Marginally bound (critical) geodesics of rapidly rotating black holes

Shahar Hod

The Ruppin Academic Center, Emeq Hefer 40250, Israel

and

The Hadassah Institute, Jerusalem 91010, Israel

(Dated: March 12, 2018)

Abstract

One of the most important geodesics in a black-hole spacetime is the marginally bound spherical orbit. This critical geodesic represents the innermost spherical orbit which is bound to the central black hole. The radii $r_{mb}(\tilde{a})$ of the marginally bound equatorial circular geodesics of rotating Kerr black holes were found analytically by Bardeen et. al. more than four decades ago (here $\tilde{a} \equiv J/M^2$ is the dimensionless angular-momentum of the black hole). On the other hand, no closed-form formula exists in the literature for the radii of generic (non-equatorial) marginally bound geodesics of the rotating Kerr spacetime. In the present study we analyze the critical (marginally bound) orbits of rapidly rotating Kerr black holes. In particular, we derive a simple analytical formula for the radii $r_{mb}(\tilde{a} \approx 1; \cos i)$ of the marginally bound spherical orbits, where $\cos i$ is an effective inclination angle (with respect to the black-hole equatorial plane) of the geodesic. We find that the marginally bound spherical orbits of rapidly-rotating black holes are characterized by a critical inclination angle, $\cos i = \sqrt{2/3}$, above which the coordinate radii of the geodesics approach the black-hole radius in the extremal $\tilde{a} \to 1$ limit. It is shown that this critical inclination angle signals a transition in the physical properties of the orbits: in particular, it separates marginally bound spherical geodesics which lie a finite proper distance from the black-hole horizon from marginally bound geodesics which lie an infinite proper distance from the horizon.
I. INTRODUCTION

The geodesic motions of test particles in the rotating Kerr black-hole spacetime have attracted much attention since the pioneering work of Carter [1], see also [2–9] and references therein. Among the various geodesics which characterize the black-hole spacetime, the single most important family of geodesics are the spherical orbits — orbits with constant coordinate radii on which test particles can travel around the central black hole.

As emphasized in [2], not all spherical orbits are bound to the black hole. An unbound geodesic is characterized by \( E/\mu > 1 \), where \( E \) and \( \mu \) are the total (conserved) energy and rest mass of the orbiting particle, respectively. Given an infinitesimal outward perturbation, a particle on an unbound geodesic will escape to infinity [2]. The unbound spherical orbits are separated from the bound orbits by a critical geodesic called the marginally bound spherical geodesic. This important geodesic is characterized by a zero binding energy \( E_{\text{binding}} \equiv E(\infty) - E(r_{\text{mb}}) = 0 \), or equivalently

\[
E(r_{\text{mb}}) = \mu .
\]  

(1)

The critical orbit so defined is sometimes referred to as the IBSO (innermost bound spherical orbit) [6, 7].

The marginally bound spherical orbit is interesting from both an astrophysical and theoretical points of view. In particular, this critical geodesic plays an important role in the evolution and dynamics of star clusters around supermassive galactic black holes [7, 8]. As emphasized in [2], any parabolic orbit which penetrates below the innermost bound spherical orbit must plunge directly into the central black hole.

It is well-known that astrophysically realistic black holes generally possess angular momentum. An astrophysically realistic model of particle-dynamics in a black-hole spacetime should therefore involve a non-spherical Kerr geometry [10]. It should be emphasized that, for rotating Kerr black holes, the physical properties (in particular, the characteristic radii and angular-momentum) of the marginally bound spherical orbits must be computed numerically. The only known exceptions are the co-rotating and counter-rotating marginally bound circular geodesics in the equatorial plane of the Kerr black hole, in which case a closed analytical formula [see Eq. (20) below] for the radii of these special orbits has been given in [2].
To the best of our knowledge, no closed-form formula exists in the literature for the radii of generic (non-equatorial) marginally bound geodesics which characterize the rotating Kerr black-hole spacetime. As we shall show below, for rapidly-rotating black holes one can obtain a simple and compact analytical formula for the characteristic radii of the marginally bound spherical geodesics. In particular, we shall show below that rapidly-rotating Kerr black holes are characterized by a significant fraction of marginally bound geodesics whose radii approach the black-hole radius in the near-extremal limit.

The rest of the paper is devoted to the investigation of the physical properties of non-equatorial marginally bound spherical geodesics of the rotating Kerr black-hole spacetime. In Sec. II we describe the dynamical equations which determine the geodesics of test particles in the rotating Kerr spacetime. In Sec. III we derive the characteristic equation which determines the radii of the generic (non-equatorial) marginally bound spherical orbits. In Sec. IV we focus on near-extremal black holes and analyze the marginally bound geodesics of these rapidly-rotating black holes. We conclude in Sec. V with a brief summary of our results.

