K3 SURFACES WITH A PAIR OF COMMUTING
NON-SYMPLECTIC INVOLUTIONS

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Abstract. We study K3 surfaces with a pair of commuting involutions
that are non-symplectic with respect to two anti-commuting complex
structures that are determined by a hyper-Kähler metric. One motiva-
tion for this paper is the role of such Z_2-actions for the construction of
G_2-manifolds. We find a large class of smooth K3 surfaces with such
pairs of involutions, but we also pay special attention to the case that
the K3 surface has ADE-singularities. Therefore, we introduce a special
class of non-symplectic involutions that are suitable for explicit calcu-
lations and find 320 examples of pairs of involutions that act on K3
surfaces with a great variety of singularities.

Contents
1. Introduction 1
2. K3 surfaces and their moduli spaces 3
3. Singular K3 surfaces 10
4. Non-symplectic involutions 11
5. K3 surfaces with singularities and a non-symplectic involution 18
6. K3 surfaces with two involutions 23
References 29

1. Introduction

A non-symplectic involution of a K3 surface S is a holomorphic involution
\( \rho : S \to S \) such that \( \rho \) acts as \(-1\) on \( H^{2,0}(S) \). Any K3 surface with a
non-symplectic involution admits a Kähler metric that is invariant under \( \rho \).
Since any Kähler metric on a K3 surface is in fact hyper-Kähler, there are

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three complex structures $I$, $J$ and $K$ and three Kähler forms $\omega_I$, $\omega_J$ and $\omega_K$ on $S$. If $\rho$ is holomorphic with respect to $I$, we have

$$
\rho^* \omega_I = \omega_I, \quad \rho^* \omega_J = -\omega_J, \quad \rho^* \omega_K = -\omega_K
$$

In this paper, we search for K3 surfaces that admit two non-symplectic involutions $\rho^1$ and $\rho^2$. We require that $\rho^1$ and $\rho^2$ commute and that they are non-symplectic with respect to two different complex structures from the triple $(I, J, K)$. Without loss of generality, we can assume that

$$
\begin{align*}
\rho^1 \omega_I &= \omega_I, & \rho^1 \omega_J &= -\omega_J, & \rho^1 \omega_K &= -\omega_K \\
\rho^2 \omega_I &= -\omega_I, & \rho^2 \omega_J &= -\omega_J, & \rho^2 \omega_K &= -\omega_K
\end{align*}
$$

A motivation to study these pairs of involutions is their relation to the construction of $G_2$-manifolds. $\rho^1$ and $\rho^2$ generate a group that is isomorphic to $\mathbb{Z}_2^2$ and acts isometrically on $S$. In the habilitation thesis of the author [17] we have described how such an action can be extended to products $S \times T^3$ of a K3 surface and a 3-torus such that the quotients $(S \times T^3)/\mathbb{Z}_2^2$ carry a $G_2$-structure. In a forthcoming paper we resolve the singularities of those quotients by the methods of Karigiannis and Joyce [11] and thus obtain compact $G_2$-manifolds. In [17] we have also explained that pairs of involutions with the above properties can be used to solve the so called matching problem in Kovalev’s and Lee’s construction of compact $G_2$-manifolds by twisted connected sums [13].

A non-symplectic involution is determined by its action on the lattice $H^2(S, \mathbb{Z})$. Nikulin [14, 15, 16] has classified the non-symplectic involutions of K3 surfaces in terms of invariants of their fixed lattices. By embedding the direct sum of two possible fixed lattices into $H^2(S, \mathbb{Z})$ we are able to find a large class of pairs $(\rho^1, \rho^2)$ with the desired properties. If the hyper-Kähler metric on $S$ is chosen generically, it has no singularities. In this paper, we also pay special attention to the case that $S$ has ADE-singularities. If we start one of the above constructions of $G_2$-manifolds with a K3 surface with singularities as its starting point, we obtain a $G_2$-orbifold with ADE-singularities. Such orbifolds are studied as compactifications of M-theory since the ADE-singularities are needed to explain the presence of non-abelian gauge fields [1, 2].

In order to construct K3 surfaces with a pair $(\rho^1, \rho^2)$ satisfying [1] that have as many types of ADE-singularities as possible, we restrict ourselves to a special class of non-symplectic involutions that are suitable for explicit calculations. We call these involutions simple. We show that 28 out of 75 types of non-symplectic involutions are simple. Moreover, we find 320 different kinds of pairs $(\rho^1, \rho^2)$ such that $\rho^1$ and $\rho^2$ are simple. Each of them
acts on a K3 surface with 3 $A_1$- and 2 $E_8$-singularities. Furthermore, there exist plenty of K3 surfaces with fewer and milder singularities that admit the same kind of involutions.

This paper is organised as follows. In Section 2-4, we present the necessary background material that can be found in the literature. The definition of a simple non-symplectic involution and their classification can be found at the end of Section 4. The main results of this paper are proven in Section 5 and 6. Section 5 deals with K3 surfaces with singularities that admit one involution and Section 6 deals with K3 surfaces that admit a pair of involutions.

2. K3 surfaces and their moduli spaces

Since some of the readers of this article may have a background in Riemannian rather than algebraic geometry, we provide a short introduction to the theory of K3 surfaces and their moduli spaces. We refer the reader to [6, Chapter VIII] and references therein for a more detailed account. First of all, we define what a K3 surface is.

**Definition 2.1.** A K3 surface is a compact, simply connected, complex surface with trivial canonical bundle.

At the beginning of this section, we consider only smooth K3 surfaces. Later on, we allow ADE-singularities, too. The underlying manifold of any K3 surface is of a fixed diffeomorphism type. Therefore, the topological invariants of all K3 surfaces are the same. The second cohomology with integer coefficients together with the intersection form is a lattice. Since we have to work with the second cohomology and its various sublattices, we need some concepts from lattice theory. The content of the following pages can be found in any reference on this subject, for example in [6, Chapter I.2], [7] or in [9].

**Definition 2.2.**

1. A lattice is a free abelian group $L$ of finite rank together with a symmetric bilinear form $\cdot : L \times L \to \mathbb{Z}$. We write $x^2$ for $x \cdot x$. The rank of a lattice is the same as the rank of the underlying abelian group. $L$ is called even if $x^2 \in 2\mathbb{Z}$ for all $x \in L$. Let $(e_1, \ldots, e_n)$ be a basis of $L$. The $n \times n$-matrix with coefficients $e_i \cdot e_j$ is called the Gram matrix $G(L)$ of $L$ with respect to $(e_1, \ldots, e_n)$. $L$ is called unimodular if $|\det G(L)| = 1$.

2. The $\mathbb{Z}$-bilinear form on $L$ can be extended to an $\mathbb{R}$-bilinear form on $L \otimes \mathbb{Z} \mathbb{R}$. Terms as non-degenerate lattice and signature of a lattice will always be defined with respect to the extended form.

3. Let $L$ and $L'$ be lattices and let $\cdot_L$ and $\cdot_{L'}$ be the corresponding bilinear forms. A lattice isomorphism of $L$ and $L'$ is a bijective $\mathbb{Z}$-linear map $\phi : L \to L'$ with $x \cdot_L y = \phi(x) \cdot_{L'} \phi(y)$ for all $x, y \in L$. 

If \( L = L' \), \( \phi \) is called an automorphism. We denote the group of all automorphisms of \( L \) by \( \text{Aut}(L) \).

**Remark 2.3.** The number \( |\text{det} G(L)| \) from the above definition is independent of the choice of the basis \((e_1, \ldots, e_n)\).

**Definition 2.4.**

1. An element \( x \) of a lattice \( L \) is called primitive if there exists no \( k > 1 \) and \( y \in L \) such that \( x = k \cdot y \).
2. A sublattice \( K \subseteq L \) is called primitive if the quotient \( L/K \) has no torsion.
3. A lattice \( N \) is primitively embedded in \( L \) if \( L \) has a primitive sublattice that is isomorphic to \( N \).

The dual of a lattice \( L \) is defined as

\[
L^* := \{ \phi : L \to \mathbb{Z}|\phi \text{ is } \mathbb{Z}\text{-linear} \}.
\]

From now on, we assume that \( L \) is a non-degenerate lattice. \( L^* \) can be equipped with the dual bilinear form, which takes its values in \( \mathbb{Q} \) but not necessarily in \( \mathbb{Z} \). The Gram matrix of \( L^* \) with respect to the dual basis is given by \( G(L)^{-1} \). If \( L \) is unimodular, \( L^* \) is thus a lattice, too. The map \( \iota : L \to L^* \) that is defined by \( \iota(x)(y) := x \cdot y \) is an injection. The quotient group \( L^*/\iota(L) \) is called the discriminant group of \( L \).

**Lemma 2.5.** The discriminant group of a lattice \( L \) is a finite group of order \( |\text{det} G(L)| \). The minimal number \( \ell(L) \) of generators of the discriminant group satisfies \( \ell(L) \leq \text{rank}(L) \).

The invariant \( \ell(L) \) allows us to formulate a theorem on primitive embeddings that can be found in [7] or [15].

**Theorem 2.6.** Let \( K \) be an even non-degenerate lattice of signature \((k_+, k_-)\) and \( L \) be an even unimodular lattice of signature \((l_+, l_-)\). We assume that \( k_+ \leq l_+ \) and \( k_- \leq l_- \) and that

1. \( 2 \cdot \text{rank}(K) \leq \text{rank}(L) \) or
2. \( \text{rank}(K) + \ell(K) < \text{rank}(L) \).

Then there exists a primitive embedding \( i : K \to L \). If in addition \( k_+ < l_+ \) and \( k_- < l_- \) and one of the following conditions holds

1. \( 2 \cdot \text{rank}(K) \leq \text{rank}(L) - 2 \),
2. \( \text{rank}(K) + \ell(K) \leq \text{rank}(L) - 2 \),

the embedding \( i \) is unique up to an automorphism of \( L \).

We return to K3 surfaces and describe their topology.
Theorem 2.7. Let $S$ be a K3 surface.

1. The Hodge numbers of $S$ are determined by $h^{0,0}(S) = h^{2,0}(S) = 1$, $h^{1,0}(S) = 0$ and $h^{1,1}(S) = 20$.
2. The second integral cohomology $H^2(S, \mathbb{Z})$ together with the intersection form is an even unimodular lattice of signature $(3, 19)$. Up to isometries, the only lattice with these properties is

$$L := 3H \oplus 2(-E_8),$$

where $H$ is the hyperbolic plane lattice with the bilinear form

$$
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
$$

and $-E_8$ is the root lattice of $E_8$ together with the negative of the usual bilinear form.

These facts motivate the following definitions.

Definition 2.8. (1) The lattice $L$ from the above theorem is called the K3 lattice.

(2) A K3 surface $S$ together with a lattice isometry $\phi : H^2(S, \mathbb{Z}) \to L$ is called a marked K3 surface.

(3) Two marked K3 surfaces $(S, \phi)$ and $(S', \phi')$ are called isomorphic if there exists a biholomorphic map $f : S \to S'$ such that $\phi \circ f^* = \phi'$, where $f^* : H^2(S', \mathbb{Z}) \to H^2(S, \mathbb{Z})$ is the pull-back.

