Regularity of Weak Solutions of Elliptic and Parabolic Equations with Some Critical or Supercritical Potentials

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Abstract
We prove Hölder continuity of weak solutions of the uniformly elliptic and parabolic equations

\[ \partial_i(a_{ij}(x)\partial_j u(x)) - \frac{A}{|x|^{2+\beta}} u(x) = 0 \quad (A > 0, \quad \beta \geq 0), \]  \hspace{1cm} (0.1)

\[ \partial_i(a_{ij}(x,t)\partial_j u(x,t)) - \frac{A}{|x|^{2+\beta}} u(x,t) - \partial_t u(x,t) = 0 \quad (A > 0, \quad \beta \geq 0), \]  \hspace{1cm} (0.2)

with critical or supercritical 0-order term coefficients which are beyond De Giorgi-Nash-Moser’s Theory. We also prove, in some special cases, weak solutions are even differentiable.

Previously P. Baras and J. A. Goldstein\textsuperscript{3} treated the case when \( A < 0 \), \((a_{ij}) = I\) and \( \beta = 0 \) for which they show that there does not exist any regular positive solution or singular positive solutions, depending on the size of \(|A|\). When \( A > 0 \), \( \beta = 0 \) and \((a_{ij}) = I\), P. D. Milman and Y. A. Semenov\textsuperscript{7,8} obtain bounds for the heat kernel.

Keywords: weak solutions, elliptic, parabolic, Hölder continuity, critical, supercritical potential

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1. Introduction

In this paper, we consider regularity of weak solutions of divergence form elliptic equations

\[ \partial_i(a_{ij}\partial_j u) - \frac{A}{|x|^{2+\beta}} u = 0 \]  \hspace{1cm} (1.1)

and parabolic equation

\[ \partial_i(a_{ij}\partial_j u) - \frac{A}{|x|^{2+\beta}} u - \partial_t u = 0 \]  \hspace{1cm} (1.2)

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in the unit ball \( B := B(0,1) \) (or \( B \times \mathbb{R}_+ \) in \( \mathbb{R}^d \), with \( d \geq 3 \), \( A > 0 \), \( \beta \geq 0 \). Here \( a_{ij} \in L^\infty(B) \) (or \( L^\infty(\mathbb{R}_+, L^\infty(B)) \)), and the second order coefficient matrix \( \left( a_{ij} \right)_{1 \leq i,j \leq d} \) satisfies the uniformly elliptic condition:

\[
\lambda I \leq \left( a_{ij} \right)_{1 \leq i,j \leq d} \leq \Lambda I, \text{ for some } 0 < \lambda \leq \Lambda < \infty.
\] (1.3)

Here and below, we use the Einstein summation convention. We say \( u \in H^1(B) \) is a weak solution of the elliptic equation of (1.1), if \( \forall \psi \in C_0^\infty(B) \), there holds

\[
\int_B a_{ij}(x)\partial_i\psi(x)\partial_j u(x)dx + \int_B \frac{A}{|x|^{2+\beta}} u(x)\psi(x)dx = 0,
\] (1.4)

where \( \partial_i \) indicates \( \partial_{x_i} \) here and below. Similarly, for the parabolic equation (1.2), we say \( u \in L^2([0,T], H_0^1(B)) \) is a weak solution, if \( \forall \psi \in C_0^\infty(B \times [-T,T]) \), there holds

\[
- \int_0^T \int_B u(x,t)\partial_t\psi(x,t)dx + \int_0^T \int_B a_{ij}(x,t)\partial_i\psi(x,t)\partial_j u(x,t)dxdt
+ \int_0^T \int_B \frac{A}{|x|^{2+\beta}} u(x,t)\psi(x,t)dxdt = \int_B \psi(x,0)u(x,0)dx.
\] (1.5)

In the middle of the last century, De Giorgi[5], Nash[11] and Moser[9][10] developed new methods on the studying of elliptic and parabolic equation, which opened a new area on the study of regularity of weak solutions of elliptic and parabolic equations in divergence form:

\[
\partial_t(a^{ij}\partial_j u) + b^i\partial_i u + cu = f + \partial_t f^i,
\] (1.6)

\[
\partial_t(a^{ij}\partial_j u) + b^i\partial_i u + cu - \partial_t u = f + \partial_t f^i.
\] (1.7)

They proved that, under certain integrable conditions of the coefficients \( b^i, c \) and non-homogeneous term \( f \) and \( f^i \), weak solutions of equation (1.6),(1.7) have \( C^\alpha \) Hölder continuity. A key condition for their theory for elliptic equation is that the coefficient of the 0-order term \( c \) must belong to the Lebesgue space \( L^p \), with \( p > \frac{d}{2} \). Obviously, the 0-order terms in our equations (1.1) and (1.2) do not satisfy this assumption. Actually the case when \( \beta = 0 \) is the critical borderline case where the theory of De Giorgi, Nash and Moser fails. See for example of Baras and Goldstein [3] in the case \( A < 0 \). When \( A > 0 \), even though it is easily seen that weak solutions are locally bounded, it is not clear these weak solutions have any regularity.

However, such equations are closely related to several physical equations. For instance the
3-dimensional axially-symmetric incompressible Navier-Stokes equations in fluid dynamics are

\[
\begin{aligned}
\frac{\partial}{\partial t}v^r + b \cdot \nabla v^r - \frac{v^\theta v^\theta}{r} + \partial_r p &= \left( \Delta - \frac{1}{r^2} \right) v^r \\
\frac{\partial}{\partial t}v^\theta + b \cdot \nabla v^\theta + \frac{r v^r v^\theta}{r^2} &= \left( \Delta - \frac{1}{r^2} \right) v^\theta \\
\frac{\partial}{\partial t}v^z + b \cdot \nabla v^z + \partial_z p &= \Delta v^z \\
b &= v^r r^r + v^z e_z, \quad \nabla \cdot b = \partial_r v^r + \frac{v^r}{r} + \partial_z v^z = 0
\end{aligned}
\]  

(1.8)

where

\[v(x,t) = v^r(r,z,t)e_r + v^\theta(r,z,t)e_\theta + v^z(r,z,t)e_z.\]  

(1.9)

Observe that the linear parts of the first and second equation of (1.8) are related to equation (1.2) with \( \beta = 0 \).

We point out here that the case when \( A < 0 \) was studied in [3] Baras and Goldstein, who proved that the Cauchy problem of heat equation

\[
\begin{aligned}
\Delta u(x,t) - \frac{A}{|x|^2} u(x,t) - \partial_t u(x,t) &= 0, \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}_+ \\
u(x,0) &= u_0
\end{aligned}
\]  

(1.10)

have no weak nonnegative solution if \(-A > \left( \frac{d-2}{2} \right)^2 \). They also prove if \( 0 < -A \leq \left( \frac{d-2}{2} \right)^2 \), (1.10) has unbounded positive weak solutions.

The case when \( A > 0 \), \( \beta = 0 \) and the leading operator being the Laplacian was first studied in Milman and Semenov [7], where the authors obtained a sharp upper bound for the fundamental solution of (1.2). Their method is to use explicit special solutions of the elliptic equation as weights and convert the studying of the problem to that of a weighted equation via Doob’s transform. Based on this bound, in this special case, one can prove Hölder continuity of solutions quite easily. Here we observe further that when \( A \) is sufficiently large, weak solutions are even differentiable. In the variable coefficient case that we are working on, it is hard or impossible to find an explicit solution of the elliptic equation. So a different method is needed to prove Hölder continuity of weak solutions.

The following are the main results of the paper. The first one pertains the elliptic equation in the case when the leading operator is the Laplacian but the result is stronger, including differentiability in some situations. This is a little unexpected since it is well known that singular potential terms usually mess up the derivative bound for solutions. We also obtain a similar result for the corresponding parabolic cases with \( \beta = 0 \). The second result deals with both elliptic and parabolic equations whose leading coefficients are just bounded. We prove Hölder continuity of weak solutions.

**Theorem 1.1.** A weak solution \( u = u(x) \in H^1_0(B) \) of \( \Delta u - \frac{A}{|x|^2} u = 0 \) has the following regularity properties. Let \( \alpha = \alpha(A) = -\frac{d^2+2+\sqrt{d^2-4d+4A}}{2} \in ]n, n+1] \) for a nonnegative integer \( n \).

(1) If \( \beta = 0 \), then \( u \in C^{n,\alpha(A)-n}(B_{1/4}) \):
(II) If \( \beta > 0 \), then \( u \in C^\infty(\mathbb{B}_{1/4}) \).

In addition,

(III) A weak solution \( u = u(x, t) \in L^2([0, T], H^1_0(B)) \) of \( \Delta u - \frac{A}{|x|^{2+\beta}} u - \partial_t u = 0 \) satisfies
\[
\partial_t^{m_1} \nabla^{m_2} u \in C^{(\alpha(A) - n - (\alpha(A) - n)/2)}(\mathbb{B}_{1/4} \times [t_0, T]),
\]
for \( 2m_1 + m_2 = n \). Here, the Hölder norms above depend on \( d, A \) and the \( L^2 \) norm of \( u \). \( T, t_0 \) are given positive constants, and \( C_\cdot \) defines any number smaller than but close to the constant \( C \).

□

Theorem 1.2. If \( \beta = 0 \), the weak solution \( u \) of the elliptic equation (1.1) is Hölder continuous, i.e.
\[
\|u\|_{C^\alpha(B_{1/4})} \leq C(\lambda, \Lambda, d, A) \|u\|_{L^2(B)}
\] (1.11)
with \( \alpha = \alpha(\lambda, \Lambda, d, A) > 0 \). Moreover any weak solution \( u = u(x, t) \) of the parabolic equation (1.2) is Hölder continuous when \( t \) is away from 0, i.e.
\[
\|u\|_{C^{\alpha, 2}(B_{1/4} \times [t_0, T])} \leq C(\lambda, \Lambda, d, A, t_0)^{-\left(d/2+1+\alpha\right)} \|u\|_{L^2}
\] (1.12)
with \( \alpha = \alpha(\lambda, \Lambda, d, A) > 0, t_0 > 0 \).

□

This theorem provides an interior estimate which deteriorates near initial time. However, this is necessary since no Hölder regularity assumption on the initial datum is made.

We also mention that the Harnack inequality could not hold for solutions of these equations, because one can find a class of non-negative solutions which do not satisfy it. See section 2 e.g. Moreover, these special solutions are instrumental in studying the regularity of the solutions of
\[
\Delta u - \frac{A}{|x|^{2+\beta}} u = 0, \quad (\beta \geq 0),
\] (1.13)
the elliptic case where the leading operator is the Laplacian. It helps to prove an \( \alpha \)-order decay estimate of the weak solution \( u = u(x) \) when \( \beta = 0 \) at \( 0 \in \mathbb{R}^d \), namely:
\[
|u(x)| \leq C|x|^{\alpha}
\] (1.14)
for \( \alpha \in (0, 1) \) and \( C \in \mathbb{R}_+ \) is a constant. When \( \beta > 0 \), the decay of \( u(x) \) at \( 0 \in \mathbb{R}^d \) turns exponential.

For the variable 2nd order coefficients case (1.1), (1.2), the situation is more complicated. Roughly speaking, we could not find a good enough special solution as the Laplacian case (1.13). However, if \( \beta = 0 \), we find a weighted mean value inequality, which is motivated by (13) and (12). The weight, decaying at certain rate near the origin, plays the same role as the special solution in the Laplacian case (1.13), giving similar \( \alpha \)-decay estimate.

