Hermite–Hadamard Inclusions for Co-Ordinated Interval-Valued Functions via Post-Quantum Calculus

Jessada Tariboon 1,*,†, Muhammad Aamir Ali 2,*,†, Hüseyin Budak 3,† and Sotiris K. Ntouyas 4,5,†

1 Intelligent and Nonlinear Dynamic Innovations Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut’s University of Technology North Bangkok, Bangkok 10800, Thailand
2 Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China
3 Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce 81620, Turkey; hsyn.budak@gmail.com
4 Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece; sntouyas@uoi.gr
5 Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

* These authors contributed equally to this work.

Abstract: In this paper, the notions of post-quantum integrals for two-variable interval-valued functions are presented. The newly described integrals are then used to prove some new Hermite–Hadamard inclusions for co-ordinated convex interval-valued functions. Many of the findings in this paper are important extensions of previous findings in the literature. Finally, we present a few examples of our new findings. Analytic inequalities of this nature and especially the techniques involved have applications in various areas in which symmetry plays a prominent role.

Keywords: Hermite–Hadamard inequality; Hermite–Hadamard inclusion; $(p,q)$-integral; quantum calculus; co-ordinated convexity; interval-valued functions

1. Introduction

The modern name for the study of calculus without limits is quantum calculus, or $q$-calculus. It has been studied since the early eighteenth century. Euler, a prominent mathematician, invented $q$-calculus, and F. H. Jackson [1] discovered the definite $q$-integral known as the $q$-Jackson integral in 1910. Orthogonal polynomials, combinatorics, number theory, simple hypergeometric functions, quantum theory, dynamics, and theory of relativity are only a few of the applications of quantum calculus in mathematics and physics; see, for example, [2–19] and the references therein. V. Kac and P. Cheung’s book [20] discusses the fundamentals of quantum calculus as well as the basic theoretical terms.

J. Tariboon and S. K. Ntouyas [21] described and proved some of the properties of the $q$-derivative and $q$-integral of a continuous functions on finite intervals in 2013. Moreover, they proved Hermite–Hadamard-type inequalities and many others for convex functions in the setting of quantum calculus; for more information, see [22].

M. Tunç and E. Göv [23] presented the $(p,q)$-derivative and $(p,q)$-integral on finite intervals in 2016, proved some of their properties, and proved a number of integral inequalities using the $(p,q)$-calculus. Many researchers have recently begun working in this direction, based on the works of M. Tunç and E. Göv, and some further findings on the analysis of $(p,q)$-calculus can be found in [24–27].

In [28], S. S. Dragomir proved the following inequalities, which are Hermite–Hadamard type inequalities for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$. 

Symmetry 2021, 13, 1216. https://doi.org/10.3390/sym13071216
Theorem 1. Suppose that $f : [a, b] \times [c, d] \to \mathbb{R}$ is co-ordinated convex; then we have the following inequalities:

$$ f\left(\frac{a + b}{2}, \frac{c + d}{2}\right) \leq \frac{1}{2} \left[ \frac{1}{b - a} \int_a^b f\left(x, \frac{c + d}{2}\right) dx + \frac{1}{d - c} \int_c^d f\left(\frac{a + b}{2}, y\right) dy \right] $$ (1)

$$ \leq \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) dy dx $$

$$ \leq \frac{1}{4} \left[ \frac{1}{b - a} \int_a^b f(x, c) dx + \frac{1}{b - a} \int_a^b f(x, d) dx \right. $$

$$ \left. + \frac{1}{d - c} \int_c^d f(a, y) dy + \frac{1}{d - c} \int_c^d f(b, y) dy \right] $$

$$ \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} $$

The above inequalities are sharp. The inequalities in (1) hold in the reverse direction if the mapping $f$ is a co-ordinated concave mapping.

The quantum variant of above inequality (1) was given by M. Kunt et al. in [29], S. Bermudo et al. [30] recently used $q$-calculus to describe new $q^b$-derivative and $q^b$-integral, as well as to give the Hermite–Hadamard inequality. H. Budak et al. [31] defined some new $q^b$-integrals for co-ordinates and Hermite–Hadamard inequalities for co-ordinated convex functions as a result of this. F. Wannalookkhee et al. [32] in 2021 gave some new definitions of $(p, q)^b$-integrals and used them to prove the following Hermite–Hadamard inequalities:

$$ f\left(\frac{q_1 a + p_1 b}{[2]_{p_1,q_1}}, \frac{p_2 c + q_2 d}{[2]_{p_2,q_2}}\right) $$ (2)

\[
\leq \frac{1}{2} \left[ \frac{1}{p_1(b - a)} \int_a^{p_1b + (1 - p_1)a} f\left(x, \frac{p_2c + q_2d}{[2]_{p_2,q_2}}\right) d_{p_1,q_1} x \right.
\]

\[
+ \frac{1}{p_2(d - c)} \int_c^{p_2c + (1 - p_2)c} f\left(\frac{q_1a + p_1b}{[2]_{p_1,q_1}}, y\right) d_{p_2,q_2} y \right]
\]

\[
\leq \frac{1}{p_1p_2(b - a)(d - c)} \int_a^{p_1b + (1 - p_1)a} \int_c^{p_2c + (1 - p_2)c} f(x, y) d_{p_2,q_2} y \, d_{p_1,q_1} x \]

\[
\leq \frac{1}{2p_2[2]_{p_1,q_1}(d - c)} \left[ q_1 \int_a^{d_{p_2,q_2} + (1 - p_2)c} f(a, y) d_{p_2,q_2} y + p_1 \int_a^{d_{p_2,q_2} + (1 - p_2)c} f(b, y) d_{p_2,q_2} y \right]
\]

\[
+ \frac{1}{2p_1[2]_{p_2,q_2}(b - a)} \left[ p_2 \int_a^{p_1b + (1 - p_1)a} f(x, c) d_{p_1,q_1} x + q_2 \int_a^{p_1b + (1 - p_1)a} f(x, d) d_{p_1,q_1} x \right]
\]

\[
q_1p_2f(a, c) + q_1q_2f(a, d) + p_1p_2f(b, c) + q_2p_2f(b, d)
\]

\[
\leq \frac{f\left(\frac{p_1a + q_1b}{[2]_{p_1,q_1}}, \frac{q_2c + p_2d}{[2]_{p_2,q_2}}\right)}{[2]_{p_1,q_1}[2]_{p_2,q_2}}
\]

\[
\leq \frac{1}{2} \left[ \frac{1}{p_1(b - a)} \int_a^{p_1b + (1 - p_1)b} f\left(x, \frac{q_2c + p_2d}{[2]_{p_2,q_2}}\right) d_{p_1,q_1} x \right.
\]

\[
+ \frac{1}{p_2(d - c)} \int_c^{p_2d + (1 - p_2)c} f\left(\frac{p_1a + q_1b}{[2]_{p_1,q_1}}, y\right) d_{p_2,q_2} y \right] \]
\[ \leq \frac{1}{p_1p_2(b-a)(d-c)} \int_b^{b+1-p_1} \int_c^{2d+(1-p_2)c} f(x,y) \; c \; dp_{2,q_1}^b \; dy \; dp_{1,q_1}^x \]

\[ \leq \frac{1}{2p_2[2]p_{1,q_1}^a} \left[ \frac{1}{p_1 \int_{p_2+a+(1-p_1)b}^{p_2+b+(1-p_2)c} f(a,y) \; c \; dp_{2,q_1}^b \; dy + \frac{1}{q_1 \int_{p_2+a+(1-p_1)b}^{p_2+b+(1-p_2)c} f(b,y) \; c \; dp_{2,q_1}^b \; dy} \right] \]

\[ + \frac{1}{2p_1[2]p_{2,q_1}^b(b-a)} \left[ q_2 \int_b^{b+1-p_1} f(x,c) \; b \; dp_{1,q_1}^x + p_2 \int_b^{b+1-p_1} f(x,d) \; b \; dp_{1,q_1}^x \right] \]

\[ \leq \frac{p_1q_2f(a,c) + p_1r_2f(a,d) + q_1q_2f(b,c) + q_1r_2f(b,d)}{[2]p_{1,q_1}^a[2]p_{2,q_1}^b} \]

\[ f \left( \frac{p_1a + q_1b}{[2]p_{1,q_1}^a}, \frac{p_2c + q_2d}{[2]p_{2,q_1}^b} \right) \]

\[ \leq \frac{1}{2} \left[ \frac{1}{p_2(b-a) \int_{p_2+a+(1-p_1)b}^{p_2+b+(1-p_2)c} f(x,y) \; c \; dp_{2,q_1}^b \; dy + \frac{1}{p_2(d-c) \int_{p_2+c+(1-p_1)d}^{p_2+b+(1-p_2)c} f(x,y) \; c \; dp_{2,q_1}^b \; dy} \right] \]

