SEMILINEAR STOCHASTIC EQUATIONS WITH
BILINEAR FRACTIONAL NOISE

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Abstract. In the paper, we study existence and uniqueness of solutions to
semilinear stochastic evolution systems, driven by a fractional Brownian mo-
tion with bilinear noise term, and the long time behavior of solutions to such
equations. For this purpose, we study at first the random evolution operator
deﬁned by the corresponding bilinear equation which is later used to deﬁne the
mild solution of the semilinear equation. The mild solution is also shown to be
weak in the PDE sense. Furthermore, the asymptotic behavior is investigated
by using the Random Dynamical Systems theory. We show that the solution
generates a random dynamical system that, under appropriate stability and
compactness conditions, possesses a random attractor.

1. Introduction and preliminaries. In this paper we investigate the following
type of semilinear stochastic evolution equations:
\[ du(t) = (Au(t) + F(u(t)))dt + Bu(t) \circ dB^H(t), \quad u(0) = u_0 \in V, \]
(1)
where \((V, \| \cdot \|_V)\) is a separable Hilbert space, \(B^H\) denotes an one–dimensional frac-
tional Brownian motion (fBm) with Hurst parameter \(H \in (0, 1)\) and the stochastic
integral is understood in the Stratonovich sense, which is described as usual by
the symbol \(\circ\). The operators \(A\), \(B\) and the nonlinear function \(F\) satisfy adequate
properties which will be specified later on.

Recall that \(B^H(t), t \in \mathbb{R}\), is a centered Gaussian process deﬁned on a probability
space \((\Omega, \mathcal{F}, \mathbb{P})\). Its law is characterized by its covariance function, deﬁned by
\[ \mathbb{E}B^H(t)B^H(s) = \frac{1}{2}(\|t\|^{2H} + \|s\|^{2H} - \|t-s\|^{2H}), \quad t, s \in \mathbb{R}. \]
For \(H = 1/2\), fBm is the usual Brownian motion, so the family \(\{B^H, H \in (0, 1)\}\)
may be seen as one of the most natural generalizations of this classical process.

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Applications for stochastic equations driven by fBm can be found in so many fields, which include biophysics, financial modeling, electrical engineering, to name a few.

In the current article we assume that the Hurst coefficient satisfies $H > 1/2$ and at first we focus on the study of the existence of a solution to (1). Existence of solutions have been addressed e.g. in [2], [4], [11], [18] or [21]. Stochastic bilinear PDEs have been addressed e.g. in [2], [4], [11], [18] or [21]. Stochastic bilinear PDEs are considered in [7], [20] and [26]. In these papers stochastic integration is under-
tstood in the Skorokhod sense (so Section 1 of the present paper is a counterpart of these results for Stratonovich type integration) and in particular, in [20] it is shown that the mild solutions to the semilinear equations defined by means of the random evolution operator corresponding to the equation without nonlinearity are not the same as weak solutions (and hence the concept of mild solutions defined in this way is not reasonable). It is also shown that the equation does not define a random dynamical system. Therefore, the results obtained in the present paper show that if we replace the Skorokhod integral in the equation by the Stratonovich one, the system behaves much more like in the classical case of white noise perturbations.

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tion. 

\[
\int_a^b f(x) \circ dg(x) = (-1)\alpha \int_a^b D_{a+}^\alpha f(x) D_{b-}^{1-\alpha} g_b^\prime(x) dx, \tag{2}
\]

where

\[
g_b^\prime(x) := \chi_{(a,b)}(x)\left(g(x) - g(b^-)\right),
\]

(g(b−) denotes the left–sided limit at b, which is supposed to exist), $f \in I_{a+}^{\alpha}(L^p([a,b]))$ such that $f(a^\prime)$ exists, $g_{b^-} \in I_{b-}^{1-\alpha}(L^q([a,b]))$, with $1/p + 1/q \leq 1$, $\alpha p < 1$ and $0 < \alpha < 1$.

Note that the definition is independent of the choice of $\alpha \in (0, 1)$, $I_{a+}^{\alpha}(L^p([a,b]))$ and $I_{b-}^{1-\alpha}(L^q([a,b]))$ stand for the image of $L^p([a,b])$ and $L^q([a,b])$ under left– and right–sided fractional Riemann–Liouville integral (for the definition see e.g. [23]), respectively. The corresponding Weyl derivatives appearing in (2) are defined as

\[
D_{a+}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(x)}{(x-a)^\alpha} + \alpha \int_a^x \frac{f(x) - f(y)}{(x-y)^{\alpha+1}} dy \right) \chi_{(a,b)}(x)
\]

and

\[
D_{b-}^{1-\alpha} g_{b^-}(x) = \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left( \frac{g(x) - g(b^-)}{(b-x)^{1-\alpha}} + (1-\alpha) \int_x^b \frac{g(x) - g(y)}{(y-x)^{2-\alpha}} dy \right) \chi_{(a,b)}(x)
\]

for $f \in I_{a+}^{\alpha}(L^p([a,b]))$ and $g_{b^-} \in I_{b-}^{1-\alpha}(L^q([a,b]))$, respectively.

For a detailed construction of the pathwise (stochastic) integral (2) as well as its main properties, we refer the reader to the paper [27].

In our further estimates, we will make use of the following change of variable formula (see [28], Theorem 3.1): let $0 < \alpha < 1/2$, $f \in I_{a+}^{\alpha}(L^2([a,b]))$ be bounded, $g_{b^-} \in I_{b-}^{1-\alpha}(L^2([a,b]))$ and

\[
h(t) = h(s) + \int_s^t f(r) \circ dg(r), \quad s < t \in [a,b].
\]
Then for any $G \in C^1([a, b] \times \mathbb{R}; \mathbb{R})$ such that $\frac{\partial G}{\partial x}(t, \cdot) \in C^1(\mathbb{R}; \mathbb{R})$ and for any $t_0 \leq t \in [a, b]$, 

$$G(t, h(t)) = G(t_0, h(t_0)) + \int_{t_0}^t \frac{\partial G}{\partial x}(s, h(s)) f(s) \circ dg(s) + \int_{t_0}^t \frac{\partial G}{\partial t}(s, h(s)) ds.$$  

(3)

Let $C^\beta([a, b]; \mathbb{R})$ be the Banach space of Hölder continuous functions with exponent $\beta > 0$ having values in $\mathbb{R}$. A norm on this space is given by

$$\|u\|_\beta = \|u\|_{\beta, a, b} = \|u\|_{\infty, a, b} + \|u\|_{\beta, a, b},$$

with

$$\|u\|_{\infty, a, b} = \sup_{s \in [a, b]} |u(s)|; \quad \|u\|_{\beta, a, b} = \sup_{\alpha \leq s \leq t \leq b} \frac{|u(t) - u(s)|}{|t - s|^\beta}.$$

and for any $\alpha \leq s \leq t \leq b$ there exists a constant $c$ depending only on $b - a$, $\beta$, $\beta'$ such that

$$\left| \int_s^t f \circ dg \right| \leq c \|f\|_{\beta, a, b} \|g\|_{\beta', a, b} (t - s)^{\beta'}.$$

Proof. The existence of the integral follows by [27]. Let us prove the estimate. Taking arbitrary $\alpha \in (1 - \beta', \beta)$ we have

$$|D_\alpha^x f(x)| \leq c \left| \frac{f(x)}{(x - s)\alpha} + \alpha \int_{x}^s \frac{f(y) - f(y)}{(x - y)^{\alpha + 1}} dy \right|$$

$$\leq c \|f\|_{\infty, a, b} (x - s)^{-\alpha} \|f\|_{\beta, a, b} \int_{x}^s \frac{(x - y)^{\beta}}{(x - y)^{\alpha + 1}} dy$$

$$\leq c \|f\|_{\infty, a, b} (x - s)^{-\alpha} \|f\|_{\beta, a, b} \int_{x}^s (x - y)^{\beta - \alpha} dy$$

$$\leq c \|f\|_{\beta, a, b} (x - s)^{-\alpha}.$$

Similarly, for $g \in C^{\beta'}([a, b]; \mathbb{R})$, it is straightforward to obtain that

$$|D^{1 - \alpha} g(-t)(x)| \leq c \left| \frac{g(x) - g(t)}{(t - x)^{1 - \alpha}} + \int_x^t \frac{g(x) - g(y)}{(y - x)^{2 - \alpha}} dy \right|$$

$$\leq c \|g\|_{\beta', a, b} (t - x)^{\beta' + \alpha - 1}.$$

Then

$$\left| \int_s^t f \circ dg \right| \leq c \|f\|_{\beta, a, b} \|g\|_{\beta', a, b} \int_{s}^t (x - s)^{-\alpha} (t - x)^{\beta' + \alpha - 1} dx$$

$$\leq c \|f\|_{\beta, a, b} \|g\|_{\beta', a, b} (t - s)^{\beta'}.$$

Let us point out that the relationship between the different parameters ensures that all the previous integrals are well-defined. Also, the left hand side is independent of the choice of $\alpha$. $\square$
Remark 1. Kolmogorov–Chentsov Theorem ensures that the fBm $B^H$ has a version, denoted from now on by $\omega$, which is $\beta'$–Hölder continuous on any interval $[-k,k]$ for $\beta' < H$, see [16], Theorem 1.4.1. As a consequence, we can consider pathwise integrals in the sense of (2) with the fBm $\omega$ as integrator and the Hurst parameter $H \in (1/2, 1)$.

