NEW APPROACH TO NONLINEAR ELECTRODYNAMICS:
DUALITIES AS SYMMETRIES OF INTERACTION

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Abstract

We elaborate on the duality-symmetric nonlinear electrodynamics in a new formulation with auxiliary tensor fields. The Maxwell field strength appears only in bilinear terms of the corresponding generic Lagrangian, while the self-interaction is presented by a function $E$ depending on the auxiliary fields. Two types of dualities inherent in the nonlinear electrodynamics models admit a simple off-shell characterization in terms of this function. In the standard formulation, the continuous $U(1)$ duality symmetry is nonlinearly realized on the Maxwell field strength. In the new setting, the same symmetry acts as linear $U(1)$ transformations of the auxiliary field variables. The nonlinear $U(1)$ duality condition proves to be equivalent to the linear $U(1)$ invariance of the self-interaction $E$. The discrete self-duality (or self-duality by Legendre transformation) amounts to a weaker reflection symmetry of $E$. For a class of duality-symmetric Lagrangians we introduce an alternative representation with the auxiliary scalar field and find new explicit examples of such systems.

1 Introduction

It is well known that the on-shell $SO(2)$ ($U(1)$) duality symmetry (or self-duality) of Maxwell equations can be generalized to the whole class of the nonlinear electrodynamics models, including the famous Born-Infeld theory. The condition of $SO(2)$ duality can be formulated as a nonlinear differential constraint on the Lagrangians of these models \[1, 2, 3\]. Using a non-analytic change of basic field variables of the Lagrangian the $SO(2)$ duality condition can be transformed to the well-known Courant-Hilbert equation \[4, 5, 6\]. Recently, it has been observed that the requirement of analyticity of the initial Lagrangian implies an additional algebraic constraint which selects the proper subclass of solutions of this Courant-Hilbert equation \[7\].

In this paper we elaborate on another approach to the $U(1)$ duality-symmetric Lagrangians, in which the manifest analyticity is guaranteed at each step.\(^1\) It makes use of the auxiliary tensor fields. The starting point is the generic nonlinear electrodynamics Lagrangian

\[
L(F^2, \bar{F}^2) = -\frac{1}{2}(F^2 + \bar{F}^2) + L^{int}(F^2, \bar{F}^2), \tag{1.1}
\]

\(^1\)A preliminary version of this approach was presented in \[8, 9\].
where \( F^2 = F_{\alpha\beta}F^{\alpha\beta}, \bar{F}^2 = F^{\dot{\alpha}\dot{\beta}}\bar{F}_{\dot{\alpha}\dot{\beta}} \) and \( F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}} \) are the mutually conjugated \((1,0)\) and \((0,1)\) components of the Maxwell field strength in the two-component spinor notation. Its new representation involves, apart from the Maxwell field strength, also unconstrained auxiliary symmetric bispinor (tensor) fields \( V_{\alpha\beta}, \bar{V}_{\dot{\alpha}\dot{\beta}} \) and their squares \( V^2 \equiv \nu \) and \( \bar{V}^2 \equiv \bar{\nu} \)

\[
\mathcal{L}(F, V) = \frac{1}{2}(F^2 + \bar{F}^2) + (\nu + \bar{\nu}) - 2(V \cdot F + \bar{V} \cdot F) + E(\nu, \bar{\nu}),
\]

where \( E(\nu, \bar{\nu}) \) codifies the entire self-interaction. The generic Lagrangian (1.1) is recovered as a result of eliminating the auxiliary fields in (1.2) by their algebraic equations of motion. The basic advantage of this novel representation for \( U(1) \) duality-symmetric systems is related to its following remarkable feature. In contradistinction to nonlinear \( U(1) \) duality transformations of \( F^2, \bar{F}^2 \), the transformations of the new auxiliary variables are linear. As a consequence, the \( SO(2) \) duality condition is linearized and can be explicitly solved in this new setting. The general Lagrangian solving this constraint is specified by the interaction term \( E^{\mu\nu}(\nu, \bar{\nu}) = E(\nu, \bar{\nu}) \) which includes only the \( U(1) \)-invariant scalar combination of the auxiliary fields \( \nu \bar{\nu} = V^2\bar{V}^2 \) as an argument. More general nonlinear electrodynamics Lagrangians respecting the so-called discrete self-duality (or duality by Legendre transformation) also admit a simple off-shell characterization in terms of the function \( E \). In this case it should be even, \( E(\nu, \bar{\nu}) = E(-\nu, -\bar{\nu}) \), and otherwise arbitrary.

The paper is organized as follows. In Sect. 2 we give a brief account of the continuous and “discrete” dualities in nonlinear electrodynamics in the conventional approach. A novel representation of the appropriate Lagrangians via bispinor auxiliary fields is discussed in Sect. 3. An explicit solution of the algebraic equations of motion relating the initial and auxiliary variables can be immediately found only for a restricted class of the interaction functions \( E(\nu, \bar{\nu}) \), e.g. for the text-book case of Born-Infeld theory (though their perturbative solution always exists). In order to construct new explicit examples of duality-symmetric models we introduce, in Sect. 4 an alternative representation for the important subclass of interactions \( L^{int}(F^2, \bar{F}^2) \), namely, those containing terms of the 4th order in the field strengths. This representation makes use of a different linearly transforming auxiliary scalar variable \( \mu \) which is related to the variable \( \nu = V^2 \) by a sort of Legendre transformation. The ansatz for the appropriate class of solutions of the \( U(1) \) duality condition contains an invariant analytic function \( I(\mu, \bar{\mu}) \), and the expression for the corresponding Lagrangians in the \( \mu \)-representation is parametrized by this function. The algebraic equation for the auxiliary variable \( \mu \) can be explicitly solved in terms of the initial variables \( F^2, \bar{F}^2 \) for a wide class of the functions \( I(\mu, \bar{\mu}) \). Explicit examples of duality-symmetric analytic Lagrangians, including the Born-Infeld Lagrangian and some new ones constructed here for the first time, are collected in Sect. 5.

## 2 Dualities in nonlinear electrodynamics

We start by recapitulating the basic facts about nonlinear 4D electrodynamics models which reveal duality properties and include the free Maxwell theory and Born-Infeld theory as particular cases. Detailed motivations why such models are of interest to study can be found e.g. in [3].
2.1 Continuous on-shell $SO(2)$ duality

In the two-component spinor notation, the Maxwell field strengths are defined by

\[
F^\beta_\alpha(A) \equiv \frac{1}{4} (\sigma^n)^{\alpha\beta} (\bar{\sigma}^n)^{\dot{\alpha}\dot{\beta}} F_{mn} = \frac{1}{4} (\partial^\beta_\alpha A^\alpha_{\beta\dot{\beta}} + \partial^\dot{\beta}_\alpha A_{\alpha\beta}) ,
\]

(2.1)

\[
\bar{F}^{\dot{\alpha}}_\dot{\beta}(A) \equiv F^\beta_\alpha(A) , \quad F_{mn} = \partial_m A_n - \partial_n A_m ,
\]

where $\sigma^m, \bar{\sigma}^n$ are the Weyl matrices of the group $SL(2, C)$, $\partial_{\alpha\dot{\beta}} = (\sigma^m)^{\alpha\beta} \partial_m$ and $A_{\alpha\beta} = (\sigma^m)_{\alpha\beta} A_m$ is the corresponding vector gauge potential. Below we shall sometimes treat $F^\alpha_\beta$ ($\bar{F}^{\dot{\alpha}}_\dot{\beta}$) as independent variables, without assuming them to be expressed through $A_m$.

Let us introduce the Lorentz-invariant complex variables

\[
\varphi \equiv F^2 = F^{\alpha\beta} F_{\alpha\beta} , \quad \bar{\varphi} = \bar{F}^{\dot{\alpha}}_\dot{\beta} \bar{F}_{\dot{\alpha}}^{\dot{\beta}} .
\]

(2.2)

Two independent real invariants which one can construct out of the Maxwell field strength in the standard vector notation take the following form in these complex variables:

\[
F^m F_n = 2(\varphi + \bar{\varphi}) , \quad \frac{1}{2} \epsilon^{mnpq} F_{mn} F_{pq} = -2i(\varphi - \bar{\varphi}) .
\]

(2.3)

It will be convenient to deal with dimensionless $F^\alpha_\beta, \bar{F}^{\dot{\alpha}}_\dot{\beta}$ and $\varphi, \bar{\varphi}$, introducing a coupling constant $f$, $[f] = 2$. Then the generic nonlinear Lagrangian can be represented as

\[
f^{-2}L(\varphi, \bar{\varphi}) ,
\]

where

\[
L(\varphi, \bar{\varphi}) = -\frac{1}{2} (\varphi + \bar{\varphi}) + L^{\text{int}}(\varphi, \bar{\varphi})
\]

(2.4)

and the real analytic self-interaction $L^{\text{int}}(\varphi, \bar{\varphi})$ collects all possible higher-order terms $\varphi^k \bar{\varphi}^m, (k + m) \geq 2$. This analyticity requirement rules out, for instance, terms with radicals of the type $\sqrt{\varphi}$ or $\sqrt{\varphi} \pm \bar{\varphi}$.

