Comments of Stephen Parrott concerning

“An experimental test of non-local realism”

by

Simon Gröblacher, Tomasz Paterek, Rainer Kaltenbaek, Časlav Brukner, Marek Żukowski, Markus Aspelmeyer, and Anton Zeilinger

Nature 446, 871-875 (2007)

1 Introduction

I found this paper very interesting, in fact so interesting that I was motivated to think carefully about its assumptions and to check some tedious mathematics. I noticed what looked like a serious error in the paper’s proof of its key inequality (9). In searching for an alternative to (9), I found a simple, straightforward proof of this inequality (based on ideas in the paper, which in turn is based on [1]). This is presented in Sections 3 and 4.

The paper seems fairly clearly written, but since it is not completely explicit (e.g., there are symbols whose meaning the reader has to guess), I was worried that I might have misinterpreted something. To reduce this possibility, the following explains my interpretation of its content in greater detail than usual.

I thank the authors for their comments and for pointing out a slip, which I have corrected. Of course, I take responsibility for any further errors.

I do assume that the reader is somewhat familiar with the paper and has it at hand. The notation follows the paper as much as possible. Any undefined symbols are as in the paper. Page numbers refer to the version www.arXiv.org/quant-ph/0704.2529v1. I have not seen the published version, but since the arXiv version is dated April 19, 2007 and the published version appeared days later, I assume that they are identical, or nearly so.

2 My interpretation of the paper’s setup

For ease of language when introducing the definitions, it will be sometimes be convenient to pretend that probability distributions arising are discrete. For example, the paper considers pairs of photons with polarizations \( \vec{u}, \vec{v} \), occurring with probability density \( F(\vec{u}, \vec{v}) \). I will sometimes refer to \( F(\vec{u}, \vec{v}) \) as the probability that photon 1 has polarization \( \vec{u} \) and photon 2 has polarization \( \vec{v} \), which would be correct language if \( F \) were a discrete probability distribution.

A source emits pairs of photons in different directions, as depicted in Figure 2 of the paper. One photon goes to Alice, and the other to Bob.

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1 Current contact information can be found on my web page, www.math.umb.edu/~sp.

2 The authors have since sent me a revised proof avoiding the error.
The probability that Alice’s photon has polarization $\vec{u}$ and Bob’s has polarization $\vec{v}$ is denoted $F(\vec{u}, \vec{v})$. Here $\vec{u}, \vec{v}$ represent points on the unit sphere in three-dimensional space $R^3$. The standard angular polar coordinates of a vector like $\vec{u}$ are denoted $\theta_{\vec{u}}, \phi_{\vec{u}}$, so that $\vec{u} = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)$.

This corresponds to a photon represented quantum-mechanically by the ray in the two-dimensional complex Hilbert space $C^2$, represented by the vector

\[
\begin{bmatrix}
\cos \theta/2 \\
e^{i\phi} \sin \theta/2
\end{bmatrix}.
\]

The paper considers a “hidden variable” $\lambda$ associated with the source. Presumably, this can be thought of as a classical label attached by the source to each of the pair of emitted photons. The same label is attached to each of the photons in an emitted pair, but the label can vary from pair to pair.

My first impression was that the authors were thinking of the source as emitting two photons with polarizations $\vec{u}, \vec{v}$ with an additional label $\lambda$ attached to each photon, as in their Appendix I example of an explicit non-local hidden-variable model. (The set of possible labels $\lambda$ is allowed to depend on $\vec{u}$ and $\vec{v}$, as in the example.) However, this seems inconsistent with some of their later notation, so I eventually settled on the interpretation to be described below. The two interpretations are essentially equivalent (modulo technicalities), so the choice of either is a matter of taste and notation.

The nature of the label $\lambda$ is not specified and is irrelevant to the proofs. It could be a real number in a certain range (depending on $\vec{u}$ and $\vec{v}$), as in the Appendix I example, or something more complicated.

We could use a new label $\lambda'$ defined as a triple $\lambda' := (\lambda, \vec{u}, \vec{v})$, where $\lambda$ is the “old label” in the viewpoint above. This is conceptually simpler in that there is now only one label $\lambda'$ rather than three. In order to stay close to the paper’s notation, from now on we write $\lambda$ instead of $\lambda'$ and work with only one label.

