The second homology of $SL_2$ of $S$-integers

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Abstract. We calculate the structure of the finitely generated groups $H_2(SL_2(\mathbb{Z}[1/m]), \mathbb{Z})$ when $m$ is a multiple of 6. Furthermore, we show how to construct homology classes, represented by cycles in the bar resolution, which generate these groups and have prescribed orders. When $n \geq 2$ and $m$ is the product of the first $n$ primes, we combine our results with those of Jun Morita to show that the projection $St(2, \mathbb{Z}[1/m]) \to SL_2(\mathbb{Z}[1/m])$ is the universal central extension. Our methods have wider applicability: The main result on the structure of the second homology of certain rings is valid for rings of $S$-integers with sufficiently many units. For a wide class of rings $A$, we construct explicit homology classes in $H_2(SL_2(A), \mathbb{Z})$, functorially dependent on a pair of units, which correspond to symbols in $K_2(2, A)$.

1. Introduction

We calculate the structure of the finitely generated groups $H_2(SL_2(\mathbb{Z}[1/m]), \mathbb{Z})$ when $m$ is a multiple of 6 (Theorem 6.12 below). Furthermore, we show how to construct explicit homology classes, in the bar resolution, which generate these groups and have prescribed orders (sections 7 and 8). Our methods have wider applicability, however: The main result on the structure of the second homology of certain rings is valid for rings of $S$-integers with sufficiently many units. For a wide class of rings $A$, we construct explicit homology classes in $H_2(SL_2(A), \mathbb{Z})$, functorially dependent on a pair of units, which correspond to symbols in $K_2(2, A)$.

For a ring $A$ satisfying some finiteness conditions the homology groups $H_2(SL_n(A), \mathbb{Z})$ are naturally isomorphic to the $K$-theory group $K_2(A)$ when $n$ is sufficiently large. However, $n = 2$ is rarely sufficiently large, even when $A$ is a field.

We review some background results (see Milnor [8] for details). For a commutative ring $A$, the unstable $K_2$-groups of the ring $A$, $K_2(n, A)$, are defined to be the kernel of a surjective homomorphism $St(n, A) \to E_n(A)$ where $St(n, A)$ is the rank $n - 1$ Steinberg group of $A$ and where $E_n(A)$ is the subgroup of $SL_n(A)$ generated by elementary matrices. There are compatible homomorphisms $St(n, A) \to St(n + 1, A)$, $E_n(A) \to E_{n+1}(A)$, and taking direct limits as $n \to \infty$, we obtain a surjective map $St(A) \to E(A)$ whose kernel is $K_2(A) := \lim K_2(n, A)$. In fact, $K_2(A)$ is central in $St(A)$ and the extension
\[
1 \to K_2(A) \to St(A) \to E(A) \to 1
\]
is the universal central extension of $E(A)$ and hence $H_2(E(A), \mathbb{Z}) \cong K_2(A)$. Furthermore, for a commutative ring $A$, $E(A) = SL(A) = \lim SL_n(A)$.

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When $A$ satisfies some reasonable finiteness conditions these statements remain true when $K_2(A), St(A)$ and $E(A)$ are replaced with $K_2(n, A), St(n, A)$ and $E_n(A)$ for all sufficiently large $n$. In particular, when $F$ is a field with at least 10 elements $H_2(SL_2(F), \mathbb{Z}) \cong K_2(2, F)$.

When $F$ is a global field and when $S$ is a nonempty set of primes of $F$ containing the infinite primes, we let $O_S$ denote the corresponding ring of $S$-integers. (For example if $F = \mathbb{Q}$ and $1 < m \in \mathbb{Z}$, we have $\mathbb{Z}[1/m] = O_S$ where $S$ consists of the primes dividing $m$ and the infinite prime.) Now the groups $H_2(SL_2(O_S), \mathbb{Z})$ and $K_2(2, O_S)$ are finitely-generated abelian groups which satisfy

$$\lim_{S} H_2(SL_2(O_S), \mathbb{Z}) = H_2(SL_2(F), \mathbb{Z}) \text{ and } \lim_{S} K_2(2, O_S) = K_2(2, F).$$

It is natural to guess that we might have $H_2(SL_2(O_S), \mathbb{Z}) \cong K_2(2, O_S)$ when $S$ is sufficiently large in some appropriate sense. The example of $O_S = \mathbb{Z}$, when $H_2(SL_2(\mathbb{Z}), \mathbb{Z}) = 0$ while $K_2(2, \mathbb{Z}) = \mathbb{Z}$ shows that some condition on $S$ will be required.

In the current paper, rather than comparing $H_2(SL_2(O_S), \mathbb{Z})$ to $K_2(2, O_S)$ directly, we introduce a convenient proxy for $K_2(2, O_S)$ which we denote $\tilde{K}_2(2, O_S)$ (see section 6 below for definitions). There are natural maps

$$H_2(SL_2(O_S), \mathbb{Z}) \to \tilde{K}_2(2, O_S), \quad K_2(2, O_S) \to \tilde{K}_2(2, O_S)$$

and the structure of the group $\tilde{K}_2(2, O_S)$ is easy to describe (see Lemma 6.3):

$$\tilde{K}_2(2, O_S) \cong K_2(O_S)_+ \oplus \mathbb{Z}^r$$

where $K_2(O_S)_+$ is the subgroup of totally positive elements of $K_2(O_S)$ and $r$ is the number of real embeddings of $F$.

Our main theorem (6.10) states that when $S$ is sufficiently large (see the statement for more details) that the natural map $H_2(SL_2(O_S), \mathbb{Z}) \to \tilde{K}_2(2, O_S)$ is an isomorphism. In the case $F = \mathbb{Q}$, the condition that $S$ be sufficiently large reduces to the requirement that $2, 3 \in S$. In particular, when $6|m$, we obtain isomorphisms

$$H_2(SL_2(\mathbb{Z}[1/m]), \mathbb{Z}) \cong \tilde{K}_2(2, \mathbb{Z}[1/m]) \cong \mathbb{Z} \oplus \bigoplus_{p|m} \mathbb{Z}_p^s.$$ 

Jun Morita ([13]) proved isomorphisms of the form

$$K_2(2, \mathbb{Z}[1/m]) \cong \tilde{K}_2(2, \mathbb{Z}[1/m])$$

for certain integers $m$ (eg. if $m$ is the product of the first $n$ prime numbers). Combining Morita’s results with those above we deduce that

$$H_2(SL_2(\mathbb{Z}[1/m]), \mathbb{Z}) \cong K_2(2, \mathbb{Z}[1/m])$$

for such $m$, and that, consequently, the extension

$$1 \to K_2(2, \mathbb{Z}[1/m]) \to St(2, \mathbb{Z}[1/m]) \to SL_2(\mathbb{Z}[1/m]) \to 1$$

is a universal central extension.

The main tool we use to prove Theorem 6.10 is the expression of $SL_2(O_{S \cup \{p\}})$ as an amalgamated product

$$SL_2(O_S) *_{\Gamma_0(O_S, p)} H(p)$$

associated to the action of $SL_2(O_{S \cup \{p\}})$ on the Serre tree corresponding to the discrete valuation of the prime ideal $p$. This decomposition gives a Mayer-Vietoris sequence in homology. Analysis of the terms and the maps in low dimension yields, for $S$ sufficiently large, an exact sequence

$$H_2(SL_2(O_S), \mathbb{Z}) \longrightarrow H_2(SL_2(O_{S \cup \{p\}}), \mathbb{Z}) \quad \delta \quad H_1(k(p), \mathbb{Z}) \longrightarrow 0.$$
where the map $\delta$ is essentially the tame symbol of $K$-theory (see Theorem 5.17). This analysis requires, in particular, the deep and beautiful theorem of Vaserstein and Liehl ([21] and [5]) and the solution of the congruence subgroup problem for $SL_2$ (Serre, [14]).

In the later part of the paper, we tackle an old question in $K_2$-theory; namely, how to write down natural homology classes in $H_2(SL_2(A), \mathbb{Z})$, depending functorially on a pair of units $u, v \in A^\times$, which correspond, under the map $H_2(SL_2(A), \mathbb{Z}) \to K_2(2, A)$ when it exists, to the symbols $c(u, v) \in K_2(2, A)$. The answer to the corresponding question for $H_2(SL_3(A), \mathbb{Z})$ and $K_2(3, A)$ is well-known, namely the homology class (in the bar resolution)

$$\left( [\text{diag}(u, u^{-1}, 1)\text{diag}(v, 1, v^{-1})] - [\text{diag}(v, 1, v^{-1})\text{diag}(u, u^{-1}, 1)] \right) \otimes 1$$

corresponds to the symbol $\{u, v\} \in K_2(3, A)$, at least up to sign. There is no such simple expression in the case of $K_2(2, A)$. The symbols $c(u, v)$ are easily and naturally described in terms of the generators of the Steinberg group, but the corresponding natural homology classes, even in the case of a field, have no known simple construction. Since $K_2(2, \mathbb{Z})$ is infinite cyclic with generator $c(-1, -1)$ while $H_2(SL_2(\mathbb{Z}), \mathbb{Z}) = 0$ it follows that there can be no simple universal expression defined over the ring $\mathbb{Z}$. The homology classes, $C(u, v)$, that we construct in section 7 below are not very elegant (though it seems unlikely that they can be greatly improved on).

To begin with, the construction of the representing cycles requires the presence of a unit $\lambda$ such that $\lambda^2 - 1$ is also a unit, although the resulting homology classes can be shown quite generally to be independent of the choice of $\lambda$. Furthermore, the representing cycles consist usually of $32$ terms and hence are far from simple.

However, the cycles we construct are explicit and functorial for homomorphisms of rings. We prove (Theorem 7.8) that they map to the symbols $c(u, v) \in K_2(2, A)$ when $A$ is a field. We can thus use them to write down provably non-trivial homology classes in $H_2(SL_2(A), \mathbb{Z})$ for more general rings $A$. In particular, in section 8, we use them to write down explicit elements of the groups $H_2(SL_2(O_S), \mathbb{Z})$ with given order and to construct generators of the groups $H_2(SL_2(\mathbb{Z}[1/m]), \mathbb{Z})$ when $m$ is divisible by $6$.

2. Preliminaries and notation

2.1. Notation. For a Dedekind Domain $A$ with field of fractions $F$, $\text{Cl}(A)$ denotes the ideal classgroup of $A$. If $p$ is a nonzero prime ideal of $A$, $v_p : F^\times \to \mathbb{Z}$ denotes the corresponding discrete value. For a global field $F$ and a nonempty set of primes $S$ of $F$ we let $O_S$ denote the ring of $S$-integers:

$$O_S := \{a \in F^\times | v_p(a) \geq 0 \text{ for all } p \not\in S\}.$$  

For a finite abelian group $M$, $M(p)$ denotes the Sylow $p$-subgroup of $M$.

For a commutative ring $A$, we let $R_A := \mathbb{Z}[A^\times/(A^\times)^2]$ be the group ring of the group of square classes of units. For $a \in A^\times$, the square class of $a$ will be denoted $\langle a \rangle \in R_A$. Furthermore, the element $\langle a \rangle - 1$ in the augmentation ideal, $I_A \subset R_A$, will be denoted $\langle\langle a \rangle\rangle$.

2.2. Elementary matrices. We will have occasion to refer to the following facts:

For a commutative ring $A$, and any $x \in A$ we define the elementary matrices

$$E_{12}(x) := \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, \quad E_{21}(x) := \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \in SL_2(A).$$

Let $E_2(A)$ be the subgroup of $SL_2(A)$ generated by $E_{12}(x), E_{21}(y), x, y \in A$.

The following theorem of Vaserstein and Liehl will be essential below. Its proof relies on the resolution of the congruence subgroup problem for $SL_2$ (see Serre [14]).
Theorem 2.1 (Vaserstein [21], Liehl [5]). Let $K$ be a global field and let $S$ be a set of places of $K$ of cardinality at least 2 and containing all archimedean places. Let

$$O_S := \{x \in K \mid \nu(x) \geq 0 \text{ for all } \nu \not\in S\}$$

be the ring of $S$-integers of $K$. Let $I_1$ and $I_2$ be nonzero ideals of $O_S$. Let

$$\tilde{\Gamma}(I_1, I_2) := \left\{ \begin{array}{c} a & b \\ c & d \end{array} \right\} \in \text{SL}_2(O_S) \mid b \in I_1, c \in I_2, a - 1, b - 1 \in I_1I_2 \right\}$$

Then $\tilde{\Gamma}(I_1, I_2)$ is generated by the elementary matrices

$$E_{12}(x), x \in I_1 \text{ and } E_{21}(y), y \in I_2.$$

Proposition 2.2. Let $A$ be a commutative ring.

1. $E_2(A) = \text{SL}_2(A)$ if $A$ is a field or a Euclidean domain or if $A = O_S$ is the ring of $S$-integers in a global field and $|S| \geq 2$.
2. $E_2(A)$ is perfect if there exists $\lambda_1, \ldots, \lambda_n \in A^\times$ and $b_1, \ldots, b_n \in A$ such that

$$\sum_{i=1}^n b_i(\lambda_i^2 - 1) = 1 \text{ in } A.$$

In particular, $E_2(A)$ is perfect if there exists $\lambda \in A^\times$ such that $\lambda^2 - 1 \in A^\times$ also.

Proof. (1) This is standard linear algebra in the case of a Euclidean Domain or a field, and the theorem of Vaserstein-Liehl in the case of $S$-integers.

(2) For $\lambda \in A^\times$, let

$$D(\lambda) := \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \in \text{SL}_2(A).$$

Note that $D(\lambda) \in E_2(A)$ since

$$D(\lambda) = w(\lambda)w(-1) \text{ where } w(\lambda) := \begin{bmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{bmatrix} = E_{12}(\lambda)E_{21}(-\lambda^{-1})E_{12}(\lambda).$$

Then

$$D(\lambda)E_{12}(x)D(\lambda)^{-1} = E_{12}(\lambda^2 x)$$

and hence, for any $b \in A$ we have

$$[D(\lambda), E_{12}(bx)] = D(\lambda)E_{12}(bx)D(\lambda)^{-1}E_{12}(-bx) = E_{12}((\lambda^2 - 1)bx).$$

Thus

$$E_{12}(x) = E_{12}\sum_i (\lambda_i^2 - 1)b_i x = \prod_i E_{12}((\lambda_i^2 - 1)b_i x) = \prod_i [D(\lambda_i), E_{12}(b_i x)].$$

Remark 2.3. On the other hand, the groups $E_2(\mathbb{F}_2) = \text{SL}_2(\mathbb{F}_2)$ and $E_2(\mathbb{F}_3) = \text{SL}_2(\mathbb{F}_3)$ are not perfect. It follows that if the ring $A$ admits a homomorphism to $\mathbb{F}_2$ or $\mathbb{F}_3$ then $E_2(A)$ is not perfect. In particular, the group $E_2(\mathbb{Z})$ is not perfect.

Remark 2.4. In [18], R. Swan showed that $E_2(A) \neq \text{SL}_2(A)$ for $A = \mathbb{Z}[\sqrt{-5}]$.

Indeed, when $A$ is the ring of integers in a quadratic imaginary number field then $E_2(A) \neq \text{SL}_2(A)$ except in the five cases that $A$ is a Euclidean Domain (see [21]).
2.3. Homology of Groups. For any group $G$, $F_*(G)$ will denote the (right) bar resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$: i.e. for $n \geq 1$, $F_n(G)$ is the free right $\mathbb{Z}[G]$-module with generators $[g_n]\cdots [g_1]$, $\gamma_i \in G$, and $F_0(G) = \mathbb{Z}[G]$ (regarded as a right $\mathbb{Z}[G]$-module). The boundary homomorphism $d_i: F_i(G) \to F_{i-1}(G)$ is given by

$$d_i([g_n]\cdots [g_1]) = [g_n]\cdots [g_2]g_1 + \sum_{i=1}^{n-1} (-1)^{n-i}[g_{n-1}\cdots [g_{i+1}g_i]\cdots [g_1] + (-1)^i [g_{n-1}\cdots [g_1]$$

for $n \geq 2$ and $d_i([g]) := g - 1$.

