Indecomposable and Noncrossed Product Division Algebras over Function Fields of Smooth $p$-adic Curves

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Abstract

We construct indecomposable and noncrossed product division algebras over function fields of connected smooth curves $X$ over $\mathbb{Z}_p$. This is done by defining an index preserving morphism $s : \text{Br}(\overline{K(X)})' \to \text{Br}(K(X))'$ which splits $\text{res} : \text{Br}(K(X)) \to \text{Br}(\overline{K(X)})$, where $\overline{K(X)}$ is the completion of $K(X)$ at the special fiber, and using it to lift indecomposable and noncrossed product division algebras over $\overline{K(X)}$.

Keywords: Brauer groups; division algebras; noncrossed products; indecomposable division algebras; ramification; function fields of smooth curves.

1. Introduction

Let $X$ be a connected smooth projective curve over $S = \text{Spec} \mathbb{Z}_p$, let $F = K(X)$ be its function field, and let $\overline{K(X)}$ denote the completion of $K(X)$ with respect to the discrete valuation on $K(X)$ defined by the special fiber $X_0$. We define an index-preserving homomorphism

$$\text{Br}(\overline{K(X)})' \to \text{Br}(K(X))'$$

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that splits the restriction map \( \text{res} : \text{Br}(K(X))' \to \text{Br}(\hat{K}(X))' \). Here the “prime”
denotes “prime-to-\( p \)”. The field \( K(\hat{X}) \) is not unlike a power series field over
a number field, and using the methods of \([7]\) and \([8]\), we construct certain
exotic kinds of division algebras over \( K(\hat{X}) \), and transfer these constructions to
\( K(X) \) using our homomorphism. In particular, we have a new construction of
noncrossed product division algebras and indecomposable division algebras of
unequal period and index over the rational function field \( \mathbb{Q}_p(t) \) (see Theorem
4.23 and Corollary 4.8). The indexes of our noncrossed product examples are as
low as \( q^2 \), for \( q \) an odd prime not equal to \( p \), and 8.

Recall if \( K \) is a field, a \( K \)-division algebra \( D \) is a division ring that is finite-
dimensional and central over \( K \). The period of \( D \) is the order of the class \([D]\)
in \( \text{Br}(K) \), and the index \( \text{ind}(D) \) is the square root of \( D \)'s \( K \)-dimension. A
noncrossed product is a \( K \)-division algebra whose structure is not given by a
Galois 2-cocycle. Noncrossed products were first constructed by Amitsur in \([1]\),
settling a longstanding open problem. Since then there have been several other
constructions, including \([34]\), \([24]\), \([7]\), \([9]\), \([33]\), \([20]\) and \([21]\). Saltman recently
showed that all division algebras of prime degree over our fields are cyclic \([37]\);
the indexes of our examples are all divisible by the square of a prime.

A \( K \)-division algebra is indecomposable if it cannot be expressed as the tensor
product of two nontrivial \( K \)-division algebras. It is easy to see that all division
algebras of equal period and index are indecomposable, and that all division
algebras of composite period are decomposable, so the problem of producing an
indecomposable division algebra is only interesting when the period and index
are unequal prime-powers. Albert constructed decomposable division algebras of
unequal (2-power) period and index in the 1930’s, but indecomposable division
algebras of unequal period and index did not appear until \([35]\) and \([2]\). Since
then there have been several constructions, including \([42]\), \([24]\), \([23]\), \([39]\), \([25]\),
\([8]\), and \([30]\). In \([6]\) two of the authors proved that over the function field of a
\( p \)-adic curve, any division algebra of (odd) prime period \( q \) not equal to \( p \) and index \( q^2 \) is decomposable, completing the proof that all division algebras of prime period \( q \) are crossed products over such fields (the index \( q \) case is \([37]\)).

Noncrossed products over a rational function field \( K(t) \) were constructed in \([9]\), for any \( p \)-adic field \( K \). However the construction here is much more general,
and our fields constitute a much larger class. For example, our methods apply
to fields such as \( K(X) = \mathbb{Q}_p(t)((\sqrt{t^3 + at + b})) \), where \( a, b \in \mathbb{Z}_p \), and \( p \neq 2, 3 \)
does not divide the discriminant \( 4a^3 + 27b^2 \). For here \( K(X) \) is the function field
of the elliptic curve \( X = \text{Proj} \mathbb{Z}_{p}[x, y, z]/(y^2z - x^3 - axz^2 - bz^3) \) (with \( t = x/z \)),
which is smooth over \( \mathbb{Z}_p \) by \([28]\), IV.3.30 and IV.3.35. Nevertheless, it is well
known that not all finite extensions of \( \mathbb{Q}_p(t) \) are function fields of smooth curves
over \( \mathbb{Z}_p \), as we will indicate; we do not consider such fields in this paper.

In his Ph.D. thesis (see \([10]\)), Feng Chen has constructed an index preserving
homomorphism \( \text{Br}(K(\hat{X})) \to \text{Br}(K(X)) \) over function fields of connected
smooth curves, this time over an arbitrary complete discrete valuation ring.
Chen’s approach is quite different from ours, building on patching techniques
developed by Harbater and Hartmann \([22]\). We believe that both methods are of
interest, and may in the future complement each other in the study of division algebras over function fields of curves over complete rings. Finally, we mention that it should be possible to transfer Hanke-Sonn’s comprehensive analysis of noncrossed products in [21] to our situation.

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**Notation.** Throughout this paper we let \((c)\) denote the image of \(c \in K^*\) in \(H^1(K, \mu_n)\). In general we write \(a \cdot b\) for the cup product of cohomology classes \(a\) and \(b\), unless \(a \in H^1(K, \mathbb{Q}/\mathbb{Z})\) and \(b = (c)\), in which case for historical reasons we write

\[(a, c) = a.(c) \in \text{Br}(K).\]

2. Tamely ramified covers of smooth curves

In this section we review some facts about smooth curves over complete discrete valuation rings and tamely ramified covers of them.

2.1. Smooth Curves and Marks

Let \(R\) be a noetherian ring. By a smooth curve \(X\) over \(R\) we mean a scheme \(X\) which is projective and smooth of relative dimension 1 over \(\text{Spec} \, R\). In particular, \(X\) is flat and of finite presentation over \(\text{Spec} \, R\).

By a mark \(D\) on \(X\) we mean an effective étale-relative Cartier divisor \(D\) on \(X\), that is, a closed subscheme of \(X\) that is étale over \(\text{Spec} \, R\) and whose defining ideal is invertible as an \(\mathcal{O}_X\)-module.

Note that the definitions of smooth curves, effective relative Cartier divisors, and marks are stable under arbitrary base change (see [18] 17.3.3 (iii), [16] 5.5.5 (iii), and [26] 1.1.4).

In this paper we work with smooth curves over complete discrete valuation rings. In the next lemma we collect some useful facts about them.

**Lemma 2.1.** Let \((R, \mathfrak{m}, k)\) be a complete discrete valuation ring with maximal ideal \(\mathfrak{m}\), residue field \(k = R/\mathfrak{m}\), and field of fractions \(K = \text{Frac} \, R\). Let \(X\) be a smooth curve over \(R\) and write \(X_0 \overset{df}{=} X \times_{\text{Spec} \, R} \text{Spec} \, k\) for its special fiber (a smooth curve over \(k\)). For any effective relative Cartier divisor \(D\) on \(X\), denote its restriction to \(X_0\) by \(D_0 \overset{df}{=} D \times_{\text{Spec} \, R} \text{Spec} \, k\).

1. Both \(X\) and \(X_0\) are regular.
2. \(X\) is connected if and only if \(X_0\) is connected.
3. Any effective relative Cartier divisor \(D\) on \(X\) is finite over \(\text{Spec} \, R\). In particular, we may write \(D = \text{Spec} \, S\) where \(S\) is a product of finite free local \(R\)-algebras.
4. Let \(D\) be an effective relative Cartier divisor \(D\) on \(X\). Then

\[D\text{ is a mark on } X \iff D_0\text{ is a mark on } X_0\]
v. Let $D$ be a mark on $X$. Then

$D$ is irreducible $\iff$ $D$ is integral $\iff$ $D$ is connected

Hence there is a 1-1 correspondence between irreducible components of a mark $D$ and those of $D_0$, and in particular, if $D$ is an integral mark, then so is $D_0$.

vi. If $D$ is an integral mark then $[k(D) : K] = [k(D_0) : k]$ where $K(D)$ and $k(D_0)$ denote the function fields of $D$ and $D_0$ respectively.

vii. Any irreducible effective Cartier divisor on $X$ other than the irreducible components of $X_0$ is relative. Moreover any mark $D_0$ on $X_0$ lifts to a mark $D$ on $X$.

Proof. Since $X$ and $X_0$ are smooth over $\text{Spec} \ R$ and $\text{Spec} \ k$ respectively, (i) follows from [18] 17.5.8 (iii). On the other hand (ii) is just a special case of [18] 18.5.19.

The structure map $D \to \text{Spec} \ R$ is proper as the composition of the closed immersion $D \hookrightarrow X$ and the projective morphism $X \to \text{Spec} \ R$, so the first assertion of (iii) follows from [26] 1.2.3. The second assertion follows from the fact that (by definition) finite morphisms are affine, that any finite algebra $S$ over a henselian ring $R$ is a product of finite local $R$-algebras (see [31] I.4.2 (b)), and that a finitely generated module over a local ring is flat if and only if it is free (see [29] 7.10). This proves (iii).

To prove (iv) we may assume by (iii) that $D = \text{Spec} \ S$ for some finite free (hence flat) local $R$-algebra $S$, and it remains to show that $S$ is unramified over $R$ if and only if $S \otimes_R k$ is unramified over $k$. This follows from [18] 17.4.1 (a),(d) since $S$, being a local ring, is unramified over $R$ if and only if it is unramified over $R$ at its maximal ideal (c.f. ibid, Définition 17.3.7).

