Two-Component Generalization of a Generalized the Short Pulse Equation

Mohammed Allami
Department of Mathematics, College of Education, Misan University, Misan, Iraq

Abstract
In this article, we introduce a two-component generalization for a new generalization type of the short pulse equation was recently found by Hone and his collaborators. The coupled of nonlinear equations is analyzed from the viewpoint of Lie’s method of a continuous group of point transformations. Our results show the symmetries that the system of nonlinear equations can admit, as well as the admitting of the three-dimensional Lie algebra. Moreover, the Lie brackets for the independent vectors field are presented. Similarity reduction for the system is also discussed.

Keywords: Generalized the short pulse equation, Two-component generalization, Lie symmetry analysis, Similarity reduction.

1. Introduction
The short pulse equation appears in many fields of sciences, and it becomes a source of interest in nonlinear optics, among others. The derivation of the equation came out in the differential geometry in the work of [1, 2]. The equation is given by

$$u_{xt} = u + \frac{1}{6} (u^3)_{xx},$$

where $u$ is a function of two independent variables $x$ and $t$, and the subscripts denote to the partial derivatives regarding independent variables. Sakovich [3] brought in a generalized version of equation (1), and later he gave a generalization for the generalized equation [4]. The super extensions of equation (1) are inspected in [5]. Pietrzyk and al. [6] suggested a vector generalization of the short pulse type equation, which is later studied in [7]. Along the same line, Mastsuno [8] came up with a multi-component type of equation (1), and the author also succeeded in getting some type of solutions. Feg [9] also derived an integrable coupled short pulse equations, and the multi-component generalization of modified type of equation (1) that obtained as a reduction of Feg’s system is considered in [10]. The Lax representations of Mastsuno’s and Feg’s systems were examined by Popowicz [11]. More recently,
Hone et al. [12] classified a general form of partial differential equations of order two and they found a new equation that generalizing equation (1). This is given by

\[ u_{xt} = u + (v^2 - 4uv^2)u_x, \]

where \( u \) is a field of one spatial dimension \( x \) plus time \( t \). In the current work, we propose a generalization for equation (2) to two-component system of nonlinear equations, that is

\[ u_{xt} = u + (v^2 - 4uv^2)u_x, \quad v_{xt} = v + (u^2 - 4uv^2)u_x, \]

where \( u \) and \( v \) are two fields of one spatial dimension \( x \) plus time \( t \). Clearly, the system consists of six nonlinear terms \( v_v, v_{xx}, v^2u_{xx}, u_{ux}, u_{ux}v_x \) and \( u^2v_{xx} \), and four linear terms \( u_{xt}, u, v_{xt} \) and \( v \).

As far as we know, the two-component system (3)-(4) that generalizes the generalized type of the short pulse equation that has introduced in the current work seen nowhere in the literature. We shall look at the system, which represents the main problem, in the coming sections. The remaining of the paper is lined up in such a way. In section two, the Lie group analysis is overviewed in order to make the work is self-contained. We proceed, in section three, to analyse the two-component system of equations by Lie’s method of a continuous group of point transformations. Section four is stressed to discuss the similarity reduction of the main problem. The conclusions are given in the last section.

2. Lie Symmetry Analysis

The Lie’s method of a continuous group of point transformations is a well-known approach that has been applied to several types of differential equations. The start point of this approach goes back to Sophus Lie, and from since the method has been noticed and developed. The approach can be briefly summarized, following the same descriptions in [13] and also can be found in [14-18], as follows:

Consider a system of equations

\[ \Delta^n \left( x_i, u^p, u^{p (i_1)}, u^{p (i_2)}, \ldots, u^{p (i_1i_2 \ldots i_k)} \right) = 0, \quad n = 1, 2, 3, \ldots, N \]

where \( x_i, i = 1, 2, 3, \ldots, s \) are independent variables and \( u^p, p = 1, 2, 3, \ldots, r \) are dependent variables, and \( u^{p (i_1)}, \ldots, u^{p (i_1i_2 \ldots i_k)} = \frac{\partial^k u^p}{\partial x_{i_1} \ldots \partial x_{i_k}}, i_m = 1, 2, \ldots, s \) for \( m = 1, 2, \ldots, k \) refer to the partial derivatives regarding independent variables, \( N \) is the number of equations, and \( k \) is the number of the highest derivatives. A general form of group of point transformations [13], reads

