Extensions of the MapDE algorithm for mappings relating differential equations

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Abstract

This paper is a sequel of our previous work in which we introduced the MapDE algorithm to determine the existence of analytic invertible mappings of an input (source) differential polynomial system (DPS) to a specific target DPS, and some times by heuristic integration an explicit form of the mapping. A particular feature was to exploit the Lie symmetry invariance algebra of the source, without integrating its equations, to facilitate MapDE, making algorithmic an approach initiated by Bluman and Kumei. In applications, however, the explicit form of a target DPS is not available, and a more important question is, can the source be mapped to a more tractable class. This aspect was illustrated by giving an algorithm to determine the existence of a mapping of a linear differential equation to the class of constant coefficient linear differential equations, again algorithmically realizing a method of Bluman and Kumei. Key for this application was the exploitation of a commutative sub-algebra of symmetries corresponding to translations of the independent variables in the target.

In this paper, we extend MapDE to determine if a source nonlinear DPS can be mapped to a linear differential system. The methods combine aspects of the Bluman-Kumei mapping approach, together with techniques introduced by Lyakhov, Gerdt and Michels, for the determination of exact linearizations of ODE. The Bluman-Kumei approach which is applied to PDE, focuses on the fact that such linearizable systems must admit an infinite Lie sub-pseudogroup corresponding to the linear superposition of solutions in the target. In contrast, Lyakhov et al., focus on ODE, and properties of the so-called derived sub-algebra of the (finite) dimensional Lie algebra of symmetries of the ODE. Examples are given to illustrate the approach, and a heuristic integration method, some times gives explicit forms of the maps. We also illustrate the powerful maximal symmetry groups facility as a natural tool to be used in conjunction with MapDE.

Keywords: Symmetry, Lie algebra, defining equations, structure constants, algorithm, differential algebra, differential elimination, involutivity, linearization, numerical

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1 Introduction

This paper is a sequel to [27], and is part of a series in which we explore algorithmic aspects of exact and approximate mappings of differential equations. We are interested in mapping less
tractable differential equations into more tractable ones, in particular in this article focusing on mapping nonlinear systems to linear systems. This builds on progress in [27] where we considered mappings from a specific differential system to a specific target system; and also mappings from a linear to a linear constant coefficient differential equation.

As in [27] we consider systems of (partial or ordinary) differential equations with \( n \) independent variables and \( m \) dependent variables which are local analytic functions of their arguments. Suppose \( \hat{R} \) has independent variables \( \hat{x} = (\hat{x}_1, \ldots, \hat{x}_n) \) and dependent variables \( \hat{u} = (\hat{u}_1, \ldots, \hat{u}_m) \). In particular, we consider local analytic mappings \( \Psi: (\hat{x}, \hat{u}) = \Psi(x, u) = (\psi(x, u), \phi(x, u)) \), so that \( \hat{R} \) is locally and invertibly mapped to \( \hat{R} \):

\[
\hat{x}_j = \psi_j(x, u), \quad \hat{u}_k = \phi_k(x, u) \tag{1}
\]

where \( j = 1, \ldots, n \) and \( k = 1, \ldots, m \). The mapping is locally invertible, so the determinant of the Jacobian of the mapping is nonzero:

\[
\text{Det Jac}(\Psi) = \text{Det} \left( \frac{\partial(\psi, \phi)}{\partial(x, u)} \right) \neq 0, \tag{2}
\]

where \( \frac{\partial(\psi, \phi)}{\partial(x, u)} \) is the usual Jacobian \((n + m) \times (n + m)\) matrix of first order derivatives of the \( (n + m) \) functions \((\psi, \phi)\) with respect to the \( (n + m) \) variables \((x, u)\). Note throughout this paper, we will call \( \hat{R} \) the Target system of the mapping, which will generally have some more desirable features than \( R \), which we call the Source system. A well-known direct approach to forming the equations satisfied by \( \Psi \) is roughly to substitute the general change of variables into \( \hat{R} \), evaluate the result modulo \( R \), appending equations that express the independence of \( \Psi \) on derivative jet variables (or equivalently decomposing in independent expressions in the jet variables). The resulting equations for \( \Psi \) are generally nonlinear overdetermined systems.

A very general approach to such problems, concerning maps \( \Psi \) from \( R \) to \( \hat{R} \), is Cartan’s famous Method of Equivalence which finds invariants, that label the classes of systems, equivalent under the pseudogroup of such mappings. See especially the texts [30] and [25]. The fundamental importance and computational difficulty of such equivalence questions, has attracted attention from the symbolic computation community [28]. For recent developments and extensions of Cartan’s moving frames for equivalence problems see [13], [40] and [3]. The DifferentialGeometry package [24], available in Maple and has been applied to equivalence problems [17]. Underlying these calculations, is that overdetermined PDE systems, with some non-linearity, must be reduced to forms that enable the statement of a local existence and uniqueness theorem [16, 14, 38, 37, 9].

Our methods here and in [27] are based on the mapping approach initiated by Bluman and Kumei [18], which focuses on the interaction between such mappings and Lie symmetries via their infinitesimal form on the source and target. In particular, let \( \mathcal{G} \) be the Lie group of transformations leaving \( R \) invariant. Also, \( \hat{\mathcal{G}} \) be the Lie group of transformations leaving \( \hat{R} \) invariant. Locally such Lie groups are characterized by their linearizations in a neighborhood of their identity, that is by their Lie algebras \( \mathcal{L} \), \( \hat{\mathcal{L}} \). If an invertible map \( \Psi \) exists then \( \mathcal{G} \simeq \hat{\mathcal{G}} \) and \( \mathcal{L} \simeq \hat{\mathcal{L}} \). This yields a subsystem of linear equations for \( \Psi \) which we call the Bluman-Kumei equations.

It is a significant challenge to translate the methods of Bluman and Kumei, into procedures that are algorithmic (i.e., guaranteed to succeed on a defined class of inputs in finitely many steps). Please see [11, 17, 11, 12] for progress in their approach, and also some (heuristic) integration based
computer implemented methods. Our methods are also inspired by recent progress on this question for ODE, by Lyakhov et al. [23] for the question of determining when an ODE is linearizable; since they used a very old method of one us (see Reid [31]), which has been dramatically improved and extended [32] [22], with the latest improvements in the LAVF package [21] [15].

In our previous work [27], we gave an algorithm to determine the existence of a mapping of a linear differential equation to the class of constant coefficient linear homogeneous differential equations. Key for this application was the exploitation of a commutative sub-algebra of symmetries of $\mathcal{L}$ corresponding to translations of the independent variables in the target. The main contribution of this paper, is to present an algorithmic method for determining existence of a mapping $\Psi$ a nonlinear system $R$ to a linear system $\hat{R}$. Using a technique of Bluman and Kumei, we exploit the fact that $\hat{R}$ must admit a sub-pseudo group, corresponding the superposition property that linear systems by definition must satisfy. Once existence is established, a second stage can determine features of the map, and some times by integration, explicit forms of the mapping.

As we mentioned in our previous paper, for an algorithmic treatment in these series papers, we limit the systems considered to being differential polynomials, with coefficients from computable subfield of $\mathbb{C}$ (e.g., $\mathbb{Q}$). Some non-polynomial systems can be converted to differential polynomial form by the use of the Maple command, \texttt{dpolyform}.

In §2 and §3, we discuss the implementation of the algorithm when TargetClass = LinearDE. We give examples of application uses MapDE, the linearization of a third order ODE and the linearization of a system of nonlinear PDE, in §4. Finally, we provide a discussion in §5.

2 Symmetries & Mapping Equations

2.1 Symmetries

Various differential systems will be considered, both linear and nonlinear. Algorithmic reduction of these to forms for which a local existence and uniqueness theorem is available will be key to this paper. We note that even requiring smoothness of a differential equation in its arguments does not guarantee local existence of smooth, or weaker forms of a solution. Lewy’s famous counterexample of a single linear differential equation in 3 variables, of order 1, with the smooth inhomogeneous term, provides such a counter example [20]. Moreover, Lie’s classical theory of groups and their algebras requires local analyticity in its defining equations.