We shall use the following notation for the derivatives of the Lagrangian $L(\varphi, \bar{\varphi})$ \footnote{In these and some subsequent relations it is assumed that the functional argument $F$ stands for both $F^\alpha_\beta$ and $\bar{F}^{\dot{\alpha}}_\dot{\beta}$; we hope that this short-hand notation will not give rise to any confusion.}

\[
P_{\alpha\beta}(F) \equiv i \partial L/\partial F^{\alpha\beta} = 2i F_{\alpha\beta} L_\varphi ,
\]

(2.5)

\[
L_\varphi = \partial L/\partial \varphi , \quad L_{\bar{\varphi}} = \partial L/\partial \bar{\varphi}
\]

and for the bilinear combinations of them

\[
\pi \equiv P^2 = P^{\alpha\beta} P_{\alpha\beta} = -4 \varphi (L_\varphi)^2 , \quad \bar{\pi} = \bar{P}^2 = \bar{P}^{\dot{\alpha}}_\dot{\beta} \bar{P}_{\dot{\alpha}}^{\dot{\beta}} = -4 \bar{\varphi} (L_{\bar{\varphi}})^2 .
\]

(2.6)

In the vector notation, the same quantities read

\[
\bar{P}^{mn} \equiv \frac{1}{2} \epsilon^{mnpq} P_{pq} = 2 \partial L/\partial F_{mn} , \quad \frac{i}{2} P_{mn} \bar{P}^{mn} = \pi - \bar{\pi} .
\]
The nonlinear equations of motion have the following form in the spinor notation:

$$\partial^\alpha \dot{P}_{\alpha\beta}(F) - \partial^\beta \dot{P}_{\alpha\beta}(F) = 0 .$$  \hfill (2.7)

These equations, together with the Bianchi identities

$$\partial^\alpha \dot{F}_{\alpha\beta} - \partial^\beta \dot{F}_{\alpha\beta} = 0 ,$$  \hfill (2.8)

constitute a set of first-order equations in which one can treat $F_{\alpha\beta}$ and $\bar{F}_{\dot{\alpha}\dot{\beta}}$ as unconstrained conjugated variables.

This set is said to be duality-symmetric if the Lagrangian $L(\varphi, \bar{\varphi})$ satisfies certain nonlinear condition \[1, 2, 3\]. The precise form of this $SO(2)$ duality condition is as follows

$$F^2 + P^2 - \bar{F}^2 - \bar{P}^2 \equiv \frac{i}{4} \varepsilon^{mnqp}(F_{mn}F_{pq} + P_{mn}P_{pq})$$

$$= \varphi + \pi - \bar{\varphi} - \bar{\pi} = \varphi - \bar{\varphi} - 4 [\varphi(L_{\varphi})^2 - \bar{\varphi}(L_{\bar{\varphi}})^2] = 0 .$$  \hfill (2.9)

To clarify the meaning of (2.9), let us define the nonlinear transformations

$$\delta_\omega F_{\alpha\beta} = \omega P_{\alpha\beta}(F) = 2i \omega F_{\alpha\beta}L_{\varphi} ,$$  \hfill (2.10)

$$\delta_\omega \varphi = 4i \omega \varphi L_{\varphi}$$  \hfill (2.11)

where $\omega$ is a real parameter. Then eq. (2.9) ensures that this transformation constitutes a nonlinear realization of the $SO(2)$ group. Indeed, given (2.9), $F_{\alpha\beta}$ and $P_{\alpha\beta}(F)$ form an $SO(2)$ vector

$$\delta_\omega P_{\alpha\beta}(F) = -\omega F_{\alpha\beta} .$$  \hfill (2.12)

The set of equations (2.7), (2.8) and the constraint (2.9) itself are clearly invariant under these transformations. Thus they are an obvious generalization of the $SO(2)$ duality transformation in the Maxwell theory:

$$\delta_\omega F_{\alpha\beta} = -i \omega F_{\alpha\beta} , \quad \delta_\omega \bar{F}_{\dot{\alpha}\dot{\beta}} = i \omega \bar{F}_{\dot{\alpha}\dot{\beta}} ,$$  \hfill (2.13)

which is a symmetry of the vacuum Maxwell equation $\partial^\beta \dot{F}_{\alpha\beta} = 0$.

Using (2.9), the following important relations can be derived:

$$\delta_\omega L = i \omega (\varphi - \bar{\varphi}) , \quad \delta L_{\varphi} = \frac{i}{2} \omega - 2i \omega L_{\varphi}^2 .$$  \hfill (2.14)

It should be pointed out that these transformations make sense only on the mass shell defined by eqs. (2.7), (2.8).

The general solution of the $SO(2)$ duality condition (2.9) has been considered earlier in Refs.\[2, 4, 5, 6, 7\]. Using the nonanalytic change of variables

$$p = \frac{1}{4}(\varphi + \bar{\varphi}) + \frac{1}{2}\sqrt{\varphi \bar{\varphi}} , \quad q = \frac{1}{4}(\varphi + \bar{\varphi}) - \frac{1}{2}\sqrt{\varphi \bar{\varphi}}$$  \hfill (2.15)
one can cast equation (2.9) in the form of the well-known Courant-Hilbert equation
\[ L_p L_q = 1 . \] (2.16)
The general solution of this equation is parametrized by a real analytic function \( v(s) \),
\[ L(p, q) = v(s) + \frac{2p}{v'(s)} , \quad q = s + \frac{p}{[v'(s)]^2} \] (2.17)
and it is completely specified by an algebraic equation for the auxiliary variable \( s \). The authors of Ref. [7] have shown that the natural requirement of analyticity of the Lagrangian with respect to the initial variables \( \varphi, \bar{\varphi} \) can be rephrased as the additional constraint on the function \( \Psi(s) = -s[v'(s)]^2 \)
\[ \Psi[\Psi(s)] = s . \] (2.18)
The perturbative analysis shows that the whole class of duality-symmetric analytic solutions \( L[p(\varphi, \bar{\varphi}), q(\varphi, \bar{\varphi})] \) exists. However, the only solution explicitly worked out so far is the familiar Born-Infeld example. Nonphysical solutions of eq.(2.16) contain nonanalytic terms \( \sqrt{\varphi \bar{\varphi}} \) (see, e.g. [6]).

In Sect. 3 and 4 we shall discuss two complementary approaches to solving the \( SO(2) \) duality equation which guarantee analyticity and covariance of solutions at each stage of calculations. Based on this, in Sect. 5 we shall present several new examples of duality-symmetric Lagrangians which meet the analyticity criterion.

It is worth pointing out once more that the \( SO(2) \) duality transformations in the standard setting described above cannot be realized on the vector potential \( A_m \); they provide a symmetry between the equations of motion and Bianchi identity and as such define on-shell symmetry. The manifestly \( SO(2) \) duality-invariant off-shell Lagrangians can be constructed in the formalism with additional vector and auxiliary fields [10]. We are planning to discuss a relation to this extended formalism elsewhere.

The Lagrangian \( L(\varphi, \bar{\varphi}) \) satisfying (2.9) is not invariant with respect to transformation (2.10). Yet one can construct, out of \( \varphi \) and \( \bar{\varphi} \), the \( SO(2) \) invariant function
\[ I(\varphi, \bar{\varphi}) \equiv L + \frac{i}{2} (F \cdot P - \bar{F} \cdot \bar{P}) = L - \varphi L_{\varphi} - \bar{\varphi} L_{\bar{\varphi}} , \] (2.19)
where \( F \cdot P = F^{\alpha\beta} P_{\alpha\beta} \). However, \( I(\varphi, \bar{\varphi}) \) starts with the 4-th order term \( \varphi \bar{\varphi} \), so this invariant cannot be interpreted as a Lagrangian.

