by Lemma 4.4. \qed

**Remarks 4.2.** (i) Proposition 4.5 says that if $\mu \neq 0$ then $E L_n$ is of order $n$. If $\mu = 0$, it says $E L_n = o(n)$. We will show later in Proposition 4.9 that under mild extra conditions in the $\mu = 0$ case, $n^{-1/2}E L_n$ has a limit.

(ii) Snyder and Steele [57, p. 1168] showed that if $E (\|Z_1\|^2) < \infty$ and $\mu \neq 0$, then in fact $n^{-1}L_n \to 2\|\mu\|$ a.s. as $n \to \infty$. We give a proof of this in Proposition 5.5 below.

For the zero drift case $\mu = 0$, we have the following.

**Lemma 4.6.** Suppose $E (\|Z_1\|^2) < \infty$ and $\mu = 0$, then $E \|S_n\| = O(n)$ and $E \|S_n\|^2 = O(n^{1/2})$.

**Proof.** Consider $\|S_n\|^2$,

\[
\|S_{n+1}\|^2 = \|S_n + Z_{n+1}\|^2 = \|S_n\|^2 + 2S_n \cdot Z_{n+1} + \|Z_{n+1}\|^2. \quad (4.2)
\]

So,

\[
E (\|S_{n+1}\|^2) - E (\|S_n\|^2) = E (\|Z_1\|^2),
\]

since $S_n$ and $Z_{n+1}$ are independent and $Z_{n+1}$ has mean 0, so $E (S_n \cdot Z_{n+1}) = E S_n \cdot E Z_{n+1} = 0$. Then sum from $n = 0$ to $m - 1$ to get

\[
E (\|S_m\|^2) - E (\|S_0\|^2) = mE (\|Z_1\|^2).
\]

Hence, $E (\|S_n\|^2) = O(n)$. The last result is given by Jensen’s inequality, $E \|S_n\| \leq (E [\|S_n\|^2])^{1/2}$. \qed

**Remark 4.3.** Lemma 4.6 only gives the upper bound for the order of $E \|S_n\|$. Under the mild assumption $P (\|Z_1\| = 0) < 1$, $n^{-1/2}E \|S_n\|$ in fact has a positive limit, as we will see in the proof of Proposition 4.9 below. This extra condition is of course necessary for the positive limit, since if $Z_1 \equiv 0$ then $E \|S_n\| \equiv 0$.

**Proposition 4.7.** Suppose $E (\|Z_1\|^2) < \infty$ and $\mu = 0$, then $E L_n = O(n^{1/2})$. 

Proof. By Lemma 4.6 and Spitzer–Widom formula (1.1), for some constant C,

$$\mathbb{E} L_n \leq 2 \sum_{i=1}^{n} \frac{C\sqrt{i}}{i} = 2C \sum_{i=1}^{n} i^{-1/2} = O(n^{1/2}).$$

Lemma 4.8. Let $p > 1$. Suppose that $\mathbb{E} [\|Z_1\|^p] < \infty$.

(i) For any $e \in \mathbb{S}_1$ such that $e \cdot \mu = 0$, $\mathbb{E} [\max_{0 \leq m \leq n} |S_m \cdot e|^p] = O(n^{1/p})$.

(ii) Moreover, if $\mu = 0$, then $\mathbb{E} [\max_{0 \leq m \leq n} \|S_m\|^p] = O(n^{1/p})$.

(iii) On the other hand, if $\mu \neq 0$, then $\mathbb{E} [\max_{0 \leq m \leq n} |S_m \cdot \hat{\mu}|^p] = O(n^p)$.

Proof. Given that $\mu \cdot e = 0$, $S_n \cdot e$ is a martingale, and hence, by convexity, $|S_n \cdot e|$ is a non-negative submartingale. Then, for $p > 1$,

$$\mathbb{E} \left[ \max_{0 \leq m \leq n} |S_m \cdot e|^p \right] \leq \left( \frac{p}{p-1} \right)^p \mathbb{E} [\|S_n \cdot e\|^p] = O(n^{1/p}),$$

where the first inequality is Doob’s $L^p$ inequality (see Lemma 2.17) and the second is the Marcinkiewicz–Zygmund inequality (see Lemma 2.19). This gives part (i).

Part (ii) follows from part (i): take $\{e_1, e_2\}$ an orthonormal basis of $\mathbb{R}^2$ and apply (i) with each basis vector. Then by the triangle inequality

$$\max_{0 \leq m \leq n} \|S_m\| \leq \max_{0 \leq m \leq n} |S_m \cdot e_1| + \max_{0 \leq m \leq n} |S_m \cdot e_2|$$

together with Minkowski’s inequality (see Lemma 2.16), we have

$$\mathbb{E} \left[ \max_{0 \leq m \leq n} \|S_m\|^p \right] \leq \mathbb{E} \left[ \left( \max_{0 \leq m \leq n} |S_m \cdot e_1| + \max_{0 \leq m \leq n} |S_m \cdot e_2| \right)^p \right]$$

$$= \left\| \max_{0 \leq m \leq n} |S_m \cdot e_1| + \max_{0 \leq m \leq n} |S_m \cdot e_2| \right\|^p$$

$$\leq \left( \left\| \max_{0 \leq m \leq n} |S_m \cdot e_1| \right\|^p + \left\| \max_{0 \leq m \leq n} |S_m \cdot e_2| \right\|^p \right)^p$$

$$= O(n^{1/p}).$$

Part (iii) follows from the fact that

$$\max_{0 \leq m \leq n} |S_m \cdot \hat{\mu}| \leq \sum_{k=1}^{n} |Z_k \cdot \hat{\mu}| \leq \sum_{k=1}^{n} \|Z_k\|$$
and an application of Rosenthal’s inequality (see Lemma 2.20) to the latter sum gives

\[
\mathbb{E} \left[ \max_{0 \leq m \leq n} \| S_m \cdot \hat{\mu} \|^p \right] \leq \mathbb{E} \left[ \left( \sum_{k=1}^{n} \| Z_k \| \right)^p \right] \\
\leq \max \left\{ 2^p \sum_{k=1}^{n} \mathbb{E} \| Z_k \|^p, 2^p \left( \sum_{k=1}^{n} \mathbb{E} \| Z_k \| \right)^p \right\} \\
\leq \max \{ O(n), O(n^p) \} \\
\leq O(n^p).
\]

Proposition 4.7 gives the order of \( \mathbb{E} L_n \). Now we can have the exact limit by the following result, the statement of which is similar to an example on p. 508 of [58].

**Proposition 4.9.** Suppose \( \mathbb{E} (\| Z_1 \|^2) < \infty \) and \( \mu = 0 \). Then, for \( Y \sim \mathcal{N}(0, \Sigma) \),

\[
\lim_{n \to \infty} n^{-1/2} \mathbb{E} L_n = \mathbb{E} \mathcal{L}(\Sigma^{1/2} h_1) = 4 \mathbb{E} \| Y \|.
\]

**Proof.** The finite point-set case of Cauchy’s formula gives

\[
L_n = \int_{S_1} \max_{0 \leq k \leq n} (S_k \cdot e) \, de \leq 2\pi \max_{0 \leq k \leq n} \| S_k \|. \quad (4.3)
\]

Then by Lemma 4.8(ii) we have \( \sup_n \mathbb{E} [(n^{-1/2} L_n)^2] < \infty \). Hence \( n^{-1/2} L_n \) is uniformly integrable, so that Theorem 3.6 yields \( \lim_{n \to \infty} n^{-1/2} \mathbb{E} L_n = \mathbb{E} \mathcal{L}(\Sigma^{1/2} h_1) \).

It remains to show that \( \lim_{n \to \infty} n^{-1/2} \mathbb{E} L_n = 4 \mathbb{E} \| Y \| \). One can use Cauchy’s formula to compute \( \mathbb{E} \mathcal{L}(\Sigma^{1/2} h_1) \); instead we give a direct random walk argument, following [58]. The central limit theorem for \( S_n \) implies that \( n^{-1/2} \| S_n \| \to \| Y \| \) in distribution. Under the given conditions, \( \mathbb{E} [\| S_{n+1} \|^2] = \mathbb{E} [\| S_n \|^2] + \mathbb{E} [\| Z_{n+1} \|^2] \), so that \( \mathbb{E} [\| S_n \|^2] = O(n) \). It follows that \( n^{-1/2} \| S_n \| \) is uniformly integrable, and hence

\[
\lim_{n \to \infty} n^{-1/2} \mathbb{E} \| S_n \| = \mathbb{E} \| Y \|.
\]

So for any \( \varepsilon > 0 \), there is some \( n_0 \in \mathbb{N} \) such that \( |k^{-1/2} \mathbb{E} \| S_k \| - \mathbb{E} \| Y \| | < \varepsilon \) for all \( k \geq n_0 \). Then by the S–W formula (1.1), we have

\[
\left| \frac{\mathbb{E} L_n}{\sqrt{n}} - 2 \mathbb{E} \| Y \| \frac{1}{\sqrt{n}} \sum_{k=1}^{n} k^{-1/2} \right|.
\]
\[
\frac{2}{\sqrt{n}} \sum_{k=1}^{n} \left( \frac{\mathbb{E} \|S_k\|}{k} - \mathbb{E} \|Y\|^{k^{-1/2}} \right) \leq \frac{2}{\sqrt{n}} \sum_{k=1}^{n} \left| \frac{\mathbb{E} \|S_k\|}{\sqrt{k}} - \mathbb{E} \|Y\| \right|^{k^{-1/2}} \\
\leq D \frac{2}{\sqrt{n}} + \frac{2}{\sqrt{n}} \sum_{k=n_0+1}^{n} \left| \frac{\mathbb{E} \|S_k\|}{\sqrt{k}} - \mathbb{E} \|Y\| \right|^{k^{-1/2}} \\
\leq D \frac{2}{\sqrt{n}} + \frac{2\varepsilon}{\sqrt{n}} \sum_{k=n_0+1}^{n} k^{-1/2},
\]

for some constant \(D\) and the \(n_0\) mentioned above.

Also notice the fact that \(\lim_{n \to \infty} n^{-1/2} \sum_{k=1}^{n} k^{-1/2} = 2\). This can be proved by the monotonicity,

\[
2 \left( (n + 1)^{1/2} - 1 \right) = \int_{1}^{n+1} x^{-1/2} \, dx \leq \sum_{k=1}^{n} k^{-1/2} \leq \int_{0}^{n} x^{-1/2} \, dx = 2n^{1/2}.
\]

Taking \(n \to \infty\) in the displayed inequality gives

\[
\limsup_{n \to \infty} \left| \frac{\mathbb{E} L_n}{\sqrt{n}} - 2\mathbb{E} \|Y\| \frac{1}{\sqrt{n}} \sum_{k=1}^{n} k^{-1/2} \right| \leq 4\varepsilon.
\]

Since \(\varepsilon > 0\) was arbitrary, it follows that

\[
\lim_{n \to \infty} \left| \frac{\mathbb{E} L_n}{\sqrt{n}} - 2\mathbb{E} \|Y\| \frac{1}{\sqrt{n}} \sum_{k=1}^{n} k^{-1/2} \right| = 0.
\]

Therefore,

\[
\lim_{n \to \infty} \frac{\mathbb{E} L_n}{\sqrt{n}} = \lim_{n \to \infty} 2\mathbb{E} \|Y\| \frac{1}{\sqrt{n}} \sum_{k=1}^{n} k^{-1/2} = 4\mathbb{E} \|Y\|. \quad \Box
\]

Cauchy’s formula applied to the line segment from 0 to \(Y\) with Fubini’s theorem implies \(2\mathbb{E} \|Y\| = \int_{S_1} \mathbb{E} [(Y \cdot e)^+] \, de\). Here \(Y \cdot e = e^\top Y\) is univariate normal with mean 0 and variance \(e^\top \Sigma e = \|\Sigma^{1/2} e\|^2\), so that \(\mathbb{E} [(Y \cdot e)^+]\) is \(\|\Sigma^{1/2} e\|\) times one half of the mean of the square-root of a \(\chi^2\) random variable. Hence

\[
\mathbb{E} \|Y\| = (8\pi)^{-1/2} \int_{S_1} \|\Sigma^{1/2} e\| \, de,
\]
which in general may be expressed via a complete elliptic integral of the second kind in terms of the ratio of the eigenvalues of $\Sigma$. In the particular case $\Sigma = I$, $E \|Y\| = \sqrt{\pi/2}$ so then Proposition 4.9 implies that

$$\lim_{n \to \infty} n^{-1/2} E L_n = \sqrt{8\pi},$$

matching the formula $E \ell_1 = \sqrt{8\pi}$ of Letac and Takács [39, 61] (see Lemma 4.10 below). We also note the bounds

$$\pi^{-1/2} \sqrt{\text{tr} \Sigma} \leq E \|Y\| \leq \sqrt{\text{tr} \Sigma}; \quad (4.4)$$

the upper bound here is from Jensen’s inequality and the fact that $E [\|Y\|^2] = \text{tr} \Sigma$. The lower bound in (4.4) follows from the inequality

$$E \|Y\| \geq \sup_{e \in S_1} E |Y \cdot e| = \sqrt{2/\pi} \sup_{e \in S_1} (\text{Var}[Y \cdot e])^{1/2}$$

together with the fact that

$$\sup_{e \in S_1} \text{Var}[Y \cdot e] = \sup_{e \in S_1} \|\Sigma^{1/2} e\|^2 = \|\Sigma^{1/2}\|_{\text{op}}^2 = \|\Sigma\|_{\text{op}} = \lambda_{\Sigma} \geq \frac{1}{2} \text{tr} \Sigma,$$

where $\|\cdot\|_{\text{op}}$ is the matrix operator norm and $\lambda_{\Sigma}$ is the largest eigenvalue of $\Sigma$; in statistical terminology, $\lambda_{\Sigma}$ is the variance of the first principal component associated with $Y$.

We give a proof of the formula of Letac and Takács [39, 61].

**Lemma 4.10.** Let $\ell_1 = L(h_1)$ (see equation (3.5)) be the perimeter length of convex hull of a standard Brownian motion on $[0, 1]$ in $\mathbb{R}^2$. Then, $E \ell_1 = \sqrt{8\pi}$.

**Proof.** Applying Fubini’s theorem (Lemma 2.21) in Cauchy formula (2.5) for $\ell_1$,

$$\ell_1 = \int_0^{2\pi} \sup_{t \in [0, 1]} (b(t) \cdot e_\theta) d\theta,$$

we have

$$E \ell_1 = \int_0^{2\pi} \mathbb{E} \sup_{t \in [0, 1]} (b(t) \cdot e_\theta) d\theta$$

$$= 2\pi \mathbb{E} \sup_{t \in [0, 1]} (b(t) \cdot e_\theta),$$

where $b(t) \cdot e_\theta$ is a 1 dimensional Brownian motion,

$$= 2\pi \mathbb{E} \sup_{t \in [0, 1]} w(t).$$
Here $w(t)$ is defined as a standard 1-dimensional Brownian motion, which is the same as in Corollary 2.11. Then we have

\[
\mathbb{E} \sup_{t \in [0,1]} w(t) = \int_0^\infty \mathbb{P} \left( \sup_{t \in [0,1]} w(t) > r \right) dr
\]

\[
= 2 \int_0^\infty \mathbb{P} (w(1) > r) dr, \text{ by Reflection principle (Corollary 2.11)},
\]

\[
= 2 \int_0^\infty \frac{dr}{\sqrt{2\pi}} \int_r^\infty e^{-y^2/2} dy
\]

\[
= \sqrt{\frac{2}{\pi}} \int_0^\infty dy \int_0^y e^{-y^2/2} dr, \text{ by changing orders of integrals},
\]

\[
= \sqrt{\frac{2}{\pi}}
\]

Hence, the result follows. \qed
Chapter 5

Asymptotics for perimeter length of the convex hull

5.1 Overview

To start this chapter we discuss some simulations. We considered a specific form of random walk with increments \(Z_i - \mathbb{E}[Z_i] = (\cos \Theta_i, \sin \Theta_i)\), where \(\Theta_i\) was uniformly distributed on \([0, 2\pi)\), corresponding to a uniform distribution on a unit circle centred at \(\mathbb{E}[Z_i] = \mu\). We took one example with \(\mu = 0\), and two examples with \(\mu \neq 0\) of different magnitudes.

