Comments on D-Instantons in $c < 1$ Strings

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ABSTRACT

We suggest that the boundary cosmological constant $\zeta$ in $c < 1$ unitary string theory be regarded as the one-dimensional complex coordinate of the target space on which the boundaries of world-sheets can live. From this viewpoint we explicitly construct analogues of D-instantons which satisfy Polchinski’s “combinatorics of boundaries.” We further show that our operator formalism developed in the preceding articles is powerful in evaluating D-instanton effects, and also demonstrate for simple cases that these effects exactly coincide with the stringy nonperturbative effects found in the exact solutions of string equations.
Recent progress has revealed the vital role of D-branes in the nonperturbative aspects of string theory. For understanding nonperturbative dynamics which arises from the D-branes, it has been particularly important to consider the combinatorics of boundaries of world-sheets. In fact, Polchinski made an interesting observation sometime ago that the stringy non-perturbative effects of the form $e^{-C/g}$ ($g$ is the coupling constant of closed strings) could be explained in terms of the combinatorics of boundaries in a target space with D-instanton background. The point is that when summing up connected diagrams in such target space, one should take into account not only connected world-sheets, but also configurations where disconnected world-sheets are attached together at a D-instanton. Then in the weak coupling limit, the dominant contribution comes from the disk amplitude $-C/g$ for each world-sheet, which will be accumulated to the form $e^{-C/g}$ after summing over the number of attached world-sheets.

In order to investigate such “many-boundary systems,” however, it should be more natural and convenient to use quantum string fields which can create and annihilate the boundaries of world-sheets. Furthermore, if we can introduce the quantum fields that create loop boundaries of Dirichlet type, then the interactions and combinatorics of D-branes can be dealt with in a second-quantized way. Though for critical strings it seems difficult to fully carry out this program in string-field language at hand there might be a chance to do it for $c \leq 1$ unitary case, because almost all the dynamical variables can be gauged away with 2D diffeomorphism, leaving the external lines only zero-modes (the positions). This implies that the string field theory could reduce to a local field theory in this case. We will show that actually the string field theory for $c = 1 - 6/p(p + 1)$ ($p = 2, 3, ...$) reduces to a chiral 2D conformal field theory by identifying the target-space coordinate with the so-called boundary cosmological constant $\zeta$. In particular, the solitons $D_{ab}$ constructed by the present authors (to be described below) are interpreted as analogues of D-instantons and shown to satisfy combinatorics similar to that of Polchinski.

We start our discussion with recalling that noncritical $c = 1$ string theory is a $D = 2$ critical string theory with linear dilaton background [7]: $G_{\mu\nu}(X) = \eta_{\mu\nu} = \text{diag}(-1, +1)$, $B_{\mu\nu}(X) = 0$ and $\Phi(X) = (Q/2)X^1$, where $X^\mu$ ($\mu = 0, 1$) are coordinates of target space, and $G_{\mu\nu}(X)$, $B_{\mu\nu}(X)$ and $\Phi(X)$ are, respectively, the background metric, antisymmetric 2-tensor and dilaton. Upon requiring the NL$\sigma$ model with this background to be invariant under the Weyl transformation as well as the 2D diffeomorphism, the value of $Q$ is uniquely determined up to sign (which can be absorbed into a redefinition of $X^1$) as $Q = \sqrt{2(26 - D)/3\alpha'} = 4/\sqrt{\alpha'}$, and

