Uncertainty, Monogamy and Locking of Quantum Correlations

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Squashed entanglement and entanglement of purification are quantum mechanical correlation measures and defined as certain minimisations of entropic quantities. In this paper, we present the first non-trivial calculations of both quantities. Our results lead to the conclusion that both measures can drop by an arbitrary amount when only a single qubit of a local system is lost. This property is known as “locking” and has previously been observed for other correlation measures such as accessible information, entanglement cost and logarithmic negativity.

In the case of squashed entanglement, the results are obtained using an inequality that can be understood as a quantum channel analogue of well-known entropic uncertainty relations. This inequality may prove a useful tool in quantum information theory.

The regularised entanglement of purification is known to equal the entanglement needed to prepare many copies of quantum state by local operations and a sublinear amount of communication. Here, monogamy of quantum entanglement (i.e., the impossibility of a system being maximally entangled with two others at the same time) leads to an exact calculation for all quantum states that are supported either on the symmetric or on the antisymmetric subspace of a $d \times d$-dimensional system.

I. INTRODUCTION

Von Neumann entropy plays an important role in many areas of quantum information theory, most fundamentally as the asymptotic rate of quantum source coding [21, 24]. Von Neumann entropy also appears in the study of correlation properties of bipartite quantum states, though less directly and only via difficult optimisations and regularisation. A well-known example is the entanglement of formation,

$$E_F(\rho) = \min \left\{ \sum_i p_i E(\psi_i) : \rho = \sum_i p_i \psi_i \right\},$$

where the minimisation is performed over all pure state ensembles \{\(p_i, \psi_i = |\psi_i\rangle \langle \psi_i|\)\} of \(\rho\), and for a pure state \(\psi\), \(E(\psi^{AB}) = S(\text{Tr}_B\psi^{AB})\) is the entropy of entanglement. Throughout this paper we denote the restrictions of states to subsystems by appropriate superscripts. Entanglement measures as well as correlation measures of a state are understood as being relative to the bipartite cut between all “A” systems (\(A, A', A_1, \text{etc.}\)) and all “B” systems. The entanglement cost \(E_C\), the asymptotic cost to prepare a quantum state from singlets, is well-known to be given by the regularised entanglement of formation [16],

$$E_C(\rho) = \lim_{n \to \infty} \frac{1}{n} E_F(\rho^{\otimes n}).$$

In this paper, however, we are concerned with a different type of minimisation, namely a minimisation over arbitrary extensions of the system. This minimisation occurs in two quantities that are relevant for measuring the amount of correlation in a system.

Squashed entanglement \[9\] is an entanglement monotone, i.e. a non-negative functional on state space, which decreases under local operations and classical communication (LOCC), and is given by

$$E_{sq}(\rho^{AB}) = \inf \left\{ \frac{1}{2} I(A; B|E) : \rho^{AB} = \text{Tr}_E \rho^{ABE} \right\},$$

where the minimisation is taken over all extensions \(\rho^{ABE}\) of \(\rho^{AB}\), and \(I(A; B|E) = S(\rho^{AE}) + S(\rho^{BE}) - S(\rho^{E}) - S(\rho^{ABE})\) is the quantum conditional mutual information. In this respect, we will always denote entropies of reduced states as the entropy of the corresponding subsystem, if the underlying global state is clear: e.g., \(S(\rho^{AE}) = S(\rho^{AE})_\rho = S(\rho^{AE})\). A priori the minimisation cannot be restricted to quantum states on a system \(E\) of bounded size, which is the reason why a numerical algorithm has not been found to date. Interestingly, precisely this unboundedness leads to very simple proofs of the additivity and superadditivity properties of squashed entanglement \[9\].

