Optimal Investment in an Illiquid Market with Search Frictions and Transaction Costs

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Abstract
We consider an optimal investment problem to maximize expected utility of the terminal wealth, in an illiquid market with search frictions and transaction costs. In the market model, an investor’s attempt of transaction is successful only at arrival times of a Poisson process, and the investor pays proportional transaction costs when the transaction is successful. We characterize the no-trade region describing the optimal trading strategy. Our asymptotic analysis implies that the effects of the transaction costs are more pronounced (more widening effect of the no-trade region and more diminishing effect of the value function) in the market with less search frictions.

Keywords Stochastic control · Optimal investment · Illiquidity · Transaction costs · Search frictions

1 Introduction

Understanding the effects of liquidity on optimal investment is one of the main topics in mathematical finance and financial economics. According to [52], stated simply, liquidity is the ease of trading a security. Various sources of illiquidity include exogenous transaction costs, search frictions such as the difficulty of locating a counterparty...
with whom to trade, and price impacts due to private information.\textsuperscript{1} This paper studies an optimal investment problem in the market model with two different types of illiquidity: search frictions and transaction costs.

Under the assumption of the perfect liquidity,\textsuperscript{2} the pioneering papers [41, 42] formulate the optimal investment problem (so-called Merton’s portfolio problem) with a geometric Brownian motion and a CRRA (constant relative risk aversion) investor, and show that the optimal solution is to keep the constant fraction of wealth invested in the risky asset. With the same assumption of perfect liquidity, more general stochastic processes and utility functions have been considered to obtain more general characterizations of the optimal investment strategies (e.g., [28, 30, 31, 33, 34]).

In the optimal investment problems, the assumption of perfect liquidity can be relaxed by considering search frictions in the market. As Table 1 in [3] shows, many financial assets are illiquid in the sense that it is difficult to find a counterparty who is willing to trade. One way to incorporate this type of illiquidity, search frictions, into the optimal investment problem is to impose some restrictions on trade times. In the classical Merton framework, [48] considers an investor who is allowed to change portfolio only at times which are multiple of a constant $h > 0$, and [3, 40, 49] assume that an illiquid asset can only be traded on the arrival of a randomly occurring trading opportunity that is represented by jump times of a Poisson process. [13, 44] consider an optimal investment/consumption problem under the assumption that the asset price is observed only at the random trade times. [22] complicates the model by using random intensity of trade times, regime-switching, and liquidity shocks. In [17], a risky asset can be traded only on deterministic time intervals and the investor pays proportional transaction costs.

Transaction costs (such as order processing fees or transaction taxes) are another source of market illiquidity, and the optimal investment problems with transaction costs have been extensively studied in mathematical finance community. [19, 39, 51] study the model in [42] with the assumption that proportional transaction costs are levied on each transaction, and show (with different level of mathematical rigorosity) that it is optimal to keep the fraction of wealth invested in the risky asset in an interval so called no-trade region. The boundaries of the no-trade region are characterized in terms of the free-boundaries determined by the HJB (Hamilton-Jacobi-Bellman) equation of the control problem. The models with transaction costs and multiple risky assets have been studied (e.g., [1, 9, 38, 43] for costs on all assets and [7, 11, 16, 27] for costs on only one assets) to characterize the value function or no-trade region. More general stochastic processes have been also considered in the framework of optimal investment with transaction costs (e.g., [5, 14, 15]).

In this paper, we merge the aforementioned frameworks and analyze an optimal investment problem in a market model with both search frictions and transaction costs. We consider the classical Merton’s portfolio problem with a log-utility investor, whose goal is to maximize the expected utility of wealth at the terminal time $T > 0$. We

\textsuperscript{1} For instance, [4, 12, 20, 37] study how asymmetric information effects price impact and optimal trading strategy in equilibrium. [2, 23, 46, 47] consider optimal order execution problems with exogenously given price impacts.

\textsuperscript{2} Here, perfect liquidity assumption means that assets can be traded in any quantity and at any moment in time, without any transaction costs.
assume that the investor’s attempt of trading is successful only when a Poisson process with intensity $\lambda$ jumps (as in [40, 49]), and the investor needs to pay proportional transaction costs for successful trading (as in [19, 51]). We show that there exists a unique classical solution of the HJB equation and provide the standard verification argument. As in the aforementioned models for transaction costs, the optimal trading strategy is characterized by a no-trade region: if the investor can trade at time $t \in [0, T)$, then there are two constants $0 \leq y(t) \leq \bar{y}(t) \leq 1$ such that the investor should minimally trade to keep the fraction of wealth invested in the risky asset inside of the interval $[y(t), \bar{y}(t)]$. Strict concavity of the value function uniquely determines the boundaries $y(t)$ and $\bar{y}(t)$ of the no-trade region.

We provide asymptotic expansions of the value function and the no-trade boundaries for small transaction costs. The coefficients of the expansions are not explicit, as they are expressed in terms of solutions of some partial differential equations. For the transaction cost parameter $\epsilon$, in our model with the search frictions, both the width of the no-trade region and the decrease of the value function are the order of $\epsilon^3$. We analyze the coefficients and find that the effects of the transaction costs are more pronounced (more widening effect of the no-trade region and more diminishing effect of the value function) in the market with less search frictions.

Our modeling assumption of the Poisson arrivals of the successful trading times has a similar taste to the assumptions for the liquidity provision in some models on limit order markets. [21, 50] derive equilibrium order placement strategies in a limit order market model, where patient/impatient agents arrive at the market according to a Poisson process. [26] solves an optimal market-making model with execution/inventory risks, where the limit order book is modeled by a Markov chain that jumps according to a stochastic clock described by a Poisson process. [36] studies Merton’s portfolio problem where the investor has the choice between market orders (immediate transactions) and limit orders (Poisson arrivals of execution times). In the model, trading with limit orders has a flavor of negative proportional transaction costs.

The remainder of the paper is organized as follows. Section 2 describes the model. In Sect. 3, we provide the verification argument and the strict concavity of the value function. In Sect. 4, we characterize the optimal trading strategy in terms of the no-trade region, and present some properties of the no-trade region. In Sect. 5, we provide asymptotic analysis for small transaction costs. Section 6 summarizes this paper. Proof of the technical lemmas can be found in Appendix.

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3 In the similar model set up without the search frictions, it is well known that the width of the no-trade region is the order of $\epsilon^{\frac{1}{3}}$ and the decrease of the value function is the order of $\epsilon^{\frac{2}{3}}$ (e.g., see [7, 8, 10, 11, 25, 29, 45, 51]).

4 To be more specific, let $0 < \lambda_1 < \lambda_2$ and consider two markets $\mathcal{M}_1$ and $\mathcal{M}_2$ with search friction parameters $\lambda_1$ and $\lambda_2$, respectively. In words, $\mathcal{M}_2$ has less search frictions than $\mathcal{M}_1$. Our result implies that if we increase the transaction costs, the widening speed of the no-trade region in $\mathcal{M}_2$ is faster than that in $\mathcal{M}_1$. Similarly, if we increase the transaction costs, the diminishing speed of the optimal value in $\mathcal{M}_2$ is faster than that in $\mathcal{M}_1$. 
2 The Model

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) be a filtered probability space satisfying the usual conditions. Under the filtration, we assume that \((B_t)_{t \geq 0}\) is a standard Brownian motion and \((P_t)_{t \geq 0}\) is a Poisson process with intensity \(\lambda > 0\). Then \((B_t)_{t \geq 0}\) and \((P_t)_{t \geq 0}\) are independent, since they are Levy processes with zero quadratic covariation.

We consider a market that has two different types of illiquidity: (i) the investor’s attempt of trading is successful only when the Poisson process \((P_t)_{t \geq 0}\) jumps,\(^5\) and (ii) the investor needs to pay proportional transaction costs for successful trading.

To be more specific, we consider a financial market consisting of a bond and a stock, whose price processes \((S_t^{(0)})_{t \geq 0}\) and \((S_t)_{t \geq 0}\) are given by the following stochastic differential equations (SDEs):

\[
dS_t^{(0)} = S_t^{(0)} r dt \\
dS_t = S_t (\mu dt + \sigma dB_t),
\]

(2.1) (2.2)

where \(\mu, r, \sigma, S_t^{(0)}, S_0\) are constants and \(\sigma, S_t^{(0)}, S_0\) are assumed to be strictly positive.

The proportional transaction costs are described by two constants \(\epsilon \in [0, 1)\) and \((1 + \bar{\epsilon})S_t\) for purchasing one share of the stock, and pays \((1 + \bar{\epsilon})S_t\) for selling one share of the stock.

Let \(W_t^{(1)}\) be the amount of wealth invested in the stock and \(W_t^{(0)}\) be the amount of wealth in the bond, at time \(t \geq 0\). If the investor tries to obtain the stock worth \(M_s\) at time \(s \in [0, t]\), then

\[
W_t^{(1)} = W_0^{(1)} + \int_0^t W_s^{(1)} (\mu ds + \sigma dB_s) + \int_0^t M_s dP_s,
\]

\[
W_t^{(0)} = W_0^{(0)} + \int_0^t W_s^{(0)} r ds + \int_0^t (1 - \epsilon)M_s^- - (1 + \bar{\epsilon})M_s^+ dP_s,
\]

(2.3)

where we use notation \(x^\pm = \max\{0, \pm x\}\) for \(x \in \mathbb{R}\), and the constants \(W_0^{(1)} \geq 0\) and \(W_0^{(0)} \geq 0\) are given model parameters that represent the initial position of the investor. We assume that the initial total wealth, \(w_0 := w_0^{(1)} + w_0^{(0)}\), is strictly positive: \(w_0 > 0\).

The trading strategy \((M_t)_{t \geq 0}\) is called admissible if it is a predictable process and the corresponding total wealth process \(W := W^{(0)} + W^{(1)}\) is nonnegative all the time. This nonnegativity condition is equivalent to \(W_t^{(0)} \geq 0\) and \(W_t^{(1)} \geq 0\) for all \(t \geq 0\), because the rebalancing times are discrete.\(^6\) Therefore, an admissible strategy \(M\) satisfies

\[
-W_{t^-} \leq M_t \leq \frac{W_{t^-}}{1 + \bar{\epsilon}}, \quad t \geq 0.
\]

(2.4)

\(^5\) Therefore, bigger \(\lambda\) implies more frequent trading opportunities (less search frictions), on average.

\(^6\) Indeed, for \(s > t\) and \(A = \{W_t^{(1)} < 0, W_t^{(0)} \geq 0\}\) (or \(A = \{W_t^{(1)} \geq 0, W_t^{(0)} < 0\}\)), we observe that \(\mathbb{P}(W_s < 0 \mid A) \geq \mathbb{P}(P_t = P_s\) and \(W_s < 0 \mid A) > 0\).
We observe that $W_t > 0$ for $t \geq 0$ almost surely, as long as $M$ satisfies (2.4).\footnote{To check this, let $\tau_n := \inf\{t \geq 0 : P_t = n\}$ and $\tau_0 = 0$. For $\tau_n \leq t < \tau_{n+1}$, the dynamics (2.3) produce $W_t = W_{\tau_n} e^{(t-\tau_n)\sigma + \int_{\tau_n}^t \lambda dB_s} + W_{\tau_n} (t-\tau_n)^{\frac{\alpha^2}{2}} e^{(t-\tau_n)\sigma + \int_{\tau_n}^t \lambda dB_s}$. If $W_{\tau_n} > 0$, $W_{\tau_n}^{(0)} > 0$ and $W_{\tau_n}^{(1)} > 0$ hold. By this way, we can inductively show that $W_{\tau_n} > 0$, $W_{\tau_n}^{(0)} > 0$ and $W_{\tau_n}^{(1)} > 0$ for all $n$, almost surely. Now the expression of $W_t$ above implies that $W_t > 0$ for all $t \geq 0$, almost surely.}

For an admissible strategy $M$ and the corresponding solutions $W^{(1)}$ and $W^{(0)}$ of the SDEs (2.3), let $X_t := W_t^{(1)}/W_t$ be the fraction of the total wealth invested in the stock market at time $t$. Then, the inequalities in (2.4) imply that $0 \leq X_t \leq 1$ and the application of Ito’s formula produces the following SDE for $(W, X)$:

\[
\begin{align*}
\frac{dW_t}{W_t} &= (r - \bar{\epsilon} M_t^+ + \bar{\epsilon} M_t^-) dt + \sigma X_t \frac{dW_t}{W_t}, \\
\frac{dX_t}{X_t} &= X_t \left( \left( \mu - \frac{\sigma^2}{2} X_t \right) dt + \sigma dW_t \right) + \frac{1}{W_t} \left( \bar{\epsilon} \left( \bar{M}_t^+ - \bar{M}_t^- \right) - \bar{\epsilon} X_t \right) dt,
\end{align*}
\]

(2.5)

where the initial conditions are $W_0 = w_0$ and $X_0 = x_0 := w_0^{(1)}/w_0$. The nonnegativity of $w_0^{(0)}$ and $w_0^{(1)}$ implies $0 \leq x_0 \leq 1$.

Let a constant $T > 0$ represent the terminal time of trading. We assume that the investor’s goal is to maximize the expected log-utility of the total wealth at the terminal time. That is, we analyze the following optimal investment problem:

\[
\sup_{(M_t)_{t \in [0, T]}} \mathbb{E}[\ln(W_T)],
\]

(2.6)

where the supremum is taken over all admissible trading strategies.

Remark 2.1 If $\lambda = \infty$ and $\bar{\epsilon} = \bar{\epsilon} = 0$, then our model becomes the classical Merton’s portfolio problem. The case of $\bar{\epsilon} = \bar{\epsilon} = 0$ and $\lambda < \infty$ is studied in [40].

3 Value Function

Let $V$ be the value function of the control problem (2.6):

\[
V(t, x, w) = \sup_{(M_t)_{t \in [0, T]}} \mathbb{E}[\ln(W_T)|F_t] |_{(X_t, W_t) = (x, w)},
\]

(3.1)

As usual, the scaling property of the wealth process and the property of log-utility enable us to conjecture the form of the value function as

\[
V(t, x, w) = \ln(w) + v(t, x)
\]
for a function $v$ (we verify this in Theorem 3.5). Then, the HJB equation for (3.1) becomes

$$
\begin{align*}
0 &= v(T, x), \\
0 &= v_t(t, x) + (\mu - r)x + r - \frac{1}{2}\sigma^2 x^2 - \lambda v(t, x) \\
&\quad + \lambda \sup_{y \in [0, 1]} \left( v(t, y) - \ln \left( \frac{1 + \bar{\epsilon} y}{1 + \bar{\epsilon} x} \right) 1_{\{x < y\}} - \ln \left( \frac{1 - \bar{\epsilon} y}{1 - \bar{\epsilon} x} \right) 1_{\{x > y\}} \right),
\end{align*}
$$

(3.2)

where $v_t, v_x, v_{xx}$ are partial derivatives. The following lemma ensures that there exists a unique classical solution of the above PDE (partial differential equation).

**Lemma 3.1** There exists a unique $v \in C([0, T] \times [0, 1]) \cap C^{1,2}([0, T] \times (0, 1))$ that satisfies the followings:

(i) $v$ satisfies the HJB equation (3.2) for $(t, x) \in (0, T) \times (0, 1)$.

(ii) For $x \in [0, 1]$, the map $t \mapsto v(t, x)$ is continuously differentiable on $[0, T]$ and satisfies

$$
\begin{align*}
0 &= v(T, x), \\
0 &= v_t(t, x) + (\mu - r)x + r - \frac{1}{2}\sigma^2 x^2 - \lambda v(t, x) \\
&\quad + \lambda \sup_{y \in [0, 1]} \left( v(t, y) - \ln \left( \frac{1 + \bar{\epsilon} y}{1 + \bar{\epsilon} x} \right) 1_{\{x < y\}} - \ln \left( \frac{1 - \bar{\epsilon} y}{1 - \bar{\epsilon} x} \right) 1_{\{x > y\}} \right).
\end{align*}
$$

(3.3)

(iii) $v_t(t, x), x(1-x)v_x(t, x), x^2(1-x)^2v_{xx}(t, x)$ are uniformly bounded on $(t, x) \in (0, T) \times (0, 1)$.

**Proof** See Appendix. \qed

**Remark 3.2** In Lemma 3.1, we present the differential equations for $x \in (0, 1)$ and $x \in \{0, 1\}$ as (3.2) and (3.3). The reason we separate these two cases is that $v_x$ and $v_{xx}$ may not be continuously extended to the endpoints $x \in \{0, 1\}$.

**Remark 3.3** The model with the search frictions can be seen as a generalization of [18]. However, our analysis has some technically easier features than that of [18], because our solvency region is the first quadrant and we have no subtle issues regarding the regularity of the value function at $x = 1$ or $x = 0$.

**Remark 3.4** If the investor is forced to liquidate the stock at the terminal time $T$, then the objective function of the investor becomes $\mathbb{E}[\ln(W_T^{(0)} + (1 - \bar{\epsilon})W_T^{(1)})]$. In this case, we may define new state processes $\tilde{W}_t := W_t^{(0)} + (1 - \bar{\epsilon})W_t^{(1)}$ and $\tilde{X}_t := \frac{(1-\bar{\epsilon})W_t^{(1)}}{W_t}$, whose dynamics are

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\[ d\tilde{W}_t = \tilde{W}_t \left[ \left( r + (\mu - r)\tilde{X}_t \right) dt + \sigma \tilde{X}_t dB_t \right] - (\epsilon + \overline{\epsilon}) \tilde{M}_t^+ dP_t, \]
\[ d\tilde{X}_t = \tilde{X}_t \left( 1 - \tilde{X}_t \right) \left[ (\mu - r - \sigma^2 \tilde{X}_t) dt + \sigma dB_t \right] + \left( \frac{M_t + (\epsilon + \overline{\epsilon})\tilde{X}_t M_t^+}{\tilde{W}_t - (\epsilon + \overline{\epsilon}) \tilde{M}_t^+} \right) dP_t. \]

(3.4)

Note that the same class of admissible strategies as in the original problem is considered. We observe that \( \tilde{W}_t > 0 \) for \( t \geq 0 \) almost surely, as long as \( M \) satisfies (2.4), by the same way as in Footnote 7.

Using (3.4), one can derive the corresponding HJB equation below:

\[
\begin{cases}
0 = v(T, x), \\
0 = v_t + x(1-x)(\mu - r - \sigma^2 x)v_x + \frac{1}{2}\sigma^2 x^2(1-x)^2 v_{xx} \\
\quad + (\mu - r)x + r - \frac{1}{2}\sigma^2 x^2 - \lambda v \\
\quad + \lambda \sup_{y \in [0,1]} \left( v(t, y) - \ln \left( \frac{1+(\epsilon + \overline{\epsilon})y}{1+(\epsilon + \overline{\epsilon})x} \right) 1_{\{x < y\}} \right).
\end{cases}
\]

(3.5)

If we replace \((\epsilon, \overline{\epsilon})\) by \((\epsilon + \overline{\epsilon}, 0)\) in (3.2), then (3.2) becomes the above PDE. Therefore, the analysis for the forced liquidation case is essentially the same as the analysis of (2.6).

