Derivative expansion of the heat kernel in curved space

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The heat kernel in curved space-time is computed to fourth order in a strict expansion in the number of covariant derivatives. The computation is made for arbitrary non abelian gauge and scalar fields and for the Riemann connection in the coordinate sector. The expressions obtained hold for arbitrary tensor representations of the matter field. Complete results are presented for the diagonal matrix elements and for the trace of the heat kernel operator. In addition, Chan’s formula is extended to curved space-time. As a byproduct, the bosonic effective action is also obtained to fourth order.

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I. INTRODUCTION

The heat kernel operator, the exponential of the Klein-Gordon operator, is a useful tool in quantum field theory since it is ultraviolet finite, one-valued and gauge covariant and allows to obtain the propagator and the effective action \[1, 2, 3, 4\]. Quite different applications of the heat kernel (such as spectral densities, index theorems, \(\zeta\)-function, quantum anomalies, chiral gauge theories, effective theories of QCD, Casimir effect, black hole entropies, membranes, etc.) are illustrated in \[3, 6, 7, 8, 9, 10, 11, 12, 13, 14\]. The exact evaluation of the heat kernel is not possible in general (see however, \[15, 16\] and \[17, 18, 19\] for particular cases). Nevertheless the series expansion in powers of the proper time is available with computable coefficients, the so-called Hadamard-Minakshisundaram-DeWitt-Seeley (HMDS) or heat kernel coefficients \[4, 20\]. These coefficients have been computed with different techniques in several setups and to dimension ten \[21, 22, 23, 24, 25, 26, 27\]. For reviews on spectral geometry and field theory on curved space see \[28, 29, 30, 31, 32, 33, 34, 35, 36\]. For an alternative approach in the curved case and develop our own approach based on the use of covariant symbols \[45, 46\]. The covariant symbols are in fact multiplicative operators (with respect to \(x\)) which define a faithful representation of the algebra of pseudodifferential operators. This technique is similar to that of symbols for pseudodifferential operators \[20, 47, 48, 49\] and shares with it the feature of providing diagonal matrix elements of generic operators, not necessarily related to the heat kernel, since it is not based on recurrence relations. However there is an important difference between both techniques: in the standard method of symbols covariance is not manifest prior to momentum integration whereas with covariant symbols covariance (under both gauge and coordinate transformations) is manifest at every step of the calculation. In that section results are provided for the diagonal matrix elements to four derivatives. These results are presented in compact form using the technique of labeled operators, also used in \[14\]. In this notation the mass term carries a label indicating its position in the expression; in this way it becomes effectively a c-number and momentum integrals can be carried out explicitly.

Explicit formulas for the derivative expansion of the trace of the heat kernel operator are presented in Section \[IV\]. This is of interest in the computation of the effective action. It is noted that many, in fact most, of the new terms introduced by the curvature can be eliminated by
a suitable redefinition of the mass term in the Klein-Gordon operator.

In Section V, we obtain the expressions for the diagonal matrix elements and trace of the heat kernel in the form first derived by Chan [54] for the effective action to four derivatives and extended in [51] to six derivatives. This is done following a rather indirect path since Chan’s derivation is not used. Instead a Chan’s form is proposed and the coefficients are adjusted to reproduce the results previously derived using covariant symbols. The question of whether Chan’s elegant method can be used in curved manifolds is left open. In any case, as expected from the experience in the flat case, Chan’s form is truly much more compact. Drawbacks are that the expressions are less explicit because one parametric integral is left undone, and for the same reason there is a by parts integration ambiguity in the formulas.

Section VI is devoted to computing the bosonic effective action in an explicit way, with the help of labeled integration ambiguity in the formulas. Somewhat more explicit details on the calculation with covariant symbols are given in Appendix A. Trace cyclic property and integration by parts identities are derived in Appendix B.

II. DERIVATIVE EXPANSION OF THE HEAT KERNEL

We assume a compact boundaryless Riemannian manifold of dimension $d$ and Euclidean metric $g_{\mu\nu}(x)$ to represent the space-time upon Wick rotation. Further, for the world sector indices we take the Levi-Civita connection (i.e., torsionless and metric preserving). The Klein-Gordon operator will act on wavefunctions $\psi(x)$ (matter fields) defined on the space-time manifold. The wave-functions are vectors with respect to some representation of a certain gauge group. Without loss of generality we will assume that $\psi$ carries a single gauge (or internal) index. $\psi$ is also allowed to carry world indices, that is, we do not assume $\psi$ to be a world scalar. Following for instance [34], we use a single covariant derivative $\nabla_\mu$ which acts on all indices with the appropriate connection, $\nabla = \partial + \Gamma + \omega$. $\Gamma^\lambda{}_{\mu\nu}$ is the connection on world indices and $\omega_\mu(x)$ the connection on gauge indices, a matrix in internal space. Our convention for the Riemann tensor is such that, if $\psi$ is a world vector,

$$[\nabla_\mu, \nabla_\nu]\psi_\lambda = \Omega_{\mu\nu\lambda}\psi_\sigma + R_{\mu\nu\lambda\sigma}\psi_\sigma$$

(2.1)

where

$$\Omega_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + [\omega_\mu, \omega_\nu]$$

(2.2)

is the gauge field strength tensor. In addition, for the Ricci tensor and scalar curvature

$$R_{\mu\nu} = R_{\lambda\mu\nu\lambda}, \quad R = R_{\mu\nu}.$$  

(2.3)

In the previous formulas we use the same notation for contravariant and covariant indices. We will follow this convention throughout unless an ambiguity arises.

The Klein-Gordon operator is of the form

$$K = g^{\mu\nu}\nabla_\mu \nabla_\nu + X.$$  

(2.4)

$X(x)$ is a scalar field with respect to world indices and a matrix with respect the internal space. As is well-known the heat kernel operator $e^{\tau K}$ is ultraviolet finite for Re ($\tau$) $> 0$, and its matrix elements admit an asymptotic expansion in powers of $\tau$. For the diagonal matrix elements

$$\langle x| e^{\tau K} | x \rangle = \frac{1}{(4\pi \tau)^{d/2}} \sum_{n=0}^{\infty} \tau^n a_n(x) .$$

(2.5)

The coefficients $a_n$ are still operators with respect to gauge and world indices since the brackets $\langle x|, | x \rangle$ refer only to $x$-space. To lowest orders, the well-known result is

$$a_0 = 1,$$

$$a_1 = X + \frac{1}{6} R,$$

$$a_2 = \frac{1}{2} X^2 + \frac{1}{6} X_{\mu\nu} + \frac{12}{15} Z_{\mu\nu}^2 + \frac{1}{6} RX + \frac{1}{30} R_{\mu\nu}$$

$$+ \frac{1}{72} R^2 - \frac{1}{180} R_{\mu\nu}^2 + \frac{1}{180} R_{\mu\nu\alpha\beta}^2 .$$

