BOUNDARY LAYERS IN SMOOTH CURVILINEAR DOMAINS: PARABOLIC PROBLEMS

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Abstract. The goal of this article is to study the boundary layer of the heat equation with thermal diffusivity in a general (curved), bounded and smooth domain in \(\mathbb{R}^d\), \(d \geq 2\), when the diffusivity parameter \(\epsilon\) is small. Using a curvilinear coordinate system fitting the boundary, an asymptotic expansion, with respect to \(\epsilon\), of the heat solution is obtained at all orders. It appears that unlike the case of a straight boundary, because of the curvature of the boundary, two correctors in powers of \(\epsilon\) and \(\epsilon^{1/2}\) must be introduced at each order. The convergence results, between the exact and approximate solutions, seem optimal. Beside the intrinsic interest of the results presented in the article, we believe that some of the methods introduced here should be useful to study boundary layers for other problems involving curved boundaries.

1. Introduction. In this article, we study the boundary layer of the heat equation with small thermal diffusivity in a general bounded and smooth domain; we aim to simplify and extend, in different ways, the results presented in [33] (see also [20] and [36]). We consider the heat equation in a bounded smooth domain \(\Omega \subset \mathbb{R}^d\), \(d \geq 2\), with boundary \(\partial \Omega\):

\[
\begin{align*}
\partial_t u'(x,t) - \epsilon \Delta u'(x,t) &= f(x,t), \quad (x,t) \in \Omega \times (0,T), \\
u'(x,t) &= 0 \quad \text{on} \quad \partial \Omega, \\
u'|_{t=0} &= u_0(x),
\end{align*}
\]

where \(f\) and \(u_0\) are given smooth functions, and \(\epsilon\) is a small strictly positive parameter. We will specify the regularities of \(\partial \Omega\), \(f\) and \(u_0\) below when we perform the error analysis although the emphasis in this article is not on optimal regularity requirements. Moreover, we impose a consistency condition on the data, namely:

\[
u_0 = 0 \quad \text{on} \quad \partial \Omega.
\]

As we see below, due to the curvature of the boundary, the usual expansion in powers of \(\epsilon\) will not give a suitable approximation. Indeed, as we show below, the usual expansion has to be adapted to the current situation by introducing terms
of order $\epsilon^{j+1/2}$ in the expansion; these terms appear because, for a general curved domain, unlike a channel or a cube domain, the normal direction is changing along the boundary. Using the techniques of differential geometry and continuing the work in [12], we find explicit expressions of the correctors at all orders and obtain the optimal convergence rate as stated in Theorems 3.1 and 4.1 hereafter.

Using the techniques developed in this work, we intend in the future to study the linearized and non-linear Navier-Stokes equations in a general bounded and smooth domain; this will constitute a continuation of e.g. [32]-[35], [16], [17] and [19]. It is known that, at small viscosity, the solution of the time-dependent linearized Navier-Stokes equations (LNSE) behaves, to some (limited) extent, in parts of the domain under consideration, like the heat solution. This is especially the case when the boundary is characteristic, i.e. the homogeneous Dirichlet boundary conditions are imposed. In this direction, and in order to study the LNSE, Temam and Wang considered first a channel domain to avoid the geometric complexity of a general domain; see [32]. Later, they focused on the study of the boundary layer of the heat equation and the LNSE in a two-dimensional general domain. More precisely, they proposed in [33] the use of a curvilinear coordinate system adapted to the geometry of the boundary, which was also used in the convergence result for the fully non-linear Navier-Stokes problem; see [34]. We recall here that, when studying the boundary layers, the non-characteristic case for the Navier-Stokes problem does not involve the heat equation; the readers may consult [16] and [35].

The article is organized as follows: first, in Section 2 we propose a formal asymptotic expansion of $u^\epsilon$, solution of (1.1), and express the Laplace operator in terms of a curvilinear coordinate system adapted to the boundary. Then, in Section 3, we explicitly give the correctors at order $\epsilon^0$ and $\epsilon^{1/2}$ and perform the error estimates; in fact, we will see that, by adding the corrector at order $\epsilon^{1/2}$ in the expansion of $u^\epsilon$, we recover the optimal convergence rate of the remainder. Finally, in Section 4, we expand our results to all orders $\epsilon^N$ and $\epsilon^{N+1/2}$, $N \geq 1$.

2. Asymptotic expansions. In this article, $(x_1, \ldots, x_d)$ denotes the Cartesian coordinates of a point $x \in \mathbb{R}^d$ and $\Delta$ is the Laplace operator with respect to the $x$ variable. We assume that the domain $\Omega$ satisfies the following property:

$$\partial \Omega = \bigcup_{i=0}^n \Gamma_i,$$

where each $\Gamma_i$ is a connected component of $\partial \Omega$ which is a smooth Jordan surface in $\mathbb{R}^{d-1}$ with $\Omega$ lying locally on one side of $\Gamma_i$.

We choose a curvilinear coordinate system $\xi = (\xi', \xi_d)$, $\xi' = (\xi_1, \ldots, \xi_{d-1})$, adapted to $\partial \Omega$, such that, more precisely:

$$\partial \Omega \subset \{ \xi \in \mathbb{R}^d \mid \xi_d = 0 \text{ and } \Omega \text{ is located on the side } \xi_d > 0 \}. \quad (2.2)$$

For $\delta$ “small”, we will consider the $3\delta$-neighborhood $\Omega_{3\delta}$ of $\partial \Omega$:

$$\Omega_{3\delta} = \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) < 3\delta \}, \quad (2.3)$$

and assume that this set is diffeomorphic to the following set in the space $\mathbb{R}^d_{\xi'}$:

$$\Omega_{3\delta, \xi} = \{ (\xi', \xi_d) \in \mathbb{R}^d_{\xi'} \mid 0 < \xi_d < 3\delta \}, \quad (2.4)$$

where $\omega'$ is an open bounded set in $\mathbb{R}^{d-1}_{\xi'}$.

We will also consider

$$\Omega_{2\delta} = \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) < 2\delta \}, \quad (2.5)$$
and
\[ \Omega_{2\delta, \xi} = \{(\xi', \xi_d) \in \mathbb{R}^d | \xi' \in \omega', \ 0 < \xi_d < 2\delta\}. \] (2.6)

In this article, we assume that \( \epsilon \ll \delta \), which is reasonable since we aim to study the asymptotic behavior of the solution of (1.1) when the parameter \( \epsilon \) tends to 0.

Without loss of generality, we assume that our curvilinear coordinate system \( \xi \) is orthogonal and satisfies
\[ \frac{\partial \mathbf{x}}{\partial \xi_1} \cdot \frac{\partial \mathbf{x}}{\partial \xi_d} = \sum_{i=1}^{d} \left( \frac{\partial x_i}{\partial \xi_d} \right)^2 = 1, \] (2.7)

and we introduce the following classical geometrical notations for the coordinates \( \xi \) (see e.g. [5] and [37]):
\[
\begin{align*}
g_\alpha &= \frac{\partial \mathbf{x}}{\partial \xi_\alpha} = \left( \frac{\partial x_1}{\partial \xi_\alpha}, \ldots, \frac{\partial x_d}{\partial \xi_\alpha} \right), 1 \leq \alpha \leq d, \\
\{g_{\alpha \beta}\}_{1 \leq \alpha, \beta \leq d} &= \{g_\alpha, g_\beta\}_{1 \leq \alpha, \beta \leq d} = \text{diag}(g_{11}, \ldots, g_{d-1d-1}, 1), \\
g &= \det\{g_{\alpha \beta}\}_{1 \leq \alpha, \beta \leq d}, \\
\{g^{\alpha \beta}\}_{1 \leq \alpha, \beta \leq d} &= \{g_{\alpha \beta}\}_{1 \leq \alpha, \beta \leq d}^{-1} = \text{diag}\left( \frac{1}{g_{11}}, \ldots, \frac{1}{g_{d-1d-1}}, 1 \right).
\end{align*}
\] (2.8)

If we set \( h = \sqrt{\epsilon} > 0 \), \( h' = \partial h/\partial \xi_d \), then, using (2.8), we can write the Laplace operator in the \( \xi \) variable in the following form:
\[
\Delta = \sum_{1 \leq \alpha, \beta \leq d} \frac{1}{h} \frac{\partial}{\partial \xi_\alpha} \left( h g_{\alpha \beta} \frac{\partial}{\partial \xi_\beta} \right) = \sum_{1 \leq \gamma \leq d-1} \frac{1}{h} \frac{\partial}{\partial \xi_\gamma} \left( h g_{\gamma \gamma} \frac{\partial}{\partial \xi_\gamma} \right) + \frac{1}{h} \frac{\partial}{\partial \xi_d} \left( h' \frac{\partial}{\partial \xi_d} \right) + \frac{\partial^2}{\partial \xi_d^2} \] (2.9)

see [5] and [37] for more details.

To study the singularly perturbed problem (1.1), we will look for an asymptotic expansion of \( u^\epsilon \) in the form:
\[ u^\epsilon \simeq \sum_{j=0}^{\infty} (\epsilon^j w^j + \epsilon^j \theta^j + \epsilon^{j+\frac{1}{2}} \theta^{j+\frac{1}{2}}), \] (2.10)

where the \( w^j \) correspond to the external expansion (outside of the boundary layer) and the correctors \( \theta^j, \theta^{j+\frac{1}{2}} \) correspond to the inner expansion (inside the boundary layer); see e.g. [9], [18], [20] or [26] for general results on singular perturbation problems.

To obtain the external expansion of \( u^\epsilon \), we formally set \( u^\epsilon \simeq \sum_{j=0}^{\infty} \epsilon^j w^j \) and insert this expression in (1.1)\textsubscript{1} and (1.1)\textsubscript{3}. By identifying all the terms of order \( \epsilon^j \), for each \( j \) in each equation, we obtain the following set of equations and initial conditions:
\[
\begin{align*}
\partial_t u^0(x, t) &= f(x, t), \quad u^0(x, 0) = u_0(x), \\
\partial_t w^j(x, t) - \Delta w^{j-1}(x, t) &= 0, \quad w^j(x, 0) = 0, \quad j \geq 1.
\end{align*}
\] (2.11)
Integrating (2.11) over $(0, t)$, we recursively obtain the $u^j$, $j \geq 0$ in the form:

\[
\begin{align*}
\begin{cases}
  u^0(x, t) = u_0(x) + \int_0^t f(x, s)ds, \\
  u^j(x, t) = \frac{1}{j!} \Delta^j u_0(x) + \int_0^t \frac{1}{j!} (t-s)^j \Delta^j f(x, s)ds, & j \geq 1;
\end{cases}
\end{align*}
\]  

(2.12)

note that the $u^j$ are well-defined for all $j \geq 0$ under the regularity assumption

\[
u_0 \in H^2(\Omega) \text{ and } f \in L^\infty(0, T; H^2(\Omega)).
\]  

(2.13)

In the boundary layer, $u' - \sum_{j=0}^\infty \epsilon^j u^j \simeq \sum_{j=0}^\infty \left(\epsilon^j \theta^j + \epsilon^{j+\frac{1}{2}} \theta^{j+\frac{1}{2}}\right)$, which gives

\[
\sum_{j=0}^\infty \left(\epsilon^j \partial_\theta \theta^j + \epsilon^{j+\frac{1}{2}} \partial_\theta \theta^{j+\frac{1}{2}}\right) - \sum_{j=0}^\infty \left(\epsilon^{j+1} \Delta \theta^j + \epsilon^{j+\frac{3}{2}} \Delta \theta^{j+\frac{3}{2}}\right) = 0;
\]  

(2.14)

as we will see, the terms $\epsilon^{j+\frac{3}{2}} \theta^{j+\frac{3}{2}}$ appear because of the geometry; we will explain below the need to introduce the correctors $\theta^{j+\frac{1}{2}}$ which ensure optimal convergence results (see Remark 2.2 and Theorems 3.1 and 4.1).

