Extensions of the Algorithmic Lovász Local Lemma

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Abstract

We consider recent formulations of the algorithmic Lovász Local Lemma by Achlioptas-Iliopoulos-Kolmogorov [2] and by Achlioptas-Iliopoulos-Sinclair [3]. These papers analyze a random walk algorithm for finding objects that avoid undesired “bad events” (or “flaws”), and prove that under certain conditions the algorithm is guaranteed to find a “flawless” object quickly. We show that conditions proposed in these papers are incomparable, and introduce a new family of conditions that includes those in [2, 3] as special cases.

Secondly, we extend our previous notion of “commutativity” in [15] to this more general setting, and prove that it allows to use an arbitrary strategy for selecting the next flaw to address. In the special case of primary flaws we prove a stronger property: the flaw selection strategy does not affect at all the expected number of steps until termination, and also does not affect the distribution induced by the algorithm upon termination. This applies, in particular, to the single-clause backtracking algorithm for constraint satisfaction problems considered in [3].

1 Introduction

The Lovász Local Lemma (LLL), due to Erdős and Lovász [7], is a powerful tool for proving the existence of combinatorial objects with certain properties. Informally, it can be stated as follows: given a probability measure $\omega$ over a finite set $\Omega$ and a collection of bad events, LLL asserts that the probability of a good event (i.e. an event that avoids all bad events) is positive provided that each bad event has a sufficiently small probability and depends only on a small number of other bad events. Extensions of the original LLL formulation include Shearer’s condition [19], lopsided LLL [8, 4], and LLL with the “cluster expansion” condition [5].

Note that LLL does not say how to find good objects (whose probability can be exponentially small). In a breakthrough paper Moser and Tardos showed [16] that in the variable model a simple local search algorithm is guaranteed to find a good object quickly under LLL conditions. A large number of follow-up work proposed different extensions and generalizations [14, 13, 6, 17, 9, 10, 1, 11, 15, 2, 8].

We focus on two recent works by Achlioptas-Iliopoulos-Kolmogorov [2] and by Achlioptas-Iliopoulos-Sinclair [3]. Both use a random walk algorithm of the following form: given current state $\sigma$, select flaw $f$ present in $\sigma$ and “address” it, i.e. sample a new state according to distribution $\rho(\cdot|f,\sigma)$ provided by the user. Under some conditions the algorithm is guaranteed to find a flawless object after a small number of steps with high probability.

The conditions need to reason about interactions between flaws. The work [2] uses a potential causality graph for this purpose. This graph has edge from $f$ to $g$ if addressing $f$ may cause $g$ to appear. The condition in [3] uses a more refined information about the problem: it tracks which sets of flaws may appear when addressing $f$. Furthermore, [3] introduces the notion of primary flaws: these are flaws that are never eradicated by addressing other flaws. As observed in [3], primary flaws appear in some backtracking algorithms.
Contribution 1: new condition

The first question that we study is the relationship between criteria in [2] and [3]. On the one hand, [2] takes into account only pairwise interactions, while the criterion in [3] goes beyond pairwise interactions. One the other hand, the criterion in [2] sums only over independent subsets of flaws (resembling the “cluster expansion” condition of Bissacot [5]); this does not have an analogue in [3]. We show on a toy example that conditions in [2] and [3] are incomparable: there exist problems in which one of the criteria works but not the other one.

Motivated by this fact, we introduce a new family of conditions that includes the two previous conditions as special cases. For each pair of flaws \((f, g)\) the user needs to choose number \(\alpha(f, g) \in [0, 1]\), which we call the strength of interaction between \(f\) and \(g\). If \((f, g)\) do not interact (or more precisely if \((f, g)\) is not in the potential causality graph) then the best choice is to set \(\alpha(f, g) = 0\). Setting \(\alpha(f, g) = 1\) for remaining pairs recovers the condition in [2], while setting \(\alpha(f, g) = 0\) recovers the condition in [3].

Note that the strength of interaction between bad events also appeared in the “soft LLL version” of Scott and Sokal [18]. We are not aware of any deeper connection between the two approaches.

We remark that at the moment we do not have a concrete application where the new condition would improve on previous ones. We thus view the significance of the new condition as mainly theoretical, i.e. as clarifying the relationship between the conditions in [2] and [3]. A more practical aspect of our work is given by the next contribution.

Contribution 2: commutativity

Observe that the algorithms in [2, 3] require a very specific rule for choosing the next flaw to address (such as choosing the lowest available flaw w.r.t. to some fixed total order on flaws). In our earlier paper [15] we introduced the commutativity condition and showed, among other things, that it allows to use an arbitrary rule for selecting flaws under the condition from [2].

In this work we introduce a new commutativity condition applicable to our more general framework (and in particular to the framework in [3]). This condition generalizes the one in [15]. We show that it allows an arbitrary flaw selection rule to be used. In the special case of primary flaws we prove a much stronger property: the flaw selection rule does not affect at all the expected runtime of the algorithm, as well as the distribution induced by the algorithm upon termination (independent of whether an algorithmic LLL condition holds or not). These results apply, in particular, to the backtracking algorithm for constraint satisfaction problems considered in [3].

Next, in Section 2 we introduce some technical background and notation, and review conditions in [2, 3]. In Section 3 we formally state our results, and prove them in the remaining sections.

2 Background and preliminaries

Let \(\Omega\) be a (large) finite set of objects and \(F\) be a set of flaws, where a flaw \(f \in F\) is a non-empty set of “bad” objects, i.e. \(f \subseteq \Omega\). Flaw \(f\) is said to be present in \(\sigma\) if \(\sigma \in f\). Let \(F_\sigma = \{f \in F | \sigma \in f\}\) be the set of flaws present in \(\sigma\). Object \(\sigma\) is called flawless if \(F_\sigma = \emptyset\). We will consider the following algorithm.

\begin{verbatim}
Algorithm 1: Random Walk. Input: distribution \(\omega^\text{init}\), flaw selection strategy \(\Lambda\).
1 sample \(\sigma \in \Omega\) according to \(\omega^\text{init}\)
2 while \(F_\sigma\) non-empty do
3     select a flaw \(f \in F_\sigma\) according to strategy \(\Lambda\)
4     sample \(\sigma' \in \Omega\) according to some distribution \(\rho(\sigma' | f, \sigma)\), set \(\sigma \leftarrow \sigma'\).
\end{verbatim}
Clearly, if the algorithm terminates then it produces a flawless object $\sigma$. The choice of distributions $\omega^{\text{init}}(\cdot)$ and $\rho(\cdot|f,\sigma)$ for $\sigma \in f \in F$ depends heavily on the application. For the algorithm to be efficient, it should be possible to sample from these distributions efficiently.

Strategy $\Lambda$ is a (possibly randomized) function that outputs flaw $f \in F_\sigma$ based on the entire past execution history. One popular strategy is choose some permutation $\pi$ beforehand, and then output the lowest flaw in $F_\sigma$ w.r.t. $\pi$. We call this the $\pi$-strategy.

It is known that under certain conditions Algorithm 1 terminates with high probability after a small number of steps. Below we review three existing conditions to which we refer as conditions (A), (B) and (C). The first two are due to Achlioptas-Iliopoulos-Kolmogorov [2] and the third one is due to Achlioptas-Iliopoulos-Sinclair [3]. All conditions assume that we have fixed some probability distribution $\omega$ over $\Omega$ with $\text{supp}(\omega) = \Omega$.

### 2.1 Conditions (A),(B)

To formulate the conditions, we need some additional notation and terminology. One step of the algorithm can be described by a transition $\sigma \xrightarrow{f} \tau$ where $\sigma \in \Omega$ is the old state, $f$ is a flaw in $F_\sigma$, and $\tau \in \supp(\rho(\cdot|f,\sigma)) \subseteq \Omega$ is the new state. Whenever we write $\sigma \xrightarrow{f} \tau$, we assume that $\sigma, f, \tau$ satisfy the conditions above. This step is said to “address” flaw $f$ at state $\sigma$, in the hope of eliminating flaw $f$. We say that transition $\sigma \xrightarrow{f} \tau$ introduces flaw $g$ if $g \in F_\tau$ and either $g \not\in F_\sigma$ or $g = f$. The set of flaws introduced by transition $\sigma \xrightarrow{f} \tau$ will be denoted as

$$\Delta(\sigma \xrightarrow{f} \tau) = F_\tau - (F_\sigma - \{f\})$$

(1)

We say that flaw $f$ causes flaw $g$ (and write it as $f \sim g$) if there exists at least one transition $\sigma \xrightarrow{f} \tau$ with $g \in \Delta(\sigma \xrightarrow{f} \tau)$. The directed graph $(F, \sim)$ is called the causality graph for Algorithm 1.

In some applications it is more convenient to work with a potential causality graph, which is any directed graph $(F, \rightarrow)$ containing $(F, \sim)$ as a subgraph. We will denote $\Gamma(f) = \{g \mid f \rightarrow g\}$ to be the set of out-neighbors of flaw $f \in F$ in the graph $(F, \rightarrow)$, and for a subset $S \subseteq F$ denote

$$\text{Ind}(S) = \{U \subseteq S \mid \pi(g) > \pi(h) \text{ for all distinct } g, h \in U \text{ with } g \rightarrow h\}$$

Here the potential causality graph $(F, \rightarrow)$ and permutation $\pi$ are assumed to be fixed, and thus not reflected in the notation. Note, if relation $\rightarrow$ is symmetric then $\text{Ind}(S)$ is simply the set of subsets of $S$ which are independent in the undirected graph $(F, \sim)$ where $g \sim h$ if $g \rightarrow h$ and $g \neq h$.

For a flaw $f \in F$ define

$$\gamma_f = \max_{\tau \in \Omega} \sum_{\sigma \in f} \frac{\omega(\sigma)\rho(\tau|f,\sigma)}{\omega(\tau)} = \omega(f) \cdot \max_{\tau \in \Omega} \frac{\omega^f(\tau)}{\omega(\tau)}$$

(2)

where $\omega^f(\cdot)$ is the probability distribution over $\Omega$ obtained by first sampling $\sigma \sim \omega(\cdot)$ conditioned on event $f$ and then sampling $\tau \sim \rho(\cdot|f,\sigma)$:

$$\omega^f(\tau) = \sum_{\sigma \in f} \omega(\sigma|f)\rho(\tau|f,\sigma) = \sum_{\sigma \in f} \frac{\omega(\sigma)\rho(\tau|f,\sigma)}{\omega(f)}$$

(3)

(Here and below we use notation $\nu(A) = \mathbb{P}_\nu[A]$ and $\nu(A|B) = \mathbb{P}_\nu[A|B]$ for a probability distribution $\nu$ over $\Omega$ and events $A, B \subseteq \Omega$ with $\nu(B) > 0$). It can be seen from (2) that $\gamma_f \geq \omega(f)$. Furthermore, $\gamma_f = \omega(f)$ if and only if distributions $\omega^f$ and $\omega$ are identical. This special case was studied by Harvey and Vondrák [4]; distributions $\rho(\cdot|f,\sigma)$ were then called regenerating oracles.

We are now ready to formulate conditions (A) and (B).
(A) There exist positive real numbers \( \{\mu_f\}_{f \in F} \) and constant \( \theta \in (0,1) \) such that
\[
\gamma_f \frac{\sum_{S \subseteq \Gamma(f)} \mu(S)}{\mu_f} \leq \theta \quad \forall f \in F
\] (4)
where \( \mu(S) = \prod_{f \in S} \mu_f \).

(B) There exist positive real numbers \( \{\mu_f\}_{f \in F} \) and constant \( \theta \in (0,1) \) such that
\[
\gamma_f \frac{\sum_{S \in \text{Ind}(\Gamma(f))} \mu(S)}{\mu_f} \leq \theta \quad \forall f \in F
\] (5)

Clearly, condition (B) is weaker than (A).

**Theorem 1** ([2]). Suppose one of the following is true:

(a) Condition (A) holds and Algorithm 1 uses \( \pi \)-strategy.
(b) Condition (B) holds and Algorithm 1 uses a “Recursive Walk” strategy (that depends on \( \rightarrow, \pi \)).

Then the algorithm will terminate within \( (T_0 + s)/(1 - \theta) \) steps with probability at least \( 1 - 2^{-s} \), where
\[
T_0 = \log_2 \left( \max_{\sigma \in \Omega} \frac{\omega_{\text{init}}(\sigma)}{\omega(\sigma)} \right) + \log_2 \left( \sum_{S \subseteq \text{init}} \mu(S) \right) \quad \text{in the case (a)}
\] (6a)
\[
T_0 = \log_2 \left( \max_{\sigma \in \Omega} \frac{\omega_{\text{init}}(\sigma)}{\omega(\sigma)} \right) + \log_2 \left( \sum_{S \in \text{Ind}(\text{init})} \mu(S) \right) \quad \text{in the case (b)}
\] (6b)
and \( F^{\text{init}} = \bigcup_{\sigma \in \text{supp}(\omega_{\text{init}})} F_\sigma \) is the set of flaws that can be present in the initial state.

### 2.2 Commutativity

It has been shown in [15] that the conclusion of Theorem 1 holds for an arbitrary flaw selection strategy assuming the following commutativity condition.

**Definition 2.** Algorithm 1 is called commutative if relation \( \rightarrow \) is symmetric (let us denote it as \( \sim \)), and there exists injective mapping \( \text{Swap} \) that sends walks of the form \( \sigma_1 \xrightarrow{f} \sigma_2 \xrightarrow{g} \sigma_3 \) with \( f \sim g \) to walks of the form \( \sigma_1 \xrightarrow{g} \sigma_2' \xrightarrow{f} \sigma_3' \) such that \( \rho(\sigma_2'|f,\sigma_1)\rho(\sigma_3|g,\sigma_2) = \rho(\sigma_3'|g,\sigma_1)\rho(\sigma_3|f,\sigma_2') \).

Note that commutativity is a property of distributions \( \rho(\cdot|f,\sigma) \) and relation \( \rightarrow \). Calling Algorithm 1 “commutative” is thus somewhat imprecise, but we choose to do so for brevity.

### 2.3 Condition (C)

Next, we review condition (C) from [3]. This condition takes into account which subsets of flaws can be introduced by transitions \( \sigma \xrightarrow{f} \tau \). Specifically, for a flaw \( f \in F \), subset \( S \subseteq F \) and state \( \tau \in \Omega \) let \( \Omega_f(S,\tau) \) be the set of states that can “lead” to \( (S,\tau) \) by addressing \( f \):
\[
\Omega_f(S,\tau) = \{ \sigma \in f \mid S \subseteq \Delta(\sigma \xrightarrow{f} \tau) \}
\] (7)
Define
\[
\gamma_f(S) = \max_{\tau \in \Omega} \sum_{\sigma \in \Omega_f(S,\tau)} \frac{\omega(\sigma)\rho(\tau|f,\sigma)}{\omega(\tau)}
\] (8)
Condition (C) can now be formulated as follows.
There exist positive real numbers \( \{\mu_f\}_{f \in F} \) and constant \( \theta \in (0, 1) \) such that

\[
\frac{1}{\mu_f} \sum_{S \subseteq F} \gamma_f(S)\mu(S) \leq \theta \quad \forall f \in F
\]

As observed in [3], condition (C) is weaker than (A). Indeed, suppose that \( S \) contains flaw \( g \) which is not in \( \Gamma(f) \). By the definition of a potential causality graph, there are no transitions \( \sigma \xrightarrow{f} \tau \) that introduce \( g \), and so \( \Omega_f(S, \tau) = \emptyset \) and \( \gamma_f(S) = 0 \). Thus, the summation in (9) is effectively over subsets \( S \subseteq \Gamma(f) \), and for each such \( S \) we have \( \gamma_f(S) \leq \gamma_f(\emptyset) = \gamma_f \).