\[ \leq \frac{1}{p_1p_2(b-a)(d-c)} \int_b^{b+1-p_1} \int_c^{2d+(1-p_2)c} f(x,y) \; d \; dp_{2,q_1}^b \; dy \; dp_{1,q_1}^x \]

\[ \leq \frac{1}{2p_2[2]p_{1,q_1}^a} \left[ \frac{1}{p_1 \int_{p_2+c+(1-p_1)d}^{p_2+b+(1-p_2)c} f(a,y) \; d \; dp_{2,q_1}^b \; dy + \frac{1}{q_1 \int_{p_2+c+(1-p_1)d}^{p_2+b+(1-p_2)c} f(b,y) \; d \; dp_{2,q_1}^b \; dy} \right] \]

\[ + \frac{1}{2p_1[2]p_{2,q_1}^b(b-a)} \left[ q_2 \int_b^{b+1-p_1} f(x,c) \; b \; dp_{1,q_1}^x + q_1 \int_b^{b+1-p_1} f(x,d) \; b \; dp_{1,q_1}^x \right] \]

\[ \leq \frac{p_1p_2f(a,c) + p_1q_2f(a,d) + q_1p_2f(b,c) + q_1q_2f(b,d)}{[2]p_{1,q_1}^a[2]p_{2,q_1}^b} \]

2. Interval Calculus

In this section, we provide notation and background information on interval analysis.

The space of all closed intervals of \( \mathbb{R} \) is denoted by \( I_c \), and \( \Delta \) is a bounded element of \( I_c \).

We have the representation

\[ \Delta = [\Theta_1, \Theta_1] = \{ \Theta \in \mathbb{R} : \Theta_1 \leq \Theta \leq \Theta_1 \} \]

where \( \Theta_1, \Theta_1 \in \mathbb{R} \) and \( \Theta_1 \leq \Theta_1 \). \( L(\Delta) = \Theta_1 - \Theta_1 \) can be used to express the length of the interval \( \Delta = [\Theta_1, \Theta_1] \). The left and right endpoints of interval \( \Delta \) are denoted by the numbers \( \Theta_1 \) and \( \Theta_1 \), respectively. The interval \( \Delta \) is said to be degenerate when \( \Theta_1 = \Theta_1 \), and the form \( \Delta = [\Theta_1, \Theta_1] \) is used. In addition, if \( \Theta_1 > 0 \), we can say \( \Delta \) is positive, and if \( \Theta_1 < 0 \), we can say \( \Delta \) is negative. \( I_c^+ \) and \( I_c^- \) denote the sets of all closed positive intervals and closed negative intervals of \( \mathbb{R} \), respectively. Between the intervals \( \Delta \) and \( \Lambda \), the Pompeiu–Hausdorff distance is defined by

\[ d_H(\Delta, \Lambda) = d_H([\Theta_1, \Theta_1], [\Theta_2, \Theta_2]) = \max \{|\Theta_1 - \Theta_2|, |\Theta_1 - \Theta_2|\}. \]

(\( I_c, d \)) is a complete metric space, as far as we know (see, [33]).

\( |\Delta| \) denotes the absolute value of \( \Delta \), which is the maximum of the absolute values of its endpoints:

\[ |\Delta| = \max \{|\Theta_1|, |\Theta_1|\}. \]

The following are the concepts for fundamental interval arithmetic operations for the intervals \( \Delta \) and \( \Lambda \):

\[ \Delta + \Lambda = [\Theta_1 + \Theta_2, \Theta_1 + \Theta_2], \]
The interval $\Delta$'s scalar multiplication is defined by
\[
\mu \Delta = \mu [\Omega, \Theta] = \begin{cases} 
[\mu \Omega, \mu \Theta], & \mu > 0; \\
\{0\}, & \mu = 0; \\
[\mu \Theta, \mu \Omega], & \mu < 0,
\end{cases}
\]
where $\mu \in \mathbb{R}$.

The opposite of the interval $\Delta$ is
\[
-\Delta := (-1)\Delta = [-\Theta, -\Omega],
\]
where $\mu = -1$.

In general, $-\Delta$ is not additive inverse for $\Delta$, i.e., $\Delta - \Delta \neq 0$.

**Definition 1** ([2]). For some kind of the intervals $\Delta, \Lambda \in I_c$, we denote the the H-difference of $\Delta$ and $\Lambda$ as the $\Omega \in I_c$, and we have
\[
\Delta \Theta \Lambda = \Omega \iff \begin{cases} 
(i) \Delta = \Lambda + \Omega \\
(ii) \Delta = \Lambda + (-\Omega).
\end{cases}
\]

It seems uncontroversial that
\[
\Delta \Theta \Lambda = \begin{cases} 
[\Theta_1 - \Theta_2, \Theta_1 - \Theta_2], & \text{if } L(\Delta) \geq L(\Lambda) \\
[\Theta_1 - \Theta_2, \Theta_1 - \Theta_2], & \text{if } L(\Lambda) \leq L(\Lambda),
\end{cases}
\]
where $L(\Delta) = \Theta_1 - \Theta_2$ and $L(\Lambda) = \Theta_2 - \Theta_2$.

The definitions of operations generate a large number of algebraic properties, enabling $I_c$ to be a quasilinear space (see [34]). The following are some of these characteristics (see [33–36]):

1. (Law of associative under $+$) $(\Delta + \Lambda) + \mathcal{C} = \Delta + (\Lambda + \mathcal{C})$ for all $\Delta, \Lambda, \mathcal{C} \in I_c$,
2. (Additivity element) $\Delta + 0 = 0 + \Delta = \Delta$ for all $\Delta \in I_c$,
3. (Law of commutative under $+$) $\Delta + \Lambda = \Lambda + \Delta$ for all $\Delta, \Lambda \in I_c$,
4. (Law of cancellation under $+$) $\mathcal{C} + (\Delta + \Lambda) = \mathcal{C} + \Lambda \Rightarrow \Delta = \Lambda$ for all $\Delta, \Lambda, \mathcal{C} \in I_c$,
5. (Law of associative under $\times$) $(\Delta \cdot \Lambda) \cdot \mathcal{C} = \Delta \cdot (\Lambda \cdot \mathcal{C})$ for all $\Delta, \Lambda, \mathcal{C} \in I_c$,
6. (Law of commutative under $\times$) $\Lambda \cdot \Delta = \Delta \cdot \Lambda$ for all $\Delta, \Lambda \in I_c$,
7. (Multiplicativity element) $\Lambda \cdot 1 = 1 \cdot \Lambda = \Lambda$ for all $\Lambda \in I_c$,
8. (The first law of distributivity) $\lambda(\Delta + \Lambda) = \lambda\Delta + \lambda\Lambda$ for all $\Lambda, \Delta \in I_c$ and all $\lambda \in \mathbb{R}$,
9. (The second law of distributivity) $(\lambda + \mu)\Delta = \lambda\Delta + \mu\Delta$ for all $\Delta \in I_c$ and all $\lambda, \mu \in \mathbb{R}$.

Aside from any of these characteristics, the distributive law does not always apply to intervals. As an example, $\Delta = [1, 2], \Lambda = [2, 3]$ and $\mathcal{C} = [-2, -1]$.

\[
\Delta \cdot (\Lambda + \mathcal{C}) = [0, 4],
\]
whereas
\[
\Delta \cdot \Lambda + \Delta \cdot \mathcal{C} = [-2, 5].
\]
Another distinct feature is the inclusion \( \subseteq \), which is described by
\[
\Delta \subseteq \Lambda \iff \Theta_1 \geq \Theta_2 \quad \text{and} \quad \Theta_1 \leq \Theta_2.
\]

In [37], Zhao et al. gave the notions about the co-ordinated convex interval-valued functions and inclusions of Hermite–Hadamard type.

**Definition 2 ([37]).** A function \( F = \left[ F, \mathcal{F} \right]: [a, b] \times [c, d] \to I_+^q \) is said to be co-ordinated convex interval-valued function if the following inclusion holds:
\[
F(tx + (1 - t)y, su + (1 - s)w) \supseteq tsF(x, u) + t(1 - s)F(x, w) + s(1 - t)F(y, u) + (1 - s)(1 - t)F(y, w),
\]
for all \((x, y), (u, w) \in [a, b] \times [c, d] \) and \( s, t \in [0, 1] \).

**Remark 1.** A function \( F = \left[ F, \mathcal{F} \right]: [a, b] \times [c, d] \to I_+^q \) is said to be co-ordinated convex interval-valued function if and only if \( F \) and \( \mathcal{F} \) are co-ordinated convex and concave, respectively.

