SOME REMARKS ON BLUEPRINTS AND $\mathbb{F}_1$-SCHEMES

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Abstract. Over the past two decades several different approaches to defining a geometry over $\mathbb{F}_1$ have been proposed. In this paper, relying on Toën and Vaquié’s formalism [23], we investigate the category $\text{Sch}_B$ of schemes relative to the category of blueprints introduced by Lorscheid [14]. A blueprint, that may be thought of as a pair consisting of a monoid $M$ and a relation on the semiring $M \otimes_{\mathbb{F}_1} \mathbb{N}$, is a monoid object in a certain symmetric monoidal category $B$, which is shown to be complete, cocomplete, and closed. We prove that every object of $\text{Sch}_B$, that we name a $B$-scheme, can be associated, through adjunctions, with both a scheme over $\mathbb{Z}$ and a scheme over $\mathbb{F}_1$ in the sense of Deitmar [2]. Furthermore, we show that the category of “$\mathbb{F}_1$-schemes” defined by A. Connes and C. Consani in [1] can be naturally merged with that of $B$-schemes to obtain a larger category, whose objects we call “$\mathbb{F}_1$-schemes with relations”.

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1. Introduction

1.1. A quick overview of $\mathbb{F}_1$-geometry. The nonexistent field $\mathbb{F}_1$ made its first appearance in Jacques Tits’s 1956 paper "Sur les analogues algébriques des groupes semi-simples complexes" [22]. According to Tits, it was natural to call “$n$-dimensional projective space over $\mathbb{F}_1$” a set of $n + 1$ points, on which the symmetric group $\Sigma_{n+1}$ acts as the group of projective transformations. So, $\Sigma_{n+1}$ was thought of as the group of $\mathbb{F}_1$-points of $SL_{n+1}$, and more generally it was conjectured that, for each algebraic group $G$, one ought to have $W(G) = G(\mathbb{F}_1)$, where $W(G)$ is the Weyl group of $G$.

A further strong motivation to seek for a geometry over $\mathbb{F}_1$ was the hope, based on the multifarious analogies between number fields and function fields, to find some pathway to attack Riemann’s hypothesis by mimicking André Weil’s celebrated proof. The idea behind that, as explicitly stated in Yuri Manin’s influential 1991–92 lectures [18] and in Kapranov and Smirnov’s unpublished paper [11], was to regard Spec $\mathbb{Z}$, the final object of the category of schemes, as an arithmetic curve over the “absolute point” Spec $\mathbb{F}_1$. Manin’s work drew inspiration from Kurokawa’s paper [12] together with Deninger’s results about “representations of zeta functions as regularized infinite determinants [5, 6, 7] of certain ‘absolute Frobenius operators’ acting upon a new cohomology theory”. Developing these insights, Manin suggested a conjectural decomposition of the classical complete Riemann zeta function of the form [18, eq. (1.5)]

$$Z(\text{Spec} \mathbb{Z}, s) := 2^{-1/2} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \prod_{\rho}^{\text{reg}} \left(\frac{s - \rho}{s - \rho - 2\pi}\right) = \frac{\text{det}^{\text{reg}}\left(\frac{1}{2\pi}(s \cdot \text{Id} - \Phi)\right) |H^1_{\text{reg}}(\text{Spec} \mathbb{Z})|}{\text{det}^{\text{reg}}\left(\frac{1}{2\pi}(s \cdot \text{Id} - \Phi)\right) |H^0_{\text{reg}}(\text{Spec} \mathbb{Z})|} \text{det}^{\text{reg}}\left(\frac{1}{2\pi}(s \cdot \text{Id} - \Phi)\right) |H^2_{\text{reg}}(\text{Spec} \mathbb{Z})|, \quad (1.1)$$

where the notation $\prod_{\rho}^{\text{reg}}$ and $\text{det}^{\text{reg}}$ refers to “zeta regularization” of infinite products and the last hypothetical equality “postulates the existence of a new cohomology theory $H^*_\text{reg}$, endowed with a canonical ‘absolute Frobenius’ endomorphism $\Phi$”. He conjectured, moreover, that the functions of the form $\frac{s - n}{2\pi}$ in eq. 1.1 could be interpreted as zeta functions according to the definition

$$Z(\mathbb{T}^n, s) = s - n \frac{1}{2\pi}, \quad n \geq 0,$$

where “Tate’s absolute motive” $\mathbb{T}$ was to be “imagined as a motive of a one-dimensional affine line over the absolute point, $\mathbb{T}^0 = \bullet = \text{Spec} \mathbb{F}_1$.”

*For a more detailed and exhaustive account of the development of $\mathbb{F}_1$-geometry we refer to [13] and [15].
The first full-fledged definition of variety over “the field with one element” was proposed by Christophe Souleé in the 1999 preprint [20]; five years later such definition was slightly modified by the same author in the paper [21]). Taking as a starting point Kapranov and Smirnov’s suggestion that $F_1$ should have an extension $F_{1n}$ of degree $n$, Souleé insightfully posited that

$$F_{1n} \otimes_{F_1} Z = Z[T]/(T^n - 1) =: R_n.$$ 

Let $R$ be the full subcategory of the category $\text{Ring}$ of commutative rings generated by the rings $R_n$, $n \geq 1$ and their finite tensor products. An affine variety $X$ over $F_1$ is then defined as a covariant functor $R \to \text{Set}$ plus some extra data such that there exists a unique (up to isomorphism) affine variety $X_Z = X \otimes_{F_1} Z$ over $Z$ along with an immersion $X \hookrightarrow X_Z$ satisfying a suitable universal property [21, Définition 3]. In particular, one has a natural inclusion $X(F_{1n}) \subset (X \otimes_{F_1} Z)(R_n)$ for each $n \geq 1$. A notable result proven by Souleé was that smooth toric varieties can always be defined over $F_1$.

To formalize $F_1$-geometry Anton Deitmar adopted, in 2005, a different approach, which can be dubbed as “minimalistic” (using the evocative terminology introduced by Manin in [19]). In his terse paper [2], he associates to each commutative monoid $M$ its “spectrum over $F_1$” $\text{Spec} M$ consisting of all prime ideals of $M$, i.e. of all submonoids $P \subset M$ such that $xy \in P$ implies $x \in P$ or $y \in P$. The set $\text{Spec} M$ can be endowed with a topology and with a structure (pre)sheaf $O_M$ via localization, just as in the usual case of commutative rings. A topological space $X$ with a sheaf $O_X$ of monoids is then called a “scheme over $F_1$”, if for every point $x \in X$ there is an open neighborhood $U \subset X$ such that $(U, O_X|_U)$ is isomorphic to $(\text{Spec} M, O_M)$ for some monoid $M$. The forgetful functor $\text{Ring} \to \text{Mon}$ has a left adjoint given by $M \mapsto M \otimes_{F_1} Z$ (in Deitmar’s notation), and this functor extend to a functor $\otimes_{F_1} Z$ from the category of schemes over $F_1$ to the category of classical schemes over $Z$. Tit’s 1957 conjecture stating that $GL_n(F_1) = \Sigma_n$ can be easily proven in Deitmar’s theory. Indeed, since $F_1$-modules are just sets and $F_{1n} \otimes_{F_1} Z$ has to be isomorphic $\mathbb{Z}^n$, it turns out that $F_{1n}$ can be identified with the set $\{1, \ldots, n\}$ of $n$ elements. Hence

$$GL_n(F_1) = \text{Aut}_{F_1}(F_{1n}) = \text{Aut}(1, \ldots, n) = \Sigma_n.$$ 

It is not hard to show, moreover, that the functor $GL_n$ on rings over $F_1$ is represented by a scheme over $F_1$ [2, Prop. 5.2]. As for zeta functions, Deitmar defines, for a scheme $X$ over $F_1$ and for a prime $p$, the formal power series

$$Z_X(p, T) = \exp \left( \sum_{n=1}^{\infty} \frac{T^n}{n} \# X(F_{p^n}) \right),$$

where $F_{p^n}$ stands for the field of $p^n$ elements with only its monoidal multiplicative structure and $X(F_{p^n})$ denotes the set of $F_{p^n}$-valued points of $X$, and proves that $Z_X(p, T)$ coincides with the Hasse–Weil zeta function of $X \otimes_{F_1} F_{p^n}$ [2, Prop. 6.3]. Albeit elegant, this result is a bit of a letdown, for — as the author himself is ready to admit — it is clear that “this type of zeta function [...] does not give new insights”.

A natural and extremely general formalism for $F_1$-geometry was elaborated by Bertrand Toën and Michel Vaquié in their 2009 paper [23], tellingly entitled Au dessous de Spec $\mathbb{Z}$,
whose approach appears to be largely inspired by Monique Hakim’s work [9]. The authors
there showed how to construct an “algebraic geometry” relative to any symmetric monoidal
category $\mathcal{C} = (\mathcal{C}, \otimes, 1)$, which is supposed to be complete, cocomplete and to admit internal
homs. The basic idea is that the category $\text{CMon}_\mathcal{C}$ of commutative (associative and unitary)
monoid objects in $\mathcal{C}$ can be taken as a substitute for the category of commutative rings
(the monoid objects in the category $\text{Ab} = \text{Z-Mod}$ of Abelian groups) to the end of defining
a suitable notion of “scheme over $\mathcal{C}$”. Each object $V$ of $\text{CMon}_\mathcal{C}$ gives rise to the category
$V\text{-Mod}$ of $V$-modules and each morphism $V \to W$ in $\text{CMon}_\mathcal{C}$ determines a change of basis
functor $- \otimes_V W : V\text{-Mod} \to W\text{-Mod}$; the category of commutative $V$-algebras can be
realized as the category of commutative monoids in $V\text{-Mod}$ and is naturally equivalent
to the category $V/\text{CMon}_\mathcal{C}$. An affine scheme over $\mathcal{C}$ is, by definition, an object of the
opposite category $\text{Aff}_\mathcal{C} = \text{CMon}_\mathcal{C}^{\text{op}}$ and the tautological contravariant functor
$\text{CMon}_\mathcal{C} \to \text{Aff}_\mathcal{C}$ is called $\text{Spec}(\cdot)$. By means of the pseudo-functor $M$ that maps an object $V$ in
$\text{CMon}_\mathcal{C}$ to the category of $V$-modules and a morphism $\text{Spec} V \to \text{Spec} W$ to the functor
$- \otimes_V W : V\text{-Mod} \to W\text{-Mod}$, one may introduce the notions of “Zariski cover” and “flat
cover” (“$M$-faithfully flat in Toën and Vaquié’s terminology; see Def. 2.4 and Remark
2.5 below) and use such notions to equip $\text{Aff}_\mathcal{C}$ with two distinct Grothendieck topologies,
called, respectively, the flat and the Zariski topology. These topologies determine two
categories of sheaves on $\text{Aff}_\mathcal{C}$, namely $\text{Sh}^{\text{flat}}(\text{Aff}_\mathcal{C}) \subset \text{Sh}^{\text{Zar}}(\text{Aff}_\mathcal{C}) \subset \text{Presh}(\text{Aff}_\mathcal{C})$. At this
point, mimicking what is done in classical algebraic geometry, a “scheme over $\mathcal{C}$” is defined
as a sheaf in $\text{Sh}^{\text{Zar}}(\text{Aff}_\mathcal{C})$ that admits an affine Zariski cover (see Def.
2.6 and 2.7 below). 

Deitmar’s schemes appear therefore to constitute the very core of $\mathbb{F}_1$-geometry, not just
because their definition is rooted in the basic notion of prime spectrum of a monoid, but
especially because they naturally fit into the categorical framework established by Toën and
Vaquié in [23], which admits of generalizations in many directions (e.g. towards a derived
algebraic geometry over $\mathbb{F}_1$). Nonetheless, they are affected by some intrinsic limitations,
which are clearly revealed by a result proven by Deitmar himself in 2008 [4, Thm. 4.1]: 

\textbf{Theorem.} Let $X$ be a connected integral $\mathbb{F}_1$-scheme of finite type.\footnote{A Deitmar’s $\mathbb{F}_1$-scheme $X$ is said to be of finite type, if it has a finite covering by affine schemes $U_i = \text{Spec} M_i$ such that each $M_i$ is a finitely generated monoid. Deitmar proved in [3] that an $\mathbb{F}_1$-scheme $X$ is of finite type if and only if $X_Z$ is a $\mathbb{Z}$-scheme of finite type.} Then every irreducible
component of $X_\mathcal{C} = X_Z \otimes_\mathbb{Z} \mathcal{C}$ is a toric variety. The components of $X_\mathcal{C}$ are mutually
isomorphic as toric varieties.

Since every toric variety is the lift $X_\mathcal{C}$ of an $\mathbb{F}_1$-scheme $X$, the previous theorem entails
that integral $\mathbb{F}_1$-schemes of finite type are essentially the same as toric varieties. Now,
semisimple algebraic groups are not toric varieties, so it is apparent that Deitmar’s $\mathbb{F}_1$-schemes are too little flexible to implement Tits’s conjectural program.

A possible generalization of Deitmar’s geometry over $\mathbb{F}_1$ was proposed by Olivier Lorscheid, who introduced the notions of “blueprint” and “blue scheme” [14]. The basic idea can be illustrated through the following example. The affine group scheme $(SL_2)_\mathbb{Z}$ over the integers is defined as

$$(SL_2)_\mathbb{Z} = \text{Spec}(\mathbb{Z}[T_1, T_2, T_3, T_4]/(T_1T_4 - T_2T_3 - 1)).$$

As the relation $T_1T_4 - T_2T_3 = 0$ does not make sense in the monoid $\mathbb{F}_1[T_1, T_2, T_3, T_4]$, any naive attempt to adapt the previous definition to get a scheme over $\mathbb{F}_1$ will necessarily be unsuccessful. The notion of “blueprint” just serves serves the purpose of getting rid of this difficulty:

**Definition.** A blueprint is a pair $B = (R, A)$, where $R$ is a semiring and $A$ is a multiplicative subset of $R$ containing $0$ and $1$ and generating $R$ as a semiring. A blueprint morphism $f : B_1 = (R_1, A_1) \to B_2 = (R_2, A_2)$ is a semiring morphism $f : R_1 \to R_2$ such that $f(A_1) \subset A_2$.