**Theorem 3** ([3]). Suppose that condition (C) holds and Algorithm \( \mathcal{A} \) uses \( \pi \)-strategy. Then the algorithm will terminate within \((T_0 + s)/(1 - \theta)\) steps with probability at least \(1 - 2^{-s}\), where

\[
T_0 = \log_2 \left( \max_{\sigma \in \Omega} \frac{\omega^{\text{init}}(\sigma)}{\omega(\sigma)} + \log_2 \left( \sum_{S \subseteq F^{\text{init}}} \mu(S) \right) + \log_2 \left( \max_{S \subseteq F} \frac{1}{\mu(S)} \right) \right)
\]

### 2.4 Primary flaws and backtracking algorithms

Note that if transition \( \sigma \xrightarrow{f} \tau \) introduces flaw \( f \) it does not necessarily mean that this flaw will be later addressed; it may happen that \( f \) is eradicated “collaterally” by addressing some other flaw \( g \neq f \). This fact explains why the condition in eq. (7) has containment \( S \subseteq \Delta(\sigma \xrightarrow{f} \tau) \) instead of equality. As observed in [3], there are applications in which flaws are never eradicated collaterally. Such flaws are called primary. Formally, flaw \( f \in F \) is primary if for any transition \( \sigma \xrightarrow{g} \tau \) with \( g \neq f \) and \( f \in F_r \) we have \( f \in F_r \). Let \( P \) be the set of primary flaws in \( F \). We will use the following notation throughout the paper: we write \( A \sqsubseteq B \) for subsets \( A, B \subseteq F \) if \( A \subseteq B \) and \( B \cap P \subseteq A \cap P \). In particular, if all flaws are primary then condition \( A \sqsubseteq B \) means that \( A = B \). Define

\[
\Omega_f(S, \tau) = \{ \sigma \in f \mid S \subseteq \Delta(\sigma \xrightarrow{f} \tau) \}
\]

Note that if \( P = \emptyset \) then expressions (7) and (11) coincide.

**Theorem 4** ([3]). Theorem 3 continues to hold if expression (7) is replaced with (11) in the definition of condition (C).

In [3, Corollary 3.7] the authors also mention that if all flaws are primary then the sum over \( S \subseteq F^{\text{init}} \) in the definition of \( T_0 \) can be restricted to subsets of the form \( S = F_{\sigma} \) for some \( \sigma \in \omega^{\text{init}} \).

One application of this theorem considered in [3] is a single-clause backtracking algorithm for solving satisfiability problems. The goal is to find an assignment \( x = (x_1, \ldots, x_n) \) that satisfies a given set of constraints \( C_1, \ldots, C_m \). Each variable \( x_i \) can take values in some discrete set \( \mathcal{X}_i \), and each constraint \( C \) is specified by a tuple of variables \( \text{vars}(C) = (v_1, \ldots, v_m) \in [n]^m \) and a \( m \)-ary relation \( R_C \subseteq \mathcal{X}_{v_1} \times \cdots \times \mathcal{X}_{v_m} \). Assignment \( x \) satisfies \( C \) if \( x_{\text{vars}(C)} \in R_C \).

Given such input, define \( \Omega \) to be the set of all partial assignments (i.e. vectors \( x \) with \( x_i \in \mathcal{X}_i \cup \{\text{unassigned}\} \)) that satisfy all constraints. In other words, we have \( x \in \Omega \) if the following holds for each \( C \): if all entries in \( x_{\text{vars}(C)} \) are assigned then \( x_{\text{vars}(C)} \) satisfies \( R_C \). For each variable \( v \in [n] \) define flaw \( f_v \) as the set of all partial assignments \( x \in \Omega \) with \( x_v = \text{unassigned} \). Clearly, flawless objects are precisely satisfying assignments of the input instance. For a flaw \( f_v \) the sampling distribution \( \rho(\cdot | f_v, x) \) is defined as follows: (i) sample \( x_v \in \mathcal{X}_v \) according to some prespecified distribution \( p_v \) over \( \mathcal{X}_v \); (ii) if some constraints become violated then pick the lowest indexed such clause \( C \) and unassign all variables in \( \text{vars}(C) \). Clearly, flaw \( f_v \) can only be eliminated by addressing \( f_v \), i.e. all flaws are primary.
3 Our results

Although condition (C) indeed improves on (A), it is not immediately clear whether it also improves on (B), at least for non-primary flaws. (Note, all applications of (C) considered in [3] involve primary flaws). We show in Appendix A that conditions (B) and (C) are incomparable: there exist examples in which (B) works while (C) does not, and vice versa. Motivated by this fact, in Section 3.1 we propose a new condition (⋆) which in a certain sense “interpolates” between (B) and (C), and includes both conditions as special cases.

As our second contribution, we introduce a new notion of commutativity that is applicable to condition (⋆), and prove several results under this condition. This is described in Section 3.2.

3.1 New condition

We assume that each pair of flaws \((f,g)\) is assigned a number \(\alpha(f,g) \in [0,1]\) called a strength of interaction \((f,g)\). We require that \(\alpha(f,g) = 0\) if flaw \(g\) is primary. For a pair \((f,\sigma)\) with \(\sigma \in f \in \Omega\), subset \(S \subseteq F\) and state \(\tau\) define the following expressions:

\[
\alpha^-(S) = \prod_{(g,h): g,h \in S, \pi(g) < \pi(h)} (1 - \alpha(g,h))
\]

\[
\alpha^+(S,\tau) = \begin{cases} 
\prod_{g \in S - \Delta(\sigma \overset{f}{\rightarrow} \tau)} \alpha(f,g) & \text{if } \Delta(\sigma \overset{f}{\rightarrow} \tau) \cap P \subseteq S \\
0 & \text{otherwise}
\end{cases}
\]

\[
\alpha_f(\sigma,\tau) = \alpha^-(S) \cdot \alpha^+(f,\sigma)(S,\tau)
\]

\[
\lambda_f(S) = \max_{\tau \in \Omega} \sum_{\sigma \in f} \omega(\sigma) \rho(\tau|f,\sigma) \cdot \alpha_f(\sigma,\tau)
\]

We can now formulate our new condition.

\((\star)\) There exist positive real numbers \(\{\mu_f\}_{f \in F}\) and constant \(\theta \in (0,1)\) such that

\[
\frac{1}{\mu_f} \sum_{S \subseteq F} \lambda_f(S) \mu(S) \leq \theta \quad \forall f \in F
\]

Let us make a few remarks:

• If \(\alpha(f,g) = \alpha(g,f)\) for all \(f,g\) then condition \((\star)\) does not depend on permutation \(\pi\).

• We can assume that \(\alpha(f,f) = 0\) for all \(f\): setting these values to zero can only improve the condition, i.e. make it weaker.

• Suppose that \(P = \emptyset\) and \(\alpha(f,g) = [f \rightarrow g]\) where we use the Iverson bracket notation: \([\phi] = 1\) if \(\phi\) is true, and \([\phi] = 0\) otherwise. Then we have \(\alpha^-(S) = [S \in \text{Ind}(F)]\) and \(\alpha^+_f(\sigma,\tau) = [S \subseteq \Gamma(f)]\), and therefore \((\star)\) is equivalent to condition (B).

• Suppose that \(\alpha(f,g) = 0\) for all \(f,g\). Then \(\alpha^- = 1\) for all \(S\) and \(\alpha^+_f(\sigma,\tau) = [\sigma \in \Omega_f(S,\tau)]\), and therefore \((\star)\) is equivalent to condition (C).
For subsets $S \subseteq F$ and $\Omega' \subseteq \Omega$ we write $S \subseteq F_{\Omega'}$ if $S \subseteq F_\sigma$ for some $\sigma \in \Omega'$. Our first result is as follows.

**Theorem 5.** Suppose that condition (\ast) holds and Algorithm $[\Box]$ uses $\pi$-strategy. Then the algorithm will terminate within $(T_0 + s)/\log_2 \frac{1}{\theta} \leq (T_0 + s)/(1 - \theta)$ steps with probability at least $1 - 2^{-s}$, where

$$T_0 = \log_2 \left( \max_{\sigma \in \Omega} \frac{\omega^{\text{init}}(\sigma)}{\omega(\sigma)} \right) + \log_2 \left( \sum_{S \subseteq F_{\supp(\omega^{\text{init}})}} \alpha^-(S) \mu(S) \right)$$  \hspace{1cm} (14)

Note got rid of the last term in $[10]$. Also note that Theorems $[11] (b)$ and $[5]$ use different strategies for selecting flaws, namely a “Recursive Walk” and a $\pi$-strategy, respectively. The latter strategy is arguably simpler, and also does not need a knowledge of the causality graph $(F, \rightarrow)$. 

### 3.2 Commutativity

For a transition $\varphi = \sigma \overset{f}{\rightarrow} \tau$ we denote $\blacksquare(\varphi) = \Delta(\varphi) \cup \{ g \in F | \alpha(f, g) = 1 \}$. Let $\Phi^*$ be the set of walks of the form $\sigma_1 \overset{f_1}{\rightarrow} \sigma_2 \overset{g}{\rightarrow} \sigma_3$ such that $g \notin \blacksquare(\sigma_1 \overset{f}{\rightarrow} \sigma_2)$. We also denote $\rho(\sigma \overset{f}{\rightarrow} \tau) = \rho(\tau | f, \sigma)$. We now introduce a new notion of commutativity.

**Definition 6.** Algorithm $[\Box]$ is called commutative if there exists an injective mapping $\text{Swap} : \Phi^* \rightarrow \Phi^*$ such that for any $\sigma_1 \overset{f}{\rightarrow} \sigma_2 \overset{g}{\rightarrow} \sigma_3 \in \Phi^*$ we have $\text{Swap}(\sigma_1 \overset{f}{\rightarrow} \sigma_2 \overset{g}{\rightarrow} \sigma_3) = \sigma_1 \overset{g'}{\rightarrow} \sigma_2 \overset{f'}{\rightarrow} \sigma_3$ and the following holds for transitions $\varphi_1 = \sigma_1 \overset{f}{\rightarrow} \sigma_2, \varphi_2 = \sigma_2 \overset{g}{\rightarrow} \sigma_3, \varphi'_1 = \sigma_1 \overset{g'}{\rightarrow} \sigma_2, \varphi'_2 = \sigma_2 \overset{f'}{\rightarrow} \sigma_3$:

$$\rho(\varphi_1)\rho(\varphi_2) = \rho(\varphi'_1)\rho(\varphi'_2)$$ \hspace{1cm} (15a)

$$\blacksquare(\varphi_1) \cup \blacksquare(\varphi_2) = \blacksquare(\varphi'_1) \cup \blacksquare(\varphi'_2)$$ \hspace{1cm} (15b)

Note that commutativity is a property of distributions $\rho(\cdot | f, \sigma)$ and values $\alpha(\cdot, \cdot)$. We again choose to call Algorithm $[\Box]$ “commutative” for brevity. Let us make a few remarks.

- Suppose that $P = \emptyset$, relation $\rightarrow$ is symmetric (let us denote it as $\sim$), and $\alpha(f, g) = [f \sim g]$ (so that condition (\ast) is equivalent to (B)). It can be checked that $\blacksquare(\sigma \overset{f}{\rightarrow} \tau) = \{ g | g \sim f \}$ (since $(F, \sim) = (F, \rightarrow)$ is a potentially causality graph) and $\Phi^* = \{ \sigma_1 \overset{f}{\rightarrow} \sigma_2 \overset{g}{\rightarrow} \sigma_3 | f \sim g \}$. It can now be seen that the new definition is equivalent to the one given in Definition $[2]$. Thus, the new notion of commutativity generalizes the previous one from $[15]$, and so using the same name is justified.

- Suppose all flaws are primary. Then we have $\blacksquare(\sigma \overset{f}{\rightarrow} \tau) = \Delta(\sigma \overset{f}{\rightarrow} \tau) = F_\tau - (F_\sigma - \{ f \})$ (recall that $\alpha(f, g) = 0$ if $g$ is primary). We claim that condition (15b) is then redundant, and can be removed from the definition of commutativity. Indeed, it can be checked that if $\sigma_1 \overset{f}{\rightarrow} \sigma_2 \overset{g}{\rightarrow} \sigma_3 \in \Phi^*$ then

$$\blacksquare(\sigma_1 \overset{f}{\rightarrow} \sigma_2) \cup \blacksquare(\sigma_2 \overset{g}{\rightarrow} \sigma_3) = F_{\sigma_3} - (F_{\sigma_1} - \{ f, g \})$$

A similar expression holds for $\sigma_1 \overset{g}{\rightarrow} \sigma_2 \overset{f}{\rightarrow} \sigma_3 \in \Phi^*$, which implies the claim.

- It is straightforward to check that the backtracking algorithm described in Section $[2.4]$ is commutative.
Theorem 7. (a) If Algorithm 1 is commutative then Theorem 5 holds for an arbitrary flaw selection strategy.

(b) Suppose in addition that all flaws are primary. For integer \( t \geq 0 \) and a flawless state \( \sigma \) let \( p_t(\sigma|\omega^{\text{init}}, \Lambda) \) be the probability that Algorithm 1 with parameters \( \omega^{\text{init}}, \Lambda \) reaches \( \sigma \) (and thus terminates) after exactly \( t \) steps. Then \( p_t(\sigma|\omega^{\text{init}}, \Lambda) \) does not depend on \( \Lambda \), whether or not condition \((\ast)\) holds. Consequently, the probability that Algorithm 1 terminates after exactly \( t \) steps does not depend on \( \Lambda \), as well as the distribution over flawless states upon termination.

One surprising consequence is that in the backtracking algorithm described in Section 2.3 it does not matter how we select the next flaw to address: all choices will lead to the same probability of success, and the same expected number of steps until termination.

Let us say a few words about our proof technique for Theorem 7(a). As in [15], we use mapping Swap to transform walks produced by Algorithm 1 to some canonical walks (we call them \( \pi \)-walks). There are, however, significant technical differences. The swaps for a given walk that we used in [15] had one useful invariant, namely they preserved a certain tree (in other papers such tree was called a “witness tree”). Accordingly, we could divide the initial set of walks into groups, where walks in each group had the same witness tree. The latter property was essential for showing that swapping operations applied to each group are injective.

In the current paper we also define a certain tree (or rather a forest) for each walk. However, applying swaps to a walk will in general modify this forest, so the approach of [15] had to be adapted. The main challenge was to find the right invariant to carry the induction argument. Roughly speaking, we start grouping walks “on the fly”; as we perform swapping operations, one part of the walk (namely a prefix) gets fixed, and we use this incrementally growing prefix (together with some “boundary” information) to define groups.

Remark 1. In [15] we used the commutativity condition to design a parallel algorithm with a small (e.g. polylogarithmic) number of rounds. Commutativity was also used by Iliopoulos [12] to show that the distribution induced by the algorithm approximates in a certain sense the “LLL distribution” (i.e. the distribution over flawless states induced by \( \omega \)). It is natural to ask whether such extensions are possible for our framework. Unfortunately, the techniques from [15, 12] do not seem to apply. The main difference is that [15, 12] used backward analysis (the root of the witness tree was on the right), while in our case only forward analysis seem to work, where the roots of the forest are on the left. We do not see a natural way to define a backward forest or tree. The existence of an efficient parallel algorithm thus remains an open problem. We refer to Appendix B for a further discussion of parallelization.

4 Proof overview

A run of Algorithm 1 can be described by a sequence \( \varphi = \sigma_1 \xrightarrow{w_1} \sigma_2 \xrightarrow{w_2} \cdots \sigma_t \xrightarrow{w_t} \sigma_{t+1} \) where \( \sigma_i \xrightarrow{w_i} \sigma_{i+1} \) is a valid transition for each \( i \in [t] \). Such a sequence will be called a walk of length \( t \) (denoted as \( t = \text{length}(\varphi) \)). Unless noted otherwise, we will always assume that walk \( \varphi \) has the form shown above.