(2.6)

In these formulas (and hereafter) we use the following notational convention: the covariant derivative of an object is represented by adding a world index to it and further covariant derivatives add further indices to the left. So for instance,

$$X_\mu = [\nabla_\mu, X], \quad X_{\mu\nu} = [\nabla_\mu, [\nabla_\nu, X]],$$

$$R_{\alpha\mu\nu} = \nabla_\alpha R_{\mu\nu} .$$

(2.7)

We have also introduced the quantity

$$Z_{\mu\nu} := [\nabla_\mu, \nabla_\nu] .$$

(2.8)

Let us emphasize that $Z_{\mu\nu}$ is different from $\Omega_{\mu\nu}$. The operators $Z_{\mu\nu}$ and $\Omega_{\mu\nu}$ are both multiplicative with respect

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1 In this work we will use the label world interchangeably with coordinate or space-time in expressions like “world tensor”, “world index”, etc, to refer to properties tied to indices $\mu, \nu, \ldots$ associated to natural bases, $\partial/\partial x^\mu$, of the tangent space of the Riemannian manifold.

2 Of course one can choose to transform such world indices into internal or gauge indices using a tetrad field. This is the standard approach [34]. Our formulas hold whether this choice is made or not.
to $x$-space and matrices in internal space, however, unlike $\Omega_{\mu\nu}$, $Z_{\mu\nu}$ acts also on world indices (cf. (2.1)). This implies, for instance, that $Z_{\mu\nu}$ and $R_{\alpha\beta}$ do not commute:

$$[Z_{\mu\nu}, R_{\alpha\beta}] = R_{\mu\nu\sigma\gamma}R_{\sigma\gamma\beta} + R_{\mu\nu\sigma\beta}R_{\sigma\gamma\alpha}.$$  \hspace{1cm} (2.9)

Likewise, $[Z_{\mu\nu}, X_\lambda] = [\Omega_{\mu\nu}, X_\lambda] + R_{\mu\nu\lambda\sigma}X_\sigma$, etc. The derivative of $Z_{\mu\nu}$ will also be needed,\(^3\)

$$Z_{\alpha\mu\nu} := [\nabla_\alpha, Z_{\mu\nu}] - \frac{1}{2}([\nabla_\lambda, R_{\lambda\mu\nu}]).$$  \hspace{1cm} (2.10)

In our formulas we will use $Z_{\mu\nu}$ rather than $\Omega_{\mu\nu}$. Nevertheless, if desired one can always move the $Z$'s to the right using their commutation properties and apply them to the wavefunction $\psi$ to produce $\Omega$'s and $R$'s. (Of course, the result so obtained will depend on the concrete tensor representation of the wavefunction.) For instance, the term $Z_{\mu\nu}/12$ in $a_0$ is equivalent to more standard $\Omega_{\mu\nu}/12$ when the wavefunction happens to be a world scalar. The advantage of using $Z_{\mu\nu}$ is that the expressions take the same form regardless of the world-tensor representation of $\psi$ and yet the original connections $\Gamma$ and $\omega$ are used. (See footnote 2) Fuller details can be found in \[40\].

By dimensional counting it is clear that the standard heat kernel expansion \[25\] is an expansion with coefficients ordered by their mass dimension: if $g_{\mu\nu}$ carries dimension 0, $\nabla_\mu$ carries dimension 1, and $X$ carries dimension 2, the coefficient $a_0$ has dimension $2n$. In what follows we will set $\tau = 1$ since dimensional counting allows to restore the parameter $\tau$ at any time if needed.

In this work we consider a different classification of terms in the heat kernel, namely, operators are classified by the number of covariant derivatives they carry, so $\nabla_\mu$ counts as order 1 while $X$ and $g_{\mu\nu}$ count as order zero. Thus, e.g., $XX_\mu$ is of first order, $R_{\mu\nu\alpha\beta}$ and $Z_{\mu\nu}$ of second order, $Z_{\alpha\mu\nu}$ of third order, and so on. In this expansion

$$\langle x | e^K | x \rangle \approx \frac{1}{(4\pi)^{d/2}} \sum_{n=0}^{\infty} A_n(x),$$  \hspace{1cm} (2.11)

where the coefficient $A_n$ collects all operators with $2n$ derivatives and any number of $X$. In turn, the $A_n$ can be reexpanded using operators classified by their mass dimension,

$$A_0 = 1 + X + \frac{1}{2} X^2 + \cdots,$$

$$A_1 = \frac{1}{6} R + \frac{1}{6} X_{\mu\nu} + \frac{1}{6} R X + \cdots,$$

$$A_2 = \frac{1}{12} Z^2 + \frac{1}{30} R_{\mu\nu} + \frac{1}{12} R^2 - \frac{1}{180} R_{\mu\nu}^2 + \frac{1}{180} R_{\mu\nu}^3 + \cdots.$$  \hspace{1cm} (2.12)

(Of course, when $\tau$ is not set to unity these coefficients are also functions of $\tau$.)

In \[14\] we presented complete formulas (i.e., valid to all orders in $X$) for $A_0$, $A_1$ and $A_2$, as well as $B_0$, $B_1$, $B_2$ and $B_3$ (defined below), for the flat case. Presently we extend those results (except $B_3$) to curved manifolds.

The key tool to write the result to all orders in $X$ in closed form is the use of labeled operators. For instance, all terms in $A_1$ of the type $X^n X_{\mu\nu} X^m$, with $n, m = 0, 1, \ldots$, can be collected as

$$I_{2,2} X_{\mu\nu} = \left( \frac{e^{X_1} + e^{X_2}}{(X_1 - X_2)^2} - 2 \frac{e^{X_1} - e^{X_2}}{(X_1 - X_2)^3} \right) X_{\mu\nu}.$$  \hspace{1cm} (2.13)

The label 1 in $X_1$ indicates that the corresponding $X$ should be placed just before (i.e. to the left of) the fixed operator $X_{\mu\nu}$, likewise, $X_2$ indicates that that $X$ is to be put just after (i.e. to the right of) the fixed operator. Upon a series expansion in powers of $X_1$ and $X_2$

$$I_{2,2} X_{\mu\nu} = \left( \frac{1}{6} + \frac{1}{12} X_1 + \frac{1}{12} X_2 ight.$$  

$$\left. + \frac{1}{40} X_2^2 + \frac{1}{40} X_1 X_2 + \cdots \right) X_{\mu\nu}$$

$$= \frac{1}{6} X_{\mu\nu} + \frac{1}{12} X X_{\mu\nu} + \frac{1}{12} X_{\mu\nu} X$$

$$+ \frac{1}{40} X^2 X_{\mu\nu} + \frac{1}{40} X_{\mu\nu} X^2 + \frac{1}{30} X X_{\mu\nu} X$$

$$+ \cdots.$$  \hspace{1cm} (2.14)

Likewise, $X_2^2 X_2 X_3^2$ would stand for $X^2 X_{\mu\nu} X^2 X_{\mu\nu}$, etc. The labels always refer to the position of $X$'s with respect to “fixed operators” such as $X_{\alpha\mu\nu}$, $Z_{\mu\nu}$, etc. Because $X$ commutes with $R_{\mu\nu\alpha\beta}$ we will not need to include the Riemann tensor among the set of fixed operators. So for instance, $I_{2,2} R X_{\mu\nu}$ will be used to mean $R$ multiplied by $I_{2,2} X_{\mu\nu}$.