**Lemma 1 ([37]).** A function \( F: [a, b] \times [c, d] \to I_+^q \) is an interval-valued convex on co-ordinates if and only if there exist two functions \( F_x: [c, d] \to I_+^q \), \( F_x(w) = F(x, w) \) and \( F_y: [a, b] \to I_+^q \), \( F_y(u) = F(y, u) \) are interval-valued convex.

It is easy to prove that an interval-valued convex function is an interval-valued co-ordinated convex, but the converse may not be true. For this, we can see the following example.

**Example 1.** An interval-valued function \( F: [0, 1]^2 \to I_+^q \) defined as \( F(x, y) = [xy, (6 - x^2)(6 - ey)] \) is an interval-valued convex on co-ordinates, but it is not an interval-valued convex on \([0, 1]^2\).

For more recent inclusions of Hermite–Hadamard type for co-ordinated convex interval-valued functions one can read [37,38].

**3. Basics of Quantum and Post-Quantum Calculus**

In this section, we review some necessary definitions about \( q \) and \( (p, q) \)-calculus for real-valued and interval-valued functions. Moreover, here and further, we use the following notations with \( 0 < q < p \leq 1 \):
\[
[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + q^2 + \ldots + q^{n-1},
\]
\[
[n]_{p,q} = \frac{p^n - q^n}{p - q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \ldots + q^{n-1}.
\]

**Definition 3 ([23]).** For a function \( f: [a, b] \to \mathbb{R} \), the definite \((p, q)\)-integral of \( f \) is stated as:
\[
\int_a^b f(t) \, dp_{q,t} = (p - q)(x - a) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f \left( \frac{q^n}{p^{n+1}} x + \left( 1 - \frac{q^n}{p^{n+1}} \right) a \right)
\]
(6)
where \( 0 < q < p \leq 1 \) and \( x \in [a, pb + (1 - p)a] \).

**Definition 4 ([24]).** For a function \( f: [a, b] \to \mathbb{R} \), the definite \((p, q)\)-integral of \( f \) is stated as:
\[
\int_a^b f(t) \, dp_{q,t} = (p - q)(b - x) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f \left( \frac{q^n}{p^{n+1}} x + \left( 1 - \frac{q^n}{p^{n+1}} \right) b \right)
\]
(7)
with \( 0 < q < p \leq 1 \) and \( x \in [pa + (1 - p)b, b] \).
Remark 2. If $f$ is a symmetric function, that is, $f(t) = f(b + a - t)$, for $t \in [a, b]$, then we have
\[
\int_{a}^{b + (1-p)a} f(t) d_p, a = \int_{pa + (1-p)b}^{b} f(t) d_p, a.
\]

Definition 5 ([26,32]). For a function $f : [a, b] \times [c, d] \to \mathbb{R}$,
1. The $(p, q)_a^d$ integral of $f$ is given as:
\[
\int_{a}^{b} \int_{y}^{d} f(t, s) d_{p_2, q_2} s a_{p_1, q_1} t = (p_1 - q_1)(p_2 - q_2)(x - a)(d - y)
\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q_n^{m+1}}{p_1^{m+1}} \frac{q_m^{n+1}}{p_2^{n+1}} f \left( \frac{q_1^n}{p_1^{n+1}} x + \left( 1 - \frac{q_1^n}{p_1^{n+1}} \right) a, \frac{q_2^m}{p_2^{m+1}} y + \left( 1 - \frac{q_2^m}{p_2^{m+1}} \right) d \right),
\]
where $x, y \in [a, p_1 b + (1 - p_1) a] \times [p_2 c + (1 - p_2) d, d]$.
2. The $(p, q)_b^c$ integral of $f$ is given as:
\[
\int_{a}^{b} \int_{y}^{d} f(t, s) c_{p_2, q_2} b d_{p_1, q_1} t = (p_1 - q_1)(p_2 - q_2)(b - x)(y - c)
\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q_n^{m+1}}{p_1^{m+1}} \frac{q_m^{n+1}}{p_2^{n+1}} f \left( \frac{q_1^n}{p_1^{n+1}} x + \left( 1 - \frac{q_1^n}{p_1^{n+1}} \right) b, \frac{q_2^m}{p_2^{m+1}} y + \left( 1 - \frac{q_2^m}{p_2^{m+1}} \right) c \right)
\]
where $x, y \in [p_1 a + (1-p_1) b, b] \times [p_2 c + (1 - p_2) d, d]$.
3. The $(p, q)_c^d$ integral of $f$ is given as:
\[
\int_{a}^{b} \int_{y}^{d} f(t, s) a_{p_2, q_2} d_{p_1, q_1} t = (p_1 - q_1)(p_2 - q_2)(b - x)(d - y)
\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q_n^{m+1}}{p_1^{m+1}} \frac{q_m^{n+1}}{p_2^{n+1}} f \left( \frac{q_1^n}{p_1^{n+1}} x + \left( 1 - \frac{q_1^n}{p_1^{n+1}} \right) b, \frac{q_2^m}{p_2^{m+1}} y + \left( 1 - \frac{q_2^m}{p_2^{m+1}} \right) d \right),
\]
where $x, y \in [p_1 a + (1-p_1) b, b] \times [c, p_2 d + (1 - p_2) c]$.
4. The $(p, q)_ac$ integral of $f$ is given as:
\[
\int_{a}^{b} \int_{y}^{d} f(t, s) c_{p_2, q_2} a d_{p_1, q_1} t = (p_1 - q_1)(p_2 - q_2)(x - a)(y - c)
\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q_n^{m+1}}{p_1^{m+1}} \frac{q_m^{n+1}}{p_2^{n+1}} f \left( \frac{q_1^n}{p_1^{n+1}} x + \left( 1 - \frac{q_1^n}{p_1^{n+1}} \right) a, \frac{q_2^m}{p_2^{m+1}} y + \left( 1 - \frac{q_2^m}{p_2^{m+1}} \right) c \right)
\]
where $x, y \in [a, p_1 b + (1 - p_1) c] \times [c, p_2 d + (1 - p_2) c]$.

Recently, in [39], the authors gave the notions of quantum integral for the interval-valued functions and stated the following:

Definition 6 ([39]). For an interval-valued function $F = [I, F] : [a, b] \to I_c$, the $Iq_a$-definite integral is defined by
\[
\int_{a}^{b} F(s) a d_p, s = (1-q)(x-a) \sum_{n=0}^{\infty} q^n F(q^n x + (1-q^n)a)
\]
for all $x \in [a, b]$. 

Definition 7 ([40]). For an interval-valued function $F = [\mathcal{E}, \mathcal{T}] : [a, b] \to I_c$, the $Iq^b$-definite integral is defined by

$$\int_x^b F(s) \, d_{q^b} s = (1 - q)(b - x) \sum_{n=0}^{\infty} q^n F(q^n x + (1 - q^n)b)$$ (9)

for all $x \in [a, b]$.

In [41], Ali et al. gave the post-quantum version of Definition 7 and defined it as:

Definition 8. For an interval-valued function $F = [\mathcal{E}, \mathcal{T}] : [a, b] \to I_c$, the $I(p, q)^b$-definite integral is defined by

$$\int_x^b F(s) \, d_{p, q^b} s = (p - q)(b - x) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} F \left( \frac{q^n}{p^{n+1}} x + \left(1 - \frac{q^n}{p^{n+1}} \right)b \right)$$ (10)

for all $x \in [pa + (1 - p)b, b]$.

In [40], Ali et al. gave the co-ordinated version of the quantum integrals for interval-valued functions and defined it as:

Definition 9 ([40]). Suppose that $F = [\mathcal{E}, \mathcal{T}] : [a, b] \times [c, d] \to I_c$ is an interval-valued function. Then, the definite $q_{ac}$, $q_{bd}$ and $q_{cd}$ integrals on $[a, b] \times [c, d]$ are defined by

$$\int_a^x \int_c^y F(t, s) \, d_{q_{ac}}^s \, d_{q_{bd}}^t = (1 - q_1)(1 - q_2)(x - a)(y - c)$$

$$\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m F(q_1^n x + (1 - q_1^n)a, q_2^m y + (1 - q_2^m)c),$$

$$\int_a^x \int_y^d F(t, s) \, d_{q_{bd}}^s \, d_{q_{ac}}^t = (1 - q_1)(1 - q_2)(x - a)(d - y)$$

$$\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m F(q_1^n x + (1 - q_1^n)a, q_2^m y + (1 - q_2^m)d),$$

$$\int_x^b \int_c^y F(t, s) \, d_{q_{cd}}^s \, d_{q_{bd}}^t = (1 - q_1)(1 - q_2)(b - x)(y - c)$$

$$\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m F(q_1^n x + (1 - q_1^n)b, q_2^m y + (1 - q_2^m)c),$$

and

$$\int_x^b \int_y^d F(t, s) \, d_{q_{bd}}^s \, d_{q_{cd}}^t = (1 - q_1)(1 - q_2)(b - x)(d - y)$$

$$\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m F(q_1^n x + (1 - q_1^n)b, q_2^m y + (1 - q_2^m)d),$$

respectively, for $(x, y) \in [a, b] \times [c, d]$. 
Remark 3. It is very easy to observe that

\[
\int_a^b \int_c^d F(t, s) \ cd^{t}_q s \ \ ad^{t}_q t = \int_a^b \int_c^d F(t, s) \ cd^{t}_q s \ \ ad^{t}_q t = \int_a^b \int_c^d F(t, s) \ cd^{t}_q s \ \ b d^{t}_q t = \int_a^b \int_c^d F(t, s) \ cd^{t}_q s \ \ b d^{t}_q t
\]

by taking the limits \(q_1, q_2 \to 1^-\) (see, [42]).

