Involution of polynomially parametrized surfaces

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Abstract

We provide an algorithm for detecting the involutions leaving a polynomially parametrized surface invariant. As a consequence, symmetry axes, symmetry planes or symmetry centers for these surfaces can be determined directly from the parametrization. For this purpose, we show that any involution of the surface comes in fact from an involution of the parameter space, \( \mathbb{R}^2 \), in this case; so, by determining the latter, the former can be found. As a by-product, we also provide a method to detect if a polynomially parametrized surface is a revolution surface. The algorithm has been implemented in the computer algebra system Maple 17. Evidence of its efficiency for moderate degrees, examples and a complexity analysis are also provided.

1. Introduction

Symmetry detection in 3D objects is an important matter in fields like Computer Graphics or Computer Vision. In Computer Graphics, it is useful in order to gain understanding when analyzing pictures, and also to perform tasks like compression, shape editing or shape completion. In Computer Vision, symmetry is important for object detection and recognition. Many techniques have been tried to solve the problem. Some of them involve statistical methods and, in particular, clustering; see for example the papers \cite{10,11,27,34}, where the technique of transformation voting is used, or \cite{42}, based on the Extended Gauss Image. Other techniques are robust auto-alignment \cite{41}, spherical harmonic analysis \cite{26}, feature points \cite{25}, primitive fitting \cite{37}, and spectral analysis \cite{24}, to quote just a few. In addition, there are algorithms for computing the symmetries of 2D and 3D discrete objects \cite{5,8,13,21} and for boundary-representation models \cite{21,22,13}. The list of all papers addressing the subject is really very long, and the interested reader is referred to the bibliographies in these papers to get a more complete list.

In the references on the topic, the object to be analyzed is quite commonly a point cloud or a mesh, sometimes with missing parts, so that little structure

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is assumed on it. One exception here is the case of tensor product surfaces [5]. In this case the geometry of the object, in particular its symmetry, follows from that of a discrete object behind, the control polyhedron, that controls its shape. So, the symmetries of the object can be found by applying methods for discrete objects [5, 8, 18, 21].

In this paper we consider the problem of computing involutions, i.e. symmetries with respect to a point, line or plane, for objects with a stronger structure, namely the set \( S \) of all points defined by a polynomial parametrization

\[
\mathbf{x}(t, s) = (x(t, s), y(t, s), z(t, s))
\]

with \((t, s) \in \mathbb{R}^2\). Such objects, well-known in Constructive Algebraic Geometry and Computer Aided Geometric Design, are called polynomially parametrized algebraic surfaces. Certainly a tensor product surface corresponds to this description whenever \((t, s)\) is restricted to a compact rectangle \([a, b] \times [c, d] \subset \mathbb{R}^2\). However, in our case \((t, s)\) takes values over the whole plane \(\mathbb{R}^2\). Therefore we deal with the global surface \(S\), not just with a piece of it, and an approach like [5, 8, 18, 21] is not applicable here.

In order to solve the problem we assume “good” properties on the parametrization, namely injectivity, except perhaps for a closed subset of (possibly singular) points of \(S\), and surjectivity. Under these conditions, we prove that any involution of the surface is in fact the result of lifting an involution of the plane to \(S\) via the parametrization of the surface. This way, the problem is translated to the parameter space, \(\mathbb{R}^2\) in this case, and in turn shown to be solvable by applying bivariate real polynomial system solving. As a by-product, we also obtain an algorithm to detect whether a polynomially parametrized surface in the above conditions is a revolution surface or not, since this is proven to be equivalent to the existence of infinitely many symmetry planes.

The method can be seen as the generalization to surfaces of some ideas recently applied to compute symmetries of planar or space rational curves [2, 3, 4]. Furthermore, the problem treated here is related to the more general question of extracting geometric invariants from a surface parametrization. This question appears as one of the eight open problems on the interplay between Algebraic Geometry and Geometric Modeling posed by Prof. Ron Goldman in [15].

The structure of the paper is the following. In Section 2 we introduce some generalities on surface parametrizations and isometries, and we prove some results on symmetries of surfaces; although rotational symmetry is not addressed in the paper, some properties of this type of symmetries are considered here, and then used to prove certain facts on involutions. The method itself is presented in Section 3. Section 4 briefly addresses the special case of cylindrical surfaces. Finally, in Section 5 we provide two detailed examples, we address complexity issues, and we report on the practical implementation of the algorithm carried out in the computer algebra system Maple 17.
2. Generalities

2.1. Properness and normality.

Along the paper we consider an algebraic surface $S \subset \mathbb{R}^3$ different from a plane, polynomially parametrized by $x : \mathbb{R}^2 \to \mathbb{R}^3$, where

$$x(t, s) = (x(t, s), y(t, s), z(t, s))$$

and $x(t, s), y(t, s), z(t, s)$ are polynomials in the variables $t, s$ with coefficients in $\mathbb{Q}$. Since $S$ admits a rational, and in fact polynomial, parametrization then in particular it is irreducible. The functions $x(t, s), y(t, s), z(t, s)$ are called the components of $x$, while $t, s$ are referred to as the parameters of $x$. We call the total degree of the parametrization $x$, and we denote it by $n$, to the maximum of the total degrees of the components of $x$. Furthermore, we will assume that $x$ is proper, i.e. birational or equivalently injective for almost all points of $S$ except for at most a closed subset of $S$. In particular, this implies that $x^{-1}$ is a rational map. One can check properness by using the algorithms in [29, 30]; for reparametrization questions one may see [6, 23, 31, 32]. Additionally, for technical reasons we will require $x(0)$ to be a regular point of $S$; this requirement can always be satisfied by applying a random linear change of parameters, if necessary. Also, although we assume to be working with a real parametrization $x : \mathbb{R}^2 \to \mathbb{R}^3$, at certain points of the paper we will implicitly assume that the parametrization can be also considered as $x : \mathbb{C}^2 \to \mathbb{C}^3$, so that the surface can be embedded into the complex space. Similarly for other real mappings in the paper.

We say that the parametrization $x(t, s)$ is normal if it is surjective, i.e. if every point of $S$ is reached via $x$ by some pair of parameters $(t, s) \in \mathbb{C}^2$. This problem has been well studied for the case of rational curves [38]. However, the same problem for surfaces is not completely well understood yet. The question has been addressed in [7, 9] for special kinds of surfaces, and also in [33], where partial results on the problem are presented. In particular, in [33] a sufficient condition for a polynomial parametrization to be normal is given (see Corollary 3.15 and Corollary 4.4 therein). In addition to the above hypotheses, along the paper we will assume that the parametrization $x$ we work with is normal.

2.2. Isometries of algebraic surfaces.

Let us recall some facts from Euclidean geometry [12]. An isometry of $\mathbb{R}^n$ is a map $f : \mathbb{R}^n \to \mathbb{R}^n$ preserving Euclidean distances. Any isometry $f$ of $\mathbb{R}^n$ is linear affine, taking the form

$$f(x) = Qx + b, \quad x \in \mathbb{R}^n,$$

with $b \in \mathbb{R}^n$ and $Q \in \mathbb{R}^{n \times n}$ an orthogonal matrix. In particular $\det(Q) = \pm 1$. For $n = 3$, the isometries of the space form a group under composition that is generated by reflections, i.e., symmetries with respect to a hyperplane, also called mirror symmetries. An isometry is called direct when it preserves the
orientation, and opposite when it does not. In the former case \( \det(Q) = 1 \),
while in the latter case \( \det(Q) = -1 \). The identity map \( \text{id}_{\mathbb{R}^n} \) of \( \mathbb{R}^n \) is called the trivial symmetry. An isometry \( f(x) = Qx + b \) of \( \mathbb{R}^n \) is called an involution if \( f \circ f = \text{id}_{\mathbb{R}^n} \), in which case \( Q^2 = I \) is the identity matrix and \( b \in \ker(Q + I) \).

