Trajectory of a light ray in Kerr field: A material medium approach

Saswati Roy\textsuperscript{\textcopyright} and A.K. Sen\textsuperscript{\textcopyright}

Physics Department, Assam University, Silchar-788011
India

(Dated: August 15, 2014)

The deflection of light ray as it passes around a gravitational mass can be calculated by different methods. Such calculations are generally done by using the null geodesics under both strong field and weak field approximation. However, several authors have studied the gravitational deflection of light ray using material medium approach. For a static, non-rotating spherical mass, one can determine the deflection in Schwarzschild field, by expressing the line element in an isotropic form and calculating the refractive index to determine the trajectory of the light ray. In this paper, we draw our attention to the refractive index of light ray in Kerr field using the material medium approach. The frame dragging effects in Kerr field was considered to calculate the velocity of light ray and finally the refractive index in Kerr field geometry was determined. Hence the deflection of light ray in Kerr field was calculated, assuming far field approximation and compared the results with those calculations done earlier using Null geodesics.

PACS numbers: 95.30.Sf,04.25.D-j,04.70.Bw,04.70.-s

I. INTRODUCTION

Gravitational deflection of light ray is the most important consequence of Einstein’s General theory of Relativity. Scientists have worked to find out the light deflection angle due to a gravitating body using different approaches. One of the approaches is the null geodesics, where based on any of the forms of the line element, either by using the perturbation [1,2] or by integrating the null geodesic equations [3,4,5], the deflection of light ray is calculated. In this paper we have considered the material medium approach, to find out the trajectory of the light ray in the gravitational field of a rotating body (also known as Kerr field). In this approach, the effect of the gravitation on the ray of light is estimated by considering the propagation of the electromagnetic waves be reduced to the problem of wave propagation in material medium in flat space time. This concept can be defined as the equivalent material medium approach. This method is attractive because it suggests that the classical optics is as suitable as that of the Riemannian geometry for the study of the electromagnetic phenomena in a gravitational field.

Material medium approach was first used by the author Tamm [6] in 1924 and then by Balaz [7] in 1958, to calculate the effect of a rotating body on the polarization of light. The same concept had also been utilized by Plebanski [8] in 1960 to study the light scattering by gravitational field. Felice [9] in 1971 had also discussed the optical phenomena for the deflection of a electromagnetic wave by gravitational field. Mashhoon [10,11] had calculated the deflection and polarization due to the Schwarzschild and Kerr black holes. Later Kopeikin and Mashhoon [12] corrected the light deflection angle of the Shapiro time delay caused by the rotation of gravitating bodies, by this approach. Fishbach and Freeman [13] derived the effective refractive index of the material medium in Schwarzschild field and calculated the second order contribution to the gravitational deflection, where the value of the refractive index of any medium indicates the strength of the gravitational field [14]. In a recent series of articles, Evans, Rosenquist, Nandi and Islam [15-18] derived and used the effective refractive index to calculate the gravitational time delay and trajectories of light rays in Schwarzschild geometry. The authors also showed that the Opto-mechanical analogy of general relativity reproduces the equation of GR and matches with the classical equations. P. M. Alsing [19] has extended the Newtonian formalism of Evans, Nandi and Islam [18] to the case of stationary metrics, typical of rotating space times.

Ishihara, Takahashi and Tanimatsu [20] studied how the polarization vector of a linearly polarized electromagnetic wave propagates in a curved space-time in the presence of Kerr black hole. They have showed that, in the weak field limit the rotation angle of the plane of polarization is proportional to the line of sight component of the black hole’s angular momentum. The above fact is very similar to the Faraday Effect, if the angular momentum is replaced by the magnetic field. This justified that, the rotation of the plane of polarization due to the angular momentum of the Kerr black hole is just the Gravitational Faraday Rotation. Sereno[21,22] has also used the similar idea to derive the time delay function and deflection angle by drawing the trajectory of the light ray by Fermat’s principle and also discussed the Gravitational Faraday Rotation in the weak field limit.

Very recently, Sen [23] also used the same approach to calculate the light deflection for a static non rotating mass in Schwarzschild geometry, by expressing the line element in an isotropic form to determine the trajectory of light rays. In this approach, the value of the refractive index of any medium indicates the strength of the gravitational field [14]. In a recent series of articles, Evans, Rosenquist, Nandi and Islam [15-18] derived and used the effective refractive index to calculate the gravitational time delay and trajectories of light rays in Schwarzschild geometry. The authors also showed that the Opto-mechanical analogy of general relativity reproduces the equation of GR and matches with the classical equations. P. M. Alsing [19] has extended the Newtonian formalism of Evans, Nandi and Islam [18] to the case of stationary metrics, typical of rotating space times.

Ishihara, Takahashi and Tanimatsu [20] studied how the polarization vector of a linearly polarized electromagnetic wave propagates in a curved space-time in the presence of Kerr black hole. They have showed that, in the weak field limit the rotation angle of the plane of polarization is proportional to the line of sight component of the black hole’s angular momentum. The above fact is very similar to the Faraday Effect, if the angular momentum is replaced by the magnetic field. This justified that, the rotation of the plane of polarization due to the angular momentum of the Kerr black hole is just the Gravitational Faraday Rotation. Sereno[21,22] has also used the similar idea to derive the time delay function and deflection angle by drawing the trajectory of the light ray by Fermat’s principle and also discussed the Gravitational Faraday Rotation in the weak field limit.

Very recently, Sen [23] also used the same approach to calculate the light deflection for a static non rotating mass in Schwarzschild geometry, by expressing the line element in an isotropic form to determine the trajectory of light rays.
of the light ray. In this paper we shall follow a similar approach corresponding to Kerr field.

On the other hand, in a more conventional manner, using null geodesic method many authors have worked to find out the deflection angle solutions in the Kerr field. In 2009, Iyer and Hansen[24] calculated the deflection angle using the null geodesic in the equatorial plane. They also concluded that the deflection angle due to Kerr field is greater than the Schwarzschild value for pro-grade or direct orbit and smaller for retrograde orbit. But as in the schwarzschild case, when the deflection angle exceeds 2π, it forms in multiple loops and relativistic images are produced. For the higher rotation case, the effect is much more pronounced.

In 1986, Bray [25] presented approximate solutions to the equation of motion for a ray of light in the Kerr field and considered multi-imaging aspect of the gravitational lens effect. Bozza, Luca, Scarpetta, Sereno[26] presented the exact solution of Einstein’s Field Equation of General Relativity for a stationary, axially symmetric gravitational field of an uncharged rotating body is given by the Kerr metric. This celebrated solution was first given by R.P. Kerr in 1963 [32]. In the Boyer Lindquist form, in the (ct, r, θ, φ) co-ordinate system, the line element of Kerr metric is as follows [31]:

\[ ds^2 = (1 - \frac{2GM}{r})c^2 dt^2 - \frac{\Sigma^2}{\Delta} dr^2 - \Sigma^2 d\theta^2 - (r^2 + \alpha^2 + \frac{r_g r \alpha}{\Sigma^2} \sin^2 \theta) \sin^2 \theta d\phi^2 + \frac{2r_g r \alpha}{\Sigma^2} \cos^2 \theta d\phi dt \] (1)

where

\[ \Sigma^2 = r^2 + \alpha^2 \cos^2 \theta \] (2a)

and

\[ \Delta = r^2 - r_g r + \alpha^2 \] (2b)

The constants \( r_g \) and \( \alpha \) are the schwarzschild radius and the rotation parameter of the Kerr gravitating body. As usual \( r_g = \frac{4GM}{c^2} \) and \( \alpha \) is defined as \( \alpha = \frac{J}{Mr^2} \), where \( J \) is the angular momentum of the gravitating body, \( M \) is the total mass of the gravitating body and \( c \) is the velocity of light.

