Homogeneous Locally Nilpotent Derivations of Nonfactorial Trinomial Algebras

Yu. I. Zaitseva 1*

1 Lomonosov Moscow State University, Moscow, 119991 Russia

Received August 3, 2017; in final form, June 7, 2018; accepted September 12, 2018

Abstract—We describe homogeneous locally nilpotent derivations of the algebra of regular functions for a class of affine trinomial hypersurfaces. This class comprises all nonfactorial trinomial hypersurfaces.

DOI: 10.1134/S0001434619050201

Keywords: affine hypersurface, torus action, graded algebra, derivation.

1. INTRODUCTION

Let $K$ be an algebraically closed field of characteristic zero, and let $R$ be an algebra over $K$. A derivation of $R$ is a $K$-linear map $\delta : R \to R$ satisfying the Leibniz rule:

$$\delta(fg) = \delta(f)g + f\delta(g) \quad \text{for all } f, g \in R.$$ 

A derivation $\delta$ is called locally nilpotent if, for every $f \in R$, there is a positive integer $m$ such that $\delta^m(f) = 0$.

Let $X$ be an irreducible affine algebraic variety over $K$, and let $G_a = (K, +)$ be the additive group of the ground field. The locally nilpotent derivations of the algebra $K[X]$ are known to be in one-to-one correspondence with the regular $G_a$-actions on the variety $X$; see, e.g., [1, Sec. 1.5].

Suppose that $R$ is graded by a finitely generated Abelian group $K$:

$$R = \bigoplus_{w \in K} R_w.$$ 

A derivation $\delta : R \to R$ is said to be homogeneous if it maps homogeneous elements to homogeneous ones. In this case, there exists an element $\deg \delta \in K$ satisfying $\delta(R_w) \subseteq R_{w+\deg \delta}$ for all $w \in K$. The element $\deg \delta \in K$ is called the degree of the derivation $\delta$.

Recall that an affine algebraic group is called a quasitorus if it is isomorphic to the direct product of a torus and a finite Abelian group. Suppose that a quasitorus $H$ acts on the variety $X$. Such an action corresponds to a grading on the algebra $K[X]$ by the finitely generated Abelian group $K$ of characters of the quasitorus $H$:

$$K[X] = \bigoplus_{w \in K} K[X]_w,$$

where $K[X]_w = \{ f \mid h \circ f = w(h)f \ \forall h \in H \}$.

It is easily shown that a locally nilpotent derivation of $K[X]$ is homogeneous with respect to this grading if and only if the quasitorus $H$ normalizes the corresponding $G_a$-action on $X$. A description of homogeneous locally nilpotent derivations enables us to describe the automorphism group of an algebraic variety; see, e.g., [2, Theorem 5.5] and [3, Theorem 3].

*E-mail: yuliazaitseva@gmail.com
Fix positive integers \( n_0, n_1, \) and \( n_2 \) and let \( n = n_0 + n_1 + n_2 \). For each \( i = 0, 1, 2, \) we also fix an \( n_i \)-tuple \( l_i = (l_{ij} \mid j = 1, \ldots, n_i) \) of positive integers \( l_{ij} \) and define a monomial \( T_{i}^{l_i} = T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}} \) in the polynomial algebra \( \mathbb{K}[T_{ij}, i = 0, 1, 2, j = 1, \ldots, n_i] \). By a *trinomial* we mean a polynomial of the form

\[
g = T_0^{l_0} + T_1^{l_1} + T_2^{l_2}.
\]

A *trinomial hypersurface* \( X(g) \) is the zero set \( g = 0 \) in the affine space \( \mathbb{A}^n \). By \( R(g) \) we denote the algebra \( \mathbb{K}[X(g)] \) of regular functions on \( R(g) \).

Our motivation to study trinomials comes from toric geometry. Consider an effective action \( T \times X \to X \) of an algebraic torus \( T \) on an irreducible variety \( X \). The *complexity* of such an action is the codimension of a general \( T \)-orbit in \( X \). It equals \( \dim X - \dim T \).

Actions of complexity zero are torus actions with an open orbit. A normal variety admitting such an action is called a *toric variety*. If \( X \) is a toric (not necessary affine) variety with acting torus \( T \), then \( \mathbb{G}_m \)-actions on \( X \) normalized by \( T \) can be described in terms of the Demazure roots of the fan corresponding to \( X \); see [4] and [5, Sec. 3.4] for the original approach and [6], [7], and [2] for generalizations.

Let \( T \times X \to X \) be a torus action of complexity one. A classification of the \( \mathbb{G}_m \)-actions on \( X \) normalized by \( T \) in terms of proper polyhedral divisors can be found in [8] and [6]. It is an interesting problem to find their explicit form in particular cases.

The study of toric varieties is related to binomials; see, e.g., [9, Chap. 4]. At the same time, Cox rings also enable us to give a new proof of a rigidity criterion for factorial trinomial hypersurfaces, which was a criterion for the existence of homogeneous locally nilpotent derivations of trinomial algebras. This enables us to give a new proof of a rigidity criterion for factorial trinomial hypersurfaces, which was proved earlier in [14, Theorem 1] (see Corollary 3).

### 2. Preliminaries

In this section, the “finest” deg-grading and elementary derivations of a trinomial algebra are defined.

**Construction 1.** Fix positive integers \( n_0, n_1, \) and \( n_2 \) and let \( n = n_0 + n_1 + n_2 \). For each \( i = 0, 1, 2, \) we also fix an \( n_i \)-tuple \( l_i = (l_{ij} \mid j = 1, \ldots, n_i) \) of positive integers and define the monomial

\[
T_i^{l_i} = T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}} \in \mathbb{K}[T_{ij}, i = 0, 1, 2, j = 1, \ldots, n_i].
\]

By a *trinomial* we mean a polynomial of the form

\[
g = T_0^{l_0} + T_1^{l_1} + T_2^{l_2}.
\]
A trinomial hypersurface $X(g)$ is the zero set $g = 0$ in the affine space $\mathbb{A}^n$. It can be checked that the polynomial $g$ is irreducible, and hence the algebra $R(g) := \mathbb{K}[T_{ij}] / (g)$ of regular functions on $X(g)$ has no zero divisors. We refer to such algebras as trinomial. We use the same notation for elements of $\mathbb{K}[T_{ij}]$ and their projections on $R(g)$.