The first Chern class on $S$ is a bijective map between the Picard group and $H^{1,1}(S) \cap H^2(S, \mathbb{Z})$. Therefore, we introduce the following terms:

Definition 2.9. The lattice $H^{1,1}(S) \cap H^2(S, \mathbb{Z})$ is called the Picard lattice and its rank is called the Picard number. The orthogonal complement of the Picard lattice in $H^2(S, \mathbb{Z})$ is called the transcendental lattice.

Convention 2.10. The maximal value of the Picard number is 20. In the literature, a K3 surface with maximal Picard number is often called singular and a compact, simply connected, complex surface with trivial canonical bundle that may admit ADE-singularities is sometimes called a Gorenstein K3 surface. In this article, we use a different convention and call K3 surfaces with ADE-singularities singular.

Any K3 surface $S$ admits a Kähler metric. Since $S$ has trivial canonical bundle, there exists a unique Ricci-flat Kähler metric in each Kähler class. The holonomy group $SU(2)$ is isomorphic to $Sp(1)$. Therefore, the Ricci-flat Kähler metrics are in fact hyper-Kähler. When we talk about isomorphisms between K3 surfaces, we usually mean biholomorphic maps with respect to fixed complex structures on the K3 surfaces. Another natural class of maps
between K3 surfaces, which will be studied later on, are isometries between K3 surfaces with hyper-Kähler metrics. It should be noted that there are isometries between K3 surfaces that are not holomorphic.

We need some background knowledge on the moduli spaces of K3 surfaces. There are several related moduli spaces whose points represent K3 surfaces with an extra structure. We denote them all by \( \mathcal{K}^3 \) with an appropriate index. The first of them is the moduli space of marked K3 surfaces \( \mathcal{K}^3_m \) that is defined as the set of all marked K3 surfaces modulo isomorphisms. We describe \( \mathcal{K}^3_m \) in more detail below.

On any K3 surface, there exists a holomorphic \((2,0)\)-form that is unique up to multiplication with a constant. We denote it by \( \omega_J + i\omega_K \), where \( \omega_J \) and \( \omega_K \) are real-valued 2-forms. This observation motivates the following definition.

**Definition 2.11.** Let \( (S, \phi) \) be a marked K3 surface. Moreover, let \( \mathbb{K} \in \{ \mathbb{R}, \mathbb{C} \} \), \( L_{\mathbb{K}} := L \otimes_{\mathbb{Z}} \mathbb{K} \) and \( \phi_{\mathbb{K}} : H^2(S, \mathbb{K}) \to L_{\mathbb{K}} \) be the \( \mathbb{K} \)-linear extension of \( \phi \). The complex line that is spanned by \( \phi_C([\omega_J + i\omega_K]) \), where the square brackets denote the cohomology class, defines a point \( p(S, \phi) \in \mathbb{P}(L_C) \), where \( \mathbb{P}(L_C) \) is the projective space of all complex lines in \( L_C \). \( p(S, \phi) \) is called the period point of \( (S, \phi) \). This assignment defines a map \( p : \mathcal{K}^3_m \to \mathbb{P}(L_C) \), which is called the period map for K3 surfaces.

It is not difficult to prove that \( p(S, \phi) \) is always contained in the following subset of \( \mathbb{P}(L_C) \).

**Definition 2.12.** We denote the complex line that is spanned by \( x \in L_C \setminus \{0\} \) by \( \ell_x \). The set

\[
\Omega := \{ \ell_x \in \mathbb{P}(L_C) | x \cdot x = 0, \ x \cdot \overline{x} > 0 \}
\]

is called the period domain.

We reduce the target set of the period map such that from now on \( p : \mathcal{K}^3_m \to \Omega \). An important theorem in the theory of K3 surfaces is that the period map is surjective. Moreover, it is a local isomorphism of complex manifolds, but it is not injective. Therefore, \( \mathcal{K}^3_m \) and \( \Omega \) are not isomorphic. In order to describe \( \mathcal{K}^3_m \) explicitly, we need some further definitions.

**Definition 2.13.**

1. Let \( S \) and \( S' \) be K3 surfaces. A lattice isometry \( \psi : H^2(S, \mathbb{Z}) \to H^2(S', \mathbb{Z}) \) is called a Hodge-isometry if its \( \mathbb{C} \)-linear extension preserves the Hodge decomposition \( H^2(S, \mathbb{C}) = H^{2,0}(S) \oplus H^{1,1}(S) \oplus H^{0,2}(S) \).

2. A class \( x \in H^2(S, \mathbb{Z}) \) is called effective if there exists an effective divisor \( D \) of \( S \) with \( c_1(\mathcal{O}_S(D)) = x \). An effective class \( x \) is called nodal if \( x^2 = -2 \).
(3) The connected component of the set \( \{x \in H^{1,1}(S, \mathbb{R}) | x \cdot x > 0\} \) which contains a Kähler class is called the positive cone of \( S \).

(4) A Hodge-isometry \( \psi : H^2(S, \mathbb{Z}) \to H^2(S', \mathbb{Z}) \) is called effective if it maps the positive cone of \( S \) to the positive cone of \( S' \) and effective classes in \( H^2(S, \mathbb{Z}) \) to effective classes in \( H^2(S', \mathbb{Z}) \).

Remark 2.14. Since \( H^{1,1}(S) = \overline{H^{1,1}(S)} \), \( H^{1,1}(S) \) is a complex vector space. We denote its real part \( H^{1,1}(S) \cap H^2(S, \mathbb{R}) \) by \( H^{1,1}(S, \mathbb{R}) \). The restriction of the intersection form to \( H^{1,1}(S, \mathbb{R}) \) has signature \((1, 19)\). The set \( \{x \in H^{1,1}(S, \mathbb{R}) | x \cdot x > 0\} \) thus has exactly two connected components. Exactly one of them contains a Kähler class and the definition of the positive cone therefore makes sense.

The following lemma often helps to decide if a Hodge-isometry is effective.

Lemma 2.15. (See [6, p. 313]) Let \( S \) and \( S' \) be K3 surfaces and \( \psi : H^2(S, \mathbb{Z}) \to H^2(S', \mathbb{Z}) \) be a Hodge-isometry. If \( \psi \) maps at least one Kähler class of \( S \) to a Kähler class of \( S' \), then \( \psi \) is effective.

With help of the terms that we have defined above, we are able to state the following theorem.

Theorem 2.16. (Torelli theorem) Let \( S \) and \( S' \) be two unmarked K3 surfaces. If there exists an effective Hodge-isometry \( \psi : H^2(S', \mathbb{Z}) \to H^2(S, \mathbb{Z}) \), \( \psi \) is the pull-back of a unique biholomorphic map \( f : S \to S' \).

The converse of the above theorem is also true. If \( f : S \to S' \) is a biholomorphic map, its pull-back is an effective Hodge-isometry. Since effective Hodge-isometries are closely related to the set of all Kähler classes, we are able to describe \( \mathcal{K}3^m \) with help of an explicit description of the Kähler cone.

Theorem 2.17. Let \( S \) be a K3 surface and let \( \mathcal{C}_S \subseteq H^{1,1}(S, \mathbb{R}) \) be its Kähler cone, i.e. the set of all cohomology classes representing a Kähler form. Then we have

\[
\mathcal{C}_S = \{x \in H^{1,1}(S, \mathbb{R}) | x \cdot x > 0 \text{ and } x \cdot d > 0 \text{ for all nodal classes } d\}
\]

We introduce further terms that allow us to describe the Kähler cone more algebraically.

Definition 2.18. (1) Let \( \ell_x \in \mathbb{P}(L_C) \). We define the root system of \( \ell_x \) as

\[
\triangle_x := \{d \in L | d \cdot d = -2, x \cdot d = 0\}.
\]
(2) We define the Kähler chambers of \( \ell_x \) as the connected components of
\[
\{ z \in L_\mathbb{R} | z \cdot z > 0, \ z \cdot x = 0, \ z \cdot d \neq 0 \ \forall d \in \triangle_x \}.
\]

**Theorem 2.19.** The subgroup of Aut(\( L \)) that preserves \( \ell_x \) acts transitively on the set of all Kähler chambers of \( \ell_x \). The image \( \phi_\mathbb{R}(C_S) \) of the Kähler cone of a marked K3 surface \((S, \phi)\) with period point \( x \) is one of the Kähler chambers of \( \ell_x \).

Finally, we arrive at the description of \( \mathcal{K}^m \).

**Definition 2.20.** We define the augmented period domain as
\[
\widetilde{\Omega} = \{ (\ell_x, C) | \ell_x \in \Omega, \ C \subseteq L_\mathbb{R} \text{ is a Kähler chamber of } \ell_x \}
\]
and the augmented period map \( \widetilde{p} : \mathcal{K}^m \to \widetilde{\Omega} \) by
\[
\widetilde{p}(S, \phi) := (p(S, \phi), \phi_\mathbb{R}(C_S)).
\]

**Theorem 2.21.** The augmented period map \( \widetilde{p} : \mathcal{K}^m \to \widetilde{\Omega} \) is bijective.

In order to determine a Ricci-flat Kähler metric, we need to specify a Kähler class and not only the complex structure. Therefore, we need another moduli space that takes this additional information into account.

**Definition 2.22.**
1. A marked pair is a pair of a marked K3 surface \((S, \phi)\) and a Kähler class \( y \in H^{1,1}(S, \mathbb{R}) \). We usually write a marked pair as \((S, \phi, y)\).
2. Two marked pairs \((S, \phi, y)\) and \((S', \phi', y')\) are called isomorphic if there exists a biholomorphic map \( f : S \to S' \) that satisfies \( \phi \circ f^* = \phi' \) and \( f^*y' = y \).
3. The moduli space of marked pairs \( \mathcal{K}^m \) is the set of all marked pairs modulo isomorphisms.

Moreover, we define the following two sets:
\[
\begin{align*}
K\Omega & := \{ (\ell_x, y) \in \Omega \times L_\mathbb{R} | x \cdot y = 0, y \cdot y > 0 \} \\
K\Omega^0 & := \{ (\ell_x, y) \in K\Omega | y \cdot d \neq 0 \ \forall d \in L \text{ with } d^2 = -2, x \cdot d = 0 \}
\end{align*}
\]
and the refined period map
\[
\begin{align*}
p' : \mathcal{K}^m & \to \Omega \times L_\mathbb{R} \\
p'(S, \phi, y) & := (p(S, \phi), \phi_\mathbb{R}(y))
\end{align*}
\]
Theorem 2.23. \( p' \) takes its values in \( K\Omega^0 \). Moreover, it is a bijection between \( \mathcal{X}^{3mp} \) and \( K\Omega^0 \). As a consequence, \( \mathcal{X}^{3mp} \) is a real analytic Hausdorff manifold of dimension 60.