Entanglement of purification \[23\] is a measure of the total correlation in a bipartite state and defined as

$$E_p(\rho^{AB}) = \min \left\{ S(\rho^{AE}) : \rho^{AB} = \text{Tr}_E \rho^{ABE} \right\},$$

where the minimisation is taken over all extensions \(\rho^{ABE}\) of \(\rho^{AB}\). Due to the concavity of the entropy, it suffices to restrict the minimisation to systems \(E\) of bounded size.
In fact, the asymptotic cost of preparing many copies of $\rho$ from singlets using only local operations and a sublinear amount of quantum or classical communication is given by the regularisation,

$$E_P^\gamma(\rho) = \lim_{n \to \infty} \frac{1}{n} E_P(\rho^\otimes n).$$

Based on an information inequality derived in section [11] in section [11] we evaluate squashed entanglement for a family of states introduced in [20], and show that it can exhibit a property known as “locking”. This means that squashed entanglement can drop by an arbitrary amount when a single qubit is removed from the quantum state. Then, in section [14] we evaluate entanglement of purification and its regularisation for states which are invariant under exchange of systems $A$ and $B$; i.e., states that are supported on either the symmetric or the antisymmetric subspace. Our result easily implies locking for the asymptotic entanglement of purification, too.

The property of locking is shared by a number of other correlation measures such as intrinsic information [22], accessible information [12], entanglement of formation, entanglement cost and logarithmic negativity [20] and is closely related to the irreversibility of information-theoretical tasks. In contrast, the secret key rate of a so-called ccq-state, with respect to the Holevo information, is not lockable [22]. A ccq-state is defined in [11] as a correlation where Alice and Bob have classical information, while the eavesdropper has a quantum system. To be precise, information passing to the eavesdropper can decrease the key rate by no more than the entropy of the compromised data. It is an open question whether or not distillable entanglement is lockable. It is known, however, that the relative entropy of entanglement is not lockable, because it can drop by at most 2 units when a single qubit is lost [20].

So far, entropic uncertainty relations constitute the only known tool to prove locking for entanglement measures. They provide lower bounds on the sum of entropies when two incommensurable measurements are performed on identical systems and were first considered by Bialynicki-Birula and Mycielski [2], and Deutsch [10].

The generalisation by Maassen and Uffink [22] proved useful in the present context [12]. The inequality that we derive in section [11] can be interpreted as a quantum channel analogue of an entropic uncertainty relation. We provide two proofs for it. The first is operationally motivated and interprets the information quantities as superdense coding capacities of quantum channels. The second is analytical and makes use of the strong subadditivity of von Neumann entropy.

In contrast, our calculation of the entanglement of purification relies on the “monogamy” of entanglement: that fact that quantum mechanics limits the possible entanglement between two systems if one of them is already entangled with a third one. As in the case of squashed entanglement, we provide two proofs, one operational, the other analytical.

II. QUANTUM CHANNEL UNCERTAINTY RELATIONS

In this section we prove an inequality, which generalises the setting of the Maassen-Uffink entropic uncertainty relation [22], though it is more special in other respects. Entropic uncertainty relations lower bound the sum of the noise of complementary (or more generally non-commuting) measurements on quantum states. An equivalent viewpoint is to consider one measurement applied to two possible states that are related by the basis transform that changes between the measurements. Such statements can be rewritten in such a way that the roles of states and measurement are swapped, and then they imply upper bounds on the sum of the classical mutual information obtained by a measurement of an ensemble of states and the complementary ensemble (see corollary [4] below and [12]).

We extend the formulation from measurements to completely positive and trace preserving (CPTP) quantum channels $\Lambda$. The resulting inequality relates Holevo information quantities of the channel output to the quantum mutual information of the channel.

More precisely, we consider a uniform ensemble $E_0 = \{\frac{1}{d}, |i\rangle \}_{i=1}^d$ of basis states of a Hilbert space $H$ and the rotated ensemble $E_1 = \{\frac{1}{\sqrt{d}}, U|i\rangle \}_{i=1}^d$, with a unitary $U$. Applying the map $\Lambda$ (with output in a potentially different Hilbert space) we obtain ensembles

$$\Lambda(E_0) = \left\{ \frac{1}{d}, \Lambda(|i\rangle\langle i|) \right\},$$

$$\Lambda(E_1) = \left\{ \frac{1}{d}, \Lambda(U|i\rangle\langle i|U^\dagger) \right\},$$

with the Holevo information

$$\chi(\Lambda(E_0)) = S \left( \frac{1}{d} \sum_i \Lambda(|i\rangle\langle i|) \right) - \frac{1}{d} \sum_i S(\Lambda(|i\rangle\langle i|)),$$

and similarly for $E_1$. We also consider the quantum mutual information of $\Lambda$ relative to the maximally mixed state $\tau = \frac{1}{d} I$, which is the average state of either $E_0$ or $E_1$:

$$I(\tau; \Lambda) = S(\tau) + S(\Lambda(\tau)) - S((\id \otimes \Lambda) \Phi_d),$$

where $\Phi_d$ is a maximally entangled state of Schmidt rank $d$ purifying $\tau$.