**Theorem 3.5** Let \( V \) be the value function in (3.1) and \( v \) be as in Lemma 3.1. Then, for \((t, x) \in [0, T] \times [0, 1], \)

\[ V(t, x, w) = \ln(w) + v(t, x). \]

(3.6)

**Proof** Without loss of generality, we prove \( V(0, x_0, w_0) = \ln(w_0) + v(0, x_0) \). Let \( M \) be an admissible trading strategy and \((W, X)\) be the corresponding solution of (2.5). Let \( \tau_n := \inf\{t \geq 0 : P_t = n\} \) for \( n \in \mathbb{N} \) and \( \tau_0 := 0 \). If \( X_{\tau_n} = 0 \) (resp., \( X_{\tau_n} = 1 \)), then for \( \tau_n \leq t < \tau_{n+1}, \)

\[ X_t = 0, \quad dW_t = rW_t dt \quad \text{(resp.,} X_t = 1, \quad dW_t = \mu W_t dt + \sigma W_t dB_t). \]

(3.7)

We apply Ito’s formula to \( \ln(W_t) + v(t, X_t) \) with (2.5) and (3.7), and use the fact that \( X_t \) and \( W_t \) can only jump at \( t = \tau_n \) for \( n \in \mathbb{N} \) to obtain

\[
\ln(W_{\tau_{n+1}}) + v(\tau_{n+1}, X_{\tau_{n+1}}) - \ln(W_{\tau_n}) - v(\tau_n, X_{\tau_n}) = \begin{cases}
\int_{\tau_n}^{\tau_{n+1}} \left( v_t(t, x) + x(1-x)(\mu - r - \sigma^2 x)v_x(t, x) \\
+ \frac{1}{2}\sigma^2 x^2(1-x)^2 v_{xx}(t, x) \\
+ (\mu - r)x + r - \frac{1}{2}\sigma^2 x^2 \right) |_{x=X_t} dt, & \text{if } 0 < X_{\tau_n} < 1 \\
\int_{\tau_n}^{\tau_{n+1}} v(t, 0) d\tau, & \text{if } X_{\tau_n} = 0 \\
\int_{\tau_n}^{\tau_{n+1}} v(t, 1) + \mu - \frac{1}{2}\sigma^2 dt + \sigma d\tau, & \text{if } X_{\tau_n} = 1 \\
\end{cases}
\]

\[
+ \ln(W_{\tau_{n+1}}) + v(\tau_{n+1}, X_{\tau_{n+1}}) - \ln(W_{\tau_{n+1}^-}) - v(\tau_{n+1}, X_{\tau_{n+1}^-}).
\]
In the above equation, we can replace $\tau_n$ and $\tau_{n+1}$ by $\tau_n \wedge T$ and $\tau_{n+1} \wedge T$, and sum them up for $n$ to obtain

$$\ln(W_T) - \ln(w_0) - v(0, x_0)$$

$$= \sum_{n=0}^{\infty} \left( \ln(W_{\tau_n \wedge T}) + v(\tau_n \wedge T, X_{\tau_n \wedge T}^+) - \ln(W_{\tau_n \wedge T}) - v(\tau_n \wedge T, X_{\tau_n \wedge T}^-) \right)$$

$$= \int_0^T \left( v_t(t, x) + (\mu - r) x + r - \frac{1}{2} \sigma^2 x^2 \right) \cdot 1_{[0 < x < 1]} \, dt$$

$$+ \int_0^T \left( v_t(t, x) + (\mu - r) x + r - \frac{1}{2} \sigma^2 x^2 \right) \cdot 1_{[x \in [0,1]]} \, dB_t$$

$$+ \sum_{0 < t \leq T} \left( \ln(W_t) - \ln(W_{t-}) + v(t, X_t) - v(t, X_{t-}) \right).$$

The stochastic integral term above is a true martingale, because Lemma 3.1 (iii) implies that the integrand is uniformly bounded. The sum of jumps term above can be written as

$$\int_0^T \left( v(t, y) - v(t, x) - \ln \left( \frac{1 + \epsilon y}{1 + \epsilon x} \right) 1_{\{x < y\}} - \ln \left( \frac{1 - \epsilon y}{1 - \epsilon x} \right) 1_{\{x > y\}} \right) \, dB_t.$$

Considering that the integrand above is bounded and $(P_t - \lambda t)_{t \in [0, T]}$ is a martingale, we can write the expected value of the above expression as

$$\mathbb{E} \left[ \int_0^T \lambda \left( v(t, y) - v(t, x) - \ln \left( \frac{1 + \epsilon y}{1 + \epsilon x} \right) 1_{\{x < y\}} - \ln \left( \frac{1 - \epsilon y}{1 - \epsilon x} \right) 1_{\{x > y\}} \right) \, dB_t \right].$$

Combining these observations, we obtain
\[ \mathbb{E}[\ln(W_T)] - \ln(w_0) - v(0, x_0) \]
\[ = \mathbb{E}\left[ \int_0^T \left( v_t(t, x) + x(1 - x)(\mu - r - \sigma^2 x)v_x(t, x) + \frac{1}{2}\sigma^2 x^2(1 - x)^2v_{xx}(t, x) + (\mu - r)x \right. \right. \]
\[ + r - \frac{1}{2}\sigma^2 x^2 + \lambda\left( v(t, y) - v(t, x) - \ln\left( \frac{1 + \tilde{\epsilon} y}{1 + \tilde{\epsilon} x} \right) \right) 1_{[x,y]} \]
\[ - \ln\left( \frac{1 - \epsilon y}{1 - \epsilon x} \right) 1_{[x,y]} \left. \right) \cdot 1_{[0 < x < 1]} \]
\[ + \left( v_t(t, x) + (\mu - r)x + r - \frac{1}{2}\sigma^2 x^2 + \lambda\left( v(t, y) - v(t, x) \right. \right. \]
\[ - \ln\left( \frac{1 + \tilde{\epsilon} y}{1 + \tilde{\epsilon} x} \right) 1_{[x,y]} \]
\[ - \ln\left( \frac{1 - \epsilon y}{1 - \epsilon x} \right) 1_{[x,y]} \left) \right| (x, y) = (x_t, \frac{x_t - W_t + M_t}{W_t - \tilde{\epsilon} M_t + \tilde{\epsilon} M_t}) \right] dt \]. \hspace{1cm} (3.8)

where the integrability is due to Lemma 3.1. To be more specific, we first recall that the admissibility condition (2.4) ensures \( 0 \leq \frac{x_t - W_t + M_t}{W_t - \tilde{\epsilon} M_t + \tilde{\epsilon} M_t} \leq 1 \), so the dummy variable \( y \) in (3.8) is in \([0, 1]\). Lemma 3.1 (iii) (Lemma 3.1 (ii), resp.) and the boundedness of \( v \) imply that the term with the indicator \( 1_{[0 < x < 1]} \) \( (1_{[x\in[0,1]}), \) resp.) is bounded. All in all, the integrand in (3.8) is bounded.

The equality (3.8) and Lemma 3.1 (i) and (ii) imply that for any admissible trading strategy \( M \),

\[ \mathbb{E}[\ln(W_T)] \leq \ln(w_0) + v(0, x_0). \] \hspace{1cm} (3.9)

To complete the proof, we find an optimal strategy \( \hat{M} \) that satisfies the equality in (3.9). We observe that the map

\[ (t, x, y) \mapsto v(t, y) - \ln\left( \frac{1 + \tilde{\epsilon} y}{1 + \tilde{\epsilon} x} \right) 1_{[x,y]} \]

is continuous on \([0, T] \times [0, 1]^2\). Therefore, according to Lemma D.1, we can choose a measurable function \( \hat{y} : [0, T] \times [0, 1] \rightarrow [0, 1] \) such that

\[ \hat{y}(t, x) \in \arg\max_{y \in [0,1]} \left( v(t, y) - \ln\left( \frac{1 + \tilde{\epsilon} y}{1 + \tilde{\epsilon} x} \right) 1_{[x,y]} - \ln\left( \frac{1 - \epsilon y}{1 - \epsilon x} \right) 1_{[x,y]} \right). \] \hspace{1cm} (3.10)

Using \( \hat{y} \), we define a measurable function \( m : [0, T] \times [0, \infty) \times [0, 1] \rightarrow \mathbb{R} \) as

\[ m(t, w, x) := \frac{w(\hat{y}(t, x) - x)}{1 + \tilde{\epsilon} \hat{y}(t, x)} \cdot 1_{\{\hat{y}(t, x) > x\}} + \frac{w(\hat{y}(t, x) - x)}{1 - \tilde{\epsilon} \hat{y}(t, x)} \cdot 1_{\{\hat{y}(t, x) < x\}}. \] \hspace{1cm} (3.11)
Let \( (\hat{W}, \hat{X}) \) be the unique solution$^8$ of SDE (2.5) with \( M_t = m(t, W_t, X_t) \). Now we define \( \hat{M}_t := m(t, W_{t-}, \hat{X}_{t-}) \), then it is a predictable process.

From (3.11) and (2.5), we observe that
\[
\Delta \hat{X}_t = \left( \hat{y}(t, \hat{X}_{t-}) - \hat{X}_{t-} \right) \Delta P_t.
\]

Then the structure of (2.5) and \( 0 \leq \hat{y} \leq 1 \) imply that \( 0 \leq \hat{X}_t \leq 1 \) and \( \hat{W}_t \geq 0 \) for all \( t \in [0, T] \). Therefore, we conclude that \( \hat{M} \) is an admissible trading strategy.

Finally, we substitute \((\hat{X}, \hat{W}, \hat{M})\) for \((X, W, M)\) in (3.8). Then, the differential equations (3.2) and (3.3) for \( \hat{y} \), the optimality of \( \hat{y} \) in (3.10), and the observation
\[
\frac{\hat{X}_{t-}\hat{W}_{t-}+\hat{M}_{t-}}{\hat{W}_{t-}\hat{M}_{t-} = \hat{y}(t, \hat{X}_{t-})}
\]
produce
\[
\mathbb{E}[\ln(\hat{W}_T)] = \ln(w_0) + v(0, x_0).
\] (3.12)

By (3.9) and (3.12), we conclude \( V(0, x_0, w_0) = \ln(w_0) + v(0, x_0) \) and the optimality of \( \hat{M} \).

In the proof of Theorem 3.5, the optimal trading strategy is described by the function \( \hat{y} \) in (3.10). In the next section, we show that the maximizer in (3.10) is unique, and we characterize the form of the unique optimal strategy in detail. For this purpose, we use the strict concavity of the value function. To be more specific, we define \( \tilde{V} : [0, T] \times ([0, \infty)^2 \setminus \{(0,0)\}) \to \mathbb{R} \) as
\[
\tilde{V}(t, a, b) := \sup_{(M_s)_{s \in [t,T]}} \mathbb{E} \left[ \ln \left( W_T^{a,b,M} \right) \right],
\] (3.13)

where \( W_T^{a,b,M} \) represents the total wealth at time \( T \) with \( (W_t^{(0)}, W_t^{(1)}) = (a, b) \) and the trading strategy \( M \). In other words, \( \tilde{V} \) is the value function in terms of the wealth amount invested in the bond and the stock. Then, (3.1) and (3.13) imply that
\[
\tilde{V}(t, a, b) = V \left( t, \frac{b}{a+b}, a+b \right).
\] (3.14)

**Proposition 3.6** For \( t \in [0, T) \), the maps \((a, b) \mapsto \tilde{V}(t, a, b)\) and \( x \mapsto v(t, x) \) are strictly concave.

**Proof** Without loss of generality, we prove the statement for \( t = 0 \) case. Let \((a_0, b_0) \neq (a_1, b_1)\) be elements of \([0, \infty)^2 \setminus \{(0,0)\}\) and \( \theta \in (0, 1) \). According to the proof of Theorem 3.5 and the relation (3.14), we can find optimal trading strategies for the initial positions \((a_0, b_0)\) and \((a_1, b_1)\), and we denote them by \( \hat{M}^0 \) and \( \hat{M}^1 \), respectively. Then the structure of the SDE in (2.3) implies that \( M^0 := (1-\theta)\hat{M}^0 + \theta\hat{M}^1 \) is an admissible

---

$^8$ Indeed, the SDE in (2.5) without \( dP_t \) term has a unique (explicit) solution. The unique solution of (2.5) can be obtained by patching the unique solutions on time intervals between the jump times of the Poisson process, with the jump size described by the coefficient of \( dP_t \) term.
trading strategy with initial position \((a_\theta, b_\theta) := ((1 - \theta)a_0 + \theta a_1, (1 - \theta)b_0 + \theta b_1)\)
and satisfies
eq (1 - \theta) W^a_0, b_0, \hat{M}_0^0 + \theta W^a_1, b_1, \hat{M}_1^1 \leq W^a_\theta, b_\theta, M_\theta^0.
\tag{3.15}

We also observe that
\[
\mathbb{P} \left( W^{a_0, b_0, \hat{M}_0}_T \neq W^{a_1, b_1, \hat{M}_1}_T \right) 
\geq \mathbb{P} \left( W^{a_0, b_0, \hat{M}_0}_T \neq W^{a_1, b_1, \hat{M}_1}_T \text{ and } P_T = 0 \right) 
= \mathbb{P}(P_T = 0) \cdot \mathbb{P} \left( W^{a_0, b_0, \hat{M}_0}_T \neq W^{a_1, b_1, \hat{M}_1}_T \mid P_T = 0 \right) 
= e^{-\lambda T} \cdot \mathbb{P} \left( (a_1 - a_0)e^{\sigma T} + (b_1 - b_0)e^{(\mu - \frac{\sigma^2}{2})T + \sigma B_T} \neq 0 \right) 
= e^{-\lambda T} > 0,
\tag{3.16}
\]
where the second equality is due to the independence of \(B\) and \(P\), and the last equality is due to \((a_0, b_0) \neq (a_1, b_1)\) and the fact that \(B_T\) is a continuous random variable. By these observations, we obtain the strict concavity of \(\tilde{V}\):
\[
(1 - \theta) \tilde{V}(0, a_0, b_0) + \theta \tilde{V}(0, a_1, b_1)
= \mathbb{E} \left[ (1 - \theta) \ln \left( W^{a_0, b_0, \hat{M}_0}_T \right) + \theta \ln \left( W^{a_1, b_1, \hat{M}_1}_T \right) \right]
\leq \mathbb{E} \left[ \ln \left( (1 - \theta) W^{a_0, b_0, \hat{M}_0}_T + \theta W^{a_1, b_1, \hat{M}_1}_T \right) \right]
\leq \mathbb{E} \left[ \ln \left( W^{a_\theta, b_\theta, M_\theta}_T \right) \right]
\leq \tilde{V}(0, a_\theta, b_\theta),
\]
where the first inequality is due to the strict concavity of logarithm and (3.16), and the second inequality is from (3.15). Therefore, the map \((a, b) \mapsto \tilde{V}(0, a, b)\) is strictly concave. This also implies that the map \(x \mapsto \tilde{V}(0, 1 - x, x)\) is strictly concave. Finally, Theorem 3.5 and the relation (3.14) connect \(\tilde{V}\) and \(v\) as
\[
\tilde{V}(0, 1 - x, x) = V(0, x, 1) = v(0, x),
\]
and we conclude that the map \(x \mapsto v(0, x)\) is strict concave. \(\square\)

\textsuperscript{9} The inequality (3.15) becomes strict on the event \(\omega \in \Omega : \exists t \in [0, T] \text{ such that } \Delta P_t(\omega) = 1 \text{ and } \hat{M}_t^0(\omega) < 0\).
4 Optimal Trading Strategy

In this section, we show that the optimal trading strategy can be characterized in terms of the no-trade region. We start with the construction of the candidate boundary points \( \bar{y} \) and \( \tilde{y} \) of the no-trade region in the following lemma.

**Lemma 4.1** For \( t \in [0, T) \), there exist \( 0 \leq \bar{y}(t) \leq \tilde{y}(t) \leq 1 \) such that

\[
\begin{align*}
\{ \bar{y}(t) \} &= \operatorname{argmax}_{y \in [0, 1]} \left( v(t, y) - \ln(1 + \bar{\varepsilon} y) \right), \\
\{ \tilde{y}(t) \} &= \operatorname{argmax}_{y \in [0, 1]} \left( v(t, y) - \ln(1 - \varepsilon y) \right).
\end{align*}
\]

To be more specific, the following statements hold:

(i) The map \( y \mapsto v(t, y) - \ln(1 + \bar{\varepsilon} y) \) strictly increases (decreases) on \( y \in [\bar{y}(t), 1] \).

If \( 0 < \bar{y}(t) < 1 \), then \( \bar{y}(t) \) is the unique solution of the equation \( v_x(t, x) = \frac{\varepsilon}{1 + \varepsilon x} \).

(ii) The map \( y \mapsto v(t, y) - \ln(1 - \varepsilon y) \) strictly increases (decreases) on \( y \in [0, \tilde{y}(t)] \).

If \( 0 < \tilde{y}(t) < 1 \), then \( \tilde{y}(t) \) is the unique solution of the equation \( v_x(t, x) = -\frac{\varepsilon}{1 - \varepsilon x} \).

**Proof** Let \( t \in [0, T) \) be fixed. We consider the map \( z \in [0, \frac{1}{1 + \varepsilon}] \mapsto \tilde{V}(t, 1 - (1 + \bar{\varepsilon})z, z) \), where \( \tilde{V} \) is defined in (3.13). Theorem 3.5 and (3.14) imply that this map is differentiable (we denote the partial derivatives as \( \tilde{V}_a \) and \( \tilde{V}_b \)), and the map is strictly concave due to Proposition 3.6. Therefore, its derivative

\[
D(t, z) := -(1 + \bar{\varepsilon})\tilde{V}_a(t, 1 - (1 + \bar{\varepsilon})z, z) + \tilde{V}_b(t, 1 - (1 + \bar{\varepsilon})z, z) \tag{4.1}
\]

is strictly decreasing in \( z \in (0, \frac{1}{1 + \varepsilon}) \), and there exists a unique \( z(t) \in [0, \frac{1}{1 + \varepsilon}] \) such that

\[
\begin{align*}
D(t, z) > 0 & \quad \text{for } z \in (0, z(t)), \\
D(t, z) < 0 & \quad \text{for } z \in (z(t), \frac{1}{1 + \varepsilon}).
\end{align*} \tag{4.2}
\]

Obviously, in case \( z(t) \in (0, \frac{1}{1 + \varepsilon}) \), \( z(t) \) is the unique solution of \( D(t, z) = 0 \).

By Theorem 3.5 and (3.14), we observe that for \( y \in [0, 1] \),

\[
v(t, y) - \ln(1 + \bar{\varepsilon} y) = V \left( t, y, \frac{1}{1 + \varepsilon y} \right) = \tilde{V} \left( t, \frac{1 - y}{1 + \varepsilon y}, \frac{y}{1 + \varepsilon y} \right).
\]

We take derivative with respect to \( y \) above and use (4.1) to obtain

\[
\frac{\partial}{\partial y} \left( v(t, y) - \ln(1 + \bar{\varepsilon} y) \right) = \frac{1}{(1 + \bar{\varepsilon} y)^2} D \left( t, \frac{y}{1 + \varepsilon y} \right). \tag{4.3}
\]
Now we define $\underline{y}(t) := \frac{z(t)}{1 + \bar{\varepsilon}z(t)} \in [0, 1]$. Since the map $y \mapsto \frac{y}{1 + \varepsilon y}$ is strictly increasing on $[0, 1]$, the definition of $\underline{z}(t)$ in (4.2) implies that

$$\begin{align*}
D \left( t, \frac{y}{1 + \varepsilon y} \right) > 0 & \quad \text{for } y \in \left( 0, \underline{y}(t) \right), \\
D \left( t, \frac{y}{1 + \varepsilon y} \right) < 0 & \quad \text{for } y \in \left( \underline{y}(t), 1 \right).
\end{align*}$$

(4.4)

Also, in case $\underline{y}(t) \in (0, 1)$, $\underline{y}(t)$ is the unique solution of $D \left( t, \frac{y}{1 + \varepsilon y} \right) = 0$. From (4.3) and (4.4), we conclude that statement (i) holds and

$$\{ \underline{y}(t) \} = \arg\max_{y \in [0, 1]} \left( v(t, y) - \ln(1 + \varepsilon y) \right).$$

By the same way, we conclude that $\max_{y \in [0, 1]} \left( v(t, y) - \ln(1 - \varepsilon y) \right)$ has the unique maximizer denoted by $\tilde{y}(t)$ and statement (ii) holds.