\[ \xi^\# = X_i(x_i, u^p; \alpha), \quad (u^p)^\# = U^p(x_i, u^p; \alpha), \]

where the parameter \( \alpha \) is considered to be too small \( \alpha \ll 1 \). The linearization of the Lie group around the identity (\( \alpha = 0 \)) shapes the infinitesimals form of Lie group, this is given by [13],

\[ x_i^\# = X_i(x_i, u^p; \alpha) \approx x_i + \alpha \xi_i(x_i, u^p) + O(\alpha^2), \]

\[ (u^p)^\# = U^p(x_i, u^p; \alpha) \approx u^p + \alpha \eta_p(x_i, u^p) + O(\alpha^2), \]

and the infinitesimals \( \xi_i \) and \( \eta_p \) are therefore taken to be

\[ \frac{dx_i^\#}{\alpha} = \xi_i(x_i, u^p) \quad and \quad \frac{d(u^p)^\#}{\alpha} = \eta_p(x_i, u^p), \]

with the initial conditions

\[ (x_i^\#, (u^p)^\#)|_{\alpha=0} = (x_i, u^p). \]

The infinitesimal generator of (6) is expressed by

\[ Y = \xi_i(x_i, u^p) \frac{\partial}{\partial x_i} + \eta_p(x_i, u^p) \frac{\partial}{\partial u^p}, \]

and the extension of the infinitesimal generator (the Prolongations) to include the derivatives is [13],

\[ \text{Pr}^{(k)}Y = \xi_i(x_i, u^p) \frac{\partial}{\partial x_i} + \eta_p(x_i, u^p) \frac{\partial}{\partial u^p} \eta_p^{(k)}(x_i, u^p, u^{p (i_1)}, u^{p (i_2)}, \ldots, u^{p (i_1i_2 \ldots i_k)}) \frac{\partial}{\partial u^{p (i_1i_2 \ldots i_k)}}, \]

1761
Consider the system of nonlinear equations (3)-(4), that reads
\[
\begin{align*}
\Delta^{(1)} &= \Delta^{(1)}(x, t, u, v) = u_{xt} - (v^2 - 4v^2 u_x) - u = 0, \\
\Delta^{(2)} &= \Delta^{(2)}(x, t, u, v) = v_{xt} - (u^2 - 4u^2 v_x) - v = 0,
\end{align*}
\] (9) (10)
we start off by setting group of infinitesimals for the system above as
\[
\begin{align*}
x^\# &= x + \alpha \xi_1(x, t, u, v), \\
t^\# &= t + \alpha \xi_2(x, t, u, v), \\
u^\# &= u + \alpha \eta_1(x, t, u, v), \\
v^\# &= v + \alpha \eta_2(x, t, u, v),
\end{align*}
\]
where the parameter \( \alpha \ll 1 \), and \( \xi_1, \xi_2, \eta_1 \) and \( \eta_2 \) are functions of \( x, t, u \) and \( v \) are taken to be
\[
\begin{align*}
\frac{dx^\#}{d\alpha} &= \xi_1(x, t, u, v), \\
\frac{dt^\#}{d\alpha} &= \xi_2(x, t, u, v), \\
\frac{du^\#}{d\alpha} &= \eta_1(x, t, u, v), \\
\frac{dv^\#}{d\alpha} &= \eta_2(x, t, u, v),
\end{align*}
\]
subject to the conditions
\[
(x^\#, t^\#, u^\#, v^\#)|_{\alpha=0} = (x, t, u, v),
\]
and the corresponding infinitesimals generator is the linear first order differential operator
\[
Y = \xi_1(x, t, u, v) \partial x + \xi_2(x, t, u, v) \partial t + \eta_1(x, t, u, v) \partial u + \eta_2(x, t, u, v) \partial v
\] (11)
and the extended of transformation to include derivatives is
\[
\begin{align*}
\Pr^{(2)}Y &= Y + \eta_1^{(1)} \partial u_t + \eta_2^{(1)} \partial v_t + \eta_1^{(2)} \partial u_x + \eta_2^{(2)} \partial v_x + \eta_1^{(x)} \partial u_{xx} + \eta_2^{(x)} \partial v_{xx} + \eta_1^{(xx)} \partial u_{xxx} + \eta_2^{(xx)} \partial v_{xxx} + \eta_1^{(x)} \partial u_{xt} + \eta_2^{(x)} \partial v_{xt}.
\end{align*}
\] (12)
From acting the second prolongation (12) on the system of equations (9)-(10), this is written by
\[
\begin{align*}
\Pr^{(2)}Y(\Delta^{(1)}(x, t, u, v)) &= 0 \quad \text{whenever} \quad \Delta^{(1)}(x, t, u, v) = 0, \\
\Pr^{(2)}Y(\Delta^{(2)}(x, t, u, v)) &= 0 \quad \text{whenever} \quad \Delta^{(2)}(x, t, u, v) = 0,
\end{align*}
\]
From acting the second prolongation (12) on the system of equations (9)-(10), this is written by
\[
\begin{align*}
\eta_1^{(x)} - 2n_2 v_x - 2n_1^{(2)} + 4v^2 \eta_1^{(x)} + 8n_2 v_x u_x + 8n_1^{(x)} v_x u_x + 8n_2^{(x)} u_x + 8n_2 v_x \eta_1^{(x)} - \eta_1 = 0, \\
\eta_2^{(x)} - 2n_1 u_x - 2u n_2^{(x)} + 4u^2 \eta_2^{(x)} + 8n_1 u_x v_x + 8n_1^{(x)} u_x v_x + 8n_1 v_x \eta_2^{(x)} - \eta_2 = 0,
\end{align*}
\] (13) (14)
where $\eta_1, \eta_2, \eta_1^{(t)}, \eta_1^{(x)}, \eta_1^{(xx)}, \eta_1^{(xt)}$ and $\eta_2^{(xt)}$ are the coefficients of the second prolongation (12), and they are given by