The polarization $\vec{u}$ of the photon received by Alice is assumed to be a function $\vec{u} = \alpha(\lambda)$ of the hidden variable label attached to her photon, and similarly the polarization of Bob’s photon is $\vec{v} = \beta(\lambda)$. The functions $\alpha(\cdot), \beta(\cdot)$ (which are not part of the paper’s notation) are introduced for later convenience instead of writing $\vec{u}(\lambda), \vec{v}(\lambda)$; certain distinctions are hard to make in the latter notation.

This could give a classical explanation for correlations between the polarizations of Alice’s and Bob’s photons. The paper’s aim is to show that such a classical explanation of observed correlations contradicts both quantum mechanics and experiment.

The set of possible labels is a probability space, whose probability measure will not be named. Since Alice’s polarization is a function $\vec{u} = \alpha(\lambda)$ of the hidden variable $\lambda$, this induces a probability distribution $F(\vec{u}, \vec{v})$ on the set of polarization pairs $\vec{u}, \vec{v}$ as follows. When the set of $\lambda$ is discrete, the probability $F(\vec{u}, \vec{v})$ of a particular polarization pair $\vec{u}, \vec{v}$ is the probability of the set of all $\lambda$ such that $\alpha(\lambda) = \vec{u}$ and $\beta(\lambda) = \vec{v}$.

When $\lambda$ is a continuous variable, the mathematical object corresponding to $F(\vec{u}, \vec{v})$ is a probability measure which might be denoted $F(\vec{u}, \vec{v}) d\vec{u} d\vec{v}$ in the special case in which it is given by a probability density function, where $d\vec{u}$ and $d\vec{v}$ represent Lebesgue measure on the unit sphere. We follow the paper by using
the notation of a probability density function, with the understanding that the measure might have a singular part (e.g., concentrated at a point or on a line). A precise mathematical definition might be cumbersome, but the discrete case above gives the idea.

The paper defines “Malus’ law” as “the well-known cosine dependence of the intensity of a polarized beam after an ideal polarizer”. I take this to mean the following. Alice has an instrument to measure polarization in any chosen direction \( \vec{a} \). The only possible results of the measurement are \( \pm 1 \). A reading of \( +1 \) means that the observed polarization was in the direction \( \vec{a} \) and \( -1 \) means that it was in the opposite direction \( -\vec{a} \). If she receives many photons with polarization \( \vec{u} \), then the average reading is \( \vec{a} \cdot \vec{u} \) (which is the cosine of the angle between \( \vec{a} \) and \( \vec{u} \)).

The paper introduces a symbol \( \rho_{\vec{u},\vec{v}} \), giving only the cryptic explanation: “Each emitted pair is fully defined by the subensemble distribution \( \rho_{\vec{u},\vec{v}}(\lambda) \).” I take this to mean that \( \rho_{\vec{u},\vec{v}}(\cdot) \) is a conditional probability density function: in the discrete case, \( \rho_{\vec{u},\vec{v}}(\lambda) \) is the probability of \( \lambda \) given that the polarizations of the emitted pair was \( \vec{u},\vec{v} \). A precise mathematical definition in the generality considered by the paper might be cumbersome, but the idea is clear in the discrete case: Given a particular \( \vec{u},\vec{v} \) and \( \lambda_0 \) with \( \alpha(\lambda_0) = \vec{u} \) and \( \beta(\lambda_0) = \vec{v} \), \( \rho_{\vec{u},\vec{v}}(\lambda_0) \) is defined as the probability of \( \lambda_0 \) divided by the probability of the set of all \( \lambda \) such that \( \alpha(\lambda) = \vec{u} \) and \( \beta(\lambda) = \vec{v} \).

Suppose Alice sets her instrument to measure polarization in the \( \vec{a} \) direction, Bob sets his to measure in the \( \vec{b} \) direction, and the hidden variable attached to each of their photons is \( \lambda \). The paper denotes the outcome of Alice’s measurement (either \( +1 \) or \( -1 \)) as \( A(a,b,\lambda) \) and Bob’s as \( B(a,b,\lambda) \). The assumption that Malus’ law holds is then given by the paper’s equations (1) and (2):

\[
\begin{align*}
\bar{A}(\vec{u}) & := \int d\lambda \rho_{\vec{u},\vec{v}}(\lambda) A(\vec{a},\vec{b},\lambda) = \vec{u} \cdot \vec{a} , \\
\bar{B}(\vec{v}) & := \int d\lambda \rho_{\vec{u},\vec{v}}(\lambda) B(\vec{a},\vec{b},\lambda) = \vec{v} \cdot \vec{b} .
\end{align*}
\]

(I changed the paper’s first “\( := \)” to the definition symbol “\( := \)” because I think it is helpful to the reader to explicitly distinguish between equality by definition and assertions of equality between separately defined quantities.)

