WILSONIAN FLOW AND MASS-INDEPENDENT RENORMALIZATION

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ABSTRACT

We derive the Gell-Mann and Low renormalization group equation in the Wilsonian approach to renormalization of massless $g\phi^4$ in four dimensions, as a particular case of a non-linear equation satisfied at any scale by the Wilsonian effective action.

We give an exact expression for the $\beta$ and $\gamma_\phi$ functions in terms of the Wilsonian effective action at the Wilsonian renormalization scale $\Lambda_R$; at the first two loops they are simply related to the gradient of the flow of the relevant couplings and have the standard values; beyond two loops this relation is spoiled by corrections due to irrelevant couplings.

We generalize this analysis to the case of massive $g\phi^4$, introducing a mass-independent Wilsonian renormalization scheme; using the flow equation technique we prove renormalizability and we show that the limit of vanishing mass parameter exists. We derive the corresponding renormalization group equation, in which $\beta$ and $\gamma_\phi$ are the same as in the massless case; $\gamma_m$ is also mass-independent; at one loop it is the gradient of a relevant coupling and it has the expected value.

*Work supported in part by M.U.R.S.T.
INTRODUCTION

The renormalization group equation describes the response of the parameters in the renormalization conditions due to a change in the renormalization scale \([1]\). It provides useful information on the asymptotic behavior of the Green functions at large momenta.

A standard way of introducing the renormalization scale is to impose the renormalization conditions at a momentum subtraction point \([1,2]\).

In the Wilsonian renormalization group approach \([3]\) the renormalization scale \(\Lambda_R\) separates the hard modes with \(p > \Lambda_R\) from the soft modes with \(p < \Lambda_R\); roughly speaking, only hard modes propagate in the internal lines of the graphs of the Wilsonian effective action. The exact way in which the soft modes are frozen depends on the form of the cut-off function.

In the hard-soft (HS) renormalization schemes, first introduced in \([4]\) with the purpose of studying in a simple way the renormalizability of massless theories with BPHZ, a splitting in hard and soft fields is made at a scale \(\Lambda_R\), in such a way that the renormalization conditions can be chosen at zero momentum even in massless theories. The independence of the physical quantities from such a splitting is guaranteed by the renormalization group equation obtained using the Quantum Action Principle \([5]\).

A recent discussion of the HS schemes in the Wilsonian approach can be found in \([6]\). The problem of deducing the renormalization group equation within these Wilsonian HS schemes has not yet been solved for a generic HS cut-off function.

In the case of step-function cut-off, the renormalization group equation has been obtained in \(g\phi^4\) in \([7]\); the beta function and \(\gamma_\phi\) are simply related to the gradient of the flow of the relevant couplings. It is an interesting question whether this property holds for smoother cut-off functions, which are more suitable in perturbative Quantum Field Theory.

Using the flow equation \([3,8]\) in the case of smooth cut-off with compact support, an approximate derivation of the renormalization group equation has been given in massive \(g\phi^4\) in a HS scheme in \([8]\). Their formulae for the beta and gamma functions have the above-mentioned relation to the flow of the relevant couplings. They verified this gradient flow relation explicitly at the first non-trivial loop order for the beta function and \(\gamma_\phi\), showing that in the massless limit they have the standard values, which are expected to be scheme-independent \([10]\).

The main results of this paper are the following:

i) we introduce an ‘effective renormalization group equation’ for the Wilsonian effective action, in terms of which we obtain in the Wilsonian HS renormalization schemes exact formulae (in perturbation theory) for the beta and the gamma functions in terms of the Wilsonian effective action at scale \(\Lambda_R\);

ii) we introduce a mass-independent HS scheme for massive \(g\phi^4\), which admits zero mass limit; we deduce the corresponding mass-independent renormalization group equation. These results are first discussed heuristically in a general setting, then are proven in the framework of the flow equation technique \([8]\).

We start observing that, assuming the validity of the renormalization group equation, it is possible to obtain its extension to the case of the Wilsonian effective action at scale \(\Lambda > 0\). This is an effective renormalization group equation which is non-linear; in the limit of \(\Lambda \to 0\) it becomes the usual renormalization group equation. The situation is analogous
to the Ward identities in gauge theory; at scale \( \Lambda \neq 0 \) the Wilsonian effective action satisfies ‘effective Ward identities’ \([1]\), which in the limit \( \Lambda \to 0 \) become the usual Ward identities. In both cases there are non-linear terms due to the fact that, in general, field transformations (respectively rescalings and gauge transformations) do not commute with the Wilsonian renormalization flow.

In the massive case we study in detail the mass-independent renormalization group equation, first introduced by Weinberg \([12]\). This equation has the advantage, in comparison with the Gell-Mann and Low \([1]\) or the Callan-Symanzik \([13]\) equations, that it can be solved exactly; this is a useful property in studying the asymptotic behavior of the Green functions. The HS schemes have, by definition, renormalization conditions at zero momentum even in massless theories; it is a natural step to impose renormalization conditions at zero momentum and zero mass in a generic theory. We introduce a mass-independent HS renormalization scheme in massive \( g\phi^4 \), which is similar in spirit to the mass-independent scheme introduced by Weinberg \([12]\); in the latter case there are problems in dealing with scalars, due to the fact that mass insertions on the massless theory are infrared-singular. In the HS schemes these infrared singularities are absent; in fact mass insertions are made on the Wilsonian vertices at \( \Lambda_R \), which are trivially infrared finite. The HS mass-independent renormalization scheme described here is also related to the mass-independent scheme in \([14]\), which however applies specifically to massive theories, while the HS schemes apply equally well to massless theories which cannot be considered the massless limit of a massive theory, as originally remarked in \([7]\). Using the effective renormalization group equation at scale \( \Lambda_R \), we obtain in these schemes exact formulae (in perturbation theory) for the beta and the gamma functions in terms of the Wilsonian effective action at scale \( \Lambda_R \). \( \beta \) and \( \gamma_\phi \) are the same as in the massless case. At the first two loops they are simply related to the gradient of the flow of the relevant couplings. As a check, we verify by explicit computation that for a class of analytic cut-off functions they have the standard values, which is expected to be scheme independent \([10]\). \( \gamma_m \) is also mass-independent; at one loop it has the gradient flow expression and the standard value. Beyond two loops this gradient flow property fails for generic cut-off functions.

In the particular case in which the cut-off function characterizing the HS scheme is smooth and with compact support, these results can be put on a rigorous footing using the flow equation technique \([8]\). First of all we prove the renormalization group equation for massless \( g\phi^4 \); this proof involves a version of the Quantum Action Principle \([5]\), discussed with the flow equation technique in \([11]\). Then we treat the massive case in a similar way, using our mass-independent HS renormalization scheme. Using the flow equation we prove the ultraviolet and infrared convergence of the theory using appropriate bounds. We prove the effective mass-independent renormalization group equation and we show that for \( \Lambda \to 0 \) it becomes the usual mass-independent renormalization group equation.

In the first section we introduce the mass-independent HS schemes; we obtain the effective renormalization group equation for the Wilsonian effective action, leading to exact formulae for the beta and gamma functions in terms of the Wilsonian vertices at scale \( \Lambda_R \); we compute them at low loop orders. In the second section we prove, using the flow equation, the renormalization group equation for the massless \( g\phi^4 \) theory. In the third section and in the Appendix we prove the consistency of the mass-independent HS schemes and we derive the renormalization group equation for massive \( g\phi^4 \).
I. EFFECTIVE RENORMALIZATION GROUP EQUATION AND
MASS-INDEPENDENT RENORMALIZATION

A. Effective renormalization group equation

Consider the $g\phi^4$ theory in Euclidean four-dimensional space; the path-integral is

$$Z_{0\Lambda_0}[J] \equiv \int \mathcal{D}\Phi e^{-\frac{1}{\hbar}[S(\Phi) - J\Phi]}$$

with the bare action

$$S(\Phi) = \frac{1}{2}\Phi D_{0\Lambda_0}^{-1}\Phi + S^I(\Phi)$$

$$S^I(\Phi) = \frac{1}{2} \int \phi(-p)[c_1 + c_2 p^2]\phi(p) + \frac{c_3}{4!} \int_{p_1 p_2 p_3} \phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)$$

where $p_4 = -p_1 - p_2 - p_3$ and $S^I$ is the interacting part of the bare action. We use a compact notation in which $\Phi = \{\phi(p)\}$ and a matrix notation is used for integrations on the momenta; furthermore we will use the trace operation meaning integration over momentum $\int_p \equiv \int \frac{d^4p}{(2\pi)^4}$.

$D_{0\Lambda_0}(m, p)$ is the propagator with ultraviolet cut-off $\Lambda_0$, which for $\Lambda_0 \to \infty$ converges to $D(m, p) = \frac{1}{p^2 + m^2}$. For the moment we will not specify the ultraviolet regulator used, since our argument will not depend on it. Let us split the propagator in two parts characterized by a scale $\Lambda > 0$

$$D_{0\Lambda_0} = D_S + D_H ; \quad D_H = D_{\Lambda \Lambda_0}$$

where the ‘hard’ propagator $D_H$ behaves as $D_{0\Lambda_0}$ for large momenta, while it is regular in $p = 0$ in the massless case, $\Lambda$ acting as an infrared cut-off on this propagator; the ‘soft’ propagator $D_S$ converges fast enough to zero for large momenta, and it behaves as $D_{0\Lambda_0}$ for small momenta in the massless case. Consider a cut-off function $K_\Lambda(m, p)$ satisfying $K_\Lambda(0, 0) = 1$ and going to zero at least as $\Lambda^4/(p^2)^2$ for $p^2/\Lambda^2 \to \infty$. We require that

$$D_H = D_{\Lambda \Lambda_0} \to D(1 - K_\Lambda)$$

for $\Lambda_0 \to \infty$. For the considerations in this section this function is understood to be analytic.

Let us make an ‘incomplete integration’ over the hard modes

$$Z_{\Lambda \Lambda_0}[J] \equiv \int \mathcal{D}\Phi e^{-\frac{\hbar}{\Delta}[D_{\Lambda \Lambda_0}^{-1}\Phi + S^I(\Phi) - J\Phi]}$$

The flow of this functional from $\Lambda$ to zero can be represented as

$$Z_{0\Lambda_0}[J] = e^{\frac{\hbar}{\Delta}[D_{0\Lambda_0}^{-1} - D_{\Lambda \Lambda_0}^{-1}]} Z_{\Lambda \Lambda_0}[J]$$

$Z_{\Lambda \Lambda_0}[J]$ is infrared finite for $\Lambda > 0$ even in the massless limit. In all the formulae involving functionals used in this paper, only the source (or field) dependent terms are well-defined.

The 1-PI functional generator corresponding to $Z_{\Lambda \Lambda_0}$ is
$$\Gamma_{\Lambda_0}[\Phi] = \frac{1}{2} \Phi D^{-1}_{\Lambda_0} \Phi + \Gamma'_{\Lambda_{00}}[\Phi] \tag{7}$$

The $n$-point 1-PI effective Green function $\Gamma_n^{\Lambda_0}(m; p_1, ..., p_{n-1})$ depends on the mass $m$ and on $n-1$ independent momenta.

Let us introduce a renormalization scheme in which some renormalization scale $\Lambda_R$ appears; according to general arguments the Gell-Mann and Low renormalization group equation on $Z[J] \equiv \lim_{\Lambda_0 \to \infty} Z_{\Lambda_0}[J]$ holds

$$(\Lambda_R \frac{\partial}{\partial \Lambda_R} + \beta \frac{\partial}{\partial g} + \gamma_\phi J \frac{\delta}{\delta J} - \gamma m \frac{\partial}{\partial m^2}) Z[J] = 0 \tag{8}$$

where $\beta, \gamma_\phi$ and $\gamma_m$ are in general functions of $g$ and $\frac{m}{\Lambda_R}$. From eq.(3) it follows that

$$Z_{\Lambda}[J] = e^{\hat{W}_{\Lambda}[J]} \equiv Z_{\Lambda_{\infty}}[J] \tag{9}$$

satisfies the ‘effective renormalization group equation’

$$(\Lambda_R \frac{\partial}{\partial \Lambda_R} + \beta \frac{\partial}{\partial g} - \gamma_\phi \Phi \frac{\delta}{\delta \Phi} - \gamma m \frac{\partial}{\partial m^2}) \Gamma_{\Lambda} + \Phi A_{\Lambda} \Phi + htr A_{\Lambda} \frac{\delta^2 \Gamma_{\Lambda}}{\delta \Phi^2} = 0 \tag{10}$$

where

$$A_{\Lambda} \equiv (\gamma_\phi + \frac{1}{2} \gamma_m \frac{\partial}{\partial m^2}) (\frac{D^{-1} K_{\Lambda}}{1 - K_{\Lambda}}) \tag{11}$$

In terms of $W_{\Lambda}$ the effective renormalization group equation reads

$$(\Lambda_R \frac{\partial}{\partial \Lambda_R} + \beta \frac{\partial}{\partial g} + \gamma_\phi J \frac{\delta}{\delta J} - \gamma m \frac{\partial}{\partial m^2}) W_{\Lambda} - \frac{\delta W_{\Lambda}}{\delta J} A_{\Lambda} \frac{\delta W_{\Lambda}}{\delta J} - htr A_{\Lambda} \frac{\delta^2 W_{\Lambda}}{\delta J^2} = 0 \tag{12}$$

Observe that the source field rescaling operator $J \frac{\delta}{\delta J}$ and the mass rescaling operator $m^2 \frac{\partial}{\partial m^2}$ do not commute with the flow, leading to the last two terms in the above equation; analogous terms appear in the effective gauge Ward identity [11].