Now, we define \(I(p, q)\)-integrals for the functions of two variables as:

**Definition 10.** For an interval-valued function \(F = [F, \Gamma] : [a, b] \times [c, d] \to I_c,

1. The \(I(p, q)\)\(a\)-integral of \(F\) is given as:

\[
\int_a^b \int_c^d F(t, s) \ cd^{t}_p s \ cd^{t}_q t = (p_1 - q_1)(p_2 - q_2)(x - a)(d - y) \\
\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q_1^n}{p_1^{n+1}} \frac{q_2^m}{p_2^{m+1}} F(\frac{q_1^n}{p_1^{n+1}} x + (1 - \frac{q_1^n}{p_1^{n+1}}) a, \frac{q_2^m}{p_2^{m+1}} y + (1 - \frac{q_2^m}{p_2^{m+1}}) c)
\]

where \(x, y \in [a, p_1 b + (1 - p_1) a] \times [c, p_2 d + (1 - p_2) c].

2. The \(I(p, q)\)\(b\)-integral of \(F\) is given as:

\[
\int_a^b \int_c^d F(t, s) \ cd^{t}_p s \ cd^{t}_q t = (p_1 - q_1)(p_2 - q_2)(b - x)(y - c) \\
\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q_1^n}{p_1^{n+1}} \frac{q_2^m}{p_2^{m+1}} F(\frac{q_1^n}{p_1^{n+1}} x + (1 - \frac{q_1^n}{p_1^{n+1}}) b, \frac{q_2^m}{p_2^{m+1}} y + (1 - \frac{q_2^m}{p_2^{m+1}}) c)
\]

where \(x, y \in [p_1 a + (1 - p_1) c] \times [c, p_2 d + (1 - p_2) c].

3. The \(I(p, q)\)\(ac\)-integral of \(F\) is given as:

\[
\int_a^b \int_c^d F(t, s) \ cd^{t}_p s \ cd^{t}_q t = (p_1 - q_1)(p_2 - q_2)(x - a)(y - c) \\
\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q_1^n}{p_1^{n+1}} \frac{q_2^m}{p_2^{m+1}} F(\frac{q_1^n}{p_1^{n+1}} x + (1 - \frac{q_1^n}{p_1^{n+1}}) a, \frac{q_2^m}{p_2^{m+1}} y + (1 - \frac{q_2^m}{p_2^{m+1}}) c)
\]

where \(x, y \in [a, p_1 b + (1 - p_1)c] \times [c, p_2 d + (1 - p_2) c].

4. The \(I(p, q)\)\(bd\)-integral of \(F\) is given as:

\[
\int_a^b \int_c^d F(t, s) \ cd^{t}_p s \ cd^{t}_q t = (p_1 - q_1)(p_2 - q_2)(b - x)(d - y) \\
\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q_1^n}{p_1^{n+1}} \frac{q_2^m}{p_2^{m+1}} F(\frac{q_1^n}{p_1^{n+1}} x + (1 - \frac{q_1^n}{p_1^{n+1}}) b, \frac{q_2^m}{p_2^{m+1}} y + (1 - \frac{q_2^m}{p_2^{m+1}}) d)
\]

where \(x, y \in [p_1 a + (1 - p_1)b, b] \times [c, p_2 d + (1 - p_2) d].

**Example 2.** Define an interval-valued mapping \(F = [F, \Gamma] : [0, 1] \times [0, 1] \to I_c\) by \(F(t, s) = [t^2 s^2, ts]\). Then, by Definition 10, for \(p_1 = p_2 = \frac{1}{2}\) and \(q_1 = q_2 = \frac{1}{2}\), we have

1. From \(I(p, q)\)\(a\)-integral:

\[
\int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^{1} F(t, s) \ cd^{t}_p s \ cd^{t}_q t
\]
\[
\int_1^4 \int_0^3 F(t,s) \, s^2 \, d\tau \, s^2 \, dt
\]
\[
= \left[ \frac{1}{4} \right] \left[ \frac{1}{4} \right] \left[ \frac{3}{4} \right] \left[ \frac{3}{4} \right]
\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left( \frac{4}{3} \right)^n \left( \frac{2}{3} \right)^m \left( \frac{2}{3} \right) \cdot \left( 1 - \left( \frac{2}{3} \right)^n \right)^2 \left( \frac{2}{3} \right)^m \left( 1 - \left( \frac{2}{3} \right)^n \right)
\]
\[
= [0.0729, 0.135].
\]

2. From \( I(p,q)_b \)-integral:
\[
\int_1^4 \int_0^3 F(t,s) \, s^2 \, d\tau \, s^2 \, dt
\]
\[
= \left[ \frac{1}{4} \right] \left[ \frac{1}{4} \right] \left[ \frac{3}{4} \right] \left[ \frac{3}{4} \right]
\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left( \frac{4}{3} \right)^n \left( \frac{2}{3} \right)^m \left( \frac{2}{3} \right) \cdot \left( 1 - \left( \frac{2}{3} \right)^n \right)^2 \left( \frac{2}{3} \right)^m \left( 1 - \left( \frac{2}{3} \right)^n \right)
\]
\[
= [0.0729, 0.135].
\]

3. From \( I(p,q)_{ac} \)-integral:
\[
\int_0^2 \int_0^2 F(t,s) \, s^2 \, d\tau \, s^2 \, dt
\]
\[
= \left[ \frac{1}{4} \right] \left[ \frac{1}{4} \right] \left[ \frac{3}{4} \right] \left[ \frac{3}{4} \right]
\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left( \frac{4}{3} \right)^n \left( \frac{2}{3} \right)^m \left( \frac{2}{3} \right) \cdot \left( 1 - \left( \frac{2}{3} \right)^n \right)^2 \left( \frac{2}{3} \right)^m \left( 1 - \left( \frac{2}{3} \right)^n \right)
\]
\[
= [0.1262, 0.2025].
\]

4. From \( I(p,q)_{bd} \)-integral
\[
\int_0^1 \int_0^1 F(t,s) \, s^2 \, d\tau \, s^2 \, dt
\]
\[
= \left[ \frac{1}{4} \right] \left[ \frac{1}{4} \right] \left[ \frac{3}{4} \right] \left[ \frac{3}{4} \right]
\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left( \frac{4}{3} \right)^n \left( \frac{2}{3} \right)^m \left( 1 - \left( \frac{2}{3} \right)^n \right)^2 \left( 1 - \left( \frac{2}{3} \right)^n \right) \left( 1 - \left( \frac{2}{3} \right)^n \right)
\]
\[
= [0.0421, 0.09].
\]

4. Some New (\( p, q \))-Hermite–Hadamard Inclusions

In this section, we deal with the Hermite–Hadamard-type inclusions for co-ordinated convex interval-valued functions using the newly defined interval-valued (\( p, q \))-integrals in the last section.