**Lemma 1.** Let $U$ be the Fourier transform of dimension $d$, i.e. of the Abelian group $\mathbb{Z}_d$ of integers modulo $d$. Then, for all CPTP maps $\Lambda$,

$$\chi(\Lambda(E_0)) + \chi(\Lambda(E_1)) \leq I(\tau; \Lambda). \quad (3)$$

**Proof.** Define $\rho = (\id \otimes \Lambda) \Phi_d$, and let $M_0$ be the projection onto the basis $\{|i\rangle\}$,

$$M_0(\varphi) = \sum_{i=1}^d |i\rangle\langle i| \varphi |i\rangle \langle i|.$$
and $M_1$ the projection onto the conjugate basis $\{U|i\}\) with the states

$$
\rho_0 := (M_0 \otimes \text{id})\rho = (M_0 \otimes \Lambda)\Phi_d, \\
\rho_1 := (M_1 \otimes \text{id})\rho = (M_1 \otimes \Lambda)\Phi_d,
$$

the inequality \((4)\) can be equivalently expressed with the help of the relative entropy:

$$
D(\rho_0\|\tau \otimes \Lambda(\tau)) + D(\rho_1\|\tau \otimes \Lambda(\tau)) \leq D(\rho\|\tau \otimes \Lambda(\tau)). \quad (4)
$$

Now, let $X$ be the cyclic shift operator of the basis $\{|i\}\)$, and $Z = UXU^\dagger$ the cyclic shift of the conjugate basis $\{|i\}\)$ (these are just the discrete Weyl operators). The significance for our proof of taking $U$ as the Fourier transform lies in the fact that $\{|i\}\)$ is the eigenbasis of $Z$, and $\{U|i\}\)$ is the eigenbasis of $X$. Hence,

$$
M_0(\varphi) = \frac{1}{d} \sum_{b=1}^{d} Z^b \varphi Z^{-b}, \quad (5) \\
M_1(\varphi) = \frac{1}{d} \sum_{a=1}^{d} X^a \varphi X^{-a}. \quad (6)
$$

The idea of the proof is now to interpret the relative entropies in inequality \((4)\) as (asymptotic) dense coding capacities using the states

$$
\rho_{ab} := (X^a Z^b \otimes 1)(Z^{-b} X^{-a} \otimes 1).
$$

The left hand side is an achievable rate for uniform random coding when the decoder separately infers $a$ and $b$. The right hand side is an upper bound on the rate of any code using the signal states with equal frequency and equals the Holevo quantity of this ensemble \([4, 3, 17]\). This is sufficient to prove the inequality.

We now give a second proof, in which the random coding argument is replaced with analytic reasoning. Simply define the correlated state

$$
\Omega := \frac{1}{d^2} \sum_{a,b=1}^{d} |a\rangle\langle a|^{A} \otimes |b\rangle\langle b|^{B} \otimes \rho_{C,b} ^{C}.
$$

It is straightforward to verify, using eqs. \((5)\) and \((6)\), that

$$
D(\rho\|\tau \otimes \Lambda(\tau)) = I(AB;C)\alpha, \\
D(\rho_0\|\tau \otimes \Lambda(\tau)) = I(A;C)\alpha, \\
D(\rho_1\|\tau \otimes \Lambda(\tau)) = I(B;C)\alpha,
$$

and we can show eq. \((1)\) as follows:

$$
I(AB;C) = I(A;C) + I(B;C|A) \\
= I(A;C) + I(B;AC) \\
\geq I(A;C) + I(B;C),
$$

where we have used only standard identities and strong subadditivity and where in the second line the independence of $A$ and $B$ expresses itself as $I(B;AC) = I(B;A) + I(B;C|A) = I(B;C|A)$.