These equations seem sensible in terms of the interpretation just described in which the source emits two particles, each with just one label (the same label \( \lambda \), which implicitly contains the polarization information. If one is thinking of emission of two polarizations \( \vec{u},\vec{v} \) along with an additional label \( \lambda \), then in equation (1), \( A(\vec{a},\vec{b},\lambda) \) should be written \( A(\vec{a},\vec{b},\vec{u},\vec{v},\lambda) \) (or, less generally, \( A(\vec{a},\vec{b},\vec{u},\vec{v},\lambda) \)). In more physical language, what Alice measures is expected to depend explicitly on the polarization of the photon she receives. Indeed, the Appendix I example writes \( A = A(\vec{a},\vec{b},\vec{u},\lambda) \).

The interpretation above (with just one label \( \lambda \) which contains the polarization information) was developed to make sense of equations (1) and (2). But the two interpretations are equivalent, modulo technicalities and notation.
We are interested in the following two questions.

1. Can the hidden variable theory described in the previous section reproduce the results of quantum mechanics?

2. If not, how can we experimentally distinguish between quantum mechanics and the hidden variable theory?

This section presents a simple proof that the hidden variable theory cannot reproduce the results of quantum mechanics. This conclusion will also follow from the results of the next section, which answers question 2, but we present it separately because it is a little easier and the result is simpler than the paper’s (9). The proof of the next section is not much longer than the proof of this section, but it seems less motivated. The present section provides the motivation, notational preliminaries, and a few simple calculations which enter into the proof.

Before starting, I should acknowledge that the proof’s ideas are mostly contained in the paper under discussion, which is based on [1]. Although in retrospect, the proof seems simple, I think it would have taken me a long time to find it had I been given the problem without the solution hints contained in these two references. Any mathematician knows that the first proof is always the hardest to construct, and in retrospect is often unnecessarily complicated.

For given vectors $\vec{a}, \vec{b}$, define a “correlation function” $C(\vec{a}, \vec{b})$ by

$$C(\vec{a}, \vec{b}) := \int d\vec{u} d\vec{v} d\lambda \rho_{\vec{u}, \vec{v}}(\lambda) F(\vec{u}, \vec{v}) A(\vec{a}, \vec{b}, \lambda) B(\vec{a}, \vec{b}, \lambda).$$

Here $\rho_{\vec{u}, \vec{v}}(\lambda), F(\vec{u}, \vec{v}), A(\vec{a}, \vec{b}, \lambda),$ and $B(\vec{a}, \vec{b}, \lambda)$ are as defined in the paper and in the first section above, and $\int d\vec{u}$ represents the integral over the unit sphere in three-dimensional real Euclidean space (similarly for $\int d\vec{v}$).

The correlation $C(\vec{a}, \vec{b})$ is called $\langle AB \rangle$ in the paper (its equation (4)); we introduce the new notation because we shall need to display the dependence of $\langle AB \rangle$ on the “setting vectors” $\vec{a}$ and $\vec{b}$.

Let $\alpha := \cos^{-1} \vec{a} \cdot \vec{b}$ be the angle between $\vec{a}$ and $\vec{b}$. For a system in the singlet state (the case considered by the paper), quantum mechanics predicts that $C(\vec{a}, \vec{b}) = -\vec{a} \cdot \vec{b}$. In the following, it will be helpful to think of $\alpha$ as an acute angle (though the proof does not assume this), so that it is expected that $C(\vec{a}, \vec{b}) \leq 0$. For this case, it is a little easier to work with $-C(\vec{a}, \vec{b}) \geq 0$.