**Theorem 2.** Let \( F = [F, \mathcal{T}] : [a,b] \times [c,d] \rightarrow I^+_c \) be a co-ordinated convex interval-valued function on \([a,b] \times [c,d]\). Then, the following inclusions of Hermite–Hadamard type hold for \((p,q)_{ub}^s\)-integral:
\[
F\left( \frac{q_1 a + p_1 b}{2}, \frac{p_2 c + q_2 d}{2} \right)
\geq \frac{1}{2} \frac{1}{p_1 (b-a)} \int_a^{b+(1-p_1)a} F(x, \frac{p_2 c + q_2 d}{2}) \, d\tau \, x
\]
\[
\begin{align*}
&+ \frac{1}{p_2(d-c)} \int_{p_2c+(1-p_2)d} F \left( \frac{q_1a + p_1b}{2} \right) d_{p_2a_1} y \\
&\subseteq \frac{1}{p_1p_2(b-a)(d-c)} \int_a^{\frac{p_1b+(1-p_1)a}{2}} F(x,y) d_{p_2a_1} y a_{p_1a_1} x \\
&\subseteq \frac{1}{2p_2 \left[ \frac{p_1}{p_2a_1}, \frac{d}{p_2a_1} \right]} \left[ q_1 \int_{p_2c+(1-p_2)d} F(a,y) d_{p_2a_1} y + p_1 \int_{p_2c+(1-p_2)d} F(b,y) d_{p_2a_1} y \right] \\
&\subseteq \frac{1}{2p_1 \left[ p_2a_1, (b-a) \right]} \left[ p_2 \int_{a}^{p_1b+(1-p_1)a} F(x,c) a_{p_1a_1} x + q_2 \int_{a}^{p_1b+(1-p_1)a} F(x,d) a_{p_1a_1} x \right] \\
&\subseteq q_1 p_2 F(a,c) + q_1 q_2 F(a,d) + p_1 p_2 F(b,c) + q_2 p_1 F(b,d)
\end{align*}
\]

**Proof.** Since \( F = [F,F] : [a,b] \times [c,d] \to I^+_1 \) is a co-ordinated convex interval-valued function on co-ordinates \([a,b] \times [c,d] \), \( F \) and \( F \) are co-ordinated convex and concave on co-ordinates \([a,b] \times [c,d] \), respectively. Hence, from co-ordinated convexity of \( F \) and using (2), we have

\[
F \left( \frac{q_1a + p_1b}{2} \right) \leq \frac{1}{2} \left[ 1 \int_{a}^{p_1b+(1-p_1)a} F(x,c) a_{p_1a_1} x \\
+ \frac{1}{p_2(d-c)} \int_{p_2c+(1-p_2)d} F(x,y) d_{p_2a_1} y a_{p_1a_1} x \right] \leq \frac{1}{2p_2 \left[ \frac{p_1}{p_2a_1}, \frac{d}{p_2a_1} \right]} \left[ q_1 \int_{p_2c+(1-p_2)d} F(a,y) d_{p_2a_1} y + p_1 \int_{p_2c+(1-p_2)d} F(b,y) d_{p_2a_1} y \right] \\
\leq \frac{1}{2p_1 \left[ p_2a_1, (b-a) \right]} \left[ p_2 \int_{a}^{p_1b+(1-p_1)a} F(x,c) a_{p_1a_1} x + q_2 \int_{a}^{p_1b+(1-p_1)a} F(x,d) a_{p_1a_1} x \right]
\]

From co-ordinated concavity of \( F \) and again using (2), we have

\[
F \left( \frac{q_1a + p_1b}{2} \right) \leq \frac{1}{2} \left[ 1 \int_{a}^{p_1b+(1-p_1)a} F(x,c) a_{p_1a_1} x \\
+ \frac{1}{p_2(d-c)} \int_{p_2c+(1-p_2)d} F(x,y) d_{p_2a_1} y a_{p_1a_1} x \right] \leq \frac{1}{2p_2 \left[ \frac{p_1}{p_2a_1}, \frac{d}{p_2a_1} \right]} \left[ q_1 \int_{p_2c+(1-p_2)d} F(a,y) d_{p_2a_1} y + p_1 \int_{p_2c+(1-p_2)d} F(b,y) d_{p_2a_1} y \right] \\
\leq \frac{1}{2p_1 \left[ p_2a_1, (b-a) \right]} \left[ p_2 \int_{a}^{p_1b+(1-p_1)a} F(x,c) a_{p_1a_1} x + q_2 \int_{a}^{p_1b+(1-p_1)a} F(x,d) a_{p_1a_1} x \right]
\]
\[
\frac{q_1 p_2 \mathcal{F}(a, c) + q_1 q_2 \mathcal{F}(a, d) + p_1 p_2 \mathcal{F}(b, c) + q_2 p_1 \mathcal{F}(b, d)}{[\mathcal{F}]_{p_1 q_1} [\mathcal{F}]_{p_2 q_2}}.
\]

Now, from the inequalities (12) and (13), we have following inclusions:

\[
\frac{1}{2} \left[ \frac{1}{p_1 (b - a)} \right] p_{1b + (1 - p_1)a} \int_a^d p_{1b + (1 - p_1)a} \mathcal{F} \left( x, \frac{p_{2c + q_2d}}{[\mathcal{F}]_{p_2 q_2}} \right) x p_{1q_1} \int_a^d p_{1b + (1 - p_1)a} \mathcal{F} \left( x, \frac{p_{2c + q_2d}}{[\mathcal{F}]_{p_2 q_2}} \right) x p_{1q_1} \right]
\]
\[
+ \frac{1}{p_2 (d - c)} \left[ \int_{p_{2c} + (1 - p_2)d} p_{2b + (1 - p_2)a} \mathcal{F} \left( x, \frac{p_{2c + q_2d}}{[\mathcal{F}]_{p_2 q_2}} \right) x p_{1q_1} \right]
\]

\[
\frac{1}{2} \left[ \frac{1}{p_1 (b - a)} \right] p_{1b + (1 - p_1)a} \int_a^d p_{1b + (1 - p_1)a} \mathcal{F} \left( x, \frac{p_{2c + q_2d}}{[\mathcal{F}]_{p_2 q_2}} \right) x p_{1q_1} \int_a^d p_{1b + (1 - p_1)a} \mathcal{F} \left( x, \frac{p_{2c + q_2d}}{[\mathcal{F}]_{p_2 q_2}} \right) x p_{1q_1} \right]
\]
\[
+ \frac{1}{p_2 (d - c)} \left[ \int_{p_{2c} + (1 - p_2)d} p_{2b + (1 - p_2)a} \mathcal{F} \left( x, \frac{p_{2c + q_2d}}{[\mathcal{F}]_{p_2 q_2}} \right) x p_{1q_1} \right]
\]
\[
\frac{1}{2p_1[2]p_2(d - c)} \left[ \int_a^{p_1b + (1 - p_1)a} F(x, c) \, d_{p_1,q_1}^1 x + \int_a^{p_1b + (1 - p_1)a} F(x, d) \, d_{p_1,q_1}^1 x \right]
\]

\[
+ \frac{1}{2p_2[2]p_1(d - c)} \left[ \int_a^{p_2c + (1 - p_2)d} F(a, y) \, d_{p_2,q_2}^1 y + \int_a^{p_2c + (1 - p_2)d} F(b, y) \, d_{p_2,q_2}^1 y \right]
\]

\[
+ \frac{1}{2p_1[2]p_2(b - a)} \left[ \int_a^{p_1b + (1 - p_1)a} F(x, c) \, d_{p_1,q_1}^1 x + \int_a^{p_1b + (1 - p_1)a} F(x, d) \, d_{p_1,q_1}^1 x \right]
\]

and

\[
\frac{1}{2p_2[2]p_1(d - c)} \left[ \int_a^{d} F(a, y) \, d_{p_2,q_2}^1 y + \int_a^{d} F(b, y) \, d_{p_2,q_2}^1 y \right]
\]

\[
+ \frac{1}{2p_1[2]p_2(b - a)} \left[ \int_a^{d} F(x, c) \, d_{p_1,q_1}^1 x + \int_a^{d} F(x, d) \, d_{p_1,q_1}^1 x \right]
\]

Remark 4. In Theorem 2, if \( F = F \), then inclusions (11) reduce to inequalities (2).

Remark 5. In Theorem 2, if we set \( p_1 = p_2 = 1 \), then Theorem 2 becomes ([40], Theorem 12).

Theorem 3. Let \( F = [\underline{F}, \overline{F}] : [a, b] \times [c, d] \rightarrow I^+_1 \) be a co-ordinated convex interval-valued function on \([a, b] \times [c, d]\). The following inclusions of Hermite–Hadamard type hold for \( I(p, q)^b_1 \)-integral:

\[
F \left( \frac{p_1a + q_1b}{2}, \frac{p_2d}{2} \right)
\]

\[
\geq \frac{1}{2} \left( \int_a^{b} F(x, c) \, d_{p_1,q_1}^1 x \right) + \frac{1}{2} \left( \int_a^{d} F(a, y) \, d_{p_2,q_2}^1 y \right)
\]

\[
+ \frac{1}{2} \left( \int_a^{d} F(x, d) \, d_{p_1,q_1}^1 x \right) + \frac{1}{2} \left( \int_a^{d} F(b, y) \, d_{p_2,q_2}^1 y \right)
\]