In the case \( n = 3 \), the classification of the nontrivial isometries of Euclidean space includes reflections (in a plane), rotations (about an axis), and translations, which combine in commutative pairs to form twists (compositions of a rotation about an axis and a translation), glide reflections (composition of a reflection in a plane and a translation), and rotatory reflections (composition of a rotation and a reflection). Composing three reflections in mutually perpendicular planes through a point \( P \), yields a central inversion, also called central symmetry, with center \( P \), i.e., a symmetry with respect to the point \( P \), that will be called the symmetry center. The special case of rotation by an angle \( \pi \) is of special interest, and it is called an axial symmetry. Central inversions, reflections, and axial symmetries are involutions. Furthermore, axial symmetries are direct, while reflections and central inversions are opposite.

In the sequel, we will examine some questions related to algebraic surfaces that are invariant through isometries. For this purpose, we will consider a real, irreducible algebraic surface \( S \). The term “real” means that the real part of \( S \) is 2-dimensional, so that it does not reduce to its singular locus. For instance, the surface defined by \( x^2 + y^2 + z^2 = 0 \) is not real, since its real part reduces to the origin; also \( x^2 + y^2 = 0 \) is not real either because its real part reduces to the \( z \)-axis. Furthermore, we will say that \( S \) is a cylindrical surface (or a cylinder) if it is a ruled surface whose generatrices are all of them parallel to a given direction; additionally, we will say that it is a circular cylinder if all the generatrices are equidistant to a line, called the cylinder axis.

**Lemma 1.** If there exists a line \( \ell \) and a circular cylinder \( \text{Cyl}_\ell \) of axis \( \ell \) which is tangent to \( S \) at every real non-isolated point \( P \in S \), then \( S = \text{Cyl}_\ell \).

**Proof.** By applying if necessary a linear change of coordinates, we can assume that \( \ell \) is the \( z \)-axis. Also, let \( g(x, y, z) = 0 \) be the implicit equation of \( S \). Then for every real point \( P \in S \), it holds that \( \nabla(g)(P) \) is normal to the \( z \)-axis, implying that \( g_z(P) = 0 \). Since \( S \) is real, \( g(x, y, z) = 0 \) and \( g_z(x, y, z) = 0 \) have infinitely many common points defining a 2-dimensional set of \( \mathbb{R}^3 \). Since \( g \) is irreducible by hypothesis then we deduce that \( g \) divides \( g_z \); but since the degree of \( g_z \) is less than that of \( g \), this means that \( g_z \) must be identically 0, and therefore \( g = g(x, y) \). Hence, \( S \) is a cylinder itself and from here we deduce that \( S = \text{Cyl}_\ell \).

**Proposition 2.** If \( S \) is invariant under any of the following transformations:

(i) a translation of vector \( \bar{a} \in \mathbb{R}^3 \),

(ii) a twist operation about an axis \( \ell \), with translation vector \( \bar{a} \),

(iii) a glide reflection about a plane \( \pi \), with translation vector \( \bar{a} \),
then \( S \) is a cylindrical surface and its generatrices are parallel to the vector \( \vec{a} \). More than that, in (ii) \( S \) is a circular cylinder of axis \( \ell \).

**Proof.** As for (i) or (iii), we observe that for every \( P \in S \), the line \( P + \lambda \cdot \vec{a} \) has infinitely many points in common with \( S \); therefore it must be contained in \( S \), and the result follows. So, let us address (ii). Let \( T \) be the twist operation about an axis \( \ell \), with translation vector \( \vec{a} \). If \( S \) is a circular cylinder of axis \( \ell \), then we have finished. Otherwise, by Lemma 1 there exists a real, non-isolated point \( P \in S \) such that the circular cylinder \( \text{Cyl}_{\ell,P} \) of axis \( \ell \), passing through \( P \), is not tangent to \( S \) at \( P \). Furthermore, \( C = \text{Cyl}_{\ell,P} \cap S \) is a non-empty, algebraic space curve. Since \( C \) is invariant under \( T \) and algebraic, it must be the union of several parallel lines. Finally, because \( P \) is generic we deduce that \( S \) and \( \text{Cyl}_{\ell,P} \) have infinitely many lines in common; so, \( \text{Cyl}_{\ell,P} \subset S \), and from the irreducibility of \( S \), \( \text{Cyl}_{\ell,P} = S \).

We will say that \( S \) has **translational symmetry**, **twist symmetry** or **glide reflection-symmetry** if it is invariant with respect to a transformation of the types (i), (ii), (iii) in the above lemma, respectively. From Proposition 2, any surface showing these symmetries must be cylindrical. So, in the rest of the section we focus on non-cylindrical surfaces.

**Proposition 3.** If \( S \) is not cylindrical, the symmetry center of \( S \), if it exists, is unique.

**Proof.** Assume that \( S \) has two symmetry centers, i.e. that it is invariant under two different central symmetries. Since the composition of two central symmetries of centers \( p_1, p_2 \) is a translation of vector \( p_1 - p_2 \), then \( S \) is invariant under a translation. But then from Proposition 2 it must be cylindrical, which cannot happen by hypothesis.

Now we consider some results on the rotation symmetries of \( S \).

**Lemma 4.** Let \( C \) be a planar algebraic curve of degree \( d \) with rotation symmetry, and let \( P, \theta \) be the rotation center and the rotation angle, respectively. If \( C \) is not a circle centered at \( P \), then \( \theta = \frac{2\pi}{m} \) where \( 0 < m \leq 2d \), \( m \in \mathbb{N} \).

**Proof.** Since \( C \) has rotation symmetry, it consists of \( x \) copies of a certain 1-dimensional subset of \( \mathbb{R}^2 \), which is rotated several times to generate the whole \( C \). Furthermore, by considering a real circle \( C^* \) centered at \( P \), the number of real intersection points of \( C^* \) and \( C \) is, by Bezout’s Theorem, at most \( 2d \). Hence, \( x \leq 2d \). On the other hand, by Lemma 2 in [19] it holds that \( \theta = \frac{2\pi}{m} \), with \( m \in \mathbb{N} \); so, \( x \cdot \frac{2\pi}{m} = 2\pi \) and therefore \( m = x \leq 2d \).

**Lemma 5.** Let \( S \) be an algebraic surface of degree \( d \), invariant under a rotation of axis \( L \) and a non-trivial angle \( \theta \) (i.e. \( \theta \neq 2k\pi \), with \( k \in \mathbb{Z} \)). If \( L \) is not a revolution axis, then \( \theta = \frac{2\pi}{m} \), where \( 0 < m \leq 2d \), \( m \in \mathbb{N} \).
Proof. Since by hypothesis \( \mathcal{L} \) is not a revolution axis, there exists a plane \( \Pi \), normal to \( \mathcal{L} \), such that the intersection \( \mathcal{C} = \Pi \cap S \) is not a circle centered at \( P = \Pi \cap \mathcal{L} \). Now \( \mathcal{C} \) is an algebraic planar curve, of degree at most \( d \); furthermore, since \( S \) has rotation symmetry about \( \mathcal{L} \) then \( \mathcal{C} \) has rotation symmetry with center \( P = \Pi \cap \mathcal{L} \), with rotation angle \( \theta \). But then the result follows from Lemma 4.