It only remains to check the inequality $\underline{y}(t) \leq \tilde{y}(t)$. If $\tilde{y}(t) = 1$, then $\underline{y}(t) \leq \tilde{y}(t)$ is obvious. If $\tilde{y}(t) < 1$, then $v_x(t, \tilde{y}(t)) \leq \frac{-\bar{\varepsilon}}{1 + \bar{\varepsilon} \tilde{y}(t)} \leq \frac{\tilde{\varepsilon}}{1 + \varepsilon \tilde{y}(t)}$ by (ii). From (4.3) and (4.4), we obtain $\underline{y}(t) \leq \tilde{y}(t)$.

In the next theorem, we explicitly characterize the optimizer $\widehat{y}$ in (3.10) in terms of $\underline{y}$ and $\tilde{y}$ in Lemma 4.1.

**Theorem 4.2** For $t \in [0, T)$, the argmax in (3.10) is a singleton, and $\widehat{y}$ has the following expression:

$$\hat{y}(t) = \begin{cases} 
\underline{y}(t), & \text{if } x \in \left( 0, \underline{y}(t) \right) \\
x, & \text{if } x \in \left( \underline{y}(t), \tilde{y}(t) \right) \\
\tilde{y}(t), & \text{if } x \in (\tilde{y}(t), 1]
\end{cases}$$

(4.5)

where $\underline{y}(t)$ and $\tilde{y}(t)$ are uniquely determined in Lemma 4.1.

**Proof** We may rewrite the maximization in (3.10) as

$$\max_{y \in [0, 1]} \left( v(t, y) - \ln \left( \frac{1 + \varepsilon y}{1 + \varepsilon x} \right) 1_{x < y} - \ln \left( \frac{1 - \varepsilon y}{1 - \varepsilon x} \right) 1_{x > y} \right) = \max \left\{ \max_{y \in [0, x]} \left( v(t, y) - \ln \left( \frac{1 - \varepsilon y}{1 - \varepsilon x} \right) \right), \max_{y \in [x, 1]} \left( v(t, y) - \ln \left( \frac{1 + \varepsilon y}{1 + \varepsilon x} \right) \right) \right\},$$

(4.6)

Using Lemma 4.1, we observe that

$$\max_{y \in [0, x]} \left( v(t, y) - \ln \left( \frac{1 - \varepsilon y}{1 - \varepsilon x} \right) \right) = \begin{cases} 
v(t, x), & \text{if } x \leq \underline{y}(t) \\
v(t, \underline{y}(t)) - \ln \left( \frac{1 - \varepsilon \underline{y}(t)}{1 - \varepsilon x} \right), & \text{if } x > \underline{y}(t)
\end{cases}$$

(4.7)

and

$$\max_{y \in [x, 1]} \left( v(t, y) - \ln \left( \frac{1 + \varepsilon y}{1 + \varepsilon x} \right) \right) = \begin{cases} 
v(t, \tilde{y}(t)) - \ln \left( \frac{1 + \varepsilon \tilde{y}(t)}{1 + \varepsilon x} \right), & \text{if } x < \tilde{y}(t) \\
v(t, x), & \text{if } x \geq \tilde{y}(t)
\end{cases}$$
Fig. 1 The left graph shows $y(t)$ and $\bar{y}(t)$ as functions of $t$, and the right graph describes $\hat{y}(t, x)$ as a function of $x$ for fixed $t = 0.5$. In both graphs, the dashed line is the Merton fraction $y_{\infty} = \frac{\mu - r}{\sigma^2}$. The parameters are $\mu = 0.4$, $r = 0.1$, $\sigma = 1$, $\lambda = 3$, $\epsilon = \bar{\epsilon} = 0.05$, and $T = 1$

where the maximizers are unique. Combining (4.6) and (4.7), we obtain

$$
\max_{y \in [0, 1]} \left( v(t, y) - \ln \left( \frac{1 + \bar{\epsilon} y}{1 + \epsilon x} \right) 1_{\{x < y\}} - \ln \left( \frac{1 - \bar{\epsilon} y}{1 - \epsilon x} \right) 1_{\{x > y\}} \right) =
\begin{cases}
  v(t, y(t)) - \ln \left( \frac{1 + \bar{\epsilon} y(t)}{1 + \epsilon x} \right), & \text{if } x \in [0, y(t)] \\
  v(t, x), & \text{if } x \in [y(t), \bar{y}(t)] \\
  v(t, \bar{y}(t)) - \ln \left( \frac{1 - \bar{\epsilon} \bar{y}(t)}{1 - \epsilon x} \right), & \text{if } x \in (\bar{y}(t), 1]
\end{cases}
$$

and conclude that the corresponding unique maximizer is as in (4.5). $\square$

Theorem 4.2 implies that the optimal trading strategy is characterized by the no-trade region: if the investor can trade at time $t \in [0, T)$, then the investor should minimally trade to keep the fraction of wealth invested in the stock inside of the interval $[y(t), \bar{y}(t)]$. To be specific, if the fraction $X_t$ is less (more, resp.) than $y(t)$ ($\bar{y}(t)$, resp.), then the investor should buy (sell, resp.) the stock and adjust the fraction to $y(t)$ ($\bar{y}(t)$, resp.). If the fraction $X_t$ is inside of the interval $[y(t), \bar{y}(t)]$, then the investor should not trade. Figure 1 illustrates the no-trade region and the optimal trading strategy.

If $\bar{\epsilon} = \epsilon = 0$, then Lemma 4.1 implies that $v(t) = \bar{y}(t)$, hence the no-trade region becomes a singleton. For $\bar{\epsilon} > 0$, one may expect that the investor would not want to buy the stock at times close to the terminal time $T$ due to the transaction costs. If $\bar{\epsilon} = 0$ and the Merton fraction $\frac{\mu - r}{\sigma^2}$ is greater than zero, then one may expect that the investor would want to hold strictly positive shares of the stock all the time. Our next task is to examine and prove this type of trading behaviors.

For detailed analysis, we first provide stochastic representations of $v$ and $v_x$. We apply the Feynman-Kac formula (i.e., see Theorem 5.7.6 in [32]) and the expression of the optimizer $\hat{y}$ in Theorem 4.2 to Lemma 3.1, and obtain the following representation
for \( v \):

\[
v(t, x) = \int_t^T e^{-\lambda(s-t)} \mathbb{E} \left[ (\mu - r) Y^{(t,x)}_s + r - \frac{1}{2} \sigma^2 (Y^{(t,x)}_s)^2 + \lambda L(s, Y^{(t,x)}_s) \right] ds,
\]

where for \((s, x) \in [t, T) \times [0, 1],\)

\[
Y^{(t,x)}_s := \frac{x \cdot \exp \left( (\mu - r - \frac{1}{2} \sigma^2) (s - t) + \sigma (B_s - B_t) \right)}{x \cdot \exp \left( (\mu - r - \frac{1}{2} \sigma^2) (s - t) + \sigma (B_s - B_t) \right) + (1 - x)}.
\]

\[
L(s, y) := v(s, \hat{y}(s, y)) - \ln \left( \frac{1 + \epsilon \hat{y}(s, y)}{1 - \epsilon y} \right) 1_{\{y < \hat{y}(s, y)\}} - \ln \left( \frac{1 - \epsilon \hat{y}(s, y)}{1 - \epsilon y} \right) 1_{\{y > \hat{y}(s, y)\}}.
\]

The representation of \( v_x \) is given in the following lemma.

**Lemma 4.3** The function \( L \) in \((4.9)\) is continuously differentiable with respect to \( y \),

\[
L_y(t, x) = \begin{cases} \tilde{\epsilon} & x \in (0, y(t)) \\ \frac{v_x(t, x)}{y(t)} & x \in (y(t), \tilde{y}(t)) \text{ for } (t, x) \in [0, T) \times (0, 1), \\ -\frac{\epsilon}{1 - \epsilon x} & x \in [\tilde{y}(t), 1) \end{cases}
\]

and \( v_x(t, x) \) has the following representation: for \((t, x) \in [0, T) \times (0, 1),\)

\[
v_x(t, x) = \int_t^T e^{-\lambda(s-t)} \mathbb{E} \left[ \left( \frac{\partial}{\partial x} Y^{(t,x)}_s \right) \left( \mu - r - \sigma^2 Y^{(t,x)}_s + \lambda L_y(s, Y^{(t,x)}_s) \right) \right] ds.
\]

**Proof** Combining \((4.5)\) and \((4.9)\), we rewrite \( L \) as

\[
L(t, x) = \begin{cases} v(t, y(t)) - \ln \left( \frac{1 + \epsilon y(t)}{1 + \epsilon x} \right), & x \in [0, y(t)] \\ v(t, x), & x \in (y(t), \tilde{y}(t)) \\ v(t, \tilde{y}(t)) - \ln \left( \frac{1 - \epsilon \tilde{y}(t)}{1 - \epsilon x} \right), & x \in [\tilde{y}(t), 1] \end{cases}
\]

We take derivative with respect to \( x \) above and obtain the expression \((4.10)\) for \( x \in (0, 1) \setminus \{y(t), \tilde{y}(t)\} \). If \( 0 < y(t) < 1 \) (resp., \( 0 < \tilde{y}(t) < 1 \)), then \( v_x(t, y(t)) = \frac{\epsilon}{1 + \epsilon y(t)} \) (resp., \( v_x(t, \tilde{y}(t)) = -\frac{\epsilon}{1 - \epsilon \tilde{y}(t)} \)). Therefore, we conclude that \( L \) is continuously differentiable with respect to \( y \) and \((4.10)\) is valid.

Since \(-\frac{\epsilon}{1 - \epsilon} < v_x(t, x) < \frac{\epsilon}{1 + \epsilon x}\) for \( x \in (y(t), \tilde{y}(t)) \), we observe that for \((t, x) \in [0, T) \times (0, 1),\)

\[
-\frac{\epsilon}{1 - \epsilon} \leq L_y(t, x) \leq \tilde{\epsilon}.
\]
We also observe that for \((t, x) \in [0, T) \times (0, 1),\)
\[
e^{-r-\mu+\frac{\sigma^2}{2}(s-t)-\sigma(B_s-B_t)} \leq \frac{\partial}{\partial x} Y_s^{(t,x)} \leq e^{-r-\mu+\frac{\sigma^2}{2}(s-t)+\sigma(B_s-B_t)}.
\] (4.14)

Now we take derivative with respect to \(x\) in (4.8). The mean value theorem and the dominated convergence theorem, together with the inequalities (4.13) and (4.14), allow us to take derivative inside of the expectation. By (4.10) and the chain rule, we obtain the representation (4.11).

Using these representations, we extract some properties about the boundaries \(y(t)\) and \(\bar{y}(t)\) of the no-trade region.

**Proposition 4.4** (Properties of no-trade region)

Let \(y_\infty := \frac{\mu - r}{\sigma^2}\) denote the Merton fraction.

(i) If \(0 < y_\infty < \bar{y}(t) > 0 < y(t) < \bar{y}(t)\) for \(t \in [0, T)\).

(ii) If \(0 < y_\infty < 1\) and at least one of \(\bar{y}\) and \(\bar{y}\) is strictly positive, then \(y(t) < \bar{y}(t)\) for \(t \in [0, T)\).

(iii) If \(0 < y_\infty < \bar{y}(t)\) for \(t \in [0, T)\), where \(\bar{t} = \left(T - \frac{\ln(1+\bar{y})}{\mu-r} \right)^+\).

If \(y_\infty < 1\), then \(\bar{y}(t) = 1\) for \(t \in [\bar{t}, T)\), where \(\bar{t} = \left(T - \frac{\ln(1+\bar{y})}{\mu-r} \right)^+\).

**Proof** (i) Suppose that \(0 < y_\infty\) (the case of \(y_\infty < 1\) can be treated similarly). Let \(t \in [0, T)\) be fixed. To check \(\bar{y}(t) > 0\), we observe from (4.11) and the dominated convergence theorem that

\[
\lim_{x \downarrow 0} v_x(t, x) = \int_{\bar{t}}^{T} e^{-\lambda(s-t)} \mathbb{E} \left[ \lim_{x \downarrow 0} \left( \frac{\partial}{\partial x} Y_s^{(t,x)} \right) \left( \mu - r - \sigma^2 Y_s^{(t,x)} + \lambda L_y(s, Y_s^{(t,x)}) \right) \right] ds
\]

\[
= \int_{\bar{t}}^{T} e^{-\lambda(s-t)} \mathbb{E} \left[ e^{(\mu-r-\frac{1}{2}\sigma^2)(s-t)+\sigma(B_s-B_t)} \right]
\]

\[
\cdot \left( \mu - r + \lambda \left( \bar{\epsilon} \cdot 1_{\{\bar{\gamma}(s) > 0\}} + \lim_{x \downarrow 0} v_x(s, x) \cdot 1_{\{\bar{\gamma}(s) = 0 < \bar{\gamma}(s)\}} - \epsilon \cdot 1_{\{\bar{\gamma}(s) = 0\}} \right) \right) ds
\]

\[
\geq \int_{\bar{t}}^{T} e^{(\mu-r-\lambda)(s-t)} (\mu - r - \lambda \bar{\epsilon}) ds,
\]

where the second equality is from (4.10) and (4.14), and the inequality is due to the fact that \(-\frac{\bar{\epsilon}}{1-\bar{\epsilon}x} < v_x(t, x)\) for \(x < \bar{y}(t)\). Obviously, the last integral in (4.15) is nonnegative if \(\mu - r - \lambda \bar{\epsilon} \geq 0\). If \(\mu - r - \lambda \bar{\epsilon} < 0\), then \(\mu - r - \lambda \bar{\epsilon} < 0\) due to \(\epsilon \in [0, 1)\) and we observe that

\[
\int_{\bar{t}}^{T} e^{(\mu-r-\lambda)(s-t)} (\mu - r - \lambda \bar{\epsilon}) ds = \frac{\mu - r - \lambda \bar{\epsilon}}{\mu - r - \lambda} \left( e^{(\mu-r-\lambda)(T-t)} - 1 \right) > -\bar{\epsilon}.
\]
where we use $y_\infty > 0$ for the inequality. The above inequality and (4.15) imply
\[
\lim_{x \downarrow 0} \frac{\partial}{\partial x} \left( v(t, x) - \ln(1 - \epsilon x) \right) > 0,
\]
and we conclude $\tilde{y}(t) > 0$ by Lemma 4.1.

(ii) Suppose that $0 < y_\infty < 1$ and $\bar{\epsilon} > 0$ (the case of $\epsilon > 0$ can be treated similarly). According to (i), $y(t) < 1$ and $\bar{y}(t) > 0$. Therefore, there are only two possibilities: $\overline{y}(t) = 0$ or $0 < \overline{y}(t) < 1$. In case $\overline{y}(t) = 0$, we immediately obtain $y(t) < \bar{y}(t)$ since $\overline{y}(t) > 0$. In case $0 < \overline{y}(t) < 1$, by Lemma 4.1, we have
\[
v_x(t, \overline{y}(t)) = \frac{\bar{\epsilon}}{1 + \bar{\epsilon} \overline{y}(t)} > \frac{-\epsilon}{1 - \epsilon \overline{y}(t)},
\]
and this inequality, together with (4.3) and (4.4), implies $y(t) < \bar{y}(t)$.

(iii) Suppose that $0 < y_\infty$ (the case of $y_\infty < 1$ can be treated similarly). For convenience, let
\[
\beta(t) := \begin{cases} 
\lim_{x \downarrow 0} v_x(t, x) & \text{for } t \in [0, T), \\
0 & \text{for } t = T.
\end{cases}
\]
Lemma 4.1 implies that for $t \in [0, T)$,
\[
\overline{y}(t) > 0 \quad \text{if and only if} \quad \beta(t) > \bar{\epsilon}.
\]
The second equality in (4.15) can be written as
\[
\beta(t) = \int_t^T e^{(\mu - r - \lambda)(s - t)} \left( \mu - r + \lambda \bar{\epsilon} \cdot 1_{\beta(s) > \bar{\epsilon}} + \lambda \beta(s) \cdot 1_{\beta(s) \leq \bar{\epsilon}} \right) ds,
\]
where we use $\bar{y}(s) > 0$ by part (i) and the equivalence (4.16). From (4.17), we observe that $\beta$ is differentiable and satisfies the following differential equation:
\[
\beta'(t) = - (\mu - r)(\beta(t) + 1) - \lambda (\bar{\epsilon} - \beta(t)) \cdot 1_{\beta(t) > \bar{\epsilon}}, \quad \beta(T) = 0.
\]
Let’s consider the case of $\bar{\epsilon} > 0$ first. We define $\xi$ as
\[
\xi := \inf \{ t \in [0, T) : \beta(s) \leq \bar{\epsilon} \text{ for all } s \in [t, T) \}.
\]
Note that the set in (4.19) is non-empty due to $\beta(T) = 0 < \bar{\epsilon}$ and the continuity of $\beta$. To find the explicit expression of $\xi$, we solve (4.18) on the interval $[\xi, T)$ to obtain
\[
\beta(t) = e^{(\mu - r)(T - t)} - 1 \quad \text{for } t \in [\xi, T).
\]
The above expression implies that if \( T \leq \frac{\ln(1+\bar{\epsilon})}{\mu - r} \), then \( \beta(t) \leq \bar{\epsilon} \) for \( t \in [0, T) \) and \( t = 0 \). Otherwise, we solve \( \beta(t) = \bar{\epsilon} \) with the above expression to obtain \( \bar{t} = T - \frac{\ln(1+\bar{\epsilon})}{\mu - r} \).

All in all, \( z = \left( T - \frac{\ln(1+\bar{\epsilon})}{\mu - r} \right)^+ \).