For an algorithmic treatment using differential elimination (differential algebra), we limit the systems considered to being differential polynomials, with coefficients from computable subfield of $\mathbb{C}$ (e.g., $\mathbb{Q}$). Some non-polynomial systems can be converted to differential polynomial form by the use of the Maple command, \texttt{dpolyform}. Moreover, a central input in such algorithms are rankings of derivatives [36]. Indeed let $\Omega(R)$ be all the derivatives of dependent variables for $R$. Throughout this paper by derivatives we also include 0-order derivatives (i.e. dependent variables). A ranking on $\Omega(R)$ is a total order $\prec$ that satisfies the axioms in [36]. The ranking and the algorithms in [37] yields a determination of initial data, that yields existence and uniqueness of formal power series solutions satisfying the initial data. Additionally if the ranking is orderly and of Riquier type (i.e. ordered first by total order of derivative, with a ranking specified by a Riquier ranking matrix) analytic initial data, yields local analytic solutions. See Lemaire [19] for a proof of this result. In our paper we will need some block elimination rankings, that eliminate
groups of dependent variables in favor of others. Enforcing the block order via the first row of the Riquier Matrix, and then enforcing total order of the derivative as the next criterion for each block, enables an easy extension of the analytic data yields analytic solutions, sufficient for our purposes in this paper.

The geometric approach to DEs centers on the jet locus, the solution set of the equations obtained by replacing derivatives with formal variables, yielding systems of polynomial equations and inequations and differences of varieties. The union of prolonged graphs of local solutions is a subset of the jet locus in \( J^q \), the jet space of order \( q \). For details concerning the Jet geometry see [29, 38].

The infinitesimal Lie point symmetries for \( R \) are found by seeking vector fields

\[
V = \sum_{i=1}^{n} \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{j=1}^{m} \eta^j(x, u) \frac{\partial}{\partial u^j}
\]  

whose associated one-parameter group of transformations

\[
x^* = x + \xi(x, u) \epsilon + O(\epsilon^2)
\]

\[
u^* = u + \eta(x, u) \epsilon + O(\epsilon^2)
\]

which away from exceptional points preserve the jet locus of such systems - mapping solutions to solutions. See [5, 6] for applications. The infinitesimals \( (\xi^i, \eta^j) \) of a symmetry vector field (3) for a system of DEs are found by solving an associated system of linear homogeneous defining equations \( S \) (or determining equations) for the infinitesimals. The defining system \( S \) is derived by explicit prolongation formulae, for which numerous computer implementations are available [11, 12, 35].

The resulting vector space of vector fields is closed under its commutator. The commutator of two vector fields for vector fields \( X = \sum_{i=1}^{m+n} \nu^i \frac{\partial}{\partial z^i}, Y = \sum_{i=1}^{m+n} \mu^i \frac{\partial}{\partial z^i} \) in a Lie algebra \( \mathcal{L} \) and \( z = (x, u) \), is:

\[
[X, Y] = XY - YX = \sum_{i=1}^{m+n} \omega^i \frac{\partial}{\partial z^i}
\]

where \( \omega^k = \sum_{i=1}^{m+n} (\nu^i \mu^{k}_{zi} - \mu^i \nu^{k}_{zi}) \).

Similarly, we suppose that the Target admits symmetry vector fields

\[
\hat{V} = \sum_{i=1}^{n} \hat{\xi}^i(\hat{x}, \hat{u}) \frac{\partial}{\partial \hat{x}^i} + \sum_{j=1}^{m} \hat{\eta}^j(\hat{x}, \hat{u}) \frac{\partial}{\partial \hat{u}^j}
\]

in the Target infinitesimals \( (\hat{\xi}, \hat{\eta}) \), which correspondingly satisfies a linear homogeneous defining system \( \hat{S} \) generating a Lie algebra \( \hat{\mathcal{L}} \). Computations with defining systems of both systems will be essential in our approach. We have implemented our algorithms in Huang and Lisle’s powerful object oriented LAVF, Maple package [21].

**Example 2.1.** Consider as a simple example the third order nonlinear ODE which is in rif-form with respect to an orderly ranking

\[
u_{xxx} = \frac{3(uu_{xx} + u^2 + 1)^2}{u(uu_x + x)} - \frac{3u_x u_{xx}}{u} + \frac{8x(uu_x + x)^4 (u^2 + x^2 + 1)}{u(u^2 + x^2)}
\]
at points $u \neq 0, uu_x + x \neq 0, u^2 + x^2 \neq 0$. When Maple’s $\text{dsolve}$ is applied to (7) it yields no result. Later in this section, we will discover important information about (7) using symmetry aided mappings. It will be used as a simple running example to illustrate the techniques of the article.

The defining system for Lie point symmetries of form $\xi(x,u) \frac{\partial}{\partial x} + \eta(x,u) \frac{\partial}{\partial u}$ of (7) has RIF form with respect to an orderly ranking given by:

$$S = \left[ \xi = -\frac{\eta u}{x}, \quad \eta_{x,u} = \frac{(u - x)(u + x)}{ux^2} \eta_u + \frac{(u^2 + x^2)}{xu} \eta_x - \frac{x\eta_{u,u}}{u}, \right]$$

$$\eta_{x,x} = -\frac{(2u^4 - x^2u^2 + x^4)}{u^3x^2} \eta_x + \frac{(u^2 + x^2)}{xu} \eta_u - \frac{x^2\eta_{u,u}}{u^2}, \quad \eta_{u,u,u} = -\frac{(16u^8 + 24u^6x^2 + 8u^4x^4 + 16u^6 + 8u^4x^2 + 3u^2 + 3x^2)}{2u^3x(u^2 + x^2)^2} \eta_u - \frac{(8u^6x^2 + 8u^4x^4 + 8u^4x^2 - 3u^2 - 3x^2)}{u^2(u^2 + x^2)} \eta_x + 8 \frac{u^3x(u^2 + x^2 + 1)}{u^2 + x^2} \eta_x$$

(8)

It’s corresponding initial data is

$$\text{ID}(S) = [\eta(x_0,u_0) = c_1, \quad \eta_x(x_0,u_0) = c_2, \quad \eta_u(x_0,u_0) = c_3, \quad \eta_{u,u}(x_0,u_0) = c_4]$$

(9)

There are 4 arbitrary constants in the initial data at regular points $(x_0,u_0)$, so (7) has a 4 dimensional local Lie algebra of symmetries $\mathcal{L}$ in a neighborhood of such points: $\text{dim } \mathcal{L} = 4$.

The structure of $\mathcal{L}$ of (8) can be algorithmically determined without integrating the defining system [32, 22, 21, 19, 15]:

$$\begin{align*}
[Y_1,Y_2] &= -Y_1 - 2Y_2, \quad [Y_1,Y_3] = Y_1 - 2Y_3, \quad [Y_1,Y_4] = -2Y_4, \\
[Y_2,Y_3] &= Y_2 + Y_3, \quad [Y_2,Y_4] = Y_4, \quad [Y_3,Y_4] = -Y_4
\end{align*}$$

(10)

where a non-regular point $(x_0 = 1, u_0 = 1)$ has been substituted into the relations.