To prove (v), first observe that if $D$ is a mark, then it is reduced by [17] I.9.2 since $R$ is a domain. Hence a mark is irreducible if and only if it is integral. Clearly if $D$ is irreducible then it must be connected; conversely, since $D \to \text{Spec} \ R$ is étale and $R$ is normal, $D$ is also normal (17 I.9.10), hence if $D$ is connected it must be irreducible. Therefore connected and irreducible components of $D$ agree, and since $D \to \text{Spec} \ R$ is proper and $R$ is henselian the rest of (v) follows directly from [18] 18.5.19 (or [31] I.4.2).

To prove (vi), write $D = \text{Spec} \ S$ for some finite free local $R$-algebra $S$ using (iii). Note that $[S \otimes_R K : K] = [S \otimes_R k : k]$ equals the rank of $S$ over $R$, hence it is enough to show that $S \otimes_R K = K(D)$ and $S \otimes_R k = k(D_0)$. Since $S$ is étale over $R$, $mS$ is the maximal ideal of $S$ and $S/mS = S \otimes_R k = k(D_0)$ is its residue field; on the other hand, $S \otimes_R K \subset \text{Frac} \ S$ is a localization of $S$ that contains $S$ and is étale over $K$, hence we must have $S \otimes_R K = \text{Frac} \ S = K(D)$.

Finally the first fact in (vii) follows from [28] IV.3.10. The second assertion is then a consequence of (iv) and [28] VIII.3.35 (see also [18] 21.9.11 (i) and 21.9.12).

2.2. Tamely ramified covers

Let $K$ be a field and $v : K \to \mathbb{Z} \cup \{\infty\}$ be a discrete valuation with residue field of characteristic $p$. Let $L/K$ be a finite separable field extension and $L'$ be
the Galois closure of $L$ in some separable closure of $K$ containing $L$. Let $\{w_i\}$ be the discrete valuations of $L'$ extending $v$ and denote by $I_i$ their inertia groups (see [45] V.2.3 or [27] VII.2). Recall that $L/K$ is said to be tamely ramified with respect to $v$ if $p$ does not divide $|I_i|$ for all $i$.

Let $X$ be an integral smooth curve over a complete discrete valuation ring $(R, m, k)$. By Lemma [2.1] (i) $X$ is regular, hence normal, so that each irreducible effective Weil (or Cartier) divisor $E$ defines a discrete valuation on the function field $K(X)$ of $X$, which we will denote by $v_E$. Now let $D$ be a mark on $X$ and $\rho: Y \to X$ be a finite (Spec $R$)-morphism of integral smooth curves over $R$. We say that $\rho$ is a tamely ramified cover of the pair $(X, D)$ if it is étale over $X - D$ and tamely ramified along $D$, that is, the function field $K(Y)$ of $Y$ is a tamely ramified extension of the function field $K(X)$ of $X$ with respect to the valuations defined by irreducible components of $D$. Étale locally, tamely ramified covers have the following description (see [45] 2.3.4 and [13] A.I.11): for each geometric closed point $y$: $\text{Spec } \Omega \to Y$ with image $x = \rho \circ y$: $\text{Spec } \Omega \to X$ there exist affine étale neighborhoods $\text{Spec } B \to Y$ and $\text{Spec } A \to X$ of $y$ and $x$ such that $B = A[w]/(w^n - z)$ for some $z \in A$ (an étale local coordinate of $D$) and some integer $n$ prime to the characteristic of $k$.

**Lemma 2.2.** Let $X$ be an integral smooth curve over a complete discrete valuation ring $(R, m, k)$, and $D$ be a mark on $X$. Let $\rho: Y \to X$ be a tamely ramified cover of $(X, D)$. Let $E$ be a mark on $X$ such that either $E \cap D = \emptyset$ or $E \subset D$. Then

i. $Y$ is flat over $X$ and equals the normalization of $X$ in $K(Y)$;
ii. $(\rho^{-1}E)_{\text{red}}$ is a mark;
iii. if $E$ is irreducible and $F$ is an irreducible mark on $Y$ lying over $E$, then the ramification (resp. the inertia) degree of $v_F$ over $v_E$ equals the ramification (resp. inertia) degree of $v_{F_0}$ over $v_{E_0}$.

**Proof.** The restriction $\rho_0: Y_0 \to X_0$ of $\rho$ to the special fibers is a finite generically étale map between smooth curves over a field, which is flat by [17] IV.1.3 (ii) for instance. Hence $\rho$ is also flat by the local criterion of flatness (see [17] IV.5.9). To finish the proof of (i), note that $Y$ is regular (Lemma [2.1] (i)) and thus normal, and since it is also integral over $X$, it equals the normalization of $X$ in $K(Y)$.

To prove (ii), assume first that $E \cap D = \emptyset$. Since $\rho^{-1}(X - D) \to X - D$ is étale by assumption, $\rho^{-1}E \to E$ is étale by base change. Therefore since $E$ is reduced, $\rho^{-1}E$ is reduced ([17] I.9.2), and since $E \to \text{Spec } R$ is already étale, the composition $\rho^{-1}E \to E \to \text{Spec } R$ is étale, hence $(\rho^{-1}E)_{\text{red}} = \rho^{-1}E$ is a mark.

Now suppose that $E \subset D$; we may assume without loss of generality that $E$ is a connected and hence irreducible component of $D$ (see Lemma [2.1] (v)). We first show that each connected component of $\rho^{-1}E$ is irreducible. We start by understanding the situation locally.

Let $x_0$ be the closed point of $E$ and $y_0 \in Y_0$ be such that $\rho(y_0) = x_0$. Write $A = \mathcal{O}_{X, x_0}$ and $B = \mathcal{O}_{Y, y_0}$; both are 2-dimensional noetherian regular local
rings and hence also factorial domains by Auslander-Buchsbaum’s theorem. We have an exact sequence

\[ 0 \rightarrow I_{E,x_0} \rightarrow \mathcal{O}_{X,x_0} \rightarrow \mathcal{O}_{E,x_0} \rightarrow 0 \]

which can be rewritten as

\[ 0 \rightarrow A \rightarrow A \rightarrow A/(z) \rightarrow 0 \]

where \( z \in A \) is a prime element defining \( E \) so that \( E = \text{Spec} A/(z) \) (recall that \( E \) is a local affine scheme by Lemma 2.1 (iii)). Since \( A/(z) \) is étale over the complete discrete valuation ring \( R \), \( A/(z) \) is normal, hence it is a complete discrete valuation ring, and it follows that \( z \) is part of a regular system of parameters of \( A \), and that \( (z) + mA \) is the maximal ideal of \( A \). Write \( A_0 \stackrel{\text{def}}{=} \mathcal{O}_{X_0,x_0} = A \otimes_R k = A/mA \), and let \( z_0 \) be the image of \( z \) in \( A_0 \). Then \( A_0 = (A_0, (z_0), k) \) is a discrete valuation ring whose maximal ideal \((z_0)\) defines \( x_0 = E_0 \).

The closed subscheme \( \rho^{-1}E \) of \( Y \) is a relative Cartier divisor by flat pull-back (see [26], 1.1.4), hence by Lemma 2.1 (iii) there is a 1-1 correspondence between the connected components of \( \rho^{-1}E \) and its closed points. Showing that the connected component of \( \rho^{-1}E \) going through \( y_0 \) is irreducible is now equivalent to showing that \( z \) is divisible by a single prime factor \( w \) in \( B \), so that it can be written as \( z = u \cdot w^e \), \( u \in B^\times \). This follows from the étale local description of \( \rho: Y \rightarrow X \) (see [13] 2.3.4 and [12], 1.3.2 and A.1.11): since \( A \) is a regular local ring, \( B \) is the localization of the normalization of \( A \) in \( K(Y) \) with respect to one of its maximal ideals, and \( B \) is tamely ramified just along \( z \), for some integer \( e \) prime to \( \text{char} \ k \) we have a commutative diagram

\[
\begin{array}{ccc}
B & \rightarrow & B_{sh} = A_{sh}[T] / (T^e - z) \\
\downarrow & & \downarrow \\
A & \rightarrow & A_{sh}
\end{array}
\]

where \( A_{sh} \) denotes the strict henselization of \( A \), and \( B_{sh} \) that of \( B \). Observe that all four maps are faithfully flat (see [13] 18.8.8 (iii)), and that all four rings are noetherian regular local rings, and thus factorial domains. In particular, the associated maps between spectra are surjective (see [31] 1.2.7 (c) ) and preserve height 1 prime ideals (by [26] 15.1), all of which are principal (ibid. 20.1). Hence in order to show that there is a single prime \( (w) \) in \( B \) lying over \( (z) \), it is enough to show that there is a single prime in \( B_{sh} \) lying over \( (z) \). Notice that \( z \) stays prime in \( A_{sh} \). For as shown above, \( z \) is part of a regular system of parameters of \( A \). Since \( A_{sh} \) is unramified over \( A \), the maximal ideal of \( A_{sh} \) is generated by the maximal ideal of \( A \), and therefore \( z \) is also part of a regular system of parameters of \( A_{sh} \), which implies that \( z \) is prime in \( A_{sh} \). The only prime in \( B_{sh} \) lying above \( (z) \) is \((T)\), since the fiber \( \text{Spec} B_{sh} \otimes_{A_{sh}} \mathcal{O}(z) = \text{Spec} \mathcal{O}(z)[T]/(T^e) \)
consists of a single point (here \( \kappa(z) = \text{Frac} A/(z) \) denotes the residue field of the prime \((z)\)). The image of \((\overline{T})\) in Spec \(B\) is the unique prime \((w)\) lying over \((z)\), and since \(B_{sh}\) is unramified over \(B\), \((w)B_{sh} = (\overline{T})\) in \(B_{sh}\). This completes the proof that each connected component of \(\rho^{-1}E\) is irreducible.

