Categorical Foundations for Physics - I: Program at a Glance

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Measures in the context of Category Theory lead to various relations, even differential relations, of categories that are independent of the mathematical structure forming objects of a category. Such relations, which are independent of mathematical structure that we may represent a physical body or a system of reference with, are, precisely, demanded to be the Laws of Physics by the General Principle of Relativity. This framework leads to a theory for the physical entirety.

I. INTRODUCTION

We describe here a program of the categorical basis for Physics. The overall motivation for this program arises out of following considerations.

As Physics is our attempt to conceptually grasp the happenings around us, and as Mathematics is a language for succinctly expressing the associated conceptions, the most general Mathematics would help us express the most general of such conceptions. As a result, we would hopefully encompass the entirety of physical phenomena within any such most general description.

This point of principle was advocated along with the Universal Theory of Relativity whose developmental stages can be found in [1]. We also note that this universal relativity is based on Einstein’s general principle of relativity that the Laws of Physics have the same mathematical form irrespective of (the state of motion of) the system of reference, any physical body. In difference with other attempts [2, 3] at providing a foundation for the physical entirety, we seek to base universal relativity on the most general categorical considerations for reasons arising out of this point of principle.

Then, in summary of the contents of [1], what is really needed is a way of dealing with and extracting the needed “information” from the most general category because it is the currently known most general mathematical structure. In this context, an appropriate notion of (real-valued or not) measures [1, 4] over a generic category is the way.

A. General Description of This Program

The following part of Introduction specifically keeps in sight the purpose of Mathematics for Physical Theories, rather than losing it with details of mathematical considerations. In what follows, we therefore discuss relevant conceptions without precise mathematical details, which follow the end of this general discussion of the involved issues.

To understand Nature, we “associate” a mathematical structure, let it be any, with physical bodies, and study changes in that mathematical structure to “model” changes in physical bodies. When observations of Nature or results of concerned experiments agree with the “predictions” of our model, we claim to successfully “explain” the associated observable physical phenomena.

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Using different mathematical structures to represent physical bodies, we also construct various such models, and compare them. Then, a mathematical model explaining the largest body of observational data with the least number of associated physical conceptions is our fundamental understanding about Nature. We also advance, mathematically easy, models that “approximate” the fundamental model. The grand dream of physicists is of a fundamental mathematical model, the Theory of Everything, which explains all observations of Nature - the physical entirety.

As is quite well known, Newtonian theories, even though they were quite successful in explaining many observations of Nature, failed to “naturally” explain observations related to electromagnetic radiation, and various (quantum) experiments.

Einstein, in a remarkable insight, realized that Newtonian theories were all based on the Special Principle of Relativity. By additional principle of the constancy of the speed of light for inertial systems of reference, he then formulated his Special Relativity. Newtonian model is an approximation of this Special Relativity. Successes of Special Relativity then convinced Einstein that the “strategy” of “Relativity” for formulating model of the Physical World is truly an appropriate one.

As a straightforward step, Einstein then proposed the General Principle of Relativity, a statement of “strategy” that the Laws of Physics be such that they have the same (mathematical) form irrespective of the state of (motion of) the system of reference, any physical body. With this strategy, Einstein, likewise with Newton, aimed at a comprehensive theoretical description of the physical entirety.

But, “changes” to mathematical structure representing physical bodies also “mean” changes to reference physical bodies. The (General or Universal) Principle of Relativity then requires those mathematical laws, to be the Laws of Physics, which do not depend on even the mathematical structure that we may associate with a physical body. This is then “the” meaning of General Covariance.

Now, any mathematical structure can be an “object” or an “element” of a collection of similar mathematical structures. For example, a group, a “structured” set, can be an element of a collection of all groups. Of course, we then have to avoid set-theoretic paradoxes, like Russell’s paradox, while making such collections. This is always achievable by adopting a suitable definition of a set such that a collection of all sets is not a set, but what we “name” as a class. Furthermore, the collection of all classes is also not a class, but what we “name” as a conglomerate; and so on. With these definitions, Russell’s paradox is then a harmless statement that a “collection” of all elements, gathered according to a defining property, does not have the “defining” property of its elements.

A change of one mathematical structure to another of its collection is called as an “arrow” connecting “objects” (of that collection). An arrow then has a source (domain) and a target (co-domain) as objects. Every arrow (relation) need not be a function in the mathematical sense. An identity arrow of an object is then an identity transformation of the associated mathematical structure.

Mathematically, the collection of all such arrows forms a partial binary algebra, and with, remarkably naturally arising, additional compatibility properties, a Category. It is a mathematical structure completely specified by only the arrows. For a category, we also form sub-collections, called as the hom-collections, of all the arrows from one to another of its objects. This categorical structure is then quite “separate” from the issues of the set theory.

