Time-series observation of the effects of kinetic energy conservation error on isotropic and anisotropic steady incompressible turbulence

Ryoma Honda¹, Hiroki Suzuki²,³ and Shinsuke Mochizuki¹

¹Graduate School of Sciences and Technology for Innovation, Yamaguchi University, 2-16-1 Tokiwadai, Ube-shi, Yamaguchi 755-8611, Japan
²Graduate School of Natural Science and Technology, Okayama University, 3-1-1 Tsushima-naka, Kita-ku, Okayama-shi, Okayama 700-8530, Japan
³E-mail: h.suzuki@okayama-u.ac.jp

Abstract. The purpose of this study was to examine the effects of kinetic energy conservation error on isotropic and anisotropic steady incompressible turbulence fields. Results were obtained from time series of turbulent kinetic energy and static pressure fluctuation. A standard large-eddy simulation (LES) was used in the numerical analysis. The conservation error was generated using the Crank-Nicolson method. The result obtained using the fourth-order Runge-Kutta method was used to set results of reference. The effects of the conservation error on the mean value of the global turbulent kinetic energy were found to be small. On the other hand, it was found that the time series and mean values of static pressure fluctuation root mean square (rms) were sensitive to the conservation error. This discrepancy in the static pressure fluctuation rms was able to be reduced by adjusting the value of the model constant used in the LES. However, this adjusted model constant caused an error in the global turbulent kinetic energy. These results were found in both isotropic and anisotropic turbulence fields.

1. Introduction

Incompressible flows that are found in industrial equipment are often in a turbulent state. Turbulence phenomena essentially affect the nature of the flow [1]. Turbulent diffusion, which is due to turbulent phenomena, is used for the turbulent mixing of heat and mass (e.g., [2, 3]). Large-eddy simulation (LES) [1] is often used to analyze such turbulent fields. LES is also used in software such as OpenFOAM for analyzing turbulent flow fields. When analyzing a turbulent field with LES, it is necessary to accurately analyze the conservation law of turbulent energy, which is derived from the governing equations (e.g., [4, 5]). The accuracy of LES can be maintained to a sufficiently high level by using a discretization scheme that can maintain the conservation law of the turbulent kinetic energy with high accuracy.

Time-integration methods can be categorized as explicit or implicit methods. The high-order Runge-Kutta (RK) method is an example of an explicit method that has been used in previous studies (e.g., [5, 6]). Implicit methods such as the CN method have also been used in previous studies [7-12]. Implicit methods are used to analyze both isotropic and anisotropic turbulence. These previous studies used implicit methods to analyze both viscous and non-linear convection terms. The method proposed by [8] integrated such a convection term over time using an implicit method while constraining the kinetic energy conservation error to be sufficiently small. When the convection term is analyzed using
an implicit method without using this approach, the conservation error may be non-negligible. For example, when using software such as OpenFOAM [9], the observed results may be affected by the conservation error. Therefore, we were motivated to investigate the effect of the kinetic energy conservation error on isotropic and anisotropic turbulent flows.

Static pressure fluctuations are often used as the fluctuations that characterize turbulent fields. In previous studies, turbulent kinetic energy was mainly analyzed, but static pressure fluctuations were less investigated. The static pressure fluctuation is considered to have the same dimension as the turbulent kinetic energy with density. Nevertheless, the effect of turbulent kinetic energy conservation error on the static pressure fluctuation is seen to have hardly been investigated. In this study, we provide isotropic and anisotropic external force terms using the linear forcing method and numerically study the basic characteristics of the generated steady turbulence. By focusing on the Reynolds number dependence of steady turbulence that is maintained by an external force so that the large-scale turbulent field becomes isotropic, the error is disturbed when a kinetic energy conservation error occurs. One can then investigate the effect on the nature of the flow field.

We clarify in this study the effect of the conservation error of kinetic energy on turbulent flow fields by using results obtained from time series of static pressure as well as turbulent kinetic energy. Specifically, isotropic and anisotropic steady turbulence fields are used to examine the characteristics of turbulent kinetic energy and static pressure fluctuations [13]. An LES-based method is used to numerically analyze these turbulent fields. The kinetic energy conservation error is generated using the CN method, which is an implicit method. By using the fourth-order RK method, a reference analysis with a negligible conservation error is set. The effects of the conservation error on the turbulent kinetic energy and static pressure fluctuations are examined, focusing on the value of the LES model constant.

