MPS: An R package for modelling new families of distributions

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Abstract: We introduce an R package, called MPS, for computing the probability density function, computing the cumulative distribution function, computing the quantile function, simulating random variables, and estimating the parameters of 24 new shifted families of distributions. By considering an extra shift (location) parameter for each family more flexibility yields. Under some situations, since the maximum likelihood estimators may fail to exist, we adopt the well-known maximum product spacings approach to estimate the parameters of shifted 24 new families of distributions. The performance of the MPS package for computing the cdf, pdf, and simulating random samples will be checked by examples. The performance of the maximum product spacings approach is demonstrated by executing MPS package for three sets of real data. As it will be shown, for the first set, the maximum likelihood estimators break down but MPS package find them. For the second set, adding the location parameter leads to acceptance the model while absence of the location parameter makes the model quite inappropriate. For the third set, presence of the location parameter yields a better fit.

Keywords: Cumulative distribution function; Maximum likelihood estimation; Method of maximum product spacings; Probability density function; Quantile function; R package; Simulation;

1 Introduction

Over the last two decades, generalization of the statistical distributions has attracted much attention in the literature. Most of these extensions have been spawned by applications found in analyzing lifetime data. The generalized distributions not only have great potentials to provide families which incorporate more flexible probability density function (pdf), but also exhibit flexible hazard rate function. It is well known that hazard rate function plays the main role in survival and lifetime analysis. Depending on the model which is under study, this function can be constant, decreasing, increasing, upside-down bathtub, and bathtub-shaped. So, the new introduced distributions may have different appeals for different users.

In this work we mainly focus on new generalized families of statistical distributions whose pdf has positive support. Up to now, we are aware of 24 generalized families of distributions with applications in lifetime analysis. We introduce a quite efficient R package, called MPS, for statistical modelling of 24 generalized families of distributions when they are equipped with the location parameter. The statistical modelling involves computing pdf, computing cumulative distribution function (cdf), simulating random realizations, and estimating the parameters via maximum product spacings (MPS) approach introduced by Cheng and Amin (1983). This paper is organized
In what follows we mention 24 new families of statistical distributions (known in the literature as $G$ families of distributions). A general description about the method of MPS and details for using the MPS package for users who are familiar with R will be given in Section 2. Section 3 is devoted for checking the MPS package through examples and real data applications. We conclude the paper in Section 4.

Suppose $G$ is a valid cdf defined on the real line. The general way for introducing a new cdf, $F$, say, is to put the $G$ into the domain of an increasing function such as $h$ with the following form.

$$F(x) = h(G(x, \theta)),$$

where $h : [0, 1] \rightarrow [0, 1]$ and $\theta$ is parameter space of $G$ distribution. Several candidates exist in the literature for $h$. In the following we review 24 approaches for producing new family of distributions.

1. Beta exponential $G$ (betaexpg) family: Consider the $T - X$ family of distributions introduced by Alzaatreh et al. (2013b). The betaexpg family is in fact beta-X family. The cdf and pdf of betaexpg family are given as

$$F_{\text{betaexpg}}(x, \Theta) = 1 - \frac{\int_0^1 (1 - G(x, \theta))^d y^{a-1}(1 - y)^{b-1} dy}{B(a, b)},$$

$$f_{\text{betaexpg}}(x, \Theta) = \frac{d g(x, \theta) [1 - (1 - G(x, \theta))^d]^{b-1} (1 - G(x, \theta))^{ad-1}}{B(a, b)},$$

where $\Theta = (a, b, d, \theta^T)^T$ is the parameter space of betaexpg family, $a > 0$, $b > 0$, and $d > 0$ are the new induced shape parameters, and $\theta$ is the parameter space of distribution of $G$.

2. Beta $G$ (betag) family: The betag family of distributions introduced by Eugene et al. (2002). The cdf and pdf of the betag family are given by

$$F_{\text{betag}}(x, \Theta) = \frac{1}{B(a, b)} \int_0^{G(x, \theta)} y^{a-1}(1 - y)^{b-1} dy,$$

$$f_{\text{betag}}(x, \Theta) = \frac{1}{B(a, b)} g(x, \theta)(G(x, \theta))^{a-1}(1 - G(x, \theta))^{b-1},$$

where $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$ in which $\Gamma(.)$ is the ordinary gamma function defined as $\Gamma(a) = \int_0^\infty y^{a-1}e^{-y}dy$ and $\Theta = (a, b, \theta^T)^T$ is the parameter space of the betag family. Here, $a > 0$ and $b > 0$ are the new induced shape parameters and $\theta$ is the parameter space of distribution of $G$. This family can be used for modelling the failure time of a $a$-out-of-$a + b - 1$ system when the failure times of the components are independent and identically distributed random variables whose distribution is $G$. Many candidates have been considered in the literature for distribution of $G$ that among them we refer to Pareto [8], Cauchy [14], generalized exponential [27], Fréchet [28], generalized normal [39], Birnbaum-Saunders [43], half Cauchy [44], Laplace [45], power [46], moyal [47], extended Weibull [53], generalized gamma [51], generalized Rayleigh [52], exponentiated Weibull [53], Domma [62], normal [69], Burr III [77], inverse Weibull [84], weighted Weibull [85], Gompertz [87], linear failure rate [88], inverse Rayleigh [102], Weibull [105, 70], gamma [96], Laplace [97], generalized Pareto [106, 139], Lindley [115], lognormal [119], generalized logistic [120], Gumbel [123], Fréchet [126], exponential [130], generalized Lindley [144], Burr XII [146], generalized half-normal [152], Lomax [158], Nakagami [179], modified Weibull [180], generalized Weibull [184], and exponentiated Pareto [190].
3. Exponentiated exponential Poisson \( G \) (expexppg) family: The general form for the cdf and pdf of the expexppg family due to Ristic and Nadarajah (2014) are given by

\[
\begin{align*}
F_{\text{expexppg}}(x, \Theta) &= 1 - e^{-b(G(x, \theta))^a} \\
f_{\text{expexppg}}(x, \Theta) &= \frac{ab g(x, \theta)(G(x, \theta))^{a-1}e^{-b(G(x, \theta))^a}}{1 - e^{-b}},
\end{align*}
\]

(1.5)

(1.6)

where \( \Theta = (a, b, \theta^T)^T \) is the parameter space of the expexppg family, \( a > 0 \) and \( b > 0 \) are the new induced shape parameters, and \( \theta \) is the parameter space of distribution of \( G \). Ristic and Nadarajah (2014) used this family for modelling the time to failure of the first out of a Poisson number of systems functioning independently.

4. Exponentiated \( G \) family (expg): This family first time introduced by Mudholkar et al. (1991). Contrary to the Weibull distribution that can accommodate just the monotone hazard rates, the hazard rate of the exponentiated Weibull distribution can take unimodal, bathtub shaped, and monotone forms. The general form for cdf and pdf of the expg family are given by

\[
\begin{align*}
F_{\text{expg}}(x, \Theta) &= (G(x, \theta))^a, \\
f_{\text{expg}}(x, \Theta) &= ag(x, \theta)(G(x, \theta))^{a-1},
\end{align*}
\]

(1.7)

(1.8)

where \( \Theta = (a, \theta^T)^T \) is the parameter space of the expg family, \( a > 0 \) is the new induced shape parameter, and \( \theta \) is the parameter space of distribution of \( G \). This family have been used for several distributions of \( G \) among them we refer to Lomax [1], modified Weibull [33], generalized class of distributions [54], generalized Birnbaum-Saunders [57], generalized inverse Weibull ([66], [81], [82], [104]), Weibull ([122], [123], [124], and [127]), general exponentiated type [128], Gumbel [129], gamma [131], Lomax [169], and Pareto [178].

5. Exponentiated generalized \( G \) (expgg) family: General form for the cdf and pdf of the expgg family are given by

\[
\begin{align*}
F_{\text{expgg}}(x, \Theta) &= \left[1 - (1 - G(x, \theta))^a\right]^b, \\
f_{\text{expgg}}(x, \Theta) &= ab g(x, \theta)(1 - G(x, \theta))^{a-1}[1 - (1 - G(x, \theta))^a]^{b-1},
\end{align*}
\]

(1.9)

(1.10)

where \( \Theta = (a, b, \theta^T)^T \) is the parameter space of the expgg family, \( a > 0 \) and \( b > 0 \) are the new induced shape parameters, and \( \theta \) is the parameter space of distribution of \( G \). For being familiar with application of this family see [54] and [137].

6. Exponentiated Kumaraswamy \( G \) (expkumg) family: Lemonte et al. (2013) introduced expkumg family of distributions to model the lifetimes. The cdf and pdf of this family are given by

\[
\begin{align*}
F_{\text{expkumg}}(x, \Theta) &= \left\{1 - [1 - (G(x, \theta))^a]^b\right\}^d, \\
f_{\text{expkumg}}(x, \Theta) &= abc g(x, \theta)(G(x, \theta))^{a-1}[1 - (G(x, \theta))^a]^{b-1}d \left\{1 - [1 - (G(x, \theta))^a]^b\right\}^{d-1},
\end{align*}
\]

(1.11)

(1.12)

where \( \Theta = (a, b, d, \theta^T)^T \) is the parameter space of the expkumg family, \( a > 0 \), \( b > 0 \), and \( d > 0 \) are the new induced shape parameters, and \( \theta \) is the parameter space of distribution of \( G \). Some efforts have been made for investigating the properties and applications of this family. We refer readers to [86], [164], and [165], when distributions of \( G \) are supposed to be Dagum, exponential, and inverse Weibull, respectively.
7. Gamma $G$ family ($\text{gammag}$): Zografos and Balakrishnan (2009) introduced the $\text{gammag}$ family of distributions which is similar to that introduced by [69] and [89]. The only difference is that, here, the generator is the cdf of a gamma distribution with shape parameter $a$. General form of the cdf and pdf of $\text{gammag}$ family are given as

$$F_{\text{gammag}}(x, \Theta) = \frac{\gamma(- \log(1 - G(x, \theta)), a)}{\Gamma(a)},$$

$$f_{\text{gammag}}(x, \Theta) = \frac{g(x, \theta)}{\Gamma(a)} \left[ - \log(1 - G(x, \theta)) \right]^{a-1},$$

where $\gamma(x, a) = \int_0^x y^{a-1}e^{-y}dy$; for $a > 0$, denotes the incomplete gamma function. Now $\Theta = (a, \theta^T)^T$ is the parameter space of the $\text{expkung}$ family, $a > 0$ is the new induced shape parameter, and $\theta$ is the parameter space of distribution of $G$. The $\text{gammag}$ family has been studied for several distributions of $G$. Those include Pareto [15], half normal [16], normal [19], exponentiated Weibull [34], logistic [35], Dagum [143], log-logistic [159], and extended Frechet [181].

8. Gamma$^1$ $G$ family ($\text{gammag}^1$): Ristic and Balakrishnan (2012) proposed $\text{gammag}^1$ family of distributions whose cdf and pdf are given by

$$F_{\text{gammag}^1}(x, \Theta) = 1 - \frac{\gamma(- \log(G(x, \theta)), a)}{\Gamma(a)},$$

$$f_{\text{gammag}^1}(x, \Theta) = \frac{g(x, \theta)}{\Gamma(a)} \left[ - \log(G(x, \theta)) \right]^{a-1},$$

where $\Theta = (a, \theta^T)^T$ is the parameter space of $\text{gammag}^1$ family, $a > 0$ is the new induced shape parameter, and $\theta$ is the parameter space of distribution of $G$. This family has been studied by [25], [153], and [148] and when distributions of $G$ are exponentiated Weibull, inverse Weibull, and Lindley, respectively.

9. Gamma$^2$ $G$ family ($\text{gammag}^2$): An extension of $\text{gammag}$ family, called here $\text{gammag}^2$, introduced by Torabi and Montazeri (2012). The cdf and pdf of $\text{gammag}^2$ family are given by

$$F_{\text{gammag}^2}(x, \Theta) = \frac{\gamma \left( \frac{G(x, \theta)}{1 - G(x, \theta)} \right), a}{\Gamma(a)},$$

$$f_{\text{gammag}^2}(x, \Theta) = \frac{g(x, \theta)}{\Gamma(a) (1 - G(x, \theta))^2} \left( \frac{G(x, \theta)}{1 - G(x, \theta)} \right)^{a-1},$$

where $\Theta = (a, \theta^T)^T$ is the parameter space of $\text{gammag}^2$ family, $a > 0$ is the new induced shape parameter, and $\theta$ is the parameter space of distribution of $G$. Torabi and Montazeri (2012) pointed out that the (1.17) family provides great flexibility in modelling the negative and positive skewed, convex-concave shape, and reverse ‘J’ shaped distributions. Also, $\text{gammag}^2$ family has been studied by [58] and [59] when distributions of $G$ are linear failure rate and Lomax, respectively.