Making a Legendre transformation we get

$$(\Lambda_R \frac{\partial}{\partial \Lambda_R} + \beta \frac{\partial}{\partial g} - \gamma_\phi \Phi \frac{\delta}{\delta \Phi} - \gamma m \frac{\partial}{\partial m^2}) \Gamma_{\Lambda} + \Phi A_{\Lambda} \Phi + htr A_{\Lambda} \frac{\delta^2 \Gamma_{\Lambda}}{\delta \Phi^2}^{-1} = 0 \tag{13}$$

Using (5) we get the effective renormalization group equation on $\Gamma'_\Lambda[\Phi] \equiv \Gamma'_\Lambda_{\infty}[\Phi]$ in the form

$$(\Lambda_R \frac{\partial}{\partial \Lambda_R} + \beta \frac{\partial}{\partial g} - \gamma_\phi \Phi \frac{\delta}{\delta \Phi} - \gamma m \frac{\partial}{\partial m^2}) \Gamma'_\Lambda[\Phi] =$$

$$= \Phi (\gamma_\phi D^{-1} + \frac{m^2}{2} \gamma_m) \Phi - \gamma_\phi T^\Lambda_{\phi}[\Phi] - \gamma_m T^\Lambda_m[\Phi] \tag{14}$$

with the following non-linear terms

$$T^\Lambda_{\phi}[\Phi] \equiv htr \frac{D^{-1} K_{\Lambda}}{1 - K_{\Lambda}} \frac{\delta^2 \Gamma_{\Lambda}}{\delta \Phi^2}^{-1} ; \ T^\Lambda_m[\Phi] \equiv \frac{h}{2} tr \left[ m^2 \frac{\partial}{\partial m^2} \left( \frac{D^{-1} K_{\Lambda}}{1 - K_{\Lambda}} \right) \right] \frac{\delta^2 \Gamma_{\Lambda}}{\delta \Phi^2}^{-1} \tag{15}$$
The $n$-point functions $T_n^A(m; p_1, ..., p_{n-1})$ are defined factorizing as usual the delta function for momentum conservation. From (14) one obtains the effective renormalization group equations on $\Gamma_n^A(m; p_1, ..., p_{n-1})$; we do not give any meaning to the field independent part of (14).

Imposing Wilsonian renormalization conditions at $\Lambda = \Lambda_R > 0$, one can use the effective renormalization group equation (14) to give an expression of the beta and gamma functions in terms of the Wilsonian effective action. A standard set of zero-momentum renormalization conditions at a Wilsonian scale $\Lambda_R > 0$ for the massive $g\phi^4$ theory is the following [84]:

$$\Gamma_2^{I\Lambda_R\Lambda_0(l)}(m; 0) = 0 \ ; \ \partial_{p^2}|_{p=0}\Gamma_2^{I\Lambda_R\Lambda_0(l)}(m; p) = 0 \ ; \ \Gamma_4^{I\Lambda_R\Lambda_0(l)}(m; 0, 0, 0) = g \delta^{l,0} \tag{16}$$

where $l$ is the loop index in the perturbative expansion. In the following we will call for short HS (hard-soft) scheme any renormalization scheme in which zero-momentum renormalization conditions at a Wilsonian scale $\Lambda_R > 0$ are used. For a discussion of these schemes see [6,15].

For the beta and gamma functions one finds, using (14,16)

$$\beta = \beta_3 + \gamma_\phi [4g + \alpha_3] + \gamma_m \alpha_4 \ ; \ \gamma_\phi = -\frac{\beta_2 + \gamma_m \alpha_5}{2 + \alpha_2} \ ; \ \gamma_m = -\frac{\beta_1 + \gamma_\phi (2 + \alpha_1)}{1 + \alpha_6} \tag{17}$$

where

$$\beta_1 = \Lambda \frac{\partial}{\partial \Lambda}|_{\Lambda_R} m^{-2} \Gamma_2^L(m; 0) \ ; \ \beta_2 = \Lambda \frac{\partial}{\partial \Lambda}|_{\Lambda_R} \partial_{p^2}|_{0} \Gamma_2^L(m; p) \ ; \ \beta_3 = \Lambda \frac{\partial}{\partial \Lambda}|_{\Lambda_R} \Gamma_4^L|_{p=0} \tag{18}$$

are the gradients of the flow of the relevant terms at $\Lambda_R$, and the alpha coefficients are related to the non-linear terms of (14)

$$\alpha_1 = -m^{-2} \mathcal{T}_2^L(m; 0) \ ; \ \alpha_2 = -\partial_{p^2}|_{0} \mathcal{T}_2^L(m; p) \ ; \ \alpha_3 = -\mathcal{T}_4^L(m; 0, 0, 0) \ ; \ \alpha_4 = -\mathcal{T}_4^L(m; 0, 0, 0) \ ; \ \alpha_5 = -\partial_{p^2}|_{0} \mathcal{T}_2^L(m; p) \ ; \ \alpha_6 = -m^{-2} \mathcal{T}_2^L(m; 0) \tag{19}$$

Although for simplicity we have derived the above equations in the case of analytic cut-off, it is straightforward to generalize them to the case of smooth functions $K_A(m, p)$ with compact support, which we will use in the following sections.

In the limit of step-function cut-off $K_A(m; p) \to \theta(\Lambda - |p|)$ the $n$-point functions $\mathcal{T}_n^A(m; p_1, ..., p_{n-1})$ vanish since $K_A(1 - K_A) \to 0$, so that the Wilsonian vertices satisfy in this limit the ordinary renormalization group equation; the beta and the gamma functions are gradients of the flow of relevant couplings, since their alpha terms vanish, in agreement with [7].

For a generic cut-off function the alpha terms in the beta and the gamma functions are absent at lowest orders in perturbation theory, so that the gradient flow property holds in this approximation, as checked in [4]; however this property fails at higher loops.

To our knowledge, exact relations (in perturbation theory) for the beta and gamma functions in terms of the Wilsonian effective action at the renormalization scale have not been given before; for instance in [9] the alpha terms in (17) are absent.
B. Mass-independent HS renormalization scheme

In the following we will restrict our attention to mass-independent HS renormalization schemes which admit a straightforward massless limit. The reason of interest of mass-independent renormalization schemes, as pointed out by Weinberg [12], is that the beta and gamma functions are mass-independent, allowing a simple solution to the renormalization group equation (8).

In these renormalization schemes the bare action \( (2) \) depends only polynomially on the mass; in particular \( c_2 \) and \( c_3 \) are mass-independent, while the mass counterterm has the following form

\[
c_1 = \hat{c}_1 \Lambda_0^2 + \tilde{c}_1 m^2
\]

where \( \hat{c}_1 \) and \( \tilde{c}_1 \) are mass-independent.

Let us introduce the following mass-independent zero-momentum renormalization conditions at a renormalization scale \( \Lambda_R > 0 \):

\[
\Gamma_{I}^{\Lambda=0 \Lambda_0(l)}(0; 0) = 0 \quad (21)
\]

\[
\partial \mu^2 |_{\mu^2=0} \Gamma^{\Lambda_R \Lambda_0(l)}_2(0; \mu) = 0 \quad ; \quad \Gamma_{4}^{\Lambda_R \Lambda_0(l)}(0; 0, 0, 0) = g \delta^{4 \mu} \quad (22)
\]

and

\[
\partial m^2 |_{m=0} \Gamma^{\Lambda_R \Lambda_0(l)}_2(m; 0) = 0 \quad (23)
\]

In (21-23) there is one more condition than in (16); on the other hand in (20) an extra bare parameter appears, so that the number of bare parameters matches again with the number of renormalization conditions; the fourth renormalization condition (23) fixes the extra parameter \( \tilde{c}_1 \).

We will prove later (see Appendix B) that the theory defined by (21-23) becomes, for \( m \to 0 \), the massless theory associated with the conditions (21,22) only.

The condition (24) is the one which is necessary to impose in massless theories; it guarantees that a self-energy subdiagram insertion provides the \( p^2 \) factor needed to cancel the extra \( 1/p^2 \) propagator. In the conditions (22) the scale \( \Lambda_R \) plays a role similar to the non-zero momentum subtraction point in a massless theory. The condition (23) has no equivalent at \( \Lambda = 0 \) and non-zero momentum subtraction point; in fact \( \partial_m \Gamma_{2}^{\Lambda=0 \Lambda_0(l)}(m; \mu^2=\mu^2) \) does not exist at \( m = 0 \), since mass insertions on the internal lines of a massless graph diverge in the infrared region of loop momenta for any value of the external momenta. Obviously for \( \Lambda_R > 0 \) these divergences are absent. In theories without scalars, analogous renormalization conditions at \( \Lambda = 0 \) and non-zero momentum subtraction point can be imposed [12]: e.g. on the fermionic two-point function in QED one can choose the condition

\[
\partial_m \Gamma_{2}^{\Lambda=0 \Lambda_0(l)}(m; \mu^2=\mu^2) |_{m=0} = 0.
\]

Let us discuss in a qualitative way the consistency of this HS scheme; it is useful to treat separately the ultraviolet and the infrared problems. To show that it leads to ultraviolet renormalization, observe that substituting the condition (21) with

\[
\Gamma^{\Lambda_R \Lambda_0(l)}_2(0; 0) = \rho_1^{(l)} \quad (24)
\]
(with $\rho_1^{(0)} = 0$) and keeping the conditions \( \rho_1 \), the Wilsonian vertices $\Gamma_{\Lambda R \Lambda 0}$, constructed with the bare action \( \rho_1 \) and the scalar propagator $D_{\Lambda R \Lambda 0}$, are renormalized. The condition \( \rho_1 \) can be consistently imposed since $\Lambda_R$ acts as an infrared cut-off on the propagator. This condition is a satisfactory renormalization condition for the massive theory since in this scheme the derivatives with respect to $m^2$, applied to a single graph in which subdivergences have been removed, lower by two the degree of divergence. In a scheme which is not mass-independent this fact would not be true, since the mass derivative can act on the factors $\ln m^2$ in the counterterms. The theory at $\Lambda = 0$ can be constructed with these Wilsonian vertices and soft propagators $D_{\Lambda R \Lambda 0}$. Since the Wilsonian vertices are renormalized and the soft propagators go to zero sufficiently fast (at least as $\Lambda_R^4/(p^2)^3$) at large momentum, by power counting it follows that the graphs are superficially convergent, so that by the usual arguments one expects that at $\Lambda = 0$ the theory is ultraviolet finite (a proof of this fact will be given in the case of compact-support cut-off in Appendix A). For $m \neq 0$ there are no infrared problems in this procedure; however for generic values of $\rho_1$ in \( \rho_1 \) the quantity $\Gamma_{\Lambda R \Lambda 0}(m;0)$ will not admit limit for $m \to 0$, so that the condition \( \rho_1 \) is not satisfied, and the massless limit of the theory does not exist.

It is interesting to notice that, choosing $\rho_1^{(0)} = \Lambda_R f^{(0)}(g;\Lambda_R)$ in such a way that \( \rho_1 \) holds order by order in loops, the limit $m \to 0$ can be made on the Green functions with non-exceptional momenta, obtaining the corresponding Green functions of the massless theory (a proof of this fact will be given in the case of compact-support cut-off in Appendix B).

In this HS scheme the bare coefficients $\hat{c}_1, \tilde{c}_1, c_2$ and $c_3$ in \( \rho_1 \), determined loop by loop in terms of the renormalization conditions \( \rho_1 \), are mass-independent.

Using the renormalization conditions \( \rho_1 \) in equation \( \rho_1 \) at $\Lambda = \Lambda_R$ we get the following exact expressions for the coefficient functions of the renormalization group equation in terms of the Wilsonian action:

$$
\gamma_\phi = \frac{-\beta_2}{2 + \alpha_2}; \quad \beta = \beta_3 + \gamma_\phi(4g + \alpha_3); \quad \gamma_m = -\frac{\beta_1 + \gamma_\phi(2 + \alpha_1)}{1 + \alpha_m}
$$

(25)

where now the gradients of the flow of the relevant couplings are

$$
\beta_1 = \Lambda \frac{\partial}{\partial \Lambda} |_{\Lambda_R} \partial_{m^2} \Gamma^2_{\phi} \Lambda |_0; \quad \beta_2 = \Lambda \frac{\partial}{\partial \Lambda} |_{\Lambda_R} \partial_{p^2} \Gamma^2_{\phi} \Lambda |_0; \quad \beta_3 = \Lambda \frac{\partial}{\partial \Lambda} |_{\Lambda_R} \Gamma^A_{4} |_0
$$

(26)

and $(..)|_0$ indicates evaluation at zero mass and momenta; the alpha coefficients

$$
\alpha_1 \equiv -\partial_{m^2} \mathcal{T}_{\phi 2}^{\Lambda_R} |_0; \quad \alpha_m \equiv -\partial_{m^2} \mathcal{T}_m^{\Lambda_R} |_0; \quad \alpha_2 \equiv -\partial_{p^2} \mathcal{T}_{\phi 2}^{\Lambda_R} |_0; \quad \alpha_3 \equiv -\mathcal{T}_{\phi 4}^{\Lambda_R} |_0
$$

(27)

represent the correction due to the non-linear terms in the effective renormalization group equation \( \rho_1 \); they are vanishing at tree level.

C. Low order computations

According to general arguments \( \rho_1 \) one expects that $\gamma_\phi^{(1)}$, $\beta^{(1)}$, $\gamma_m^{(2)}$ and $\beta^{(2)}$ are the same in all mass-independent renormalization schemes (while $\gamma_\phi^{(1)}$ is trivially zero). We will verify this explicitly for a class of HS schemes. Observe that from \( \rho_1 \), $\gamma_\phi^{(1)} = 0$ and the vanishing
at tree level of the alpha coefficients in (27): it follows that $\gamma^{(1)}_m, \beta^{(1)}, \gamma^{(2)}_m$ and $\beta^{(2)}$ depend in a simple way on the gradient of the flow of the relevant couplings at $\Lambda_R$, in agreement with analogous formulae in [4].

$\beta^{(1)}$ and $\gamma^{(2)}_m$ have been computed in [4] in a HS scheme using a generic smooth compact-support cut-off, giving the standard value in the massless limit. Let us compute $\gamma^{(1)}_m, \beta^{(1)}, \beta^{(2)}$ and $\gamma^{(2)}_m$ in a class of mass-independent HS schemes with analytic cut-off.

Using the renormalization conditions (21)-(23) the non-vanishing one-loop bare parameters are the following

$$
\bar{c}^{(1)}_1 = -\frac{g}{2 \Lambda_0} \int_D D_{\Lambda_0}(q) ; \quad \bar{c}^{(1)}_1 = -\frac{g}{2} \int_{D_R} \partial_{m^2} [D_{\Lambda_0}(m; q) ; \quad \bar{c}^{(1)}_3 = \frac{3g^2}{2} \int_D D_{R \Lambda_0}^2(q) \tag{28}
$$

where we define $D_{\Lambda_0}(p) \equiv D_{\Lambda_0}(0, p)$. Observe that $\bar{c}^{(1)}_1$ would be ill-defined for $\Lambda_R = 0$ due to an infrared divergence. This is the simplest indication that it is not possible to impose the mass-independent renormalization condition $\partial_{m^2} \Gamma_{2}^{\Lambda = \Lambda_0}(m; \mu) = 0$, as discussed in the previous subsection.