For the expected perimeter length, the simulations (see Figure 5.1) are consistent with the Spitzer–Widdom–Baxter result (see the argument below (1.1)), Proposition 4.9 and Proposition 5.5. In the case of \(\mu = 0\), the result in Proposition 4.9 take the form: \(\lim_{n \to \infty} n^{-1/2} \mathbb{E} L_n = 4 \mathbb{E} \|Y\| = 4\). In the case of \(\mu \neq 0\), the result in Proposition 5.5 take the form: \(n^{-1} L_n \xrightarrow{a.s.} 2\|\mu\| = 0.4 \text{ or } 0.72\).

For the variance of perimeter length with drift, the result in Theorem 5.13 take the form: \(\lim_{n \to \infty} \mathbb{V} \text{ar}[L_n] = 4 \mathbb{E} [\cos^2 \Theta_1] = 2\) and in Theorem 5.14 \((2n)^{-1/2}(L_n - \mathbb{E}[L_n])\) converges in distribution to a standard normal distribution. The corresponding pictures in Figures 5.2 and 5.3 show an agreement between the simulations and the theory. In the zero drift case, the simulations (the leftmost plot in Figure 5.2) suggest that \(\lim_{n \to \infty} n^{-1} \mathbb{V} \text{ar}[L_n]\) exists but Figure 5.3 does not appear to be consistent with a normal distribution as a limiting distribution.
Figure 5.1: Plots of \( y = \mathbb{E} [L_n] \) estimates against \( x = (\text{left to right}) n^{1/2}, n, n \) for about 25 values of \( n \) in the range \( 10^2 \) to \( 2.5 \times 10^5 \) for 3 examples with \( \|\mu\| = (\text{left to right}) 0, 0.2, 0.36 \). Each point is estimated from \( 10^3 \) repeated simulations. Also plotted are straight lines \( y = 3.532x \) (leftmost plot), \( y = 0.40x \) (middle plot) and \( y = 0.721x \) (rightmost plot).

We will show in Proposition 5.15 that

\[
\text{if } \mu = 0 : \lim_{n \to \infty} n^{-1} \text{Var} L_n = u_0(\Sigma),
\]

where \( u_0(\cdot) \) is finite and positive provided \( \sigma^2 < \infty \). For the constant \( u_0(I) \) (\( I \) being the identity matrix), Table 5.1 gives numerical evaluation of rigorous bound that we prove in Proposition 5.16 below, plus estimate from simulations. See also Section 7.2 for an explicit integral expression for \( u_0(I) \).
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Figure 5.3: Simulated histogram estimates for the distribution of \( \frac{L_n - E[L_n]}{\sqrt{\text{Var}[L_n]}} \) with \( n = 5 \times 10^3 \) in the three examples described in Figure 5.1. Each histogram is compiled from \( 10^3 \) samples.

| \( u_0(I) \) | lower bound | simulation estimate | upper bound |
|--------------|-------------|---------------------|-------------|
|              | 2.65 \times 10^{-3} | 1.08                 | 9.87        |

Table 5.1: The simulation estimate is based on \( 10^5 \) instances of a walk of length \( n = 10^5 \). The final decimal digit in the numerical upper (lower) bounds has been rounded up (down).

### 5.2 Upper bound for the variance

Assuming that \( E[\|Z_1\|^2] < \infty \), Snyder and Steele \([57]\) obtained an upper bound for \( \text{Var}[L_n] \) using Cauchy’s formula together with a version of the Efron–Stein inequality. Snyder and Steele’s result (Theorem 2.3 of \([57]\)) can be expressed as

\[
n^{-1} \text{Var}[L_n] \leq \frac{\pi^2}{2} \left( E[\|Z_1\|^2] - \|E[Z_1]\|^2 \right), \quad (n \in \mathbb{N} := \{1, 2, \ldots\}). \tag{5.1}
\]

As far as we are aware, there are no lower bounds for \( \text{Var}[L_n] \) in the literature. According to the discussion in \([57, \S5]\), Snyder and Steele had “no compelling reason to expect that \( O(n) \) is the correct order of magnitude” in their upper bound for \( \text{Var}[L_n] \), and they speculated that perhaps \( \text{Var}[L_n] = o(n) \) (maybe with a distinction between the cases of zero and non-zero drift). Our first main result settles this question under minimal conditions, confirming that (5.1) is indeed of the correct order, apart from in certain degenerate cases, while demonstrating that the constant on the right-hand side of (5.1) is not, in general, sharp.

The first step in looking for the variance upper bound is a martingale difference argument, based on resampling members of the sequence \( Z_1, \ldots, Z_n \), to get an
expression for $\mathbb{V} \text{ar}[L_n]$ amenable to analysis: see Section 2.2. Let $\mathcal{F}_0$ denote the trivial $\sigma$-algebra, and for $n \in \mathbb{N}$ set $\mathcal{F}_n := \sigma(Z_1, \ldots, Z_n)$, the $\sigma$-algebra generated by the first $n$ steps of the random walk. Then $S_n$ is $\mathcal{F}_n$-measurable, and for $n \in \mathbb{N}$ we can write $L_n = \Lambda_n(Z_1, \ldots, Z_n)$ for $\Lambda_n : \mathbb{R}^{2n} \to [0, \infty)$ a measurable function.

Let $Z'_1, Z'_2, \ldots$ be an independent copy of the sequence $Z_1, Z_2, \ldots$. Fix $n \in \mathbb{N}$. For $i \in \{1, \ldots, n\}$, we ‘resample’ the $i$th increment, replacing $Z_i$ with $Z'_i$, as follows. Set

$$S^{(i)}_j := \begin{cases} S_j & \text{if } j < i \\ S_j - Z_i + Z'_i & \text{if } j \geq i; \end{cases}$$

then $(S^{(i)}_j; 0 \leq j \leq n)$ is a modification of the random walk $(S_j; 0 \leq j \leq n)$ that keeps all the components apart from the $i$th step which is independently resampled. We let $L^{(i)}_n$ denote the perimeter length of the corresponding convex hull for this modified walk, namely $\text{hull}(S^{(i)}_0, \ldots, S^{(i)}_n)$, i.e.,

$$L^{(i)}_n := \Lambda_n(Z_1, \ldots, Z_{i-1}, Z'_i, Z_{i+1}, \ldots, Z_n).$$

For $i \in \{1, \ldots, n\}$, define

$$D_{n,i} := \mathbb{E}[L_n - L^{(i)}_n \mid \mathcal{F}_i];$$

in other words, $-D_{n,i}$ is the expected change in the perimeter length of the convex hull, given $\mathcal{F}_i$, on replacing $Z_i$ by $Z'_i$. The point of this construction is the following result.

**Lemma 5.1.** Let $n \in \mathbb{N}$. Then (i) $L_n = \mathbb{E}[L_n] = \sum_{i=1}^{n} D_{n,i}$; and (ii) $\mathbb{V} \text{ar}[L_n] = \sum_{i=1}^{n} \mathbb{E}[D_{n,i}^2]$, whenever the latter sum is finite.

**Proof.** Take $W_n = L_n$ in Lemma 2.8. Then the results follow. \hfill \Box

**Remark 5.1.** Lemma 5.1 with the conditional Jensen’s inequality gives the bound

$$\mathbb{V} \text{ar}[L_n] \leq \sum_{i=1}^{n} \mathbb{E} \left[ (L^{(i)}_n - L_n)^2 \right],$$

which is a factor of 2 larger than the upper bound obtained from the Efron–Stein inequality: $\mathbb{V} \text{ar}[L_n] \leq 2^{-1} \sum_{i=1}^{n} \mathbb{E} \left[ (L^{(i)}_n - L_n)^2 \right]$ (see equation (2.3) in [57]).
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Let \( e_\theta = (\cos \theta, \sin \theta) \) be the unit vector in direction \( \theta \in (-\pi, \pi] \). For \( \theta \in [0, \pi] \), define

\[
M_n(\theta) := \max_{0 \leq j \leq n} (S_j \cdot e_\theta), \quad \text{and} \quad m_n(\theta) := \min_{0 \leq j \leq n} (S_j \cdot e_\theta).
\]

Note that since \( S_0 = 0 \), we have \( M_n(\theta) \geq 0 \) and \( m_n(\theta) \leq 0 \), a.s. In the present setting (see equation (2.7)), Cauchy’s formula for convex sets yields

\[
L_n = \int_0^\pi (M_n(\theta) - m_n(\theta)) \, d\theta = \int_0^\pi R_n(\theta) \, d\theta,
\]

where \( R_n(\theta) := M_n(\theta) - m_n(\theta) \geq 0 \) is the parametrized range function. Similarly, when the \( i \)th increment is resampled,

\[
L_n^{(i)} = \int_0^\pi (M_n^{(i)}(\theta) - m_n^{(i)}(\theta)) \, d\theta = \int_0^\pi R_n^{(i)}(\theta) \, d\theta,
\]

where \( R_n^{(i)}(\theta) = M_n^{(i)}(\theta) - m_n^{(i)}(\theta) \), defining

\[
M_n^{(i)}(\theta) := \max_{0 \leq j \leq n} (S_j^{(i)} \cdot e_\theta), \quad \text{and} \quad m_n^{(i)}(\theta) := \min_{0 \leq j \leq n} (S_j^{(i)} \cdot e_\theta).
\]

Thus to study \( D_{n,i} = \mathbb{E} [L_n - L_n^{(i)} \mid \mathcal{F}_i] \) we will consider

\[
L_n - L_n^{(i)} = \int_0^\pi (R_n(\theta) - R_n^{(i)}(\theta)) \, d\theta = \int_0^\pi \Delta_n^{(i)}(\theta) \, d\theta, \tag{5.4}
\]

where \( \Delta_n^{(i)}(\theta) := R_n(\theta) - R_n^{(i)}(\theta) \). For \( \theta \in [0, \pi] \), let

\[
\mathcal{J}_n(\theta) := \arg \min_{0 \leq j \leq n} (S_j \cdot e_\theta), \quad \text{and} \quad \mathcal{J}_n(\theta) := \arg \max_{0 \leq j \leq n} (S_j \cdot e_\theta),
\]

so \( m_n(\theta) = S_{\mathcal{J}_n(\theta)} \cdot e_\theta \) and \( M_n(\theta) = S_{\mathcal{J}_n(\theta)} \cdot e_\theta \). Similarly, recalling (5.2), define

\[
\mathcal{J}_n^{(i)}(\theta) := \arg \min_{0 \leq j \leq n} (S_j^{(i)} \cdot e_\theta), \quad \text{and} \quad \mathcal{J}_n^{(i)}(\theta) := \arg \max_{0 \leq j \leq n} (S_j^{(i)} \cdot e_\theta).
\]

(Apply the following conventions in the event of ties: \( \arg \min \) takes the maximum argument among tied values, and \( \arg \max \) the minimum.)

We will use the following simple bound repeatedly in the arguments that follow. This upper bound for \( |\Delta_n^{(i)}(\theta)| \) is also given in Lemma 2.1 of [57]. But we have a different way to prove here.

**Lemma 5.2.** Almost surely, for any \( \theta \in [0, \pi] \) and any \( i \in \{1, 2, \ldots, n\} \),

\[
|\Delta_n^{(i)}(\theta)| \leq |(Z_i - Z'_i) \cdot e_\theta| \leq \|Z_i\| + \|Z'_i\|. \tag{5.5}
\]
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Proof. Consider the effect on \( S_k \cdot e_\theta \) when \( Z_i \) is replaced by \( Z'_i \). If \( i > k \), then
\[
S_k \cdot e_\theta = S_k^{(i)} \cdot e_\theta.
\]
If \( i \leq k \), then
\[
S_k \cdot e_\theta = S_k^{(i)} \cdot e_\theta + (Z_i - Z'_i) \cdot e_\theta.
\]
Hence, for all \( i \),
\[
S_k \cdot e_\theta \leq S_k^{(i)} \cdot e_\theta + ((Z_i - Z'_i) \cdot e_\theta \lor 0).
\]
Therefore,
\[
\max_{1 \leq k \leq n} S_k \cdot e_\theta \leq \max_{1 \leq k \leq n} S_k^{(i)} \cdot e_\theta + ((Z_i - Z'_i) \cdot e_\theta \lor 0).
\]
Similarly, we have
\[
\min_{1 \leq k \leq n} S_k \cdot e_\theta \geq \min_{1 \leq k \leq n} S_k^{(i)} \cdot e_\theta + ((Z_i - Z'_i) \cdot e_\theta \land 0).
\]
Combining these two inequalities with maximum and minimum, we get
\[
R_n(\theta) - R_n^{(i)}(\theta) \leq ((Z_i - Z'_i) \cdot e_\theta \lor 0) - ((Z_i - Z'_i) \cdot e_\theta \land 0)
\]
\[
= |(Z_i - Z'_i) \cdot e_\theta|.
\]
Also similarly, we can get
\[
R_n^{(i)}(\theta) - R_n(\theta) \leq |(Z'_i - Z_i) \cdot e_\theta|.
\]
Thus, the result follows from the triangle inequality.

The following is Lemma 2.2 in [57].

Lemma 5.3. For all \( 1 \leq i \leq n \),
\[
\mathbb{E} \left[ \left( \int_0^\pi |(Z_i - Z'_i) \cdot e_\theta| \, d\theta \right)^2 \right] \leq \pi^2 \left( \mathbb{E} \left| Z_1 \right|^2 - \|\mu\|^2 \right) = \pi^2 \sigma^2.
\]

Proof. By Cauchy-Schwarz Inequality, we have
\[
\mathbb{E} \left[ \left( \int_0^\pi |(Z_i - Z'_i) \cdot e_\theta| \, d\theta \right)^2 \right] \leq \pi \mathbb{E} \left( \int_0^\pi |(Z_i - Z'_i) \cdot e_\theta|^2 \, d\theta \right).
\]
Then, since \( Z_i, Z'_i \) are identically and independently distributed,
\[
\mathbb{E} \left[ |Z_i \cdot e_\theta - Z'_i \cdot e_\theta|^2 \right] = \mathbb{E} \left[ (Z_i \cdot e_\theta)^2 \right] + \mathbb{E} \left[ (Z'_i \cdot e_\theta)^2 \right] - 2 \mathbb{E} \left[ (Z_i \cdot e_\theta)(Z'_i \cdot e_\theta) \right]
\]
\[
= 2 \mathbb{V} \text{ar}[Z_1 \cdot e_\theta]
\]
\[
= 2 \left( \sigma^2 \cos^2 \theta + \sigma^2 \mu_\perp \cos^2 \theta + 2 \cos \theta \sin \theta \rho_{\mu \perp} \sigma_\mu \sigma_{\mu_\perp} \right),
\]
where \( \rho_{\mu \perp} \) is the covariance of \((Z_1 - \mu) \cdot \hat{\mu} \) and \((Z_1 - \mu) \cdot \hat{\mu}_\perp \). So,
\[
\mathbb{E} \int_0^\pi |(Z_i - Z'_i) \cdot e_\theta|^2 \, d\theta = 2 \left( \sigma^2 \mu_\perp \int_0^\pi \cos^2 \theta \, d\theta + \sigma^2 \mu_\perp \int_0^\pi \sin^2 \theta \, d\theta \right)
\]
This proves the lemma.

The next result is a version of Theorem 2.3 in [57]. But they get better right-hand side by using Efron–Stein inequality

**Proposition 5.4.** Suppose $\mathbb{E}(\|Z\|^2) < \infty$. Then

$$\text{Var}(L_n) \leq \frac{\pi^2 \sigma^2}{2n}. \quad (5.6)$$

**Proof.** By Lemma 2.9, equation (5.4) and (5.5),

$$\text{Var}[L_n] \leq \frac{1}{2} \sum_{i=1}^{n} \mathbb{E} \left[ \left( \int_{0}^{\pi} \Delta_n^{(i)}(\theta)d\theta \right)^2 \right]$$

$$\leq \frac{1}{2} \sum_{i=1}^{n} \mathbb{E} \left[ \left( \int_{0}^{\pi} |(Z_i - Z_i') \cdot \mathbf{e}_{\theta}|d\theta \right)^2 \right]$$

$$\leq \frac{1}{2} \sum_{i=1}^{n} \pi^2 \sigma^2$$

$$= \frac{n \pi^2 \sigma^2}{2},$$

since $Z_i$ are independent identically distributed. \qed

### 5.3 Law of large numbers

As we mentioned earlier in Remarks 4.2, Snyder and Steele [57] has shown the asymptotic behaviour of $L_n/n$. They state their law of large numbers only for $\mu \neq 0$ but the case with $\mu = 0$ works equally well. Here we give a different proof of the law of large numbers by using the variance bound.