We let $\bar{F}_*(G)$ denote the complex $\{\bar{F}_n(G)\}_{n \geq 0}$ where

$$\bar{F}_n(G) := F_n(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}.$$ 

Thus $H_n(G, \mathbb{Z}) \cong H_n(\bar{F}_*(G)).$

We will require the following standard “centre kills” argument from group homology:

**Lemma 2.5.** Let $G$ be a group and let $M$ be a $\mathbb{Z}[G]$-module. Suppose that $g \in Z(G)$ has the property that $g - 1$ acts as an automorphism on $M$. Then $H_i(G, M) = 0$ for all $i \geq 0$.

3. The functor $K_2(2, A)$

In this section, we review some of the theory of the functor $K_2(2, A)$ for commutative rings $A$.

3.1. Definitions. Let $A$ be a commutative ring.

We let $A^\times$ act by automorphisms on $\text{SL}_2(A)$ as follows: Let

$$M(a) := \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \in \text{GL}_2(A).$$

and define

$$a \ast X := XM(a)M(a)^{-1}XM(a)$$

for $a \in A^\times$, $X \in \text{SL}_2(A)$.

In particular, we have

$$a \ast E_{12}(x) = E_{12}(a^{-1}x) \text{ and } a \ast E_{21}(x) = E_{21}(ax)$$

for all $a \in A^\times$, $x \in A$.

The rank one Steinberg group $\text{St}(2, A)$ is defined by generators and relations as follows: The generators are the terms $x_{12}(t)$ and $x_{21}(t)$, $t \in A$

and the defining relations are

1. $x_{ij}(s)x_{ij}(t) = x_{ij}(s + t)$

   for $i \neq j \in \{1, 2\}$ and all $s, t \in A$, and

2. For $u \in A^\times$, let

   $$w_{ij}(u) := x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u)$$

   for $i \neq j \in \{1, 2\}$. Then

   $$w_{ij}(u)x_{ij}(t)w_{ij}(-u) = x_{ji}(-u^{-2}t)$$

   for all $u \in A^\times$, $t \in A$. 

There is a natural surjective homomorphism $\phi : \text{St}(2, A) \to E_2(A)$ defined by $\phi(x_{ij}(t)) = E_{ij}(t)$ for all $t$. It is easily verified that the formulae
\[
a \ast x_{12}(t) = x_{12}(a^{-1} t) \text{ and } a \ast x_{21}(t) = x_{21}(at)
\]
define an action of $A^\times$ on $\text{St}(2, A)$ by automorphisms. Clearly the homomorphism $\phi$ is equivariant with respect to this action.

By definition $K_2(2, A)$ is the kernel of $\phi$. It inherits an action of $A^\times$.

For $u \in A^\times$ and for $i \neq j \in \{1, 2\}$, we let
\[
h_{ij}(u) := w_{ij}(u)w_{ij}(-1).
\]
Note that
\[
\phi(w_{12}(u)) = \begin{bmatrix} 0 & u \\ -u^{-1} & 0 \end{bmatrix} \quad \text{and} \quad \phi(h_{12}(u)) = \begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix}.
\]
Note that, from the definitions and defining relation (1), for any $a \in A$ and any unit $u$ we have
\[
x_{ij}(a)^{-1} = x_{ij}(-a) \text{ and } w_{ij}(u)^{-1} = w_{ij}(-u).
\]
The defining relation (2) above thus immediately gives the following conjugation formula.

**Lemma 3.1.** Let $A$ be a commutative ring. Let $a \in A$ and $u \in A^\times$. For $i \neq j \in \{1, 2\}$
\[
x_{ij}(a)^{w_{ij}(-u)} = x_{ji}(-u^{-2}a).
\]
Since the right-hand-side is unchanged by $u \to -u$, we deduce:

**Corollary 3.2.** Let $A$ be a commutative ring. Let $a \in A$ and $u \in A^\times$. For $i \neq j \in \{1, 2\}$
\[
x_{ij}(a)^{w_{ij}(u)}^{-1} = x_{ji}(-u^{-2}a) = x_{ij}(a)^{w_{ij}(u)}.
\]
and
\[
x_{ji}(a)^{w_{ij}(u)} = x_{ij}(-u^2a).
\]
Form the definition of $h_{ij}(u)$, we then obtain:

**Corollary 3.3.** Let $A$ be a commutative ring. Let $a \in A$ and $u \in A^\times$. For $i \neq j \in \{1, 2\}$
\[
x_{ij}(a)^{h_{ij}(u)} = x_{ij}(u^{-2}a) \text{ and } x_{ij}(a)^{h_{ij}(u)}^{-1} = x_{ij}(u^2a).
\]

### 3.2. Symbols.

In particular, for $u, v \in A^\times$ the symbols
\[
c(u, v) := h_{12}(u)h_{12}(v)h_{12}(uv)^{-1}
\]
lie in $K_2(2, A)$.

The elements $c(u, v)$ are central in $\text{St}(2, A)$. We let $C(2, A)$ denote the subgroup of $K_2(2, A)$ generated by these symbols.

Note that for $a, u \in A^\times$ we have
\[
a \ast w_{12}(u) = w_{12}(a^{-1}u) \text{ and } a \ast w_{21}(u) = w_{21}(au)
\]
and hence
\[
a \ast h_{12}(u) = h_{12}(a^{-1}u)h_{12}(a^{-1})^{-1} \text{ and } a \ast h_{21}(u) = h_{21}(au)h_{21}(a)^{-1}.
\]
It follows easily that
\[
a \ast c(u, v) = c(u, a^{-1})^{-1}c(u, a^{-1}v).
\]
Thus the abelian group $C(2, A)$ is a module over the group ring $\mathbb{Z}[A^\times]$ with this action.
**Lemma 3.4.** Let $A$ be a commutative ring. Then
\[ a^2 \ast c(u, v) = c(u, v) \]
for all $a, u, v \in A^\times$.

In particular, $C(2, A)$ is naturally an $R_A$-module.

**Proof.** We have $h_{ij}(u) = h_{ji}(u)^{-1}$ in St(2, $A$). Thus
\[ c(u, v) = h_{12}(u)h_{12}(v)h_{12}(uv)^{-1} = h_{21}(u)^{-1}h_{21}(v)^{-1}h_{21}(uv). \]

Thus
\[ a \ast c(u, v) = h_{21}(a)h_{21}(au)^{-1}h_{21}(a)h_{21}(av)^{-1}h_{21}(auv)h_{21}(a)^{-1} \]
\[ = h_{21}(a)h_{21}(au)^{-1}h_{21}(a) \cdot \left( h_{21}(u)h_{21}(u)^{-1} \right) \cdot h_{21}(av)^{-1}h_{21}(auv)h_{21}(a)^{-1} \]
\[ = h_{21}(a)c(u, a)^{-1}c(u, av)h_{21}(a)^{-1} \]
\[ = c(u, a)^{-1}c(u, av) \]
\[ = a^{-1} \ast c(u, v). \]

□

The symbols $c(u, v)$ satisfy the following properties (see [6], or also [16]):

**Proposition 3.5.** Let $A$ be a commutative ring. Then

1. $c(u, v) = 1$ if $u = 1$ or $v = 1$.
2. $c(u, v) = c(v^{-1}, u)$ for all $u, v \in A^\times$.
3. $c(u, vw)c(v, w) = c(uv, w)c(u, v)$ for all $u, v, w \in A^\times$.
4. $c(u, v) = c(u, -uv)$ for all $u, v \in A^\times$.
5. $c(u, v) = c(u, (1 - u)v)$ whenever $u, 1 - u, v \in A^\times$.

**Remark 3.6.** Combining the result of Lemma 3.4 with Proposition 3.5 (3), we see that the square class $\langle a \rangle \in R_A$ acts on $C(2, A)$ via
\[ \langle a \rangle c(u, v) = c(u, a)^{-1}c(u, av) = c(au, v)c(a, v)^{-1}. \]

Furthermore Proposition 3.5 (4) is equivalent to
\[ \langle v \rangle c(u, -u) = 1 \text{ for all } u, v \in A^\times \]
and Proposition 3.5 (5) is equivalent to
\[ \langle v \rangle c(u, 1 - u) = 1 \text{ for all } u, v \in A^\times. \]

We will use the following property of symbols ([6]):

**Lemma 3.7.** If $u, v, w$ are units in $A$, then
\[ c(u, v^2w) = c(u, v^2)c(u, w) \]
and
\[ c(u, v^2) = c(u, v)c(v, u)^{-1} = c(u^2, v). \]

Furthermore, we have the following theorem of Matsumoto and Moore ([6], [10]):

**Theorem 3.8.** Let $F$ be an infinite field. Then
(1) The sequence

$$1 \to K_2(2, F) \to \text{St}(2, F) \to \text{SL}_2(F) \to 1$$

is the universal central extension of the perfect group $\text{SL}_2(F)$.

In particular, $K_2(2, F) \cong H_2(\text{SL}_2(F), \mathbb{Z})$ naturally.

(2) $K_2(2, F)$ has the following presentation: It is generated by the symbols $c(u, v)$, $u, v \in F^\times$, subject to the five relations of Proposition 3.5

3.3. The stabilization homomorphism $K_2(2, F) \to K_2(F)$. For a field $F$, the Theorem of Matsumoto also gives a presentation of $K_2(n, F)$ for all $n \geq 3$. In particular, it follows that $K_2(F) = K^M_2(F)$, the second Milnor $K$-group of the field $F$. The stabilization map $K_2(2, F) \to K_2(F)$ is surjective and sends the symbols $c(u, v)$ to the symbols $\{u, v\}$ of algebraic $K$-theory.

Let $GW(F)$ be the Grothendieck-Witt ring of isometry classes of nondegenerate quadratic forms over $F$. It is generated by the classes $\langle a \rangle$ of 1-dimensional forms and the map $R_F \to GW(F)$ sending $\langle a \rangle \to \langle a \rangle$ is a surjection of rings. The fundamental ideal $I(F)$ of $GW(F)$ is the ideal generated by the elements $\langle a \rangle := \langle a \rangle - 1$.

There is a natural surjective homomorphism of $R_F$-modules

$$K_2(2, F) \to I^2(F), \quad c(u, v) \mapsto \langle u \rangle \langle v \rangle.$$

Furthermore, by a theorem of Milnor ([9]) there is also a surjective map $K^M_2(F) \to I^2(F)/I^3(F)$ sending the symbol $\{u, v\}$ to the class of $\langle u \rangle \langle v \rangle$. The kernel of this map is precisely $2K^M_2(F)$.

By a result essentially due to Suslin ([17], but see also [7]) for an infinite field $F$, we also have the following description of $K_2(2, F)$:

**Theorem 3.9.** Let $F$ be an infinite field. The maps $K_2(2, F) \to K_2(F)$, $K_2(2, F) \to I^2(F)$ induce an isomorphism of $R_F$-modules

$$K_2(2, F) \to K^M_2(F) \times_{I^2(2)} I^2(F), \quad c(u, v) \mapsto [u, v] := ([u, v], \langle u \rangle \langle v \rangle).$$

**Corollary 3.10.** Let $F$ be an infinite field. There is a natural short exact sequence of $GW(F)$-modules

$$0 \to I^2(F) \to K_2(2, F) \to K^M_2(F) \to 0.$$

3.4. Milnor-Witt $K$-theory. The homology of the special linear group of a field is related to the Milnor-Witt $K$-theory of the field (see, for example, [4]).

Milnor-Witt $K$-theory of a field $F$ is a $\mathbb{Z}$-graded algebra $K^{\text{MW}}_*(F)$ generated by symbols $[u]$, $u \in F^\times$ in degree 1 and a symbol $\eta$ in degree $-1$, satisfying certain relations (see [11] for details). It arises naturally as a ring of operations in stable $\mathbb{A}^1$-homotopy theory.

A deep theorem of Morel asserts:

**Theorem 3.11.** ([12]) There is a natural isomorphism of graded rings

$$K^{\text{MW}}_*(F) \cong K^M_*(F) \times_{I^*(F)/I^{n+1}(F)} I^*(F).$$

(Here, when $n < 0$, $K^\text{MW}_n(F) := 0$ and $I^n(F) := W(F)$, the Witt ring of the field.)

The theorem of Suslin on the structure of $K_2(2, F)$ quoted above, implies

**Proposition 3.12.** There is a natural isomorphism $K_2(2, F) \cong K^\text{MW}_2(F)$, sending $c(u, v)$ to $[u][v]$. 
4. The map from $H_2(\text{SL}_2(F), \mathbb{Z})$ to $K_2(2, F)$

Let $A$ be a commutative ring for which $E_2(A) = \text{SL}_2(A)$ is a perfect group. Suppose further that the group extension

$$1 \longrightarrow K_2(2, A) \longrightarrow \text{St}(2, A) \xrightarrow{\phi} \text{SL}_2(A) \longrightarrow 1$$

is a central extension.

Let $s : \text{SL}_2(A) \to \text{St}(2, A)$ be a section of $\phi$. Then there is a corresponding 2-cocycle $f_s : \text{SL}_2(A) \times \text{SL}_2(A) \to K_2(2, A)$ defined by

$$f_s(X, Y) := s(X)s(Y)s(XY)^{-1}.$$  

This yields a cohomology class $f \in H^2(\text{SL}_2(A), K_2(2, A))$ which is independent of the choice of section $s$.

However, since $H_1(\text{SL}_2(A), \mathbb{Z}) = 0$, the universal coefficient theorem tells us that there is a natural isomorphism

$$H^2(\text{SL}_2(A), K_2(2, A)) \cong \text{Hom}(H_2(\text{SL}_2(A), \mathbb{Z}), K_2(2, A))$$

described as follows: Let $z \in H^2(\text{SL}_2(A), K_2(2, A))$ be represented by the 2-cocycle $h$. Then $h$ induces a homomorphism

$$\tilde{h} : H_2(\text{SL}_2(A), \mathbb{Z}) \to K_2(2, F).$$

In particular, the cocycle $f_s$ above induces the homomorphism

$$H_2(\text{SL}_2(A), \mathbb{Z}) \to K_2(2, F), \quad \sum n_i[X_i|Y_i] \mapsto \prod f_s(X_i, Y_i)^{n_i}$$

This homomorphism is an isomorphism precisely when the central extension is universal. In particular, it is an isomorphism when $A$ is an infinite field, by the theorem of Matsumoto-Moore.

We now specialise to the case of a field $F$.

For our calculations, we will use the following section $s : \text{SL}_2(F) \to \text{St}(2, F)$:

$$s \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) := \begin{cases} x_{12}(ab)h_{12}(a), & \text{if } c = 0, \\ x_{12}(ac^{-1})w_{12}(-c^{-1})x_{12}(dc^{-1}), & \text{if } c \neq 0. \end{cases}$$

Note that, in particular, we have

$$s(E_{ij}(a)) = x_{ij}(a) \text{ and } s(D(u)) = h_{12}(u)$$

when $i \neq j \in \{1, 2\}$, $a \in A$ and $u \in F^\times$.

Furthermore, functoriality of the constructions above guarantee that the induced homomorphism

$$\tilde{f} : H_2(\text{SL}_2(F), \mathbb{Z}) \to K_2(2, F)$$

is a map of $\mathbb{Z}[F^\times]$-modules. Recall that this homomorphism is induced by the homomorphism

$$\bar{f}_s(\text{SL}_2(F)) \to K_2(2, F), [X|Y] \mapsto f_s(X, Y) = s(X)s(Y)s(XY)^{-1}.$$
LEMMA 4.1. Let $F$ be a field. Let $u, v \in F^\times$ and $a, b \in F$. Let
\[ X = \begin{bmatrix} u & a \\ 0 & u^{-1} \end{bmatrix}, \quad Y = \begin{bmatrix} v & b \\ 0 & v^{-1} \end{bmatrix} \]
Then $f_s(X, Y) = c(u, v)$.