Finally, notice that, given some \( L^{ds}(\varphi, \bar{\varphi}) \) obeying (2.9), the following Lagrangian related to \( L^{ds} \) by the simple rescaling
\[ L^{ds}(\varphi, \bar{\varphi}) \Rightarrow r L^{ds}(r^{-1}\varphi, r^{-1}\bar{\varphi}) , \] (2.20)
with \( r \neq 0 \) being an arbitrary real number, also obeys (2.9) and so yields a duality-symmetric model. Clearly, rescaling the coupling constant as \( f^2 \rightarrow |r|f^2 \) and properly rescaling \( F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}} \), one can always choose \( |r| = 1 \), so only the sign of \( r \) actually matters in (2.20). Thus
\[ L^-(\varphi, \bar{\varphi}) = -L^{ds}(-\varphi, -\bar{\varphi}) \] (2.21)
gives a non-equivalent duality-symmetric Lagrangian for each given \( L^{ds} \). In Sect 5 we shall consider this transformation for the Lagrangian of the Born-Infeld theory.
2.2 Self-duality by Legendre transformation

To explain what the "discrete duality" means we shall need a first-order representation of the action corresponding to the Lagrangian (2.4). It is such that the Bianchi identities (2.8) are implemented in the action with a Lagrange multiplier and so $F_{\alpha\beta}$, $\bar{F}_{\dot{\alpha}\dot{\beta}}$ are unconstrained complex variables off shell. This form of the action is given by

$$\frac{1}{f^2} \int d^4x L^D(F, F^D) = \frac{1}{f^2} \int d^4x \left[ L(\phi, \bar{\phi}) + i(F \cdot F^D - \bar{F} \cdot \bar{F}^D) \right],$$

(2.22)

where

$$F^D_{\alpha\beta} \equiv \frac{1}{4}(\partial_{\alpha} A^D_{\beta\dot{\beta}} + \partial_{\beta} A^D_{\alpha\dot{\alpha}}).$$

(2.23)

Varying with respect to the Lagrange multiplier $A^D_{\alpha\dot{\beta}}$, one obtains just the Bianchi identities for $F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}}$ (2.8). Solving them in terms of the gauge potential $A_{\alpha\dot{\beta}}$ and substituting the result into (2.22), we come back to (2.4). On the other hand, the multiplier $A^D_{\alpha\dot{\beta}}$ is defined up to the standard Abelian gauge transformation, which suggests interpreting $A^D_{\alpha\dot{\beta}}$ and $F^D_{\alpha\beta}$ as the dual gauge potential and gauge field strength, respectively. Using the algebraic equations of motion for the variables $F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}}$, one can express the action (2.22) in terms of $F^D_{\alpha\beta}, \bar{F}^D_{\dot{\alpha}\dot{\beta}}$. If the resulting action has the same form as the original one in terms of $F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}}$, the corresponding model is said to enjoy the discrete duality.

This sort of duality should not be confused with the on-shell continuous $SO(2)$ duality discussed earlier. However, as we shall see soon, any $L(\phi, \bar{\phi})$ solving the constraint (2.9) defines a system possessing the discrete duality. The inverse statement is not generally true, so the class of nonlinear electrodynamics actions admitting $SO(2)$ duality of equations of motion forms a subclass in the variety of actions which are duality-symmetric in the "discrete" sense.

Let us elaborate on this in some detail. The dual picture is achieved by varying (2.22) with respect to the independent variables $F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}}$, which yields the equation

$$F^D_{\alpha\beta} = i\partial L/\partial F^{\alpha\beta} \equiv P_{\alpha\beta}(F) = 2iF_{\alpha\beta}L_\phi,$$

(2.24)

where $P_{\alpha\beta}(F)$ is the same as in (2.23). Substituting the solution of this algebraic equation, $F_{\alpha\beta} = F_{\alpha\beta}(F^D)$, into (2.22) gives us the dual Lagrangian $L'(F^D)$

$$L'(\phi^D, \bar{\phi}^D) \equiv L^D[F(F^D), F^D],$$

(2.25)

where $\phi^D \equiv F^{\alpha\beta}_{\alpha\beta}F^D_{\alpha\beta} = \pi(F)$ and $\pi, \bar{\pi}$ were defined in (2.6). Then the discrete self-duality defined above amounts to the condition

$$L'(\phi^D, \bar{\phi}^D) = L(\phi^D, \bar{\phi}^D),$$

(2.26)

or, equivalently, to

$$L'(\pi, \bar{\pi}) = L(\pi, \bar{\pi}).$$

(2.27)

Using (2.24) and its conjugate, as well as the definitions (2.22), (2.25), one can explicitly check the property

$$F_{\alpha\beta} = -i\partial L'(\phi^D, \bar{\phi}^D)/\partial F^{\alpha\beta} \quad \text{(and c.c.)}.$$

(2.28)
Due to this relation, and keeping in mind the inverse one (2.24), one can treat the equation
\[ L'(P^2, \bar{P}^2) = L(F^2, \bar{F}^2) + i(F \cdot P - \bar{F} \cdot \bar{P}) = L(\varphi, \bar{\varphi}) - 2\varphi L_\varphi - 2\bar{\varphi} L_{\bar{\varphi}} \] (2.29)
as setting the Legendre transforms \( L \leftrightarrow L' \) between two functions of complex variables. Thus the discrete duality (2.20), (2.27) can be equivalently called “self-duality by Legendre transformation”.

On the level of equations of motion (2.7) and (2.8), the discrete self-duality (2.27) can be equivalently defined as their invariance with respect to the special finite \( SO(2) \) transformation \( \Lambda \)
\[ F_{\alpha\beta} \rightarrow \Lambda F_{\alpha\beta} = P_{\alpha\beta}, \quad P_{\alpha\beta} \rightarrow \Lambda P_{\alpha\beta} = -F_{\alpha\beta}. \] (2.30)
This invariance is manifested in the following on-shell transformation properties of the Lagrangian and its derivative
\[ \Lambda L(\varphi, \bar{\varphi}) = L(\varphi, \bar{\varphi}) + iP \cdot F - i\bar{P} \cdot \bar{F} \equiv L(\pi, \bar{\pi}), \] (2.31)
\[ \Lambda L_\varphi = \frac{1}{4} L^{-1}. \]

Let us show that the \( SO(2) \) duality condition (2.29) indeed guarantees the discrete duality (2.27). The simplest proof of this statement (see e.g. [3]) makes use of the special \( SO(2) \) transformation \( \Lambda \), eq. (2.30), and the invariance of function (2.19) under the global version of the general \( SO(2) \) transformations (2.10)
\[ \Lambda I(\varphi, \bar{\varphi}) \equiv L(\pi, \bar{\pi}) - \frac{i}{2} F \cdot P + \frac{i}{2} \bar{F} \cdot \bar{P} = I(\varphi, \bar{\varphi}). \] (2.32)
Comparing this relation with (2.29), we arrive at the condition (2.27). Clearly, the \( \Lambda \)-invariance of \( I(\varphi, \bar{\varphi}) \) is a weaker condition than its \( SO(2) \) invariance, so the Lagrangians revealing the property of \( SO(2) \) duality form a subclass of those which are self-dual in the discrete sense.

3 The nonlinear electrodynamics and dualities revisited

3.1 A new setting for Lagrangians of nonlinear electrodynamics
The recently constructed \( N = 3 \) supersymmetric extension of the Born-Infeld theory [8] suggests a new representation for the actions of nonlinear electrodynamics discussed in the previous Section.

The infinite-dimensional off-shell \( N = 3 \) vector multiplet contains gauge field strengths (2.1) and auxiliary fields \( V_{\alpha\beta} \) and \( \bar{V}_{\dot{\alpha}\dot{\beta}} \).

The gauge field part of the off-shell super \( N = 3 \) Maxwell component Lagrangian is
\[ \mathcal{L}_2(V, F) = \nu + \bar{\nu} - 2(V \cdot F + \bar{V} \cdot \bar{F}) + \frac{1}{2} (\varphi + \bar{\varphi}), \] (3.1)

\( ^3 \)In the rest of the paper we put the overall coupling constant \( f \) equal to 1.
where
\[ \nu \equiv V^2 = V^{\alpha\beta}V_{\alpha\beta}, \quad \bar{\nu} \equiv \bar{V}^2 = \bar{V}^{\dot{\alpha}\dot{\beta}}\bar{V}_{\dot{\alpha}\dot{\beta}}, \]
\[ V \cdot F \equiv V^{\alpha\beta}F_{\alpha\beta}, \quad \bar{V} \cdot \bar{F} \equiv \bar{V}^{\dot{\alpha}\dot{\beta}}\bar{F}_{\dot{\alpha}\dot{\beta}}. \tag{3.2} \]

Eliminating \( V^{\alpha\beta} \) by its algebraic equation of motion,
\[ V^{\alpha\beta} = F_{\alpha\beta}, \quad \bar{V}^{\dot{\alpha}\dot{\beta}} = \bar{F}^{\dot{\alpha}\dot{\beta}}, \tag{3.3} \]
we arrive at the free Maxwell Lagrangian
\[ L_2(F) = -\frac{1}{2}(\varphi + \bar{\varphi}). \tag{3.4} \]

Our aim will be to find a nonlinear extension of the free Lagrangian \((3.1)\), such that this extension becomes the generic nonlinear Lagrangian \(L(F^2, \bar{F}^2)\), eq. \((2.4)\), upon eliminating the auxiliary fields \(V^{\alpha\beta}, \bar{V}^{\dot{\alpha}\dot{\beta}}\) by their algebraic (nonlinear) equations of motion.