\( \square \)

**Lemma 6.** The surface \( S \) cannot have two different parallel rotation axes.

**Proof.** Let \( \mathcal{L} \) be a rotation axis of \( S \). For a given plane \( \Pi \) normal to \( \mathcal{L} \), a rotation of \( S \) around \( \mathcal{L} \) induces a rotation of the planar curve \( S \cap \Pi \) around the point \( \mathcal{L} \cap \Pi \). By contradiction, if \( S \) has another rotation axis \( \mathcal{L}' \neq \mathcal{L} \), parallel to \( \mathcal{L} \), then \( S \cap \Pi \) exhibits rotation symmetry around two different rotation centers (the intersections of \( \Pi \) with \( \mathcal{L}, \mathcal{L}' \)). However, the rotation center of an algebraic curve, if any, is unique (see Theorem 5.3 in [20]).

\( \square \)

**Proposition 7.** Let \( S \) be a non-cylindrical real, irreducible, algebraic surface, different from a sphere. Then it has finitely many rotation axes, that must intersect at a point.

**Proof.** Assume by contradiction that \( S \) has infinitely many rotation axes. By Lemma 6 there is no pair of parallel axes among them. Since the composition of two rotations whose axes do not intersect is a twist, and \( S \) is not cylindrical, then all the axes must intersect at a point \( P \). Now we have infinitely many axes going through \( P \); as a consequence, the number of different angles these axes form with each other is also infinite. On the other hand, the composition of two rotations with concurrent axes is another rotation, about an axis perpendicular to the concurrent rotation axes. Furthermore, if the rotations have rotation angles \( \alpha, \beta \) and the angle between them is \( \Phi \), then the composition is a rotation of rotation angle \( \gamma \), where [17]

\[
\cos \left( \frac{\gamma}{2} \right) = \cos \left( \frac{\alpha}{2} \right) \cdot \cos \left( \frac{\beta}{2} \right) - \sin \left( \frac{\alpha}{2} \right) \cdot \sin \left( \frac{\beta}{2} \right) \cdot \cos(\Phi)
\]

So, since from Lemma 6 there are finitely many \( \alpha, \beta \)'s and however we have infinitely many \( \Phi \)'s, we get infinitely many \( \gamma \)'s too. But this is a contradiction with Lemma 5.

\( \square \)

**Corollary 8.** Let \( S \) be a non-cylindrical real, irreducible, algebraic surface, different from a sphere. Then \( S \) has finitely many symmetry axes, that must intersect at a point.

**Proposition 9.** Let \( S \) be an irreducible, real algebraic revolution surface, different from a sphere. Then the revolution axis of \( S \) is unique.

**Proof.** Assume by contradiction that \( S \) has two revolution axes, \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \). If they do not intersect, then \( S \) has screw symmetry, which implies that \( S \) is cylindrical; in that case, since \( S \) is a revolution surface and is irreducible, then we deduce that it is a circular cylinder, and the statement follows. If \( S \) is not
cylindrical, then $L_1$ and $L_2$ intersect at a point $P$. Since the composition of two rotations with concurrent axes is a rotation about an axis perpendicular to the plane formed by $L_1$ and $L_2$, with angle equal to the sum of the angles of the rotations we are composing, we deduce that in fact $S$ has a third revolution axis, $L_3$, normal to $L_1, L_2$. If $L_1, L_2, L_3$ are not mutually perpendicular, then by applying the same reasoning as before, since $L_1, L_3$ are perpendicular then we can find another revolution axis, $L_4$, perpendicular to both. In any case, we are left with three perpendicular revolution axes, that we will denote as $A_1, A_2, A_3$. Now if we take the plane $\Pi$ containing $A_1, A_2$, since it is normal to $A_3$, that is a revolution axis, the intersection $S \cap \Pi$ will be a circle centered at $P_0 = A_1 \cap A_2$. But now since, say, $A_2$, is a revolution axis, then by rotating the circle around $A_2$ we get a sphere. Since $S$ is irreducible, the statement follows.

Finally, we address planar symmetry.

**Proposition 10.** If $S$ is not a revolution surface, then it has finitely many symmetry planes, that must intersect. Furthermore, $S$ is a revolution surface if and only if it has infinitely many symmetry planes.

**Proof.** The composition of two planar symmetries is either a translation, if the planes are parallel, or a rotation about the common line, with angle equal to twice the angle between the planes, if the planes are concurrent. Since $S$ is not cylindrical, by Proposition 2 every two symmetry planes must intersect at a line, which will be a rotation axis of $S$. Now by Proposition 7 either the number of symmetry planes is finite, or there are infinitely many symmetry planes intersecting at a certain line $L$. However, in this second case these planes determine infinitely many different angles, and $S$ is invariant under the rotations about $L$ with these rotation angles; so, by Lemma 5 $S$ must be a revolution surface. Finally, let us see ($\Rightarrow$). For this purpose, if $S$ is a revolution surface then any plane containing the revolution axis is a symmetry plane; this finishes the proof.

The special case of cylindrical surfaces will be treated in Section 4 of the paper. In the next section we will assume to be working with a non-cylindrical surface admitting a polynomial parametrization in the conditions formulated at the beginning of this section.

### 3. Involutions of Polynomally Parametrized Surfaces.

Our goal is, first, to detect if $S$ exhibits central symmetry, axial symmetry or symmetry about a plane, and second, in the affirmative case, compute the elements of the symmetry (the symmetry center, the symmetry axes and the symmetry planes, respectively). The surface $S$ exhibits some of these symmetries if and only if there exists an affine mapping $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $f(x) = Qx + b$, with $Q$ orthogonal, such that $f^2 = \text{id}_{\mathbb{R}^3}$ and $f(S) = S$. Furthermore, since $x$
is proper then $x^{-1}$ exists and we have a mapping $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ making the following diagram commute:

\[
\begin{array}{c}
S \\
\downarrow f \\
\uparrow S \\
\end{array}
\begin{array}{c}
x \downarrow \\
1 \\
R^2 \\
\end{array}
\begin{array}{c}
\varphi \downarrow \\
1 \varphi \\
R^2 \xrightarrow{\varphi} \mathbb{R}^2 \\
\end{array}
\]

**Theorem 11.** Let $S$ be a surface properly and polynomially parametrized by $x$ and let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear mapping $f(x) = Qx + b$, with $Q$ orthogonal, such that $f(S) = S$. Then $f$ is an involution of $S$ if and only if there exists a unique mapping $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying the following conditions: (1) $x \circ \varphi = f \circ x$; (2) it is linear affine; (3) $\varphi^2 = \text{id}_{\mathbb{R}^2}$.

**Proof.** “$\Rightarrow$”: Condition (1) must hold because $x$ is invertible, and therefore we can define $\varphi = x^{-1} \circ f \circ x$. As for condition (2), we observe the following:

(i) $\varphi(t,s)$ is a real, rational mapping: since $x$ is proper then $x^{-1}$ is a real, rational mapping. So, $\varphi = x^{-1} \circ f \circ x$ is a composition of real rational mappings, and therefore it is also real rational itself.

(ii) $\varphi(t,s)$ is polynomial: indeed, if it is not polynomial then we can find infinitely many (possibly complex) affine points in the $(t,s)$-plane such that the extension $\tilde{\varphi}$ of $\varphi$ to the complex projective plane $\mathbb{P}^2(\mathbb{C})$ maps them to points at infinity. Let $P$ be one of these points. Since $\tilde{x}$ is polynomial, $P$ is mapped to an affine point $Q = x(P)$ on the surface. Following the diagram (2), the symmetry $f$ maps $Q$ to an affine point $Q' = f(x(P))$. Since $\tilde{x}$ is normal, $Q'$ must be generated by some point in the parameter space. And since $Q'$ is affine and $\tilde{x}$ is polynomial, in fact it must be generated by an affine pair $(t',s')$, and not by a point at infinity. Hence, $\varphi$ must be polynomial.

