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Twisted waveguide with a Neumann window

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Dedicated to Pavel Exner on the occasion of his 70th birthday

Abstract

This paper is concerned with the study of the existence/non-existence of the discrete spectrum of the Laplace operator on a domain of \( \mathbb{R}^3 \) which consists in a twisted tube. This operator is defined by means of mixed boundary conditions. Here we impose Neumann Boundary conditions on a bounded open subset of the boundary of the domain (the Neumann window) and Dirichlet boundary conditions elsewhere.

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1 Introduction

In this work, we would like to study the influence of a geometric twisting on trapped modes which occur in certain waveguides. Here the waveguide consists in a straight tubular domain \( \Omega_0 := \mathbb{R} \times \omega \) having a Neumann window on its boundary \( \partial \Omega_0 \).

The cross section \( \omega \) is supposed to be an open bounded connected subset of \( \mathbb{R}^2 \) of diameter \( d > 0 \) which is not rotationally invariant. Moreover \( \omega \) is supposed to have smooth boundary \( \partial \omega \).

It can be shown that the Laplace operator associated to such a straight tube has bound states [8].

Let us introduce some notations. Denote by \( \mathcal{N} \) the Neumann window. It is an open bounded subset of the boundary \( \partial \Omega_0 \). Let \( \mathcal{D} \) be its complement set in \( \partial \Omega_0 \). When \( \mathcal{N} \) is an annulus of size \( l > 0 \) we will denote it by,

\[
\mathcal{A}_a(l) := I_a(l) \times \partial \omega, I_a(l) := (a, l + a), a \in \mathbb{R}.
\]

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Consider first the self-adjoint operator $H_0^N$ associated to the following quadratic form. Let $D(Q^N) = \{ \psi \in H^1(\Omega_0) \mid \psi|_\mathcal{D} = 0 \}$ and for $\psi \in D(Q^N)$,

$$Q^N(\psi) = \int_{\Omega_0} |\nabla \psi|^2 dx$$

i.e. the Laplace operator defined on $\Omega_0$ with Neumann boundary conditions (NBC) on $\mathcal{N}$ and Dirichlet boundary conditions (DBC) on $\mathcal{D}$ [5, 11].

It is actually shown in the Section 2 of this paper that if $\mathcal{N}$ contains an annulus of size $\ell$ large enough then $H_0^N$ has at least one discrete eigenvalue. In fact it is proved in [8] that this holds true if $\mathcal{N}$ contains an annulus of any size $\ell > 0$.

The question we are interested in is the following: is it possible that the discrete spectrum of $H_0^N$ disappears when we apply a geometric twisting on the guide? This question is motivated by the results of [6, 10] where it is shown that this phenomenon occurs in some bent tubes when they are subjected to a twisting defined from an angle function $\theta$ having a derivative $\dot{\theta}$ with a compact support.

In this paper we consider the situation described above which is very different from the one of [6, 10].

Let us now define the twisting [4, 7]. Choose $\theta \in C^1_c(\mathbb{R})$ and introduce the diffeomorphism

$$\mathcal{L} : \Omega_0 \longrightarrow \mathbb{R}^3$$

$$(s,t_2,t_3) \longmapsto (s,t_2 \cos \theta(s) - t_3 \sin \theta(s), t_2 \sin \theta(s) + t_3 \cos \theta(s)).$$

The twisted tube is given by $\Omega_\theta := \mathcal{L}(\Omega_0)$. Let $D(Q^N_\theta) = \{ \psi \in H^1(\Omega_\theta) \mid \psi|_\mathcal{L}(\mathcal{D}) = 0 \}$ and consider the following quadratic form

$$Q^N_\theta(\psi) := \int_{\Omega_\theta} |\nabla \psi|^2 dx, \ \psi \in D(Q^N_\theta).$$

Through unitary equivalence, we then have to consider

$$q^N_\theta(\psi) := Q^N_\theta(\psi \mathcal{L}^{-1}) = \| \nabla' \psi \|^2 + \| \partial_\nu \psi + \dot{\theta} \partial_\tau \psi \|^2,$$

$\psi \in D(q^N_\theta) := \{ \psi \in H^1(\Omega_0) \mid \psi|_\mathcal{D} = 0 \}$ and where

$$\nabla' := t_2 \partial_{t_2} - t_3 \partial_{t_3}.$$ 

Denote by $H^N_\theta$ the associated self-adjoint operator. It is defined as follows (see [5, 11]). Let $D(H^N_\theta) = \{ \psi \in D(q^N_\theta), \ H^N_\theta \psi \in L^2(\Omega_0) \ \frac{\partial \psi}{\partial n}[\mathcal{N}] = 0 \}$ with

$$H^N_\theta \psi = (-\Delta_\omega - (\dot{\theta} \partial_\tau + \partial_\nu)^2)\psi,$$

where the transverse Laplacian $\Delta_\omega := \partial_{t_2}^2 + \partial_{t_3}^2$. If $\mathcal{N} = \mathcal{A}_\alpha(l), l > 0$, we will denote these forms respectively as $Q^N_\theta, q^N_\theta$ and the corresponding operator as $H^N_\theta$ and if $\mathcal{N} = \emptyset$ we denote the associated operator by $H_\theta$.