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1 An attempt in critical string field theory can be found in [4], where the interaction between a D-brane and a closed string field are investigated by coupling the string field to the boundary state.
this solution has no higher \( \alpha' \) corrections since the system is essentially Gaussian (Hereafter we take the CFT unit, \( \alpha' = 2 \)). The NL\( \sigma \)-model action with this linear dilaton term has exactly the same form with that of Liouville gravity coupled to \( c = 1 \) conformal matter if a spatial coordinate is regarded as Liouville field, \( \phi = X^1 \):

\[
S[X^\mu] = \int d^2\sigma \sqrt{\hat{g}} \left\{ \frac{1}{8\pi} \hat{g}^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu + \frac{\hat{R}}{8\pi} Q^\mu X_\mu \right\}, \tag{1}
\]

with \( Q^\mu = (0, Q) \). (Here we have dropped the boundary term, or taken the world-sheet metric such that its geodesic curvature vanishes.) In fact, this is obtained \cite{8,9} if we impose only the 2D diffeomorphism on a NL\( \sigma \) model with a scalar field \( X^0 \) (with negative metric) and express the world-sheet metric as \( g_{ab} = e^\phi \hat{g}_{ab} \) which fluctuates around a fixed metric \( \hat{g}_{ab} \). Furthermore, the \( c < 1 \) case can be realized from \( c = 1 \) (\( D = 2 \)) by making a Lorentz boost on \( (X^0, X^1) \rightarrow (X^0 \cosh \omega - X^1 \sinh \omega, -X^0 \sinh \omega + X^1 \cosh \omega) \) as well as on \( Q^\mu \) with hyperbolic angle \( e^\omega = (\sqrt{1-c} + \sqrt{25-c})/2\sqrt{6} \) \cite{10}, and the corresponding NL\( \sigma \)-model action with vanishing cosmological constant would be (after the Wick rotation \( X = iX^0 \)):

\[
S[\phi, X] = \int d^2\sigma \sqrt{\hat{g}} \left\{ \frac{1}{8\pi} \hat{g}^{ab} (\partial_a \phi \partial_b \phi + \partial_a X \partial_b X) + \frac{\hat{R}}{4\pi} \left( \frac{Q}{2} \phi + i\alpha_0 X \right) \right\}, \tag{2}
\]

where \( c = 1 - 12\alpha_0^2 \) and now \( Q = \sqrt{(25-c)/3} \). Note that after the Wick rotation it has an invariance under the shift \( X \rightarrow X + 2\pi/\alpha_0 \) (including the case when boundaries exist).

In 2D gravity we can introduce the macroscopic-loop operator \( \hat{\Psi}(l) \) \cite{11} that is defined effectively as creating a loop boundary of length \( l \) on the random surface. Its Laplace transform is defined with the boundary cosmological constant \( \zeta \) as \( \hat{\Psi}(\zeta) \equiv \int_0^\infty dl \ e^{-l\zeta} \hat{\Psi}(l) \), and when \( c = 1 - 6/p(p+1) \) it is expanded around \( \zeta = \infty \) (equivalently \( l = 0 \)) as

\[
\hat{\Psi}(\zeta) = \frac{1}{p} \sum_n \mathcal{O}_n \zeta^{-n/p-1}, \tag{3}
\]

where \( \mathcal{O}_n \)'s are scaling operators and have the following form \cite{8,9}: 

\[
\mathcal{O}_n = \int d^2\sigma \sqrt{\hat{g}} \ e^{\beta_n \phi(\sigma)} \ e^{i\alpha_n X(\sigma)}, \tag{4}
\]

with \( \beta_n = -\alpha_0(n-2p-1) \) and \( \alpha_n = -\alpha_0(n-1) \) \cite{8}. Note that these operators are also invariant under the shift \( X \rightarrow X + 2\pi/\alpha_0 \). This implies that the external on-shell states effectively live on a target space with the \( X \)-direction compactified with radius \( 1/\alpha_0 \), while the bulk of the world-sheet lives whole in the 2D target space. This is a specialty of the discrete unitary series.