The important point is that the labeled operators can be treated as $\epsilon$-numbers, e.g., $X_1 X_2 = X_2 X_1$; the true position of the labeled operator is given by its label. This is similar to what happens within normal or chronological orders. Labeled operators were introduced in \[53, 54\].

The function $I_{2,2}$ belongs to the family of functions $I_{r_1, \ldots, r_n}$, with arguments $X_1, \ldots, X_n$. They are defined as

$$I_{r_1, r_2, \ldots, r_n} := \int_\Gamma \frac{dz}{2\pi i} e^{z N_1^r + N_2^r + \cdots + N_n^r},$$  \hspace{1cm} (2.15)
there is an ambiguity in the definition of $N_i = (z - X_i)^{-1}$, and $\Gamma$ is a positively oriented closed simple path on the complex plane enclosing the eigenvalues of $X$ (at the given point $x$). These functions enjoy a number of properties described in [44]. In particular, they are entire functions of the $X_i$ and can be computed by recurrence relations starting from $I_1 = e^{X_1}$. They are not linearly independent; integration by parts implies the relation

$$I_{r_1, r_2, \ldots, r_n} = \sum_{j=1}^{n} r_j I_{r_1, r_2, \ldots, r_j+1, \ldots, r_n}.$$  

(2.17)

The trace of the heat kernel operator,

$$\text{Tr}(e^K) = \int d^d x \sqrt{g} \text{tr}(e^K),$$  

(2.18)

is also of great interest in applications, such as the computation of the effective action (see Section [VI]). The trace $\text{tr}$ refers to gauge and world indices. The trace of the heat kernel admits an asymptotic expansion with terms classified by their mass dimension:

$$\text{Tr}(e^K) \sim \frac{1}{(4\pi)^{d/2}} \sum_{n=0}^{\infty} \int d^d x \sqrt{g} \text{tr}(b_n(x)),$$  

with

$$b_0 = 1,$$

$$b_1 = X + \frac{1}{6} R,$$

$$b_2 = \frac{1}{2} \partial^2 + \frac{1}{12} Z_{\mu \nu} + \frac{1}{6} RX + \frac{1}{12} R^2 - \frac{1}{180} R_{\mu \nu} R_{\alpha \beta}.$$

(2.19)

Likewise, the terms can be classified by their number of derivatives

$$\text{Tr}(e^K) \sim \frac{1}{(4\pi)^{d/2}} \sum_{n=0}^{\infty} \int d^d x \sqrt{g} \text{tr}(B_n(x)).$$  

(2.20)

Due to the trace cyclic property and integration by parts, there is an ambiguity in the definition of $B_n(x)$. This allows to bring $B_n(x)$ to a simpler form starting from $A_n(x)$. In turn, functional derivation allows one to obtain $A_n$ from $B_n$

$$A_n(x) = \frac{\delta}{\delta X(x)} \int d^d x \sqrt{g} \text{tr}(B_n(x)).$$  

(2.21)

(2.22)

Before closing this section, let us comment on the nature of the derivative expansion in the present context. [We reiterate $\tau$ for this discussion.] As is well-known, the standard expansion in powers of $\tau$ is an asymptotic one, reliable for small $\tau$ only. Intuitively, this is because the coefficients $a_n(x)$ are local, i.e., they depend on a finite number of derivatives of the external fields (namely, the metric, the gauge connection and the scalar field $X$). Consequently, the expansion at a given point $x_0$ is not sensitive to modifications of these external fields taking place outside a fixed neighborhood of $x_0$. Such modifications affect the exact matrix element at $x_0$ (and hence its $\tau$ dependence) but this is not seen by the asymptotic expansion. By the same token, the derivative expansion is expected to be asymptotic too, since the coefficients $A_n(x; \tau)$ are also local. To make mathematical sense of the asymptotic series it is necessary to write it as a power series expansion. The derivative expansion can be viewed as a power series as follows: for a given point $x_0$ consider a deformation of the external fields such that within a fixed neighborhood of $x_0$ the deformation is just a dilatation by a parameter $\lambda$

$$X(x) \mapsto X(x_{\lambda}),$$

$$g_{\mu \nu}(x) \mapsto g_{\mu \nu}(x_{\lambda}),$$

$$\omega_{\mu}(x) \mapsto \lambda \omega_{\mu}(x_{\lambda}),$$

$$x_{\lambda}^\mu = x_0^\mu + \lambda(x^\mu - x_0^\mu).$$

(2.23)

The deformation is smoothly continued outside of the neighborhood. This produces a family of Klein-Gordon operators $K_{\lambda}$. By construction the parameter $\lambda$ counts the number of derivatives, that is,

$$A_n(x_0; \tau, \lambda) = \lambda^{2n} A_n(x_0; \tau).$$  

(2.24)

Therefore the derivative expansion can be read off from an expansion of $\langle x_0 | e^{\tau K_{\lambda}} | x_0 \rangle$. This construction suggests that, although the series in $\lambda$ is asymptotic, the coefficients $A_n(x; \tau)$ themselves are well defined quantities, as is also the case for the coefficients $a_n(x)$. This point is to be settled by a rigorous mathematical approach. Note that this definition of the coefficients (from an expansion in $\lambda$) is not identical to the alternative definition

$$A_n(x; \tau) = \sum_{m \geq n} a_{m,n}(x) \tau^{m-n}$$  

(2.25)

(2.26)

(using $a_{m,n}(x)$ to denote the terms of $a_m(x)$ with $2n$ derivatives). We expect both definitions to coincide. This expectation relies on (i) the fact that the functions $I_{r_1, r_2, \ldots, r_n}$ are entire functions of $X_i$ and so of $\tau$, thus their expansion in $\tau$ is absolutely convergent, and (ii) the obvious reason for the series in $\tau$ to be asymptotic does not apply here since $A_n(x; \tau)$ is itself local. In summary, we expect the derivative expansion to be valid for finite (not small) $\tau$, provided $\lambda$ is sufficiently small.

