On low rank perturbations of complex matrices and some discrete metric spaces.

Lev Glebsky and Luis Manuel Rivera

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1 Introduction

The article is devoted to different aspects of the question: "What can be done with a complex-valued matrix by a low rank perturbation?"

From the works of Thompson [15] we know how the Jordan normal form can be changed by a rank $k$ perturbation, see Theorem 2. Particularly, it follows that one can do everything with a geometrically simple spectrum by a rank 1 perturbation, see Corollary 1. But the situation is quite different if one restricts oneself to normal matrices, see Theorem 3 and Corollary 2. We think that Corollary 2 may be considered as a finite dimension analogue of the continuous spectrum conservation under compact perturbations in Hilbert spaces. For unitary and self-adjoint matrices the inequality of Corollary 2 is the only restrictions on "what can be done with a spectrum by a rank $k$ perturbation", see Theorem 4.

We don’t know if there is an analogue of Theorem 4 for normal matrices. It is worth to mention that Corollary 2 for self-adjoint matrices follows from Cauchy interlacing theorem [2]. Theorem 4 is related with the converse Cauchy interlacing theorem [6].

The spectrum of $H_1 + H_2$ with known spectra of self-adjoint matrices $H_1$ and $H_2$ is studied a lot, see [9] and the bibliography therein. Although the complete set of restrictions on the spectrum $H_1 + H_2$ known in this situation, we are not sure that there is an easy proof of Theorem 4 using results of [9].

Although Theorem 2 should be known (see, for example, [13], where Theorem 2 formulated in one direction), we will give a proof here, mainly because our proof falls in a general framework, which is also used in the proof of Theorem 4. Let us describe the framework. The set $C_{n \times n}$ of all complex $n \times n$-matrices (set of self-adjoint matrices) we equip with the arithmetic distance, $d(A, B) = \text{rank}(A - B)$ (see [3]). The arithmetic distance is geodesic for these cases. The spectral properties of matrices, such as Weyr characteristics and spectra (multiset) also may be considered as a metric spaces with distance, related to the arithmetic distance on matrices, see Section 2. These distances also turn out to be geodesic. Then we prove Theorem 2 (Theorem 4) for $\text{rank}(A - B) = 1$ and the general results will follow from Proposition 1.

**Proposition 1.** Let $X$ and $Y$ be geodesic metric spaces, let $O^X_n(x)$ denote the closed ball of radius $n$ around $x$ in $X$, let $\phi : X \to Y$ be such that $\phi(O^X_n(x)) = O^Y_n(\phi(x))$ for all $x \in X$. Then $\phi(O^X_n(x)) = O^X_n(\phi(x))$ for any $n \in \mathbb{N}$ and $x \in X$.

**Proof.** The proof is by induction. For $n = 1$ there is nothing to prove. Step $n \to n + 1$: It follows that $O^X_{n+1}(x) = \bigcup_{z \in O^X_n(x)} O^X_1(z)$ (X is geodesic), then

$$\phi(O^X_{n+1}(x)) = \bigcup_{z \in O^X_n(x)} \phi(O^X_1(z)) = \bigcup_{z \in \phi(O^X_n(x))} O^Y_1(z) = \bigcup_{z \in \phi(O^X_1(x))} O^Y_{n+1}(\phi(x))$$

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In Section 5 and Section 6 we use the normalized arithmetic distance \( d_r(A, B) = \frac{\text{rank}(A-B)}{n} \), where \( n \) is the size of the matrices. We are interested in the following questions: "Suppose that matrices almost satisfy some equations (in the sense of \( d_r(\cdot, \cdot) \)). If close to that matrices there exist matrices satisfying the equations (uniformly with respect to \( n \))?" We manage to answer only the following: close to an almost unitary (self-adjoint) matrix there exists an unitary (self-adjoint) matrix. We do not know if the same is true for normal matrices. (This question has the affirmative answer for norm distance \( d_n(A, B) = \|A - B\| \), see [10]. It is equivalent to the following: "close to any pair of almost commuting self-adjoint matrices there exists a pair of commuting self-adjoint matrices (with respect to the distance \( d_n(\cdot, \cdot) \)). It is interesting that there are almost commuting (with respect to \( d_n(\cdot, \cdot) \)) matrices, close to which there are no commuting matrices, [4, 5, 16]. The similar problem for the pairs of almost commuting matrices, as far as we know, is open. Precisely, if for any \( \epsilon > 0 \) there exists \( \delta > 0 \) such that for any \( A, B \), \( d_r(AB, BA) < \epsilon \) there exists \( (A, B) \) satisfying \( AB = BA \) and \( d_r(A, A), d_r(B, B) < \delta \) (\( \delta \) does not depends on size of the matrices).

We think that low rank perturbations of matrices may related to sofic groups. The following question seems to be interesting from this point of view (although it seems to goes beyond the scope of the present article). One can show that all solutions of equation \( C^{-1}A^{-1}CAC^{-1}AC = A^2 \) in finite unitary matrices are trivial in \( A = E \). On the other hand, it is true that for any \( \epsilon > 0 \) there exist \( A, C \), \( d(A, E) = d(C, E) = 1 \) and \( d(C^{-1}A^{-1}CAC^{-1}AC, A^2) < \epsilon \). If the above assertion is true with additional requirements \( C^4 = 1? \) If not, it gives an example of non-sofic group.

Note. All linear spaces are supposed to be finite dimensional in the rest of the article. \( C_{n \times n} \) will denote the set of all complex \( n \times n \)-matrices, \( \mathbb{N} = \{0, 1, 2, \ldots\} \).

2 Some discrete geodesic spaces.

2.1 Arithmetic distance on \( C_{n \times n} \)

Lemma 1. The arithmetic distance \( \text{rank}(A - B) \) is geodesic on

- Set of all \( n \times n \) matrices.
- Set of all self-adjoint \( n \times n \) matrices.
- Set of all unitary \( n \times n \) matrices.

Proof. It is clear that a rank \( k \) matrix (self-adjoint matrix) may be represented as sum of \( k \) matrices (self-adjoint matrices) of rank 1. The first two items follow from the fact that set of matrices (self-adjoint matrices) is closed with respect to summation. For unitary matrices. Let \( \text{rank}(U_1 - U_2) = k \), or, the same, \( \text{rank}(E - U_1^{-1}U_2) = k \). It means that, in a proper basis, \( U_1^{-1}U_2 = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k, 1, 1, \ldots) \). Now the sequence \( U_1, U_1\text{-diag}(\lambda_1, 1, 1, \ldots, 1), U_1\text{-diag}(\lambda_1, \lambda_2, 1, 1, \ldots, 1) \ldots U_1\text{-diag}(\lambda_1, \lambda_2, \ldots, \lambda_k, 1, 1, \ldots) = U_2 \) give us the geodesic needed.

