A sharp $L^p$-Hardy type inequality on the $n$-sphere

Songting Yin, Yu Ren, Caiyun Liu*

Department of Mathematics and Computer Science, Tongling University, Tongling 244000 Anhui, China

*Corresponding author, e-mail: freeliucy@163.com

ABSTRACT: We obtain an $L^p$-Hardy inequality on the $n$-sphere and give the corresponding sharp constant. Furthermore, the obtained inequalities are used to derive an uncertainty principle inequality and some corollaries. The results generalize and improve some related inequalities in recent literature.

KEYWORDS: Hardy inequality, sphere, sharp constant

MSC2010: 26D10 46E36

INTRODUCTION

For $n \geq 3$, $p \geq 1$, and all $f \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$, the classical $L^p$-Hardy inequality is given by

$$
\int_{\mathbb{R}^n} |\nabla f|^p \, dx \geq \left( \frac{n-p}{p} \right)^p \int_{\mathbb{R}^n} |f|^p \, dx,
$$

where the constant $\left( \frac{n-p}{p} \right)^p$ is sharp. This inequality has been studied extensively in the Euclidean spaces (see [1–3]) due to its applications in different fields such as harmonic analysis, physics, spectral theory, geometry, and partial differential equations. For this line of research, we refer to [4–6] and the references therein.

In the case of Riemannian manifolds, there are also many valuable research (see [4,7] and so on) in Hardy inequality. Carron [8] studied the weighted $L^2$-Hardy inequalities under several geometric assumptions. More specifically, he proved that

$$
\int_M \rho^a |\nabla f|^2 \, dV \geq \left( \frac{C + \alpha - 1}{2} \right)^2 \int_M \rho^a \frac{f^2}{\rho^2} \, dV,
$$

for any function $f \in C_c^\infty(M \setminus \rho^{-1}\{0\})$ and $C + \alpha - 1 > 0$, where the weight function $\rho$ must satisfy both $|\nabla \rho| = 1$ and $\Delta \rho \geq C/\rho$. In particular, Kombe-Ozaydin [9] extended Carron’s results to the general case for $1 \leq p < C + 1 + \alpha$ and derived that

$$
\int_M \rho^a |\nabla f|^p \, dV \geq \left( \frac{C + \alpha - p}{p} \right)^p \int_M \rho^a \frac{|f|^p}{\rho^p} \, dV.
$$

For more generalizations see [5, 10, 11].

There are several important known results (see [12–15]) which reveal an importance of the scale invariance of the classical Hardy inequality in a ball. For more details, we further refer to [16–19] and the references therein.

As is well known, the $n$-sphere is of constant curvature and possess the delicate symmetry. However, there are only a few Hardy inequalities obtained on the $n$-sphere so far. Recently, Dai-Xu [20] and Xiao [21] discussed this issue and established some $L^2$-Hardy inequalities. By introducing the tangent function, Yin [22] acquired the following

$$
\frac{n-2}{2} \int_{S^n} f^2 \, dV + \int_{S^n} |\nabla f|^2 \, dV \geq \frac{(n-2)^2}{4} \int_{S^n} \frac{f^2}{\tan^2 d(p, x)} \, dV,
$$

where $p$ is a fixed point in $S^n$, and the constant $\frac{(n-2)^2}{4}$ is sharp. Based on the results above, Abolarinwa-Apata [1], Abolarinwa-Rauf-Yin [23], and Sun-Pan [24] further gave some $L^p$-Hardy inequalities on the sphere.

MAIN RESULT

In this paper, we present a more general version of $L^p$ Hardy inequalities on the unit $n$-sphere and show that the associated constant is the best possible, which is a generalization of those in [23] and [22]. Applications of the obtained inequality yield an uncertainty principle inequality and some corollaries. The main theorem is stated as follows:

**Theorem 1** Let $S^n$ be the standard $n$-sphere with sectional curvature 1. Then for any function $f \in$
To prove the result, we follow the arguments in [22] (see also [23]) with some modifications. First, we construct an auxiliary function by utilizing the symmetry of the sphere, and then using the antipodal points. We will carry out the calculation in two hemispheres. Considering that the auxiliary function can only be continuous, we use an approximation of smooth functions to show the sharpness of the constant. Since the introduction of general $p$ and $\alpha$ makes the calculation more complicated than that in [22], we need some techniques in the cumbersome computation below to estimate some terms in the $L^p$ case.