By Lorentz covariance, the off-shell \((F, V)-\)representation of the nonlinear Lagrangian \((2.4)\) has the following general form:
\[ \mathcal{L}[V, F(A)] = L_2[V, F(A)] + E(\nu, \bar{\nu}) \tag{3.5} \]
where \(E\) is a real analytic function of two variables which encodes self-interaction. Varying the action with respect to \(V^{\alpha\beta}\), we derive the analytic relation between \(V\) and \(F(A)\) in this formalism
\[ F_{\alpha\beta}(A) = V_{\alpha\beta}(1 + E_{\nu}) \quad \text{(and c.c.)}, \tag{3.6} \]
where \(E_{\nu} \equiv \partial E(\nu, \bar{\nu})/\partial \nu\). The corresponding algebraic relations between the scalar functions are
\[ \varphi = \nu(1 + E_{\nu})^2, \quad F \cdot V = \nu(1 + E_{\nu}). \tag{3.7} \]

The relation \((3.6)\) can be used to eliminate the auxiliary variable \(V_{\alpha\beta}\) in terms of \(F_{\alpha\beta}\) and \(\bar{F}^{\dot{\alpha}\dot{\beta}}\), \(V_{\alpha\beta} \Rightarrow V_{\alpha\beta}[F(A)]\) (see eq. \((3.11)\) below). The natural restrictions on the interaction function \(E(\nu, \bar{\nu})\) are
\[ E(0, 0) = 0, \quad E_{\nu}(0, 0) = E_{\bar{\nu}}(0, 0) = 0. \tag{3.8} \]

They mean that the \((\nu, \bar{\nu})\)-expansion of \(E(\nu, \bar{\nu})\) does not contain constant and linear terms. Clearly, given some analytic interaction Lagrangian \(L^{int}(\varphi, \bar{\varphi})\) in \((2.4)\), one can pick up the appropriate function \(E(\nu, \bar{\nu})\), such that the elimination of \(V_{\alpha\beta}, \bar{V}^{\dot{\alpha}\dot{\beta}}\) by \((3.6)\) yields just this self-interaction. Thus \((3.5)\) with an arbitrary (non-singular) interaction function \(E\) is an alternative form of generic nonlinear electrodynamics Lagrangian \((2.4)\). The second equation of motion in this representation, obtained by varying \((3.5)\) with respect to \(A_{\alpha\dot{\alpha}}\), has the form
\[ \partial_{\dot{\alpha}}[F_{\alpha\beta}(A) - 2V_{\alpha\beta}] + \text{c.c.} = 0. \tag{3.9} \]
After substituting \(V_{\alpha\beta} = V_{\alpha\beta}[F(A)]\) from \((3.6)\), eq. \((3.9)\) becomes the dynamical equation for \(F_{\alpha\beta}(A), \bar{F}^{\dot{\alpha}\dot{\beta}}(A)\) corresponding to the generic Lagrangian \((2.4)\). Comparing \((3.9)\) with \((2.7)\) yields the important relation
\[ P_{\alpha\beta}(F) = i[F_{\alpha\beta} - 2V_{\alpha\beta}(F)], \tag{3.10} \]

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where $P_{\alpha\beta}(F)$ was defined in (2.5).

Let us elaborate in more detail on how the $(F, V)$-representation of the nonlinear electrodynamics Lagrangians is related to the original “minimal” one (2.4). The general solution of the algebraic equation (3.6) for $V_{\alpha\beta}$ can be written as

$$V_{\alpha\beta}(F) = F_{\alpha\beta} G(\varphi, \bar{\varphi}) .$$

The relation of the transition functions $G, \bar{G}$ to $E(\nu, \bar{\nu})$ follows from eq. (3.6)

$$G^{-1} = 1 + E_{\nu}, \quad G^{-1} = 1 + E_{\bar{\nu}} .$$

Eq. (3.11) gives us the relations

$$\nu = \varphi G^2, \quad \bar{\nu} = \bar{\varphi} \bar{G}^2 ,$$

$$V(F) \cdot F = \varphi G, \quad \bar{V}(F) \cdot F = \bar{\varphi} \bar{G} .$$

which, taking into account (3.12), coincide with (3.7).

The transition function $G(\varphi, \bar{\varphi})$ can be found from the basic requirement that (3.5) coincides with the initial nonlinear action after eliminating $V_{\alpha\beta}, \bar{V}_{\alpha\beta}$:

$$\mathcal{L}[V(F), F] = L(\varphi, \bar{\varphi}) .$$

Using eqs. (3.10), (3.14) and the definition (2.5), it is easy to obtain the simple expression for the transition function in terms of the Lagrangian (2.4)

$$G(\varphi, \bar{\varphi}) = \frac{1}{2} - L_{\varphi} .$$

A useful corollary of this formula and of eqs. (3.8), (3.16) is

$$\nu E_{\nu} = \frac{1}{4} \varphi (1 - 4L_{\varphi}^2) .$$

Given a fixed $L(\varphi, \bar{\varphi})$, one can express $\varphi, \bar{\varphi}$ (and then $G, \bar{G}$) in terms of $\nu, \bar{\nu}$ from eqs. (3.13), (3.16) and restore the explicit form of $E(\nu, \bar{\nu})$ from (3.5), (3.11),

$$E = L(\varphi, \bar{\varphi}) - \frac{1}{2} (\varphi + \bar{\varphi}) - \nu - \bar{\nu} + 2(\varphi G + \bar{\varphi} \bar{G}) ,$$

via the substitution $\varphi, \bar{\varphi} \rightarrow \varphi(\nu, \bar{\nu}), \bar{\varphi}(\nu, \bar{\nu})$. Conversely, given $E(\nu, \bar{\nu})$, one can restore $L(\varphi, \bar{\varphi})$, by expressing $\nu$ through $\varphi, \bar{\varphi}$ from the first of eqs. (3.7). In practice, finding such explicit relations is a rather complicated task (see Sect. 5).

### 3.2 Duality symmetries as invariance of self-interaction

So far we did not discuss dualities in the $(F, V)$-representation. A link with the consideration in the previous Section is established by eq. (3.10) which relates the functions $P_{\alpha\beta}(F)$ and $V_{\alpha\beta}(F)$.
Using this identification, the realization of the $SO(2)$ duality transformations (2.10), (2.12) on independent variables $F_{\alpha \beta}$ and $V_{\alpha \beta}$ is easily found to be
\[
\delta_\omega V_{\alpha \beta} = -i\omega V_{\alpha \beta}, \\
\delta_\omega F_{\alpha \beta} = i\omega [F_{\alpha \beta} - 2V_{\alpha \beta}].
\] (3.19)

We see that, before effecting the algebraic equation (3.6) which expresses $V_{\alpha \beta}$ in terms of $F_{\alpha \beta}$ and $\bar{F}_{\dot{\alpha} \dot{\beta}}$, $SO(2)$ duality symmetry is realized linearly.

Next, substituting (3.10) into the $SO(2)$ duality condition (2.9) and making use of eq. (3.17), we find
\[
\frac{1}{4} \varphi(1 - 4L^2_{\varphi}) - \frac{1}{4} \bar{\varphi}(1 - 4L^2_{\bar{\varphi}}) = \nu E_\nu - \bar{\nu} E_{\bar{\nu}} = 0.
\] (3.20)

Thus passing to the $(F, V)$-representation allows one to rewrite the nonlinear differential equation (2.9) as a linear differential equation for the function $E(\nu, \bar{\nu})$. It is important to emphasize that the new form (3.20) of the constraint (2.9) admits a transparent interpretation as the condition of invariance of $E(\nu, \bar{\nu})$ under the $U(1)$ transformations (3.19)
\[
\delta_\omega E = 2i\omega (\bar{\nu} E_{\bar{\nu}} - \nu E_\nu) = 0.
\] (3.21)

The general solution of (3.20) is an analytic function $E(a)$ depending on the single real $U(1)$ invariant variable $a = \nu \bar{\nu}$ which is quartic in the auxiliary fields $V_{\alpha \beta}$ and $\bar{V}_{\dot{\alpha} \dot{\beta}}$:
\[
E_{ds}(\nu, \bar{\nu}) = E(a) = E(\nu \bar{\nu}), \ E(0) = 0.
\] (3.22)

We come to the notable result that in the representation (3.5) the whole class of nonlinear extensions of the Maxwell action admitting the on-shell $SO(2)$ duality is parametrized by an arbitrary $SO(2)$ invariant real function of one argument $E_{ds} = E(\nu \bar{\nu})$. A remarkable property of $E_{ds}$ is that its power expansion collects only terms $\sim \nu^n \bar{\nu}^n$, i.e. those of $4n$-th order in the fields. Below we shall present this expansion for a few examples, including the notorious case of Born-Infeld theory.