4 The dilatation depends not only on $x_0$ but also on the coordinate system used. Eq. (2.24) is coordinate independent.
III. DIAGONAL COEFFICIENTS

In \cite{44} the calculation of \( A_n \) in flat space-time was based on that of \( B_n \). In turn \( B_n \) was adapted from the result of Chan \cite{50} for the effective action to four derivatives and its extension in \cite{51} to six derivatives. Unfortunately, it is not obvious how the elegant approach of \cite{50} is to be extended to the curved case. As we may recall, in that approach a symbol method (\( D_\mu \to D_\mu + p_\mu \)) is applied, to \( \text{Tr}(\log(K)) \). The expression is then formally expanded in the number of derivatives and brought to a canonical form. The covariant symbols were introduced \( \text{on} \) matrix elements of general operators, no necessarily gauge invariant expressions may have the same canonical form. The trouble is that a similar statement does not hold in the curved case. For instance, \[
\text{Tr}(Z_{\mu\nu}[p_\mu, p_\nu N^2]) = 0 \tag{3.1}
\]
(with \( N = (p^2 - X)^{-1} \)), since \( p_\mu \) and \( X \) commute. However, under formal cyclic property the same expression would be equivalent to
\[
\text{Tr}(p_\nu N^2[Z_{\mu\nu}, p_\mu]) = \text{Tr}(p_\nu N^2 R_{\mu\nu\sigma\rho} p_\sigma) = \text{Tr}(p_\mu p_\nu R_{\mu\nu} N^2). \tag{3.2}
\]
This is equivalent to \( -\text{Tr}(p^2 R N^2)/d \) and does not vanish.

Since Chan’s approach is not available, we will compute \( A_n \) from scratch and then obtain \( B_n \) from it. The starting point is the representation
\[
e^K = \int \frac{dz}{2\pi i} \frac{1}{z} \int \frac{d^3p}{(2\pi)^3} e^z \tag{3.3}
\]
where we can apply the method of covariant symbols \cite{45,46}. This is the second key ingredient of our approach. This gives
\[
\langle x|e^K|x \rangle = \int \frac{dz}{2\pi i} \frac{1}{z} \int \frac{d^3p}{(2\pi)^3} \frac{e^z}{z - \nabla^2 - X}. \tag{3.4}
\]
The covariant symbols are designed for computing diagonal matrix elements of general operators, no necessarily the heat kernel. The covariant symbols were introduced by Pletnev and Banin in \cite{45} for the gauge connection and extended to curved space-time in \cite{46} where they have been computed to four derivatives. Explicitly, to two derivatives
\[
\nabla = X - X_\alpha \partial^\alpha + \frac{1}{2!} X_{\alpha\beta} \partial^\alpha \partial^\beta + \cdots, \tag{3.5}
\]
where \( \partial^\alpha = \partial / \partial p^\alpha \) and the dots refer to terms with three derivatives or more. Note that, unlike ordinary symbols, these operators are multiplicative and covariant already before momentum integration. The calculation is straightforward along the lines of the examples presented in \cite{46}. Details are provided in Appendix A. One obtains
\[
A_0 = I_1 = e^X \tag{3.6}
\]
to zeroth order and
\[
A_1 = (I_{1,2} - 2I_{1,3})X_{\mu\mu} + (2I_{1,1,2} - 4I_{1,1,3} - 2I_{1,2,2})X_{\mu}^2 \tag{3.7}
\]
\[
+ \frac{1}{3} I_3 R.
\]

Let us emphasize that the functions \( f(X_1, X_2) \) multiplying \( X_{\mu\mu} \) and \( f(X_1, X_2, X_3) \) multiplying \( X_{\mu}^2 \), are well defined and unambiguous. The only ambiguity enters in how they are written in terms of the overcomplete basis \( I_{r_1, \ldots, r_n} \).

The two terms without \( R \) are identical to those of the flat case in \cite{44}, only with the “minimal coupling” replacement \( D = \partial + \omega \to \nabla = \partial + \Gamma + \omega \). These are then minimal terms required by covariance (under general coordinate transformations). The term with \( R \) is non minimal; covariant but not required by covariance. In general we will obtain coefficients of the form
\[
A_n = A^m_n + A^R_n. \tag{3.9}
\]

The terms in \( A^R_n \) are those which contain explicitly the Riemann tensor and so vanish in the flat space case. On the other hand, the minimal terms \( A^m_n \) can be reconstructed by minimal coupling from the flat space expressions. Let us warn, however, that this separation is not an unambiguous one: in general \( A^m_n \) will depend of the concrete \( A_n \) of flat space used to apply the minimal coupling. This is because reordering of world indices in the flat space expression may introduce Riemann tensors in the curved case.\(^5\)

The calculation of \( A_2 \) gives
\[ A^{\alpha\beta\mu\nu} = 2I_{2,1,2} Z_{\mu\nu} Z_{\alpha\nu} + (2I_{2,2,2} - 4I_{3,0,3} + 4I_{2,1,3}) Z_{\mu\alpha\nu} X_{\nu} + (2I_{2,2,2} - 4I_{3,0,3} + 4I_{3,1,2}) X_{\mu} Z_{\alpha\nu} + (4I_{3,1,2} - 16I_{3,0,1,3} - 8I_{7,0,2,2} + 8I_{3,1,1,2}) X_{\mu} Z_{\alpha\nu} X_{\nu} + (16I_{3,0,1,3} - 8I_{2,1,1,3} - 4I_{2,1,2,2} + 8I_{3,0,2,2}) Z_{\mu\nu} X_{\alpha} X_{\nu} + (16I_{3,1,0,3} - 8I_{3,1,1,2} - 4I_{2,2,1,2} + 8I_{2,2,0,3}) X_{\mu} X_{\alpha} Z_{\nu} + 2I_{3,3} X_{\mu\nu\alpha} X_{\nu} + (2I_{3,3} - 4I_{3,1,3} + 2I_{2,2,3}) X_{\mu} X_{\nu} + 8I_{3,1,3} X_{\mu\nu} + (8I_{3,1,3} + 4I_{3,2,2}) X_{\mu} X_{\nu} + (8I_{3,1,3} + 4I_{2,2,3}) X_{\mu} X_{\nu} + (4I_{2,2,2} + 16I_{3,1,1,3} + 8I_{2,2,1,3} + 8I_{3,1,2,2}) X_{\mu} X_{\nu} + (4I_{2,2,2} + 8I_{3,1,1,3} + 4I_{2,2,1,3} + 4I_{3,1,1,2} + 4I_{3,2,1,2}) X_{\mu} X_{\nu} + (2I_{2,2,2} + 8I_{3,1,1,3} + 4I_{2,2,1,3} + 4I_{2,2,1,2} + 8I_{3,1,1,2}) X_{\mu} X_{\nu} + (16I_{3,1,1,3} + 8I_{3,1,2,2}) X_{\mu} X_{\nu} + (16I_{3,1,1,3} + 8I_{3,2,1,3}) X_{\mu} X_{\nu} + (4I_{2,2,1,2} + 16I_{3,1,1,3} + 8I_{2,2,1,3} + 8I_{3,1,2,1,2} + 4I_{2,2,2,1,2} + 4I_{2,2,1,2,2} + 8I_{2,1,1,3} + 8I_{3,1,1,2,2}) X_{\mu} X_{\nu} X_{\nu} X_{\nu} + (4I_{2,2,1,2,2} + 16I_{3,1,1,1,3} + 8I_{2,2,1,1,3} + 8I_{3,1,1,2,2}) X_{\mu} X_{\nu} X_{\nu} X_{\nu} + (4I_{2,2,1,2,2} + 16I_{3,1,1,1,3} + 8I_{2,2,1,1,3} + 8I_{3,1,1,2,2}) X_{\mu} X_{\nu} X_{\nu} X_{\nu} \] (3.10)