**Proof of Theorem 1**

Let $f = \psi^p\phi$ with $\gamma < 0$ be a smooth function in $C^\infty(S^n)$. Then we have

$$|\nabla f|^p = |\phi^\gamma \psi^{-1}\nabla \psi + \psi^\gamma \nabla \phi|^p.$$  

Notice that the following inequality is valid for any $a, b \in \mathbb{R}^n$ and $p \geq 1$:

$$|a + b|^p - |a|^p \geq p|a|^{p-2} \langle a, b \rangle.$$  

Therefore, one obtains

$$|\nabla f|^p \geq |\phi^p \psi^{\gamma p - p} \gamma |\nabla \psi|^p + p|\gamma|^{p-2} \phi^{p-2} \psi^{\gamma p - 2p - 2p + 2} \langle \phi \gamma \psi^{-1} \nabla \psi, \psi^\gamma \nabla \phi \rangle = |\phi^p \psi^{\gamma p - p} |\nabla \psi|^p + p|\gamma|^{p-2} \phi^{p-2} \psi^{\gamma p - p + 1} \langle \nabla \psi, \nabla \phi \rangle.$$  

Compute

$$\psi^\alpha |\nabla f|^p \geq |\phi^p \psi^{\gamma p - p + \alpha} |\nabla \psi|^p$$

$$\geq |\phi^p \psi^{\gamma p - p + \alpha} |\nabla \psi|^p + \frac{|\gamma|^{p-2} \phi^{p-2} \psi^{\gamma p - p + 1 + \alpha} \langle \nabla \psi, \nabla \phi \rangle}{\gamma p - p + \alpha + 2}$$

$$= |\phi^p \psi^{\gamma p - p + \alpha} |\nabla \psi|^p + \frac{|\gamma|^{p-2} \phi^{p-2} \psi^{\gamma p - p + 2}}{\gamma p - p + \alpha + 2} \langle \nabla \psi, \nabla \phi \rangle + \frac{\gamma p - p + \alpha + 2}{\gamma p - p + \alpha + 2} \Delta (\psi^\gamma \nabla \phi), \hspace{1cm} (2)$$

and

$$\Delta (\sin r)^{-\beta} = \frac{\sin (\sin r)^{-\beta}}{\sin (\sin r)^{-\beta}} = \frac{\sin (-\beta (\sin r))^{-\beta-1} \cos r \nabla r}{\sin (\sin r)^{-\beta-1} \cos r \Delta r}$$

$$= -\beta (\sin r)^{-\beta-1} \cos r \Delta r + \beta (\beta + 1)(\sin r)^{-\beta-2} \cos^2 r + \beta (\sin r)^{-\beta}. \hspace{1cm} (3)$$

