Monopole-like Quantum Excitations in the Non-abelian Vacuum

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It is offered to consider monopoles in Abelian Projection as quantum excitations which are solutions of the quantized Yang-Mills equations. According to the Heisenberg quantization method these equations are equivalent an infinite set of equations for Green’s functions. A procedure for cutting off these infinite series of differential equations after some assumptions is offered. The received equations are identical to equations describing a dyon.

I. INTRODUCTION

The Abelian Projection method (for review see Ref. [1]) is well known in the lattice QCD simulations. The essence of this method is that any calculations on the lattice with the big accuracy can be made if only field distributions with monopoles is taken into account. The Abelian Projection allows us to answer on the following question: is there monopoles in given field configuration, and says that the field configurations with monopoles give an essential contribution for calculations. We would like to emphasize that in the general case these monopoles are not dynamical ’t Hooft-Polyakov monopoles, i.e. they are not solutions of Yang-Mills equations. Only one property they have: a non-zero flux of magnetic field across a closed surface. The role of Abelian Projection is to detect the presence of monopoles in given field distribution. Every such non-dynamical monopole (i.e. it is not ’t Hooft-Polyakov monopole) can be considered as a quantum excitation in quantized gauge field.

In this paper I would like to show that the quantization of non-abelian gauge field theory leads to the appearance of a static spherically symmetric quantized field distribution which in fact is the ’t Hooft-Polyakov monopole and can be considered as an excitation in the non-abelian vacuum. In Ref. [2] some arguments was presented that singular spherically symmetric classical solutions of Yang-Mills equations after quantization give us an asymptotically well behaved field configuration with a flux of magnetic field.

Our main idea is that the non-abelian quantum field theory has non-perturbative effects which can not be described on the language of Feynmann diagrams techniques. What we can expect in this case? Probably it can be (a) a static distribution of the quantized field (for example, it can be a flux tube stretched between the quark-antiquark pair in QCD); (b) quantum excitations in the vacuum (for example, it can be spherically symmetric monopole-like quantum fluctuations of the gauge field in the vacuum). Such static field configurations are characterized by \( \langle A_{\mu} \rangle \neq 0 \), \( \langle A_{\mu} A_{\nu} \rangle \neq 0 \) and so on (here \( \langle \rangle \) is a quantum average). These effects are non-perturbative and the corresponding field configurations are not a cloud of quanta.

For the quantization in this case we will follow to Heisenberg idea [3] [4]. It means that we declare \( A_{\mu} \) as an operator which satisfies to the corresponding Yang-Mills operator equations:

\[
D_\nu \tilde{F}^{\alpha\mu\nu} = 0
\]

where \( \tilde{F}^{\alpha\mu\nu} = \partial_\mu \hat{A}^{\alpha}_\nu - \partial_\nu \hat{A}^{\alpha}_\mu + f_{abc} \hat{A}^{a}_\mu \hat{A}^{b}_\nu \) is an operator of the field strength tensor for the non-abelian gauge field and \( f_{abc} \) is the structural constants. Of coarse this operator equation has a big mathematical difficulties but following to Heisenberg [3] [4] we will apply this equation in order to obtain an infinite set of equations connecting Green’s functions of different orders. On the language of Feynmann diagrams techniques it is the Ward identities connecting the Green’s functions of different orders. The operator equation (1) and the above-mentioned infinite set of equations for the Green’s functions mathematically is equivalent. The main difficulty by solving these equations for the Green’s functions is cutting off of the equations system to a finite set. It is a skill of a physicist.

Now we would like to consider some partial case: a spherically symmetric field distribution of the quantized SU(2) non-abelian gauge field. These words allows us to simplify Yang-Mills operator equations (1). In order to do this we will follow by this way: (a) on the first step we will write the spherically symmetric ansatz for the classical SU(2) field; (b) on the second step we will write the classical Yang-Mills equations for this ansatz; (c) on the third step we replace the classical functions by the quantum operators; (d) on the fourth step we will write equations for the Green's...
functions and cut off the derived infinite equations system; (e) and on the last step we will offer this field distribution as quantum excitation in the non-abelian vacuum. Evidently that quantum excitations can have an arbitrary field distribution inside itself but the distributions obeying the Yang-Mills equations are more probable.