10. Generalized beta $G$ family ($\text{gbetag}$): Alexander et al. (2012) introduced the $\text{gbetag}$ family by replacing the generalized beta distribution of the first kind, see [110], with the beta distribution in definition of $\text{betag}$ family given in [1.3]. The cdf and pdf of $\text{gbetag}$ family are
11. Geometric exponential Poisson (gexppg) family: Nadarajah et al. (2013) introduced the gexppg family. The family of distributions proposed by Kus (2007) is special case of gexppg family. General form of the cdf and pdf of gexppg family are given by

\[
F_{gexppg}(x, \Theta) = \frac{e^{-a-aG(x,\theta)} - e^{-a}}{1 - e^{-a} - b + be^{-a+aG(x,\theta)}},
\]

\[
f_{gexppg}(x, \Theta) = \frac{a(1-b)g(x,\theta)(1-e^{-a})e^{-a+aG(x,\theta)}}{(1-e^{-a} - b + be^{-a+aG(x,\theta)})^2},
\]

where \(\Theta = (a,b,\theta^T)^T\) is the parameter space of the gexppg family, \(a > 0\) and \(0 < b < 1\) are the new induced shape parameters, and \(\theta\) is the parameter space of distribution of \(G\). This family is used for modelling the time to failure of the first out of a geometric number of systems functioning independently where the number of parallel units in each system has Poisson distribution and failure times for each units follow independently a \(G\) distribution, see [132].

12. Gamma-X family of modified beta exponential \(G\) (gmbetaexpg) distribution: The cdf and pdf of gmbetaexpg family are given by

\[
F_{gmbetaexpg}(x, \Theta) = \left(1 - e^{-b \frac{G(x,\theta)}{G(x,\theta)}}\right)^a,
\]

\[
f_{gmbetaexpg}(x, \Theta) = abg(x,\theta)(1-G(x,\theta))^{-2}e^{-b \frac{G(x,\theta)}{G(x,\theta)}} \left[1 - e^{-b \frac{G(x,\theta)}{G(x,\theta)}}\right]^{-a-1},
\]

where \(\Theta = (a,b,\theta^T)^T\) is the parameter space of the gmbetaexpg family, \(a > 0\) and \(b > 0\) are the new induced shape parameters, and \(\theta\) is the parameter space of distribution of \(G\). The gmbetaexpg is in fact the gamma-X family due to Alzaatreh et al. (2013b). We address readers to Alzaatreh et al. (2012), Alzaatreh and Knight (2013), and Alzaatreh et al. (2014) for properties and applications of the gamma-Pareto, gamma-half normal, and gamma-normal families, respectively.

13. Generalized transmuted-G (gtransg) family: The functional combination of the cdf of a given distribution with the inverse cdf of another distribution known in the literature as the transmutation map. The cdf and pdf of the generalized transmuted-G, called here gtransg, due to Merovci et al. (2017) are given by

\[
F_{gtransg}(x, \Theta) = (G(x,\theta))^a[1 + b(1 - G(x,\theta))]^a,
\]

\[
f_{gtransg}(x, \Theta) = a g(x,\theta)(G(x,\theta))^{a-1} \left[1 + b - 2bG(x,\theta)\right][1 + b(1 - G(x,\theta))]^{a-1},
\]
where $\Theta = (a, b, \theta^T)^T$ is the parameter space of the gtrans family, $a > 0$ and $-1 < b < 1$ are the new induced shape parameters, and $\theta$ is the parameter space of distribution of $G$. The transmutation map has been applied in the literature to many candidates for distribution of $G$. Those include inverse Rayleigh [7], extreme value [21], Weibull [22], log-logistic [23], Rayleigh [113], Lindley [112], generalized Rayleigh [114], Pareto [116], and Lindley-geometric [117]. It should be noted that there are other generalizations of the transmuted-$G$ family including: exponentiated transmuted Weibull [83], beta transmuted Weibull [145], transmuted exponentiated generalized-G family [188], generalizations of the transmuted-$G$ family [140], transmuted geometric-$G$ family [4], Kumaraswamy transmuted-$G$ family [5], beta transmuted-$G$ family [6], and complementary generalised transmuted Poisson-$G$ [13].

14. Log-logistic-$X$ family of $G$ (gxlogistic) distribution: The gxlogistic family is a special case of $T - X$ family due to Alzaatreh et al. (2013b). If we let $T$ follow a log-logistic distribution with shape parameter $a$ and $W(F(.)) = -\log(1 - F(.))$, then the gxlogistic family is obtained. The cdf and pdf of the gxlogistic family are given by

$$F_{gxlogistic}(x, \Theta) = \frac{1}{1 + [-\log(1 - G(x, \theta))]^{-a}}, \quad (1.27)$$

$$f_{gxlogistic}(x, \Theta) = \frac{ag(x, \theta) [-\log(1 - G(x, \theta))]^{-a-1}}{(1 - G(x, \theta)) \{1 + [-\log(1 - G(x, \theta))]^a\}^2}, \quad (1.28)$$

where $\Theta = (a, \theta^T)^T$ is the parameter space of the gxlogistic family, $a > 0$ is the new induced shape parameter, and $\theta$ is the parameter space of distribution of $G$.

15. Kumaraswamy $G$ family (kumg): Based on Kumaraswamy (1980) distribution, Jones (2009) introduced a new family of distributions which is known as Kumaraswamy $G$ family in the literature. The cdf and pdf of kumg family are given by

$$F_{kumg}(x, \Theta) = 1 - [1 - (G(x, \theta))^a]^b, \quad (1.29)$$

$$f_{kumg}(x, \Theta) = a b g(x, \theta) (G(x, \theta))^{a-1} [1 - (G(x, \theta))^{a}]^{b-1}, \quad (1.30)$$

where $\Theta = (a, b, \theta^T)^T$ is the parameter space of the gtrans family, $a > 0$ and $-1 < b < 1$ are the new induced shape parameters, and $\theta$ is the parameter space of distribution of $G$. Many candidates have been considered in the literature for distribution of $G$. Among them we refer to modified inverse Weibull [24], Pareto [31], Lindley [32], Weibull [11, 42], Gumbel ([32, 48]), normal [32], inverse-Gaussian [42], generalized half-normal [50], modified Weibull [50], generalized (Stacy) gamma [42, 60], log-logistic [61], generalized linear failure rate [63], exponentiated Pareto [64], quasi Lindley [65], Kumaraswamy [68], generalized Rayleigh [78], half-Cauchy [74], generalized Pareto [133], inverse exponential [141], Burr XII distribution [147], generalized gamma distribution [149], inverse Rayleigh [160], log-logistic [171], Birnbaum-Saunders [172], inverse Weibull [174], generalized exponentiated Pareto [175], and generalized Lomax [176].

16. Log-gamma1 $G$ (loggammag1) family: This family introduced by Amini et al. (2013). The cdf and pdf of the loggammag1 family are given by

$$F_{loggamma1}(x, \Theta) = \int_0^{-b \log(1 - G(x, \theta))} \frac{y^{a-1} e^{-y}}{\Gamma(a)} dy, \quad (1.31)$$

$$f_{loggamma1}(x, \Theta) = \frac{b^a}{\Gamma(a)} g(x, \theta) [-\log(1 - G(x, \theta))]^{a-1} (1 - G(x, \theta))^{b-1}, \quad (1.32)$$
where \( \Theta = (a, b, \theta^T) \) is the parameter space of the loggamma family, \( a > 0 \) and \( b > 0 \) are the new induced shape parameters, and \( \theta \) is the parameter space of distribution of \( G \). Amini et al. (2013) applied this family to model the earnings of workers from the US Bureau of Labor Statistics.

17. Log gamma type II \( \text{loggamma}_2 \) family: General form for the cdf and pdf of the \( \text{loggamma}_2 \) family due to Amini et al. (2013) are given by

\[
F_{\text{loggamma}_2}(x, \Theta) = 1 - \int_0^{-b \log(G(x, \theta))} \frac{y^{a-1}e^{-y}}{\Gamma(a)} \, dy, \tag{1.33}
\]

\[
f_{\text{loggamma}_2}(x, \Theta) = \frac{b^a}{\Gamma(a)} g(x, \theta) \left[ -\log(G(x, \theta)) \right]^{a-1} (G(x, \theta))^{b-1}, \tag{1.34}
\]

where \( \Theta = (a, b, \theta^T) \) is the parameter space of the \( \text{loggamma}_2 \) family, \( a > 0 \) and \( b > 0 \) are the new induced shape parameters, and \( \theta \) is the parameter space of distribution of \( G \). Amini et al. (2013) applied this family to model the earnings of workers from the US Bureau of Labor Statistics.

18. Modified beta \( \text{betagen} \) family: General form for the cdf and pdf of the \( \text{betagen} \) family are given by

\[
F_{\text{betagen}}(x, \Theta) = \int_0^d \frac{dG(x, \theta)}{1 - d G(x, \theta)} \, \frac{y^{a-1}(1-y)^{b-1}}{B(a, b)} \, dy, \tag{1.35}
\]

\[
f_{\text{betagen}}(x, \Theta) = \frac{d^a g(x, \theta) (G(x, \theta))^{a-1} (1 - G(x, \theta))^{b-1}}{B(a, b)[1 - (1 - d) G(x, \theta)]^{a+b}}, \tag{1.36}
\]

where \( \Theta = (a, b, d, \theta^T) \) is the parameter space of the \( \text{betagen} \) family, \( a > 0 \), \( b > 0 \), and \( d > 0 \) are the new induced shape parameters, and \( \theta \) is the parameter space of distribution of \( G \). The \( \text{betagen} \) family was used to model S&P/IFC (Standard & Poor’s/International Finance Corporation) global daily price indices in United States dollars for South Africa, see [136]. Also, a slightly different of this family is the \( \text{betagen} \)-geometric family that has been investigated when distributions of \( G \) are exponential ([3], [29]), Kumaraswamy [9], and Weibull ([55], [30]).

19. Marshal-Olkin \( \text{mog} \) family: Marshall and Olkin (1997) proposed a new approach for adding a parameter to a family of distributions and then applied it exponential and Weibull families. General form for the cdf and pdf of the \( \text{mog} \) family are given by

\[
F_{\text{mog}}(x, \Theta) = 1 - \frac{a \left( 1 - G(x, \theta) \right)}{[1 - (1 - a) (1 - G(x, \theta))]}, \tag{1.37}
\]

\[
f_{\text{mog}}(x, \Theta) = \frac{ag(x, \theta)}{[1 - (1 - a) (1 - G(x, \theta))]^2}, \tag{1.38}
\]

where \( \Theta = (a, \theta^T) \) is the parameter space of the \( \text{mog} \) family, \( a > 0 \) is the new induced shape parameter, and \( \theta \) is the parameter space of distribution of \( G \). Rubio and Mark (2012) studied the Marshall and Olkin’s (1997) approach as a skewing mechanism. Also, properties and applications of this family have been studied for many distributions of \( G \) including extended Burr type XII [11], generalized (Stacy) gamma ([26], [60]), log-logistic [61], exponential Pareto [67], Esscher transformed Laplace [73], extended Weibull [75], extended Lomax [76], power
log-normal [79], extended log-logistic [80], beta [91, 136], q-Weibull [92], extended uniform [93], Morgenstern Weibull [94], generalized asymmetric Laplace [98], Fréchet [99], Birnbaum-Saunders [103], inverse Weibull [132], Zipf [151], gamma [162], and discrete uniform [170].

20. Marshall-Olkin Kumaraswamy $G$ (mokum) family: General form for the cdf and pdf of the mokum family due to Roshini and Thobias (2017) are given by

$$F_{\text{mokum}}(x, \Theta) = 1 - \frac{d[1 - (G(x, \theta))^a]^b}{1 - (1 - d)[1 - (G(x, \theta))^a]^b},$$

$$f_{\text{mokum}}(x, \Theta) = \frac{abd g(x, \theta)(G(x, \theta))^{a-1}[1 - (G(x, \theta))^a]^{b-1}}{[1 - (1 - d)[1 - (G(x, \theta))^a]^b]^2},$$

where $\Theta = (a, b, d, \theta^T)^T$ is the parameter space of the mokum family, $a > 0$, $b > 0$, and $d > 0$ are the new induced shape parameters, and $\theta$ is the parameter space of distribution of $G$.

21. Odd log-logistic $G$ (ologlogg) family: Gauss et al. (2017) introduced the ologlogg family. General form for the cdf and pdf of this family are given by

$$F_{\text{ologlogg}}(x, \Theta) = \frac{a b d g(x, \theta)(G(x, \theta))^{a-1}[\bar{G}(x, \theta)]^{d-1}}{(G(x, \theta))^d - (\bar{G}(x, \theta))^d} \left\{ \left[ \frac{(G(x, \theta))^d}{(G(x, \theta))^d - (\bar{G}(x, \theta))^d} \right]^{a-1} \right\}^{b-1},$$

$$f_{\text{ologlogg}}(x, \Theta) = 1 - \left\{ 1 - \left[ \frac{(G(x, \theta))^d}{(G(x, \theta))^d - (\bar{G}(x, \theta))^d} \right]^{a-1} \right\}^{b-1},$$

where $\bar{G}(x, \theta) = 1 - G(x, \theta)$, $\Theta = (a, b, d, \theta^T)^T$ is the parameter space of the ologlogg family, $a > 0$, $b > 0$, and $d > 0$ are the new induced shape parameters, and $\theta$ is the parameter space of distribution of $G$.

22. Truncated-exponential skew-symmetric $G$ (texpsg) family: General form for the cdf and pdf of the texpsg family are given by

$$F_{\text{texpsg}}(x, \Theta) = \frac{1 - e^{-aG(x, \theta)}}{1 - e^{-a}},$$

$$f_{\text{texpsg}}(x, \Theta) = \frac{a}{1 - e^{-a}} g(x, \theta) e^{-aG(x, \theta)},$$

where $\Theta = (a, \theta^T)^T$ is the parameter space of the texpsg family, $a > 0$ is the new induced shape parameter, and $\theta$ is the parameter space of distribution of $G$. This family was used for modelling the annual maximum daily rainfall of 14 locations in west central Florida, see [135].