Remark 1. The methods used in the above proof are not applied for normal matrices – the set of normal matrices is not closed with respect neither summation nor multiplication. In fact, an example from [6] hints that arithmetic distance might be non geodesic on the set of normal matrices.

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Proposition 2. Let $\phi(x) = (ax + b)^{-1}(cx + d)$ be a Möbius transformation of $\mathbb{C}^{n \times n}$, defined on $A, B$. Then $\text{rank}(A - B) = \text{rank}(\phi(A) - \phi(B))$.

Proof. A Möbius transformation is a composition of linear transformations $A \to aA + b$ ($a, b \in \mathbb{C}$) and taking inverse $A \to A^{-1}$. Those transformations (if defined) clearly conserve arithmetic distance, for example, $\text{rank}(A^{-1} - B^{-1}) = \text{rank}(A^{-1}(B - A)B^{-1}) = \text{rank}(A - B)$. ($A^{-1}$ and $B^{-1}$ is of full rank.) □

2.2 Distance on the spaces of the Weyr characteristics.

Having in mind the Weyr characteristics of complex matrices (see below), we introduce the spaces $\mathcal{S}_n$ of the Weyr characteristics. Where $\mathcal{S}_n$ is the space of functions $\mathbb{Z}^+ \times \mathbb{C} \to \mathbb{N}$, $(i, \lambda) \to \eta_i(\lambda)$ such that

- $\eta_i(\lambda) \neq 0$ for finitely many $(i, \lambda)$ only, and $\sum_{\lambda \in \mathbb{C}} \sum_{i \in \mathbb{N}} \eta_i(\lambda) = n$.
- $\eta_i(\lambda) \geq \eta_{i+1}(\lambda)$.

On $\mathcal{S}_n$ define a metric $d(\eta, \mu) = \max_{(i, \lambda)} \{|\eta_i(\lambda) - \eta_i(\lambda)|\}$. First of all let us note that $d(\cdot, \cdot)$ is indeed a metric. Trivially, $d(\eta, \mu) = 0$ implies $\eta = \mu$ and $d(\cdot, \cdot)$ satisfies triangle inequality as supremum (maximum) of semimetrics. It is clear, that $d(\mu, \nu)$ is also well defined for $\mu$ and $\nu$ in different spaces of Weyr characteristics (for different n). We will need the following

Proposition 3. Let $\mu \in \mathcal{S}_m$ and $n > m$. Then there exists $\nu \in \mathcal{S}_n$ such that for any $\eta \in \mathcal{S}_n$, the inequality $d(\mu, \eta) \geq d(\nu, \eta)$ holds.

Proof. We can do as follows. Let $\mu_i(0) \neq 0$ and $\mu_{i+1}(0) = 0$. We can take $\nu_{i+1}(0) = \nu_{i+2}(0) = \ldots = \nu_{i+n-m}(0) = 1$ and $\nu_{j}(\lambda) = \mu_{j}(\lambda)$ for all other pairs $(j, \lambda)$.

Proposition 4. $\mathcal{S}_n$ are geodesic metric spaces.

Proof. Let $\eta, \mu \in \mathcal{S}_n$ and $d(\eta, \mu) = k > 1$ it is enough to find $\nu \in \mathcal{S}_n$ such that either $d(\eta, \nu) = 1$ and $d(\nu, \mu) = k - 1$, or $d(\eta, \nu) = k - 1$ and $d(\nu, \mu) = 1$, moreover, by Proposition 3 it is enough to find $\nu \in \mathcal{S}_m$ for $m \leq n$. Let $S_+ = \{(j, \lambda) \in \mathbb{Z}^+ \times \mathbb{C} \mid \eta_j(\lambda) - \mu_j(\lambda) = k\}$ and $S_- = \{(j, \lambda) \in \mathbb{Z}^+ \times \mathbb{C} \mid \eta_j(\lambda) - \mu_j(\lambda) = -k\}$. Suppose, that $|S_+| \geq |S_-|$ (if not, we can change $\eta \leftrightarrow \mu$). Now let

$$
\nu_i(\lambda) = \begin{cases} 
\eta_i(\lambda) - 1 & \text{if } (i, \lambda) \in S_+ \\
\eta_i(\lambda) + 1 & \text{if } (i, \lambda) \in S_- \\
\eta_i(\lambda) & \text{if } (i, \lambda) \notin S_+ \cup S_-
\end{cases}
$$

We have to show that $\nu \in \mathcal{S}_m$ for $m = n - |S_+| + |S_-|$. It is enough to show that $\nu_{j+1}(\lambda) \leq \nu_j(\lambda)$. Suppose contrary, that $\nu_{j+1}(\lambda) > \nu_j(\lambda)$. There are three possibility:

- $\eta_{j+1}(\lambda) = \eta_j(\lambda), (j, \lambda) \in S_+$ and $(j + 1, \lambda) \notin S_+$, but then $k > \eta_{j+1}(\lambda) - \mu_{j+1}(\lambda) \geq \eta_j(\lambda) - \mu_j(\lambda) = k$, a contradiction.

- $\eta_{j+1}(\lambda) = \eta_j(\lambda), (j + 1, \lambda) \notin S_-$, but then $-k < \eta_j(\lambda) - \mu_j(\lambda) \leq \eta_j(\lambda) - \mu_j(\lambda) = k$, a contradiction.

- $\eta_{j+1}(\lambda) = \eta_j(\lambda) - 1, (j, \lambda) \in S_+$ and $(j + 1, \lambda) \in S_-$, but then $-k = \eta_{j+1}(\lambda) - \mu_{j+1}(\lambda) \geq \eta_j(\lambda) - \mu_j(\lambda) = k$, so $-k \geq k - 1$ and $1/2 \geq k$, a contradiction with $k > 1$.

Now, by construction, $d(\nu, \eta) = 1$ and $d(\nu, \mu) = k - 1$. □
2.3 Distances $dc$ and $\tilde{dc}$ on finite multisets of the complex numbers.