But what can such symmetry information tell us about nonlinear systems $R$ such as the above ODE using mappings? In particular, we focus in this paper on the question of when a system $R$ can be mapped to a linear system $\hat{R}$. Throughout this paper, we maintain blanket local analyticity assumptions. So the case of a single differential equation $\hat{R}$ has the form $\mathcal{H} \hat{u} = f(\hat{x})$ where $\mathcal{H}$ is a linear differential operator with coefficients that are analytic functions of $\hat{x}$ and $f$ is also analytic. Lewy’s famous counterexample of a single linear differential equation in 3 variables, of order 1, where $\mathcal{H}$ is analytic with smooth inhomogeneous term, without smooth solutions, provides a counterexample in the smooth case. See [20] for a recent review of related results. Moreover, Lie’s classical theory of groups and their algebras, requires local analyticity in its defining equations. Then supposing we have the existence of a local analytic solution $\hat{u}$ in a neighborhood of $\hat{x}_0$, in this neighborhood the point transformation $\hat{u} \rightarrow \hat{u} - \hat{u}$ implies that without loss we can consider $\hat{R}$ to be a homogeneous linear differential equation $\mathcal{H} \hat{u} = 0$ where $\hat{u} \in \mathcal{A}(\hat{x}_0, \delta)$, the set of analytic functions on some sufficiently small disk $|\hat{x} - \hat{x}_0| < \delta$. Then solutions of $\hat{R}$ satisfies the superposition property $\mathcal{H}(\hat{v} + \hat{w}) = \mathcal{H} \hat{v} + \mathcal{H} \hat{w} = 0$. This corresponds to point symmetries generated by the Lie algebra vectorfields

$$\mathcal{L}^* := \left\{ \hat{v}(\hat{x}) \frac{\partial}{\partial \hat{u}} : \mathcal{H} \hat{v}(\hat{x}) = 0 \text{ and } \hat{v} \in \mathcal{A}(\hat{x}_0, \delta) \right\}$$

(11)
Consequently assuming the existence of a local analytic map, \( \dim \hat{L}^* = \dim \hat{R} = \dim R \). If \( R \) is an ODE of order \( d \geq 2 \) then \( \dim \hat{L}^* = \dim \hat{R} = \dim R = d \). Similarly the superposition property \( \mathcal{H}(\hat{u}) = \mathcal{H}(\hat{v}) \) corresponds to a 1 parameter family of scalings with symmetry vectorfield \( \hat{u} \frac{\partial}{\partial \hat{u}} \). So we get the well-known and obvious result that an ODE of order \( d \) that can be mapped to a linear ODE must have \( \dim \mathcal{L} = \dim \hat{\mathcal{L}} \geq d + 1 \). Similarly if \( R \) is linearizable and \( \dim R = \infty \) then \( \dim \mathcal{L} = \dim \hat{\mathcal{L}} = \infty \) with similar properties for systems.

The Lie sub-algebra \( \hat{L}^* \) is easily shown to be ‘abelian’ by direct computation of commutator \[30\], in both the finite and infinite dimensional case. Indeed consider the so-called derived algebra \( \mathcal{L}' = \text{DerivedAlgebra}(\mathcal{L}) \), which is the Lie subalgebra of \( \mathcal{L} \) generated by commutators of members of \( \mathcal{L} \); and similarly for \( \hat{\mathcal{L}}' \).

Again by direct computation of commutators (e.g. \([\hat{v}(\hat{x}) \frac{\partial}{\partial \hat{x}}, \hat{u} \frac{\partial}{\partial \hat{u}}] = \hat{v}(\hat{x}) \frac{\partial}{\partial \hat{u}} \in \hat{\mathcal{L}} \) so that \( \hat{\mathcal{L}}^* \) is a sub-algebra of \( \hat{\mathcal{L}}' \). Thus a necessary condition for the existence of a map \( \Psi \) to a linear target is that \( \hat{\mathcal{L}}' \) has a \( d \)-dimensional abelian subalgebra both the finite and infinite dimensional cases (see Olver \[30\]).

Lyakhov, Gerdt and Michels \[23\] use this to give an algorithm to determine the existence of a linearization for a single ODE of order \( d \) which we briefly summarize in the pseudo-code in Algorithm 1. Also see \[24\]. There are two main cases. The first is when the nonlinear ODE has maximal dimension, as shown in Step 5 of Algorithm 1 and these are all linearizable. These occur for \( d = 1, 2 \) the maximal dimensions of \( \mathcal{L} \) are \( \infty \) and 8 respectively, and for \( d > 2 \) where the maximal dimension of \( \mathcal{L} \) is \( d + 4 \). The submaximal cases occur for \( d > 2 \), and interestingly occur when \( \dim \mathcal{L} = d + 1 \), \( \dim \mathcal{L} = d + 2 \).

Algorithm 1: LGMLinTest(\( R \))

- **Input**: a DPS ODE \( R \) solved for its highest derivative.
- **Output**: Lin = true if \( R \) linearizable otherwise Lin = false

1: Lin := false
2: Compute \( S = \text{DiffThomasDec}(\text{DetSys}(R)) \)
3: Find \( \dim \mathcal{L} := \dim S, d := \text{difforder}(R) \)
4: ComRels(S) := Structure(S)
5: if \( d = 1 \) or \((d = 2 \text{ and } \dim \mathcal{L} = 8) \) or \((d > 2 \text{ and } \dim \mathcal{L} = d + 4) \) then Lin := true
   else if \( d > 2 \) and \((\dim \mathcal{L} = d + 1 \text{ or } \dim \mathcal{L} = d + 2) \) then
   \( \mathcal{L}' := \text{DerivedAlgebra}(\text{ComRels}(S)) \)
   if IsAbelian(\( \mathcal{L}' \)) and \( d = \text{dim}(\mathcal{L}') \) then Lin := true
   endif
   endif
6: return Lin

**Example 2.2.** We illustrate the above discussion and Algorithm 1 by a continuation of Example 2.1. For that example \( d = 3 \) and \( \dim \mathcal{L} = 4 \). Then from the commutation relations the derived algebra \( \mathcal{L}' \) is generated by

\[
[Z_1 = Y_1 - 2Y_3, \ Z_2 = Y_2 + Y_3, \ Z_3 = Y_4]
\] (12)

Thus \( \text{dim} \text{DerivedAlgebra}(\mathcal{L}) = \text{dim} \mathcal{L}' = 3 \) and also its structure is easily found as

\[
[Z_1, Z_2] = [Z_1, Z_3] = [Z_2, Z_3] = 0
\] (13)
so $\mathcal{L}'$ is abelian and by Algorithm 1 that (7) is exactly linearizable.

The Algorithms introduced by Lyakhov et al. [23] have two stages: the first given above is for the existence of linearization and the second is for construction of the mapping as a compatible system; and when possible explicit forms for the mapping. Both existence and construction employ their differential elimination algorithm \texttt{DifferentialThomasDecompostion} which is available, in distributed Maple 18. Their construction step also involves heuristic integration. Indeed the algorithm that they use to construct a compatible system for the mapping to linear is essentially a special case of the algorithm \texttt{EquivDetSys} in our introductory paper [27]. It is expensive as we illustrate later with examples, and one of the contributions of our paper, is to find a more efficient algorithm, that avoids the application of \texttt{EquivDetSys}.

2.2 Bluman-Kumei Mapping Equations

Assume the existence of a local analytic invertible map $\Psi = (\psi, \phi)$ between the \textit{Source} system $\mathcal{R}$ and the \textit{Target} system $\hat{\mathcal{R}}$, with Lie symmetry algebras $\mathcal{L}$, $\hat{\mathcal{L}}$ respectively. Applying $\Psi$ to the infinitesimals $(\hat{\xi}, \hat{\eta})$ of a vectorfield in $\hat{\mathcal{L}}$ yields what we will call the Bluman-Kumei (BK) mapping equations:

$$M_{BK}(\mathcal{L}', \hat{\mathcal{L}}') = \begin{cases} \hat{\xi}^k(\hat{x}, \hat{u}) & = \sum_{i=1}^{n} \xi^i(x, u) \frac{\partial \psi^k}{\partial x^i} + \sum_{j=1}^{m} \eta^j(x, u) \frac{\partial \psi^k}{\partial u^j} \\ \hat{\eta}^\ell(\hat{x}, \hat{u}) & = \sum_{i=1}^{n} \xi^i(x, u) \frac{\partial \phi^\ell}{\partial x^i} + \sum_{j=1}^{m} \eta^j(x, u) \frac{\partial \phi^\ell}{\partial u^j} \end{cases}$$

where $1 \leq k \leq n$ and $1 \leq \ell \leq m$, and $(\xi, \eta)$ are infinitesimals of Lie symmetry vectorfields in $\mathcal{L}$. See Bluman and Kumei [4, 8] for details and generalizations (e.g. to contact transformations). Note that all quantities on the RHS of the BK mapping equations (14) are functions of $(x, u)$ including $\phi$ and $\psi$. See [27, Example 1] for a simple introductory example of mappings; as well as the examples in [8].