Then the main result of this paper is
Theorem 1.1. i) Under conditions stated above on $\omega$ and $\theta$, there exists $l_{\min} := l_{\min}(\omega, d) > 0$ such as if for some $a \in \mathbb{R}$ and $l > l_{\min}$, $N \subset \mathcal{A}_a(l)$ then

$$\sigma(\mathcal{H}_0^{N'}) \neq \emptyset.$$  \hspace{1cm} (6)

ii) Suppose $\theta$ is a non zero function satisfying the same conditions as in i) and has a bounded second derivative. Then there exists $d_{\max} := d_{\max}(\theta, \omega) > 0$ such that for all $0 < d < d_{\max}$ there exists $l_{\max} := l_{\max}(\omega, d, \theta)$ such as for all $0 < l \leq l_{\max}$, if $N \subset \mathcal{A}_a(l)$ and $\text{supp}(\theta) \cap I_a(l) = \emptyset$ for some $a \in \mathbb{R}$ then

$$\sigma(\mathcal{H}_0^{N'}) = \emptyset.$$  \hspace{1cm} (7)

Roughly speaking this result implies that for $d$ small enough, the discrete spectrum disappears when the width of the Neumann window decreases.

Let us describe briefly the content of the paper. In the Section 2 we give the proof of the Theorem 1.1 i). The section 3 is devoted to the proof of the second part of the Theorem 1.1, this proof needs several steps. In particular we first establish a local Hardy inequality. This allows us to reduce the problem to the analysis of a one dimensional Schrödinger operator from which the Theorem 1.1 ii) follows. Finally in the Appendix of the paper we give partial results we use in previous sections.

2 Existence of bound states

First we prove the following. Denote by $E_1, E_2, \ldots$ the eigenvalues (transverse modes) of the Laplacian $-\Delta_\omega$ defined on $L^2(\omega)$ with DBC on $\partial \omega$. Let $\chi_1, \chi_2, \ldots$ be the associated eigenfunctions. Then we have

Proposition 2.1. $\sigma_{\text{ess}}(\mathcal{H}_0^{N'}) = [E_1, \infty)$.

Proof. We know that $\sigma(\mathcal{H}_0) = [E_1, \infty)$ see e.g. [2]. But by usual arguments [12], $\mathcal{H}_0^{N'} \leq \mathcal{H}_\theta$, then

$$[E_1, \infty) \subset \sigma_{\text{ess}}(\mathcal{H}_0^{N'}).$$  \hspace{1cm} (8)

Let $a' \in \mathbb{R}$ and $l' > 0$ large enough such that $N \subset \mathcal{A}_{a'}(l') = I_{a'}(l') \times \partial \omega$ and $\text{supp}(\theta) \subset I_{a'}(l')$. Let $\mathcal{H}_{0'}$ be the operator defined as in (5) but with additional Neumann boundary conditions on $\{a'\} \times \omega \cup \{a' + l'\} \times \omega$. So $\mathcal{H}_0^{N'} \geq \mathcal{H}_{0'}$ and then $\sigma_{\text{ess}}(\mathcal{H}_0^{N'}) \subset \sigma_{\text{ess}}(\mathcal{H}_{0'})$ [12].

But $\mathcal{H}_{0'} = \mathcal{H}_i \oplus \mathcal{H}_e$. The interior operator $\mathcal{H}_i$ is the corresponding operator defined on $L^2(\mathcal{I}_{a'} \omega(l') \times \omega)$ with NBC on $\{a'\} \times \omega \cup \{a' + l'\} \times \omega$ and DBC elsewhere on $\mathcal{A}_{a'}(l')$. By general arguments of [12] it has only discrete spectrum consequently $\sigma_{\text{ess}}(\mathcal{H}_i^{N'}) = \sigma_{\text{ess}}(\mathcal{H}_e)$.

