Coulomb problem in non-commutative quantum mechanics - Exact solution

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Abstract

We investigate consequences of space non-commutativity in quantum mechanics of the hydrogen atom. We introduce rotationally invariant noncommutative space $\hat{\mathbb{R}}^3_0$ - an analog of the hydrogen atom (H-atom) configuration space $\mathbb{R}^3_0 = \mathbb{R}^3 \setminus \{0\}$. The space $\hat{\mathbb{R}}^3_0$ is generated by noncommutative coordinates realized as operators in an auxiliary (Fock) space $\mathcal{F}$. We introduce the Hilbert space $\hat{\mathcal{H}}$ of wave functions $\hat{\psi}$ formed by properly weighted Hilbert-Schmidt operators in $\mathcal{F}$. Finally, we define an analog of the $H$-atom Hamiltonian in $\hat{\mathbb{R}}^3_0$ and explicitly determine the bound state energies $E^\lambda_n$ and the corresponding eigenstates $\hat{\psi}^\lambda_{n,j,m}$. The Coulomb scattering problem in $\hat{\mathbb{R}}^3_0$ is under study.
1 Introduction

Basic ideas of non-commutative geometry have been developed in [1] and, in a form of matrix geometry, in [2]. The main applications have been considered

- in the area of quantum quantum field theory in order to understand, or even to remove, UV singularities, and
- eventually to formulate a proper base for the quantum gravity.

The analysis performed in [3] led to the conclusion that quantum vacuum fluctuations and Einstein gravity could create (micro)black holes which prevent localization of space-time points. Mathematically this requires non-commutative coordinates $x^\mu$ in space-time satisfying specific uncertainty relations. The simplest set of operators $\hat{x}^\mu$ representing $x^\mu$ in an auxiliary Hilbert space should satisfy Heisenberg-Moyal commutation relations

$$[\hat{x}^\mu, \hat{x}^\nu] = i \theta^{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 3,$$

where $\theta^{\mu\nu}$ are given numerical constants that specify the non-commutativity of the space-time in question.

Later in [4] it was shown that field theories in NC spaces with (1) can emerge as effective low energy limits of string theories. These results supported a vivid development of non-commutative QFT. However, such models contain various unpleasant and unwanted features. The divergences are not removed, on the contrary, UV-IR mixing appears, [5]. The Lorentz invariance is broken down to $SO(2) \times SO(1, 1)$, but even this is sufficient to prove the classical CPT and Spin-statistics theorems, [6]. This was not accidental and led to the twisted Poincaré reinterpretation of NC space-time symmetries, [7].

However, it could be interesting to reverse the approach. Not to use the NC geometry to improve the foundation of QFT, what is a very complicated task, but to test the effect of non-commutativity of the space on the deformation of the well-defined quantum mechanics (QM):

- Various QM systems have been investigated in 3D space with Heisenberg-Moyal commutation relations $[\hat{x}_i, \hat{x}_j] = i \theta^{ij}$, $i, j = 1, 2, 3$, e.g. harmonic oscillator, Aharonov-Bohm effect, Coulomb problem, see [8], [9]. However, in such 3D NC space the rotational symmetry is violated and there are systems, such as $H$-atom, that are tightly related to the rotational symmetry.
- The rotational symmetry survives in 2D Heisenberg-Moyal space with NC coordinates $\hat{x}_1, \hat{x}_2$ satisfying the $F$ commutation relations $[\hat{x}_1, \hat{x}_2] = i \theta$
in an auxiliary Hilbert space. In [10] a planar spherical well was described in detail:

(i) First, the Hilbert space $\mathcal{H}$ of operator wave functions $\hat{\psi} = \psi(\hat{x}_1, \hat{x}_2)$ was defined;

(ii) Further, the Hamiltonian was defined as an operator acting in $\mathcal{H}$. It was nice to see how the persisted rotational symmetry helps to solve exactly the problem in question.

The presented list of references is incomplete and we apologize for that. We restricted ourselves to those which initiated progress or are close to our approach.

Our aim is to extend this scheme to the QM problems with rotationally symmetric potentials $V(r)$ in the configuration space $R^3_0 \equiv R^3 \setminus \{0\}$. We restrict ourselves to the Coulomb potential which, in the usual (commutative) setting, is a solution of the Poisson equation finite at infinity:

$$\Delta V(r) = 0 \Rightarrow V(r) = -\frac{q}{r} + q_0.$$  

For $H$-atom $q$, in a Gaussian system of units, is a square of electric charge $e^2$, and we put the inessential constant $q_0 = 0$. In this case we are dealing with Schrödinger equation

$$\frac{-\hbar^2}{2m} \Delta \psi(x) - \frac{e^2}{r}\psi(x) = E\psi(x), \quad r = |x| > 0$$

in the Hilbert space $\mathcal{H}$ specified by the norm

$$\|\psi\|_0^2 = \int d^3x \: |\psi(x)|^2.$$  

Expressing the wave function as

$$\psi(x) = R_j(r) H_{jm}(x), \quad H_{jm}(x) \sim r^j Y_{jm}(\vartheta, \varphi),$$

and putting $\alpha = 2me^2/\hbar^2$ and $\kappa = \sqrt{-2mE}/\hbar$, we obtain the radial Schrödinger equation:

$$r R_j''(r) + 2(j + 1)R_j'(r) + \alpha R_j(r) = \kappa^2 r R_j(r).$$

Its solution is

$$R_j(r) = e^{-\kappa r} F\left(j + 1 - \frac{\alpha}{2\kappa}, 2j + 2; 2\kappa r\right),$$
where $F(a, c; x)$ is the confluent hypergeometric function.