The one and two-loop contributions to the four-point relevant Wilsonian vertex at scale $\Lambda$ are given respectively by:

$$
\Gamma^{\Lambda_0(1)}_4(0; 0, 0, 0) = -\frac{3g^2}{2} \int \partial_{m^2} [D_{\Lambda_0}^2 - D_{R \Lambda_0}^2](q)
$$

$$
\Gamma^{\Lambda_0(2)}_4(0; 0, 0, 0) = 3g^3 \int_{pq} [D_{\Lambda_0}^2(q)D_{\Lambda_0}(p + q + p) - D_{R \Lambda_0}^2(p)] + \frac{3g^3}{4} \int_{pq} [(D_{\Lambda_0} - D_{0 \Lambda_0})(p)D_{R \Lambda_0}^2(q)] + g^3 A(\frac{\Lambda_R}{\Lambda_0}) \tag{29}
$$

where $g^3 A(\frac{\Lambda_R}{\Lambda_0}) = c^{(2)}_3 - \frac{2g^3}{4} \int [D_{R \Lambda_0}^2(p)]^2$ is the constant necessary to satisfy the renormalization condition (22).

The other non-trivial relevant couplings we need to compute are

$$
\partial_{m^2} \Gamma^{\Lambda_0(1)}_2(m; 0) = \frac{g}{2} \int \partial_{m^2} [D_{\Lambda_0}(m; q) - D_{R \Lambda_0}(m; q)] \tag{30}
$$

and

$$
\partial_{p^2} \Gamma^{\Lambda_0(2)}_2(p; 0) = -\frac{g^2}{6} \int D_{\Lambda_0}(q)D_{\Lambda_0}(r) \frac{1}{8} \frac{\partial^2}{\partial p_\mu \partial p_\mu} \partial_{m^2} \partial_{\Lambda_0}(p + q + r) + c^{(2)}_2 \tag{31}
$$

Let us consider the class of HS schemes characterized by a propagator of the form

$$
D_{\Lambda_0}(m; p) = \int_{\Lambda_0^2} d^4 x e^{-\alpha(p^2 + m^2)} \rho(\alpha \Lambda^2) \tag{32}
$$

where the function $\rho(x)$ satisfies $\rho(0) = 1, \rho'(0) = 0$ and goes sufficiently fast to zero for $x \to \infty$. The first condition guarantees that the hard propagator satisfies $D_{\Lambda_\infty}(m; p) \sim D(m; p)$ for $p^2 \gg \Lambda^2$; the second condition (which will not be used in the following computations) is chosen in such a way that for $\Lambda_0 \to \infty$ the soft propagator $D_S(m; p) = DK_\Lambda(m; p)$ goes to zero at least as fast as $\frac{\Lambda^4}{(p^2 + m^2)^2}$ for $p^2 \gg \Lambda^2$. For instance $\rho(x) = \theta(1 - x)$ leads to
$K_\Lambda(m; p) = e^{-(p^2 + m^2)/\Lambda^2}$ and $\rho(x) = (1 + x)e^{-x}$ leads to $K_\Lambda(m; p) = \frac{\Lambda^4}{(p^2 + m^2 + \Lambda^2)^2}$, which have been used in HS schemes in [15] and [3] respectively.

At one loop we get

$$\gamma_{m}^{(1)} = -\beta_{1}^{(1)} = \lim_{\Lambda_0 \to \infty} \frac{g}{16\pi^2} \int_{\frac{\Lambda_0}{\Lambda_0}}^{\infty} d\alpha \frac{d\rho}{d\alpha} = -\frac{g}{16\pi^2}$$

and

$$\beta^{(1)} = \beta_{3}^{(1)} = \lim_{\Lambda_0 \to \infty} \frac{-6g^2}{16\pi^2} \int_{\frac{\Lambda_0}{\Lambda_0}}^{\infty} \frac{d\alpha_1 d\alpha_2}{(\alpha_1 + \alpha_2)^2} \frac{d\rho}{d\alpha_1} \rho(\alpha_2) = \frac{3g^2}{16\pi^2}$$

where in the last step we have integrated by parts and used $\rho(0) = 1$.

The last integral in (29) is constant in $\Lambda$ in the limit of infinite ultraviolet cut-off, so that eqs. (29) give

$$\beta_{3}^{(2)} = \lim_{\Lambda_0 \to \infty} \frac{\partial}{\partial \Lambda} \Lambda \beta_{R}^{3} g^{3} \int_{\rho_{q}}^{\infty} D_{x}^{2} \Lambda_{\Lambda} (q) \left[D_{\Lambda} \Lambda_{\Lambda} (p) D_{\Lambda} \Lambda_{\Lambda} (q + p) - D_{x}^{2} \Lambda_{\Lambda} (p) \right]$$

$$= \lim_{\Lambda_0 \to \infty} \frac{-6g^3}{(16 \pi^2)^2} \int_{\frac{\Lambda_0}{\Lambda_0}}^{\infty} \left[\prod_{i=1}^{4} d\alpha_i \right] f(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \sum_{i=1}^{4} \alpha_i \frac{d}{d\alpha_i} \left[\prod_{j=1}^{4} \rho(\alpha_j)\right]$$

Integrating by parts, using $\rho(0) = 1$ and the homogeneity of the rational function $f$ in the above formula, we get

$$\beta_{3}^{(2)} = \frac{-12g^3}{(16 \pi^2)^2} \lim_{\epsilon \to 0} \int_{1}^{\infty} dx dy dz \rho(\epsilon x) \rho(\epsilon y) \rho(\epsilon z) \left[f(1, x, y, z) + f(x, y, 1, z)\right] = \frac{-6g^3}{(16 \pi^2)^2}$$

where, after reordering the terms in the integrand, the limit can be taken inside the integral, which is reduced to elementary integrals

$$\int_{1}^{\infty} dx dy dz \frac{1}{(1 + x)(y + z) + x^2} + \int_{1}^{\infty} dx dy dz \frac{1}{(1 + x)(y + z) + x^2} - \frac{1}{(1 + x)^2(y + z)^2} = \frac{1}{2}$$

From (28), (26) and (31)

$$\gamma_{\phi}^{(2)} = \frac{-\beta_{3}^{(2)}}{2} = \lim_{\Lambda_0 \to \infty} \frac{-g^2}{4(16 \pi^2)^2} \int_{\frac{\Lambda_0}{\Lambda_0}}^{\infty} \left[\prod_{i=1}^{4} d\alpha_i \right] \rho(\alpha_1) \alpha_2 \frac{d\rho}{d\alpha_2} \rho(\alpha_3) \frac{d}{d\alpha_3} \left[\frac{\alpha^3}{(\alpha_2 + \alpha_3 - \alpha_1 - \alpha_2 \alpha_3)^2}\right]$$

Integrating by parts, using $\rho(0) = 1$ and eq. (25) we get the standard results [16]

$$\gamma_{\phi}^{(2)} = \frac{g^2}{12(16 \pi^2)^2} ; \beta^{(2)} = \frac{-17}{3} \frac{g^3}{(16 \pi^2)^2}$$

It is easy to see that the gradient flow property fails at three loops; in fact from (25) it follows that
\[ \beta^{(3)} = \beta_3^{(3)} - 2g\beta_2^{(3)} - \frac{1}{2}\alpha_3^{(1)}\beta_2^{(2)} \]  

(35)

where

\[ \alpha_3^{(1)} = -6g^2 \int_p \frac{K_{\Lambda R}(p)(1 - K_{\Lambda R}(p))^2}{(p^2)^2} \]  

(36)

is non-vanishing for a generic cut-off function.

Let us finally discuss the case of the renormalization conditions (16), which apply only to the massive theory; the beta and gamma functions are mass-dependent; at the first two loops \( \beta \) and \( \gamma_\phi \) coincide, in the massless limit, with those given above. The function \( \gamma_m \) in (17) does not admit for a generic cut-off the massless limit; in fact

\[ \gamma_m^{(1)} = \frac{g}{16\pi^2m^2} \int_0^\infty \frac{q^3}{q^2 + m^2} \Lambda \frac{\partial}{\partial \Lambda} |_{\Lambda R} K_\Lambda(m; q) \]  

(37)

has a singular term for \( m \to 0 \).

II. PROOF OF THE RENORMALIZATION GROUP EQUATION IN THE MASSLESS CASE

In this section we begin with a short review on the renormalization and infrared finiteness in the massless case, including the results concerning composite operators, in the framework of the flow equation. Some of the proofs needed are given in Appendices A and B, as the particular case \( m = 0 \) of the massive theory defined in Section III. Using the same technique we will prove the renormalization group equation in the massless case. The flow equation technique requires some restriction on the form of the hard propagator; in [8,17] a smooth cut-off function with compact support is used; another possible choice is the exponential cut-off [18] (corresponding to the choice \( \rho(x) = \theta(1 - x) \) in eq.(32)).

From now on we will use the cut-off function \( K_\Lambda(m, p) = K\left(\frac{p^2 + m^2}{\Lambda^2}\right) \) where \( K(x) \) is a smooth function, satisfying \( K(x) = 1 \) for \( x < 1 \) and \( K(x) = 0 \) for \( x > 4 \). The propagator with ultraviolet cut-off \( \Lambda_0 \) and running cut-off \( \Lambda \) will be chosen of the form

\[ D_{\Lambda\Lambda_0}(m, p) = D(m, p)(K_{\Lambda_0}(m; p) - K_\Lambda(m; p)) \]  

(38)

where \( D(m; p) = 1/(p^2 + m^2) \) is the propagator.

Introduce \( L_{\Lambda\Lambda_0} \) satisfying

\[ e^{-\frac{1}{\hbar}L_{\Lambda\Lambda_0}[\Phi]} = e^{\hbar \Delta_{\Lambda\Lambda_0}} e^{-\frac{1}{\hbar}L_{\Lambda_0}[\Phi]} \]  

(39)

with

\[ L_{\Lambda_0}[\Phi] = S^I(\Phi) \]  

(40)

where \( S^I \) is the interacting part of the bare action given in eqs. (2,20) with \( m = 0 \), and

\[ \Delta_{\Lambda\Lambda_0} = \frac{1}{2} \frac{\delta}{\delta \Phi} D_{\Lambda\Lambda_0} \frac{\delta}{\delta \Phi} \]  

(41)
The Wilsonian effective action (5) is given by [17]

$$Z_{\Lambda\Lambda_0}[J] = e^{iW_{\Lambda\Lambda_0}[J]} = e^{i\omega_{\Lambda\Lambda_0}[D_{\Lambda\Lambda_0},J]}$$  \hspace{1cm} (42)$$

where

$$\omega_{\Lambda\Lambda_0}[\Phi] \equiv \frac{1}{2} \Phi D_{\Lambda\Lambda_0}^{-1} \Phi - L_{\Lambda\Lambda_0}[\Phi]$$  \hspace{1cm} (43)$$

$L_{\Lambda\Lambda_0}[\Phi]$ is the functional generator of the truncated connected Green functions (apart from the tree-level two-point function) of the Wilsonian theory with propagator $D_{\Lambda\Lambda_0}$ and bare vertices $S_I$. Formulae (39) and (42) are understood as relations between formal power series in $\bar{\hbar}$ in which only the terms depending on the fields are well-behaved. The same considerations apply to all following functionals; therefore all the formulae hold up to terms constant in the fields.

$L_{\Lambda\Lambda_0}$ satisfies the flow equation

$$\delta_{\Lambda} L_{\Lambda\Lambda_0} = \bar{\hbar} (\delta_{\Lambda} \Delta_{\Lambda\Lambda_0}) L_{\Lambda\Lambda_0} - \frac{1}{2} L'_{\Lambda\Lambda_0} \left( \partial_{\Lambda} D_{\Lambda\Lambda_0} \right) L'_{\Lambda\Lambda_0}$$  \hspace{1cm} (44)$$

where $L'_{\Lambda\Lambda_0} \equiv \frac{\delta L_{\Lambda\Lambda_0}}{\delta \Phi}$. The flow equation can be solved in perturbation theory assigning the relevant couplings at $\Lambda = \Lambda_R$. Define the relevant couplings at scale $\Lambda > 0$:

$$\rho^{\Lambda_0}_{1} = L_{2}^{\Lambda_0}(0) ; \quad \rho^{\Lambda_0}_{2} = \partial_{\rho^{2}} L_{2}^{\Lambda_0} |_{p=0} ; \quad \rho^{\Lambda_0}_{3} = L_{4}^{\Lambda_0} |_{p=0}$$  \hspace{1cm} (45)$$

All the other couplings are irrelevant and are fixed to be zero at $\Lambda = \Lambda_0$ by eq.(40). Using the flow equation one can compute the connected Green functions at scale $\Lambda$ using these boundary conditions; indeed conditions (33) and (12) are in a bimivocal relation in perturbation theory. This observation is the starting point for proving renormalizability with the method of the flow equation [8]. While the relevant couplings $\rho^{\Lambda_0}_{2}$ and $\rho^{\Lambda_0}_{3}$ are defined only for $\Lambda > 0$, $\rho^{\Lambda_0}_{1}$ can be extended to $\Lambda = 0$, and it must vanish there to avoid infrared divergences in the effective Green functions $L_{n}^{\Lambda_0}(p_1, ..., p_{n-1})$ for non-exceptional momenta in the limit $\Lambda \to 0$. This limit will be studied in detail in the next section and in Appendix B, as the particular case in which the mass $m$ is taken to be zero.

Choose the renormalization conditions

$$\rho^{\Lambda_0}_{1} = 0 \ ; \quad \rho^{\Lambda_0}_{2} = 0 \ ; \quad \rho^{\Lambda_0}_{3} = g \delta_{l,0}$$  \hspace{1cm} (46)$$

which are the same as those chosen in the first section in eqs.(21-23) (this is true only for compact-support cut-off; for an analytic cut-off there is an extra term in the renormalization condition on $p_2$ in (20), to get the renormalization conditions (22) on the two-point vertex).

In order to prove renormalizability with the flow equation technique, an appropriate norm must be defined.

Given a function $f(m; p_1, ..., p_{n-1})$, and $p_n \equiv -\sum_{i}^{n-1} p_i$, we define the norm

$$\|f\|_{\Lambda} = \text{Sup}_{\Lambda M}|f|$$  \hspace{1cm} (47)$$

where $M = \{p, m : |p_i| \leq \text{Max}(\eta, 2\Lambda), i = 1, ..., n, \ m \leq \eta\}$ and $\eta$ is a fixed positive quantity. In this section the mass $m$ is zero.
In Appendix A we will prove that
\[ ||\partial_p^w \partial_a^z L^{A\Lambda_0}_n||_{A} \leq \Lambda^{4-n-w-z} P(\frac{\Lambda}{\Lambda_R}) \]  
(48)
where \( P \) is a polynomial; using this bound it is easy to prove renormalizability.