For example, a category of a group has only one object and group elements as arrows. Every arrow of a group category is an iso-arrow (isomorphism) of the only object of the group category, for an inverse of every group element is a member of the group. A category of a monoid also has only one object and monoid elements as arrows. But, only the monoid identity is a n iso-arrow of the only object of this category, for the inverses of general monoid elements are not members of the monoid.

The object-free structure of a category is then the fundamental mathematical structure of all the mathematical structures, as we can always form a collection of mathematical structures similar to any chosen one and consider changes of one to others of that collection as a category.

Different categories can now be “related” to each other by functors, which are the partial binary algebra preserving maps that also preserve identities and compositions of arrows. Functors from one to another category always exist.

Notably, categorical can then be the mathematical structure of the collection of all categories, when we form the category of all categories, for a category is a mathematical structure with a functor changing it to another similar structure.

Now, we can “reorganize” arrows of any category in different collections that can themselves be treated as (new) arrows, and by ensuring that the conditions of the definition of a category are satisfied, form another category. “New” categories from “known” categories are then obtainable.
Functors connecting two (fixed) categories can also be collected to form a partial binary algebra, and a category, called as a functor category, is also obtainable with “natural transformations” or “functor morphisms” as arrows connecting such functors.

Now, in the setting of the standard set theory, a set-theoretic (standard or usual) measure is a set-function that forms a commutative monoid or an abelian group of addition over (countable) collections of pairwise mutually disjoint sets.

Fundamental to set-theoretic measures are the additivity of measures and the pairwise disjointedness of involved sets. It is only when these hold that a measure is a unique association of a set with an element of the (commutative monoid or) abelian group of addition. We then use “other” properties of the monoid or group of addition, for example, topological ones, to formulate our (general) notions of the continuity, the differentiability, the metric etc.

For an abelian group of addition, the addition function, denoted by +, is defined over the arrows with the same source and the same target, and the composition of arrows of this group category is left and right distributive over +, with the identity element of the group of addition behaving as a “zero arrow” of this category with only one object. A category with this additive structure is an additive category. An abelian group and a commutative monoid of addition are its obvious examples.

In general, a category need not possess an “additive structure” over its collection of arrows, “addition” being a very specific function on the collection of categorical arrows. Furthermore, no natural notion of the complement of a categorical object is available in a general category, even though sub-object of an object corresponds to a subset of a set. Such difficulties had prevented any definition of measures in the general categorical context, till the work in [4], whose strategy for defining measures on a general category is what is described below.

A general category, a collection of arrows, always has a sub-collection of identity arrows of its partial binary algebra of arrows. Then, we can always form sub-collections (families) of the objects (identity arrows) of any category. Such a sub-collection can also be empty, i.e., we can have an empty family. Intuitively, when we “combine” families to form a larger family, we have the conception of “addition” implicit in this operation.

Then, if we “reorganize” the arrows of a category in conjunction with the formation of the families of its objects, we obtain an additive structure of a commutative monoid over the hom-collections of “new” arrows connecting the families.

Now, co-product is a categorical generalization of disjoint union of sets. In forming families of objects of any category, we also generate “objects” (families themselves) that are always the co-products of some other objects (families) of their category. Then, a specific additive category, to be called as a pointed family category of a given category, can always be obtained from any category by forming families of its arrows, and by forming correspondingly “new” arrows for these families.

Next, the functor category of functors from the pointed family category of a given category to any additive category has the additive structure of the latter category. Then, the functor category of functors from the pointed family category of a category to the category of a group (monoid) of addition of real numbers has the additive structure of a group (monoid). It is a unique association of an object of the former category with an element of the concerned group (monoid). Such functors, henceforth to be called as measure functors, are then (additive category-based) categorical measures. Such categorical measures are evidently definable for every category.

Notice here that the category of (group or monoid of) addition of real numbers can also be replaced with any additive category. Because functors map identities to identities, notice furthermore that any measure functor assigns identity arrows of all the objects of the pointed family category of a given category to only one arrow, the identity arrow, which is also the only zero arrow, of the additive category. Moreover, notice that functors preserve compositions of arrows.

Measure functors that are “naturally isomorphic” to each other form an equivalence class within the collection of all the measure functors acting on a given category. All members of this equivalence class provide an association of the “same” arrow of the pointed family category of a category with the “same” arrow of the additive category. Therefore, we need to consider a measure functor, categorical measure, only modulo its equivalence class.

Now, a given category is “embedded” in its pointed family category as a sub-category, and an “inclusion functor” describes this embedding. (It is a one-one association of the arrows of the involved categories.) Then, a composition of inclusion functor and a measure functor is a functor, call it an m-i functor (notice that an inclusion functor acts first, and then a measure functor), from that given category to the additive category. Then, such a composition of the involved functors, an m-i functor is also an assignment of a
(unique) element of additive category with an object of that given category, it being independent of the mathematical structure of the objects of that category.

