Insertion and Sorting in a Sequence of Numbers Minimizing the Maximum Sum of a Contiguous Subsequence

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Abstract

Let A be a sequence of \( n \geq 0 \) real numbers. A subsequence of A is a sequence of contiguous elements of A. A maximum scoring subsequence of A is a subsequence with largest sum of its elements, which can be found in \( O(n) \) time by Kadane’s dynamic programming algorithm. We consider in this paper two problems involving maximal scoring subsequences of a sequence. Both of these problems arise in the context of buffer memory minimization in computer networks. The first one, which is called insertion in a sequence with scores (ISS), consists in inserting a given real number \( x \) in A in such a way to minimize the sum of a maximum scoring subsequence of the resulting sequence, which can be easily done in \( O(n^2) \) time by successively applying Kadane’s algorithm to compute the maximum scoring subsequence of the resulting sequence corresponding to each possible insertion position for \( x \). We show in this paper that the ISS problem can be solved in linear time and space with a more specialized algorithm. The second problem we consider in this paper is the sorting in a sequence by scores (SSS) one, stated as follows: find a permutation \( A' \) of A that minimizes the sum of a maximum scoring subsequence. We show that the SSS problem is strongly NP-Hard and give a 2-approximation algorithm for it.

1 Introduction

Let the elements of a sequence \( A \) of \( n \geq 0 \) real numbers be denoted by \( a_1, a_2, \ldots, a_n \). Then, \( A \) is the sequence \( (a_1, a_2, \ldots, a_n) \) (which is \( (\) if \( n = 0 \)) and its size is \( |A| = n \). A subsequence of \( A \) defined by indices \( 0 \leq i \leq j \leq n \) is denoted by \( A^j_i \), which equals either \( (\) if \( i = j \), or the sequence \( \{a_{i+1}, \ldots, a_j\} \) of contiguous elements of \( A \), otherwise (see Figure 1 for an example). Let \( \text{score}(A^j_i) = \sum_{k=i+1}^{j} a_k \) stand for the sum of elements of \( A^j_i \) (we consider \( \text{score}(\{\}) = 0 \)). A maximum scoring subsequence of \( A \) is a subsequence with largest score. The maximum scoring subsequence (MSS) problem is that of finding a maximum scoring subsequence of a given sequence \( A \). The MSS problem can be solved in \( O(n) \) time by Kadane’s dynamic programming algorithm, whose essence is to consider \( A \) as a concatenation \( (A^h_{1}, A^h_{2} = j, \ldots, A^h_{k}) \) of appropriate subsequences, called intervals, and to determine \( S_k \) as a maximum scoring subsequence of \( A^h_{k} \), for all \( k \in \{1,2,\ldots,\ell\} \). Defining each interval \( A^h_{k} \) – with the possible exception of the last one – to be such that \( \text{score}(A^h_{k}) < 0 \) and \( \text{score}(A^h_{j}) \geq 0 \), for all \( i_k \leq j < k \), then the largest score subsequence among \( \{S_1, S_2, \ldots, S_k\} \) is a maximum scoring subsequence of \( A \). The value of \( A \) is \( \text{score}^*(A) = \text{score}(S) \), for any maximum scoring subsequence \( S \) of \( A \).

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The MSS problem has several applications in practice, where maximum scoring subsequences correspond to various structures of interest. For instance, in Computational Biology, in the context of certain amino acid scoring schemes and several other applications mentioned in [3,4]. In such a context, it may also be useful to find not only one but a maximal set of non-overlapping maximum scoring subsequences of a given sequence $A$. This can be formalized as the ALL MAXIMAL SCORING SUBSEQUENCES problem, for which have been devised a linear sequential algorithm [4], a PRAM EREW work-optimal algorithm that runs in $O(\log n)$ time and makes $O(n)$ operations [5] and a BSP/CGM parallel algorithm which uses $p$ processors and takes $O(|A|/p)$ time and space per processor [6]. The MSS problem has also been generalized in the direction of finding a list of $k$ (possibly overlapping) maximum scoring subsequences of a given sequence $A$. This is known as the $k$ MAXIMUM SUMS PROBLEM [7] and for a generalization of it an optimal $O(n+k)$ time and $O(k)$ space algorithm has been devised [8,9]. An optimal $O(n \cdot \max\{1, \log (k/n)\})$ algorithm has also been developed for the related problem of selecting the subsequence with the $k$-th largest score [9].

We consider in this paper two problems related to the MSS. The first one, which is called INSERTION IN A SEQUENCE WITH SCORES (ISS), consists in inserting a given real number $x$ in $A$ in such a way to minimize the maximum score of a subsequence of the resulting sequence. The operation of inserting $x$ in $A$ is associated with an insertion index $p \in \{0, \ldots, n\}$ and the resulting sequence $A^{(p)} = \langle A^p_0, x, A^p_n \rangle$, that is, the sequence obtained by the concatenation of $A^p_0$, $x$, and $A^p_n$. The objective of the ISS problem is to determine an insertion index $p^*$ that minimizes $\text{score}^*(A^{(p^*)})$, which can be easily done in $O(n^2)$ time and $O(n)$ space by successively using Kadane’s algorithm to compute the maximum scoring subsequence of $A^{(0)}$, $\ldots$, $A^{(n)}$. We show in this paper that we can do better. More precisely, we show that the ISS problem can be solved in linear time.

The ISS problem can be approached more specifically depending on the value of $x$. The case $x = 0$ is trivial since $\text{score}^*(A^{(p)}) = \text{score}^*(A)$ independently of the value of $p$, which means that any insertion index $p$ is optimal for $A$. If $x < 0$, then $\text{score}^*(A^{(p)}) < \text{score}^*(A)$, for all insertion indices $p \in \{0, 1, \ldots, n\}$. Intuitively, then, $x$ has to be inserted inside some maximum scoring subsequence $S = A^j$ of $A$, in an attempt to reduce the value of $A^{(p)}$ with respect to that of $A$. Even though the value of $A^{(p)}$ cannot be smaller than $\text{score}^*(A)$ in certain cases (for instance, if $S$ has only one positive element, or $\text{score}^*(A^j) = \text{score}(S)$, or $\text{score}^*(A_{n_1}^j) = \text{score}(S)$, then all insertion indices are equally good for $A$ since $\text{score}^*(A^{(p)}) = \text{score}^*(A)$ for any particular choice of $p$), we describe an $O(n)$ time and space algorithm to determine a best insertion position in a maximum scoring subsequence of $A$, provided that $x$ is negative.