PROOF. We have,
\[ s(X) = x_{12}(au)h_{12}(u), \quad s(Y) = x_{12}(bv)h_{12}(v) \text{ and } s(XY) = x_{12}(bu^2v + au)h_{12}(uv). \]
Thus
\[ f(X, Y) = x_{12}(au)h_{12}(u)x_{12}(bv)h_{12}(v)h_{12}(uv)^{-1}x_{12}(-bu^2v - au) \]
\[ = x_{12}(au)x_{12}(bv)^{-1}h_{12}(u)h_{12}(v)h_{12}(uv)^{-1}x_{12}(-bu^2v - au) \]
\[ = x_{12}(au)x_{12}(bu^2v)c(u, v)x_{12}(-bu^2v)x_{12}(-au) \quad \text{by Corollary 3.3} \]
\[ = c(u, v) \quad \text{since } c(u, v) \text{ is central}. \]

\[ \square \]

COROLLARY 4.2. Let $F$ be a field. Let $a, b \in F^\times$. Then
\[ ([D(a)|D(b)] - [D(b)|D(a)]) \otimes 1 \in F_2(SL_2(F)) \otimes \mathbb{Z} \]
is a cycle and the corresponding homology class maps to $c(a^2, b)$ under the natural isomorphism \( H_2(SL_2(K), \mathbb{Z}) \cong K_2(2, F) \) induced by $f_s$.

PROOF. The first statement in immediate since $D(a)D(b) = D(ab) = D(b)D(a)$.
The image of this cycle is
\[ f_s(D(a), D(b)) \cdot f_s(D(b), D(a))^{-1} = c(a, b)c(b, a)^{-1} = c(a^2, b) \]
by Lemma 3.7

\[ \square \]

5. The Mayer-Vietoris sequence

Throughout this section $A$ will denote a Dedekind Domain with field of fractions $K$.

5.1. The groups $H(I)$. We collect together some basic and well-known facts about certain subgroups of $SL_2(K)$ (see for example [14, p. 520]).

Let $I$ be a fractional ideal of $A$.

We consider the lattice $\Lambda = \Lambda_I := A \oplus I \subset K \oplus K = K^2$.

Let $H(I)$ denote the subgroup
\[ \{ M \in SL_2(K) \mid M \cdot \Lambda = \Lambda \} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(K) \mid a, d \in A, c \in I, d \in I^{-1} \right\} = \Gamma(I, I^{-1}). \]

Note that, in particular, $H(A) = SL_2(A)$.

We also note that if $J$ is any nonzero fractional ideal of $A$, then
\[ H(I) = \{ M \in SL_2(K) \mid M \cdot (JA) = JA \} \]
where
\[ JA = J \cdot (A \oplus I) = J \oplus IJ. \]

LEMMA 5.1. Let $I$ be a fractional ideal of the $A$. 

(1) Suppose that $I' = aI$ where $0 \neq a \in K$. Then $H(I') = H(I)^{M(a)}$ where
\[
M(a) = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \in \text{GL}_2(K).
\]

(2) Suppose $I$ is an integral ideal. Let
\[
A' = \{ r \in K \mid \nu_q(r) \geq 0 \text{ for all } q \mid I \}.
\]
Then there exists $M \in \text{SL}_2(A')$ such that $H(I^2) = \text{SL}_2(A)^M$. In particular, $H(I^2) \cong \text{SL}_2(A)$.

**Proof.**

(1) This follows from the observation that multiplication by $M(a)$ induces an isomorphism of lattices $A \oplus I' \cong a \cdot (A \oplus I)$, and hence conjugation by $M(a)$ induces an isomorphism of the stabilizers.

(2) We first observe that, since $I^{-1} \cdot \Lambda_I = I^{-1} \oplus I$, $H(I^2)$ is the stabilizer of $I^{-1} \oplus I$.

There exists an integral ideal $J$ of $A$ satisfying: $I + J = A$ and $IJ = xA$ for some nonzero $x \in A$. So $J = xI^{-1}$. Thus multiplication by $M(x)$ induces an isomorphism $I^{-1} \oplus I \cong J \oplus I$.

Choose $a \in I, b \in J$ with $a+b = 1$. Consider the short exact sequence of $A$-modules
\[
0 \longrightarrow xA \overset{f}{\longrightarrow} J \oplus I \overset{g}{\longrightarrow} A \longrightarrow 0
\]
where $g(y) = (y, -y)$ and $f(y, z) = y + z$. There is a splitting $A \rightarrow J \oplus I$ given by $y \mapsto (by, ay)$. This gives an isomorphism of $A$-modules
\[
J \oplus I \cong xA \oplus A, \quad (y, z) \mapsto (ay - bz, y + z);
\]
i.e. multiplication by
\[
N := \begin{bmatrix} a & -b \\ 1 & 1 \end{bmatrix} \in \text{SL}_2(A)
\]
induces an isomorphism of lattices $J \oplus I \cong xA \oplus A$.

Now, multiplication by $M(x)^{-1}$ induces an isomorphism $xA \oplus A \cong A \oplus A$.

Putting all of this together, multiplication by
\[
M := M(x)^{-1}NM(x) = \begin{bmatrix} a & -b/x \\ x & 1 \end{bmatrix} \in \text{SL}_2(K)
\]
induces an isomorphism of lattices $I^{-1} \oplus I \cong A \oplus A$, and thus conjugation by $M$ induces an isomorphism of stabilizers as required.

Finally, we note that since $xA = IJ$ and $bA = JK$ for some integral ideal $K$, $(b/x)A = KI^{-1}$ and hence $b/x \in A'$. Thus $M \in \text{SL}_2(A')$ as claimed.

\[
\square
\]

**Corollary 5.2.** Let $I$ be a fractional ideal of $A$. Suppose that the class of $I$ in $\text{Cl}(A)$ is a square. Then $H(I) \cong \text{SL}_2(A)$.

**Remark 5.3.** In particular, the ideal $\mathfrak{p} := \mathfrak{p}A_\mathfrak{p}$ in $A_\mathfrak{p}$ is a principal ideal with generator $\pi$, say. It follows from Lemma [5.1] that
\[
H(\mathfrak{p}) = \text{SL}_2(A_\mathfrak{p})^{M(\pi)}.
\]

Let $p$ be a nonzero prime ideal of $A$. Let $n \geq 1$ and let $\pi \in A$ satisfy $v_p(\pi) = 1$. We let $\gamma_{\pi,n} : H(p) \to \text{SL}_2(A/p^n)$ be the composite

$$H(p) \xrightarrow{\cong} H(\bar{p}) \xrightarrow{\cong} \text{SL}_2(A_p) \xrightarrow{\cong} \text{SL}_2(A/p^n) \xrightarrow{\cong} \text{SL}_2(A/p^n)$$

**Lemma 5.4.** The map $\gamma_{\pi,n}$ is surjective for all $n$ and the kernel of this map is independent of the choice of $\pi$.

**Proof.** By definition, we have

$$\gamma_{\pi,n} \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} \tilde{a} & \tilde{\pi} b \\ c/\pi & \tilde{d} \end{bmatrix}$$

where

$$\tilde{x} := x + \tilde{p}^n \in A_p/\tilde{p}^n \cong A/p^n.$$

Since $\text{SL}_2(A/p^n)$ is generated by elementary matrices, we need only show how to lift these. We begin by observing that $\pi A = pJ$ where $J$ is an ideal not contained in $p$. It follows that $A = p^n + J$ for any $n \geq 1$; i.e. the map $J \to A/p^n$ is surjective.

Thus, given any $x \in A$ there exists $x' \in J$ with $\tilde{x}' = \tilde{x}$. Since $x' \in J$ it follows that $x'/\pi \in J \cdot (pJ)^{-1} = p^{-1}$. Hence $E_{12}(x'/\pi) \in H(p)$ and

$$\gamma_{\pi,n}(E_{12}(x'/\pi)) = E_{12}(\tilde{x}') = E_{12}(\tilde{x}).$$

Of course, we also have $E_{21}(\pi x) \in H(p)$ and $\gamma_{\pi,n}(E_{21}(\pi x)) = E_{21}(\tilde{x})$. This proves the surjectivity statement.

For the second part, suppose that $\pi' \in A$ also satisfies $v_p(\pi') = 1$. Then $\pi' = \pi \cdot u$ for some $u \in A_p^\times$. From the definition, we have

$$\gamma_{\pi',n} = f \circ \gamma_{\pi,n}$$

where $f$ is conjugation by $M(\tilde{u}^{-1})$ on $\text{SL}_2(A/p^n)$. It follows at once that $\text{Ker}(\gamma_{\pi',n}) = \text{Ker}(\gamma_{\pi,n})$ as claimed.

We let $\tilde{\Gamma}(A, p^n)$ denote the kernel of the $\gamma_{\pi,n}$ (for any choice of $\pi$). Thus, for all $n \geq 1$, there is a short exact sequence

$$1 \to \tilde{\Gamma}(A, p^n) \to H(p) \to \text{SL}_2(A/p^n) \to 1.$$ 

Note that

$$\tilde{\Gamma}(A, p^n) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in H(p) \mid a - 1, d - 1 \in p^n, c \in p^{n+1}, b \in p^{n-1} \right\}.$$ 

In particular, for all $n \geq 1$ we have

$$\Gamma(A, p^{n+1}) \subset \tilde{\Gamma}(A, p^n) \subset \Gamma_0(A, p^n) \subset \text{SL}_2(A).$$

For a field $F$, we will use the notation

$$B(F) := \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \in \text{SL}_2(F) \right\} \text{ and } B'(F) := \left\{ \begin{bmatrix} a & 0 \\ c & a^{-1} \end{bmatrix} \in \text{SL}_2(F) \right\}.$$ 

Of course, these two subgroups of $\text{SL}_2(F)$ are naturally isomorphic.

We will need the following result below.

**Lemma 5.5.** There is a natural short exact sequence

$$1 \to \tilde{\Gamma}(A, p) \to \Gamma_0(A, p) \to B'(k(p)) \to 1.$$
Thus there is also an induced decomposition \( \text{SL}_2(k(p)) \) under the map \( \gamma_{\pi,1} \) is precisely \( B'(k(p)) \).

### 5.2. The Mayer-Vietoris sequence.

Let \( p \) be a nonzero prime ideal of \( A \) and let \( v = v_p \) be the associated discrete valuation. We let \( k(p) \) or \( k(v) \) denote the residue field \( A/p \). We will further suppose that the class of \( p \) has finite order in \( \text{Cl}(A) \). Thus \( p^n = xA \) for some \( n \geq 1 \) and \( x \in A \). (This condition is automatically satisfied when \( K \) is a global field.)

Let 
\[
\Gamma(A, p) := \text{Ker}(\text{SL}_2(A) \to \text{SL}_2(k(\pi))) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(A) : 1 - a, 1 - d, b, c \in p \right\}
\]

and let 
\[
\Gamma_0(A, p) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(A) : c \in p \right\}.
\]

We let \( \tilde{p} \) denote the extension of \( p \) to the localization \( A_p \), which is thus a discrete valuation ring with unique (principal) nonzero prime ideal \( \tilde{p} \).

The action of \( \text{SL}_2(K) \) on the Serre tree associated to the valuation \( v \) ([15 Chapter II]) yields a decomposition

\[
\text{SL}_2(K) = \text{SL}_2(A_p) \star_{\Gamma_0(A_p, \tilde{p})} H(\tilde{p})
\]

of \( \text{SL}_2(K) \) as the sum of \( \text{SL}_2(A_p) \) and \( H(\tilde{p}) \) amalgamated along their intersection \( \text{SL}_2(A_p) \cap H(\tilde{p}) = \Gamma_0(A_p, \tilde{p}) \).

Let 
\[
A' := \{ a \in K \mid v_q(a) \geq 0 \text{ for all prime ideals } q \neq p \}.
\]

Note that since \( p^n = xA \) by assumption, \( A' = A[1/x] \).

Since \( A[1/x] \) is dense in \( K \) in the \( v \)-adic topology, and since 
\[
\text{SL}_2(A[1/x]) \cap \text{SL}_2(A_p) = \text{SL}_2(A), \quad \text{SL}_2(A[1/x]) \cap H(\tilde{p}) = H(p)
\]

there is also an induced decomposition

\[
\text{SL}_2(A[1/x]) = \text{SL}_2(A) \star_{\Gamma_0(A, p)} H(p).
\]

For convenience, in the remainder of this section we will set 
\[
G := \text{SL}_2(A[1/x]), \quad G_1 := \text{SL}_2(A), \quad G_2 := H(p) \text{ and } \Gamma_0 := \Gamma_0(A, p).
\]

Thus \( G = G_1 \star_{\Gamma_0} G_2 \) and this decomposition gives rise to a short exact sequence of \( \mathbb{Z}[G] \)-modules:

\[
\begin{array}{c}
0 \rightarrow \mathbb{Z}[G/\Gamma_0] \xrightarrow{\alpha} \mathbb{Z}[G/G_1] \oplus \mathbb{Z}[G/G_2] \xrightarrow{\beta} \mathbb{Z} \rightarrow 0.
\end{array}
\]

where \( \alpha \) is the map 
\[
\alpha : \mathbb{Z}[G/\Gamma_0] \to \mathbb{Z}[G/G_1] \oplus \mathbb{Z}[G/G_2], \quad g\Gamma_0 \mapsto (gG_1, gG_2)
\]

and \( \beta \) is the unique \( \mathbb{Z}[G] \)-homomorphism 
\[
\beta : \mathbb{Z}[G/G_1] \oplus \mathbb{Z}[G/G_2] \to \mathbb{Z}, \quad (G_1, 0) \mapsto -1, (0, G_2) \mapsto 1.
\]

This short exact sequence of \( \mathbb{Z}[G] \)-modules gives rise to a long exact sequence in homology. Combining this with the isomorphisms of Shapiro’s lemma, \( H_r(G, \mathbb{Z}[G/H]) \cong H_r(H, \mathbb{Z}) \), gives us the Mayer-Vietoris exact sequence of the amalgamated product:

\[
\cdots \rightarrow H_r(\Gamma_0, \mathbb{Z}) \xrightarrow{\alpha} H_r(G_1, \mathbb{Z}) \oplus H_r(G_2, \mathbb{Z}) \xrightarrow{\beta} H_r(G, \mathbb{Z}) \xrightarrow{\delta} \cdots
\]
The maps $\alpha$ and $\beta$ in this sequence can be described as follows: Let $\iota_1 : \Gamma_0 \to G_1$ and $\iota_2 : \Gamma_0 \to G_2$ be the natural inclusions. Then

$$\alpha(z) = (\iota_1(z), \iota_2(z)) \text{ for all } z \in H_r(\Gamma_0, \mathbb{Z}).$$

Likewise, let $j_1 : G_1 \to G$ and $j_2 : G_2 \to G$ be the natural inclusions. Then

$$\beta(z_1, z_2) = j_2(z_2) - j_1(z_1) \text{ for all } z_1 \in H_r(G_1, \mathbb{Z}), z_2 \in H_r(G_1, \mathbb{Z}).$$

The amalgamated product decomposition $\Delta$ – i.e. taking the case $A = A_p$ – also gives rise to a Mayer-Vietoris sequence

$$\cdots \to H_r(\Gamma_0(A_p, \tilde{\psi}), \mathbb{Z}) \to H_r(\Gamma_0(A_p, \tilde{\psi}), \mathbb{Z}) \oplus H_r(\Gamma_0(A_p, \tilde{\psi}), \mathbb{Z}) \delta \to H_r(\Gamma_0(A_p, \mathbb{Z}) \oplus H_r(\Gamma_0(A_p, \mathbb{Z}) \oplus \cdots$$

5.3. The connecting homomorphism. As above, let $p$ be a prime ideal of the Dedekind Domain $A$ and let

$$\delta : H_2(\Gamma_0(A_p, \tilde{\psi}), \mathbb{Z}) \to H_1(\Gamma_0(A_p, \tilde{\psi}), \mathbb{Z})$$

be the connecting homomorphism in the Mayer-Vietoris sequence associated to the decomposition

$$\text{SL}_2(K) = \text{SL}_2(A_p) \ast_{\Gamma_0(A_p, \tilde{\psi})} H(\tilde{\psi}) = \text{SL}_2(A_p) \ast_{\Gamma_0(A_p, \tilde{\psi})} \text{SL}_2(A_p)^M(\pi).$$

Proposition 5.6. Let $\rho : \Gamma_0(A_p, \tilde{\psi}) \to k(\tilde{\psi})^\times$ be the (surjective) map

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto a \pmod{\tilde{\psi}}.$$

Then the composite homomorphism, $\Delta$ say,

$$K_2(2, K) \cong H_2(\text{SL}_2(K), \mathbb{Z}) \delta H_1(\Gamma_0(A_p, \tilde{\psi}), \mathbb{Z}) \delta H_1(k(\tilde{\psi})^\times, \mathbb{Z}) \cong k(\tilde{\psi})^\times$$

is the map

$$c(a, b) \mapsto (-1)^{v(a)v(b)} \frac{b^{v(a)}}{a^{v(b)}} \pmod{\tilde{\psi}}.$$

Remark 5.7. In fact, the isomorphisms in the statement of Proposition 5.6 are canonical only up to sign. We have made our choices so that the sign is $+1$; but the choice of sign does not materially affect our main results.

