Generalized information theoretic measure to discern the quantumness of correlations

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A novel measure, quantumness of correlations, $Q_{AB}$, is introduced here for bipartite states, by incorporating the required measurement scheme crucial in defining any such quantity. Quantumness coincides with the previously proposed measures in special cases and it vanishes for separable states, a feature not captured by the measures proposed earlier. It is found that an optimal generalized measurement on one of the parts leaves the overall state in its closest separable form, which shares the same marginal for the other part, implying that $Q_{AB}$ is non-zero for all entangled bipartite states and it serves as an upper bound to the relative entropy of entanglement.

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In classical probability theory, two random variables $A$ and $B$ are said to be correlated if their probability distribution, $P(a,b)$ cannot be expressed as a mere product of the marginal probabilities $P(a)$ and $P(b)$. There are several equivalent quantitative measures of testing these classical correlations (CC). In the quantum description, probability distributions are replaced by density operators and the statement, a bipartite density matrix is correlated, if it cannot be expressed in a simple product form of its constituent density matrices provides a natural extension of the idea of correlations. Several counter-intuitive features raise their heads due to quantum correlations (QC) exhibited by subsystems of a composite state and have led to philosophical debates in the conceptual understanding of quantum theory, following Einstein-Podolsky-Rosen criticism. The rapid progress of quantum information science has stimulated intense activity recently on the characterization of genuine QC. It may be emphasized here that the notion of correlation per se does not set a borderline between classical or quantum descriptions. Following Werner, a bipartite density matrix $\hat{\rho}_{AB}$ is classically correlated or separable if it admits a convex combination of product states as

$$\hat{\rho}_{AB}^{\text{(sep)}} = \sum_i p_i \hat{\rho}_A^{(i)} \otimes \hat{\rho}_B^{(i)}, \quad 0 \leq p_i \leq 1, \quad \sum_i p_i = 1. \quad (1)$$

This provides a generalization of the simplest uncorrelated state, $\hat{\rho}_{AB} = \hat{\rho}_A \otimes \hat{\rho}_B$, and forms a basis for the characterization and quantification of QC that exist over and above the classical ones. However, several research groups claim that separability (absence of quantum entanglement) does not necessarily imply CC. In particular, Ollivier and Zurek [OZ] argued that any measure of QC must be based on the fact that it is impossible, through measurement on one part of a composite state, to ascribe independent reality to the other part and hence, a measurement scheme, which verifies this basic feature, forms an essential ingredient of this characterization. A bipartite state is classically correlated, if there exists an optimal measurement on a part of the correlated system, such that the overall state remains unperturbed. It has been found that only a subset of separable states show such insensitivity to measurements performed on one of their subsystems and only this subset of separable states is qualified to be classically correlated. However, OZ employed perfect orthogonal projective measurements on one of the subsystem to quantify QC via quantum discord, a measure proposed by them. Henderson and Vedral considered general measurements through local positive operator valued measures (POVM) to separate pure CC from total correlations in a bipartite state. Quantum deficit, another alternative measure proposed by Rajagopal and Rendell, determines genuine QC by finding how close a given quantum state is to its decohered classical counterpart.

In this Letter, we address the basic issue of discerning quantumness of correlations, by introducing a novel theoretical measure quantumness, $Q_{AB}$, where we employ generalized orthogonal projective measurements on one part of a bipartite state, in an extended Hilbert space. In this new framework, quantum discord and quantum deficit, are found to be particular cases of quantumness. Separable states are shown to have zero quantumness - a significant feature not identified by the measures proposed earlier - thus establishing that absence of entanglement and classicality are synonymous. Furthermore, quantumness is shown to be (i) non-zero for all entangled states and (ii) it provides an upper bound to the relative entropy of entanglement of the bipartite state. This provides the much needed connection between the results of measurement theory with those characterizing quantum entanglement.