Showing that the ISS problem can be solved in linear time is a more complex task when $x > 0$. Inserting $x$ inside a maximum scoring subsequence $S$ of $A$ will certainly lead to a subsequence $S'$ of $A^{(p)}$ such that $\text{score}(S') > \text{score}(S)$ (this may happen even if $x$ is inserted outside $S$). Intuitively, therefore, we should choose an insertion position where $x$ can only “contribute” to subsequences whose scores are as small as possible. Computing the necessary information for this in $O(n)$ time may seem hard at first, but we can make things simpler by considering the partition into intervals of $A$ (the same used in Kadane’s algorithm). The idea is to determine the interval $A^h_{k}$ having an optimal insertion index. The difficulty to accomplish this task in linear time stems from the fact that computing $\text{score}^*(A^{(p^*)})$ when $p$ is an insertion index in an interval $A^h_{k}$ may involve one or more intervals other than $A^h_{k}$. We overcome this difficulty by means of a dynamic programming approach.

The second problem we consider in this paper is the SORTING A SEQUENCE BY SCORES (SSS), stated as follows: given the sequence $A$, find a permutation $A'$ of $A$ that minimizes $\text{score}^*(A')$. The SSS problem is referred to as the SEQUENCING TO MINIMIZE THE MAXIMUM RENEWAL CUMULATIVE
Cost in [10]. Among other applications, this latter problem models buffer memory usage in a node of a computer network. In this case, the absolute value of a number models the local memory space required to store a corresponding message after its reception and before it is resent through the network (in practice, there are additional cases in which the message is produced or consumed locally; these situations are ignored in this high level description for the sake of simplicity of exposition). This behavior can be seen as the execution of tasks (sending or receiving messages), each of which is associated with a (positive or negative) cost that corresponds to the additional units of resources (local memory space) that are occupied after its execution. Receiving a message results in a positive cost, while sending a message can be viewed as effecting a negative cost. In this context, finding maximum scoring subsequences of sequences defining communications between the nodes of a network corresponds to finding the greatest buffer usage in each node [11]. Moreover, when the intention is to find an ordering for these communications with the aim of minimizing the resulting memory usage, then we are left with the problem of sorting the communications so as to minimize the maximum renewal cumulative cost.

It is mentioned in [10] that the SSS problem has been proved to be strongly NP-hard by means of a transformation from the 3-PARTITION problem. Indeed, a straightforward reduction from 3-PARTITION yields that the SSS problem remains NP-hard in the strong sense even if all negative elements in A are equal to a value $-s$ and every positive element is in a certain range depending on $s$ (more details are given in Section 5). It is known that the SSS problem becomes polynomially solvable if the negative elements are $-s$ and the positive elements are all equal to some value $s'$ [10]. In this paper, we devise a $(1 + M/\text{score}'(A))$-approximation algorithm for the SSS problem, where $M$ is the maximum element in $A$, which runs in $O(n \log n)$ time. For the general case of the SSS problem, since $\text{score}'(A) \geq M$, this algorithm has approximation factor of 2, and we show that this factor is tight. However, for the aforementioned more particular case where the elements of $A$ are bounded, the approximation factor of this same algorithm becomes $3/2$, for $n \geq 3$.

We organize the remainder of the text as follows. Section 2 states some useful properties of maximum score subsequences for later use. In Section 3 and Section 4 we then present our solutions to the ISS problem for the cases where the inserted number $x$ is negative and positive, respectively. Section 5 contains our results on the SSS problem, and Section 6 finally provides conclusions and directions for further investigations.

## 2 Preliminaries on the ISS problem

Let us establish some simple and useful properties of sequence $A$ and a subsequence $A^j_i$, for $0 \leq i \leq j \leq n$. We start with three properties that give a view of minimal (with respect to inclusion) maximum scoring subsequences. Let a prefix (suffix) of $A^j_i$ be a subsequence $A^j_{i'}$ ($A^j_{i'}$), with $i \leq i' \leq j$ ($i \leq \ell \leq j$).

**Fact 1.** If $A^j_i$ is a maximum scoring subsequence of $A$, then its prefixes and suffixes have all nonnegative scores, otherwise a larger scoring subsequence can be obtained by deleting a prefix or a suffix of negative score. Conversely, $\text{score}(X) \leq 0$, where $X$ is any suffix of $A^0_0$ or prefix of $A^n_n$, otherwise a larger scoring subsequence can be obtained by concatenating $A^j_i$ with a suffix of $A^0_0$ or prefix of $A^n_n$ of positive score.

**Fact 2.** If $A^j_i$ is a maximum scoring subsequence of $A$, then there is a maximum scoring subsequence of $A^j_i$ which is a prefix (suffix) of $A^j_i$.

The definitions in the sequel are illustrated in Figure 2. The subsequence $A^j_i$ is an interval if $\text{score}(A^j_i) < 0$ or $j = n$, and $\text{score}(A^j_i) \geq 0$, for all $i \leq j' < j$. The partition into intervals of $A$ is the concatenation $\langle I_1 = A^0_0, I_2 = A^j_{i_2}, \ldots, I_k = A^n_n \rangle$ of the maximal intervals of $A$. Such a partition is explored in Kadane’s algorithm due to the fact that a maximum scoring subsequence of $A$ is a subsequence of some of its intervals.

**Fact 3.** If $A^j_i$ is a maximum scoring subsequence of interval $I_k$ and $A^j_{i'}$ is a prefix (suffix) of $A^j_i$ such that $\text{score}(A^j_{i'}) = 0$, then $A^j_i \setminus A^j_{i'}$ is a maximum scoring subsequence of $I_k$.
Figure 2: Partition into intervals of the sequence in Figure 1. For each interval, the score of its prefixes is indicated, as well as its maximum scoring subsequence.