The paper (following [1]) shows that:

$$-1 + \int d\vec{u} d\vec{v} F(\vec{u}, \vec{v}) [\vec{a} \cdot \vec{u} - \vec{b} \cdot \vec{v}] \leq -C(\vec{a}, \vec{b}) \leq 1 - \int d\vec{u} d\vec{v} F(\vec{u}, \vec{v}) [\vec{a} \cdot \vec{u} + \vec{b} \cdot \vec{v}].$$

Only the right-hand inequality will be used below, which will essentially result in establishing half of the paper’s inequality (9). The other half follows similarly from the left inequality in [4], as will be indicated in the next section.
According to quantum mechanics, for all $\vec{a}$,

$$1 = -C(\vec{a}, \vec{a}) \leq 1 - \int d\vec{u} \, d\vec{v} \, F(\vec{u}, \vec{v}) |\vec{a} \cdot (\vec{u} + \vec{v})|, \quad (5)$$

so the integral on the right must vanish. Since the integrand is non-negative, this implies that $F(\vec{u}, \vec{v})$ must be concentrated on the singular set of all $\vec{u}, \vec{v}$ such that $\vec{v} = -\vec{u}$. Restricting to this set, the probability distribution can be symbolically represented by a probability density function of just one sphere variable $\vec{u}$. We denote this new probability density function as $F_s(\vec{u})$ and rewrite inequality (4) as:

$$-C(\vec{a}, \vec{b}) \leq 1 - \int d\vec{u} \, F_s(\vec{u}) |(\vec{a} - \vec{b}) \cdot \vec{u}|. \quad (6)$$

Suppose temporarily that unit vectors $\vec{b} \neq \pm \vec{a}$, so that $\vec{a}$ and $\vec{b}$ are contained in a unique plane. Following the paper and [1], we obtain more tractable inequalities by averaging $C(\vec{a}, \vec{b})$ over rotations in the plane determined by $\vec{a}, \vec{b}$ (i.e., rotations about the $\vec{a} \times \vec{b}$ axis). The result, which depends only on the plane of rotation and the angle $\alpha := \cos^{-1}((\vec{a} \cdot \vec{b})$, will be denoted $E(\alpha)$. More explicitly, if $R(\sigma)$ denotes a rotation through the angle $\sigma$ about the $\vec{a} \times \vec{b}$ axis, then

$$E(\alpha) := \frac{1}{2\pi} \int d\sigma \, C(R(\sigma)\vec{a}, R(\sigma)\vec{b}) \quad . \quad (7)$$

In this notation, $E(\alpha)$ implicitly depends on the plane of $\vec{a}$ and $\vec{b}$. When we want to include in the notation that this plane is the $x$-$y$ plane, we write $E_{xy}(\alpha)$ instead of $E(\alpha)$, and similarly $E_{xz}(\alpha)$ denotes $E(\alpha)$ when $\vec{a}$ and $\vec{b}$ lie in the $x$-$z$ plane.

Next we derive (following the paper and [1]) an inequality for $E_{xy}(\alpha)$. For any vector $\vec{u} = (u_x, u_y, u_z)$ on the unit sphere, write $\vec{u}_{xy} := (u_x, u_y, 0)$ to denote the projection of $\vec{u}$ to the $x$-$y$ plane. Then for any vector $\vec{q}$ in the $x$-$y$ plane, $\vec{q} \cdot \vec{u} = \vec{q} \cdot \vec{u}_{xy} = |\vec{q}| |\vec{u}_{xy}| \cos \beta$, where $\beta$ is the angle between $\vec{q}$ and $\vec{u}_{xy}$. Hence for $\vec{a}, \vec{b}$ in the $x$-$y$ plane, the average of $|\vec{a} - \vec{b} \cdot \vec{u}|$ over rotations in that plane is

$$\frac{1}{2\pi} \int_0^{2\pi} d\sigma \, |R(\sigma)(\vec{a} - \vec{b}) \cdot \vec{u}| = |\vec{a} - \vec{b}| |\vec{u}_{xy}| \frac{1}{2\pi} \int_0^{2\pi} d\tau \, |\cos \tau| = \frac{2}{\pi} |\vec{a} - \vec{b}| |\vec{u}_{xy}| \quad , \quad (8)$$

where the integration variable was changed from $\sigma$ to $\tau := \beta - \sigma$, with $\beta$ the angle between $\vec{u}$ and $\vec{a} - \vec{b}$. Combining this with inequality (6) gives