II. EQUATIONS

Now we would like to present equations describing quantum excitations in the vacuum of SU(2) non-abelian gauge field. Every such excitation can have an inner structure, i.e. they can be solutions of some differential equations. In fact here we suppose that (according to Heisenberg) these equations for averaged quantum field approximately can be presented as classical Yang-Mills equations with quantum corrections coming from equations for the Green’s functions of higher orders. The spherically symmetric ansatz is

\[ A^a_i = \frac{x^a}{r^2} g(r), \]
\[ A^a_i = \frac{\varepsilon_{a i j} x^j}{r^2} [1 - f(r)] \]

where \( i = 1, 2, 3 \). Yang-Mills equations for this ansatz have the following form

\[ r^2 f'' = f^3 - f g^2 - f, \]
\[ r^2 g'' = 2 g f^2. \]

At first I would like to describe what we can expect from our next calculations. In the Ref. [5] it was shown that the general solution for the spherically symmetric ansatz in SU(3) non-abelian gauge theory has the following asymptotical behavior:

\[ A^a_0 \propto r^\alpha \text{ and } A^a_i \text{ is strongly oscillating (by } r \to \infty). \]

In our case the situation is the same

\[ f(x) \approx f_0 \sin (x^\alpha + \phi_0), \]
\[ g(x) \approx \alpha x^\alpha + \frac{\alpha - 1}{4} \cos (2 x^\alpha + 2 \phi_0) \]

where \( x = r/r_0 \) is the dimensionless radius; \( r_0, f_0, \alpha > 1 \) and \( \phi_0 \) are some constants and functions \( f(r) \) and \( g(r) \) are the classical solutions of corresponding Yang-Mills equations. We see that the frequency of the space oscillations of \( f(r) \) is increased by \( r \to \infty \). But after the quantization the situation can be changed. The reason is that the quantum uncertainty \( \Delta F^a_{\mu \nu} \) should obey the Heisenberg’s uncertainty relation

\[ \Delta S \approx \Delta F^2 \Delta t \Delta V \approx \hbar \]

where \( \Delta S \) and \( \Delta F \) is uncertainties of the action \( S \) and the non-abelian field \( F^a_{\mu \nu} \) respectively; \( \Delta V = \Delta z \Delta r \Delta l \) (\( \Delta l = r \Delta \phi \)). Let us define \( \Delta r \) as the distance between two adjacent maxima of the function \( f(r) \)

\[ (x + \Delta x)^2 \approx x^2 + 2 x \Delta x \approx x^2 + 2 \pi, \quad x = \frac{r}{r_0} \gg 1 \]
\[ r \approx \frac{2 \pi r_0^2}{\Delta r} \]

It means that inside of the small volume \( \Delta V \) (which is on the distance \( r \approx 2 \pi r_0^2 / \Delta r \) from the origin) the field fluctuations can be very big

\[ \Delta F^2 \Delta t \approx \frac{\hbar}{\Delta z \Delta r \Delta l} \]

and for \( \Delta r \approx 2 \pi r_0^2 / r \)

\[ \Delta F^2 \Delta t \approx \frac{\hbar}{r_0^3 \Delta z \Delta l} r, \]

i.e. there is \( \tilde{r}_0 \) where \( \Delta F \approx F \). The consequence of this is that the quantization of strongly oscillating fields should smooth these fluctuations. But it is not all : the functions \( f(r) \) and \( g(r) \) are connected with the Yang-Mills equations. In Ref. [3] it is shown that in the consequence of the field equations the bad asymptotical behavior of these functions \( f(r) \) and \( g(r) \) will be changed. In this paper we will try to describe this process more exactly.
Following to Heisenberg we replace classical functions \( f(r) \) and \( g(r) \) by quantum operators \( \hat{f}(r) \) and \( \hat{g}(r) \)

\[
\begin{align*}
    r^2 \hat{f}'' &= \hat{f}^3 - \hat{f} \hat{g}^2 - \hat{f}, \\
    r^2 \hat{g}'' &= 2\hat{g}f^2.
\end{align*}
\]