23. Weibull extended $G$ (weibullextg) family: The weibullextg is in fact the Weibull-X family introduced by Alzaatreh et al. (2013b). General form for the cdf and pdf of the weibullextg family are given by

$$F_{\text{weibullextg}}(x, \Theta) = 1 - \exp \left\{ -a \left( \frac{G(x, \theta)}{1 - G(x, \theta)} \right)^{\frac{1}{b}} \right\},$$

$$f_{\text{weibullextg}}(x, \Theta) = \frac{a g(x, \theta)}{b(1 - G(x, \theta))^2} \left( \frac{G(x, \theta)}{1 - G(x, \theta)} \right)^{\frac{1}{b} - 1} \exp \left\{ -a \left( \frac{G(x, \theta)}{1 - G(x, \theta)} \right)^{\frac{1}{b}} \right\},$$

where $\Theta = (a, \theta^T)^T$ is the parameter space of the weibullextg family.
where $\Theta = (a, b, d, \theta^T)^T$ is the parameter space of the \textit{weibullextg} family, $a > 0$, $b > 0$, and $d > 0$ are the new induced shape parameters, and $\theta$ is the parameter space of distribution of $G$. For more details about this family and its properties we refer readers to [17] and [18].

24. Weibull $G$ (\textit{weibullg}) family: The \textit{weibullg} is in fact the Weibull-X family of distributions introduced by Alzaatreh et al. (2013b). The cdf and pdf of the \textit{weibullg} family are given by

$$F_{\text{weibullg}}(x, \Theta) = 1 - e^{-\left( \frac{-\log(1-G(x, \theta))}{b} \right)^a},$$

(1.47)

$$f_{\text{weibullg}}(x, \Theta) = \frac{a}{b^a} \frac{g(x, \theta)}{1-G(x, \theta)} \left[ -\log (1 - G(x, \theta)) \right]^{a-1} e^{-\left( \frac{-\log(1-G(x, \theta))}{b} \right)^a},$$

(1.48)

where $\Theta = (a, b, \theta^T)^T$ is the parameter space of the \textit{weibullg} family, $a > 0$ and $b > 0$ are the new induced shape parameters, and $\theta$ is the parameter space of distribution of $G$. Some works have been devoted to investigate the properties and applications of \textit{weibullg} family, see [137].

2 MPS package: A guide to use in applications

Cheng and Amin (1979, 1983) and independently Ranneby (1984) developed the maximum product of spacings (MPS) estimators. The MPS approach can be considered as an alternative to the maximum likelihood (ML) method for estimating the parameters of a continuous univariate distribution. Cheng and Amin (1979) proved the asymptotic property of the MPS estimators and proved that MPS estimators are as efficient as the ML estimators when they break down. Coolen and Newby (1991) proved that the MPS estimators have invariance property. For applications in statistical inference, we refer reader to Shah and Gokhale (1993) (for Burr XII Distributions), Fitzgerald (1996) (for generalized Pareto and log-logistic), Rahman and Pearson (2002) (for two-parameter exponential), Rahman and Pearson (2003) (for two-parameter Pareto), Wong and Li (2006) (for extreme value), Rahman et al. (2007) (for two-parameter gamma), Abouammoh and Alshingiti (2009), and Singh et al. (2014) (for generalized inverse exponential, and Singh et al. (2016) (for generalized inverse exponential under progressive type II censoring scheme) among them. Suppose $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ are the ordered random observations of size $n$ drawn from a population with cdf $F(\cdot, \theta)$ with unknown parameter space $\theta$. The MPS approach works on the basis of maximizing the mean of log-spacing function

$$S(\theta) = \frac{1}{m} \sum_{i=1}^{m} \log \left[ F(x_{(i)}, \theta) - F(x_{(i-1)}, \theta) \right],$$

with respect to $\theta$ in which $m = n + 1$, $F(x_{(0)}, \theta) = 0$ and $F(x_{(m)}, \theta) = 1$ with $m = n + 1$. It can be shown that the Moran’s statistic ($S(\theta)$ when $\theta$ is known) has asymptotic normal distribution. Also, a chi-square approximation exists for small samples whose mean and variance approximately are $m(\log(m) + 0.57722) - 0.5 - 1/(12m)$ and $m(\pi^2/6 - 1) - 0.5 - 1/(6m)$, respectively, see [38]. Based on what mentioned above, the MPS approach is quite efficient in estimating the parameters of distributions with a shifted origin. So, hereafter we assume that all $24$ $G$ families introduced in the previous section have an extra location parameter called $\mu$, and hence the cdf and pdf of the $G$ distribution are generally shown by $G(x, \theta^*)$ and $g(x, \theta^*)$, respectively, where $\theta^* = (\theta, \mu)^T$ in which $\theta$ is the parameter space of $G$ distribution.

The MPS package has been developed for five tasks including: computing the cdf, computing the pdf, computing the quantile, generating random samples and estimating the parameters (using the
MPS approach) of 24 $G$ families introduced in the previous section. For each of these $G$ families, distribution of $G$ is freely chosen from 15 standard distributions whose probability density functions, i.e., $g(x, \theta^*)$ are given by the following.

- Birnbaum-Saunders ("birnbaum-saunders") with pdf
  \[
g(x, \theta^*) = \frac{\sqrt{\frac{x-\mu}{\beta}} + \sqrt{\frac{\beta}{x-\mu}}}{2}\alpha(x-\mu) \phi\left(\frac{\sqrt{\frac{x-\mu}{\beta}} - \sqrt{\frac{\beta}{x-\mu}}}{\alpha}\right),
  \]
  where $\phi(.)$ is the standard normal pdf, $x > \mu$ and $\theta^* = (\alpha, \beta, \mu)^T$ in which $\alpha > 0$, $\beta > 0$, $\mu \in \mathbb{R}$ are the shape, scale, and location parameters, respectively.

- Burr XII ("burrxii") with pdf
  \[
g(x, \theta^*) = \alpha \beta (x-\mu)^{\alpha-1} (1+(x-\mu)^{\beta})^{-\alpha-1},
  \]
  where $x > \mu$ and $\theta^* = (\alpha, \beta, \mu)^T$ in which $\alpha > 0$ and $\beta > 0$ are the first and second shape parameters and $\mu \in \mathbb{R}$ is location parameter.

- Chen ("chen") with pdf
  \[
g(x, \theta^*) = \alpha \beta (x-\mu)^{\alpha-1} \exp\{-(x-\mu)^\alpha\} \exp\{-\beta \left[ \exp((x-\mu)^\alpha) - 1 \right]\},
  \]
  where $x > \mu$ and $\theta^* = (\alpha, \beta, \mu)^T$ in which $\alpha > 0$ and $\beta > 0$ are the first and second shape parameters and $\mu \in \mathbb{R}$ is location parameter.

- Chi-square ("chisq") with pdf
  \[
g(x, \theta^*) = \Gamma^{-1}\left(\frac{\alpha}{2}\right) 2^{-\frac{\alpha}{2}} (x-\mu)^{\frac{\alpha}{2}-1} \exp\left(-\frac{x-\mu}{2}\right),
  \]
  where $x > \mu$ and $\theta^* = (\alpha, \mu)^T$ in which $\alpha > 0$ and $\mu \in \mathbb{R}$ are degrees of freedom and location parameter, respectively.

- Exponential ("exp") with pdf
  \[
g(x, \theta^*) = \alpha \exp(-\alpha(x-\mu)),
  \]
  where $x > \mu$ and $\theta^* = (\alpha, \mu)^T$ in which $\alpha > 0$ and $\mu \in \mathbb{R}$ are the rate and location parameters, respectively.

- F ("f") with pdf
  \[
g(x, \theta^*) = B^{-1}\left(\frac{\alpha}{2}, \frac{\beta}{2}\right) \left(\frac{\alpha}{\beta}\right)^{\frac{\alpha}{2}} (x-\mu)^{\frac{\alpha}{2}-1} \left(1 + \frac{x-\mu}{\beta}\right)^{-\left(\frac{\alpha+\beta}{2}\right)},
  \]
  where $x > \mu$ and $\theta^* = (\alpha, \beta, \mu)^T$ in which $\alpha > 0$ and $\beta > 0$ are the first and second degrees of freedom parameters and $\mu \in \mathbb{R}$ is location parameter.

- Frechet ("frechet") with pdf
  \[
g(x, \theta^*) = \frac{\alpha}{\beta} \left(\frac{x-\mu}{\beta}\right)^{-\alpha-1} \exp\left\{-\left(\frac{x-\mu}{\beta}\right)^{-\alpha}\right\},
  \]
  where $x > \mu$ and $\theta^* = (\alpha, \beta, \mu)^T$ in which $\alpha > 0$, $\beta > 0$, and $\mu \in \mathbb{R}$ are the shape, scale, and location parameters, respectively.
• Gamma ("gamma") with pdf
\[ g(x, \theta^*) = \left[ \beta^\alpha \Gamma(\alpha) \right]^{-1} (x - \mu)^{\alpha - 1} \exp\left( -\frac{x - \mu}{\beta} \right), \] (2.56)
where \( x > \mu \) and \( \theta^* = (\alpha, \beta, \mu)^T \) in which \( \alpha > 0, \beta > 0 \), and \( \mu \in \mathbb{R} \) are the shape, scale, and location parameters, respectively.

• Gompertz ("gompertz") with pdf
\[ g(x, \theta^*) = \alpha \exp \left\{ \beta (x - \mu) - \frac{\alpha}{\beta} \left[ \exp(\beta(x - \mu)) - 1 \right] \right\}, \] (2.57)
where \( x > 0 \) and \( \theta^* = (\alpha, \beta, \mu)^T \) in which \( \alpha > 0, \beta > 0 \), and \( \mu \in \mathbb{R} \) are the first, second, and location parameters, respectively.

• Linear failure rate ("lfr") with pdf
\[ g(x, \theta^*) = (\alpha + \beta(x - \mu)) \exp \left\{ -\alpha x - \frac{\beta(x - \mu)^2}{2} \right\}, \] (2.58)
where \( x > 0 \) and \( \theta^* = (\alpha, \beta, \mu)^T \) in which \( \alpha > 0, \beta > 0 \), and \( \mu \in \mathbb{R} \) are the first, second, and location parameters, respectively.

• Log-logistic ("log-logistic") with pdf
\[ g(x, \theta^*) = \frac{\alpha \beta}{\beta^\alpha \Gamma(2\alpha)} (x - \mu)^{\alpha - 1} \left[ \frac{(x - \mu)}{\beta} + 1 \right]^{-2}, \] (2.60)
where \( x > \mu \) and \( \theta^* = (\alpha, \beta, \mu)^T \) in which \( \alpha > 0, \beta > 0 \), and \( \mu \in \mathbb{R} \) are the shape, rate, and location parameters, respectively.

• Lomax ("lomax") with pdf
\[ g(x, \theta^*) = \alpha \beta \left( 1 + \beta(x - \mu) \right)^{-(\alpha + 1)}, \] (2.60)
where \( x > \mu \) and \( \theta^* = (\alpha, \beta, \mu)^T \) in which \( \alpha > 0, \beta > 0, \mu \in \mathbb{R} \) are the shape, rate, and location parameters, respectively.

• Rayleigh ("rayleigh") with pdf
\[ g(x, \theta^*) = \frac{2(x - \mu)}{\beta^2} \exp \left\{ -\left( \frac{x - \mu}{\beta} \right)^2 \right\}, \] (2.61)
where \( x > \mu \) and \( \theta^* = (\beta, \mu)^T \) in which \( \beta > 0 \) and \( \mu \in \mathbb{R} \) are the scale and location parameters, respectively.
Weibull ("weibull") with pdf

\[ g(x, \theta^*) = \frac{\alpha}{\beta} \left(\frac{x - \mu}{\beta}\right)^{\alpha-1} \exp\left\{-\left(\frac{x - \mu}{\beta}\right)^{\alpha}\right\}, \]

where \( x > \mu \) and \( \theta^* = (\alpha, \beta, \mu)^T \) in which \( \alpha > 0, \beta > 0, \mu \in \mathbb{R} \) are the shape, scale, and location parameters, respectively.

### 2.1 R command for computing the pdf of G families

In this subsections, we give the general format of commands to compute the pdf of 24 G families introduced in the Section 1 including `betaexpg`, `betag`, `expexppg`, `expg`, `expgg`, `expkumg`, `gammag`, `gammag1`, `gammag2`, `gbetag`, `gexppg`, `gmbetaexpg`, `gtransg`, `gxlogisticg`, `kumg`, `loggammag1`, `loggammag2`, `mbetag`, `mog`, `mokumg`, `ologlogg`, `texpsg`, `weibullextg`, and `weibullg`. The commands for computing the pdf are `dbetaexpg(...)`, `dbetag(...)`, `dexpexppg(...)`, `dexpg(...)`, `dexpgg(...)`, `dexpkumg(...)`, `dgammag(...)`, `dgammag1(...)`, `dgammag2(...)`, `dgbetag(...)`, `dgexppg(...)`, `dgmbetaexpg(...)`, `dgtransg(...)`, `dgxlogisticg(...)`, `dkumg(...)`, `dloggammag1(...)`, `dloggammag2(...)`, `dmbetag(...)`, `dmog(...)`, `dmokumg(...)`, `dologlogg(...)`, `dtexpsg(...)`, and `dweibullextg(...)`, respectively. In the following, for instance, general format for computing the pdf of `betaexpg` family and details about its arguments are given.

