Canaries in a coal mine: using globular clusters to place limits on massive black holes in the Galactic halo

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ABSTRACT

We explore the possibility that massive black holes comprise a significant fraction of the dark matter of our Galaxy by studying the dissolution of Galactic globular clusters bombarded by them. In our simulations, we evolve the clusters along a sequence of King models determined by changes of state resulting from collisions with the black holes. We include mass loss in collisions as well as the heating of the remaining bound stars, and determine the role that a finite number of stars plays in the variance of the energy input and mass loss. Several methods are used to determine the range of black hole masses and abundances excluded by survival of Galactic globular clusters: simple order-of-magnitude estimates; collision-by-collision simulations of the energy input and mass loss of a stellar cluster; and a ‘smoothed’ Monte Carlo calculation of the evolution of cluster energy and mass. The results divide naturally into regimes of ‘small’ and ‘large’ black hole mass. ‘Small’ black holes do not destroy clusters in single collisions; their effect is primarily cumulative, leading to a relation between $M_{\text{bh}}$ and $f_{\text{halo}}$, the fraction of the halo in black holes of mass $M_{\text{bh}}$, which is $f_{\text{halo}} M_{\text{bh}} < \text{constant}$ (up to logarithmic corrections). For $f_{\text{halo}} = 1$, we find $M_{\text{bh}} \lesssim 10^5 M_\odot$ by requiring survival of the same clusters studied by Moore, who neglected cluster evolution, mass loss, and stochasticity of energy inputs in his estimates, but reached a similar conclusion. ‘Large’ black holes may not penetrate a cluster without disrupting it; their effect is mainly catastrophic (close collisions), but also partly cumulative (distant collisions). In the large-$M_{\text{bh}}$ limit, $f_{\text{halo}}$ (but not $M_{\text{bh}}$) can be constrained by computing the probability that a cluster survives a combination of close, destructive encounters and distant, non-destructive encounters. We find that it is unlikely that $f_{\text{halo}} \lesssim 0.3$ by requiring 50 per cent survival probability for Moore’s clusters over $10^{10}$ yr.

Key words: Galaxy: general – globular clusters: general – Galaxy: halo – Galaxy: kinematics and dynamics – dark matter.

1 INTRODUCTION

In this paper we re-examine the idea that black holes compose a substantial fraction of the dark matter in the halo of our Galaxy. Ostriker & Lacey (1985) originally proposed that heating by a significant population of halo black holes with masses $M_{\text{bh}} \sim 10^6 M_\odot$ could explain Wielen’s (1977) inference that the velocity dispersions of disc stars of age $t_*$ follow the trend $\sigma_v \propto t_*^{1/2}$. Although subsequent work questioned the basis of this argument, both because it is unclear that $\sigma_v$ rises as fast as $t_*^{1/2}$ (e.g. Carlberg et al. 1985; Strömgren 1987; Gomez et al. 1990), and that black holes are needed to explain the observed trend of $\sigma_v$ with $t_*$ (e.g. Lacey 1991), the heating of disc stars remains a constraint on the masses and abundance of any hypothetical population of massive halo objects; in the notation of Wasserman & Salpeter (1994), if the halo consists of a fraction $f_{\text{halo}}$ in the form of massive black holes, then the Ostriker & Lacey argument implies $f_{\text{halo}} M_{\text{bh}} \lesssim 10^6 M_\odot$. Limits on the masses and abundances of massive black holes could have important cosmological implications (e.g. Loeb 1993; Umemura, Loeb & Turner, 1993; Loeb & Rasio 1994).

Carr (1994) has reviewed various limits on baryonic or black hole dark matter in the halo of our Galaxy. (See also Carr & Sakellariadou 1999.) One of the most powerful constraints on black hole properties yet proposed was put forth by Moore (1993), who concluded that the survival of a set of relatively tenuous, low-mass ($M \lesssim 4 \times 10^4 M_\odot$) globular clusters over a time-span $\gtrsim 7 \times 10^9$ yr would be possible only if $M_{\text{bh}} \lesssim 10^3 M_\odot$. The argument employed by Moore was first applied to this problem by Wielen (1985), who explored perturbations of more massive clusters ($M \sim 10^6 M_\odot$) by heavier black holes, in the range advocated by Ostriker & Lacey (1985). (Klessen & Burkert 1996 extended Moore’s and Wielen’s calculations to a range of globular cluster properties.) The idea is to compute the heating of a globular cluster by passing black holes over a chosen time-scale comparable to the cluster age (as noted above,
Moore used $7 \times 10^9$ yr, roughly half the age inferred for typical clusters. If the total energy imparted by black hole perturbations is sufficiently large, then the cluster is said to be disrupted.

In view of the importance of this problem, we have begun a more comprehensive study of the disruption of the same set of globular clusters considered by Moore (1993) by a hypothetical population of halo black holes. This investigation aims to tighten up Moore’s argument in several different ways. Some of the improvements are technical (e.g. Moore used a simple analytic form for the energy input to a cluster by a passing black hole which is only very approximate; Klessen & Burkert 1996 improved on his formula), but others are qualitative. Of particular importance are the following. (1) Moore computed the energy input for a ‘static’ cluster, whose structure was held fixed. Our calculations evolve the clusters along a King sequence. (2) Qualitatively, one expects a cluster that gains a large amount of energy compared to its initial binding energy from encounters with passing black holes to lose a large fraction of its mass, too. Our calculations include mass loss by the clusters. (3) Although one can get a rough idea of the survival probability by considering the mean heating of a cluster, the energy transfer process is actually stochastic, and the variance in the energy input may play an important role in final estimates of critical masses for disruption to be likely. (4) Moore’s calculations pertain to black holes of relatively low mass, which are incapable of disrupting a cluster in a single perturbation. For black hole masses $M_{bh} \approx M V_{rel}/\sigma_\text{cl}$, where $\sigma_\text{cl}$ is the characteristic velocity dispersion of the cluster, and $V_{rel}$ its characteristic speed relative to the approaching black hole ($V_{rel} \approx 330$ km s$^{-1}$ is a fair estimate), a single encounter at the cluster’s tidal radius will likely destroy it. In this regime, one can obtain limits on the halo mass density in massive black holes, but not on $M_{bh}$ (e.g. Wielen 1988; Wasserman & Salpeter 1994).

Our study combines analytic and Monte Carlo calculations. Our most complete results come from Monte Carlo simulations in which we simulate each encounter between a cluster and a black hole separately. For simplicity, we shall consider clusters at fixed Galactocentric radius, as was done by Moore (1993). We model clusters using $N$ point masses whose positions and velocities are chosen from a King distribution (see, e.g., Binney & Tremaine 1987). Velocity perturbations are computed for a given black hole impact parameter and speed relative to the cluster centre of mass using the impulse approximation (e.g. Binney & Tremaine 1987). The change in velocity of the cluster centre of mass is subtracted from each individual velocity perturbation to determine the change in cluster energy and mass as a consequence of the collision. Determining which stars are ejected is tricky in any scheme that does not employ a direct $N$-body simulation of interparticle interactions, but as long as the perturbations in individual encounters are not excessive, it should be sufficient to designate for ejection those stars whose post-collision velocities exceed their local pre-collision escape speed. To find the new King model that describes the remaining cluster, we need its tidal radius in addition to its mass and energy after the black hole encounter; we get this by assuming that the tidal radius is proportional to $M^{1/3}$, consistent with our assumption of fixed Galactocentric radius. Within these ‘rules of the game’ we simulate the evolution of the globular clusters studied by Moore over a time-span of $10^9$ yr, or until they are disrupted, whichever comes first. Our simulations are more comprehensive than the Monte Carlo calculations of Klessen & Burkert (1996), who chose discrete black hole encounter times, relative velocities $V_{rel}$ and impact parameters $b$ from the appropriate probability distributions, but merely updated the cluster velocity dispersion by a completely deterministic amount $\Delta \sigma(b, V_{\text{rel}})$, without accounting for stochasticity of heating, mass loss or the change in internal cluster structure in individual encounters.

Several different criteria are employed to decide whether or not a cluster has been destroyed. Some of these may be called ‘global’ in that they depend on integrated properties of the cluster, such as total energy or total mass. We can regard a cluster as having been destroyed, for example, when its mass or energy per mass has changed by a fractional amount in excess of some pre-set values (e.g., 0.5). Since we evolve models along a sequence of King models that is limited, clusters may also die when they reach the end of the sequence (see also Chernoff, Kochanek & Shapiro 1986, hereafter CKS). The other criteria for cluster disruption to be used are ‘local’, in that they depend on changes in the properties of a cluster as a consequence of a single collision. Thus, if the mass or energy of a cluster changes by more than some pre-set fractional amounts in an encounter with a black hole, we shall regard it as disrupted. This should also allow us to control the inaccuracy of our criteria for determining the mass loss per encounter somewhat. We evaluate survival probabilities for the various criteria separately.

The Monte Carlo calculations outlined above allow us to study both the large and small-$M_{bh}$ regimes. In the large-$M_{bh}$ regime, where destruction occurs only after relatively few encounters, the Monte Carlo calculations ought to be reasonably fast computationally. However, for small $M_{bh}$ where the encounters are more frequent (the regime focused on by Moore 1993), we expect the Monte Carlo calculations to be more cumbersome, thus limiting the number of stars we can use in realizing clusters. Ideally, one would like to be able to represent the stars ‘one-by-one’ since the variances in energy input depend on the number of cluster particles. Fortunately, the perturbations due to individual encounters are relatively gentle in the small-$M_{bh}$ regime, and the program lends itself to a Fokker–Planck treatment, which promises to be faster (at the price – justified in our view – of losing the ability to resolve short time-scale structure in the cluster evolution). We present a two-dimensional Fokker–Planck scheme in which we follow cluster evolution in mass and energy.

The calculations reported here suffer from two or three principal deficiencies. Most important is their reliance on the sequence of King models and on the impulse approximation. In addition, one would like to interweave perturbations by a hypothetical population of black holes with well-established sources of heating, such as disc shocking (e.g. Ostriker, Spitzer & Chevalier 1972; Spitzer & Chevalier 1973; Chernoff et al. 1986; Binney & Tremaine 1987); in the similar problem of wide-binary evolution, the interplay of perturbations by stars, molecular clouds and dark matter is known to be important (e.g. Ritterer & King 1982; Bahcall, Hut & Tremaine 1985; Weinberg, Shapiro & Wasserman 1987; Wasserman & Weinberg 1991). Moreover, globular clusters evolve on their own as a consequence of internal relaxation, resulting in evaporation and energy changes even without external perturbations; limits on halo black hole properties may be altered when internally induced changes in state are accounted for properly. These important effects will be ignored here in order to concentrate solely on the cluster evolution due to the collisions with black holes. We shall study some of these issues in a subsequent paper (Murali et al., in preparation).

2 Qualitative overview of the collision process and evolutionary scenarios

2.1 Approximations

Throughout this paper we employ the impulse approximation and
ignore deflection of the perturbing black hole orbit from a straight line. For non-penetrating encounters at impact parameter $b$ and relative velocity $V_{\text{rel}}$, these approximations are valid when $bV_{\text{rel}}/c \ll 1$. The characteristic frequency is roughly $\Omega \sim (GM/b_{\text{hm}} c^2)^{1/2}$, where $M$ is the cluster mass and $b_{\text{hm}}$ is the cluster half-mass radius, for the outer parts of the cluster where the tidal perturbation is strongest. If we set $V_{\text{rel}} = \xi_{\text{c}} V_{\text{c}}$, where $V_{\text{c}}$ is the Galactic circular speed, and let $\rho_0 = \xi_{\text{c}} (GM/b_{\text{hm}})^{1/2}$ be the cluster’s central one-dimensional velocity dispersion, then the impulse and straight-line approximations should hold for $b/r_{\text{hm}} \ll (\xi_{\text{c}}^2) V_{\text{c}}/\rho_0$. For the clusters studied in this paper, $V_{\text{c}}/\rho_0 \approx 10-100$, and the approximations fail only far outside $r_{\text{hm}}$.

For penetrating encounters, this assessment remains valid for ‘typical’ collisions, but may fail for perturbations of particles deep in the cluster core. This is because the characteristic frequency near the centre of the cluster where the perturbations are largest is $\Omega = (4\pi G \rho_0/3)^{1/2}$, where $\rho_0$ is the central mass density of the cluster. Adopting $b = r_{\text{hm}}$ as typical, we find that the impulse and straight-line approximations hold for $V_{\text{rel}} \approx 0.1 \times r_{\text{hm}} (pc/\rho_0/M_\odot)^{1/2} km^{-1}s^{-1}$. Although our approximations would fail for the most concentrated clusters, they remain true for the relatively tenous ones studied here (and indeed for many ‘normal’ clusters).

We also ignore all processes influencing cluster evolution except perturbations by black holes, even though our limits are based on survival probabilities over time-scales long enough for tidal shocking and internal relaxation to be important. In more realistic simulations, including these effects could tighten limits on properties of hypothetical halo black holes.