The rest of the paper is organized as follows. In section 2, we give the proof of Theorem 1.1. In section 3, we state and prove the aforementioned weighted mean value inequality for
general parabolic equations (1.2). In section 4, we give the proof of the variable coefficient case for weak solutions of (1.1) with critical 0-order term coefficient. Finally in section 5, we extend our conclusion in section 4 to the parabolic case. Some elementary but useful works, giving the proof of the existence of weak solutions, local boundedness of weak solution, maximum principle, and an introduction of the modified Bessel’s equation, could be found in the Appendix.

2. Laplacian Case

In this section, we will prove Theorem 1.1. First we need a simple lemma on certain special solutions of (1.13), which will serve as a benchmark for comparison with other solutions. As mentioned in the introduction, the case when \( \beta = 0 \), Hölder continuity of solutions can also be proven by the bound in [7]. Here we give a direct proof based on the maximal principle.

Lemma 2.1. (i) If \( \beta = 0 \), then

\[
  u(x) = |x|^{\alpha}, \quad \alpha = \alpha(A) = \frac{-d + 2 + \sqrt{d^2 - 4d + 4 + 4A}}{2}
\]

is a weak solution of (1.13).

(ii) If \( \beta > 0 \), then

\[
  u(x) = |x|^{-\frac{d}{\beta} + 1} \mathcal{K}_{(d-2)/\beta}(\frac{2}{\beta} \sqrt{A|x|^{-\frac{d}{\beta}}})
\]

is a weak solution of (1.13), where \( \mathcal{K}_{(d-2)/\beta} \) is the modified Bessel’s function of second kind mentioned above.

Proof. Since we are looking for radially symmetric solution of (1.13) here, we can just solve the corresponding ODE. Define \( r = |x| \), we find the solution \( u = u(r) \). Thus, (1.13) turns to

\[
  u'' + \frac{d - 1}{r} u' - \frac{A}{r^{2+\beta}} u = 0
\]

If \( \beta = 0 \), this is an Euler type ODE. Set \( u = r^\alpha \), and take this into the equation (2.3), we have

\[
  \alpha^2 + (d - 2)\alpha - A = 0
\]

Solve this equation with a positive number, we have

\[
  \alpha = \alpha(A) = \frac{-d + 2 + \sqrt{d^2 - 4d + 4 + 4A}}{2}.
\]

If \( \beta > 0 \), suppose \( B_\lambda(r) \) satisfies the modified Bessel’s equation

\[
  r^2 B''_\lambda(r) + r B'_\lambda(r) - (r^2 + \lambda^2) B_\lambda(r) = 0.
\]
Change \( r \) into \( \nu \cdot r^\mu \), where \( \nu \neq 0 \) and \( \mu \neq 0 \) are real numbers to be determined later, we have

\[ g(r) := B_\lambda(\nu \cdot r^\mu) \]

satisfies the following equation by direct calculation:

\[ g'' + \frac{1}{r}g' - \frac{\mu^2 \nu^2}{r^{2-2\mu}} g - \frac{\mu^2 \lambda^2}{r^2} g = 0. \] (2.7)

Observe (2.3), we choose \( \mu = -\frac{\beta}{2} \) and \( \nu = \frac{\beta A}{2} \sqrt{A} \), thus (2.7) becomes

\[ g'' + \frac{1}{r}g' - \frac{A}{r^{2+\beta}} g - \left(\frac{\beta \lambda}{2r}\right)^2 g = 0. \] (2.8)

Now, to eliminate the last term and modify the coefficient of the second term on the left of (2.8), we set \( h(r) := r^\theta \cdot g(r) \), where \( \theta \) is a real number to be determined later. By direct calculation, we have \( h \) satisfies

\[ h'' + \frac{1 - 2\theta}{r} h' - \frac{A}{r^{2+\beta}} h + \left[\theta^2 - \left(\frac{\beta \lambda}{2}\right)^2 \right] r^{-2} h = 0. \] (2.9)

Compare (2.9) to (2.3), we have

\[
\begin{aligned}
\theta^2 - \left(\frac{\beta \lambda}{2}\right)^2 &= 0 \\
-2\theta + 1 &= d - 1
\end{aligned}
\] (2.10)

Thus we have \( \theta = -\frac{d}{2} + 1 \), and \( \lambda = \frac{d-2}{\beta} \), which we have \( h(r) = r^{-\frac{d}{2} + 1} B_{\frac{d-2}{\beta}}(\frac{\beta}{2} \sqrt{A} r^{-\frac{\beta}{2}}) \) solves (2.3). Since we are looking for local bounded solution, we choose

\[ B_{\frac{d-2}{\beta}}(r) = K_{\frac{d-2}{\beta}}(r), \] (2.11)

the modified Bessel’s function of second kind, which is exponentially growing at \( 0 \in \mathbb{R}^d \).

As for (ii) in Lemma 2.1, we have

**Lemma 2.2.** The function \( |x|^{-\frac{d}{2} + 1} K_{(d-2)/\beta} \left( \frac{2}{\beta} \sqrt{A} |x|^{-\frac{\beta}{2}} \right) \) is smooth in \( B \), and decays exponentially to 0 at \( x = 0 \).

This is a direct corollary of the property of modified Bessel’s function, see [1] for more details.

Now we start the proof of Theorem 1.1, Case (I).
We denote by $J_\beta = J_\beta(x)$ the special solutions of (1.13) mentioned above, namely

$$J_\beta(x) = \begin{cases} |x|^{-\frac{d+1}{2}} K_{(d-2)/\beta} \left(\frac{2\sqrt{A}|x|^{-\frac{d}{2}}}{\beta}\right), & \beta > 0 \\ |x|^{-\frac{d+2+\sqrt{d^2-4d+4}}{2}}, & \beta = 0. \end{cases} \quad (2.12)$$

Then, on $B_{1/2}$, there exists a constant $C = C(d, \|u\|_{L^2(B)})$, such that the functions $v_1 = u(x) + CJ_\beta(x)$ and $v_2 = u(x) - CJ_\beta(x)$ satisfy

$$\begin{cases} \Delta v_i(x) - \frac{1}{|x|^{2+\beta}} v_i(x) = 0, & \text{in } B_{1/2}, \quad i = 1, 2, \\ v_1(x) \geq 0, \quad v_2(x) \leq 0, & \text{on } \partial B_{1/2}. \end{cases} \quad (2.13)$$

By the maximum principle in Lemma 6.2 in the Appendix e.g., we have

$$-C \cdot J_\beta(x) \leq u(x) \leq C \cdot J_\beta(x). \quad (2.14)$$

According to the Green’s Representation formula (c.f. [4]), we have, for $x \in B_{1/2}$

$$u(x) = \int_{B_{1/2}} \Gamma(x-y) \frac{1}{|y|^{2+\beta}} u(y) dy + H(x) \quad (2.15)$$

where $\Gamma(x-y) = \frac{|x-y|^{2-d}}{d(2-d)\omega_d}$ is the fundamental solution of the Laplace equation ($\omega_d$ is the volume of $d$-dimensional unit ball), and $H = H(x)$ is harmonic in $B_{1/2}$. Since the second term $H(x)$ of (2.15) is regular enough in $B_{1/4}$, we only need to consider the regularity of

$$w(x) = \int_{B_{1/2}} \Gamma(x-y) \frac{1}{|y|^{2+\beta}} u(y) dy. \quad (2.16)$$

We divide the rest of the proof into several cases, firstly:
2.1. Case (I) $\beta = 0$, $\alpha(A) \leq 1$

By (2.14) $\forall x_1, x_2 \in B_{1/4}$, define $z = \frac{1}{2}(x_1 + x_2)$, $\delta = |x_1 - x_2|.$

$$|w(x_1) - w(x_2)| \leq C \int_{B_{1/2}} |\Gamma(x_1 - y) - \Gamma(x_2 - y)| \frac{1}{|y|^{2-\alpha(A)}} \, dy$$

$$\leq C \int_{B_{1/2} \cap B(z, \delta)} |\Gamma(x_1 - y)| \frac{1}{|y|^{2-\alpha(A)}} \, dy$$

$$+ C \int_{B_{1/2} \cap B(z, \delta)} |\Gamma(x_2 - y)| \frac{1}{|y|^{2-\alpha(A)}} \, dy$$

$$+ C \int_{B_{1/2} \cap B(z, \delta)} |\Gamma(x_1 - y) - \Gamma(x_2 - y)| \frac{1}{|y|^{2-\alpha(A)}} \, dy$$

$$= C(I_1 + I_2 + I_3) \tag{2.17}$$

As to $I_1$, for $\frac{d}{2} < p_1 < \frac{d}{2-\alpha(A)}$, by Hölder inequality

$$I_1 \leq C \left( \int_{B(x_1, \delta/2)} |x_1 - y|^{(2-d)p_1/p_1} \, dy \right)^{1-1/p_1} \cdot \left( \int_{B_{1/2}} |y|^{(\alpha(A)-2)p_1/p_1} \, dy \right)^{1/p_1}$$

$$\leq C \delta^{2p_1-d/p_1} \cdot \left[ \frac{1}{(\alpha(A)-2)p_1 + d} \right]^{1/p_1} \cdot \left[ \frac{2p_1 - d}{p_1 - 1} \right]^{-1+1/p_1} \tag{2.18}$$

$$\leq C \cdot \delta^{\alpha(A)-}, \quad \text{by choosing } p_1 \to \left( \frac{d}{2-\alpha(A)} \right)_{-}.$$

Here and below, we use $C_-$ to denote an arbitrary number close but smaller than $C$. Similarly, we have $I_2$ satisfies the same estimate. As for $I_3$, by mean value inequality, there exists an $\hat{x}$ lies between $x_1$ and $x_2$, and for $1 < p_2 < \frac{d}{2-\alpha(A)},$

$$I_3 \leq C \delta \int_{B_{1/2} \cap B(z, \delta)} \frac{1}{|y|^{2-\alpha(A)}} \, dy$$

$$\leq C \delta \left( \int_{B(0, \delta/2)} |y|^{(1-d)p_2/p_2} \, dy \right)^{1-1/p_2} \cdot \left( \int_{B_{1/2}} |y|^{(\alpha(A)-2)p_2} \, dy \right)^{1/p_2}$$

$$\leq C \delta^{2p_2-d/p_2} \cdot \left[ \frac{1}{(\alpha(A)-2)p_2 + d} \right]^{1/p_2} \tag{2.19}$$

$$\leq C \cdot \delta^{\alpha(A)-}, \quad \text{by choosing } p_2 \to \left( \frac{d}{2-\alpha(A)} \right)_{-}.$$

This means $w = w(x)$ is $\alpha(A)_-$ Hölder continuous in $B_{1/4}$. This and (2.15) imply that

$$\|u\|_{C^{\alpha(A)-}(B_{1/4})} \leq C(d, \|u\|_{L^2(B)}).$$
This proves Case (I) of the theorem when \( n = 0 \).

We point out that this estimate is almost optimal since the special solution \( J_0(x) = |x|^{\alpha(A)} \) has only \( \alpha(A) \) Hölder continuity in \( B_{1/2} \).

**Remark 2.1.** Let us pay attention to a special case when \( A = 1, \beta = 0, d = 3 \). According to the results above, we know that the weak solution of

\[
\Delta u - \frac{1}{|x|^2} u = 0 \tag{2.20}
\]

in \( B_{1/2} \) is Hölder continuous with exponent \( \left( \frac{\sqrt{5} - 1}{2} \right)_- \approx 0.618 \), which is the golden ratio.