An object of a category can itself be viewed as a category. Then, there are “internal” measures as well as “external” measures definable for it. Notice however that, as a collection of measures definable for a category, the internal and external measures belong to the same collection. [That these measures belong to the same collection can be expected, for example, from the fact that we can consider distance between two physical bodies, as well as distance between “parts” internal to any physical body. These are essentially the “same” mathematical notions, except that we call one as “external” and the other as an “internal” distance (measure). The same applies to other notions. Distinction between internal and external measures is then a matter of our convenience.]

Consider now an endo-functor, \(ie\), a functor from a given category to itself. Notice here that an endo-functor acts first, then the inclusion functor and, in the end, the measure functor. Endo-functors of any given category can then be classified as “measure-preserving” and “measure non-preserving” ones, obviously, relative to the measure functor under considerations. Then, the same endo-functor need not be preserving all the measures of its category.

Now, topological structure of the additive (group or monoid) category (of real numbers) permits definitions of continuity, differentiability, metric, distance · · · Essentially, categorical measures can be varied as per these notions, which are available for every category. This is, quite naturally, as it is with the usual or standard theory of measures.

An interplay of measure-preservation and non-preservation now emerges as the mathematical basis underlying physical phenomena, when we associate measures with various of our physical conceptions. For example, consider a positive real-valued “external” measure to “correspond” to our notion of “physical distance” between bodies. (We use topological properties of the additive category while making any such correspondence.) Mathematical law of measure non-preservation under the action of a categorical endo-functor would then correspond to change in this physical distance between corresponding physical bodies. Such a law would, clearly, be a “differential” law, because the topological structure of the additive category can be used to write the action of an endo-functor in that manner.

Now, any such “categorical differential structure” too is independent of the mathematical structure of the objects of a category. This is [9] then the precise “mathematical reason” as to why the physical world is well describable using differential equations.

Thus, we obtain categorical interrelationships that are, and only such interrelationships can perhaps be, “free” of mathematical structures forming a category. It may then be emphasized that these interrelations arising out of the considerations of categorical measures are functorial in character. This is the primary reason for these categorical interrelationships to be entirely independent of the mathematical structure of the objects forming a category.

These categorical “relations” are then exactly what we needed to mathematically implement Einstein’s General Principle of Relativity, for these are independent of the mathematical structure that we associate with any physical body, and therefore, free of any physical system of reference whatsoever. Clearly, such “relations” (of Universal Relativity) constitute then the most general Laws about Nature [9]. This is also the reason why we can expect such laws to encompass the physical entirety.

Now, the notion of time is that of the periodic motion of a “clock” body relative to an “observer” body. Within the present categorical framework, this notion of “time” is obtained, using the notion of “distance” in this context, from the “periodic” variations of the “location” of one (categorical) object, the “clock” body, in relation to any other (categorical) object, the “observer” body. [Notably, there does not exist any other notion of “time” within this general categorical context.] Categorical “velocity”, “acceleration”, etc. then use such a notion of time.

Of categorical measures is then also the mathematical way using which we can [9] now “understand” how one model of Nature “approximates or not” another model, because measures provide mathematically precise sense of “closeness” of models.

With this introduction, we now turn to mathematical considerations of this program for the categorical basis of the physical entirety.

In Section III we consider measures in the categorical context. In Section III we first work out an explicit example of the physical distance measure to establish the use of present framework for physical considerations. We also briefly mention the general procedure for obtaining the Laws of Physics from the considerations of categorical measures. But, the general procedure of using categorical measures for assigning properties to physical bodies is quite involved. It therefore deserves separate presentation [9]. We end this article with some remarks in Section IV.
II. CATEGORICAL MEASURES

A fundamental property of standard or usual or set-theoretic measures is their (countable) additivity. Categorically, only the “additivity” is defined as follows:

**Definition 1** A monoid (group) additivity structure on any pointed category $\mathcal{A}$ is a function $+$ that associates with each pair $A \rightarrow B$ of $\mathcal{A}$-arrows with common source (or domain) $A$ and common target (or co-domain) $B$, another $\mathcal{A}$-arrow, denoted by $g + h : A \rightarrow B$ or by $A \xrightarrow{g + h} B$, such that

(M1) for each pair $(A, B)$ of $\mathcal{A}$-objects, the function $+$ induces on the corresponding hom-collection $[A, B]$ a commutative structure of a monoid (an abelian group) of addition,

(M2) the composition of arrows in $\mathcal{A}$ is left and right distributive over $+$, ie, whenever $C \xrightarrow{k} A \xrightarrow{g} B \xrightarrow{h} D$ are $\mathcal{A}$-arrows, then $f \circ (g + h) = (f \circ g) + (f \circ h)$ and $(g + h) \circ k = (g \circ k) + (h \circ k)$. That is, the composition functions $[B, C] \times [A, B] \rightarrow [A, C]$ are bilinear.