Now the exterior operator $\mathcal{H}_e$ is defined on $L^2((-\infty, a') \times \omega \cup (a' + l', \infty) \times \omega)$ with DBC on $(-\infty, a') \times \partial \omega \cup (a' + l', \infty) \times \partial \omega$ and NBC on $\{a'\} \times \omega \cup \{a' + l'\} \times \omega$. Since $\theta = 0$ for $x < a'$ and $x > a' + l'$, it is easy to see that

$$\mathcal{H}_e = \bigoplus_{n \geq 1} (-\partial^2 + E_n)(\chi_n, \ldots)\chi_n.$$

Hence $\sigma(\mathcal{H}_e) = \sigma_{\text{ess}}(\mathcal{H}_e) = [E_1, +\infty)$. \hfill \Box

The Theorem 1.1 i) follows from
Proposition 2.2. Under conditions of the Theorem 1.1 i), there exists \( l_{\text{min}} := l_{\text{min}}(\omega, d) > 0 \) such as for all \( l > l_{\text{min}} \) we have
\[
\sigma_d(H^l_\theta) \neq \emptyset.
\]

Proof. Let \( \varphi_{l,a} \) be the following function
\[
\varphi_{l,a}(s) := \begin{cases} 
10(s-a), & \text{on } [a, a + \frac{l}{10}); \\
1, & \text{on } [a + \frac{l}{10}, a + \frac{9l}{10}); \\
-10(s-a), & \text{on } [a + \frac{9l}{10}, a + l); \\
0, & \text{elsewhere.}
\end{cases}
\]

It is easy to see that \( \varphi_{l,a} \in D(q^l_\theta) \) and \( \| \varphi_{l,a} \|_2^2 = \frac{13}{15} |\omega| \). Let us calculate
\[
q^l_\theta(\varphi_{l,a}) - E_1 \| \varphi_{l,a} \|_2^2 = \| \nabla \varphi_{l,a} \|_2^2 + \| \hat{\theta} \partial_r \varphi_{l,a} + \partial_s \varphi_{l,a} \|_2^2 - E_1 \| \varphi_{l,a} \|_2^2.
\]

Evidently the first term on the r.h.s of (10) is zero. For the second term on the r.h.s of (10) we get,
\[
\| \hat{\theta} \partial_r \varphi_{l,a} + \partial_s \varphi_{l,a} \|_2^2 = \| \partial_s \varphi_{l,a} \|_2^2 = \frac{20}{l} |\omega|.
\]

Then
\[
q^l_\theta(\varphi_{l,a}) - E_1 \| \varphi_{l,a} \|_2^2 = \| \partial_s \varphi_{l,a} \|_2^2 = \frac{20}{l} |\omega| - \frac{15}{13} E_1
\]

and thus if \( l \geq l_{\text{min}} := \sqrt{\frac{300}{13E_1}} \) we have \( q^l_\theta(\varphi_{l,a}) - E_1 \| \varphi_{l,a} \|_2^2 \leq 0 \)

2.1 Proof of the Theorem 1.1 i)

Using the same notation as in the Theorem 1.1 i), then \( H^N_\theta \leq H^l_\theta \). Moreover these operators have the same essential spectrum, then by the min-max principle the assertion follows.

3 Absence of bound state

In this section we want to prove the second part of the Theorem 1.1. Denote by \( \theta_m = \inf(\text{supp}(\hat{\theta})) \) and \( \theta_M = \sup(\text{supp}(\hat{\theta})) \) and \( L = \theta_M - \theta_m \). Here \( L > 0 \). We first consider the case where the Neumann window is an annulus, \( A_a(l) = I_a(l) \times \omega \).

Proposition 3.1. Suppose \( A_a(l) \) is such that \( \theta_m \leq \theta_M \). Assume that conditions of the Theorem 1.1 ii) hold. Then there exists \( d_{\text{max}} := d_{\text{max}}(\omega, \theta) > 0 \), such that for all \( 0 < d \leq d_{\text{max}} \) there exists \( l_{\text{max}}(d, \theta, \omega) > 0 \) such as for all \( 0 < l \leq l_{\text{max}} \) we have
\[
\sigma_d(H^l_\theta) = \emptyset.
\]

Remark 3.2. the case where \( l+a \leq \theta_m \) follows from same arguments developed below.

This proof is based on the fact that under conditions of the Proposition 3.1, for every \( \psi \in D(q^l_\theta) \) it holds,
\[
Q(\psi) := q^l_\theta(\psi) - E_1 \| \psi \|^2 \geq 0.
\]

The proof of (13) involves several steps.
3.1 A local Hardy inequality

The aim of this paragraph is to show a Hardy type inequality needed for the proof of the Proposition 3.1. It is the first step of the proof of (13). Let $g$ be the following function

$$g(s) := \begin{cases} 0, & \text{on } I_0(l); \\ E_1, & \text{elsewhere.} \end{cases} \quad (14)$$

Choose $p \in (\theta_m, \theta_M)$ s.t. $\dot{\theta}(p) \neq 0$ and let

$$\rho(s) := \begin{cases} \frac{1}{1 + (s - p)^2}, & \text{on } (-\infty, p]; \\ 0, & \text{elsewhere.} \end{cases} \quad (15)$$

