NON-SUBELLIPTIC ESTIMATES FOR THE TANGENTIAL CAUCHY-RIEMANN SYSTEM

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Abstract. We prove non-subelliptic estimates for the tangential Cauchy-Riemann system over a weakly “q-pseudoconvex” higher codimensional submanifold $M$ of $\mathbb{C}^n$. Let us point out that our hypotheses do not suffice to guarantee subelliptic estimates, in general. Even more: hypoellipticity of the tangential C-R system is not in question (as shows the example by Kohn of [14] in case of a Levi-flat hypersurface). However our estimates suffice for existence of smooth solutions to the inhomogeneous C-R equations in certain degree.

The main ingredients in our proofs are the weighted $L^2$ estimates by Hörmander [12] and Kohn [14] of §2 and the tangential $\bar{\partial}$-Neumann operator by Kohn of §4; for this latter we also refer to the book [5]. As for the notion of $q$ pseudoconvexity we follow closely Zampieri [19]. The main technical result, Theorem 2.1, is a version for “perturbed” $q$-pseudoconvex domains of a similar result by Ahn [1] who generalizes in turn Chen-Shaw [5].

1. $q$-PSEUDOCONVEXITY IN HIGHER CODIMENSION

Let $M$ be a real generic submanifold of $\mathbb{C}^n$ of codimension $l$ and of class $C^i$, $i \geq 4$, defined by a system of $l$ independent equations $\rho^h = 0$, $h = n - l + 1, \ldots, n$. We denote by $\rho$ the vector valued function with components $\rho^h$. Let $TM$ denote the tangent bundle to $M$, $T^cM = TM \cap iTM$ the complex tangent bundle, $T^{1,0}M$ and $T^{0,1}M$ the subbundles of $\mathbb{C} \otimes_R T^cM$ of holomorphic and antiholomorphic forms respectively. Let $\mathcal{L}_\rho$, resp. $\mathcal{L}_M$, be the Levi form of $\rho$, resp. $M$, which is the Hermitian form defined, in a system of complex coordinates $z = x + iy$ for $\mathbb{C}^n$, by the matrix $\left(\frac{\partial^2}{\partial z^j \bar{z}^i} \rho\right)$, resp. $\left(\frac{\partial^2}{\partial z^j \bar{z}^i} \rho\right)_{T^cM}$. Let $T^*_M\mathbb{C}^n$ denote the conormal bundle to $M$ consisting of those $(0,1)$ forms whose real part vanish over $TM$ and set $\hat{T}^*_M\mathbb{C}^n = T^*_M\mathbb{C}^n \setminus \{0\}$. Note that the set of the $\partial \rho^h$’s are a basis for $T^*_M\mathbb{C}^n$. Identify in this basis $\mathbb{R}^l \mapsto (T^*_M\mathbb{C}^n)|_z z \in M$ by $a \mapsto \xi : = \sum h a h \partial \rho^h(z)$; this yields an identification $T^*_M\mathbb{C}^n \simeq M \times \mathbb{R}^l$ and $T^*_M\mathbb{C}^n / \mathbb{R}^l \simeq M \times S^{l-1}$ where $S^{l-1}$ is the spherical surface of dimension $l - 1$. Set $\partial^2_{z^i \bar{z}^j} \rho^k = \sum h a h \partial^2_{z^i \bar{z}^j} \rho^h$ and...
define $\mathcal{L}_\rho(z) = \left(\partial_{z_i z_j}^2 \rho(z)\right)_{ij}$ and

\begin{equation}
\mathcal{L}_\rho(z) = \left(\partial_{z_i z_j}^2 \rho(z)\right)_{ij} \bigg|_{T^c M} (z, \xi) \in M \times S^{l-1}.
\end{equation}

The form $\mathcal{L}_\rho(z)$ is called the “microlocal” Levi form of $M$ at $z$ in codirection $\xi$. Note that the Levi form is independent of the choice of a system of equations $\rho = 0$ for $M$.

**Definition 1.1.** We will deal with the assumption that there exists a smooth subbundle $\mathcal{V}^q_o = \mathcal{V}^q_{(z, \xi)}$ of $T^c M$ of rank $q_o \leq q_0$ such that for any bundle $\mathcal{V}^{q+1}$ of rank $q + 1$ we have

\begin{equation}
\text{trace} \left(\mathcal{L}_\rho(z)\right) \bigg|_{\mathcal{V}^{q+1}} - \text{trace} \left(\mathcal{L}_\rho(z)\right) \bigg|_{\mathcal{V}^{q_0}} \geq 0 \forall (z, \xi) \in M \times S^{l-1}.
\end{equation}

We will deal also with the local version of (1.2) at $z_0$ in which the condition holds for any $(z, \xi) \in M' \times S^{l-1}$ where $M'$ is a neighborhood of $z_0$ in $M$. Let us denote by $\lambda_j = \lambda_j^z(z)$ the eigenvalues of $\mathcal{L}_M = \mathcal{L}_\rho(z)$ ordered as $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-l}$, and by $s^+ = s^+(z, \xi)$, $s^- = s^-(z, \xi)$, $s^0 = s^0(z, \xi)$ the numbers of its respectively positive, negative, and null eigenvalues; note that $s^+(z, -\xi) = s^-(-z, +\xi)$. We consider an orthonormal basis $\{\omega_j\}_{j \leq n}$ of (1, 0) vector fields. We make our choice so that $\omega_j = \partial \rho^{l-n+1+j}$ for any $j \geq n - l + 1$ and decompose the basis into $\{\omega_j\}_{j \leq n-l}$ and $\{\omega_j\}_{j \geq n-l+1}$, and use the similar decomposition for the dual basis $\{\partial^* \omega_j\}_{j \leq n-l}$ and $\{\partial^* \omega_j\}_{j \geq n-l+1}$. If we change the partial basis $\{\omega_j\}$ so that $\mathcal{V}^{q_0} = \text{Span}\{\text{Re} \partial^* \omega_j\}_{j \leq q_o}$, then (1.2) reads as

\begin{equation}
\sum_{j \leq q+1} \lambda_j^z(z) - \sum_{j \leq q_o} \rho_{j j}^z(z) \geq 0 \forall (z, \xi) \in M \times S^{l-1}.
\end{equation}

Note that (1.3) implies the similar property with $q + 1$ replaced by any $k \geq q + 1$. In fact if (1.3) holds, then $\lambda_{q+1} \geq 0$ and hence $\lambda_j \geq 0 \forall j \geq q + 1$. Thus the terms $\lambda_j$ with $q + 2 \leq j \leq k$ can be added to the left hand side of the inequality without destroying it. In case $q = 0$, condition (1.3) reduces to $\lambda_1 \geq 0$ which means that $\mathcal{L}_M$ is positive semi-definite: thus $M$ is pseudoconvex in the classical sense. For $\mathcal{L}_M$ non-degenerate and with $q = s^-$, we regain the classical notion of strong $s$-pseudoconvexity (cf. for instance [12]). In the weak case, that is when in (1.2) or (1.3) we have weak inequalities, our condition goes back to (1.1) and, more closely, to [19]; it was also recently refined by [1]. Before entering algebraic details about (1.2) or (1.3) we wish to discuss some examples. In all of them we have $q = q_o$. 


Example 1.2. We fix a point $z_0$ and set $q = \sup_{\xi} (s^- (z_0, \xi) + s^0 (z_0, \xi))$. Note that $s^- + s^0$ is upper semicontinuous in $z$ and $\xi$ and integer valued; hence it attains a local maximum at any point. In particular $q$ remains unchanged if we take the supremum also with respect to $z$ in a neighborhood of $z_0$. We choose $\xi_0$ where $s^- + s^0$ attains the global maximum in $S^{l-1}$. Let $\mathcal{V}_{(z_0, \xi_0)}^q$ be the span of the negative and null eigenvectors at $(z_0, \xi_0)$ that we identify to the span of the first $q$ coordinate vectors. We have

$$\sum_{j \leq q+1} \lambda_{j}^{\xi_0} (z_0) - \sum_{j \leq q} \rho_{j}^{\xi_0} (z_0) \geq \lambda_{q+1}^{\xi_0} (z_0) > 0.$$  

Hence if we move $(z, \xi)$ near $(z_0, \xi_0)$ it remains true, by continuity, that (1.4) is $> 0$. In general, for any $(z, \xi)$, we can choose $\mathcal{V}_{(z, \xi)}^q$ such that $\sum_{j \leq q+1} \lambda_{j}^{\xi}(z) - \sum_{j \leq q} \rho_{j}^{\xi}(z) > 0$, though the difference in the left side needs not to coincide with $\lambda_{q+1}^{\xi}(z)$. By a partition argument over the unit circle $S^{l-1}$, we get (1.4) for all $(z, \xi) \in M' \times S^{l-1}$ where $M'$ is a neighbourhood of $z_0$. The above condition is considered by Nacinovich in [17] where existence theorems for the tangential $\bar{\partial}$ system are derived. Our task is to refine the above criterion and move $q$ to lower values.