If an invertible map $\Psi$ exists mapping $\mathcal{R}$ to $\hat{\mathcal{R}}$ then it most generally depends on $\dim(\mathcal{L'}) = \dim(\hat{\mathcal{L}}')$ parameters. But we only need one such $\hat{\Psi}$. So reducing the number of such parameters, e.g., by restricting to a Lie subalgebra $\mathcal{L}'$ of $\mathcal{L}$ with corresponding Lie subalgebra $\hat{\mathcal{L}}'$ of $\hat{\mathcal{L}}$ that still enables the existence of such $\hat{\Psi}$, is important in reducing the computational difficulty of such methods. We will use the notation $S', S''$ denote the symmetry defining systems of Lie sub-algebras $\mathcal{L}', \hat{\mathcal{L}}'$ respectively. See [8, 30] discussion on this matter.

For mapping from nonlinear to linear systems, a natural candidate for $\mathcal{L}'$ is the derived algebra DerivedAlgebra$(\mathcal{L}')$, and the natural target is for $\hat{\mathcal{L}}^*$ to correspond to the superposition as defined in (11).

\textbf{Example 2.3.} This is a continuation of Examples 2.1, 2.2 concerning (7) and use the BK mapping equations (14) where $\mathcal{L}' =$ DerivedAlgebra$(\mathcal{L}')$. Our goal is to efficiently characterize the compatible form for the mapping equations in $\Psi$ and as building blocks towards the main algorithm of our paper given in (11).

For construction of $\Psi$ we actually need differential equations for $\mathcal{L}'$, in addition its structure, and these are provided by the algorithm DerivedAlgebra in the LAVF package which for (7) yields
its rif-form:

\[
S' = [\xi = -\frac{\eta u}{x}, \eta_x = \frac{(u^2 + x^2)}{xu^2} \eta + \frac{\eta u x}{u}, \\
\eta_{u,u,u} = -\frac{(8 u^6 + 8 u^6 x^2 + 8 u^6 + 3 u^2 + 3 x^2)}{(u^2 + x^2) u^3} \eta + 3 \frac{\eta u}{u^2}]
\]  

(15)

The derived algebra is then shown by LAVF commands be both 3 dimensional and abelian. Moreover its determining system (15) much simpler than the determining system of \( L \) given in (8).

Crucially it means we can exploit this determining system using the BK mapping equations. Since the target infinitesimal generator is \( \hat{\xi} \frac{\partial}{\partial \hat{x}} + \hat{\eta} \frac{\partial}{\partial \hat{u}} = 0 \cdot \frac{\partial}{\partial \hat{x}} + \hat{\eta}(\hat{x}) \frac{\partial}{\partial \hat{u}} \) where \( \mathcal{H} \hat{u} = 0 \). So \( \hat{\xi} = 0 \) and \( \mathcal{H} \hat{u} = 0 \) and the BK equations are:

\[
M_{BK}(L', \hat{L}^*) = \begin{cases} \\
0 = \xi(x,u) \frac{\partial \psi}{\partial x} + \eta(x,u) \frac{\partial \psi}{\partial u} \\
\hat{\eta}(\hat{x}, \hat{u}) = \xi(x,u) \frac{\partial \phi}{\partial x} + \eta(x,u) \frac{\partial \phi}{\partial u}
\end{cases}
\]  

(16)

where \( \hat{\eta} = 0 \) and \( \mathcal{H} \hat{u} = 0 \). Remarkably these equations essentially determine the necessary and sufficient conditions for the linearization of (7) and this will be exploited in Section §3 in the computation of the mapping.

### 3 Algorithms

In this section we extend the MapDE algorithm introduced in our previous paper [27], to determining if there exists a mapping of a nonlinear source \( R \) to some linear target \( \hat{R} \), using the target input option \( \text{Target} = \text{LinearDE} \). This builds on progress in [27] in which we gave an algorithm for mapping a specific \( R \) to a specific target \( \hat{R} \), and also an algorithm for determining existence of a mapping of a linear source \( R \) to a \( \hat{R} \) in the class of constant coefficient linear differential equations; plus further methods that sometimes determine an explicit mapping.

#### 3.1 Symmetries of the linear target and the derived algebra

We summarize and generalize some aspects of the discussion in [1] and [2]. The following theorem is a straightforward consequence of the necessary conditions in [8] where we have also required that the target system is in rif-form.

**Theorem 3.1** (Superposition symmetry for linearizable systems). Suppose that the analytic system \( R \) is exactly linearizable by a holomorphic diffeomorphism \( \hat{x} = \psi(x,u), \hat{u} = \phi(x,u) \), to yield a linear target system, \( \hat{R} \) locally takes the form

\[
\hat{R} : \mathcal{H} \hat{v}(\hat{x}) = 0
\]  

(17)

where \( \mathcal{H} \) is a vector partial differential operator, with coefficients that are analytic functions of \( \hat{x} \) and the system (17) is in rif-form wrt an orderly ranking. Moreover \( \hat{R} \) admits the symmetry vector field \( \sum_{i=1}^n 0 \frac{\partial}{\partial \hat{x}^i} + \sum_{j=1}^m \hat{\eta}^j(\hat{x}) \frac{\partial}{\partial \hat{u}^j}, j = 1, \cdots, m: \)

\[
\hat{S}^* := \left\{ \hat{\xi} = 0, \mathcal{H} \hat{\eta} = 0, \hat{\eta}^j_{ak} = 0 : 1 \leq i \leq n, 1 \leq j, k \leq m \right\}
\]  

(18)
A simple consequence of the commutator formula (5) is that the commutators generate a Lie algebra, which is called the derived algebra. Lisle and Huang [21] implement efficient algorithm in the LAVF package to compute the determining system for the derived algebra for finite dimensional Lie algebras of vectorfields. We have made a first implementation in the infinite dimensional case, together with the Lie pseudogroup structure relations. First each of the \( \nu, \mu, \omega \) in the commutation relations (5) must satisfy the determining system of \( L \) so we enter three copies of those determining systems. We then reduce the combined system using a block elimination ranking which ranks any derivative of \( \omega \) strictly less those of \( \mu, \nu \). The resulting block elimination system for \( \omega \) generates the derived algebra in the infinite case.

The commutator between any superposition generator and the scaling symmetry admitted by linear systems yields

\[
\left[ \sum_{i=1}^{m} \hat{v}^i(\hat{x}) \frac{\partial}{\partial \hat{u}^i}, \sum_{i=1}^{m} \hat{u}^i \frac{\partial}{\partial \hat{u}^i} \right] = \sum_{i=1}^{m} \hat{v}^i(\hat{x}) \frac{\partial}{\partial \hat{u}^i}
\]  

(19)

So we have the following result as an easy consequence (See Olver [30] for related discussion in both the finite and infinite case).

**Theorem 3.2.** Suppose that the analytic system \( R \) is exactly linearizable by a holomorphic diffeomorphism \( \hat{x} = \psi(x,u), \hat{u} = \phi(x,u) \), to yield a linear target system (18) and \( L, L' \) are the Lie symmetry algebra and its derived algebra for \( R \). Also, let \( \hat{L}, \hat{L}' \) be the corresponding algebras for \( \hat{R} \). Let \( \hat{L}^*, \hat{L}^* \) be the superposition algebras under \( \Psi \). Then \( \hat{L}^* \) is a subalgebra of \( \hat{L}' \) and \( L^* \) is a subalgebra of \( L' \). Moreover \( L^* \) and \( \hat{L}^* \) are abelian.