It remains to show that each connected (irreducible) component of \((\rho^{-1}E)_{\text{red}}\) is a mark. Write \(B_0 \overset{\text{df}}{=} \mathcal{O}_{Y_0,y_0} = B \otimes_R k = B/mB\) and let \(w_0\) be the image of \(w\) in \(B_0\). By Lemma 2.1 (iv), to show that the connected component Spec \(B/(w)\) of \((\rho^{-1}E)_{\text{red}}\) is a mark, it is enough to show that its restriction Spec \(B_0/(w_0)\) to the special fiber is a mark, i.e., that \(w_0\) is a uniformizer of the discrete valuation ring \(B_0\). Now observe that the extension of the ideal \(b \overset{\text{df}}{=} (w) + mB\) of \(B\) to \(B_{sh}\) is the maximal ideal \(bB_{sh} = (\overline{T}) + mB_{sh}\) of \(B_{sh}\): in fact, by direct computation we have an isomorphism

\[
\frac{B_{sh}}{(T) + mB_{sh}} = \frac{A_{sh}}{(z) + mA_{sh}}
\]

and since \((z) + mA\) is the maximal ideal of \(A\), \((z) + mA_{sh}\) is the maximal ideal of \(A_{sh}\). On the other hand, \(B_{sh}\) is faithfully flat over \(B\), hence \(b\) must be the maximal ideal of \(B\) (see [29] 7.5 (ii)). Therefore \((w_0)\), the image of \(b\) in \(B_0\), is the maximal ideal of \(B_0\), as was to be shown.

We now prove (iii). Note that \(E_0\) and \(F_0\) are irreducible marks by Lemma 2.1 (vi) so that \(v_{E_0}\) and \(v_{F_0}\) are well-defined. Denoting \(K = \text{Frac} R\), and by \(K(F), K(E), k(F_0), k(E_0)\) the function fields of \(F, E, F_0, E_0\), we have by Lemma 2.1 (vi) that

\[
[K(F) : K(E)] = \frac{[K(F) : k]}{[K(E) : k]} = \frac{[k(F_0) : k]}{[k(E_0) : k]} = [k(F_0) : k(E_0)]
\]

showing that the inertia degree of \(v_F\) over \(v_E\) equals that of \(v_{F_0}\) over \(v_{E_0}\).

To show equality of ramification degrees, we keep the notation in the proof of (ii). If \(E \cap D = \emptyset\), then \(F \to E\) and \(F_0 \to E_0\) are both étale, so the ramification degree is 1 in both cases. Now assume that \(E \subseteq D\); observe that \(z\) and \(w\) are uniformizers of \(v_E\) and \(v_F\), respectively. We showed above that \((z_0)\) is the maximal ideal of the discrete valuation ring \(A_0\), i.e., \(z_0\) is a uniformizer of \(v_{E_0}\), and similarly \(w_0\) is a uniformizer of \(v_{F_0}\). Since \(z = w^e \cdot u\), where \(u \in B^\times\) and \(e\) is the ramification degree of \(v_F\) over \(v_E\), we have \(z_0 = w_0^e \cdot u_0\), where \(u_0\) is the image of \(u\) in \(B_0^\times\), and thus \(v_{F_0}(z_0) = e\) as desired. □

2.3. An equivalence of categories

Let \((R, m, k)\) be a complete discrete valuation ring, \(X\) be a smooth integral curve over \(R\), and \(D\) be a mark on \(X\). We write Rev\(_R\)^\(_D\)(\(X\)) for the category whose objects are the tamely ramified covers of \((X, D)\) and whose arrows are the \(X\)-morphisms. We have a restriction functor Rev\(_R\)^\(_D\)(\(X\)) \to Rev\(_k\)^\(_D\)(\(X_0\)) taking a tamely ramified cover \(Y\) of \((X, D)\) to the tamely ramified cover \(Y_0\) of \((X_0, D_0)\), and a map \(f: Y \to Z\) to its restriction \(f_0 \overset{\text{df}}{=} f \times_{\text{Spec} R} \text{Spec} k: Y_0 \to Z_0\) to
the special fibers. Observe that by Lemma 2.1 (i) and the definition of tamely ramified cover all objects in $\text{Rev}_{D}^{R}(X)$ and $\text{Rev}_{D}^{0}(X_0)$ are regular schemes.

Amazingly, this functor $\text{Rev}_{D}^{R}(X) \to \text{Rev}_{D}^{0}(X_0)$ is an equivalence of categories (see [43] 3.1.3 for the proof):

**Theorem 2.3.** (Grothendieck) Let $(R, m, k)$ be a complete discrete valuation ring, $X$ be a smooth integral curve over $R$, and $D$ be a mark on $X$. Then restriction to the special fibers gives an equivalence of categories

$$\text{Rev}_{D}^{R}(X) \xrightarrow{\approx} \text{Rev}_{D}^{0}(X_0)$$

For any scheme $X$ and effective Cartier divisor $D$ we write $\pi^t_{1}(X, D, \overline{x})$ for the tame fundamental group of $X$ with respect to $D$ with geometric base point $\overline{x}: \text{Spec } \Omega \to X - D$ (see [43] 4.1.2, [17] XIII.2.1.3 or [13] A.I.13). By definition, $\pi^t_{1}(X, D, \overline{x})$ classifies pointed tamely ramified covers of $(X, D)$, and thus we obtain the following (c.f. [17] X.2.1)

**Corollary 2.4.** With the notation and hypotheses of the previous theorem, let $\overline{x}_0$ be a geometric point of $X_0 - D_0$. Then the natural map

$$\pi^t_{1}(X_0, D_0, \overline{x}_0) \xrightarrow{\approx} \pi^t_{1}(X, D, \overline{x}_0)$$

of tame fundamental groups is an isomorphism.

2.4. The residue map

In what follows, all cohomology groups are étale cohomology groups. For a ring $R$ and étale sheaf $F$ on $\text{Spec } R$ we write $H^a(R, F)$ instead of $H^a(\text{Spec } R, F)$. In particular, for a field $K$, $H^a(K, F)$ agrees with the Galois cohomology group $H^a(G_K, F)$ where $G_K = \text{Gal}(K_{\text{sep}}/K)$ denotes the absolute Galois group of $K$ and where we still write $F$ for the corresponding $G_K$-module.

Let $K$ be any field, let $\nu: K \to \mathbb{Z} \cup \{\infty\}$ be a discrete valuation on $K$, and let $k$ be its residue field. Recall that for any integer $r$ and any integer $n$ prime to the characteristic of $k$ there is a group morphism

$$\partial_{\nu}: H^a(K, \mu_n^{\otimes r}) \to H^{a-1}(k, \mu_n^{\otimes (r-1)})$$

called the residue or ramification map (see [14] II.7.9 or [13] VI.8). The residue map has the following functorial behavior: if $L$ is a finite extension of $K$ and $\nu: L \to \mathbb{Z} \cup \{\infty\}$ is a discrete valuation with residue field $l$ such that $\nu$ extends $v$ then we have a commutative diagram

$$\begin{array}{ccc}
H^a(L, \mu_n^{\otimes r}) & \xrightarrow{\partial_{\nu}} & H^{a-1}(l, \mu_n^{\otimes (r-1)}) \\
\text{res} & & \text{res} \\
H^a(K, \mu_n^{\otimes r}) & \xrightarrow{\partial_{\nu}} & H^{a-1}(k, \mu_n^{\otimes (r-1)})
\end{array}$$
where $e_{w/v}$ denotes the ramification degree of $w$ over $v$, and res denotes cohomological restriction.

If $X$ is a normal integral scheme and $D \subset X$ is an irreducible Weil divisor then we write

$$\partial_D : \mathcal{H}^a(K(X), \mu^\otimes_n) \to \mathcal{H}^{a-1}(K(D), \mu^\otimes_n(r-1))$$

for the residue map with respect to the discrete valuation $v_D$.

**Lemma 2.5.** Let $X$ be a smooth curve over a complete discrete valuation ring, and let $n$ be an invertible integer on $X$ (i.e., $n$ is prime to all residue characteristics on $X$). Let $D$ be a mark on $X$, $U = X - D$, and denote by $j : U \hookrightarrow X$ and $i : D \hookrightarrow X$ the corresponding open and closed immersions. We have an exact Gysin sequence

\[ 0 \to \mathcal{H}^1(X, \mu^\otimes_n) \to \mathcal{H}^1(U, \mu^\otimes_n) \to \mathcal{H}^0(D, \mu^\otimes_n(r-1)) \to \cdots \]

where $\mathcal{H}^a(X, \mu^\otimes_n) \to \mathcal{H}^a(U, \mu^\otimes_n)$ are the natural restriction maps, and the maps $\mathcal{H}^a(U, \mu^\otimes_n) \to \mathcal{H}^{a-1}(D, \mu^\otimes_n(r-1))$ are compatible with the residue maps in the sense that the following diagram commutes up to sign:

\[ \begin{array}{ccc} \mathcal{H}^0(U, \mu^\otimes_n) & \to & \mathcal{H}^1(D, \mu^\otimes_n(r-1)) \\ & \downarrow & \downarrow \\ \mathcal{H}^0(K(X), \mu^\otimes_n) & \xrightarrow{\partial_D} & \mathcal{H}^1(K(D), \mu^\otimes_n(r-1)) \end{array} \]

**Proof.** Note the conclusions make sense even if $D$ is reducible, for in this case $D$ is the disjoint union of its irreducible components and $K(D)$ is a direct product of the corresponding function fields. The long exact Gysin sequence will follow once we show that

\[ R^q j_\ast \mu^\otimes_n,U = \begin{cases} \mu^\otimes_n,X & \text{if } q = 0 \\ i_\ast \mu^\otimes_n,D & \text{if } q = 1 \\ 0 & \text{if } q \geq 2 \end{cases} \]

For then the Leray spectral sequence

\[ \mathcal{H}^p(X, R^q j_\ast \mu^\otimes_n,U) \Rightarrow \mathcal{H}^{p+q}(U, \mu^\otimes_n,U) \]

degenerates, and as $i_\ast$ is an exact functor we may substitute $\mathcal{H}^{q-1}(D, \mu^\otimes_n,D)$ for $\mathcal{H}^{q-1}(X, i_\ast \mu^\otimes_n,D)$, by the Leray spectral sequence for $i_\ast$.