Example 1.3. Let $M$ be a hypersurface; in this situation $M \times S^0$ consists of just two components $(z, \pm \xi)$. We write $\lambda_{j}^{\pm}(z)$ instead of $\lambda_{j}^{\pm\xi}(z)$, $s^{\pm}(z)$ instead of $s^{\pm}(z, +\xi)$ and so on; note that $s^{\pm}(z) = s^{\mp}(z, \mp \xi)$. In this situation (1.3) means the existence of two bundles $\mathcal{V}_+^{q_0}$ and $\mathcal{V}_-^{q_0}$ resp., such that in the two systems in which these bundles are reduced to the span of the first $q_0$ coordinate vectors, we have

$$\sum_{j \leq q+1} \lambda_{j}^{\pm}(z) - \sum_{j \leq q_0} \rho_{j}^{\pm}(z) \geq 0.$$  

According to Example 1.2 a first rough index $q$ for which (1.3) holds in $M' \times S^{l-1}$ for a neighbourhood $M'$ of $z_0$, is

$$q = \sup (s^-(z_0), s^+(z_0)) + s^0(z_0).$$  

In some cases we can do better. For instance, assume that $s^-(z)$ is constant for $z$ close to $z_0$. Then $\lambda_{s^-}^{+} < 0 \leq \lambda_{s^-+1}^{+}$ and hence the negative eigenvectors span a bundle $\mathcal{V}_-^{s^-}$ that, identified to the span of the first $s^-$ coordinate vectors, yields $\sum_{j \leq s^-+1} \lambda_{j}^{+}(z) - \sum_{j \leq s^-} \rho_{j}^{+}(z) \geq 0$. Of course, the same can be said in case $s^+(z)$ is constant. For the bundle $\mathcal{V}_+^{s^+}$ of the positive eigenvectors, identified to the first $s^+$ vectors, we have $\sum_{j \leq s^++1} \lambda_{j}^{-}(z) - \sum_{j \leq s^+} \rho_{j}^{-}(z) \geq 0$. Thus if both $s^\pm(z)$ are constant,
or equivalently if the corank $s^0(z)$ is constant, then we have (1.3) at $z_o$ for
\[ q = \text{sup}(s^-, s^+). \]
Thus we succeeded in decreasing by $s^0$ the value of $q$ with respect to (1.6).

We want to consider a variant of conditions (1.2) or (1.3) that we need first to express in new terms. For ordered multiindices $J = j_1 < j_2 < ... < j_k$ of length $|J| = k$, let us consider vectors $w = (w_J)$. Decompose $J = jK$ with $|K| = k - 1$ and write $w_{jK} = \text{sign}(iK_j) w_i$ where $(iK_j)$ is the permutation which orders $jK$. We will deal with the class of tangential forms; these are the forms $w = (w_J)$ such that any coefficient $w_J$ is 0 if $J$ contains some index $j = n - l + 1, \ldots, n$.

Sometimes we denote these forms by the alternative notation $w^\tau$. We will denote by $\sum'$ summation over ordered indices. One checks that (1.3) is equivalent to
\begin{equation}
(1.7) \quad \sum'_{|K|=k-1} \sum_{|j|=k} \rho^\xi_{ij}(z) w_i K w_j K - \sum'_{|K|=k} \sum_{|j|=q_o} \rho^\xi_{jj}(z) |w_j K|^2 \geq 0
\end{equation}
for any tangential form $w$ of length $k \geq q + 1$, and $\forall (z, \xi) \in M \times S^{l-1}$.

Along with (1.7) we will also consider the condition
\begin{equation}
(1.8) \quad \sum'_{|K|=k-1} \sum_{|j|=n-l} \rho^\xi_{ij}(z) w_i K w_j K - \sum'_{|K|=k} \sum_{|j|=q_o} \rho^\xi_{jj}(z) |w_J|^2 \geq 0
\end{equation}
for any tangential form $w$ of length $k \geq q + 1$, and $\forall (z, \xi) \in M \times S^{l-1}$.

One can also consider some intermediate condition between (1.7) and (1.8) in which for part of the indices $j \leq q_o$ one takes $J = jK$ and for the remaining indices one takes all $J$ without requiring $j \in J$. For $q = \text{sup} (s^-, s^+)$, (1.7) holds according to Example 1.2 in this case one sees that (1.8) is also fulfilled. But we can also discuss some cases of (1.8) which do not fit (1.7).

Example 1.4. Let $M_1 \times M_2 \subset \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$ be quadric hypersurfaces given by diagonal equations. Thus $L^\pm_{M_i}, i = 1, 2$ are diagonal at any $z$. We define $q_i = \text{sup}(s^+_M, s^-_M)$, denote by $V^q_i$ the span in $T^{1,0} M$ of the non-null eigenvectors, and put
\[ q = \text{sup}(n_1 - 1 + q_2, n_2 - 1 + q_1). \]
Consider a point, say \( \xi = (0, +\xi_2) \), and take \( \mathcal{V}^{n_1+q_2} = T^cM_1 \oplus \mathcal{V}^{q_2} \) where \( \mathcal{V}^{q_2} \) contains the span of the \( s^- \) negative eigenvectors. We have

\[
\sum_{|K|=k-1} \sum_{ij \leq n_1-1} \rho_i^j(z)w_{iK}\bar{w}_{jK} - \sum_{|K|=k-1} \sum_{ij \leq n_1-1} \rho_{jK}(z)|w_{jK}|^2 \\
+ \sum_{|K|=k-1} \sum_{n_1+1 \leq i \leq n_1+n_2-2} \rho_i^j(z)w_{iK}\bar{w}_{jK} - \sum_{|K|=k-1} \sum_{n_1+1 \leq j \leq n_1+n_2} \rho_{jK}(z)|w_{jK}|^2 \\
\geq \lambda_{q_2+1}^+|w_{n_1+q_2+1}|^2 + \cdots + \lambda_{k-n_1}^+|w_k|^2 \geq 0.
\]

The above discussion applies for instance to the manifold in \( \mathbb{C}^{n_1+n_2} \) defined by the equations

\[
\begin{align*}
y_{n_1} &= |z_1|^2 - |z_2|^2, \\
y_{n_1+n_2} &= |z_{n_1+1}|^2 - |z_{n_1+2}|^2.
\end{align*}
\]

Here \( q_i = 1 \) for \( i = 1, 2 \) and therefore (1.8) is satisfied for \( q = \sup(n_1, n_2) + 1 \).

We refine now our choice of the basis of (1, 0) forms to make it better adapted to \( M \). We first choose our equations \( \rho^h = 0 \ h = n - l + 1, \ldots, n \) having orthonormal differentials along \( M \). We extend the system of the \( \partial\rho^h \big|_M \) to an orthonormal system \( \{\omega''\} \) which spans \( \text{Span}\{\partial\rho^h\} \) even outside of \( M \). We then take an orthonormal completion \( \{\omega'\} \) of \( \{\omega''\} \) and denote by \( \{\partial_{\omega'}, \partial_{\omega''}\} \) the dual system of (1, 0) vector fields. Note that by our choice we have \( \forall j, k \)

\[
\partial_{\omega'}^j \rho^k \equiv 0 \quad \partial_{\omega''}^j \rho^k \equiv \nu_{jk} \text{ on } \mathbb{C}^n,
\]

where \( \nu_{jk} \) is the Kronecker symbol. We introduce now the spaces of forms of type \( (0, k) \); in a basis \( \{\omega_j\} \) they can be written as \( u = \sum_{|J|=k} u_J \omega_J \ |J| = k \) with coefficients in spaces of various kind such as \( C^\infty(\mathbb{C}^n) \) or \( L^2(\mathbb{C}^n) \) or, for a positive function \( \varphi \), \( L^2(\mathbb{C}^n) \) that is the space of functions which satisfy \( ||u_J||_\varphi := \left( \int e^{-\varphi}|u_J|^2dV \right)^{\frac{1}{2}} < \infty \). We will denote by \( C_k^\infty, L^2_k, (L^2_\varphi)_k \) the above defined spaces, and also denote them by the common symbol \( \Lambda_k \) when we want not to stress attention to the kind of the coefficients. We denote by \( \mathcal{C}_k \) the restriction to \( M \) of the ideal of \( \Lambda_k \) engendered by \( \rho \) and \( \partial\rho \), and define the space of tangential forms on \( M \) as the orthogonal complement of \( \mathcal{C}_k \) in \( \Lambda_k \):

\[
\mathcal{T}_k = \mathcal{C}_k^\perp.
\]
Any tangential form can be represented as the restriction to \( M \) of a form satisfying the \( \bar{\partial} \)-Neumann conditions on \( M \):

\[
\sum_{j=1,\ldots,n} \rho_j^h u_j K \big|_M = 0 \quad \forall h = n - l + 1, \ldots, n \ \forall K,
\]

where we have used the notation \( \rho_j^h \) for \( \partial \omega_j \rho^h \). Let us take an orthonormal frame \( \{ \omega_j \} \) satisfying the above conditions and in particular (1.11).