**Proposition 3.3.** Under same conditions of the Proposition 3.1, then there exists a constant $C > 0$ depending on $p$ and $\omega$ and $\dot{\theta}$ such that for any $\psi \in D(q^1_0)$,

$$\| \nabla^t \psi \|^2 + \| \dot{\theta} \partial_t \psi + \partial_s \psi \|^2 - \int_{\Omega_0} g(s) | \psi |^2 \, dsdt \geq C \int_{\Omega_0} \rho(s) | \psi |^2 \, dsdt. \quad (16)$$

We first show the following lemma. Denote by $\Omega_p := (-\infty, p) \times \omega$.

**Lemma 3.4.** Under same conditions of the Proposition 3.3. Then for any $\psi \in D(q^1_0)$ we have

$$\int_{\Omega_p} | \nabla^t \psi |^2 + \| \dot{\theta} \partial_t \psi + \partial_s \psi \|^2 - E_1 | \psi |^2 \, dsdt \geq C \int_{\Omega_p} \rho(s) | \psi |^2 \, dsdt. \quad (17)$$

In the following we will use notations suggested in [6]. For $A \subset \mathbb{R}$ denote by $\chi_A$ the characteristic function of $A \times \omega$. Let $\psi \in D(q^1_0)$ and define,

$$q^1_A(\psi) := \| \chi_A \nabla^t \psi \|^2 - E_1 \| \chi_A \psi \|^2, \quad q^2_A(\psi) := \| \chi_A \dot{\theta} \partial_t \psi \|^2, \quad q^3_A(\psi) := 2 \Re(\chi_A \dot{\theta} \partial_t \psi). \quad (18)$$

Denote also by $Q^A(\psi) = q^1_A(\psi) + q^2_A(\psi) + q^3_A(\psi)$.

Here and hereafter we often use the fact that for any $\psi \in D(q^1_0)$

$$q^1_A(\psi) \geq 0, \quad (19)$$

for every $A \subset \mathbb{R}$ such that $A \cap I_0(l) = \emptyset$.

**Proof.** Choose $r > 0$ such that $\dot{\theta}(s) \neq 0$ for any $s \in [p - r, p]$. Let $f$ be the following function:

$$f(s) := \begin{cases} 0, & \text{on } (p, \infty); \\ \frac{p - s}{r}, & \text{on } (p - r, p]; \\ 1, & \text{elsewhere.} \end{cases} \quad (20)$$

For any $\psi \in D(q^1_0)$, simple estimates lead to:

$$\int_{\Omega_p} \frac{| \psi(s, t) |^2}{1 + (s - p)^2} \, dsdt = \int_{\Omega_p} \frac{| \psi(s, t)f(s) + (1 - f(s))\psi(s, t) |^2}{1 + (s - p)^2} \, dsdt \quad (21)$$

$$\leq 2 \left( \int_{\Omega_p} \frac{| f(s)\psi(s, t) |^2}{(s - p)^2} \, dsdt + \| \chi_{(p - r, p)} \psi \|^2 \right).$$
Since \( f(p)\psi(p, \cdot) = 0 \), we can use the usual Hardy inequality (see e.g. [9]), then we get,

\[
\int_{\Omega_p} \frac{|\psi(s, t)|^2}{1 + (s - p)^2} ds dt \leq 8q_2^{(-\infty, p)}(f\psi) + 2\|\chi_{(p-r, p)}\psi\|^2. \tag{22}
\]

Note that with our choice \([p - r, p] \cap [a, a + \ell] = \emptyset\). Hence to estimate the second term on the r.h.s of (22) we use the Theorem 6.5 of [10], then there exists \( \lambda_0 = \lambda_0(\theta, p, r) > 0 \) s.t. for any \( \psi \in D(q_0^l) \) we have

\[
\|\chi_{(p-r, p)}\psi\|^2 \leq \frac{1}{\lambda_0} Q^{(p-r, p)}(\psi) \leq \frac{1}{\lambda_0} Q^{(-\infty, p)}(\psi). \tag{23}
\]

We now want to estimate the first term on the right hand side of (22). We have

\[
q_2^{(-\infty, p)}(f\psi) = \int_{\Omega_p} |\partial_s(f\psi)|^2 \, ds dt = q_2^{(-\infty, \theta_m)}(f\psi) + q_2\bigl(\Theta_{\psi,m}\bigr)(f\psi). \tag{24}
\]

Evidently since \( \hat{\theta} = 0 \) and \( f = 1 \) in \((-\infty, \theta_m)\), from (19), we have

\[
q_2^{(-\infty, \theta_m)}(f\psi) \leq Q^{(-\infty, \theta_m)}(\psi). \tag{25}
\]

