New half supersymmetric solutions of the heterotic string

G Papadopoulos

Department of Mathematics, King’s College London, Strand, London WC2R 2LS, UK

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Abstract
We describe all supersymmetric solutions of the heterotic string which preserve eight supersymmetries and show that these are distinguished by the holonomy, hol(\nabla), of the connection, \nabla, with skew-symmetric torsion. The hol(\nabla) \subseteq SU(2) solutions are principal bundles over a four-dimensional hyper-Kähler manifold equipped with an anti-self-dual connection and fibre group \( G \) which has a Lie algebra, \( \mathfrak{Lie}(G) = \mathbb{R}^5 \oplus \mathfrak{su}(2) \) or \( \mathfrak{so}_6 \). Some of the solutions have the interpretation as 5-branes wrapped on \( G \) with transverse space any hyper-Kähler four-dimensional manifold. We construct new solutions for \( \mathfrak{Lie}(G) = \mathfrak{so}(2, \mathbb{R}) \oplus \mathfrak{su}(2) \) and show that they are characterized by three integers and have continuous moduli. There is also a smooth family in this class with one asymptotic region and the dilaton is bounded everywhere on the spacetime. We also demonstrate that the worldvolume theory of the backgrounds with holonomy \( SU(2) \) can be understood in terms of gauged WZW models for which the gauge fields are composite. The hol(\nabla) = \{1\} heterotic string backgrounds which preserve eight supersymmetries are Lorentzian group manifolds.

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1. Introduction
It has been well known for sometime that branes have a low-energy description in terms of half supersymmetric solutions of 10- and 11-dimensional supergravity theories. As such they have been instrumental in understanding the superstring dualities and in formulating the proposal for M-theory, see [1, 2]. In type II theories, apart from the various branes [3–10], half supersymmetric solutions include brane bound states, see e.g. [11], and backgrounds which have applications in AdS/CFT correspondence, see e.g. [12, 13]. Other half supersymmetric supergravity solutions are the vacua of string theory and M-theory compactifications like those on \( K_3 \). In type I and heterotic theories, it is known that half supersymmetric solutions include...
the fundamental string [4], the pp-wave and the 5-brane [7]. Despite these developments, our understanding of half supersymmetric solutions of 10- and 11-dimensional supergravity theories is rather limited. In particular, it is not known what kind of geometry the half supersymmetric solutions have or a systematic way to describe them.

In this paper, we construct all the half supersymmetric solutions of the heterotic\(^1\) supergravity. This is based on the results of [14, 15], where the Killing spinor equations of heterotic supergravity have been solved in all cases using the spinorial geometry technique of [16]. We find that there are three classes of solutions distinguished by the holonomy, hol\(\nabla\), of the connection, \(\nabla\), with skew-symmetric torsion, \(\mathcal{H}\). These are hol\((\nabla)\) \(\subseteq SU(2)\), hol\((\nabla)\) \(\subseteq \mathbb{R}^8\) and hol\((\nabla)\) = \{1\}.

The solutions of the first class, hol\((\nabla)\) \(\subseteq SU(2)\), can be described in terms of the data of a principal bundle. In particular, we show that the spacetime \(M\) is a principal bundle, \(M = P(G, B_{hk}; \pi)\), equipped with a principal bundle connection \(\lambda\). In addition, the base space \(B_{hk}\) is a four-dimensional hyper-Kähler manifold, and the curvature of \(\lambda\) is (1,1) with respect to all three complex structures, i.e. \(\lambda\) is anti-self-dual. The fibre (gauge) group \(G\) is a six-dimensional Lorentzian, self-dual\(^2\) Lie group. The associated self-dual Lie algebras have been classified and have been shown to be isomorphic to \(\mathbb{R}^{5,1}\), \(sl(2, \mathbb{R}) \oplus su(2)\) or \(cw_6\) [17]. In addition, the gaugino Killing spinor equation implies that the gauge connection \(A\) of the heterotic string is also an anti-self-dual connection over \(B_{hk}\). The Killing spinor equations determine the hol\((\nabla)\) \(\subseteq SU(2)\) solutions up to an arbitrary function \(h\) of spacetime. This in turn can be specified by either solving the field equations or the (anomalous) Bianchi identity of \(H\) (3.13). Therefore, the data needed to construct all half supersymmetric solutions in this class of the heterotic string up and including 2-loops in the sigma model perturbation theory are