\[ A^{\alpha\beta\mu\nu} = 2I_{2,1,2} Z_{\mu\nu} Z_{\alpha\nu} + \frac{1}{30} I_{1} R_{\mu\nu} + \frac{1}{72} I_{1} R^2 + \frac{1}{6} I_{2,2} R X_{\mu\nu} + \frac{1}{4} I_{2,2} R X_{\mu\nu} + \frac{4}{3} I_{3,3} R_{\mu\nu} X_{\mu\nu} + \frac{1}{3} I_{2,1,2} R X^2_{\mu\nu} + \left( \frac{2}{3} I_{2,2,2} + \frac{4}{3} I_{2,3,2} - \frac{16}{3} I_{3,1,3} \right) R_{\mu\nu} X_{\mu} X_{\nu} - \frac{1}{180} I_{1} R^2_{\mu\nu} + \frac{1}{180} I_{1} R^2_{\mu\nu\alpha\beta} \] (3.11)

\( A^{\alpha\beta\mu\nu} \) is formally identical to the expression in (4.10) of [44]. (There mirror symmetry of \( A_{\alpha} \) was exploited to write \( A_{2} \) in a shorter, less explicit form.) As said before, in the formula for \( A_{2}^{\alpha\beta\mu\nu} \) the Riemann tensor is not to be considered as one of the fixed operators.

The coefficients \( A_{1} \) and \( A_{2} \), for curved space and non-abelian gauge group, with contributions to all orders in \( X \) are computed here for the first time. Upon expansion in powers of \( X \) they reproduce the corresponding terms with up to four derivatives of the standard expansion \( a_{n} \). In particular the coefficients quoted in Eqs. (4.26-29) of [34] are correctly reproduced. (The comparison with [34] is achieved by restricting our results to the case of world scalar wavefunction and using the trace cyclic property, but not integration by parts.) Results to all orders in \( X \) and up to four derivatives for curved space are also available from [55] for the so-called minimal case, i.e., \( X(x) \) a c-number and \( \omega_{\mu} = 0 \), and \( \psi \) a world scalar. Our formulas also check Eq. (22) of [55].

**IV. COEFFICIENTS FOR THE TRACE**

Application of integration by parts and the trace cyclic property allows to obtain the simpler coefficients \( B_{n} \).
The trace of the heat kernel can then be expanded as
terms can be generated by a modified “minimal coupling” contribution is added by the extra term. This equation four derivatives they take the form

\[ B_0 = I_1, \]
\[ B_1 = -\frac{1}{2} I_{2,2} X_\mu X_\nu + \frac{1}{6} I_1 R, \]
\[ B_2 = (-I_{2,2,2,2} + 4 I_{3,1,3,1} X_\mu X_\nu X_\rho X_\sigma + 4 I_{3,1,3} X_\mu X_\nu X_\rho + 4 I_{3,1,3} X_\mu X_\nu X_\rho X_\sigma + 4 I_{3,1,3} X_\mu X_\nu X_\rho X_\sigma X_\tau) + \frac{1}{2} I_{2,2} Z_{\mu \nu} Z_{\mu \nu}. \]

The last term would not be present under the standard trace cyclic property of matrices. The rationale of the extra term is that in \( X_\mu Z_{\mu \nu} X_\nu \), \( Z_{\mu \nu} \) acts on the index \( \nu \) of \( X_\nu \) (as well as on the subsequent world indices in the wavefunction) while in \( X_\mu X_\nu Z_{\mu \nu} \) it does not; the missing contribution is added by the extra term. This equation is derived in Appendix 13.

It can be verified that \( A_n \) and \( B_n \) are indeed equivalent inside \( \int d^d x \sqrt{g} \text{tr}(\ ) \) and that (2.22) is fulfilled.

To this order one can see that most of the non minimal terms can be generated by a modified “minimal coupling” prescription. Namely, \( \partial + \omega \rightarrow \partial + \omega + \Gamma \) and \( X \rightarrow X' \), with the new scalar field

\[ X' = X + \frac{1}{6} R + \frac{1}{180} (R_{\mu \nu} - R^2_{\mu \nu}) + \mathcal{O}(\nabla^6). \]

The trace of the heat kernel can then be expanded as

\[ \text{Tr}(e^K) \sim \frac{1}{(4\pi)^{d/2}} \sum_{n=0}^{\infty} \int d^d x \sqrt{g} \text{tr}(B_n'(x)), \]

with

\[ B'_0 = I_1', \]
\[ B'_1 = -\frac{1}{2} I_{2,2} X'_\mu X'_\nu, \]
\[ B'_2 = (-I'_{2,2,2,2} + 4 I'_{3,1,3,1} X'_\mu X'_\nu X'_\rho X'_\sigma + 4 I'_{3,1,3} X'_\mu X'_\nu X'_\rho + 4 I'_{3,1,3} X'_\mu X'_\nu X'_\rho X'_\tau) + \frac{1}{2} I'_{2,2} Z'_{\mu \nu} Z'_{\mu \nu}. \]

(\( I'_{1,\ldots,r} \) being defined as in (2.15) but using \( X' \) instead of \( X \).) There are corresponding coefficients \( \lambda'_n \). The relations \( \int d^d x \sqrt{g} \text{tr}(A_n) = \int d^d x \sqrt{g} \text{tr}(B_n) \) and (2.22) hold also for the primed coefficients. The redefinition \( \lambda' = \lambda + \frac{1}{6} R \) is quite standard in the literature [15] to eliminate some of the terms. Note that beyond the primed expansion is no longer a strict expansion in the number of covariant derivatives (and \( \lambda' \) will depend on \( \tau \) when \( \tau \) is restored).