**Proposition 5.5.** If $\mathbb{E}(\|Z\|^2) < \infty$, then $n^{-1}L_n \to 2\|\mu\|$ a.s. as $n \to \infty$.

**Proof.** We have $n^{-1} \mathbb{E} L_n \to 2\|\mu\|$ by Proposition 4.5 and the variance bound $\text{Var} L_n \leq Cn$ by Proposition 5.4. Chebyshev’s inequality says, for any $\varepsilon > 0$,

$$\mathbb{P} \left( \left| \frac{L_n}{n} - \frac{\mathbb{E} L_n}{n} \right| > \varepsilon \right) \leq \frac{\text{Var}(n^{-1}L_n)}{\varepsilon^2} \leq \frac{C}{\varepsilon^2 n}.$$
Take \( n = n_k = k^2 \), then
\[
\sum_{k=1}^{\infty} \mathbb{P} \left( \left| \frac{L_{n_k}}{n_k} - \mathbb{E} L_{n_k} \right| > \varepsilon \right) \leq C \varepsilon^2 \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.
\]

So the Borel–Cantelli lemma (see Lemma 2.3) implies that
\[n^{-1} L_{n_k} - n^{-1} \mathbb{E} L_{n_k} \to 0 \text{ a.s. as } k \to \infty.\]
Hence
\[
\left| \frac{L_{n_k}}{n_k} - 2\|\mu\| \right| \leq \left| \frac{L_{n_k}}{n_k} - \mathbb{E} L_{n_k} \right| + \left| \mathbb{E} L_{n_k} - 2\|\mu\| \right| \to 0 \text{ a.s. as } k \to 0.
\]

For any \( n \), let \( k = \lfloor \sqrt{n} \rfloor \). Then \( n_k \leq n < n_{k+1} \). Since \( L_n \) is non-decreasing in \( n \) by (2.8), we have
\[
\frac{L_n}{n} \leq \frac{L_{n_{k+1}}}{n_{k+1}} \leq \frac{n_{k+1}}{n_k} \leq \frac{L_{n_{k+1}}}{n_{k+1}} \cdot \frac{n_{k+1}}{n_k},
\]
and also
\[
\frac{L_n}{n} \geq \frac{L_{n_k}}{n_k} \geq \frac{n_k}{n} \geq \frac{L_{n_k}}{n_k} \cdot \frac{n_k}{n_{k+1}}.
\]
Then as \( n \to \infty, k \to \infty \) so
\[
\frac{L_{n_k}}{n_k} \xrightarrow{a.s.} 2\|\mu\| \quad \text{and} \quad \frac{n_k}{n_{k+1}} = \frac{\left( \lfloor \sqrt{n} \rfloor \right)^2}{\left( \lfloor \sqrt{n} \rfloor + 1 \right)^2} \to 1.
\]
Therefore \( n^{-1} L_n \to 2\|\mu\| \) a.s.

Proposition 5.5 says that if \( \mathbb{E} \|Z_1\|^2 < \infty \) and \( \mu = 0 \), then \( n^{-1} L_n \to 0 \) a.s. But Proposition 4.7 says that \( \mathbb{E} L_n = O(n^{1/2}) \), so we might expect to be able to improve on this ‘law of large numbers’. Indeed, we have the following.

**Proposition 5.6.** Suppose \( \mathbb{E} \|Z_1\|^2 < \infty \).

(i) For any \( \alpha > 1/2 \), as \( n \to \infty \),
\[
\frac{L_n - \mathbb{E} L_n}{n^\alpha} \to 0, \text{ in probability.}
\]

(ii) If, in addition, \( \mu = 0 \), then for any \( \alpha > 1/2 \), \( n^{-\alpha} L_n \to 0 \) a.s. as \( n \to \infty \).

**Proof.** Similarly to the proof of Proposition 5.5, Chebyshev’s inequality gives, for \( \varepsilon > 0 \),
\[
\mathbb{P} \left( \left| \frac{L_n - \mathbb{E} L_n}{n^\alpha} \right| > \varepsilon \right) \leq \frac{C}{\varepsilon^2 n^{1-2\alpha}}.
\]
The right-hand side here tends to 0 as \( n \to \infty \) provided \( \alpha > 1/2 \), giving (i).

For part (ii), take \( n = n_k = 2^k \) in (5.7). Then

\[
\sum_{k=1}^{\infty} \mathbb{P} \left( \left| \frac{L_{n_k} - \mathbb{E} L_{n_k}}{n_k^{\alpha}} \right| > \varepsilon \right) < \infty,
\]

provided \( \alpha > 1/2 \). So

\[
\lim_{k \to \infty} \frac{|L_{n_k} - \mathbb{E} L_{n_k}|}{n_k^{\alpha}} = 0, \text{ a.s.}
\]

But

\[
\lim_{k \to \infty} \frac{\mathbb{E} L_{n_k}}{n_k^{\alpha}} = \lim_{n \to \infty} \frac{\mathbb{E} L_n}{n^{\alpha}} = 0,
\]

by Proposition 4.7 and hence

\[
\lim_{k \to \infty} \frac{L_{n_k}}{n_k^{\alpha}} = 0, \text{ a.s.}
\]

For every positive integer \( n \), there exists \( k(n) \in \mathbb{Z}_+ \) for which \( 2^{k(n)} \leq n < 2^{k(n)+1} \) and \( k(n) \to \infty \) as \( n \to \infty \). Hence, by (2.8),

\[
\frac{L_n}{n^{\alpha}} \leq \frac{L_{2^{k(n)+1}}}{(2^{k(n)})^{\alpha}} = 2^{\alpha} \cdot \frac{L_{2^{k(n)+1}}}{(2^{k(n)+1})^{\alpha}},
\]

which tends to 0 a.s. as \( n \to \infty \). \( \square \)

Moreover, \((L_n - \mathbb{E} L_n)n^{-\alpha} \) in Proposition 5.6(i) is also convergent to 0 almost surely, if we assume \( \|Z_1\| \) is upper bounded by some constant. To show this, we need to use Azuma–Hoeffding inequality (see Lemma 2.18).

**Lemma 5.7.** Assume \( \|Z_1\| \leq B \) a.s. for some constant \( B \). Then, for any \( t > 0 \),

\[
\mathbb{P} \left( \left| L_n - \mathbb{E} L_n \right| > t \right) \leq 2 \exp \left( -\frac{t^2}{8\pi^2 B^2 n} \right).
\]

**Proof.** Let \( D_{n,i} = \mathbb{E} \left[ L_n - L_n^{(i)} \mid \mathcal{F}_i \right] \), where \( \mathcal{F}_0 \) denote the trivial \( \sigma \)-algebra, and for \( i \in \mathbb{N}, \mathcal{F}_i = \sigma(Z_1, \ldots, Z_i) \) is the \( \sigma \)-algebra generated by the first \( n \) steps of the random walk. So \( D_{n,i} \) is \( \mathcal{F}_i \)-measurable. Since \( L_n^{(i)} \) is independent of \( Z_i \),

\[
\mathbb{E} \left[ L_n^{(i)} \mid \mathcal{F}_i \right] = \mathbb{E} \left[ L_n^{(i)} \mid \mathcal{F}_{i-1} \right] = \mathbb{E} \left[ L_n \mid \mathcal{F}_{i-1} \right],
\]

so that \( D_{n,i} = \mathbb{E} \left[ L_n \mid \mathcal{F}_i \right] - \mathbb{E} \left[ L_n \mid \mathcal{F}_{i-1} \right] \). Hence, \( \mathbb{E} [D_{n,i} \mid \mathcal{F}_{i-1}] = 0 \).
By using equation (5.5) and our assumption that $\|Z_1\| \leq B$ a.s., we can deduce an upper bound for $|D_{n,i}|$ as follows.

$$|D_{n,i}| \leq \mathbb{E} \left[ \int_0^\pi |\Delta_n^{(i)}(\theta)|d\theta \middle| \mathcal{F}_i \right] \leq \pi (\|Z_i\| + \|Z_i'\|) \leq 2\pi B.$$ 

Hence, the result follows Lemma 2.18 with $d_\infty = 2\pi B$. \hfill \Box

**Proposition 5.8.** Suppose $\|Z_1\| \leq B$ for some constant $B$. Then for any $\alpha > 1/2$, 

$$\frac{L_n - \mathbb{E} L_n}{n^\alpha} \rightarrow 0 \text{ a.s.}$$

*Proof.* The result follows Lemma 5.7 by using Borel–Cantelli Lemma (see Lemma 2.3). \hfill \Box

## 5.4 Central limit theorem for the non-zero drift case

### 5.4.1 Control of extrema

For the remainder of this section, without loss of generality, we suppose that $\mathbb{E}[Z_1] = \mu e_{\pi/2}$ with $\mu \in (0, \infty)$. Observe that $(S_j \cdot e_\theta; 0 \leq j \leq n)$ is a one-dimensional random walk: indeed, $S_j \cdot e_\theta = \sum_{k=1}^j Z_k \cdot e_\theta$. The mean drift of this one-dimensional random walk is

$$\mathbb{E}[Z_1 \cdot e_\theta] = \mathbb{E}[Z_1] \cdot e_\theta = \mu \sin \theta. \quad (5.8)$$

Note that the drift $\mu \sin \theta$ is positive if $\theta \in (0, \pi)$. This crucial fact gives us control over the behaviour of the extrema such as $M_n(\theta)$ and $m_n(\theta)$ that contribute to (5.4), and this will allow us to estimate the conditional expectation of the final term in (5.4) (see Lemma 5.10 below).

For $\gamma \in (0, 1/2)$ and $\delta \in (0, \pi/2)$ (two constants that will be chosen to be suitably small later in our arguments), we denote by $E_{n,i}(\delta, \gamma)$ the event that the following occur:

- for all $\theta \in [\delta, \pi - \delta]$, $J_n(\theta) < \gamma n$ and $\bar{J}_n(\theta) > (1-\gamma)n$;
- for all $\theta \in [\delta, \pi - \delta]$, $\bar{J}_n^{(i)}(\theta) < \gamma n$ and $\bar{J}_n^{(i)}(\theta) > (1-\gamma)n$. 


We write $E_{n,i}^{c} (\delta, \gamma)$ for the complement of $E_{n,i} (\delta, \gamma)$. The idea is that $E_{n,i} (\delta, \gamma)$ will occur with high probability, and on this event we have good control over $\Delta_{n}^{(i)} (\theta)$. The next result formalizes these assertions. For $\gamma \in (0, 1/2)$, define $I_{n, \gamma} := \{1, \ldots, n\} \cap [\gamma n, (1 - \gamma)n]$.

**Lemma 5.9.** For any $\gamma \in (0, 1/2)$ and any $\delta \in (0, \pi/2)$, the following hold.

(i) If $i \in I_{n, \gamma}$, then, a.s., for any $\theta \in [\delta, \pi - \delta]$,

$$\Delta_{n}^{(i)} (\theta) \mathbf{1} (E_{n,i} (\delta, \gamma)) = (Z_{i} - Z_{i}') \cdot \mathbf{e}_{\theta} \mathbf{1} (E_{n,i} (\delta, \gamma)).$$

(5.9)

(ii) If $\mathbb{E} \|Z_{1}\| < \infty$ and $\|\mathbb{E} [Z_{1}]\| \neq 0$, then $\min_{1 \leq i \leq n} \mathbb{P} [E_{n,i} (\delta, \gamma)] \to 1$ as $n \to \infty$.

*Proof.* First we prove part (i). Suppose that $i \in I_{n, \gamma}$, so $\gamma n \leq i \leq (1 - \gamma)n$. Suppose that $\theta \in [\delta, \pi - \delta]$. Then on $E_{n,i} (\delta, \gamma)$, we have $J_{n} (\theta) < i < \bar{J}_{n} (\theta)$ and $\bar{J}_{n}^{(i)} (\theta) < i < \bar{J}_{n}^{(i)} (\theta)$. Then from (5.2) it follows that in fact $J_{n} (\theta) = \bar{J}_{n}^{(i)} (\theta)$ and $\bar{J}_{n} (\theta) = \bar{J}_{n}^{(i)} (\theta)$. Hence $m_{n} (\theta) = m_{n}^{(i)} (\theta)$ and

$$M_{n}^{(i)} (\theta) = S_{\bar{J}_{n}^{(i)} (\theta)} (\theta) \cdot \mathbf{e}_{\theta} = M_{n} (\theta) + (Z_{i}' - Z_{i}) \cdot \mathbf{e}_{\theta}, \text{ by } [5.2].$$

Equation (5.9) follows.

Next we prove part (ii). Suppose that $\mu = \mathbb{E} [Z_{1}] > 0$. Since $\mathbb{E} \|Z_{1}\| < \infty$, the strong law of large numbers implies that $\|n^{-1} S_{n} - \mathbb{E} [Z_{1}]\| \to 0$, a.s., as $n \to \infty$. In other words, for any $\varepsilon_{1} > 0$, there exists $N := N (\varepsilon_{1})$ such that $\mathbb{P} [N < \infty] = 1$ and $\|n^{-1} S_{n} - \mathbb{E} [Z_{1}]\| < \varepsilon_{1}$ for all $n \geq N$. In particular, for $n \geq N$, by (5.8),

$$|n^{-1} S_{n} \cdot \mathbf{e}_{\theta} - \mu \sin \theta| = |n^{-1} S_{n} \cdot \mathbf{e}_{\theta} - \mathbb{E} [Z_{1}] \cdot \mathbf{e}_{\theta}| \leq \|n^{-1} S_{n} - \mathbb{E} [Z_{1}]\| < \varepsilon_{1},$$

(5.10)

for all $\theta \in [0, 2\pi)$.

Take $\varepsilon_{1} < \mu \sin \delta$. If $n \geq N$, then, by (5.10),

$$S_{n} \cdot \mathbf{e}_{\theta} > (\mu \sin \theta - \varepsilon_{1})n \geq (\mu \sin \delta - \varepsilon_{1})n,$$

provided $\theta \in [\delta, \pi - \delta]$. By choice of $\varepsilon_{1}$, the last term in the previous display is strictly positive. Hence, for $n \geq N$, for any $\theta \in [\delta, \pi - \delta]$, $S_{n} \cdot \mathbf{e}_{\theta} > 0$. But, $S_{0} \cdot \mathbf{e}_{\theta} = 0$. So $J_{n} (\theta) < N$ for all $\theta \in [\delta, \pi - \delta]$, and

$$\mathbb{P} \left[ \cap_{\theta \in [\delta, \pi - \delta]} \{J_{n} (\theta) < \gamma n\} \right] \geq \mathbb{P} [N < \gamma n] \to 1,$$
as \( n \to \infty \), since \( N < \infty \) a.s.

Now,

\[
\max_{0 \leq j \leq (1 - \gamma)n} S_j \cdot e_\theta \leq \max \left\{ \max_{0 \leq j \leq N} S_j \cdot e_\theta, \max_{N \leq j \leq (1 - \gamma)n} S_j \cdot e_\theta \right\}. \tag{5.11}
\]

For the final term on the right-hand side of (5.11), (5.10) implies that

\[
\max_{N \leq j \leq (1 - \gamma)n} S_j \cdot e_\theta \leq \max_{0 \leq j \leq (1 - \gamma)n} (\mu \sin \theta + \varepsilon_1)j \leq (\mu \sin \theta + \varepsilon_1)(1 - \gamma)n.
\]

On the other hand, if \( n \geq N \), then (5.10) implies that \( S_n \cdot e_\theta \geq (\mu \sin \theta - \varepsilon_1)n \).