### 2.2 Single Collisions

Consider a single collision between a globular cluster and a black hole that takes place in the halo of the Galaxy. Let the cluster have tidal radius $r_t$, $N$ stars, King model parameter $W_0 = \psi_0$, and total energy $E$. The black hole passes the cluster with a speed $V_{\text{rel}}$ and an impact parameter $b$. Define the ‘collision parameter’, $\eta_i = (M_{\text{bh}}/M_\odot)\alpha_i V_{\text{rel}}$. In Section 5.2 it will be shown that in the impulsive limit the energy input, mass loss and their variances are

$$\Delta E(b, V_{\text{rel}})/E = \eta_i^2 R_E(b, \eta_i, \psi_0),$$

$$\Delta M(b, V_{\text{rel}})/M = -\eta_i^2 R_M(b, \eta_i, \psi_0),$$

$$\sigma_{\Delta E}^2/E^2 = \eta_i^4 R_{EE}(b, \eta_i, \psi_0),$$

$$\sigma_{\Delta M}^2/M^2 = \eta_i^4 R_{MM}(b, \eta_i, \psi_0),$$

(1)

where the $b$-dependent radial functions have the approximate limits (see Section 5.3)

$$R_E = C_E(\psi_0) (r_t/b)^{1/4},$$

$$R_M = C_M(\psi_0) (r_t/b)^{1/4},$$

$$R_{EE} = C_{EE}(\psi_0) (r_t/b)^{1/4},$$

$$R_{MM} = C_{MM}(\psi_0) (r_t/b)^{1/4},$$

(2)

when $b \gg r_t$. As much of the mass in the cluster is contained within the core radius, one can actually use the $b^{-2}$ dependence all the way into the core as a first approximation. Examples of the radial functions $R_E$, $R_M$, $R_{EE}$ and $R_{MM}$ for impacts inside the cluster are given in Section 5.2.

For large enough $M_{\text{bh}}$, a single collision will suffice to disrupt the cluster. This will be referred to as the ‘high black hole mass limit’. Define ‘disruption’ to occur for an energy input of size $f[E]$. Then the cluster is destroyed in a single collision for

$$b \leq b_d = \frac{C_E}{f} \eta_i^{1/2}. \tag{3}$$

A safe overestimate of the mass, $M_{\text{high}}$, at which the cluster may be disrupted by the black hole mass at which $b_d = r_t$. Using $V_{\text{rel}} = \xi_{\text{c}} V_{\text{c}}$, $\xi_{\text{c}} = 1.5$, and the Galactic circular speed $V_{\text{c}} = 220 km^{-1}s^{-1}$, the result is

$$M_{\text{high}} = \frac{V_c}{\rho_0} \frac{(\xi_{\text{c}} V_{\text{c}})^{1/2}}{C_E} \tag{4}$$

where $f$ is much larger than $M$ by a factor of $V_{\text{c}}/\rho_0 \gg 1$.

The probability distribution for energy input (and mass loss) can be understood in the following qualitative terms. The ratio of the variance to the mean energy input is

$$\sigma_{\Delta E}^2/E^2 = \frac{R_{EE}}{NC_E^2} \frac{R_{EE}}{\eta_i^4} \frac{1}{E} \tag{5}$$

This becomes small for large $N$ and small $M_{\text{bh}}$, but is also proportional to $b^4$, ensuring that $\sigma_{\Delta E} \geq \Delta E$ for $b \geq b_{\text{diff}}$, and become ‘fuzzy’ for $b \approx b_{\text{diff}}$. Indeed, the energy input will always be sharply peaked about the mean at the destructive radius $b_d$, because $b_d < b_{\text{diff}}$. Hence the probability of getting an energy input which is not destructive when $b < b_d$ is quite small.

### 2.3 Evolution over many Collisions

If $M_{\text{bh}} \ll M_{\text{high}}$ (‘the small-$M_{\text{bh}}$ limit’), it will take many collisions to disrupt the cluster. Individual collisions only ‘tickle’ the cluster, and the evolution can be approximated by averaging over the effects of many collisions. In Section 5.4, we find that the mean rate of energy input, averaged over $b$ and $V_{\text{rel}}$, takes the form

$$\langle E \rangle/E = d_E(\eta_0, \psi_0) \Gamma_0,$$  

(6)

where $\Gamma_0 = n_{\text{bh}} \pi r_t^2 V_c$, $n_{\text{bh}}$ is the number density of black holes, $\eta_0 = (M_{\text{bh}}/M_\odot)\alpha_i V_{\text{rel}}$, and $d_E(\eta_0, \psi_0) = \kappa_E(\psi_0) \ln (1/\eta_0) \eta_0^2$. The mean time for disruption, which we define here to occur at $\Delta E = f[E]$, is

$$T_{\text{disrupt}} = \frac{f[E]}{\langle E \rangle} = \frac{f}{\Gamma_0 \eta_0 \ln (1/\eta_0) \eta_0^2} \propto \frac{1}{M_{\text{bh}}}, \tag{8}$$

so that for very small $M_{\text{bh}}$ it takes a long time to disrupt the cluster. Given a value for $T_{\text{disrupt}}$, there is a certain critical value of $M_{\text{bh}}$, called $M_{\text{bh},\text{crit}}$, above which the cluster is disrupted. The variance of the averaged energy input in a time $T$ takes on the form (see Section 5.4)

$$\sigma_{\Delta E}^2/E^2 = \frac{d_{EE}(\eta_0, \psi_0)}{N} \propto \frac{1}{M_{\text{bh}}} \Gamma_0 T,$$  

(9)

where $d_{EE}(\eta_0, \psi_0) = \kappa_{EE}(\psi_0) \ln (1/\eta_0) \eta_0^2$. 

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Given an ensemble of clusters with initial energy $E_0$, a time $T \gg 1/\alpha_0$ later the cluster energies will be roughly $E(T) = E_0 + \langle \Delta(E(T)) \rangle = \sigma_T(T)$. For small times the variance will dominate the mean corresponding to a very broad spread in cluster states, but for sufficiently long times the ensemble will peak sharply about the mean. We find that for times $T = T_{\text{disrupt}}$

$$
\sigma_{\Delta E}^2(T_{\text{disrupt}}) = \frac{\kappa_{EE}}{N\kappa_{E}} \frac{|E|}{\langle \Delta(E(T_{\text{disrupt}})) \rangle} \approx \frac{\kappa_{EE}}{N\kappa_{E}} \frac{1}{N} \ll 1, \quad (10)
$$

so that the range of final states is always sharply defined about the mean for times long enough to disrupt the cluster when $M_{bh} \ll M_{\text{high}}$. The probability of survival, $P_s$, is given (here) by the fraction of clusters with $\Delta E < f|E_0|$. This fraction will change from unity to a minimum value in a `transition region' with size, $\delta M_{bh,crit}$, dictated by the ratio $\sigma_{\Delta E}(T_{\text{disrupt}})/\langle \Delta(E(T_{\text{disrupt}})) \rangle$, so that $\delta M_{bh,crit}/M_{bh,crit} \sim N^{-1/2} \ll 1$.

When $M_{bh} \simeq M_{\text{high}}$, any collision inside the destructive radius $b_d$ will destroy the cluster. In the tidal limit, the expected number of destructive encounters, $N_d$, in the time $T$ is

$$
N_d = \left( \frac{C_k}{f} \right)^{1/2} \frac{\rho_{bh}}{M} t_{1/2}^{1/2} T, \quad (11)
$$

where $\rho_{bh} = M_{bh} r_{bh}$ is the mass density in black holes. For an ensemble of clusters with initial energy $E_0$, the destructive $b < b_d$ collisions act as a `sink' for clusters, while the non-destructive $b > b_d$ encounters give rise to a slower, diffusive energy change. The probability of survival, $P_s$ (which takes on its minimum value in the $M_{bh} \simeq M_{\text{high}}$ limit) is the product of $\exp(-N_d)$, the probability that no single destructive encounters occur, and the probability that all the $b > b_d$ collisions combined give $\Delta E < f|E_0|$. Since $N_d$ does not depend on $M_{bh}$, only $\rho_{bh}$ (or equivalently $f_{\text{halo}}$) can be constrained in the $M_{bh} \geq M_{\text{high}}$ limit, not the black hole mass. In addition, the cluster evolution is quite stochastic in this regime, as it depends on whether the destructive collisions do or do not occur.

The model for cluster evolution described in this section is summarized in the (purely illustrative) diagram of Fig. 1. The fraction of clusters destroyed in time $T$, called $f_{\text{destroyed}} = 1 - P_s$, changes from 0 to $\sim 0.8$ in a region of width $\sim 2000 M_0$ centred on $M_{bh,crit} \sim 5000 M_0$. Note that $f_{\text{destroyed}}$ in this example does not asymptote at one, but instead reaches $f_{\text{destroyed}} \sim 0.8$ corresponding to $N_d \sim 1$.

### 3 THE CLUSTER MODEL AND EVOLUTION OF THE CLUSTERS

The non-dimensional King models (e.g. Binney & Tremaine 1987) are uniquely determined by the normalized central potential $\tilde{\psi}_0 = \psi(r=0)/\sigma_T^2$, which is the parameter $W_0$ of King (1966). The distribution function is given by

$$
\tilde{f}_{\text{King}} (r, v) d^3r d^3v = d^3r d^3v \frac{\rho_1}{(2\pi\sigma_0^2)^{3/2}} \left[ \frac{(v^2 - v_0^2)^2}{\sigma_0^2} \right]^{1/2} \left[ \frac{(v^2 - v_0^2)}{\sigma_0^2} - 1 \right] \quad (12)
$$

for stellar positions $r$, velocities $v$, and local escape speed $v_0(r)$. The dimensional King models can be specified by three independent quantities such as $E$, $M$ and $r_1$. Define $v^s(\tilde{\psi}_0)$ by $E = \psi(0)(GM/r_1)$. For $0.0 < \tilde{\psi}_0 < 8.5$, or $-0.60 < v > -2.13$, $v^s(\tilde{\psi}_0)$ is single-valued so that the King model is known, given $E$, $M$ and $r_1$. A physical reason for excluding large $\tilde{\psi}_0$ is that simulations have shown King models become susceptible to gravothermal instability at $\tilde{\psi}_0 \sim 7.40$ (Wiyanto, Kato & Inagaki 1985). The clusters we study in this paper are not core-collapsed.

A further restriction on the clusters is that they be tidally limited by the Galaxy. To include the time-dependent effect of tidal stripping due to the Galaxy would require detailed restricted three-body simulations for a range of both globular cluster orbits and orbits of stars in the clusters. In this paper we consider a first approximation in which the Galactic tidal field provides a relationship between $r_1$ and $M$, but does not contribute a time-dependent perturbing force. For circular orbits, if we define $r_1$ as the distance from the cluster centre to the Lagrange point of the cluster plus galaxy potential, we get

$$
\frac{M}{r_1^3} = \frac{M_g}{R_g^3}, \quad (13)
$$

for a point-mass galaxy with mass $M_g$ and Galactocentric radius $R_g$, and we adopt $M_g R_g^3 = \text{constant}$ for our clusters as they evolve due to collisions with black holes.

A weakness of our paper is its dependence on the King model sequence. As we shall see in Section 8, for some clusters black hole collisions may force $\tilde{\psi}_0 = 0$ after only modest energy input and mass loss, leading to very tight (but somewhat artificial) bounds on black hole properties. We shall rectify this deficiency in a subsequent paper, where cluster structure is not restricted to the King sequence (Murali et al., in preparation).

We shall use two different sets of globular clusters to determine a maximum allowed black hole mass, $M_{bh,crit}$. First, we examine the set of loosely bound globular clusters found in Moore (1993) and listed in Table 1, but then we also investigate a larger cluster perhaps more representative of the initial cluster population.

**Table 1. Parameters of the globular clusters.**

| Cluster Name | $M$ ($M_\odot$) | $r_1$ (pc) | $r_{core}$ (pc) | $R_g$ (kpc) |
|--------------|-----------------|------------|-----------------|-------------|
| AM 4         | 700             | 11.7       | 3.7             | 30.0        |
| Arp 2        | 18000           | 75.0       | 9.5             | 20.4        |
| NGC 5035     | 37700           | 72.0       | 10.9            | 16.7        |
| NGC 7492     | 10400           | 45.4       | 4.5             | 18.7        |
| Pal 4        | 24900           | 92.3       | 15.4            | 96.0        |
| Pal 5        | 13700           | 76.4       | 13.9            | 16.5        |
| Pal 13       | 3000            | 30.0       | 3.0             | 25.7        |
| Pal 14       | 10400           | 117.5      | 22.4            | 69.9        |
| Pal 15       | 15000           | 41.9       | 10.5            | 30.0        |
4 THE BLACK HOLE MODEL

Black holes of mass $M_{bh}$ are assumed to compose a spherical halo with an isotropic velocity distribution. A fraction $f_{bh}$ of the total halo mass is presumed to be in the black holes. When no value of $f_{bh}$ is explicitly stated, $f_{bh} = 1$ is assumed. We model the black hole population as a singular isothermal sphere with one-dimensional velocity dispersion $\sigma_{bh}$ and mass density

$$\rho_{bh}(r_g) = f_{bh}\frac{\sigma_{bh}^2}{2\pi G R_g^2},$$

(e.g. Binney & Tremaine 1987, Ch.4) with $\sigma_{bh} = V_c/\sqrt{2} = 220\sqrt{2} \text{ km s}^{-1} = 156 \text{ km s}^{-1}$. The number density of black holes at $r_g$ is

$$n_{bh}(r_g) = f_{bh}\frac{V_c^2}{4\pi GM_{bh} R_g^4}.$$  

We ignore rotation of the black hole halo, so their velocity distribution is $f(V_{rel}) = (\pi V_c^2)^{-3/2} \exp\left[-(V_{rel}/V_c)^2\right]$. For computing the impulsive mass loss and energy input to the cluster due to the collision, we need the distribution of relative speeds, $V_{rel}$. Let $V_c$ be the cluster velocity, so $V_{rel} = V_{bh} - V_c$. The distribution of relative speeds between the halo of black holes and the cluster becomes

$$f(V_{rel})dV_{rel} = \frac{1}{\sqrt{\pi} V_c} \left[\exp\left(-\frac{(V_{rel} - V_c)^2}{V_c^2}\right) - \exp\left(-\frac{(V_{rel} + V_c)^2}{V_c^2}\right)\right] \frac{V_{rel}dV_{rel}}{V_c^2}.$$  

In this paper the clusters are on circular orbits with $V_c = V_{cl}$. The mean relative speed given by this distribution is

$$(V_{rel}) = V_c \left[\frac{x + 1}{2}\right] \text{erf}(x) + \frac{1}{\sqrt{\pi}} \exp(-x^2),$$  

where $x = V_{rel}/V_c$ and erf$(x)$ is $(2/\sqrt{\pi}) \int_0^x \exp(-t^2)dt$. As $x \rightarrow 0$, $(V_{rel}) \rightarrow 2V_c/\sqrt{\pi}$ (the Gaussian result), as $x \rightarrow \infty$, $(V_{rel}) \rightarrow V_c$, and for $V_{rel} = V_c$ we find $(V_{rel}) = 1/\sqrt{6} = 0.18$. Note that the distribution in equation (16) is different from that used by Moore (1993) and Klessen & Burkert (1996). In addition, they chose to approximate the differential rate of collisions $d\tau = 3\rho_{bh}\sigma_{bh}dV_{rel}dV_{rel}$ by $3\rho_{bh}\sigma_{bh}(V_{rel})dV_{rel}$, which will lead to errors in the number of collisions and the rate of energy input and mass loss to the cluster.