- a hyper-Kähler four-dimensional manifold \(B_{hk}\),
- an anti-self-dual connection \(\lambda\) on \(B_{hk}\) with gauge group \(G\) such that \(\mathfrak{sl}(G)\) is the Lorentzian self-dual Lie algebra \(\mathbb{R}^{5,1}\), \(sl(2, \mathbb{R}) \oplus su(2)\) or \(cw_6\),
- an anti-self-dual connection \(A\) on \(B_{hk}\) and
- a solution of (3.13) for function \(h\).

Given these data, the solutions can be written\(^3\) as

\[
\begin{align*}
\text{d}s^2 &= \eta_{ab}\lambda^a\lambda^b + h\text{d}x_5^2, \\
\text{e}^{2\phi} &= h, \\
H &= \frac{1}{3}\eta_{ab}\lambda^a \wedge \lambda^b + \frac{2}{3}\eta_{ab}\lambda^a \wedge \mathcal{F}^b - \star_{hk} dh,
\end{align*}
\]

where \(\mathcal{F}\) is the curvature of \(\lambda\).

We mostly explore explicit solutions for the class of backgrounds for which the contribution from the anomaly cancellation mechanism to the Bianchi identity of the 3-form field strength, \(H\), vanishes, and so \(H\) is closed \(dH = 0\). Although our formalism applies to the case that \(dH \neq 0\) as well.

The most straightforward case to consider is that for which the principal bundle connection \(\lambda\) is trivial. Then the solutions up to discrete identifications are

\[
G \times B_{hk}, \quad S_{\text{adS}_5[G]}: \quad G = \mathbb{R}^{5,1}, \quad \text{AdS}_5 \times S^3, \quad cW_6.
\]

\(^1\) The Killing spinor equations of type I supergravity are identical to those of the heterotic string. Consequently, the half supersymmetric solutions of type I supergravity will be constructed as well.

\(^2\) Self-duality in this context means that the 3-form associated with the structure constants of the Lorentzian Lie algebra, \(\mathfrak{sl}(G)\), of \(G\) is self-dual.

\(^3\) From now on, the subscript \(hk\) indicates that the associated operation or space is taken with respect to the hyper-Kähler metric.
where $B_{hk}$ is any four-dimensional hyper-Kähler manifold and $5_{B_{hk}}[G]$ denotes the 5-brane solution with transverse space $B_{hk}$ and worldvolume geometry $G$. For the 5-brane solution of [7], $B_{hk} = R^4$ and $G = R^{5,1}$. In the $AdS_3 \times S^3 \times B_{hk}$ and $5_{B_{hk}}[AdS_3 \times S^3]$ solutions, the radius of $AdS_3$ is equal to that of $S^3$ because of the self-duality condition. The solutions $CW_6 \times B_{hk}$ and $5_{B_{hk}}[G]$, $G \neq R^{5,1}$, are new.

New families of solutions can be constructed whenever $\lambda$ has non-vanishing curvature. Such solutions have the interpretation of wrapped 5-branes on $G$ with transverse space any hyper-Kähler four-dimensional manifold $B_{hk}$, where $G = R^{5,1}, AdS_3 \times S^3$ or $CW_6$ up to discrete identifications. In particular, we explicitly construct the solution for $G = AdS_3 \times S^3$ and $B_{hk} = R^4$ in the case that only $S^3$ is gauged. The background is as in (1.1) and $h$ is given in (5.23). These solutions are characterized by three integers $(k, \nu, p)$, where $k$ is the flux of the 3-form in $S^3$, $\nu$ is the instanton number and $p$ is the asymptotic 5-brane charge. They also depend on at least $8\nu - 3$ continuous parameters, the moduli of $SU(2)$ instantons. In addition, there is a family of solutions within this class, with $h$ given in (5.14) and $\nu = 1$, which are smooth, and the dilaton is bounded over the whole of spacetime. In this case $p = 4$. For this family, the spacetime metric has an $AdS_3 \times S^3 \times R^4$ asymptotic region and there is no throat at the position of the 5-branes. We propose an M-theoretic interpretation of the $\nu = 1$ solution as the near M2-brane geometry of an M2- with -M5 brane intersection on a self-dual string. In particular, we identify the near horizon geometry of the M2-brane as it emerges from the prescribed data. It is also likely that the more general solution based on $\nu$ instantons, with $h$ given in (5.21), is also smooth, and the dilaton is bounded over the whole of spacetime. This solution has $p = 4\nu$ and similar asymptotic properties to the solution based on a single instanton.