Walk \( \varphi \) is called terminal if it ends at a flawless state \( (F_{\sigma_{t+1}} = \emptyset) \), and non-terminal otherwise. We will use the following notation: \( \varphi_{i:j} = \sigma_i \xrightarrow{w_i} \sigma_{i+1} \xrightarrow{w_{i+1}} \cdots \xrightarrow{w_j} \sigma_{j+1} \) for integers \( i, j \), and also

\[
p^{\text{init}}(\varphi) = \omega^{\text{init}}(\sigma_1) \prod_{i=1}^{t} \rho(\sigma_{i+1}|w_i, \sigma_i) \quad p(\varphi) = \omega(\sigma_1) \prod_{i=1}^{t} \rho(\sigma_{i+1}|w_i, \sigma_i)
\]
For a set of walks $\mathcal{X}$ we will denote $p^{\text{init}}(\mathcal{X}) = \sum_{\varphi \in \mathcal{X}} p^{\text{init}}(\varphi)$ and $p(\mathcal{X}) = \sum_{\varphi \in \mathcal{X}} p(\varphi)$.

Walk $\varphi$ of length $t$ is a prefix of walk $\varphi'$ of length $t' \geq t$ if $\varphi = \varphi'_i, t'$. It is a proper prefix if in addition $t' > t$. We will need the notion of a nonconflicting set of walks (which is a minor variation of a valid set from [13]).

**Definition 8.** Walks $\varphi, \varphi'$ are called conflicting if they have the same length and have the form $\varphi = \xi \xrightarrow{f} \ldots$ and $\varphi' = \xi' \xrightarrow{f} \ldots$ for some walk $\xi$ and flaws $f \neq f'$. A set of walks $\mathcal{X}$ is conflicting is it contains conflicting walks $\varphi, \varphi'$. Otherwise $\mathcal{X}$ is nonconflicting.

We assume in the analysis that Algorithm 1 uses a deterministic strategy for selecting a flaw, i.e. $w_{i+1}$ is uniquely determined by prefix $\varphi_{1:i}$. This assumption can be made w.l.o.g.: if the strategy is randomized (i.e. a distribution over some set of deterministic strategies) the claim of Theorems 5 and 7 can be obtained by taking the appropriate expectation over strategies (whose number is finite for walks of bounded length). Formally, a strategy is a mapping $\Lambda$ that maps non-terminal walk $\varphi$ that ends at state $\sigma_{t+1}$ to a flaw in $F_{\sigma_{t+1}}$. We say that walk $\varphi$ follows strategy $\Lambda$ if $w_{i+1} = \Lambda(\varphi_{1:i})$ for all $i \in [0, t - 1]$.

For an interval $t = [t^-, t^+]$ with $t^- \leq t^+$, strategy $\Lambda$ and subsets $\Omega_1, \Omega_{\text{last}} \subseteq \Omega$ we denote $\text{Runs}_\Lambda(\Omega_1, t, \Omega_{\text{last}})$ to be the set of all walks $\varphi$ that follow $\Lambda$, start at state from $\Omega_1$, end at a state from $\Omega_{\text{last}}$, and satisfy the following conditions: length($\varphi$) $\in [t^-, t^+]$, and if length($\varphi$) $< t^+$ then $\varphi$ is terminal. If one of the sets $\Omega_1, \Omega_{\text{last}}$ equals $\Omega$ then we will omit it from the notation, e.g. $\text{Runs}_\Lambda(\Omega_1, t)$ will mean $\text{Runs}_\Lambda(\Omega_1, t, \Omega)$. Let $\text{Runs}_\Lambda'(...)$ be the set of terminal walks in $\text{Runs}_\Lambda(...)$. If $\Lambda$ is the strategy used by Algorithm 1 then we will usually omit subscript $\Lambda$ from these notations. If $\Lambda$ is the $\pi$-strategy for a permutation $\pi$ then we will write $\text{Runs}_\pi(...)$ instead of $\text{Runs}_\Lambda(...)$. It can be checked that all sets defined in this paragraph are nonconflicting.

Observe that set $\mathcal{X} = \text{Runs}(\text{supp}(\omega^{\text{init}}), [t^-, t^+])$ can be equivalently described as follows: take runs of Algorithm 1 of positive probability that make at least $t^-$ steps, follow them until they either terminate or make $t^+$ steps, and add the corresponding sequence $\varphi$ to $\mathcal{X}$. It can now be seen that the probability that Algorithm 1 does not terminate within $t^-$ steps equals $p^{\text{init}}(\mathcal{X})$. We will be using the bound $p^{\text{init}}(\mathcal{X}) \leq \gamma^{\text{init}} \cdot p(\mathcal{X})$ where

$$
\gamma^{\text{init}} = \max_{\sigma_1 \in \Pi} \frac{\omega^{\text{init}}(\sigma_1)}{\omega(\sigma_1)}
$$

Our goal will thus be to upper bound expression $p(\mathcal{X})$ for the set $\mathcal{X} = \text{Runs}(\text{supp}(\omega^{\text{init}}), [t^-, t^+])$.

We will work with directed labeled forests $T = (V_T, E_T, \ell_T)$ whose edges are oriented away from the roots, and the label $\ell_T(v)$ of node $v \in V_T$ belongs to $F$. The parent of a non-root node $v \in V_T$ will be denoted as $\text{parent}_{T}(v) \in V_T$. For roots $r \in V_T$ we define $\text{parent}_{T}(r) = 0$ where we assume that $0 \notin V_T$. Thus, we have $E_T = \{ (\text{parent}(v), v) \mid \text{parent}(v) \neq 0 \}$. Since set $E_T$ and mapping $\text{parent}_{T} : V_T \to V_T \cup \{0\}$ uniquely determine each other, with some abuse of notation we will usually write $T = (V_T, \text{parent}_{T}, \ell_T)$ instead of $T = (V_T, E_T, \ell_T)$. Forest $T$ will be called proper if its roots have distinct labels, and so are the children of each node. The set of labels of the roots of $T$ will be denoted as $\text{Roots}(T) \subseteq F$, and the set of labels of children of node $v \in V_T$ as $\text{children}_{T}(v) \subseteq F$. If $T$ is understood from the context we may drop subscript $T$.

Throughout the paper we use the following notation for a walk $\varphi = \sigma_1 \xrightarrow{w_1} \sigma_2 \ldots \sigma_t \xrightarrow{w_t} \sigma_{t+1}$:

- $\Delta_i = \Delta(\sigma_i \xrightarrow{w_i} \sigma_{i+1})$ and $\square_i = \square(\sigma_i \xrightarrow{w_i} \sigma_{i+1})$ for $i \in [t]$;
Let us now assume that the algorithm is commutative. Consider walk \( \varphi = \sigma_1 \xrightarrow{w_1} \sigma_2 \xrightarrow{w_2} \cdots \xrightarrow{w_t} \sigma_t \xrightarrow{w_{t+1}} \sigma_{t+1} \) and index \( i \in [2, t] \) with \( \text{parent}(i) \neq i - 1 \), or equivalently \( w_i \notin \square(\sigma_{i-1} \xrightarrow{w_{i-1}} \sigma_i) \). We have \( \varphi_{i-1:i} = \sigma_{i-1} \xrightarrow{w_{i-1}} \sigma_i \xrightarrow{w_i} \sigma_{i+1} \in \Phi^* \), so we can apply mapping Swap from the definition of commutativity. We define Swap, to be the walk obtained from \( \varphi \) by replacing subwalk \( \varphi_{i-1:i} \) with \( \text{Swap}(\varphi_{i-1:i}) = \sigma_{i-1} \xrightarrow{w_{i-1}} \sigma_i \xrightarrow{w_i} \sigma_{i+1} \). Note that \( p^{\text{init}}(\text{Swap}(\varphi)) = p^{\text{init}}(\varphi) \) and \( p(\text{Swap}(\varphi)) = p(\varphi) \) by the property of mapping Swap. A mapping \( \Lambda \) from walks to walks will be called a swapping mapping if \( \Lambda(\varphi) \) is obtained from \( \varphi \) by a sequence of such operations. In Section 6 we prove the following result.
Theorem 13. Consider set of walks \( X = \text{Runs}(\Omega_1, t, \Omega_{\text{last}}) \) for an interval \( t = [t^-, t^+] \), and assume that Algorithm [7] is commutative.

(a) There exists an injective swapping mapping \( \Pi \) that sends walks \( \varphi \in X \) to \( \pi \)-walks.

(b) If all flaws are primary then \( \Pi(\text{Runs}(\Omega_1, t, \Omega_{\text{last}})) = \text{Runs}_\pi(\Omega_1, t, \Omega_{\text{last}}) \).

Corollary 14. In the commutative case the conclusion of Corollary [12] holds for any flaw selection strategy.

Proof. As before, pick interval \( t = [t^-, t^+] \) with \( t^+ \geq t \), and define \( X = \Pi(\text{Runs}(\Omega_1, t)) \) where \( \Omega_1 = \text{supp}(\omega^{\text{init}}) \) and \( \Pi \) is the mapping for \( \text{Runs}(\Omega_1, t) \) from Theorem [13] Clearly, we have \( p(X) = p(\text{Runs}(\Omega_1, t)) \). Observe that swapping mappings preserve the length and the first and the last state, and in particular terminal walks go to terminal walks. By applying Theorem [11] to \( X \) we obtain the claim in the same way as in the proof of Corollary [12].

Corollary 15. Suppose Algorithm [4] is commutative and all flaws are primary. Then \( p_t(\sigma|\omega^{\text{init}}, \Lambda) = p_t(\sigma|\omega^{\text{init}}, \Lambda_{\pi}) \) for any flawless state \( \sigma \in \Omega \), where \( p_t(\cdot) \) is as defined in Theorem [7] (b) and \( \Lambda_{\pi} \) is the \( \pi \)-strategy.

Proof. The claim follows by applying Theorem [13] to the set \( X = \text{Runs}_{\Lambda}(\Omega_1, [t, t^+], \{\sigma\}) \) with \( \Omega_1 = \text{supp}(\omega^{\text{init}}) \), and observing that \( p_t(\sigma|\omega^{\text{init}}, \Lambda) = p_t^{\text{init}}(\text{Runs}_{\Lambda}(\Omega_1, [t, t^+], \{\sigma\})) \) for any strategy \( \Lambda \).

5 Counting \( \pi \)-walks: Proof of Theorem [11]

In this section we will work with tuples \( T = (V_T, \text{parent}_T, \ell_T, Q_T) \) where \( (V_T, \text{parent}_T, \ell_T) \) is a forest and \( Q_T \) is a subset of \( P \). We will refer to such \( T \) as an augmented forest (or sometimes just as a forest).

Let us fix walk \( \varphi = \sigma_1 \triangleright_{w_1} \sigma_2 \triangleright_{w_2} \cdots \triangleright_{w_{t+1}} \sigma_{t+1} \). Define \( Q_\varphi = F_{\sigma_{t+1}} \cap P \) and \( V_\varphi = [t] \cup Q_\varphi \). Let us also extend mappings \( w : [t] \to F \) and \( \text{parent} : [t] \to [0, t] \) introduced in the previous section as follows: for a flaw \( f \in Q_\varphi \) set \( w_f = f \) and \( \text{parent}(f) = \max\{i \in [t] \mid f \in \Delta_i\} \). As before, \( \max \emptyset \) is assumed to be 0. We denote \( T_\varphi^{\text{augmented}} = (V_\varphi, \text{parent}, w, Q_\varphi) \), and for \( j \in V_\varphi \) denote

\[
(j^-, j^0) = \begin{cases} (j - 1, j) & \text{if } j \in [t] \\ (t, t + 1) & \text{if } j \in Q_\varphi \end{cases}
\]

We define \( \text{WF}(\varphi) \) ("witness forests") to be the set of augmented forests \( T = (V_\varphi, \text{parent}^*, w, Q_\varphi) \) that satisfy the following constraints for each \( j \in V_\varphi \): if flaw \( w_j \) is primary then \( \text{parent}^*(j) = \text{parent}(j) \), otherwise \( \text{parent}^*(j) = [\text{parent}(j), j - 1] \).

Proposition 16. (a) If \( j \in V_\varphi \) then \( w_j \in F_{\sigma_{p+1}} \cap \cdots \cap F_{\sigma_{j-1}} \cap F_{\sigma_j} \) and \( w_j \not\in \{w_{p+1}, \ldots, w_{i-1}\} \) where \( p = \text{parent}(j) \). Consequently, if \( w_i = w_j \) for \( j \in V_\varphi, i \in [j^-] \) then \( \text{parent}(j) = i \).

(b) Forests \( T \in \text{WF}(\varphi) \) are proper, and satisfy \( \text{Roots}(T) \subseteq F_{\sigma_1} \).

Proof. (a) Condition \( w_j \in F_{\sigma_j} \) holds both if \( j \in [t] \) (since \( \sigma_j \triangleright_{w_j} \sigma_{j+1} \) is a valid transition) and if \( j \in Q_\varphi \) (since \( w_j = \sigma_j \)). For every \( i \in [\text{parent}(j) + 1, j^-] \) we have \( w_j \not\in \Delta_i \). Induction on \( i = j^- - 1, \ldots, \text{parent}(j) + 1 \) now yields the claim.

(b) For the first claim we need to prove that \( \text{parent}^*(i) \neq \text{parent}^*(j) \) for any distinct nodes \( i, j \in V_\varphi \) with \( w_i = w_j \). Note that \( i^0 \neq j^0 \) (otherwise we would have \( i, j \in Q_\varphi \) and so \( w_i \neq w_j \)).
Lemma 18. proven in Sections 5.1 and 5.2, respectively.

As shown in part (a), we must have \( \text{parent}(j) \geq i \), and therefore \( \text{parent}^*(j) \geq \text{parent}(j) \geq i > \text{parent}^*(i) \).

Now consider a root \( j \) of \( T \) with the label \( w_j \in \text{Roots}(T) \), then \( \text{parent}^*(j) = 0 \) and so \( \text{parent}(j) = 0 \). As shown in part (a), we have \( w_j \in F_{\sigma_1} \). This shows that \( \text{Roots}(T) \subseteq F(\sigma_1) \).

It remains to show that for every primary flaw \( f \in F_{\sigma_1} \) we have \( f \in \text{Roots}(T) \). Let \( j \) be the minimum index in \([t]\) with \( w_j = f \), or \( j = f \) if such index does not exist. Define \( j^* = j \) in the first case and \( j^* = t + 1 \) in the second case. Since primary flaws are never eradicated by addressing other flaws, we have \( f \in F_{\sigma_1} \cap \ldots \cap F_{\sigma_j} \). Therefore, \( j \in V_\varphi \) and \( f = w_j \notin \Delta_1 \cap \ldots \cap \Delta_{j-} \). Also, we have \( \alpha(g,f) = 0 \) for all \( g \in F \) (since \( f \) is primary). This implies that \( \text{parent}(j) = 0 \) and so \( \text{parent}^*(j) = 0 \) (since \( w_j \) is primary), thus proving the claim. \( \square \)

Let \( \text{GenerateWF}(\varphi) \) be a randomized procedure for generating \( T = (V_\varphi, \text{parent}^*, w, Q_\varphi) \in \text{WF}(\varphi) \) in which \( \text{parent}^*(j) \) for \( j \in [t] \) with non-primary \( w_j \) is produced as follows (independently for each \( j \)):

1. for \( i = j - 1 \) downto \( \text{parent}(j) \) do
2.     if \( i = \text{parent}(j) \) or \( \text{Bernoulli}(\alpha(w_i,w_j)) = 1 \) then
3.         set \( \text{parent}^*(j) = i \) and terminate

Here \( \text{Bernoulli}(p) \) returns value 1 with probability \( p \) and value 0 with probability \( 1 - p \). For an augmented forest \( T \) let \( \mathbb{P}[T|\varphi] \) be the probability that \( \text{GenerateWF}(\varphi) \) returns \( T \). For \( T \in \text{WF}(\varphi) \) we also define

\[
\alpha(T|\varphi) = \alpha^-(\text{Roots}(T)) \cdot \prod_{i \in [t]} \alpha_{w_i,\sigma_i}(\text{children}_T(i),\sigma_{i+1}) \tag{18}
\]

\[
\lambda_T = \prod_{i \in V_\varphi} \lambda_{w_i}(\text{children}_T(i)) \tag{19}
\]

where \( \alpha_{f,\sigma}(S,\tau) \) and \( \lambda_{f}(S) \) are the numbers from condition (⋆). The next two lemmas will be proven in Sections 5.1 and 5.2 respectively.

