Black hole shadow and black hole thermodynamics

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We investigate the relations between the black hole shadow and black hole thermodynamics. We first show that phase structures of the black holes can be characterized by the shadow radii of the black holes. Then according to the second law of the black hole thermodynamics, we propose that the shadow radius of the black hole does not decrease with time. Lastly, the minimal sizes of the black hole shadows are given considering the third law of the black hole thermodynamics.

I. INTRODUCTION

Recently, the event horizon telescope (EHT) has given the first image of the supermassive M87\textsuperscript{*} black hole in the central giant elliptical galaxy [1–6], which gave a direct support of the Einstein’s general relativity and the existence of the black hole in our universe. The image of the black hole can provide us with information of jets and matter accretion of black hole. Moreover, the shadow of a black hole can tell us information of the black hole, such as the mass and the rotation parameters of the black hole.

The shadow of the black hole, due to the gravitational lensing of light, is defined as the observer’s dark sky, where the light sources are distributed everywhere except of the region between the observer and the black hole. The shadow of a spherically symmetric black hole is a black circular disk. The shadow of a Schwarzschild black hole, which is dependent of the black hole mass and the position of the observer, was first investigated in [7, 8]. The shadow of a rotating Kerr black hole, which is elongated silhouette-like due to the rotation’s dragging effect, was firstly worked out in [9]. The shadow of the Kerr-Neuman black hole was studied in [10, 11]. For the whole class of Plebański-Demiański spacetime, the shadow was studied in [12, 13]. The shadows of various other black holes have been studied in [12–37]. Moreover, the shadows of the wormholes are investigated [38–42]. The shadow perturbed by gravitational wave was studied in [43]. For a brief review, one can see [44].

Black hole thermodynamics has been widely researched [45–71] since the seminal work [72, 73]. Black holes are viewed as thermodynamic systems, which own thermodynamic properties that are quite similar to those of ordinary objects. There are four thermodynamic laws for the black holes [74]. The first law can be written as

\begin{equation}
\delta U = T \delta S + X_i \delta Y_i,
\end{equation}

where \( U \) is the internal energy, \( S \) is the Bekenstein-Hawking entropy, \( X_i \) are the generalized displacements and \( Y_i \) are the generalized forces. The second law claims that the event horizon area of the black hole will not decrease with time, i.e.,

\begin{equation}
\delta A \geq 0.
\end{equation}

The third law says that the surface gravity \( \kappa \) cannot be reduced to zero by finite operations. The zeroth law reads the surface gravity is constant over the event horizon for stationary black holes.

One can see that all the four thermodynamic laws for the black hole are related with the event horizon, a null hyper-surface which holds the symmetric characteristics of the spacetime. The event horizon plays a key role reflecting the physical properties of the black hole. Nonetheless, there are some endeavours trying to replace the role of event horizon with the photon orbit around the black hole. Based on the case studies in [75, 76], it was found that the circular orbit radius of the photon can be a characteristic quantity to reflect the thermodynamic phase structure of a spherically symmetric black hole [77]. Related extensions can also be seen in [78, 79]. The first image of the black hole makes us think about whether it is possible to use the black hole shadow, which is observable, to reflect the thermodynamics of the black hole. In this compact paper, we will introduce our thoughts on this. Besides the event horizon and circular orbit radius of the photon, we use a third quantity, the radius of the shadow, to uncover the thermodynamic information of the black holes, both for spherically symmetric one and axially symmetric one. The motivation of using the radius of the black hole shadow as a characteristic quantity to detect the thermodynamics of the black hole is that, this new quantity can be observed, such that thermodynamics of the black hole can be tested by observable data.

This work is organized as follows. In Sec. II, we will show that, like the event horizon, the shadow radius of the black hole can be a characteristic quantity to reflect the phase structure of the spherically symmetric or axially symmetric black hole (RN black hole and Kerr
black hole are used as representative models). In Sec. III, based on the thermodynamic second law, we tenta-

tively give relations which shadow radius of the black hole should comply with. These relations, on the other

hand, make it possible for us to check the second law of the black hole thermodynamics by astronomical obser-

vation. In Sec. IV, realizing that the thermodynamic third law yields that the temperature of a black hole should be

positive, we deduce the minimal sizes of the black hole shadows that an observer can detect.