The rationale behind this definition can be explained by considering the following situation: if one is given a monoid $A$ and some relation which does not makes sense in $A$ but becomes meaningful in the semiring $A \otimes_{\mathbb{F}_1} \mathbb{N}$, then one can look at the blueprint $(A \otimes_{\mathbb{F}_1} \mathbb{N}, A)$.

In the same vein as Deitmar’s approach, Lorscheid [14] associates to each blueprint $B$ its spectrum $\text{Spec} B$, which turns out to be a locally blueprinted space (i.e. a topological spaces endowed with a sheaf of blueprints, such all stalks have a unique maximal ideal). An affine blue scheme is then defined as a locally blueprinted space that is isomorphic to the spectrum of a blueprint, and a blue scheme as a locally blueprinted space that has a covering by affine blue schemes. Deitmar’s schemes over $\mathbb{F}_1$ and classical schemes over $\mathbb{Z}$ are recovered as special cases of this definition.

1.2. **About the present paper.** A natural question arises: do blue schemes fit into Toën and Vaquié’s framework? This problem was addressed by Lorscheid himself in his 2017 paper [16] and answered in the negative. Nonetheless, it is possible — as pointed out in [16] — to define a category of schemes relative (in Toën and Vaquié’s sense) to the category of blueprints. The main purpose of the present paper is to study this category by introducing the notion of blueprint in a purely functorial way, and to investigate — via natural adjunctions — its relationship with the category of Deitmar’s schemes and with that of “$\mathbb{F}_1$-schemes” defined by A. Connes and C. Consani in [1]. More in detail the present paper is organized as follows.

After briefly recalling in § 2 the fundamental notions of “relative algebraic geometry” and fixing our notation, in § 3 we define the full subcategory $\mathbf{B}$ of the category $\mathbb{N}[-]/\text{Mon}_0$ (where the functor $\mathbb{N}[-] : \text{Set}_* \to \text{Mon}_0$ is left adjoint to the forgetful functor $|-|$ from the

\[\text{This overview is complemented by Appendix A where we review some basic facts about fibered categories, pseudo-functors, and stacks.}\]
category $\textbf{Set}_*$ of pointed sets to the category of monoids with “absorbent object”; see § 2.2), whose objects $(X, \mathbb{N}[X] \to M)$ satisfy the conditions:

a) the morphism $\mathbb{N}[X] \to M$ is an epimorphism;

b) the composition $X \to |\mathbb{N}[X]| \to |M|$ is a monomorphism.

As proven in Theorem 3.5, the category $\mathcal{B}$ — which corresponds to the category of pointed set endowed with a pre-addition structure introduced in [16, §4] — carries a natural structure of symmetric monoidal category. Moreover, this structure is closed, complete, and cocomplete, so that $\mathcal{B}$ possesses all the properties necessary to carry out Toën and Vaquié’s program.

It is quite straightforward to show (Prop. 3.6) that the category $\text{Blp}$ of monoid objects in $\mathcal{B}$ coincides with the category of blueprints (this result was already stated, in equivalent terms, in [16, Lemma 4.1], but we provide a detailed and completely functorial proof). Thus, by applying Toën and Vaquié’s formalism to the category $\mathcal{B}$, we define the category $\text{Aff}_{\mathcal{B}}$ of affine $\mathcal{B}$-schemes and then the category $\text{Sch}_{\mathcal{B}}$ of $\mathcal{B}$-schemes.

The natural adjunction between the category $\textbf{Mon}_0$ and the category $\textbf{Set}_*$ gives rise to an adjunction $\text{Aff}_{\textbf{Mon}_0} \dashv \text{Aff}_{\textbf{Set}_*}$ that factorizes as shown in the following diagram

\[
\begin{array}{ccc}
\text{Aff}_{\textbf{Mon}_0} & \dashv & \text{Aff}_{\textbf{Set}_*} \\
\downarrow F & & \downarrow \sigma \\
\text{Aff}_{\mathcal{B}} & \downarrow \rho & \text{Aff}_{\textbf{Set}_*}
\end{array}
\]

In Prop. 4.4 it is proven that the functors $F$ and $\sigma$ above induce functors, respectively, $\hat{F}: \text{Sch}_{\mathcal{B}} \to \text{Sch}_{\text{Mon}_0}$ and $\hat{\sigma}: \text{Sch}_{\textbf{Set}_*} \to \text{Sch}_{\mathcal{B}}$, whose adjoint functors are denoted, respectively, by $\hat{G}$ and $\hat{\rho}$. Moreover, there is also an adjunction between the category of $\mathcal{B}$-schemes and the category $\text{Sch}_{\text{Ab}}$ of classical schemes over $\mathbb{Z}$, namely $\text{Sch}_{\mathcal{B}} \xrightarrow{\hat{F}_Z} \text{Sch}_{\text{Ab}} \xrightarrow{\hat{G}_Z} \text{Sch}_{\mathcal{B}}$. By means of these adjunctions, we show that each $\mathcal{B}$-scheme $\Sigma$ determines the following geometric data:

- a monoidal scheme $\Sigma = \hat{\rho}(\Sigma)$;
- a scheme $\Sigma_Z = \hat{F}_Z(\Sigma)$ over $\mathbb{Z}$;
- a natural transformation $\Lambda: \Sigma_Z \to \Sigma \odot |-| \cong \Sigma \otimes_{\mathbb{F}_1} \mathbb{Z}$.

§What we call a “$\mathcal{B}$-scheme” was named a “subcanonical blue scheme” in [16].
The data associated with a $B$-scheme appear to be akin to those involved in the notion of $F_1$-scheme as introduced by Alain Connes and Caterina Consani in their paper [1]. According to this definition [1, Def. 4.7], an $F_1$-scheme is a triple $(\Xi, \Xi_Z, \Phi)$, where $\Xi$ is a monoidal scheme, $\Xi_Z$ is a scheme over $\mathbb{Z}$, and $\Phi$ is natural transformation $\Xi \to \Xi_Z \circ (- \otimes_{F_1} \mathbb{Z})$, such that the induced natural transformation $\Xi \circ | \cdot | \to \Xi_Z$, when evaluated on fields, gives isomorphisms (of sets). Thus, the category of $B$-schemes and that of $F_1$-schemes can be combined into a larger category, namely their fibered product over the category of monoidal schemes, whose objects will be called $F_1$-schemes with relations (Def. 5.5). In more explicit terms, a $B$-scheme $\Sigma$ determining the pair $(\Sigma, \Sigma_Z)$ and an $F_1$-scheme $(\Sigma, \Sigma'_Z, \Phi)$ will give rise to a $F_1$-scheme with relations denoted by the quadruple $(\Sigma, \Sigma_Z, \Sigma'_Z, \Phi)$. The main motivation behind this notion is to combine in a single geometric object both the advantages of blueprint approach and the benefits of Connes and Consani’s definition (cf. Remark 5.15 for a better explanation).

As we show in § 5, each $F_1$-scheme with relations $(\Sigma, \Sigma_Z, \Sigma'_Z, \Phi)$ determines a natural transformation $\Psi_1: \Sigma_Z \to \Sigma'_Z$ and a natural transformation $\Psi_2: \Sigma'_B \to \Sigma'_Z$, where $\Sigma'_B$ is a certain pullback sheaf on the category $\text{Ring}$ (defined by the diagram 5.7). This implies that, given a $B$-scheme $\Sigma$ underlying a $F_1$-scheme with relations, we can think of its “$F_1$-points” in two different senses, and therefore count them in two different ways, as stated in Prop. 5.6 and in Theorem 5.7.

An interesting case is when the $F_1$-points of the underlying monoidal scheme $\Sigma$ are counted by a polynomial in $n$. Theorem 4.10 of [1] shows that, if $(\Sigma, \Sigma_Z, \Phi)$ is an $F_1$-scheme such that the monoidal scheme $\Sigma$ is noetherian and torsion-free, then $\#(\Sigma(F_1^n)) = P(\Sigma, n)$, where

$$P(\Sigma, n) = \sum_{x \in \Sigma} \# \text{Hom}(O_{\Sigma, x}(\mathbb{F}_1^n), \mathbb{F}_1^n).$$

For a $F_1$-scheme with relations $(\Sigma, \Sigma_Z, \Sigma'_Z, \Phi)$ such that the underlying $B$-scheme $\Sigma$ is noetherian and torsion-free (Def. 5.11), we introduce the polynomial

$$Q(\Sigma, n) = \sum_{x \in \Sigma} \# \text{Hom}_B(O_{\Sigma, x}(\mathbb{F}_1^n), \mathbb{F}_1^n),$$

and prove (Prop. 5.14) that $Q(\Sigma, n) \leq P(\Sigma, n)$.

Finally, we would like to emphasize that our approach to blueprints, being entirely functorial, seems to be appropriate to carry out a “derived version” of the category of $B$-schemes. In fact, in quite general terms, a definition of “derived $B$-scheme” could be obtained by replacing, in our definition of $B$-scheme, the category $\text{Set}$ (resp. $\text{Set}_*$) by the category $\text{S}$ of spaces (resp. $\text{S}_*$ of pointed spaces) and the notion of monoid object by that of $\mathbb{E}_\infty$-algebra. This issue will be the object of future work.
2. The general setting

2.1. Schemes over a monoidal category. For the reader’s convenience, we start by
giving a quick résumé of some of the basic constructions of the “relative algebraic geometry”
developed in \[23, \S 2\].

Let \( C = (C, \otimes, 1) \) be a symmetric monoidal category (1 is the unit object), and denote
by \( \text{CMon}_C \) the category of commutative (associative and unitary) monoid objects in \( C \).

We assume that \( C \) is complete, cocomplete, and closed (i.e., for every pair of objects \( X, Y \),
the contravariant functor \( \text{Hom}_C(- \otimes X, Y) \) is represented by an “internal hom” set \( \text{Hom}(X, Y) \)).

The assumptions on \( C \) imply, in particular, that the forgetful functor
\[ |\cdot| : \text{CMon}_C \rightarrow C \]
admits a left adjoint
\[ L : C \rightarrow \text{CMon}_C, \]
which maps an object \( X \) to the free commutative monoid object \( L(X) \) generated by \( X \).

For each commutative monoid \( V \) in \( \text{CMon}_C \) one may introduce the notion of \( V \)-module
(cf. \[10, p. 478\]). The category \( V\Mod \) of such objects has a natural symmetric monoidal
structure given by the “tensor product” \( \otimes_V \); this structure turns out to be closed. Given
a morphism \( V \rightarrow W \) in \( \text{CMon}_C \), there is a change of basis functor
\[ - \otimes_V W : V\Mod \rightarrow W\Mod, \]
whose adjoint is the forgetful functor \( W\Mod \rightarrow V\Mod \). Note that the category of
commutative monoids in \( V\Mod \) — i.e. the category of commutative \( V \)-algebras — is
naturally equivalent to the category \( V/\text{CMon}_C \).

The category \( \text{Aff}_C \) of affine schemes over \( C \) is, by definition, the category \( \text{CMon}_C^{\text{op}} \).
Given an object \( V \) in \( \text{CMon}_C \) the corresponding object in \( \text{Aff}_C \) will be denoted by \( \text{Spec} V \).

To define, in full generality, the category of schemes over \( C \) one follows the standard
procedure of glueing together affine schemes. To this end, one first endows \( \text{Aff}_C \) with
a suitable Grothendieck topology. Let us recall the general definition.

**Definition 2.1.** Let \( G \) be any category. A Grothendieck topology on \( G \) is the assignment
to each object \( U \) of \( G \) of a collection of sets of arrows \( \{U_i \rightarrow U\} \) called coverings of \( U \) so
that the following conditions are satisfied:

i) if \( V \rightarrow U \) is an isomorphism, then the set \( \{V \rightarrow U\} \) is a covering;
ii) if \( \{U_i \rightarrow U\} \) is a covering and \( V \rightarrow U \) is any arrow, then there exist the fibered
products \( \{U_i \times_U V\} \) and the collection of projections \( \{U_i \times_U V \rightarrow V\} \) is a covering;
iii) if \( \{U_i \rightarrow U\} \) is a covering and for each index \( i \) there is a covering \( \{V_{ij} \rightarrow U_i\} \) (where
\( j \) varies in a set depending on \( i \)), each collection \( \{V_{ij} \rightarrow U_i \rightarrow U\} \) is a covering of
\( U \).
A category with a Grothendieck topology is called a site.

Remark 2.2. As it is clear from the definition above, a Grothendieck topology on a category \( G \) is introduced with the aim of gluing objects locally defined, and what really matters is therefore the notion of covering. So, in spite of its name, a Grothendieck topology could better thought of as a generalization of the notion of covering rather than of the notion of topology (notice, for example, that, though the maps \( U_i \to U \) in a covering can be seen as a generalization of open inclusions \( U_i \subset U \), no condition generalizing the topological requirement about unions of open subsets is prescribed).

Given a site \( G \) and a covering \( \mathcal{U} = \{U_i \to U\}_{i \in I} \), we denote by \( h_U \) the presheaf represented by \( U \) and by \( h_U \subset h_U \) the subpresheaf of those maps that factorise through some element of \( \mathcal{U} \).

Definition 2.3. Let \( G \) be a site. A presheaf \( F : G^{op} \to \text{Set} \) is said to be a sheaf if, for every covering \( \mathcal{U} = \{U_i \to U\}_{i \in I} \), the restriction map \( \text{Hom}(h_U, F) \to \text{Hom}(h_{\mathcal{U}}, F) \) is an isomorphism.

