Quantum Function Computation Using Sublogarithmic Space

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We examine the power of quantum Turing machines (QTM’s) which compute string-valued functions of their inputs using small workspace. A (probabilistic or quantum) Turing machine has three tapes: A read-only input tape, a read/write work tape, and a “1.5-way” write-only output tape. Consider a partial function \( f : \Sigma_1^* \rightarrow \Sigma_2^* \), where \( \Sigma_1 \) and \( \Sigma_2 \) are the input and output tape alphabets, respectively. We say that such a machine \( M \) computes the function \( f \) with bounded error if, when started with string \( w \in \Sigma_1^* \) on its input tape,

1. the probability that \( M \) halts by arriving at an accept state, having written the string \( f(w) \) on its output tape, is at least \( \frac{2}{3} \), if \( f(w) \) is defined, and,

2. the probability that \( M \) halts by arriving at a reject state is at least \( \frac{2}{3} \), if \( f(w) \) is undefined.

We compare QTM’s with probabilistic Turing machines (PTM’s) under the same space bounds. It is straightforward to show that any function computable by a PTM in space \( s \) can be computed in space \( s \) by a QTM, for any space bound \( s \). We prove that QTM’s are strictly superior to PTM’s in this regard for any \( s = o(\log n) \), by demonstrating that there exists a quantum finite state transducer (QFST), that is, a QTM which does not use its work tape, and moves the input head to the right in every step of its execution, which computes the function

\[
f_1(x) = \begin{cases} w, & \text{if } x = w2w, \text{ where } w \in \{0, 1\}^*, \\ \text{undefined}, & \text{otherwise} \end{cases}
\]

with bounded error. This function cannot be computed with bounded error by any PTM using \( o(\log n) \) space.

Our QFST for \( f_1 \) is also interesting as a language recognition device, in the sense that it is the first example for a quantum finite-state machine which traverses its input with a one-way head, and is able to recognize a nonregular language with bounded error.

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Models
A (probabilistic or quantum) Turing machine (PTM or QTM) has a finite register and three tapes:
1. A read-only input tape,
2. a read/write work tape, and
3. a “1.5-way” write-only output tape.
The finite register is observed after each transition in order to check whether the computation has halted by accepting or rejecting, or not. The register is refreshed before the next transition. This mechanism allows QTM’s to implement general quantum operators.

A quantum finite state transducer (QFST) is a QTM which does not use its work tape, and moves the input head to the right in every step of its execution.

Function Computation
Consider a partial function \( f : \Sigma_1^* \rightarrow \Sigma_2^* \), where \( \Sigma_1 \) and \( \Sigma_2 \) are the input and output tape alphabets, respectively. We say that a machine \( M \) computes the function \( f \) with bounded error if, when started with string \( w \in \Sigma_1^* \) on its input tape,
1. the probability that \( M \) halts by observing an accept symbol in the register, having written the string \( f(w) \) on its output tape, is at least \( \frac{2}{3} \), if \( f(w) \) is defined, and,
2. the probability that \( M \) halts by observing a reject symbol in the register is at least \( \frac{2}{3} \), if \( f(w) \) is undefined.

A QFST algorithm
We compute the function
\[
f_1(x) = \begin{cases} 
  w, & \text{if } x = wcw, \text{ where } w \in \{a, b\}^*, \\
  \text{undefined}, & \text{otherwise}
\end{cases}
\]
with bounded error by a QFST as follows:

1. The computation splits into three paths, \( \text{path}_1 \), \( \text{path}_2 \), and \( \text{path}_3 \), with equal probability at the beginning.
2. \( \text{path}_3 \) rejects immediately.
3. \( \text{path}_1 \) (\( \text{path}_2 \)) outputs \( w_1 \) (\( w_2 \)) if the input is of the form \( w_1cw_2 \), where \( w_1, w_2 \in \{a, b\}^* \).
   (a) If the input is not of the form \( w_1cw_2 \), both paths reject.
   (b) Otherwise, at the end of the computation, \( \text{path}_1 \) and \( \text{path}_2 \) perform the following QFT:

\[
\text{path}_1 \rightarrow \frac{1}{\sqrt{2}} |q, "Accept", w_1\rangle + \frac{1}{\sqrt{2}} |q, "Reject", w_1\rangle \\
\text{path}_2 \rightarrow \frac{1}{\sqrt{2}} |q, "Accept", w_2\rangle - \frac{1}{\sqrt{2}} |q, "Reject", w_2\rangle
\]

The configurations belonging to \( \text{path}_1 \) and \( \text{path}_2 \) interfere with each other, i.e., the machine accepts with probability \( \frac{2}{3} \), if and only if the input is of the form \( wcw \), \( w \in \{a, b\}^* \), the case where \( f_1 \) is defined.

Main Result
Fact 1. Any function computable by a PTM in space \( s \) can be computed in space \( s \) by a QTM, for any space bound \( s \).
Fact 2. No PTM can recognize \( L_{pal} = \{ w \mid w = a^k, w \in \{a, b\}^* \} \) with bounded error using \( o(\log n) \) space.
Fact 3. Function \( f_1 \) cannot be computed with bounded error by any PTM using \( o(\log n) \) space. Otherwise, we would have a PTM recognizing the language \( \{wcw \mid w \in \{a, b\}^* \} \) and also \( L_{pal} \) with bounded error using \( o(\log n) \) space. This would contradict Fact 2.

Main Result. We prove that QTM’s are strictly superior to PTM’s in function computation for any space \( o(\log n) \) due to the QFST algorithm and Facts 1 and 3.

Remark. Our QFST for \( f_1 \) is also interesting as a language recognition device, in the sense that it is the first example for a quantum finite-state machine which traverses its input with a one-way head, and is able to recognize a nonregular language with bounded error.