Following [12], we build a $2 \times n$ matrix $L$ from the trinomial $g$ as follows:

$$L = \begin{pmatrix} -l_0 & l_1 & 0 \\ -l_0 & 0 & l_2 \end{pmatrix}. $$

Let $L^*$ be the transpose of $L$. We denote the quotient group $\mathbb{Z}^n / \text{Im} L^*$ by $K$ and the projection onto it by $Q: \mathbb{Z}^n \to K$. Let $e_{ij} \in \mathbb{Z}^n$, $i = 0, 1, 2$, $j = 1, \ldots, n$, be the canonical basis vectors. The relations $

\text{deg } T_{ij} = Q(e_{ij})

(1)

define a $K$-grading on the algebra $\mathbb{K}[T_{ij}]$.

Since $g$ is a homogeneous polynomial of degree $\mu = l_1 Q(e_{i1}) + \cdots + l_{in_i} Q(e_{in_i})$ for some $\mu \in K$ and any $i = 0, 1, 2$, it follows that relations (1) also define a $K$-grading on the algebra $R(g) = \mathbb{K}[T_{ij}] / (g)$. By deg we denote degree with respect to this grading.

**Example 1.** Let

$$g = T_{01} T_{02}^3 + T_{11}^3 + T_{21}^2,$$

i.e.,

$$L = \begin{pmatrix} -1 & -3 & 3 & 0 \\ -1 & -3 & 0 & 2 \end{pmatrix}.$$

(see [2, Example 3.3]). Since the matrix $L$ can be reduced by integer elementary row and column operations to the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

it follows that the grading group $K = \mathbb{Z}^4 / \text{Im } L^*$ is isomorphic to $\mathbb{Z}^2$. The grading can be defined explicitly as

$$\text{deg } T_{01} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}, \quad \text{deg } T_{02} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{deg } T_{11} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad \text{deg } T_{21} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$ 

It is unique up to the choice of a basis in $\mathbb{Z}^4$.

**Example 2.** Let

$$g = T_{01} T_{02} + T_{11} T_{12} + T_{21}^2,$$

$$L = \begin{pmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{pmatrix}.$$ 

The matrix $L$ can be reduced by elementary row and column operations to the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

and hence the group $K = \mathbb{Z}^5 / \text{Im } L^*$ is isomorphic to $\mathbb{Z}^2$. The grading can be defined explicitly as

$$\text{deg } T_{01} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad \text{deg } T_{02} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad \text{deg } T_{11} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$
\[
\deg T_{12} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \quad \deg T_{21} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.
\]

**Example 3.** Let

\[
g = T_{01}^2 + T_{11}^2 + T_{21}^2, \quad L = \begin{pmatrix} -2 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix}.
\]

Since \( L \) can be reduced to the form

\[
\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix},
\]

it follows that \( K = \mathbb{Z}^3 / \text{Im} L^* \) is isomorphic to \( \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \). We set \( \mathbb{Z}_2 = \{ [0]_2, [1]_2 \} \). The grading can be defined explicitly as

\[
\deg T_{01} = \begin{pmatrix} 1 \\ [0]_2 \\ [0]_2 \end{pmatrix}, \quad \deg T_{11} = \begin{pmatrix} 1 \\ [1]_2 \\ [0]_2 \end{pmatrix}, \quad \deg T_{21} = \begin{pmatrix} 1 \\ [0]_2 \\ [1]_2 \end{pmatrix}.
\]

The following lemma shows that the \( \deg \)-grading constructed above is the “finest” grading on the algebra \( R(g) \) with respect to which all generators \( T_{ij} \) are homogeneous.

**Lemma 1.** Let \( \deg \) be the \( K \)-grading on the algebra \( R(g) \) defined in Construction 1, and let \( \hat{\deg} \) be any \( \hat{K} \)-grading on \( R(g) \) by some Abelian group \( \hat{K} \) such that all generators \( T_{ij} \) are homogeneous. Then \( \hat{\deg} = \psi \circ \deg \) for some homomorphism \( \psi : K \to \hat{K} \), and any derivation homogeneous with respect to the \( \deg \)-grading is also homogeneous with respect to the \( \hat{\deg} \)-grading.

**Proof.** The weights \( \hat{w}_{ij} = \hat{\deg} T_{ij} \) determine a well-defined grading on the algebra \( R(g) \) if and only if the polynomial \( g \) is homogeneous with respect to this grading, i.e., the sums

\[
l_{i1} \hat{w}_{i1} + \cdots + l_{in_i} \hat{w}_{in_i} \in \hat{K}
\]

are the same for all \( i = 0, 1, 2 \). The factorization of \( \mathbb{Z}^n = \langle e_{ij} \rangle \) by \( \text{Im} L^* \) ensures that the images \( e_{ij} \) in \( K \) satisfy this condition. Any other grading on \( R(g) \) with homogeneous generators \( T_{ij} \) is obtained by a further factorization of the group \( K = \mathbb{Z}^n / \text{Im} L^* \); this factorization is the map \( \psi \) in the statement of the lemma. The homogeneity with respect to the \( \hat{\deg} \)-grading of any derivation homogeneous with respect to the \( \deg \)-grading easily follows from the definition of homogeneity.

The following construction is borrowed from [2]; it is described below in the particular case of trinomial hypersurfaces; in the notation of [2],

\[
r = 2, \quad A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \quad g = g_I, \quad I = \{0, 1, 2\}, \quad R(g) = R(A, R_0).
\]

**Construction 2.** Let us define a derivation \( \delta_{C, \beta} \) of \( R(g) \). Suppose given

- a sequence \( C = (c_0, c_1, c_2) \), where \( c_i \in \mathbb{Z}, 1 \leq c_i \leq n_i \);
- a vector \( \beta = (\beta_0, \beta_1, \beta_2) \) such that \( \beta_i \in \mathbb{K}, \beta_0 + \beta_1 + \beta_2 = 0 \).
It is clear that, for $\beta \neq 0$, either all entries $\beta_i$ differ from zero or there is a unique $i_0$ for which $\beta_{i_0} = 0$. Let us consider these two cases.