**Lemma 17.** If \( \varphi \) is a \( \pi \)-walk then \( \mathbb{P}[T|\varphi] \leq \alpha(T|\varphi) \).

**Lemma 18.** Consider proper augmented forest \( T \) with \( V_T = [t] \cup Q_T \) and state \( \tau \in \Omega \). Let \( (T|\tau) \) be the set of walks \( \varphi \) that end at \( \tau \) and satisfy \( \text{WF}(\varphi) \supseteq T \), and let \( (T) = \bigcup_{\tau \in \Omega} (T|\tau) \). Then

\[
\sum_{\varphi \in (T|\tau)} p(\varphi) \alpha(T|\varphi) \leq \alpha^-(\text{Roots}(T)) \cdot \frac{\lambda_T}{\lambda(Q_T)} \cdot \omega(\tau) \tag{20a}
\]

\[
\sum_{\varphi \in (T)} p(\varphi) \alpha(T|\varphi) \leq \alpha^-(\text{Roots}(T)) \cdot \frac{\lambda_T}{\lambda(Q_T)} \tag{20b}
\]

where for a subset \( Q \subseteq P \) we defined

\[
\lambda(Q) = \prod_{q \in Q} \lambda_q(\emptyset) \tag{21}
\]

We say that augmented forest \( T \) is realizable if \( T \in \text{WF}(\varphi) \) for some \( \pi \)-walk \( \varphi \). We emphasize that the “identities” of nodes in \( V_\varphi \) are a part of the definition of \( T \). This means that the sequence \( W = w_1 \ldots w_t \) can be trivially reconstructed from a realizable \( T \), since \( W = \ell_T(1) \ldots \ell_T(t) \). As we will see later, we will be able to reconstruct \( T \) (and thus \( W \)) even if we “remove” node identities.
More formally, write \( T_1 \equiv T_2 \) for proper augmented forests \( T_1, T_2 \) if \( Q_{T_1} = Q_{T_2} \) and there exists a label-preserving isomorphism between \( V_{T_1} \) and \( V_{T_2} \). An equivalence class w.r.t. \( \equiv \) will be called an unnamed forest. For a proper augmented forest \( T \) let \( [T] \) be the unnamed forest to which \( T \) belongs. “Removing node identities” can now be defined as replacing \( T \) with \([T]\).

The reconstruction algorithm will use the following building blocks for a proper (named or unnamed) augmented forest \( T \).

- Node \( v \in V_T \) is said to be frozen in \( T \) if \( \ell_T(v) \in Q_T \) and there are no other nodes in \( V_T \) with the label \( \ell_T(v) \). Otherwise \( v \) is non-frozen.

- Let \( \text{MinNode}(T) \) be the non-frozen root \( r \) of \( T \) with the smallest value of \( \pi(\ell_T(r)) \).

- Let \( T^- \) be the forest obtained from \( T \) by removing root \( r = \text{MinNode}(T) \) together with outgoing edges and setting \( Q_{T^-} = Q_T \).

Note, if all roots of \( T \) are frozen then \( \text{MinNode}(T) \) and \( T^- \) are undefined; we set \( \text{MinNode}(T) = T^- = \perp \) in this case.

For a forest \( T \) and integer \( \delta \) we define \( \text{shift}_\delta(T) \) to be the forest obtained from \( T \) by renaming each node \( v \in V_T \) as follows: if \( v \in \mathbb{Z} \) then rename \( v \mapsto v + \delta \), otherwise keep \( v \) unchanged.

**Proposition 19.** Consider \( \pi \)-walk \( \varphi \) of length \( t \geq 1 \) and forest \( T = (V_\varphi, \text{parent}^+, w, Q_\varphi) \in \text{WF}(\varphi) \). (a) \( \text{MinNode}(T) = 1 \). (b) \( \text{shift}_{-1}(T^-) \in \text{WF}(\varphi^-) \) where \( \varphi^- = \varphi_{2t} \). (c) \( \varphi^- \) is a \( \pi \)-walk.

**Proof.** (a) It can be seen that all nodes \( i \in [t] \), and in particular node \( i = 1 \), are non-frozen in \( T \) (recall that \( V_T = [t] \cup Q_\varphi = [t] \cup Q_T \)). Also, node 1 is a root of \( T \). Consider another non-frozen root \( j \in V_T - \{1\} \) of \( T \). We have \( \text{parent}^*(j) = 0 \), and therefore \( \text{parent}(j) = 0 \). If \( j \in Q_\varphi \) then there must exist \( k \in [t] \) with \( w_k = w_j \) since \( j \) is non-frozen, but then we would have \( \text{parent}(j) \geq k \) by Proposition 19(b,c), a contradiction. Thus, \( j \in [t] \). Since \( \varphi \) is a \( \pi \)-walk and \( 1 \in [\text{parent}(j),j^-] \), we have \( \pi(w_1) < \pi(w_j) \). This shows that \( \text{MinNode}(T) = 1 \).

(b) Denote \( U = [2,t] \cup Q_\varphi \), and let \( p^+, p : U \to \{0\} \cup U \) be the parent mappings of forests \( T^- \) and \( \text{shift}_{+1}(T^-_{\text{augmented}}) \), respectively. Note that \( T^- = (U, p^+, w, Q_\varphi) \) and \( \text{shift}_{+1}(T^-_{\text{augmented}}) = (U, p, w, Q_\varphi) \). It can be checked that for \( j \in U \) we have

\[
p^+(j) = \begin{cases} \text{parent}^*(j) & \text{if } \text{parent}^*(j) \geq 2 \\ 0 & \text{if } \text{parent}^*(j) \in \{0,1\} \end{cases}
\]

\[
p(j) = \begin{cases} \text{parent}(j) & \text{if } \text{parent}(j) \geq 2 \\ 0 & \text{if } \text{parent}(j) \in \{0,1\} \end{cases}
\]

Therefore, \( p^+(j) = p(j) \) if flaw \( w_j \) is primary, and \( p^+(j) \in [p(j),j^-] \) otherwise (since analogous conditions hold for mappings \( \text{parent}^* \) and \( \text{parent} \)). This implies that \( \text{shift}_{-1}(T^-) \in \text{WF}(\varphi^-) \).

(c) This claim follows easily from definitions.

**Corollary 20.** If forest \( T \) is realizable then it can be uniquely reconstructed from \([T]\). Consequently, \([T] \cap [T'] = \emptyset\) for distinct realizable forests \( T, T' \).

**Proof.** We need to give an algorithm that renames nodes of a given forest \( \bar{T} \equiv T \) so that we get \( \bar{T}^- = T^- \) after renaming. If \( |V_T| = |Q_T| \) then we rename \( q \mapsto \ell_T(q) \) for each \( q \in V_T \). Suppose that \( |V_T| > |Q_T| \). First, we find node \( r = \text{MinNode}(T) \in V_T \) and rename \( r \mapsto 1 \). Now consider forest \( T^- \). Note that \( \bar{T}^- \equiv T^- \equiv \text{shift}_{-1}(T^-) \) and forest \( \text{shift}_{-1}(T^-) \) is realizable by Proposition 19(b,c). Thus, we can apply the reconstruction algorithm to \( T^- \) recursively. This algorithm renames nodes of \( T^- \) so that we get \( \bar{T}^- = \text{shift}_{-1}(T^-) \). Let us rename it once more via \( \bar{T}^- \leftarrow \text{shift}_{+1}(\bar{T}^-) \), then we get \( \bar{T}^- = T^- \). Now for each \( v \in V_{T^-} - \{r\} \) take node \( u \in V_{\bar{T}^-} \) corresponding to \( v \) and rename \( v \mapsto u \).

\[ \square \]
Lemma 21. For subsets $R \subseteq F$, $Q \subseteq P$ and integer $t \geq 0$ let $\text{Forests}_t(R, Q)$ be the set of realizable forests $T$ with $|V_T| \geq t$, $\text{Roots}(T) = R$ and $Q_T = Q$. If condition $(\ast)$ holds then

$$\sum_{T \in \text{Forests}_t(R, Q)} \lambda_T \leq \theta^t \cdot \mu(R)$$

Proof. We closely follow the presentation in [16]. Consider the following multitype Galton-Watson branching process for generating a proper unnamed augmented forest $\hat{T}$. In the first round, produce $|R|$ singleton vertices labeled with flaws from $R$. Then in each subsequent round, for each vertex $v$ produced in the previous round generate subset $S \subseteq F$ with probability proportional to $\nu_f(S) = \lambda_f(S)\mu(S)$ where $f$ is the label of $v$, and for each $g \in S$ add to $v$ a child node carrying label $g$. The process continues until it does out naturally because no new vertices are born in some round (depending on the probabilities used, there is, of course, the possibility that this never happens). Finally, we set $Q_\hat{T} = Q$.

Now fix forest $T \in \text{Forests}_t(R, Q)$ with $t' = |V_T| \geq t$. The probability of event $[\hat{T} \equiv T]$ equals

$$p_T = \prod_{v \in V_T, \ell_T(v) = f} \frac{\nu_f(\text{children}_T(v))}{\sum_{S \subseteq F} \nu_f(S)} \geq \prod_{v \in V_T, \ell_T(v) = f} \frac{\mu(\text{children}_T(v))}{\mu_f} = \frac{\lambda_T}{\theta^{t'}} \prod_{v \in V_T, \ell_T(v) = f} \frac{1}{\mu(R)} \geq \frac{\lambda_T}{\theta^{t'}} \cdot \frac{1}{\mu(R)}$$

By Corollary 20 events $[\hat{T} \equiv T]$ and $[\hat{T} \equiv T']$ are disjoint for distinct $T, T' \in \text{Forests}_t(R, Q)$. We thus have $1 \geq \sum_{T \in \text{Forests}_t(R, Q)} p_T \geq \sum_{T \in \text{Forests}_t(R, Q)} \frac{\lambda_T}{\theta^{t'}} \cdot \frac{1}{\mu(R)}$, which yields the claim. \hfill $\square$

By combining previous results we obtain

Theorem 22. Let $\mathcal{X}_t(\Omega_1, Q)$ be the set of $\pi$-walks $\varphi$ of length at least $t$ that start at a state $\sigma_1 \in \Omega_1$ and satisfy $Q_\varphi = Q$. Then

$$p(\mathcal{X}_t(\Omega_1, Q)) \leq \frac{\theta^t}{\lambda(Q)} \cdot \sum_{R \in F_{\Omega_1}} \alpha^{-}(R) \mu(R)$$

Proof. We can write

$$p(\mathcal{X}_t(\Omega_1, Q)) = \sum_{\varphi \in \mathcal{X}_t(\Omega_1, Q)} \sum_{T \in \text{supp}(\mathcal{W}(\varphi))} p(\varphi)\mathbb{P}[T|\varphi] \quad (\text{since } \sum_T \mathbb{P}[T|\varphi] = 1)$$

$$\leq \sum_{\varphi \in \mathcal{X}_t(\Omega_1, Q)} \sum_{T \in \text{supp}(\mathcal{W}(\varphi))} p(\varphi)\alpha(T|\varphi) \quad (\text{by Lemma 17})$$

$$\leq \sum_{R \subseteq F_{\Omega_1}} \sum_{T \in \text{Forests}_t(R, Q)} \sum_{\varphi \in (T)} p(\varphi)\alpha(T|\varphi) \quad (\ast)$$

$$\leq \sum_{R \subseteq F_{\Omega_1}} \sum_{T \in \text{Forests}_t(R, Q)} \alpha^{-}(R) \cdot \frac{\lambda_T}{\lambda(Q)} \quad (\text{by Lemma 18})$$

$$\leq \sum_{R \subseteq F_{\Omega_1}} \alpha^{-}(R) \cdot \frac{\theta^t \cdot \mu(R)}{\lambda(Q)} \quad (\text{by Lemma 21})$$

To see (\ast), observe that for every $\varphi \in \mathcal{X}_t(\Omega_1, Q)$ and $T \in \mathcal{W}(\varphi)$ we have $\text{Roots}(T) \subseteq F_{\Omega_1}$ by Proposition 16 and $Q_\varphi = Q$ by definition. \hfill $\square$
We are now ready to prove Theorem 11. By assumptions, set \( X \) satisfies
\[
X \subseteq \mathcal{X}_t - (\Omega_1, \emptyset) \cup \bigcup_{Q \subseteq P} \mathcal{X}_t^*(\Omega_1, Q)
\]
Applying Theorem 22 now gives eq. (17) with constant \( C = \sum_{Q \subseteq P} \frac{1}{\lambda(Q)} \).

5.1 Proof of Lemma 17

Below we prove that \( \mathbb{P}[T|\varphi] \leq \alpha(T|\varphi) \) for \( T \in \mathsf{WF}(\varphi) \) assuming that \( \alpha(f, g) < 1 \) for all \( f, g \). By continuity, this will imply the claim in the general case. Indeed, for a value \( \theta \in [0, 1] \) define weights \( \alpha_\theta(f, g) = \min\{\alpha(f, g), \theta\} \), and let \( \mathbb{P}_\theta[T|\varphi] \), \( \alpha_\theta(T|\varphi) \), \( \mathsf{WF}_\theta(\varphi) \) be the corresponding quantities for weights \( \alpha_\theta(\cdot, \cdot) \). It can be checked that as \( \theta \to 1 \), we have \( \mathbb{P}_\theta[T|\varphi] \to \mathbb{P}_1[T|\varphi] \) and \( \alpha_\theta(T|\varphi) \to \alpha_1(T|\varphi) \) for any \( T \in \mathsf{WF}_1(\varphi) \), and also \( \mathbb{P}_1[T|\varphi] = 0 \) for any \( T \in \mathsf{WF}_\theta(\varphi) - \mathsf{WF}_1(\varphi) \). The claim follows.

From now on we fix a \( \pi \)-walk \( \varphi \) and forest \( T = (V_\varphi, \text{parent}, w, Q_\varphi) \in \mathsf{WF}(\varphi) \). Note, \( \text{parent}^*(i) \) is no longer treated as a random variable; the random variable defined in procedure \texttt{GenerateWF}(\varphi) will be denoted as parent(i) instead. We denote \( E = E_T \) and \( E^+ = \{ (\text{parent}^*(i), i) \mid i \in V_\varphi \} \). By plugging expression (12) for \( \alpha_{f,\sigma}(S, \tau) \), the definition of \( \alpha(T|\varphi) \) can be rewritten as follows:
\[
\alpha(T|\varphi) = \begin{cases} 
\prod_{\langle j, k \rangle \in X} (1 - \alpha(w_j, w_k)) \cdot \prod_{(i, k) \in Y} \alpha(w_i, w_k) & \text{if } \Delta_i \cap P \subseteq \text{children}_T(i) \quad \forall i \in [t] \\
0 & \text{otherwise}
\end{cases}
\] (22)
where
\[
X = \{ \langle j, k \rangle \mid (i, j), (i, k) \in E^+ \text{ for some } i \in [0, t], \pi(w_j) < \pi(w_k), w_k \notin P \} \quad (23a)
\]
\[
Y = \{ \langle i, k \rangle \mid (i, k) \in E, w_k \notin \Delta_i \} \quad (23b)
\]
Note that we have added condition “\( w_k \notin P \)” in (23a). This does not change expression (22) since for primary flaws \( w_k \in P \) we have \( \alpha(w_j, w_k) = 0 \).

