Space-time correlations of a Gaussian interface

François Dunlop
Laboratoire de Physique Théorique et Modélisation (CNRS – UMR 8089)
Université de Cergy-Pontoise, 95302 Cergy-Pontoise, France

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Abstract

The serial harness introduced by Hammersley [7] is equivalent, in the Gaussian case, to the Gaussian Solid-On-Solid interface model with parallel heat bath dynamics. Here we consider sub-lattice parallel dynamics, and give exact results about relaxation dynamics, based on the equivalence to the infinite time limit of a time periodic random field. We also give a numerical comparison to the harness process in continuous time studied by Hsiao [8] and by Ferrari, Niederhauser and Pechersky [4, 5].

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1 Introduction

Let $L$ be a positive even integer and let the initial condition $h^0 = \{h^0_i : i \in \mathbb{Z}/L\mathbb{Z}\}$ be distributed according to the un-normalized measure

$$
\mu(dh^0) = \prod_i e^{-\frac{1}{2} \sum (h^0_{i+1} - h^0_i)^2} dh^0_i
$$

(1)

where $dh^0_i$ is the Lebesgue measure over $\mathbb{R}$. The index $i$ runs over $\mathbb{Z}/L\mathbb{Z}$, which corresponds to periodic boundary conditions. The measure (1) may be considered as a finite volume Gibbs measure with Hamiltonian

$$
H(h^0) = \frac{1}{2} \sum (h^0_{i+1} - h^0_i)^2
$$

The corresponding sub-lattice parallel heat bath dynamics is defined by

$$
\mathbb{P}(dh^t | h^{t-1}) = \prod_{i+t \text{ even}} e^{-\left(h^t_i - \frac{1}{2}(h^t_{i+1} - h^t_{i-1})\right)^2} \prod_{i+t \text{ odd}} \delta(h^t_i - h^{t-1}_i) \prod_i dh_i^t / \text{norm.}
$$

(2)
where the normalization of the probability is a finite constant, independent of \( h^{t-1} \). The stochastic process defined by (2) is intermediate between Hammersley’s original serial harness [7] and the harness process in continuous time [8, 4, 5]. Various sub-lattice parallel stochastic dynamics for interface models have been studied, e.g. in [1, 3], showing a closer similarity with continuous time dynamics than with fully parallel dynamics.

The heat bath dynamics leaves invariant the Gibbs measure which motivated it:

\[
\int \mu(dh^{t-1}) \mathbb{P}(dh^t | h^{t-1}) = \mu(dh^t)
\]  

(3)

As the initial condition \( h^0 \) is already distributed with the measure \( \mu \), we have a stationary problem. Our main result is a computation of space-time correlations, in the thermodynamic limit \( L \to \infty \). The correlation function of two space gradients at time and space separation \((2t, j)\) will be denoted \( g_{11}(t, j) \), the correlation function of two time gradients at time and space separation \((2t, j)\) will be denoted \( g_{22}(t, j) \). The time separation \( 2t \) corresponds to \( t \) updates at each site between the two events:

\[
g_{11}(t, j) = \lim_{L \to \infty} \mathbb{E}(h_{j+2}^{2t} - h_j^{2t})(h_0^0 - h_j^0), \quad t \geq 0
\]  

(4)

\[
g_{22}(t, j) = \lim_{L \to \infty} \mathbb{E}(h_j^{2t+2} - h_j^{2t})(h_0^0 - h_j^0), \quad t \geq 0
\]  

(5)

and similarly

\[
g_{12}(t, j) = \lim_{L \to \infty} \mathbb{E}(h_{j+1}^{2t} - h_{j-1}^{2t})(h_0^0 - h_j^0), \quad t \geq 1
\]  

(6)

\[
g_{21}(t, j) = \lim_{L \to \infty} \mathbb{E}(h_0^{2t} - h_{j+1}^{2t-2})(h_{j+1}^0 - h_{j-1}^0), \quad t \geq 1
\]  

(7)

**Proposition 1.** Let \( h^{[0,t]} \) be distributed according to (1)(2), for each \( t \in \mathbb{Z}_+ \). Then for each \( t \in \mathbb{Z}_+ \) and \( j \in \mathbb{Z} \) the limits (4)(5)(6)(7) exist and satisfy

\[
g_{11}(t, j) = 2^{2t+1}(2t)! \left[ \frac{2^{t+1/2}}{(t - \frac{j}{2})!(t + \frac{j}{2})!} \right] \quad \text{if} \quad 0 \leq j \leq 2t,
\]  

(8)

\[
g_{11}(t, -j) = -g_{11}(t, j) \quad \text{if} \quad j \geq 1
\]  

(9)

\[
g_{11}(t, j) = 0 \quad \text{if} \quad |j| > 2t \geq 2.
\]  

(10)

\[
g_{22}(t, j) = -\frac{1}{4} \left[ g_{11}(t - 1, |j|) - g_{11}(t, |j|) \right] \quad \text{if} \quad t \geq 1
\]  

(11)

\[
g_{12}(t, j) = -g_{12}(t, -j) = -g_{21}(t, j) = g_{21}(t, -j) = g_{11}(t - 1, |j + 1|) - g_{11}(t - 1, |j - 1|)
\]  

(12)

Moreover, as \( t \to \infty \), uniformly in \( j \in \mathbb{Z} \),

\[
g_{11}(t, j) = \frac{2}{\sqrt{\pi t}} e^{-t^2} + \mathcal{O}(t^{-2})
\]  

(14)
\[ g_{22}(t, j) = -\frac{1}{4t^{3/2}} e^{-\frac{j^2}{2t}} + O(t^{-2}) \] (15)

\[ g_{12}(t, j) = -\frac{2j}{t^{3/2}} e^{-\frac{j^2}{2t}} + O(t^{-2}) \] (16)

\[ \lim_{L \to \infty} E(h^t_j - h^0_j)(h^0_t - h^0_0) = \sqrt{\frac{2t}{\pi}} \left[ e^{-\frac{j^2}{4t}} + j \int_{\frac{1}{\sqrt{t}}}^{\infty} du \ e^{-u^2/2} \right] + O(\ln t) \] (17)

Remark 1. (9)(10)(12) follow respectively from space symmetry, causality, and the detailed balance condition. More identities can be extracted from (8) and also from the loop condition (the sum of gradients around a closed loop is identically zero). In particular \( g_{22}(t, 0) = 4g_{12}(t, 1) \) \( \forall t \).