In what follows, in the situation of Lemma 1.1 with $\omega = g$, if $1/2 < \beta < \beta' < H$, let $\Omega$ be the set of paths $\omega: \mathbb{R} \to \mathbb{R}$ which are $\beta'$–Hölder continuous on any compact subinterval of $\mathbb{R}$, being zero at zero.

In this paper we study the solution of (1) based on a previous investigation of the solution of the associated linear problem, i.e., the problem (1) with $F \equiv 0$. We assume that $A$ generates an analytic semigroup $S_A$, $B$ generates a group $S_B$, satisfying the commutativity assumption $AS_B(t)x = S_B(t)Ax, x \in D(A), t \in \mathbb{R}$, cf. Section 2 for more details. We find a formula for the weak solution to the linear problem. In the formula, the noise $B^H$ does not appear as the integrand of any stochastic integral but as an argument of the group $S_B$. Then under the Lipschitz continuity of the nonlinear unbounded mapping $F$, the existence of a mild solution of (1) is obtained, which is also shown to be a weak solution.

Beside the existence of a solution to equation (1) we also investigate its long time behavior, analyzing the existence of a random attractor for (1). In virtue of the pathwise character of the stochastic integral defined above, there are not exceptional sets which would prevent to establish the cocycle property for the solution, which is an essential part in the definition of a random dynamical system. This fact is in contrast to the case $H = 1/2$, i.e., to the case of equations driven by Brownian motion, where the integrals are only defined almost surely and exceptional sets may depend on the integrand. Large time behavior of solutions for equations with fBm and $H > 1/2$ based on the existence of random attractors is to some extent still in its infancy, although there are some papers like [10], [12] and [14] in the finite dimensional setting, or [9] and [15] in the infinite–dimensional case. However, more effort is needed to describe the inner structure of random attractors, which would give more information on the large time behavior of the solutions.

Let us comment that, in order to obtain the existence of an absorbing set, we additionally assume $F$ to be decomposed as $F = aI + G$ where $G$ is a bounded nonlinear function and $a \in \mathbb{R}$. With the additional hypotheses $a + \lambda < 0$, where $\lambda \in \mathbb{R}$ is related to the exponential bound of $S_A$, we are able to find an absorbing ball. We need to find, however, a compact absorbing set and at this point we need to assume that the semigroup $S_A$ is compact (or, equivalently, $A$ has compact resolvents). This guarantees a suitable compactness embedding making possible to find a compact absorbing set. Then the main result on the asymptotic behavior of the solution of (1) is that there exists a unique random attractor in $V$ associated to the random dynamical system generated by the solution of (1).

The remainder of this article is structured as follows: in Section 2 we study the existence and uniqueness of the mild solutions to the equation (1) by studying first the associated linear problem. For (1) it is shown that there exists a mild solution that is also weak in the PDE sense. Section 3 is devoted to establish that these solutions define a cocycle. Section 4 concerns the analysis of the existence of a unique attractor. Finally, in Section 5 we provide a couple of examples fitting the abstract results obtained in the paper. As examples, we consider semilinear stochastic parabolic equations with the nonlinear term in the drift and with bilinear diffusion of the type described above.
2. Existence and uniqueness of solutions. In order to study the problem (1), consider first the linear problem given by

\[ dv(t) = Av(t)dt + Bv(t) \circ d\omega(t), \quad v(s) = u_0 \in V, \] (4)

for \( t \geq s \geq 0. \)

Throughout the paper, the following assumptions are imposed:

(A) The linear operator \( A : D(A) \subset V \to V \) is closed, densely defined and generates an analytic semigroup \( \{S_A(t)\}_{t \in \mathbb{R}^+} \) on \( V. \)

(B) The linear operator \( B : D(B) \subset V \to V \) is closed, densely defined and generates a strongly continuous group \( \{S_B(t)\}_{t \in \mathbb{R}} \) on \( V. \)

(AB) \( D(A) \subset D(B) \) and the operators satisfy the commutativity assumption

\[ AS_B(t)x = S_B(t)Ax \]

for all \( t \in \mathbb{R} \) and \( x \in D(A). \) Moreover, \( D(A^*) \subset D((B^*)^2). \)

As a consequence of (AB) \( S_A \) and \( S_B \) may be shown to commute on \( V, \) i.e.

\[ S_A(s)S_B(t)x = S_B(t)S_A(s)x, \quad x \in V, t \in \mathbb{R}, s \geq 0. \] (5)

Note that for analytic semigroups (see e.g. [22], Theorem 2.6.13) for any \( \alpha \in (0, 1] \) there exists a constant \( K_\alpha > 0 \) such that

\[ \|S_A(t-s) - S_A(r-s)\|_{L(V)} \leq K_\alpha \left( \frac{t-r}{r-s} \right)^{\alpha}, \quad 0 \leq s < r \leq t. \] (6)

Moreover, since there exists \( \beta_0 \in \mathbb{R} \) such that operator \( \beta_0I - A \) is strictly positive, the fractional powers

\[ A^\delta := (\beta_0I - A)^\delta, \quad D(A^\delta) = D((\beta_0I - A)^\delta), \]

are well-defined for all \( \delta \in (0, 1] \) (e.g. [22]). The domains \( D(A^\delta) \) are equipped with the graph norm, i.e.

\[ \|x\|_{D(A^\delta)} = \|x\|_V + \|(\beta_0I - A)^\delta x\|_V. \]

Let us note that in the definition of \( A^\delta \) the constant \( \delta \) is understood as an index, whereas \( \delta \) in the expression \( (\beta_0I - A)^\delta \) denotes the fractional power of the operator \( \beta_0I - A. \)

**Definition 2.1.** Given \( T > 0, \) a stochastic process \( v = \{v(t), t \in [0, T]\} \) is said to be a weak solution to the equation (4) if for any \( \zeta \in D(A^*) \)

\[ \langle v(t), \zeta \rangle_V = \langle u_0, \zeta \rangle_V + \int_s^t \langle v(r), A^* \zeta \rangle_V dr + \int_s^t \langle v(r), B^* \zeta \rangle_V \circ d\omega(r) \]

for all \( t \in [s, T]. \) Similarly, a stochastic process \( v = \{v(t), t \in [0, T]\} \) is said to be a weak solution to the equation (1) if for any \( \zeta \in D(A^*) \)

\[ \langle v(t), \zeta \rangle_V = \langle u_0, \zeta \rangle_V + \int_0^t \langle v(r), A^* \zeta \rangle_V dr + \int_0^t \langle F(v(r)), \zeta \rangle_V dr \]

\[ + \int_0^t \langle v(r), B^* \zeta \rangle_V \circ d\omega(r) \]

is satisfied for all \( t \in [0, T], \) provided that all integrals are meaningful.

The following existence theorem for the linear problem is proved:
Theorem 2.2. Assume that (A), (B), (AB) hold. Then there exists a weak solution \( v \) to (4) that for \( t \geq s \geq 0 \) is given by

\[
U(t, \omega, s)u_0 := S_B(\omega(t) - \omega(s))S_A(t - s)u_0.
\]

(7)

To simplify the notation, the argument \( \omega \) is omitted in \( U = U(t, s) \) if no confusion is possible.