Here

\[
\mu \sin \theta - \varepsilon_1 \geq (\mu \sin \theta + \varepsilon_1)(1 - \gamma) \text{ if } \varepsilon_1 < \frac{\gamma \mu \sin \theta}{2 - \gamma}.
\]

Now we choose \( \varepsilon_1 < \frac{\gamma \mu \sin \delta}{2} \). Then, for any \( \theta \in [\delta, \pi - \delta] \), we have that, for \( n \geq N \),

\[
S_n \cdot e_\theta > \max_{N \leq j \leq (1 - \gamma)n} S_j \cdot e_\theta.
\]

Hence, by (5.11),

\[
P\left[ \bigcap_{\theta \in [\delta, \pi - \delta]} \{ \bar{J}_n(\theta) > (1 - \gamma)n \} \right] \geq P \left[ \bigcap_{\theta \in [\delta, \pi - \delta]} \left\{ S_n \cdot e_\theta > \max_{0 \leq j \leq (1 - \gamma)n} S_j \cdot e_\theta \right\} \right]
\]

\[
\geq P \left[ N \leq n, \bigcap_{\theta \in [\delta, \pi - \delta]} \left\{ S_n \cdot e_\theta > \max_{0 \leq j \leq N} S_j \cdot e_\theta \right\} \right].
\]

Also, for \( n \geq N \), \( S_n \cdot e_\theta > (1 - \frac{\gamma}{2}) \mu n \sin \delta \), so we obtain

\[
P\left[ \bigcap_{\theta \in [\delta, \pi - \delta]} \{ \bar{J}_n(\theta) > (1 - \gamma)n \} \right] \geq P \left[ N \leq n, \max_{0 \leq j \leq N} \| S_j \| \leq \left( 1 - \frac{\gamma}{2} \right) \mu n \sin \delta \right],
\]

using the fact that \( \max_{0 \leq j \leq N} S_j \cdot e_\theta \leq \max_{0 \leq j \leq N} \| S_j \| \) for all \( \theta \).

Now, as \( n \to \infty \), \( P[N > n] \to 0 \), and

\[
P\left[ \max_{0 \leq j \leq N} \| S_j \| > \left( 1 - \frac{\gamma}{2} \right) \mu n \sin \delta \right] \to 0,
\]

since \( N < \infty \) a.s. So we conclude that

\[
P\left[ \bigcap_{\theta \in [\delta, \pi - \delta]} \{ J_n(\theta) < \gamma n, \bar{J}_n(\theta) > (1 - \gamma)n \} \right] \to 1,
\]

as \( n \to \infty \), and the same result holds for \( J_n^{(i)}(\theta) \) and \( \bar{J}_n^{(i)}(\theta) \), uniformly in \( i \in \{1, \ldots, n\} \), since resampling \( Z_i \) does not change the distribution of the trajectory.
5.4.2 Approximation for the martingale differences

The following result is a key component to our proof. Recall that $D_{n,i} = \mathbb{E} [L_n - L_n^{(i)} | \mathcal{F}_i]$.

**Lemma 5.10.** Suppose that $\mathbb{E} \|Z_i\| < \infty$, $\gamma \in (0, 1/2)$, and $\delta \in (0, \pi/2)$. For any $i \in I_{n,\gamma}$,

$$
\left| D_{n,i} - \frac{2(Z_i - \mathbb{E} [Z_i]) \cdot \mathbb{E} [Z_i]}{\|\mathbb{E} [Z_i]\|} \right| \leq 4\delta\|Z_i\| + 4\delta\mathbb{E} \|Z_1\| + 3\pi\|Z_i\|\mathbb{P} [E_{n,i}^\gamma (\delta, \gamma) | \mathcal{F}_i] + 3\pi \mathbb{E} [\|Z_i\| \mathbb{1} (E_{n,i}^\gamma (\delta, \gamma)) | \mathcal{F}_i], \text{ a.s.} \quad (5.12)
$$

**Proof.** Taking (conditional) expectations in (5.4), we obtain

$$
D_{n,i} = \int_0^\pi \mathbb{E} [\Delta_n^{(i)} (\theta) \mathbb{1} (E_{n,i} (\delta, \gamma)) | \mathcal{F}_i]d\theta + \int_0^\pi \mathbb{E} [\Delta_n^{(i)} (\theta) \mathbb{1} (E_{n,i}^c (\delta, \gamma)) | \mathcal{F}_i]d\theta. \quad (5.13)
$$

For the second term on the right-hand side of (5.13), we have

$$
\left| \int_0^\pi \mathbb{E} [\Delta_n^{(i)} (\theta) \mathbb{1} (E_{n,i}^c (\delta, \gamma)) | \mathcal{F}_i]d\theta \right| \leq \int_0^\pi \mathbb{E} [\|\Delta_n^{(i)} (\theta)\| \mathbb{1} (E_{n,i}^c (\delta, \gamma)) | \mathcal{F}_i]d\theta. \quad (5.14)
$$

Applying the bound (5.5), we obtain

$$
\int_0^\pi \mathbb{E} [\|\Delta_n^{(i)} (\theta)\| \mathbb{1} (E_{n,i}^c (\delta, \gamma)) | \mathcal{F}_i]d\theta \leq \pi\mathbb{E} [\|Z_i\| + \|Z_i^2\|] \mathbb{1} (E_{n,i}^c (\delta, \gamma)) | \mathcal{F}_i]
$$

$$
= \pi\|Z_i\|\mathbb{P} [E_{n,i}^c (\delta, \gamma) | \mathcal{F}_i] + \pi \mathbb{E} [\|Z_i\| \mathbb{1} (E_{n,i}^c (\delta, \gamma)) | \mathcal{F}_i], \quad (5.15)
$$

since $Z_i$ is $\mathcal{F}_i$-measurable with $\mathbb{E} \|Z_i\| < \infty$.

We decompose the first integral on the right-hand side of (5.13) as $I_1 + I_2 + I_3$, where

$$
I_1 := \int_0^\delta \mathbb{E} [\Delta_n^{(i)} (\theta) \mathbb{1} (E_{n,i} (\delta, \gamma)) | \mathcal{F}_i]d\theta,
$$

$$
I_2 := \int_\delta^{\pi-\delta} \mathbb{E} [\Delta_n^{(i)} (\theta) \mathbb{1} (E_{n,i} (\delta, \gamma)) | \mathcal{F}_i]d\theta,
$$

$$
I_3 := \int_{\pi-\delta}^\pi \mathbb{E} [\Delta_n^{(i)} (\theta) \mathbb{1} (E_{n,i} (\delta, \gamma)) | \mathcal{F}_i]d\theta.
$$

First we deal with $I_1$ and $I_3$. We have

$$
|I_1| \leq \int_0^\delta \mathbb{E} [\|\Delta_n^{(i)} (\theta)\| | \mathcal{F}_i]d\theta \leq \delta\mathbb{E} [\|Z_i\| + \|Z_i^2\| | \mathcal{F}_i], \text{ a.s.,}
$$
by another application of (5.5). Here $\mathbb{E}[\|Z_i\| \mid F_i] = \|Z_i\|$, since $Z_i$ is $F_i$-measurable, and, since $Z_i'$ is independent of $F_i$, $\mathbb{E}[\|Z_i'\| \mid F_i] = \mathbb{E}\|Z_i'\| = \mathbb{E}\|Z_i\|$.

A similar argument applies to $I_3$, so that

$$|I_1 + I_3| \leq 2\delta\|Z_i\| + 2\delta\mathbb{E}\|Z_1\|, \text{ a.s.} \quad (5.16)$$

We now consider $I_2$. From (5.9), since $i \in I_{n,\gamma}$, we have

$$I_2 = \int_\delta^{\pi-\delta} \mathbb{E}[(Z_i - Z_i') \cdot e_\theta 1(E_{n,i}(\delta, \gamma)) \mid F_i] d\theta$$

$$= \int_\delta^{\pi-\delta} \mathbb{E}[(Z_i - Z_i') \cdot e_\theta 1(E_{n,i}^c(\delta, \gamma)) \mid F_i] d\theta$$

Here, by the triangle inequality,

$$\left| \int_\delta^{\pi-\delta} \mathbb{E}[(Z_i - Z_i') \cdot e_\theta 1(E_{n,i}(\delta, \gamma)) \mid F_i] d\theta \right|$$

$$\leq \int_0^\pi \mathbb{E}[\|Z_i\| + \|Z_i'\|] 1(E_{n,i}^c(\delta, \gamma)) \mid F_i] d\theta$$

$$= \pi\|Z_i\|\mathbb{P}[E_{n,i}^c(\delta, \gamma) \mid F_i] + \pi\mathbb{E}[\|Z_i'\| 1(1 - E_{n,i}(\delta, \gamma)) \mid F_i], \quad (5.17)$$

similarly to (5.15). Finally, similarly to (5.16),

$$\left| \int_\delta^{\pi-\delta} \mathbb{E}[(Z_i - Z_i') \cdot e_\theta \mid F_i] d\theta - \int_\delta^{\pi} \mathbb{E}[(Z_i - Z_i') \cdot e_\theta \mid F_i] d\theta \right|$$

$$\leq 2\delta\mathbb{E}[\|Z_i\| + \|Z_i'\| \mid F_i] = 2\delta (\|Z_i\| + \mathbb{E}\|Z_1\|). \quad (5.18)$$

We combine (5.13) with (5.14) and the bounds in (5.15)–(5.18) to give

$$\left| D_{n,i} - \int_0^\pi \mathbb{E}[(Z_i - Z_i') \cdot e_\theta \mid F_i] d\theta \right| \leq 4\delta\|Z_i\| + 4\delta\mathbb{E}\|Z_1\| + 3\pi\mathbb{E}[\|Z_i'\| 1(1 - E_{n,i}(\delta, \gamma)) \mid F_i]$$

$$+ 3\pi\mathbb{E}[\|Z_i'\| 1(E_{n,i}^c(\delta, \gamma)) \mid F_i], \text{ a.s.} \quad (5.19)$$

To complete the proof of the lemma, we compute the integral on the left-hand side of (5.19). First note that $\mathbb{E}[(Z_i - Z_i') \cdot e_\theta \mid F_i] = (Z_i - \mathbb{E}[Z_i]) \cdot e_\theta$, since $Z_i$ is $F_i$-measurable and $Z_i'$ is independent of $F_i$, so that

$$\int_0^\pi \mathbb{E}[(Z_i - Z_i') \cdot e_\theta \mid F_i] d\theta = \int_0^\pi (Z_i - \mathbb{E}[Z_i]) \cdot e_\theta d\theta.$$

To evaluate the last integral, it is convenient to introduce the notation $Z_i - \mathbb{E}[Z_i] = R_i e_{\Theta_i}$, where $R_i = \|Z_i - \mathbb{E}[Z_i]\| \geq 0$ and $\Theta_i \in [0, 2\pi)$. Then

$$\int_0^\pi (Z_i - \mathbb{E}[Z_i]) \cdot e_\theta d\theta = \int_0^\pi R_i e_{\Theta_i} \cdot e_\theta d\theta = R_i \int_0^\pi \cos(\theta - \Theta_i) d\theta.$$
\[ = 2R_i \sin \Theta_i = 2R_i e^{\Theta_i} \cdot e_{\pi/2}. \]

Now (5.12) follows from (5.19), and the proof is complete. \(\square\)

5.4.3 Proofs for the central limit theorem

For ease of notation, we write
\[ Y_i := 2\|E[Z_1]\|^{-1}(Z_i - E[Z_1]) \cdot E[Z_1], \]
and define
\[ W_{n,i} := D_{n,i} - Y_i. \]

The upper bound for \(|W_{n,i}|\) in Lemma 5.10 together with Lemma 5.9(ii) will enable us to prove the following result, which will be the basis of our proof of Theorem 5.12.

Lemma 5.11. Suppose that \(E[\|Z_1\|^2] < \infty\) and \(E[Z_1] \neq 0\). Then
\[
\lim_{n \to \infty} n^{-1} \sum_{i=1}^{n} E[W_{n,i}^2] = 0.
\]

Proof. Fix \(\varepsilon > 0\). We take \(\gamma \in (0, 1/2)\) and \(\delta \in (0, \pi/2)\), to be specified later. We divide the sum of interest into two parts, namely \(i \in I_{n,\gamma}\) and \(i \notin I_{n,\gamma}\). Now from (5.4) with (5.5) we have
\[ |L_n(i) - L_n| \leq \pi (\|Z_i\| + \|Z'_i\|), \text{ a.s.}, \]
so that
\[ |D_{n,i}| \leq \pi E[\|Z_i\| + \|Z'_i\| | F_i] = \pi (\|Z_i\| + E[\|Z_i\|]). \]

It then follows from the triangle inequality that
\[ |W_{n,i}| \leq |D_{n,i}| + 2\|Z_i - E[Z_1]\| \leq (\pi + 2)(\|Z_i\| + E[\|Z_i\|]). \]

So provided \(E[\|Z_1\|^2] < \infty\), we have \(E[W_{n,i}^2] \leq C_0\) for all \(n\) and all \(i\), for some constant \(C_0 < \infty\), depending only on the distribution of \(Z_1\). Hence
\[
\frac{1}{n} \sum_{i \notin I_{n,\gamma}} E[W_{n,i}^2] \leq \frac{1}{n} 2\gamma n C_0 = 2\gamma C_0,
\]
using the fact that there are at most \(2\gamma n\) terms in the sum. From now on, choose \(\gamma > 0\) small enough so that \(2\gamma C_0 < \varepsilon\).

Now consider \(i \in I_{n,\gamma}\). For such \(i\), (5.12) shows that, for some constant \(C_1 < \infty\),
\[ |W_{n,i}| \leq C_1 (1 + \|Z_i\|) \delta + C_1 \|Z_i\| P[E_{n,i}^c(\delta, \gamma) | F_i] \]

using the fact that there are at most \(2\gamma n\) terms in the sum. From now on, choose \(\gamma > 0\) small enough so that \(2\gamma C_0 < \varepsilon\).
+ \mathcal{C}_{1} \mathbb{E} \left[ \| Z_i' \| \mathbf{1}(E_{n,i}^{c}(\delta, \gamma)) \mid \mathcal{F}_i \right], \text{ a.s.} \tag{5.20}

Here, for any \( B_1 \in (0, \infty) \), a.s.,

\[
\mathbb{E} \left[ \| Z_i' \| \mathbf{1}(E_{n,i}^{c}(\delta, \gamma)) \mid \mathcal{F}_i \right] \leq \mathbb{E} \left[ \| Z_i' \| \mathbf{1}\{ \| Z_i' \| > B_1 \} \mid \mathcal{F}_i \right] + B_1 \mathbb{P}[E_{n,i}^{c}(\delta, \gamma) \mid \mathcal{F}_i]
\]

= \mathbb{E} \left[ \| Z_i' \| \mathbf{1}\{ \| Z_i' \| > B_1 \} \right] + B_1 \mathbb{P}[E_{n,i}^{c}(\delta, \gamma) \mid \mathcal{F}_i],

since \( Z_i' \) is independent of \( \mathcal{F}_i \). Here, since \( \mathbb{E} \| Z_i' \| = \mathbb{E} \| Z_i \| < \infty \), the dominated convergence theorem (see Lemma 2.1) implies that \( \mathbb{E} \| Z_i' \| \mathbf{1}\{ \| Z_i' \| > B_1 \} \to 0 \) as \( B_1 \to \infty \). So we can choose \( B_1 = B_1(\delta) \) large enough so that

\[
\mathbb{E} \left[ \| Z_i' \| \mathbf{1}(E_{n,i}^{c}(\delta, \gamma)) \mid \mathcal{F}_i \right] \leq \delta + B_1 \mathbb{P}[E_{n,i}^{c}(\delta, \gamma) \mid \mathcal{F}_i], \text{ a.s.}
\]

Combining this with (5.20) we see that there is a constant \( C_2 < \infty \) for which

\[
|W_{n,i}| \leq C_2(1 + \| Z_i \|) \left( \delta + B_1 \mathbb{P}[E_{n,i}^{c}(\delta, \gamma) \mid \mathcal{F}_i] \right), \text{ a.s.}
\]