For bound states $E < 0$, i.e. real-valued $\kappa$, the solution should have a finite norm in $\mathcal{H}_0$. This is the case when the first argument of the degenerated hypergeometric function is zero or a negative integer, and this determines the discrete energy eigenvalues:

$$\frac{\alpha}{2\kappa_n} = n = j + 1, j + 2, \ldots \Rightarrow E_n = -\frac{\hbar^2}{2m} \kappa_n^2 = -\frac{me^4}{2\hbar^2 n^2}. \tag{8}$$

The hypergeometric function then reduces to a polynomial of degree $n - j - 1$ in the variable $x = 2r\kappa$:

$$F(j + 1 - n, 2j + 2; x) = \sum_{k=0}^{n-j-1} c_{nj}^k \frac{(-x)^k}{k!}, \quad c_{nj}^k = \frac{(n - 1)!(2j + 1)!}{(n - 1 - k)!(2j + 1 + k)}. \tag{9}$$

In this paper we extend the QM solution (2)-(8) of the Coulomb problem in $R^3_0$ to the non-commutative rotationally invariant space. In Section 2 we define $\hat{R}^3_0$, a rotationally invariant NC generalization of the configuration space $R^3_0$, we introduce the generators of rotations and the NC analog of their eigenfunctions. In Section 3 we define Hilbert space $\hat{\mathcal{H}}$ - the NC analog of $\mathcal{H}$, and we introduce the NC analog of the Coulomb problem Hamiltonian acting in $\hat{\mathcal{H}}$. In Section 4 we exactly solve the corresponding NC analog of the Schrödinger equation. Last Section 5 contains conclusions and perspectives.

## 2 The noncommutative space $\hat{R}^3_0$

In this section we define the noncommutative space $\hat{R}^3_0$, possessing full rotational invariance, as a sequence of fuzzy spheres introduced, in various contexts, in [11]. Different fuzzy spheres are related in such a way that at large distances we recover space $R^3_0$ with the usual flat geometry. A similar construction of a 3D noncommutative space, as a sequence of fuzzy spheres, was proposed in [12]. However, various fuzzy spheres are related to each other differently (not leading to the flat 3D geometry at large distances).

We realize the noncommutative coordinates in $\hat{R}^3_0$ in terms of 2 pairs of boson annihilation and creation operators $\hat{a}_\alpha$, $\hat{a}^\dagger_\alpha$, $\alpha = 1, 2$, satisfying the following commutation relations, see [13]:

$$[\hat{a}_\alpha, \hat{a}^\dagger_\beta] = \delta_{\alpha\beta}, \quad [\hat{a}_\alpha, \hat{a}_\beta] = [\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\beta] = 0. \tag{10}$$
They act in an auxiliary Fock space $\mathcal{F}$ spanned by normalized vectors

$$
|n_1,n_2\rangle = \frac{(\hat{a}_1^\dagger)^{n_1}(\hat{a}_2^\dagger)^{n_2}}{\sqrt{n_1!n_2!}} |0\rangle.
$$

(11)

Here, $|0\rangle \equiv |0,0\rangle$ denotes the normalized vacuum state: $\hat{a}_1 |0\rangle = \hat{a}_2 |0\rangle = 0$.

The noncommutative coordinates $\hat{x}_j$, $j = 1, 2, 3$, in the space $\hat{R}_0^3$ are given as

$$
\hat{x}_j = \lambda \hat{a}^+ \sigma_j \hat{a} \equiv \lambda \sigma^j_{\alpha\beta} \hat{a}_\alpha^\dagger \hat{a}_\beta, \quad j = 1, 2, 3,
$$

(12)

where $\lambda$ is a universal length parameter. The coordinates $\hat{x}_j$ satisfy rotationally invariant commutation rules:

$$
[\hat{x}_i, \hat{x}_j] = 2i \lambda \varepsilon_{ijk} \hat{x}_k, \quad [\hat{x}_i, \hat{\varrho}] = 0,
$$

(13)

where $\hat{\varrho} = \lambda \hat{N}$, and $\hat{N} = \hat{a}^+ \hat{a} \equiv \hat{a}_\alpha^\dagger \hat{a}_\alpha$.

The operator that approximates the Euclidean distance from the origin in an optimal way is $\hat{r} = \lambda (\hat{N} + 1) = \hat{\varrho} + \lambda$, and not $\hat{\varrho}$. Namely, it holds

$$
\hat{r}^2 - \hat{x}_j^2 = \lambda^2,
$$

where $\hat{\varrho}^2 - \hat{x}_j^2 = o(\lambda)$. In Section 3 we give a strong argument supporting the exceptional role of $\hat{r}$.

