Cosmic Wheels:  
From integrability to the Galois coaction  

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We argue that the description of Feynman loop integrals as integrable systems is intimately connected with their motivic properties and the action of the Cosmic Galois Group. We show how in the case of a family of fishnet graphs, coaction relations between them follow directly from iterative constructions of Q-functions in the Quantum Spectral Curve formalism. Using this observation we conjecture a “differential equation for numbers” that enter these periods.

INTRODUCTION

One of the lessons from recent developments in perturbative Quantum Field Theory is that the most efficient way to deal with Feynman loop integrals is to avoid them where possible. This point of view is prevailingly supported by the achievements of the cluster bootstrap programme for scattering amplitudes in maximally-supersymmetric Yang-Mills theory (MSYM), in which loop integration is traded with linear constraints on a linear space of functions conjectured to contain the amplitude [1]. These methods rely heavily on the coalgebra structure exhibited by Feynman integrals as iterated period integrals. In particular the symbol, the central ingredient of cluster bootstrap, is essentially an iteration of the Galois coaction on multiple polylogarithms [2, 3].

A number of powerful conjectures on periods in QFT are expressed in terms of a motivic Galois coaction once they are lifted to motivic periods [4–6]. The (motivic) periods in φ4 theory are expected to be closed under a “cosmic” Galois group that acts on these periods, and this was the guiding principle of studies of these periods through 11 loops [7]. Dimensionally-regulated one- and two-loop integrals observe this principle order-by-order in the expansion around an integer dimension, where the coaction is expressed in terms of the original graph with cut and contracted edges [8]. Furthermore, evidence for the Galois coaction relations have been observed in six–particle amplitudes in MSYM evaluated on special kinematical datapoints [1] and in results for the electron anomalous magnetic moment. And most recently, the coaction on general one-loop graphs in four dimensions has been studied in [9]. All these studies that reveal fascinating properties of Feynman periods employ their often-complicated integral representations, and the underlying geometry determined by the integrands. More often than not, this complexity is in contrast with the simplicity of the emergent structures.

On the other hand, the description of certain Feynman integrals as integrable systems paints an entirely different picture [10–13]. This is for example the case in the Fishnet Model [10], which is an integrable QFT of two complex scalars φ1 and φ2 interacting with a particular quartic coupling of the form Trφ1φ2φ1†φ2†. The virtue of this coupling is that it only allows up to a single Feynman diagram to contribute to several quantities at each loop order, and this provides an opportunity to realise such descriptions and the possibility of computing the periods with no mention of loop integration whatsoever. The state-of-the-art approach on integrability in QFT is formulated in terms of the Quantum Spectral Curve [14, 15] and it has been successfully adapted to describe observables in the Fishnet Model [16].

The purpose of this Letter is to argue that there is a close link between the integrability of the Fishnet Model and the coaction properties of its periods: We will present a problem in which coaction relations can be straightforwardly derived from the properties of the quantum spectral curve.

To this end, we will focus on the periods of wheel graphs with three spokes in the planar limit of the Fishnet Model. These graphs enter the perturbative expansion of the correlator

$$\langle \text{Tr}\phi_1^0(0) \text{Tr}\phi_1^3(x) \rangle = [x^2]^{-D}, \quad D = 3 - \xi^3 \delta$$

where D is the quantum scaling dimension of the operator Trφ1. The anomalous dimension δ has a perturbative expansion in the cube of the fishnet coupling ξ:

$$\delta = \sum_{i=0}^{\infty} \delta_m \xi^{3m}.$$
TABLE I: First few wheel periods in an $f$-alphabet.

This expansion is justified from the diagrams that contribute to these correlators (depicted in Figure 1a) by noting that the number of interactions is always a multiple of three. In the Feynman diagram, the solid and dashed lines represent propagators of complex scalars $\phi_1$ and $\phi_2$, respectively. One of the operators in (1) is placed at the origin while the other is imagined to reside at infinity. Up to a normalisation the numbers $\delta_m$ are identified [17] with the periods of the corresponding "wheel" Feynman graph in Figure 1b.

Direct calculations are only available for $\delta_0$ [18] and for $\delta_1$ [7], and in general it is proven that they are linear combinations of Multiple Zeta Values (MZVs) [6, 19]. Owing to the integrability of the Fishnet Model, powerful techniques of integrability can be employed to algorithmically determine further anomalous dimensions $\delta_m$ [16] in terms of these numbers.