Now let us write down the expression for \( \mathbb{P}[T|\varphi] \). By construction, for each \( k \in V_\varphi \) we have
\[
\mathbb{P}[^\text{parent}(k) = i] = \begin{cases} 
\prod_{j=i+1}^{k^-} (1 - \alpha(w_j, w_k)) & \text{if } i = \text{parent}(k) \\
\prod_{j=i+1}^{k^-} (1 - \alpha(w_j, w_k)) \alpha(w_i, w_k) & \text{if } i \in [\text{parent}(k) + 1, k^-]
\end{cases}
\]
We can now write
\[
\mathbb{P}[T|\varphi] = \prod_{(i, k) \in E^+} \mathbb{P}[^\text{parent}(k) = i] = \prod_{\langle j, k \rangle \in X'} (1 - \alpha(w_j, w_k)) \cdot \prod_{(i, k) \in Y'} \alpha(w_i, w_k)
\] (24)
where
\[
X' = \{ \langle j, k \rangle \mid (i, k) \in E^+ \text{ for some } i \in [t], j \in [i + 1, k^-] \} \quad (25a)
\]
\[
Y' = \{ (i, k) \mid (i, k) \in E, i \neq \text{parent}(k) \}
\] (25b)
We will show next that \( X \subseteq X', Y \subseteq Y' \) and the value \( \alpha(T|\varphi) \) is determined by the first expression in (22); this will prove Lemma 17.

First, consider \( (j, k) \in X \). By definition, there exists \( i \in [0, t] \) such that \( (i, j), (i, k) \in E^+ \), \( \pi(w_j) < \pi(w_k) \) and \( w_k \notin P \) (implying that \( k \in [t] \)). Condition (16) for \( \pi \)-walks gives that \( k \notin \)
[parent(j) + 1, j−]. We also have k > i ≥ parent(j), and therefore k > j− (implying j ∈ [k]). Since k ≠ j, we get j ∈ [i + 1, k−], and therefore (j, k) ∈ X′.

Now consider ⟨i, k⟩ ∈ Y. By definition, we have ⟨i, k⟩ ∈ E and w_k ≠ Δ_i. The latter condition implies that parent(⟨i, k⟩) ≠ i, and therefore ⟨i, k⟩ ∈ Y′.

Finally, let us prove that p(T|φ) is determined by the first expression in [22]. Consider primary flaw g ∈ Δ_i for i ∈ [t]. We need to show that E contains edge ⟨i, j⟩ with w_j = g.

We know that g ∈ F_{σ_i} − {w_i} and g ∈ F_{σ_{i+1}}. Let j ∈ [i + 1, t] be the minimum index with w_j = g; if such j does not exist, then set j = g. Since g is a primary flaw, transitions σ_k →_i σ_{k+1} for k ∈ [i + 1, j−] do not eliminate g (note that w_k ≠ g). Therefore, g ∈ F_{σ_{k+1}} for all k ∈ [i, j−]. We thus have j ∈ V_φ and w_j = g (if j = g then g ∈ F_{σ_{i+1}} and so g ∈ Q_φ).

To summarize, we showed that w_j ∈ Δ_i and w_j ≠ Δ_k for k ∈ [i + 1, j−]. Therefore, parent(⟨i, j⟩) = i. Since flaw w_j is primary, we must have parent*(j) = parent(⟨i, j⟩) = i and thus ⟨i, j⟩ ∈ E.

5.2 Proof of Lemma 18

Summing (20a) over τ ∈ Ω gives (20b), so it suffices to prove the former inequality. We assume that V_T = [t] ∪ Q_T for some t ≥ 0 and the nodes q ∈ Q_T are leaves of T, otherwise (T|τ) = ∅ and the claim is trivial. For i ∈ [0, t] and τ ∈ Ω define (T|τ)_i = {φ_1:i | φ = σ_1 →_i σ_2 →_i ... →_i σ_t →_i σ_{i+1}, σ_1 ∈ Ω_1, WF(φ) ⊇ T, σ_{i+1} = τ}. We will prove by induction on i = 0, 1, ..., t that

\[ \sum_{φ ∈ (T|τ)_{i+1}} p(φ) α_{1:i}(φ) \leq λ_{1:i} \cdot ω(σ_{i+1}) \]  

(26)

where

\[ α_{1:i}(σ_1 →_1 σ_2 →_i σ_{i+1}) = \prod_{k=1}^{i} α_k(σ_k →_k σ_{k+1}) \]

\[ α_k(σ_k →_k σ_{k+1}) = \prod_{i ∈ [i]} \lambda_{w_k}(children(k)) \]

This will imply (20a) since ⟨T|τ⟩ = ⟨T|τ⟩_i, α(⟨T|φ⟩) = α_{1:i} (φ) · α(⟨Roots(T)⟩) and λ_T = λ_{1:t} · λ(Q_T).

The base case i = 0 is trivial (note that p(σ_1) = ω(σ_1) and α_{0:1}(φ) = λ_{1:0} = 1). Let us prove it for i ∈ [t] assuming that it holds for i − 1. Applying eq. (13) with f = w_i, σ = σ_i, S = children(i), τ = σ_{i+1} yields

\[ \sum_{σ_i ∈ Ω} p(σ_{i+1} | w_i, σ_i) α_i(σ_i →_i σ_{i+1}) ω(σ_i) \leq λ_{w_i}(children(i)) \cdot ω(σ_{i+1}) \]  

(27)

We can now prove the claim as follows:

\[ \sum_{φ ∈ (T|σ_{i+1})_1} p(φ) α_{1:i}(φ) \leq \sum_{σ_i ∈ Ω} \sum_{ψ ∈ (T|σ_i)_{i+1}} p(ψ →_i σ_{i+1}) α_{1:i}(ψ →_i σ_{i+1}) \]

\[ = \sum_{σ_i ∈ Ω} p(σ_{i+1} | w_i, σ_i) α_i(σ_i →_i σ_{i+1}) \sum_{ψ ∈ (T|σ_i)_{i+1}} p(ψ) α_{1:i-1}(ψ) \]

(induction hypothesis)

\[ \leq \sum_{σ_i ∈ Ω} p(σ_{i+1} | w_i, σ_i) α_i(σ_i →_i σ_{i+1}) \cdot λ_{1:i-1} \cdot ω(σ_i) \]

(by eq. (27))

\[ \leq λ_{1:i} \cdot λ_{w_i}(children(i)) \cdot ω(σ_{i+1}) \]

\[ = λ_{1:i} \cdot ω(σ_{i+1}) \]
6 Commutativity: Proof of Theorem 13

From now on we assume that the algorithm is commutative. To simplify notation, we will prove Theorem 13 in the case when \( \Omega_1 = \Omega_{\text{last}} = \Omega \). Clearly, this will imply Theorem 13 for arbitrary \( \Omega_1 \) and \( \Omega_{\text{last}} \), since swapping mappings preserve the first and the last state of the walk. Accordingly, we assume that \( \mathcal{X} = \text{Runs}(t) = \text{Runs}([t^-, t^+]) \).

For a walk \( \varphi = \sigma_1 \underbrace{w_1}_s \sigma_2 \ldots \sigma_t \underbrace{w_t}_s \sigma_{t+1} \) and integer \( s \geq 1 \) we define

\[
I_s(\varphi) = \{ i \in [s, t] \mid \text{parent}(i) < s \} \quad (28a)
\]

\[
R_s(\varphi) = \{ w_i \mid i \in I_s(\varphi) \} \quad (28b)
\]

\[
\text{MinNode}_s(\varphi) = \begin{cases} 
\arg\min_{i \in I_s(\varphi)} \pi(w_i) & \text{if } I_s(\varphi) \neq \emptyset \\
0 & \text{otherwise}
\end{cases} \quad (28c)
\]

Note that we cannot have \( w_i = w_j \) for distinct \( i, j \in I_s(\varphi) \) (since then we would have \( \text{parent}(j) \geq i \geq s \), if \( i < j \)). Thus, taking “arg min” in eq. (28c) is well-defined. Note, if \( s \leq t \) then \( s \in I_s(\varphi) \) and \( \text{MinNode}_s(\varphi) \in [s, t] \), and if \( s > t \) then \( I_s(\varphi) = \emptyset \) and \( \text{MinNode}_s(\varphi) = 0 \).

Observe that if index \( i = \text{MinNode}_s(\varphi) \in I_s(\varphi) \) satisfies \( i > s \) then \( \text{parent}(i) \neq i - 1 \), and therefore we can apply operation \( \text{Swap}_i(\varphi) \). Now consider the following algorithm for transforming a nonconflicting set of walks \( \mathcal{X} \).

**Algorithm 2: Transform(\( \mathcal{X}, t_{\text{max}} \)).** Input: a nonconflicting set of walks \( \mathcal{X} \), integer \( t_{\text{max}} \).

1. for \( s = 1, \ldots, t_{\text{max}} \) do
2.     while \( i = \max \{ \text{MinNode}_s(\varphi) \mid \varphi \in \mathcal{X} \} > s \) do
3.         update \( \mathcal{X} \leftarrow \{ \Pi(\varphi) \mid \varphi \in \mathcal{X} \} \) where \( \Pi(\varphi) = \begin{cases} \text{Swap}_i(\varphi) & \text{if } \text{MinNode}_s(\varphi) = i \\
\varphi & \text{if } \text{MinNode}_s(\varphi) < i \end{cases} \)

This algorithm is analyzed in the next two sections. Namely, in Section 6.1 we study a swapping operation for a single walk \( \varphi \), and prove that Algorithm 2 terminates and produces \( \pi \)-walks upon termination if it is run with \( t_{\text{max}} = t^+ \). Then in Section 6.2 we prove that all mappings \( \Pi \) in line 3 are injective. This will mean that Algorithm 2 defines a “cumulative” injective mapping \( \Pi \) on the input set \( \mathcal{X} \) (which is the composition of individual mappings in line 3). We will thus establish Theorem 13(a).

It is easy to see that \( \Pi(\varphi) \) for a given walk \( \varphi \in \mathcal{X} \) depends on \( \varphi \) and \( t_{\text{max}} \) but not on the input set \( \mathcal{X} \). This fact will be implicitly used in the proof of Theorem 13(b), which will be given in Section 6.3. In this proof we will apply Algorithm 2 to the set \( \mathcal{X} = \text{Runs}([t^-, t^+]) \) for some \( t^* \geq t^+ \), and set \( t_{\text{max}} = t^+ \).

6.1 Analyzing individual swaps

In this section we prove the following result.

**Lemma 23.** Consider walk \( \varphi \in \mathcal{X} \) with \( i = \text{MinNode}_s(\varphi) > s \) in line 3 of Algorithm 2 and let \( \varphi' = \Pi(\varphi) = \text{Swap}_i(\varphi) \).

(a) Walk \( \varphi' \) satisfies \( R_s(\varphi') = R_s(\varphi) \) and \( \text{MinNode}_s(\varphi') = \text{MinNode}_s(\varphi) - 1 \). Consequently, each iteration of Algorithm 2 will terminate after a finite number of steps.

(b) The following invariant holds at each step of Algorithm 2 for walks \( \varphi \in \mathcal{X} \):

\[
\pi(w_i) < \pi(w_j) \quad \forall j \in [t], i \in [\text{parent}(j) + 1, j - 1] \cap [s - 1] \quad (29)
\]

In particular, \( \varphi \) is a \( \pi \)-walk at iterations \( s \geq t \).
In the proof below we label all quantities related to \( \varphi \) with a “prime”: \( \sigma'_i, w'_i, \Delta'_i, \square'_i, \text{parent}' \). In particular, we assume that

\[
\varphi = \sigma_1 \xmapsto{w_1} \sigma_2 \xmapsto{w_2} \ldots \xmapsto{w_t} \sigma_{t+1}, \quad \varphi' = \sigma'_1 \xmapsto{w'_1} \sigma'_2 \xmapsto{w'_2} \ldots \xmapsto{w'_{t+1}} \sigma'_{t+1}
\]

Let \( \diamondsuit : [t] \to [t] \) be the mapping that swaps \( i - 1 \) and \( i \), and is the identity for other integers. Its result for index \( j \in [t] \) will be denoted as \( \diamondsuit^j \). Note that \( w_j = w'_j \) and \( \diamondsuit^j = j \) for any \( j \in [t] \).

We claim that the following implications hold for each \( j \in [t] \):

\[
\begin{align*}
\text{parent}(j) \notin \{i - 1, i\} & \quad \Rightarrow \quad \text{parent}'(\diamondsuit^j) = \text{parent}(j) \quad (30a) \\
\text{parent}(j) \in \{i - 1, i\} & \quad \Rightarrow \quad \text{parent}'(\diamondsuit^j) \in \{i - 1, i\} \text{ and } \diamondsuit^j = j \quad (30b)
\end{align*}
\]

To prove (30), observe that \( \square_k = \square_k' \) for all \( k \in [t] - \{i - 1, i\} \), and also \( \square_{i-1} \cup \square_i = \square_{i-1} \cup \square_i' \) since the algorithm is commutative. Also, we have \( \text{parent}(i) \neq i - 1 \) (by the choice of \( i \)) and \( \text{parent}'(i) \neq i - 1 \) (since we have \( \sigma'_{i-1} \xmapsto{w'_{i-1}} \sigma'_i \xmapsto{w'_i} \sigma'_{i+1} = \text{Swap}(\sigma_{i-1} \xmapsto{w_{i-1}} \sigma_i \xmapsto{w_i} \sigma_{i+1}) \in \Phi^* \) by the definition of mapping \( \text{Swap} \)). From the facts above it should now be easy to verify (30).

One consequence of (30) is the following implications for \( j \in [t] \):

\[
\begin{align*}
\text{parent}(j) < s & \quad \Rightarrow \quad \text{parent}'(\diamondsuit^j) = \text{parent}(j) \quad (31a) \\
\text{parent}(j) \geq s & \quad \Rightarrow \quad \text{parent}'(\diamondsuit^j) \geq s \quad (31b)
\end{align*}
\]

We are now ready to prove parts (a)-(b) of Lemma 6.1.

(a) From (31) we conclude that \( I_s(\varphi') = \{j^* \mid j \in I_s(\varphi)\} \). This in turn implies that \( R_s(\varphi') = R_s(\varphi) \) and \( \text{MinNode}_s(\varphi') = \text{MinNode}_s(\varphi) - 1 \).

(b) The claim holds trivially for \( s = 1 \). Using (31), one can verify that the update \( \varphi' = \Pi(\varphi) \) in line 3 preserves this invariant. It remains to prove the following: if (29) holds at the end of iteration \( s \) for walk \( \varphi \in \mathcal{X} \) then it still holds for \( \varphi \) when \( s \) is increased by 1. Suppose not, then there must exist \( j \in [t], i \in [\text{parent}(j) + 1, j - 1] \cap \{s\} \) with \( \pi(w_i) \geq \pi(w_j) \). We have \( \text{parent}(j) + 1 \leq i = s \leq j - 1 \), so \( j \geq s + 1 \) and \( \text{parent}(j) < s \). Therefore, \( j \in I_s(\varphi) \). Conditions above imply that \( \text{MinNode}_s(\varphi) > s \) (we cannot have \( \text{MinNode}_s(\varphi) = s \) since \( \pi(w_s) \geq \pi(w_j) \)). But then iteration \( s \) would not have terminated - a contradiction.

### 6.2 Transforming a set of walks

In this section we show that mapping \( \Lambda \) in line 3 of Algorithm 2 is injective on \( \mathcal{X} \); in fact, we will establish a stronger property. The assumption that the input set of \( \mathcal{X} \) is nonconflicting will be essential.

First, we describe an alternative characterization of conflicting walks from [15]. A **generalized walk** is a formal finite sequence \( \varphi = \sigma_1 \xmapsto{w_1} \sigma_2 \xmapsto{w_2} \ldots \) with \( w_i \in F_i \) and \( \sigma_{i+1} \in \text{supp}(\rho(\langle w_i, \sigma_i \rangle)) \) for all \( i \). Note that \( \varphi \) can either end with a state \( (\varphi = \ldots \sigma_t) \), or end with a flaw \( (\varphi = \ldots \sigma_t \xmapsto{w_t} \ldots) \), or be empty \( (\varphi = \epsilon) \). In the first case \( \varphi \) is a usual walk. To indicate this case, we will write \( \varphi = \ldots \Omega \). We emphasize that by a “walk” we always mean a sequence of the form \( \varphi = \ldots \Omega \), unless we explicitly use the word “generalized”. For two generalized walks \( \varphi, \bar{\varphi} \) their largest common prefix is denoted as \( \varphi \land \bar{\varphi} \) (it is itself a generalized walk). It can now be seen that walks \( \varphi, \bar{\varphi} \) are conflicting if and only if they are distinct, have the same length and satisfy \( \varphi \land \bar{\varphi} = \ldots \Omega \).