We obtain the required result (11) by combining the inclusions (14)–(17). □
\[ \frac{1}{2p_2[2]_{p_1,q_1}(d-c)} \left[ p_1 \int_c^{p_2[d+(1-p_2)c]} F(a,y) \, da_{p_2,q_2} + q_1 \int_c^{p_2[d+(1-p_2)c]} F(b,y) \, da_{p_2,q_2} \right] \\
+ \frac{1}{2p_1[2]_{p_1,q_1}(b-a)} \left[ q_2 \int_{p_1[a+(1-p_1)b]}^{b} F(x,c) \, dq_{p_1,q_1} + p_2 \int_{p_1[a+(1-p_1)b]}^{b} F(x,d) \, dq_{p_1,q_1} \right] \\
p_1q_2F(a,c) + p_1p_2F(a,d) + q_1q_2F(b,c) + q_1p_2F(b,d) \]
\[
\frac{1}{p_2(d-c)} \int_{c} F\left(\frac{q_1a + p_1b}{[2]_{p_1,q_1}}, y\right) c d_{p_2,q_2} y d_{p_1,q_1} x
\]

\[
\frac{1}{p_1} \int_{a} F\left(x, \frac{q_2c + p_2d}{[2]_{p_2,q_2}}\right) c d_{p_2,q_2} y d_{p_1,q_1} x
\]

\[
\frac{1}{p_2} \int_{c} F\left(\frac{q_1a + p_1b}{[2]_{p_1,q_1}}, \frac{q_2c + p_2d}{[2]_{p_2,q_2}}\right) c d_{p_2,q_2} y d_{p_1,q_1} x
\]

\[
\frac{1}{p_2} \int_{a} F\left(x, \frac{q_2c + p_2d}{[2]_{p_2,q_2}}\right) c d_{p_2,q_2} y d_{p_1,q_1} x
\]

\[
\frac{1}{p_2} \int_{a} F\left(x, \frac{q_2c + p_2d}{[2]_{p_2,q_2}}\right) c d_{p_2,q_2} y d_{p_1,q_1} x
\]

\[
\frac{1}{p_2} \int_{a} F\left(x, \frac{q_2c + p_2d}{[2]_{p_2,q_2}}\right) c d_{p_2,q_2} y d_{p_1,q_1} x
\]

\[
\frac{1}{p_2} \int_{a} F\left(x, \frac{q_2c + p_2d}{[2]_{p_2,q_2}}\right) c d_{p_2,q_2} y d_{p_1,q_1} x
\]

**Proof.** Following arguments similar to those in the proof of Theorem 2 and the concepts of inequalities (2)–(4), by taking into account the \(I(p,q)_{ac}\)-integral, the desired inclusion can be attained. \(\square\)

**Remark 10.** In Theorem 5, if \(F = T\), then we have the following inequality:

\[
F\left(\frac{q_1a + p_1b}{[2]_{p_1,q_1}}, \frac{q_2c + p_2d}{[2]_{p_2,q_2}}\right)
\leq
\frac{1}{2} \int_{a} F\left(x, \frac{q_2c + p_2d}{[2]_{p_2,q_2}}\right) c d_{p_2,q_2} y d_{p_1,q_1} x
\]

\[
\frac{1}{p_1} \int_{a} F\left(x, \frac{q_2c + p_2d}{[2]_{p_2,q_2}}\right) c d_{p_2,q_2} y d_{p_1,q_1} x
\]

\[
\frac{1}{p_2} \int_{c} F\left(\frac{q_1a + p_1b}{[2]_{p_1,q_1}}, \frac{q_2c + p_2d}{[2]_{p_2,q_2}}\right) c d_{p_2,q_2} y d_{p_1,q_1} x
\]

\[
\frac{1}{p_2} \int_{a} F\left(x, \frac{q_2c + p_2d}{[2]_{p_2,q_2}}\right) c d_{p_2,q_2} y d_{p_1,q_1} x
\]

\[
\frac{1}{p_2} \int_{a} F\left(x, \frac{q_2c + p_2d}{[2]_{p_2,q_2}}\right) c d_{p_2,q_2} y d_{p_1,q_1} x
\]

This can be found as a special case in [26].

**Remark 11.** In Theorem 5, if we set \(p_1 = p_2 = 1\), then Theorem 4 becomes ([40], Theorem 11).

**Corollary 1.** Let \(F = [F, T] : [a, b] \times [c, d] \rightarrow I^+_1\) be a co-ordinated convex interval-valued function on \([a, b] \times [c, d]\). The following inclusions of Hermite–Hadamard type hold for \(I(p,q)_{ac}\), \(I(p,q)_{ad}\), \(I(p,q)_{bd}\) and \(I(p,q)_{cd}\)-integrals:

\[
\frac{1}{4} \left[ F\left(\frac{q_1a + p_1b}{[2]_{p_1,q_1}}, \frac{p_2c + q_2d}{[2]_{p_2,q_2}}\right) + F\left(\frac{p_1a + q_1b}{[2]_{p_1,q_1}}, \frac{q_2c + p_2d}{[2]_{p_2,q_2}}\right) + F\left(\frac{p_1a + q_1b}{[2]_{p_1,q_1}}, \frac{p_2c + q_2d}{[2]_{p_2,q_2}}\right) + F\left(\frac{q_1a + p_1b}{[2]_{p_1,q_1}}, \frac{q_2c + p_2d}{[2]_{p_2,q_2}}\right) \right]
\]

\[
\frac{1}{8} \int_{a} F\left(x, \frac{q_2c + p_2d}{[2]_{p_2,q_2}}\right) c d_{p_2,q_2} y d_{p_1,q_1} x
\]
\[ + \int_{p_1a+(1-p_1)b}^{b} \left[ F \left( x, \frac{q_2c + p_2d}{2}, p_2d \right) + F \left( x, \frac{p_2c + q_2d}{2}, p_2d \right) \right] \, b \, d_{p_1a}^1 \]
\[ + \frac{1}{8p_2(d-c)} \int_{c} \left[ p_2d+(1-p_2)c \right] \left[ F \left( q_1a + p_1b, \frac{p_1a + q_1b}{2}, y \right) + F \left( p_1a + q_1b, \frac{p_1a + q_1b}{2}, y \right) \right] \, c \, d_{p_2d}^1 \]
\[ + \frac{1}{4p_1p_2(b-a)(d-c)} \int_{a} \left[ p_1b+(1-p_1)a \right] \int_{c} \left[ p_2d+(1-p_2)c \right] \left[ F \left( x, y \right) + d_{p_1a}^1 \right] \, b \, d_{p_1a}^1 \]
\[ \geq \frac{1}{4p_1p_2(b-a)(d-c)} \int_{a} \left[ p_1b+(1-p_1)a \right] \int_{c} \left[ p_2d+(1-p_2)c \right] \left[ F \left( x, y \right) + d_{p_1a}^1 \right] \, b \, d_{p_1a}^1 \]
\[ \geq \frac{1}{4p_1p_2(b-a)(d-c)} \int_{a} \left[ p_1b+(1-p_1)a \right] \int_{c} \left[ p_2d+(1-p_2)c \right] \left[ F \left( x, y \right) + d_{p_1a}^1 \right] \, b \, d_{p_1a}^1 \]
\[ \geq \frac{1}{4p_1p_2(b-a)(d-c)} \int_{a} \left[ p_1b+(1-p_1)a \right] \int_{c} \left[ p_2d+(1-p_2)c \right] \left[ F \left( x, y \right) + d_{p_1a}^1 \right] \, b \, d_{p_1a}^1 \]

**Remark 12.** In Corollary 1, if we set \( F = T \), then Corollary 1 becomes ([32], Corollary 1).

**Remark 13.** In Corollary 1, if we set \( p_1 = p_2 = 1 \), then Corollary 1 reduces to ([40], Corollary 3).

5. Examples

**Example 3.** We define a convex interval-valued function \( F = [F,T] \) \( \Rightarrow I^*_c \) by \( F(t,s) = [t^2s^2, ts] \). From Theorem 2, for \( p_1 = p_2 = \frac{3}{4} \) and \( q_1 = q_2 = \frac{3}{4} \), we have
\[
F \left( \frac{q_1a + p_1b}{p_1 + q_1}, \frac{p_2c + q_2d}{p_2 + q_2} \right) = \left[ \frac{36}{625}, \frac{6}{25} \right],
\]
\[
\frac{1}{2} \left[ \frac{1}{p_1(b-a)} \int_{a}^{b} \left( \frac{p_2c + q_2d}{p_2 + q_2} \right) a \, d_{p_1a}^1 \right]
\]
\[
\frac{1}{p_2(d - c)} \int_{p_2c + (1 - p_2)d}^{d} F \left( \frac{q_1a + p_1b}{p_1 + q_1}, y \right) d_{p_1,q_1}x = \left[ \frac{207}{2375 \cdot 25} \right],
\]

\[
\frac{1}{p_1p_2(b - a)(d - c)} \int_{p_1a + (1 - p_1)b}^{b} \int_{c} F(x, y) c d_{p_2,q_2}y b d_{p_1,q_1}x = \left[ \frac{234}{1805 \cdot 25} \right],
\]

\[
q_1p_2F(a, c) + q_1q_2F(a, d) + p_1p_2F(b, c) + q_2p_1F(b, d) = \left[ \frac{6 \cdot 6}{25 \cdot 25} \right].
\]

It is obvious that

\[
\left[ \frac{36 \cdot 6}{625 \cdot 25} \right] \supset \left[ \frac{207}{2375 \cdot 25} \right] \supset \left[ \frac{234}{1805 \cdot 25} \right] \supset \left[ \frac{252}{1425 \cdot 25} \right] \supset \left[ \frac{6 \cdot 6}{25 \cdot 25} \right],
\]

which shows that the results of Theorem 2 are correct.