V. CHAN’S FORM OF THE COEFFICIENTS

Let us call the coefficients \( A_n \) and \( B_n \) just derived the coefficients in \( X \)-form, to distinguish them from their Chan’s or \( N \)-form, to be discussed in this section.

Let us briefly summarize Chan’s method in flat space [50, 51]. In this method the results are obtained in terms of derivatives of \( N = (z - X)^{-1} \), instead of derivatives of \( X \), that is, \( N_\mu = [\nabla_\mu N] \), \( N_{\mu \nu} \), etc. Because \( z \) appears inside \( N_\mu_1 \ldots \mu_n \), the integral over \( z \) cannot be carried out explicitly and is left undone. Integration by parts (with respect to \( z \)) allows to reorder terms so that in each term of \( B_n \) the quantity \( N \) (derivated or not) appears exactly \( 2n \) times. A virtue of this approach is that for each \( B_n \) there is only a limited number of available covariant structures constructed with \( 2n \) \( N \)’s and \( 2n \) \( \nabla \)’s, thus the expressions so obtained are quite compact. To pass a result given in \( N \)-form to \( X \)-form is, of course, straightforward using the relation \( N_\mu = N X_\mu N \) and its derivatives. As pointed out before, the result in \( X \)-form is free from ambiguities.

Because the extension of Chan’s method to curved space is not known, the existence of a Chan’s form for the heat kernel coefficients in the curved case is not obvious. In principle, undoing the \( z \) integrals in the \( I_{r_1, \ldots, r_n} \) and using identities of the type

\[ X_\mu = N^{-1} N_\mu N^{-1}, \]
\[ X_{\mu \nu} = N^{-1} N_{\mu \nu} N^{-1} - N^{-1} N_{\mu \nu} N^{-1} N_\mu N^{-1} - N^{-1} N_{\mu \nu} N^{-1} N_\mu N^{-1} \]

would allow to bring the coefficients computed in \( X \)-form to an almost Chan’s form, except that, in general, negative powers of \( N \) will be present. Moreover the precise

\[ \text{Tr}(e^K) \sim \frac{1}{(4\pi)^{d/2}} \sum_{n=0}^{\infty} \int d^d x \sqrt{g} \text{tr}(A_n'(x)), \]

(4.1)

(4.2)

(4.3)

(4.4)

(5.1)
result will be subject to ambiguities due to integration by parts on \( z \) and not algorithm is available to bring it to a compact form. This method works for \( B_0 \) and \( B_1 \).

The procedure that we have followed to obtain an \( N \)-form for the coefficient \( B_2 \) is as follows. In the flat case the \( N \)-form is known for \( B_2 \) and hence for \( A_2 \) (by functional variation with respect to \( X \)), therefore we apply minimal coupling there. This already reproduces most of the terms of the known full \( A_2 \) in \( X \)-form, \(3.10-3.11\). For the few remaining terms of \( A_2 \) it is relatively easy to bring them to a rather compact \( N \)-form by hand. This remainder is the functional variation of the non minimal remainder in \( B_2 \). \( B_2^R \), not yet determined. To obtain \( B_2^R \) we simply write down the most general terms having four derivatives, at least one Riemann tensor and no more than four \( N \)'s and with arbitrary numerical coefficients. These coefficients are then chosen to reproduce the non minimal remainder of \( A_2 \). This procedure gives \(7\)

\[
B_0 = \langle N \rangle_z, \\
B_1 = \langle -\frac{1}{2} N^2 + \frac{1}{6} R N \rangle_z, \\
B_2 = \langle -N^2 N^2 + \frac{1}{2} (N N^2)^2 + (NN_{\mu\nu})^2 + (NN_{\mu\nu\gamma})^2 \\
+ 2NN_{\mu\nu}N_{\gamma\mu\nu} + \frac{1}{2} (NZ_{\mu\nu\gamma} N)^2 \\
- \frac{1}{2} R N^2 - \frac{3}{2} \mathcal{R}_{\mu\nu} N N_{\mu\nu} N_{\gamma\mu\nu} + \frac{1}{30} \mathcal{R}_{\mu\nu} N \\
+ \frac{1}{72} R^2 N - \frac{1}{180} \mathcal{R}^2_{\mu\nu} N + \frac{1}{180} \mathcal{R}^2_{\mu\nu\alpha\beta} N \rangle_z,
\]

where we use the shorthand notation

\[
\langle \cdot \rangle_z := \int_\Gamma \frac{dz}{2\pi i} e^{z \cdot (\cdot)}.
\]

\( B_2 \) in \( X \)-form, \(4.1\), has been obtained from this \( N \)-form. Eqs. \(5.2\) are the extension of Chan’s formulas to curved space-time. Once again, to this order, the only non minimal term surviving is \(-\frac{1}{2} \mathcal{R}_{\mu\nu} N N_{\mu\nu} N_{\gamma\mu\nu} \) if \( X' \) is used throughout.

Using \(2.22\), \( A_n \) in \( N \)-form is easily obtained from \( B_n \).

\[\text{This gives}^8\]

\[
A_0 = \langle N \rangle_z, \\
A_1 = \langle NN_{\mu\nu} N + \frac{1}{6} R N \rangle_z, \\
A_2 = \langle 2N^2 NN_{\mu\nu} N^2 \\
+ 4NN_{\mu\nu} N_{\gamma\mu\nu} N + 2NN_{\mu\nu\gamma\mu\nu} N \\
+ 2N2N_{\mu\nu} N_{\gamma\mu\nu} N + 2NN_{\mu\nu\gamma\mu\nu} N^2 \\
+ 4NN_{\mu\nu\gamma\mu\nu} N + 4NN_{\mu\nu} Z_{\mu\nu\gamma\mu\nu} N^2 \\
+ 4NN_{\mu\nu} Z_{\mu\nu\gamma\mu\nu} N \\
+ 2N^2 Z_{\mu\nu\gamma\mu\nu} N_{\mu\nu} N + 2NN_{\mu\nu} Z_{\mu\nu\gamma\mu\nu} N^2 \\
+ 2N^2 Z_{\mu\nu\gamma\mu\nu} N_{\mu\nu} N^2 \rangle_z.
\]

As noted, the existence of a Chan’s form for the coefficients was not completely obvious a priori in the curved case. The fact that this Chan’s form exists suggests that perhaps Chan’s method could find a suitable extension in the case of curved space.