Next we prove Case (I) of the theorem when \( n = 1 \). In this case, we first claim \( w \in C^1(B_{1/4}) \) and

\[
\partial_i w(x) = \int_{B_{1/2}} \partial_i \Gamma(x - y) \frac{u(y)}{|y|^2} dy, \quad i = 1, 2, ..., d. \tag{2.21}
\]

Here goes the proof.

Since \( \nabla^n \Gamma \) satisfies the following estimate

\[
|\nabla^n \Gamma(x - y)| \leq C|x - y|^{2 - d - n}, \quad n = 1, 2, \ldots \tag{2.22}
\]

where \( C = C(d, n) \), the following function

\[
\xi(x) = \int_{B_{1/2}} \partial_i \Gamma(x - y) \frac{u(y)}{|y|^2} dy \tag{2.23}
\]

is well defined by the Hölder inequality. The reason is:

\[
\partial_i \Gamma(x - y) \in L^p(B_{1/2}), \quad 1 \leq p < \frac{d}{d - 1};
\]

\[
\frac{|u(y)|}{|y|^2} \leq C|y|^\alpha(A) - 2 \in L^q(B_{1/2}), \quad 1 \leq q < \frac{d}{2 - \alpha(A)}. \tag{2.24}
\]

Thus

\[
1 - \frac{\alpha(A) - 1}{d} < \frac{1}{p} + \frac{1}{q} \leq 2 \tag{2.25}
\]

and we get

\[
\partial_i \Gamma(x - y) \frac{|u(y)|}{|y|^2} \in L^{1 + \frac{\alpha(A) - 1}{d - \alpha(A) + 1}}(B_{1/2}). \tag{2.26}
\]

Therefore \( \xi \) is well defined. By usual approximation argument, one can prove easily that \( \partial_i w(x) = \xi(x) \). This proves the claim.

Similarly as in Case (I), \( n = 0, \forall x_1, x_2 \in B_{1/4} \), define \( z = \frac{1}{2}(x_1 + x_2), \quad \delta = |x_1 - x_2| \). We
have the inequality for the gradient of $w$:

$$
|\partial_i w(x_1) - \partial_i w(x_2)| \leq C \int_{B_{1/2}} |\partial_i \Gamma(x_1 - y) - \partial_i \Gamma(x_2 - y)| \frac{1}{|y|^{2-\alpha(A)}} \, dy
$$

$$
\leq C \int_{B_{1/2} \cap B(z,\delta)} |\partial_i \Gamma(x_1 - y)| \frac{1}{|y|^{2-\alpha(A)}} \, dy
$$

$$
+ C \int_{B_{1/2} \cap B(z,\delta)} |\partial_i \Gamma(x_2 - y)| \frac{1}{|y|^{2-\alpha(A)}} \, dy
$$

$$
+ C \int_{B_{1/2} \cap B(z,\delta)} |\partial_i \Gamma(x_1 - y) - \partial_i \Gamma(x_2 - y)| \frac{1}{|y|^{2-\alpha(A)}} \, dy
$$

$$
\equiv C(I_1 + I_2 + I_3).
$$

As to $I_1$, for $d < p_1 < \frac{d}{2-\alpha(A)}$, by Hölder inequality

$$
I_1 \leq C \left( \int_{B(1,\frac{3\delta}{4})} |x_1 - y|^{\frac{1-d}{p_1}} \, dy \right)^{1-1/p_1} \left( \int_{B_{1/2}} |y|^{\alpha(A)-2p_1} \, dy \right)^{1/p_1}
$$

$$
\leq C \delta^{\frac{p_1-d}{p_1}} \cdot \left[ \frac{1}{(\alpha(A)-2)p_1 + d} \right]^{1/p_1} \cdot \left[ \frac{2p_1 - d}{p_1 - 1} \right]^{-1+1/p_1}
$$

$$
\leq C \cdot \delta^{(\alpha(A)-1)-}, \quad \text{by choosing } p_1 \to \left( \frac{d}{2-\alpha(A)} \right)^-.
$$

Likewise, we have that $I_2$ satisfies the same estimate. As for $I_3$, by mean value inequality, there exists an $\hat{x}$ lies between $x_1$ and $x_2$, and for $1 < p_2 < \frac{d}{2-\alpha(A)}$:

$$
I_3 \leq C \delta \int_{B_{1/2} \cap B(z,\delta)} |\nabla^2 \Gamma(\hat{x} - y)| \frac{1}{|y|^{2-\alpha(A)}} \, dy
$$

$$
\leq C \delta \left( \int_{B(0,\frac{\delta}{2})} |y|^{\frac{dp_2}{p_2-1}} \, dy \right)^{1-1/p_2} \left( \int_{B_{1/2}} |y|^{\alpha(A)-2p_2} \, dy \right)^{1/p_2}
$$

$$
\leq C \delta^{1-\frac{d}{p_2}} \cdot \left[ \frac{1}{(\alpha(A)-2)p_2 + d} \right]^{1/p_2}
$$

$$
\leq C \cdot \delta^{\alpha(A)-1-}, \quad \text{by choosing } p_2 \to \left( \frac{d}{2-\alpha(A)} \right)^-.
$$

This means $\nabla w$ is $\alpha(A) - 1$- H"older continuous in $B_{1/4}$ and $w \in C^{1, (\alpha(A)-1-)}(B_{1/4})$. Moreover

$$
\|\nabla u\|_{C^{(\alpha(A)-1-)}(B_{1/4})} \leq C(d, \|u\|_{L^2(B)}).
$$

This shows Case (I) of the theorem when $n = 1$ holds.

Now we prove Case (I) of the theorem when $n > 1$. First by induction, it is easy to see the
following lemma, we list here without proof.

**Lemma 2.3.** If \( u \in C^{m,\gamma}(B) \), \( \gamma \in (0,1) \) and \( |u(x)| \leq C|x|^{n+\gamma} \) with \( n \geq m \), we have

\[
|\nabla^k u(x)| \leq C(\|u\|_{C^{m,\gamma}(B)}) \cdot |x|^{n-k+\gamma}, \forall k \in [1,m] \text{ and } k \text{ is an integer},
\]

(2.31)

for all \( x \in B \).

Moreover,

\[
\partial_{x_j} \int_{B_{1/2}} \partial_{x_i} \Gamma(x-y) \nabla^{n-2} \left( \frac{u(y)}{|y|^2} \right) dy = -P.V. \int_{B_{1/2}} \partial_{x_j} \partial_{y_i} \Gamma(x-y) \nabla^{n-2} \left( \frac{u(y)}{|y|^2} \right) dy
\]

(2.32)

\[
= \int_{B_{1/2}} \partial_{x_j} \Gamma(x-y) \partial_{y_i} \nabla^{n-2} \left( \frac{u(y)}{|y|^2} \right) dy - \int_{\partial B_{1/2}} \partial_{x_j} \Gamma(x-y) \nabla^{n-2} \left( \frac{u(y)}{|y|^2} \right) n_i dy.
\]

Note the last term above is smooth since \( x_1, x_2 \in B_{1/4} \). By (2.15), in order to prove Hölder continuity of \( \nabla^n u \), we only need to prove that of

\[
w_n(x) := \int_{B_{1/2}} \partial_{x_j} \Gamma(x-y) \partial_{y_i} \nabla^{n-2} \left( \frac{u(y)}{|y|^2} \right) dy
\]

(2.33)

for each \( n \). Just as the situation \( n = 0 \) and 1, since \( u \) and its derivatives satisfy the decay property (2.31), we get the following regularity result by induction.

\[
\|\nabla^n w\|_{C^{(\alpha(A)-n)}-\left(B_{1/4}\right)} \leq C(d, \|u\|_{L^2(B)}).
\]

(2.34)

Consequently

\[
\|\nabla^n u\|_{C^{(\alpha(A)-n)}-\left(B_{1/4}\right)} \leq C(d, \|u\|_{L^2(B)}).
\]

(2.35)

Thus we finished the proof of Case I of Theorem 1.

We also mention here that we could use the classical Schauder estimate to get the regularity of higher derivatives, by treating the term \( Au(x)/|x|^2 \) as the inhomogeneous term and using the decay property of \( u \) near 0.

For Case (II), i.e., \( \beta > 0 \), one can use (2.14) and the exponential decay property of \( J_\beta \) to conclude that \( u \) decays exponentially at 0. Then following an analogous argument as in Case (I), we know that \( u \in C^{\infty}(B_{1/4}) \).

Before commencing the case of variable second order coefficients, we prove part (III) of the Theorem [1.1] namely the corresponding result for the parabolic equations. The result is based on a bound of the fundamental solution \( \Gamma_0 = \Gamma_0(x,t; y, s) \) of the parabolic equation with
leading term being Laplace
\[ \Delta u - \frac{A}{|x|^2} u - \partial_t u = 0, \tag{2.36} \]
which was proved in \[8\]. It states that
\[ \Gamma_0(x, t; y, 0) \leq C \frac{t^{d/2}}{t^{d/2}} \left( 1 + \frac{\sqrt{t}}{|x|} \right)^{-\alpha} \left( 1 + \frac{\sqrt{t}}{|y|} \right)^{-\alpha} e^{-c|x-y|^2}. \tag{2.37} \]
Where \( C \) and \( c \) are positive constants, \( \alpha = \alpha(A) = -\frac{d+2+\sqrt{d^2-4d+4A}}{2} \), just as before. See also \[\[6\] for more details. We claim here that this bound of fundamental solution leads to a higher regularity of weak solutions of (2.36) in a neighbourhood of \( x = 0 \) when \( t \) is away from zero, by a similar method of Lemma 2.2 in \[12\], also Lemma 3.2 in the next section, we have a mean value inequality of \( u \):
\[ u^2(x, t) \leq C \frac{|Q_1(x, t)|}{|Q_r(x, t)|} \left( \frac{|x|}{r} \right)^{2\alpha} \int_{Q_2(x, t)} u^2(y, s) dy ds, \tag{2.38} \]
where \( C > 0 \) and \( Q_r(x, t) = B(x, r) \times [t-r^2, t] \subset \mathbb{R}^d \times \mathbb{R}_+ \). This inequality gives an "\( \alpha \)-order" decay of \( u \) when \( t \) is away from 0, namely:
\[ |u(x, t)| \leq C(d, \|u\|_{L^2}) t^{-(d/2+1+\alpha)} |x|^\alpha. \tag{2.39} \]
Now we are going to consider the regularity of \( u = u(x, t) \) with \( t > t_0 > 0 \). By Duhamel’s Principle (see also section 5 for more details), we only need to consider the regularity of
\[ w(x, t) \equiv \int_{t_0/2}^t \int_{B_{1/2}} e^{-\frac{|x-y|^2}{4(t-s)}} \frac{u(y, s)}{(t-s)^{d/2}} \frac{1}{|y|^2} dy ds. \tag{2.40} \]
Since \( \alpha \in ]0, 1[ \) will be considered in section 5, we only consider \( \alpha \in ]n, n+1[ \) for an integer \( n \geq 1 \). Firstly, if \( n = 1 \), similarly as (2.21), we have
\[ \partial_{x_i} w(x, t) = \int_{t_0/2}^t \int_{B_{1/2}} \partial_{x_i} \left( \frac{e^{-\frac{|x-y|^2}{4(t-s)}}}{(t-s)^{d/2}} \right) \frac{u(y, s)}{|y|^2} dy ds \]
\[ = -\frac{1}{2} \int_{t_0/2}^t \int_{B_{1/2}} e^{-\frac{|x-y|^2}{4(t-s)}} \frac{(x_i - y_i)}{(t-s)^{d/2+1}} \frac{u(y, s)}{|y|^2} dy ds. \tag{2.41} \]
Using the same method in section 5 (from (5.7) to (5.22)), we have the Hölder continuity of
\[ \nabla_x u, \text{ hence we have:} \]
\[ \| \nabla_x u \|_{C^{(\alpha(A)-1)-(\alpha(A)-1)/2}(B_{1/4} \times [t_0, T])} \leq C(d, t_0, \| u \|_{L^2}). \quad (2.42) \]

