AUSLANDER-REITEN SEQUENCES FOR GORENSTEIN RINGS OF DIMENSION ONE

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ABSTRACT. Let $R$ be a complete local Gorenstein ring of dimension one, with maximal ideal $m$. We show that if $M$ is a Cohen-Macaulay $R$-module which begins an AR-sequence, then this sequence is produced by a particular endomorphism of $m$ corresponding to a minimal prime ideal of $R$. We apply this result to determining the shape of some components of stable Auslander-Reiten quivers, which in the considered examples are shown to be tubes.

1. INTRODUCTION

The theory of Auslander-Reiten (AR) quivers is central in the study of artin algebras. Regarding AR theory for maximal Cohen-Macaulay modules over a complete Cohen-Macaulay local ring, the cases of finite AR quivers have been studied thoroughly (see [17]), but in the more common case of infinite type, shapes of AR quivers seem to be largely unknown.

The paper [1] agrees with this assessment (cf. its introduction), and begins to bridge this gap. It establishes a variety of lemmas in the context of symmetric orders over a DVR, $\mathcal{O}$, and applies these lemmas to proving the shape (namely, a tube) of some components of the AR quiver of a truncated polynomial ring $\mathcal{O}[X]/(X^n)$. The work of the latter calculation consisted largely in proving that a module is indecomposable, but also in proving that a sequence is an AR sequence. Our main result, Theorem 4.8, makes the latter task much easier, in the setting of a complete local Gorenstein ring $R$ of dimension one, with maximal ideal $m$. Namely it shows that there exists a set of elements of $\text{End}_R m$, corresponding to the minimal primes of $R$, which produce the AR sequences in a concrete way. We apply the theorem in Section 6, where we establish the shapes of AR components (again, tubes) over a graded hypersurface of the form $k[x,y]/((bx^p + y^q)f)$ where $f \in k[x,y]$ is an arbitrary homogeneous polynomial.

Regarding the structure of this paper, Sections 2 and 3, as well as the Appendix, consist of supporting material for the proof of our main result in Section 4. In Section 5 we give background concerning the abstract notion of stable translation quivers, and in Section 6 we record lemmas concerning the AR quiver of $R$. Specifically, Proposition 6.23 establishes a criterion, based on material in [1], for proving that an AR component is a tube. In Section 7 we apply this proposition and our main result, to an example.

We would like to thank the developers of Singular [7]; we used it to compute many examples testing Theorem 4.8 and Proposition 8.7 before proving them.

2. PRELIMINARIES

Notation 2.1. Throughout this paper, all rings are assumed noetherian. A ring which is described with any subset of the words {reduced, Cohen-Macaulay, Gorenstein, regular} is implicitly commutative. By a graded ring we will mean a $\mathbb{Z}$-graded ring, that is, a ring $A = \bigoplus_{i \in \mathbb{Z}} A_i$ satisfying $A_i A_j \subseteq A_{i+j}$. If $A$ has not been referred to as a graded ring, $\mathcal{J}(A)$ will denote the Jacobson radical of $A$, whereas if $A$ is explicitly graded, $\mathcal{J}(A)$ will denote
the intersection of all maximal graded left ideals of \( A \) (in our situations this will always coincide with the intersection of all maximal graded right ideals). Similarly, but when \( A \) is commutative, if \( A \) is not given a grading then \( Q(A) = A[\text{nonzerodivisors}]^{-1} \) (the total quotient ring of \( A \)), whereas if \( A \) is graded then \( Q(A) = A[\text{homogeneous nonzerodivisors}]^{-1} \).

A (graded) ring \( A \) is said to be (graded-) local if \( A/J(A) \) is a division ring. By a connected graded ring we shall mean a commutative \( \mathbb{N} \)-graded ring \( R = \bigoplus_{i \geq 0} R_i \) such that \( R_0 \) is a field. In this case \( \hat{R} \) will denote the \( m \)-adic completion of \( R \), where \( m = \bigoplus_{i \geq 1} R_i \). If \( R \) is any commutative ring, \( \text{min} R \) will denote its set of minimal primes. If \( R \) is furthermore local or graded-local, we usually write \( m_R \) instead of \( J(R) \).

We will say that an \( R \)-module \( M \) has rank (specifically, rank \( n \)), if \( M \otimes_R Q(R) \) is a free \( Q(R) \)-module (of rank \( n \)). If \( R \) is reduced, then \( \overline{R} \) will denote the integral closure of \( R \) in \( Q(R) \).

### 2.1. Trace lemmas

We establish some preliminary lemmas regarding trace. Observations of this general type have certainly been made before; see [2, proposition 5.4]. First, we define the trace of an endomorphism of an arbitrary finitely generated projective module, as in [10].

**Definition 2.2.** Let \( A \) be a commutative ring, and let \( P \) be a finitely generated projective \( A \)-module. Then the map \( \mu_P : \text{Hom}_A(P, A) \otimes_A P \rightarrow \text{End}_A P \) given by \( f \otimes x \mapsto (y \mapsto f(y)x) \) is an isomorphism, by Lemma 2.14. Let \( \varepsilon : \text{Hom}_A(P, A) \otimes_A P \rightarrow A \) denote the map given by \( f \otimes x \mapsto f(x) \). For \( h \in \text{End}_A P \), we define \( \text{trace}(h) = \varepsilon(\mu_P^{-1}(h)) \). If \( e_1, \ldots, e_n \) and \( \varphi_1, \ldots, \varphi_n \in \text{Hom}_A(P, A) \) are such that \( \mu_P(\sum_{i=1}^n \varphi_i \otimes e_i) = \text{id}_P \), then \( \text{trace}(h) \) furthermore equals \( \sum_{i=1}^n \varphi_i(h(e_i)) \). From this, and using that \( P = \sum_i A e_i \), it follows that trace is symmetric, in the sense that \( \text{trace}(gh) = \text{trace}(hg) \) for all \( g, h \in \text{End}_A P \).

**Remark 2.3.** We can see that the above definition of trace specializes to the usual one when \( P \) is free, by taking the aforementioned \( \{e_i, \varphi_i\}_i \) to be a free basis and the corresponding projection maps. If \( A = k_1 \times \cdots \times k_s \) is a product of fields \( k_i \), then by a similar argument we see that for any \( h \in \text{End}_A P \), we have \( \text{trace}(h) = (\text{trace}(h \otimes_A k_1), \ldots, \text{trace}(h \otimes_A k_s)) \).

Let \( R \) be a commutative ring and set \( Q = Q(R) \). Recall that if \( R \) is an ungraded reduced ring, then \( Q \) is the product of fields \( R_p = Q(R/p) \) where \( p \) ranges over \( \text{min} R \). In particular, each \( R_p \) is an ideal of \( Q \), and a \( Q \)-algebra.

**Lemma 2.4.** Let \( R \) be a reduced ring (possibly graded), let \( M \) be a finitely generated \( R \)-module such that \( M \otimes_R Q \) is \( Q \)-projective, and let \( h \in \text{End}_R M \). Then \( \text{trace}(h \otimes_R Q) \in \overline{R} \). (In the ungraded case, the condition that \( M \otimes_R Q \) is \( Q \)-projective is automatically satisfied, since \( Q \) is semisimple.)

**Proof.** First suppose the graded case. Let \( Q' = R[\text{nonzerodivisors}]^{-1} \); thus \( Q' \) is a localization of \( Q \) is a localization of \( R \). and \( R \subseteq Q \subseteq Q' \). As \( M \otimes_R Q \) is \( Q \)-projective, there exists a finite set \( \{e_i \in M \otimes_R Q\} \) and corresponding \( \{\varphi_i : M \otimes_R Q \rightarrow Q\} \) such that \( y = \sum_i \varphi_i(y)e_i \) for all \( y \in M \otimes_R Q \). Then the images of the \( e_i \) in \( M \otimes_R Q' \) have the property that \( y = \sum_i (\varphi_i \otimes_Q Q')(y)e_i \) for all \( y \in M \otimes_R Q' \). Therefore \( \text{trace}(h \otimes_R Q) = \sum_i \varphi_i((h \otimes_R Q)(e_i)) = \sum_i (\varphi_i \otimes_Q Q')(h \otimes_R Q'(e_i)) \). Since \( \overline{R} \) is equal to the integral closure of \( R \) in \( Q' \) by [11, Corollary 2.3.6], we are thus reduced to the ungraded case.

Since \( \overline{R} = \prod_{p \in \text{min} R} \overline{R}/p \), we see by Remark 2.3 that it suffices to show \( \text{trace}(h \otimes_R R_p) \in \overline{R}/p \), for each \( p \in \text{min} R \). As \( h \otimes_R R_p = (h \otimes_R R/p) \otimes_R R_p \), we may assume \( R \) is a domain. By [15, Theorem 2.1], \( h \) satisfies a monic polynomial with coefficients in \( R \), say \( f(X) \in R[X] \). Let
\( H = h \otimes_R Q \), and let \( \mu(X) \in Q[X] \) be the minimal polynomial of \( H \). Let \( \chi(X) \in Q[X] \) be the characteristic polynomial of \( H \), and take a field extension \( L \supseteq Q \) over which \( \chi(X) \) splits, say \( \chi(X) = (X - \alpha_1)(X - \alpha_2) \cdots (X - \alpha_s) \), \( \alpha_i \in L \). Each \( \alpha_i \) is also a root of \( \mu(X) \), and therefore of \( f(X) \). Therefore \( R[\alpha_1, \ldots, \alpha_s] \) is an integral extension of \( R \). Thus \( \overline{R} \supseteq Q \cap R[\alpha_1, \ldots, \alpha_s] \), which contains the coefficients of \( \chi(X) \). Finally, recall that \( \text{trace}(H) \) is the degree-\((s-1)\) coefficient of \( \chi \).

**Lemma 2.5.** In the situation of Lemma 2.4, assume that \( \dim R = 1 \) and that \( R \) is either a complete local ring or a connected graded ring. If some power of \( h \) lies in \( m_R \text{End}_R M \), then \( \text{trace}(h \otimes_R Q) \in \mathcal{J}(\overline{R}) \).

**Proof:** As \( \mathcal{J}(\overline{R}) = \prod_{p \in \min_R} \mathcal{J}(\overline{R}/p) \), we may again assume \( R \) is a domain. In the connected graded case, we have \( \mathcal{J}(\overline{R}) \cap \overline{R} = \mathcal{J}(\overline{R}) \) by Lemma 9.4, and we can therefore assume the complete local case. We can also assume \( M \subseteq M \otimes_R Q \), i.e., replace \( M \) by its image in \( M \otimes_R Q \).

Now let \( M\overline{R} \) denote the \( \overline{R} \)-module of \( M \otimes_R Q \) generated by \( M \). Note that \( M\overline{R} \) is a free \( \overline{R} \)-module, since all torsion-free \( \overline{R} \)-modules are free. Since \( \overline{R} \) is local, we can choose a basis for \( M\overline{R} \) which consists of elements of \( M \), say \( e_1, \ldots, e_n \). (Indeed, setting \( n = \text{rank}(M\overline{R}) \), Nakayama’s lemma allows us to find a set \( \{e_1, \ldots, e_n\} \subseteq M \) such that \( M\overline{R} = \sum_i \overline{R} e_i \). Then we have a surjective endomorphism of \( M\overline{R} \), equivalently an automorphism, mapping a basis onto \( \{e_1, \ldots, e_n\} \).

By fixing this basis, we can identify \( \text{End}_R M \) as an \( R \)-subalgebra of the ring of \( n \times n \) matrices \( \text{Mat}_{n \times n}(\overline{R}) \), in the obvious way. By assumption on \( h \), some power of \( h \) lies in \( m_R \text{Mat}_{n \times n}(\overline{R}) \subseteq \mathcal{J}(\overline{R})\text{Mat}_{n \times n}(\overline{R}) \). Thus the image of \( h \) in \( \text{Mat}_{n \times n}(\overline{R}/\mathcal{J}(\overline{R})) \) is nilpotent. The lemma now follows from the fact that over a field, any nilpotent matrix has zero trace.

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### 2.2. Cohen-Macaulay Modules and Gorenstein Rings

For the remainder of this section, assume \( R \) is a Cohen-Macaulay ring which is either a complete local ring or a connected graded ring. Let \( m = m_R \).

**Notation 2.6.** In the complete local case, let \( \text{CM}(R) \) denote the category of finitely generated maximal Cohen-Macaulay \( R \)-modules, and (following [3]) let \( L_p(R) \) denote the full subcategory of \( \text{CM}(R) \) whose objects have the property that \( M_p \) is \( R_p \)-free for all nonmaximal prime ideals \( p \). If \( R \) is instead a Cohen-Macaulay connected graded ring, we define \( \text{CM}(R) \) and \( L_p(R) \) in the same way but restricted to graded modules (keeping the full Hom sets).

Let \( k = R/m_R \). Then \( R \) is Gorenstein if and only if it is Cohen-Macaulay and \( \dim_k(\text{Ext}^{\dim_R(k,R)}_{R}(k,R)) = 1 \). If \( R \) is Gorenstein, and \( M \in \text{CM}(R) \), then ([6, Theorem 3.3.10]): \( \text{Ext}^{i}_{R}(M,R) = 0 \) for all \( i \geq 1 \), and the map \( M \rightarrow \text{Hom}_R(\text{Hom}_R(M,R),R) \) given by \( m \mapsto (f \mapsto f(m)) \) is an isomorphism. We denote \( \text{Hom}_R(M,R) \) by \( M^* \).

### 2.3. Auslander-Reiten Sequences

**Definition 2.7.** Let \( N \) be an indecomposable in \( \text{CM}(R) \). Then (cf. [17] Lemma 2.9') we may define an Auslander-Reiten (AR) sequence starting from \( N \) to be a short exact sequence

\[
(2.1) \quad 0 \longrightarrow N \overset{p}{\longrightarrow} E \overset{q}{\longrightarrow} M \longrightarrow 0
\]

in \( \text{CM}(R) \) such that \( M \) is indecomposable and the following property is satisfied: Any map \( N \rightarrow L \) in \( \text{CM}(R) \) which is not a split monomorphism factors through \( p \). Equivalently, \( N \)
is indecomposable and any map \( L \rightarrow M \) in \( \text{CM}(R) \) which is not a split epimorphism factors through \( q \). The sequence (2.1) is unique if it exists, and is also called the AR sequence ending in \( M \). Given an AR sequence (2.1), \( N \) is called the Auslander-Reiten translate of \( M \), written \( \tau(M) \); and \( \tau^{-1}(N) \) denotes \( M \).