For a globular cluster at $R_g$, there will be a certain value of $M_{bh}$ below which there will be more than one black hole inside the cluster on average at any time (the `many-body' limit). The number of black holes inside a cluster is $N_{inside} = \rho_{bh}\pi r_i^3/3 = (V_c^2 r_i^3)/(3GM_{bh} R_g^2)$. Since $r_i \simeq GM_{bh}/\sigma_{bh}^2$, equations (13) and (14) give $3(V_c^2 R_g^2)/(2\pi GM_{bh}) \simeq 1$, and so $N_{inside} \simeq M(10M_{bh})$. This exceeds one for $M_{bh} < 0.1M$, which we will see is close to black hole mass limits for some clusters. The mathematical description of the energy input is complicated in this regime, since the duration of a collision $\tau_{bh}/V_{rel}$ is longer than the time between collisions $\tau_{bh}/V_{rel}^{-1}$.

5 THE IMPULSIVE ENERGY INPUT FOR A SINGLE ENCOUNTER

In the impulse and straight-line approximations, the velocity change of a star due to the passage of a black hole is

$$\Delta v = -\frac{2GM_{bh}s}{V_{rel}^2},$$

where $s \perp V_{rel}$ is the projected vector from the black hole to the star at closest approach. Since the stellar velocity, $v \ll V_{cl}$ and $v \ll V_{bh}$, we may neglect $v$ in the relative velocity, so $V_{rel} = V_{bh} - V_{cl}$. In this approximation, all stars receive a velocity kick in the same plane perpendicular to $V_{rel}$. The density of cluster stars can be projected on to this plane; then for a star at projected position $R$ relative to the cluster centre, $s = R - b$, where $b$ is the impact parameter of the black hole relative to the cluster centre.

The cluster is destroyed for impacts with $\Delta v \simeq \sigma_0$. An order-of-magnitude estimate of the ratio of these two speeds is

$$\frac{\Delta v}{\sigma_0} = \frac{M_{bh}}{M_{bh}/V_{rel}b^2} = \eta, \frac{b^2}{V_{rel}^2}.$$  

For a penetrating impact with $b \approx r_i$, the cluster will be destroyed if $\eta \approx 1$. As the velocity kick, energy input, etc. in the impulsive limit must scale as $M_{bh}/V_{rel}$, $\eta$ is a convenient dimensionless measure of the destructiveness of the collision.

Next, let star $i$ have a mass $m_i$ and a velocity $v_i$ with respect to the centre of mass before the collision. The energy of the cluster before the collision is then

$$E_{before} = \sum_{i=1}^{N} m_i \left(\frac{1}{2}v_i^2 + \frac{1}{2}\phi_i\right),$$  

where $\phi_i = -GM/r_i - \psi_i$ is the gravitational potential, and $\psi_i = \sqrt{2}\eta v_i$ is the escape speed for star $i$. The velocity kick relative to the centre of mass is

$$\delta v_i = \Delta v_i - \Delta v = \Delta v_i - \frac{1}{N} \sum_{i=1}^{N} \Delta v_i,$$  

where $\Delta v$ is computed for a continuous cluster in Appendix A. Star $i$ is ejected if $(v_i + \delta v_i)^2 > 2\psi_i$. The total amount of mass ejected from the cluster can then be formally written down as

$$\Delta M = -\sum_{i=1}^{N} m_i \theta_i(ej).$$  

where $\theta_i(ej) = \theta \left[(v_i + \delta v_i)^2 - 2\psi_i\right]$ is a useful shorthand notation; similarly, $\theta_i(ej) = \theta \left[(v_i + \delta v_i)^2 - 2\psi_i\right]$ for the bound stars. We neglect the possibility that stars unbound by our criterion remain near the cluster for a long time, possibly to become bound once again in a subsequent encounter with a black hole (Spitzer 1987 discusses orbits of this type).

To find the change in cluster energy we need the kinetic and potential energy changes, $\Delta T$ and $\Delta V$ respectively. The change in kinetic energy arises both from the kicks to the bound stars and the loss of the energy of the ejected stars:

$$\Delta T = -\sum_{j=1}^{N} \sum_{i=1}^{N} m_i \frac{v_i v_j}{r_{ij}} \theta_i(ej) + \sum_{i=1}^{N} m_i (v_i \delta v_i + \frac{1}{2} \delta v_i^2) \theta_i(b).$$  

The change in the potential energy is $\Delta V = V_{after} - V_{before}$, where the potential energy before the collision is

$$V_{before} = -\frac{GM_{bh}}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \theta_i(b) \theta_j(b) \frac{m_i m_j}{r_{ij}} \theta_i(ej).$$  

and potential energy of the bound stars remaining afterwards is

$$V_{after} = -\frac{GM_{bh}}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \theta_i(b) \theta_j(b) \frac{m_i m_j}{r_{ij}} \theta_i(ej).$$  

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for separation vectors $r_{ij}$. Consequently, 
\[
\Delta V = \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{G m_i m_j}{r_{ij}} - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \theta_0 \theta_{ij} \left(\frac{G m_i m_j}{r_{ij}}\right)
\]
(26)
\[
= - \sum_{i=1}^{N} \theta_0 m_i \phi_i - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \theta_0 \theta_{ij} \left(\frac{G m_i m_j}{r_{ij}}\right),
\]
(27)
where $\phi_i$ still denotes the potential of star $i$ from before the collision. The first term in equation (27) is just the pre-collision potential energy of the ejected stars, and is $O(\Delta M/M)$, the fractional mass loss. The second term is $O(\Delta M/M)^2$, and will be smaller than the first as long as $\Delta M/M$ is small. In our calculations we neglect the smaller (and difficult to compute) second term, and evaluate the first term using the King model potential at the positions of the ejected stars. To protect against gross inaccuracies, we terminate our simulations if $|\Delta M/M| > 0.2$ in a single collision.

With these approximations, the energy change is
\[
\Delta E = - \sum_{i=1}^{N} m_i \left(\frac{1}{2} \dot{v}^2_i + \phi_i\right) \theta_0 (\epsilon) + \sum_{i=1}^{N} m_i \left(\dot{v}_i \delta v_i + \frac{1}{2} \delta v^2_i\right) \theta_0 (\epsilon).
\]
(28)

It should be noted that the energy change due to ejected stars is always positive, and hence mass loss always heats the cluster. In the sum over bound stars, the $\dot{v}_i \delta v_i$ term can have either sign and hence can heat or cool. The mean of this term is non-zero but small, since it depends on the mass loss to make the final distribution slightly anisotropic; it also contributes substantial variance. The $\frac{1}{2} \delta v^2_i$ term gives rise to the familiar mean heating (King 1966; Binney & Tremaine 1987).

5.1 The model for energy input and mass loss

For the analytic estimates and Fokker–Planck calculations of cluster survival, we shall need formulae for the mean energy and mass changes, $\langle \Delta E \Delta M \rangle$, their variances, and the cross-term $\langle \Delta E \Delta M \rangle$. To obtain them, we use a model due to CKS for the energy input and mass loss. In this model, the portion of phase-space from which mass escapes is identified. The energy and mass of the remaining cluster are then found as integrals over the distribution function of the remaining stars. This model relies only on the information given by the distribution function from before the collision; no detailed solution of the Vlasov equation is attempted. The higher order effects which arise from the detailed alteration of the distribution function by the perturbing black holes are estimated, and define the error in our method.

We shall also extend the the CKS model slightly by considering the fluctuations in the number of stars lost in any collision resulting from a finite number $N$ of stars. Before the collision, the phase-space density of the cluster is given by the King model in equation (12). After the collision, we do not have a full expression for the phase-space density including the effect of the change in velocity to all the stars, but we do know that all parts of phase-space for which $\theta_0 = 1$ will have their mass ejected. Define $f_c = f_{\text{King}}(\theta, b)$ and $f_s = f_{\text{King}}(\theta, 1)$, which are non-zero only for bound and unbound stars respectively. The average of a quantity $x$ over the bound, or ejected, stars will be written
\[
\langle x \rangle_c = \frac{\int d^3T f_c x}{\int d^3T f_c}, \quad \langle x \rangle_s = \frac{\int d^3T f_s x}{\int d^3T f_s}.
\]
(29)

respectively. If a symbol $< >$ is not specified, it means that the quantity is averaged over the entire phase-space.

The probability that a star is in the portion of phase-space from which mass is ejected is $p_{\text{ej}} = \langle \theta_0 \rangle$, which is an integral over the cluster model and is independent of $N$. We then suppose that the distribution for losing $N_{\text{ej}}$ stars out of the total $N$ is given by
\[
P(N_{\text{ej}}; p_{\text{ej}} N) = \frac{(p_{\text{ej}} N)^{N_{\text{ej}}}}{N_{\text{ej}}!} e^{-p_{\text{ej}} N}.
\]
(30)

In averaging the expressions of Section 5, we will need only the moments $\langle N_{\text{ej}} \rangle = N p_{\text{ej}}$ and $\langle N_{\text{ej}}^2 \rangle = (N p_{\text{ej}})^2 + N p_{\text{ej}}$.

5.2 Moments of $\Delta E$ and $\Delta M$ for individual collisions

In this section we integrate the first and second moments of $\Delta E$ and $\Delta M$ from Section 5 over the King cluster model and the distribution for the number of ejected stars $P(N_{\text{ej}})$. These averages will be denoted by an overbar to distinguish them from the averages over bound and ejected stars defined in the previous section.

The mean mass loss and energy input are
\[
\overline{\Delta M(b, \eta, \hat{\psi}_0)} = - \langle \theta_0 \rangle
\]
(31)
and
\[
\overline{\Delta E(b, \eta, \hat{\psi}_0) \langle E \rangle} = M \left[ -p_{\text{ej}} \left(\frac{1}{2} \dot{v}^2 + \phi\right) \right] + (1 - p_{\text{ej}}) \left(\dot{v} \delta v + \frac{1}{2} \delta v^2\right) \langle E \rangle
\]
\[
= M \left[ -p_{\text{ej}} \left(\frac{1}{2} \dot{v}^2 + \phi\right) + \left(\dot{v} \delta v + \frac{1}{2} \delta v^2\right) \langle E \rangle \right] + O(\eta^4).
\]
(32)

Consistent with our neglect of terms $\propto (\Delta M/M)^2$ (and smaller), we only retain contributions at $O(\eta^3)$ and $O(\eta^4)$; note that $p_{\text{ej}} \propto \eta^2$, $|\delta v| \propto \eta$, and consequently $(\dot{v} \delta v) \propto \eta$ and $(\dot{v} \delta v^2) \propto \eta^2$, which vanish in absence of mass loss -- is $\propto \eta^3$. To the same accuracy, the variance of $\Delta M$ is
\[
\overline{\sigma_{\Delta M}^2(b, \eta, \hat{\psi}_0, N)} = \frac{p_{\text{ej}}}{N},
\]
(33)
and the variance of $\Delta E$ is
\[
\overline{\sigma_{\Delta E}^2(b, \eta, \hat{\psi}_0, N)} = \frac{M^2}{N E^2} \left[ (1 - p_{\text{ej}}) \left(\dot{v} \delta v + \frac{1}{2} \delta v^2\right) \right] + p_{\text{ej}} \left(\frac{1}{2} \dot{v}^2 + \phi\right)
\]
\[
+ (1 - p_{\text{ej}}) \left(\dot{v} \delta v + \frac{1}{2} \delta v^2\right) \langle E \rangle
\]
\[
= \frac{M^2}{N E^2} \left[ (1 - p_{\text{ej}}) \left(\dot{v} \delta v + \frac{1}{2} \delta v^2\right) \right] + p_{\text{ej}} \left(\frac{1}{2} \dot{v}^2 + \phi\right) \langle E \rangle.
\]
(34)

Finally, the mixed moment is
\[
\overline{\sigma_{\Delta M \Delta E}^2(b, \eta, \hat{\psi}_0, N)} = \frac{\langle \Delta M \Delta E(b, \eta, \hat{\psi}_0) \rangle}{M \langle E \rangle} = \frac{\langle \Delta M \Delta E(b, \eta, \hat{\psi}_0) \rangle}{M \langle E \rangle}
\]
\[
= \frac{M}{N} \left[ p_{\text{ej}} \left(\frac{1}{2} \dot{v}^2 + \phi\right) \right],
\]
(35)
keeping only terms $\propto \eta^3$ and larger.

