Non-locality sharing for a three-qubit system via multilateral sequential measurements

Changliang Ren,1,* Xiaowei Liu,† Wenlin Hou, Tianfeng Feng, and Xiaoqi Zhou

1Key Laboratory of Low-Dimensional Quantum Structures and Quantum Control of Ministry of Education,
Department of Physics and Synergetic Innovation Center for Quantum Effects and Applications,
Hunan Normal University, Changsha 410081, China

2State Key Laboratory of Optoelectronic Materials and Technologies and School of Physics,
Sun Yat-sen University, Guangzhou, People’s Republic of China

Non-locality sharing for a three-qubit system via multilateral sequential measurements was deeply discussed. Different from 2-qubit cases, it is shown that non-locality sharing between Alice1 – Bob1 – Charlie, and Alice2 – Bob2 – Charlie2 in a 3-qubit system can be observed, where two Mermin-Ardehali-Belinskii-Klyshko (MABK) inequalities can be violated simultaneously. What’s more, a complete non-locality sharing with all of 8 MABK inequalities simultaneous violations can be also observed. Compared with 2-qubit cases, the nonlocal sharing in a 3-qubit system shows more novel characteristics. Finally a general interpretation of nonlocal sharing according to joint conditional probability was discussed.

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I. INTRODUCTION

Local realism indicates the nature of the world that measurement outcomes are pre-deterministic, and the measurement on one party of a multipartite system does not affect the other parties. However, quantum mechanics predicts that there are stronger correlations than the correlations of local hidden variables because of inherent non-locality of quantum theory [1]. The so-called Bell inequality is exploited to distinguish the differences between classical correlation and quantum correlation [2]. Subsequently, Bell-type inequalities have been studied extensively from various perspectives [3–10] and experimentally verified in many different quantum systems [11–19]. These kinds of research, are not only important to deeply understand quantum theory, but also play an crucial role in quantum information protocols, such as quantum key distribution [20], randomness generation [21–25], and entanglement certification [26]. For a background on Bell inequalities, readers could refer to [27], and references therein.

Inspired by Bell’s work, Clauser, Horne, Shimony, and Holt (CHSH) derived a modified inequality [3], which provides a faithful way for experimentally testing the non-locality property in 2-qubit composite systems. However, most discussions of non-locality based on CHSH inequality focus on one pair of entangled qubits distributed to only two separated observers. Recently, a surprising result that non-locality can actually be shared among more than two observers using weak measurements, has been reported by Silva et al. [28]. In Silva’s scenario, a pair of maximally-entangled qubits is distributed to three observers Alice, Bob1, and Bob2, in which Alice accesses one qubit and the two Bobs access the other qubit. Alice performs a strong measurement on her own qubit, while Bob1 performs a weak measurement on his qubit and passes it to Bob2. Finally Bob2 carries out a strong measurement. The measurement results reveal that it is possible to observe a simultaneous violation of CHSH inequality in Alice-Bob1 and Alice-Bob2. To date, a series of fruitful related theoretical researches [29–44] have been proposed by tracking this path and several experimental demonstrations have also been performed [45–47]. Especially, in Ref.[36, 47], it shows a observation of non-locality sharing in a wide area even if Bob1’s measurement is close to a strong measurement, which is impossible in the original protocol [28]. Nevertheless, almost all discussions are limited to one-sided sequential case, i.e., one entangled pair is distributed to one Alice and multiple Bobs. Recently, Zhu et.al explored the non-locality sharing in two-sided sequential measurements case in which one entangled pair is distributed to multiple Alices and Bobs [43]. But non-locality sharing between Alice1 – Bob1 and Alice2 – Bob2 is impossible in such scenario [43]. In this letter, we explored the non-locality sharing for a three-qubit system via multilateral sequential measurements. Different from 2-qubit cases, it is shown that non-locality sharing between Alice1 – Bob1 – Charlie1 and Alice2 – Bob2 – Charlie2 in 3-qubit system can be observed, where two MABK inequalities can be violated simultaneously. Not only that, a complete non-locality sharing with all of 8 MABK inequalities simultaneous violations can be also observed. These results not only shed new light on the interplay between non-locality and quantum measurements, especially the emergence of non-locality sharing via weak measurements, but also can be applied in unbounded randomness certification [48], quantum coherence [33] and quantum steering [34].
FIG. 1: Scenario of three-sided sequential case: a 3-qubit entangled state is distributed to three sides, and each side has two observers, in which the first observers perform weak measurements and the second perform strong measurements. Two Alices occupy one-third of the state and Bobs, Charlies respectively. Those observers in the same side will measure their shared qubit sequentially. The communication is forbidden between the observers, and the measurement choice of these observers are independent. Each observer randomly chooses one of two observables to measure, which can be defined as $X_i = A_{i,l}$ for Alice, $Y_j = B_{j,m}$ for Bob, and $Z_k = C_{k,n}$ for Charlie, respectively, where $A_{i,l}$ is the $l$-th measurement chosen by the observer Alice, similar definition for $B_{j,m}$ and $C_{k,n}$, $\{i,j,k\} \in \{1,2\}$ and $\{l,m,n\} \in \{1,2\}$. The binary outcomes of the dichotomic measurement for these observers are given by $a_i, b_j, c_k$ with $\{a_i,b_j,c_k\} \in \{-1,1\}$. Such a scenario is characterized by the joint conditional probabilities of the outcomes $P(a_1,a_2,b_1,b_2,c_1,c_2 \mid X_1,X_2,Y_1,Y_2,Z_1,Z_2)$.