Now, we introduce the stretched variable $\xi_d$ of $\xi$:

\[
\xi_d = \epsilon^{-\frac{3}{2}} \xi_d,
\]  

(2.15)

and, using (2.15), we rewrite the Laplacian as follows:

\[
\begin{cases}
  \Delta = S + \epsilon^{-\frac{3}{2}} L + \epsilon^{-1} \frac{\partial^2}{\partial \xi^2_d}, \\
  S = \sum_{1 \leq \gamma \leq d-1} \frac{1}{h} \frac{\partial}{\partial \xi_\gamma} \left(hg^{\gamma\gamma} \frac{\partial}{\partial \xi_\gamma}\right), \quad L = \frac{1}{h} \frac{\partial}{\partial \xi_d}.
\end{cases}
\]  

(2.16)

By considering $h(\xi', \xi_d) = h(\xi', \epsilon^{1/2} \xi_d)$, with $\xi'$, $\xi_d$ variables of order one in the boundary layer, a dependency of $S$ and $L$ on $\epsilon$ appears in (2.16). To address this dependency, we first introduce a notation: for any $\phi \in C^\infty(\Omega_d)$, we set

\[
\phi^{(j)}(0) = \frac{\partial^j \phi}{\partial \xi_d^j}(\xi', \xi_d = 0), \quad j \geq 0, \quad \phi(0) = \phi^{(0)}(0),
\]  

(2.17)

and we write the Taylor expansion of $\phi(\xi', \xi_d) = \phi(\xi', \epsilon^{1/2} \xi_d)$ in $\xi_d$ at $\xi_d = 0$ in the form:

\[
\phi(\xi', \xi_d) \simeq \phi(\xi', \epsilon^{1/2} \xi_d) \simeq \sum_{j=0}^\infty \epsilon^{j} \xi_d^j \phi_j, \quad \phi_j = \frac{1}{j!} \phi^{(j)}(0).
\]  

(2.18)

In particular, we write

\[
\begin{cases}
  h(\xi', \xi_d) \simeq \sum_{j=0}^\infty \epsilon^{j} \xi_d^j (h)_j, \quad \frac{1}{h}(\xi', \xi_d) \simeq \sum_{j=0}^\infty \epsilon^{j} \xi_d^j (\frac{1}{h})_j, \\
  h'(\xi', \xi_d) \simeq \sum_{j=0}^\infty \epsilon^{j} \xi_d^j (h')_j, \quad g^{\gamma\gamma}(\xi', \xi_d) \simeq \sum_{j=0}^\infty \epsilon^{j} \xi_d^j (g^{\gamma\gamma})_j,
\end{cases}
\]  

(2.19)

where $(h)_j$, $(1/h)_j$, $(h')_j$ and $(g^{\gamma\gamma})_j$ can be made explicit by using the Faà di Bruno formula; see [6] and also [11], [12].
Thanks to (2.16) and (2.19), we now find the following expansions of $S$ and $L$:

$$
S \simeq \sum_{j=0}^{\infty} \epsilon^j \xi_d^j S_j, \quad L \simeq \sum_{j=0}^{\infty} \epsilon^j \xi_d^j L_j,
$$

$$
S_j = \sum_{1 \leq \gamma \leq d-1} \sum_{j_1+j_2+j_3=j} \left( \frac{1}{h} \right)_{j_1} \frac{\partial}{\partial \xi_{\gamma}} \left( (h)_{j_2} (g^{\gamma\gamma})_{j_3} \frac{\partial}{\partial \xi_{\gamma}} \right), \quad j \geq 0,
$$

$$
L_j = \sum_{j_1+j_2=j} \left( \frac{1}{h} \right)_{j_1} (h')_{j_2} \frac{\partial}{\partial \xi_d} = \epsilon^{j/2} \sum_{j_1+j_2=j} \left( \frac{1}{h} \right)_{j_1} (h')_{j_2} \frac{\partial}{\partial \xi_d}, \quad j \geq 0;
$$

(2.20)

note that, for all $j \geq 0$, $S_j$, $L_j$, which are independent of $\epsilon$ and $\xi_d$, are well-defined if

$$
\partial \Omega \text{ is of class } C^{j+2}.
$$

It is noteworthy to observe that the $S_{j/2}$ are tangential operators near $\partial \Omega$ and that the operators $L_{j/2}$ are proportional to $\epsilon^{j/2} \partial/\partial \xi_d$.

Of course, if we just want to study the asymptotic expansion of $u^\epsilon$ at order $\epsilon^0$ (this is what we will do in Section 3), we do not need the expansions (2.20). However, they are needed for the higher-order cases starting from $\epsilon^{1/2}$.

Using (2.16) and (2.20), (2.14) yields

$$
\sum_{j=0}^{\infty} \left( \epsilon^j \partial_t \theta^j + \epsilon^{j+\frac{1}{2}} \partial_t \theta^{j+\frac{1}{2}} \right) - \sum_{j=0}^{\infty} \left( \epsilon^{j+1} S \theta^j + \epsilon^{j+\frac{1}{2}} L \theta^j + \epsilon^j \frac{\partial^2 \theta^j}{\partial \xi_d^2} \right) \right) - \sum_{j=0}^{\infty} \left( \epsilon^{j+\frac{1}{2}} S \theta^{j+\frac{1}{2}} + \epsilon^{j+1} L \theta^{j+\frac{1}{2}} + \epsilon^{j+\frac{1}{2}} \frac{\partial^2 \theta^{j+\frac{1}{2}}}{\partial \xi_d^2} \right) = 0,
$$

(2.22)

which allows us to find the equations for the correctors $\theta^j$ and $\theta^{j+\frac{1}{2}}$, $j \geq 0$; these equations will be presented in the following sections.

**Remark 2.1.** We will use the stretched variable $\xi_d$ to “weight” the different terms in the equation (2.22) and other similar equations. Otherwise we will generally revert to the initial variable $\xi_d$.

**Remark 2.2.** If we study the problem (1.1) in a domain with a flat boundary, we notice that all the terms of order $\epsilon^{j+\frac{1}{2}}$, $j \geq 0$, disappear in the expansion (2.20); that is, in (2.16), $h$ and $g^{\gamma\gamma}$, $1 \leq \gamma \leq 1-d$, are independent of $\xi_d$. Hence we do not require the correctors $\theta^{j+\frac{1}{2}}$, $j \geq 0$, to obtain the optimal estimates in Theorems 3.1 and 4.1 below.

3. **Analysis at lower orders $\epsilon^0$ and $\epsilon^\frac{1}{2}$.**

3.1. **Correctors $\theta^0$ and $\theta^{\frac{1}{2}}$.** To derive an equation for $\theta^0$, we collect all the terms of order $\epsilon^0$ in (2.22) and find:

$$
\frac{\partial \theta^0}{\partial t} - \frac{\partial^2 \theta^0}{\partial \xi_d^2} = 0, \text{ in } \Omega, \text{ or at least in } \Omega_{3d}.
$$

(3.1)

To make (3.1) well-posed, we need to impose two boundary conditions which are not available (see (1.1)2 and (2.2)). To overcome this difficulty, we replace (3.1) by...
the following equation (3.2) and add the boundary and initial conditions:

\[
\begin{aligned}
\partial_t \theta^0 - \epsilon \frac{\partial}{\partial \xi_d} \left\{ \sigma(\xi_d) \frac{\partial \theta^0}{\partial \xi_d} \right\} &= 0, \quad \text{in } \Omega, \\
\theta^0 &= -u^0, \quad \text{at } \xi_d = 0, \\
\theta^0|_{t=0} &= 0.
\end{aligned}
\] (3.2)

Here \(\sigma(\cdot)\) is a cut-off function such that \(\sigma(\xi_d) \in [0, 1]\) and,

\[
\sigma(\xi_d) = \begin{cases} 
1, & 0 \leq \xi_d \leq \delta, \\
0, & \xi_d \geq 2\delta.
\end{cases}
\] (3.3)

The existence and uniqueness of a solution for (3.2) is established in Section 3.4.

For \(\theta^\frac{1}{2}\), we collect all the terms of order \(\epsilon^\frac{1}{2}\) in (2.22) and find

\[
\frac{\partial \theta^\frac{1}{2}}{\partial t} - \frac{\partial^2 \theta^\frac{1}{2}}{\partial \xi_d^2} = L_0 \theta^0, \quad \text{in } \Omega,
\] (3.4)

where (see (2.19) and (2.20))

\[
L_0 \theta^0 = \left( \frac{1}{h} \right)_0 (h')_0 \frac{\partial \theta^0}{\partial \xi_d} = \epsilon^\frac{1}{2} \left( \frac{1}{h} \right)_0 (h')_0 \frac{\partial \theta^0}{\partial \xi_d}.
\] (3.5)

We modify (3.4) as for the corrector \(\theta^0\), and impose the boundary and initial conditions; we then obtain the equation for \(\theta^\frac{1}{2}\):

\[
\begin{aligned}
\partial_t \theta^\frac{1}{2} - \epsilon \frac{\partial}{\partial \xi_d} \left\{ \sigma(\xi_d) \frac{\partial \theta^\frac{1}{2}}{\partial \xi_d} \right\} &= f^\frac{1}{2} := L_0 \theta^0, \quad \text{in } \Omega, \\
\theta^\frac{1}{2} &= 0, \quad \text{at } \xi_d = 0, \\
\theta^\frac{1}{2}|_{t=0} &= 0.
\end{aligned}
\] (3.6)

Note that, due to the presence of \(\sigma\) in (3.2) and (3.6), the problems (3.2) and (3.6) are well-posed with only one boundary condition at \(\xi_d = 0\). Furthermore, we notice that, on \(\Omega \setminus \Omega_{2\delta}\), i.e. for \((\xi', \xi_d) \in \Omega\) such that \(\xi_d \geq 2\delta\),

\[
\frac{\partial^k \theta^0}{\partial \xi_i^k} = \frac{\partial^k \theta^\frac{1}{2}}{\partial \xi_i^k} = 0, \quad k \geq 0, \quad 1 \leq i \leq d.
\]

To perform the error analysis at orders \(\epsilon^0\) and \(\epsilon^\frac{1}{2}\), we define the remainders:

\[
w^0_e = u^e - u^0 - \theta^0, \\
w^\frac{1}{2}_e = u^e - u^0 - \theta^0 - \epsilon^\frac{1}{2} \theta^\frac{1}{2}.
\] (3.8)

The equation for \(w^0_e\) read

\[
\begin{aligned}
\partial_t u^0_e - \epsilon \Delta u^0_e &= \epsilon \Delta u^0 + R^0_e, \\
w^0_e(\xi', \xi_d = 0, t) &= 0, \\
w^0_e|_{t=0} &= 0,
\end{aligned}
\] (3.9)

where

\[
R^0_e = R^0_e(\theta) = \epsilon S \theta^0 + \epsilon^\frac{1}{2} L \theta^0 + \epsilon \frac{\partial}{\partial \xi_d} \left\{ (1 - \sigma) \frac{\partial \theta^0}{\partial \xi_d} \right\}.
\] (3.10)

\(^1\)For boundary value problems involving elliptic operators that degenerate at the boundary, see e.g. [1], [7] and [13].
Similarly, computing the difference between (1.1) and the sum of (2.11)_1, (3.2) and $\epsilon^\frac{1}{2}(3.6)$, we obtain the equation for $w^\frac{1}{2}$:

$$
\begin{align*}
\partial_t w^\frac{1}{2} - \epsilon \Delta w^\frac{1}{2} &= \epsilon \Delta u^0 + R^\frac{1}{2}_\epsilon, \\
\int_0^t \left( \frac{1}{2} \Delta u^0 \right)^2 d\xi + \frac{1}{2} |R^\frac{1}{2}_\epsilon|^2 d\xi &= 0.
\end{align*}
$$

(3.11)

where

$$
R^\frac{1}{2}_\epsilon = R^\frac{1}{2}_\epsilon(\theta) = \epsilon S \theta^0 + \epsilon^\frac{1}{2} (L - L_0) \theta^0 + \epsilon \frac{\partial}{\partial \xi} \left\{ (1 - \sigma) \frac{\partial \theta^0}{\partial \xi} \right\} + \epsilon^\frac{3}{2} \left[ \epsilon S \theta^\frac{1}{2} + \epsilon^\frac{1}{2} L \theta^\frac{1}{2} + \epsilon \left\{ (1 - \sigma) \frac{\partial \theta^\frac{1}{2}}{\partial \xi} \right\} \right].
$$

(3.12)

Multiplying (3.9)_1 by $w^0$ and integrating over $\Omega$, we find

$$
\frac{1}{2} \frac{d}{dt} |w^0|^2_{L^2(\Omega)} + \epsilon |\nabla w^0|^2_{L^2(\Omega)} \leq |w^0|^2_{L^2(\Omega)} + \epsilon^2 |\Delta u^0|^2_{L^2(\Omega)} + \frac{1}{2} |R^0_\epsilon|^2_{L^2(\Omega)}.
$$

(3.13)

Hence, using the Gronwall inequality, we obtain

$$
|w^0(t)|^2_{L^2(\Omega)} \leq \kappa T \int_0^T \left( \epsilon^2 |\Delta u^0|^2_{L^2(\Omega)} + |R^0_\epsilon|^2_{L^2(\Omega)} \right) dt,
$$

(3.14)

where $\kappa T$ denotes a constant depending on $T$ and the data, but independent of $\epsilon$, and which may be different at different occurrences.