**Case 1.** If $\beta_{i_0} \neq 0$ for all $i = 0, 1, 2$ and there is at most one $i_1$ for which $l_{i_1 c_1} > 1$, then we set

$$\delta_{C,\beta}(T_{ij}) = \begin{cases} 
\beta_i \prod_{k \neq i} \frac{\partial T_{kj}}{\partial T_{kk}}, & j = c_i, \\
0, & j \neq c_i.
\end{cases}$$

**Case 2.** If $\beta_{i_0} = 0$ for a unique index $i_0$ and there is at most one $i_1$ for which $i_1 \neq i_0$ and $l_{i_1 c_1} > 1$, then we set

$$\delta_{C,\beta}(T_{ij}) = \begin{cases} 
\beta_i \prod_{k \neq i, i_0} \frac{\partial T_{kj}}{\partial T_{kk}}, & j = c_i, \\
0, & j \neq c_i.
\end{cases}$$

These assignments define a map $\delta_{C,\beta}$ on the generators $T_{ij}$. It can be extended uniquely to a derivation on $\mathbb{K}[T_{ij}]$ by the Leibniz rule. Clearly, we have $\delta_{C,\beta}(g) = 0$, and hence the constructed map induces a well-defined derivation of the quotient algebra $R(g)$.

**Lemma 2.** Every derivation $\delta_{C,\beta}$ of the algebra $R(g)$ is primitive and locally nilpotent.

**Proof.** The proof is given in [2, Construction 4.3].

Let $h \in R(g)$ be a homogeneous element in the kernel of a derivation $\delta_{C,\beta}$. It is easy to see that the derivation $h\delta_{C,\beta}$ a homogeneous locally nilpotent derivation of $R(g)$ as well.

**Definition 1.** We say that a derivation of a trinomial algebra $R(g)$ is *elementary* if it has the form $h\delta_{C,\beta}$, where $h$ is a homogeneous element in the kernel of $\delta_{C,\beta}$. *Elementary derivations of Type I* are elementary derivations with $\delta_{C,\beta}$ corresponding to Case 1 in Construction 2, and *elementary derivations of Type II* are elementary derivations with $\delta_{C,\beta}$ corresponding to Case 2.

**Example 4.** Consider $g = T_{01}T_{02}^3 + T_{11}^3 + T_{21}^3$ (see Example 1 and [2, Example 4.7]). Let us find all elementary derivations of the algebra $R(g)$. Note that there is only one variable, $T_{01}$, with exponent 1 in $g$. Hence only Case 2 with $i_0 \neq 0$ and $C = (1, 1, 1)$ is possible. Thus, we have two cases for elementary derivations $h\delta_{C,\beta}$ of Type II:

(a) $i_0 = 1$, i.e., $\beta = (\beta_0, 0, -\beta_0)$ for some $\beta_0 \in \mathbb{K}$; in this case,

$$\delta_{C,\beta}(T_{01}) = 2\beta_0 T_{21}, \quad \delta_{C,\beta}(T_{02}) = 0, \quad \delta_{C,\beta}(T_{11}) = 0, \quad \delta_{C,\beta}(T_{21}) = -\beta_0 T_{02}^3,$$

i.e.,

$$\delta_{C,\beta} = 2\beta_0 T_{21} \frac{\partial}{\partial T_{01}} - \beta_0 T_{02}^3 \frac{\partial}{\partial T_{21}}, \quad h \in \text{Ker} \delta_{C,\beta};$$

(b) $i_0 = 2$, i.e., $\beta = (\beta_0, -\beta_0, 0)$ for some $\beta_0 \in \mathbb{K}$; in this case,

$$\delta_{C,\beta}(T_{01}) = 3\beta_0 T_{11}^2, \quad \delta_{C,\beta}(T_{02}) = 0, \quad \delta_{C,\beta}(T_{11}) = -\beta_0 T_{02}^3, \quad \delta_{C,\beta}(T_{21}) = 0,$$

i.e.,

$$\delta_{C,\beta} = 3\beta_0 T_{11}^2 \frac{\partial}{\partial T_{01}} - \beta_0 T_{02}^3 \frac{\partial}{\partial T_{11}}, \quad h \in \text{Ker} \delta_{C,\beta}.$$
Example 5. Let \( g = T_{01}T_{02} + T_{11}T_{12} + T_{21}^2 \) (see Example 2). There is only one variable \( T_{ij} \) with exponent \( i_j > 1 \) in the trinomial. Hence the algebra \( R(g) \) admits elementary derivations of both types: we can take any sequence \( C = (c_0, c_1, 1), \ c_0, c_1 \in \{1, 2\} \), and any vector \( \beta = (\beta_0, \beta_1, \beta_2) \) satisfying the condition \( \beta_0 + \beta_1 + \beta_2 = 0 \). An example of an elementary derivation of Type I is

\[
\delta_{C,\beta} = T_{11}T_{21} \frac{\partial}{\partial T_{02}} + T_{01}T_{21} \frac{\partial}{\partial T_{12}} - T_{01}T_{11} \frac{\partial}{\partial T_{21}}.
\]

Example 6. Let \( g = T_{01}^2 + T_{11}^2 + T_{21}^2 \) (see Example 3). All exponents in this trinomial are greater than 1. Hence there exists no elementary derivation of the form \( \delta_{C,\beta} \).

3. AUXILIARY LEMMAS

We use the notation introduced in the previous section.

Let us recall some definitions. Let \( K \) be an Abelian group, and let \( R \) be a \( K \)-graded algebra. By a \( K \)-prime element of \( R \) we mean a homogeneous nonzero nonunit \( f \in R \) such that if \( f \mid gh \) for homogeneous \( g, h \in R \), then \( f \mid g \) or \( f \mid h \). We say that \( R \) is factorially \( K \)-graded if every nonzero homogeneous nonunit of \( R \) is a product of \( K \)-primes. It is clear that this decomposition is unique up to a permutation of factors and multiplication by units.