Hence

\[
W_{n,i}^2 \leq C_2^2(1 + \| Z_i \|)^2 \left[ \delta^2 + 2B_1 \delta \mathbb{P}[E_{n,i}^{c}(\delta, \gamma) \mid \mathcal{F}_i] + B_1^2 \mathbb{P}[E_{n,i}^{c}(\delta, \gamma) \mid \mathcal{F}_i]^2 \right]
\]

\[
\leq C_2^2(1 + \| Z_i \|)^2 \left( \delta + B_1^2 \mathbb{P}[E_{n,i}^{c}(\delta, \gamma) \mid \mathcal{F}_i] \right),
\]

for some constant \( C_3 < \infty \), using the facts that \( \delta < \pi/2 < 2 \) and \( \mathbb{P}[E_{n,i}^{c}(\delta, \gamma) \mid \mathcal{F}_i] \leq 1 \). Taking expectations we get

\[
\mathbb{E} \left[ W_{n,i}^2 \right] \leq C_2^2 \delta \mathbb{E} \left[ (1 + \| Z_i \|)^2 \right] + C_2^2 B_1^2 \mathbb{E} \left[ (1 + \| Z_i \|)^2 \mathbb{P}[E_{n,i}^{c}(\delta, \gamma) \mid \mathcal{F}_i] \right].
\]

Provided \( \mathbb{E} \| Z_1 \|^2 < \infty \), there is a constant \( C_4 < \infty \) such that the first term on the right-hand side of the last display is bounded by \( C_4\delta \). Now fix \( \delta > 0 \) small enough so that \( C_4\delta < \varepsilon \); this choice also fixes \( B_1 \). Then

\[
\mathbb{E} \left[ W_{n,i}^2 \right] \leq \varepsilon + C_3^2 B_1^2 \mathbb{E} \left[ (1 + \| Z_i \|)^2 \mathbb{P}[E_{n,i}^{c}(\delta, \gamma) \mid \mathcal{F}_i] \right]. \tag{5.21}
\]

For the final term in (5.21), observe that, for any \( B_2 \in (0, \infty) \), a.s.,

\[
(1 + \| Z_i \|)^2 \mathbb{P}[E_{n,i}^{c}(\delta, \gamma) \mid \mathcal{F}_i] \leq (1 + B_2)^2 \mathbb{P}[E_{n,i}^{c}(\delta, \gamma) \mid \mathcal{F}_i]
\]

\[
+ (1 + \| Z_i \|)^2 \mathbf{1}\{ \| Z_i \| > B_2 \}. \tag{5.22}
\]
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Here $\mathbb{E}[(1 + \|Z_i\|)^2 1\{\|Z_i\| > B_2\}] \to 0$ as $B_2 \to \infty$, provided $\mathbb{E}[\|Z_1\|^2] < \infty$, by the dominated convergence theorem. Hence, since $\delta$ and $B_1$ are fixed, we can choose $B_2 = B_2(\varepsilon) \in (0, \infty)$ such that

$$C_2^2 B_1^2 \mathbb{E} [(1 + \|Z_i\|)^2 1\{\|Z_i\| > B_2\}] < \varepsilon.$$  

Then taking expectations in (5.22) we obtain from (5.21) that

$$\mathbb{E}[W_{n,i}^2] \leq 2\varepsilon + C_2^2 B_1^2 (1 + B_2)^2 \mathbb{P}[E_{n,i}^c(\delta, \gamma)].$$

Now choose $n_0$ such that $C_2^2 B_1^2 (1 + B_2)^2 \mathbb{P}[E_{n,i}^c(\delta, \gamma)] < \varepsilon$ for all $n \geq n_0$, which we may do by Lemma 5.9(ii). So for the given $\varepsilon > 0$ and $\gamma \in (0, 1/2)$, we can choose $n_0$ such that for all $i \in I_{n,\gamma}$ all $n \geq n_0$, $\mathbb{E}[W_{n,i}^2] \leq 3\varepsilon$. Hence

$$\frac{1}{n} \sum_{i \in I_{n,\gamma}} \mathbb{E}[W_{n,i}^2] \leq 3\varepsilon,$$

for all $n \geq n_0$.

Combining the estimates for $i \in I_{n,\gamma}$ and $i \notin I_{n,\gamma}$, we see that

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[W_{n,i}^2] \leq 2\gamma C_0 + 3\varepsilon \leq 4\varepsilon,$$

for all $n \geq n_0$. Since $\varepsilon > 0$ was arbitrary, the result follows.

Now we can claim and prove our main theorems.

**Theorem 5.12.** Suppose that $\mathbb{E}[\|Z_1\|^2] < \infty$ and $\|\mathbb{E}[Z_1]\| \neq 0$. Then, as $n \to \infty$,

$$n^{-1/2} \left| L_n - \mathbb{E}[L_n] - \sum_{i=1}^{n} \frac{2(Z_i - \mathbb{E}[Z_1]) \cdot \mathbb{E}[Z_1]}{\|\mathbb{E}[Z_1]\|} \right| \to 0, \text{ in } L^2.$$  

**Proof.** First note that

$$\mathbb{E}[W_{n,i} | \mathcal{F}_{i-1}] = \mathbb{E}[D_{n,i} | \mathcal{F}_{i-1}] - \mathbb{E}[Y_i | \mathcal{F}_{i-1}] = 0 - \mathbb{E}[Y_i],$$

since $D_{n,i}$ is a martingale difference sequence and $Y_i$ is independent of $\mathcal{F}_{i-1}$. Here, by definition, $\mathbb{E}[Y_i] = 0$, and so $W_{n,i}$ is also a martingale difference sequence. Therefore, by orthogonality,

$$n^{-1} \mathbb{E} \left[ \left( \sum_{i=1}^{n} W_{n,i} \right)^2 \right] = n^{-1} \sum_{i=1}^{n} \mathbb{E} \left[ W_{n,i}^2 \right] \to 0 \text{ as } n \to \infty, \text{ by Lemma 5.11}.$$

In other words, $n^{-1/2} \sum_{i=1}^{n} W_{n,i} \to 0$ in $L^2$, which, with Lemma 5.1(i), implies the statement in the theorem. □
**Theorem 5.13.** Suppose that $\mathbb{E} [\| Z_1 \|^2] < \infty$ and $\| \mathbb{E} [Z_1] \| \neq 0$. Then
\[
\lim_{n \to \infty} n^{-1} \text{Var} [L_n] = \frac{4\mathbb{E} [((Z_1 - \mathbb{E} [Z_1]) \cdot \mathbb{E} [Z_1])^2]}{\| \mathbb{E} [Z_1] \|^2} = 4\sigma_\mu^2.
\]

**(5.23)**

**Remarks 5.2.** (i) The assumptions $\mathbb{E} [\| Z_1 \|^2] < \infty$ and $\| \mathbb{E} [Z_1] \| \neq 0$ ensure $4\sigma_\mu^2 < \infty$.

(ii) To compare the limit result (5.23) with Snyder and Steele’s upper bound (5.1), observe that
\[
4\sigma_\mu^2 = 4 \left( \frac{\mathbb{E} [(Z_1 \cdot \mathbb{E} [Z_1])^2] - \| \mathbb{E} [Z_1] \|^4}{\| \mathbb{E} [Z_1] \|^2} \right) \leq 4 \left( \mathbb{E} [\| Z_1 \|^2] - \| \mathbb{E} [Z_1] \|^2 \right).
\]

(iii) The limit $4\sigma_\mu^2$ is zero if and only if $(Z_1 - \mathbb{E} [Z_1]) \cdot \mathbb{E} [Z_1] = 0$ with probability 1, i.e., if $Z_1 - \mathbb{E} [Z_1]$ is always orthogonal to $\mathbb{E} [Z_1]$. In such a degenerate case, (5.23) says that $\text{Var} [L_n] = o(n)$. This is the case, for example, if $Z_1$ takes values $(1, 1)$ and $(1, -1)$ each with probability $1/2$. Note that the Snyder–Steele bound (5.1) applied in this example says only that $\text{Var} [L_n] \leq (\pi^2/2)n$, which is not the correct order. Here, the two-dimensional trajectory can be viewed as a space-time trajectory of a one-dimensional simple symmetric random walk. We conjecture that in fact $\text{Var} [L_n] = O(\log n)$. Steele [59] obtains variance results for the number of faces of the convex hull of one-dimensional simple random walk, and comments that such results for $L_n$ seem “far out of reach” [59, p. 242].

**Proof.** Write
\[
\xi_n = \frac{L_n - \mathbb{E} [L_n]}{\sqrt{n}}; \quad \text{and} \quad \zeta_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n Y_i, \quad \text{where} \quad Y_i = \frac{2(Z_i - \mathbb{E} [Z_1]) \cdot \mathbb{E} [Z_1]}{\| \mathbb{E} [Z_1] \|}.
\]

(5.24)

Then Theorem 5.12 shows that $|\xi_n - \zeta_n| \to 0$ in $L^2$ as $n \to \infty$. Also, with $4\sigma_\mu^2$ as given by (5.23), $\mathbb{E} [\zeta_n^2] = 4\sigma_\mu^2$. Then a computation shows that
\[
n^{-1} \text{Var} [L_n] = \mathbb{E} [\xi_n^2] = \mathbb{E} [\xi_n^2 - (\xi_n - \zeta_n)^2] + \mathbb{E} [\zeta_n^2] + 2\mathbb{E} [(\xi_n - \zeta_n)\zeta_n].
\]

Here, by the $L^2$ convergence, $\mathbb{E} [\xi_n^2 - (\xi_n - \zeta_n)^2] \to 0$ and, by the Cauchy–Schwarz inequality (see Lemma 2.14),
\[
|\mathbb{E} [(\xi_n - \zeta_n)\zeta_n]| \leq (\mathbb{E} [\xi_n^2] \mathbb{E} [\zeta_n^2])^{1/2} \to 0 \text{ as well.}
\]

So $\mathbb{E} [\zeta_n^2] \to 4\sigma_\mu^2$ as $n \to \infty$. \(\square\)
In the case where $E[\|Z_1\|^2] < \infty$ and $\|E[Z_1]\| = \mu > 0$, Snyder and Steele deduce from their bound (5.1) a strong law of large numbers for $L_n$, namely $\lim_{n \to \infty} n^{-1}L_n = 2\mu$, a.s. (see [57, p. 1168]). Given this and the variance asymptotics of Theorem 5.13, it is natural to ask whether there is an accompanying central limit theorem. Our next result gives a positive answer in the non-degenerate case, again with essentially minimal assumptions.

In the proof of Theorem 5.14 we will use two facts about convergence in distribution that we now recall (see Lemma 2.4). First, if sequences of random variables $\xi_n$ and $\zeta_n$ are such that $\zeta_n \to \zeta$ in distribution for some random variable $\zeta$ and $|\xi_n - \zeta_n| \to 0$ in probability, then $\xi_n \to \zeta$ in distribution (this is Slutsky’s theorem). Second, if $\zeta_n \to \zeta$ in distribution and $\alpha_n \to \alpha$ in probability, then $\alpha_n\zeta_n \to \alpha\zeta$ in distribution.

**Theorem 5.14.** Suppose that $E[\|Z_1\|^2] < \infty$, $\|E[Z_1]\| \neq 0$ and $\sigma_\mu^2 > 0$. Then for any $x \in \mathbb{R}$,

$$
\lim_{n \to \infty} P \left[ \frac{L_n - E[L_n]}{\sqrt{\text{Var}[L_n]}} \leq x \right] = \lim_{n \to \infty} P \left[ \frac{L_n - E[L_n]}{\sqrt{4\sigma_\mu^2 n}} \leq x \right] = \Phi(x),
$$

(5.25)

where $\Phi$ is the standard normal distribution function.

**Proof.** Use the notation for $\xi_n$ and $\zeta_n$ as given by (5.24). Then, by Theorem 5.12 $|\xi_n - \zeta_n| \to 0$ in $L^2$, and hence in probability.

In the sum $\zeta_n$, the $Y_i$ are i.i.d. random variables with mean 0 and variance $E[Y_i^2] = 4\sigma_\mu^2$. Hence the classical central limit theorem (see e.g. [17, p. 93]) shows that $\zeta_n$ converges in distribution to a normal random variable with mean 0 and variance $4\sigma_\mu^2$. Slutsky’s theorem then implies that $\xi_n$ has the same distributional limit. Hence, for any $x \in \mathbb{R}$,

$$
\lim_{n \to \infty} P \left[ \frac{\xi_n}{\sqrt{4\sigma_\mu^2}} \leq x \right] = \lim_{n \to \infty} P \left[ \frac{L_n - E[L_n]}{\sqrt{4\sigma_\mu^2 n}} \leq x \right] = \Phi(x),
$$

where $\Phi$ is the standard normal distribution function. Moreover,

$$
P \left[ \frac{L_n - E[L_n]}{\sqrt{\text{Var}[L_n]}} \leq x \right] = P \left[ \frac{\xi_n\alpha_n}{\sqrt{4\sigma_\mu^2 \text{Var}[L_n]}} \leq x \right],
$$

where $\alpha_n = \frac{\sqrt{4\sigma_\mu^2 n}}{\text{Var}[L_n]} \to 1$ by Theorem 5.13. Thus we verify the limit statements in (5.25). \qed
5.5 Asymptotics for the zero drift case

Recall that \( h_1 \) is defined in (3.4) and \( \Sigma \) is a covariance matrix (see Section 3.3), which is positive semidefinite and symmetric. Let

\[
u_0(\Sigma) := \text{Var} \mathcal{L}(\Sigma^{1/2} h_1), \tag{5.26}\]

we have the following results.

**Proposition 5.15.** Suppose that \((M_p)\) holds for some \( p > 2 \), and \( \mu = 0 \). Then

\[
\lim_{n \to \infty} n^{-1} \text{Var} L_n = u_0(\Sigma).
\]

**Proof.** From (4.3) and Lemma 4.8(ii), for \( p > 2 \) we have \( \sup_n \mathbb{E} [(n^{-1} L_n^2)^{p/2}] < \infty \). Hence \( n^{-1} L_n^2 \) is uniformly integrable, and we deduce convergence of \( n^{-1} \text{Var} L_n \) in Corollary 3.7. \( \square \)

The next result gives bounds on \( u_0(\Sigma) \) defined in (5.26).

**Proposition 5.16.**

\[
\frac{263}{1080} \pi^{-3/2} e^{-144/25} \text{tr} \Sigma \leq u_0(\Sigma) \leq \frac{\pi^2}{2} \text{tr} \Sigma. \tag{5.27}\]

In addition, if \( \Sigma = I \) we have the following sharper form of the lower bound:

\[
\text{Var} \varphi_1 = u_0(I) \geq \frac{2}{5} \left( 1 - \frac{8}{25\pi} \right) e^{-25\pi/16} > 0.
\]

For the proof of this result, we rely on a few facts about one-dimensional Brownian motion, including the bound (see e.g. equation (2.1) of [33]), valid for all \( r > 0 \),

\[
P \left[ \sup_{0 \leq s \leq 1} |w(s)| \leq r \right] \geq \frac{4}{\pi} \left( e^{-\pi^2/(8r^2)} - \frac{1}{3} e^{-9\pi^2/(8r^2)} \right). \tag{5.28}\]

We let \( \Phi \) denote the distribution function of a standard normal random variable; we will also need the standard Gaussian tail bound (see e.g. [17, p. 12])

\[
1 - \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-y^2/2} dy \geq \frac{1}{x\sqrt{2\pi}} \left( 1 - \frac{1}{x^2} \right) e^{-x^2/2}, \text{ for } x > 0. \tag{5.29}\]

We also note that for \( e \in S_1 \) the diffusion \( e \cdot (\Sigma^{1/2} b) \) is one-dimensional Brownian motion with variance parameter \( e^\top \Sigma e \).
The idea behind the variance lower bounds is elementary. For a random variable $X$ with mean $\mathbb{E}X$, we have, for any $\theta \geq 0$,

$$\text{Var}X = \mathbb{E}[(X - \mathbb{E}X)^2] \geq \theta^2 \mathbb{P}[|X - \mathbb{E}X| \geq \theta].$$

If $\mathbb{E}X \geq 0$, taking $\theta = \alpha \mathbb{E}X$ for $\alpha > 0$, we obtain

$$\text{Var}X \geq \alpha^2 (\mathbb{E}X)^2 (\mathbb{P}[X \leq (1 - \alpha)\mathbb{E}X] + \mathbb{P}[X \geq (1 + \alpha)\mathbb{E}X]), \quad (5.30)$$

and our lower bounds use whichever of the latter two probabilities is most convenient.