In the scenario, the first observer of each side performs the optimal weak measurements, while the second observer of each side will carry out strong measurements. We assumed that they share a 3-qubit state, and the density matrix is defined as $\rho$. In the whole measurement process, we can always obtain the quantum state after measurement according to the selection of measurement and it’s outcome. Without loss of generality, Alice firstly performs a weak measurement $X_1$ on her received qubit with the quality factor $F_1$ and precision factor $G_1$ of the measurement, where $X_i = A_{i,l} = \xi_{i,l} \cdot \hat{\sigma}$, $\hat{\sigma}$ is a vector consists of three Pauli matrices $\hat{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and $\xi_{i,l}$ is a direction vector on the Bloch sphere, $\xi_{i,l} = (\sin \theta_{i,l} \cos \phi_{i,l}, \sin \theta_{i,l} \sin \phi_{i,l}, \cos \theta_{i,l})$. As is introduced in [28], there exists a trade-off between measurement disturbance and information gain, where the optimal weak measurement requires that $F_1^2 + G_1^2 = 1$, which means that more information can be extracted with same disturbance. We assume that the weak measurement process in this scenario is the optimal pointer case. When the measurement outcome is $a_i$, according to the discussion in [28], the state changes to

$$\rho_{X_1}^{a_i} = \frac{F_1}{2} \rho + \frac{1 + a_iG_1 - F_1}{2} [U_{X_1}^{-1} \rho (U_{X_1}^{-1})^\dagger]$$

$$+ \frac{1 - a_iG_1 - F_1}{2} [U_{X_1}^{-1} \rho (U_{X_1}^{-1})^\dagger]$$

(1)

where $U_{X_1}^{a_i} = \Pi_{X_1}^{a_i} \otimes I \otimes I$ and $\Pi_{X_1}^{a_i} = I + a_i X_1$. Subsequently, if Alice$_2$ performs a strong measurement $X_2$ with the outcome $a_2$, the 3-qubit state will change to

$$\rho_{X_2}^{a_2} = U_{X_2}^{a_2} \rho_{X_1}^{a_i} U_{X_2}^{a_2 \dagger}.$$  

(2)

Then, suppose that the measurements of Bobs are later than that of Alice$_2$, similarly the measurements of Charlies are later than that of Bob$_2$. Bob$_1$ performs a weak measurement $Y_1$ on his received qubit with the quality factor $F_2$ and precision factor $G_2$ of the measurement, where the optimal weak measurement requires that $F_2^2 + G_2^2 = 1$. When the measurement outcome is $b_1$, the 3-qubit state becomes

$$\rho_{Y_1}^{b_1} = \frac{F_2}{2} \rho_{X_2}^{a_2} + \frac{1 + b_1G_2 - F_2}{2} [U_{X_2}^{-1} \rho_{X_2}^{a_2} (U_{X_2}^{-1})^\dagger]$$

$$+ \frac{1 - b_1G_2 - F_2}{2} [U_{X_2}^{-1} \rho_{X_2}^{a_2} (U_{X_2}^{-1})^\dagger].$$  