Similarly one can study the functionals representing the insertion of an operator. In our framework the flowing functional generator of the connected and amputated Green functions with one insertion of an operator \( O \) is [19]
\[ O_{\Lambda\Lambda_0}[\Phi] = e^{\frac{1}{2} L^{A\Lambda_0}_n[\Phi]} e^{h\Delta A_{\Lambda_0}(\frac{1}{2} \Phi)} O^{\Lambda_0}[\Phi] e^{-\frac{1}{2} L^{A_{\Lambda_0}}[\Phi]} \]  
(49)
which satisfies
\[ \partial_\Lambda O_{\Lambda\Lambda_0} = h (\partial_\Lambda \Delta A_{\Lambda_0}) O_{\Lambda\Lambda_0} - L^{A}_{\Lambda\Lambda_0} (\partial_\Lambda D A_{\Lambda_0}) O'_{\Lambda\Lambda_0} \]  
(50)
\( O^{A\Lambda_0}[\Phi] \) interpolates between the bare operator \( O^{A_0}[\Phi] \), boundary condition of eq.(50) and the physical functional generator of the insertion \( O^{0\Lambda_0}[\Phi] \). An operator of dimension \( D \) is associated to boundary conditions on \( O^{A_0}_n(p_1, ..., p_{n-1}) \) such that
\[ ||\partial_p^w O^{A_0}_n(l)[l]||_{A_0} \leq \Lambda_0^{D-n-z} P(\frac{\Lambda_0}{\Lambda_R}) \]  
(51)
for \( D - n - z < 0 \) (see [19]), \( P \) being a polynomial, and satisfies renormalization conditions at \( \Lambda_R \) on its relevant part, defined at vanishing momentum and at scale \( \Lambda = \Lambda_R \), up to dimension \( D \).

In this paper we are interested only in the case of integrated scalar composite operators of dimension four. The simplest choice for the boundary condition at \( \Lambda = \Lambda_0 \) would be \( O^{A_0}_n = 0 \) for \( D - n - z < 0 \), but for our purposes it is useful to keep the more general condition \( [51] \). The renormalization conditions for a \( D = 4 \) operator have the form
\[ O_2^{A\Lambda_0}(0) = a_1^{(l)} ; \quad \partial_p^z |_{p=0} O_2^{AR\Lambda_0}(p) = a_2^{(l)} ; \quad O_4^{AR\Lambda_0}(0) = a_3^{(l)} \]  
(52)
In order to guarantee the existence of the limit \( \Lambda \to 0 \), the coefficients \( a_1^{(l)} \) must be tuned in such a way that
\[ O_2^{0\Lambda_0}(0) = 0 \]  
(53)

Let us now discuss the renormalization group equation. Making the theory flow from \( \Lambda_R \) to \( \Lambda'_R \) the new relevant couplings are functions of \( \Lambda_R, \Lambda_0 \) and \( g \) which are determined by the flow equation
\[ \rho_2^{A'_{R\Lambda_0}} = \partial_p^z |_{p=0} L_2^{A'_{R\Lambda_0}} (\Lambda_R, 0; g; p) ; \quad \rho_3^{A'_{R\Lambda_0}} = L_4^{A'_{R\Lambda_0}} (\Lambda_R, 0; g; 0, 0, 0) \]  
(54)
while the renormalization condition on \( \rho_1 \) remains the same as in \( [10] \). Interpreting \( \Lambda'_R \) as the new renormalization point and \( \rho_2^{A'_{R\Lambda_0}} \) and \( \rho_3^{A'_{R\Lambda_0}} \) as the new relevant couplings, the theory is unchanged, since these two conditions are on the same renormalization group trajectory:
\[ L^{A\Lambda_0}[\Lambda'_R, \rho_2^{A'_{R\Lambda_0}}, \rho_3^{A'_{R\Lambda_0}}; \Phi] = L^{A\Lambda_0}[\Lambda_R, 0, g; \Phi] \]  
(55)
This is an exact renormalization group equation, holding for any $\Lambda$, $\Lambda'_R$ and $\Lambda_0$. Differentiating this equation with respect to $\Lambda'_R$ and setting $\Lambda'_R = \Lambda_R$ we get the following exact renormalization group equation [9]:

$$\Lambda_R \frac{\partial}{\partial \Lambda_R} L^{\Lambda_0}[\Lambda_R, 0, g; \Phi] + \beta_2 \frac{\partial}{\partial \rho_2}|_{\rho_2=0} L^{\Lambda_0}[\Lambda_R, \rho_2, g; \Phi] + \beta_3 \frac{\partial}{\partial g} L^{\Lambda_0}[\Lambda_R, 0, g; \Phi] = 0 \quad (56)$$

which holds for any $\Lambda$ and $\Lambda_0$, with

$$\beta_2^{(l)}(g, \Lambda_R, \Lambda_0) = \Lambda \frac{\partial}{\partial \Lambda}|_{\Lambda=\Lambda_R} \rho_2^{\Lambda_0(l)} ; \quad \beta_3^{(l)}(g, \Lambda_R, \Lambda_0) = \Lambda \frac{\partial}{\partial \Lambda}|_{\Lambda=\Lambda_R} \rho_3^{\Lambda_0(l)} \quad (57)$$

This discussion on the exact renormalization group equation is rather formal; the equation (56) must be justified in perturbation theory; this is done in Appendix C.

Let us write equation (56) in the form

$$(\Lambda_R \frac{\partial}{\partial \Lambda_R} + \beta_3 \frac{\partial}{\partial g}) L^{\Lambda_0}[\Lambda_R, 0, g; \Phi] + \beta_2 \mathcal{O}_k^{\Lambda_0}[\Phi] = 0 \quad (58)$$

where

$$\mathcal{O}_k^{\Lambda_0}[\Phi] \equiv \frac{\partial}{\partial \rho_2}|_{\rho_2=0} L^{\Lambda_0}[\Lambda_R, \rho_2, g; \Phi] \quad (59)$$

Since $\Lambda_R$ is the mass scale parameter introduced in the renormalization procedure, eq.(58) should correspond to a generalization of the Gell-Mann and Low renormalization group equation. To prove this correspondence starting from (58) we have to obtain an equation in which the last term $\mathcal{O}_k^{\Lambda_0}$ is replaced by the ‘number operator’ $\Phi \frac{\partial}{\partial \Phi} \omega^{\Lambda_0}[\Phi]$; besides that, we expect that the usual renormalization group equation is recovered only in the limits $\Lambda_0 \to \infty$ and $\Lambda \to 0$. To this aim we have to prove some results concerning the Quantum Action Principle [5] derived in the context of the flow equation technique in [11]. We want to prove that the functional obtained by differentiation of the functional $\omega^{\Lambda_0}[\Phi]$ with respect to its parameters can be obtained as insertions of suitable local operators of dimension four.

Equation (58) can now be understood as a relation concerning three generators of insertions; indeed the functionals

$$\mathcal{O}^{\Lambda_0}_{\Lambda_R} [\Phi] \equiv \Lambda_R \frac{\partial}{\partial \Lambda_R} L^{\Lambda_0}[\Lambda_R, 0, g; \Phi] ; \quad \mathcal{O}^{\Lambda_0}_{g} = \frac{\partial}{\partial g} L^{\Lambda_0}[\Lambda_R, 0, g; \Phi] \quad (60)$$

and $\mathcal{O}_k^{\Lambda_0}$ defined in (59) satisfy (50) and (51) for $D = 4$, indeed only the first two functionals are in a direct relation with the generator $\omega^{\Lambda_0}[\Lambda_R; 0, g; \Phi]$, being $\mathcal{O}^{\Lambda_0}_{\Lambda_R} = -\Lambda_R \partial_{\Lambda_R} \omega^{\Lambda_0}$ and $\mathcal{O}^{\Lambda_0}_g = -\partial_g \omega^{\Lambda_0}$, and in this sense they explicitly agree with the Quantum Action Principle. In order to express $\mathcal{O}_k^{\Lambda_0}$ in a similar way we start by considering the ‘number operator’ $\Phi \frac{\partial}{\partial \Phi}$ acting on the generator of the connected and amputated Green functions

$$\mathcal{O}_N^{0\Lambda_0}[\Phi] = e^{-\frac{\delta}{\delta \omega^{0\Lambda_0}[\Phi]} h \Phi K_{\Lambda_0} \frac{\delta}{\delta \Phi} e^{\frac{\delta}{\delta \omega^{0\Lambda_0}[\Phi]}} \quad (61)$$

where, for technical reasons, we introduced the factor $K_{\Lambda_0}$ in the number operator ($K_{\Lambda_0} \to 1$ for $\Lambda_0 \to \infty$). The functional $\mathcal{O}_N^{0\Lambda_0}$ can be computed as the $\Lambda \to 0$ limit of the following functional
\[ O^\Lambda_{N\Lambda_0} = e^{\frac{i}{\hbar}\lambda^\Lambda_{\Lambda_0}[\Phi]} e^{-\hbar\Delta_{\Lambda_0}}[\Phi D^{-1}\Phi + \hbar\Phi K^\Lambda_{\Lambda_0} \frac{\delta}{\delta \Phi} - h^2 \frac{\delta}{\delta \Phi} D_{\lambda\Lambda_0} K^\lambda_{\Lambda_0} \frac{\delta}{\delta \Phi}] e^{\frac{i}{\hbar}\lambda^\Lambda_{\Lambda_0}}[\Phi] \]

= \frac{\hbar}{\delta \Phi} \frac{\delta}{\delta \Phi} D_{\lambda\Lambda_0} K^\lambda_{\Lambda_0} \frac{\delta}{\delta \Phi} e^{\frac{i}{\hbar}\lambda^\Lambda_{\Lambda_0}}[\Phi] \]

which solves, because of its definition, the flow equation of an operator insertion (50). Expanding the above expression we get

\[ O^\Lambda_{N\Lambda_0} = \Phi D^{-1}\Phi - L'_{\Lambda\Lambda_0} D_{\lambda\Lambda_0} K^\lambda_{\Lambda_0} L'_{\Lambda\Lambda_0} + \hbar tr D_{\lambda\Lambda_0} K^\lambda_{\Lambda_0} L''_{\Lambda\Lambda_0} - \Phi(K_{\Lambda_0} - 2K^\lambda) L'_{\Lambda\Lambda_0} \]

Using this equation for \( \Lambda = \Lambda_0 \) and the bounds (48) satisfied by \( L_{\lambda\Lambda_0} \) one checks the validity of eq. (51) for \( D = 4 \).

From (53) and (40) one gets

\[ O^\Lambda_{N\Lambda_0}(0) = 0 \; \; \partial_{p^2} O^\Lambda_{N\Lambda_0} \big|_{p=0} = 2 + \alpha_2 \; \; O^\Lambda_{N\Lambda_0} \big|_{p=0} = 4g + \alpha_3 \]

where

\[ \alpha_2 = h \partial_{p^2} \int q D_{\lambda\Lambda_0}(q) K_{\Lambda_R}(q) L'_{\Lambda\Lambda_0}(q, -q, p) \]

\[ \alpha_3 = h \int q D_{\lambda\Lambda_0}(q) K_{\Lambda_R}(q) L''_{\Lambda\Lambda_0}(q, -q, 0, 0, 0) \]

are the same quantities defined in (27) (as above, this correspondence is exact only for a compact-support cut-off function). We can conclude that the functional (52) can be computed as the insertion of a ‘local operator’ of dimension \( D = 4 \) whose renormalization conditions are given in (53).

The same considerations apply in a simpler way to the other two insertions appearing in eq. (58). For \( O^\Lambda_{\Lambda_0} \) the renormalization conditions are

\[ O^\Lambda_{2\Lambda_0}(0) = 0 \; \; \partial_{p^2} O^\Lambda_{2\Lambda_0} \big|_{p=0} = 1 \; \; O^\Lambda_{4\Lambda_0} \big|_{p=0} = 0 \]

so that \( O^\Lambda_{\Lambda_0} \) is the operator insertion which at tree-level is equal to \( \Phi D^{-1}\Phi \); for the other operator,

\[ O^\Lambda_{2\Lambda_0}(0) = 0 \; \; \partial_{p^2} O^\Lambda_{2\Lambda_0} \big|_{p=0} = 0 \; \; O^\Lambda_{4\Lambda_0} \big|_{p=0} = 1 \]

so that \( O^\Lambda_{\Lambda_0} \) is the operator insertion which at tree-level is equal to \( \Phi^4 \).

A generic operator \( O^\Lambda_{\Lambda_0} \) of dimension four, satisfying the renormalization conditions

\[ O^\Lambda_{2\Lambda_0}(0) = 0 \; \; \partial_{p^2} O^\Lambda_{2\Lambda_0} \big|_{p=0} = a \; \; O^\Lambda_{4\Lambda_0} \big|_{p=0} = b \]

can be decomposed in terms of \( O^\Lambda_{\Lambda_0} \) and \( O^\Lambda_{\Lambda_0} \); in fact

\[ E^\Lambda_{\Lambda_0} \equiv O^\Lambda_{\Lambda_0} - aO^\Lambda_{\Lambda_0} - bO^\Lambda_{\Lambda_0} \]

is an operator insertion satisfying eq. (50), the bounds (51) and it has vanishing renormalization conditions

\[ E^\Lambda_{2\Lambda_0}(0) = 0 \; \; \partial_{p^2} E^\Lambda_{2\Lambda_0} \big|_{p=0} = 0 \; \; E^\Lambda_{4\Lambda_0} \big|_{p=0} = 0 \]
so that it is evanescent (see [19] and Appendix A).

Therefore (64) implies

\[ O_N^{Λ_0} = (2 + α_2)O_k^{Λ_0} + (4g + α_3)O_g^{Λ_0} + E^{Λ_0} \tag{71} \]

where \( E^{Λ_0} \) is evanescent.

Eq. (71) is the relation expressing \( O_k^{Λ_0} \) in terms of \( L^{Λ_0} \) (or \( ω^{Λ_0} \)) which we were looking for. Introducing in (58) the expression for \( O_k^{Λ_0} \) in eq. (71) we get

\[ \Lambda_R \frac{∂}{∂Λ_R} + β \frac{∂}{∂g} \omega^{Λ_0} + γ_φ[O_N^{Λ_0} - E^{Λ_0}] = 0 \tag{72} \]

where

\[ β = β_3 + γ_φ(4g + α_3) \quad ; \quad γ_φ = \frac{-β_2}{2 + α_2} \tag{73} \]

with \( β_2, β_3 \) defined in (57) and \( α_2, α_3 \) defined in (55). Using (72) we obtain

\[ \Lambda_R \frac{∂}{∂Λ_R} + β \frac{∂}{∂g} + γ_φ(Φ_0^{Λ_0} \frac{δ}{δΦ} - h_0 \frac{δ}{δΦ} D^{Λ_0} K^{Λ_0} \frac{δ}{δΦ} - E^{Λ_0})e^kΩ^{Λ_0}[Φ] = 0 \tag{74} \]

Notice that in the limit \( Λ_0 \to ∞ \) \( β \) and \( γ_φ \) have the same expressions as in eq. (25) (in the limit of cut-off function with compact support): using (12) in (14) we obtain the effective renormalization group equation (10) (without the mass term), as we claimed. In particular, in the limit \( Λ \to 0 \) we get the renormalization group equation

\[ (Λ_R \frac{∂}{∂Λ_R} + β_0(4g + α_3)) \frac{∂}{∂g} Z[J] = 0 \tag{75} \]

The existence of this limit will be proven in the next section and in Appendix B.

