Learning to Match Mathematical Statements with Proofs

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Abstract

We introduce a novel task consisting in assigning a proof to a given mathematical statement. The task is designed to improve the processing of research-level mathematical texts. Applying Natural Language Processing (NLP) tools to research level mathematical articles is both challenging, since it is a highly specialized domain which mixes natural language and mathematical formulae. It is also an important requirement for developing tools for mathematical information retrieval and computer-assisted theorem proving (Mathematical Sciences, 2014). We release a dataset for the task, consisting of over 180k statement-proof pairs extracted from mathematical research articles. We carry out preliminary experiments to assess the difficulty of the task. We first experiment with two bag-of-words baselines. We show that considering the assignment problem globally and using weighted bipartite matching algorithms helps a lot in tackling the task. Finally, we introduce a self-attention-based model that can be trained either locally or globally and outperforms baselines by a wide margin.

1 Introduction

Research-level mathematical discourse is a challenging domain for Natural Language Processing (NLP). Indeed, mathematical articles switch frequently between natural language and mathematical formulae. A semantic analysis of mathematical text needs to solve relationships (e.g. coreference) between mathematical symbols and concepts. Moreover, mathematical writing follows a lot of conventions, such as variable naming or typography, that are implicit, and may differ from a subfield to another.

However, mathematical research can benefit from NLP (Mathematical Sciences, 2014), in particular as concerns bibliographical research: researchers need tools to find work relevant for their research. Indeed, prior NLP work on mathematical research articles focused on Mathematical Information Retrieval (MIR) and related tools or data (Zanibbi et al., 2016; Stathopoulos and Teufel, 2016, 2015).

In this paper, we introduce a task aimed at improving the processing of research-level mathematical articles and make a step towards the modeling of mathematical reasoning. Given a collection of mathematical statements and a collection of mathematical proofs of the same size, the task consists in finding and assigning a proof to each mathematical statement. We construct and release a dataset for the task, by collecting over 180k statement-proof pairs from mathematical research articles (an example is given in Figure 1).

There are multiple motivations for the design of the task. We believe it may help MIR by serving as a proxy for the search for the existence of a mathematical result, or for theorems and proofs related to one another (e.g. using the same proof technique), an important search tool for any digital mathematical library (Mathematical Sciences, 2014). Learning to match statements and proofs would also benefit computer-assisted theorem proving, as it is akin to tasks such as premise selection, also recently addressed with NLP methods (Piotrowski and Urban, 2019). More generally, finding supporting informa-

Figure 1: Example of a statement-proof pair.

Theorem 1.3. Suppose that $|\text{Sing}(S)| < (2r - 1)r$. Then $X$ is factorial.

Proof. The subset $\text{Sing}(S) \subset \mathbb{P}^3$ is a set-theoretic intersection of surfaces of degree $2r - 1$, which implies that $X$ is factorial by Theorem 1.1.
tion for or against a given statement, is integral to tasks such as question answering or fact-checking (Vlachos and Riedel, 2014). Our mathematical statement-proof assignment task can be thought of as the transposition of such problem to the very specific domain of mathematical research articles.

We provide preliminary results on our proposed task with (i) two bag-of-words baselines (ii) a neural model based on a self-attentive encoder and a bilinear similarity function. Though the neural model outperforms the baselines when using local decoding, i.e. assigning the best-scoring proof to each statement, we found that it performs even better with global decoding, i.e. finding the best bipartite matching between the sets of statements and proofs. Therefore we also design a global training procedure with a structured max-margin objective. Such an architecture may have applications to other NLP problems that can be cast as maximum bipartite matching problems, which is the case, for example, for some alignment problems (Taskar et al., 2005b; Padó and Lapata, 2006).

In summary, our contributions are three-fold:

- The definition of a mathematical statement-proof matching task;
- The construction and release of a corresponding dataset;
- A self-attention-based model for maximum weighted bipartite matching problems, that can be trained either locally or globally.

2 Related Work

Processing mathematical articles Most NLP work on mathematical discourse focuses on improving Mathematical Information Retrieval (Zanibbi et al., 2016, MIR) by establishing connections between mathematical formulae and natural language text in order to improve the representation of formulae.