VI. THE EFFECTIVE ACTION

After functional integration, the effective action of a complex bosonic field is given by \(-\text{Tr}(\log K)\). This can be related to the heat kernel by

\[
-\text{Tr}(\log K) = \int_0^\infty \frac{d\tau}{\tau} \text{Tr}(e^{\tau K}).
\]

Upon restoring \( \tau \) in the expressions, a contribution \( I_{r_1,\ldots, r_n} \) is in \( B_n \) picks up a factor \( \tau^{d/2-d/2} \), where \( 2\gamma \) is the mass dimension of the operator \( \mathcal{O} \) and \( \rho = \sum_{i=1}^n r_i \), (e.g., \( \tau^{d/2-d/2} \) for \( I_{2,2} RX_{\mu} X_{\mu} \) in \( B_2 \)). After carrying out the integral over \( \tau \), the integral over \( z \) in \( I_{r_1,\ldots, r_n} \) can be traded by a momentum integral. This gives the replacement rule for going from the heat kernel to the effective action

\[
\frac{1}{(4\pi)^{d/2}} I_{r_1,\ldots, r_n} \rightarrow I_{r_1,\ldots, r_n}^{\rho^{-\gamma}} \mathcal{O}
\]

8 Let us note that the minimal parts of \( A_2 \) in \( X \)-form and in \( N \)-form are different:

\[
A_{2,N}^0 = A_{2,X}^0 - 8I_{3,1,3} \mathcal{R}_{\mu\nu} X_{\mu} X_{\nu}.
\]
where we have defined
\[ I^k_{r_1, \ldots, r_n} := \frac{\Gamma(d/2)}{\Gamma(k + d/2)} \int d^d q \left( \frac{q^2}{2\pi} \right)^k N^{r_1} \cdots N^{r_n}, \]  
(6.3)

with \( N = (q^2 - X)^{-1} \). (Note that \( I^k_{r_1, \ldots, r_n} \) depends also on \( d \).) The contributions to the effective action may be ultraviolet and infrared divergent. Dimensional regularization applies here. These integrals have been computed in [32] using minimal subtraction.

The replacement rule gives for the effective action expanded in derivatives

\[ -\text{tr} \log K = \int d^d x \sqrt{g} \sum_{n=0}^{\infty} \text{tr} W_n \]  
(6.4)

with

\[
\begin{align*}
W_0 &= I^1_1, \\
W_1 &= -\frac{1}{2} I^2_{2,2} X_{\mu} X_{\mu} + \frac{1}{6} I^3_1 R, \\
W_2 &= -(I^2_{2,2,2,2} + 4 I^2_{3,1,3,1}) X_{\mu} X_{\nu} X_{\mu} X_{\nu} \\
&\quad + \frac{1}{2} I^2_{2,2,2,2} X_{\mu} X_{\nu} X_{\mu} X_{\nu} + 4 I^2_{3,1,3,1} X_{\mu} X_{\mu} X_{\nu} X_{\nu} \\
&\quad + I^3_{4,3} X_{\mu\nu} X_{\mu} X_{\nu} + 2 I^3_{2,2,2} X_{\mu} X_{\nu} Z_{\mu\nu} + \frac{1}{2} I^2_{2,2} Z_{\mu\nu} Z_{\mu\nu} \\
&\quad + \frac{1}{12} I^2_{2,2} R X_{\mu} X_{\mu} - \frac{2}{3} I^2_{3,3} R_{\mu\nu} X_{\mu} X_{\nu} \\
&\quad + \frac{1}{30} I^3_2 R_{\mu\nu} + \frac{1}{72} I^3_2 R^2 \\
&\quad - \frac{1}{180} I^2_2 R^2_{\mu\nu} + \frac{1}{180} I^2_2 R_{\mu\nu\alpha\beta}.
\end{align*}
\]

VII. SUMMARY

We have derived, for the first time, expressions valid to all orders in \( X \) and to four covariant derivatives for the diagonal matrix elements and also for the trace of the heat kernel operator in a Riemannian curved manifold. The expressions presented check previously available results for the so-called minimal case [35] and, when reexpanded in powers of \( X \), they reproduce the known HMDS coefficients to four derivatives. We also extend Chan’s formula, originally derived for the effective action, to include curvature. As in the flat case, the expressions in Chan’s form are remarkably simple also in the curved case. This simplicity suggests a direct calculation of the energy-momentum tensor taking a variation of the effective action with respect to the metric. Such a calculation has not been addressed here. The method of covariant symbols allows to consider more general coordinate connections, including torsion. This would be of interest in the derivation of the Lorentz group generators since the coordinate connection couples to them. A virtue of our formulas is that they are equally simple for scalar wavefunctions and for tensor ones, without redefining the gauge connection to include the parallel transport of the new tetrad fields indices. This is achieved through a consistent use of the operators \( Z_{\mu_1, \ldots, \mu_n} \) which are defined so that they are multiplicative. Remarkably the formulas obtained are independent of the tensor representation of the wavefunction even for the coefficients \( B_n(x) \). This is because the general formulas for the trace cyclic property and integration by parts can also be written in a tensor representation independent way (see Appendix [32]).

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APPENDIX A: COVARIANT SYMBOLS

Let us give some details on the calculation of \( A_n \) using covariant symbols. Fuller details on the use of covariant symbols with derivative expansions can be found in [56] for flat space-time and [46] for curved space-time. Using the expansions (3.5) in (3.4), reexpanding the result and keeping terms with at most two covariant derivatives, gives

\[
\langle x'|e^K|x \rangle = \int \frac{dz}{2\pi i} \frac{1}{\sqrt{|g|}} \frac{d^d p}{(2\pi)^d} e^z \times \left[ N - N X_{\alpha} \partial^\alpha N + N X_{\alpha} \partial^\alpha N X_{\beta} \partial^\beta N \right. \\
+ N \left( \frac{1}{2!} X_{\alpha\beta} \partial_{\alpha} \partial_{\beta} + \frac{1}{6} R + Z_{\alpha\beta} p_{\alpha} \partial_{\beta} \right. \right. \\
- \frac{1}{3} R_{\alpha\beta} p_{\alpha} \partial_{\beta} + \left. \frac{1}{3} R_{\alpha\beta\gamma} p_{\alpha} p_{\beta} \partial_{\gamma} \partial_{\mu} \right) N \\
\left. \left. + O(\nabla^4) \right] , \tag{A1} \right.
\]

with \( N = (z - p^2 - X)^{-1} \). (Let us warn that we are using a purely imaginary \( p_{\mu} \) to avoid the proliferation of \( i \)'s in the formulas.) The derivatives with respect to \( p_{\mu} \) are then carried out (using \( \partial_{\mu} N = 2p_{\mu} N^2 \)). After that, the shift \( z \rightarrow z + p_{\mu}^2 \) allows to isolate the \( p_{\mu} \) dependence in integrals of the type \( \int d^d \rho e^{\rho^2 p_{\mu_1} \cdots p_{\mu_n}} \) which are easily evaluated. These steps produce

\[
\langle x'|e^K|x \rangle = \int \frac{dz}{(4\pi)^{d/2}} \frac{1}{\sqrt{|g|}} e^z \times \left[ N + 2N X_{\mu\nu} N - 2N X_{\mu\nu} N^3 \right. \\
+ 2N X_{\mu} N^2 - 4N X_{\mu} N X_{\nu} N^3 \\
\left. - 2N X_{\mu} N^2 X_{\nu} N^2 + 2N X_{\mu} N^3 R \\
+ O(\nabla^4) \right] , \tag{A2} \right.

with $N = (z - x)^{-1}$. This expression immediately translates into those in \((\text{3.6})\) and \((\text{3.7})\).