II. DESCRIPTION OF THE SYSTEM

We shall analyze the spherical geodesics of test particles in the spacetime of a rapidly-rotating Kerr black hole. In Boyer-Lindquist coordinates the metric is given by (we use gravitational units in which $G = c = 1$) \cite{3, 11}

$$ds^2 = -(1 - \frac{2Mr}{\rho^2})dt^2 - \frac{4Mar}{\rho^2}dtd\phi + \frac{\rho^2}{\Delta}dr^2 + \rho^2d\theta^2 + \left(r^2 + a^2 + \frac{2Ma^2r \sin^2 \theta}{\rho^2}\right)\sin^2 \theta d\phi^2,$$  \hspace{1cm} (2)

where $M$ and $a$ are the mass and angular momentum per unit mass of the black hole, respectively. Here $\Delta \equiv r^2 - 2Mr + a^2$ and $\rho \equiv r^2 + a^2 \cos^2 \theta$. The black-hole (event and inner) horizons are located at the zeroes of $\Delta$:

$$r_{\pm} = M \pm (M^2 - a^2)^{1/2}.$$ \hspace{1cm} (3)

The general geodesics of test particles in the Kerr black-hole spacetime are characterized by four constants of the motion \cite{1}. In terms of the covariant Boyer-Lindquist components
of the 4-momentum, these conserved quantities are [2]:

\[ E \equiv -p_t = \text{total energy} \, , \] (4)

\[ L_z \equiv p_\phi = \text{component of angular momentum parallel to the symmetry axis} \, , \] (5)

\[ Q \equiv p_\theta^2 + \cos^2 \theta [a^2(\mu^2 - p_t^2) + p_\phi^2/\sin^2 \theta] \, , \] (6)

and

\[ \mu = \text{the rest mass of the particle} \, . \] (7)

The geodesics in the black-hole spacetime are governed by the following set of equations [2]

\[ \rho \frac{dr}{d\lambda} = \pm \sqrt{V_r} \, , \] (8)

\[ \rho \frac{d\theta}{d\lambda} = \pm \sqrt{V_\theta} \, , \] (9)

\[ \rho \frac{d\phi}{d\lambda} = \left( \frac{L_z}{\sin^2 \theta} - aE \right) + \frac{aT}{\Delta} \, , \] (10)

\[ \rho \frac{dt}{d\lambda} = a\left( L_z - aE \sin^2 \theta \right) + \left( r^2 + a^2 \right) \frac{T}{\Delta} \, , \] (11)

where \( \lambda \) is related to the particle’s proper time by \( \lambda = \tau/\mu \). Here

\[ T \equiv E(r^2 + a^2) - L_z a \, , \] (12)

\[ V_r \equiv T^2 - \Delta \left[ \mu^2 r^2 + (L_z - aE)^2 + Q \right] \, , \] (13)

\[ V_\theta \equiv Q - \cos^2 \theta \left[ a^2(\mu^2 - E^2) + L_z^2/\sin^2 \theta \right] \, . \] (14)

The effective potentials \( V_r \) and \( V_\theta \) determine the orbital motions in the \( r \) and \( \theta \) directions, respectively [6, 12].

Circular equatorial orbits are characterized by \( Q = 0 \) [2]. It is convenient to use an effective inclination angle \( i \) to quantify the deviation of a generic (non-equatorial) orbit from the equatorial plane of the black hole. The effective inclination angle is defined by [6, 13–15]

\[ \cos i \equiv \frac{L_z}{L} \, , \] (15)

where

\[ L \equiv \sqrt{L_z^2 + Q} \, . \] (16)
Note that $L$ and $i$ are constants of the motion. For spherical black-hole spacetimes (with $a = 0$), $L$ is the total angular momentum of the orbiting particle \[2\]. The extensively studied equatorial orbits are characterized by $\cos^2 i = 1$, where $\cos i = +1/-1$ correspond to co-rotating/counter-rotating geodesics, respectively.

\section{MARGINALLY BOUND SPHERICAL GEODESICS OF THE BLACK-HOLE SPACETIME}

Spherical geodesics in the black-hole spacetime are characterized by the two conditions \[2, 3\]
\[V_r = 0 \quad \text{and} \quad V_r' = 0 . \tag{17}\]
Substituting (13) into (17) and using the condition $E = \mu$ [see Eq. (11)] for the marginally bound orbits, one obtains the dimensionless angular-momentum-to-energy ratio
\[\frac{L}{M\mu} = \sqrt{\frac{4r^3}{M(r^2 - a^2 \sin^2 i)}} \tag{18}\]
and the characteristic equation
\[r^4 - 4Mr^3 - a^2(1 - 3 \sin^2 i)r^2 + a^4 \sin^2 i + 4a \cos i \sqrt{Mr^5 - Ma^2r^3 \sin^2 i} = 0 \tag{19}\]
for the radii $r_{mb}(M, a; \cos i)$ of the marginally bound spherical orbits.