The interpretation of variables is highly dependent on the context. For example, the symbol $E$ could denote an expectation in a statistics article, or the energy in a physics article. Some studies use the surrounding context of a formula to assign a definition or a type to the whole formula, or to specific variables. Nghiem Quoc et al. (2010) focus on identifying coreferences between mathematical formulae and mathematical concepts in Wikipedia articles. Kristianto et al. (2012) extract definitions of mathematical expressions. Grigore et al. (2009), Wolska et al. (2011) and Schubotz et al. (2016) disambiguate mathematical identifiers, such as variables, using the surrounding textual context. Stathopoulos et al. (2018) infer the type of a variable in a formula from the textual context of the formula.

Another line of work focused on identifying specialized terms or concepts to improve MIR (Stathopoulos and Teufel, 2015, 2016).

Some work adapts standard NLP tools to the specificity of mathematical discourse, e.g. POS taggers (Schöneberg and Sperber, 2014), with the objective of using linguistic features to improve the search for definitions of mathematical expressions (Pagel and Schubotz, 2014).

Maximum bipartite matching in NLP Global models for maximum weighted bipartite matching problems have been explored in NLP for the task of word alignments, a traditional component of machine translation systems (Matusov et al., 2004; Taskar et al., 2005b; Bhagwani et al., 2012; Wang and Lepage, 2016), or for assigning arguments to predicates (Lluis et al., 2013). In particular, Taskar et al. (2005b) introduced a discriminative global model with a max-margin objective.

In these articles, the bipartite graph is usually formed by two sentences. In contrast, we predict matchings on graphs that are an order of magnitude larger and each node in our bipartite graph is a complete text (a statement or a proof), i.e. a highly structured object, from which we learn fixed size vector representations.

3 Task Description

Given a collection of mathematical statements $\{s_i\}_{i \leq N}$, and a separate equal-size collection of mathematical proofs $\{p_i\}_{i \leq N}$, we are interested in the problem of assigning a proof to each statement.

Evaluation We use two evaluation metrics. Assuming that a system predicts a ranking of proofs, instead of providing only a single proof, we evaluate its output with the Mean Reciprocal Rank (MRR) measure:

$$\text{MRR}(\{\hat{r}_i\}_{i \in \{1,...,N\}}) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\hat{r}_i},$$

where $N$ is the number of examples and $\hat{r}_i$ is the rank of the gold proof for statement number $i$, as predicted by the system.
As a second evaluation metric, we use a simple accuracy, i.e. the proportion of statements whose first-ranked proof is correct.

By construction (see Section 4), it is possible though unlikely that the same mathematical statement occurs several times in the dataset. It is all the more unlikely that several occurrences have exactly the same formulation and use the same variable names. Therefore, we consider a match to be correct if and only if it is associated with its original proof.

**Task variation** We propose three variations of the task, depending on the input of the system:

1. Natural language text and mathematical formulae;
2. Natural language text only;
3. Mathematical formulae only.

The comparison of these settings is meant to provide insight into which type of information is crucial to the task.

4 Dataset Construction

This section describes the construction of a dataset\(^1\) of statement-proof pairs (see Figure 1).

**Source corpus** We use the MREC corpus\(^2\) (Liška et al., 2011) as a source. The MREC corpus contains around 450k articles from ArXMiLV (Stamerjohans et al., 2010), an on-going project aiming at converting the arXiv\(^3\) repository from \LaTEX\ to XML, a format more suited to machine processing. In this collection, mathematical formulae are represented in the MathML\(^4\) format, a markup language.

**Statement-proof identification** For each XML document (corresponding to a single arXiv article), we extract pairs of consecutive \(<\text{div}>\) tags such that: (i) the \textit{class} attribute of the first \texttt{div} node contains the string "theorem"; (ii) the \textit{class} attribute of the second \texttt{div} node is the string "proof". Documents that do not contain such pairs of tags are discarded, as well as documents that are not written in English (representing 143 articles in French, 11 in Russian, 5 in German, 2 in Portuguese and 1 in Ukrainian), as identified by the polyglot Python package\(^5\).