The worldvolume theory of the hol($\hat{\mathcal{V}}$) $\subseteq SU(2)$ class of solutions can be related to gauged WZW models. In particular, the principal bundle connection can be thought of as the gauge field which arises from the gauging of the right action of a $R^{5,1}$, $sl(2, R) \oplus su(2)$ or $cw_6$ WZW model. Such a gauging is anomalous but in this case the gauge fields are composite and depend on the scalars of the base space $B_{hk}$. As a result, the anomaly is cancelled because there is an additional contribution from $B_{hk}$.

The worldsheet supersymmetry of the resulting theory depends on the particular solution. If the gauge connection is identified with the spacetime 1, i.e. the contribution of the anomaly vanishes and so $dH = 0$, the minimal amount of worldvolume supersymmetry is (1,1) which in some sectors can enhance up to (4,4). All the solutions are finite up and including 2 loops. In addition, an indirect argument, based on (4,1) worldvolume supersymmetry [18], suggests that most of, and perhaps all, such holonomy $SU(2)$ backgrounds are ultraviolet finite to all orders in perturbation theory. In addition, we comment on the integrability of strings propagating in such backgrounds.

Alternatively, if $dH \neq 0$, i.e. there is an anomalous contribution in the Bianchi identity of $H$, then the minimal worldvolume supersymmetry of the solutions is (1,0) which in some sectors can enhance to (4,0). Again an indirect argument, based on (4,0) worldvolume supersymmetry [18], suggests that some of these backgrounds are again ultraviolet finite to all orders in perturbation theory. However, the couplings may receive finite local counterterm corrections which arise from changing the renormalization scheme from a manifestly (1,0) to a (4,0) supersymmetric one [19].

The hol($\hat{\mathcal{V}}$) $\subseteq R^4$ class of solutions include the fundamental string of [4] and the pp-wave solutions in flat space. We show that the most general solutions in this class are superpositions of the fundamental string, the pp-wave solutions together with a null rotation. Because of the chiral nature of these models, $dH = 0$, i.e. the anomalous contribution to the Bianchi identity of $H$ vanishes. The worldvolume theory of generic such backgrounds exhibits (1,0)
supersymmetry. The ultraviolet properties of the worldvolume theories of these backgrounds have been investigated in [20], and it has been found that some of them are not finite in the context of heterotic string.

The solutions of the third class of backgrounds that preserve eight supersymmetries, \( \text{hol}(\tilde{\nabla}) = \{1\} \), are WZW models. The spacetime is a Lorentzian group manifold. The gravitino Killing spinor equation admits 16 \( \tilde{\nabla} \)-parallel spinors but the dilatino and the gaugino Killing spinor equations preserve only half of the parallel spinors. The spacetime is a product, up to the appropriate dimension, of one of the Lorentzian groups \( SL(2, \mathbb{R}) \), \( \mathbb{R} \) and \( CW_{2n} \) with the Riemannian groups \( \mathbb{R}, SU(2) \) and \( SU(3) \). The complete list can be found in table 4 of [21].

This paper is organized as follows. In section 2, we state the Killing spinor equations of heterotic string. In section 3, we solve the Killing spinor equations for the hol(\( \tilde{\nabla} \)) \( \subseteq \) \( SU(2) \) class of backgrounds. In sections 4 and 5, we give examples of new solutions and explore their properties. In section 6, we give an M-theoretic interpretation for the hol(\( \tilde{\nabla} \)) \( \subseteq \) \( SU(2) \) class of backgrounds. In section 7, we investigate the worldvolume theories associated with the hol(\( \tilde{\nabla} \)) \( \subseteq \) \( SU(2) \) backgrounds. In section 8, we give the solutions of hol(\( \tilde{\nabla} \)) \( \subseteq \) \( \mathbb{R}^8 \) class of backgrounds, and in section 9 we present our conclusions.