In equatorial plane \( \theta = \frac{\pi}{2} \), which implies \( \Sigma^2 = r^2 \). Under far field approximation we can assume, \( \frac{\alpha^2}{r^2} \ll 1 \). Thus the linearized form of the Kerr metric in terms of spherical polar co-ordinates \( r, \theta, \phi \) can be obtained with the first power in \( \frac{\alpha}{r} \) as [33,34]:

\[ ds^2 = (1 - \frac{r_g}{r})c^2 dt^2 - \frac{1}{(1 - \frac{r_g}{r})} dr^2 - r^2 d\theta^2 - r^2 (1 + \frac{\alpha^2}{r^2} + \frac{2r_g r \alpha}{r^3}) d\phi^2 + \frac{2r_g r \alpha}{r} c \sin^2 \theta d\phi dt \]

or, \( ds^2 \cong [(1 - \frac{r_g}{r}) + \frac{2r_g r \alpha}{r} \frac{d\phi}{dt} ] c^2 dt^2 - \frac{1}{(1 - \frac{r_g}{r})} dr^2 - r^2 (d\phi^2 + d\theta^2) \) (3)

In most practical purposes the linearized Kerr metric satisfactorily describes the gravitational field around a rotating star or planet. To derive the expression for re-
fractive index, we follow a procedure similar to the one adopted by Sen[23] for a static field (Schwarzschild geometry). Thus to express the above line element in an isotropic form we introduce a new radius co-ordinate $(\rho)$ with the following transformation equation [14] as

$$\rho = \frac{1}{2}(r - \frac{r_g}{2}) + r^{1/2}(r - r_g)^{1/2}$$

(4)

The above equation may be also written as:

$$r = \rho(1 + \frac{r_g}{4\rho})^2$$

(5)

As was done by Sen[23] from Eqn. (5) the value of $\frac{dr}{d\rho}$ can be calculated as

$$\frac{dr}{d\rho} = (1 + \frac{r_g}{4\rho})^2 - \frac{r_g}{2\rho}(1 + \frac{r_g}{4\rho})$$

$$= 1 - \frac{r_g^2}{16\rho^2}$$

(6)

Substituting the value of $r$ and $dr^2$ from Eqn. (5) and (6) in Eqn. (3) (which has far field or slow rotation approximation) we get:

$$ds^2 = [1 - \frac{r_g}{\rho(1 + \frac{r_g}{4\rho})^2} + \frac{2r_g}{\rho(1 + \frac{r_g}{4\rho})^2} \frac{d\phi}{c^2} dt^2 - \frac{(1 - \frac{r_g^2}{16\rho^2})}{1 - \frac{r_g}{\rho(1 + \frac{r_g}{4\rho})^2}}] \rho^2 - \rho^2(1 + \frac{r_g}{4\rho})^2(d\phi^2 + d\theta^2)$$

$$= \frac{(1 - \frac{r_g^2}{16\rho^2})}{1 - \frac{r_g}{\rho(1 + \frac{r_g}{4\rho})^2}}(1 + \frac{r_g}{4\rho})^2(d\phi^2 + d\theta^2)$$

(7)

The above result expressed by Eqn.(7) gives the isotropic form of Kerr solution.

Now in spherical co-ordinate system the quantity $(d\rho^2 + \rho^2(d\phi^2 + d\theta^2))$ has the dimension of square of infinitesimal length vector $d\vec{p}$.

By setting $ds = 0$, the velocity of light $(v(\rho, \theta))$ can be identified from the expression of the form $ds^2 = f(\rho, \theta) dt^2 - d\vec{p}^2$, as $v(\rho, \theta) = \sqrt{f(\rho, \theta)}$. Therefore the velocity of light in the present case (characterized by Schwarzschild radius $r_g$ and rotation parameter $\alpha$) can be expressed as:

$$v(\rho, \theta) = v(\rho, \theta) \frac{dr}{d\rho}$$

$$= v(\rho, \theta) [1 - \frac{r_g^2}{16\rho^2}]$$

$$= \frac{(1 - \frac{r_g}{4\rho})\sqrt{(1 - \frac{r_g}{4\rho})^2 + \frac{2\alpha}{c} \frac{d\phi}{c^2}}}{(1 + \frac{r_g}{4\rho})^2}$$

$$= \frac{(1 - \frac{r_g}{4\rho})\sqrt{(1 - \frac{r_g}{4\rho})^2 + \frac{2\alpha}{c} \frac{d\phi}{c^2}}}{(1 + \frac{r_g}{4\rho})^2}$$

$$= \frac{(4\rho - r_g)^2}{(4\rho + r_g)^2} \sqrt{1 + \frac{8\alpha}{c} \frac{d\phi}{c^2} \frac{4\rho r_g}{(4\rho + r_g)^2}}$$

(9)

Substituting the value of $\rho$ from Eqn. (4) as $4\rho = 2r - r_g + 2\sqrt{r(r - r_g)}$ and then replacing $r/r_g$ by $x$, we can write the above expression for velocity of light as:

$$v^2(x, \theta) = c^2(1 - \frac{1}{x})^2(1 + \frac{2\alpha}{c} \frac{1}{x} \frac{d\phi}{c^2})$$

(10)

Similarly, the refractive index $n(x, \theta)$ can be expressed by the relation:
\[ n(x, \theta) = \frac{x}{x-1} \left[ 1 + 2 \alpha \frac{1}{c} \frac{d\phi}{dt} \right] - \frac{1}{2} \]  

(11)

when \( \alpha = 0 \), we find the central gravitational mass is static and in that case the above expression of refractive index goes over to that for Schwarzschild mass, which is:

\[ n(r) = \frac{x}{x-1} = \frac{r}{r - r_g} \]  

(12)

This is exactly same as the refractive index calculated by Sen [23] for a static non rotating mass (Schwarzschild geometry).

### A. Calculation of \( \frac{d\phi}{dt} \) in the expression of refractive index

The value of \( \frac{d\phi}{dt} \) can be calculated by following a procedure from Landau and Lifshitz [14]. Below we outline this procedure, which can be used to calculate the expression for \( \frac{d\phi}{dt} \).

In the gravitational field of a rotating spherical mass, the relativistic action function \( S \) for a particle with the time \( t \) and the angle \( \phi \) as cyclic variables, can be expressed as:

\[ S = -E_0 t + L\phi + S_r(r) + S_\theta(\theta) \]  

(13)

where \( E_0 \) is the conserved energy and \( L \) denotes the component of the angular momentum along the axis of the symmetry of the field.

