Propositional Encoding of Constraints over Tree-Shaped Data

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Abstract. We present a functional programming language for specifying constraints over tree-shaped data. The language allows for Haskell-like algebraic data types and pattern matching. Our constraint compiler CO4 translates these programs into satisfiability problems in propositional logic. We present an application from the area of automated analysis of (non-)termination of rewrite systems.

1 Motivation

The paper presents a high-level declarative language CO4 for describing constraint systems. The language includes user-defined algebraic data types and recursive functions defined by pattern matching, as well as higher-order and polymorphic types. This language comes with a compiler that transforms a high-level constraint system into a satisfiability problem in propositional logic. This is motivated by the following.

Recent years have seen a tremendous development of constraint solvers for propositional satisfiability (SAT solvers). Based on the Davis-Putnam-Logemann-Loveland algorithm and extended with conflict-driven clause learning, SAT solvers like Minisat are able to find satisfying assignments for conjunctive normal forms with $10^6$ and more clauses in a lot of cases. SAT solvers are used in industrial-grade verification of hardware and software.

With the availability of powerful SAT solvers, propositional encoding is a promising method to solve constraint systems that originate in different domains. In particular, this approach had been used for automatically analysing (non-)termination of rewriting successfully, as can be seen from the results of International Termination Competitions (most of the participants use propositional encodings).

So far, these encodings are written manually: the programmer has to construct explicitly a formula in propositional logic that encodes the desired properties. This has the advantage that the formula can be optimized in clever ways, but also the drawback that correctness of the formula is not evident, so the process is error-prone.
This is especially so if the data domain for the constraint system is remote from the “sequence of bits” domain that naturally fits propositional logic. In typical applications, data is tree-structured (e.g., terms, and lists of terms) and one wants to write constraints on such data in a direct way.

Our language is similar to Haskell \cite{Haskell} in the following sense: CO4 syntactically is a subset of Haskell (including data declarations, case expressions, higher order functions, polymorphism, but no type classes), and semantically CO4 is evaluated strictly.

The advantages of re-using a high level declarative language for expressing constraint systems are: the programmer can rely on established syntax and semantics, does not have to learn a new language, can re-use his experience and intuition, and can re-use actual code.

For instance, the (Haskell) function that describes the application of a rewrite rule at some position in some term can be directly used in a constraint system that describes a rewrite sequence with a certain property. We treat this application in detail in Section \ref{section:compilation} but need some preparation first.

A constraint programming language needs some way of parametrizing the constraint system to data that is not available when writing the program. For instance, a constraint program for finding looping derivations for a rewrite system \(R\), will not contain a fixed system \(R\), but will get \(R\) as run-time input.

To accommodate for such applications, CO4 programs are handled and executed in two stages: The input program defines a function of type

\[
f : K \times U \rightarrow \{ \text{False}, \text{True} \}
\]

where \(K\) is some parameter domain (e.g., rewrite systems) and \(U\) is the domain of the unknown object (e.g., derivations). In the first processing stage (at compile-time), the program for \(f\) is translated into a program

\[
g : K \rightarrow (F, \Sigma \rightarrow U)
\]

with \(F\) being the set of formulas of propositional logic, and \(\Sigma\) being the set of assignments from variables of \(F\) to truth values.

In the second stage (at run-time), a parameter value \(p \in K\) is given, and \(g\) \(p\) is evaluated to produce a pair \((v, d) \in (F, \Sigma \rightarrow U)\). An external SAT solver then tries to determine a satisfying assignment \(\sigma \in \Sigma\) of \(v\). On success, \(d(\sigma)\) is evaluated to a solution value \(s \in U\). Proper compilation ensures that \(f\) \(p\) \(s = \text{True}\).

A formal specification of compilation is given in Section \ref{section:compilation} and a concrete realization of compilation of first-order programs using algebraic data types and pattern matching is given in Section \ref{section:implementation}. In these sections, we assume that data
types are finite (e.g., composed from `Bool`, `Maybe`, `Either`), and programs are total. We then extend this in section 4 to handle infinite (that is, recursive) data types (e.g., lists, trees).

We then treat briefly two ideas that serve to improve writing and executing CO4 programs: In Section 5 we discuss the compilation of higher-order and polymorphic features in CO4 programs. In Section 6 we show that memoization of function calls improves efficiency since it allows to share sub-formulas.

With these preparations, we give the CO4 formulation of looping derivations in term rewriting systems in Section 7. Propositional encodings for string rewrite sequences have appeared in the literature [7]. To our knowledge, the propositional encoding of term rewriting is new, and it looks quite an insurmountable task to write such an encoding without the help of a compilation system.