Multisets and operations. The language of multisets is very convenient to deal with spectrums. We will need only finite multisets. For a multiset $A$ let $\text{set}(A)$ denote the set of elements of $A$, forgetting multiplicity. It is clear that a multiset maybe considered as the multiplicity function $\chi_A : \text{set}(A) \to \mathbb{N}$, for any $x \not\in A$ we will suppose $\chi_A(x) = 0$. (For all cases, considered here, $\text{set}(A) \subset \mathbb{C}$, so we can consider $\chi_A : \mathbb{C} \to \mathbb{N} = \{0, 1, 2, \ldots\}$.) As far as the authors aware, there are several generalizations of the set-theoretical operations to multisets. We will need the following operations:

- Difference of two multiset $A \setminus B$, $\chi_{A \setminus B}(x) = \max\{0, \chi(A) - \chi(B)\}$.
- Intersection $A \cap X$ of a set $X$ and a multiset $A$,
  $$\chi_{A \cap X}(x) = \begin{cases} \chi_A(x), & \text{if } x \in X \\ 0, & \text{if } x \notin X \end{cases}$$
- Union $A \cup B$, $\chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x)$.

Let $S(a, r) = \{x \in \mathbb{C} : |x - a| \leq r\}$ and $S = \{S(a, r) \mid a \in \mathbb{C}, \ r \in \mathbb{R}^+\}$ be the set of circles. For $A, B \subset M$ let $dc(A, B) = \max_{S \in S}(||A \cap S| - |B \cap S||)$. Let as extend $S$ to $\tilde{S}$ which include interior of complements of circles and semiplains: $\tilde{S} = S \cup \{\{x \in \mathbb{C} : |x - a| \geq r\} \mid a \in \mathbb{C}, \ r \in \mathbb{R}^+\} \cup \{\{x \in \mathbb{C} : \text{Im}(\frac{a-b}{a}) \geq 0\} \mid a, b \in \mathbb{C}\}$; and introduce new metric $\tilde{dc}$: $\tilde{dc}(A, B) = \max_{S \in S}(||A \cap S| - |B \cap S||)$. 

Proposition 5. $dc$ and $\tilde{dc}$ are metrics on the finite multisubsets of $\mathbb{C}$.

- $dc(A, B) = dc(A \setminus B, B \setminus A)$, $\tilde{dc}(A, B) = \tilde{dc}(A \setminus B, B \setminus A)$.
- If $|A| = |B|$, then $\tilde{dc}(A, B) = dc(A, B)$.

Proof. The same as for the spaces of Weyr characteristics.

- Let $\sum_S(A, B) = |A \cap S| - |B \cap S| = \sum_{x \in S} (\chi_A(x) - \chi_B(x))$. Then $\sum_S(A \setminus B, B \setminus A) = \sum_{x \in S} (\max\{0, \chi_A(x) - \chi_B(x)\} - \max\{0, \chi_B(x) - \chi_A(x)\}) = \sum_{x \in S} (\chi_A(x) - \chi_B(x))$. Now the item follows by definition of $dc$ ($\tilde{dc}$).
- First of all, due to $A$ and $B$ are finite multisets, for any semiplain $p$ we can find a circle $c$ such that $\sum_p(A, B) = \sum_p(A, B)$. Also for any closed circle $c$, there exists an open circle $c_0$ such that $\sum_{c_0}(A, B) = \sum_{c_0}(A, B)$. Now, under assumption of the item $\sum_S(A, B) = \sum_{c_0}(A, B)$ and the result follows.

Proposition 6. Let $\phi(x) = \frac{ax + b}{cx + d}$ be a Möbius transformation of $\mathbb{C}$, defined on $\text{set}(A) \cup \text{set}(B)$. Then $\tilde{dc}(A, B) = \tilde{dc}(\phi(A), \phi(B))$.

Proof. A Möbius transformation defines a bijection of $\tilde{S}$.

We don’t know if the metric $dc$ is geodesic on the multisets with fixed cardinality, but its restriction on any circle $o$ line is:

Proposition 7. Let $l \subset \mathbb{C}$ be a circumference or a straight line. Let $A, B \subset M$, $|A| = |B| = n$ and $dc(A, B) = k \geq 2$. Then there exists $C \subset M$, $|C| = n$ such that $dc(A, C) = 1$ and $dc(C, B) = k - 1$. 

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Proof. By Proposition 6 it is enough to proof it for the unit circle. Let us start with the case when set(\(A\)) \(\cap\) set(\(B\)) = \(\emptyset\). Let \(\Gamma = \text{set}(A) \cup \text{set}(B) \subset C^1\). Let \(|\Gamma| = r\). We will cyclically anticlockwise order \(\Gamma = \{\gamma_0, \gamma_1, \ldots, \gamma_{r-1}\}\) by elements of \(Z_r\). To construct \(C\) we move each element of \(A\) to the next element in \(\Gamma\), precisely, set(\(C\)) \(\subseteq\) \(\Gamma\) and

\[
\chi_C(\gamma_i) = \max\{0, \chi_A(\gamma_i) - 1\} + \chi_{\text{set}(A)}(\gamma_{i-1}),
\]

the other words

\[
\chi_C(\gamma_i) = \begin{cases} 
\chi_A(\gamma_i) - 1 & \text{if } \gamma_i \in \text{set}(A) \text{ and } \gamma_{i-1} \notin \text{set}(A) \\
1 & \text{if } \gamma_i \notin \text{set}(A) \text{ and } \gamma_{i-1} \in \text{set}(A) \\
\chi_A(\gamma_i) & \text{for the other cases}
\end{cases}
\]

We will check that \(C\) satisfies our needs. For \(x, y \in \Gamma\) let \([x, y]\) denote the closed segment of \(C^1\), starting from \(x\) and going anticlockwise to \(y\) (so \([x, y]\) \(\cup\) \([y, x]\) = \(C^1\)). It is clear that for \(X, Y \subset M\) one has \(dc(X, Y) = \max\{|X \cap [\alpha, \beta]| - |Y \cap [\alpha, \beta]| : \alpha, \beta \in \text{set}(X) \cup \text{set}(Y)\}\). Denote by \(\sum_{[\alpha, \beta]}(X, Y) = |X \cap [\alpha, \beta]| - |Y \cap [\alpha, \beta]|\).