Suppose that the following is not true:

\[
\beta(t) > \bar{\epsilon} \quad \text{for} \quad t \in [0, T). \tag{4.21}
\]

Then, we should have \( \bar{t} > 0 \), \( \beta(t) = \bar{\epsilon} \) and \( \beta'(t) < 0 \) by (4.18). This observation and the negation of (4.21) guarantee the existence of \( t^* \in [0, T) \) such that \( \beta(t) > \bar{\epsilon} \) for \( t \in (t^*, \bar{t}) \) and \( \beta(t^*) = \bar{\epsilon} \). This implies \( \beta'(t^*) \geq 0 \), which contradicts to \( \beta'(t^*) = - (\mu - r)(\bar{\epsilon} + 1) < 0 \) by (4.18). Therefore, we conclude (4.21). Now we combine (4.16), (4.19) and (4.21) to conclude that

\[
in \text{case} \ \bar{\epsilon} > 0, \quad y(t) = \begin{cases} 
0 & \text{for} \quad t \in [\bar{t}, T), \\
> 0 & \text{for} \quad t \in [0, \bar{t}),
\end{cases} \quad \text{where} \quad \bar{t} = \left( T - \frac{\ln(1+\bar{\epsilon})}{\mu - r} \right)^+. \tag{4.22}
\]

It remains to consider the case of \( \bar{\epsilon} = 0 \). Suppose that the following is not true:

\[
\beta(t) > 0 \quad \text{for} \quad t \in [0, T). \tag{4.23}
\]

We have \( \beta(T) = 0 \) and \( \beta'(T) = -(\mu - r) < 0 \) by (4.18). This observation and the negation of (4.23) guarantee the existence of \( t^* \in [0, T) \) such that \( \beta(t) > 0 \) for \( t \in (t^*, T) \) and \( \beta(t^*) = 0 \). This implies \( \beta'(t^*) \geq 0 \), which contradicts to \( \beta'(t^*) = - (\mu - r) < 0 \) by (4.18). Therefore, we conclude (4.23). Now we combine (4.16) and (4.23) to conclude that

\[
in \text{case} \ \bar{\epsilon} = 0, \quad y(t) > 0 \quad \text{for} \quad t \in [0, T). \tag{4.24}
\]

We combine (4.22) and (4.24) to complete the proof.

\( \square \)

**Remark 4.5** The left graph of Fig. 1 demonstrates that \( y(t) = 0 \) after certain threshold \( \bar{t} \) and \( \bar{y}(t) = 1 \) after certain threshold \( \bar{t} \), as Proposition 4.4 (iii) indicates.

**Remark 4.6** Straightforward interpretations of Proposition 4.4 are as follows:

(i) If \( 0 < y_\infty \) (\( y_\infty < 1 \), resp), then the investor never rebalances to the zero-holding of the stock (bond, resp.). However, if the initial holding of the stock (bond, resp.) is zero, then the investor may not try to leave the state of zero-holding of the stock (bond, resp.), depending on the size of the transaction costs.

(ii) If \( 0 < y_\infty < 1 \) and there exist transaction costs, then the no-trade interval is non-trivial (with strictly positive length) all the time.

(iii) The existence of the transaction costs for buying (selling, resp.) the stock makes the investor not to buy (sell, resp.) the stock when it is close to the terminal time. This is natural, because for short period of time, the benefit of rebalancing is small. On the other hand, if there is no cost for trading the stock, then the investor tries to rebalance until the end.
Remark 4.7 Obviously, if $y_{\infty} \leq 0$ ($y_{\infty} \geq 1$, resp.), then there is no reason for buying (selling, resp.) the stock, so $ar{y}(t) = 0$ ($\tilde{y}(t) = 1$, resp.) for $t \in [0, T)$.

5 Asymptotic Analysis

In this section, we provide asymptotic analysis for small transaction costs. To be specific, we focus on the first order approximation of the no-trade region and the value function with respect to the transaction cost parameter around zero. To consider non-trivial cases (see Remark 4.7), we assume that the Merton fraction $y_{\infty}$ is between zero and one, and we set $\bar{\epsilon} = \epsilon = \epsilon$ for convenience.

Assumption 5.1 In this section, we assume that $0 < y_{\infty} < 1$ and $\bar{\epsilon} = \epsilon = \epsilon$ for $\epsilon \in [0, 1)$.

Notation 5.2 (Current section only)

(i) To emphasize their dependence on the transaction cost parameter $\epsilon$, we denote $v, y, \bar{y}, \hat{y}, L, L_y$ by $v_{\epsilon}, y_{\epsilon}, \bar{y}_{\epsilon}, \hat{y}_{\epsilon}, L_{\epsilon}, L_{\epsilon y}$. In particular, when $\epsilon = 0$, they are denoted by $v_0, y_0, \bar{y}_0, \hat{y}_0, L_0, L_{\epsilon y}$.

(ii) Under Assumption 5.1, Lemma 4.1 and Proposition 4.4 imply that

$$0 < \hat{y}_0(t, x) = \bar{y}_0(t) = \tilde{y}_0(t) < 1 \text{ for } (t, x) \in [0, T) \times [0, 1). \quad (5.1)$$

In words, $\hat{y}_0(t, x)$ is independent of variable $x$ (just a function of $t$) and its value equals $\bar{y}_0(t)$ and $\tilde{y}_0(t)$. For convenience, we abuse notation and write $\hat{y}_0(t)$ for $\hat{y}_0(t, x)$ (i.e., $\hat{y}_0(t) = \bar{y}_0(t) = \tilde{y}_0(t)$).

We start with the technical lemma that is used in the proof of the asymptotic result.

Lemma 5.3 (i) Let $\epsilon = 0$. For $t \in [0, T)$,

$$v_{\epsilon x}^0(t, \hat{y}_0(t)) = 0, \quad (5.2)$$

$$\sup_{x \in (0, 1)} v_{\epsilon x x}^0(t, x) < 0. \quad (5.3)$$

(ii) Let $F : [0, T) \times (0, 1) \rightarrow \mathbb{R}$ be defined as

$$F(t, x) := \lambda \int_t^T e^{-\lambda(s-t) \epsilon} \mathbb{E} \left[ \left( \frac{\partial}{\partial x} y_{\epsilon}^{(t,x)} \right) \cdot \text{sgn} \left( \hat{y}_{\epsilon}^0(s) - Y_{\epsilon}^{(t,x)}(s) \right) \right] ds. \quad (5.4)$$

Then, for $t \in [0, T)$,

$$-1 < F(t, \hat{y}_0(t)) < 1. \quad (5.5)$$
(iii) Suppose that \( x_0 \in (0, 1) \) and \( \lim_{\epsilon \downarrow 0} x_\epsilon = x_0 \). Then, for \( t \in [0, T) \),
\[
\lim_{\epsilon \downarrow 0} v^\epsilon_x(t, x_\epsilon) = v^0_x(t, x_0),
\]
\[
\lim_{\epsilon \downarrow 0} v^\epsilon_{xx}(t, x_\epsilon) = v^0_{xx}(t, x_0).
\]

**Proof** See Appendix. \( \square \)

The following theorem provides the first order approximation of the no-trade boundaries.

**Theorem 5.4** For \( t \in [0, T) \),
\[
y^\epsilon(t) = \tilde{y}^0(t) - \frac{F(t, \tilde{y}^0(t)) - 1}{v^0_{xx}(t, \tilde{y}^0(t))} \cdot \epsilon + o(\epsilon),
\]
\[
\tilde{y}^\epsilon(t) = \tilde{y}^0(t) - \frac{F(t, \tilde{y}^0(t)) + 1}{v^0_{xx}(t, \tilde{y}^0(t))} \cdot \epsilon + o(\epsilon),
\]
where \( F \) is defined in (5.4). In particular, for small enough \( \epsilon > 0 \), we have
\[
0 < y^\epsilon(t) < \tilde{y}^0(t) < \tilde{y}^\epsilon(t) < 1.
\]

**Proof** For \( (t, x) \in [0, T) \times (0, 1) \), the expression (4.11) implies that
\[
\left| v^\epsilon_x(t, x) - v^0_x(t, x) \right| = \left| \int_t^T e^{-\lambda(s-t)} \mathbb{E} \left[ \left( \frac{\partial}{\partial x} Y^{(t,s)} \right) L^\epsilon_y(s, Y^{(t,s)}_s) \right] ds \right| \\
\leq \lambda \int_t^T e^{-\lambda(s-t)} \mathbb{E} \left[ \left( \frac{\partial}{\partial x} Y^{(t,s)} \right) \right] ds \cdot \frac{\epsilon}{1-\epsilon},
\]
where the first inequality is due to (4.13) and the positivity of \( \frac{\partial}{\partial x} Y^{(t,s)} \) in (4.14), and the second inequality is due to Assumption 5.1.

Proposition 4.4 and Assumption 5.1 imply that \( y^\epsilon(t) < 1 \) and \( \tilde{y}^\epsilon(t) > 0 \) for \( t \in [0, T) \). By Lemma 4.1, if \( 0 < y^\epsilon(t) < 1 \), then \( v^\epsilon_x(t, y^\epsilon(t)) = \frac{\epsilon}{1+\epsilon-y^\epsilon(t)} \), and if \( y^\epsilon(t) = 0 \), then \( -\frac{\epsilon}{1-\epsilon} < v^\epsilon_x(t, y^\epsilon(t)) < \frac{\epsilon}{1+\epsilon} \) for \( x \in (0, y^\epsilon(t)) \). In any case, we have
\[
\left| v^\epsilon_x(t, y^\epsilon(t)) \right| \leq \frac{\epsilon}{1-\epsilon},
\]
where \( v^\epsilon_x(t, 0) := \lim_{x \downarrow 0} v^\epsilon_x(t, x) \) is well-defined (see (4.15) for details) for the case of \( y^\epsilon(t) = 0 \). We use the mean value theorem and (5.2) to obtain
\[
\inf_{x \in (0, 1)} \left| v^0_{xx}(t, x) \right| \cdot \left| y^\epsilon(t) - \tilde{y}^0(t) \right| \leq \left| v^0_x(t, y^\epsilon(t)) - v^0_x(t, \tilde{y}^0(t)) \right| \\
\leq \left| v^0_x(t, \tilde{y}^\epsilon(t)) - v^\epsilon_x(t, y^\epsilon(t)) \right| + \left| v^\epsilon_x(t, \tilde{y}^\epsilon(t)) \right|.
\]
From the above inequality, together with (5.3), (5.11), and (5.12), we conclude that
\[ y^\epsilon(t) - \hat{y}^0(t) = O(\epsilon) \text{ for } t \in [0, T). \] (5.14)

By the same way, we also obtain
\[ \bar{y}^\epsilon(t) - \hat{y}^0(t) = O(\epsilon) \text{ for } t \in [0, T). \] (5.15)

The expression of \( L_y \) in (4.10), together with (5.14) and (5.15), implies the following limit:
\[ \lim_{\epsilon \downarrow 0} L_{y}^\epsilon(t, x) = \begin{cases} 1, & \text{if } x \in (0, \hat{y}^0(t)) \\ -1, & \text{if } x \in (\hat{y}^0(t), 1) \end{cases} \text{ for } t \in [0, T). \] (5.16)

For \((s, x) \in (t, T) \times (0, 1)\), the observation \( P\left(Y_s^{(t, x)} = \hat{y}^0(t)\right) = 0 \) and (5.16) produce
\[ \lim_{\epsilon \downarrow 0} L_{y}^\epsilon(s, Y_s^{(t, x)}) = sgn \left( \hat{y}^0(t) - Y_s^{(t, x)} \right) \text{ almost surely.} \] (5.17)

For \((t, x) \in [0, T) \times (0, 1)\), the limit (5.17) and the inequalities (4.13) and (4.14) enable us to use the dominated convergence theorem to obtain
\[ \frac{v^\epsilon_x(t, x) - v^0_x(t, x)}{\epsilon} = \lambda \int_t^T e^{-\lambda(s-t)} E \left[ \left( \frac{\partial}{\partial x} Y_s^{(t, x)} \right) \frac{L_{y}^\epsilon(s, Y_s^{(t, x)})}{\epsilon} \right] ds \rightarrow F(t, x). \] (5.18)

The inequality \( 0 < \hat{y}^0(t) < 1 \) and (5.14) imply that \( 0 < y^\epsilon(t) < 1 \) for small enough \( \epsilon > 0 \). Hence, by Lemma 4.1, we obtain
\[ v^\epsilon_x(t, y^\epsilon(t)) = \frac{\epsilon}{1 + \epsilon y^\epsilon(t)} \text{ for small enough } \epsilon > 0. \] (5.19)

By the mean value theorem, there exists \( k(\epsilon) \) such that
\[ v^\epsilon_x(t, y^\epsilon(t)) - v^\epsilon_x(t, \hat{y}^0(t)) = v^\epsilon_{xx}(t, k(\epsilon)) \left( y^\epsilon(t) - \hat{y}^0(t) \right) \text{ and } \lim_{\epsilon \downarrow 0} k(\epsilon) = \hat{y}^0(t). \] (5.20)
Now we obtain (5.8) as follows:

\[
\lim_{\epsilon \downarrow 0} \frac{y^\epsilon(t) - \hat{y}^0(t)}{\epsilon} = \lim_{\epsilon \downarrow 0} \frac{v_x^\epsilon(t, y^\epsilon(t)) - v_x^\epsilon(t, \hat{y}^0(t))}{\epsilon \cdot v_{xx}^\epsilon(t, k(\epsilon))} = \lim_{\epsilon \downarrow 0} \frac{1}{v_{xx}^\epsilon(t, k(\epsilon))} \left( \frac{1}{1 + \epsilon y^\epsilon(t)} - \frac{v_x^\epsilon(t, \hat{y}^0(t)) - v_x^0(t, \hat{y}^0(t))}{\epsilon} \right) = - \frac{1}{v_{xx}^0(t, \hat{y}^0(t))}.
\]

(5.21)

where the first equality is from (5.20), the second equality is due to (5.19) and (5.2), and the third equality is due to (5.18) and (5.7) with the limit in (5.20). We also obtain (5.9) by the same way.

Due to (5.3) and (5.5), we observe that 

\[-F(t, \hat{y}^0(t)) - \frac{1}{v_{xx}^0(t, \hat{y}^0(t))} < 0 \] and

\[-F(t, \hat{y}^0(t)) + \frac{1}{v_{xx}^0(t, \hat{y}^0(t))} > 0.\]

Therefore, (5.8) and (5.9) imply that (5.10) holds for small enough \(\epsilon > 0.\)

The correction terms in Theorem 5.4 are not explicit in general (see Remark 5.8 for a special case of \(y_\infty = \frac{1}{2}\)). Therefore, one may want to estimate the coefficients of the correction terms:

\[
C_{\text{corr}}^y(t) := - \frac{F(t, \hat{y}^0(t)) - 1}{v_{xx}^0(t, \hat{y}^0(t))},
\]

\[
C_{\text{corr}}^{\bar{y}}(t) := - \frac{F(t, \hat{y}^0(t)) + 1}{v_{xx}^0(t, \hat{y}^0(t))}.
\]

(5.22)

**Proposition 5.5** Let \(C_{\text{corr}}^y(t)\) and \(C_{\text{corr}}^{\bar{y}}(t)\) be as in (5.22). For \(t \in [0, T)\), we have the following limits:

\[
\lim_{\lambda \to \infty} \frac{C_{\text{corr}}^y(t)}{\lambda} = - \frac{1}{\sigma^2},
\]

\[
\lim_{\lambda \to \infty} \frac{C_{\text{corr}}^{\bar{y}}(t)}{\lambda} = \frac{1}{\sigma^2}.
\]

(5.23)

**Proof** Equation (2.7) in [40] and the inequality \(|\hat{y}^0(t) - y_\infty| \leq 1\) imply that there is a constant \(C\) such that

\[
|\hat{y}^0(t) - y_\infty| \leq \frac{C}{\lambda} \quad \text{for} \quad (t, \lambda) \in [0, T) \times (1, \infty).
\]

(5.24)

We define a function \(\Gamma : [0, T) \times (0, 1) \to \mathbb{R}\) as

\[
\Gamma(t, x) := \mathbb{E} \left[ Y_t^{(0,x)} \left( 1 - Y_t^{(0,x)} \right) \left( \mu - r - \sigma^2 Y_t^{(0,x)} \right) \right].
\]

(5.25)
Observe that $\Gamma$ does not depend on $\lambda$. In the proof of Lemma 5.3 (i), we obtain the stochastic representation of $v^0_{xx}$ (see (B.6) in Appendix). This representation can be written in terms of $\Gamma$ as below:

$$v^0_{xx}(t, x) = \int_0^{T-t} e^{-\lambda s} \Gamma_x(s, x) ds. \quad (5.26)$$

Using $Y_{t,x} = x$, direct computations produce

$$\Gamma_x(0, x) = -\sigma^2. \quad (5.27)$$

With (5.27) and (5.24), we apply Lemma D.2 to (5.26) to obtain the following:

$$\lim_{\lambda \to \infty} \lambda v^0_{xx}(t, \hat{y}^0(t)) = -\sigma^2. \quad (5.28)$$

Now we move on to the limit of $F(t, \hat{y}^0(t))$ part. From (5.4), we have

$$F(t, \hat{y}^0(t)) = \lambda \int_t^T e^{-\lambda (s-t)} \mathbb{E} \left[ \left( \frac{\partial}{\partial x} Y_{s}^{(t,x)} \right) \cdot \text{sgn} \left( \hat{y}^0(s) - Y_{s}^{(t,x)} \right) \right] ds \bigg|_{x=\hat{y}^0(t)}$$

$$= \lambda \int_t^T e^{-\lambda (s-t)} \times \mathbb{E} \left[ \frac{e^{(\mu - r - \sigma^2/2) (s-t) + \sigma (B_s - B_t)}}{(x e^{(\mu - r - \sigma^2/2) (s-t) + \sigma (B_s - B_t)} + 1 - x)^2} \cdot \text{sgn} \left( \hat{y}^0(s) - Y_{s}^{(t,x)} \right) \right] ds \bigg|_{x=\hat{y}^0(t)}$$

$$= \int_0^{\lambda(T-t)} e^{-u} \mathbb{E} \left[ \frac{e^{(\mu - r - \sigma^2/2) u + \sigma (B_{t+\frac{u}{\lambda}} - B_t)}}{(x e^{(\mu - r - \sigma^2/2) u + \sigma (B_{t+\frac{u}{\lambda}} - B_t)} + 1 - x)^2} \cdot \text{sgn} \left( \hat{y}^0(t + \frac{u}{\lambda}) - Y_{s}^{(t,x)} \right) \right] du \bigg|_{x=\hat{y}^0(t)}, \quad (5.29)$$

where the second equality is due to the explicit expression of $Y_{s}^{(t,x)}$ in (4.9) and the third equality is obtained by the change of variables as $u = \lambda(s-t)$.