Let \( |Q\rangle \) is a quantum state which describes (in some approximation) the spherically symmetric distribution of quantized field. It means that the following quantum average exists \( \langle f(r) \rangle = \langle Q | \hat{f}(r)|Q\rangle \), \( \langle f^3(r) \rangle = \langle Q | \hat{f}^3(r)|Q\rangle \) and so on. For these Green’s functions we have the following two equations which are the first in the infinite series of equations

\[
\begin{align*}
    r^2 \langle f'' \rangle &= \langle f^3 \rangle - \langle fg^2 \rangle - \langle f \rangle, \\
    r^2 \langle g'' \rangle &= 2 \langle gf^2 \rangle.
\end{align*}
\]

The above-mentioned difficulty is that we must have the corresponding equations for \( \langle f^3 \rangle, \langle fg^2 \rangle, \langle gf^2 \rangle \) and so on up to infinity. Now we would like to cut off this infinite process.

### III. CUTTING OFF

Let us to introduce a new function \( \varphi(r) \) which will describe the influence of the Green’s functions of higher orders in Eq. (15) by such a way

\[
\langle f(r_1)g(r_2)g(r) \rangle = \langle f(r_1) \rangle \langle g(r_2) \rangle \langle g(r) \rangle - \langle f(r_1) \rangle \varphi(r_2)g(r).
\]

In order to obtain an equation for \( \varphi(r) \) we act with the operator \( r^2 \frac{d^2}{dr^2} \) on the left-hand and right-hand sides of this definition. In the consequence we have

\[
\langle f(r_1) \rangle \varphi(r_2)r^2 \varphi''(r) = 2 \left( \langle f(r_1) \rangle \langle g(r_2) \rangle \langle g(r) f^2(r) \rangle - \langle f(r_1) g(r_2) g(r) f^2(r) \rangle \right).
\]

But we are interested only for the case \( r_{1,2} \rightarrow r \)

\[
\langle f \rangle \varphi r^2 \varphi'' = 2 \left( \langle f \rangle \langle g \rangle \langle f^2 g \rangle - \langle f^3 g^2 \rangle \right)
\]

here we suppose that \( \hat{f}(r)\hat{g}(r) = \hat{g}(r)\hat{f}(r) \). It is interesting to emphasize here the absence of the \( \delta- \) function. Heisenberg wrote in this occasion [4]: “The states of Hilbert space II change the commutator in such a manner that the \( \delta- \) and \( \delta' - \) functions on the light-cone disappear, and that actually the wave function \( \psi \) and \( \psi^+ \) anti-commute everywhere on the subspace \( t = \text{const.} \).” It is necessary to note that such calculations are like to Heisenberg’s quantization of a non-linear spinor field [3] and on the classical level to the average of the product of turbulence velocities \( \langle v_i v_j \rangle, \langle v_i v_j v_k \rangle \) and so on [6]. Our main assumption is that

\[
\begin{align*}
    \langle f^3 g^2 \rangle &\approx \langle f^2 \rangle \langle fg^2 \rangle, \\
    \langle f^2 g \rangle &\approx \langle f^2 \rangle \langle g \rangle.
\end{align*}
\]

In this case we have

\[
\langle f \rangle \varphi r^2 \varphi'' = 2 \left( \langle f \rangle \langle g \rangle \langle f^2 \rangle - \langle fg^2 \rangle \right)
\]

From Eq. (15) we see that

\[
\langle f(r) \rangle \varphi^2(r) = \langle f(r) \rangle \langle g(r) \rangle^2 - \langle f(r) g^2(r) \rangle.
\]