This shows the result holds when \( n = 1 \). When \( n > 1 \), by a similar induction method as in the elliptic case (see (2.32) for more details), we have the Hölder continuity of higher order spatial derivatives, namely:
\[ \| \nabla_x^n u \|_{C^{(\alpha(A)-n)-(\alpha(A)-n)/2}(B_{1/4} \times [t_0, T])} \leq C(d, t_0, \| u \|_{L^2}). \quad (2.43) \]

Finally, we will estimate time derivatives of \( u \). By equation (2.36), \( \partial_t u = \Delta u - \frac{A}{|x|^2} u \), we can eliminate all the time derivatives, for \( 2m + |L| = n \)
\[ \partial_t^m \partial_x^L u = \sum_{|J|+|K|+2l=n} C_{J,K,l,d} \frac{x^J}{|x|^{2|l+|J|}} (\partial_x^K u)(x, t). \quad (2.44) \]

Where \( J, K, L \) above are multi-indexes, e.g. \( L = (L_1, L_2, \ldots, L_d), |L| = L_1 + L_2 + \ldots + L_d \)
\[ \partial_x^L = \partial_{x_1}^{L_1} \partial_{x_2}^{L_2} \ldots \partial_{x_d}^{L_d}, \quad x^L = x_1^{L_1} x_2^{L_2} \ldots x_d^{L_d}. \quad (2.45) \]
And \( C_{J,K,l,d} \) is a constant. Then we will give a Hölder estimate of each term on the right hand side of (2.44). We choose \( x_1, x_2 \in B_{1/4} \) and \( T > t_1 > t_2 > t_0 > 0 \). We suppose \( |x_1| \geq |x_2| \) without loss of generality. Then
\[ \left| \frac{x_1^J}{|x_1|^{2|l+|J|}} \partial_x^K u(x_1, t_1) - \frac{x_2^J}{|x_2|^{2|l+|J|}} \partial_x^K u(x_2, t_2) \right| \leq \left| \frac{x_1^J}{|x_1|^{2|l+|J|}} \partial_x^K u(x_1, t_1) - \frac{x_1^J}{|x_1|^{2|l+|J|}} \partial_x^K u(x_1, t_2) \right| + \left| \frac{x_1^J}{|x_1|^{2|l+|J|}} \partial_x^K u(x_1, t_2) - \frac{x_2^J}{|x_2|^{2|l+|J|}} \partial_x^K u(x_2, t_2) \right| + \left| \frac{x_2^J}{|x_2|^{2|l+|J|}} \partial_x^K u(x_2, t_1) - \frac{x_2^J}{|x_2|^{2|l+|J|}} \partial_x^K u(x_2, t_2) \right| \quad (2.46) \]
\[ := I + II + III \]

We only consider cases when \( |K| < n \), otherwise \( \frac{x_1^J}{|x_1|^{2|l+|J|}} \partial_x^K u = \partial_x^K u = \partial_x^n u \) which was considered before. Actually, by the structure of equation (2.36), we have \( |K| \leq n - 2 \) and
\[ \partial_t \partial^K_x u(x_1, \cdot) \] is well defined according to the equation (2.36). Then, if \( t_1 - t_2 \leq |x_1|^2 \),

\[
I \leq \frac{C}{|x_1|^{2|l|+|J|}} \sup_{\tau \in [t_0, T]} |\partial_t \partial^K_x u(x_1, \tau)| \cdot (t_1 - t_2)
\]

\[
\leq \frac{C}{|x_1|^{2|l|+|J|}} |x_1|^{n+2-\alpha(A)} \left( \sup_{\tau \in [t_0, T]} |\Delta \partial^K_x u(x_1, \tau)| + \sup_{\tau \in [t_0, T]} \left| \frac{\partial^K_x u(x_1, \tau)}{|x_1|^2} \right| \right) \cdot (t_1 - t_2)^{(\alpha(A)-n)/2}
\]

\[
\leq \frac{C(t_0)}{|x_1|^{2|l|+|J|}} \cdot |x_1|^{n+2-\alpha(A)} \cdot |x_1|^{|\alpha(A)-(|K|+2)|} (t_1 - t_2)^{(\alpha(A)-n)/2}
\]

\[
= C(t_0) (t_1 - t_2)^{(\alpha(A)-n)/2}.
\]

If \( t_1 - t_2 > |x_1|^2 \),

\[
I \leq \frac{C}{|x_1|^{2|l|+|J|}} \cdot \frac{(t_1 - t_2)^{(\alpha(A)-n)/2}}{|x_1|^{|\alpha(A)-n|}} \left( |\partial^K_x u(x_1, t_1)| + |\partial^K_x u(x_1, t_2)| \right)
\]

\[
\leq \frac{C(t_0)}{|x_1|^{2|l|+|J|}} \cdot \frac{(t_1 - t_2)^{(\alpha(A)-n)/2}}{|x_1|^{|\alpha(A)-n|}} \cdot |x_1|^{|\alpha(A)-|K|} \leq C(t_0) (t_1 - t_2)^{(\alpha(A)-n)/2}.
\]

Thus we finished the estimate of I. As for II and III, we have, for a \( \hat{x} \) lies between \( x_1 \) and \( x_2 \),

\[ II \leq \frac{C}{|x_1|^{2|l|+|J|}} |\nabla_x \partial^K_x u(\hat{x}, t_2)| \cdot |x_1 - x_2| \leq C |\hat{x}|^{|\alpha(A)-|K|} \frac{|x_1 - x_2|}{|x_1|^{2|l|+|J|}} |x_1 - x_2| \]

\[
\leq \frac{C}{|x_1|^{n-\alpha(A)+1}} \left( |x_1| + |x_2| \right)^{n-\alpha(A)+1} |x_1 - x_2|^{\alpha(A)-n} \leq C 2^{n-\alpha(A)+1} |x_1 - x_2|^{\alpha(A)-n};
\]

\[ (2.49) \]

\[ III \leq |\partial^K_x u(x_2, t_2)| \cdot \left| \frac{x^j_1}{|x_1|^{2(l+|J|)}} - \frac{x^j_2}{|x_2|^{2(l+|J|)}} \right| \leq C \cdot |x_1 - x_2|^{\alpha(A)-|K|} \left| \frac{x^j_1}{|x_1|^{2(l+|J|)}} - \frac{x^j_2}{|x_2|^{2(l+|J|)}} \right|
\]

\[
\leq C \cdot |x_1 - x_2|^{\alpha(A)-n}.
\]

\[ (2.50) \]

This proves that, for \( 2m + |L| = n \):

\[
\| \partial^m_t \partial^L_x u \|_{C^{(\alpha(A)-n)-}(\alpha(A)-n)-/2(B_{1/4} \times [t_0, T])} \leq C (d, t_0, \|u\|_{L^2}).
\]

\[ (2.51) \]

We can also draw the same conclusion by using the fact that \( \partial^m_t u \) is also solution of (2.36) in some sense. But certain approximation procedure is needed since we do not know a priori any \( L^p \) bounds for the time derivatives.
3. A Mean Value Inequality

Now we start treating equations with variable coefficients.

In this section, we state and prove a mean value inequality for solutions of equations (0.2), which has extra decay comparing with the standard mean value inequality. This will be used in the following sections. The proof uses an iteration process involving the potential and a boosting process by the Feynman-Kac formula. To start with, we need a crude mean value inequality, which is similar to that in \[12\].

**Lemma 3.1.** Let \( u \) be a weak solution to the equation

\[
\partial_{x_i} (a_{ij} \partial_{x_j} u) - Vu - \partial_t u = 0
\]

in \( Q_{2r}(x,t) \). Here \( a_{ij} \in L^\infty \), \( \lambda I \leq (a_{ij})_{1 \leq i,j \leq d} \leq \Lambda I \) for some \( 0 < \lambda \leq \Lambda \leq \infty \). If \( a \geq a_0 := \frac{192(d+4)^2(1+\Lambda)}{\lambda} \) and

\[
V = \frac{a}{1 + |x|^2},
\]

then \( \exists C > 0 \), depending on \( a, \lambda, \Lambda \) but independent of \( r \), such that:

\[
u^2(x,t) \leq C \left[ \max \left\{ \frac{r}{1 + |x|}, 1 \right\} \right]^{-2} \int_{Q_r(x,t)} u^2(y,s) \, dy \, ds.
\]

**Proof.** We pick a Lipschitz cut-off function \( \phi \) such that \( \phi(y,s) = 1 \) if \( (y,s) \in Q_r(x,t) \), \( \phi(y,s) = 0 \) if \( (y,s) \in Q^c_{r}(x,t) \), and \( |\nabla \phi| \leq 1/((\tau - 1)r) \), a.e. \( |\partial_t \phi| \leq 1/((\tau - 1)r)^2 \), a.e. Using \( \phi^2 u \) as a test function, after routine calculation,

\[
\lambda \int |\nabla (\phi u)|^2 \, dy \, ds + \int V(y) u^2 \phi^2 \, dy \, ds \leq (1 + \Lambda) \int u^2 \left[ |\nabla \phi|^2 + |\partial_s \phi| \right] \, dy \, ds.
\]

Therefore:

\[
\int_{Q_{rr}(x,t)} V(y) u^2 \phi^2 \, dy \, ds \leq \frac{1 + \Lambda}{\lambda ((\tau - 1)r)^2} \int_{Q_{rr}(x,t)} u^2 \, dy \, ds.
\]

When \( y \in B(x,r) \), we have \( |y|^2 \leq 2(|x - y|^2 + |x|^2) \) and hence, without loss of generality, we assume \( |x| \geq 1 \) and \( r \geq 1 \). Otherwise (3.1) is the standard mean value inequality.

\[
V(y) \geq \frac{a}{3(|x - y|^2 + |x|^2)} \geq \frac{a}{3(r^2 + |x|^2)}.
\]

It follows that

\[
\int_{Q_r(x,t)} u^2 \, dy \, ds \leq \frac{3(1 + \Lambda)(r^2 + |x|^2)}{a\lambda(\tau - 1)^2 r^2} \int_{Q_{rr}(x,t)} u^2 \, dy \, ds.
\]
For each $r > 1$ we take $\tau > 1$ s.t.

$$\frac{3(1 + \Lambda)(r^2 + |x|^2)}{a\lambda(\tau - 1)^2\tau^2} = \frac{1}{2}. \quad (3.8)$$

This implies

$$\tau r = r + \left[6a^{-1}\lambda^{-1}(1 + \Lambda)(r^2 + |x|^2)\right]^{1/2}. \quad (3.9)$$

Under such choice of $\tau$, we have

$$\int_{Q_r(x,t)} u^2 dy ds \leq \frac{1}{2} \int_{Q_{\tau r}(x,t)} u^2 dy ds. \quad (3.10)$$