`dbetaexpg(mydata, g, param, location = TRUE, log=FALSE)`

Details for command arguments are:

- `mydata`: Vector of observations.
- `g`: The name of family’s pdf including: "birnbaum-saunders", "burrxi", "chisq", "chen", "exp", "f", "frechet", "gamma", "gompertz", "lfr", "log-normal", "log-logistic", "lomax", "rayleigh", and "weibull".
- `param`: The parameter space can be of the form \( \Theta = (a, \theta^* T)^T \), \( \Theta = (a, b, \theta^* T)^T \), or \( \Theta = (a, b, d, \theta^* T)^T \), where \( \theta^* \) is the parameter space of shifted G distribution as mentioned before. The general form for \( \theta^* \) can be \( \theta^* = (\alpha, \mu)^T \), \( \theta^* = (\alpha, \beta, \mu)^T \), or \( \theta^* = (\beta, \mu)^T \). As it is seen, the location parameter is placed in the last component of \( \theta^* \). The induced parameters \( a, b, \) and \( d \) are listed before \( \theta^* T \) in parameter space \( \Theta \).
- `location`: If FALSE, then the location parameter is absent.
- `log`: If TRUE, then the logarithm of pdf is returned.

### 2.2 R command for computing the cdf of G families

In this subsections, we give the general format of commands to compute the cdf of 24 G families introduced in the Section 1 including `betaexpg`, `betag`, `expexppg`, `expg`, `expgg`, `expkumg`, `gammag`, `gammag1`, `gammag2`, `gbetag`, `gexppg`, `gmbetaexpg`, `gtransg`, `gxlogisticg`, `kumg`, `loggammag1`, `loggammag2`, `mbetag`, `mog`, `mokumg`, `ologlogg`, `texpsg`, `weibullextg`, and `weibullg`. The commands for computing the cdf are `pbetaexpg(...)`, `pbetag(...)`, `pexpexppg(...)`, `pexpg(...)`, `pexpgg(...)`, `pexpkumg(...)`, `pgammag(...)`, `pgammag1(...)`, `pgammag2(...)`, `pgbetag(...)`, and `dweibullg(...)`, respectively.
pgexppg(...), pgmbetaexpg(...), pgtransg(...), pgxlogisticg(...), pkumg(...), ploggammag1(...), ploggammag2(...), pmetag(...), pmog(...), pmokumg(...), plogloggg(...), ptxpsg(...), pweibullextg(...), and pweibullg(...), respectively. In the following, for instance, general format for computing the cdf of betaexpg family and details about its arguments are given.

\[
pbetaexpg(mydata, g, \text{param}, \text{location} = \text{TRUE}, \text{log.p} = \text{FALSE}, \text{lower.tail} = \text{TRUE})
\]

Details for command arguments are:

- **mydata**: Vector of observations.
- **g**: The name of family’s pdf including: "birnbaum-saunders", "burrxii", "chisq", "chen", "exp", "f", "frechet", "gamma", "gompertz", "lfr", "log-normal", "log-logistic", "lomax", "rayleigh", and "weibull".
- **\text{param}**: The parameter space can be of the form \( \Theta = (a, \theta^* T)^T \), \( \Theta = (a, b, \theta^* T)^T \), or \( \Theta = (a, b, d, \theta^* T)^T \), where \( \theta^* \) is the parameter space of shifted \( G \) distribution as mentioned before. The general form for \( \theta^* \) can be \( \theta^* = (\alpha, \mu)^T \), \( \theta^* = (\alpha, \beta, \mu)^T \), or \( \theta^* = (\beta, \mu)^T \). As it is seen, the location parameter is placed in the last component of \( \theta^* \). The induced parameters \( a \), \( b \), and \( d \) are listed before \( \theta^* T \) in parameter space \( \Theta \).
- **location**: If FALSE, then the location parameter is absent.
- **\text{log.p}**: If TRUE, then the logarithm of cdf is returned and quantile is computed for \( \exp(-p) \).
- **\text{lower.tail}**: If FALSE, then \( 1 - \text{cdf} \) is returned and quantile is computed for \( 1 - p \).

### 2.3 R command for computing the quantile of \( G \) families

Here, we give the general format of commands to compute the quantiles of 24 \( G \) families introduced in the Section 1 including betaexpg, betag, expexppg, expg, expgg, expkumg, gammag, gammag1, gammag2, gbeta, gexppg, gmbetaexpg, gtransg, gxlogisticg, kumg, loggammag1, loggammag2, mbetag, mog, mokumg, ologlogg, texpsg, weibullextg, and weibullg. The commands for computing the quantile are qbetaexpg(...), qbetag(...), qexpexppg(...), qexpg(...), qexpg(...), qexpgg(...), qexpkumg(...), qgammag(...), qgammag1(...), qgammag2(...), qgbetag(...), qgexppg(...), qgmbetaexpg(...), qgtransg(...), qxlogisticg(...), qkumg(...), qloggammag1(...), qloggammag2(...), qmbetag(...), qmog(...), qmokumg(...), qologlogg(...), qtexpsg(...), qweibullextg(...), and qweibullg(...), respectively. In the following, for instance, general format for computing the quantile of betaexpg family and details about its arguments are given.

\[
qbetaexpg(p, g, \text{param}, \text{location} = \text{TRUE}, \text{log.p} = \text{FALSE}, \text{lower.tail} = \text{TRUE})
\]

Details for command arguments are:

- **\( p \)**: A vector of value(s) between 0 and 1 at which the quantile needs to be computed at those points.
- **\( g \)**: The name of family’s pdf including: "birnbaum-saunders", "burrxii", "chisq", "chen", "exp", "f", "frechet", "gamma", "gompertz", "lfr", "log-normal", "log-logistic", "lomax", "rayleigh", and "weibull".

13
In this subsection we give the general format of commands for estimating the parameters of each of 24 $G$ families introduced in the Section 1. These include betaexpg, betag, expexppg, expg, expgg, expkung, gammag, gammag1, gammag2, gbetag, gexppg, gmbetaexpg, gtransg, gxlogisticg, kumg, loggammag1, loggammag2, mbetag, mog, mokumg, ologlogg, texpsg, weibullextg, and weibullg. The commands for generating realizations are: rbetapexpg(...), rbetag(...), rexpg(...), rexppg(...), rxkung(...), rgammag(...), rgammag1(...), rgammag2(...), rgbetag(...), rgexppg(...), rgmbeaexpg(...), rgtransg(...), rgxlogisticg(...), rloggammag1(...), rloggammag2(...), rmbea(...), rmog(...), rmokumg(...), rologlogg(...), rtxpsg(...), rheulplx(...), and rweibullg(...), respectively. In the following, for instance, general format for simulating realizations from betaexpg family and details about its arguments are given.

\[
\text{rbetapexpg}(n, g, \text{param}, \text{location} = \text{TRUE})
\]

Details for command arguments are:

- **n**: The number of realizations needed for generation.
- **g**: The name of family's pdf including: "birnbaum-saunders", "burrxii", "chisq", "chen", "exp", "f", "frechet", "gamma", "gompertz", "lfr", "log-normal", "log-logistic", "lomax", "rayleigh", and "weibull".
- **param**: The parameter space can be of the form $\Theta = (a, \theta^* T)^T$, $\Theta = (a, b, \theta^* T)^T$, or $\Theta = (a, b, d, \theta^* T)^T$, where $\theta^*$ is the parameter space of shifted $G$ distribution as mentioned before. The general form for $\theta^*$ can be $\theta^* = (\alpha, \mu)^T$, $\theta^* = (\alpha, \beta, \mu)^T$, or $\theta^* = (\beta, \mu)^T$. As it is seen, the location parameter is placed in the last component of $\theta^*$. The induced parameters $a$, $b$, and $d$ are listed before $\theta^* T$ in parameter space $\Theta$.
- **location**: If FALSE, then the location parameter is absent.
- **log.p**: If TRUE, then the logarithm of cdf is returned and quantile is computed for $\exp(-p)$.
- **lower.tail**: If FALSE, then 1-cdf is returned and quantile is computed for $1 - p$.

### 2.4 R command for simulating random generation from $G$ families

Here, we give the general format of commands for simulating realizations from each of 24 $G$ families introduced in the Section 1. These include betaexpg, betag, expexppg, expg, expgg, expkung, gammag, gammag1, gammag2, gbetag, gexppg, gmbetaexpg, gtransg, gxlogisticg, kumg, loggammag1, loggammag2, mbetag, mog, mokumg, ologlogg, texpsg, weibullextg, and weibullg. The commands for generating realizations are: rbetapexpg(...), rbetag(...), rexpg(...), rexppg(...), rxkung(...), rgammag(...), rgammag1(...), rgammag2(...), rgbetag(...), rgexppg(...), rgmbeaexpg(...), rgtransg(...), rgxlogisticg(...), rloggammag1(...), rloggammag2(...), rmbea(...), rmog(...), rmokumg(...), rologlogg(...), rtxpsg(...), rheulplx(...), and rweibullg(...), respectively. In the following, for instance, general format for simulating realizations from betaexpg family and details about its arguments are given.

\[
\text{rbetapexpg}(n, g, \text{param}, \text{location} = \text{TRUE})
\]

Details for command arguments are:

- **n**: The number of realizations needed for generation.
- **g**: The name of family's pdf including: "birnbaum-saunders", "burrxii", "chisq", "chen", "exp", "f", "frechet", "gamma", "gompertz", "lfr", "log-normal", "log-logistic", "lomax", "rayleigh", and "weibull".
- **param**: The parameter space can be of the form $\Theta = (a, \theta^* T)^T$, $\Theta = (a, b, \theta^* T)^T$, or $\Theta = (a, b, d, \theta^* T)^T$, where $\theta^*$ is the parameter space of shifted $G$ distribution as mentioned before. The general form for $\theta^*$ can be $\theta^* = (\alpha, \mu)^T$, $\theta^* = (\alpha, \beta, \mu)^T$, or $\theta^* = (\beta, \mu)^T$. As it is seen, the location parameter is placed in the last component of $\theta^*$. The induced parameters $a$, $b$, and $d$ are listed before $\theta^* T$ in parameter space $\Theta$.
- **location**: If FALSE, then the location parameter is absent.

### 2.5 R command for estimating the parameters of the $G$ families

In this subsections we give the general format of commands for estimating the parameters of 24 $G$ families introduced in the Section 1. These include betaexpg, betag, expexppg, expg, expgg, expkung, gammag, gammag1, gammag2, gbetag, gexppg, gmbetaexpg, gtransg, gxlogisticg, kumg, loggammag1, loggammag2, mbetag, mog, mokumg, ologlogg, texpsg, weibullextg, and weibullg. The commands for generating realizations are: rbetapexpg(...), rbetag(...), rexpg(...), rexppg(...), rxkung(...), rgammag(...), rgammag1(...), rgammag2(...), rgbetag(...), rgexppg(...), rgmbeaexpg(...), rgtransg(...), rgxlogisticg(...), rloggammag1(...), rloggammag2(...), rmbea(...), rmog(...), rmokumg(...), rologlogg(...), rtxpsg(...), rheulplx(...), and rweibullg(...), respectively. In the following, for instance, general format for simulating realizations from betaexpg family and details about its arguments are given.

\[
\text{rbetapexpg}(n, g, \text{param}, \text{location} = \text{TRUE})
\]

Details for command arguments are:

- **n**: The number of realizations needed for generation.
- **g**: The name of family's pdf including: "birnbaum-saunders", "burrxii", "chisq", "chen", "exp", "f", "frechet", "gamma", "gompertz", "lfr", "log-normal", "log-logistic", "lomax", "rayleigh", and "weibull".
- **param**: The parameter space can be of the form $\Theta = (a, \theta^* T)^T$, $\Theta = (a, b, \theta^* T)^T$, or $\Theta = (a, b, d, \theta^* T)^T$, where $\theta^*$ is the parameter space of shifted $G$ distribution as mentioned before. The general form for $\theta^*$ can be $\theta^* = (\alpha, \mu)^T$, $\theta^* = (\alpha, \beta, \mu)^T$, or $\theta^* = (\beta, \mu)^T$. As it is seen, the location parameter is placed in the last component of $\theta^*$. The induced parameters $a$, $b$, and $d$ are listed before $\theta^* T$ in parameter space $\Theta$.
- **location**: If FALSE, then the location parameter is absent.
The commands for estimating the parameters are: `mpsbetapart(...)`, `mpsbetag(...)`, `mpsexpexp(...)`, `mpsexpg(...)`, `mpsexpkumg(...)`, `mpsgammag(...)`, `mpsgammag1(...)`, `mpsgammag2(...)`, `mpsgbetapart(...)`, `mpsgexpp(...)`, `mpsggamb(...)`, `mpsgtra(...)`, `mpsgxlogistic(...)`, `mpskumg(...)`, `mpslgammag1(...)`, `mpslgammag2(...)`, `mpsmbetapart(...)`, `mpsmog(...)`, `mpsmokumg(...)`, `mpsolnlog(...)`, `mpstexp(...)`, and `mpsw ebull(...)`, respectively. In the following, for instance, general format for estimating the parameters of `betaexpg` family and details about its arguments are given.