2. Killing spinor equations

The gravitino, dilatino and gaugino Killing spinor equations of the heterotic supergravity (and type I) are

\[
\begin{align*}
\tilde{\nabla}_A \psi &= 0 + \mathcal{O}(\alpha'^2), \\
\left( \Gamma^A \partial_A \Phi - \frac{1}{12} H_{ABC} \Gamma^{ABC} \right) \psi &= 0 + \mathcal{O}(\alpha'^2) \\
F_{AB} \Gamma^{AB} \psi &= 0 + \mathcal{O}(\alpha'^2),
\end{align*}
\]

(2.1)

where \( A, B, C = 0, \ldots, 9 \) are frame indices, \( \Phi \) is the dilaton, \( H \) is the NS \( \otimes \) NS three-form field strength, \( F \) is the gauge field strength and

\[
\tilde{\nabla}_B Y^A = \nabla_B Y^A + \frac{1}{2} H_{ABC} Y^C
\]

(2.2)

is a metric connection with torsion \( H \). These Killing spinor equations are valid [22] up and including 2-loops in the sigma model perturbation theory [23]. As has already been indicated that they are expected to receive \( \alpha' \) corrections from higher loops. Taking into account the 1-loop anomaly cancellation condition, \( H \) is not closed but instead

\[
dH = -\frac{\alpha'}{4} \varepsilon (\hat{R}^2 - F^2) + \mathcal{O}(\alpha'^2),
\]

(2.3)

where \( \hat{R} \) is the curvature of the metric connection \( \tilde{\nabla} \) which has torsion \( -H \). We follow the conventions of [24].

The Killing spinor equations (2.1) have been solved in complete generality in [14, 15], and all the conditions on the geometry of the spacetime required for the existence of Killing spinors have been found. Here we shall investigate the special case of backgrounds with strictly eight supersymmetries. A consequence of the results of [14, 15] is that such backgrounds are completely characterized by the holonomy of the \( \tilde{\nabla} \) connection which can be either \( SU(2) \) or \( \mathbb{R}^8 \), i.e. \( \text{hol}(\tilde{\nabla}) \subseteq SU(2), \mathbb{R}^8 \). In what follows, we shall solve the field equations for these backgrounds and we shall obtain a complete description of the solutions.
3. Holonomy $SU(2)$ solutions

3.1. Geometry

The backgrounds with eight supersymmetries and $\text{hol}(\hat{\nabla}) \subseteq SU(2)$ [14] admit an orthonormal frame $e^A = (e^i, e^l)$, $a = 0, 1, 2, 7, 3, 8, 4, 9$, such that the six 1-forms $e^a$ and three 2-forms $\omega_r$, $r = 1, 2, 3$

$$
e^a, \omega_r = \frac{1}{2} (\omega_r)_{ij} e^i \wedge e^j,
$$

are $\hat{\nabla}$-parallel, where $(\omega_r)_{ij} = \delta_{jk} (I_r)^i_j$ and the three endomorphisms $I_r$ satisfy the algebra of imaginary unit quaternions, $I_r I_s = -\delta_{rs} 1_{3 \times 4} + \epsilon_{rst} I_t$. In addition, $e^0$ is chosen to be time-like and the other five are space-like.

All the conditions that arise from the Killing spinor equations can be expressed as restrictions on the forms $(e^a, \omega_r)$. Introducing a Hermitian basis with respect to $I_1$, i.e. $e^i = (e^n, e^\beta), n = 3, 4,$ and $\omega_0 = -i \sum_n e^n \wedge e^\beta$, the conditions on the geometry imposed by gravitino and dilatino Killing spinor equations can be expressed as

$$
\hat{\nabla}_M e^n_r = 0, \quad \hat{\nabla}_M \omega_0 = 0, \quad (3.2)
$$

$$
[e_a, e_b], [e_a, \omega_0] = 0, \quad H_{a b c d} + \frac{1}{3!} \epsilon_{a b c d} h_{b c d} b_i b_j b_k H_{b c d} = 0,
$$

$$
(d e^a)^n = 0, \quad (d e^a)_{mn} = 0,
$$

$$
2 \partial_\Phi - (\theta \omega_0) = 0,
$$

$$
\partial_\Phi = 0,
$$

respectively, where $M, N = 0, \ldots, 9$ are spacetime coordinate indices and the indices of the rest of the formulae have been converted to frame indices using the frame, and $e_{051627} = 1$. The 1-form $\theta \omega_0$ is the Lee form associated with $\omega_0$ and its definition can be found in [14].