The four momentum of the particle is

\[ p^i = mc \frac{dx^i}{ds} = g^{ik} p_k = -g^{ik} \frac{\partial S}{\partial x^k} \]  

(14)

where \( i \) and \( k \) have the values 0,1,2,3 which stand for the coordinates \( ct, r, \theta, \phi \) respectively,[14] (page 17,23,29,264). Now, for the variables \( t \) and \( \phi \) one can write:

\[ mc \frac{dx^0}{ds} = -g^{00} \frac{\partial S}{\partial x^0} - g^{01} \frac{\partial S}{\partial x^1} - g^{02} \frac{\partial S}{\partial x^2} - g^{03} \frac{\partial S}{\partial x^3} \]  

(15)

and

\[ mc \frac{dx^3}{ds} = -g^{30} \frac{\partial S}{\partial x^0} - g^{31} \frac{\partial S}{\partial x^1} - g^{32} \frac{\partial S}{\partial x^2} - g^{33} \frac{\partial S}{\partial x^3} \]  

(16)

Comparing with the Kerr line element expressed by Eqn. (1) one can write:

\[ g^{00} = (1 - \frac{r_g r}{\Sigma}), \quad g^{11} = -\frac{\Sigma}{\Delta}, \quad g^{22} = -\frac{\Sigma}{\Delta}, \quad g^{33} = -\frac{\Sigma - \frac{r_g r}{\Sigma}}{\Delta \sin^2 \theta} \sin^2 \theta, \quad g^{03} = g^{30} = \frac{\frac{r_g r}{\Sigma}}{\Delta} \sin^2 \theta \]

The determinant of the linearized form of the Kerr metric tensor is given by:

\[ |g| = -\Sigma^4 \sin^4 \theta \]  

(17)

Thus using the formula \( g^{ij} = \frac{\text{co-factor of } g_{ij}}{|g|} \) the contravariant components are:

\[ g^{00} = \frac{1}{\Delta \sin^2 \theta} (r^2 + \alpha^2 + \frac{r_g r \alpha}{\Sigma} \sin^2 \theta), \]

\[ g^{33} = -\frac{\Sigma^2 - \frac{r_g r}{\Sigma}}{\Delta \sin^2 \theta}, \]

\[ g^{03} = g^{30} = \frac{\frac{r_g r \alpha}{\Sigma}}{\Delta \sin^2 \theta} \]

and other components are zero.

Using the above values of the components of the metric tensors, the Eqn.(15) and (16) become

\[ mc \frac{dt}{ds} = -\frac{1}{\Delta \sin^2 \theta} (r^2 + \alpha^2 + \frac{r_g r \alpha}{\Sigma} \sin^2 \theta) \left( \frac{E_0}{c} \right) - \frac{\frac{r_g r \alpha}{\Sigma} \sin^2 \theta}{\Delta \sin^2 \theta} L \]  

(18)

\[ mc \frac{d\phi}{ds} = -\frac{r_g r \alpha}{\Sigma^2 \Delta \sin^2 \theta} \left( \frac{E_0}{c} \right) + \frac{\Sigma^2 - \frac{r_g r}{\Sigma}}{\Sigma^2 \Delta \sin^2 \theta} L \]  

(19)

Therefore, the value of \( \frac{d\phi}{dt} \) is
In the propagation of a light ray (or photon like particle), the momentum \( p \) and the conserved energy \( E_0 \) can be expressed by the relation

\[
E_0 = pc \tag{21a}
\]

Further, restricting the light ray in the equatorial plane \( \theta = \frac{\pi}{2} \), the angular momentum \( L \) can be expressed as

\[
L = pb \tag{21b}
\]

where \( b \) = impact parameter, which implies

\[
\frac{Lc}{E_0} = b \tag{22}
\]

So, for equatorial plane, \( \Sigma^2 = r^2 \) and the Eqn. \( 20 \) becomes

\[
\frac{d\phi}{dt} = \frac{r_g r \alpha + (r^2 - r_g r)b}{r^2 (r^2 + \alpha^2 + \frac{r_g r^2 \alpha^2}{r^2}) - r_g r \alpha b} \frac{c}{c}
\]

Now under far field approximation (corresponding to the line element as in Eqn. \( 3 \)), the above expression becomes

\[
\frac{d\phi}{dt} = \frac{r_g r \alpha + (r - r_g)b}{r^3 - r_g \alpha b} \frac{c}{c} = \frac{r b + r_g (\alpha - b)}{r^3 - r_g \alpha b} \frac{c}{c} = \frac{x v + (u - v)}{r_g (x^3 - u v)} \tag{24}
\]

where we define \( \frac{r}{r_g} = x \) (as was done earlier), \( \frac{v}{r_g} = u \) and \( \frac{b}{r_g} = v \).

Now by substituting the value of \( \frac{d\phi}{dt} \) from Eqn. \( 24 \) into Eqn. \( 10 \), the velocity of propagation of light ray in Kerr geometry can be expressed as:

\[
v^2(x, \frac{\pi}{2}) = c^2 \left[ 1 - \frac{1}{x} \right]^2 \left( 1 + 2 \frac{\alpha}{c} - \frac{1}{(x - 1)} - \frac{u}{r_g (x^3 - u v)} \right) \tag{25}
\]

In the above expression of velocity of light we find, the first term refers to the velocity due to Schwarzschild geometry alone \( [23] \) and the second term refers to the contribution due to rotation (under Kerr field geometry).

From Eqn. \( 25 \), we can write the refractive index \( n(x, \frac{\pi}{2}) \) at an arbitrary point on equatorial plane in the Kerr field as:

\[
n(x, \frac{\pi}{2}) = \frac{x}{x - 1} \left[ 1 + 2 u \frac{x v + u - v}{(x - 1) (x^3 - u v)} \right]^{-\frac{1}{2}} = n_0(x) \left[ 1 + 2 S_x \right]^{-\frac{1}{2}} = n_0(x) . \eta_x \tag{26}
\]

In the above, we introduced the parameters \( n_0(x) = \frac{\alpha}{x - 1} \), \( S_x = \frac{u (x v + u - v)}{(x - 1) (x^3 - u v)} \) and \( \eta_x = \left[ 1 + 2 S_x \right]^{-\frac{1}{2}} \).

Here we can also show for all \( r > r_g \), we get \( S_x << 1 \). This is explained in Appendix A. Therefore, one can also write the expression of refractive index in terms of the following converging series:

\[
n(x, \frac{\pi}{2}) = \frac{x}{x - 1} \left( 1 - S_x + \frac{3}{2} S_x^2 \ldots \right) \tag{27}
\]

Now at \( \alpha = 0 \), \( S_x = 0 \) so that the refractive index becomes \( n(x, \frac{\pi}{2}) = \frac{x}{x - 1} = n_0(x) \), which is exactly the same result calculated by Sen\([23]\).
III. CALCULATION OF DEFLECTION IN KERR FIELD

The Eqn. (27) provides the general expression for refractive index on the equatorial plane in Kerr field. Thus, the trajectory of the light ray can be written as [23,35]:

\[ \Delta \psi = 2 \int_{b}^{\infty} \frac{dr}{r \sqrt{\left( \frac{n(r)}{n(b)} \right)^2 - 1}} - \pi \]  \hspace{1cm} (28)

As had been already discussed by Sen[23], here in our present problem the light is approaching from asymptotic infinity ( \( r = -\infty \) or \( x = -\infty \)) towards the rotating gravitational mass, which is placed at the origin and characterized by Schwarzschild radius \( r_s \) and rotation parameter \( \alpha \). Then the ray goes to \( r = +\infty \), or \( x = +\infty \) after undergoing certain amount of deflection (\( \Delta \psi \)). Here, the closest distance of approach, for the approaching ray is \( b \). (we note that, in actual case the impact parameter and closest distance of approach are only approximately equal.) When the light ray passes through the closest distance of approach (i.e. \( r = b \)), the tangent to the trajectory becomes perpendicular to the vector \( \hat{r} \) (which is \( \hat{b} \)).

Now we change the variable to \( x = \frac{r}{r_s} \) so that \( dr = r_s dx \). The corresponding limit changes from \( x = v \) to \( x = \infty \), as the limit of \( r \) changes from \( r = b \) to \( r = \infty \).