Now, \(dc(A, C) = dc(A \setminus C, C \setminus A) = 1\), for \(A \setminus C\) and \(C \setminus A\) are interlacing sets on \(C^1\). Suppose further, \(dc(C, B) = dc(C \setminus B, B \setminus C) \geq k\), then there exists \([\gamma_i, \gamma_j]\) such that either

1. \(\sum_{[\gamma_i, \gamma_j]}(C \setminus B, B \setminus C) \geq k\)
2. \(\sum_{[\gamma_i, \gamma_j]}(C \setminus B, B \setminus C) \leq -k\).

In the first case we may assume that \(\gamma_i, \gamma_j \in C \setminus B\) and \(\gamma_{i-1}, \gamma_{j+1} \notin C \setminus B\). Now, changing interval if necessary \((i^n = i \mp 1\) and \((or) j^n = j - 1\) we may, keeping \(\sum_{[\gamma_i, \gamma_j]}(C \setminus B, B \setminus C)\), achieve that \(\gamma_i, \gamma_j \in A\) and \(\gamma_{i-1}, \gamma_{j+1} \notin A\). Then \(\sum_{[\gamma_i, \gamma_j]}(A, B) = \sum_{[\gamma_i, \gamma_j]}(C \setminus B, B \setminus C) + 1 \geq k + 1\), a contradiction. The second case may be considered similarly.

If set(\(A\)) \(\cap\) set(\(B\)) \(\neq 0\) then we can find \(C'\) for \(A \setminus B\) and \(B \setminus A\) and then take \(C = C' \cup X\), where \(X = A \setminus (A \setminus B) = B \setminus (B \setminus A)\).

\[\Box\]

3 On spectrum of low rank perturbations.

**Theorem 1** (Thompson). Let \(n \times n\) matrix \(A\) over a field \(F\) have similarity invariants \(h_n(A) \mid h_{n-1}(A) \mid \ldots \mid h_1(A)\). Then: as column \(n\) tuple \(x\) and row \(n\)-tuple \(y\) range over all vectors entries in \(F\), the similarity invariants assumed by the matrix

\[
B = A + xy
\]

are precisely the monic polynomials \(h_n(B) \mid \ldots \mid h_1(B)\) over \(F\) for which degree(\(h_1(B) \cdots h_n(B)\)) = \(n\) and

\[
\begin{align*}
\{h_n(B) \mid h_{n-1}(A) \mid h_{n-2}(B) \mid h_{n-3}(A) \mid \ldots, \\
\{h_n(A) \mid h_{n-1}(B) \mid h_{n-2}(A) \mid h_{n-3}(B) \mid \ldots
\end{align*}
\]

We are going to reformulate Theorem \(1\) for the field \(C\) using Weyr characteristic. Let \(\eta_m(A, \lambda)\) denote the number of \(\lambda\)-Jordan blocks in \(A\) of size greater or equal to \(m\) \((m \in N)\).

\[
\eta_m(A, \lambda) = \dim \text{Ker}(\lambda E - A)^m - \dim \text{Ker}(\lambda E - A)^{m-1}
\]

This sequence of numbers \(\eta_1(A, \lambda), \ldots, \eta_q(A, \lambda)\) is called the Weyr characteristic for the eigenvalue \(\lambda\) of matrix \(A\), see [14].

**Theorem 2.** Let \(A \in C_{n \times n}\) with Weyr invariants \(\eta_m(A, \lambda)\). Then as \(R\) ranges over all \(n \times n\) complex matrices of rank less or equal \(k\), the Weyr invariants assumed by the matrix \(B = A + R\) are precisely those, that satisfy both of the following conditions:

\[\Box\]
Corollary 1. If the geometric multiplicity of any eigenvalue \( \lambda \) of \( A \) (number of \( \lambda \)-Jordan cells) is 1, then for any multiset \( M \) of size \( n \) there is a rank 1 matrix \( B \) such that \( sp(A + B) = M \).

4 Case of normal matrices.

We will say that the vector \( x \) is an \( \alpha \)-eigenvector if \( Ax = \alpha x \). We will denote by \( R(A, \lambda, \epsilon) \) the space generated by all the \( \alpha \)-eigenvectors of \( A \) with \( |\lambda - \alpha| \leq \epsilon \). For the case of normal matrices, the

\[
\begin{array}{cccccccc}
q_1(B, \lambda) & \eta_1(B) & \eta_2 & \eta_3 & \eta_4 & \cdots & \eta_k \\
q_2(B, \lambda) & \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\
q_3(B, \lambda) & \bullet & \bullet & \bullet & \cdots & \bullet \\
\vdots \\
q_{n-1}(B, \lambda) & \bullet & \cdots & \bullet \\
q_n(B, \lambda) & \bullet & \cdots & \bullet \\
\eta_1(A) & \eta_2 & \eta_3 & \eta_4 & \cdots & \eta_k \\
q_2(A, \lambda) & \bullet & \bullet & \bullet & \cdots & \bullet \\
q_3(A, \lambda) & \bullet & \bullet & \cdots & \bullet \\
q_4(A, \lambda) & \bullet & \cdots & \bullet \\
\vdots \\
q_{n}(A, \lambda) & \bullet & \cdots & \bullet \\
\end{array}
\]

Figure 1: Relation between Ferrers diagrams of \((B, \lambda)\) and \((A, \lambda)\)

- For any \( \lambda \in \mathbb{C} \) and any \( m \in \mathbb{N} \)
  \[
  | \eta_m(A, \lambda) - \eta_m(B, \lambda) | \leq k.
  \]

- \[
  \sum_{\lambda \in \mathbb{C}} \sum_{m \in \mathbb{N}} \eta_m(B, \lambda) = n.
  \]

Proof. It is enough to prove the theorem for \( k = 1 \). Indeed, assume that the theorem is valid for \( k = 1 \). Then we may consider the Weyr characteristics as a map \( \psi : \mathbb{C}_{n \times n} \rightarrow \mathbb{Z}_n \), which satisfies Proposition \( \ref{prop:weyr-characteristics} \). So, Theorem \( \ref{thm:weyr-characteristics} \) follows, though it states that \( \psi(O_k(A)) = O_k(\psi(A)) \).

Let us prove Theorem \( \ref{thm:weyr-characteristics} \) for \( k = 1 \). For a given eigenvalue \( \lambda \in sp(A) \) the sequence of numbers

\[
q_1(A, \lambda) \geq q_2(A, \lambda) \geq \ldots
\]

corresponding to the sizes of the \( \lambda \)-Jordan blocks in the Jordan normal form of \( A \) are known as the Segre characteristics of \( A \) relative to \( \lambda \).