**Proof of Proposition 5.16.** We start with the upper bounds. Snyder and Steele’s bound [5.6] with the statement for $\text{Var}L_n$ in Proposition 5.15 gives the upper bound in (5.27).

We now move on to the lower bounds. Let $e_{\Sigma} \in S_1$ denote an eigenvector of $\Sigma$ corresponding to the principal eigenvalue $\lambda_{\Sigma}$. Then since $\Sigma_{1/2}h_1$ contains the line segment from 0 to any (other) point in $\Sigma_{1/2}h_1$, we have from monotonicity of $L$ that

$$L(\Sigma_{1/2}h_1) \geq 2 \sup_{0 \leq s \leq 1} ||\Sigma_{1/2}b(s)|| \geq 2 \sup_{0 \leq s \leq 1} (e_{\Sigma} \cdot (\Sigma_{1/2}b(s))).$$

Here $e_{\Sigma} \cdot (\Sigma_{1/2}b)$ has the same distribution as $\lambda_{\Sigma}^{1/2}w$. Hence, for $\alpha > 0$,

$$\mathbb{P} \left[ L(\Sigma_{1/2}h_1) \geq (1 + \alpha)= \mathbb{E} L(\Sigma_{1/2}h_1) \right] \geq \mathbb{P} \left[ \sup_{0 \leq s \leq 1} w(s) \geq \frac{1 + \alpha}{2} \lambda_{\Sigma}^{-1/2} \mathbb{E} L(\Sigma_{1/2}h_1) \right]$$

$$\geq \mathbb{P} \left[ \sup_{0 \leq s \leq 1} w(s) \geq 2(1 + \alpha)\sqrt{2} \right],$$

using the fact that $\lambda_{\Sigma} \geq \frac{1}{2} \text{tr} \Sigma$ and the upper bound in (4.4). Applying (5.30) to $X = L(\Sigma_{1/2}h_1) \geq 0$ gives, for $\alpha > 0$,

$$\text{Var}L(\Sigma_{1/2}h_1) \geq \alpha^2 (\mathbb{E} L(\Sigma_{1/2}h_1))^2 \mathbb{P} \left[ \sup_{0 \leq s \leq 1} w(s) \geq 2(1 + \alpha)\sqrt{2} \right]$$

$$\geq \frac{32}{\pi} \alpha^2 (\text{tr} \Sigma) \left( 1 - \Phi(2(1 + \alpha)\sqrt{2}) \right),$$

using the lower bound in (4.4) and the fact that $\mathbb{P} [\sup_{0 \leq s \leq 1} w(s) \geq r] = 2\mathbb{P} [w(1) \geq r] = 2(1 - \Phi(r))$ for $r > 0$, which is a consequence of the reflection principle.
Numerical curve sketching suggests that $\alpha = 1/5$ is close to optimal; this choice of $\alpha$ gives, using (5.29),

\[
\text{Var} \mathcal{L}(\Sigma^{1/2} h_1) \geq \frac{32}{25\pi} (\text{tr} \Sigma) \left(1 - \Phi(12\sqrt{2}/5)\right) \geq \frac{263}{1080} \pi^{-3/2} (\text{tr} \Sigma) \exp \left\{ -\frac{144}{25} \right\},
\]

which is the lower bound in (5.27). We get a sharper result when $\Sigma = I$ and $\mathcal{L}(h_1) = \ell_1$, since we know $\mathbb{E} \ell_1 = \sqrt{8\pi}$ explicitly. Then, similarly to above, we get

\[
\text{Var} \ell_1 \geq 8\pi \alpha^2 \mathbb{P} \left[ \sup_{0 \leq s \leq 1} w(s) \geq (1 + \alpha) \sqrt{2\pi} \right], \quad \text{for } \alpha > 0,
\]

which at $\alpha = 1/4$ yields the stated lower bound. \qed
Chapter 6

Results on area of the convex hull

6.1 Overview

The aims of the present chapter are to provide first and second-order information for $A_n$ in both the cases $\mu = 0$ and $\mu \neq 0$. We start by some simulations. We considered the same form of random walk as in Section 5.1.

For the expected area, the simulations (see Figure 6.1) are consistent with Theorem 6.8 and Theorem 6.9. In the case of $\mu = 0$, Theorem 6.8 implies:

$$\lim_{n \to \infty} n^{-1} E A_n = \frac{\pi}{2} \sqrt{\det \Sigma} = 0.785.$$  

In the case of $\mu \neq 0$, Theorem 6.9 takes the form:

$$\lim_{n \to \infty} n^{-3/2} E A_n = \frac{1}{3} \|\mu\| \sqrt{2\pi \sigma_{\mu_\perp}^2} = 0.236 \text{ or } 0.425.$$  

For the variance of area, Proposition 6.12 and 6.13 show that the limits for variance exist in both zero and non-zero drift cases. For example, we will show that

$$\lim_{n \to \infty} n^{-3} \text{Var} A_n = v_+ \|\mu\|^2 \sigma_{\mu_\perp}^2;$$

$$\lim_{n \to \infty} n^{-2} \text{Var} A_n = v_0 \det \Sigma,$$  

where $v_0$ and $v_+$ are finite and positive, and these quantities are in fact variances associated with convex hulls of Brownian scaling limits for the walk. These scaling limits provide the basis of the analysis in this chapter; the methods are necessarily quite different from those in [63]. For the constants $v_0$ and $v_+$, Table 6.1 gives numerical evaluations of rigorous bounds that we prove in Proposition 6.14 below, plus estimates from simulations. The variance limits we deduced in the simulations (see Figure 6.2) are indeed lie in the variance bounds given by Proposition 6.14.
Figure 6.1: Plots of $y = \mathbb{E}[A_n]$ estimates against $x = (\text{left to right}) n, n^{3/2}, n^{3/2}$ for about 25 values of $n$ in the range $10^2$ to $2.5 \times 10^5$ for 3 examples with $\|\mu\| = (\text{left to right}) 0, 0.4, 0.72$. Each point is estimated from $10^3$ repeated simulations. Also plotted are straight lines $y = 0.781x$ (leftmost plot), $y = 0.236x$ (middle plot) and $y = 0.425x$ (rightmost plot).

|       | lower bound | simulation estimate | upper bound |
|-------|-------------|---------------------|-------------|
| $v_0$ | $8.15 \times 10^{-7}$ | 0.30                | 5.22        |
| $v_+$ | $1.44 \times 10^{-6}$ | 0.019               | 2.08        |

Table 6.1: Each of the simulation estimates is based on $10^5$ instances of a walk of length $n = 10^5$. The final decimal digit in each of the numerical upper (lower) bounds has been rounded up (down).

6.2 Upper bound for the expected value and variance for the area

Proposition 6.1. Let $p \geq 1$. Suppose that $\mathbb{E}[\|Z_1\|^{2p}] < \infty$.

(i) We have $\mathbb{E}[A_n^p] = O(n^{3p/2})$. Suppose in addition $\mathbb{E}(\|Z_1\|^{4p}) < \infty$, then $\text{Var}(A_n^p) = O(n^{3p})$.

(ii) Moreover, if $\mu = 0$ we have $\mathbb{E}[A_n^p] = O(n^p)$. Suppose in addition $\mathbb{E}(\|Z_1\|^{4p}) < \infty$, then $\text{Var}(A_n^p) = O(n^{2p})$.

Proof. For part (i), it suffices to suppose $\mu \neq 0$. Then, bounding the convex hull
by a rectangle,

\[ A_n \leq \left( \max_{0 \leq m \leq n} S_m \cdot \hat{\mu} - \min_{0 \leq m \leq n} S_m \cdot \hat{\mu} \right) \left( \max_{0 \leq m \leq n} S_m \cdot \hat{\mu}_\perp - \min_{0 \leq m \leq n} S_m \cdot \hat{\mu}_\perp \right) \]

\[ \leq 4 \left( \max_{0 \leq m \leq n} |S_m \cdot \hat{\mu}| \right) \left( \max_{0 \leq m \leq n} |S_m \cdot \hat{\mu}_\perp| \right). \]

Hence, by the Cauchy–Schwarz inequality, we have

\[ \mathbb{E}[A_n^p] \leq 4^p \left( \mathbb{E} \left[ \max_{0 \leq m \leq n} |S_m \cdot \hat{\mu}|^2 \right] \right)^{1/2} \left( \mathbb{E} \left[ \max_{0 \leq m \leq n} |S_m \cdot \hat{\mu}_\perp|^2 \right] \right)^{1/2}. \]

Now an application of Proposition 4.8(i) and (iii) gives \( \mathbb{E}[A_n^p] = O(n^{3p/2}). \)

Suppose in addition \( \mathbb{E}(\|Z_1\|^{4p}) < \infty. \) By the same process as above, we have

\[ A_{n}^{2p} \leq 4^{2p} \left( \max_{0 \leq m \leq n} |S_m \cdot \hat{\mu}|^2 \right) \left( \max_{0 \leq m \leq n} |S_m \cdot \hat{\mu}_\perp|^2 \right), \]

and \( \mathbb{E}(A_{n}^{2p}) = O(n^{3p}). \) Hence, \( \text{Var}(A_{n}^{p}) = \mathbb{E}(A_{n}^{2p}) - (\mathbb{E} A_{n}^{p})^2 = O(n^{3p}). \)

For part (ii), \( \mu = 0. \) Since the convex hull\((S_0, \ldots, S_n)\) is contained in the disk of radius \( \max_{0 \leq m \leq n} \|S_m\| \) and centre \( 0, \) \( A_{n}^{p} \leq \pi^p(\max_{0 \leq m \leq n} \|S_m\|^{2p}) \) a.s. Proposition 4.8(ii) then yields \( \mathbb{E}[A_{n}^{p}] = O(n^p). \)

Suppose in addition \( \mathbb{E}(\|Z_1\|^{4p}) < \infty. \) By the same process as above, we have \( \mathbb{E}[A_{n}^{2p}] = O(n^{2p}). \) Therefore, \( \text{Var}(A_{n}^{p}) = O(n^{2p}). \)

\[ \square \]

**Remark 6.1.** We will show below in Theorem 6.9 \( n^{-3/2} \mathbb{E} A_n \) has a limit in the non-zero drift case and, in Proposition 6.8 \( n^{-1} \mathbb{E} A_n \) has a limit in the zero drift case.
6.3 Asymptotics for the expected area

Let \( T(u, v) \) \((u, v \in \mathbb{R}^2)\) be the area of a triangle with sides of \( u, v \) and \( u + v \). Then,
\[
T(u, v) = \frac{1}{2} \sqrt{\|u\|^2\|v\|^2 - (u \cdot v)^2}.
\]

For \( \alpha, \beta > 0 \), \( T(\alpha u, \beta v) = \alpha \beta T(u, v) \).

**Lemma 6.2.** Suppose \( \mathbb{E}(\|Z_1\|^2) < \infty \), \( \mathbb{E}Z_1 = 0 \) and \( \mathbb{E}(Z_1^T Z_1) = \Sigma \). Then as \( m \to \infty \) and \( (k - m) \to \infty \),
\[
\frac{\mathbb{E} T(S_m, S_k - S_m)}{\sqrt{m(k - m)}} \to \mathbb{E} T(Y_1, Y_2),
\]
where \( Y_1, Y_2 \) are iid. rvs. \( Y_1, Y_2 \sim \mathcal{N}(0, \Sigma) \).

**Proof.** By Central Limit Theorem in \( \mathbb{R}^2 \) (see [17]), \( n^{-1/2} S_n \overset{d}{\to} \mathcal{N}(0, \Sigma) \). Since \( S_m \) and \( S_k - S_m \) are independent, as \( m \) and \( k - m \to \infty \),
\[
\left( \frac{S_m}{\sqrt{m}}, \frac{S_k - S_m}{\sqrt{k - m}} \right) \overset{d}{\to} T(Y_1, Y_2).
\]

Using the fact \( T \) is continuous,
\[
\frac{T(S_m, S_k - S_m)}{\sqrt{m(k - m)}} \overset{d}{=} T\left( \frac{S_m}{\sqrt{m}}, \frac{S_k - S_m}{\sqrt{k - m}} \right) \to T(Y_1, Y_2).
\]

Also, by Lemma 4.6,
\[
\mathbb{E} \left( \left[ \frac{\mathbb{E} T(S_m, S_k - S_m)}{\sqrt{m(k - m)}} \right]^2 \right) \leq \frac{\mathbb{E}(\|S_m\|^2\|S_k - S_m\|^2)}{m(k - m)}
\]
\[
\leq \frac{\mathbb{E}\|S_m\|^2}{m} \cdot \frac{\mathbb{E}\|S_k - S_m\|^2}{k - m} < \infty.
\]

That means \( m^{-1/2}(k - m)^{-1/2}T(S_m, S_k - S_m) \) is uniformly integrable over \((m, k)\) with \( m \geq 1, k \geq m + 1 \). So the result follows.

We state the following result without proof. It is a higher dimensional analogue of S–W formula (1.1). See Barndorff-Nielson and Baxter [9] for the proof.

**Lemma 6.3** (Barndorff Nielsen & Baxter).
\[
\mathbb{E}(A_n) = \sum_{k=2}^{n} \sum_{m=1}^{k-1} \frac{\mathbb{E}[T(S_m, S_k - S_m)]}{m(k - m)}.
\]
Lemma 6.4.

\[
\lim_{k \to \infty} \sum_{m=1}^{k-1} \frac{1}{m^{1/2}(k-m)^{1/2}} = \pi.
\]

Proof. Let \( f(m,k) = m^{-1/2}(k-m)^{-1/2} \). For any \( \delta \in (0,1) \), we have \( f(m,k) \leq f(m-\delta,k) \) if \( m \leq k/2 \) and \( f(m,k) \geq f(m-\delta,k) \) if \( m \geq k/2 \). Consider the sum as two parts,

\[
\sum_{m=1}^{k-1} f(m,k) = \left( \sum_{m=1}^{\lfloor k/2 \rfloor} + \sum_{m=\lfloor k/2 \rfloor + 1}^{k-1} \right) f(m,k).
\]

Then,

\[
\sum_{m=1}^{k-1} f(m,k) \geq \int_1^{\lfloor k/2 \rfloor} f(m,k) \, dm + \int_{\lfloor k/2 \rfloor + 1}^{k-1} f(m-1,k) \, dm,
\]

by letting \( u = \frac{m}{k} \) and \( v = \frac{m-1}{k} \),

\[
= \int_{1/k}^{\lfloor \frac{k}{2} \rfloor / k} \frac{1}{\sqrt{u(1-u)}} \, du + \int_{\lfloor \frac{k}{2} \rfloor / k}^{1-\frac{2}{k}} \frac{1}{\sqrt{v(1-v)}} \, dv \\
= \int_{1-2/k}^{1} \frac{1}{\sqrt{u(1-u)}} \, du.
\]

Also,

\[
\sum_{m=1}^{k-1} f(m,k) \leq \int_1^{\lfloor k/2 \rfloor} f(m-1,k) \, dm + \int_{\lfloor k/2 \rfloor + 1}^{k-1} f(m,k) \, dm
\]

\[
= \int_0^{\lfloor \frac{k}{2} \rfloor / k} \frac{1}{\sqrt{u(1-u)}} \, du + \int_{\lfloor \frac{k}{2} \rfloor / k + 1 / k}^{1-\frac{1}{k}} \frac{1}{\sqrt{v(1-v)}} \, dv \\
\leq \int_0^{1-1/k} \frac{1}{\sqrt{u(1-u)}} \, du.
\]

Therefore,

\[
\lim_{k \to \infty} \sum_{m=1}^{k-1} f(m,k) = \int_0^{1} (u(1-u))^{-1/2} \, du = B\left(\frac{1}{2}, \frac{1}{2}\right) = \Gamma\left(\frac{1}{2}\right)^2 = \pi, \quad (6.3)
\]

where \( B(\cdot, \cdot) \) is the Beta function and \( \Gamma(\cdot) \) is the Gamma function.

Lemma 6.5.

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=2}^{n} \sum_{m=1}^{k-1} \frac{1}{m^{1/2}(k-m)^{1/2}} = \pi.
\]
Proof. The result follows from Lemma 2.22 and Lemma 6.4.