**Definition 2.8.** A morphism \( f : X \rightarrow Y \) in \( \text{CM}(R) \) is called an irreducible morphism if (1) \( f \) is neither a split monomorphism nor a split epimorphism, and (2) Given any pair of morphisms \( g \) and \( h \) in \( \text{CM}(R) \) satisfying \( f = gh \), either \( g \) is a split epimorphism or \( h \) is a split monomorphism.

2.9. Let \( L \) be an indecomposable in \( \text{CM}(R) \), and assume we have the AR sequence (2.1). Then the following are equivalent (cf. [17, 2.12 and 2.1'-]):

- (a) \( L \) is isomorphic to a direct summand of \( E \).
- (b) There exists an irreducible morphism \( N \rightarrow L \).
- (c) There exists an irreducible morphism \( L \rightarrow M \).

**Theorem 2.10.** (cf. [17, Theorem 3.4], and [3, Theorem 3]) Let \( M \not\cong R \) be an indecomposable in \( \text{CM}(R) \). Then \( M \in L_p(R) \) if and only if there exists an AR sequence ending in \( M \).

Notice also that if \( R \) is Gorenstein, applying \((_)^\ast\) shows that there exists an AR sequence ending in \( M \) if and only if there exists an AR sequence starting from \( M \). The appendix of [11] contains a nice proof of Theorem 2.10 in a setting which includes Gorenstein rings of dimension one.

**Lemma 2.11.** Assume \( \dim R = 1 \), and let \( N \in \text{CM}(R) \). Then \( N \in L_p(R) \) if and only if \( N \otimes_R Q \) is a projective \( Q \)-module.

**Proof.** The prime ideals of \( Q \) correspond to \( \text{min}R \). Now use the fact that, since \( Q \) is noetherian, a \( Q \)-module is projective precisely when it is free at all maximal ideals of \( Q \) (cf. [8, Exercise 4.11]).

2.12. For the remainder of this section assume furthermore that \( R \) is Gorenstein of dimension one, and \( M \not\cong R \) is an indecomposable in \( L_p(R) \). Then (ignoring a graded shift, in the graded case; it will not concern us) \( \tau(M) = \text{syz}_R(M) \) (cf. [17, 3.11]), where \( \text{syz}_R(M) \) denotes the syzygy module of \( M \), which is defined to be the kernel of a minimal surjection onto \( M \) by a free \( R \)-module. The module \( \tau^{-1}(M) = \text{syz}_{R}^{-1}(M) \in L_p(R) \) is determined by \( \text{syz}_R(\text{syz}_{R}^{-1}(M)) \cong M \), and can be computed via \( \text{syz}_R^{-1} \equiv (\text{syz}_{R}(M^\ast))^\ast \).

**Definition 2.13.** Given a ring \( A \), and \( A \)-modules \( X \) and \( Y \), \( \text{Hom}_A(X,Y) \) denotes \( \text{Hom}_A(X,Y)/(\text{maps factoring through projective } A \text{-modules}) \); and \( \text{End}_A(X) \) denotes \( \text{Hom}_A(X,X) \). An \( A \)-homomorphism is said to be stably zero if it factors through a projective \( A \)-module.

**Lemma 2.14.** ([17, Lemma 3.8]) Let \( A \) be a commutative ring, and let \( X \) and \( Y \) be finitely generated \( A \)-modules. The sequence

\[
\text{Hom}_A(X,A) \otimes_A Y \xrightarrow{\mu} \text{Hom}_A(X,Y) \longrightarrow \text{Hom}_A(X,Y) \longrightarrow 0
\]

is exact, where \( \mu : \text{Hom}_A(X,A) \otimes_A Y \rightarrow \text{Hom}_A(X,Y) \) is given by \( f \otimes y \mapsto (x \mapsto f(x)y) \).

**Lemma 2.15.** \( \text{End}_R(M) \cong \text{Ext}_R^1(\text{syz}_{R}^{-1}(M),M) \) as left \( \text{End}_R(M) \)-modules.
Remark 2.16. Let \( M \in L_p(R) \) be a nonfree indecomposable. Then \( \text{End}_R M \) is a (graded-)local ring (cf. [3] proposition 8), and therefore so is \( \overline{\text{End}}_R M \). It follows from Lemma 2.15 and Theorem 2.10 that the ring \( \text{End}_R(M) \) has a simple socle when considered as a left module over itself, and that if \( h : M \to M \) generates this socle then the AR sequence starting from \( M \) equals the pushout via \( h \) of the short exact sequence \( 0 \to M \to F \to \text{syz}_R^1(M) \to 0 \) where \( F \) is free. In particular, if \( i \) denotes the given injective map \( M \to F \), and \( 0 \to M \to X \to N \to 0 \) is the AR sequence starting from \( M \), then \( X \cong (M \oplus F)/((-h(m),i(m)))m \in M \).

3. Testing stable-vanishing with trace.

In this section, let \( R \) simply be a commutative ring, let \( Q = Q(R) \), and let \( M \) be a finitely generated \( R \)-module such that \( M \otimes_R Q \) is a projective \( Q \)-module. Let \( (_,^*) \) denote \( \text{Hom}_R(_,R) \).

Notation 3.1. Given an \( R \)-algebra \( B \), let \( D_B(_,) \) denote \( \text{Hom}_R(_,B) \). Let \( v_B \) denote \( D_B(_,^*) = D_B \circ D_R(_,) \), and let \( \lambda_B \) denote the Hom-Tensor adjoint isomorphism \( \lambda_B : D_B(M^* \otimes_R _,-) \to \text{Hom}_R(_,B_M) \). We also let \( \mu_M \) denote the natural transformation \( \mu_M : M^* \otimes_R _,- \to \text{Hom}_R(_,M) \) given by \( f \otimes x \mapsto (m \mapsto f(m) x) \). For future reference, we note that for a given \( R \)-module \( X \), the map \( \lambda_B \circ (D_B \mu_M) : D_B \text{Hom}_R(M,X) \to \text{Hom}_R(X,v_B M) \) is given by the rule

\[
\lambda_B \circ (D_B \mu_M)(\sigma)(x)(f) = \sigma(\mu_M(f \otimes x)), \text{ for all } \sigma \in D_B \text{Hom}_R(M,X), x \in X, f \in M^*.
\]

Let \( E = Q/R \). The exact sequence \( 0 \longrightarrow R \xrightarrow{i} Q \xrightarrow{q} E \longrightarrow 0 \) induces the exact commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & D_R \text{Hom}_R(M,_) \\
\downarrow \lambda_B \circ D_B \mu_M & & \downarrow \lambda_B \circ D_Q \mu_M \\
0 & \longrightarrow & \text{Hom}_R(_,v_R M) \\
\end{array}
\]

We now show that \( D_Q \mu_M \) is an isomorphism on the category of finitely generated \( R \)-modules, so that the second map in the composable pair

\[
D_R \text{Hom}_R(M,_) \xrightarrow{\lambda_B \circ D_R \mu_M} \text{Hom}_R(_,v_R M) \xrightarrow{q_*(\lambda_B \circ D_Q \mu_M)^{-1} i_*} D_E \text{Hom}_R(M,_)\]

is well-defined.

Lemma 3.2. [1] Appendix]

1. The map \( D_Q \mu_M \) is an isomorphism on finitely generated \( R \)-modules, and the sequence is exact.

2. If \( R \) is Gorenstein of dimension one, and both \( M \) and the input module \( X \) lie in \( \text{CM}(R) \), then the image of \( \lambda_R \circ D_R \mu_M \) consists of the stably zero maps \( X \longrightarrow v_R M \).

Proof. We can identify \( \mu_M \otimes_R Q \) with \( \mu_{M \otimes Q} : \text{Hom}_Q(M \otimes_R Q,Q) \otimes_Q (_,\otimes_R Q) \to \text{Hom}_Q(M \otimes_R Q,Q) \), which is an isomorphism because \( M \otimes_R Q \) is a projective \( Q \)-module. Thus \( D_Q \mu_M \)
is an isomorphism, since it can be viewed as $D_Q(\mu_M \otimes_R Q)$. The exactness of (3.3) is seen by chasing the diagram (3.2).

Now we assume the hypotheses of (2). Let $0 \rightarrow \text{syz}_R(M) \rightarrow F \overset{p}{\rightarrow} M \rightarrow 0$ be a short exact sequence, where $F$ is a free $R$-module. Consider the commutative diagram

\[
\begin{array}{cccccc}
H \otimes_R M, & H \otimes_R M, & H \otimes_R M, & H \otimes_R M, & H \otimes_R M, & H \otimes_R M, \\
\downarrow \lambda & \downarrow \lambda & \downarrow \lambda & \downarrow \lambda & \downarrow \lambda & \downarrow \lambda,
\end{array}
\]

where the vertical maps are induced by $p: F \rightarrow M$, and the horizontal maps on the right are the isomorphisms induced by $M \cong M^{**}$ and $F \cong F^{**}$. It is easy to see that the image of the rightmost vertical map consists of the stably zero maps $X \rightarrow M$, and it follows that the third vertical map consists of the stably zero maps $X \rightarrow M^{**}$. Let $H$ denote the map $\text{Hom}_R(M,X) \rightarrow \text{Hom}_R(F,X)$ induced by $p$. Since the top row of diagram (3.4) consists of isomorphisms, establishing surjectivity of the leftmost vertical map, namely $D_RH$, is sufficient for proving (2). Let $N = \text{cok}H$. By left-exactness of $\text{Hom}$, we have a left-exact sequence

\[
0 \rightarrow \text{Hom}_R(M,X) \xrightarrow{H} \text{Hom}_R(F,X) \rightarrow \text{Hom}_R(\text{syz}_R(M),X),
\]

and therefore $N$ embeds into $\text{Hom}_R(\text{syz}_R(M),X)$. Thus $N \in \text{CM}(R)$, so $\text{Ext}_1^R(N,R) = 0$.

Therefore the sequence $0 \rightarrow N^* \rightarrow \text{Hom}_R(F,X)^* \xrightarrow{D_RH} \text{Hom}_R(M,X)^* \rightarrow 0$ is exact, so (2) is proved.

**Lemma 3.3.** Assume $R$ is Gorenstein of dimension one, and let $M \in \text{CM}(R)$ and $h \in \text{End}_R M$. Then $h$ is stably zero if and only if trace($gh \otimes Q$) $\in R$ for all $g \in \text{End}_R M$. (Recall the definition of trace, Definition 2.2)

**Proof.** Let $\eta$ denote the isomorphism $\text{End}_R M \rightarrow \text{Hom}_R(M,M^{**})$ induced by $M \cong M^{**}$, and let $\theta$ denote $(\lambda_Q \circ D_Q\mu_M)^{-1}_{\text{tr}} : \text{Hom}_R(M,M^{**}) \rightarrow \text{Hom}_R(\text{End}_R M, Q)$. It follows from Lemma 3.2 that $h$ is stably zero if and only if $[\theta(\eta h)](g) \in R$ for all $g : M \rightarrow M$. Thus, by the symmetry of trace, it suffices to show that $[\theta(\eta h)](g) = \text{trace}(hg \otimes Q)$. Let $\sigma : \text{End}_R M \rightarrow Q$ denote the map sending $g \in \text{End}_R M$ to trace($hg \otimes Q$). Thus, we wish to show $\theta(\eta h) = \sigma$; equivalently, $\iota_*(\eta h) = (\lambda_Q \circ D_Q\mu_M)(\sigma)$.

Take a finite collection of maps $\{\phi_i : M \otimes_R Q \rightarrow Q\}_i$ and elements $\{e_i\}_i \in M \otimes_R Q$, such that $w = \sum_i \phi_i(w)e_i$ for all $w \in M \otimes_R Q$; thus trace($hg \otimes Q$) $= \sum_i \phi_i((h \otimes Q)(e_i))$, as we mentioned in Definition 2.2. Given $m \in M$, and $f \in M^*$, let $g$ denote the endomorphism $\mu_M(f \otimes m)$. By equation 3.1 $(\lambda_Q \circ D_Q\mu_M)(\sigma)(m)(f) = \sigma(g) = \text{trace}(hg \otimes Q) = \sum_i \phi_i((h \otimes Q)(g \otimes Q)(e_i)))$. Now using firstly that $g \otimes Q$ is given by $w \rightarrow (f \otimes Q)(w)m$, and then that $f \otimes Q$ and the $\phi_i$’s have output in $Q$, we have

\[
(\lambda_Q \circ D_Q\mu_M)(\sigma)(m)(f) = \sum_i \phi_i((h \otimes Q)(f \otimes Q)(e_i)m)) = \sum_i (f \otimes Q)(e_i)\phi_i(h(m))
\]

\[
= (f \otimes Q)(\sum_i \phi_i(h(m))e_i) = f(h(m)) = \iota_*(\eta h)(m)(f).
\]

□
4. Our main result.

Our goal now is to prove Theorem 4.8, which is really a formula for the AR sequence beginning at $M$. For this section, let $R$ be a one-dimensional Gorenstein ring which is either a complete local ring or a connected graded ring. Let $\mathfrak{p}$ be a minimal prime of $R$ and let $R' = R/\mathfrak{p}$. Set $Q = Q(R)$, $Q' = Q(R')$, and $m = m_R$.

Recall that for a commutative ring $A$, if $M$ and $N$ are finitely generated $A$-submodules of $Q(A)$, and $M$ contains a faithful element $w$, i.e. $\text{Ann}_A(w) = 0$, then $\text{Hom}_A(M,N)$ is naturally identified with $(N : qQ(A))M = \{ q \in Q(A) | qM \subseteq N \}$. Essentially Theorem 4.8 will assert that the AR sequence beginning at any $M$ supported at $\mathfrak{p}$ is induced by an element $\gamma \in \text{End}_RF_m$ which may be found by the following recipe: Pick $\gamma' \in (R' : qQ \mathcal{J}(\overline{R'})) \setminus R'$, lift $\gamma'$ from $Q'$ to $\tilde{\gamma}' \in Q$, and find $z \in R$ such that $\mathfrak{p} = \text{Ann}_R(z)$ and $z\gamma' \notin R$; finally, set $\gamma = z\tilde{\gamma}'$. We first show that these steps for finding $\gamma$ are well-defined.