While the previous properties are general for every sequence, the next one is more specific to the resulting sequence of an insertion. Recall that $x$ stands for the real number given as input to the ISS problem. Assume that the insertion index $p$ is such that $i_k \leq p < j_k$, which means that $x$ is inserted in $I_k$.

**Fact 4.** The score of all elements of $I_k$ whose indices are greater than $p$ are affected by the insertion of $x$ in the following way: for every $p < q \leq j_k + 1$, $\text{score}(A(p)^q)_{i_k} = \text{score}(A(q-1)^{p})_{i_k} + x$.

This fact is the reason why the discussion of cases $x < 0$ and $x > 0$ is carried out separately in the two next sections. For the positive case, since all prefixes of $I_k$ have nonnegative scores (Fact 1), consecutive intervals may be merged in the resulting sequence, provided that $x$ is large enough to make $\text{score}(A(p)^q)_{i_k} > 0$. For instance, consider interval $I_1$ in Figure 2 The insertion of $x = 6$ at the very end of this interval (i.e., at insertion position $p = j_1 - 1 = 5$) creates the subsequence $(A_{j_1}^5, 6, -4)$ and the new interval $(A_{j_1}^5, 6, -4, I_2, I_3)$. On the other hand, for the negative case, the insertion of $x$ may split $I_k$ into two or more intervals if there exists $p \leq q \leq j_k$ such that $\text{score}(A(p)^q)_{i_k} < 0$, in which case $A(p)^q_{i_k}$ is an interval of $A(p)$ but $A(p)^q_{i_k}$ is not. Again in Figure 2 the insertion of $x = -6$ between the elements -2 and 5 of interval $I_4$ splits it into 3 intervals, namely $(2, 4, -2, -6)$, $(5, 3, 0, -6, -4)$, and $(3, 2, -4, -6)$.

### 3 Inserting $x < 0$

As already mentioned in the Introduction, solving the ISS problem when $x < 0$ corresponds to insert $x$ in some maximum scoring subsequence $A_i^j$. According to Fact 3 we assume that $A_i^j$ is minimal with respect to inclusion. What remains to be specified is the way to find an appropriate insertion index in $A_i^j$. The cases $n = 0$, $j \leq i + 1$, and $\text{score}(A_i^j) = 0$ are trivial. Then, assume that $n > 0$, $j > i + 1$, and $\text{score}(A_i^j) > 0$. Inserting $x$ inside $A_i^j$ divides the latter in its left (a prefix of $A_i^j$) and right (a suffix of $A_i^j$) parts, and different choices of $p$ may lead to different values of $A(p)$, as depicted in Figure 3. Using Fact 2 the algorithm computes the insertion index $i < p < j$ such that the maximum between $\text{score}^*(A_i^j)$ and $\text{score}^*(A_{i_k}^q)$ is as small as possible. Such a computation can be carried out by simply performing a left-to-right traversal of $A_i^j$ to compute (and store) the values of all possible prefixes of $A_i^j$, and a further right-to-left traversal to compute the values of all possible suffixes of $A_i^j$. This strategy is materialized in Algorithm INSERTIONOFNEGATIVE, which receives as input an array with the elements of $A$ and the number $x < 0$, and returns $p$ computed as above.

**Lemma 1.** Algorithm INSERTIONOFNEGATIVE($A, x$) returns an optimal insertion index, provided that $x < 0$. In addition, it runs in $O(n)$ time and space.

**Proof.** Let $p \in \{i + 1, \ldots, j - 1\}$ be the value computed by the algorithm and $p' \neq p$ be another arbitrary insertion index. We show that $\text{score}^*(A(p)) \leq \text{score}^*(A(p'))$. Let in addition $T$ be a maximum scoring
subsequence of \( A^{(p)} \), minimal with respect to inclusion. Note that \( T \neq \emptyset \) since \( \text{score}^* (A) = \text{score} (A_T^j) > 0 \). Moreover, by Fact 1, \( x \) is neither the first nor the last element of \( T \). So, let \( y \) and \( z \) be such that \( T_0^1 = \langle a_{y+1} \rangle \) and \( T_{|T|-1}^1 = \langle a_z \rangle \). The first case to be analyzed is when \( x \) is in \( T \), i.e. \( y < p < z \) (Figure 4(a)).

In this case, by Fact 1 and the minimality of \( A_T^j \) and \( T \), \( y = i \) and \( z = j \) or, in other words, \( T = \langle A^p_T, x, A^p \rangle \). The elements of \( A_T^j \) also form, perhaps with the occurrence of \( x \) at some position, a subsequence \( T' \) of \( A^{(p)} \), and since \( x < 0 \), we conclude that \( \text{score} (T') \geq \text{score} (T) \) (equality holds if \( y < p' < z \)). Then \( \text{score}^* (A^{(p)}) = \text{score} (T) \leq \text{score} (T') \leq \text{score}^* (A^{(p')}) \), as claimed.

Assume that \( p \notin \{y, \ldots, z\} \). If \( T \)’s elements also form a subsequence of \( A^{(p')} \) (more precisely, \( p' \notin \{y + 1, \ldots, z - 1\} \)), then \( \text{score}^* (A^{(p)}) = \text{score} (T) \leq \text{score} (A^{(p')}) \), as desired. Then, assume that \( p' \in \{y + 1, \ldots, z - 1\} \). If \( A_T^j \) and \( T \) are disjoint, then \( A_T^j \) is also a subsequence of \( A^{(p')} \). It turns out that \( \text{score}^* (A^{(p')}) \leq \text{score}^* (A) = \text{score} (A_T^j) \) yields \( \text{score}^* (A^{(p')}) = \text{score} (A_T^j) = \text{score}^* (A) \geq \text{score}^* (A^{(p)}) \).

Finally, we are left with the case when \( A_T^j \) and \( T \) are not disjoint (Figure 4(b)). By Fact 1 and the minimality of \( T \), either \( y = i \) or \( z = j \). Without loss of generality, let us suppose the first equality, since the other one is analogous. We have that \( \max \{ \text{score}^* (A_T^j), \text{score}^* (A^p) \} \geq \max \{ \text{score}^* (A_T^j), \text{score}^* (A^p) \} \geq \text{score}^* (A_T^j) = \text{score} (T) \). The result follows since both \( A_T^j \) and \( A^p \) are subsequences of \( A^{(p')} \).