(iii) $\varphi(t,s)$ is linear affine: let $\varphi(t,s) = (P(t,s),Q(t,s))$, with $P,Q$ polynomials, and let $\ell = \max\{\deg(P),\deg(Q)\}$. We want to prove that $\ell = 1$. For this purpose, let $L$ be a generic line of the plane $(t,s)$, i.e. $s = a + b t$ with $a,b$ generic. Since $L$ is generic, then $x(L)$ is a space curve contained in $S$ of degree $n$, where $n$ is the total degree of the parametrization $x$. Additionally, since $f$ is an affine map we have that $(f \circ x)(L) = x(\varphi(L))$ is also a space curve of degree $n$. Now let

$$\varphi(L) = \varphi(t, at + bt) = (P(t, at + b), Q(t, at + b)) = (p(t), q(t))$$

If the degree of either $p(t)$ or $q(t)$ is not 1, again by the genericity of $L$ we have that $x(\varphi(L))$ is a space curve of degree higher than $n$. So, $\deg(p) = \deg(q) = 1$. But again because of the genericity of $a,b$, this implies that the degrees of $P(t,s),Q(t,s)$ must be one. This completes the proof of the condition (2).

As for condition (3), we have that $\varphi^2 = x^{-1} \circ f^2 \circ x$; since $f^2 = \text{id}_{\mathbb{R}^3}$ then $\varphi^2 = \text{id}_{\mathbb{R}^2}$ too.

“$\Leftarrow$”: since $f = x \circ \varphi \circ x^{-1}$, it follows that $f^2 = x \circ \varphi^2 \circ x^{-1}$; but $\varphi^2 = \text{id}_{\mathbb{R}^2}$ and thus $f^2 = \text{id}_{\mathbb{R}^3}$.

The uniqueness of $\varphi$ follows also from the relationship $f = x \circ \varphi \circ x^{-1}$. \qed
From Theorem [11] any involution of \( S \) is the result of lifting an involution of the plane, defined by \( \varphi(t, s) \), to \( S \) by means of the parametrization \( x \). Furthermore, if \( S \) has involution symmetry then also from Theorem [11] it holds that

\[
Qx(t, s) + b = x(\varphi(t, s))
\]

for appropriate \( Q, b, \varphi \), with \( Q \) orthogonal. The main idea of our method is, first, to write all the parameters of \( \varphi \) in terms of at most two of them, and then to write \( Q, b \) also in terms of these parameters. Then (3) gives rise to a bivariate polynomial system, whose solutions can be isolated by applying existing methods (see [1, 35, 36]). The consistency of the system guarantees the existence of symmetry.

The map \( \varphi \) can be written as

\[
\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad t \mapsto A t + c = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} t \\ s \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},
\]

Notice that from condition (3) in Theorem [11] it holds that \( \Delta = ad - bc \neq 0 \). In this expression we have 6 variables. The next lemma allows to reduce the number of variables to at most 3.

**Lemma 12.** The matrix \( A \) and the vector \( c \) satisfy one of the following:

(a) \( A = -I \) and \( c \in \mathbb{R}^2 \),

(b) \( A = \begin{bmatrix} 1 & b \\ 0 & -1 \end{bmatrix}, \; c = c_2 \begin{bmatrix} -b/2 \\ 1 \end{bmatrix} \),

(c) \( A = \begin{bmatrix} -1 & b \\ 0 & 1 \end{bmatrix}, \; c = \begin{bmatrix} c_1 \\ 0 \end{bmatrix} \),

(d) \( A = \begin{bmatrix} a & (1 - a^2)/c \\ c & -a \end{bmatrix}, \; c = c_2 \begin{bmatrix} (a - 1)/c \\ 1 \end{bmatrix}, \; c \neq 0 \).

**Proof.** Since \( t = \varphi^2(t) = A^2 t + A c + c \), it follows that \( (A + I) ((A - I) t + c) = 0 \) for all \( t \). Picking \( t = 0 \) shows that \( c \in \ker(A + I) \), and therefore \( t = \varphi^2(t) = A^2 t \) for all \( t \). It follows that \( A^2 = I \) and the eigenvalues \( \lambda, \mu \) of \( A \) are 1 or -1. Then

\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = \begin{bmatrix} a^2 + bc & b(a + d) \\ c(a + d) & d^2 + bc \end{bmatrix}.
\]

We distinguish two cases.

**Case I:** \( a + d \neq 0 \). Then \( b = c = 0 \) and \( a^2 = d^2 = 1 \). Since \( a + d \neq 0 \), we find \( A = I \) or \( A = -I \). In the former case, \( c \in \ker(A + I) = \ker(2I) \) yields \( c = 0 \); this implies that \( \varphi \) is the identity, which can be discarded as a trivial case. In the latter case any \( c \in \mathbb{R}^2 \) will satisfy \( \varphi \circ \varphi = \text{Id} \).

**Case II:** \( a + d = 0 \). Since \( \mu + \lambda = \text{Tr}A = a + d = 0 \), we find \( \mu = \pm 1 \) and \( \lambda = \mp 1 \). Since \( -1 = \det A = -a^2 - bc \), we also have \( a^2 + bc = 1 \). If \( c = 0 \), then \( a^2 = 1 \) and we obtain

\[
A = \begin{bmatrix} 1 & b \\ 0 & -1 \end{bmatrix}, \; c = c_2 \begin{bmatrix} -b/2 \\ 1 \end{bmatrix}, \quad \text{or} \quad A = \begin{bmatrix} -1 & b \\ 0 & 1 \end{bmatrix}, \; c = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}.
\]
If $c \neq 0$, then $b = (1 - a^2)/c$ and
\[
\mathcal{A} = \begin{bmatrix}
    a & (1 - a^2)/c \\
    c & -a
\end{bmatrix}, \quad c = c_2 \begin{bmatrix}
    (a - 1)/c \\
    1
\end{bmatrix}.
\]

Throughout the paper we will refer to the cases in the above lemma as cases (a), (b), (c), (d), respectively. In the first three cases, $\mathcal{A}$ and $c$ depend on 2 variables; in the last one, they depend on 3 variables. Let us find some extra relationships, that will allow to reduce the number of variables to 2 variables also in the last case. For this purpose we will make use of the first fundamental form of $x$. Recall that this is a form defined in the tangent space of $S$ by means of the matrix:
\[
\mathbf{I}_x = \begin{bmatrix}
    E & F \\
    F & G
\end{bmatrix} = \begin{bmatrix}
    x_t \cdot x_t & x_t \cdot x_s \\
    x_t \cdot x_s & x_s \cdot x_s
\end{bmatrix}
\]

Now if $\xi$ is an isometry between two surfaces $S_1$ and $S_2$ then the first fundamental forms of $S_1, S_2$ are equal at corresponding points (see [13 4.2]); i.e. if $P' = \xi(P)$, then $\mathbf{I}_{\xi(P')} = \mathbf{I}_x(P)$. Any symmetry is an isometry: so, if $f$ is a symmetry of $S$, since $f(S) = S$ is also parametrized by $x \circ \varphi$ we get that for $P, P' \in S$ fulfilling $P' = f(P)$, it holds that $\mathbf{I}_{x \circ \varphi}(P') = \mathbf{I}_x(P)$. Let $\tilde{x} = x \circ \varphi$, and let
\[
\mathbf{I}_{\tilde{x}} = \begin{bmatrix}
    \tilde{E} & \tilde{F} \\
    \tilde{F} & \tilde{G}
\end{bmatrix} = \begin{bmatrix}
    \tilde{x}_t \cdot \tilde{x}_t & \tilde{x}_t \cdot \tilde{x}_s \\
    \tilde{x}_t \cdot \tilde{x}_s & \tilde{x}_s \cdot \tilde{x}_s
\end{bmatrix}
\]