$$
\eta_1^{(t)} = D_t(\eta_1) - u_x D_t(\xi_1) - u_t D_t(\xi_2),
$$
$$
\eta_2^{(t)} = D_t(\eta_2) - v_x D_t(\xi_1) - v_t D_t(\xi_2),
$$
$$
\eta_1^{(x)} = D_x(\eta_1) - u_x D_x(\xi_1) - u_t D_x(\xi_2),
$$
$$
\eta_2^{(x)} = D_x(\eta_2) - v_x D_x(\xi_1) - v_t D_x(\xi_2),
$$
and

$$
\eta_1^{(R)} = D_x(\eta_1^{(R)}) - u_{Rx} D_x(\xi_1) - u_{Rt} D_x(\xi_2),
$$
$$
\eta_2^{(R)} = D_x(\eta_2^{(R)}) - v_{Rx} D_x(\xi_1) - v_{Rt} D_x(\xi_2),
$$
the R’s are alternated by x’s and t’s, and

$$
D_x = \partial_x + u_x \partial_u + u_{xx} \partial_{u_x} + u_{xt} \partial_{u_t} + \cdots + v_x \partial_v + v_{xx} \partial_{v_x} + u_{xt} \partial_{v_t} + \cdots
$$
and

$$
D_t = \partial_t + u_t \partial_u + u_{tt} \partial_{u_t} + u_{tt} \partial_{u_t} + \cdots + v_t \partial_v + v_{xt} \partial_{v_x} + v_{tt} \partial_{v_t} + \cdots.
$$

Simplifying equations (13)-(14) with the benefit of equations (9)-(10), and due to the tremendous calculations, the computer software Maple 18 and the DesolvII package [19] on a PC with Core i7 Intel 3.6 GHz and 32M RAM is used, to get the following twenty-one overdetermined equations