**Notation 4.1.** Let $\mathfrak{J}(R')$ denote $(R' : qQ \mathcal{J}(\overline{R'}))$.

**Lemma 4.2.** We have $\mathfrak{J}(R') \subseteq \text{Hom}_R(m_{R'}, R')$, while $\mathfrak{J}(R') \not\subseteq R'$.

**Proof.** Let $D = R'$. As $m_D \subseteq \mathcal{J}(\overline{D})$, we have $\mathfrak{J}(D) \subseteq \text{Hom}_D(m_D, D)$. In the complete local case $\overline{D}$ is a DVR, while in the connected graded case $\overline{D}$ is a polynomial ring over a field by Lemma 9.5. Let $\pi$ denote a generator for $m_D$, and let $n$ be the positive integer such that the conductor ideal $(D : \pi \overline{D})$ equals $\pi^n\overline{D}$. It is clear that $\pi^{n-1}\overline{D} \subseteq \mathfrak{J}(D)$ and $\pi^{n-1}\overline{D} \not\subseteq D$. \hfill $\square$ \hfill $\square$

**Lemma 4.3.** We have an $R$-algebra isomorphism $Q/\mathfrak{p}Q \cong Q'$.

**Proof.** If $x \in R$ is a (homogeneous) nonzerodivisor, we have $R_x = Q$. To see this, it suffices to check that a given (homogeneous) nonzerodivisor $y \in R$ becomes a unit in $R_x$. As $Ry$ is $m$-primary, we have $x^l = ry$ for some $i \geq 1$ and some $r \in R$. Therefore $y$ is a unit in $R_x$; hence $R_x = Q$. Now $Q/\mathfrak{p}Q = Q \otimes_R R' = R_x \otimes_R R' = R'_x$, which equals $Q'$ by the same argument as above. \hfill $\square$ \hfill $\square$

**Lemma 4.4.** Let $\gamma'$ be a (homogeneous) element of $\mathfrak{J}(R') \setminus R'$ (which exists by Lemma 4.2) and let $\tilde{\gamma}'$ be a lift of $\gamma'$ to $Q$ (see Lemma 4.3). Then there exists (homogeneous) $z \in R$ such that $\mathfrak{p} = \text{Ann}_R(z)$ and $z\gamma' \notin R$.

**Proof.** Let $\omega$ denote the ideal $\text{Ann}_R(\mathfrak{p})$. Now $\omega \subseteq \text{Hom}_R(R', R)$ is, up to a graded shift, a canonical module for $R'$ (cf. [6, Theorem 3.3.7] and [6, proposition 3.6.12]), and therefore we have $\text{End}_R\omega \cong R'$ (cf. [6, Theorem 3.3.4] and the proof of [6, proposition 3.6.9b]). We will also use that $\mathfrak{p} = \text{Ann}_R(\omega) = \text{Ann}_R(z)$ for each nonzero $z \in \omega$, which is true because all associated primes of $R$ are minimal, so that any ideal strictly larger than $\mathfrak{p}$ contains a nonzerodivisor.

Regarding $\omega$ as a subset of $Q$ via $\omega \subseteq R \subseteq Q$, suppose that $\gamma'\omega \subseteq \omega$. Then the action of $\gamma'$ on $\omega$ agrees with the multiplication on $\omega$ by some $r \in R$, so $\gamma' - r \in \text{Ann}_Q(\omega) = \mathfrak{p}Q$. But then $\gamma' \in R'$ is a contradiction. So there must exist $z \in \omega$ such that $\gamma'z \notin \omega$. As $\text{Ann}_Q(\mathfrak{p}) \cap R = \omega$, we thus have $z\gamma' \notin R$. \hfill $\square$ \hfill $\square$

**Lemma 4.5.** For $\tilde{\gamma}' \in Q$ and $z \in R$ as in Lemma 4.4 we have $z\tilde{\gamma}' \in \text{End}_R m$.

**Proof.** From Lemma 4.2 we have $\gamma'mR' \subseteq R'$. Since $zR \supseteq R'$, it follows that $\tilde{\gamma}'zM \subseteq zR$, thus $z\tilde{\gamma}' \in m^*$. It remains to observe that $m$ has no free direct summand. But any proper direct summand of an ideal has nonzero annihilator; and if $m$ were free $R$ would be regular. \hfill $\square$ \hfill $\square$
We have now shown that the steps for for finding $\gamma$, given in the introduction of this section, are well-defined. We need one more preliminary lemma.

**Lemma 4.6.** Let $M \in L_p(R)$, and $h \in \text{End}_R M$. Then $\text{trace}(h \otimes Q') = \text{trace}(h \otimes Q) + pQ$.

**Proof.** Take $\{\phi_i : M \otimes_R Q \rightarrow Q\}_i$ and $\{\epsilon_i : M \otimes_R Q\}_i$ such that $w = \sum_i \phi_i(w)e_i$ for all $w \in M \otimes_R Q$. If $\phi'_i = \phi \otimes_R R' : M \otimes_R Q' \rightarrow Q'$ and $\epsilon'_i$ denotes the image of $\epsilon_i$ in $M \otimes_R Q'$, then $w' = \sum_i \phi'_i(w')e'_i$ for all $w' \in M \otimes_R Q'$. Now $\text{trace}(h \otimes Q') = \sum_i \phi'((h \otimes Q)(e'_i)) = \sum_i \phi(h \otimes Q(e_i)) + pQ = \text{trace}(h \otimes Q) + pQ$. □ □

**Notation 4.7.** Let $\gamma'$ be a (homogeneous) element of $\mathfrak{g}(R') \setminus R'$, let $\tilde{\gamma}'$ be a lift of $\gamma'$ to $Q$ and let $z \in R$ (homogeneous) such that $p = \text{Ann}_R(z)$ and $\tilde{\gamma}'z \notin R$. (Those steps are well-defined by the above lemmas.) Let $\gamma = z\tilde{\gamma}'$. Assume $M \in L_p(R)$ has no free direct summands. Then there exists no surjection $M \rightarrow R$, so $M^* = \text{Hom}_R(M, m)$, hence $M \cong \text{Hom}_R(M, m)^*$ is a module over the ring $\text{End}_R m$. Therefore $\gamma$ induces an endomorphism of $M$, by Lemma 4.5. Denote it by $\gamma_M$. Denote by $[\gamma_M]$ the class of $\gamma_M$ in $\text{End}_R M$.

**Theorem 4.8.** Assume $M \in L_p(R)$ is a nonfree indecomposable. Then, using Notation 4.7 we have $[\gamma_M] \in \text{soc}(\text{End}_R M)$. Moreover, $[\gamma_M] \neq 0$ provided $\dim_{R_p}(M \otimes_R R'_1)$ is a unit in $R$. Thus in this case $\gamma_M$ induces the AR sequence beginning at $M$ (see Remark 2.16).

**Proof.** First we show $[\gamma_M] \in \text{soc}(\text{End}_R M)$, which by Lemma 3.3 is equivalent to having $\text{trace}(\gamma h \otimes Q) \in R$ for an arbitrary nonisomorphism $h : M \rightarrow M$. As $\text{End}_R M/(\text{mEnd}_R M)$ is an artinian local ring, there exists some $i \geq 1$ such that $h^i \in \text{mEnd}_R M$, and thus $h^i \otimes_R R' \in \text{mEnd}_R(M')$. So $\text{trace}(h \otimes Q') \in \mathfrak{j}(R')$, by Lemma 2.5. Now using Lemma 4.6, $\tilde{\gamma}' \text{trace}(h \otimes Q) + pQ \in \gamma' \mathfrak{j}(R') \subset R + pQ$, whence $\gamma \text{trace}(h \otimes Q) \in zR + zpQ = zR \subset R$.

Now suppose $n := \dim_{R'_p}(M \otimes_R R'_1)$ is a unit in $R$. We have $\text{trace}(\gamma_M \otimes Q) = \gamma \text{trace}(1_{M \otimes Q}) \in \gamma(n + pQ)$ by Lemma 4.6, while $\gamma(n + pQ) = n\gamma$, since $\gamma p = \gamma' zp = 0$. Since $\gamma \notin R$ by definition, we have $n\gamma \notin R$, and thus $[\gamma_M] \neq 0$ by Lemma 3.3. □ □

The following lemma, which we use in Section 6, is for locating $\gamma'$ when $R'$ is Gorenstein.

**Lemma 4.9.** Assume $R'$ is Gorenstein and connected graded, and let $g \in \text{Hom}_R(m_{R'}, R')$ be a homogeneous element, and $a = \text{deg} g$. If $R'_0 = 0$, then $g \in \mathfrak{g}(R') \setminus R'$.

**Proof.** Let $D = R'$. Since $D_a = 0$, $g$ is not an element of $D$, and it remains to show that $g \in \mathfrak{g}(D)$. As $D$ is Gorenstein, $\dim_k \text{Ext}_D^1(k, D) = 1$, where $k = D_0$. Then applying $(\cdot)^* = \text{Hom}_D((\cdot), D)$ to the short exact sequence $0 \rightarrow m_D \rightarrow D \rightarrow k \rightarrow 0$ yields $\dim_k (m_D^*(D)/D) = 1$. By Lemma 4.2, there exists some homogeneous $\gamma' \in \mathfrak{g}(D) \setminus D$. Then $\gamma' \in m_D^* \setminus D$, so $m_D^* = D + k \gamma'$. Therefore $g = d + d'\gamma'$ for some $d \in D_a$ and $d' \in D$. But $d = 0$ since $D_a = 0$. Thus $g \in d' \mathfrak{g}(D) \subset \mathfrak{g}(D)$. □ □

5. **Stable AR Quivers.**

In this section we provide the background for those unfamiliar with stable translation quivers and their tree classes. Confer, e.g., [1] and [5], although the meaning of “valued” is different in [5].

**Definition 5.1.** A quiver is a directed graph $\Gamma = (\Gamma_0, \Gamma_1)$, where $\Gamma_0$ is the set of vertices and $\Gamma_1$ is the set of arrows. A morphism of quivers $\phi : \Gamma \rightarrow \Gamma'$ is a pair $(\phi_0 : \Gamma_0 \rightarrow \Gamma'_0, \phi_1 : \Gamma_1 \rightarrow \Gamma'_1)$ such that $\phi_1$ applied to an arrow $x \rightarrow y$ is an arrow $\phi(x) \rightarrow \phi(y)$. For $x \in \Gamma_0$, $x^-$ denotes the
set \( \{ y \in \Gamma_0 \mid \exists \text{arrow } y \rightarrow x \text{ in } \Gamma_1 \} \); and \( x^+ = \{ y \in \Gamma_0 \mid \exists \text{arrow } x \rightarrow y \text{ in } \Gamma_1 \} \). \( \Gamma \) is \textit{locally finite} if \( x^+ \) and \( x^- \) are finite sets for each \( x \in \Gamma_0 \). A \textit{loop} is an arrow from a vertex to itself. A \textit{multiple arrow} is a set of at least two arrows from a given vertex to another given vertex.

A \textit{valued quiver} is a quiver \( \Gamma \) together with a map \( v : \Gamma_1 \rightarrow \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1} \). By a \textit{graph} we mean an undirected graph. A \textit{valued graph} is a graph \( G \) together with specified integers \( d_{xy} \geq 1 \) and \( d_{yx} \geq 1 \) for each edge \( x \rightarrow y \).

**Definition 5.2.** A \textit{stable translation quiver} is a locally finite quiver together with a quiver automorphism \( \tau \) called the \textit{translation}, such that:

- \( \Gamma \) has no loops and no multiple arrows.
- \( \text{For } x \in \Gamma_0, \ x^- = \tau(x)^+ \).

Given a stable translation quiver \((\Gamma, \tau)\) and a map \( v : \Gamma_1 \rightarrow \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \), the triple \((\Gamma, v, \tau)\) is called a \textit{valued stable translation quiver} if \( v(x \rightarrow y) = (a, b) \Leftrightarrow v(\tau(y) \rightarrow x) = (b, a) \). A stable translation quiver is \textit{connected} if it is non-empty and cannot be written as disjoint union of two subquivers each stable under the translation.

**Definition 5.3.** Let \( C \) be a full subquiver of a quiver \( \Gamma \) which satisfies Definition 5.2 except possibly for the no-loop condition. We call \( C \) a \textit{component} of \( \Gamma \) if:

1. For all vertices \( x \in C \), we have \( \tau x \in C \) and \( \tau^{-1} x \in C \).
2. \( C \) is a union of connected components of the underlying undirected graph of \( \Gamma \).
3. There is no proper subquiver of \( C \) that satisfies (1) and (2).

**Definition 5.4.** By a \textit{directed tree} we shall mean a quiver \( T \), with no loops or multiple arrows, such that the underlying undirected graph of \( T \) is a tree, and for each \( x \in T \), the set \( x^- \) has at most one element.

Given a directed tree \( T \), there is an associated stable translation quiver \( \mathbb{Z}T \) defined as follows. The vertices of \( \mathbb{Z}T \) are the pairs \((n, x)\) with \( n \in \mathbb{Z} \) and \( x \) a vertex of \( T \). The arrows of \( \mathbb{Z}T \) are determined by the following rules: Given vertices \( x, y \in T \), and \( n \in \mathbb{Z} \),

- \( (n, x) \rightarrow (n, y) \in \mathbb{Z}T \Leftrightarrow x \rightarrow y \in T \Leftrightarrow (n, y) \rightarrow (n-1, x) \in \mathbb{Z}T \);
- If \( n' \notin [n, n-1] \), there is no arrow \((n, x) \rightarrow (n', y)\).

**Remark 5.5.** Let \( T \) be a valued quiver which is also a directed tree. Then there is a unique extension of \( v \) to \( \mathbb{Z}T \) such that the latter becomes a valued stable translation quiver. Namely, if \( v(x \rightarrow y) = (a, b) \), then \( v((n, x) \rightarrow (n, y)) = (a, b) \), and \( v((n, y) \rightarrow (n-1, x)) = (b, a) \).

**Lemma 5.6.** Let \( T \) and \( T' \) be (valued) directed trees. Then \( \mathbb{Z}T \cong \mathbb{Z}T' \) as (valued) stable translation quivers if and only \( T \cong T' \) as (valued) graphs.