Since $\varphi(0) = (c_1, c_2) = c$, in particular we get that
\[
E(0) = \tilde{E}(c), \quad F(0) = \tilde{F}(c), \quad G(0) = \tilde{G}(c)
\]

In order to exploit the above relationships, we need to write $\tilde{E}(c), \tilde{F}(c), \tilde{G}(c)$ in terms of $E(c), F(c), G(c)$ respectively. For this purpose, we observe that
\[
\nabla(\tilde{x}) = \nabla(x \circ \varphi) = \begin{bmatrix}
    \tilde{x}_t \\
    \tilde{x}_s
\end{bmatrix} = \begin{bmatrix}
    (x \circ \varphi)_t \\
    (x \circ \varphi)_s
\end{bmatrix} = \begin{bmatrix}
    a & c \\
    b & d
\end{bmatrix} \cdot \begin{bmatrix}
    x_t \\
    x_s
\end{bmatrix} = \begin{bmatrix}
    ax_t + cx_s \\
    bx_t + dx_s
\end{bmatrix}
\]

Using this together with [1], we reach the relationships:
\[
\begin{align*}
E(0) &= E(c) \cdot a^2 + 2F(c) \cdot ac + G(c) \cdot c^2 \\
F(0) &= E(c) \cdot ab + F(c) \cdot (ad + bc) + G(c) \cdot cd \\
G(0) &= E(c) \cdot b^2 + 2F(c) \cdot bd + G(c) \cdot d^2
\end{align*}
\]

Finally, by sake of convenience, let us denote
\[
E(0) = A, \quad F(0) = B, \quad G(0) = C
\]

Also, let us recall the notation $\Delta = ad - bc \neq 0$. Then we can solve [5] for $E(c), F(c)$ and $G(c)$, to get
\[
\begin{align*}
E(c) &= \frac{Cc^2 + Ad^2 - 2Bcd}{\Delta^2} \\
F(c) &= \frac{B(bc + \Delta^2) - Abd - Cac}{\Delta^2} \\
G(c) &= \frac{Ab^2 - 2Bab + Ca^2}{\Delta^2}
\end{align*}
\]
Lemma 13. The possible configurations for $A$, $b$ are:

(a) $A = -I$, $E(c) = A$, $F(c) = B$, $G(c) = C$.

(b) $A = \begin{bmatrix} 1 & b \\ 0 & -1 \end{bmatrix}$, $c = c_2 \begin{bmatrix} -b/2 \\ 1 \end{bmatrix}$, $E(c) = A$, $F(c) = Ab - B$, $G(c) = b^2 - 2Bb + C$.

(c) $A = \begin{bmatrix} -1 & b \\ 0 & 1 \end{bmatrix}$, $c = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$, $E(c) = A$, $F(c) = -Ab - B$, $G(c) = Ab^2 + 2Bb + C$.

(d) $A = \begin{bmatrix} a & (1 - a^2)/c \\ c & -a \end{bmatrix}$, $c = c_2 \begin{bmatrix} (a - 1)/c \\ 1 \end{bmatrix}$, $c \neq 0$, with two possible sub-cases:

\( (d.1) \) $c = 0$.

\( (d.2) \) $c \neq 0$, $a = \frac{-c_2^2E(c) - c_1c_2[F(c) - B] + Cc_2^2}{Ac_1^2 + 2Bc_1c_2 + Cc_2^2}$, where $Ac_1^2 + 2Bc_1c_2 + Cc_2^2 \neq 0$.

Proof. The first three configurations follow in a very straightforward way from the relations \[6\] and Lemma \[12\]. So, let us deduce the last one. In the case \( (d) \) of Lemma \[12\], by using the first relation in \[6\] we get

$$E(c) = Aa^2 + 2Bac + Cc^2$$

Multiplying by $c_1^2$, and taking into account that $c_1c = c_2(a - 1)$, we get that

$$c_1^2E(c) = (c_2^2 + 2Bc_1c_2 + Ac_1^2) \cdot a^2 - (2c_2C + 2c_1c_2B) \cdot a + Cc_2^2 \quad (7)$$

On the other hand, from the second relation in \[6\], and since $b = \frac{1 - a^2}{c}$ (notice that $c \neq 0$) and $d = -a$,

$$F(c) = A \frac{1 - a^2}{c}a + B(1 - 2a^2) - Cac$$

Furthermore, since $c \cdot c_1 = c_2 \cdot (a - 1)$, after multiplying the above equation by $c_1 \cdot c_2$ we can write

$$c_1c_2F(c) = -Ac_1^2a(1 + a) + Bc_1c_2(1 - 2a^2) - Cc_2^2a(a - 1)$$

and hence we get

$$c_1c_2F(c) = (-Ac_1^2 - 2Bc_1c_2 - Cc_2) \cdot a^2 - (Ac_1^2 + Cc_2^2) \cdot a + Bc_1c_2 \quad (8)$$

By adding up \[7\] and \[8\], the terms in $a^2$ cancel, and we obtain

$$(Ac_1^2 + 2Bc_1c_2 + Cc_2^2) \cdot a = -c_2^2E(c) - c_1c_2[F(c) - B] + Cc_2^2$$

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Notice that
\[ Ac_1^2 + 2Bc_1c_2 + Cc_2^2 = c^T \cdot I_{x(0)} \cdot c. \]
So, since \(x(0)\) is regular and because of the positive-definiteness of the first fundamental form, whenever \(c \neq 0\) we have that \(Ac_1^2 + 2Bc_1c_2 + Cc_2^2 \neq 0\). Then the result follows.

So, in case (a) we are left with the variables \(c_1, c_2\); in case (b), with \(b, c_2\); in case (c), with \(b, c_1\). In the case (d), if \(c = 0\) then we are left with \(a, c\); if \(c \neq 0\), we distinguish two subcases: (i) if \(a = 1\) then \(c_1 = b = 0\), and we are left with \(a, c\); (ii) if \(a \neq 1\) then \(c_1 \neq 0\) and we can write \(c = c_2 \cdot \frac{a-1}{ct}\), so that we are left with \(c_1, c_2\).