Let us decompose any \( u \) as \( u = u^\tau + u^\nu \) where in \( u^\tau \) we collect coefficients corresponding to indices \( J \) such that \( n - l + 1, \ldots, n \notin J \) and in \( u^\nu \) the remaining ones. The fact that \( u \big|_M \) satisfies the \( \bar{\partial} \)-Neumann conditions reads

\[
(1.12) \quad u^\nu(z) \big|_M \equiv 0 \quad \forall z \in M.
\]

It is obvious that (1.12) implies \( \partial_{\omega_j} u^\nu \big|_M \equiv 0 \forall K \). We can see that we may choose, among representatives of a tangential form, one which satisfies

\[
(1.13) \quad u^\nu(z) \equiv 0 \quad \forall z \in \mathbb{C}^n.
\]

We also choose the extension of \( u \) from \( M \) to \( \mathbb{C}^n \) so that for all coefficients we have that \( \partial_{\omega_j} u_J \big|_M \) is nearly 0 according to the following considerations.

**Proposition 1.5.** Let \( M \) have class \( C^{m+1} \) and be locally defined at 0 by \( y''_h = g_h \) for \( g_h(0) = 0 \), \( \partial g_h(0) = 0 \). Then there is a local system of vector fields \( \bar{L}_h, h \geq n - l + 1 \) of class \( C^m \) and type \( (0,1) \) with \( \bar{L}_h(0) = \partial z_h \) such that for any function \( f \in C^m(M) \) there exists an extension \( \tilde{f} \) in \( \mathbb{C}^n \) such that

\[
\begin{cases}
\bar{L}_h \tilde{f} \big|_M \equiv O^m, \\
\tilde{f} \big|_M \equiv f,
\end{cases}
\]

(where the symbol \( O^m \) denotes an infinitesimal of order \( m \) with respect to the distance to \( M \)).

**Proof.** We consider the parametrization of \( M \):

\[
G : \mathbb{C}^{n-l} \times \mathbb{R}^l \to M, \ (z', x'') \mapsto (z', x'' + ig(z', x'')).
\]

We extend \( G \) to \( \tilde{G} \) which is \( m \)-holomorphic along \( M \) that is

\[
\tilde{G} : \mathbb{C}^{n-l} \times \mathbb{C}^l \to \mathbb{C}^n, \ (z', z'') \mapsto (z', z'' + i\tilde{g}(z', z'')),
\]

such that

\[
(1.14) \quad \partial_{x''} \tilde{g} \big|_{\mathbb{C}^{n-l} \times \mathbb{R}^l} = O^m.
\]
This statement belongs to the family of Whitney’s extension theorems. Given \( f \), we define \( f_o := f \circ G \), extend \( f_o \) to \( \tilde{f} \) from \( \mathbb{C}^{n-l} \times \mathbb{R}^l \) to \( \mathbb{C}^{n-l} \times \mathbb{C}^l \) with the property

\[
\partial'^\prime \bar{z} \tilde{f}_o|_{\mathbb{C}^{n-l} \times \mathbb{R}^l} = O^m,
\]

and set \( \tilde{f} := \tilde{f}_o \circ \tilde{G}^{-1} \). We also define \( \tilde{L}_h = \tilde{G}_* \partial'^\prime \bar{z} h \geq n - l + 1 \).

It is clear that each \( \tilde{L}_h \) is of type \((0,1)\) along \( M \) due to (1.14). We also have

\[
\begin{align*}
\tilde{L}_h &\sim \partial'^\prime \bar{z} h \text{ due to } \partial g_j(0) = 0 \forall j, \\
\tilde{L}_h \tilde{f}|_M &\sim \partial'^\prime \bar{z} f_o|_{\mathbb{C}^{n-l} \times \mathbb{R}^l} = O^m \text{ due to (1.15)}.
\end{align*}
\]

Remark 1.6. Let \( \{\omega_j\} \) be an orthonormal system of \((1,0)\)-forms with \( \omega_h'' = \partial \rho^h, h \geq n - l + 1 \) for a system of equations \( \rho^h = 0 \) of \( M \) such that \( \partial \rho^h(0) = dz_h \). If \( z \in \mathbb{C}^n \) is close to \( M \) and \( z^* \) is the point on \( M \) of minimal distance, we have

\[
\partial \omega_k(z) = \partial \omega_k(z^*) + O(|z - z^*|), \forall k \text{ such that } 1 \leq k \leq n.
\]

For \( h \geq n - l + 1 \), write

\[
\partial'^\prime \omega_h|_M = \sum_{i\geq n-l+1} b_i L_i|_M + \sum_{j\leq n-l} a_j \partial \omega_j|_M;
\]

note that the \( a_j \)'s and the \( b_i \)'s for \( i \neq h \) are small. Thus, if \( \tilde{f} \) satisfies the conclusions of the preceding proposition we have

\[
\partial'^\prime \omega_h \tilde{f}(z) = \sum_{j\leq n-l} a_j \partial \omega_j f(z) + O(|z - z^*|),
\]

for small \( a_j \)'s.

We let

\[
\tilde{\rho} = \frac{|\rho|^2 - \epsilon^2}{2\epsilon},
\]

and define the system of “tuboidal” neighborhoods of \( M \) adapted to the frame \( \omega \) by

\[
U_\epsilon = \{z \in \mathbb{C}^n : \tilde{\rho}(z) < 0\}.
\]

Let \( |\rho(z)| = \epsilon \) and \( a = \epsilon^{-1}(\rho^h(z))h \); recall that we are identifying \( a \) to a cotangent vector \( \xi \) the one with coordinates \( a \) in the system of \( 1 \)-forms \( \partial \rho^h, h = n - l + 1, ..., n \); note that \( \xi \) is conormal to \( \partial U_\epsilon \). Let \( \mathbb{C}^n \to M, z \mapsto z^* \) be any transversal projection. We have, if \( \rho \) is \( C^k \)
and keeping the assumption that $z$ belongs to $\partial U_\epsilon$ and $z^*$ is the point of minimal distance on $M$ all through the sequel:

\[(1.18)\quad |\rho_{ij}^h(z) - \rho_{ij}^h(z^*)| = O(\epsilon) \forall h, \forall ij.\]

We also have

\[(1.19)\quad \mathcal{L}_\rho = \epsilon^{-1} \sum_h \partial \rho \otimes \bar{\partial} \rho + \epsilon^{-1} \sum_h \rho \mathcal{L}_{\rho h}.
\]

We write $u'_K = (u_{iK})_{i\leq n-1}$, $u''_K = (u_{iK})_{i\geq n-l+1}$. It follows for any $K$

\[(1.20)\quad \mathcal{L}_\rho(u_K, \bar{u}_K) \geq \epsilon^{-1}|u''_K|^2 + \mathcal{L}_\rho^\xi(u'_K, \bar{u}'_K) - c_1|u'_K||u''_K| - c_2|u''_K|^2,
\]

and hence

\[(1.21)\quad \mathcal{L}_\rho(z)(u_K, \bar{u}_K) \geq \epsilon^{-1}|u''_K|^2 + \mathcal{L}_\rho^\xi(z^*)(u'_K, \bar{u}'_K) - O(\epsilon)|u'_K|^2.
\]

By combining (1.18) and (1.21) and by taking summation on $K$, we get the proof of the following statement which describes how (1.7) is affected when $z$ is no more a point of $M$, and $u$ is not necessarily a tangential form.