$$-E_{xy}(\alpha) \leq 1 - \frac{2}{\pi} |\vec{a} - \vec{b}| \int d\vec{u} \, F_s(\vec{u}) |\vec{u}_{xy}| \quad , \quad (9)$$

and

$$= 1 - \frac{4}{\pi} |\sin \frac{\alpha}{2}| \int d\vec{u} \, F_s(\vec{u}) |\vec{u}_{xy}| \quad , \quad (10)$$
where the last line follows from the routine calculation
\[ |\vec{a} - \vec{b}|^2 = 2 - 2\vec{a} \cdot \vec{b} = 2(1 - \cos \alpha) = 4 \sin^2 \frac{\alpha}{2}. \]

It is hard to deduce more from inequality (11) without specific knowledge of
the probability density \( F_s(\vec{u}) \). But adding the \( x-y \) and \( x-z \) versions of (11) gives
something useful:
\[
- E_{xy}(\alpha) - E_{xz}(\alpha) \leq 2 - \frac{4}{\pi} |\sin \frac{\alpha}{2}| .
\] (11)

Here we have used the facts that \( \int F_s(\vec{u}) \, d\vec{u} = 1 \) and that \( |\vec{u}_{xy}| + |\vec{u}_{xz}| \geq 1 \).
(Proof: \(|\vec{u}_{xy}| + |\vec{u}_{xz}| \geq |\vec{u}_{xy}|^2 + |\vec{u}_{xz}|^2 = u_x^2 + u_y^2 + u_x^2 + u_z^2 \geq u^2 = 1\).)

The argument just given assumed that \( C(\vec{a}, \vec{a}) = -1 \), which implies that
\( F(\vec{u}, \vec{v}) \) is concentrated on \( \vec{v} = -\vec{u} \). If \( F(\vec{u}, \vec{v}) \) is not concentrated on \( \vec{v} = -\vec{u} \),
then \( E_{xy}(0) \) gives some information about \( F(\vec{u}, \vec{v}) \) for \( \vec{v} \neq -\vec{u} \). This suggests that
it might be productive to look at
\[
- E_{xy}(\alpha) - E_{xy}(0) ,
\]
as the paper does.

4 Testing the hidden-variable theory

Finally, we give a proof of the paper’s (9) without assuming that \( C(\vec{a}, \vec{a}) = -1 \).
We use the notation of the last section, along with some simple facts established
there.

Apply inequality (11) to obtain
\[
- C(\vec{a}, \vec{b}) - C(\vec{a}, \vec{a}) \leq 2 - \int d\vec{u} d\vec{v} F(u, v)[|\vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{v}| + |\vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{v}|]
= 2 - \int F(\vec{u}, \vec{v})[|\vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{v}| + |\vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{v}|]
\leq 2 - \int F(\vec{u}, \vec{v})(\vec{b} - \vec{a}) \cdot \vec{v}| ,
\] (12)

where the last line comes from the triangle inequality, \(|\vec{p}| + |\vec{q}| \geq |\vec{p} + \vec{q}|\).

Let \( \alpha := \cos^{-1} \vec{a} \cdot \vec{b} \) be the angle between \( \vec{a} \) and \( \vec{b} \). Average over rotations in
the \( x-y \)plane to obtain
\[
- E_{xy}(\alpha) - E_{xy}(0) \leq 2 - |\vec{b} - \vec{a}| \frac{2}{\pi} \int d\vec{u} d\vec{v} F(\vec{u}, \vec{v})|\vec{v}_{xy}|
= 2 - \frac{4}{\pi} |\sin \frac{\alpha}{2}| \int d\vec{u} d\vec{v} F(\vec{u}, \vec{v})|\vec{v}_{xy}| .
\] (13)

The same procedure using the left inequality in (11) yields
\[
C(\vec{a}, \vec{b}) + C(\vec{a}, \vec{a}) \leq 2 - \int d\vec{u} d\vec{v} F(u, v)[|\vec{a} \cdot \vec{a} - \vec{b} \cdot \vec{v}| + |\vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{v}|]
\leq 2 - \int F(\vec{u}, \vec{v})(\vec{b} - \vec{a}) \cdot \vec{v}| ,
\]
\[ E_{xy}(\alpha) + E_{xy}(0) \leq 2 - \frac{4}{\pi} \sin \frac{\alpha}{2} \int d\vec{u} d\vec{v} F(\vec{u}, \vec{v}) |\vec{v}_{xy}|. \]