Coming back to our symmetric monoidal category \( \mathcal{C} \), the associated category of affine schemes \( \text{Aff}_\mathcal{C} \) can be equipped with two different Grothendieck topologies by means of the following ingenious definitions (which, of course, generalize the corresponding usual definitions in “classical” algebraic geometry).

One says [23, Def. 2.9, 1), 2), 3)] that a morphism \( f : \text{Spec} \, W \to \text{Spec} \, V \) in \( \text{Aff}_\mathcal{C} \) is

- **flat** if the functor \( - \otimes_V W : V\text{-Mod} \to W\text{-Mod} \) is exact;
- **an epimorphism** if, for any \( Z \) in \( \text{CMon}_\mathcal{C} \), the functor
  \[
  f^* : \text{Hom}_{\text{CMon}_\mathcal{C}}(W, Z) \to \text{Hom}_{\text{CMon}_\mathcal{C}}(V, Z)
  \]
  is injective;
- **of finite presentation** if, for any filtrant diagram \( \{Z_i\}_{i \in I} \) in \( V/\text{CMon}_\mathcal{C} \), the natural morphism
  \[
  \lim_{\rightarrow} \text{Hom}_{V/\text{CMon}_\mathcal{C}}(V, Z_i) \to \text{Hom}_{V/\text{CMon}_\mathcal{C}}(W, \lim_{\rightarrow} Z_i)
  \]
  is an isomorphism.

Definition 2.4. [23, Def. 2.9, 4); Def. 2.10] a) A collection of morphisms \( \{f_j : \text{Spec} \, W_j \to \text{Spec} \, V\}_{j \in J} \)

in \( \text{Aff}_\mathcal{C} \) is a flat cover if

i) each morphism \( f_j : \text{Spec} \, W_j \to \text{Spec} \, V \) is flat and

ii) there exists a finite subset of indices \( J' \subset J \) such that the functor
  \[
  \prod_{j \in J'} - \otimes_V W_j : V\text{-Mod} \to \prod_{j \in J'} W_j\text{-Mod}
  \]
  is conservative.
(b) A morphism \( f : \text{Spec} W \to \text{Spec} V \) in \( \text{Aff}_C \) is an open Zariski immersion if it is a flat epimorphism of finite presentation.

(c) A collection of morphisms \( \{ f_j : \text{Spec} W_j \to \text{Spec} V \}_{j \in J} \) in \( \text{Aff}_C \) is a Zariski cover if it is a flat cover and each \( f_j : \text{Spec} W_j \to \text{Spec} V \) is an open Zariski immersion.

**Remark 2.5.** The previous definition is actually a particular case of a more general construction. Indeed, as shown in [23], to define a topology on a complete and cocomplete category \( D \) is enough to assign a pseudo-functor \( M : D^{\text{op}} \to \text{Cat} \) satisfying the following conditions:

i) for each morphism \( q : X \to Y \) in \( D \), the functor \( M(q) = q^* : M(Y) \to M(X) \) has a right adjoint \( q_* : M(X) \to M(Y) \) which is conservative

ii) for each Cartesian diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{q'} & Y' \\
\downarrow r & & \downarrow r' \\
X & \xrightarrow{q} & Y
\end{array}
\]

in \( D \), the natural transformation \( q^* r'_* \Rightarrow r_* q'^* \) is an isomorphism.

In terms of such a functor one can define the notion of \( M \)-faithfully flat cover [23, Def. 2.3] and the associated pretopology [23, Prop. 2.4], which induces a topology on \( D \).

In the classical theory of schemes, \( D \) is the category \( \text{Ring}^{\text{op}} \) of affine schemes and, for each \( X = \text{Spec} A \), \( M(A) \) is the category of quasi-coherent sheaves on \( X \). When starting with a monoidal category \( C \) satisfying our assumptions, \( D \) is the category \( \text{Aff}_C \) and the pseudo-functor \( M \) maps an object \( V \) in \( \text{CMon}_C \) to the category of \( V \)-modules and a morphism \( \text{Spec} V \to \text{Spec} W \) to the functor \( - \otimes_V W : V\text{-Mod} \to W\text{-Mod} \). What we have called “flat covers” correspond to Toën-Vaquié’s “\( M \)-faithfully flat covers” (cf. [23, Def. 2.8, Def. 2.10]). When \( D \) is endowed with a topology, a natural question that arises is how the pseudo-functor \( M \) behaves with respect to it. It can be proven ([23, Th. 2.5]) that \( M \) is a stack with respect to that topology. For the reader’s convenience, we review the notion of a stack in the Appendix.  

By making use of flat covers and Zariski covers introduced in Definition 2.4 we may equip the category \( \text{Aff}_C \) with two distinct Grothendieck topologies, called, respectively, the flat and the Zariski topology. Correspondingly, there are two categories of sheaves on \( \text{Aff}_C \), namely

\[
\text{Sh}^{\text{flat}}(\text{Aff}_C) \subset \text{Sh}^{\text{Zar}}(\text{Aff}_C) \subset \text{Presh}(\text{Aff}_C)
\]

Notice that, for each affine scheme \( \Xi \), the presheaf \( Y(\Xi) \) given by the Yoneda embedding \( Y(-) : \text{Aff}_C \to \text{Presh}(\text{Aff}_C) \) is actually a sheaf in \( \text{Sh}^{\text{flat}}(\text{Aff}_C) \subset \text{Sh}^{\text{Zar}}(\text{Aff}_C) \) [23, Cor. 2.11, 1]); this sheaf will be denoted again by \( \Xi \).

The next and final step is to define the category of schemes over the category \( C \). We first have to introduce the notion of affine Zariski cover in the category \( \text{Sh}^{\text{Zar}}(\text{Aff}_C) \).
Definition 2.6. [23, Def. 2.12] a) Let $\Xi$ be an affine scheme in $\text{Aff}_C$. A subsheaf $F \subset \Xi$ is said to be a Zariski open of $\Xi$ if there exists a collection of open Zariski immersions $\{\Xi_i \to \Xi\}_{i \in I}$ such that $F$ is the image of the sheaf morphism $\coprod_{i \in I} \Xi_i \to \Xi$.

(b) A morphism $F \to G$ in $\text{Sh}^{\text{Zar}}(\text{Aff}_C)$ is said to be an open Zariski immersion if, for any affine scheme $\Xi$ and any sheaf morphism $\Xi \to G$, the induced morphism $F \times_G \Xi \to \Xi$ is a monomorphism whose image is a Zariski open of $\Xi$.

(c) Let $F$ be a sheaf in $\text{Sh}^{\text{Zar}}(\text{Aff}_C)$. A collection of open Zariski immersions $\{\Xi_i \to F\}_{i \in I}$, where each $\Xi_i$ is an affine scheme over $\text{Aff}_C$, is said to be an affine Zariski cover of $F$ if the resulting morphism

$$\prod_{i \in I} \Xi_i \to F$$

is a sheaf epimorphism.

It should be noted that, in the case of affine schemes over $C$, the definition of open Zariski immersion in Definition 2.6, (b) does coincide with that previously introduced in Definition 2.4, (b) [23, Lemma 2.14].

Definition 2.7. A scheme over the category $C$ is a sheaf $F$ in $\text{Sh}^{\text{Zar}}(\text{Aff}_C)$ that admits an affine Zariski cover. The category of schemes over $C$ will be denoted by $\text{Sch}_C$.

2.2. Notation and examples. Primarily to the purpose of fixing our notational conventions, we now briefly describe the basic examples of symmetric monoidal categories we shall work with in the sequel of the present paper.

- The category $\text{Set}$ of sets can be endowed with a monoidal product given by the Cartesian product. Then $(\text{Set}, \times, \ast)$ is a symmetric monoidal category and $\text{CMon}_{\text{Set}} = \text{Mon}$ is the usual category of commutative, associative and unitary monoids.

- The category $\text{Set}_*$ of pointed sets can be endowed with a monoidal product given by the smash product $\wedge$; in this case, the unit object is the pointed set $S^0$ consisting of two elements. Then $(\text{Set}_*, \wedge, S^0)$ is a symmetric monoidal category and $\text{CMon}_{\text{Set}_*} = \text{Mon}_0$ is the category of commutative, associative and unitary monoids with “absorbent object” (such an object will be denoted by 0 in multiplicative notation and by $-\infty$ in additive notation).

- The category $\text{Mon}$ can be endowed with a monoidal product $\otimes$ defined in the following way: $R \otimes R'$ is the quotient of the product $R \times R'$ by the relation $\sim$ such that $(nr, r') \sim (r, nr')$ for each $(n, r, r') \in \mathbb{N} \times R \times R'$. Clearly, the unit object is the additive monoid $(\mathbb{N}, +)$. Then $(\text{Mon}, \otimes, \mathbb{N})$ is a symmetric monoidal category and $\text{CMon}_{\text{Mon}} = \text{SRing}$ is the category of commutative, associative and unitary semirings.

- The category $\text{Ab} = \mathbb{Z} \cdot \text{Mod}$ of Abelian groups can be endowed with a monoidal product $\otimes_{\mathbb{Z}}$ given by the usual tensor product of $\mathbb{Z}$-modules. Then $(\text{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$ is a symmetric monoidal category and $\text{CMon}_{\text{Ab}} = \text{Ring}$ is the category of commutative, associative and unitary rings.
Some remarks on blueprints and $F_1$-schemes

For the functor $L : C \to \text{CMon}_C$ defined in eq. 2.1 as left adjoint to the forgetful functor $|-| : \text{CMon}_C \to C$ we shall adopt the following special conventions:

• if $C = \text{Set}$, $L$ will be denoted by
  \[ \mathbb{N}[-] : \text{Set} \to \text{Mon} ; \]

• if $C = \text{Mon}$, $L$ will be denoted by
  \[ - \otimes_\mathbb{U} \mathbb{N} : \text{Mon} \to \text{SRing} , \]
  where $\mathbb{U}$ is the monoid consisting of just one element (the notation being motivated by the identity $\mathbb{U} \otimes_\mathbb{U} \mathbb{N} = \mathbb{N}$);

• if $C = \text{Mon}_0$, $L$ will be denoted by
  \[ - \otimes_{F_1} \mathbb{N} : \text{Mon}_0 \to \text{SRing} , \]
  where $F_1$ is the object of $\text{Mon}_0$ consisting of two elements, namely $F_1 = \{0, 1\}$ in multiplicative notation (also in this case, the notation is motivated by the identity $F_1 \otimes_{F_1} \mathbb{N} = \mathbb{N}$);

• if $C = \text{Ab}$, $L$ will be denoted by
  \[ \mathbb{Z}[-] : \text{Ab} \to \text{Ring} . \]

All symmetric monoidal categories $\text{Set}$, $\text{Set}_*$, $\text{Mon}$, $\text{Mon}_0$, $\text{Ab}$ described above are complete, cocomplete, and closed, so we can apply the machinery of Toën-Vaquié’s theory illustrated in Subsection 2.1 and define, for each of these categories, the corresponding category of schemes over it. In this way, when $C = \text{Ab}$, one unsurprisingly recovers the usual notion of classical scheme. A more intriguing example is provided by the case of $C = \text{Set}$.

Example 2.8. Monoidal schemes An object of the category $\text{Sch}_{\text{Set}}$ is a “scheme over $F_1$” in the sense of [2]. The equivalence between the two definitions was proved in [24]. We recall that, if $M$ is a commutative monoid, its “spectrum over $F_1$” $\text{Spec} M$ can be realized as the set of prime ideals of $M$ and given a topological space structure.

In the present paper we shall call an object in $\text{Sch}_{\text{Set}}$ a monoidal scheme and use the name of “$F_1$-scheme” for a different kind of algebro-geometric structures (see Definition 5.3).

3. B-schemes

The notion of blueprint was introduced by Olivier Lorscheid in his 2012 paper [14].
Definition 3.1. A blueprint is a pair $B = (R, A)$, where $R$ is a semiring and $A$ is a multiplicative subset of $R$ containing 0 and 1 and generating $R$ as a semiring. A blueprint morphism $f : B_1 = (R_1, A_1) \to B_2 = (R_2, A_2)$ is a semiring morphism $f : R_1 \to R_2$ such that $f(A_1) \subset A_2$.

Notice that, given a blueprint morphism $f : B_1 = (R_1, A_1) \to B_2 = (R_2, A_2)$, its restriction $f|_{A_1} : A_1 \to A_2$ is a monoid morphism that uniquely determines $f$ on the whole of $R_1$.

The idea underlying the notion of blueprint can be illustrated as follows. Some equivalence relations that do not make sense in a monoid $A$ may be expressed in the semiring $A \otimes F_1 \mathbb{N}$. Now, any equivalence relation $R$ on a semiring $S$ induces a projection $S \to S/R$ and can indeed be recovered by such a map. So, the assignment of a pair $(A, A \otimes F_1 \mathbb{N} \to R)$ is to be interpreted as the datum of a monoid $A$ plus the relation on $A \otimes F_1 \mathbb{N}$ given by the epimorphism $A \otimes F_1 \mathbb{N} \to R$.

Example 3.2. Consider the monoid $A_T = \mathbb{N} \cup \{-\infty\}$ (in additive notation, corresponding to $\{T^i\}_{i \in \mathbb{N} \cup \{-\infty\}}$ in multiplicative notation) and the corresponding free semiring $A_T \otimes F_1 \mathbb{N}$ of polynomials in $T$ with coefficient in $\mathbb{N}$ (the functor $- \otimes F_1 \mathbb{N}$ has been introduced in eq. 2.4).