**Theorem 1.7.** Let $M$ satisfy (1.3); then

\[(1.22)\quad \sum_{|K|=k-1} \sum_{i=1}^{n-1} \sum_{j=1}^{n-l} \bar{\rho}_{ij}(z)u_{iK}u_{jK} - \sum_{|J|=k} \sum_{i=1}^{n-1} \sum_{j=1}^{n-l} \bar{\rho}_{jj}(z)u_{jJ}^2 \geq -O(\epsilon)|u|^2
\]

\[\forall z \in \partial U_\epsilon \text{ and } \forall u \text{ of length } k \geq q + 1.\]

One has also a local version of Theorem 1.7 in a neighborhood of $z_0$.

**Remark 1.8.** If, instead of (1.3), we assume (1.8), then we have the similar conclusion as (1.22) but with the second term in the left containing only the indices for which $j \in J$, or equivalently those in the form $J = jK$.

**Remark 1.9.** The coefficients $a_{ij}$ of the basis of forms $\{\omega_i\}$ in which (1.22) holds are singular in $M$. In particular for the normal vector fields we have that $(\partial_{\omega_h}'' + \partial_{\omega_h}')a_{ij} \forall h \geq n - l$ grow as $|\rho|^{-1}$. However, for the tangential vector fields, we have that $\partial_{\omega_h}a_{ij}, \partial_{\omega_h}'a_{ij} \forall k \leq n - l$ and $\frac{1}{2}(\partial_{\omega_h} - \partial_{\omega_h}')a_{ij} \forall h \geq n - l + 1$ are bounded.

**Remark 1.10.** It follows from Theorem 1.2 that $M$ has a fundamental system of neighborhoods which are “almost” $q$-pseudoconvex. In general these neighborhoods cannot be $q$-pseudoconvex as shows the example by Diederich-Fornaess of non-trivial “nebenhülle”.
Recall that $\partial_{\alpha_k}^\rho \rho^h \equiv 0$ and that $\partial_{\alpha_k}^\rho \rho^h = \omega_{\alpha k}$. It follows
\begin{equation}
(1.23) \quad \sum_h \rho^h \rho^h_h|_z = \sum_h \rho^h \rho^h \rho^h \rho^h \rho^h |_z \rho^h \rho^h + \rho^h \rho^h \rho^h |_z = \rho^h \rho^h \rho^h |_z \rho^h \rho^h |_z = \rho^h \rho^h \rho^h |_z \rho^h \rho^h |_z.
\end{equation}
Choose any transversal projection $z \mapsto z^*$; we have
\begin{equation}
(1.24) \quad \sum_h \rho^h (z) \rho^h (u, \bar{u}) = \rho^h (z^*) (u, \bar{u}) + O(|u|^2) \text{ for } z \in \partial U \epsilon.
\end{equation}
This gives the proof of the following

**Proposition 1.11.** Let $M$ satisfy $\text{(1.3)}$. Then
\begin{equation}
(1.25) \quad \sum' \sum_{|K|=k-1} \sum_{i=1} \sum_h \rho^h (z) \rho^h (u, \bar{u}) = \rho^h (z^*) (u, \bar{u}) + O(|u|^2),
\end{equation}
for any $z \in \partial U \epsilon$ and for any form $u = u(z)$ (not necessarily satisfying $\partial$-Neumann conditions on $M$) of order $k \geq q + 1$.

Again, we have a local version at $z_0$ of this statement and also a variant under the assumption $\text{(1.8)}$.

**Definition 1.12.** We say that $M$ is $q$-pseudoconvex, resp. locally $q$-pseudoconvex at $z_0$, when $\text{(1.2)}$ or $\text{(1.3)}$ are fulfilled for any $(z, \xi) \in M \times S^{q-1}$, resp. for any $(z, \xi) \in M' \times S^{q-1}$ for a neighborhood $M'$ of $z_0$.

## 2. $L^2$ ESTIMATES FOR THE AMBIENT $\overline{\partial}$ SYSTEM

We denote by $u(z) = (u_j(z)) \in M \subset \mathbb{C}^n$, a form of type $(0, k)$ satisfying the $\overline{\partial}$-Neumann conditions; most of times its coefficients are supposed to be smooth. We also suppose that the orthonormal frame $\{\omega', \omega''\}$ and the extension $u$ satisfy all conditions listed in §1 including Proposition $\text{(1.5)}$ and the related remark. In particular $u'' = 0$ also outside $M$ and $u'' = 0$ also outside $M$ and
\begin{equation}
(2.1) \quad \partial_{\omega_h} u_j = \sum_{j \leq n-l} a_j \partial_{\omega_h} u_j + O(\rho),
\end{equation}
with small coefficients $a_j$. We denote by $|| \cdot ||_{H^0(M)}$ or $|| \cdot ||_{H^0(U \epsilon)}$ the $H^0 = L^2$ norms on $M$ and $U \epsilon$ respectively; for any real positive function $\varphi$ we denote by $H^0_\varphi$ the $L^2$ norms with weight $e^{-\varphi}$. We will make our choice of $\varphi$ as $\varphi = (t+c)|z|^2$ for a large parameter $t$ and for a constant $c$
depending on the coefficients of the $\omega_j$'s. We denote by $\bar{\partial}$, resp. $\bar{\partial}'$, the complex on antiholomorphic forms associated to all antiholomorphic vector fields $\partial_{\omega_j}$, $1 \leq j \leq n$, resp. to $\partial'_{\omega_j}$, $1 \leq j \leq n - l$. We denote by $\bar{\partial}^*$, resp. $\bar{\partial}'^*$ the $H^0_\varphi$-transposed; note that $\bar{\partial}^* = \bar{\partial}'^* + O(|\rho|)$ over $\bar{\partial}$-Neumann forms. We will still denote by $U_\varepsilon$ the intersection of the tube $U_\varepsilon$ with a suitable sphere centered at $z_0$.

**Theorem 2.1.** Let $M$ be $q$-pseudoconvex at $z_0$. Then for any $\bar{\partial}$-Neumann form $u$ of degree $k \geq q + 1$ with support whose coefficients satisfy $(2.1)$, and for any large real $t$, we have

$$(2.2) \quad \frac{t}{2} ||u||^2_{H^0_\varphi(U_\varepsilon)} \leq ||\bar{\partial}'^* u||^2_{H^0_\varphi(U_\varepsilon)} + ||\bar{\partial}' u||^2_{H^0_\varphi(U_\varepsilon)} + o(\varepsilon).$$

**Proof.** We will only give the proof under the assumption $(1.7)$ in local form, the case of $(1.8)$ being analogous. We also point out that by cutting the tube $U_\varepsilon$ by a sphere we still have a domain which satisfies $(1.22)$ and $(1.25)$ in each smooth part of the boundary. Also, in the integrations by parts, some integrals in the 2-codimensional strata appear. But these are positive and so we can neglect them or equivalently we can assume from the beginning that $U_\varepsilon$ is compact, smooth and satisfies $(1.22)$ and $(1.25)$. The proof is closely related to that by Ahn [1] who deals with a $q$ pseudoconvex domain and gets the similar estimate as $(2.2)$ without the error term $o(\varepsilon)$. We simplify our notation and write $|| \cdot ||_{\varphi}$ instead of $|| \cdot ||_{H^0_\varphi(U_\varepsilon)}$ all through the proof. We also drop the symbol $'$ in most of notations: it will be understood that our indices will generally vary between 1 and $n - l$. We set $\varphi_j = \partial_{\omega_j} \varphi$ and define

$$\delta_{\omega_j} = \partial_{\omega_j} - \varphi_j;$$

Hence $\delta_{\omega_j}$ is the transposed of $-\partial_{\omega_j}$ in the weighted $H^0_\varphi$ scalar product apart from a 0-order operator which depends on tangential derivatives of the coefficients of the forms $\omega_j$'s. We have

$$(2.3) \quad \sum'_{|K|=k-1} \sum_{j=1,...,n-l} \int_{U_\varepsilon} e^{-\varphi} \left( \delta_{\omega_j} u_{iK} \delta_{\omega_j} u_{jK} - \partial_{\omega_j} u_{iK} \bar{\partial}_{\omega_j} u_{jK} \right) dV$$