In the remaining collection of pairs of statements and proofs, we filter out pairs for which either the statement or the proof is too short.\(^6\) Indeed, the short texts were often empty (only consisting of a title, e.g. “5.26 Lemma.”), which we attribute to the noise inherent to the conversion to XML, or not self-contained. In particular, we identified several prototypical cases:

- Omitted (or easy) proofs contain usually a single word (‘omitted’, ‘straightforward’, ‘well-known’, ‘trivial’, ‘evident’), but are sometimes more verbose (‘This is obvious and will be left to the readers’).
- Proofs that consist of a single reference to
  - An appendix (‘See Appendix A’);
  - Another theorem (‘This follows immediately from Proposition 4.4 (ii).’);
  - The proof method of another theorem (‘Similar to proof of Lemma 6.1’);
  - Another article (‘See [BK3, Theorem 4.8]’);
  - Another part of the article (‘The proof will appear elsewhere.’, ‘See above.’, ‘Will be given in section 5.’).

Filtering on the number of tokens also exclude self-contained short proofs, such as ‘Take $Q_i' = p_{i'} - p_i$.’ However, such proofs were very infrequent on manual inspection of the discarded pairs (2 in a manually inspected random sample of 100 discarded proofs).

**Preprocessing: linearizing equations** Mathematical formulae in the XML documents are enclosed in a \(<\text{math}>\) markup tag, that materializes the switch to the MathML format, and whose internal structure represents the formula as an XML tree. As a preprocessing step, we linearize each formula to a raw sequence of strings.

In MathML, an equation can be encoded in a content-based (semantic) way or in a presentation way, using different sets of markup tags. We first convert all MathML trees to presentation MathML using the XSL stylesheet from the Con-
tent MathML Polyfill repository.\footnote{https://github.com/fred-wang/webextension-content-mathml-polyfill} Then we perform a depth-first search on each tree rooted in a \texttt{math} tag to extract the text content of the whole tree.

During this preprocessing, we tested several processing choices:

- **Font information.** In mathematical discourses, fonts play an important role. Their semantics depend on conventions shared by researchers. If both $x$ and $x$ appear in the same article, they are most likely to represent different mathematical objects, e.g. a scalar and a vector. Therefore, we use distinct symbols for tokens that are in distinct fonts.

- **Math-English ambiguity.** Some symbols can be used both in natural language text and in formulae. For example, ‘a’ can be a determiner in English, or a variable name in a formula. To avoid increasing ambiguity when linearizing formula, we type each symbol (as math or text) to make the mathematical vocabulary completely disjoint from the text vocabulary.

Both these preprocessing steps had a beneficial effect on the baselines in preliminary experiments.

**Statistics** We report in Table 1 some statistics about the dataset. The extracted articles were from a diverse set of mathematical subdomains, and connected domains, such as computer science (746 articles from 30 subcategories) and mathematical physics (2562 from 31 subcategories). There are in average 6.6 statement-proof pairs per article.

We report statistics about the size of statements and proofs in number of tokens in Table 2. We report the number of tokens in formulae (math), in the text itself (text) and in both (text+math). On average, proofs are much longer than statements. Statements and proofs have approximately the same proportion of text and math. Overall, the variation in number of tokens across statements and proofs is extremely high, as illustrated by the standard deviation (SD) of all presented metrics.

5 **Self-Attentive Bilinear Similarity Model**

We propose a system based on a self-attentive encoder \cite{vaswani2017attention} that constructs fixed-size vector representations for statements and proofs, and a similarity function that scores the relatedness of a statement-proof pair.

**Self-attentive encoder** We encode each text with a token-level self-attentive encoder. We first project a text to a sequence of token embeddings of dimension $w$. Then we run $\ell$ self-attention layers \cite{vaswani2017attention}, to obtain a contextualized embedding for each token. Finally we construct a vector representation for the text with a max-pooling layer over the contextualized embeddings of the last self-attention layer.

The hyperparameters of the encoder are the dimension of the token embeddings $w$, the number of self-attentive layers $\ell$, the dimension of the en-
coder $d$ (size of contextualized embeddings), the number of heads for each self-attentive layer $h$ and the dimension of query and key vectors $d_k$.

**Trainable bilinear similarity function** Given the encoded representations of a statement $s = \text{enc}(s)$ and a proof $p = \text{enc}(p)$, we compute an association score with the following bilinear form:

$$\text{score}(s, p) = s^\top \cdot W \cdot p + b,$$

where $W$ and $b$ are parameters that are learned together with the self-attentive module parameters.

