On Monopoles and Domain Walls

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Abstract

The purpose of this paper is to describe a relationship between maximally supersymmetric domain walls and magnetic monopoles. We show that the moduli space of domain walls in non-abelian gauge theories with $N$ flavors is isomorphic to a complex, middle dimensional, submanifold of the moduli space of $U(N)$ magnetic monopoles. This submanifold is defined by the fixed point set of a circle action rotating the monopoles in the plane. To derive this result we present a D-brane construction of domain walls, yielding a description of their dynamics in terms of truncated Nahm equations. The physical explanation for the relationship lies in the fact that domain walls, in the guise of kinks on a vortex string, correspond to magnetic monopoles confined by the Meissner effect.
1 Introduction

Domain walls in gauge theories with eight supercharges have rather special properties. These walls were first studied by Abraham and Townsend [1] who showed that in two-dimensions, where domain walls are known as kinks, they exhibit dyonic behaviour reminiscent of magnetic monopoles. Further similarities between kinks and magnetic monopoles, at both the classical and quantum level, were uncovered in [2]. The physical explanation for this relationship was presented in [3], where new BPS solutions were described corresponding to magnetic monopoles in a phase with fully broken gauge symmetry. The Meissner effect ensures that monopoles are confined: the magnetic flux is unable to propagate through the vacuum and leaves the monopole in two collimated tubes. From the perspective of the flux tube, the monopole appears as a kink. The idea of describing confined monopoles as kinks in $Z_N$ strings occurred previously in [4]. The relationship between the confined magnetic monopoles and the kink was further explored in [5–7] and related systems were studied in [8–13].

In this paper we use D-brane techniques to study the moduli space of multiple domain walls. This allows us to develop a description of the domain wall dynamics in terms of a linearized Nahm equation, providing a direct relationship to the dynamics of

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monopoles. Specifically, we show that the moduli space of domain walls, which we denote as $W_{\vec{g}}$, is isomorphic to a middle dimensional submanifold of the moduli space of magnetic monopoles $M_{\vec{g}}$. This submanifold describes magnetic monopoles lying along a line, and can be described as the fixed point of an $S^1$ action $\hat{k}$, rotating the monopoles in a plane,

$$W_{\vec{g}} \cong M_{\vec{g}}|_{\hat{k}=0} \quad (1.1)$$

The correspondence captures the topology and asymptotic metric of the domain wall moduli space $W_{\vec{g}}$. It does not extend to the full metric on $W_{\vec{g}}$. Nevertheless, as we shall explain, it does correctly capture the most important feature of domain walls: their ordering along the line.

The relationship (1.1) plays companion to the result of [14], where the moduli space of vortices was shown to be a middle dimensional submanifold of the moduli space of instantons. Indeed, upon dimensional reduction, the self-dual instanton equations become the monopole equations, while the vortex equations descend to the domain wall equations.

We start in the following section by describing the domain walls in question, together with a review of their moduli spaces. We pay particular attention to the crudest physical feature of domain walls, namely the rules governing their spatial ordering along the line. Section 3 contains a brief review of magnetic monopoles in higher rank gauge groups, primarily in order to fix notation, allowing us to elaborate on the relationship (1.1). We also describe the Nahm construction of the monopole moduli space as it arises from D-branes. The meat of the paper is in Section 4. We present a D-brane embedding of domain wall solitons which gives a description of their dynamics in terms of a linear Nahm equation. This equation is somewhat trivial, with the content hidden in various boundary conditions. We show how these boundary conditions capture the prescribed ordering of domain walls.

2 Domain Walls

In this paper we will study a class of BPS domain wall solutions occurring in maximally supersymmetric theories with multiple, isolated vacua. The Lagrangian for these models includes a $U(k)$ gauge field $A_\mu$, a real adjoint scalar $\sigma$ and $N$ fundamental scalars $q_i$, each with real mass $m_i$

$$\mathcal{L} = \text{Tr} \left[ \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2e^2} |D_\mu \sigma|^2 + \frac{e^2}{2} (q_i \otimes q_i^\dagger - v^2)^2 \right] + \sum_{i=1}^{N} \left[ |D_\mu q_i|^2 + q_i^\dagger (\sigma - m_i)^2 q_i \right]$$

where there is an implicit sum over the flavor index $i$ in the adjoint valued term $q_i \otimes q_i^\dagger$. This Lagrangian can be embedded in a theory with 8 supercharges in any spacetime
dimension $1 \leq d \leq 5$ (e.g. $\mathcal{N} = 2$ SQCD in $d = 3 + 1$). Such theories include further scalar fields which can be shown to vanish on the domain wall solutions\(^1\). The fermions do contribute zero modes but will not be important here.

When the Higgs expectation value $v^2$ is non-vanishing, and the masses $m_i$ are distinct ($m_i \neq m_j$ for $i \neq j$), the theory has a set of isolated vacua. Each vacuum is labelled by a set $\Xi$ of $k$ distinct elements, chosen from a possible $\mathcal{N} = 1$.

$$\Xi = \{\xi(a) : \xi(a) \neq \xi(b) \text{ for } a \neq b\} \quad (2.1)$$

Here $a = 1, \ldots, k$ runs over the color index, while $\xi(a) \in \{1, \ldots, N\}$. Up to a gauge transformation, the vacuum associated to this set is given by,

$$\sigma = \text{diag}(m_{\xi(1)}, \ldots, m_{\xi(k)}) \quad \text{and} \quad q_i^a = \delta_{i=\xi(a)}^a \quad (2.2)$$

For $N < k$ there are no supersymmetric vacua; for $N \geq k$, the number of vacua is

$$N_{\text{vac}} = \binom{N}{k} = \frac{N!}{k!(N-k)!} \quad (2.3)$$

Each of these vacua is isolated and exhibits a mass gap. There are $k^2$ non-BPS massive gauge bosons and quarks with masses $m^2_\gamma \sim e^2 v^2 + |m_{\xi(a)} - m_{\xi(b)}|^2$ and $k(N-k)$ BPS massive quark fields with masses $m_q \sim |m_{\xi(a)} - m_i|$ (with $i \notin \Xi$).

For vanishing masses $m_i = 0$ the theory enjoys an $SU(N)$ flavor symmetry, rotating the $q_i$. When distinct masses are turned on this is broken explicitly to the Cartan-sub-algebra $U(1)^{N-1}$. Meanwhile, the $U(k)$ gauge group is broken spontaneously in the vacuum by the expectation values (2.2).