It is evident that the bilinear part of the duality-symmetric Lagrangian in the $(F, V)$-representation (3.1) is not invariant
\[
\delta_\omega L_2(F, V) = i\omega (F^2 + 2V^2 - 2F \cdot V - c.c) = i\omega (\varphi - \bar{\varphi}).
\] (3.23)

Thus the continuous $SO(2)$ duality in the $(F, V)$-representation amounts to a “partial” $SO(2)$ symmetry of the entire Lagrangian: it is a symmetry of its interaction part $E_{ds}(\nu, \bar{\nu})$. It should be pointed out that the auxiliary field $V_{\alpha \beta}, \bar{V}_{\dot{\alpha} \dot{\beta}}$ is not subjected off shell to any constraint (as distinct from the Maxwell field strength which is subjected to the Bianchi identity), so the characterization of the $SO(2)$ duality-symmetric systems in the $(F, V)$-representation as those with the $SO(2)$ invariant self-interaction is valid off shell.

Let us consider the general $U(1)$ invariant interaction $E(\nu \bar{\nu})$. In order to construct the corresponding Lagrangian $L(\varphi, \bar{\varphi})$ one should solve the algebraic equations for $V_{\alpha \beta}(F)$ or $[V(F)]^2 = \nu(\varphi, \bar{\varphi})$
\[
F_{\alpha \beta} = V_{\alpha \beta}(1 + \bar{\nu} E_a) \Rightarrow \varphi = \nu(1 + \bar{\nu} E_a)^2.
\] (3.24)
Using eq. \((3.24)\), one can derive the general equations relating the auxiliary variables \(\nu\) and \(a = \nu \bar{\nu}\) to the original variables \(\varphi, \bar{\varphi}\)

\[
\nu(1 - a^2 \mathcal{E}_a^4) = \varphi - \bar{\varphi} a \mathcal{E}_a^2 + 2a \mathcal{E}_a(a \mathcal{E}_a^2 - 1),
\]

\[
(1 + a \mathcal{E}_a^2)^2 \varphi \bar{\varphi} = a[\mathcal{E}_a(\varphi + \bar{\varphi}) + (1 - a \mathcal{E}_a^2)^2]^2.
\]

Note that it is not easy to find examples of the function \(\mathcal{E}(a)\) for which the algebraic equation for \(\nu(\varphi, \bar{\varphi})\) becomes explicitly solvable. In the next Section we shall consider an alternative choice of the auxiliary scalar variables which simplifies the explicit construction of duality-symmetric Lagrangians.

Finally, let us examine which restrictions on the interaction Lagrangian \(E(\nu, \bar{\nu})\) are imposed by the requirement of the discrete self-duality with respect to the exchange \(F(A) \leftrightarrow F^D(A^D)\). We shall do it in two ways.

We shall need a first-order representation of the Lagrangian \((3.5)\) analogous to \((2.22)\). Let us treat \(L(V, F)\) in eq.\((3.5)\) as a function of two independent complex variables \(V_\alpha, F_\alpha\) and implement the Bianchi identities for \(F_\alpha, \bar{F}_\alpha\) (amounting to the expressions \((2.4)\)) in the Lagrangian via the dual field-strength \(F^D_\alpha, \bar{F}^D_\alpha\) \((2.23)\):

\[
\tilde{L}[V, F, F^D] \equiv L(V, F) + i[F^D \cdot F - \bar{F}^D \cdot \bar{F}] .
\]

The algebraic equation of motion for \(V^\alpha\), i.e. \(\partial \tilde{L}/\partial V^\alpha = 0\), is just the relation \((3.6)\). On the other hand, since \(F_\alpha, \bar{F}_\alpha\) enter only bilinear part of the full Lagrangian in \((3.25)\), varying \((3.25)\) with respect to \(F_\alpha\) (with keeping \(V_\alpha, \bar{V}_\alpha\) off-shell) yields the exact linear relation

\[
F_\alpha - 2V_\alpha = -i F^D_\alpha (A^D)\quad \text{(and c.c.)}
\]

as the corresponding equation of motion. As the result, one can explicitly find the dual form of \((3.25)\) in terms of \(F_\alpha, \bar{F}_\alpha, V_\alpha, \bar{V}_\alpha\), expressing \(F_\alpha\) and \(\bar{F}_\alpha\) from eq. \((3.26)\):

\[
\tilde{L}[V, F(V, F^D), F^D] \equiv \tilde{L}(U, F^D) = \mathcal{L}_2(U, F^D) + E(-u, -\bar{u}) ,
\]

where

\[
U_\alpha \equiv \Lambda V_\alpha = -iV_\alpha , \quad u = U^\alpha \Lambda U_\alpha .
\]

The discrete self-duality now amounts to demanding the Lagrangian \((3.27)\) to have the same form in the variables \(U, F^D\) as the original Lagrangian \(\mathcal{L}(V, F)\) has in terms of \(V, F\). Comparing the dual Lagrangian \((3.27)\) with the original one \((3.5)\), one firstly observes that \(\mathcal{L}_2\) in \((3.27)\) looks the same in terms of the variables \(U, F^D\) as the original \(\mathcal{L}_2\), eq. \((3.1)\), in terms of \(V, F\). Then the necessary and sufficient condition of the discrete self-duality is the following simple restriction on the interaction function \(E\)

\[
E(\nu, \bar{\nu}) = E(-\nu, -\bar{\nu}) .
\]

Another proof is an analog of the on-shell consideration based on eqs. \((2.30)\), \((2.31)\), \((2.32)\) in the standard formulation. Let us consider the transformation of \(\mathcal{L}(V, F)\) \((3.5)\) with respect to a discrete version of the \(U(1)\) transformations \((3.19)\)

\[
\Lambda F_\alpha = i(F_\alpha - 2V_\alpha) = P_\alpha , \quad \Lambda V_\alpha = -iV_\alpha ,
\]

\[
\Lambda \mathcal{L}(V, F) = \mathcal{L}_2(V, F) + E(-\nu, -\bar{\nu}) + iP \cdot F - i\bar{P} \cdot \bar{F} .
\]

\[11\]
By analogy with the condition (2.32) the requirement of discrete self-duality in the \((F, V)\)-representation can now be reformulated as the \(\Lambda\)-invariance of the following function:

\[
I(F, V) = \mathcal{L}(F, V) + \frac{i}{2} P \cdot F - \frac{i}{2} \bar{P} \cdot \bar{F}.
\] (3.31)

We end up with the same condition (3.28) for \(E(\nu, \bar{\nu})\).

Obviously, an arbitrary \(SO(2)\)-invariant function \(E_{ds}(\nu, \bar{\nu}) = E(\nu, \bar{\nu})\) corresponding to a \(SO(2)\) duality-symmetric system automatically satisfies the discrete self-duality condition (3.28). This elementary consideration provides us with a simple proof of the fact (mentioned in Sect. 2) that the \(SO(2)\) duality-symmetric systems constitute a subclass in the set of those revealing the discrete self-duality.

4 An alternative auxiliary field representation

Eq. (3.6) (or eqs. (3.7)) can be treated as an algebraic relation between two independent arguments of the function \(\mathcal{L}(F, V)\) (3.5). Eliminating variables \(F_{\alpha\beta}\) in this function, one can define an on-shell \(\nu\)-representation of the general nonlinear Lagrangian

\[
L[\phi(\nu, \bar{\nu}), \bar{\phi}(\nu, \bar{\nu})] \equiv \tilde{L}(\nu, \bar{\nu}) = E + \frac{1}{2} \nu(E_\nu^2 - 2E_\nu - 1) + \frac{1}{2} \bar{\nu}(E_{\bar{\nu}}^2 - 2E_{\bar{\nu}} - 1).
\] (4.1)

However, this representation with \(E = E(a)\) is not much helpful for finding explicit examples of Lagrangians \(L_{ds}(\phi, \bar{\phi})\) in terms of the initial variables (2.2). It proves useful to define an alternative representation for the duality-symmetric Lagrangians, \(\tilde{L}(\mu, \bar{\mu}) \equiv \tilde{L}[\nu(\mu, \bar{\mu}), \bar{\nu}(\mu, \bar{\mu})]\), introducing new scalar auxiliary variables \(\mu, \bar{\mu}\). Basic quantities of this \(\mu\)-representation are related to the corresponding quantities of the \(\nu\)-representation via the Legendre transformation. In Sect 5 we shall see that the defining algebraic equation of this \(\mu\)-representation is more convenient for constructing explicit solutions of the \(U(1)\) duality condition than the analogous one in the \(\nu\)-representation.