We wish to determine if a system \( R \) is linearizable and if so, characterize the target \( \hat{R} \), i.e \( \mathcal{H} \hat{\eta} = 0 \). But initially we don’t know \( \mathcal{H} \). One approach is to write a general form for this system, that specifies \( \hat{S}^* \) with undetermined coefficient functions, whose form is established in further computation. See, for example, [23] that pursue this approach in the case of a single ODE, but don’t consider the Bluman-Kumei mapping system. That approach appears to contribute to significant growth, in the size of calculations in the examples we’ve tried, as the number and order of variables increases. Instead, we only include \( \xi^i = 0, \hat{\eta}^i_{\hat{u}k} = 0 \) and don’t include \( \mathcal{H} \hat{\eta} = 0 \). Thus we only include a subset \( \hat{S}^* \) of \( \hat{S}^* \), denoting the truncated system as

\[
\hat{S}^* := \left\{ \hat{\xi}^i = 0, \hat{\eta}^i_{\hat{u}k} = 0 : 1 \leq i \leq n, 1 \leq j, k \leq m \right\} \subseteq \hat{S}^*
\]  

(20)

and allow \( \mathcal{H} \hat{\eta} = 0 \) to be found naturally later in the algorithm. We note that \( \hat{S}^* \) are the defining equations of an infinite Lie pseudogroup.

Finally we note that the initial data function determines a set of parametric derivatives in the formal power series solution of a system in rif-form at a point \( x = x^0 \), taken with respect to an orderly Riquier ranking. Let \( c_k \) be the number of parametric derivatives at each derivative order \( k \), then \( \text{HS}(s) := \sum_{k=0}^{\infty} c_k s^k \) is called the Differential Hilbert Series for \( R \). And the output of the initial data which essentially partitions the parametric derivatives into disjoint cones of various dimensions \( \leq n \), can express this in the rational function form \( \text{HS}(s) = \frac{\text{P}(s)}{(1-s)^d} \) where \( d = \text{d}(R) \) is the differential dimension of \( R \). It corresponds to the maximum number of free independent variables appearing the functions for the initial data. For further information on differential Hilbert Series see [26].
3.2 The MapDE algorithm with Target = LinearDE

Algorithm 2 described in this section converges in finitely many steps for input DPS in rif-form, accompanied by inequations, due to finiteness of each of the sub-algorithms used; the most frequently used being the rif algorithm. Note that diffalg or DifferentialThomasDecomposition could be used in place of rif. Theoretically the algorithm also depends on some results of Bluman and Kumei, in particular their necessary and sufficient conditions for linearization of nonlinear DE, and the previous discussion.

Algorithm 2 MapDE with Target = LinearDE

\begin{verbatim}
MapDE(Source, Target, Map)
Input:
    Source: a leading nonlinear DPS system in rif-form
        wrt an orderly Riquier ranking; vars [x, u], [ξ, η], Opt
    Target: Target = LinearDE, Opt
    Map:  Ψ, Opt
Output:
    ∅ if no nontrivial rif-form map cases computed
    else rif-form case \( P_k, \hat{R}, \Psi = \text{PDsolve}(P_k), \hat{R}|\Psi \)
1: Compute ID(\( R \)), \( d := \text{dim}(R) \)
2: Let \( \mathcal{L}' = \text{DerivedAlgebra}(\mathcal{L}) \) and compute:
    \begin{align*}
    S &:= \text{rif}(\text{DetSys}(\mathcal{L})), \quad S' := \text{rif}(\text{DetSys}(\mathcal{L}')) \\
    \text{ID}(S), \text{ID}(S'), \text{dim}(S), \text{dim}(S')
    \end{align*}
3: if PreEquivTest(\( R, \text{Linear} \)) \( \neq \) true then
    return \( \Psi = \emptyset \)
end if
4: Set \( \hat{S}^* := \{ \hat{\xi} = 0, \hat{\eta}^j_{\hat{a}_k} = 0 : 1 \leq i \leq n, 1 \leq j, k \leq m \} \subseteq \hat{S}^* \)
    \begin{align*}
    M &:= S' \cup \hat{S}^*|\Psi \cup M_{\text{BK}}(\mathcal{L}', \hat{\mathcal{L}}^*) \cup \{\text{DetJac}(\Psi) \neq 0\}
    \end{align*}
5: Compute list of consistent cases \( P = [P_1, \cdots, P_{\text{nc}}] \):
    \begin{align*}
    P &:= \text{rif}(M, \prec, \text{casesplit}, \text{mindim} = d)
    \end{align*}
6: if \( P = \emptyset \) then return \( \Psi = \emptyset \) end if
7: \( \hat{R} := \emptyset \)
8: for \( k = 1 \) to \( \text{nc} \) while \( \hat{R} = \emptyset \) do
9:    if InvariantCount(\( P_k, \psi \)) \( \neq n \) or HS(\( R \)) \( \neq \) HS(\( P_k \))
        then break
    end if
10: \( \hat{R} := \text{ExtractTarget}(P_k, \hat{\eta}) \)
end do
11: if \( \hat{R} \neq \emptyset \) then return \( P_k, \hat{R}, \Psi = \text{PDsolve}(P_k), \hat{R}|\Psi \)
    else return \( \Psi = \emptyset \)
end if
\end{verbatim}
3.3 Notes on the MapDE Algorithm with Target = LinearDE

We briefly list some main aspects of Algorithm 2.

Input: Due to current limitations of Maple’s DeterminingPDE we restrict to input a single system $R$, in rif-form, with no leading nonlinear equations. Other Maple packages could be used; such as DifferentialThomasDecomposition, or DifferentialAlgebra, Casesplit which offers a uniform interface to these 3 packages, could also be used.

In the input Opt refers to additional options. For example, for Map, Opt can contain additional user input constraints for the mapping, and strategy options for the computation. Other instructions such as including OutputDetails, in Opt can yield more detailed outputs.

Step 1: Note that ID is valid at points satisfying the inequations of rif-form.

Step 3: As discussed in §2, $R$ being linearizable implies that the superposition is in its Lie symmetry algebra and is a coordinate change of $R$, implying some fairly well-known efficient tests for screening out non-linearizable cases. For linearization necessarily dim $S \geq d + 1$ and dim $S' \geq d$ for finite $d$. For $d = \infty$, necessarily dim $S = \infty = \dim S'$ and in terms of differential dimensions $d(S) \geq d(S') \geq d(R)$.

We also apply Algorithm [1] for the LGMLinTest [23] for the case of $R$ being an ODE which unlike the above tests give NASCs for $R$ to be linearizable.

Step 4: $\hat{S}^*|_\psi$ is $\hat{S}^*$ evaluated in $(x, u)$ coordinates via $\Psi$ using differential reduction.

Step 5: Here mindim = dim($R$) = $d$, as computed by the Maple command initialdata of the rif form of $R$. This is a preliminary screen, by dimension, rejecting cases of dimension $< d$. The block elimination ranking $\prec$ ranks all infinitesimals and their derivatives for the first block $[\xi, \eta, \hat{\xi}, \hat{\eta}]$ strictly greater than the second block of the $\phi$ map variables, which are strictly greater than all derivatives of the third block of $\psi$ variables. This maintains linearity in the variables $[\xi, \eta, \hat{\xi}, \hat{\eta}]$, and the mindim dimension is computed with respect to these variables, and not the degrees of freedom in the map variables ($\psi, \phi$). The subsequent block structure also reflects the geometric structure and facilitates the later integration phase. Each case $P_k$ consists of equations and inequations.

Step 9: InvariantCount($P_k, \psi$), a command from the LAVF package, isolates and reduces the differential system for the $\psi$ map variables, and and without integrating it, determines the number of invariants. If InvariantCount($P_k, \psi$) $\neq n$ (the number of independent variables) then this case is rejected as not being linearizable.

The Hilbert Series of $R$ and $P_k$ (disregarding the equations that don’t involve infinitesimals) should be equal if the system is linearizable.