Before proving Proposition 5.6 we require

Lemma 5.8. Let $K$ be a field with discrete valuation $v$. Then $K_2(2, K)$ is generated by the set $C_v := \{c(x, u) \mid v(u) = 0, v(x) = 1\}$.

Proof. Let $D$ be the subgroup of $K_2(2, K)$ generated by $C_v$. Let $a, b \in K^\times$. We must prove that $c(a, b) \in D$.

Since

$$c(a, b) = c(b^{-1}, b) = c(a^{-1}, b) = c(b, a^{-1})$$

we can assume that $v(a), v(b) \geq 0$.

We will prove the result by induction on $n = v(a) + v(b) \geq 0$.

If $n = 0$, then $v(a) = v(b) = 0$ and choosing $\pi \in K^\times$ with $v(\pi) = 1$ we have

$$c(a, b) = c(\pi a, b)^{-1} c(\pi a, b) c(\pi, a) \in D.$$

On the other hand, suppose that $v(a), v(b) > 0$. If $0 < v(b) \leq v(a)$ then $a = bc$ with $0 \leq v(c) < v(a)$ and hence

$$c(a, b) = c(bc, b) = c(-c, b) \in D.$$
by the inductive hypothesis. An analogous argument applies to the case \( 0 < v(a) < v(b) \).

Since \( c(a, b) = c(b^{-1}, a) \), we can reduce to the case where \( v(b) = 0 \) and \( v(a) \geq 2 \). Then let \( a = a' \pi \) where \( v(\pi) = 1 \) and \( 1 \leq v(a') < v(a) \). We have

\[
c(a, b) = c(a' \pi, b) = c(a', \pi b) c(\pi, b) c(a', \pi)^{-1}
\]

which lies in \( D \) by the induction hypothesis (using the argument for the case \( v(a), v(b) > 0 \) for the first term).

**Proof of Proposition 5.6.** By Lemma 5.8 we must prove that

\[
\Delta(c(x, u)) = u \quad \text{(mod } \bar{p})
\]

whenever \( v(u) = 0, v(x) = 1 \).

We note that it is enough to prove that \( \Delta(c(x, u^2)) = u^2 \) (mod \( \bar{p} \)) whenever \( v(u) = 0, v(x) = 1 \). For if \( u \in K \) is not a square, choose an extension \( v' \) of \( v \) to \( K' := K(\sqrt{u}) \). Then there is a natural map of Mayer-Vietoris exact sequences inducing a commutative square

\[
\begin{array}{ccc}
H_2(\SL_2(K), \mathbb{Z}) & \xrightarrow{\delta} & k(v)^x \\
\downarrow & & \uparrow i \\
H_2(\SL_2(K'), \mathbb{Z}) & \xrightarrow{\delta'} & k(v')^x
\end{array}
\]

so that \( i(\Delta(c(x, u)) = \Delta'(c(x, u)) = \bar{u} \in k(v')^x \) since \( u \) is a square in \( K' \), and thus \( \Delta(c(x, u)) = \bar{u} \in k(v)^x \).

Now, by Corollary 4.2 the symbol \( c(x, u^2) \in K(2, K) \) corresponds to the homology class represented by the cycle

\[
Z := ([D(x)[D(u)] - [D(u)[D(x)]]) \otimes 1 \in F_2(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}
\]

where \( G = \SL_2(K) \).

Recall that the Mayer-Vietoris sequence is the long exact homology sequence derived from the short exact sequence of complexes

\[
0 \rightarrow F_\bullet(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G/G_1] \rightarrow F_\bullet(G) \otimes_{\mathbb{Z}[G]} (\mathbb{Z}[G/G_1] \oplus \mathbb{Z}[G/G_2]) \rightarrow F_\bullet(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z} \rightarrow 0.
\]

Now the cycle \( Z \) lifts to

\[
([D(x)[D(u)] - [D(u)[D(x)]]) \otimes (1 \cdot G_1, 0) \in F_2(G) \otimes (\mathbb{Z}[G/G_1] \oplus \mathbb{Z}[G/G_2])
\]

Under the boundary map \( d_2 \), this is sent to

\[
[D(u)] \otimes (D(x) \cdot G_1 - 1 \cdot G_1, 0) \in F_1(G) \otimes (\mathbb{Z}[G/G_1] \oplus \mathbb{Z}[G/G_2])
\]

since \( D(u) \in \Gamma_0 \subset G_1 \).

Now let

\[
w := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in G_1.
\]

Then

\[
w \cdot D(x) = w^{M(x)} \in G_2.
\]

Thus \( (D(x) \cdot G_1 - 1 \cdot G_1, 0) \) is the image of

\[
w^{-1} \cdot (w^{M(x)} \Gamma_0 - \Gamma_0) = D(x)\Gamma_0 - w^{-1}\Gamma_0
\]

under the map \( \alpha : \mathbb{Z}[G/\Gamma_0] \rightarrow \mathbb{Z}[G/G_1] \oplus \mathbb{Z}[G/G_2] \).
Thus the homology class $\delta(Z) \in H_1(\Gamma_0, \mathbb{Z})$ is represented by the cycle

$$[D(u)] \otimes (D(x)\Gamma_0 - w^{-1}\Gamma_0) = \left([D(u)]D(x) - [D(u)]w^{-1}\right) \otimes \Gamma_0 \in F_1(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G/\Gamma_0].$$

This, in turn, is the image of

$$\left([D(u)]D(x) - [D(u)]w^{-1}\right) \otimes 1 \in F_1(G) \otimes_{\mathbb{Z}[\Gamma_0]} \mathbb{Z}$$

under the natural isomorphism

$$F_\bullet(G) \otimes_{\mathbb{Z}[\Gamma_0]} \mathbb{Z} \cong F_\bullet(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G/\Gamma_0].$$

For a group $H$ we let $C_\bullet(H)$ denote the right homogeneous resolution of $H$. The isomorphism $F_\bullet(H) \to C_\bullet(H)$ of complexes of right $\mathbb{Z}[H]$-modules is given by

$$[h_n \cdots | h_1] \mapsto (h_n \cdot h_{n-1} \cdots h_1, \ldots, h_1, 1).$$

Thus the cycle $\left([D(u)]D(x) - [D(u)]w^{-1}\right) \otimes 1 \in F_1(G) \otimes_{\mathbb{Z}[\Gamma_0]} \mathbb{Z}$ corresponds to the cycle

$$\left((D(ux), D(u)) - (D(u)w^{-1}, w^{-1})\right) \otimes 1 \in C_1(G) \otimes_{\mathbb{Z}[\Gamma_0]} \mathbb{Z}.$$

To construct an augmentation-preserving map of $\mathbb{Z}[\Gamma_0]$-resolutions from $C_\bullet(G)$ to $C_\bullet(\Gamma_0)$, we choose any set-theoretic section $s: G/\Gamma_0 \to G$ of the natural surjection $G \to G/\Gamma_0, g \mapsto g\Gamma_0$. For $g \in G$ we let $\bar{g} := s(g\Gamma_0)^{-1}g \in \Gamma_0$. Then the map

$$\tau: C_\bullet(G) \to C_\bullet(\Gamma_0), (g_n, \ldots, g_0) \mapsto (\bar{g}_n, \ldots, \bar{g}_0)$$

is an augmentation preserving map of $\mathbb{Z}[\Gamma_0]$-complexes.

We further specify that we the section $s$ satisfies

$$s(D(u)w^{-1}\Gamma_0) = w^{-1}$$

and $s(D(x)\Gamma_0) = D(x)$

for all $u$ with $v(u) = 0$. Then

$$\tau \left(\left((D(ux), D(u)) - (D(u)w^{-1}, w^{-1})\right)\right) = (D(u), 1) - (D(u^{-1}), 1) \in C_1(\Gamma_0)$$

since $wD(u)w^{-1} = D(u^{-1})$ in $G$.

Finally, the homology class

$$\left((D(u), 1) - (D(u^{-1}), 1)\right) \otimes 1 \in C_1(\Gamma_0) \otimes_{\mathbb{Z}[\Gamma_0]} \mathbb{Z}$$

corresponds to the element

$$D(u) \cdot D(u^{-1})^{-1} = D(u^2) \in \Gamma_0/[\Gamma_0, \Gamma_0]$$

under the isomorphism $H_1(\Gamma_0, \mathbb{Z}) \cong \Gamma_0/[\Gamma_0, \Gamma_0]$, and hence maps to $u^2 \pmod{\bar{p}} \in k(\bar{p})^\times$ under the map $\rho$.

5.4. The abelianization of some congruence subgroups.

**Proposition 5.9.** Let $A$ be a ring of $S$-integers in a global field $K$. Suppose that $|S| \geq 2$ and that there exists $\lambda \in A^\times$ such that $\lambda^2 - 1 \in A^\times$ also. Let $\mathfrak{p}$ be a nonzero prime ideal of $A$.

Then the map $\rho: \Gamma_0(A, \mathfrak{p}) \to k(\mathfrak{p})^\times$ induces an isomorphism

$$H_1(\Gamma_0(A, \mathfrak{p}), \mathbb{Z}) \cong k(\mathfrak{p})^\times.$$

There are natural isomorphisms of groups in the notation of Theorem 2.1.

Let $A$ be a Dedekind domain and let $p$ be a maximal ideal. Then $\hat{\Gamma}(A, p)$ is generated by elementary matrices $E_{12}(x), x \in A, E_{21}(y), y \in p$. However,

$E_{12}(x) = [D(\lambda), E_{12}(x/(\lambda^2 - 1))], E_{21}(y) = [D(\lambda), E_{21}(y/(\lambda^2 - 1))] \in [\Gamma_0(A, p), \Gamma_0(A, p)]$.

So $[\Gamma_0(A, p), \Gamma_0(A, p)] = \Gamma_1(A, p)$ as required.

For a field $k$ we let $sl_2(k)$ denote the 3-dimensional vector space of $2 \times 2$ trace zero matrices.

**Lemma 5.10.** Let $A$ be a Dedekind domain and let $p$ be a maximal ideal. Then for any $m \geq 1$ there are natural isomorphisms of groups

$$\frac{\hat{\Gamma}(A, p^m)}{\hat{\Gamma}(A, p^{m+1})} \cong sl_2(k(p)) \cong \frac{\Gamma(A, p^m)}{\Gamma(A, p^{m+1})}$$

**Proof.** From the definitions of $\Gamma(A, p^m)$ and $\hat{\Gamma}(A, p^m)$ we have

$$\frac{\hat{\Gamma}(A, p^m)}{\hat{\Gamma}(A, p^{m+1})} \cong \text{Ker}(\text{SL}_2(A/p^{m+1}) \to \text{SL}_2(A/p^m)) \cong \frac{\Gamma(A, p^m)}{\Gamma(A, p^{m+1})}.$$

Let $\pi \in p \setminus p^2$. For any $n \geq 1$, the group $p^n/p^{n+1}$ is a 1-dimensional $k(p)$-vector spaces with basis $\{\pi^n + p^{n+1}\}$.

The required isomorphism

$$sl_2(k(p)) \to \text{Ker}(\text{SL}_2(A/p^{n+1}) \to \text{SL}_2(A/p^n))$$

is then the map

$$\begin{bmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{bmatrix} \mapsto \begin{bmatrix} 1 + a\pi^n & b\pi^n \\ c\pi^n & 1 + d\pi^n \end{bmatrix}$$

where $a, b, c, d \in A$ map to $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in k(p)$.

**Corollary 5.11.** Let $A$ be a Dedekind domain and let $p$ be a maximal ideal. Suppose that $k(p)$ is a finite field with $q$ elements. Then

$$[\hat{\Gamma}(A, p) : \hat{\Gamma}(A, p^m)] = q^{3(n-1)} = [\Gamma(A, p) : \Gamma(A, p^n)]$$

for all $n \geq 1$.

**Lemma 5.12.** Suppose that $I$ and $J$ are comaximal ideals in $A$; i.e. $I + J = A$. Then the composite map $\hat{\Gamma}(A, I) \to \text{SL}_2(A) \to \text{SL}_2(A/J)$ is surjective.
PROOF. By the Chinese Remainder Theorem the map $A/IJ \to (A/I) \times (A/J)$, $a \mapsto (a + I, a + J)$ is an isomorphism of rings. It follows that the map

$$\text{SL}_2(A/IJ) \to \text{SL}_2(A/I) \times \text{SL}_2(A/J), \quad X \mod IJ \mapsto (X \mod I, X \mod J)$$

is an isomorphism of groups and hence that

$$\text{SL}_2(A) \to \text{SL}_2(A/I) \times \text{SL}_2(A/J), \quad X \mapsto (X \mod I, X \mod J)$$

is a surjective group homomorphism. This implies the statement of the Lemma. \qed

**Lemma 5.13.** Suppose that $k(p)$ is a finite field with $q$ elements. We have

$$[\text{SL}_2(A) : \Gamma(A, p)] = q(q^2 - 1) = [\text{SL}_2(A) : \tilde{\Gamma}(A, p)]$$

**Proof.** The first equality follows from the isomorphism

$$\frac{\text{SL}_2(A)}{\Gamma(A, p)} \cong \text{SL}_2(k(p)).$$

For the second inequality, denote by $C$ the image of the map

$$\tilde{\Gamma}(A, p) \to \text{SL}_2(A) \to \text{SL}_2(A/p^2).$$

Then $C$ fits into a short exact sequence

$$1 \to W \to C \to T \to 1$$

where

$$T = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \in \text{SL}_2(A/p^2) \mid a - 1, d - 1 \in p \right\} \cong k(p)$$

and

$$W = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \in \text{SL}_2(A/p^2) \right\} \cong A/p^2.$$"
Thus \([\Gamma, \Gamma]/\Gamma(A, I)\) surjects onto \(SL_2(A/J)\) and hence
\[
|SL_2(A/J)| \text{ divides } |([\Gamma, \Gamma] : \Gamma(A, I))|.
\]
It follows that
\[
|SL_2(A) : [\Gamma, \Gamma]| |SL_2(A/p^m)| = (q^2 - 1)q^{3m-2}.
\]
Since \([SL_2(A) : \Gamma] = q(q^2 - 1)\) by Lemma 5.13 it follows that \([\Gamma/\Gamma, \Gamma]\) divides \(q^{3m-3}\), and so is a power of \(p\) as claimed. \(\square\)

5.5. The second homology of congruence subgroups.

Lemma 5.15. Let \(k\) be a finite field of characteristic \(p\) and let \(M\) be an \(SL_2(k)\)-module. Then, for all \(i \geq 0\), the natural map
\[
H_i(B(k), M)_{(p)} \to H_i(SL_2(k), M)_{(p)}
\]
is an isomorphism.