**Proof.** See [5, proposition 4.15.3].

A group \( \Pi \) of automorphisms (commuting with the translation) of a stable translation quiver \( \Gamma \) is said to be \textit{admissible} if no orbit of \( \Pi \) on the vertices of \( \Gamma \) intersects a set of the form \( \{ x \} \cup x^+ \) or \( \{ x \} \cup x^- \) in more than one point. The quotient quiver \( \Gamma / \Pi \), with vertices the \( \Pi \)-orbits of \( \Gamma_0 \), and with the induced arrows and translation, is also a stable translation quiver. A surjective morphism of stable translation quivers \( \phi : \Gamma \rightarrow \Gamma' \) is called a \textit{covering} if, for each \( x \in \Gamma_0 \), the induced maps \( x^- \rightarrow \phi(x)^- \) and \( x^+ \rightarrow \phi(x)^+ \) are bijective. Note that if \( \Pi \) is an admissible group of automorphisms of \( \Gamma \),

\[
\text{(5.1) the canonical projection } \Gamma \rightarrow \Gamma / \Pi \text{ is a covering.}
\]
Theorem 5.7. (Riedtmann Structure Theorem; see [5, Theorem 4.15.6]) Given a connected stable translation quiver $\Gamma$, there is a directed tree $T$ and an admissible group of automorphisms $\Pi \subseteq \text{Aut}(\mathbb{Z}T)$ such that $\Gamma \cong \mathbb{Z}T/\Pi$. In particular, we have a covering $\mathbb{Z}T \rightarrow \Gamma$. The underlying undirected graph of $T$ is uniquely determined by $\Gamma$, up to isomorphism.

The underlying undirected graph of $T$ is called the tree class of $\Gamma$.

Remark 5.8. Formally, the tree class $T$ of $\Gamma$ is constructed as follows (as in the proof of Theorem 5.7, which we will not reproduce here). Choose any vertex $x \in \Gamma$, and define the vertices of $T$ to be the set of paths

$$(x = y_0 \rightarrow y_1 \rightarrow \cdots \rightarrow y_n) \quad (n \geq 0)$$

for which no $y_i = \tau(y_{i+2})$. The arrows of $T$ are

$$(x = y_0 \rightarrow y_1 \rightarrow \cdots \rightarrow y_{n-1}) \rightarrow (x = y_0 \rightarrow y_1 \rightarrow \cdots \rightarrow y_n).$$

Remark 5.9. Suppose $\Gamma$ is a valued stable translation quiver, and let $\phi : \mathbb{Z}T \rightarrow \Gamma$ be a covering, which exists by the Theorem. Now $\mathbb{Z}T$ becomes a valued stable translation quiver, by setting $v(x \rightarrow y) = v(\phi(x \rightarrow y))$. In particular, $T$ becomes a valued quiver.

Definition 5.10. The valued tree class of a stable translation quiver $\Gamma$ is a valued graph $(T, v)$ where $T$ denotes the tree class of $\Gamma$, and $v$ : (edges of $T$) $\rightarrow \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is given as in Remark 5.9.

Definition 5.11. Let $(\Gamma, v)$ be a valued, locally finite quiver without multiple arrows. For $x \rightarrow y$ in $\Gamma$, we write $v(x \rightarrow y) = (d_{xy}, d_{yx})$. If there is no arrow between $x$ and $y$, we set $d_{xy} = d_{yx} = 0$. Let $Q_{>0}$ be the set of positive rational numbers.

(i) A subadditive function on $(\Gamma, v)$ is a $Q_{>0}$-valued function $f$ on the set of vertices of $\Gamma$ such that $2f(x) \geq \sum_{y \in \Gamma} d_{xy}f(y)$, for each vertex $x$.

(ii) An additive function on $(\Gamma, v)$ is a $Q_{>0}$-valued function $f$ on the set of vertices of $\Gamma$ such that $2f(x) = \sum_{y \in \Gamma} d_{xy}f(y)$, for each vertex $x$.

Lemma 5.12. [5, Theorem 4.5.8] Let $(\Gamma, v)$ be a connected valued quiver without loops or multiple arrows. Suppose $f$ is a subadditive function on $\Gamma$, and assume $\Gamma$ has infinitely many vertices. Then:

1. The underlying valued graph of $\Gamma$ is an infinite Dynkin diagram.
2. If $f$ is unbounded, or if $f$ is not additive, then the underlying valued graph of $\Gamma$ is $A_{\infty}$.

6. THE COHEN-MACAULAY SETTING

The over-arching ideas of this section are largely from [1], which in turn is based partly on [9]. Since we confine our arguments to the commutative setting, they are sometimes slightly easier. Also, the proof of [1, Lemma 1.23] is flawed, since its penultimate sentence is false, and we give a correct version as Lemma 5.14. A principal goal of this section is to provide some sufficient conditions for guaranteeing that a given component of the stable AR quiver of a hypersurface is a tube; see Proposition 5.23.

Assume $R$ is a Cohen-Macaulay complete local ring, with maximal ideal $m = m_R$. (But the same results hold when $R$ is connected graded instead of complete local.)

Definition 6.1. If $M$ and $N$ are indecomposables in $\text{CM}(R)$, let $\text{Irr}(M, N)$ denote the module of nonisomorphisms $M \rightarrow N$ modulo those which are not irreducible. Let $k_M$ denote the division ring $(\text{End}_R M)/\mathcal{J}(\text{End}_R M)$. Thus $\text{Irr}(M, N)$ is a right $k_M$-space, and a left $k_N$-space.
Definition 6.2. The Auslander-Reiten quiver of $R$ is the valued quiver defined as follows:
- Vertices are isoclasses of indecomposables in $\text{CM}(R)$.
- There is an arrow $M \rightarrow N$ if and only if there exists an irreducible morphism $M \rightarrow N$, i.e. $\text{Irr}(M, N) \neq 0$. The value $\nu(M \rightarrow N)$ of the arrow $M \rightarrow N$ is $(a, b)$ where $a$ is the dimension of $\text{Irr}(M, N)$ as a right $k_M$-space, and $b$ is the dimension of $\text{Irr}(M, N)$ as a left $k_N$-space.

Recall that we use $\tau$ to denote the AR-translate (defined at the end of Definition 2.7).

Lemma 6.3. Let $M$ and $N$ be indecomposables in $L_p(R)$.

1. If $0 \rightarrow \tau N \rightarrow E \rightarrow N \rightarrow 0$ is an AR sequence, the number of copies of $M$ appearing in a direct sum decomposition of $E$ equals the dimension of $\text{Irr}(M, N)$ as a right $k_M$-space.
2. If $0 \rightarrow M \rightarrow E' \rightarrow \tau^{-1}M \rightarrow 0$ is an AR sequence, then the number of copies of $N$ appearing in a direct sum decomposition of $E'$ equals the dimension of $\text{Irr}(M, N)$ as a left $k_N$-space.

Proof. See [17, Lemmas 5.5 and 5.6].

Remark 6.4. Suppose that $k = R/m$ is algebraically closed. Then in the notation of Lemma 6.3 we have $k = k_M = k_N$, and it therefore follows from Lemma 6.3 that the number of copies of $N$ appearing in a decomposition of $E'$ equals the number of copies of $M$ appearing in a decomposition of $E$.

Notationally, we allow $\tau$ to be a partially-defined morphism on the AR quiver of $R$; $\tau x$ is defined precisely when the vertex $x$ corresponds to a non-projective module in $L_p(R)$, by [17, Theorem 3.4]. The following fact is used in [1], and the proof essentially follows that of [4, VII 1.4].

Lemma 6.5. Let $x \rightarrow y$ be an arrow in the AR quiver of $R$, and let $(a, b) = \nu(x \rightarrow y)$. If $\nu y$ is defined, then $\nu(\tau y \rightarrow x) = (b, a)$. If $\nu x$ and $\nu y$ are both defined, then $\nu(\tau x \rightarrow \tau y) = \nu(x \rightarrow y)$.

Proof. We need not prove the last sentence, as it follows from the previous. Let $M$ and $N \in \text{CM}(R)$ be the modules corresponding to $x$ and $y$ respectively. We first show $k_N$ and $k_{\tau N}$ are isomorphic $k$-algebras, where $k = R/m$. Let $0 \rightarrow \tau N \overset{p}{\rightarrow} E \overset{q}{\rightarrow} N \rightarrow 0$ be an AR sequence. Given $h \in \text{End}_R N$, there exists a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \tau N & \overset{p}{\rightarrow} & E & \overset{q}{\rightarrow} & N & \rightarrow & 0 \\
& & \uparrow h' & & \uparrow h & & \downarrow h & & \\
0 & \rightarrow & \tau N & \overset{p}{\rightarrow} & E & \overset{q}{\rightarrow} & N & \rightarrow & 0
\end{array}
\]

Indeed, note that $hq$ is not a split epimorphism, because if $h$ is surjective, then $h$ is an isomorphism, and thus $hq$ is not a split epimorphism because $q$ is not. Therefore, by Definition 2.7 there exists $u : E \rightarrow E$ such that $hq = qu$, and the existence of $h'$ follows.

By the dual argument, any given $h' \in \text{End}_R(\tau N)$ can be fit into a similar commutative diagram.

We wish to show that $h \mapsto h'$ induces a well-defined map $k_N \rightarrow k_{\tau N}$. If so then it is a surjective ring map from a division ring, hence an isomorphism, so we will be done. It suffices to show that, given any commutative diagram (6.1) such that $h$ is a nonisomorphism, it follows that $h'$ is also a nonisomorphism. Suppose, to the contrary, that $h$ is an nonisomorphism.
and \( h' \) is an isomorphism. We may assume \( h' \) is the identity map, since we could certainly compose the diagram \((6.1)\) with a similar diagram which has \((h')^{-1}\) on the left. As \( h \) is not a split epimorphism, it factors through \( q \). But then the top sequence in \((6.1)\) splits, cf. [13] Ch. III, Lemma 3.3; and this of course is a contradiction. Thus \( k_N \cong k_{\tau N} \) as \( k \)-algebras.

In particular, \( \dim_k(k_N) = \dim_k(k_{\tau N}) \). As \( \dim_k \text{Irr}(M,N) = \dim_{k_N} \text{Irr}(\tau N,M) \) is an immediate consequence of Lemma 6.3, our aim is to show \( \dim_{k_N} \text{Irr}(M,N) = \dim_{k_{\tau N}} \text{Irr}(\tau N,M) \). By the former, we have \( \dim_k \text{Irr}(M,N) = \dim_k \text{Irr}(\tau M,N) \). Thus, \( \dim_{k_N} \text{Irr}(M,N) = \dim_k \text{Irr}(M,N)/\dim_{k_N} = \dim_k \text{Irr}(\tau N,M)/\dim_{k_{\tau N}} = \dim_{k_{\tau N}} \text{Irr}(\tau N,M) \).

**Definition 6.6.** If \( R \) is Gorenstein, the *stable Auslander-Reiten (AR) quiver* of \( R \) is the valued quiver defined as in Definition [6.2] except that the vertices are only the isoclasses of nonfree indecomposable modules \( M \in L_p(R) \). By a *stable AR component*, we shall mean a component (Definition 5.3) of the stable AR quiver.

**Definition 6.7.** Let \( (\Gamma, \tau) \) be a translation quiver, and \( x \) a vertex of \( \Gamma \). If \( x = \tau^n x \) for some \( n > 0 \), we say that \( x \) is \( \tau \)-periodic. A module \( M \in \text{CM}(R) \) is said to be \( \tau \)-periodic if it corresponds to a \( \tau \)-periodic vertex in the AR quiver of \( R \), i.e., \( M \cong \tau^n M \). When \( R \) is Gorenstein of dimension one, we will omit the prefix “\( \tau \)-” and just say \( M \) is periodic.

The following is well-known.

**Lemma 6.8.** If \( (\Gamma, \tau) \) is a connected translation quiver containing a \( \tau \)-periodic vertex, then all of its vertices are \( \tau \)-periodic.

**Proof.** If \( x \) is a vertex in \( \Gamma \) and \( \tau^n x = x \), then \( \tau^n \) induces a permutation on the finite set \( x^- \), and so some power of \( \tau^n \) stabilizes \( x^- \) pointwise. Thus each vertex in \( x^- \) is \( \tau \)-periodic; likewise for \( x^+ \), so every vertex in \( \Gamma \) is \( \tau \)-periodic by induction.

**Definition 6.9.** We say that a connected translation quiver is periodic if one, equivalently all, of its vertices is \( \tau \)-periodic.

**Definition 6.10.** A valued stable translation quiver \( \Gamma \) is called a tube if \( \Gamma \cong ZA_{\infty}/\langle \tau^n \rangle \) for some \( n \geq 1 \). If \( n = 1 \), \( \Gamma \) is called a homogeneous tube.

**Remark 6.11.** Let \( \Gamma \) be a connected periodic stable translation quiver, and suppose the valued tree class of \( \Gamma \) is \( A_{\infty} \). Then \( \Gamma \) is a tube. To see this, let \( \Pi \) be an admissible group of automorphisms of \( ZA_{\infty} \) such that \( \Gamma \cong ZA_{\infty}/\Pi \). Note that every automorphism of the stable translation quiver \( ZA_{\infty} \) is of the form \( \tau^n \) for some \( n \geq 0 \). Thus \( \Pi = \langle \tau^n \rangle \) for some \( n \geq 0 \); and the periodicity implies \( n \geq 1 \).

**Notation 6.12.** If \( R \) is Gorenstein of dimension one, and \( M \) is an indecomposable in \( L_p(R) \), define an \( R \)-module \( \text{push}(M) \) as follows. If \( M \) is free, let \( \text{push}(M) = 0 \). Otherwise let \( \text{push}(M) \) denote the unique module (up to isomorphism) such that there exists an AR sequence \( 0 \rightarrow M \rightarrow \text{push}(M) \rightarrow \text{syz}_{R}^{-1}(M) \rightarrow 0 \). More generally, if \( M = \bigoplus_{i=1}^{n} M_i \) with each \( M_i \) in \( L_p(R) \), then we set \( \text{push}(M) = \bigoplus_{i=1}^{n} \text{push}(M_i) \).

**Notation 6.13.** (See, e.g., [15] 14.1-14.6.) For an \( R \)-module \( M \), let \( e(M) \) denote the multiplicity of \( M \). This can be defined as \( e(M) = \lim_{n \rightarrow \infty} \frac{d}{n^d} \text{length}(M/m^n M) \), where \( d = \dim R \), but the reader may ignore this definition; we use only the following properties:

- If \( 0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0 \) is exact, then \( e(M) = e(M') + e(M'') \).
- For all \( M \in \text{CM}(R) \), \( e(M) \) is a positive integer.
**Notation 6.14.** Define a function $e_{\text{avg}}$ from $\tau$-periodic maximal Cohen-Macaulay $R$-modules to $\mathbb{Q}_{>0}$ as follows: If $M$ is $\tau$-periodic of period $n$, let $e_{\text{avg}}(M) = \frac{1}{n} \sum_{i=0}^{n-1} e(\tau^i(M))$.