Back to (3.13), we also notice that

$$
\int_0^T |\nabla w^0|^2_{L^2(\Omega)} dt \leq \epsilon^{-1} \int_0^T \left( |w^0|^2_{L^2(\Omega)} + \epsilon^2 |\Delta u^0|^2_{L^2(\Omega)} + \frac{1}{2} |R^0_\epsilon|^2_{L^2(\Omega)} \right) dt.
$$

(3.15)

For $w^\frac{1}{2}$, we multiply the equation (3.11) by $w^\frac{1}{2}$, perform the same computation as for $w^0$, and obtain

$$
|w^\frac{1}{2}(t)|^2_{L^2(\Omega)} \leq \kappa T \int_0^T \left( \epsilon^2 |\Delta u^0|^2_{L^2(\Omega)} + |R^\frac{1}{2}_\epsilon|^2_{L^2(\Omega)} \right) dt,
$$

$$
\int_0^T |\nabla w^\frac{1}{2}|^2_{L^2(\Omega)} dt \leq \epsilon^{-1} \int_0^T \left( |w^\frac{1}{2}|^2_{L^2(\Omega)} + \epsilon^2 |\Delta u^0|^2_{L^2(\Omega)} + \frac{1}{2} |R^\frac{1}{2}_\epsilon|^2_{L^2(\Omega)} \right) dt.
$$

(3.16)

Note that, using (2.13) with $j = 0$, (3.14)-(3.16) are valid when

$$
\begin{align*}
\theta^0 \in H^2(\Omega), \\
\epsilon \in L^\infty(0, T; H^2(\Omega)).
\end{align*}
$$

(3.17)

3.2. **Approximations $\overline{\theta}^0$ and $\overline{\theta}^\frac{1}{2}$ of the correctors $\theta^0$ and $\theta^\frac{1}{2}$.** To go further, we need information on the correctors $\theta^0$ and $\theta^\frac{1}{2}$ that is not available because the boundary value problems defining them are not easy to solve. To derive this information, we now introduce the approximations $\overline{\theta}^0$ and $\overline{\theta}^\frac{1}{2}$ of the correctors $\theta^0$.
and $\theta^\pm$ defined as the solutions of the following heat equations on $\mathbb{R}_+$:

\[
\begin{aligned}
&\begin{cases}
\partial_t \overline{\theta}^0 - \frac{\partial^2 \overline{\theta}^0}{\partial \xi_d^2} = 0, & \xi' \in \omega', \, \xi_d > 0, \\
\overline{\theta}^0 = -u^0, & \text{at } \xi_d = 0, \\
\overline{\theta}^0 (\xi', \xi_d, t) \to 0 & \text{as } \xi_d \to \infty, \\
\overline{\theta}^0 |_{t=0} = 0,
\end{cases}
\end{aligned}
\]

and

\[
\begin{aligned}
&\begin{cases}
\partial_t \overline{\theta}^\pm - \frac{\partial^2 \overline{\theta}^\pm}{\partial \xi_d^2} = \mathcal{J}_\epsilon (\xi', \xi_d, t) := L_0 \overline{\theta}^0, & \xi' \in \omega', \, \xi_d > 0, \\
\overline{\theta}^\pm = 0, & \text{at } \xi_d = 0, \\
\overline{\theta}^\pm (\xi', \xi_d, t) \to 0 & \text{as } \xi_d \to \infty, \\
\overline{\theta}^\pm |_{t=0} = 0.
\end{cases}
\end{aligned}
\]

Due to the compatibility condition (1.2), the explicit expressions of $\overline{\theta}^0$ and $\overline{\theta}^\pm$ are given in the form (see [3]):

\[
\overline{\theta}^0 (\xi', \xi_d, t) = - \int_0^t \mathcal{I} (\xi_d, t-s) \frac{\partial u^0}{\partial t} (\xi', 0, s) ds,
\]

and

\[
\begin{aligned}
&\begin{cases}
\overline{\theta}^\pm (\xi', \xi_d, t) = J_- (\xi', \xi_d, t) - J_+ (\xi', \xi_d, t), & \text{with} \\
J_{\pm} (\xi', \xi_d, t) = - \frac{1}{2} \int_0^t \int_0^\infty \frac{\partial \mathcal{I}}{\partial \xi_d} (\xi_d \pm \eta, t-s) \mathcal{J}_\epsilon (\xi', \eta, s) d\eta ds,
\end{cases}
\end{aligned}
\]

where $\mathcal{J}_\epsilon^{1/2} = L_0 \overline{\theta}_0$ has been defined in (3.19); see also (3.5).

In (3.20), $\mathcal{I} (\xi_d, t) = 2 \text{erfc}(\xi_d/\sqrt{2\epsilon t})$ where

\[
\begin{aligned}
\text{erf}(z) &= \frac{1}{\sqrt{2\pi}} \int_0^z e^{-y^2/2} dy, \quad \text{and} \\
\text{erfc}(z) &= \frac{1}{2} - \text{erf}(z) = \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-y^2/2} dy.
\end{aligned}
\]

Using polar coordinates, one can verify the following useful inequality: for $z > 0$,

\[
(\text{erfc}(z))^2 = \frac{1}{4\pi} \int_z^\infty \int_z^\infty e^{-(y_1^2+y_2^2)/2} dy_1 dy_2 
\]

\[
\leq \frac{1}{4} \int_\sqrt{z}^\infty e^{-r^2/2} r dr \leq \frac{1}{4} e^{-z^2},
\]

and hence

\[
\mathcal{I} (\xi_d, t) \leq \exp \left( - \frac{\xi_d^2}{4\epsilon t} \right).
\]

For the sake of convenience, we introduce a notation for a function $\phi$ in $C^\infty((0, T) \times \Omega_{2\delta})$:

\[
\phi_{j+k+m} := \frac{\partial^{j+k+m} \phi}{\partial \tau^j \partial \xi_d^k \partial \xi_d^m}, \ j, k, m \geq 0,
\]
where $\partial^k / \partial r^k$ is any tangential operator of order $k$,
\[ \frac{\partial^k}{\partial r^k} = \frac{\partial^{k_1 + \cdots + k_{d-1}}}{\partial \xi_1^{k_1} \cdots \partial \xi_{d-1}^{k_{d-1}}}, \quad k_1 + \cdots + k_{d-1} = k. \tag{3.26} \]

Now we state and prove the following lemmas:

**Lemma 3.1.** For $(t, \xi_d) \in (0, T) \times \mathbb{R}^+$, the function $I(\xi_d, t)$, in (3.22), satisfies the following pointwise estimates: for $m = 1, 2$,
\[ |I_d^m(\xi_d, t)| \leq \kappa_m (et)^{-m + \frac{t}{2}} |\xi_d|^{m-1} \exp \left( -\frac{\varepsilon^2}{4et} \right), \tag{3.27} \]
and, for $m \geq 3$,
\[ |I_d^m(\xi_d, t)| \leq \kappa_m \varepsilon^{-m + \frac{t}{2}} (1 + t^{-m + \frac{t}{2}})(1 + |\xi_d|^{m-1}) \exp \left( -\frac{\varepsilon^2}{4et} \right), \tag{3.28} \]
where $\kappa_m$ is a constant depending on $m$, but independent of $\varepsilon$, and which may be different at different occurrences.

**Proof.** For (3.27), using (3.22), we notice
\[ I_d(\xi_d, t) = -\frac{1}{\sqrt{\pi}} (et)^{-\frac{t}{2}} \exp \left( -\frac{\varepsilon^2}{4et} \right), \]
\[ I_d^2(\xi_d, t) = \frac{1}{2\sqrt{\pi}} (et)^{-\frac{t}{2}} \xi_d \exp \left( -\frac{\varepsilon^2}{4et} \right); \tag{3.29} \]
hence (3.27) follows.

Moreover, by differentiating (3.29)$_2$ $(m - 2)$-times with respect to $\xi_d$, and using the Leibnitz formula for the $m$-th derivative of a product, we find, for $m \geq 3$,
\[ I_d^m(\xi_d, t) = -\frac{1}{\sqrt{\pi}} (-2)^{-m+1}(et)^{-m+\frac{t}{2}} \xi_d^{m-1} \exp \left( -\frac{\varepsilon^2}{4et} \right) + r_{0,m}(\xi_d, t; \varepsilon), \tag{3.30} \]
where $r_{0,m}$ is given in the form: for some strictly positive integers $a_{i,m}$, $0 \leq i \leq n-2$,
\[ \left\{ \begin{array}{l}
 r_{0,2n-1}(\xi_d, t; \varepsilon) = \frac{-1}{\sqrt{\pi}} \sum_{i=0}^{n-2} a_{i,2n-1} (-2)^{-2n+i+3}(et)^{-2n+i+\frac{t}{2}} \xi_d^{2n-2i-4} \exp \left( -\frac{\varepsilon^2}{4et} \right), \\
 r_{0,2n}(\xi_d, t; \varepsilon) = \frac{-1}{\sqrt{\pi}} \sum_{i=0}^{n-2} a_{i,2n} (-2)^{-2n+i+2}(et)^{-2n+i+\frac{t}{2}} \xi_d^{2n-2i-3} \exp \left( -\frac{\varepsilon^2}{4et} \right). 
\end{array} \right. \tag{3.31} \]

Using (3.31), we can bound the lower order term $r_{0,m}$, in (3.30), with respect to $\varepsilon$:
\[ |r_{0,m}(\xi_d, t; \varepsilon)| \leq \kappa_m \varepsilon^{-m+\frac{t}{2}} (1 + t^{-m+\frac{t}{2}})(1 + |\xi_d|^{m-3}) \exp \left( -\frac{\varepsilon^2}{4et} \right), \tag{3.32} \]
and, from (3.30) and (3.32), we obtain (3.28) for $m \geq 3$. \hfill \Box

**Lemma 3.2.** For any $p \geq 0$ and $q \geq 1$, we have
\[ \left( e^{-\frac{t}{2}\xi_d} \right)^p \left| \right. \left|_{L^2(\Omega_{2t})} \right| _{L^q(\Omega_{2t})} \leq \kappa_T \varepsilon^q, \tag{3.33} \]
where $\kappa_T$ is a constant depending on $T$, but independent of $\varepsilon$. 
Proof. To verify (3.33), since the Jacobian determinant $h$ is bounded on $\Omega_{2\delta}$, we write
\[
\int_{\Omega_{2\delta}} \left( e^{-\frac{1}{2} \xi_d} \right)^{2p} \exp \left( - \frac{\xi_d^2}{2q\epsilon t} \right) d\Omega_{2\delta} = \int_{\Omega_{2\delta}, \xi} \left( e^{-\frac{1}{2} \xi_d} \right)^{2p} \exp \left( - \frac{\xi_d^2}{2q\epsilon t} \right) h(\xi', \xi_d) d\xi_d d\xi' 
\leq (\text{setting } \eta = \xi_d/\sqrt{q\epsilon t}) 
\leq K_T \epsilon^{\frac{1}{2}} \int_0^\infty \eta^2 e^{-\eta^2/2} d\eta \leq K_T \epsilon^{\frac{1}{2}};
\]
(3.34) follows. \qed

Lemma 3.3. Assume that $\partial \Omega$ is of class $C^2$, and that, for $k \geq 0$, $u_0$ and $f$ belong to $H^k(\Omega)$ and $L^\infty(0, T; H^k(\Omega))$ respectively. Then, the approximate corrector $\overline{\theta}^0$ given by (3.20) satisfies the following pointwise estimates: for $(t, \xi) \in (0, T) \times \Omega_{2\delta, \xi}$, and for $m = 0, 1$, we have
\[
\left| \overline{\theta}^0_{r+k} (\xi', \xi_d, t) \right| \leq K_{T, k, m} \epsilon^{-\frac{1}{2}} \exp \left( - \frac{\xi_d^2}{4\epsilon s} \right). \tag{3.35}
\]
Moreover, for $(t, \xi) \in (0, T) \times \Omega_{2\delta, \xi}$, and for $j = 0$ and $m \geq 2$, and $j \geq 1$ and $m \geq 0$, we have
\[
\left| \overline{\theta}^0_{r+j+k} (\xi', \xi_d, t) \right| \leq K_{T, j, k, m} \epsilon^{-j-m+\frac{1}{2}} \int_0^t (1 + s^{-2j-m+\frac{1}{2}}) \exp \left( - \frac{\xi_d^2}{4\epsilon s} \right) ds, \tag{3.36}
\]
where $K_{T, k, m}$ and $K_{T, j, k, m}$ are constants depending on $T$, $j$, $k$, $m$, and the other data, but independent of $\epsilon$.

Proof. Using (3.20), we write, for $k, m \geq 0$,
\[
\overline{\theta}^0_{r+k} (\xi', \xi_d, t) = - \int_0^t I_d (\xi_d, t-s) u^0_{r+k} (\xi', 0, s) ds, \tag{3.37}
\]
and, due to (3.24), the estimate (3.35) follows for $m = 0$.

For $m \geq 1$, using (3.27) and (3.28), and setting $s' = t-s$, (3.37) yields
\[
\left| \overline{\theta}^0_{r+k} (\xi', \xi_d, t) \right| \leq K_{T, k, m} \epsilon^{-m+\frac{1}{2}} \int_0^t \left\{ 1 + (s')^{-m+\frac{1}{2}} \right\} \left\{ 1 + |\xi_d|^{m-1} \right\} \exp \left( - \frac{\xi_d^2}{4\epsilon s} \right) ds'. \tag{3.38}
\]
Since $s^{-1/2}$ is integrable over $(0, T)$, we obtain (3.35) for $m = 1$, and (3.36) also follows for $j = 0$ and $m \geq 2$.

To verify (3.36) for $j \geq 1$ and $m \geq 0$, we use (3.18) and write
\[
\overline{\theta}^0_{r+j+k} = \epsilon^j \overline{\theta}^0_{r+k} \overset{d}{=} \overline{\theta}^0_{r+k, d^{2j+m+2}}. \tag{3.39}
\]

More generally, we can recursively obtain, for $j \geq 1$,
\[
\overline{\theta}^0_{r+j+k} = \epsilon^j \overline{\theta}^0_{r+k} \overset{d}{=} \overline{\theta}^0_{r+k, d^{2j+m}}. \tag{3.40}
\]
Then, thanks to (3.38), (3.36) follows for $j \geq 1$ and $m \geq 0$. \qed

Lemma 3.4. Assume that, for $k \geq 0$, $\partial \Omega$ is of class $C^{2+k}$, and that $u_0$ and $f$ belong to $H^k(\Omega)$ and $L^\infty(0, T; H^k(\Omega))$ respectively. Then the approximate corrector $\overline{\theta}^2$,
satisfies the following pointwise estimates: for \((t, \xi) \in (0, T) \times \omega' \times \mathbb{R}_+\), and for \(m = 0, 1\), we have
\[
| \tilde{\tau}^{\frac{1}{2}}_{T^2, d,m}(\xi', \xi_d, t) | \leq \kappa_{T, k, m} e^{-\frac{m}{8} \exp \left( - \frac{\epsilon^2}{8(1 + m)e^t} \right)}. \tag{3.41}
\]
Moreover, for \((t, \xi) \in (0, T) \times \Omega_{2\delta, \xi}\), and for \(j = 0\) and \(m \geq 2\), and \(j \geq 1\) and \(m \geq 0\), we have
\[
| \tilde{\tau}^{\frac{1}{2}}_{j, T^2, d,m}(\xi', \xi_d, t) | \leq \kappa_{T, j, k, m} e^{-j - m - \frac{1}{2}} \int_0^t \{1 + s^{-2j - m - \frac{1}{2}}\} \exp \left( - \frac{\epsilon^2}{8e^s} \right) ds, \tag{3.42}
\]
where \(\kappa_{T, j, k, m}\) and \(\kappa_{T, j, k, m}\) are constants depending on \(T, j, k, m\), and the other data, but independent of \(\epsilon\).