$$+ \sum'_{|J|=k} \sum_{j=1,...,n} \int_{U_\varepsilon} e^{-\varphi} |\partial_{\omega_j} u_j|^2 dV \leq 2(||\bar{\partial}'^* u||^2_{\varphi} + ||\bar{\partial}' u||^2_{\varphi}) + R^1,$$

where $R^1$ is an error term which only involves integration of $|u|^2$ and not of its derivatives. We will use the notation “s.c.”, resp. “l.c.”, to
denote small constants, resp. large constants. We have

\[ \| \partial_{\omega j} u_j \|_p^2 = \| \partial_{\omega j} u_j \|_p^2 + \int_{U_\epsilon} e^{-\varphi} \left[ \delta''_{\omega j}, \partial_{\omega j} \right] u_j \bar{u}_j dV + R_{jj}^2 \quad \forall j \leq n - l, \]

where \( R_{jj}^2 \) can be estimated both by s.c. \( \| \partial_{\omega j} u_j \|_p^2 + 1.c. \| u_j \|_p^2 \) or s.c. \( \| \partial_{\omega j} u_j \|_p^2 + 1.c. \| u \|_p^2 \). In fact the boundary integrals which arise in the integrations by parts for interchanging \( \partial_{\omega j} \) with \( \delta'_j \) are 0 due to (1.11) that is \( \partial_{\omega_j} \rho \equiv 0 \). We rewrite now the integrals of \( \partial_{\omega_j} u_{iK} \partial_{\omega_i} u_{jK} \) in the left side of (2.3). Integration by parts yields

\[ \int_{U_\epsilon} e^{-\varphi} \partial_{\omega_j} u_{iK} \partial_{\omega_i} u_{jK} = \int_{+\partial U_\epsilon} e^{-\varphi} \partial_{\omega_j} (u_{iK}) \tilde{\rho}_i \bar{u}_{jK} dV - \int_{U_\epsilon} e^{-\varphi} \delta_{\omega_i} \partial_{\omega_j} (u_{iK}) \bar{u}_{jK} dV + R_{ij}^3, \]

where \( R_{ij}^3 \) is an error which involves integrals of \( \bar{u}_{jK} \partial_{\omega_j} u_{iK} \). Again, in (2.5) the boundary integral is 0: in fact, since \( i \leq n - l \), then \( \tilde{\rho}_i \equiv 0 \) (where we are using as always the notation \( \tilde{\rho}_i = \partial_{\omega_i} \rho \)). We also have

\[ \int_{U_\epsilon} e^{-\varphi} \delta_{\omega_i} u_{iK} \partial_{\omega_j} u_{jK} dV = \int_{+\partial U_\epsilon} e^{-\varphi} \tilde{\rho}_j \bar{u}_{jK} \delta_{\omega_i} u_{iK} dV - \int_{U_\epsilon} e^{-\varphi} \partial_{\omega_j} \delta_{\omega_i} u_{iK} \bar{u}_{jK} dV + R_{ij}^4, \]

where \( R_{ij}^4 \) involves integrals of \( \delta_{\omega_i} u_{iK} \bar{u}_{jK} \). Again, the boundary integral in (2.6) is 0 due to (1.11) and \( \| \bar{u}_{jK} \|_p \). Thus in the left side of (2.3) we use (2.4), (2.6) in the first two terms for any \( i \) and \( j \) and next (2.4) in the third, but now only for \( j \leq p \). In this way we can rewrite the left side of (2.3) as

\[ \left( \sum' \sum'_{|K|=k-1} \int_{U_\epsilon} e^{-\varphi} [\delta_{\omega_i}, \partial_{\omega_j}] u_{iK} \bar{u}_{jK} dV \right) \]

\[ \left( -\sum' \sum_{|J|=k} \int_{U_\epsilon} e^{-\varphi} [\delta_{\omega_j}, \partial_{\omega_i}] u_j \bar{u}_j dV \right) \]

\[ + \left( \sum' \sum_{|J|=k} \int_{U_\epsilon} e^{-\varphi} |\delta_{\omega_j} u_j|^2 dV + \sum' \sum_{|J|=k+1} \int_{U_\epsilon} e^{-\varphi} |\partial_{\omega_j} u_j|^2 dV \right) + R^5, \]
where \( R^k \) is the sum of the \( R^k_{ij} \)'s (for \( j \leq q \)), the \( R^3_{ij} \)'s and the \( R^4_{ij} \)'s. We denote by \( S \) the second term in \( (2.8) \) that is \( (\sum'_{|J| = k_2 \leq q_2} + \sum'_{|J| = k_2 \geq p + 1}) \).

The terms \( R^2 \) were already estimated. As for the remaining we clearly have an analogous estimate

\[
R^i \leq \text{s.c.} \, S + \text{l.c.} \, ||u||_{\psi}^2 \quad \forall i \geq 2.
\]

Clearly an estimate of the type \( (2.8) \) also holds for \( R^1 \). We pass now to compute the commutators \([\delta_{\omega_i}, \partial_{\omega_j}]\). Let \((c^h_{ij})\) be the matrix of the 2-form \( \partial_{\omega_i} \); note that since for \( h \geq n - l + 1 \), we have \( \omega_h = \partial \rho^h \), then \((c^h_{ij}) = (\rho^h_{ij})\) is the matrix of the Levi-form \( L_{\rho^h} \). The identity \( \partial \partial = -\partial \bar{\partial} \) yields

\[
(2.9) \quad [\partial_{\omega_i}, \partial_{\omega_j}] = \sum_{h=1}^n c^h_{ij} \partial_{\omega_h} - \sum_{h=1}^n c^h_{ij} \partial_{\omega_h}.
\]

We denote by \((\varphi_{ij})\) the matrix of \( L_{\varphi} \) which coincides, up to an error term, with \((t + 2c)\, \kappa_{ij}\). We get

\[
(2.10) \quad [\delta_{\omega_i}, \partial_{\omega_j}] = [\partial_{\omega_i} - \varphi_{i}, \partial_{\omega_j}]
\]

\[
= [\partial_{\omega_i}, \partial_{\omega_j}] - [\varphi_{i}, \partial_{\omega_j}]
\]

\[
= \varphi_{ij} + \sum_{h} c^h_{ij} \delta_{\omega_h} - \sum_{h} c^h_{ij} \partial_{\omega_h}
\]

\[
= \varphi_{ij} + \sum_{h \geq n - l + 1} \rho^h_{ij} (\delta_{\omega_h} - \partial_{\omega_h}) + \sum_{h \leq n - l} (c^h_{ij} \delta_{\omega_h} - c^h_{ij} \partial_{\omega_h}).
\]

Integration by parts yields, on account of \( (1.1) \):

\[
\int_{U_\varepsilon} e^{-\varphi} |c^h_{ij} \delta_{\omega_h} u_{J}(\bar{u}_{I})|dV \leq \text{l.c.} \, ||u||_{\psi}^2 + \text{s.c.} \, S \quad \forall h \leq n - l,
\]

and

\[
\int_{U_\varepsilon} e^{-\varphi} |c^h_{ij} \delta_{\omega_h} u_{J}(\bar{u}_{I})|dV \leq \text{l.c.} \, ||u||_{\psi}^2 + \text{s.c.} \, S \quad \forall h \leq n - l.
\]

For \( h \geq n - l + 1 \), we want to interchange \( \delta_{\omega_h} \) with \( \partial_{\omega_h} \) in our integrals; we have

\[
(2.11) \quad \sum_{h \geq n - l + 1} \int_{U_\varepsilon} e^{-\varphi} \rho^h_{ij} \delta_{\omega_h} u_{J}(\bar{u}_{I})dV = \sum_{h \geq n - l + 1} \int_{\partial U_\varepsilon} e^{-\varphi} \rho^h_{ij} \rho^h_{ij} u_{J}(\bar{u}_{I})dV
\]

\[
- \sum_{h \geq n - l + 1} \int_{U_\varepsilon} e^{-\varphi} \rho^h_{ij} \overline{\delta_{\omega_h} u_{I}}dV + R^7,
\]
Here, for the error term we have the estimate $R^7 \leq c\|u\|^2 + o(\epsilon')$. In fact the coefficients of the vector fields $\partial_{\omega_h}$ for $h \geq n - l + 1$ are nonsingular at $M$. The key point is that the boundary integral in (2.11) is positive due to our assumption of $q$-pseudoconvexity as restated in Proposition 1.11. By discarding the positive boundary integrals we are thus reduced to integrals involving only terms of type $\partial_{\omega_h} u_j \bar{u}_l$ for $h \geq n - l + 1$. These latter are in turn reduced to integrals of type $\partial_{\omega_j} u_j \bar{u}_l$ for $j \leq n - l$ due to the choice of the distinguished representative of the form $u$ whose coefficients satisfy in particular (2.1). Summarizing up, (2.7) can be rewritten as