**APPENDIX B: TRACE AND INTEGRATION BY PARTS**

In this work (and in particular in this appendix), all formulas hold for arbitrary tensor representations of the wavefunctions $\psi(x)$ on which the Klein-Gordon operator acts. The space of world tensors of rank $r$ is spanned by $e^{\alpha_1}_a \cdots e^{\alpha_r}_a$, where $e^\alpha_a(x)$ is a local basis of the space-time tangent space.

The trace $tr$ in \((\text{2.18})\) refers to gauge indices and to world indices, $tr = tr_{\text{gauge}} tr_{\text{world}}$. We need to consider only multiplicative operators $\hat{O}$ (that is, $\hat{O}\psi$ at $x$ does not depend on $\psi$ at $x' \neq x$). All operators in $A_n$ and $B_n$ are multiplicative. In this case, the trace on world indices in the representation of tensors of rank $r$ will be

$$tr_{\text{world}}(\hat{O}) = e^{\alpha_1}_a \cdots e^{\alpha_r}_a (\hat{O} e^{\alpha_1}_a \cdots e^{\alpha_r}_a) \quad (\text{B1})$$

where $e^\alpha_a$ is the dual basis, $e^\alpha_b e^\beta_b = \delta_{ab}$. Note that $tr_{\text{world}}(\hat{O})$ may only be non vanishing when $\hat{O}$ is itself a world scalar, that is, it maps tensors of rank $r$ to tensors of rank $r$. As illustration, in the space of tensors of rank $r$, an easy calculation yields

$$tr \left( \frac{1}{12} Z_{\mu \nu}^2 \right) = d^r \frac{1}{12} Z_{\mu \nu}^2$$

$$- r d^{r-1} n_g \frac{1}{12} R^2_{\mu \nu \alpha \beta}, \quad (\text{B2})$$

$n_g$ being the dimension of the gauge representation.

The use of the trace cyclic property and integration by parts requires some care. These properties work as usual when the operators involved are (i) multiplicative and (ii) *they do not act on world indices*. (For a scalar operator $\hat{O}$, this means that the component $\mu$ of $\hat{O}$ applied to $e^\nu_a(x)$ does not depend on $e^\nu_{\nu}(x)$, for $\nu \neq \mu$.) Instances of such operators are $X$, $X_{\mu}$ and $\Omega_{\mu \nu}$. In the notation of \((\text{4.4})\), these are the operators in the class $\mathcal{C}(\nabla, \bar{Z})$. On the other hand, modifications occur in the trace cyclic property and integration by parts when multiplicative operators acting on world indices are involved (class $\mathcal{C}(\bar{Z})$). An instance of this is $Z_{\mu \nu}$, since $(Z_{\mu \nu} e^\nu_a)^{\lambda} = R_{\mu \nu \lambda \sigma} e^\sigma_a + \Omega_{\mu \nu} e^\lambda_a$. (However, $[\nabla_\mu, A]$ and $[Z_{\mu \nu}, A] \in \mathcal{C}(\nabla, \bar{Z})$ provided $A \in \mathcal{C}(\nabla, \bar{Z})$.)

To write down the correct relations, let us define $Z^R_{\mu_1 \cdots \mu_n}$ as the curvature parts of $Z_{\mu_1 \cdots \mu_n}$, that is, obtained by dropping $\omega_\mu$ in the covariant derivative. In particular,

$$Z_{\mu \nu} = Z^R_{\mu \nu} + \Omega_{\mu \nu}. \quad (\text{B3})$$

Furthermore, let us introduce the shorthand notation

$$\langle \hat{O} \rangle := \int d^d x \sqrt{g} tr(\hat{O}) \quad (\text{B4})$$

The two following useful properties are easily established

$$\langle AZ^R_{\mu \nu} \rangle = 0, \quad A \in \mathcal{C}(\nabla, \bar{Z})$$

$$\langle AZ^R_{\alpha \mu \nu} \rangle = \langle - \frac{1}{2} R_{\sigma \alpha \mu \nu} A \rangle, \quad A \in \mathcal{C}(\nabla, \bar{Z}) \quad (\text{B5})$$

which hold for arbitrary multiplicative operators $A$ not acting on world indices.

Using these properties, one can prove the following relations for arbitrary operators $A, B, \ldots, \in \mathcal{C}(\nabla, \bar{Z})$:

$$\langle [\nabla_\mu, A] B \rangle = \langle - A [\nabla_\mu, B] \rangle, \quad (\text{B6})$$

$$\langle [\nabla_\alpha, A] B Z_{\mu \nu} C \rangle = \langle - A (B_\alpha Z_{\nu} C + B Z_{\mu \nu} C + B Z_{\mu \nu} C + R_{\sigma \alpha \mu \nu} BC + \frac{1}{2} R_{\sigma \alpha \mu \nu} BC) \rangle, \quad (\text{B7})$$

$$\langle A B Z_{\mu \nu} C \rangle = \langle B Z_{\mu \nu} C A - B C [Z^R_{\mu \nu}, A] \rangle, \quad (\text{B8})$$

$$\langle A B Z_{\alpha \mu} C \rangle = \langle B Z_{\alpha \mu} C A - B C [Z^R_{\alpha \mu}, A] \rangle, \quad (\text{B9})$$

$$\langle A B Z_{\mu \nu} C Z_{\alpha \beta} D \rangle = \langle B Z_{\mu \nu} C Z_{\alpha \beta} D A - B Z_{\mu \nu} C D [Z^R_{\alpha \beta}, A] - B C Z_{\alpha \beta} D [Z^R_{\mu \nu}, A] + B C D [Z^R_{\alpha \beta}, [Z^R_{\mu \nu}, A]] \rangle, \quad (\text{B10})$$

$$\langle Z_{\mu \nu} A Z_{\alpha \beta} B \rangle = \langle A Z_{\alpha \beta} B Z_{\mu \nu} - R_{\alpha \mu \nu \sigma} A B Z_{\sigma \nu} - R_{\alpha \nu \mu \sigma} A B Z_{\mu \sigma} \rangle. \quad (\text{B11})$$

The first two identities refer to integration by parts while the other allow to apply the trace cyclic property. The relation \((\text{1.2})\) is a consequence of \((\text{B8})\).

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