To treat the sign term above, we observe that the explicit expression of $Y_{s}^{(t,x)}$ in (4.9) implies

$$Y_{s}^{(t,x)} < y \iff B_s - B_t < \gamma(s-t, x, y),$$

where $\gamma(t, x, y) := \frac{1}{\sigma} \left( \ln \left( \frac{y}{1-y} \right) - \ln \left( \frac{x}{1-x} \right) - (\mu - r - \frac{\sigma^2}{2}) t \right). \quad (5.30)$$
Then, for fixed \( u > 0 \), (5.24) implies that

\[
\sqrt{\frac{u}{\lambda}} \cdot \gamma \left( \frac{u}{\lambda}, \hat{y}^0(t), \hat{y}^0(t + \frac{u}{\lambda}) \right) = \frac{1}{\sigma} \sqrt{\frac{u}{\lambda}} \left( \ln \left( \frac{\hat{y}^0(t + \frac{u}{\lambda})}{1 - \hat{y}^0(t)} \right) - \ln \left( \frac{\hat{y}^0(t)}{1 - \hat{y}^0(t)} \right) \right) - \frac{\mu - r - \sigma^2}{2} \frac{1}{\sqrt{\frac{u}{\lambda}}}
\]

\[
\lambda \to \infty \quad \longrightarrow 0.
\]

(5.31)

We apply (5.30) and \( \left( B_t + \frac{u}{\lambda} - B_t \right) \sim \sqrt{\frac{u}{\lambda}} \cdot \mathcal{N}(0, 1) \) to (5.29) and produce

\[
F(t, \hat{y}^0(t)) = \int_0^{\lambda(T-t)} \int_{-\infty}^{\hat{y}^0(t + \frac{u}{\lambda})} e^{-u} \cdot \frac{e^{(\mu - r + \sigma^2/2)u + \sigma \sqrt{\frac{u}{\lambda}z}}}{\sigma \sqrt{2\pi} \sqrt{\frac{u}{\lambda}z + 1-x}} \cdot \frac{e^{\frac{-z^2}{2}}}{\sqrt{2\pi}} dz du \bigg|_{x=\hat{y}^0(t)} \]

\[
- \int_0^{\lambda(T-t)} \int_{\hat{y}^0(t + \frac{u}{\lambda})}^{\infty} e^{-u} \cdot \frac{e^{(\mu - r + \sigma^2/2)u + \sigma \sqrt{\frac{u}{\lambda}z}}}{\sigma \sqrt{2\pi} \sqrt{\frac{u}{\lambda}z + 1-x}} \cdot \frac{e^{\frac{-z^2}{2}}}{\sqrt{2\pi}} dz du \bigg|_{x=\hat{y}^0(t)} \]

\[
\lambda \to \infty \quad \longrightarrow \int_0^{\infty} \int_{-\infty}^{0} e^{-u} \cdot \frac{e^{\frac{-z^2}{2}}}{\sqrt{2\pi}} dz du - \int_0^{\infty} \int_{0}^{\infty} e^{-u} \cdot \frac{e^{\frac{-z^2}{2}}}{\sqrt{2\pi}} dz du = 0,
\]

(5.32)

where the convergence is due to (5.24), (5.31), and the dominated convergence theorem.

Finally, the limits in (5.28) and (5.32) produce (5.23). \( \square \)

The following theorem provides the first order approximation of the value function.

**Theorem 5.6** For \( t \in [0, T) \),

\[
v^\epsilon(t, \hat{y}^0(t)) = v^0(t, \hat{y}^0(t)) - \left( G(t) + \lambda \int_t^T G(s) ds \right) \cdot \epsilon + o(\epsilon),
\]

(5.33)

where \( G : [0, T] \to \mathbb{R} \) is defined as

\[
G(t) := \lambda \int_t^T e^{-\lambda(s-t)} \mathbb{E} \left[ \left| Y_s(y^0(t)) - y^0(s) \right| \right] ds.
\]

(5.34)
**Proof** For $\epsilon > 0$, we define a function $\psi^\epsilon : [0, T) \to \mathbb{R}$ as

$$
\psi^\epsilon (t) := \frac{v^\epsilon (t, \hat{y}^0(t)) - v^0 (t, \check{y}^0(t))}{\epsilon} = \frac{1}{\epsilon} \int_t^T e^{-\lambda(s-t)} \mathbb{E} \left[ \lambda \left( L^\epsilon (s, Y_s(t, \hat{y}^0(t))) - L^0(s, Y_s(t, \check{y}^0(t))) \right) \right] ds
$$

(5.35)

where the second equality is due to the expression of $v^\epsilon$ in (4.8). Using the expression of $L^\epsilon$ in (4.12) and the fact that $L^0(s, y) = v^0(s, \check{y}^0(s))$, we observe

$$
L^\epsilon (s, Y_s(t, \hat{y}^0(t))) - L^0(s, Y_s(t, \check{y}^0(t))) = v^\epsilon (s, \hat{y}^0(s)) - v^0(s, \check{y}^0(s))
$$

$$
= \begin{cases} 
  v^\epsilon (s, \hat{y}^0(s)) - v^0 (s, \check{y}^0(s)) - \ln \left( \frac{1+\epsilon \psi^\epsilon (s)}{1+\epsilon \psi^0(s)} \right), & \text{if } Y_s(t, \hat{y}^0(t)) \leq Y_s(t, \check{y}^0(t)) \\
  v^\epsilon (s, Y_s(t, \check{y}^0(t))) - v^\epsilon (s, \hat{y}^0(s)), & \text{if } \psi^\epsilon (s) < Y_s(t, \hat{y}^0(t)) < \check{y}^\epsilon (s) \\
  v^\epsilon (s, \check{y}^\epsilon (s)) - v^\epsilon (s, \check{y}^0(s)) - \ln \left( \frac{1-\epsilon \check{y}^\epsilon (s)}{1-\epsilon \check{y}^0(s)} \right), & \text{if } \check{y}^\epsilon (s) \leq Y_s(t, \hat{y}^0(t)) 
\end{cases}
$$

We apply the above expression to (5.35) to obtain

$$
\psi^\epsilon (t) = \lambda \int_t^T e^{-\lambda(s-t)} \psi^\epsilon (s) ds 
\quad + \lambda \int_t^T e^{-\lambda(s-t)} \mathbb{E} \left[ I_1^{(t,s,\epsilon)} + I_2^{(t,s,\epsilon)} + I_3^{(t,s,\epsilon)} \right] ds,
$$

(5.36)

where the random variables $I_1^{(t,s,\epsilon)}$, $I_2^{(t,s,\epsilon)}$, $I_3^{(t,s,\epsilon)}$ are defined as

$$
I_1^{(t,s,\epsilon)} := \left( \frac{v^\epsilon (s, \hat{y}^\epsilon (s)) - v^\epsilon (s, \check{y}^0(s))}{\epsilon} \right) - \frac{1}{\epsilon} \ln \left( \frac{1+\epsilon \psi^\epsilon (s)}{1+\epsilon \psi^0(s)} \right) \cdot 1 \left\{ Y_s(t, \hat{y}^0(t)) \leq \check{y}^\epsilon (s) \right\};
$$

$$
I_2^{(t,s,\epsilon)} := \left( \frac{v^\epsilon (s, Y_s(t, \check{y}^0(t))) - v^\epsilon (s, \hat{y}^0(s))}{\epsilon} \right) \cdot 1 \left\{ \psi^\epsilon (s) < Y_s(t, \hat{y}^0(t)) < \check{y}^\epsilon (s) \right\};
$$

$$
I_3^{(t,s,\epsilon)} := \left( \frac{v^\epsilon (s, \check{y}^\epsilon (s)) - v^\epsilon (s, \check{y}^0(s))}{\epsilon} \right) - \frac{1}{\epsilon} \ln \left( \frac{1-\epsilon \check{y}^\epsilon (s)}{1-\epsilon \check{y}^0(s)} \right) \cdot 1 \left\{ \check{y}^\epsilon (s) \leq Y_s(t, \hat{y}^0(t)) \right\}.
$$

(5.37)

The mean value theorem produces

$$
\frac{1}{\epsilon} \ln \left( \frac{1+\epsilon \psi^\epsilon (s)}{1+\epsilon x} \right) \leq 1 \quad \text{and} \quad \frac{1}{\epsilon} \ln \left( \frac{1-\epsilon \check{y}^\epsilon (s)}{1-\epsilon x} \right) \leq \frac{1}{1-\epsilon} \quad \text{for } x \in (0, 1).
$$

(5.38)
By Assumption 5.1 and Proposition 4.4 (ii), we obtain $y^\epsilon(s) < 1$ and $\bar{y}^\epsilon(s) > 0$ for $s \in [0, T)$. The definition of $Y_s^{(t,x)}$ in (4.9) indicates that $0 < Y_s^{(t,x)} < 1$. Therefore,

$$\frac{1}{1 - y^\epsilon(s)} \leq \frac{1}{1 - \bar{y}^\epsilon(s)} \leq \frac{1}{1 - \hat{y}^\epsilon(s)} \leq 1$$ \hspace{1cm} (5.39)

The concavity of $v^\epsilon$ on $x$ variable (see Proposition 3.6) and (5.38) and (5.39) imply

$$|I_1^{(t,s,\epsilon)}| \leq \left( \max \left\{ \frac{v^\epsilon_x(s, y^\epsilon(s))}{\epsilon}, \frac{v^\epsilon_x(s, \bar{y}^\epsilon(s))}{\epsilon} \right\} \cdot \left| y^\epsilon(s) - y^0(s) \right| + 1 \right) \cdot \frac{1}{1 - \epsilon} \cdot 1_{\{0 < y^\epsilon(s) < 1\}}.$$ \hspace{1cm} (5.40)

Combining (5.11) and (5.2), we obtain

$$\left| v^\epsilon_x(s, \bar{y}^\epsilon(s)) \right| \leq \lambda T \cdot e^{(\mu - r + \sigma^2)T} \cdot \frac{\epsilon}{1 - \epsilon}.$$ \hspace{1cm} (5.41)

By Lemma 4.1, we have the following inequalities:

$$\left| v^\epsilon_x(s, y^\epsilon(s)) \right| \cdot \frac{1}{1 - y^\epsilon(s)} \leq \epsilon, \quad \left| v^\epsilon_x(s, \bar{y}^\epsilon(s)) \right| \cdot \frac{1}{1 - \bar{y}^\epsilon(s)} \leq \frac{\epsilon}{1 - \epsilon},$$

$$\left| v^\epsilon_x(s, x) \right| \cdot \frac{1}{1 - \hat{y}^\epsilon(s)} \leq \frac{\epsilon}{1 - \epsilon} \quad \text{for} \quad x \in (0, 1).$$ \hspace{1cm} (5.42)

Now we apply (5.41) and (5.42) to (5.40) and obtain

$$|I_1^{(t,s,\epsilon)} + I_2^{(t,s,\epsilon)} + I_3^{(t,s,\epsilon)}| \leq C \cdot \frac{1}{1 - \epsilon},$$ \hspace{1cm} (5.43)

where $C$ is a constant independent of $(t, s, \epsilon)$. Also, Theorem 5.4, the mean value theorem, and (5.6) imply that

$$\lim_{\epsilon \downarrow 0} \left( I_1^{(t,s,\epsilon)} + I_2^{(t,s,\epsilon)} + I_3^{(t,s,\epsilon)} \right) = - \left| Y_s^{(t,\hat{y}^0(t))} - \bar{y}^0(s) \right| \quad \text{almost surely.}$$ \hspace{1cm} (5.44)

The uniform boundedness in (5.43) and the convergence in (5.44) allow us to apply the dominated convergence theorem and conclude

$$\lim_{\epsilon \downarrow 0} J^\epsilon(t) = -G(t) \quad \text{and} \quad \lim_{\epsilon \downarrow 0} \int_t^T J^\epsilon(s) ds = - \int_t^T G(s) ds,$$ \hspace{1cm} (5.45)
where $G$ is defined in (5.34) and the function $J^\epsilon : [0, T] \to \mathbb{R}$ is defined as

$$J^\epsilon(t) := \lambda \int_t^T e^{-\lambda(s-t)} \mathbb{E} \left[ I_1^{(t,s,\epsilon)} + I_2^{(t,s,\epsilon)} + I_3^{(t,s,\epsilon)} \right] ds.$$  

We rewrite (5.36) as

$$J^\epsilon(t) = \psi^\epsilon(t) - \lambda \int_t^T e^{-\lambda(s-t)} \psi^\epsilon(s) ds = -\frac{d}{dt} \left( \int_t^T e^{-\lambda(s-t)} \psi^\epsilon(s) ds \right).$$

We integrate both sides above to obtain

$$\int_t^T e^{-\lambda(s-t)} \psi^\epsilon(s) ds = \int_t^T J^\epsilon(s) ds. \quad (5.46)$$

Using (5.46), the equation (5.36) becomes

$$\psi^\epsilon(t) = J^\epsilon(t) + \lambda \int_t^T J^\epsilon(s) ds. \quad (5.47)$$

Finally, (5.45) and (5.47) imply (5.33). \qed

The correction term in Theorem 5.6 is not explicit in general (see Remark 5.8 for a special case of $y_\infty = \frac{1}{2}$). As in the no-trade boundary case, we try to estimate the coefficient of the correction term:

$$C^v_{\text{corr}}(t) := -\left( G(t) + \lambda \int_t^T G(s) ds \right). \quad (5.48)$$

**Proposition 5.7** Let $C^v_{\text{corr}}(t)$ be as in (5.48). For $t \in [0, T)$, we have the following limits:

$$\lim_{\lambda \to \infty} \frac{C^v_{\text{corr}}(t)}{\sqrt{\lambda}} = -\frac{\sigma}{\sqrt{2}} y_\infty (1 - y_\infty)(T - t). \quad (5.49)$$

**Proof** Explicit computations using the expression of $Y_s^{(t,x)}$ in (4.9) produce

$$E \left[ Y_s^{(t,x)} - x \right] \sqrt{s - t} = \int \frac{x (1 - x)}{x + (1 - x) e^{(r - \mu + \frac{\sigma^2}{2}) u^2 - \sigma u z}} \cdot \left| 1 - e^{(r - \mu + \frac{\sigma^2}{2}) u^2 - \sigma u z} \right| \cdot e^{-\frac{z^2}{2}} \sqrt{2\pi} dz \bigg|_{u=\sqrt{s-t}} \quad (5.50)$$

$$\leq \int e^{(r - \mu + \frac{\sigma^2}{2}) |w^2 + \sigma u z|} - e^{-(r - \mu + \frac{\sigma^2}{2}) |w^2 - \sigma u z|} \cdot e^{-\frac{z^2}{2}} \sqrt{2\pi} dz \bigg|_{u=\sqrt{s-t}}$$

$$e^{(r - \mu + \frac{\sigma^2}{2}) |u^2|} \left( 1 + \int_0^{\frac{\sigma u}{\sqrt{2}}} \frac{2e^{-z^2}}{\sqrt{\pi}} dz \right) - e^{-(r - \mu + \frac{\sigma^2}{2}) |u^2|} \left( 1 - \int_0^{\frac{\sigma u}{\sqrt{2}}} \frac{2e^{-z^2}}{\sqrt{\pi}} dz \right) \bigg|_{u=\sqrt{s-t}}.$$
where we use $|1 - e^a| \leq e^{|a|} - e^{-|a|}$ for $a \in \mathbb{R}$ to obtain the inequality. The last expression converges to $\frac{2\sqrt{2\pi}}{\sqrt{2\pi}}$ as $u \downarrow 0$, therefore, it is bounded on $u \in (0, \sqrt{T}]$. We conclude that there is a constant $C$ such that

$$\frac{\mathbb{E} \left[ Y_s^{(t,x)} - x \right]}{\sqrt{s - t}} \leq C \text{ for } (t, s, x) \in [0, T] \times (t, T] \times (0, 1).$$

(5.51)

Observe that the integrand in (5.50) is bounded by $e^{C|z|} \cdot \frac{e^{-z^2}}{\sqrt{2\pi}}$ for a constant $C$ independent of $(u, x, z) \in (0, \sqrt{T}] \times (0, 1) \times \mathbb{R}$. Since this bound is integrable with respect to $z$, we apply the dominated convergence theorem to (5.50) and obtain

$$\lim_{s \downarrow t} \frac{\mathbb{E} \left[ Y_s^{(t,x)} - x \right]}{\sqrt{s - t}} = \int_{\mathbb{R}} x(1 - x)\sigma|z| \cdot \frac{e^{-z^2}}{\sqrt{2\pi}} dz = \sigma \sqrt{\frac{2}{2\pi}}(1 - x).$$

(5.52)

Using (5.24) and (5.52), we apply Lemma D.2 to conclude that for $t \in [0, T)$,

$$\lambda^\frac{3}{2} \int_t^T e^{-\lambda(s-t)} \mathbb{E} \left[ Y_s^{(t,y^0(t))} - y^0(t) \right] ds = \lambda^\frac{3}{2} \int_0^{T-t} e^{-\lambda u} u^\frac{1}{2} \mathbb{E} \left[ Y_{u+t}^{(t,y^0(t))} - y^0(t) \right] du \xrightarrow{\lambda \to \infty} \frac{\sigma}{\sqrt{2}} y_\infty (1 - y_\infty).$$

(5.53)

Also, (5.24) implies that for $t \in [0, T)$,

$$\lambda^\frac{3}{2} \int_t^T e^{-\lambda(s-t)} \left| y^0(s) - y^0(t) \right| ds \xrightarrow{\lambda \to \infty} 0.$$

(5.54)

We combine (5.53) and (5.54) to obtain

$$\lim_{\lambda \to \infty} \lambda^\frac{1}{2} G(t) = \frac{\sigma}{\sqrt{2}} y_\infty (1 - y_\infty) \text{ for } t \in [0, T).$$

(5.55)

To treat the limit of the integral term in (5.48), we need suitable bound of the integrand to apply the dominated convergence theorem. Inequalities (5.24) and (5.51) imply

$$\mathbb{E} \left[ Y_s^{(t,y^0(t))} - y^0(s) \right] \leq C \left( \sqrt{s-t} + \frac{1}{\lambda} \right) \text{ for } (t, s, \lambda) \in [0, T) \times (t, T] \times [1, \infty),$$

(5.56)
for a constant $C$. The above inequality, together with the bound in (D.5), produces
\[
\left| \lambda \frac{1}{2} G(s) \right| \leq C \quad \text{for} \quad (s, \lambda) \in [0, T] \times [1, \infty).
\] (5.57)
for a constant $C$. Therefore, we apply the dominated convergence theorem to obtain
\[
\lim_{\lambda \to \infty} \frac{C_{\text{corr}}(t)}{\sqrt{\lambda}} = \lim_{\lambda \to \infty} \left( -\frac{G(t)}{\sqrt{\lambda}} - \int_t^T \lambda \frac{1}{2} G(s) ds \right) = -\frac{\sigma}{\sqrt{2}} y_\infty (1 - y_\infty)(T - t),
\]
where we use (5.55) and (5.57).