Since $D$ is a mark, $(X, D)$ is a smooth $(\textrm{Spec } R)$-pair of codimension $c = 1$, and hence by purity ([31] VI.5.1) we already know that $R^q j_\ast \mu^\otimes_n,U = 0$ for $q \neq 0,1$, and that $j_\ast \mu^\otimes,n,U = \mu^\otimes_n,X$. It remains to compute $R^1 j_\ast \mu^\otimes_n,U$. 


By XIX.3 we know that \( R^1 j_* \mu_{n,U} = i_*(\mathbb{Z}/n_D) \). For the general case, consider the cup product map
\[
p_{n,X} \otimes i_*(\mathbb{Z}/n_D) = R^0 j_* p_{n,U} \otimes R^1 j_* \mu_{n,U} \xrightarrow{\cup} R^1 j_* p_{n,U}
\]
We see this is an isomorphism by looking at stalks. Since \( i^* \mu_{n,X} = \mu_{n,D} \), we obtain a sequence of maps
\[
R^1 j_* \mu_{n,U} \xrightarrow{\cup} \mu_{n,X} \otimes i_*(\mathbb{Z}/n_D) \xrightarrow{\text{can}} i_* i^* \mu_{n,X} \otimes i_*(\mathbb{Z}/n_D) \xrightarrow{\cup} i_* \mu_{n,D}
\]
which we see are isomorphisms, again by looking at stalks. This yields the required isomorphism \( R^1 j_* \mu_{n,U} \approx i_* \mu_{n,D} \).

Finally, to prove the compatibility with the residue map, we may assume that \( D \) is connected. Observe that \( K(D) \) is the residue field of \( \mathcal{O}_{v_D} \). By the naturality of the Leray spectral sequence we have a commutative diagram
\[
\begin{array}{cccccc}
\cdots & \rightarrow & H^n(X, \mu_n^{\otimes r}) & \rightarrow & H^n(U, \mu_n^{\otimes r}) & \rightarrow & H^{n-1}(D, \mu_n^{\otimes (r-1)}) & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \rightarrow & H^n(\mathcal{O}_{v_D}, \mu_n^{\otimes r}) & \rightarrow & H^n(K(X), \mu_n^{\otimes r}) & \xrightarrow{(*)} & H^{n-1}(K(D), \mu_n^{\otimes (r-1)}) & \rightarrow & \cdots 
\end{array}
\]
whose rows are Gysin sequences, and (*) is known to be the residue map with respect to the valuation \( v_D \) (see XIII.3.3) possibly up to sign.

**Remark 2.6.** Let \( K \) be a field. We apply the previous lemma to \( X = \text{Spec } K[t] \). Since \( X \rightarrow \text{Spec } K \) is acyclic (XI.4.20) we have \( H^n(X, \mu_n^{\otimes r}) = H^n(K(\mu_n^{\otimes r})) \). Moreover, any mark \( D \) is a disjoint union of closed points, therefore we have \( H^{n-1}(D, \mu_n^{\otimes (r-1)}) = \bigoplus_{P \in D} H^{n-1}(K(P), \mu_n^{\otimes (r-1)}) \). Thus the Gysin sequence for \( X \) reads
\[
\cdots \rightarrow H^n(K, \mu_n^{\otimes r}) \rightarrow H^n(U, \mu_n^{\otimes r}) \rightarrow \bigoplus_{P \in D} H^{n-1}(K(P), \mu_n^{\otimes (r-1)}) \rightarrow \cdots
\]
where \( U = X - D \). On the other hand, since étale cohomology commutes with projective limits of schemes (III.1.16) and \( \text{Spec } K(t) = \text{proj lim}_D X - D \), where \( D \) runs over all marks of \( X \), by taking limits we obtain
\[
\cdots \rightarrow H^n(K, \mu_n^{\otimes r}) \rightarrow H^n(K(t), \mu_n^{\otimes r}) \rightarrow \bigoplus_{P \in X^{(1)}} H^{n-1}(K(P), \mu_n^{\otimes (r-1)}) \rightarrow \cdots
\]
where \( X^{(1)} \) denotes the set of closed points (i.e. points of codimension 1) of \( X \). This is just the familiar (affine) Faddeev sequence with finite coefficients (IX.6.9.3), which splits into short exact sequences
\[
0 \rightarrow H^n(K, \mu_n^{\otimes r}) \rightarrow H^n(K(t), \mu_n^{\otimes r}) \rightarrow \bigoplus_{P \in X^{(1)}} H^{n-1}(K(P), \mu_n^{\otimes (r-1)}) \rightarrow 0
\]
via the coreidue maps
\[
\psi_P : H^{n-1}(K(P), \mu_n^{\otimes (r-1)}) \to H^n(K(t), \mu_n^{\otimes r})
\]
\[\xi \mapsto \text{cor}_{K(P)(t)/K(t)}(\xi(t - \tau_P))\]
where \(\tau_P\) denotes the image of \(t\) in \(K(P)\) (so that \(K(P) = K(\tau_P)\)) and \((t - \tau_P)\) is the image of \(t - \tau_P\) in \(H^1(K(P)(t), \mu_n)\).

3. Splitting the restriction map

3.1. Setup and conventions

Henceforth we write
- \((R, m, k) = \text{complete discrete valuation ring with finite residue field } k \text{ of characteristic } p, \text{ and fraction field } K = \text{Frac } R \text{ (a local field)};\]
- \(\pi = \text{a uniformizer of } R;\]
- \(n = \text{integer prime to } p;\]
- \(X = \text{a smooth integral curve over } R;\]
- \(X_0 = \text{the special fiber of } X \text{ (a smooth integral curve over } k);\]
- \(K(X) = \text{the function field of } X.\]
- \(k(X_0) = \text{the function field of } X_0 \text{ (a global field);}\]
- \(\hat{K}(X) = \text{completion of } K(X) \text{ with respect to the valuation defined by the}\]
  \(\text{special fiber } X_0. \text{ Observe that } \pi \text{ is also a uniformizer of } \hat{K}(X) \text{ and that}\]
  \(\text{its residue field is } k(X_0);\]
- \(V = \text{a fixed set of marks on } X \text{ lifting each mark (i.e closed point) of } X_0, \text{ see Lemma 2.1 (vii).}\]

By 28 VIII.3.4, the set \(V\) is in 1-1 correspondence with a subset of closed points of the generic fiber \(X_0 \overset{df}{=} X \times_{\text{Spec } R} \text{Spec } K\). In what follows, we will identify these two sets and refer to the unique mark \(D \in V\) (or closed point \(P \in X_0\) whose closure equals \(D\)) lifting a closed point \(P_0 \in X_0\) as the \(V\)-lift of \(P_0\). For instance, if \(X = \mathbb{P}^1_R = \text{Proj } R[x, y]\) and we choose the mark defined by \(y\) to be the \(V\)-lift of the “infinite point” of \(X_0 = \mathbb{P}^1_k = \text{Proj } k[x, y]\) defined by \(y\), then specifying the remaining \(V\)-lifts amounts to choosing a monic lift in \(R[t]\) for each monic irreducible polynomial in \(k[t]\) (where \(t = x/y\)).
3.2. Splitting the restriction map

In this section we construct a map

$$s = s_{V, \pi} : \text{Br}(K(X))' \rightarrow \text{Br}(K(X))'$$

splitting the restriction map

$$\text{res} : \text{Br}(K(X))' \rightarrow \text{Br}(K(X))'$$

Here ' denotes the prime-to-p part of the corresponding group. In the next section we show that this map preserves the index.

**Lemma 3.1.** (Tame lifting) The choice of $V$ defines, for each $a \geq 0$ and $r \in \mathbb{Z}$, a group morphism

$$\lambda_V : H^a(k(X_0), \mu_n^{\otimes r}) \rightarrow H^a(K(X), \mu_n^{\otimes r})$$

compatible with the residue maps: for each irreducible mark $D \in V$,

$$H^a(k(X_0), \mu_n^{\otimes r}) \xrightarrow{\partial_D} H^{a-1}(k(D_0), \mu_n^{\otimes (r-1)}) \xrightarrow{\text{can}} H^{a-1}(K(D), \mu_n^{\otimes (r-1)})$$

commutes up to sign, where the bottom arrow is given by the composition

$$H^{a-1}(k(D_0), \mu_n^{\otimes (r-1)}) \xrightarrow{\text{can}} H^{a-1}(D, \mu_n^{\otimes (r-1)}) \xrightarrow{\text{can}} H^{a-1}(K(D), \mu_n^{\otimes (r-1)}).$$

**Proof.** Let $D$ be a mark with support in $V$, and set $U = X - D$. Consider the commutative diagram

$$\cdots \rightarrow H^a(X, \mu_n^{\otimes r}) \xrightarrow{\approx} H^a(U, \mu_n^{\otimes r}) \xrightarrow{\approx} H^{a-1}(D, \mu_n^{\otimes (r-1)}) \xrightarrow{\approx} \cdots$$

where the rows are the exact Gysin sequences for $(X, D)$ and $(X_0, D_0)$ respectively (see Lemma 2.5), and the vertical arrows are the natural ones (restrictions to the fibers). Since $R$ is henselian, the left and right arrows are isomorphisms by proper base change ([31] VI.2.7), hence so is the middle one by the 5-lemma.

