Analytic-bilinear approach to integrable hierarchies.
I. Generalized KP hierarchy.

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Abstract

Analytic-bilinear approach for construction and study of integrable hierarchies, in particular, the KP hierarchy is discussed. It is based on the generalized Hirota identity. This approach allows to represent generalized hierarchies of integrable equations in a condensed form of finite functional equations. Resolution of these functional equations leads to the $\tau$-function and addition formulae to it. General discrete transformations of the $\tau$-function are presented in the determinant form. Closed one-form and other formulae also arise naturally within the approach proposed. Generalized KP hierarchy written in terms of different invariants of Combescure symmetry transformations coincides with the usual KP hierarchy and the mKP hierarchy.

1 Introduction

The Kadomtsev-Petviashvili (KP) equation and the whole KP hierarchy of equations are significant parts of the theory of integrable equations. They arise in various fields of physics from hydrodynamics to string theory. They are also the tools to solve several problems in mathematics from the differential geometry of surfaces to an algebraic geometry.

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The KP hierarchy has been described and studied within the framework of different approaches. The Sato approach [1] (see also [2]-[4]) and the ∂-dressing method [5]-[8] are, perhaps, the most beautiful and powerful among them. The infinite-dimensional Grassmannian, pseudo-differential operators, Hirota bilinear identity and τ-function are the basic ingredients of the Sato approach which is in essence algebraic. In contrast, the ∂-method, based on the nonlocal ∂-problem for the wave function, mostly uses the analytic properties of the wave function. These two approaches look like completely different. On the other hand, each of them has its own advantages. So one could expect that their marriage could be rather profitable.

A bridge between the Sato approach and the ∂-dressing method has been established by the observation that the Hirota bilinear identity can be derived from the ∂-dressing method [8], [9]. Elements of the approach which combines the characteristic features of both methods, namely, the Hirota bilinear identity from the Sato approach and the analytic properties of solutions from the ∂-dressing method have been considered in [10], [11].

This paper is devoted to the analytic-bilinear approach to integrable hierarchies. It is based on the generalized Hirota bilinear identity for the wave function with simple analytic properties (Cauchy-Baker-Akhiezer (CBA) function). This approach allows us to derive generalized hierarchies of integrable equations in terms of the CBA function \( \psi(\lambda, \mu, x) \) in a compact, finite form. Such a generalized equations contain integrable equations in their usual form, their modified partners and corresponding linear problems.

The generalized Hirota bilinear identity provides various functional equations for the CBA function. The resolution of some of them leads to the introduction of the \( \tau \)-function, while others give rise to the addition theorems for the \( \tau \)-function. The properties of the \( \tau \)-function, such as the associated closed 1-form and its global definition are also arise in a simple manner within the approach under consideration. Generic discrete transformations of the CBA function and of the \( \tau \)-function are presented in the determinant form. These transformations are the generalized form of the usual iterated Darboux transformations.

The generalized KP hierarchy is presented also in the ‘moving’ frame depending on the parameter. This generalized KP hierarchy written in the different ‘moving’ frames contains the Darboux system of equations.

In addition to the usual infinite-dimensional symmetries the generalized KP hierarchy possess the symmetries given by the Combescure transformations. The invariants of these symmetry transformations have the compact forms in terms of the CBA function. The generalized KP hierarchy written
in terms of these invariants coincide with the usual KP hierarchy, mKP hierarchy and SM-KP hierarchy.

The present paper is devoted to the one-component KP hierarchy. The authors plan to consider multi-component KP hierarchy and the 2-dimensional Toda lattice in subsequent papers.

The paper is organized as follows. Generalized Hirota identity is introduced in section 2. The generalized KP hierarchy is derived in section 3. Combescure symmetry transformations are discussed in section 4. Generalized KP hierarchy in the ‘moving’ frame is considered in section 5. The \( \tau \)-function is introduced in section 6. The addition formulae for the \( \tau \)-function are also obtained here. Transformations of the CBA function and the \( \tau \)-function given by the determinant formulae and the Darboux transformations are discussed in section 7. Closed one-form variational formulae for the \( \tau \)-function are presented in section 8.

2 Generalized Hirota identity

The famous Hirota bilinear identity provides us condensed and compact form of the integrable hierarchies. Here we will derive the generalized Hirota bilinear identity in frame of the \( \bar{\partial} \)-dressing method.

The \( \bar{\partial} \)-dressing method (see [3]-[8]) is based on the nonlocal \( \bar{\partial} \)-problem of the form

\[
\bar{\partial}_\lambda(\chi(x, \lambda) - \eta(x, \lambda)) = \int\!\int_C \, d\mu \wedge d\bar{\mu} \chi(\mu) \chi^{-1}(\mu) R(\mu, \lambda) g(\lambda) \quad (1)
\]

where \( \lambda \in \mathbb{C} \), \( \bar{\partial}_\lambda = \partial / \partial \bar{\lambda} \), \( \eta(x, \lambda) \) is a rational function of \( \lambda \) (normalization).

In general case the function \( \chi(\lambda) \) and the kernel \( R(\lambda, \mu) \) are matrix-valued functions.

The dependence of the solution \( \chi(\lambda) \) of the problem (1) on the dynamical variables is hidden in the function \( g(\lambda) \). Here we will consider only the case of continuous variables, for which \( g_i = \exp(K_i x_i) \), where \( K_i(\lambda) \) are, in general, matrix meromorphic functions.

It is assumed that the problem (1) is uniquely solvable. The \( \bar{\partial} \)-dressing method allows us to construct and solve wide classes of nonlinear PDEs which correspond to the different choice of the functions \( K_i(\lambda) \).

Here we will assume that the kernel \( R(\lambda, \mu) \) is equal to zero in some open subset \( G \) of the complex plane with respect to \( \lambda \) and to \( \mu \). This subset should
typically include all zeroes and poles of the considered class of functions \(g(\lambda)\) and a neighborhood of infinity. In this case the solution of the problem (1) normalized by \(\eta\) is the function

\[
\chi(\lambda) = \eta(x, \lambda) + \varphi(x, \lambda),
\]

where \(\eta(\lambda)\) is a rational function of \(\lambda\) (normalization), all poles of \(\eta(\lambda)\) belong to \(G\), \(\varphi(\lambda)\) decreases as \(\lambda \to \infty\) and is analytic in \(G\).

The special class of solutions of the \(\bar{\partial}\) problem (1) normalized by \((\lambda - \mu)^{-1}\) \((\eta(\lambda) = (\lambda - \mu)^{-1}, \mu \in G)\) is of particular importance for the whole \(\bar{\partial}\)-dressing method [6]. Let us consider the \(\bar{\partial}\)-problem (1) for such solutions and corresponding dual problem for the dual function \(\chi^*(\lambda, \mu)\):

\[
\bar{\partial}_\lambda \chi(\lambda, \mu) = 2\pi i \delta(\lambda - \mu) + \int \int_C d\nu \wedge d\bar{\nu} \chi(\nu, \mu) g_1(\nu) R(\nu, \lambda) g_1(\nu)^{-1},
\]

\[
\bar{\partial}_\mu \chi^*(\lambda, \mu) = -2\pi i \delta(\lambda - \mu) - \int \int_C d\nu \wedge d\bar{\nu} g_2(\nu) R(\lambda, \nu) g_2(\nu)^{-1} \chi^*(\nu, \mu). \tag{2}
\]

where \(g_1 = g(\lambda, x), g_2 = g(\lambda, x')\) After simple calculations, one gets

\[
\int \int_G d\nu \wedge d\bar{\nu} \frac{\partial}{\partial \bar{\nu}} \left( \chi(\nu, \lambda; g_1(\nu) g_2(\nu)^{-1} \chi^*(\nu, \mu; g_2) \right) = 
\int_{\partial G} \chi(\nu, \lambda; g_1(\nu) g_2(\nu)^{-1} \chi^*(\nu, \mu; g_2) d\nu = 0. \tag{3}
\]

