THE $f$-VECTOR OF A REALIZABLE MATROID COMPLEX IS STRICTLY LOG-CONCAVE

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Abstract. We show that $f$-vectors of matroid complexes of realizable matroids are strictly log-concave. This was conjectured by Mason in 1972. Our proof uses the recent result by Huh and Katz who showed that the coefficients of the characteristic polynomial of a realizable matroid form a log-concave sequence. We also prove a statement on log-concavity of $h$-vectors which strengthens a result by Brown and Colbourn.

In the last two sections, we give a brief introduction to zonotopal algebra and we explain how it relates to our log-concavity results and various matroid/graph polynomials.

1. Introduction

Let $M = (E, \Delta)$ be a matroid of rank $r$. $E$ denotes the ground set and $\Delta \subseteq 2^E$ denotes the matroid complex, i.e. the abstract simplicial complex of independent sets. Let $f = (f_0, \ldots, f_r)$ be the $f$-vector of $\Delta$, i.e. $f_i$ is the number of sets of cardinality $i$ in $\Delta$. Dominic Welsh conjectured in 1969 [33] that the $f$-vector of a matroid complex is unimodal, i.e. there exists $j \in \{0, 1, \ldots, r\}$ s.t. $f_0 \leq f_1 \leq \ldots \leq f_j \geq \ldots \geq f_r$. Three successive strengthenings of this conjecture were proposed by John Mason in 1972 [23]. The weakest of them is log-concavity of the $f$-vector, i.e.

$$f_i^2 \geq f_{i-1}f_{i+1} \text{ for } i = 1, \ldots, r - 1. \quad (1.1)$$

If the inequalities in (1.1) are strict, we say that the $f$-vector is strictly log-concave. Since then, those conjectures have received considerable attention. See for example [6, 10, 12, 14, 19, 22, 26, 27, 32, 34]. Carolyn Mahoney proved log-concavity for cycle matroids of outerplanar graphs in 1985 [22]. David Wagner [32] describes further partial results, several stronger variants of Mason’s conjecture, and other sequences of integers that are associated to a matroid and that are conjectured to be log-concave. Log-concave sequences arising in combinatorics have been studied by many authors. For an overview, see the surveys by Francesco Brenti and Richard Stanley [5, 28].

Our main result is the following theorem:

**Theorem 1.1.** The $f$-vector of the matroid complex of a realizable matroid is strictly log-concave.
The fact that we are able to prove strict log-concavity indicates that the $f$-vector of a matroid complex might satisfy stronger inequalities.

The strongest of Mason’s three conjectures \cite{Mason1978} is ultra-log-concavity, i.e. the conjecture that the following inequalities hold:

$$\frac{f_i^2}{(f_i)^2} \geq \frac{f_{i-1}}{(i-1)} \frac{f_{i+1}}{(i+1)}$$

for $i = 1, \ldots, r - 1$. \hfill (1.2)

This conjecture is one of the main topics of an upcoming workshop at AIM\footnote{Workshop on Stability, hyperbolicity, and zero localization of functions, December 5 to December 9, 2011 at the American Institute of Mathematics, Palo Alto, California. Organized by Petter Brändén, George Csordas, Olga Holtz, and Mikhail Tyaglov. \url{http://www.aimath.org/ARCC/workshops/hyperbolicpoly.html}}.

Finding inequalities satisfied by $f$-vectors of matroid complexes is interesting because it is a step towards the classification of $f$-vectors and $h$-vectors of matroid complexes. Johnson, Kontoyiannis, and Madiman \cite{JohnsonKontoyiannisMadiman2018} show that Theorem 1.1 implies a bound on the entropy of the cardinality of a random independent set in a matroid. A possible application to network reliability is explained in Section 7.

Our log-concavity results might also help to prove statements about coefficients and zeroes of various graph polynomials.