The complexities stem directly from the facts that the algorithm employs one additional array of size \( O(n) \) (for the left-to-right traversal of \( A \)) and performs, in addition to a call to a version of Kadane’s algorithm as a sub-routine returning the indices \( i \) and \( j \) and the score of the minimal maximum scoring subsequence considered, two disjoint \( O(n) \)-time loops.

\[\square\]

4 Inserting \( x > 0 \)

The discussion in this section is based on the partition into intervals \( (I_1, I_2, \ldots, I_ℓ) \) of \( A \). For the sake of convenience, we assume that \( a_n = 0 \) (observe that this can be done without loss of generality since appending a new null element to \( A \) does not alter the scores of the suffixes of \( A \)), which means that
The first two terms in (1) and (2) indicate the best insertion index in graphically aligning the scores of the prefixes of the extended intervals with respect to the intervals of the exhaustive search takes quadratic time in the worst case. However, as depicted in Figure 5, by best interval in if
addition, write
Observation 1. Let \( a \in I_k \) be the element of indices \( j \) in \( I(k) \) and \( j' \) in \( I(k') \), \( k' \geq k + 1 \) (an assumption that is tacitly made here is that \( I_{k'} \) is a subinterval of \( I(k) \)). Then,

\[
\text{score}(I_{k}) \geq 0. \quad \text{A particularity of this positive case, which is derived from Fact 4, is the following: for every interval \( I_k \), index \( j_k - 1 \) is at least as good as any other insertion index in this interval. Thus, an optimal insertion index exists among \( j_k - 1, j_2 - 1, \ldots, j_k - 1 \), corresponding each one of these indices to one interval of the partition into intervals of \( A \). If \( p = j_k - 1 \) is chosen as the insertion index, then the resulting interval in \( A(p) \) (which may correspond to a merge of several contiguous intervals of \( A \) in the sense of Fact 4) is referred to as an extended interval, relative to \( I_k \) and denoted by \( I(k) \). If \( I_{k'} \) is one of the intervals which are merged to produce \( I(k) \), then \( I_{k'} \) is a subinterval of \( I(k) \). In the remaining of this section, we show a linear time algorithm to compute \( \text{score}^*(I(k')) \), for all \( k \in \{1, 2, \ldots \} \). Clearly, the smallest of these values is associated with the optimal insertion index for \( x \).

For each \( k \), computing \( \text{score}^*(I(k)) \) by means of Kadane’s algorithm takes \( \Theta(n) \) time. Therefore, the exhaustive search takes quadratic time in the worst case. However, as depicted in Figure 5, by graphically aligning the scores of the prefixes of the extended intervals with respect to the intervals of \( A \), one can visualize some useful observations in connection with these curves which are explored in the algorithm described in the sequel. Let the sequence of negative elements composed by intervals’ scores be denoted by \( N = \langle \text{score}(I_1), \text{score}(I_2), \ldots, \text{score}(I_t) \rangle \).

**Observation 1.** Let \( a \in I_k \) be the element of indices \( j \) in \( I(k) \) and \( j' \) in \( I_{k'} \), \( k' \geq k + 1 \) (an assumption that is tacitly made here is that \( I_{k'} \) is a subinterval of \( I(k) \)). Then,

\[
\text{score}(I(k)_j) = \text{score}(A_{h_k}) + x + a + \text{score}(A_{h_k} + j') \\
= x + \text{score}(A_{h_k} + j') + \text{score}(I(k')) \\
= x + \text{score}(A_{h_k} + j') + \text{score}(I(k'))
\]

As an example, take \( a = 4 \), \( I(k) = \langle I(1) \rangle \), and \( I_{k'} = I_4 \) in Figure 5. The equality above indicates the distance of \( I \) between the curves of \( I(1) \) and \( I_{k'} \) for the element \( 4 \in I_k \).

A first consequence of Observation 1 is a recurrence relation which is used to govern our dynamic programming algorithm. If \( k < \ell \), let \( I(k) \cap I(k+1) \) stand for the concatenation of the common subintervals of \( I(k) \) and \( I(k+1) \) (for the sake of illustration, \( I(1) \cap I(2) = \langle I_2, I_3, I_4 \rangle \) in the example of Figure 5). In addition, write \( I_{k'} \subseteq I(k) \cap I(k+1) \) to say that interval \( I_{k'} \) is a common subinterval of \( I(k) \) and \( I(k+1) \). The recurrence for \( \text{score}^*(I(k)) \) is given by

\[
\text{score}^*(I(k)) = \max \{ \text{score}^*(I_k), x + \text{score}(A_{h_k}) \}, \tag{1}
\]

if \( k = \ell \) (considering that the last element of \( A \) is null) or \( k < \ell \) and \( I(k) \cap I(k+1) = \emptyset \) or, otherwise,

\[
\max \{ \text{score}^*(I_k), x + \text{score}(A_{h_k}), x + \max_{I_{k'} \subseteq I(k) \cap I(k+1)} \{ \text{score}(N_{k-1}) + \text{score}^*(I_{k'}) \} \} \tag{2}
\]

The first two terms in (1) and (2) indicate the best insertion index in \( I_k \), while the third one in (2) gives the best interval in \( I(k) \cap I(k+1) \) (if any). The crucial point is then the computation of \( \max_{I_{k'} \subseteq I(k) \cap I(k+1)} \{ \text{score}(} \)
\( N_{k-1}^{k-1} + \text{score}^*(I_k) \) when \( I_{k+1} \) is a subinterval of \( I^{(k)} \) (i.e. \( I^{(k)} \cap I^{(k+1)} \neq \emptyset \)), which is performed in the light of the following additional observations.