$$(2.12)$$

$$\sum'_{|K|=k-1} \left( \sum_{ij=1,\ldots,n-l} \int_{U_\kappa} e^{-\varphi} \varphi_{ij} u_{iK} \bar{u}_{jK} dV - \sum_{j<q_0} \int_{U_\kappa} e^{-\varphi} \varphi_{jj} |u_j|^2 dV \right)$$

$$(2.13)$$

$$+ \sum'_{|K|=k-1} \left( \sum_{ij=1,\ldots,n-l} \int_{\partial U_\kappa} e^{-\varphi} \rho_{ij} u_{iK} \bar{u}_{jK} - \sum_{j<q_0} \int_{\partial U_\kappa} e^{-\varphi} \rho_{jj} |u_j|^2 dV \right)$$

$$(2.14)$$

$$+ S + R^8,$$

with $R^8$ having the same estimate as prior error terms and with $\xi = \partial \tilde{\rho}$. Finally, by Proposition 1.11 the term in (2.13) is bigger than $-O(\epsilon) \int_{\partial U_\kappa} |u|^2 dV = -O(\epsilon')\|u\|^2_{H^q(M)} + o(\epsilon') = -c\|u\|^2_{H^q(U_\kappa)} + o(\epsilon')$. Note that the term in (2.12) is bigger than $(t + c\|u\|^2_{H^q(U_\kappa)})$ for large $t$. If we then choose $c$ which takes care of $c'$ and of the large constant for the estimate of $R^8$, we get from (2.3) the conclusion of the theorem.

\[\square\]

3. **Tangential estimates**

We recall that we are choosing an orthonormal basis of $(1, 0)$ forms $\{\omega\} = \{\omega', \omega''\}$ satisfying

$$\partial'_{\omega_j} \rho^h \equiv 0, \partial''_{\omega_j} \rho^h = \kappa_{jh}.$$ 

We recall that $\partial'_{\omega_j} |_M$ and $\partial''_{\omega_j} |_M$ for $j \leq n - l$ are the tangential vector fields of type $(1, 0)$ and $(0, 1)$ respectively and that the $T_h := \partial''_{\omega_h} - \partial''_{\omega_h}$ and $N_h := \partial'_{\omega_h} + \partial'_{\omega_h}$ for $h \geq n - l$ are the vector fields totally real tangential and normal to $M$ respectively. We also choose the extension of our forms $u$ from $M$ to $\mathbb{C}^n$ such that $u'' \equiv 0$ and

$$\partial''_{\omega_h} u_j = \sum_{j \leq n-l} a_j \partial_{\omega_j} u_j + O(|\rho|),$$
for small coefficients $a_j$. By the $C^1$ regularity of the extensions, we then get

$$
(3.1) \quad u^\tau = u^\tau |_M + O(|\rho|), \quad \partial_{\omega_j} u^\tau = \partial_{\omega_j} u^\tau |_M + O(|\rho|), \quad T_j u^\tau = T_j u^\tau |_M + O(|\rho|).
$$

We note that (3.1) implies for $u$ the following relations between its coefficients $u_J$ and their restrictions $(u_J)|_M$

$$
(3.2) \quad ||u_J||_{H^p_0(\mathcal{U})} = \epsilon^j ||u_J||_{H^p_0(M)} + o(\epsilon^j),
$$

$$
||\partial_{\omega_j} u_J||_{H^p_0(\mathcal{U})} = \epsilon^j ||\partial_{\omega_j} u_J||_{H^p_0(M)} + o(\epsilon^j)
$$

and so on. We denote by $\bar{\partial}_b$ and $\bar{\partial}_b^*$ the tangential complexes to $M$ associated to $\bar{\partial}$ and $\bar{\partial}$ respectively. (3.2) immediately yields

**Lemma 3.1.** In the above situation we have

$$
(3.3) \quad ||u||_{H^p_0(\mathcal{U})} = \epsilon^j ||u||_{H^p_0(M)} + o(\epsilon^j),
$$

$$
(3.4) \quad ||\bar{\partial} u||_{H^p_0(\mathcal{U})} = \epsilon^j ||\bar{\partial} u||_{H^p_0(M)} + o(\epsilon^j),
$$

$$
(3.5) \quad ||\bar{\partial}^* u||_{H^p_0(\mathcal{U})} = \epsilon^j ||\bar{\partial}^* u||_{H^p_0(M)} + o(\epsilon^j).
$$

**Proof.** (3.3) is obvious. As for (3.4), we have

$$
\bar{\partial} u = \sum_{|J|=k} \sum_{j \notin J} \partial_{\omega_j} u_j \bar{\omega}_j \wedge \bar{\omega}_J,
$$

and

$$
\bar{\partial}_b u = \sum_{|J|=k} \sum_{j \notin J} \partial_{\omega_j} u_j |_M \bar{\omega}_j \wedge \bar{\omega}_J.
$$

Since

$$
(3.6) \quad \partial_{\omega_j} u_J = \partial_{\omega_j} u_J |_M + o(|\rho|^j),
$$

then (3.4) immediately follows. Similarly

$$
\bar{\partial}^* u = - \sum_{|K|=k-1} \sum_{j=1}^{n} \delta_{\omega_j} u_j K \bar{\omega}_K,
$$

and

$$
\bar{\partial}_b^* u = \sum_{|K|=k-1} \sum_{j=1}^{n} \delta_{\omega_j} u_j K \bar{\omega}_K,
$$

where we remember that $\delta_{\omega_j} = \partial_{\omega_j} - \varphi_j$. (Note here that $\bar{\partial}_b^* = \bar{\partial}^* |_M$ over $\bar{\partial}$-Neumann forms.)
We go back to Theorem 3.1. We recall that $U_\epsilon$ denotes the intersection of the tube defined by $\tilde{\rho} < 0$ with a ball $B$ centered at $z_0$. We will consider the neighborhood of $z_0$ defined by $M' = M \cap B$. If we multiply both sides of (2.2) by $\epsilon^{-1}$ and go to the limit for $\epsilon \to 0$ we get for any large $t$ and for any tangential form $u$ of degree $k \geq q + 1$

\begin{equation}
(3.7) \quad \frac{t}{3} ||u||^2_{H^s(M')} \leq ||\partial_\nu u||^2_{H^s(M')} + ||\partial^*_b u||^2_{H^s(M')}.
\end{equation}

We deal now with the (unweighted) Sobolev spaces $H^s$ (for $s$ integer). We will emphasize from now on the dependence of $\varphi_t$ on $t$. We will assume also that $M$ is $C^\infty$. The main result of the section is the following

**Theorem 3.2.** Let $M$ be q-pseudoconvex at $z_0$. Then for any $s$, for any sufficiently large $t = t_s$, for suitable $c = c_{t_s}$ and for a suitable neighborhood $M'$ of $z_0$ we have

\begin{equation}
(3.8) \quad ||u||^2_{H^s(M')} \leq c(||\partial_\nu u||^2_{H^s(M')} + ||\partial^*_b u||^2_{H^s(M')}),
\end{equation}

for any tangential form $u$ of length $k \geq q + 1$, resp. $k \leq p - 1$. (Here we write $\partial^*_b$ to emphasize the dependence on the weight $\varphi_{t_s}$.)

Note that the weight $\varphi_{t_s}$, which is eliminated in the norms, reappears in an essential way in the operation of adjunction.