Notice that Spec $A_T$ has two points, namely the prime ideals $\{-\infty\}$ and $(\mathbb{N} \setminus \{0\}) \cup \{-\infty\}$, which embed in Spec $A_T \otimes F_1 \mathbb{N}$ (we are loosely thinking of Spec $A_T \otimes F_1 \mathbb{N}$ as the underlying topological space).

Now, if one takes a closed subset of Spec $A_T \otimes F_1 \mathbb{N}$ and intersects it with Spec $A_T$, one could naively think that the intersection is nonempty only when the chosen closed subset is defined by some relation in $A_T$. However, this is not the case: for instance, the relation $2T = 1$, which makes the ideal $(T)$ trivial, cannot be expressed in the monoid $A_T$. According to Lorscheid’s idea, one can represent this affine “monoidal scheme” by considering the pair $(A_T, A_T \otimes F_1 \mathbb{N} \to A_T \otimes F_1 \mathbb{N}/(2T = 1))$. $\triangle$

The category of blueprints can be given a handier description, which makes it easier to characterise it as the category of commutative monoids in a suitable symmetric monoidal category.

Let us consider the functor $- \otimes F_1 \mathbb{N} : \text{Mon}_0 \to \text{SRing}$ (introduced in eq. 2.4)

Definition 3.3. The category $\text{Blp}$ is the full subcategory of $- \otimes F_1 \mathbb{N}/\text{SRing}$ whose objects $(A, A \otimes F_1 \mathbb{N} \to R)$ satisfy the conditions:

a) the morphism $A \otimes F_1 \mathbb{N} \to R$ is an epimorphism;

b) the composition $A \to |A \otimes F_1 \mathbb{N}| \to |R|$, is a monomorphism (the first map being the unit of the adjunction).

(3.1)

It is immediate that the category $\text{Blp}$ is equivalent to the category of blueprints introduced in Definition 3.1.

Consider now the forgetful functor $| - | : \text{Mon}_0 \to \text{Set}_*$; for each monoid $M$ with absorbent object 0 (in multiplicative notation), the base point of the associated set $|M|$ is clearly the
element corresponding to 0. Its adjoint functor is the functor

\[ \mathbb{N}[-] : \text{Set}_* \to \text{Mon}_0. \]

We can now form the full subcategory \( B \) of \( \mathbb{N}[-] / \text{Mon}_0 \) whose objects \((X, \mathbb{N}[X] \to M)\) are described by conditions formally identical to those in eq. 3.1

\begin{align*}
\text{a) the morphism } &\mathbb{N}[X] \to M \text{ is an epimorphism;} \\
\text{b) the composition } &X \to |\mathbb{N}[X]| \to |M| \text{ is a monomorphism.}
\end{align*}

(3.2)

**Remark 3.4.** The category \( B \) above corresponds to the category of pointed set endowed with a pre-addition structure, as described in [16, §4].

**Theorem 3.5.** The category \( B \) carries a natural structure of symmetric monoidal category. Moreover, this structure is closed, complete, and cocomplete.

**Proof.** In the category \( B \) there is a natural symmetric monoidal product given by

\[ (X, \mathbb{N}[X] \to M) \otimes (X', \mathbb{N}[X'] \to M') = (X \wedge X', \mathbb{N}[X \wedge X'] \to M \otimes M'), \]

(3.3)

where the map \( \mathbb{N}[X \wedge X'] \to M \otimes M' \) is the composition

\[ \mathbb{N}[X \wedge X'] \to \mathbb{N}[X] \otimes \mathbb{N}[X'] \to M \otimes M'; \]

the first morphism maps \( n(x, x') \) to \( nx \otimes x' \) and is an isomorphism (in other words, the functor \( \mathbb{N}[-] \) is monoidal).

Since \( M \otimes M' \) is generated as a monoid by elements of the form \( x \otimes x' \), and since the two maps \( \mathbb{N}[X] \to M \) and \( \mathbb{N}[X'] \to M' \) are surjective, the map \( \mathbb{N}[X \wedge X'] \to M \otimes M' \) is also surjective. Moreover, by the definition of tensor product in the category \( \text{Mon} \), for any \( x, y \in X \setminus \{\ast\} \) and \( x', y' \in X' \setminus \{\ast\} \) one has \( x \otimes x' = y \otimes y' \) if and only if \((x, x') = (y, y')\), so that the map

\[ X \wedge X' \to |M \otimes M'| \]

is a monomorphism. Conditions 3.2 are therefore satisfied.

We now show that the monoidal category \( B \) is closed. Let us define the internal hom functor by setting

\[ \text{Hom}((X, \mathbb{N}[X] \to M), (Y, \mathbb{N}[Y] \to N)) = (Y^X \times_{|\mathbb{N}|^X} |N^M|, \mathbb{N}[Y^X \times_{|\mathbb{N}|^X} |N^M|] \to \overline{N^M}), \]

(3.4)

where \( \overline{N^M} \) is the image of the map

\[ \mathbb{N}[Y^X \times_{|\mathbb{N}|^X} |N^M|] \to \mathbb{N}[|N^M|] \to N^M \]

(the second map above is the counit of the adjunction). Let us check the adjunction property. For each map

\[ (X, \mathbb{N}[X] \to M) \otimes (Y, \mathbb{N}[Y] \to N) = (X \wedge Y, \mathbb{N}[X \wedge Y] \to M \otimes N) \to (Z, \mathbb{N}[Z] \to L), \]

(3.5)
the first component corresponds, by the exponential law in $\text{Set}_*$, to a map $X \to Z^Y$, while the second component is given by a commutative square

$$
\begin{array}{ccc}
\text{N}[X \wedge Y] & \rightarrow & \text{N}[Z] \\
\downarrow & & \downarrow \\
M \otimes N & \rightarrow & L
\end{array}
$$

where the arrow on the left is the product map $\text{N}[X] \otimes \text{N}[Y] \rightarrow M \otimes N$ and the top arrow is the image of the map in the first component through the functor $\text{N}[-]$. By using the property that $\text{N}[-]$ is the left adjoint to the forgetful functor and by noticing that the bottom arrow in A.1 corresponds to a map $M \rightarrow L^N$, it is immediate that to assign the commutative diagram A.1 is equivalent to assign the two commutative diagrams

$$
\begin{array}{ccc}
X & \rightarrow & Z^Y \\
\downarrow & & \downarrow \\
|L|^Y & \rightarrow & |M| \rightarrow |L|^N
\end{array}
$$

together with the condition that the diagonal morphism of the first coincides with the composition of the diagonal morphism of the second and the morphisms $|L|^N \hookrightarrow |L|^N \rightarrow |L|^Y$ (the second map being induced by the map $Y \rightarrow |N|$). Summing up, a map as in eq. 3.5 is equivalent to a map from $X$ to the pullback defined by the diagram

$$
\begin{array}{ccc}
Z^Y & \rightarrow & |L|^N \\
\downarrow & & \downarrow \\
|L|^N & \rightarrow & |L|^Y
\end{array}
$$
along with a compatible map $M \rightarrow L^N$ in such a way that the following diagram commutes:

$$
\begin{array}{ccc}
X & \rightarrow & |L|^N \times_{|L|^Y} Z^Y \\
\downarrow & & \downarrow \\
|M| & \rightarrow & |L|^N \rightarrow |L|^Y
\end{array}
$$

This shows that the internal hom functor in eq. 3.4 is indeed a right adjoint to the monoidal product functor in eq. 3.3.

We wish now to show that the category $\mathcal{B}$ is complete and cocomplete. First we prove that it admits colimits. Given a diagram whose objects are $(X_i, \text{N}[X_i] \rightarrow M_i)$, we claim that its colimit is the object

$$
B = (\varinjlim X_i, \text{N}[\varinjlim X_i] \rightarrow \varinjlim M_i),
$$

where $\varinjlim X_i$ denotes the image of the natural map $\varinjlim X_i \rightarrow |\varinjlim M_i|$; the maps from the diagram to $B$ are the obvious ones. It is immediate that $B$ is an object of $\mathcal{B}$. The injectivity
condition is satisfied by definition. As for the surjectivity condition, one has that, since
the functor $\mathbb{N}[-]$ preserves colimits (being a left adjoint), the map $\mathbb{N}[\text{lim} \ X_i] \to \text{lim} \ M_i$ is
surjective (it is enough to show that for the cases of coproducts and coequalizers, in which
it is a consequence of the surjectivity of the maps $\mathbb{N}[X_i] \to M_i$), so that the image of $\text{lim} \ X_i$
generates $\text{lim} \ M_i$; hence, the map $\mathbb{N}[\text{lim} \ X_i] \to \text{lim} \ M_i$ is surjective.

Consider a map from the given diagram to an object $C$ of $\mathcal{B}$. In the category $\mathbb{N}[-]/\text{Mon}_0$
such a map factorises in a unique way through the object $(\text{lim} \ X_i, \mathbb{N}[\text{lim} \ X_i] \to \text{lim} \ M_i)$
because of the colimit properties in the categories $\text{Set}_*$ and $\text{Mon}_0$ and because the functor
$\mathbb{N}[-]$ preserves colimits. If two elements $x, y \in \text{lim} \ X_i$ have the same image $m \in \text{lim} \ M_i$,
then their images in the first component of $C$ are mapped by the morphism in the second
component to the same element. So, the images of $x$ and $y$ do coincide, just because $C$ is
an object of $\mathcal{B}$. It follows that he map from the diagram in $C$ uniquely factorises through $B$, so that our claim is proved.

Second we prove that $\mathcal{B}$ admits limits. Given a diagram as above, we claim that its limit
is the object

$$B' = (\text{lim} \ X_i, \mathbb{N}[\text{lim} \ X_i] \to \text{lim} \ M_i),$$

where $\text{lim} \ M_i$ is the image of the natural map $\mathbb{N}[\text{lim} \ X_i] \to \text{lim} \ M_i$, which is adjoint to the
map $\text{lim} \ X_i \to \text{lim} |M_i| \cong |\text{lim} \ M_i|$ (the last isomorphism holds since $|-|$ preserves limits,
being a right adjoint) induced by the maps $X_i \to |M_i|$; the maps from $B'$ to the diagram
are the obvious ones. It is clear that $B'$ is an object of $\mathcal{B}$: the surjectivity condition holds
by definition, while for the injectivity condition it is enough to note that it holds when
the limit is either a product or an equalizer. Consider now a map from an object $C$ to
the given diagram. In the category $\mathbb{N}[-]/\text{Mon}_0$ such a map uniquely factorises through the
object $(\text{lim} \ X_i, \mathbb{N}[\text{lim} \ X_i] \to \text{lim} \ M_i)$, because of the limit properties in the categories $\text{Set}_*$
and $\text{Mon}_0$. Since the second component of $C$ is a surjective morphism, this map uniquely
factorises through $B'$. Thus, $B'$ satisfies the limit condition, as claimed. \qed

**Proposition 3.6.** The category $\text{Blp}$ of blueprints is equivalent to the category $\text{CMon}_B$ of
monoids in the symmetric monoidal category $\mathcal{B}$.

**Proof.** To begin with, notice that, for each monoid object $(X, \mathbb{N}[X] \to M)$ in $\mathcal{B}$, the product
declared in eq. 3.3, namely

$$(X, \mathbb{N}[X] \to M) \otimes (X, \mathbb{N}[X] \to M) = (X \wedge X, \mathbb{N}[X \wedge X] \to M \otimes M)$$

induces a map $\mu: (X \wedge X, \mathbb{N}[X \wedge X] \to M \otimes M) \to (X, \mathbb{N}[X] \to M)$. So the first component
of $\mu$ is a map

$$m : X \wedge X \to X$$
which defines a (multiplicative) monoid structure on the set $X$, while the second component of $\mu$ yields a commutative diagram

$$
\begin{array}{ccc}
N[X \land X] & \overrightarrow{N[m]} & N[X] \\
\downarrow & & \downarrow \\
M \otimes M & \rightarrow & M
\end{array}
$$

whose bottom arrow induces an associative and commutative multiplication on the monoid $M$ compatible with its monoidal sum; in other words, it induces a semiring structure on $M$.

Similarly, the top arrow induces a semiring structure one the monoid $N[X]$. In this case, since the multiplication is given by the application of the free monoid functor $N[-]$ to the multiplication $m$ of $A$, the resulting semiring is nothing but the free semiring $X \otimes_{F_1} N$ generated by the monoid $(X, m)$. The commutativity of the diagram ensures that the multiplication on $X$ is consistent with that on $M$, so that $X$ can still be seen as a subobject of $|M|$.

In conclusion, a monoid object in the category $B$ is a blueprint, and it is also obvious that any blueprint can be obtained this way. $\square$

**Remark 3.7.** Theorem 3.5 and Proposition 3.6 should hopefully provide a full elucidation of [16, Lemma 4.1]. $\triangle$

We have shown that the category of blueprints fits in with the general framework proposed by Toën and Vaquié, so we can apply the formalism of Subsection 2.1 to define the category of schemes over $B$.

**Definition 3.8.** An affine $B$-scheme is an object of the category $\text{Aff}_B = \text{Blp}^{op}$, a $B$-scheme an object of the category $\text{Sch}_B$ (see Definition 2.7).

4. Adjunctions

This section aims to show that the natural adjunction between the category $\text{Sch}_{\text{Set}}$, and the category $\text{Sch}_{\text{Mon}_0}$ factorises through an adjunction between $\text{Sch}_{\text{Mon}_0}$ and $\text{Sch}_B$ and an adjunction $\text{Sch}_B$ and $\text{Sch}_{\text{Set}}$.