In the other hand since \( f(p)\psi(p, \cdot) = 0 \), we can apply the Lemma 4.1 of the Appendix. So for any \( 0 < \alpha < 1 \) there exists \( \gamma_{\alpha,1} > 0 \) such that

\[
|q_2\bigl(\Theta_{\psi,m}\bigr)(f\psi)| \leq \gamma_{\alpha,1} q_1^{(\theta_m,p)}(f\psi) + \alpha q_2^{(\theta_m,p)}(f\psi) + q_3^{(\theta_m,p)}(f\psi). \tag{26}
\]

Let \( \gamma := \max(1, \gamma_{\alpha,1}) \). Then

\[
\gamma^{-1} |q_2\bigl(\Theta_{\psi,m}\bigr)(f\psi)| \leq q_1^{(\theta_m,p)}(f\psi) + \alpha \gamma^{-1} q_2^{(\theta_m,p)}(f\psi) + \gamma^{-1} q_3^{(\theta_m,p)}(f\psi). \tag{27}
\]

Hence with the decomposition, \( q_2\bigl(\Theta_{\psi,m}\bigr) = \gamma^{-1} q_2^{(\theta_m,p)} + (1 - \gamma^{-1}) q_2^{(\theta_m,p)} \) and (27) we have,

\[
Q^{(\theta_m,p)}(f\psi) \geq (1 - \gamma^{-1}) \left(q_1^{(\theta_m,p)}(f\psi) + q_2^{(\theta_m,p)}(f\psi) + q_3^{(\theta_m,p)}(f\psi)\right) \tag{28}
\]

\[
+ \gamma^{-1} (1 - \alpha) q_2^{(\theta_m,p)}(f\psi)
\]

and since \( q_3^{(\theta_m,p)} + q_2^{(\theta_m,p)} + q_2^{(\theta_m,p)} \geq 0 \), we arrive at,

\[
q_2^{(\theta_m,p)}(f\psi) \leq \frac{\gamma}{(1 - \alpha)} Q^{(\theta_m,p)}(f\psi). \tag{29}
\]

Now by using that, \( q_1^{(\theta_m,p)}(f\psi) \leq q_1^{(\theta_m,p)}(\psi) \),

\[
\|\chi_{(\theta_m,p)}(\partial_s + \hat{\theta}\partial_r)(f\psi)\|^2 \leq 2\|\chi_{(\theta_m,p)}(\partial_s + \hat{\theta}\partial_r)\psi\|^2 + \frac{1}{r^2}\|\chi_{(p-r, p)}\psi\|^2
\]

and (23), we get,

\[
q_2^{(\theta_m,p)}(f\psi) \leq \frac{2\gamma}{(1 - \alpha)} Q^{(\theta_m,p)}(\psi) + \frac{1}{\lambda_0 r^2} Q^{(p-r, p)}(\psi) \leq c' Q^{(\theta_m,p)}(\psi) \tag{30}
\]
with \( c' = \frac{2\gamma}{(1 - \alpha)} (1 + \frac{1}{3\sigma_0^p}) \). Then (25) and (30) imply
\[
q_2^{(-\infty,p)}(f\psi) \leq (1 + c')Q^{(-\infty,p)}(\psi).
\] (31)
Hence (31) and (23) prove the lemma with
\[
C^{-1} = 8(1 + c') + \frac{2}{\lambda_0}.
\] (32)

Proof of the proposition 3.3. To prove the proposition we note that for any \( \psi \in D(q_p') \) and for \( p' \in \mathbb{R} \) we have
\[
\int_\omega \int_0^\infty |\nabla' \psi|^2 + |\hat{\theta} \partial_r \psi + \partial_s \psi|^2 \, dsdt \geq \int_\omega \int_0^\infty g(s) |\psi|^2 \, dsdt.
\] (33)
Then (33) with \( p' = p \) and Lemma 3.4 imply (16).

3.2 Reduction to a one dimensional problem

We now want to prove the following result,

**Proposition 3.5.** Under conditions of the Proposition 3.1, then a sufficient condition in order to get (13) is given by
\[
\int_{\mathbb{R}} |\psi'(s)|^2 + 2C\rho(s) |\psi(s)|^2 \, ds - 4E_1 \int_a^{a+l} |\psi(s)|^2 \, ds \geq 0,
\] (34)
for any \( \psi \in H^1(\mathbb{R}) \) where the constant \( C \) is defined in (32).