In the particular case \(g_1 = g_2\) from (3) it follows that in \(G\) the function \(\chi(\lambda, \mu)\) is equal to \(-\chi^*(\lambda, \mu)\) (see also [12]). So finally we have

\[
\int_{\partial G} \chi(\nu, \mu; g_1(\nu) g_2(\nu)^{-1}(\nu) \chi(\lambda, \nu; g_2) d\nu = 0. \tag{4}
\]

Here the function \(\chi(\lambda, \mu)\) possesses the following analytical properties:

\[
\bar{\partial}_\lambda \chi(\lambda, \mu) = 2\pi i \delta(\lambda - \mu), \quad -\bar{\partial}_\mu \chi(\lambda, \mu) = 2\pi i \delta(\lambda - \mu),
\]

where \(\delta(\lambda - \mu)\) is a \(\delta\)-function, or, in other words, \(\chi \to (\lambda - \mu)^{-1}\) as \(\lambda \to \mu\) and \(\chi(\lambda, \mu)\) is analytic function of both variables \(\lambda, \mu\) for \(\lambda \neq \mu\).

In the particular case \(\lambda = \mu = 0\) the relation (4) is nothing but the usual Hirota identity for the KP wave function \(\chi(\nu, 0, x)\) and the dual KP wave function \(\chi^*(\nu, 0, x')\) (see e.g. [1]-[4]). Just in this case \((\lambda = \mu = \infty)\) the Hirota bilinear identity has been derived from the \(\bar{\partial}\)-problem in [9].

The identity (4) represents itself the generalization of the Hirota identity which is bilocal with respect to the dynamical variables \(x\) and the spectral
variables \((\lambda, \mu)\). It can be considered as the point of departure without any reference to the \(\bar{\partial}\)-dressing method. Namely, the generalized Hirota bilinear relation (4) is the starting point of the analytic-bilinear approach which we will consider in this paper. The double bilocality (with respect to \(x\) and \((\lambda, \mu)\) provides us an additional freedom which will allow us to represent integrable hierarchies in a unified and condensed form.

Introducing the function \(\psi(\lambda, \mu, x)\) via

\[
\psi(\lambda, \mu; g) = g^{-1}(\mu)\chi(\lambda, \mu; g)g(\lambda),
\]

one gets another form of the generalized Hirota equation

\[
\int_{\bar{\partial}G} \psi(\nu, \mu; g_1)\psi(\lambda, \nu; g_2)d\nu = 0.
\]

Note that in the framework of algebro-geometric technique the function \(\psi(\lambda, \mu)\) corresponds to the Cauchy-Baker-Akhiezer kernel on the Riemann surface (see [13]). We will refer to the function \(\psi(\lambda, \mu)\) as the Cauchy-Baker-Akhiezer (CBA) function.

We will assume in what follows that we are able to find solutions for the relation (4) somehow. In particular, it can be done by the \(\bar{\partial}\)-dressing method.

The general setting of the problem of solving (4) requires some modification of Segal-Wilson Grassmannian approach [4]. Let us consider two linear spaces \(W(g)\) and \(\tilde{W}(g)\) defined by the function \(\chi(\lambda, \mu)\) (satisfying (4)) via equations connected with equation (4)

\[
\int_{\bar{\partial}G} f(\nu; g)\chi(\lambda, \nu; g)d\nu = 0,
\]

\[
\int_{\bar{\partial}G} \chi(\nu, \mu; g)h(\nu; g)d\nu = 0,
\]

here \(f(\lambda) \in W, \ h(\lambda) \in \tilde{W}; f(\lambda), h(\lambda)\) are defined in \(\bar{G}\).

It follows from the definition of linear spaces \(W, \ \tilde{W}\) that

\[
f(\lambda) = 2\pi i \int_{\bar{G}} \eta(\nu)\chi(\lambda, \nu)d\nu \wedge d\bar{\nu}, \quad \eta(\nu) = \left(\frac{\partial}{\partial \nu} f(\nu)\right),
\]

\[
h(\mu) = -2\pi i \int_{\bar{G}} \chi(\nu, \mu)\bar{\eta}(\nu)d\nu \wedge d\bar{\nu}, \quad \bar{\eta}(\nu) = \left(\frac{\partial}{\partial \nu} h(\nu)\right).
\]

These formulae in some sense provide an expansion of the functions \(f, h\) in terms of the basic function \(\chi(\lambda, \mu)\). The formulae (4) readily imply that
linear spaces \( W, \tilde{W} \) are transversal to the space of holomorphic functions in \( G \) (transversality property).

From the other point of view, these formulae define a map of the space of functions (distributions) on \( \tilde{G} \eta, \tilde{\eta} \) to the spaces \( W, \tilde{W} \). We will call \( \eta (\tilde{\eta}) \) a normalization of the corresponding function belonging to \( W (\tilde{W}) \).

The dynamics of linear spaces \( W, \tilde{W} \) looks very simple

\[
W(g) = W_0 g^{-1}; \quad \tilde{W}(g) = g \tilde{W}_0, \quad (10)
\]

here \( W_0 = W(g = 1), \tilde{W}_0 = \tilde{W}(g = 1) \) (the formulae (11) follow from identity (4) and the formulae (9)).

To introduce a dependence on several variables (may be of different type), one should consider a product of corresponding functions \( g(\lambda) \) (all of them commute).

The formula (4) is a basic tool for our construction. Analytic properties of the CBA kernel accompanied with the different choices of the functions \( g_1 \) and \( g_2 \) will provide us various compact and useful relations.

3 Generalized KP hierarchy

In the present paper we will consider the scalar KP hierarchy. It is generated by the generalized Hirota formula (3) where \( G \) is a unit disk and

\[
g(x, \lambda) = \exp \left( \sum_{i=1}^{\infty} x_i \lambda^{-i} \right). \quad (11)
\]

Let us consider the formula (3) with

\[
g_1 g_2^{-1} = g(x, \nu) g^{-1}(x', \nu) = \exp \left( \sum_{i=1}^{\infty} (x_i - x'_i) \nu^{-i} \right) = \frac{\nu - a}{\nu - b}. \quad (12)
\]

where \( a \) and \( b \) are arbitrary complex parameters, \( a, b \in G \). Since \( \log(1 - \epsilon) = \sum_{i=1}^{\infty} \epsilon^i / i \), one has

\[
x'_i - x_i = \frac{1}{i} a^i - \frac{1}{i} b^i.
\]

Substituting the expression (12) into (4), one gets

\[
\left( \frac{\mu - a}{\mu - b} \right) \chi(\lambda, \mu, x + [a]) - \left( \frac{\lambda - a}{\lambda - b} \right) \chi(\lambda, \mu, x + [b]) +

(b - a) \chi(\lambda, b, x + [a]) \chi(b, \mu, x + [b]) = 0, \quad \lambda \neq \mu
\]

(13)
where \( x + [a] = x_i + [a]_i, \ 0 \leq i < \infty, [a]_i = \frac{1}{i}a_i \). Equation (13) is the simplest functional equation for \( \chi(\lambda, \mu, x) \) which follows from the generalized Hirota equation (4).