Remark 2. Proposition 1 conveys information for \( |j| \ll \sqrt{t \ln t} \).

Proposition 1 is proven in Section 5. It is based on the equivalence in law of the space-time field \( h^{[0, t]} \) with the infinite time limit of a space and time periodic random field, which is naturally diagonalized by Fourier transform. This random field is defined in Section 2. The Fourier transform diagonalization is performed in Section 3. The proof of equivalence is completed in Section 4. Generalization to arbitrary dimension is outlined in Section 6. A numerical comparison to the harness process in continuous time is given in Section 7.

2 Space-time periodic field

For \( T \) a positive even integer, the marginal space time field \( h^{[0, t]} \) is easily checked to be distributed according to the un-normalized measure

\[ \mu_{free}^{TL}(dh) = \left( \prod_{t+i \text{ even}} e^{-\left( h^t_i - \frac{1}{2}(h^{t-1}_i + h^{t+1}_i) \right)^2 dh^t_i} \right) \left( \prod_{i \text{ even}} e^{-\frac{1}{4}(h^0_i - h^0_{i+2})^2 dh^0_i} \right) \] (19)

where “free” refers to the time \( T - 1 \) final condition and the range of \( t \) in the product is \( 1 \leq t \leq T - 1 \). The corresponding “space-time Hamiltonian” is

\[ \mathcal{H}_{free}^{TL} = \sum_{t+i \text{ even}} \left( h^t_i - \frac{1}{2} h^{t-1}_{i+1} - \frac{1}{2} h^{t+1}_{i+1} \right)^2 + \frac{1}{4} \sum_{i \text{ even}} \left( h^0_i - h^0_{i+2} \right)^2 \] (20)

A good feature of \( \mu_{free}^{TL} \) is that its marginal at time \( t \) is known exactly. However, in order to compute time correlations by Fourier transform, we are going to use periodic boundary conditions in the time variable also: let

\[ h = \left\{ h^t_i : (t, i) \in (\mathbb{Z}/T\mathbb{Z} \times \mathbb{Z}/L\mathbb{Z}) \cap \{ t + i \text{ even} \} \right\} \] (21)
be distributed according to the un-normalized measure

\[ \mu_{\text{per}}^{TL}(dh) = \prod_{t+i \text{ even}} e^{-\left(h_i^t - \frac{1}{2}h_{i-1}^{t-1} + h_{i+1}^{t-1}\right)^2} dh_i^t \]  

(22)

The corresponding “space-time Hamiltonian” is

\[ H_{\text{per}}^{TL} = \sum_{t+i \text{ even}} \left(h_i^t - \frac{1}{2}h_{i-1}^{t-1} - \frac{1}{2}h_{i+1}^{t-1}\right)^2 \]

\[ = \sum_{t+i \text{ even}} \left(h_i^t - \frac{1}{2}h_{i-1}^{t-1} - \frac{1}{2}h_{i+1}^{t-1}\right)^2 + \sum_{i \text{ even}} \left(h_0^i - \frac{1}{2}h_{i-1}^{T-1} - \frac{1}{2}h_{i+1}^{T-1}\right)^2 \]  

(23)

The last term in (23) or (20) is necessary in order to have only one mode distributed according to the Lebesgue measure, for uniform global translations of the system. Fig. 1 shows in solid line the interaction terms common to (23) and (20), and, in dashed line, the interaction terms corresponding to the last term in (23) or (20).

\[ \mu_{\text{per}}^{TL} \]

\[ \mu_{\text{free}}^{TL} \]

Proposition 2. Let \( L, T_1, T \) be positive even integers. Let \( h_{\text{per}}^{TL} \) and \( h_{\text{free}}^{T_1L} \) be random fields distributed according to \( \mu_{\text{per}}^{TL} \) and \( \mu_{\text{free}}^{T_1L} \) respectively. Then, as \( T \to \infty \), the marginal \( h_{\text{per}}^{TL}|_{0 \leq t \leq T_1-1} \) converges in distribution to \( h_{\text{free}}^{T_1L} \). In particular, the one-time marginal \( h_0^{\text{per}} \) converges in distribution to \( h_0^{\text{free}} \), distributed
according to $\mu(\text{d}h^0)$. And, extending the random fields to the full lattice with $(h_{\text{per}}^t)_j = (h_{\text{per}}^{t+1})_j$ for $t + j$ odd, and similarly for $h_{\text{free}}^0$, then $h_{\text{per}}^0$ converges in distribution to $h_{\text{free}}^0$, distributed according to $\mu(\text{d}h^0)$.

**Proof.** Each random field has one real component distributed according to the Lebesgue measure, the same for all random fields. We need only consider the gradient fields. The statements about the one-time marginals follow from the convergence of the one-time covariance matrix, e.g. $\mathbb{E}(h_i^0 - h_{i-1}^0)(h_j^0 - h_{j-1}^0)$ for $i, j = 1, \ldots, L - 1$, which are linear combinations of $\mathbb{E}(h_j^0 - h_{j-1}^0)^2$ for $j = 1, \ldots, L - 1$. This is a computation, given in Section 4. We thus have, for the gradient fields,

$$
\mathbb{P}_{\text{per}}^T(L \text{d}h^0) \rightarrow \mathbb{P}_{\text{free}}^L(\text{d}h^0) \quad \text{as } T \rightarrow \infty \quad (24)
$$

On the other hand,

$$
\mathbb{P}_{\text{per}}^T(\text{d}h^{[0, T-1]} | h^0) = \mathbb{P}_{\text{free}}^L(\text{d}h^{[0, T-1]} | h^0, h^T = h^0)
\rightarrow \mathbb{P}_{\text{free}}^L(\text{d}h^{[0, T-1]} | h^0) \quad \text{as } T \rightarrow \infty \quad (25)
$$