We have calculated the integrals needed to evaluate these moments using Monte Carlo methods. The velocity kick is evaluated using equation (18), and the centre-of-mass velocity kick is given in equation (A1). A sample of the results for
\[ \Psi_0 = 2.0, 0.0 \text{ and } \eta_c = 10^{-2} \text{ are plotted in Figs 2, 3 and 4. For } \Psi_0 = 2.0, 0.0 \text{ and } 0.0, \] 
the core radii are at \( r_{\text{core}}/r_t = 0.2, 0.1 \) and 0.05 respectively, while the half-mass radii are at \( r_{\text{hm}}/r_t = 0.3, 0.2 \) and 0.15 respectively. The Monte Carlo integrals were computed to an estimated fractional error of 5 per cent.

The separate contributions to the energy input in equation (32) have been plotted in Fig. 2. The contribution of mass loss is comparable to that due to the \( \delta v^2/2 \) heating usually considered.

Note too that the \( h v \delta v^2 \) term is surprisingly large for small-\( b \) encounters, contributing \( 5 \) per cent to the energy input for impacts near the cluster centre. The variance in equation (34) is also broken up into the heating part, \( \delta E/\Delta M \), and the mass loss part, \( \partial_\ell \langle (\delta v^2)^2 \rangle \), which are comparable in size.

### 5.3 The tidal limit

The calculation of \( \delta E/\Delta M = \sqrt{N} \delta E/\Delta M \), \( \delta E/\Delta E \), and \( \delta E/\Delta M \) is simplified considerably in the tidal limit, partly because mass loss is restricted to particles with speeds very close to the escape speed. Spitzer (1958) computed \( \delta E/\Delta E \) without mass loss; however, mass loss contributes significantly to \( \delta E/\Delta E \), sometimes exceeding the Spitzer term. The expressions found below illustrate the importance of mass loss explicitly, and are also useful for understanding disruption of clusters by high-mass black holes.

Formally, we should expand in the two parameters \( \eta \) and \( r_t/b \ll 1 \). Here, we keep only the lowest powers of \( \eta \) and \( r_t/b \). In this approximation, we find

\[
\frac{\Delta M(b, \eta_c, \Psi_0)}{M} = -C_M \frac{\eta_c^2}{\beta^2},
\]

where

\[
C_M = \frac{8}{9} \int_0^1 \left( \frac{\Psi_0}{\Psi_1} \right)^2 \frac{r_1^4}{f_1 f_0} d\xi \Theta(\xi)^{3/2},
\]

and \( \xi = r/a, \xi_1 = r_1/a, \) \( a = (\Psi_0/4\pi G r_0)^{1/2}, \) \( \mu_i = M_i/M_0, \) \( M_0 = 4\pi \rho_0 a^3, \) \( f_i = \rho_i/\rho_1, \) \( g_i = \sigma_i^2/\sigma_1^2, \) and \( \Theta(\xi) = \Psi(\xi)/\Psi_0 \) \( \rho_1 \)

| \( \Psi_0/\sigma_i^2 = 2.0 \) | \( \eta_c = 10^{-2} \) |
|-----------------------------|------------------|
| \( \delta M \)             | \( \delta v^2/2 \) |
| \( \delta E/\Delta M \)    | \( \delta E/\Delta E \) |

**Figure 2.** \( \langle \delta E(b, \eta_c) \rangle / |E| \).
and $\sigma_i$ are defined in equation (12)]. Hence $\sigma_{E,M}^2/M^2 = p_0/N = (C_{MM}/N)(\sigma_i^2/\beta^2)$, where $C_{MM} = C_M$. To $O(\eta_i^2)$, 

$$ \frac{\Delta E(b, \eta_i, \psi_0)}{|E|} = \frac{M}{|E|} \left( \frac{p_0}{r_i} \frac{GM}{r_i} + \left( \frac{1}{2} \hat{b} \nu \right) \right) $$

$$ = \frac{p_0}{|E|} \frac{GM}{r_i} + 4 \frac{G^2 M^2 \langle \sigma_i^2 \rangle}{|E|} = C_E \eta_i^2/\beta^2 $$

with

$$ C_E = C_M \frac{GM^2}{r_i |E|} + \frac{4 G^2 M^2 \langle \sigma_i^2 \rangle}{|E|} $$

In addition to the Spitzer energy loss $\langle \delta \nu^2/2 \rangle$, $C_E$ includes the energy carried away by the mass-loss, $\Delta M = -p_0 M$, from near the escape surface, where the energy per mass is $-GM/r_i$. Note that the ejected mass can come from any spatial position in the cluster as long as the velocity is close enough to the escape velocity. Indeed, the mean radius from which mass is lost depends weakly on cluster concentration, and is roughly $2r_i/3$ for $0 \leq \psi_0 \leq 8.5$. The variance to the energy input is

$$ \sigma_{E,E,M}^2 = \frac{1}{N^2 \eta_i} \left( \frac{GM^2}{|E| r_i} \right)^2 $$

$$ + 8 \frac{1}{N} \frac{GM^2 M^2}{9 E^2 V_{esc}^2} \left( \nu^2 \right)^2 $$

and the mixed moment is

$$ \sigma_{E,E,M}^2 = \frac{M}{N |E|} \left( \frac{1}{r_i} \frac{GM}{r_i} \right)^2 + \frac{4}{N} \frac{G^2 M^2 \langle \sigma_i^2 \rangle}{|E|} $$

$$ = - \frac{M}{N |E|} \left( \frac{1}{r_i} \frac{GM}{r_i} \right)^2 $$

with $C_{EM} = C_M(GM^2)/(|E| r_i)$. 

Figure 3. $|\Delta M(b, \eta_i)|/M$. 

$\psi_0/\sigma_i^2 = 2.0$

$\eta_e = 10^{-2}$

$\psi_0/\sigma_i^2 = 4.0$

$\eta_e = 10^{-2}$

$\psi_0/\sigma_i^2 = 6.0$

$\eta_e = 10^{-2}$

$\psi_0/\sigma_i^2 = 8.0$

$\eta_e = 10^{-2}$

$\psi_0/\sigma_i^2 = 10.0$

$\eta_e = 10^{-2}$
Another extremely important quantity will be the change in $n$; $\Delta n = GM^2/\Theta t$, in the tidal limit. Using the fact that the cluster is tidally limited, $n$ can be expressed as

$$n = \frac{GM^2}{\Theta t}$$

where the subscript refers to some reference value. The change in $n$ is then

$$\Delta n = \frac{GM^2}{\Theta t}$$

which can be written in the usual form

$$\Delta n = C_n \eta_c^2 \frac{r_t}{b^4}$$

The variance of the mean change in $n$ is given by

$$\sigma_n^2 = \frac{GM^2}{\Theta t}$$

The seven coefficients for the tidal limit are plotted in Figs 5 and 6 as a function of $\Psi_0$. All the curves show a monotonic decrease as $\Psi_0$ increases over the range $\Psi_0 \in (0.0, 8.5)$. Most noticeable is that $C_n$ becomes less than zero at roughly $\Psi_0 = 5.5$. Since $\nu = \nu(\Psi_0)$, whether an impact will make the cluster more or less concentrated is determined by the value of $\Psi_0$ for that cluster. Clusters with $\Psi_0 < 5.5$ are driven toward dissolution and clusters with $\Psi_0 > 5.5$ are driven toward core collapse. This agrees well with CKS, who derived the same result in the context of perturbations by giant molecular clouds.

For penetrating encounters, the situation is quite different. Over almost the entire range of $\Psi_0$ and $\eta_c$, the diffusion coefficients yield positive changes in $\nu$, implying dissolution.

### Figure 4. $Na_{\Delta E}(b, \eta_c)/E^2$.

In Section 5.2 we found $\Delta E$, $\Delta M$, $\sigma_{\Delta E}$, $\sigma_{\Delta M}$ and $\sigma_{\Delta E \Delta M}$ for individual collisions as a function of impact parameter $b$ and...
relative velocity $V_{rel}$. When $M_{bh}$ is small enough that the changes in the cluster properties are always small for single impacts, then we may average over many encounters to find the changes in $E$ and $M$ over a time period which includes many collisions, but for which the changes in cluster properties are still small.

We weight the formulae for the various moments by the differential rate of collisions
\[ d\Gamma = m_{bh}^2 \pi h f(V_{rel}) V_{rel} dV_{rel} \quad (46) \]
and integrate over $b$ and $V_{rel}$, where $f(V_{rel})$ is given in equation (16).

Table 2. Fitting formula coefficients for $dE$, $dM$, $dEE$ and $dEM$.

| $\hat{\psi}_0$ | $\kappa_E$ | $\kappa_M$ | $\kappa_{EE}$ | $\kappa_{EM}$ |
|----------------|------------|------------|----------------|-------------|
| 0.0            | 12.8       | 70.5       | 4.14           | 22.8        |
| 0.5            | 13.4       | 80.5       | 4.34           | 26.1        |
| 1.0            | 11.0       | 73.2       | 3.38           | 22.5        |
| 1.5            | 10.1       | 75.3       | 3.10           | 23.1        |
| 2.0            | 7.05       | 60.1       | 2.06           | 17.6        |
| 2.5            | 7.68       | 76.2       | 2.24           | 22.3        |
| 3.0            | 6.42       | 75.8       | 1.85           | 21.9        |
| 3.5            | 5.42       | 78.2       | 1.51           | 21.8        |
| 4.0            | 4.85       | 88.0       | 1.39           | 25.2        |
| 4.5            | 3.80       | 89.5       | 1.06           | 24.9        |
| 5.0            | 2.85       | 90.2       | 0.740          | 23.4        |
| 5.5            | 2.33       | 102        | 0.702          | 30.8        |
| 6.0            | 1.88       | 117        | 0.603          | 37.7        |
| 6.5            | 1.40       | 126        | 0.499          | 45.0        |
| 7.0            | 1.06       | 137        | 0.379          | 48.8        |
| 7.5            | 0.836      | 141        | 0.363          | 61.4        |
| 8.0            | 0.677      | 134        | 0.320          | 63.2        |
| 8.5            | 0.647      | 126        | 0.362          | 70.7        |

Note: The dimensionless diffusion coefficients are approximated by
\[ d = \hat{\psi} \ln(1/h_{\psi_0}) h_{\psi_0}^2 \]
and
\[ \hat{\psi}_0 = \eta_0 GM/\ln r_{\psi_0} \]
where $\hat{\psi}$ was taken to be $10^{-0.5}$. Due to the extreme long integration times for small values of $\eta_0$ and large $\hat{\psi}_0$, a few values were obtained using analytic fitting formulae of the form
\[ d_{EM} = \kappa_{EM} \ln(1/\eta_0) h_{\psi_0}^2, \]
which represented the data well; the coefficients for these fits are given in Table 2. The fits were generally good to $\pm 5 - 30$ per cent and are useful for quick estimates, although for more precise numerical work interpolation on tabulated values was used. The values generated using the fitting formulae were used only for the very smallest values of $\eta_0$ for the cluster Pal 5, where the fitting formulae were most accurate. Fig. 7 displays the various coefficients for several values of $\eta_0$, which are shown increasing from bottom to top. The triangles represent data.
which was generated by the Monte Carlo program itself, while the open circles represent the data estimated using the fitting formula.

It can be seen that for a fixed value of \( \psi_0 \), the diffusion coefficients in Fig. 7 vary by about an order of magnitude over the range \( \psi_0 = 0.0 - 8.5 \). The coefficients for the fitting formulae show a similar range of variation. This spread can be reduced considerably if we recall that many quantities vary little over the King sequence when expressed in terms of the half-mass radius, \( r_{\text{hm}} \). Choose a new unit of rate \( \tilde{\psi}_0 = \psi_0 r_{\text{hm}}^2 \tilde{V}_c \), and a new collision parameter \( \tilde{\eta}_{\text{coll}} = \eta_{\text{coll}} (G M / r_{\text{hm}}^2 \tilde{\psi}_0) \). The constants \( \tilde{k} \) in the new dimensionless diffusion coefficients \( \tilde{d} = \tilde{k} \ln(1/\tilde{\eta}_{\text{coll}}) \tilde{\psi}_0^2 = \tilde{d}(r_i^2/r_{\text{hm}}^2) \) then vary by about a factor of 2 over the range \( \tilde{\psi}_0 = 0.0 - 8.5 \), as can be seen in Table 2.