**Remark 3.6.** This proposition means that the positivity needed here is given by the positivity of the effective one dimensional Schrödinger operator on \( L^2(\mathbb{R}) \),
\[
-\frac{d^2}{ds^2} + 2C\rho(s) - 4E_1 1_{I_a(l)}.
\] (35)
where \( 1_{I_a(l)} \) is the characteristic function of \( I_a(l) \).

**Proof.** Evidently we have
\[
Q(\psi) = \frac{1}{2} (Q(\psi) - \int_{\Omega_0} (E_1 - g(s)) |\psi|^2 \, dsdt + \int_{\Omega_0} g(s) |\psi|^2 \, dsdt),
\] (36)
where \( g \) is defined in (14). By using (16), then
\[
Q(\psi) \geq \frac{1}{2} (q_p'(\psi) - E_1 \|\psi\|^2 + C \int_{\Omega_0} \rho(s) |\psi|^2 \, dsdt - E_1 \|\chi_{(a,a+l)}\psi\|^2)
\] (37)
Rewrite the expression of \( q_p' \) given by (3) as follows:
\[
q_p'(\psi) = \|\nabla' \psi\|^2 + \|\partial_s \psi\|^2 + \|\hat{\theta} \partial_r \psi\|^2 + 2R(\partial_s \psi, \hat{\theta} \partial_r \psi).
\] (38)
We estimate the last term of the r.h.s. of (38). By using the formula (49) of the Appendix,

\[ |q_{2,3}(\psi)| = |q_{2,3}^{(\theta_{m}, \theta_{M})}(\psi)| \leq \gamma_{\frac{1}{2}+}q_{1}^{(\theta_{m}, \theta_{M})}(\psi) + \frac{1}{2}q_{2}^{(\theta_{m}, \theta_{M})}(\psi) + \frac{1}{2}q_{3}^{(\theta_{m}, \theta_{M})}(\psi) \]  

where

\[ \gamma_{\frac{1}{2}+} := \bar{\gamma}_{\frac{1}{2}+} + 4d^2 \| \hat{\theta} \|_{\infty}^2 \]  

with \( \bar{\gamma}_{\frac{1}{2}+} := \max\left\{ \frac{d\|\theta\|_{\infty}}{\theta_0\sqrt{\lambda}}, \frac{d^2\|\theta\|_{\infty}^2 f(L)}{\lambda\theta_0}, 2d^2 \| \hat{\theta} \|_{\infty}^2 f(L) \right\} \) for some constant \( \lambda > 0 \) depending only on the section \( \omega \) and \( f(L) := \max\{2 + \frac{16d^2}{\nu}, 4L^2\} \).

Hence (38) together with (39) give:

\[ q'_{\theta}(\psi) \geq \| \nabla' \psi \|^2 + \frac{1}{2} \| \partial_s \psi \|^2 + \frac{1}{2} \| \hat{\theta} \partial_s \psi \|^2 - \gamma_{\frac{1}{2}+}q_{1}^{(\theta_{m}, \theta_{M})}(\psi). \]  

In view of (19) we have

\[ \| \nabla' \psi \|^2 - E_1 \| \psi \|^2 \geq q_{1}^{(\theta_{m}, \theta_{M})}(\psi) + q_{1}^{I_{l}(t)}(\psi) \geq q_{1}^{(\theta_{m}, \theta_{M})}(\psi) - E_1 \| \chi_{(a,l+a)} \psi \|^2. \]

Thus this last inequality together with (41) in (37) give

\[
Q(\psi) \geq \frac{1}{2} \left( \frac{1}{2} \| \partial_s \psi \|^2 + \frac{1}{2} \| \hat{\theta} \partial_s \psi \|^2 + C \int_{\Omega_l} \rho(s) | \psi |^2 ds dt - 2E_1 \| \chi_{(a,l+a)} \psi \|^2 \right.
\]
\[+ \left. (1 - \gamma_{\frac{1}{2}+})q_{1}^{(\theta_{m}, \theta_{M})}(\psi) \right). \]

Now if \( 0 < d \leq d_{\text{max}} \) then \( \gamma_{\frac{1}{2}+} \leq 1 \) so the Proposition 3.5 follows.

### 3.3 The one dimensional Schrödinger operator

In this part, under our conditions, we want to show that the one dimensional Schrödinger operator (35) is a positive operator. In view of the Proposition 3.5 this will imply the Proposition 3.1. Here we follow a similar strategy as in [1].