**Observation 2.** Let \( a \in I^{(k)} \) be the element of indices \( j \) and \( j' \) in, respectively, \( I^{(k)} \) and \( I^{(k)}' \), \( k' \geq k + 1 \). Write \( I_v \) for the interval containing \( a \), and \( j'' \) for the index of \( a \) in \( I_v \). Assuming that \( k'' \neq k' \), then

\[
\text{score}(I^{(k)} j) - \text{score}(I^{(k)} j') = \text{score}(N_{k-1}^{k-1}) + \text{score}(I_v j') - \text{score}(N_{k-1}^{k-1}) - \text{score}(I_v j')
\]

Thus, the respective curves of \( I^{(k)} \) and \( I^{(k)}' \) remain at a constant distance for all intervals \( I_v \subseteq I^{(k)} \cap I^{(k+1)} \), \( k' \neq k + 1 \), with the curve of \( I^{(k)}' \) above that of \( I^{(k)} \).

The last observation before going into the details of the algorithm is useful to decide whether a given interval \( I_v \) is a subinterval of \( I^{(k)} \).

**Observation 3.** Observation 2 implies that if interval \( I_v \), \( k' \geq k + 1 \), is contained in \( I^{(k)} \), then \( x + \text{score}(N_{k-1}^{k-1}) \geq 0 \). The converse is also true since \( x + \text{score}(N_{k-1}^{k-1}) \geq 0 \) yields \( x + \text{score}(N_{k-1}^{k-1}) \geq 0 \), for all \( k < k'' < k' \), because all members of \( N \) are negative.

![Figure 5: Scores of prefixes of all possible extended intervals resulting from the insertion of \( x = 9 \) in the sequence in Figure 1](image)

For each interval \( I_k \), the points corresponding to \( \text{score}^*(I_k) \) and \( \text{score}^*(I^{(k)}) \) are highlighted. The last null element of the sequence is omitted.

The computation of the largest scores of prefixes of extended intervals \( I^{(k)} \) is divided into two phases. The first phase is a modification of the Kadane’s algorithm and its role is twofold. First, it determines the largest scores of prefixes of \( I_1, I_2, \ldots, I_\ell \) and, then, it sets the initial values of the arrays that are used in the second phase. Such arrays are the following:

- \( SN \) suffix sums of \( N \), i.e. \( SN[k] \) equals \( \text{score}(N_{k-1}^k) \), for all \( k \in \{1, 2, \ldots, \ell\} \). By definition, \( \text{score}(N_{k-1}^k) = SN[k] - SN[k'] \), for all \( k' \geq k \).

- \( INTSCR \) largest intervals’ scores, i.e. \( INTSCR[k] = \text{score}^*(I_k) \), for all \( k \in \{1, 2, \ldots, \ell\} \).

- \( XSCR \) for each interval \( k \in \{1, 2, \ldots, \ell\} \), this array stores the score of the subsequence ending at \( x \), provided that \( x \) is inserted in \( I_k \), i.e. \( XSCR[k] = x + \text{score}(A_{k-1}^k) \).

The second phase is devoted to the computation of the extended interval containing the best insertion position for \( x \). This is done iteratively from \( k = 1 \) until \( k = \ell \). For each \( k \), the recurrence relation is used to start the computation of \( \text{score}^*(I^{(k)}) \) and to update the maximum score of extended intervals started in previous iterations as described in Algorithm 1. Such information is stored as follows. The
Algorithm 1: Second phase for the case \( x > 0 \)

**Input:** Arrays \( SN, INTSCR, \) and \( XSCR \) computed in the first phase

**Output:** An optimal insertion interval for \( A \)

1. \( k \leftarrow 1 \)
2. \( EXTSCR[k] \leftarrow \max\{INTSCR[k], XSCR[k]\} \)
3. \( Q \leftarrow 1 \)
4. \( INTQ[Q] \leftarrow k \)
5. for \( k \leftarrow 2, \ldots, \ell \) do
6. \( DIST \leftarrow x + SN[INTQ[Q]] - SN[k] \)
7. while \( DIST \geq 0 \) and \( DIST + INTSCR[k] > EXTSCR[INTQ[Q]] \) do
8. \( EXTSCR[INTQ[Q]] \leftarrow DIST + INTSCR[k] \)
9. if \( Q > 1 \) and \( EXTSCR[INTQ[Q]] \geq EXTSCR[INTQ[Q - 1]] \) then
10. \( Q \leftarrow Q - 1 \)
11. \( DIST \leftarrow x + SN[INTQ[Q]] - SN[k] \)
12. \( EXTSCR[k] \leftarrow \max\{INTSCR[k], XSCR[k]\} \)
13. if \( EXTSCR[k] < EXTSCR[INTQ[Q]] \) then
14. \( Q \leftarrow Q + 1 \)
15. \( INTQ[Q] \leftarrow k \)
16. return \( INTQ[Q] \)

array \( EXTSCR \) contains the maximum scores of prefixes of the extended intervals \( I^{(k)} \), for all \( k' \in \{1, 2, \ldots, k\} \). The intervals with best prefix scores obtained so far are kept in the queue \( INTQ \). \( Q \) is the rear of the queue \( INTQ \), initialized at 0.

The correctness of the two-phase algorithm stems from the following lemma.

**Lemma 2.** For every iteration \( k \) (just before execution of line 5 of Algorithm 1), let \( I_k \) be an interval and \( k'' = INTQ[Q] \). Then, the following conditions hold:

1. \( EXTSCR[k''] = \text{score}^*(I^{(k''\setminus A_{j_k^k})}) \);
2. if \( Q > 1 \) and \( k' \) appears in \( INTQ \) but \( k'' \neq k' \), then \( k' < k'' \) and \( \text{score}^*(I^{(k''\setminus A_{j_k^k})}) < \text{score}^*(I^{(k\setminus A_{j_k^k})}) \); and
3. if \( k' < k \) does not appear in \( INTQ \), then \( k'' \) is such that \( \text{score}^*(I^{(k''\setminus A_{j_k^k})}) \leq \text{score}^*(I^{(k\setminus A_{j_k^k})}) \).