The similarity invariant factors of \( A \in \mathbb{C}_{n \times n} \) are sequence of monic polynomials in \( x \), \( h_n(A) | h_{n-1}(A) | h_{n-2}(A) \ldots | h_1(A) \). It is known that \( h_i(A) = \prod_{\lambda} (\lambda - x)^{\eta(A, \lambda)} \), where \( \lambda \in sp(A) \) and \( q_i(A, \lambda) \) is a Segre characteristic corresponding to \( \lambda \). So, if \( rank(A - B) = 1 \), by Thompson’s theorem

\[
q_1(B, \lambda) \geq q_2(B, \lambda) \geq \ldots
\]

As Weyr characteristic is the conjugate partition of Segre characteristic, we can use Ferrers diagram to compute \( \eta_m(B, \lambda) \) (see, \( \ref{fig:weyr-characteristics} \)) as the number of points of column \( m \) in Ferrer diagram of the Segre characteristic of \( B \) relatively to \( \lambda \) (by short the Ferrer diagram of \((B, \lambda)\)). Precisely, the Ferrer diagram for \( q(B, \lambda) \) is the set \( F_B^\lambda = \{(i, j) \in \mathbb{Z}^+ \times \mathbb{Z}^+ | j \leq q_i(B, \lambda)\} \), see Figure \( \ref{fig:weyr-characteristics} \) (the numbering from top to bottom and left to right). The Weyr characteristics is related with Ferrer diagram by the formula \( \eta_j(B, \lambda) = |\{(x, y) \in F_B^\lambda | x = j\}| \). From inequalities \( \ref{ineq:weyr-characteristics} \) we have that \( q_i(B, \lambda) \geq q_{i+1}(A, \lambda) \). This inequality is equivalent to the statement \( \forall i \neq 1 \ (i, j) \in F_A^\lambda \rightarrow (i - 1, j) \in F_B^\lambda \) (Figure \( \ref{fig:weyr-characteristics} \)), which is equivalent to the fact that \( \eta_j(B, \lambda) \geq \eta_j(A, \lambda) - 1 \). In similar form, from inequalities \( \ref{ineq:weyr-characteristics} \) we can observe that \( q_i(A, \lambda) \geq q_{i+1}(B, \lambda) \) with \( 1 \leq i \leq n - 1 \). Therefore, \( \eta_j(A, \lambda) \geq \eta_j(B, \lambda) - 1 \), and the theorem follows.

\[\Box\]

Corollary 1. If the geometric multiplicity of any eigenvalue \( \lambda \) of \( A \) (number of \( \lambda \)-Jordan cells) is 1, then for any multiset \( M \) of size \( n \) there is a rank 1 matrix \( B \) such that \( sp(A + B) = M \).
following theorem shows that the difference between the dimension of $\mathcal{R}(A, \lambda, \epsilon)$ and the dimension $\mathcal{R}(B, \lambda, \epsilon)$ is bounded by the rank of the difference matrix $A - B$.

**Theorem 3.** If $A$ and $B$ are normal matrices, then for any $\lambda$, and for any $\epsilon \geq 0$,

$$| \dim(\mathcal{R}(A, \lambda, \epsilon)) - \dim(\mathcal{R}(B, \lambda, \epsilon)) | \leq \text{rank}(A - B)$$

Let $X^\perp$ be the orthogonal complement of subspace $X$ and $P_X$ be an orthogonal projection on $X$.

**Lemma 2.** Let $N : L \to L$ be a normal operator, and $X$ be a subspace of $L$ such that $\|(N - \lambda)x\| \leq \epsilon\|x\|$ for any $x \in X$, then (we will write $\mathcal{R}(\lambda, \epsilon)$ for $\mathcal{R}(N, \lambda, \epsilon)$)

1. $P_{\mathcal{R}(\lambda, \epsilon)} x \neq 0$ for any $x \in X$, $x \neq 0$.
2. $\|P_{\mathcal{R}(\lambda, \epsilon)} x\| \geq \sqrt{1 - \frac{1}{a^2}} \|x\|$ for any $x \in X$.
3. $\dim(\mathcal{R}(\lambda, \epsilon)) \geq \dim(X)$

**Proof.** It is clear that (1) implies (3).

(1) Let $e_1, e_2, ..., e_n$ be a diagonal orthonormal basis for $N$, and $\lambda_1, \lambda_2, ..., \lambda_n$ corresponding eigenvalues ($Ne_i = \lambda_i e_i$). Let $x = \alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n \in X$ and $\|x\| = \sqrt{\sum_{i=1}^{n} |\alpha_i|^2} = 1$. Let $x = x_1 + x_2$ where $x_1 \in \mathcal{R}(\lambda, \epsilon)$ and $x_2 \in \mathcal{R}^\perp(\lambda, \epsilon)$. Now, $\|(N - \lambda)x\|^2 = \sum_{i=1}^{n} |\alpha_i|^2|\lambda_i - \lambda|^2 \leq \epsilon^2$ implies that

$$\sum_{i \mid |\lambda_i - \lambda| > \epsilon} |\alpha_i|^2 < 1.$$ 

So, $P_{\mathcal{R}(\lambda, \epsilon)}(x) = x_1 = \sum_{i \mid |\lambda_i - \lambda| \leq \epsilon} \alpha_i e_i \neq 0$.

(2) Similarly,

$$\sum_{i \mid |\lambda_i - \lambda| > \epsilon} |\alpha_i|^2 < \frac{1}{a^2}$$ and

$$\sum_{i \mid |\lambda_i - \lambda| \leq \epsilon} |\alpha_i|^2 \geq (1 - \frac{1}{a^2}),$$

so, $\|P_{\mathcal{R}(\lambda, \epsilon)} x\| \geq \sqrt{1 - \frac{1}{a^2}} \|x\|$.

Now we are ready to prove Theorem 3. Let $\text{rank}(A - B) = r$ then there exists $X = \mathcal{R}(A, \lambda, \epsilon) \cap \ker(A - B)$ with $\dim(X) \geq \dim(\mathcal{R}(A, \lambda, \epsilon)) - r$ and $A|_X = B|_X$, so $\|(B - \lambda)|_X\| \leq \epsilon$ and, by Lemma 2 $\dim(\mathcal{R}(B, \lambda, \epsilon)) \geq \dim(\mathcal{R}(A, \lambda, \epsilon)) - r$. By symmetry, we get Theorem 3.