The existence of multiple, gapped, isolated vacua is sufficient to guarantee the existence of co-dimension one domain walls (otherwise known as kinks). These walls are BPS objects, satisfying first order Bogomol’nyi equations which can be derived in the usual manner by completing the square. We first choose a flat connection $F_{\mu\nu} = 0$, and allow the fields to depend only on a single coordinate, say $x^3$. Then the Hamiltonian is given by

$$\mathcal{H} = \text{Tr} \left[ \frac{1}{2e^2} |\mathcal{D}_3 \sigma|^2 + \frac{e^2}{2}(q_i \otimes q_i^\dagger - v^2)^2 \right] + \sum_{i=1}^{N} \left[ |\mathcal{D}_3 q_i|^2 + q_i^\dagger (\sigma - m_i)^2 q_i \right]$$

$$= \frac{1}{2e^2} \text{Tr} \left[ \mathcal{D}_3 \sigma - e^2 (q_i \otimes q_i^\dagger - v^2)^2 \right] + \sum_{i=1}^{N} |\mathcal{D}_3 q_i - (\sigma - m_i) q_i|^2$$

$$+ \text{Tr} \left[ \mathcal{D}_3 \sigma (q_i \otimes q_i^\dagger - v^2) \right] + \sum_{i=1}^{N} \left[ q_i^\dagger (\sigma - m_i) \mathcal{D}_3 q_i + \mathcal{D}_3 q_i^\dagger (\sigma - m_i) q_i \right] \quad (2.4)$$

\(^1\)If we promote the scalar field $\sigma$ and the masses $m_i$ to complex variables, then the theories admit an interesting array of domain wall junctions [15] and dyonic walls [16].
Our domain wall interpolates between a vacuum $\Xi^-$ at $x^3 = -\infty$, as determined by a set (2.1), and a distinct vacuum $\Xi^+$ at $x^3 = +\infty$. The minus signs above have been chosen under the assumption that $\Delta m > 0$, where $\Delta m = \sum_{i \in \Xi^-} m_i - \sum_{i \in \Xi^+} m_i$, so that a BPS domain wall satisfies the Bogomoln’yi equations,

$$D_3 \sigma = e^2 (q_i \otimes q_i^\dagger - v^2), \quad D_3 q_i = (\sigma - m_i) q_i$$

and has tension given by $T = v^2 \Delta m$. Analytic solutions to these equations can be found in the $e^2 \to \infty$ limit [17–19], which give smooth approximations to the solution at large, but finite $e^2$ [20].

### 2.1 Classification of Domain Walls

Domain walls in field theories are classified by the choice of vacuum $\Xi^-$ and $\Xi^+$ at left and right infinity. However, our theory contains an exponentially large number of vacua (2.3) and one may hope that there is a coarser, less unwieldy, classification which captures certain relevant properties of a given domain wall. Such a classification was offered in [21].

Firstly define the $N$-vector $\vec{m} = (m_1, \ldots, m_N)$. The tension of the BPS domain wall can then be written as

$$T_{\vec{g}} = v^2 \Delta m \equiv v^2 \vec{m} \cdot \vec{g}$$

where the $N$-vector $\vec{g} \in \Lambda_R(su(N))$, the root lattice of $su(N)$. Note that there do not exist domain wall solutions for all $\vec{g} \in \Lambda_R(su(N))$; the only admissible vectors are of the form $\vec{g} = (p_1, \ldots, p_N)$ with $p_i = 0$ or $\pm 1$. Note also that a choice of $\vec{g}$ does not specify a unique choice of vacua $\Xi^-$ and $\Xi^+$ at left and right infinity. Nor, in fact, does it specify a unique domain wall moduli space $W_{\vec{g}}$. Nevertheless, domain walls in sectors with the same $\vec{g}$ share common traits.

The dimension of the moduli space of domain wall solutions was computed in [21] using an index theorem, following earlier results in [22, 19]. To describe the dimension of the moduli space, it is useful to decompose $\vec{g}$ in terms of simple roots\(^1\) $\vec{\alpha}_i$,

$$\vec{g} = \sum_{i=1}^{N-1} n_i \vec{\alpha}_i, \quad n_i \in \mathbb{Z}$$

\(^1\)The basis of simple roots is fixed by the requirement that $\vec{m} \cdot \vec{\alpha}_i > 0$ for each $i$. A unique basis is defined in this way if $\vec{m}$ lies in a positive Weyl chamber, which occurs whenever the masses are distinct so that $SU(N) \to U(1)^{N-1}$. If we choose the ordering $m_1 > m_2 > \ldots > m_N$ we have simple roots $\vec{\alpha}_1 = (1, -1, 0, \ldots, 0)$ and $\vec{\alpha}_2 = (0, 1, -1, 0, \ldots, 0)$ through to $\vec{\alpha}_{N-1} = (0, \ldots, 1, -1)$. 
The index theorem of [21] reveals that domain wall solutions to (2.5) exist only if \( n_i \geq 0 \) for all \( i \). If this holds, the number of zero modes of a solution is given by

\[
\text{dim} (W_{\vec{g}}) = 2 \sum_{i=1}^{N-1} n_i
\]

(2.8)

where \( W_{\vec{g}} \) denotes the moduli space of any set of domain walls with charge \( \vec{g} \). This result has a simple physical interpretation. There exist \( N - 1 \) types of “elementary” domain walls corresponding to a \( \vec{g} = \vec{\alpha}_i \), the simple roots. Each of these has just two collective coordinates corresponding to a position in the \( x^3 \) direction and a phase. As first explained in [1], the phase coordinate is a Goldstone mode arising because the domain wall configuration breaks the \( U(1)^{N-1} \) flavor symmetry as we review below.

In general, a domain wall labelled by \( \vec{g} \) can be thought to be constructed from \( \sum_i n_i \) elementary domain walls, each with its own position and phase collective coordinate. Let us now turn to some examples.

2.2 The Moduli Space of Domain Walls: Some Examples

**Example 1:** \( \vec{g} = \vec{\alpha}_1 \)

The simplest system admitting a domain wall is the abelian \( k = 1 \) theory with \( N = 2 \) charged scalars \( q_1 \) and \( q_2 \). The \( N_{\text{vac}} = 2 \) vacua of the theory are given by \( \sigma = m_i \) and \( |q_j|^2 = v^2 \delta_{ij} \) for \( i = 1, 2 \). There is a single domain wall in this theory with \( \vec{g} = \vec{\alpha}_1 \) interpolating between these two vacua. Under the \( U(1)_F \) flavor symmetry, \( q_1 \) has charge +1 and \( q_2 \) has charge −1. In each of the vacua, the \( U(1)_F \) symmetry coincides with the \( U(1) \) gauge action but, in the core of the domain wall, both \( q_1 \) and \( q_2 \) are non-vanishing, and \( U(1)_F \) acts non-trivially. The resulting goldstone mode is the phase collective coordinate. The moduli space of the domain wall is simply

\[
W_{\vec{g} = \vec{\alpha}_1} \cong \mathbb{R} \times S^1
\]

(2.9)

where the \( \mathbb{R} \) factor describes the center of mass of the domain wall and the \( S^1 \) the phase. One can show that the \( S^1 \) has radius \( 2\pi v^2/T_{\vec{g}} = 2\pi/(m_1 - m_2) \).