Residues of the l.h.s. of (13) at the poles \( \mu = b \) and \( \lambda = b \) vanish identically. Evaluating the l.h.s. of (13) at \( \mu = a \) and \( \lambda = a \), one gets the equations

\[
\left( \frac{\lambda - a}{\lambda - b} \right) \chi(\lambda, a, x + [b]) = (b - a)\chi(\lambda, b, x + [a])\chi(b, a, x + [b]), \quad (14)
\]

\[
\left( \frac{\mu - a}{\mu - b} \right) \chi(a, \mu, x + [a]) = (a - b)\chi(a, b, x + [a])\chi(b, \mu, x + [b]), \quad (15)
\]

\( a \neq b \).

Equations (14) and (15) imply that

\[
(\lambda - \mu)\chi(\lambda, \mu, x) = \frac{(\lambda - a)\chi(\lambda, a, x - [a] + [\mu])}{(\mu - a)\chi(\mu, a, x - [a] + [\mu])} = \frac{(a - \mu)\chi(a, \mu, x + [a] - [\lambda])}{(a - \lambda)\chi(a, \lambda, x + [a] - [\lambda])}. \quad (16)
\]

Since \( (\mu - a)\chi(\mu, a) \to 1 \) as \( \mu - a \to 0 \), one gets from (13)

\[
(\lambda - \mu)^2\chi(\lambda, \mu, x + [\lambda])\chi(\mu, \lambda, x + [\mu]) = -1. \quad (17)
\]

We will solve the equations (14) - (17) in the next section. Now, let us consider the particular form of equation (13) for \( b = 0 \). In terms of the CBA function it reads

\[
\psi(\lambda, \mu, x + [a]) - \psi(\lambda, \mu, x) = a\psi(\lambda, 0, x + [a])\psi(0, \mu, x); \quad x_i' - x_i = \frac{1}{i}a_i. \quad (18)
\]

This equation is a condensed finite form of the whole KP-mKP hierarchy. Indeed, the expansion of this relation over \( a \) generates the KP-mKP hierarchies (and dual hierarchies) and linear problems for them. To demonstrate this, let us take the first three equations given by the expansion of (18) over \( a \)

\[
a: \quad \psi(\lambda, \mu, x)_x = \psi(\lambda, 0, x)\psi(0, \mu, x), \quad (19)
\]

\[
a^2: \quad \psi(\lambda, \mu, x)_y = \psi(\lambda, 0, x)_x\psi(0, \mu, x) - \psi(\lambda, 0, x)\psi(0, \mu, x)_x, \quad (20)
\]

\[
a^3: \quad \psi(\lambda, \mu, x)_t = \frac{1}{4}\psi(\lambda, \mu, x)_{xxx} - \frac{3}{4}\psi(\lambda, 0, x)_x\psi(0, \mu, x)_x + \frac{3}{4}\psi(\lambda, 0, x)_y\psi(0, \mu, x) - \psi(\lambda, 0, x)\psi(0, \mu, x)_y \quad (21)
\]

\( x = x_1; \quad y = x_2; \quad t = x_3 \).
In the order $a^2$ equation (18) gives rise equivalently to the equations
\begin{align*}
\psi(\lambda, \mu, x)_y - \psi(\lambda, \mu, x)_{xx} &= -2\psi(\lambda, 0, x)\psi(0, \mu, x), \quad (22) \\
\psi(\lambda, \mu, x)_y + \psi(\lambda, \mu, x)_{xx} &= 2\psi(\lambda, 0, x)\psi(0, \mu, x), \quad (23)
\end{align*}
Evaluating the first equation at $\mu = 0$, the second at $\lambda = 0$ one gets
\begin{align*}
f(x)_y - f(x)_{xx} &= u(x)f(x), \quad (24) \\
\tilde{f}(x)_y + \tilde{f}(x)_{xx} &= -u(x)\tilde{f}(x) \quad (25)
\end{align*}
where $u(x) = -2\psi(0, 0)_x$ and
\begin{align*}
f &= \int \psi(\lambda, 0)\rho(\lambda) d\lambda, \\
\tilde{f} &= \int \tilde{\rho}(\mu)\psi(0, \mu) d\mu
\end{align*}
$\rho(\lambda)$ and $\tilde{\rho}(\mu)$ are some arbitrary functions.

In a similar manner, one obtains from (19)-(21) the equations
\begin{align*}
f_t - f_{xxx} &= \frac{3}{2} uf_x + \frac{3}{4}(u_x + \partial_x^{-1} u_y)f, \quad (26) \\
\tilde{f}_t - \tilde{f}_{xxx} &= \frac{3}{2} u\tilde{f}_x + \frac{3}{4}(u_x - \partial_x^{-1} u_y)\tilde{f}. \quad (27)
\end{align*}
Both the linear system (24), (26) for the wave function $f$ and the linear system (25), (27) for the wave function $\tilde{f}$ give rise to the same KP equation
\begin{align*}
u_t &= \frac{1}{4} u_{xxx} + \frac{3}{2} uu_x + \frac{3}{4}\partial_x^{-1} u_{yy}. \quad (28)
\end{align*}
To derive linear problems for the mKP and dual mKP equations, we integrate equations (19), (22), (23) and (21) with the two arbitrary functions $\rho(\lambda), \tilde{\rho}(\mu)$
\begin{align*}
\Phi(x)_x &= f(x)\tilde{f}(x), \quad (29) \\
\Phi(x)_y - \Phi(x)_{xx} &= -2f(x)\tilde{f}(x)_x, \quad (30) \\
\Phi(x)_y + \Phi(x)_{xx} &= 2f(x)x\tilde{f}(x), \quad (31) \\
\Phi(x)_t - \Phi(x)_{xxx} &= -\frac{3}{2} f(x)x\tilde{f}(x)_x - \frac{3}{4}(f(x)\tilde{f}(x)_y - f(x)y\tilde{f}(x)) \quad (32)
\end{align*}
where
\begin{align*}
\Phi &= \int\int \tilde{\rho}(\mu)\psi(\lambda, \mu)\rho(\lambda) d\lambda d\mu.
\end{align*}
Using the first equation to exclude \( f \) from the second (and \( \tilde{f} \) from the third), we obtain

\[
\begin{align*}
\Phi_y - \Phi_{xx} &= v(x)\Phi_x, \quad (33) \\
\Phi_y + \Phi_{xx} &= -\tilde{v}(x)\Phi_x, \quad (34)
\end{align*}
\]

where \( v = -2\frac{\tilde{f}(x)x}{f(x)} \), \( \tilde{v} = 2\frac{f(x)x}{\tilde{f}(x)} \).

Similarly, one gets from (21)

\[
\begin{align*}
\Phi_t - \Phi_{xxx} &= \frac{3}{2}v(x)\Phi_{xx} + \frac{3}{4}(v_x + v^2 + \partial_x^{-1}v_y)\Phi_x, \quad (35) \\
\Phi_t - \Phi_{xxx} &= \frac{3}{2}\tilde{v}(x)\Phi_{xx} + \frac{3}{4}(\tilde{v}_x + \tilde{v}^2 - \partial_x^{-1}\tilde{v}_y)\Phi_x. \quad (36)
\end{align*}
\]

The system (33), (35) gives rise to the mKP equation

\[
v_t = v_{xxx} + \frac{3}{4}v^2v_x + 3v_x\partial_x^{-1}v_y + 3\partial_x^{-1}v_{yy}, \quad (37)
\]

while the system (34), (36) leads to the dual mKP equation, which is obtained from the (37) by the substitution \( v \rightarrow \tilde{v}, \ t \rightarrow -t, \ y \rightarrow -y, \ x \rightarrow -x \).