2. Numerical methods

The governing equations of the flow field to be analyzed are the continuity equation and the Navier-Stokes equation, the latter of which includes an external forcing term $F_i$ for applying the linear forcing method. The external forcing term is given as follows:

$$F_i = Q u^F_i.$$  \hspace{1cm} (1)

The linear forcing method (e.g., [14]) generates steady turbulence by providing a component of the external forcing term that is proportional to the velocity component. The components of the analytical solution $u^F (u_1^F, u_2^F, u_3^F)$ given by [15] involve setting the external forcing term such that anisotropic steady turbulence is generated:

$$u_1^F = -\cos(x_1) \sin(x_2), u_2^F = \sin(x_1) \cos(x_2), \text{ and } u_3^F = 0,$$  \hspace{1cm} (2)

where $x_1$, $x_2$, and $x_3$ are the respective velocity components for the streamwise, transverse, and spanwise directions. These velocity components are based on Taylor's analytical solution. Therefore, the external force term can be regarded to provide, through the use of the linear forcing method, the velocity component of Taylor's analytical solution to the external forcing term component. The external force term that generates isotropic steady turbulence is also set by combining the components of the analytical solution. The above velocity fields, used in this study to generate anisotropic and isotropic steady turbulent fields, satisfy the continuity equation.

In this study, we use the steady turbulence fields that are maintained by an external forcing term such that the large-scale turbulence field is isotropic or anisotropic, and the effect of the conservation error of kinetic energy on the turbulence field can then be examined. The numerical results, obtained using the fourth-order RK method and the second-order CN method, are compared in the present analysis. As mentioned, there are explicit and implicit methods of time integration. Because the conservation law of the discretization scheme is explicit, conservation error may occur when the time-integration method is implicit. Therefore, if implicit time integration is used over an extended time
There is a possibility that the conservation error will have a non-negligible magnitude. Applying the CN method without using the previous technique causes a significant kinetic energy conservation error. In this analysis, a numerical technique based on LES is applied. LES analyzes only large-scale turbulent structures directly.

In this study, the computational grid was set as the staggered grid. All of the quantities are taken to be nondimensional. The periodic cubic box is used as the present computational domain. Therefore, the periodic boundary condition is set for all boundaries. The size of the computational domain is $L^3 = (2\pi)^3$. The governing equations were analyzed until nondimensional time $t = 3000$. The value of the constant of the external forcing term was set to $Q = 1$. The initial velocity field was set to an isotropic velocity field. The Reynolds number was set to $Re = 200$. For this Reynolds number, the turbulent Reynolds number, based on Taylor’s microscale, is about 100. The number of grid points was set to $N^3 = 32^3$. The present numerical code is based on those of the previous works [5,6]. The second-order central difference was used as the spatial discretization scheme. The fractional step method was used to integrate the governing equations. The fourth-order RK and second-order CN schemes were used as the time-integration methods. The skew-symmetric form was used as the form of the convection term [4]. A direct solver using the fast Fourier transform [5,6] was used to solve the pressure Poisson equation. The standard Smagorinsky model [1] was used as the sub-grid scale (SGS) model for the LES analysis. Initially, $C_s = 0.1$, widely used as a standard value, was used as the model constant. The model constants for the LES result were also set to agree with those of DNS at a sufficiently high Reynolds number. Specifically, $C_s = 0.0375$ was set in the isotropic turbulent field and $C_s = 0.12$ in the anisotropic field. These optimized values are given only for cases without kinetic energy conservation error.

3. Results and discussion

In the inviscid homogeneous isotropic fluctuation field, the results of the CN method, which is an implicit time-integration method, were compared with those of the RK method. Figure 1 shows the temporal evolution of turbulent kinetic energy in the inviscid flow field. As shown in the figure, in the results obtained by the RK method, the value of the normalized kinetic energy was constant over time. On the other hand, in the results obtained by the CN method, the value of kinetic energy increased over time. We investigated the effects of this conservation error on the turbulent flow field. As shown by this
result, the conservation error was observed to be sufficient in the results of the CN method but was sufficiently small in the results of the RK method.

In this analysis, the time series of global turbulent kinetic energy was investigated. The global turbulent kinetic energy is the spatial mean value of turbulent kinetic energy at each time point. Also, the global turbulent kinetic energy is a temporal function. In Figure 2, the time series of the global turbulent kinetic energy was compared in the presence and absence conservation error. Figure 2 also shows the mean value of these time series. As shown, when the optimized value of the model constant was used, the mean value obtained by the CN method was in good agreement with that obtained by the RK method. Also, from these two numerical results, the amplitude of the time series obtained by the CN method was also comparable to that obtained by the RK method. These agreements in results suggest that the conservation error set using the CN method has little effect on the time series and mean value of the global turbulent kinetic energy. When a widely-used value is set as the model constant, the mean value may be slightly smaller than that obtained using the optimized value.