**Proof.** By induction on \( k \). For \( k = 1 \), condition 1 holds trivially due to line 2 while conditions 2 and 3 hold by vacuity. Let \( k > 1 \). We need to analyze the changes in \( INTQ \). We start with the intervals that are removed from \( INTQ \). At line 6 Observation 1 is used to compute the distance between the curves of \( I^{(k)} \) and \( I_k \). If this distance is negative, then \( I_k \) is not a subinterval of \( I^{(k)} \). Otherwise, condition 1 of the induction hypothesis is used in the comparison of line 7 and \( EXTSCR[k''] \) is updated at line 8 according to 2. So, condition 1 remains valid for \( k \) up to this point of the execution. If \( EXTSCR[k''] \) increases (i.e., line 8 is executed), then Observation 2 and condition 2 of the induction hypothesis are evoked to remove \( I^{(k')} \) from the queue respecting condition 3 in case a point of \( I_k \) in the curve of \( I^{(k)} \) overcomes that of an interval that precedes \( I_k \). \( EXTSCR[k''] \) is updated again according to 2 in order to satisfy condition 1. This procedure is repeated until condition 2 is valid for the intervals still in \( INTQ \).

Finally, lines 12–15 correspond to the insertion in \( INTQ \). The maximum score of the prefix of \( I^{(k)} \) containing \( I_k \) and \( x \) only is updated at line 12 and \( I^{(k)} \) enters the queue only if such maximum score is below the maximum score of the prefix of \( I^{(INTQ[Q])} \) considered so far. This implies that conditions 2 and 3 are also valid for \( k \). □
Theorem 1. The ISS problem can be solved in $O(n)$ time and space.

5 Sorting

We now turn our attention to the SSS problem. Its hardness is analyzed considering the following derived problem.

Restricted version of the SSS problem: we denote by SSS($k, s$) the restricted version of the SSS problem where, for some two positive integers $k$ and $s$, $n = 4k - 1$, the elements in $A$ are integers bounded by a polynomial function of $k$, $k - 1$ elements are negative, every negative element is equal to $-s$, every positive element $a_i$ is such that $s/4 < a_i < s/2$, and $\text{score}(A) = s$.

A consequence of the fact that sorting a sequence is similar to accommodate the positive elements in order to create an appropriate partition into intervals leads to the following result.

Theorem 2. The SSS($k, s$) problem is strongly NP-hard.

Proof. By reduction from the 3-PARTITION decision problem, stated as follows: given $3k$ positive integers $a_1, \ldots, a_{3k}$, all polynomially bounded in $k$, and a threshold $s$ such that $s/4 < a_i < s/2$ and $\sum_{i=1}^{3k} a_i = ks$, there exist $k$ disjoint triples of $a_1$ to $a_{3k}$ such that each triple sums up to exactly $s$? The 3-PARTITION problem is known to be NP-complete in the strong sense \cite{12}.

Given an instance $C$ of the 3-PARTITION problem, an instance of the SSS($k, s$) problem is defined by an arbitrary permutation $A$ of the multiset $C'$ obtained from $C$ by the inclusion of $k - 1$ occurrences of $-s$. A solution for the SSS instance is to choose elements of $C$ for each negative element of $C'$, which gives a partition of $C$. Since $a_i > s/4$, for all $i \in \{1, \ldots, 3k\}$, every sequence of $4$ positive elements chosen from $C'$ has value greater than $s$. Thus, $C$ is a “yes” instance of the 3-PARTITION problem if and only if there exists a permutation $A'$ of $A$ such that $\text{score}^*(A') = s$.

We show in the sequel that Algorithm 2 is a parameterized approximation algorithm for the SSS problem. Such an algorithm builds a permutation of $A$ keeping the maximum scoring subsequence of all intervals, except the last one, bounded by the input parameter plus the largest element of $A$. For the last interval, the following holds for every sequence $A$.

Observation 4. If $N = \langle \text{score}(I_1), \text{score}(I_2), \ldots, \text{score}(I_l) \rangle$ is the sequence of negative elements composed by intervals’ scores, then $\text{score}(N^l_{l-1}) = \text{score}(A) - \text{score}(N^l_{l-1})$. Considering that $I_i$ is a subsequence of $A$ and that $\text{score}(N^l_{l-1}) < 0$, we conclude that $\text{score}(N^l_{l-1})$ is a lower bound for $\text{score}^*(A)$ at least as good as $\text{score}(A)$.

Algorithm 2 gets as input, in addition to the instance $A$ (with size $n$), the parameter $L$, which depends on $M = \max\{0, \max_{a \in A} a\}$. A variable $S$ is used to keep the score of the interval being currently constructed. Just after step 10 is executed, it turns out that $L + M \geq S \geq L$. On the other hand, execution of step 14 leads to $S \leq L$ or includes all remaining negative elements in $A'$. Moreover, if $S + \text{score}(Q) + \text{score}(R) < 0$, then a new interval $I_k$ is established and $S$ is incremented by $-\text{score}(I_k)$ (and becomes 0). A straightforward consequence is that $\text{score}^*(A') > L + M$ only if step 14 is executed with positive elements of $A$, and this due to the last interval (in the sense of Observation 4). This leads to the following result.

Lemma 3. Let $A$ be an instance of the SSS problem, $A'$ be the sequence returned by the call $\text{PARAMETERIZEDSORTING}(A, L)$, for some $L \geq M$, and $N^l_{l-1}$ be the sequence of the $l$'th interval scores of $A'$. Then,

$$\text{score}^*(A') \leq \max\{L + M, \text{score}(N^l_{l-1}) = \text{score}(A) - \text{score}(N^l_{l-1})\}. \quad (3)$$

Moreover, $\text{PARAMETERIZEDSORTING}(A, L)$ runs in $O(n)$ time.
Algorithm 2: \textsc{ParametrizedSorting}(A, L)