Lemma 3. Given a trinomial algebra \( R(g) \), consider its grading by the Abelian group \( K \) described in Construction 1. In the notation of Construction 1, the following statements hold:

a) the generators \( T_{ij} \) are pairwise nonassociated \( K \)-prime elements of \( R(g) \);

b) the algebra \( R(g) \) is factorially \( K \)-graded.

These statements were proved in [12, Proposition 2.2(i) and Theorem 1.1(i)].

The following two lemmas describe basic properties of locally nilpotent derivations.

Lemma 4. Suppose that \( \delta: R \to R \) is a locally nilpotent derivation of a domain \( R \) and \( f, g \in R \). Then

a) if \( f \mid \delta(f) \), then \( \delta(f) = 0 \);

b) a derivation \( f \delta \) is locally nilpotent if and only if \( f \in \text{Ker} \delta \) holds;

c) if \( f \mid \delta(g) \) and \( g \mid \delta(f) \), then \( \delta(f) = 0 \) or \( \delta(g) = 0 \);

d) if \( \delta = \sum_{m \leq i \leq n} \delta_i \), where all derivations \( \delta_i \) are homogeneous with respect to some \( \mathbb{Z} \)-grading on algebra \( R \) and \( \delta_m, \delta_n \neq 0 \), then \( \delta_m \) and \( \delta_n \) are locally nilpotent.

Proof. The proof can be found, e.g., in [1, Principles 5, 7, 14 and Corollary 1.20].

Lemma 5. Let \( R \) be a finitely generated \( \mathbb{Z}^k \)-graded domain. If there exists a nonzero locally nilpotent derivation of \( R \), then \( R \) admits a nonzero homogeneous locally nilpotent derivation.

Proof. Let us denote the given locally nilpotent derivation on \( R \) by \( \delta^{(0)} \). Consider the \( \mathbb{Z} \)-grading on the algebra \( R \) induced by the first component in \( \mathbb{Z}^k \). Since \( R \) is finitely generated, we have

\[
\delta^{(0)} = \sum_{m \leq i \leq n} \delta_i^{(0)}
\]

for some homogeneous derivations \( \delta_i^{(0)} \). It follows from Lemma 4(d) that \( \delta^{(1)} = \delta_n^{(0)} \) is a nonzero locally nilpotent derivation homogeneous with respect to the \( \mathbb{Z} \)-grading by the first component of \( \mathbb{Z}^k \). Applying Lemma 4(d) \( k - 1 \) times to the \( \mathbb{Z} \)-gradings by other components of \( \mathbb{Z}^k \), we obtain a derivation \( \delta^{(k)} \) homogeneous with respect to the grading by all components of \( \mathbb{Z}^k \), i.e., with respect to the \( \mathbb{Z}^k \)-grading. \( \square \)
The following lemma was proved in [15, Lemma 3.4]. For the reader’s convenience, we give its short proof below.

**Lemma 6.** Let \( \delta \) be a deg-homogeneous locally nilpotent derivation of the algebra \( R(\mathfrak{g}) \). Then every monomial \( T_i^{l_i} \) contains at most one variable \( T_{ij} \) such that \( \delta(T_{ij}) \neq 0 \).

**Proof.** Assume the contrary. Then there is a monomial with at least two variables not belonging to the kernel of the derivation. We can assume that these variables are \( T_{01} \) and \( T_{02} \) in \( T_0^0 \).

Consider the following grading on the algebra \( \mathbb{K}[T_{ij}] \):

| \( T_{ij} \) | \( T_{01} \) | \( T_{02} \) | \( T_{03} \) | \( \ldots \) | \( T_{0n_0} \) | \( T_{11} \) | \( T_{12} \) | \( \ldots \) | \( T_{2n_2} \) |
|---|---|---|---|---|---|---|---|---|
| \( \hat{\deg} T_{ij} \) | 0 | \( -l_{01} \) | 0 | \( \ldots \) | 0 | 0 | \( \ldots \) | 0 |

The trinomial \( \mathfrak{g} \) is homogeneous (of degree 0) with respect to this grading; therefore, the \( \hat{\deg} \)-grading is a well-defined grading on the quotient algebra \( R(\mathfrak{g}) \).

By Lemma 1, it follows that the derivation \( \delta \) is \( \hat{\deg} \)-homogeneous. We have the following two cases.

**Case 1:** \( \hat{\deg} \delta \geq 0 \). In this case, \( \hat{\deg} \delta(T_{01}) = \hat{\deg} \delta + \hat{\deg} T_{01} > 0 \). Note that \( T_{01} \) is a unique variable with positive degree. Hence every monomial in \( \delta(T_{01}) \) contains \( T_{01} \) and, therefore, \( T_{01} \) divides \( \delta(T_{01}) \). But \( \delta(T_{01}) \neq 0 \) by assumption. This contradicts Lemma 4 (a).

**Case 2:** \( \hat{\deg} \delta \leq 0 \). In this case, \( \hat{\deg} \delta(T_{02}) = \hat{\deg} \delta + \hat{\deg} T_{02} < 0 \). As in the preceding case, \( T_{02} \) divides \( \delta(T_{02}) \), which contradicts Lemma 4 (a).

\[ \square \]

4. **MAIN RESULTS**

The following proposition and its proof are essentially variations of Theorem 4.4 of [2]. The only difference is that we consider only the case of trinomial hypersurfaces, but replace the assumption that the given homogeneous locally nilpotent derivation is primitive by the assumption that the images of monomials under this derivation are proportional. The latter assumption is weaker, since, according to [2, Proposition 3.5], the dimension of the homogeneous component of degree \( w \) is equal to 1, provided that \( w - \deg \mathfrak{g} \) does not lie in the weight cone.

**Proposition 1.** Let \( \delta: R(\mathfrak{g}) \to R(\mathfrak{g}) \) be a deg-homogeneous locally nilpotent derivation. Suppose that \( \delta(T_0^{l_0}), \delta(T_1^{l_1}), \) and \( \delta(T_2^{l_2}) \) lie in a subspace of \( R(\mathfrak{g}) \) of dimension 1. Then the derivation \( \delta \) is elementary.