**Proof.** Using (3.21), we write, for \(k, m \geq 0\),
\[
\begin{cases}
\tilde{\tau}^{\frac{1}{2}}_{T^2, d,m}(\xi', \xi_d, t) = \{J_- (\xi', \xi_d, t)\}_{T^2, d,m} - \{J_+ (\xi', \xi_d, t)\}_{T^2, d,m}, \\
\{J_{\pm} (\xi', \xi_d, t)\}_{T^2, d,m} = -\frac{1}{2} \int_0^t \int_0^\infty I_{d,m} (\xi_d \pm \eta, t - s) (\tilde{\tau}^{\frac{1}{2}}_{\epsilon})(\xi', \eta, s) d\eta ds.
\end{cases}
\tag{3.43}
\]

Using (2.20) and (3.19), we observe that \(\tilde{\tau}^{\frac{1}{2}}_{\epsilon}\) is the product of a bounded function with \(e^{1/2 \partial \tilde{\tau}^{\frac{1}{2}}_{\epsilon}}/\partial \xi_d\); hence we infer from Lemma 3.3 that
\[
| (\tilde{\tau}^{\frac{1}{2}}_{\epsilon})(\xi', \xi_d, t) | \leq \kappa_{T, k} e^{\frac{m}{2} \tilde{\tau}^{\frac{1}{2}}_{\epsilon}(\xi', \xi_d, t)} \leq \kappa_{T, k} \exp \left( - \frac{\epsilon^2}{4e^t} \right), \quad (t, \xi) \in (0, T) \times \omega' \times \mathbb{R}_+. \tag{3.44}
\]

Now, concerning (3.41), we only show (3.41) for \(m = 1\) which is the most difficult case.

To estimate \((J_-)_{T^2, d}^2\) pointwise, we write this term as the sum of two integrals \((J_-)_{T^2, d}^2\) on \((0, t/2)\) and \((J_-)_{T^2, d}^2\) on \((t/2, t)\), and estimate them separately:

We first consider the integral \((J_-)_{T^2, d}^2\) on \((t/2, t)\) which is more problematic:
\[
| (J_-)_{T^2, d}^2 | \leq (\text{using (3.27) for } m = 2 \text{ and (3.44)})
\]
\[
\leq \kappa_{T, k} e^{-\frac{m}{8} \int_{t/2}^t \int_0^\infty \frac{|\xi_d - \eta|}{(t - s)^{\frac{1}{2}}} \exp \left( - \frac{(\xi_d - \eta)^2}{4e^t} \right) \exp \left( - \frac{\eta^2}{4e^s} \right) ds d\eta}
\leq (\text{using the Schwarz inequality})
\]
\[
\leq \kappa_{T, k} e^{-\frac{m}{8} \int_{t/2}^t \left\{(t - s)^{-1} \left( \int_0^\infty \frac{|\xi_d - \eta|}{t - s} \exp \left( - \frac{(\xi_d - \eta)^2}{4e(t - s)} d\eta \right) \right)^{\frac{1}{2}} \right\} ds;
\tag{3.45}
\]
we have
\[
\int_0^\infty \frac{|\xi_d - \eta|^2}{t - s} \exp \left( - \frac{(\xi_d - \eta)^2}{4e(t - s)} d\eta \right) \leq (\text{setting } \eta' = (\eta - \xi_d)/\sqrt{2e(t - s)})
\leq (2e)^{\frac{3}{2}} \sqrt{t - s} \int_{-\infty}^\infty (\eta')^2 e^{(\eta')^2/2} d\eta'
\leq \kappa e^{\frac{3}{2} \sqrt{t - s}}.
\tag{3.46}
\]
and, since $t-s<s$ for $t/2<s<t$, we also find
\[
\int_0^\infty \exp \left( -\frac{(\xi_d - \eta)^2}{4\epsilon(t-s)} - \frac{\eta^2}{2\epsilon s} \right) d\eta \leq \int_0^\infty \exp \left( -\frac{(\xi_d - \eta)^2 + \eta^2}{4\epsilon s} \right) d\eta
\]
\[
\leq \exp \left( -\frac{\xi_d^2}{8\epsilon s} \right) \int_0^\infty \exp \left( -\frac{(\eta - \xi_d/2)^2}{2\epsilon s} \right) d\eta
\]
\[
\leq (\text{setting } \eta' = (\eta - \xi_d/2)/\sqrt{\epsilon s}) \leq \kappa \epsilon^{\frac{1}{2}} \sqrt{s} \exp \left( -\frac{\xi_d^2}{8\epsilon s} \right).
\]
(3.47)

Combining (3.45)-(3.47), we can bound $(J_-)^2_{\tau^k,d}$ as follows:
\[
\| (J_-)^2_{\tau^k,d} \| \leq \kappa_{T,k} \epsilon^{-\frac{1}{2}} \int_{t/2}^t (t-s)^{-\frac{3}{4}} \exp \left( -\frac{\xi_d^2}{16\epsilon s} \right) ds
\]
\[
\leq \kappa_{T,k} \epsilon^{-\frac{1}{2}} \exp \left( -\frac{\xi_d^2}{16\epsilon t} \right) \int_{t/2}^t (t-s)^{-\frac{3}{4}} ds
\]
\[
\leq \kappa_{T,k} \epsilon^{-\frac{1}{2}} \exp \left( -\frac{\xi_d^2}{16\epsilon t} \right).
\]
(3.48)

Since the other integral $(J_-)^1_{\tau^k,d}$ on $(0,t/2)$ satisfies the estimate (3.45) with $(J_-)^2_{\tau^k,d}$ and $(t/2,t)$ replaced by $(J_-)^1_{\tau^k,d}$ and $(0,t/2)$ respectively, and since, on $(0,t/2)$, $(t-s)^{-3/2}$ is bounded by $(t/2)^{-3/2}$ and $t^{-3/2}$, it is easy to see that $|(J_-)^1_{\tau^k,d}|$ is also bounded by the right-hand side of (3.48). Hence, we obtain
\[
\left| (J_-)_{\tau^k,d}(\xi', \xi_d, t) \right| \leq \kappa_{T,k} \epsilon^{-\frac{1}{2}} \exp \left( -\frac{\xi_d^2}{16\epsilon t} \right);
\]
(3.49)

with the same (but easier) proof for the term $(J_+)^1_{\tau^k,d}$, we obtain (3.41) for $m = 1$.

To show (3.42) for $j = 0$ and $m \geq 2$, we go back to (3.43), and use (3.28) and (3.44). As a result we can bound $(J_{\pm})_{\tau^k,d}^{m}$ pointwise:
\[
\| (J_{\pm})_{\tau^k,d}^{m} \| \leq \kappa_{T,k,m} \epsilon^{-\frac{m+1}{2}} \int_0^\infty \int_0^\infty \left[ \{1 + (t-s)^{-m-\frac{1}{2}}\} \{1 + |\xi_d \pm \eta|^{m}\} \exp \left( -\frac{(\xi_d \pm \eta)^2}{4\epsilon(t-s)} - \frac{\eta^2}{4\epsilon s} \right) \right] d\eta ds
\]
\[
\leq (\text{using the analog of (3.47)})
\]
\[
\leq \kappa_{T,k,m} \epsilon^{-\frac{m+1}{2}} \int_0^t \{1 + (t-s)^{-m-\frac{1}{2}}\} \exp \left( -\frac{\xi_d^2}{8\epsilon(t-s)} \right) ds;
\]
(3.50)

(3.42) follows for $j = 0$ and $m \geq 2$.

For (3.42) when $j \geq 1$ and $m \geq 0$, we use (3.19), and find
\[
\vartheta_{\tau^k,d}^{m} = \vartheta_{\tau^k,d}^{m+2} + (L_0 \vartheta^{m})_{\tau^k,d}^{m}.
\]
(3.51)

More generally, as for (3.40), we sequentially obtain, for $j \geq 1$,
\[
\vartheta_{\tau^k,d}^{j} = e^{\theta_{\tau^k,d}^{j}} \vartheta_{\tau^k,d}^{j+2} + \sum_{i=0}^{j-1} e^i (L_0 \vartheta^{i})_{\tau^k,d}^{j+2+i},
\]
(3.52)
and, using (2.20) and (3.36), we find
\[
\sum_{i=0}^{j-1} e^i (L_0^d)_{\frac{i-1}{T} d i + m} \leq \kappa \sum_{i=0}^{j-1} e^{i+\frac{1}{T}} (\varphi')_{\frac{i-1}{T} d i + m + 1} \\
\leq \kappa \epsilon^{j-m+1} \int_0^t (1 + s^{-2j-m+\frac{1}{T}}) \exp \left(-\frac{\xi_2 d}{4c} \right) ds.
\] (3.53)

Hence, from (3.50), (3.52) and (3.53), (3.42) follows for \( j \geq 1, m \geq 0 \), and the lemma is proved. \( \Box \)

We now return to our objective of respectively comparing \( \theta^0 \) and \( \varphi^0 \), and \( \theta^\pm \) and \( \varphi^\pm \). To compare \( \theta^0 \) and \( \varphi^0 \) in \( \Omega_{2b} \), we compute (3.2)–(3.18) and find that \( \Phi = \theta^0 - \varphi^0 \) satisfies the system (A.1) in the Appendix where
\[
\psi = F_0^0 (\xi', \xi_d, t) := \epsilon \{(\sigma - 1) \theta^0 \}_{d} = \epsilon \{(\sigma \varphi^0_d + (\sigma - 1) \varphi^0) \}_{d}.
\] (3.54)

Thanks to Lemma 3.3 and the definition of \( \sigma \) in (3.3), we notice that, for \( j, k, m \geq 0 \),
\[
|(F_0^0)_{t \frac{j}{T} d \frac{k}{m}} (\xi', \xi_d, t)| \leq \kappa e^{-\delta^2/\epsilon T} \leq e.s.t.,
\] (3.55)

where \( \kappa \) is a constant depending on the data, and on the indices \( j, k, m \), but not on \( \epsilon \). Essential here is the fact that \( (\sigma - 1), (\sigma') \) and the higher derivatives of \( \sigma \) vanish for \( 0 < \xi_d < \delta \). Here we call \( e.s.t. \) a function (or a constant) whose norm in all Sobolev spaces \( H^s \) (and thus spaces \( C^s \)) is exponentially small with a bound of the form \( c_1 \exp(-c_2/\epsilon^s) \), \( c_1, c_2, \alpha > 0 \), for each \( s \).

Hence, we deduce from (3.55) that \( F_0^0 \) satisfies the condition (A.2) for any fixed \( J, K \) and \( M \), and, by applying Lemma A.1 for \( \Phi = \theta^0 - \varphi^0 \), we obtain
\[
|((\Theta^0 - \varphi^0)_{t \frac{j}{T} d \frac{k}{m}})_{\Omega (0, T; L^2(\Omega_{2b}))} = e.s.t.,
\] (3.56)

Now, we compute (3.6)–(3.19) and consider the difference \( \Phi = \theta^\pm - \varphi^\pm \) as the solution of (A.1) where
\[
\psi = F_0^0 (\xi', \xi_d, t) := \epsilon \{(\sigma - 1) \theta^\pm \}_{d} + \epsilon \{\frac{1}{h} (h')_0 (h^0 - \varphi^0 - \varphi^0) \}_{d}.
\] (3.57)

For the first term in the right-hand side of (3.57), as for (3.55), observing that \( (\sigma - 1) \) and the derivatives of \( \sigma \) vanish for \( 0 < \xi_d < \delta \), we use Lemma 3.4, and find
\[
\epsilon \{(\sigma - 1) \theta^\pm \}_{t \frac{j}{T} d \frac{k}{m + 1}} = e.s.t., \quad j, k, m \geq 0.
\] (3.58)

Combining this with (3.56), we conclude that \( F_0^\pm \) satisfies the condition (A.2), and hence, applying Lemma A.1 for \( \Phi = \theta^\pm - \varphi^\pm \), we obtain
\[
|((\Theta^\pm - \varphi^\pm)_{t \frac{j}{T} d \frac{k}{m}})_{\Omega (0, T; L^2(\Omega_{2b}))} = e.s.t.,
\] (3.59)

Note that, in the error analysis at lower orders \( \epsilon^0 \) and \( \epsilon^1 \) below, we will need (3.59) for \( j = 0, 0 \leq k, m \leq 2 \), or equivalently, (3.56) for \( 0 \leq j \leq 1, 0 \leq k \leq 2 \) and \( 0 \leq m \leq 3 \), which are valid under the regularity assumptions (3.17).
3.3. Convergence results at lower orders $e^0$ and $e^\frac{\theta}{2}$. Thanks to the results in Section 3.2, we are now able to make the estimates (3.14)-(3.16) more useful.

Using (3.7) and (3.10), we write

$$|R(e^0(\theta)|_{L^2(\Omega)} = |R(e^0(\theta) + R(\theta - \overline{\theta})|_{L^2(\Omega)} + |R(e^0(\overline{\theta})|_{L^2(\Omega)}).$$

(3.60)

In (3.60) and below, we use $\theta$ and $\overline{\theta}$ to denote the whole collections of $\theta^2$ and $\overline{\theta}^2$, and do not specify those used (needed) for the expressions under consideration.