\textbf{Input}: an array $A$ of $n \geq 0$ numbers and a parameter $L \geq M$

\textbf{Output}: an array $A'$ containing a permutation of $A$

1. Let $A'$ be an array of size $n$
2. Let $A^- \subseteq A$ and $A^+ \subseteq A$ be the sequences of negative and nonnegative members of $A$, respectively
3. $j \leftarrow 1$
4. $S \leftarrow 0$
5. \textbf{while} $A^- \neq \emptyset$ \textbf{and} $A^+ \neq \emptyset$ \textbf{do}
6. \hspace{1em} Let $Q$ be a sequence of elements of $A^+$ such that $L \leq S + \text{score}(Q) \leq L + M$, if one exists, or $Q = A^+$ otherwise
7. \hspace{1em} Assign the elements of $Q$ to $A'[j \ldots j + |Q| - 1]$
8. \hspace{1em} $j \leftarrow j + |Q|$
9. \hspace{1em} $S \leftarrow S + \text{score}(Q)$
10. \hspace{1em} $A^- \leftarrow A^+ \setminus Q$
11. \hspace{1em} Let $R$ be a minimal sequence of elements of $A^-$ such that $S + \text{score}(R) < L$, if one exists, or $Q = A^-$ otherwise
12. \hspace{1em} $S \leftarrow \max\{0, S + \text{score}(R)\}$
13. \hspace{1em} Assign the elements of $R$ to $A'[j \ldots j + |R| - 1]$
14. \hspace{1em} $j \leftarrow j + |R|$
15. $A^- \leftarrow A^- \setminus R$
16. Assign the elements of $A^- \cup A^+$ to $A'[j \ldots |A^- \cup A^+| - 1]$
17. \textbf{return} $A'$

The key of our approximation algorithm is to provide Algorithm \textsc{ParametrizedSorting} with an appropriate lower bound parameter. The most immediate one is $L = \max\{M, \text{score}(A)\}$, which, however, does not capture the contribution of the negative members of $A$ whose values are smaller than $-L$ when $A$ contains at least one nonnegative element. In order to circumvent this difficult case of Lemma 3, assume that $A^*$ is an optimum solution, with $N^*$ being the sequence of $\ell^*$ scores of the corresponding partition into intervals, and $OPT = \text{score}^*(A^*)$. According to (3), we need to find a new value for $L$ such that $\text{score}(N^*_{\ell^*-1}) \leq L \leq OPT$, being $I_{\ell^*}$ the last interval of the sequence $A'$ returned by \textsc{ParametrizedSorting}(A, L), with the purpose of having $\text{score}^*(A') \leq OPT + M$.

\textbf{Lemma 4.} Let $x$ be a real number and

$$b(x) = \text{score}(A) + \sum_{a \in B_x} (-a - x),$$

where $B_x = \{a_i \in A | a_i < -x\}$ (note that $B_x$ is a multiset). Then, $x \geq b(x)$ implies $b(x) \leq OPT$.

\textbf{Proof.} By contradiction, assume that $x \geq b(x)$ and $b(x) > OPT$. Since $x > OPT$, we get $B_x \subseteq B_{OPT}$. In addition, Observation 4 gives

$$OPT \geq \text{score}(A^*) - \text{score}(N^*_{\ell^*-1}) \geq \text{score}(A) + \sum_{a \in B_{OPT}} (-a - OPT) \geq \text{score}(A) + \sum_{a \in B_x} (-a - x) = b(x),$$

which contradicts the assumption $b(x) > OPT$. \hfill \Box

Based on lemmata 3 and 4, we define the two-phase Algorithm \textsc{ApproxSorting}. Its first phase consists in determining the largest $B_L$ satisfying $L \geq M$ (Lemma 3) and $L \geq b(L)$ (Lemma 4). Set
\[ \text{L}_0 = \max\{M, \text{score}(A)\} \] and take the elements of a decreasing sequence \( P \) on the set \( \{-a \mid a \in A, a < -L_0\} \cup \{L_0\} \) (note that, by definition, all elements of \( P \) are distinct). Write this sequence as \( P = \langle p_0, p_1, \ldots, p_{|P|-1} \rangle \), which means that \( B_{p_0} = \emptyset \) and \( b(p_0) = \text{score}(A) \). Then, find the maximal index \( i \) (in the range from 0 to \( |P| - 1 \)) such that \( i = 0 \) or \( b(p_i) < p_{i-1} \). It is worth mentioning that we can have \( b(p_i) < L_0 \) when \( \text{score}(A) < M \).

The second phase is simply a call \textsc{ParametrizedSorting}(\( A, L = \max\{L_0, b(p_i)\} \)).

**Theorem 3.** \textsc{ApproxSorting} is a 2-approximation algorithm for the SSS problem and a 3/2-approximation algorithm for the SSS(\( k, s \)) problem which runs in \( O(n \log n) \) time.

**Proof.** First we show that \( L = b(p_i) \leq \text{OPT} \) (the case \( L = L_0 \) is trivial). If \( P = \langle 0 \rangle \), then \( M = 0 \) and \( \text{OPT} = 0 \). In this case, \( L = \text{OPT} = 0 \). Otherwise, there are two subcases. If \( i = 0 \), then \( b(p_0) = \text{score}(A) \leq L_0 \leq \text{OPT} \). On the other hand, if \( i > 0 \), then, by Lemma 4 it is sufficient to show that \( b(p_i) < p_{i-1} \) yields \( p_i \geq b(p_i) \) or \( b(p_i) = b(b(p_i)) \). This implication holds since if \( p_i < b(p_i) < p_{i-1} \), then \( B_{p_i} = B_{b(p_i)} \).

Let \( A' \) be the permutation of \( A \) produced by \textsc{ApproxSorting}(\( A \)), with partition into intervals \( \langle I_1', I_2', \ldots, I_m' \rangle \). Since each interval \( I_k' \) having \( \text{score}(I_k') < 0 \) has an \( a \in B_L \) as last element, we get \( \text{score}(I_k') \geq a + L \). It turns out that \( \text{score}(N_{-a-1}^0) \geq \sum_{a \in B_L (a + L) = \text{score}(A) - b(L) \geq \text{score}(A) - L \). Observation 4 leads to \( \text{score}(N_{-a-1}^0) \leq L \). Therefore, Lemma 3 gives that \( \text{score}^*(A') \leq L + M \).

The approximation factors stem directly from Lemma 3 and \( M \leq \text{score}^*(A) \). In special, for the SSS(\( k, s \)) case, the first phase of Algorithm \textsc{ApproxSorting} obtains \( L_0 = \max\{\text{score}(A) = s, M < s/2\} = s \) and, by the definition of \( L, L = s \). This leads to the approximation factor \( (L + M)/L < (s + s/2)/s = 3/2 \).

Finally, the time complexity is due to the construction of the sequence \( P \) (notice that the search for \( p_i \) in \( P \) can be easily done in linear time).