Let us consider the linear space of normal ordered polynomials containing the same number of creation and annihilation operators:

$$
\hat{\Psi} = \sum C_{m_1,m_2,n_1,n_2} (\hat{a}_1^\dagger)^{m_1} (\hat{a}_2^\dagger)^{m_2} (\hat{a}_1)^{n_1} (\hat{a}_2)^{n_2},
$$

(14)

where the summation is finite over nonnegative integers satisfying $m_1 + m_2 = n_1 + n_2$. In this space we define generators of rotations $L_j$, $j = 1, 2, 3$, as follows

$$
L_j \hat{\Psi} = \frac{i}{2} [\hat{a}^+ \sigma_j \hat{a}, \hat{\Psi}], \quad j = 1, 2, 3,
$$

(15)

obeying proper commutation relations

$$
[L_i, L_j] \hat{\psi} \equiv (L_i L_j - L_j L_i) \hat{\psi} = i \varepsilon_{ijk} L_k \hat{\psi}.
$$

(16)

With respect to the rotations (15) the doublet of annihilation (creation) operators transforms as spinor (conjugated spinor), whereas the triplet of NC coordinates as vector

$$
L_j \hat{a}_\alpha = -\frac{i}{2} \sigma^j_{\alpha\beta} \hat{a}_\beta, \quad L_j \hat{a}_\alpha^\dagger = \frac{i}{2} \sigma^j_{\beta\alpha} \hat{a}_\beta^\dagger, \quad L_i \hat{x}_j = i \varepsilon_{ijk} \hat{x}_k.
$$
The standard eigenfunctions \( \hat{\psi}_{jm} \), \( j = 0, 1, 2, \ldots \), \( m = -j, \ldots, +j \), satisfying
\[
L_i^2 \hat{\psi}_{jm} = j(j + 1) \hat{\psi}_{jm}, \quad L_3 \hat{\psi}_{jm} = m \hat{\psi}_{jm},
\]
are given by the formula
\[
\hat{\psi}_{jm} = \lambda^j \sum_{(jm)} \frac{(\hat{a}_1^\dagger)^{m_1} (\hat{a}_2^\dagger)^{m_2}}{m_1! m_2!} : R_j(\hat{\rho}) : \frac{\hat{a}_1^{m_1} (-\hat{a}_2)^{n_2}}{n_1! n_2!}
\]
with the summation over all nonnegative integers satisfying \( m_1 + m_2 = n_1 + n_2 = j \), \( m_1 - m_2 = n_1 - n_2 = 2m \). Thus \( \hat{\psi}_{jm} = 0 \) when restricted to the subspaces \( \mathcal{F}_N = \{|n_1, n_2| : n_1 + n_2 = N\} \) with \( N < j \). For any fixed \( : R_j(\hat{\rho}) : \); equation (18) defines a representation space for a unitary irreducible representation with spin \( j \).

The symbol \( : R_j(\hat{\rho}) : \) represents a normal ordered analytic function in the operator \( \hat{\rho} \):
\[
: R_j(\hat{\rho}) : = \sum_k c_k^j : \hat{\rho}^k : = \sum_k c_k^j \lambda^k \frac{\hat{N}!}{(\hat{N} - k)!}.
\]
The last equality follows from the equation
\[
: \hat{N}^k : |n_1, n_2\rangle = \frac{N!}{(N - k)!} |n_1, n_2\rangle, \quad N = n_1 + n_2
\]
(which can be proved by induction in \( k \)). Since, \( : \hat{N}^k : |n_1, n_2\rangle = 0 \) for \( k > n_1 + n_2 \), the summation in (19) is effectively restricted to \( k \leq N \) on any subspace \( \mathcal{F}_N \).

### 3 Quantum mechanics in space \( \hat{\mathbb{R}}^3_0 \)

Let \( \hat{\mathcal{H}} \) denote the Hilbert space generated by functions (18) with weighted Hilbert-Schmidt norm
\[
||\hat{\Psi}||^2 = 4\pi \lambda^3 \text{Tr}[\hat{N} + 1 \hat{\Psi}^\dagger \hat{\Psi}] = 4\pi \lambda^2 \text{Tr}[\hat{\rho} \hat{\Psi}^\dagger \hat{\Psi}], \quad \hat{\rho} = \lambda (\hat{N} + 1).
\]
The rotationally invariant weight \( w(\hat{r}) = 4\pi \lambda^2 \hat{r} \) is determined by the requirement that a ball in \( \hat{\mathbb{R}}^3_0 \) with radius \( r = \lambda (N + 1) \) should possess a
standard volume in the limit $r \to \infty$. The projector $\hat{P}_N$ on the subspace $\mathcal{F}_0 \oplus \ldots \oplus \mathcal{F}_N$ corresponds to the characteristic functions of a ball with the radius $r = \lambda(N + 1)$. Therefore, the volume of the ball in question in $\hat{\mathbb{R}}^3$ is

$$V_r = 4\pi \lambda^3 \text{Tr}[(\hat{N} + 1) \hat{P}_N] = 4\pi \lambda^3 \sum_{n=0}^{N+1} (n + 1)^2 = \frac{4\pi}{3} r^3 + o(\lambda). \quad (22)$$

Thus, the chosen weight $w(\hat{r}) = 4\pi \lambda^2 \hat{r}$ possesses the desired property.

*Note:* The weighted trace $\text{Tr}[w(\hat{r}) \ldots]$ with $w(\hat{r}) = 4\pi \lambda^2 \hat{r}$ at large distances goes over to the usual volume integral $\int d^3\vec{x} \ldots$. The 3D noncommutative space proposed in [12] corresponds to the choice $w(\hat{r}) = \text{const}$ and at large distances does not correspond to the flat space $\mathbb{R}^3$.