Substituting $\Delta r = (n-1)\cot r$ into (3) yields

$$\Delta (\sin r)^{-\beta} = \beta (\beta + 2 - n)(\sin r)^{-\beta-2} + \beta (n - \beta - 1)(\sin r)^{-\beta}.$$
Now by taking $\beta = -(\gamma p - p + \alpha + 2)$ and $\gamma = \frac{p - a - n}{p}$, we derive
\[
\Delta (\sin r)^{-\beta} = (n - 2)(\sin r)^{2-n}.
\]
Further, letting $\psi = \sin r$, it follows from (2) that
\[
(\sin^a r)|\nabla f|^p \geq (\sin r)^{-\gamma p} |\nabla (\sin r)^{-\gamma p} f|^p + \frac{1}{n-2} \left( \frac{n + a - p}{p} \right)^{p-1} \div (\sin r)^{-\gamma p} |\nabla (\sin r)^{-\gamma p} f|^p - \frac{1}{n-2} \left( \frac{n + a - p}{p} \right)^{p-1} (\sin r)^{-\gamma p} |\nabla (\sin r)^{-\gamma p} f|^p = \left( \frac{n + a - p}{p} \right)^{p} (\sin^a r) |f|^p \frac{\cos r}{\tan r} \div \left( \frac{n + a - p}{p} \right)^{p-1} |\nabla (\sin r)^{-\gamma p} f|^p - \frac{1}{n-2} \left( \frac{n + a - p}{p} \right)^{p-1} (\sin r)^{-\gamma p} |\nabla (\sin r)^{-\gamma p} f|^p + \frac{1}{n-2} \left( \frac{n + a - p}{p} \right)^{p-1} \div (\sin r)^{-\gamma p} |\nabla (\sin r)^{-\gamma p} f|^p.
\]
Integrating both sides of the above inequality on $S^n$, and applying the divergence theorem, we deduce that
\[
\int_{S^n} (\sin^a r)|\nabla f|^p dV \geq \int_{S^n} \left( \frac{n + a - p}{p} \right)^{p} (\sin^a r) |f|^p \frac{\cos r}{\tan r} dV.
\]
This completes the proof of the inequality stated in Theorem 1. Next, we show that the constant $\left( \frac{n + a - p}{p} \right)^{p}$ is sharp. Let $\zeta : R \rightarrow [0, 1]$ be a smooth function such that $0 \leq \zeta \leq 1$ and
\[
\zeta(t) = \begin{cases} 
1, & |t| \leq 1; \\
0, & |t| > 2.
\end{cases}
\]
Let $H(t) = 1 - \zeta(t)$, and for sufficiently small $\varepsilon > 0$ we construct
\[
f_{\varepsilon}(r) = \begin{cases} 
0, & r = 0; \\
H \left( \frac{\pi}{\varepsilon} \right) (\tan r)^{-\gamma p}, & 0 < r \leq \frac{\varepsilon}{2}; \\
H \left( \frac{\pi}{\varepsilon} \right) (\tan (\pi - r))^{-\gamma p}, & \frac{\varepsilon}{2} \leq r < \pi; \\
0, & r = \pi.
\end{cases}
\]
Observe that $f_{\varepsilon}(r)$ is continuous and can be approximated by smooth functions on the sphere $S^n$. For the antipodal points $q$ and $q'$ on $S^n$, let $r_q$ (respectively, $r'_q$) denote the distance function from $q$ (respectively, $q'$). Then we have $r_q + r'_q = \pi$, and thus