**Proof.** By Lemma 6, each monomial \( T_i^{l_i} \) contains at most one variable \( T_{ij} \) such that \( \delta(T_{ij}) \neq 0 \). Let \( \mathcal{R} = \{ i \mid \exists c_i: \delta(T_{ic_i}) \neq 0 \} \).

Let us prove that if \( \delta(T_{ic_i}) \neq 0 \) and \( \delta(T_{kc_k}) \neq 0 \), then \( l_{ic_i} = 1 \) or \( l_{kc_k} = 1 \). Assume that, on the contrary, \( l_{ic_i} \neq 1 \) and \( l_{kc_k} \neq 1 \). According to what was said above, we have

\[ \delta(T_i^{l_i}) = \delta(T_{ic_i}) \frac{\partial T_i^{l_i}}{\partial T_{ic_i}}. \]  

Since \( l_{ic_i} \neq 1 \), we have \( T_{ic_i} \mid \partial T_i^{l_i} / \partial T_{ic_i} \); therefore, \( T_{ic_i} \mid \delta(T_i^{l_i}) \). Similarly, \( T_{kc_k} \mid \delta(T_k^{l_k}) \). By assumption, \( \delta(T_i^{l_i}) \) and \( \delta(T_k^{l_k}) \) are proportional; hence \( T_{kc_k} \mid \delta(T_i^{l_i}) \) and \( T_{ic_i} \mid \delta(T_k^{l_k}) \). At the same time, by (2), \( \delta(T_i^{l_i}) \) is the product of \( \delta(T_{ic_i}) \) and the variables not equal to \( T_{kc_k} \), whence we have (see Lemma 3 (b)) \( T_{kc_k} \mid \delta(T_{ic_i}) \). For the same reason, \( T_{ic_i} \mid \delta(T_{kc_k}) \), which contradicts Lemma 4 (c).

Thus, \( l_{ic_i} > 1 \) holds for at most one \( i \in \mathcal{R} \).
The elements $\delta(T_i^{l_i})$ are proportional, that is, there exists an $f \in R(\mathfrak{g})$ such that $\delta(T_i^{l_i}) \in \mathbb{K} f$ for any $i$. It follows from equality (2) that $\partial T_i^{l_i}/\partial T_{ic_i}$ divides $f$ for all $i \in \mathfrak{K}$. Thus, applying Lemma 3(b), we see that the product $\prod_{i \in \mathfrak{K}} \partial T_i^{l_i}/\partial T_{ic_i}$ divides $f$. We denote this ratio by $h$. From (2) we can find $\delta(T_{ic_i})$:

$$f = h \prod_{i \in \mathfrak{K}} \frac{\partial T_i^{l_i}}{\partial T_{ic_i}}, \quad \delta(T_{ic_i}) = \begin{cases} \beta_i h \prod_{k \in \mathfrak{K} \setminus \{i\}} \frac{\partial T_k^{l_k}}{\partial T_{kc_k}}, & i \in \mathfrak{K}, \\ 0, & i \notin \mathfrak{K}. \end{cases}$$

We set $\beta_i = 0$ for all $i \notin \mathfrak{K}$. Let us show that the sum of all $\beta_i$ is equal to 0. Note that $\mathfrak{g}$ should divide

$$\delta(\mathfrak{g}) = \sum_{i \in \mathfrak{K}} \delta(T_{ic_i}) \frac{\partial T_i^{l_i}}{\partial T_{ic_i}} = \sum_{i \in \mathfrak{K}} \left( \beta_i h \prod_{k \in \mathfrak{K} \setminus \{i\}} \frac{\partial T_k^{l_k}}{\partial T_{kc_k}} \right) \frac{\partial T_i^{l_i}}{\partial T_{ic_i}} = h \left( \sum_{i \in \mathfrak{K}} \beta_i \right) \prod_{i \in \mathfrak{K}} \frac{\partial T_k^{l_k}}{\partial T_{kc_k}}$$

in the algebra $\mathbb{K}[T_{ij}]$, since $\delta$ is a well-defined derivation of the algebra $R(\mathfrak{g})$. But the trinomial $\mathfrak{g}$ and the monomial $\prod_{i \in \mathfrak{K}} (\partial T_k^{l_k}/\partial T_{kc_k})$ are coprime in the factorial algebra $\mathbb{K}[T_{ij}]$. Hence $\mathfrak{g}$ divides the sum $\sum_{i \in \mathfrak{K}} \beta_i$, whence

$$\sum_{i \in \mathfrak{K}} \beta_i = \sum_i \beta_i = 0.$$

Now, arbitrarily completing the $c_i$, $i \in \mathfrak{K}$, to a sequence $C = (c_0, c_1, c_2)$, we obtain $\delta = h\delta_{C, \beta}$, where $h$ belongs to the kernel of $\delta_{C, \beta}$ by Lemma 4(b).

The following theorem is the main result of this paper.

**Theorem 1.** Let $R(\mathfrak{g})$ be a trinomial algebra. Suppose that there is at most one monomial in $\mathfrak{g}$ containing a variable with exponent 1, i.e., $(l_{ij1} = l_{ij2} = 1) \Rightarrow (i_1 = i_2)$.

Then every deg-homogeneous locally nilpotent derivation of the algebra $R(\mathfrak{g})$ is elementary of Type II.

**Proof.** Let $\delta$ be any deg-homogeneous locally nilpotent derivation of the algebra $R(\mathfrak{g})$. By Lemma 6, every monomial $T_i^{l_i}$ contains at most one variable $T_{ic_i}$ with $\delta(T_{ic_i}) \neq 0$. Let $\mathfrak{K} = \{i \mid \exists c_i : \delta(T_{ic_i}) \neq 0\}$.

1) Fix any $i \in \mathfrak{K}$ for which $l_{ic_i} > 1$. We claim that $T_{ic_i}$ divides $\delta(T_{kc_k})$ for any $k \in \mathfrak{K} \setminus \{i\}$.