SOLVING THE $L = 3$ BAXTER EQUATION RECURSIVELY

As detailed in [16], in the QSC setup, the wheel periods are determined through a quantisation condition

$$q_2(0, \mu)q_4(-\mu) + q_2(-\mu)q_4(0, \mu) = 0,$$  

(3)

where the Q-functions $q_2(u, \mu)$ and $q_4(u, \mu)$ satisfy the Baxter equation,

$$\Box q_\alpha = \frac{\mu}{u^2}q_\alpha + \frac{(D-3)(D-1)}{4u^2}q_\alpha \quad \alpha = 2, 4$$  

(4)

and are distinguished by their asymptotics:

$$q_2(u, \mu) = u^{-D/2-1/2} + O(u^{-1})$$

$$q_4(u, \mu) = u^{-D/2+3/2} + O(u^{-1}).$$  

(5)

The $\Box$ denotes the second-order difference operator: $\Box f(u) = f(u+i) - 2f(u) + f(u-i)$. The parameter $\mu$ is related to the fishnet coupling as $\mu^2 = \xi^3$ [20].

To compute the anomalous dimensions, we consider the expansions of $q_\alpha$ in $\delta$ and $\mu$ in with coefficients $q_{\alpha}^{i,j}(u)$:

$$q_\alpha(u, \mu) = \sum_{i=-1}^\infty \sum_{j=-1}^\infty q_{\alpha}^{i,j}(u) \mu^i \delta^j,$$  

(6)

with

$$q_{\alpha}^{i,j} = 0 \quad \text{for} \quad i < -1 \text{ or } j < -1,$$

$$q_{\alpha}^{i,j} = 0 \quad \text{for} \quad i < 0 \text{ or } j < 0,$$  

(7)

while keeping in mind that $\delta$ also has a $\mu$ expansion as in equation (2). For a similar perturbative analysis of the Konishi anomalous dimensions in MSYM in the context of QSC see [21].

The Baxter equation (4) implies the following equation for the components of $q_2$ and $q_4$:

$$\Box q_{\alpha}^{i,j} = \frac{1}{2u^2}q_{\alpha}^{i-2,j-1} - \frac{1}{4u^2}q_{\alpha}^{i-4,j-2} + \frac{1}{2u^3}q_{\alpha}^{i-1,j},$$  

(8)

which can be algorithmically solved as a linear combination [22] of Multiple Hurwitz Zeta functions (MHZF's):

$$\eta_{s_1, \ldots, s_k}(u) = \sum_{n_1, \ldots, n_k \geq 0} \frac{1}{(u + i n_1)^{s_1} \cdots (u + i n_k)^{s_k}}$$  

(9)

using well-known relations that they satisfy [14-16]. Importantly, when evaluated at $u = \sqrt{-1}$, they reduce to Multiple Zeta Values (MZVs):

$$\eta_{s_1, \ldots, s_k}(\sqrt{-1}) = (-1)^{\Theta} \sum_{i=1}^{s_k} \zeta_{s_1, \ldots, s_k}.$$  

(10)
PATTERNS IN NESTED WHEELS

General structure

The poles in \( u \) in the RHS of equation (8) imply that \( \delta_m \) are linear combinations of MZVs of weight \( 4m + 3 \) and indices \( \{1, 2, 3\} \) [23]. The first few numbers, computed in an “\( f \)-alphabet” [4] using the Maple package HyperlogProcedures [24] are presented in Table I. The expressions for \( \delta \leq 3 \) match the those provided in [16] in terms of MZVs, and we obtained all periods through \( \delta \leq 7 \). Their lengthy expressions can be found in the ancillary file wheels.m.

The main point of rewriting the MZVs in an \( f \)-alphabet is that it is a basis where the Galois coaction is simply deconcatenation [4]:

\[
\Delta f_w = \sum_{w_1 w_2 = w} f_{w_1} \otimes f_{w_2},
\]

provided that \( w \) only contains odd letters, while multiplication of such words is the shuffle product

\[
f_{w_1} f_{w_2} = \sum_{w \in w_1 \shuffle w_2} f_w.
\]

Multiplication of a word by \( \zeta_2 = f_2 \) works by appending the letter 2 to it:

\[
f_2 f_w = f_{w2}.
\]

One can also define derivatives that act on the first entry of a word in an \( f \)-alphabet in the following way:

\[
\partial_{2n+1} f_w = \begin{cases} f_w & a = 2n + 1 \\ 0 & a \neq 2n + 1. \end{cases}
\]

Consequently this framework allows one to consider “differential equations” for MZVs, which we will use to relate various numbers that appear in wheel periods.