**Remark 5.8** The correction terms in Theorem 5.4 and Theorem 5.6 are expressed through the functions $F(t, \hat{y}^0(t))$ and $G(t)$ from (5.4) and (5.34). In general, these functions are not explicit because $\hat{y}^0(t)$ cannot be written explicitly. However, in case $y_\infty = \frac{1}{2}$, we can evaluate these expressions explicitly as follows:

\[
F(t, \hat{y}^0(t)) = 0, \quad G(t) = \int_0^T \int_0^\infty \lambda e^{-\lambda u} \frac{e^{\sigma z} - 1}{e^{\sigma z} + 1} \cdot \frac{e^{-\frac{2}{\sqrt{2\pi}}}}{\sqrt{2\pi}} \, dz \, du. \quad (5.58)
\]
To check this, we first use the expression of $v_0^0$ in (B.5) and $Y_s(t, x)$ in (4.9) to obtain
\[
v_x^0(t, \frac{1}{2}) = \int_t^T e^{-\lambda(s-t)} \int_{-\infty}^\infty 2\sigma^2 e^{\sigma z} (1 - e^{\sigma z}) \cdot \frac{e^{-\frac{2}{\sqrt{2\pi}}}}{\sqrt{2\pi}} \, dz \, ds. \quad (5.59)
\]
Since the integrand above is an odd function of $z$, the above integral is zero. This implies that $\hat{y}^0(t) = \frac{1}{2}$ for all $t \in [0, T)$. We substitute $\hat{y}^0(t) = \frac{1}{2}$ to (5.4) and (5.34), and obtain the desired result:

\[
F(t, \hat{y}^0(t)) = \lambda \int_t^T e^{-\lambda(s-t)} \int_{-\infty}^\infty \frac{4e^{\sigma z}}{(e^{\sigma z} + 1)^3} \, \text{sgn}(-z) \cdot \frac{e^{-\frac{2}{\sqrt{2\pi}}}}{\sqrt{2\pi}} \, dz \, ds = 0,
\]

\[
G(t) = \frac{1}{2} \lambda \int_t^T e^{-\lambda(s-t)} \int_{-\infty}^\infty \frac{e^{\sigma z} - 1}{e^{\sigma z} + 1} \, \text{sgn}(z) \cdot \frac{e^{-\frac{2}{\sqrt{2\pi}}}}{\sqrt{2\pi}} \, dz \, ds = \int_0^T \int_0^\infty \lambda e^{-\lambda u} \frac{e^{\sigma z} - 1}{e^{\sigma z} + 1} \cdot \frac{e^{-\frac{2}{\sqrt{2\pi}}}}{\sqrt{2\pi u}} \, dz \, du,
\]
where the second (fourth, resp.) equality holds because the integrand is an odd (even, resp.) function of $z$. \[ Springer \]
The left graph shows $y^\epsilon(t)$ and $\bar{y}^\epsilon(t)$ as functions of $\epsilon$, where the dashed lines are the linear approximations of them in Theorem 5.4, i.e., $\hat{y}^0_0(t) - F(t, \hat{y}^0_0(t)) - 1/v^x_0(t, \hat{y}^0_0(t)) \cdot \epsilon$ and $\bar{y}^0(t) - F(t, \bar{y}^0(t)) + 1/v^x_0(t, \bar{y}^0(t)) \cdot \epsilon$. The right graph describes $v^\epsilon(t, \hat{y}^0_0(t))$ as a function of $\epsilon$, where the dashed line is the linear approximation of it in Theorem 5.6, i.e., $v^0_0(t, \hat{y}^0_0(t)) - (G(t) + \lambda \int_t^T G(s)ds) \cdot \epsilon$. In both graphs, the parameters are $\mu = 0.4$, $r = 0.1$, $\sigma = 1$, $\lambda = 3$, $T = 1$, and $t = 0.75$.

**Fig. 2**

**Corollary 5.9** For $(t, x) \in [0, T) \times (0, 1)$,

$$v^\epsilon(t, x) = v^0(t, x) - \left(G(t) + \lambda \int_t^T G(s)ds - \int_{\hat{y}^0(t)}^x F(t, \eta)d\eta\right) \cdot \epsilon + o(\epsilon),$$

where $F$ and $G$ are defined in (5.4) and (5.34).

**Proof** The inequality (5.11) and the limit (5.18) allow us to use the dominated convergence theorem, and we obtain

$$\lim_{\epsilon \downarrow 0} \int_x^{\hat{y}^0(t)} \frac{v^\epsilon(t, \eta) - v^0_0(t, \eta)}{\epsilon} d\eta = \int_x^{\hat{y}^0_0(t)} F(t, \eta)d\eta.$$  

Using the above limit and Theorem 5.6, we obtain

$$\lim_{\epsilon \downarrow 0} \frac{v^\epsilon(t, x) - v^0(t, x)}{\epsilon} = v^\epsilon(t, \hat{y}^0_0(t)) - v^0_0(t, \hat{y}^0_0(t)) + \int_x^{\hat{y}^0_0(t)} \frac{v^\epsilon(t, \eta) - v^0_0(t, \eta)}{\epsilon} d\eta \rightarrow - \left(G(t) + \lambda \int_t^T G(s)ds\right) + \int_x^{\hat{y}^0_0(t)} F(t, \eta)d\eta.$$

The above limit is equivalent to (5.60).

Not surprisingly, Theorems 5.4 and 5.6 indicate that the no-trade region widens and the value function diminishes as the transaction cost parameter $\epsilon$ increases. See Fig. 2 for numerical illustrations.

Our next task is to investigate some intertwined effects of the search frictions and transaction costs on the no-trade region and value function. To be specific, for two different search friction parameters $\lambda_1 < \lambda_2$, we would like to compare the magnitude of the widening (diminishing, resp.) effects of the transaction costs on
the no-trade region (the value function, resp.). Figure 3 numerically describes this comparison result. The following technical lemma turns out to be useful to the proof of this comparison.

**Lemma 5.10**

(i) \( \frac{\partial}{\partial \lambda} v_0(t, x) \) and \( \frac{\partial}{\partial \lambda} v_{xx}(t, x) \) exist for \((t, x) \in [0, T] \times (0, 1)\), and

\[
\lim_{\lambda \to \infty} \lambda^{-2} \frac{\partial}{\partial \lambda} v_0(t, x) \bigg|_{x = \hat{y}^0(t)} = 0, \quad \lim_{\lambda \to \infty} \lambda^{-2} \frac{\partial}{\partial \lambda} v_{xx}(t, x) \bigg|_{x = \hat{y}^0(t)} = \sigma^2, \quad \lim_{\lambda \to \infty} \lambda v_{xx}(t, \hat{y}^0(t)) = 0.
\]  
(5.61)

(ii) \( \frac{\partial}{\partial \lambda} \hat{y}^0(t) \) exists for \( t \in [0, T) \), and there is a constant \( C \) such that

\[
\left| \frac{\partial}{\partial \lambda} \hat{y}^0(t) \right| \leq \frac{C}{\lambda^2} \quad \text{for} \quad (t, \lambda) \in [0, T) \times [1, \infty).
\]  
(5.62)

(iii) \( \frac{\partial}{\partial \lambda} \mathbb{E} \left[ \left| Y_s^t(t, \hat{y}^0(t)) - \hat{y}^0(s) \right| \right] \) exists for \( 0 \leq t \leq s \leq T \), and there is a constant \( C \) such that

\[
\left| \frac{\partial}{\partial \lambda} \mathbb{E} \left( \mathbb{E} \left[ \left| Y_s^t(t, \hat{y}^0(t)) - \hat{y}^0(s) \right| \right] \right) \right| \leq \frac{C}{\lambda^2} \quad \text{for} \quad (t, s, \lambda) \in [0, T) \times [t, T] \times [1, \infty).
\]  
(5.63)

**Proof** See Appendix. \( \square \)

**Proposition 5.11** For \( t \in [0, T) \), there exists \( \Lambda(t) > 0 \) such that \( \Lambda(t) < \lambda_1 < \lambda_2 \) implies

\[
\left\{ \begin{array}{l}
(\hat{y}^\epsilon(t) - y^\epsilon(t)) \big|_{\lambda = \lambda_1} < \left( \hat{y}^\epsilon(t) - y^\epsilon(t) \right) \big|_{\lambda = \lambda_2} \\
(v_0^\epsilon(t, \hat{y}^0(t)) - v^\epsilon(t, \hat{y}^0(t))) \big|_{\lambda = \lambda_1} < \left( v_0^\epsilon(t, \hat{y}^0(t)) - v^\epsilon(t, \hat{y}^0(t)) \right) \big|_{\lambda = \lambda_2}
\end{array} \right. \quad \text{for small enough} \ \epsilon > 0.
\]
Proof Theorem 5.4 implies that

\[ \bar{y}^\varepsilon(t) - y^\varepsilon(t) = -\frac{2}{v_{x_0}^0(t, \bar{y}^0(t))} \cdot \epsilon + o(\epsilon). \] (5.64)

Therefore, to prove the first statement, it is enough to show that \( \frac{\partial}{\partial \lambda} \left( v_{x_0}^0(t, \bar{y}^0(t)) \right) > 0 \) for sufficiently large \( \lambda \). Indeed, using (i) and (ii) in Lemma 5.10, we obtain

\[
\lambda^2 \frac{\partial}{\partial \lambda} \left( v_{xx}^0(t, \bar{y}^0(t)) \right) = \lambda^2 \frac{\partial}{\partial \lambda} v_{xx}^0(t, x) \bigg|_{x=\bar{y}^0(t)} + \lambda v_{xxx}^0(t, \bar{y}^0(t)) \cdot \lambda \frac{\partial}{\partial \lambda} \bar{y}^0(t) \xrightarrow{\lambda \to \infty} \sigma^2 > 0. \] (5.65)

To prove the second statement, it is enough check \( \frac{\partial}{\partial \lambda} \left( G(t) + \lambda \int_t^T G(s) ds \right) > 0 \) for sufficiently large \( \lambda \), according to Theorem 5.6. For this purpose, we focus on proving the following:

\[
\lim_{\lambda \to \infty} \sqrt{\lambda} \left( \frac{\partial}{\partial \lambda} \left( G(t) + \lambda \int_t^T G(s) ds \right) \right) > 0. \] (5.66)

By (iii) in Lemma 5.10, we can take derivative inside of the expectations and obtain

\[
\sqrt{\lambda} \left( \frac{\partial}{\partial \lambda} \left( G(t) + \lambda \int_t^T G(s) ds \right) \right) = \lambda^\frac{1}{2} \frac{\partial}{\partial \lambda} G(t) + \int_t^T \left( \lambda^\frac{3}{2} G(s) + \lambda^\frac{5}{2} \frac{\partial}{\partial \lambda} G(s) \right) ds, \] (5.67)

where the partial derivative \( \frac{\partial}{\partial \lambda} G(t) \) can be written as

\[
\frac{\partial}{\partial \lambda} G(t) = \int_t^T e^{-\lambda(s-t)} (1 - \lambda(s-t)) \mathbb{E} \left[ \left| Y_s(t, \bar{y}^0(t)) - \bar{y}^0(s) \right| \right] ds.
\]

We use (5.24), (5.52), (5.63) and (D.4) to obtain the following limit:

\[
\lim_{\lambda \to \infty} \lambda^\frac{3}{2} \frac{\partial}{\partial \lambda} G(s) = -\frac{\sigma}{2\sqrt{2}} y_\infty(1 - y_\infty) \quad \text{for} \quad s \in [0, T). \] (5.68)

We apply (5.63), (5.56) and (D.5) to the expressions of \( G \) and \( \frac{\partial}{\partial \lambda} G \) to obtain the boundedness:

\[
\left| \lambda^\frac{1}{2} G(s) \right| + \left| \lambda^\frac{3}{2} \frac{\partial}{\partial \lambda} G(s) \right| \leq C \quad \text{for} \quad (s, \lambda) \in [0, T] \times [1, \infty). \] (5.69)
Finally, (5.68) and (5.69) enable us to apply the dominated convergence theorem to (5.67):

$$
\lim_{\lambda \to \infty} \sqrt{\lambda} \left( \frac{\partial}{\partial \lambda} \left( G(t) + \lambda \int_t^T G(s) ds \right) \right) = \frac{\sigma}{2\sqrt{2}} y_{\infty}(1 - y_{\infty})(T - t) > 0. \quad (5.70)
$$

Therefore, we conclude (5.66) and the proof is done. \(\Box\)

**Remark 5.12** Proposition 5.11 implies that the effects of the transaction costs are more pronounced (more widening effect of the no-trade region and more diminishing effect of the value function) in the market with less search frictions. For better understanding of the result, it would be helpful to observe that the following two extreme cases ($\lambda = 0$ and $\lambda = \infty$) are consistent with the result:

(i) If we consider an extreme case of $\lambda = 0$ (i.e., no trading opportunity), the transaction costs do not affect the investor anyway. In words, in the market with very severe search frictions, the effects of the transaction costs are negligible.

(ii) If we consider an extreme case of $\lambda = \infty$ (i.e., continuous trading opportunities), it is well known that the width of the no-trade region is the order of $\epsilon^{\frac{1}{3}}$ and the decrease of the value function is the order of $\epsilon^{\frac{2}{3}}$. In contrast, when $\lambda < \infty$, Theorems 5.4 and 5.6 say that both the width of the no-trade region and the decrease of the value function are the order of $\epsilon$. Obviously, for small enough $\epsilon$, we have $\epsilon < \epsilon^{\frac{1}{3}}$ and $\epsilon < \epsilon^{\frac{2}{3}}$. This implies that in the market with the search frictions ($\lambda < \infty$), the effects of the transaction costs are less pronounced, compared to the market with no search friction ($\lambda = \infty$).

**Remark 5.13** In [24], wealth turnover is considered as a benchmark for trading volume. We can investigate how the search frictions affect wealth turnover using the asymptotic result we obtain in the proof of Proposition 5.11. To focus on the search friction parameter $\lambda$, we set $\epsilon = 0$ and $X_0 = \hat{y}^0(0)$ in the discussion below. As in [24], we define aggregate wealth turnover as

$$
aggregate \ wealth \ turnover := \int_0^T \left| \frac{\hat{M}_t}{W_t} \right| dP_t
$$

$$
= \sum_{n=0}^{\infty} \left| Y^{(\tau_n, \hat{y}^0(\tau_n))} - \hat{y}^0(\tau_{n+1}) \right| \cdot 1_{\{\tau_{n+1} \leq T\}}, \quad (5.71)
$$

where $\tau_n = \inf\{t \geq 0 : P_t = n\}$ for $n \in \mathbb{N}$ and $\tau_0 = 0$. Observe that

$$
\mathbb{E} \left[ \left| Y^{(0, \hat{y}^0(0))} - \hat{y}^0(\tau_1) \right| \cdot 1_{\{\tau_1 \leq T\}} \right] = \int_0^T \mathbb{E} \left[ \left| Y^{(0, \hat{y}^0(0))} - \hat{y}^0(t) \right| \right] \cdot P(\tau_1 = t) \, dt
$$

$$
= \int_0^T \lambda e^{-\lambda t} \mathbb{E} \left[ \left| Y^{(0, \hat{y}^0(0))} - \hat{y}^0(t) \right| \right] dt, \quad (5.72)
$$
where the first equality is due to the independence of \( B \) and \( P \). Similarly, for \( n \in \mathbb{N} \),

\[
\mathbb{E} \left[ \left| Y_{t_{n+1}}(\tau_n) - \hat{y}_0(\tau_{n+1}) \right| \cdot 1_{\{\tau_{n+1} \leq T\}} \right] \\
= \int_0^T \left( \int_t^T \mathbb{E} \left[ \left| Y_{s}(t, \hat{y}_0(t)) - \hat{y}_0(s) \right| \right] \cdot \mathbb{P}(\tau_{n+1} = s | \tau_n = t) \, ds \right) \cdot \mathbb{P}(\tau_n = t) \, dt \\
= \int_0^T \left( \int_t^T \lambda e^{-\lambda(s-t)} \mathbb{E} \left[ \left| Y_{s}(t, \hat{y}_0(t)) - \hat{y}_0(s) \right| \right] \, ds \right) \cdot \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} \, dt \quad (5.73)
\]

We combine (5.71), (5.72), (5.73), and (5.34) to express the expected aggregate wealth turnover:

\[
\mathbb{E}[\text{aggregate wealth turnover}] = G(0) + \lambda \int_0^T G(t) \, dt.
\]

By (5.55) and the above expression, we observe that for large enough \( \lambda \),

\[
\mathbb{E}[\text{aggregate wealth turnover}] \approx \frac{\sigma}{\sqrt{2}} y_\infty (1 - y_\infty) T \cdot \sqrt{\lambda}, \\
\mathbb{E}[\text{number of trades}] \approx \frac{\sigma}{\sqrt{2}} y_\infty (1 - y_\infty) \cdot \frac{1}{\sqrt{\lambda}}.
\]

Therefore, as \( \lambda \to \infty \), total trading volume tends to infinity, but each trading size tends to zero.

### 6 Conclusion

This paper presents a utility maximization problem of the terminal wealth, in a market with two different types of the illiquidity: search frictions and transaction costs. We show the existence of the solution to the HJB equation that is regular enough, and provide the verification argument. The optimal trading strategy is characterized by a no-trade region, where the boundary points of the no-trade region are uniquely determined by the strict concavity of the value function. In Proposition 4.4, we provide some conditions under which the investor wants to achieve zero/positive stock/bond holdings. We provide the asymptotic expansions of the boundaries of the no-trade region and the value function for small transaction costs. We further show that reduction of the search frictions amplifies the effects of the transaction costs (more widening effect of the no-trade region and more diminishing effect of the value function).

As a future research, we plan to investigate a joint limiting behavior of the no-trade region and the value function for small transaction costs and large arrival rate. For
that purpose, we think it could be useful to consider some stationary versions of the problem with a hope to obtain explicit formulas independent of the time variable.

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**Declarations**

**Conflict of interest** The authors have no competing interests to declare that are relevant to the content of this article.

**Appendix A. Proof of Lemma 3.1**

Since the parabolic type PDE (3.2) is not uniformly elliptic, we change variable as \( x = h(z) := \frac{e^z}{1+e^z} \) and consider the PDE for \( v(t, h(z)) \).

To handle the nonlinear term in the PDE, we first consider the following map \( \phi \) from \( C_b([0, T] \times \mathbb{R}) \) (equipped with the uniform norm) to itself:

\[
\phi(f)(t, z) := \int_t^T e^{-\lambda(s-t)} \mathbb{E} \left[ K_f(s, Z_s(t, z)) \right] ds,
\]

where for \( (s, z) \in [t, T] \times \mathbb{R} \),

\[
Z_s(t, z) := z + \left( \mu - r - \frac{\sigma^2}{2} \right) (s-t) + \sigma (B_s - B_t),
\]

\[
K_f(s, z) := (\mu - r) h(z) + r - \frac{1}{2} \sigma^2 h(z)^2 + \lambda \sup_{\zeta \in \mathbb{R}} \left( f(s, \zeta) - \ln \left( \frac{1+\bar{\epsilon} h(\zeta)}{1+\epsilon h(z)} \right) I_{\{z<\zeta\}} - \ln \left( \frac{1-\epsilon h(\zeta)}{1-\bar{\epsilon} h(z)} \right) I_{\{z>\zeta\}} \right).
\]

We observe that for \( f, g \in C_b([0, T] \times \mathbb{R}) \),

\[
\|\phi(f) - \phi(g)\|_{\infty} \leq \lambda \int_t^T e^{-\lambda(s-t)} \left( \sup_{\zeta \in \mathbb{R}} |f(s, \zeta) - g(s, \zeta)| \right) ds \leq \left( 1 - e^{-\lambda T} \right) \|f - g\|_{\infty},
\]

where the first inequality is due to the triangular inequality for supremum. Therefore, the map \( \phi \) in (A.1) is a contraction map and there exists a unique \( \hat{u} \in C_b([0, T] \times \mathbb{R}) \) such that \( \phi(\hat{u}) = \hat{u} \), by the Banach fixed point theorem.

**Claim:** For all \( \delta \in (0, 1) \), \( K_{\hat{u}} \in C^{\frac{1}{2}, \delta}([0, T] \times \mathbb{R}) \).

(Proof of Claim): We first check that \( K_{\hat{u}} \) is bounded. Since \( 0 < h(z) < 1 \), we observe that

\[
|K_{\hat{u}}(t, z)| \leq |\mu| + 2|r| + \frac{\sigma^2}{2} + \lambda \left( \|\hat{u}\|_{\infty} + \ln \left( \frac{1+\epsilon}{1-\epsilon} \right) \right),
\]
where we used the mean value theorem and the bounds $0 < h < 1$ and $0 < h' < 1$. We can treat $\Delta < 0$ by the same way, and conclude that $K_{\hat{u}}$ is uniformly Lipschitz with respect to $z$.