Step 10: Applies ExtractTarget to extract a system ExtractTarget($P_k, \hat{\eta}$) for $\hat{\eta}$ by a block elimination ranking $\prec'$ that ranks all derivatives of $\hat{\eta}^i$ strictly lower than derivatives of $\xi, \eta$. The system is then simplified by applying rif and determining 0 and nonzero coefficients using differential algebra, without needing to explicitly integrate the mapping equations.
The following correspondence between Jacobians in \((x, u)\) and \((\hat{x}, \hat{u})\) coordinate systems

\[
\frac{\partial(\hat{\psi}, \hat{\phi})}{\partial(\hat{x}, \hat{u})} = \left[ \frac{\partial(\psi, \phi)}{\partial(x, u)} \right]^{-1}
\]

is exploited to swap back and forth between the systems and use differential algebra tools. In particular the first order derivatives of the mapping functions \(\psi, \phi\) and \(\hat{\psi}, \hat{\phi}\) are relabeled as new dependent variables. Then (21) is a point change of variables in the new space, and Maple’s \texttt{dchange} is then used to execute the coordinate changes, yielding the target in \(\hat{\eta}\) derivatives wrt \(\hat{x}\) with coefficients in terms of \(x, u, \psi, \phi\).

Step 11: If \(\hat{R} \neq \emptyset\) then \(P_k, \hat{R}\) are returned in rif-form. But with coefficients in terms of \(\psi, \phi, x, u\). In addition the heuristic integration routine \texttt{PDsolve}, a simple interface to Maple’s \texttt{pdsolve}, is applied to \(P_k\) to attempt to find an explicit form of \(\Psi\) and if so use \(\Psi\) explicitly evaluate the coefficients of \(\hat{R}\) in terms of \(\hat{x}\).

4 Examples

To illustrate the \texttt{MapDE} Algorithm we consider some examples.

Example 4.1. (Continuation and Conclusion for Examples 2.1, 2.2, 2.3) We can now complete this example and describe the steps of Algorithm.

The input is (7) which is in rif-form with respect to the orderly ranking \(u \prec u_x \prec u_{xx} \prec \cdots\), together with the inequations \(u \neq 0, uu_x + x \neq 0, u^2 + x^2 \neq 0\). This can be regarded as derived from the leading linear DPS which results from multiplying through by the inequation factors.

Step 1: \(\text{ID}(R) = [u(x_0) = c_1, u_x(x_0) = c_2, u_{xx}(x_0) = c_3]\) and \(\dim R = 3\).

Step 2: See Example 2.1 for \(S := \text{rif}(\text{DetSys}(L))\) in (8), together with its \(\text{ID}(S)\) and \(\dim L = \dim(S) = 4\). See Example 2.3 and in particular (15) for \(S' := \text{rif}(\text{DetSys}(L'))\) which yields in that example \(\dim L' = \dim(S') = 3\).

Step 3: Since \(\dim S = 4 \geq d + 1 = 4\) and \(\dim S' = 3 \geq d = 3\), our most basic necessary condition for linearizability is passed. Also \(d(S) = d(S') = d(R) = 0\). Application of Algorithm for the LGMLinTest in Example 2.2 shows that \(R\) is linearizable.

Step 4: \(\text{DetJac}(\Psi) = \psi_x \phi_u - \psi_u \phi_x \neq 0\), \(\hat{S}^* := \{ \xi = 0, \eta \neq 0 \}\) and \(\hat{L}^*\) is replaced with \(\hat{L}^*\) in (16) to yield:

\[
M_{BK}(L', \hat{L}^*) = \left\{ \xi(x, \hat{u}) = 0 = \xi \psi_x + \eta \psi_u, \eta(x, \hat{u}) = \xi \phi_x + \eta \phi_u \right\}
\]

Evaluate \(\hat{S}^*\) modulo \(\Psi: \hat{x} = \psi(x, u), \hat{u} = \phi(x, u)\) to obtain \(\hat{S}^* |_{\psi}\). This yields \(\hat{S}^* |_{\psi} = \{ \hat{\xi} = 0, \psi_u \hat{\eta}_x - \psi_x \hat{\eta}_u = 0 \} \). Note for brevity of notation have replaced \(\hat{\xi}(x, u)\) with \(\xi(x, u)\) and \(\hat{\eta}(x, u)\) with \(\eta(x, u)\). Thus the
mapping system $M = S' \cup \hat{S}^*|_{\psi} \cup M_{BK}(\mathcal{L}', \hat{\mathcal{L}}^*) \cup \{\text{DetJac}(\Psi) \neq 0\}$ is:

$$M = [\xi = \frac{\eta u}{x}, \eta_x = \frac{(u^2 + x^2) \eta}{xu^2} + \frac{\eta u x}{u}, \eta_{u,u,u} = -\frac{(8 u^8 + 8 u^6 x^2 + 8 u^6 + 3 u^2 + 3 x^2) \eta}{(u^2 + x^2) u^3} + \frac{3 \eta u}{u^2}, \hat{\xi} = 0, \psi_u \hat{\eta}_x - \psi_x \hat{\eta}_u = 0, \hat{\xi} = \xi \psi_x + \eta \psi_u, \hat{\eta} = \xi \phi_x + \eta \phi_u, \psi_x \phi_u - \psi_u \phi_x \neq 0] \quad (23)$$

**Step 5:** Compute $P := \text{RIF}(M, \prec, \text{casesplit}, \text{mindim} = d)$ where $d = 3$. This results in 3 cases, two of which are rejected before their complete calculation since an upper bound in the computation drops below $\text{mindim} = d = 3$. The output for the single consistent case $P_1$ found is:

$$P_1 = [\xi = \frac{\eta u}{x}, \eta_x = \frac{(u^2 + x^2) \eta}{xu^2} + \frac{\eta u x}{u}, \eta_{u,u,u} = -\frac{(8 u^8 + 8 u^6 x^2 + 8 u^6 + 3 u^2 + 3 x^2) \eta}{(u^2 + x^2) u^3} + \frac{3 \eta u}{u^2}, \phi_{x,x} = \frac{2 \phi_{x,u} xu^2 - \phi_{u,u} x^2 u + \phi_{u} u^2 + \phi_{x} x^2}{u^3}, \psi_x = \psi_u x, \hat{\eta} = \frac{(u \phi_x - x \phi_u) \eta}{x}, \hat{\xi} = 0, x \phi_u - u \phi_x \neq 0, \psi_u \neq 0] \quad (24)$$

**Step 6:** $P \neq \emptyset$ so proceed.

**Step 7:** Initialize $\hat{R} := \emptyset$

**Step 8:** The number of cases found is $nc = 1$

**Step 9:** The $\psi$ system here is $\psi_x = \frac{\psi u x}{u}$ so $\text{InvariantCount}(P_1, \psi) = n = 1$. Also $HF(R) = 1 + s + s^2$ and the ID for $P_1$ yields $HF(P_1) = 1 + s + s^2$. So $HF(R) = HF(P_1)$.

**Step 10:** To extract the target $\hat{R} := \text{ExtractTarget}(P_k, \hat{\eta})$, first apply RIF with an elimination ranking $\prec'$ that ranks any derivatives of $\hat{\eta}$ strictly lower than derivatives of $\xi, \eta$. Note that all derivatives of the map variables $\psi, \phi$ are also strictly ranked lower than any derivative of $\xi, \eta, \hat{\eta}$.

The [21] is a point change of variables in the new space, then to execute the coordinate changes using Maple’s dchange, yields without integration, together with $\frac{\partial}{\partial u} \hat{\eta}(\hat{x}, \hat{u}) = 0$, the following expression for the target

$$\left(\frac{\partial}{\partial x}\right)^3 \hat{\eta}(\hat{x}) = a_2(\hat{x}) \left(\frac{\partial}{\partial x}\right)^2 \hat{\eta}(\hat{x}) + a_1(\hat{x}) \frac{\partial}{\partial x} \hat{\eta}(\hat{x}) + a_0(\hat{x}) \hat{\eta}(\hat{x}) \quad (25)$$

where

$$a_2(\hat{x}) = -3 \frac{\psi_{u,u} \phi_{x} u - \psi_{u} \phi_{u,u} u - \psi_{u,u,u} \phi_{x} x + \psi_{u} \phi_{u,u} x - \psi_{u} \phi_{x} x}{\psi_{u}^2 (\phi_{x} u - \phi_{u} x)} \quad (26)$$

together with longer expressions for the other coefficients $a_1(\hat{x})$ and $a_0(\hat{x})$. Together with the conditions obtained in the previous steps this satisfies the necessary and sufficient conditions given in [58] for linearization.