(M3) The zero arrows of $\mathcal{A}$ act as monoid (group) identities with respect to $+$, ie, for each $\mathcal{A}$-arrow $f$, $0 + f = f + 0 = f$. That is, the identity elements of the monoids (abelian groups) behave as zero arrows whenever the compositions are defined.

The category $\mathcal{C}$ of the commutative group as well as the category $\mathcal{M}$ of a commutative monoid are obvious examples of categories having the above additive structure(s). In general, we will denote by $\mathcal{A}$ any category possessing a (monoid or group) additive structure.

To “construct” an appropriate category $\text{Family}(\mathcal{C})$ of the families of arrows of a generic category $\mathcal{C}$, let $A_I$ be the family $(A_i)_{i \in I}$ of objects of $\mathcal{C}$, indexed by some collection $I$. With families $A_I$ as objects, an arrow from $A_I$ to $B_J$ is then a pair $(f, i)$ with $f : I \rightarrow J$ as a map of collections and $i$ as a family $\{ A_i \}_{i \in I}$ of arrows in $\mathcal{C}$. For arrows $(f, i) : A_I \rightarrow B_J$ and $(g, j) : B_J \rightarrow C_K$ in the category $\text{Family}(\mathcal{C})$, the composition $(g, j) \circ (f, i)$ is defined as the arrow $(h, \eta) : A_I \rightarrow C_K$ such that $h = gf$ and $\eta_i = g(f(i))$. An identity arrow for $A_I$ in $\text{Family}(\mathcal{C})$ is the identity map $\text{id}_I : I \rightarrow I$ and the family, $\{ \text{id}_{A_i} \}_{i \in I}$, of identity arrows for $\mathcal{C}$-objects $A_i, i \in I$. By this construction, any category $\mathcal{C}$ is always fully embedded in the category $\text{Family}(\mathcal{C})$, its family category, as a sub-category.

Every object $A_I$ of the category $\text{Family}(\mathcal{C})$ is a co-product of its constituent objects $\{ A_i \}_{i \in I}$ (viewed as one-member families) with $1_i = (i, 1_{A_i}) : A_i \rightarrow (A_i)_{i \in I}$ being co-product injections. An initial object of $\text{Family}(\mathcal{C})$ is an “empty” family, while its terminal object is the “singleton” family, and these two are distinct objects of category $\text{Family}(\mathcal{C})$. Hence, $\text{Family}(\mathcal{C})$ is not a pointed category. It therefore does not have additive structure defined above.

A pointed category $p\text{Family}(\mathcal{C})$, freely constructible from the category $\text{Family}(\mathcal{C})$, has objects as pairs $(A_I, A_i)$ where $A_I$ is an object of $\text{Family}(\mathcal{C})$ indexed by $I$ with (fixed) $A_i \in A_I$ being a “base” object of this pair. As its arrows, this category has the arrows of $\text{Family}(\mathcal{C})$ of the form $(f, i) : A_I \rightarrow B_J$ such that we also have $(f, i)A_i = B_j$ with $A_i$ and $B_j$ being taken as one-member families. That is, we have $f(i) = j$, and there always exists an arrow $f_i : A_i \rightarrow B_j$ in the collection $f$. We say that any such arrow “preserves” the bases of the families. Its hom-collection, $[(A_I, A_i) : (B_J, B_j)]$, is a collection of all such arrows in $\text{Family}(\mathcal{C})$. An identity arrow $(\text{id}_I, \{ \text{id}_{A_i} \}_{i \in I}) : (A_I, A_i) \rightarrow (A_I, A_i)$ also exists in such collections with obviously $(\text{id}_I, \{ \text{id}_{A_i} \}_{i \in I})A_i = A_i$. An object $(\emptyset, \emptyset)$ is a zero object of $p\text{Family}(\mathcal{C})$, that was mentioned earlier as the pointed family category of a category $\mathcal{C}$. The zero arrows of $p\text{Family}(\mathcal{C})$ are the arrows with empty domain, and these are its only zero arrows. The category $p\text{Family}(\mathcal{C})$ therefore has the additive structure defined before. Every object $(A_I, A_i)$ of the category $p\text{Family}(\mathcal{C})$ is also a co-product of its constituent objects $\{ A_i \}_{i \in I}$.