Finally, we describe the moduli space of all hyper-Kähler structures on K3 surfaces. A hyper-Kähler structure on a marked K3 surface is a tuple \((S, \phi, g, \omega_I, \omega_J, \omega_K)\), where \( g \) is the hyper-Kähler metric and \( \omega_I, \omega_J, \omega_K \) are the Kähler forms with respect to the complex structures \( I, J \) and \( K \) that satisfy \( IJK = -1 \). The forms \( \omega_I, \omega_J, \omega_K \) determine the metric and an orientation that makes \( \omega_I \wedge \omega_J \) positive. Moreover, the cohomology classes of \( \omega_I, \omega_J, \omega_K \) already determine \( g \). \( \phi_C([\omega_I] + i[\omega_K]) \) yields a period point of a K3 surface and \( \phi_R([\omega_I]) \) determines a Kähler chamber. This information determines the complex structure \( I \) and then the Kähler class \([\omega_I]\) determines the hyper-Kähler metric. Conversely, \( g \) alone determines the span of \( \omega_I, \omega_J, \omega_K \), but yields no basis of that space. The above observations motivate the following lemma on isometries of K3 surfaces that will be useful later on.

Lemma 2.24. Let \( S_j \) with \( j \in \{1, 2\} \) be K3 surfaces together with hyper-Kähler metrics \( g_j \) and Kähler forms \( \omega_I^{(j)}, \omega_J^{(j)} \) and \( \omega_K^{(j)} \). Moreover, let \( V_j \subset H^2(S_j, \mathbb{R}) \) be the subspace that is spanned by \([\omega_I^{(j)}], [\omega_J^{(j)}] \) and \([\omega_K^{(j)}] \).

1. Let \( f : S_1 \rightarrow S_2 \) be an isometry. The pull-back \( f^* : H^2(S_2, \mathbb{Z}) \rightarrow H^2(S_1, \mathbb{Z}) \) is a lattice isometry. Its \( \mathbb{R} \)-linear extension maps \( V_2 \) to \( V_1 \).
2. Let \( \psi : H^2(S_1, \mathbb{Z}) \rightarrow H^2(S_2, \mathbb{Z}) \) be a lattice isometry such that \( \psi_\mathbb{R}(V_1) = V_2 \). Moreover, \( \psi_\mathbb{R} \) shall map the positive cone of \( S_1 \) to the positive cone of \( S_2 \). Then there exists an isometry \( f : S_2 \rightarrow S_1 \) such that \( f^* = \psi \).
3. Let \( f : S \rightarrow S \) be an isometry that acts as the identity on \( H^2(S, \mathbb{Z}) \). Then, \( f \) itself is the identity map. As a consequence, the isometry from \( S \) is unique.

Proof. The first claim is obvious and the third one follows from Proposition 11.3 in Chapter VIII in [3]. The second claim is a consequence of the Torelli theorem. More precisely, the fact that \( \omega_I^{(j)} + i\omega_K^{(j)} \) is a \((2, 0)\)-form determines a splitting of \( H^2(S_j, \mathbb{C}) \) into \( H^{2,0}(S_j) \oplus H^{1,1}(S_j) \oplus H^{0,2}(S_j) \). Since \( \psi_\mathbb{R} \) preserves the positive cone, it follows from our explicit description of \( C_{S_j} \) that it preserves the Kähler cone, too. The Torelli theorem thus yields a biholomorphic map \( f : S_2 \rightarrow S_1 \) with \( f^* = \psi \). The triples \([\omega_I^{(2)}], [\omega_J^{(2)}], [\omega_K^{(2)}] \) and \((\psi([\omega_I^{(1)}]), \psi([\omega_J^{(1)}]), \psi([\omega_K^{(1)}])) \) yield unique hyper-Kähler metrics on \( S_2 \). Since both triples span the same subspace, these metrics are the same and we have \( f^* g_1 = g_2 \). \( \square \)
Remark 2.25. If we had omitted the condition that $\psi_R$ preserves the positive cone, the second part of our lemma would have been slightly more complicated. In that situation $\psi := -\text{Id}_{H^2(S,\mathbb{Z})}$ would satisfy all conditions from the lemma. The corresponding isometry $f : S \to S$ would be the identity map, but it would have to be interpreted as an anti-holomorphic map between $(S,I)$ and $(S,-I)$.

Finally, we describe the moduli space $\mathcal{M}^{3hk}$ of all marked hyper-Kähler structures $(S,\phi,g,\omega_I,\omega_J,\omega_K)$. As a consequence of Theorem 2.23 and Lemma 2.24 (see also [10, p. 161]), it follows that $\mathcal{M}^{3hk}$ is diffeomorphic to the hyper-Kähler period domain

$$\Omega^{hk} := \{(x,y,z) \in L^3_{\mathbb{R}} | x^2 = y^2 = z^2 > 0, x \cdot y = x \cdot z = y \cdot z = 0, \exists d \in L \text{ with } d^2 = -2 \text{ and } x \cdot d = y \cdot d = z \cdot d = 0 \}.$$  

3. Singular K3 surfaces

In this section, we discuss singular K3 surfaces and their relation to smooth ones. The results that we present here were originally proven in [3, 4, 12]. A short overview can also be found in [10, p.161 - 162].

Let $S$ be a K3 surface and let $w \in H^2(S,\mathbb{Z})$ be a class with $w^2 = -2$ that represents a submanifold $Z$ of $S$. We do not assume that $w \in H^{1,1}(S)$ and thus $Z$ is not necessarily a divisor. $S$ shall carry a hyper-Kähler structure $(g,\omega_I,\omega_J,\omega_K)$. It can be shown that $Z$ can be chosen as a sphere that is minimal with respect to $g$. Its area $A$ is given by

$$A^2 = ([\omega_I] \cdot w)^2 + ([\omega_J] \cdot w)^2 + ([\omega_K] \cdot w)^2$$

We choose a marking $\phi$ of $S$. If we move within the hyper-Kähler period domain towards a triple $(x,y,z) \in L^3_{\mathbb{R}}$ with

$$x \cdot \phi(w) = y \cdot \phi(w) = z \cdot \phi(w) = 0,$$

the volume of the sphere shrinks to zero. In other words, we obtain a singularity. This is in fact the geometric meaning of the condition in the definition of $\Omega^{hk}$ that there shall be no $d \in L$ with $d^2 = -2$ and $x \cdot d = y \cdot d = z \cdot d = 0$. We assume that there is exactly one $d \in L$ with this property. In this situation, we obtain the singularity by collapsing a single sphere with self-intersection $-2$ to a point. Since this is the reversal of blowing up an $A_1$-singularity, the K3 surface has an $A_1$-singularity at a single point. Next, we assume that there exists an arbitrary number of $d$s with $d^2 = -2$ and $x \cdot d = y \cdot d = z \cdot d = 0$. We define the set
\[ \tilde{\Omega}^{hk} := \{ (x, y, z) \in L^3_{\mathbb{R}} | x^2 = y^2 = z^2 > 0, x \cdot y = x \cdot z = y \cdot z = 0 \} \]

and for any \( \alpha = (x, y, z) \in \tilde{\Omega}^{hk} \) we define

\[ D_\alpha := \{ d \in L | d^2 = -2, x \cdot d = y \cdot d = z \cdot d = 0 \} . \]

By joining \( d_1, d_2 \in D_\alpha \) with \( d_1 \neq d_2 \) by \( d_1 \cdot d_2 \) edges, we obtain a graph \( G \). This graph is the disjoint union of simply laced Dynkin diagrams. As the hyper-Kähler structure approaches \( \alpha \), a set of 2-spheres whose intersection numbers are given by \( d_i \cdot d_j \) collapses, which means that the Dynkin diagrams describe the type of the singularities. For example, if \( G \) consists of one Dynkin diagram of type \( E_8 \) and 2 isolated nodes, the singularities of the K3 surface are at 3 different points. At one of them we have a singularity of type \( E_8 \) and at the other two ones we have \( A_1 \)-singularities. Our considerations show that the singular and the smooth marked K3 surfaces with a hyper-Kähler structure can be combined into a larger moduli space that is diffeomorphic to \( \tilde{\Omega}^{hk} \).

4. Non-symplectic involutions

In this section, we introduce the most important results about non-symplectic involutions. These results were proven by Nikulin [14, 15, 16] and are also summed up in [5, 13]. Moreover, we define a class of non-symplectic involutions that are well suited for explicit calculations and we classify them.

**Definition 4.1.** Let \( S \) be a K3 surface. A non-symplectic involution is a biholomorphic map \( \rho : S \to S \) such that

1. \( \rho^2 = \text{Id} \), but \( \rho \neq \text{Id} \).
2. The pull-back \( \rho^* : H^{2,0}(S) \to H^{2,0}(S) \) is not the identity map, or equivalently \( \rho^*(\omega_J + i\omega_K) = -(\omega_J + i\omega_K) \).

From now on, let \( S \) be a K3 surface and \( \rho : S \to S \) be a non-symplectic involution. We define the fixed lattice of \( \rho \) by

\[ L^\rho := \{ x \in H^2(S, \mathbb{Z}) | \rho^* x = x \} . \]

\( L^\rho \) is a primitive sublattice of \( H^2(S, \mathbb{Z}) \). Since \( \rho^* \) acts as \(-1\) on \( H^{2,0}(S) \) and \( H^{0,2}(S) \), \( L^\rho \) is a sublattice of the Picard lattice. A K3 surface with a non-symplectic involution admits an integral Kähler class \( x \) and is thus algebraic by the Kodaira embedding theorem. Moreover, it admits an integral \( \rho \)-invariant Kähler class since \( x + \rho^* x \) is \( \rho \)-invariant.
We choose a marking \( \phi : H^2(S, \mathbb{Z}) \to L \) and abbreviate \( \phi(L^\rho) \) by \( L^\rho \). It can be shown that \( L^\rho \) is a non-degenerate sublattice of \( L \) with signature \((1, t)\). A lattice with that kind of signature is called hyperbolic. The rank \( r = 1 + t \) is an invariant of \( L^\rho \). \( L^\rho \) is 2-elementary which means that \( L^\rho / L^\rho \) is isomorphic to a group of type \( \mathbb{Z}_2^2 \). The number \( a \in \mathbb{N}_0 \) is a second invariant of \( L^\rho \). We define a third invariant \( \delta \) by

\[
\delta := \begin{cases} 
0 & \text{if } x^2 \in \mathbb{Z} \text{ for all } x \in L^\rho^* \\
1 & \text{otherwise}
\end{cases}
\]

**Theorem 4.2.** (Theorem 4.3.2 in [16]) For each triple \((r, a, \delta) \in \mathbb{N}_0 \times \mathbb{N}_0 \times \{0, 1\}\) there is up to isometries at most one even, hyperbolic, 2-elementary lattice with invariants \((r, a, \delta)\).