$$
\xi_{2,x} = \xi_{1,uu} = \xi_{2,v} = 0,
$$
$$
\eta_{1,v} = \eta_{2,u} = \eta_{1,uu} = \eta_{2,vv} = \eta_{2,vx} = 0,
$$
$$
\nu\xi_{2,t} - \nu\xi_{1,t} + \nu\eta_2 = 0,
$$
$$
4u\eta_{2,x} + u\eta_{2,v} - \eta_2 = 0,
$$
$$
8\eta_{2,u} + u^2 \xi_{2,t} - 4u^2 \xi_{1,x} - \xi_{1,t} = 0,
$$
$$
-\eta_2 - \nu\xi_{1,x} - \nu\xi_{1,t} + 4u^2 \eta_{2,vx} + \nu\eta_2 - 2\eta_{1,u} = 0,
$$
$$
\eta_{2,uv} + 8u\eta_{1,x} - 4u^2 \xi_{1,xx} - \xi_{1,xt} = 0,
$$
$$
8\eta_{2,v} + 4u^2 \xi_{2,t} - 4v^2 \xi_{1,x} - \xi_{1,t} = 0,
$$
$$
\eta_{1,uu} + 8u\eta_{1,v} - 4u^2 \xi_{1,xx} - \xi_{1,xt} = 0,
$$
$$
\nu\xi_{2,u} - \nu\xi_{1,t} + \nu\eta_1 = 0,
$$
$$
-\nu\eta_{2,v} + \nu\eta_{1,u} + 4\nu\eta_{1,v} - \nu\xi_{2,t} - \eta_2 = 0,
$$
$$
\eta_{1,uu} + 8u\eta_{2,x} - 4v^2 \xi_{1,xx} - \xi_{1,xt} = 0,
$$
$$
-\eta_1 + \eta_{1,xt} + \nu\eta_{1,u} - \nu\xi_{2,t} + 4v^2 \eta_{1,vx} - 2\nu\eta_{2,u} - \nu\xi_{1,x} = 0,
$$
where $\xi_{1,\sigma} = \frac{\partial \xi_1}{\partial \sigma}, \eta_{1,\sigma} = \frac{\partial \eta_1}{\partial \sigma}, \xi_{2,\sigma\rho} = \frac{\partial^2 \xi_2}{\partial \sigma \partial \rho}$ and $\eta_{2,\sigma\rho} = \frac{\partial^2 \eta_2}{\partial \sigma \partial \rho}$, for $i = 1, 2$ and $\sigma, \rho = t, x, v, u$. Solving then the last equations to gain

$$
\xi_1 = c_1 x + c_2, \quad \xi_2 = -c_1 t + c_3, \quad \eta_1 = c_1 u \quad \text{and} \quad \eta_2 = c_4 v,
$$
where $c_i, i = 1, 2, 3$ are arbitrary constants; that give Lie algebra of three dimensions. The Lie algebra of point symmetry generators is spanned by the three vector fields

$$
Y_1 = x \partial_x - t \partial_t + u \partial_u + v \partial_v, \quad Y_2 = \partial_t, \quad Y_3 = \partial_x,
$$
and the Lie bracket (or Commutator) is given by [20], $[Y_i, Y_j] = Y_i Y_j - Y_j Y_i$, $i, j = 1, 2, 3$. The Commentator of the Lie algebra is listed in Table-1.

| Table 1-The Commentator table of the Lie algebra |
|----------------------|----------------------|----------------------|----------------------|
| $Y_1$ | $Y_2$ | $Y_3$ |
| $Y_1$ | 0 | $-Y_2$ | $Y_3$ |
| $Y_2$ | $Y_2$ | 0 | 0 |
| $Y_3$ | $-Y_3$ | 0 | 0 |

The transformed point that comes from the entries is $\exp(\alpha Y_i)(x, t, u, v) = (x^\alpha, t^\alpha, u^\alpha, v^\alpha)$, and the groups of symmetries of the one parameter for the system (9)-(10) is therefore written by

$$
\Gamma^1_1: (x, t, u, v) \rightarrow (x + \alpha, t, u, v), \quad \text{space translation},
$$
$$
\Gamma^2_1: (x, t, u, v) \rightarrow (x, t + \alpha, u, v), \quad \text{time translation},
$$
$$
\Gamma^3_1: (x, t, u, v) \rightarrow (e^{\alpha x}, e^{-\alpha t}, e^{\alpha u}, e^{\alpha v}), \quad \text{scaling}.
$$

1763
Based on the relations [15], \( \text{Ad}(\exp(\alpha Y_i))Y_j = \sum_{r=0}^{\infty} \frac{\alpha^r}{r!} (ad Y_i)^r Y_j \), and \( ad Y_i |_{Y_j} = [Y_j, Y_i] = -[Y_i, Y_j] \), \( i, j = 1, 2, 3 \). The adjoint construction for the Lie algebra is listed in Table-2.