**Lemma 6.15.** Assume $R$ is Gorenstein of dimension one, and $M \in L_p(R)$ is indecomposable and periodic. If $\text{push} M = X \oplus F$ where $X$ has no free direct summands and $F$ is a (possibly zero) free module, then $X$ is periodic, and $e_{\text{avg}}(\text{push} M) \leq 2 e_{\text{avg}}(M)$.

**Proof.** We know $X$ is periodic from Lemma 6.8. Note that if $N \in \text{CM}(R)$ is periodic, then for any $j \in \mathbb{Z}$, and $n \in \mathbb{N}$ a multiple of the period of $N$, $\sum_{i=j}^{n+j-1} e(\tau^i N) = ne_{\text{avg}}(N)$. For each integer $i$, we have by Lemma 6.5 an AR sequence $0 \rightarrow \tau^{i+1} M \rightarrow F_i \oplus \tau^i X \rightarrow \tau^i M \rightarrow 0$, where $F_i$ is a (possibly zero) free module. So $e(\tau^i X) \leq e(\tau^{i+1} M) + e(\tau^i M)$, hence $\sum_{i=1}^{n} e(\tau^i X) \leq \sum_{i=1}^{n} e(\tau^{i+1} M) + \sum_{i=1}^{n} e(\tau^i M)$ for each $n \in \mathbb{N}$. This inequality gives the desired result by taking $n$ to be a common multiple of the periods of $M$ and $X$, and dividing both sides by $n$. \qed

The following goes back at least to [9] (in a slightly different setting).

**Lemma 6.16.** Let $C$ be a connected $\tau$-periodic valued stable translation quiver which is a subquiver of the stable AR quiver of $R$. Then the valued tree class of $C$ admits a subadditive function (Definition 5.11).

**Proof.** Let $T$ denote the valued tree class (Definition 5.10) of $C$. By definition of $T$, we have a value-preserving covering $\phi: \mathbb{Z}T \rightarrow C$. Define a function $f: \mathbb{Z}T \rightarrow \mathbb{Q}_{>0}$ by the rule $f(x) = e_{\text{avg}}(\phi(x))$. We claim that the restriction of $f$ to $T$ is a subadditive function. That is, $2f(x) \geq \sum_{y \in T} d_{yx} f(y)$, for each vertex $x$ of $T$. By Lemma 6.5, $d_{yx} = d_{(\tau^{-1} y)x}$ for all $x, y \in C$, hence for all $x, y \in \mathbb{Z}T$. In what follows, for any $x \in T$, the sets $x^{-}$ and $x^{+}$ will always be taken with respect to $\mathbb{Z}T$; to signify the predecessors of $x$ with respect to $T$ we can use $x^{-} \cap T$. If $x \in T$, then $x^{+}$ equals the disjoint union of $x^{+} \cap T$ and $\tau^{-1}(x^{-} \cap T)$. Now, we have

$$
\sum_{y \in T} d_{yx} f(y) = \sum_{y \in x^{-} \cap T} d_{yx} f(y) + \sum_{y \in x^{+} \cap T} d_{yx} f(y) = \sum_{y \in x^{-}} d_{(\tau^{-1} y)x} f(\tau^{-1} y) + \sum_{y \in x^{+}} d_{yx} f(y) = \sum_{y \in x^{+}} d_{yx} f(y).
$$

So subadditivity of $f$ is equivalent to $2f(x) \geq \sum_{y \in x^{+}} d_{yx} f(y)$. Since $\phi$ is a covering, $\sum_{y \in x^{+}} d_{yx} f(y) = \sum_{y \in \phi(x)^{+}} d_{\phi(y)x} e_{\text{avg}}(y)$, which is bounded by $2 e_{\text{avg}}(\phi(x))$ by Lemma 6.15. So $f$ is subadditive. \qed

**Lemma 6.17.** Assume $R$ is Gorenstein, let $M \in L_p(R)$ be a nonfree indecomposable, and suppose there exists an irreducible map from $M$ to itself. Let $C$ denote the component of the stable AR quiver containing $M$, and assume $C$ is infinite. Then $C$ is a homogeneous tube with a loop at the end:

$$
\underbrace{M = X_0 \hspace{1cm} X_1 \hspace{1cm} X_2 \hspace{1cm} X_3 \hspace{1cm} \ldots}
$$

In particular, $\tau X_i \cong X_i$ for all $X_i \in C$.

**Proof.** First we show that $M \cong \tau M$. If not, then the AR sequence ending in $M$ is $0 \rightarrow \tau M \rightarrow M \oplus \tau M \oplus N \rightarrow M \rightarrow 0$ for some $N \in \text{CM}(R)$. Then $e(N) = 0$, hence $N = 0$. Now Miyata’s Theorem [16, Theorem 1] says that the given AR sequence splits, which is a contradiction. So $\tau M \cong M$. 

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Since \( C \) has a loop, it does not satisfy the definition of stable translation quiver (Definition 5.2). But removing the loops in \( C \) (and keeping all vertices and all non-loop arrows), we get a \( \tau \)-periodic connected stable translation quiver; call it \( \Gamma \), and let \( T \) denote valued tree class of \( \Gamma \). Now \( T \) admits a subadditive function given by \( e_{\text{avg}} \), as in the proof of 6.16 From the fact that \( \Gamma \) is not a full subquiver of the AR quiver of \( R \), it follows that \( e_{\text{avg}} \) is strictly subadditive (i.e., not additive). As \( \Gamma \) is infinite and \( \tau \)-periodic, \( T \) must be infinite. Therefore \( T \cong A_\infty \) by Lemma 5.12 and \( \Gamma \cong A_\infty/(\tau) \), by Remark 6.11 So \( \Gamma \) has the form

\[
X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow \ldots
\]

Suppose \( M = X_i \) for some \( i > 0 \). Then we have an AR sequence \( 0 \rightarrow X_i \rightarrow X_i \oplus X_{i-1} \oplus X_{i+1} \oplus F \rightarrow X_i \rightarrow 0 \), for some free module \( F \), so \( e(X_i) \geq e(X_{i-1}) + e(X_{i+1}) \). But the AR sequences ending in \( X_{i-1} \) and \( X_{i+1} \) give us \( 2e(X_{i+1}) > e(X_i) \) and \( 2e(X_{i-1}) \geq e(X_i) \). These inequalities contradict the previous one, so \( M = X_0 \).

The following “Maranda-type result” corresponds to Lemma 1.24 in [1]. In our setting, namely that of a Cohen-Macaulay complete local ring, this result is well-known (but possibly has only been stated for the case when the ring is an isolated singularity). The following proof can be found, for example, in [12, proposition 15.8] and its corollaries.

**Lemma 6.18.** Let \( M \) and \( N \) be nonisomorphic indecomposables in \( L_p(R) \), and let \( x \in m \) be a nonzerodivisor. Then there exists \( i \geq 1 \) such that \( M/x^iM \) and \( N/x^iN \) are nonisomorphic indecomposable modules.

**Proof.** Since \( M \) lies in \( L_p(R) \), \( \text{Ext}_R^1(M,N) \) has finite length (since for any nonmaximal prime \( p \), we have \( 0 = \text{Ext}_R^1(M_p,N_p) = \text{Ext}_R^1(M,N)_p \)). Therefore we may assume, after replacing \( x \) by a suitable power of itself, that \( x \text{Ext}_R^1(M,N) = 0 \). By applying \( \text{Hom}_R(M,-) \) to the commutative exact diagram

\[
\begin{array}{ccc}
0 & \rightarrow & N \\
\downarrow x & \downarrow & \downarrow x^2 N \\
0 & \rightarrow & N/x^2 N
\end{array}
\]

we obtain a commutative exact diagram

\[
\begin{array}{ccc}
\text{Hom}_R(M,N) & \rightarrow & \text{Hom}_R(M,N/x^2 N) \\
\downarrow & & \downarrow x \\
\text{Hom}_R(M,N) & \rightarrow & \text{Ext}_R^1(M,N)
\end{array}
\]

Consider the maps \( \theta : \text{Hom}_R(M,N) \rightarrow \text{Hom}_R(M/x^2 M,N/x^2 N) \) and \( \theta_2 : \text{Hom}_R(M/x^2 M,N/x^2 N) \rightarrow \text{Hom}_R(M/x^2 M,N/x^2 N) \) given by tensoring all maps with \( R/(x) \). Notice that in diagram (6.2), the horizontal and vertical maps into \( \text{Hom}_R(M,N/x^2 M) \) can be identified with \( \theta \) and \( \theta_2 \) respectively, while the rightmost vertical map is zero. Therefore a diagram chase yields

\[
\text{im}(\theta) = \text{im}(\theta_2).
\]

We claim \( i = 2 \) will suffice. Suppose \( M/x^2 M \) is not indecomposable. Then there exists a nontrivial idempotent \( e \in \text{End}_R(M/x^2 M) \). Consider the equation (6.3) in the case \( M = N; \)
now \( \theta \) and \( \theta_2 \) are of course ring homomorphisms. Since \( \text{End}_R M \) is (noncommutative-) local, so is \( \text{im} \theta \), and therefore \( \theta_2(e) \) is either 0 or 1. Since \( 1-e \in \text{End}_R (M/x^2 M) \) is also a nontrivial idempotent, we may assume \( \theta_2(e) = 0 \), i.e. \( \text{im} e \subseteq xM/x^2 M \). But then \( e^2 = 0 \) is a contradiction.

Now suppose \( \varphi: M/x^2 M \to N/x^2 N \) is an isomorphism, with inverse \( \psi: N/x^2 N \to M/x^2 M \). By (6.3), there exist \( \tilde{\varphi}: M \to N \) such that \( \tilde{\varphi} \circ_R (R/x) = \varphi \circ_R (R/x) \), and \( \tilde{\psi}: M \to N \) such that \( \tilde{\psi} \circ_R (R/x) = \psi \circ_R (R/x) \). By Nakayama’s Lemma, \( \tilde{\varphi} \) and \( \tilde{\psi} \) are surjective. Thus \( \tilde{\psi} \tilde{\varphi} \) is a surjective endomorphism, equivalently, an isomorphism; and thus \( \tilde{\varphi} \) is an isomorphism. \( \square \)

**Lemma 6.19.** Assume \( \text{dim} R = 1 \), and let \( M \) be an indecomposable in \( \text{CM}(R) \). Then there exists an irreducible morphism \( M \to R \) if and only if \( M \) is isomorphic to a direct summand of \( m \). If \( R \) is Gorenstein, there exists an irreducible morphism \( R \to M \) if and only if \( M \) is isomorphic to a direct summand of \( m^* \).

**Proof.** Write \( m \) as a direct sum of indecomposables, \( m = \bigoplus_i m_i \). Let \( i_i \) denote the inclusion map \( m_i \to R \). To see that \( i_i \) is irreducible, take a factorization \( i_i = h g \) in \( \text{CM}(R) \), where \( h \) is not a split epimorphism. Then \( h \) is not onto, so \( \text{im} h \subseteq m \). If \( h' \) denotes the map into \( m \) given by \( x \to h(x) \), and \( p_i \) denotes the projection \( m \to m_i \), we have that \( p_i h' g = 1_{m_i} \), so \( g \) is a split monomorphism; hence \( i_i \) is irreducible. Now let \( M \) be an indecomposable in \( \text{CM}(R) \) and let \( f: M \to R \) be an irreducible morphism. Let \( i \) denote the inclusion map \( m \to R \). Since \( f \) is not a split epimorphism, \( \text{im} f \subseteq m \), hence \( f = ig \) for some \( g: M \to m \). As \( i \) is certainly not a split epimorphism, \( g \) is a split monomorphism.

For the last sentence of the statement, note that the irreducible maps from \( R \) are obtained by dualizing the irreducible maps into \( R \). \( \square \)

We recall the Harada-Sai Lemma:

**Lemma 6.20.** If \( \Lambda \) be an artin algebra (e.g. a commutative artinian ring). If \( f_i: M_i \to M_{i+1} \) are nonisomorphisms between indecomposable modules \( M_i \) for \( i = 1, \ldots, 2^n - 1 \) and \( \text{length} (M_i) \leq n \) for all \( i \), then \( f_{2^n-1} \cdots f_1 = 0 \).

**Lemma 6.21.** Assume \( R \) is Gorenstein of dimension one, and \( m \) is indecomposable; and suppose \( R \) has a stable AR component \( C \) which is finite. Then \( C \) consists of all isoclasses of non-projective indecomposables in \( \text{CM}(R) \).

**Proof.** As \( C \) is finite, Lemma 6.18 implies that we can take \( x \in m \) such that for each pair \( M \not\cong N \in C \), \( M/xM \) and \( N/xN \) are nonisomorphic indecomposable modules.

We may assume \( R \) is not regular, and therefore \( m \) is not free. Now first we show \( m \in C \). Suppose not; then there are no irreducible maps to \( R \) from any module in \( C \) (Lemma 6.19). Therefore if \( N \in C \) and \( N \to N' \) is any irreducible map in \( \text{CM}(R) \), \( N' \) must lie in \( C \) (since \( L_p(R) \) is closed under \( \text{syz} \), and therefore under irreducible maps by consideration of AR sequences). Pick a module \( M \in C \). By replacing \( x \) by a power of itself if necessary, we can choose \( f: M \to R \) such that \( f(M) \not\subseteq xR \), i.e. \( f \circ_R (R/x) \neq 0 \). Since \( f \) is not a split monomorphism, and there exists an AR sequence beginning in \( M \), \( f \) equals a sum of maps of the form \( gh \), where \( h \) is an irreducible map between modules in \( C \). Since \( g \in \text{Hom}_R (N, R) \) for some \( N \in C \), \( g \) is not a split monomorphism, and can in turn be written as a sum of maps of the form \( kl \) where \( l \) is an irreducible map in \( C \); now \( f = \sum klh \). Continue this process until we have written \( f \) as a sum \( \sum_i g_i h_{2^{n-1}, i} \cdots h_{1,i} \) where each \( h_{j,i} \) is an irreducible map in \( C \), and \( n = \max(\text{length}(N/xN)/N \in C) \). Note that each \( h_{j,i} \circ_R (R/x) \) is a nonisomorphism by our
assumption on \( x \) together with Lemma 6.17. Therefore, Lemma 6.20 implies \( f \otimes_R (R/x) = 0 \), contradiction. Thus \( m \in C \).