II. SHADOWS AND THERMODYNAMIC PHASE STRUCTURES OF BLACK HOLES

The specific heat of a black hole, according to the first law of the black hole thermodynamics, can be written as

\[ C = \frac{dU}{dT} = T \left( \frac{\partial S}{\partial T} \right)_{\mathcal{Y}_i}, \tag{3} \]

\( C > 0, \) \( C < 0 \) correspond to the thermodynamic unstable state and stable state individually. The point making \( C = 0 \) is the phase transition point. In general, the entropy \( S \) is related to the event horizon \( r_h \) of the black hole with

\[ \frac{dS}{dr_h} > 0. \tag{4} \]

Accordingly, we can know that

\[ \text{Sgn}(C) = \text{Sgn} \left( \frac{\partial T}{\partial r_h} \right), \tag{5} \]

where \( \text{Sgn} \) is the sign function. Thus, in the \( r_h - T \) diagram, a positive slope means that the black hole is in a thermodynamically stable state and a negative slope corresponds to a thermodynamically unstable black hole. In the following, we will further show how the shadow radii of the black holes can be used to mirror the phase structures of the black holes, first for spherically symmetric case and then for the axially symmetric case.

A. Shadow and phase structure of the spherically symmetric black hole

The static spherically symmetric spacetime background can be described by the line element

\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2d\theta^2 + r^2\sin^2 \theta d\phi^2, \]

where \( f(r) \) and \( g(r) \) are functions of coordinate \( r \), this metric ansatz covers the black holes with hairs \cite{80, 81}. The Hamiltonian of the photon moving in the static spherically symmetric spacetime is

\[ 2H = g^{ij}p_ip_j = 0. \tag{6} \]

Without loss of generality, due to the spherically symmetric characteristics of the black hole, we can consider photons moving in the equatorial plane with \( \theta = \pi/2 \). Then (6) can be explicitly written as

\[ \frac{1}{2} \left[ -\frac{p_t^2}{f(r)} + g(r)p_r^2 + \frac{p_\phi^2}{r^2} \right] = 0. \tag{7} \]

Thinking of the relations

\[ \dot{p}_t = -\frac{\partial H}{\partial t} = 0 \tag{8} \]

and

\[ \dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0, \tag{9} \]

we can know that \( p_t \) and \( p_\phi \) are constants of motion. We can define \( -p_t = e \) and \( p_\phi = j \), which can be physically interpreted as the energy and angular momentum of the photon.

The equations of motion for the photon can be obtained as

\[ \dot{t} = \frac{\partial H}{\partial p_t} = -\frac{p_r}{f(r)}, \tag{10} \]

\[ \dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{r^2}, \tag{11} \]

\[ \dot{r} = \frac{\partial H}{\partial p_r} = p_r g(r), \tag{12} \]

where \( p_r \) is the radial momentum. The effective potential of the photon can be defined by

\[ V_e + \dot{r}^2 = 0. \tag{13} \]

As a result, the effective potential can be expressed as

\[ V_e = g(r) \left[ \frac{f^2}{r^2} - \frac{e^2}{f(r)^2} \right]. \tag{14} \]

The radius of the photon can be obtained from the relation

\[ V_e = V_e' = 0. \tag{15} \]

The orbit equation for the photon is

\[ \frac{dr}{d\phi} = \frac{\dot{r}^2 g(r)p_r}{\dot{\phi}}. \tag{16} \]

Substituting the solution of \( p_r \) from (7) into (16), we can obtain

\[ \frac{dr}{d\phi} = \pm r\sqrt{g(r) \left[ \frac{r^2 e^2}{f(r)^2} - 1 \right]}. \tag{17} \]
At the turning point of the photon orbit, we should have
\[ \frac{dr}{d\phi} \bigg|_{r=R} = 0, \quad (18) \]
which gives
\[ \frac{e^2}{j^2} = \frac{f(R)}{R^2}. \quad (19) \]
Then we can know
\[ \frac{dr}{d\phi} = \pm r \sqrt{g(r)} \left[ \frac{r^2 f(R)}{f(r) R^2} - 1 \right]. \quad (20) \]

For a light ray sending from a static observer at position \( r_o \) and transmitting into the past with an angle \( \alpha \) with respect to the radial direction, we have
\[ \cot \alpha = \frac{\sqrt{g_{rr}} \cdot \frac{dr}{d\phi}}{\sqrt{g_{\phi\phi}}} \bigg|_{r=r_o} = \frac{1}{r} \frac{1}{\sqrt{g(r)}} \frac{dr}{d\phi} \bigg|_{r=r_o}. \quad (21) \]