Let \( N \) be a hyperbolic lattice such that there exists a primitive embedding of \( N \) into \( L \). We assume that \( N^*/N \) is 2-elementary and that \( N \subset L \) contains a Kähler class. Then there exists an involution \( \rho_N \) of \( L \) with fixed lattice \( N \). \( \rho_N \) acts as \(-1\) on \( N^\perp_R \subseteq L_R \) and \( N^\perp_R \) contains a positive plane \( P \) with an orthonormal basis \((x, y)\). The surjectivity of the period map and the Torelli theorem guarantee that there exists a K3 surface \( S \) together with a non-symplectic involution \( \rho \) such that \( \rho^* = \rho_N \) and \( H^2(S, \mathbb{R}) \cap (H^{2,0}(S) \oplus H^{0,2}(S)) = P \). The period point of that K3 surface is the complex line that is spanned by \( x + iy \).

There is up to isometries of \( L \) at most one primitive embedding of a lattice with invariants \((r, a, \delta)\) into \( L \) and it follows that the deformation classes of K3 surfaces with a non-symplectic involution can be classified in terms of triples \((r, a, \delta)\). Nikulin [16] has shown that there exist 75 possible triples that satisfy

\[
1 \leq r \leq 20, \quad 0 \leq a \leq 11 \quad \text{and} \quad r - a \geq 0.
\]

A figure with a graphical representation of all possible values of \((r, a, \delta)\) can be found in [13, 16]. Next, we describe the moduli space of all K3 surfaces with a non-symplectic involution whose fixed lattice is of a given isomorphism type. In order to do this, we need the following concept.

**Definition 4.3.** (cf. Dolgachev [8]) Let \( N \) be a hyperbolic lattice that is primitively embedded into \( L \).

1. A marked ample \( N \)-polarised K3 surface is a K3 surface \( S \) together with a marking \( \phi : H^2(S, \mathbb{Z}) \to L \) such that \( \phi^{-1}(N) \) is a sublattice of the Picard lattice. Moreover, \( \phi^{-1}(N) \) shall contain an integral ample class, which is since \( S \) is a compact Kähler manifold, the same as an integral Kähler class.
(2) Two marked ample $N$-polarised K3 surfaces $(S, \phi)$ and $(S', \phi')$ are called isomorphic if there exists a biholomorphic map $f : S \to S'$ such that $\phi' = \phi \circ f^*$.

(3) We denote the moduli space that consists of all marked ample $N$-polarised K3 surfaces modulo isomorphisms by $\mathcal{K}^3m(N)$.

We denote the lattice with invariants $(r, a, \delta)$ by $L(r, a, \delta)$. The moduli space of all marked K3 surfaces with a non-symplectic involution whose fixed lattice is isomorphic to $L(r, a, \delta)$ is the same as $\mathcal{K}^3m(N)$, which we abbreviate by $\mathcal{K}^3m(r, a, \delta)$. We remark that this moduli space is the same as the moduli space $K^3m(r, a, \delta)$ in [13]. There is a nice explicit description of $\mathcal{K}^3m(r, a, \delta)$.

Theorem 4.4. (Corollary 3.2 in [8]) Let $N$ be a hyperbolic lattice that can be primitively embedded into $L$. We denote the orthogonal complement of $N$ in $L$ by $M$ and define the following sets:

$$
\Omega_N := \{ \ell_x \in \mathbb{P}(M_C) | x = 0, x \cdot \overline{\tau} > 0 \}
$$

$$
\Delta(M) := \{ d \in M | d^2 = -2 \}
$$

$$
H_d := \{ \ell_z \in \mathbb{P}(M_C) | z \cdot d = 0 \}
$$

$$
\Omega'_N := \Omega_N \setminus \bigcup_{d \in \Delta(M)}(H_d \cap \Omega_N)
$$

$\mathcal{K}^3m(N)$ is isomorphic to $\Omega'_N$ and the isomorphism is given by the period map.

Remark 4.5. Since $N$ contains a Kähler class, any element of $H_d \cap \Omega_N$ would correspond to a K3 surface $S$ with the property that $d$ is orthogonal to the real and imaginary part of the $(2, 0)$-form and to a Kähler class. In other words, $S$ would carry a singular hyper-Kähler metric. This is the reason why we have to remove the set $\bigcup_{d \in \Delta(M)}(H_d \cap \Omega_N)$ from $\Omega_N$.

The topology of the fixed locus $S^\rho := \{ x \in S | \rho(x) = x \}$ of a non-symplectic involution $\rho$ can be described in terms of the invariants $r$ and $a$.

Theorem 4.6. (cf. [13 16]) Let $\rho : S \to S$ be a non-symplectic involution of a K3 surface and let $(r, a, \delta)$ be the invariants of the fixed lattice. The fixed locus $S^\rho$ of $\rho$ is a disjoint union of complex curves.

(1) If $(r, a, \delta) = (10, 10, 0)$, $S^\rho$ is empty.

(2) If $(r, a, \delta) = (10, 8, 0)$, $S^\rho$ is the disjoint union of two elliptic curves.

(3) In the remaining cases, we have

$$
S^\rho = C_g \cup E_1 \cup \ldots \cup E_k
$$

where $C_g$ is a curve of genus $g = \frac{22 - r - a}{2}$ and the $E_i$ are $k = \frac{r - a}{2}$ curves that are biholomorphic to $\mathbb{C}P^1$, i.e. they are rational curves.
We define a class of non-symplectic involutions whose action on $L$ has a very simple matrix representation. In order to do this, we have to fix a basis of $L$. We write

$$ L = H_1 \oplus H_2 \oplus H_3 \oplus (-E_8)_1 \oplus (-E_8)_2 $$

in order to distinguish between the different summands. We choose a basis $(u_i^1, u_i^2)$ of each $H_i$ such that

$$ u_i^1 \cdot u_i^1 = u_i^2 \cdot u_i^2 = 0, \quad u_1^1 \cdot u_2^2 = 1. $$

Moreover, $(v_1^i, \ldots, v_8^i)$ shall be a basis of $(-E_8)_i$ such that the bilinear form has the matrix representation

$$
\begin{pmatrix}
-2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2
\end{pmatrix}
$$

We call

$$ (w_1, \ldots, w_{22}) = (u_1^1, u_1^2, u_2^1, u_2^2, u_3^1, u_3^2, v_1^1, \ldots, v_8^1, v_1^2, \ldots, v_8^2) $$

the standard basis of $L$. With help of this basis, we are able to define our class of non-symplectic involutions.

**Definition 4.7.** Let $S$ be a K3 surface and let $\rho : S \to S$ be a non-symplectic involution. We call $\rho$ a simple non-symplectic involution if there exists a marking $\phi : H^2(S, \mathbb{Z}) \to L$ such that for all $i \in \{1, \ldots, 22\}$ there exists a $j \in \{1, \ldots, 22\}$ with $\rho(w_i) = \pm w_j$, where $\phi \circ \rho^* \circ \phi^{-1}$ is abbreviated by $\rho$.

Let $\rho$ be a simple non-symplectic involution. Since $\rho : L \to L$ is a lattice isometry and we have $\rho(w_i) = \pm w_j$, $\rho$ maps any of the sublattices $H_k \subseteq L$ to an $H_l$. There are four possibilities for the value of $\rho(u_1^k)$ and of $\rho(u_2^k)$. We check for each combination if $\rho|_{H_k} : H_k \to H_l$ is a lattice isometry and see that $\rho|_{H_k}$ is given by one of the following maps:

1. $\rho|_{H_k}(u_1^k) = u_1^l$, $\rho|_{H_k}(u_2^k) = u_2^l$
2. $\rho|_{H_k}(u_1^k) = -u_1^l$, $\rho|_{H_k}(u_2^k) = -u_2^l$
(3) $\rho|_{H_k}(u^1_k) = u^2_k$, $\rho|_{H_k}(u^2_k) = u^1_k$
(4) $\rho|_{H_k}(u^1_k) = -u^2_k$, $\rho|_{H_k}(u^2_k) = -u^1_k$

Since $\rho$ is non-symplectic, its fixed lattice is hyperbolic. Therefore, $\rho|_{3H} : 3H \rightarrow 3H$ has to preserve exactly one positive vector. By enumerating all possibilities for $\rho|_{3H}$ with this property and comparing the invariants of the fixed lattices, we can conclude that $\rho|_{3H}$ is up to conjugation one of the following maps $\rho^i_1 : 3H \rightarrow 3H$ with $i = 1, \ldots, 7$:

| $i$ | Matrix representation | Basis of the fixed lattice | $r$ | $a$ | $\delta$ |
|-----|-----------------------|-----------------------------|-----|-----|--------|
| 1   | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$ | $(u^1_1, u^2_1)$ | 2   | 0   | 0      |
| 2   | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -1 & 0 \end{pmatrix}$ | $(u^1_1, u^2_1, u^1_2 - u^2_2)$ | 3   | 1   | 1      |
| 3   | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -1 & 0 \end{pmatrix}$ | $(u^1_1, u^2_1, u^2_1 - u^2_2, u^3_1 - u^3_2)$ | 4   | 2   | 1      |
| 4   | $\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$ | $(u^1_1 + u^1_2)$ | 1   | 1   | 1      |
We study the restriction of $\rho$ to $2(-E_8)$. Let $i \in \{7, \ldots, 22\}$. If $\rho(w_i) = w_j$ with $i \neq j$, we have $\rho(w_j) = w_i$ since $\rho$ is an involution. If $\rho(w_i) = -w_j$ with $i \neq j$, we have $\rho(w_j) = -w_i$ for the same reason. Therefore, there exists a permutation $\sigma$ of $\{7, \ldots, 22\}$ such that the basis $(w'_1, \ldots, w'_{16}) := (w_{\sigma(7)}, \ldots, w_{\sigma(22)})$ satisfies:

1. $\rho(w'_i) = w'_i$ for $i \in \{1, \ldots, k_1\}$,
2. $\rho(w'_i) = -w'_i$ for $i \in \{k_1 + 1, \ldots, k_2\}$,
3. $\rho(w'_{2i-1}) = w'_{2i}$ and $\rho(w'_{2i}) = w'_{2i-1}$ for $i \in \{\frac{k_2}{2} + 1, \ldots, k_3\}$ and
4. $\rho(w'_1) = -w'_2$ and $\rho(w'_{2i}) = -w'_{2i-1}$ for $i \in \{k_3 + 1, \ldots, 8\}$.

for suitable $k_1, k_2, k_3 \in \mathbb{N}_0$. Let $i \in \{k_1 + 1, \ldots, k_2\}$, which means that $\rho(w'_i) = -w'_i$. The number $i$ corresponds to a node of one of the two Dynkin diagrams of type $E_8$. Let $j$ be a node that is connected to $i$ by an edge.

The restriction of the bilinear form to $\text{span}(w'_i, w'_j)$ is given by

\[
\begin{pmatrix}
-2 & 1 \\
1 & -2
\end{pmatrix}
\]
If \( \rho(w'_j) = w'_j \), \( \rho \) does not preserve the bilinear form. Therefore, we have \( \rho(w'_j) = \pm w'_k \) with \( k \neq i, j \) or \( \rho(w'_j) = -w'_j \). We assume that \( \rho(w'_j) = \pm w'_k \).