**Proposition 3.7.** for all \( \varphi \in H^1(\mathbb{R}) \), then there exists \( l_{\text{max}} > 0 \) such that for any \( 0 < l \leq l_{\text{max}} \) we have

\[ \int_{\mathbb{R}} | \varphi'(s) |^2 + 2C \rho(s) | \varphi(s) |^2 ds \geq 4E_1 \int_{I_{l}(t)} | \varphi(s) |^2 ds. \]  

**Proof.** Introduce the following function:

\[ \Phi(s) := \left\{ \begin{array}{ll}
\left( \frac{\pi}{2} + \arctan (s-p) \right), & \text{if } s < p; \\
\frac{\pi}{2}, & \text{if } s \geq p.
\end{array} \right. \]  

where \( p \) is the same real number as in (15). So clearly \( \Phi' = \rho \). For any \( t \in I_{a}(l) \) and \( \varphi \in H^1(\mathbb{R}) \), we have:

\[
\frac{\pi}{2} \varphi(t) = \Phi(t) \varphi(t) = \int_{-\infty}^{t} (\Phi(s) \varphi(s))' ds
\]
\[= \int_{-\infty}^{t} \rho(s) \varphi(s) ds + \int_{-\infty}^{t} \Phi(s) \varphi'(s) ds \]
and since \( \rho(s) = 0 \) for any \( s \in (p, \infty) \), we get,
\[
\frac{\pi}{2} \varphi(t) = \int_{-\infty}^{p} \rho(s)\varphi(s)ds + \int_{-\infty}^{t} \Phi(s)\varphi'(s)ds. \tag{45}
\]

Then some straightforward estimates lead to,
\[
\frac{\pi^2}{4} \varphi^2(t) \leq 2 \left( \left( \int_{-\infty}^{p} \rho(s)\varphi(s)ds \right)^2 + \left( \int_{-\infty}^{t} \Phi(s)\varphi'(s)ds \right)^2 \right) \tag{46}
\]
\[
\leq 2 \left( \int_{-\infty}^{p} \rho(s)ds \int_{-\infty}^{p} \rho(s)\varphi^2(s)ds + \int_{-\infty}^{t} \Phi^2(s)ds \int_{-\infty}^{t} \varphi'^2(s)ds \right).
\]

By direct calculation \( \int_{-\infty}^{p} \rho(s)ds = \frac{\pi}{2} \) and \( \int_{-\infty}^{p} \Phi^2(s)ds + \int_{p}^{t} \Phi^2(s)ds = \pi \ln 2 + \frac{\pi^2}{4}(t - p) \). Hence we get,
\[
|\varphi(t)|^2 \leq \frac{4}{\pi} \int_{-\infty}^{p} \rho(s)\varphi^2(s)ds + \left( \frac{8 \ln 2}{\pi} + 2(t - p) \right) \int_{-\infty}^{t} |\varphi'(s)|^2 ds \tag{47}
\]

We integrate both sides of (47) over \( I_a(l) \), then
\[
\int_{I_a(l)} |\varphi(t)|^2 dt \leq \frac{4l}{\pi} \int_{-\infty}^{p} \rho(s)\varphi^2(s)ds + \left( \frac{8 \ln 2}{\pi} + 2(a - p) \right) \int_{-\infty}^{t} |\varphi'(s)|^2 ds \]
\[
\leq c'' \int_{-\infty}^{t} 2C\rho(s)\varphi^2(s) + |\varphi'(s)|^2 ds
\]
where \( c'' = 2l\left( \frac{1}{\pi C} + \frac{4 \ln 2}{\pi} + a - p \right) + l^2 \). Finally we get,
\[
4E_1 \int_{a}^{l+a} |\varphi(t)|^2 dt \leq 4E_1 c'' \int_{-\infty}^{t} 2C\rho(s) |\varphi(s)|^2 + |\varphi'(s)|^2 ds. \tag{48}
\]

So choose \( 0 < l \leq l_{max} \) with
\[
l_{max} := \left( \frac{1}{\pi C} + \frac{4 \ln 2}{\pi} + a - p \right) + \sqrt{\left( \frac{1}{\pi C} + \frac{4 \ln 2}{\pi} + a - p \right)^2 + (4E_1)^{-1}}
\]
then \( 4E_1 c'' \leq 1 \) and the proposition 3.7 follows.

\[\square\]

### 3.4 proof of the Theorem 1.1 ii)

Under assumptions of the Theorem 1.1 ii) \( H_{p}^{\infty} \geq H_{p} \). These two operators have the same essential spectrum so the Theorem 1.1 ii) is proved by applying the Proposition 3.1 and the min-max principle.