### III. MASS-INDEPENDENT RENORMALIZATION AND THE RENORMALIZATION GROUP

The discussion of the (effective) renormalization group equation of the previous section could be straightforwardly extended to the massive case, however in a generic scheme one would get a dependence on \( \frac{Δω}{m} \) of the \( β \) and \( γ \) functions. This is an unwanted complication from the point of view of the utilization of the renormalization group equation in studying the asymptotic behavior of the Green functions. The mass-independent renormalization scheme introduced in [12] overcomes this problem but it has some difficulties in the theories with scalars due to infrared divergences. In later treatments [20] the problem is in some way solved by defining a generalized functional of the massless theory with a new source \( ∫ k(x)φ^2(x)dx \), which is supposed to be summed exactly to all orders in \( k(x) \), making at the end the replacement \( k(x) \to m^2 \). In this way the infrared singularities which are present for a finite number of \( ∫ φ^2(x)dx \) insertions are avoided. In [14] a mass-independent approach similar to the HS approach originally proposed in [4] is developed, but it applies specifically to massive theories, while the HS schemes apply equally well to massless theories.
In this section we consider a mass-independent HS scheme, outlined in Section I, in which the above-mentioned infrared problems are overcome in a more natural way: instead of defining mass-insertions, it will be sufficient to make derivatives with respect to the mass; the mass parameter will be considered more like a ‘momentum’ than as a coupling constant. The mass-independent renormalization group equation is then obtained in a rather straightforward way. Technical details of the proof are postponed to the appendixes.

The bare coefficients $\tilde{c}_1, \tilde{c}_2, c_3, c_4$ in the bare action (74,75) do not depend on $m^2$. Perturbatively these parameters are in an invertible relation with the following mass-independent renormalization conditions in a formal loop expansion:

$$L^2_{0}(0;0) = \rho_1^{(0)}; \quad \partial_{m^2}|_{m=0} L^2_{0}(m^2;0) = \rho_1^{(1)}$$
$$\partial_{p^2}|_{p=0} L^4_{0}(0;p) = \rho_2^{(1)}; \quad L^4_{0}(0;0,0,0) = \rho_3^{(l)}$$

(76)

with $\rho_1^{(0)} = \rho_1^{(0)} = \rho_2^{(0)} = 0$ (so that $L^2_{0}(0;0) = 0$).

We have to prove that eqs.(70) lead to ultraviolet renormalization. The proof is a modification of the one made in (17), in which one considers derivatives with respect to the mass besides those with respect to the momenta. Indeed the functions $L^\Lambda_{0}(m;p_1,\ldots,p_{n-1})$ for $\Lambda > 0$ and $\Lambda_0 < \infty$ are, due to (23,40,70), $C^\infty$ functions of $n-1$ independent momenta and also of the mass $m$. One can prove by induction in the loop index $l$ and in the number of external legs $n$ the following bounds ( $\Lambda \geq \Lambda_R$ and $z$ is the total order of the derivatives with respect to the momenta):

$$||\partial^w_{m^2} \partial^2_p L^\Lambda_{0}||_A \leq \Lambda^{4-n-w-2r-z} P(\ln A / \Lambda_R)$$

(77)

where $w = 0, 1$ and $P$ is a polynomial with positive coefficients which are independent from $\Lambda, \Lambda_0$ and $m^2$; in general $P$ will denote a different polynomial whenever it appears.

Eq.(77) is essentially the renormalization theorem; using it one can show that

$$||\partial_{\Lambda}^r \partial^m_{m^2} \partial^2_p L^\Lambda_{0}(l)||_A \leq \frac{\Lambda^{5-n-z-2r}}{\Lambda_0^z} P(\ln \Lambda \Lambda_0 / \Lambda_R)$$

(78)

and from this the proof of the existence of $\lim_{\Lambda_0 \to \infty} \partial^r_{m^2} \partial^2_p L^\Lambda_{0}(l)$ follows easily (uniformly on the compact sets of momenta and mass). The details of the proof are given in appendix A. The only difference with the quoted standard proof is that for the relevant interacting Green function we use Taylor reconstruction with respect to the mass too.

The theory at $\Lambda = 0$ is obtained collecting graphs with $L^\Lambda_{0}$ vertices and soft propagator $D_{0}\Lambda_R(m;p)$, which has compact support. From the existence of $\lim_{\Lambda_0 \to \infty} L^\Lambda_{0}(l)$ it follows that $\lim_{\Lambda_0 \to \infty} L^0_{0}(l)$ exists too, and is a $C^\infty$ function of the momenta for $m > 0$. In order that the parameters of the Gell-Mann and Low equation for the massive case be mass-independent, they must be the same as in the massless theory, so that we want to show that $\lim_{m \to 0} L^0_{0}(l)(m;p_1,\ldots,p_{n-1}) = L^0_{0}(0;p_1,\ldots,p_{n-1})$ are the Green functions considered in the previous section. On the other hand in Section II we did not prove that $\lim_{\Lambda \to 0} L^\Lambda_{0}(l)(0;p_1,\ldots,p_{n-1})$ exists for non-exceptional momenta. This fact could be proven directly, however we find it more economical to solve together these two problems by proving the existence of the global limit $(\Lambda, m, \Lambda_0) \to (0,0,\infty)$ of the Green functions for non-exceptional momenta.
In the course of the proof, using the flow equation, it will be necessary to establish bounds on Green functions with some sets of $k$ exceptional momenta, with $0 \leq k \leq n - 1$.

Let $\eta_1$ and $\eta_2$ be fixed positive quantities. We want to consider the sets of $n - 1$ independent momenta $p_i$, $k$ of which are ‘small’ (i.e. $|p_i| < 2\Lambda$), while the remaining $n - 1 - k$ momenta, and all their partial sums, are larger than $\eta_1$. We will not need to use the more general sets of momenta, in which a subset of momenta is large, but partial sums of large momenta are not necessarily large. In this sense our treatment is similar to the infrared treatment of the massless case in [21]. More general sets of exceptional momenta have been considered in [22].

Let us define norms $|| \cdot ||^k_\Lambda$, for $0 \leq k \leq n - 1$. Define for $k$ with $0 \leq k < n - 1$ the following norm for a smooth function $f$ of $n - 1$ independent momenta $\{p_1, \ldots, p_{n-1}\}$

$$||\partial_{p_{z'}} f||^k_\Lambda \equiv \text{MaxSup}_{X(k,\Lambda)} \left| \frac{\partial^{z_1}_{p_1} \ldots \partial^{z_k}_{p_k} \partial^{z_{k+1}}_{p_{k+1}} \ldots \partial^{z_{n-1}}_{p_{n-1}} f}{\partial p_{i_1} \ldots \partial p_{i_k} \partial p_{i_{k+1}} \ldots \partial p_{i_{n-1}}} \right|$$

(79)

where Max is taken over $\sum_1^k z_i = z$ and $\sum_{k+1}^{n-1} z_i = z'$ and over the permutations of the first $n - 1$ integers, whose generic element is called $i_1, \ldots, i_{n-1}$, and the set $X(k,\Lambda)$ is so defined:

$$X(k,\Lambda) \equiv \{p_1 \ldots p_{n-1} : |p_i| < 2\Lambda, r = 1, \ldots, k; \eta_1 < |p_{i_s}| < \eta_2, s \geq k + 1;$$

$$p_{i_1} + p_{i_2} > \eta_1, s_1, s_2 \geq k + 1; \ |p_{i_{s_1}} + p_{i_{s_2}} + p_{i_{s_3}}| > \eta_1, s_1, s_2, s_3 \geq k + 1; \ldots$$

$$p_{i_1} + \ldots + p_{i_{n-1-k}} > \eta_1, s_1, \ldots, s_{n-1-k} \geq k + 1 \}$$

(80)

For $k = n - 1$, define the norm

$$||\partial_{p} f||^{n-1}_\Lambda \equiv \text{MaxSup}|\partial_{p} f|$$

(81)

where Max is the same as above and Sup is taken over $\{p_1 \ldots p_{n-1} : |p_i| < 2\Lambda; |\sum_{i=1}^{n-1} p_i| < 2\Lambda\}$. For $k = 0$ the index $\Lambda$ in $||\partial_{p} f||^0_\Lambda$ is meaningless and it can be omitted. Using these norms, in the appendix we will prove the following theorem: it is possible loop by loop to choose coefficients $\rho_i^{(l)} = \Lambda_R^2 f_i^{(l)}(g; \Lambda_0)$ in such a way that

$$\lim_{\Lambda \to 0} L_2^{\Lambda_0 l}(0; 0) = 0$$

(82)

Furthermore, for arbitrary values of $\Lambda$ and $m$ (with $\Lambda < \Lambda_R$) the following bounds hold: consider $w = 0, 1$ and let $z$ be the overall order of the derivatives with respect to the momenta; then

$$||\partial_{\Lambda}^w \partial_{m_z}^r \partial_{p_{z'}} L_n^{\Lambda_0 l}||^k_\Lambda \leq \begin{cases} (\Lambda + \frac{m}{2})^{-1-n-z-2r-w} & k = n - 1 \\ (\Lambda + \frac{m}{2})^{-2|k-1|z-2r-w} & 0 < k < n - 1 \\ \text{const.} & k = 0, w = r = 0 \\ (\Lambda + \frac{m}{2})^{-2w-2r} & k = 0, w + r > 0 \end{cases}$$

(83)

where $P \equiv P(ln \frac{\Lambda_R + \frac{m}{2}}{\Lambda + \frac{m}{2}})$ is a polynomial whose coefficients do not depend on $\Lambda$ and $m$ and where $[x]$ represents the integer part of $x$. 

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From (88) and (83) ultraviolet convergence and infrared finiteness for non-exceptional momenta immediately follow; indeed

\[ |\partial_p^z L_{m_1}^{\Lambda_0} (m_1; p_i) - \partial_p^z L_{m_2}^{\Lambda_0} (m_2; p_i)| \leq \int_{\Lambda_0}^{\Lambda_1} d\Lambda' |\partial_{\Lambda'} \partial_p^z L_{m_2}^{\Lambda_0} (m_2; p_i)| + \int_{m_1^2}^{m_2^2} \frac{1}{d^2 m_2} |\partial_{p^2} \partial_p^z L_{m_1}^{\Lambda_0} (m_1; p_i)| \to 0 \] (84)

for \((\Lambda_{1,2}, m_{1,2}, \Lambda_0, \Lambda_0') \to (0, 0, \infty, \infty)\).

One can discuss the renormalization of the generating functional of a composite operator which satisfies (51) in an analogous way. In this paper we are actually interested only in integrated composite operators. In a mass-independent scheme one starts with a bare operator \(O^{\Lambda_0}\) which, like \(L^{\Lambda_0}\), has a local term of dimension \(D\) which depends polynomially on \(m^2\). In general there is also a remainder of irrelevant terms, namely terms of \(O^{\Lambda_0}\) which are generally non-local and are \(C^\infty\) functions of \(m^2\) and of the momenta, but which in general do not depend in a polynomial way on \(m^2\) and such that the following inequality is satisfied

\[ ||\partial_{m^2}^r \partial_p^z O_n^{\Lambda_0}||_{\Lambda_0} \leq \Lambda_0^{D-n-2r-z} P(\frac{\Lambda_0}{\Lambda_R}) \] (85)

for \(D-n-2r-z<0\). \(O_n^{\Lambda_0}(m_1, p_1, ..., p_{n-1})\) will be \(C^\infty\) functions of the momenta and of \(m^2\) for \(\Lambda > 0\). The relevant part of \(O^{\Lambda_0}\) is in a biunivocal relation with the following renormalization conditions, which we write down only in the case \(D=4\), the generalization to operators of arbitrary dimensions being obvious:

\[ O_2^{\Lambda R_{\Lambda_0}(l)}(0; 0) = a_1^{(l)} \quad \partial_{m^2}|_{m=0} O_2^{\Lambda R_{\Lambda_0}(l)}(m; 0) = a_1^{(l)} \]

\[ \partial_{p^2}|_{p=0} O_2^{\Lambda R_{\Lambda_0}(l)}(0; p) = a_2^{(l)} \quad O_4^{\Lambda R_{\Lambda_0}(l)}(0; 0, 0, 0) = a_3^{(l)} \] (86)

From (86) the ultraviolet part of the proof follows along the lines discussed for \(L^{\Lambda_0}\). In particular for an operator of dimension \(D\), for \(\Lambda \geq \Lambda_R\),

\[ ||\partial_{m^2}^r \partial_p^z O_n^{\Lambda_0(l)}||_{\Lambda} \leq \Lambda^{D-n-2r-z} P(\frac{\Lambda}{\Lambda_R}) \] (87)

If \(D>2\), by choosing \(a_1^{(l)}\) in (86) loop by loop in such a way that

\[ \lim_{\Lambda \to 0} O_2^{\Lambda_0(l)}(0; 0) = 0 \] (88)

one can prove suitable infrared bounds which in the case \(D=4\) coincides with (83), and then the existence of \(O_n^{\Lambda_0(l)}(m; p_1, ..., p_{n-1})\) with non-exceptional momenta in the limit \((m, \Lambda) \to (0, 0)\) follows.