**Lemma 4.1.** The functor $\tilde{F} : N[-]/\text{Mon}_0 \to \text{Mon}_0$ mapping an object $(X, N[X] \to M)$ to the monoid $M$ admits a right adjoint

$$
\tilde{G} : \text{Mon}_0 \to N[-]/\text{Mon}_0,
$$

mapping a monoid $M$ to the object $(|M|, N[|M|] \to M)$, where the second component is the counit of the adjunction $N[-] \dashv | - |$. The adjunction $\tilde{F} \dashv \tilde{G}$ induces an adjunction between

$$
\begin{array}{ccc}
N[X \land X] & \overrightarrow{N[m]} & N[X] \\
\downarrow & & \downarrow \\
M \otimes M & \rightarrow & M
\end{array}
$$

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In conclusion, a monoid object in the category $B$ is a blueprint, and it is also obvious that any blueprint can be obtained this way. $\square$

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$$
\tilde{G} : \text{Mon}_0 \to N[-]/\text{Mon}_0,
$$

mapping a monoid $M$ to the object $(|M|, N[|M|] \to M)$, where the second component is the counit of the adjunction $N[-] \dashv | - |$. The adjunction $\tilde{F} \dashv \tilde{G}$ induces an adjunction between

$$
\begin{array}{ccc}
N[X \land X] & \overrightarrow{N[m]} & N[X] \\
\downarrow & & \downarrow \\
M \otimes M & \rightarrow & M
\end{array}
$$

whose bottom arrow induces an associative and commutative multiplication on the monoid $M$ compatible with its monoidal sum; in other words, it induces a semiring structure on $M$.

Similarly, the top arrow induces a semiring structure one the monoid $N[X]$. In this case, since the multiplication is given by the application of the free monoid functor $N[-]$ to the multiplication $m$ of $A$, the resulting semiring is nothing but the free semiring $X \otimes_{F_1} N$ generated by the monoid $(X, m)$. The commutativity of the diagram ensures that the multiplication on $X$ is consistent with that on $M$, so that $X$ can still be seen as a subobject of $|M|$.

In conclusion, a monoid object in the category $B$ is a blueprint, and it is also obvious that any blueprint can be obtained this way. $\square$

**Remark 3.7.** Theorem 3.5 and Proposition 3.6 should hopefully provide a full elucidation of [16, Lemma 4.1]. $\triangle$

We have shown that the category of blueprints fits in with the general framework proposed by Toën and Vaquié, so we can apply the formalism of Subsection 2.1 to define the category of schemes over $B$.

**Definition 3.8.** An affine $B$-scheme is an object of the category $\text{Aff}_B = \text{Blp}^{op}$, a $B$-scheme an object of the category $\text{Sch}_B$ (see Definition 2.7).
the associated categories of monoids

\[ \text{SRing} \begin{array}{c} F \leftarrow \bigotimes_{\mathbb{F}_1} \mathbb{N}/\text{SRing} \end{array} G \rightarrow \]

where \( F \) maps an object \((A, A \otimes_{\mathbb{F}_1} \mathbb{N} \rightarrow R)\) to the semiring \( R \) and its right adjoint \( G \) maps a semiring \( R \) to the object \((|R|, |R| \otimes_{\mathbb{F}_1} \mathbb{N} \rightarrow R)\), where the second component is the counit of the adjunction \(- \otimes_{\mathbb{F}_1} \mathbb{N} \dashv \cdot\).

**Proof.** Let \((X, \mathbb{N}[X] \rightarrow M)\) be an object of \(\mathbb{N}[-]/\text{Mon}_0\) and \(N\) a monoid. Let us consider a morphism

\( (X, \mathbb{N}[X] \rightarrow M) \rightarrow (|N|, \mathbb{N}[|N|] \rightarrow N) \)

in the category \(\mathbb{N}[-]/\text{Mon}_0\) and denote by \(f: X \rightarrow |N|\) the induced set morphism. In the commutative square

\[ \begin{array}{c} \mathbb{N}[X] \xrightarrow{\mathbb{N}[f]} \mathbb{N}[|N|] \\ \downarrow \quad \quad \quad \downarrow \\ M \quad \quad \quad N \end{array} \] (4.3)

the map \(\mathbb{N}[f]\), because of the property of the vertical arrow on the right (which is the counit of the adjunction), amounts to the same as a map \(\mathbb{N}[X] \rightarrow N\). Such a map, by adjunction, must be induced by the map \(f: X \rightarrow |N|\). Thus, the assignment of the map \(f\) and the commutative square 4.3 are equivalent to the assignment of the commutative triangle

\[ \begin{array}{c} \mathbb{N}[X] \\ \downarrow \quad \quad \quad \downarrow \\ M \quad \quad \quad N \end{array} \]

But this diagram is equivalent to the assignment of a map \(M \rightarrow N\), since the vertical map is given. We have therefore the adjunction \(\tilde{F} \dashv \tilde{G}\), as claimed. The last statement is now straightforward. \(\square\)

Since image of the functor \(\tilde{G}: \text{Mon}_0 \rightarrow \mathbb{N}[-]/\text{Mon}_0\) is contained in the subcategory \(\mathcal{B}\), the adjunction 4.2 restricts to the adjunction

\[ \text{SRing} \begin{array}{c} F \leftarrow \text{Blp} \end{array} G \rightarrow \]

(4.4)
It is immediate that the adjunction $\text{SRing} \xrightarrow{\dashv} \text{Mon}_0$ factorises through the adjunction

$\text{Mon}_0 \xrightarrow{\rho} \text{Blp}$, \hspace{1cm} (4.5)

where $\rho(A, A \otimes_{F_1} N \rightarrow R) = A$ and $\sigma(A) = (A, A \otimes_{F_1} N \rightarrow A \otimes_{F_1} N)$.

The adjunctions above induce opposite adjunctions between the corresponding categories of affine schemes. We have therefore the following diagram

\[ \begin{array}{ccc}
\text{Aff}_{\text{Mon}_0} & \xrightarrow{\sim} & \text{Aff}_{\text{Set}^*} \\
F & \downarrow{\rho} & \uparrow{\sigma} \\
\text{Aff}_{\tilde{B}} & \xrightarrow{\tilde{F}} & \text{Set}^* \\
\sigma & \downarrow{\tilde{\sigma}} & \uparrow{\tilde{\rho}} \\
\text{Set}^* & \xrightarrow{\sim} & \tilde{\text{B}} \\
\end{array} \] \hspace{1cm} (4.6)

associated to the diagram

\[ \begin{array}{ccc}
\text{Mon}_0 & \xrightarrow{\sim} & \text{Set}^* \\
\tilde{\rho} & \downarrow{\tilde{\sigma}} & \uparrow{\tilde{\rho}} \\
\tilde{\text{B}} & \xrightarrow{\sim} & \text{Set}^* \\
\end{array} \] \hspace{1cm} (4.7)

We now wish to show that the functors in diagram (4.7) satisfy the conditions that are required to apply [23, Cor. 2.1, Cor. 2.2]. Of course, it will be enough to check that for the adjunctions $\tilde{F} \dashv \tilde{G}$ and $\hat{\rho} \dashv \hat{\sigma}$.

**Lemma 4.2.** In the adjunction $\text{Mon}_0 \xrightarrow{\sim} \tilde{\text{B}}$

(1) the left adjoint $\tilde{F}$ is monoidal;

(2) the right adjoint $\tilde{G}$ is conservative;

(3) the functor $\tilde{G}$ preserves filtered colimits.

**Proof.** (1) and (2) are straightforward.

As for (3), we have to show that the right adjoint preserves filtered colimits, which is also quite obvious. The colimit of a filtered diagram $(X_i, N[X_i] \rightarrow M_i)$ is indeed given by $(\lim_{\longrightarrow} X_i, N[\lim_{\longrightarrow} X_i] \rightarrow \lim_{\longrightarrow} M_i)$ provided that it belongs to our category (notice that $N[\lim_{\longrightarrow} X_i] \cong \lim_{\longrightarrow} N[X_i]$ since $N[-]$ is a left adjoint). But it does, because the map $N[\lim_{\longrightarrow} X_i] \rightarrow \lim_{\longrightarrow} M_i$ is surjective due to the
fact that so are the maps $N[X] \to M_i$ and the injectivity condition is satisfied since the
diagram is filtrant. □

Lemma 4.3. In the adjunction $B \xrightarrow{\hat{\sigma}} \text{Set}_*$

(1) the left adjoint $\hat{\sigma}$ is monoidal;
(2) the right adjoint $\hat{\rho}$ is conservative;
(3) the functor $\hat{\rho}$ preserves filtered colimits.

Proof. The functors $\hat{\sigma}, \hat{\rho}$ are defined as follows: $\hat{\sigma}(X) = (X, N[X] \xrightarrow{\pi} N[X])$ and

$\hat{\rho}(X, N[X] \to M) = X$. (1) is then straightforward. As for (2), we know that a map

$(X, N[X] \to M) \to (Y, N[Y] \to N)$ is determined by the first component, so that $\hat{\rho}$ is

conservative. Finally, (3) is proved by proceeding as in the proof of Lemma 4.2. □

Proposition 4.4. a) The functor $F: \text{Aff}_B \to \text{Aff}_{\text{Mon}_0}$ is continuous w.r.t. the Zariski and

the flat topology; moreover, the functor

\[ F: \text{Sh(Aff)} \to \text{Sh(Aff}_{\text{Mon}_0}) \] (4.8)

preserves the subcategories of schemes and so induces a functor

\[ F: \text{Sch}_B \to \text{Sch}_{\text{Mon}_0} \]

\[ \Sigma \mapsto F(\Sigma) \] (4.9)

b) The functor $\sigma: \text{Aff}_{\text{Set}_*} \to \text{Aff}_B$ is continuous w.r.t. the Zariski and the flat topology;

moreover, the functor

\[ \hat{\sigma}: \text{Sh(Aff)}_{\text{Set}_*} \to \text{Sh(Aff}_B \] (4.10)

preserves the subcategories of schemes and so induces a functor

\[ \hat{\sigma}: \text{Sch}_{\text{Set}_*} \to \text{Sch}_B \]

\[ \Xi \mapsto \hat{\sigma}(\Xi) \] (4.11)

Proof. a) We first note that, given objects $X_M = (X, N[X] \to M), X'_M = (X, N[X] \to M')$
in $B$, if $X_M \to X'_M$ is a flat morphism in $B$, then in the associated diagram

\[ \begin{array}{ccc}
M \text{- Mod} & \xrightarrow{f} & X_M \text{- Mod} \\
\downarrow & & \downarrow \\
M' \text{- Mod} & \xrightarrow{g} & X'_M \text{- Mod}
\end{array} \]

the natural transformation between the two compositions is an isomorphism. We wish
to prove that an analogous property holds when one considers a flat morphism in the
category $\text{Blp}$. As usual, it will be enough to work in the category $\otimes_{F_1} \text{N/SRing}$. Let

$A_R = (A, A \otimes_{F_1} \text{N} \to R)$ and $A_S = (A, A \otimes_{F_1} \text{N} \to S)$ be objects in this category, and

consider a flat morphism $A_R \to A_S$. An $A_R$-module is given by a pair

$(N, M) \in \text{Set}_* \times \text{Mon}_0$
such that $N$ is a subset of $|M|$ and generates it as a module, together with an action of $A$ on $N$ and an action of $R$ on $M$, such that the former is the restriction of the latter. If $M$ is an $R$-module $M$, its associated $A_R$-module is the $(R, R \otimes_{F_1} N \to R)$-module $(|M|, M)$, whose $A_R$-module structure is induced by the map

$$A_R \to (R, R \otimes_{F_1} N \to R)$$

given by the pair of immersions $\iota: A \hookrightarrow R$ and $\iota \otimes_{F_1} \text{id}: A \otimes_{F_1} N \to R \otimes_{F_1} N$, where the latter fits in the commutative square

$$\begin{array}{ccc}
A \otimes_{F_1} N & \xrightarrow{\iota \otimes_{F_1} \text{id}} & R \otimes_{F_1} N \\
\downarrow & & \downarrow \\
R & \xrightarrow{\text{id}_R} & R
\end{array}$$

The category $R$-$\text{Mod}$ can therefore be identified with the full subcategory of the category of

$$(A \otimes_{F_1} N, A \otimes_{F_1} N \to R) - \text{Mod}$$

whose underlying objects in $\text{Mon}_0/\text{Mon}_0$ are of the kind $(M, M = M)$.

We have now to show that, for any flat morphism $A_R \to A_S$ in $- \otimes_{F_1} N/\text{SRing}$, in the associated diagram

$$\begin{array}{ccc}
R \text{-Mod} & \xrightarrow{} & A_R \text{-Mod} \\
\downarrow & & \downarrow \\
S \text{-Mod} & \xrightarrow{} & A_S \text{-Mod}
\end{array}$$

the natural transformation between the two compositions is an isomorphism. As for the first component, the commutativity up isomorphism of the above diagram is straightforward. As for the second component, it can be easily shown by adapting the argument in proof of Prop. 3.6 of [23].

The statement then follows from [23, Cor. 2.22].

b) Consider a flat morphism $A \to B$ in the category $\text{Mon}_0$, and denote by $A_{A \otimes_{F_1} N}$ the object $(A, A \otimes_{F_1} N = A \otimes_{F_1} N)$ in $- \otimes_{F_1} N/\text{SRing}$. Each $A_{A \otimes_{F_1} N}$-module is given by a pair $(N, M) \in \text{Set}_* \times \text{Mon}_0$ together with an action of $A$ on $N$ and an action of $A \otimes_{F_1} N$ on $M$, the two actions being compatible in the obvious sense. In the diagram

$$\begin{array}{ccc}
A_{A \otimes_{F_1} N} \text{-Mod} & \xrightarrow{} & A \text{-Mod} \\
\downarrow & & \downarrow \\
B_{B \otimes_{F_1} N} \text{-Mod} & \xrightarrow{} & B \text{-Mod}
\end{array}$$

the horizontal map sends an object $(N, M)$ the set $N$ endowed with an action of the monoid $A$. Since tensor products are defined “componentwise”, the diagram commutes. $\square$
Some remarks on blueprints and \( F_1 \)-schemes

5. B-schemes, \( F_1 \)-schemes, and \( F_1 \)-schemes with relations

Consider the functor

\[
- \otimes_{F_1} Z : \text{Mon}_0 \rightarrow \text{Ring}
\]

which is the left adjoint to the forgetful functor, and define the category \((- \otimes_{F_1} Z)/\text{Ring}\). We shall denote by \( \mathbb{Z} \text{-Blp} \) the full subcategory of \((- \otimes_{F_1} Z)/\text{Ring}\) formally defined in the same way as the subcategory blueprints \( \text{Blp} \) of \(- \otimes_{F_1} \text{N}/\text{SRing}\).