The first condition in (3.2) implies that the six vector fields $e_a$ dual to the 1-forms $e^a$ are Killing and that the twist $d e^a$ of these vector fields is determined in terms of $H$. This together with $\hat{\nabla}_M \omega_0 = 0$ implies that $\text{hol}(\hat{\nabla}) \subseteq SU(2)$ as expected. The (3.3) conditions arise from the dilatino Killing spinor equation. In particular, the first two conditions imply that the Lie brackets of any two of $e_a$ vector fields associated with the 1-forms $e^a$ close in the span of the vector fields, i.e. the six parallel vector fields span a metric Lorentzian Lie algebra. The structure constants of this Lie algebra are the $H_{a b c}$ components of $H$. Moreover, the structure constants are self-dual. The six-dimensional self-dual Lorentzian Lie algebras have been classified [17] and they are isomorphic to

$$
\mathbb{R}^{5,1}, \quad \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{su}(2), \quad \mathfrak{so}_6.
$$

In particular, the non-vanishing Lie algebra relations of $\mathfrak{so}_6$ are

$$
[e_p, e_q] = \beta_{pq} k, \quad [t, e_p] = \beta_{pq} e_q, \quad \beta_{pq} = \frac{1}{2} \epsilon_{p q q' q'} \beta_{q' q},
$$

where $\beta$ is a constant 2-form $p, q, p', q' = 1, 6, 2, 7$ and the Lorentzian metric is

$$
[e_p, e_q] = \delta_{pq}, \quad \langle k, t \rangle = 1.
$$

The condition on $\beta$ arises from the requirement of self-duality of the structure constants of $\mathfrak{so}_6$.

The conditions on $d e^a$ in (3.3) restrict the twist of the parallel vector fields $e_a$ along the $e^i$ directions to be $(1, 1)$ with respect to all three endomorphisms $I_r$. Observe that $d e^a_{ij} = H^a_{ij}$.
These restrictions will be interpreted as an anti-self-duality condition for a curvature. The condition on the Lee form $\theta$ will also be explained later.

The last condition in (3.3) implies that the dilaton is invariant under the parallel vector fields. Since the metric is also invariant, it remains to discuss the invariance of $H$. The Killing spinor equations alone do not imply that $H$ is invariant. However if $H$ is closed, then it is straightforward to show that $H$ is also invariant, $\mathcal{L}_a H = 0$. Moreover in a perturbation scenario, assuming that at the zeroth order $H$ is invariant, because it is closed, the corrections to $dH$ are constructed from curvature tensors and so are invariant. So it is reasonable to choose $H$ to be invariant as well. It is also the case that

$$i_{\omega} \omega_{\nu} = 0, \quad \mathcal{L}_a \omega_{\nu} = 0. \quad (3.7)$$

The latter condition follows because $\omega_{\nu}$ is $\nabla$-parallel and that $d\epsilon_{ij}^a$ is $(1,1)$ with respect to all three $I_r$.

3.2. Solution of the Killing spinor and field equations

3.2.1. Solution of the Killing spinor equations. Taking $H$ to be invariant under all isometries $e_{\nu}$, the conditions in (3.2) and (3.3) have been interpreted in terms of principal bundles and their connections [14]. In particular, provided that the infinitesimal action generated by the $\nabla$-parallel vector fields can be integrated to a group action, the spacetime, $M$, is a principal bundle $M = P(B, G; \pi)$, where $\mathfrak{Lie} G$ is one of the self-dual algebras in (3.4). It is clear from (3.7) that $\omega_{\nu}$ are the pull-backs of 2-forms on $B$ and so one can write $\omega_{\nu} = \pi^* \omega_{\nu}$. As a consequence, the base space $B$ is an HKT four-dimensional manifold, $(d\tilde{\omega}_{12}^r, \hat{H})$ equipped with the Hermitian forms $\tilde{\omega}_r$. The condition $\theta = 2 d\Phi$ implies that $B$ is conformally balanced.