Now define $\lambda_D$ as the composition

$$\lambda_D : H^a(U_0, \mu_n^{\otimes r}) \xrightarrow{\approx} H^a(U_0, \mu_n^{\otimes r}) \xrightarrow{\text{can}} H^a(U, \mu_n^{\otimes r})$$

$$\cdots \rightarrow H^a(U_0, \mu_n^{\otimes r}) \xrightarrow{\approx} H^a(U, \mu_n^{\otimes r}) \xrightarrow{\approx} H^{a-1}(D_0, \mu_n^{\otimes (r-1)}) \xrightarrow{\approx} \cdots$$

where the rows are the exact Gysin sequences for $(X, D)$ and $(X_0, D_0)$ respectively (see Lemma 2.5), and the vertical arrows are the natural ones (restrictions to the fibers). Since $R$ is henselian, the left and right arrows are isomorphisms by proper base change ([31] VI.2.7), hence so is the middle one by the 5-lemma.
Consider the set \( V \) of all marks with support in \( V \) and order them by inclusion. Since étale cohomology commutes with projective limits of schemes (\cite{10} III.1.16) and
\[
\text{Spec } k(X_0) = \text{proj lim } U_0
\]
Taking the direct limit of the \( \lambda_D \) over all \( D \in V \) we obtain the desired map
\[
\lambda' : H^a(k(X_0), \mu_n^\otimes r) \rightarrow H^a(K(X), \mu_n^\otimes r).
\]
Since by Lemma 2.5 the Gysin sequences are compatible with residue maps up to sign, and the arrow
\[
H^{a-1}(k(D_0), \mu_n^\otimes (r-1)) \xrightarrow{\text{can}} H^{a-1}(D, \mu_n^\otimes (r-1))
\]
is invertible, we see that \( \lambda_V \) is also compatible with residue maps. \( \Box \)

**Remark 3.2.** In case \( X = \mathbb{P}^1_k \), we can give a more explicit description of the tame lifting using the Faddeev sequence (see Remark 2.6). Lifting the point at infinity as in the example of Section 2.1, the map \( \lambda_V \) can be defined by the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & H^a(k, \mu_n^\otimes r) & \rightarrow & H^a(k(X_0), \mu_n^\otimes r) & \rightarrow & \bigoplus_{P_0 \in X_0^{(1)}} H^{a-1}(k(P_0), \mu_n^\otimes (r-1)) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & H^a(K, \mu_n^\otimes r) & \rightarrow & H^a(K(X), \mu_n^\otimes r) & \rightarrow & \bigoplus_{P \in X^{(1)}} H^{a-1}(K(P), \mu_n^\otimes (r-1)) & \rightarrow & 0
\end{array}
\]

where each row is the split exact Faddeev sequence of Remark 2.6. The left vertical arrow is the natural one while the right vertical arrow sends, via the natural map \( H^{a-1}(k(P_0), \mu_n^\otimes (r-1)) \rightarrow H^{a-1}(K(P), \mu_n^\otimes (r-1)) \), the \( P_0 \)-th component to the \( P \)-th component where \( P \) denotes the generic point of the \( V \)-lift of \( P_0 \). Explicitly, using the splitting given by the coresidence maps \( \psi_{P_0} \), we may write an element of \( H^a(k(X_0), \mu_n^\otimes r) \) as \( \alpha_0 + \sum_{P_0} \psi_{P_0}(\xi_{P_0}) \) with \( \alpha_0 \in H^a(k, \mu_n^\otimes r) \) and \( \xi_{P_0} \in H^{a-1}(k(P_0), \mu_n^\otimes (r-1)) \). Then
\[
\lambda_V \left( \alpha_0 + \sum_{P_0} \psi_{P_0}(\xi_{P_0}) \right) = \alpha + \sum_P \psi_P(\xi_P)
\]
where \( P \) is the closed point of \( X_n \) corresponding to the \( V \)-lift of \( P_0 \) and \( \alpha \in H^a(K, \mu_n^\otimes r) \) and \( \xi_P \in H^{a-1}(K(P), \mu_n^\otimes (r-1)) \) denote the unramified lifts of \( \alpha_0 \) and \( \xi_{P_0} \) respectively.

**Lemma 3.3.** Let \( \chi_0 \in H^1(k(X_0), \mathbb{Z}/n) \), and let \( D_0 \subset X_0 \) be the ramification locus of \( \chi_0 \). Denote by \( Y_0 \) the cyclic tamely ramified cover of \((X_0, D_0)\) defined by \( \chi_0 \). Let \( \chi = \lambda_V(\chi_0) \in H^1(K(X), \mathbb{Z}/n) \) be as in the previous lemma. Then \( \chi \) defines the tamely ramified cover \( Y \) of \((X, D)\) lifting \( Y_0 \) in Theorem 2.5, where \( D \) is the \( V \)-lift of \( D_0 \).
Proof. By definition of $\lambda_V$, $\chi \in H^1(X - D, \mathbb{Z}/n) \subset H^1(K(X), \mathbb{Z}/n)$ is the unique character that restricts to $\chi_0 \in H^1(X_0 - D_0, \mathbb{Z}/n) \subset H^1(k(X_0), \mathbb{Z}/n)$. Since the groups $H^1(X - D, \mathbb{Z}/n) = \text{Hom}_{\text{cont}}(\pi_1'(X, D), \mathbb{Z}/n)$ and $H^1(X_0 - D_0, \mathbb{Z}/n) = \text{Hom}_{\text{cont}}(\pi_1'(X_0, D_0), \mathbb{Z}/n)$ classify degree $n$ (tame) cyclic Galois covers of $(X, D)$ and $(X_0, D_0)$ (see [13] I.2.11), and the restriction map $\text{res}: H^1(X - D, \mathbb{Z}/n) \to H^1(X_0 - D_0, \mathbb{Z}/n)$ is given by the natural map $\pi_1'(X_0, D_0) \xrightarrow{\sim} \pi_1'(X, D)$ induced by the functor $Y \to Y_0$ (see Corollary [21]), the cyclic Galois cover $Y$ of $(X, D)$ defined by $\chi$ restricts to the cyclic Galois cover $Y_0$ of $(X_0, D_0)$ defined by $\chi_0$, and we are done.

Theorem 3.4. Let $X$, $K(X)$, $\hat{K}(X)$ and $n$ be as in Section 3.1. Each choice of $\pi$ and $V$ as in Section 3.1 defines, for each $a \geq 0$ and all $r \in \mathbb{Z}$, a group morphism

$$s = s_{V, \pi}: H^a(\hat{K}(X), \mu_n^{\otimes r}) \to H^a(K(X), \mu_n^{\otimes r})$$

splitting $\text{res}: H^a(K(X), \mu_n^{\otimes r}) \to H^a(\hat{K}(X), \mu_n^{\otimes r})$, that is, such that $\text{res} \circ s$ is the identity.

Proof. Let $A = O_{X, \eta_0}$ where $\eta_0$ denotes the generic point of $X_0 \subset X$. Then $A$ is a discrete valuation ring; let $\hat{A}$ be its completion, so that $\hat{K}(X) = \text{Frac} \hat{A}$. Observe that the residue fields of both $A$ and $\hat{A}$ are equal to $k(X_0)$, and that $\pi$ is a uniformizer for both discrete valuation rings. We have an exact Witt sequence (see [14] II.7.10 and II.7.11)

$$0 \to H^a(k(X_0), \mu_n^{\otimes r}) \xrightarrow{\partial_{X_0}} H^{a-1}(k(X_0), \mu_n^{\otimes (r-1)}) \to 0$$

split by the cup product with $(\pi) \in H^1(\hat{K}(X), \mu_n)$:

$$H^{a-1}(k(X_0), \mu_n^{\otimes (r-1)}) \xrightarrow{-\cdot (\pi)} H^a(\hat{K}(X), \mu_n^{\otimes r})$$

Hence each element of $H^a(\hat{K}(X), \mu_n^{\otimes r})$ can be uniquely written as a sum $\alpha_0 + \chi_0(\pi)$ with

$$\alpha_0 \in H^a(k(X_0), \mu_n^{\otimes r}) = H^a(\hat{A}, \mu_n^{\otimes r}) \subset H^a(\hat{K}(X), \mu_n^{\otimes r})$$

and

$$\chi_0 \in H^{a-1}(k(X_0), \mu_n^{\otimes (r-1)}) = H^{a-1}(\hat{A}, \mu_n^{\otimes (r-1)}) \subset H^{a-1}(\hat{K}(X), \mu_n^{\otimes (r-1)})$$