Since \(-w'_i \cdot \pm w'_k = \rho(w'_i) \cdot \rho(w'_j) = w'_i \cdot w'_j = 1\) and all off-diagonal coefficients of the Cartan matrix are positive, we have \( \rho(w'_j) = -w'_k \) and \( i \) and \( k \) have to be connected by an edge. Analogously, we can conclude that any node that is connected to \( j \) is mapped to a node that is connected to \( k \). By repeating this argument, it follows that \( \rho \) acts as a non-trivial graph automorphism on the diagram \( E_8 \) to which \( i \) belongs. Since \( E_8 \) has no symmetries, this is impossible and we have \( \rho(w'_j) = -w'_j \). Again, we can repeat this argument and conclude that \( \{k_1 + 1, \ldots, k_2\} \) consists of zero, one or both connected components of \( 2E_8 \).

Next, let \( i \in \{\frac{k_3}{2} + 1, \ldots, k_3\} \), which means that \( w'_{2i-1} \) is mapped to another basis element \( w'_{2i} \). By the same argument as above, we see that all nodes that are connected to \( 2i - 1 \) are mapped to nodes that are connected to \( 2i \). The restriction of \( \rho \) to \( \text{span}(w'_{k_2+1}, \ldots, w'_{2k_3}) \) thus maps connected components of \( 2E_8 \) to other connected components. It follows that either \( \{\frac{k_3}{2} + 1, \ldots, k_3\} \) is empty or \( \rho \) interchanges both copies of \( E_8 \). Finally, let \( i \in \{k_3 + 1, \ldots, 8\} \). In this case, we have \( \rho(w'_{2i-1}) = -w'_{2i} \) and it follows that if \( \{k_3 + 1, \ldots, 8\} \) is not empty, the first \( E_8 \) is mapped to the second \( E_8 \) such that \( v'_k \) is mapped to \(-v'_k\). All in all, the restricted map \( \rho_{2(-E_8)} : 2(-E_8) \to 2(-E_8) \) is up to conjugation one of the involutions \( \rho^j_2 \) below. Let \( x_1 \in (-E_8)_1 \) and \( x_2 \in (-E_8)_2 \). We define

\[
\begin{align*}
\rho^1_2(x_1, x_2) &:= (x_1, x_2), \\
\rho^2_2(x_1, x_2) &:= (-x_1, x_2), \\
\rho^3_2(x_1, x_2) &:= (-x_1, -x_2), \\
\rho^4_2(x_1, x_2) &:= (x_2, x_1).
\end{align*}
\]

Moreover, any conjugate \( \psi : 2(-E_8) \to 2(-E_8) \) of the \( \rho^j_2 \) that is still simple is given either by

\[
\psi(x_1, x_2) = (x_1, -x_2) \quad \text{or} \quad \psi(x_1, x_2) = (-x_2, -x_1)
\]

The fixed lattices and invariants of the \( \rho^j_2 \) can be found in the following table:

| \( j \) | Fixed lattice | \( r \) | \( a \) | \( \delta \) |
|---|---|---|---|---|
| 1 | \( 2(-E_8) \) | 16 | 0 | 0 |
| 2 | \( -E_8 \) | 8 | 0 | 0 |
| 3 | \{0\} | 0 | 0 | 0 |
| 4 | \( -E_8(2) \) | 8 | 8 | 0 |

Any \( \rho^i_1 \oplus \rho^j_2 \) with \( 1 \leq i \leq 7 \) and \( 1 \leq j \leq 4 \) is an involution of \( L \). Since the complement of the fixed lattice contains a positive plane, we can conclude
with help of the Torelli theorem or with Lemma 2.24 that these lattice involutions are pull-backs of non-symplectic involutions. Finally, we compute the invariants of the involutions that we have found. Let $K = K_1 \oplus K_2$ be a direct sum of even, hyperbolic, 2-elementary lattices. We denote the invariants of $K$ by $(r, a, \delta)$ and those of the $K_i$ by $(r_i, a_i, \delta_i)$. It is easy to see that $r = r_1 + r_2$, $a = a_1 + a_2$ and $\delta = \max\{\delta_1, \delta_2\}$. Therefore, we have proven the following theorem:

**Theorem 4.8.** Let $S$ be a K3 surface and let $\rho : S \to S$ be a non-symplectic involution. $\rho$ is simple if and only if its invariants $(r, a, \delta)$ can be found in the table below. Moreover, the action of $\rho$ on the K3 lattice $L$ is conjugate to an involution $\rho_1 \oplus \rho_2$ that we have defined above. The values of $i$ and $j$ that correspond to an involution with invariants $(r, a, \delta)$ are also included in the following table.

| $(i, j)$ | $(r, a, \delta)$ | $(i, j)$ | $(r, a, \delta)$ |
|---------|-----------------|---------|-----------------|
| (1, 1)  | (18, 0, 0)      | (4, 3)  | (1, 1, 1)       |
| (1, 2)  | (10, 0, 0)      | (4, 4)  | (9, 9, 1)       |
| (1, 3)  | (2, 0, 0)       | (5, 1)  | (18, 2, 1)      |
| (1, 4)  | (10, 8, 0)      | (5, 2)  | (10, 2, 1)      |
| (2, 1)  | (19, 1, 1)      | (5, 3)  | (2, 2, 1)       |
| (2, 2)  | (11, 1, 1)      | (5, 4)  | (10, 10, 1)     |
| (2, 3)  | (3, 1, 1)       | (6, 1)  | (19, 3, 1)      |
| (2, 4)  | (11, 9, 1)      | (6, 2)  | (11, 3, 1)      |
| (3, 1)  | (20, 2, 1)      | (6, 3)  | (3, 3, 1)       |
| (3, 2)  | (12, 2, 1)      | (6, 4)  | (11, 11, 1)     |
| (3, 3)  | (4, 2, 1)       | (7, 1)  | (18, 2, 0)      |
| (3, 4)  | (12, 10, 1)     | (7, 2)  | (10, 2, 0)      |
| (4, 1)  | (17, 1, 1)      | (7, 3)  | (2, 2, 0)       |
| (4, 2)  | (9, 1, 1)       | (7, 4)  | (10, 10, 0)     |

5. K3 surfaces with singularities and a non-symplectic involution

In this section, we study which kinds of ADE-singularities a K3 surface with a non-symplectic involution may have. We focus on the case where the non-symplectic involution is simple. Let $(S, \phi)$ be a marked K3 surface with a hyper-Kähler structure and a distinguished complex structure. Moreover, let $\ell_{x+iy}$ be its period point and let $z \in L$ be the image of the Kähler class with respect to $\phi$. We assume that $S$ admits a non-symplectic involution $\rho$ with invariants $(r, a, \delta)$ that leaves the metric invariant. This implies that

$$\rho(x) = -x, \quad \rho(y) = -y, \quad \rho(z) = z.$$  

We recall that the set
\[ D := \{ d \in L | d^2 = -2, x \cdot d = y \cdot d = z \cdot d = 0 \} \]

is a root system that determines the number and type of the singular points. In order to study the possible singularities of \( S \), we choose \( x, y \) and \( z \) in such a way that \( D \) is large. \( x \) and \( y \) have to be positive elements in the orthogonal complement of the fixed lattice \( L^\rho \). Our description of the moduli space \( K3m(r,a,\delta) \) guarantees that any choice of \( x, y \in L^\rho \) with \( x^2 = y^2 > 0 \) and \( x \cdot y = 0 \) yields a period point of a (possibly singular) K3 surface with a non-symplectic involution with invariants \((r,a,\delta)\). Moreover, we can choose \( z \) as an arbitrary element of \( L^\rho \) with \( z^2 = x^2 \) and \( z \cdot x = z \cdot y = 0 \).

We assume that \( \rho \) is a simple non-symplectic involution and that we have chosen the marking such that \( \rho \) acts as \( \rho_1^i \oplus \rho_2^j \) on \( L \). Depending on \( i \) we choose \( z \in L_{\mathbb{R}} \) as follows:

\[
\begin{aligned}
z &:= \begin{cases} 
    u_1^1 + u_2^1 & \text{if } 1 \leq i \leq 6, \\
    u_1^1 + u_2^1 + u_1^2 + u_2^2 & \text{if } i = 7.
\end{cases}
\end{aligned}
\]

If \( i = 7 \), we have \( z^2 = 4 \) and we have \( z^2 = 2 \) otherwise. We choose \( x \) and \( y \) as:

\[
\begin{aligned}
x &:= \begin{cases} 
    u_1^2 + u_2^2 & \text{if } 1 \leq i \leq 6, \\
    u_1^1 + u_2^1 - u_1^2 - u_2^2 & \text{if } i = 7.
\end{cases}
\end{aligned}
\quad \text{and} \quad
\begin{aligned}
y &:= \begin{cases} 
    u_3^3 + u_2^2 & \text{if } 1 \leq i \leq 6, \\
    \sqrt{2}(u_3^1 + u_2^2) & \text{if } i = 7.
\end{cases}
\end{aligned}
\]

\( x, y \) and \( z \) satisfy \( x^2 = y^2 = z^2 \) and the three vectors are pairwise orthogonal.

By a short calculation, we see that for any value of \( i \) we have \( z \in L^\rho \) and \( x \) as well as \( y \) is orthogonal to \( L^\rho \). The orthogonal complement of \( \text{span}_\mathbb{Z}(x,y,z) \) is for all values of \( i \) given by

\[
\text{span}_\mathbb{Z}(u_1^1 - u_2^1, u_1^2 - u_2^2, u_1^3 - u_2^3) \oplus (-E_8)_1 \oplus (-E_8)_2.
\]

A K3 surface \( S \) with a hyper-Kähler structure that is determined by \( x, y \) and \( z \) thus has 3 singular points of type \( A_1 \) and 2 singular points of type \( E_8 \). Since \( \rho \) acts on \( x, y \) and \( z \) as in equation (6), there exists a non-symplectic involution of \( S \) with fixed lattice \( L^\rho \). All in all, we have proven the following theorem.

**Theorem 5.1.** Let \((r,a,\delta) \in \mathbb{N} \times \mathbb{N}_0 \times \{0,1\}\) be a triple such that there exists a K3 surface with a simple non-symplectic involution with invariants
(r, a, δ). Then there exists a K3 surface which has 3 singular points with $A_1$-singularities and 2 singular points with $E_8$-singularities and carries a hyper-Kähler metric that is invariant with respect to a non-symplectic involution with the same values of (r, a, δ).

**Remark 5.2.** The Picard lattice of the K3 surface from the above theorem is the direct sum of the lattice (7) and span$_\mathbb{Z}(z)$ and $S$ therefore has maximal Picard number.