**Example 2:** \( \vec{g} = \vec{\alpha}_1 + \vec{\alpha}_2 \)

The simplest system admitting multiple domain walls is the abelian \( k = 1 \) theory with \( N = 3 \) charged scalars. There are now three vacua, given by \( \sigma = m_i \) and \( |q_j|^2 = v^2 \delta_{ij} \). In
each a $U(1)_{F_1} \times U(1)_{F_2}$ flavor symmetry is unbroken, under which the scalars have charge

$$U(1)_{F_1} \times U(1)_{F_2} : \begin{cases} q_1 : (1,0) \\ q_2 : (-1,1) \\ q_3 : (0,-1) \end{cases}$$

(2.10)

The first elementary domain wall $\vec{g} = \vec{\alpha}_1$ interpolates between vacuum 1 and vacuum 2, breaking $U(1)_{F_1}$ along the way. The second elementary domain wall interpolates between vacuum 2 and vacuum 3, breaking $U(1)_{F_2}$ along the way. Of interest here is the domain wall $\vec{g} = \vec{\alpha}_1 + \vec{\alpha}_2$ interpolating between vacuum 1 and vacuum 3. It can be thought of as a composite of two domain walls. The moduli space for these two domain walls was studied in [17, 23] and is of the form,

$$W_{\vec{g}=\vec{\alpha}_1+\vec{\alpha}_2} \cong \mathbb{R} \times \frac{\mathbb{R} \times \tilde{W}_{\vec{\alpha}_1+\vec{\alpha}_2}}{\mathbb{Z}}$$

(2.11)

The first factor of $\mathbb{R}$ corresponds to the center of mass of the two domain walls; the second factor corresponds to the combined phase associated to the two domain walls. When the ratio of tensions of the two elementary domain walls $T_{\vec{\alpha}_1}/T_{\vec{\alpha}_2}$ is rational, the ratio of the periods of the two phases are similarly rational and the second $\mathbb{R}$ factor collapses to $S^1$, while the quotient $\mathbb{Z}$ reduces to a finite group. The relative moduli space $W_{\vec{\alpha}_1+\vec{\alpha}_2}$ corresponds to the separation and relative phases of the two domain walls. Importantly, and unlike other solitons of higher co-dimension, the domain walls must obey a strict ordering on the $x^3$ line: the $\vec{g} = \vec{\alpha}_1$ domain wall must always be to the left of $\vec{g} = \vec{\alpha}_2$ domain wall. The separation between the walls is therefore the halfline $\mathbb{R}^+$. It was shown in [17] that the relative phase is fibered over $\mathbb{R}^+$ to give rise to a smooth cylinder, with the tip corresponding to coincident domain walls. The resulting moduli space is shown in Figure 1.

Note that the moduli space (2.11) is toric, inheriting two isometries from the $U(1)_{F_1} \times U(1)_{F_2}$ symmetry. In an abelian gauge theory with arbitrary number of flavors $N$, the domain wall charge is always of the form $\vec{g} = \sum_i n_i \vec{\alpha}_i$ with $n_i = 0,1$, and the moduli space is always toric, meaning that half of the dimensions correspond to $U(1)$ isometries.
Example 3: \( \vec{g} = \vec{\alpha}_1 + 2\vec{\alpha}_2 + \vec{\alpha}_3 \)

In non-abelian theories, the domain wall moduli spaces are no longer toric. The simplest such theory has a \( U(2) \) gauge group with \( N = 4 \) fundamental scalars. The 6 vacua, and 15 different domain walls, of this theory were detailed in [21]. Under the \( U(1)^3_F \) flavor symmetry, the fundamental scalars transform as

\[
U(1)_{F_1} \times U(1)_{F_2} \times U(1)_{F_3} : \begin{cases} 
q_1 : (1,0,0) \\
q_2 : (-1,1,0) \\
q_3 : (0,-1,1) \\
q_4 : (0,0,-1)
\end{cases} \tag{2.12}
\]

With this convention, the elementary domain wall \( \vec{g} = \vec{\alpha}_i \) picks up its phase from the action of the \( U(1)_{F_i} \) flavor symmetry.

Here we concentrate on the domain wall system with the maximal number of zero modes which arises from the choice of vacua \( \Xi^- = (1,2) \) and \( \Xi^+ = (3,4) \) so that \( \vec{g} = \vec{\alpha}_1 + 2\vec{\alpha}_2 + \vec{\alpha}_3 \). This system can be separated into four constituent domain walls. As explained in [19, 21], the ordering of domain walls is no longer strictly fixed in this example. The two outer elementary domain walls, on the far left and far right, are each of the type \( \vec{g} = \vec{\alpha}_2 \). However, the relative positions of the middle two domain walls, \( \vec{g} = \vec{\alpha}_1 \) and \( \vec{\alpha}_3 \) are not ordered and they may pass through each other.

Unlike the situation for abelian gauge theories, the 8 dimensional domain wall moduli space for this example is no longer toric; \( W_{\vec{g}} \) inherits only three \( U(1) \) isometries from (2.12). Physically this means that the two phases associated to the \( \vec{\alpha}_2 \) domain walls are not both Goldstone modes and they may interact as the domain walls approach. This behaviour is familiar from the study of the Atiyah-Hitchin metric describing the dynamics of two monopoles in \( SU(2) \) gauge theory; we shall make the analogy more precise in the following.

2.3 The Ordering of Domain Walls

As we stressed in the above examples, in contrast to other solitons domain walls must obey some ordering on the line. This will be an important ingredient when we come to extract domain wall data from the linearized Nahm’s equations in Section 4. Here we linger to review this ordering.

The ordering of the elementary domain walls in non-abelian theories was studied in detail in [19]. One can derive the result by considering the possible sequences of vacua as we move over each domain wall. For example, we could consider the “maximal domain wall”, interpolating between \( \Xi^- = \{1, 2, \ldots, k\} \) and \( \Xi^+ = \{N - k + 1, \ldots, N\} \). From the
left, the first elementary domain wall that we come across must be $\vec{g} = \vec{\alpha}_k$, corresponding to $\Xi_- = (1, 2, \ldots, k-1, k) \rightarrow (1, 2, \ldots, k-1, k+1)$. The next elementary domain wall may be either $\vec{\alpha}_{k-1}$ or $\vec{\alpha}_{k+1}$. These two walls are free to pass through each other, but cannot move further to the left than the $\vec{\alpha}_k$ wall. And so on. Iterating this procedure, one finds that two neighbouring elementary domain walls $\vec{\alpha}_i$ and $\vec{\alpha}_j$ may pass through each other whenever $\vec{\alpha}_i \cdot \vec{\alpha}_j = 0$, but otherwise have a fixed ordering on the line. The net result of this analysis is summarized in Figure 2. The $x^3$ positions of the domain walls are shown on the vertical axis; the position on the horizontal axis denotes the type of elementary domain wall, starting on the left with $\vec{\alpha}_1$ and ending on the right with $\vec{\alpha}_{N-1}$.