Remark 2. Since \( S_A \) and \( S_B \) are a strongly continuous semigroup and group, respectively, and the fBm is continuous, there exists a random constant \( C_U(\omega) = C_U > 0 \) such that

\[
\|U(t, \omega, s)\|_{L(V)} \leq C_U
\]

(8)

for any \( 0 \leq s \leq t \leq T \).

Proof. of Theorem 2.2. First, assume that \( u_0 \in D(A) \). Let us consider \( \zeta \in D(A^*) \) and define \( G : [s, T] \times \mathbb{R} \to \mathbb{R} \) by

\[
G(t, x) = \langle S_A(t - s)u_0, S_B^*(x)\zeta \rangle_V,
\]

and

\[
h(t) = \omega(t) - \omega(s),
\]

for \( t \in [s, T] \). We would like to apply the change of variable formula (3). Observe that \( G \in C^1([s, T] \times \mathbb{R}; \mathbb{R}) \), \( \frac{\partial G}{\partial t}(t, \cdot) \in C^1(\mathbb{R}; \mathbb{R}) \) for any \( t \in [s, T] \),

\[
h(t) = \int_s^t \chi(s, T)(r) \circ d\omega(r),
\]

where \( g = \omega, \, g_{-T} \in I_{-T}^{-\alpha}(L^2([s, T])) \) and \( f = \chi(s, T) \in I_{-T}^{\alpha}(L^2([s, T])) \) is bounded for any \( \alpha \in (1 - H, 1/2) \). Therefore, thanks to (3),

\[
G(t, \omega(t) - \omega(s)) = \langle u_0, \zeta \rangle_V \\
+ \int_s^t \langle S_A(r - s)u_0, S_B^*(\omega(r) - \omega(s))B^*\zeta \rangle_V \circ d\omega(r) \\
+ \int_s^t \langle A\omega(r) - \omega(s)\rangle_V \, d\omega(r)
\]

Using (7) and the commutativity assumption (AB) in the third summand,

\[
\langle U(t, s)u_0, \zeta \rangle_V = \langle u_0, \zeta \rangle_V + \int_s^t \langle U(r, s)u_0, B^*\zeta \rangle_V \circ d\omega(r) \\
+ \int_s^t \langle U(r, s)u_0, A^*\zeta \rangle_V \, d\omega(r)
\]

holds for any \( t \in [s, T] \), which completes the proof for \( u_0 \in D(A) \).

Now, let \( u_0 \in V \) and consider a sequence \( \{x_n\}_{n \in \mathbb{N}} \subset D(A) \) such that \( \|x_n - u_0\|_V \xrightarrow{n \to +\infty} 0 \). Taking into account the previous considerations, for \( t \in [s, T] \)

\[
\langle U(t, s)x_n, \zeta \rangle_V = \langle x_n, \zeta \rangle_V + \int_s^t \langle U(r, s)x_n, B^*\zeta \rangle_V \circ d\omega(r) \\
+ \int_s^t \langle U(r, s)x_n, A^*\zeta \rangle_V \, d\omega(r)
\]
holds for all $n \in \mathbb{N}$ and $\zeta \in D(A^*)$. The aim is to pass to the limit in the above equation. Clearly, using (8)

$$\|x_n, \zeta\|_V - \|u_0, \zeta\|_V \xrightarrow{n \to +\infty} 0,$$

$$\|U(t, s)x_n, \zeta\|_V - \|U(t, s)u_0, \zeta\|_V \xrightarrow{n \to +\infty} 0,$$

$$\left|\int_s^t \langle U(r, s)x_n, A^*\zeta\rangle_d r - \int_s^t \langle U(r, s)u_0, A^*\zeta\rangle_d r \right| \xrightarrow{n \to +\infty} 0.$$

For the stochastic term, on account of Lemma 1.1 we know that

$$\sup_{s,T} \langle U(\tau, s)(x_n - u_0), B^*\zeta\rangle_V \leq C_U \|B^*\zeta\|_V \|x_n - u_0\|_V \xrightarrow{n \to +\infty} 0.$$  

Second, for $s \leq \tau_1 < \tau_2 \leq T$

$$\|\langle U(\tau_2, s) - U(\tau_1, s)(x_n - u_0), B^*\zeta\rangle_V\|

= \|\langle (S_B(\omega(\tau_2) - \omega(s)) - S_B(\omega(\tau_1) - \omega(s)))S_A(\tau_2 - s)(x_n - u_0), B^*\zeta\rangle_V\|

+ \|\langle S_B(\omega(\tau_1) - \omega(s))(S_A(\tau_2 - s) - S_A(\tau_1 - s))(x_n - u_0), B^*\zeta\rangle_V\| =: I_1 + I_2.$$  

Let us estimate each of these terms. Using [22], Theorem 2.4.d,

$$I_1 \leq \|\langle S_A(\tau_2 - s)(x_n - u_0), (S_B^*(\omega(\tau_2) - \omega(s)) - S_B^*(\omega(\tau_1) - \omega(s)))B^*\zeta\rangle_V\|

\leq c\|x_n - u_0\|_V \int_{s}^{\tau_2 - s} \|S_B^*(z) (B^*)^2 \zeta\|_V d\zeta

\leq c\|x_n - u_0\|_V \|B^*\zeta\|_V \|\omega\|_{\beta', \tau_1, \tau_2} (\tau_2 - \tau_1)^{\beta'},$$

hence

$$\sup_{s \leq \tau_1 < \tau_2 \leq T} \frac{I_1}{(\tau_2 - \tau_1)\beta} \leq c\|x_n - u_0\|_V \|B^*\zeta\|_V \|\omega\|_{\beta', \tau_1, \tau_2} (T - s)^{\beta - \beta} \xrightarrow{n \to +\infty} 0,$$

because $\beta' > \beta$, see Remark 1.

On the other hand, thanks to (6),

$$I_2 \leq \|\langle (S_A(\tau_2 - s) - S_A(\tau_1 - s))(x_n - u_0), S_B^*(\omega(\tau_1) - \omega(s))B^*\zeta\rangle_V\|

\leq c(\tau_1 - s)^{-\beta} (\tau_2 - \tau_1)^{\beta} \|x_n - u_0\|_V \|B^*\zeta\|_V \|\omega\|_{\beta', \tau_1, \tau_2} (\tau_1 - s)^{\beta'},$$

and therefore,

$$\sup_{s \leq \tau_1 < \tau_2 \leq T} \frac{I_2}{(\tau_2 - \tau_1)\beta} \leq c\|x_n - u_0\|_V \|B^*\zeta\|_V \|\omega\|_{\beta', \tau_1, \tau_2} (T - s)^{\beta - \beta} \xrightarrow{n \to +\infty} 0,$$

which concludes the proof. \(\Box\)

Now, consider the problem (1). The mapping $F : V \to V$ is assumed to be Lipschitz continuous, i.e. there exists $L > 0$ such that

$$\|F(u) - F(v)\|_V \leq L\|u - v\|_V, \ u, v \in V.$$  

(9)
Definition 2.3. A stochastic process \( u = \{u(t), t \in [0,T]\} \) is said to be a mild solution to the equation (1) if
\[
u(t) = U(t,0)u_0 + \int_0^t U(t,r)F(u(r))dr \quad (10)
\]
for any \( t \in [0,T], \) where \( \{U(t,s), 0 \leq s \leq t \leq T\} \) is defined by (7).

We can establish the following result concerning the existence of solutions to (1):

Theorem 2.4. Given \( T > 0, \) under the conditions (A), (B), (AB) and (9) there exists a unique mild solution \( u \in C([0,T];V) \) to (1) for every \( u_0 \in V. \)

Proof. Step 1. Assume additionally that \( F \) is bounded, i.e. there exists a constant \( L > 0 \) such that
\[
\|F(u)\|_V \leq L, \ u \in V, \quad (11)
\]
and fix \( u_0 \in V. \) Define the operator \( \Phi \) given by
\[
(\Phi(y))(t) = U(t,0)u_0 + \int_0^t U(t,r)F(y(r))dr, \ t \in [0,T].
\]
We want to show that \( \Phi \) is a continuous contraction mapping from \( C([0,T];V) \) into itself.