2 Semantics of Propositional Encodings

In this section give the specification for compilation of CO4 expressions, in the form of an invariant (it should hold for all sub-expressions). When applied to the full input program, the specification implies that the compiler works as expected: a solution for the constraint system can be found via the external SAT solver. We defer discussion of our implementation of this specification to Section 3 and give here a more formal, but still high-level view of the CO4 language and compiler.

*Evaluations on concrete data.* We denote by \( P \) the set of expressions in the input language. It is a first-order functional language with

- algebraic data types,
- pattern matching,
- global and local function definitions (using `let`) that may be recursive.

The concrete syntax is a subset of Haskell. We give examples—which may appear unrealistically simple but at this point we cannot use higher-order or polymorphic features. These will be discussed in see Section 5.

```haskell
data Bool = False | True
and2 :: Bool -> Bool -> Bool
and2 x y = case x of { False -> False ; True -> y }

data Maybe_Bool = Nothing | Just Bool
f :: Maybe_Bool -> Maybe_Bool -> Maybe_Bool
f p q = case p of
  Nothing -> Nothing
  Just x -> case q of
    Nothing -> Nothing
    Just y -> Just (x `and2` y)
```


Nothing -> Nothing
Just y -> Just (and2 x y)

For instance, \(f\) (Just \(x\)) Nothing is an expression of \(\mathbb{P}\), containing a variable \(x\). We allow only simple patterns (a constructor followed by variables), and we require that pattern matches are complete (there is exactly one pattern for each constructor of the respective type). It is obvious that nested patterns can be translated to this form.

Evaluation of expressions is defined in the standard way: The domain of concrete values \(\mathbb{C}\) is the set of data terms. For instance, Just False \(\in\) \(\mathbb{C}\). A concrete environment is a mapping from program variables to \(\mathbb{C}\). A concrete evaluation function \(\text{concrete-value}: \mathbb{E} \times \mathbb{P} \rightarrow \mathbb{C}\) computes the value of a concrete expression \(p \in \mathbb{P}\) in a concrete environment \(e\). Evaluation of function and constructor arguments is strict. This is where we deviate from Haskell's lazy evaluation.

Evaluations on abstract data. The CO4 compiler transforms an input program that operates on concrete values, to an abstract program that operates on abstract values. An abstract value contains propositional logic formulas that may contain free propositional variables. An abstract value represents a set of concrete values. Each assignment of the propositional values produces a concrete value.

We formalize this in the following way: the domain of abstract values is called \(\mathbb{A}\). The set of assignments (mappings from propositional variables to truth values \(\mathbb{B} = \{0, 1\}\)) is called \(\Sigma\), and there is a function \(\text{decode}: \mathbb{A} \times \Sigma \rightarrow \mathbb{C}\).

We now specify abstract evaluation. (The implementation is given in Section [3].) We use abstract environments \(E_{\mathbb{A}}\) that map program variables to abstract values, and an abstract evaluation function \(\text{abstract-value}: E_{\mathbb{A}} \times \mathbb{P} \rightarrow \mathbb{A}\).

Allocators. As explained in the introduction, the constraint program receives known and unknown arguments. The compiled program operates on abstract values.

The abstract value that represents a (finite) set of concrete values of an unknown argument is obtained from an allocator. For a property \(q: \mathbb{C} \rightarrow \mathbb{B}\) of concrete values, a \(q\)-allocator constructs an object \(a \in \mathbb{A}\) that represents all concrete objects that satisfy \(q\):

\[
\forall c \in \mathbb{C} : q(c) \iff \exists \sigma \in \Sigma : c = \text{decode}(a, \sigma).
\]

We use allocators for properties \(q\) that specify \(c\) uses constructors that belong to a specific type. Later (with recursive types, see Section [4]) we also specify a size bound for \(c\). An example is an allocator for lists of booleans of length \(\leq 4\).

As a special case, an allocator for a singleton set is used for encoding a known concrete value. This constant allocator is given by a function \(\text{encode}: \mathbb{C} \rightarrow \mathbb{A}\) with the property that \(\forall c \in \mathbb{C}, \sigma \in \Sigma: \text{decode}(\text{encode}(c), \sigma) = c\).
Correctness of constraint compilation. The semantical relation between an expression \( p \) (a concrete program) and its compiled version \( \text{compile}(p) \) (an abstract program) is given by the following relation between concrete and abstract evaluation:

**Definition 1.** We say that \( p \in \mathbb{P} \) is compiled correctly if

\[
\forall e \in E_A \forall \sigma \in \Sigma : \text{decode}(\text{abstract-value}(e, \text{compile}(p)), \sigma) = \text{concrete-value}(\text{decode}(e, \sigma), p)
\]

Here we used \( \text{decode}(e, \sigma) \) as notation for lifting the decoding function to environments, defined element-wise by

\[
\forall e \in E_A \forall v \in \text{dom}(e) \forall \sigma \in \Sigma : \text{decode}(e, \sigma)(v) = \text{decode}(e(v), \sigma).
\]

Application of the Correctness Property. We are now in a position to show how the stages of CO4 compilation and execution fit together.

