Inversions of the windowed ray transform

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Abstract

The windowed ray transform is a natural generalization of the “Analytic-Signal Transform” which is developed to extend arbitrary functions from $\mathbb{R}^n$ to $\mathbb{C}^n$. The X-ray transform is also the special case of this transform. Similarly to the X-ray transform, the problem of inverting this transform is overdetermined. Hence it is highly possible to existence of several inversion formulas.

Kaiser obtained inversion formula in 1993. We present several inversion formulas here. One of them is similar to that of Kaiser, but require weaker condition. The others are new.

1 Introduction

The windowed ray transform was introduced in [6] by Kaiser and Streater. It is a natural generalization of the “Analytic-Signal Transform” [5] arising from a method for extending arbitrary functions from $\mathbb{R}^n$ to $\mathbb{C}^n$ in a semi-analytic way in relativistic quantum theory. Namely, the Analytic-Signal Transform of $f \in \mathcal{S}(\mathbb{R}^n)$ is the function $g : \mathbb{C}^n \rightarrow \mathbb{C}$ defined by

$$g(u + iv) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(u + \tau v)}{\tau - i} d\tau.$$ 

Its generalization, the windowed ray transform, is defined as

$$P_h f(u, v) = \int_{\mathbb{R}} f(u + tv)h(t)dt, \text{ for } (u, v) \in \mathbb{R}^n \times \mathbb{R}^n \setminus 0.$$ 

Here $h$ is regarded as a window, which explains the terminology “windowed ray transform.” When $h = 1$ and $||v|| = 1$, it becomes the usual X-ray transform. In order to minimize analytical subtleties, we assume that $h$ is smooth with rapid decay, i.e., $h \in \mathcal{S}(\mathbb{R})$.

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While $f$ depends upon $n$-dimensional variable, $P_hf$ depends upon $2n$-dimensional variable. Hence the problem of inverting the windowed ray transform is $n$-dimensions overdetermined. Here we will study the inversion of the windowed ray transform. In next section 2, we present several inversion formulas for the transform. In fact, one of our inversions is similar to an inversion formula Kaiser and Streater already derived in [6], but requires weaker conditions.

2 Reconstruction

Theorem 1. Let $h \in S(\mathbb{R})$ be non-zero. Then $f \in S(\mathbb{R}^n)$ can be reconstructed from $P_hf$ as follows:

$$f(x) = \pi^{-\frac{n+1}{2}} \Gamma(n/2) \left( \int_{\mathbb{R}} |\hat{h}(-t)|^2 dt \right)^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}^n} P_hf(x - vt, v) I^{-1} h(-t)|v|^{-n} dtdv,$$

where $I^{-1} h(\eta) = |\eta| \hat{h}(\eta)$.

Proof. Clearly, $P_h$ is invariant under a shift with respect to the first $n$ variables. Hence taking the Fourier transform with respect to $u$ looks reasonable. Doing this, we get

$$\hat{P_hf}(\xi, v) = \hat{f}(\xi) \int_{\mathbb{R}} h(t) e^{i\xi \cdot v} dt = \hat{f}(\xi) \hat{h}(-\xi \cdot v), \quad (1)$$

where $\hat{P_hf}$ is the Fourier transform of $P_hf$ with respect to first $n$ dimensional variable $u$.

Note that the complex conjugate of $\hat{h}(-\xi)$ is $\hat{h}(\xi)$. Multiplying (1) by $|v|^{-n} \hat{h}(\xi \cdot v)|\xi \cdot v|$ and integrating with respect to $v$ yield

$$\int_{\mathbb{R}^n} \hat{P_hf}(\xi, v) \hat{h}(\xi \cdot v)|v|^{-n} |\xi \cdot v| dv = \hat{f}(\xi) \int_0^\infty \int_{S^{n-1}} |\hat{h}(r\xi \cdot \theta)|^2 |\xi \cdot \theta| d\theta dr$$

$$= |S^{n-2}| \hat{f}(\xi) \int_0^\infty \int_{S^{n-1}} |\hat{h}(r\xi|t)|^2 (1 - t^2)^{(n-3)/2} |t||\xi| dtdr,$$

where in the first line, we switched from the Cartesian coordinate $v \in \mathbb{R}^n$ to the polar coordinates $(r, \theta) \in [0, \infty) \times S^{n-1}$ and in the second line, we used the known relation