Also, the category $\mathcal{C}$ is embedded in the category $p\text{Family}(\mathcal{C})$, and the corresponding inclusion functor $\mathcal{J} : \mathcal{C} \rightarrow p\text{Family}(\mathcal{C})$ is given by the one-one association of any arrow $A \xrightarrow{f} B$ of category $\mathcal{C}$ with an arrow $\langle (A), A \xrightarrow{(f, g)} \{ B \} \rangle, B$ with $(f, \{ g \})A = B$ of the category $p\text{Family}(\mathcal{C})$. However, the inclusion of a sub-category $\mathcal{C}$ in its pointed family category or its pointed free co-product completion category $p\text{Family}(\mathcal{C})$ is not, always, a full embedding, even though the inclusion of a sub-category $\mathcal{C}$ in its family category $\text{Family}(\mathcal{C})$ is always a full embedding.
Now, if a category $A$ is additive, then, for any category $B$, the functor category $A^{B}$, inherits the additive structure of $A$. Thus, a functor category $\mathcal{F}^{\mathcal{F}amily(C)}$ has the group additive structure, while the functor category $\mathcal{M}_{\mathcal{F}amily(C)}$ has a monoid additive structure.

The additive structure of $\mathcal{F}^{\mathcal{F}amily(C)}$ is precisely that of the natural transformations of functors from $p\mathcal{F}amily(C)$ to $A$. The additive structure of $\mathcal{F}^{\mathcal{F}amily(C)}$ is then the categorical analogue of the “countable additivity” of usual measures. Therefore, $A$-based measures, also called $A$-based measure functors or simply as categorical measures, on an arbitrary category $C$ are now defined to be the objects of the functor category $A^{\mathcal{F}amily(C)}$.

Notably, what we require for the definition of categorical measures are the “additivity” of category $A$, relative to which measures are defined, and the “co-product completion character” of “additive” category $p\mathcal{F}amily(C)$ that is “freely constructible” from any arbitrary category $C$. Categorical analogue of the “countable additivity” of these categorical measures trivially follows because such a functor category is ensured to have an appropriate additive structure.

In general, a symbol $\mathcal{M}$ will be used to denote an $A$-based categorical measure or a measure functor $\mathcal{M} : p\mathcal{F}amily(C) \to A$. All functors, whenever used, will be considered modulo all those functors that are naturally isomorphic to each other.

Now, composition of a measure functor $p\mathcal{F}amily(C) \xrightarrow{\mathcal{M}} A$ and an inclusion functor $C \xrightarrow{\alpha} p\mathcal{F}amily(C)$ is a functor $C \xrightarrow{\mathcal{M} \circ \alpha} A$. Because an inclusion functor associates with an arrow of category $C$ a unique arrow of category $p\mathcal{F}amily(C)$, and a measure functor associates with an arrow of $p\mathcal{F}amily(C)$ a unique arrow of the additive category $A$, the functor $\mathcal{M} \circ \alpha$, to be called as an $m$-i functor, associates a unique arrow of category $A$ with an arrow of category $C$. An $m$-i functor is then an association of a unique element of the additive monoid or group with an arrow of the category $A$. When the arrow under consideration is an identity arrow, also called as a unit of the partial binary algebra of arrows, in the category $p\mathcal{F}amily(C)$, a measure functor associates it with an identity arrow of $A$, with this association being clearly independent of the mathematical structure of the object, which only “labels” that identity arrow. Then, we will also call an $m$-i functor as an $A$-based measure or categorical measure on the objects of category $C$. An $m$-i functor $\mathcal{M} \circ \alpha : C \to A$ is a partial binary algebra preserving, identity-preserving and compositions-preserving function from the collection, $C(C)$, of all the arrows of a category $C$ to the collection, $C(A)$, of all the arrows of an additive category $A$.

In terms of the standard ways of measure theory, we have then effectively isolated, as corresponding measures, a collection of countably additive functions $\mathcal{M} \circ \alpha : C(C) \to C(A)$, which are partial binary algebra preserving, identity preserving and compositions preserving.

Families of the units of the partial binary algebra of the category $C$, families of its objects, do not however correspond to Borel sets $\mathcal{B}$. We can, from an object $A$ in $C$, form a family $A_{I} = \{A, A, A, \ldots\}$ that is non-Borel for the usual ways of measure theory.

The usual Borel structure however exists with the hom-collections $[(A_{I}, A_{I}), (B_{I}, B_{I})]$ of the category $p\mathcal{F}amily(C)$, as we are allowed to form a Borel structure for every such hom-collection because of the very definition $\mathcal{R}$ of a category. The construction of the category $p\mathcal{F}amily(C)$ serves here to “ensure” therefore only the “consistency of structures” (generally unavailable for any arbitrary category) of hom-collections of the category $p\mathcal{F}amily(C)$ and that of an additive category $A$. We are therefore allowed the use of the standard measure theory, limited to aforementioned considerations of the hom-collections, of course. Categorical measures, then, provide the overall consistency of categorical structures containing all the hom-collections of the involved categories.

Of particular interest are the hom-collections $[(\{A\}, A), (\{A\}, A)]$ for an arbitrary object $A$ of the category $C$. In the case that this hom-collection has only a single arrow, an identity arrow, any categorical measure (functor) must map it to the additive identity of the additive category. For categorical measures, such are then the situations of categorical measure zero.