Finally we need to write \(Q, b\) in \(f(x) = Qx + b\) in terms of the parameters of \(\varphi\). For this purpose, by differentiating (3) with respect to \(t, s\) respectively, we get:
\[
\begin{align*}
Q \cdot x_t(t, s) &= x_t(\varphi(t, s)) \cdot a + x_s(\varphi(t, s)) \cdot c \\
Q \cdot x_s(t, s) &= x_t(\varphi(t, s)) \cdot b + x_s(\varphi(t, s)) \cdot d \\
\end{align*}
\]
(Equation 9)
Evaluating (9) at \((t, s) = (0, 0)\) yields
\[
\begin{align*}
Q \cdot x_t(0, 0) &= x_t(c) \cdot a + x_s(c) \cdot c \\
Q \cdot x_s(0, 0) &= x_t(c) \cdot b + x_s(c) \cdot d \\
\end{align*}
\]
(Equation 10)
Additionally, the normal to \(S\) at \(x(0,0)\) is parallel to \(x_t(0,0) \times x_s(0,0) \neq 0\) (remember that \(x(0)\) is regular by hypothesis). From (10),
\[
Q \cdot (x_t(0,0) \times x_s(0,0)) = \operatorname{det}(Q) \cdot \Delta \cdot (x_t \times x_s)(c) \\
\]
(Equation 11)
where \(\operatorname{det}(Q) = \pm 1\) depending on whether \(Q\) preserves orientation (axial symmetries) or not (central and planar symmetries). So, by using (10) and (11), we can derive the matrix \(Q\) from its action on \(x_t(0,0), x_s(0,0)\) and \(x_t(0,0) \times x_s(0,0)\). Multiplying \(Q\) by the matrix
\[
M = [x_t(0,0), x_s(0,0), x_t(0,0) \times x_s(0,0)]
\]
gives the matrix
\[
L = [x_t(c) \cdot a + x_s(c) \cdot c, x_t(c) \cdot b + x_s(c) \cdot d, \operatorname{det}(Q) \cdot \Delta \cdot (x_t \times x_s)(c)] \\
\]
(Equation 12)
So, \(Q = LM^{-1}\), and therefore we have the elements of \(Q\) written in terms of the parameters of \(\varphi\). By evaluating (3) at \(t = 0\), we deduce that
\[
b = x(c) - Qx(0) \\
\]
(Equation 13)

3.1. Detection of direct involutions.

In order to detect orientation-preserving involutions, i.e. axial symmetries, we must set \(\operatorname{det}(Q) = 1\) in (12), and then check if each polynomial system obtained from (3) for each possible configuration of \(A, b\) (see Lemma 13) has
real solutions. Notice that by Corollary, each system can only have finitely many real solutions.

In the affirmative case, one also wants to find the symmetry axis $L$. Since $L$ is the set of fixed points of the symmetry $f(x) = Qx + b$, once that $Q, b$ have been found one can find $L$ as the solution set of the system $(Q - I)x = -b$. Observe that the direction of $L$ corresponds to the eigenspace associated with $\lambda = 1$, that is an eigenvalue of $Q$. However, in some cases there is an alternative to find $L$, based on the analysis of the involution $\varphi$ of the plane $f(x)$ comes from. For this purpose, we observe first that the fixed points of $\varphi$ can be found by solving $(A - I)t + c = 0$. So, we have the following result, that can be deduced after easy calculations.

**Lemma 14.** The following statements are true:

(i) In case (a), $\varphi$ has just one fixed point, namely $c/2$.

(ii) In case (b), $\varphi$ has: (i) one fixed point, namely $(c_2/2, c_2/2)$, if $b \neq 0$; (ii) a line of fixed points, namely $s = \frac{c_2}{2}$, if $b = 0$.

(iii) In case (c), $\varphi(t, s)$ has a line of fixed points, namely $t = \frac{b}{2}s + \frac{c_2}{2}$.

(iv) In case (d), $\varphi(t, s)$ has a line of fixed points, namely $t = \frac{1}{c}(a + 1)s - \frac{c_2}{c}$.

Notice that the set of fixed points $M$ of $\varphi$ is non-empty in all the cases. Since $x(\varphi(t, s))$ is the symmetric point of $x(t, s)$, any fixed point of $\varphi$ leads to a fixed point of $f(x)$. The converse is not necessarily true, since $f(x)$, as a mapping from $\mathbb{R}^3$ to $\mathbb{R}^3$, can have fixed points that do not belong to $S$. However, $M$ certainly provides the fixed points of $f(x)$ reached by the parametrization $x(t, s)$. Since $x(t, s)$ is normal by hypothesis (see Subsection 2.1), in our case $M$ corresponds to the fixed points of $f(x)$ contained in $S$. Hence, we have the following result.

**Proposition 15.** Let $S$ be polynomially, properly and normally parametrized, and let $f(x) = Qx + b$ be an axial symmetry of $S$ with symmetry axis $L$. Also, let $M$ be the set of fixed points of the mapping $\varphi$ corresponding to $f$. The following statements are true:

(i) If $x(M)$ is a line, then $x(M) = L$.

(ii) If $x(M)$ is a regular point $P$, then $L$ is normal to $S$ through $P$.

**Proof.** From Lemma 14 we have that $M$ is either a point or a line, and therefore $x(M)$ is also either a point or a curve contained in $S$; however, since $x(M)$ is included in the set of fixed point of $S$ with respect to the symmetry, which is at most a line, then in fact $x(M)$ is either a point or a line. If $x(M)$ is a line, then it coincides with the set of fixed points of $f$, i.e. $x(M) = L$. So, (i) holds. Now let us see (ii). In order to prove this, observe that $f(x) = Qx + b$ induces a symmetry of the same kind $n_{f(x)} = Q : n_x$ between the normal vectors to $S$ at corresponding points $x(t, s)$ and $f(x(t, s))$. Since $f(P) = P$, we deduce that
the normal vector to \( S \) at \( P \), \( n_P \), which is well-defined because \( P \) is regular, satisfies \((Q - I) \cdot n_P = 0\); so, \( n_P \) is an eigenvector of \( Q \), associated with the eigenvalue \( \lambda = 1 \), and therefore its direction coincides with that of \( \mathcal{L} \).

3.2. Detection of opposite involutions.

Here we have central and planar symmetries. In order to detect them, one sets \( \det(Q) = -1 \) in (12), and, as in Section 3.1, checks whether the polynomial systems obtained from (3) for each possible configuration of \( A, b \) (see Lemma 13) have real solutions, or not. In order to distinguish whether the symmetry is central or planar, and also to find the elements of the symmetry (the symmetry center or the symmetry plane), one can compute \( Q, b \), and then study the solution set of \((Q - I)x = -b\). In particular, an opposite involution \( f(x) \) is a central symmetry if and only if the set of fixed points reduces to a point, i.e. iff \( \det(Q - I) \neq 0 \); furthermore, in that case the fixed point is the symmetry center. If \( \det(Q - I) = 0 \), then the involution has a plane of fixed points, and that plane is the symmetry plane \( \Pi \). In fact, \( \Pi \) corresponds to the eigenspace of \( Q \) associated with the eigenvalue \( \lambda = 1 \).

However, as in Subsection 3.1, we by exploiting Lemma 14 we get an alternative, useful in several cases.

**Proposition 16.** Let \( S \) be polynomially, properly and normally parametrized, and let \( f(x) = Qx + b \) be an opposite involution of \( S \). Also, let \( \mathcal{M} \) be the set of fixed points of the mapping \( \varphi \) corresponding to \( f \). The following statements are true:

(i) If \( x(\mathcal{M}) \) reduces to a regular point \( P \), then \( f(x) \) is a central symmetry and \( P \) is the symmetry center.

(ii) If \( x(\mathcal{M}) \) is not a point, then \( f(x) \) is a planar symmetry, and the symmetry plane \( \Pi \) contains \( x(\mathcal{M}) \). Furthermore, if \( x(\mathcal{M}) \) is a line containing some regular point of \( S \), then \( \Pi \) is normal to \( S \) at \( x(\mathcal{M}) \).