**Organization of the article.** In Section 2 we introduce the $f$-polynomial and the characteristic polynomial of a matroid. Recently, June Huh and Eric Katz \cite{HuhKatz2018} proved that the characteristic polynomial of a realizable matroid is log-concave (a univariate polynomial is log-concave if its coefficient form a log-concave sequence). In Section 3 we establish a connection between the characteristic polynomial and the $f$-polynomial. In conjunction with the result by Katz and Huh, this implies log-concavity of the $f$-polynomial of realizable matroids. In Section 4 we deduce that $h$-vectors of certain thickenings of a realizable matroid are log-concave, which strengthens a result by Jason Brown and Charles Colbourn. As we will see in Section 6 this implies Theorem 1.1.

In Section 6 we give a brief introduction to zonotopal algebra and explain how the $f$-polynomial and the characteristic polynomial are related to it. Zonotopal algebra is the study of several classes of vector spaces of polynomials that can be associated with a realization of a matroid. The Hilbert series of those spaces are matroid invariants. In Section 7 we explain the relationship between various graph/matroid polynomials, zonotopal algebra and our log-concavity results.

### 2. Matroid polynomials

In this section, we review the definitions of some matroid polynomials. We assume that the reader is familiar with matroid theory. Good references are the book by James Oxley \cite{Oxley2011} and the Wikipedia articles on matroids and the Tutte polynomial.

Recall that we denote by $M = (E, \Delta)$ a matroid of rank $r$. The **Tutte polynomial** \cite{Tutte1960} of $M$ is defined as

$$T_M(x, y) = \sum_{A \subseteq E} (x - 1)^{r - \text{rk}(A)} (y - 1)^{|A| - \text{rk}(A)}.$$ \hfill (2.1)
An important specialization of the Tutte polynomial is the characteristic polynomial
\[ \chi_M(q) = (-1)^r T_M(1 - q, 0) = \sum_{A \subseteq E} (-1)^{|A|} q^{r - \text{rk}(A)}. \] (2.2)

The reduced characteristic polynomial is defined as
\[ \bar{\chi}_M(q) = \frac{1}{q-1} \chi_M(q). \] (2.3)

Note that \( \chi_M(q) \) vanishes for \( q = 1 \), so \( \bar{\chi}_M(q) \) is indeed a polynomial. Huh and Katz proved the following theorem, extending an earlier theorem by the first author [17]:

**Theorem 2.1** (18).

If \( M \) is a realizable matroid, then the coefficients of its reduced characteristic polynomial \( \bar{\chi}_M(q) \) form a log-concave sequence.

It is easy to see that log-concavity of \( \bar{\chi}_M(q) \) implies log-concavity of \( \chi_M(q) \). We are interested in the \( f \)-polynomial of the matroid given by
\[ f_M(q) = T_M(1 + q, 1) = \sum_{A \subseteq \Delta} q^{r - \text{rk}(A)} = \sum_{i=0}^{r} f_i q^{r-i}. \] (2.4)

### 3. Free (Co-)Extensions

In this section, we introduce free (co-)extensions of matroids. This helps us to establish a connection between the characteristic polynomial and the \( f \)-polynomial. In conjunction with Theorem 2.1, this connection implies log-concavity of the \( f \)-polynomial of realizable matroids.

**Definition 3.1.** Let \( M = (E, \Delta) \) be a matroid of rank \( r \) and let \( e \notin E \). The free extension of \( M \) (by \( e \)) is the matroid \( M + e = (E \cup \{e\}, \Delta + e) \), where
\[ \Delta + e := \Delta \cup \{I \cup \{e\} : I \in \Delta \text{ and } |I| \leq r - 1 \}. \] (3.1)

Several properties of the free extension are described in [7, 7.3.3. Proposition].

**Remark 3.2.** If \( M \) is realized over the field \( \mathbb{K} \) by the list of vectors \( X \subseteq \mathbb{K}^r \), then \( M + e \) is realized by the list \( (X, x) \), where \( x \in \mathbb{K}^r \) is a vector that is not contained in any (linear) hyperplane spanned by the vectors in \( X \). If \( \mathbb{K} \) is a finite field, such a vector might not exist. However, if \( M \) is realizable over the field \( \mathbb{K} \), it is also realizable over the infinite field \( \mathbb{K}(t) \) of rational functions in \( t \) with coefficients in \( \mathbb{K} \).