**Local decoding** For a collection of $n$ statements and proofs, we first score all possible pairs $(s, p)$, and construct a matrix $M = (m_{ij}) \in \mathbb{R}^{n \times n}$, with

$$m_{ij} = \text{score}(s^{(i)}, p^{(j)}),$$

where $s^{(i)}$ and $p^{(j)}$ are the encoded representations of, respectively, the $i^{th}$ statement and the $j^{th}$ proof. Then we can straightforwardly sort each row by decreasing order and assign the proof ranking to the corresponding statement. The best ranking proof $\hat{p}_i$ for statement $i$ satisfies:

$$\hat{p}_i = \arg \max_j m_{ij}.$$ We call this decoding method ‘local’, since it does not take into account dependencies between assignments. In particular, several statements may have the same highest-ranking proof.

**Global decoding** The local decoding method overlooks a crucial piece of information: a proof should correspond to a single statement. In a worst-case situation, a small number of proofs may score high with most statements and be systematically assigned as highest-ranking proof by the local decoding method.

During preliminary experiments, we analysed the output of our system with local decoding on the development set, focusing on the distribution of the single highest-ranking proof for each statement. It turned out that around 23% of proofs were assigned to at least two different statements, whereas more than 40% of proofs were assigned to no statement (Table 3).

We propose a second decoding method based on a global constraint on the output: a proof can be assigned only to a single statement. Intuitively, the constraint models the fact that if a proof is assigned by the system to a certain statement with high confidence, we can rule it out as a candidate for other statements. Under this constraint, the decoding problem reduces to a classical maximum weighted bipartite matching problem, or equivalently, a Linear Assignment Problem (LAP). In more realistic scenarios (e.g. if the input sets of statements and proofs do not have the same size), the method would require some adaptation.

Formally, we define an assignment $A$ as a boolean matrix $A = (a_{ij}) \in \{0, 1\}^{n \times n}$ with the following constraints:

$$\forall i, j \sum_j a_{ij} = \sum_i a_{ij} = 1,$$

i.e. each row and each column of $A$ contains a single non-zero coefficient. The score of an assignment $A$ is the sum of scores of the chosen edges:

$$\text{score}(A, M) = \sum_i \sum_j a_{ij} m_{ij}.$$ Finally, global decoding consists in solving the following LAP:

$$\hat{A}(M) = \arg \max_{A \in \{0, 1\}^{n \times n}} \text{score}(A, M),$$

s.t. $\forall i, j \sum_j a_{ij} = \sum_i a_{ij} = 1$

| Statements | Proofs | %  |
|------------|--------|----|
| $\geq 20$  | 7      | 0.0|
| $\geq 10$  | 80     | 0.2|
| $\geq 5$   | 1027   | 1.9|
| $\geq 2$   | 11949  | 22.6|
| = 1        | 19531  | 37.0|
| < 1        | 21275  | 40.3|

Table 3: Cumulative distribution of proofs in the development set, by number of statements to which they are assigned with the local decoding method.
We would like to train our model to assign a high similarity to the gold statement-proof pair, and a low similarity to all other statement-proof pairs. This corresponds to the following objective, for a single statement \( s \) and its gold proof \( p \):

\[
L_{LOC}(s, p, P; \theta) = - \log P[p | s; \theta]
\]

\[
= - \log \left( \frac{e^{\text{score}(s, p)}}{\sum_{p' \in P} e^{\text{score}(s, p')}} \right),
\]

where \( P \) is the set of proofs, and \( \theta \) are the parameters of the model. Directly optimizing this loss function requires the computation of \( p = \text{enc}(p) \) for every proof in the dataset, for a single optimization step. This is not realistic considering memory limitations, the size of the train set, and the fact that the self-attentive encoder is the most computationally expensive part of the network.

Instead, we sample minibatches of \( b \) pairs and optimize the following proxy loss for the sequence \( S' = (s_1, \ldots, s_n) \) of statements and the sequence \( P' = (p_1, \ldots, p_n) \) of corresponding proofs:

\[
L'_{LOC}(S', P'; \theta) = \sum_{i=1}^{b} L_{LOC}(s_i, p_i, P'; \theta).
\]

In practice, we sample uniformly and without replacement \( b \) pairs from the training set at each stochastic step.