Based on the turbulent kinetic energy that was mentioned in the previous paragraph, static pressure fluctuation was used. In Figure 2, the time series of the root mean square (rms) of static pressure fluctuation is shown. The rms value was calculated based on the spatial average at each time point. Figure 2 shows that the static pressure fluctuation obtained using the RK method fluctuates with time. Also, the mean value of the static pressure fluctuation when the conservation error was added using the CN method was larger than that obtained by using the RK method. This larger mean value was found in both of the time series, obtained using the model constants of $C_s = 0.0573$ and $C_s = 0.1$. As shown in Figure 2(a), the mean value of the global turbulent kinetic energy was not noticeably affected by the conservation error. On the other hand, as shown in Figure 2(b), the effects of the conservation error on the mean value of the temporally varying rms value of static pressure fluctuation may be non-negligible. Therefore, the static pressure fluctuation rms was found to be more sensitive to

Figure 2. Time series and mean values of the global turbulent kinetic energy (a) and the spatial rms value of static pressure fluctuation (b) in the isotropic turbulence field.

![Figure 2](image_url)
the conservation error than the turbulent kinetic energy. This result suggests that both turbulence kinetic energy and static pressure fluctuation are needed to be investigated to ensure sufficiently small conservation error.

By changing the model constant, the mean values of the global turbulent kinetic energy and static pressure fluctuation rms can be altered. Setting the model constant to \( C_s = 0.466 \), the mean value of the rms of the static pressure fluctuation obtained by using the CN method can be compared to that obtained by using the RK method at \( Re = 300 \). The time series of the global turbulent kinetic energy and static pressure fluctuation rms using this model constant is also shown in the figure. Figure 2(b) shows that, with this model constant, the mean value of the static pressure fluctuation rms affected by the conservation error can be comparable to that without such error. Therefore, by optimizing the value of the model constant, the effects of the conservation error on the static pressure fluctuation rms value were able to be reduced. The global turbulent kinetic energy obtained using this optimized value is shown in Figure 2(a). Figure 2(a) also shows that the mean value of the global turbulent kinetic energy obtained using this optimized value was significantly smaller than the reference value obtained using the RK method. This result indicates that the model constant value that minimizes the static pressure fluctuation rms caused an error in the mean value of turbulent kinetic energy.

Figures 3(a) and 3(b) show the time series and mean values of the global turbulent kinetic energy and spatial rms value of static pressure fluctuation in the anisotropic turbulence field. The figures also show the time series of the global turbulent kinetic energy in the presence and absence of conservation error. Similar to the results of the isotropic turbulence field shown in the previous figure, the effect of the conservation error on the mean value of the global turbulent kinetic energy was found to be small. As shown in Figure 3(b), the conservation error had a significant effect on the rms value of the static pressure fluctuation. Specifically, the conservation error increased the mean value of the rms value of the static pressure fluctuation. Therefore, the responses of the global turbulent kinetic energy and the static pressure rms of the anisotropic turbulence field to the conservation error are qualitatively equivalent to those of the isotropic turbulence field.
Also, in the anisotropic turbulence field, the deviation of the static pressure fluctuation rms value due to the conservation error was reduced. Under the condition that Re = 300, the optimized value of the model constant that matches the static pressure fluctuation rms value affected by the conservation error with that obtained without this error was also obtained in the anisotropic turbulence field. As shown in the figure, by using the optimized value of the model constant, the static pressure fluctuation rms value affected by the conservation error roughly matched that obtained without this error. Figure 3(a) also shows the time series and the mean of the global turbulent kinetic energy analyzed using this optimized value. As shown in the figure, the mean value of the global turbulent kinetic energy obtained using this optimized value was significantly smaller than the reference value that is not affected by the conservation error. Therefore, in an anisotropic turbulence field, minimizing the error of the static pressure fluctuation rms value by optimizing the model constant value also caused an error in the global turbulent kinetic energy. This result obtained in the anisotropic turbulence field was qualitatively consistent with the results obtained in the isotropic turbulence field.