Accordingly, the value of deflection (\( \Delta \psi \)), can be written as:

\[ \Delta \psi = 2 \int_{v}^{\infty} \frac{dx}{x \sqrt{\left( \frac{n(x)}{n(v)} \right)^2 - 1}} - \pi \]

\[ = 2 - \pi \]  \hspace{1cm} (29)

Using Eqn. (26) and substituting \( D_k = n(v) \cdot v \), like it was done by Sen[23], we can re-write the above equation as:

\[ I = n(v) \cdot v \int_{v}^{\infty} \frac{dx}{x \sqrt{\left( n(x) \cdot x \right)^2 - (n(v) \cdot v)^2}} \]

\[ = D_k \int_{v}^{\infty} \frac{dx}{x \sqrt{n_0(x)(1 + 2S_x)^{-\frac{1}{2}} \cdot x^2 - D_k^2}} \]

\[ = D_k \int_{v}^{\infty} \frac{dx}{x \sqrt{n_0^2(x)x^2(1 + 2S_x)^{-1} - D_k^2}} \]

\[ = D_k \int_{v}^{\infty} \frac{dx}{x \sqrt{n_0^2(x)x^2 - D_0^2 + n_0^2(x)x^2(1 + 2S_x)^{-1} - n_0^2(x)x^2 + D_0^2 - D_k^2}} \]

\[ = D_k \int_{v}^{\infty} \frac{dx}{x \sqrt{n_0^2(x)x^2 - D_0^2[1 + J(x)]^{-\frac{1}{2}}} - D_k^2} \]

\[ = D_k \int_{v}^{\infty} \frac{dx}{x \sqrt{n_0^2(x)x^2 - D_0^2[1 + J(x)]^{-\frac{1}{2}}}} \]

where \( D_0 = n_0(v) \cdot v \) (corresponding to schwarzschild deflection). And we have also denoted

\[ J(x) = \frac{n_0^2(x)x^2(1 + 2S_x)^{-1} - 1 + D_0^2 - D_k^2}{n_0^2(x)x^2 - D_0^2} \]  \hspace{1cm} (31)

At this stage we can show that \( J(x) << 1 \) (discussed in Appendix B). Therefore, from Eqns.(29) and (30) one can write:
\[ \begin{align*}
\Delta \psi = & 2D_k \int_v^\infty \frac{dx}{x \sqrt{n_0^2(x)x^2 - D_0^2}} [1 - \frac{1}{2} J(x) + \frac{3}{8} J^2(x) - \frac{5}{16} J^3(x) + \frac{35}{128} J^4(x) - \frac{63}{254} J^5(x) + \ldots . ] - \pi \\
= & 2[I_0 + I_1 + I_2 + I_3 + \ldots . ] - \pi 
\end{align*} \]

(32)

where, we have introduced additional notations:

\[ I_0 = D_k \int_v^\infty \frac{dx}{x \sqrt{n_0^2(x)x^2 - D_0^2}} \] (33a)

\[ I_1 = D_k \int_v^\infty \frac{dx}{x \sqrt{n_0^2(x)x^2 - D_0^2}} (-1/2 J(x)) \] (33b)

\[ I_2 = D_k \int_v^\infty \frac{dx}{x \sqrt{n_0^2(x)x^2 - D_0^2}} (3/8 J^2(x)) \] (33c)

and so on.

Again we follow a procedure same as what was followed by Sen [23] to evaluate a similar integral.

Thus

\[ I_0 = D_k \int_v^\infty \frac{dx}{x \sqrt{(x^2 + 1)^2 - D_0^2}} \]

\[ = D_k \int_v^\infty \frac{dx}{x \sqrt{x^4(1 - D_0^2)}} \]

\[ = D_k \int_v^\infty \frac{(x - 1)dx}{x^2 \sqrt{1 - D_0^2(x-1)^2}} \]

\[ = D_k \int_v^\infty \frac{(x - 2)dx}{x^3 \sqrt{1 - D_0^2(x-1)^2}} + D_k \int_v^\infty \frac{dx}{x^3 \sqrt{1 - D_0^2(x-1)^2}} \]

\[ = I_0 + I_{02} \text{ (say)} \]

The first part in above (viz \( I_{01} \)) contains an integral, which has been already evaluated earlier by Sen [23]. Now we introduce a new variable \( y = D_0 x^2(x - 1) \), so that \( dy = - D_0 x^{-3}(x - 2)dx \). Therefore, the limits of integration change as \( y = 0 \) and

\[ y = D_0 \cdot v^{-2}(v - 1) \]

\[ = n_0(v) \cdot v \cdot v^{-2}(v - 1) \]

\[ = \frac{v}{v - 1} \cdot v \cdot v^{-2}(v - 1) \]

\[ = 1 \]

Thus,

\[ I_{02} = D_k \int_v^\infty \frac{dx}{x^3 \sqrt{1 - D_0^2(x-1)^2}} \]

\[ = D_k \int_v^\infty \frac{dx}{\sqrt{x^6 - D_0^2 x^2(x - 1)^2}} \]

\[ = D_k \int_0^\infty \frac{dz}{\sqrt{1 - D_0^2 z^2(1 - z)^2}} \] (36)

This can be evaluated in terms of Elliptical function as expressed by Eqn.(18) of Sen [23]. And finally for a given value of \( a \), its numerical value can be obtained.

Now to evaluate the value of \( J(x) \) we evaluate the value of \((1 + 2S_x)^{-1} - 1 \) first as follows:

\[ (1 + 2S_x)^{-1} - 1 = \frac{1}{1 + 2S_x} - 1 \]

\[ = \frac{2S_x}{1 + 2S_x} \]

\[ = \frac{2u(xv + u - v)}{x(x - 1)^2(x^2 - uv)} \]

\[ = \frac{2u(xv + u - v)}{(x - 1)(x^3 - uv) + 2u(xv + u - v)} \]

\[ = \frac{2u(xv + u - v)}{(x - 1)(x^3 + uv) + 2u^2} \] (37)

Now in Eqn. (31) above, we substitute the value of \((1 + 2S_x)^{-1} - 1 \) (from Eqn. (37)) and \( n_0(x) = x/(x-1) \).
As a result, we can write the following expressions for \( J(x) \):

\[
J(x) = \frac{n_0^2(x)x^2\{(1 + 2S_x)^{-1} - 1\} + D_0^2 - D_0^2}{n_0^2(x)x^2 - D_0^2} = \frac{n_0^2(x)x^2\{(1 + 2S_x)^{-1} - 1\} + D_0^2\{1 - (1 + 2S_x)^{-1}\}}{n_0^2(x)x^2 - D_0^2} = \frac{x^4\{\frac{2u(x\cdot u - v)}{x^2 - D_0^2} + D_0^2\{1 - (1 + 2S_x)^{-1}\}\}}{x^4 - D_0^2(x - 1)^2}
\]

Further, substituting the value of \( n_0(x) = \frac{x}{x^2} \) and \( J(x) \) from Eqn. (38), the integral \( I_1 \) becomes

\[
I_1 = -\frac{1}{2} D_k \int_0^\infty \frac{J(x)}{x \sqrt{n_0^2(x)x^2 - D_0^2}} \, dx = -\frac{1}{2} D_k \int_0^\infty \frac{1}{x^4 - D_0^2(x - 1)^2} \left\{ -x^4 \left( \frac{2u(x\cdot u - v)}{(x-1)(x^2 + 4u^2 + 2uv)} + D_0^2(x - 1)^2 \left\{ \frac{2u(x^2 + u^2)}{(x-1)(x^2 + 4u^2 + 2uv)} \right\} \right) \right\} dx \]
\]