For $\Delta > 0$, using (A.5) and the mean value theorem, we observe that

\[
\mathbb{E} \left| e^{\lambda \Delta} K_{\hat{u}}(s, Z_s(t+\Delta, z)) - K_{\hat{u}}(s, Z_s(t, z)) \right| \\
\leq \lambda e^{\lambda T} \| K_{\hat{u}} \|_{\infty} \Delta + \mathbb{E} \left| K_{\hat{u}}(s, Z_s(t+\Delta, z)) - K_{\hat{u}}(s, Z_s(t, z)) \right| \\
\leq \lambda e^{\lambda T} \| K_{\hat{u}} \|_{\infty} \Delta + \left( |\mu - r| + \sigma^2 + \lambda \left( \bar{\epsilon} + \frac{\epsilon}{1-\epsilon} \right) \right) \mathbb{E} \left| Z_s(t+\Delta, z) - Z_s(t, z) \right| \\
\leq \lambda e^{\lambda T} \| K_{\hat{u}} \|_{\infty} \Delta + \left( |\mu - r| + \sigma^2 + \lambda \left( \bar{\epsilon} + \frac{\epsilon}{1-\epsilon} \right) \right) \\
\left( |\mu - r - \sigma^2 \Delta | + \sigma \sqrt{\frac{2}{\pi}} \Delta^{\frac{1}{2}} \right),
\]

where the last inequality is due to

\[
\mathbb{E} \left| Z_s(t+\Delta, z) - Z_s(t, z) \right| \leq \left| \mu - r - \sigma^2 \right| \Delta + \sigma \mathbb{E} |B_{t+\Delta} - B_t| \\
= \left| \mu - r - \sigma^2 \right| \Delta + \sigma \sqrt{\frac{2}{\pi}} \Delta^{\frac{1}{2}}.
\]

Using $\hat{\mu} = \phi(\hat{u})$ and triangular inequality, we obtain

\[
|K_{\hat{u}}(t + \Delta, z) - K_{\hat{u}}(t, z)| \leq \lambda \sup_{\zeta \in \mathbb{R}} |\hat{\mu}(t + \Delta, \zeta) - \hat{\mu}(t, \zeta)| \\
= \lambda \sup_{\zeta \in \mathbb{R}} \left| \int_{t+\Delta}^{T} e^{-\lambda(s-t-\Delta)} \mathbb{E} \left[ K_{\hat{u}}(s, Z_s(t+\Delta, \zeta)) \right] ds \\
- \int_{t}^{T} e^{-\lambda(s-t)} \mathbb{E} \left[ K_{\hat{u}}(s, Z_s(t, \zeta)) \right] ds \right|.
\]
where the generic constant $C$ only depends on the market parameters and $\|\hat{u}\|_\infty$ and the last inequality is due to (A.4) and (A.6). We can treat $\Delta < 0$ by the same way, and conclude that $K_{\hat{u}}$ is $\frac{1}{2}$-Hölder continuous with respect to $t$ variable. (End of the proof of Claim).

Let $\delta \in (0, 1)$ be fixed. Then, the above claim and Theorem 9.2.3 in [35] ensure that there exists a unique function $u \in C^{1+\frac{\delta}{2}, 2+\delta}([0, T]) \times \mathbb{R})$ such that it satisfies the following PDE on $(t, z) \in (0, T) \times \mathbb{R}$.

\[
\begin{align*}
0 &= u(T, z), \\
0 &= u_t + (\mu - r - \frac{\sigma^2}{2})u_z + \frac{1}{2}\sigma^2u_{zz} - \lambda u + K_{\hat{u}}.
\end{align*}
\]

Since $u$ admits a unique continuous extension on $[0, T] \times \mathbb{R}$ (i.e., see chapter 8.5 in [35]), we let $u \in C^{1+\frac{\delta}{2}, 2+\delta}([0, T] \times \mathbb{R})$. By the Feynman-Kac formula (i.e., see Theorem 5.7.6 in [32]), the solution $u$ of the parabolic PDE (A.7) has the stochastic representation $u = \phi(\hat{u})$, where $\phi$ is defined in (A.1). Since $\hat{u}$ is chosen as the unique fixed point of the map $\phi$, we conclude that $u = \hat{u}$.

Our next task is to define $u(t, \pm \infty)$ for $t \in [0, T]$. Using $\lim_{z \to \pm \infty} h(Z_s(t, z)) = 1$ and $\lim_{z \to -\infty} h(Z_{s}(t, z)) = 0$ almost surely, we obtain

\[
\begin{align*}
\lim_{z \to \infty} K_u(s, Z_s(t, z)) &= \mu - \frac{1}{2}\sigma^2 + \lambda \sup_{\xi \in \mathbb{R}} \left( u(s, z) - \ln \left( \frac{1 - \epsilon h(\xi)}{1 - \epsilon} \right) \right) \quad a.s. \\
\lim_{z \to -\infty} K_u(s, Z_s(t, z)) &= r + \lambda \sup_{\xi \in \mathbb{R}} \left( u(s, z) - \ln \left( 1 + \epsilon h(\xi) \right) \right)
\end{align*}
\]

(A.8)

The above convergence and $\|K_u\|_\infty < \infty$ enable us to apply the dominated convergence theorem:

\[
\lim_{z \to \pm \infty} u(t, z) = \lim_{z \to \pm \infty} \phi(u)(t, z) = \lim_{z \to \pm \infty} \int_t^T e^{-\lambda(s-t)} E\left[ K_u(s, Z_s(t, z)) \right] ds
\]

\[
= \begin{cases} 
\int_t^T e^{-\lambda(s-t)} \left( \mu - \frac{1}{2}\sigma^2 \right) ds + \lambda \sup_{\xi \in \mathbb{R}} \left( u(s, \xi) - \ln \left( \frac{1 - \epsilon h(\xi)}{1 - \epsilon} \right) \right) ds, & \text{for } z \to \infty \\
\int_t^T e^{-\lambda(s-t)} \left( r + \lambda \sup_{\xi \in \mathbb{R}} \left( u(s, \xi) - \ln \left( 1 + \epsilon h(\xi) \right) \right) \right) ds, & \text{for } z \to -\infty
\end{cases}
\]
Therefore, we can continuously extend \( u \) to \( z = \pm \infty \) and \( u(t, \infty) \) and \( u(t, -\infty) \) are defined by the above limit. We observe that for \( z \in (\infty, -\infty) \), \( u(t, z) \) satisfies

\[
0 = u_t(t, z) + (\mu - r)h(z) + r - \frac{1}{2}\sigma^2 h(z)^2 - \lambda u(t, z) \\
+ \lambda \sup_{\xi \in \mathbb{R}} \left( u(t, \xi) - \ln \left( \frac{1+\epsilon h(\xi)}{1-\epsilon h(\xi)} \right) 1_{\{z<\xi\}} - \ln \left( \frac{1-\epsilon h(\xi)}{1+\epsilon h(\xi)} \right) 1_{\{z>\xi\}} \right),
\]

(A.9)

where the function \( h \) is continuously extended as \( h(\infty) := 1 \) and \( h(-\infty) := 0 \).

Now we define \( v \) as \( v(t, x) := u(t, h^{-1}(x)) \) for \( (t, x) \in [0, T] \times [0, 1] \). Such \( v \) is well-defined because \( h : \mathbb{R} \cup (-\infty, \infty) \rightarrow [0, 1] \) is bijective. We observe that for \( (t, x) \in (0, T) \times (0, 1) \) and \( z = h^{-1}(x) \),

\[
v_t(t, x) = u_t(t, z), \\
x(1-x)v_x(t, x) = u_z(t, z), \\
x^2(1-x)^2v_{xx}(t, x) = u_{zz}(t, z) - (1-2x)u_z(t, z).
\]

(A.10)

The PDE for \( u \) in (A.7) with \( \hat{u} \) replaced by \( u \) and the equalities in (A.10) produce the PDE for \( v \), which is (3.2). Therefore, statement (i) is valid.

To check statement (ii), we observe that \( v(t, 0) = u(t, -\infty) \) and \( v(t, 1) = u(t, \infty) \). Then, the continuous differentiability of \( v(t, 0) \) and \( v(t, 1) \) with respect to \( t \) is followed by that of \( u(t, -\infty) \) and \( u(t, \infty) \), and (A.9) produces (3.3).

Finally, statement (iii) is a direct consequence of (A.10) and \( u \in C^{1+\frac{\delta}{2}, 2+\delta}([0, T] \times \mathbb{R}) \).

\section*{Appendix B. Proof of Lemma 5.3}

(i) When \( \epsilon = 0 \), Lemma 4.1 and (5.1) produce (5.2).

To prove (5.3), we first check that \( Y_s^{(t,x)} \) in (4.9) satisfies

\[
dY_s^{(t,x)} = Y_s^{(t,x)}(1 - Y_s^{(t,x)}) \left( (\mu - r - \sigma^2 Y_s^{(t,x)}) ds + \sigma dB_s \right).
\]

(B.1)

Then application of Ito’s formula produces that for \( (s, x) \in [t, T) \times (0, 1) \),

\[
\left( Y_s^{(t,x)} - x \right)^2 = \int_t^s Y_u^{(t,x)}(1 - Y_u^{(t,x)}) \left( 2(Y_u^{(t,x)} - x)(\mu - r - \sigma^2 Y_u^{(t,x)}) \right) du \\
+ \sigma^2 Y_u^{(t,x)}(1 - Y_u^{(t,x)}) d\mu \\
+ \int_t^s 2\sigma (Y_u^{(t,x)} - x) Y_u^{(t,x)}(1 - Y_u^{(t,x)}) dB_u.
\]

\( \square \) Springer
Since $0 < Y_s^{(t,x)} < 1$, the local martingale part above is a true martingale and we obtain

\[
\frac{\partial}{\partial x} \left( \mathbb{E} \left[ \left( Y_s^{(t,x)} - x \right)^2 \right] \right) \nonumber = \mathbb{E} \left[ Y_s^{(t,x)} (1 - Y_s^{(t,x)}) \left( 2(Y_s^{(t,x)} - x)(\mu - r - \sigma^2 Y_s^{(t,x)}) \right) + \sigma^2 Y_s^{(t,x)} (1 - Y_s^{(t,x)}) \right] = -x^2 (1 - x)^2 \nonumber \\
\cdot \mathbb{E} \left[ \frac{\partial}{\partial x} \left( \frac{Y_s^{(t,x)} (1 - Y_s^{(t,x)}) (\mu - r - \sigma^2 Y_s^{(t,x)})}{x(1-x)} \right) \right], \tag{B.2}
\]

where the second equality is from elementary computations using the definition of $Y_s^{(t,x)}$ in (4.9).

For $x \in (0, 1)$, we observe that

\[
\frac{Y_s^{(t,x)} - x}{x(1-x)} = \frac{A^{(t,x)} - 1}{A^{(t,x)}(x(1-x)} \quad \text{with} \quad A^{(t,x)} := e^{(\mu - r - \frac{\sigma^2}{2}) (s-t) + \sigma (B_s - B_t)}, \tag{B.3}
\]

and the above expression is decreasing in $x$, therefore,

\[
1 - \frac{1}{A^{(t,x)}} < \frac{Y_s^{(t,x)} - x}{x(1-x)} < A^{(t,x)} - 1 \quad \text{for} \quad 0 < x < 1. \tag{B.4}
\]

When $\epsilon = 0$, the representation of $v_x$ in (4.11) becomes

\[
v_x^0(t, x) = \int_t^T e^{-\lambda(s-t)} \mathbb{E} \left[ \frac{Y_s^{(t,x)} (1 - Y_s^{(t,x)}) (\mu - r - \sigma^2 Y_s^{(t,x)})}{x(1-x)} \right] ds, \tag{B.5}
\]

We take derivative with respect to $x$ in the above expression. Then, the mean value theorem and the dominated convergence theorem, together with the inequalities (4.14) and (B.4), allow us to take derivative inside of the expectation and obtain that for $(t, x) \in [0, T) \times (0, 1), \nonumber$

\[
v_{xx}^0(t, x) = \int_t^T e^{-\lambda(s-t)} \mathbb{E} \left[ \frac{\partial}{\partial x} \left( \frac{Y_s^{(t,x)} (1 - Y_s^{(t,x)}) (\mu - r - \sigma^2 Y_s^{(t,x)})}{x(1-x)} \right) \right] ds \tag{B.6}

\]

\[
= -\int_t^T e^{-\lambda(s-t)} \frac{\partial}{\partial s} \left( \mathbb{E} \left[ \left( \frac{Y_s^{(t,x)} - x}{x(1-x)} \right)^2 \right] \right) ds 

\]

\[
= -e^{-\lambda(T-t)} \mathbb{E} \left[ \left( \frac{Y_s^{(t,x)} - x}{x(1-x)} \right)^2 \right] - \lambda \int_t^T e^{-\lambda(s-t)} \mathbb{E} \left[ \left( \frac{Y_s^{(t,x)} - x}{x(1-x)} \right)^2 \right] ds, \tag{B.7}
\]

where the second equality is due to (B.2), and the third equality is from integration by parts. Obviously (B.7) implies that $v_{xx}^0(t, x) < 0$ for $(t, x) \in [0, T) \times (0, 1).$
To conclude (5.3), it only remains to check that \( \lim_{x \uparrow 1} v^0_{xx}(t, x) < 0 \) and \( \lim_{x \downarrow 0} v^0_{xx}(t, x) < 0 \). Indeed, (B.3) and (B.4) enable us to apply the dominated convergence theorem to (B.7) and obtain

\[
\lim_{x \uparrow 1} v^0_{xx}(t, x) = -e^{-\lambda(T-t)} \mathbb{E} \left[ \left( 1 - \frac{1}{A(t, s)} \right)^2 \right] - \lambda \int_t^T e^{-\lambda(s-t)} \mathbb{E} \left[ \left( 1 - \frac{1}{A(t, s)} \right)^2 \right] ds,
\]

\[
\lim_{x \downarrow 0} v^0_{xx}(t, x) = -e^{-\lambda(T-t)} \mathbb{E} \left[ (A(t, T) - 1)^2 \right] - \lambda \int_t^T e^{-\lambda(s-t)} \mathbb{E} \left[ (A(t, s) - 1)^2 \right] ds,
\]

and we conclude that \( \lim_{x \uparrow 1} v^0_{xx}(t, x) < 0 \) and \( \lim_{x \downarrow 0} v^0_{xx}(t, x) < 0 \).

(ii) The SDE for \( Y_s^{(t,x)} \) in (B.1) and \( 0 < Y_s^{(t,x)} < 1 \) imply that for \( (s, x) \in [t, T) \times (0, 1) \),

\[
\frac{\partial}{\partial s} \left( \mathbb{E} \left[ Y_s^{(t,x)} \right] \right) = \mathbb{E} \left[ Y_s^{(t,x)} \left( 1 - Y_s^{(t,x)} \right) \left( \mu - r - \sigma^2 Y_s^{(t,x)} \right) \right].
\]

We divide both sides of (B.8) by \( x(1-x) \) and take derivative with respect to \( x \). Then, we can put the derivative inside of the expectation as in the proof of part (i), and obtain

\[
\mathbb{E} \left[ \frac{\partial}{\partial x} \left( Y_s^{(t,x)} \left( 1 - Y_s^{(t,x)} \right) \left( \mu - r - \sigma^2 Y_s^{(t,x)} \right) / x(1-x) \right) \right] = \frac{\partial}{\partial s} \left( \mathbb{E} \left[ \frac{\partial}{\partial x} Y_s^{(t,x)} / x(1-x) \right] \right).
\]

We rearrange the above equation and use (B.8), (B.5), and (B.7) to obtain

\[
\int_t^T e^{-\lambda(s-t)} \frac{\partial}{\partial s} \left( \mathbb{E} \left[ \frac{\partial}{\partial x} Y_s^{(t,x)} \right] \right) ds = x(1-x) v^0_{xx}(t, x) + (1-2x) v^0_x(t, x). \tag{B.9}
\]

Now we conclude that \( F(t, \hat{y}^0(t)) < 1 \) by the following way:

\[
F(t, \hat{y}^0(t)) \leq \lambda \int_t^T e^{-\lambda(s-t)} \mathbb{E} \left[ \frac{\partial}{\partial x} Y_s^{(t,x)} \right] ds \bigg|_{x=\hat{y}^0(t)} \leq \left( 1 - e^{-\lambda(T-t)} \right) \mathbb{E} \left[ \frac{\partial}{\partial x} Y_T^{(t,x)} \right] + x(1-x) v^0_{xx}(t, x) + (1-2x) v^0_x(t, x) \bigg|_{x=\hat{y}^0(t)} < 1,
\]

where the first inequality is from the definition of \( F \) in (5.4) and the positivity of \( \frac{\partial}{\partial x} Y_s^{(t,x)} \) (see (4.14)), and the equality is due to integration by parts and (B.9), and the last inequality is due to the positivity of \( \frac{\partial}{\partial x} Y_T^{(t,x)} \), \( v^0_x(t, \hat{y}^0(t)) = 0 \), and \( v^0_{xx}(t, \hat{y}^0(t)) < 0 \).
We can check that $F(t, z^0(t)) > -1$ by the same way as above. (iii) The expression in (4.11) produces

$$v^\epsilon_x(t, x) - v^0_x(t, x) = \lambda \int_t^T e^{-\lambda(s-t)} \mathbb{E} \left[ \left( \frac{\partial}{\partial x} y_s(t, x) \right) L_y^\epsilon(s, Y_s(t, x)) \right] ds. \quad (B.10)$$

In the above expression, when we take limit as $\epsilon \downarrow 0$, the inequalities (4.13) and (4.14) enable us to use the dominated convergence theorem to conclude that

$$\lim_{\epsilon \downarrow 0} \left( v^\epsilon_x(t, x) - v^0_x(t, x) \right) = 0.$$

The above limit and the continuity of $v^0_x$ imply (5.6).

