THE GELFAND-KAZHDAN CRITERION AS A NECESSARY AND SUFFICIENT CRITERION

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Abstract. We show that under certain conditions the Gelfand-Kazhdan criterion for the Gelfand property is a necessary condition. We work in the generality of finite groups, however part of the argument carries over to p-adic and real groups.

1. Introduction

In this note we study the Gelfand-Kazhdan criterion for the Gelfand property (see [GK75]) and show that under some conditions it is not only a sufficient condition but also a necessary one. We discuss mostly finite groups, but we hope that some of these methods can be pushed to the generality of p-adic groups (and even Lie groups). The Gelfand-Kazhdan criterion was originally developed as a version of the Gelfand trick that is valid for p-adic groups and not only for compact groups. However, even for finite groups the Gelfand-Kazhdan criterion is slightly more informative than the Gelfand trick.

The main result of this note is the following:

Theorem 1. Let $G$ be a finite group, $\theta : G \to G$ be an involution and $H \subset G$ be a $\theta$-stable subgroup. Assume that for any $x \in G$ there exists $g \in G$ s.t. $gx^{-1}g^{-1} = \theta(x)$. Then the following are equivalent:

1. $(G, H)$ is a Gelfand pair.
2. For any $x \in G$ there are $h_1, h_2 \in H$ s.t. $h_1x^{-1}h_2 = \theta(x)$.

We also have slightly more general version of this theorem.

Theorem 2. Let $G$ be a finite group, $\theta : G \to G$ be an involution and $H \subset G$ be a $\theta$-stable subgroup. Then the following are equivalent:

1. $(G, H)$ is a Gelfand pair and any $H$-distinguished representation $\pi$ of $G$ (i.e. a representation satisfying $(\pi)^H \neq 0$) satisfies $\pi \circ \theta \cong \pi^*$.
2. For any $g \in G$ there are $h_1, h_2 \in H$ s.t. $h_1g^{-1}h_2 = \theta(g)$

This theorem implies the previous one.
We can generalize this Theorem further:

Theorem 3. Let $G$ be a finite group, $\theta : G \to G$ be an involution and $H \subset G$ be a subgroup. Then the following are equivalent:

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(1) \((G, H)\) is a Gelfand pair and any \(H\)-distinguished representation possesses a symmetric non-zero bilinear form \(B\) satisfying \(B(\pi(g)v, w) = B(v, \pi(\theta(g^{-1}))w)\).

(2) For any \(g \in G\) there are \(h_1, h_2 \in H\) s.t. \(h_1g^{-1}\theta(h_2) = \theta(g)\).

Theorem 3 implies Theorem 2 by [Vin06, Proposition 3]. The last theorem follows from Proposition 3 below which is a reinterpretation of the original proof of the Gelfand-Kazhdan criterion. In order to formulate this proposition, we need to recall the definition of the twisted Frobenius-Schur indicator.

Notation 1. Let \(\pi \in \text{irr}(G)\) and let \(\theta : G \to G\) be an involution. We denote

\[
\varepsilon_\theta(\pi) = \begin{cases} 
0, & \pi \not\simeq \pi^* \circ \theta, \\
1, & \pi \text{ possesses a non-zero symmetric bilinear form } B \text{ satisfying } \\
B(\pi(g)v, w) = B(v, \pi(\theta(g^{-1}))w), \\
-1, & \pi \text{ possesses a non-zero anti-symmetric bilinear form } B \text{ satisfying } \\
B(\pi(g)v, w) = B(v, \pi(\theta(g^{-1}))w).
\end{cases}
\]

Proposition 4. Let \(G\) be a finite group, \(\theta : G \to G\) be an involution and \(H \subset G\) be a subgroup. Let \(V\) be the space of functions on \(G\) which are left invariant w.r.t. \(H\), right invariant w.r.t. \(\theta(H)\) and anti-invariant w.r.t. \(\sigma := \theta \circ \text{inv}\), where \(\text{inv} : G \to G\) is the inversion. Then

\[
V \cong \left( \bigoplus_{\varepsilon_\theta(\pi) = 0} \pi^H \otimes (\pi^*)^{\theta(H)} \right) S_2 \text{sign} \oplus \left( \bigoplus_{\varepsilon_\theta(\pi) = 1} \Lambda^2(\pi^H) \right) \oplus \left( \bigoplus_{\varepsilon_\theta(\pi) = -1} \text{Sym}^2(\pi^H) \right),
\]

where the action of \(S_2\) on \(\bigoplus_{\varepsilon_\theta(\pi) = 0} \pi^H \otimes (\pi^*)^{\theta(H)}\) is given by the involution \(s(v \otimes w) \mapsto w \otimes v\) where \(v \otimes w \in \pi^H \otimes (\pi^*)^{\theta(H)}\) and \(w \otimes v \in (\pi^*)^{\theta(H)} \otimes \pi^H \cong (\pi^* \circ \theta)^H \otimes (\pi^* \circ \theta)\).

Corollary 5. Using the notations above we have,

\[
\# \{ O \in H \setminus G/\theta(H) : \sigma(O) \neq O \} = \sum_{\pi \in \text{irr}(G)} \dim(\pi^H) \left( \dim(\pi^{\theta(H)}) - \varepsilon_\theta(\pi) \right).
\]

1.1. The case of \(p\)-adic and real groups. Some of the arguments above work also for \(l\)-groups and even real reductive groups. First of all, as in the original Gelfand-Kazhdan criterion, the second condition in all three theorems should be replaced by a condition on distributions. Similarly, the space \(V\) as above should be replaced by a space of distributions.