Theorem 3 implies that any circle in complex plain, containing $m$ spectral points of $A$ ($B$) should contain at least $m - k$ spectral points of $B$ ($A$). So, we have

**Corollary 2.** If $A$ and $B$ are normal matrices then $\text{dc}(\text{sp}(A), \text{sp}(B)) \leq \text{rank}(A - B)$.

**Proof.** It is just a reformulation of Theorem 3.

If the condition of Corollary 2 describes all accessible by rank k perturbation spectra? We are going to show that the answer is "yes" for self-adjoint and unitary matrices.

**Theorem 4.** Let $A$ be a self-adjoint (unitary) $n \times n$-matrix. Let $B \subset M \subset M(C^1)$, $|B| = n$. Then there exists self-adjoint (unitary) matrix $B$ such that $\text{sp}(B) = B$ and $\text{rank}(A - B) = \text{dc}(\text{sp}(A), B)$.

In fact the following, more general result is valid:

**Theorem 5.** Let $l \subset \mathbb{C}$ be a circumference or straight line. Let $A$ be a normal $n \times n$-matrix, $\text{sp}(A) \subset M \subset l$. Let $B \subset M \subset l$, $|B| = n$. Then exists a normal matrix $B$ such that $\text{sp}(B) = B$ and $\text{rank}(A - B) = \text{dc}(\text{sp}(A), B)$. 

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• It is enough to prove Theorem 5 for self adjoint matrices. Indeed, let \( sp(A), B \subset_R l, |B| = n \) for a circle \((l) \subset C\). Then there exists a Möbius transformation \( \phi \), defined on \( sp(A) \cup B \), which map \( l \) to the real line. Then \( \phi(A) \) is a self-adjoint matrix and we can apply Theorem 4 to \( \phi(A) \) and \( \phi(B) \) to find \( \tilde{B} \) with \( sp(\tilde{B}) = \phi(B) \) and \( rank(\phi(A) - \tilde{B}) = dc(\phi(sp(A)), \phi(B)) \). Now take \( B = \phi^{-1}(\tilde{B}) \) and results follows, for the Möbius transformations conserve arithmetic distance on \( C_{n \times n} \) and the distance \( dc \) on multisets (Proposition 2 and Proposition 6).

• It is enough to prove Theorem 5 for \( dc(sp(A), B) = 1 \) and the rest will follow from Proposition 1, Proposition 7 and Lemma 1.

• Also w.l.g. we may assume that \( set(sp(A)) \cap set(B) = \emptyset \). For if \( X = sp(A) \setminus (sp(A) \setminus B) \) we can write \( A = A_1 \oplus A_2 \) with \( sp(A_1) = X \) and \( sp(A_2) = sp(A) \setminus X \). We can find \( B_2 \) with \( sp(B_2) = B \setminus X \) and \( rank(A_2 - B_2) = 1 \). Now, take \( B = A_1 \oplus B_2 \).

• Let \( A, B \subset M \), \( set(A) \cap set(B) = \emptyset, |A| = |B| \) and \( dc(A, B) = 1 \). Then, in fact, \( A \) and \( B \) are interlacing sets. It means that if \( A = \{\alpha_1, \alpha_2, ..., \alpha_n\} \), \( B = \{\beta_1, \beta_2, ..., \beta_n\} \) then \( \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < ... \) or \( \beta_1 < \alpha_1 < \beta_2 < \alpha_2 < ... \).

So, we need to prove only

**Lemma 3.** Let \( A \in C_{n \times n} \) be a self-adjoint matrix with a simple spectrum. Let \( B \subset R \) with \( |B| = n \). If \( sp(A) \) and \( B \) are interlacing then there exists a self-adjoint matrix \( B \) with \( sp(B) = B \) and \( rank(A - B) = 1 \).

Let \( sp(A) = A = \{\alpha_1, \alpha_2, ..., \alpha_n\} \) and \( B = \{\beta_1, \beta_2, ..., \beta_n\} \). As \( A \) and \( B \) can be put in diagonal normal form \( A = diag(\alpha_1, \alpha_2, ..., \alpha_n) \) and \( B = diag(\beta_1, \beta_2, ..., \beta_n) \) by unitary transformations and unitary transformations map \((\phi)\) self-adjoint matrices to self adjoint matrices, Lemma 3 is equivalent to the fact that under our assumptions on \( A \) and \( B \) the equation

\[
\tilde{A}X - X\tilde{B} = R
\]  

has a solution in \((X, R)\) for unitary \( X \) and \( R \) of rank 1. Before solving Eq 2 let us introduce some notations and prove a proposition. For a finite \( A \subset R \) let \( P_A(\lambda) = \prod_{\alpha \in A}(\lambda - \alpha) \). Let \( A \) and \( B \) be finite subsets of \( R \) of equal cardinality. It follows from interpolation that there exist unique \( x : A \rightarrow A \), such that

\[
P_B = P_A - \sum_{\alpha \in A} x_{\alpha} P_{A\setminus{\alpha}}
\]  

(3)

(we write \( x_{\alpha} \) not \( x(\alpha) \)). Moreover, \( x_{\alpha} = P_B(\alpha)/P_{A\setminus{\alpha}}(\alpha) \). Studying signs of \( P_B(\alpha) \) and \( P_{A\setminus{\alpha}}(\alpha) \) we trivially get

**Proposition 2.** If \( A \) and \( B \) interlacing, then all \( x_{\alpha} \) in Eq 3 have the same sign.

**Remark.** In fact inverse of this proposition is also valid, see Lemma 1.20 of [8].

Now let us go back to solutions of Eq 2. For \( R = \{r_{ij}\} \) fixed the equation has the unique solution \( X = \{x_{ij}\} \), with \( x_{ij} = \frac{r_{ij}}{\alpha_i - \beta_j} \). Now, suppose that \( rank(R) = 1 \) or, the same \( r_{ij} = y_iz_j \) for some \( y, z \in C^n, y, z \neq 0 \). When the matrix \( X \) is unitary? When its columns (rows) are orthonormal, or

\[
z_jz_k^* \sum_{i} \frac{|y_i|^2}{(\alpha_i - \beta_j)(\alpha_i - \beta_k)} = \delta_{jk}
\]  

(4)