We shall iterate the above inequality according to the formula:

$$\tau_{k+1} = \tau_k r_k := r_k + \left[6a^{-1}\lambda^{-1}(1 + \Lambda)(r_k^2 + |x|^2)\right]^{1/2} \quad (3.11)$$

with $r_0 = |x|$. Writing $\mu := \left(12a^{-1}\lambda^{-1}(1 + \Lambda)\right)^{1/2}$, we claim that:

$$r_k \leq (1 + \mu)^{2^k}(1 + |x|). \quad (3.12)$$

Obvious (3.12) holds for $k = 1$. Suppose it holds for $k$, then

$$r_{k+1} \leq r_k + \mu(r_k + |x|) = (1 + \mu)r_k + \mu|x|$$

$$\leq (1 + \mu)^{2^{k+1}}(1 + |x|) + (1 + \mu)^{2^k} \mu(1 + |x|) \quad (3.13)$$

This implies that to reach $r$ from $|x|$ one needs at least

$$k = \frac{\ln(r/(1 + |x|))}{2 \ln(1 + \mu)}. \quad (3.14)$$

number of iterations (round up to an integer). Iterating (3.10) $k$ times we have

$$u^2(x,t) \leq C \left( \frac{r}{1 + |x|} \right)^{-\ln 2/[2 \ln(1 + \mu)]} \left( \frac{r}{1 + |x|} \right)^{d+2} \int_{Q_r(x,t)} u^2 dy ds \quad (3.15)$$

Simplifying the above, we reach

$$u^2(x,t) \leq C \left( \frac{r}{1 + |x|} \right)^{-\ln 2/[2 \ln(1 + \mu)]} \left( \frac{r}{1 + |x|} \right)^{d+2} \frac{1}{|Q_r(x,t)|} \int_{Q_r(x,t)} u^2 dy ds. \quad (3.16)$$
Recall that \( \mu = \left(12\alpha^{-1}\lambda^{-1}(1 + \Lambda)\right)^{1/2} \). When \( a \geq \frac{192(d+4)^2(1+\Lambda)}{\lambda} \), we have
\[
\mu \leq \frac{1}{4(d + 4)}.
\] (3.17)

Hence
\[
\frac{\ln 2}{2\ln(1 + \mu)} \geq \frac{\ln 2}{2\mu} = (d + 4)2\ln 2 \geq d + 4.
\] (3.18)

This proves the lemma.

Based on this crude mean value inequality, using the Feymann-Kac product formula similarly as in Proposition 2.1 of [12], denoting \( t - s \) by \( l \), we obtain the global bounds of \( \Gamma_1 \), the fundamental solution of (3.21).

\[
\Gamma_1(x, t; y, s) = \Gamma_1(x, l; y, 0) \leq c_1 \frac{w(x, l)}{|B(x, \sqrt{l})|} e^{-c_2|x-y|^2/l},
\] (3.20)

for any \( y \in Q_{3r/2}(x, t) \). Here \( w(x, l) = [\max\{\sqrt{l}, 1\}]^{-\alpha} \) with \( \alpha = \alpha(\lambda, \Lambda, a, d) > 0 \). Since the proof is the same, we omit the details here. Using this bound, we will give a refined mean value formula. Notice that there is no restriction on the size of the positive number \( A \) comparing with Lemma 3.1

**Lemma 3.2.** Let \( u \) be a weak solution to the parabolic equation
\[
\partial_x, (a_{ij} \partial_{x_j} u) - \frac{A}{1 + |x|^2} u - \partial_t u = 0
\] (3.21)

with \( a_{ij}(x, t) \in L^\infty \) satisfying the elliptic condition \( \lambda I \prec (a_{ij}(x, t)) \prec \Lambda I \) with \( 0 < \lambda < \Lambda < \infty \) in the parabolic cube \( Q_{2r}(x, t) = B(x, 2r) \times [t - 4r^2, t], \forall r > 0 \). Then there exists \( C > 0 \), \( \alpha > 0 \), depending only on \( \lambda, \Lambda, A, d \), such that
\[
u(3r/2)^2 = 0
\] (3.22)

**Proof.** Select a cut-off function \( \eta \in C_0^\infty (Q_{3r/2}(x, t)) \) such that \( \eta = 1 \) in \( Q_r(x, t) \), \( 0 \leq \eta \leq 1 \) and \( |\nabla \eta| \leq \frac{C}{r}, |\nabla^2 \eta| \leq \frac{C}{r^2}, |\eta_t| \leq \frac{C}{r^2} \). We have \( \eta u \) satisfies
\[
\begin{cases}
\partial_j (a_{ij} \partial_i (\eta u)) - \frac{A}{1 + |x|^2} \eta u - \partial_t (\eta u) = u \partial_j (a_{ij} \partial_i \eta) + 2a_{ij} \partial_j \eta \partial_i u - u \partial_t \eta := f; \\
\eta u(y, s) = 0, (y, s) \in \partial B(x, 3r/2) \times [t - (3r/2)^2, t]; \\
\eta u(y, t - (3r/2)^2) = 0
\end{cases}
\] (3.23)
Let \( \Gamma_1(x, t; y, s) \) be the fundamental solution of \( \partial_j (a_{ij} \partial_i u) - \frac{A}{1+|x|^2} u - \partial_t u = 0 \). Then

\[
\begin{align*}
u(x, t) &= -\int_{Q_{3r/2}(x,t)} \Gamma_1(x, t; y, s) (a_{ij} \partial_j \eta \partial_i u - u \partial_t \eta) (y, s) dy ds \\
&\quad + \int_{Q_{3r/2}(x,t)} \partial_j \Gamma_1(x, t; y, s) (a_{ij} u \partial_i \eta) (y, s) dy ds \\
&= I + II.
\end{align*}
\] (3.24)

For I, we have:

\[
I^2 \leq C \int_{Q_{3r/2}(x,t)-Q_r(x,t)} \Gamma_1^2(x, t; y, s) dy ds \int_{Q_{3r/2}(x,t)-Q_r(x,t)} \left( \frac{1}{r^4} u^2 + \frac{1}{r^2} |\nabla u|^2 \right) dy ds. \tag{3.25}
\]

Since \( \partial_j (a_{ij} \partial_i u) - \partial_t u = \frac{A}{1+|x|^2} u \), it is well known that

\[
\int_{Q_{3r/2}(x,t)-Q_r(x,t)} |\nabla u|^2 dy ds \leq C \int_{Q_{2r}(x,t)-Q_{r/2}(x,t)} u^2 dy ds. \tag{3.26}
\]

Combine (3.25) and (3.26) we have

\[
I^2 \leq C \int_{Q_{3r/2}(x,t)-Q_r(x,t)} \Gamma_1^2(x, t; y, s) dy ds \int_{Q_{2r}(x,t)} u^2 dy ds. \tag{3.27}
\]

As for II:

\[
\begin{align*}II^2 &\lesssim \frac{1}{r^2} \int_{Q_{3r/2}(x,t)-Q_r(x,t)} |\nabla \Gamma_1(x, t; y, s)|^2 dy ds \cdot \int_{Q_{3r/2}(x,t)-Q_r(x,t)} u^2 dy ds. \tag{3.28}
\end{align*}
\]

Similarly as (3.26), since \( \Gamma_1(x, t; y, s) \) is a well-defined weak solution of equation (3.21) in \( Q_{3r/2}(x, t) - Q_r(x, t) \), we have:

\[
\begin{align*}II^2 &\lesssim \frac{1}{r^4} \int_{Q_{2r}(x,t)-Q_{r/2}(x,t)} \Gamma_1(x, t; y, s) dy ds \cdot \int_{Q_{3r/2}(x,t)-Q_r(x,t)} u^2 dy ds. \tag{3.29}
\end{align*}
\]

By the estimate of I and II above, we arrive:

\[
\begin{align*}u(x, t)^2 &\lesssim \frac{1}{r^4} \int_{Q_{2r}(x,t)-Q_{r/2}(x,t)} \Gamma_1(x, t; y, s) dy ds \cdot \int_{Q_{3r/2}(x,t)-Q_r(x,t)} u^2 dy ds. \tag{3.30}
\end{align*}
\]

By (3.20), denoting \( t - s \) by \( l \), we obtain

\[
\Gamma_1(x, t; y, s) = \Gamma_1(x, l; y, 0) \leq c_1 \frac{w(x, l)}{|B(x, \sqrt{l})|} e^{-c_2|x-y|^2/l}. \tag{3.31}
\]
Here \( w(x, l) = [\max\{\frac{\sqrt{l}}{1+|x|}, 1\}]^{-\alpha} \) with \( \alpha = \alpha(\lambda, \Lambda, A, d) > 0 \).

When \((y, s) \in Q_{2r}(x, t) - Q_{r/2}(x, t)\) and \( t - (r/2)^2 \leq s \leq t \), we have \( 0 \leq s \leq (r/2)^2 \leq |x - y|^2 \). Hence

\[
\Gamma_1(x, t; y, s) \leq c_1 \frac{\max\{\frac{\sqrt{l}}{1+|x|}, 1\}^{-\alpha}}{l^{d/2}} e^{-(c_2 s^2)/(4l)} \leq c_1 \frac{\max\{\frac{r}{1+|x|}, 1\}^{-\alpha}}{r^d}.
\] (3.32)

On the other hand, when \((y, s) \in Q_{2r}(x, t) - Q_{r/2}(x, t)\) and \( t - (2r)^2 \leq s \leq t - (r/2)^2 \), we have \( l = t - s \geq (r/2)^2 \). Therefore

\[
\Gamma_1(x, t; y, s) \leq c_1 \frac{w(x, l)}{l^{d/2}} e^{-c_2|x-y|^2/l} \leq c \frac{\max\{\frac{r}{1+|x|}, 1\}^{-\alpha}}{r^d} \] (3.33)

Therefore, \( \forall (y, s) \in Q_{2r}(x, t) - Q_{r/2}(x, t) \), we have

\[
\Gamma_1(x, t; y, s) \leq c \frac{\max\{\frac{r}{1+|x|}, 1\}^{-\alpha}}{r^d} \] (3.34)

Substituting this into (3.27) to get

\[
u(x, t)^2 \leq \max\{\frac{r}{1+|x|}, 1\}^{-2\alpha} \frac{C}{r^{2+d}} \int_{Q_{2r}(x, t)} u^2 dy ds\] (3.35)

\( \square \)

4. Variable Second Order Coefficients Case

In this section we prove Theorem 1.2, the elliptic case. Namely

\[
\partial_i (a_{ij}(x) \partial_j u(x)) - \frac{A}{|x|^2} u(x) = 0
\] (4.1)

where \( a_{ij} \in L^\infty(B) \) and satisfies uniformly elliptic condition. Although this elliptic case is a special case of the parabolic one in the next section, we present a proof since it is more transparent.