### 6 ‘SLOW-HEATING’ LIFETIMES

#### 6.1 The fixed cluster approximation

When \( M_{\text{bh}} < M_{\text{high}} \), the properties of a given cluster change only slightly over many encounters with black holes. In this limit, cluster survival implies an upper bound to \( dE dt \) and \( dM dt \) and \( d\psi dt \) are independent of time, and \( \langle \Delta E(T) \rangle = \langle \Delta M(T) \rangle = \langle \Delta W(T) \rangle = \langle \Delta E(T) \rangle = \langle \Delta M(T) \rangle = \langle \Delta W(T) \rangle = -\tilde{\psi}_0 \tilde{d}_E \). To go from \( \eta_{\text{coll}}(G M / r_{\text{hm}}^2 \tilde{\psi}_0) \) and \( \tilde{d}_E \) are independent of time and \( \tilde{d}_M \), and \( \tilde{d}_\psi \), we defined three distinct values of \( M_{\text{bh,crit}} \) by \( \langle \Delta E(T, M_{\text{bh,crit}}) \rangle = \langle \Delta M(T, M_{\text{bh,crit}}) \rangle = \langle \Delta W(T, M_{\text{bh,crit}}) \rangle = \langle \Delta E(T) \rangle = \langle \Delta M(T) \rangle = \langle \Delta W(T) \rangle = \langle \Delta E(T) \rangle = \langle \Delta M(T) \rangle = \langle \Delta W(T) \rangle = -\tilde{\psi}_0 \).

Choosing a time of \( T = 10^{10} \) yr as the time over which the clusters have been subjected to collisions, we defined three distinct values of \( M_{\text{bh,crit}} \) by \( \langle \Delta E(T, M_{\text{bh,crit}}) \rangle = \langle \Delta M(T, M_{\text{bh,crit}}) \rangle = \langle \Delta W(T, M_{\text{bh,crit}}) \rangle = \langle \Delta E(T) \rangle = \langle \Delta M(T) \rangle = \langle \Delta W(T) \rangle = -\tilde{\psi}_0 \). The results are presented in Table 3 for the sample of nine weakly bound clusters employed by Moore (1993). Moore’s results correspond most closely to \( M_{\text{bh,crit}}(E) \), as he used the criterion \( \langle \Delta E(T) = 7 \times 10^9 \rangle \) yr, \( M_{\text{bh,crit}}(E) = \langle \Delta E(T) \rangle \). When comparing with

| Cluster Name | \( M_{\text{bh,crit}}(E)/M_\odot \) | \( M_{\text{bh,crit}}(M)/M_\odot \) | \( M_{\text{bh,crit}}(\psi)/M_\odot \) |
|--------------|----------------|----------------|----------------|
| AM 4         | 1500           | 7000           | 710            |
| ARP 2        | 3100           | 13,000         | 4900           |
| NGC 5053     | 4100           | 18,000         | 5200           |
| NGC 7492     | 3400           | 15,000         | 6500           |
| Pal 4        | 170,000        | 8,500,000      | 350,000        |
| Pal 5        | 1100           | 4300           | 1100           |
| Pal 13       | 3100           | 13,000         | 6300           |
| Pal 14       | 14,000         | 68,000         | 15,000         |
| Pal 15       | 9100           | 39,000         | 6100           |

Note: \( M_{\text{bh,crit}}(E)/M_\odot \) is the critical black hole mass for the criterion \( \psi(E)/E = 0.5 \). \( M_{\text{bh,crit}}(M)/M_\odot \) is the critical black hole mass for the criterion \( |\Delta E|/|\Delta M| = 0.5 \). \( M_{\text{bh,crit}}(\psi)/M_\odot \) is the critical black hole mass for \( \psi \) to go from its initial value to \( \psi = -0.6 \).
Moore’s results, one should multiply $M_{bh,crit}$ by a factor of 20/7 to account for the larger value of $T$ and smaller $\Delta E$ used here. Except for Pal 13, our results agree with Moore’s to within a factor of order a few, which is reasonably close considering the improvements made here (e.g., inclusion of mass loss, correct $V_{rel}$ distribution).

### 6.2 Gaussian model with evolution

In Section 6.1 two approximations were made. First, the cluster was treated as having a fixed profile for which the diffusion coefficients did not change over time. The second approximation was that the energy input and mass loss had sharply defined values over any time interval. In this section the evolution of the cluster is followed over appropriately chosen intervals $\delta t$, and a Gaussian distribution of energy input and mass loss is assumed.

The characteristic time for $N_{min}$ collisions to occur inside $b_{max}$ is

$$T_{coll} = \frac{N_{min}}{\eta \sigma_0 b_{max}^2 \hat{V}} \propto M_{bh} R_{*}^2 N_{min}^{1/2}. \quad (48)$$

As most of the energy input is given by the penetrating encounters, a conservative estimate is to set $b_{max} = r_i$ in $T_{coll}$. If $\eta_{max}$ is the largest change allowed for the cluster in a time period, then a time $T_{change}$ over which the cluster evolves significantly is given by (Section 5.4)

$$T_{change} = \frac{\hat{V}_{max}}{\eta_{max} \sigma_0 \hat{V} c} \propto \frac{\max R_{*}^{1/2}}{M_{bh}}. \quad (49)$$

The Gaussian approach is justified if there exists a $\delta t$ such that $T_{coll} < \delta t < T_{change}$, so that the cluster properties change little over $\delta t$, but enough collisions occur that the distribution of energy inputs and mass loss is approximately Gaussian. An estimate of the largest black hole mass, $M_{bh,fp}$, for which the Gaussian approach is justified is

$$M_{bh,fp} \approx M_{bh} \sqrt{\frac{\hat{V}}{\sigma_0 \max R_{*}}} N_{min}^{1/2}, \quad (50)$$

or $\eta_{1,0} \approx (\max / N_{min})^{1/2}$. The critical mass can only be found in the present 'diffusion' approximation when $\max \ll 1$ and $N_{min} \gg 1$, so that the cluster must be destroyed for $\eta_{1,0} < 1$.

Let the vector $\tau = (\tau_1, \tau_2) = (\Delta E, \Delta M)$, where $\Delta E$ and $\Delta M$ are the energy input and mass loss over the time interval $\delta t$; the expected values of $\Delta E$ and $\Delta M$ over the time $\delta t$ are $(\tau_1) = (\Delta E(\delta t))$ and $(\tau_2) = (\Delta M(\delta t))$. The normalized distribution of energy input and mass loss is then

$$P(\tau) d \tau = \frac{\sqrt{det(\mathbf{Z})}}{2\pi} d \tau \exp \left[ -\frac{1}{2} (\tau - \langle \tau \rangle)^T \mathbf{Z} (\tau - \langle \tau \rangle) \right], \quad (51)$$

where $\mathbf{Z}(\delta t)$ is the covariance matrix. The matrix $\mathbf{Z}^{-1}$ may be shown to be

$$\mathbf{Z}^{-1} = \begin{bmatrix}
(\tau_1 - \langle \tau_1 \rangle)^2 & (\tau_1 - \langle \tau_1 \rangle)(\tau_2 - \langle \tau_2 \rangle) \\
(\tau_1 - \langle \tau_1 \rangle)(\tau_2 - \langle \tau_2 \rangle) & (\tau_2 - \langle \tau_2 \rangle)^2
\end{bmatrix},$$

inverting implies

$$\mathbf{Z} = \frac{1}{\det(\mathbf{Z}^{-1})} \begin{bmatrix}
(\tau_2 - \langle \tau_2 \rangle)^2 & -(\tau_1 - \langle \tau_1 \rangle)(\tau_2 - \langle \tau_2 \rangle) \\
-(\tau_1 - \langle \tau_1 \rangle)(\tau_2 - \langle \tau_2 \rangle) & (\tau_1 - \langle \tau_1 \rangle)^2
\end{bmatrix}.$$

These expressions reduce to the standard sum of Gaussian terms when the correlation $\langle \tau_1 - \langle \tau_1 \rangle \rangle (\tau_2 - \langle \tau_2 \rangle)$ is zero. The eigenvalues of $\mathbf{Z}$ are $\lambda_{\pm} = \pm \frac{1}{2} \sqrt{\text{Tr}(\mathbf{Z}) \pm \sqrt{\text{Tr}(\mathbf{Z})^2 - 4 \det(\mathbf{Z})}}$.

In order to choose values for $\Delta E$ and $\Delta M$, we define a new set of variables $\xi = (\xi_+, \xi_-)$ along the eigenvectors of $\mathbf{Z}$; the distribution of $\xi$ is

$$P(\xi) d^2 \xi = \frac{\sqrt{\lambda_+ \lambda_-}}{2\pi} d^2 \xi \exp(-\lambda_+ \xi_+^2/2 - \lambda_- \xi_-^2/2), \quad (52)$$

and $\tau - \langle \tau \rangle = \Delta \xi$, where the unitary matrix

$$\mathbf{A} = \begin{pmatrix}
\lambda_+ & \lambda_- \\
\lambda_- & \lambda_+
\end{pmatrix} = \frac{1}{\sqrt{(\lambda_+ - \lambda_-)^2 + \lambda_{12}^2}}$$

$$\begin{pmatrix}
\lambda_+ - \lambda_- & \lambda_{12} \\
-\lambda_{12} & \lambda_- - \lambda_+
\end{pmatrix}.$$

After choosing $\xi_{\pm}$ from this Gaussian distribution, we multiply by $\mathbf{A}$ to get

$$\Delta E = (\Delta E) + \lambda_+ \xi_+ + \lambda_- \xi_-,$$

$$\Delta M = (\Delta M) + \lambda_+ \xi_+ + \lambda_- \xi_-.$$

(53)

In the limit of infinite numbers of collisions $N_i$ in time $\delta t$ (the limit in which the Gaussian model is equivalent to a two-dimensional Fokker–Planck equation)

$$\langle \tau_1 \rangle = \langle \Delta E \rangle + \frac{1}{N_i} \mathcal{E}_0 d\mathcal{E} \xi_+,$$

$$\langle \tau_2 \rangle = -\frac{1}{N_i} \mathcal{E}_0 M \mathcal{E}_{M} \xi_+,$$

$$\xi_{\pm} \sim \frac{1}{\mathcal{E}_0 M} \mathcal{E}_{M} \xi_+.$$

(54)

For finite $N_\epsilon$, these expressions are accurate to $O(N_\epsilon^{-1})$.

Let us ignore the correlations of the change in mass and energy and focus solely on changes in cluster energy in a qualitative feel for the ‘distribution of cluster states’. The Fokker–Planck equation for the distribution of energies, $P(E,t)$, of an ensemble of clusters is

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial E} (D_E P) + \frac{1}{2} \frac{\partial^2}{\partial E^2} (D_{EE} P),$$

(55)

where we have chosen the initial condition $E = E_0$ at $t = 0$. For small times $t < D_{EE}/D_E$, the width of the Gaussian is much larger than the mean, implying a very spread out set of cluster states, and vice versa for times $t > D_{EE}/D_E$. As we have already shown in Section 2.3, for times $t = T_{disrupt}$ the set of final states is sharply defined about the mean. Using the survival criterion $\Delta E < f |E_0|$, we find that the probability of survival after time $t$ is given by

$$P_s = \int_{E_0}^{E_0 + f |E_0|} dE P(E,t) = \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{f |E_0| - D_E t}{\sqrt{2D_{EE} t}} \right) \right], \quad (57)$$

The same equation would have resulted if we had used the two-dimensional Fokker–Planck equation for $E$ and $M$. In terms of the fitting formulas of Section 5.4, the characteristic mass range, $M_{bh,crit}$ over which $P_s$ changes from one to zero is

$$\frac{\delta M_{bh,crit}}{M_{bh,crit}} = \frac{\mathcal{S}_{EE} \sqrt{\frac{N_f}{N_{M_\epsilon}}}}{\mathcal{E}_0 \mathcal{E}_{M}} \ll 1,$$

(58)

so that in the $M_{bh} \ll M_{high}$ limit the value of $M_{bh,crit}$ found from just one history is likely to be close to the statistical mean.

### 6.3 Simulations and results

We chose the accuracy parameters from the previous section to be $\varepsilon_{max} = 0.01$ and $N_{min} = 10$. The time-step was chosen so that the
estimated change in energy would be 0.01. As the diffusion coefficients tended to increase as $\tilde{v}_0 \to 0$ and $M$ decreased, the time step was decreased if two events in a row occurred with fractional changes in $\nu$, $M$ or $E$ greater than 0.01. Also, the time step was increased if the estimated number of events in $\tilde{\Delta}t$ was less than $N_{\text{rand}}$. If $M_{\text{bh}} > M_{\text{bh,dir}}$, as described in the previous section, the simulation was stopped, because the approximations had broken down.

For each of the nine weakly bound clusters from Moore (1993), values of $M_{\text{bh}}$ were chosen and the cluster was evolved to one of the following outcomes: (1) the Fokker–Planck assumption broke down; (2) $T = 10^{10}$ yr was reached and the cluster survived; (3) the quantity $\nu$ went out of the range $(-2.13, -0.6)$, signalling either core collapse or ‘dissolution’. Survival probabilities and their uncertainties were derived from the number of surviving clusters out of 1000 simulations using Bayesian arguments. The results are shown in Fig. 8 for Moore’s clusters as well as one more ‘normal’ cluster. Note that all curves asymptote to $f_{\text{dest}} = 1.0$ as $M_{\text{bh}} \to \infty$, because single-event destruction has been ignored in these simulations.

Pal 4 did not satisfy the Fokker–Planck assumption at the black hole mass required to disrupt it in $T = 10^{10}$ yr. Since Pal 4 lies at such a great Galactocentric distance, encounters are so infrequent that disruption in $10^{10}$ yr requires individually destructive collisions. Hence a critical black hole mass $M_{\text{bh,crit}}$ cannot be determined by the slow heating approximation for Pal 4; the more detailed collision-by-collision history of Section 8 is warranted for this case.