**Remark 2.** The statement of lemma \(1\) holds more generally for any finite Abelian group labeling the ensemble $E_0$ and $U$ the Fourier transform of that group; e.g. for $d = 2^t$, $U = H^\otimes t$, with the Hadamard transform $H$ of a qubit, corresponding to the group $\mathbb{Z}_2^t$. The proof goes through basically unchanged; one only has to replace the operators $X$ and $Z$ by the regular representation of the group and its conjugate via the Fourier transform. Except for the slightly more awkward notation, the randomisation formulas \((5)\) and \((6)\) for the projections $M_0$ and $M_1$ still hold true, and from there the proof follows the one given above literally.

This result implies the following corollary, which has previously been proved in \([12]\) using the entropic uncertainty relation $S(M_0(\rho)) + S(M_1(\rho)) \geq \log d$, for all $\rho$. The latter is an instance of more general entropic uncertainty relation derived in \([22]\).

**Corollary 3.** For $U$ the Fourier transform as above, and the ensemble $E = \frac{1}{2}E_0 + \frac{1}{2}E_1$,

$$
I_{\text{acc}}(E) = \frac{1}{2} \log d.
$$

**Proof.** Let $X$ denote a random variable uniformly distributed over the labels $ij$ ($i = 1, \ldots, d, j = 0, 1$) of the ensemble $E$. The left hand side of inequality \((3)\) then equals $2I(X;Y)$ in the special case where the CPTP map $\Lambda$ is a measurement with outcome $Y$, while the right hand side, $I(\tau;\Lambda)$, is upper bounded by $\log d$. Clearly, a measurement performed in one of the two bases will achieve this bound.

**III. SQUASHED ENTANGLEMENT**

The tools derived in the previous section now enable us to calculate squashed entanglement for the states considered in \([20]\).

**Proposition 4.** For “flower states” \([24]\) $\rho^{AA'BB'}$ with a purification of the form

$$
|\Psi\rangle^{AA'BB'EC} = \frac{1}{\sqrt{2^d}} \sum_{i=0,1} \sum_{j=0,1} |i\rangle^{A} |j\rangle^{A'} |i\rangle^{B} |j\rangle^{B'} U_j \rho^{AB} |i\rangle^{C}, \quad (7)
$$

where $U_0 = 1$ and $U_1$ is a Fourier transform, we have

$$
E_{\text{sq}}(\rho^{AA'BB'}) = 1 + \frac{1}{2} \log d \quad \text{and} \quad E_{\text{sq}}(\rho^{AB}) = 0.
$$

**Proof.** The minimisation over state extensions $\rho^{ABE}$ in squashed entanglement is equivalent to a minimisation over CPTP channels $\Lambda : C \rightarrow E$, acting on the purifying system $C$ for $\rho^{AA'BB'}$ \([3]\):

$$
\rho^{AA'BB'E} = (\text{id}^{AA'BB'} \otimes \Lambda)|\Psi^{AA'BB'EC}.
$$
The reduced state of $\Psi$ on $C$ is maximally mixed:

$$\text{Tr}_{AA'BB'} \Psi = \tau = \frac{1}{d} \mathbb{1},$$

and hence

$$S(\rho^E) = S(\Lambda(\tau)),$$

$$S(\rho^{AA'BB'E}) = S((\text{id} \otimes \Lambda)\Phi_d). \quad (8)$$

Since the state $\rho$ is maximally correlated we can write the reduced states of $\rho$ onto $ZZ'E$, for $ZZ' \in \{AA', BB'\}$:

$$\rho^{ZZ'E} = \frac{1}{2d} \sum_{i,j} |i\rangle\langle i| \otimes |j\rangle\langle j| \otimes \Lambda(U_j|i\rangle\langle U_j^1|)^E.$$

Thus we can calculate the other entropy terms of the quantum conditional mutual information,

$$S(\rho^{AA'E}) = S(\rho^{BB'E})$$

$$= \log d + 1 + \frac{1}{2d} \sum_{i,j} S(\Lambda(U_j|i\rangle\langle U_j^1|))$$

$$= 1 + S(\tau) + S(\Lambda(\tau))$$

$$\geq 2 + \log d,$$

where the last inequality is an application of lemma 1.