Recall that the dual matroid \( M^* = (E, \Delta^*) \) is given by
\[ \Delta^* = \{ A : \text{rk}(E \setminus A) = r \}. \] (3.2)

The dual matroid has rank \( r^* = |E| - r \) and its rank function is given by \( \text{rk}^*(A) = |A| + \text{rk}(E \setminus A) - r \). The Tutte polynomial satisfies \( T_M(x, y) = T_{M^*}(y, x) \). We will use the free coextension \( M \times e \) of a matroid \( M \) which is defined as
\[ M \times e := (M^* + e)^*. \] (3.3)

Equivalently, the free coextension of \( M \) is the extension by a non-loop \( e \) which is contained in every dependent flat [25, Section 7.3].

**Proposition 3.3.** Let \( M \) be a matroid of rank \( r \) and let \( M \times e \) denote its free coextension. Then,
\[ (-1)^{r+1} \chi_{M \times e}(-q) = (1 + q) f_M(q). \] (3.4)
Proof. For the proof of this statement, we use the fact that both the characteristic polynomial and the \( f \)-polynomial are evaluations of the Tutte polynomial. Note that the matroid \( M \times e \) has rank \( r + 1 \). To simplify notation, the rank functions of \( M^* \) and \( M^* + e \) are both denoted by \( \mathrm{rk}^* \).

\[
(-1)^{r+1} \chi_{M \times e}(-q) = T_{M \times e}(1 + q, 0) = T_{M^* + e}(0, 1 + q) = \sum_{A \subseteq E} (-1)^{r^* - \mathrm{rk}^*(A)} q^{|A| - \mathrm{rk}^*(A)} \quad (3.5)
\]

\[
= \sum_{A \subseteq E} \left( (-1)^{r^* - \mathrm{rk}^*(A)} q^{|A| - \mathrm{rk}^*(A)} + (-1)^{r^* - \mathrm{rk}^*(A \cup e)} q^{|A| + 1 - \mathrm{rk}^*(A \cup e)} \right) \quad (3.6)
\]

\[
= (1 + q) \sum_{A \subseteq E, \mathrm{rk}^*(A) = r^*} q^{|A| - r^*} = (1 + q)T_{M^*}(1, 1 + q) \quad (3.7)
\]

\[
(1 + q)f_{M^*}(q) = (1 + q)T_{M^*}(1 + q, 1) = (1 + q) f_M(q) \quad (3.8)
\]

\[(3.8) \text{ is equal to } (3.7) \text{ because } \mathrm{rk}^*(A) < r^* \text{ implies } \mathrm{rk}^*(A \cup e) = \mathrm{rk}^*(A) + 1. \text{ For those } A, \text{ the summands vanish.} \hfill \square \]

Corollary 3.4. The \( f \)-vector of the matroid complex of a realizable matroid is log-concave.

Proof. Combine Proposition 3.3 and Theorem 2.1. Keep in mind that free coextensions of realizable matroids are realizable (cf. Remark 3.2). \hfill \square

Remark 3.5. Proposition 3.3 appeared implicitly in an article by Thomas Brylawski on (reduced) broken-circuit complexes [9].

In Section 6 we give another proof of Proposition 3.3 for matroids that are realizable over a field of characteristic zero. This proof uses zonotopal algebra.

Example 3.6. We consider the uniform matroid \( U_{2,6} \), i.e., the matroid on six elements where every set of cardinality at most two is independent. Note that \( U_{2,6} \times e = (U_{4,6} + e)^* = U_{4,7}^* = U_{3,7} \).

\[
f_{U_{2,6}}(q) = q^2 + 6q + 15
\]

\[
(3.8) \chi_{U_{3,7}}(-q) = q^3 + 7q^2 + 21q + 15 = (q + 1)f_{U_{2,6}}(q)
\]

4. Log-concavity of some \( h \)-vectors

In this section, we strengthen a result by Jason Brown and Charles Colbourn [6]. They showed in a nonconstructive way that every matroid has a thickening whose \( h \)-vector is log-concave. Thickening denotes an operation where additional copies of some elements of the ground set are added. We prove that the \( h \)-vector of a \( k \)-fold thickening (i.e., every element of the ground set is replaced by \( k \) copies of itself) of a realizable matroid is log-concave for sufficiently large \( k \).