First we show that \( \Phi : C([0,T];V) \to C([0,T];V). \) Take \( y \in C([0,T];V) \) and \( 0 \leq s, t \leq T. \) Then
\[
\|\Phi(y)(t) - \Phi(y)(s)\|_V \leq \|U(t,0)u_0 - U(s,0)u_0\|_V + \int_0^t \int_s^t U(r,s)F(y(r))dr \int_s^t U(s,r)F(y(r))dr \|_V =: I_1 + I_2,
\]
where
\[
I_1 = \|(S_B(\omega(t))S_A(t) - S_B(\omega(s))S_A(s))u_0\|_V
\leq \|(S_B(\omega(t)) - S_B(\omega(s)))S_A(t)u_0\|_V
+ \|S_B(\omega(s))(S_A(t) - S_A(s))u_0\|_V \xrightarrow{s \to t} 0.
\]

Let \( s < t. \) Then, by (8) and (11)
\[
I_2 \leq \int_0^s \int_s^t (U(t,r) - U(s,r))F(y(r))dr \|_V + \int_s^t U(t,r)F(y(r))dr \|_V
\leq \int_0^s \int_s^t (S_B(\omega(t) - \omega(r)) - S_B(\omega(s) - \omega(r)))S_A(t-r)F(y(r))dr \|_V
+ \int_0^s S_B(\omega(s) - \omega(r))(S_A(t-r) - S_A(s-r))F(y(r))dr \|_V + \int_s^t C_Ldr
=: I_3 + I_4 + C_L(t-s) \to 0
\]
as \( t \to s+ \) or \( s \to t-, \) provided that \( I_3 \) and \( I_4 \) tend to zero as \( t \to s+ \) or \( s \to t-. \) Since
\[
\|S_B(\omega(s) - \omega(r))(S_A(t-r) - S_A(s-r))F(y(r))\|_V \to 0
\]
as \( t \to s+ \) or \( s \to t-, \) and using (11) and (6), for any \( \alpha \in (0,1] \)
\[
I_4 \leq cL \int_0^s (t-s)^\alpha(s-r)^{-\alpha}dr \leq cLT^{\alpha} \int_0^s (s-r)^{-\alpha}dr \leq cLT < +\infty,
\]
thus \( I_4 \to 0 \) as \( t \to s+ \) or \( s \to t- \) by the Lebesgue dominated convergence Theorem.
On the other hand, define the set
\[ \mathcal{K} := \left\{ z \in V; \exists 0 \leq s_1 \leq t_1 \leq T \text{ such that } z = \int_0^{s_1} S_B(-\omega(r))S_A(t_1-r)F(y(r))dr \right\}. \]
This is a compact set since it is the image of a compact set by a continuous mapping.
Therefore,
\[ \lim_{t \to s} \sup_{z \in \mathcal{K}} \left\| (S_B(\omega(t)) - S_B(\omega(s)))z \right\|_V = 0, \]
because the pointwise convergence becomes uniform convergence on compact sets.
Hence
\[ I_3 = \left\| (S_B(\omega(t)) - S_B(\omega(s))) \int_0^s S_B(-\omega(r))S_A(t-r)F(y(r))dr \right\|_V \]
\[ \leq \sup_{z \in \mathcal{K}} \left\| (S_B(\omega) - S_B(\omega(s)))z \right\|_V \to 0 \]
as \( t \to s+ \) or \( s \to t- \) which completes the first part of the proof.
Let \( y_1, y_2 \in C([0,T];V) \). Applying (8) and (9) one gets
\[ \| \Phi(y_1) - \Phi(y_2) \|_{\infty,0,T} = \sup_{t \in [0,T]} \left\| \int_0^t U(t,s)(F(y_1(r)) - F(y_2(r)))dr \right\|_V \]
\[ \leq C_{U,L}T \| y_1 - y_2 \|_{\infty,0,T}. \]
If \( T > 0 \) is small enough, namely \( T < (C_{U,L})^{-1} \), then \( \Phi \) is a contraction mapping.
Hence, by the Banach fixed point Theorem there exists a unique mild solution to the equation (1) for \( T \) small enough. A unique mild solution to (1) for any \( T > 0 \) can be obtained using the standard methods.

**Step 2.** Now we drop the condition (11). For \( N > \| u_0 \|_V \), set
\[ K(N) := N\left( \| u_0 \|_V + T\| F(0) \|_V \right)e^{LNT} \]
and define \( \Omega_N \subset \Omega \),
\[ \Omega_N := \{ \omega \in \Omega; \| U(t,s) \|_{L(V)} \leq N \}. \]
Furthermore, let
\[ F_K(x) = \begin{cases} F(x), & \| x \|_V \leq K, \\ F \left( \frac{Kx}{\| x \|_V} \right), & \| x \|_V > K. \end{cases} \]
Obviously \( F_K \) is bounded by \( \| F(0) \|_V + LK \) and Lipschitz continuous with Lipschitz constant \( L \), hence by Step 1 of the proof there exists a unique solution to the equation
\[ u_K(t) = U(t,0)u_0 + \int_0^t U(t,s)F_K(u_K(s))ds, \ t \in [0,T]. \]
For \( \omega \in \Omega_N \) we have that
\[ \| u_K(t,\omega) \|_V \leq N\| u_0 \|_V + \int_0^t N\left( \| F(0) \|_V + L\| u_K(s,\omega) \|_V \right)ds, \ t \in [0,T], \]
which by Gronwall Lemma yields
\[ \| u_K(t,\omega) \|_V \leq K = K(N). \]
Therefore \( F_K(u_K(t,\omega)) = F(u_K(t,\omega)) \) and \( u_K \) solves the equation (1) on the set \( \Omega_N \). Setting
\[ u(t,\omega) = u_K(t,\omega), \ \omega \in \Omega_N, \]
and using the fact that \( \bigcup_{n \in \mathbb{N}, n > \| u_0 \|_V} \Omega_n = \Omega \) the proof is completed. \( \square \)
Remark 3. From Step 2 of the proof of Theorem 2.4 it follows that there exist constants $C_1(\omega), C_2(\omega) > 0$ (depending on $\omega$) such that
\[ \|u(t)\|_V \leq C_1(\omega) + C_2(\omega)\|u_0\|_V \] (12)
for any $t \in [0, T]$.

Now we prove that the mild solution from Theorem 2.4 is a weak one.

Theorem 2.5. Let the assumptions of Theorem 2.4 be satisfied and let $u$ be the mild solution to the equation (1). Then $u$ is also a weak solution to (1).

Proof. Take $t \in [0, T]$, $\zeta \in D(A^*)$. From the expression of the mild solution $u$ (see (10)) and the fact that $U$ is a weak solution to the equation (4), we get
\[
\langle u(t), \zeta \rangle_V = \langle U(t, 0)u_0, \zeta \rangle_V + \int_0^t \langle U(t, r)F(u(r)), \zeta \rangle_V \, dr
\]
\[
= \langle u_0, \zeta \rangle_V + \int_0^t \langle U(t, 0)u_0, A^*\zeta \rangle_V \, dr + \int_0^t \langle U(t, 0)u_0, B^*\zeta \rangle_V \circ d\omega(t)
\]
\[
+ \int_0^t \langle F(u(r)), \zeta \rangle_V \, dr + \int_0^t \int_0^t \langle U(\tau, r)F(u(r)), A^*\zeta \rangle_V \, d\tau \, dr
\]
\[
+ \int_0^t \int_0^t \langle U(\tau, r)F(u(r)), B^*\zeta \rangle_V \circ d\omega(\tau) \, dr
\]
\[
\overset{(\Delta)}{=} \langle u_0, \zeta \rangle_V + \int_0^t \langle F(u(r)), \zeta \rangle_V \, dr
\]
\[
+ \int_0^t \langle U(\tau, 0)u_0, A^*\zeta \rangle_V \, d\tau + \int_0^t \int_0^t \langle U(\tau, r)F(u(r)), A^*\zeta \rangle_V \, d\tau \, dr
\]
\[
+ \int_0^t \langle U(\tau, 0)u_0, B^*\zeta \rangle_V \circ d\omega(\tau)
\]
\[
+ \int_0^t \int_0^t \langle U(\tau, r)F(u(r)), B^*\zeta \rangle_V \, dr \circ d\omega(\tau)
\]
\[
= \langle u_0, \zeta \rangle_V + \int_0^t \langle u(\tau), A^*\zeta \rangle_V \, d\tau + \int_0^t \langle F(u(r)), \zeta \rangle_V \, dr
\]
\[
+ \int_0^t \langle u(\tau), B^*\zeta \rangle_V \circ d\omega(\tau),
\]
where in the last equality the definition of $u$ was applied again. In the equality $(\Delta)$ Fubini Theorem was used. Indeed, Fubini Theorem can be applied to the deterministic integral because using (12)
\[
\left\| \int_0^t \int_0^t \langle U(\tau, r)F(u(r)), A^*\zeta \rangle_V \, d\tau \, dr \right\|_V
\]
\[
\leq CT^2\|A^*\zeta\|_V \left( \|F(0)\|_V + L(C_1(\omega) + C_2(\omega))\|u_0\|_V \right) < +\infty
\]
holds and with the stochastic integral one can proceed as in Theorem 2.2. Altogether, the proof is complete. □

Remark 4. The above results can be also extended to the case of time-dependent operator $A$, that is, if $A(t), t \in \mathbb{R}^+$, generates a strongly continuous evolution operator. For simplicity we consider here just the time homogeneous case.
3. **Random dynamical system.** In what follows we would like to study the random dynamical system generated by the solution of our bilinear problem.