This bound is achieved for trivial $E$, since $I(A; B) = 2 + \log d$.

On the other hand, $\rho^{ABB'}$ is evidently separable and thus has zero squashed entanglement.

**Remark 5.** It is an open question whether or not the minimisation in squashed entanglement can be taken only over POVMs. If so, the simpler argument $I(\Lambda(AA'; B')E) \geq I(\Lambda(A; B')E) = 2 \log d + 2 - \log d$, only using corollary 3 proves proposition 4.

In [20] it was observed that $E_C(\rho^{AA'BB'}) \geq \frac{1}{2} \log d$.

We remark that the argument given in [20] actually proves

$$E_C(\rho^{AA'BB'}) = E_F(\rho^{AA'BB'}) = 1 + \frac{1}{2} \log d,$$

via corollary 3 with the easy relation

$$E_F(\rho^{AA'BB'}) = S(\rho^A) - \max_M \chi = S(\rho^A) - I_{\text{acc}}(\mathcal{E}),$$

where the maximisation is over all measurements $M$ on $E$ and $\chi$ is the Holevo quantity of the induced ensemble on $AA'$. In fact, we can obtain this directly as a corollary of proposition 4 by observing that $E_C(\rho) \geq E_{\text{sq}}(\rho)$, and that equality is achieved (even for $E_F$) for $\Lambda$ being a complete measurement in one of the mutually conjugate bases.

The gap between entanglement of formation and squashed entanglement, as well as between squashed entanglement and distillable entanglement, can be made simultaneously large. This is shown below, where we use an idea of [14] to bound the entanglement cost.

**Proposition 6.** Let $\rho^{AA'BB'}$ be defined by the purification

$$|\Psi\rangle = \frac{1}{\sqrt{2dm}} \sum_{i \in [d], j \in [m], k \in [1...m]} |ijk\rangle^{AA'} |ijk\rangle^{BB'} V_k U_j |i\rangle^C,$$

where $U_0 = \mathbb{1}$ and $U_1$ is a Fourier transform. For all $\epsilon > 0$ and large enough $d$ there exists a set of $m = \lfloor (\log d)^3 \rfloor$ unitaries $V_k$ such that

$$E_C(\rho^{AA'BB'}) \geq (1 - \epsilon) \log d + 3 \log \log d - 3,$$

$$E_{\text{sq}}(\rho^{AA'BB'}) = \frac{1}{2} \log d + 3 \log m + 1$$

$$= \frac{1}{2} \log d + 3 \log \log d + 1 + o(1),$$

$$E_D(\rho^{AA'BB'}) \leq 6 \log \log d + 2.$$

Hence, $E_D \ll E_{\text{sq}} \ll E_C$ is possible.

**Proof.** Define ensembles $\mathcal{E} = \{\frac{1}{m^d}, V_k U_j |i\rangle\}_{i,j,k}$ and $\tilde{\mathcal{E}} = \{\frac{1}{m^d}, \tilde{V}_k |i\rangle\}_{i,j,k}$. As already observed for the states under consideration,

$$E_F(\rho^{AA'BB'}) = S(\rho^A) - \max_M \chi$$

$$= \log d + \log m + 1 - I_{\text{acc}}(\mathcal{E}),$$

and since $I_{\text{acc}}(\mathcal{E})$ is additive (see also [13]),

$$E_C(\rho^{AA'BB'}) = \log d + \log m + 1 - I_{\text{acc}}(\mathcal{E}).$$

As is shown in [15], for all $\epsilon > 0$ and large enough $d$ there exists a set of $m = \lfloor (\log d)^3 \rfloor$ unitaries $V_k$ such that $I_{\text{acc}}(\tilde{\mathcal{E}}) \leq \epsilon \log d + 3$. Clearly the mixing of two such ensembles cannot increase the accessible information by more than 1 (operationally, even if the bit identifying the ensemble was known, a measurement would still face an ensemble isomorphic to $\mathcal{E}$): $I_{\text{acc}}(\mathcal{E}) \leq I_{\text{acc}}(\mathcal{E}) + 1$.