In summary, we see that for $n_i = n_{i+1} - 1$, the $\vec{\alpha}_i$ domain walls are trapped between the $\vec{\alpha}_{i+1}$ domain walls. The reverse holds when $n_i = n_{i+1} + 1$. Finally, when $n_i = n_{i+1}$ the positions of the $\vec{\alpha}_i$ domain walls are interlaced with those of the $\vec{\alpha}_{i+1}$ domain walls. The last $\vec{\alpha}_{i+1}$ domain wall is unconstrained by the $\vec{\alpha}_i$ walls in its travel in the positive $x^3$ direction, although it may be trapped in turn by a $\vec{\alpha}_{i+2}$ wall.

While we have discussed the maximal domain wall above, other sectors can be reached either by removing some of the outer domain walls to infinity, or by taking non-interacting products such subsets. It’s important to note that labelling a topological sector $\vec{g}$ does not necessarily determine the ordering of domain walls\(^{1}\). We shall show that domain walls with the same $\vec{g}$, but different orderings, descend from different submanifolds of the same monopole moduli space.

\(^{1}\)An example: the $\vec{g} = \vec{\alpha}_1 + \vec{\alpha}_2 + \vec{\alpha}_3$ domain wall. In the abelian theory with $k = 1$ and $N = 4$, the ordering is $\vec{\alpha}_1 < \vec{\alpha}_2 < \vec{\alpha}_3$. However, in the non-abelian theory with $k = 2$ and $N = 4$, the ordering is $\vec{\alpha}_1, \vec{\alpha}_3 < \vec{\alpha}_2$. 

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Figure 2: The ordering for the maximal domain wall. The $x^3$ spatial direction is shown horizontally. The position in the vertical direction denotes the type of domain wall. Domain walls of neighbouring types have their positions interlaced.
3 Monopoles

The main goal of this paper is to show how the moduli space of domain walls introduced in the previous section is isomorphic to a submanifold of a related monopole moduli space. In this section we review several relevant aspects of these monopole moduli spaces.

It will turn out that the domain walls of Section 2 are related to monopoles in an $SU(N)$ gauge theory. Note that the flavor group from Section 2 has been promoted to a gauge group; we shall see the reason behind this in Section 5. The Bogomol’nyi monopole equations are

$$B_\mu = D_\mu \phi$$

(3.1)

where $B_\mu$, $\mu = 1, 2, 3$ is the $SU(N)$ magnetic field and $\phi$ is an adjoint valued real scalar field. The monopoles exist only if $\phi$ takes a vacuum expectation value,

$$\langle \phi \rangle = \text{diag}(m_1, \ldots, m_N)$$

(3.2)

where we take $m_i \neq m_j$ for $i \neq j$, ensuring breaking to the maximal torus, $SU(N) \to U(1)^{N-1}$. It is not coincidence that we’ve denoted the vacuum expectation values by $m_i$, the same notation used for the masses in Section 2; it is for this choice of vacuum that the correspondence holds. (Specifically, the masses of the kinks will coincide with the masses of monopoles, ensuring that the asymptotic metrics on $\mathcal{W}_{\vec{g}}$ and $\mathcal{M}_{\vec{g}}$ also coincide).

As described long ago by Goddard, Nuyts and Olive [24], the allowed magnetic charges under each unbroken $U(1)^{N-1}$ are specified by a root vector\(^1\) of $su(N)$, $\vec{g} = (p_1, \ldots, p_N)$. It is customary to decompose this in terms of simple roots $\vec{\alpha}_i$,

$$\vec{g} = \sum_{i=1}^{N-1} n_i \vec{\alpha}_i , \quad n_i \in \mathbf{Z}$$

(3.3)

Once again, the notation is identical to that used for domain walls (2.7) for good reason. Solutions to the monopole equations (3.1) exist for all values of $n_i \geq 0$. This is in contrast to domain walls where, as we have seen, configurations only exist in a finite number of sectors defined by $p_i = 0$ or $p_i = \pm 1$. The mass of the magnetic monopole is $M_{\text{mono}} = (2\pi/e^2) \vec{m} \cdot \vec{g}$.

The monopole moduli space $\mathcal{M}_{\vec{g}}$ is the space of solutions to (3.2) in a fixed topological sector $\vec{g}$. The dimension of this space, equal to the number of zero modes of given solution, was computed by E. Weinberg in [25] using Callias’ version of the index theorem. The

\(^1\)We ignore the factor of 2 difference between roots and co-roots. For simply laced groups, such as $SU(N)$, it can be absorbed into convention.
result is:
\[ \dim (\mathcal{M}_{\vec{g}}) = 4 \sum_{i=1}^{N-1} n_i \] (3.4)
which is to be compared with (2.8).

### 3.1 The Relationship between $\mathcal{M}_{\vec{g}}$ and $\mathcal{W}_{\vec{g}}$

We are now in a position to describe the relationship between the moduli space of domain walls $\mathcal{W}_{\vec{g}}$ and the moduli space of magnetic monopoles $\mathcal{M}_{\vec{g}}$. We will show that $\mathcal{W}_{\vec{g}}$ is a complex, middle dimensional, submanifold of $\mathcal{M}_{\vec{g}}$, defined by the fixed point set of the action rotating the monopoles in a plane, together with a suitable gauge action. To do this, we first need to describe the symmetries of $\mathcal{M}_{\vec{g}}$.

The monopole moduli space $\mathcal{M}_{\vec{g}}$ admits a natural, smooth, hyperKähler metric [26, 27]. For generic $\vec{g}$ this metric enjoys $(N - 1)$ tri-holomorphic isometries arising from the action of the $U(1)^{N-1}$ abelian gauge group. Further the metric has an $SU(2)_{R}$ symmetry, arising from rotations of the monopoles in $\mathbb{R}^3$, which acts on the three complex structures of $\mathcal{M}_{\vec{g}}$. In other words, any $U(1)_{R} \subset SU(2)_{R}$ is a holomorphic isometry, preserving a single complex structure while revolving the remaining two. Let us choose $U(1)_{R}$ to rotate the monopoles in the $(x^2 - x^3)$ plane. In what follows we will be interested in a specific holomorphic $\hat{U}(1)$ action which acts simultaneously by a $U(1)_{R}$ rotation and a linear combination of the gauge rotations $U(1)^{N-1}$ (to be specified presently). We denote the Killing vector on $\mathcal{M}_{\vec{g}}$ associated to $\hat{U}(1)$ as $\hat{k}$. We claim
\[ \mathcal{W}_{\vec{g}} = \mathcal{M}_{\vec{g}}|_{\hat{k}=0} \] (3.5)
This result holds at the level of topology and asymptotic metric of the spaces. The manifold $\mathcal{W}_{\vec{g}}$ inherits a metric from $\mathcal{M}_{\vec{g}}$ by this reduction: it does not coincide with the domain wall metric in the interior on $\mathcal{W}_{\vec{g}}$. (For example, corrections to the asymptotic metric on $\mathcal{W}_{\vec{g}}$ are exponentially suppressed while those of $\mathcal{M}_{\vec{g}}$ have power law behaviour). It would be interesting to examine if $\mathcal{W}_{\vec{g}}$ inherits the correct Kähler class and/or complex structure from $\mathcal{M}_{\vec{g}}$.