Proof. As in the proof of [3 Cor 3.10.2]. \(\square\)

Proposition 5.16. Let \(A\) be a ring of \(S\)-integers in a global field \(K\) where \(|S| \geq 2\). Let \(p\) be a nonzero prime ideal and let \(p > 0\) be the characteristic of the residue field \(k(p)\). Suppose that \(p^m = xa\) for some \(m \geq 1\), \(x \in A\). Suppose further that there exist \(\lambda \in A[1/x]^\times\) such that \(\lambda^2 - 1 \in A[1/x]^\times\) also.

Then the natural maps
\[
t_1 : H_2(\Gamma_0(A, p), \mathbb{Z}) \to H_2(SL_2(A), \mathbb{Z})
\]
and
\[
t_2 : H_2(\Gamma_0(A, p), \mathbb{Z}) \to H_2(H(p), \mathbb{Z})
\]
are surjective.

Proof. Let \(k = k(p)\). There is a commutative diagram of group extensions
\[
\begin{array}{cccccc}
1 & \longrightarrow & \Gamma(A, p) & \longrightarrow & \Gamma_0(A, p) & \longrightarrow & B(k) & \longrightarrow & 1 \\
\downarrow{id} & & \downarrow{t_0} & & \downarrow{t_1} & & \downarrow{t} \\
1 & \longrightarrow & \Gamma(A, p) & \longrightarrow & SL_2(A) & \longrightarrow & SL_2(k) & \longrightarrow & 1 \\
\end{array}
\]
and (using Lemma 5.5)
\[
\begin{array}{cccccc}
1 & \longrightarrow & \Gamma(A, p) & \longrightarrow & \Gamma_0(A, p) & \longrightarrow & B'(k) & \longrightarrow & 1 \\
\downarrow{id} & & \downarrow{t_2} & & \downarrow{t} & & \downarrow{\gamma_{x,1}} \\
1 & \longrightarrow & \Gamma(A, p) & \longrightarrow & H(p) & \longrightarrow & SL_2(k) & \longrightarrow & 1 \\
\end{array}
\]
We give the argument for \(t_1\). The analogous argument for \(t_2\) is achieved by replacing \(B(k)\) with \(B'(k)\).

The top group extension gives rise to a spectral sequence
\[
E_{i,j}^2(\Gamma_0(A, p)) = H_i(B(k), H_j(\Gamma(A, p), \mathbb{Z})) \Rightarrow H_{i+j}(\Gamma_0(A, p), \mathbb{Z})
\]
and the lower one gives rise to the spectral sequence
\[
E_{i,j}^2(SL_2(A)) = H_i(SL_2(k), H_j(\Gamma(A, p), \mathbb{Z})) \Rightarrow H_{i+j}(SL_2(A), \mathbb{Z}).
\]
The map of extensions induces a natural map of spectral sequences compatible with the map \(t_1\) on abutments.
For $H = \Gamma_0(A, p)$ or $\text{SL}_2(A)$, the image of the edge homomorphism $E^\infty_{0,j}(H) \to H_j(H, \mathbb{Z})$ is equal to the image of $H_j(\Gamma(A, p), \mathbb{Z}) \to H_j(H, \mathbb{Z})$. Thus, comparing the $E^\infty$-terms of total degree 2, we obtain a commutative diagram of the form

$$
\begin{array}{c}
\text{H}_2(\Gamma(A, p), \mathbb{Z}) \longrightarrow \text{H}_2(\Gamma_0(A, p), \mathbb{Z}) \longrightarrow \text{C}(\Gamma_0(A, p)) \longrightarrow 0 \\
\text{id} \hspace{1cm} \text{i} \hspace{1cm} \text{id} \\
\text{H}_2(\Gamma(A, p), \mathbb{Z}) \longrightarrow \text{H}_2(\text{SL}_2(A), \mathbb{Z}) \longrightarrow \text{C}(\text{SL}_2(A)) \longrightarrow 0
\end{array}
$$

where, for $H = \Gamma_0(A, p)$ or $\text{SL}_2(A)$, $C(H)$ is a group fitting into an exact sequence

$$0 \to E^\infty_{1,1}(H) \to C(H) \to E^\infty_{2,0}(H) \to 0.$$

Since $H_1(\Gamma(A, p), \mathbb{Z})$ is a finite abelian $p$-group, it follows from Lemma 5.15 that the natural maps

$$E^2_{1,1}(\Gamma_0(A, p)) = \text{H}_1(\mathbb{B}(k), \text{H}_1(\Gamma(A, p), \mathbb{Z})) \to \text{H}_1(\text{SL}_2(k), \text{H}_1(\Gamma(A, p), \mathbb{Z})) = E^2_{1,1}(\text{SL}_2(A))$$

are all isomorphisms.

Since, furthermore, all these groups $E^2_{i,1}$ are finite abelian $p$-groups, it follows that the differentials

$$d^2_{i,0} : E^2_{i,0}(H) \to E^2_{i-1,1}(H)$$

factor through $E^2_{i,0}(H)_{(p)}$, for $H = \Gamma_0(A, p)$ or $\text{SL}_2(A)$.

By Lemma 5.15 again, we have natural isomorphisms

$$E^2_{i,0}(\Gamma_0(A, p))_{(p)} = \text{H}_1(\mathbb{B}(k), \mathbb{Z})_{(p)} \cong \text{H}_1(\text{SL}_2(k), \mathbb{Z})_{(p)} = E^2_{i,0}(\text{SL}_2(A))_{(p)}.$$

Thus, we have

$$E^\infty_{1,1}(\Gamma_0(A, p)) \cong E^\infty_{1,1}(\text{SL}_2(A))$$

since

$$E^\infty_{1,1}(H) = \text{Coker}(d^2 : E^2_{3,0}(H)_{(p)} \to E^2_{1,1}(H))$$

when $H = \Gamma_0(A, p)$ or $\text{SL}_2(A)$.

Finally, for $H = \Gamma_0(A, p)$ or $\text{SL}_2(A)$, we have

$$E^\infty_{2,0}(H) = \text{Ker}(d^2 : E^2_{2,0}(H) \to E^2_{0,1}(H)).$$

But straightforward calculations (see, for example, [3, Section 3]) show that

$$E^2_{2,0}(\Gamma_0(A, p)) = \text{H}_2(\mathbb{B}(k), \mathbb{Z}) = \text{H}_2(\mathbb{B}(k), \mathbb{Z})_{(p)} = \text{H}_2(\text{SL}_2(k), \mathbb{Z})_{(p)} = \text{H}_2(\text{SL}_2(k), \mathbb{Z}) = E^2_{2,0}(\text{SL}_2(A)).$$

Thus

$$E^\infty_{2,0}(\Gamma_0(A, p)) \cong E^\infty_{2,0}(\text{SL}_2(A)).$$

Hence the map $C(\Gamma_0(A, p)) \to C(\text{SL}_2(A))$ is an isomorphism, and the result follows. \hfill \Box

### 5.6. An exact sequence for the second homology of $S$-integers.

Let $K$ be a global field and let $S \subset T$ be nonempty sets of primes of $K$ containing the infinite primes. Then there is a natural short exact sequence

$$0 \longrightarrow K_2(\mathcal{O}_S) \longrightarrow K_2(\mathcal{O}_T) \longrightarrow \sum_{p \in T \setminus S} k(p)^\times \longrightarrow 0.$$

In this section, we demonstrate an analogous exact sequence for $H_2(\text{SL}_2(\mathcal{O}_S), \mathbb{Z})$, at least when $S$ is sufficiently large.
Theorem 5.17. Let $A$ be a ring of $S$-integers in a global field $K$ where $|S| \geq 2$. Let $\mathfrak{p}$ be a nonzero prime ideal and let $p > 0$ be the characteristic of the residue field $k(\mathfrak{p})$.

Suppose also that there exists $\lambda \in A^\times$ such that $\lambda^2 - 1 \in A^\times$ also.

Let $x \in A$ and $m \geq 1$ such that $\mathfrak{p}^m = xA$.

Then there is a natural exact sequence

$$
\xymatrix{
\text{H}_2(\text{SL}_2(A), \mathbb{Z}) \ar[r] & \text{H}_2(\text{SL}_2(A[1/x]), \mathbb{Z}) \ar[r]^\delta & \text{H}_1(k(\mathfrak{p})^\times, \mathbb{Z}) \ar[r] & 0
}
$$

Here the map $\delta$ fits into a commutative diagram

$$
\xymatrix{
\text{H}_2(\text{SL}_2(A[1/x]), \mathbb{Z}) \ar[r]^\delta \ar[d] & \text{H}_1(k(\mathfrak{p})^\times, \mathbb{Z}) \ar[d] \\
\text{H}_2(\text{SL}(K), \mathbb{Z}) = K^M_2(K) \ar[r]^{\tau_\mathfrak{p}} & k(\mathfrak{p})^\times
}
$$

where $\tau_\mathfrak{p} : K^M_2(K) \to k(\mathfrak{p})^\times$ is the tame symbol

$$
\tau_\mathfrak{p}([x, y]) = (-1)^{v(x)v(y)}x^{v(x)}y^{v(y)} \pmod{\mathfrak{p}} \in k(\mathfrak{p})^\times.
$$

Proof. By Proposition 5.16 the map $\tau_2 : \text{H}_2(\Gamma_0(A, \mathfrak{p}), \mathbb{Z}) \to \text{H}_2(H(\mathfrak{p}), \mathbb{Z})$ is surjective.

Thus the Mayer-Vietoris sequence yields the exact sequence

$$
\xymatrix{
\text{H}_2(\text{SL}_2(A), \mathbb{Z}) \ar[r] & \text{H}_2(\text{SL}_2(A[1/x]), \mathbb{Z}) \ar[r]^\delta & \text{H}_1(\Gamma_0(A, \mathfrak{p}), \mathbb{Z}) \ar[r] & 0
}
$$

The remaining statements of the theorem follow from Proposition 5.6 and Proposition 5.9. □

Remark 5.18. In this proof, the hypothesis that $\lambda, \lambda^2 - 1 \in A^\times$ is needed so that Proposition 5.9 is validly applied.

The first part of the proof only requires the weaker condition that $\lambda, \lambda^2 - 1 \in A[1/x]^\times$. For example, taking $A = \mathbb{Z}[1/3], \mathfrak{p} = 2A, x = 2$ and $\lambda = 3$, we obtain an exact sequence

$$
\xymatrix{
\text{H}_2(\text{SL}_2(\mathbb{Z}[1/3]), \mathbb{Z}) \ar[r] & \text{H}_2(\text{SL}_2(\mathbb{Z}[1/6]), \mathbb{Z}) \ar[r] & \text{H}_1(\Gamma_0(\mathbb{Z}[1/3], 2), \mathbb{Z}) \ar[r] & 0
}
$$

In this sequence,

$$
\text{H}_2(\text{SL}_2(\mathbb{Z}[1/3]), \mathbb{Z}) \cong \mathbb{Z} \text{ and } \text{H}_2(\text{SL}_2(\mathbb{Z}[1/6]), \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/2
$$

by the calculations of Adem-Naffah, [1], and Tuan-Ellis, [19].

Thus

$$
\text{H}_1(\Gamma_0(\mathbb{Z}[1/3], 2), \mathbb{Z}) \neq 0
$$

while $\text{H}_1(k(2)^\times, \mathbb{Z}) = \text{H}_1(\Gamma^2, \mathbb{Z}) = 0$, so that the conclusion of Proposition 5.9 is false in the case $A = \mathbb{Z}[1/3]$ and $\mathfrak{p} = 2A$.

Remark 5.19. Theorem 5.17 is not valid for more general Dedekind Domains $A$, even when there is a unit $\lambda$ such that $\lambda^2 - 1$ is also a unit.

For example, let $k$ be an infinite field and let $K = k(t), A = k[t], \mathfrak{p} = tA, x = t$. It is shown in [2] Theorem 4.1] that the cokernel of the natural map

$$
\text{H}_2(\text{SL}_2(A), \mathbb{Z}) \to \text{H}_2(\text{SL}_2(A[1/t]), \mathbb{Z})
$$

is isomorphic to $K^M_1(k)$, the first Milnor-Witt $K$-group of the residue field $k$. It seems reasonable to suppose that this statement should be true for a larger class of Dedekind Domains.

Note that there is a natural surjective map $K^M_1(k) \to K^M_1(k) \cong k^\times$ which is an isomorphism when the field $k$ is finite. However, in general, the kernel of this homomorphism is $\tilde{I}^2(k)$ (see section 3.4 above).
Corollary 5.20. Let $K$ be a global field. Let $S$ be a set of primes of $K$ containing the infinite primes. Suppose that $|S| \geq 2$ and that $O_S$ contains a unit $\lambda$ such that $\lambda^2 - 1$ is also a unit. Let $T$ be any set of primes containing $S$. Then there is a natural exact sequence
\[ H_2(SL_2(O_S), \mathbb{Z}) \to H_2(SL_2(O_T), \mathbb{Z}) \to \bigoplus_{p \in T \setminus S} k(p)^\times \to 0. \]

Proof. We proceed by induction on $|T \setminus S|$. The case $|T \setminus S| = 1$ is just Theorem 5.17. The inductive step follows immediately by applying the snake lemma to the commutative diagram with exact rows
\[
\begin{array}{c}
H_2(SL_2(O_{T'}), \mathbb{Z}) \\ \\
\downarrow \\ 0 \\
\downarrow \\
\bigoplus_{p \in T'} k(p)^\times \\
\downarrow \\
k(q)^\times \\
\downarrow \\
0
\end{array}
\]
where $q$ is any element in $T \setminus S$ and $T' = T \setminus \{q\}$. □

Corollary 5.21. Let $K$ be a global field. Let $S$ be a set of primes of $K$ containing the infinite primes. Suppose that $|S| \geq 2$ and that $O_S$ contains a unit $\lambda$ such that $\lambda^2 - 1$ is also a unit. Then there is a natural exact sequence
\[ H_2(SL_2(O_S), \mathbb{Z}) \to H_2(SL_2(K), \mathbb{Z}) \to \bigoplus_{p \notin S} k(p)^\times \to 0. \]

Proof. Since
\[ K = \lim_{S \subseteq T} O_T \]
this follows from Corollary 5.20 by taking (co)limits. □

6. The main theorem

Let $K$ be a global field. In this section, we use the results above to prove our main theorem which identifies $H_2(SL_2(O_S), \mathbb{Z})$ with a certain subgroup of $K_2(2, K)$, which we now describe. For a prime $p$ of $K$, we denote by $T_p$ the composite
\[ K_2(2, K) \to K_2^M(K) \to k(p)^\times \]
where $\tau_p$ is the tame symbol, as above.

When $S$ is a nonempty set of primes of $K$ containing the infinite primes, we set
\[ \tilde{K}_2(2, O_S) := \text{Ker}(K_2(2, K) \to \bigoplus_{p \notin S} k(p)^\times). \]

We begin by noting that this group is closely related to $K_2(O_S)$:

Lemma 6.1. For any global field $K$ and for any nonempty set $S$ of primes which contains the infinite primes there is a natural exact sequence
\[ 0 \to I^3(K) \to \tilde{K}_2(2, O_S) \to K_2(O_S) \to 0. \]

In particular, $\tilde{K}_2(2, O_S) \cong K_2(O_S)$ if $K$ is of positive characteristic or is a totally imaginary number field.
Proof. Apply the snake lemma and Corollary 3.10(1) to the map of short exact sequences

\[0 \rightarrow \tilde{K}_2(2, O_S) \rightarrow K_2(2, K) \rightarrow \bigoplus_{p \in S} k(p)^	imes \rightarrow 0\]

\[0 \rightarrow K_2(O_S) \rightarrow K_2^M(K) \rightarrow \bigoplus_{p \in S} k(p)^	imes \rightarrow 0.\]

The second statement follows from the fact that, for a global field \(K\), \(I^3(K) \cong \mathbb{Z}^r(K)\) where \(r(K)\) is the number of distinct real embeddings of \(K\).