By the expression (3.10) of $R^0$, we see that

$$|R^0(e^0(\theta)|_{L^2(\Omega)} \leq \kappa\epsilon|\sigma(\theta)|_{L^2(\Omega)} + \kappa\epsilon|\overline{\theta}|_{L^2(\Omega)} + \kappa\epsilon\left(\frac{1}{2}|\sigma|_{L^2(\Omega)} + \frac{1}{2}|\overline{\theta}|_{L^2(\Omega)}\right).$$

(3.61)

Observing that $\sigma$ is a tangential differential operator and that $L$ is the product of a bounded function with $\epsilon^{1/2}\partial/\partial \xi_d$, we infer from (3.55) and Lemmas 3.2 and 3.3 that

$$|R^0(e^0(\theta)|_{L^2(\Omega)} \leq \kappa\epsilon^{1+\frac{1}{2}} + \kappa\epsilon^{1+\frac{1}{2}} + e.s.t. \leq \kappa\epsilon^{1+\frac{1}{2}}.$$ (3.62)

Since, using (3.56), $|R^0(e^0(\theta)|_{L^2(\Omega)}$ is an e.s.t., the last bound is also valid for $R^0(e^0(\theta)$ and thus:

$$|R^0(e^0(\theta)|_{L^2(\Omega)} \leq \kappa\epsilon^{1+\frac{1}{2}}.$$ (3.63)

Using (3.7) and (3.12), we estimate $R^0(e^\frac{\theta}{2}(\theta)$ similarly:

$$|R^0(e^\frac{\theta}{2}(\theta)|_{L^2(\Omega)} = |R^0(e^\frac{\theta}{2}(\theta)|_{L^2(\Omega)} \leq |R^0(e^\frac{\theta}{2}(\theta)|_{L^2(\Omega)} + |R^0(e^\frac{\theta}{2}(\theta)|_{L^2(\Omega)}.$$ (3.64)

Due to the expression (3.12) of $R^0(e^\frac{\theta}{2}(\theta)$ and (3.59), we promptly see that $|R^0(e^\frac{\theta}{2}(\theta)|_{L^2(\Omega)}$ is an e.s.t. and, for $|R^0(e^\frac{\theta}{2}(\theta)|_{L^2(\Omega)}$, we have

$$|R^0(e^\frac{\theta}{2}(\theta)|_{L^2(\Omega)} \leq \kappa\epsilon|\sigma(\theta)|_{L^2(\Omega)} + \epsilon^{1+\frac{1}{2}}|L|_{L^2(\Omega)} + \epsilon^{1+\frac{1}{2}}|\overline{\theta}|_{L^2(\Omega)} + \epsilon^{1+\frac{1}{2}}|\overline{\theta}|_{L^2(\Omega)}.$$ (3.65)

Moreover, using (2.20), we notice that

$$\epsilon^{1+\frac{1}{2}}|L|_{L^2(\Omega)} \leq \epsilon^{1+\frac{1}{2}}\epsilon L_{\Omega^2}|\overline{\theta}|_{L^2(\Omega)};$$ (3.66)

hence, using (3.55), (3.56), (3.59) and Lemmas 3.2 and 3.4, (3.65) yields

$$|R^0(e^\frac{\theta}{2}(\theta)|_{L^2(\Omega)} \leq \kappa\epsilon^{1+\frac{1}{2}}.$$ (3.67)

and, finally we obtain

$$|R^0(e^\frac{\theta}{2}(\theta)|_{L^2(\Omega)} \leq \kappa\epsilon^{1+\frac{1}{2}}.$$ (3.68)

Now, by inserting (3.63) and (3.68) into (3.14)-(3.16), we obtain the convergence results at orders $e^0$ and $e^\frac{\theta}{2}$ that we summarize in the following theorem:

**Theorem 3.1.** Assume that $\partial \Omega$ is of class $C^4$, and that $u_0$ and $f$ satisfy (1.2) and the regularity assumptions (3.17). Then the remainders $w^0 = u^0 - u^0\theta$ and $w^\frac{\theta}{2} = u^\frac{\theta}{2} - u^\frac{\theta}{2}\theta$ satisfy the estimates:

$$|w^0|_{L^\infty(0, T; L^2(\Omega))} \leq \kappa\epsilon^{1+\frac{1}{2}}, \quad |w^0|_{L^2(0, T; H^1(\Omega))} \leq \kappa\epsilon^{1+\frac{1}{2}},$$ (3.69)

and

$$|w^\frac{\theta}{2}|_{L^\infty(0, T; L^2(\Omega))} \leq \kappa\epsilon^{1+\frac{1}{2}}, \quad |w^\frac{\theta}{2}|_{L^2(0, T; H^1(\Omega))} \leq \kappa\epsilon^{1+\frac{1}{2}}.$$ (3.70)
where \( \kappa_T \) is a constant depending on \( T \) and the other data, but independent of \( \epsilon \). Moreover, \( \mathbf{w}_r^\epsilon := u^\epsilon - u^0 - \mathbf{\theta}^0 \) and \( \mathbf{w}_\theta^\epsilon := u^\epsilon - u^0 - \epsilon \frac{1}{2} \mathbf{\theta}^2 \) also satisfy the estimates (3.69) and (3.70) with \( u_0^\epsilon, \mathbf{w}_r^\epsilon \) replaced by \( \mathbf{w}_r^\epsilon, \mathbf{w}_\theta^\epsilon \) respectively.

**Remark 3.1.** Note that, for (3.69) in Theorem 3.1, it is enough to assume that \( \partial \Omega \) is of class \( C^2 \).

From Theorem 3.1 and Remark 3.1, we promptly obtain the following convergence result of \( u^\epsilon \) to \( u^0 \):

**Corollary 3.1.** If \( \partial \Omega \) is of class \( C^2 \), and if \( u_0 \) and \( f \) satisfy (1.2) and the regularity assumptions (3.17), then we have

\[
|u^\epsilon - u^0|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa_T \epsilon^{\frac{1}{2}}.
\]

**Proof.** To show (3.71), we use (3.8) and write

\[
|u^\epsilon - u^0|_{L^\infty(0,T;L^2(\Omega))} \leq |u_\epsilon^0|_{L^\infty(0,T;L^2(\Omega))} + |\theta^0|_{L^\infty(0,T;L^2(\Omega))}
\]

(\text{thanks to (3.7)})

\[
\leq |u_\epsilon^0|_{L^\infty(0,T;L^2(\Omega))} + |\theta^0|_{L^\infty(0,T;L^2(\Omega_\epsilon))}
\]

(3.72)

\[
\leq |u_\epsilon^0|_{L^\infty(0,T;L^2(\Omega))} + |\theta^0 - \tilde{\theta}^0|_{L^\infty(0,T;L^2(\Omega_\epsilon))}
\]

hence (3.71) follows using (3.56), (3.69) and Lemma 3.2.

**Remark 3.2.** Theorem 3.1 shows that

\[
u^\epsilon \simeq u^0 + \mathbf{\theta}^0, \quad \mathbf{\theta}^0 \simeq \Theta(u^0) \exp \left( - \frac{(\text{dist}(x, \partial \Omega))^2}{c \epsilon t} \right)
\]

(3.73)

where \( \Theta \) is a function of \( u^0 \), and \( c > 0 \) depends on the function space considered.

**3.4. Existence and uniqueness of a solution for (3.2).** In this section, we briefly establish the existence and uniqueness of a solution for the system (3.2). Firstly, to make the boundary condition homogeneous, we reuse the same cut-off function \( \sigma(\xi_d) \) from (3.3), and consider the function

\[
\varphi(\xi', \xi_d, t) := \varphi^0(\xi', \xi_d, t) + \sigma(\xi_d)u^0(\xi', \xi_d, t).
\]

(3.74)

The function \( \varphi \) satisfies the following system:

\[
\left\{ \begin{array}{l}
\frac{\partial \varphi}{\partial t} - \frac{\partial}{\partial \xi_d} \left( \sigma(\xi_d) \frac{\partial \varphi}{\partial \xi_d} \right) = \tilde{g}(\xi', \xi_d, t), \text{ in } \Omega, \\
\varphi = 0, \text{ at } \xi_d = 0, \\
\varphi|_{t=0} = \varphi_0(\xi', \xi_d),
\end{array} \right.
\]

(3.75)

where

\[
\tilde{g}(\xi', \xi_d, t) := \sigma \frac{\partial u^0}{\partial t} - \frac{\partial}{\partial \xi_d} \left( \sigma \frac{\partial (\sigma u^0)}{\partial \xi_d} \right),
\]

(3.76)

\[
\varphi_0(\xi', \xi_d) := \sigma u_0.
\]
We set
\[ H = L^2(Ω_{2δ,ξ}), \]
\[ W = \left\{ v \in L^2(Ω_{2δ,ξ}) \mid \sqrt{σ} \frac{∂v}{∂ξ_d} \in L^2(Ω_{2δ,ξ}) \text{ and } v = 0 \text{ at } ξ_d = 0 \right\}. \] (3.77)

Note that, if \( v \) and \( \sqrt{σ} \frac{∂v}{∂ξ_d} \) belong to \( L^2(Ω_{2δ,ξ}) \) (that is \( \frac{∂v}{∂ξ_d} \in L^2(Ω_{δ,ξ}) \)), the trace of \( v \), at \( ξ_d = 0 \), is well-defined and belongs to \( L^2(ω′) \).

Then \( ϕ \) is looked for as a function from \((0,T]\) into \( W \) satisfying the following weak form of (3.75), that is
\[
\begin{align*}
\frac{d}{dt}(ϕ, \tilde{ϕ})_H + ε \int_{Ω_{2δ,ξ}} σ(ξ_d) \frac{∂ϕ}{∂ξ_d} \frac{∂\tilde{ϕ}}{∂ξ_d} dξ_d dξ = (\tilde{g}, \tilde{ϕ})_H, \quad ∀ \tilde{ϕ} ∈ W, \\
ϕ(0) = ϕ_0.
\end{align*}
\] (3.78)

The problem (3.78) is a standard linear evolution problem for which we easily obtain the existence and uniqueness of a solution \( ϕ \) such that
\[
ϕ ∈ L^2(0,T;W) \cap C([0,T];H); \quad (3.79)
\]
see e.g. [23] and [31].

Further regularity properties of \( ϕ \) (and thus of \( θ^0 \)) are obtained by inspection of (3.2) in a routine way.

4. Analysis at general orders \( ε^N \) and \( ε^{N+\frac{1}{2}} \), \( N ≥ 0 \).

4.1. Correctors \( θ^N \) and \( θ^{N+\frac{1}{2}} \). To derive an equation for \( θ^N \), \( N ≥ 0 \), we collect all terms of order \( ε^N \) in (2.22) and find
\[
\frac{∂θ^N}{∂t} - \frac{∂^2θ^N}{∂ξ_d^2} = f^N_ε, \quad (4.1)
\]
where
\[
f^N_ε := \sum_{j=0}^{2N-2} ε^{-\frac{j}{2}} ξ_d^j S_d^2 θ^{N-1-\frac{j}{2}} + \sum_{j=0}^{2N-1} ε^{-\frac{j}{2}} ξ_d^j L_d^2 θ^{N-\frac{j}{2}}. \quad (4.2)
\]
We modify (4.1) as for the lower orders and impose the proper boundary and initial conditions. Then, as a result, we obtain (propose) the following equation for \( θ^N \):
\[
\begin{align*}
\partial_t θ^N - ε \frac{∂}{∂ξ_d} \left( σ(ξ_d) \frac{∂θ^N}{∂ξ_d} \right) &= f^N_ε, &\text{in } Ω, \\
θ^N &= -u^N, &\text{at } ξ_d = 0, \tag{4.3} \\
θ^N|_{t=0} &= 0.
\end{align*}
\]

For \( θ^{N+\frac{1}{2}} \), we collect all terms of order \( ε^{N+\frac{1}{2}} \) in (2.22) and find:
\[
\frac{∂θ^{N+\frac{1}{2}}}{∂t} - \frac{∂^2θ^{N+\frac{1}{2}}}{∂ξ_d^2} = f^{N+\frac{1}{2}}_ε, \quad (4.4)
\]
where
\[
f^{N+\frac{1}{2}}_ε := \sum_{j=0}^{2N-1} ε^{-\frac{j}{2}} ξ_d^j S_d^2 θ^{N-1-\frac{j}{2}} + \sum_{j=0}^{2N} ε^{-\frac{j}{2}} ξ_d^j L_d^2 θ^{N-\frac{j}{2}}; \quad (4.5)
\]
after we modify (4.4) and impose the boundary and initial conditions, we obtain the following proposed equation for $\theta^{N+\frac{1}{2}}$:

$$
\begin{aligned}
\begin{cases}
\partial_t \theta^{N+\frac{1}{2}} - \epsilon \frac{\partial}{\partial \xi_d} \left\{ \sigma(\xi_d) \frac{\partial \theta^{N+\frac{1}{2}}}{\partial \xi_d} \right\} = f^{N+\frac{1}{2}}, & \text{in } \Omega, \\
\theta^{N+\frac{1}{2}} = 0, & \text{on } \xi_d = 0, \\
\theta^{N+\frac{1}{2}} \big|_{t=0} = 0.
\end{cases}
\end{aligned}
$$

(4.6)