It leads to the following equation

\[
r^2 \varphi'' = 2 \varphi \langle f^2 \rangle
\]

and we have

\[
\begin{align*}
    r^2 \langle f'' \rangle &= \langle f^3 \rangle - \langle f \rangle \langle g^2 \rangle - \langle f \rangle \varphi^2 - \langle f \rangle, \\
    r^2 \langle g'' \rangle &= 2 \langle g \rangle \langle f^2 \rangle, \\
    r^2 \varphi'' &= 2 \varphi \langle f^2 \rangle
\end{align*}
\]
with the following approximations
\[ \varphi^2 = \langle g \rangle^2 - \frac{\langle fg^2 \rangle}{\langle f \rangle}, \quad (28) \]
\[ \langle f^3 g^2 \rangle \approx \langle f^2 \rangle \langle fg^2 \rangle, \quad (29) \]
\[ \langle f^2 g \rangle \approx \langle f^2 \rangle \langle g \rangle. \quad (30) \]

If we suppose that
\[ \langle f^3 \rangle \approx \langle f \rangle \langle f^2 \rangle \quad (31) \]
then the first equation will be
\[ r^2 \langle f \rangle'' = \langle f \rangle \left( \langle f^2 \rangle - \langle g \rangle^2 + \varphi^2 - 1 \right) \quad (32) \]
and finally if \( \langle f^2 \rangle \approx \langle f \rangle^2 \) (it means that \( \langle \Delta f^2 \rangle \ll \langle f \rangle^2 \) and fluctuations of the field \( f \) is very small) we have the following set of equations
\[ r^2 \langle f \rangle'' = \langle f \rangle^3 - \langle f \rangle \langle g \rangle^2 + \langle f \varphi \rangle^2 - \langle f \rangle \quad (33) \]
\[ r^2 \langle g \rangle'' = 2 \langle g \rangle \langle f \rangle^2, \quad (34) \]
\[ r^2 \varphi'' = 2 \varphi \langle f \rangle^2 \quad (35) \]

It is very interesting to compare these equations with the equations describing a dyon [7]
\[ r^2 f'' = f^3 - f g^2 + f \varphi^2 - f, \quad (36) \]
\[ r^2 g'' = 2 g f^2, \quad (37) \]
\[ r^2 \varphi'' = 2 \varphi f^2 + \lambda \varphi \left( \varphi^2 - r^2 \varphi_0^2 \right) \quad (38) \]
where \( f \) and \( g \) are the same as in Eq’s (2) (3) and \( \varphi(r) \) describes the spherically symmetric Higgs scalar field \( \phi^a \)
\[ \phi^a(r) = \frac{x^a}{r^2} \varphi(r). \quad (39) \]

Immediately we see that our equations coincide with dyon equations in the limit \( \lambda \to 0 \). Consequently the solution is
\[ \langle f(x) \rangle = \frac{x}{\sinh x}, \quad (40) \]
\[ \langle g(x) \rangle = \sinh \gamma \left( \frac{x}{\tanh x} - 1 \right), \quad (41) \]
\[ \varphi(x) = \cosh \gamma \left( \frac{x}{\tanh x} - 1 \right) \quad (42) \]
where \( x = r/r_0 \) and \( \gamma, r_0 \) are some constants. Thus we can say that Eq’s (36)-(38) describe the quantized spherically symmetric configurations of the SU(2) gauge field with taking into account the higher order of Green’s functions. Another words the function \( \varphi \) is the result of the non-linearity of non-abelian field. It allows us to do the cautious assumption that the Higgs scalar field in SU(2) and SU(3) gauge theories is the consequence of the quantization of the non-linear fields.

Our interpretation of the solution (40)-(42) is the following: it describes the spherically symmetric excitation in the non-abelian vacuum.

**IV. HIGGS POTENTIAL**

Let us consider more carefully the right-hand side of Eq. (19). We can introduce a new function \( \chi(r) \) by the following way
\[ \langle f \rangle \langle g \rangle \langle f^2 g \rangle - \langle f^3 g^2 \rangle = \langle f^2 \rangle \left( \langle f \rangle \langle g^2 \rangle - \langle fg^2 \rangle \right) \left( 1 + \frac{1}{2} \chi \right) = \langle f^2 \rangle \langle f \rangle \varphi^2 \left( 1 + \frac{1}{2} \chi \right) \quad (43) \]
The corresponding equation for $\varphi$ looks as

$$r^2 \varphi'' = 2\varphi \langle f \rangle^2 + \varphi \langle f^2 \rangle \chi$$  \hspace{1cm} (44)$$