**Lemma 5.1.** There is an equivalence between the category \( \mathbb{Z} \text{-Blp} \) and the category \( \text{Blp} \).

**Proof.** Notice that there is a natural functor

\[
\text{Blp} \rightarrow \mathbb{Z} \text{-Blp}
\]

sending \((A, A \otimes_{F_1} \text{N} \rightarrow R)\) to \((A, A \otimes_{F_1} Z \rightarrow R \otimes_{\text{N}} Z)\). This is well defined: notice indeed that, if \( R \) is the quotient of \( A \otimes_{F_1} \text{N} \) through a relation \( R \), it can be seen as the pushout

\[
\begin{array}{ccc}
(A[T_{(x,y)}]_{(x,y) \in R} \otimes_{F_1} \text{N}) & \longrightarrow & A \otimes_{F_1} \text{N} \\
\downarrow & & \downarrow \\
A \otimes_{F_1} \text{N} & \longrightarrow & R
\end{array}
\]

\( T_{(x,y)} \) being sent to \( x \) by the upper arrow and to \( y \) by the left one (the restriction of both maps to \( A \otimes_{F_1} \text{N} \) being of course the identity).

Since \(- \otimes_{\text{N}} Z\) is a left adjoint, it preserves pushouts, so that \( R \otimes_{\text{N}} Z\) is the pushout of the diagram

\[
\begin{array}{ccc}
(A[T_{(x,y)}]_{(x,y) \in R} \otimes_{F_1} Z) & \longrightarrow & A \otimes_{F_1} Z \\
\downarrow & & \downarrow \\
A \otimes_{F_1} Z & \longrightarrow & R \otimes_{\text{N}} Z
\end{array}
\]

That means that \( R \otimes_{\text{N}} Z\) is the quotient of \( A \otimes_{F_1} Z\) by the ideal generated by \( \{x - y\}_{(x,y) \in R}\). It is thus clear that \( A \otimes_{F_1} Z \rightarrow R \otimes_{\text{N}} Z\) is surjective and that \( A \rightarrow |R \otimes_{\text{N}} Z|\) is injective if so is \( A \rightarrow |R|\).

Since any element of \( A \otimes_{F_1} Z\) can be written in a unique way as a difference of elements \( \sum_{i=1}^{k} n_i a_i - \sum_{i=k+1}^{l} n_i a_i \) in \( A \otimes_{F_1} \text{N}\) with \( a_i \neq a_j \) for \( i \neq j\), the same argument shows that our functor is essentially surjective and injective on objects. Now, in both \( \text{Blp} \) and \( \mathbb{Z} \text{-Blp} \) a map is uniquely determined by its first component, and the only restriction on this component is given by the relation \( R \). So, because of the considerations above, our functor is also fully faithful. \[\square\]

It immediately follows from Lemma 5.1 the category of schemes associated with the category \( \mathbb{Z} \text{-Blp} \) is equivalent to the category \( \text{Sch}_B \) of B-schemes.
By mimicking what has been done in the previous Section (see Lemma 4.1 and eq. 4.4), one may define an adjunction

\[
\text{Ring} \xrightarrow{G_Z} \mathcal{Z} - \text{Blp}
\]

between the category $\mathcal{Z} - \text{Blp}$ and the category of rings. By proceeding as in proof of Prop. 4.4 one shows that the adjunction (5.2) induces an adjunction

\[
\text{Sch}_{\text{Ab}} \xleftarrow{\hat{F}_Z} \text{Sch}_B
\]

between the corresponding categories of schemes.

Since a morphism in $\mathcal{Z} - \text{Blp} \simeq \text{Blp}$ is given by a morphism in $\text{Mon}_0$ and a morphism in $\text{Ring}$ (plus compatibility conditions), it is not surprising that a $B$-scheme gives rise to a pair consisting of a monoidal scheme and a classical scheme.

**Definition 5.2.** Given a scheme $\Sigma$ of $\text{Sch}_B$, we set

- $\Sigma_{\mathcal{Z}} := \hat{F}_Z(\Sigma)$, which is an object of $\text{Sch}_{\text{Ab}}$ (i.e. a classical scheme);
- $\Sigma := \hat{\rho}(\Sigma)$, which is an object of $\text{Sch}_{\text{Set}}$ (i.e. a monoidal scheme).

We will say that the pair $(\Sigma, \Sigma_{\mathcal{Z}})$ is generated by the $B$-scheme $\Sigma$.

There is a natural transformation $\Sigma_{\mathcal{Z}} \rightarrow \Sigma \otimes_{F_1} \mathcal{Z}$, which is obtained via the unit of the adjunction $\hat{\rho} \dashv \hat{\sigma}$ and by applying the functor $\hat{F}_Z$. By definition, there is indeed a map

\[
F_Z \Sigma \rightarrow F_Z \sigma \rho \Sigma \cong \Sigma \otimes_{F_1} \mathcal{Z},
\]

where the isomorphism is given by the natural isomorphism $F_Z \circ \sigma = - \otimes_{F_1} \mathcal{Z}$.

In the affine case, such a map is simply realized as the bottom arrow of the map between arrows

\[
\begin{array}{ccc}
\mathcal{Z} & \xrightarrow{\Delta} & \mathcal{Z} \\
\downarrow & & \downarrow \\
\mathcal{Z} & \rightarrow & R
\end{array}
\]

where the top and the left map are identities (the square above being of course the counit of the adjunction in the object $(A, A \otimes_{F_1} \mathcal{Z} \rightarrow R)$ of $(\mathcal{Z} - \text{Blp})^{\text{op}}$).

Summing up, a $B$-scheme $\Sigma$ induces therefore the following objects:

- a monoidal scheme $\Sigma_{\mathcal{Z}}$;
- a (classical) scheme $\Sigma_{\mathcal{Z}}$ over $\mathcal{Z}$;
- a natural transformation

\[
\Lambda: \Sigma_{\mathcal{Z}} \rightarrow \Sigma \circ | - | \cong \Sigma \otimes_{F_1} \mathcal{Z}.
\]
Some remarks on blueprints and $F_1$-schemes

Such data are similar to (but different from) those used by A. Connes and C. Consani [1] in their definition of $F_1$-schemes, which we now recall.

**Definition 5.3.** [1, Def. 4.7] An $F_1$-scheme is a triple $(\Xi, \Xi_Z, \Phi)$, where

1. $\Xi$ is a monoidal scheme;
2. $\Xi_Z$ is a (classical) scheme;
3. $\Phi$ is natural transformation $\Xi \rightarrow \Xi_Z \circ (- \otimes_{F_1} \mathbb{Z})$, such that the induced natural transformation $\Xi \circ \mid \cdot \mid \rightarrow \Xi_Z$, when evaluated on fields, gives isomorphisms (of sets).

A manifest evident difference between $B$-schemes and $F_1$-schemes is, of course, the direction of the natural transformation linking the monoidal scheme and the classical scheme. Moreover, the condition on $\Phi$ in Definition 5.3.(3) may fail to be fulfilled in the case of $B$-schemes, as shown by the following example.

**Example 5.4.** Consider a pair $(A, R \rightarrow A \otimes_{F_1} \mathbb{Z})$ defining an affine $F_1$-scheme in the sense Definition 5.3. Notice that, in this case, the natural transformation $\Phi$ calculated on a field $k$ corresponds to mapping a prime ideal $P$ of $A \otimes_{F_1} \mathbb{Z}$ plus an immersion $A \otimes_{F_1} \mathbb{Z}/P \hookrightarrow k$ to their restrictions to $R$; the requirement is that this is a bijection.

On the other hand, according to the general idea underlying the notion of blueprint, if the pair $(A, R)$ is associated with an affine $B$-scheme, then the ring $R$ encodes the information of a relation $R$ intended to reduce the number of ideals of $A$. Take for instance the case $(A, A \otimes_{F_1} \mathbb{Z} \rightarrow R)$, with $A = \mathbb{N} \cup \{ -\infty \}$ (additive notation) and $R = T \otimes_{F_1} \mathbb{Z}/(2T - 1)$. Then, $\mathbb{N}$ is an ideal not coming from any ideal of $R$, since $T$ is invertible (in more algebraic terms, we are saying that the map to any field $k$ sending $T$ to 0 can not be lifted to a map from $R$ to $k$).

We shall now combine the two categories of $B$-schemes and $F_1$-schemes into a larger category.

**Definition 5.5.** The category of $F_1$-schemes with relations is the fibered product of the categories of $B$-schemes and that of $F_1$-schemes over the category of monoidal schemes. Thus, a $B$-scheme $\Sigma$ generating the pair $(\Sigma, \Sigma_Z)$ and an $F_1$-scheme $(\Sigma, \Sigma'_Z, \Phi)$ will determine a $F_1$-scheme with relations denoted by the quadruple $(\Sigma, \Sigma_Z, \Sigma'_Z, \Phi)$.

Definition 5.2 and 5.3 imply that, for every $F_1$-scheme with relations $(\Sigma, \Sigma_Z, \Sigma'_Z, \Phi)$, there is a natural transformation $\Psi_1 : \Sigma_Z \rightarrow \Sigma'_Z$ given by the composition

$$\Sigma_Z \xrightarrow{\Lambda} \Sigma \circ \mid \cdot \mid \xrightarrow{\Phi} \Sigma'_Z,$$

which will be called the first transferring map determined by the given $F_1$-scheme with relations. As its name would suggest, the natural transformation $\Psi_1$, loosely speaking, conveys information on about how many “points” of $\Sigma'_Z$ are compatible with the $B$-scheme that

---

[1] In [1] the functor $- \otimes_{F_1} \mathbb{Z}$ is denoted by $\beta$ and its right adjoint $\mid \cdot \mid$ by $\beta^*$. 


generates the pair $(\Sigma, \Sigma_Z)$. Actually, there is a different way to “transfer” this information from the $B$-scheme to the $F_1$-scheme associated with the fibered object $(\Sigma, \Sigma_Z, \Sigma', \Phi)$.

The counit of the adjunction $- \otimes_{F_1} \mathbb{Z} \dashv | \dashv -|$ induces a map
\[
\Sigma \circ | -| \to \Sigma \circ | -| \otimes_{F_1} \mathbb{Z}.
\] (5.5)
Moreover, the natural transformation 5.3 induces a map
\[
\Sigma' \circ | -| \otimes_{F_1} \mathbb{Z} \to \Sigma \circ | -| \otimes_{F_1} \mathbb{Z}.
\] (5.6)
Let $\Sigma_B'$ be the sheaf on the category $\text{Ring}$ obtained as the pullback of the maps 5.5 and 5.6, i.e.
\[
\begin{CD}
\Sigma' \circ | -| \otimes_{F_1} \mathbb{Z} @>{\Phi}>> \\
\Sigma \circ | -| \otimes_{F_1} \mathbb{Z}
\end{CD}
\] (5.7)
By composing the vertical arrow on the left with $\Phi$, we get a natural transformation
\[
\Psi_2: \Sigma_B' \to \Sigma_Z,
\] (5.8)
which will be called the second transferring map determined by the $F_1$-scheme $(\Sigma, \Sigma_Z, \Sigma', \Phi)$.

In the case of a $F_1$-scheme $(\Sigma, \Sigma_Z, \Sigma', \Phi)$, the natural transformation $\Phi$ induces an isomorphism $\Sigma(|K|) \simeq \Sigma'_Z(K)$ for every field $K$. Since for the finite field $F_q$, one has $|F_q| = F_{1q-1}$, it immediately follows, as observed in [1], that there is bijective correspondence between the set of $F_q$-points of $\Sigma'_Z$ and the set of $F_{1q-1}$-points of $\Sigma$; in other words, one has
\[
\#\Sigma'_Z(F_q) = \#\Sigma(F_{1q-1}).
\] (5.9)
This result can be extended to our setting in two different ways, because, for a $B$-scheme underlying a $F_1$-scheme with relations, we can think of its “$F_{1q-1}$-points” in two different senses.

On the one hand, the forgetful functor $| - |: \text{Ring} \to \text{Mon}_0$ admits the obvious factorization
\[
\begin{CD}
\text{Ring} @>{G_Z}>> \mathbb{Z} - \text{Blp} @>{\rho}>> \text{Mon}_0
\end{CD}
\] (5.10)
(see eq. 4.5 and eq. 5.2). Clearly, one has
\[
G_Z(F_q) = (F_{1q-1}, F_{1q-1} \otimes_{F_1} \mathbb{Z} \to F_{1q-1})
\]
and $\rho(G_Z(F_q)) = |F_q| = F_{1q-1}$. Now, by definition, the first transferring map $\Psi_1$ factorises as $\Psi_1 = \Phi \circ \Lambda$. Since $\Phi$ gives isomorphisms (of sets) when evaluated on fields and $\Lambda$ is always locally injective, it is immediate to prove the following result.