`mpsbetapart(mydata, g, location = TRUE, method, sig.level)`

Details for command arguments are:

- **mydata**: Vector of observations.
- **g**: The name of family’s pdf including: "birnbaum-saunders", "burrxi", "chisq", "chen", "exp", "f", "frechet", "gamma", "gompertz", "lfr", "log-normal", "log-logistic", "lomax", "rayleigh", and "weibull".
- **location**: If `FALSE`, then the location parameter is absent.
- **method**: The used method for maximizing the sum of log-spacing function. It will be "BFGS", "CG", "L-BFGS-B", "Nelder-Mead", or "SANN".
- **sig.level**: Significance level for the approximated chi-square goodness-of-fit test.

The details of output of `mpsbetapart(mydata, g, location = TRUE, method, sig.level)` are:

- Estimated parameter space $\hat{\Theta}$, represented by `$MPS$`.
- A sequence of goodness-of-fit statistics such as: Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), Hannan-Quinn information criterion (HQIC), Cramer-von Misses statistic (CM), Anderson Darling statistic (AD), log-likelihood statistic (log), and Moran’s statistic (Moran). These measures are represented by `$Measures$`.
- Kolmogorov-Smirnov test statistic and corresponding p-value, represented by `$KS$`.
- Chi-square test statistic, critical upper tail chi-square distribution, related p-value, represented by `$chi-square$`.
- Convergence status, represented by `$Convergence Status$`.

We note that the package is available from the Comprehensive R Archive Network (CRAN) at [https://cran.r-project.org/package=MPS](https://cran.r-project.org/package=MPS).

### 3 Examples and illustrations

Here, we provide some examples and real data applications to check the performance of the `MPS` package. Firstly, we compute the cdf and pdf of the `betaexpg` family when $G$ is three-parameter
gamma distribution. Secondly, we use the MPS package to estimate the parameters of weibullg, kumg, and mog families when these families are applied to the three sets of real data. Finally, the mechanism of random number generation will be checked for loggammag family when $G$ is supposed to be "birnbaum-saunders", "log-logistic", "lomax", and "weibull".

3.1 Computing the cdf and pdf

The following commands will produce the pdf plot of four members of betaexpg family when $G$ has distribution with pdf given in (2.56). The results are displayed in left-hand side of Figure 1.

```r
R> x <- seq(0, 20, 0.1)
R> y1 <- dbetaexpg(x, "gamma", c(1,1,1,2,1,0))
R> y2 <- dbetaexpg(x, "gamma", c(1,1,1,3,1,1))
R> y3 <- dbetaexpg(x, "gamma", c(1,1,1,4,1,2))
R> y4 <- dbetaexpg(x, "gamma", c(1,1,1,5,1,3))
R> xrange <- range(x)
R> yrange <- range(y1, y2, y3, y4)
R> plot(x, y1, type="l", xlab="x", ylab="pdf", xlim=xrange, ylim=yrange, lty=1)
R> lines(x, y2, col = "blue", lty=2)
R> lines(x, y3, col = "red", lty=3)
R> lines(x, y4, col = "green", lty=4)
R> cols<-c("black","blue","red","green")
R> legend(7.5, 0.4, legend=c("a=1, b=1, d=1, alpha=2, beta=1, mu=0",
+ "a=1, b=1, d=1, alpha=3, beta=1, mu=1",
+ "a=1, b=1, d=1, alpha=4, beta=1, mu=2",
+ "a=1, b=1, d=1, alpha=5, beta=1, mu=3"), col=cols, lty=1:4, lwd=2.5, cex=1)
```

The following commands will produce the cdf plot of four members of betaexpg family when $G$ has distribution with pdf given in (2.56). The results are displayed in right-hand side of Figure 1.

```r
R> x <- seq(0, 20, 0.1)
R> y1 <- pbetaexpg(x, "gamma", c(1,1,1,2,1,0))
R> y2 <- pbetaexpg(x, "gamma", c(1,1,1,3,1,1))
R> y3 <- pbetaexpg(x, "gamma", c(1,1,1,4,1,2))
R> y4 <- pbetaexpg(x, "gamma", c(1,1,1,5,1,3))
R> xrange <- range(x)
R> yrange <- range(y1, y2, y3, y4)
R> plot(x, y1, type="l", xlab="x", ylab="cdf", xlim=xrange, ylim=yrange, lty=1)
R> lines(x, y2, col = "blue", lty=2)
R> lines(x, y3, col = "red", lty=3)
R> lines(x, y4, col = "green", lty=4)
R> cols<-c("black","blue","red","green")
R> legend(7.5, 0.4, legend=c("a=1, b=1, d=1, alpha=2, beta=1, mu=0",
+ "a=1, b=1, d=1, alpha=3, beta=1, mu=1",
+ "a=1, b=1, d=1, alpha=4, beta=1, mu=2",
+ "a=1, b=1, d=1, alpha=5, beta=1, mu=3"), col=cols, lty=1:4, lwd=2.5, cex=1)
```
3.2 Estimating the parameters

The performance of the MPS package is demonstrated by analyzing three sets of real data. For the first set the usual ML estimators are not found while, the MPS counterparts exist and MPS package find them. Two other applications verify that presence of the location parameter in the fitted model makes it more appropriate.

As the first real data application, we focus on fatigue life in hours of 10 bearings that initially reported by McCool (1974) analyzed by several researchers. For modelling fatigue life of bearings (denoted here as bearing) data via \texttt{weibullg} family, we use the following commands.

```r
R>x<-c(152.7, 172.0, 172.5, 173.3, 193.0, 204.7, 216.5, 234.9, 262.6, 422.6)
R>mpsweibullg(x,"weibull",TRUE,"Nedler-Mead",0.05)
```

The output is

```
$MPS
[1] 0.9988519 0.9708349 0.8618143 83.4125577 147.1825435

$Measures
AIC   CAIC   BIC   HQIC   CM   AD  log Moran
116.7875 131.7875 118.3004 115.1278 0.06325825 0.3809677 -53.39375 31.37394

$KS
statistic  p-value
0.2042573 0.7266451

$'chi-square'
```

Figure 1: Plots of pdf and cdf for four members of \texttt{betaexpg} family when \( G \) is three-parameter gamma distribution with shape, scale, and location parameters as \texttt{alpha}, \texttt{beta}, and \texttt{mu}, respectively.
The estimated $\hat{\Theta}$ for weibullg family (with cdf given in (1.47) in which $G$ has a distribution with pdf given in (2.62)) is $\hat{\Theta} = (0.998851, 0.970834, 0.861814, 83.412557, 147.182543)^T$ where $\hat{a} = 0.998851$ and $\hat{b} = 0.970834$ are induced shape parameters and $\hat{\theta}^* = (0.861814, 83.412557, 147.182543)^T$ is the estimated parameter space of three-parameter Weibull distribution with $\hat{a} = 0.998851$, $\hat{b} = 0.970834$, and $\hat{\mu} = 147.182543$. Other features of the above output are given by the following. The Akaike information criterion (116.7875), consistent Akaike information criterion (131.7875), Bayesian information criterion (118.3004), Hannan-Quinn information criterion (115.1278), Cramer-von Misses statistic (0.06325825), Anderson Darling statistic (0.3809677), log-likelihood statistic (-53.39375), Kolmogorov-Smirnov test statistic (0.0245273), corresponding $p$-value (0.7266451), the chi-square test statistic (12.88606), critical upper tail chi-square distribution (18.30704), related $p$-value (0.230112), and convergence status ("Algorithm Converged"). For bearing data, as pointed out by Nagatsuka et al. (2013), the usual ML estimators break down. The estimated induced shape parameters are close to one ($\hat{a} = 0.998851$ and $\hat{b} = 0.970834$) that means a three-parameter Weibull distribution with shape, scale, and location parameters given, respectively, by $0.861814$, $83.412557$, and $147.182543$ is an appropriate model for bearing data. The latter can be shown using a likelihood ratio test.

As the second real data application, we consider the large recorded intensities (in Richter scale) of the earthquake in seismometer locations in western North America between 1940 and 1980. The related features were reported by [95]. Among the features, we focus on the 182 distances from the seismological measuring station to the epicenter of the earthquake (in km) as the variable of interest. We apply the kumg family with cdf given in (1.29) to the large recorded intensities of the earthquake (denoted here as earthquake) data in two cases including: 1- when $G$ follows a three-parameter Birnbaum-Saunders distribution with pdf given in (2.49), and 2- $G$ follows a Birnbaum-Saunders distribution. For this, we use the following commands.

```R
R>x<-c(7.5,8.8,8.9,9.4,9.7,9.7,10.5,10.5,12,12.2,12.8,14.6,14.9,17.6,23.9,25,2.9, + 3.2,7.6,17.8,10,10.8,19,21,13,22,29,31,5.8,12,12.1,20.5,20.5,25.3,35.9,36.1, + 36.3,38.5,41.4,43.6,44.4,46.1,47.1,47.7,49.2,53.1,4,10.1,11.1,17.7,22.5,26.5, + 29,30.9,37.8,48.3,62,50,16,62,1.2,1.6,9.1,3.7,5.3,7.4,17.9,19.2,23.4,30.8,9, + 10.8,15.7,16.7,20.8,28.5,33.1,40.3,32,30,31,16.1,63.6,6.6,9.3,13,17.3,105, + 112,123.5,23.5,26,0.5,0.6,1.3,1.4,2.6,3.8,4.5,1.6,2.6,8.7,5.7,6.8,4.8,5.8,5, + 10.6,12.6,12.7,12.9,14,15,16,17.7,18,22,22,23,23.2,29,32,32.7,36,43.5,49,60, + 64,105,122,141,200,48,130,147,187,197,203,211,17,19.6,20,2,21.1,21.9,66,87, + 23.4,24.6,25.7,28.6,37.4,46.7,56.9,60.7,61.4,62,64,82,88,91,12,24.2,148,42, + 85,107,109,156,224,293,359,370,25.4,32.9,92.2,45,145,300)
R> mpskumg(x,"birnbaum-saunders",TRUE,"BFGS",0.05)
$MPS
[1] 3.419681 35.382782 5.180796 222.382191 -3.728349
$Measures
  AIC  CAIC  BIC  HQIC   CM  AD  log Moran
```
When the location is present, we see $\hat{\Theta} = (3.419681, 5.180796, 222.382191, -3.728349)^T$ in which $\hat{a} = 3.419681$ and $\hat{b} = 5.180796$ are induced shape parameters. The estimated parameter space of three-parameter Birnbaum-Saunders distribution is $\hat{\theta}^* = (5.180796, 222.382191, -3.728349)^T$ with $\hat{\alpha} = 5.180796$, $\hat{\beta} = 222.382191$, and $\hat{\mu} = -3.728349$. Also, when the location is absent, we have $\hat{\Theta} = (2.2185211, 0.3324036, 1.7054161, 3.2352482)^T$ in which $\hat{a} = 2.2185211$ and $\hat{b} = 0.3324036$ are induced shape parameters and $\hat{\theta} = (1.7054161, 3.2352482)^T$ is estimated parameter space of two-parameter Birnbaum-Saunders distribution in which $\hat{\alpha} = 1.7054161$ and $\hat{\beta} = 3.2352482$. Based on above output, in the absence of location parameter, the Kolmogorov-Smirnov test statistic (0.1044829) and corresponding p-value (0.03760838) suggest that the kumg family is not appropriate model, while in the presence of location parameter, the kumg family is accepted. As Figure 2 shows, when the location parameter is present, the fitted cdf captures well the general shape of the empirical distribution function.

Steen and Stickler (1976) reported the beach pollution level (measured in number of coliform per 100 ml) over 20 days in South Wales. As the third application, we apply the mog family with
Figure 2: Histogram and fitted probability density functions for earthquake data are displayed in left hand-side subfigure. The empirical distribution function and fitted cumulative distribution functions for earthquake data are displayed is the right hand-side subfigure. In each subfigure two cases are considered. Those include the presence of the location parameter (location=TRUE) and absence of the location parameter (location=FALSE).

cdf given in (1.37) to this set of data in two cases: 1- when $G$ follows a two-parameter exponential distribution with pdf given in (2.53), and 2- $G$ follows an exponential distribution. For this, we use the following commands.

```r
R>x<-c(1364,2154,2236,2518,2527,2600,3009,3045,4109,5500,5800,7200,8400,8400,8900, + 11500,12700,15300,18300,20400)
R> mpsmog(x,"exp",TRUE,"Nedler-Mead",0.05)
$MPS
[1] 7.668608e-01 1.300979e-04 1.075007e+03

$Measures
 AIC  CAIC  BIC  HQIC  CM  AD  log Moran
 395.7932 397.2932 398.7804 396.3763 0.04093273 0.2959096 -194.8966 78.72329

$KS
 statistic  p-value
 0.1241431 0.9175147

$`chi-square`
 statistic  chi-value  p-value
 28.18183  31.41043  0.1051644

$`Convergence Status`
 [,1]
```
$MPS$
\[
\begin{array}{ll}
1.7785378951 & 0.0001715355 \\
\end{array}
\]

$Measures$
\[
\begin{array}{cccccccc}
AIC & CAIC & BIC & HQIC & CM & AD & \log & Moran \\
398.5125 & 399.2183 & 400.5039 & 398.9012 & 0.06436965 & 0.4949034 & -197.2562 & 80.00278 \\
\end{array}
\]

$KS$
\[
\begin{array}{ll}
statistic & p-value \\
0.1508478 & 0.7529763 \\
\end{array}
\]

$'chi-square'$
\[
\begin{array}{lll}
statistic & chi-value & p-value \\
29.54727 & 31.41043 & 0.07752922 \\
\end{array}
\]

$'Convergence Status'$
\[
\begin{array}{c}
[,1] \\
[1,] "Algorithm Converged" \\
\end{array}
\]

Clearly, when the location parameter is present, the mog family yields a better fit for beach pollution data. Plots of histogram fitted density functions, empirical distribution function, and fitted distribution functions are displayed in Figure 3. This fact that presence of the location parameter yields a better fit for the beach pollution data is verified by statistics given in $Measure$ part of related outputs.