In addition, the principal bundle is equipped with a connection $\lambda^a \equiv e^a$, see e.g. [25]. The curvature

$$\mathcal{F}^a \equiv d\lambda^a - \frac{1}{2} H^a_{bc} \lambda^b \lambda^c = \frac{1}{2} H^a_{ij} e^i \wedge e^j \quad (3.8)$$

of $\lambda^a$ is $(1,1)$ with respect to all the complex structures of the HKT manifold. Now if the orientation of $B$ is chosen with respect to the $\omega_i$ complex structure, then the $(1,1)$ condition on $\mathcal{F}$ with respect to $I_r$ is compatible with the restriction that $\mathcal{F}$ is anti-self-dual,

$$\mathcal{F}^a_{ij} = \frac{1}{2} \epsilon_{ijkl} \mathcal{F}^a_{kl}, \quad d\text{vol} = \frac{1}{2} \omega_i \wedge \omega_i = \frac{1}{4!} \epsilon_{i_1 \ldots i_4} e^{i_1} \wedge \cdots \wedge e^{i_4}. \quad (3.9)$$

Given these data, the metric and $H$ of spacetime can be written as

$$ds^2 = \eta_{ab} \lambda^a \lambda^b + \pi^* d\tilde{\omega}_{ab}^2 \quad H = \frac{1}{4} \eta_{ab} \lambda^a \wedge \lambda^b + \frac{3}{2} \eta_{ab} \lambda^a \wedge \mathcal{F}^b + \pi^* \hat{H}, \quad (3.10)$$

where $\eta$ is a bi-invariant metric on $\mathfrak{Lie} G$. We can always choose the frame $\lambda$ such that $\eta$ is the standard Lorentzian metric. Observe that $H$ can be written as a sum of the Chern–Simons form of $\lambda$ and the 3-form on $B$.

The solution to the Killing spinor equations (3.10) can be simplified further. In particular, since $B$ is a conformally balanced HKT manifold, there is a function $h$ of $B$ such that

$$d\tilde{\omega}_{ab}^2 = h d\tilde{s}_{\mathfrak{hk}}^2, \quad \hat{H} = - \star_{\mathfrak{hk}} dh, \quad e^{2\Phi} = h, \quad (3.11)$$

where $d\tilde{s}_{\mathfrak{hk}}$ is a four-dimensional hyper-Kähler metric. Since the anti-self-duality condition of the curvature of $\lambda$ is conformally invariant, $\lambda$ can be thought of as an anti-self-dual principal bundle connection over the hyper-Kähler manifold $B_{\mathfrak{hk}}$. Thus, all such solutions of the Killing

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4 The opposite orientation was chosen in [14, 15].

5 Our form conventions are $l = \frac{1}{2!} l_{i_1 \ldots i_4} dx^{i_1} \wedge \cdots \wedge dx^{i_4}$ and Hodge dual is taken as $\star_{l_{i_1 \ldots i_4}} = \frac{1}{e^{l_{i_1 \ldots i_4}}} e^{i_1 \ldots i_4} l_{i_1 \ldots i_4}$. 

6
spinor equations can be constructed from a hyper-Kähler four-dimensional manifold \( B_{hk} \) and an anti-self-dual principal bundle connection on \( B_{hk} \) with gauge group a six-dimensional Lorentzian group \( G \) which has a self-dual Lie algebra (3.4). To summarize, the solution to the Killing spinor equations can be written as in (1.1). To find the explicit solutions, one has to determine the function \( h \).

3.2. Solution of the field equations. It remains to solve the field equations to determine \( h \). For this it suffices to solve the (anomalous) Bianchi identity for \( H \) which can now be written as

\[
dH = d\pi^*\hat{H} + \eta_{ab}\mathcal{F}^a \wedge \mathcal{F}^b = -\frac{\alpha'}{4} \text{tr}(\hat{R}^2 - F^2) + O(\alpha'^2). \tag{3.12}
\]