Of course, for the definition of $\chi$ we should have another equation from the infinite set of equations for the Green’s functions. Such chain of differential equations have need for break. Evidently the next equation for the function $\chi$ will be differential equation. In order to avoid this differential equation we should introduce some algebraic relation between new function $\chi$ and old functions $f, g, \varphi$. For this algebraic relation we suppose that this addition should not destroy the monopole-like behavior of the initial solution (without $\chi(r)$). It means that $\langle f^2 \rangle \chi$ should be the Higgs potential

$$\langle f^2 \rangle \chi \approx \lambda \left( \varphi^2 - \varphi^2_\infty \right)$$  \hspace{1cm} (45)$$

where $\lambda$ is some constant and $\varphi(r) \to \varphi_\infty(r)$ by $r \to \infty$. More concretely we see from Eq. (42)

$$\varphi_\infty(r) = \frac{\sinh \gamma}{\gamma_0} r = \varphi_0 r$$  \hspace{1cm} (46)$$

where $\varphi_0 = \sinh \gamma/\gamma_0$. Thus we have

$$r^2 \langle f \rangle'' = \langle f \rangle^3 - \langle f \rangle \langle g \rangle^2 + \langle f \rangle \varphi^2 - \langle f \rangle$$  \hspace{1cm} (47)$$

$$r^2 \langle g \rangle'' = 2\langle g \rangle \langle f \rangle^2,$$  \hspace{1cm} (48)$$

$$r^2 \varphi'' = 2\varphi \langle f \rangle^2 + \lambda \varphi \left( \varphi^2 - \varphi^2_\infty \right).$$  \hspace{1cm} (49)$$

That completely coincides with the dyon equations. From equation (43) we see that the function $\chi$ is the next quantum correction for our equations, i.e. it is the correction $\chi$ to the correction $\varphi$. It is necessary to note that Eq. (49) is similar to the Ginzburg-Landau equation in the superconductivity theory. In both theories they are pure quantum equations connected with the non-linearity of initial classical equations and are the consequence of the Heisenberg quantization method applied for the non-abelian gauge theory and for the quantum solid theory.

### V. DISCUSSION AND CONCLUSIONS

The above-mentioned discussion about function $\varphi(r)$ and an analog of Higgs potential $\langle f^2 \rangle \chi \approx \lambda \left( \varphi^2 - \varphi^2_\infty \right)$ allows us to present such interpretation of the Higgs field : (a) it describes quantum corrections to the classical equations; (b) unstable vacuum $\varphi = 0$ describes classical singular solutions; (c) stable vacuum $\pm \varphi_\infty$ describes stable state of Yang-Mills field in the quantum region. It means that the **Higgs field describes the transition from unstable classical state to the quantum steady-state**. It is possible that our interpretation of the function $\varphi$ as an analog of the Higgs scalar field is correct in the general case.

Finally we can conclude

1. Among quantum excitations (which are indicated by Abelian Projection) in the non-abelian vacuum there are the dynamical monopoles. These excitations exist some time which can be defined from the uncertainty principle $\Delta t \approx h/Mc^2 \approx 10^{-27}\text{s}$ where $M \approx 10^9\text{eV}$ is the monopole mass.
2. They are the spherically symmetric configurations of the quantized field which can be obtained from quantum Yang-Mills equations.
3. Quantum corrections of the classical Yang-Mills equations in the first approximation is connected with the Higgs scalar field. It is interesting to note that if it is true then the Higgs mechanism for electro-weak interactions and confinement in color chromodynamics have one origin: they are quantum corrections in the consequence of the non-linearity of the initial classical Yang-Mills equations.

In Ref. [5] is shown that SU(3) gauge field theory has very similar classical equations for the spherically symmetric field configuration. It allows us to suppose that in the SU(3) chromodynamics quantum excitations also are the consequence of quantized SU(3) Yang-Mills theory.
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