We define

$$s(\alpha_0 + \chi_0(\pi)) = \alpha + \chi(\pi)$$

where

$$\alpha = \lambda_V(\alpha_0) \in H^a(K(X), \mu_n^{\otimes r})$$

and

$$\chi = \lambda_V(\chi_0) \in H^{a-1}(K(X), \mu_n^{\otimes (r-1)})$$

are the tame lifts given by Lemma 3.1.
In order to show that \( \text{res} \circ s = \text{id} \) it is enough to prove that \( \alpha \big|_{K(X)} = \alpha_0 \) and \( \chi \big|_{K(X)} = \chi_0 \). But this follows from the functoriality of cohomology: for instance, for \( \alpha_0 \), let \( U_0 \) be an open set on which \( \alpha_0 \) is defined (i.e., \( \alpha_0 \) belongs to the image of \( H^a(U_0, \mu_n^{\otimes r}) \rightarrow H^a(k(X_0), \mu_n^{\otimes r}) \)), let \( D_0 = X_0 - U_0 \), let \( D \) be the \( V \)-lift of \( D_0 \), and let \( U = X - D \). Observe that the generic point of \( X_0 \) belongs to \( U \) so that the natural map \( H^a(U, \mu_n^{\otimes r}) \rightarrow H^a(k(X), \mu_n^{\otimes r}) \) factors through \( H^a(A, \mu_n^{\otimes r}) \). Consequently we have a commutative diagram

\[
\begin{array}{c}
\xymatrix{
H^a(U_0, \mu_n^{\otimes r}) \ar[r]_{\text{res}} & H^a(U, \mu_n^{\otimes r}) \\
H^a(k(X_0), \mu_n^{\otimes r}) \ar[r] & H^a(A, \mu_n^{\otimes r}) \ar[r] & H^a(K(X), \mu_n^{\otimes r}) \\
H^a(\hat{A}, \mu_n^{\otimes r}) \ar[r] & H^a(\hat{K}(X), \mu_n^{\otimes r})
}
\end{array}
\]

and \( \alpha_0 \), viewed as an element of \( H^a(K(\hat{X}), \mu_n^{\otimes r}) \), is obtained by following the path given by \( U_0, k(X_0), \hat{A}, \) and \( K(\hat{X}) \), while \( \alpha \big|_{K(X)} \) can be obtained by following the path given by \( U_0, U, K(X), \) and \( K(X) \). Both paths yield the same element, so this completes the proof.

3.3. The index does not change

In section 3.2 we constructed \( s = s_{V, \pi} : H^a(K(\hat{X}), \mu_n^{\otimes r}) \rightarrow H^a(K(X), \mu_n^{\otimes r}) \) a map splitting the restriction. In particular, since

\[
\text{Br}(K(X))' = \text{inj lim}_{n \not\equiv 0 \pmod{p}} H^2(K(X), \mu_n)
\]

and similarly for \( \text{Br}(K(\hat{X}))' \), we automatically obtain a map

\[
s = s_{V, \pi} : \text{Br}(K(\hat{X}))' \rightarrow \text{Br}(K(X))'
\]

that also splits the restriction. In this section we show that this map preserves the index. First let us recall some facts about Brauer groups of regular schemes.

**Lemma 3.5.** Let \( X \) be an integral regular scheme of dimension at most 2.

i. The Brauer group \( \text{Br}(X) \) of classes of Azumaya algebras on \( X \) coincides with the cohomological Brauer group \( H^2(X, \mathbb{G}_m) \).

ii. There is an exact sequence

\[
0 \rightarrow \text{Br}(X)' \rightarrow \text{Br}(K(X))' \oplus \bigoplus_D H^1(K(D), \mathbb{Q}/\mathbb{Z})'
\]

where \( D \) runs over all irreducible Weil (or Cartier) divisors of \( X \).
iii. If \( X \) is projective over a henselian ring \((A, m, k)\) and the special fiber \( X_0 \equiv X \times_{\text{Spec } A} \text{Spec } k \) has dimension at most 1 then

\[
\text{Br}(X) = \text{Br}(X_0)
\]

In particular, if \( X_0 \) is a projective smooth curve over a finite field \( k \) then both groups are trivial.

**Proof.** For (i), see [31] IV.2.16. The injectivity of \( \text{Br}(X) \to \text{Br}(K(X)) \) in (ii) is proven in [31] IV.2.6, while the exactness in the middle term follows from the purity of the Brauer group (see [4] 7.4 or [31] IV.2.18 (b), and also [38], Lemma 6.6). Finally (iii) is [19] 3.1 (see also [12] 1.3 for a proof using proper base change in the prime to \( p \) case), together with the fact that for any projective smooth curve \( C \) over a finite field we have \( \text{Br}(C) = 0 \), as follows by comparing the sequence in (ii) with the one from Class Field Theory (see [15] 6.5):

\[
0 \to \text{Br}(K(C)) \oplus \bigoplus_{P \in C^{(1)}} H^1(K(P), \mathbb{Q}/\mathbb{Z}) \to \sum_{P \in C^{(1)}} \mathbb{Q}/\mathbb{Z} \to 0
\]

(here \( P \) runs over all irreducible Weil divisors of \( C \), namely, over all its closed points).

Now we are ready to show

**Theorem 3.6.** The map

\[
s = s_{V, \pi} : \text{Br}(K(X))' \to \text{Br}(K(X))'
\]

preserves the index.

**Proof.** Let \( n \) be prime to \( p \). Given an arbitrary element

\[
\hat{\gamma} = \alpha_0 + (\chi_0, \pi) \in \pi \text{Br}(K(X)) = H^2(K(X), \mu_n),
\]

where \( \alpha_0 \in \pi \text{Br}(k(X_0)) = H^2(k(X_0), \mu_n) \) and \( \chi_0 \in H^1(k(X_0), \mathbb{Z}/n) \), let

\[
\gamma = s(\hat{\gamma}) = \alpha + (\chi, \pi) \in \pi \text{Br}(K(X)) = H^2(K(X), \mu_n)
\]

where \( \alpha = \lambda_V(\alpha_0) \in H^2(K(X), \mu_n) \) and \( \chi = \lambda_V(\chi_0) \in H^1(K(X), \mathbb{Z}/n) \) are the tame lifts of \( \alpha_0 \) and \( \chi_0 \).

Since \( \text{res} \circ s = \text{id} \), we have that \( \text{res} \gamma = \hat{\gamma} \) and therefore \( \text{ind} \gamma = \text{ind} \hat{\gamma} \). To prove that \( \text{ind} \gamma | \text{ind} \hat{\gamma} \) we now construct a splitting field for \( \gamma \) of degree \( \text{ind} \hat{\gamma} \) over \( K(X) \).

The character \( \chi_0 \) defines a cyclic extension \( L \) of \( k(X_0) \) of degree equal to the order \(|\chi_0|\). Since \( k \) is perfect, the normalization \( Y_0 \) of \( X_0 \) in \( L \) is a smooth curve over \( k \), namely ramified over \( X_0 \) (since \(|\chi_0|\) is prime to \( p = \text{char } k \)) along some mark \( D_0 \) of \( X_0 \) (the ramification locus of \( \chi_0 \)). By the Nakayama-Witt index formula (see [24] 5.15(a)) we have that

\[
\text{ind} \hat{\gamma} = |\chi_0| \cdot \text{ind}(\alpha_0|_{k(Y_0)})
\]

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But since \( k(Y_0) \) is a global field, the Albert-Brauer-Hasse-Noether theorem (32, Corollary 9.2.3, p. 461, and the functoriality of Corollary 9.1.8, p. 458) tells us that \( \alpha_0|_{k(Y_0)} \) is cyclic, hence there is a cyclic extension of \( k(Y_0) \) of degree \( \text{ind}(\alpha_0|_{k(Y_0)}) \) that splits \( \alpha_0|_{k(Y_0)} \). Corresponding to this extension there is a cyclic cover \( Z_0 \) of \( Y_0 \), tamely ramified along some mark \( E_0 \) of \( Y_0 \).

Let \( D \subset X \) be the \( V \)-lift of \( D_0 \). Let \( \rho: Y \to X \) be the tamely ramified cover of \((X,D)\) lifting the tamely ramified cover \( \rho_0: Y_0 \to X_0 \) of \((X_0,D_0)\), as in Theorem 2.3. Now by Lemma 2.2 (ii) the set \( (\rho^{-1}V)_{\text{red}} \) defines a choice of marks on \( Y \) lifting the closed points of \( Y_0 \). Let \( E \) be the mark on \( Y \) that lifts \( E_0 \) and whose support belongs to \((\rho^{-1}V)_{\text{red}} \). Finally define \( \sigma: Z \to Y \) to be the tamely ramified cover of \((Y,E)\) lifting the tamely ramified cover \( \sigma_0: Z_0 \to Y_0 \) of \((Y_0,E_0)\). Since

\[
[K(Z) : K(X)] = \left[ K(Z) : K(Y) \right] \cdot \left[ K(Y) : K(X) \right]
\]

\[
= \left[ k(Z_0) : k(Y_0) \right] \cdot \left[ k(Y_0) : k(X_0) \right]
\]

\[
= \text{ind}(\alpha_0|_{k(Y_0)}) \cdot |\chi_0|
\]

\[
= \text{ind} \gamma
\]

it is enough to show that \( K(Z) \) splits \( \gamma \).

Since \( Z \) is integral and regular of dimension 2, to show that \( \gamma|_{K(Z)} = 0 \) it is enough to show, by Lemma 5.5 that \( \gamma|_{K(Z)} \) is unramified with respect to all Weil divisors on \( Z \). On the other hand, \( K(Y) \) splits \( \chi \) by Lemma 6.3 hence \( \gamma|_{K(Z)} = \alpha|_{K(Z)} \) and it remains to show \( \alpha|_{K(Z)} \) is unramified with respect to the Weil divisors on \( Z \). Moreover, by the construction of \( \lambda_V \) in the proof of Lemma 6.4 \( \alpha \in H^2(U, \mu_n) \) for some open set \( U \subset X \) that is the complement of a mark with support in \( V \). Consequently, \( \alpha \) only ramifies along marks in \( V \).