Example 6.2. Consider the global field \(K = \mathbb{Q}\).

For any set \(S\) of prime numbers, we will set

\[\mathbb{Z}_S := \mathbb{Z}[[1/p]_{p \in S}] = O_{S \cup \{\infty\}}.\]

The kernel of the surjective map

\[K_2(2, \mathbb{Q}) \rightarrow \bigoplus_{p \in S} \mathbb{F}_p^{\times}\]

is an infinite cyclic direct summand with generator \(c(-1, -1)\).

It follows that for any set \(S\) of prime numbers

\[\tilde{K}_2(2, \mathbb{Z}_S) \cong \mathbb{Z} \oplus \left(\bigoplus_{p \in S} \mathbb{F}_p^{\times}\right).\]

More generally, we have the following description of the groups \(\tilde{K}_2(2, O_S)\):

For a global field \(K\), let \(\Omega\) be the set of real embeddings of \(K\). For \(\sigma \in \Omega\), there is a corresponding homomorphism

\[T_\sigma : K_2(K) \rightarrow \mu_2, \quad \{a, b\} \mapsto \begin{cases} -1, & \text{if } \text{sgn}(\sigma(a)), \text{sgn}(\sigma(b)) < 0 \\ 1, & \text{otherwise} \end{cases}\]

Let

\[K_2(K)_+ := \text{Ker}(\bigoplus_{\sigma \in \Omega} T_\sigma : K_2(K) \rightarrow \mu_2^\Omega)\]

and let \(K_2(O_S)_+ = K_2(O_S) \cap K_2(K)_+\).

Lemma 6.3. Let \(K\) be a global field. Let \(S\) be a nonempty set of primes of \(K\) including the infinite primes. Then

\[\tilde{K}_2(2, O_S) \cong K_2(O_S)_+ \oplus \mathbb{Z}^\Omega.\]

Proof. By classical quadratic form theory, the group \(I^\bullet(\mathbb{R})\) is infinite cyclic with generator \(\langle -1 \rangle^n = (-2)^n \langle -1 \rangle\).

It is shown in [2] that for a global field \(K\) the natural surjective map

\[K_2(2, K) \rightarrow I^2(\mathbb{R})^\Omega \cong \mathbb{Z}^\Omega, \quad (c(u, v) \mapsto \langle \langle \text{sgn}(\sigma(u)) \rangle \langle \text{sgn}(\sigma(u)) \rangle \rangle)_{\sigma \in \Omega}\]

has kernel isomorphic to \(K_2(K)_+\), where this isomorphism is realised by restricting the natural map \(K_2(2, K) \rightarrow K_2(K)\). Furthermore, the composite map \(I^3(K) \rightarrow K_2(2, K) \rightarrow I^2(\mathbb{R})^\Omega\) induces an isomorphism

\[I^3(K) \cong I^3(\mathbb{R})^\Omega = 2 \cdot (I^2(\mathbb{R})^\Omega).\]

Since \(I^3(K) \subset \tilde{K}_2(2, O_S)\), the image of the map

\[\tilde{K}_2(2, O_S) \rightarrow I^2(\mathbb{R})^\Omega \equiv \mathbb{Z}^\Omega\]

contains a full sublattice.

On the other hand, the kernel of this map is isomorphic – via the map \(K_2(2, K) \rightarrow K_2(K)\) – to \(K_2(O_S) \cap K_2(K)_+\).
It is natural to ask, of course, about the relation between $\tilde{K}_2(2, O_S)$ and $K_2(2, O_S)$.

It is a theorem of van der Kallen ([20]) that when $K$ is a global field and when $S$ contains all infinite places and $|S| \geq 2$ then the stabilization map

$$K_2(2, O_S) \to K_2(O_S)$$

is always surjective.

We deduce:

Lemma 6.4. Let $K$ be a global field and let $S$ be a nonempty set of primes of $K$ containing the infinite primes. Then the image of the natural map $K_2(2, O_S) \to K_2(2, K) \cong K_2(2, K)$ lies in $\tilde{K}_2(2, O_S)$.

Furthermore, when $|S| \geq 2$, and when there exist units $u_\sigma \in O_S^\times$, $\sigma \in \Omega$ satisfying

$$\text{sgn}(\tau(u_\sigma)) = (-1)^{\delta_{\sigma,r}},$$

the resulting natural map $K_2(2, O_S) \to \tilde{K}_2(2, O_S)$ is surjective; i.e. the image of the map $K_2(2, O_S) \to K_2(2, K)$ is precisely $\tilde{K}_2(2, O_S)$.

Proof. The diagram

$$
\begin{array}{cccc}
K_2(2, O_S) & \longrightarrow & K_2(2, K) & \longrightarrow \bigoplus_{p \in S} k(p)^\times \\
\downarrow & & \downarrow & \\
0 & \longrightarrow & K_2(O_S) & \longrightarrow \tilde{K}_2^S(K) \longrightarrow \bigoplus_{p \in S} k(p)^\times \longrightarrow 0
\end{array}
$$

commutes.

Our hypothesis on units ensures that the map

$$\tilde{K}_2(2, O_S) \to K_2(2, K) \to \Omega^{2}(\mathbb{R}^\Omega)$$

is surjective.

Since we also have

$$K_2(O_S) \subset K_2(2, O_S) \subset \tilde{K}_2(2, O_S)$$

by the result of van der Kallen, the second statement follows.

One would expect that the resulting map $K_2(2, O_S) \to \tilde{K}_2(2, O_S)$ is very often an isomorphism. It seems to be difficult, however, to prove this in any given instance. In the case $K = \mathbb{Q}$, Jun Morita, [13] Theorems 2,3] has proved:

Theorem 6.5. Let $S$ be any of the following sets of primes numbers:

$S = \{p_1, \ldots, p_n\}$, the set of the first $n$ successive prime numbers, or $S$ is one of $\{2, 5\}$, $\{2, 3, 7\}$, $\{2, 3, 11\}$, $\{2, 3, 5, 11\}$, $\{2, 3, 13\}$, $\{2, 3, 7, 13\}$, $\{2, 3, 17\}$, $\{2, 3, 5, 19\}$.

Then $K_2(2, \mathbb{Z}_S)$ is central in $\text{St}(2, \mathbb{Z}_S)$ and the natural map

$$K_2(2, \mathbb{Z}_S) \to \tilde{K}_2(2, \mathbb{Z}_S) \cong \mathbb{Z} \oplus \left(\bigoplus_{p \in S} \mathbb{F}_p^\times\right)$$

is an isomorphism.

Lemma 6.6. Let $K$ be a global field and let $S$ be a nonempty set of primes of $K$ containing the infinite primes. Then the image of the natural map

$$H_2(\text{SL}_2(O_S), \mathbb{Z}) \to H_2(\text{SL}_2(K), \mathbb{Z}) \longrightarrow K_2(2, K)$$

lies in $\tilde{K}_2(2, O_S)$. 


Proof. The diagram

\[
\begin{array}{cccccc}
H_2(\text{SL}_2(\mathcal{O}_S), \mathbb{Z}) & \longrightarrow & H_2(\text{SL}(\mathcal{O}_S), \mathbb{Z}) & \longrightarrow & \oplus_{p \in S} k(p)^\times & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \text{id} & & \\
H_2(\text{SL}_2(K), \mathbb{Z}) & \longrightarrow & H_2(\text{SL}(K), \mathbb{Z}) & \longrightarrow & \oplus_{p \in S} k(p)^\times & \longrightarrow & 0 \\
\cong & & \cong & & & & \\
K_2(2, K) & \longrightarrow & K_2^M(K)
\end{array}
\]

commutes. But \(H_2(\text{SL}(\mathcal{O}_S), \mathbb{Z}) \cong K_2(\mathcal{O}_S)\) and the natural map \(K_2(\mathcal{O}_S) \rightarrow K_2^M(K)\) induces an isomorphism

\[
K_2(\mathcal{O}_S) \cong \text{Ker}(K_2^M(k) \rightarrow \oplus_{p \in S} k(p)^\times).
\]

\[\square\]

If \(K\) is a global field and and if \(S\) is a nonempty set of primes containing the infinite primes we will let

\[
\mathcal{K}_S := \text{Ker}(H_2(\text{SL}_2(\mathcal{O}_S), \mathbb{Z}) \rightarrow H_2(\text{SL}_2(K), \mathbb{Z})).
\]

Note that

\[
\mathcal{K}_S := \text{Ker}(H_2(\text{SL}_2(\mathcal{O}_S), \mathbb{Z}) \rightarrow \tilde{K}_2(2, \mathcal{O}_S))
\]

since \(\tilde{K}_2(2, \mathcal{O}_S) \subset K_2(2, K) \cong H_2(\text{SL}_2(K), \mathbb{Z})\).

Remark 6.7. In general, the kernels \(\mathcal{K}_S\) can be arbitrarily large, even in the case \(K = \mathbb{Q}\):

The calculations of Adem-Naffah, [H], show that the ranks of the groups \(H_2(\text{SL}_2(\mathbb{Z}[1/p]), \mathbb{Z})\) grow with linearly \(p\) when \(p\) is a prime number. On the other hand, the rank of \(H_2(\text{SL}_2(\mathbb{Q}), \mathbb{Z})\) is 1.

Lemma 6.8. Let \(K\) be a global field. Let \(S\) be a set of primes of \(K\) containing the infinite primes. Suppose that \(|S| \geq 2\) and that \(\mathcal{O}_S\) contains a unit \(\lambda\) such that \(\lambda^2 - 1\) is also a unit.

Then

1. The natural map

\[
H_2(\text{SL}_2(\mathcal{O}_S), \mathbb{Z}) \rightarrow \tilde{K}_2(2, \mathcal{O}_S)
\]

is surjective.

2. If \(T \supseteq S\), then the natural map \(\mathcal{K}_S \rightarrow \mathcal{K}_T\) is surjective.

Proof.

1. By Corollary 5.21, we have a commutative diagram with exact rows

\[
\begin{array}{cccccc}
H_2(\text{SL}_2(\mathcal{O}_S), \mathbb{Z}) & \longrightarrow & H_2(\text{SL}_2(K), \mathbb{Z}) & \longrightarrow & \oplus_{p \in S} k(p)^\times & \longrightarrow & 0 \\
\downarrow & & \downarrow \text{id} & & \downarrow \text{id} & & \\
0 & \longrightarrow & \tilde{K}_2(2, \mathcal{O}_S) & \longrightarrow & K_2(2, K) & \longrightarrow & \oplus_{p \in S} k(p)^\times & \longrightarrow & 0.
\end{array}
\]

2. Apply the snake lemma to the diagram

\[
\begin{array}{cccccc}
H_2(\text{SL}_2(\mathcal{O}_S), \mathbb{Z}) & \longrightarrow & H_2(\text{SL}_2(\mathcal{O}_T), \mathbb{Z}) & \longrightarrow & \oplus_{p \in T \setminus S} k(p)^\times & \longrightarrow & 0 \\
\downarrow & & \downarrow \text{id} & & \downarrow \text{id} & & \\
0 & \longrightarrow & \tilde{K}_2(2, \mathcal{O}_S) & \longrightarrow & \tilde{K}_2(2, \mathcal{O}_T) & \longrightarrow & \oplus_{p \in T \setminus S} k(p)^\times & \longrightarrow & 0.
\end{array}
\]

\[\square\]
REMARK 6.9. Note, on the other hand, that the map
\[ 0 = H_2(SL_2(\mathbb{Z}), \mathbb{Z}) \to \tilde{K}_2(2, \mathbb{Z}) \cong K_2(2, \mathbb{Z}) \cong \mathbb{Z} \]
cannot be surjective.

**Theorem 6.10.** Let \( K \) be a global field.

(1) There exists a finite set \( S \) of primes of \( K \) satisfying
   (a) \( S \) contains all infinite primes and \( |S| \geq 2 \).
   (b) There exists a unit \( \lambda \) of \( \mathcal{O}_S \) such that \( \lambda^2 - 1 \) is also a unit.
   (c) The natural map \( H_2(SL_2(\mathcal{O}_S), \mathbb{Z}) \to \tilde{K}_2(2, \mathcal{O}_S) \) is an isomorphism.

(2) If \( T \) is any set of primes containing \( S \) then \( H_2(SL_2(\mathcal{O}_T), \mathbb{Z}) \cong \tilde{K}_2(2, \mathcal{O}_T) \); i.e. there is a natural short exact sequence
\[ 0 \to H_2(SL_2(\mathcal{O}_T), \mathbb{Z}) \to K_2(2, \mathcal{O}_T) \to \bigoplus_{p \in T} k(p)^\times \to 0. \]

**Proof.**

(1) Let \( S_0 \) be any set of primes satisfying (a) and (b). Since \( H_2(SL_2(\mathcal{O}_S), \mathbb{Z}) \) is a finitely-generated abelian group, so also is \( K_{S_0} \). Since \( H_2(SL_2(K), \mathbb{Z}) = \lim_T H_2(SL_2(\mathcal{O}_T), \mathbb{Z}) \), the limit being taken over finite sets \( T \) of primes, it follows that there is a finite set of primes \( S \) containing \( S_0 \) for which \( K_{S_0} = 0 \) and hence that \( H_2(SL_2(\mathcal{O}_S), \mathbb{Z}) \cong \tilde{K}_2(2, \mathcal{O}_S) \) as required.

(2) This is immediate from Lemma 6.8. \( \square \)

**Lemma 6.11.** Let \( K = \mathbb{Q} \) and let \( S = \{2, 3, \infty\} \). Then \( S \) satisfies conditions (a)-(c) of Theorem 6.10 (1).

**Proof.** The set \( S = \{2, 3, \infty\} \) clearly satisfies conditions (a) and (b).

Observe that
\[ \mathcal{O}_S = \mathbb{Z}_{\{2,3\}} = \mathbb{Z} \left[ \frac{1}{2}, \frac{1}{3} \right] = \mathbb{Z} \left[ \frac{1}{6} \right]. \]

By Lemma 6.8, the natural map
\[ H_2(SL_2(\mathbb{Z}[1/6]), \mathbb{Z}) \to \tilde{K}_2(2, \mathbb{Z}[1/6]) \cong \mathbb{Z} \oplus \mathbb{F}_3^\times \]
(see Example 6.2) is surjective.

On the other hand, the calculations of Tuan and Ellis, [19], show that
\[ H_2(SL_2(\mathbb{Z}[1/6]), \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}. \]

It follows that the natural map above is an isomorphism. \( \square \)

In view of Theorem 6.10 (2) and Example 6.2, we immediately deduce:

**Theorem 6.12.** Let \( T \) be any set of prime numbers containing 2, 3. Then there is an isomorphism
\[ H_2(SL_2(\mathbb{Z}[1/6]), \mathbb{Z}) \cong \mathbb{Z} \oplus \bigoplus_{p \in T} \mathbb{F}_p^\times. \]

In particular, if \( m \in \mathbb{Z} \) and if \( 6|m \) then
\[ H_2(SL_2(\mathbb{Z}[1/m]), \mathbb{Z}) \cong \mathbb{Z} \oplus \bigoplus_{p|m} \mathbb{F}_p^\times. \]
Combining this with Morita’s Theorem (6.5) we deduce:

**Proposition 6.13.** Let \( S \) be any of the following sets of primes numbers: 
\( S = \{ p_1, \ldots, p_n \} \), the set of the first \( n \) successive prime numbers, or \( S \) is one of \( \{2, 3, 7\}, \{2, 3, 11\}, \{2, 3, 5, 11\}, \{2, 3, 13\}, \{2, 3, 7, 13\}, \{2, 3, 17\}, \{2, 3, 5, 19\} \).

Then the natural map 
\[
H_2(\text{SL}_2(\mathbb{Z}_S), \mathbb{Z}) \to K_2(2, \mathbb{Z}_S)
\]

is an isomorphism and 
\[
1 \to K_2(2, \mathbb{Z}_S) \to \text{St}(2, \mathbb{Z}_S) \to \text{SL}_2(\mathbb{Z}_S) \to 1
\]
is the universal central extension of \( \text{SL}_2(\mathbb{Z}_S) \).