Combining this with (13) gives

\[ |E_{xy}(\alpha) + E_{xy}(0)| \leq 2 - \frac{4}{\pi} \sin \frac{\alpha}{2} \int d\vec{u} d\vec{v} F(\vec{u}, \vec{v}) |\vec{v}_{xy}|. \quad (14) \]

Do the same for the \(x-z\) plane and add the results, recalling from the last section that \(|\vec{v}_{xy}| + |\vec{v}_{xz}| \geq 1\), to obtain the paper’s (9):

\[ |E_{xy}(\alpha) + E_{xy}(0)| + |E_{xz}(\alpha) + E_{xz}(0)| \leq 4 - \frac{4}{\pi} \sin \frac{\alpha}{2} |. \]

for the particular choice of orthogonal planes \(x-y\) and \(x-z\).

Of course, the proof just given applies to any two orthogonal planes—the particular choice of planes was made to simplify the notation. The paper’s statement of its (9) appears to apply to any two planes, not necessarily orthogonal. However, its proof does explicitly assume orthogonal planes (on the top of its page 13), so I assume this was intended.

## 5 Statistical methods

The paper does not completely explain its statistical methods, and I’m not sure I can agree with what is explained. I have questions about the standard deviations claimed. The paper states that “the errors [presumably meaning standard deviations] are calculated assuming that the counts follow a poissonian distribution”. I don’t understand this assumption. I’m not sure precisely what it means, and under all interpretations which have occurred to me, it seems questionable.

If we were measuring the number of counts observed by Alice in a given time interval (say the 10 sec. mentioned on p. 5, during which Alice observes about 95,000 counts), \(that\) would be expected to follow a Poisson distribution:\[ 4 \]

\[ p(k) = (\mu^k e^{-\mu})/k!, \]

where \(p(k)\) is the probability of exactly \(k\) counts and \(\mu\) is the mean of the distribution. Also, if we were measuring the number of times that Alice and Bob “simultaneously” observe a photon in that 10 seconds, that would be expected to follow a Poisson distribution (with a different mean). Here “simultaneously” means that Alice and Bob both observe photons at times differing by less than some preassigned constant \(\delta > 0\); e.g., they both observe a photon at times differing by less than 1 microsecond. But these are not what we are measuring.

What we are measuring is the following. First we select all the occasions on which Alice and Bob receive a photon “simultaneously” (as defined in the last

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\[ \text{The Poisson distribution was invented to describe the the number of random events expected to occur in a given time interval. One of the first uses of it was to describe the number of Prussian cavalry which would be kicked to death by horses in a given year! The actual numbers matched the distribution very closely.} \]
paragraph). Then for each such occasion, we observe the value of a “yes-no” random variable which takes the value “yes” if and only if (Alice observes spin +1 (relative to her instrument set at $\vec{a}$) and Bob observes spin +1 (relative to his instrument set at $\vec{b}$)) or (Alice observes spin −1 and Bob also observes −1). Then we calculate the relative frequency of “yes” answers (the number of occurrences of “yes” divided by the total number of simultaneous pairs), a statistic $S$ called the “sample mean” (to distinguish it from the usually unknown mean of the probability distribution from which the random sample is drawn). The sample mean $S$ estimates the probability (call it $q$) of “yes”. Routine calculation reveals that when $n$ simultaneous pairs are observed, the sample mean has standard deviation $\sqrt{q(1-q)}/\sqrt{n}$. Hence it seems reasonable to estimate the standard deviation of the sample mean by 

$$\sqrt{S(1-S)}/\sqrt{n}.$$ 

From this, follows easily an estimate for the correlations $C := C(\vec{a}, \vec{b}) = E(\vec{a}, \vec{b})$. Suppose that we observe $n$ photon pairs with $n_+$ “yes” results and $n_-$ “no”, $n_+ + n_- = n$. Then the sample mean $S = n_+ / n$, and the measured correlation $C = n_+ / n - n_- / n = (2n_+ - n) / n = 2S - 1$. Hence the estimated standard deviation of $E = C$ is twice the estimated standard deviation $\sqrt{S(1-S)}/n$ for $S$.