\[
\int_{S^n} \frac{|f_r|^p}{(\sin r)^{p-a-2}} dV = \int_{B_\varepsilon(\xi)} \frac{|f_r|^p}{(\sin r)^{p-a-2}} dV + \int_{B'_\varepsilon(\xi)} \frac{|f_r|^p}{(\sin r)^{p-a-2}} dV,
\]
where
\[
\int_{B_\varepsilon(\xi)} \frac{|f_r|^p}{(\sin r)^{p-a-2}} dV = \text{Vol}(S^{n-1}) \int_{-\varepsilon}^{\varepsilon} \frac{H}(\zeta) (\tan r)^{p-a-n} \times (\sin (\pi - r))^2 (\sin (\pi - r))^{-1} dr \leq \text{Vol}(S^{n-1}) \int_{-\varepsilon}^{\varepsilon} r_q dr = \frac{\text{Vol}(S^{n-1})}{2} \left( \frac{\pi^2}{4} - \varepsilon^2 \right)
\]
and
\[
\int_{B'_\varepsilon(\xi)} \frac{|f_r|^p}{(\sin r)^{p-a-2}} dV = \text{Vol}(S^{n-1}) \int_{-\varepsilon}^{\varepsilon} \frac{H}{(\zeta)} (\tan r)^{p-a-n} \times (\sin (\pi - r))^{2+a-p} (\sin (\pi - r))^{-1} dr \leq \text{Vol}(S^{n-1}) \int_{-\varepsilon}^{\varepsilon} r_q dr = \frac{\text{Vol}(S^{n-1})}{2} \left( \frac{\pi^2}{4} - \varepsilon^2 \right).
\]
Combining the above two inequalities, we obtain
\[
\int_{S^n} \frac{|f_r|^p}{(\sin r)^{p-a-2}} dV \leq \text{Vol}(S^{n-1}) \left( \frac{\pi^2}{4} - \varepsilon^2 \right).
\]
On the other hand, we have
\[
\int_{B_\varepsilon(\xi)} \frac{|f_r|^p}{(\tan r)^{p-a-2}} dV = \text{Vol}(S^{n-1}) \times \int_{-\varepsilon}^{\varepsilon} H(\zeta) (\tan r)^{p-a-n} (\sin r_q)^p (\tan r_q)^{-1} dr \geq \text{Vol}(S^{n-1}) \times \int_{-2\varepsilon}^{2\varepsilon} H(\zeta) (\tan r_q)^{a+n} (\sin r_q)^p (\tan r_q)^{-1} dr = \text{Vol}(S^{n-1}) \int_{2\varepsilon}^{2\varepsilon} (\tan r_q)^{a+n} (\sin r_q)^p \frac{\cos r}{\tan r} dV.
\]
and
\[
\int_{B_{r_{\theta}}(\frac{\pi}{2})} (\sin r_{q})^a |f_{q}|^p (\tan r_{q})^p dV
\]
\[
= \text{Vol}(S^{n-1}) \int_{\epsilon}^{\frac{\pi}{2}} H^p \left( \frac{r_{\theta}}{r_{q}} \right) (\tan(\pi - r_{q}))^{p-a-n} d\rho
\times \frac{(\sin(\pi - r_{q}))^a}{(\tan(\pi - r_{q}))^p} (\sin(\pi - r_{q}))^{-n} d\rho
\]
\[
= \text{Vol}(S^{n-1}) \int_{\epsilon}^{\frac{\pi}{2}} H^p \left( \frac{r_{\theta}}{r_{q}} \right) (\tan r_{q})^{p-a-n} d\rho
\times (\sin r_{q})^a (\sin r_{q})^{-n} d\rho
\]
\[
\geq \text{Vol}(S^{n-1}) \int_{2\epsilon}^{\frac{\pi}{2}} H^p \left( \frac{r_{\theta}}{r_{q}} \right) (\tan r_{q})^{-a-n} d\rho
\times (\sin r_{q})^{a+n-1} d\rho
\]
\[
= \text{Vol}(S^{n-1}) \int_{2\epsilon}^{\frac{\pi}{2}} (\tan r_{q})^{-a-n} (\sin r_{q})^{a+n-1} d\rho.
\]