(3)

where $Y_j = B_{j,m} = \xi_{j,m} \cdot \hat{\sigma}$, $U_{Y_j}^{b_1} = I \otimes \Pi_{Y_j}^{b_1} \otimes I$. Similarly, if Bob$_2$ performs a strong measurement $Y_2$ with the outcome $b_2$, the 3-qubit state becomes $\rho_{Y_2}^{b_2} = U_{X_1}^{b_2} \rho_{X_1}^{a_i} U_{X_1}^{b_2 \dagger}$. Subsequently, Charlie$_1$ performs a weak measurement $Z_k$ on his received qubit with the quality factor $F_3$ and precision factor $G_3$ of the measurement, where the optimal weak measurement requires that $F_3^2 + G_3^2 = 1$. When the measurement outcome is $c_1$, then the 3-qubit state changes to

$$\rho_{Z_1}^{c_1} = \frac{F_3}{2} \rho_{Y_2}^{b_2} + \frac{1 + c_1G_3 - F_3}{2} [U_{Z_1}^{-1} \rho_{Z_1}^{b_2} (U_{Z_1}^{-1})^\dagger]$$

$$+ \frac{1 - c_1G_3 - F_3}{2} [U_{Z_1}^{-1} \rho_{Z_1}^{b_2} (U_{Z_1}^{-1})^\dagger].$$  

(4)

where $Z_k = B_{k,n} = \xi_{k,n} \cdot \hat{\sigma}$ and $U_{Z_k}^{c_k} = I \otimes I \otimes \Pi_{Z_k}^{c_k}$. Finally, when Charlie$_2$ performs a strong measurement $Z_2$ with the outcome $c_2$, the 3-qubit state will turn to $\rho_{Z_2}^{c_2} = U_{Z_2}^{c_2} \rho_{Z_1}^{c_1} U_{Z_2}^{c_2 \dagger}$. So a cyclic measurement process of this scenario has been completely described. After repeating such process over and over
again, we can obtain a complete joint probability distribution \( P(a_1, a_2, b_1, b_2, c_1, c_2 \mid X_1, X_2, Y_1, Y_2, Z_1, Z_2) \). From the unnormalized postmeasurement state \( \rho_{Z_2}^x \), the joint conditional probability distribution is given as
\[
P(a_1, a_2, b_1, b_2, c_1, c_2 \mid X_1, X_2, Y_1, Y_2, Z_1, Z_2) = \text{Tr}[\rho_{Z_2}^x]
\]

Here we need to point out that only the two observers on the same side should measure sequentially, such as Alice1 should measure before Alice2, the same for Bob1−Bob2 and Charlie1−Charlie2. However, there is no assumption of measurement order between observers on different sides. In other words, the sequence of local measurements between the observers in three different sides does not change the final joint conditional probability distribution \( P(a_1, a_2, b_1, b_2, c_1, c_2 \mid X_1, X_2, Y_1, Y_2, Z_1, Z_2) \). For simplicity, we follow the sequence of Alice1−Alice2−Bob1−Bob2−Charlie1−Charlie2 to describe the measurement process.

![Figure 2: Plot of MABK quantity \( B_1 \) for Alice1−Bob1−Charlie1 and \( B_3 \) for Alice2−Bob2−Charlie2 when the state is GHZ state. The thin red line describes \( B_1 \) and the thick blue line describes \( B_3 \). When we choose \( G_1 = G_2 = G_3 = G \), they both exceed the bound of 2 in a narrow range. The picture above is the magnification of the violation part.](image)

![Figure 3: Plot of MABK quantities \( B_2 \) to \( B_7 \) for Alice−Bob−Charlie when the state is GHZ state. When we choose \( G_1 = G_2 = G_3 = G \), \( B_2 = B_3 = B_4 \) (purple dotted line) and \( B_5 = B_6 = B_7 \) (green dashed line). They can exceed the classical bound simultaneously in a narrow range.](image)

According to the content discussed in this article, we are more concerned about the joint conditional probabilities for the measurement of any three different-side observers. Supposed that the observable choosing is unbiased for each observer, which requires that each measurement setting of every observer should be chosen with equal probability. The joint conditional probability \( P(a_i, b_j, c_k \mid \hat{X}_i, \hat{Y}_j, \hat{Z}_k) \) is obtained via marginalizing the corresponding variables.

\[
P(a_i, b_j, c_k \mid \hat{X}_i, \hat{Y}_j, \hat{Z}_k) = \frac{1}{2^3} \sum_{\hat{X}_i', \hat{Y}_j', \hat{Z}_k'} \sum_{a_i', b_j', c_k'} P(a_i, b_j, c_k, a_i', b_j', c_k' \mid \hat{X}_i, \hat{Y}_j, \hat{Z}_k, \hat{X}_i', \hat{Y}_j', \hat{Z}_k').
\]