We begin with the considerations of OZ, regard-
ing perfect measurements on the subsystem $A$ of a bipartite state $\hat{\rho}_{AB}$, defined by a set of one dimensional orthogonal projectors $\{\Pi^A_\alpha\}$: $\sum_\alpha \hat{\Pi}^A_\alpha = I_A$ (completeness), $\hat{\Pi}^A_\alpha \hat{\Pi}^A_{\alpha'} = \delta_{\alpha \alpha'}$ (orthogonality), where $\alpha$ distinguishes different outcomes of measurement. The conditional density matrix of the subsystem $B$ when measurement on $A$ is known to have led to the value $\alpha$ is given by

$$
\hat{\rho}^{(\alpha)}_{AB} = \frac{\hat{\Pi}^A_\alpha \otimes I_B \hat{\rho}_{AB} \hat{\Pi}^A_\alpha \otimes I_B}{\text{Tr}[\hat{\Pi}^A_\alpha \otimes I_B \hat{\rho}_{AB}]},
$$
(2)

which occurs with a probability $p_\alpha = \text{Tr}[\hat{\Pi}^A_\alpha \otimes I_B \hat{\rho}_{AB}]$. The entropy $S(\hat{\rho}^{(\alpha)}_{AB}) = -\text{Tr}[\hat{\rho}^{(\alpha)}_{AB} \log \hat{\rho}^{(\alpha)}_{AB}]$ of the state $\hat{\rho}^{(\alpha)}_{AB}$ of Eq. (2) gives the uncertainty in interpreting the state of the system $B$, when the outcome $\alpha$ has been realized for the subsystem $A$. Given the results of the complete measurement $\{\Pi^A_\alpha\}$, the conditional information entropy [10] is given by

$$
S(B|A_{(\Pi^A_\alpha)}) = \sum_\alpha p_\alpha S(\hat{\rho}^{(\alpha)}_{AB}).
$$
(3)

On the other hand, a structural generalization of Shannon conditional information $H(B|A) = \sum_{a,b} P(a,b) \log P(a|b) = H(A,B) - H(B)$ (which is a consequence of the Bayes rule $P(b|a) = P(a,b)/P(a)$ for classical conditional probabilities) in terms of von Neumann entropies leads to an equivalent expression for the conditional entropies i.e.,

$$
S(B|A) = S(A,B) - S(A) = -\text{Tr}[\hat{\rho}_{AB} \log \hat{\rho}_{AB}] - \text{Tr}[\hat{\rho}_A \log \hat{\rho}_A].
$$
(4)

Quantum discord [3] is defined as the minimum value of the difference of two classically identical expressions Eqs. (3) and (4) for conditional entropies:

$$
\delta(A,B)_{(\Pi^A_\alpha)} = \min_{\{\Pi^A_\alpha\}} S(B|A_{(\Pi^A_\alpha)}) - S(B|A).
$$
(5)

(Here, the minimization is taken over the set $\{\Pi^A_\alpha\}$ of all orthogonal projectors, and it intends to find a scheme, which leaves the overall state of the system with least disturbance after measurement). Quantum discord vanishes iff $S(\hat{B}|\hat{A}_{(\Pi^A_\alpha)}) = S(\hat{B}|\hat{A})$; in such a situation, the correlations are well within the domain of Bayes rule of probability theory and are therefore classical. Also,

$$
\delta(A,B)_{(\Pi^A_\alpha)} = 0 \iff \hat{\rho}_{AB}^{\text{classical}} = \sum_\alpha \hat{\Pi}^A_\alpha \otimes I_B \hat{\rho}_{AB}^{\text{classical}} \hat{\Pi}^A_\alpha \otimes I_B,
$$
(6)

i.e., an optimal choice of measurement $\{\hat{\Pi}^A_\alpha\}$ on $A$ leaves the overall state $\hat{\rho}_{AB}^{\text{classical}}$ of a classically correlated system unperturbed. Quantum discord does not necessarily vanish for all separable states of the form Eq. (1) and this strongly suggests that genuine QC imply more than entanglement. A natural question would be to verify if this identification is true, even when we employ more general measurements - the ones considered by OZ.

Quantumness of correlations $Q_{AB}$: Let us consider the set of all tripartite density operators $\hat{\rho}_{A'AB}$ in an extended Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_A \otimes \mathcal{H}_B$, such that the bipartite state $\hat{\rho}_{AB}$ under investigation is a marginal of this extended system: $\text{Tr}_A[\hat{\rho}_{A'AB}] = \hat{\rho}_{AB}$. Carrying out an orthogonal projective measurement $\Pi^A_i; i = 1, 2, \ldots$, on one of the subsystems $A' \setminus A$ of the three party state $\hat{\rho}_{A'AB}$, we find that