We defer a derivation of (3.5) to the following section, but first present some simple examples.

**Example 1:** $\vec{g} = \vec{a}_1$

Monopoles in $SU(2)$ gauge theories are labelled by a single topological charge $\vec{g} = n_1 \vec{a}_1$. For a single monopole ($n_1 = 1$) the moduli space is simply
\[ \mathcal{M}_{\vec{g}=\vec{a}_1} \cong \mathbb{R}^3 \times S^1 \] (3.6)
where the $\mathbb{R}^3$ factor denotes the position of the monopole, while the $S^1$ arises from global gauge transformations under the surviving $U(1)$. The radius of the $S^1$ is $2\pi/(m_1 - m_2)$. In this case the $\hat{U}(1)$ action coincides with the rotation $U(1)_R$ in the $(x^2 - x^3)$ plane and we have trivially

$$\mathcal{W}_{\vec{\alpha}_1} \cong \mathbb{R} \times S^1 \cong \mathcal{M}_{\vec{\alpha}_1}\big|_{k=0}$$ (3.7)

The similarity between the domain wall and monopole moduli spaces for a single soliton was noted by Abraham and Townsend [1]. In both cases, motion in the $S^1$ factor gives rise to dyonic solitons.

Note that monopole moduli spaces for charges $\vec{g} = n_1 \vec{\alpha}_1$ exist for all $n_1 \in \mathbb{Z}^+$. For example, the $n_1 = 2$ monopole moduli space is home to the famous Atiyah-Hitchin metric [27]. However, there is no domain wall moduli space with this charge in the class of theories we discuss in Section 2.

**Example 2:** $\vec{g} = \vec{\alpha}_1 + \vec{\alpha}_2$

Our second example is the $\vec{g} = \vec{\alpha}_1 + \vec{\alpha}_2$ monopole in $SU(3)$ gauge theories (sometimes referred to as the $(1, 1)$ monopole). The moduli space was determined in [28–30] to be of the form

$$\mathcal{M}_{\vec{g} = \vec{\alpha}_1 + \vec{\alpha}_2} \cong \mathbb{R}^3 \times \mathbb{R} \times \tilde{\mathcal{M}}_{\vec{\alpha}_1 + \vec{\alpha}_2} / \mathbb{Z}$$ (3.8)

where the relative moduli space $\tilde{\mathcal{M}}_{\vec{\alpha}_1 + \vec{\alpha}_2}$ is the Euclidean Taub-NUT space, endowed with the metric

$$ds^2 = \left(1 + \frac{1}{r}\right) \left(dr^2 + r^2 d\theta^2 + 2 \sin^2 \theta \, d\phi^2\right) + \left(1 + \frac{1}{r}\right)^{-1} (d\psi + \cos \theta d\phi)^2$$ (3.9)

Here $r, \theta$ and $\phi$ are the familiar spherical polar coordinates. The coordinate $\psi$ arises from $U(1)$ gauge transformations. The manifold has a $SU(2)_R \times U(1)$ isometry, of which only a $U(1)_R \times U(1)$ are manifest in the above coordinates. The holomorphic $U(1)_R$ isometry acts by rotating the two monopoles: $\phi \to \phi + c$. The tri-holomorphic $U(1)$ isometry changes the relative phase of the monopoles: $\psi \to \psi + c$. Both of these actions have a unique fixed point at $r = 0$, the “nut” of Taub-NUT. However, the combined action with Killing vector $\partial_\psi + \partial_\phi$ has a fixed point along the half-line $\theta = \pi$, with $\psi$ fibered over this line to produce the cigar shown in Figure 1. This is the relative moduli space $\tilde{\mathcal{W}}_{\vec{\alpha}_1 + \vec{\alpha}_2}$.

Similar calculations hold for monopoles of charge $\vec{g} = \sum_{i=1}^{N-1} \vec{\alpha}_i$, whose dynamics is described by a class of toric hyperKähler metrics, known as the Lee-Weinberg-Yi metrics [31]. Once again, a suitable $S^1$ action on these spaces can be identified such that the fixed points localize on $\mathcal{W}_{\vec{g}}$, the moduli space of domain walls in $U(1)$ gauge theories with $N$ charged scalars.
Example 3: $\vec{g} = 2\vec{\alpha}_1 + \vec{\alpha}_2 + \vec{\alpha}_3$

As described in the previous section, the simplest domain wall charge $\vec{g} = \sum_i n_i \vec{\alpha}_i$ with some $n_i > 1$ occurs for $\vec{g} = \vec{\alpha}_1 + 2\vec{\alpha}_2 + \vec{\alpha}_3$, and corresponds to a monopole in a $SU(4)$ gauge theory. No explicit expression for the metric on this monopole moduli space is known although, given the results of [32], such a computation may be feasible. Without an explicit expression for the metric in this, and more complicated examples, we need a more powerful method to describe the moduli space. This is provided by the Nahm construction, which we now review.

### 3.2 D-Branes and Nahm’s Equations

The moduli space of magnetic monopoles is isomorphic to the moduli space of Nahm data. Here we review the Nahm construction of the monopole moduli space [33] and, in particular, the embedding within the framework of D-branes due to Diaconescu [34]. This will be useful to compare to the domain walls of the next section.

In the D-brane construction, Nahm’s equations arise as the low-energy description of D-strings suspended between D3-branes [34]. The $SU(N)$ Yang-Mills theory lives on the worldvolume of $N$ D3-branes separated in, say, the $x_6$ direction, with the $i^{th}$ D3-brane placed at position $(x_6)_i = m_i$ in accord with the adjoint expectation value (3.2). The monopole of charge $\vec{g} = \sum_i n_i \vec{\alpha}_i$ corresponds to suspending $n_i$ D-strings between the $i^{th}$ and $(i+1)^{th}$ D3-brane. This configuration is shown in Figure 3.

The motion of the D-strings in each segment $m_i \leq x_6 \leq m_{i+1}$ is governed by four hermitian $n_i \times n_i$ matrices, $X_1, X_2, X_3$ and $A_6$ subject to the covariant version of Nahm’s equations,

$$\frac{dX_\mu}{dx_6} - i[A_6, X_\mu] - \frac{i}{2} \epsilon_{\mu\nu\rho}[X_\nu, X_\rho] = 0 \quad m_i \leq x_6 \leq m_{i+1} \quad (3.10)$$
modulo $U(n_i)$ gauge transformations acting on the interval $m_i \leq x_6 \leq m_{i+1}$, and vanishing at the boundaries. The $X_\mu$ form the triplet representation of the $SU(2)_R$ symmetry which rotates monopoles in $\mathbb{R}^3$. The $U(1)^{N-1}$ surviving gauge transformations acts on the Nahm data by constant shifts of the $(N-1)$ “Wilson lines” $A_6 \to A_6 + c_{1_{n_i}}$.