In general, each of the hom-collections $[(A_{I}, A_{I}), (B_{I}, B_{I})]$ of $p\mathcal{F}amily(C)$ is therefore a Borel space with associated Borel structure. It is of course the “same” measure that gets “defined” here for all the hom-collections of the category $p\mathcal{F}amily(C)$.

Then, under conditions of the Lebesgue-Radon-Nikodym (LRN) Theorem $\mathcal{R}$, for each of these hom-collections, there exists a unique finite valued measurable function, $f = d\ell/dt$, the LRN-derivative, where the $\ell$-finite measure $\ell$ is absolutely continuous with respect to measure $t$, and all properties of the “differential” hold modulo a set of $t$-measure zero, ie, $t$-almost everywhere or $t$-a.e.

Any categorical measure “carries” this “differential structure” consistently to the category $p\mathcal{F}amily(C)$ and, thence, to the category $C$ by way of the composition $\mathcal{M} \circ \alpha$. 

Because an object of a category can itself be looked upon as a category in its own right, we have “categorical measures” or $A$-based measures that are “internal” to an object, and “measures” that are “external” to it. Internal and external measures of any categorical object belong however to the same collection of $A$-based categorical measures definable for any category. As was remarked earlier, this feature is not surprising because “externally” usable notions can also be used “internally” to a (physical as well as mathematical) object. Nevertheless, internal measures “characterize” categorical objects within this most general categorical framework.

A group of addition of real numbers, whose set is denoted by $\mathbb{R}$, is a topological group $(\mathbb{R}, +)$ with respect to the usual metric topology. A monoid of addition of real numbers, $\mathbb{R}_+$ being the set of strictly positive (or strictly negative) real numbers, is a topological monoid $(\mathbb{R}_+, +)$ with respect to the usual metric topology. Both, $(\mathbb{R}, +)$ and $(\mathbb{R}_+, +)$ are locally compact. $\mathbb{R}^+$ denotes the group category, and $\mathbb{R}^+_\times$ the monoid category, of addition of real numbers.

Then, the collection $\mathcal{C}(A)$ of all the arrows of an additive category $A$ has the structure of a locally compact topological monoid, for example, of $(\mathbb{R}_+, +)$ under the usual metric topology. Due to its construction, the collection $\mathcal{C}(p\text{Family}(\mathbb{C}))$ of the arrows of the category $p\text{Family}(\mathbb{C})$ has the structure of the product of locally compact topological monoids, and has the associated product topology. Therefore, categorical measures $\mathfrak{M} : \mathcal{C}(p\text{Family}(\mathbb{C})) \to \mathcal{C}(A)$ are the partial binary algebra preserving, identity preserving, and compositions-preserving continuous functions from $\mathcal{C}(p\text{Family}(\mathbb{C}))$ to $\mathcal{C}(A)$.

III. CATEGORICAL FOUNDATIONS FOR PHYSICS

Now, we may “visualize” a mathematical structure, let it be any, representing physical bodies as being an object of a category, and the changes or the transformations of physical bodies as the arrows of that category. Then, entirely independently of the mathematical structure that is chosen to represent physical bodies, categorical measures of categorical objects represent various of the physical properties of physical bodies within this categorical framework. Consequently, categorical measures are then “fundamental” to formation of any of our physical conceptions.

The characterization of categorical objects by internal measures, and the overall framework of category theory, then provide “relations” that are, in a definite sense, “free” of the mathematical structure that we choose to represent physical bodies with. This is the categorical general covariance. In complete conformity with the General Principle of Relativity, such relations then constitute the most general Laws about Nature. We can expect such categorical laws to encompass the physical entirety. In particular, such laws turn out to be differential laws. This is then the precise mathematical reason as to why the physical world is well describable using differential equations.

Aforementioned characteristics of (categorical) objects are the “observable characteristics” of physical bodies within this most general mathematical framework. Changes in these characteristics are then the changes to physical bodies. The essential aim of a Program of Categorical Foundations for Physics is then that of obtaining definite mathematical laws for the changes (modulo their “equivalence” classes) of aforementioned characteristics (measures) of categorical objects.

Now, the categorical procedure for obtaining “object-independent” relations using categorical measures is required to be such as to be applicable to all the relations so obtainable. This, in fact, is the pivotal issue, which underlies the categorical general covariance.

To fix ideas, consider therefore any categorical measure $\mathfrak{M} : \mathcal{C}(p\text{Family}(\mathbb{C})) \to \mathbb{R}^+_\times$. Its additivity property allows the corresponding “metric” structure to be constructed. We now “actually construct” a metric structure that is the one of (physical) distance separating the objects.