We search for K3 surfaces with a simple non-symplectic involution whose singularities are of a different kind. More precisely, let $G$ be a Dynkin diagram that can be obtained by deleting some nodes from the union of three Dynkin diagrams of type $A_1$ and two of type $E_8$. We investigate if there exists a K3 surface with a simple non-symplectic involution whose singularities are described by $G$. We denote the lattice (7) by $N$ and fix a basis

$$(8) \quad (\tilde{w}_1, \ldots, \tilde{w}_{19}) := (u_1^1 - u_2^2, u_1^2 - u_2^3, u_1^3 - u_2^4, u_1^4, \ldots, v_1^1, \ldots, v_8^2)$$

of $N$. Let $S$ be a K3 surface with a simple non-symplectic involution $\rho$ whose invariants are $(r, a, \delta)$. We choose a marking such that $\rho(w_1) = \pm w_j$. It is easy to see that $\rho(N) = N$ and that for any $i \in \{1, \ldots, 19\}$ there exists a $j$ such that $\rho(\tilde{w}_i) = \tilde{w}_j$. For the same reasons as in Section 4 there exists a permutation $\sigma$ of $\{1, \ldots, 19\}$ such that $(\tilde{w}_1^\prime, \ldots, \tilde{w}_{19}^\prime) := (\tilde{w}_{\sigma(1)}, \ldots, \tilde{w}_{\sigma(19)})$ satisfies:

1. $\rho(\tilde{w}_i^\prime) = \tilde{w}_i^\prime$ for $i \in \{1, \ldots, k_1\}$,
2. $\rho(\tilde{w}_i^\prime) = -\tilde{w}_i^\prime$ for $i \in \{k_1 + 1, \ldots, k_2\}$,
3. $\rho(\tilde{w}_{2i}^\prime) = \tilde{w}_{2i+1}^\prime$ and $\rho(\tilde{w}_{2i+1}^\prime) = \tilde{w}_{2i}^\prime$ for $i \in \{\frac{k_2 + 1}{2}, \ldots, k_3\}$ and
4. $\rho(\tilde{w}_{2i}^\prime) = -\tilde{w}_{2i+1}^\prime$ and $\rho(\tilde{w}_{2i+1}^\prime) = -\tilde{w}_{2i}^\prime$ for $i \in \{k_3 + 1, \ldots, 9\}$.

for suitable $k_1, k_2, k_3 \in \mathbb{N}_0$. We choose four arbitrary subsets $M_1 \subseteq \{1, \ldots, k_1\}$, $M_2 \subseteq \{k_1 + 1, \ldots, k_2\}$, $M_3 \subseteq \{\frac{k_2 + 1}{2}, \ldots, k_3\}$ and $M_4 \subseteq \{k_3 + 1, \ldots, 9\}$. Moreover, we choose for any $j \in M_i$ an $\alpha_{ij} \in \mathbb{R}$ such that the family

$$(1, \alpha_{1 \min M_1}, \ldots, \alpha_{1 \max M_1}, \ldots, \alpha_{4 \min M_4}, \ldots, \alpha_{4 \max M_4})$$

is $\mathbb{Q}$-linearly independent. We replace $x, y, z \in L_\mathbb{R}$ that we have defined in the proof of Theorem 5.1 by
\[ x' = x + \sum_{j \in M_2} \alpha_{2j} \tilde{w}'_j + \sum_{j \in M_3} \alpha_{3j} (\tilde{w}'_{2j} - \tilde{w}'_{2j+1}) + \sum_{j \in M_4} \alpha_{4j} (\tilde{w}'_{2j} + \tilde{w}'_{2j+1}) \]
\[ y' = \left( \frac{x'^2}{y'^2} \right) y \]
\[ z' = z + \sum_{j \in M_1} \alpha_{1j} \tilde{w}'_j \]

\( x' \) and \( y' \) are still in the \((-1)\)-eigenspace of \( \rho \) and \( z' \) is still \( \rho \)-invariant. If the \( \alpha_{ij} \) are sufficiently small, \( x', y' \) and \( z' \) are positive. We have

\[ x'^2 = x^2 - 2 \sum_{j \in M_2} \alpha_{2j}^2 - 4 \sum_{j \in M_3} \alpha_{3j}^2 - 4 \sum_{j \in M_4} \alpha_{4j}^2 = y'^2, \quad z'^2 = z^2 - 2 \sum_{j \in M_1} \alpha_{ij}^2, \]

since \( x, y \) and \( z \) are orthogonal to \( N \). It is possible to choose the \( \alpha_{ij} \) such that

\[ 2 \sum_{j \in M_2} \alpha_{2j}^2 + 4 \sum_{j \in M_3} \alpha_{3j}^2 + 4 \sum_{j \in M_4} \alpha_{4j}^2 = 2 \sum_{j \in M_1} \alpha_{ij}^2 \]

and thus we can assume that \( x'^2 = y'^2 = z'^2 > 0 \). Moreover, we have \( x' \cdot y' = x' \cdot z' = y' \cdot z' = 0 \). If \( M_1 = \emptyset \) or \( M_2 \cup M_3 \cup M_4 = \emptyset \), we can define \( z' = \lambda z \) or \( x' = \mu x \) and \( y' = \mu y \) for appropriate \( \lambda, \mu \in \mathbb{R} \) such that \( x'^2 = y'^2 = z'^2 \). Therefore, we obtain a triple \((x', y', z')\) with the same properties as above in that case.

All in all, the triple \((x', y', z')\) defines a new hyper-Kähler structure on \( S \). Since \( x' \) and \( y' \) remain in the \((-1)\)-eigenspace of \( \rho \), \( S \) admits a non-symplectic involution with the same fixed lattice as before. Since \( z' \) is \( \rho \)-invariant, \( \rho \) is the pull-back of an isometry with respect to the new hyper-Kähler metric. The set

\[ D' := \{ d \in L | d^2 = -2, x' \cdot d = y' \cdot d = z' \cdot d = 0 \} \]
\[ = \left\{ \tilde{w}'_i \mid i \notin M_1 \cup M_2 \land \frac{i}{2} \notin M_3 \cup M_4 \land \frac{i - 1}{2} \notin M_3 \cup M_4 \right\} \]

is a root system that describes the number and type of the singular points of the new hyper-Kähler metric.

We interpret \( D' \) geometrically. Any \( \tilde{w}'_i \) with \( i \notin M_1 \cup M_2 \) corresponds to a sphere \( S^2 \) with vanishing area. The isometry \( \rho : S \to S \) maps such an \( S^2 \) to another \( S^2 \) with vanishing area. Since \( \rho(\tilde{w}'_i) = \pm \tilde{w}'_i \), the sphere is mapped to itself and the sign determines if \( \rho \) acts orientation-preserving on the sphere. Analogously, the \( \tilde{w}'_i \) with \( i \notin M_3 \cup M_4 \) and the \( \tilde{w}'_{2i+1} \) with \( i \notin M_3 \cup M_4 \) correspond to sets of spheres with area 0 that are mapped to
each other. Since the hyper-Kähler metric shall be \( \rho \)-invariant, we have to blow up the singularities that are described by the \( \bar{w}_{2l+1} \), too, if we blow up the singularities that are described by the \( \bar{w}_2 \). With help of this geometric interpretation, we are able to formulate our corollary.

**Corollary 5.3.** Let \( (r, a, \delta) \in \mathbb{N} \times \mathbb{N}_0 \times \{0, 1\} \) be a triple such that there exists a K3 surface with a simple non-symplectic involution with invariants \( (r, a, \delta) \). Moreover, let \( S \) be the K3 surface from Theorem 5.2 that has 3 points with \( A_1 \)-singularities and 2 points with \( E_8 \)-singularities and let \( \rho \) be the non-symplectic involution from the same theorem. Moreover, let \( G_1, \ldots, G_k \) be the connected components of \( 3A_1 \cup 2E_8 \) that are mapped to itself by \( \rho \) and let \( G'_1, \ldots, G'_{k_2} \) be a set of connected components that are not invariant under \( \rho \) such that

\[
G_1 \cup \ldots \cup G_{k_1} \cup G'_1 \cup \ldots \cup G'_{k_2} \cup \rho(G'_1) \cup \ldots \cup \rho(G'_{k_2}) = 3A_1 \cup 2E_8
\]

Finally, let \( \tilde{G}_1, \ldots, \tilde{G}_{l_1} \) be connected Dynkin diagrams that can be obtained by deleting some nodes of \( G_1 \cup \ldots \cup G_{k_1} \) and let \( \tilde{G}'_1, \ldots, \tilde{G}'_{l_2} \) be connected Dynkin diagrams that can be obtained by deleting some nodes of \( G'_1 \cup \ldots \cup G'_{k_2} \). Then there exists a K3 surface with a hyper-Kähler metric that admits an isometric non-symplectic involution with invariants \( (r, a, \delta) \) that has \( l_1 \) singular points of type \( \tilde{G}_1, \ldots, \tilde{G}_{l_1} \) and \( 2l_2 \) singular points of type \( \tilde{G}'_1, \ldots, \tilde{G}'_{l_2} \).

**Example 5.4.** Let \( (r, a, \delta) = (10, 10, 0) \). We recall that this is the case where the fixed locus is empty. It is possible to choose the marking such that \( \rho \) acts as \( \rho_1^7 \oplus \rho_2^4 \) on \( L \). More explicitly, we have

\[
\rho(u_j^i) = u_j^{3-i}, \quad \rho(u_j^3) = -u_j^3, \quad \rho(v_k^i) = v_k^{3-i}
\]

for all \( i, j \in \{1, 2\} \) and \( k \in \{1, \ldots, 8\} \). \( \rho \) interchanges the two Dynkin diagrams of type \( E_8 \) and two of the Dynkin diagrams of type \( A_1 \). The third Dynkin diagram of type \( A_1 \) is preserved by \( \rho \) since \( \rho(\bar{w}_3) = \rho(u_3^3 - u_2^3) = -\bar{w}_3 \) and \( -\bar{w}_3 \) is another root of the lattice \( A_1 \). We delete the node from \( E_8 \) that is connected to three other nodes. The remaining diagram is of type \( A_1 \cup A_2 \cup A_4 \). Corollary 4.3 guarantees that there exists a singular K3 surface with a non-symplectic involution \( \rho \) with invariants \( (10, 10, 0) \) that has 5 singular points of type \( A_1 \), 2 of type \( A_2 \) and 2 of type \( A_4 \). Both points of type \( A_2 \) and \( A_4 \) are mapped by \( \rho \) to each other. Moreover, there exist 2 points with \( A_1 \)-singularities that are mapped to 2 other points with \( A_1 \)-singularities and one point \( p \in S \) with an \( A_1 \)-singularity is fixed by \( \rho \).

We remove the singularity at \( p \) such that \( \rho : L \to L \) is still induced by a non-symplectic involution. This is only possible if we add a term \( \lambda \bar{w}_3 \) to \( x \) or \( y \). Afterwards, \( \bar{w}_3 \) is not contained in the Picard lattice anymore and thus it does not correspond to a complex curve on the K3 surface.
Technically speaking, the family of K3 surfaces with \( x_t := x + t \tilde{w}_3 \) defines a one-parameter family of hyper-Kähler metrics that converges to the singular one, but our construction is not a resolution in the sense of algebraic geometry. Since \( \rho \) acts orientation-reversing on the 2-sphere that represents \( \tilde{w}_3 \), our example does not contradict the fact that an involution with invariants \((10, 10, 0)\) of a smooth K3 surface does not have any fixed points.