**Proof.** We denote by a common symbol $T$ all tangent vector fields that is any combination of the $\partial'_{\nu_j}$, $\partial'_{\omega_h}$ and $(\partial'_{\nu_j} - \partial'_{\omega_h})$’s. If $\alpha$ is a multiindex, we set $T^\alpha = T_1^{\alpha_1}\ldots T_n^{\alpha_n}$. We write the commutators $[\partial^*_b, T^\alpha] = A_s + A_{t_s-1}$ where $A_s$ is an operator of degree $s$ independent of $t$ and $A_{t_s-1}$ is of degree $s - 1$; thus the coefficients of $A_{t_s-1}$ are estimated by $t$. It follows that $||A_s u||_{H^0} \leq a_s ||u||_{H^s}$ and $||A_{t_s-1} u||_{H^0} \leq a_s t ||u||_{H^{s-1}}$ for a suitable constant $a_s$. We apply (3.7) to all terms of the type $T^\alpha u$ for $||\alpha|| = s$; we have

\begin{equation}
(3.9) \quad \frac{t}{3} ||T^\alpha u||^2_{H^0} \leq ||\partial_\nu T^\alpha u||^2_{H^0} + ||\partial^*_b T^\alpha u||^2_{H^0}
\end{equation}

\begin{equation}
\leq ||T^\alpha \partial_\nu u||^2_{H^0} + ||T^\alpha \partial^*_b u||^2_{H^0} + a_s ||u||_{H^s} + a_s t ||u||_{H^{s-1}}.
\end{equation}

Now, by inductive assumption we have

$a_s ||u||^2_{H^s} + a_s t ||u||^2_{H^{s-1}} \leq a_s ||u||^2_{H^s} + a_s c_{s-1} t (||\partial_\nu u||^2_{H^{s-1}} + ||\partial^*_b u||^2_{H^{s-1}}).

If we take $t$ so large that $\frac{t}{3} - a_s c_{s-1} \geq 1$ (in such a way that the term involving $||u||_{H^s}$ in the right side of (3.9) can be “absorbed” in the left), and define $c_s := 1 + a_s c_{s-1} t$, we get the conclusion of the proof of (3.8).
We point out that only the use of the weight $\varphi_{ts}$ produces a big constant on the left side of (3.7) which makes it possible to pass through derivatives absorbing the constants $a_s$ and $c_{s-1}$ in the above proof. Once this is carried out, we come back to unweighted estimates (since the spaces $H^0$ and $H^0_{\varphi_{ts}}$ coincide and have equivalent norms). Thus, we did eventually get rid of the weights from our norms. However they did a great service and gave the control of the derivatives of the coefficients of our forms $u$.

4. Existence theorems for $\bar{\partial}_b$

The main applications of the tangential estimates of §3 consist in results of local existence of $C^\infty$ solutions for $\bar{\partial}_b$. We will follow here closely the theory by Kohn. If $s$ is any Sobolev index, we take $t = ts$ such that the conclusions of Theorem 3.2 hold: thus (3.8) is satisfied. We recall that we are denoting by $\bar{\partial}_{b,ts}$ the transposed of $\bar{\partial}_b$ in the $H^0_{\varphi_{ts}}$ scalar product. We set

$$\Box_{b,ts} = \bar{\partial}_b \bar{\partial}_{b,ts}^* + \bar{\partial}_{b,ts}^* \bar{\partial}_b.$$ 

We remark that with this notation, (3.8) can be rewritten as

$$\frac{t}{3} ||u||^2_{H^0_{\varphi_{ts}}} \leq (\Box_{b,ts} u, u)_{H^0_{\varphi_{ts}}} \leq ||\Box_{b,\varphi_{ts}} u||_{H^0_{\varphi_{ts}}} \langle u ||u||_{H^0_{\varphi_{ts}}},$$

for any tangential form $u$ of degree $k \geq q + 1$. Denote by $R_{\Box_{b,ts}}$ and $D_{\Box_{b,ts}}$ the range and the domain of $\Box_{b,ts}$ respectively. It follows from (4.1) that $R_{\Box_{b,ts}}$ is closed and $\Box_{b,ts}$ is injective. From the orthogonal decomposition $H^0_{\varphi_{ts}} = R_{\Box_{b,ts}} \oplus \text{Ker} \Box_{b,ts} = R_{\Box_{b,ts}}$, we conclude that there is a well defined “weighted $\bar{\partial}$-Neumann operator”

$$N_{b,ts} : L^2 \rightarrow D_{\Box_{b,ts}},$$

such that $N_{b,ts} \Box_{b,ts} = \Box_{b,ts} N_{b,ts} = \text{id}$ and which satisfies

$$t ||N_{b,ts} f||_{H^0_{\varphi_{ts}}} \lesssim ||f||^2_{H^0_{\varphi_{ts}}} \forall f \in C^\infty,$$

where “$\lesssim$” denotes estimation up to a multiplicative constant. We can also rephrase the conclusions of Theorem 3.2 in terms of the weighted Neumann operator: for any $s$ and for a suitable $ts$ we have

$$||N_{b,ts} f||_s \lesssim ||f||_{H^s} \text{ if } f \text{ and } N_{b,ts} f \text{ are } C^\infty.$$

We want to get rid of the condition $N_{b,ts} f \in C^\infty$ from equation (4.3). For this purpose we define an elliptic perturbation $\Box_{b,ts}^e := \Box_{b,ts} + \Box_{b,ts}^e$. We remark that this is an elliptic operator and satisfies

$$t ||N_{b,ts} f||_{H^0_{\varphi_{ts}}} \lesssim ||f||^2_{H^0_{\varphi_{ts}}} \forall f \in C^\infty,$$

which allows us to use the weighted $\bar{\partial}$-Neumann operator $N_{b,ts}^e$ to get rid of the condition $N_{b,ts} f \in C^\infty$.
\( \sigma(\sum T^2) \) where the sum is extended to a full set of tangential vector fields. This yields an inverse “regularizing” operator

\[
N^\sigma_{b,t_s} : H^s \to D_{\square_{b,t_s}} \cap H^{s+1},
\]

which satisfies

\[
||N^\sigma_{b,t_s} f||_{H^s} + \sigma ||N^\sigma_{b,t_s} f||_{H^{s+1}} \lesssim ||f||_{H^s}.
\]

It follows that for some \( \sigma_j \to 0 \), the sequence \( N^\sigma_{b,t_s} f \) has a weak \( H^s \)-limit. Hence for \( f \in H^s \), we have \( N_{b,t_s} f \in H^s \) and \( N^\sigma_{b,t_s} f \to N_{b,t_s} f \); in particular

\[
||N_{b,t_s} f||_{H^s} \sim ||f||_{H^s} \quad \forall f \in C^\infty,
\]

and, in fact, for any \( f \in H^s \) by density. By using the above construction we get the following statement

**Proposition 4.1.** Fix \( s \) and assume that for suitable \( t = t_s \) \((3.8)\) holds for forms of a certain degree \( k \). Let \( f \) be a \( C^\infty \) form on \( M' \) of degree \( k \) satisfying \( \bar{\partial}_b f = 0 \), and define \( u := \bar{\partial}_b^* N_{b,t_s} f \); then \( u \) belongs to \( H^s \) and satisfies

\[
\begin{aligned}
\bar{\partial}_b u &= f, \\
||u||_{H^s} &\lesssim ||f||_{H^s}.
\end{aligned}
\]

The afore-defined \( u \) is orthogonal to \( \text{Ker} \ \bar{\partial}_b \) and it is also clear that under such condition there is uniqueness for the solution. Note that according to Theorem \( 3.2 \) the hypotheses of Proposition 4.1 are fulfilled, for any \( s \), for any degree \( k \geq q + 1 \) and for a suitable neighborhood \( M' \) of \( z_o \), when \( M \) is \( q \)-pseudoconvex at \( z_o \).

**Proof.** We have

\[
\bar{\partial}_b g = \bar{\partial}_b \square_{b,t_s} N_{b,t_s} g = \bar{\partial}_b \left( \bar{\partial}_b^* \bar{\partial}_b + \bar{\partial}_b \bar{\partial}_b^* \right) N_{b,t_s} g
= \left( \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b \right) \bar{\partial}_b N_{b,t_s} g.
\]

Hence, if \( f \) satisfies \( \bar{\partial}_b f = 0 \), we have

\[
0 = N_{b,t_s} \bar{\partial}_b f = N_{b,t_s} \square_{b,t_s} \bar{\partial}_b N_{b,t_s} f
= \bar{\partial}_b N_{b,t_s} f.
\]

It follows that for \( u := \bar{\partial}_b^* N_{b,t_s} f \) we have

\[
f = \left( \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b \right) N_{b,t_s} f
= \bar{\partial}_b \left( \bar{\partial}_b^* N_{b,t_s} f \right)
= \bar{\partial}_b \left( \bar{\partial}_b N_{b,t_s} f \right).
\]
This completes the proof of the first of (4.7).