(5)

Based on the joint conditional probability distribution, we can calculate the expected value \( E(\hat{X}_i, \hat{Y}_j, \hat{Z}_k) \), which is given as

\[
E(\hat{X}_i, \hat{Y}_j, \hat{Z}_k) = \sum_{a_i, b_j, c_k} a_i b_j c_k P(a_i, b_j, c_k \mid \hat{X}_i, \hat{Y}_j, \hat{Z}_k) = \frac{1}{2^3} \sum_{a_i, b_j, c_k} a_i b_j c_k \sum_{\hat{X}_i', \hat{Y}_j', \hat{Z}_k'} a_i', b_j', c_k' P(a_i, b_j, c_k, a_i', b_j', c_k' \mid \hat{X}_i, \hat{Y}_j, \hat{Z}_k, \hat{X}_i', \hat{Y}_j', \hat{Z}_k').
\]

(6)

The quantum non-locality can be witnessed via violations of corresponding inequalities. To explore the phenomenon of nonlocal sharing for a 3-qubit system via multilateral sequential measurements, we will use the typical N-qubit Bell-type inequality, Mermin-Ardehali-Belinskii-Klyshko (MABK) inequality, which can be described as

\[
| - E(A_{i,1}, B_{j,1}, C_{k,1}) + E(A_{i,2}, B_{j,1}, C_{k,2}) + E(A_{i,1}, B_{j,2}, C_{k,1}) + E(A_{i,1}, B_{j,2}, C_{k,2}) | \leq 2.
\]

(7)

Obviously, by choosing different observers in each side, we can discuss the violations of eight MABK inequalities. For clarity of discussions, we denote MABK quantity as \( B_w \), which is the value on the left side of Eq. (7) for the combination of different observers, where \( B_1 \) for \( (i = j = k = 1) \), \( B_2 \) for \( (i = k = 1, j = 2) \), \( B_3 \) for \( (i = j = 1, k = 2) \), \( B_4 \) for \( (j = k = 1, i = 2) \), \( B_5 \) for \( (j = k = 2, i = 1) \), \( B_6 \) for \( (i = k = 2, j = 1) \), \( B_7 \) for \( (i = j = k = 2) \). Each expected value in every MABK inequality can be obtained from Eq. (6). Thus it is possible to check whether there exists multiple violation via calculation results, which will be analyzed in Sec. III.

III. NONLOCAL SHARING IN MULTI-SIDED SEQUENTIAL MEASUREMENTS CASE

As is known, non-locality sharing between Alice1−Bob1 and Alice2−Bob2 is impossible in a 2-qubit system [43]. We firstly explore whether nonlocal sharing between Alice1−Bob1−Charlie1 and
Alice\(_2\) – Bob\(_2\) – Charlie\(_2\) exists in a 3-qubit system or not.

Without loss of generality, we assume that the observers in the three different sides share a three-qubit GHZ state, which is

\[
| \psi \rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle). \tag{8}
\]

Each observer has two measurement directions to choose, and the directions of the dichotomic measurements are denoted as \((\{\theta_{11}, \phi_{11}\}, \{\theta_{12}, \phi_{12}\})\) for Alice\(_1\), \((\{\theta_{13}, \phi_{13}\}, \{\theta_{14}, \phi_{14}\})\) for Alice\(_2\), \((\{\theta_{21}, \phi_{21}\}, \{\theta_{22}, \phi_{22}\})\) for Bob\(_1\), \((\{\theta_{23}, \phi_{23}\}, \{\theta_{24}, \phi_{24}\})\) for Bob\(_2\), \((\{\theta_{31}, \phi_{31}\}, \{\theta_{32}, \phi_{32}\})\) for Charlie\(_1\), \((\{\theta_{33}, \phi_{33}\}, \{\theta_{34}, \phi_{34}\})\) for Charlie\(_2\).