Taking \( H \) to be closed in the zeroth order in \( \alpha' \), the Bianchi identities imply that \( \hat{R}_{AB,CD} = \hat{R}_{CD,AB} \). Since \( \text{hol}(\hat{\nabla}) \subseteq SU(2) \), the non-vanishing components of \( \hat{R} \) are \( \hat{R}_{ij,AB} \) and moreover \( \hat{R}_{ij,AB} \) is \((1,1)\) with respect to \( I_r \). Similarly, the gaugino Killing spinor equation implies that the non-vanishing components of \( F \) are \( F_{ij} \) and moreover \( F \) is \((1,1)\) with respect to \( I_r \). Using these, we find that

\[
-\nabla^2_{\text{hk}} h - \frac{1}{2} \eta_{ab}F_{ij}F^{abij} = \frac{\alpha'}{8} (\text{tr} \hat{R}_{ij} \hat{R}^{ij} - \text{tr} F_{ij} F^{ij}) + O(\alpha'^2), \tag{3.13}
\]

where all indicated contractions of \( i, j \) indices are with respect to the hyper-Kähler metric. To find a solution, this equation must be solved for function \( h \).

In what follows, we shall mostly explore solutions for which the \( \hat{\nabla} \) connection is identified with the gauge connection \( A \). As a result, the curvature square terms in the right-hand side of (3.13) cancel and there is no anomalous contribution in the Bianchi identity for \( H \), i.e. \( dH = 0 \). This identification can always be made if the associated backgrounds are thought of as solutions of the common sector of type II supergravities. However some care is required when the \( \hat{\nabla} \) connection is identified with \( A \) in the context of heterotic strings, see also [20] for a similar discussion. This is because for generic backgrounds \( \text{hol}(\hat{\nabla}) \subseteq SO(9,1) \) and so it cannot be embedded in the \( SO(32) \) or \( SO(16) \times SO(16) \) internal symmetry of the fermionic sector of the worldvolume action necessary for the construction of \( \text{Spin}(32)/\mathbb{Z}_2 \) and \( E_8 \times E_8 \) heterotic strings, respectively. In the explicit examples that we shall consider below for the \( \text{hol}(\hat{\nabla}) \subseteq SU(2) \) class of backgrounds, \( \text{hol}(\hat{\nabla}) \) is always an orthogonal compact group which can be embedded in both \( SO(32) \) and \( SO(16) \times SO(16) \) and so this difficulty does not arise. We shall give the holonomy of \( \text{hol}(\hat{\nabla}) \) in most examples for completeness.

4. Trivial principal bundle solutions

4.1. \( B_{hk} = \mathbb{R}^4 \)

Solution

4.1.1. WZW models and the self-dual string. Suppose that the principal bundle connection is trivial. This implies that \( \mathcal{F} = 0 \) and so one recovers the Maurer–Cartan equations

\[
d\lambda^a = \frac{1}{2} H^a_{bc} \lambda^b \wedge \lambda^c. \tag{4.1}
\]

So \( \lambda \) are the left-invariant vector fields on the fibre group \( G \). Moreover, suppose that \( \hat{R} = F = 0, \text{hol}(\hat{\nabla}) = \{1\} \), so that the contribution from the curvature square terms in (3.13) vanishes. The resulting equation gives

\[
\nabla^2_{\text{hk}} h = 0. \tag{4.2}
\]
One solution of this is to take $h$ constant, e.g. $h = 1$. In such a case, one finds the solution

$$ds^2 = \eta_{ab} \lambda^a \lambda^b + ds^2(\mathbb{R}^4), \quad H = \frac{1}{2} H_{abc} \lambda^a \wedge \lambda^b \wedge \lambda^c, \quad \Phi = \text{const.} \quad (4.3)$$

For the three choices of self-dual Lie algebras (3.4), one finds the Minkowski vacuum solution, $\mathbb{R}^{9,1}$, $\text{AdS}_3 \times S^1 \times \mathbb{R}^4$ and $\text{CW}_6 \times \mathbb{R}^4$ up to discrete identifications, respectively. Note that $SL(2, \mathbb{R}) = \text{AdS}_3$ and $SU(2) = S^3$. All these solutions have constant dilaton.

The invariant metric on any $\text{CW}$ space can be written as

$$ds^2(\text{CW}) = 2 dv (du + \frac{1}{8} y^2 \beta^2 y dv) + dy^i dy^j. \quad (4.4)$$

The solution requires that $\beta$ be self-dual 2-form. So it can be written as a sum of the three self-dual forms on $\mathbb{R}^4$. As a result,

$$ds^2(\text{CW}_6) = 2 dv (du - \frac{\mu^2}{8} y^2 \beta y dv) + dy^i dy^j, \quad (4.5)$$

where $\mu$ is the length of $\beta$. In fact, the $\text{CW}_6 \times \mathbb{R}^4$ solution preserves 12 supersymmetries [26] and it is the Penrose limit of $\text{AdS}_3 \times S^3 \times \mathbb{R}^4$ [27].