The generators of rotations $L_j$, $j = 1, 2, 3$, are hermitian (self-adjoint) operators in $\hat{\mathcal{H}}$, and consequently, the two operators $\hat{\Psi}_{jm}$ and $\hat{\Psi}_{jm'}$, with arbitrary factors: $R_j(\hat{\varrho})$ : and : $\hat{R}_{j'}(\hat{\varrho})$ :, are in $\hat{\mathcal{H}}$ orthogonal. It is sufficient to calculate $\|\hat{\Psi}_{jm}\|^2 = \|\hat{\Psi}_{jj}\|^2$ (this equality follows from the rotational invariance of the norm in question):

$$\|\hat{\Psi}_{jm}\|^2 = 4\pi \lambda^3 \sum_{N=j}^{\infty} \sum_{n=0}^{N} (N + 1) \langle n, N-n | (\hat{N} + 1) \hat{\Psi}_{jj}^\dagger \hat{\Psi}_{jj} | n, N-n \rangle, \quad (23)$$

We benefit from the fact that $\hat{\Psi}_{jj}$ has a simple form

$$\hat{\Psi}_{jj} = \frac{\lambda_j}{(j!)^2} (\hat{a}_1^\dagger)^j : R_j(\hat{\varrho}) : (-\hat{a}_2)^j. \quad (24)$$

The matrix element we need to calculate is

$$\langle n, N-n | (\hat{a}_2^\dagger)^j : R_j(\hat{\varrho}) : \hat{a}_1^j (\hat{a}_1^\dagger)^j : R_j(\hat{\varrho}) : \hat{a}_2^j | n, N-n \rangle$$

$$= \frac{(n+j)! (N-n)!}{n! (N-j-n)!} |R_j(N-j)|^2; \quad (25)$$

where

$$R_j(N) = \langle n, N-n | : R_j(\hat{\varrho}) : | n, N-n \rangle \quad (26)$$

(the expression on the r.h.s. is $n$ - independent). Inserting (24), (25) into (23) and using the identity

$$\sum_{n=0}^{N} \binom{n+j}{j} \binom{N-n}{j} = \binom{N+j+1}{2j+1},$$
we obtain

$$
\|\hat{\Psi}_{jm}\|^2 = \frac{4\pi\lambda^{3+2j}}{(j!)^2} \sum_{N=0}^{\infty} (N+j+1) \left(\frac{N+j+1}{2j+1}\right) |\mathcal{R}_j(N)|^2. 
$$

(27)

This expression represents, up to an eventual normalization, the square of a norm of the radial part of the operator wave function.

Now we are ready to define the Coulomb problem Hamiltonian in the noncommutative case.

The kinetic term in $\hat{R}_0$. In the first we define the NC analog of the Laplacian in $\hat{R}_0$ as follows:

$$
\Delta_\lambda \hat{\Psi} = -\frac{1}{\lambda \hat{r}} [\hat{a}_\alpha^\dagger, [\hat{a}_\alpha, \hat{\Psi}]] = -\frac{1}{\lambda^2 (N+1)} [\hat{a}_\alpha^\dagger, [\hat{a}_\alpha, \hat{\Psi}]]. 
$$

(28)

This choice is motivated by the following facts:

- A double commutator is an analog of a second order differential operator;
- The factor $\hat{r}^{-1}$ guarantees that the operator $\Delta_\lambda$ is hermitian (self-adjoint) in $\hat{\mathcal{H}}$, and finally,
- The factor $\lambda^{-1}$, or $\lambda^{-2}$ respectively, guarantees the correct physical dimension of $\Delta_\lambda$ and its non-trivial commutative limit.

Calculating the action of (31) on $\hat{\Psi}_{jm}$ given in (18) we can check whether the postulate (31) is a reasonable choice. The corresponding formula is derived in Appendix A:

$$
- [\hat{a}_\alpha^\dagger, [\hat{a}_\alpha, \hat{\Psi}]] = \lambda^j \sum_{(jm)} \frac{(\hat{a}_1^\dagger)^{m_1}(\hat{a}_2^\dagger)^{m_2}}{m_1! m_2!} 
\times [\hat{\rho} R''(\hat{\varphi}) + 2(j+1) R'(\hat{\varphi})] : \hat{a}_1^{n_1}(\hat{a}_2)^{n_2} : n_1! n_2!.
$$

(29)

Here the symbols $R'(\hat{\varphi})$ and $R''(\hat{\varphi})$ are defined as:

$$
R(\hat{\varphi}) = \sum_{k=0}^{\infty} c_k \hat{\varphi}^k \quad \Rightarrow \quad R'(\hat{\varphi}) = \sum_{k=1}^{\infty} k c_k \hat{\varphi}^{k-1},
$$

$$
R''(\hat{\varphi}) = \sum_{k=2}^{\infty} k(k-1) c_k \hat{\varphi}^{k-2}. 
$$

(30)
Thus, the prime corresponds exactly to the usual derivative $\partial_\phi$. We see that the angular dependence in (29) remains untouched, since $\Delta_\lambda$ is rotation invariant. In the commutative limit $\lambda \to 0$ formally $\hat{\phi} \to r$, and we see that (29) guarantees that $\Delta_\lambda$ reduces just to the standard Laplacian.