The top-level parametric constraint is given by a function declaration \( \text{main} \ k \ u = b \) where \( b \) (the body, a concrete program) is of type \( \text{Bool} \). It will be processed in the following stages:

1. **compilation** produces an abstract program \( \text{compile}(b) \),
2. **abstract computation** takes a concrete parameter value \( p \in C \) and a \( q \)-allocator \( a \in A \), and computes the formula
   \[
   F = \text{abstract-value}({k \mapsto \text{encode}(p), u \mapsto a}, \text{compile}(b))
   \]
3. **solving** calls the backend SAT solver to determine \( \sigma \in \Sigma \) with \( \text{decode}(F, \sigma) = \text{True} \). If this was successful,
4. **decoding** produces a concrete value \( s = \text{decode}(a, \sigma) \),
5. and optionally, **testing** checks that \( \text{concrete-value}({k \mapsto p, u \mapsto s}, b) = \text{True} \).

The last step is just for reassurance against implementation errors, since the invariant implies that the test returns True. This highlights another advantage of re-using Haskell for constraint programming: one can easily check the correctness of a solution candidate.

### 3 Implementation of a Propositional Encoding

In this section, we give a realization for abstract values, and show how compilation creates programs that operate correctly on those values, as specified in Definition 1.
Encoding and Decoding of Abstract Values. The central idea is to represent an abstract value as a tree, where each node contains an encoding for a symbol (a constructor) at the corresponding position, and the list of concrete children of the node is a prefix of the list of abstract children (the length of the prefix is the arity of the constructor).

The encoding of constructors is by a sequence of formulas that represent the number of the constructor in binary notation.

We denote by $F$ the set of propositional logic formulas. At this point, we do not prescribe a concrete representation. For efficiency reasons, we will allow some form of sharing, by representing formulas as directed acyclic graphs (e.g., and/inverter graphs). Our implementation (satchmo-core) assigns names to subformulas by doing the Tseitin transform on-the-fly, creating a fresh propositional literal for each subformula.

Definition 2. The set of abstract values $\mathcal{A}$ is the smallest set with $\mathcal{A} = F^* \times \mathcal{A}^*$.

An element $a \in \mathcal{A}$ thus has shape $(\overrightarrow{f}, \overrightarrow{a})$ where $\overrightarrow{f}$ is a sequence of formulas, called the flags of $a$, and $\overrightarrow{a}$ is a sequence of abstract values, called the arguments of $a$.

We introduce notation

- $\text{flags} : \mathcal{A} \rightarrow F^*$ gives the flags of an abstract value
- $\text{flags}_i : \mathcal{A} \rightarrow F$ gives the $i$-th flag of an abstract value
- $\text{arguments} : \mathcal{A} \rightarrow \mathcal{A}^*$ gives the arguments of an abstract value,
- $\text{argument}_i : \mathcal{A} \rightarrow \mathcal{A}$ gives the $i$-th argument of an abstract value

Equivalently, in Haskell notation,

```haskell
data A = A { flags :: [F] , arguments :: [A] }
```

The sequence of flags of an abstract value encodes the number of its constructor. We use the following variant of a binary encoding: For each data type $T$ with $c$ constructors, we use as flags a set of sequences $S \subseteq \{0,1\}^*$ with $|S| = c$ and such that each long enough $w \in \{0,1\}^*$ does have exactly one prefix in $S$.

We could have $S$ dependent on $T$, but this is not necessary. In practice we use a fixed encoding

$$S_1 = \{\epsilon\}; \quad \text{for } n > 1: \quad S_n = 0 \cdot S_{[n/2]} \cup 1 \cdot S_{[n/2]}$$

For example, $S_2 = \{0,1\}, S_3 = \{00,01,1\}, S_5 = \{000,001,01,10,11\}$. The lexicographic order of $S_c$ induces a bijection $\text{numeric}_c : S_c \rightarrow \{1,\ldots,c\}$.

The encoding function (from concrete to abstract values) is defined by

$$\text{encoder}_T(C(v_1,\ldots)) = (\text{numeric}_c^-(i), [\text{encoder}_T(v_1),\ldots])$$
where $C$ is the $i$-th constructor of type $T$, and $T_j$ is the type of the $j$-th argument of $C$. Note that here, $\text{numeric}_c(i)$ denotes a sequence of constant flags (formulas) that represents the corresponding binary string.