Then, for the gradient fields,

$$
\mathbb{P}_{\text{per}}^T(\text{d}h^{[0, T-1]} ) = \mathbb{P}_{\text{per}}^T(\text{d}h^{[0, T-1]} | h^0) \mathbb{P}_{\text{per}}^T(h^0)
\rightarrow \mathbb{P}_{\text{free}}^L(\text{d}h^{[0, T-1]} | h^0) \mathbb{P}_{\text{free}}^L(h^0)
= \mathbb{P}_{\text{free}}^L(\text{d}h^{[0, T-1]} ) \quad (26)
$$

\[ \qed \]

### 3 Fourier transform

In order to compute the Fourier transform of the space-time periodic Gaussian field, it is convenient to set

$$
h_{2i}^{2t+1} = 0 \quad \text{and} \quad h_{2i}^{2t+2} = 0 \quad \forall \, t, i \quad (27)
$$

Then, for $(\nu, k) \in (\mathbb{Z}/T\mathbb{Z} \times \mathbb{Z}/L\mathbb{Z})$,

$$
\hat{h}_k^{\nu} = \frac{1}{\sqrt{LT}} \sum_{j=0}^{L-1} \sum_{i=0}^{T-1} \exp \left( -2i\pi \frac{kj}{L} + 2i\pi \frac{\nu}{T} h_j^i \right), \quad \hat{h}_{L-k}^{\nu} = \overline{\hat{h}_k^{\nu}}, \quad \hat{h}_{k+L/2}^{\nu + T/2} = \hat{h}_k^{\nu}
$$

and

$$
h_j^i = \frac{1}{\sqrt{LT}} \sum_{k=0}^{L-1} \sum_{\nu=0}^{T-1} \exp \left( -2i\pi \frac{kj}{L} - 2i\pi \frac{\nu}{T} h_k^{\nu} \right) \quad (28)
$$

\[ ]
and

\[ H_{\text{per}}^{TL} = \sum_{j=0}^{L-1} \sum_{t=0}^{T-1} \left( h_j^t - \frac{1}{2} h_{j-1}^t - \frac{1}{2} h_{j+1}^t \right)^2 \]

\[ = \frac{1}{LT} \sum_{j,t,\nu,k,\nu',k'} \hat{h}_k^\nu \hat{h}_{k'}^\nu' \exp(-2i\pi(k+k')\frac{j}{L} - 2i\pi(\nu + \nu')\frac{t}{T}). \]

\[ \cdot \left( \frac{1}{2} e^{2i\pi k \frac{j}{L} + 2i\pi \nu \frac{t}{T} - \frac{1}{2} e^{-2i\pi k \frac{j}{L} + 2i\pi \nu' \frac{t}{T}} \right) \]

\[ \cdot \left( \frac{1}{2} e^{2i\pi k' \frac{j}{L} + 2i\pi \nu' \frac{t}{T} - \frac{1}{2} e^{-2i\pi k' \frac{j}{L} + 2i\pi \nu \frac{t}{T}} \right) \]

\[ = \sum_{\nu,k} |\hat{h}_k^\nu|^2 \left( 1 - 2\cos 2\pi \frac{\nu}{T} \cos 2\pi \frac{k}{L} + \cos^2 2\pi \frac{k}{L} \right) \]

\[ = \sum_{\nu,k} |\hat{h}_k^\nu|^2 \gamma_k^\nu \]

(30)

The Fourier transform may be cast into an orthogonal transformation of the LT/2 random variables \( h_j^t \). Setting \( \hat{h}_k^\nu = a_k^\nu + ib_k^\nu \), the new LT/2 random variables \( a_k^\nu \)'s and \( b_k^\nu \)'s are chosen as follows: the orbit of any \((\nu, k)\) under possible combinations of \( (\nu, k) \to (T - \nu, L - k) \) and \( (\nu, k) \to (\nu + T/2, k + L/2) \)

(31)

has 4 elements, except for the 2-element orbits

\[ \{(0,0), (T/2, L/2)\}, \quad \{(0,L/2), (T/2,0)\} \]

(32)

\[ \{(T/4, L/4), (3T/4, 3L/4)\}, \quad \{(T/4,3L/4), (3T/4, L/4)\} \]

(33)

corresponding to real \( \hat{h}_k^\nu \)'s. Choosing one element per orbit, we get

\[ h_j^t = \frac{2}{\sqrt{LT}} \left[ a_0^0 + (-)^j a_{L/2}^0 + (-)^{j/2} a_{T/4}^0 + (-)^{(j+1)/2} a_{3T/4}^0 \right] \]

\[ + \frac{4}{\sqrt{LT}} \sum_{k=1}^{L/2-1} \cos 2\pi \frac{kj}{L} a_0^k + \sin 2\pi \frac{kj}{L} b_0^k \]

\[ + \frac{4}{\sqrt{LT}} \sum_{k=L/4+1}^{3L/4-1} \left[ \cos(2\pi \frac{kj}{L} + \pi t/2) a_k^{T/4} + \sin(2\pi \frac{kj}{L} + \pi t/2) b_k^{T/4} \right] \]

(34)

\[ + \frac{4}{\sqrt{LT}} \sum_{\nu=1}^{T/4-1} \sum_{k=0}^{L-1} \left[ \cos(2\pi \frac{kj}{L} + 2\pi \frac{\nu t}{T}) a_0^\nu + \sin(2\pi \frac{kj}{L} + 2\pi \frac{\nu t}{T}) b_0^\nu \right] \]
The Jacobian of the transformation from $h_j$’s to $a_k$’s and $b_k$’s is actually $2^{T/L}/4$. The new measure is

$$\exp(-\mathcal{H}_{\text{per}}^{TL}) \prod_{\nu,k} da_k \prod_{\nu,k} db_k / \text{norm.}$$

(35)

where the index sets for the $a_k$’s and $b_k$’s are as in the formula (34) for $h_j$, with a total of $LT/2$, and

$$\mathcal{H}_{\text{per}}^{TL} = 2 \sum_{2\text{-orbits}} \gamma_k(a_k')^2 + 4 \sum_{4\text{-orbits}} \gamma_k [(a_k')^2 + (b_k')^2]$$