Thinking of (20), we can have
\[ \cot^2 \alpha = \frac{r_o^2 f(R)}{f(r_o) R^2} - 1. \quad (22) \]

Using the relation \( 1 + \cot^2 \alpha = \sin^{-2} \alpha \), we further have
\[ \sin^2 \alpha = \frac{f(r_o) R^2}{r_o^2 f(R)}. \quad (23) \]

The angular radius of the black hole shadow can be obtained by letting \( R \rightarrow r_p \), with \( r_p \) the circular orbit radius of the photon. We can obtain the shadow radius of the black hole observed by a static observer at position \( r_o \) as
\[ r_s = r_o \sin \alpha = R \sqrt{\frac{f(r_o)}{f(R)}} \bigg|_{R=r_p}. \quad (24) \]

We now consider using the black hole shadow to reflect the phase structure of the black hole. As a non-trivial example, we here specifically investigate the relation between the shadow and the phase structure of the Reissner–Nordström (RN) black hole with
\[ f(r) = g(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \quad (25) \]
where \( M, Q \) are mass and electric charge of the black hole. The event horizon and temperature of the RN black hole are
\[ r_h = M + \sqrt{M^2 - Q^2}, \quad T = \frac{r_h^2 - Q^2}{r_h^3}. \quad (26) \quad (27) \]

Using (15), we can obtain the circular orbit radius of the photon as
\[ r_p = \frac{1}{2} \left( 3M + \sqrt{9M^2 - 8Q^2} \right). \quad (28) \]

According to (24), we have the radius of the RN black hole shadow as
\[ r_s = \frac{\sqrt{f(r_0)} \left( \sqrt{9M^2 - 8Q^2} + 3M \right)^2}{2 \sqrt{2M \left( \sqrt{9M^2 - 8Q^2} + 3M \right) - 4Q^2}} \]
\[ = \frac{\left( \sqrt{9M^2 - 8Q^2} + 3M \right)^2}{2 \sqrt{2M \left( \sqrt{9M^2 - 8Q^2} + 3M \right) - 4Q^2}} \quad (29) \]
where \( f(r_0) = 1 \) in the second “\( \Rightarrow \)”, since we hereby consider an observer at spatial infinity.

For constant \( Q \), we have
\[ \frac{dr_h}{ds} = \frac{2 \sqrt{2} \sqrt{9M^2 - 8Q^2} \left( \sqrt{9M^2 - Q^2} + M \right)}{\sqrt{M^2 - Q^2} \left( \sqrt{9M^2 - 8Q^2} + 3M \right)^2} \]
\[ \times \frac{3M \sqrt{9M^2 - 8Q^2} + 9M^2 - 2Q^2)^{3/2}}{9M^2 - 8Q^2 + 9M^2 - 8Q^2} \quad (30) \]

Based on (30), we have \( dr_h/dr_s > 0 \). Also, for the temperature with fixed charge \( Q \), we have
\[ \frac{dT}{dr_h} = \frac{dT}{dr_s} \frac{dr_s}{dr_h}, \quad (31) \]
which means
\[ \frac{dT}{dr_h} > 0, \quad \frac{dT}{dr_s} = 0, \quad \frac{dT}{dr_h} < 0 \quad (32) \]
correspond to
\[ \frac{dT}{dr_h} > 0, \quad \frac{dT}{dr_s} = 0, \quad \frac{dT}{dr_s} < 0 \quad (33) \]
respectively. That is, the radius of the black hole shadow can be a characteristic quantity to reflect the phase structure of the black hole.

In addition, when the condition \( dT/dr_h = 0 \) holds, we can find the critical charge-to-mass ratio \( Q/M = \sqrt{3}/2 \). The critical radius of the event horizon and the shadow can also be obtained as
\[ r_h^c = \sqrt{3}Q, \quad (34) \]
\[ r_s^c = 2\sqrt{2\sqrt{3} + 3}Q. \quad (35) \]

According to Fig. 1, if the radius of the shadow observed is bigger than \( r_s^c \), for fixed \( Q \), the RN black hole is in a thermodynamically unstable phase; otherwise, it’s in a thermodynamically stable phase. Thus, we can observe the phase structure of the RN black hole by observing its shadow radius.
B. Shadow and phase structure of the axially symmetric black hole