The exact (analytical) solution of the characteristic equation (19) is only known for the simple case of circular geodesics in the equatorial plane of the black hole. As mentioned above, these special orbits are characterized by $\cos i = \pm 1$, where the upper sign corresponds to the co-rotating circular orbit while the lower sign corresponds to the counter-rotating circular orbit. In this simple case one finds \[2\]
\[r_{mb}(M, a; \cos i = \pm 1) = 2M \mp a + 2M^{1/2}(M \mp a)^{1/2} . \tag{20}\]

To the best of our knowledge, no closed-form formula exists in the literature for the radii $r_{mb}(M, a; \cos i)$ of generic (non-equatorial, $\cos i \neq \pm 1$) marginally bound spherical geodesics of the Kerr spacetime. It is worth mentioning that Will \[7\] also studied the marginally bound spherical geodesics of the Kerr spacetime. In particular, Ref. \[7\] provides an expansion of the physical quantities in powers of the dimensionless ratio $a/M$. As emphasized in \[7\],
this expansion in powers of $a/M$ works extremely well (to better than 0.5%) in the regime $0 \leq a/M \leq 0.9$, but is less accurate in the regime of rapidly-rotating (near-extremal) black holes. [Note that [7] also provides results for the marginally bound geodesics of exactly extremal (with $a$ exactly equals $M$) black holes.] In the present paper we shall perform an analysis which is complementary to the one presented in [7]. In particular, we shall provide an alternative expansion of the physical quantities in the small parameter $(r_+ - M)/M \ll 1$. This expansion is valid for rapidly-rotating (but not necessarily extremal) black holes (see [7] for the exactly extremal $a = M$ case). We shall derive a compact analytical expression for the characteristic radii of these critical (marginally-bound) orbits in the spacetime of rapidly-rotating black holes. In particular, we shall show that the dependence of $r_{\text{mb}}(a \simeq M; \cos i)$ on the effective inclination angle of the orbit exhibits an interesting “phase transition” at the critical inclination angle $\cos i = \sqrt{2/3}$.

Before proceeding further, it is worth emphasizing that similar phase transitions (with respect to the effective inclination angle) are also found for other geodesics, including the stable circular orbits [16–20]. In particular, Wilkins [16] studied the spherical geodesics of a maximally rotating ($a = M$) black hole. It was found in [16] that, for exactly extremal $a = M$ black hole, there is a very interesting family of orbits whose coordinate radius coincides with the coordinate radius of the (extremal) black-hole horizon, $r = r_+ = M$. Johnston and Ruffini [17] extended the results of [16] to the case of charged Kerr-Newman black holes. Our analysis extends the results of [16] to the regime of rapidly-rotating (but not necessarily extremal) black holes. It is worth mentioning that an analysis analogous to the one presented here reveals that the innermost stable circular orbit (ISCO) is characterized by a similar phase transition which, for near-extremal black holes, occurs at the effective inclination angle $\cos i = \sqrt{4/5}$. In addition, the null spherical geodesics [3, 21, 22] of near-extremal Kerr black holes are also characterized by an analogous phase transition which occurs at the effective inclination angle $\cos i = \sqrt{4/7}$ [9].

**IV. RAPIDLY-ROTATING (NEAR-EXTREMAL) BLACK HOLES**

For rapidly-rotating (near-extremal) black holes it is convenient to define

$$r_\pm \equiv M(1 \pm \epsilon) \quad \text{and} \quad r_{\text{mb}} \equiv M(1 + \delta_{\text{mb}}),$$

(21)
where $\epsilon, \delta_{mb} \ll 1$. From (21) one finds
\[ a = M[1 - \epsilon^2/2 + O(\epsilon^4)]. \] (22)