To prove (5.7), we first observe that for $(t, x) \in [0, T) \times (0, 1)$ and $(s, z) \in (t, T) \times (0, 1)$, the density function for $Y_s^{(t, x)}$ is given by

$$\varphi(s, z; t, x) := \frac{\partial}{\partial z} \mathbb{P}(Y_s(t, x) \leq z) = \exp \left( -\frac{1}{2\sigma^2(s-t)} \left( (r-\mu+\frac{z^2}{2})(s-t)+\ln \left( \frac{z(1-z)}{r-\mu+\frac{z^2}{2}} \right) \right)^2 \right) / \sigma z(1-z) \sqrt{2\pi(s-t)}.$$

Then, the expression in (4.11) and $\frac{\partial}{\partial x} Y_s^{(t, x)} = \frac{y_s^{(t, x)}(1-y_s^{(t, x)})}{x(1-x)}$ imply that

$$v^\epsilon_x(t, x) - v^0_x(t, x) = \lambda \int_t^T e^{-\lambda(s-t)} \mathbb{E} \left[ \frac{y_s^{(t, x)}(1-y_s^{(t, x)})}{x(1-x)} L_y^\epsilon(s, Y_s(t, x)) \right] ds\quad (B.12)$$

For $x \in (0, 1)$, direct computations produce

$$\frac{\partial}{\partial x} \left( \frac{z(1-z)}{x(1-x)} \varphi(s, z; t, x) \right) = \frac{(1-z)z(r-\mu+(2x-\frac{1}{2})\sigma^2+\frac{1}{2}\ln \left( \frac{z(1-z)}{r-\mu+\frac{1}{2}\sigma^2} \right) ) \varphi(s, z; t, x)}{(1-x)^2 x^2 \sigma^2}. \quad (B.13)$$

Assumption 5.1 implies that

$$\left| r - \mu + \frac{\sigma^2}{2} \right| \leq \sigma^2, \quad \left| r - \mu + (2x - \frac{1}{2})\sigma^2 \right| \leq 2\sigma^2 \quad \text{for} \quad x \in (0, 1). \quad (B.14)$$
Then, (B.11) and (B.13) produce the following:

\[
\int_0^1 \left| \frac{\partial}{\partial x} \left( \frac{z(1-z)}{x(1-x)} \varphi(s, z; t, x) \right) \right| \, dz \\
\leq \int_0^1 \left( \frac{2\sigma^2 + \frac{1}{x-t} \ln \left( \frac{z(1-x)}{(1-z)x} \right) \right)}{\sigma^3(1-x)^2x^2} \exp \left( -\frac{\ln \left( \frac{z(1-x)}{(1-z)x} \right)}{2\sigma^2(s-t)} \right)^2 + \left| \ln \left( \frac{z(1-x)}{(1-z)x} \right) \right| \right) \, dz \\
= \int_{-\infty}^\infty \left( \frac{2\sigma^2 + \frac{1}{x-t} \ln (\zeta + |\xi|)}{\sigma^3(1-x)^2x^2} \right) \frac{x}{(1 + \frac{1}{x-t} \exp(\xi))} \, d\zeta \\
\leq \int_{-\infty}^\infty \left( \frac{2\sigma^2 + \frac{1}{x-t} \ln (\zeta + |\xi|)}{\sigma^3(1-x)^2x^2} \right) \exp \left( -\frac{4\sigma^2(s-t)}{x^4 \sqrt{2\pi} \sqrt{s-t}} \right) \, d\zeta \\
= \frac{2\sqrt{\zeta}}{(1-x)^3x} \left( 1 + \frac{1}{\sigma \sqrt{\pi} (s-t)} \right) e^{4\sigma^2(s-t)},
\]

where the first inequality is due to (B.14), and the first equality is obtained by change
of variables as \( \zeta = \ln \left( \frac{z(1-x)}{(1-z)x} \right) \). The second inequality is due to the inequality of
arithmetic and geometric means, and the second equality is obtained by direct com-
putations.

By (B.15), we conclude that

\[
\int_t^T e^{-\lambda(s-t)} \int_0^1 \left| \frac{\partial}{\partial x} \left( \frac{z(1-z)}{x(1-x)} \varphi(s, z; t, x) \right) \right| \, dz \, ds < \infty, \quad (B.16)
\]

and the function \( H : [0, T] \times (0, 1) \rightarrow \mathbb{R} \) given by

\[
H(t, x) := \lambda \int_t^T e^{-\lambda(s-t)} \int_0^1 \left| \frac{\partial}{\partial x} \left( \frac{z(1-z)}{x(1-x)} \varphi(s, z; t, x) \right) \right| L_{y}^\epsilon(s, z) \, dz \, ds \quad (B.17)
\]
is well-defined due to (B.16) and the boundedness \( |L_{y}^\epsilon| \leq \frac{\epsilon}{1-\epsilon} \) in (4.13). Then, (B.15)
implies that

\[
|H(t, x)| \leq \frac{2\sqrt{\zeta} e^{4\sigma^2T}}{(1-x)^3x} \left( T + \frac{2\sqrt{\zeta} e^{4\sigma^2T}}{\sigma \sqrt{\pi}} \right) \cdot \frac{\epsilon}{1-\epsilon} \quad \text{for} \quad (t, x) \in [0, T] \times (0, 1). \quad (B.18)
\]

Now, let’s check that

\[
H(t, x) = \psi_{x}^\epsilon(t, x) - \psi_{x}^0(t, x). \quad (B.19)
\]
Indeed, for \((t, x) \in [0, T) \times (0, 1),\)

\[
H(t, x) = \lim_{\delta \to 0} \frac{1}{\delta} \int_x^{x+\delta} H(t, \eta) d\eta
\]

\[
= \lim_{\delta \to 0} \frac{1}{\delta} \lambda \int_t^T e^{-\lambda (s-t)} \int_0^1 \left( \frac{z(1-z)\varphi(s, z; t, x+\delta)}{(x+\delta)(1-x-\delta)} - \frac{z(1-z)\varphi(s, z; t, x)}{x(1-x)} \right) L^0_{g}(s, z) dz \, ds
\]

\[
= \lim_{\delta \to 0} \frac{1}{\delta} \left( (v^\epsilon(t, x + \delta) - v^0_x(t, x + \delta)) - (v^\epsilon_x(t, x) - v^0_{xx}(t, x)) \right)
\]

where the second equality is due to Fubini’s theorem and the fundamental theorem of calculus, and the third equality is from (B.12).

Finally, we conclude (5.7) by the following observation:

\[
\limsup_{\epsilon \downarrow 0} \left| v^\epsilon_{xx}(t, x_\epsilon) - v^0_{xx}(t, x_0) \right| \\
\leq \limsup_{\epsilon \downarrow 0} \left| v^\epsilon_{xx}(t, x_\epsilon) - v^0_{xx}(t, x_\epsilon) \right| + \limsup_{\epsilon \downarrow 0} \left| v^0_{xx}(t, x_\epsilon) - v^0_{xx}(t, x_0) \right|
\]

\[
\leq \limsup_{\epsilon \downarrow 0} \frac{2 \sqrt{2} e^{\sigma^2 T}}{(1-x_\epsilon)^3 x_\epsilon} \left( T + \frac{2 \sqrt{T}}{\sigma \sqrt{\pi}} \right) \cdot \frac{\epsilon}{1-\epsilon} = 0,
\]

where the second inequality is due to (B.18), (B.19), and the continuity of \(v^0_{xx}\).

**Appendix C. Proof of Lemma 5.10**

(i) Using \(\Gamma\) in (5.25), the equations (B.5) and (B.6) can be written as

\[
v^0_x(t, x) = \int_0^T e^{-\lambda s} \Gamma(s, x) ds, \quad v^0_{xx}(t, x) = \int_0^T e^{-\lambda s} \Gamma_x(s, x) ds. \tag{C.1}
\]

We can do the similar argument to obtain representations for \(v^0_{xxx}\) and partial derivatives of \(v^0_x\) and \(v^0_{xx}\) with respect to \(\lambda\), with the observation that \(\Gamma, \Gamma_x, \Gamma_{xx}\) are continuous & bounded maps on \([0, T] \times (0, 1)\). The result is summarized as follows:

\[
\frac{\partial}{\partial x} v^0_x(t, x) = -\int_0^T e^{-\lambda s} s \Gamma(s, x) ds, \quad \frac{\partial}{\partial x} v^0_{xx}(t, x) = -\int_0^T e^{-\lambda s} s \Gamma_x(s, x) ds
\]

\[
v^0_{xx}(t, x) = \int_0^T e^{-\lambda s} \Gamma_x(s, x) ds, \quad v^0_{xxx}(t, x) = \int_0^T e^{-\lambda s} \Gamma_{xx}(s, x) ds. \tag{C.2}
\]

Using \(Y_{t}^{(t, x)} = x\), direct computations produce

\[
\Gamma(0, x) = \mu - r - \sigma^2 x, \quad \Gamma_x(0, x) = -\sigma^2, \quad \Gamma_{xx}(0, x) = 0. \tag{C.3}
\]
With (C.3) and (5.24), we apply Lemma D.2 to (C.2) and conclude (5.61).

(ii) The mean value theorem and (5.2) produce

\[
\hat{y}^{0,\lambda+\delta}(t) - \hat{y}^{0,\lambda}(t) = -\frac{v_x^{0,\lambda+\delta}(t,\hat{y}^{0,\lambda}(t)) - v_x^{0,\lambda}(t,\hat{y}^{0,\lambda}(t))}{\partial_x v_x^{0,\lambda+\delta}(t,\lambda)} \quad \text{for} \, \lambda, \delta \text{ between } \hat{y}^{0,\lambda}(t) \text{ and } \hat{y}^{0,\lambda+\delta}(t),
\]

where we specify the dependence on \( \lambda \) for clarity. Since \( \partial_\lambda v_x^{0} \) exists (see (C.2)), the above equality and (5.3) ensure that \( \hat{y}^{0,\lambda}(t) \) is differentiable with respect to \( \lambda \) and

\[
\frac{\partial}{\partial \lambda} \hat{y}^{0}(t) = -\frac{\partial}{\partial x} v_x^{0}(t, x) \bigg|_{x=\hat{y}^{0}(t)}.
\]

(C.4)

Observe that the bounds for \( \frac{\partial}{\partial x} Y_t^{0,x} \) and \( \frac{y_t^{0,x}-x}{x(1-x)} \) in (4.14) and (B.4) do not depend on the variable \( x \). Therefore, the following convergence is uniform on \( x \in (0, 1) \):

\[
\Gamma_x(t, x) = -\mathbb{E}\left[ \sigma^2 \left( \frac{\partial}{\partial x} Y_t^{0,x} \right)^2 \right] + 2 \left( \mu - r - \sigma^2 Y_t^{0,x} \right) \left( \frac{y_t^{0,x}-x}{x(1-x)} \right) \left( \frac{\partial}{\partial x} Y_t^{0,x} \right) \bigg|_{t=0} \rightarrow -\sigma^2.
\]

Hence, there is a constant \( \tilde{T} \in (0, T) \) such that \( \Gamma_x(t, x) \leq -\frac{\sigma^2}{2} \) for \( (t, x) \in [0, \tilde{T}] \times (0, 1) \). This observation and the expression of \( v_{xx} \) in (C.2) imply

\[
\lambda v_{xx}(t, \hat{y}^{0}(t)) \leq -\frac{\sigma^2}{2} \left( 1 - e^{-\lambda(T-t)} \right) \quad \text{for} \, \, (t, x) \in [T - \tilde{T}, T) \times (0, 1).
\]

(C.5)

We obtain \( \| \Gamma_{xt} \|_\infty < \infty \) by using Ito’s formula with (B.1) and the bounds (4.14) and (B.4). Then, (C.2) and (C.3) imply

\[
\left| \lambda v_{xx}(t, \hat{y}^{0}(t)) + \sigma^2 \right| = \left| \lambda \int_0^{T-t} e^{-\lambda s} \left( \frac{\Gamma_x(s, \hat{y}^{0}(t))-\Gamma_x(0, \hat{y}^{0}(t))}{s} \right) ds + \sigma^2 e^{-\lambda(T-t)} \right| \leq \| \Gamma_{xt} \|_\infty \left( \frac{1-e^{-\lambda(T-t)}}{\lambda} - (T-t)e^{-\lambda(T-t)} \right) + \sigma^2 e^{-\lambda(T-t)}.
\]

This implies that there exists a constant \( \tilde{\Lambda} \) (may depend on \( \tilde{T} \)) such that

\[
\lambda v_{xx}(t, \hat{y}^{0}(t)) \leq -\frac{\sigma^2}{2} \quad \text{for} \, \, (t, \lambda) \in [0, T - \tilde{T}] \times [\tilde{\Lambda}, \infty).
\]

(C.6)
Using (C.2) and (C.3), we obtain

\[ \left| \lambda^3 \frac{\partial}{\partial \lambda} v_0(t, x) \right|_{x=\hat{y}_0(t)} \leq \lambda^3 \int_0^{T-t} e^{-\lambda s} s^2 \left( \frac{\Gamma(s, \hat{y}_0(t)) - \Gamma(0, \hat{y}_0(t))}{s} \right) ds \]

\[ + \left| \sigma^2 (y_\infty - \hat{y}_0(t)) \lambda^3 \int_0^{T-t} e^{-\lambda s} s ds \right| \]

\[ \leq C \left( 1 - e^{-\lambda(T-t)} + \lambda(T-t) \left( 1 + \lambda(T-t) \right) e^{-\lambda(T-t)} \right) \]

for \((t, \lambda) \in [0, T] \times [1, \infty)\), \((C.7)\)

where the second inequality is due to \(\| \Gamma \|_\infty < \infty \) and (5.24).

From (C.4), we obtain the boundedness of \(\left| \lambda^2 \frac{\partial}{\partial \lambda} \hat{y}_0(t) \right|\):

\[ \sup_{(t, \lambda) \in [0, T] \times [1, \infty]} \left| \lambda^2 \frac{\partial}{\partial \lambda} \hat{y}_0(t) \right| < \infty \text{ due to (C.6), (C.7), sup } x(1+x)e^{-x} < \infty, \]

\[ \sup_{(t, \lambda) \in [0, T] \times [1, \infty]} \left| \lambda^2 \frac{\partial}{\partial \lambda} \hat{y}_0(t) \right| < \infty \text{ due to the continuity on compact set,} \]

\[ \sup_{(t, \lambda) \in [T-\tilde{T}, T] \times [1, \infty]} \left| \lambda^2 \frac{\partial}{\partial \lambda} \hat{y}_0(t) \right| < \infty \text{ due to (C.5), (C.7), sup } \frac{x(1+x)e^{-x}}{1-e^{-x}} < \infty. \]

Therefore, we conclude (5.62).

(iii) The bounds (4.14) and (5.62) enable us to use Leibniz integral rule to obtain

\[ \frac{\partial}{\partial \lambda} \mathbb{E} \left[ \left| Y_s(t, \hat{y}_0(t)) - \hat{y}_0(s) \right| \right] = \mathbb{E} \left[ \left( \frac{\partial}{\partial \lambda} \hat{y}_0(t) \cdot \frac{\partial}{\partial x} Y_s(t, x) \right)_{x=\hat{y}_0(t)} \right] \frac{x}{1-e^{-x}} < \infty. \]

The above expression, together with the bounds (4.14) and (5.62), implies (5.63).

**Appendix D. Supplementary Lemmas**

**Lemma D.1** Let \( F : [0, T] \times [0, 1]^2 \rightarrow \mathbb{R} \) be a continuous function. We define \( f : [0, T] \times [0, 1] \rightarrow [0, 1] \) as

\[ f(t, x) := \max \left\{ z : z \in \text{argmax}_{y \in [0,1]} F(t, x, y) \right\}, \quad \text{(D.1)} \]

then \( f \) is upper semicontinuous (which is obviously Borel-measurable).

**Proof** This type of result is well-known (e.g., see p. 153 in [6]), but we give a short proof here for the sake of self-containedness.
Therefore, Lemma D.2 simple computations and (D.6) produce

\[ F(t_n, x_n, f(t_n, x_n)) \leq F(t_n, x_n, f(t_n, x_n)). \quad \text{(D.2)} \]

We let \( n \to \infty \) above and using the continuity of \( F \) to obtain

\[ F(t_\infty, x_\infty, f(t_\infty, x_\infty)) \leq F(t_\infty, x_\infty, \lim_{n \to \infty} f(t_n, x_n)). \quad \text{(D.3)} \]

This implies that \( \lim_{n \to \infty} f(t_n, x_n) \in \text{argmax}_{y \in [0, 1]} F(t_\infty, x_\infty, y) \), and the definition of \( f \) ensures

\[ f(t_\infty, x_\infty) \geq \lim_{n \to \infty} f(t_n, x_n). \]

Therefore, \( f \) is upper semicontinuous. \( \square \)

**Lemma D.2** Let \( f(s, x) : [0, 1] \times (0, 1) \to \mathbb{R} \) be a continuous function, and \( g(\lambda) : [1, \infty) \to (0, 1) \) be a function satisfying \( \lim_{\lambda \to \infty} g(\lambda) = x_\infty \in (0, 1) \). Then, for \( \alpha \in \{0, 1, 2, 3, 4\} \) and \( t > 0 \),

\[ \lim_{\lambda \to \infty} \lambda^{\frac{\alpha}{2} + 1} \int_0^t e^{-\lambda s} s^{\frac{\alpha}{2}} f(s, g(\lambda)) ds = c_\alpha \cdot f(0, x_\infty), \quad \text{(D.4)} \]

where \( c_0 = 1, c_1 = \frac{\sqrt{\pi}}{2}, c_2 = 1, c_3 = \frac{3\sqrt{\pi}}{4}, c_4 = 2. \) Also, there exists a constant \( C \) such that

\[ \lambda^{\frac{\alpha}{2} + 1} \int_0^t e^{-\lambda s} s^{\frac{\alpha}{2}} ds \leq C, \quad \text{for} \quad (t, \lambda, \alpha) \in [0, \infty) \times [1, \infty) \times \{0, 1, 2, 3, 4\}. \quad \text{(D.5)} \]

**Proof** Let \( \eta > 0 \) be a given constant. The uniform continuity of \( f \) on a compact set containing the point \((0, x_\infty)\), together with \( \lim_{\lambda \to \infty} g(\lambda) = x_\infty \in (0, 1) \), implies that there exists \( \delta > 0 \) such that

\[ |f(s, g(\lambda)) - f(0, x_\infty)| \leq \eta \quad \text{for any} \quad (s, \lambda) \in [0, \delta) \times \left[\frac{1}{\delta}, \infty\right). \quad \text{(D.6)} \]

Simple computations and (D.6) produce

\[
\begin{align*}
\lim_{\lambda \to \infty} \sup \left| \int_0^t e^{-\lambda s} f(s, g(\lambda)) ds - f(0, x_\infty) \right| &= \lim_{\lambda \to \infty} \sup \left| \int_0^\delta e^{-\lambda s} (f(s, g(\lambda)) - f(0, x_\infty)) ds \right| \\
&\quad + \int_\delta^t \lambda e^{-\lambda s} (f(s, g(\lambda)) - f(0, x_\infty)) ds - e^{-\lambda t} f(0, x_\infty) \leq \eta.
\end{align*}
\]
Since \( \eta > 0 \) can be arbitrary small, we conclude (D.4) for the case of \( \alpha = 0 \). The other cases in (D.4) can be obtained by the same way as above, using the following expressions:

\[
\lambda^2 \int_0^t e^{-\lambda s} s^1 ds = -\frac{\sqrt{\lambda t} e^{-\lambda t}}{\lambda \to \infty} + \frac{\sqrt{\lambda t} e^{-\lambda t}}{2} \to \frac{\sqrt{\pi}}{2},
\]

\[
\lambda^2 \int_0^t e^{-\lambda s} s ds = 1 - (1 + \lambda t) e^{-\lambda t} \to 1, \tag{D.7}
\]

\[
\lambda^2 \int_0^t e^{-\lambda s} s^2 ds = -\frac{(3 + 2\lambda t) \sqrt{\lambda t} e^{-\lambda t}}{2} + \frac{3}{2} \int_0^{\sqrt{\lambda t}} e^{-s^2} ds \to \frac{3\sqrt{\pi}}{4},
\]

\[
\lambda^3 \int_0^t e^{-\lambda s} s^3 ds = 2 - (2 + \lambda t (2 + \lambda t)) e^{-\lambda t} \to 2.
\]

One can easily observe that \( \sup_{x > 0} x^\alpha e^{-x} < \infty \) for \( \alpha \in \{0, 1, 2, 3, 4\} \). This observation and the explicit expressions in (D.7) produce the bound (D.5). \( \square \)

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