Therefore,

$$E_C(\rho^{AA'BB'}) \geq (1 - \epsilon) \log d + 3 \log \log d - 3.$$
the relative entropy of entanglement by more than $2(1 + \log m)$ \cite{shor1995additivity} and since the latter is a bound on distillable entanglement \cite{horodecki1998additivity}. \[E_D(\rho) \leq 2(3 \log d + 1). \]

Squashed entanglement can be regarded as a quantum analogue to intrinsic information. Intrinsic information is a classical information-theoretic quantity that provides a bound on the secret-key rate. Interestingly, the flower states, eq. \eqref{eq:flower}, can be understood as quantum analogues of the distributions analysed in \cite{renner2008security}.

\section{Entanglement of Purification}

Here we calculate entanglement of purification for symmetric and antisymmetric states and prove their additivity. We remind the reader that in the case where both systems $A$ and $B$ are of $d$-dimensional Hilbert spaces, the Hilbert space of $AB$ falls into two parts, the symmetric and the antisymmetric space

$$C^d \otimes C^d = \mathcal{H}_{\text{sym}} \oplus \mathcal{H}_{\text{anti}}.$$ \[ \]

\textbf{Proposition 7.} For all states $\rho^{AB}$ with support entirely within the symmetric or the antisymmetric subspace,

$$E_P^\infty(\rho^{AB}) = E_P(\rho^{AB}) = S(\rho^A).$$

In fact, for another such state, $\rho'^{AB}$,

$$E_P(\rho^{AB} \otimes \rho'^{AB}) = E_P(\rho^{AB}) + E_P(\rho'^{AB}).$$

\textbf{Proof.} To every quantum state that is entirely supported on the symmetric subspace we can find a purification of the form $|\Psi\rangle = \sum_i \sqrt{p_i} |\zeta_i\rangle^A |\varphi_i\rangle^C$, with $F|\zeta_i\rangle = |\zeta_i\rangle$, where $F$ stands for flip and denotes the operator swapping the two systems and $A$ and $B$. A similar form exists for states on the support of the antisymmetric subspace, $|\Psi\rangle = \sum_i \sqrt{p_i} |\alpha_i\rangle^A |\beta_i\rangle^C$ with $F|\alpha_i\rangle = -|\alpha_i\rangle$. In either case, the state $\Psi$ is invariant under exchanging $A$ and $B$:

$$(F_{AB} \otimes I)_C \Psi (F_{AB} \otimes I)_C = \Psi.$$ \[ \]

We obtain any other state extension of $\rho^{AB}$ by the application of a CPTP map $\Lambda : C \rightarrow E$, i.e.$$ho_{AE}^{AB} = (\text{id} \otimes \Lambda) \Psi^{ABC},$$

and clearly $\rho$ inherits the exchange symmetry from $\Psi$:

$$\langle F_{AB} \otimes I_E \rangle \rho (F_{AB} \otimes I_E) = \rho.$$ \[ \]

From the symmetry of $\rho_{AE}^{AB}$ and $\rho_{BE}^{AB}$ it immediately follows that $S(E|A) = S(E|B)$ and by weak monotonicity of the von Neumann entropy (which is equivalent to strong subadditivity), we have $2S(E|A) = S(E|A) + S(E|B) \geq 0$. Hence for every extension $\rho_{AE}^{AB}$, $S(E) \geq S(A)$ holds, with equality for the trivial extension.

Another way of arriving at this conclusion is via the no-cloning principle: if $\rho^{AE}$ is one-way distillable from Eve to Alice, so is $\rho^{BE}$ from Eve to Bob by symmetry, whereby Eve uses the same qubits and the same instrument for both directions. Hence, Eve would share the same maximally entangled state with both Alice and Bob, which is impossible by the monogamy of entanglement. By the hashing inequality \cite{renner2008security}, vanishing one-way distillability implies $S(E|A) \geq 0$ and $S(E|B) \geq 0$ and the conclusion on $E_P$ follows.

Since the same reasoning applies to a tensor product of two (anti-)symmetric states, this being (anti-)symmetric as well, we obtain the additivity of $E_P$ for such states, and hence $E_P^\infty$.