Here we will use Kerr geometry to show that the shadow radius can be a characteristic quantity to reflect the phase structure of the axially symmetric black hole. The Kerr black hole can be described by the line element using Boyer-Lindquist coordinates as

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2d\theta^2 - \frac{4Mr\rho\sin^2\theta}{\rho^2}dtd\phi + \frac{\Delta(\rho^2 + \Delta a^2 \sin^2\theta)\sin^2\theta}{\rho^2}d\phi^2,$$

where

$$\Delta = r^2 - 2Mr + a^2, \quad \rho^2 = r^2 + a^2 \cos^2\theta,$$

(37)

$M$ is the mass of the black hole and $a$ is the rotation parameter defined by $a \equiv J/M$ with $J$ the angular momentum of the black hole. The event horizon of the Kerr black hole is

$$r_h = M + \sqrt{M^2 - J^2 \over M},$$

(38)

which can be got from the restriction $\Delta(r_h) = 0$. The temperature of the Kerr black hole is

$$T = -\frac{a^2 - r_h^2}{4\pi a^2 r_h + 4\pi r_h^3} = \frac{M^2 \sqrt{M^4 - J^2}}{4\pi M^3 \left(\sqrt{M^2 - J^2 \over 2\pi} + M\right)^2}.$$

(39)

The null geodesics around the Kerr black hole is

$$\rho^2\frac{dt}{d\lambda} = a(l - aE \sin^2\theta) + \frac{\rho^2 + a^2}{\Delta} \left[ E(r^2 + a^2) - a l \right],$$

$$\rho^2\frac{dr}{d\lambda} = \sqrt{\mathcal{R}},$$

$$\rho^2\frac{d\theta}{d\lambda} = \sqrt{\Theta},$$

$$\rho^2\frac{d\phi}{d\lambda} = (l \csc^2\theta - aE) + \frac{a}{\Delta} \left[ E(r^2 + a^2) - a l \right],$$

(40)

(41)

(42)

(43)

with $\lambda$ the affine parameter, $l$ the angular momentum of the photon, $E$ the energy of the photon (we here use different denotations from the RN case), $Q$ the Carter constant and

$$\mathcal{R} = \left( a^2E^2 - aE - a^2l \right) - \Delta \left[ Q + (l - aE)^2 \right],$$

$$\Theta = Q - (l \csc\theta - aE \sin\theta)^2 - (l - aE)^2.$$

(44)

(45)

The effective potential of the photon is

$$V_e = -\frac{a^2(2M + r) + 4aLM + l^2(r - 2M) - r^3}{r^3}.$$

(46)

By using the condition (15), the circular orbit radius $r_p$ of the photon can be obtained and the parameters $\xi \equiv l/E$ and $\eta \equiv Q/E^2$ can thus be expressed as

$$\xi = \frac{(3M - r_p)^2 - a^2(M + r_p)}{a(r_p - M)},$$

$$\eta = \frac{r_p^2(4a^2M - r_p(3M - r_p)^2)}{a^2(r_p - M)^2}.$$

(47)

(48)

One constraint for the value of the photon’s circular orbit radius is $\mathcal{R}(r_p) > 0$.

The shape of the shadow seen by an observer at spatial infinity can be obtained from the geodesics of the photons and described by the celestial coordinates \([20, 82]\)

$$x = -\xi \csc\theta_0,$$

$$y = \pm\sqrt{\eta + a^2 \cos^2\theta_0 - \xi^2 \cot^2\theta_0},$$

(49)

(50)

where $\theta_0$ is the inclination angle of the observer.

Referring to Fig. 2, a “typical radius” of the Kerr black hole shadow can be defined as

$$r_s = \frac{x_{rm} - x_{lm}}{2},$$

(51)
FIG. 2. The shadow of the Kerr black hole for $M = 1$, $J = 0.99$, $\theta_0 = \pi/2$. The coordinates of the leftmost, rightmost, topmost and lowest points are individually denoted by $(x_{rm}, 0)$, $(x_{lm}, 0)$, $(x_{tm}, y_{tm})$, $(x_{tm}, -y_{tm})$

where $x_{rm}$ and $x_{lm}$ can be got from

$$0 = y(r_p = r_1) = y(r_p = r_2)$$

and

$$x_{rm} = x(r_p = r_1), \quad x_{lm} = x(r_p = r_2).$$

Here the “typical radius” of the Kerr black hole shadow (which we would like to call it shadow radius of the Kerr black hole) is used to make the study in what follows be illustrative. Note that there are other definitions of the shadow radius, such as the widely-used one proposed in [83].