Now we are going to argue that this \( \zeta \) could be regarded as a coordinate of target space. One justification comes from the 2D-gravity side. In fact, in the matrix-model regularization
of this system, the $\Psi(\zeta)$ is obtained as the continuum limit of the resolvent of a random matrix $M$, $f(P) = \text{tr}(P - M)^{-1}$, by setting $P = P_c e^{\zeta a}$ with $P_c$ the convergence radius of $\langle f(P) \rangle$ around $P = \infty$ and $a$ the lattice spacing. On the other hand, according to our experience in any matrix models which generate random surfaces in $c$-dimensional target space (see [12] for example), the argument of the resolvent is always identified with an extra coordinate of target space (thus totally $D = c + 1$-dimensions). This motivates us to identify $\zeta$ with a coordinate of target space.

This idea can be elaborated more if we stand on the string-theory side. In fact, substituting eq. (4) into (3), we have

$$\Psi(\zeta) \sim \int d^2\sigma \sqrt{g} \sum_n e^{-na_0(\phi(\sigma) + X(\sigma))} e^{-in\alpha_0(\phi(\sigma))} \zeta^{-n/p}$$

$$\sim \int d^2\sigma \sqrt{g} \zeta^{1/p - \frac{1}{\alpha_0(\phi(\sigma) + X(\sigma))}}.$$  \hfill (5)

Here we have used $e^{-na_0(\phi(\sigma) + iX(\sigma))} e^{-ma_0(\phi(\sigma))} \sim e^{-(n+m)a_0(\phi(\sigma) + iX(\sigma))}$, since the singularities from contact interaction both for $\phi$ and $X$ cancel each other. We also neglected possible coefficients on the right hand side of (4) which would not give essential changes to the singularity behavior in (3). Noticing that the whole expression includes only $\zeta$ (not $\bar{\zeta}$),\footnote{This is also a specialty of $c < 1$. For $c = 1$ there will appear both $\zeta$ and $\bar{\zeta}$ as well as extra discrete states. The relationship between $c = 1$ and $c < 1$ is reminiscent of the one between AdS$_3$ and AdS$_2$ in the AdS/CFT correspondence [13].} and also that the integrand is a delta function in a complex plane, we thus see that the role of $\Psi(\zeta)$ is substantially to create a loop boundary in one-dimensional complex target space with the (exponentiated) coordinate $\zeta$. Furthermore, due to the form $\zeta^{1/p}$, any observables should be invariant under the rotation $\zeta \rightarrow e^{2\pi ip} \zeta$ which corresponds to the shift $X \rightarrow X - 2\pi/\alpha_0$. Although the above arguments might require to be refined more, we may conclude that the on-shell boundaries live on a complex one-dimensional target space with a chiral coordinate $\zeta$ with periodicity $\zeta \rightarrow e^{2\pi ip} \zeta$, and the role of the operator $\Psi(\zeta)$ is to pin the world-sheet at the point $\zeta^{1/p} = e^{-\alpha_0(\phi + iX)}$.

An operator formalism of $c < 1$ string field theory is developed in [13, 14]. There the string field $\Psi(\zeta)$ is expressed as the first derivative of a scalar field $\varphi_0(\zeta)$, and has the following mode expansion under the $\mathbb{Z}_p$-twisted vacuum $|\hat{\sigma}\rangle$: $\Psi(\zeta) \equiv \partial \varphi_0(\zeta) = (1/p) \sum_{n \in \mathbb{Z}} \alpha_n \zeta^{-n/p-1}$ with $[\alpha_n, \alpha_m] = n \delta_{n+m,0}$ and $\alpha_n |\hat{\sigma}\rangle = 0 \ (n \geq 0)$. It thus raises a monodromy among $p$ scalar fields $\partial \varphi_a(\zeta)$ ($a = 0, 1, \ldots, p - 1$) as $\partial \varphi_a(e^{2\pi i} \zeta) = \partial \varphi_{[a+1]}(\zeta)$ with $[a] \equiv a \ (\text{mod } p)$. The correlation functions (generally disconnected) are then given by