**Proposition 6.6.** Suppose $\mathbb{E} (\|Z\|^2) < \infty$ and $\mu = 0$. Then,

$$\lim_{n \to \infty} \mathbb{E} \frac{A_n}{n} = \pi \mathbb{E} T(Y_1, Y_2),$$

where $Y_1, Y_2$ are iid. rvs. $Y_1, Y_2 \sim \mathcal{N}(0, \Sigma)$ and $\Sigma = \mathbb{E} (Z_1^T Z_1)$.

Proof. In (6.2), denote $g(k, m) := m^{-1/2}(k - m)^{-1/2}\mathbb{E} [T(S_m, S_k - S_m)]$. Then,

$$\mathbb{E} A_n = \sum_{k=2}^{n} \sum_{m=1}^{k-1} \frac{g(k, m)}{m^{1/2}(k - m)^{1/2}}$$

(6.4)

and by Lemma 6.2,

$$\lim_{m \to \infty, k \to \infty} g(k, m) = \mathbb{E} T(Y_1, Y_2) := \lambda.$$ (6.5)

So, for every $\varepsilon > 0$, there exists $m_0 \in \mathbb{Z}^+$ such that for any $m \geq m_0$ and $k-m \geq m_0$ we have $|g(k, m) - \lambda| \leq \varepsilon$.

For the upper bound of $\mathbb{E} A_n$, separate the inner sum as

$$\mathbb{E} A_n = \left( \sum_{k=2}^{m_0} + \sum_{k=m_0+1}^{n} \right) \sum_{m=1}^{k-1} \frac{g(k, m)}{m^{1/2}(k - m)^{1/2}}$$

$$= \sum_{k=m_0+1}^{n} \sum_{m=1}^{k-1} \frac{g(k, m)}{m^{1/2}(k - m)^{1/2}} + O(1)$$

$$= \sum_{k=m_0+1}^{n} \left( \sum_{m=1}^{m_0} + \sum_{m=m_0+1}^{k-1} + \sum_{m=m_0+1}^{k-m_0-1} \right) \frac{g(k, m)}{m^{1/2}(k - m)^{1/2}} + O(1),$$

where

$$\sum_{k=m_0+1}^{n} \left( \sum_{m=1}^{m_0} + \sum_{m=m_0+1}^{k-1} \right) \frac{g(k, m)}{m^{1/2}(k - m)^{1/2}} \leq m_0 \sum_{k=m_0+1}^{n} \frac{\max_1 \leq m \leq m_0 g(k, m)}{(k - m_0)^{1/2}} + m_0 \sum_{k=m_0+1}^{n} \frac{\max_{k-m_0 \leq m \leq k} g(k, m)}{(k - m_0)^{1/2}}$$

$$\leq \lambda \sum_{k=m_0+1}^{n} \frac{2m_0}{(k - m_0)^{1/2}}, \quad \text{since } \max_{1 \leq k, m \leq n} g(k, m) < \infty,$$

$$\leq O(n^{1/2}), \quad (6.6)$$
where \( \lambda' \) is some constant, and
\[
\sum_{k=m_0+1}^{n} \sum_{m=m_0+1}^{k-m_0-1} \frac{g(k, m)}{m^{1/2}(k-m)^{1/2}} \leq (\lambda + \varepsilon) \sum_{k=2}^{n} \sum_{m=1}^{k-1} \frac{1}{m^{1/2}(k-m)^{1/2}}.
\]

By Lemma 6.5,
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{k=m_0+1}^{n} \sum_{m=m_0+1}^{k-m_0-1} \frac{g(k, m)}{m^{1/2}(k-m)^{1/2}} \leq (\lambda + \varepsilon) \pi.
\]

Hence, \( \limsup_{n \to \infty} n^{-1} \mathbb{E} A_n \leq (\lambda + \varepsilon) \pi \) by (6.6). So \( \limsup_{n \to \infty} n^{-1} \mathbb{E} A_n \leq \lambda \pi \), since \( \varepsilon > 0 \) was arbitrary.

For the lower bound
\[
\mathbb{E} A_n \geq \sum_{k=2}^{n} \sum_{m=m_0}^{k-m_0} \frac{g(k, m)}{m^{1/2}(k-m)^{1/2}}
\]
\[
\geq (\lambda - \varepsilon) \sum_{k=2}^{n} \sum_{m=m_0}^{k-m_0} \frac{1}{m^{1/2}(k-m)^{1/2}}
\]
\[
\geq (\lambda - \varepsilon) \sum_{k=2}^{n} \left( \sum_{m=1}^{k-1} - \sum_{m=1}^{m_0-1} - \sum_{m=m_0+1}^{k-1} \right) \frac{1}{m^{1/2}(k-m)^{1/2}}
\]
\[
\geq (\lambda - \varepsilon) \sum_{k=2}^{n} \frac{1}{m_0^2(k-m_0)^{1/2}} - (\lambda - \varepsilon) \sum_{k=2}^{n} \frac{2(m_0 - 1)}{(k-1)^{1/2}}.
\]

By Lemma 6.5, \( \liminf_{n \to \infty} n^{-1} \mathbb{E} A_n \geq (\lambda - \varepsilon) \pi \). Therefore \( \liminf_{n \to \infty} n^{-1} \mathbb{E} A_n \geq \lambda \pi \), since \( \varepsilon > 0 \) was arbitrary. Then the result follows.

**Lemma 6.7.** If \( Y_1, Y_2 \) are iid. rvs. \( Y_1, Y_2 \sim \mathcal{N}(0, \Sigma) \) and \( \Sigma = \mathbb{E} (Z_1^T Z_1) \) Then,
\[
\mathbb{E} T(Y_1, Y_2) = \frac{1}{2} \sqrt{\det \Sigma}.
\]

**Proof.** With \( \Sigma = (\Sigma^{1/2})^2 \), we have that \( (Y_1, Y_2) \) is equal in distribution to \( (\Sigma^{1/2} W_1, \Sigma^{1/2} W_2) \) where \( W_1 \) and \( W_2 \) are independent \( \mathcal{N}(0, I) \) random vectors. Since \( \Sigma^{1/2} \) acts as a linear transformation on \( \mathbb{R}^2 \) with Jacobian \( \sqrt{\det \Sigma} \),
\[
\mathbb{E} T(Y_1, Y_2) = \mathbb{E} T(\Sigma^{1/2} W_1, \Sigma^{1/2} W_2) = \sqrt{\det \Sigma} \mathbb{E} T(W_1, W_2).
\]

Here
\[
\mathbb{E} T(W_1, W_2) = \frac{1}{2} \mathbb{E} [\|W_1\| \|W_2\| \sin \Theta],
\]
where the minimum angle $\Theta$ between $W_1$ and $W_2$ is uniform on $[0, \pi]$, and 
$(\|W_1\|, \|W_2\|, \Theta)$ are independent. Hence

$$
\mathbb{E} T(W_1, W_2) = \frac{1}{2} (\mathbb{E} \|W_1\|)^2 (\mathbb{E} \sin \Theta) = \frac{1}{2},
$$

using the fact that $\mathbb{E} \sin \Theta = \frac{2}{\pi}$ and $\|W_1\|$ is the square-root of a $\chi_2^2$ random variable, so $\mathbb{E} \|W_1\| = \sqrt{\frac{\pi}{2}}$ and the result follows.

**Theorem 6.8.** Suppose that $\mathbb{E} \|Z_1\|^2 < \infty$ and $\mu = 0$. Then,

$$
\lim_{n \to \infty} n^{-1} \mathbb{E} A_n = \frac{\pi}{2} \sqrt{\det \Sigma}.
$$

**Proof.** The result follows from Proposition 6.6 combining with Lemma 6.7.

**Theorem 6.9.** Suppose that $\{M_p\}$ holds for some $p > 2$, $\mu \neq 0$, and $\sigma_{\mu_\perp}^2 > 0$. Then

$$
\lim_{n \to \infty} n^{-3/2} \mathbb{E} A_n = \|\mu\| (\sigma_{\mu_\perp}^2)^{1/2} \mathbb{E} \tilde{a}_1 = \frac{1}{3} \|\mu\| \sqrt{2\pi \sigma_{\mu_\perp}^2}.
$$

In particular, $\mathbb{E} \tilde{a}_1 = \frac{1}{3} \sqrt{2\pi}$.

**Proof.** Recall that $\tilde{a}_1 = A(h_1)$ is the convex hull area of the space-time diagram of one-dimensional Brownian motion run for unit time.

Given $\mathbb{E} [\|Z_1\|^p] < \infty$ for some $p > 2$, Proposition 6.1(i) shows that $\mathbb{E} [A_n^{p/2}] = O(n^{3p/4})$, so that $\mathbb{E} [(n^{-3/2} A_n)^{p/2}]$ is uniformly bounded. Hence $n^{-3/2} A_n$ is uniformly integrable, so Corollary 3.9 implies that

$$
\lim_{n \to \infty} n^{-3/2} \mathbb{E} A_n = \|\mu\| (\sigma_{\mu_\perp}^2)^{1/2} \mathbb{E} \tilde{a}_1.
$$

(6.7)

In light of (6.7), it remains to identify $\mathbb{E} \tilde{a}_1 = \frac{1}{3} \sqrt{2\pi}$. It does not seem straightforward to work directly with the Brownian limit; it turns out again to be simpler to work with a suitable random walk. We choose a walk that is particularly convenient for computations.

Let $\xi \sim \mathcal{N}(0, 1)$ be a standard normal random variable, and take $Z$ to be distributed as $Z = (1, \xi)$ in Cartesian coordinates. Then $S_n = (n, \sum_{k=1}^n \xi_k)$ is the space-time diagram of the symmetric random walk on $\mathbb{R}$ generated by i.i.d. copies $\xi_1, \xi_2, \ldots$ of $\xi$.

For $Z = (1, \xi)$, $\mu = (1, 0)$ and $\sigma^2 = \sigma_{\mu_\perp}^2 = \mathbb{E} [\xi^2] = 1$. Thus by (6.7), to complete the proof of Theorem 6.9 it suffices to show that for this walk $\lim_{n \to \infty} n^{-3/2} \mathbb{E} A_n =$
If $u, v \in \mathbb{R}^2$ have Cartesian components $u = (u_1, u_2)$ and $v = (v_1, v_2)$, then we may write $T(u, v) = \frac{1}{2} |u_1 v_2 - v_1 u_2|$. Hence

$$T(S_m, S_k - S_m) = \frac{1}{2} \left| (k - m) \sum_{j=1}^{m} \xi_j - m \sum_{j=m+1}^{k} \xi_j \right|.$$ 

By properties of the normal distribution, the right-hand side of the last display has the same distribution as $\frac{1}{2} |\xi| \sqrt{km(k-m)}$. Hence

$$\frac{\mathbb{E}(T(S_m, S_k - S_m))}{\sqrt{m(k-m)}} = \frac{1}{2} \mathbb{E} |\xi| \sqrt{k} = \frac{1}{2} \sqrt{2k/\pi},$$

using the fact that $|\xi|$ is distributed as the square-root of a $\chi^2_1$ random variable, so $\mathbb{E} |\xi| = \sqrt{2/\pi}$. Hence, by (6.4), this random walk enjoys the exact formula

$$\mathbb{E} A_n = \frac{1}{\sqrt{2\pi}} \sum_{k=2}^{n} \sum_{m=1}^{k-1} \frac{\sqrt{k}}{\sqrt{m(k-m)}}.$$ 

Then from (6.3) we obtain $\mathbb{E} A_n \sim \sqrt{\pi/2} \sum_{k=2}^{n} k^{1/2}$, which gives the result. \hfill \Box

**Remark 6.2.** The idea used in the proof of Theorem 6.9, first establishing the existence of a limit for a class of models and then choosing a particular model for which the limit can be conveniently evaluated, goes back at least to Kac; see [34, p. 293].

### 6.4 Law of large numbers for the area

**Proposition 6.10.** Suppose $\mathbb{E}(\|Z_1\|^{4}) < \infty$ and $\|\mathbb{E} Z_1\| = 0$. Then for any $\alpha > 1$, $n^{-\alpha} A_n \to 0$ a.s. as $n \to \infty$.

**Proof.** By Chebyshev’s inequality for $A_n$,

$$\mathbb{P} \left( \frac{|A_n - \mathbb{E} A_n|}{n^\alpha} \geq \varepsilon \right) = \mathbb{P}(|A_n - \mathbb{E} A_n| \geq \varepsilon n^\alpha) \leq \frac{\text{Var}(A_n)}{\varepsilon^2 n^{2\alpha}}.$$ 

Since $\text{Var}(A_n) = O(n^2)$ by Proposition 6.1(ii), for any $\alpha > 1$, as $n \to \infty$ we have

$$\mathbb{P} \left( \frac{|A_n - \mathbb{E} A_n|}{n^\alpha} \geq \varepsilon \right) = O(n^{2-2\alpha}).$$

So $n^{-\alpha} (A_n - \mathbb{E} A_n) \to 0$ in probability.
Take \( n = n_k = 2^k \) for \( k \in \mathbb{N} \), we have
\[
P \left( \left| A_{n_k} - \mathbb{E} A_{n_k} \right| \geq \varepsilon \right) = O(n_k^{2-2\alpha}) = O(4^k(1-\alpha)).
\]
So for any \( \varepsilon > 0 \),
\[
\sum_{k=1}^{\infty} P \left( \left| A_{n_k} - \mathbb{E} A_{n_k} \right| \geq \varepsilon n_k^\alpha \right) < \infty.
\]
By Borel–Cantelli Lemma (Lemma 2.3), as \( k \to \infty \)
\[
\frac{A_{n_k} - \mathbb{E} A_{n_k}}{n_k^\alpha} \to 0 \quad \text{a.s.}
\]
By Proposition 6.1(ii), \( n_k^{-\alpha} \mathbb{E} A_{n_k} \to 0 \) as \( n \to \infty \), we get
\[
\frac{A_{n_k}}{n_k^\alpha} \to 0 \quad \text{a.s. as } k \to \infty.
\]
For any \( n \in \mathbb{N} \), there exists \( k(n) \in \mathbb{N} \) such that \( 2^{k(n)} \leq n < 2^{k(n)+1} \). By monotonicity of \( A_n \),
\[
2^{-\alpha} A_{n(n)} = \frac{A_{2^{k(n)}}}{(2^{k(n)+1})^\alpha} \leq \frac{A_n}{n^\alpha} \leq \frac{A_{2^{k(n)+1}}}{(2^{k(n)})^\alpha} = 2^\alpha \frac{A_{2^{k(n)+1}}}{n_k^{(k(n)+1)}}.
\]
The result follows by the Squeezing Theorem. \( \square \)

**Proposition 6.11.** Suppose \( \mathbb{E} (\|Z_1\|^4) < \infty \). Then, for any \( \alpha > 3/2 \), \( n^{-\alpha} A_n \to 0 \) a.s. as \( n \to \infty \).

**Proof.** By Chebyshev’s inequality for \( A_n \),
\[
P \left( \left| A_n - \mathbb{E} A_n \right| \geq \varepsilon \right) = P \left( \left| A_n - \mathbb{E} A_n \right| \geq \varepsilon n^\alpha \right) \leq \frac{\text{Var}(A_n)}{\varepsilon^2 n^{2\alpha}}.
\]
Since \( \text{Var}(A_n) = O(n^3) \) by Proposition 6.1(i), for any \( \alpha > 3/2 \), as \( n \to \infty \) we have
\[
P \left( \left| A_n - \mathbb{E} A_n \right| \geq \varepsilon \right) = O(n^{3-2\alpha}).
\]
So \( n^{-\alpha}(A_n - \mathbb{E} A_n) \to 0 \) in probability.