Again applying the change of variable as \( z = \frac{1}{x} \) (like it was done for \( I_{02} \)) we may write the integral \( I_1 \) as

\[
I_1 = -\frac{1}{2} D_k \int_0^a \frac{(1 - z)}{\sqrt{1 - D_0^2 z^2}(1 - z)^2} \left\{ \frac{2u z^3(x\cdot u - v)z}{(1 - z)(1 + wu z^2 + 2u^2 z^4)} + D_0^2 z^2(1 - z)^2 \left\{ \frac{2u(x^2 + u^2)}{(x-1)(x^2 + 4u^2 + 2uv)} \right\} \right\} dz
\]

Similarly, \( I_2 , I_3 \) etc. can be written as

\[
I_2 = \frac{3}{8} D_k \int_0^a \frac{(1 - z)}{\sqrt{1 - D_0^2 z^2}(1 - z)^2} \left\{ \frac{2u z^3(x\cdot u - v)z}{(1 - z)(1 + wu z^2 + 2u^2 z^4)} + D_0^2 z^2(1 - z)^2 \left\{ \frac{2u(x^2 + u^2)}{(x-1)(x^2 + 4u^2 + 2uv)} \right\} \right\}^2 dz
\]

\[
I_3 = -\frac{5}{16} D_k \int_0^a \frac{(1 - z)}{\sqrt{1 - D_0^2 z^2}(1 - z)^2} \left\{ \frac{2u z^3(x\cdot u - v)z}{(1 - z)(1 + wu z^2 + 2u^2 z^4)} + D_0^2 z^2(1 - z)^2 \left\{ \frac{2u(x^2 + u^2)}{(x-1)(x^2 + 4u^2 + 2uv)} \right\} \right\}^3 dz
\]

\[
I_4 = \frac{35}{64} D_k \int_0^a \frac{(1 - z)}{\sqrt{1 - D_0^2 z^2}(1 - z)^2} \left\{ \frac{2u z^3(x\cdot u - v)z}{(1 - z)(1 + wu z^2 + 2u^2 z^4)} + D_0^2 z^2(1 - z)^2 \left\{ \frac{2u(x^2 + u^2)}{(x-1)(x^2 + 4u^2 + 2uv)} \right\} \right\}^4 dz
\]
Thus form Eqn. (32), the expression for deflection of light ray in Kerr geometry can be expressed as:

$$\Delta \psi = 2\frac{D_k}{D_0} \pi + 2D_k \int_0^a \frac{z \, dz}{\sqrt{1 - D_0 z^2(1-z)^2}}$$

$$- \frac{1}{2} \int_0^a \frac{(1-z)}{\sqrt{1 - D_0 z^2(1-z)^2}} \left[ \frac{2u z^2 [(v+(u-v)z)]}{(1-z)(1+uvz^2)+2uz^2} \right] + D_0 z^2 (1-z)^2 \left[ \frac{2u(z^2+uv)}{(v-1)(v^3+uv)+2uz^2} \right] \, dz$$

$$+ \frac{3}{8} \int_0^a \frac{(1-z)}{\sqrt{1 - D_0 z^2(1-z)^2}} \left[ \frac{2u z^2 [(v+(u-v)z)]}{(1-z)(1+uvz^2)+2uz^2} \right] + D_0 z^2 (1-z)^2 \left[ \frac{2u(z^2+uv)}{(v-1)(v^3+uv)+2uz^2} \right] \, dz$$

$$+ \frac{5}{16} \int_0^a \frac{(1-z)}{\sqrt{1 - D_0 z^2(1-z)^2}} \left[ \frac{2u z^2 [(v+(u-v)z)]}{(1-z)(1+uvz^2)+2uz^2} \right] + D_0 z^2 (1-z)^2 \left[ \frac{2u(z^2+uv)}{(v-1)(v^3+uv)+2uz^2} \right] \, dz$$

$$+ \frac{35}{64} \int_0^a \frac{(1-z)}{\sqrt{1 - D_0 z^2(1-z)^2}} \left[ \frac{2u z^2 [(v+(u-v)z)]}{(1-z)(1+uvz^2)+2uz^2} \right] + D_0 z^2 (1-z)^2 \left[ \frac{2u(z^2+uv)}{(v-1)(v^3+uv)+2uz^2} \right] \, dz$$

$$- \frac{63}{256} \int_0^a \frac{(1-z)}{\sqrt{1 - D_0 z^2(1-z)^2}} \left[ \frac{2u z^2 [(v+(u-v)z)]}{(1-z)(1+uvz^2)+2uz^2} \right] + D_0 z^2 (1-z)^2 \left[ \frac{2u(z^2+uv)}{(v-1)(v^3+uv)+2uz^2} \right] \, dz$$

(44)

The above expression has been obtained for gravitational deflection in the equatorial plane of a rotating body considering the Kerr line element.

### IV. CALCULATION OF NUMERICAL VALUES FOR DEFLECTION FOR SOME GRAVITATIONAL OBJECTS

Considering Sun as a rotating body, we can calculate the deflection angle for pro-grade ($b = -ve$) and retrograde ($b = +ve$) direction of the light ray with respect to the gravitating body. From Eqn. (45) after actual numerical calculations, it was found that for pro-grade direction, the deflection will be greater and for retro-grade direction, the deflection will be smaller than that of the Schwarzschild geometry. This has been also confirmed by Iyer et al. in their previous work [24]. A more recent work by Werner [43], also confirms this phenomena where the author has considered terms only up to first order in $\alpha$ (rotation parameter).

For a Sun grazing ray, we can calculate the deflection of a light ray using the Eqn. (45) obtained in this present work. Here, we may consider the closest distance of ap-
TABLE I. Refractive index due to different Gravitational Objects.

| Name of the Gravitational Objects | Schwarzschild Radius $r_g$ (Km) | Rotation Parameter $\alpha$ (Km) | Physical Radius $R$ (b (Km)) | $u$ $(\alpha/r_g)$ | $v$ $(b/r_g)$ | Refractive index n $(\alpha = 0)$ | Refractive index n $(Pro – grade)$ $(b = -ve)$ | Refractive index n $(Retro – grade)$ $(b = +ve)$ |
|-----------------------------------|---------------------------------|---------------------------------|-----------------------------|-------------------|-----------------|---------------------------------|---------------------------------|---------------------------------|
| SUN                              | 2.9554                          | 1.6762                          | 6.955e+05                   | 0.5671            | 235331.2911     | 1.00000424934                  | 1.00000424935                  | 1.00000424933                  |
| PSR B 1919+21 (Hewish et al. 1968)[37] | 4.1375                          | 2.5075                          | 2.000e+01                   | 0.6060            | 4.8337          | 1.26084097201                  | 1.2928546990                  | 1.22753613865                  |
| PSR J 1748-2446 ad (Hessel et al. 2006)[38] | 3.9898                          | 2.4261                          | 2.010e+01                   | 0.6080            | 5.0378          | 1.24765683468                  | 1.2769806817                  | 1.21726163358                  |
| PSR B 1937+21 (Ashworth et al. 1983)[39] | 3.9898                          | 2.1970                          | 2.020e+01                   | 0.5506            | 5.0629          | 1.24612905064                  | 1.2724450476                  | 1.21895467231                  |
| PSR J 1909-3744 (Jacoby et al. 2003)[40] | 4.2498                          | 2.7486                          | 3.110e+01                   | 0.6467            | 7.3178          | 1.15828146469                  | 1.1721555862                  | 1.14418491095                  |
| PSR 1855+09                       | 3.9898                          | 3.4339                          | 4.690e+01                   | 0.8606            | 11.7549         | 1.09298024705                  | 1.09976248166                 | 1.08614772138                  |
| PSR J 0737-3039 A (Lyne et al. 2004)[41] | 3.9602                          | 6.5918                          | 1.336e+02                   | 1.6645            | 33.7352         | 1.03054808987                  | 1.03205416015                 | 1.02903968046                  |
| PSR 0531+21                       | 3.9898                          | 6.8042                          | 1.644e+02                   | 1.7054            | 41.2050         | 1.02487248899                  | 1.02590137268                 | 1.02384251987                  |
| PSR B 1534+12 (Strairs et al. 1998)[42] | 3.9602                          | 6.0879                          | 1.659e+02                   | 1.5372            | 41.8913         | 1.02445506278                  | 1.02535206812                 | 1.02355723286                  |