6. K3 surfaces with two involutions

Finally, we study K3 surfaces with two commuting involutions that are non-symplectic with respect to two anti-commuting complex structures. As before, let \( x, y, z \in L_{\mathbb{R}} \) be the images of the 3 Kähler classes with respect to the marking. We denote the two non-symplectic involutions by \( \rho^1, \rho^2 : S \to S \).

Without loss of generality, \( \rho^1 \) and \( \rho^2 \) shall act on the Kähler classes as

\[
\begin{align*}
\rho^1(x) &= -x & \rho^1(y) &= -y & \rho^1(z) &= z \\
\rho^2(x) &= -x & \rho^2(y) &= y & \rho^2(z) &= -z
\end{align*}
\]

The composition \( \rho^1 \rho^2 \) is a third involution that satisfies

\[
\begin{align*}
\rho^1 \rho^2(x) &= x & \rho^1 \rho^2(y) &= -y & \rho^1 \rho^2(z) &= -z
\end{align*}
\]

There is a straightforward method to construct pairs \((\rho^1, \rho^2)\) with the above properties. Let \((r_i, a_i, \delta_i)\) with \( i = 1, 2 \) be triples of invariants that belong to non-symplectic involutions such that the direct sum \( L^1 \oplus L^2 \) of the fixed lattices can be primitively embedded into \( L \). Theorem 2.6 guarantees that this is possible if

\[ r_1 + r_2 \leq 11 \quad \text{or} \quad r_1 + r_2 + a_1 + a_2 < 22 \]

Since the embedding is primitive, there exists a basis

\[
(u_1, \ldots, u_{r_1}, v_1, \ldots, v_{r_2}, w_1, \ldots, w_{22-r_1-r_2})
\]

of \( L \) such that \((u_1, \ldots, u_{r_1})\) is a basis of \( L^1 \) and \((v_1, \ldots, v_{r_2})\) is a basis of \( L^2 \). The maps \( \rho^1 \) and \( \rho^2 \) that are defined by

\[
\begin{align*}
\rho^1(u_i) &= u_i & \rho^1(v_i) &= -v_i & \rho^1(w_i) &= w_i \\
\rho^2(u_i) &= -u_i & \rho^2(v_i) &= v_i & \rho^2(w_i) &= -w_i
\end{align*}
\]

commute and they induce non-symplectic involutions with respect to suitable complex structures. Since \( L^1 \) and \( L^2 \) are hyperbolic lattices and \( L \)
exists a smooth K3 surface $S$.

The sets of all positive elements in $L$ are non-symplectic with respect to different complex structures. Since the surface with a hyper-Kähler structure and two commuting involutions that assume that $(3, 19)$ exist non-symplectic involutions with invariants (10). All in all, we have constructed a K3 surface with a hyper-Kähler structure and two commuting involutions that are non-symplectic with respect to different complex structures. Since the sets of all positive elements in $L^1, L^2$ or $(L^1, L^2)$ are open, we can choose

$$x = \sum_{i=1}^{22-r_1-r_2} \gamma_i w_i, \quad y = \sum_{i=1}^{r_2} \beta_i v_i, \quad z = \sum_{i=1}^{r_1} \alpha_i u_i$$

such that

$$(\alpha_1, \ldots, \alpha_{r_1}, \beta_1, \ldots, \beta_{r_2}, \gamma_1, \ldots, \gamma_{22-r_1-r_2})$$

is $\mathbb{Q}$-linearly independent. Since any $d \in L$ has integer coefficients with respect to the basis (12), this condition guarantees that there exists no $d \in L$ with $d^2 = -2$ and $x \cdot d = y \cdot d = z \cdot d = 0$. Therefore, it is possible to choose $S$ as a smooth K3 surface. All in all, we have proven the following theorem.

**Theorem 6.1.** Let $(r_1, a_1, \delta_1), (r_2, a_2, \delta_2) \in \mathbb{N} \times \mathbb{N}_0 \times \{0, 1\}$ such that there exist non-symplectic involutions with invariants $(r_1, a_1, \delta_1)$. Moreover, we assume that $r_1 + r_2 \leq 11$ or $r_1 + r_2 + a_1 + a_2 < 22$. In this situation, there exists a smooth K3 surface $S$ with a hyper-Kähler structure that admits two commuting involutions $\rho^1$ and $\rho^2$ that are non-symplectic with respect to different complex structures $I_1$ and $I_2$ with $I_1I_2 = -I_2I_1$ and have invariants $(r_1, a_1, \delta_1)$ and $(r_2, a_2, \delta_2)$.

**Remark 6.2.** An important step in Kovalev’s and Lee’s construction of $G_2$-manifolds (13) is to find two K3 surfaces $S_1$ and $S_2$ with non-symplectic involutions $\rho^1$ and $\rho^2$ and a so called matching. A matching is defined as an isometry $f : S_1 \to S_2$ such that

$$f^* \omega_{I_2} = \omega_{J_1}, \quad f^* \omega_{J_2} = \omega_{I_1}, \quad f^* \omega_{K_2} = -\omega_{K_1},$$

where $\omega_{I_k}, \omega_{J_k}$ and $\omega_{K_k}$ are the three Kähler forms on $S_k$. Let $S$ be a K3 surface with two involutions that satisfy (10). If we choose the triple of complex structures on $S$ first as $(I, J, K)$ and then as $(J, I, -K)$, the identity map becomes a matching. Therefore, the above theorem shows that a matching exists if the invariants of $\rho^1$ and $\rho^2$ satisfy $r_1 + r_2 \leq 11$ or $r_1 + r_2 + a_1 + a_2 < 22$. This fact is also shown in (13).
We are especially interested in constructing K3 surfaces with ADE-singularities that admit a pair of commuting involutions with the above properties. Unfortunately, it is hard to tell how the set $D$ that describes the singular set can look like in this general situation. The reason for this is that we have an existence result for the basis (12) but no further information. If $\rho^1$ and $\rho^2$ are simple, it is possible to choose the hyper-Kähler structure such that we can determine $D$ explicitly.

Therefore, we assume from now on that $\rho^1$ and $\rho^2$ are simple. Let $\phi_k : H^2(S, \mathbb{Z}) \to L$ with $k = 1, 2$ be markings such that $\rho^k_w = \pm w_j$. The matrix representation of the pull-back map $\rho^1 : H^2(S, \mathbb{Z}) \to H^2(S, \mathbb{Z})$ with respect to the basis $\phi^{-1}_1(w_i)$ is a convenient matrix whose columns are unit vectors multiplied with $\pm 1$. If $\phi_2 \neq \phi_1$, the matrix representation of $\rho^2$ with respect to $\phi^{-1}_1(w_i)$ may be more complicated. Since that case is rather difficult to handle, we restrict ourselves to the case $\phi_1 = \phi_2$.

Up to conjugation, $\rho^k : L \to L$ can be written as $\rho^k \oplus \rho^k_2$, where $\rho^k_1 : 3H \to 3H$ and $\rho^k_2 : 2(-E_8) \to 2(-E_8)$ are two of the maps that we have defined in Section 4. By a direct calculation we see that $\rho^1_1$ and $\rho^2_1$ commute if and only if $(j_1, j_2) \in \{(2, 4), (4, 2)\}$. By adjusting the marking $\phi_1$, we can assume that the restriction of $\rho^1$ to $2(-E_8)$ actually is one of the maps $\rho^2_1$ with $j_1 \in \{1, \ldots, 4\}$. Nevertheless, the restriction of $\rho^2$ may be a conjugate of a map $\rho^2_2$ such that we still have $\rho^2(w_i) = \pm w_j$ for $i \in \{7, \ldots, 22\}$. As we have remarked in Section 4, the only additional possibilities for $\rho^2|_{2(-E_8)}$ are

$$\rho^2|_{2(-E_8)}(x_1, x_2) = (x_1, -x_2)$$

if $j_2 = 2$ or

$$\rho^2|_{2(-E_8)}(x_1, x_2) = (-x_2, -x_1)$$

if $j_2 = 4$. If we take account of these additional possibilities, it is still not possible that $\rho^1|_{2(-E_8)}$ and $\rho^2|_{2(-E_8)}$ commute if $(j_1, j_2) \in \{(2, 4), (4, 2)\}$. Nevertheless, this idea will be helpful in the next case. Let $i_1, i_2 \in \{1, \ldots, 7\}$. First, we assume that $i_1, i_2 \neq 7$. We see that $\rho^{i_1}_1$ and $\rho^{i_2}_1$ always commute, since the smaller matrix blocks

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

commute pairwisely. In Section 5 we have defined a hyper-Kähler structure by
The involution \( \rho^1 \) preserves \( z \) and acts as \(-1\) on \( x \) and \( y \). Unfortunately, the same is true for \( \rho^2 \), although \( \rho^2 \) should preserve \( y \) and act as \(-1\) on \( x \) and \( z \). In order to solve this problem, we conjugate \( \rho^{i_2}_1 \) by the map \( \tau : 3H \to 3H \) that is defined by

\[ \tau(u^l_k) := u^{4-l}_k \quad \forall k \in \{1, 2\}, l \in \{1, 2, 3\}. \]

In other words, we permute the first and the third block of the matrices that define \( \rho^{i_2}_1 \). We obtain a map that is still an isometry of \( 3H \) and maps any \( u_i \) to a \( \pm w_j \). After this conjugation, \( \rho^{i_1}_1 \) and \( \rho^{i_2}_1 \) still commute and the maps \( \rho^1, \rho^2 : L \to L \) satisfy the relations (10).

If \( i_1 = 7 \) and \( i_2 \in \{1, \ldots, 6\} \), \( \rho^{i_1}_1 \) has a \( 4 \times 4 \)-block in the upper left corner that interchanges \( H_1 \) and \( H_2 \). Therefore, \( \rho^{i_2}_1 \) and \( \tau^{-1}\rho^{i_2}_1\tau \) commute if and only if the last two \( 2 \times 2 \)-blocks of \( \rho^{i_2}_1 \) are the same. This is the case for all values of \( i_2 \) except 2 and 5. We consider the second hyper-Kähler structure from Section 5 that is defined by

\[ x := u^1_1 + u^1_2 - u^2_1 - u^2_2, \quad y := \sqrt{2}(u^3_1 + u^3_2), \quad z := u^1_1 + u^1_2 + u^2_1 + u^2_2. \]

After a short calculation, we see that \( \rho^1 \) and \( \rho^2 \) satisfy the relations (10) again. All in all, we have proven the following sufficient condition for the existence of a pair \((\rho^1, \rho^2)\) of non-symplectic involutions.

**Theorem 6.3.** Let \((i_1, j_1), (i_2, j_2) \in \{1, \ldots, 7\} \times \{1, \ldots, 4\}\) such that \((j_1, j_2) \notin \{(2, 4), (4, 2)\}\) and \((i_1, i_2) \notin \{(2, 7), (5, 7), (7, 2), (7, 5), (7, 7)\}\). Moreover, let \((r_k, a_k, \delta_k)\) with \(k \in \{1, 2\}\) be the triples of invariants that characterise the non-symplectic involutions that act as \(\rho^{i_1}_1 \oplus \rho^{i_2}_2\) on \(L\). In this situation, there exists a possibly singular \(K3\) surface \(S\) that admits two commuting involutions \(\rho^1\) and \(\rho^2\) that are non-symplectic with respect to different complex structures \(I_1\) and \(I_2\) with \(I_1I_2 = -I_2I_1\) and have invariants \((r_1, a_1, \delta_1)\) and \((r_2, a_2, \delta_2)\).