So the function \( \Phi \) is simultaneously a wave function for the mKP and dual mKP linear problems with different potentials, defined by the dual KP (KP) wave functions.

Using the equation (21) and relations (33) and (34), one also obtains an equation for the function \( \Phi \)

\[
\Phi_t - \frac{1}{4}\Phi_{xxx} - \frac{3}{8}\frac{\Phi_y^2 - \Phi_x^2}{\Phi_x} + \frac{3}{4}\Phi_x W_x = 0, \quad W_x = \frac{\Phi_y}{\Phi_x}. \quad (38)
\]

This equation first arose in Painleve analysis of the KP equation as a singularity manifold equation \(^{[14]}\).

It is tedious but absolutely straightforward check that the expansion of (18) in higher orders of \( a \) generates

1) the whole hierarchy of KP singularity manifold equations for \( \psi(\lambda, \mu) \) (or \( \Phi(x) \))

2) the hierarchy of linear problems for the mKP and dual mKP equations, where \( \psi(\lambda, \mu) \) (or \( \Phi(x) \)) is the common wave function and \( v = -2(\log \psi(0, \lambda, x))_x \), \( \tilde{v} = -2(\log \psi(\lambda, 0, x))_x \) are the potentials

3) mKP hierarchy for \( v \) and dual mKP hierarchy for \( \tilde{v} \)

4) the hierarchies of KP linear problems for \( \psi(\lambda, 0, x) \) and dual KP linear
problems for $\tilde{\psi}(\lambda, 0, x)$
and, finally
5) the KP hierarchy of equations for $u = -2\psi(0, 0)_x$.

Note also one interesting consequence of the formula (17)

$$\chi(0, \lambda, x) = -\frac{1}{\lambda^2} \chi^{-1}(\lambda, 0, x + [\lambda]).$$  (39)

4 KP hierarchy in the 'moving frame'. Darboux equations as the horizontal subhierarchy

Now let us consider the expansion of the l.h.s. of (13) over $\epsilon = a - b$, where $\epsilon \to 0$. In the first order in $\epsilon$ one gets

$$\Delta_1(b)\psi(\lambda, \mu, x) = \psi(b, \mu, x)\psi(\lambda, b, x)$$  (40)

where

$$\Delta_1(b) = \sum_{n=1}^{\infty} b^{n-1} \frac{\partial}{\partial x_n}.$$  

In the higher orders in $\epsilon$ one obtains the hierarchy of equations of the form (19)-(21) and their higher analogues with the substitution $\psi(\lambda, 0, x) \to \psi(\lambda, b, x)$, $\psi(0, \mu, x) \to \psi(b, \mu, x)$ and

$$\frac{\partial}{\partial x_i} \to \Delta_i(b) = \sum_{n=1}^{\infty} \frac{n!}{n!(n-i)!} b^{n-i} \frac{\partial}{\partial x_i}.$$  

Such a substitution is in fact nothing but the change of dynamical variables (or the coordinates on the group of functions $g$). Indeed, it is not difficult to show that $\Delta_i(b) = \frac{\partial}{\partial x_i(b)}$, where the dynamical variables $x_i(b)$ are defined by the relation

$$\sum_{i=1}^{\infty} \frac{x_i(b)}{(\lambda - b)^i} = \sum_{i=1}^{\infty} \frac{x_i}{(\lambda)^i}.$$  (41)

It is clear that

$$\left[ \frac{\partial}{\partial x_i(\lambda')}, \frac{\partial}{\partial x_i(\lambda)} \right] = 0.$$  (42)

Note one interesting property of the derivatives $\frac{\partial}{\partial x_i(\lambda)}$, namely

$$\left[ \frac{\partial}{\partial \lambda}, \frac{\partial}{\partial x_i(\lambda)} \right] = (i + 1) \frac{\partial}{\partial x_{i+1}(\lambda)}.$$  (43)
So the operator $\frac{\partial}{\partial \lambda}$ is a ‘mastersymmetry’ for all vector fields $\frac{\partial}{\partial x_i(\lambda)}$.

The expansion of equation (13) up to the third order in $\epsilon$ gives the equations

$$\frac{\partial}{\partial x_1(b)} \psi(\lambda, \mu, x(b)) = \psi(b, \mu, x(b)) \psi(\lambda, b, (x(b)), \quad \text{(44)}$$

$$\frac{\partial}{\partial x_2(b)} \psi(\lambda, \mu, x(b)) = \frac{\partial}{\partial x_1(b)} \psi(\lambda, b) \cdot \psi(b, \mu) - \psi(\lambda, b) \cdot \frac{\partial}{\partial x_1(b)} \psi(b, \mu), \quad \text{(45)}$$

$$\frac{\partial}{\partial x_3(b)} \psi(\lambda, \mu, x(b)) = \frac{1}{4} \frac{\partial^2}{\partial x_1(b)^2} \psi(\lambda, \mu) - \frac{3}{4} \frac{\partial}{\partial x_1(b)} \psi(\lambda, b) \cdot \frac{\partial}{\partial x_1(b)} \psi(b, \mu) + \frac{3}{4} \left( \psi(b, \mu) \cdot \frac{\partial}{\partial x_2(b)} \psi(b, \mu) - \psi(\lambda, b) \cdot \frac{\partial}{\partial x_2(b)} \psi(b, \mu) \right). \quad \text{(46)}$$

The analogues of equations (22), (23) have the form

$$\frac{\partial}{\partial x_2(b)} \psi(\lambda, \mu, x(b)) - \frac{\partial^2}{\partial x_1(b)^2} \psi(\lambda, \mu) + 2 \psi(\lambda, b) \frac{\partial}{\partial x_1(b)} \psi(b, \mu) = 0, \quad \text{(47)}$$

$$\frac{\partial}{\partial x_2(b)} \psi(\lambda, \mu, x(b)) + \frac{\partial^2}{\partial x_1(b)^2} \psi(\lambda, \mu) - 2 \frac{\partial}{\partial x_1(b)} \psi(\lambda, b) \cdot \psi(b, \mu) = 0. \quad \text{(48)}$$

Equations (44)-(46) and higher equations again give rise to the generalized KP hierarchy but now in coordinates $x_i(b), i = 1, 2, 3,...$. For such KP hierarchy written in the ‘moving’ frame the parameter $b$ is an arbitrary one, but fixed.

Let us consider now equations of the type (44) written for several values of $b$. We denote $x_1(b_\alpha) = \xi_\alpha, \alpha = 1, 2, ..., n$. Equations (44) taken for $b = b_\alpha, \lambda = b_\beta, \mu = b_\gamma (\alpha \neq \beta \neq \gamma)$, look like

$$\frac{\partial}{\partial \xi_\alpha} \psi_{\beta\gamma} = \psi_{\beta\alpha} \psi_{\alpha\gamma}, \quad \alpha \neq \beta \neq \gamma \quad \text{(49)}$$

where $\psi_{\alpha\beta} = \psi(b_\alpha, b_\beta, x)$. The system (13) is just well-known system of $N^2 - N$ resonantly interacting waves.

Integrating equations (44) over $\mu$ with the function $\rho(\mu)$ and evaluating the result at $b = b_\alpha, \gamma = b_\beta$, one gets

$$\frac{\partial f_\beta}{\partial \xi_\alpha} = \psi_{\beta\alpha} f_\alpha, \quad (\alpha \neq \beta) \quad \text{(50)}$$
where $f_\beta = \int d\mu \psi(b_\beta, \mu) \rho(\mu)$. Analogously one gets
\[
\frac{\partial f^*_\beta}{\partial \xi_\alpha} = \psi_{\alpha\beta} f^*_\alpha, \quad (\alpha \neq \beta)
\]
where $f^*_\beta = \int d\lambda \psi(\lambda, b_\beta) \rho^*(\lambda)$ and $\rho^*(\lambda)$ is an arbitrary function. The systems (50) and (51) are the linear problem and dual linear problem for equations (49), respectively.