To discuss the renormalization group equation we have now to generalize the procedure followed in (57). Consider standard renormalization conditions in which the parameters in (76) assume the values \(\rho_1 = \rho_2 = 0\) and \(\rho_3^{(l)} = g \delta^{l,0}\), while \(\rho_1\) is fixed by (72); then

\[ L^{\Lambda_0}[\Lambda_R; m; \rho_1^{\Lambda_0}, \rho_2^{\Lambda_0}, \rho_3^{\Lambda_0}; \Phi] = L^{\Lambda_0}[\Lambda_R; m; 0, 0, g; \Phi] \] (89)

Applying \(\frac{\Lambda_R^{\beta \Lambda_0}}{\partial \Lambda_0}\) to \(\Lambda_R = \Lambda_0\) to the left hand side of this equation we get
where \( \beta_2 \) and \( \beta_3 \) are the same as in the massless case (\[27\]) and

\[
\beta_1(g, \Lambda_R, \Lambda_0) = \lambda \frac{\partial}{\partial \Lambda} \bigg|_{\Lambda = \Lambda_R} \frac{\partial}{\partial m^2} \bigg|_0 L^{\Lambda_0}[\Lambda_R; m, \hat{\rho}_1, 0; g; \Phi]
\]

The exact renormalization group equation (89) can be written in the form

\[
(\lambda \frac{\partial}{\partial \Lambda} + \beta_2 \frac{\partial}{\partial g}) L^{\Lambda_0}[\Lambda_R; m, 0, 0; g; \Phi] + \beta_1 \mathcal{O}_{m^2}^{\Lambda_0} + \beta_2 \mathcal{O}_k^{\Lambda_0} = 0
\]

where \( \mathcal{O}_k \) is defined as in (59) but with mass, while

\[
\mathcal{O}_{m^2}^{\Lambda_0} \equiv \frac{\partial}{\partial \hat{\rho}_1} \bigg|_0 L^{\Lambda_0}[\Lambda_R; m, \hat{\rho}_1, 0; g; \Phi]
\]

satisfies, as well as \( \mathcal{O}_{A_2}, \mathcal{O}_g \) and \( \mathcal{O}_k \), the flow equation for operator insertions (50). These operators, which are defined in term of \( L^{\Lambda_0} \), satisfy eq.(83) and the following mass-independent renormalization conditions:

\[
\mathcal{O}_{k_2}^{\Lambda_0} (0; 0) = 0 ; \quad \partial_{m^2} \mathcal{O}_{k_2}^{\Lambda_0} (0; 0) = 0
\]

\[
\mathcal{O}_{k_2}^{\Lambda_0} (0; 0) = 1 ; \quad \partial_{m^2} \mathcal{O}_{k_2}^{\Lambda_0} (0; 0) = 0
\]

\[
\mathcal{O}_{g_2}^{\Lambda_0} (0; 0) = 0 ; \quad \partial_{m^2} \mathcal{O}_{g_2}^{\Lambda_0} (0; 0) = 0
\]

\[
\mathcal{O}_{g_2}^{\Lambda_0} (0; 0) = 1 ; \quad \partial_{m^2} \mathcal{O}_{g_2}^{\Lambda_0} (0; 0) = 0
\]

\[
\mathcal{O}_{m^2}^{\Lambda_0} (0; 0) = 0 ; \quad \partial_{m^2} \mathcal{O}_{m^2}^{\Lambda_0} (0; 0) = 0
\]

Observe that \( \mathcal{O}_{m^2}^{\Lambda_0} \) is a functional insertion of the form (19) which at \( \Lambda_0 \) is the bare term \( \mathcal{O}_{m_0^2}^{\Lambda_0} = m^2 B(\frac{\Lambda_0}{\lambda^2}) \Phi^2 \), with \( B(0) = 1 \). Using the Quantum Action Principle we can express \( \mathcal{O}_k \) and \( \mathcal{O}_{m^2} \) in terms of \( \Phi \frac{\delta}{\delta \Phi} \) and \( m^2 \frac{\partial}{\partial m^2} \) acting on a functional. In the case of the number operator eqs. (92, 93, 94) remain true, provided \( D \) is the massive propagator. From (93) we get

\[
\mathcal{O}_{N_2}^{\Lambda_0} (0; 0) = 0 ; \quad \partial_{m^2} |_0 \mathcal{O}_{N_2}^{\Lambda_0} (0; p) = 2 + \alpha_2
\]

\[
\partial_{m^2} |_0 \mathcal{O}_{N_2}^{\Lambda_0} (m; 0) = 2 + \alpha_1 ; \quad \mathcal{O}_{N_4}^{\Lambda_0} (0; 0, 0, 0) = 4g + \alpha_3
\]

with

\[
\alpha_1 = \hbar \partial_{m^2} |_0 \int_q K_{\Lambda_R} (m; q) D_{\Lambda_R \Lambda_0} (m; q) L_4^{\Lambda_0} (m; q, -q, 0)
\]

while \( \alpha_2 \) and \( \alpha_3 \) are the same as in eq.(\[66\]). Eq.(\[63\]) shows that \( \mathcal{O}_N \) satisfies eq. (83) and then it can be considered as a mass-independent operator of dimension four.

Using eqs.(\[64\], \[67\]) we can write
\[ \mathcal{O}_N^{\Lambda_0}[\Phi] = (2 + \alpha_1)\mathcal{O}_{m^2}^{\Lambda_0}[\Phi] + (2 + \alpha_2)\mathcal{O}_k^{\Lambda_0}[\Phi] + (4g + \alpha_3)\mathcal{O}_g^{\Lambda_0}[\Phi] + \mathcal{E}^{\Lambda_0}[\Phi] \]  

(99)

where \( \mathcal{E}^{\Lambda_0} \) is an evanescent flowing functional. Indeed \( \mathcal{E} \) defined in (93) is a functional which satisfies the flow equation (50), it has vanishing mass-independent renormalization conditions and it fulfils the condition (85) for \( D = 4 \); \( \mathcal{E} \) is not identically zero due to the remainder of irrelevant terms in \( \mathcal{O}_N \).

To complete our proof of the Quantum Action Principle we now consider

\[ \mathcal{O}^{0\infty}_{M^2}[DJ] = \exp\left\{ -\frac{1}{\hbar}\omega^{0\infty}[DJ; m^2] \right\} \hbar m^2 \partial m^2 \exp\left\{ \frac{1}{\hbar}\omega^{0\infty}[DJ; m^2] \right\} \]  

(100)

where the function \( \omega^{0\Lambda_0} \) was introduced in the massless case in eq. (93). For finite \( \Lambda_0 \) we define (with some arbitrariness) a functional which in the limit \( \Lambda_0 \to \infty \) gives the functional defined in eq. (100); thus we define

\[ \mathcal{O}^{0\Lambda_0}_{M^2}[\Phi] = e^{\frac{1}{\hbar}L^{\Lambda_0}[\Phi, m^2]} \left[ -\frac{m^2}{2} \Phi^2 + \hbar m^2 \partial_m - h\Phi m^2 \frac{\delta}{\delta \Phi} \right] e^{-\frac{1}{\hbar}L^{\Lambda_0}[\Phi, m^2]} \]  

(101)

Note that in these two last formulas, differently from the previous one, \( \partial_m \) does not act on the external legs \( \Phi = D^{\Lambda_0}J \). The corresponding flowing functional is

\[ \mathcal{O}^{\Lambda_0}_{M^2}[\Phi] = e^{\frac{1}{\hbar}L^{\Lambda_0}[\Phi, m^2]} e^{-\hbar\Delta^{\Lambda_0}[\Phi, m^2]} \left[ -\frac{m^2}{2} \Phi^2 + \hbar m^2 \partial_m - h\Phi m^2 \frac{\delta}{\delta \Phi} \right] e^{-\frac{1}{\hbar}L^{\Lambda_0}[\Phi, m^2]} \]

\[ = -\frac{m^2}{2} \Phi^2 - m^2 \partial_m \mathcal{L}_{\Lambda_0} + m^2 \Phi (1 - K_\Lambda) D\mathcal{L}_{\Lambda_0} + \frac{m^2}{2} \frac{\partial}{\partial m} \mathcal{L}_{\Lambda_0} D[1 - K_\Lambda] K_\Lambda + (\partial_m K_\Lambda)] \mathcal{L}_{\Lambda_0} + \frac{m^2}{2} \hbar m \partial_m \mathcal{L}_{\Lambda_0} \mathcal{L}_{\Lambda_0} K_\Lambda + (\partial_m K_\Lambda)] \mathcal{L}_{\Lambda_0} \]  

(102)

\( \mathcal{O}^{\Lambda_0}_{M^2} \) satisfies the condition (85) for \( D = 4 \). From eq. (102) we see that \( \mathcal{O}^{\Lambda_0}_{M^2} \) satisfies the renormalization condition (85) and

\[ \partial_m^2 \mathcal{O}^{\Lambda_0}_{M^2}(m; 0) = -1 - \alpha_m \]

\[ \alpha_m \equiv \frac{\hbar}{2} \int_q D(q) \left[ D(1 - K_\Lambda R) K_\Lambda + \frac{K_\Lambda}{\Lambda^2 R} \right] (q) \mathcal{L}_{\Lambda_0}(q) \mathcal{L}_{\Lambda_0}(0; q, -q, 0) \]  

(103)

the parameters in the other renormalization conditions for an operator of dimension four being zero. In the above equation \( K'_\Lambda(p) \) is the derivative of the function \( K \) with respect to its argument. We can then write

\[ \mathcal{O}^{\Lambda_0}_{M^2}[\Phi, m^2] = -(1 + \alpha_m) \mathcal{O}_{m^2}^{\Lambda_0} + \mathcal{E}^{\Lambda_0} \]  

(104)

where \( \mathcal{E}^{\Lambda_0} \) in a new evanescent functional. Now from (85), (99) and (104) we obtain

\[ (\Lambda_R \frac{\partial}{\partial \Lambda_R} + \beta \frac{\partial}{\partial g}) L^{\Lambda_0} - \gamma_\phi \mathcal{O}_N^{\Lambda_0} + \gamma_m \mathcal{O}_{M^2}^{\Lambda_0} = \mathcal{E}^{\Lambda_0} \]  

(105)

where

\[ \beta = \beta_3 + \gamma_\phi(4g + \alpha_3) \quad ; \quad \gamma_\phi = -\frac{\beta_2}{2 + \alpha_2} \quad ; \quad \gamma_m = -\frac{\beta_1 + \gamma_\phi(2 + \alpha_1)}{1 + \alpha_m} \]  

(106)
\( \beta, \gamma_\phi \) and \( \gamma_m \) in (106) are independent from \( m^2 \) so that, for \( \Lambda_0 \to \infty \), they become functions of \( g \) only and coincide with those in (25). The operator insertion \( \mathcal{E}^{\Lambda_0} \) is evanescent and, for \( \Lambda_0 \to \infty \) and \( \Lambda \to 0 \), one gets from (63, 100) and (104) the mass-independent renormalization group equation (5).

In the appendix we will show that the last term in the left hand side of (104) is vanishing in the limit \( (\Lambda, m) \to (0, 0) \) and we get eq.(72) from eq.(105).

Let us discuss the relation between the mass-independent renormalization group equation and the Callan-Symanzik equation [13]. Note that eq.(104) gives a simple relation between the differentiation with respect to the mass and the insertion of the composite operator \( m^2 \Phi^2 \) acting on the Green functions. In eq.(96) this operator was renormalized, in our mass-independent scheme, as an operator of dimension four, but it is trivial to check that

\[
\mathcal{O}^{\Lambda_0}_{m^2} \left[ \Phi \right] = m^2 \mathcal{O}^{\Lambda_0}_{\phi^2} \left[ \Phi \right] \tag{107}
\]

where \( \mathcal{O}^{\Lambda_0}_{\phi^2} \left[ \Phi \right] \) satisfies eq.(84) for \( D = 2 \) and the renormalization condition

\[
\mathcal{O}^{\Lambda_0}_{\phi^2} \left[ \Phi \right] (\Lambda_0, m) = \delta(0, 0)
\]

Observe that \( \mathcal{O}^{\Lambda_0}_{\phi^2} \left[ \Phi \right] \) is infrared singular, in the sense that it does not exist in the limit \( (\Lambda, m) \to (0, 0) \).

In the language of Zimmermann normal product [23] (at scale \( \Lambda_R \) and in our scheme) eq.(107) could be written as

\[
N_4^{\Lambda_R} \left[ m^2 \Phi^2 \right] = m^2 N_2^{\Lambda_R} \left[ \Phi^2 \right] \tag{108}
\]

The Callan-Symanzik equation can be obtained by taking a linear combination of \( O_g, O_N, O_{\Lambda_R} \) and \( O_{M^2} \) in such a way that their sum is equal to an operator of dimension two and the scaling operator \( \Lambda_R \partial_{\Lambda_R} + m \partial_m \) is obtained. As a consequence of eq.(107) it has the form

\[
(\Lambda_R \frac{\partial}{\partial \Lambda_R} + m \frac{\partial}{\partial m} + \beta \frac{\partial}{\partial g} + \gamma_\phi J \frac{\delta}{\delta J}) W[J] = m^2 \sigma \mathcal{O}^{0\infty}_{\phi^2}[D,J] \tag{109}
\]

where the beta and gamma functions are the same as in eq.(106) and

\[
\sigma = -(2 + \gamma_m)(1 + \alpha_m) \tag{110}
\]

We conclude making an observation on the mass-independent renormalization conditions (76) in which \( \rho_1 \) is not tuned according to (82) but it is simply set to zero and \( \tilde{\rho}_1 = \rho_2 = 0 \) and \( \rho_3 = g \) as before. The proof of ultraviolet renormalization is unaffected, so that \( L_n^{0\infty}(m; p) \) exists for \( m > 0 \); however the limit for \( m \to 0 \) is singular. The bare lagrangian (2) has still a polynomial dependence on \( m \), the renormalization group equation (8) holds, with the same formulae (27) for \( \beta(g) \) and \( \gamma_\phi(g) \), while \( \gamma_m(g, m) \) has a simple dependence on the mass:

\[
\gamma_m = \gamma_m,1(g) + \frac{1}{m^2} \gamma_m,2(g).
\]

The singularity of its limit for \( m \to 0 \) reflects the corresponding singularity of the Green functions \( L_n^{0\infty}(m; p) \) in this limit.
IV. CONCLUSION

In the Wilson-Polchinski approach to renormalization it is natural to set renormalization conditions at zero momentum at a Wilsonian scale $\Lambda_R > 0$, even in massless theories. This renormalization scheme is similar to the hard-soft (HS) renormalization scheme proposed in [4] to renormalize in a simple way massless theories with BPHZ, though differing from it in the choice of the cut-off function separating soft and hard modes.

In this paper we have shown that the renormalization group equation in a Wilsonian HS scheme in massless and in massive $g\phi^4$ can be deduced from the exact renormalization flow and the use of the Quantum Action Principle; at scale $\Lambda$ one obtains an effective renormalization group equation, which reduces to the usual renormalization group equation at $\Lambda = 0$. We give exact formulae for the beta and gamma functions in terms of the Wilsonian vertices at scale $\Lambda_R$ and we verify that their lowest order coefficients have the expected values [10]. In the massive case we have introduced a mass-independent hard-soft renormalization scheme, and we have proven its consistency. The beta function and $\gamma_{\phi}$ are the same as in the massless case. For non-exceptional momenta the massless limit of the massive Green functions exists and it coincides to the massless Green functions.

The relevant terms of the massive theory are given by the Maclaurin expansion in momenta and in the mass of the Wilsonian effective action at scale $\Lambda_R$, up to dimension four. In this scheme the renormalization of the massless theory is the particular case $m = 0$ of the massive theory and its relevant terms are simply given by the $m = 0$ part of the above mentioned Maclaurin expansion.