Now just suppose \( C \) omits some indecomposable nonfree \( M \in \text{CM}(R) \). Again choose \( f : M \to R \) such that \( f \otimes_R (R/x) \neq 0 \). Note that any map to \( R \) which is not a split epimorphism factors through \( m \). Whereas in the previous paragraph we reached a contradiction via Lemma 6.20, by “stacking irreducible maps while moving forwards through \( C \)”, we now obtain a contradiction by “stacking irreducible maps while moving backwards through \( C \cup \{ R \} \). □ □

**Remark 6.22.** Assume \( R \) is Gorenstein and let \( C \) be a stable AR component without loops. Then \( C \) is a valued stable translation quiver (by Lemma 6.5) and therefore has a valued tree class \( T \) (Definition 5.10). Then \( T \) carries the information of how many nonfree direct summands \( \text{push}(M) \) and \( \text{push}(\text{push}(M)) \) (in general, \( \text{push}^i(M) \)) have for modules \( M \in C \). Let us explain further. Let \( x \) be the vertex in \( C \) corresponding to \( M \), and let \( n = \sum_{(x,y) \in C} d_{yx} \).

Then \( n \) is the number of nonfree summands in \( \text{push}(M) \); that is, \( \text{push}(M) = F \oplus \bigoplus_{i=1}^n X_i \) where \( F \) is a (possibly zero) free module, and the \( X_i \) are (not necessarily nonisomorphic) nonfree indecomposables in \( L_R(R) \). We have a value-preserving covering \( \phi : ZT \to C \), and after possibly composing \( \phi \) with a power of \( r \), we have \( x \in \phi(T) \), say \( x = \phi(u) \). Since \( \phi : ZT \to C \) is a covering, \( \sum_{(x,y) \in C} d_{yx} = \sum_{(u,w) \in ZT} d_{wu} \), and by definition of \( ZT \) this equals \( \sum_{w \in T} d_{wu} \).

Thus \( n = \sum_{w \in T} d_{wu} \). Likewise, \( \sum_{w,z \in T} d_{zw}d_{wu} \) is the number of nonfree direct summands in \( \text{push}(\text{push}(M)) \).

**Proposition 6.23.** (cf. [11] Lemma 1.23 and Theorem 1.27) Assume \( R \) is Gorenstein of dimension one, \( m \) is indecomposable, and \( \text{CM}(R) \) has infinitely many isoclasses of indecomposables. Let \( C \) be a periodic component of the stable AR quiver of \( R \), and suppose that either \( R \) is a reduced hypersurface and \( C \) has no loops, or that there exists some \( M \in C \) such that \( \text{push}(\text{push}(M)) = X \oplus Y \oplus F \) for some indecomposables \( X \) and \( Y \), and some possibly-zero free module \( F \). Then, \( C \) is a tube.

**Proof.** If \( C \) has a loop, then by Lemma 6.17 for every \( M \in C \), the module \( \text{push}M \) has two nonfree indecomposable summands, and therefore \( \text{push}(\text{push}(M)) \) has four. So we may assume \( C \) has no loops. Thus \( C \) is a valued stable translation quiver, and we have a valued directed tree \( T \) and a value-preserving covering \( \phi : ZT \to C \). Let the function \( f : ZT \to \mathbb{Q}_{>0} \) be given by \( f(x) = e_{\text{avg}}(\phi(x)) \). As seen in Lemma 6.16 \( f \) restricts to a subadditive function on \( T \). Since \( \phi \) is surjective, every vertex of \( C \) lies in the \( r \)-orbit of a vertex in \( \phi(T) \). Note also that \( C \) has infinitely many vertices, by Lemma 6.21. Therefore \( T \) is infinite, so it is an infinite Dynkin diagram by Lemma 5.12. If \( R \) is a reduced then \( \{e(M) | M \in C\} \) is unbounded (see [17] Theorem 6.2)); and so if \( R \) is a reduced hypersurface (and thus all modules in \( C \) have period 2) then \( f \) is unbounded. Then \( T \cong A_\infty \), by Lemma 5.12. If the alternate condition holds, we get \( T \cong A_\infty \) by eliminating the other infinite Dynkin diagrams, in light of Remark 6.22. Thus \( C \) is a tube, by Remark 6.11. □ □

7. AN EXAMPLE.

In this section, we apply the results of the previous sections to determine the shape (namely, a tube) of some components of the Auslander-Reiten quiver of the ring \( \bar{R} \) defined
in [7.4] below. Recall that a hypersurface, i.e. a regular (graded-) local ring modulo a nonzerodivisor, is always Gorenstein.

7.1. Let \( S \) be a regular (graded-) local ring, and \( f \in S \) a nonzero element. Let \( R = S/fS \). A matrix factorization of \( f \) is a pair of matrices \((\varphi, \psi)\), with entries in \( S \), such that \( \varphi \psi = \psi \varphi = f \text{id}_{1 \times 1} \) for some \( l > 0 \). As consequences of the definition, we have \( \text{cok} \varphi \cong \text{cok}(\varphi \otimes_S R) \), and ([17] 7.2.2))

\[
\text{im}(\varphi \otimes_S R) = \ker(\psi \otimes_S R) \quad \text{and} \quad \text{im}(\psi \otimes_S R) = \ker(\varphi \otimes_S R).
\]

In particular, \( \text{cok} \varphi \) and \( \text{cok} \psi \) are periodic \( R \)-modules, of period two.

Remark 7.2. Let \((\varphi, \psi)\) and \((\varphi', \psi')\) be matrix factorizations of \( f \). Let \( n_1 \) and \( n_2 \) be the integers such that \( \varphi \) is \( n_1 \)-by-\( n_1 \) and \( \varphi' \) is \( n_2 \)-by-\( n_2 \). Given \( h : \text{cok} \varphi \to \text{cok} \varphi' \), there of course exist \( \alpha : S^{(n_1)} \to S^{(n_2)} \) and \( \beta : S^{(n_1)} \to S^{(n_2)} \) making the diagram

\[
\begin{array}{ccc}
S^{(n_1)} & \xrightarrow{\varphi} & S^{(n_1)} \\
\beta \downarrow & & \alpha \downarrow \\
S^{(n_2)} & \xrightarrow{\varphi'} & S^{(n_2)}
\end{array}
\]

commute. Now it is easy to see that \( \left( \begin{pmatrix} \varphi' & -\alpha \\ 0 & \psi \end{pmatrix}, \begin{pmatrix} \psi' & \beta \\ 0 & \varphi \end{pmatrix} \right) \) is a matrix factorization of \( f \).

If \((\varphi, \psi)\) is a matrix factorization such that \( \varphi \) and \( \psi \) each contains no unit entry, then it is called a reduced matrix factorization. If \((\varphi, \psi)\) is a reduced matrix factorization, then neither \( \text{im} \varphi \) nor \( \text{im} \psi \) contains a free summand (cf. [17] 7.5.1)).

7.3. Let \((\varphi, \psi)\) be a reduced matrix factorization, let \( M = \text{cok} \varphi \), and assume that \( R, M \) and \( \gamma \) satisfy the hypotheses of Theorem [4.8]. Pick \( \alpha \) and \( \beta \) lifting \( \gamma_M \in \text{End}_R M \) in the sense of Remark 7.2. One may check that the valid choices for \( \alpha \) are precisely those choices such that \( \psi \alpha = \gamma \psi \) after passing to \( \hat{R} \). By Remark 2.16 \( \text{push}(M) \cong (\text{im}(\psi \otimes_S R) \oplus R^{(n)})/((-\gamma c, c) \in \text{im}(\psi \otimes_S R)) \), where \( n \) denotes the side length of the matrices \( \varphi \) and \( \psi \). Then we see that \( \text{push}(M) = \text{cok} \begin{pmatrix} \varphi & -\alpha \\ 0 & \psi \end{pmatrix} \).

7.4. Let \( k \) be a field, of characteristic not equal to 2, and let us set up a connected graded hypersurface \( R \) as follows. Let \( p \) and \( q \) be relatively prime integers \( \geq 3 \), and let \( S = k[x, y] \) be the graded polynomial ring such that \( S_0 = k, \deg x = q, \) and \( \deg y = p \). Let \( f \in S \) be a homogeneous polynomial which is not divisible by \( x \). Let \( g = (bx^p + y^q)f \), where \( b \in k \), and \( b \) is allowed to be zero. Now, let \( R = k[x, y] \langle g \rangle \). The \( m \)-adic completion of \( R \) is \( \hat{R} = k[[x, y]] \langle g \rangle \). Let \( v = \deg(f)/p \), which is an integer because \( x \nmid f \). We assume that \( f - y^v \in xS \). Lastly, assume that there are infinitely many isoclasses of indecomposables in \( \text{CM}(R) \).

Now fix an ideal of \( R \) of the form \( I = (x^m, y^n) \), where \( 1 \leq m < p - 1 \) and \( 2 \leq n < q \). We will show that the component of \( \Gamma_R \) containing \( I \) is a tube, by showing that \( \text{push}(\text{push}(I)) \) has only two indecomposable summands, and applying Proposition 6.23. However, we will work over \( R \):

Remark 7.5. Let \( C \) be a component of \( \Gamma_R \). Now consider the valued translation quiver \( C' \) obtained from \( C \) by identifying vertices \( x \) and \( y \) when they correspond to modules which are merely graded-shifts of one another, where a “graded-shift” of a module \( M \) means a module
$M(i)$ defined by $M(i)_j = M_{i+j}$. By [3] Theorem 3], $C'$ is naturally identified with a component of $\Gamma_{\mathfrak{R}}$. Therefore we might as well work over $R$, and just try not to keep track of the grading on $\bar{M}$ and the grading on $\text{push}(M)$ simultaneously.

**Notation 7.6.** Let $\gamma = y^{q-1}/f \in Q(R)$. If $b \neq 0$, set $R' = S/(bx^p + y^q)S$; if $b = 0$, set $R' = S/yS \cong k[x]$. In either case, $R'$ is a domain:

**Lemma 7.7.** If $b \neq 0$, then $S/(bx^p + y^q)S$ is a domain.

*Proof.* As $S$ is factorial, it suffices to show $bx^p + y^q$ is irreducible. Since a product $ss'$ fails to be homogeneous when either $s$ or $s'$ does, $bx^p + y^q$ is either irreducible or equal to a product of homogeneous nonunits. Let $s$ and $s'$ be homogeneous elements satisfying $ss' = bx^p + y^q$, and assume $s$ is a nonunit. Then $s$ has a term of the form $ax^i$ for $a \in k \setminus \{0\}$, so that $q \text{deg} s$. Likewise $p \text{deg} s$, and thus $\text{deg} s = \text{deg}(bx^p + y^q)$, hence $\text{deg} s' = 0$, so $s'$ is a unit. \qed \qed

We will use the following piece of arithmetic several times. We omit the easy proof.

**Lemma 7.8.** If $b_1 < q$ and $b_2 < 0$, or if $b_1 < 0$ and $b_2 < p$, then $b_1 p + b_2 q \notin p\mathbb{N} + q\mathbb{N}$.

**Lemma 7.9.** Let $(\varphi, \psi)$ be a reduced matrix factorization of $g$ and such that each indecomposable direct summand of $\text{cok}\varphi$ has rank, and char $k$ does not divide any of these ranks. Let $a$ be a matrix such that $\psi a = \gamma \varphi$ after passing to $R$. Then, $\text{push}(\text{cok}\varphi) = \text{cok} \begin{pmatrix} \varphi & -a \\ 0 & \psi \end{pmatrix}$.

*Proof.* By [7,3] we only need to check that $\gamma$ agrees with Notation 4.7 and the indecomposable summands of $\text{cok}\varphi$ satisfy the hypotheses in Theorem 4.8. If $b \neq 0$ then $R' = S/(bx^p + y^q)S$ and we take $\gamma' = y^{q-1}/x \in \text{Hom}_{R'}(m_{R'}, R')$, and set $z = f$. Let $Q = Q(R)$ and $Q' = Q(R')$. As $\text{deg}(y^{q-1}x) = p(q-1) - q \notin p\mathbb{N} + q\mathbb{N}$ by Lemma 7.8 we have $\gamma' \in (R'/Q' \cdot J(R')) \setminus R'$ by Lemma 4.9. So $\gamma = y^{q-1}f/x$ agrees with Notation 4.7. If $b = 0$, then $R' = S/YS$ and we take $\gamma' = 1/x \in (R'/Q' \cdot J(R')) \setminus R'$ and set $z = y^{q-1}f$. Again $\gamma = y^{q-1}f/x$ agrees with Notation 4.7. It only remains to note that $M \otimes_R Q'$ is a free $Q'$-module of rank equal to that of $M \otimes_R Q$, by Lemma 4.3. \qed

In preparation for what immediately follows, let us observe that $g - y^{q+v} \in x^m S$. Indeed, we have by assumption $f - y^v \in xS$, and $\text{deg} f = \text{deg}(y^v) = pv$. So if $x^i y^j$ is a monomial occurring in $f - y^v$, then we have $i > 0$, and $qi + pj = pv$. Since $\gcd(p, q) = 1$, $i$ is therefore a positive multiple of $p$; in particular, $i > m$. Thus, if $\equiv$ denotes congruence modulo $x^m$, we have $f - y^v \equiv 0$, and $g - y^{q+v} = (bx^p + y^q)f - y^{q+v} \equiv y^q(f - y^v) \equiv 0$.

Let

\begin{equation}
\varphi = \begin{pmatrix} (g - y^{q+v})/x^m & -y^n \\ y^{q+v-n} & x^m \end{pmatrix}, \quad \psi = \begin{pmatrix} x^m & -yq+v-n \\ -y^{q+v-n} & (g - y^{q+v})/x^m \end{pmatrix},
\end{equation}

then $I \equiv \text{cok}\varphi$, and $(\varphi, \psi)$ is a matrix factorization of $g$. Let

\begin{equation}
\alpha = \begin{pmatrix} -bx^{p-m-1}y^{n-1}f & 0 \\ x^{m-1}y^{q-n-1}f & 0 \end{pmatrix},
\end{equation}

and note that $\psi \alpha = \gamma \psi$ after passing to $R$. Therefore if we let $\xi = \begin{pmatrix} \varphi & -a \\ 0 & \psi \end{pmatrix}$, it follows from Lemma 7.9 that $\text{cok}\xi = \text{push}(I)$.