As shown in these results, the deviation of the static pressure fluctuation rms value, caused by the conservation error, was minimized using the SGS component in the isotropic and anisotropic turbulence fields. However, this optimization caused an error in the mean value of the global turbulent kinetic energy. This result suggests that it is difficult to maintain the accuracy of both the turbulent kinetic energy and static pressure fluctuation in common LES analysis when conservation error occurs. As shown in the figures, the form shapes of the time series of the global turbulent kinetic energy and static pressure fluctuation rms were different between the isotropic and anisotropic turbulence fields. As shown in the time series result, the turbulence structure was significantly different between the isotropic and anisotropic turbulence fields. As shown in this study, we consider that differences in the form shape and turbulent flow structure of the time series did not qualitatively influence the effects of the conservation error on the mean values of turbulent kinetic energy and static pressure fluctuation.

4. Conclusions

The purpose of this study was to use time series signals to clarify the effects of conservation error on the isotropic and anisotropic turbulence fields. The conservation error was generated using the CN method. Time series signals of the global turbulent kinetic energy and the rms of static pressure fluctuation at each time point were used. A standard large-eddy simulation based on the Smagorinsky model was used and the second-order-accurate central difference scheme was applied to discretize the governing equations. The external forcing term of the governing equation was used to generate the isotropic and anisotropic steady turbulence fields.

In the inviscid homogeneous isotropic fluctuation field, the conservation error of the kinetic energy using the CN method was verified to be significant. However, in this analysis using the RK method, the squared storage error was negligible. In the isotropic turbulence field, the effect of the conservation error on the time-series signal and the mean value of the global turbulent kinetic energy were small. However, the mean value of the static pressure fluctuation rms increased due to the conservation error. This increase was able to be reduced by adjusting the SGS model constant. However, the optimization of the model constant values caused a discrepancy between the mean value of global turbulent kinetic energy and the reference value. These results of the anisotropic turbulence field were qualitatively the same as those of the isotropic field.

In this study, the effects of the conservation error on the global turbulent kinetic energy and static pressure fluctuation were examined. If the Reynolds number is not sufficient, the characteristics of the turbulence field will change depending on the Reynolds number. For example, low Reynolds number anisotropic turbulence can be found near the wall surface [6,16,17]. Based on the results of this study, further research should be conducted from the viewpoint of Reynolds number dependence. Also, the turbulence fields analyzed in this study were only steady. The present results will need to be examined in unsteady turbulence fields. Fluid acceleration of mean flow [5,18,19] can provide the nature of unsteady turbulence in a straightforward manner. In future work, we also intend to investigate whether isotropic and anisotropic fields are affected by the freestream fluid acceleration.
Acknowledgments
This work was partly supported by the Japanese Ministry of Education, Culture, Sports, Science and Technology through Grants-in-Aid (Nos. 18H01369, 18K03932, 20H02069, and 21K03859).

References
[1] Pope S B 2000 Turbulent Flows. Cambridge University Press
[2] Suzuki H, Nagata K, Sakai Y, Hasegawa Y 2013 Phys. Scr. 2013 (T155) 014062
[3] Suzuki H, Nagata K, Sakai Y 2012 J. Vis. 15 109-117
[4] Morinishi Y, Lund T S, Vasilyev O V, Moin P 1998 J. Comput. Phys. 143 90-124
[5] Suzuki H, Nagata K, Sakai Y, Hayase T, Hasegawa Y, Ushijima T 2013 Fluid Dyn. Res. 45 061409
[6] Suzuki H, Nagata K, Sakai Y, Hayase T, Hasegawa Y, Ushijima T 2013 Int. J. Numer. Meth. Fluids 73 509-522
[7] Choi H, Moin P 1994 J. Comput. Phys. 113 1-4
[8] Ham F E, Lien F S, Strong A B 2002 J. Comput Phys. 177 117-133
[9] Komen E, Shams A, Camilo L, Koren B. 2014 Comput. Fluids 96 87-104
[10] Moin P, Mahesh K 1998 Ann. Rev. Fluid Mech. 30 539-578
[11] Cho C K, Kim S 2008 Int. J. Numer. Meth. Fluids 56 1351-1357
[12] Kasbaoui M H, Patel R G, Koch D L, Desjardins O 2017) J. Fluid Mech 833 687-716
[13] Pumir A 1994 Phys. Fluids 6 2071-2083
[14] Carroll P L, Blanquart G 2013 Phys. Fluids 25 105114
[15] Goto S, Vassilicos J C 2015 Phys. Lett. A 379 1144-1148
[16] Moser R D, Kim J, Mansour N N 1999 Phys. Fluid 11 943-945
[17] Lee M, Moser R D 2015 J. Fluid Mech. 774 395-415
[18] Suzuki H, Mochizuki S, Hasegawa Y 2018 Flow Meas. Instrum. 62 1-8
[19] Suzuki H, Mochizuki S, Hasegawa Y 2020 Adv. Mech. Eng. 12 1-13