**Proposition 5.6.** Let $(\Sigma, \Sigma_Z, \Sigma'_Z, \Phi)$ be a $F_1$-scheme with relations. The first transferring map $\Psi_1: \Sigma_Z \to \Sigma'_Z$, when evaluated on a field, gives an injective map (of sets). In particular, the set of $G_Z(F_q)$-points of the underlying $B$-scheme naturally injects into the set of $F_q$-points of the scheme $\Sigma''_Z$ (which isomorphic to the set of $F_{1q-1}$-points of the monoidal scheme $\Sigma$).
On the other hand, one has the immersion $\sigma: \text{Mon}_0 \rightarrow \text{Bl}_p$, with

$$
\sigma(F_{19-1}) = (F_{19-1}, F_{19-1} \otimes_{F_1} \mathbb{Z} \xrightarrow{id} F_{19-1} \otimes_{F_1} \mathbb{Z}).
$$

Notice that $G_Z(F_q) \neq \sigma(F_{19-1})$, while $|G_Z(F_q)| = |\sigma(F_{19-1})| = F_{19-1}$.

**Theorem 5.7.** Let $(\Sigma, \Sigma_Z, \Sigma \Phi)$ be a $\mathbb{F}_1$-scheme with relations. The set of $\sigma(F_{19-1})$-points of the underlying $\mathbb{B}$-scheme is in natural bijection with the set of $\mathbb{F}_q$-points of the subpresheaf of $\Sigma_Z$ given by the image of $\Psi_2: \Sigma_B \rightarrow \Sigma_Z$.

**Proof.** Since we can work locally, we assume that the underlying scheme is given by a monoid $M$, a ring $R$, and a map $M \otimes_{F_1} \mathbb{Z} \rightarrow R$ satisfying the usual conditions. An $F_{19-1}$-point is given by a commutative square

$$
\begin{array}{ccc}
M \otimes_{F_1} \mathbb{Z} & \xrightarrow{\sim} & F_{19-1} \otimes_{F_1} \mathbb{Z} \\
\downarrow & & \downarrow \text{id} \\
R & \xrightarrow{\sim} & F_{19-1} \otimes_{F_1} \mathbb{Z}
\end{array}
$$

such that the arrow on the top is induced by a map $M \rightarrow F_{19-1}$.

The datum of a generic commutative square as above is equivalent to the datum of an $F_q$-point in $\text{Spec} R \circ (|\cdot| \otimes_{F_1} \mathbb{Z})$.

The fact that the map on the top has the required property is equivalent to the fact that the image of the point above through the restriction map

$$
\text{Spec} R(|F_q| \otimes_{F_1} \mathbb{Z}) \rightarrow \text{Spec}(M \otimes_{F_1} \mathbb{Z})(|F_q| \otimes_{F_1} \mathbb{Z})
$$

is in the image of the map

$$
\text{Spec} M(|F_q|) \rightarrow \text{Spec}(M \otimes_{F_1} \mathbb{Z})(|F_q| \otimes_{F_1} \mathbb{Z})
$$

induced by the functor $- \otimes_{F_1} \mathbb{Z}$. \(\square\)

We are now interested in the case where the $\mathbb{F}_1$-points of the underlying monoidal scheme $\Sigma$ are counted by a polynomial in $n$. Some preliminary definitions and results are in order.

A monoidal scheme $\Sigma$ is said to be **noetherian** if it admits a finite open cover by representable subfunctors $\{\text{Spec}(A_i)\}$, with each $A_i$ a noetherian monoid. Recall that, as it is proved in [8, Theorem 5.10 and 7.8], a monoid is noetherian if and only if it is finitely generated. This immediately implies that, for any prime ideal $p \subset M$, the localized monoid $M_p$ is noetherian and the abelian group $M_p^\times$ of invertible elements in $M_p$ is finitely generated.

**Remark 5.8.** Notice that, given an $\mathbb{F}_1$-scheme $(\Sigma, \Sigma_Z, \Phi)$, the fact that the monoidal scheme $\Sigma$ is noetherian does not entail that the scheme $\Sigma_Z$ is noetherian as well. Let us consider, for instance, the affine $\mathbb{F}_1$-scheme given by $\mathbb{Z}[X, \varepsilon]/(\varepsilon_i^2) \rightarrow \mathbb{Z}[X]$, with $i \in \mathbb{N}$. The monoidal scheme is noetherian, while the ascending chain of ideals $\ldots \subset (\varepsilon_0, \ldots, \varepsilon_i) \subset (\varepsilon_0, \ldots, \varepsilon_{i+1}) \subset \ldots$ does not have a maximal element. Observe that, as for the points of
the classical scheme, the presence of the \(\varepsilon_i\)'s is immaterial; hence, one has the required isomorphism \(\mathbb{Z}[X][[\mathbb{K}]] \simeq \mathbb{Z}[X, \varepsilon_i]/(\varepsilon_i^2(\mathbb{K}))\) for any field \(\mathbb{K}\).

Let \(\tilde{\Sigma}\) the geometrical realization of the monoidal scheme \(\Sigma\). Following Connes-Consani’s definition \([1, p. 25]\), we shall say that \(\Sigma\) is torsion-free if, for any \(x \in \tilde{\Sigma}\), the abelian group \(\mathcal{O}_{\Sigma, x}^\times\) is torsion-free.

**Lemma 5.9.** A noetherian monoidal scheme \(\Sigma\) is torsion-free if and only if, for any finite group \(G\) with \(#G = n\), the number \(#\text{Hom}(\mathcal{O}_{\Sigma, x}^\times, G)\) is polynomial in \(n\).

**Proof.** Since \(\Sigma\) is noetherian, the abelian group \(\mathcal{O}_{\Sigma, x}^\times\) is finitely generated by the remark above. So, if \(\Sigma\) is also torsion-free, then \(\mathcal{O}_{\Sigma, x}^\times\) is free of rank \(N(x)\), and, for any finite group \(G\) with \(#G = n\), we have \(#\text{Hom}(\mathcal{O}_{\Sigma, x}^\times, G) = n^{N(x)}\).

For the converse, suppose there is a point \(x\) such that \(\mathcal{O}_{\Sigma, x}^\times\) is not torsion-free. Being noetherian, \(\mathcal{O}_{\Sigma, x}^\times\) decomposes as a product \(\mathbb{Z}^n \times \prod_{i \in \{1, \ldots, m\}} \mathbb{Z}_{n_i}\). For each prime number \(p_0\) not dividing any of the \(n_1, \ldots, n_m\), say \(p_0 > \text{LCM}(n_1, \ldots, n_m)\), the number of elements of \(\text{Hom}(\mathcal{O}_{\Sigma, x}^\times, \mathbb{Z}_{p_0})\) is then \(p_0^n\). Since there are infinitely many such prime numbers, were \(#\text{Hom}(\mathcal{O}_{\Sigma, x}^\times, \mathbb{Z}_{p})\) polynomial in \(p\), it would be the polynomial \(p^n\). Take now a prime number \(p_1\) dividing \(n_1\); in that case, the number of elements of \(\text{Hom}(\mathcal{O}_{\Sigma, x}^\times, \mathbb{Z}_{p_1})\) is greater than \(p_1^n\). In conclusion, \(#\text{Hom}(\mathcal{O}_{\Sigma, x}^\times, \mathbb{Z}_{p})\) cannot be polynomial in \(p\). \(\square\)

By Lemma 5.9, for each noetherian and torsion-free monoidal scheme \(\Sigma\), one can define the polynomial

\[
P(\Sigma, n) = \sum_{x \in \tilde{\Sigma}} #\text{Hom}(\mathcal{O}_{\Sigma, x}^\times, \mathbb{F}_1^n).
\]

(5.11)

The following result is proved in \([1]\) (Theorem 4.10, (1) and (2)).

**Theorem 5.10.** Let \((\Sigma, \Sigma', \Phi)\) be an \(\mathbb{F}_1\)-scheme such that the monoidal scheme \(\Sigma\) is noetherian and torsion-free. Then

1. \(#\Sigma(\mathbb{F}_1^n) = P(\Sigma, n)\);
2. for each finite field \(\mathbb{F}_q\) the cardinality of the set of points of the scheme \(\Sigma'_{\mathbb{Z}}\) that are rational over \(\mathbb{F}_q\) is equal to \(P(\Sigma, q - 1)\).

Note that the last statement immediately follows from eq. 5.9, which holds true without any additional assumption on the monoidal scheme.

For each \(B\)-scheme \(\Sigma\) and each abelian group \(G\) (in multiplicative notation, with absorbing element 0), we denote by

\[\text{Hom}_B(\mathcal{O}_{\Sigma, x}^\times, G)\]

the subset of \(\text{Hom}(\mathcal{O}_{\Sigma, x}^\times, G)\) given by the morphisms satisfying the relations encoded in the \(B\)-structure of \(\Sigma\). Lemma 5.9 prompts us to introduce the following definition.
Definition 5.11. A B-scheme \( \Sigma \) is said to be noetherian if the monoidal scheme \( \Sigma \) is noetherian. A noetherian B-scheme \( \Sigma \) is said to be torsion-free if for any finite group \( G \), the number \( \# \text{Hom}_B(\mathcal{O}_\Sigma^x, G) \) is polynomial in \( \# G \).

Remark 5.12. While in the case of a noetherian torsion-free monoidal scheme \( \Sigma \) the polynomial \( \# \text{Hom}_B(\mathcal{O}_\Sigma^x, G) \) is always a monic monomial, this is not always the case for a noetherian torsion-free B-scheme. The next example illustrates this point. \( \triangle \)

Example 5.13. Consider the affine B-scheme \( \Sigma \) given by the free monoid \( M = \langle T_1, T_2, T_3, T_4 \rangle \) generated by four elements with relations given by the natural projection
\[
\mathbb{Z}[T_1, T_2, T_3, T_4] \to \mathbb{Z}[T_1, T_2, T_3, T_4]/(T_1 - T_3 + T_2 - T_4).
\]
Let \( G \) be a finite group (in multiplicative notation, with absorbing element 0); we look for maps \( f: M \to G \) together with compatible maps
\[
\begin{array}{c}
\mathbb{Z}[T_1, T_2, T_3, T_4] \\
\downarrow
\end{array}
\begin{array}{c}
\mathbb{Z}[T_1, T_2, T_3, T_4]/(T_1 - T_3 + T_2 - T_4) \\
\downarrow \text{id}
\end{array}
\begin{array}{c}
G \otimes_{\mathbb{F}_1} \mathbb{Z} \\
\end{array}
\]
Since \( G \otimes_{\mathbb{F}_1} \mathbb{Z} \) is free, to ensure the compatibility of \( f \) with the relation \( T_1 + T_2 = T_3 + T_4 \) one must have that either \( f(T_1) = f(T_2) \) and \( f(T_2) = f(T_3) \), or \( f(T_1) = f(T_4) \) and \( f(T_2) = f(T_3) \). There are therefore only 3 possible cases for the polynomial expressing the cardinality of \( \text{Hom}_B(\mathcal{O}_\Sigma^x, G) \):

- \( f(T_1) = f(T_2) = f(T_3) = f(T_4) = 0 \); in this case the polynomial is the constant polynomial 1;
- either \( f(T_1) = 0 \) and \( f(T_2) \neq 0 \) or \( f(T_1) \neq 0 \) and \( f(T_2) = 0 \), each case giving rise to two possible cases; therefore, in each of the four possible cases the polynomial is \( n \);
- \( f(T_1) \neq 0 \) and \( f(T_2) \neq 0 \); in this case the polynomial is \( 2n^2 - n \) (the term \( 2n^2 \) accounts for 2 possible free nonzero choices on \( f(T_1) \) and \( f(T_2) \), that have to be counted twice since either \( f(T_1) = f(T_3) \) or \( f(T_1) = f(T_4) \), and the term \(-n \) accounts for the case \( f(T_1) = f(T_2) \)). \( \triangle \)

Let \( (\Sigma, \Sigma Z, \Sigma Z', \Phi) \) be a \( \mathbb{F}_1 \)-scheme with relations such that the underlying B-scheme \( \Sigma \) is noetherian and torsion-free. We define the polynomial
\[
Q(\Sigma, n) = \sum_{x \in \Sigma} \# \text{Hom}_B(\mathcal{O}_\Sigma^x, \mathbb{F}_1^n).
\]

Proposition 5.14. In the above hypotheses one has the inequality \( Q(\Sigma, n) \leq P(\Sigma, n) \).

Proof. It is clear that \( \text{Hom}_B(\mathcal{O}_\Sigma^x, \mathbb{F}_1^n) \subset \text{Hom}(\mathcal{O}_\Sigma^x, \mathbb{F}_1^n) \), since the first set contains only the monoid morphisms that are compatible with the blueprint structure locally defined around \( x \). \( \square \)
Remark 5.15. Recall that the aim of Lorscheid’s definition of blueprint is to increase the amount of closed subschemes of a monoidal scheme. If we loosely refer to the features of the underlying topological space as “shape” of the scheme, we could say that category of B-schemes adds “extra shapes” to Deitmar’s category of monoidal schemes.

Consider now $\mathbb{F}_1$-schemes, and let us restrict our attention to the affine case. So, we just have a ring $R$, a monoid $M$, and a map $R \to M \otimes_{\mathbb{F}_1} \mathbb{Z}$. Since it is required, by definition, that points remain the same, the monoid is not enriched with “extra shapes”. However, if we think of the given map as a restriction map between the spaces of functions of the affine schemes $M \otimes_{\mathbb{F}_1} \mathbb{Z}$ and $R$, we can interpret the datum of the $\mathbb{F}_1$-scheme as an enlargement of the space of functions of the affine monoidal scheme $M$.

In conclusion, a $\mathbb{F}_1$-scheme with relation, according to the definition 5.5, allows us both to add “extra shapes” to the underlying monoidal scheme and to enlarge its space of functions.

Appendix A. Fibered categories and stacks

To give the reader a better understanding of Toën and Vaquié’s general construction presented in Section 2.1, we briefly review some basic facts about fibered categories, pseudo-functors, and stacks, closely following the exposition in [25] (to which the reader is referred for further details).