$$\langle \Psi(\zeta_1) \cdots \Psi(\zeta_n) \rangle \equiv \int_0^\infty dl_1 \cdots dl_n e^{-l_1 \zeta_1 - \cdots - l_n \zeta_n} \langle \bar{\Psi}(l_1) \cdots \bar{\Psi}(l_n) \rangle.$$
\[
\langle -\frac{B}{g} | : \partial \phi_0(\zeta_1) \cdots \partial \phi_0(\zeta_n) : | \Phi \rangle \rangle,
\]
and the connected correlation functions are obtained as their cumulants. Here the normal ordering respects \( SL(2, \mathbb{C}) \)-invariant vacuum, and the state \( \langle -\frac{B}{g} | = \langle \hat{\sigma} | \exp \left\{ -\left(\frac{1}{g}\right) \sum_{n=1}^{2p+1} B_n \alpha_n \right\} \) characterizes the theory and corresponds schematically to the NL\(\sigma\)-model action \( S = \sum_{n=1}^{2p+1} B_n O_n \).

The state \( | \Phi \rangle \) satisfies the \( W_{1+\infty} \) constraints [14, 15, 16, 17]:
\[
W_n^k | \Phi \rangle = 0 \quad (k \geq 1, \ n \geq -k + 1).
\]
The generators of the \( W_{1+\infty} \) algebra [18] are given by the mode expansion
\[
W^k(\zeta) \equiv \sum_{n \in \mathbb{Z}} W_n^k \zeta^{-n-k} = \sum_{a=0}^{p-1} : \bar{c}_a(\zeta) \partial^{-1}_\zeta c_a(\zeta) : ,
\]
Here \( c_a(\zeta) \) and \( \bar{c}_a(\zeta) \) \( (a = 0, 1, \ldots, p - 1) \) are fermions constructed from the scalars by bosonization:
\[
\bar{c}_a(\zeta) = K_a : e^{\bar{\phi}_a(\zeta)} :, \quad c_a(\zeta) = K_a : e^{-\phi_a(\zeta)} :,
\]
where \( K_a \) is a cocycle factor that ensures the correct anticommutation relations between different indices \( a \neq b \), and all the operators are again normal-ordered with respect to \( SL(2, \mathbb{C}) \) invariant vacuum. In addition to the \( W_{1+\infty} \) constraints, we further require that \( | \Phi \rangle \) be a decomposable state[^3].

Now we try to construct the D-instanton operator in this operator formalism. By definition it should be expressed as an operator that specifies a point \( \zeta \) at which boundaries of world-sheets are glued together (see Fig. 1). Since there exists a redundancy reflecting the invariance under one-dimensional diffeomorphism along each boundary, this should be gauge-fixed if one would like to count only independent configurations. This can be simply done by creating strings at \( \zeta \) not with \( \partial \phi_0(\zeta) \) but with \( -\phi_0(\zeta) \), since for the latter the redundancy that is proportional to loop length is automatically divided out: \( -\phi_0(\zeta) \sim \int_0^\infty d\ell e^{-l\zeta} \left(1/l\right) \tilde{\Psi}(l) \). Taking also into account the monodromy of scalar fields, we should equally treat all the \( -\phi_a(\zeta) 's \) as such gauge-fixed string fields. Thus, after summing over the number of the boundaries glued at \( \zeta \), we have \( \sum_n (q^n/n!) \left\{ -\phi_a(\zeta) \right\}^n = \exp \left\{ -q_a \phi_a(\zeta) \right\} \), where \( 1/n! \) is a statistical factor and we assume

[^3]: A state \( | \Phi \rangle \) is called decomposable if it is written as \( | \Phi \rangle = e^H | \hat{\sigma} \rangle \), where \( H \) is a bilinear form of the fermions. This is equivalent to the statement \( \tau(x) = \langle \hat{\sigma} | \exp\{\sum_{n=1}^\infty x_n \alpha_n\} | \Phi \rangle \) is a \( \tau \) function of the KP hierarchy [14]. It is proved in [17] that this set of conditions (\( W_{1+\infty} \) constraints and decomposability) is equivalent to the Douglas equation [20], \( [P, Q] = 1 \).
A D-instanton is here.