3.3 Random realization mechanism accuracy

Here, we perform a simulation study to check the accuracy of the MPS package for generating realizations from 24 $G$ families. To save the space, we confine ourselves to the study the simulation mechanism from the loggammag1 family when $G$ is one of "birnbaum-saunders", "log-logistic", "lomax", and "weibull" distributions. For this aim, we follow the algorithm for each of four earlier mentioned $G$ distributions given by the following.

1. Generate a random sample of size $n$ from loggammag1 family using the routines provided in subsection 2.4
2. Compute the p-value of the one-sample Kolmogorov-Smirnov hypothesis test that whether the sample follows the loggammag1 family distribution or not,
3. Repeat steps 1 and 2 for 100 times for each $n$ that ranges from 5 to 100, giving p-values $p_1, p_2, \ldots, p_{100}$ say,

The result of simulations are depicted in Figure 4. It should be noted that, for implementing the algorithm, all five components of $\hat{\Theta} = (a, b, \alpha, \beta, \mu)^T$ are generated from uniform distribution over $(0.5, 5)$ in each iteration. It follows, from Figure 4, that almost all of depicted boxplots are above 0.05 for all $n = 1, 2, \ldots, 100$. 

![Image](image-url)
Figure 3: Histogram and fitted probability density functions for pollution data are displayed in left hand-side subfigure. The empirical distribution function and fitted cumulative distribution function for pollution data are displayed is the right hand-side subfigure. In each subfigure two cases are considered. Those include the presence of the location parameter (location=TRUE) and absence of the location parameter (location=FALSE).

4 Conclusion

We have introduced an R package, called MPS, for statistical modelling of 24 shifted $G$ families of distributions. The statistical modelling involves computing the probability density function, computing the cumulative distribution function, computing the quantile function, simulating random realizations, and estimating the parameters via the maximum product spacings (MPS) approach introduced by Cheng and Amin (1983). The performance of MPS package have been demonstrated through examples and real data applications. Adding a new shift (location) parameter to the 24 $G$ families of distributions made them more flexible and appropriate for modelling in practice. We have shown by the first real data application, when the maximum likelihood estimators break down, the MPS estimators exist and the MPS package gives them. The MPS package dose not depend on any other packages developed for R environment and uploaded in Comprehensive R Archive Network (CRAN) at https://cran.r-project.org/package=MPS. The MPS package can be updated for any new family of distributions in the future.

References

[1] Abdul-Moniem, I. and Abdel-Hameed, H. (2012). On exponentiated Lomax distribution, International Journal of Mathematical Archive, 3, 2144-2150.

[2] Abouammoh, A. M., and Alshingiti, A. M. (2009). Reliability estimation of generalized inverted exponential distribution, Journal of Statistical Computation and Simulation, 79 (11), 1301-1315.
Figure 4: Plots of p-value for testing the null hypothesis whether realizations come from the loggamma1 family with specified $G$ or not. The name of $G$ distribution is presented under each subfigure.

[3] Adamidis, K., Loukas, S. (1998). A lifetime distribution with decreasing failure rate, *Statistics and Probability Letters*, 39, 35-42.

[4] Afify, A. Z., Alizadeh, M., Yousof, H. M., Aryal, G., and Ahmad, M. (2016). The transmuted geometric-$G$ family of distributions: Theory and applications, *Pakistan Journal of Statistics*, 32 (2), 139-160.

[5] Afify, A. Z., Cordeiro, G. M., Yousof, H. M., Nofal, Z. M., and Alzaatreh, A. (2016). The Kumaraswamy transmuted-$G$ family of distributions: Properties and applications, *Journal of Data Science*, 14, 245-270.
Afify, A.Z., Yousof, H. M., and Nadarajah, S. (2017). The beta transmuted-$H$ family of distributions: properties and applications, *Statistics and its Inference*, 10, 505-520.

Ahmad, A., Ahmad, S., and Ahmed, A. (2014). Transmuted inverse Rayleigh distribution: a generalization of the inverse Rayleigh distribution, *Mathematical Theory and Modeling*, 4 (7), 90-98.

Akinsete, A., Famoye, F., and Lee, C. (2008). The beta Pareto distribution, *Statistics*, 42, 547-563.

Akinsete, A., Famoye, F., and Lee, C. (2014). The Kumaraswamy-geometric distribution, *Journal of Statistical distributions and Applications*, 1-17, doi:10.1186/s40488-014-0017-1.

Al-Babtain, A. A., Merovci, F., and Ibrahim Elbatal, I. (2015). The McDonald exponentiated gamma distribution and its statistical properties, *SpringerPlus*, 4 (1): 2, doi: 10.1186/2193-1801-4-2.

Al-Saiari, A. Y., Baharith, L. A., and Mousa, S. A. (2014). Marshall-Olkin extended Burr type XII distribution, *International Journal of Statistics and Probability*, 3(1), 78-84.

Alexander, C., Cordeiro, G. M., Ortega, E. M. M., and Sarabia, J. M. (2012). Generalized beta-generated distributions, *Computational Statistics & Data Analysis*, 56 (6), 1880-1897.

Alizadeh, M., Yousof, H. M., Afify, A. Z., Cordeiro, G. M., and Mansoor, M. (2018). The complementary generalized transmuted Poisson-G family of distributions, *Austrian Journal of Statistics*, 47, 51-71.

Alshawarbeh, E., Famoye, F., and Lee, C. (2013). Beta-Cauchy distribution: some properties and applications, *Journal of Statistical Theory and Applications*, 12 (4), 378-391.

Alzaatreh, A., Famoye, F., and Lee, C. (2012). Gamma-Pareto distribution and its applications, *Journal of Modern Applied Statistical Methods*, 11 (1), 78-94.

Alzaatreh, A. and Knight, K. (2013). On the gamma-half normal distribution and its applications, *Journal of Modern Applied Statistical Methods*, 12, 103-119.

Alzaatreh, A., Famoye, F., and Lee, C. (2013). Weibull-Pareto distribution and its applications, *Communications in Statistics-Theory and Methods*, 42 (9), 1673-1691.

Alzaatreh, A., Lee, C., and Famoye, F. (2013). A new method for generating families of continuous distributions, *Metron*, 71, 63-79.

Alzaatreh, A., Famoye, F., and Lee, C. (2014). The gamma-normal distribution: properties and applications, *Computational Statistics & Data Analysis*, 69, 67-80.

Amini, M., MirMostafaee, S., and Ahmadi, J. (2013). Log-gamma-generated families of distributions, *Statistics*, 48 (4), 913-932.

Aryal, G. R. and Tsokos, C. P. (2009). On the transmuted extreme value distribution with application, *Nonlinear Analysis: Theory, Methods and Applications*, 71 (12), e1401-e1407.

Aryal, G. R. and Tsokos, C. P. (2011). Transmuted Weibull distribution: A generalization of the Weibull probability distribution, *European Journal of Pure and Applied Mathematics*, 4 (2), 89-102.
[23] Aryal, G. R. (2013). Transmuted log-logistic distribution, *Journal of Statistics Applications and Probability, 2 (1)*, 11-20.

[24] Aryal, G. R. and Elbata, I. (2015). Kumaraswamy modified inverse Weibull distribution: theory and application, *Applied Mathematics and Information Sciences, 9*, 651-660.

[25] Bhati, D., Malik, M. A., and Vaman, H. J. (2015). Lindley-Exponential distribution: properties and applications, *Metron, 73*(3), 335-357.

[26] Barrigaa, G. D. C., Cordeiro, G. M., Dey, D. K., Canchod, V. G., Louzada, F., and Suzuki, A. K. (2018). The Marshall-Olkin generalized gamma distribution, *Communications for Statistical Applications and Methods, 25*, 245-261.

[27] Barreto-Souza, W., Santos, A. H. S., and Cordeiro, G. M. (2010). The beta generalized exponential distribution, *Journal of Statistical Computation and Simulation, 80*, 159-172.

[28] Barreto-Souza, W., Cordeiro, G. M., and Simas, A. B. (2011). Some results for beta Frechet distribution, *Communications in Statistics-Theory and Methods, 40*, 798-811.

[29] Bidram, H. (2012). The beta exponential-geometric distribution, *Communications in Statistics-Theory and Methods, 41*, 1606-1622.

[30] Bidram, H., Behboodian, J., and Towhidi, M. (2013). The beta Weibull geometric distribution, *Journal of Statistical Computation and Simulation, 83*, 52-67.

[31] Bourguignon, M., Silva, R. B., Zea, L. M., and Cordeiro, G. M. (2013). The Kumaraswamy Pareto distribution, *Journal of Statistical Theory and Applications, 12* (2), 129-144.

[32] Cakmakyapan, S. and Kadilar, G. A. (2014). A new customer lifetime duration distribution: The Kumaraswamy Lindley distribution, *International Journal of Trade, Economics and Finance, 5* (5), 441-444.

[33] Carrasco, J., M. F., Ortega, E. M. M., and Cordeiro, G. M. (2008). A generalized modified Weibull distribution for lifetime modeling, *Computational Statistics & Data Analysis, 53*, 450-462.

[34] Castellares, F. and Lemonte, A. (2015). A new generalized Weibull distribution generated by gamma random variables, *Journal of the Egyptian Mathematical Society, 23* (2), 382-390.

[35] Castellares, F., Santos, M. A. C., Montenegro, L., and Cordeiro, G. M. (2015). A gamma-generated Logistic distribution: Properties and inference, *American Journal of Mathematical and Management Sciences, 34*, 14-39.

[36] Cheng, R. C. H., Amin, N. A. K. (1979). Maximum product-of-spacings estimation with applications to the lognormal distribution, *University of Wales IST, Math Report, 79-1*.

[37] Cheng, R. C. H., Amin, N. A. K. (1983). Estimating parameters in continuous univariate distributions with a shifted origin, *Journal of the Royal Statistical Society. Series B, 45* (3), 394-403.

[38] Cheng, R. C. H. and Stephens, M. A. (1989). A goodness-of-fit test using Moran’s statistic with estimated parameters, *Biometrika, 76* (2), 385-392.
[39] Cintra, R. J., Rego, L. C., Cordeiro, G. M., and Nascimento, A. D. C. (2014). Beta generalized normal distribution with an application for SAR Image Processing, *Statistics*, 48, 279-294.

[40] Coolen, F., and Newby, M. J. (1991). A note on the use of the product of spacings in Bayesian inference, *Kwantitatieve Methoden*, 37, 19-32.

[41] Cordeiro, G. M., Ortega, E. M. M., and Nadarajah, S. (2010). The Kumaraswamy Weibull distribution with application to failure data, *Journal of the Franklin Institute*, 347, 1399-1429.

[42] Cordeiro, G. M. and Castro, M. (2011). A new family of generalized distributions, *Journal of Statistical Computation and Simulation*, 81 (7), 883-898.

[43] Cordeiro, G. M. and Lemonte, A. J. (2011a). The beta Birnbaum-Saunders distribution: an improved distribution for fatigue life modeling, *Computational Statistics & Data Analysis*, 55, 1445-1461.

[44] Cordeiro, G. M. and Lemonte, A. J. (2011b). The beta-half-Cauchy distribution, *Journal of Probability and Statistics*, 2011, 1-18. doi.org/10.1155/2011/904705.

[45] Cordeiro, G. M. and Lemonte, A. J. (2011c). The beta Laplace distribution, *Statistics and Probability Letters*, 81, 973-982.

[46] Cordeiro, G. M. and Brito, R. S. (2012). The beta power distribution, *Brazilian Journal of Probability and Statistics*, 26, 88-112.

[47] Cordeiro, G. M., Nobre, J. S., Pescim, R. R., and Ortega, E. M. M. (2012). The beta moyal: A useful skew distribution, *International Journal of Research and Reviews in Applied Sciences*, 10 (2), 1-22.

[48] Cordeiro, G. M., Nadarajah, S., and Ortega, E. M. M. (2012). The Kumaraswamy Gumbel distribution, *Statistical Methods and Applications*, 21, 139-168.

[49] Cordeiro, G. M., Ortega, E. M. M., and Silva, G. (2012). The beta extended Weibull family, *Journal of Probability and Statistical Science*, 10, 15-40.

[50] Cordeiro, G. M., Pescim, R. R., and Ortega, E. M. M. (2012). The Kumaraswamy generalized half-normal distribution for skewed positive data, *Journal of Data Science*, 10, 195-224.

[51] Cordeiro, G., Castellares, F., Montenegro, L. C., and Castro, M. (2013). The beta generalized gamma distribution, *Statistics*, 47, 888-900.

[52] Cordeiro, G. M., Cristino, C. T., Hashimoto, E. M., and Ortega, E. M. M. (2013). The beta generalized Rayleigh distribution, *Statistical Papers*, 54, 133-161.