The temperature of the Kerr black hole can be a function of not only the event horizon radius $r_h$, but also the radius of the shadow $r_s$, i.e.,

$$T(r_h) = T[r_h(r_p)] = T[r_p(r_s)] = T(r_s).$$

So we have

$$\frac{dT}{dr_h} = \frac{dT}{dr_s} \frac{dr_s}{dr_h}.$$  \hspace{1cm} (55)

By numerical calculation, one can find that the variation relation between the event horizon radius $r_h$ and the shadow radius $r_s$ satisfies

$$\frac{dr_s}{dr_h} > 0,$$

which can be shown in the upper panel of Fig. 3.

As a result, we can know

$$\frac{dT}{dr_h} > 0, \quad \frac{dT}{dr_h} = 0, \quad \frac{dT}{dr_h} < 0$$

correspond to

$$\frac{dT}{dr_s} > 0, \quad \frac{dT}{dr_s} = 0, \quad \frac{dT}{dr_s} < 0.$$  \hspace{1cm} (58)

Hence, similar to the RN black hole case, the thermodynamic phase structure of the Kerr black hole can also be observed by analyzing the shadow of the Kerr black hole. The critical angular momentum-to-mass ratio is $J/M^2 = (3 - \sqrt{5}) \sqrt{5/2} + 2/2$ and the critical event horizon can be obtained as $r_h^c = \sqrt{5/2} + 2a$. The tedious expression of critical shadow radius can be got by using (15), (46), (47), (48) and (51) and we won’t show it here.

FIG. 3. The upper diagram shows the variation of the shadow radius with respect to the event horizon radius and the lower diagram shows the variation of the black hole temperature with respect to the shadow radius for the Kerr black hole, whose angular momentum is set to be $J = 0.9$.

III. BLACK HOLE SHADOWS WILL NOT DECREASE DURING THE BLACK HOLES MERGER

According to the second law of the black hole thermodynamics, the entropy of final black hole will not be less than the total entropies of former two black holes. Mathematically, we can express this as $S_U > S_{U_1} + S_{U_2}$. For the RN black hole in Einstein gravity, the entropy can be obtained by the area law, so we have $S = \pi r_h^2$. This
means

\[ r_h^2 > r_{h1}^2 + r_{h2}^2. \]  \hspace{1cm} (59)

From Fig. 1, we can see that the shadow radius is almost proportional to the event horizon for the RN black hole with large \( M \) (Strictly speaking, \( dr_s/dr_h \to 0 \) when \( M \to Q \), but for large \( M \), \( dr_s/dr_h \) grows asymptotically to a constant which is \( 3\sqrt{3}/2 \) for infinite \( M \) and \( M \gg Q \), see Fig. 4). So we can get a relation for the black hole shadows in the massive black hole merger as

\[ r_s^2 > r_{s1}^2 + r_{s2}^2, \]  \hspace{1cm} (60)

where \( r_s \) is the shadow radius of the final black hole originating from coalescing of two primordial black holes with shadow radii \( r_{s1} \) and \( r_{s2} \). For the rotating Kerr black hole case, the situation becomes subtler and we do not plan to elaborate upon it here. However, we can confirm that the relation (60) is not violated for merger of black holes with (anti-)parallel angular momentum in the regime \( M \gg 1 \) and \( M \gg J \).

The second law of the black hole thermodynamics also points out that the entropy of a black hole will not decrease in the clockwise direction, which can be denoted as

\[ \frac{dS}{dt} > 0. \]  \hspace{1cm} (61)

Based on the relation \( dr_s/dr_h > 0 \) we have got for black holes, we can obtain that the shadow radius of the black hole will also not decrease in the clockwise direction, that is,

\[ \frac{dr_s}{dt} > 0. \]  \hspace{1cm} (62)

This formula, being as a deduction of the second law of the black hole thermodynamics, directly reveals the impact of the thermodynamics of the black hole on the astronomical observation.