Substituting (21) and (22) into the characteristic equation (19), one finds
\[
(\cos^2 i - 2)(3 \cos^2 i - 2)\delta^2 + [(\cos^6 i - 18 \cos^4 i + 20 \cos^2 i - 8)/2 \cos^2 i]\delta^3
+ 2 \cos^2 i(2 - \cos^2 i)\epsilon^2 + O(\delta^4, \epsilon^4) = 0 \] (23)
for the marginally bound orbits of rapidly-rotating black holes. The qualitative behavior of $\delta_{mb}(\epsilon; \cos i)$ in the near-extremal $\epsilon \to 0$ limit depends on whether the coefficient of the $O(\delta^2)$ term in Eq. (23) is positive, negative, or zero. Note that this coefficient vanishes at the critical inclination angle
\[ \xi \equiv \cos i - \sqrt{2/3} = 0. \] (24)

The solution of the characteristic equation (23) is given by
\[
\delta_{mb}(\epsilon; \cos i) = \begin{cases} 
\sqrt{2\cos^2 i/3\cos^2 i - 2} \epsilon + O(\epsilon^2/\xi^2) & \text{for } \cos i - \sqrt{2/3} \gg \epsilon^{2/3}; \\
\epsilon^{2/3} + O(\xi) & \text{for } -\epsilon^{2/3} \ll \cos i - \sqrt{2/3} \ll \epsilon^{2/3}; \\
9\sqrt{6}(\sqrt{2/3} - \cos i) + O(\epsilon^2/\xi^2) & \text{for } \epsilon^{2/3} \ll \sqrt{2/3} - \cos i \ll 1.
\end{cases} \] (25)

From (25) one learns that the solution $\delta_{mb}(\epsilon; \cos i)$ of the characteristic equation (23) exhibits a “phase transition” [from a $\delta_{mb}(\epsilon \to 0) \to 0$ behavior to a finite $\delta_{mb}(\epsilon \to 0)$ behavior], which occurs in the extremal $\epsilon \to 0$ limit at the critical inclination angle $\cos i = \sqrt{2/3}$.

We shall now show that the critical inclination angle, $\cos i = \sqrt{2/3}$, separates marginally bound orbits which are characterized by finite proper distances to the black-hole horizon from marginally bound orbits which are characterized by infinite proper distances to the horizon. The proper radial distance between the black-hole horizon [at $r_+ = M(1 + \epsilon)$] and the intersection point of the marginally bound orbit [of radius $r_{mb} = M(1 + \delta_{mb})$] with the equatorial plane of the black hole is given by (we emphasize that we use here the Boyer-Lindquist coordinates) (24)
\[
\Delta \ell = M[\sqrt{\delta^2 - \epsilon^2} + \ln (\delta + \sqrt{\delta^2 - \epsilon^2}) - \ln \epsilon]. \] (26)
Substituting \( \delta_{mb}(\epsilon; \cos i) \) from Eq. (25) into Eq. (26), one finds

\[
\Delta \ell(\epsilon \to 0) = M \times \begin{cases} 
\ln \left( \frac{\sqrt{2} \cos i + \sqrt{2 - \cos^2 i}}{\sqrt{3} \cos^2 i - 2} \right) + O(\epsilon) & \text{for } \cos i - \sqrt{2/3} \gg \epsilon^{2/3} ; \\
-\frac{1}{3} \ln \epsilon + O(1) & \text{for } -\epsilon^{2/3} \ll \cos i - \sqrt{2/3} \ll \epsilon^{2/3} ; \\
-\ln \epsilon + O(\ln \xi) & \text{for } \epsilon^{2/3} \ll \sqrt{2/3} - \cos i \ll 1 ,
\end{cases}
\]

in the extremal \( \epsilon \to 0 \) limit. One therefore concludes that the critical inclination angle, which occurs at \( \cos i = \sqrt{2/3} \) in the near-extremal limit, signals a transition from finite to infinite proper distances of the marginally bound orbits from the black-hole horizon [25].

Each marginally bound orbit is bounded in some strip \([\theta^-, \theta^+]\) of the polar angle \( \theta \), where \( \theta^- = \pi - \theta^+ \). The two polar turning-points are determined from the requirement

\[
V_\theta(\theta^\pm) = 0 ,
\]

see Eq. (14). We shall now evaluate the polar boundaries \( \{\theta^-, \theta^+\} \) of the marginally bound near-horizon geodesics (the marginally bound orbits which are characterized by finite proper distances to the black-hole horizon in the near-extremal limit). Taking cognizance of Eqs. (18), (21), and (25), one finds the dimensionless ratio

\[
\frac{L}{M \mu} = \frac{2}{\cos i} \left[ 1 + \sqrt{\frac{3 \cos^2 i - 2}{2 \cos^2 i}} \epsilon + O(\epsilon^2) \right]
\]

for the marginally bound spherical orbits in the regime \( \cos i - \sqrt{2/3} \gg \epsilon^{2/3} \) [26]. Substituting (29) into (28) [see Eq. (14) for \( V_\theta \)], one finds that the polar turning-points (which characterize the maximal polar-deviation of a near-horizon geodesic from the equatorial plane of the black hole) are given by the simple relation [27]

\[
\cos^2 \theta^\pm = \sin^2 i .
\]