To this end, call the families $A_I^{SF} = \{A, A, \ldots\}$ as self-families of the object $A$ of $\mathbb{C}$. Consider now a categorical measure $\mathfrak{D}$ that, for every object $A$ of the category $\mathbb{C}$, maps every of the hom-collections $\{((A_I^{SF}, A), (A_I^{SF}, A))\}$ of $p\text{Family}(\mathbb{C})$ to the “same” element, additive identity, of the additive category $\mathbb{R}^+_\times$. Since the hom-collection $\{(A), (A), (A), A\}$ also gets mapped to zero of real numbers, we call this as the property of vanishing of the “self-distance” of objects.

Moreover, for this physical distance measure, $\mathfrak{D}$, and for non-identical (which could be isomorphic) objects $A$ and $B$ of $\mathbb{C}$, we require that all the hom-collections $\{(A_I^{SF}, A), (B_I^{SF}, B)\}$ are mapped by it to the “same” non-identity element, say $a$, of the additive category $\mathbb{R}^+_\times$. Consistency with the vanishing self-distance property is then self-evident. The element $a$ of $\mathbb{R}^+_\times$ then defines metrical distance, $d(A, B)$, between objects $A$ and $B$ of category $\mathbb{C}$, with metrical properties of $d(A, B)$ following from the additivity
of the physical distance measure $\mathcal{D}$. We associate the metric structure of measure $\mathcal{D}$ with that of the physical distance separating physical bodies represented by the objects of category $\mathcal{C}$.

Physically, the aforementioned has the significance that the “distance” of a physical body from itself would then be vanishing always, and furthermore making any family out of that physical body would be of no physical relevance to this vanishing self-distance.

Such a physical distance measure, a functor, evidently exists, and the metric structure corresponding to its additivity defines the physical distance separating physical bodies.

The aforementioned is an instance of a general procedure for defining “characterizing” properties for physical bodies. Evidently, it involves “classifying” categorical measures according to their actions on the objects of the category $p\text{Family}(\mathcal{C})$. It will be discussed in [9].

Now, changes to categorical measures $\mathcal{M} : \mathcal{C}(p\text{Family}(\mathcal{C})) \rightarrow \mathbb{R}_+^*$ can occur when an endo-functor of the category $\mathcal{C}$, a functor $\mathcal{E} : \mathcal{C} \rightarrow \mathcal{C}$, “changes” the assignments of its arrows (with identity arrows). An endo-functor is 1-1 and onto on the collection $\mathcal{C}(\mathcal{C})$ of arrows of the category $\mathcal{C}$.

Let the physical distance measure possess the assignments: $d(A, B) = a$, $d(B, C) = b$, $d(C, A) = c$. Endo-functor can, for example, contain a “cycle” of the form $\mathcal{E}A = B$, $\mathcal{E}B = C$, $\mathcal{E}C = A$, while mapping all the other objects of $\mathcal{C}$ to themselves. Then, under the action of endo-functor $\mathcal{E}$, we have $d(\mathcal{E}A, \mathcal{E}B) = b$, $d(\mathcal{E}B, \mathcal{E}C) = c$, $d(\mathcal{E}C, \mathcal{E}A) = a$, ie, distance between $A$ and $B$ thus changes from being $a$ to being $b$ etc. Distances of $A$, $B$, $C$ from all other objects also change, without any changes to mutual distances of all other objects. Now, if $\mathcal{F}$ is another endo-functor that restores “original” distances, and such a functor obviously exists, then we have an “instance” of periodic motion under the action of composition of endo-functors $\mathcal{F} \circ \mathcal{E}$.

A general endo-functor then causes changes to $p\text{Family}(\mathcal{C})$, and therefore to measures from $p\text{Family}(\mathcal{C})$ to $\mathbb{R}_+^*$. Periodic behavior of a general endo-functor can then result in a periodic change in the “physical distance” of one from other objects. This describes a periodic “motion” of an object. By the continuity of the distance measure, these are continuous changes. The period of such a motion then provides us the notion of time, importantly relative to that periodic motion. Then, time is also a suitable categorical measure within this framework.

An “absolute zero” of time is thus the situation when there is no motion “whatsoever” of any of the objects of category $\mathcal{C}$. It can evidently be “reached” more than once, ie, when some motions of objects take place, then all the motions of all bodies stop, and then some motions occur; again and again. [There is then no “origin” of the “Universe” of physical bodies within this framework, just as it was the situation with Newton’s theoretical framework.]

Alternatively, we may consider Borel automorphisms [10] of the (Borel) space $\mathcal{C}(p\text{Family}(\mathcal{C}))$. A Borel automorphism is [10] then “periodic” if every point of the Borel space is periodic, but the period may differ from point to point of the Borel space. [Every point of the space $\mathcal{C}(p\text{Family}(\mathcal{C}))$ is periodic because every “distance” related to an object undergoing periodic motion changes periodically, and it is “object-symmetric”.] Then, we have the following.