$$
\hat{\rho}_{A''AB} \rightarrow \hat{\rho}_{A''AB}^{(i)} = \frac{1}{p_i} \left[ \hat{\Pi}^A_i \otimes I_B \hat{\rho}_{A''AB} \hat{\Pi}^A_i \otimes I_B \right],
$$
(7)

and

$$
\hat{\rho}_{AB} \rightarrow \hat{\rho}_{AB}^{(i)} = \frac{1}{p_i} \text{Tr}_A \left[ \hat{\Pi}^A_i \otimes I_B \hat{\rho}_{A''AB} \hat{\Pi}^A_i \otimes I_B \right],
$$

with $p_i = \text{Tr}[\hat{\rho}_{A''AB} \hat{\Pi}^A_i \otimes I_B \hat{\rho}_{A''AB}]$ denoting the probability of occurrence. We define Quantumness $Q_{AB}$ associated with a bipartite state $\hat{\rho}_{AB}$ as the minimum Kullback-Liebler relative entropy [11]

$$
Q_{AB} = \min_{(\hat{\Pi}^A_i),\hat{\rho}_{A''AB}} S(\hat{\rho}_{AB}||\hat{\rho}_{AB}^{(i)}) = \min_{(\hat{\Pi}^A_i),\hat{\rho}_{A''AB}} \left( \text{Tr}[\hat{\rho}_{AB} \log \hat{\rho}_{AB}] - \text{Tr}[\hat{\rho}_{AB} \log \hat{\rho}_{AB}^{(i)}] \right),
$$
(8)

where $\hat{\rho}_{AB}^{(i)}$ denotes the residual state of the bipartite system, after the generalized measurement is performed: $\hat{\rho}_{AB} = \text{Tr}_A \left[ \sum_i \hat{\Pi}^A_i \otimes I_B \hat{\rho}_{A''AB} \hat{\Pi}^A_i \otimes I_B \right]$. The minimum in Eq. (8) is taken over the set $\{\hat{\Pi}^A_i\}$ of projectors on the subsystems $A' \setminus A$ of all possible extended states $\{\hat{\rho}_{A''AB}\}$, which contain $\hat{\rho}_{AB}$ as their marginal system.

The quantumness, $Q_{AB} \geq 0$ (by definition), for all generalized measurements - the equality sign holding iff $\hat{\rho}_{AB} = \hat{\rho}_{AB}$ i.e., quantumness vanishes iff the bipartite state $\hat{\rho}_{AB}$ remains insensitive to generalized measurement $\{\hat{\Pi}^A_i\}$. It is easy to find that with every separable state of the form Eq. (1), there exists a tripartite density operator $\hat{\rho}_{A''AB} = \sum_i p_i \hat{\Pi}^A_i \otimes \hat{\rho}^{(i)}_B$, with $\text{Tr}_A[\hat{\Pi}^A_i \otimes \hat{\rho}^{(i)}_B] = \hat{\rho}^{(i)}_A$, so that $\text{Tr}_A[\hat{\rho}_{A''AB}] = \hat{\rho}^{(\text{sep})}_A = \sum_i p_i \hat{\rho}^{(i)}_A \otimes \hat{\rho}^{(i)}_B$. Clearly, a separable state is left unperturbed under the generalized measurements $\{\hat{\Pi}^A_i\}$ on one end (i.e., on the subsystem $A' \setminus A$ of the tripartite state $\hat{\rho}_{A''AB}$ i.e., $\hat{\rho}_{A''AB}^{(\text{sep})} = \hat{\rho}_{AB}^{(\text{sep})}$). Hence, quantumness $Q_{AB}$ vanishes for all separable states. This is a striking aspect - not identified by the measures proposed previously - which establishes that separability and classicality imply each other.