By Remark 7.2 we can pick a matrix $\beta$, with entries in $S$, such that

\begin{equation}
\alpha \varphi = \varphi \beta.
\end{equation}
In fact
\[ \beta = \begin{pmatrix} y^{-1} (f - y^v)/x & -x^{m-1} y^{n-1} \\ y^{q-1} (b x^{p-m-1} y^v f + (f - y^v)(g - y^{q+v+1})/x^{m+1}) & -y^{q-1} (f - y^v)/x \end{pmatrix}. \]

We will never need to refer to the actual entries of \( \beta \), though we will use that \( \beta \) has no unit entries. By equation (7.5), the pair

\[ (\xi, \eta) \]

forms a matrix factorization of \( g \), where \( \xi = \begin{pmatrix} \psi & -\alpha \\ 0 & \psi \end{pmatrix} \), and \( \eta = \begin{pmatrix} \psi & \beta \\ 0 & \psi \end{pmatrix} \).

Furthermore, \( (\xi, \eta) \) is a reduced matrix factorization.

To avoid extreme clutter, we will henceforth abuse notation!

**Caveat 7.10.** Regarding all matrices in this section, we from now on always take the entries as living in \( \mathbb{R} \) rather than \( \mathbb{S} \), unless stated otherwise.

The reader can check directly that \( \alpha \psi = -\gamma \psi \). In other words,

\[ \phi \beta = -\gamma \psi. \]

**Definition 7.11.** We choose a matrix \( W \) such that \( \eta W = \gamma \eta \). Such \( W \) exists by (7.3) Let \( Z \) and \( Z' \) be 2-by-2 matrices such that \( W = \begin{pmatrix} \alpha & Z' \\ 0 & -\beta + \psi Z \end{pmatrix} \).

We explain why \( W \) can be chosen to be of this form. To begin with, let \( W \) be an arbitrary matrix such that \( \eta W = \gamma \eta \), and let \( A, B, C \) and \( D \) be 2-by-2 matrices such that

\[ W = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \]

The equation \( \begin{pmatrix} \psi & \beta \\ 0 & \phi \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \gamma \psi & \gamma \beta \\ 0 & \gamma \phi \end{pmatrix} \) implies \( \phi D = \gamma \phi \), which equals \( -\phi \beta \) (equation (7.7)). Therefore \( \phi (D + \beta) = 0 \), and this implies \( D + \beta = \psi Z \) for some matrix \( Z \). That we may choose \( \begin{pmatrix} A \\ C \end{pmatrix} \) to be \( \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \) follows from the equation \( \psi \alpha = \gamma \psi \).

Now, let \( \theta \) denote the 8-by-8 matrix \( \theta = \begin{pmatrix} \xi & -W \\ 0 & \eta \end{pmatrix} \). As \( \text{rank}(\text{cok} \eta) = \text{rank}(\text{cok} \xi) = \text{rank}(\text{push}(I)) = 2 \), Lemma (7.9) gives \( \text{cok} \theta = \text{push}(\text{cok} \xi) = \text{push}(\text{push}(I)) \). By Proposition (6.28), in order to show the component of \( \Gamma \) containing \( I \) is a tube, it suffices to show that \( \text{cok} \theta = X \oplus Y \oplus F \), for some indecomposables \( X \) and \( Y \) and some possibly-zero free module \( F \). It suffices to do this for \( \text{im} \theta \) instead of \( \text{cok} \theta \). We clarify that the term the term “indecomposable” is unambiguous:

**Lemma 7.12.** [3, Lemma 1] Given an indecomposable \( N \) in \( L_p(R) \) (i.e., \( N \) has no proper graded direct summand), we have that \( \hat{N} \) is indecomposable in \( L_p(\hat{R}) \). In particular, \( \hat{N} \) is indecomposable as an \( R \)-module.

We state the above discussion as a lemma.

**Lemma 7.13.** In order to establish that the component of the AR quiver containing \( \hat{I} \) is a tube, it suffices to show that \( \text{im} \theta = X \oplus Y \) for some graded modules \( X \) and \( Y \) each having no proper graded direct summand.

We begin by multiplying \( \theta \) on the left and on the right by invertible matrices. Let \( \text{id} \) denote the 2-by-2 identity matrix, and let \( H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). Let \( P' \) denote the 8-by-8 matrix
the goal of showing that $S$ to $j$ all
Therefore, since $\xi \neq \eta$, we know that $\tilde{\theta}$ is part of a matrix factorization $(\tilde{\theta}, \tilde{\theta}')$ where
Let $\tilde{\theta}$ be the lift of $\theta$, $\tilde{\theta} = \begin{pmatrix} \xi & -\tilde{W} \\ 0 & \eta \end{pmatrix}$. By the same reasoning used for the matrix factorization $(\xi, \eta)$, we know that $\tilde{\theta}$ is part of a matrix factorization $(\tilde{\theta}, \tilde{\theta}')$ where
Let $r_3, \ldots, r_8 \in R$ be homogeneous elements such that $\sum_{j=3}^8 r_j c_j = \pi(c_5)$ and $\deg(r_j) = \deg(c_5) - \deg(c_j)$. Then each of $\deg(r_6) = -np + mq$, $\deg(r_7) = -(v+1)p + (m+1-p)q$, and $\deg(r_8) = -(n+1)p + q$ does not lie in $\mathbb{N}p + \mathbb{N}q$ by Lemma 7.8 and so $r_6 = r_7 = r_8 = 0$.
For a brief moment let us consider matrices with entries in $S$. Namely let $\tilde{W}$ denote a “lift to $S$” of the matrix $W$, and let $\tilde{\theta}$ be the lift of $\theta$, $\tilde{\theta} = \begin{pmatrix} \xi & -\tilde{W} \\ 0 & \eta \end{pmatrix}$. By the same reasoning used for the matrix factorization $(\xi, \eta)$, we know that $\tilde{\theta}$ is part of a matrix factorization $(\tilde{\theta}, \tilde{\theta}')$ where
$\tilde{\theta}' = \begin{pmatrix} \eta & \tilde{W}' \\ 0 & \xi \end{pmatrix}$ for some 4-by-4 matrix $\tilde{W}'$. Let $\theta' = \tilde{\theta}' \otimes S$. R.
As $\theta \theta' = 0$, each column of matrix $P^{-1} \theta'$ is a syzygy relation for the columns of $P \theta P$. We can compute that the last four entries of the column $P^{-1} \theta'_{(4)}$ are, in order, $-\frac{1}{2}y^n, \frac{1}{2}x^m, 0, 0$. Therefore $\frac{1}{2}x^m c_6 \in \frac{1}{2}y^n c_5 + \sum_{j=3}^4 R c_j$. Then, $\pi(c_6) = \frac{\chi^n}{\chi^m} \pi(c_5) = \sum_{j=3}^5 \chi^n \pi(c_j)$, and in particular $R$ must contain the fourth entry of this column: $\frac{\chi^n}{\chi^m}(r_3 y^{q+v-n} - r_4 x^m - 2 r_5 x^{m-1} y^{q-n-1} f) \in R$. Therefore, since $y^{q+v} / x^n \in R$, we have $2 r_5 y^{q-1} f / x \in R$. Since $r_5 \in k$ and char $k \neq 2$, this implies that either $r_5 = 0$ or $y^{q-1} f / x \in R$. If the latter were true, then $r x = y^{q-1} f$ for some $r \in R$, and lifting $r$ to a preimage $s \in S$ we would have $sx - y^{q-1} f \in gS$. But $sx - y^{q-1} f$
has nonzero $y^{q+v-1}$-term, whereas $\deg g = \deg y^{q+v} > \deg y^{q+v-1}$, so this is a contradiction. Hence $r_5 = 0$. Therefore $\pi(c_5) = r_3c_3 + r_4c_4 \in \ker(\pi)$, hence $\pi(c_5) = 0$ as $\pi$ is idempotent, and $\pi(c_6) = (y^n/x^m)\pi(c_5) = 0$.

Now we simply repeat the argument in order to show that $\pi(c_8) = \pi(c_7) = 0$. For $r_3', ..., r_8' \in R$ homogeneous such that $\Sigma_{j=3}^8 r_j'c_j = \pi(c_8)$ and $\deg(r_j') = \deg(c_8) - \deg(c_j)$, each of $\deg(r_5') = (n-1)p - q$, $\deg(r_6') = -p + (m-1)q$, and $\deg(r_7') = (-v+n)p + (m-p)q$ does not lie in $\mathbb{N}p + \mathbb{N}q$ by Lemma 7.8, so $r_5' = r_6' = r_7' = 0$. The last two entries of $P^{-1} \theta_{1,7}'$ are $x^m$ and $-y^{q+v-n}$, so we obtain $x^m c_7 \in y^{q+v-n}c_8 + \sum_{j \leq 8} Rc_j$, and therefore $\pi(c_7) = (y^{q+v-n}/x^m)\pi(c_8) = (y^{q+v-n}/x^m)(r_5'c_3 + r_6'c_4 + r_7'c_8)$, whose fifth entry is $-r_5'(y^{q+v-n}/x^m)W_{34}$. If $r_5' = 0$ then $\pi(c_7) \in \ker(\pi)$ whence 0 = $\pi(c_7) = \pi(c_8)$, so, showing $r_5' = 0$ is the last step. If $r_5' \neq 0$ then it is a unit, and therefore $(y^{q+v-n}/x^m)W_{34} \in R$. Then the lemma below would imply $y^{q+v-1}/x \in R$, and the reader can check that this is false.

Lemma 7.14. $W_{34}$, the $(3,4)$-th entry of the matrix $W$, lies in $kx^{m-1}y^{n-1} \setminus \{0\}$.

Proof. Recall that $\eta W = \gamma \eta$ by definition of $W$. As $\eta_{4,4} = x^m$, we get $\gamma x^m = \eta_{4,4}, W_{4,4} = y^{q+v-n}W_{34} + x^mW_{44}$. As $x$ is a nonzerodivisor and $\gamma \notin R$, we have $W_{34} \neq 0$. We naturally choose $W$ so that $\deg(\eta_{ij}) + \deg(W_{jj}) = \deg(\gamma_{ij})$ for each $i, j, j'$. Setting $i = 4$, $j = 3$, $j' = 4$, we have $\deg(W_{34}) = \deg(\gamma_{4,4}) - \deg(\eta_{4,3}) = \deg(\gamma x^m) - \deg(y^{q+v-n}) = \deg(y^{q-1}f x^{m-1}) - \deg(y^{q+v-n}) = (n-1)p + (m-1)q$. Since $p$ and $q$ are coprime, it follows that $W_{34} \in kx^{m-1}y^{n-1}$.□□

8. Another Observation: $\text{soc}([\gamma_M])$.

In this section, assume $R$ is a reduced complete local Gorenstein (but not regular) ring of dimension one, let $m = m_R$, $Q = Q(R)$, and fix some indecomposable nonfree $M \in L_p(R)$. We aim to prove Proposition 8.7, which (after some additional assumptions) states the relationship between the socle elements $[\gamma_M]$ and $[\gamma_{\text{soc}(M)}]$ with respect to the $R$-algebra isomorphism $\text{soc}_R : \mathbf{End}_R M \rightarrow \mathbf{End}_R (\text{soc}_R(M))$. Let $M\overline{R}$ denote the $\overline{R}$-submodule of $M \otimes_R Q$ generated by $M$, and assume the following: $M\overline{R}$ is a free $\overline{R}$-module which possesses a basis consisting of elements in $M$. This is true if $R$ is a domain, since $\overline{R}$ is in that case a DVR.

Notation 8.1. Fix $\gamma \in J(\overline{R}) \setminus R$, and fix elements $e_1, ..., e_n \in M$ forming a free $\overline{R}$-basis for $M\overline{R}$. Given $h \in \text{End}_R M$, let $\overline{h}$ denote the unique $\overline{R}$-linear endomorphism of $M\overline{R}$ extending $h$. We regard $\overline{h}$ is an $n$-by-$n$ matrix with entries in $\overline{R}$. Let $I^{cd} = (R :_R \overline{R})$, the conductor ideal.

Lemma 8.2. We have $\gamma M\overline{R} \subseteq M$, and $I^{cd}(M\overline{R}) \subseteq \bigoplus_i Re_i$.

Proof. As $(R :_R J(\overline{R})) = (I^{cd} :_R J(\overline{R}))$, we have $\gamma J(\overline{R}) \subseteq I^{cd}$. Therefore $(\overline{R} \gamma)m \subseteq (\overline{R} \gamma)J(\overline{R}) \subseteq I^{cd} \subseteq m$, which says that $\overline{R} \gamma \subseteq \text{End}_R m$. Since $M \cong \text{Hom}(M, m)^*$ is an $\text{End}_R m$-module, we obtain $(\overline{R} \gamma)M \subseteq M$, equivalently $\gamma M\overline{R} \subseteq M$. That $I^{cd}(M\overline{R}) \subseteq \bigoplus_i Re_i$ is obvious.□□

We have the following immediate consequence.

Lemma 8.3. Let $A \in \text{End}_R(M\overline{R})$, i.e. $A$ is an $n \times n$ matrix with entries in $\overline{R}$ (recall that we have a fixed basis, $\{e_1, ..., e_n\}$). If each entry of $A$ lies in $\gamma \overline{R}$, then $A$ sends $M$ into $M$. If each entry of $A$ lies in $I^{cd}$, then $A|_M : M \rightarrow M$ is stably zero.

Lemma 8.4. There exists $f \in \text{End}_R M$ satisfying the following conditions:

(i) $[f]$ generates $\text{soc}(\text{End}_R M)$;
(ii) all nonzero entries of $\bar{f}$ lie in $\overline{Ry}$.
(iii) the first column of $\bar{f}$ is its only nonzero column.
(iv) $\bar{f}_{1,1} = \gamma$.

**Proof.** If we take an $n \times n$ matrix $A$ with $A_{1,1} = \gamma$ and all other entries zero, then by Lemma 8.3, $A = \overline{h}$ for some endomorphism $h \in \text{End}_R M$. As $\text{trace}(h \otimes Q) = \text{trace}(\overline{h}) = \gamma \in R$, $h$ is stably nonzero by Lemma 8.3. Therefore by essentiality of the socle of $\text{End}_R M$, there exists $g \in \text{End}_R M$ such that $[gh]$ generates $\text{soc}(\text{End}_R M)$. By Lemma 8.3, there exists $h' \in \text{End}_R M$ such that $\text{trace}(ghh' \otimes_R Q) \in R$, i.e. $\text{trace}(h'gh \otimes_R Q) \in R$. As $h'gh$ is stably nonzero by Lemma 8.3 once more, $[h'gh]$ generates $\text{soc}(\text{End}_R M)$. Let $f = h'gh$. Now trace($f \otimes_R Q$) = trace($\bar{f}$) = $\bar{f}_{1,1} \in (\gamma R) \setminus R$. Therefore $\bar{f}_{1,1} = u\gamma$ for some unit $u \in R$. Finally, replacing $\bar{f}$ by $u^{-1}\bar{f}$, the result still sends $M$ into $M$, by Lemma 8.3. □ □

Note that $\text{syz}_R$ is in general a well-defined functor on the stable category. In particular it gives an isomorphism of $R$-algebras $\text{End}_R M \to \text{End}_R(\text{syz}_R(M))$.