**Proof.** Since \( K_2(2, \mathbb{Z}_S) \) is central in \( \text{St}(2, \mathbb{Z}_S) \), there is a natural homomorphism \( H_2(\text{SL}_2(\mathbb{Z}_S), \mathbb{Z}) \to K_2(2, \mathbb{Z}_S) \) through which the map \( H_2(\text{SL}_2(\mathbb{Z}_S), \mathbb{Z}) \to K_2(2, \mathbb{Q}) \) factors.

Since \( H_2(\text{SL}_2(\mathbb{Z}_S), \mathbb{Z}) \) and \( K_2(2, \mathbb{Z}_S) \) are both isomorphic to \( \tilde{K}_2(2, \mathbb{Z}_S) \subset K_2(2, \mathbb{Q}) \), the result follows immediately. \( \square \)

7. Some 2-dimensional homology classes

In this section we construct explicit cycles in the bar resolution of \( \text{SL}_2(A) \) which represent homology classes in \( H_2(\text{SL}_2(A), \mathbb{Z}) \). We show that these classes map to the symbols \( (u, v) \in K_2(2, A) \), when \( A \) is a field.

7.1. The homology classes \( C(a, b) \). Let \( A \) be a commutative ring and let \( a \in A^\times \). We define the following elements of \( \text{SL}_2(A) \):

\[
w := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad G_a := \begin{bmatrix} 0 & -1 \\ 1 & a + a^{-1} \end{bmatrix}, \quad H_a := E_{21}(a) = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}.
\]

Note that 
\[
wG_a = \begin{bmatrix} 1 & a + a^{-1} \\ 0 & 1 \end{bmatrix} = E_{12}(a + a^{-1}).
\]

We also define 
\[
R_a := H_a G_a H_a^{-1} = H_a G_a H_{-a} = \begin{bmatrix} a & -1 \\ 0 & a^{-1} \end{bmatrix}.
\]

Thus, by definition, 
\[
H_a G_a = R_a H_a = \begin{bmatrix} 0 & -1 \\ 1 & a^{-1} \end{bmatrix}.
\]

Let 
\[
\Theta_a := [H_a]G_a - [R_a]H_a + [w^{-1}]wG_a \in \tilde{F}_2(\text{SL}_2(A)) = \tilde{F}_2.
\]

Then 
\[
d_2(\Theta_a) = [w^{-1}] + [wG_a] - [R_a] \in \tilde{F}_1.
\]

Now let \( a, b \in A^\times \). Then 
\[
d_2(\Theta_{ab} - \Theta_a - \Theta_b + \Theta_1) = ([R_a] + [R_b] - [R_{ab}]) + ([wG_{ab}] - [wG_a] - [wG_b] + [wG_1] - [R_1]).
\]

Now 
\[
[R_a] + [R_b] = [R_a R_b] + d_2([R_a]R_b)
\]

and 
\[
[R_{ab}] = [R_a R_b] + ([R_a R_b]^{-1}) - d_2\left([R_a R_b](R_a R_b)^{-1}(R_{ab})\right).
\]
Hence
\[ [R_a] + [R_b] - [R_{ab}] = -(R_aR_b)^{-1}R_{ab} + d_2([R_a] + [R_aR_b](R_aR_b)^{-1}(R_{ab})) \]
where
\[ (R_aR_b)^{-1}R_{ab} = \begin{bmatrix} 1 & (ab)^{-1}(a + b^{-1} - 1) \\ 0 & 1 \end{bmatrix} = E_{12}((ab)^{-1}(a + b^{-1} - 1)). \]

Putting this together, we deduce
\[
d_2(\Theta_{ab} - \Theta_a - \Theta_b + \Theta_1 - [R_a]R_b) - [R_aR_b](R_aR_b)^{-1}(R_{ab})) \]
\[ = [wG_{ab}] - [wG_a] - [wG_b] + [wG_1] - [R_1] + [(R_aR_b)^{-1}(R_{ab})] \]
\[ = [E_{12}(ab + (ab)^{-1})] - [E_{12}(a + a^{-1})] - [E_{12}(b + b^{-1})] \]
\[ + [E_{12}(2)] - [E_{12}(-1)] + [E_{12}((ab)^{-1}(a + b^{-1} - 1))]. \]

Now suppose that there exists \( \lambda \in A^\times \) such that \( \lambda^2 - 1 \in A^\times \). Let
\[ D(\lambda) := \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \in SL_2(A). \]
Recall that for any \( x \in A \)
\[ D(\lambda)E_{12}(x)D(\lambda)^{-1} = E_{12}(\lambda^2 x) \]
and hence for any \( x \in A \) we have
\[ E_{12}(x) = D(\lambda)E_{12}(x')D(\lambda)^{-1}E_{12}(x')^{-1} \]
\[ = D(\lambda)(E_{12}(x')(E_{12}(x')D(\lambda))^{-1} = [D(\lambda), E_{12}(x')]. \]

where \( x' := x/(\lambda^2 - 1) \).

Now if \( G \) is any group and if \( g, h \in G \) then
\[ D_2([(gh)(hg)^{-1}|hg] - [gh] + [h|g]) = [(gh)(hg)^{-1}] = [[g, h]]. \]

Thus, we define
\[ \Psi_x = \Psi_{x, \lambda} := [E_{12}(x)|E_{12}(x')D(\lambda)] - [D(\lambda)|E_{12}(x')] + [E_{12}(x')|D(\lambda)] \in \bar{F}_2. \]

By the preceding remarks, we have \( d_2(\Psi_{x, \lambda}) = [E_{12}(x)] \in \bar{F}_1 \) for any \( x \in A \).

From all of these calculations we deduce:

**Proposition 7.1.** Let \( A \) be a commutative ring. Suppose that there exists \( \lambda \in A^\times \) such that \( \lambda^2 - 1 \in A^\times \). Let \( a, b \in A^\times \). Then
\[ F(a, b)_\lambda := [R_a]R_b + [R_aR_b](R_aR_b)^{-1}(R_{ab})] + \Theta_a + \Theta_b - \Theta_{ab} - \Theta_1 \]
\[ + \Psi_{ab+(ab)^{-1}} - \Psi_{a+a^{-1}} - \Psi_{b+b^{-1}} + \Psi_2 - \Psi_1 + \Psi_{(ab)^{-1}(a+b^{-1} - 1)} \in \bar{F}_2 \]
is a cycle, representing a homology class \( C(a, b)_\lambda \in H_2(SL_2(A), \mathbb{Z}). \)

**Remark 7.2.** The cycles \( F(a, b)_\lambda \) are clearly functorial in the sense that if \( \psi : A \rightarrow B \) is a homomorphism of commutative rings and if \( a, b, \lambda, \lambda^2 - 1 \in A^\times \) then
\[ \psi_*(F(a, b)_\lambda) = F(\psi(a), \psi(b))_{\psi(\lambda)} \in \bar{F}_2(SL_2(B)). \]

**Remark 7.3.** More generally, suppose that \( \Lambda = (\lambda_1, \ldots, \lambda_n, b_1, \ldots, b_n) \in (A^\times)^n \times (A^n) \) satisfies
\[ \sum_{i=1}^n (\lambda_i^2 - 1)b_i = 1 \]
Then for any \( x \in A \)
\[
E_{12}(x) = \prod_i [D(\lambda_i), E_{12}(b_i x)].
\]
by the proof of Proposition \ref{2.2}.

Since
\[
\left[ \prod_{i=1}^n c_i \right] = \sum_{i=1}^n [c_i] - d_2 \left( \sum_{k=1}^{n-1} [c_k \cdots c_k c_{k+1}] \right)
\]
in \( \bar{F}_1(A) \), we can easily write down an element \( \Psi_{x,\lambda} \in \bar{F}_2(A) \) satisfying \( d_2(\Psi_{x,\lambda}) = [E_{12}(x)] \) and thus construct cycles \( F(a, b)_\lambda \).

**Remark 7.4.** Specialising to the case \( a = b = -1 \) we obtain:
\[
F(-1, -1)_{\lambda} = [R_{-1}^{-1}] + [E_{12}(2)] [E_{12}(-3)] + 2(\Theta_{-1} - \Theta_{1} + \Psi_{2} - \Psi_{-2}).
\]

As we will see, when \( A \) is a field with at least four elements, the homology class \( C(a, b)_\lambda \) does not depend on the choice of \( \lambda \). In fact, this is the case for many commutative rings. For example, we have:

**Lemma 7.5.** Let \( A \) be a commutative ring. Suppose there exists \( n \in \mathbb{Z} \) such that \( n, n^4 - 1 \in A^\times \). Then, for any \( a, b \in A^\times \), the homology class \( C(a, b)_\lambda \in H_2(SL_2(A), \mathbb{Z}) \) is independent of the choice of \( \lambda \).

**Proof.** Suppose that \( \lambda, \mu \in A^\times \) satisfy the condition \( \lambda^2 - 1, \mu^2 - 1 \in A^\times \).

Let \( a, b \in A^\times \). Note that \( F(a, b)_\lambda - F(a, b)_\mu \) is a sum or difference of terms of the form \( \Psi_{x,\lambda} - \Psi_{x,\mu} \), \( x \in A \). We will show that each such term is a boundary.

We begin by observing that, for any \( x \in A \), the elements \( \Psi_{x,\lambda} \) and \( \Psi_{x,\mu} \) lie in \( \bar{F}_2(B) \) where
\[
B := \left\{ \begin{bmatrix} u & y \\ 0 & u^{-1} \end{bmatrix} \in SL_2(A) \big| u \in A^\times \right\}
\]
is the subgroup of upper-triangular matrices in \( SL_2(A) \).

Note that
\[
d_2(\Psi_{x,\lambda} - \Psi_{x,\mu}) = [E_{12}(x)] - [E_{12}(x)] = 0
\]
so that \( \Psi_{x,\lambda} - \Psi_{x,\mu} \) represents a homology class in \( H_2(B, \mathbb{Z}) \). We will show that it represents the trivial class.

Let \( T := \{ D(u) \mid u \in A^\times \} \) be the group of diagonal matrices and let \( \pi : B \to T \) be the natural surjective homomorphism sending \( D(u)E_{12}(z) \) to \( D(u) \). Then
\[
U := \text{Ker}(\pi) = \{ E_{12}(y) \mid y \in A \}
\]
is the group of unipotent matrices.

We have \( T \cong A^\times \) via \( D(u) \leftrightarrow u \) and \( U \cong A \), via \( E_{12}(x) \leftrightarrow x \).

Note that
\[
\pi(\Psi_{x,\lambda}) = \pi([E_{12}(x)]E_{12}(x')D(\lambda)] - [D(\lambda)]E_{12}(x'))
\]
\[= [I]D(\lambda) - [D(\lambda)]I + [I]D(\lambda) \in \bar{F}_2(T).
\]

Since
\[
d_3([X]I[I]) = [I][I] - [I][X] \text{ and } d_3([I][I]X) = [X][I] - [I][I]
\]
it follows easily that \( \pi(\Psi_{x,\lambda} - \Psi_{x,\mu}) \in d_3(\bar{F}_3(T)) \). Thus \( \pi(\Psi_{x,\lambda} - \Psi_{x,\mu}) \) represents the trivial homology class in \( H_2(T, \mathbb{Z}) \).
To conclude, we will show that our hypotheses are enough to ensure that \( \pi \) induces an isomorphism \( H_2(B, \mathbb{Z}) \cong H_2(T, \mathbb{Z}) \).

We consider the Hochschild-Serre spectral sequence associated to the short exact sequence

\[ 1 \to U \to B \to T \to 1. \]

This has the form

\[ E^2_{i,j} = H_i(T, H_j(U, \mathbb{Z})) \Rightarrow H_{i+j}(B, \mathbb{Z}) \]

\( D(u) \in T \) acts by conjugation on \( U \cong A \) as multiplication by \( u^2 \). Thus the induced action of \( D(u) \) on \( H_2(U, \mathbb{Z}) \cong U \hat{\otimes} U \cong A \hat{\otimes} A \) is multiplication by \( u^4 \).

In particular, \( D(n) \) acts as multiplication by \( n^2 \) on \( H_1(U, \mathbb{Z}) \), and as multiplication by \( n^4 \) on \( H_2(U, \mathbb{Z}) \).

Since \( T \) is abelian, and since \( n^2 - 1, n^4 - 1 \) are units in \( A \), it follows from the “centre kills” argument that \( H_i(T, H_j(U, \mathbb{Z})) = 0 \) for \( 1 \leq j \leq 2 \).

Thus, from the spectral sequence, the map \( \pi \) induces an isomorphism \( H_n(B, \mathbb{Z}) \cong H_n(T, \mathbb{Z}) \) for \( n \leq 2 \).

**Remark 7.6.** Since \( 2^4 - 1 = 3 \cdot 5 \), the condition of the Lemma 7.5 is satisfied by any ring in which 2, 3 and 5 are units.

**7.2. A variation.** We describe a slightly more compact 2-cycle \( \tilde{F}(a, b)_\lambda \) in the case where \( a^2 - 1, b^2 - 1 \) and \( (ab)^2 - 1 \) are all units in \( A \).

Suppose that \( a \in A \) is a unit such that \( a^2 - 1 \in A^\times \) also. Let

\[ \tilde{H}_a = \left[ \begin{array}{cc} \frac{1}{1-a} & \frac{a}{1-a} \\ \frac{1}{1+a} & \frac{1}{1+a} \end{array} \right] \in \text{SL}_2(A). \]

Then

\[ \tilde{H}_a G_a \tilde{H}_a^{-1} = \left[ \begin{array}{cc} a & 0 \\ 0 & a^{-1} \end{array} \right] = D(a). \]

Thus if we let

\[ \bar{\Theta}_a := [\tilde{H}_a G_a] - [D(a)\tilde{H}_a] + [w^{-1}wG_a] \in \tilde{F}_2 \]

then

\[ d_2(\bar{\Theta}_a) = [w^{-1}] + [wG_a] - [D(a)]. \]

If \( a^2 - 1, b^2 - 1, (ab)^2 - 1 \in A^\times \) then

\[ d_2(\bar{\Theta}_a + \bar{\Theta}_b - \bar{\Theta}_{ab} - \Theta_1) = [D(ab)] - [D(a)] - [D(b)] \]

\[ + [E_{12}(a + a^{-1})] + [E_{12}(b + b^{-1})] - [E_{12}(ab + (ab)^{-1})] + [E_{12}(-1)] - [E_{12}(2)] \]

\[ = d_2([-D(a)D(b)] + \Psi_{a+a^{-1}} + \Psi_{b+b^{-1}} - \Psi_{ab+(ab)^{-1}} + \Psi_{-1} - \Psi_2). \]

We deduce:

**Proposition 7.7.** If \( a, b, \lambda, a^2 - 1, b^2 - 1, (ab)^2 - 1, \lambda^2 - 1 \in A^\times \) then

\[ \tilde{F}(a, b)_\lambda := [D(a)]D(b)] + \bar{\Theta}_a + \bar{\Theta}_b - \bar{\Theta}_{ab} - \Theta_1 + \Psi_{ab+(ab)^{-1}} - \Psi_{a+a^{-1}} - \Psi_{b+b^{-1}} + \Psi_2 - \Psi_{-1} \]

is a cycle representing a homology class \( \tilde{C}(a, b)_\lambda \in H_2(\text{SL}_2(A), \mathbb{Z}) \).
7.3. **Symbols as homology classes.** In this section, the map of sets \( s : SL_2(F) \to St(2, F) \) and the homomorphism \( \tilde{f} : H_2(SL_2(F), \mathbb{Z}) \to K_2(2, F) \) are those described in section 4 above.