To explore nonlocal sharing between Alice\(_1\) – Bob\(_1\) – Charlie\(_1\) and Alice\(_2\) – Bob\(_2\) – Charlie\(_2\), it is necessary to determine whether the MABK quantities, \(B_1\) and \(B_8\), can surpass the classical bounds simultaneously or not. To simplify calculation, we require that the measurement directions of every observers will be always in the \(X – Y\) plane, where \(\theta_{11} = \theta_{12} = \theta_{13} = \theta_{14} = \theta_{21} = \theta_{22} = \theta_{23} = \theta_{24} = \theta_{31} = \theta_{32} = \theta_{33} = \theta_{34} = \frac{\pi}{2}\). Unfortunately, it is still too complex to obtain the optimal measurement settings which can show the maximal nonlocal sharing between Alice\(_1\) – Bob\(_1\) – Charlie\(_1\) and Alice\(_2\) – Bob\(_2\) – Charlie\(_2\). But we can easily show such nonlocal sharing by simple measurement settings even though they are suboptimal measurement settings. Interestingly, when we chose such simple measurement settings, \(\phi_{11} = \phi_{21} = \phi_{31} = 0\), \(\phi_{12} = \phi_{22} = \phi_{32} = –\phi_{14} = –\phi_{24} = –\frac{\pi}{2}\), \(\phi_{13} = –\phi_{33} = \phi_{34} = \pi\), \(\phi_{14} = \frac{3\pi}{2}\), the MABK quantities, \(B_1\) and \(B_8\), turn to

\[
B_1 = 4G_1G_2G_3 \tag{9}
\]

\[
B_8 = \frac{1}{2} (1 + F_1)(1 + F_2)(1 + F_3). \tag{10}
\]

For simplicity, when \(G_1 = G_2 = G_3 = G\), \(B_1\) and \(B_8\) changes to \(B_1 = 4G^3\) and \(B_8 = \frac{1}{8}(1 + \sqrt{1 – G^2})^2\). The MABK quantities, \(B_1\) and \(B_8\), can exceed 2 simultaneously in the narrow range of \(G \in (\sqrt{2(\frac{3}{2} – \frac{2}{2})}, \frac{3}{2})\) (approximate \(G \in (0.793, 0.809)\)). As illustrated in Fig. 2, when \(G = 0.8\), \(B_1 = B_8 = 2.048\), which is the maximal simultaneous violation for \(B_1\) and \(B_8\). Different from a 2-qubit case, it shows that non-locality sharing between Alice\(_1\) – Bob\(_1\) – Charlie\(_1\) and Alice\(_2\) – Bob\(_2\) – Charlie\(_2\) in 3-qubit system can be observed, where two MABK inequalities can be violated simultaneously.

Secondly, we can explore whether nonlocal sharing still exists or not for other different observers combinations. Certainly, it can be analyzed by discussing the simultaneous violation for these MABK quantities, \(B_2\) to \(B_7\). When the same measurement directions mentioned above are used, the measurement directions of every observer will be always in the \(X – Y\) plane, the MABK quantities, \(B_2\) to \(B_7\), can be written as

\[
B_2 = 2(1 + F_3)G_1G_3 \tag{11}
\]

\[
B_3 = 2(1 + F_3)G_1G_2
\]

\[
B_4 = 2(1 + F_1)G_2G_3
\]

\[
B_5 = (1 + F_2)(1 + F_3)G_1
\]

\[
B_6 = (1 + F_1)(1 + F_3)G_2
\]

\[
B_7 = (1 + F_1)(1 + F_2)G_3.
\]

Similarly, when these MABK quantities in Eq. (11) can exceed 2 simultaneously, the nonlocal sharing phenomena can be observed. For simplicity, when \(G_1 = G_2 = G_3 = G\), the MABK quantities \(B_2\) to \(B_4\) will change to the same value \(2G^2(1 + \sqrt{1 – G^2})\), while the MABK quantities \(B_5\) to \(B_7\) will also change to another value \(G(1 + \sqrt{1 – G^2})^2\). As illustrated in Fig. 3, it is easily to find the MABK quantities \(B_2\) to \(B_7\) will exceed 2 simultaneously in the range of \(G \in (\sqrt{\frac{\sqrt{3} – 1}{2}}, 1)\).

When \(G = 2\sqrt{3}\), the MABK quantities \(B_2\) to \(B_4\) achieves the maximal value 2.37 by choosing these measurement settings. The MABK quantities \(B_5\) to \(B_7\) will exceed 2 simultaneously in the range of \(G \in (0.638, 0.839)\), and the MABK quantities \(B_5\) to \(B_7\) achieves the maximal value 2.07 by choosing these measurement settings when \(G = \frac{2\sqrt{3}}{3}\). Hence, the MABK quantities \(B_2\) to \(B_7\) will exceed 2 simultaneously in the range of \(G \in (\sqrt{\frac{\sqrt{3} – 1}{2}}, 0.839)\). When \(G = 0.8\), \(B_2 = B_3 = B_4 = B_5 = B_6 = B_7 = 2.048\), which is the maximal simultaneous violation for \(B_2\) to \(B_7\).