As for the second, we first recall that \( \|N^b_{s,t,s} f\|_{H^{s}_{\tilde{\nu}ts}} \lesssim \|f\|_{H^{s}_{\tilde{\nu}ts}} \). Next, we remark that

\[
(\bar{\partial}_b N_{b,t,s} f, \bar{\partial}_b N_{b,t,s} f) + (\bar{\partial}^*_{b,t,s} N_{b,t,s} f, \bar{\partial}^*_{b,t,s} N_{b,t,s} f) = (\Box_{b,t,s} N_{b,t,s} f, N_{b,t,s} f) \leq \|f\|_{H^{s}_{\tilde{\nu}ts}} \lesssim \|f\|_{H^{s}_{\tilde{\nu}ts}}^2.
\]

This implies immediately the second of (4.7). □

Theorem 4.2. Let \( M \) be \( C^\infty \) and \( q \)-pseudoconvex at \( z_0 \). Then for any \( f \) in \( C^\infty \) of degree \( k \geq q + 1 \) with \( \bar{\partial}_b f = 0 \), we can find a \( C^\infty \) solution \( u \) of \( \bar{\partial}_b u = f \) at \( z_0 \).

Proof. According to Theorem 3.2 for any \( t = t_s \) holds for forms of any degree \( k \geq q + 1 \) and for a suitable neighborhood \( M' \) of \( z_0 \). According to Proposition 4.1 we can find for any \( s \) an \( H^s \) solution \( u_s \) in \( M' \) with the estimate \( (4.7) \). We want to carry on our proof by showing by induction that there is a sequence of solutions \( u_\nu \in H^\nu \) of \( \bar{\partial}_b u_\nu = f \) which satisfies

\[
(4.13) \quad \|u_{\nu+1} - u_\nu\|_{H^\nu} \lesssim 2^{-\nu}.
\]

In fact, once \( u_1, \ldots, u_\nu \) have been found, we take \( \bar{u}_{\nu+1} \in C^\infty \) and \( v_{\nu+1} \in H^{\nu+1} \) such that

\[
(4.14) \quad \begin{cases}
\|\bar{u}_{\nu+1} - u_\nu\|_{H^\nu} \leq 2^{-(\nu+1)} \\
\bar{\partial}_b v_{\nu+1} = f - \bar{\partial}_b \bar{u}_{\nu+1}, \\
\|v_{\nu+1}\|_{H^\nu} \leq \|f - \bar{\partial}_b \bar{u}_{\nu+1}\|_{H^\nu} \leq 2^{-(\nu+1)}.
\end{cases}
\]

If we then set

\[
u_{\nu+1} := \bar{u}_{\nu+1} + v_{\nu+1},
\]

we have

\[
(4.15) \quad \|u_{\nu+1} - u_\nu\|_{H^\nu} \leq \|\bar{u}_{\nu+1} - u_\nu\|_{H^\nu} + \|v_{\nu+1}\|_{H^\nu} \leq 2^{-(\nu+1)} + 2^{-(\nu+1)} = 2^{-\nu}.
\]
Thus $u_{\nu+1}$ is a solution of $\bar{\partial}_b u_{\nu+1} = f$ in $M'$ which satisfies (4.13). □

5. The Hypersurface Case

We want to end by discussing in greater detail the case of a hypersurface $M$. We have already seen that by setting $q = \max(s^-(z_0) + s^0(z_0), s^+(z_0) + s^0(z_0))$ we have local $q$-pseudoconvexity at $z_0$. Also, if $s^0$ is locally constant when we move $z$, then it has been proved that in fact local $q$-pseudoconvexity holds for the lower choice $q = \max(s^-(z_0), s^+(z_0))$. In both cases the equation $\bar{\partial}_b u = f$ has local $C^\infty$ solution $u$ for any $C^\infty$ datum $f$ with $\bar{\partial}_b f = 0$ in any degree $k$ bigger than the corresponding $q$. However, we can improve much our existence theorems. To this end we denote by $U_\pm$ the two components of $\mathbb{C}^n \setminus M$ with outward conormals $\pm \xi$. We still assume that $s^\pm$ are constant and notice that $s^-(z, -\xi) = s^+(z, +\xi)$. The argument of the above sections can be applied separately to each domain $\bar{U}^+$ and $\bar{U}^-$ which is $s^-$ and $s^+$ pseudoconvex with respect to its respective conormal:

**Proposition 5.1.** Let $s^0$ be constant in a neighborhood of $z_0$. Then for any $f$ with $C^\infty(\bar{U}^\pm)$ coefficients in a neighbourhood of $z_0$, satisfying $\bar{\partial}_b f = 0$ and of degree $k \geq s^\pm + 1$, there exists a solution $u$ of $\bar{\partial}_b u = f$ in a neighbourhood of $z_0$ with coefficients in $C^\infty(\bar{U}^\pm)$.

We pass to consider the equation $\bar{\partial}_b u = f$ for $k \leq s^\mp - 1$. In this case, we have the so called local $s^\mp$-pseudoconcavity. Similar arguments as in Section 2 go through without need of a weight $t|z|^2$ and yield the so called “subelliptic estimates”

\begin{equation}
||u||^2_{H^1(\bar{U}^\pm)} \lesssim ||\bar{\partial}u||^2_{H^0(\bar{U}^\pm)} + ||\bar{\partial}^* u||^2_{H^0(\bar{U}^\pm)} + ||u||^2_{H^0(\bar{U}^\pm)}
\end{equation}

∀$u$ of degree $k \leq s^\mp - 1$.

The estimate (5.1) yields “gain” of regularity for the solution $u$ with respect to the datum $f$; in particular it implies the hypoellipticity of the system $(\bar{\partial}, \bar{\partial}^*)$. Moreover, by replacing in the calculations of Section 2 the weight $t|z|^2$ by $-t \sum_{j \leq s^-} |z_j|^2 + t \sum_{j > s^+ + 1} |z_j|^2$ in case of $\bar{U}^+$, resp. $-t \sum_{j \leq s^+} |z_j|^2 + t \sum_{j > s^- + 1} |z_j|^2$ for $\bar{U}^-$, we can prove $H^0$ estimates for $\bar{U}^+$, resp. $\bar{U}^-$, of the type of those in Theorem 2.1 which imply local existence of $H^0$ solutions (cf. [12] Theorem 3.3.1). In combination with the afore-mentioned hypoellipticity this implies that the equation $\bar{\partial}_b u = f$ with $\bar{\partial}_b f = 0$ is locally solvable in $C^\infty(\bar{U}^\pm)$ for any degree $k \leq s^\mp - 1$. On the other hand it is classical that the tangential $\bar{\partial}$-problem for the hypersurface $M$ can be split into the $\bar{\partial}$ problems for the half-spaces $\bar{U}^\pm$. In fact, any germ of $C^\infty$ form $f$ satisfying $\bar{\partial}_b f = 0$
on $M$ can be decomposed into the sum $f = f^+ \oplus f^-$ with $f^\pm$ satisfying $ar{\partial} f^\pm = 0$ on $U^\pm$. This yields

**Theorem 5.2.** (Cf. [21]) Let $M$ be a $C^\infty$ hypersurface such that $s^0$ is constant in a neighborhood of $z_0$. Then for any germ of $C^\infty(M)$ form $f$ at $z_0$ of degree $k \neq s^-, s^+$, satisfying $\bar{\partial}_b f = 0$, there exists a germ of $C^\infty(M)$ form $u$ which solves $\bar{\partial}_b u = f$.

Let us point out that according to [21] the equation $\bar{\partial}_b u = f$ is not solvable in the two critical degrees $s^-$ and $s^+$; when the Levi form of $M$ is non-degenerate, that is $s^0 = 0$, the result was already proved in [2]. If we go back to the literature, the solvability of the system $\bar{\partial}_b$ in degree $k$ is related to the so-called $Y(k)$-condition by Kohn and Hörmander: the Levi form of $M$ has $\max(k + 1, n - k)$ eigenvalues of the same sign or $\min(k + 1, n - k)$ pairs of eigenvalues of opposite sign at each point. Another equivalent formulation of $Y(k)$ is that: $k \notin [s^-, s^- + s^0] \cup [s^+, s^+ + s^0]$. Under this condition Kohn and Hörmander proved tangential estimates of subelliptic type [5.1] which yield existence of smooth solutions in degree $k$, except for a finite-dimensional set of $f$.

Indeed, by an argument similar to the one which led to Theorem 5.2, they proved that there are no exceptions at all. If we compare with our Theorem 5.2 we see that when $s^0$ is constant, then we have got new results of solvability for all indices $k \in (s^-, s^- + s^0) \cup (s^+, s^+ + s^0)$.

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