The proof of [Vin06] works also for \(l\)-groups (see [Vin06, Lemma 3]), and the same argument seems to work for real reductive groups. Thus, the main difference is in Proposition 3, Proposition 4 does not work as is in those cases. However, the construction of the spherical (a.k.a. relative) character gives an embedding

\[
\nu : \left( \bigoplus_{\varepsilon_\theta(\pi) = 0} (\pi^*)^H \otimes (\pi^*)^{\theta(H)} \right) S_2 \text{sign} \oplus \left( \bigoplus_{\varepsilon_\theta(\pi) = 1} \Lambda^2((\pi^*)^H) \right) \oplus \left( \bigoplus_{\varepsilon_\theta(\pi) = -1} \text{Sym}^2((\pi^*)^H) \right) \to V.
\]

Using this, the implication (2) \(\Rightarrow\) (1) of Theorems 1, 2 and 3 follows. In fact, this is a reformulation of the classical proof of the Gelfand-Kazhdan criterion, and its extension that was proven in [JR96]. It is reasonable to expect that in many cases \(\nu\) has dense image. If this is the case, then Theorems 1, 2 and 3 hold in the \(p\)-adic and real settings. Namely, consider the following:
**Definition.** Let \( H_1, H_2 \subset G \) be subgroups of an l-group or of a real reductive group and let \( S^*(G) \) denote the space of Schwartz distributions on \( G \). We say that \( (G, H_1, H_2) \) satisfies spectral density if the space spanned by spherical characters of irreducible (admissible) representations of \( G \) w.r.t. \( H_1, H_2 \) is dense in \( S^*(G)^{H_1 \times H_2} \).

We prove that a weaker property is satisfied in the \( p \)-adic case for many cases in [AGS15, Theorems C and D].

The argument above show the following:

**Theorem 6.** If \( G \) is an l-group (or real reductive group) and \( H \) is a subgroup s.t. \((G, H, H)\) satisfies spectral density, then Theorems [4, 2 and 3 hold for \( G, H \) with the above mentioned changes.

2. **Double invariant functions and multiplicities (proof of Proposition 4)**

Let \( X = G/H' \), and set \( H' = \theta(H) \) and \( X' = G/H' \). Let \( \sigma \) be the involution of \( X \times X' \) given by \(([g], [h]) \mapsto ([\theta(h)], [\theta(g)]) \). We have

\[
\mathbb{C}[X] = \bigoplus_{\pi \in \text{irr}(G)} \pi \otimes (\pi^*)^H \quad \text{and} \quad \mathbb{C}[X'] = \bigoplus_{\pi \in \text{irr}(G)} \pi \otimes (\pi^*)^{H'}.
\]

Thus

\[
W := \mathbb{C}[G]^{H \times H'} \cong \mathbb{C}[X \times X']^{\Delta G} \cong \bigoplus_{\pi \in \text{irr}(G)} (\pi^*)^H \otimes (\pi)^{H'},
\]

where \( \Delta G \) denotes the diagonal embedding of \( G \) into \( G \times G \). Its remains to understand the action of \( \sigma \) on \( W \). For this let us first analyze the action of \( \sigma \) on \( \mathbb{C}[X \times X'] \). We have

\[
\mathbb{C}[X \times X'] \cong \bigoplus_{\pi, \tau \in \text{irr}(G)} \pi \otimes (\pi^*)^H \otimes (\tau^*)^{H'} \cong \bigoplus_{\pi, \tau \in \text{irr}(G)} (\pi \circ \theta) \otimes (\pi^*)^{H'} \otimes (\tau \circ (\tau^*)^{H'}
\]

\[
\cong \bigoplus_{\pi, \tau \in \text{irr}(G)} (\pi \circ \theta) \otimes \tau \otimes (\pi^*)^{H'} \otimes (\tau^*)^{H'}
\]

\[
\cong \left( \bigoplus_{\pi \in \text{irr}(G)} (\pi \circ \theta) \otimes \pi \otimes (\pi^*)^{H'} \otimes (\pi^*)^{H'} \right) \oplus \left( \bigoplus_{\pi \in \text{irr}(G)} (\pi \circ \theta) \otimes \tau \otimes (\pi^*)^{H'} \otimes (\tau^*)^{H'} \right).
\]

The action of \( \sigma \) on

\[
\bigoplus_{\pi \neq \tau \in \text{irr}(G)} (\pi \circ \theta) \otimes \tau \otimes (\pi^*)^{H'} \otimes (\tau^*)^{H'}
\]

is given by interchanging the summand corresponding to \((\pi, \tau)\) with the summand corresponding to \((\tau, \pi)\). The action of \( \sigma \) on

\[
\bigoplus_{\pi \in \text{irr}(G)} (\pi \circ \theta) \otimes \pi \otimes (\pi^*)^{H'} \otimes (\pi^*)^{H'}
\]

is by acting on each summand separately, and is given by

\[
v \otimes w \otimes \alpha \otimes \beta \mapsto w \otimes v \otimes \beta \otimes \alpha.
\]

Now, let us restrict this action to \(((\pi \circ \theta) \otimes \pi \otimes (\pi^*)^{H'} \otimes (\pi^*)^{H'})^{\Delta G} \cong ((\pi \circ \theta) \otimes \pi)^{\Delta G} \otimes (\pi^*)^{H'} \otimes (\pi^*)^{H'} \). We see that the space \((\pi \circ \theta) \otimes \pi)^{\Delta G}

is either 0 or 1-dimensional, and in the latter case, the action of \( \sigma \) on it is given by \( \varepsilon_\theta(\pi) \).
Finally we use the fact that $V = \{ w \in W : \sigma(w) = -w \}$ and obtain the required identity.

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