Let $\mathbb{Z}_m = \{[0]_m, [1]_m, \ldots, [m - 1]_m\}$ be the cyclic group of order $m$. Consider the following $\mathbb{Z}_{l_{ic_i}}$-grading on $R(\mathfrak{g})$:

$$\begin{array}{cccccccc}
T_{ks} & T_{01} & T_{02} & \cdots & T_{ic_i} & \cdots & T_{2n_2} \\
\hat{\deg} T_{ks} & [0]_{l_{ic_i}} & [0]_{l_{ic_i}} & \cdots & [1]_{l_{ic_i}} & \cdots & [0]_{l_{ic_i}}
\end{array}$$

It is well defined, since $l_{ic_i}[1]_{l_{ic_i}} = [0]_{l_{ic_i}}$, i.e., all monomials in $\mathfrak{g}$ have the same degree $[0]_{l_{ic_i}}$. By Lemma 1, the derivation $\delta$ is deg-homogeneous.

If $\hat{\deg} \delta(T_{ic_i}) \neq [0]_{l_{ic_i}}$, then $T_{ic_i}$ divides $\delta(T_{ic_i})$. Indeed, this inequality implies that the degrees of all monomials in $\delta(T_{ic_i})$ do not equal $[0]_{l_{ic_i}}$, and this is possible only if every monomial contains $T_{ic_i}$ (since $T_{ic_i}$ is a unique variable with nonzero degree).

On the other hand, the fact that $T_{ic_i}$ divides $\delta(T_{ic_i})$, in conjunction with $\delta(T_{ic_i}) \neq 0$, contradicts Lemma 4(a). Thus, $\hat{\deg} \delta(T_{ic_i}) = [0]_{l_{ic_i}}$; it follows that $\hat{\deg} \delta = [l_{ic_i} - 1]_{l_{ic_i}} \neq [0]_{l_{ic_i}}$. Therefore, we have $\hat{\deg} \delta(T_{ks}) \neq [0]_{l_{ic_i}}$ for all $(k, s) \neq (i, c_i)$. In particular, $\hat{\deg} \delta(T_{kc_k}) \neq [0]_{l_{ic_i}}$ for $k \in \mathfrak{K} \setminus \{i\}$, i.e., $T_{ic_i}$ divides $\delta(T_{kc_k})$.
(2) Let us show that $|\mathcal{R}| \leq 2$. Assume the contrary. By the assumptions of the theorem, the number of $i$ such that $l_{ic_i} = 1$, does not exceed 1, whence $\{|i \in \mathcal{R} \mid l_{ic_i} > 1\} \geq 2$. Take any distinct $i$ and $k$ in this set. According to (1), $T_{ic_i}$ divides $\delta(T_{ic_k})$ and $T_{kc_k}$ divides $\delta(T_{ic_i})$, which contradicts Lemma 4 (c).

(3) Thus, $|\mathcal{R}| \leq 2$. Hence there are at most two nonzero $\delta(T_i^l)$. On the other hand,

$$\delta(T_i^{l_0}) + \delta(T_i^{l_1}) + \delta(T_i^{l_2}) = \delta(g) = 0$$

in $R(g)$. Consequently, the $\delta(T_i^l)$, $i = 0, 1, 2$, lie in a subspace of dimension 1. By Proposition 1, the derivation $\delta$ is elementary. In addition, there are at most two nonzero $\delta(T_i^l)$. This implies that $\delta$ is elementary of Type II. \hfill $\square$

If a trinomial contains a linear term (i.e., $n_i l_{i1} = 1$ for some $i = 0, 1, 2$), then the corresponding trinomial hypersurface is isomorphic to the affine space $\mathbb{K}^{n-1}$. In what follows, we assume that $n_i l_{i1} > 1$ for all $i = 0, 1, 2$. According to [12, Theorem 1.1(ii)], in this case, the following statement holds.

**Proposition 2.** The following conditions are equivalent:

a) the algebra $R(g)$ is factorial;

b) the group $K$ in Construction 1 is torsion free;

c) the numbers $d_i := \text{GCD}(l_{i1}, \ldots, l_{in_i})$ are pairwise coprime.

This proposition enables us to prove some consequences of Theorem 1.

**Corollary 1.** If an algebra $R(g)$ is nonfactorial, then any deg-homogeneous locally nilpotent derivation of this algebra is elementary of Type II.

**Proof.** It suffices to show that all algebras that do not satisfy the condition of Theorem 1 are factorial. By assumption, $g$ has at least two monomials containing variables with exponent 1. Therefore, at least two of the numbers $d_0$, $d_1$, and $d_2$ are equal to 1, and hence $d_0$, $d_1$, and $d_2$ are pairwise coprime. By Proposition 2, the algebra $R(g)$ is factorial. \hfill $\square$

**Corollary 2.** The algebra $R(g)$ admits a nonzero deg-homogeneous locally nilpotent derivation if and only if $l_{ij} = 1$ for some $i = 0, 1, 2$ and $j = 1, \ldots, n_i$.

**Proof.** If the condition $l_{ij} = 1$ holds for some pair $(i, j)$, then there exists a derivation of the form $\delta_{C, \beta}$ (see Construction 2, Case 2). It is deg-homogeneous and locally nilpotent.

We prove the reverse implication by contradiction. Let $l_{ij} \geq 2$ for all $i = 0, 1, 2$ and $j = 1, \ldots, n_i$. Together with Theorem 1, this implies that all deg-homogeneous locally nilpotent derivations of $R(g)$ are of the form $h \delta_{C, \beta}$ (where $h \in \text{Ker} \delta_{C, \beta}$). However, it follows from Construction 2 that there is no derivation of the form $\delta_{C, \beta}$ if $l_{ij} \geq 2$ for all $i = 0, 1, 2$ and $j = 1, \ldots, n_i$. This contradiction proves the corollary. \hfill $\square$

An affine variety is said to be **rigid** if its algebra of regular functions admits no nonzero locally nilpotent derivation. Geometrically, this means that the variety admits no nontrivial $G_a$-action. The automorphism group of a rigid trinomial variety was described in [3, Theorem 3].

Corollary 2 enables us to obtain a new proof of a result obtained earlier in [14, Theorem 1].