Obviously, this situation is different from the Laplacian case in the section before, since we could not find a special solution as in section 2. But fortunately, for the critical case \( \beta = 0 \), we can still prove that weak solutions of (1.1) vanish at 0 \( \in \mathbb{R}^d \) in the order of \( |x|^\alpha \) for some \( \alpha > 0 \) by the mean value inequality in Lemma 3.2. First we need the following two lemmas:

**Lemma 4.1.** Let \( u \in H^1(B) \) be a weak solution of the equation

\[
\partial_i (a_{ij}(x) \partial_j u(x)) - \frac{A}{|x|^2} u(x) = 0
\] (4.2)
\(x_0 \in B_{1/2} \subset \mathbb{R}^d\). Here \(a_{ij} \in L^\infty(B)\), \(\lambda I \leq (a_{ij})_{1 \leq i,j \leq d} \leq \Lambda I\) for some \(0 < \lambda \leq \Lambda \leq \infty\). Then there exists an \(\alpha > 0\), \(C > 0\) depending only on \(\lambda, \Lambda, A, d\) and the \(L^2(B)\) norm of \(u\), such that

\[|u(x_0)| \leq C|x_0|^\alpha\]

(4.3)

**Proof.** Since \(1/|x|^2\) is bounded except in \(B(0, \delta), \delta > 0\) fixed, by standard theory, \(u\) is Hölder continuous on \(B(0, \delta)^c\). Note that this Hölder exponent may not be uniform when \(\delta \to 0\). Pick a positive integer \(k, r \geq |x_0| \geq 0\) and consider the equation

\[
\begin{cases}
\partial_i (a_{ij}(x) \partial_j u_k(x)) - \frac{A}{|x|^{2+k-2}} u_k(x) = 0 \\
u_k \big|_{\partial B(x_0, 2r)} = u \big|_{\partial B(x_0, 2r)}
\end{cases}
\]

(4.4)

By boundedness of the potentials \(A/(|x|^2 + k^{-2})\), the problem above has a unique solution which is Hölder continuous on \(B(x_0, 2r) - B(0, \delta), \forall \delta > 0\) fixed, with Hölder exponent and Hölder norms that are uniform with respect to \(k\) for each fixed \(\delta\). By Arzela-Ascoli theorem, there exists a subsequence, still denoted by \(\{u_k\}\) for simplicity, such that

\[u_k(x) \to u(x) \quad \text{uniformly in } B(x_0, 2r) - B(0, \delta)\]

(4.5)

Next we do a scaling \(x = y/k\), define \(\bar{u}_k(y) = u_k(y/k)\), then \(\bar{u}_k\) satisfies

\[
\partial_i \left( a_{ij}(\frac{y}{k}) \partial_j \bar{u}_k(y) \right) - \frac{A}{|y|^2 + 1} \bar{u}_k(y) = 0, \quad y \in B(kx_0, 2kr)
\]

(4.6)

By Lemma 3.2, which applies to the elliptic case, we know that, for \(y_0 = kx_0\):

\[
\bar{u}_k^2(y_0) \leq C \cdot \max\left\{ \frac{kr}{1+k|x_0|}, 1 \right\}^{-2\alpha} \int_{B(y_0,kr)} \bar{u}_k^2(y)dy.
\]

(4.7)

Changing back to \(x\)-coordinate, we deduce

\[
u_k^2(x_0) \leq C \cdot \max\left\{ \frac{r}{(1/k)+|x_0|}, 1 \right\}^{-2\alpha} \int_{B(x_0,r)} u_k^2(x)dx.
\]

(4.8)

By (4.5), since \(r > |x_0|\) by assumption, after taking limit and using the dominated convergence theorem, we have:

\[
u^2(x_0) \leq C \cdot \left( \frac{|x_0|}{r} \right)^{2\alpha} \frac{1}{r^d} \int_{B(x_0,r)} u^2(x)dx.
\]

(4.9)

Taking \(r = 1/2\), we know that \(u\) decays to 0 as \(x \to 0\) in an \(\alpha-\)Hölder sense.

We define \(\Gamma_{\alpha}(x, y)\) to be fundamental solution of the elliptic equation (1.1) without the potential term, namely

\[
\partial_i \left( a_{ij}(x) \partial_j \Gamma_{\alpha}(x, y) \right) = \delta(x - y).
\]

(4.10)
Since the elliptic coefficients $a_{ij}$ are not smooth enough, we cannot hope for the gradient estimate of the related fundamental solution $\Gamma_a(x, y)$ as the Laplacian case. However, we have the following lemma on its Hölder estimate, which is a direct consequence of the De Giorgi-Nash-Moser theory. It plays a similar role as the gradient estimate in the proof of section 2.1, estimate of $I_1$ to $I_3$.

**Lemma 4.2.** $\Gamma_a(x, y)$ satisfies the following pointwise estimate

$$|\Gamma_a(x, y)| \leq C_1 |x - y|^{2-d} \quad (4.11)$$

and

$$|\Gamma_a(x_1, y) - \Gamma_a(x_2, y)| \leq C_2 |x_1 - x_2| \alpha \left[ |x_1 - y|^{2-d-\alpha} + |x_2 - y|^{2-d-\alpha} \right] \quad \forall y \in B(z, \delta)^c \quad (4.12)$$

where $z = \frac{x_1 + x_2}{2}$, $\delta = |x_1 - x_2|$ as in section 2, $C_1$ and $C_2$ are two a priori constant depending only on $d$, $\lambda$, $\Lambda$.

**Proof.** The bound (4.11) can be find in [2], section 5. Thus we omit the proof here.

Now we prove (4.12) for completeness. Consider $\Gamma_a(x, y)$ with $x \in B(z, \frac{3|y - z|}{4})$ and $y \in B(z, \delta)^c$. Since $\Gamma_a(x, y)$ is a well defined weak solution of the homogeneous elliptic equation

$$\partial_{x_i} (a_{ij}(x) \partial_{x_j} \Gamma_a(x, y)) = 0 \quad (4.13)$$

in domain $B(z, \frac{3|y - z|}{4})$, according to the classical Hölder estimate of weak solutions, (c.f.[5][9][11]), we have:

$$|\Gamma_a(x_1, y) - \Gamma_a(x_2, y)| \leq C \sup_{x \in B(z, \frac{3|y - z|}{4})} |\Gamma_a(x, y)| \cdot \left( \frac{|x_1 - x_2|}{d(x_1, x_2)} \right)^\alpha \quad (4.14)$$

here

$$d(x_1, x_2) = \min\{dist(x_1, \partial B(z, \frac{3|y - z|}{4})), dist(x_2, \partial B(z, \frac{3|y - z|}{4}))\}$$

$$= \frac{3|y - z|}{4} - \frac{\delta}{2} \geq C_0 \max\{|y - x_1|, |y - x_2|\}, \quad \forall y \in B(z, \delta)^c. \quad (4.15)$$

Here $C_0$ is a constant. Now by (4.11)

$$\sup_{x \in B(z, \frac{3|y - z|}{4})} |\Gamma_a(x, y)| \leq C_1 \sup_{x \in B(z, \frac{3|y - z|}{4})} |x - y|^{2-d}$$

$$\leq \tilde{C} \left( |x_1 - y|^{2-d} + |x_2 - y|^{2-d} \right) \forall y \in B(z, \delta)^c \quad (4.16)$$
where $\hat{C} = \hat{C}(d, \lambda, \Lambda)$. By (4.14), (4.15) and (4.16), we have $\forall y \in B(z, \delta)^c$

$$|\Gamma_a(x_1, y) - \Gamma_a(x_2, y)|$$

$$\leq \hat{C} \left( |x_1 - y|^{2-d} + |x_2 - y|^{2-d} \right) \left( C_0^{-1} |x_1 - x_2| \right)^\alpha \cdot \min\{|x_1 - y|^{-\alpha}, |x_2 - y|^{-\alpha}\}$$

(4.17)

Thus we get (4.12). □

Now we continue with the proof of the theorem. Suppose $u$ is a weak solution of (1.1), then it can be divided into

$$u(x) = u_{a,0}(x) + \int_{B_1/2} \Gamma_a(x, y) A \cdot u(y) \frac{1}{|y|^2} dy$$

(4.18)

where $u_{a,0}(x)$ satisfies the homogeneous elliptic equation

$$\partial_{x_i} (a_{ij}(x) \partial_{x_j} u_{a,0}(x)) = 0$$

(4.19)

weakly in $B_{1/2}$. By the standard De Giorgi-Nash-Moser’s theory, $u_0 \in C^{\alpha_0}$, for some $\alpha_0 = \alpha_0(d, \lambda, \Lambda) \in (0, 1)$. Thus we only need to prove the Hölder continuity of

$$w_a(x) = \int_{B_{1/2}} \Gamma_a(x, y) A \cdot u(y) \frac{1}{|y|^2} dy$$

(4.20)

Similar to the Laplacian case in section 2.1, we have: using Lemma 4.1

$$|w_a(x_1) - w_a(x_2)| \leq C \int_{B_{1/2}} |\Gamma_a(x_1, y) - \Gamma_a(x_2, y)| \frac{1}{|y|^{2-\alpha}} dy$$

$$= C \int_{B_{1/2} \cap B(z, \delta)} |\Gamma_a(x_1, y)| \frac{1}{|y|^{2-\alpha}} dy$$

$$+ C \int_{B_{1/2} \cap B(z, \delta)} |\Gamma_a(x_2, y)| \frac{1}{|y|^{2-\alpha}} dy$$

(4.21)

$$+ C \int_{B_{1/2} - B(z, \delta)} |\Gamma_a(x_1, y) - \Gamma_a(x_2, y)| \frac{1}{|y|^{2-\alpha}} dy$$

$$= C(I_{a,1} + I_{a,2} + I_{a,3})$$

where $\alpha$ is identically same as in Lemma 4.1. By (4.11) in Lemma 4.2, we have the estimate of $I_{a,1}$ and $I_{a,2}$ is identically same as the estimate of $I_1$ and $I_2$ in section 3.1 since the fundamental solutions $\Gamma(x, y)$ and $\Gamma_a(x, y)$ share the same bound (4.11). Now we use estimate (4.12) in
Lemma 4.2 to estimate $I_{a,3}$, for $p < d/(2 - \alpha)$:

$$I_{a,3} \leq C\delta^\alpha \left( \int_{\mathcal{B}(0,\frac{d}{2})} |y|^{\frac{(2-\alpha-d)p}{p-1}} dy \right)^{1-1/p} \cdot \left( \int_{\mathcal{B}_{1/2}} |y|^{(\alpha-2)p} dy \right)^{1/p}$$

(4.22)

As mentioned before, we use $C_-$ to denote an arbitrary number close but smaller than $C$. Then we get the Hölder continuity of the potential $w_a(x)$, thus we finished the proof of the elliptic part of Theorem 2.

5. Parabolic Case

We consider

$$\partial_t(a_{ij}(x,t)\partial_j u(x,t)) - \frac{A}{|x|^2} u(x,t) - \partial_t u(x,t) = 0$$

(5.1)

in $B \times \mathbb{R}_+$, where $A > 0$ is a constant and $a_{ij}(x,t) \in L^\infty$ satisfies the elliptic condition $\lambda I < (a_{ij}(x,t)) < \Lambda I$ with $0 < \lambda < \Lambda < \infty$ as before. By classical De Giorgi’s result, the solution is bounded on $B_{1/2} \times [0,T]$ for some $0 < T < \infty$. Similar to the elliptic case, treating the 0-order term $\frac{A}{|x|^2} u(t,x)$ as an external force term, and by Duhamel’s Principle, for $t > t_2 > 0$, we could write a weak solution of (5.1) in the following form

$$u(x,t) = u_{a,0}(x,t) + \int_{t_2/2}^t \int_{B_{1/2}} \Gamma_a(x,t;y,s) \frac{A \cdot u(y,s)}{|y|^2} dy ds$$

(5.2)

where $u_{a,0}$ is a weak solution of (5.1) in $B_{1/2} \times \mathbb{R}_+$ without potential ($A = 0$) for $t > \frac{t_2}{2}$, namely

$$\partial_t(a_{ij}(x,t)\partial_j u(x,t)) - \partial_t u(x,t) = 0,$$

(5.3)

and $\Gamma_a(x,t;y,s)$ is the fundamental solution of (5.1) with the source point at $(y,s)$. The first term is a weak solution of (5.3) and it is obviously Hölder continuous when $t > t_2/2 > 0$. Similarly as before, we only need to estimate

$$w_a(x,t) = \int_{t_2/2}^t \int_{B_{1/2}} \Gamma_a(x,t;y,s) \frac{A \cdot u(y,s)}{|y|^2} dy ds$$

(5.4)
and get its Hölder continuity in a space-time cube. Let \( t_1 > t_2 > 0, \ x_1, x_2 \in B_{1/2} \), and set \( z = \frac{t_1 + t_2}{2}, \ \delta = |x_1 - x_2| \) as before. We firstly give the parabolic version of Lemma 4.1

**Lemma 5.1.** Let \( u \) be a weak solution of the equation

\[
\partial_t (a_{ij}(x,t) \partial_i u(x,t)) - \frac{A}{|x|^2} u(x,t) - \partial_i u(x,t) = 0
\]

(5.5)

\( x_0 \in B_{1/2}, \ \sqrt{t}/3 \geq r \geq |x_0| \geq 0. \) Here \( a_{ij} \in L^\infty (B \times \mathbb{R}_+), \ \lambda I \leq (a_{ij}(x,t))_{1 \leq i,j \leq d} \leq \Lambda I \) for some \( 0 < \lambda \leq \Lambda < \infty. \) Then there exists an \( \alpha > 0, \ C > 0 \) depending only on \( \lambda, \ \Lambda, \ A, \ d \) and the \( L^2 \) module of \( u \) on \( Q_r(x_0, t) \), such that

\[
|u(x_0, t)|^2 \leq C t^{-(d/2 + 1 + \alpha)} |x_0|^{2\alpha}
\]

(5.6)

We could get this lemma similarly as the elliptic case (Lemma 4.1 the mean value inequality in section 3 is originally for parabolic case). We omit the details here.