(36)

with $\gamma_k$ defined in (30), so that $\mathbb{E}|\hat{h}_k'|^2 = 1/(4\gamma_k^2)$ for all $(\nu,k) \neq (0,0)$ or $(T/2, L/2)$. We have one zero mode $a_0^T = a_{L/2}^T$ distributed with Lebesgue measure, and soft modes, Gaussians of large variance, around the zero mode:

**Lemma 3.** $\hat{h}_0^T = \hat{h}_{L/2}^T$ is distributed according to the Lebesgue measure. All other $\hat{h}_k'$ are independent centred real or complex Gaussian variables with

$$\mathbb{E}|\hat{h}_k'|^2 = \frac{1}{4} \frac{1}{1 - 2\cos2\pi \frac{k}{L} + \cos^2 \frac{2\pi k}{L}}$$

(37)

$\mathbb{E}\hat{h}_k'\hat{h}_k'$ is non zero only if $(\nu,k)$ and $(\nu',k')$ belong to the same orbit, with $\hat{h}_k'$ and $\hat{h}_{k'}'$ complex conjugate: $(\nu',k') = (T - \nu, L - k)$ or $(\nu',k') = (T/2 - \nu, L/2 - k)$.

### 4 Equal time covariance

The equilibrium measure (1) can also be diagonalized by Fourier transform, which yields

$$\mathbb{E}(h_j^0 - h_0^0)^2 = \frac{1}{L} \sum_{k \neq 0} \frac{1 - \cos \frac{2\pi kj}{L}}{1 - \cos \frac{2\pi k}{L}}$$

(38)

where $k \in \mathbb{Z}/L\mathbb{Z}$. A change of summation index $k \rightarrow \frac{L}{2} - k$ leads to equivalent formulas, differing slightly according to the parity of $j$. Averaging the two formulas yields

$$\mathbb{E}(h_j^0 - h_0^0)^2 = \frac{1}{L} \sum_{k \neq 0, L/2} \frac{1 - \cos \frac{2\pi kj}{L}}{1 - \cos^2 \frac{2\pi k}{L}} \quad \text{if } j \text{ even}$$

$$= \frac{1}{L} + \frac{1}{L} \sum_{k \neq 0, L/2} \frac{1 - \cos \frac{2\pi kj}{L} \cos \frac{2\pi k}{L}}{1 - \cos^2 \frac{2\pi k}{L}} \quad \text{if } j \text{ odd}$$

(39)

Here of course $L = \infty$ would be simpler, with $\mathbb{E}(h_j^0 - h_0^0)^2 = j$, but our aim is to complete the proof of Proposition 2, where $L$ is finite. Let us now compute
the analogue for the space and time periodic field:

\[ E_{\text{per}}(h_0^j - h_0^j)^2 = \frac{1}{LT} \sum_{\nu,k,\nu',k'} \left( e^{-\frac{2i\pi k_j}{L}} - 1 \right) \left( e^{-\frac{2i\pi \nu^j}{T}} - 1 \right) \mathbb{E} \hat{h}_\nu^j \hat{h}_{\nu'}^j, \]  

(40)

Therefore, with \( j \) even, using Lemma 3,

\[ E_{\text{per}}(h_0^j - h_0^j)^2 = \frac{2}{LT} \sum_{\nu,k} \left( e^{-\frac{2i\pi k_j}{L}} - 1 \right) \left( e^{\frac{2i\pi k_j}{L}} - 1 \right) \mathbb{E} |\hat{h}_\nu^j|^2 \]

\[ = \frac{1}{LT} \sum_{k \neq 0, L/2} \sum_{\nu} \frac{1 - \cos \frac{2\pi k_j}{L}}{1 - 2 \cos \frac{2\pi \nu}{T} \cos \frac{2\pi k_j}{L} + \cos^2 \frac{2\pi k_j}{L}} \]

\[ \to \frac{1}{L} \sum_{k \neq 0, L/2} \frac{1}{2\pi} \int_0^{2\pi} d\omega \frac{1 - \cos \frac{2\pi k_j}{L}}{1 - 2 \cos \omega \cos \frac{2\pi k_j}{L} + \cos^2 \frac{2\pi k_j}{L}} \quad \text{as} \quad T \to \infty \]

\[ = \frac{1}{L} \sum_{k \neq 0, L/2} \frac{1 - \cos \frac{2\pi k_j}{L}}{1 - \cos^2 \frac{2\pi k_j}{L}} \]  

(41)

where the last step comes from the identity [6], p 366,

\[ \frac{1}{2\pi} \int_0^{2\pi} d\omega \frac{\cos n\omega}{1 - 2a \cos \omega + a^2} = \frac{a^n}{1 - a^2} \]  

(42)
The covariance (41) is indeed the same as the covariance (39). Similarly, for $j$ odd,

$$
\mathbb{E}_{\text{per}}(h_j^1 - h_0^0)^2 = \frac{1}{LT} \sum_{\nu,k} \left( e^{\frac{2\pi i k}{L}} - \frac{2\pi \nu}{T} - 1 \right) e^{\frac{2\pi i k'}{L}} - 1) \mathbb{E}[\hat{h}_\nu^k \hat{h}_{\nu'}^{k'}]
$$

$$= \frac{2}{LT} \sum_{\nu,k} \left( e^{\frac{-2\pi i k}{L}} - \frac{2\pi \nu}{T} - 1 \right) e^{\frac{2\pi i k'}{L}} + 1) \mathbb{E}[\hat{h}_\nu^k | h_{\nu'}^{k'}]
$$

$$= \frac{1}{LT} \sum_{(k,\nu) \neq (0,0), (\frac{L}{2}, \frac{T}{2})} \frac{1 - \cos(2\pi k j L + 2\pi \nu T)}{1 - 2 \cos 2\pi \nu T + \cos^2 2\pi \frac{T}{L}}
$$

$$= \frac{1}{L} \frac{T - 1}{T} + \frac{1}{L} \sum_{k \neq 0, L/2} \sum_{\nu \neq 0, \frac{T}{2}} \frac{1 - \cos(2\pi k j L + 2\pi \nu T)}{1 - 2 \cos 2\pi \nu T + \cos^2 2\pi \frac{T}{L}}
$$

$$\rightarrow \frac{1}{L} \left[ 1 + \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{1 - \cos(2\pi j L \omega + \phi)}{1 - 2 \cos \omega \cos \frac{2\pi \omega T}{L} + \cos^2 \frac{2\pi \omega}{L}} d\omega d\phi \right] \quad \text{as} \quad T \rightarrow \infty
$$

$$(43)
$$

which again is the same as (39). This completes the proof of Proposition 2.