**Corollary 3.** A factorial trinomial hypersurface $X(g)$ is rigid if and only if $l_{ij} \geq 2$ for all $i = 0, 1, 2$ and $j = 1, \ldots, n_i$. 
**Proof.** If the condition $l_{ij} \geq 2$ does not hold for some $(i, j)$, then there exists a derivation of the form $\delta_{C,\beta}$ (see Construction 2, Case 2). This implies the only if part.

Let us prove the if part. By Corollary 2, there exists no nonzero deg-homogeneous locally nilpotent derivation of the algebra $R(g)$. Since the trinomial hypersurface is factorial, it follows by Proposition 2 that the group $K$ is torsion free. By Lemma 5, the nonexistence of nonzero deg-homogeneous locally nilpotent derivations implies the nonexistence of any nonzero locally nilpotent derivations of $R(g)$, which means rigidity.

**Example 7.** Consider

$$g = T_{01}T_{02}^3 + T_{11}^3 + T_{21}^2$$

(see Examples 1 and 4). This trinomial satisfies the conditions of Theorem 1; hence all deg-homogeneous locally nilpotent derivations $\delta$ of $R(g)$ are divided into two classes:

(a) $\delta = 2h\beta_0T_{21}\frac{\partial}{\partial T_{01}} - h\beta_0T_{02}^3\frac{\partial}{\partial T_{21}}$, 

where $h \in \ker \delta_{C,\beta}$ for $C = (1, 1, 1), \beta = (\beta_0, 0, -\beta_0)$,

(b) $\delta = 3h\beta_0T_{11}^2\frac{\partial}{\partial T_{01}} - h\beta_0T_{02}^3\frac{\partial}{\partial T_{11}}$, 

where $h \in \ker \delta_{C,\beta}$ for $C = (1, 1, 1), \beta = (\beta_0, -\beta_0, 0)$.

Some of them are not primitive, i.e., $\deg \delta = \deg h\delta_{C,\beta}$ may lie in the weight cone $\omega$ for some $h \in \ker \delta_{C,\beta}$. Let us consider an example. The weight monoid is generated by vectors

$$\deg T_{01} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}, \quad \deg T_{02} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \deg T_{11} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad \deg T_{21} = \begin{pmatrix} 0 \\ 3 \end{pmatrix},$$

and therefore the weight cone $\omega$ is the angle

$$\{-u \leq v \leq u\}, \quad \text{where} \quad \deg = \begin{pmatrix} u \\ v \end{pmatrix}$$

(see the figure).

In case (a), we have

$$\deg \delta_{C,\beta} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$
Since, for any $k \in \mathbb{N}$, the polynomial $h = T_{11}^k$ belongs to the kernel of $\delta_{C, \beta}$, it follows that the derivation $T_{11}^k \cdot \delta_{C, \beta}$ is homogeneous and locally nilpotent. Its degree equals

$$k \begin{pmatrix} 0 \\ 2 \\ 3 \\ 0 \\ 2k \end{pmatrix}$$

and lies in the weight cone for $k \geq 2$.

**Example 8.** Let

$$g = T_{01}^2 + T_{11}^2 + T_{21}^2$$

(see Examples 3 and 6). By Corollary 2, there exists no deg-homogeneous locally nilpotent derivation of $R(g)$.

Nevertheless, the algebra $R(g)$ admits a nonzero locally nilpotent derivation. For example, consider the following derivation $\delta$:

$$\delta(T_{01}) = i T_{21}, \quad \delta(T_{11}) = -T_{21}, \quad \delta(T_{21}) = -i T_{01} + T_{11},$$

where $i \in \mathbb{K}, i^2 = -1$. The derivation $\delta$ is locally nilpotent, since

$$\delta(-i T_{01} + T_{11}) = 0, \quad \delta^2(T_{21}) = 0, \quad \delta^3(T_{01}) = 0.$$

### 5. OPEN QUESTIONS

In this section, we discuss several open questions related to locally nilpotent derivations of trinomial algebras.

**Conjecture 1.** All deg-homogeneous locally nilpotent derivations of a trinomial algebra are elementary.

In the proof of Theorem 1, we were able to apply Proposition 1 and, as a consequence, prove the conjecture for the trinomial algebras $R(g)$ corresponding to trinomials $g$ containing at most one monomial including variables with exponent 1. We wish to prove that Proposition 1 applies also in the case where there exist at least two monomials in $g$ including variables with exponent 1. In this case, by Proposition 2, the algebra $R(g)$ is factorial, the group $K$ is torsion free, and the quasitorus whose action corresponds to the deg-grading is a torus of dimension $n - 2$. In [16], all deg-homogeneous locally nilpotent derivations for several trinomial hypersurfaces not satisfying the conditions of Theorem 1 were described; in particular, the following statements were proved (Theorems 3.22 and 3.24 in [16]).

**Example 9 (cf. Example 5).** Any deg-homogeneous locally nilpotent derivation of the algebra $R(g)$, where $g = T_{01}T_{02} + T_{11}T_{12} + T_{21}^2$, has the form

$$\lambda T_{01}^k T_{1j} T_{21}^p \left( T_{1j} \frac{\partial}{\partial T_{1j}} - T_{0i} \frac{\partial}{\partial T_{1j}} \right)$$

or

$$T_{0i}^k T_{1j} (\beta T_{01}T_{02} - \alpha T_{11}T_{12})^p \left( \alpha T_{1j}T_{21} \frac{\partial}{\partial T_{0i}} + \beta T_{0i}T_{21} \frac{\partial}{\partial T_{1j}} - \frac{\alpha + \beta}{2} T_{0i}T_{1j} \frac{\partial}{\partial T_{21}} \right),$$

where $k, l, p \in \mathbb{Z}_{\geq 0}, \{i, \bar{i}\} = \{j, \bar{j}\} = \{1, 2\}$ and $\alpha, \beta, \lambda \in \mathbb{K}, \alpha + \beta \neq 0$.