Based on that, we postulate the kinetic term of the Hamiltonian as follows

$$H_0 = -\frac{\hbar^2}{2m} \Delta_\lambda \hat{\Psi} = \frac{\hbar^2}{2m\lambda r} [\hat{a}_\alpha^\dagger, [\hat{a}_\alpha, \hat{\Psi}]].$$

(31)

The potential term in $\hat{R}_0^3$ is defined by left multiplication of the operator wave function by potential $\hat{V} = V(\hat{\phi})$: $\hat{\Psi} \mapsto \hat{V} \hat{\Psi}$. The potential is central if $\hat{V} = V(\hat{N})$, or equivalently, $\hat{V} = V(\hat{r})$.

4 The Coulomb problem in $\hat{R}_0^3$

In the commutative case the Coulomb potential is a radial solution of the equation (2) finite at infinity. Due to our choice of the noncommutative Laplacian $\Delta_\lambda$ the equivalent equation in $\hat{R}_0^3$ is

$$[\hat{a}_\alpha^\dagger, [\hat{a}_\alpha, V(\hat{N})]] = 0.$$  

It can be rewritten as a simple recurrent relation

$$(\hat{N} + 2) V(\hat{N} + 1) - (\hat{N} + 1) V(\hat{N}) = (\hat{N} + 1) [V(\hat{N}) - \hat{N} V(\hat{N} - 1)].$$

(32)

Putting

$$(\hat{M} + 1) V(\hat{M}) - \hat{M} V(\hat{M} - 1) = q_0, \quad V(0) = q_0 - \frac{q}{\lambda},$$

and summing up the first equation over $\hat{M} = 1, \ldots, N$, we obtain the general solution:

$$V(\hat{N}) = -\frac{q}{\lambda(\hat{N} + 1)} + q_0 = -\frac{q}{\hat{r}} + q_0,$$

(33)

where, $q$ and $q_0$ are arbitrary constants ($\lambda$ is introduced for convenience).

For $H$-atom $q = e^2$ and we put $q_0 = 0$. We see that the dependence $\hat{r}^{-1}$ of the NC Coulomb potential is inevitable.
Thus, the noncommutative analog of the Schrödinger equation with the Coulomb potential in $\mathbb{R}^3$ is

$$\frac{\hbar^2}{2m\lambda r} \left[ \hat{a}^\dagger_\alpha, [\hat{a}_\alpha, \hat{\Psi}] \right] - \frac{q}{\hat{r}} \hat{\Psi} = E \hat{\Psi} \iff \frac{1}{\lambda} \left[ \hat{a}^\dagger_\alpha, [\hat{a}_\alpha, \hat{\Psi}] \right] - \alpha \hat{\Psi} = -\kappa^2 \hat{r} \hat{\Psi},$$ (34)

where $\alpha = \frac{2me^2}{\hbar^2}$ and $\kappa = \sqrt{-2mE/\hbar}$. The noncommutative corrections coming from $\hat{r} \hat{\Psi}$ are calculated in Appendix A:

$$\hat{r} \hat{\Psi}_{jm} = \sum_{(jm)} \frac{\left( \hat{a}_1^\dagger \right)^{m_1} \left( \hat{a}_2^\dagger \right)^{m_2}}{m_1! m_2!} : [\hat{\rho} + \lambda j + \lambda] \hat{R}_j + \lambda \hat{\rho} \hat{R}'_j : \frac{\hat{a}_1^{n_1} (-\hat{a}_2)^{n_2}}{n_1! n_2!}. \quad (35)$$

where $\hat{R}_j \equiv R_j(\hat{\rho})$ and similarly for derivatives, e.g., $\hat{R}'_j \equiv R'_j(\hat{\rho})$. The angular dependence is again untouched as the multiplication by $\hat{r}$ represents a rotation invariant operator.

Inserting (29) and (35) into (34) we obtain the NC analog of radial Schrödinger equation:

$$: [\hat{\rho} \hat{R}'_j + 2(j+1) \hat{R}'_j + \alpha \hat{R}_j] : = \kappa^2 : [\hat{\rho} \hat{R}_j + \lambda ((j+1) \hat{R}_j + \hat{\rho} \hat{R}'_j)] :. \quad (36)$$

The term proportional to $\lambda$ represents the NC correction. For $\lambda \to 0$ this NC radial Schrödinger equation reduces to the standard radial Schrödinger equation.

We associate the following ordinary differential equation to the mentioned operator radial Schrödinger equation (36):

$$\rho \mathcal{R}''_j + 2(j+1) \mathcal{R}'_j + \alpha \mathcal{R}_j = \kappa^2 \left[ \rho \mathcal{R}_j + \lambda ((j+1) \mathcal{R}_j + \rho \mathcal{R}'_j) \right]. \quad (37)$$

If the function $\mathcal{R}_j = \mathcal{R}_j(\rho) = \sum_k c_k^j \rho^k$ solves the associated ordinary differential equation (37), then

$$\hat{R}_j = : \mathcal{R}_j(\hat{\rho}) : = \sum_k c_k^j \hat{\rho}^k = \sum_k c_k^j \lambda^k \frac{\hat{N}!}{(\hat{N} - k)!}. \quad (38)$$

solves the operator radial equation (36). Moreover, the operator function $\hat{R}_j = : \mathcal{R}_j(\hat{\rho}) :$ possesses a finite norm in $\mathcal{H}$ provided the function $\mathcal{R}_j = \mathcal{R}_j(\rho)$ has finite norm in $\mathcal{H}$ (since the norm (21) asymptotically reduces to the usual QM norm).