It should be possible to generalize our results to other field theories, in particular to gauge theories, using HS renormalization conditions compatible with the effective Ward identities [11]. It is obvious that in these theories one can compute the beta and gamma functions in terms of the Wilsonian vertices at scale $\Lambda_R$. It is an interesting question whether the mass-independent Wilsonian HS scheme introduced here can be generalized to any renormalizable theory involving mass parameters.

Appendix A: Renormalization.

In this Appendix we prove the ultraviolet finiteness of the theory in the mass-independent scheme; we will consider generic renormalization conditions (76), that is we will not need to choose a particular value for $\rho_1$ to prove the convergence of the effective Green functions at scale $\Lambda \geq \Lambda_R$. This will be done in two steps. We start by proving that eq.(76) implies the bounds (77); to this end we write down explicitely the flow equation for the $n$-point Green functions at $l$ loops:

$$\partial_{\Lambda}L_n^{\Lambda_0(l)}(m; p_1, \ldots, p_{n-1}) = \frac{1}{2} \int_q \partial_{\Lambda} D_{\Lambda_0}(m; p) L_{n+2}^{\Lambda_0(l-1)}(m; q, -q, p_1, \ldots, p_{n-1}) +$$

$$- \frac{1}{2} \sum_{n_1 + n_2 = n+2} L_{n_1}^{\Lambda_0(l_1)}(m; p_{i_1}, \ldots, p_{i_{n_1}-1}) \partial_{\Lambda} D_{\Lambda_0}(m; P) L_{n_2}^{\Lambda_0(l_2)}(m; P, p_{i_{n_1}}, \ldots, p_{i_{n_2}-1})$$

(111)
where \( P = - \sum_{i=1}^{n-1} p_i \) and \( l_1 + l_2 = l \); in the sum there is a symmetrization with respect to \( p_1, \ldots, p_{n-1} \) and \(- \sum_{i=1}^{n-1} p_i\). In the proof a key role is played by the following bound on the propagator

\[
\| \partial_{m^2} \partial_p^z \partial_A D_{\Lambda A_0} \| \Lambda < C \Lambda^{-3-z-2r}
\]

where the norm was introduced in \( \| \cdot \| \). Here and in the following \( C \) will denote constants whose value is not important. Applying \( \partial_{m^2} \partial_p^z \) to both sides of \( \| \cdot \| \), using \( \| \cdot \| \) one easily arrives at

\[
\| \partial_A \partial_{m^2} \partial_p^z L_{n_0}^{\Lambda A_0(l)} \| \Lambda < C_1 \sum_{r_1, r_2 = r}^{L^{\Lambda A_0(l-1)}} \Lambda^{1-2r_1} \| \partial_{m^2} \partial_p^z L_{n_0}^{\Lambda A_0(l-1)} \| \Lambda
\]

\[+ C_2 \sum_{r_1 = r}^{L^{\Lambda A_0(l)}} \Lambda^{-3-z-2r_1} \| \partial_{m^2} \partial_p^z L_{n_0}^{\Lambda A_0(l)} \| \Lambda \| \partial_{m^2} \partial_p^z L_{n_0}^{\Lambda A_0(l)} \| \Lambda \]

where in the second term the sum is also over \( \sum r_i = r \) and \( \sum z_i = z \). The proof is made by induction in the loop index \( l \) and in the \( l \)-th step we will use an induction in the index \( n \) of the number of legs. In general eq.(77) will be proven first in the case \( w = 1 \) and then, with a suitable integration, for \( w = 0 \). From \( \| \cdot \| \), which for \( l = 0 \) has not the first term, one gets \( L_2^{\Lambda A_0(0)} = 0 \) and \( L_4^{\Lambda A_0(0)} = g = \rho_3^{(0)} \), which fulfills trivially eq.(77). Let us consider now \( n > 4 \): in the right hand side, because of \( L_2^{\Lambda A_0(0)} = 0 \), only the terms with \( L_{n'}^{\Lambda A_0(0)} \) with \( n' < n \) are involved, so that we use an induction in \( n \), and we arrive at

\[
\| \partial_A \partial_{m^2} \partial_p^z L_{n_0}^{\Lambda A_0(0)} \| \Lambda < C \Lambda^{3-n-z-2r}
\]

From this, integrating from \( \Lambda = \Lambda_0 \),

\[
\| \partial_{m^2} \partial_p^z L_{n_0}^{\Lambda A_0(0)} \| \Lambda \leq \int_{\Lambda_0}^{\Lambda} d\Lambda' \| \partial_A \partial_{m^2} \partial_p^z L_{n_0}^{\Lambda A_0(0)} \| \Lambda' \leq C \Lambda^{4-n-z-2r}
\]

For \( l > 0 \) we assume eq.(77), for every \( z, r, w \) and \( l' < l \). We start considering the case \( n = 2 \); only \( L_{l'}^{(0)} \) with \( l' < l \) are present in the right hand side of eq.(113) so that, from the induction hypothesis, we get

\[
\| \partial_{m^2} \partial_p^z L_{l}^{\Lambda A_0(l)} \| \Lambda \leq \Lambda^{1-2r} P(\ln(\frac{\Lambda}{\Lambda_R})))
\]

Now for \( 2r + z > 2 \) we integrate in \( \Lambda \) from \( \Lambda = \Lambda_0 \) arriving at eq.(77) as shown in the case \( l = 0 \). Consider now the case \( 2r + z = 2 \). Using eq.(76) we can write

\[
\partial_{m^2} \partial_p^z L_{2}^{\Lambda A_0(l)}(m; p) = \delta_{r,1} \rho_1^{(l)}(0) + \int_0^p \int_0^p d^{m^2} \partial_{m^2} \partial_p^z \partial^{z+1} L_{2}^{\Lambda A_0(l)}(0; p') + \int_{\Lambda}^{m^2} \partial_{m^2} \partial_p^z \partial_{m^2} \partial_p^z \partial^{z+1} L_{2}^{\Lambda A_0(l)}(m'; p)
\]

In the integrands we can use eq.(77) for \( 2r + z > 2 \) so that we arrive again at eq.(77) for \( 2r + z = 2 \).

At this point analogous considerations can be applied to the case \( n = 2, r = z = 0 \), where now the renormalization condition on \( L_{2}^{\Lambda A_0(l)}(0; 0) \) plays the fundamental role.

For \( n = 4 \) we observe that the only addendum in the right hand side of eq.(113) which contains terms with \( l \) loops is \( \| \partial_{m^2} \partial_p^z L_{2}^{\Lambda A_0(l)} \| \) for which we have just now proven
the bound, therefore \[ \| \partial_\Lambda \partial^r_{m^2} \partial^z_p L^{\Lambda_0(t)}_4 \|_\Lambda \leq \Lambda^{-1-z-2r} P(\ln(\frac{\Lambda}{\Lambda_R})). \] From this it follows that \[ \| \partial_{m^2}^2 \partial^r_p L^{\Lambda_0(t)}_4 \|_\Lambda \leq \Lambda^{-1-z-2r} P(\ln(\frac{\Lambda}{\Lambda_R})), \] which in the case \( z + 2r > 0 \) it is obtained using integration from \( \Lambda = \Lambda_0 \), while for \( z = r = 0 \) it is obtained using the Taylor reconstruction formula:

\[
L^{\Lambda_0(t)}_4(m; p_1, p_2, p_3) = \rho^{(l)}_3 + \int_{\Lambda_R}^\Lambda d\Lambda' \partial_\Lambda' L^{\Lambda_0(t)}_4(0; 0) + \int_0^{m^2} dm'^2 \frac{\partial}{\partial m'^2} L^{\Lambda_0(t)}_4(m'; 0) 
+ \int_0^1 dt \sum_l q^l \frac{\partial}{\partial q^l} L^{\Lambda_0(t)}_4(m; q) \big|_{q=tp}, \tag{118}
\]

For \( n > 4 \) one realizes, as in the previous case, that in the right hand side of eq.(113) are contained only \( L^{(l')}_n \) for \( l' < l \) or, if \( l' = l \), with \( n' < n \); thus we arrive at \( \| \partial_\Lambda \partial^r_{m^2}^2 \partial^z_p L^{\Lambda_0(t)}_n \|_\Lambda \leq \Lambda^{3-n-z-2r} P(\ln(\frac{\Lambda}{\Lambda_R})) \) and then, integrating from \( \Lambda = \Lambda_0 \), at eq.(77) also for \( w = 0 \).

The second step consists in proving eq.(78). To this aim let us differentiate with respect to \( \Lambda_0 \) both sides of eq.(111). The resulting equation

\[
\partial_\Lambda O^{\Lambda_0(t)}_n(p_1, ..., p_{n-1}) = \frac{1}{2} \int_q \partial_\Lambda D_\Lambda(m; q) O^{\Lambda_0(t-1)}_{n+2}(m; q, -q; p_1, ..., p_{n-1}) - 
\sum L^{(l_1)}_{n_1}(m; p_1, ..., p_{n_1-1}) \partial_\Lambda D_\Lambda(P) O^{\Lambda_0(t_2)}_{n_2}(m; P, p_{n_1}, ..., p_{n-1}) \tag{119}
\]

coincides with the flow equation of the operator insertion \([5]()\) with \( O^{\Lambda_0} \equiv \partial_\Lambda L^{\Lambda_0}_n \). Now the boundary conditions on \( O^{\Lambda_0} \) from its definition are, for \( n + 2r + z \geq 5 \) using (77)

\[
\| \partial^r_{m^2} \partial^z_p O^{\Lambda_0(t)}_n \|_{\Lambda_0} = \| \partial_\Lambda |_{\Lambda=\Lambda_0} \partial^r_{m^2} \partial^z_p L^{\Lambda_0(t)}_n \|_{\Lambda_0} \leq \Lambda^{3-n-z-2r} P(\ln(\frac{\Lambda_0}{\Lambda_R})) \tag{120}
\]

while the renormalization conditions, as operator of dimension 4, are vanishing.

From eq.(111) one easily gets, using eq.(77),

\[
\| \partial_\Lambda \partial^r_{m^2} \partial^z_p O^{\Lambda_0(t)}_n \|_\Lambda \leq C_1 \sum \Lambda^{1-2r_1} \| \partial^r_{m^2} \partial^z_p O^{\Lambda_0(t-1)}_{n+2} \|_\Lambda 
+ C_2 \sum \Lambda^{4-n_1-z_1-2r_1} P(\ln(\frac{\Lambda}{\Lambda_R})) \Lambda^{3-z_2-2r_2} \| \partial^r_{m^2} \partial^z_p O^{\Lambda_0(t_2)}_{n_2} \|_\Lambda \tag{121}
\]

Eq. (78) is proven by induction in \( l \). For \( l = 0 \) one has \( O^{\Lambda_0(t)}_n = O^{\Lambda_0(t)}_n = 0 \), and for \( n \geq 5 \) one uses an induction in \( n \) as in the first theorem. Analogously for \( l > 0 \) starting from \( n = 2, 2r + z > 2 \), then \( 2r + z = 2 \) and \( 2r + z = 0 \) using vanishing renormalization conditions. Subsequently we go to the case \( n = 4, 2r + z > 0 \) and then, using the renormalization conditions, to the case \( n = 4, 2r + z = 0 \). For the cases with \( n \geq 5 \), using the induction hypothesis one has

\[
\| \partial_\Lambda \partial^r_{m^2} \partial^z_p O^{\Lambda_0(t)}_n \|_\Lambda \leq \Lambda^{4-n-2r-z} P(\ln(\frac{\Lambda_0}{\Lambda_R})) \tag{122}
\]

and then integrating from \( \Lambda = \Lambda_0 \) one arrives at the bound (78).

Note that if the renormalization conditions are not vanishing but are consistent with the behaviour \( \Lambda^{-2} \) for \( \Lambda_0 \to \infty \) we still arrive at eq. (78); with few modifications one can prove that if the flow of a composite operator of dimension \( D \) satisfies
\[ ||\partial_m^r \partial_p^z O_n^{\Lambda_0}||_{\Lambda_0} \leq \frac{\Lambda_0^{D-n-2r-z} P(ln(\frac{\Lambda_0}{\Lambda_R}))}{\Lambda_0} \] (123)

and their renormalization conditions go to zero at least as \( \Lambda_0^{-1} \), then

\[ ||\partial_m^r \partial_p^z O_n^{\Lambda_0}||_{\Lambda} \leq \frac{\Lambda^{D+1-n-2r-z} \Lambda_0}{\Lambda_0} P(ln(\frac{\Lambda_0}{\Lambda_R})) \rightarrow 0 \] (124)

for \( \Lambda_0 \rightarrow \infty \).

### Appendix B: Infrared finiteness.

In this appendix we want to prove the infrared regularity of the mass-independent scheme. The last goal will be to prove that with suitable renormalization conditions on \( L_2^{\Lambda R \Lambda_0}(0;0) \) arbitrary in Appendix A - the vertices \( L_n^{\Lambda_0}(m; p_1, ..., p_{n-1}) \) with non-exceptional momenta converge for \( (m, \Lambda, \Lambda_0) \rightarrow (0,0,\infty) \).

The first step consists in proving the bounds (83). In this appendix we will pay more attention to the dependence on the mass \( m \). We begin by noticing that

\[ |\partial_\Lambda \partial_m^r \partial_p^z D_{\Lambda \Lambda_0}(m; p)| \leq C(\Lambda + \frac{m}{2})^{-3-z-2r} (4\Lambda^2 - m^2 - p^2) \]

\[ \leq C(\Lambda + \frac{m}{2})^{-3-z-2r} (\Lambda - \frac{m}{2}) \] (125)

Moreover from the flow equation, for \( \Lambda \leq \eta_1/2 \)

\[ ||\partial_\Lambda \partial_m^r \partial_p^z z L_n^{\Lambda_0(l)}||_{\Lambda} \leq C_1 \sum (\Lambda + \frac{m}{2})^{-3-z-2r} ||\partial_m^r \partial_p^z L_{n+2}^{\Lambda_0(l-1)}||_{\Lambda}^k \]

\[ + C_2 \sum_{n_1 \leq k+1} (\Lambda + \frac{m}{2})^{-3-z-2r} ||\partial_m^r \partial_p^z L_{n_1}^{\Lambda_0(l_1)}||_{\Lambda}^k ||\partial_m^r \partial_p^z L_{n_2}^{\Lambda_0(l_2)}||_{\Lambda}^{k-n_1+2} \] (126)

where \( z_1 + z_2 + z_3 = z \) and where for the second term in the right hand side of (126) we have used the fact that, because of the first line of (125), the terms in which \( ||\partial_m^r \partial_p^z L_{n_1}^{\Lambda_0(l_1)}||_{\Lambda}^k \) should appear, with \( k_1 < n_1 - 1 \), are vanishing for \( \Lambda \leq \eta_1/2 \).