$$\int_{S^{n-1}} f(\omega \cdot \theta) d\theta = |S^{n-2}| \int_{-1}^1 f(t)(1 - t^2)^{(n-3)/2} dt.$$
for any integrable function $f$ on $\mathbb{R}$ and $\omega \in S^{n-1}$ \cite{7}. Using the Fubini-Tonelli theorem, we continue to compute equation \cite{2} as following:

$$
\int_{\mathbb{R}^n} \widehat{P_h f}(\xi, v) \hat{h}(\xi \cdot v) |v|^{-n} |\xi \cdot v| dv 
$$

$$
= |S^{n-2}| \hat{f}(\xi) \left\{ \int_0^1 \int_0^1 |\hat{h}(r|\xi|t)|^2 (1-t^2)^{(n-3)/2} t|\xi| dr dt + \int_0^{\infty} \int_0^{\infty} |\hat{h}(r|\xi|t)|^2 (1-t^2)^{(n-3)/2} t|\xi| dr dt \right\} 
$$

$$
= |S^{n-2}| \hat{f}(\xi) \left\{ \int_0^1 |\hat{h}(r)|^2 dr \left(\int_0^1 (1-t^2)^{(n-3)/2} dt \right) - \int_0^1 |\hat{h}(r)|^2 dr \left(\int_0^1 (1-t^2)^{(n-3)/2} dt \right) \right\} 
$$

$$
= |S^{n-2}| \hat{f}(\xi) \int_\mathbb{R} |\hat{h}(r)|^2 dr \left(\int_0^1 (1-t^2)^{(n-3)/2} dt \right) = \pi^{(n+1)/2} \Gamma(n/2)^{-1} \hat{f}(\xi) \int_\mathbb{R} |h(t)|^2 dt, 
$$

where in the third line, we changed the variable $r$ to $r/|\xi|t$ and in the last equation, we used the Plancherel formula. Thus we have

$$
\hat{f}(\xi) = \pi^{-(n+1)/2} \Gamma(n/2) \left( \int_\mathbb{R} |h(t)|^2 dt \right)^{-1} \int_{\mathbb{R}^n} \widehat{P_h f}(\xi, v) \hat{h}(\xi \cdot v) |v|^{-n} |\xi \cdot v| dv. 
$$

Taking the inverse Fourier transform completes the proof. \hfill \Box

**Remark 2.** This inversion approach is similar to that of \cite{6}. We, however, multiply \cite{1} by $|v|^{-n} \hat{h}(\xi \cdot v)|\xi \cdot v|$, unlike the factor $|v|^{-n} \hat{h}(\xi \cdot v)$ in \cite{6}. This makes it unnecessary to require that $h$ is admissible ($h(0) = 0$).

Now we present another inversion formula.

**Theorem 3.** Let $h \in S(\mathbb{R})$ be non-zero. Then we have for $f \in S(\mathbb{R}^n)$

$$
f(x) = 2^{-n-1} \pi^{-n} \left( \int_\mathbb{R} |h(t)|^2 dt \right)^{-1} \int_\mathbb{R} \int_\mathbb{R} \widehat{P_h f}(\xi, r\xi/|\xi|) \hat{h}(r|\xi|) e^{i\xi x} |\xi| dr d\xi.
$$

**Proof.** Let us consider $P_h f(u + \tau v, v)$ for $u \cdot v = 0$ and $\tau \in \mathbb{R}$. Then we have

$$
P_h f(u + \tau v, v) = \int_\mathbb{R} f(u + \tau v + tv) h(t) dt = \int_\mathbb{R} f(u + tv) h(t - \tau) dt.
$$
Switching from the Cartesian coordinate $v \in \mathbb{R}^n$ to the polar coordinates $(r, \theta) \in [0, \infty) \times S^{n-1}$, we get

$$P_h f(u + \tau r\theta, r\theta) = \int_{\mathbb{R}} f(u + tr\theta)h(t - \tau)dt.$$ 