Figure 1: Geometrical meaning of the D-instanton operator. All the points on the boundary are mapped to a single point $\zeta$ in the target space.

that a boundary of a connected world-sheet is accompanied with weight $q_a$ for each $-\varphi_a(\zeta)$. Furthermore, we need to make an integration over the collective coordinate $\zeta$ of D-instanton. The path of the integration surrounds $\zeta = \infty$ $p$ times, which corresponds to the integration over $X$ with $0 \leq X \leq 2\pi/\alpha_0$. We are thus lead to the following partition function of $N$ D-instantons:

$$Z_N = \oint d\zeta_1 \cdots \oint d\zeta_N \left\langle \prod_{i=1}^{N} e^{-\vec{q} \cdot \vec{\varphi}(\zeta)} \right\rangle,$$

(10)

where $\vec{q} \cdot \vec{\varphi}(\zeta_i) = \sum_{a=0}^{p-1} q_a \varphi_a(\zeta_i)$ and $\zeta_i$ represents the collective coordinate of the $i$-th D-instanton. Note that this equation actually realizes Polchinski’s combinatorics of boundaries $Z$. After summing over $N$, we obtain the grand-canonical partition function:

$$Z = \sum_{N=0}^{\infty} \frac{\theta^N}{N!} Z_N = \left\langle \exp \left\{ \theta \oint d\zeta e^{-\vec{q} \cdot \vec{\varphi}(\zeta)} \right\} \right\rangle,$$

(11)

where $1/N!$ is a statistical factor and $\theta$ is a fugacity. Equations (10), (11) imply that our D-instanton can be identified with an object which locally couples to the scalar fields $\varphi_a(\zeta)$ at the position $\zeta$ with “charges” $q_a$.

This is not the end of story. In fact, the charges $q_a$ in (11) cannot take arbitrary values, since in order for the expression to give a correct background, the state $\exp \left\{ \theta \oint d\zeta e^{-\vec{q} \cdot \vec{\varphi}(\zeta)} \right\} \Phi$
must satisfy the $W_{1+\infty}$ constraints when $|\Phi\rangle$ does. This is equivalent to the condition that the operator $\int p \, d\zeta \, e^{-\bar{\Phi}(\zeta)}$ should commute with any generators of the $W_{1+\infty}$ algebra, and it is shown in [3] that, for this condition to hold, only two charges among $q_a$ can take nonvanishing values with $\pm 1$. This is actually the combination for which the operator can be expressed as a fermion bilinear, and has exactly the same form with the operator $D_{ab}$ that was introduced in [5, 6]:

$$D_{ab} = \oint p \, d\zeta \, e^{-\bar{\Phi}(\zeta)} \sim \oint p \, d\zeta \, e^{\varphi_a(\zeta)-\varphi_b(\zeta)} \quad (a \neq b). \quad (12)$$

Furthermore, since there are various ways to choose the pairs $a$ and $b$, we can introduce the corresponding fugacities $\theta_{ab}$, and finally have the following form instead of (11):

$$Z = \prod_{a \neq b} \exp \{\theta_{ab} D_{ab}\} = \exp \left\{ \prod_{a \neq b} \left[ \theta_{ab} \oint p \, d\zeta \, e^{\varphi_a(\zeta)-\varphi_b(\zeta)} \right] \right\}. \quad (13)$$

Thus, we conclude that the $D_{ab}$ is the operator that creates a $D$-instanton and the $\prod_{a \neq b} e^{\theta_{ab} D_{ab}}$ the creation operator of multi $D$-instantons. Notice that in [3] the form of the multi $D$-instanton operator was originally determined by requiring that the decomposability be preserved when acting on $|\Phi\rangle$.