We start by introducing the definition of a random dynamical system (RDS).

**Definition 3.1.** Consider the probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where the measure \(\mathbb{P}\) is assumed to be invariant and ergodic with respect to a flow \(\theta\). Then \((\Omega, \mathcal{F}, \mathbb{P}, \theta)\) is called a metric dynamical system.

**Definition 3.2.** A random dynamical system over the metric dynamical system \((\Omega, \mathcal{F}, \mathbb{P}, \theta)\) is a \(\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}(V), \mathcal{B}(V)\)–measurable mapping such that the cocycle property holds

\[
\varphi(t + \tau, \omega, u_0) = \varphi(t, \theta_\tau \omega, \varphi(\tau, \omega, u_0)), \quad \varphi(0, \omega, u_0) = u_0,
\]

for all \(t \geq \tau \in \mathbb{R}^+, u_0 \in V\) and \(\omega \in \Omega\).

First of all, we remind that Kolmogorov–Chentsov Theorem ensures that \(B^H\) has a continuous version. Hence we can consider the canonical interpretation of an fBm: let \(C_0(\mathbb{R}; \mathbb{R})\) be the space of continuous functions on \(\mathbb{R}\) into itself vanishing at zero, equipped with the compact open topology. Let \(\mathcal{F} = \mathcal{B}(C_0(\mathbb{R}; \mathbb{R}))\) be the associated Borel \(\sigma\)–algebra, \(\mathbb{P}\) the distribution of the fBm \(B^H\), and \(\theta = (\theta_t)_{t \in \mathbb{R}}\) be the flow of Wiener shifts given by

\[
\theta_t \omega(\cdot) = \omega(t + \cdot) - \omega(t), \quad \text{for any } t \in \mathbb{R}.
\]

Then \((C_0(\mathbb{R}; \mathbb{R}), \mathcal{F}, \mathbb{P}, \theta)\) is an ergodic metric dynamical system, see [19] and [13].

As we previously said in Remark 1, this (canonical) process has a version, denoted by \(\omega\), which is \(\beta'\)–Hölder continuous on any interval \([-k, k]\) for \(\beta' < H\).

As we did in the previous section, for \(1/2 < \beta < \beta' < H\), we denote by \(\Omega \subset C_0(\mathbb{R}; \mathbb{R})\) the set of functions which are \(\beta'\)–Hölder continuous on any interval \([-k, k]\), \(k \in \mathbb{N}\), and are zero at zero. Then \(\Omega \in \mathcal{F}\) and \(\mathbb{P}(\Omega) = 1\). In addition, \(\Omega\) is \((\theta_t)_{t \in \mathbb{R}}\)–invariant, see [4]. This means that we can restrict the above metric dynamical system to the set \(\Omega\) getting a new ergodic metric dynamical system.

Since we consider different fibers of the noise, from now on we show the \(\omega\)–dependence of the operator given by (7) explicitly in the notation \(U(t, \omega, s)\). Next we examine some properties of the evolution operator \(U\) that are needed for establishing the cocycle property.

**Lemma 3.3.** For \(U\) defined by (7) the following properties hold:

1. \(U(t + \tau, \omega, r + \tau) = U(t, \theta_\tau \omega, r), \quad \text{for every } t, \tau, r \in \mathbb{R}^+\).
2. \(U(0, \omega, 0) = \text{id}_V\).
3. \(U(t + \tau, \omega, r) = U(t, \theta_\tau \omega, 0) \circ U(\tau, \omega, r), \quad \text{for every } t, \tau, r \in \mathbb{R}^+\).

**Proof.** It is straightforward but we present the proof for the sake of completeness.

1. The first property follows by the definition of the Wiener shift:

\[
U(t + \tau, \omega, r + \tau) = S_B\left(\omega(t + \tau) - \omega(r + \tau)\right)S_A(t + \tau - r - \tau) = S_B\left(\theta_\tau \omega(t) - \theta_\tau \omega(r)\right)S_A(t - r) = U(t, \theta_\tau \omega, r).
\]

2. This is an immediate consequence of the properties of the semigroups and groups: \(U(0, \omega, 0) = S_B(0)S_A(0) = \text{id}_V\).
3. Finally
\[
U(t + \tau, \omega, r) = S_B(\omega(\tau + \tau) - \omega(r))S_A(t + \tau) - r
\]
\[
= S_B(\theta_\tau \omega(t) - \theta_\tau \omega(0) + \omega(\tau) - \omega(r))S_A(t + \tau - r)
\]
\[
= S_B(\theta_\tau \omega(t) - \theta_\tau \omega(0))S_B(\omega(\tau) - \omega(r))S_A(t + \tau - r)
\]
\[
= S_B(\theta_\tau \omega(t) - \theta_\tau \omega(0))S_A(t)S_B(\omega(\tau) - \omega(r))S_A(t + \tau - r)
\]
\[
= U(t, \theta_\tau \omega, 0) \circ U(\tau, \omega, r).
\]

In the penultimate step the commutativity of the group and the semigroup (5) is necessary.

\[\square\]

**Theorem 3.4.** Under the conditions of Theorem 2.4, for every \( u_0 \in V \) the unique mild solution \( u \) of (1) generates a random dynamical system \( \varphi : \mathbb{R}^+ \times \Omega \times V \rightarrow V \) defined by

\[
\varphi(t, \omega, u_0) = U(t, \omega, 0)u_0 + \int_0^t U(t, \omega, r)F(u(r))dr.
\]

**Proof.** In order to prove that \( \varphi \) is a cocycle we will make use of Lemma 3.3. Then, for \( t, \tau \in \mathbb{R}^+ \) and \( \omega \in \Omega \),

\[
\varphi(t + \tau, \omega, u_0) = U(t + \tau, \omega, 0)u_0 + \int_0^{t+\tau} U(t + \tau, \omega, r)F(u(r))dr
\]
\[
= U(t, \theta_{\tau} \omega, 0) \left( U(\tau, \omega, 0)u_0 + \int_0^{\tau} U(\tau, \omega, r)F(u(r))dr \right)
\]
\[
+ \int_{\tau}^{t+\tau} U(t + \tau, \omega, r + \tau)F(u(r + \tau))dr.
\]

Therefore, setting \( y(\cdot) = u(\cdot + \tau) \) on \([0, t]\), on account of the property (1) of Lemma 3.3,

\[
\varphi(t + \tau, \omega, u_0) = U(t, \theta_{\tau} \omega, 0)y(0) + \int_0^t U(t, \theta_{\tau} \omega, r)F(y(r))dr
\]
\[
= \varphi(t, \theta_{\tau} \omega, \varphi(\tau, \omega, u_0)).
\]

On the other hand, trivially \( \varphi(0, \omega, u_0) = u_0 \).