### 4 Appendix

In this appendix we give a slight extension of the lemma 3 of [6] which states that under our conditions, for all \( \psi \in D(q_{0}) \) we have for any \( \alpha, \beta > 0 \) there exists \( \gamma_{a,\beta} > 0 \) such that:
\[
|q_{2,3}(\psi)| \leq \gamma_{a,\beta} q_{1}(\psi) + \alpha q_{2}(\psi) + \beta q_{3}(\psi). \tag{49}
\]

Then we have
Lemma 4.1. Let $p \in (\theta_m, \theta_M)$. For all $\psi \in D(q_0^p)$ such that $\psi(p,.) = 0$, then for any $\alpha, \beta > 0$ there exists $\gamma_{\alpha, \beta} > 0$ such that:

$$| q_2^{(\theta_m, p)}(\psi) | \leq \gamma_{\alpha, \beta} q_1^{(\theta_m, p)}(\psi) + \alpha q_2^{(\theta_m, p)}(\psi) + \beta q_3^{(\theta_m, p)}(\psi).$$

(50)

Proof. Let $\psi \in D(q_0^p)$ such that $\psi(p,.) = 0$. Then $\psi \in H_0^1(\Omega_p)$. We know that we may first consider vectors $\psi_s(t) = \chi_1(t)\phi(s, t)$, where $\phi \in C_0^\infty(\Omega_p)$. For such a vector $\psi$ we have,

$$q_1^{(\theta_m, p)}(\psi) = \| \chi_{(\theta_m, p)}\chi_1 \nabla' \phi \|^2, \quad q_2^{(\theta_m, p)}(\psi) = \| \chi_{(\theta_m, p)}\chi_1 \partial_s \phi \|^2$$

(51)

and

$$q_3^{(\theta_m, p)}(\psi) = \| \chi_{(\theta_m, p)}\tilde{\phi}(\chi_1 \partial_r \phi + \phi \partial_r \chi_1) \|^2$$

(52)

Integrating by parts twice and using the fact that $\dot{\chi}_1(0) = 0$ and $\ddot{\chi}_1(0) = 0$ then (53) and (58) imply

$$\| \chi_{(\theta_m, p)}\tilde{\phi}\chi_1, \chi_1 \partial_s \phi \| = \| \chi_{(\theta_m, p)}\tilde{\phi}\chi_1, \chi_1 \partial_r \phi \|.$$  

(54)

Then the Cauchy Schwartz inequality implies,

$$\| (\chi_{(\theta_m, p)}\tilde{\phi}\chi_1, \chi_1 \partial_r \phi) \|^2 \leq \| \tilde{\phi} \|^2 \| q_1^{(\theta_m, p)} \chi_{(\theta_m, p)}\chi_1 \|^2.$$  

(55)

Let $p' \in \mathbb{R}$ and $r' > 0$ such that $(p' - r', p') \subset (\theta_m, p)$ and for $s \in (p' - r', p')$, $|\tilde{\phi}(s)| \geq \tilde{\phi}_0$ for some $\tilde{\phi}_0 > 0$. As in the proof of the Lemma 3 of [6] we have,

$$\| \chi_{(\theta_m, p)}\chi_1 \|^2 \leq c_2 \left( q_2^{(\theta_m, p)}(\psi) + \tilde{\phi}_0^{-2} \| \chi_{(p' - r', p')}\tilde{\phi}\chi_1 \|^2 \right)$$

(56)

where $c_2 := \max\left\{ 2 + 16\left(\frac{p - \theta_m}{r'}\right)^2, 4(p - \theta_m)^2 \right\}$.

Moreover, for any $s \in \mathbb{R}$, $\tilde{\phi}(s) \chi_1(\theta, s) \in H_0^1(\Omega_p)$, then by using the Lemma 1 of [6] there exists $\lambda > 0$ depending on $\omega$ such that :

$$\| \chi_{(p' - r', p')}\tilde{\phi}\chi_1 \|^2 \leq \chi_{(\theta_m, p)}\tilde{\phi}\chi_1 \| \chi_{(\theta_m, p)}\tilde{\phi}\chi_1 \|^2 \leq \lambda^{-1} \left( q_2^{(\theta_m, p)}(\psi) + \tilde{\phi}_0 \right)^2.$$  

(57)

Hence (56), (57) and (54) give

$$\| (\chi_{(\theta_m, p)}\tilde{\phi}\chi_1, \chi_1 \partial_r \phi) \|^2 \leq \left( c_3 q_1^{(\theta_m, p)}(\psi) + \alpha \| q_2^{(\theta_m, p)}(\psi) + \beta q_3^{(\theta_m, p)}(\psi) \right)^2$$

(58)

where $c_3 := \max\left\{ d \| \tilde{\phi} \|_\infty \sqrt{2}, d \| \tilde{\phi} \|_\infty^2, d \| \tilde{\phi} \|_\infty^2 \right\}$. Then (53) and (58) imply (50) with $\gamma_{\alpha, \beta} := c_1 + c_3$.

Note that we can choose $c_1 > 0$ on $\omega$. So that (50) holds for every $\psi \in C_0^\infty(\Omega_p)$ and by a density argument this is even true for $\psi \in H_0^1(\Omega_p)$.
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