A final remark that can be made in connection with algorithm \textsc{ApproxSorting} is that the approximation factor of 2 is tight. To see this, consider \( x > 0 \) and \( x/2 < y < x \). The sequence \( A \) returned by the call \textsc{ParametrizedSorting}(\( \langle y, -x, y, -x, x, y \rangle, x \)) is either \( \langle y, y, -x, x, -x \rangle, \) or \( \langle y, x, -x, y, -x \rangle, \) or \( \langle x, -x, y, y, -x \rangle \). It follows that \( 2y \leq \text{score}^*(A) \leq y + x \). Then, since \( \text{OPT} = x, \frac{\text{score}^*(A)}{\text{OPT}} \to 2 \) as \( x - y \to 0 \).

# 6 Concluding remarks

We motivated two problems related to maximum scoring subsequences of a sequence, namely the Insertion in a Sequence with Scores (ISS) and Sorting a Sequence by Scores (SSS) problems. For the ISS problem, we presented a linear time solution, and for the SSS one we proved its NP-hardness (in the strong sense) and gave a 2-approximation algorithm.

The SSS problem is also closely related to another set partitioning problem, called Multicore Processor Scheduling problem, stated as follows: given a multiset \( C \) of positive integers and a positive integer \( m \), find a partition of \( C \) into \( m \) subsets \( C_0, C_1, \ldots, C_{m-1} \) such that \( \max_{i \in \{0,1,\ldots,m-1\}} \{\sum_{a \in C_i} a\} \) is minimized. Not surprisingly, given an instance \( (C, m) \) of \textsc{Multicore Processor Scheduling}, an instance of the SSS problem can be defined as an arbitrary partition \( A \) of the multiset \( C' \) obtained from \( C \) by the inclusion of \( m - 1 \) occurrences of the negative integer \( -\text{score}(C) - 1 \), indicating that a solution of the SSS problem for \( A \) induces a solution of the \textsc{Multicore Processor Scheduling} problem for \( (C, m) \). This problem admits a polynomial time approximation scheme (PTAS) [13][14] as well as list scheduling heuristics producing a solution which is within a factor of \( 2 - 1/n \) (being \( n \) the number of elements in the input multiset \( C \)) from the optimal [15]. On the other hand, MAX-3-PARTITION, the optimization version of the problem used in the proof of Theorem 3 is known to be in APX-hard [16]. A natural open question is, thus, whether there exist a polynomial time approximation algorithm with factor smaller than 2 for the SSS problem. In this regard, note that although transferring our approximation factor from the SSS problem to the \textsc{Multicore Processor Scheduling} problem is easy, the converse appears
harder to be done, since we do not know in advance how many intervals there should be in an optimal permutation $A'$ of $A$.

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**References**

[1] J. Bentley. Programming pearls: algorithm design techniques. *Communications of the ACM*, 27(9):865–873, 1984.

[2] D. Gries. A note on a standard strategy for developing loop invariants and loops. *Science of Computer Programming*, 2(3):207–214, 1982.

[3] M. Csűrős. Maximum-scoring segment sets. *IEEE/ACM Transactions on Computational Biology and Bioinformatics*, 1:139–150, 2004.

[4] W. L. Ruzzo and M. Tompa. A linear time algorithm for finding all maximal scoring subsequences. In *Proc. of International Conference on Intelligent Systems for Molecular Biology*, pages 234–241, 1999.

[5] H.-K. Dai and H.-C. Su. A parallel algorithm for finding all successive minimal maximum subsequences. In J. Correa and A. Hevia M. Kiwi, editors, *LATIN 2006: Theoretical Informatics*, volume 3887 of *Lecture Notes in Computer Science*, pages 337–348. 2006.

[6] C. E. R. Alves, E. N. Cáceres, and S. W. Song. A BSP/CGM algorithm for finding all maximal contiguous subsequences of a sequence of numbers. In W. E. Nagel, W. V. Walter, and W. Lehner, editors, *Proc. of Euro-Par*, volume 4128 of *Lecture Notes in Computer Science*, pages 831–840, 2006.

[7] S. E. Bae and T. Takaoka. Algorithms for the problem of $K$ maximum sums and a VLSI algorithm for the $K$ maximum subarrays problem. In *7th International Symposium on Parallel Architectures, Algorithms and Networks (I-SPAN 2004)*, pages 247–253, 2004.

[8] G. Brodal and A. Jørgensen. A linear time algorithm for the $k$ maximal sums problem. In L. Kucera and A. Kucera, editors, *Mathematical Foundations of Computer Science 2007*, volume 4708 of *Lecture Notes in Computer Science*, pages 442–453. 2007.

[9] G. Brodal and A. Jørgensen. Selecting sums in arrays. In S.-H. Hong, H. Nagamochi, and T. Fuku-naga, editors, *Algorithms and Computation*, volume 5369 of *Lecture Notes in Computer Science*, pages 100–111. 2008.

[10] Li-Hui Tsai. Sequencing to minimize the maximum renewal cumulative cost. *Operations Research Letters*, 12(2):117–124, 1992.

[11] F. R.J. Vieira, José F. de Rezende, V. C. Barbosa, and S. Fdida. Scheduling links for heavy traffic on interfering routes in wireless mesh networks. *Computer Networks*, 56(5):1584 – 1598, 2012.

[12] Michael R. Garey and David S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, New York, NY, USA, 1979.

[13] D. S. Hochbaum and D. S. Shmoys. Using dual approximation algorithms for scheduling problems: Theoretical and practical results. *Journal of the Association for Computing Machinery*, 34(1):144–162, 1987.
[14] H. Kellerer, U. Pferschy, and D. Pisinger. *Knapsack problems*. Springer, 2004.

[15] R. L. Graham. Bounds on multiprocessing timing anomalies. *SIAM Journal on Applied Mathematics*, 17(2):416–429, 1969.

[16] A. Feldmann and L. Foschini. Balanced partitions of trees and applications. In C. Dürr and T. Wilke, editors, *Proceedings of the 29th Symposium on Theoretical Aspects of Computer Science (STACS’12)*, Leibniz International Proceedings in Informatics, pages 100–111, 2012.