It remains to establish the proper measurability conditions for the mapping \( \varphi \), that is, the \( (\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}(V), \mathcal{B}(V)) \) measurability. It is easy to see that \( \varphi(t, \omega, u_0) \) is continuous in \( (t, u_0) \) when \( \omega \) is fixed and that \( \varphi(t, \omega, u_0) \) is measurable with respect to \( \omega \). These last two considerations together with Lemma III.14 in [3] allows to claim that \( \varphi(t, \omega, u_0) \) is jointly measurable. \[\square\]

4. **Existence of random attractors.** In this section we are going to study the long time behavior of the random dynamical system defined in Theorem 3.4, by showing that it possesses a random attractor. First of all, in order to deal with random attractors we have to introduce some random variables. In the sequel, let \( (\Omega, \mathcal{F}, \mathbb{P}, \theta) \) be an abstract ergodic metric dynamical system. A random variable \( X \) in \( V \) is called tempered if

\[
\lim_{t \to \pm \infty} \frac{\| \log^+ X(\theta\omega) \|_V}{|t|} = 0
\]
or equivalently if $t \mapsto \|X(\theta_t \omega)\|_V$ has a sub–exponential growth for $t \to \pm \infty$. In other words, for $\epsilon > 0$ and $\omega \in \Omega$ there exists a $t_0(\epsilon, \omega) \geq 0$ such that for $|t| \geq t_0(\epsilon, \omega)$ it holds

$$\|X(\theta_t \omega)\|_V \leq e^{\epsilon t}.$$  

Let $\omega \to D(\omega)$ be a set–valued mapping from $\Omega$ into the space of non–empty closed subsets from $V$. This mapping is called a random set if for any $y \in V$ the mapping

$$\Omega \ni \omega \mapsto \inf_{x \in D(\omega)} \|x - y\|_V$$

is a random variable. A random set $D$ is called tempered if the random variable

$$\Omega \ni \omega \mapsto \sup_{x \in D(\omega)} \|x\|_V$$

is tempered. In particular, we denote by $D$ the subset of all tempered sets such that the convergence relation

$$\lim_{t \to \pm \infty} \frac{\log^+ \sup_{x \in D(\theta_t \omega)} \|x\|_V}{|t|} = 0$$

holds for all $\omega \in \Omega$.

**Definition 4.1.** A random set $A = \{A(\omega)\}_{\omega \in \Omega} \subset D$ is called a random attractor for the random dynamical system $\varphi$ given in Theorem 2.4 if, for any $\omega \in \Omega$, $A(\omega)$ is compact, $A(\omega)$ is invariant in the sense that

$$\varphi(t, \omega, A(\omega)) = A(\theta_t \omega), \quad \text{for all } \omega \in \Omega, \ t \geq 0,$$

and morever satisfies the pullback attractiveness property

$$\lim_{t \to \infty} \text{dist}_V(\varphi(t, \omega, D(\theta_{-t} \omega)), A(\omega)) = 0, \quad \text{for all } D \in D, \ \omega \in \Omega,$$

where $\text{dist}_V$ denotes the Hausdorff semidistance in $V$.

The following conditions ensure the existence of a random attractor, see [8], [24] or [25].

**Theorem 4.2.** Let $\varphi$ be a continuous random dynamical system. Suppose that $\varphi$ has a pullback $D$–absorbing set $C \in D$, that is, for any $D \in D$ and $\omega \in \Omega$ there exists a $t_0 = t_D(\omega)$ such that

$$\varphi(t, \omega, D(\theta_{-t} \omega)) \subset C(\omega) \quad \text{for all } t \geq t_0.$$

In addition, suppose that $C(\omega)$ is compact. Then the random dynamical system $\varphi$ has a random attractor which is unique in $D$, given for any $\omega \in \Omega$ by

$$A(\omega) := \bigcap_{s \geq 0} \bigcup_{t \geq s} \varphi(t, \theta_{-t} \omega, C(\theta_{-t} \omega))^\vee.$$

According to the previous theorem, we need to find a compact absorbing set. In the sequel, we are going to assume that the unbounded nonlinear mapping $F$ can be rewritten as $F := aI + G$, where $a \in \mathbb{R}$ and $G : V \to V$ is a Lipschitz continuous function that is bounded as well. We will denote by $C_G$ a bound of $G$. Hence, we can rewrite system (1) as

$$du(t) = (\hat{A}u(t) + G(u(t)))dt + Bu(t) \circ d\omega(t), \quad u(0) = u_0 \in V.$$  


with $\hat{A} := aI + A$. This operator generates the analytic semigroup\( S_\hat{A}(t) = e^{at}S_A(t) \) for \( t \geq 0 \). Then, on account of the results of Section 2, the equation (15) has a unique mild solution given by

\[
u(t) = \hat{U}(t,0)u_0 + \int_0^t \hat{U}(t,r)G(u(r))dr,
\]

where \( \hat{U}(t,s) = S_B(\omega(t) - \omega(s))S_\hat{A}(t - s) = e^{a(t-s)}U(t,s) \). We can easily check that the operator \( \hat{U} \) satisfies the properties of Lemma 3.3, and therefore, the mild solution of (15) generates a random dynamical system that we also denote by \( \varphi \), since it does not lead to any confusion.

Assume now that for given \( M_A \geq 1 \) and \( \lambda \in \mathbb{R} \) we have that

\[
\|S_\hat{A}(t)\|_{L(V)} \leq M_Ae^{\lambda t}, \quad t \geq 0.
\]

Also, let \( M_B \geq 1 \) and \( \mu \in \mathbb{R} \) be such that

\[
\|S_B(0)\|_{L(V)} \leq M_Be^{\mu t}, \quad t \in \mathbb{R},
\]

(see e.g. [22]). Note that the constants \( M_B \) and \( \mu \) will not enter our statement on existence of the attractor and we may take them arbitrary while the constant \( \lambda \) in (16) enters an important stability condition.

**Lemma 4.3.** Assume that the stability condition \( a + \lambda < 0 \) is satisfied, and let \( D = \{D(\omega)\}_{\omega \in \Omega} \) be the family of tempered sets in \( V \). Under the above assumptions, the ball \( B_{V}(\omega) := B(0,R(\omega)) \), with

\[
R(\omega) = 2CGMA_M \int_{-\infty}^{0} e^{\mu(\omega(r)-(a+\lambda)r)}dr
\]

is a \( D \)-absorbing set.

**Proof.** We start by obtaining an a priori estimate of our solution in \( V \), assuming that the fiber is given for \( \theta_{-t}\omega \) for \( t \geq 0 \). Note that

\[
\|\varphi(t,\theta_{-t}\omega,u_0)\|_V
\]

\[
= \|S_B(\theta_{-t}\omega(t))S_\hat{A}(\theta_{-t}\omega(t))u_0 + \int_0^t S_B(\theta_{-t}\omega(t) - \theta_{-t}\omega(r))S_\hat{A}(t - r)G(u(r))dr\|_V
\]

\[
\leq M_AM_Be^{\mu(\theta_{-t}\omega(t) + (a+\lambda)t)}\|u_0\|_V + CGMA_M \int_{-\infty}^{0} e^{\mu(\theta_{-t}\omega(t) - \theta_{-t}\omega(r)) + (a+\lambda)(t-r)}dr
\]

\[
= M_AM_Be^{\mu(-\omega(-t) + (a+\lambda)t)}\|u_0\|_V + CGMA_M \int_{-\infty}^{0} e^{\mu(-\omega(r) - (a+\lambda)r)}dr.
\]

From this a priori estimate we shall derive an absorbing set. If we define \( R(\omega) \) by (17), take \( D \in D \) and replace \( \|u_0\|_V \) by \( \sup_{u_0 \in D(\theta_{-t}\omega)} \|u_0\|_V \) in the previous expression, then we get

\[
\|\varphi(t,\theta_{-t}\omega,u_0)\|_V \leq M_AM_Be^{\mu(-\omega(-t) + (a+\lambda)t)} \sup_{u_0 \in D(\theta_{-t}\omega)} \|u_0\|_V + R(\omega).
\]

From the law of iterated logarithm for the fractional Brownian motion (see for instance [17], Theorem 7.2.15), \( \omega(t) = o(|t|) \) for \( t \to \infty \). Combining this property with the assumption \( a + \lambda < 0 \), we get that for \( u_0 \in D(\theta_{-t}\omega) \) there exists \( t_0 = t_D(\omega) \) such that

\[
\|\varphi(t,\theta_{-t}\omega,u_0)\|_V \leq R(\omega), \quad t \geq t_0.
\]
In fact, given \( \epsilon > 0 \) small enough such that \( \epsilon \mu + (a + \lambda) < 0 \),

\[
\lim_{t \to \infty} e^{\epsilon|\omega(t)|+(\alpha+\lambda)t} \sup_{u_0 \in D(\theta,\omega)} \|u_0\|_V \leq \lim_{t \to \infty} e^{\epsilon(u+\alpha+\lambda)t} \sup_{u_0 \in D(\theta,\omega)} \|u_0\|_V = 0.
\]