For the remainder, assume $R$ is a domain, and assume $k = R/m$ is algebraically closed.

**Proposition 8.5.** If $f \in \text{End}_R M$ and $g \in \text{End}_R(\text{syz}_R M)$ are given such that $[f] \in \text{soc}(\text{End}_R M)$ and $[g] = \text{syz}_R([f])$, then $\text{trace}(f + \text{trace} \overline{f})$. 

**Proof.** By Lemma 8.3 trace induces well-defined maps $\text{End}_R M \to \overline{R}/R$ and $\text{End}_R(\text{syz}_R M) \to \overline{R}/R$. As $\text{syz}_R$ gives an isomorphism of $R$-algebras $\text{End}_R M \to \text{End}_R(\text{syz}_R M)$, it restricts to an isomorphism on socles, which are $R$-simple due to $k$ being algebraically closed. Because of these remarks, we can take our pick of $f$ and $g$, as long as $[f] \neq 0$ and $[g] = \text{syz}_R([f])$; we will choose $f$ as in Proposition 8.4. Let $n = \text{rank}(M)$, and $v > n$ be the minimal number of generators of $M$. Let $\xi_1, ..., \xi_v$ be a set of generators for $M$, such that $\{e_1 = \xi_1, ..., e_n = \xi_n \}$ is an $\overline{R}$-basis for $\overline{M}$. For each $\xi_j$ we have an equation $\xi_j = \sum_{i=1}^{n} w_{i,j} e_i$, for $w_{i,j} \in R$. Since $\overline{R} = R + J(\overline{R})$ (due to $k$ being algebraically closed), we may assume that for each $j > n$, and for each $i$, we have $w_{i,j} \in J(\overline{R})$ and therefore $w_{i,j} \gamma \in I^{cd}$.

Take a free cover $\pi : F \to M$ sending $i$-th basis element to $\xi_i$. Since $f \in \text{End}_R M$ is as in Proposition 8.4, there is a $v \times v$ matrix $A : F \to F$ such that $\pi A = \overline{f} \pi$, with the following properties. Columns 2 through $n$ of $A$ are zero. In addition, $A_{ij} = w_{i,j} f_{i,1}$ for $(i, j) \in \{1, ..., n \} \times \{n+1, ..., v \}$, and $A_{ij} = 0$ for $(i, j) \in \{n+1, ..., v \} \times \{n+1, ..., v \}$. Set $N = \ker(\pi)$, and let $\overline{f} = [r_1, ..., r_v]^T \in N$, that is, $\sum_{j=1}^{v} r_j \xi_j = 0$. Recalling that $M \overline{R}$ is free, and projecting onto the basis element $e_1$, we get $r_1 + \sum_{j=n+1}^{v} r_j w_{1,j} = 0$. If we set $\overline{f}_{i,1} = 0$ for $i > n$, then by definition of $A$ we have that the $i$-th entry of $A \overline{f}$ is $A_{i1} r_1 + \sum_{j=n+1}^{v} w_{i,j} \overline{f}_{i,1} r_j = A_{i1} r_1 + \overline{f}_{i,1} \sum_{j=n+1}^{v} r_j w_{1,j}$, and by the above equation this equals $(A_{i1} - \overline{f}_{i,1}) r_1$. In other words, $A \overline{f} = r_1 \overline{v}$ where $\overline{v} = [v_1, ..., v_v]^T \in F$ is given by $v_i = A_{i1} - \overline{f}_{i,1}$. So if we let $g \in \text{End}_R N$ be the restriction of $A$, we see that the image of $g$ has rank 1. We also see that the $A^2 \overline{v} = A(r_1 \overline{v}) = r_1 v_1 \overline{v}$, so that $v_1$, which equals $A_{1,1} - \gamma$, is an eigenvalue for $g$. Our goal is to show that trace($\overline{g} + \gamma \in R$). Since trace($\overline{g}$) = trace($g \otimes_R Q$) and $\text{im}(g \otimes_R Q) \subseteq Q$, the following lemma finishes the proof. □ □

**Lemma 8.6.** If $\phi : F \to F$ is an endomorphism of a free module over a domain $D$, with $\text{im}(\phi) \subseteq D$, and $\lambda$ is an eigenvalue for $\phi$, then $\lambda = \text{trace}(\phi)$.
Proof. Let \( \bar{x} = [x_1, ..., x_s]^T \in F \) generate the image of \( \phi \). It is easily checked that \( \phi(\bar{x}) = \lambda \bar{x} \). Let \( y_1, ..., y_s \in D \) such that \( \phi \cdot j = y_j \bar{x} \). Then \( \lambda \bar{x} = \phi(\bar{x}) = \sum_{j=1}^{s} x_j \phi \cdot j = \sum_{j=1}^{s} x_j y_j \). So \( \lambda = \sum_{j=1}^{s} x_j y_j = \sum_j \phi \cdot j = \text{trace}(\phi) \).

\[ \text{Theorem 9.1.} \quad \text{The following theorem is well-known.} \]

\[ \text{Lemma 9.3.} \quad \text{Let} \quad R \quad \text{be a connected graded ring, let} \quad \hat{R} \quad \text{denote the completion of} \quad R \quad \text{with respect to its graded maximal ideal,} \quad m. \]

\[ \text{Notation 9.2.} \quad \text{If} \quad R \quad \text{is a connected graded ring, let} \quad \hat{R} \quad \text{denote the completion of} \quad R \quad \text{with respect to its graded maximal ideal,} \quad m. \]

\[ \text{Lemma 9.3.} \quad \text{Let} \quad R \quad \text{be a reduced connected graded ring. Then:} \]

\[ \begin{align*} 
(1) & \quad \text{The integral closure of} \quad R \quad \text{in} \quad R[\text{nonzerodivisors}]^{-1} \quad \text{coincides with the integral closure of} \quad R \quad \text{in} \quad Q = R[\text{graded nonzerodivisors}]^{-1}, \quad \text{our definition of} \quad \hat{R}. \quad \text{Moreover,} \quad \hat{R} = \bigoplus_{i \geq 0} \hat{R}_i \\
(2) & \quad \text{We have} \quad \hat{R} = \prod_{i \geq 0} R_i. \\
(3) & \quad \text{The completion,} \quad \hat{R}, \quad \text{is also reduced. If} \quad R \quad \text{is a domain, then} \quad \hat{R} \quad \text{is a domain.} \\
(4) & \quad \text{The integral closure,} \quad \hat{R}, \quad \text{is finitely generated as an} \quad R \quad \text{-module.} \\
(5) & \quad \text{The integral closure of the completion,} \quad \hat{R}, \quad \text{is finitely generated as an} \quad \hat{R} \quad \text{-module.} 
\end{align*} \]

\[ \text{Proof.} \quad \text{Statement (1) is [11, Corollary 2.3.6]. Statement (2) can be checked by noting that} \quad \{m^i\}_i \quad \text{is cofinal with} \quad \{\bigoplus_{j \geq i} R_j\}_i, \quad \text{and checking that the completion of} \quad R \quad \text{with respect to the latter filtration is isomorphic to} \quad \prod_{i \geq 0} R_i. \quad \text{From (2) we see that} \quad \hat{R} \quad \text{is reduced, resp. a domain, if} \quad R \quad \text{is such. As} \quad R \quad \text{is a finitely generated algebra over the field} \quad R_0, \quad (4) \quad \text{is a consequence of [14, Theorem 72]. The last assertion is a consequence of Theorem 9.1 (alternatively, it follows from (4)).} \]

\[ \text{Lemma 9.4.} \quad \text{Let} \quad D \quad \text{be a connected graded domain of dimension one, and let} \quad q = \bigoplus_{i \geq 1} D_i, \quad \text{and} \quad n = \bigoplus_{i \geq 1} D_i. \quad \text{Then} \]

\[ \begin{align*} 
(a) & \quad \hat{D}_0 \quad \text{is a field, and} \\
(b) & \quad \prod_{i \geq 0} D_i = \hat{D} = \hat{D}^n = \hat{D}. 
\end{align*} \]

\[ \text{Proof.} \quad \text{The notation} \quad \hat{D}_i \quad \text{means} \quad \langle D \rangle_i, \quad \text{and makes sense due to Lemma 9.3 as does} \quad \hat{D}. \quad \text{Since} \quad \hat{D} \quad \text{is an} \quad \mathbb{N} \text{-graded domain,} \quad n \quad \text{is a prime ideal, and is thus maximal since} \quad \dim \hat{D} = \dim D = 1. \]
Remark 9.6. If $X \neq 0$ for all graded $\overline{D}$-modules $X \neq 0$. Now $\overline{D}_n/(q\overline{D})_n$ is an artinian local ring, so there exists $i \geq 1$ such that $((n^i + q\overline{D})/q\overline{D})_n = 0$, hence $(n^i + q\overline{D})/q\overline{D} = 0$. Thus $\{n^i\}_i$ and $\{q^n\}_i$ are cofinal, so $\overline{D}_n^n = \overline{D}_n^n$.

Lastly we show $\overline{D_0}$ is a DVR; let $\pi \in D$ be a uniformizing parameter. So $\pi D = \prod_{i \geq 0} D_i$. Then $\pi t = u\pi^i$ for some unit $u \in \prod_{i \geq 0} D_i$, and it follows that $i = 1$, hence $t$ is a uniformizing parameter for $\overline{D}$. It follows that $D_i = 0$ for $i \notin N\ell$, and $D_i = D_0^{(\ell)}$ for $i \in N\ell$. The lemma follows.

Lemma 9.5. Let $D$ be a connected graded domain of dimension one, and let $l = \min(i > 0) \overline{D}_i \neq 0$. Let $t$ be any nonzero element of $\overline{D}_1$. Then $\overline{D} = \bigoplus_{i \geq 0} \overline{D}_0 t^i$ is the polynomial ring over the field $\overline{D}_0$ in the variable $t$; and $\overline{D} = \prod_{i \geq 0} \overline{D}_0 t^i$ is the power series ring.

Proof. By the previous lemmas, $\overline{D}$ is connected graded, so we may assume $D = \overline{D}$ to improve notation. Then the previous lemma also shows that $\overline{D} = \prod_{i \geq 0} D_i$ is a normal domain. Thus it is a DVR, let $\pi \in \overline{D}$ be a uniformizing parameter. So $\pi D = \prod_{i \geq 1} D_i$. Then $t = u\pi^i$ for some unit $u \in \prod_{i \geq 0} D_i$, and it follows that $i = 1$, hence $t$ is a uniformizing parameter for $\overline{D}$. It follows that $D_i = 0$ for $i \notin N\ell$, and $D_i = D_0^{(\ell)}$ for $i \in N\ell$. The lemma follows.

Remark 9.6. If $D_0$ is algebraically closed, Lemmas 9.4 and 9.5 show that $D$ has the form $k[t^i, ..., t^n]$.

REFERENCES

1. Ariki, S., Kase, R., Miyamoto, K.: On components of stable Auslander-Reiten quivers that contain Heller lattices: the case of truncated polynomial rings. Nagoya Mathematical Journal pp. 1–42 (2016). DOI 10.1017/nmj.2016.53
2. Auslander, M.: Rational singularities and almost split sequences. Trans. Amer. Math. Soc. 293(2), 511–531 (1986)
3. Auslander, M., Reiten, I.: Cohen-Macaulay modules for graded Cohen-Macaulay rings and their completions. Commutative algebra (1987)
4. Auslander, M., Reiten, I., Smalø, S.O.: Representation theory of Artin algebras, Cambridge Studies in Advanced Mathematics, vol. 36. Cambridge University Press, Cambridge (1997). Corrected reprint of the 1995 original
5. Benson, D.J.: Representations and Cohomology, Cambridge Studies in Advanced Mathematics, vol. 1. Cambridge University Press (1991). DOI 10.1017/CBO9780511623615
6. Bruns, W., Herzog, J.: Cohen-Macaulay rings, Cambridge Studies in Advanced Mathematics, vol. 39. Cambridge University Press, Cambridge (1993)
7. Decker, W., Greuel, G.M., Pfister, G., Schönemann, H.: SINGULAR 4-1-0 — A computer algebra system for polynomial computations. http://www.singular.uni-kl.de (2016)
8. Eisenbud, D.: Commutative algebra, Graduate Texts in Mathematics, vol. 150. Springer-Verlag, New York (1995). With a view toward algebraic geometry
9. Happel, D., Preiser, U., Ringel, C.M.: Vinberg’s characterization of Dynkin diagrams using subadditive functions with application to DTr-periodic modules, pp. 280–294. Springer Berlin Heidelberg, Berlin, Heidelberg (1980). DOI 10.1007/BFb0088469. URL https://doi.org/10.1007/BFb0088469
10. Hattori, A.: Rank element of a projective module. Nagoya Math. J. 25, 113–120 (1965). URL https://projecteuclid.org/euclid.nmj/1118801428
11. Huneke, C., Swanson, I.: Integral closure of ideals, rings, and modules, London Mathematical Society Lecture Note Series, vol. 336. Cambridge University Press, Cambridge (2006)
12. Leuschke, G.J., Wiegand, R.: Cohen-Macaulay representations, Mathematical Surveys and Monographs, vol. 181. American Mathematical Society, Providence, RI (2012). URL http://www.leuschke.org/research/MCMBook
13. Mac Lane, S.: Homology. Die Grundlehren der mathematischen Wissenschaften, Bd. 114. Springer-Verlag, Berlin-Heidelberg-New York (1963)
14. Matsumura, H.: Commutative Algebra. W.A. Benjamin, New York (1970)
15. Matsumura, H.: Commutative ring theory, Cambridge Studies in Advanced Mathematics, vol. 8, second edn. Cambridge University Press, Cambridge (1989). Translated from the Japanese by M. Reid
16. Miyata, T.: Note on direct summands of modules. J. Math. Kyoto Univ. 7, 65–69 (1967)
17. Yoshino, Y.: Cohen-Macaulay modules over Cohen-Macaulay rings, *London Mathematical Society Lecture Note Series*, vol. 146. Cambridge University Press, Cambridge (1990)