\( \square \)

We commence with the proof of the parabolic part of the Theorem 2.

\[
|w_a(x_1, t_1) - w_a(x_2, t_2) |
\]

\[
= \left| \int_{t_2}^{t_1} \int_{B_{1/2}} \Gamma_a(x_1, t_1; y, s) \frac{A \cdot u(s, y)}{|y|^2} dy \ ds \right|
\]

\[
- \left| \int_{t_2}^{t_1} \int_{B_{1/2}} \Gamma_a(x_2, t_2; y, s) \frac{A \cdot u(s, y)}{|y|^2} dy \ ds \right|
\]

\[
\leq \int_{t_2}^{t_1} \int_{B_{1/2}} \left| \Gamma_a(x_1, y; t_1, s) \right| \left| \frac{A \cdot u(s, y)}{|y|^2} \right| dy \ ds
\]

\[
+ \int_{t_2}^{t_1} \int_{B_{1/2} - B(z, \delta)} \left| \Gamma_a(x_1, t_1; y, s) - \Gamma_a(x_2, t_2; y, s) \right| \left| \frac{A \cdot u(s, y)}{|y|^2} \right| dy \ ds
\]

\[
+ \int_{t_2}^{t_1} \int_{B(z, \delta)} \left| \Gamma_a(x_1, t_1; y, s) \right| \left| \frac{A \cdot u(s, y)}{|y|^2} \right| dy \ ds
\]

\[
+ \int_{t_2}^{t_1} \int_{B(z, \delta)} \left| \Gamma_a(x_2, t_2; y, s) \right| \left| \frac{A \cdot u(s, y)}{|y|^2} \right| dy \ ds
\]

\[
:= I + II + III + IV.
\]

To bound term I, we choose \( \alpha_1 \in (0, \alpha), \) by the bound of \( \Gamma_a. \) We first bound the following
integral:
\[
\int_{t_2}^{t_1} \frac{1}{(t_1 - s)^{d/2}} \exp \left( - \frac{|x_1 - y|^2}{C(t_1 - s)} \right) s^{-(d/2+1+\alpha)} ds
\]
\[
\leq t_2^{-(d/2+1+\alpha)} \int_{0}^{t_1-t_2} \frac{1}{s^{d/2}} \exp \left( - \frac{|x_1 - y|^2}{Cs} \right) ds
\]
\[
\leq t_2^{-(d/2+1+\alpha)} \cdot (t_1 - t_2)^{\alpha_1/2} \cdot \left( \int_{0}^{t_1-t_2} \frac{1}{s^{d/2}} \exp \left( - \frac{|x_1 - y|^2}{Cs} \right) \frac{2}{2-\alpha_1} ds \right)^{1-\alpha_1/2}
\]
\[
\leq t_2^{-(d/2+1+\alpha)} \cdot (t_1 - t_2)^{\alpha_1/2} \cdot \left( \int_{0}^{\infty} \frac{1}{s^{d/2}} \exp \left( - \frac{|x_1 - y|^2}{Cs} \right) \frac{2}{2-\alpha_1} ds \right)^{1-\alpha_1/2}
\]
\[
\lesssim t_2^{-(d/2+1+\alpha)} \cdot (t_1 - t_2)^{\alpha_1/2} \cdot |x_1 - y|^{2-\alpha_1-d}
\]

Thus, by (5.6), term I in (5.7) satisfies the following estimate:
\[
I \lesssim t_2^{-(d/2+1+\alpha)} \cdot (t_1 - t_2)^{\alpha_1/2} \cdot \int_{B_{1/2}} |x_1 - y|^{2-d-\alpha_1} \frac{1}{|y|^{2-\alpha}} dy.
\]

Choose \(p > 1\) such that \(\frac{d}{2-\alpha_1} < p < \frac{d}{2-\alpha}\). By Hölder inequality, we have
\[
I \lesssim t_2^{-(d/2+1+\alpha)} \cdot (t_1 - t_2)^{\alpha_1/2} \cdot \left( \int_{B_{1/2}} |x_1 - y|^{(2-d-\alpha_1)p} dy \right)^{1-1/p}
\]
\[
\cdot \left( \int_{B_{1/2}} |y|^{p(\alpha-2)} dy \right)^{1/p}
\]
\[
\lesssim t_2^{-(d/2+1+\alpha)} \cdot (t_1 - t_2)^{\alpha_1/2} \cdot \left( \int_{B} |y|^{(2-d-\alpha_1)p} dy \right)^{1-1/p}
\]
\[
\cdot \left( \int_{B_{1/2}} |y|^{p(\alpha-2)} dy \right)^{1/p}
\]
\[
\lesssim t_2^{-(d/2+1+\alpha)} \cdot \left( \frac{p - 1}{(2 - \alpha_1)p - d} \right)^{1-1/p} \cdot \left( \frac{1}{(\alpha - 2)p + d} \right)^{1/p} \cdot (t_1 - t_2)^{\alpha_1/2}
\]
\[
\lesssim t_2^{-(d/2+1+\alpha)} \cdot (t_1 - t_2)^{\alpha_1/2}, \quad \text{as } \alpha_1 \to \alpha
\]

Before the estimation of II, we first give two identity of the heat kernel function, which will be used later. Next we recall the following well-known facts for heat kernel.

Let
\[
f(t) = \frac{1}{t^{d/2}} \exp \left( - \frac{a^2}{t} \right), \quad a > 0.
\]

Then
\[
\int_{0}^{\infty} f(t) dt = \Gamma(d/2 - 1)a^{2-d} \sim a^{2-d},
\]
where $\Gamma$ is the usual $\Gamma$ function, and

$$
\sup_{t \in (0, \infty)} f(t) = \exp\left(-\frac{d}{2}\right) \cdot \left(\frac{d}{2}\right)^{d/2} \cdot a^{-d} \sim a^{-d},
$$

(5.13)

and $f$ increases when $t \in (0, \frac{2a^2}{d})$, decreases when $t \in (\frac{2a^2}{d}, \infty)$.

Since $y \in B(z, \delta)^c$, then for fixed $y$ and $s$, $\Gamma_a$ is a well-defined weak solution of equation

$$
\partial_{x_i}(a_{ij}(x, t) \partial_{x_j}u(x, t)) - \partial_t u(x, t) = 0
$$

(5.14)

for $(x, t) \in B(z, \frac{3|z-y|}{4}) \times [t_2 - \tau_0, t_1]$, where $\tau_0$ is to be defined later. By the classical De Giorgi-Nash-Moser estimate, we have

$$
\left| \Gamma_a(x_1, t_1; y, s) - \Gamma_a(x_2, t_2; y, s) \right|
\leq \sup_{(x, t) \in B(z, \frac{3|z-y|}{4}) \times [t_2 - \tau_0, t_1]} \left| \Gamma_a(x, t; y, s) \right| \cdot \left( \frac{|x_1 - x_2| + \sqrt{t_1 - t_2}}{\min\{|y - x_1|, |y - x_2|, \sqrt{\tau_0}\}} \right)\alpha
\leq \sup_{(x, t) \in B(z, \frac{3|z-y|}{4}) \times [t_2 - \tau_0, t_1]} \frac{1}{(t - s)^{d/2}} \exp\left(-\frac{|x - y|^2}{C(t - s)}\right) \cdot \left( \frac{|x_1 - x_2| + \sqrt{t_1 - t_2}}{\min\{|y - x_1|, |y - x_2|, \sqrt{\tau_0}\}} \right)\alpha
\leq \sup_{t \in [t_2 - \tau_0, t_1]} \frac{1}{(t - s)^{d/2}} \left( \exp\left(-\frac{|x_1 - y|^2}{C(t - s)}\right) + \exp\left(-\frac{|x_2 - y|^2}{C(t - s)}\right) \right) \cdot \left( \frac{|x_1 - x_2| + \sqrt{t_1 - t_2}}{\min\{|y - x_1|, |y - x_2|, \sqrt{\tau_0}\}} \right)\alpha
$$

(5.15)

The second "$\leq$" is due to the bound of fundamental solution of parabolic equation, which can be found in [2]. Now choose $\tau_0 = \min\{|x_1 - y|^2, |x_2 - y|^2\}/C$, where $C$ is the constant in the last line of (5.15). Since in the domain $y \in B(z, \delta)^c$, $\Gamma_a(x, t; y, s)$ could do a well-defined 0 extension to $t < s$, we need not worry about $t_2 < \tau_0$. In order to proceed, we need to bound the following integral:

$$
J := \int_{t_2}^{t_1} \sup_{t \in [t_2 - \tau_0, t_1]} \frac{1}{(t - s)^{d/2}} \left( \exp\left(-\frac{|x_1 - y|^2}{C(t - s)}\right) + \exp\left(-\frac{|x_2 - y|^2}{C(t - s)}\right) \right) \cdot s^{-(d/2 + 1 + \alpha)} ds
\leq t_2^{(d/2 + 1 + \alpha)} \left[ \int_{t_2}^{t_1} \sup_{t \in [t_2 - \tau_0, t_1]} \frac{1}{(t - s)^{d/2}} \left( \exp\left(-\frac{|x_1 - y|^2}{C(t - s)}\right) \right) ds
\right]
\quad + \int_{t_2}^{t_1} \sup_{t \in [t_2 - \tau_0, t_1]} \frac{1}{(t - s)^{d/2}} \left( \exp\left(-\frac{|x_2 - y|^2}{C(t - s)}\right) \right) ds
\quad := t_2^{-(d/2 + 1 + \alpha)} \cdot (J_1 + J_2).
$$

(5.16)

Note that this integral has a "sup" inside, we will need to split the interval of integration here.
Let $C_0 = \frac{1}{C} \left(1 + \frac{2}{d}\right)$, for $i = 1, 2$, we have, when $s \in \left[\frac{t_2}{2}, t_2 - C_0|x_i - y|^2\right]$ and $t \in [t_2 - \tau_0, t_1]$, $t - s \geq \frac{2}{d} \cdot \frac{|x_i - y|^2}{C}$.