**Definition 24.** Walks \( \varphi, \bar{\varphi} \) are called s-conflicting if they are conflicting, have length \( t \geq s \), and satisfy \( \varphi_{1:s} = \bar{\varphi}_{1:s} \) and \( R_s(\varphi) = R_s(\bar{\varphi}) \). A set of walks \( \mathcal{X} \) is called s-conflicting if it contains s-conflicting walks \( \varphi, \bar{\varphi} \). Otherwise \( \mathcal{X} \) is s-nonconflicting.


Lemma 25. Suppose set $\mathcal{X}$ in line 3 of Algorithm 2 is s-nonconflicting. Then mapping $\Pi$ in line 3 is injective, and set $\Pi(\mathcal{X})$ is also s-nonconflicting.

Proof. Assume that the lemma is false, then $\mathcal{X}$ contains distinct s-nonconflicting walks

$$\varphi = \sigma_1 \xrightarrow{w_1} \sigma_2 \cdots \sigma_t \xrightarrow{w_t} \sigma_{t+1} \quad \bar{\varphi} = \bar{\sigma}_1 \xrightarrow{\bar{w}_1} \bar{\sigma}_2 \cdots \bar{\sigma}_t \xrightarrow{\bar{w}_t} \bar{\sigma}_{t+1}$$

that were transformed to $\eta = \Pi(\varphi)$ and $\bar{\eta} = \Pi(\bar{\varphi})$ such that either $\eta = \bar{\eta}$ or walks $\eta, \bar{\eta}$ are s-conflicting. At least one of the walks must have changed; assume w.l.o.g. that $\eta \neq \varphi$, then $\eta = \text{Swap}_i(\varphi)$. We thus have $\text{MinNode}_s(\varphi) = i > s$, and also $\text{MinNode}_s(\bar{\varphi}) \leq i$ by construction. Note that $\bar{\eta} \in \{\bar{\varphi}, \text{Swap}_i(\bar{\varphi})\}$.

We know that walks $\eta, \bar{\eta}$ are either equal or conflicting, have the same length $t > s$ and satisfy $\eta_{1:s} = \bar{\eta}_{1:s}$ and $R_s(\eta) = R_s(\bar{\eta})$. We can conclude that $\eta \wedge \bar{\eta} = \ldots \Omega$, $\varphi_{1:s} = \bar{\eta}_{1:s} = \bar{\varphi}_{1:s}$ and $R_s(\varphi) = R_s(\eta) = R_s(\bar{\varphi})$ (the latter is by Lemma 23(a)). As a consequence, walks $\varphi, \bar{\varphi}$ cannot be conflicting (otherwise they would also be s-conflicting contradicting the choice of these walks).

We can write

$$\varphi = \alpha \xrightarrow{a} \sigma_i \xrightarrow{b} \beta \quad \eta = \alpha \xrightarrow{b} \sigma_i' \xrightarrow{a'} \beta$$

for some walks $\alpha, \beta$, with $\alpha = \varphi_{1:i-2}$. We have $\varphi \wedge \bar{\varphi} \neq \ldots \Omega$ since $\varphi, \bar{\varphi}$ are nonconflicting. If $\varphi \wedge \bar{\varphi}$ is a proper prefix of $\alpha$ then $\eta \wedge \bar{\eta} = \varphi \wedge \bar{\varphi}$, contradicting the condition that $\eta \wedge \bar{\eta} = \ldots \Omega$. Thus, $\alpha$ is a prefix of $\varphi \wedge \bar{\varphi}$. Condition $\varphi \wedge \bar{\varphi} \neq \ldots \Omega$ means that

$$\bar{\varphi} = \alpha \xrightarrow{a} \gamma$$

for some walk $\gamma$.

We claim that $\eta = \alpha \xrightarrow{a} \sigma_i \xrightarrow{b} \ldots$ and $\bar{\eta} = \text{Swap}_i(\eta)$. Indeed, since $\text{MinNode}_s(\varphi) = i$, we have $b = w_i \in R_s(\varphi)$ and $\pi(b) = \min_{f \in R_s(\varphi)} \pi(f)$. Since $R_s(\varphi) = R_s(\bar{\varphi})$, $\text{MinNode}_s(\bar{\varphi})$ returns the leftmost index $j \geq s$ with $\bar{w}_j = b$. We have $j \leq i$ by the choice of $i$ in Algorithm 2. Since $\text{MinNode}_s(\bar{\varphi}) = i$, we have $b \notin \{w_s, \ldots, w_{i-1}\} = \{\bar{w}_s, \ldots, \bar{w}_{i-1}\}$, and therefore $j \geq i$. This shows that $j = \text{MinNode}_s(\bar{\varphi}) = i$ and $\bar{w}_i = b$, thus yielding the claim.

To summarize, we have shown that

$$\varphi = \alpha \xrightarrow{a} \sigma_i \xrightarrow{b} \sigma_{i+1} \zeta \quad \eta = \alpha \xrightarrow{b} \sigma_i' \xrightarrow{a} \sigma_{i+1} \zeta$$

$$\bar{\varphi} = \alpha \xrightarrow{a} \sigma_i \xrightarrow{b} \sigma_{i+1} \bar{\zeta} \quad \bar{\eta} = \alpha \xrightarrow{b} \sigma_i' \xrightarrow{a} \sigma_{i+1} \bar{\zeta}$$

for appropriate sequences $\zeta, \bar{\zeta}$. Condition $\eta \wedge \bar{\eta} = \ldots \Omega$ implies that $\sigma_i' = \bar{\sigma}_i$ and $\sigma_{i+1} = \bar{\sigma}_{i+1}$. Injectiveness of mapping $\text{Swap} : \Phi^* \to \Phi^*$ from Definition 7 implies that $\sigma_i = \bar{\sigma}_i$. It can now be seen that we cannot have simultaneously $\varphi \wedge \bar{\varphi} \neq \ldots \Omega$ and $\eta \wedge \bar{\eta} = \ldots \Omega$. We have obtained a contradiction.

Next, we will need the following auxiliary result.

Lemma 26. Consider walk $\varphi = \sigma_1 \xrightarrow{w_1} \sigma_2 \cdots \sigma_s \xrightarrow{w_s} \sigma_{s+1} \ldots$ of length $t \geq s \geq 1$. Then

$$R_s(\varphi) = (R_{s+1}(\varphi) - \square_s) \cup \{w_s\}$$

where $\square_s = \square(s \xrightarrow{w_s} \sigma_{s+1})$. Thus, $R_s(\varphi)$ can be uniquely reconstructed from $\varphi_{1:s}$ and $R_{s+1}(\varphi)$.

Proof. We show below that $R_s(\varphi) \subseteq (R_{s+1}(\varphi) - \square_s) \cup \{w_s\}$ and $(R_{s+1}(\varphi) - \square_s) \cup \{w_s\} \subseteq R_s(\varphi)$.
• Consider \( f \in R_s(\varphi) - \{w_s\} \), and let \( i \in I_s(\varphi) \subseteq [s,t] \) be the index with \( w_i = f \). Condition \( w_i \neq w_s \) means that \( i \neq s \), and so \( i \in [s+1,t] \). By definition, we have \( \text{parent}(i) < s \). We can conclude that \( w_i \in R_{s+1}(\varphi) \) and \( w_i \notin \square_s \), and therefore \( f = w_i \in R_{s+1}(\varphi) - \square_s \).

• Note that we always have \( w_s \in R_s(\varphi) \). Now consider \( f \in R_{s+1}(\varphi) - \square_s \), and let \( i \in I_{s+1}(\varphi) \subseteq [s+1,t] \) be the index with \( w_i = f \). By definition, we have \( \text{parent}(i) \leq s \). Condition \( w_i \notin \square_s \) means that \( \text{parent}(i) \neq s \). We obtain that \( \text{parent}(i) < s \), and so \( i \in I_s(\varphi) \) and \( f = w_i \in R_s(\varphi) \).

\[ \square \]

**Lemma 27.** (a) If set \( \mathcal{X} \) is nonconflicting then it is 1-nonconflicting. (b) If set \( \mathcal{X} \) is s-nonconflicting for \( s \geq 1 \) then it is also \( (s + 1) \)-nonconflicting.

**Proof.** Part (a) follows directly from definitions; let us show part (b). Suppose that \( \mathcal{X} \) contains \( (s+1) \)-conflicting walks \( \varphi, \bar{\varphi} \). This means they are conflicting, \( \varphi_{1:s+1} = \bar{\varphi}_{1:s+1} \) (implying \( \varphi_{1:s} = \bar{\varphi}_{1:s} \)) and \( R_{s+1}(\varphi) = R_{s+1}(\bar{\varphi}) \). From Lemma 26 we obtain that \( R_s(\varphi) = R_s(\bar{\varphi}) \). Therefore, \( \varphi, \bar{\varphi} \) are s-conflicting, contradicting the assumption of the lemma.

From Lemmas 25 and 27 we obtain

**Corollary 28.** At every step of Algorithm 2 set \( \mathcal{X} \) is s-nonconflicting, and all mappings \( \Pi \) in line 3 are injective.

This establishes Theorem 13(a).

### 6.3 Primary flaws: Proof of Theorem 13(b)

In this section we assume additionally that all flaws in \( F \) are primary. In this case we have \( R_1(\varphi) = \text{Roots}(Tϕ) = F_{σ_1} \) for a terminal walk \( \varphi = σ_1 \rightarrow \ldots \rightarrow σ_{t+1} \) (by Proposition 9(b)). More generally, \( R_s(\varphi) = \text{Roots}(ϕ_{s:t}) = F_{σ_s} \) for \( s \in [t] \). This key property will be used many times in the proof. However, we will need such property for non-terminal walks as well. (Even though the statement of Theorem 13(b) is only about terminal walks, we will work with arbitrary walks to make the induction hypothesis work.) To tackle this issue, we will extend walks in \( \text{Runs}([t^−, t^+]) \) by “letting them run” a bit longer with a strategy that addresses available flaws in a rotating manner. Formally, choose an ordering of flaws \( F = \{f_0, \ldots, f_{n-1}\} \), and for each \( i \in [0, n-1] \) choose permutation \( π_{f_i} \) in which flaw \( f_i \) is the lowest. When proving Theorem 13(b) for the set \( \text{Runs}^c([t^−, t^+]) = \text{Runs}^c(ϕ_{s:t}) = \text{Roots}(ϕ_{s:t}) = F_{σ_s} \), we can assume w.l.o.g. that strategy \( Λ \) for a walk \( \varphi = \ldots σ_{t+1} \) of length \( t \geq t^+ \) returns the lowest flaw in \( F_{σ_{t+1}} \). In the remainder of this section we will analyze the behaviour of Algorithm 2 applied to the set \( \mathcal{X} = \text{Run}_Λ(t^*) \) with \( t_{max} = t^+ \) where \( t^* = [t^−, t^+] \), \( t = t^* + pn \), \( p = t^+ + 1 \).

**Lemma 29.** The following holds at each step of Algorithm 2: (a) Walks \( \varphi = σ_1 \rightarrow \ldots \rightarrow σ_{t+1} \in \mathcal{X} \) of length \( t \geq s \) satisfy \( R_s(\varphi) = F_{σ_s} \). (b) Set \( \mathcal{X} \) is nonconflicting.

**Proof.** (a) As we remarked earlier, for terminal walks the claim \( R_s(\varphi) = \text{Roots}(ϕ_{s:t}) = F_{σ_s} \) follows from Proposition 9(b). Consider non-terminal walk \( \varphi \) of length \( t = t^* \), and let us “track” how it evolves as the algorithm progresses. Its initial value will be denoted as \( \varphi^0 \). Define intervals \( t_0 = [t^+] \) and \( t_r = [t^+ + 1 + (r-1)n, t^+ + rn] \) for \( r \in [p] \), so that \( t^* \) is a disjoint union of \( t_0, t_1, \ldots, t_p \). For an interval \( t_r = [i, j] \) denote \( ϕ_{t_r} = ϕ_{i:j} \). We claim that \( ϕ = σ_1 \rightarrow \ldots \rightarrow σ_{t+1} \) satisfies the following for each \( s \in [t^+] \):
(A_s) \ (φ_{t_1}, \ldots, φ_{t_s}) = (φ_{t_1}^\circ, \ldots, φ_{t_s}^\circ) \ at \ the \ beginning \ of \ iteration \ s.

(B_s) \ R_s(φ) = F^\sigma(φ) \ at \ the \ beginning \ of \ iteration \ s \ (and \ thus \ at \ any \ point \ of \ iteration \ s).

Property (A_1) trivially holds. We will prove next implication \ (A_s) ⇒ (B_s) ∧ (A_{s+1}); \ clearly, \ this \ will \ imply \ the \ lemma. \ Below \ we \ denote \ σ_{(r)} \ to \ be \ the \ first \ state \ of \ φ_{t_r} \ (and \ the \ last \ state \ of \ walk \ φ_{t_{r-1}}). \ Such \ state \ for \ walk \ φ^\circ \ will \ be \ denoted \ as \ σ^\circ_{(r)}. \ We \ will \ need \ the \ following \ observation:

(*) \ If \ F^\sigma(φ_{(r)}) \ contains \ flaw \ f \ then \ f \ is \ addressed \ in \ φ^\circ_{t_r}. \ Indeed, \ suppose \ not, \ then \ f \ is \ present \ in \ all \ states \ of \ walk \ φ^\circ_{t_r} \ (since \ primary \ flaws \ are \ never \ eradicated \ by \ addressing \ other \ flaws).

By construction, walk φ^\circ \ follows \ strategy \ Λ, \ so \ φ^\circ_{t_r} \ must \ contain \ a \ state \ (excluding \ the \ last \ one) \ at \ which \ Λ \ would \ produce \ the \ lowest \ available \ flaw \ w.r.t. \ permutation \ π_f, \ which \ would \ be \ f. \ Thus, \ f \ would \ be \ addressed \ in \ φ^\circ_{t_r} - \ a \ contradiction.

Now \ suppose \ that \ (A_s) \ holds \ for \ a \ walk \ φ \ at \ the \ beginning \ of \ iteration \ s. \ Consider \ flaw \ f \ ∈ \ F^\sigma(φ). \ We \ claim \ that \ f \ is \ addressed \ in \ the \ walk \ φ_{s:max t_s}. \ Indeed, \ suppose \ not, \ then \ f \ is \ present \ in \ all \ states \ of \ this \ walk \ (since \ primary \ flaws \ are \ never \ eradicated \ by \ addressing \ other \ flaws). \ By \ (A_s), \ we \ have \ σ_{(s)} = σ^\circ_{(s)} \ and \ φ_{t_s} = φ^\circ_{t_s}. \ Thus, \ f \ is \ present \ in \ σ^\circ_{(s)} \ and \ is \ not \ addressed \ in \ φ^\circ_{t_s}, \ which \ contradicts \ (*). \ Now \ pick \ smallest \ index \ i ∈ [s,t] \ with \ w_i = f. \ We \ have \ f \ ∈ \ \{w_s, w_{s+1}, \ldots, w_{i-1}\} \ and \ f \ ∈ \ F^\sigma_s \ ∩ \ F^\sigma_{s+1} \ ∩ \ldots \ ∩ \ F^\sigma_t \ (since \ f \ ∈ \ F^\sigma_s \ and \ primary \ flaws \ are \ not \ eradicated \ by \ addressing \ other \ flaws). \ Therefore, \ f \ ∈ \ Δ_j = \{j\} \ for \ j ∈ [s, i - 1]. \ This \ implies \ that \ parent(i) < s, \ and \ therefore \ i ∈ I_s(φ) \ and \ f = w_i ∈ R_s(φ).