In the rest of this article, we briefly explain how the nonperturbative effects due to $D$-instantons can be calculated in our operator formalism, and show that they coincide with those effects that were found in exact solutions of string equations. Most of the argument here will closely follow [3, 4], and to make the comparison most easily, we mainly consider the string susceptibility in the $D$-instanton background:

$$u(t, g, \theta) \equiv g^2 \partial_t^2 \log Z$$

$$= g^2 \partial_t^2 \log \left\langle -\frac{B}{g} \left| \prod_{a \neq b} e^{\theta_{ab} D_{ab}} \right| \Phi \right\rangle, \quad (14)$$

where $t$ and $g$ are the cosmological and string coupling constant, respectively, and we have set $B_1 = t$, $B_{2p+1} = -4p/((p+1)(2p+1))$, $B_n = 0 \ (n \neq 1, 2p + 1)$. The free energy $\log \left\langle -\frac{B}{g} \left| \prod_{a \neq b} e^{\theta_{ab} D_{ab}} \right| \Phi \right\rangle$ in (14) can be evaluated around $\theta = 0$ by rewriting it as follows:

$$\log \left\langle -\frac{B}{g} \prod_{a \neq b} e^{\theta_{ab} D_{ab}} \right| \Phi \right\rangle = \log \left\langle -\frac{B}{g} \right| \Phi \right\rangle + \log \left\langle \prod_{a \neq b} e^{\theta_{ab} D_{ab}} \right\rangle, \quad (15)$$

where

$$\left\langle \prod_{a \neq b} e^{\theta_{ab} D_{ab}} \right\rangle = \frac{\left\langle -\frac{B}{g} \prod_{a \neq b} e^{\theta_{ab} D_{ab}} \right| \Phi \right\rangle}{\left\langle -\frac{B}{g} \right| \Phi \right\rangle}. \quad (16)$$
Thus, expanding $e^{\theta_{ab} D_{ab}}$ in (15) with respect to $\theta_{ab}$ and picking up only the first-order term, we get the string susceptibility which includes one-D-instanton effect:

$$u(t, g, \theta) = u_{\text{pert}}(t, g) + g^2 \sum_{a \neq b} \theta_{ab} \partial_t^2 \langle D_{ab} \rangle + O(\theta_{ab}^2).$$  (17)

Here $u_{\text{pert}}(t, g)$ is the string susceptibility which is perturbatively evaluated at $\theta_{ab} = 0$. We will make a comment on multi-D-instanton effects later.

The expectation value of $D_{ab}$ in (17) can be represented as

$$\langle D_{ab} \rangle = \int \frac{d\zeta}{2\pi i} \langle e^{\varphi_a(\zeta) - \varphi_b(\zeta)} \rangle$$

$$= \int \frac{d\zeta}{2\pi i} \exp \left\{ \langle e^{\varphi_a(\zeta) - \varphi_b(\zeta) - 1} \rangle \right\}$$

$$= \int \frac{d\zeta}{2\pi i} \exp \left\{ \langle \varphi_a(\zeta) - \varphi_b(\zeta) \rangle + \frac{1}{2} \langle (\varphi_a(\zeta) - \varphi_b(\zeta))^2 \rangle_c + \cdots \right\}.$$  (18)

Since a connected $n$-point function has the following expansion in $g$:

$$\langle \varphi_{a_1}(\zeta_1) \cdots \varphi_{a_n}(\zeta_n) \rangle_c = \sum_{h=0}^{\infty} \langle \varphi_{a_1}(\zeta_1) \cdots \varphi_{a_n}(\zeta_n) \rangle_{c(h)} g^{-2h+n},$$  (19)

we know that in the weak coupling limit, leading contributions to the exponent come from spherical topology ($h = 0$):

$$\langle D_{ab} \rangle = \int \frac{d\zeta}{2\pi i} e^{(1/g) \Gamma_{ab}(\zeta) + (1/2) K_{ab}(\zeta) + O(g)}$$  (20)