If $t_2 > 2C_0|x_i - y|^2$, namely $t_2 - C_0|x_i - y|^2 > t_2/2$, we have

$$J_i \leq \int_{\frac{t_2 - C_0|x_i - y|^2}{2}}^{t_2} \sup_{t \in [t_2 - \tau_0, t_1]} \frac{1}{(t - s)^{d/2}} \exp\left(-\frac{|x_i - y|^2}{C(t - s)}\right) ds$$

$$+ \int_{t_2 - C_0|x_i - y|^2}^{t_2} \sup_{t \in [t_2 - \tau_0, t_1]} \frac{1}{(t - s)^{d/2}} \exp\left(-\frac{|x_i - y|^2}{C(t - s)}\right) ds$$

(5.17)

By (5.11), the monotonicity of $f$, we have

$$\sup_{t \in [t_2 - \tau_0, t_1]} \frac{1}{(t - s)^{d/2}} \exp\left(-\frac{|x_i - y|^2}{C(t - s)}\right) = \frac{1}{(t_2 - \tau_0 - s)^{d/2}} \exp\left(-\frac{|x_i - y|^2}{C(t_2 - \tau_0 - s)}\right)$$

$$\forall s \in \left[\frac{t_2}{2}, t_2 - C_0|x_i - y|^2\right] \quad i = 1, 2.$$  

(5.18)

Thus, by (5.12) and (5.13), we have:

$$J_i \lesssim \int_{0}^{\infty} \frac{1}{s^{d/2}} \exp\left(-\frac{|x_i - y|^2}{Cs}\right) ds + C_0|x_i - y|^{2-d}$$

$$\lesssim |x_i - y|^{2-d}.$$  

(5.19)

Else, if $t_2 \leq 2C_0|x_i - y|^2$, we have:

$$J_i \lesssim |x_i - y|^2 \cdot \sup_{s \in \mathbb{R}^+} \frac{1}{s^{d/2}} \exp\left(-\frac{|x_i - y|^2}{Cs}\right)$$

$$\lesssim |x_i - y|^{2-d}.$$  

(5.20)
Therefore, term II satisfies the following estimate

\[
II \lesssim t_2^{-(d/2+1+\alpha)} \int_{B_1/2-B(z,\delta)} \frac{J_1 + J_2}{|y|^{2-\alpha}} dy \\
\lesssim t_2^{-(d/2+1+\alpha)} \cdot (|x_1 - x_2| + \sqrt{t_1 - t_2})^\alpha \\
\cdot \int_{B_1/2-B(z,\delta)} \left(|x_1 - y|^{2-\alpha-d} + |x_2 - y|^{2-\alpha-d}\right) \frac{1}{|y|^{2-\alpha}} dy \\
\lesssim t_2^{-(d/2+1+\alpha)} \cdot (|x_1 - x_2| + \sqrt{t_1 - t_2})^\alpha \\
\cdot \left( \int_{B(0,\frac{\delta}{2})} |y|^{(2-\alpha-d)p} dy \right)^{1/p} \cdot \left( \int_{B_1/2} |y|^{(\alpha-2)p} dy \right)^{1/p} \\
\lesssim t_2^{-(d/2+1+\alpha)} \cdot (|x_1 - x_2| + \sqrt{t_1 - t_2})^{2-\frac{d}{p} \cdot (p-1)^{1-1/p}} \cdot (\alpha-2)p + d \\
\lesssim t_2^{-(d/2+1+\alpha)} \cdot (|x_1 - x_2| + \sqrt{t_1 - t_2})^\alpha, \quad \text{by choosing } p \rightarrow \left(\frac{d}{2-\alpha}\right)^-.
\]

Finally, III and IV are essentially the same, thus we estimate them together

\[
III, IV \lesssim t_2^{-(d/2+1+\alpha)} \cdot \int_0^\infty \int_{B(z,\delta)} \frac{1}{s^{d/2}} \exp\left(-\frac{|x_i - y|^2}{Cs}\right) \frac{1}{|y|^{2-\alpha}} |dy| ds, \quad i = 1, 2 \\
\lesssim t_2^{-(d/2+1+\alpha)} \cdot \int_{B(z,\delta)} |x_i - y|^{2-d} \frac{1}{|y|^{2-\alpha}} dy, \quad i = 1, 2 \\
\lesssim t_2^{-(d/2+1+\alpha)} \cdot |x_1 - x_2|^\alpha. 
\]

The last step is the same as (2.28) in section 3. Thus we get the Hölder continuity of \( u_\alpha(t, x) \) with \( x_1, x_2 \in B_1/2 \) and \( t_1 > t_2 > 0 \). The Hölder norm of \( u \) respect to \( \lambda, \Lambda, d, A, t_2 \), and the \( L^2 \) norm of \( u \), i.e.

\[
\|u\|_{C^{\alpha-\frac{\alpha}{d-1}}((t_0,\infty)\times B_1/2)} \leq C(\lambda, \Lambda, d, A)t_0^{-(d/2+1+\alpha)}\|u\|_{L^2} 
\]

where \( \alpha = \alpha(\lambda, \Lambda, d, A) > 0 \).

6. Appendix

6.1. Existence Results

Since the Potential \( \frac{A}{|x|^{d+\alpha}} \) is much more singular than the classical case, the existence and uniqueness of weak solutions of (1.1) can not be taken for granted. Let \( u_k \) be the unique solution
to the following parabolic problem:

\[ \partial_t (a_{ij} \partial_j u_k) - Ac_k u_k - \partial_t u_k = 0 \quad \text{in } B \times [0, T] \]
\[ u_k(x, 0) = u_0 \in L^2(B) \quad \text{on } B \]
\[ u_k = 0 \quad \text{on } \partial B \times [0, T]. \]  (6.1)

Here \( a_{ij} \) is as in (1.2), \( c_k = c_k(x) \in C_0^\infty(B) \) satisfies

\[ c_k \to \frac{1}{|x|^{2+\beta}}, \quad \text{strongly in } L^p(B). \]  (6.2)

for a fixed \( p \in [1, \frac{d}{2+\beta}] \). This is possible since \( \frac{1}{|x|^{2+\beta}} \in L^{\frac{d}{2+\beta}}(B) \). Multiplying Eq. (6.1) by \( u_k \), one easily obtain

\[ \lambda \int_0^T \int_B |\nabla u_k|^2 \, dx \, dt + A \int_0^T \int_B c_k u_k^2 \, dx \, dt + \int_B u_k^2 \, dx \int_B (\cdot, T) \, dx \leq \int_B u_0^2 \, dx. \]  (6.3)

Hence there exists a function \( u \) such that \( u, \nabla u \in L^2(B \times [0, T]) \) and a subsequence of \( \{u_k\} \) such that

\[ u_k \rightharpoonup u \quad \text{weakly in } L^2(B \times [0, T]); \]
\[ \nabla u_k \rightharpoonup \nabla u \quad \text{weakly in } L^2(B \times [0, T]); \]
\[ u_k \rightharpoonup u \quad \text{weakly in } L^2([0, T], L^{\frac{2d}{d-2}}(B)). \]  (6.4)

Now we are going to prove \( u \) is a weak solution of (1.2) with the same initial and boundary condition as (6.1). Clearly, \( \forall \phi \in C_0^\infty(B \times [0, T]) \), \( u_k \) satisfies

\[ \int_0^T \int_B a_{ij} \partial_i u_k \partial_j \phi \, dx \, dt + A \int_0^T \int_B c_k u_k \phi \, dx \, dt - \int_0^T \int_B u_k \partial_t \phi \, dx \, dt = \int_B u_0 \phi(x, 0) \, dx. \]  (6.5)

By the weak convergence of \( u_k \) and \( \nabla u_k \), we have

\[ \int_0^T \int_B a_{ij} \partial_i u_k \partial_j \phi \, dx \, dt - \int_0^T \int_B u_k \partial_t \phi \, dx \, dt \to \int_0^T \int_B a_{ij} \partial_i u \partial_j \phi \, dx \, dt - \int_0^T \int_B u \partial_t \phi \, dx \, dt, \]
\[ \text{as } k \to \infty. \]  (6.6)
Next, notice that
\[
\int_0^T \int_B c_k u_k \phi dxdt - \int_0^T \int_B \frac{u_0 \phi}{|x|^{2+\beta}} dxdt
= \int_0^T \int_B u_k \phi \left( c_k - \frac{1}{|x|^{2+\beta}} \right) dxdt + \int_0^T \int_B \frac{1}{|x|^{2+\beta}} (u_k - u) \phi dxdt.
\] (6.7)

For \( \beta \in [0, d - 2] \), by the strong convergence of \( c_k \) and weak convergence of \( u_k \), we have
\[
\int_0^T \int_B c_k u_k \phi dxdt - \int_0^T \int_B \frac{u_0 \phi}{|x|^{2+\beta}} dxdt \to 0 \quad \text{as} \quad k \to \infty.
\] (6.8)

By (6.5) and (6.8), we obtain
\[
\int_0^T \int_B a_{ij} \partial_i u \partial_j \phi dxdt + A \int_0^T \int_B \frac{1}{|x|^{2+\beta}} u \phi dxdt - \int_0^T \int_B u \partial_t \phi dxdt = \int_B u_0 \phi(x, 0) dx.
\] (6.9)

i.e. \( u \) is a weak solution to (1.2). Since elliptic problem is a time-independent parabolic case, we get the existence results for both.

6.2. Local Boundedness of the Weak Solution and Maximum Principle

**Lemma 6.1.** The weak solution of (1.1) in \( B \) is bounded in \( B_{1/2} \).

**Proof.** Since the potential term coefficient \( \frac{A}{|x|^{2+\beta}} \geq 0 \), we could get boundedness of \( u \) by classical De Giorgi iteration method. We omit the details here.

**Lemma 6.2** (weak maximum principle for elliptic equation). Let \( u \in H^\beta(B) \) be a weak solution of elliptic equation (1.1) in \( B \), then
\[
\sup_{B_{1/2}} u \leq \sup_{\partial B_{1/2}} u_+ , \quad \inf_{B_{1/2}} u \geq \inf_{\partial B_{1/2}} u_- \quad \text{(6.10)}
\]

Here \( u_+ = \max\{u, 0\} \) and \( u_- = \min\{u, 0\} \).

**Proof.** Choose test function \( v = \max\{u - l, 0\} \), where \( l = \sup_{B_{1/2}} u_+ \), pay attention that \( v \in H^\beta_0(B_{1/2}) \), \( \frac{1}{|x|^{2+\beta}} u \cdot v \geq 0 \), then we have
\[
0 \geq \int_{B_{1/2}} a_{ij}(x) \partial_i u(x) \partial_j v(x) dx
= \int_{B_{1/2}} a_{ij}(x) \partial_i (u(x) - l)_+ \partial_j (u(x) - l)_+ dx
\geq \lambda \int_{B_{1/2}} |\nabla (u - l)_+|^2 dx
\] (6.11)
which means $v \equiv 0$ in $B_{1/2}$, which we get the first inequality. The second one is similar.

6.3. An Introduction to the Modified Bessel’s Equation

We call an important ordinary differential equation which was used in this paper: the Modified Bessel’s equation

$$t^2 x''(t) + tx'(t) - (t^2 + \lambda^2) x(t) = 0.$$ \tag{6.12}

This equation has two linearly independent solution, i.e. $I_\lambda(t), K_\lambda(t)$, which are exponentially growing and decaying as $t \to +\infty$ and which are referred to as modified Bessel’s function of first and second kind, respectively. For more detailed information of Modified Bessel’s equation and modified Bessel’s function, we refer to [1], a well-known handbook of mathematical special functions.

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