For decoding, we need to take care of extra flags and arguments that may have been created by the function $\text{merge}$ (Definition 6) that is used in the compilation of case expressions.

We extend the mapping $\text{numeric}_c$ to longer strings by $\text{numeric}_c(uv) := \text{numeric}_c(u)$ for each $u \in S_c, v \in \{0, 1\}^*$. This is possible because of the unique-prefix condition.

Given the type declarations

```plaintext
data Bool = False | True
data Maybe_Bool = Nothing | Just Bool
data Ordering = LT | EQ | GT
data Either_Bool_Ordering = Left Bool | Right Ordering
```

the concrete value $\text{True}$ can be represented by the abstract value $a_1 = ([x], [])$ and assignment $\{x = 1\}$, since $\text{True}$ is the second (of two) constructors, and $\text{numeric}_2([1]) = 2$. The same concrete value $\text{True}$ can also be represented by the abstract value $a_2 = ([x,y],[a_1])$ and assignment $\{x = 1, y = 0\}$, since $\text{numeric}_2([1,0]) = 2$. This shows that extra flags and extra arguments are ignored in decoding.

We give a formal definition: for a type $T$ with $c$ constructors, $\text{decode}_T((f, a), \sigma)$ is the concrete value $v = C_i(v_1, \ldots)$ where $i = \text{numeric}_c(f \sigma)$, and $C_i$ is the $i$-th constructor of $T$, and $v_j = \text{decode}_{T_j}(a_j, \sigma)$ where $T_j$ is the type of the $j$-th argument of $C_i$.

As stated, this is a partial function, since any of $f, a$ may be too short. For this Section, we assume that abstract values always have enough flags and arguments for decoding, and we defer a discussion of partial decodings to Section 4.

Allocators for Abstract Values. Since we consider (in this section) finite types only, we restrict to complete allocators: for a type $T$, a complete allocator is an abstract value $a \in A$ that can represent each element of $T$: for each $e \in T$, there is some $\sigma$ such that $\text{decode}_T(a, \sigma) = e$.

For the types given above, complete allocators are

| type             | complete allocator |
|------------------|--------------------|
| Bool             | $a_1 = ([x], [])$  |
| Ordering         | $a_2 = ([x_1, x_2], [])$ |
| Either_Bool_Ordering | $a_3 = ([x_1], [([x_2, x_3], [])])$ |

where $x_1, \ldots$ are (boolean) variables. We compute $\text{decode}(a_3, \sigma)$ for $\sigma = \{x_1 = 0, x_2 = 1, x_3 = 0\}$: Since $\text{numeric}_2([0]) = 1$, the top constructor is $\text{Left}$. It has one argument, obtained as $\text{decode}_{\text{Bool}}(([x_2, x_3], []), \sigma)$. For this we compute
numeric₂([1, 0]) = 2, denoting the second constructor \( \text{True} \) of \( \text{Bool} \). Thus, \( \text{decode}(a_3, \sigma) = \text{Left True} \).

**Compilation of Programs.** In the following we illustrate the actual transformation of the input program (that operates on concrete values) to an abstract program (operating on abstract values) and prove its soundness according to invariant (Definition 1).

Generally, compilation keeps structure and names of the program intact. For instance, if the original program defines functions \( f \) and \( g \), and the implementation of \( g \) calls \( f \), then the transformed program also defines functions \( f \) and \( g \), and the implementation of \( g \) calls \( f \).

The crucial exception is that compilation removes pattern matches. This is motivated as follows. Concrete evaluation of a pattern match (in the input program) consists of choosing a branch according to a concrete value (of the discriminant expression). Abstract evaluation cannot access this concrete value (since it will only be available after the SAT solver determines an assignment). This means that we cannot abstractly evaluate pattern matches. Therefore, they must be removed by compilation.

Compilation of variables, bindings, and function calls is straightforward, and we deal with them first.

**Definition 3 (Compilation, easy cases).**

- **a name is compiled into itself:**
  
  if \( v \) is a variable, then \( \text{compile}(v) = v \).

- **a local binding is compiled structurally:**
  
  \[ \text{compile}(\text{let } v = a \ \text{in} \ b) = \text{let } v = \text{compile}(a) \ \text{in} \ \text{compile}(b) \]

- **a function call is compiled structurally:**
  
  \[ \text{compile}(f(a_1, \ldots, a_n)) = f(\text{compile}(a_1), \ldots, \text{compile}(a_n)) \]

  Here, compilation creates an application of \( f \). It is executed during abstract evaluation.