with $\Gamma_{ab}(\zeta) \equiv \langle \varphi_a(\zeta) - \varphi_b(\zeta) \rangle^{(0)}$ and $K_{ab}(\zeta) \equiv \langle (\varphi_a(\zeta) - \varphi_b(\zeta))^2 \rangle^{(0)}_c$. These functions $\Gamma_{ab}(\zeta)$ and $K_{ab}(\zeta)$ can be calculated by integrating the disk and cylinder amplitudes ($\langle \Psi(\zeta) \rangle^{(0)}_c$ and $\langle \Psi(\zeta_1) \Psi(\zeta_2) \rangle^{(0)}_c$) \[11\] followed by analytic continuation. For example, $\Gamma_{ab}(\zeta)$ can be evaluated as

$$\Gamma_{ab}(\zeta) = \langle \varphi_a(\zeta) \rangle^{(0)}_{\zeta \rightarrow e^{2\pi i a} \zeta} - \langle \varphi_b(\zeta) \rangle^{(0)}_{\zeta \rightarrow e^{2\pi i b} \zeta}$$

$$= \int \frac{d\zeta'}{2\pi i} \langle \Psi(\zeta') \rangle^{(0)}_{\zeta \rightarrow e^{2\pi i a} \zeta} - \int \frac{d\zeta'}{2\pi i} \langle \Psi(\zeta') \rangle^{(0)}_{\zeta \rightarrow e^{2\pi i b} \zeta}.$$  (21)

Thus, the leading contribution in the weak coupling limit can be calculated by applying the saddle point method in the complex $\zeta$ plane. To do so, it is convenient to introduce a new coordinate $s$ which is defined by

$$s(\zeta) = \frac{1}{\sqrt{t}} \left( \zeta + \sqrt{\zeta^2 - t} \right),$$  (22)
for which $\Gamma_{ab}(s)$ is expressed as

$$
\Gamma_{ab}(s) = \gamma_p \left[ \frac{1}{r+1} \left\{ (\omega^a - \omega^b)s^{r+1} + (\omega^{-a} - \omega^{-b})s^{-(r+1)} \right\} 
- \frac{1}{r-1} \left\{ (\omega^a - \omega^b)s^{r-1} + (\omega^{-a} - \omega^{-b})s^{-(r-1)} \right\} \right],
$$

(23)

where $\gamma_p = 2^{-1/p} t^{(2p+1)/2p}/(p+1)$, $\omega = e^{2\pi i/p}$ and $r = (p+1)/p$. The saddle points are found at $s_0 = e^{(np-a-b)\pi i/(p+1)}$, which gives

$$
\Gamma_{ab}(s_0) = \frac{8p}{2^{1/p} \cdot (2p+1)} \frac{\pi^{2n+1}}{p!} \sin \left( \frac{a+b+n}{p+1} \pi \right) \sin \left( \frac{a-b}{p} \pi \right). 
$$

(24)

Here $n$ takes integers which satisfy $\Gamma_{ab}(s_0) < 0$ (see [3, 8] for detailed investigations). They give the leading nonperturbative effects that are proportional to $e^{(1/g)\Gamma_{ab}(s_0)}$. Coefficients can also be calculated explicitly by performing an integration around the saddle points (see [3, 8]).

Next, we compare these nonperturbative effects with those that were found in the exact solutions of string equations. In the $p = 2$ (pure gravity) case, the string equation is $4u^2 + (2g^2/3)\partial_t^2 u = t$ (Painlevé I equation) [21]. The leading nonperturbative effect can be calculated by expanding the equation around $u_{pert} = -\sqrt{t/2} + O(g^2)$ and is found to be $\Delta u \equiv u - u_{pert} \propto e^{C/g}$ with $C = -4\sqrt{6} t^{5/4}/5$ [3, 21, 22]. This exponent coincides with $\Gamma_{01}(s_0)$ with $n = 0, 1$ and $\Gamma_{10}(s_0)$ with $n = 3, 4$ in (24).