It follows that \( z_j \neq 0 \) for all \( j = 1, ..., n \), changing rows by columns we get the same for \( y \). So, the difficult part is to guarantee that l.h.s. of Eq. 4 is 0 for \( j \neq k \). Putting equality

\[
\frac{|y_i|^2}{(\alpha_i - \beta_j)(\alpha_i - \beta_k)} = \frac{1}{\beta_j - \beta_k} \left( \frac{|y_k|^2}{\alpha_i - \beta_k} - \frac{|y_i|^2}{\alpha_i - \beta_j} \right).
\]
into Eq\,[\text{I}] and multiplying it by $P_A(\beta_k)P_A(\beta_j)(\beta_j - \beta_k)/(z_jz_k^*)$ we get (after some elementary transformations):

$$P_A(\beta_k)\sum_{i=1}^n |y_i|^2 P_{A\setminus\{\alpha_i\}}(\beta_j) = P_A(\beta_j)\sum_{i=1}^n |y_i|^2 P_{A\setminus\{\alpha_i\}}(\beta_k),$$

which imply that there exists $c$, such that

$$\sum_{i=1}^n |y_i|^2 P_{A\setminus\{\alpha_i\}}(\beta_j) = cP_A(\beta_j),$$

for all $j$. Or the same, $B$ is the set of roots of polynomial

$$\Phi(\lambda) = \sum_{i=1}^n |y_i|^2 P_{A\setminus\{\alpha_i\}}(\beta_j) - cP_A(\beta_j),$$

so $|y_i|^2 = P_B(\alpha_i)/cP_{A\setminus\{\alpha_i\}}(\alpha_i)$. Choosing $c = 1$ or $c = -1$, we get, by Proposition [\text{S}] that $|y_i|^2$ is well defined. Now, take, for example, $y_i = |y_i|$. From Eq\,[\text{I}] for $j = k$ we can find $|z_j|^2$. Then, taking $z_j = |z_j|$ we get needed solution of Eq\,[\text{I}]

5 Almost unitary operators are near unitary operators with respect to normalized arithmetic distance

For $A, B \in C_{n \times n}$, let $d_r(A, B)$ be the normalized arithmetic distance:

$$d_r(A, B) = \frac{\text{rank}(A - B)}{n}$$

The matrix $A$ is called an $\alpha$-self-adjoint matrix if $d_r(A, A^*) = \alpha$, where $A^*$ denotes the adjoint of $A$. The matrix $A$ is called an $\alpha$-unitary matrix if $d_r(A^*A, E) = \alpha$

The following theorem says that "near" to any $\alpha$-self-adjoint matrix there exists a self-adjoint matrix $S$, and that "near" to any $\alpha$-unitary matrix there exists an unitary matrix $U$ (for small $\alpha$).

**Theorem 6.** For any $A \in C_{n \times n}$ there exists a self-adjoint matrix $S \ (S = S^*)$ such that $d_r(A, S) \leq d_r(A, A^*)$.

**Proof.** Take $S = \frac{1}{2}(A + A^*)$. \hfill \Box

**Theorem 7.** For any $A \in C_{n \times n}$ there exists a unitary matrix $U \ (U^*U = E)$, such that $d_r(A, U) \leq d_r(A^*A, E)$.

The good illustrations for this theorem are 0-Jordan cells:

$$
\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\end{pmatrix}
\begin{pmatrix}
0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 1 \\
\end{pmatrix},
$$

but the matrix

$$
\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & \cdots & 1 \\
1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\end{pmatrix}
$$
is unitary.

**Proof.** Let $\text{rank}(A^*A - E) = r$, so there exist subspace $X \subset L$, $\text{dim}(X) = n - r$ such that $A^* A|_X = E|_X$. Consider $A|_X : X \rightarrow Y = A(X)$. Under assumptions of the theorem $A^*(Y) = X$, it follows that $(A|_X)^* = A^*|_Y : Y \rightarrow X$, so $A|_X : X \rightarrow Y$ is an unitary operator. Choose any unitary operator $B : X^\perp \rightarrow Y^\perp$ ($B^* B = E_{X^\perp}$). Then $U = A|_X \oplus B$ proves the theorem. \[ \square \]

It is not clear if this proof could be adapted for normal matrices – unitary operator from an unitary space to another unitary space is well defined, but how to define normal operators between different unitary spaces...

Question: If we define $\alpha$-normal matrices in similar form to self-adjoint and unitary matrices, the equivalent of theorems \[ \underline{[3]} \] and \[ \underline{[7]} \] are true for normal matrices?

6 Almost commuting matrices

**Theorem 8.** For every $4 \leq n \in \mathbb{N}$ and every $A \in \mathbb{C}_{n \times n}$ with simple spectrum there exists $X \in \mathbb{C}_{n \times n}$ such that $d_r(AX,XA) < 2/n$ and for any matrix $B$, commuting with $A$, $d_r(B,X) \geq \frac{1}{2}$.

Before starting the proof of the theorem we need some facts.

**Proposition 9.** Let $\{\lambda_1,\ldots,\lambda_k\}$ and $\{\alpha_1,\ldots,\alpha_k\}$ be two disjoint sets, then the matrix $M = [x_{ij}]$ with $x_{ij} = \frac{1}{\alpha_i - \lambda_j}$ is nonsingular.

**Proof.** The matrix $M$ has the form

$$
M = \begin{pmatrix}
\frac{1}{\alpha_1 - \lambda_1} & \frac{1}{\alpha_1 - \lambda_2} & \cdots & \frac{1}{\alpha_1 - \lambda_k} \\
\frac{1}{\alpha_2 - \lambda_1} & \frac{1}{\alpha_2 - \lambda_2} & \cdots & \frac{1}{\alpha_2 - \lambda_k} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{1}{\alpha_k - \lambda_1} & \frac{1}{\alpha_k - \lambda_2} & \cdots & \frac{1}{\alpha_k - \lambda_k}
\end{pmatrix}
$$

Let $P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_k)$. Multiply each row $j$ of the matrix $M$ by $P(\alpha_j)$ we obtain a matrix of the form

$$
\tilde{M} = \begin{pmatrix}
P_1(\alpha_1) & P_2(\alpha_1) & \cdots & P_k(\alpha_1) \\
P_1(\alpha_2) & P_2(\alpha_2) & \cdots & P_k(\alpha_2) \\
\cdots & \cdots & \cdots & \cdots \\
P_1(\alpha_k) & P_2(\alpha_k) & \cdots & P_k(\alpha_k)
\end{pmatrix}
$$

where $P_j(\lambda) = \frac{P(\lambda)}{\lambda - \alpha_j}$. This matrix will be nonsingular if and only if matrix $M$ is nonsingular. We will prove that matrix $\tilde{M}$ is nonsingular showing that the following system of linear equations has a unique solution:

$$
\begin{pmatrix}
P_1(\alpha_1) & P_2(\alpha_1) & \cdots & P_k(\alpha_1) \\
P_1(\alpha_2) & P_2(\alpha_2) & \cdots & P_k(\alpha_2) \\
\cdots & \cdots & \cdots & \cdots \\
P_1(\alpha_k) & P_2(\alpha_k) & \cdots & P_k(\alpha_k)
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_k
\end{pmatrix}
= 
\begin{pmatrix}
b_1 \\
b_2 \\
\vdots \\
b_k
\end{pmatrix}
$$