We first note some glaring properties of these numbers: The depths of MZVs that appear in \( \delta_m \) increase with \( m \), curiously skipping multiples of 4 as shown in Table II.

\[
\begin{array}{cccccccc}
m & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\
\text{Maximum depth in } \delta_m & 1 & 2 & 3 & 5 & 6 & 7 & 9 & 10 & \cdots
\end{array}
\]

TABLE II: Maximum depth of MZVs in \( f \)-basis that appear in each \( \delta_m \)

Moreover, they include only odd letters and are symmetric in the last two letters. While the former is a well-understood consequence of the coaction principle and the fact that \( \pi \) and \( \zeta_2 \) not being periods of this theory [5, 7], the latter, which is a property shared by other wheel families in the Fishnet Model [25] but not generic \( \phi^3 \) periods [7], still begs for a good explanation.

We should emphasise that “the \( f \)-alphabet” satisfying (11) is not unique, and it is not guaranteed that powers of \( \pi \) will not appear after a basis change. Nevertheless we observe that this is not the case in the bases available in [24].

Wheel MZVs and their coaction

The quantisation condition (3) leads to a natural decomposition of a given \( \delta_m \) as a polynomial in \( \delta_{m'} < m \) with MZV coefficients. To see this we first observe that using

\[
q_2^{i-1} q_4^{j+1} = q_2^{j-1} q_4^{i+1} \quad \forall i, j
\]

the condition (3) can be rewritten in the following form:

\[
W(\delta) := \sum_{i=0}^{\infty} W_i(\mu) \delta^i = 0.
\]

The coefficients \( W_i(\mu) \) are generating functions

\[
W_i(\mu) := \sum_{n=0}^{\infty} W_{i,n} \mu^{2n}
\]

of “wheel MZVs” \( W_{i,n} \) that are defined as bilinear combinations of \( q_2^{i,j} \) and \( q_4^{i,j} \), evaluated at \( u = 0 \):

\[
W_{m,n} = 4 \sum_{i,j} (-1)^{i+\frac{1}{2}} q_2^{i,j}(0) q_4^{2m+2-i,n-j}(0),
\]

where the sum is over all pairs \( i, j \) that give a non-zero contribution according to equation (7).

It is now easy to see the structure of the polynomials one obtains by expanding (16) in \( \mu \):

\[
W = W_{0,0} + \mu^2 (W_{0,1} + W_{1,1} \delta_0) + \mu^4 (W_{0,2} + W_{1,2} \delta_0 + 2W_{2,2} \delta_0^2 + 3W_{3,3} \delta_0^3 + W_{1,3} \delta_1) + \mu^6 (W_{0,3} + 2W_{1,3} \delta_0 + 3W_{2,3} \delta_0^2 + 4W_{3,3} \delta_0^3 + W_{1,2} \delta_1 + 2W_{2,2} \delta_0 \delta_1 + W_{1,1} \delta_2) + \mathcal{O}(\mu^8).
\]

Requiring the vanishing of (19) at each order, and noting that \( W_{1,1} = -1 \) allows us to express any \( \delta_m \) as a polynomial in \( \delta_{m'} < m \) with MZV coefficients.

As a first iterative pattern, we note that the only new numbers that enter a period \( \delta_m \) are the coefficients of the powers of \( \delta_0 \), while coefficients of other monomials will have occurred as the coefficient of some power of \( \delta_0 \) in a smaller wheel.

We will now show that the \( W_i \) have curious properties under the coaction that they inherit from the solutions of the Baxter equation. As an explicit illustration of these, we will focus on the sequence \( W_{0,m} \). First few elements of this sequence are presented in Table III.
The components $q_4^{-1} = q_4^{i,0}$ of solutions to the system of equations (4) & (5) satisfy
\[ c_k^{-1} \partial_k q_4^{i,0}(0) = \begin{cases} c_{k-4}^{-1} \partial_k q_4^{i-2,0}(0) & k = 3 \text{ mod } 4 \\ 0 & k = 1 \text{ mod } 4 \end{cases} \]  
(20)

with the same rational constant $c_k$ for all $i \geq 2$, with the first few numbers $c_k$:
\[ (c_3, c_7, c_{11}, \ldots) = \left( \frac{1}{4}, \frac{63}{4}, \frac{740907}{4}, \frac{46817062179}{4}, \frac{123200}{4}, \frac{2748660980272441072644717}{4}, \frac{94776262380800}{4}, \ldots \right). \]  
(21)

The relation (20) together with the fact that $q_2^{i-1} = q_2^{i,0}$ immediately lead to the differential equations that hold for any $k = 3 \text{ mod } 4$:
\[ \mathcal{O}_k W_0 := (c_k^{-1} \partial_k - c_{k-4}^{-1} \partial_{k-4}) W_0 = 0 \]  
(22)

while any other derivative $\partial_k$ with $k \neq 3 \text{ mod } 4$ annihilates $W_0$.