Consider first the case \( l = 0 \). For \( n = 2 \) and \( n = 4 \) all the bounds are trivially satisfied. For \( n > 4 \) the induction in \( n \) works. If \( 3 \leq k < n - 1 \) we get \( ||\partial_\Lambda \partial_m^r \partial_p^z L_n^{\Lambda_0(0)}||_{\Lambda}^k \leq \Lambda^{-1-2\frac{k-1}{n-1} z-2r} \), and then integrating from \( \Lambda = \Lambda_R \):

\[ ||\partial_m^r \partial_p^z L_n^{\Lambda_0(0)}||_{\Lambda}^k \leq \int_{\Lambda}^{\Lambda_R} d\Lambda' ||\partial_\Lambda' \partial_m^r \partial_p^z L_n^{\Lambda_0(0)}||_{\Lambda'}^k + ||\partial_m^r \partial_p^z L_n^{\Lambda_R \Lambda_0(0)}||_{\Lambda_R}^k \]

\[ \leq C(\Lambda + \frac{m}{2})^{-2(\frac{k-1}{n-1}) z-2r} \]

A similar discussion holds for the case \( k = n - 1 \). If \( k < 3 \) the right hand side of eq.(126) is vanishing, thus \( ||\partial_\Lambda \partial_m^r \partial_p^z L_n^{\Lambda_0(0)}||_{\Lambda}^k = 0 \) and \( ||\partial_m^r \partial_p^z L_n^{\Lambda_0(0)}||_{\Lambda}^k \leq C \).
Let us consider now $l > 0$. The induction hypothesis is eq. (83) for $l' < l$; the $l$-th step of the proof is made by induction in $n$. For $n = 2$ the right hand side of eq. (126) contains only terms with loop index $l' < l$; using the induction hypothesis we get the bound of eq. (83) for $w = 1$, $k = 0$ and $k = 1$. Notice that for $k = 0$ the second term of eq. (126) is absent, and that integrating from $\Lambda = \Lambda_R$ we get (83) also for $k = 0 = w$. Now consider $k = 1$. If $2r + z \geq 2$, integrating from $\Lambda = \Lambda_R$ we get the result also for $w = 0$. For the case $r = z = 0$ we notice that since $|\partial_\Lambda L_{2}^{\Lambda_0(l)}(0; 0)| < \Lambda P(\ln \frac{\Lambda}{\Lambda})$ this function is integrable in a neighborhood of $\Lambda = 0$. Thus

$$L_{2}^{\Lambda_0(l)}(0; 0) = \rho_1^{(l)} - \int_\Lambda^{\Lambda_R} d\Lambda'' \partial_\Lambda L_{2}^{\Lambda''_0(l)}(0; 0)$$

has a finite limit for $\Lambda \to 0$ and since the second term does not depend on $\rho_1^{(l)}$, this constant can be chosen such that $\lim_{\Lambda \to 0} L_{2}^{\Lambda_0(l)}(0; 0) = 0$. With this choice

$$|L_{2}^{\Lambda_0(l)}(0; 0)| \leq \int_{\Lambda}^{\Lambda_R} d\Lambda'' |\partial_\Lambda L_{2}^{\Lambda''_0(l)}(0; 0)| \leq \Lambda^2 P(\ln \frac{\Lambda_R}{\Lambda}) \leq (\Lambda + \frac{m}{2})^2 P(\ln \frac{\Lambda_R + \frac{m}{2}}{\Lambda + \frac{m}{2}})$$

Thus from

$$L_{2}^{\Lambda_0(l)}(m; p) = L_{2}^{\Lambda_0(l)}(0; 0) + \int_{0}^{p} dp' \partial_\rho L_{2}^{\Lambda_0(l)}(m; p') + \int_{0}^{m^2} dm'^2 \partial_{m'^2} L_{2}^{\Lambda_0(l)}(m'; p)$$

one gets

$$||L_{2}^{\Lambda_0(l)}||_{A}^{1} \leq (\Lambda + \frac{m}{2})^2 P(\ln \frac{\Lambda_R + \frac{m}{2}}{\Lambda + \frac{m}{2}})$$

For $n > 2$ in the right hand side of (126) there are only terms $L_{n}^{(l')}$ with $l' < l$ or, if $l' = l$, with $n' < n$.

For $k \geq 1$ every term in (126) has at least one ‘small momentum’. From the induction hypothesis we arrive at the bound (83) for $w = 1$ and then, integrating from $\Lambda = \Lambda_R$, at the bound for $w = 0$. If $k = 0$ only the first term in (126) has to be taken into account and thus

$$||\partial_\Lambda \partial_{m'^2} \partial_{p'} L_{n}^{\Lambda_0(l)}||_{A}^{1} \leq (\Lambda + \frac{m}{2})^1 P(\ln \frac{\Lambda_R + \frac{m}{2}}{\Lambda + \frac{m}{2}})$$

and then one gets easily eq. (83) also for $w = 0$.

At this point of the proof one could easily conclude that, after taking the limit $\Lambda_0 \to \infty$, the infrared part of the proof could be done starting from the renormalization point $L^{\Lambda_0 = \infty}[\Phi]$, showing that the limit $\Lambda \to 0$ exists, and moreover that the limit $\Lambda \to 0$, $m \to 0$ exists, leading to Green functions which are $C^\infty$ functions for non-exceptional momenta. But actually it is possible to prove that the global limit $(\Lambda, m, \Lambda_0) \to (0, 0, \infty)$ exists, so that no ambiguities in the definition of the theory are involved. Let us outline this proof. To this aim one should generalize the second theorem of Appendix A and prove that, if $\rho_1^{(l)}$ are chosen loop by loop according to eq. (83), then the quantities $\partial_\Lambda L_{n}^{\Lambda_0(l)}$ are such that $||\partial_\Lambda \partial_{m'^2} \partial_{p'} \partial_\Lambda L_{n}^{\Lambda_0(l)}||_{A}^{k}$ satisfy the bound of eq. (83), multiplied by $P(\ln \frac{\Lambda_R}{\Lambda_0^2})/\Lambda_0^2$.

These bounds are essentially obvious, and are proven by induction; we do not reproduce the proof in detail. They hold because: i) $\partial_\Lambda L_{n}^{\Lambda_0(l)}(p_1, \ldots, p_{n-1})$ satisfies eq. (73); ii) the flow
Therefore, parameters, that is (group equation (56); similar considerations can be repeated for eq. (90).

The functionals obtained as the massless case can be follows that

\[ \partial^{r} L^{A\Lambda_{0}(l)}(m; p) - \partial^{r} L^{A\Lambda_{0}(l)}(m; p) \leq \int_{\Lambda_{1}}^{\Lambda_{2}} L^{A\Lambda_{0}(l)}(m; p) \\ \leq \int_{\Lambda_{1}}^{\Lambda_{2}} L^{A\Lambda_{0}(l)}(m; p) \to 0 \]

for \((\Lambda_{1}, m_{1}, \Lambda_{0}, \Lambda_{0}) \to (0, 0, \infty, \infty)\).

As a last remark let us notice that from the bound (38) for \(k = w = 0\) and \(r = 1\) it follows that \(\partial^{m_{2}} L^{A\Lambda_{0}(l)}(m; p_{1}, \ldots, p_{n-1})\) has only logarithmic divergences in \(m\) for \(m \to 0\) and therefore \(m^{2} \partial^{m_{2}} L^{A\Lambda_{0}(l)}(m; p_{1}, \ldots, p_{n-1}) \to 0\) for non-exceptional momenta in the limit \(m \to 0\), so that the Gell-Mann and Low renormalization group equation in the massless case can be obtained as the \(m \to 0\) limit of the corresponding equation for the massive case.

**Appendix C: Exact renormalization group equation.**

In this appendix we examine some points regarding the proof of the exact renormalization group equation (43); similar considerations can be repeated for eq. (50).

Because of eq. (43), the functional \(L^{A\Lambda_{0}(l)}[\Phi]\) will depend on \(2l + 1\) arbitrary parameters, that is \((\rho_{2}^{(0)}, \ldots, \rho_{2}^{(l)}) = \rho_{2}^{(l)}\) and \((\rho_{3}^{(0)}, \ldots, \rho_{3}^{(l)}) = \rho_{3}^{(l)}\), and we will use the notation \(L^{A\Lambda_{0}(l)}[\Lambda_{R}, \rho_{2}^{A\Lambda_{0}}, \rho_{3}^{A\Lambda_{0}}, \Phi]\) for this functional. The standard values of \(\rho_{2}^{(l)}\) and \(\rho_{3}^{(l)}\) are determined at \(\Lambda = \Lambda_{R}\) in eq. (44). In perturbation theory it is true that

\[ L^{A\Lambda_{0}(l)}[\Lambda_{R}, \rho_{2}^{A\Lambda_{0}}, \rho_{3}^{A\Lambda_{0}}, \Phi] = L^{A\Lambda_{0}(l)}[\Lambda_{R}, \rho_{2}^{A\Lambda_{0}}, \rho_{3}^{A\Lambda_{0}}, \Phi] \]

since the relation between the bare coefficients and the parameters appearing in the renormalization conditions is invertible.

Differentiating this equation with respect to \(\Lambda'_{R}\) in \(\Lambda'_{R} = \Lambda_{R}\) we get

\[ \frac{\partial}{\partial \Lambda_{R}} \left[ \frac{\partial}{\partial \Lambda'_{R}} L^{A\Lambda_{0}(l)}[\Lambda_{R}, \rho_{2}, \rho_{3}; \Phi] \right] = 0 \]  

(128)

The functionals

\[ \mathcal{O}_{k,l'}^{A\Lambda_{0}(l)}[\Phi] \equiv \frac{\partial}{\partial \rho_{2}^{(l')}} L^{A\Lambda_{0}(l)}[\Lambda_{R}, \rho_{2}, \rho_{3}; \Phi] \]  

\[ \mathcal{O}_{g,l'}^{A\Lambda_{0}(l)}[\Phi] \equiv \frac{\partial}{\partial \rho_{3}^{(l')}} L^{A\Lambda_{0}(l)}[\Lambda_{R}, \rho_{2}, \rho_{3}; \Phi] \]  

(129)
satisfy eq. (50) and in fact define integrated composite operators of dimension four, with renormalization conditions

\[ \mathcal{O}_{k,l'}^{=0A_0(l)}(0) = 0 ; \quad \partial_{p^2}\mathcal{O}_{k,l'}^{=0A_0(l)}|_{p=0} = \delta^i,l' ; \quad \mathcal{O}_{k,l'}^{=0A_0(l)}|_{p=0} = 0 \]

and

\[ \mathcal{O}_{g,l'}^{=0A_0(l)}(0) = 0 ; \quad \partial_{p^2}\mathcal{O}_{g,l'}^{=0A_0(l)}|_{p=0} = 0 ; \quad \mathcal{O}_{g,l'}^{=0A_0(l)}|_{p=0} = \delta^i,l' \]

Using eq. (51) one could prove easily that \( \mathcal{O}_{k,l'}^{=A_0(l)}[\Phi] = \mathcal{O}_{g,l'}^{=A_0(l)}[\Phi] = 0 \) for \( l < l' \), since they are operators insertions with vanishing renormalization conditions for \( l < l' \). Therefore we can define, as a formal series in \( h \), the functionals \( h^{-l'}\mathcal{O}_{k,l'}^{=A_0}[\Phi] \) and \( h^{-l'}\mathcal{O}_{g,l'}^{=A_0}[\Phi] \), which again satisfy eq. (51). The functionals \( \mathcal{O}_{g,0}^{=A_0}[\Phi] - h^{-l'}\mathcal{O}_{g,l'}^{=A_0}[\Phi] \) have vanishing renormalization conditions and therefore are vanishing; then it follows that

\[
\frac{\partial}{\partial p_3} L^{A_0(l'-l)} = \frac{\partial}{\partial p_3} L^{A_0(0)}
\]

Using this fact we get

\[
\sum_{l'=0}^{l} \beta_3(l') \frac{\partial}{\partial p_3} L^{A_0(l)}[\Lambda_R, \rho_2, \rho_3; \Phi] = \sum_{l'=0}^{l} \beta_3(l') \frac{\partial}{\partial p_3} L^{A_0(l'-l)+1}[\Lambda_R, \rho_2, \rho_3; \Phi]
\]

which can now be interpreted as the \( l \)-th term of the product of the series \( \beta_3 \) and \( \frac{\partial}{\partial p_3} L^{A_0(0)}[\Phi] \).

Similar considerations can be performed on the second term of eq. (128). Therefore one has

\[
\sum_{l'=1}^{l} \beta_2(l') \frac{\partial}{\partial p_2} L^{A_0(l)}[\Lambda_R, \rho_2, \rho_3; \Phi] = \sum_{l'=1}^{l} \beta_2(l') \frac{\partial}{\partial p_2} L^{A_0(l'-l)+1}[\Lambda_R, \rho_2, \rho_3; \Phi]
\]

and this quantity is the \( l \)-th term of the product of the series \( \beta_2 \) and \( h^{-1} \frac{\partial}{\partial p_2} L^{A_0(0)}[\Phi] \).

Considering the case in which the only non-vanishing parameters are \( \rho_3(0) \equiv g \) and \( \rho_2(1) \equiv r \); as a consequence of the above equations, we get

\[
[\Lambda_R \frac{\partial}{\partial \Lambda_R} + h^{-1} \beta_2 \frac{\partial}{\partial r} + \beta_3 \frac{\partial}{\partial g}] L^{A_0(0)}[\Lambda_R, hr, g; \Phi] = 0 \tag{130}
\]

In the case \( r = 0 \) we obtain eq. (59). Observe that we never introduced a parameter \( \rho_2(0) \) for technical reasons.

Integrating formally eq. (130) one arrives at the integral form (55), where \( \rho_2^{A_0} \) and \( \rho_3^{A_0} \) depend on \( r \) and \( g \) through the equations

\[
\frac{d}{d\Lambda_R} \rho_2 = \beta_2(\rho_2, \rho_3) ; \quad \frac{d}{d\Lambda_R} \rho_3 = \beta_3(\rho_2, \rho_3)
\]

with initial conditions \( \rho_2^{A_0} = r h \) and \( \rho_3^{A_0} = g \).

Let us finally make a comment on the statement made in the paper, that the composite operators introduced in proving the renormalization group equation are operators of dimension four which satisfy the crucial equation (58); actually this is the only requirement which is not trivially satisfied. In all the cases considered, starting from the definition of the functionals of the operators in term of the flow \( L^{A_0(0)}[\Phi] \) it is an exercise with the usual induction scheme to prove the bound (58) for \( k = n - 1 \) and then eq. (58).
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