### 6.2 Hybrid Local and Global Training

The local training method only considers statements in isolation. Even though we expect a locally trained model to perform better with global decoding, we hypothesize that a model that is trained to predict the full structure (a bipartite matching) will be even better.

For a collection of \( n \) proofs and \( n \) statements, the size of the search space (i.e. the number of bipartite matchings) is \( n! \), since each matching corresponds to a permutation of proofs. As a result, the use of a globally normalized model is impractical. We turn to a max-margin model that does not require normalization over the full search space.

We use the following max-margin objective, for a set \( B \) of \( n \) pairs corresponding to matrix \( M \):

\[
L_{GLOBAL}(B; \theta) = \max(0, \Delta(\hat{A}, I)) + \text{score}(\hat{A}, M) - \text{score}(I, M),
\]

where \( \theta \) is the set of all parameters \( \theta \) is the predicted assignment and \( I \) is the gold assignment, i.e. the identity matrix. The structured cost

\[
\Delta(\hat{A}, I) = \sum_{ij} \max(0, (\hat{A} - I)_{ij})
\]

aims at enforcing a margin for each individual assignment. In order to compute the loss during training, we perform decoding on matrix \( M' \), which directly incorporates the cost of wrong assignments (Taskar et al., 2005a):

\[
M' = M + (1 - I).
\]

The computation of this loss requires exact decoding for each optimization step. Since exact decoding is only feasible for a small \( n \), and since we need to keep track of all intermediary vectors to compute the backpropagation step,\(^8\) we perform each stochastic optimization step on a minibatch of pairs of size \( b \). Since this global objective had a very slow convergence rate (see Section 7.1), in practice, we optimize a hybrid local-global objective: \( L_{LOC}' + L_{GLOBAL} \).

### 7 Experiments

Our experiments address several questions. First, we assess the difficulty of the task and provide preliminary results with baseline systems. Secondly, we evaluate the performance of our neural model in several settings: global or local training, global

\(^8\)We also experimented with a Noise-Contrastive Estimation approach (Gutmann and Hyv"{a}rinen, 2012). However, it exhibited a much slower convergence rate.

\(^9\)In particular, the computation graph needs to conserve all encoding layers for the \( 2n \) texts involved.
or local decoding. In particular, we are interested in assessing whether global decoding improves accuracy when training is only local, and how the more complex global training method fares with respect to local training. Finally, we are interested in the informativeness of different types of input: text, mathematical formulae, or both.

We describe the experimental protocol (Section 7.1) before discussing results (Section 7.2).

### 7.1 Experimental setup

**Dataset** We use the dataset whose construction is described in Section 4. We shuffle the collection of statement-proof pairs before performing a 80%/10%/10% train-development-test split, corresponding to 147276 pairs for the training sets and 18409 pairs for the development and tests. Due to the shuffling, pairs from a single article may be distributed across the three sections.

**Baselines** We provide two baseline systems that rank proofs according to their similarity to the statement, using classical similarity measures. The first baseline computes cosine similarities between TF-IDF representations of statements and proofs. The second baseline uses Dice’s similarity measure computed over bag-of-word representations of statements and proofs:

\[
\text{Dice}(s, p) = \frac{2|s \cap p|}{|s| + |p|},
\]

where \(s\) and \(p\) are the word multiset representations of, respectively, a statement and a proof.

Both baselines are implemented using the scikit-learn Python package (Pedregosa et al., 2011) with default parameters. We estimate the IDF metric on the training set only.

**Neural model** We implemented the neural network in Pytorch (Paszke et al., 2017). Token embedding have \(c = 300\) dimensions, we use \(\ell = 2\) self-attentive layers with 4 heads to obtain contextualized embeddings of dimension \(d = 300\). The query and key vectors have size \(d_k = 128\).

We trained each model on a single GPU using the Pytorch implementation of the Averaged Stochastic Gradient Descent algorithm (ASGD Polyak and Juditsky, 1992), with learning rate 0.02, and an exponential learning rate scheduler (the learning rate is multiplied by 0.99 after each epoch).