\textbf{Remark 8.} The above proof using monogamy and the hashing inequality has the advantage of giving a slightly more general result: assume that for a purification $\Psi^{ABC}$ of $\rho^{AB}$, $\rho^{AC}$ is not one-way distillable (from $C$ to $A$). Then for every channel $\Lambda : C \rightarrow E$, $\rho_{AE}^{AB} = (\text{id} \otimes \Lambda) \rho_{AC}^{AC}$ is still one-way nondistillable, hence $\sqrt{\rho^{AE}} \geq S(\rho^A)$. Otherwise we would have a contradiction to the hashing inequality.

\textbf{Lemma 9 (\cite{renner2008security}, Lemma 2).} Entanglement of purification is non-increasing under local operations, i.e.

$$E_P(\rho^{AB}) \geq \sum_k p_k E_P(\rho_k^{AB}),$$

where the $\rho_k$ (with probability $p_k$) have been obtained through a local quantum instrument of one of the parties, i.e. in the case of Alice,

$$p_k \rho_k^{AB} = \sum_i (A_i^{(k)} \otimes I) \rho^{AB} (A_i^{(k)} \otimes I)^\dagger,$$

with $\sum_{k,i} A_i^{(k)} A_i^{(k)^\dagger} = I$. \[ \]

This easy fact allows us to derive the following interesting consequence of proposition \cite{renner2008security}.

\textbf{Corollary 10.} Let

$$\rho^{A'AB} = p|0\rangle\langle 0|^A \otimes \sigma^{AB} + (1-p)|1\rangle\langle 1|^A \otimes \alpha^{AB},$$

with states $\sigma$ and $\alpha$ supported on the symmetric and antisymmetric subspace respectively. Then,

$$E_P^\infty(\rho^{A'AB}) = E_P(\rho^{A'AB}) \geq pS(\sigma^A) + (1-p)S(\alpha^A).$$

In particular, we have

$$E_P(\omega^{A'AB}) = \log d \quad \text{and} \quad E_P(\omega^{AB}) = 0$$

for

$$\omega^{A'AB} = \frac{d+1}{2d} |0\rangle\langle 0|^A \otimes \frac{2}{d(d+1)} P_{\text{sym}}^{AB} + \frac{d-1}{2d} |1\rangle\langle 1|^A \otimes \frac{2}{d(d-1)} P_{\text{anti}}^{AB},$$

with the projectors $P_{\text{sym}}$ and $P_{\text{anti}}$ onto the symmetric and antisymmetric subspace respectively.
Proof. Lemma 3 and proposition 7 imply that $E_p(p^{\omega_{A^B}B}) \geq pS(\sigma^{AB}) + (1 - p)S(\sigma^{AB})$, and the same for $E_p^{\infty}$.

The dimensions of the symmetric and antisymmetric subspace are given by $\frac{d(d+1)}{2}$ and $\frac{d(d-1)}{2}$. The state $\omega^{A^B}$ is constructed such that $\omega^{AB}$ is maximally mixed on $AB$, with evidently zero entanglement of purification. On the other hand, by the above,

$$E_p^{\infty}(\omega^{A^B}) \geq \frac{d + 1}{2d}E_p\left(\frac{2}{d(d+1)}P_{\text{sym}}\right) + \frac{d - 1}{2d}E_p\left(\frac{2}{d(d-1)}P_{\text{anti}}\right) = \log d.$$ 

This bound is attained since $E_p(\omega^{A^B}) \leq S(\omega^B) = \log d$. □

V. CONCLUSION

In this paper we have demonstrated the first nontrivial calculations of squashed entanglement and the regularised entanglement of purification. The former is for a family of states which came up in the context of locking of accessible information and entanglement cost, the latter for all symmetric and antisymmetric states and some affiliated states.

These calculations lead to the conclusion that both squashed entanglement and the regularised entanglement of purification are lockable, which is a significant advancement in our understanding of these quantities. We remark that both quantities obey uniform continuity properties of a form known as “asymptotic continuity”: see 1 for squashed entanglement and 20 for entanglement of purification. The locking effect of both quantities therefore does not arise for trivial reasons 20.