As for the rest of Moore’s clusters, we see that their values of $M_{\text{bh,crit}}$ agree well with the last column of Table 3; the values obtained from the Gaussian Monte Carlo simulation are lower by less than a factor of 2. As clusters are heated and lose mass, diffusion coefficients increase as $\tilde{v}_0 \to 0$, and $\tilde{v}_0$, decreases, leading to smaller $M_{\text{bh,crit}}$ than for fixed cluster structure and mass. Also, note that the fractional width, $\delta M_{\text{bh,crit}}(M_{\text{bh,crit}})$ of the ‘transition region’ over which $f_{\text{dest}} = 0.1 \to 0.9$ varies as $N^{-1/2}$, as predicted in Section 2.3.

We also simulated a large cluster with $M = 10^6 M_\odot$, $N = 1.4 \times 10^5$, $r = 50$ pc, $r_{\text{core}} = 5$ pc, and $R = 10$ kpc; hence $\tilde{v}_0 = 4.85$ and the central one-dimensional velocity dispersion $\sigma_0 = 9.64$ km s$^{-1}$. Fig. 8 shows $P_s$ as a function of $M_{\text{bh}}$ for this cluster. The critical black hole mass is quite a bit larger than that found with the small clusters from Moore (1993), because $M$ is larger. Note that $M_{\text{bh,crit}} \sim M / 10$ for this cluster, which is on the verge of the limit in which many black holes will be inside the cluster at any given time.

Klessen & Burkert (1996) recently argued that Moore’s determination of $M_{\text{bh,crit}}$ for his set of nine diffuse clusters was flawed, because deviations from the mean energy input for each encounter history would be quite large, implying a significant scatter in the values of $M_{\text{bh,crit}}$. They contend that clusters are destroyed as a consequence of a small number of encounters with large energy inputs. Instead, we find that for eight of Moore’s nine clusters, all except Pal 4, disruption is due to numerous encounters with $b \approx r_{\text{core}}$ which cause small changes individually. For these light clusters, the expected number of impacts, $N_{\text{core}}(M_{\text{bh,crit}})$, inside $r_{\text{core}}$ is large for $M_{\text{bh}} = M_{\text{bh,crit}}$. We find that five of Moore’s clusters have $N_{\text{core}}(M_{\text{bh,crit}}) > 200$, two have $N_{\text{core}}(M_{\text{bh,crit}}) > 50$, one has $N_{\text{core}}(M_{\text{bh,crit}}) = 20$, and Pal 4 has $N_{\text{core}}(M_{\text{bh,crit}}) \leq 1$. So (aside from Pal 4) the expected fluctuations in the $N_{\text{core}}(M_{\text{bh,crit}})$ are $\leq 20$ per cent for eight of Moore’s clusters, and $\leq 10$ per cent for seven of them. Only for Pal 4, which sits at a large Galactocentric radius where few encounters occur, is the evolution near $M_{\text{bh,crit}}$ very stochastic. This case will be discussed in the next section.

7 THE LARGE-$M_{\text{bh}}$ LIMIT FOR THE SURVIVAL PROBABILITY

7.1 Theory

The previous section treated the small-$M_{\text{bh}}$ regime in which many collisions slowly add energy and remove mass from a cluster. This approximation will be valid for $M_{\text{bh}} \ll M_{\text{high}}$, the mass at which the cluster can be disrupted in a single collision. In the opposite limit for which $M_{\text{bh}} \approx M_{\text{high}}$, any collision within a ‘destructive impact parameter’, $b_d$, will destroy the cluster. In addition, the cumulative effect of many collisions outside $b_d$ can also destroy the cluster.

Formulae for the changes in $E$, $M$, and $\nu$ for a single collision in the tidal limit were derived in Section 5.3; see equations (37), (38) and (44). The three formulae have the same scaling with $r$, $b$, and $\nu$, and only differ by a factor which depends on $\tilde{v}_0$. Hence, in the following derivations we only use the formula for energy input to derive results explicitly. To find the expressions for disruption by mass loss or change in $\nu$, one must only substitute $C_E \to C_M$, $C_r$.

The energy input can be rewritten as

$$\frac{\Delta E}{|E|} = C_E \frac{r^4}{b^4} = f \left( \frac{b_d^4}{b^4} \right),$$

(59)

where the impact parameter for $\Delta E/|E| = f$ is

$$b_d = r \eta^{1/2} \left( \frac{c_E}{f} \right)^{1/4}.$$  

(60)

This energy input is sharply peaked about the mean for $b = b_d$, because the impact parameter at which $\Delta E = \sigma_{\Delta,b} E$ is larger than $b_d$ by a factor of $N^{1/2}$.

A cluster has been destroyed if an impact occurs with $b < b_d$. The critical impact parameter $b_d \approx r$ for $M_{\text{bh}} \approx M_{\text{high}} = (M_{\text{rel}}/\sigma_0)/(f CE)^{1/2}$. Using $\langle V_{\text{rel}} \rangle = 1.47 V_c$ (see equation 17), and $\sigma_0 = 1 \times 10$ km s$^{-1}$ implies $M_{\text{high}} \approx 10^{-100} M_\odot$.

The mean number of destructive encounters, $N_d$, which occur in a time $T$ is given by the integrated rate of encounters inside $b_d(V_{\text{rel}})$:

$$N_d(T) = \pi \eta_{\text{bh}} T \int_{b_d(V_{\text{rel}})}^{\infty} \frac{d V_{\text{rel}}}{V_{\text{rel}}} \int_{b_d(V_{\text{rel}})}^{\infty} 2 \pi b d b \left[ f \left( \frac{b_d^4}{b^4} \right) \right] = \left( \frac{c_E}{f} \right)^{1/2} \rho_{\text{bh}} \pi^2 \eta_0^4 T.$$  

(61)

Note that $N_d(T) \propto f_{\text{halo}}$, but is independent of $M_{\text{bh}}$. Consequently, when $M_{\text{bh}} > M_{\text{high}}$, limits can be derived for $f_{\text{halo}}$ but not $M_{\text{bh}}$ (Wienlen 1988).

For given $f$ and $T$, the Poisson probability of no individually destructive events is given by

$$P_s(f_{\text{halo}}) = \exp[-N_d(f_{\text{halo}})].$$  

(62)

The expression for $P_s$ becomes more complicated when we include the cumulative effect of all the non-destructive encounters outside $b_d$. As most collisions have $b > b_d$, they do not significantly change the cluster properties. These gentle collisions can be modelled using diffusion coefficients.

The mean rate of change of $E$ due to impacts outside $b_d$ is given by

$$\frac{d E}{d T} = n_{\text{bh}} \int_0^{\infty} d V_{\text{rel}}(V_{\text{rel}}) V_{\text{rel}} \int_{b_d(V_{\text{rel}})}^{\infty} 2 \pi b d b \left[ f \left( \frac{b_d^4}{b^4} \right) \right] = f \Gamma_d,$$  

(63)

where $\Gamma_d = N_d(t) / t$. Similarly, the time derivative of the variance is given by

$$\frac{d \sigma^2}{d E} = n_{\text{bh}} \int_0^{\infty} d V_{\text{rel}}(V_{\text{rel}}) V_{\text{rel}} \int_{b_d(V_{\text{rel}})}^{\infty} 2 \pi b d b \left[ f \left( \frac{b_d^4}{b^4} \right) \right] = \frac{1}{3} f^2 \Gamma_d.$$  

(64)

where we have ignored the small variance per collision; this variance is due only to the variable number of encounters in a
Figure 8. Fraction of clusters destroyed in $T = 10^{10}$ yr plotted against black hole mass. These curves were generated using the Gaussian Monte Carlo method.
given time period. In addition, collisions inside $b_d$ act as a 'sink' for clusters. The appropriate equation for $P(E, t)$, the probability density at time $t$ and energy $E$, is

$$\frac{\partial P}{\partial t} = -\Gamma_d P - \frac{E}{E_0} \frac{\partial P}{\partial E} + \frac{1}{2} \frac{\partial^2 P}{\partial E^2},$$

where again the diffusion coefficients are treated as constants for simplicity. The solution with $E = E_0$ at $t = 0$ is

$$P(E, t) = \exp(-\Gamma_d t) \left[ \frac{(E - E_0 - Et)^2}{2\sigma_E^2 t} \right].$$

This expression reduces to the $M_{bh} < M_{high}$ solution when $b_d \to 0$.

The probability of survival is then the cumulative probability that the change in $E$ has been less than $f|E|$, or

$$P_s = \frac{1}{2} e^{-f|E|T_{1/2}} \left[ 1 + \text{erf} \left( \frac{f|E| - ET}{\sqrt{2}\sigma_E^2 T_{1/2}} \right) \right]$$

$$= \frac{1}{2} e^{-N_d} \left[ 1 + \text{erf} \left( \frac{1 - N_d}{\sqrt{2N_d / 3}} \right) \right].$$

For small $N_d$, $P_s = \exp(-N_d)$, as in equation (62), but for large $N_d$, $P_s = (6\pi N_d)^{-1/2} \exp(-5N_d/2)$, which decreases much faster. Analogous results can be derived for changes in $\nu$ and $M$.

This decrease in the survival probability below the simple exponential decline due to the effects of distant encounters has been noted by previous investigators. In Wielen (1988), fig. 5 presented a numerical determination of the survival probability in the large-$M_{bh}$ limit for clusters bombarded by both black holes and giant molecular clouds. Using $N_d(T) = 0.5 \times T/T_{1/2}$, where $T_{1/2}$ is the time over which half the clusters have dissolved, our equation (67) is a rather good fit to the figure.

The results of this section do not agree with Klessen & Burkert (1996), since the curves of $P_s$ against $M_{bh}$ in their figs 8, 9 and 10 show no systematic evidence of an asymptotic $P_s$ independent of $M_{bh}$. This is surprising, as they have included exponential adiabatic damping (Spitzer 1958, but see also Murali & Arras, in preparation, and Weinberg 1994 for a more recent version of these effects), which would decrease the energy input per encounter and hence increase $P_s$. The lack of encounters with small energy input in
Klessen & Burkert’s (1996) simulations could result if their method of choosing the collision parameters over-sampled small $b$ and $V_{\text{rel}}$ compared to our method.

Table 4 contains the allowed values of $f_{\text{halo}}$ derived from the fixed cluster approximation of this section for the nine clusters found in Moore (1993). Two different destruction criteria were used: (1) $\nu$ out of bounds in $T = 10^{10}$ yr due to the combined influence of single destructive collisions and multiple nondestructive collisions (equation 67), and (2) a single episode of 20 per cent mass loss in $T = 10^{10}$ yr (equation 62). Also given are the values of $M_{\text{high}}$, the black hole mass above which the cluster can be destroyed in a single collision, appropriate to each criterion. Three values were used for the probability of survival: $P_s = 0.1, 0.5$ and 0.9.

For 50 per cent survival probability using the criterion $\nu \rightarrow -0.6$, the limiting values of $f_{\text{halo}}$ range from 0.02 (Pal 5) to 0.90 (Pal 4). It is unlikely that $f_{\text{halo}} > 0.3$, since $P_s < 0.1$ for most of the clusters in that case. These limits depend sensitively on Galactocentric radius and cluster size, as is evident from the scatter in the values of $f_{\text{halo}}$.

The results of this section will be tested in Section 8 using the results of the full Monte Carlo simulations. One wrinkle which appears in the results is that the different criteria for destruction can compete with each other, so that the results of this section are not always good approximations to $P_s(N_b)$. Only in the cases where one method of destruction completely dominates over all the others do the results agree accurately.

## 8 MONTE CARLO SIMULATIONS OF INDIVIDUAL COLLISIONS

### 8.1 The set-up

Previously, $M_{\text{bh, crit}}$ and $f_{\text{halo}}$ were calculated for the two limiting cases of slow heating (small $M_{\text{bh}}$) and single-event destruction (large $M_{\text{bh}}$). The major simplification for both methods was the neglect of detailed, collision-by-collision evolution of the cluster.

Both calculations relied on approximations. The Gaussian Monte Carlo method relied on restricting collisions to very small energy and mass changes individually, and involved averaging over many collisions to streamline the computations. In this section we present a Monte Carlo simulation of individual black hole encounters. We make no approximations for the energy input besides the impulse and straight-line approximations; consistent with the former, we neglect displacement of cluster stars during the encounter. This treatment is simpler than an $N$-body simulation because the evolution of the cluster is mapped by the sequence of King models. As before, the King sequence limits the evolution we allow.

The calculations amount to simulating the ‘Green’s function’ of the cluster. A cluster in a given initial configuration is subjected to collisions with the halo black hole population. After each collision the King model for the cluster is altered, using its post-collision $E$ and $M$, and assuming $r_i \propto M^{1/3}$. An evolutionary history of the cluster is mapped out over $10^{10}$ yr, and a final cluster state is found. As this process depends on many random variables, a range of final states is possible for each initial state, and many realizations are needed to find the Green’s function.