We illustrate this with an example. Consider a separable mixture of non-orthogonal states of two qubits, $\hat{\rho}_{AB} = p|0_A,0_B\rangle \langle 0_A,0_B| + (1 - p)|+A,+B\rangle \langle +A,+B|$, with $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$, and $0 \leq p \leq 1$, which is found to have non-classical correlations [3]. Following our procedure, a three qubit state $\hat{\rho}_{A''AB} = p|1_A',0_A,0_B\rangle \langle 1_A',0_A,0_B| + (1 - p)|0_A',+A,+B\rangle \langle 0_A',+A,+B|$, may be constructed such that $\text{Tr}_A'[\hat{\rho}_{A''AB}] = \hat{\rho}_{AB}$. An optimal measurement on $A'$ constitutes a set of complete, orthogonal projectors $\{\hat{\Pi}^A_i\} = \{1_A',0_A\}, \{1_A',0_A\}$, $\hat{\rho}^{(i)}_{A''A} = \{1_A',1_A\}, \{1_A',1_A\}$, $\hat{\rho}^{(i)}_{A''A} = \{0_A',+A\}, \{0_A',+A\}, \hat{\rho}^{(i)}_{A''A} = \{0_A',-A\}, \{0_A',-A\}$, and this leaves the overall state unperturbed: $\sum_{i=1} \hat{\rho}^{(i)}_{A''A} \otimes I_B \hat{\rho}_{A''AB} \hat{\rho}^{(i)}_{A''A} \otimes I_B = \hat{\rho}_{AB}$, and $\hat{\rho}_{AB} = \hat{\rho}_{AB}$. Hence, $Q_{AB} = 0$ for this state [12].
There arises a question on the operational aspects of finding an optimal measurement over the set of all extended states \( \{ \hat{\rho}_{A,AB} \} \), as we have access only to the marginal bipartite state \( \hat{\rho}_{AB} \). It may be noted that when the minimization in Eq. (5) is done by confining ourselves to three party extended systems of the direct product form \( \hat{\rho}_{A} \otimes \hat{\rho}_{AB} \), the orthogonal projective measurement \( \{ \hat{\Pi}^{A} \} \) corresponds to a set of POVMs \( \{ V^{A} \} \); \( \sum_{i} V^{A}_{i} V^{A}_{i} = I_{A} \), on the subsystem \( A \) of the bipartite state \( \hat{\rho}_{AB} \), transforming it to, \( \hat{\rho}_{AB} = \sum_{i} V^{A}_{i} \otimes I_{B} \hat{\rho}_{AB} V^{A}_{i} \otimes I_{B} \), as a result of generalized measurement and no explicit reference on the extended state \( \hat{\rho}_{A} \otimes \hat{\rho}_{AB} \) is needed in such situations. There still remains a problem concerning optimization involving more general extended states \( \hat{\rho}_{A,AB} \), that do not have the direct product structure \( \hat{\rho}_{A} \otimes \hat{\rho}_{AB} \); but this will be resolved at the end of this Letter, where we prove that quantumness \( Q_{AB} = \min \{ \rho_{A}^{(\text{sep})} \} S(\rho_{AB} \| \rho_{A}^{(\text{sep})}) \), where \( \{ \rho_{A}^{(\text{sep})} \} \) is the set of all bipartite separable states sharing the same marginal system \( \hat{\rho}_{A} \) for the part which does not come under the action of generalized measurements.

**Unifying features of quantumness:** We will now show that previously proposed measures of QC are special cases of \( Q_{AB} \). First, let us consider quantum deficit \( D_{AB} \) of a bipartite state \( \hat{\rho}_{AB} \) given by,

\[
D_{AB} = S(\rho_{AB} \| \hat{\rho}_{A}^{(d)}) - \text{Tr}[\rho_{AB} \log \hat{\rho}_{AB}] - \text{Tr}[\rho_{AB} \log \hat{\rho}_{A}^{(d)}],
\]

where \( \rho_{A}^{(d)} \) corresponds to the classical (decohered) density operator with the same marginal states \( \{ \hat{\rho}_{A} \} \). The quantum deficit \( D_{AB} \) determines the quantum excess of correlations in the state \( \hat{\rho}_{AB} \), with reference to its classical counterpart \( \hat{\rho}_{A}^{(d)} \). The decohered state \( \rho_{A}^{(d)} = \sum_{a,b} p(a,b) |a\rangle \langle a| \otimes |b\rangle \langle b| \) is diagonal in the eigenbasis \( \{ |a\rangle \} \), \( \{ |b\rangle \} \) of the marginal density operators \( \rho_{A}, \rho_{B} \) (with \( p(a,b) = \rho_{\alpha,\beta a,b} \) denoting the diagonal elements of \( \hat{\rho}_{AB} \)), in the basis of its subsystems \( A \) and is easy to see that if the optimal measurements that minimize the quantumness (see Eq. (8)) are the same projectors \( \{ \Pi_{a} = |a\rangle \langle a| \} \) of the subsystem \( A \) - with no need to invoke measurements in an extended space - quantumness \( Q_{AB} \) is equal to the quantum deficit \( D_{AB} \).