Conversely, \ consider \ flaw \ f \ ∈ \ R_s(φ) \ and \ index \ i ∈ I_s(φ) \ with \ f = w_i. \ We \ have \ parent(i) < s \ by \ the \ definition \ of \ I_s(φ), \ and \ so \ f \ ∈ \ F^\sigma_s \ by \ Proposition 9(a). \ Assume \ now \ that \ f = \arg \min_{f ∈ R_s(φ)} π(f), \ then \ i = \minNode_s(φ). \ As \ shown \ above, \ f \ is \ addressed \ in \ φ_{s:max t_s}, \ and \ therefore, \ i ≤ \max t_s. \ During \ iteration \ s \ flaw \ f \ will \ be \ moved \ to \ the \ left \ from \ position \ i \ to \ position \ s. \ Such \ swaps \ do \ not \ affect \ walks \ φ_{t_{s+1}}, \ldots, φ_{t_p}, \ and \ therefore \ condition \ (A_{s+1}) \ will \ hold. \ This \ proves \ the \ claim.

(b) \ Suppose \ the \ claim \ is \ false, \ and \ consider \ the \ earliest \ moment \ when \ set \ \mathcal{X} \ became \ conflicting. \ This \ must \ have \ happened \ after \ the \ update \ in \ line \ 3 \ for \ some \ s ≥ 1 \ and \ i > s. \ Let \ φ, \ \overline{φ} \ be \ conflicting \ walks \ in \ \mathcal{X}. \ They \ must \ have \ the \ form

φ = σ_1 \xrightarrow{w_1} \ldots \xrightarrow{w_i-1} σ_j \xrightarrow{w_j} \ldots

\overline{φ} = σ_1 \xrightarrow{w_1} \ldots \xrightarrow{w_j-1} σ_j \xrightarrow{w_j} \ldots

where \ w_j \neq \overline{w}_j \ and \ j ≥ i - 1 ≥ s \ (since \ flaws \ at \ positions \ 1, 2, \ldots, i - 2 \ have \ not \ changed \ in \ the \ latest \ update). \ We \ have \ φ_{1:s} = \overline{φ}_{1:s} \ and \ R_s(φ) = F^\sigma_s = R_s(\overline{φ}), \ and \ so \ walks \ φ, \ \overline{φ} \ are \ s-conflicting. \ However, \ we \ showed \ in \ the \ previous \ section \ that \ \mathcal{X} \ is \ s-nonconflicting - \ a \ contradiction.

By \ construction, \ at \ each \ point \ of \ Algorithm 2 \ non-terminal \ walks \ φ ∈ \mathcal{X} \ have \ length \ t^*. \ This \ fact \ together \ with \ Lemma 20(b) \ gives

Corollary 30. \ At \ each \ point \ of \ Algorithm 2 \ set \ \mathcal{X} \ satisfies \ the \ following: \ there \ exists \ a \ deterministic \ strategy \ Λ \ such \ that \ \mathcal{X} \subseteq \text{Runs}_Λ(t^*).

We \ will \ show \ the \ following \ fact.

Theorem 31. \ Consider \ line \ 3 \ of \ Algorithm 2 \ for \ indices \ s \ and \ i, \ and \ let \ \mathcal{X} \ and \ \mathcal{X}' \ be \ the \ sets \ before \ and \ after \ the \ update, \ respectively. \ If \ \mathcal{X} = \text{Runs}_Λ(t^*) \ and \ \mathcal{X}' \subseteq \text{Runs}_{Λ'}(t^*) \ for \ some \ deterministic \ strategies \ Λ, Λ' \ then \ \mathcal{X}' = \text{Runs}_{Λ'}(t^*)\.
Algorithm 2. This will in turn imply Theorem 31(b). Indeed, recall that Π is injective. It is easy to check inclusions

\[
\begin{align*}
\text{Runs}_d^0(t) & \subseteq \text{Runs}(t^*) \\
\text{Runs}_d^a(t) & \subseteq \text{Runs}_A(t^*) \\
\text{Runs}^0(t) & \subseteq \text{Runs}^0_A(t) \\
\text{II}(\text{Runs}^0(t)) & \subseteq \text{Runs}_A^0(t) \\
\text{II}^{-1}(\text{Runs}_A^0(t)) & \subseteq \text{Runs}^0(t)
\end{align*}
\]

This implies that II(\text{Runs}^0(t)) = \text{Runs}^0_A(t). By Lemma 32(b), all walks in II(\text{Runs}^0(t)) are π-walks, and therefore follow the π-strategy Λ_π (as shown in the proof of Corollary 12). This means that strategy Λ acts on all walks in \text{Runs}^0_A(t) as Λ_π, and so II(\text{Runs}^0(t)) = \text{Runs}^0_A(t) = \text{Runs}^0_*(t).

In the remainder of this section we prove Theorem 31. We will need the following construction:

- Consider walk φ of length \( t \leq t^* \). If \( φ \) follows Λ then we define \( \text{follow}_A(φ) \) as a walk obtained by extending \( φ \) using strategy Λ until we either arrive at a flawless steps or get a walk of length \( t^* \). (Of course, there may be many such extensions; let us fix some determinstic rule for picking one of them). If φ does not follow Λ then we let \( i \in [l - 1] \) be the largest index such that \( \varphi_{1:i-1} \) follows Λ, and define \( \text{follow}_A(φ) = \text{follow}_A(φ_{1:i-1}) \). By construction, we have \( \text{follow}_A(φ) \in \text{Runs}_A(t^*) = Χ \).

Consider walk η ∈ \text{Runs}_A(t^*). We need to prove that there exists φ ∈ Χ such that Π(φ) = η where Π is the mapping defined in line 3. Suppose this is false. If η ∈ Χ then we must have Π(η) = η, i.e. Π(η) = \text{Swap}(η) ∈ Χ. Therefore, walks η and \text{Swap}(η) both follow strategy Λ’. It can be checked that this is impossible. We thus assume from now on that η \notin Χ. Consider walk \( φ = \text{follow}_A(η) \in Χ \). Note that Π(φ) ∈ Χ’ ⊆ \text{Runs}^*_A(t^*). Since φ follows Λ and η does not, we have

\[
\begin{align*}
φ & = σ_1 \rightarrow w_1 \rightarrow ... \rightarrow w_{k-1} \rightarrow σ_k \rightarrow a \rightarrow σ_{k+1} \rightarrow ... \\
η & = σ_1 \rightarrow w_1 \rightarrow ... \rightarrow w_{k-1} \rightarrow σ_k \rightarrow b \rightarrow τ_{k+1} \rightarrow ...
\end{align*}
\]

where \( k \in [l] \) and \( a \neq b \). We must have Π(φ) \neq φ (otherwise we would have \( φ, η \in \text{Runs}^*_A(t^*) \), but these walks follow different strategies - a contradiction). Thus, Π(φ) = \text{Swap}(φ) ∈ Χ’ ⊆ \text{Runs}^*_A(t^*).

We know that walks \text{Swap}(φ) and η follow the same strategy Λ’. It can be checked that this is only possible if \( k = i - 1 \) and \( φ, η \) have the following forms:

\[
\begin{align*}
φ & = \xi \rightarrow a \rightarrow σ_i \rightarrow b \rightarrow σ_{i+1} \rightarrow ... \\
η & = \xi \rightarrow b \rightarrow τ_i \rightarrow ...
\end{align*}
\]

where \( ξ = σ_1 \rightarrow w_1 \rightarrow ... \rightarrow w_{i-1} \rightarrow σ_{i-1} \) and \( σ_{i-1} \rightarrow a \rightarrow σ_i \rightarrow b \rightarrow σ_{i+1} \in Φ^* \).

Lemma 32. There exists walk ψ = σ_{i-1} \rightarrow a \rightarrow σ_i \rightarrow b \rightarrow σ_{i+1} \in Φ^* such that \text{Swap}(ψ) = σ_{i-1} \rightarrow b \rightarrow τ_i \rightarrow σ_{i+1}. Furthermore, if η = ξ \rightarrow b \rightarrow τ_i \rightarrow σ_{i+1} \rightarrow ... \) then such ψ can be chosen so that \( σ_{i+1} = τ_{i+1} \).

Proof. Note that flaw a is present in \( σ_{i-1} \) and thus in \( τ_i \). Therefore, there exists transition \( τ_i \rightarrow σ_{i+1} \) for some \( σ_{i+1} \). Furthermore, we can choose \( σ_{i+1} = τ_{i+1} \) if \( η = ξ \rightarrow b \rightarrow τ_i \rightarrow σ_{i+1} \rightarrow ... \). We have \( σ_{i-1} \rightarrow b \rightarrow τ_i \rightarrow σ_{i+1} \in Φ^* \) (since \( a \in F_{σ_{i-1}} - \{b\} \) and so \( a \notin Δ(σ_{i-1} \rightarrow τ_i) \)). Define \( Φ^*_ab = \{ψ \in Φ^* \mid ψ = σ_{i-1} \rightarrow a \rightarrow σ \rightarrow b \rightarrow σ_{i+1} \} \) and \( Φ^*_ba = \{ψ \in Φ^* \mid ψ = σ_{i-1} \rightarrow b \rightarrow σ \rightarrow a \rightarrow σ_{i+1} \} \). By the definition of commutativity, there exist injective mappings \( Φ^*_ab \rightarrow Φ^*_ba \) and \( Φ^*_ba \rightarrow Φ^*_ab \). Therefore, these mappings are actually bijections, and so there exists \( ψ \in Φ^*_ab \) with \( \text{Swap}(ψ) = σ_{i-1} \rightarrow b \rightarrow τ_i \rightarrow σ_{i+1} \). □
Let us now define \( \tilde{\varphi} = \text{follow}_{\Lambda}(\xi \xrightarrow{a} \tilde{\sigma}_i \xrightarrow{b} \tilde{\sigma}_{i+1}) \in \mathcal{X} \).

Lemma 33. There holds \( \tilde{\varphi} = \xi \xrightarrow{a} \tilde{\sigma}_i \xrightarrow{b} \tilde{\sigma}_{i+1} \ldots \).

Proof. We know that walk \( \xi \xrightarrow{a} \tilde{\sigma}_i \) follows strategy \( \Lambda \) (since \( \xi \xrightarrow{a} \) is a prefix of \( \varphi \in \mathcal{X} \)). Therefore, \( \varphi = \xi \xrightarrow{a} \tilde{\sigma}_i \ldots \).

Since \( \Pi(\varphi) = \text{Swap}_i(\varphi) \), we have \( i = \text{MinNode}_i(\varphi) \) and \( b = \arg \min_{f \in R_i(\varphi)} \pi(f) \). Since \( \varphi_{1:s} = \tilde{\varphi}_{1:s} \) and \( \varphi, \tilde{\varphi} \in \mathcal{X} \), from Lemma 20 we conclude that \( R_s(\varphi) = R_s(\tilde{\varphi}) \). Denote \( j = \text{MinNode}_s(\tilde{\varphi}) \); it is smallest index \( j \in [s, t^*] \) such that walk \( \tilde{\varphi} \) has flaw \( b \) at position \( j \). If \( j < i \) then walk \( \varphi \) would have flaw \( b \) at position \( j \in [s, i-1] \) (since \( \varphi_{s:i-1} = \tilde{\varphi}_{s:i-1} \)), contradicting condition \( i = \text{MinNode}_s(\varphi) \).

We also cannot have \( j = i \), implying that \( \tilde{\varphi} = \xi \xrightarrow{a} \tilde{\sigma}_i \xrightarrow{b} \tilde{\sigma}_{i+1} \ldots \). By recalling the definition of \( \tilde{\varphi} \) we can now conclude that \( \tilde{\varphi} = \xi \xrightarrow{a} \tilde{\sigma}_i \xrightarrow{b} \tilde{\sigma}_{i+1} \ldots \). \( \square \)

We are now getting closer. The form of \( \tilde{\varphi} \) implies that \( \Pi(\tilde{\varphi}) = \text{Swap}_i(\tilde{\varphi}) = \xi \xrightarrow{b} \tau_i \xrightarrow{a} \tilde{\sigma}_{i+1} \ldots \).

Since walks \( \eta = \xi \xrightarrow{b} \tau_i \ldots \) and \( \Pi(\tilde{\varphi}) \) both follow strategy \( \Lambda' \) and belong to \( \text{Runs}_{\Lambda'}(t^*) \), we must have \( \eta = \xi \xrightarrow{b} \tau_i \xrightarrow{a} \tilde{\sigma}_{i+1} \ldots \). Furthermore, we can assume that \( \tau_{i+1} = \tilde{\sigma}_{i+1} \) (see Lemma 32).

Let us write \( \eta = \xi \xrightarrow{b} \tau_i \xrightarrow{a} \tilde{\sigma}_i \xrightarrow{a} \xi' \) where \( \xi' = \tau_{i+1} \equiv u_{i+1} \ldots u_i \tau_{i+1} \). Define walk \( \varphi^* = \xi \xrightarrow{a} \tilde{\sigma}_i \xrightarrow{b} \xi' \), so that \( \text{Swap}_i(\varphi^*) = \eta \). To prove Theorem 31 it now suffices to show that \( \varphi^* \in \mathcal{X} \).

Define \( \hat{\varphi}^* = \text{follow}_{\Lambda}(\varphi^*) \in \mathcal{X} \). Walk \( \hat{\varphi} = \xi \xrightarrow{a} \tilde{\sigma}_i \xrightarrow{b} \tau_{i+1} \ldots \) follows \( \Lambda \), therefore we must have \( \hat{\varphi}^* = \xi \xrightarrow{a} \tilde{\sigma}_i \xrightarrow{b} \tau_{i+1} \ldots \) and consequently \( \Pi(\hat{\varphi}^*) = \text{Swap}_i(\hat{\varphi}^*) \). Suppose that \( \hat{\varphi}^* \neq \varphi^* \), then we must have

\[
\begin{align*}
\eta &= \xi \xrightarrow{b} \tau_i \xrightarrow{a} \tau_{i+1} \ldots u_{j+1} \\
\varphi^* &= \xi \xrightarrow{a} \tilde{\sigma}_i \xrightarrow{b} \tau_{i+1} \ldots u_{j+1} \\
\hat{\varphi}^* &= \xi \xrightarrow{a} \tilde{\sigma}_i \xrightarrow{b} \tau_{i+1} \ldots \hat{u}_{j+1} \\
\Pi(\hat{\varphi}^*) &= \xi \xrightarrow{b} \tau_i \xrightarrow{a} \tau_{i+1} \ldots \hat{u}_{j+1}
\end{align*}
\]

where \( j \geq i + 1 \) and \( u_{j+1} \neq \hat{u}_{j+1} \). But this is impossible since walks \( \eta, \Pi(\hat{\varphi}^*) \) follow the same strategy \( \Lambda' \).

A Comparison of conditions (A), (B), (\star) for non-primary flaws

In this section we assume \( P = \emptyset \) and \( \alpha(f,g) = [f \sim g] \in \{0, 1\} \) where \( \sim \) is some symmetric binary relation \( \sim \) on \( F \). In this case the definition of \( \lambda_f(S) \) from condition (\star) can be rewritten as follows:

\[
\lambda_f(S) = [S \text{ is independent in } (F, \sim)] \cdot \gamma_f(S - \{g \mid g \sim f\})
\]

The next question is how to choose relation \( \sim \). Intuitively, we believe that it is beneficial to set \( f \sim g \) if the interaction between \( f \) and \( g \) is “sufficiently strong”. To support this claim, consider the following toy example. Let \( \Omega = \{00, 10\} \) and \( F = \{f,g\} \) where \( f = \{00, 01\}, \ g = \{00, 10\} \).

Choose positive numbers \( p \in (0,1) \) and \( a \in (2p-1, p^2) \), and set \( \omega = \frac{a}{b} \begin{bmatrix} a & b & c \end{bmatrix} = \frac{a}{p-a} \begin{bmatrix} a & p-a & 1+a-2p \end{bmatrix} \).