Let \( D' = D \cup \rho(E) \) where \( \rho(E) \) is the image of the mark \( E \). By our choice of \( E \), \( \rho(E) \subset V \) and hence \( D' \subset V \). We now have that the composition \( \rho \circ \sigma: Z \to X \) is a tamely ramified cover of \((X,D')\), which is finite and flat (Lemma 2.2 (i)). Therefore the image of any irreducible Weil divisor \( F \) in \( Z \) is also a Weil divisor in \( X \) by IV.3.14 (that is, it cannot “contract” to a closed point). Moreover if \( G \subset V \) then since \( D' \subset V \) either \( G \subset D' \) or \( G \cap D' = \emptyset \), and by Lemma 2.2 (ii) \( F \) is also a mark. Therefore, since the ramification locus of \( \alpha \) is contained in \( V \), it is enough to show that \( \alpha|_{K(Z)} \) is unramified at all marks lying over marks in \( V \).

Let \( F \) be an irreducible mark on \( Z \) lying over an irreducible mark \( G \) on \( X \) whose support belongs to \( V \). Since \( G \subset D' \) or \( G \cap D' = \emptyset \), by Lemma 2.2 (iii) the ramification degree \( e \) of \( v_F \) over \( v_G \) equals the ramification degree of \( v_{F_0} \) over \( v_{G_0} \). By Lemma 5.1 and the functorial behavior of residue maps under finite extensions, we have a diagram, commutative up to sign,
Here we view $H^1(k(G_0), \mathbb{Z}/n) = H^1(G, \mathbb{Z}/n)$ as the subgroup of unramified characters of $H^1(K(G), \mathbb{Z}/n)$, and similarly $H^1(k(F_0), \mathbb{Z}/n) = H^1(F, \mathbb{Z}/n) \subset H^1(K(F), \mathbb{Z}/n)$.

If $\alpha_0 \in n\text{Br}(k(X_0))$, we obtain
\[ \partial_F(\alpha|_{K(Z)}) = \pm e \cdot \partial_{G_0}(\alpha_0)|_{K(F)} \in H^1(K(F), \mathbb{Z}/n) \]
from squares $\square 1 + \square 2$ and we obtain
\[ \partial_{F_0}(\alpha_0|_{k(Z_0)}) = \pm e \cdot \partial_{G_0}(\alpha_0)|_{k(F_0)} \in H^1(k(F_0), \mathbb{Z}/n) \]
from square $\square 4$. Hence $\partial_F(\alpha|_{K(Z)}) = \pm \partial_{F_0}(\alpha_0|_{k(Z_0)})$ by square $\square 3$, which vanishes since $\alpha_0|_{k(Z_0)} = 0$, and we are done.

4. Indecomposable and noncrossed product division algebras.

Adopt all notation from Sections 1-3. In this section we construct indecomposable division algebras over $K(X)$ and noncrossed product algebras over $K(X)$ of prime power index for all primes $q$ with $q \neq p$. Note that noncrossed product division algebras with index equal to period over $K(X)$ for $X = \mathbb{P}^1_K$ are already known to exist by [9].

4.1. Indecomposable Division Algebras over $K(X)$.

We construct indecomposable division algebras over $K(X)$ by constructing them over $\hat{K}(X)$ and using the splitting $s : \text{Br}(\hat{K}(X))' \to \text{Br}(K(X))'$ from Theorem 4.10 to lift the Brauer classes to Brauer classes over $\hat{K}(X)$ whose underlying division algebras are indecomposable. The construction over $\hat{K}(X)$ follows the
method in \[8\], where indecomposable division algebras of unequal prime-power
index and period are shown to exist over power series fields over number fields.

We start by stating a well known lemma on the invariants of a Brauer class
of a global field after a finite extension. This lemma is helpful in computing
the index reduction of the Brauer class after the finite extension.

**Lemma 4.1** (see \[10\], XIII, §3). Let \( \beta \in \text{Br}(F) \) be a Brauer class over a
global field \( F \). Let \( L/F \) be a finite Galois extension. Then for all discrete valuations
\( w \) in \( L \) lying over a fixed prime \( v \) of \( F \), \( \text{inv}_w(\beta_L) = e_v f_v \text{inv}_v(\beta) \).

We now construct indecomposable division algebras over \( K^\wedge(X) \).

**Proposition 4.2.** Let \( e \) and \( i \) be integers satisfying \( 1 \leq e \leq i \leq 2e - 1 \). For
any prime \( q \neq \text{char } k \) there exists a Brauer class \( \hat{\gamma} \in \text{Br}(K^\wedge(X)) \) satisfying
\( (\text{ind}(\hat{\gamma}), \text{per}(\hat{\gamma})) = (q^e, q^e) \) and whose underlying division algebra is indecomposable.

**Proof.** Let \( 1 \leq t \leq e \) so that \( i = 2e - t \). To prove the proposition we produce
a Brauer class \( \hat{\gamma} \in \text{Br}(K^\wedge(X)) \) such that \( (\text{ind}(\hat{\gamma}), \text{per}(\hat{\gamma})) = (q^{2e-t}, q^e) \) and
\( \text{ind}((\hat{\gamma})^r) = q^{2e-t} \). Since \( \text{ind}(\hat{\gamma}) = q^{2e-t} \) and \( \text{ind}((\hat{\gamma})^r) = q^{2e-t} \), by \([8\], Lemma 3.2\) the division algebra underlying \( \hat{\gamma} \) is indecomposable. Choose two closed
points \( x_1, x_2 \in X_0 \). Let \( v_1 \) and \( v_2 \) be the discrete valuations on \( k(X_0) \) corresponding to \( x_1 \) and \( x_2 \). Let \( \alpha_0 \in \text{Br}(k(X_0)) \) be the Brauer class whose invariants are
\[
\text{inv}_{v_1}(\alpha_0) = 1/q^e \\
\text{inv}_{v_2}(\alpha_0) = -1/q^e
\]
and at all other discrete valuations \( v \) on \( k(X_0) \), \( \partial_v(\alpha_0) = 0 \). The Brauer class
\( \alpha_0 \) exists by Hasse’s residue theorem (\([13\], 6.5.4\)) and the fact that \( k(X_0) \) is
a global field. Let \( \xi_v = \partial_v(\alpha_0) \in H^1(k(v_1), \mathbb{Q}/\mathbb{Z}) \). Let \( k(X_0)_v \) be the completion
of \( k(X_0) \) at the valuation \( v \) and choose unramified characters \( \theta_v \in H^1(k(X_0)_v, \mathbb{Q}/\mathbb{Z}) \) of order \( q^e \). By the Grunwald-Wang theorem there exists a
global character \( \theta_0 \in H^1(k(X_0), \mathbb{Q}/\mathbb{Z}) \) of order \( q^e \) with restrictions \( \theta_v \) at \( v_i \) for
\( i = 1, 2 \).

Set \( \hat{\gamma} = \alpha_0 + (\theta_0, \pi) \in \text{Br}(K^\wedge(X)) \), an element with period \( q^e \). We claim that
\( \text{ind}(\hat{\gamma}) = q^{2e-t} \) and \( \text{ind}(\hat{\gamma}) = q^{2e-t} \). By the Nakayama-Witt index formula
(see \([24\], 5.15(a)\)) we have \( \text{ind}(\hat{\gamma}) = |\theta_0| \cdot \text{ind}(\alpha_0|k(X_0)(\theta_0)) \) where \( \alpha_0|k(X_0)(\theta_0) \)
is the restriction of \( \alpha_0 \) to \( k(X_0)(\theta_0) \), the finite extension defined by the character
\( \theta_0 \). By construction, \( |\theta_0| = q^e \) so it is only left to show that \( \text{ind}(\alpha_0|\theta_0) = q^{2e-t} \).
Since \( k(X_0)(\theta_0) \) is a finite extension of \( k(X_0) \), \( k(X_0)(\theta_0) \) is a global field and
\[
\text{ind}(\alpha_0|k(X_0)(\theta_0)) = \text{per}(\alpha_0|k(X_0)(\theta_0)) = \text{lcm}(\text{inv}_w(\alpha_0|k(X_0)(\theta_0)))
\]
where the least common multiple is taken over all discrete valuations \( w \) of
\( k(X_0)(\theta_0) \). This shows, by our assumptions on \( \alpha_0 \), that for all discrete valuations \( w \) of \( k(X_0)(\theta_0) \),
\[
\text{inv}_w(\alpha_0|k(X_0)(\theta_0)) = \begin{cases} 0, & \text{if } w \text{ does not lie over } v_i \text{ for } i = 1, 2 \\ \pm |(\theta_0)_{v_i}| \cdot q^{-e}, & \text{if } w \text{ lies over } v_i \text{ for } i = 1, 2 \end{cases}
\]

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By our assumption on \( \theta_0 \), \(|(\theta_0)_{v_i}| = q^i \) for \( i = 1, 2 \) and therefore we have
\[
\text{ind}(q)_{K(X)}(\theta_0) = q^{t-t} \quad \text{and} \quad \text{ind}(\hat{\gamma}) = q^{2e-t}.
\]

A similar calculation for \( q^\hat{\gamma} \) gives \(|q \theta_0| = q^{t-1} \) and \(|q \alpha_0|_{K(X)}(q \theta_0) = q^{t-t} \)
since by the same reasoning,
\[
\text{inv}_w(q \alpha_0 |_{K(X)}(q \theta_0)) = \begin{cases} 0, & \text{if } w \text{ does not lie over } v_i \text{ for } i = 1, 2 \\ ±|(q \theta_0)_{v_i}| \cdot q^{1-e}, & \text{if } w \text{ lies over } v_i \text{ for } i = 1, 2 \end{cases}
\]

and \(|(q \theta_0)_{v_i}| = q^{t-1} \) for \( i = 1, 2 \). We conclude \( \text{ind}(q \hat{\gamma}) = q^{2e-t-1} \).