The interactions between neighbouring segments depends on the relative size of the matrices and is given by [35]

\[ n_i = n_{i+1}: \text{ In this case the } U(n_i) \text{ gauge symmetry is extended to the interval } m_i \leq x_6 \leq m_{i+2} \text{ and an impurity is added to the right-hand-side of Nahm’s equations, which now read} \]

\[
\frac{dX_\mu}{dx_6} - i[A_6, X_\mu] - \frac{i}{2} \epsilon_{\mu\nu\rho}[X_\nu, X_\rho] = \omega_\alpha \sigma^{\alpha\beta}_\mu \omega^{\dagger}_\beta \delta(x_6 - m_{i+1}) \quad (3.11)
\]

Here $\sigma_\mu$ are the Pauli matrices. The impurity degrees of freedom lie in the complex 2-vector, $\omega_\alpha = (\psi, \tilde{\psi})^\dagger$ which is a doublet under the $SU(2)_R$ symmetry. Both $\psi$ and $\tilde{\psi}^\dagger$ are themselves complex $n_i$ vectors, transforming in the fundamental representation of the $U(n_i)$ gauge group. The combination $\omega_\alpha \sigma^{\alpha\beta}_\mu \omega^{\dagger}_\beta$ is thus an $n_i \times n_i$ matrix, transforming in the adjoint representation of the gauge group. The $\omega_\alpha$ fields can be thought of as a hypermultiplet arising from $D1 - D3$ strings [36–38]

\[ n_i = n_{i+1} - 1: \text{ In this case } X_\mu \to (X_\mu)_-, \text{ a set of three } n_i \times n_i \text{ matrices, as } x_6 \to (m_i)_- \text{ from the left. To the right of } m_i, \text{ the } X_\mu \text{ are } (n_i + 1) \times (n_i + 1) \text{ matrices which must obey} \]

\[
X_\mu \to \begin{pmatrix} y_\mu & a_\mu^\dagger \\ a_\mu & (X_\mu)_- \end{pmatrix} \quad \text{as } x_6 \to (m_i)_+ \quad (3.12)
\]

where $y_\mu \in \mathbb{R}$ and each $a_\mu$ is a complex $n_i$-vector.

\[ n_i \leq n_{i+1} - 2: \text{ Once again we take } X_\mu \to (X_\mu)_- \text{ as } x_6 \to (m_i)_- \text{ but, from the other side, the matrices } X_\mu \text{ now have a simple pole at the boundary,} \]

\[
X_\mu \to \begin{pmatrix} J_\mu/(x_6 - m_i) + Y_\mu & 0 \\ 0 & (X_\mu)_- \end{pmatrix} \quad \text{as } x_6 \to (m_i)_+ \quad (3.13)
\]

where $J_\mu$ is the irreducible $(n_{i+1} - n_i) \times (n_{i+1} - n_i)$ representation of $su(2)$, and $Y_\mu$ are now constant $(n_{i+1} - n_i) \times (n_{i+1} - n_i)$ matrices.

Case 2 above is usually described as a subset of Case 3 (with the one-dimensional irreducible $su(2)$ representation given by $J_\mu = 0$). Here we have listed Case 2 separately since when we come to describe a similar construction for domain walls, only Case 1 and 2 above will appear. The conditions for $n_i < n_{i+1}$ were derived in [39] by starting with the impurity data (3.11) and taking several monopoles to infinity. Obviously, for $n_i > n_{i+1}$, one imposes the same boundary conditions described above, only flipped in the $x_6$ direction.
Figure 4: The D-brane set-up for the $U(1)$ gauge theory with $N = 3$ flavors. The vacuum is shown on the left; the elementary domain wall $\vec{g} = \vec{\alpha}_1$ on the right.

The space of solutions to Nahm’s equations, subject to the boundary conditions detailed above, is isomorphic to the monopole moduli space $\mathcal{M}_{\vec{g}}$. Moreover, there exists a natural hyperKähler metric on the solutions to Nahm’s equations which can be shown to coincide with the Manton metric on the monopole moduli space. For the $\vec{g} = \vec{\alpha}_1 + \vec{\alpha}_2$ monopole, the metric on the associated Nahm data was computed in [28] and shown to give rise to the Euclidean Taub-NUT metric (3.9). For the $\vec{g} = \sum_i \vec{\alpha}_i$ monopoles, the corresponding computation was performed in [40], resulting in the Lee-Weinberg-Yi metrics [31].

4 D-Branes and Domain Walls

In this section we would like to realize the domain walls that we described in Section 2 on the worldvolume of D-branes, mimicking Diaconescu’s construction for monopoles. From the resulting D-brane set-up we shall read off the world-volume dynamics of the domain walls to find that they are described by a truncated version of Nahm’s equations (3.10). Nahm’s equations have also arisen as a description of domain walls in $\mathcal{N} = 1^*$ theories [41], although the relationship, if any, with the current work is unclear. Domain walls of the type described in Section 2 were previously embedded in D-branes in [42, 43] and several properties of the solitons were extracted (see in particular the latter reference). However, the worldvolume dynamics of the walls is difficult to determine in these set-ups and the relationship to magnetic monopoles obscured.

We start by constructing the theory with eight supercharges on the worldvolume of D-branes [36]. For definiteness we choose to build the $\mathcal{N} = 2$, $d = 3 + 1$ theory in IIA string theory although, by T-duality, we could equivalently work with any spacetime
The construction is well known and is drawn in Figure 4. We suspend \( k \) D4-branes between two NS5-branes, and insert a further \( N \) D6-branes to play the role of the fundamental hypermultiplets. The worldvolume dimensions of the branes are

\[
\begin{align*}
NS5 : & \quad 012345 \\
D4 : & \quad 01236 \\
D6 : & \quad 0123789
\end{align*}
\]

The gauge coupling \( e^2 \) and the Higgs vev \( v^2 \) are encoded in the separation of the NS5-branes in the \( x_6 \) and \( x_9 \) directions respectively, while the masses \( m_i \) are determined by the positions of the D6-branes in the \( x_4 \) direction (we choose the D6-branes to be coincident in the \( x_5 \) direction, corresponding to choosing all masses to be real).

\[
\frac{1}{e^2} \sim \frac{\Delta x_6}{l_s g_s}igg|_{NS5}, \quad v^2 \sim \frac{\Delta x_9}{l_s^3 g_s}igg|_{NS5}, \quad m_i \sim -\frac{x_4}{l_s^2}igg|_{D6_i}
\]

(4.1)

After turning on the Higgs vev \( v^2 \), the D4-branes must split on the D6-branes in order to preserve supersymmetry. The S-rule [36] ensures that each D6-brane may play host to only a single D4-brane. In this manner a vacuum of the theory is chosen by picking \( k \) out of the \( N \) D6-branes on which the D4-branes end, in agreement with equation (2.1).