Then $P_h$ is invariant under a shift with respect to $\tau$. Taking the Fourier transform with respect to $\tau$ looks reasonable. To get $\hat{f}(\sigma \theta)$, we take the Fourier transform with respect to $\tau$ and integrate with respect to $u \in \theta^\perp$ so that

$$\widehat{P_h f}(\sigma/r\theta, r\theta) = r \int_{\theta^\perp} \int_{\mathbb{R}} P_h f(u + \tau r\theta, r\theta)e^{-i\sigma \tau} d\tau du = r \int_{\theta^\perp} \int_{\mathbb{R}} f(u + tr\theta)e^{-i\sigma t}dtdu \hat{h}(-\sigma)$$

$$= \int_{\theta^\perp} \hat{f}(u + \sigma/r\theta)du \hat{h}(-\sigma) = \hat{f}(\sigma/r\theta)\hat{h}(-\sigma),$$

or

$$\widehat{P_h f}(\sigma, r\theta) = \hat{f}(\sigma)\hat{h}(-r\sigma),$$

where $\hat{f}$ and $\widehat{P_h f}$ are the $n$-dimensional Fourier transforms of $f$ and $P_h f$ with respect to $x$ and $u$, respectively. Multiplying by $\hat{h}(r\sigma)$ and integrating equation (3) with respect to $r$ yield

$$\int_0^\infty \widehat{P_h f}(\sigma, r\theta)\hat{h}(r\sigma)dr = \int_0^\infty |\hat{h}(r\sigma)|^2 dr = \hat{f}(\sigma) - \sigma^{-1} \int_0^\infty |\hat{h}(r)|^2 dr.$$

We have

$$\hat{f}(\sigma\theta) = \left(\int_0^\infty |\hat{h}(\eta)|^2 d\eta\right)^{-1} |\sigma| \int_0^\infty \widehat{P_h f}(\sigma, r\theta)\hat{h}(r\sigma)dr,$$

or

$$f(x) = (2\pi)^{-n} \left(\int_0^\infty |\hat{h}(\eta)|^2 d\eta\right)^{-1} \int_{S^{n-1}} \left(\int_0^\infty \widehat{P_h f}(\sigma, r\theta)\hat{h}(r\sigma)dr\right) e^{i\sigma \theta \cdot x} \sigma^n d\sigma d\theta. \quad (4)$$

Note that

$$\int_0^\infty |\hat{h}(\eta)|^2 d\eta = \int_0^\infty |\hat{h}(-\eta)|^2 d\eta,$$

because of $|\hat{h}(\eta)|^2 = |\hat{h}(-\eta)|^2$. The Plancherel formula implies that

$$\frac{1}{2} \int_{\mathbb{R}} |h(t)|^2 dt = \int_0^\infty |\hat{h}(\eta)|^2 d\eta.$$

Combining this equation and equation (4) gives our assertion. \qed
Theorem 4. Let $h$ be non-vanishing at a point $a \in \mathbb{R}$. For $f \in S(\mathbb{R}^n)$, we have for $u = (u_1, u') \in \mathbb{R} \times \mathbb{R}^{n-1}$
\[
|\sigma|\hat{P}_h f(\sigma, u', a\sigma, v') = 2\pi \hat{f}(\sigma, av' + u')h(a).
\]
Here $\hat{f}$ is the Fourier transform of $f$ with respect to the first variable $x_1$ and $\hat{P}_h f$ is the 2-dimensional Fourier transform of $P_h f$ with respect to the two variable $(u_1, v_1)$.

Proof. Taking the Fourier transform of $P_h f(u, v)$ with respect to $u_1$ yields
\[
\int_{\mathbb{R}} P_h f(u, v)e^{-i\sigma u_1}du_1 = \int_{\mathbb{R}} \hat{f}(\sigma, u' + tv')e^{itu_1}h(t)dt.
\]
To get $\hat{f}$, multiplying $e^{-iav_1\sigma}$ and integrating with respect to $v_1$ gives
\[
\int_{\mathbb{R}} \int_{\mathbb{R}} P_h f(u, v)e^{-i(au_1 + uu')\sigma}du_1dv_1 = \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\sigma, tv' + u')e^{itu_1}\sigma h(t)e^{-iav_1\sigma}dt dv_1
\]
\[
= \int_{\mathbb{R}} \hat{f}(\sigma, tv' + u')h(t)\int_{\mathbb{R}} e^{i(t-a)v_1\sigma}dv_1dt
\]
\[
= 2\pi \int_{\mathbb{R}} \hat{f}(\sigma, tv' + u')h(t)\frac{\delta(t-a)}{|\sigma|}dt
\]
\[
= 2\pi \hat{f}(\sigma, av' + u')h(a)|\sigma|^{-1}.
\]

Remark 5. Theorem 4 leads naturally to a Fourier type inversion formula, supplementing the inverse Fourier transform.