**Lemma 1 (Correctness of compilation, easy cases).** Invariant \( \Box \) holds on compilation of variables, local bindings and function calls.

**Proof.** Let \( v \) be a variable of the input program and \( \text{compile}(v) = v \). As the abstract value of \( v \) only depends on the value of \( v \), i.e.

\[ \forall e \in E_a : \text{abstract-value}(e, v) = e(v), \]

and the concrete value of the original expression \( v \) only depends on the value of \( v \) as well, i.e.
∀e ∈ E_A : concrete-value(decode(e, σ), v) = decode(e, σ)(v),

by (2) we have

∀e ∈ E_A ∀σ ∈ Σ : decode_T(abstract-value(e, compile(v)), σ)
  = decode_T(abstract-value(e, v), σ)
  = decode_T(e(v), σ)
  = decode(e, σ)(v)
  = concrete-value(decode(e, σ), v).

So invariant (1) holds.

The proof of correctness of compilation of local bindings and function calls is by structural induction. □

Definition 4 (Compilation, constructor call).

For a constructor call \( C(p_1, \ldots, p_n) \) where \( C \) is the \( i \)-th constructor of a data type \( T \) (with \( c \) constructors in total) and \( p_j \) is of type \( T_j \),

\[
\text{compile}(C(p_1, \ldots, p_n)) = C'(\text{compile}(p_1), \ldots, \text{compile}(p_n))
\]

where \( C' : \mathbb{A}^* \rightarrow \mathbb{A} \) is a function that gets the abstract values \( (a_1, \ldots, a_n) \) of the compiled constructor arguments as input.

\[
C'(a_1, \ldots, a_n) = (\text{numeric}_c(i), [a_1, \ldots, a_n])
\]  \hspace{1cm} (2)

Note that \( C' \) is evaluated during the runtime of the abstract program.

Lemma 2 (Correctness of compilation, constructor call). Invariant (1) holds on compilation of constructor calls.

Proof. If \( C \) is the \( i \)-th constructor of a type \( T \), the decoding of \( \text{compile}(C(p_1, \ldots, p_n)) \)'s top level constructor is determined by the fixed flags of its abstract value (2).

\[
\forall e \in E_A \forall σ ∈ Σ : 
  \text{decode}_T(abstract-value(e, \text{compile}(C(p_1, \ldots, p_n))), σ)
  = \text{decode}_T(C'(\text{compile}(p_1), \ldots, \text{compile}(p_n)), σ)
  = C(\text{decode}_T(\text{compile}(p_1)), \ldots, \text{decode}_T(\text{compile}(p_1)))
\]  \hspace{1cm} (3)

The top-level constructor of \( C(p_1, \ldots, p_n) \)'s concrete value is independent of any environment, so
\[ \forall e \in E \forall \sigma \in \Sigma : \text{concrete-value}(\text{decode}(e, \sigma), C(p_1, \ldots, p_n)) = C(\text{concrete-value}(\text{decode}(e, \sigma), p_1), \ldots, \text{concrete-value}(\text{decode}(e, \sigma), p_n)) \quad (4) \]

The equality of (4) and (4) is proven by induction over the constructor arguments. \qed

We restrict to pattern matches where patterns are simple (a constructor followed by variables) and complete (one branch for each constructor of the type).

**Definition 5 (Compilation, pattern match).**

Consider a pattern match expression \( e \) of shape `case d of{...}`, for a discriminant expression \( d \) of type \( T \) with \( c \) constructors.

We have \( \text{compile}(e) = \text{let} \ x = \text{compile}(d) \ \text{in} \ \text{merge}(\text{flags}(x), b_1, \ldots) \) where \( x \) is a fresh variable, and \( b_i \) represents the compilation of the \( i \)-th branch.

Each such branch is of shape \( C v_1 \ldots v_n \rightarrow e_i \), where \( C \) is the \( i \)-th constructor of the type \( T \).

Then \( b_i \) is obtained as \( \text{let} \ \{v_1 = \text{argument}_1(x); \ldots\} \ \text{in} \ \text{compile}(e_i) \).

We need the following auxiliary function that combines the abstract values from branches of pattern matches, according to the flags of the discriminant.