More profoundly, both calculations employ mathematical versions of the quantum mechanical complementarity principle: in the case of $E_{sq}$ it is a new generalised entropic uncertainty relation for conjugate bases, in the case of $E_p$ and $E_p^{\infty}$ it is the monogamy of entanglement. However tempting, one cannot claim that locking is a purely quantum mechanical effect, perhaps intimately related to complementarity. Rather, locking of intrinsic information 22 and of the (classical) correlation cost of preparing a bipartite probability distribution 22 are significant counterexamples which show that classical models also suffer locking. Nevertheless, our two calculations, and perhaps even the techniques we have developed to perform them, can be taken as expressions of the principle that complementarity is an important underlying reason for locking.

We believe that lemma 4 the information-uncertainty relation, is of great independent significance; it is the first inequality of its kind which goes beyond measurements and instead considers general quantum channels. Consequently, classical (Shannon) information is replaced by quantum mutual information. The latter paves the way for an interesting application to quantum channel coding:

It is known from 2 that for a quantum channel $\Lambda$ with fidelities of the standard basis and the phase ensemble both close to 1, i.e.

$$\frac{1}{d} \sum_{i=1}^{d} F([i][i], \Lambda([i][i])) \geq 1 - \epsilon,$$

the entanglement fidelity obeys

$$F(\Phi_d, (\text{id} \otimes \Lambda)\Phi_d) \geq 1 - 2\epsilon.$$ 

Here, $|\phi\rangle := \sum_{j=1}^{d} e^{2\pi i \phi(j)}$ denotes a vector with phases given by $\phi = (\phi_1, \ldots, \phi_d)$. Note that in this case the ensemble testing the channel’s quality contains at least $3^d + d$ states. This was recently improved by Hofmann 18 to only requiring fidelity $\geq 1 - \epsilon$ for a basis ensemble and its Fourier conjugate, with the same conclusion.

Lemma 11 gives an information version of this: if the Holevo information of the basis and the conjugate basis ensembles are $\geq \log d - \epsilon$, it implies (via lemma 11) that the coherent information

$$S(\Lambda(\tau)) - S((\text{id} \otimes \Lambda)\Phi_d) \geq \log d - 2\epsilon.$$ 

As a consequence of a result in 22, we find that there exists a postprocessing map $\Lambda'$ such that the entanglement fidelity of $\Lambda' \circ \Lambda$ is not less than $1 - 2\sqrt{2\epsilon}$. What is interesting here is that we require “good behaviour” of the channel only on $2d$ states of two mutually unbiased bases and this already leads to the conclusion that $\Lambda$ can be error-corrected to be close to the identity on the whole space. In a certain sense this is also better than Hofmann’s result 18, since we do not require that the map $\Lambda$ itself produces any high-fidelity output. Instead there only have to exist highly reliable detectors of the basis and of the conjugate basis information, respectively, which may not even be quantum mechanically compatible. Only after learning that the quantum mutual information is large, we conclude that there is a quantum decoder for the map $\Lambda$.

Another application of lemma 11 is an information gain vs. disturbance tradeoff for the Holevo information 7. Alice sends states from a basis and a conjugate ensemble via Eve to Bob. If Bob detects an average disturbance less than $\epsilon$, the Holevo information gain of Eve obeys the bound

$$\chi(E_{\epsilon}) \leq 4\sqrt{\epsilon}\log d + 2\eta(2\sqrt{\epsilon}),$$

where $\eta(x) := \min\{-x log x, \frac{1}{x}\}$ and $\chi(E_{\epsilon})$ denotes the Holevo information of Eve’s ensemble $E_{\epsilon}$. Due to the large discrepancy between Holevo information and accessible information caused by locking, this result can be seen as a significant improvement of a recently established tradeoff for the accessible information 3.
Finally, it would be interesting to generalise our inequality in various directions. Firstly, we do not expect that its truth depends on ensembles related exactly by a Fourier transform. In the vein of Maassen and Uffink [22], we would rather expect an inequality for ensembles which are not perfectly mutually unbiased. In a fully quantum version of the inequality, then, we would expect to see ensembles and Holevo quantities replaced by correlated states and quantum mutual information.

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