5 Proof of Proposition 1

Proof of (8)(14): using (29) and Lemma 3 with $t$ even,

$$
\mathbb{E}_{\text{per}}(h_{j+2}^t - h_j^t)(h_0^2 - h_0^0) =
$$

$$= \frac{2}{LT} \sum_{\nu,k} \left( e^{-2\pi i \frac{k(j+2)}{L}} - 2\pi \frac{\nu}{T} - e^{-2\pi i \frac{kj}{L}} - 2\pi \frac{\nu}{T} \right) e^{4\pi i \frac{k}{T}} - 1) \mathbb{E}[\hat{h}_k^2]
$$

$$= \frac{1}{LT} \sum_{\nu} \sum_{k \neq 0, L/2} \frac{\cos 2\pi \frac{k j}{L} \cos 2\pi \frac{\nu}{T} (1 - \cos 4\pi \frac{k}{T})}{1 - 2 \cos 2\pi \nu T + \cos^2 2\pi \frac{T}{L}}
$$

$$\rightarrow \frac{1}{4\pi^2} \int_0^{2\pi} d\omega \int_0^{2\pi} d\phi \frac{\cos \phi j \cos \omega t (1 - \cos 2\phi)}{1 - 2 \cos \omega \cos \phi + \cos^2 \phi} \quad \text{as} \quad L, T \rightarrow \infty
$$

$$(44)$$
so that

\[
\lim_{L \to \infty} E(h_{j+2}^t - h_j^t)(h_2^0 - h_0^0) = \frac{1}{4\pi^2} \int_0^{2\pi} d\omega \int_0^{2\pi} d\phi \frac{\cos \phi j \cos \omega t (1 - \cos 2\phi)}{1 - 2 \cos \omega \cos \phi + \cos^2 \phi} = \frac{1}{\pi} \int_0^{2\pi} d\phi \cos \phi (\cos \phi)^t = \frac{2^{-t+1}t!}{t^2 + 1}.
\]

(45)

where we used (42) from the first to the second line, and the last line assumes \(j \leq t\), otherwise the result is zero. Changing \(t\) into \(2t\) yields (8). Stirling’s formula then leads to (14).

Proof of (11)(15):

\[
E_{\text{per}}(h_{j+2} - h_j^t)(h_2^0 - h_0^0) = 2LT \sum_{\nu,k} (e^{-2i\pi \nu t T} - 2i\pi \nu T - 2i\nu \frac{2\pi}{T} (e^{4i\pi \nu T} - 1)) E|\hat{h}_\nu|^2
\]

\[
= \frac{1}{LT} \sum_{\nu \neq 0, T/2} \sum_k \cos 2\pi \frac{kj}{L} \cos 2\pi \frac{\nu t}{T} \frac{1 - \cos 4\pi \frac{\nu t}{T}}{1 - 2 \cos 2\pi \frac{\nu t}{T} \cos^2 2\pi \frac{\nu t}{T}}
\]

\[
\longrightarrow \frac{1}{4\pi^2} \int_0^{2\pi} d\omega \int_0^{2\pi} d\phi \frac{\cos \phi j \cos \omega t (1 - \cos 2\omega)}{1 - 2 \cos \omega \cos \phi + \cos^2 \phi} \quad \text{as } L, T \to \infty
\]

(46)

so that

\[
\lim_{L \to \infty} E(h_{j+2} - h_j^t)(h_2^0 - h_0^0) = \frac{1}{4\pi^2} \int_0^{2\pi} d\omega \int_0^{2\pi} d\phi \frac{\cos \phi j \cos \omega t (1 - \cos 2\omega)}{1 - 2 \cos \omega \cos \phi + \cos^2 \phi}
\]

\[
= -\frac{1}{4\pi} \int_0^{2\pi} d\phi \cos \phi j \left[(\cos \phi)^{(t-2)} - (\cos \phi)^t\right]
\]

(47)

where (42) was used once more. Comparing with the second line of (45) gives (11), which combined with (14) gives (15).

Proof of (13):

\[
E_{\text{per}}(h_{j+1}^t - h_{j-1}^t)(h_2^0 - h_0^0) = \frac{2}{LT} \sum_{\nu,k} e^{-2i\pi \nu t T} - 2i\pi \nu T - 2i\pi \nu \frac{2\pi}{T} (e^{4i\pi \nu T} - 1) (e^{-2i\pi \nu \frac{2\pi}{L}} - e^{-2i\nu \frac{2\pi}{L}}) E|\hat{h}_\nu|^2
\]

\[
= -\frac{4}{LT} \sum_{\nu,k} e^{-2i\pi \nu \frac{2\pi}{T}} (e^{4i\pi \nu \frac{2\pi}{L} - 1}) \sin 2\pi \frac{kj}{L} \sin 2\pi \frac{k}{L} E|\hat{h}_\nu|^2
\]

(48)
where the space symmetry $j \rightarrow -j$ was used to get the last line. In order to use time reversal symmetry, coming with the detailed balance equation, let us compute similarly

$$E_{\text{per}}(h_0^j - h_0^{-j})(h_0^t - h_0^{-t-2}) =$$

$$= \frac{4}{LT} \sum_{\nu,k} 2 \pi \nu \left( e^{-4 \pi \nu \frac{t}{T}} - 1 \right) \sin 2 \pi \frac{k}{L} \sin 2 \pi \frac{k}{L} E[h_k^\nu]^2$$  \hspace{1cm} (49)