Remark 6. Even if the domain of $u$ is restricted to a line, say $x_1$-axis, then we get the analogue of Theorem 4, i.e., for $a \in \mathbb{R}$ with $h(a) \neq 0$,
\[
|\sigma|\hat{P}_h f(\sigma, a\sigma, v') = 2\pi \hat{f}(\sigma, av')h(a),
\]
so we can still reconstruct $f$ from $P_h f$.

Remark 7. The great point of Theorem 4 is that we don’t need to know the whole information of $h$. Nonzero value of $h$ at only one point is required.

When $n = 2$, we can get a series formula for an inversion of the windowed ray transform, by using circular harmonics. Consider $P_h f(u, u^\perp)$ where $u^\perp = (-u_2, u_1)$. Let $g(\rho, \theta)$ be the function $P_h f(u, u^\perp)$ where $\rho = |u|$ and $\theta = u/|u|$, and let $f(r, \phi)$ be the image function in
polar coordinates. Then the Fourier series of \( f \) and \( g \) with respect to their angular variables can be written as

\[
f(r, \phi) = \sum_{l=-\infty}^{\infty} f_l(r) e^{il\phi}, \quad g(\rho, \theta) = \sum_{l=-\infty}^{\infty} g_l(\rho) e^{il\theta},
\]

where the Fourier coefficients are given by

\[
f_l(r) = \frac{1}{2\pi} \int_{0}^{2\pi} f(r, \phi) e^{-il\phi} d\phi, \quad g_l(\rho) = \frac{1}{2\pi} \int_{0}^{2\pi} g(\rho, \theta) e^{-il\theta} d\theta.
\]

**Theorem 8.** Let \( f \in C^\infty_c(\mathbb{R}^2) \). If \( h \in C^\infty_c(\mathbb{R}) \) is not odd, then we have

\[
\mathcal{M} g_l(s) = \mathcal{M} f_l(s+1) \mathcal{M} H(s),
\]

where \( \mathcal{M} \) is the Mellin transform and

\[
H(r) = \left\{ \begin{array}{ll}
\left[ h \left( \sqrt{\frac{1}{r^2} - 1} \right) + h \left( -\sqrt{\frac{1}{r^2} - 1} \right) \right] \frac{e^{i\arccos r}}{\sqrt{1-r^2}} & \text{if } r < 1, \\
0 & \text{otherwise.}
\end{array} \right.
\]

**Proof.** We can write \( P_h f(u, u^\perp) \) as

\[
\int_{\mathbb{R}} f(u + tu^\perp) h(t) dt = \int_{\mathbb{R}^2} f(x) h \left( \frac{x \cdot u^\perp}{|u|^2} \right) \delta \left( |u| - x \cdot \frac{u}{|u|} \right) dx,
\]

where \( \delta \) is the Dirac delta function. Then we have

\[
g_l(\rho) = \frac{1}{2\pi} \int_{0}^{2\pi} g(\rho, \theta) e^{-il\theta} d\theta
\]

\[
= \frac{1}{2\pi} \int_{0}^{2\pi} \int_{\mathbb{R}^2} f(x) h \left( \frac{x \cdot (-\sin \theta, \cos \theta)}{\rho} \right) \delta(\rho - x \cdot (\cos \theta, \sin \theta)) dx e^{-il\theta} d\theta.
\]

Changing variables \( x \rightarrow (r, \phi) \in [0, \infty) \times [0, 2\pi) \) gives

\[
g_l(\rho) = \frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{\infty} \int_{0}^{2\pi} f(r, \phi) h \left( \frac{r \sin(\phi - \theta)}{\rho} \right) \delta(\rho - r \cos(\phi - \theta)) re^{-i\theta} dr d\phi d\theta
\]

\[
= \frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{\infty} \int_{0}^{2\pi} f(r, \phi) h \left( \frac{r \sin(\phi - \theta)}{\rho} \right) \delta(\rho - r \cos(\phi - \theta)) re^{-i\theta} dr d\phi d\theta
\]