Adding the two inequalities above together gives
\[
\int_{S^n} (\sin^a r) |f_{q}|^p dV \geq 2\text{Vol}(S^{n-1})
\times \int_{2\epsilon}^{\frac{\pi}{2}} (\tan r_{q})^{-a-n} (\sin r_{q})^{a+n-1} d\rho.
\] (5)

Next, we are going to estimate the integral
\[
\int_{S^n} (\sin^a r) |\nabla f_{q}|^p dV = \int_{B_{r_{\theta}}(\frac{\pi}{2})} (\sin r_{q})^a |\nabla f_{q}|^p dV
\]
\[
+ \int_{B_{r_{\theta}}(\frac{\pi}{2})} (\sin r_{q})^a |\nabla f_{q}|^p dV.
\]

By a cumbersome calculation, we have
\[
\left( \int_{B_{r_{\theta}}(\frac{\pi}{2})} (\sin r_{q})^a |\nabla f_{q}|^p dV \right)^\frac{1}{p}
\]
\[
= \text{Vol}(S^{n-1}) \int_{\epsilon}^{\frac{\pi}{2}} H^{\frac{1}{p}} \left( \frac{r_{\theta}}{r_{q}} \right) \left( \frac{p-a-n}{p} \right) (\tan r_{q})^{p-a-n} d\rho
\]
\[
+ \left( \frac{p-a-n}{p} \right) H \left( \frac{r_{\theta}}{r_{q}} \right) (\tan r_{q})^{\frac{p-a-n}{p}} \sec^2 r_{q} \left( \sin r_{q} \right)^{-n} d\rho \leq \text{Vol}(S^{n-1}) \int_{\epsilon}^{\frac{\pi}{2}} H^{\frac{1}{p}} \left( \frac{r_{\theta}}{r_{q}} \right) (\tan r_{q})^{p-a-n} d\rho
\]
\[
+ \left( \frac{p-a-n}{p} \right) H \left( \frac{r_{\theta}}{r_{q}} \right) (\tan r_{q})^{\frac{p-a-n}{p}} \sec^2 (\pi - r_{q}) \left( \sin (\pi - r_{q}) \right)^{-n} d\rho.
\]
\[
\begin{align*}
&\quad = \text{Vol}(S^{n-1})^\frac{1}{2} \left( \int_\varepsilon^\infty \left| H' \left( \frac{r}{\varepsilon} \right) \left( \frac{1}{r} \right) (\tan r_q)^p \right| dr_q \right)^{-\frac{1}{p}} \\
&\quad - \frac{p-a-n}{p} \text{Vol}(S^{n-1})^\frac{1}{2} \max_{t \in [0,2]} H'(t) \\
&\quad + \frac{n+a-p}{p} \text{Vol}(S^{n-1})^\frac{1}{2} \left( \int_\varepsilon^\infty \left( (\tan r_q)^{-a-n} (\sin r_q)^{a+n-1} \right)^\frac{1}{p} dr_q \right)^p. \\
\end{align*}
\]

Therefore, it is not difficult to obtain that

\[
\begin{align*}
&\quad \int_{S^p} \sin^a r |\nabla f_e|^{p} \, dV \\
&\quad \leq 2 \left( \frac{2^{\frac{1}{p}} - 1}{2} \right)^\frac{1}{p} \text{Vol}(S^{n-1})^\frac{1}{2} \left( \int_\varepsilon^\infty \left( (\tan r_q)^{-a-n} (\sin r_q)^{a+n-1} \right)^\frac{1}{p} dr_q \right)^p. \\
\end{align*}
\]

Define

\[
A := \inf_{f \in C^\infty(S^n), \{0\}} \left\{ \int_{S^n} \left( \sin^a r \right) |\nabla f|^{p} \, dV \right\} + \left( \frac{n+a-p}{p} \right)^{p-1} \int_{S^n} \left[ \frac{|f|^{p}}{\sin r^{p-a-2}} \right] \, dV \left/ \int_{S^n} \left( \sin^a r \right) |\nabla f|^{p} \, dV \right. \right. \left. \right. \\
\]

\[
\begin{align*}
&\quad \text{Since } f_e(r) \text{ can be approximated by smooth functions on the sphere } S^n, \text{ it follows from (4)} - (6) \text{ that} \\
&\quad A \leq \left( \frac{n+a-p}{p} \right)^p \int_{S^n} \left( \sin^a r \right) |\nabla f_e|^{p} \, dV + \left( \frac{n+a-p}{p} \right)^{p-1} \int_{S^n} \left[ \frac{|f_e|^{p}}{\sin r^{p-a-2}} \right] \, dV. \\
\end{align*}
\]

Obviously, we have

\[
\lim_{r \to 0} \int_{2\varepsilon}^\infty \left( (\tan r_q)^{-a-n} (\sin r_q)^{a+n-1} \right)^\frac{1}{p} dr_q = \infty.
\]

By L'Hôpital's rule, it holds that

\[
\lim_{r \to 0} \frac{\int_{2\varepsilon}^\infty \left( (\tan r_q)^{-a-n} (\sin r_q)^{a+n-1} \right) \, dr_q}{\int_{2\varepsilon}^\infty \left( (\tan r_q)^{-a-n} (\sin r_q)^{a+n-1} \right)^\frac{1}{p} dr_q} = \frac{1}{1}
\]

and

\[
\lim_{r \to 0} \frac{\int_{2\varepsilon}^\infty \left( (\tan r_q)^{2p-a-n} (\sin r_q)^{a+n-1} \right) \, dr_q}{\int_{2\varepsilon}^\infty \left( (\tan r_q)^{-a-n} (\sin r_q)^{a+n-1} \right)^\frac{1}{p} dr_q} = 0.
\]

This means \( I = II = IV = 0 \). Therefore, we get from (7) that

\[
A \leq 2 \left( \frac{n+a-p}{p} \right)^p = \left( \frac{n+a-p}{p} \right)^p.
\]

The reverse inequality is also valid by (2). Thus the constant \( \left( \frac{n+a-p}{p} \right)^p \) is sharp.