Definition 4.1. Let \( M \) be a matroid of rank \( r \). Its \( h \)-vector \( (h_0, \ldots, h_r) \) consists of the coefficients of the \( h \)-polynomial defined by the equation \( h_M(q) = \sum_{i=0}^r h_i q^{r-i} = \)
from (4.1) that

\[ h_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{r-i}{j-i} f_i \quad \text{for } i = 0, \ldots, r. \]  

(4.1)

**Definition 4.2.** Let \( M = (E, \Delta) \) be a matroid and let \( k \) be a positive integer. We define the \( k \)-fold thickening \( M^k \) of \( M \) to be the matroid on the ground set \( E \times \{1, \ldots, k\} \) whose independence complex is given by

\[ \Delta^k = \{ I \subseteq E \times \{1, \ldots, k\} : \pi_E(I) \in \Delta \text{ and } |\pi_E(I)| = |I| \}. \]  

(4.2)

**Remark 4.3.** If \( M \) is realized by a list of vectors \( X, M^k \) is realized by the list \( X^k \) that contains \( k \) copies of every element of \( X \).

**Theorem 4.4.** Let \( M = (E, \Delta) \) be a realizable matroid of rank \( r \) and let \( f_1 \) denote the number of elements in \( E \) that are not loops. Then, there exists an integer \( k_0 \leq (f_1 r^{3r}) \) s. t. for all \( k \geq k_0 \), the \( h \)-vector of \( M^k \), the \( k \)-fold thickening of \( M \), is log-concave.

**Remark 4.5.** We expect that a careful analysis will yield an upper bound on \( k_0 \) that is a lot stronger.

**Proof.** First, we observe the following connection between the \( f \)-polynomials of \( M \) and \( M^k \):

\[ f_{M^k}(q) = \sum_{i=0}^{r} k^i q^{r-i} = k^r f_M \left( \frac{1}{k} q \right). \]  

(4.3)

Let \((f_0, \ldots, f_r)\) denote the \( f \)-vector of \( M \) and let \((h'_0, \ldots, h'_r)\) denote the \( h \)-vector of \( M^k \). By (4.1), \( h'_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{r-i}{j-i} k^i f_i \). Hence,

\[ (h'_j)^2 = \left( \sum_{i=0}^{j} (-1)^{j-i} \binom{r-i}{j-i} k^i f_i \right)^2 = k^{2j} f_j^2 + o(k^{2j}) \]  

(4.4)

\[ h'_{j-1} h'_{j+1} = \left( \sum_{i=0}^{j-1} (-1)^{j-i} \binom{r-i}{j-i-1} k^i f_i \right) \left( \sum_{i=0}^{j+1} (-1)^{j-i} \binom{r-i}{j-i+1} k^i f_i \right) \]  

(4.5)

\[ = k^{2j} f_{j-1} f_{j+1} + o(k^{2j}). \]  

(4.6)

For large \( k \), all summands except for the ones involving \( k^{2j} \) are negligible. In particular, for large \( k \)

\[ (h'_j)^2 \geq h'_{j-1} h'_{j+1} \]  

is equivalent to \( f_j^2 \geq f_{j-1} f_{j+1} \).

The latter inequality holds by Corollary 3.4.

For the upper bound on \( k_0 \), note that Ed Swartz proved in [21] that

\[ f_i \leq \sum_{j=0}^{i} \binom{r-j}{r-i} \left( \binom{r-1}{j} h_r + \binom{r-1}{r-j} \right). \]  

(4.8)

\( h_r \) can be bounded above by the following argument: the \( h \)-vector of a matroid complex is the \( h \)-vector of a multicomplex [29 Theorem II.3.3]. It follows directly from (4.1) that \( h_1 = f_1 - 1 \). Hence, \( h_r \leq \left( \frac{f_1}{r-1} \right) \). Thus, we can deduce from (4.8) that \( f_i \leq r^{2i} f_1^2 \). Comparing this with (4.4) and (4.5) implies the upper bound. \( \square \)
5. Strict log-concavity of $f$-vectors

In this section, we show that the results in the previous section imply Theorem 1.1 and we discuss the location of the modes of the $f$-vector of a matroid complex.