The solution $\mathcal{R}_j$ of the associated radial Schrödinger equation (37) is given similarly as in the standard Coulomb problem in (7), but with particularly
scaled $\varrho$ dependence in the exponent and in the argument of the confluent hypergeometric function:

$$
R_j(\varrho) = e^{-b \kappa \varrho} F\left(j + 1 - \frac{\alpha}{2d \kappa}, 2j + 2; 2\varrho \kappa d\right).
$$

(39)

The dimensionless quantities $b$ and $d$ given as (see Appendix B)

$$
b = \sqrt{1 + \eta^2 - \eta}, \quad d = \sqrt{1 + \eta^2}, \quad \eta = \frac{1}{2} \frac{\lambda \kappa}{2}.
$$

(40)

specify the NC corrections to the usual radial function in (7). They enter $b$ and $d$ via the parameter $\eta = \lambda \kappa / 2$. So they vanish not only in the commutative limit $\lambda \to 0$, but also for $\kappa \to 0$.

We shall restrict ourselves to the determination of the bound states with $E < 0$ and $\kappa > 0$. In this case $R_j(\varrho)$ should be normalizable. This is ensured if the first argument of the confluent hypergeometric function is zero or negative integer, what determines the discrete energy eigenvalues (remember that $d$ is $\kappa$-dependent, see (40)):

$$
\frac{\alpha}{2d_n \kappa_n} = n = j + 1, j + 2, \ldots \Rightarrow
$$

$$
E_n^\lambda = -\frac{\hbar^2}{2m} \kappa_n^2 = -\frac{m e^4}{2\hbar^2 n^2} \frac{2}{1 + \sqrt{1 + \lambda^2 / a_0^2 n^2}},
$$

(41)

where $a_0 = 2/\alpha = \hbar^2 / m e^2 = 5.29 \times 10^{-11} m$ is the Bohr radius. The first factor in $E_n^\lambda$ is just the standard bound state energy of the Coulomb problem, whereas the second one represents the noncommutative correction. The NC corrections in the limit $\lambda/n \to 0$, i.e., in the commutative limit $\lambda \to 0$, or for fixed $\lambda$, in the quasi-classical limit $n \to \infty$ for highly excited states.

The solutions $\hat{\Psi}^\lambda_{njm}$ of the operator equation (36) corresponding to the energy $E_n^\lambda$ is

$$
\hat{\Psi}^\lambda_{njm} = N_{njm} \sum_{(jm)} \frac{(\hat{a}_1^\dagger)^{m_1} (\hat{a}_2^\dagger)^{m_2}}{m_1! m_2!} : R_{nj}^{\lambda \kappa \varrho} : \frac{\hat{a}_1^{n_1} (-\hat{a}_2)^{n_2}}{n_1! n_2!},
$$

(42)

where $N_{njm}$ denotes the normalization factor and

$$
R_{nj}^{\lambda \kappa \varrho} = e^{-b_n \varrho \kappa \hat{\varrho}} F\left(j + 1 - n, 2j + 2; 2\varrho \kappa \varrho d_n\right).
$$

(43)
Here, the parameters $b_n$ and $d_n$ are given by (40) with $\kappa_n = \sqrt{-2mE_n^\lambda}/\hbar$. From the equation (20) it follows directly
\[
e^{-a\hat{\varrho}^k} : = (1 - a\lambda)^{\hat{N}} \frac{\lambda^k}{(1 - a\lambda)^k} \frac{\hat{N}!}{(\hat{N} - k)!} = (1 - a\lambda)^{\hat{\varrho}/\lambda} \cdot \frac{\hat{\varrho}^k}{(1 - \lambda a)^k}. \tag{44}\]

This allows us to express the normal ordered form of the radial part of operator wave function:
\[
: R_{nj}(\hat{\varrho}) : = (1 - b_n \kappa_n \lambda)^{\hat{\varrho}/\lambda} \cdot F\left(j + 1 - n, 2j + 2, \frac{2\hat{\varrho} \kappa_n d_n}{1 - b_n \kappa_n \lambda}\right) : \tag{45}
\]
with the coefficients $c_{nj}^k$ given in (9). Inserting
\[
R_{nj}(N) = N_{jm}^{\lambda n} (1 - b_n \kappa_n \lambda)^N \sum_{k=0}^{n-j-1} c_{nj}^k \left(\frac{-2\lambda \kappa_n d_n}{1 - b_n \kappa_n \lambda}\right)^k \frac{\hat{N}!}{(\hat{N} - k)!}, \tag{46}
\]
into (27) the normalization constant $N_{jm}^{\lambda n}$ can be determined (we skip its calculation). In commutative limit the factor in front of the sum gives the usual exponential damping factor in (7). However, as the argument of the polynomial becomes scaled due to NC corrections, the separation of the asymptotic factor from the polynomial part is not perfect.

5 Conclusions

We carefully defined the NC rotationally invariant analog of the QM configuration space and the Hilbert space of operator wave functions in NC configuration space. The central point of our construction was the definition of $\Delta_{\lambda}$ the NC analog of Laplacian, supplemented by a consequent definition of the weighted Hilbert-Schmidt norm and a definition of the Coulomb potential satisfying NC Poisson equation.