It remains to prove that \( B_V(\omega) \) is tempered. Choose \( 0 < \kappa < -(\alpha + \lambda) \), then

\[
\lim_{t \to -\infty} e^{-\kappa|t|} \int_{-\infty}^0 e^{\mu|\theta_t\omega(r)|-(\alpha + \lambda)r} dr
\]

\[
= \lim_{t \to -\infty} e^{\epsilon t} \int_{-\infty}^0 e^{-\frac{\mu}{2}r} e^{-\frac{\lambda}{2}r} e^{\epsilon\tau t} e^{\mu|\omega(t+r) - \omega(t)|} dr
\]

\[
\leq \lim_{t \to -\infty} e^{\epsilon t} \int_{-\infty}^0 e^{-\frac{\mu}{2}r} e^{\epsilon\tau t + \mu|\omega(t+r)|} e^{\epsilon\tau t + \mu|\omega(t)|} dr
\]

\[
\leq \sup_{\tau \in (-\infty,0]} e^{2(\frac{\mu}{2}\tau + \mu|\omega(\tau)|)} \lim_{t \to -\infty} e^{\epsilon t} \int_{-\infty}^0 e^{-\frac{\mu}{2}r} dr = 0.
\]

The above supremum is finite due to the law of iterated logarithm. Finally, thanks to [1], Theorem 4.1.3, we can prove the previous convergence when \( t \to \infty \), hence \( B_V(\omega) \) is a tempered ball.

In the rest of the present section we additionally assume

(C) \( D(A^\gamma) \subset D(B) \) for some \( \gamma \in (0,1) \) and \( (\beta \delta I - A)^{-1} \) is a compact operator.

**Lemma 4.4.** Given \( \delta \in (0,1), \beta \in (0,H), 0 < \epsilon < T \) and \( R > 0 \), there exists a random constant \( C = C(\omega) > 0 \) such that

\[
\|u\|_{\beta,\epsilon,T} \leq C(\omega)
\]

and

\[
\|A^\delta u(t)\|_V \leq C(\omega)
\]

hold for each \( t \in [\epsilon,T] \) and \( u_0 \in V \) such that \( \|u_0\|_V \leq R \).

**Proof.** In the sequel, set

\[
h_1 = \min_{r \in [0,T]} \\{ \omega(r) \}, \quad h_2 = \max_{r \in [0,T]} \\{ \omega(r) \}.
\]

We consider the expression of \( u \) given by (10). To prove (18), note at first that for \( \epsilon \leq s < t \leq T \) we have

\[
\|U(t,0)u_0 - U(s,0)u_0\|_V = \|\left(S_B(\omega(t))S_A(t) - S_B(\omega(s))S_A(s)\right)u_0\|_V
\]

\[
\leq \|S_B(\omega(t))\|_L(V) \|\left(S_A(t) - S_A(s)\right)u_0\|_V
\]

\[
+ \|\left(S_B(\omega(t)) - S_B(\omega(s))\right)S_A(s)u_0\|_V
\]

\[
\leq \left\| \frac{\epsilon}{\epsilon^\beta} \max_{r \in [h_1,h_2]} \left\{ \|S_B(r)\|_{L(V)} \right\} (t - s)^\beta \right\|_V
\]

\[
+ \left\| \int_{\omega(s)}^{\omega(t)} S_B(r)BS_A(s)u_0dr \right\|_V,
\]

where we used (6) and the fact that \( S_A(s)u_0 \in D(A) \subset D(B) \) for \( s > 0 \).

It follows that

\[
\left\| \int_{\omega(s)}^{\omega(t)} S_B(r)BS_A(s)u_0dr \right\|_V
\]

\[
\leq |\omega(t) - \omega(s)| \max_{r \in [h_1,h_2]} \left\{ \|S_B(r)\|_{L(V)} \right\} \|B\|_{L(D(A),V)} \frac{\epsilon\|u_0\|_V}{\epsilon}
\]

(21)
because \( s \geq \varepsilon \) and \( B(\beta_0 I - A)^{-1} \in L(V) \) by the closed graph theorem. By the Hölder continuity of \( \omega \), (20) and (21) yield
\[
\|U(t, 0)u_0 - U(s, 0)u_0\|_V \leq c_1(\omega)(t - s)^\beta. \tag{22}
\]
Furthermore, for \( \varepsilon \leq s < t \leq T \) we have
\[
\left\| \int_0^t U(t, r)F(u(r))dr - \int_0^s U(s, r)F(u(r))dr \right\|_V \\
\leq \int_s^t \|U(t, r)F(u(r))\|_V dr + \int_0^s \|(U(t, r) - U(s, r))F(u(r))\|_V dr =: I_1 + I_2.
\]
In view of (12)
\[
I_1 \leq \int_s^t \|s_B(\omega(r))\|_{L(V)}\|s_A(r)\|_{L(V)}(\|F(0)\|_V + L(C_1(\omega) + C_2(\omega)\|u_0\|_V))dr \\
\leq c_2(\omega)(t - s).
\]
Moreover,
\[
I_2 \leq \int_0^s \|s_B(\omega(t) - \omega(r))(s_A(t - r) - s_A(s - r))F(u(r))\|_V dr \\
+ \int_0^s \|(s_B(\omega(t) - \omega(r)) - s_B(\omega(s) - \omega(r)))s_A(s - r)F(u(r))\|_V dr \\
=: I_{21} + I_{22}.
\]
Using again (6) and (12)
\[
I_{21} \leq \max_{r \in [h_1 - h_2, h_2 - h_1]} \{\|s_B(r)\|_{L(V)}\} \int_0^s \frac{c(t - s)^\beta}{(s - r)^\beta} \\
\times (\|F(0)\|_V + L(C_1(\omega) + C_2(\omega)\|u_0\|_V))dr \leq c_3(\omega)(t - s)^\beta.
\]
Finally, we have
\[
I_{22} \leq \int_0^s \left\| \int_{\omega(s) - \omega(r)}^{\omega(t) - \omega(r)} s_B(\tau)BS_A(s - r)F(u(r))d\tau \right\|_V dr.
\]
In virtue of (C) we obtain
\[
I_{22} \leq \int_0^s \left\| \int_{\omega(s) - \omega(r)}^{\omega(t) - \omega(r)} s_B(\tau)\|s_B(\tau)\|_{L(V)}\|B\|_{L(D(A^\gamma), V)} \frac{c}{(s - r)^\gamma} \\
\times (\|F(0)\|_V + L(C_1(\omega) + C_2(\omega)\|u_0\|_V))d\tau \right\|_V dr \\
\leq \int_0^s \frac{c_4(\omega)}{(s - r)^\gamma} |\omega(t) - \omega(s)|dr \leq c_5(\omega)(t - s)^\beta
\]
by the Hölder continuity of \( \omega \). By the above estimates on \( I_1, I_{21} \) and \( I_{22} \) we get
\[
\left\| \int_0^t U(t, r)F(u(r))dr - \int_0^s U(s, r)F(u(r))dr \right\|_V \leq c_6(\omega)(t - s)^\beta,
\]
for \( \varepsilon \leq s < t \leq T \), which together with (22) completes the proof of (18).

To prove (19), for a given \( \omega \in \Omega \) by (5) and using analyticity of \( S_A \) we have that
\[
\|A^6 U(t, 0)u_0\|_V = \|A^6 S_A(t)S_B(\omega(t))u_0\|_V \\
\leq \frac{c}{c^6} \max_{r \in [h_1, h_2]} \{\|s_B(r)\|_{L(V)}\}\|u_0\|_V \leq c_7(\omega). \tag{23}
\]
Furthermore, by (9) and (12)
\[
\left\| A^\delta \int_0^t U(t,s)F(u(s))ds \right\|_V \\
\leq \int_0^t \left\| A^\delta S_A(t-s) \right\|_{\mathcal{L}(V)} \left\| S_B(\omega(t) - \omega(s)) \right\|_{\mathcal{L}(V)} \left( \| F(0) \|_V + L \| u(s) \|_V \right) ds \\
\leq \left( \| F(0) \|_V + L(C_1(s) + C_2(s)\| u_0 \|_V) \right) \max_{r \in [h_1-h_2+h_1]} \{ \| S_B(r) \|_{\mathcal{L}(V)} \} \\
\times \int_0^t \frac{c}{(t-s)^\theta} ds \leq c_\delta(\omega),
\]
which together with (23) implies (19).