Let us introduce new complex scalar fields

\[
\mu(\nu, \bar{\nu}) = E_\nu, \quad \bar{\mu}(\nu, \bar{\nu}) = E_{\bar{\nu}}
\] (4.2)

and consider the complex Legendre transformation \(E(\nu, \bar{\nu}) \rightarrow H(\mu, \bar{\mu})\)

\[
E(\nu, \bar{\nu}) - \nu E_\nu - \bar{\nu} E_{\bar{\nu}} = H(\mu, \bar{\mu}) .
\] (4.3)

The corresponding inverse transformation is

\[
E(\nu, \bar{\nu}) = H(\mu, \bar{\mu}) - \mu H_\mu - \bar{\mu} H_{\bar{\mu}}
\] (4.4)

and

\[
\nu(\mu, \bar{\mu}) = -H_\mu, \quad \bar{\nu}(\mu, \bar{\mu}) = -H_{\bar{\mu}} .
\] (4.5)

Note that the standard conditions (3.8) for the function \(E(\nu, \bar{\nu})\) do not imply any restriction on the second derivatives of this function. However, for the transformed function \(H(\mu, \bar{\mu})\) to be analytic at the origin and, respectively, for the relation (4.2), (4.5)
to be invertible, one is led to impose the following subsidiary condition on the Jacobian
\[ J(\nu, \bar{\nu}) \equiv |E_{\nu\nu}|^2 - |E_{\nu\bar{\nu}}|^2 \] of the Legendre transformation:
\[ J(0, 0) \neq 0. \] (4.6)

It implies an analogous condition for \( H(\mu, \bar{\mu}) \) and selects those \( L^{\text{int}}(\varphi, \bar{\varphi}) \), the \((\varphi, \bar{\varphi})\)-

\[ \text{expansion of which starts with a non-degenerate 2nd order term. Below we shall limit our study to such analytic functions } H(\mu, \bar{\mu}). \]

Using eqs. (4.2), (3.12) and (3.16) one can find how \( \mu \) is mapped on the derivative \( L_{\varphi} \)
\[ \mu(L_{\varphi}) = \frac{1 + 2L_{\varphi}}{1 - 2L_{\varphi}} = G^{-1} - 1, \quad L_{\varphi} = \frac{\mu - 1}{2(\mu + 1)}. \] (4.7)

The basic algebraic relation of the \( \nu \)-representation (3.7) can be transformed as follows
\[ \varphi(\mu, \bar{\mu}) = -(1 + \mu)^2 H_{\mu} \quad \text{ (and c.c.)}. \] (4.8)

In order to find the corresponding Lagrangian \( \tilde{L}(\varphi, \bar{\varphi}) \) one should solve this basic relation for the function \( \mu(\varphi, \bar{\varphi}) \). This solution can be analyzed perturbatively for any real analytic function \( E \) (or \( H \)). However, the explicit solutions can be found only for some special cases.

Performing the Legendre transformation \( E \leftrightarrow H \) in the Lagrangian (4.1) (with the condition (4.6) imposed) one can cast it in the \( \mu \)-representation
\[ \tilde{L}(\mu, \bar{\mu}) = \tilde{L}[\nu(\mu, \bar{\mu}), \bar{\nu}(\mu, \bar{\mu})] = \frac{1}{2}(1 - \mu^2)H_{\mu} + \frac{1}{2}(1 - \bar{\mu}^2)H_{\bar{\mu}} + H(\mu, \bar{\mu}). \] (4.9)

It is interesting to note that this Lagrangian and the relation (4.8) can be reproduced from an off-shell Lagrangian with \( \mu \) as an independent complex auxiliary field
\[ \tilde{L}(\varphi, \mu) = \frac{\varphi(\mu - 1)}{2(1 + \mu)} + \frac{\bar{\varphi}(\bar{\mu} - 1)}{2(1 + \bar{\mu})} + H(\mu, \bar{\mu}). \] (4.10)

Indeed, varying (4.10) with respect to \( \mu \) one obtains just eq. (4.8). Substituting the latter back in (4.10), one recovers the on-shell representation \( \tilde{L}(\mu, \bar{\mu}) \), eq. (4.9). The off-shell Lagrangian (4.10) is an analogue of the auxiliary-field reformulations of the Born-Infeld Lagrangian [11, 12] (see Sect. 5).

Since the auxiliary fields \( \nu \) and \( \mu \) are related via the Legendre transform (4.3), (4.4), a similar off-shell Lagrangian should also exist for the on-shell \( \nu \)-representation (4.1), with the \( \varphi \leftrightarrow \nu \) relation (3.7) arising as the appropriate algebraic equation of motion for \( \nu \). However, the derivation of such a Lagrangian is not straightforward.

Let us turn to duality issues in the \( \mu \) representation. Using eq. (4.7) and the formula (2.14) for the variation \( \delta L_{\varphi} \), one can show that the \( SO(2) \) duality group acts on \( \mu \) as a linear \( U(1) \) transformation
\[ \delta_{\omega} \mu = 2i\omega \mu. \] (4.11)

Eq. (3.17) implies the relation
\[ \frac{1}{4} \varphi(1 - 4L_{\varphi}^2) = E_{\nu \nu} - \mu H_{\mu}. \] (4.12)
Then the $U(1)$ duality condition (2.9) in the $\mu$-representation is equivalent to the condition of $U(1)$-invariance of $H(\mu, \bar{\mu})$

$$\delta_\omega H = 2i\omega(\mu H_\mu - \bar{\mu} H_{\bar{\mu}}) = 0 \Rightarrow H^{ds}(\mu, \bar{\mu}) = I(b) , \quad (4.13)$$

where $I(b)$ is a real function of the invariant argument $b = \mu\bar{\mu}$. The Jacobian condition (4.6) now amounts to the one-dimensional relations

$$E_a(0) \neq 0 \iff I_b(0) \neq 0 . \quad (4.14)$$

Thus the solution of the $U(1)$ duality condition has the following form in the $\mu$-representation:

$$\nu(\mu, \bar{\mu}) = -\bar{\mu} I_b , \quad \varphi = -(1 + \mu)^2 \bar{\mu} I_b . \quad (4.15)$$

This solution is easily checked to provide the correct transformation rule for $\varphi$

$$\delta_\omega \varphi = 2i\omega \varphi \frac{\mu - 1}{\mu + 1} = 4i\omega \varphi L_\varphi . \quad (4.16)$$

From the definition of $\mu$ and the relations (4.15) it is straightforward to derive

$$a \equiv \nu \bar{\nu} = b I_b^2 , \quad \mu = \bar{\nu} E_a \Rightarrow \frac{dE(a)}{da} = - \left( \frac{dI(b)}{db} \right)^{-1} . \quad (4.18)$$

The one-dimensional Legendre transform (4.3), (4.4) for the $U(1)$-invariant functions in the $\nu$- and $\mu$-representations reads

$$E(a) = I(b) - 2b I_b , \quad I(b) = E(a) - 2a E_a . \quad (4.19)$$

Relations (4.17), (4.18) can be directly derived from (4.19).

The general expression for the $\mu$-representation of the $U(1)$ duality-symmetric Lagrangian follows by substituting $I(b)$ for $H(\mu, \bar{\mu})$ into $\tilde{L}(\mu, \bar{\mu})$ defined by eq. (4.9)

$$\tilde{L}^{ds}(\mu, \bar{\mu}) = \frac{1}{2}(\mu + \bar{\mu})(1 - b) I_b + I(b) . \quad (4.20)$$

It possesses the correct $U(1)$ transformation properties

$$\delta_\omega \tilde{L}^{ds} = i\omega(\mu - \bar{\mu})(1 - b) I_b = i\omega(\varphi - \bar{\varphi})$$

$$\delta_\omega(\varphi - \bar{\varphi}) = 4i\omega(\tilde{L}^{ds} - I) . \quad (4.21)$$

Eq. (4.10) with the substitution $H \to I(b)$ provides an off-shell description of the considered restricted class of the $U(1)$ duality-symmetric theories in the $\mu$ representation. Using eqs. (4.7) and (4.15) one can find

$$\varphi L_\varphi = \frac{1}{2}(1 - \mu^2)\bar{\mu} I_b \quad (4.22)$$
and, substituting this in (4.20), show that the analytic functions \( I(b) \) satisfying the condition (4.14) constitute a particular class of the invariant functions \( I(\varphi, \bar{\varphi}) \) defined in (2.19):

\[
I[b(\varphi, \bar{\varphi})] = L(\varphi, \bar{\varphi}) - \varphi L_\varphi - \bar{\varphi} L_{\bar{\varphi}} \equiv I(\varphi, \bar{\varphi}) .
\]

This class of functions \( I(\varphi, \bar{\varphi}) \) is characterized by the presence of nonzero term of 4th order in Maxwell field strength in their \( (\varphi, \bar{\varphi}) \)-expansion, \( I(\varphi, \bar{\varphi}) = I_b(0) \varphi \bar{\varphi} + \ldots \).