| Table 2- The adjoint table of the Lie algebra |
|-----------------------------------------------|
| ![Table content](image)|

We are now at the position to state the following, if \( \{u(x, t), v(x, t)\} \) is a solution for the two-components system of equations (3)-(4) then

\[
\begin{align*}
\Gamma^1_1 \cdot u(x, t) &= u^#(x^# - \alpha, t^#), \\
\Gamma^2_1 \cdot v(x, t) &= v^#(x^# - \alpha, t^#), \\
\Gamma^3_1 \cdot u(x, t) &= u^#(e^{-\alpha}x^#, e^\alpha t^#), \\
\Gamma^3_2 \cdot v(x, t) &= v^#(e^{-\alpha}x^#, e^\alpha t^#),
\end{align*}
\]

are also solutions. To clarify that, suppose that \( \{u(x, t) = f(x, t), v(x, t) = g(x, t)\} \) is a solution for the two-component system (3)-(4) then the new solutions \( \{u^#(x^#, t^#), v^#(x^#, t^#)\} \) follow from groups \( \Gamma^i_j, i = 1, 2, 3 \) actions, as follow:

From \( x^# = x + \alpha \) and \( t^# = t \) that gives \( x = x^# - \alpha, t = t^# \) and \( u^#(x^#, t^#) = u(x, t) = f(x, t) = f(x^# - \alpha, t^#) \) and \( v^#(x^#, t^#) = v(x, t) = g(x, t) = g(x^# - \alpha, t^#) \).

From \( x^# = x \) and \( t^# = t + \alpha \) that gives \( x = x^#, t = t^# - \alpha \) and \( u^#(x^#, t^#) = u(x, t) = f(x, t) = f(x^#, t^# - \alpha) \) and \( v^#(x^#, t^#) = v(x, t) = g(x, t) = g(x^#, t^# - \alpha) \).

From \( x^# = e^\alpha x \) and \( t^# = e^{-\alpha} t \) that gives \( x = e^{-\alpha}x^#, t = e^\alpha t^# \) and \( u^#(x^#, t^#) = e^\alpha u(x, t) = e^\alpha f(x, t) = e^\alpha f(e^{-\alpha}x^#, e^\alpha t^#) \) and \( v^#(x^#, t^#) = e^\alpha v(x, t) = e^\alpha g(x, t) = e^\alpha g(e^{-\alpha}x^#, e^\alpha t^#) \), are also solutions satisfy the system of equations (3)-(4).

4. Similarity reduction of the main problem

We focus, in this section, on the reduction of the main problem (3)-(4) relying on similarity variables; these variables come from solving characteristics equations, and as a result a coupled of ordinary differential equations are formed.

In order to gain similarity variables related to the Lie symmetries (15) we solve the characteristics equations

\[
\frac{dx}{u} = \frac{dt}{v} = \frac{du}{\theta} = \frac{dv}{\varphi},
\]

Take \( \frac{dx}{x} = \frac{dt}{-t} \) and solve it to have \( \zeta = xt \). In the same way, one can have \( ut = \theta \) and \( vt = \varphi \), and that implies \( u = \frac{1}{t} \theta(\zeta) \) and \( v = \frac{1}{t} \varphi(\zeta) \). Substituting into the coupled of nonlinear partial differential equations (9)-(10) (or (3)-(4)) one can get the following nonlinear system of equations as a reduction

\[
\begin{align*}
\theta(\zeta) - 2\varphi(\zeta) + 4\psi^2\varphi_\zeta + 8\psi\varphi_\zeta\theta(\zeta) - \delta &= 0, \\
\zeta\varphi_\zeta - 2(\theta_\zeta + 4\psi^2\varphi_\zeta + 8\psi\varphi_\zeta) - \varphi &= 0,
\end{align*}
\]

where \( \theta_\zeta = \frac{d\theta}{d\zeta}, \varphi_\zeta = \frac{d\varphi}{d\zeta} \). That leads to state the following, if \( \{\theta(\zeta), \varphi(\zeta)\} \) is a solution for the nonlinear equations (16)-(17) then \( f(x(t), v(x(t)) \) is a solution for nonlinear equations (3)-(4).