For the $p = 3$ (Ising) case, the string equation is $4u^2 + (3g^2/2)(\partial_t u)^2 + 3g^2 u \partial_t^2 u + (g^4/6)\partial_t^4 u = -t$ [23]. The leading nonperturbative effects are of the form $e^{C/g}$ with $C = -6\sqrt{6} t^{7/6}/(2^{1/3} \cdot 7)$ or $-12\sqrt{3} t^{7/6}/(2^{1/3} \cdot 7)$. The set of these exponents agrees with that of different negative values of $\Gamma_{ab}(s_0)$ in (24).

We finally consider the $p = 4$ (tricritical Ising) case. The string equations are now coupled differential equations for two unknown functions $u$ and $v$ [24]:

$$
\begin{align*}
0 &= 40u^3 + 40uv + 15g^2(\partial_t u)^2 + 10g^2 u \partial_t^2 u + 10g^2 \partial_t^2 v - 2g^4 \partial_t^4 u \\
5t &= -20u^4 + 40v^2 + 50g^2 u(\partial_t u)^2 + 20g^2 \partial_t u \partial_t v + 20g^2 w^2 \partial_t^2 u \\
&\quad - 20g^2 u \partial_t^2 v + 11g^4(\partial_t^2 u)^2 + 13g^4 \partial_t u \partial_t^3 u + 11g^4 u \partial_t^3 u + g^4 \partial_t^4 v.
\end{align*}
$$

(25)

(26)

The leading nonperturbative effects are of the form $e^{C/g}$ with $C = -2^{13/4}\sqrt{5} \pm \sqrt{5} t^{9/8}/9$ or $-2^{11/4}\sqrt{5} \pm \sqrt{5} t^{9/8}/9$. The set of these exponents completely accord with that of different negative values of $\Gamma_{ab}(s_0)$ in (24).

Moreover, we can also evaluate multi-D-instanton effects by using our formalism. As an example, the $p = 2$ (pure gravity) case was considered in detail in [6] and the string susceptibility
in the multi-D-instanton background was obtained as

\[ u(t, g, \theta) = -\frac{\sqrt{t}}{2} + 6 \theta_R \sqrt{g} t^{-1/8} e^{-4\sqrt{6} t^{3/4}/5g} \times (1 + \theta_R \sqrt{g} t^{-5/8} e^{4\sqrt{6} t^{3/4}/5g})^{-2}. \] (27)

Here we have neglected contributions from higher topologies, and \( \theta_R \) is the renormalized fugacity which absorbed an integration constant that arises when integrating cylinder amplitudes. This \( u(t, g, \theta) \) exactly reproduces a series of nonperturbative corrections in the string equation for pure gravity \[4\].

These examples confirm that the stringy nonperturbative effects found in the exact solutions of string equations can be interpreted as D-instanton effects.

In this short letter, we demonstrate that the conformal field theory that used to be a technical tool to compactly describe macroscopic-loop amplitudes, is actually the field theory that describes the target space where the boundaries of world-sheets live. We also explicitly construct analogues of D-instantons which satisfy Polchinski’s “combinatorics of boundaries.” It is surprising that our operator formalism can be naturally applied to this combinatorics. Moreover, we show that this formalism is powerful in evaluating the D-instanton effects and demonstrate for \( p = 2, 3, 4 \) that these effects coincide with the stringy nonperturbative effects found in the exact solutions of string equations. Finally we point out the similarities between noncritical strings and the Sine-Gordon theory (or more generally affine \( SU(p) \) Toda field theories) since for both the fundamental degrees of freedom are described by scalar fields \( \varphi_a \) corresponding to the fundamental weights (this is actually \( SU(p) \), not \( U(p) \), because \( \sum_{a=0}^{p-1} \partial \varphi_a = 0 \) under the \( W_{1+\infty} \) constraints), while the solitons are expressed by their exponentials in a combination associated with the roots, \( \exp(\varphi_a - \varphi_b) \).

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