**Hyperparameters** For training a local model, we perform 400 epochs over the whole training set, assuming an epoch consists in \(N/b\) stochastic steps (where \(N\) is the total number of training pairs and \(b\) is the number of pairs in each minibatch). We evaluate the model’s performance on the development set every 20 epochs and select the best model among these intermediate models. We use batches of size \(b = 60\) based on preliminary experiments.

For global training, we perform 400 epochs (around 3 days with a single GPU) and use the same model selection method as in the local training experiments. We observed in initial experiments that training only with the global objective required a very long time and had a very slow convergence rate. Therefore, we used the following global-local objective: \(L'_{\text{LOC}} + L_{\text{GLOB}}\), that we optimized by alternating one stochastic step for each loss. We use batches of size 60 for both the local loss and the global loss. Although the global model might benefit from larger batches, 60 was the maximum possible size given our memory resources.

**Global decoding** Recall that exact global decoding is only feasible for a small subset of pairs. During global training, we chose a batch size small enough to perform exact decoding. However, it is not feasible to perform exact decoding on the whole development and test corpora. Therefore, we prune the search space by keeping only the 500-best candidate proofs for each statement, and use the LAP-MOD algorithm designed for sparse matrices. In practice, we used the implementations of the LAP-JV and LAP-MOD algorithms from the lap Python package,\(^\text{10}\) for respectively exact decoding on minibatches during global training and decoding on whole datasets during evaluation.

### 7.2 Results

**Baseline vs self-attentive systems** We report baseline results in Table 4. The best baseline is the TF-IDF model considering both text and mathematical formulae as input, it achieves an MRR of 29.9 and an accuracy of 23.8 (dev set). These results suggest that the task is not trivial, and that bag-of-words model are insufficiently expressive to solve it. In contrast, our best self-attentive model (Table 5) outperforms all baselines by a wide margin, obtaining an MRR of 64.5 and an accuracy of 57.8 (dev set, local decoding). However, the neural model fails to improve over the baselines in the text-only setting, perhaps due to the fact that

\(^\text{10}\)https://github.com/gatagat/lap
Table 4: Baseline results with the TF-IDF system and the word-overlap system (Dice), with either global or local decoding. The input to the systems are either only the textual parts, only the mathematical formulae, or both.

| Input | Method | Local decoding | Global decoding |
|-------|--------|----------------|-----------------|
|       |        | MRR            | Accuracy        |
| Dev   | Both   | 16.6           | 12.7            |
|       | Dice   |               | 25.2            |
|       | Both   | 29.9           | 23.8            |
|       | TF-IDF |               | 36.3            |
| Text  | Dice   | 10.4           | 7.8             |
|       |        |               | 16.2            |
|       | Both   | 27.9           | 22.7            |
|       | TF-IDF |               | 26.3            |
| Math  | Dice   | 13.3           | 10.0            |
|       |        |               | 10.4            |
|       | Both   | 12.1           | 9.1             |
|       | TF-IDF |               | 9.5             |
| Test  | Both   | 16.8           | 12.9            |
|       | Dice   |               | 25.4            |
|       | Both   | 31.2           | 25.0            |
|       | TF-IDF |               | 35.6            |
| Text  | Dice   | 10.7           | 8.0             |
|       |        |               | 17.3            |
|       | Both   | 27.8           | 22.4            |
|       | TF-IDF |               | 26.4            |
| Math  | Dice   | 13.6           | 10.2            |
|       |        |               | 11.1            |
|       | Both   | 12.2           | 9.3             |
|       | TF-IDF |               | 9.7             |

Global decoding with local training In all settings, the use of global decoding substantially improves accuracy. This improvement is also manifested with baselines.

Global training We obtain a substantial improvement over local training when incorporating the global loss. However, the improvement is much better for models that already have high results (i.e. math-only and math-text settings).

Effect of input type For baselines, we observe that using both mathematical formulae and text gives the best results. The baseline models using only text outperform the neural models using the same input as well as the baselines in the math-only settings. The pattern is different for neural models: the models using only math input are the best and slightly outperform models with both text and math input. This result suggests that mathematical formulae are crucial to solve the task and best used with an expressive neural model.