Jeremy Dawson conjectured in [10] that the $h$-vector of a matroid is log-concave and proved that this would imply log-concavity of the $f$-polynomial. Actually, a more general statement holds (cf. also [5, Corollary 8.4]):

**Lemma 5.1.** Let $a_0, \ldots, a_r$ be non-negative integers and $a_0 \neq 0$. Suppose that the polynomial $a(q) = \sum_{i=0}^{r} a_i q^{-i}$ is log-concave. Then, the polynomial $b(q) = \sum_{i=0}^{r} b_i q^{-i} = a(q+1)$ is strictly log-concave.

**Proof.** Our proof is inspired by Dawson’s proof in [10]. For $0 < k \leq r$, we define $a^k(q) = \sum_{i=0}^{k} a_i q^{-i}$ and $b^k(q) = \sum_{i=0}^{k} b_i q^{-i} = a^k(q+1)$.

The polynomials $a^k(q)$ are by construction log-concave. We show by induction over $k$ that this implies log-concavity of the polynomials $b^k(q)$. This is sufficient since $b(q) = b^r(q)$.

For $k \leq 1$, nothing needs to be shown. For $k = 2$, we need to check one inequality:

$$b_1^2 = (a_1 + 2a_0)^2 = a_1^2 + 4a_0a_1 + 4a_0^2 \quad (5.1)$$

$$\geq a_0a_2 + 4a_0a_1 + 4a_0^2 > a_0(a_2 + a_1 + a_0) = b_0b_2. \quad (5.2)$$

Now let $k \geq 3$. Note that

$$b^{k+1}(q) = a^{k+1}(q+1) = (q+1)a^k(q+1) + a_{k+1} = (q+1)b^k(q) + a_{k+1}. \quad (5.3)$$

This polynomial is strictly log-concave if $(q+1)(qb^k(q) + a_{k+1}) = q((q+1)b^k(q) + a_{k+1}) + a_{k+1}$ is, since setting the $q^0$ coefficient to zero followed by a division by $q$ preserves strict log-concavity.

It is an easy exercise to show that multiplication by $(q+1)$ preserves strict log-concavity of a polynomial in $q$. Hence, it is sufficient to prove that $(qb^k(q) + a_{k+1})$ is strictly log-concave. By induction, we only need to check the inequality involving the term $a_{k+1}$, i.e. $b_{k,k}^2 > b_{k-1,k}a_{k+1}$:

$$b_{k,k}^2 - b_{k-1,k}a_{k+1} = (a_0 + \ldots + a_k)^2 - \sum_{j=0}^{k-1} (k-j)a_ja_{k+1} \quad (5.4)$$

$$> (a_0 + \ldots + a_k)^2 - \sum_{j=0}^{k-1} \sum_{i=1}^{k-j} a_j a_{k+1-i} \quad (5.5)$$

$$= \sum_{i+j \leq k} a_i a_j \geq a_0^2 \geq 1. \quad (5.6)$$

To see that $(5.4)$ is greater than $(5.5)$, note that strict log-concavity of the $a_j$ implies $a_ja_{k+1} < a_{j+i}a_{k+1-i}$ for $1 \leq i \leq k-j$. □

**Proof of Theorem 1.1** By Theorem 4.4 there exists an integer $k$ s.t. the $h$-polynomial of $M^k$, the $k$-fold thickening of $M$, is log-concave. By Lemma 5.1 this implies strict log-concavity of the $f$-polynomial of $M^k$. (4.3) implies that the $f$-polynomial of $M^k$ is strictly log-concave if and only if the $f$-polynomial of $M$ is strictly log-concave. □
Modes of $f$-vectors. For a unimodal sequence $f_0, \ldots, f_r$, it is interesting to find the location of its modes, i.e., the element(s) where the maximum of the sequence is attained.

Remark 5.2. The index of the smallest mode of the $f$-vector of a rank $r$ matroid is at least $\lceil r/2 \rceil$. In fact, the first half of the $f$-vector of an arbitrary matroid is strictly monotonically increasing [4, 7.5.1. Proposition]. The minimum $\lceil r/2 \rceil$ is attained by the uniform matroid $U_{r,r}$.