**Lemma 4.5.** Under the conditions of Lemma 4.3, there exists a family of compact absorbing sets \( C = \{ C(\omega) \}_{\omega \in \Omega} \in \mathcal{D} \) for the cocycle \( \varphi \).

**Proof.** This result relies upon the fact that it is possible to apply the Arzelà–Ascoli Theorem. Let us take \( u_0 \in S \subset V \), where \( S \) is bounded. By Lemma 4.4, the set of solutions \( u \) with initial conditions \( u_0 \in S \) is Hölder–continuous on \([\varepsilon, T]\), and the set \( \| A^\delta u(t) \|_V \) is uniformly bounded for some \( \delta \in (0,1) \), for \( t \in [\varepsilon, T] \). The former property implies the equicontinuity of the solution. Note that because \( (\beta_1 I - A)^{-1} \) is a compact operator, the compactness of embeddings of the domain \( \mathcal{D}(A^\delta) \) in \( V \) follows. Therefore, the set of these functions is relatively compact in \( C([\varepsilon, T];V) \) by the Arzelà–Ascoli Theorem. As a result, we can consider
\[
C(\omega) = \overline{\varphi(t,\theta_{-t}\omega,B(\theta_{-t}\omega))}^V
\]
which is compact. Moreover, since \( B \in \mathcal{D}, B \) absorbs itself, and as consequence \( C(\omega) \subset B(\omega) \) which implies \( C \in \mathcal{D} \). Finally, \( C \) is absorbing because of
\[
\varphi(s,\theta_{-s-t}\omega,D(\theta_{-s-t}\omega)) \subset B(\theta_{-t}\omega)
\]
for \( s \geq t_D(\theta_{-t}\omega) \) and \( D \in \mathcal{D} \).

In virtue of Theorem 4.2 and the previous results, we obtain the existence and uniqueness of an attractor for the equation (15):

**Corollary 1.** Under the condition (C) and the stability condition \( a + \lambda < 0 \), the random dynamical system \( \varphi \) has a unique random attractor \( \mathcal{A} = \{ A(\omega) \}_{\omega \in \Omega} \subset \mathcal{D} \).

5. **Examples.** In the last section we give some examples of stochastic partial differential equations to illustrate the obtained results.

**Example 1.** Consider the following stochastic semilinear parabolic equation of 2th order
\[
\frac{\partial u}{\partial t}(t,x) = L(x)u(t,x) + f(u(t,x)) + bu(t,x)\frac{d\omega}{dt}, \\
u(0,x) = x_0(x), \quad x \in \Omega,
\]
\[
(D^\alpha u)(t,x) = 0, \quad (t,x) \in [0,T] \times \partial \Omega, \quad |\alpha| \in \{0,1,\ldots,k-1\},\tag{24}
\]
where \( k \in \mathbb{N}, \Omega \subset \mathbb{R}^d \) is a bounded domain with the boundary of class \( C^k \), \( b \in \mathbb{R} \setminus \{0\} \) and
\[
L(x) = \sum_{|\alpha| \leq 2k} a_\alpha(x)D^\alpha
\]
is a strongly elliptic operator on $\mathcal{O}$, i.e. there exists a constant $\vartheta > 0$ such that
\[
(-1)^k \sum_{|\alpha|=2k} a_\alpha(x) \zeta^\alpha > \vartheta \|\zeta\|_{\mathbb{R}^d}^{2k}
\]
for all $x \in \mathcal{O}$ and $0 \neq \zeta = (\zeta_1, \ldots, \zeta_d) \in \mathbb{R}^d$, where $\zeta^\alpha = \zeta_1^{\alpha_1} \cdots \zeta_d^{\alpha_d}$ for $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $a_\alpha \in C^{2k}(\mathcal{O})$.

Suppose moreover that $f$ is Lipschitz continuous.

Equation (24) can be rewritten in the form of (1)
\[
dv(t) = (Av(t) + F(v(t)))dt + Bv(t) \circ d\omega(t), \ v(0) = x_0 \in V,
\]
for $t \in [0, T]$, where $V = L^2(\mathcal{O})$, $v(t) = u(t, x)$,
\[
(Av(t))(x) = L(x)u(t, x),
\]
where $D(A) = D = H^{2k}(\mathcal{O}) \cap H_0^k(\mathcal{O})$, $F(x) = f(x(\xi))$, $x \in V$, $\xi \in \mathcal{O}$ and $B = bI \in \mathcal{L}(V)$.

Since $A$ is an infinitesimal generator of an analytic semigroup on $V$ (see e.g. [22], Theorem 7.2.7), $B$ generates a strongly continuous group on $V$ which trivially commutes with $A$ on $D$ and $F$ is Lipschitz continuous on $V$, there exists a unique mild solution (in the space $C([0, T]; V)$) to the equation (25) by Theorem 2.4 which is also a weak one by Theorem 2.5.

Assume now that the nonlinear term takes the form $f(x) = ax + g(x)$, where $a \in \mathbb{R}$ and $g$ is bounded and Lipschitz continuous. The operator $(\beta_0 I - A)^{-1}$ is compact on $V$ by the Sobolev embedding theorem, so the condition (C) is satisfied. Therefore, if the semigroup generated by $A + aI$ is exponentially stable, the random dynamical system defined by the equation (24) has a random attractor by the results of Section 4.

As an example of this setting a heat equation
\[
\frac{\partial u}{\partial t}(t, x) = \Delta u(t, x) + f(u(t, x)) + bu(t, x) \frac{d\omega}{dt},
\]
\[
u(0, x) = x_0(x), \ x \in \mathcal{O},
\]
\[
u(t, x) = 0, \ (t, x) \in [0, T] \times \partial \mathcal{O},
\]
in a bounded domain $\mathcal{O} \subset \mathbb{R}^3$ with a smooth boundary is given, in which case the stability condition reads $\lambda_0 + a < 0$, where $\lambda_0 < 0$ is the first eigenvalue of Dirichlet Laplacian on the domain $\mathcal{O}$.

**Example 2.** In the second example consider the stochastic parabolic equation of the second order
\[
\frac{\partial u}{\partial t}(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x) + f(u(t, x)) + b \frac{\partial u}{\partial x}(t, x) \frac{d\omega}{dt},
\]
\[
u(0, x) = x_0(x), \ x \in \mathbb{R},
\]
where $f$ is Lipschitz continuous and $b \in \mathbb{R} \setminus \{0\}$.

Equation (26) can be rewritten in the form
\[
dv(t) = (Av(t) + F(v(t)))dt + Bv(t) \circ d\omega(t), \ v(0) = x_0 \in V,
\]
for $t \in [0, T]$, where $V = L^2(\mathbb{R})$, $v(t) = u(t, x)$, $F$ is the superposition operator as in the previous example and
\[
A = \frac{\partial^2}{\partial x^2}, \quad B = b \frac{\partial}{\partial x},
\]
with $D(A) = D = H^2(\mathbb{R})$ and $D(B) = H^1(\mathbb{R}) \supset D$, respectively. As in the previous example, the operator $A$ generates an analytic semigroup on $V$ and $B$ generates a strongly continuous group

$$
(S_B(t)v)(x) = v(x + bt), \quad x, t \in \mathbb{R},
$$

which commutes with $A$ on $H^2(\mathbb{R})$. The other assumptions of Theorems 2.4 and 2.5 are trivially satisfied, therefore the existence of a mild and weak solution to the equation (27) follows. Note, however, that the compactness of resolvent (assumed in the condition (C)) is not satisfied in the present case, so the results of Section 4 on random attractor are not applicable. It can be rectified if we consider this problem in weighted Sobolev spaces $L^2_\rho(\mathbb{R})$ (which, of course, changes the state space of solutions). For example, taking the weight $\rho(\xi) = (1 + \kappa \xi^2)^{-1}$ where $\kappa > 0$ is a parameter, we obtain the required compactness of resolvent of $A$ by the fact that the fundamental solution of the heat equation is integrable w.r.t. the measure $\rho(\xi) d\xi$. Then the semigroup $S_A$ is compact and analytic (see e.g. [5]). Moreover, we have that

$$
\|S_A(t)\|_{L(V)} \leq \exp \left( \frac{3}{4} \kappa t \right)
$$

for $t > 0$ ([6], Proposition 9.4.5 (ii)), hence the stability condition easily follows for $f(x) = ax + g(x)$, with $g$ bounded and $a < 0$ if we take $\kappa > 0$ sufficiently small.

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