For the equatorial circular orbit with \( \cos i = 1 \) one finds \( \cos \theta^\pm = 0 \) (\( \theta^\pm = \frac{\pi}{2} \)) as expected. For the critical marginally bound geodesic with \( \cos i = \sqrt{2/3} \) [see Eq. (24)] one finds from (30)

\[
\cos \theta^\pm = \pm \frac{1}{\sqrt{3}} .
\]

V. SUMMARY

We have studied analytically the critical (marginally bound) spherical geodesics of rapidly-rotating Kerr black holes. While most former studies have focused on circular orbits in the
equatorial plane of the black hole \cite{2}, in the present paper we have considered generic (non-equatorial) marginally bound geodesics. In particular, we have derived an analytical expression [see Eq. (25)] for the radii \( r_{mb}(a \simeq M; \cos i) \) of the critical (marginally bound) spherical geodesics which characterize the rapidly-rotating Kerr black-hole spacetime.

The analytical formula (25) reveals that the marginally bound spherical geodesics of rapidly-rotating black holes are characterized by a critical inclination angle, \( \cos i = \sqrt{2/3} \), above which the coordinate radii of the orbits approach the black-hole radius in the extremal limit. We have proved that this critical inclination angle signals a transition in the physical properties of the marginally bound orbits: in particular, it separates marginally bound geodesics which lie a finite proper distance from the black-hole horizon from marginally bound geodesics which lie an infinite proper distance from the horizon.

ACKNOWLEDGMENTS

This research is supported by the Carmel Science Foundation. I thank Yael Oren, Arbel M. Ongo and Ayelet B. Lata for helpful discussions.

[1] B. Carter, Phys. Rev. 174, 1559 (1968).
[2] J. M. Bardeen, W. H. Press and S. A. Teukolsky, Astrophys. J. 178, 347 (1972).
[3] S. Chandrasekhar, The Mathematical Theory of Black Holes, (Oxford University Press, New York, 1983).
[4] S. L. Shapiro and S. A. Teukolsky, Black holes, white dwarfs, and neutron stars: The physics of compact objects (Wiley, New York, 1983).
[5] V. Cardoso, A. S. Miranda, E. Berti, H. Witek and V. T. Zanchin, Phys. Rev. D 79, 064016 (2009).
[6] R. Grossman, J. Levin, and G. Perez-Giz, Phys. Rev. D 85, 023012 (2012).
[7] C. M. Will, Class. Quantum Gravit. 29, 217001 (2012).
[8] D. Merritt, T. Alexander, S. Mikkola, and C. M. Will, Phys. Rev. D 84 044024 (2011).
[9] S. Hod, Phys. Rev. D 84, 104024 (2011) [arXiv:1201.0068]; S. Hod, Phys. Rev. D 84, 124030 (2011) [arXiv:1112.3286]; S. Hod, Phys. Lett. B 718, 1552 (2013) [arXiv:1210.2486]; S. Hod, Phys. Rev. D 87, 024036 (2013).
In obtaining the solution (25) for $\delta_{mb}(\epsilon; \cos i)$ we have used the observation that, depending of the relative magnitudes of $\epsilon$ and $\xi$, one of the three terms in the characteristic equation (23) can be neglected as compared to the other two terms. It should be emphasized that one of the two remaining (non-negligible) terms must be positive while the other non-negligible term must be negative. Specifically, there are three distinct regimes: (1) For $\xi \gg \epsilon^{2/3}$ the $O(\delta^3)$ term is negligible as compared to the other two terms in Eq. (23). (2) For $-\epsilon^{2/3} \ll \xi \ll \epsilon^{2/3}$ the $O(\xi \delta^2)$ term is negligible as compared to the other two terms in Eq. (23). (3) For $\xi \ll -\epsilon^{2/3}$ the $O(\epsilon^2)$ term is negligible as compared to the other two terms in Eq. (23).

Note that one finds $\Delta \ell = M \ln(1 + \sqrt{2})$ for the prograde equatorial geodesic with $\cos i = 1$, in accord with the result of [2].

Note that one finds $L/M\mu = 2/\cos i$ for extremal black holes with $\epsilon \equiv 0$, in accord with the result of [4].

It is worth emphasizing again that the effective inclination angle $i$ is a constant of the motion, while the polar angle $\theta$ varies along the trajectory within the interval $[\theta^-, \theta^+]$, where the two boundaries are determined by the relation (30).