Finally, when all the eight MABK quantities can exceed 2 simultaneously, it will exhibit a complete non-locality sharing in such a three qubit system via multilevel sequential measurements. Actually, the eight MABK inequalities, from \(B_1\) to \(B_8\), can be simultaneously violated. We can easily show such nonlocal sharing by simple measurement settings which are mentioned above, even though they are suboptimal measurement settings. As illustrated in Fig. 4, when \(G_1 = G_2 = G_3 = G\), we show all the eight MABK quantities will exceed 2 simultaneously in the range of \(G \in (\sqrt{2(\frac{3}{2} – \frac{2}{2})}, \frac{3}{2})\), when \(G = 0.8\), \(B_1 = B_2 = B_3 = B_4 = B_5 = B_6 = B_7 = B_8 = 2.048\), which is the maximal simultaneous violation for all the eight MABK quantities. Compared with 2-qubit cases, the nonlocal sharing in a three qubit system shows more novel characteristics.

**IV. NONLOCAL SHARING PERSPECTIVE FROM JOINT CONDITIONAL PROBABILITY**

Although the investigation of nonlocal sharing scenario has made some progress, it is still in its infancy where lots of open questions are very confusing. As mentioned above, we explored the non-locality sharing for a three-qubit system via multilevel sequential measurements by investigating and analyzing the corresponding joint
Alice we choose criteria and experimental verification involved in quantum conditional probabilities, we note that almost all the criterion, inequality type, entropy criterion, geometric criterion, thought caused by a real conditional probability “. In quantum correlation problem can be summarized as “A locality, but was ignored for a long time? such nonlocal sharing scenario is worth to explore? why did it not develop with the development of Bell’s nonlocality, but was ignored for a long time?

Not only Bell’s non-locality, but also the whole EPR quantum correlation problem can be summarized as “A thought caused by a real conditional probability “. In fact, no matter which kind of quantum correlation criterion, inequality type, entropy criterion, geometric criterion, etc., can all be expressed in the form of the joint conditional probabilities.

If we understand them from the perspective of joint conditional probabilities, we note that almost all the criteria and experimental verification involved in quantum correlation can be described as follows: for a quantum state system with N particles, N observers will share and measure them. Each observer may have multiple measurement settings, but every observer will only measure the particle in its hand once in each round of measurement. After repeating such process over and over again, the joint conditional probability distribution based on N measurements will be obtained. Taking a 2-qubit system as example, all the criteria can start from a joint conditional probability $P(a, b | \hat{A}, \hat{B}, \rho)$. Then, the combination of these conditional probabilities or the physical quantities which (average value, etc.) are based on the conditional probabilities are used to exhibit the conflict between quantum prediction and classical model.

One of the key features in this scenario is that each shared particle of the quantum state will be measured once and discarded. However, this feature should not be an inevitable condition. On the contrary, the conditional probability can be derived from a higher dimensional conditional probability based on more measurements. Consider the simplest example, for a 2-qubit system, we assumed that each particle will be measured twice in each round. For instance, Alice will carry out sequential measurements of $A_1, A_2$ and Bob will carry out sequential measurements of $B_1, B_2$. Then a joint conditional probability $P(a_1, a_2, b_1, b_2 | A_1, A_2, B_1, B_2, \rho)$ can be obtained. The nonclassical correlation in a 2-qubit system will be expressed more comprehensive by using this joint conditional probability, as the previous joint conditional probability can be always obtained via marginal constraints, $P(a_1, b_1 | \hat{A}_1, \hat{B}_1, \rho) = \sum_{a_2, b_2} P(a_1, a_2, b_1, b_2 | \hat{A}_1, \hat{A}_2, \hat{B}_1, \hat{B}_2, \rho)$.