In other words, the coaction relations between wheel MZVs (22) rely on a conjectural identity for the Q-functions, which we have verified for $m \leq 7$, i.e. up to weight 31.

**A differential equation for wheel MZVs**

Equation (22) in fact follows from a remarkable relation satisfied by $W_0$. We first note that the differential equation (22) implies that $W_0$ has the form
\[ W_0 = Z \cdot S, \]  
(23)

where $\cdot$ denotes concatenation that is distributive over addition, $Z$ has a Taylor expansion in $\mu$ that involves only single zeta values:
\[ Z(\mu) = \sum_{m=1}^{\infty} c_{4m-1} f_{4m-1} \mu^m, \]  
(24)

and $S$ is a function that can be worked out a posteriori. Upon defining
\[ \tilde{Z}(\mu) = \sum_{m=1}^{\infty} \tilde{c}_{4m+1} f_{4m+1} \mu^m \]  
(25)

with
\[ (\tilde{c}_5, \tilde{c}_9, \tilde{c}_{13}, \ldots) = \left( \frac{1}{3}, \frac{31}{3}, \frac{718191}{3}, \frac{7408590567067}{3}, \frac{200}{484000}, \frac{1276765401753338789096939955193067}{216930372306720000}, \ldots \right), \]  
(26)

and solving for the function $S$, we observe that $W_0$ obeys (up to the available order) an almost homogeneous differential equation of the form:
\[ W_0 - \frac{1}{\mu^4} Z \cdot (Z \cdot \partial_3 \partial_3 + \tilde{Z} \cdot \partial_5 \partial_3) W_0 = Z \cdot P, \]  
(27)

where $P = P(\mu)$ is a function whose expansion only contains powers of $\pi$:
\[ P(\mu) = 12 + 144 \mu^2 \zeta_4 + \frac{342}{5} \mu^8 \zeta_8 + \cdots. \]  
(28)

Since the powers of $\zeta_2$ cancel out in $\delta_m$, we can ignore the $\mathcal{O}(\mu)$ terms in $P$.

To emphasise the predicitvity of (27) we remind that both $Z$ and $\tilde{Z}$ are $\mathcal{O}(\mu)$. In summary, the coactions on all $W_{k,0}$ are encoded in a pair of functions whose expansion contains only single zeta values. This is reminiscent of an observation made about string-theory amplitudes considered in [26].

While we focussed on the simplest case of $W_0(\mu)$ in this work, similar (and more involved) constructions are in principle possible for other sequences of wheel MZVs. For example the analogue of equation (22) for $W_1$ is
\[ (\mathcal{O}_k - \mu^2 \mathcal{O}_{k-4}) W_1 = 0, \]  
(29)

and this relation relies on a relation satisfied by “higher” components of the Q-functions:
\[ (\mathcal{O}_k - 2 \mu^2 \mathcal{O}_{k-4} + \mu^4 \mathcal{O}_{k-8}) q_4^{i,1} = 0 \]  
(30)
\[ (\mathcal{O}_k - 2 \mu^2 \mathcal{O}_{k-4} + \mu^4 \mathcal{O}_{k-8}) q_4^{i,0} = 0, \]  
(30)

for all $k = 3 \text{ mod } 4$ in all the available data.

We conclude the discussion with a comment on the prospects of generalisations of (22) and (29). One may be tempted to speculate equations for $W_i$ using operators $\mathcal{O}_k^{(i)}$ that are defined recursively as
\[ \mathcal{O}_k^{(i)} := \mathcal{O}_k^{(i-1)} - \mu^2 \mathcal{O}_k^{(i-1)} + \mathcal{O}_k^{(i-2)}, \quad \mathcal{O}_k^{(0)} = \mathcal{O}_k. \]  
(31)

However, given the range of depths that enter these equations being comparable to the amount of data available up to weight 31 does not allow for convincing checks. Therefore, it will be crucial to prove Conjecture 1, and its generalisations to make further statements on the coaction on wheel periods.
Subleading-weight pieces as polynomials in $\delta_m$

Separating the anomalous dimensions into the leading- and subleading-weight parts:

$$\delta_m = \delta_m + (-2)^m \tilde{\delta}_m$$  \hspace{1cm} (32)

where $\delta_m$ contains all terms of weight less than $4m - 3$, gives further insight to patterns that relate various anomalous dimensions to each other.