As we have seen in the case when $N = 0$, the problems (4.3) and (4.6) are well-posed for any $N \geq 0$. Moreover, on $\Omega \setminus \Omega_{2\delta}$, $(\xi_d \geq 2\delta)$, we notice that

$$
\frac{\partial^k \theta^N}{\partial \xi_i^k} = \frac{\partial^k \theta^{N+\frac{1}{2}}}{\partial \xi_i^k} = 0, \quad k \geq 0, \quad 1 \leq i \leq d.
$$

(4.7)

To perform the error analysis at orders $\epsilon^N$ and $\epsilon^{N+\frac{1}{2}}$, we consider the remainders:

$$
\begin{aligned}
w^N_e &= u^e - \sum_{j=0}^{N} \left( \epsilon^j u^j + \epsilon^j \theta^j \right) - \sum_{j=0}^{N-1} \epsilon^{j+\frac{1}{2}} \theta^{j+\frac{1}{2}}, \\
w^{N+\frac{1}{2}}_e &= u^e - \sum_{j=0}^{N} \left( \epsilon^j u^j + \epsilon^j \theta^j + \epsilon^{j+\frac{1}{2}} \theta^{j+\frac{1}{2}} \right).
\end{aligned}
$$

(4.8)

We find the equation for $w^N_e$, by performing (1.1) $- \sum_{j=0}^{N} \epsilon^j \left( (2.11) + (4.3) \right)$ $- \sum_{j=0}^{N-1} \epsilon^{j+\frac{1}{2}} (4.6)$, and, for the equation of $w^{N+\frac{1}{2}}_e$, we compute (1.1) $- \sum_{j=0}^{N} \epsilon^j (2.11) + \epsilon^j (4.3) + \epsilon^{j+\frac{1}{2}} (4.6)$). Then we obtain, for $r = 0, 1$,

$$
\begin{aligned}
\begin{cases}
\frac{\partial_t w^{N+\frac{1}{2}}_e - \epsilon \Delta w^{N+\frac{1}{2}}_e}{\partial \xi_d} = \epsilon^{N+1} \Delta u^N + R^{N+\frac{1}{2}}_e (\theta), \\
w^{N+\frac{1}{2}}_e (\xi', \xi_d = 0, t) = 0, \\
w^{N+\frac{1}{2}}_e \big|_{t=0} = 0,
\end{cases}
\end{aligned}
$$

(4.9)

where

$$
R^N_e (\theta) = \sum_{k=0}^{2N-2} \epsilon^{\frac{k}{2}+1} \left\{ S - \sum_{j=0}^{2N-k-1} \xi_d (L^{\frac{1}{2}}) \theta^{\frac{k}{2}} + \epsilon^{N+\frac{k}{2}} S^{\theta+N-\frac{1}{2}} + \epsilon^{N+1} S^{\theta+N} \right\}
$$

$$
+ \sum_{k=0}^{2N-1} \epsilon^{\frac{k}{2}+1} \left\{ L - \sum_{j=0}^{2N-k-1} \xi_d (L^{\frac{1}{2}}) \theta^{\frac{k}{2}} + \epsilon^{N+\frac{k}{2}} L^{\theta+N-\frac{1}{2}} + \epsilon \sum_{k=0}^{2N} \xi_d (L^{\frac{1}{2}}) \frac{\partial}{\partial \xi_d} \left\{ (1 - \sigma) \frac{\partial \theta^{\frac{k}{2}}}{\partial \xi_d} \right\} \right\},
$$

(4.10)

and

$$
R^{N+\frac{1}{2}}_e (\theta) = \sum_{k=0}^{2N-1} \epsilon^{\frac{k}{2}+1} \left\{ S - \sum_{j=0}^{2N-k-1} \xi_d (L^{\frac{1}{2}}) \theta^{\frac{k}{2}} + \epsilon^{N+1} S^{\theta+N} + \epsilon^{N+\frac{k}{2}} S^{\theta+N+\frac{1}{2}} \right\}
$$

$$
+ \sum_{k=0}^{2N} \epsilon^{\frac{k}{2}+1} \left\{ L - \sum_{j=0}^{2N-k-1} \xi_d (L^{\frac{1}{2}}) \theta^{\frac{k}{2}} + \epsilon^{N+1} L^{\theta+N+\frac{1}{2}} + \epsilon \sum_{k=0}^{2N+1} \xi_d (L^{\frac{1}{2}}) \frac{\partial}{\partial \xi_d} \left\{ (1 - \sigma) \frac{\partial \theta^{\frac{k}{2}}}{\partial \xi_d} \right\} \right\}.
$$

(4.11)
Using the analogs of (3.13)-(3.15), we find the following estimates for $w_e^{N+\frac{r}{2}}$, $r = 0, 1$:

$$
|w_e^{N+\frac{r}{2}}(t)|^2_{L^2(\Omega)} \leq \kappa \int_0^T \left( e^{2N+2}\|\Delta u^N\|^2_{L^2(\Omega)} + \|R_e^{N+\frac{r}{2}}(\theta)\|^2_{L^2(\Omega)} \right) dt,
$$

$$
\int_0^T |\nabla w_e^{N+\frac{r}{2}}|^2_{L^2(\Omega)} dt \leq \epsilon^{-1} \int_0^T \left( |w_e^{N+\frac{r}{2}}|^2_{L^2(\Omega)} + \frac{1}{2} e^{2N+2}\|\Delta u^N\|^2_{L^2(\Omega)} + \frac{1}{2} \|R_e^{N+\frac{r}{2}}(\theta)\|^2_{L^2(\Omega)} \right) dt;
$$

(4.12)

note that, using (2.13) with $j = N$, the estimates (4.12) are valid when

$$
u_0 \in H^{2N+2}(\Omega), \quad f \in L^\infty(0, T; H^{2N+2}(\Omega)).
$$

(4.13)

### 4.2. Approximations $\overline{\theta}^N$ and $\overline{\theta}^{N+\frac{r}{2}}$ of the correctors $\theta^N$ and $\theta^{N+\frac{r}{2}}$, $N \geq 0$.

To make the estimates (4.12) more useful, we introduce the approximations $\overline{\theta}^N$ and $\overline{\theta}^{N+\frac{r}{2}}$ of the correctors $\theta^N$ and $\theta^{N+\frac{r}{2}}$, defined as the solutions of the following heat equations on $\mathbb{R}_+$:

$$
\begin{align*}
\partial_t \overline{\theta}^N - \frac{\partial^2 \overline{\theta}^N}{\partial \xi_d^2} &= \overline{\mathcal{F}}^N_e(\xi', \overline{\xi}_d, t), \quad \xi' \in \omega', \overline{\xi}_d > 0, \\
\overline{\theta}^N &= -u^N, \text{ at } \overline{\xi}_d = 0, \\
\overline{\theta}^N(\xi', \overline{\xi}_d, t) &\to 0 \text{ as } \overline{\xi}_d \to \infty, \\
\overline{\theta}^N \big|_{t=0} &= 0,
\end{align*}
$$

(4.14)

where

$$
\overline{\mathcal{F}}^N_e := \sum_{j=0}^{2N-2} e^{-\frac{j}{2}\xi_d^2} S_j \overline{\theta}^{N-1-\frac{j}{2}} + \sum_{j=0}^{2N-1} e^{-\frac{j}{2}\xi_d^2} L^{\frac{j}{2}} \overline{\theta}^{N-\frac{j}{2}},
$$

(4.15)

and

$$
\begin{align*}
\partial_t \overline{\theta}^{N+\frac{r}{2}} - \frac{\partial^2 \overline{\theta}^{N+\frac{r}{2}}}{\partial \xi_d^2} &= \overline{\mathcal{F}}^{N+\frac{r}{2}}_e(\xi', \overline{\xi}_d, t), \quad \xi' \in \omega', \overline{\xi}_d > 0, \\
\overline{\theta}^{N+\frac{r}{2}} &= 0, \text{ at } \overline{\xi}_d = 0, \\
\overline{\theta}^{N+\frac{r}{2}}(\xi', \overline{\xi}_d, t) &\to 0 \text{ as } \overline{\xi}_d \to \infty, \\
\overline{\theta}^{N+\frac{r}{2}} \big|_{t=0} &= 0,
\end{align*}
$$

(4.16)

where

$$
\overline{\mathcal{F}}^{N+\frac{r}{2}}_e := \sum_{j=0}^{2N-1} e^{-\frac{j}{2}\xi_d^2} S_j \overline{\theta}^{N+\frac{r}{2}-\frac{j}{2}} + \sum_{j=0}^N e^{-\frac{j}{2}\xi_d^2} L^{\frac{j}{2}} \overline{\theta}^{N+\frac{r}{2}}.
$$

(4.17)
Due to the linearity of the equation (4.14), we find the solution \( \overline{\theta}^N \), \( N \geq 1 \) in the form:

\[
\begin{align*}
\overline{\theta}^N &= \overline{\theta}^N_h + \overline{\theta}^N_p, \\
\overline{\theta}^N_h &= (3.20) \text{ with } u^0 \text{ replaced by } u^N, \quad (4.18) \\
\overline{\theta}^N_p &= (3.21) \text{ with } \overline{f}^n_t \text{ replaced by } \overline{f}^n_t.
\end{align*}
\]

from (2.11), we see that the \( u^j \), \( j \geq 1 \) satisfy the initial conditions:

\[
|u^j|_{t=0} = 0, \quad j \geq 1.
\]

We also find the solution \( \overline{\theta}^{N+\frac{1}{2}} \), \( N \geq 1 \) of (4.16) in the form:

\[
\overline{\theta}^{N+\frac{1}{2}} = (3.21) \text{ with } \overline{f}^n_t \text{ replaced by } \overline{f}^{N+\frac{1}{2}}_t. \quad (4.20)
\]

Now, we prove the following pointwise estimates for \( \overline{\theta}^N \) and \( \overline{\theta}^{N+\frac{1}{2}} \):

**Lemma 4.1.** Assume that, for \( k \geq 0 \), \( \partial \Omega \) is of class \( C^{2N+2+k} \), and that \( u_0 \) and \( f \) belong to \( H^{2N+k}(\Omega) \) and \( L^\infty(0, T; H^{2N+k}(\Omega)) \) respectively. Then, the approximate correctors \( \overline{\theta}^{N+\frac{1}{2}}, r = 0, 1 \), in (4.18) and (4.20), satisfy the following pointwise estimates: for \((t, \xi) \in (0, T) \times \Omega_{25, \xi} \), and for \( m = 0, 1 \), we have

\[
|\overline{\theta}^{N+\frac{1}{2}}|_{t+k,m}(\xi', \xi_d, t) \leq \kappa_{T,k,m} \epsilon^{-\frac{\xi_d^2}{2m+42N+1+\epsilon\xi}} \int_0^t \exp \left( -\frac{\xi_d^2}{2m+42N+1+\epsilon\xi} \right) ds.
\]

Moreover, for \((t, \xi) \in (0, T) \times \Omega_{25, \xi} \), and for \( j = 0 \) and \( m \geq 2 \), and \( j \geq 1 \) and \( m \geq 0 \), we have

\[
|\overline{\theta}^{N+\frac{1}{2}}|_{t+k,m}(\xi', \xi_d, t) \leq \kappa_{T,j,k,m} \epsilon^{-j-m} \int_0^t \left( 1 + s^{-2j-m-\frac{1}{2}} \right) \exp \left( -\frac{\xi_d^2}{42N+r+1+\epsilon\xi} \right) ds,
\]

where \( \kappa_{T,k,m} \) and \( \kappa_{T,j,k,m} \) are constants depending on \( T, j, k, m \), and the other data, but independent of \( \epsilon \).

**Proof.** We proceed by induction on \( N \). Thanks to Lemmas 3.3 and 3.4, we first see that (4.21) and (4.22) hold true when \( N \) is equal to 0. If we assume that (4.21) and (4.22) are valid when \( N \leq k \) and \( r = 0, 1 \), then, using (4.15) and the inductive assumption, we notice that

\[
\left| \overline{f}^{k+1} \right|_{t+k}(\xi', \xi_d, t) \leq \kappa_{T,k} \left\{ 1 + \left( \epsilon^{-\frac{\xi_d}{2}} \xi_d \right)^{2k+1} \right\} \exp \left( -4^{-2k} - \frac{\xi_d^2}{2\epsilon\xi} \right).
\]

Then, using the same arguments as in the proofs of Lemmas 3.3 and 3.4, one can verify that (4.21) and (4.22) hold true when \( N = k+1 \) and \( r = 0 \). Moreover, using the results (4.21) and (4.22) for \( \overline{f}^{k+1} \) with the inductive assumption, and repeating the proof of Lemma 3.4, one can also verify that (4.21) and (4.22) hold true for \( \overline{\theta}^{N+3/2} \), and the proof is complete.