Note that the general rescaling (2.20) which preserves the \( SO(2) \) duality corresponds in the \( \mu \)-representation to the rescaling \( I(b) \to r I(b) \). In particular, (2.21) corresponds to the reflection \( I(b) \to -I(b) \). In Sect.5 we shall consider the impact of this reflection on the Lagrangian of the Born-Infeld theory.

The basic algebraic problem of the \( \mu \)-representation is to restore the function \( \mu(\varphi, \bar{\varphi}) \) and then \( b(\varphi, \bar{\varphi}) \) by the given inverse function \( \varphi(\mu, \bar{\mu}) \) in (4.15). Once the latter function is analytic, the analyticity of \( \mu(\varphi, \bar{\varphi}) \) is guaranteed by the implicit function theorem. The basic algebraic equation for the function \( b(\varphi, \bar{\varphi}) \),

\[
(b + 1)^2 \varphi \bar{\varphi} = b[(\varphi + \bar{\varphi}) - I_b(b - 1)^2]^2 ,
\]

and the corresponding representation for \( \mu(\varphi, \bar{\varphi}) \),

\[
\mu = -\frac{\bar{\varphi} - b \varphi - 2b(b - 1)I_b}{I_b(1 - b^2)} ,
\]

follow from eqs. (4.16). In the next Section we shall see that the relations (4.23) and (4.24) are helpful while seeking the explicit solutions of the \( SO(2) \) duality constraint.

Finally, let us analyze the discrete duality of \( \tilde{L}(\mu, \bar{\mu}) \) (4.9). In the \( \mu \)-representation, the appropriate discrete transformations are

\[
\Lambda \mu(\nu, \bar{\nu}) \equiv \mu(-\nu, -\bar{\nu}) = -\mu(\nu, \bar{\nu}) .
\]

The discrete self-duality of \( \tilde{L}(\mu, \bar{\mu}) \) is then equivalent to the symmetry

\[
\Lambda H(\mu, \bar{\mu}) \equiv H(-\mu, -\bar{\mu}) = H(\mu, \bar{\mu}) ,
\]

which guarantees the correct \( \Lambda \) transformation (2.31) of the full Lagrangian (4.9).

5 Examples of duality-symmetric systems

5.1 Born-Infeld theory

The Lagrangian of the Born-Infeld theory has the following form in terms of complex invariants (2.2)

\[
L(\varphi, \bar{\varphi}) = [1 - Q(\varphi, \bar{\varphi})] ,
\]

where

\[
Q(\varphi, \bar{\varphi}) = \sqrt{1 + \bar{X}} , \quad X(\varphi, \bar{\varphi}) \equiv (\varphi + \bar{\varphi}) + (1/4)(\varphi - \bar{\varphi})^2 .
\]
The power expansion of the BI Lagrangian is

\[ L = -\frac{1}{2}(\varphi + \bar{\varphi}) + \frac{1}{2}\varphi \bar{\varphi} - \frac{1}{4}\varphi \bar{\varphi}(\varphi + \bar{\varphi}) + \frac{1}{8}\varphi \bar{\varphi}(3\varphi \bar{\varphi} + \varphi^2 + \bar{\varphi}^2) + O(\varphi^5) \, . \] (5.3)

In the BI theory the function (2.5) has the following explicit form

\[ P_{\alpha\beta}(F) = i \frac{\partial L}{\partial F_{\alpha\beta}} = -iF_{\alpha\beta}Q^{-1}(\varphi, \bar{\varphi})[1 + \frac{1}{2}(\varphi - \bar{\varphi})] \] (5.4)

and the basic $U(1)$-transformations of the scalar variable is

\[ \delta_\omega \varphi = -2i\omega \frac{\varphi[1 + \frac{1}{2}(\varphi - \bar{\varphi})]}{Q} \, . \] (5.5)

The function $G(\varphi, \bar{\varphi})$ relating the variables $V_{\alpha\beta}$ and $F_{\alpha\beta}$ and defined by eq. (3.16), is given by the expression

\[ G = \frac{1}{2} \left\{ 1 + Q^{-1} \left[ 1 + \frac{1}{2}(\varphi - \bar{\varphi}) \right] \right\} \, . \] (5.6)

Let us first discuss the $\mu$-representation of BI theory. It is easy to find the relations

\[ \varphi = \frac{2\bar{\mu}(1 + \mu)^2}{(1 - \mu\bar{\mu})^2}, \quad \bar{\varphi} = \frac{2\mu(1 + \bar{\mu})^2}{(1 - \mu\bar{\mu})^2}, \] (5.7)

which correspond to the following choice of the invariant function in the $\mu$-representation (4.15):

\[ I(b) = \frac{2b}{b - 1}, \quad I_b = -\frac{2}{(b - 1)^2} \] (5.8)

(with $b = \mu\bar{\mu}$). Using this choice of the auxiliary function in (4.23) we obtain the quadratic equation for the invariant variable $b$

\[ \varphi \bar{\varphi} b^2 + [2\varphi \bar{\varphi} - (\varphi + \bar{\varphi} + 2)^2] b + \varphi \bar{\varphi} = 0 \, . \] (5.9)

The invariant and linearly transforming functions $b$ and $\mu$ obtained by solving (5.9) and using the general formulas (4.7) or (4.24) are given by the expressions

\[ b = \frac{\varphi \bar{\varphi}}{[1 + Q + \frac{1}{2}(\varphi + \bar{\varphi})]^2}, \quad \mu = G^{-1} - 1 = \frac{Q - 1 - \frac{1}{2}(\varphi - \bar{\varphi})}{Q + 1 + \frac{1}{2}(\varphi - \bar{\varphi})} \, . \] (5.10)

The corresponding representations for the off- and on-shell BI Lagrangian read

\[ \tilde{L}(\varphi, \mu) = \frac{2b}{b - 1} + \frac{\varphi(\mu - 1)}{2(1 + \mu)} + \frac{\bar{\varphi}(\bar{\mu} - 1)}{2(1 + \bar{\mu})}, \quad \tilde{L}(\mu, \bar{\mu}) = \frac{\mu + \bar{\mu} + 2b}{b - 1} \, . \] (5.11)

Note that the authors of Ref. [11] considered a polynomial off-shell representation of the BI Lagrangian with two complex auxiliary fields. The basic auxiliary field $\chi$ of this representation is related to our fields $\varphi, \mu$ and $b = \mu\bar{\mu}$ as follows

\[ \chi + \frac{1}{2} \chi \bar{\chi} = \varphi, \quad \chi = \frac{2(\mu + b)}{b - 1}, \quad \tilde{L}(\mu, \bar{\mu}) = -\frac{1}{2}(\chi + \bar{\chi}) \, . \] (5.12)
Let us also study the original \((F, V)\)-representation for the BI case. Our aim is to find \(E(a)\) as a function of the variable

\[ a = \nu \bar{\nu} = \frac{4b}{(b - 1)^4} \tag{5.13} \]

(recall eq. (4.17)). Introducing \(t \equiv (b - 1)^{-1}\), one finds that \(t\) satisfies the following quartic equation:

\[ t^4 + t^3 - \frac{1}{4} \nu \bar{\nu} = 0 \,, \ t(\nu = \varphi = 0) = -1 \, . \tag{5.14} \]

It allows one to express \(t\) in terms of \(a \equiv \nu \bar{\nu}\):

\[ t(a) = -1 - \frac{a}{4} + \frac{3a^2}{16} - \frac{15a^3}{64} + \ldots \, . \tag{5.15} \]

One can easily write a closed expression for \(t(a)\) as the proper solution of (5.14), but we do not present it here.

Now we are ready to find the invariant self-interaction \(E(\nu \bar{\nu})\) for this case. Taking into account Eqs.(5.8) and (4.19) we find a simple expression for the self-interaction through the real variables \(b\) or \(t(a)\) (see also[8])

\[ E_{BI}[a(b)] = \frac{2b(1 + b)}{(1 - b)^2} = 2[2t^2(a) + 3t(a) + 1] = \frac{a}{2} - \frac{a^2}{8} + \frac{3a^3}{32} + \ldots . \tag{5.16} \]

It is easy to show that (5.18) is reduced to quadratic equation only for the one-parameter family of functions

\[ I_b = -2r/(b - 1)^2 \, , \tag{5.17} \]

which corresponds to performing the transformation (2.20) in the BI Lagrangian (5.1). A new duality-symmetric Lagrangian (not reducible to the BI one) is obtained in the case \(r = -1\)

\[ L^{(-)} = -1 + \sqrt{1 - \varphi - \bar{\varphi} + \frac{1}{4}(\varphi - \bar{\varphi})^2} = -1 + \sqrt{1 + E^2 - \mathbf{B}^2 - (\mathbf{E} \mathbf{B})^2} \, , \tag{5.18} \]

where \(E\) and \(\mathbf{B}\) are electric and magnetic fields, respectively. This Lagrangian is obtained from the BI one (5.1) by changing its overall sign and making the replacement \(F_{mn} \to \tilde{F}_{mn} = \frac{1}{2} \epsilon^{mankl} F_{kl}\), or

\[ E \to \mathbf{B} \, , \ \mathbf{B} \to -E \, . \tag{5.19} \]

It would be interesting to find out the physical meaning and implications of this “magnetic” counterpart of the BI theory.