5. Conclusions

To sum up, in the present work, we have proposed two-component generalization of a generalized the short pulse equation. The system of nonlinear equations have introduced here does not appear to have been considered before in the literature. Based on the Lie analysis we have characterized all possible symmetry groups that the two-component system of equations can admit; in terms of the space translation, the time translation and the scaling. The symmetry algebra of the two-component system of nonlinear equations is generated by the three vector fields. The Lie brackets for the vector fields are given. The similarity variable is used to get the reduction of the main problem to the coupled of nonlinear ordinary differential equations. To be clear, the main problem results are settled in sections three and four.
We would like to pinpoint that exact solution for the two-component system is an open problem needs to be explored. Studying the behaviour of solutions in a long and short period of time for the two fields is a good task one can carry on. In addition, the searching for Lax representation for the two-component system also needs to be considered. The integrability of nonlinear equations in terms of Painlevé analysis is an interesting piece of work we intend to examine in the near future.

References
1. Beals R, Rabelo M and Tenenblat K. 1989. Bäcklund transformations and inverse scattering solutions for some pseudospherical surface equations. Studies in Applied Mathematics. 81(2):125-51.
2. Rabelo ML. 1989. On equations which describe pseudospherical surfaces. Studies in Applied Mathematics. 81(3):221-48.
3. Sakovich S. 2016. Transformation and integrability of a generalized short pulse equation. Communications in Nonlinear Science and Numerical Simulation. 39: 21-8.
4. Sakovich S. 2016. Integrability of a generalized short pulse equation revisited. arXiv preprint arXiv:161203105.
5. Brunelli J. 2018. Super extensions of the short pulse equation. Communications in Nonlinear Science and Numerical Simulation. 63: 356-64.
6. Pietrzyk, M, Kanattšikov, I. and Bandelow, U. 2008. On the propagation of vector ultra-short pulses. Journal of Nonlinear Mathematical Physics. 15(2):162-70.
7. Sakovich, S. 2008. Integrability of the vector short pulse equation. Journal of the Physical Society of Japan. 77(12): 123001.
8. Matsuno, Y. 2011. A novel multi-component generalization of the short pulse equation and its multisoliton solutions. Journal of Mathematical Physics. 52(12):123702.
9. Feng, B-F. 2012. An integrable coupled short pulse equation. Journal of Physics A: Mathematical and Theoretical. 45(8): 085202.
10. Matsuno Y. 2016. Integrable multi-component generalization of a modified short pulse equation. Journal of Mathematical Physics. 57(11):111507.
11. Popowicz, Z. 2017. Lax representations for matrix short pulse equations. Journal of Mathematical Physics. 58(10): 103506.
12. Hone, A.N.W., Novikov, V. and Wang, J.P. 2018. Generalizations of the short pulse equation. Letters in Mathematical Physics. 108(4): 927-47.
13. Bluman, G. and Anco, S. 2002. Symmetry and integration methods for differential equations: Springer-Verlag: New York, USA.
14. Bluman, G.W., Cheviakov, A.F. and Anco, S.C. 2009. Applications of symmetry methods to partial differential equations: Springer-Verlag: New York, USA.
15. Olver, P.J. 1993. Applications of Lie groups to differential equations: Springer-Verlag: New York, USA.
16. Sahoo, S. and Ray, S.S. 2017. Lie symmetry analysis and exact solutions of (3+1) dimensional Yu–Toda–Sasa–Fukuyama equation in mathematical physics. Computers & Mathematics with Applications. 73(2): 253-60.
17. Sahoo, S., Garai, G. and Ray, S.S. 2017. Lie symmetry analysis for similarity reduction and exact solutions of modified KdV–Zakharov–Kuznetsov equation. Nonlinear Dynamics. 87(3): 1995-2000.
18. Francesco, O. 2010. Lie symmetries of differential equations: classical and recent contributions, symmetry, 2: 658-706.
19. Vu, K., Jefferson, G. and Carminati, J. 2012. Finding higher symmetries of differential equations using the MAPLE package DESOLVII. Computer Physics Communications. 183(4):1044-54.
20. Hydon, P.E. 2000. Symmetry methods for differential equations: a beginner's guide: Cambridge University Press.