Time reversal shows that (49) is the opposite of (48), which therefore equals

$$E_{\text{per}}(h_t^j - h_t^{-j})(h_0^2 - h_0^{-2}) =$$

$$= \frac{4}{LT} \sum_{\nu,k} \cos 2 \pi \nu \frac{t}{T} \sin 2 \pi \frac{k}{L} \sin 2 \pi \frac{k}{L} E[h_k^\nu]^2$$

$$\rightarrow \frac{1}{\pi^2} \int_0^{2\pi} d\omega \int_0^{2\pi} d\phi \frac{\cos \omega t - \cos \omega (t - 2)}{1 - 2 \cos \omega \cos \phi + \cos^2 \phi}$$  \hspace{1cm} (50)

so that, using (42) once more,

$$\lim_{L,T \rightarrow \infty} E(h^t_j - h^t_{-j})(h^2_0 - h^{-2}_0) = \frac{1}{\pi} \int_0^{2\pi} d\phi (\cos \phi)^{t-2} (\cos \phi(j + 1) - \cos \phi(j - 1))$$

$$\equiv \int_0^{\pi/2} d\omega e^{i\omega t} g(\omega, j)$$  \hspace{1cm} (51)

Comparing with the second line of (10) gives (13), which combined with (14) gives (16).

**Second proof of (15):** We give here a second proof of (15), not relying upon (11). Let

$$g(\omega, j) = (1 - \cos 2\omega) \int_0^{2\pi} \frac{d\phi \cos \phi j}{1 - 2 \cos \omega \cos \phi + \cos^2 \phi} = g(-\omega, j) = g(\pi - \omega, j)$$

so that

$$f(t, j) \equiv \int_0^{2\pi} d\omega \int_0^{2\pi} d\phi \frac{\cos \phi j \cos \omega t (1 - \cos 2\omega)}{1 - 2 \cos \omega \cos \phi + \cos^2 \phi}$$

$$= \int_0^{2\pi} d\omega e^{i\omega t} g(\omega, j)$$

$$= -\frac{1}{2i} \int_0^{\pi/2} d\omega e^{i\omega t} g'(\omega, j)$$

$$= -\frac{2}{i} \int_0^{\pi/2} d\omega e^{i\omega t} g'(\omega, j) + \text{c.c.}$$  \hspace{1cm} (53)

In order to perform the integral over $\phi$ and estimate $g'(\omega, j)$ near $\omega = 0$, we decompose

$$\frac{1}{1 - 2 \cos \omega \cos \phi + \cos^2 \phi} = \frac{1}{1 - e^{-2i\omega}} - \frac{1}{1 - e^{i\omega} \cos \phi} + \text{c.c.}$$  \hspace{1cm} (54)
and use the residue theorem:

\[
\int_{-\pi}^{\pi} \frac{d\phi e^{i\phi j}}{1 - e^{i\omega \cos \phi}} = O(j^{-1}) + 2e^{-i\omega} \int_{-\infty}^{\infty} \frac{d\phi e^{i\phi j}}{\phi^2 - (2 - 2e^{-i\omega})} \\
= O(j^{-1}) + 2e^{-i\omega} \frac{2\pi}{2\sqrt{2} - 2e^{-i\omega}} e^{i\sqrt{2} - 2e^{-i\omega} j} \\
= O(j^{-1}) + O(\omega^{1/2}) + \sqrt{2}\pi e^{i\pi/4} e^{-i\pi/4} \omega^{-1/2} e^{-\sqrt{2}\pi e^{-i\pi/4} \omega^{1/2} j} \\
= \int_{-\pi}^{\pi} \frac{d\phi \cos \phi j}{1 - e^{i\omega \cos \phi}} \\
\]  

(55)

because the integral with \(e^{-i\phi j}\), with the contour closed in the lower complex half-plane, gives an equal contribution. The \(O(j^{-1})\) is a regular function of \(\omega\), bounded by const. \(j^{-1}\) as \(j \to \infty\). Therefore

\[
g(\omega, j) = O(\omega^{-1}) + O(\omega^{3/2}) + \sqrt{2}\pi e^{-i\pi/4} \omega^{1/2} e^{-\sqrt{2}\pi e^{-i\pi/4} \omega^{1/2} j} + c.c. \\
\]  

(56)

and

\[
g'(\omega, j) = O(j^{-1}) + O(\omega^{1/2}) + i\pi j e^{-\sqrt{2}\pi e^{-i\pi/4} \omega^{1/2} j} + \frac{\pi}{\sqrt{2}} e^{-i\pi/4} \omega^{-1/2} e^{-\sqrt{2}\pi e^{-i\pi/4} \omega^{1/2} j} + c.c. \\
\]  

(57)

In (57), the phase factor oscillating with \(\omega\) is \(\exp(i\omega^{1/2}j)\) in both terms shown in full, whereas the c.c. counterparts will have \(\exp(-i\omega^{1/2}j)\). When integrating against \(\exp(i\omega t)\) the leading contributions will therefore come from the c.c. counterparts, which will give a stationary phase region. This agrees with the result of the calculation below.

\[
\int_{0}^{\pi/2} d\omega \ e^{i\omega t - \sqrt{2}\pi e^{i\pi/4} \omega^{1/2} j} = O(t^{-1} e^{-\sqrt{2}\pi j}) + \int_{0}^{\infty} d\omega \ e^{i\omega t - \sqrt{2}\pi e^{i\pi/4} \omega^{1/2} j} \\
= O(t^{-1} e^{-\sqrt{2}\pi j}) + 2i \int_{0}^{e^{-i\pi/4}} xdx \ e^{-tx^2 - i\sqrt{2}jx} \\
= O(t^{-1} e^{-\sqrt{2}\pi j}) + 2i \int_{0}^{\infty} xdx \ e^{-tx^2 - i\sqrt{2}jx} \\
\]

(58)

\[
\int_{0}^{\pi/2} d\omega \ e^{i\omega t - \sqrt{2}\pi e^{i\pi/4} \omega^{1/2} j} = O(t^{-1}) + \int_{0}^{\infty} d\omega \ e^{i\omega t - \sqrt{2}\pi e^{i\pi/4} \omega^{1/2} j} \\
= O(t^{-1}) + 2e^{i\pi/4} \int_{0}^{e^{-i\pi/4}} xdx \ e^{-tx^2 - i\sqrt{2}jx} \\
= O(t^{-1}) + 2e^{i\pi/4} \int_{0}^{\infty} xdx \ e^{-tx^2 - i\sqrt{2}jx} \\
\]