[53] Cordeiro, G. M., Gomes, A. E., da Silva, C. Q., and Ortega, E. M. M. (2013). The beta exponentiated Weibull distribution, *Journal of Statistical Computation and Simulation*, 83, 114-138.

[54] Cordeiro, G. M., Ortega, E. M. M., and Cunha, D. C. (2013). The exponentiated generalized class of distributions, *Journal of Data Science*, 11, 1-27.

[55] Cordeiro, G. M., Silva, G. O., and Ortega, E. M. M. (2013). The beta Weibull geometric distribution, *Statistics*, 47, 817-834.
[56] Cordeiro, G. M., Ortega, E. M. M., and Silva, G. O. (2014). The Kumaraswamy modified Weibull distribution: Theory and applications, *Journal of Statistical Computation and Simulation*, 84, 1387-1411.

[57] Cordeiro, G. M. and Lemonte, A. J. (2014). The exponentiated generalized Birnbaum-Saunders distribution, *Applied Mathematics and Computation*, 247, 762-779.

[58] Cordeiro, G. M., Ortega, E. M. M., and Popovic, B. (2014). The gamma-linear failure rate distribution: Theory and applications, *Journal of Statistical Computation and Simulation*, 84, 2408-2426.

[59] Cordeiro, G. M., Ortega, E. M. M., and Popovic, B. (2015). The gamma-Lomax distribution, *Journal of Statistical Computation and Simulation*, 85, 305-319.

[60] de Pascoa, M. A. R., Ortega, E. M. M., Cordeiro, G. M. (2011). The Kumaraswamy generalized gamma distribution with application in survival analysis, *Statistical Methodology*, 8 (5), 411-433.

[61] de Santana, T. V. F., Ortega, E. M. M., Cordeiro, G. M., Silva, G. O. (2012). The Kumaraswamy-log-logistic distribution, *Journal of Statistical Theory and Applications*, 11 (3), 265-291.

[62] Domma, F. and Condino, F. (2013). The beta-Dagum distribution: Definition and properties, *Communications in Statistics-Theory and Methods*, 42, 4070-4090.

[63] Elbatal, I. (2013a). Kumaraswamy generalized linear failure rate distribution, *Indian Journal of Computational and Applied Mathematics*, 1, 61-78.

[64] Elbatal, I. (2013b). The Kumaraswamy exponentiated Pareto distribution, *Economic Quality Control*, 28, 1-8.

[65] Elbatal, I. and Elgarhy, M. (2013). Statistical properties of Kumaraswamy quasi Lindley distribution, *International Journal of Mathematics Trends and Technology*, 4, 237-246.

[66] Elbatal, I. and Muhammed, H. Z. (2014). Exponentiated generalized inverse Weibull distribution, *Applied Mathematical Sciences*, 8, 3997-4012.

[67] El-Said El-Nadi, K., Fatehy, L. M., and Ahmed, N. H. (2017). Marshall-Olkin exponential Pareto distribution with application on cancer stem cells, *American Journal of Theoretical and Applied Statistics*, 6 (5), 1-7.

[68] El-Sherpieny, E. A., and Ahmed, M. A. (2014). On the Kumaraswamy Kumaraswamy distribution, *International Journal of Basic and Applied Sciences*, 3, 372-381.

[69] Eugene, N., Lee, C., and Famoye, F. (2002). Beta-normal distribution and its applications, *Communications in Statistics-Theory and Methods*, 31, 497-512.

[70] Famoye, F., Lee, C., and Olumolade, O. (2005). The beta-Weibull distribution, *Journal of Statistical Theory and Applications*, 4, 121-136.

[71] Fitzgerald, D. L. (1996). Maximum product of spacings estimators for the generalized Pareto and log-logistic distributions, *Stochastic Hydrology and Hydraulics*, 10, 1-15.
[72] Gauss, M. C., Alizadeh, M., Ozel, G., Hosseini, B. Ortega, E. M. M., and Altunc, E. (2017). The generalized odd log-logistic family of distributions: Properties, regression models and applications, *Journal of Statistical Computation and Simulation*, 87(5), 908-932.

[73] George, D. and George, S. (2013). Marshall-Olkin Esscher transformed Laplace distribution and processes, *Brazilian Journal of Probability and Statistics*, 27, 162-184.

[74] Ghosh, I. (2014). The Kumaraswamy half-Cauchy distribution: properties and applications, *Journal of Statistical Theory and Applications*, 13, 122-134.

[75] Ghitany, M. E., Al-Hussaini, E. K., and Al-Jarallah, R. A. (2005). Marshall-Olkin extended Weibull distribution and its application to censored data, *Journal of Applied Statistics*, 32, 1025-1034.

[76] Ghitany, M. E., Al-Awadhi, F. A., and Alkhalfan, L. A. (2007). Marshall-Olkin extended Lomax distribution and its application to censored data, *Communications in Statistics-Theory and Methods*, 36, 1855-1866.

[77] Gomes, A. E., Silva, C. Q., Cordeiro, G. M., and Ortega, E. M. M. (2013). The beta Burr III model for lifetime data, *Brazilian Journal of Probability and Statistics*, 27, 502-543.

[78] Gomes, A. E., Silva, C. Q., Cordeiro, G. M., and Ortega, E. M. M. (2014). A new lifetime model: The Kumaraswamy generalized Rayleigh distribution, *Journal of Statistical Computation and Simulation*, 84, 290-309.

[79] Gui, W. (2013a). A Marshall-Olkin power log-normal distribution and its applications to survival data, *International Journal of Statistics and Probability*, 2 (1), 63-72.

[80] Gui, W. (2013b). Marshall-Olkin extended log-logistic distribution and its application in minification processes, *Applied Mathematical Sciences*, 7, 3947-3961.

[81] Gupta, R. C., Gupta, P. L., and Gupta, R. D. (1998). Modeling failure time data by Lehman alternatives, *Communications in Statistics-Theory and Methods*, 27, 887-904.

[82] Gupta, R. D. and Kundu, D. (1999). Generalized exponential distributions, *Australian and New Zealand Journal of Statistics*, 41(2), 173-188.

[83] Hady, A. and Ebraheim, N. (2014). Exponentiated transmuted Weibull distribution: A generalization of the Weibull distribution, *International Journal of Mathematical, Computational, Physical and Quantum Engineering*, 8, doi:10.12988/ams.2014.44267.

[84] Hanook, S., Shahbaz, M. Q., Mohsin, M., and Kibria, B. M. (2013). A note on beta inverse-Weibull distribution, *Communications in Statistics-Theory and Methods*, 42, 320-335.

[85] Idowu, B. N. and Ikegwu, E. M. (2013). The beta weighted Weibull distribution: Some properties and application to bladder cancer data, *Journal of Applied and Computational Mathematics*, 2 (5), 1-6.

[86] Huang, S. and Oluyede, B. O., (2014). Exponentiated Kumaraswamy-Dagum distribution with applications to income and lifetime data, *Journal of Statistical Distributions and Applications*, 1:8, doi.org/10.1186/2195-5832-1-8.

[87] Jafari, A. A., Tahmasebi, S., and Alizadeh, M. (2014). The beta Gompertz distribution, *Revista Colombiana de Estadística*, 37, 139-156.
[88] Jafari, A. A. and Mahmoudi, E. (2015). Beta-linear failure rate distribution and its applications, *Journal of Iranian Statistical Society*, 14 (1), 89-105.

[89] Jones, M. C. (2004). Families of distributions arising from distributions of order statistics, *TEST*, 13, 1-43.

[90] Jones, M. C. (2009). Kumaraswamy’s distribution: a beta-type distribution with some tractability advantages, *Statistical Methodology*, 6, 70-81.

[91] Jose, K. K., Joseph, A., and Ristic, M. M. (2009). A Marshall-Olkin beta distribution and its applications, *Journal of Probability and Statistical Science*, 7, 173-186.

[92] Jose, K. K., Naik, S. R., and Ristic, M. M. (2010). Marshall-Olkin q-Weibull distribution and max/min processes, *Statistical Papers*, 51, 837-851.

[93] Jose, K. K. and Krishna, E. (2011). Marshall-Olkin extended uniform distribution, *ProbStat Forum*, 4, 78-88.

[94] Jose, K. K. and Sebastian, R. (2013). Marshall-Olkin Morgenstern Weibull distribution: Generalizations and applications, *Economic Quality Control*, 28, 105-116.

[95] Joyner, W. B. and Boore, D. M. (1981). Peak horizontal acceleration and velocity from strong-motion records including records from the 1979 Imperial Valley, California, earthquake, *Bulletin of the Seismological Society of America*, 71, 2011-2038.

[96] Kong, L., Carl, L., and Sepanski, J. H. (2007). On the properties of beta gamma distribution, *Journal of Modern Applied Statistical Methods*, 6 (1), 187-211.

[97] Kozubowski, T. J. and Nadarajah, S. (2008). The beta Laplace distribution, *Journal of Computational Analysis and Applications*, 10, 305-318.

[98] Krishna, E., and Jose, K. K. (2011). Marshall-Olkin generalized asymmetric Laplace distributions and processes, *Statistica*, 71, 453-467.

[99] Krishna, E., Jose, K. K., Alice, T., and Ristic, M. M. (2013). Marshall-Olkin Frechet distribution, *Communications in Statistics-Theoretical and Methods*, 42, 4091-4107.

[100] Kumaraswamy, P. (1980). A generalized probability density for double-bounded random processes, *Journal of Hydrology*, 46, 79-88.

[101] Kus, C. (2007). A new lifetime distribution, *Computational Statistics & Data Analysis*, 51, 4497-4509.

[102] Leao, J., Saulo, H., Bourguignon, M., Cintra, R., Rego, L., and Cordeiro, G. M. (2013). On some properties of the beta inverse Rayleigh distribution, *Chilean Journal of Statistics*, 4 (2), 111-131.

[103] Lemonte, A. J. (2013). A new extension of the Birnbaum-Saunders distribution, *Brazilian Journal of Probability and Statistics*, 27, 133-149.

[104] Lemonte, A. J., Barreto-Souza, W., and Cordeiro, G. M. (2013). The exponentiated Kumaraswamy distribution and its log-transform, *Brazilian Journal of Probability and Statistics*, 27, 31-53.
[105] Lee, C., Famoye, F., and Olumolade, O. (2007). Beta-Weibull distribution some properties and applications to censored data, *Journal of Modern Applied Statistical Methods*, 6, 173-186.

[106] Mahmoudi, E. (2011). The beta generalized Pareto distribution with application to lifetime data, *Mathematics and Computers in Simulation*, 81, 2414-2430.

[107] Marciano, F., Nascimento, A., Santos-Neto, M., and Cordeiro, G. M. (2012). The Mc-gamma distribution and its statistical properties: an application to reliability data, *International Journal of Statistics and Probability*, 1 (1), 53-71.

[108] Marshall, A. W. and Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families, *Biometrika*, 84, 641-652.

[109] McCool, J. I. (1974). Inferential techniques for Weibull populations, *Technical Report TR 74-0180, Wright Patterson Air Force Base, Ohio*.

[110] McDonald, J. B., 1984. Some generalized functions for the size distribution of income, *Econometrica*, 52, 647-663.

[111] Mead, M. E. (2014). An extended Pareto distribution, *Pakistan Journal of Statistics and Operation Research*, 10 (3). doi:10.18187/pjsor.v10i3.766.

[112] Merovci, F. (2013). Transmuted Lindley distribution, *International Journal of Open Problems in Computer Science & Mathematics*, 6, 63-72.

[113] Merovci, F. (2013b). Transmuted Rayleigh distribution, *Austrian Journal of Statistics*, 42 (1), 21-31.

[114] Merovci, F. (2014). Transmuted generalized Rayleigh distribution, *Journal of Statistics Applications and Probability*, 3 (1), 9-20.

[115] Merovci, F. and Sharma, V. K. (2014). The beta Lindley distribution: Properties and applications, *Journal of Applied Mathematics*, doi:10.1155/2014/198951.

[116] Merovcia, F. and Puka, L. (2014). Transmuted Pareto distribution, *ProbStat Forum*, 07, 1-11

[117] Merovci, F. and Elbatal, I. (2014). Transmuted Lindley-geometric distribution and its applications, *Journal of Statistics Applications and Probability*, 3 (1), 77-91.

[118] Merovci, F., Alizadeh, M., Yousof, H. M., and Hamedani, G. G. (2017). The exponentiated transmuted-G family of distributions: Theory and applications, *Communications in Statistics-Theory and Methods*, 46 (21), 10800-10822.

[119] Montenegro, L. C. and Cordeiro, G. M. (2013). The beta lognormal distribution, *Journal of Statistical Computation and Simulation*, 83, 203-228.

[120] Morais, A. L., Cordeiro, G. M., and Cysneiros, A. H. M. A. (2013). The beta generalized logistic distribution, *Brazilian Journal of Probability and Statistics*, 27, 185-200.

[121] Mudholkar, G. S., Kollia, G. D., Lin, C. T., and Patel, K. R. (1991). A graphical procedure for comparing goodness-of-fit tests, *Journal of Royal Statistical Society, B*, 53, 221-232.
[122] Mudholkar, G. S. and Srivastava, D. K. (1993). Exponentiated Weibull family for analyzing bathtub failure rate data, IEEE Transaction on Reliability, 42, 299-302.