Consider bipartite states \( \hat{\rho}_{AB} \), which allow optimization of \( Q_{AB} \) in terms of orthogonal projective set \( \{ \Pi_{a} \} \) on the subsystem \( A \) (i.e., there is no need to invoke extended states \( \hat{\rho}_{A,AB} \), if minimization of quantumness could be achieved using actual space perfect orthogonal measurements). Using (i) the completeness \( \sum_{a} \Pi_{a} \otimes I_{B} = I_{A} \otimes I_{B} \), (ii) the orthogonality \( \Pi_{a} \Pi_{a'}^{\dagger} \otimes I_{B} = \delta_{a,a'} \delta_{a',a} \Pi_{a} \otimes I_{B} \) and (iii) the property \( [\Pi_{a} \otimes I_{B}, (\hat{\rho}_{AB})^{k}] = 0; k = 1, 2, \ldots \) (which follows readily from the structure of the bipartite state \( \hat{\rho}_{AB} = \sum_{a} \Pi_{a} \otimes I_{B} \rho_{AB} \Pi_{a} \otimes I_{B} \) after the projective measurement \( \{ \hat{\Pi}_{a}^{A} \} \)), it may be seen that,

\[
\begin{align*}
\text{Tr}[\rho_{AB} \log \hat{\rho}_{AB}}] &= \text{Tr}[\sum_{a} \Pi_{a} \otimes I_{B} \rho_{AB} \log \hat{\rho}_{AB}] \\
&= \text{Tr}[\sum_{a} \Pi_{a} \otimes I_{B} \rho_{AB} \log \hat{\rho}_{AB} \Pi_{a} \otimes I_{B} \\
&= \text{Tr}[\sum_{a} \Pi_{a} \otimes I_{B} \rho_{AB} \Pi_{a} \otimes I_{B} \log \rho_{AB}] \\
&= \text{Tr}[\rho_{AB} \log \rho_{AB}] \\
\text{Tr}[\rho_{AB} \log \hat{\rho}_{AB}] &= \text{Tr}[\sum_{a} \Pi_{a} \otimes I_{B} \rho_{AB} \log \hat{\rho}_{AB}] \\
&= \text{Tr}[\sum_{a} \Pi_{a} \otimes I_{B} \rho_{AB} \log \rho_{AB} \Pi_{a} \otimes I_{B}] \\
&= \text{Tr}[\sum_{a} \Pi_{a} \otimes I_{B} \rho_{AB} \Pi_{a} \otimes I_{B} \log \rho_{AB}] \\
&= \text{Tr}[\rho_{AB} \log \rho_{AB}]
\end{align*}
\]

(10)

In this particular case, the quantumness assumes the form,

\[
Q_{AB} = \min \{ [\Pi_{a}] \} S(\rho_{AB} \| \rho_{A}^{(d)}) = \min \{ [\Pi_{a}] \} [S(\rho_{AB}) - S(\rho_{A})]
\]

(11)

which immediately leads to

\[
Q_{AB} = \delta(A, B)(\Pi_{a}) + S(\rho_{A} \| \rho_{A}^{(d)})
\]

(12)

relating quantumness and quantum discord. Further, when the optimal measurements correspond to the set of eigenproectors \( \{ \Pi_{a} \} \) of the marginal density matrix \( \hat{\rho}_{A} \) (in which case \( \hat{\rho}_{A} \) coincides with \( \hat{\rho}_{A}^{(d)} \) and hence, \( S(\rho_{A} \| \rho_{A}^{(d)}) = 0 \)), all the three measures \( Q_{AB}, \delta(A, B)(\Pi_{a}) \) and \( D_{AB} \) of quantum correlations are identical:

\[
Q_{AB} = \delta(A, B)(\Pi_{a}) = D(A, B)
\]

(13)