Now, to compare \( \overline{\theta}^N \) and \( \overline{\theta}^{N+\frac{1}{2}} \) for each \( 0 \leq n \leq 2N+1 \), we use (4.14) and (4.16), and find that the differences \( \overline{\theta}^N - \overline{\theta}^{N+\frac{1}{2}} \) satisfy the system (A.1) in the Appendix where \( \psi = \psi_{n/2} \) is given by

\[
\psi(\xi', \xi_d, t) = (f^N - \overline{f}^N_t) + \epsilon \sum_{r=0}^n \epsilon^r \left( (\sigma - 1) \overline{\theta}^N \right) d_r.
\]
As for the lower order cases, thanks to Lemma 4.1 and the fact that \((\sigma - 1)\) and the derivatives of \(\sigma\) vanish for \(0 < \xi_d < \delta\), we find

\[
\sum_{n=0}^{2N+1} \left| \left( \sigma - 1 \right) \theta_d^{\frac{\delta}{2}} \right|_{L^2(\Omega^{2N+2})} = e.s.t., \ j, k, m \geq 0.
\] (4.25)

Then, using (3.56) to start induction on \(0 \leq n \leq 2N + 1\), and recursively using Lemma A.1 for \(\Phi = \theta_d^{\frac{\delta}{2}} - \theta_d^{\frac{\delta}{2}}\) with (4.24) and (4.25), we easily prove, for \(0 \leq n \leq 2N + 1\), that

\[
\left| \left( \theta_d^{\frac{\delta}{2}} - \theta_d^{\frac{\delta}{2}} \right) \right|_{L^2(0,T;L^2(\Omega^{2N}))} = e.s.t., \ j, k, m \geq 0.
\] (4.26)

Note that (4.26) is valid for all \(0 \leq n \leq 2N + 1\), under the regularity assumptions (4.13).

**4.3. Convergence results at general orders \(\epsilon^N\) and \(\epsilon^{N+\frac{1}{2}}\).** Due to the results in Section 4.2, we are now able to make the estimates (4.12) useful. Thanks to (4.7) and (4.26), we infer from the expressions (4.10) and (4.11) of \(R_{\epsilon^{N+\frac{3}{2}}}\), \(r = 0, 1, 2\), that

\[
|R_{\epsilon^{N+\frac{3}{2}}} (\theta)|_{L^2(\Omega^{2N+2})} = |R_{\epsilon^{N+\frac{3}{2}}} (\theta)|_{L^2(\Omega^{2N+2})} \\
\leq |R_{\epsilon^{N+\frac{3}{2}}} (\theta - \bar{\theta})|_{L^2(\Omega^{2N+2})} + |R_{\epsilon^{N+\frac{3}{2}}} (\bar{\theta})|_{L^2(\Omega^{2N+2})} + e.s.t.
\] (4.27)

By the expression (4.10) of \(R_{\epsilon^N}\), we write

\[
|R_{\epsilon^N} (\bar{\theta})|_{L^2(\Omega^{2N+2})} \\
\leq \kappa \left\{ \sum_{k=0}^{2N-2} \epsilon^{\frac{3}{2}+\frac{k}{2}} \left\{ S - \sum_{j=0}^{2N-k-2} \xi_d^{\frac{k}{2}} S_j^{\frac{k}{2}} \right\} \left| \theta_d^{\frac{\delta}{2}} \right|_{L^2(\Omega^{2N+2})} + \epsilon^{N+\frac{1}{2}} \left| S \theta_d^{\frac{1}{2}} \right|_{L^2(\Omega^{2N+2})} + e^{N+\frac{1}{2}} \left| S \theta_d^{\frac{1}{2}} \right|_{L^2(\Omega^{2N+2})} \right. \\
+ \epsilon^{N+\frac{1}{2}} \left| S \theta_d^{\frac{1}{2}} \right|_{L^2(\Omega^{2N+2})} + \left| \theta_d^{\frac{1}{2}} \right|_{L^2(\Omega^{2N+2})} \right\} + e.s.t.
\] (4.28)

Recalling that the \(S\) and \(S_{j/2}, j \geq 0\) are tangential differential operators, and that the \(L\) and \(L_{j/2}, j \geq 0\) are the products of a bounded function with \(\epsilon^{1/2} \partial / \partial \xi_d\), we
use (4.26) and Lemmas 3.2 and 4.1. As a result, we can bound $|R_e^N(\vec{\theta})|$ in (4.28):

$$
\left| R_e^N(\vec{\theta}) \right|_{L^2(\Omega_{2\delta})} \leq \kappa \left\{ \epsilon^{N+\frac{5}{2}} \right\} \left[ \sum_{k=0}^{N-2} \left( \epsilon^{-\frac{2}{5}} \xi_d \right)^{2N-2k} \left| T \right|_{L^2(\Omega_{2\delta})} + \epsilon^{N+\frac{5}{2}} \left| \theta \right|_{L^2(\Omega_{2\delta})} \right]
$$

$$
\left. + \epsilon^{N+1} \left| S \right|_{L^2(\Omega_{2\delta})} + \epsilon^{N+\frac{5}{2}} \sum_{k=0}^{2N-1} \left( \epsilon^{-\frac{2}{5}} \xi_d \right)^{2N-2k} \left| S \right|_{L^2(\Omega_{2\delta})} \right]
$$

$$
\leq \kappa_T \left( \epsilon^{N+\frac{5}{2}} + \epsilon^{N+1} \right)
$$

(4.29)

Using exactly the same method as for $R_e^N(\vec{\theta})$, we also estimate $R_e^{N+\frac{5}{2}}(\vec{\theta})$ in (4.11):

$$
\left| R_e^{N+\frac{5}{2}}(\vec{\theta}) \right|_{L^2(\Omega)} \leq \kappa \left\{ \epsilon^{N+1} \right\} \left[ \sum_{k=0}^{N-2} \left( \epsilon^{-\frac{2}{5}} \xi_d \right)^{2N-2k} \left| T \right|_{L^2(\Omega_{2\delta})} + \epsilon^{N+1} \left| \theta \right|_{L^2(\Omega_{2\delta})} \right]
$$

$$
\left. + \epsilon^{N+\frac{5}{2}} \left| S \right|_{L^2(\Omega_{2\delta})} + \epsilon^{N+1} \sum_{k=0}^{2N} \left( \epsilon^{-\frac{2}{5}} \xi_d \right)^{2N-2k} \left| S \right|_{L^2(\Omega_{2\delta})} \right]
$$

$$
\leq \kappa_T \epsilon^{N+\frac{5}{2}}.
$$

(4.30)

Thanks to (4.27)-(4.30), we finally obtain

$$
\left| R_e^N(\theta) \right|_{L^2(\Omega_{2\delta})} \leq \kappa_T \epsilon^{N+\frac{5}{2}}, \quad \left| R_e^{N+\frac{5}{2}}(\theta) \right|_{L^2(\Omega)} \leq \kappa_T \epsilon^{N+\frac{5}{2}}.
$$

(4.31)

Now, applying (4.31) to (4.12), we deduce the following convergence results at general orders $\epsilon^{N+\frac{5}{2}}, N \geq 0, r = 0, 1:

**Theorem 4.1.** If $\partial \Omega$ is of class $C^{2N+4}$, and if $u_0$ and $f$ satisfy (1.2) and the regularity assumptions (4.13), then the remainders $w^N_e$ and $w^{N+\frac{5}{2}}_e$ defined in (4.8) satisfy the estimates: for $N \geq 0$,

$$
\left| w^N_e \right|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa_T \epsilon^{N+\frac{5}{2}}, \quad \left| w^{N+\frac{1}{2}}_e \right|_{L^2(0,T;H^1(\Omega))} \leq \kappa_T \epsilon^{N+\frac{5}{2}},
$$

(4.32)

and

$$
\left| w^{N+\frac{5}{2}}_e \right|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa_T \epsilon^{N+1}, \quad \left| w^{N+\frac{5}{2}}_e \right|_{L^2(0,T;H^1(\Omega))} \leq \kappa_T \epsilon^{N+\frac{5}{2}},
$$

(4.33)

where $\kappa_T$ is a constant depending on $T$ and the other data, but independent of $\epsilon$. Moreover, $\overline{w}^N_e := u^e - \sum_{j=0}^{N-1} (\epsilon^j w^j + \epsilon^{j+\frac{5}{2}} \overline{\theta}^{j+\frac{5}{2}})$ and $\overline{w}^{N+\frac{5}{2}}_e := u^e - \sum_{j=0}^{N} (\epsilon^j w^j + \epsilon^{j+\frac{5}{2}} \overline{\theta}^{j+\frac{5}{2}})$ also satisfy the estimates (4.32) and (4.33) with $w^N_e, w^{N+\frac{5}{2}}_e$ replaced by $\overline{w}^N_e, \overline{w}^{N+\frac{5}{2}}_e$ respectively.
Remark 4.1. As in the lower order case (see Remark 3.1), for (4.32) in Theorem 4.1, it is enough to assume that $\partial\Omega$ is of class $C^{2N+2}$.

Remark 4.2. In this article, the time $T$ is assumed to be fixed and independent of $\epsilon$. However, the estimates in Theorems 3.1 and 4.1 are valid when $T = T_\epsilon < O(1/\epsilon)$.

Remark 4.3. For simplicity, here we consider the Laplacian as the elliptic operator in the problem (1.1) and also use an orthogonal curvilinear system. One can generalize this work, using any boundary fitting (non-orthogonal) coordinates, to study the boundary layer of more general elliptic equations. We believe that many of the techniques used in this article (above and below in the Appendix) will help for such problems.

Appendix. In this appendix, to compare, in $\Omega_{2\delta}$, the correctors $\theta^j$ and the approximate correctors $\overline{\theta}^j$, $j \geq 0$, we introduce a system of the form:

\[
\begin{align*}
\Phi_t - \epsilon \{\sigma \Phi_d\}_d &= \psi(\xi', \xi_d, t), \quad \text{in } \Omega_{2\delta} \times (0, T), \\
\Phi &= 0, \quad \text{at } \xi_d = 0, \\
\Phi|_{t=0} &= 0,
\end{align*}
\]  

(A.1)

and assume that $\psi$ satisfies, for $0 \leq j \leq J$, $0 \leq k \leq K$ and $0 \leq m \leq M$,

\[
|\psi_{tj+k} |_{L^\infty(\Omega_{2\delta})} = e.s.t.. \tag{A.2}
\]

Now we state and prove the following lemma:

Lemma A.1. Under the assumption (A.2), the solution $\Phi$ of the system (A.1) satisfies the following estimates: for $0 \leq j \leq J - 1$, $0 \leq k \leq K$ and $0 \leq m \leq M - 1$, such that $2j + m \leq \min(2J - 2, M)$,

\[
|\Phi_{tj+k} |_{L^\infty(0, T; L^2(\Omega_{2\delta}))} = e.s.t., \quad |\sqrt{\sigma} \Phi_{tj+k} |_{L^2(0, T; L^2(\Omega_{2\delta}))} = e.s.t.. \tag{A.3}
\]

Proof. We first consider $\Phi_{tj+k}$: we need to find the initial condition of $\Phi_{tj+k}$ at $t = 0$. By differentiating (A.1) $j$-times in $t$ and $k$-times in $\tau$, we find

\[
\Phi_{tj+k+1} = \epsilon \sigma' \Phi_{tj+k} + \epsilon \sigma \Phi_{tj+k} d^2 + \psi_{tj+k}. \tag{A.4}
\]

When $j = 0$, at $t = 0$, using (A.1) and (A.2), we find

\[
\Phi_{tj+k} |_{t=0} = \epsilon \sigma' \Phi_{tj+k} |_{t=0} + \epsilon \sigma \Phi_{tj+k} d^2 |_{t=0} + \psi_{tj+k} |_{t=0} = e.s.t.. \tag{A.5}
\]

When $j = 1$, at $t = 0$, using (A.2) and (A.5), (A.4) yields

\[
\Phi_{tj+k} |_{t=0} = \epsilon \sigma' \Phi_{tj+k} |_{t=0} + \epsilon \sigma \Phi_{tj+k} d^2 |_{t=0} + \psi_{tj+k} |_{t=0} = e.s.t.. \tag{A.6}
\]

More generally, using (A.4)-(A.6), one can recursively verify that

\[
\Phi_{tj+k} |_{t=0} = \left( \begin{array}{c}
\text{a linear combination of derivatives of } \psi \text{ in } t, \tau \text{ and } \xi_d, \\
\text{taken at } t = 0
\end{array} \right). \tag{A.7}
\]

The derivatives of $\psi$ involved in (A.7) are at most of order $j$ in $t$, $k$ in $\tau$, and $2j$ in $\xi_d$. Hence, by (A.2), we find

\[
\Phi_{tj+k} |_{t=0} = e.s.t., \quad 0 \leq j \leq \min(M/2, J - 1), \quad 0 \leq k \leq K. \tag{A.8}
\]
Thanks to (A.8), by differentiating (A.1), we find the equation for $\Phi_{\ell_j\tau^k}$: for $0 \leq j \leq \min(M/2 + 1, J)$, $0 \leq k \leq K$,

$$\begin{cases}
\Phi_{\ell_j+1\tau^k} - \epsilon \sigma \Phi_{\ell_{j+1}\tau^{k+d}} = \psi_{\ell_j\tau^k}(\xi', \xi_d, t), & \text{in } \Omega_{2\delta, \xi} \times (0, T), \\
\Phi_{\ell_j\tau^k} = 0, & \text{at } \xi_d = 0, \\
\Phi_{\ell_j\tau^k} \big|_{t=0} = e.s.t.,
\end{cases}$$

(A.9)

Multiplying (A.9) by $\Phi_{\ell_j\tau^k}$ and integrating it over $\Omega_{2\delta}$, we find

$$\frac{1}{2} \frac{d}{dt} \|\Phi_{\ell_j\tau^k}\|_{L^2(\Omega_{2\delta})}^2 - \epsilon \int_{\Omega_{2\delta, \xi}} (\sigma \Phi_{\ell_{j+1}\tau^{k+d}} h d\xi_d \xi') = (\psi_{\ell_j\tau^k}, \Phi_{\ell_j\tau^k})_{L^2(\Omega_{2\delta})};$$

(A.10)

Due to (A.9) and the definition of $\sigma$ in (3.3), we integrate by parts the second term in the left-hand side of (A.10):

$$\epsilon \int_{\Omega_{2\delta, \xi}} \sigma \Phi_{\ell_{j+1}\tau^{k+d}} (\Phi_{\ell_j\tau^k} h') d\xi_d \xi';$$

(A.11)

and

$$\epsilon \left| (\sqrt{\sigma} \Phi_{\ell_{j+1}\tau^{k+d}})_{L^2(\Omega_{2\delta})} \right| \leq \frac{\epsilon}{2} \left| \sqrt{\sigma} \Phi_{\ell_{j+1}\tau^{k+d}} \right|_{L^2(\Omega_{2\delta})}^2 + \kappa \left| \Phi_{\ell_{j+1}\tau^{k+d}} \right|_{L^2(\Omega_{2\delta})}^2.$$  

(A.12)

Using (A.2), (A.11) and (A.12), (A.10) yields

$$\frac{d}{dt} \|\Phi_{\ell_j\tau^k}\|_{L^2(\Omega_{2\delta})}^2 + \epsilon \sqrt{\sigma} \Phi_{\ell_{j+1}\tau^{k+d}} \left| L^2(\Omega_{2\delta}) \right| \leq \kappa \left| \Phi_{\ell_{j+1}\tau^{k+d}} \right|_{L^2(\Omega_{2\delta})}^2 + e.s.t.,$$

(A.13)

and, thanks to (A.9) and the Gronwall inequality, we obtain, for $0 \leq j \leq \min(M/2 + 1, J)$, $0 \leq k \leq K$,

$$\|\Phi_{\ell_j\tau^k}\|_{L^\infty(0, T; L^2(\Omega_{2\delta}))} = e.s.t., \quad |\sqrt{\sigma} \Phi_{\ell_{j+1}\tau^{k+d}}|_{L^2(0, T; L^2(\Omega_{2\delta}))} = e.s.t.,$$

(A.14)

which coincide with (A.3) for $m = 0$.