**Example 4.** We define a convex interval-valued function \( F = \left[ \frac{E}{F} \right] \) \( \rightarrow \mathbb{I}_c^+ \) by \( F(t, s) = [t^{25}, ts] \). From Theorem 3, for \( p_1 = p_2 = \frac{2}{3} \) and \( q_1 = q_2 = \frac{3}{4} \), we have

\[
F \left( \frac{p_1a + q_1b}{[2]_{p_1,q_1}}, \frac{q_2c + p_2d}{[2]_{p_2,q_2}} \right) = \left[ \frac{36}{625 \cdot 25} \right].
\]

\[
\frac{1}{2} \left[ \frac{1}{p_1(b - a)} \int_{p_1a + (1 - p_1)b}^{b} F \left( \frac{q_2c + p_2d}{[2]_{p_2,q_2}}, y \right) b d_{p_1,q_1}x + \frac{1}{p_2(d - c)} \int_{c} F \left( \frac{p_1a + q_1b}{[2]_{p_1,q_1}}, y \right) c d_{p_2,q_2}y \right] = \left[ \frac{207}{2375 \cdot 25} \right],
\]

\[
\frac{1}{p_1p_2(b - a)(d - c)} \int_{p_1a + (1 - p_1)b}^{b} \int_{c} F(x, y) c d_{p_2,q_2}y b d_{p_1,q_1}x = \left[ \frac{234}{1805 \cdot 25} \right],
\]

\[
\frac{1}{2p_2[2]_{p_1,q_1}(d - c)} \left[ \frac{p_1}{c} \int_{c} F(a, y) c d_{p_2,q_2}y + q_1 \int_{c} F(b, y) c d_{p_2,q_2}y \right] + \frac{1}{2p_2[2]_{p_1,q_1}(b - a)} \left[ q_2 \int_{p_1a + (1 - p_1)b}^{b} F(x, c) b d_{p_1,q_1}x + p_2 \int_{p_1a + (1 - p_1)b}^{b} F(x, d) b d_{p_1,q_1}x \right] = \left[ \frac{252}{1425 \cdot 25} \right].
\]
and
\[
p_{1}q_{2}F(a,c) + p_{1}p_{2}F(a,d) + q_{1}q_{2}F(b,c) + q_{1}p_{2}F(b,d) = \begin{bmatrix} 6 & 6 \\ 25 & 25 \end{bmatrix}.
\]

It is obvious that
\[
\begin{bmatrix} 36 & 6 \\ 625 & 25 \end{bmatrix} \supset \begin{bmatrix} 207 & 6 \\ 2375 & 25 \end{bmatrix} \supset \begin{bmatrix} 234 & 6 \\ 1805 & 25 \end{bmatrix} \supset \begin{bmatrix} 252 & 6 \\ 1425 & 25 \end{bmatrix} \supset \begin{bmatrix} 6 & 6 \\ 25 & 25 \end{bmatrix},
\]

which shows that the results of Theorem 3 are correct.

Example 5. We define a convex interval-valued function \( F = [F_{c}F] : \rightarrow I_{c}^{+} \) by \( F(t,s) = [t^{2}s^{2}, ts] \). From Theorem 4, for \( p_{1} = p_{2} = \frac{2}{3} \) and \( q_{1} = q_{2} = \frac{3}{4} \), we have
\[
\begin{align*}
F & \left( \frac{p_{1}a + q_{1}b}{2}, \frac{p_{2}c + q_{2}d}{2} \right) = \begin{bmatrix} 16 & 4 \\ 625 & 25 \end{bmatrix}, \\
& \frac{1}{p_{1}p_{2}(b-a)(d-c)} \int_{p_{1}a+(1-p_{1})b}^{d} \int_{p_{2}c+(1-p_{2})d}^{d} F(x,y) d_{p_{2}d_{2}} y b_{p_{1}d_{1}} x = \begin{bmatrix} 676 & 4 \\ 9025 & 25 \end{bmatrix}, \\
& \frac{1}{2p_{2}[2]_{p_{1}d_{1}}(d-c)} \left[ \frac{1}{p_{1}} \int_{p_{2}c+(1-p_{2})d}^{d} F(a,y) d_{p_{2}d_{2}} y + q_{1} \int_{p_{2}c+(1-p_{2})d}^{d} F(b,y) d_{p_{2}d_{2}} y \right] \\
& + \frac{1}{2p_{1}[2]_{p_{2}d_{2}}(b-a)} \left[ \frac{1}{p_{2}} \int_{p_{1}a+(1-p_{1})b}^{b} F(x,c) b_{p_{1}d_{1}} x + q_{2} \int_{p_{1}a+(1-p_{1})b}^{b} F(x,d) b_{p_{1}d_{1}} x \right] = \begin{bmatrix} 52 & 4 \\ 475 & 25 \end{bmatrix},
\end{align*}
\]

and
\[
\begin{align*}
p_{1}p_{2}F(a,c) + p_{1}q_{2}F(a,d) + q_{1}p_{2}F(b,c) + q_{1}q_{2}F(b,d) = \begin{bmatrix} 4 & 4 \\ 25 & 25 \end{bmatrix},
\end{align*}
\]

It is obvious that
\[
\begin{bmatrix} 16 & 4 \\ 625 & 25 \end{bmatrix} \supset \begin{bmatrix} 104 & 4 \\ 2375 & 25 \end{bmatrix} \supset \begin{bmatrix} 676 & 4 \\ 9025 & 25 \end{bmatrix} \supset \begin{bmatrix} 52 & 4 \\ 475 & 25 \end{bmatrix} \supset \begin{bmatrix} 4 & 4 \\ 25 & 25 \end{bmatrix},
\]

which shows that the results of Theorem 4 are correct.

Example 6. We define a convex interval-valued function \( F = [F_{c}F] : \rightarrow I_{c}^{+} \) by \( F(t,s) = [t^{2}s^{2}, ts] \). From Theorem 5, for \( p_{1} = p_{2} = \frac{2}{3} \) and \( q_{1} = q_{2} = \frac{3}{4} \), we have
\[
\begin{align*}
F & \left( \frac{q_{1}a + p_{1}b}{2}, \frac{q_{2}c + p_{2}d}{2} \right) = \begin{bmatrix} 81 & 9 \\ 625 & 25 \end{bmatrix}, \\
& \frac{1}{2p_{1}(b-a)} \int_{a}^{p_{1}b+(1-p_{1})a} F \left( x, \frac{q_{2}c + p_{2}d}{2} \right) d_{p_{1}d_{1}} x
\end{align*}
\]
\[
= \left[ \frac{81}{475'}, \frac{9}{25} \right],
\]
\[
1 \int_c^{p_2d+(1-p_2)c} F(x, y) \, c d_{p_2, d_2}l \, x + p_2 \int_c^{p_2d+(1-p_2)c} F(x, d) \, a d_{p_1, q_1}l \, x
\]
\[
= \left[ \frac{27}{95'}, \frac{9}{25} \right],
\]
It is obvious that
\[
\frac{81}{625}, \frac{9}{25} \supset \frac{81}{475'}, \frac{9}{25} \supset \frac{81}{361'}, \frac{9}{25} \supset \frac{27}{95'}, \frac{9}{25} \supset \frac{9}{25}, \frac{9}{25},
\]
which shows that the results of Theorem 5 are right.

6. Conclusions
In this work, for interval-valued functions of two variables, we defined \((p, q)\)-integrals. We have used newly described integrals to prove the Hermite–Hadamard-type inclusions for co-ordinated convex interval-valued functions. Other researchers’ previously reported findings are deduced as special cases of our results for \(p = 1, q \to 1^-\) and \(F = T\). Finally, some examples are given to demonstrate the findings of this article. Results for the case of symmetric interval-valued functions can be obtained by applying the concept in Remark 2, which will be studied in future work. We will look at some further refinements of the Hermite–Hadamard inclusions as well as other well-known mathematical inclusions using \((p, q)\)-integrals in the future.

Author Contributions: Conceptualization, J.T., M.A.A., H.B., S.K.N.; Formal analysis, J.T., M.A.A., H.B., S.K.N.; Funding acquisition, J.T.; Methodology, J.T., M.A.A., H.B., S.K.N. All authors have read and agreed to the published version of the manuscript.