IV. MINIMAL SHADOWS AND THERMODYNAMIC THIRD LAW OF BLACK HOLES

The third law of black hole thermodynamics yields that the temperature of the black hole should be positive. From the expression of the event horizon (26) for the RN black hole, we can know that the horizon radius decreases with increasing electric charge \( Q \); the event horizon of the extreme RN black hole is minimal. As we have shown, \( dr_h/dr_s > 0 \) for the RN black hole. So we can know that the radius of the shadow is minimal for extreme RN black hole. According to the circular orbit radius of the photon (28), we have \( r_p^c = 2M \) for the extreme RN black hole. Then according to (24), we can know that the minimal size of the shadow that can be detected by an observer at spatial infinity is

\[ r_{s\text{min}}^{\text{RN}} = 4M, \]  \hspace{1cm} (63)

the same value with the radius of the innermost stable circular orbit for a massive test particle revolving around an extreme RN black hole [84].

For the Kerr black hole, we can also analytically calculate the minimal shadow radius observed at the equatorial plane. Using (48) and (50), we can know that the circular orbit radii of the photons on the equatorial plane corresponding to the leftmost and rightmost points of the shadow are

\[ r_p^\pm = 2M \left[ 1 + \cos \left( \frac{2}{3} \arccos \left( \pm \frac{a}{M} \right) \right) \right]. \]  \hspace{1cm} (64)

Then from (49), we can know the shadow radius of the extreme Kerr black hole is

\[ r_{s\text{min}}^{\text{Kerr}} = \frac{1}{2} \left( -4M^2 + 8M^2 \cos^3 \left( \frac{2}{3} \arccos \left( -\frac{1}{M} \right) \right) + 12M^2 \cos^2 \left( \frac{2}{3} \arccos \left( -\frac{1}{M} \right) \right) + 2 \cos \left( \frac{2}{3} \arccos \left( -\frac{1}{M} \right) \right) + 3 \right) \]  \hspace{1cm} (65)

\[ + \frac{-4M^2 + 8M^2 \cos^3 \left( \frac{2}{3} \arccos \left( \frac{1}{M} \right) \right) + 12M^2 \cos^2 \left( \frac{2}{3} \arccos \left( \frac{1}{M} \right) \right) + 2 \cos \left( \frac{2}{3} \arccos \left( \frac{1}{M} \right) \right) + 3 \right) \]  \hspace{1cm} (66)
which is the minimal one according to (56). It is not difficult to check that \( r_{\text{Kerr}}^{\text{min}} \) monotonically increases with \( M \).

V. SUMMARY

Previously, black hole shadow and black hole thermodynamics were studied separately; of particular interest is the relation between them, which is the topic in this compact work. Firstly, we have shown that radius of the shadow can be a well-defined characteristic quantity to reflect the phase structure of the black hole. The spherically symmetric RN black hole and axially symmetric Kerr black hole were used as toy models. Secondly, according to the second law of the black hole thermodynamics, we obtained the relation (60) for the merger of the spherically symmetric massive black holes. We also proposed that the radius of the black hole shadow will not decrease in the clockwise direction. Thirdly, realizing that the temperature of a black hole should be bigger than zero, which is required by the third law of the black hole thermodynamics, we gave the minimal sizes of the shadows for the RN black hole and the Kerr black hole (on the equatorial plane).

Our researches may give illuminations to the studies of the black hole thermodynamics and the black hole shadow. Based on the thermodynamic first law of the black hole, we can judge the thermodynamic stability of the black hole through observations of the black hole shadow. In a precise astronomical observation era, our results on the laws that the sizes of the black hole shadows should obey [(60) and (62)] also show that the thermodynamics of the black hole can be checked by astrophysical observations. Moreover, the minimal sizes of the black hole shadows we deduced may, to some extent, help us to recognize what kind of the observed black body is there in the sky.

The shadow of a spherically symmetric black hole is also spherically symmetric, so it is easy to define their shadow radius. However, for the axially symmetric black hole, the shadow is not spherically symmetric any more (except seen on the poles). In this paper, we used an intuitive definition of the shadow radius for the Kerr black hole. As we all know, there are other methods to reflect the size of the axially symmetric black hole [83, 85–87]. However, we suspect that different definitions of the black hole shadow radius will give the same results we found. Meanwhile, we believe that the study of shadow for the Kerr black hole on the equatorial plane is also applicable to that on the plane \( \theta \neq \pi/2 \) and the results will not quantitatively change.

It is proved that our universe is in an accelerating expansion state and a positive cosmological constant should be added to the Einstein gravitational field equation. So it may be more pragmatic to investigate the shadow of the asymptotically de Sitter black holes. For those black holes, there are more abundant phase structures. Studies on the relation between the black hole shadow and the thermodynamics of asymptotically de Sitter black hole is meaningful.

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