Take \( n = n_k = 2^k \) for \( k \in \mathbb{N} \), we have
\[
P \left( \left| A_{n_k} - \mathbb{E} A_{n_k} \right| \geq \varepsilon \right) = O(n_k^{3-2\alpha}) = O(4^k(3/2-\alpha)).
\]
So for any $\varepsilon > 0$,
\[
\sum_{k=1}^{\infty} \mathbb{P}\left( \left| \frac{A_{n_k} - \mathbb{E} A_{n_k}}{n_k^\alpha} \right| \geq \varepsilon \right) < \infty.
\]
By Borel–Cantelli Lemma (Lemma 2.3), as $k \to \infty$
\[
\frac{A_{n_k} - \mathbb{E} A_{n_k}}{n_k^\alpha} \to 0 \text{ a.s.}
\]
By Proposition 6.1(i), $n_k^{-\alpha} \mathbb{E} A_{n_k} \to 0$ as $n \to \infty$, we get
\[
\frac{A_{n_k}}{n_k^\alpha} \to 0 \text{ a.s. as } k \to \infty.
\]
For any $n \in \mathbb{N}$, there exists $k(n) \in \mathbb{N}$ such that $2^{k(n)} \leq n < 2^{k(n)+1}$. By monotonicty of $A_n$,
\[
2^{-\alpha} \frac{A_{2^{k(n)}}}{n_{k(n)}^\alpha} \leq \frac{A_n}{n^\alpha} \leq \frac{A_{2^{k(n)+1}}}{n_{k(n)+1}^\alpha} = 2^\alpha \frac{A_{n_{k(n)+1}}}{n_{k(n)+1}^\alpha}.
\]
The result follows by the Squeezing Theorem.

\section{6.5 Asymptotics for the variance}

Recall that Proposition 5.15 shows $\lim_{n \to \infty} n^{-1} \text{Var} L_n = u_0(\Sigma)$. In this section, we will show that
\[
\begin{align*}
\text{if } \mu \neq 0 : & \quad \lim_{n \to \infty} n^{-3} \text{Var} A_n = v_+ \| \mu \|^2 \sigma^2_{\mu,1}; \\
\text{if } \mu = 0 : & \quad \lim_{n \to \infty} n^{-2} \text{Var} A_n = v_0 \det \Sigma.
\end{align*}
\]
(6.8)
The quantities $v_0$ and $v_+$ in (6.8) are finite and positive, as is $u_0(\cdot)$ provided $\sigma^2 \in (0, \infty)$, and these quantities are in fact variances associated with convex hulls of Brownian scaling limits for the walk.

\textbf{Proposition 6.12.} Suppose that (M$p$) holds for some $p > 4$, and $\mu = 0$. Then
\[
\lim_{n \to \infty} n^{-2} \text{Var} A_n = v_0 \det \Sigma.
\]
\textbf{Proof.} Lemma 6.1(ii) shows that $\mathbb{E}\left[ A_{n_k}^{2(p/4)} \right] = O(n^{p/2})$, so that $\mathbb{E}\left[ (n^{-2}A_n^2)^{p/4} \right]$ is uniformly bounded. Hence $n^{-2}A_n^2$ is uniformly integrable, and we deduce convergence of $n^{-2} \text{Var} A_n$ in Corollary 3.7. \qed
For the case with drift, we have the following variance result.

**Proposition 6.13.** Suppose that \((M_p)\) holds for some \(p > 4\) and \(\mu \neq 0\). Then
\[
\lim_{n \to \infty} n^{-3} \text{Var} A_n = v_+ \| \mu \|^2 \sigma^2_{\mu^\perp}.
\]

**Proof.** Given \(\mathbb{E} \| Z_1 \|^p < \infty\) for some \(p > 4\), Lemma 6.1(i) shows that \(\mathbb{E} [A_n^{2(p/4)}] = O(n^{3p/4})\), so that \(\mathbb{E} [(n^{-3} A_n^2)^{p/4}]\) is uniformly bounded. Hence \(n^{-3} A_n^2\) is uniformly integrable, so Corollary 3.9 yields the result.

### 6.6 Variance bounds

**Proposition 6.14.** We have \(u_0(\Sigma) = 0\) if and only if \(\text{tr} \Sigma = 0\). The following inequalities for the quantities defined at (5.26) hold.

\[
0 < \frac{4}{49} \left( e^{-7\pi^2/12} - \frac{1}{3} e^{-21\pi^2/4} \right)^2 \leq v_0 \leq 16(\log 2)^2 - \frac{\pi^2}{4}; \quad (6.9)
\]

\[
0 < \frac{2}{225} \left( e^{-25\pi/9} - \frac{1}{3} e^{-25\pi} \right) \leq v_+ \leq 4 \log 2 - \frac{2\pi}{9}. \quad (6.10)
\]

**Proof.** Bounding \(\tilde{a}_1\) by the area of a rectangle, we have
\[
\tilde{a}_1 \leq r_1 \leq 2 \sup_{0 \leq s \leq 1} |w(s)|, \quad \text{a.s.,} \quad (6.11)
\]
where \(r_1 := \sup_{0 \leq s \leq 1} w(s) - \inf_{0 \leq s \leq 1} w(s)\). A result of Feller states that \(\mathbb{E} [r_1^2] = 4 \log 2\). So by the first inequality in (6.11), we have \(\mathbb{E} [\tilde{a}_1^2] \leq 4 \log 2\), and by Theorem 6.9 we have \(\mathbb{E} \tilde{a}_1 = \frac{1}{3} \sqrt{2\pi}\); the upper bound in (6.10) follows.

Similarly, for any orthonormal basis \(\{e_1, e_2\}\) of \(\mathbb{R}^2\), we bound \(a_1\) by a rectangle
\[
a_1 \leq \left( \sup_{0 \leq s \leq 1} e_1 \cdot b(s) - \inf_{0 \leq s \leq 1} e_1 \cdot b(s) \right) \left( \sup_{0 \leq s \leq 1} e_2 \cdot b(s) - \inf_{0 \leq s \leq 1} e_2 \cdot b(s) \right),
\]
and the two (orthogonal) components are independent, so \(\mathbb{E} [a_1^2] \leq (\mathbb{E} [r_1^2])^2 = 16(\log 2)^2\), which with the fact that \(\mathbb{E} a_1 = \frac{\pi}{2}\) gives the upper bound in (6.9).

We now move on to the lower bounds. Tractable upper bounds for \(a_1\) and \(\tilde{a}_1\) are easier to come by than lower bounds, and thus we obtain a lower bound on the variance by showing the appropriate area has positive probability of being smaller than the corresponding mean.
Consider $a_1$; note $E a_1 = \pi/2$. Since, for any orthonormal basis $\{e_1, e_2\}$ of $\mathbb{R}^2$,

$$a_1 \leq \pi \sup_{0 \leq s \leq 1} \|b(s)\|^2 \leq \pi \sup_{0 \leq s \leq 1} \|e_1 \cdot b(s)\|^2 + \pi \sup_{0 \leq s \leq 1} \|e_2 \cdot b(s)\|^2,$$

using the fact that $e_1 \cdot b$ and $e_2 \cdot b$ are independent one-dimensional Brownian motions,

$$P[a_1 \leq r] \geq P \left[ \sup_{0 \leq s \leq 1} |w(s)|^2 \leq \frac{r}{2\pi} \right]^2, \text{ for } r > 0.$$  

We apply (5.30) with $X = a_1$ and $\alpha \in (0, 1)$, and set $r = (1 - \alpha)\frac{\pi}{2}$ to obtain

$$\text{Var } a_1 \geq \alpha^2 \frac{\pi^2}{4} P \left[ \sup_{0 \leq s \leq 1} |w(s)| \leq \frac{\sqrt{1 - \alpha}}{2} \right]^2 \geq 4\alpha^2 \left( \exp \left\{ -\frac{\pi^2}{2(1 - \alpha)} \right\} - \frac{1}{3} \exp \left\{ -\frac{9\pi^2}{2(1 - \alpha)} \right\} \right)^2,$$

by (5.28). Taking $\alpha = 1/7$ is close to optimal, and gives the lower bound in (6.9).

For $\tilde{a}_1$, we apply (5.30) with $X = \tilde{a}_1$ and $\alpha \in (0, 1)$. Using the fact that $E \tilde{a}_1 = \frac{1}{3}\sqrt{2\pi}$ (from Theorem 6.9) and the weaker of the two bounds in (6.11), we obtain

$$\text{Var } \tilde{a}_1 \geq \alpha^2 \frac{2\pi}{9} P \left[ \sup_{0 \leq s \leq 1} |w(s)| \leq \frac{(1 - \alpha)\sqrt{2\pi}}{6} \right] \geq \frac{8}{9} \alpha^2 \left( \exp \left\{ -\frac{9\pi}{4(1 - \alpha)^2} \right\} - \frac{1}{3} \exp \left\{ -\frac{81\pi}{4(1 - \alpha)^2} \right\} \right),$$

by (5.28). Taking $\alpha = 1/10$ is close to optimal, and gives the lower bound in (6.10).

Remark 6.3. The main interest of the lower bounds in Proposition 6.14 is that they are positive; they are certainly not sharp. The bounds can surely be improved. We note just the following idea. A lower bound for $\tilde{a}_1$ can be obtained by conditioning on $\theta := \sup\{s \in [0, 1] : w(s) = 0\}$ and using the fact that the maximum of $w$ up to time $\theta$ is distributed as the maximum of a scaled Brownian bridge; combining this with the previous argument improves the lower bound on $v_+$ to $2.09 \times 10^{-6}$. \qed
Chapter 7

Conclusions and open problems

7.1 Summary of the limit theorems

We summarize in general the asymptotic behaviour of the expectation and variance of \( L_n \) and \( A_n \) as the following table.

| \( \mu = 0 \) | \( \mu \neq 0 \) |
|----------------|----------------|
| \( L_n \) | \( L_n \) | \( n^{-1/2} \mathbb{E} L_n^\frac{5}{4} \) | \( n^{-1} \text{Var} L_n \) | non-Gaussian |
| \( A_n \) | \( A_n \) | \( n^{-1} \mathbb{E} A_n \) | \( n^{-2} \text{Var} A_n \) | non-Gaussian |
| \( n^{-3/2} \mathbb{E} A_n \) | \( n^{-3} \text{Var} A_n \) | Gaussian | non-Gaussian |

Table 7.1: Results originate from: \( \S \) [58]; \( \dagger \) [57]; \( \ddagger \) [63]; \( \ddagger \) [3] (in part); the rest are new. The limit laws exclude degenerate cases when associated variances vanish.

Table 7.2 collets the lower and upper bounds and simulation estimates for the constants defined at equation (5.26) and equation (6.8).

Claussen et al. [12] give some numerical estimations that \( \text{Var} l_1 \approx 1.075 \) and \( \text{Var} a_1 \approx 0.31 \), which is a good agreement with our limit estimations 1.08 and 0.30.
Table 7.2: Each of the simulation estimates is based on $10^5$ instances of a walk of length $n = 10^5$. The final decimal digit in each of the numerical upper (lower) bounds has been rounded up (down).

7.2 Exact evaluation of limiting variances

It would, of course, be of interest to evaluate any of $u_0$, $v_0$, or $v_+$ exactly. In general this looks hard. The paper [52] provides a key component to a possible approach to evaluating $u_0$. By Cauchy’s formula and Fubini’s theorem,

$$
\mathbb{E}[\ell_1^2] = \int_{S_1} \int_{S_1} \mathbb{E} \left[ \left( \sup_{0 \leq s \leq 1} (e_1 \cdot b(s)) \right) \left( \sup_{0 \leq t \leq 1} (e_2 \cdot b(t)) \right) \right] \, de_1 \, de_2.
$$

Here, the two standard one-dimensional Brownian motions $e_1 \cdot b$ and $e_2 \cdot b$ have correlation determined by the cosine of the angle $\phi$ between them, i.e.,

$$
\mathbb{E} \left[ (e_1 \cdot b(s))(e_2 \cdot b(t)) \right] = (s \wedge t) e_1 \cdot e_2 = (s \wedge t) \cos \phi.
$$

The result of Rogers and Shepp [52] then shows that

$$
\mathbb{E} \left[ \left( \sup_{0 \leq s \leq 1} (e_1 \cdot b(s)) \right) \left( \sup_{0 \leq t \leq 1} (e_2 \cdot b(t)) \right) \right] = c(\cos \phi),
$$

where the function $c$ is given explicitly in [52]. Using this result, we obtain

$$
\mathbb{E}[\ell_1^2] = 4\pi \int_{-\pi/2}^{\pi/2} c(\sin \theta) \, d\theta = 4\pi \int_{-\pi/2}^{\pi/2} \, d\theta \int_0^\infty \, du \cos \theta \frac{\cosh(u\theta)}{\sinh(u\pi/2)} \tanh \left( \frac{(2\theta + \pi)u}{4} \right).
$$

We have not been able to deal with this integral analytically, but numerical integration gives $\mathbb{E}[\ell_1^2] \approx 26.1677$, which with the fact that $\mathbb{E}\ell_1 = \sqrt{8\pi}$ gives $u_0(I) = \text{Var}\ell_1 \approx 1.0350$, in reasonable agreement with the simulation estimate in Table 6.1.

Another possible approach to evaluating $u_0$ is suggested by a remarkable computation of Goldman [27] for the analogue of $u_0(I) = \text{Var}\ell_1$ for the planar Brownian
bridge. Specifically, if $b_t'$ is the standard Brownian bridge in $\mathbb{R}^2$ with $b_0' = b_1' = 0$, and $\ell_1' = \mathcal{L}(\text{hull } b'[0,1])$ the perimeter length of its convex hull, [27, Théorème 7] states that

$$\text{Var}\ell_1' = \frac{\pi^2}{6} \left( 2\pi \int_0^\pi \frac{\sin \theta}{\theta} d\theta - 2 - 3\pi \right) \approx 0.34755. $$

### 7.3 Open problems

#### 7.3.1 Degenerate case for $L_n$ when $\mu \neq 0$ and $\sigma^2_\mu = 0$

Recall Remark 5.2(iii) for Theorem 5.13. For example, consider

$$Z_1 = \begin{cases} 
(1,1), & \text{with probability } 1/2; \\
(1,-1), & \text{with probability } 1/2.
\end{cases}$$

Then the $\sigma^2_\mu$ in Theorem 5.13 is zero and our results on the second-order properties of $L_n$ in Chapter 5 cannot be applied in this degenerate case. See Figure 7.1 for an example of random walk in this case.

![Figure 7.1: Example of the degenerate case with $n = 100$.](image)

For this example, we conjecture $\frac{\text{Var} L_n}{\log n} \to \text{constant}$, based on some simulations. See Figure 7.2 below.

A second open question is whether in this case $\frac{L_n - \mathbb{E} L_n}{\sqrt{\text{Var} L_n}}$ has a distributional limit. If so, is that limit normal? We conjecture that there is a limit, but it is not normal (see Figure 7.3).

#### 7.3.2 Heavy-tailed increments

All main results from previous chapters are based on the assumption $M_p$ for $p = 2$, that the second moments of increments are finite. But what happens in the heavy-tail problems, in which $\mathbb{E} (\|Z_1\|^2) = \infty$? We give two simulation examples.
Figure 7.2: Simulation for the degenerate case $\text{Var}L_n = 0.6612 \log(n)$.

Figure 7.3: Simulations for the degenerate case.

### 7.3.3 Centre-of-mass process

We can associate to a random walk trajectory $S_0, S_1, S_2, \ldots$ its \textit{centre-of-mass} process $G_0, G_1, G_2, \ldots$ defined by $G_0 := S_0 = 0$ and for $n \geq 1$ by $G_n = \frac{1}{n} \sum_{k=1}^{n} S_k$. By convexity, the convex hull of $\{G_0, G_1, \ldots, G_n\}$ is contained in the convex hull
of \( \{S_0, S_1, \ldots, S_n\} \). What can one say about its perimeter length or area? Note that one may express \( G_n \) as a weighted sum of the increments of the walk as

\[
G_n = \sum_{k=1}^{n} \left( \frac{n - k + 1}{n} \right) Z_k.
\]

Then, for example, we expect that the method of Section 5.4 carries through to this case; this is one direction for future work.

### 7.3.4 Higher dimensions

Most of the analysis of \( L_n \) in this thesis is restricted to \( d = 2 \) because we rely on the Cauchy formula for planar convex sets. In higher dimensions, the analogues of \( L_n \) and \( A_n \) are the intrinsic volumes of the convex body. Analogues of Cauchy’s formula are available, but these seem more difficult to use as the basis for analysis.

However, the scaling limit theories in Chapter 3 may have some relatively straightforward corollaries in higher dimensions. So, some analogous results for \( A_n \) in Chapter 6 may not be so difficult to figure out.
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