(59)
so that
\[ f(t, j) = \mathcal{O}(t^{-2}) + \frac{4\pi j}{t} \int_0^\infty x \, e^{-tx^2 - i\sqrt{2}jx} - \frac{4\pi}{t\sqrt{2}} \int_0^\infty dx \, e^{-tx^2 - i\sqrt{2}jx} + \text{c.c.} \]

(60)

Then
\[ i\frac{4\pi}{t} \int_0^\infty x \, e^{-tx^2 - i\sqrt{2}jx} + \text{c.c.} = i\frac{4\pi}{t} \int_{-\infty}^\infty x \, e^{-tx^2 - i\sqrt{2}jx} = \frac{4\pi^{3/2}j^2}{\sqrt{2}t^{5/2}} e^{-t^2/4\pi} \]

(61)

and
\[ -\frac{4\pi}{t\sqrt{2}} \int_0^\infty dx \, e^{-tx^2 - i\sqrt{2}jx} + \text{c.c.} = -\frac{4\pi}{t\sqrt{2}} \int_{-\infty}^\infty dx \, e^{-tx^2 - i\sqrt{2}jx} = -\frac{4\pi^{3/2}j^2}{\sqrt{2}t^{5/2}} e^{-t^2/4\pi} \]

(62)

Altogether
\[ f(t, j) = \mathcal{O}(t^{-2}) - \frac{4\pi^2}{\sqrt{2\pi}t^{3/2}} \left( 1 - \frac{j^2}{t} \right) e^{-t^2/4\pi} \]

(63)

which completes the second proof of (15).

**Proof of (17):**

\[ \mathbb{E}(h_t^j - h_0^j)(h_t^0 - h_0^0) = \frac{2}{LT} \sum_{\nu,k} \left( e^{2i\pi \nu \phi / L} - e^{-2i\pi \nu \phi / L} \right) \left( e^{2i\pi \nu j / T} - e^{-2i\pi \nu j / T} \right) \]

\[ = \frac{1}{LT} \sum_{\nu,k} \frac{\left( \cos 2\pi \nu j / T \right) \left( 1 - \cos 2\pi \nu t / T \right)}{2 - 2\cos 2\pi \nu t / T \cos 2\pi \nu \phi / L + \cos^2 2\pi \nu \phi / L} \]

\[ \rightarrow \frac{1}{4\pi^2} \int_0^{2\pi} d\omega \int_0^{2\pi} d\phi \frac{(1 - \cos \omega t) \cos \phi j}{1 - 2\cos \omega \cos \phi + \cos^2 \phi} \quad \text{as} \quad L, T \rightarrow \infty \]

\[ = \frac{1}{\pi^2} \int_0^{2\pi} d\omega \int_0^{2\pi} d\phi \frac{(1 - \cos \omega t) \cos \phi j}{1 - 2\cos \omega \cos \phi + \cos^2 \phi} \quad (t, j \text{ even}) \]

(64)

The integral over \( \phi \) is done like in the proof of (15). Then, using
\[ 1 - \cos \omega t = \omega \int_0^t ds \sin \omega s , \]

\[ \mathbb{E}(h_t^j - h_0^j)(h_t^0 - h_0^0) = \mathcal{O}(\ln t) + \]

\[ + \frac{1}{2^{1/2}\pi} \int_0^t ds \int_0^{\pi/2} d\omega \, e^{-i\pi/4} \omega^{-1/2} \sin \omega s \, e^{-\sqrt{2}e^{-i\pi/4}} \omega^{1/2} j + \text{c.c.} \]

(65)
Then, setting $\omega = ix^2$ and using Cauchy’s theorem,

$$ \frac{1}{2i} \int_0^\pi d\omega e^{-i\frac{\pi}{4}\omega}e^{i\omega s} - \sqrt{2} e^{-i\frac{\pi}{4}\omega} = O(s^{-1}) + \frac{1}{i} \int_0^\infty dx e^{-sx^2 - \sqrt{2} \omega x} = O(s^{-1}) + \frac{1}{i} \int_0^\infty dx e^{-sx^2 - \sqrt{2} \omega x} $$

(66)

whose main part will cancel out with its complex conjugate. Similarly, setting $\omega = -ix^2$ and using Cauchy’s theorem,

$$ -\frac{1}{2i} \int_0^\pi d\omega e^{-i\frac{\pi}{4}\omega}(-e^{-i\omega s} - \sqrt{2} e^{-i\frac{\pi}{4}\omega}) = O(s^{-1}) + \frac{1}{i} \int_0^\infty dx e^{-sx^2 + i\sqrt{2} \omega x} = O(s^{-1}) + \frac{1}{i} \int_0^\infty dx e^{-sx^2 + i\sqrt{2} \omega x} $$

(67)

and

$$ \int_0^\infty dx e^{-sx^2 + i\sqrt{2} \omega x} + c.c. = \int_{-\infty}^\infty dx e^{-sx^2 + i\sqrt{2} \omega x} = \sqrt{\frac{\pi}{s}} e^{-\frac{s^2}{2s}} $$

(68)

Altogether

$$ \mathbb{E}(h_j^t - h_j^0)(h_i^0 - h_i^0) = \frac{1}{\sqrt{2\pi}} \int_0^t ds \frac{s}{s} e^{-\frac{s^2}{2s}} + O(\ln t) $$

$$ = \sqrt{\frac{2t}{\pi}} \int_1^\infty du \frac{e^{-\frac{u^2}{2u}}}{u^2} + O(\ln t) $$

$$ = \sqrt{\frac{2t}{\pi}} \left[ e^{-\frac{t^2}{2t}} + \frac{j}{\sqrt{t}} \int_1^\infty du \frac{e^{-u^2/2}}{\sqrt{u}} \right] + O(\ln t) $$

(69)

which completes the proof of Proposition 1.