\[
= \frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{\infty} \int_{0}^{2\pi} f(r, \phi) h \left( \frac{r \sin \theta}{\rho} \right) \delta(\rho - r \cos \theta) re^{-i\theta} dr d\phi d\theta,
\]
where in the last line, we changed variables \( \theta \to \theta + \phi \). Continuing to compute \( g_l \), we get
\[
g_l(\rho) = \int_0^{2\pi} \int_0^{\infty} f_1(r) h \left( \frac{r \sin \theta}{\rho} \right) \delta(\rho - r \cos \theta) re^{i\theta} \, dr \, d\theta
\]
\[
= \int_0^{\rho} f_1(r) \left[ h \left( \frac{r}{\rho} \sqrt{1 - \left( \frac{\rho}{r} \right)^2} \right) + h \left( -\frac{r}{\rho} \sqrt{1 - \left( \frac{\rho}{r} \right)^2} \right) \right] \frac{re^{i\arccos \frac{r}{\rho}}}{\sqrt{r^2 - \rho^2}} \, dr
\]
\[
= (r f_1(r)) \times H(\rho),
\]
where
\[
f \times H(s) = \int_0^{\infty} f(r) H \left( \frac{s}{r} \right) \frac{dr}{r}.
\]

Taking the Mellin transform \( \mathcal{M} \) of \( g_l \) and the property that \( \mathcal{M}[rf(r)](s) = \mathcal{M}f(s + 1) \) complete the proof.

**Corollary 9.** Let \( f_1(r) \) be the \( l \)-th Fourier coefficient of the twice differentiable function \( f \) with compact support. Then for any \( t > 1 \) we have
\[
f_1(r) = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{t - Ti}^{t + Ti} \mathcal{M}g_l(s-1) \frac{ds}{\mathcal{M}H(s-1)}.
\]

**Proof.** For \( a > 1 \) and \( b \in \mathbb{R} \), \( |\mathcal{M}f_1(a + bi)| \) is finite because
\[
\int_0^{\infty} r^{a+bi-1}|f_1(r)|dr \leq C \int_0^{R} r^{a-1}e^{ib\ln r}dr
\]
where \( C \) is the upper bound of \( |f_1| \) and \( R \) is radius of a ball containing \( \text{supp } f \). Thus, \( \mathcal{M}f_1(s) \) is analytic on \( \{ z \in \mathbb{C} : \Re z > 1 \} \). Integrating by parts twice, we get
\[
\mathcal{M}f_1(s) = \int_0^{\infty} f_1''(r) \frac{r^{s+1}}{s(s+1)} \, dr,
\]
which implies \( \mathcal{M}f_1(s) = O(s^2) \). Hence \( \mathcal{M}f_1(t+si) \) is integrable and we can apply the inverse Mellin transform \([2, 9]\) which gives formula (6). 

\( \Box \)
3 Conclusion

We study the windowed ray transform, a general form of the analytic-signal transform. Several different inversion formulas of the windowed ray transform are provided. While in [6] the condition that $h$ is admissible is required, it is not in ours.

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References

[1] L. Ehrenpreis, P. Kuchment, and A. Pancheko. The exponential X-ray transform and fritz john’s equation. i. range description. In L. Ehrenpreis and E.L. Grinberg, editors, Analysis, Geometry, Number Theory: The Mathematics of Leon Ehrenpreis, Contemporary mathematics - American Mathematical Society, pages 173–188. American Mathematical Society, 2000.

[2] P. Flajolet, X. Gourdon, and P. Dumas. Mellin transforms and asymptotics: Harmonic sums. Theoretical Computer Science, 144:3–58, 1995.

[3] G.A. Kaiser. Quantized fields in complex spacetime. Annals of Physics, 173(2):338 – 354, 1987.

[4] G.A. Kaiser. Generalized wavelet transforms. Part I: The windowed X-ray transform. Technical report, 1990.

[5] G.A. Kaiser. Quantum Physics, Relativity, and Complex Spacetime: Towards a New Synthesis. ArXiv e-prints, October 2009.

[6] G.A. Kaiser and R.F. Streater. Wavelets: a tutorial in theory and applications. chapter Windowed Radon transforms, analytic signals, and the wave equation, pages 399–441. Academic Press, San Diego, CA, USA, 1992.

[7] F. Natterer. The Mathematics of Computerized Tomography. Classics in Applied Mathematics. Society for Industrial and Applied Mathematics, Philadelphia, 2001.
[8] F. Natterer and F. Wübbeling. *Mathematical methods in image reconstruction*. SIAM Monographs on mathematical modeling and computation. SIAM, Society of industrial and applied mathematics, Philadelphia (Pa.), 2001.

[9] E.C. Titchmarsh. *Introduction to the theory of Fourier integrals*. The Clarendon press, 1937.