With this input this Hilbert space we introduced the NC analog of $H$-atom Hamiltonian and explicitly determined the bound-state energies $E_n^\lambda$ and corresponding eigenstates $\psi_{njm}^{\lambda}$ (see equations (11), (12) and (13)).
We found that the discrete parameters \( n, j, m \) have the same meaning and range as in the standard (commutative) Coulomb problem, and moreover, the bound-state energies and eigenstates possess a smooth commutative limit \( \lambda \to 0 \). This paper does not deal with the case of the scattering in the NC configuration space - this will be discussed elsewhere.

The noncommutativity parameter \( \lambda \) is not fixed within our model. However, it can be estimated by some other physical requirement. For example, one can postulate, as was done in early days of modern physics, that the rest energy \( mc^2 \) of electron is equal to the electrostatic energy of its Coulomb field. In \( \mathbb{R}^3_0 \) this means:

\[
mc^2 = \frac{4\pi\lambda^3}{8\pi} \text{Tr} [(\hat{N} + 1) \hat{E}_j^2] \tag{47}
\]

where

\[
\hat{E}_j = \frac{e^2}{\lambda^3} \frac{1}{\hat{N}(\hat{N} + 1)(\hat{N} + 2)} \hat{x}_j, \tag{48}
\]

is the NC electric field strength corresponding to NC Coulomb potential \( \hat{\Phi} \) which was discussed in Section 4 (the details will be published, see [15]).

We stress that in the NC case the electrostatic energy of electron, determined by the trace in (47), is finite (no cut-off at short distance is needed). A straightforward calculation of the trace in (47) gives the relation:

\[
mc^2 = \frac{3}{8} \frac{e^2}{\lambda} \Rightarrow \lambda = \frac{3}{8} \frac{e^2}{mc^2} = \lambda_0. \tag{49}
\]

This \( \lambda_0 \) is fraction of the classical radius of electron \( r_0 = e^2/mc^2 \): \( \lambda_0 = 1.06 \times 10^{-15} m = 1.06 \, fm \) (the coincidence with the proton radius is purely accidental).

The NC corrections to the \( H \)-atom energy levels given in (41) are of of order \( (\lambda_0/a_0) = (9/64) \alpha_0^2 \approx 4 \times 10^{-11} \) (here \( \alpha_0 \approx 1/137 \) is fine structure constant). Such tiny corrections to energy levels are beyond any experimental evidence. Moreover, at \( \lambda_0 \approx 1 \, fm \) relativistic and QFT effects become essential.

Our investigation indicates that the noncommutativity of the configuration space is fully consistent with the general QM axioms, at least for the \( H \)-atom bound states. However, a more detailed analysis of the Coulomb problem in \( \mathbb{R}^3_0 \) would be a desirable dealing, e.g, with the following aspects:
• Coulomb scattering problem, dyon problem (electron in the electric point charge and magnetic monopole field), Pauli $H$-atom (non-relativistic spin);
• Coulomb problem in $\hat{R}_3^3$ and its dynamical symmetry, QM supersymmetry and integrability of the Coulomb system.

Besides non-relativistic $H$-atom, there are other systems that would be interesting to investigate within NC configuration space $\hat{R}_0^3=$, e.g., Dirac $H$-atom (relativistic invariance?), non-Abelian monopoles, or spherical black-holes.

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Appendix A

Here we prove two formulas, (29) and (35), we need for the calculation of the NC Coulomb Hamiltonian. Below we skip indices $j$ and $m$.

a. Let us begin with the (29):

\[
[\hat{a}_\alpha^\dagger, [\hat{a}_\alpha, \hat{\Psi}_{jm}]] = \lambda^j \sum_{(jm)} \frac{(\hat{a}_1^\dagger)^{m_1} (\hat{a}_2^\dagger)^{m_2}}{m_1! m_2!} : \hat{R} : \frac{\hat{a}_1^{n_1} (-\hat{a}_2)^{n_2}}{n_1! n_2!}]
\]

\[
= \lambda^j \sum_{(jm)} \frac{(\hat{a}_1^\dagger)^{m_1} (\hat{a}_2^\dagger)^{m_2}}{m_1! m_2!} : \hat{R} : \hat{a}_1^{n_1} (-\hat{a}_2)^{n_2} \frac{n_1! n_2!}{n_1! n_2!}
\]

\[
+ \lambda^j \sum_{(jm)} \frac{(\hat{a}_1^\dagger)^{m_1} (\hat{a}_2^\dagger)^{m_2}}{m_1! m_2!} : \hat{R} : \hat{a}_1^{n_1} (-\hat{a}_2)^{n_2} \frac{n_1! n_2!}{n_1! n_2!}
\]

\[
+ \lambda^j \sum_{(jm)} \frac{(\hat{a}_1^\dagger)^{m_1} (\hat{a}_2^\dagger)^{m_2}}{m_1! m_2!} : \hat{R} : \frac{\hat{a}_1^{n_1} (-\hat{a}_2)^{n_2}}{n_1! n_2!}
\]

\[
+ \lambda^j \sum_{(jm)} \frac{(\hat{a}_1^\dagger)^{m_1} (\hat{a}_2^\dagger)^{m_2}}{m_1! m_2!} : \hat{R} : \frac{\hat{a}_1^{n_1} (-\hat{a}_2)^{n_2}}{n_1! n_2!}
\]