The notion of time in Physics is that of the periodic motion of a “clock” body relative to an “observer” body. Within the present categorical framework, this notion of “time” is obtained, using the notion of “distance” in this context, from the “periodic” variations of the “location” of one (categorical) object, the “clock” body, in relation to any other (categorical) object, the “observer” body. [Notably, there does not exist any other notion of “time” within this general categorical framework using measures.] Categorical “velocity”, “acceleration”, etc. then use such a notion of time. These mathematical details will be communicated separately [9].

In the above, we have provided an example of the way in which categorical measures can be employed to describe the physical phenomena - physical changes. Of course, we considered only one type of a measure - the physical distance measure - for this example.

Under the action of an endo-functor of a category $\mathcal{C}$, other types of measures may also change. These changes then provide descriptions of other physical phenomena associated with those measures. An interplay of measure-preservation and non-preservation now emerges as the mathematical basis underlying physical phenomena. The aforementioned formalism uses the metric structure associated with additivity of categorical measures, and that provides the basis for relationally obtaining the Laws of Physics. There does not appear to be any another way in which Laws of Physics can be obtained within the proposed categorical framework of universal relativity. Therefore, this categorical description of the physical universe is entirely based on “mutual relationships” of objects.
Now, fundamental constants arise when we consider relations of physical bodies with each other. For example, Newton’s constant of gravitation arises when we “describe” the fall of a body to the Earth as being due to the gravitational “force” of the Earth on it. When we express this equationally, the proportionality factor is this constant. We call it as a “fundamental” constant, because it arises in considerations of the “source” property of the force of gravitation. (The concept of Force requires a physical body to have the property to generate it - the source property. For gravity, it is the gravitational mass. Such a property is an assumption, which cannot be explained by a theory using the concept of Force.) Then, fundamental constants arise, precisely, in considerations of various mutual “relations” of physical bodies, and relate to their ability to “act” on each other.

Thus, if a fundamental constant is undecidable in a theory, it is then implied that “some relationships” of physical bodies of the observable world are not within its explanatory powers. An immediate conclusion is then that the theory in question cannot describe physical entirety.

Any theory is a description of the physical world, in the language of Mathematics as a logical-deductive system of conceptions. Then, if a theory has an undecidable fundamental constant in it, it is implied that its conceptions and its (mathematical) language, both, are inadequate to describe “some relationships” of physical bodies of the observable world.

Within the proposed categorical framework, fundamental constants can only arise from the relations of (categorical) objects. This is an indication that the proposed categorical framework (using measures) of universal relativity is that of a Theory of Everything.

The program of categorical foundations for Physics then consists of using provided (and therefrom derived) notions to obtain the most general laws of Physics.

### IV. CONCLUDING REMARKS

To conclude, we have outlined here a specific categorical program for the foundations of physical entirety. It is based on the concept of categorical measures, which are definable for an arbitrary category. Evidently, it is entirely independent of the mathematical nature of categorical objects, a statement which is the categorical equivalent of the usual general covariance. This approach, that can be rightfully called as the Universal Relativity, is then in conformity with Einstein’s General Principle of Relativity. It is the currently known most general mathematical framework for Physics.

One may now “conjecture” that all the categorical velocities (obtainable from categorical measures) form, in general, a groupoid category. Provided that this “conjecture” holds, the formulation in can be one (categorical) representation of universal relativity. Of course, categorical measures, which are the fundamental mathematical notion than those derived ones such as a velocity, then provide (physical) properties of the objects of a groupoid category, which may be said to “describe” the “kinematical” part of universal relativity. The “internal” measures will, however, be outside the scope of this kinematical representation of universal relativity.

Now, an interesting laser interferometry experiment recently showed a non-detection of a frequency shift, to be specific, \( \delta \nu / \nu \approx (4.8 \pm 5.3) \times 10^{-12} \) over an observation period of \( \sim 200 \) days. This is then also the fractional change, \( \delta c / c \), in the speed of light, \( c \). Then, the non-detection of the frequency-shift in the experiment of indicates that the speed of light (in vacuum) is a fundamental constant that is independent of the system of reference.

Importantly, the constancy of the speed of light irrespective of the system of reference is an immediate consequence of the groupoid category formulation.

Discussed here are therefore some “features” of a Theory of Everything then.

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[1] See for the overall development of concerned ideas the following and references therein:
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Wagh S M (2005) *Universal Relativity and Its Mathematical Requirements*, Talk delivered at the South African Mathematical Society’s (SAMS) 48th Annual Meeting, Grahamstown, October 31 - November 2, 2005 [physics/0602038]
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[12] Then, by a collection, we will always mean a “gathering” or an “accumulation” of elements such that set-theoretic paradoxes do not arise in its considerations.

[13] That is why, in contrast to most of the mathematical literature on this subject, we will call any such structure simply as a category.