We observe that all known $\tilde{\delta}_m$, can be expressed in terms of $\delta_{m'}$ with $m' < m$:

$$\tilde{\delta}_1 = -\delta_0^2$$

$$\tilde{\delta}_2 = -8 \delta_1 \delta_0 + \frac{1}{2} (5 \delta_0^2 - 4 \delta_1) \delta_0$$

$$\tilde{\delta}_3 = -16 \delta_2 \delta_0 + 4 (7 \delta_0^2 - 4 \delta_1) \delta_1 - 7 \delta_0^4$$

$$\tilde{\delta}_4 = -40 \delta_3 \delta_0 + 10 (9 \delta_0^2 - 4 \delta_1) \delta_2 - 30 (4 \delta_0^2 - 3 \delta_1) \delta_1 \delta_0 + 21 \delta_0^6$$

$$\tilde{\delta}_5 = -96 \delta_4 \delta_0 + 24 (11 \delta_0^2 - 4 \delta_1) \delta_3 - 88 (5 \delta_0^2 - 6 \delta_1) \delta_0 \delta_2 - 44 (15 \delta_0^2 - 2 \delta_1) \delta_1^2 - 48 \delta_0^4 - 33 (2 \delta_0^2 - 15 \delta_1) \delta_0^3$$

$$\ldots$$

This motivates the following all-loop conjecture:

**Conjecture 2** The subleading-weight parts of the anomalous dimensions $\delta_m$ are always polynomials in $\delta_{m'}<m$, and are given by the following formula:

$$\tilde{\delta}_m = \frac{(2m)!}{(m+1)!} \frac{(\delta_0)^m}{m!} + (-2)^{m-A+1} \frac{m (2m - 1)}{16} \sum_{A > |a|} \delta^a$$ \hspace{1cm} (34)

The sum is over all multiplets $a = (a_1, \ldots, a_{|a|})$ of positive integers subject to the condition $A = \sum_{i=1}^{|a|} a_i > |a|$ and we used the shorthand using words $a = (a_1, a_2, \ldots)$:

$$\delta^a = \prod_{i=1}^{|a|} \frac{(\delta_{a_i-1})^{a_i}}{a_i!},$$ \hspace{1cm} (35)

where $|a|$ denotes the length of $a$, and $A = \sum_{i=1}^{|a|} a_i$ the sum of its letters, ie the degree of $\delta^a$.

We verified Conjecture 2 through $\tilde{\delta}_7$, ie up to weight 30.

**DISCUSSION**

In this Letter we demonstrated evidence for that the integrability of fishnet theory has a direct consequence on the properties of its periods under the Galois coaction. We identified a coaction principle that controls a sequence of numbers that enter the anomalous dimensions of the operator $Tr \phi_1^3$ and therefore a family of nested wheel diagrams. Using these observations we constructed a powerful equation satisfied by the these numbers, showing that complicated MZV expressions are in fact encoded in sequence of single Riemann zeta values. Furthermore we identified other recursive features of these numbers: We presented a structure that shows that numbers associated with smaller wheels recur naturally in the results for larger ones. We also made a conjecture determining the parts of anomalous dimensions of subleading transcendentality weight as polynomials in those of lower loop order.

While we analysed the anomalous dimensions order by order in the perturbative expansion of the anomalous dimension $\delta$, it would be very much in the spirit of integrability to make non-perturbative statements about its coaction. In particular the coaction on the resummed quantity $W_0(\mu)$ is reminiscent of that of hypergeometric functions, and we also remark that a non-physical toy model of (4) in [16] is solved in terms of these functions, whose coaction has been recently considered in [8, 27].

The relationship between the difference equations that these functions satisfy, and their Galois coaction is not yet clear.

We would like to point out that the cancellation of the terms involving powers of $\pi$ in $W_1$ in equation (16) relies on conspiracies between the periods $\delta_m$ and on combinatorial identities for MZVs in $\{1, 2, 3\}$.

These would be
generalisations of identities considered in number theory literature [28, 29], and their proofs would be a key to advancing the observations presented here.

An obvious question, and one of the main motivations of this letter, is on the possibility of reversing the logic presented: The Galois coaction on periods is believed to be a universal property of QFTs, and coactions on Feynman amplitudes are expected to satisfy relations that may be expressed as differential equations. It would be an exciting prospect to recast such relations into a difference equation such as (8), which could in turn imply previously-unknown integrable structures in QFTs.

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