Let $C$ be any category. Roughly speaking, a stack is a sheaf of categories on $C$ with respect to some Grothendieck topology (recall Definition 2.1).

**Definition A.1** ([25], Def. 3.1). Let $p_F : F \to C$ be a functor. An arrow $\phi : \xi \to \eta$ of $F$ is Cartesian with respect to $p_F$ if, for any arrow $\psi : \zeta \to \eta$ in $F$ and any arrow $h : p_F\zeta \to p_F\xi$ in $C$ with $p_F\phi \circ h = p_F\psi$, there exists a unique arrow $\theta : \zeta \to \xi$ with $p_F\theta = h$ and $\phi \circ \theta = \psi$, as in the following diagram:

\[
\begin{array}{ccc}
\zeta & \xrightarrow{\psi} & \eta \\
\downarrow{\theta} & & \downarrow{\phi} \\
\xi & = & \xi \\
\downarrow{h} & & \downarrow{h} \\
p_F\zeta & \xrightarrow{p_F\phi} & p_F\eta \\
\end{array}
\]
Whenever \( \xi \to \eta \) is a Cartesian arrow of \( F \) mapping to an arrow \( U \to V \) of \( C \), we shall also say that \( \xi \) is a pullback of \( \eta \) to \( U \).

**Definition A.2** ([25], Def. 3.5). A category \( F \) endowed with a functor \( p_F : F \to C \) is said to be fibered over \( C \) (with respect to \( p_F \)) if, for any map \( f : U \to V \) in \( C \) and any object \( \eta \) in \( F \) such that \( p_F \eta = V \), there exists a Cartesian map \( \phi : \xi \to \eta \) in \( F \) such that \( p_F \phi = f \).

**Definition A.3.** [25, Def. 3.9] Given a fibered category \( p_F : F \to C \) over \( C \), a cleavage is a class \( K \) of Cartesian maps in \( F \) such that, for each map \( f : X \to Y \) in \( C \) and each object \( \xi \) in \( F \) over \( Y \), there is exactly one map in \( K \) over \( f \) with codomain \( \xi \); when a cleavage is fixed, this unique map will be denoted by \( f^*_\xi \), or, by a slight abuse of notation, simply by \( f^* \), if \( \xi \) is clear from the context.

**Remark A.4.** Let \( S \) be a set and \( \text{SET} \) the set of small sets. The assignment of a map of sets \( f : S \to \text{SET} \) is obviously equivalent to the assignment of the map \( p_F : F \to S \), with \( F = \coprod_{s \in S} f(s) \) and \( p_F \) the natural projection. Notice that, for every \( s \in S \), one can recover the set \( f(s) \) as the fiber \( p_F^{-1}\{s\} \).

If we regard a set as a discrete category, then the notion of fibered category introduced in Definition A.2 can be interpreted as a generalization of the construction above to the categorical framework.

When dealing with a functor \( C^{\text{op}} \to \text{CAT} \), however, we have not only objects (namely, the categories which are images of objects of \( C \)), but also maps between them (namely, the functors which are images of maps of \( C \)). So, the fibers on objects of \( C \) with respect to the fibration \( p_F : F \to C \) have to be connected by maps. This idea is made precise by the notion of a cartesian arrow introduce in Definition A.1: the existence of a Cartesian arrow \( \phi : \xi \to \eta \) amounts to say that \( \xi \) is the image of \( \eta \) by the functor \( F_{p_F \eta} \to F_{p_F \xi} \) which is image of the map \( p_F \phi : p_F \xi \to p_F \eta \). Images of maps in \( C \) are defined likewise by imposing the Cartesian condition and using composition rules in \( F \). But there is one more issue to be considered: when weconde functors data in properties of the category \( F \) (with respect to \( p_F \)), we have to bear in mind that categorical properties make sense only up to isomorphisms, and this is reason why, in general, we may expect to recover the original functor only up to equivalences. This fact leads to the following definition.

**Definition A.5** ([25], Def. 3.10). A pseudo-functor \( \Pi : C^{\text{op}} \to \text{Cat} \) consists of the following data:

i) for each object \( U \) of \( C \), a category \( \Pi_U \);
ii) for each arrow \( f : U \to V \), a functor \( f^* : \Pi_V \to \Pi_U \);
iii) for each object \( U \) of \( C \), an isomorphism \( \alpha_U : \Pi_U \simeq \Pi_U \); of functors \( \Pi_W \to \Pi_U \).

These data are required to satisfy some natural compatibility conditions which we do not explicitly describe here(see [25, p. 47]).
It can be proven that the assignment of a fibered categories over a category $C$ is equivalent, up to isomorphism, to the assignment of a pseudo-functor $C^{op} \to \text{Cat}$. For this reason, in what follows we will tend not to distinguish between a pseudo-functor and the associated fibered category and, given a fibered category $p_F : F \to C$ and an object $X$ of $C$, we will denote by $F(X)$ the fiber over $X$.

It can also be shown that any pseudo-functor can be strictified, that is, it admits an equivalent functor ([25, Th. 3.45]). Nonetheless, it can be convenient to work with pseudo-functors because many constructions naturally arising in algebraic geometry produce non strict pseudo-functors. The point of view of fibered categories allows one to deal with pseudo-functors by remaining in the usual context of strict functors.

**Example A.6.** Let us consider Toën and Vaquié’s construction, as summarised in section 2.1, in the particular case of classical schemes. In this case, the category of interest is $\text{Ring}$, regarded as $\text{Aff}_\text{Ring}^{op}$, and there is an assignment mapping each ring $A$ to the category $A\text{-Mod}$ and each ring morphism $A \to B$ to the functor $-\otimes_A B : A\text{-Mod} \to B\text{-Mod}$. Given two consecutive morphisms $A \to B \to C$ and an object $M$ of $A\text{-Mod}$, the objects $C \otimes_B (B \otimes_A M)$ and $C \otimes_A M$ are not equal, but only isomorphic. The previous construction provides therefore a naturally defined pseudo-functor $\Pi : \text{Aff}_\text{Ring}^{op} \to \text{Cat}$.

The pseudo-functor $\Pi$ can be associated to the fibered category over $\text{Ring}^{op}$ defined (up to equivalence) in the following way. Let $\text{Mod}$ be the category whose objects are pairs $(A, M)$ with $A$ a ring and $M$ an $A$-module and whose morphisms are pairs of the form $(f, \lambda) : (A, M) \to (B, N)$, where $f : B \to A$ is a ring morphism and $\lambda : A \otimes_B N \to M$ is a morphism of $A$-modules. Then the natural projection $\text{Mod} \to \text{Ring}^{op}$ is a fibration corresponding to $\Pi$. For each map $f : \text{Spec } A \to \text{Spec } B$ and for each $B$-module $M$, a natural choice of cartesian lifting is given by $(f : B \to A, A \otimes_B M = A \otimes_B M)$.

**Example A.7.** Let $C$ be a category closed under fibered products and denote by $\text{Arr } C$ the category of arrows in $C$. Let $p_{\text{Arr } C} : \text{Arr } C \to C$ the functor mapping each arrow $X \to Y$ to its codomain $Y$ and acting in the obvious way on morphisms in $\text{Arr } C$. The $\text{Arr } C$ is a fibered category, and its associated pseudo-functor maps an object $X$ to the category $C/X$ and a morphism $X \to Y$ to the pullback functor $-\times_Y X : C/Y \to C/X$. Similarly, the functor $(p_{\text{Arr } C})^{op} : (\text{Arr } C)^{op} \to C^{op}$ is a fibration (independently of the existence of pullbacks), corresponding to the covariant functor $C \to \text{Cat}$ acting on objects as above and sending a map $f$ to the composition functor $f \circ -$.

For each object $X$ of $C$, the category $C/X$ is naturally fibered over $C$ through the projection $C/X \to C$ given by the domain functor. This fibration is associated, by identifying a set with the corresponding discrete category, to the functor $C(-, X)$, that is, the image of $X$ by the Yoneda embedding. It is thus not surprising that the pseudo-functor $(p_{\text{Arr } C})^{op}$ can be proven to induce an embedding of $C$ in the 2-category of categories fibered over $C$.

Using this embedding, the Yoneda lemma can be generalized to categories in the following way. Let us recall that the classical Yoneda lemma states that, for every presheaf $F : C^{op} \to \text{Set}$ and every object $X$ of $C$, there is a natural isomorphism $\text{Hom}(C(-, X), F) \cong F(X)$, which is obtained by sending a map of presheaves to the image of $1_X$. It can be shown...
that, for every pseudo-functor \( p_F : F \to C \) and every object \( X \) of \( C \), there is an equivalence of categories \( \text{Hom}_C(C_{/X}, F) \cong F(X) \). For a proof, see [25, 3.6.2]: we just point out that, analogously to the classical case, also the equivalence \( \text{Hom}_C(C_{/X}, F) \to F(X) \) is defined on objects by mapping a functor to the image of \( 1_X \), regarded now as an object of \( C_{/X} \) (hence, in the discrete case, this equivalence essentially gives back the Yoneda isomorphism). \( \triangle \)

Example A.7 should make clear that there is a rather strict analogy between the theory of presheaves in sets and the theory of presheaves in categories. We now see how the theory of sheaves extends to the case of categories.

Let \( C \) be a category endowed with a Grothendieck topology. Recall from Definition 2.3 that a covering \( U = \{ U_i \to U \}_{i \in I} \) is associated to the subpresheaf \( h_U \) of \( h_U = C(\cdot, U) \) given by the maps that factorise through some element of \( U \). The inclusion map induces a restriction map \( \text{Hom}(h_U, F) \to \text{Hom}(h_U, F) \) for each presheaf \( F \), and \( F \) is said to be i) separated if this map is injective for each covering \( U \); ii) a sheaf if it is bijective for each covering \( U \).

In passing from sets to categories, it is natural to replace \( h_U \) with \( C_{/U} \) (see Example A.7) and, accordingly, \( h_U \) with the full subcategory \( C_{/U} \) of \( C_{/U} \) whose objects are the maps that factorise through some element of \( U \), “monomorphism” with “embedding” (that is, “fully faithful functor”) and “bijection” with “equivalence”. So, by regarding \( C_{/U} \) as fibered over \( C \) by the composite map \( C_{/U} \to C_{/U} \to C \), we have the following definition.

**Definition A.8.** Given a site \( C \), a fibered category \( p_F : F \to C \) is said to be

i) a prestack if, for any object \( U \) of \( C \) and covering \( U \) of \( U \), the restriction functor \( \text{Hom}_C(C_{/U}, F) \to \text{Hom}_C(C_{/U}, F) \) is an embedding;

ii) a stack if, for any object \( U \) of \( C \) and covering \( U \) of \( U \), the restriction functor \( \text{Hom}_C(C_{/U}, F) \to \text{Hom}_C(C_{/U}, F) \) is an equivalence.

As already observed, the category \( \text{Hom}_C(C_{/U}, F) \) is equivalent to \( F(U) \). The category \( \text{Hom}(C_{/U}, F) \) also admits an explicit description, in terms of descent data, that can be thought of as glueing data up to isomorphism, and that we now describe.

Given a site \( C \) and a covering \( U = \{ U_i \to U \}_{i \in I} \), we shall write \( U_{i_1 \ldots i_n} \) as a shorthand for \( U_{i_1} \times_U \ldots \times_U U_{i_n} \). Notice that, whenever \( \{ i_{j_1}, \ldots, i_{j_k} \} \subseteq \{ i_1 \ldots i_n \} \), there is a natural projection map

\[
p_{i_{j_1}, \ldots i_{j_k}} : U_{i_1 \ldots i_n} \to U_{i_{j_1} \ldots i_{j_k}}.
\]

**Definition A.9.** [25, Def. 4.2] Let \( C \) be a site, \( F \) a category fibered over \( C \), and \( U = \{ U_i \to U \} \) a covering in \( C \). Let be given a cleavage \( K \) (see Definition A.3). An object with descent data \( (\{ \xi_i \}, \{ \phi_{ij} \}) \) on \( U \) is a collection of objects \( \xi_i \in F(U_i) \), together with isomorphisms \( \phi_{ij} : p_{i_{j_1}}^{*} \xi_{j_1} \cong p_{i_{j_2}}^{*} \xi_{j_2} \) in \( F(U_{j_1} \times_U U_{j_2}) \) such that the following cocycle conditions is satisfied:

\[
pr_{13}^{*} \phi_{ik} = pr_{12}^{*} \phi_{ij} \circ pr_{23}^{*} \phi_{jk} : pr_{3}^{*} \xi_k \to pr_{1}^{*} \xi_i \quad \text{for any triple of indices } i, j, k.
\]
The isomorphisms $\phi_{ij}$ are called transition isomorphisms of the object with descent data. An arrow between objects with descent data

$$\{\alpha_i\} : (\{\xi_i\}, \{\phi_{ij}\}) \to (\{\eta_i\}, \{\psi_{ij}\})$$

is a collection of arrows $\alpha_i : \xi_i \to \eta_i$ in $F(U_i)$ with the property that for each pair of indices $i, j$ the diagram

$$\begin{array}{ccc}
pr_2^*\xi_j & \xrightarrow{pr_2^*\alpha_i} & pr_2^*\eta_j \\
\downarrow \phi_{ij} & & \downarrow \psi_{ij} \\
pr_1^*\xi_i & \xrightarrow{pr_1^*\alpha_i} & pr_1^*\eta_i
\end{array}
$$

(A.1)

commutes.

**Proposition A.10.** [25, Prop. 4.5] Given a site $C$, a category $F$ fibered over $C$, and a cover $U$ in $C$, objects with descent data on $U$ with arrows between them form a category, which is equivalent to $\text{Hom}(C/_{/U}, F)$.

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