## 6 Higher dimension

In arbitrary $d \geq 1$, $h^0 = \{h_i^0 : i \in (\mathbb{Z}/L\mathbb{Z})^d \}$, the initial measure

$$ \mu(dh^0) = \prod_{[i-j]=1} e^{-\frac{1}{2}(h_i^0 - h_j^0)^2} \prod_i dh_i^0 $$

(70)

is invariant under the dynamics defined by

$$ \mathbb{P}(dh^t | h^{t-1}) = \prod_{|i|+t \text{ even}} e^{-d(h_i^t - \frac{1}{2t} \delta(h_j^t - h_j^{t-1})} \prod_i dh_i^t $$

(71)
where $\sum_j$ runs over the 2$d$ neighbors of $i$. The space-time Hamiltonian on the $d + 1$-dimensional torus $(\mathbb{Z}/T\mathbb{Z}) \times (\mathbb{Z}/L\mathbb{Z})^d$ is

$$ \mathcal{H}_{\text{per}} = d \sum_{i,t} \left( h_i^t - \frac{1}{2d} \sum_{j : |i-j|=1} h_j^{t-1} \right)^2 $$

(72)

The Fourier transform is defined as in $d = 1$, using $k = (k_1 \ldots k_d)$. Then

$$ \mathcal{H}_{\text{per}} = \sum_{\nu, k} |\hat{h}_k^\nu|^2 \left[ 1 - 2 \left( \cos 2\pi \frac{\nu}{T} \right) \frac{1}{d} \sum_{n=1}^{d} \cos 2\pi \frac{k_n}{L} + \frac{1}{d^2} \left( \sum_{n=1}^{d} \cos 2\pi \frac{k_n}{L} \right)^2 \right] $$

$$ = \sum_{\nu, k} |\hat{h}_k^\nu|^2 \gamma_k^\nu $$

$$ \sim \frac{1}{2} \sum_{\nu, k} |\hat{h}_k^\nu|^2 \left[ \frac{(2\pi \nu)^2}{T^2} \left( 1 - \frac{1}{2d} \left( \frac{2\pi k}{L} \right)^2 \right) + \frac{1}{d^2} \left( \frac{(2\pi k)^2}{2L^2} \right)^2 \right] $$

(73)

and the autocorrelation can be computed, yielding the expected $\ln(t)$ in $d = 2$ or $O(1)$ in $d \geq 3$.

7 Harness process in continuous time

The harness process in continuous time can be constructed as the $L \to \infty$ limit of the harness process with random sequential update, defined like the sub-lattice parallel dynamics but with (2) replaced by

$$ \mathbb{P}(dh^\tau \mid h^{\tau-1}) = \sum_{j=0}^{L-1} e^{-\left( h_j^\tau - \frac{1}{2}(h_j^{\tau-1} + h_{j+1}^{\tau-1}) \right)^2} \prod_{i \neq j} \delta(h_i^\tau - h_i^{\tau-1}) \prod_i dh_i^\tau / \text{norm}. $$

(74)

The time $t$ for the Poisson clocks of rate one in the harness process in continuous time is related to the microscopic time $\tau$ through $t = \tau/L$. The measure (1) is also invariant under the dynamics (74), and we still take it as initial condition.

Here we give numerical results for this model, indicating that the asymptotic forms (14)(15)(16) in Proposition 1 may still be valid, with some rescaling. The initial condition is drawn using the Fourier modes, which are independent Gaussian variables. We then run the dynamics for a time $t_1 + t$ and measure the correlations:

$$ g_{11}^{L,t_1}(t,j) = \frac{1}{L} \sum_{i=0}^{L-1} \sum_{t'=0}^{t_1-1} \left( h_i^{L+t'} - h_i^{L+t} \right) \left( h_{i+t+2}^{L+t'} - h_{i+t+2}^{L+t} \right) $$

(75)

$$ g_{22}^{L,t_1}(t,j) = \frac{1}{L} \sum_{i=0}^{L-1} \sum_{t'=0}^{t_1-1} \left( h_i^{L+t'+1} - h_i^{L+t} \right) \left( h_{i+t+1}^{L+t'+1} - h_{i+t+1}^{L+t} \right) $$

(76)
\[
g_{12}^{L,t_1}(t,j) = \frac{1}{L} \sum_{i=1}^{L} \frac{1}{t_1} \sum_{t'=0}^{t_1-1} \left( h_i^{L(t'+1)} - h_i^{L(t'+t)} \right) \left( h_{i+j+1}^{L(t'+t)} - h_{i+j-1}^{L(t'+t)} \right) \tag{77}
\]

The results are displayed in Fig. 2 and Fig. 3, where the upper indices \( L, t_1 \) have been omitted for clarity, while the values of \( L \) and \( t_1 \) appear in the captions.

Fig. 2 shows relaxation as function of time, with both the numerical results as described above, and the corresponding exact results for the (oe: odd-even) sub-lattice parallel dynamics taken from Prop. [4]. The two dynamics differ for small time but follow similar asymptotics at large time. The function \( g_{12}^{oe}(t,1) \) is not shown on Fig. 2 because, as mentioned in Remark 1, it is proportional to \( g_{22}^{oe}(t,0) \), and would therefore yield the same scaled curve.

Fig. 2: Random sequential updates, scaled empirical correlation functions \( g_{11}(t,0) \), \( g_{22}(t,0) \) and \( g_{12}(t,1) \), average taken over space \( L = 10^6 \) and time \( t_1 = 1000 \), and scaled correlation functions \( g_{11}^{oe}(t,0) \), \( g_{22}^{oe}(t,0) \) of sub-lattice parallel dynamics.

Fig. 3 shows the variation in space of the space-time correlations at a given large time \( t = 10 \), together with a fit inspired by (14) (15) (16).
Fig. 3: Random sequential updates, scaled empirical correlation functions $g_{11}(t, j)$, $g_{22}(t, j)$ and $g_{12}(t, j)$, all at time $t = 10$, average taken over space $L = 10^6$ and time $t_1 = 1000$, and conjectured asymptotics similar to sub-lattice parallel dynamics.
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