### 8.2 The probability distributions for $t, b$ and $V_{\text{rel}}$

In Section 4 we derived the distribution of relative cluster–black hole speeds assuming an isothermal black hole halo. For the Monte Carlo simulations, we need to know the probability that the next collision suffered by a cluster occurs a time interval $(0,t)$ after a given encounter, and involves an impact parameter in $(b, b+db)$ and a relative speed in $(V_{\text{rel}}, V_{\text{rel}}+dV_{\text{rel}})$. Since the number of collisions inside $b$ is $\approx b^2$, we chose a maximum impact parameter $b_{\text{max}}$, and define $\sigma_{\text{max}} = \pi b_{\text{max}}^2$. Let us first consider collisions with a single relative velocity, $V_{\text{rel}}$. The expected number of collisions in a time $t$ is

$$N = \int_0^t n_{\text{bh}} \sigma_{\text{max}} V_{\text{rel}} \, \text{d}t,$$

so the Poisson probability that there are no collisions for the time interval $(0, t)$ and then one collision in $(t, t+dt)$ is

$$dP_1(t) = \exp \left( -\int_0^t n_{\text{bh}} \sigma_{\text{max}} V_{\text{rel}} \, \text{d}t \right) \times n_{\text{bh}} \sigma_{\text{max}} V_{\text{rel}} \, \text{d}t.$$

Multiplying by the probability $2\pi bdb/\sigma_{\text{max}}$ that the collision is in

### Table 4. Limits on $f_{\text{halo}}$ in the large-$M_{\text{bh}}$ limit.

| Cluster Name | Criterion | $M_{\text{high}}/M_\odot$ | $f_{\text{halo}}(P_s = 0.9)$ | $f_{\text{halo}}(P_s = 0.5)$ | $f_{\text{halo}}(P_s = 0.1)$ |
|--------------|-----------|-----------------------------|-------------------------------|-------------------------------|-------------------------------|
| AM 4         | $\nu$     | $3.8 \times 10^3$           | 0.028                         | 0.14                          | 0.33                          |
| ARP 2        | $\nu$     | $1.6 \times 10^2$           | 0.014                         | 0.07                          | 0.16                          |
| NGC 5053     | $\nu$     | $1.8 \times 10^3$           | 0.011                         | 0.06                          | 0.13                          |
| NGC 7492     | $\nu$     | $1.5 \times 10^2$           | 0.029                         | 0.15                          | 0.34                          |
| Pal 4        | $\nu$     | $1.4 \times 10^2$           | 0.174                         | 0.90                          | $>1.0$                        |
| Pal 5        | $\nu$     | $8.6 \times 10^2$           | 0.005                         | 0.02                          | 0.05                          |
| Pal 13       | $\nu$     | $6.4 \times 10^2$           | 0.053                         | 0.27                          | 0.63                          |
| Pal 14       | $\nu$     | $8.7 \times 10^2$           | 0.035                         | 0.18                          | 0.41                          |
| Pal 15       | $\nu$     | $4.4 \times 10^2$           | 0.025                         | 0.13                          | 0.30                          |
| AM 4         | 20%M      | $4.1 \times 10^5$           | 0.031                         | 0.20                          | 0.67                          |
| ARP 2        | 20%M      | $7.6 \times 10^6$           | 0.006                         | 0.04                          | 0.14                          |
| NGC 5053     | 20%M      | $1.0 \times 10^2$           | 0.006                         | 0.04                          | 0.13                          |
| NGC 7492     | 20%M      | $4.9 \times 10^6$           | 0.009                         | 0.06                          | 0.20                          |
| Pal 4        | 20%M      | $8.9 \times 10^6$           | 0.109                         | 0.72                          | $>1.0$                        |
| Pal 5        | 20%M      | $5.8 \times 10^6$           | 0.003                         | 0.02                          | 0.07                          |
| Pal 13       | 20%M      | $2.1 \times 10^6$           | 0.018                         | 0.12                          | 0.39                          |
| Pal 14       | 20%M      | $6.1 \times 10^6$           | 0.025                         | 0.16                          | 0.54                          |
| Pal 15       | 20%M      | $3.9 \times 10^6$           | 0.023                         | 0.15                          | 0.50                          |

(a) The criterion ‘$\nu$’ means that neither single destructive events nor many non-destructive events caused $\nu \rightarrow -0.6$. The criterion ‘20%M’ means that no single events of 20 per cent mass loss occurred.
The values of \( x \) where \( \Delta V \) distributions in the first, second and third factors in square brackets can be used to choose \( h \). Also, the velocity distribution found in the middle time-independent, so equation (72) holds for any orbit, not just circular ones. We keep track of six different conditions for destruction. For two of these, the simulation is stopped at once. The first, ‘\( r \) out of bounds’, triggers if \( r \) goes out of the range \((-2.13, -0.6)\) (since all clusters simulated have \( \psi < 5.5 \), signalling that no unique member of the King sequence can be found to represent the cluster. In our simulations, clusters always went out of bounds at \( r \to -0.6 \), so below we refer to this condition as ‘\( r \to -0.6 \)’. The second criterion for an immediate halt, called ‘\( M_{20} \)’, is realized if a single event of \( \Delta M \approx 0.2M \) occurs which would greatly distort the cluster and invalidate our approximation of small \( \Delta M/M \). There are four other criteria which are recorded \( \text{the first time they occur} \), but do not stop the simulation. These are: ‘\( M_{10} \)’, a single occurrence of 10 per cent mass loss; \( \frac{1}{2} M \), decrease of the mass of the cluster to half its original value; \( \frac{1}{2} E/M \), change of the energy of the cluster up or down by half of its original value; and \( \frac{1}{2} E/M \), change of the quantity \( E/M \) up or down by half of its original value. Survival of the cluster will be called criterion ‘\( \psi \)’. Note that the criteria \( M_{10}, \frac{1}{2} E \) and \( \frac{1}{2} E/M \) can happen repeatedly before \( r \to -0.6 \). \( M_{20} \) or \( s \), but are only recorded the first time they occur. The two conditions \( \frac{1}{2} E \) and \( r \to -0.6 \) correspond most closely to the criteria used by previous investigators, but since our simulations include mass loss, the correspondence is not exact. It is possible for a cluster to be destroyed in less than \( 10^{10} \) yr by \( M_{20} \) or \( r \to -0.6 \), even though \( \frac{1}{2} M, \frac{1}{2} E, M_{10} \) or \( \frac{1}{2} E/M \) might not have had a chance to occur yet. Moreover, \( M_{20} \) may occur before \( r \to -0.6 \). When analysing the results, the competition among the various criteria must be kept in mind. This simulation ignores a number of possibly important effects. The most restrictive approximation is the use of the King sequence to model the evolution of the cluster. Relatively small changes in mass and energy may lead to \( \psi < 5.5 \), so a normal cluster could have a lifetime larger than is found here. For example, a \( M \approx 10^8 \) Msol cluster could lose 30 per cent of its binding energy and mass, but no longer be fitted well by a King model; yet you would still have a \( M \approx 7 \times 10^7 \) Msol cluster. Our treatment assumes that clusters become unstable to rapid dissolution when \( \psi \to 5.5 \).

To narrow the focus to the effects of halo black holes, we have neglected the disc and bulge components of the galaxy; destructive effects such as disc shocking and collisions with molecular clouds have also been suppressed entirely. Internal evolution of the cluster and non-spherical Galactic tidal fields are not treated. Clusters are kept at single \( R_R \), so time dependence of the density of halo black holes and relative velocities along cluster orbits is neglected. Lastly, we will make fractional errors of order \( \Delta M/M \) in our method.

Nevertheless, our model is an advance on previous attempts to constrain properties of a hypothetical population of halo black holes.

\[ \Delta \mathbf{v}_i, \] the centre-of-mass velocity kick is \( \Delta \mathbf{v} = (1/N) \sum_{i=1}^{N} \Delta \mathbf{v}_i \); even stars which are ejected are included in the sum. Star \( i \) is ejected from the cluster if \( \mathbf{v}_i + \Delta \mathbf{v}_i - \mathbf{v}_\psi \gg \mathbf{v}_\psi \), where \( \mathbf{v}_\psi \) is the pre-collision potential at \( r_i \). If \( N' \) stars remain in the cluster, then \( M' = M(N'/N) \) is the new mass of the cluster. The new energy of the cluster is the total energy input to the remaining stars (\( i = 1, 2, ..., N' \)) plus the contribution from the ejected stars (\( j = 1, 2, ..., N - N' \)):

\[ \Delta E = \sum_{i=1}^{N'} m \left( \mathbf{v}_i \cdot \delta \mathbf{v}_i + \frac{1}{2} |\delta \mathbf{v}_i|^2 \right) + \sum_{j=1}^{N-N'} \left( \frac{GM}{r_j} + \psi_j - \frac{1}{2} v_j^2 \right). \]
Figure 9. Fraction of initial AM 4 clusters destroyed according to the various criteria for destruction as a function of $M_{\text{bh}}$. These curves were made using the full Monte Carlo method.
via their effects on clusters. We include mass loss which, as was shown in Section 5.3, contributes significantly to cluster heating, sometimes slightly more than ‘Spitzer’ heating. Moreover, since the mean mass loss is comparable to the mean energy input, the evolution of \( \nu \approx E/\Delta M^3 \) is driven by both \( \Delta E \) and \( \Delta M \). Here no simplifications of the energy input are employed except the impulse approximation and the straight line orbit approximation. Hence the Monte Carlo method for finding \( \Delta E(b) \) includes the important ‘shocking’ effect of the \( \nu \delta E \). Lastly, the correct expression for the rate of collisions in equation (16) is used. We also note that many criteria for the cluster to be disrupted have been used in the past; the most popular is \( \Delta E(t)/|E| = 1 \). This choice is usually implemented with no regard to mass loss and evolution. Here, we test a variety of criteria for cluster destruction in order to find the most restrictive.

One last technical detail which must be discussed is the value of \( b_{\text{max}} \), the maximum impact parameter for our scattering experiment. Since the number of collisions that must be simulated \( \nu b_{\text{max}} \), it is essential to choose as small a \( b_{\text{max}} \) as possible without losing accuracy. First, examine the small \( M_{\text{bh}} \) case in which the cluster cannot be destroyed in a single pass. In this slow heating limit, \( \Delta E(b) \approx b^{1/2} \) outside the core, and \( \int_{b_{\text{max}}}^{b_{\text{core}}} 2\pi bdb \Delta E(b) \) converges as \( b_{\text{max}}^{1/2} \). Nearly all of the heating results from penetrating encounters, and \( b_{\text{max}} = r_1 \) will be a good approximation. However, in the large-\( M_{\text{bh}} \) case there is a critical impact parameter \( b_1^* \approx M_{\text{bh}} \) inside of which the cluster is destroyed, and we must choose \( b_{\text{max}} > b_1^* \approx M_{\text{bh}}^{1/2} \). An estimate for the \( M_{\text{bh}} \), the value of \( M_{\text{bh}} \) above which \( b_{\text{max}} \) must increase \( \approx M_{\text{bh}} \), is discussed in Section 7.

8.4 Description of the simulations

The time necessary to realize a cluster with \( N = 1000 \) stars and compute the energy input was \( \approx 1 \text{ s} \) on a Sun workstation. The computation time needed to map out the history for one cluster is \( \approx N_{\text{ev}} \times 1 \approx (80 \text{ s})(M_{\text{bh}}/10 \text{ pc}/r_1) \). This severely limited practical choices for \( N \) and \( m \). In order for the simulations to be realistic, we chose to simulate the full range of \( M_{\text{bh}} \) for AM 4 only, using \( m = 0.7 \text{ M}_\odot \) and \( N = 1000 \) initially.

To study much more massive clusters, the time limitation would force the number of stars used to be a small fraction of the physical value. As a consequence, the variance in energy input and mass loss would be unrealistically large in the simulations, and hence the Green’s function would be spread over too broad a range of final states. Too small a number of stars will decrease the number of stars in the portion of phase-space from which particles are ejected resulting in an incorrect evaluation of mass loss, especially if the total mass lost in the simulation is small. Uneven sampling of phase-space, arranged with finer spacing near the escape surface, could alleviate this problem.

For a given \( M_{\text{bh}} \), \( b_{\text{max}} \) and \( f_{\text{halo}} \), a number \( N_{\text{t}} \) trials were performed. The number of clusters destroyed, \( N_{\text{dest}} \), by each of the six criteria was recorded and the fraction \( f_{\text{dest}} = N_{\text{dest}}/N_{\text{t}} \) of clusters destroyed computed. In addition to \( f_{\text{dest}} \) for each criterion, the sum of \( f_{\text{dest}}(\nu \rightarrow -0.6) + f_{\text{dest}}(M_{\text{bh}}) \) is computed, because the sum of the fraction destroyed by these two conditions is not subject to competition effects which appear for the six criterion separately.

At least two runs with different \( b_{\text{max}} \) were done for each cluster at a certain \( M_{\text{bh}} \). The runs with the larger \( b_{\text{max}} \) are presented here. The variation in \( f_{\text{dest}} \) for the two runs was in all cases within the error bars shown in the figures. The value of \( b_{\text{max}} \) used for the results presented here was \( b_{\text{max}} = 2r_1 \) for \( M_{\text{bh}} \leq 100 \text{ M}_\odot \), and \( b_{\text{max}} = 2r_1(M_{\text{bh}}/100 \text{ M}_\odot)^{1/2} \) for \( M_{\text{bh}} > 100 \text{ M}_\odot \). These values of \( b_{\text{max}} \) are much larger than is needed, as \( M_{\text{bh}} \approx 4 \times 10^5 \text{ M}_\odot \approx 5700 \text{ M}_\odot \) (see Table 3), and the destructive radius is not outside \( r_1 \) until \( M_{\text{bh}} > M_{\text{bh}} \).

The number, \( N_{\text{t}} \), of trials ranged between \( N_{\text{t}} = 40 \) for small \( M_{\text{bh}} \) and \( N_{\text{t}} = 1000 \) for large \( M_{\text{bh}} \). The number of trials was restricted by the run time, which was greatly increased for small \( M_{\text{bh}} \) because of the large number of collisions \( \nu f_{\text{halo}}/M_{\text{bh}} \). Bayesian methods were used to compute \( P_{\text{r}} \) and its uncertainty. The error bars in the figures span the range of \( P_{\text{r}} \) around the peak of its posterior containing 68 per cent probability.