An explicit example of two qubits, \( \hat{\rho}_{AB} = p|0\rangle \langle 0|_{A} + (1-p)|1\rangle \langle 1|_{A} \), \( \hat{\rho}_{A}^{(d)} = \frac{1}{\sqrt{2}}(|0\rangle_{A} \langle 0| + |1\rangle \langle 1|_{A}) \), denoting the Bell states and \( 0 \leq p \leq 1 \) demonstrates the above unifying feature. The quantum deficit for this state is given by \( D_{AB} = \log 2 + p \log p + (1-p) \log (1-p) \). On the other hand, quantum discord is optimized \( \delta(A, B)(\Pi_{a}) \) by the set of orthogonal projectors \( \{ \Pi_{a} = |0\rangle_{A} \langle 0| + |1\rangle \langle 1|_{A} \} \) and we obtain \( \delta(A, B)(\Pi_{a}) = D_{AB} \) in this case. It may be noted that the quantum deficit and discord are equal to the relative entropy of entanglement \( E_{RE}(\hat{\rho}_{AB}) \) in this particular example \( \delta(A, B)(\Pi_{a}) \). The quantumness of correlations is also equal to the relative entropy of entanglement in this case, which follows from a more general result \( Q_{AB} \geq E_{RE}(\hat{\rho}_{AB}) \) between the two as established below.

The quantumness and the relative entropy of entanglement: Let us express the extended three party density operator \( \hat{\rho}_{A,AB} \) in terms of a complete, orthogonal set of basis states \( \{ |j_{A,A} \rangle \otimes |\beta_{B} \rangle \} \) of subsystems \( A,A' \) and \( B \) as,

\[
\hat{\rho}_{A,AB} = \sum_{j,\beta} \rho_{j,\beta} |j_{A,A} \rangle \langle j_{A,A} | \otimes |\beta_{B} \rangle \langle \beta_{B} |.
\]

(14)

Corresponding to a measurement \( \{ \hat{\Pi}_{a}^{A} \} \) on \( A \), the state gets projected to,

\[
\text{Tr}[\hat{\Pi}_{a}^{A} \otimes I_{B} \hat{\rho}_{A,AB} \hat{\Pi}_{a'}^{A} \otimes I_{B} \hat{\rho}_{A,AB}] = \sum_{a',\beta} \rho_{a',\beta} \text{Tr}[\hat{\Pi}_{a}^{A} \otimes I_{B} \hat{\rho}_{A,AB} \hat{\Pi}_{a'}^{A} \otimes I_{B} \hat{\rho}_{A,AB}]
\]

(15)
We find that generalized quantumness \( Q_{AB} \) is done over the set of all projectors \( \{\Pi_i^{(A')A}\} \), and the set of all extended states \( \{\rho_{AB}^{(sep)}\} \), it is easy to see that 
\[
\{\rho_{AB}^{(sep)}; \rho_B = \text{Tr}[\rho_{AB}^{(sep)}]\}
\]
corresponds to the set of all separable states sharing the same subsystem density matrix \( \rho_B \) for the part B, which does not come under the direct action of generalized measurements \( \{\Pi_i^{(A')A}\} \).

Thus, we obtain
\[
Q_{AB} = \min_{\{\rho_{AB}^{(sep)}\}} S(\rho_{AB}; \rho_{AB}^{(sep)})
\]
with minimization taken over the set of all separable states \( \{\rho_{AB}^{(sep)}; \rho_B = \text{Tr}[\rho_{AB}^{(sep)}]\} \).

In other words, quantumness \( Q_{AB} \) is the minimum distance of the bipartite state \( \hat{\rho}_{AB} \) with the set of all separable states \( \{\rho_{AB}^{(sep)}; \rho_B = \text{Tr}[\rho_{AB}^{(sep)}]\} \).

15] It is evident that quantumness \( Q_{AB} \) is non-zero for all entangled states \( \rho_{AB} \) and it serves as an upper bound to the relative entropy of entanglement \( E_{RE}(\hat{\rho}_{AB}) \).

This is a significant theoretical result as it provides a natural link between the concepts of quantum entanglement with those based on quantum measurement theory. Most importantly, this establishes a flawless merging of quantumness of correlations with quantum entanglement itself.