If $M$ is realizable, Theorem 1.1 implies that $f_M$ has at most two modes. Some matroids have monotonically increasing $f$-vectors. In fact, it follows from (4.3) that for an arbitrary matroid $M$ and sufficiently large $k$, the $f$-vector of the $k$-fold thickening of $M$ is strictly monotonically increasing.

6. Zonotopal Algebra and Matroid Polynomials

Zonotopal algebra is the study of several classes of graded vector spaces of polynomials that can be associated with a realization of a matroid over a field of characteristic zero. The Hilbert series of those spaces are matroid invariants. The spaces can be described in various ways and each space has a dual counterpart with the same Hilbert series.

The theory of zonotopal algebra was developed by Olga Holtz and Amos Ron [15], extending various previous results e.g. on polynomial spaces spanned by box splines [11]. Related work includes [2, 3, 16, 20, 21, 24, 30].

Let $K$ be a field of characteristic zero and let $X = (x_1, \ldots, x_N) \subseteq K^r$ be a list of vectors spanning $K^r$. The two zonotopal spaces that are of interest to us in this paper are the central $P$-space $P(X)$ and the internal $P$-space $P^-(X)$.

Given $x \in X$, we denote by $p_x$ the linear polynomial in $K[t_1, \ldots, t_r]$ whose $t_i$ coefficient is the $i$th coordinate of the vector $x$. We define

$$P(X) := \text{span} \left\{ \prod_{x \in Y} p_x : Y \subseteq X, \text{rk}(X \setminus Y) = r \right\}$$

$$P^-(X) := \text{span} \left\{ \prod_{x \in Y} p_x : Y \subseteq X, \text{rk}(X \setminus (Y,y)) = r \text{ for all } y \in X \right\}.$$ (6.1) (6.2)

The Hilbert series of those two spaces are evaluations of the Tutte polynomial $T_X(x,y)$ of the matroid defined by $X$ [11 2 15]:

$$\text{Hilb}(P(X), q) = q^{N-r}T_X(1, \frac{1}{q}),$$

$$\text{Hilb}(P^-(X), q) = q^{N-r}T_X(0, \frac{1}{q}).$$ (6.3) (6.4)

$X^* \in K^{(N-r)\times r}$ denotes a list of vectors realizing the matroid dual to the matroid realized by $X$. In the central case, we obtain

$$q^r \text{Hilb}(P(X^*), \frac{1}{q}) = T_X(q, 1)$$

by dualizing and by reversing the order of the coefficients. In the internal case, we obtain

$$q^r \text{Hilb}(P^-(X^*), \frac{1}{q}) = T_X(q, 0).$$ (6.5) (6.6)
Let $K$ be a field of characteristic zero and let $X \subseteq \mathbb{K}^r$ be a list of vectors spanning $\mathbb{K}^r$. Then,

$$f_X(q) = T_X(q + 1, 1) = (q + 1)^r \text{Hilb}(\mathcal{P}(X^*), \frac{1}{q + 1})$$

(6.7)

$$(-1)^r \chi_X(-q) = T_X(q + 1, 0) = (q + 1)^r \text{Hilb}(\mathcal{P}_-(X^*), \frac{1}{q + 1})$$

(6.8)

Example 6.2. Let $X = ((1, 0), (0, 1), (1, 1))$. $X$ realizes the uniform matroid $U_{2,3}$ and $X^* = (1, 1, 1)$. The Tutte polynomial is $T_X(x, y) = x^2 + x + y$.

$$\mathcal{P}(X^*) = \text{span}\{1, t, t^2\} \quad \mathcal{P}_-(X^*) = \text{span}\{1, t\}$$

$$q^2 \text{Hilb}(\mathcal{P}(X^*), 1/q) = q^2 + q + 1 \quad q^2 \text{Hilb}(\mathcal{P}_-(X^*), 1/q) = q^2 + q$$

$$f_X(q) = q^2 + 3q + 3 \quad \chi_X(-q) = q^2 + 3q + 2$$

Proposition 6.3. Let $\mathbb{K}$ be a field of characteristic zero and let $X \subseteq \mathbb{K}^r$ be a list of vectors spanning $\mathbb{K}^r$. Let $x \in \mathbb{K}^r$ be generic, i.e. $x$ is not contained in any (linear) hyperplane spanned by the vectors in $X$. Then,

$$\mathcal{P}_-(X, x) = \mathcal{P}(X).$$

(6.9)