**Theorem 7.8.** Let \( F \) be a field with at least four elements. Let \( \lambda \in F^\times \setminus \{\pm 1\} \).

1. Let \( a, b \in F^\times \). Then
   \[ \tilde{f}(C(a, b)) = c(a, b). \]
2. Suppose further that \( a, b, ab \notin \{\pm 1\} \). Then
   \[ \tilde{f}(\tilde{C}(a, b)) = c(a, b). \]

**Proof.** We begin by noting that, by Lemma 4.1, we have
   \[ \tilde{f}(\Psi_x) = 1 \text{ for all } x \in F \text{ and } \tilde{f}([R_a R_b (R_a R_b)^{-1}(R_{ab})]) = 1 \]
   since \( c(1, v) = c(u, 1) = 1 \) in \( K_2(2, F) \).

Also, by Lemma 4.1
   \[ \tilde{f}([R_a | R_b]) = \tilde{f}([D(a) | D(b)]) = c(a, b). \]

1. For any \( u \in F^\times \)
   \[
   \tilde{f}([H_u | G_u]) = s(H_u) s(G_u) s(H_u G_u)^{-1} \\
   = x_{21}(u) \cdot w_{12}(-1) x_{12}(u + u^{-1}) \cdot x_{12}(-u^{-1}) w_{12}(1) \\
   = x_{21}(u) \cdot (w_{12}(-1) x_{12}(u) w_{12}(1)) \\
   = x_{21}(u) x_{12}(u)^{w_{12}(1)} \\
   = x_{21}(u) x_{21}(-u) = 1 \quad \text{by Lemma 3.1}
   \]

and
   \[
   \tilde{f}([w^{-1} | w G_u]) = s(w^{-1}) s(w G_u) s(G_u)^{-1} \\
   = w_{12}(-1) x_{12}(u + u^{-1}) \cdot (w_{12}(-1) x_{12}(u + u^{-1}))^{-1} = 1.
   \]

Furthermore
   \[
   \tilde{f}([R_u | H_u]) = s(R_u) s(H_u) s(R_u H_u)^{-1} \\
   = x_{12}(-u) h_{12}(u) x_{21}(u) x_{12}(-u^{-1}) w_{12}(1) \\
   = h_{12}(u) x_{21}(-u^{-1}) x_{21}(u) x_{12}(-u^{-1}) w_{12}(1) \quad \text{since } x_{12}(-u) h_{12}(u) = x_{12}(-u^{-1}) \\
   = h_{12}(u) w_{12}(-u^{-1}) w_{12}(1).
   \]

Now
   \[ w_{12}(-u^{-1}) w_{12}(1) = w_{12}(-u^{-1}) w_{12}(-1) w_{12}(-1)^{-1} w_{12}(1)^{-1} = h_{12}(-u^{-1}) h_{12}(-1)^{-1} \]
   and hence
   \[ \tilde{f}([R_u | H_u]) = c(u, -u^{-1}) = c(-u, u) = 1. \]

Thus
   \[ \tilde{f}(\Theta_u) = 1 \]
   for all units \( u \).

Putting all of this together gives \( \tilde{f}(F(a, b), \lambda) = c(a, b) \) as required.
(2) We must show that $\tilde{f}(\tilde{\Theta}_a) = 1$ whenever $a, a^2 - 1 \in F^\times$.
As above, we have $\tilde{f}([w^{-1}|wG_a]) = 1$.

Now,

$$s(D(a)) = h_{12}(a), \quad s(\tilde{H}_a) = x_{12} \left( \frac{1 + a}{a(1 - a)} \right) w_{12} \left( \frac{1 + a}{-a} \right) x_{12}(a^{-1}),$$

and

$$s(D(a)\tilde{H}_a) = x_{12} \left( \frac{a(1 + a)}{1 - a} \right) w_{12}(-1 + a)x_{12}(a^{-1}).$$

Thus

$$\tilde{f}([D(a)|\tilde{H}_a]) = s(D(a))s(\tilde{H}_a)s(D(a)\tilde{H}_a)^{-1}$$

$$= h_{12}(a)x_{12} \left( \frac{1 + a}{a(1 - a)} \right) w_{12} \left( \frac{1 + a}{-a} \right) x_{12}(a^{-1})w_{12}(1 + a)x_{12} \left( \frac{a(1 + a)}{a - 1} \right)$$

$$= h_{12}(a)x_{12} \left( \frac{1 + a}{a(1 - a)} \right) w_{12} \left( \frac{1 + a}{-a} \right) w_{12}(1 + a)x_{12} \left( \frac{a(1 + a)}{a - 1} \right)$$

$$= h_{12}(a)x_{12} \left( \frac{1 + a}{-a} \right) x_{21} \left( \frac{-a}{1 - a^2} \right) w_{12}(1 + a)x_{12} \left( \frac{a(1 + a)}{a - 1} \right)$$

using

$$x_{12} \left( \frac{1 + a}{a(1 - a)} \right) w_{12}(1 + a) = x_{21} \left( \frac{-a}{1 - a^2} \right).$$

Since, by Lemma 3.1,

$$x_{21} \left( \frac{-a}{1 - a^2} \right) w_{12}(1 + a) = x_{12} \left( \frac{a(1 + a)}{1 - a} \right),$$

this gives

$$\tilde{f}([D(a)|\tilde{H}_a]) = h_{12}(a)x_{12} \left( \frac{1 + a}{-a} \right) w_{12}(1 + a)x_{12} \left( \frac{a(1 + a)}{1 - a} \right) x_{12} \left( \frac{a(1 + a)}{a - 1} \right)$$

$$= h_{12}(a)x_{12} \left( \frac{1 + a}{-a} \right) w_{12}(1 + a)$$

$$= h_{12}(a)h_{12} \left( \frac{1 + a}{-a} \right) h_{12}(-1 + a)^{-1}$$

$$= c(a, -(1 + a)a^{-1}) = c(a, 1 + a).$$

Now

$$s(\tilde{H}_aG_a) = s \left( \left[ \frac{a}{1 - a} \frac{a^2}{1 + a} \frac{1}{a} \frac{1}{a^2} \right] \right) = x_{12} \left( \frac{a(1 + a)}{1 - a} \right) w_{12}(-1 + a)x_{12}(a^{-1}).$$
So
\[ f([\tilde{H}_a|G_a]) = s(\tilde{H}_a)s(G_a)s(\tilde{H}_a|G_a)^{-1} \]
\[ = x_{12} \left( \frac{1 + a}{a(1 - a)} \right) w_{12} \left( \frac{1 + a}{-a} \right) x_{12}(a^{-1})w_{12}(-1)x_{12}(a + a^{-1})w_{12}(-a^{-1})w_{12}(1 + a)x_{12} \left( \frac{a(1 + a)}{a - 1} \right) \]
\[ = x_{12} \left( \frac{1 + a}{a(1 - a)} \right) w_{12} \left( \frac{1 + a}{-a} \right) x_{12}(a^{-1})w_{12}(-1)x_{12}(a)w_{12}(1 + a)x_{12} \left( \frac{a(1 + a)}{a - 1} \right) \]
\[ = w_{12} \left( \frac{1 + a}{-a} \right) x_{21} \left( \frac{-a}{1 - a^2} \right) w_{12}(-1)x_{21}(a^{-1})x_{12}(a)w_{12}(1 + a)x_{12} \left( \frac{a(1 + a)}{a - 1} \right) \]

using
\[ x_{12} \left( \frac{1 + a}{a(1 - a)} \right) w_{12} \left( \frac{1 + a}{-a} \right) = x_{21} \left( \frac{-a}{1 - a^2} \right) \text{ and } x_{12}(a^{-1})w_{12}(-1) = x_{21}(-a^{-1}). \]

Since \( x_{12}(-a)w_{12}(a) = x_{21}(-a^{-1})x_{12}(a) \), we thus have

\[ f([\tilde{H}_a|G_a]) = w_{12} \left( \frac{1 + a}{-a} \right) x_{21} \left( \frac{-a}{1 - a^2} \right) w_{12}(-1)x_{12}(a)w_{12}(1 + a)x_{12} \left( \frac{a(1 + a)}{a - 1} \right) \]
\[ = w_{12} \left( \frac{1 + a}{-a} \right) w_{12}(-1)x_{12} \left( \frac{a}{1 - a^2} \right) x_{12}(a)w_{12}(1 + a)x_{12} \left( \frac{a(1 + a)}{a - 1} \right). \]

Since
\[ x_{12} \left( \frac{a}{1 - a^2} \right) x_{12}(-a) = x_{12} \left( \frac{a}{1 - a^2} - a \right) = x_{12} \left( \frac{a^3}{1 - a^2} \right), \]

we obtain
\[ f([\tilde{H}_a|G_a]) = w_{12} \left( \frac{1 + a}{-a} \right) w_{12}(-1)x_{12} \left( \frac{a^3}{1 - a^2} \right) w_{12}(a)w_{12}(1 + a)x_{12} \left( \frac{a(1 + a)}{a - 1} \right). \]

The conjugation rules of Corollary 3.2 give
\[ x_{12} \left( \frac{a^3}{1 - a^2} \right) w_{12}(a)w_{12}(1 + a) = w_{12}(a)x_{12} \left( \frac{-a}{1 - a^2} \right) w_{12}(1 + a) = w_{12}(a)w_{12}(1 + a)x_{12} \left( \frac{a(1 + a)}{1 - a} \right). \]

Therefore
\[ f([\tilde{H}_a|G_a]) = w_{12} \left( \frac{1 + a}{-a} \right) w_{12}(-1)x_{12} \left( \frac{a(1 + a)}{1 - a} \right) x_{12} \left( \frac{a(1 + a)}{a - 1} \right) \]
\[ = w_{12} \left( \frac{1 + a}{-a} \right) \]
\[ = w_{12}(-1)x_{12} \left( \frac{a}{1 - a^2} \right) x_{12}(a)w_{12}(1 + a) \]
\[ = h_{12} \left( \frac{1 + a}{-a} \right) h_{12}(a)h_{12}(-1) \]
\[ = c(-1 + a)a^{-1}, a) = c(1 + a, a). \]

Putting this together, we get
\[ f([\tilde{H}_a]) = f([\tilde{H}_a|G_a]) \cdot f([D(a)|\tilde{H}_a])^{-1} = c(1 + a, a)c(a, 1 + a)^{-1} \]
\[ = c(a^2, 1 + a) = c((-a)^2, 1 + a) = c(-a, 1 + a)c(1 + a, -a) = 1. \]
8. Applications: generators for $H_2(\text{SL}_2(\mathbb{Z}[1/m]), \mathbb{Z})$

The general principle is the following:

Lemma 8.1. Let $m = q_1q_2 \cdots q_n$ where $q_1, \ldots, q_n$ are distinct primes. Suppose that the positive integers $u_2, \ldots, u_n \in \mathbb{Z}[1/m]^\times$ satisfy the conditions

1. $u_i$ is a primitive root modulo $q_i$ for $i \geq 2$,
2. When $i \neq j \in \{2, \ldots, n\}$, $q_i^{v_{q_j}(u_i)} \equiv 1 \pmod{q_j}$.

Then there is a direct sum decomposition

$$\tilde{K}_2(2, \mathbb{Z}[1/m]) \cong \mathbb{Z} \oplus \mathbb{Z}/(q_2 - 1) \oplus \cdots \oplus \mathbb{Z}/(q_n - 1)$$

with the property that infinite cyclic factor is generated by $c(-1, -1)$ and the factor $\mathbb{Z}/(q_i - 1)$ is generated by $c(u_i, q_i)$.

Proof. The isomorphism

$$\tilde{K}_2(2, \mathbb{Z}[1/m]) \cong \mathbb{Z} \oplus \left( \bigoplus_{i=2}^{n} \mathbb{F}_{q_i}^\times \right)$$

is induced by the map

$$\sigma : \tilde{K}_2(2, \mathbb{Z}[1/m]) \to \mathbb{Z}, \quad c(a, b) \mapsto \begin{cases} 1, & a < 0 \text{ and } b < 0 \\ 0, & \text{otherwise} \end{cases}$$

and the tame symbols $T_{p_i} : K(2, \mathbb{Q}) \to \mathbb{F}_{p_i}^\times$.

Now

$$T_{p_i}(c(u_i, q_i)) = \tau_{q_i}(\{u_i, q_i\}) = u_i \pmod{q_i} = w_i$$

while for $j \neq i$

$$T_{q_j}(c(u_i, q_j)) = q_i^{v_{q_j}(u_i)} \pmod{q_j} \equiv 1 \pmod{q_j}.$$

Remark 8.2. It is not known whether there must exist units satisfying condition (1) in general, but exceptions, if they exist, are rare.

If units $u_i$ are found satisfying condition (1), then it can always be arranged for condition (2) to hold; namely, multiply $u_i$ by a high power of $m_i$ where $m_i = (\prod_{k=1}^{n} q_k)/q_i$.

Combining Lemma 8.1 with Theorems 6.12 and 7.8, we deduce:

Corollary 8.3. Let $m = q_1 \cdots q_n$ be distinct primes satisfying $q_1 < q_2 < \cdots < q_n$ and $q_1 = 2, q_2 = 3$. Let $u_2, \ldots, u_n$ be as in Lemma 8.1. There is a direct sum decomposition

$$H_2(\text{SL}_2(\mathbb{Z}[1/m]), \mathbb{Z}) \cong \mathbb{Z} \oplus (\bigoplus_{i=2}^{n} \mathbb{Z}/(q_i - 1)\mathbb{Z})$$

where the first summand corresponds to the subgroup of $H_2(\text{SL}_2(\mathbb{Z}[1/m]), \mathbb{Z})$ generated by the homology class $C(-1, -1)$, and the summand $\mathbb{Z}/(q_i - 1)\mathbb{Z}$ corresponds to the subgroup generated by the homology class $C(u_i, q_i)$.

Example 8.4. In the case $m = 6$, we can take $u_2 = 2$. We deduce that the cyclic factors of

$$H_2(\text{SL}_2(\mathbb{Z}[1/6]), \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/2$$

are generated by the homology classes $C(-1, -1)$ and $C(2, 3)$.
Example 8.5. In the case $m = 30$, then the units $u_2 = 2$, $u_3 = 2$ satisfy the necessary congruences. Thus the cyclic factors of

$$H_2(SL_2(\mathbb{Z}[1/30]), \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4$$

are generated by the homology classes $C(-1, -1)$, $C(2, 3)$ and $C(2, 5)$.

Example 8.6. By Theorem 6.12 we have

$$H_2(\mathbb{Z}[1/42], \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{F}_3^x \oplus \mathbb{F}_7^x \cong \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/6.$$  

The first factor is generated by the homology class $C(-1, -1)$. Furthermore, $u_2 = 2 = u_3$ satisfy the congruences of Lemma 8.1. It follows that the homology classes $C(2, 3)$ and $C(2, 7)$ generate the second and third cyclic factors.

Example 8.7. Let $\omega$ be a primitive cube root of unity and let $p$ be a rational prime which is congruent to 1 modulo 3. Let $O = \mathbb{Z}[\omega, \frac{1}{3p}]$.

Observe that $\omega \in O^\times$ and $\omega^2 - 1 = -\sqrt{3}\omega \in O^\times$ also. Then $pO = p_1p_2$ where $k(p_i) \cong \mathbb{F}_p$ for $i = 1, 2$. Since $K_2(\mathbb{Z}[\omega]) = 0$, we have

$$K_2(O) \cong \tilde{K}_2(2, O) \cong k(p_1)^x \oplus k(p_2)^x \oplus k(q)^x \cong \mathbb{F}_p^x \oplus \mathbb{F}_p^x \oplus \mathbb{F}_3^x$$

where $q = \sqrt{-3}O$.

By Lemma 6.8 and Theorem 7.8 the natural map

$$H_2(SL_2(O), \mathbb{Z}) \to K_2(O)$$

is surjective and the homology class $C(-\omega, p)$ maps, via the tame symbol, to the element $-\bar{\omega} \in k(p)^x$ of order 6, while the class $C(3, p)$ maps to $\bar{3} \in k(p)^x \cong \mathbb{F}_p^x$.

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