**Example 10.** Any deg-homogeneous locally nilpotent derivation of the algebra $R(g)$, where $g = T_{01}T_{02} + T_{11}T_{12} + T_{21}T_{22}$, has the form

$$T_{0i}^k T_{1j}^k T_{21}^p (\gamma T_{11}T_{12} - \beta T_{21}T_{22})^p \left( \alpha T_{1i}T_{21} \frac{\partial}{\partial T_{0i}} + \beta T_{0i}T_{21} \frac{\partial}{\partial T_{1j}} + \gamma T_{0i}T_{1i} \frac{\partial}{\partial T_{21}} \right),$$

where $k_0, k_1, k_2, p \in \mathbb{Z}_{\geq 0}, \{i_0, \bar{i}_0\} = \{i_1, \bar{i}_1\} = \{i_2, \bar{i}_2\} = \{1, 2\}, \alpha, \beta, \gamma \in \mathbb{K}, \alpha + \beta + \gamma = 0$ and no two of the numbers $\alpha, \beta$, and $\gamma$ vanish simultaneously.
One can check that, for all derivation $\delta$ in these examples, the images of the monomials $\delta(T_0^0)$, $\delta(T_1^1)$, and $\delta(T_2^2)$ are proportional, i.e., Proposition 1 applies and the conjecture is true.

In addition to the “finest” deg-grading of the algebra $R(\mathfrak{g})$ by the group $K$, one can also consider the grading of $R(\mathfrak{g})$ by the torsion free component $K_0$ of $K$. Note that any locally nilpotent derivation homogeneous with respect to the deg-grading by the group $K$ is homogeneous with respect to the grading by $K_0$. By Proposition 2, the deg-grading by $K$ coincides with the grading by $K_0$ if and only if the algebra $R(\mathfrak{g})$ is factorial.

Example 11. Let

$$\mathfrak{g} = T_{01}^2 + T_{11}^2 + T_{21}^2$$

(see Examples 3, 6 and 8). As shown in Example 3, we have $K = \mathbb{Z} + \mathbb{Z}_2 + \mathbb{Z}_2$. Therefore, $K_0 = \mathbb{Z}$, and degree with respect to the grading by $K_0$ is equal to the standard degree of a polynomial (the sum of exponents). Hence the derivation $\delta$ in Example 8 is homogeneous with respect to the grading by $K_0$. At the same time, according to Example 8, there is no nonzero locally nilpotent derivation homogeneous with respect to the $K$-grading.

If the algebra $R(\mathfrak{g})$ admits no nonzero $K_0$-homogeneous locally nilpotent derivation, then, by Lemma 5, the algebra $R(\mathfrak{g})$ admits no nonzero locally nilpotent derivation, i.e., the variety $X(\mathfrak{g})$ is rigid.

Question 1. Describe all locally nilpotent derivations of a trinomial algebra $R(\mathfrak{g})$ that are homogeneous with respect to the $K_0$-grading.

 Whereas the $K$-grading corresponds geometrically to the action of a quasitorus on the trinomial hypersurface, the $K_0$-grading corresponds to the action of its neutral component, i.e., of the maximal torus. This means that Question 1 reduces to the problem of describing the $G_\mathfrak{a}$-actions normalized by the maximal torus, that is, of the root subgroups in the automorphism group of a trinomial hypersurface.

Question 2. Describe all (not necessary homogeneous) locally nilpotent derivations of a trinomial algebra $R(\mathfrak{g})$.

ACKNOWLEDGMENTS

The author is grateful to her supervisor I. V. Arzhantsev for posing the problem and constant support and to S. A. Gaifullin for useful discussions and comments.

FUNDING

This work was supported by the Foundation for the Advancement of Theoretical Physics and Mathematics “BASIS.”

REFERENCES

1. G. Freudenburg, Algebraic Theory of Locally Nilpotent Derivations, in Encyclopaedia Math. Sci. (Springer, Berlin, 2006), Vol. 136.
2. I. Arzhantsev, J. Hausen, E. Herppich, and A. Liendo, “The automorphism group of a variety with torus action of complexity one,” Mosc. Math. J. 14 (3), 429–471 (2014).
3. I. Arzhantsev and S. Gaifullin, “The automorphism group of a rigid affine variety,” Math. Nachr. 290 (5-6), 662–671 (2017).
4. M. Demazure, “Sous-groupes algébriques de rang maximum du groupe de Cremona,” Ann. Sci. École Norm. Sup. (4) 3, 507–588 (1970).
5. T. Oda, Convex Bodies and Algebraic Geometry. An Introduction to the Theory of Toric Varieties (Springer-Verlag, Berlin, 1988).
6. A. Liendo, “Affine $T$-varieties of complexity one and locally nilpotent derivations,” Transform. Groups 15 (2), 398–425 (2010).
7. I. Arzhantsev and A. Liendo, “Polyhedral divisors and SL$_2$-actions on affine $T$-varieties,” Michigan Math. J. 61 (4), 731–762 (2012).
8. A. Liendo, “$\mathbb{G}_a$-actions of fiber type on affine $T$-varieties,” J. Algebra 324 (12), 3653–3665 (2010).
9. B. Sturmfels, Gröbner Bases and Convex Polytopes, in Univ. Lecture Ser. (Amer. Math. Soc., Providence, RI, 1996), Vol. 8.
10. J. Hausen and H. Süss, “The Cox ring of an algebraic variety with torus action,” Adv. Math. 225 (2), 977–1012 (2010).
11. J. Hausen, E. Herppich and H. Süss, “Multigraded factorial rings and Fano varieties with torus action,” Doc. Math. 16, 71–109 (2011).
12. J. Hausen and E. Herppich, “Factorially graded rings of complexity one,” in Torsors, Étale Homotopy and Applications to Rational Points, London Math. Soc. Lecture Note Ser., (Cambridge Univ. Press, Cambridge, 2013), Vol. 405, pp. 414–428.
13. J. Hausen and M. Wrobel, “Non-complete rational $T$-varieties of complexity one,” Math. Nachr. 290 (5-6), 815–826 (2017).
14. I. Arzhantsev, “On rigidity of factorial trinomial hypersurfaces,” Internat. J. Algebra Comput. 26 (5), 1061–1070 (2016).
15. S. Gaifullin, Automorphisms of Danielewski Varieties, arXiv: 1709.09237 (2017).
16. P. Yu. Kotenkova, Torus actions and locally nilpotent derivations, Cand. Sci. (Phys.–Math.) Dissertation (Moskov. Univ., Moscow, 2014) [in Russian].