Expressing $\psi_{\alpha\beta}$ via $f_\alpha$ and $f^*_\alpha$, one gets from (50) and (51) (using (49)) the same system for $f_\alpha$ and $f^*_\alpha$
\[
\frac{\partial^2 H_\alpha}{\partial \xi_\beta \partial \xi_\gamma} = \frac{1}{H_\beta} \frac{\partial H_\beta}{\partial \xi_\gamma} \frac{\partial H_\alpha}{\partial \xi_\beta} + \frac{1}{H_\gamma} \frac{\partial H_\gamma}{\partial \xi_\beta} \frac{\partial H_\alpha}{\partial \xi_\gamma}, \quad (\alpha \neq \beta \neq \gamma \neq \alpha).
\]

The system (52) is the Darboux system which was introduced for the first time in the differential geometry of surfaces [14] and then was rediscovered in the matrix form within the $\partial-$dressing method in the paper [5]. Note that the Darboux equations in the variables of the type $x_1(b_\alpha)$ have appeared also in the paper [16] within completely different approach.

One can treat the Darboux equations (52) with different $n$ as the horizontal subhierarchy of the whole generalized KP hierarchy. Note that equations (14)-(16) and their higher analogues give rise to the higher resonantly interacting waves equations.

5 Combescure symmetry transformations for the generalized KP hierarchy

Let us consider now the symmetries of the equations derived above. All the higher equations of the hierarchy are, as usual, the symmetries of each member of the hierarchy. Here we will discuss another type of symmetries.

Since $\rho(\lambda)$ and $\bar{\rho}(\mu)$ are arbitrary functions, equation (38) and the hierarchy (18) possess the symmetry transformation
\[
\Phi(\rho(\lambda), \bar{\rho}(\mu)) \rightarrow \Phi'(\rho'(\lambda), \bar{\rho}'(\mu)).
\]

This transformation is, in fact, the transformation which changes the normalization of the wave functions. The fact that such transformations are connected with the so-called Combescure transformations, known for a long time in differential geometry, was pointed out in [10].
The Combescure transformation was introduced last century within the study of the transformation properties of surfaces (see e.g. [15], [17]). It is a transformation of surface such that the tangent vector at a given point of the surface remains parallel. The Combescure transformation is essentially different from the well-known Bäcklund and Darboux transformations. The Combescure transformation plays an important role in the theory of the systems of hydrodynamical type [18]. It is also of great interest for the theory of (2+1)-dimensional integrable systems [19]. Combescure symmetry transformations are essential part of the analytic-bilinear approach.

The Combescure transformation can be characterized in terms of the corresponding invariants. The simplest of these invariants for the mKP equation is just the potential of the KP equation L-operator expressed through the wave function
\[ u = \frac{f(x)y - f(x)xx}{f(x)}, \] (53)
or, in terms of the solution for the mKP (dual mKP) equation
\[ v'y + v'xx - \frac{1}{2}((v')^2)_x = v_y + v_{xx} - \frac{1}{2}(v^2)_x, \] (55)
\[ \tilde{v}'y - \tilde{v}'xx - \frac{1}{2}((\tilde{v}')^2)_x = \tilde{v}_y - \tilde{v}_{xx} - \frac{1}{2}(\tilde{v}^2)_x. \] (56)
The solutions of the mKP equations are transformed only by a subgroup of the Combescure symmetry group corresponding to the change of the weight function \( \tilde{\rho}(\mu) \) (left subgroup) and they are invariant under the action of the subgroup corresponding to \( \rho(\lambda) \) (vice versa for the dual mKP).

All the hierarchy of the Combescure transformation invariants is given by the expansion over \( \epsilon \) near the point \( x \) of the relation (18) rewritten in the form
\[ \frac{\partial}{\partial \epsilon} \left( \frac{\tilde{f}(x') - \tilde{f}(x)}{\epsilon f(x)} \right) = -\frac{1}{2} \frac{\partial}{\partial \epsilon} \frac{1}{\epsilon} u(x'), \quad x'_i - x_i = \frac{1}{t} \epsilon^i; \] (57)
\[ \frac{\partial}{\partial \epsilon} \left( \frac{f(x) - f(x')}{\epsilon f(x)} \right) = \frac{1}{2} \frac{\partial}{\partial \epsilon} \frac{1}{\epsilon} u(x'), \quad x'_i - x_i = -\frac{1}{t} \epsilon^i. \] (58)
The expansion of the left part of these relations gives the Combescure transformation invariants in terms of the wave functions \( \tilde{f}, f \). To express them
in terms of mKP equation (dual mKP equation) solution, one should use the formulae

\[ v = -2 \tilde{f} \frac{\xi}{f}, \quad \tilde{f} = \exp(-\frac{1}{2} \partial_x^{-1} v); \quad (59) \]

\[ \tilde{v} = 2 f \frac{f_x}{f}, \quad f = \exp(-\frac{1}{2} \partial_x^{-1} \tilde{v}). \quad (60) \]

It is also possible to consider special Combescure transformations keeping invariant the KP equation (dual KP equation) wave functions (i.e. solutions for the dual mKP (mKP) equations). The first invariants of this type are

\[ \frac{\Phi_x'(x)}{f'(x)} = \frac{\Phi_x(x)}{f(x)}, \quad (61) \]

\[ \frac{\Phi_x'(x)}{f'(x)} = \frac{\Phi_x(x)}{f(x)}. \quad (62) \]

All the hierarchy of the invariants of this type is generated by the expansion of the left part of the following relations over \( \epsilon \)

\[ \frac{\Phi(x') - \Phi(x)}{f(x)} = \epsilon f(x'), \quad x'_i - x_i = \frac{1}{i} \epsilon^i; \quad (63) \]

\[ \frac{\Phi(x) - \Phi(x')}{f(x)} = \epsilon \tilde{f}(x'), \quad x'_i - x_i = -\frac{1}{i} \epsilon^i. \quad (64) \]

Now let us consider the equation (38) and all the hierarchy given by the relation (18). This equation admits the Combescure group of symmetry transformations \( \Phi(\rho(\lambda), \tilde{\rho}(\mu)) \rightarrow \Phi' = \Phi(\rho'(\lambda), \tilde{\rho}'(\mu)) \) consisting of two subgroups (right and left Combescure transformations). These subgroups have the following invariants

\[ v = \frac{\Phi_y - \Phi_{yy}}{\Phi_x} \quad (65) \]

and

\[ \tilde{v} = \frac{\Phi_y + \Phi_{yy}}{\Phi_x}. \quad (66) \]

From (38), (39) it follows that they just obey the mKP and dual mKP equation respectively. The invariant for the full Combescure transformation can be obtained by the substitution of the expression for \( v \) via \( \Phi \) (65) to the formula (55). It reads

\[ u = \partial_x^{-1} \left( \frac{\Phi_y}{\Phi_x} \right)_y - \frac{\Phi_{xxx}}{\Phi_x} + \frac{\Phi_{xx} - \Phi_y^2}{2\Phi_x^2}. \quad (67) \]
From (24), (25), (53), (54), (55), (56) it follows that $u$ solves the KP equation.