However, previous studies ignored the research path based on the above joint conditional probability. The crucial factor is about the measurements. In previous scenario, the observer’s measurements are usually strong measurements. Taking a 2-qubit system scenario as an example, Alice and Bob perform strong measurement in their first measurement, which means $A_1, B_1$ are strong measurements. Even if the subsequent first measurements are carried out, the joint conditional probability can be always written as $P(a_1, a_2, b_1, b_2 | A_1, A_2, B_1, B_2, \rho) = P(a_1, b_1 | A_1, B_1, \rho)P(a_2 | A_2, \Pi_{A_1}^{P_{a_1}})P(b_2 | B_2, \Pi_{B_1}^{P_{b_1}})$, $\Pi_{A_1}^{P_{a_1}}$ and $\Pi_{B_1}^{P_{b_1}}$ are the eigenstates of $A_1$ and $B_1$. In this scenario, the later measurement results will not carry any correlation information of the initial state, so it is meaningless to discuss sequential measurements. However, if the decomposition of the conditional probability can not take the equal sign, the problem will become not trivial. From the physical point of view, in order to satisfy this requirement, it is necessary to protect the correlation information of the initial state as much as possible in the former measurements. Fortunately, weak measurement as an important nonperturbative measurements satisfies such a condition. Nonlocal sharing emerges in such scenario. Hence the observation of nonlocal sharing usually associate with weak measurement or POVM measurement.

V. CONCLUSION

The phenomenon of non-locality sharing for a 3-qubit system via multilateral sequential measurements has been completely discussed. In order to compare with a 2-qubit case, we firstly explored non-local sharing in a 3-qubit system between Alice1 − Bob1 − Charlie1 and Alice2 − Bob2 − Charlie2. Interestingly, the corresponding MABK inequalities, $B_1$ and $B_8$, can exceed 2 simultaneously in the narrow range of $G \in (\sqrt{2(2^\frac{1}{2} − 2^\frac{1}{14})}, 2^{-\frac{1}{3}})$. Hence, nonlocal sharing in a 3-qubit system between
Alice – Bob1 – Charlie1 and Alice2 – Bob2 – Charlie2 can be observed, while it is impossible for a 2-qubit case. Secondly, we also investigated nonlocal sharing for the other different observers combinations. It is shown that the MABK quantities \(B_2\) to \(B_8\) will exceed 2 simultaneously in the range of \(G \in \left(\sqrt{\frac{3}{2}}, 1\right)\). Finally, all the eight possible MABK inequalities in this scenario were fully explored. Actually, the eight MABK inequalities, from \(B_1\) to \(B_8\), can be violated simultaneously. We can easily show such nonlocal sharing by simple measurement settings which are mentioned above, even though they are suboptimal measurement settings. When \(G_1 = G_2 = G_3 = G\), we show all the eight MABK quantities will exceed 2 simultaneously in the range of \(G \in \left(\sqrt{2(2^4 - 2^3)}, 2^{-\frac{1}{2}}\right)\). When \(G = 0.8\), \(B_1 = B_2 = B_3 = B_4 = B_5 = B_6 = B_7 = B_8 = 2.048\), which is the maximal simultaneous violation for all the eight MABK quantities. Compared with a 2-qubit case, the nonlocal sharing in a 3-qubit system shows more novel characteristics. Besides, a deep understanding of the phenomenon of nonlocal sharing from the view of joint conditional probabilities were exhibited, which can give some beneficial viewpoints for deeply understanding general nonlocal sharing phenomenon.