Funding: This research was supported by the Thailand Research Fund under the project RSA6180059.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References
1. Jackson, D.O.; Fukuda, T.; Dunn, O.; Majors, E. On \(q\)-definite integrals. Quart. J. Pure Appl. Math. 1910, 41, 193–203 [CrossRef]
2. Ahmad, B. Boundary-value problems for nonlinear third-order \(q\)-difference equations. Electron. J. Differ. Equ. 2011, 2011, 1–7. [CrossRef]
3. Ahmad, B.; Alsaeedi, A.; Ntouyas, S.K. A study of second-order \(q\)-difference equations with boundary conditions. Adv. Differ. Equ. 2012, 2012, 35. [CrossRef]
4. Ahmad, B.; Ntouyas, S.K.; Pumaras, I.K. Existence results for nonlinear \(q\)-difference equations with nonlocal boundary conditions. Commun. Appl. Nonlinear Anal. 2012, 19, 59–72.
5. Ahmad, B.; Nieto, J.J.; Alsaedi, A.; Shah, K.; Al-Rabgahai, A.T. Existence results for nonlocal boundary value problems of fractional q-difference equations. *Adv. Differ. Equ.* 2012, 2012, 140. [CrossRef]

6. Anh, N.T.; Bui, P.Q.; Dang, H. *q*-Fractional Calculus and Equations; Springer: Berlin, Germany, 2012; Volume 2056.

7. Aral, A.; Gupta, V.; Agarwal, R.P. *Applications of q-Calculus in Operator Theory*; Springer: New York, NY, USA, 2013.

8. Băleanu, D.; Iriţan, H. *Fractional Calculus for Non-integer Orders and its Applications*; Springer: Berlin/Heidelberg, Germany; New York, NY, 2012.

9. Băleanu, D.; Telesh, V.K. New fractional models for systems of the type of ordinary multivariable differential equations. *Adv. Differ. Equ.* 2013, 2013, 443. [CrossRef]

10. Bohner, M.; Guseinov, G.S. The $q$-Laplace and $q$-Laplace transforms. *J. Math. Anal. Appl.* 2010, 365, 75–92. [CrossRef]

11. Bukkely-Kyemba, J.D.; Hounkonnou, M.N. Quantum deformed algebras: Coherent states and special functions. arXiv 2013, arXiv:1301.0116.

12. Dobrogowska, A.; Odzijewicz, A. Second order $q$-difference equations solvable by factorization method. *J. Comput. Anal. Math.* 2006, 9, 319–346. [CrossRef]

13. Ernst, T. *The History of $q$-Calculus and a New Method*; Citeeseer: University Park, PA, US, 2000.

14. Ernst, T. *A Comprehensive Introduction to $q$-Calculus*; Springer Science & Business Media: Berlin, Germany, 2012.

15. Exton, H. *q-Hypergeometric Functions and Applications*; Horwood: Bristol, UK, 1983.

16. Ferreira, R. Nontrivial solutions for fractional $q$-difference boundary value problems. *Electron. J. Qual. Theory Differ. Equ.* 2010, 2010, 70.

17. Gasper, G.; Rahman, M. Some systems of multivariable orthogonal $q$-Racah polynomials. *Ramanujan J.* 2007, 13, 389–405. [CrossRef]

18. Gauchman, H. Integral inequalities in $q$-calculus. *Comput. Math. Appl.* 2004, 47, 281–300. [CrossRef]

19. Jackson, F.H. $q$-difference equations. *Am. J. Math.* 1910, 32, 305–314. [CrossRef]

20. Kac, V.; Cheung, P. *Quantum Calculus*; Springer Science & Business Media: Berlin, Germany, 2001.

21. Tariboon, J.; Ntouyas, S.K. Quantum calculus on finite intervals and applications to impulsive difference equations. *Adv. Differ. Equ.* 2013, 2013, 282. [CrossRef]

22. Tariboon, J.; Ntouyas, S.K. Quantum integral inequalities on finite intervals. *J. Ineq. Appl.* 2014, 2014, 121. [CrossRef]

23. Tunç, M.; Gök, E. Some integral inequalities via $(p, q)$-calculus on finite intervals. *RGmia Res. Rep. Coll.* 2016, 19, 95.

24. Vivas-CortezM.; Ali, M.A; Budak, H.; Kalsoom, H.; Agarwal P. Post-quantum Hermite–Hadamard inequalities involving newly defined $(p, q)$-integral. *Entropy* 2021, 23, 828. [CrossRef]

25. Chu, Y-M.; Awan, M.U.; Talib, S.; Noor, M.A.; Noor, K.I. New post quantum analogues of Ostrowski-type inequalities using new definitions of left-right $(p, q)$ -derivatives and definite integrals. *Adv. Differ. Equ.* 2020, 2020, 634. [CrossRef]

26. Kalsoom, H.; Rashid, S.; Idrees, M.; Saifdar, F.; Akram, S.; Baleanu, D.; Chu, Y-M. Post quantum integral inequalities of Hermite–Hadamard type associated with co-ordinated higher-order generalized strongly pre-inex and quasi-pre-inex mappings. *Symmetry* 2020, 12, 443. [CrossRef]

27. Kunt, M.; İşcan, İ.; Alp, N.; Sarıkaya, M. $(p, q)$-Hermite–Hadamard inequalities and $(p, q)$-estimates for midpoint type inequalities via convex and quasi-convex functions. *RACSAM* 2018, 112, 969–992. [CrossRef]

28. Dragomir, S.S. On the Hadamard’s inequality for convex functions on the co-ordinates in a rectangle from the plane. *Taiwanese J. Math.* 2001, 5, 775–788. [CrossRef]

29. Kunt, M.; Latif, M.A.; İşcan, İ; Dragomir, S.S. Quantum Hermite-Hadamard type inequality and some estimates of quantum midpoint type inequalities for double integrals. *Sigma J. Eng. Nat. Sci.* 2019, 37, 207–223.

30. Bermudo, S.; Körüs, P.; Valdés, J.E.N. On $q$-Hermite–Hadamard inequalities for general convex functions. *Acta Math. Hungar.* 2020, 162, 364–374. [CrossRef]

31. Budak, H.; Ali, M.A.; Tarhanaci, M. Some new quantum Hermite–Hadamard-like inequalities for coordinated convex functions. *J. Optim. Theory Appl.* 2020, 186, 899–910. [CrossRef]

32. Wannalookkhee, F.; Nonlaopoont, K.; Tariboon, J.; Ntouyas, S.K. On Hermite-Hadamard inequalities for coordinated convex functions via $(p, q)$-calculus. *Mathematics* 2021, 9, 698. [CrossRef]

33. Aubin, J.-P.; Cellina, A. *Differential Inclusions: Set-Valued Maps and Viability Theory*; Springer Science & Business Media: Berlin/Heidelberg, Germany; New York, NY, Tokyo, Japan, 2012.

34. Markov, S. On the algebraic properties of convex bodies and some applications. *J. Convex Anal.* 2000, 7, 129–166.

35. Lupaş, V. Fractional calculus for interval-valued functions. *Fuzzy Sets Syst.* 2015, 265, 63–85. [CrossRef]

36. Moore, R.E. *Interval Analysis*; Prentice-Hall: Englewood Cliffs, NJ, USA, 1966.

37. Zhao, D.; Ali, M.A.; Murtaza, G.; Zhang, Z. On the Hermite Hadamard inequalities for interval-valued co-ordinated convex functions. *Adv. Differ. Equ.* 2020, 2020, 670. [CrossRef]

38. Kara, H.; Ali, M.A.; Budak, H. Hermite-Hadamard-type inequalities for interval-valued coordinated convex functions involving generalized fractional integrals. *Math. Methods Appl. Sci.* 2021, 44, 104–123. [CrossRef]

39. Lou, T.; Ye, G.; Zhao, D.; Liu, W. $q$-calculus and $q$-ermite–Hadamard inequalities for interval-valued functions. *Adv. Differ. Equ.* 2020, 2020, 446. [CrossRef]

40. Ali, M.A.; Budak, H.; Kara, H.; Qaisar, S. $q$-Hermite-Hadamard inclusions for the interval-valued functions of two variables. Preprint.

41. Ali, M.A.; Budak, H.; Murtaza, G.; Chu, Y-M. Post-quantum $H$ ermite–Hadamard inequalities for interval-valued convex functions. *J. Ineq. Appl.* 2021, 2021, 84. [CrossRef]

42. Zhao, D.F.; An, T.Q.; Ye, G.J.; Liu, W. Chebyshev type inequalities for interval-valued functions. *Fuzzy Sets Syst.* 2020, 396, 82–101. [CrossRef]