**COROLLARIES**

Choosing some special \( \alpha = 0 \) and \( \alpha = p \) in Theorem 1, we obtain the following results.

**Corollary 1.** Let \( S^n \) be the standard n-sphere as in Theorem 1. Then

\[
\int_{S^n} \left| \nabla f \right|^{p} \, dV + \left( \frac{n-p}{p} \right)^{p-1} \int_{S^n} \left( \frac{|f|^{p}}{\sin r^{p-a-2}} \right) \, dV \geq \left( \frac{n-p}{p} \right)^p \int_{S^n} \left( \frac{|f|^{p}}{\sin r^{p-a-2}} \right) \, dV
\]

and the constant \( \left( \frac{n-p}{p} \right)^p \) is sharp.
Theorem 1. Then

\[
\int_{\mathcal{S}^n} (\sin r|\nabla f|^p dV + \left(\frac{n}{p}\right)^{p-1} \int_{\mathcal{S}^n} (\sin^2 r)|f|^p dV)
\geq \left(\frac{n}{p}\right)^{p} \int_{\mathcal{S}^n} (\cos r|f|^p dV),
\]

and the constant \(\left(\frac{n}{p}\right)^{p}\) is sharp.

Remark 4 When \(\alpha = 0\) the corresponding inequality is obtained in [23], where it is divided into two cases: \(1 \leq p < 2\) and \(p \geq 2\) due to some technical reasons. In fact, we can combine them into a unified inequality as above, and thus Corollary 1 is in a form more concise than that in [23].

Corollary 2 Let \(\mathcal{S}^n\) be the standard \(n\)-sphere as in Theorem 1. Then

\[
\int_{\mathcal{S}^n} (\sin r|\nabla f|^p dV + \left(\frac{n}{p}\right)^{p-1} \int_{\mathcal{S}^n} (\sin^2 r)|f|^p dV)
\geq \left(\frac{n}{p}\right)^{p} \int_{\mathcal{S}^n} (\cos r|f|^p dV),
\]

which yields that

\[
\int_{\mathcal{S}^n} |\nabla f|^2 dV + \frac{n}{2} \int_{\mathcal{S}^n} f^2 \sin^2 r dV \geq \frac{n^2}{4} \int_{\mathcal{S}^n} f^2 \cos^2 r dV,
\]

while in [22], the coefficient in the right-hand side is \(n(n-2)/4\).

The classical uncertainty principle as introduced in quantum mechanics says that the position and the momentum of a particle can not be exactly determined at the same time, but only with an “uncertainty”. There are various forms of the uncertainty principle. At present we shall apply Theorem 1 to derive a new form as follows.

Corollary 3 Let \(\mathcal{S}^n\) be the standard \(n\)-sphere as in Theorem 1. Then

\[
\left(\int_{\mathcal{S}^n} |f|^p \sin^q r |\tan r|^q dV\right)^\frac{1}{p} \left(\int_{\mathcal{S}^n} \sin^q r |\nabla f|^p dV\right)^\frac{1}{p} \geq \left(\frac{n+\alpha-p}{p}\right)^{p} \left(\int_{\mathcal{S}^n} |f|^p \sin^q r dV\right)^\frac{1}{p},
\]

where \(1/p + 1/q = 1\).

Proof: By Hölder's inequality, we have

\[
\int_{\mathcal{S}^n} |f|^p \sin^q r |\tan r|^q dV
\leq \left(\int_{\mathcal{S}^n} |f|^p \sin^q r dV\right)^\frac{1}{p} \left(\int_{\mathcal{S}^n} |f|^p \sin^q r |\tan r|^q dV\right)^\frac{1}{q}.
\]

A simple calculation yields

\[
\int_{\mathcal{S}^n} |f|^p \sin^q r |\tan r|^q dV
\geq \left(\int_{\mathcal{S}^n} |f|^p \sin^q r dV\right)^\frac{1}{p} \left(\int_{\mathcal{S}^n} |f|^p \sin^q r |\tan r|^q dV\right)^\frac{1}{q}.
\]

Combining it with Theorem 1, the result follows directly.