**Proof.** Let us see (i). If \( f \) is a planar symmetry, by reasoning as in statement (ii) of Proposition 15, we deduce that the normal vector to \( S \) at \( P \), \( n_P \), is an eigenvector of \( Q \), and therefore the normal line to \( S \) at \( P \) is contained in the symmetry plane \( \Pi \). So, \( \Pi \) intersects \( S \) at a curve \( \mathcal{C} \), which is a curve of fixed points contained in \( S \). However, this is a contradiction because \( x(\mathcal{M}) = P \). So, \( f \) must be a central symmetry, where the symmetry center is \( P \). Now let us see (ii). The first part is clear. So, assume that \( x(\mathcal{M}) \) is a line \( \mathcal{L} \). This line is contained in the symmetry plane, \( \Pi \). Furthermore, if \( \mathcal{L} \) contains some regular point \( P \), as before we have that the normal vector to \( S \) at \( P \), \( n_P \), is also contained in \( \Pi \). Therefore, the result follows.

Notice that from Proposition 3 the symmetry center, if it exists, is unique. Therefore, we can have at most one central inversion leaving \( S \) invariant. However, from Proposition 10 we might have infinitely many symmetry planes, implying that \( S \) is a revolution surface. This gives rise to the following theorem.
**Theorem 17.** Let \( S \) be a non-cylindrical, polynomially, properly and normally parametrized surface. Then \( S \) is a revolution surface if and only if some of the polynomial systems corresponding to the configurations in Lemma 13 have a 1-dimensional real solution set.

Since \( S \) is polynomially parametrized then it cannot be a sphere; so, by Proposition 9 the revolution axis is unique. In order to find it, notice that the revolution axis belongs to the infinitely many symmetry planes of the surface, and additionally it is itself a symmetry axis of \( S \).

### 3.3. Summary of the algorithm

The following algorithm SymSurf follows from the ideas in this section.

**Algorithm SymSurf**

**Require:** A proper and normal parametrization \( x : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) of a non-cylindrical surface \( S \), where \( x(0) \) is a regular point.

**Ensure:** The involutions leaving \( S \) invariant, and their characteristic elements.

1. **Direct involutions:** for each configuration in Lemma 13 do:
   2. Find \( Q, b \) in terms of the parameters of \( \varphi(t, s) \) from (12) and (13) respectively.
   3. Derive the (in general, bivariate) polynomial system in the parameters of \( \varphi(t, s) \) from equation (3), taking into account \( \det(Q) = 1 \).
   4. Check whether the polynomial system has real solutions, or not.
   5. Derive the characteristic elements of the involution from Proposition 16 or by solving the system \((Q - I)x = -b\).

6. **Opposite involutions:** for each configuration in Lemma 13 do:
   7. Find \( Q, b \) in terms of the parameters of \( \varphi(t, s) \) from (12) and (13) respectively.
   8. Derive the (in general, bivariate) polynomial system in the parameters of \( \varphi(t, s) \) from equation (3), taking into account \( \det(Q) = -1 \).
   9. Check whether the polynomial system has real solutions, or not.
   10. If the system has finitely many real solutions, derive the characteristic elements of the involution from Proposition 16 or by solving the system \((Q - I)x = -b\).
   11. If the system has infinitely many real solutions, return *The surface is a revolution surface.*

### 4. The case of cylindrical surfaces

If \( S \) is a rational (not necessarily polynomial) surface, one can detect that it is cylindrical by applying the results of [40]. Furthermore, in that case one can also find [40] a rational parametrization of \( S \) of the form

\[
y(t, \lambda) = w(t) + \lambda v,
\]
where \( \mathbf{v} \) denotes the direction all the generatrices of \( S \) are parallel to. A first observation is that any plane normal to \( \mathbf{v} \) is a symmetry plane; so, in this case we always have infinitely many symmetry planes. The other involutions leaving \( S \) invariant can be found by studying a normal section of the surface. Indeed, let \( \Pi \equiv Ax + By + Cz + D = 0 \) be a plane normal to the direction \( \mathbf{v} \), and let \( \mathcal{E} = S \cap \Pi \) be the normal section of \( S \) corresponding to \( \Pi \). By plugging the parametrization \( y(t, \lambda) \) into the equation of \( \Pi \), we can solve for \( \lambda \) to get \( \lambda = \lambda(t) \); then, by substituting this back into \( y(t, \lambda) \), we get a rational parametrization of \( E \). Now \( S \) has axial symmetry if and only if \( E \) has central symmetry, and the symmetry axis is normal to \( \Pi \) through the symmetry center of \( E \); \( S \) is symmetric w.r.t. a plane if and only if \( E \) has symmetry with respect to a line, and the symmetry plane is normal to \( \Pi \) through the symmetry axis of \( E \). In order to determine the symmetries of \( E \) we can use the algorithms in [3, 2, 4].

5. Experimentation and implementation.

In this section we present some examples, and we report on the complexity and practical performance of the algorithm. The properness of the parametrizations tested here can be examined by using the techniques in [29]. Furthermore, all the tested parametrizations fulfill the hypotheses in Corollary 3.15 of [33], which gives a sufficient condition for normality.

5.1. An example: finding the involutions of an Enneper surface.

Consider the Enneper surface \( S \), a minimal surface of degree 9, parametrized as

\[
\mathbf{x}(t, s) = (-s^3 + 3st^2 + 3s, 3s^2t - t^3 + 3t, 3s^2 - 3t^2)
\]

We explore first direct symmetries. For this purpose, we have to test each case in Lemma 13. Case (a) succeeds, producing the values \( c_1 = 0, c_2 = 0 \). From Lemma 14 it follows that the corresponding \( \varphi(t, s) \) has just one fixed point, namely \( (0, 0) \). We get that \( \mathbf{x}(0, 0) = 0 \); since \( \mathbf{x}_t(0, 0) \times \mathbf{x}_s(0, 0) = (0, 0, -9) \neq 0 \), then \( \mathbf{x}(0, 0) \) is a regular point. So, from Proposition 15 it follows that the \( z \)-axis is a symmetry axis of \( S \). Case (d.1) also succeeds, and produces two solutions, namely \( \{ a = 0, c = -1 \} \) and \( \{ a = 0, c = 1 \} \). In the first case, \( \varphi(t, s) \) has a line of fixed points, \( t = s \), which maps to the line \( \{ x - y = 0, z = 0 \} \) on the surface. In the second case, \( \varphi(t, s) \) has also a line of fixed points, \( t = -s \), which maps to the line \( \{ x + y = 0, z = 0 \} \) on the surface. So, all in all we get three symmetry axes (see Fig. 1, center).

As for opposite symmetries, case (b) provides the solution \( \{ b = 0, c_2 = 0 \} \). From Lemma 14 the corresponding \( \varphi(t, s) \) has a line of fixed points, namely \( s = 0 \). Since \( \mathbf{x}(t, 0) = (0, -t^3 + 3t, -3t^2) \), which is a planar curve contained in the plane \( x = 0 \), from Proposition 16 we deduce that \( x = 0 \) is a symmetry plane of \( S \). In the case (c) we also get a solution, \( \{ b = 0, c_1 = 0 \} \); here, \( t = 0 \) is the line of fixed points of \( \varphi(t, s) \), and we have \( \mathbf{x}(0, s) = (-s^3 + s, 0, 3s^2) \), which is a planar curve contained in the plane \( y = 0 \); so, we get a planar symmetry too,
with respect to the plane $y = 0$ this time. Notice that the $z$-axis is precisely the intersection of the symmetry planes $x = 0, y = 0$. The symmetry planes of the surface are shown in Fig. 1 (right).