proach is equal to the solar radius as $r_\odot = 6.955 \times 10^5$ km, solar mass as $M_\odot = 1.99 \times 10^{30}$ kg and solar time period as $T = 28$ days.

In addition to Sun, we shall also consider some millisecond pulsars, to show the effect of rotation on light deflections, as we know the pulsars are fast rotating objects. Nunez et al. [36] have calculated the red-shift and preferred radius for some fast millisecond pulsars. We derive some of the input parameters (like $b$, $\alpha$, $r_g$ etc.) from their published work and perform some sample calculations. We note that, the calculations performed here on Sun and these pulsars are for demonstrative purpose only. The aim is to show that, like the Null Geodesic method, the Material Medium approach can predict equally well the gravitational deflection of the light ray in Kerr field. However, these calculated values may not be very accurate, as only the gross values of input parameters are taken for our purpose.

In table I, we have shown the refractive index along with other parameters for different gravitating body, due to pro-grade and retro-grade orbits of the light ray. The refractive index values have been expressed with 11 places after the decimal. We can note for Sun the refractive index value changes at 11th place after the decimal as we change from pro-grade to retro-grade orbit of light ray. However, for fast rotating pulsars these refractive index values vary at the second place after the decimal between pro-grade and retrograde orbits (cf. Table I). From this table it is totally clear that, the refractive index is greater for pro-grade direction and smaller for retro-grade direction as compared to Schwarzschild one. Thus as compared to the Schwarzschild case, the deflection angle for light ray should be also greater for pro-grade and smaller for retro-grade orbits of light ray.

In table II, we have calculated the deflection of a light
ray (in arc-sec) due to Sun and other pulsars considering the impact parameter as the physical radius of the gravitating body and considering up to fifth order term i.e. $D$ in the Eqns. (32) and (45). We further, note that Sun is a slow rotating object and pulsars are fast rotating. To calculate the deflection angle of these rotating gravitational objects, we used numerical integration by Simpson’s one third rule. Here it is also shown that, the value of deflection angle continuously decreases as we calculate higher order terms. In this table the deflection angle values due to the gravitational body are calculated considering no rotation ($\alpha = 0$) of the body, pro-grade ($b = -ve$) and retro-grade ($b = +ve$) orbits of light ray. In case of no rotation ($\alpha = 0$), the first column (representing $(2I_0 - \pi)$) will be zero as there is no difference between $D_k$ and $D_0$. Also the values of $I_1$, $I_2$, $I_3$ etc. will be zero, when there is no rotation (i.e. $\alpha = 0$) as can be seen in Table II and equally confirmed through Eqn (45). The light deflection values obtained for Sun, by material medium method here for pro-grade and retro-grade orbits are in good agreement with those other similar calculations done using null geodesic method [24].

In Fig. 1 we have plotted the refractive index as a function of $x$ considering the gravitational body to be Sun. If we make this plot for other pulsars, we get the similar trend in the curves. As the value of $x$ i.e. $\frac{r}{r_g}$ increases, (i.e. as we move towards asymptotically flat space), these curves merge into each other. At $x = 1$ i.e. $r = r_g$, the value of refractive index is infinite for all the gravitational bodies and this is physically expected.

In Fig. 2, we have plotted the deflection angle as a function of $u(= \frac{x}{r_g})$ for Sun. Thus, Fig. 2 illustrates the dependence of deflection angle on the rotation parameter. Here the solid line parallel to the $u$-axis, indicates the Schwarzschild geometry.

The pulsars are highly compact objects, with their physical radii being very close to the corresponding Schwarzschild radii in most of the cases. Thus it is necessary to consider the light ray to be passing at various other close distances from the pulsars. In order to understand the dependence of deflection on the impact parameter, we selected one of the pulsars from our list viz. PSRB 1919+21. In Fig. 3, we have plotted the amount of deflection as a function of $v = b/r_g$, under three cases no-rotation, pro-grade orbit and retro-grade orbit. From Fig. 3, it is also clear that the deflection angle is greater for pro-grade and smaller for retro-grade orbit as compared to the Schwarzschild one. This phenomena has been also observed by Iyer et al. [24] as discussed earlier.

From Figs 2 and 3, and also from Table II, it is clear that the deviations of light orbit from the Schwarzschild case for pro-grade and retro-grade orbit are not symmetric. Such asymmetries of deflection values for pro-grade and retro-grade orbits have been also reported by previous authors [24].

V. CONCLUSIONS

We have presented here in detail the calculations for light deflection angle on the equatorial plane of a rotating objects (viz Kerr field), by following material medium approach. The following have been observed:

(i) The material medium approach gives same values of deflection as compared to that obtained by other most conventional method of Null geodesic. This has been verified by taking Sun as a test case. Further, the cases of some millisecond pulsars were considered to understand the effect of rotation more objectively on the deflection angle.

(ii) For pro-grade orbit of the deflection angle is greatest and for retro-grade orbit, the deflection angle is smallest. The one corresponding to no-rotation (Schwarzschild case) lies in between.

(iii) The deviations of pro-grade and retro-grade orbits from the Schwarzschild deflection angle are not symmetric. The deviation is slightly higher for retro-garde orbit.

Appendix A: Proof of $S_x << 1$

We have,

$$S_x = \frac{u(xv + u - v)}{(x-1)(x^3 - uv)} \quad (A1)$$

where, $x = \frac{r}{r_g}$, $u = \frac{\alpha}{r_g}$ and $v = \frac{b}{r_g}$. As $r >> r_g$, $r >> \alpha$ so we must have $x >> 1$ and $x >> u$. Again, $\alpha < b$ so, $u < v$.