Acknowledgements: The work was supported in part by NNSFC (No.11971253), AHNSF (No.1608085MA03), TLXYXM (No.2018tlyxdz02, 2018tlydxs154) and AH-PUNSF (No.KJ2019A0701).

REFERENCES

1. Abolarinwa A, Apata T (2018) \(L^p\)-Hardy-Rellich and uncertainty principle inequalities on the sphere. Adv Oper Theory 3, 745–762.
2. Baras P, Goldstein J (1984) The heat equation with a singular potential. Trans Amer Math Soc 284, 121–139.
3. Costa D (2009) On Hardy-Rellich type inequalities in \(\mathbb{R}^n\). Appl Math Lett 22, 902–905.
4. D’Ambrosio L, Dipierro S (2014) Hardy inequalities on Riemannian manifolds and applications. Ann Inst Henri Poincaré, Anal Non Linéaire 31, 449–475.
5. Grillo G (2003) Hardy and Rellich-type inequalities for metrics defined by vector fields. Potential Anal 18, 187–217.
6. Sulaiman W (2012) Some Hardy type integral inequalities. Appl Math Lett 25, 520–525.
7. Du E, Mao J (2015) Hardy and Rellich type inequalities on metric measure spaces. J Math Anal Appl 429, 354–365.
8. Carron G (1997) Inégalités de Hardy sur les variétés Riemanniennes non-compactes. J Math Pures Appl 76, 883–891.
9. Kombe I, Özaydin M (2009) Improved Hardy and Rellich inequalities on Riemannian manifolds. Trans Amer Math Soc 361, 6191–6203.
10. Garcia Azorero J, Peral I (1998) Hardy inequalities and some critical elliptic and parabolic problems. J Differ Equations 144, 441–476.
11. Kombe I (2004) The linear heat equation with a highly singular, oscillating potential. Proc Am Math Soc 132, 2683–2691.
12. Ioku N (2019) Attainability of the best Sobolev constant in a ball. *Mathematische Annalen* **375**, 1–16.
13. Ioku N, Ishiwata M, Ozawa T (2017) Hardy type inequalities in $L^p$ with sharp remainders. *J Inequal Appl* **5**, 1–7.
14. Ioku N, Ishiwata M, Ozawa T (2016) Sharp remainder of a critical Hardy inequality. *Archiv Mathematik* **106**, 65–71.
15. Ioku N, Ishiwata M (2015) A scale invariant form of a critical Hardy inequality. *Int Math Res Not* **18**, 8830–8846.
16. Horiuchi T, Kumlin P (2012) On the Caffarelli-Kohn-Nirenberg type inequalities involving critical and supercritical weights. *Kyoto J Math* **52**, 661–742.
17. Machihara S, Ozawa T, Wadade H (2013) Hardy type inequalities on balls. *Tohoku Math J* **65**, 321–330.
18. Machihara S, Ozawa T, Wadade H (2017) Remarks on the Rellich inequality. *Math Z* **286**, 1367–1373.
19. Takahashi F (2015) A simple proof of Hardy’s inequality in a limiting case. *Arch Math* **104**, 77–82.
20. Dai F, Xu Y (2014) The Hardy-Rellich inequality and uncertainty principle on the sphere. *Constr Approx* **40**, 141–171.
21. Xiao Y (2016) Some Hardy inequalities on the sphere. *J Math Inequal* **10**, 793–805.
22. Yin S (2018) A sharp Hardy type inequality on the sphere. *New York J Math* **24**, 1101–1110.
23. Abolarinwa A, Rauf K, Yin S (2019) Sharp $L^p$ Hardy type and uncertainty principle inequalities on the sphere. *J Math Inequal* **13**, 1011–1022.
24. Sun X, Pan F (2017) Hardy type inequalities on the sphere. *J Inequal Appl* **148**, 1–8.