**Theorem 4.3.** Let \( k \), \( X_0 \), \( K \) and \( X \) be as in Section 4.3 and let \( q \) be a prime with \( q \neq \text{char } k \). Fix integers \( e \) and \( i \) satisfying \( 1 \leq e \leq i \leq 2e - 1 \). Then there exists an indecomposable division algebra \( D \) over \( K(X) \) satisfying \((\text{ind}(D), \text{per}(D)) = (q^e, q^t)\).

**Proof.** Choose \( e \) and \( i \) so that \( 1 \leq e \leq i \leq 2e - 1 \). By Proposition 4.2 there exists a Brauer class \( \hat{\gamma} \in Br(K(X)) \) satisfying \((\text{ind}(\hat{\gamma}), \text{per}(\hat{\gamma})) = (q^e, q^t)\)
and whose underlying division algebra is indecomposable. By Theorem 3.6 \( \gamma = s(\hat{\gamma}) \in Br(K(X)) \) has index \( q^i \). Since \( s \) is a splitting of the restriction map, we also have \( \text{per}(\gamma) = q^e \). To finish the proof we show the division algebra underlying \( \gamma \) is indecomposable. If \( \gamma = \beta_1 + \beta_2 \) with \( \text{ind}(\beta_1) \text{ind}(\beta_2) = \text{ind}(\gamma) \) represents a nontrivial decomposition of the division algebra underlying \( \gamma \), then \( \hat{\gamma} = \text{res}_{K(X)}(\beta_1) + \text{res}_{K(X)}(\beta_2) \). Since the index can only decrease under \( \text{res}_{K(X)} \)
we have \( \text{ind}(\hat{\gamma}) = \text{ind}(\text{res}_{K(X)}(\beta_1)) \text{ind}(\text{res}_{K(X)}(\beta_2)) \). This represents a nontrivial decomposition of the division algebra underlying \( \hat{\gamma} \), a contradiction.

**Remark 4.4.** In the case \( X = \mathbb{P}^1_k \), it is not hard to construct \( \hat{\gamma} \) which satisfies the conclusions of Proposition 4.2 and can be seen to have \( \text{ind}(\hat{\gamma}) = \text{ind}(s(\hat{\gamma})) \)
without the use of Theorem 3.6. Choose \( e \), \( i \), \( t \) so that \( 1 \leq e \leq i \leq 2e - 1 \) and \( i = 2e - t \). Then, as in the proof of Proposition 4.2 choose a single closed point \( x_0 \) in \( X_0 = \mathbb{P}^1_k \) of degree \( q^{c-t} \). Let \( \xi \in H^1(k, \mathbb{Z}/n) \) be a character of order \( q^{2e-t} \)
where \( n \) is an integer prime to \( p \) with \( q^i \mid n \). Set \( \alpha_0 = (\xi, \pi_{x_0}) \) where \( \pi_{x_0} \) is the irreducible polynomial corresponding to the closed point \( x_0 \). Then,
\[
\partial_x(\alpha_0) = \begin{cases} 0, & \text{if } x \neq x_0 \text{ and } x \neq \text{ the point at infinity} \\ \text{res}_{k[x]}(\xi), & \text{if } x = x_0 \end{cases}
\]
Set \( \theta_0 = q^{c-t} \xi \in H^1(k, \mathbb{Z}/n) \hookrightarrow H^1(k(t), \mathbb{Z}/n) \). Set \( \hat{\gamma} = \alpha_0 + (\theta_0, p) \). Since \( \text{per}(\alpha_0) = |\text{inv}_x \alpha_0| = q^{e} \) and \( \text{per}(\theta_0, p) = q^e \), \( \text{per}(\hat{\gamma}) = q^{e} \). Using the same strategy as Proposition 4.2 shows that \( \text{ind}(\hat{\gamma}) = q^{2e-t} \) and \( \text{ind}(q \hat{\gamma}) = q^{2e-t-1} \). Therefore, \( \hat{\gamma} \) satisfies the conclusions of Proposition 4.2. We now check \( \text{ind}(s(\hat{\gamma})) = q^{2e-t} \). Let \( \theta = s(\theta_0) \) which is the unique lift of the constant extension \( \theta_0 \) to \( H^1(K(t), \mathbb{Z}/n) \). The character \( \theta \) defines a \( p \)-unramified extension \( L/K(t) \) of degree \( q^e \). Then, \( s(\hat{\gamma})_L = (s(\xi), s((\pi_{x_0}))_L + (\theta, p)_L = (s(\xi), s((\pi_{x_0}))_L). \) Thus \( \text{ind}(s(\hat{\gamma})_L) = \text{ind}(s(\xi), s((\pi_{x_0}))_L) \leq |\xi|/|\theta| = q^{2e-t} \)
since \( L \) is contained in the \( p \)-unramified constant extension defined by \( s(\xi) \).
which is a lift of $\xi$. Therefore, $\text{ind}(s(\hat{\gamma})) \leq [L : K(t)]q^{e-t} = q^{2e-t} = \text{ind}(\hat{\gamma})$. Since $\text{ind}(s(\hat{\gamma})) \geq \text{ind}(\hat{\gamma})$, we get the equality $\text{ind}(s(\hat{\gamma})) = \text{ind}(\hat{\gamma})$.

**Remark 4.5.** Set $R = \mathbb{Z}_p$ and $K = \mathbb{Q}_p$ and let $X$ be as in [4]. By [36] the index of any Brauer class in $\text{Br}(K(X))$ divides the square of its period. Let $q$ be a prime with $q \neq p$. Theorem 3.3 shows that over $K(X)$ there exist indecomposable division algebras of index-period combination $(q^i, q^e)$ for all $1 \leq e \leq i \leq 2e - 1$ and all primes $q \neq p$. In [4], Suresh builds on the work of [37] to show that if $L/\mathbb{Q}_p(t)$ is a finite extension containing the $q$-th roots of unity, then every element in $H^2(L, \mu_q)$ is a sum of at most two symbols. In particular, a division algebra over $L$ of index $q^2$ and period $q$ must be decomposable as it is the sum of two symbols each of index $q$. In a forthcoming paper by Brussel and Tengan, [4], the dependence on an $q$-th root of unity is removed, showing that all division algebras of index-period combination $(q^2, q)$ over $L$ are decomposable for any finite extension $L/\mathbb{Q}_p(t)$.

### 4.2. Noncrossed products over $\hat{K}(X)$

In this section we construct noncrossed product division algebras over $K(X)$. Throughout this section we adopt all notation from Section 3.1. In particular, $K$ is the fraction field of $R$, a complete discrete valuation ring with uniformizer $\pi$ and residue field $k$, a field of characteristic $p$ and $X$ is a smooth curve over $R$. We use the same strategy as in Section 4.1 that is, we construct noncrossed products of $q$-power index $(q^e, q^f)$ over $\hat{K}(X)$ and use the splitting $s : \text{Br}(K(X))' \to \text{Br}(K(X))'$ from Theorem 4.3 to lift the noncrossed products to $K(X)$.

The method of constructing the noncrossed products over $\hat{K}(X)$ follows the method in [7] where noncrossed products over $\mathbb{Q}(t)$ and $\mathbb{Q}((t))$ are constructed. In order to mimic the construction in [7] we need only note that both the Cebotarev density theorem, and the Grunwald-Wang theorem hold for global fields which are characteristic $p$ function fields. After noting these two facts, the reader can check that the arguments in [7] apply directly to obtain noncrossed products over $\hat{K}(X)$ of index and period given below.

**Index and Period Setup 4.6.** Let $K$, $R$, $k$, $X$ and $X_0$ be as in Section 3.1. For any positive integer $a$, let $\varepsilon_a$ denote a primitive $a$-th root of unity. Set $r$ and $s$ to be the maximum integers such that $\mu_{q^r} \subset k(X_0)\times$ and $\mu_{q^s} \subset k(X_0)(\varepsilon_{q^{r+1}})\times$. Let $n$ and $m$ be integers such that $n \geq 1$, $n \geq m$, and $n, m \in \{r\} \cup \{s, \infty\}$. Let $a$ and $l$ be integers such that $l \geq n + m + 1$ and $0 \leq a \leq l - n$. See [7], p.384-385 for more information regarding these constraints.

**Theorem 4.7.** Let $K$, $R$, $k$, $X$ and $X_0$ be as in Section 3.1. Let $q$ be a prime, $q \neq p = \text{char } k$ and let $a$ and $l$ be integers satisfying the properties of 4.6. Then there exists noncrossed product division algebras over $K(X)$ of index $q^{l+a}$ and period $q^l$. 

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Corollary 4.8. Let $K$, $R$, $k$, $X$, $X_0$, $q$, $a$ and $l$ be as Theorem 4.7. Then, there exists noncrossed product division algebras over $K(X)$ of index $q^{l+a}$ and period $q^l$.

Proof. Let $\hat{D}$ be a noncrossed product over $K(X)$ of index $q^{l+a}$, period $q^l$. Let $D$ be the division algebra in the class of $s(\hat{D}) \in \text{Br}(K(X))$. By Theorem 3.6 we know that $\text{ind}(D) = \text{ind}(\hat{D})$. Assume by way of contradiction that $D$ is a crossed product with maximal Galois subfield $M/K$. Then $MK(X)$ splits $\hat{D}$, of degree $\text{ind}(\hat{D})$ and is Galois. This contradicts the fact that $\hat{D}$ is a noncrossed product.

Remark 4.9. Noncrossed products were already known to exist over $\mathbb{Q}_p(t)$ by [9]. In the noncrossed products of [3] the index always equals the period. This is not the case in the above construction.

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