Proof. By [1] and [15], $\mathcal{P}_-(X, x) = \bigcap_{y \in X} \mathcal{P}((X, x) \setminus y)$. This implies $\mathcal{P}(X) \supseteq \mathcal{P}_-(X, x)$. Equality can be established by a dimension argument: in [15], it is shown that the dimension of $\mathcal{P}(X)$ is equal to the number of bases that can be selected from $X$ and that the dimension of $\mathcal{P}_-(X)$ equals the number of internal bases in $X$, i.e. bases that have no internally active elements. It can easily be seen that $B \subseteq (X, x)$ is an internal basis if and only if $B$ is a basis and $x \notin B$. \hfill \square

Remark 6.4. Proposition 6.1 and Proposition 6.3 imply Proposition 3.3 for matroids that are realizable over a field of characteristic zero. This is how we (re-)discovered the connection between the characteristic polynomial and the Tutte polynomial. We believe that in the future, zonotopal algebra will help to solve further problems in matroid theory.

Question 6.5. We have seen that for $\mathcal{P}_*(X) \in \{\mathcal{P}_-(X), \mathcal{P}(X)\}$, the coefficients of the polynomial $(q + 1)^{X^* - r} \text{Hilb}(\mathcal{P}_*(X), 1/(q + 1))$

(a) have a combinatorial interpretation and

(b) form a log-concave sequence.

For which other zonotopal spaces does this hold?

7. Graph polynomials, zonotopal algebra, and log-concavity

In this section, we present some graph polynomials that are related to the internal and central $\mathcal{P}$-space. In all cases, the connection is made via the Tutte polynomial. Even though this connection is rather straightforward, it has never been stated in the literature. A good survey on graph polynomials that are related to the Tutte polynomial is [15] by Joanna Ellis-Monaghan and Criel Merino.

\footnote{In fact, [15] defines $\mathcal{P}(X) := \bigcap_{y \in X} \mathcal{P}(X \setminus y)$ and shows that $\mathcal{P}_-(X)$ is the kernel/inverse system of a certain ideal $I_-(X)$. In [1], it is shown that the kernel of this ideal is equal to (6.2).}
Let \( G = (V, E) \) be a graph, possibly with multiple edges and loops. Let \( M(G) \) denote the cycle matroid of \( G \). If \( \kappa(G) \) denotes the number of connected components of \( G \), then \( M(G) \) has rank \( \text{rk}(M(G)) = |V| - \kappa(G) \). \( X(G) \) denotes the reduced oriented incidence matrix of \( G \) which realizes the matroid \( M(G) \).

### 7.1. Chromatic and flow polynomials

The chromatic polynomial and the flow polynomial of a graph are related to the internal space \( \mathcal{P}_-(X) \).

The chromatic polynomial \( \chi_G \) of \( G \) evaluated at \( q \in \mathbb{N} \) equals the number of proper colorings of the graph \( G \) with \( q \) colors. \( \chi_G \) is equal to the characteristic polynomial of \( M(G) \) up to a factor:

\[
\chi_G(q) = (-1)^{\text{rk}(M(G))} q^{\kappa(G)} T_{M(G)}(1 - q, 0).
\] (7.1)

Hence, the coefficients of \( \chi_G(q) \) form a log-concave sequence and

\[
(-1)^{\text{rk}(M(G))} \chi_G(-q) = (q + 1)^{\text{rk}(M(G))} q^{\kappa(G)} \text{Hilb}(\mathcal{P}_-(X(G)^*), \frac{1}{q + 1}).
\] (7.2)

Let \( \bar{E} \) denote an orientation of the edges of \( G \) and let \( q \geq 2 \). A nowhere-zero \( q \)-flow is an assignment \( E \to \{1, \ldots, q - 1\} \) s.t. for each vertex, the sum over the incoming edges equals the sum over the outgoing edges modulo \( q \). The function \( \phi_G(q) \) which counts the number of nowhere zero \( q \)-flows is a polynomial and independent of the orientation \( \bar{E} \):

\[
\phi_G(q) = (-1)^{|E| - \text{rk}(M(G))} T_{M(G)}(0, 1 - q).
\] (7.3)