5.2. An example: finding the involutions of a circular paraboloid.

Consider the circular paraboloid $S$, parametrized as

$$\mathbf{x}(t, s) = (t, s, t^2 + s^2)$$

When exploring direct symmetries we observe that only case (a) succeeds, providing the solution $\{c_1 = c_2 = 0\}$. As in Example 1, we observe that $\mathbf{x}(0, 0) = 0$; since $\mathbf{x}_t(0, 0) \times \mathbf{x}_s(0, 0) = (0, 0, 1) \neq 0$, we get an axial symmetry with respect to the $z$-axis. As for opposite symmetries, case (b) provides the solution $\{b = 0, c_2 = 0\}$; the line of fixed points of the corresponding $\varphi(t, s)$ is $s = 0$, which maps to the curve $(t, 0, t^2)$. This is a planar curve contained in the plane $y = 0$, which is therefore a symmetry plane of $S$. Case (c) also succeeds, providing $\{b = 0, c_1 = 0\}$; the line of fixed points of $\varphi(t, s)$ is then $t = 0$, which gives rise to the curve $(0, s, s^2)$. Since this curve is contained in the plane $x = 0$, we deduce that $x = 0$ is another symmetry plane of $S$. Finally, case (d.1) succeeds too, but here we obtain infinitely many real solutions; more precisely, these solutions satisfy $a^2 + c^2 = 1$. Therefore, by Theorem 17 we recognize $S$ as a revolution surface. Furthermore, we observe that the lines of fixed points of the corresponding $\varphi(t, s)$’s are $t = \frac{a+1}{c^2}, s$. Each of these lines gives rise to the curve

$$((a + 1)s/c, s, s^2 + (a + 1)^2s^2/c^2),$$

which belongs to the plane $x - \frac{a+1}{c}y = 0$. So, we deduce that the symmetry planes of $S$ are $x - ky = 0$, with $k \in \mathbb{Z}$. All these planes share the $z$-axis, which therefore corresponds to the revolution axis. Notice also that the revolution axis is in particular a symmetry axis of $S$. 

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5.3. Observations on the complexity.

Let us consider the complexity of Algorithm SymmSurf. For this purpose we will analyze the case of direct involutions, and we will focus on the case (d.2) of Lemma 13 which is the most consuming one. The complexity is not modified when one includes the other cases of Lemma 13 or the case of opposite involutions. Along this section, in addition to the standard Big O notation $O$, we use the Soft O notation $\tilde{O}$ to ignore any logarithmic factors in the complexity analysis. Also, here we will speak about the “degree of a rational function” to mean the maximum of the total degrees of the numerator and the denominator of the function.

Let us consider first step 2, i.e. the construction of $Q, b$. Denoting by $d$ the total degree of $x$, we have that $x_t, x_s$ have total degrees bounded by $d - 1$, and therefore the degree of $x_t \times x_s$ is bounded by $2d - 2$. In the case (d.2) it holds that

$$a = \frac{-c_1^2 E(c) - c_1 c_2 [F(c) - B] + C c_2^2}{A c_1^2 + 2 B c_1 c_2 + C c_2^2}, \quad b = \frac{-(1 + a) c_1}{c_2}, \quad c = \frac{c_2 (a - 1)}{c_1}, \quad d = -a$$

Since $E(c) = x_t(c) \cdot x_t(c)$ it follows that its degree is bounded by $2d - 2$, and similarly for $F(c)$. So, $a$ is a rational function in $c_1, c_2$ with degree bounded by $2d$. Similarly we get that $b, c, d$ are also rational functions with $O(d)$ degrees. By using the expression \cite{12} for the matrix $L$, we get that the entries of $L$ are rational functions of degrees $O(d)$ in $c_1, c_2$, and similarly for the entries of the matrix $Q = L \cdot M^{-1}$. From \cite{13}, we observe that $b$ is a rational function of $c_1, c_2$ with degree $O(d)$. The operations involved in this step are essentially multiplication and addition of bivariate polynomials of $O(d)$ degrees, that can be done in $\tilde{O}(d^2)$ time \cite{28}.

We address now step 3, i.e. the derivation of the polynomial system in $c_1, c_2$ from equation \cite{3}. Let us denote

$$x(t, s) = \sum_{i, j = 0}^{d} \bar{\alpha}_{i, j} \cdot t^i s^j$$

where $i, j \leq d$

We want to compute

$$x(\varphi(t, s)) = \sum_{i, j = 0}^{d} \bar{\alpha}_{i, j} \cdot (at + bs + c_1)^i (ct + ds + c_2)^j$$

where $a, b, c, d$ are rational functions of $c_1, c_2$ of $O(d)$ degrees. Each $(at + bs + c_1)^i$ or $(ct + ds + c_2)^j$ can be computed in $\tilde{O}(d^4)$ time by using binary exponentiation and FFT-based multiplication \cite{14} \S 8.2, and produces a polynomial where the coefficients are rational functions of $c_1, c_2$ of degrees bounded by $O(d^2)$. The product $(at + bs + c_1)^i \cdot (ct + ds + c_2)^j$ is computed after $\tilde{O}(d^4)$ operations. Since
each component of $x(t, s)$ has at most $O(d^2)$ terms (as a polynomial in $t, s$), we have to repeat this process $O(d^2)$ times, therefore yielding a total complexity of $O(d^n)$ for this step. The bivariate polynomial system in $c_1, c_2$ derived this way has degree $O(d^2)$ and consists of $O(d^2)$ equations.

Finally we consider step 4, i.e. solving the system. The complexity of determining the real solutions of a (possibly overdetermined, and non necessarily zero-dimensional) polynomial system of $k$ equations in $n$ variables, with degrees bounded by $D$, is $O\left((kD)^n\right) \[16\]. Since in our case $k = O(d^2)$, $n = 2$, $D = O(d^2)$, we get a complexity of $O(d^{16})$ for this step.

Step 5 does not add any complexity to the previous steps. So, we get an overall complexity of $O(d^{16})$. This complexity is dominated by that of step 4, which is certainly the bottleneck of the algorithm.

5.4. Performance

We have implemented the algorithm SymSurf in the computer algebra system Maple 17, and examples have been run on an intel Core i7, revving up to 2.90 GHz, with 8 Gb RAM. We enlist the features of some of these examples in Table 1. More precisely, in each case we provide the bidegree $(d_1, d_2)$ of the parametrization, the absolute value of the maximum coefficient of the parametrization, the timing, and the involutions found.

The table shows a good performance for surfaces of moderate bidegrees. The bottleneck of the algorithm, as shown in the complexity section, is the isolation of the real roots of the bivariate systems corresponding to the cases in Lemma \[13\]; this explains the explosion in the time as the bidegree grows.

| Surface               | Bidegree | Coeffs. | Timing | Obs.       |
|-----------------------|----------|---------|--------|------------|
| Elliptic paraboloid   | (2,2)    | 18      | 0.374  | Planar, axial. |
| Hyperbolic paraboloid | (2,2)    | 16      | 0.234  | Planar, axial. |
| Circular paraboloid   | (2,2)    | 1       | 0.078  | Revol. surf. |
| Enneper surface       | (3,3)    | 3       | 0.141  | Planar, axial. |
| Example 8             | (3,3)    | 3       | 0.187  | Planar, axial. |
| Example 9             | (4,4)    | 6       | 0.827  | Planar, axial. |
| Example 10            | (5,5)    | 10      | 5.912  | Planar, axial. |
| Example 2             | (6,6)    | 20      | 3.073  | Planar, axial. |
| Example 1             | (7,7)    | 35      | 8.565  | Central.  |
| Example 11            | (8,8)    | 70      | 82.213 | Planar, axial. |
| Revol.2               | (8,8)    | 6       | 0.640  | Revol. surf. |
| Example 6             | (9,11)   | 924     | 196.25 | Central.  |

Table 1: Average CPU time (seconds) for involutions of several polynomially parametrized surfaces.
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