Secondly, we consider $\Phi_{\ell_{j+1}\tau^{k,d}}$ to find the initial condition of $\Phi_{\ell_{j+1}\tau^{k,d}}$ at $t = 0$, we differentiate (A.9) $m$-times in $\xi_d$, and write

$$\Phi_{\ell_{j+1}\tau^{k,d}} = \epsilon (\sigma \Phi_{\ell_{j+1}\tau^{k,d}} d^{m+1} + \psi_{\ell_{j+1}\tau^{k,d}}).$$

(A.15)

When $j = 0$, at $t = 0$, we infer from (A.1),

$$\Phi_{\ell_{j}\tau^{k,d}} \big|_{t=0} = \epsilon \sum_{r=0}^{m+1} (C(m+1, r) \sigma_d \Phi_{\ell_{r}\tau^{k,d}+2-r} \big|_{t=0}) + \psi_{\ell_{r}\tau^{k,d}} \big|_{t=0} = e.s.t.,$$

(A.16)

where $C(m+1, r) = (m+1)! / ((m+1-r)!r!)$.

When $j = 1$, at $t = 0$, using (A.15) and (A.16), we notice

$$\Phi_{\ell_{j}\tau^{k,d}} \big|_{t=0} = \epsilon \sum_{r=0}^{m+1} (C(m+1, r) \sigma_d \Phi_{\ell_{r}\tau^{k,d}+2-r} \big|_{t=0}) + \psi_{\ell_{r}\tau^{k,d}} \big|_{t=0} = e.s.t.,$$

(A.17)
More generally, as for (A.7), we find
\[ \Phi_{i+j+\tau+k}d_{m}|_{t=0} = \begin{cases} \text{a linear combination of derivatives of } \psi \text{ in } t, \tau \text{ and } \xi_d, & \text{taken at } t = 0 \\ \end{cases}, \]
and, to claim \( \Phi_{i+j+\tau+k}d_{m}|_{t=0} \) is an e.s.t., we need \( \psi_{i+j+k+d+2j}|_{t=0} \) and \( \psi_{i+j+k}d_{m}|_{t=0} \) to be e.s.t.s. Hence we find, with the assumption (A.2): for \( 0 \leq j \leq \min((M-m)/2, J-1) \), \( 0 \leq k \leq K \) and \( 0 \leq m \leq M \),
\[ \Phi_{i+j+\tau+k}d_{m}|_{t=0} = e.s.t.. \]

To find the boundary condition of \( \Phi_{i+j+k}d_{m} \) at \( \xi_d = 0 \), we differentiate (A.9)_1 \( m \)-times in \( \xi_d \), and write
\[ -\varepsilon(\sigma \Phi_{i+j+k}d_{m})_{d+1} = \psi_{i+j+k}d_{m} - \Phi_{i+j+\tau+k}d_{m}. \]

We first integrate (A.20) with \( m = 0 \), over \( 0 \leq \xi_d \leq 2\delta \), and find
\[ -\varepsilon(\sigma \Phi_{i+j+k}d_{m})_{d+1} = \int_{0}^{2\delta} (\psi_{i+j+k} - \Phi_{i+j+\tau+k})d\xi_d, \]
and, using (3.3), (A.2) and (A.14), we obtain
\[ \Phi_{i+j+k}d_{m}|_{\xi_d=0} = e.s.t., \quad 0 \leq j \leq \min(M/2, J-1), \quad 0 \leq k \leq K. \]

Back to (A.20) with \( m = 0 \), using (3.3), (A.2) and (A.14) again, we also find
\[ \Phi_{i+j+k}d_{m}|_{\xi_d=0} = e.s.t., \quad 0 \leq j \leq \min(M/2, J-1), \quad 0 \leq k \leq K. \]

Moreover, from (A.20) with \( m \geq 1 \), using (3.3), we notice
\[ \Phi_{i+j+k}d_{m+2}|_{\xi_d=0} = \varepsilon^{-1} \left( \Phi_{i+j+\tau+k+d_{m}}|_{\xi_d=0} - \psi_{i+j+k}d_{m}|_{\xi_d=0} \right), \]
and therefore, using (A.22)-(A.24), we recursively find, for \( 0 \leq j+m \leq \min(M/2, J-1) \), \( 0 \leq k \leq K \) and \( 0 \leq m \leq (M-3)/2 \),
\[ \Phi_{i+j+k}d_{m+2}|_{\xi_d=0} = e.s.t., \quad \Phi_{i+j+k}d_{m+2}|_{\xi_d=0} = e.s.t.. \]

Combining all the information from (A.19) and (A.25), by differentiating (A.1), we find the equation for \( \Phi_{i+j+k}d_{m} \): for \( 0 \leq j \leq J-1 \), \( 0 \leq k \leq K \), \( 0 \leq m \leq M-1 \) such that \( 2j+m \leq \min(2J-2, M) \),
\[ \begin{cases} \Phi_{i+j+k}d_{m} - \varepsilon(\sigma \Phi_{i+j+k}d_{m})_{d+1} = \psi_{i+j+k}d_{m}(\xi', \xi_d, t), & \text{in } \Omega_{2\delta, \xi} \times (0, T), \\ \Phi_{i+j+k}d_{m} = e.s.t., & \text{at } \xi_d = 0, \\ \Phi_{i+j+k}d_{m+1} = e.s.t., & \text{at } \xi_d = 0, \\ \Phi_{i+j+k}d_{m}|_{t=0} = e.s.t.; \end{cases} \]
note that (A.26)_3 will be useful in the analysis below.

To verify (A.3) for all indices, we use the induction on \( m \geq 0 \): from (A.9) and (A.14), we notice that (A.3) is valid when \( m = 0 \). Now, we assume that (A.3) holds true for every positive integer less than \( m \). Then, to obtain (A.3) for \( m \), we consider the system (A.26), multiply it by \( \Phi_{i+j+k}d_{m} \) and integrate over \( \Omega_{2\delta, \xi} \). As a result, we find
\[ \frac{1}{2} \frac{d}{dt} \left| \Phi_{i+j+k}d_{m} \right|_{L^2(\Omega_{2\delta})}^2 - \varepsilon \int_{\Omega_{2\delta, \xi}} (\sigma \Phi_{i+j+k}d_{m})_{d+1} \Phi_{i+j+k}d_{m} \eta d\xi d\xi' \]
\[ = (\psi_{i+j+k}d_{m}, \Phi_{i+j+k}d_{m})_{L^2(\Omega_{2\delta})}. \]
Thanks to (3.3), (A.26) and (A.26), we integrate by parts the second term in the left-hand side of (A.27) and find

\[ \varepsilon \int_{\Omega_{2\delta,\xi}} \left( \frac{d}{d\xi} \Phi_{(j \tau^k d)} \right) d\xi d\xi' - \varepsilon \int_{\omega'} \left( \frac{d}{d\xi} \Phi_{(j \tau^k d)} \right)_{\xi_d=0} d\xi' \]

\[ = \int_{\Omega_{2\delta,\xi}} \left( \frac{d}{d\xi} \Phi_{(j \tau^k d)} \right) d\xi d\xi' + \text{e.s.t.} \]

\[ = \sum_{p=0}^{m} \sum_{q=0}^{m} N_{p,q} + \text{e.s.t.}, \quad \text{(A.28)} \]

where

\[ N_{p,q} = \varepsilon \int_{\Omega_{2\delta,\xi}} \left\{ C(m,p) \sigma_{d+1} \Phi_{(j \tau^k d m+1-p)} \right\} \left\{ \Phi_{(j \tau^k d m+1-q h_d)} \right\} d\xi d\xi'. \quad \text{(A.29)} \]

We first consider the terms \( N_{p,q} \) when \( p + q \leq 1 \):

\[ N_{0,0} = \varepsilon \int_{\Omega_{2\delta,\xi}} \sigma_{d+1} \Phi_{(j \tau^k d m+1)} h d\xi d\xi' = \varepsilon \left| \sqrt{\sigma} \Phi_{(j \tau^k d m+1)} \right|^2_{L^2(\Omega_{2\delta})}. \quad \text{(A.30)} \]

\[ |N_{0,1}| = \varepsilon \int_{\Omega_{2\delta,\xi}} \sigma_{d+1} \Phi_{(j \tau^k d m+1)} \Phi_{(j \tau^k d m)} h' d\xi d\xi' \leq \left( \sqrt{\sigma} \Phi_{(j \tau^k d m+1)} \right)_{L^2(\Omega_{2\delta})} \]

\[ \leq \frac{\varepsilon}{2} \left| \sqrt{\sigma} \Phi_{(j \tau^k d m+1)} \right|^2_{L^2(\Omega_{2\delta})} + \kappa \left| \Phi_{(j \tau^k d m)} \right|^2_{L^2(\Omega_{2\delta})}. \quad \text{(A.31)} \]

\[ |N_{1,0}| = m \varepsilon \int_{\Omega_{2\delta,\xi}} \sigma' \Phi_{(j \tau^k d m)} \Phi_{(j \tau^k d m+1)} h d\xi d\xi' \leq 2^{-1} m \varepsilon \int_{\Omega_{2\delta,\xi}} \sigma' \left( \Phi_{(j \tau^k d m)} \right) h d\xi d\xi' \leq 2^{-1} m \varepsilon \int_{\Omega_{2\delta,\xi}} \sigma' h' \Phi_{(j \tau^k d m)} d\xi d\xi' \leq \kappa \left| \Phi_{(j \tau^k d m)} \right|^2_{L^2(\Omega_{2\delta})}. \quad \text{(A.32)} \]
Using the inductive assumptions (A.3) for 1, · · · , m−1, we also bound the remaining terms of $N_{p,q}$:
\[
\sum_{p=2}^{m} |N_{p,0}| \leq \kappa \epsilon \sum_{p=2}^{m} \left| \int_{\Omega_{2k,\xi}} \sigma_{d^p} \Phi_{\ell \tau k d^{m+1-p}} \Phi_{\ell \tau k d^m} h d\xi d\xi' \right|
\leq \left( \text{since, for } p \geq 2, \sigma_{d^p} = 0 \text{ at } \xi_d = 0, 2\delta, \right)
\leq \kappa \epsilon \sum_{p=2}^{m} \left| \sigma_{d^p} h \Phi_{\ell \tau k d^{m+1-p}} \Phi_{\ell \tau k d^m} d\xi d\xi' \right| \quad (A.33)
\leq \kappa |\Phi_{\ell \tau k d^m}|_{L^2(\Omega_{2k})}^2 + e.s.t.,
\sum_{p=1}^{m} |N_{p,1}| \leq \kappa \epsilon \sum_{p=1}^{m} \left| \int_{\Omega_{2k,\xi}} \sigma_{d^p} \Phi_{\ell \tau k d^{m+1-p}} \Phi_{\ell \tau k d^m} h' d\xi d\xi' \right|
\leq \kappa \epsilon \sum_{p=1}^{m} \left| \Phi_{\ell \tau k d^{m+1-p}} \Phi_{\ell \tau k d^m} \right|_{L^2(\Omega_{2k})}
\leq \kappa |\Phi_{\ell \tau k d^m}|_{L^2(\Omega_{2k})}^2 + e.s.t., \quad (A.34)
\]

Using (A.2) and (A.28)-(A.34), (A.27) yields
\[
\frac{d}{dt} |\Phi_{\ell \tau k d^m}|_{L^2(\Omega_{2k})}^2 + \epsilon \sigma |\Phi_{\ell \tau k d^{m+1}}|_{L^2(\Omega_{2k})}^2 \leq \kappa |\Phi_{\ell \tau k d^m}|_{L^2(\Omega_{2k})}^2 + e.s.t., \quad (A.35)
\]
and, thanks to (A.26), and the Gronwall inequality, we finally obtain (A.3) for m. The proof of Lemma A.1 is now complete.

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