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[1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47, 777 (1935).
[2] J. S. Bell, Physics (Long Island City, N.Y.) 1, 195 (1964).
[3] J. Clauser, M. Horne, A. Shimony, R. Holt, Phys. Rev. Lett. 23, 880 (1969).
[4] M. Żukowski and Č. Brukner, Phys. Rev. Lett. 88, 210401 (2002).
[5] N. D. Mermin, Phys. Rev. Lett. 65, 1838 (1990).
[6] A. V. Belinskii and D. N. Klyshko, Phys. Usp. 36, 653 (1993).
[7] M. Ardehali, Phys. Rev. A 46, 5375 (1992).
[8] D. Collins, N. Gisin, N. Linden, S. Massar, S. Popescu, Phys. Rev. Lett. 88, 040404 (2002).
[9] Č. Brukner, M. Žukowski, and A. Zeilinger, Phys. Rev. Lett. 89, 197901 (2002).
[10] S. M. Lee, M. Kim, H. Kim, H. S. Moon, and S. W. Kim, Quantum Science and Technology, 3(4), 045006 (2018).
[11] A. Aspect, J. Dalibard, and G. Roger, Phys. Rev. Lett. 49, 1804 (1982).
[12] G. Weihs, T. Jennewein, C. Simon, H. Weinfurter, and A. Zeilinger, Phys. Rev. Lett. 81, 5039 (1998).
[13] M. A. Rowe, D. Kielpinski, V. Meyer, C. A. Sackett, W. M. Itano, C. Monroe, and D. J. Wineland, Nature 409, 791 (2001).
[14] J. Hofmann et al., Science 337, 72 (2012).
[15] M. Giustina et al., Nature 497, 227 (2013).
[16] E. G. Christensen et al., Phys. Rev. Lett. 111, 130406 (2013).
[17] B. Hensen et al., Nature 526, 682 (2015).
[18] M. Giustina, M. A. M. Versteegh, S. Wengerowsky, J. Handsteiner, A. Hochrainer, K. Phelan et al., Phys. Rev. Lett. 115, 250401 (2015).
[19] L. K. Shalm, E. Meyer-Scott, B. G. Christensen, P. Bierhorst, M. A. Wayne, M. J. Stevens, et al., Phys. Rev. Lett. 115, 250402 (2015).
[20] A. Acín, Phys. Rev. Lett. 98, 230501 (2007).
[21] R. Colbeck, Ph.D. thesis, University of Cambridge (2007), arXiv:0911.3814.
[22] S. Pironio et al., Nature 464, 1021-1024 (2010).
[23] S. Pirandola et al., arXiv: 1906.01645.
[24] Y. Liu, et al., Nature 562, 548 (2018).
[25] P. Bierhorst, et al., Nature 556, 223 (2018).
[26] J. Bowles, I. Supic, D. Cavalcanti, A. Acín, Phys. Rev. Lett. 121, 180503 (2018).
[27] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Rev. Mod. Phys. 86, 419 (2014).
[28] R. Silva, N. Gisin, Y. Guryanova, and S. Popescu, Phys. Rev. Lett. 114, 250401 (2015).
[29] D. Das, A. Ghosal, S. Sasmal, S. Mal, and A. S. Majumdar, Physical Review A 99, 022305 (2019).
[30] S. Mal, A. S. Majumdar, and D. Home, Mathematics, 4(3), 48, (2016).
[31] S. Sasmal, D. Das, S. Mal, and A. S. Majumdar, Phys. Rev. A 98, 012305 (2018).
[32] A. Bera, S. Mal, A. SenDe, and U. Sen, Phys. Rev. A 98, 062304 (2018).
[33] S. Datta and A. S. Majumdar, Phys. Rev. A 99, 042311 (2019).
[34] A. Shenoy H., S. Designolle, F. Hirsch, R. Silva, N. Gisin, and N. Brunner, Phys. Rev. A 99, 022317 (2019).
[35] A. Kumari and A. K. Pan, Phys. Rev. A 100, 062130 (2019).
[36] C. L. Ren, T. Feng, D. Yao, H. Shi, J. Chen, and X. Zhou, Phys. Rev. A, 100, 052121(2019).
[37] S. Saha, D. Das, S. Sasmal, D. Sarkar, K. Mukherjee, A. Roy, and S. S. Bhattacharya, Quantum Inf. Processing 18, 42 (2019).
[38] K. Mohan, A. Tavakoli and N. Brunner, New J. Phys. 21, 083034 (2019).
[39] S. Roy, A. Bera, S. Mal, A. Sen De, U. Sen,
[40] C. Srivastava, S. Mal, A. Sen De, U. Sen, arXiv:1911.02908.
[41] S. Kanjilal, C. Jebarathinam, T. Paterek, D. Home, arXiv:1912.09900.
[42] D. Yao, C. L. Ren, Phys. Rev. A, 103, 052207(2021).
[43] J. Zhu, M-J. Hu, G-C. Guo, C-F. Li, and Y-S. Zhang, arXiv: 2102.02550.
[44] S. Cheng, L. Liu, and M. J. W. Hall, arXiv: 2102.11574.
[45] M. J. Hu, Z. Y. Zhou, X. M. Hu, C. F. Li, G. C. Guo, and Y. S. Zhang, npj Quantum Inform. 4, 63 (2018).
[46] M. Schiavon, L. Calderaro, M. Pittaluga, G. Vallone, and P. Villoresi, Quantum Sci. Technol. 2, 015010 (2017).
[47] T. Feng, C. Ren, Y. Tian, M. Luo, H. Shi, J. Chen, and X. Zhou, Phys. Rev. A 102, 032220 (2020).
[48] F. J. Curchod, M. Johansson, R. Augusiak, M. J. Hoban, P. Wittek, A. Acin, Phys. Rev. A 95, 020102(R) (2017).