Now, $S_x$ may also be written as,

$$S_x = \frac{uv(x - 1) + u^2}{(x-1)(x^3 - uv)} \quad (A2)$$

As $x >> 1$ or, $x - 1 \approx x$

$$S_x \approx \frac{uvx + u^2}{x(x^3 - uv)} = \frac{uv(x + \frac{u}{x})}{x^3(x - \frac{uv}{x^2})} \quad (A3)$$

Under above conditions of $x >> 1, x >> u$ and $v > u$, we may note that, we will have $\frac{u}{v} < 1 << x$ and $\frac{uv}{x^2} << x$. Therefore,

$$S_x \approx \frac{uv}{x^3} \quad (A4)$$

Thus,

$$S_x << 1 \quad (A5)$$
| Name of Objects | $2I_{01} - \pi$ (arc-sec) | $2I_{02}$ (arc-sec) | $2I_{1}$ (arc-sec) | $2I_{2}$ (arc-sec) | $2I_{3}$ (arc-sec) | $2I_{4}$ (arc-sec) | $2I_{5}$ (arc-sec) | Total Deflection (a+b+...+f+g) $\Delta \phi$ (arc-sec) |
|----------------|--------------------------|------------------|------------------|------------------|------------------|------------------|------------------|-----------------------------------------------|
| Sun $\alpha = 0$ | 0.00000 | 1.752008089976 | | | | | | 1.752008089976 |
| $\propto (b = -ve)$ | 0.66364e-05 | 1.752008089994 | -0.18122e-05 | 0.10665e-16 | -0.40393e-28 | 0.12069e-38 | -0.50226e-50 | 1.752012914194 |
| Retro $\propto (b = +ve)$ | -0.66365e-05 | 1.752008089959 | -0.18122e-05 | 0.10665e-16 | -0.76563e-28 | 0.12069e-38 | -0.50226e-50 | 1.751999641259 |
| B1919 $\alpha = 0$ | 0.00000 | 119735.4072 | | | | | | 119735.4072 |
| $\propto (b = -ve)$ | 16453.1545 | 122775.5695 | -10481.7664 | 307.7444 | -5.3394 | 0.8887 | -0.1876e-01 | 129047.2325 |
| Retro $\propto (b = +ve)$ | -17116.7756 | 116572.6231 | -9955.0483 | 292.1964 | -10.5747 | 0.8438 | -0.1781e-01 | 89783.2468 |
| J1748 $\alpha = 0$ | 0.00000 | 112974.7176 | | | | | | 112974.7176 |
| $\propto (b = -ve)$ | 15230.0314 | 113629.9776 | -9290.5744 | 243.6810 | -3.7975 | 0.5612 | -0.1057e-01 | 121809.687 |
| Retro $\propto (b = +ve)$ | -15786.4644 | 110222.4469 | -8856.0931 | 232.2850 | -7.5082 | 0.5350 | -0.1008e-01 | 85805.1911 |
| B1937 $\alpha = 0$ | 0.00000 | 112197.2694 | | | | | | 112197.2694 |
| $\propto (b = -ve)$ | 13684.5905 | 114566.6120 | -8286.1164 | 194.5455 | -2.7143 | 0.3591 | -0.6060e-02 | 120157.2703 |
| Retro $\propto (b = +ve)$ | -14130.9579 | 109750.5211 | -8751.2927 | 226.4006 | -7.2176 | 0.5072 | -0.9428e-02 | 87087.9512 |
| J1909 $\alpha = 0$ | 0.00000 | 69386.1605 | | | | | | 69386.1605 |
| $\propto (b = -ve)$ | 7761.8550 | 70217.2798 | -3538.9772 | 37.3630 | -0.2419 | 0.1382e-01 | -0.1043e-03 | 74477.2924 |
| Retro $\propto (b = +ve)$ | -7886.3101 | 68541.7148 | -3454.5281 | 36.4714 | -0.4728 | 0.1349e-01 | -0.1008e-03 | 57236.8885 |
| 1855 $\alpha = 0$ | 0.00000 | 39691.1980 | | | | | | 39691.1980 |
| $\propto (b = -ve)$ | 4021.0132 | 39937.4925 | -1486.8124 | 6.8396 | -0.1971e-01 | 0.4775e-03 | -0.1563e-05 | 42478.5136 |
| Retro $\propto (b = +ve)$ | -4050.8295 | 39443.0772 | -1468.4061 | 6.7549 | -0.3805e-01 | 0.4715e-03 | -0.1543e-05 | 33930.5589 |
| J0737 $\alpha = 0$ | 0.00000 | 12734.8996 | | | | | | 12734.8996 |
| $\propto (b = -ve)$ | 947.0043 | 12753.5108 | -285.9891 | 0.2616 | -0.1527e-03 | 0.7180e-06 | -0.4655e-09 | 13414.7874 |
| Retro $\propto (b = +ve)$ | -948.4751 | 12716.2596 | -285.1537 | 0.2608 | -0.2915e-03 | 0.7159e-06 | -0.4642e-09 | 11482.8913 |
| 0531 $\alpha = 0$ | 0.00000 | 10347.3213 | | | | | | 10347.3213 |
| $\propto (b = -ve)$ | 650.5361 | 10357.7091 | -192.8309 | 0.1193 | -0.4718e-04 | 0.1496e-06 | -0.6561e-01 | 10815.5335 |
| Retro $\propto (b = +ve)$ | -651.2224 | 10336.9225 | -192.4439 | 0.1190 | -0.8998e-04 | 0.1493e-06 | -0.6548e-01 | 9493.3751 |
| 1834 $\alpha = 0$ | 0.00000 | 10172.1154 | | | | | | 10172.1154 |
| $\propto (b = -ve)$ | 567.3840 | 10181.0220 | -167.9483 | 0.9054e-01 | -0.3119e-04 | 0.8622e-07 | -0.3293e-01 | 10580.5482 |
| Retro $\propto (b = +ve)$ | -567.9056 | 10163.2006 | -167.6543 | 0.9038e-01 | -0.5950e-04 | 0.8607e-07 | -0.3287e-01 | 9427.7310 |
In Table III, we tabulate the values of $S_x$ at $x = +v$ and $x = -v$. It is clearly seen that, these values are smaller than 1. However, as the field becomes weaker or as we have $x > v$, the values of $S_x$ will become still smaller. For $S_x$, these values are reported upto 8th place after decimal. However, for Sun we report upto 27th place after the decimal to show differences in $S_x$ values for $x = +v$ and $x = -v$. 

FIG. 1. Refractive index ($n(x, \frac{r}{r_g})$) as a function of $x(= \frac{r}{r_g})$ for Sun.

FIG. 2. Deflection ($\Delta \psi$) as a function of $u(= \frac{\alpha}{r_g})$ for Sun.
FIG. 3. Deflection ($\Delta \psi$) as a function of $v = \frac{b}{r g}$ for pulsar PSRB 1919+21.

Appendix B: Proof of $J(x) << 1$

We can follow an approach similar to what was followed in Appendix A, to show $J(x) << 1$. However, as $J(x)$ becomes discontinuous at $x = \pm v$, we follow a different approach here. As $J(x)$ is discontinuous at $x = \pm v$, we can remove its discontinuity and evaluate its values at $x = \pm v$, by applying L'Hospital's rule. From Eqn (31), we can write:

$$J(x) = \frac{n_0^2(x)v^2(1 + 2S_x)^{-1} - 1 + D_0^2 - D_k^2}{n_0^2(x)v^2 - D_0^2}$$  

Substituting the values of $n_0(x)$, $S_x$, $D_0$, $D_k$, the above equation becomes

$$J(x) = \frac{-2uvx^4}{(x-1)^2} + \frac{2uvx^4}{(x-1)^2} {\frac{v^3 + vu}{(v-1)^2} - \frac{v^3 + vu}{(v-1)^2}}$$  

The above has discontinuity at $x = \pm v$. By applying L'Hospital's rule we can write:

$$J(x = +v) = \frac{u(2u - v)(v - 2)(v - 1)}{(v - 2)(2u^2 + (v - 1)(v^3 + vu)} - \frac{uv(1)(4v^3 - 3v^2 + vu)(u + (v - 1)v)}{(v - 2)(2u^2 + (v - 1)(v^3 + vu)}$$  

and

$$J(x = -v) = -\frac{u(2u + v)(v + 2)(v + 1)}{(v + 2)(2u^2 + (v + 1)(v^3 + vu)} - \frac{uv(1)(4v^3 + 3v^2 + vu)(u + (v + 1)v)}{(v + 2)(2u^2 + (v + 1)(v^3 + vu)}$$