So there is an interesting connection between equation (38), mKP-dual mKP equations and KP equation. Equation (38) is the unifying equation. It possesses a Combesure symmetry transformations group. After the factorization of equation (38) with respect to one of the subgroups (right or left), one gets the mKP or dual mKP equation in terms of the invariants for the subgroup (53), (54). The factorization of equation (38) with respect to the full Combesure transformations group gives rise to the KP equation in terms of the invariant of group (57).

In other words, the invariant of equation (38) under the full Combesure group is described by the KP equation, while the invariants under the action of its right and left subgroups are described by the mKP or dual mKP equations.

6 $\tau$-function and addition formulae

Now we will analyze the functional equations (13)-(17). Equation (16), evaluated at $\mu = 0$ for some $a = a_0$ gives

$$\lambda \chi(\lambda, 0, x) = \frac{(\lambda - a) \chi(\lambda, a_0, x - [a_0])}{(-a) \chi(\mu, a, x - [a_0])} = \frac{Z(\lambda, x)}{Z(0, x)}$$

(68)

where we denote $Z(\lambda, x) = (\lambda - a_0) \chi(\lambda, a_0, x - [a_0])$. Substituting the expression (68) into equation (16), we get

$$(\lambda - \mu) \chi(\lambda, \mu, x) = \frac{Z(\lambda, x + [\mu])}{Z(\mu, x + [\mu])}.$$  

(69)

It is easy to check that in virtue of (69) equation (14) is satisfied identically, while equation (15) takes the form

$$R(a, \lambda)R(\lambda, b)R(b, a) = R(a, b)R(b, \lambda)R(\lambda, a)$$

(70)

where $R(a, b, x) = Z(a, x + [a] + [b])$. In terms of $R(a, b)$ we have

$$(\lambda - \mu) \chi(\lambda, \mu, x) = \frac{R(\lambda, \mu, x - [\lambda])}{R(\mu, \mu, x - [\mu])}.$$  

(71)

Thus the problem of resolving equations (14), (15) is reduced to the single functional equation (70), which is of the form of the triangle (Yang-Baxter)
equation, well-known in the quantum theory of solvable models (see e.g. [20]).

From the definition of $R(a,b,x)$ it follows that it has a certain special structure. Indeed, since $Z(a,x) = R(a,b,x - [a] - [b])$, one has $R(a,b,x - [a] - [b]) = R(a,0,x - [a])$. Consequently $R(a,b,x) = R(a,0,x + [b])$.

So we should solve the triangle equation (70) within the class of $R$ of the form $R(a,b,x) = \Xi(a)(x + [b])$, where $\Xi(a)$ is some function. Taking the logarithm of both parts of (70), one gets

$$\Theta(a,\lambda) + \Theta(a,\lambda) + \Theta(b,a) = \Theta(a,b) + \Theta(b,\lambda) + \Theta(\lambda,a) \quad (72)$$

where $\Theta = \log R$. Representing $\Theta$ as $\Theta(a,b,x) = \Theta(a,b,x) + \Theta(a,\lambda) + \Theta(b,\lambda)$, where $\Theta_+$ and $\Theta_-$ are respectively symmetric and antisymmetric parts of $\Theta$, one easily concludes that $\Theta_+$ solves (72) identically while $\Theta_-$ satisfies the equation

$$\Theta_-(a,\lambda) + \Theta_-(a,\lambda) + \Theta_-(b,a) = 0. \quad (73)$$

Taking equation (73) at $b = 0$, one gets

$$\Theta_-(a,\lambda) = \Theta_-(0,\lambda) - \Theta_-(0,a). \quad (74)$$

Then since $\Theta(a,b,x)$ (as $R(a,b,x)$) has the form $\Theta(a,b,x) = Z_a(x + [b])$, where $Z_a$ are some functions, it follows from (74) that

$$\Theta_-(a,\lambda) = Z_0-(x + [\lambda]) - Z_0-(x + [a]).$$

Then for the symmetric part of $\Theta_+$ one has $Z_0+(x + [b]) = Z_{0+}(x + [a])$. Taking $b = 0$, one concludes that $Z_0+(x) = Z_0+(x + [a])$. So $\Theta_+(a,b) = Z_0+(x + [a] + [b])$. Thus general solution of (72) has the form

$$\Theta(a,b,x) = Z_0+(x + [a] + [b]) + Z_0-(x + [b]) - Z_0-(x + [a]).$$

Consequently, the general solution of (70) for our class of $R(a,b,x)$ reads

$$R(a,b,x) = R_s(x + [a] + [b]) + \tau(x + [b]) \tau(x + [a]) \quad (75)$$

where $R_s$ and $\tau$ are arbitrary functions. Substituting now the expression (73) into the expression (72), we get

$$\chi(\lambda,\mu,x) = \frac{1}{(\lambda - \mu)} \frac{\tau(x - [\lambda] + [\mu])}{\tau(x)} \quad (76)$$
This formula coincides with the formula introduced in the paper [1] in a completely different context. Note that in our approach the function \( \tau(x) \) is still an arbitrary function giving a general solution of the functional equations (14), (15) through the formula (76).

Now we will use the general equation (13). Substituting (76) into (13), one gets

\[
(a - \mu)(\lambda - b)\tau(x + [a] + [\mu])\tau(x + [\lambda] + [b]) + \\
(\lambda - a)(b - \mu)\tau(x + [\lambda] + [a])\tau(x + [b] + [\mu]) + \\
(b - a)(\lambda - \mu)\tau(x + [b] + [a])\tau(x + [\lambda] + [\mu]) = 0.
\]

(77)

It is nothing but the simplest addition formula for the \( \tau \)-function derived in [1], which is closely connected with the Fay’s trisecant formula [21].

Generalized Hirota identity gives rise also to other addition formulae from [1]. Indeed, let us choose in (4)

\[
g(x)g^{-1}(x') = \prod_{\alpha=1}^{n} \frac{\nu - a_{\alpha}}{\nu - b_{\alpha}}
\]

(78)

where \( n \) is an arbitrary integer and \( x' - x = \sum_{\alpha=1}^{n}[a_{\alpha}] - [b_{\alpha}] \). In this case equation (4) gives

\[
\prod_{\alpha=1}^{n} \frac{\mu - a_{\alpha}}{\mu - b_{\alpha}} \chi \left( \lambda, \mu, x + \sum_{\alpha=1}^{n}[a_{\alpha}] \right) - \prod_{\alpha=1}^{n} \frac{\lambda - a_{\alpha}}{\lambda - b_{\alpha}} \chi \left( \lambda, \mu, x + \sum_{\alpha=1}^{n}[b_{\alpha}] \right) + \\
\sum_{\alpha=1}^{n} (b_{\alpha} - a_{\alpha}) \prod_{\beta, \gamma \neq \alpha} \frac{b_{\alpha} - a_{\beta}}{b_{\alpha} - b_{\gamma}} \chi \left( \lambda, b_{\alpha}, x + \sum_{\delta=1}^{n}[a_{\delta}] \right) \chi \left( b_{\alpha}, \mu, x + \sum_{\delta=1}^{n}[b_{\delta}] \right) = 0.
\]

(79)