We can’t apply this directly to the results of the paper because the value of $n$ (number of photon pairs used to calculate the sample mean) is not given. However, we can ask what value of $n$ would yield the paper’s claimed error of .0118 for $E(\vec{a}_2, \vec{b}_3) = -0.9902 \pm 0.0118$ (bottom of p. 6). The claimed error [standard deviation] of .0118 for $C = E := E(\vec{a}_2, \vec{b}_3)$ corresponds to a standard deviation of .0059 for $S$, so we need to solve the equation

$$\sqrt{S(1-S)}/\sqrt{n} = .0059$$

with $S := (C + 1)/2 = (E + 1)/2 = 0.0049$.

The solution is $n \approx 140$, which seems rather small. The paper mentions approximately 3000 photon pairs received in 10 sec. If this were the true value of $n$, then the claimed error of .0018 for $E(\vec{a}_2, \vec{b}_2)$, which scales with $1/\sqrt{n}$, would be about 5 times smaller. I wonder if the paper may have inadvertently overstated the errors.

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4 All of this is standard statistics. For simplicity, I am glossing over some statistical subtleties which are unimportant in the present context. For example, calculation reveals that the estimator $S(1-S)/n$ of the variance of the sample mean is (surprisingly) not “unbiased”; to get an unbiased estimator one replaces $S(1-S)/n$ by $S(1-S)/(n-1)$. For large $n$, the difference is negligible. It is usual to estimate the standard deviation of the sample mean as the square root of the estimator for the variance even though this estimator is not unbiased with either estimator of the variance.

5 I am following the paper in assuming that $C(\vec{a}, \vec{b}) = E(\vec{a}, \vec{b})$, where $E(\vec{a}, \vec{b})$ denotes the average of $C(\vec{a}, \vec{b})$ over the plane of $\vec{a}, \vec{b}$. The next section wonders about this assumption.
6 Final comments

As a mathematician who is largely self-taught in physics, I am unsure of the correspondence between the physical measurements described in the paper and the mathematics of the Poincaré sphere. Is this well-established physics, or is it a kind of guess, based on mathematical analogies between complex polarization vectors in classical electrodynamics and the two-dimensional complex state space describing quantum-mechanical photons?

I am uneasy about the paper’s justification for its assumption that the average over a great circle on the Poincaré sphere can be confidently replaced by an evaluation of the single correlation \( C(\vec{a}, \vec{b}) \) for \( \vec{a}, \vec{b} \) on the circle. The paper justifies this assumption as follows: (bottom of p. 5):

“So far, no experimental evidence against the rotational invariance of the singlet state exists. We therefore replace the rotation averaged correlation functions in inequality (9) with their values measured for one pair of settings (in the given plane).”

It seems dangerous to assume that something is true on the sole grounds that no one has proved it false. That risks overlooking potentially important new physics.

My impression is that \( C(\vec{a}, \vec{b}) = -\vec{a} \cdot \vec{b} \) is experimentally well established for correlations \( C(\vec{a}, \vec{b}) \) with \( \vec{a} \) and \( \vec{b} \) in the \( x\text{-}z \) plane, i.e., linear polarizations. I’m not aware of any experiments explicitly validating it for \( \vec{a}, \vec{b} \) lying in some other plane. Are there any? If so, it would be helpful if the paper gave references.

The results of the paper suggest its confirmation for the \( y\text{-}z \) plane in that correlations in the \( y\text{-}z \) plane are used in calculating \( S_{NLHV} \) on the left side of inequality (9), and the measured values of \( S_{NLHV} \) are consistent with quantum mechanics. However, the actual measured correlations \( C(\vec{a}, \vec{b}) \) are not given in the paper, except for a few special cases at the bottom of p. 6.

Enough data to suggestively confirm \( C(\vec{a}, \vec{b}) = -\vec{a} \cdot \vec{b} \) for the \( y\text{-}z \) plane was probably gathered in the course of the experiment. It would have been helpful had it been presented, if not in the Nature article (which might have had length constraints), then in an arXiv report. These experiments are probably hard to do, and print is cheap.

References

[1] A. J. Leggett, “Nonlocal Hidden-Variable Theories and Quantum Mechanics: An Incompatibility Theorem”, Found. Phys. 33 (2003), 1469-1493