Hence, \( \phi_G(q) \) is equal to the characteristic polynomial of the dual matroid. This implies that the coefficients of \( \phi_G(q) \) form a log-concave sequence and \( \phi_G(q) = (q - 1)^{|E| - \text{rk}(M(G))} \text{Hilb}(\mathcal{P}_-(X(G)), 1/(1 - q)) \).

### 7.2. Chip-firing games, shellings, and reliability

Three graph/matroid polynomials are related to the central space \( \mathcal{P}(X) \): the critical configuration polynomial, the shelling polynomial, and the reliability polynomial.

The critical configuration polynomial \( P_G(q) := T_{M(G)}(1, q) \) is related to chip-firing games played on the graph \( G \). Its \( q \) coefficient equals the number of critical configurations of level \( i \) in the chip-firing game played on the graph \( G \).

The polynomial \( h_M(q) := T_M(q, 1) \) that we defined in Section 4 is also called the shelling polynomial of the matroid \( M \). This polynomial encodes certain combinatorial properties of shellings of the independence complex of the matroid \( M \).

By [6,3], the shelling polynomial \( h_M(q) \) and the critical configuration polynomial \( P_G(q) \) are evaluations of the Hilbert series of the central \( \mathcal{P} \)-space of \( X(G) \) resp. of a realization of \( M^* \). For further information on those two polynomials, see [4] and [3] Sections 6.4 and 6.6.

**Remark 7.1.** By Theorem 2.4 the shelling polynomial of a \( k \)-fold thickening of a matroid is log-concave for large \( k \). By duality, for large \( k \), the coefficients of the critical configuration polynomial are log-concave for a \( k \)-fold subdivision of the graph \( G \). By a \( k \)-fold subdivision we mean the operation of subdividing each edge of \( G \) into \( k \) edges. This operation is dual to replacing an edge by \( k \) parallel copies of itself.

Let \( G = (V, E) \) be a connected graph on \( n \) vertices. Let \( R_G(p) \) denote the probability that \( G \) is connected if each edge is independently removed with probability
$R_G(p)$ is a polynomial [6]. It is called \textit{reliability polynomial} of $G$ and it can be expressed in the following way:

$$R_G(p) = (1 - p)^{n-1} \sum_{i=0}^{|E|-n+1} h_i p^i$$  \hspace{1cm} (7.4)

$$= (1 - p)^{n-1} p^{|E|-n+1} T_G(1, \frac{1}{p}).$$  \hspace{1cm} (7.5)

The $h_i$ denote the coefficients of the $h$-polynomial of the cycle matroid of $G$. The relationship between the $h$-vector and the reliability polynomial implies that proving bounds for the $h$-vector might have some real-world applications in determining the reliability of a network. Brown and Colbourn [6, p. 117] state that if log-concavity of the $h$-vector “holds for matroids arising in reliability problems, it would imply stronger constraints on the relation between coefficients in the $h$-vector than does Stanley’s conditions. These conditions can be incorporated in the Ball-Provan strategy for computing reliability bounds and, hence, would lead to an efficient bounding technique of the reliability polynomial.”

Example 7.2. Let $G$ be the complete graph on three vertices. Its cycle matroid is realized by the matrix $X$ in Example 6.2. Recall that the Tutte polynomial of this matroid is $T_G(x,y) = x^2 + x + y$.

$$\chi_G(q) = q^3 - 3q^2 + 2q \hspace{1cm} \Phi_G(q) = q - 1$$

$$h_G(q) = q^2 + q \hspace{1cm} P_G(q) = q + 2 \hspace{1cm} R_G(q) = (1 - p)^2(2p^2 + p)$$

Acknowledgments

I would like to thank Olga Holtz, Felipe Rincón, and Luca Moci for many stimulating conversations about Mason’s conjecture. I also thank June Huh whose comments on an earlier version of this paper lead to a simplification of the proof of Corollary [3,4].

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