Further, by differentiating $J(x)$ and subsequently applying L'Hospital’s rule, it can be shown that $J(x)$ has maxima at $x = +v$ and $x = -v$ and all other values of $J(x)$ within the domain $x = +\infty$ to $x = -\infty$ are less than this maxima. These maxima values are tabulated in Table III. We can note these values are much less than 1 and actually become still smaller compared to 1, when either $x >> v$ or the field is weaker or both. So we can safely assume $J(x) << 1$ for all $x$ for all practical purposes. In Table III, we find for Sun the difference in
TABLE III. Values of $S_x$ and $J(x)$ of different Gravitating Objects for $x = -v$ and $x = +v$

| Name of Objects          | $S_x$ $(x = -v)$       | $S_x$ $(x = +v)$       | $J$ $(x = -v)$  | $J$ $(x = +v)$  |
|--------------------------|------------------------|------------------------|----------------|----------------|
| SUN                      | 0.10241529958672695e-10| 0.10241529958672697e-10| 0.307243e-10   | 0.307248e-10   |
| PSR B 1919+21 (Hewish et al. 1968)[37] | 0.02720094            | 0.02749950             | 0.05714        | 0.12267        |
| PSR J 1748-2446 ad (Hessel et al. 2006)[38] | 0.02503882            | 0.02528190             | 0.05342        | 0.11053        |
| PSR B 1937+21 (Ashworth et al. 1983)[39] | 0.02234780            | 0.02254167             | 0.04809        | 0.09890        |
| PSR J 1909-3744 (Jacoby et al. 2003)[40] | 0.01235494            | 0.01239606             | 0.02940        | 0.04735        |
| PSR 1855+09               | 0.00630369             | 0.00631039             | 0.01633        | 0.02181        |
| PSRJ 0737-3039 A (Lyne et al. 2004)[41] | 0.00146678            | 0.00146691             | 0.00417        | 0.00461        |
| PSR 0531+21               | 0.00100644             | 0.00100648             | 0.00289        | 0.00313        |
| PSR B 1534+12 (Strairs et al. 1998)[42] | 0.00087751            | 0.00087755             | 0.00252        | 0.00273        |

$J(x)$ values for $x = +v$ and $x = -v$, occurs only at 16th place after the decimal. For other objects we report $J(x)$ values up to 5th place after the decimal.

[1] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, (W. H. Freeman and Company, New York, 1972)
[2] d’Inverno Ray, *Introducing Einstein’s Relativity*, (Oxford University Press, New York, 1998 (Reprint))
[3] S. Chandrasekhar, *The Mathematical Theory of Black Holes*, (Oxford University Press, New York, 1983)
[4] S. Weinberg, *Principles and Applications of the General Theory of Relativity*, (John Wiley & Sons Inc., 1972)
[5] R. M. Wald, *General Relativity*, (University of Chicago Press, Chicago and London, 1984)
[6] J. E. Tamm, J. Russ. Phys.-Chem. Soc. 56,2-3, 284, (1924)
[7] N. L. Balazs, Phys. Rev. 110, No. 1, 236, (1958)
[8] J. Plebanski, Phys. Rev. 118, No.5, 1396, (1960)
[9] F. de Felice, Gen. Relativ. Gravit. 2, No. 4, 347, (1971)
[10] B. Mashhoon, Phys. Rev. D 7, No. 10, 2807, (1973)
[11] B. Mashhoon, Phys. Rev. D 11, No. 10, 2679, (1975)
[12] S. Kopeikin and B. Mashhoon, Phys. Rev. D 65, No.6, 064025 (2002)
[13] E. Fischbach and B. S. Freeman, Phys. Rev. D 22, No.12, 2950, (1980)
[14] J. L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields volume 2*; (1st edition Pergamon Press, 1951; 4th edition Butterworth-Heinemann 1980)
[15] J. Evans and M. Rosenquist, Am. J. Phys. 54, No.10, 876-883, (1986)
[16] K. K. Nandi and A. Islam, Am. J. Phys. 63, No.3, 251-256, (1995)
[17] J. Evans, K. K. Nandi and A. Islam, Am. J. Phys. 64, No.11, 1404-1415, (1996)
[18] J. Evans, K. K. Nandi and A. Islam, Gen. Relativ. Gravit. 28, No. 4, 413-439 (1996)
[19] P. M. Alsing, Am. J. Phys. 66, No. 9, 779-790 (1998)
[20] H. Ishihara, M. Takahashi and A. Tanimatsus, Phys. Rev. D 38, No. 2, 472 (1988)
[21] M. Sereno, Phys. Rev. D 67, No. 6, 064007 (2003)
[22] M. Sereno, Phys. Rev. D 69, No. 8, 087501 (2004)
[23] A. K. Sen, Astrofizika 53, No. 4, 560-569 (2010)
[24] S. V. Iyer and E. C. Hansen, Phys. Rev. D 80, No. 12, 124023 (2009)
[25] I. Bray, Phys. Rev. D 34, No. 2, (1986)
[26] V. Bozza, F. de Luca, G. Scarpetta and M. Sereno, Phys. Rev. D 72, No. 10, 083003 (2005)
[27] C. R. Keeton and A.O. Petters, Phys. Rev. D 72, No. 10, 104006 (2005)
[28] C. R. Keeton and A.O. Petters, Phys. Rev. D 73, No. 4, 044024 (2006)
[29] C. R. Keeton and A. O. Petters, Phys. Rev. D 73, No. 10, 104032 (2006)
[30] X-H. Ye and Q. Lin, Journal Mod. Opt. 55, Issue 7, 1119 (2008)
[31] R. H. Boyer and R. W. Lindquist, J. Math. Phys., 8, No. 2, 265 (1967)
[32] R. P. Kerr, Phys. Rev. Letters, 11, No. 5, 237 (1963)
[33] J. N. Islam, Rotating Fields in General Relativity, (Cambridge University Press, Newyork, 2009)
[34] D. Vogt and P. S. Letelier, R. Astron. Soc. 363, 268 (2005)
[35] M. Born and E. Wolf 1947, Principles of Optics (7th Edition, Cambridge University Press, Cambridge,1999)p121
[36] P. D. Nunez and M. Nowakowski, Journal Astrophys. Astr. 31, 105-119, 2010
[37] A. Hewish; S. J. Bell; J. D. H. Pilkington; P. F. Scott; R. A. Collins, Nature, 217, 709-713, 1968
[38] J. W. T. Hessels; S. M. Ransom; I. H. Stairs; P. C. C. Freire; V. M. Kaspi; F. Camilo, Science, 311, 1901-1904, 2006
[39] M. Ashworth; A. G. Lyne; Smith, Nature, 301, 313, 1983
[40] B. A. Jacoby; M. Bailes; M. H. van Kerkwijk; S. Ord; A. Hotan; S. R. Kulkarni; S. B. Anderson, ApJ, 599, L99-L102, 2003
[41] A. G. Lyne; M. Burgay; M. Kramer, Science, 303, 1153, 2004
[42] I. H. Stairs; Z. Arzoumanian; Z. Camilo; A. G. Lyne; D. J. Nice; J. H. Taylor; S. E. Thorsett; A. Wolszczan, ApJ, 505, 352, 1998
[43] M. C. Werner, GRG. 3047-3057, 44, 2012