Separating total correlations into classical and quantum parts: In classical information theory, Shannon mutual information is an unequivocal measure of correlations. Its quantum analogue is the von Neumann mutual information, \( S(A : B) = S(\rho_{AB} || \rho_A \otimes \rho_B) \) quantifies the total correlations in a bipartite state \( \rho_{AB} \), with \( \rho_A \), \( \rho_B \) denoting the subsystem density operators. HV \[4\] proposed that \( C_A(\hat{\rho}_{AB}) = \max_{V_A}(S(\rho_B) - \sum_i q_i S(\rho_i) \), \( q_i = \text{Tr}[V_i^A \otimes I_B \rho_{AB} V_i^A \otimes I_B] \), \( S(\rho) \), \( q_i \), and \( \rho_B \) serve as a measure of CC. By analyzing some examples they found that classical and entangled correlations do not add up to give total correlations \[6\]: \( C_A(\rho_{AB}) + E_{RE}(\hat{\rho}_{AB}) \neq S(A : B) \). This leaves open the question, “Are different types of correlations not additive?” We find that generalized projective measurements \( \{\Pi_i^{(A')A}\} \) on extended three party states of the product form, \( \hat{\rho}_{A'AB} = \hat{\rho}_{A'} \otimes \hat{\rho}_{AB} \) lead to
\[
\min_{\{\rho_{A'AB}^{(sep)}\}} S(\rho_{A'AB}^{(sep)}; \rho_{A' \otimes \hat{\rho}_{AB}}^{(sep)}), \rho_{A'}^{(sep)} = \sum_i \rho_i \log \rho_i \leq S(A : B) \), giving us
\[
\min_{\{\rho_{A'AB}^{(sep)}\}} S(\rho_{A'AB}^{(sep)}; \rho_{A' \otimes \hat{\rho}_{AB}}^{(sep)}) \leq S(A : B) - C_A(\hat{\rho}_{AB}) - \sum_i q_i \log q_i.
\]

And, as \( E_{RE}(\hat{\rho}_{AB}) \leq Q_{AB} \), we obtain the inequality \( C_A(\hat{\rho}_{AB}) + E_{RE}(\hat{\rho}_{AB}) + \sum_i q_i \log q_i \leq S(A : B) \). On the other hand, a consistent separation of total correlations into classical and quantum correlations as \( S(A : B) = C_{AB} + Q_{AB} \) would require a generalization of the HV measure \( C_A(\hat{\rho}_{AB}) \) of CC as,
\[
C_{AB} = S(\rho_{AB} || \rho_A \otimes \rho_B) - \min_{\{\rho_{A'AB}^{(sep)}\}} S(\rho_{A'AB}^{(sep)}; \rho_{A' \otimes \hat{\rho}_{AB}}^{(sep)})
\]
with the set \( \{\rho_{A'AB}^{(sep)}\} \) exhausting all possible three party extensions of the quantum state \( \hat{\rho}_{AB} \) - there being no restriction on the product structure, \( \hat{\rho}_{A'AB} = \hat{\rho}_{A'} \otimes \hat{\rho}_{AB} \).

In conclusion, this is a contribution towards resolving the difficulties faced in quantifying the residual quantum correlations in bipartite separable states, through the introduction of a novel measure called quantumness, \( Q_{AB} \). This brings out the need for the crucial inclusion of a generalized measurement scheme in quantifying quantum correlations. Separable states are insensitive to an optimal generalized measurement and are shown to have zero quantumness - an important feature, which the measures proposed previously failed to recognize. An entangled bipartite state gets projected into its closest separable product form, which shares the same marginal for one of its subsystems - consequent to an optimal measurement on the other part - showing that quantumness of correlations is non-zero for all entangled states and it serves as an upper bound to the relative entropy of entanglement of the bipartite state.

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so, the orthogonal projective measurement \( \{ \Pi_{i}^{A'} \} \) can
not be expressed as a POVM transformation on the state
\( \hat{\rho}_{AB} \). A POVM optimization had shown \[6, 8\] that the
state exhibits quantum correlations.

[13] Classically correlated states, which share the same set
of marginals as the given quantum state have been em-
ployed as reference states in formulating measures of en-
tanglement by J. Eisert, K. Audenaert, and M. B. Plenio,
J. Phys. A 36, 5605 (2003); M. H. Partovi, Phys. Rev.
Lett. 92, 077904 (2004).

[14] Quantum discord may also be expressed as \[5, 8\],
\[ \delta(A, B)_{\{ \Pi_{i}^{A} \}} = \min_{\{ \Pi_{i}^{A} \}} [(S(\hat{\rho}_{AB}) - S(\hat{\rho}_{A})) - (S(\hat{\rho}_{AB}) -
S(\hat{\rho}_{A}))] = \min_{\{ \Pi_{i}^{A} \}} [S(\hat{\rho}_{AB}||\hat{\rho}_{AB}) - S(\hat{\rho}_{A}||\hat{\rho}_{A})]. \]

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