**Step 11:** Applying pdsolve first the $\psi$ equation, $\psi_x = \frac{\psi u x}{u}$ in the algorithm using Invariants from the LAVF package yields $x^2 + u^2$ and $\psi = x^2 + u^2$. That is $\psi = F(x^2 + u^2)$ more generally.
Then substitution and solution of the \( \phi \) equation yields \( \phi = G(x^2 + u^2) \). The program specializes the arbitrary functions and constants to satisfy the inequations including the Jacobian condition, and in this case yields:

\[
\begin{align*}
\dot{x} &= \psi = x^2 + u^2 \\
\dot{u} &= \phi = x
\end{align*}
\] (27)

Substitution of (27) into the target (25) and replacing \( \dot{\eta} = \dot{u} \) yields it explicitly as:

\[
\left( \frac{\partial}{\partial x} \right)^3 \dot{u}(\dot{x}) = -\frac{(\dot{x} + 1)}{\dot{x}} \dot{u}(\dot{x})
\] (28)

From the output we also subsequently explored how far we could make the Target explicit before the integration of the map equations. In particular we exploited the transformation (as do [23]) that any such ODE is point equivalent to one with its highest coefficients (here \( a_2, a_1 \)) being zero. This yields additional equations on \( \psi, \phi \) and the target takes the very simple form:

\[
\left( \frac{\partial}{\partial \dot{x}} \right)^3 \dot{u}(\dot{x}) = -\frac{8u^3(u^2 + x^2 + 1)}{(u^2 + x^2)\psi_u^3} \dot{u}(\dot{x})
\] (29)

The rif-form of system for \( \phi, \psi \) is:

\[
\begin{align*}
\psi_x &= \psi_u x \\
\psi_{u,u,u} &= -\frac{3}{2} \psi_{u,x}^2 u^2 + 3 \psi_u^2 \psi_u x \\
\phi_{x,x} &= \frac{2}{\psi_u^2} \frac{x\psi_u (\phi_{x} u - \phi_{u} x)}{\psi_u x^2 u + \phi_u u^2 - 2 \phi_u u x + \phi_{u} x^2} \\
\phi_{x,u} &= \frac{\psi_{u,u} \phi_{x} u - \psi_{u,u} \phi_{u} x + \psi_{u} \phi_{u,u} + \psi_{u} \phi_{u,x}}{\psi_u x}
\end{align*}
\] (30)

The general solution of the system is found by Maple and yields the same particular solution as before for \( \psi, \phi \).

**Example 4.2.** (Lyakhov, Gerdt and Michels Test Set) Here is one of the test sets from Lyakhov and et. al [23] for their algorithm to test the existence of linearization of a single ODE; then a separate program which uses Maple’s `pdsolve` to try to construct the linearization. They give a test set of ODE of order \( d \), for \( 1 \leq d \leq 15 \):

\[
\left( \frac{d}{dx} \right)^d (u(x)^2) + u(x)^2 = 0
\] (31)

By inspection this has the linearization for any \( d \):

\[
\Psi = \{ \dot{x} = x, \dot{u} = u^2 \}
\] (32)
They report times from Intel(R)Xeon(R) X5680 CPU clocked at 3.33 GHz and 48GB RAM, ranging from 0.2 secs for \( d = 3 \) to about 150 secs for \( d = 15 \) to test existence of the linearization. Our run of the same tests on 2.61 GHz I7-6600U processor with 16 GB of RAM ranges from 0.3 secs when \( d = 3 \) to 4.2 secs when \( d = 15 \) in Table 1. Their method to construct the linearization, reports 7512.9 secs for \( d = 9 \) and out of memory for \( d \geq 10 \). In contrast, we report times for construction that are only slightly longer than our existence times for \( 1 \leq d \leq 15 \).

| Order ODE | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-----------|---|---|---|---|---|---|---|----|----|----|----|----|----|
| Existence | .30 | .25 | .2 | .3 | .52 | .56 | .91 | 1.0 | 1.3 | 1.8 | 2.4 | 3.1 | 4.2 |
| Existence and Construction | 1.3 | 1.3 | 1.4 | 1.6 | 1.5 | 1.7 | 2.1 | 2.3 | 2.7 | 3.0 | 3.5 | 3.9 | 5.0 |

Table 1: Table presents the CPU times for \((\frac{d}{dx})^d(u(x))^2 + u(x)^2 = 0\). The timings correspond to the existence (3rd row) and construction (4th row) of linearization by MapDE.

**Example 4.3.** Consider Burger’s equation, modeling the simplest nonlinear combination of convection and diffusion:

\[ u_{x,x} = u_t - uu_x \]

Using our algorithm MapDE with TargetClass = LinearDE shows that it has finite dimensional Lie symmetry algebra \( \text{dim} \mathcal{L} = 5 < \infty \). Thus by our early preliminary equivalence test PreEquivTest, it is not linearizable by point transformation. However rewriting this equation in conserved form \( \frac{\partial}{\partial x}(u_x + \frac{1}{2}u^2) = \frac{\partial}{\partial t}u \) implies that there exists \( v \):

\[ v_x = u, \quad v_t = u_x + \frac{1}{2}u^2 \] \hfill (33)

Applying MapDE with TargetClass = LinearDE to (33) shows that this new system is linearizable, with the rif-form of the \( \Psi \) system given by:

\[ \phi_{u,u} = 0, \quad \varphi_{u,u} = 0, \quad \phi_{u,v} = -1/2 \phi_u, \]
\[ \varphi_{u,v} = -1/2 \varphi_u, \quad \phi_{v,v} = -1/2 \phi_v, \]
\[ \varphi_{v,u} = -1/2 \varphi_v, \quad \Upsilon_u = 0, \quad \psi_u = 0, \quad \Upsilon_v = 0, \quad \psi_v = 0 \] \hfill (34)

After integration this yields \( \Phi \):

\[ \hat{x} = \psi = x, \quad \hat{t} = \Upsilon = t, \quad \hat{u} = \varphi = u \exp \left( -\frac{v}{2} \right), \quad \hat{v} = \phi = \exp \left( -\frac{v}{2} \right) \]

and the Target system \( \hat{R} \) is \( \hat{u}_{\hat{z}} = -2\hat{v}_{\hat{t}}, \hat{v}_{\hat{z}} = -\hat{u}_{\hat{t}}/2 \) so that \( \hat{v}_{\hat{t}} = \hat{v}_{\hat{z}} \hat{z}, \hat{u}_{\hat{t}} = \hat{u}_{\hat{z}} \hat{z} \). This implies that the original Burger’s equation, is apparently also linearizable, through the introduction of the auxiliary nonlocal variable \( v \). This paradox is resolved in that the resulting very useful transformation is not a point transformation, since it effectively involves an integral. For extensive developments regarding such nonlocally related systems see [8].