**Remark 6.4.** The above theorem yields 320 different sets \((r_k, a_k, \delta_k)\) of invariants of pairs \((\rho^1, \rho^2)\) with the desired properties. We remark that our result is mainly an existence theorem. For one set of invariants there may exist more than one pair of simple non-symplectic involutions with the same invariants. If we choose for example \(\rho^{i_2}_2\) as one of the maps \(5\) or modify \(\rho^{i_2}_1\) by permuting the three summands \(H_1, H_2\) and \(H_3\), we could easily obtain further examples with the same invariants but a different action of \(\mathbb{Z}_4\) on \(L\). Since we have restricted ourselves to the case that the \(\rho^k\) are simple and both markings \(\phi_k : H^2(S, \mathbb{Z}) \to L\) are the same, it is even possible
that examples with further sets of invariants exist. The investigation of these questions is beyond the scope of this paper.

Since the hyper-Kähler structure on $S$ that we have introduced in the proof of the theorem is the same as in Section 5, we immediately obtain the following corollary.

**Corollary 6.5.** In the situation of the above theorem, $S$ can be chosen as a K3 surface that has 3 singular points with $A_1$-singularities and 2 singular points with $E_8$-singularities.

Our next step is to investigate if there exist K3 surfaces with further kinds of singularities that admit involutions $\rho^1$ and $\rho^2$ with the same properties as in Theorem 6.3. Let $(\tilde{w}_i)_{i=1,...,19}$ be the basis (8) of the lattice that we have introduced in (1). We recall that $\tilde{w}_i^2 = -2$ for all $i$ and that the $\tilde{w}_i$ correspond to the nodes of the Dynkin diagram $3A_1 \cup 2E_8$. The involutions $\rho^1$ and $\rho^2$ generate a group that is isomorphic to $\mathbb{Z}_2^2$. We denote the span of the orbit of $\tilde{w}_i$ by $W_i$. The dimension of $W_i$ is either 1, 2 or 4. For the same reasons as in Section 5, $\mathbb{Z}_2^2$ acts on $3A_1 \cup 2E_8$ and maps connected components to connected components. Since $3A_1 \cup 2E_8$ does not contain 4 components of the same type, the dimension of $W_i$ has to be 1 or 2. We call a $\tilde{w}_i$ of type

- $(1, 1)$ if $\rho^1(\tilde{w}_i) = \rho^2(\tilde{w}_i) = \tilde{w}_i$,
- $(1, -1)$ if $\rho^1(\tilde{w}_i) = \tilde{w}_i$ and $\rho^2(\tilde{w}_i) \neq \tilde{w}_i$,
- $(-1, 1)$ if $\rho^1(\tilde{w}_i) \neq \tilde{w}_i$ and $\rho^2(\tilde{w}_i) = \tilde{w}_i$,
- $(-1, -1)$ if $\rho^1(\tilde{w}_i) \neq \tilde{w}_i$ and $\rho^2(\tilde{w}_i) \neq \tilde{w}_i$.

Since $\rho^1$ and $\rho^2$ are involutions that preserve exactly one positive vector, their eigenvalues are precisely 1 and $-1$. Moreover, they commute and therefore we have a decomposition

$$L_R = V_{1,1} \oplus V_{1,-1} \oplus V_{-1,1} \oplus V_{-1,-1}$$

where

$$V_{\epsilon_1, \epsilon_2} = \{ v \in L_R | \rho^1(v) = \epsilon_1 v, \rho^2(v) = \epsilon_2 v \}.$$

We have $x \in V_{-1,-1}$, $y \in V_{-1,1}$ and $z \in V_{1,-1}$. If $\tilde{w}_i$ is of type $(1, -1)$, we define a $\tilde{w}_i' \in L_R$ by

$$\tilde{w}_i' = \begin{cases} 
\tilde{w}_i & \text{if } \rho^2(\tilde{w}_i) = -\tilde{w}_i \\
\tilde{w}_i - \tilde{w}_j & \text{if } \rho^2(\tilde{w}_i) = \tilde{w}_j \text{ with } i \neq j \\
\tilde{w}_i + \tilde{w}_j & \text{if } \rho^2(\tilde{w}_i) = -\tilde{w}_j \text{ with } i \neq j
\end{cases}$$
If \( \tilde{w}_i \) is of type \((-1, 1)\), we define \( \tilde{w}_i' \) analogously but replace \( \rho^2 \) by \( \rho^1 \). Finally, if \( \tilde{w}_i \) is of type \((-1, -1)\), we define

\[
\tilde{w}_i' = \begin{cases}
\tilde{w}_i & \text{if } \rho^1(\tilde{w}_i) = \rho^2(\tilde{w}_i) = -\tilde{w}_i \\
\tilde{w}_i - \tilde{w}_j & \text{if } \rho^1(\tilde{w}_i) = -\tilde{w}_i \text{ and } \rho^2(\tilde{w}_i) = \tilde{w}_j \text{ with } i \neq j \\
\tilde{w}_i + \tilde{w}_j & \text{if } \rho^1(\tilde{w}_i) = -\tilde{w}_i \text{ and } \rho^2(\tilde{w}_i) = -\tilde{w}_j \text{ with } i \neq j \\
\tilde{w}_i - \tilde{w}_j & \text{if } \rho^2(\tilde{w}_i) = -\tilde{w}_i \text{ and } \rho^1(\tilde{w}_i) = \tilde{w}_j \text{ with } i \neq j \\
\tilde{w}_i + \tilde{w}_j & \text{if } \rho^2(\tilde{w}_i) = -\tilde{w}_i \text{ and } \rho^1(\tilde{w}_i) = -\tilde{w}_j \text{ with } i \neq j
\end{cases}
\]

Since \( \dim W_i \neq 4 \), these are the only possibilities that can happen for a \( \tilde{w}_i \) of type \((-1, -1)\). By our construction \( \tilde{w}_i' \in V_{\epsilon_1, \epsilon_2} \) if \( \tilde{w}_i' \) is of type \((\epsilon_1, \epsilon_2)\).

We choose arbitrary subsets

- \( P \subseteq \{1 \leq i \leq 19 | \tilde{w}_i \text{ is of type } (-1, -1)\} \)
- \( Q \subseteq \{1 \leq i \leq 19 | \tilde{w}_i \text{ is of type } (-1, 1)\} \)
- \( R \subseteq \{1 \leq i \leq 19 | \tilde{w}_i \text{ is of type } (1, -1)\} \)

such that for any pair \((i, j)\) with \( i \neq j \) from one the three sets we have \( W_i \cap W_j = \{0\} \). Let \((x, y, z)\) be the triple of Kähler classes that determines the hyper-Kähler structure with \(3 A_1\)- and \(2 E_8\)-singularities. We define a new hyper-Kähler structure by

\[
\begin{align*}
x' &= \mu x + \sum_{i \in P} \alpha_i \tilde{w}_i' \\
y' &= \nu y + \sum_{i \in Q} \beta_i \tilde{w}_i' \\
z' &= \lambda z + \sum_{i \in R} \gamma_i \tilde{w}_i'
\end{align*}
\]

The coefficients in the above definition are chosen such that

- (1) the family that consists of 1, the \( \alpha_i \), the \( \beta_i \) and the \( \gamma_i \) is \( \mathbb{Q} \)-linearly independent,
- (2) \( x'^2 = y'^2 = z'^2 > 0 \).

The hyper-Kähler structure that is defined by \( x', y' \) and \( z' \) still satisfies the equation \((\ref{eq:hyperkahler})\). Moreover, the set \( D \) that determines the number and type of the singular points can be obtained from \(3A_1 \cup 2E_8\) by deleting all nodes that correspond to an element of the \( \mathbb{Z}_2^2\)-orbit of an \( i \in P \cup Q \cup R \). In other words, we have constructed a (partial) resolution of the singularities that is still invariant under \( \mathbb{Z}_2^2 \). We remark that in general there is a minimal singularity that cannot be resolved without destroying the \( \mathbb{Z}_2^2\)-symmetry. Its Dynkin diagram is given by all \( i \) such that \( \tilde{w}_i \) is invariant under \( \mathbb{Z}_2^2 \). If we add a multiple of such an \( \tilde{w}_i \) to \( x, y \) or \( z \), we obtain a new hyper-Kähler structure that no longer satisfies \((\ref{eq:hyperkahler})\). All in all, we have proven the following theorem.

**Theorem 6.6.** Let \( S \) be one of the K3 surfaces from Theorem \((\ref{thm:k3surfaces})\) that
admits a pair \((\rho^1, \rho^2)\) of commuting simple involutions that are non-
symplectic with respect to two complex structures \(I_1\) and \(I_2\) with
\(I_1I_2 = -I_1I_2\) and
has 3 points with \(A_1\)-singularities and 2 points with \(E_8\)-singularities.
\[
\rho^1 \text{ and } \rho^2 \text{ generate a group that is isomorphic to } \mathbb{Z}_2^2 \text{ and acts on the Dynkin diagram } 3A_1 \cup 2E_8. \text{ Let } M \text{ be a } \mathbb{Z}_2^2 \text{-invariant subset of the nodes of } 3A_1 \cup 2E_8 \text{ such that no node from } M \text{ corresponds to a } \tilde{w}_i \in L \text{ that is fixed by } \mathbb{Z}_2^2. \text{ In this situation, there exists a K3 surface } S' \text{ that}
\]
(1) admits a pair of commuting simple involutions that are non-symplectic with respect to two complex structures \(I'_1\) and \(I'_2\) with \(I'_1I'_2 = -I'_1I'_2\) and whose invariants \((r_i', a_i', \delta_i')\) are the same as of \(\rho^1\) and 
(2) whose singular set is described by the Dynkin diagram that we obtain by deleting the set \(M\) of nodes from \(3A_1 \cup 2E_8\).

In particular, \(S'\) can be chosen as a smooth K3 surface if there is no \(\tilde{w}_i\) that is fixed by \(\mathbb{Z}_2^2\).

**Example 6.7.** Let \(\rho^i : L \to L\) with \(i = 1, 2\) be the lattice isometries that act as the identity on \(H_i \oplus 2(-E_8)\) and as \(-1\) on the other two summands that are isometric to \(H\). \(\rho^1\) and \(\rho^2\) commute and are both of type \(\rho^1 \oplus \rho^2\). Corollary 6.5 guarantees that there exists a K3 surface with two \(E_8\)- and three \(A_1\)-singularities and two non-symplectic involutions that correspond to \(\rho^1\) and \(\rho^2\). Theorem 6.6 allows us to resolve one or two of the \(A_1\)-singularities, but the two \(E_8\)-singularities and the third of the \(A_1\)-singularities cannot be resolved without destroying the invariance of the hyper-Kähler metric with respect to \(\rho^1\) and \(\rho^2\).

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