Substituting the expression (76) into (79) and shifting \( x \rightarrow x + [\lambda] \), one gets

\[
\prod_{\alpha=1}^{n} \frac{\mu - a_{\alpha}}{\mu - b_{\alpha}} \tau \left( x + [\mu] + \sum_{\gamma=1}^{n}[a_{\gamma}] \right)\tau \left( x + [\lambda] + \sum_{\gamma=1}^{n}[b_{\gamma}] \right) - \\
\prod_{\alpha=1}^{n} \frac{\lambda - a_{\alpha}}{\lambda - b_{\alpha}} \tau \left( x + [\mu] + \sum_{\gamma=1}^{n}[b_{\gamma}] \right)\tau \left( x + [\lambda] + \sum_{\gamma=1}^{n}[a_{\gamma}] \right) + \\
\sum_{\alpha=1}^{n} \frac{(b_{\alpha} - a_{\alpha})}{(b_{\alpha} - \mu)(\lambda - b_{\alpha})} \prod_{\beta, \gamma \neq \alpha} \frac{b_{\alpha} - a_{\beta}}{b_{\alpha} - b_{\gamma}} \times \\
\tau \left( x + [b_{\alpha}] + \sum_{\beta=1}^{n}[a_{\beta}] \right)\tau \left( x + [\lambda] + [\mu] + \sum_{\beta, \beta \neq \alpha}[b_{\beta}] \right) = 0.
\]

(80)
It is not difficult to check (after some renotations) that equation (80) coincide with the Plücker’s relations for universal Grassmannian manifold (see Theorem 3 of [1]). That means, according to the Theorems 1-3 of the paper [1] that \( \tau \) is the \( \tau \) function of the KP hierarchy. Note that the formulae (76), (80) provide solutions simultaneously for the KP, mKP (dual mKP) and SM-KP hierarchies.

7 Determinant formulae for transformations.

The analytic-bilinear approach allows to represent in a compact from not only the integrable hierarchies but also rather general transformations acting in the space of solutions.

We will consider here the transformations which are equivalent to the action of an arbitrary meromorphic function \( g(\lambda) \) on the CBA function \( \chi(\lambda, \mu, x) \). Let \( g(\lambda) \) be a meromorphic function in \( G \) which has simple poles at the points \( a_i \) \( (i = 1, 2, ..., n) \) and simple zeros at the points \( b_i \) \( (i = 1, 2, ..., n) \), i.e. \( g(\lambda) = \prod_{i=1}^{n} \frac{\lambda - a_i}{\lambda - b_i} \). To construct the transformed CBA function \( \chi^\dagger(\lambda, \mu) \) it sufficient to find a solution of the equation

\[
\int_{\partial G} \chi(\nu, \mu)g(\nu)\chi^\dagger(\lambda, \nu)d\nu = 0
\]

where \( \chi^\dagger \) has the same normalization \((\lambda - \mu)^{-1}\) as \( \chi \).

The simplest way to find \( \chi^\dagger \) consists in the use the following consequence of equation (4)

\[
\int_{\partial G} \int_{\partial G} d\nu d\rho \chi(\nu, \mu)g(\nu)\chi^\dagger(\rho, \nu, g)g^{-1}(\rho)\chi(\lambda, \rho) = 0.
\]

Using this formula, one finds

\[
\chi^\dagger(\lambda, \mu) = g^{-1}(\lambda)g(\mu)\frac{\det \Delta_{n+1}}{\det \Delta_n}
\]

(81)

where

\[
\Delta_{m, \alpha, \beta} = \chi(a_{\alpha}, b_{\beta}), \quad \alpha, \beta = 1, 2, ..., m
\]

and \( a_{n+1} = \lambda, b_{n+1} = \mu \).

The formula (81) defines the generic discrete transformation of \( \chi \). In terms of the \( \tau \)–function this transformation has the form

\[
\frac{\tau^\dagger(x - [\lambda] + [\mu])}{\tau^\dagger(x)} = (\lambda - \mu)g^{-1}(\lambda)g(\mu)\frac{\det F_{n+1}}{\tau(x) \det F_n}
\]
In the simplest case $n = 1$ we have $(b_1 = b, a_1 = a)$

$$\chi^\dagger(\lambda, \mu) = \frac{(\lambda - b)(\mu - a)}{(\lambda - a)(\mu - b)} \begin{vmatrix} \chi(a, b) & \chi(a, \mu) \\ \chi(\lambda, b) & \chi(\lambda, \mu) \end{vmatrix}$$

(82)

and

$$\frac{\tau^\dagger(x - [\lambda] + [\mu])}{\tau^\dagger(x)} = \frac{(\lambda - \mu)(a - b)(\lambda - b)(\mu - a)}{(\lambda - a)(\mu - b)} \begin{vmatrix} \tau(x - [a] + [b]) & \tau(x - [a] + [\mu]) \\ \tau(x - [\lambda] + [b]) & \tau(x - [\lambda] + [\mu]) \end{vmatrix}$$

where $|A| = \det A$. One can represent the transformations (81) also in terms of the function $\psi(\lambda, \mu)$. In that form the determinant formulae (81) taken at $\lambda = 0$ or $\mu = 0$ are very similar to the determinant formulae for the iterated Darboux transformations (see e.g. [23]). The formulae (81) give us the Darboux transformations for all subhierarchies (KP, mKP, dual mKP, SM-KP hierarchies) of the generalized KP hierarchy.

The determinant formulae (81) provide us also the multilineal relations for the $\tau$-function. Since $\tau^\dagger(x) = \tau(x + \sum_{i=1}^n (-[a_i] + [b_i]))$, the formula (81) is the $n$-linear relation for the $\tau$-function. It is easy to check that in the simplest case $n = 1$ this formula gives the simplest addition formula (87). At $n = 2$ the formula looks like

$$\tau(x)\tau(x - [\lambda] + [\mu] + [b_1] - [a_1] + [b_2] - [a_2]) \times \begin{vmatrix} \tau(x - [a_1] + [b_1]) & \tau(x - [a_1] + [b_2]) \\ \tau(x - [a_2] + [b_1]) & \tau(x - [a_2] + [b_2]) \end{vmatrix} =$$

$$\frac{(\lambda - \mu)(a - \mu)(a - 2)(\lambda - b_1)(\lambda - b_2)}{(\mu - b_1)(\mu - b_2)(\lambda - a_1)(\lambda - a_2)} \times \begin{vmatrix} \tau(x - [\lambda] + [\mu]) & \tau(x - [\lambda] + [b_1]) & \tau(x - [\lambda] + [b_2]) \\ \tau(x - [a_1] + [\mu]) & \tau(x - [a_1] + [b_1]) & \tau(x - [a_1] + [b_2]) \\ \tau(x - [a_2] + [\mu]) & \tau(x - [a_2] + [b_1]) & \tau(x - [a_2] + [b_2]) \end{vmatrix}$$

(83)

It is cumbersome but straightforward check that the relation (83) is satisfied due to the higher addition formulae (80) with $n = 2$. 

where

$$F_n,\alpha\beta = \frac{\tau(x - [a_\alpha] + [b_\beta])}{a_\alpha - b_\beta}$$
8 Global expression for the $\tau$-function and the closed one-form

In this section we will investigate in what sense the formula (76) defines the $\tau$-function. Using this formula, it is possible to prove that the $\tau$-function is defined through the CBA function in terms of the closed 1-form, which can be written both for differentials of the variables $x$ and for variations of the function $g$ ($g$ is defined on the unit circle). Moreover, it is possible to show that the formula (76) and the definition in terms of the closed 1-form are equivalent.