So we have to solve the system $\sum_{i=1}^k a_i P_i(\alpha_j) = b_j$. Consider the polynomial $\Phi(\lambda) = \sum_{i=1}^k a_i P_i(\lambda)$, note that $\Phi(\lambda_i) = a_i P_i(\lambda_i)$ because $P_i(\lambda_j) = 0$ for $i \neq j$. We have $k$ points $b_j$, therefore, we can use Lagrange interpolation to find the unique polynomial $\Phi(\lambda)$ of degree $k - 1$ such that $\Phi(\alpha_j) = b_j$, and then we can compute the values $a_i = \frac{\Phi(\lambda_i)}{P_i(\lambda_i)}$ ($P_i(\lambda_i) \neq 0$ for all $\lambda_i$ are different). \[ \square \]

Now we are ready to prove Theorem \[ \underline{[8]} \]
Proof. Consider the matrix equation

\[ AX - XA = \{ c_{ij} \}, \]  

(5)

with \( c_{ij} = i + j \mod 2 \). This matrix has the following form

\[
AX - XA = \begin{pmatrix}
0 & 1 & 0 & 1 & \cdots & 1 + n \mod 2 \\
1 & 0 & 1 & 0 & \cdots & n \mod 2 \\
0 & 1 & 0 & 1 & \cdots & 1 + n \mod 2 \\
1 & 0 & 1 & 0 & \cdots & n \mod 2 \\
\vdots & & & & & \ddots \\
n + 1 \mod 2 & & & & & 0
\end{pmatrix}
\]

Let \( A \) be a diagonal matrix with simple spectrum \( A = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \). One matrix \( X \) that satisfies Eq. (5) is \( X = \{ x_{ij} \} \) with

\[
x_{ij} = \begin{cases}
\frac{c_{ij}}{\lambda_i - \lambda_j} & \text{for } i \neq j \\
0 & \text{for } i = j
\end{cases}
\]

Every matrix \( B \) that commute with \( A \) should be necessarily a diagonal matrix \( B = \text{diag}(b_1, b_2, \ldots, b_n) \), then \( X - B = \{ x_{ij}' \} \) with

\[
x_{ij}' = \begin{cases}
\frac{c_{ij}}{\lambda_i - \lambda_j} & \text{for } i \neq j \\
-b_i & \text{for } i = j
\end{cases}
\]

If we delete from \( X - B \) the odd columns and the even rows, we obtain a submatrix \( X' \) of size \( \lceil \frac{n}{2} \rceil \times \lceil \frac{n}{2} \rceil \) of the form

\[
\{ x_{ij}' \} = \frac{1}{\lambda_i^* - \lambda_j^*}
\]

with \( i^* = 2i - 1 \), \( j^* = 2j \). By proposition 9 this matrix is nonsingular. Therefore, \( \text{rank}(X - B) \geq \frac{n}{2} \), and we obtain \( d_r(X, B) \geq \frac{1}{2} \).

References

[1] Atkinson, K.E. An introduction to Numerical Analysis, 2nd ed. John Wiley & Sons, Ltd., 1989.

[2] Cauchy, A.L. Sur l’équation à l’aide de laquelle on détermine les inégalités séculaires, in Oeuvres Complètes de A.L.Cauchy, 2nd Ser., V IX, Gauthier-Villars, Paris 1891, pp. 174-195

[3] Che-Hsien Wan, Zhe-Xian Wan, Lo-Keng Hua Geometry of Matrices: In Memory of Professor L.K. Hua (1910-1985) World Scientific, 1996, ISBN 9810226381, 9789810226381

[4] Choi, Man Duen, Almost commuting matrices need not be nearly commuting, Proc. Amer. Math. Soc., 102, (1988), N.3, pp. 529-533

[5] Exel, R. and Loring, T. Almost Commuting Unitary Matrices, Proceedings of the American Mathematical Society, Vol. 106, No. 4. (Aug., 1989), pp. 913-915.

[6] Fan K. and Pall G. Imbedding conditions for hermitian and normal matrix Canada J. Math., 9 (1957) pp. 298-304

[7] Farah, Ilijas, All automorphisms of the calkin algebra are inner, Preprint arXiv:0705.3085v3

[8] Fisk, Steve, Polynomials, roots, and interlacing, Preprint ArXiv: math/0612833
[9] KLYACHKO, ALEXANDER, Vector Bundles, Linear Representations, and Spectral Problems, Preprint ArXiv: math/0304325

[10] LIN, HUAXIN, Almost commuting selfadjoint matrices and applications, in Operator algebras and their applications (Waterloo, ON, 1994/1995), Fields Inst. Commun., 13, pp. 193–233, Amer. Math. Soc..

[11] MORO, J. DOPICO, F. M. Low Rank Perturbation of Jordan Structure, SIAM J. Matrix Anal. Appl., vol. 25, No 2, 2003, pp. 495–506.

[12] SAVCHENKO, S. V. On the change in the spectral properties of a matrix under a perturbation of a sufficiently low rank, (Russian) Funktsional. Anal. i Prilozhen., 38 (2004), no. 1, pp. 85–88; translation in Funct. Anal. Appl. 38 (2004), no. 1, pp 69–71

[13] SAVCHENKO, S. V. Weyr characteristic under low rank perturbation. Preprtin, L.D. Landau Institute for Theoretical Physics, Russian Academy of Science Kosygina str. 2, Moscow, 119334, Russia

[14] SHAPIRO, HELENE, The Weyr characteristic. The American Mathematical Monthly Vol. 106, No. 10, (1999), pp. 919–929.

[15] THOMPSON, ROBERTG Invariant factors under rank one perturbations. Can. J. Math., XXXII, (1980), N.1. pp.240-245

[16] VOICULESCU, DAN, Asymptotically commuting finite rank unitary operators without commuting approximants, Acta Sci. Math. (Szeged), 45, (1983), N.1-4, pp. 429–431,