[123] Mudholkar, G. S., Srivastava, D. K., and Freimer, M. (1995). Exponentiated Weibull family: A reanalysis of the bus motor failure data, Technometrics, 37, 436-445.

[124] Mudholkar, G. S. and Hutson, A. D. (1996). The exponentiated Weibull family: some properties and a flood data application, Communications in Statistics-Theory and Methods, 25, 3059-3083.

[125] Nadarajah, S. and Kotz, S. (2004). The beta Gumbel distribution, Mathematical Problems in Engineering, 10, 323-332. doi.org/10.1155/S1024123X04403068

[126] Nadarajah, S. and Gupta, A. K. (2004). The beta Fréchet distribution, Far East Journal of Theoretical Statistics, 14 (1), 15-24

[127] Nadarajah, S., Gupta, A. K., (2005). On the moments of the exponentiated Weibull distribution, Communications in Statistics Theory and Methods, 35, 253-256.

[128] Nadarajah, S. and Kotz, S. (2006a). The exponentiated type distributions, Acta Applicandae Mathematica, 92, 97-111.

[129] Nadarajah, S. (2006). The exponentiated Gumbel distribution with climate application, Environmetrics, 17, 13-23

[130] Nadarajah, S. and Kotz, S. (2006b). The beta exponential distribution, Reliability Engineering and System Safety, 91, 689-697.

[131] Nadarajah, S. and Gupta, A. K. (2007). The exponentiated gamma distribution with application to drought data, Calcutta Statistical Association Bulletin, 59, 29-54.

[132] Nadarajah, S. and Cancho, V. G., Ortega, E. M. M. (2013). The geometric exponential Poisson distribution, Statistical Methods and Applications, 22, 355-380.

[133] Nadarajah, S. and Eljabri, S. (2013). The Kumaraswamy GP distribution, Journal of Data Science, 11, 739-766.

[134] Ristic, M. M. and Nadarajah, S. (2014). A new lifetime distribution, Journal of Statistical Computation and Simulation, 84 (1), 135-150.

[135] Nadarajah, S., Nassiri, V., and Mohammadpour, A. (2014). Truncated-exponential skew-symmetric distributions, Statistics, 48 (4), 872-895.

[136] Nadarajah, S., Teimouri, M., and Shih, S. H. (2014). Modified beta distributions, Sankhya B, 76 (1), 19-48.

[137] Nadarajah, S. and Rocha, R. (2016). Newdistns: An R Package for new families of distributions, Journal of Statistical Software, 69 (10), doi: 10.18637/jss.v069.i10.

[138] Nagatsuka, H., Toshinari, K., and Balakrishnan, N. (2013). A consistent method of estimation for the three-parameter Weibull distribution, Computational Statistics and Data Analysis, 58, 210-226.
[139] Nassar, M. M. and Nada, N. K. (2011). The beta generalized Pareto distribution, *Journal of Statistics: Advances in Theory and Applications*, 6, 1-17.

[140] Nofal, Z. M., Afify, A. Z., Yousof, H. M., and Cordeiro, G. M. (2016). The generalized transmuted-G family of distributions, *Communications in Statistics-Theory and Methods*, DOI:10.1080/03610926.2015.1078478.

[141] Oguntunde, P., Babatunde, O., and Ogunmola, A. (2014). Theoretical analysis of the Kumaraswamy-inverse exponential distribution, *International Journal of Statistics and Applications*, 4, 113-116.

[142] Okasha, H. M., El-Baz, A. H., Tarabia, A. M. K., and Basheer, A. M. (2017). Extended inverse Weibull distribution with reliability application, *Journal of the Egyptian Mathematical Society*, 25 (3), 343-349.

[143] Oluyede, B. O., Huang, S., and Pararai, M. (2014). A new class of generalized Dagum distribution with applications to income and lifetime data, *Journal of Statistical and Econometric Methods*, 3, 125-151.

[144] Oluyede, B. O. and Yang, T. (2015). A new class of generalized Lindley distributions with applications, *Journal of Statistical Computation and Simulation*, 85 (10), 2072-2100.

[145] Pal, M. and Tiensuwan, M. (2014). The beta transmuted Weibull distribution, *Austrian Journal of Statistics*, 43, 133-149.

[146] Paranaiba, P. F., Ortega, E. M. M., Cordeiro, G. M., and Pescim, R. R. (2011). The beta Burr XII distribution with application to lifetime data, *Computational Statistics & Data Analysis*, 55, 1118-1136.

[147] Paranaiba, P. F., Ortega, E. M. M., Cordeiro, G. M., and Pascoa, M. A. R. (2013). The Kumaraswamy Burr XII distribution: Theory and practice, *Journal of Statistical Computation and Simulation*, 83, 2117-2143.

[148] Pararai, M., Warahena-Liyanage, G., and Oluyede, B. O. (2014). A new class of generalized inverse Weibull distribution with applications, *Journal of Applied Mathematics and Bioinformatics*, 4, 17-35.

[149] Pascoa, M., Ortega, E. M. M., and Cordeiro, G. M. (2011). The Kumaraswamy generalized gamma distribution with application in survival analysis, *Statistical Methodology*, 8, 411-433.

[150] Percontini, A., Blas, B., and Cordeiro, G. M. (2013). The beta Weibull Poisson distribution, *Chilean Journal of Statistics*, 4, 3-26.

[151] Pérez-Casany, M. and Casellas, A. (2014). Marshall-Olkin extended Zipf distribution. arXiv:1304.4540 [stat.AP].

[152] Pescim, R. R., Demetrio, C. G. B., Cordeiro, G. M., Ortega, E. M. M., and Urbano, M. R. (2007). The beta generalized half-normal distribution, *Computational Statistics & Data Analysis*, 54, 945-957.

[153] Pinho, L. G. B., Cordeiro, G. M., and Nobre, J. S. (2012). The gamma-exponentiated Weibull distribution, *Journal of Statistical Theory and Applications*, 11, 379-395.
[154] R Core Team (2016). R: A Language and Environment for Statistical Computing. R Foundation for Statistical Computing, Vienna, Austria. URL https://www.R-project.org/

[155] Rahman, M. and Pearson, L. M. (2002). Estimation in two-parameter exponential distributions, Journal of Statistical Computation and Simulation, 70 (4), 371-386.

[156] Rahman, M. and Pearson, L. M. (2003). A note on estimating parameters in two-parameter Pareto distributions, International Journal of Mathematical Education in Science and Technology, 34 (2), 298-306.

[157] Rahman, M., Pearson, L. M., and Martinovic, U. R. (2007). Method of product of spacings in the two-parameter gamma distribution, Journal of Statistical Research Bangladesh, 41, 51-58.

[158] Rajab, M., Aleem, M., Nawaz, T., and Daniya, M. (2013). On five parameter beta Lomax distribution, Journal of Statistics, 20, 102-118.

[159] Ramos, M. W. A., Cordeiro, G. M., Marinho, P. R. D., Dias, C. R. B., and Hamedani, G. G. (2013). The Zografos-Balakrishnan log-logistic distribution: Properties and applications, Journal of Statistical Theory and Applications, 12, 225-244.

[160] Ranneby, B. O. (1984). The maximum spacing method. An estimation method related to the maximum likelihood method, Scandinavian Journal of Statistics, 11 (2), 93-112.

[161] Ristic, M. M. and Balakrishnan, N. (2012). The gamma exponentiated exponential distribution, Journal of Statistical Computation and Simulation, 82, 1191-1206.

[162] Ristic, M. M., Jose, K. K., and Ancy, J. (2007). A Marshall-Olkin gamma distribution and minification process, STARS: International Journal (Sciences), 1, 11-21.

[163] Ristic, M. M. and Nadarajah, S. (2014). A new lifetime distribution, Journal of Statistical Computation and Simulation, 84 (1), 135-150.

[164] Rodrigues, J. A. and Silva, A. P. C. M. (2015). The exponentiated Kumaraswamy-exponential distribution, British Journal of Applied Science and Technology, 10 (5), 1-12, 10.9734/BJAST/2015/16935.

[165] Rodrigues, J. A., Silva, A. P. C. M., and Hamedani, G. G. (2016). The exponentiated Kumaraswamy inverse Weibull distribution with application in survival analysis, Journal of Statistical Theory and Applications, 15 (1), 8-24.

[166] Roges, D. L., Gusmao, F. R. S., and Diniz, C. A. R. (2014). The Kumaraswamy inverse Rayleigh distribution, Unpublished manuscript.

[167] Roshini, G. and Thobias, S. (2017). Marshall-Olkin Kumaraswamy Distribution, International Mathematical Forum, 12 (2), 47-69.

[168] Rubio, F. J. and Mark, F. J. (2012). On the Marshall-Olkin transformation as a skewing mechanism, Computational Statistics & Data Analysis, 56 (7), 2251-2257.

[169] Salem, H. M. (2014). The exponentiated Lomax distribution: Different estimation methods, American Journal of Applied Mathematics and Statistics, 2, 364-368.
[170] Sandhya, E. and Prasanth, C. B. (2014). Marshall-Olkin discrete uniform distribution, *Journal of Probability*, 1-10, doi:10.1155/2014/979312.

[171] Santana, T. V. F., Ortega, E. M. M., Cordeiro, G. M., and Silva, G. O. (2012). The Kumaraswamy-log-logistic distribution, *Journal of Statistical Theory and Applications*, 11, 265-291.

[172] Saulo, H., Leao, J., and Bourguignon, M. (2012). The Kumaraswamy Birnbaum-Saunders distribution, *Journal of Statistical Theory and Practice*, 6, 745-759.

[173] Shah, A. and Gokhale, D. V., (1993). On maximum product of spacings estimation for Burr XII distributions, *Communications in Statistics-Simulation and Computation*, 22 (3), 615-641.

[174] Shahbaz, M. Q., Shahbaz, S., and Butt, N. S. (2012). The Kumaraswamy inverse Weibull distribution, *Pakistan Journal of Statistics and Operation Research*, 8, 479-489.

[175] Shams, T. M. (2013a). The Kumaraswamy generalized exponentiated Pareto distribution, *International Journal of Statistics and Applications*, 5, 92-99.

[176] Shams, T. M. (2013b). The Kumaraswamy generalized Lomax distribution, *Middle-East Journal of Scientific Research*, 17, 641-646.

[177] Shao, Y. (2001). Consistency of the maximum product of spacings method and estimation of a unimodal distribution, *Statistica Sinica*, 11, 1125-1140.

[178] Shawky, A. and Abu-Zinadah, H. (2009). Exponentiated Pareto distribution: Different methods of estimations, *International Journal of Contemporary Mathematical Sciences*, 4, 677-693.

[179] Shittu, O. I. and Adepoju, K. A. (2013). On the beta-Nakagami distribution, *Progress in Applied Mathematics*, 5, 49-58.

[180] Silva, G. O., Ortega, E. M. M., and Cordeiro, G. M. (2010). The beta modified Weibull distribution, *Lifetime Data Analysis*, 16, 409-430.

[181] Silva, R. V., Andrade, T. A. N., Maciel, D. B. M., Campos, R. P. S., and Cordeiro, G. M. (2013). A new lifetime model: The gamma extended Frechet distribution, *Journal of Statistical Theory and Applications*, 12, 39-54.

[182] Singh, U., Singh, S. K., and Singh, R. K. (2014). Comparative study of traditional estimation method and maximum product spacing method in generalized inverted exponential distribution, *Journal of Statistics Applications and Probability*, 3(2), 153-169.

[183] Singh, R. K., Singh, S. K., and Singh, U. (2016). Maximum product spacings method for the estimation of parameters of generalized inverted exponential distribution under progressive type II censoring, *Journal of Statistics and Management Systems*, 19 (2), 219-245.

[184] Singla, N., Jain, K., and Sharma, S. K. (2012). The beta generalized Weibull distribution: Properties and applications, *Reliability Engineering and System Safety*, 102, 5-15.

[185] Steen, P. J. and Stickler, D. J. (1976). A Sewage Pollution Study of Beaches from Cardiff to Ogmore. *Report January 1976, Cardiff: Department of Applied Biology, UWIST*. 

34
[186] Tahir, M. H., Mansoor, M., Zubair, M., Hamedani, G. G. (2014). McDonald log-logistic distribution with an application to breast cancer data, *Journal of Statistical Theory and Applications*, 13, 65-82.

[187] Torabi, H. and Montazeri, N. H. (2012). The gamma-uniform distribution and its applications, *Kybernetika*, 48, 16-30.

[188] Yousof, H. M., Afify, A. Z., Alizadeh, M., Butt, N. S., Hamedani, G. G., and Ali, M. M. (2015). The transmuted exponentiated generalized-G family of distributions, *Pakistan Journal of Statistics and Operation Research*, 11, 441-464.

[189] Wong, T. S. T. and Li, W. K. (2006). A note on the estimation of extreme value distributions using maximum product of spacings, *IMS Lecture Notes Monograph Series*, 52, 272-283.

[190] Zea, L. M., Silva, R. B., Bourguignon, M., Santos, A. M., and Cordeiro, G. M. (2012). The beta exponentiated Pareto distribution with application to bladder cancer susceptibility, *International Journal of Statistics and Probability*, 1 (2), 8-19.

[191] Zografos, K. and Balakrishnan, N. (2009). On families of beta- and generalized gamma-generated distributions and associated inference, *Statistical Methodology*, 6, 344-362.