For calculation in terms of variations, it is more convenient to use the formula (76) in the form

$$\chi(\lambda, \mu) = \frac{\tau(g(\nu) \times \left(\frac{\nu-\lambda}{\nu-\mu}\right))}{\tau(g(\nu))(\lambda - \mu)}$$

(84)

(see [11]). Differentiating (84) with respect to $\lambda$, one obtains

$$-\frac{1}{\tau(g)} \oint \frac{\delta \tau(g(\nu))}{\delta \log g} \frac{1}{\nu - \lambda} d\nu = \frac{\partial}{\partial \lambda} (\lambda - \mu) \chi(\lambda, \mu; g),$$

(85)
or for $\lambda = \mu$

$$-\frac{1}{\tau(g)} \oint \frac{\delta \tau(g(\nu))}{\delta \log g} \frac{1}{\nu - \lambda} d\nu = \chi_r(\lambda, \lambda; g)$$

(86)

where $\chi_r(\lambda, \mu; g) = \chi(\lambda, \mu; g) - (\lambda - \mu)^{-1}$. This equation can be rewritten in the form

$$-\frac{1}{\tau(g)} \oint \frac{\delta \tau(g(\nu))}{\delta \log g} \frac{1}{\nu - \lambda} d\nu = \oint \chi_r(\nu, \nu; g) \frac{1}{\nu - \lambda} d\nu.$$

(87)

Relation (87) implies that the functionals $-\frac{1}{\tau(g)} \frac{\delta \tau(g(\nu))}{\delta \log g}$ and $\chi_r(\nu, \nu; g)$ are identical for the class of functions analytic outside the unit circle and decreasing at infinity. Thus for $\frac{\delta g}{g}$ belonging to this class one has

$$-\delta \log \tau(g(\nu)) = \oint \chi_r(\nu, \nu; g) \frac{\delta g(\nu)}{g(\nu)} d\nu.$$

(88)

This expression defines a variational 1-form defining the $\tau$-function. It is easy to prove using the identity [3] that this form is closed. Indeed, according to [4]

$$\delta \chi(\lambda, \mu; g) = \oint \chi(\nu, \mu; g) \frac{\delta g(\nu)}{g(\nu)} \chi(\lambda, \nu; g) d\nu,$$
\[
\delta \chi_r(\lambda, \lambda; g) = \oint \chi_r(\nu, \lambda; g) \frac{\delta g(\nu)}{g(\nu)} \chi_r(\lambda, \nu; g) d\nu .
\] (89)

So the variation of the (88) gives

\[
- \delta^2 \log \tau(g) = \oint \oint \chi_r(\nu, \lambda; g) \chi_r(\lambda, \nu; g) \frac{\delta' g(\nu)}{g(\nu)} \frac{\delta g(\lambda)}{g(\lambda)} d\nu d\lambda .
\] (90)

The symmetry of the kernel of second variation with respect to \( \lambda, \nu \) implies that the form (88) is closed.

So the formula (88) gives the definition of the \( \tau \)-function in terms of the closed 1-form. For the standard KP coordinates

\[
\frac{\delta g(\lambda)}{g(\lambda)} = \sum_{n=1}^{\infty} \frac{dx_n}{\lambda^n} .
\]

This formula allows us to obtain a closed 1-form in terms of \( dx_n \)

\[
- \delta \log \tau(g(\nu)) = \sum_{n=0}^{\infty} \frac{\partial^n}{\partial \nu^n} \chi_r(\nu; g) \bigg|_{\nu=0} dx_n .
\] (91)

For \( x = x_1 \) this formula immediately gives the standard formula

\[
\frac{\partial^2}{\partial x^2} \log \tau = \frac{1}{2} u
\]

where \( u \) is a solution for the KP equation.

In fact it is possible to prove that the function \( \tau \) defined as the solution of the relation (88) satisfies the global formula (76). To do this, we will show using the formula (88) that the derivatives of difference of logarithms of the l.h.s. and the r.h.s. of the expression (84) with respect to \( \lambda, \mu, \bar{\lambda}, \bar{\mu} \) and the variation with respect to \( g \) are equal to zero for arbitrary \( \lambda, \mu, g \). That means that l.h.s. and r.h.s. of (76) could differ only by the factor, and the normalization of the function \( \chi \) implies that this factor is equal to 1.

First we will calculate the derivative with respect to \( \lambda \)

\[
\frac{\partial}{\partial \lambda} \left( \log \chi(\lambda, \mu)(\lambda - \mu) - \log \tau \left( g(\nu) \times \left( \frac{\nu - \lambda}{\nu - \mu} \right) \right) \right) + \frac{\partial \tau(g(\nu))}{\partial \lambda} =
\]

\[
\frac{\partial}{\partial \lambda} \log \chi(\lambda, \mu)(\lambda - \mu) + \oint \chi_r(\nu, \lambda; g(\nu)) \times \left( \frac{\nu - \lambda}{\nu - \mu} \right) \frac{1}{\lambda - \nu} d\nu =
\]

\[
\frac{\partial}{\partial \lambda} \log \chi(\lambda, \mu)(\lambda - \mu) + \chi_r(\nu, \lambda; g(\nu)) \times \left( \frac{\nu - \lambda}{\nu - \mu} \right) .
\] (92)
The second term can be found in terms of $\chi(\lambda, \mu; g)$ using the determinant formula (82). The formula (82) gives

$$\chi(\lambda, \mu; g \times \frac{\nu - \lambda}{\nu - \mu'}) = \frac{(\lambda - \mu')(\mu - \lambda)}{\mu - \mu'} \det \left( \begin{array}{cc} \chi(\lambda, \mu; g) & \chi(\lambda, \mu'; g) \\ \frac{\partial}{\partial \lambda} \chi(\lambda, \mu; g) & \frac{\partial}{\partial \lambda} \chi(\lambda, \mu'; g) \end{array} \right).$$ (93)

Taking the regular part of this formula at $\mu = \lambda$, one immediately obtains that the expression (92) is equal to zero.

The case with the derivative over $\mu$ is analogous; derivatives over $\bar{\lambda}$ and $\bar{\mu}$ immediately give zero. So now we proceed to the calculation of variation $\delta (\log \chi(\lambda, \mu)(\lambda - \mu)) = \left. \frac{\partial}{\partial \lambda} \chi(\lambda, \mu, x) \right|_{\lambda=0}$.

Below we will give a brief sketch of derivation of 1-form in terms of Baker-Akhiezer function. Differentiation of (76) (with shifted arguments) with respect to $\lambda$ and $\mu$ gives

$$\left( \frac{\partial}{\partial \lambda} + \frac{\partial}{\partial x_1(\lambda)} \right) \log[(\lambda - \mu)\chi(\lambda, \mu; x)] = -\frac{\partial}{\partial x_1(\lambda)} \log\tau(x),$$ (95)

$$\left( \frac{\partial}{\partial \mu} - \frac{\partial}{\partial x_1(\mu)} \right) \log[(\lambda - \mu)\chi(\lambda, \mu; x)] = \frac{\partial}{\partial x_1(\mu)} \log\tau(x)$$ (96)

where $\frac{\partial}{\partial x_1(\lambda)} = \sum_{n=1}^{\infty} \frac{ \partial }{ \partial x_n } \lambda^{n-1}$. Equation (95) implies that

$$\frac{\partial}{\partial x_n} \log\tau(x) = -\frac{1}{(n-1)!} \frac{\partial^{n-1}}{\partial \lambda^{n-1}} \left( \left( \frac{\partial}{\partial \lambda} + \frac{\partial}{\partial x_1(\lambda)} \right) \log[(\lambda - \mu)\chi(\lambda, \mu; x)] \right) \bigg|_{\lambda=0}. $$

Hence, we have the closed 1-form $\omega(x, dx) = \omega(x, dx) = d \log\tau$. Similar expression for $\omega$ can be obtained using (96). At $\mu = 0$ the formula (97) is equivalent to the formula found in [3].
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