**Definition 6 (Combining function).** \( \text{merge} : F^s \times A^c \rightarrow A \) combines abstract values so that \( \text{merge}(\overrightarrow{f}, a_1, \ldots, a_c) \) is an abstract value \( (\overrightarrow{g}, z_1, \ldots, z_n) \), where

- \( n = \max(|\text{arguments}(a_1)|, \ldots, |\text{arguments}(a_c)|) \)
- \( |\overrightarrow{g}| = \max(|\text{flags}(a_1)|, \ldots, |\text{flags}(a_c)|) \)
- for \( 1 \leq i \leq |\overrightarrow{g}|, \)
  
  \[ g_i \leftrightarrow (\text{numeric}_c(\overrightarrow{f}) = 1 \Rightarrow \text{flags}_i(a_1)) \]
  
  \[ \wedge (\text{numeric}_c(\overrightarrow{f}) = 2 \Rightarrow \text{flags}_i(a_2)) \]
  
  \[ \wedge \ldots \]
  
  \[ \wedge (\text{numeric}_c(\overrightarrow{f}) = c \Rightarrow \text{flags}_i(a_c)) \quad (5) \]

- for each \( 1 \leq i \leq n, \ z_i = \text{merge}(\overrightarrow{f}, \text{argument}_i(a_1), \ldots, \text{argument}_i(a_c)) \).

**Lemma 3 (Correctness of compilation, pattern match).** Invariant (1) holds on compilation of pattern matches.
Proof. Let \( m \) be a pattern match in the original program and \( m' \) the result of the corresponding merge. The decoding of \( m' \) depends on an assignment \( \sigma \). For any assignment \( \sigma \) there is a \( k \in [1,c] \) so that \( \text{numeric}_c(\text{flags}(m')) = k \). In this case, \( \square \) shows that for all flags of \( m' \) \( \text{flags}_i(m') \leftrightarrow \text{flags}_i(a_k) \) holds, with \( b_k \) being the \( k \)-th branch of \( e \) and \( a_k = \text{compile}(b_k) \). So, for a fixed \( \sigma \) (and hence \( k \)) property

\[
\forall e \in E_A : \text{decode}(m', \sigma) = \text{decode}(\text{abstract-value}(e, a_k), \sigma)
\]

holds.

As evaluating the concrete value of the original pattern match \( m \) under an environment decoded by \( \sigma \) leads to the evaluation of \( b_k \), i.e.

\[
\forall e \in E_A : \text{concrete-value}(\text{decode}(e, \sigma), m) = \text{concrete-value}(\text{decode}(e, \sigma), b_k)
\]

Invariant \( \square \) holds by induction over \( b_k \):

\[
\forall e \in E_A : \text{decode}(\text{abstract-value}(e, a_k), \sigma) = \text{concrete-value}(\text{decode}(e, \sigma), b_k)
\]

4 Partial encoding of Infinite Types

We discuss the compilation and abstract evaluation for constraints over infinite types, like lists and trees. Consider declarations (and recall that functions are still monomorphic)

\[
\begin{align*}
data N &= Z \mid S N \\
double :: N \rightarrow N \\
double x &= \text{case } x \text{ of } \{ Z \rightarrow Z ; S x' \rightarrow S (S (\text{double } x')) \}
\end{align*}
\]

Assume we have an abstract value \( a \) to represent \( x \). It consists of a flag (to distinguish between \( Z \) and \( S \)), and of one child (the argument for \( S \)), which is another abstract value. At some depth, recursion must stop, since the abstract value is finite (it can only contain a finite number of flags). Therefore, there is a child with no arguments, and it must have its flag set to \( \text{FALSE} \) (it must represent \( Z \)).

There is another option: if we leave the flag open (it can take on values \( \text{FALSE} \) or \( \text{TRUE} \)), then we have an abstract value with (possibly) a constructor argument missing. When evaluating the concrete program, the result of accessing a non-existing component gives a bottom value. This corresponds to the Haskell semantics where each data type contains bottom, and values like \( S (S \bot) \) are valid.
Definition 7. The set of abstract values $\mathbb{A}_\bot$ is the smallest set with $\mathbb{A}_\bot = F^* \times \mathbb{A}_\bot^* \times F$, i.e., an abstract value is a triple of flags and arguments (cf. definition 2) extended by an additional definedness constraint.

We write $\text{def}: \mathbb{A}_\bot \rightarrow F$ to give the definedness constraint of an abstract value, and keep flags and argument notation of Definition 2.

The decoding function is modified accordingly: $\text{decode}_T(a, \sigma)$ for a type $T$ with $c$ constructors is $\bot$ if $\text{def}(a)\sigma = \text{FALSE}$, or $\text{numeric}_c(\text{flags}(a))$ is undefined (because of “missing” flags), or $|\text{arguments}(a)|$ is less than the number of arguments of the decoded constructor.

The correctness invariant for compilation (Eq. 1) is still the same, but we now interpret it in the domain $\mathbb{C}_\bot$, so the equality says that if one side is $\bot$, then both must be.

Consequently, for the application of the invariant, we now require that the abstract value of the top-level constraint under the assignment is defined and $\text{TRUE}$.