**Example 4.4.** (Nonlinearizable examples with infinite groups) Consider the KP equation

\[ u_{x,x,x,x} = -6uu_{x,x} - 6u_x^2 - 4u_{x,t} - 3u_{y,y} \] \hfill (35)
which has

\[
ID(R) = \{u(x_0, y, t) = F_1(y, t), \quad u_x(x_0, y, t) = F_2(y, t), \\
u_{x, x}(x_0, y, t) = F_3(y, t), \quad u_{x, x, x}(x_0, y, t) = F_4(y, t)\}.
\] (36)

Applying \texttt{MapDE}, shows that the defining system \(S\) for symmetries \(\xi \partial_x + \eta \partial_y + \tau \partial_t + \beta \partial_u\) has initial data which is union of infinite data along the Hyperplane \(p = (x_0, y_0, t, u_0)\) and finite initial data at the point \(z_0 = (x_0, y_0, t_0, u_0)\):

\[
ID(S) = \{\beta(p) = H_1(t), \quad \beta_y(p) = H_2(t), \quad \beta_{y, y}(p) = H_3(t)\}
\]

\[
\bigcup \{\beta_x(z_0) = c_1, \quad \beta_u(z_0) = c_2, \quad \eta(z_0) = c_3, \quad \eta(z_0) = c_4, \quad \tau(z_0) = c_5, \quad \xi(z_0) = c_6\}
\] (37)

So both the KP equation and its symmetry system have infinite dimensional solution spaces: \(\dim R = \dim S = \infty\) since both have arbitrary functions in their data. However the KP equation has differential dimension \(d(R) = 2\) since there are a max of 2 free variables \((y, t)\) in its initial data; while its symmetry system has \(d(S) = 1\) since it has a max of one free variable in its data. Thus \(d(S) = 1 < 2 = d(R)\) and by the \texttt{PreEquivTest} the KP equation is not linearizable by point transformation.

Consider Liouville’s equation \(u_{x, x} + u_{y, y} = e^u\) which we rewrite as a DPS using \texttt{Maple’s} function \texttt{dpolyform}. That yields \(v = e^u\) and the Liouville equation in the form \(v_{x, x} = -v_{y, y} + v_x^2/v + v_y^2/v + v^2\). \texttt{MapDE} determines that \(\dim \mathcal{L} = \infty = \dim R\), and also that the Liouville equation is not linearizable by point transformation. Interestingly it is known that Liouville’s equation is linearizable by contact transformation (a more general transformation involving derivatives). For extensive developments regarding such contact related systems see [8].

**Example 4.5.** Given that exactly linearizable systems are not generic among the class of nonlinear systems, a natural question is how to identify such linearizable models. Since linearizability requires large symmetry groups (e.g. \(\infty\) dimensional for PDE) a natural approach to embed a model in a large class of systems, and seek the members of the class with largest symmetry groups.

Indeed in Wittkopf and Reid [23], such an approach was developed. We now illustrate how this approach can be used here. For a nonlinear telegraph equation one might embed it the general class of spatially dependent nonlinear telegraph systems

\[
v_x = u_t, \quad v_t = C(x, u)u_x + B(x, u)
\] (38)

where \(B_u \neq 0, C_u \neq 0, B_x \neq 0, C_x \neq 0\). Then applying the \texttt{maxdimsystems} algorithm available in \texttt{Maple} with \(\dim = \infty\) a quick calculation yields 11 cases, only 4 of which satisfy the dimension restriction; which we further narrow by requiring restriction to those that have the greatest freedom in \(C(x, u), B(x, u)\). The explicit results, successful after integration yield the linearizable class:

\[
v_x = u_t, \quad v_t = \frac{1}{q_x} u \left( \frac{u}{q_x} \right) u_x - \frac{q_{xx}}{q_x^2} f \left( \frac{u}{q_x} \right)
\] (39)

and yields after integration the linearizing transformation

\[
\hat{x} = \frac{u}{q_x}, \quad \hat{t} = v, \quad \hat{u} = q(x), \quad \hat{v} = t
\] (40)
Also in the same vein we can request the maximal dimensional symmetry group for the normalized linear Schrödinger Equation

\[ i\hbar \varphi_t = -\frac{\hbar^2}{2m} \nabla^2 \varphi + V(x,t)\varphi \]  

which restricting to 2 space plus one time yields \( V(x,y,t) = \omega(t)(x^2 + y^2) + b(t)x + c(t)y + d(t) \), and satisfies the conditions for mapping to constant coefficient DE via the methods of our earlier paper.

5 Discussion

In [27] we introduced the MapDE algorithm and its implementation in Maple for mappings relating an input DPS \( R \) to a target system \( \hat{R} \) via an analytic mapping \( \Psi \). The mappings are local holomorphic diffeomorphisms of the dependent and independent variables. A key goal of the work are algorithms, i.e. procedures that are guaranteed to achieve a task in a finite number of steps, for a specific input class (in our case, a DPS with coefficients from a computable subfield of \( \mathbb{C} \) such as an extension of \( \mathbb{Q} \)). Among the algorithmic outputs of MapDE, are the mapping equations in rif-form, obtainable by a finite number of differentiations and eliminations, from which existence of such a mapping can be decided. We emphasize, that we are not opposed to integration methods which necessarily are heuristic for multivariate systems, and we provide a further integration phase to attempt to find the mappings explicitly, based on Maple’s pdsolve.

The main contribution of our current paper an algorithmic extension of MapDE to decide whether an input DPS can be mapped by holomorphic diffeomorphism to a linear system. This work is based on creating algorithms that exploit results due to Bluman and Kumei [4, 8] and some aspects of [23]. This is a natural partner to the algorithm for deciding existence of a linear DPS to a constant coefficient linear DE given in our paper [27].

In the algorithms for mappings of ODE to linear ODE, [23] explicitly introduce a target linear target system \( \hat{R} \) with undetermined coefficients, then use the full nonlinear determining equations (a subroutine that we call EquivMapDets in our package). In particular they applied differential elimination to this determining system using the ThomasDecomposition Algorithm [39, 34], to decide the existence of such a mapping. In contrast, like Bluman and Kumei, we exploit the fact that the target appears implicitly as a subalgebra of the Lie symmetry algebra \( \mathcal{L} \) of \( R \). Unlike Bluman and Kumei, who depend on extracting this subalgebra by explicit non-algorithmic integration, we use algorithmic differential algebra. Indeed we combine aspects of [23], by using the fact that this is a subalgebra of the derived algebra \( \mathcal{L}' \) of \( \mathcal{L} \). Instead of using the BK mapping equations (14), [23] apply the transformations directly to the ODE. In contrast, our method works at the linearized Lie algebra level, instead of the nonlinear Lie Group level used in [23]; which is a factor in their increased space and time data for their test set compared to our timings (See Table 1).

The important heuristic integration phase of MapDE will be developed in future work. Even if the transformations can’t be determined explicitly, they can implicitly identify important features. Furthermore, they are available for the application of symbolic and symbolic-numeric approximation methods, a possibility that we will also explore in future work.
Currently, our algorithm for constructing the determining equations for the derived algebra $L'$ and its structure the infinite case, proceeds by a direct but inefficient elimination ranking, to isolate the equations for the commutators as lowest in the ranking. We are currently implementing a more efficient algorithm based on generalizing techniques for the finite case.

It is important to develop simple, efficient tests to reject the existence of mappings, based on structural and dimensional information. In addition to existing known tests [23], [8, 1, 42] we introduced a refined dimension test based on Hilbert Series. We will extend these tests in future work. We note that the potentially expensive change of rankings needed by our algorithms could be more efficiently be accomplished by the change of rankings approach given in [10] using Differential Algebra.

Mappings are of fundamental importance, and of great difficulty both theoretically and in terms of computational difficulty. Multiple approaches are required. Finding appropriate setups is important. For example [1, 42] use multipliers for conservation laws to facilitate the determination of linearization mappings and will be investigated in future work. Also Wolf’s approach [42] enables the determination of partially linearizable systems.

Given that nonlinear systems are usually not linearizable, a fundamental problem for future research is how to identify such linearizable models. One method we suggest is to embed the given model in a class of models, and then efficiently seek the members of the class with the largest symmetry groups and most freedom in the functions/parameters of the class. See Example 4.5 for an illustration of this strategy. Other strategies are to embed the model, in spaces have a natural relation to the original space in terms of solutions, but not related by invertible point transformation (e.g. see [8] and our see Example 4.3). Finally a model that is not exactly linearizable, maybe close to a linearizable model or other attractive target, providing motivation for our future work on approximate mapping methods.

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