The domain walls correspond to a configuration of D4-branes which start life at \( x^3 = -\infty \) in a vacuum configuration \( \Xi_{-} \), and end up at \( x^3 = +\infty \) in a distinct vacuum \( \Xi_{+} \). As is clear from Figure 4, as D4-branes walls interpolate in \( x^1 \), they must also move in both the \( x^4 \) direction and the \( x^9 \) direction [45]. The NS-branes and D6-branes are linked, meaning that a D4-brane is either created or destroyed as they pass the NS5-branes in the \( x^6 \) direction [36]. In the domain wall background, which of these possibilities occurs differs if we move the D6-branes to the left or right since the D4-brane charge varies from one end of the domain wall to the other.

As it stands, it is difficult to read off the dynamics of the D4-branes in Figure 4. However, we can make progress by taking the \( e^2 \to \infty \) limit, in which the two NS5-branes become coincident in the \( x^6 \) direction. After rotating our viewpoint, the system of branes now looks like the ladder configuration shown in Figure 5 (note that we have also rotated the branes relative to Figure 4, so the horizontal is the \( x^4 \) direction). We are left with a series of D4-branes, now with worldvolume 02349, stretched between \( N \) D6-branes, while simultaneously sandwiched between two NS5-branes. Following these manoeuvres, one finds that the domain wall \( \vec{g} = \sum_i n_i \vec{\alpha}_i \) results in \( n_i \) D4-branes stretched between the \( i^\text{th} \) and \( (i + 1)^\text{th} \) D6-branes (counting from the top, since we have chosen the ordering \( m_i > m_{i+1} \)).

\footnote{In fact, as explained in [44], the overall \( U(1) \subset U(k) \) is decoupled in the IIA brane set-up after lifting to M-theory. This effect will not concern us here.}
Figure 5: The D-brane set-up for the $U(2)$ gauge theory with $N = 6$ flavors. The maximal $\vec{g} = \vec{\alpha}_1 + \vec{\alpha}_5 + 2(\vec{\alpha}_2 + \vec{\alpha}_3 + \vec{\alpha}_4)$ domain wall is shown.

It may be worth describing how the domain wall charges arise directly in the set-up of Figure 5. We start in a chosen vacuum $\Xi^-$, denoted by placing $k$ pairs of white dots on $N$ distinct D6-branes, as shown in the figure. A domain wall arises every time a pair of dots proceeds to another D6-brane, dragging a D4-brane behind it like clingwrap. The S-rule translates to the fact that two pairs of dots may not simultaneously lie on the same D6-brane. The final vacuum $\Xi^+$ is denoted by the black dots in the figure and the domain wall charges $n_i$ are given by the number of times a D4-brane has been pulled between the $i^{th}$ and $(i+1)^{th}$ D6-branes.

4.1 Domain Wall Dynamics

We are now in a position to read off the dynamics of the domain walls. In the absence of the NS5-branes, the D4-branes would stretch to infinity in the $x_9$ direction, and the resulting D-brane set-up in Figure 5 is T-dual to the monopoles in Figure 3. The presence of the NS5-branes projects out half the degrees of freedom of the monopoles, leaving a simple linear set of equations. In each segment $m_i \leq x_4 \leq m_{i+1}$ the domain walls are described by two $n_i \times n_i$ matrices $X_3$ and $A_4$ satisfying

$$\frac{dX_3}{dx_4} - i[A_4, X_3] = 0 \quad (4.2)$$

modulo $U(n_i)$ gauge transformations acting on the interval $m_i \leq x_4 \leq m_{i+1}$, and vanishing at the boundaries. As in the case of monopoles, the interactions between neighbouring segments depends on the relative size of the matrices:
Again, the $U(n_i)$ gauge symmetry is extended to the interval $m_i \leq x_4 \leq m_{i+2}$ and an impurity is added to the right-hand-side of Nahm’s equations, which now read

$$\frac{dX_3}{dx_4} - i[A_4, X_3] = \pm \psi\psi^\dagger \delta(x_4 - m_{i+1})$$

(4.3)

where the impurity degree of freedom $\psi$ transforms in the fundamental representation of the $U(n_i)$ gauge group, ensuring the combination $\psi\psi^\dagger$ is a $n_i \times n_i$ matrix transforming, like $X_3$, in the adjoint representation. These $\psi$ degrees of freedom are chiral multiplets which survive the NS5-brane projection. We shall see shortly that the choice of $\pm$ sign will dictate the relative ordering of the domain walls along the $x_3$ direction.

In this case $X_3 \to (X_3)_-$, an $n_i \times n_i$ matrix, as $x_4 \to (m_i)_-$ from the left. To the right of $m_i$, $X_3$ is a $(n_i + 1) \times (n_i + 1)$ matrix obeying

$$X_3 \to \begin{pmatrix} y & \alpha \dagger \\ a & (X)_- \end{pmatrix}$$

as $x_4 \to (m_i)_+$

(4.4)

where $y_\mu \in \mathbb{R}$ and each $a_\mu$ is a complex $n_i$-vector. The obvious analog of this boundary condition holds when $n_i = n_{i+1} + 1$.

These boundary conditions obviously descend from the original Nahm boundary conditions for monopoles. Just as the space of Nahm data is isomorphic to the moduli space of magnetic monopoles, we conjecture that the moduli space of linearized Nahm data described above is isomorphic to the moduli space of domain walls. We shall shortly show that it indeed captures the most relevant aspect of domain walls: their ordering. In fact, the linearized Nahm equations (4.2) are rather trivial to solve. We first employ the \( \prod_i U(n_i) \) gauge transformations to make $A_4(x_4)$ a constant in each interval $m_i \leq x_4 \leq m_{i+1}$. This can be achieved by first diagonalizing $A_4$, and subsequently acting with the $U(1)^{n_i}$ transformation $A_4 \to A_4 - \partial_4 \alpha$ where, in each segment $m_i \leq x_4 \leq m_{i+1}$, $\alpha$ is given by

$$\alpha(x_4) = \int_{m_i}^{x_4} A_4(x'_4) \, dx'_4 - \int_{m_i}^{m_{i+1}} A_4(x'_4) \, dx'_4 \frac{m_i - x_4}{m_{i+1} - m_i}$$

(4.5)

which has the property that $\alpha(m_i) = \alpha(m_{i+1}) = 0$. Further gauge transformations with non-zero winding on the interval ensure that $A_4$ is periodic, with each eigenvalue lying in $A_4 \in [0, 2\pi/(m_i - m_{i+1})]$. These $N - 1$ “Wilson lines” will play the role of the phases associated to domain wall system. Note that when $n_i = n_{i+1}$, the above choice of gauge leaves a residual $U(n_i)$ gauge symmetry acting only on the chiral impurity $\psi$. In this gauge we can now easily integrate (4.2) in each interval,

$$X_3(x_4) = e^{iA_4 x_4} \hat{X}_3 e^{-iA_4 x_4}$$

(4.6)
where the eigenvalues of $X_3$ are independent of $x_4$ in each interval. We identify these $n_i$ eigenvalues with the positions of the $n_i$ elementary domain walls.