(Here \( \omega \) is a vector whose components correspond to elements of \( \Omega \) in a natural way). Note that \( \omega(f) = \omega(g) = p \). Also, if \( a = p^2 \) then events \( f \) and \( g \) are independent. Sampling distributions \( \rho \)
will be controlled by two parameters \( q, r \in [0,1] \) where \( q \) is the probability of going from 00 to \( \{00,10\} \) when addressing \( f \), and \( r \) is the probability of going from 01 to \( \{00,10\} \) when addressing \( f \). Given these parameters, define sampling distributions \( \rho \) as follows:

\[
\rho(\cdot|f, 00) = \begin{cases} \frac{qa}{p} & \text{if } p = \frac{qa}{b} \\ \frac{qb}{p} & \text{if } q = \frac{1-q}{1-p} \end{cases} \quad \rho(\cdot|g, 00) = \begin{cases} \frac{qa}{p} & \text{if } p = \frac{qa}{b} \\ \frac{qb}{p} & \text{if } q = \frac{1-q}{1-p} \end{cases}
\]

\[
\rho(\cdot|f, 01) = \frac{ra}{p} & \text{if } p = \frac{qa}{b} \\ \frac{rb}{p} & \text{if } q = \frac{1-q}{1-p} \end{cases} \quad \rho(\cdot|g, 10) = \begin{cases} \frac{ra}{p} & \text{if } p = \frac{qa}{b} \\ \frac{rb}{p} & \text{if } q = \frac{1-q}{1-p} \end{cases}
\]

To get regenerating resampling oracles, we set \( r = \frac{q^2 - qa}{b} \). After some calculations we get

\[
(\lambda_f(\omega), \lambda_f(\{f\}), \lambda_f(\{g\}), \lambda_f(\{f, g\})) = \begin{cases} (p, p, p, 0) & \text{if } f \sim g \\ (p, p, x, x) & \text{if } f \sim g \end{cases}
\]

where \( x = p - \frac{qa}{b} \in [0,p] \). The expressions for \( \lambda_q(S) \) are analogous (note, \( f \) and \( g \) are completely symmetric). Value \( x \) can be viewed as a measure of “interaction strength” between \( f \) and \( g \). In particular, if \((a, q) = (p^2, 1)\) then \( x = 0 \). Condition \( a = p^2 \) means that events \( f, g \) are independent, and \( q = 1 \) means that sampling oracles \( \rho \) “respect” this independence. One can now check the following:

- If \( f \sim g \) then condition \( (\ast) \) (equivalently, \( (B) \)) holds iff \( p < \frac{1}{2} \), assuming that \( \mu_f = \mu_g \).
- If \( f \sim g \) then condition \( (\ast) \) (equivalently, \( (C) \)) holds iff \( x < (1 - \sqrt{p})^2 \), assuming that \( \mu_f = \mu_g \).

It is now easy to check that there are regimes when \( (B) \) works but \( (C) \) does not, and vice versa. One can take, for example, the following parameters (one can check that they define valid distributions \( \omega \) and \( \rho(\cdot|f, \sigma) \)):

\[
p = 0.4 \quad (a, b, c) = (0.1, 0.3, 0.3) \quad (q, r) = (1, 0.2) \quad x = 0.15
\]

\[
p = 0.6 \quad (a, b, c) = (0.35, 0.25, 0.15) \quad (q, r) = (1, 0.04) \quad x = 0.01666\ldots
\]

Note, if \( q = 1 \) then flaws \( f, g \) are primary. It is also possible to get an example with non-primary flaws by taking \( q = 1 - \varepsilon \) for sufficiently small \( \varepsilon > 0 \).

Some applications may have both “weak” and “strong” interactions, in which case condition \( (\ast) \) could improve on both \( (B) \) and \( (C) \). While constructing an artificial example should be straightforward, at the moment we do not have a “real” application.

To conclude this section, we give some simple bound on values \( \gamma_f(\{g\}) \) (which is tight for the example above). Note, for practical applications computing \( \gamma_f(S) \) for \(|S| \geq 2 \) may be a difficult task, and one may have to resort to bounds \( \gamma_f(S) \leq \gamma_f(\{g\}) \) for some \( g \in S \).

**Theorem 34.** Let \( f, g \) be distinct flaws in \( F \) with \( f \cap g \neq \emptyset \). There holds \( \gamma_f(\{g\}) \geq \frac{\omega(f) \cdot \omega(f) - q \cdot \omega(f|g)}{\omega(g)} \) where \( q \in [0,1] \) is defined as follows: sample \( \hat{\sigma} \sim \omega(\cdot|f \cap g) \), then sample \( \hat{\tau} \sim \rho(\cdot|f, \hat{\sigma}) \). Then \( q = \mathbb{P}[\hat{\tau} \in g] \).

**Proof.** Consider random variable \( (\sigma, \tau) \) generated by the following process: \( \sigma \sim \omega(\cdot|f) \), \( \tau \sim \rho(\cdot|f, \sigma) \). Variable \( \sigma \) conditioned on event \( \sigma \in g \) has the same distribution as variable \( \hat{\sigma} \), therefore \( \mathbb{P}[\tau \in g | \sigma \in g] = q \) and so \( \mathbb{P}[\sigma \in g, \tau \in g] = \mathbb{P}[\tau \in g | \sigma \in g] \cdot \mathbb{P}[\sigma \in g] = q \cdot \omega(g|f) \). Denote \( p = \mathbb{P}[\sigma \notin g, \tau \in g] \), then

\[
\omega^f(g) = \mathbb{P}[\tau \in g] = \mathbb{P}[(\sigma \in g, \tau \in g) + (\sigma \notin g, \tau \in g)] = q \cdot \omega(g|f) + p
\]
Let us fix $\tau \in \Omega$. By definition, $\Omega_f(\{g\}, \tau)$ is the set of states $\sigma \in f$ such that $\sigma \xrightarrow{f} \tau$ is a valid transition with $\sigma \not\in g$ and $\tau \in g$. In particular, if $\tau \not\in g$ then $\Omega_f(\{g\}, \tau) = 0$. Assume that $\tau \in g$, and define

$$\nu(\tau) = \sum_{\sigma \in \Omega_f(\{g\}, \tau)} \omega(\sigma) \rho(\tau|f, \sigma) = \omega(f) \sum_{\sigma \in \Omega_f(\{g\}, \tau)} \omega(\sigma|f) \rho(\tau|f, \sigma)$$

We can now prove the claim as follows:

$$\sum_{\tau \in g} \nu(\tau) = \omega(f) \sum_{\tau \in g} \sum_{\sigma \in \Omega_f(\{g\}, \tau)} \omega(\sigma|f) \rho(\tau|f, \sigma) = \omega(f) \cdot \sum_{\sigma \in \Omega_f(\{g\}, \tau)} \omega(\sigma|f) = \omega(f) \cdot \sum_{\sigma \in \Omega_f(\{g\}, \tau)} \omega(\sigma|f) = \omega(f) \cdot \sum_{\sigma \in \Omega_f(\{g\}, \tau)} \omega(\sigma)$$

$$\gamma_f(\{g\}) = \max_{\tau \in g} \frac{\nu(\tau)}{\omega(\tau)} \geq \frac{\sum_{\tau \in g} \nu(\tau)}{\omega(g)} = \frac{\omega(f|g)}{\omega(g)} \cdot \omega(f) - q \cdot \frac{\omega(f \cap g)}{\omega(g)} = \frac{\omega(f|g)}{\omega(g)} \cdot \omega(f) - q \cdot \omega(g|f)$$

\[\square\]

B Discussion of parallelization

In the previous work the commutativity condition has led to efficient parallel algorithms [15], and was also used to show that the output of Algorithm 1 approximates in a certain sense the “LLL distribution” (i.e. distribution $\omega$ conditioned on flawless states) [12]. It is natural to ask whether similar extensions are possible in our setting. Unfortunately, there appear to be significant obstacles. In this section we describe one approach that we explored, and discuss associated difficulties. Since the results are mostly negative, we keep it on an informal level and do not provide rigorous proofs.

We focus on the case when all flaws are primary (with the hope to capture the backtracking algorithm described in Section 2.4). A natural candidate for the parallel version would be the following algorithm.

**Algorithm 3:** Parallel Random Walk.

1. sample $\sigma \in \Omega$ according to some distribution $\omega^{\text{init}}$
2. while $F_\sigma$ non-empty do
   3. let $S = F_\sigma$
   4. foreach $f \in S$ in some order do
      5. sample $\sigma' \in \Omega$ according to some distribution $\rho(\sigma'|f, \sigma)$, set $\sigma \leftarrow \sigma'$.

Note that in line 5 we must have $f \in F_\sigma$, since all flaws are primary and addressing other flaws does not eliminate $f$. Thus, Algorithm 3 is well-defined. We refer to one pass through lines 3-5 as a round. In the case of the backtracking algorithm each round could be implemented as follows: (i) sample all unassigned variables; (ii) consider violated constraints, and find a maximal independent subset of such constraints; (iii) unassign all variables involved this subset. It can be checked that there exists an ordering of flaws for which this becomes equivalent to one round of Algorithm 3 (but we need to assume that sampling oracles $\rho(\cdot|f_v, \sigma)$ for variables $v \in [n]$ are allowed to pick an arbitrary violated clause involving $v$, not just the lowest indexed clause).

One approach for analyzing Algorithm 3 is as follows. Consider flaw $r \in F$ and forest $T$ with $V_T = [\ell]$. Define forest $T[r]$ as follows: find the smallest index $i \in [\ell]$ with $\ell_T(i) = r$, and let $T[r]$ be the subtree of $T$ rooted at $i$. If such $i$ does not exist then $T[r]$ is the empty forest. Note that $T[r]$ is in fact a tree. Clearly, if $T$ is proper then so is $T[r]$.

Now let $\mathcal{X}$ be the set of terminal walks in Algorithm 3 that make at least $s$ rounds (and at most $t^+$ steps, for some large $t^+$). For a flaw $r$ let $\mathcal{X}_r$ be the set of walks $\varphi \in \mathcal{X}$ with $|V_{T_{\varphi}[r]}| \geq s$. It is
easy to show that $X \subseteq \bigcup_{r \in F} X_r$. Let us apply Algorithm 2 to $X_r$. In general, tree $T_\varphi[r]$ for a walk $\varphi \in X_r$ may shrink during swapping operations. It is possible to show, however, that $|V_{T_\varphi[r]}|$ will never decrease under two assumptions: (i) $r = \arg \max_{f \in F} \pi(f)$; (ii) mapping $\text{Swap}$ in Definition 6 satisfies additional property: $\Box(\varphi_1) - \Box(\varphi_2) \subseteq \Box(\varphi_2')$. Condition (i) can be satisfied by choosing permutation $\pi$ appropriately, and condition (ii) holds for the backtracking algorithm.

After the transformation $X_r$ becomes a set of $\pi$-walks $\varphi$ such that $V_{T_\varphi[r]} = [k, t] \subseteq [t]$ where $t = \text{length}(\varphi)$, $k \in [t]$ and $t - k + 1 \geq s$. Define distribution $\omega_r$ over $\Omega_r^d = \{ \sigma \in \Omega \mid F_\sigma = \{ r \} \}$ as follows: run Algorithm 1 with the $\pi$-strategy until first encountering state $\sigma \in \Omega_r$, and let $\omega_r$ be the resulting distribution. With some reasoning one can show that

$$p(X) \leq \theta^s \sum_{r \in F} \max_{\sigma \in \Omega_r} \frac{\omega_r(\sigma)}{\omega(\sigma)} \mu_r \leq \theta^s \cdot |F| \cdot \max_{r \in F, \sigma \in \Omega_r} \frac{\omega_r(\sigma)}{\omega(\sigma)} \cdot \max_{r \in F} \mu_r$$

Therefore, the algorithm will terminate within $(T_{\text{par}}^0 + s)/\log 2 \frac{1}{\theta}$ rounds with probability at least $1 - 2^{-s}$, where

$$T_{\text{par}}^0 = \log_2 |F| + \log_2 \left( \max_{r \in F, \sigma \in \Omega_r} \frac{\omega_r(\sigma)}{\omega(\sigma)} \right) + \log_2 \left( \max_{r \in F} \mu_r \right)$$

While the last term improves on the analogous term in (14), the middle term becomes worse. We do not see a way to get a meaningful bound on it. So this approach appears to fail.

Note in [15] we used a backward analysis to analyze the parallel algorithm. Unfortunately, we do not see how to use a backward analysis for the framework used in this paper. There appears to be no natural way to define a “backward” forest (with the roots on the right rather than on the left). We found one “unnatural” way to define such a forest, but it does not appear to lead anywhere.

In summary, we do not know whether Algorithm 3 has a polylogarithmic bound on the number of rounds. We leave this as an open question.

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References

[1] Dimitris Achlioptas and Fotis Iliopoulos. Random walks that find perfect objects and the lovász local lemma. J. ACM, 63(3):22:1–22:29, 2016.

[2] Dimitris Achlioptas, Fotis Iliopoulos, and Vladimir Kolmogorov. A local lemma for focused stochastic algorithms. arXiv:1809.01537, September 2018. To appear in SICOMP.

[3] Dimitris Achlioptas, Fotis Iliopoulos, and Alistair Sinclair. Beyond the Lovász local lemma: Point to set correlations and their algorithmic applications. In FOCS, 2019.

[4] Michael Albert, Alan Frieze, and Bruce Reed. Multicolored Hamilton cycles. Electronic Journal of Combinatorics, 2, 1995.

[5] R. Bissacot, R. Fernández, A. Procacci, and B. Scoppola. An improvement of the Lovász local lemma via cluster expansion. Combin. Probab. Comput., 20:709–719, 2011.
[6] Karthekeyan Chandrasekaran, Navin Goyal, and Bernhard Haeupler. Deterministic algorithms for the lovász local lemma. *SIAM Journal on Computing*, 42(6):2132–2155, 2013.

[7] P. Erdős and L. Lovász. Problems and results on 3-chromatic hypergraphs and some related questions. In *Colloq. Math. Soc. J. Bolyai, Infinite and Finite Sets*, volume 10, pages 609–627. North-Holland, 1975.

[8] P. Erdős and J. Spencer. The lopsided Lovász local lemma and latin transversals. *Discrete Applied Mathematics*, 30:151–154, 1991.

[9] D. Harris and A. Srinivasan. The Moser-Tardos framework with partial resampling. *J. ACM*, 66(5), 2019.

[10] David G. Harris and Aravind Srinivasan. A constructive algorithm for the Lovász local lemma on permutations. In *SODA*, pages 907–925, 2014.

[11] Nicholas Harvey and Jan Vondrák. An algorithmic proof of the Lovász local lemma via resampling oracles. In *FOCS*, 2015.

[12] Fotis Iliopoulos. Commutative algorithms approximate the LLL-distribution. In *RANDOM*, 2018.

[13] Kashyap Kolipaka, Mario Szegedy, and Yixin Xu. A sharper local lemma with improved applications. In A. Gupta, K. Jansen, J. Rolim, and R. Servedio, editors, *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques. APPROX 2012, RANDOM 2012*, volume 7408. Lecture Notes in Computer Science. Springer, Berlin, Heidelberg, 2012.

[14] Kashyap Babu Rao Kolipaka and Mario Szegedy. Moser and Tardos meet Lovász. In *Proceedings of the Forty-third Annual ACM Symposium on Theory of Computing, STOC*, pages 235–244, 2011.

[15] Vladimir Kolmogorov. Commutativity in the algorithmic Lovász local lemma. *SICOMP*, 47(6):2029–2056, 2018.

[16] Robin A. Moser and Gábor Tardos. A constructive proof of the general Lovász local lemma. *J. ACM*, 57(2), 2010.

[17] Wesley Pegden. Highly nonrepetitive sequences: Winning strategies from the local lemma. *Random Structures and Algorithms*, 38(1-2):140–161, 2011.

[18] Alexander D. Scott and Alan D. Sokal. The repulsive lattice gas, the independent-set polynomial, and the Lovász local lemma. *Journal of Statistical Physics*, 118(5-6):1151–1261, 2005.

[19] James B. Shearer. On a problem of Spencer. *Combinatorica*, 5(3):241–245, 1985.