Qualitative analysis Upon inspection of our global model’s incorrect predictions (‘both’ setting) on the development set, we found that a common source of confusion is due to the proof often introducing discourse-new concepts and new variables, while not necessarily repeating discourse-given concepts that occur in the statement. As a result, the set of variables and concepts in a proof might better match those of another statement. We provide examples of the model’s output in Appendix A (supplementary material). Finally, incorrectly predicted proofs often contain highly polysemous words (linearly, components) that also occur in the statement.

8 Conclusion

We have introduced a new task focusing on the domain of mathematical research articles. The task consists in assigning a proof to a mathematical statement. We have constructed a dataset made of 184k statement-proof pairs for the task and assessed its difficulty with two classical bag-of-words baselines. Finally, we have introduced a global neural model for addressing the structured prediction problem of maximum weighted bipartite matching. The model is based on a self-attentive encoder and a bilinear similarity function. Our experiments show that bag-of-words baselines are insufficient to solve the task, and are outperformed by our proposed model by a wide margin. We found that decoding is crucial to achieve high results, and is further enhanced by a global training loss. Finally, our results show that mathematical formulae are the most informative source of information for the task but are best taken into account with the self-attentive neural model.
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A Output Examples

We provide examples of incorrect outputs by the globally trained model (‘both’ setting) in Figures 2 and 3. In both cases, the predicted proof contains variable names or concepts from the statement that do not occur in the gold proof.
Corollary 6.13. If $G$ is a compact connected Lie group, then for any maximal abelian connected closed subgroup of $H < G$, $G$ is the union of the conjugates of $H$.

Gold proof

Proof. The only thing to observe is that we do not need any definability assumptions. The definability comes for free since any compact Lie group $G$ is isomorphic to a compact subgroup $K$ of $GL(n, \mathbb{R})$ for some $n$ (see [4, Ch. 3, Thm. 4.1, p. 136]) and any such $K$ is a (real)algebraic subgroup of $GL(n, \mathbb{R})$ [5, Prop. 2, p. 230], hence it is definable in the o-minimal structure $(\mathbb{R}, <, +, \cdot)$.

Figure 2: Example of a wrong prediction, word overlaps are highlighted in orange (present in both gold and predicted proof, red (only in predicted proof), blue (only in gold proof)).

3.10 Lemma Let $G^\circ$ be the identity component of $G$, let $H$ be a closed, connected subgroup of $G$, and let $\Gamma$ be a discrete subgroup of $G$. Then:

1. $\Gamma$ acts properly on $G/H$ if and only if $\Gamma \cap G^\circ$ acts properly on $G^\circ/H$.

2. $\Gamma \backslash G/H$ is compact if and only if $(\Gamma \cap G^\circ) \backslash G^\circ/H$ is compact.

Gold proof

Proof. (1) Because every element of the Weyl group of $G$ has a representative in $G^\circ$ [BT1, Cor. 14.6], we see that $G$ and $G^\circ$ have the same positive Weyl chamber $A^+$, and the Cartan projection $G^\circ \to A^+$ is the restriction of the Cartan projection $G \to A^+$. Thus, the desired conclusion is immediate from Corollary 3.9. (2) This is an easy consequence of the fact that $G/G^\circ$ is finite [Mo2, Appendix].

Predicted proof

Proof. Because $G$ is reductive, there is a subgroup $H$ of $G$ which is semi-simple and such that the quotient $G/H$ is an extension of a finite group by a torus. Note that the quotient group $G/H$ acts on the ring of invariants $S^H$ for the semi-simple group: $g \in G/H$ acts on $f \in S^H$ by $g \cdot h$ where $g$ is any lifting to $G$ of $\bar{g}$. It is easy to verify that $S^G = (S^H)^{G/H}$. Because $H$ is semi-simple, the ring $S^H$ is Gorenstein. Thus by the preceding lemma, it is strongly F-regular. On the other hand, $G/H$ is linearly reductive and thus the inclusion $(S^H)^{G/H} \to S^H$ is split by the Reynolds operator. This splitting descends to characteristic $p$ for all $p > 0$. Therefore, because $S^H$ is strongly F-regular in almost all fibers, so is its direct summand $S^G = (S^H)^{G/H}$.

Figure 3: Example of a wrong prediction, word overlaps are highlighted in orange (present in both gold and predicted proof, red (only in predicted proof), blue (only in gold proof)).