We are now in a position to derive the linearized Nahm equations (4.2) from the original Nahm equations (3.10) in terms of a fixed point set of a $\hat{U}(1)$ action. Consider first the action of the $U(1)_R \subset SU(2)_R$ isometry on the Nahm data, which rotates $X_1$ and $X_2$ while leaving $X_3$ fixed. This rotation also acts on the impurity $\omega = (\psi, \tilde{\psi}^\dagger)$ by $(\psi, \tilde{\psi}) \to e^{i\alpha}(\psi, \tilde{\psi})$. To retain half of the impurities for the domain wall equations (4.3), we need to compensate for this transformation with the residual $U(1) \subset U(n_i)$ transformation acting on the appropriate impurity $\omega$ by $\omega \to e^{i\beta}\omega$. By choosing $\beta = \pm \alpha$ we can pick a $\hat{U}(1)$ action which leaves either the $\psi$ or the $\tilde{\psi}$ impurity invariant. Which we choose to save is correlated with the choice of minus sign in (4.3) which, in turn, dictates the ordering of neighbouring domain walls as we shall now demonstrate.

To summarize, we have shown that the description of domain wall dynamics (4.2) arises from the fixed point of a $\hat{U}(1)$ on the original Nahm equations (3.10). This action descends to a $\hat{U}(1)$ isometry on the monopole moduli space $M_{\vec{g}}$, the fixed points of which coincide with the domain wall moduli space $W_{\vec{g}}$. A physical explanation for this correspondence follows along the lines of [3]: in theories in the Higgs phase, confined magnetic monopoles with charge $\vec{g}$ exist, emitting $k$ multiple vortex strings. When these vortex strings coincide, the worldvolume theory is of the form described in Section 2 [14] and the monopoles appear as charge $\vec{g}$ kinks.

### 4.2 The Ordering of Domain Walls Revisited

As explained in Section 2, in contrast to monopoles, domain walls must satisfy a specific ordering on the $x^3$ line. We will now show that this ordering is encoded in the boundary conditions described above. Suppose first that $n_i = n_{i+1}$. The positions of the $\vec{\alpha}_i$ domain walls are given by the eigenvalues of $X_3$ restricted to the interval $m_i \leq x_4 \leq m_{i+1}$. Let us denote this matrix as $X_3^{(i)}$ and the eigenvalues as $\lambda_m^{(i)}$, where $m = 1, \ldots, n_i$. The impurity (4.3) relates the two sets of eigenvalues by the jumping condition

$$X_3^{(i+1)} = X_3^{(i)} + \psi\tilde{\psi}^\dagger \quad (4.7)$$

where we have chosen the positive sign for definiteness. However, from the discussion in Section 2 (see, in particular, figure 3) we know that the domain walls cannot have arbitrary position but must be interlaced,

$$\lambda_1^{(i)} \leq \lambda_1^{(i+1)} \leq \lambda_2^{(i)} \leq \cdots \leq \lambda_{n_i-1}^{(i+1)} \leq \lambda_{n_i}^{(i)} \leq \lambda_{n_i}^{(i+1)} \quad (4.8)$$

We will now show that the ordering of domain walls (4.8) follows from the impurity jumping condition (4.7).
To see this, consider firstly the situation in which $\psi^\dagger \psi \ll \Delta \lambda_m^{(i)}$ so that the matrix $\psi \psi^\dagger$ may be treated as a small perturbation of $X^{(i)}_3$. The positivity of $\psi^\dagger \psi$ ensures that each $\lambda_m^{(i+1)} \geq \lambda_m^{(i)}$. Moreover, it is simple to show that the $\lambda_m^{(i+1)}$ increase monotonically with $\psi^\dagger \psi$. This leaves us to consider the other extreme, in which $\psi^\dagger \psi \to \infty$. In this limit $\psi$ becomes one of the eigenvectors of $X^{(i+1)}_3$ with eigenvalue $\lambda_n^{(i+1)} = \psi^\dagger \psi \to \infty$ which reflects the fact that this limit corresponds to the situation in which the last domain wall is taken to infinity. What we want to show is that the remaining $(n_i - 1)$ $\vec{\alpha}_{i+1}$ domain walls are trapped between the $n_i$ $\vec{\alpha}_i$ domain walls as depicted in Figure 3. Define the $n_i \times n_i$ projection operator

$$P = 1 - \hat{\psi} \hat{\psi}^\dagger$$

(4.9)

where $\hat{\psi} = \psi / \sqrt{\psi^\dagger \psi}$. The positions of the remaining $(n_i - 1)$ $\vec{\alpha}_{i+1}$ domain walls are given by the (non-zero) eigenvalues of $PX^{(i)}_3 P$. We must show that, given a rank $n$ hermitian matrix $X$, the eigenvalues of $PX X P$ are trapped between the eigenvalues of $X$. This elementary property of hermitian matrices can be seen as follows:

$$\det(PXP - \mu) = \det(XP - \mu)$$
$$= \det(X - \mu - X \hat{\psi} \hat{\psi}^\dagger)$$
$$= \det(X - \mu) \det(1 - (X - \mu)^{-1} X \hat{\psi} \hat{\psi}^\dagger)$$

Since $\hat{\psi} \hat{\psi}^\dagger$ is rank one, we can write this as

$$\det(PXP - \mu) = \det(X - \mu) [1 - \text{Tr}((X - \mu)^{-1} X \hat{\psi} \hat{\psi}^\dagger)]$$
$$= -\mu \det(X - \mu) \text{Tr}((X - \mu)^{-1} \hat{\psi} \hat{\psi}^\dagger)$$
$$= -\mu \left[ \prod_{m=1}^{n} (\lambda_m - \mu) \right] \left[ \sum_{m=1}^{n} |\hat{\psi}_m|^2 / \lambda_m - \mu \right]$$

(4.10)

where $\hat{\psi}_m$ is the $m^{th}$ component of the vector $\psi$. We learn that $PXP$ has one zero eigenvalue while, if the eigenvalues $\lambda_m$ of $X$ are distinct, then the eigenvalues of $PXP$ lie at the roots the function

$$R(\mu) = \sum_{m=1}^{n} \frac{|\hat{\psi}_m|^2}{\lambda_m - \mu}$$

(4.11)

The roots of $R(\mu)$ indeed lie between the eigenvalues $\lambda_m$. This completes the proof that the impurities (4.3) capture the correct ordering of the domain walls.

The same argument shows that the boundary condition (4.4) gives rise to the correct ordering of domain walls when $n_{i+1} = n_i + 1$, with the $\vec{\alpha}_i$ domain walls interlaced between the $\vec{\alpha}_{i+1}$ domains walls. Indeed, it is not hard to show that (4.4) arises from (4.3) in the limit that one of the domain walls is taken to infinity.
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