8.5 Results and discussion

The first set of runs was for \( f_{\text{halo}} = 1 \). The fractions of trial clusters destroyed by our various criteria are plotted in Fig. 9.

The dominant destruction mechanism by far is \( \nu \rightarrow -0.6 \). For this mechanism, the black hole mass at which 50 per cent of the clusters were destroyed is

\[
M_{\text{bh,crit}}(50\%) = 600 \text{ M}_\odot \approx 0.86 M_{\odot}.
\]  

(75)

The other destruction mechanisms only come into play once \( M_{\text{bh}} \approx 5 \times 10^5 \text{ M}_\odot \). At this point, \( \nu \rightarrow -0.6 \) still destroys 100 per cent of the clusters (as predicted in Table 4 for \( f_{\text{halo}} = 1.0 \)); even the approximately 40 per cent that die via \( M_{\text{BH}} \) do so in the same encounter that pushes \( \nu \rightarrow -0.6 \). If \( \nu \rightarrow -0.6 \) were not stopping the simulations, the other destruction criteria would be more important, and \( f_{\text{dest}} \) would not necessarily asymptote to small values as in Fig. 9.

We can compare the results in Fig. 9 for the criterion \( \nu \rightarrow -0.6 \) with the Gaussian Monte Carlo simulations in Fig. 8. There the critical black hole mass is \( M_{\text{bh,crit}}(50\%) \approx 650 \text{ M}_\odot \), which is different from the full Monte Carlo simulation by \( \approx 10 \) per cent. Considering that the error bars in \( f_{\text{dest}} \) are of order 10 – 15 per cent, the two numbers agree. In addition, the mass range for which \( 0.9 \approx f_{\text{dest}} \approx 0.1 \) is \( M_{\text{bh,crit}} \approx 250 \text{ M}_\odot \) for both the Gaussian Monte Carlo and full Monte Carlo cases. The agreement of these two methods shows that the assumptions of (1) a Gaussian distribution of energy input and mass loss, and (2) smooth Fokker–Planck evolution, are accurate in the small-\( M_{\text{bh}} \) regime. In addition, the ansatz for the variance of the number of stars ejected in Section 5.1

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Figure 9 – continued
Comparison of full Monte Carlo simulations against theory for AM 4 with $M_{bh} = 5000M$. The circles are the points of $P_s$ versus $N_d$ for the destruction criterion $\nu = -0.6$. The triangles are for $P_s$ versus $N_d$ with the destruction criterion of 20 per cent mass loss in a single encounter.

The second set of runs explores the large-$M_{bh}$ limit of Section 7. Two runs for AM 4 at $M_{bh} = 1000M = 7 \times 10^5 M_\odot$ and $M_{bh} = 5000M = 3.5 \times 10^6 M_\odot$ were performed as discussed in the previous section. The results are shown in Figs 10 and 11. In addition, we also tried NGC 5053 since it has a fairly large mass, $M = 37700 M_\odot$, and yet it restricts $f_{halo}$ to be small. Because the run time would have been prohibitively long using the correct number of stars in this cluster, we chose $N = 1000$ for NGC 5053. As discussed previously, this should not change the values of $P_s$ much, since we will be in the limit in which the collisions are quite strong, and the variances are expected to be small compared to the means. A single black hole mass of $M_{bh} = 1000M = 3.77 \times 10^5 M_\odot$ was used to get the results shown in Fig. 12.

The values of $f_{halo}$ used in Figs 10 and 11 are (from left to right) $f_{halo} = 0.05, 0.1, 0.2, 0.3, 0.4, 0.5$ and 0.6. The values of $N_d$ computed with equation (61) for each cluster will depend on the criterion for destruction through the values of $f$ and $C$ chosen. This is why the two curves for the two criteria $\nu = -0.6$ and $\Delta M/M = 20$ per cent do not have the same values of $N_d$ for each data point of a given $f_{halo}$. We have converted the values of $f_{halo}$ to the values of $N_d$ shown in the graphs using equations (60) and (61). The values of $f_{halo}$ used for NGC 5053 in Fig. 12 are (from left to right) $f_{halo} = 0.01, 0.055, 0.128, 0.17$ and 0.273.

As the results for AM 4 and NGC 5053 were somewhat different, let us discuss AM 4 first. Notice that the full Monte Carlo $P_s$ curves in Figs 10 and 11 agree quite well with each other. This was predicted in Section 7 from the fact that $N_d$, the number of destructive encounters, is independent of $M_{bh}$. Next, notice that the curves for the $\nu = -0.6$ destruction criterion agree quite well with equation (67), which was used to draw the solid lines in the figures. The predicted $P_s$ for $\Delta M/M = 20$ per cent did not, however, agree very well with equation (62). This can be attributed to the fact that $\nu = -0.6$ generally occurs well before any collision with $\Delta M/M = 20$ per cent. In fact, the good agreement of the numerical results with equation (67) for $\nu = -0.6$ is partly a consequence of the relative rarity of collisions with $|\Delta M/M| = 0.2$ for AM 4.

For NGC 5053, the fraction of clusters destroyed by $\Delta M/M = 20$ per cent is much larger, and hence neither curve agrees well with either equation (67) or equation (62). For $N_d \approx 1.5$, since few clusters were destroyed by $\Delta M/M = 20$ per cent the agreement for $\nu = -0.6$ was good, but competition between the two destruction criteria is apparent for $N_d \approx 1.5$. However, note that the survival probability against $\nu = -0.6$ is smaller than for $|\Delta M/M| = 0.2$. Since the latter

Figure 10. Comparison of full Monte Carlo simulations against theory for AM 4 with $M_{bh} = 5000M$. The circles are the points of $P_s$ versus $N_d$ for the destruction criterion $\nu = -0.6$. The triangles are for $P_s$ versus $N_d$ with the destruction criterion of 20 per cent mass loss in a single encounter.

Figure 11. Comparison of full Monte Carlo simulations against theory for AM 4 with $M_{bh} = 1000M$. The circles are the points of $P_s$ versus $N_d$ for the destruction criterion $\nu = -0.6$. The triangles are for $P_s$ versus $N_d$ with the destruction criterion of 20 per cent mass loss in a single encounter.

Figure 12. Comparison of full Monte Carlo simulations against theory for NGC 5053 with $M_{bh} = 1000M$. The circles are the points of $P_s$ versus $N_d$ for the destruction criterion $\nu = -0.6$. The triangles are for $P_s$ versus $N_d$ with the destruction criterion of 20 per cent mass loss in a single encounter.
occurs only in a single catastrophic encounter, whereas the former also involves the cumulative effect of numerous gentle collisions, it is incorrect to ascribe cluster destruction at large $M_{bh}$ entirely to the effect of the single most destructive collision.

9 CONCLUSIONS AND PROSPECTS FOR FUTURE WORK

Our results confirm the basic picture of cluster evolution proposed by Wielen (1988), in which there are two distinct regimes based on the size of the black hole mass; for $M_{bh}$ too small to destroy the cluster in a single collision, the evolution can be modelled by a smooth average over many collisions, while for $M_{bh}$ large enough to destroy the cluster in a single collision, the survival probability over $10^{10}$ yr depends both on the heating from non-destructive encounters and the stochastic effect of the individually destructive collisions.

Several technical improvements have been made over previous investigations for the $M_{bh} < M_{high}$ limit. The structure of the cluster was allowed to change by evolving it along a King sequence; comparison of the values of $M_{bh,crit}$ in Fig. 8 and the last column of Table 3 show that evolution accelerates the dissolution process for Moore’s weakly bound clusters, leading to a stricter limit. The inclusion of mass loss gave energy changes comparable to the usual $\alpha v^2/2$ heating. Finite-$N$ effects have been included, and shown to be relatively unimportant in determining $M_{bh,crit}$ for the $M_{bh,crit} < M_{high}$ case since ‘the range of final states’ is quite small when the cluster is evolved over times long enough to disrupt it. The Fokker–Planck model (Section 6.2) for the evolution of an ensemble of initial clusters agreed closely with the full Monte Carlo simulations (Section 8) showing that the ‘slow heating’ approximation is indeed an accurate representation of the evolution. Indeed, the simple model of Section 6.2 with constant diffusion coefficients given by the convenient formulae in Table 2 gives quick, analytical results accurate to within a factor of a few (Table 3).

The final fate of the clusters studied in this paper was always dissolution. This occurred for three reasons. First, we ignored internal relaxation which tends to drive a cluster to core collapse. The role of internal relaxation is currently being studied (Murali et al., in preparation). Second, when the cluster is heated slowly by many penetrating encounters, the final fate is dissolution independent of cluster concentration. Third, in the large-$M_{bh}$ limit in which all non-destructive encounters are tidal, clusters with $q_0 \leq 5.5$ dissolve and those with $q_0 \geq 5.5$ core collapse. All the clusters examined here had $q_0 \leq 5.5$.

Our results for the slow-heating, $M_{bh} < M_{high}$ limit are given in Table 3 and Figs 8 and 9. The strictest limit on $M_{bh}$ comes from the full Monte Carlo calculation for AM 4 with $M_{bh,crit} = 600 M_\odot$. For the $P_r$ out of bounds criterion, several of Moore’s clusters are disrupted at $M_{bh} \sim 1000 M_\odot$ and seven of nine are disrupted for $M_{bh} \sim 6000 M_\odot$. For the $\Delta E/|E| = 0.5$ criterion, six clusters die at $M_{bh} \sim 5000$ and two die for $M_{bh} \sim 1500$. To summarize, in this regime it is extremely unlikely that all the clusters could survive unscathed for $10^{10}$ yr if $M_{bh}$ is greater than a few thousand solar masses.

For the $M_{bh} > M_{high}$ limit, the probability of survival $P_r$ does not tend to zero as $M_{bh} \to \infty$, but instead asymptotes at a non-zero value (which does not have to be small). A simple Fokker–Planck model to determine $P_r$ has been developed (Section 7), in which both the close, destructive collisions and the distant, non-destructive collisions are included. The distribution of energies for an initial ensemble of clusters is shown to depend only on the expected number of destructive encounters for the cluster as a function of time, and the resultant $P_r$ gives good agreement with the full Monte Carlo simulations (when the competition of the various survival criteria is small). Inclusion of the distant, non-destructive encounters leads to smaller values of $P_r$, which in turn gives much tighter limits on the allowed fraction of the halo in black holes.

The results for the $M_{bh} > M_{high}$ limit are presented in Table 4 and Figs 9, 10, 11 and 12. For a particular cluster and a given value of $P_r$, one can place limits on $f_{halo}$. For $P_r = 0.5$, Table 4 gives values of $f_{halo}$ for Moore’s clusters ranging from 0.02 to 0.9. In this ‘tidal’ regime, it is unlikely that $f_{halo} > 0.3$, since then $P_r < 0.1$ for most of Moore’s clusters.

The existence for $10^{10}$ yr of the set of nine tenuous globular clusters studied by Moore (1993) places severe restrictions the allowed mass and fraction of the halo mass in massive black holes. Left unanswered in our paper is the question of whether Moore’s clusters were initially similar in size to what we see today, of if they have been ‘whittled down’ to their present small stature. Indeed, even in isolation, the least massive of these clusters, AM 4, Pal 13, and perhaps NGC 7492 might be expected to have evaporated in $\sim 10^9$ yr, suggesting that they are relatively young (Murali, personal communication; Murali et al., in preparation; Spitzer 1987). To answer this question fully, one must evolve a representative population of initial clusters and attempt to reproduce the observed population today. The work for this project has already begun (Murali et al., in preparation). Significantly tighter limits may result from these new investigations by the introduction of other sources of cluster evolution, such as internal relaxation, evaporation, and disc shocking.

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Appendix A: The Centre-of-Mass Velocity Kick

For a continuous cluster with spherical mass density $\rho(r)$, the velocity kick to the cluster centre of mass takes on a very simple form. For $b = be_x$, $V_{rel} = V_{rel}e_z$, and projected cluster position $R = R\cos(\phi)e_x + R\sin(\phi)e_y$, equation (18) gives the mean velocity kick

$$\langle \Delta v \rangle = \frac{2GM_{bh}}{V_{rel}M} \int d^3xp(r) \frac{[b - R\cos(\phi)]e_x - R\sin(\phi)e_y}{b^2 + R^2 - 2bR\cos(\phi)}$$

$$= \frac{2GM_{bh}}{V_{rel}M} \int_0^{\pi} d\phi \frac{[b - R\cos(\phi)]e_x - R\sin(\phi)e_y}{b^2 + R^2 - 2bR\cos(\phi)}$$

$$= \frac{2GM_{bh}}{V_{rel}M} \int_0^{\pi} d\phi \frac{\theta(b - R)}{b} e_x$$

$$= \frac{2GM_{bh}M_{cyl}(b)}{bV_{rel}M} e_x,$$

where $\Sigma(r) = 2 \int_0^{\sqrt{r^2 + \hat{r}^2}} d\rho \sqrt{r^2 + \hat{r}^2}$, and $M_{cyl}(b) = \int_0^{b} dR2\pi\rho \Sigma(R)$ is the mass enclosed within a cylindrical distance $b$ from the centre of the cluster. The $b$-dependent function $M_{cyl}(b)/bM$ is zero at the cluster centre, reaches a maximum inside the cluster, and is equal to 1/b outside the cluster since $M_{cyl}(b \approx r_c) = M$, recovering the usual velocity kick for a structureless particle.

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