Abstract evaluation with bottoms. For working with recursive types, we need recursive programs. If the input program is recursive, then so is the abstract program. Pattern matching is crucial to terminate recursion (e.g., to detect the end of a list), but the abstract program cannot pattern match, as explained earlier.

We introduce a limited form of matching: in the abstract evaluation of $\text{let } x = \text{compile}(d) \in \ldots$ (see Compilation, pattern match), we consider $\text{let}$ to be strict: if $\text{abstract-value}(E, \text{compile}(d))$ has a definedness flag that is constant $\text{FALSE}$, then the whole expression’s abstract value is bottom, and is represented by ([], [], $\text{FALSE}$).

If definedness is not constantly $\text{FALSE}$, then abstract evaluation will execute $\text{merge}$, modified as follows: the definedness flag of result $m$ of a $\text{merge}$ is

$$\text{def}(m) \leftrightarrow (\text{numeric}_c(\overrightarrow{f}) = 1 \Rightarrow \text{def}(a_1))$$
$$\land (\text{numeric}_c(\overrightarrow{f}) = 2 \Rightarrow \text{def}(a_2))$$
$$\land \ldots$$
$$\land (\text{numeric}_c(\overrightarrow{f}) = c \Rightarrow \text{def}(a_c))$$

Note that (definedness and other) flags are formulas, and in general we cannot determine their value (without an assignment). The given method relies on some form to detect that a formula denotes a constant.
5 Higher order functions and polymorphism

For formulating the constraints, expressiveness in the language is welcome. Since we base our design on Haskell, it is natural to include some of its features that go beyond first-order programs: higher order functions and polymorphic types.

Our program semantics is first-order: we cannot (easily) include functions as result values or in environments, since we have no corresponding abstract values for functions. Therefore, we instantiate all higher-order functions in a standard preprocessing step, starting from the main program.

Polymorphic types do not change the compilation process. The important information is the same as with monomorphic typing: the total number of constructors of a type, and the number (the encoding) of one constructor.

6 Memoization

We describe another optimization: in the abstract program, we use memoization for all subprograms. That is, during execution of the abstract program, we keep a map from (function name, argument tuple) to result. Note that arguments and result are abstract values.

This allows to write “natural” specifications and still get a reasonable implementation.

Example 1. The textbook definition of the lexicographic path order $\triangleright_{lpo}$ (cf. [I]) defines an order over terms according to some precedence. Its textbook definition is recursive, and leads to an exponential time algorithm, if implemented literally. By calling $s \triangleright_{lpo} t$ the algorithm still does only compare subterms of $s$ and $t$, and in total, there are $|s| \cdot |t|$ pairs of subterms, and this is also the cost of the textbook algorithm with memoization.

The next example is similar, but it additionally shows that abstract execution may increase cost, but memoization may reduce it again.

Example 2. The following function determines whether $xs$ is a (scattered) subword of $ys$.

\[
\text{subword} :: \text{Eq } a \Rightarrow [a] \rightarrow [a] \rightarrow \text{Bool}
\]

\[
\text{subword } xs \ ys = \text{case } xs \ of
\]

\[
\begin{array}{l}
\quad [\ ] \rightarrow \text{True} \\
\quad x : xs' \rightarrow \text{case } ys \ of \\
\quad \quad [\ ] \rightarrow \text{False} \\
\quad \quad y : ys' \rightarrow \text{case } x == y \ of \\
\quad \quad \quad \text{False} \rightarrow \text{subword } xs \ ys' \\
\quad \quad \quad \text{True} \rightarrow \text{subword } xs' \ ys'
\end{array}
\]
As a program on concrete values, this has linear complexity, since in each recursive call, the length of the second argument decreases.

In the compiled program for abstract values, for each `case`, each branch is executed, and the results are merged. In particular, both branches of `case x==y of ..` will be executed, so the resulting cost is exponential in the size of `ys`.

With memoization, the compiled program runs in polynomial time (and produces a polynomially sized formula) since in each subprogram call that happens during the evaluation of `subword xs0 ys0`, the actual arguments `xs`, `ys` are suffixes of the respective initial arguments, and there are `(length xs * length ys)` pairs of suffixes.

7 Case study: Loops in Term Rewriting

As an application, we use CO4 for compiling constraint systems that describe looping derivations. This is motivated by automated analysis of programs. A loop is an infinite computation, which may be unwanted behaviour, indicating an error in the program’s design. In general, it is undecidable whether a rewriting system admits a loop. By enumerating finite derivations, one can hope to find loops.

Our approach is to write the predicate “the derivation `d` conforms to a rewrite system `R` and `d` is looping” as a Haskell function, and solve the resulting constraint system, after putting bounds on the sizes of the terms that are involved.