5.2 Exact duality-symmetric Lagrangians corresponding to solvable algebraic equations

The initial data for restoring Lagrangians \(L^{ds}(\varphi, \bar{\varphi})\) by the known function \(I(b)\) in the \(\mu\)-representation is the equation (4.23) for \(b\) and the following representation for the
Lagrangian (4.20)

\[ \tilde{L}^{ds} = -\frac{1}{2} \left( \frac{1 - b}{1 + b} \right) [\varphi + \bar{\varphi} + 4bI_b(b)] + I(b), \quad (5.20) \]

which is obtained by substituting the expression (4.24) for \( \mu \) into (4.20). The key idea of finding out new explicit examples of duality-symmetric models is to pick up those \( I(b) \) for which the basic eq. (4.23) is simplified as much as possible. Below we analyze several examples of the function \( I(b) \) which make (4.23) a solvable algebraic equation for \( b(\varphi, \bar{\varphi}) \).

As already mentioned, the quadratic equation is obtained only in the case of BI theory (5.8) and its “magnetic” counterpart (5.17) (with \( r = -1 \)).

Next in complexity is the following ansatz for \( I_b \)

\[ I_b = -\frac{2 - cb}{(1 - b)^2} \Rightarrow I(b) = (c - 2) \frac{b}{1 - b} + c \ln(1 - b), \quad (5.21) \]

where \( c \) is some constant and we employed the conditions \( I(0) = 0, \ I_b(0) = -2 \). Being substituted into (4.23), this ansatz gives the cubic algebraic equation for the unknown \( b(\varphi, \bar{\varphi}) \)

\[ c^2 b^3 - b^2 [4c + 2c(\varphi + \bar{\varphi}) + \varphi \bar{\varphi}] + b[(\varphi + \bar{\varphi} + 2)^2 - 2\varphi \bar{\varphi}] - \varphi \bar{\varphi} = 0. \quad (5.22) \]

It is straightforward to write down the explicit solution of this equation for \( b(\varphi, \bar{\varphi}) \) as the appropriate analytic function of \( \varphi, \bar{\varphi} \) (vanishing at \( \varphi = \bar{\varphi} = 0 \)) and to find the precise expression for the related \( SO(2) \) duality-symmetric Lagrangian by substituting this solution into (5.20).

The case \( c = 0 \) yields the BI theory, while for any other value of \( c \) we obtain new examples of the duality-invariant systems. In the special case \( c = 2, \ I_b \) and \( I \) in eq.(5.21) are simplified to \( I_b = -2(1 - b)^{-1}, \ I(b) = 2 \ln(1 - b) \). With this choice, the relation between different auxiliary-fields representations is also simplified

\[ a = \frac{4b}{(b - 1)^2}, \quad b(a) = \frac{a + 2 - 2\sqrt{1 + a}}{a}, \]
\[ \mathcal{E}(a) = 2(\sqrt{1 + a} - 1) - 2 \ln \frac{1}{2}(1 + \sqrt{1 + a}). \quad (5.23) \]

A different ansatz for \( I_b \) also leading to a comparatively simple algebraic equation for \( b \) is as follows

\[ I_b = -\frac{2\sqrt{1 - cb}}{(b - 1)^2}. \quad (5.24) \]

Eq. (4.23) is reduced to

\[ (b + 1)^2 \varphi \bar{\varphi} - b(\varphi + \bar{\varphi})^2 - 4b(1 - cb) = 4(\varphi + \bar{\varphi})b\sqrt{1 - cb}, \quad (5.25) \]

which is equivalent to a quartic equation. In the limit \( c \to 0 \) eq. (5.24) becomes quadratic and one recovers the BI theory.
One more solvable ansatz for $I_b$ is

$$I_b = -2 \frac{1}{(1 - cb)(b - 1)^2}. \quad (5.26)$$

It reduces to the following quartic equation:

$$(1 - cb)^2 (b + 1)^2 \varphi \bar{\varphi} = b[(1 - cb)(\varphi + \bar{\varphi}) + 2]^2.$$

5.3 Discrete duality examples

Let us firstly consider, directly in the original $\varphi, \bar{\varphi}$ representation, two simple examples of the Lagrangians exhibiting discrete self-duality.

The first example is the Lagrangian which depends on a single real variable $\phi = \varphi + \bar{\varphi}$

$$L = 1 - \sqrt{1 + \phi}. \quad (5.27)$$

Another example is the holomorphic nonlinear Lagrangians

$$L_h(\varphi, \bar{\varphi}) = l(\varphi) + \bar{l}(\bar{\varphi}), \quad l(\varphi) = 1 - \sqrt{1 + \varphi}.$$  \quad (5.28)

It is a straightforward exercise to check that both these Lagrangians respect self-duality under Legendre transformation as it was defined in Subsect. 2.2. At the same time, they are not $SO(2)$ duality-symmetric.

Two other examples of systems with a discrete duality can be introduced in the framework of $\nu$-representation. The first one corresponds to the choice

$$E = \frac{1}{2} N^2, \quad N = \nu + \bar{\nu}, \quad (5.29)$$

then after eliminating auxiliary fields the final Lagrangian $L(\varphi, \bar{\varphi})$ is a function of the single real variable $\phi = \varphi + \bar{\varphi}$. The basic algebraic equation is cubic

$$\phi = N(1 + N)^2, \quad (5.30)$$

and it can be solved in radicals

$$N(\phi) = -\frac{2}{3} + A_+(\phi) + A_-(\phi) = \phi - 2\phi^2 + 7\phi^3 - 24\phi^4 + \ldots,$$

$$A_{\pm}(\phi) = \frac{1}{27} + B(\phi), \quad B(\phi) = \frac{1}{27}\phi + \frac{1}{4}\phi^2.$$  \quad (5.31)

Despite the presence of radical $\sqrt{B(\phi)}$ in $A_{\pm}(\phi)$, the function $N(\phi)$ is analytic.

More complicated Lagrangian corresponds to the following choice of the even self-interaction function $E(\nu, \bar{\nu}) = E(-\nu, -\bar{\nu})$:

$$E(\nu, \bar{\nu}) = \frac{1}{2}(\nu^2 + \bar{\nu}^2), \quad E_{\nu} = \nu = \mu, \quad (5.32)$$

$$L^{int}(\varphi, \bar{\varphi}) = \frac{1}{2}(\nu^2 + \bar{\nu}^2) + \nu^3 + \bar{\nu}^3.$$
The holomorphic algebraic equation

$$\varphi = \nu + 2\nu^2 + \nu^3$$  \hfill (5.33)

can be explicitly solved similarly to eq. (5.30). The corresponding Lagrangian $L(\varphi, \bar{\varphi})$ is holomorphic like (5.28).

6 Conclusion

We introduced a new $(F, V)$-representation for the Lagrangians of nonlinear electrodynamics and showed that it provides a simple description of systems exhibiting the properties of $U(1)$ duality or/and discrete self-duality in terms of real function of auxiliary bispinor complex fields, $E(V, \bar{V})$. This function encodes the entire self-interaction in the $(F, V)$-representation. The duality properties prove to be related to some linear off-shell symmetries of this general function $E$. We also defined an alternative $\mu$-representation and demonstrated its convenience and efficiency for constructing new explicit examples of duality-symmetric Lagrangians.

The auxiliary linearly transforming variables have also been used to construct the general solution of the $U(n)$ duality constraint for the interaction of $n$ Abelian gauge fields in [9]. The generalization to the $U(n)$ case is a straightforward extension of the formalism described above, so we do not present it here and send the interested reader to Ref. [9].

It is the interesting task to extend our consideration to the case of $N = 1$ and $N = 2$ supersymmetric extensions of nonlinear electrodynamics [3] in order to obtain a general characterization of the corresponding duality-symmetric systems. One more urgent problem is to define an analog (if existing) of the $(F, V)$-representation for non-Abelian BI theory and its superextensions. It could shed more light on the structure of these theories which have deep implications in string theory and still remain to be completely understood.

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