(50)

where $\hat{R} = \sum_{k=0}^{\infty} \hat{c}_k^j \hat{N}^k$, i.e., $\hat{c}_k^j = \lambda^j \hat{c}_k^j$. Now we shall use the following
commutation relations

\[
\begin{align*}
[\hat{a}_\alpha^\dagger, \hat{N}_k^\dagger : & \hat{N}^{k-1} : ] = -k \hat{a}_\alpha^\dagger : \hat{N}_k : \Rightarrow [\hat{a}_\alpha^\dagger, \hat{R} : ] = -\hat{a}_\alpha^\dagger : \partial_\hat{N} \hat{R} : , \\
[\hat{a}_\alpha, \hat{N}_k : & ] = k : \hat{N}^{k-1} : \hat{a}_\alpha \Rightarrow [\hat{a}_\alpha, \hat{R} : ] = : \partial_\hat{N} \hat{R} : \hat{a}_\alpha , \\
\end{align*}
\]  

(51)

where \( \partial_\hat{N} \) denotes the derivatives with respect to \( \hat{N} \): \( \partial_\hat{N} \hat{R} = \sum_{k=1}^\infty k \hat{c}_k \hat{N}^{k-1} \).

It is easy to see that the second line in (50) vanish, and the the first and third line give the same contribution

\[
\sum (jm) \frac{(\hat{a}_1^\dagger)^{m_1} (\hat{a}_2^\dagger)^{m_2}}{m_1! m_2!} (-j : \partial_\hat{N} \hat{R} : ) \frac{\hat{a}_1^{n_1} (-\hat{a}_2)^{n_2}}{n_1! n_2!} .
\]

(52)

From (51) the double commutator \([\hat{a}_\alpha^\dagger, [\hat{a}_\alpha, \hat{R} : ]]\) follows directly, and this gives the value of the third line in (50)

\[
\sum (jm) \frac{(\hat{a}_1^\dagger)^{m_1} (\hat{a}_2^\dagger)^{m_2}}{m_1! m_2!} (- : \hat{N} \partial_\hat{N} \hat{R} : + 2 : \partial_\hat{N} \hat{R} : ) \frac{\hat{a}_1^{n_1} (-\hat{a}_2)^{n_2}}{n_1! n_2!} .
\]

(53)

Introducing parameter \( \lambda \) and switching to the derivatives with respect to \( \hat{\rho} \) the last two equations yields (29).

b. Proof of (35) is straightforward. From equation (20) it follows easily

\[
\hat{N} = \hat{N}^k = : \hat{N}^{k+1} : + k : \hat{N}^k : \Rightarrow \hat{N} = \hat{R} = : \hat{N} \hat{R} : + : \hat{N} \partial_\hat{N} \hat{R} : .
\]

(54)

This relation gives directly

\[
(\hat{N} + 1) \sum (jm) \frac{(\hat{a}_1^\dagger)^{m_1} (\hat{a}_2^\dagger)^{m_2}}{m_1! m_2!} : \hat{R} : \frac{\hat{a}_1^{n_1} (-\hat{a}_2)^{n_2}}{n_1! n_2!}
\]

\[
= \sum (jm) \ldots [(\hat{N} + j + 1) : \hat{R} : ] \ldots
\]

\[
= \sum (jm) \ldots : [(\hat{N} + j + 1) \hat{R} + \hat{N} \partial_\hat{N} \hat{R} : ] \ldots ,
\]

(55)

where we have replaced both untouched factors containing annihilation and creation operators by dots. Introducing parameter \( \lambda \) again and switching to the derivatives with respect to \( \hat{\rho} \) we recover (35).
Appendix B

It is known that the solution \( R = R(\varrho) \) of equation

\[
\varrho R'' + (a_1 \varrho + b_1) R' + (a_2 \varrho + b_2) R = 0 \tag{56}
\]
can be expressed in terms of a confluent hypergeometric function

\[
F(a, c; x) = 1 + \frac{a}{c} x + \frac{a(a + 1)}{c(c + 1)} \frac{x^2}{2!} + \ldots .
\]
The formula, given e.g. in [14], reads

\[
R = e^{\frac{j}{2}(\varrho - a_1)\varrho} F(a, c; -D \varrho), \tag{57}
\]
where \( D \) is determined by \( D^2 = a_1^2 - 4a_2 \) and

\[
a = \frac{1}{D} \left[ \frac{1}{2} (D - a_1) b_1 + b_2 \right], \quad c = b_1. \tag{58}
\]

In our case

\[
a_1 = -\lambda \kappa^2, \quad b_1 = 2j + 2, \quad a_2 = -\kappa^2, \quad b_2 = \alpha - (j + 1) \lambda \kappa^2, \tag{59}
\]
what gives

\[
R(\varrho) = e^{\frac{j}{2}(D + \lambda \kappa^2)\varrho} F \left( j + 1 + \frac{\alpha}{D}, 2j + 2; -D \varrho \right). \tag{60}
\]

There are two solutions \( R_{\pm}(\varrho) \) depending on the sign of \( D = \pm 2\kappa \sqrt{1 + \eta^2} \), \( \eta = \frac{1}{2} \lambda \kappa \). However, \( R_+(\varrho) = R_-(\varrho) \) due the Kummer relation

\[
F(a, c; x) = e^x F(c - a, c; -x).
\]

We took the solution (43) of the associated radial NC Schrödinger equation (39) with negative \( D \) as it is more convenient for the determination of bound states.
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