Previous work uses several heuristics for enumerations resp. hand-written propositional encodings for finding loops in string rewriting systems [7].

We extend to (1) systematic compilation and (2) term rewriting.

In the following, we show the data declarations we use, and give code examples.

- we fix a signature, and a set of variables, and define the set of terms

```haskell
data Term = V Name | F Term Term Term | A | B | C

data Name = X | Y
```

- a rule is pair of terms, a rewrite system is list of rules

```haskell
data Rule = Rule Term Term

data List a = Nil | Cons a (List a)
type TRS = List Rule
```

- a rewrite step is a tuple `(t0, (l, r), P, σ, t1)` where `t0`, `t1` are terms, `(l, r)` is a rule, `p` is a position, `σ` is a substitution with `lσ = t0[p]` and `t0[p := rσ] = t1`

```haskell
data Pair a b = Pair a b
type Substitution = List (Pair Name Term)
type Step = Step Term Rule (List Pos) Substitution Term
```
– a derivation w.r.t. a TRS \( \text{trs} \) is a list of steps

\[
\text{type Derivation} = \text{List Step}
\]

where

• the result term of one step is the input term of the next step

\[
\text{derive_ok :: TRS} \rightarrow \text{Term} \rightarrow \text{Derivation} \rightarrow \text{Maybe Term}
\]

\[
\text{derive_ok trs term deriv} = \text{case deriv of}
\]

\[
\text{Nil} \rightarrow \text{Just term}
\]

\[
\text{Cons s deriv'} \rightarrow \text{case s of}
\]

\[
\text{Step t0 rule pos sub t1} \rightarrow \text{case equalTerm term t0 of}
\]

\[
\text{False} \rightarrow \text{Nothing}
\]

\[
\text{True} \rightarrow \text{case step_ok trs s of}
\]

\[
\text{False} \rightarrow \text{Nothing}
\]

\[
\text{True} \rightarrow \text{derive_ok trs t1 deriv'}
\]

• each step’s rule is from the \( \text{trs} \)

– a looping derivation’s output of the last step has a subterm that is a substitution instance of the input of the first step

Overall, the complete CO4 code (available at https://github.com/apunktbau/co4/blob/master/CO4/Test/TRS) consists of roughly 300 lines of code including the definition of all involved data types and auxiliary functions. The code snippets above shows that the constraint system literally follows the textbook definitions.

Our test case is the following term rewriting system, where \( X, Y \) are variables,

\[
\{ f(a, b, X) \rightarrow f(X, X, X), f(X, Y, c) \rightarrow X, f(X, Y, c) \rightarrow Y \},
\]

(corresponding to the classical example from [6]). We use allocators that restrict to derivations of length 3, and terms of depth 2. Abstract evaluation of the compiled program results in a propositional formula with 774663 variables and 2301608 clauses. On a standard Intel Core 2 Duo CPU with 2.20 GHz, Minisat SAT solver finds the following loop in around 10 seconds:

\[
f(a, b, f(a, b, c)) \Rightarrow f(f(a, b, c), f(a, b, c), f(a, b, c))
\]

\[
\Rightarrow f(a, f(a, b, c), f(a, b, c)) \Rightarrow f(a, b, f(a, b, c))
\]

CNF finished (#variables: 774663, #clauses: 2301608)
Solver finished in 10.823333 seconds (result: True)
Solution: Looping_Derivation ...
8 Discussion

In this paper we described the CO4 constraint language and compiler that allows to write constraints on tree-shaped data in a natural way, and to solve them via propositional encoding. We presented an outline of a correctness proof for our implementation, and gave an example that shows that the compiler actually works.

In this example, the resulting formula is huge. Still, the SAT solver can handle it rather quickly. This indicates that there is room for improving the efficiency of both compilation and abstract interpretation, in order to obtain smaller, equivalent, formulas from constraint systems.

We have several plans for this, for instance, hard-wiring improved implementations of basic boolean operations. We leave this as a subject of further research and implementation, for which the present report shall serve as a basis.

We mention two additional application areas of the concepts presented here:

- Different back-ends: In our example application, formulas (in abstract values) are ultimately represented as conjunctive normal forms, as this suits the SAT solver best. By changing the implementation of abstract values (but keeping the compiler), our system can output circuit descriptions for hardware design; and also Binary Decision Diagrams, which can be used for counting models of constraint systems.

- Complexity analysis: from an (automated) analysis of the input program, one can obtain the (asymptotic) size of the resulting propositional formula. For instance, if it is found to be polynomial (in the size of the input parameter), then satisfiability of the constraint problem is (automatically shown to be) in NP.

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