**Flux quantization**

We shall examine the worldvolume theory of strings in such backgrounds in more detail later. Here we remark that since the relative radius of $\text{AdS}_3$ and $S^3$ in the $\text{AdS}_3 \times S^3 \times \mathbb{R}^4$ background is fixed because of the self-duality condition, there is only one overall coupling constant. Moreover appealing to a Dirac quantization condition

$$\int_{S^3} H = k \quad (4.6)$$

is an integer, $k \in \mathbb{Z}$, in some units.

**Flat worldvolume**

4.1.2. 5-branes with flat and curved worldvolume. There are three types of 5-brane solutions with different worldvolume geometry for each choice of the Lie algebra (3.4). The solution found in [7] corresponds to the case that $\mathcal{E}eG = \mathbb{R}^{5,1}$. Allowing for delta function sources and identifying the $\tilde{\nabla}$ connection with the gauge connection $A$, one finds that a solution of (4.2) is

$$h = 1 + \sum_i \frac{Q_i}{|x - x_i|^2}. \quad (4.7)$$

Observe that $\text{hol}(\tilde{\nabla}) \subseteq SO(4)$. This solution can be rewritten as

$$ds^2 = ds^2(\mathbb{R}^{5,1}) + h ds^2(\mathbb{R}^4), \quad H = - \star dh, \quad e^{2\phi} = h \quad (4.8)$$

and it is the multiple 5-brane solution of the heterotic string. The 5-branes are located at the $x_i$ positions in $\mathbb{R}^4$, and $\mathbb{R}^{5,1}$ is the worldvolume.

The solution has two asymptotic regions. As $|x| \rightarrow \infty$, the spacetime approximates the Minkowski vacuum. On the other hand as $|x - x_i| \rightarrow 0$, i.e. close to the position of a 6

Observe that $h + h_i x^i$ is also a harmonic function if $h$ is. So there is a more general solution which however is not asymptotically Minkowski.

8
5-brane, the solution becomes the WZW model $\mathbb{R}^{5,1} \times S^3 \times \mathbb{R}$ with a linear dilaton. The dilaton is infinite at $x = x_\ell$, i.e. string theory is strongly coupled, but this point lies infinite affine distance away from any other point of the spacetime.

The 5-brane charge $p$, per unit worldvolume, is

$$p = \frac{1}{2\text{Vol}(S^3)} \int_{S^3} H,$$

(4.9)

where $\text{Vol}(S^3) = 2\pi^2$ is the volume of unit round $S^3$ and the integral is over the $S^3_\infty \subset \mathbb{R}^4$ at infinity, $|x| \rightarrow \infty$, of the transverse $\mathbb{R}^4$ to the worldvolume directions of the 5-brane.

A straightforward computation reveals that for the solution (4.8), the 5-brane charge is

$$p = \sum_\ell Q_\ell.$$

(4.10)

Appealing to the Dirac-like quantization condition, $p$ must also be quantized in some units.

**Curved worldvolume**

There are two generalizations of the multiple 5-brane solution by replacing the worldvolume $\mathbb{R}^{5,1}$ geometry of the above solution with either $G = \text{AdS}_3 \times S^3$ or $G = CW_6$. These solutions are

$$ds^2 = ds^2(G) + hds^2(\mathbb{R}^4), \quad H = \frac{1}{2} H_{abc} \lambda^a \wedge \lambda^b \wedge \lambda^c - *dh, \quad e^{2\phi} = h,$$

(4.11)

where $\tilde{\nabla}$ is again identified with the gauge connection $A$, $\text{hol}(\tilde{\nabla}) \subseteq \text{SO}(4)$, and $h$ is given in (4.7). These solutions have again two asymptotic regions. As $|x| \rightarrow \infty$, these solutions become either $\text{AdS}_3 \times S^3 \times \mathbb{R}$ or $CW_6 \times \mathbb{R}^4$ with a constant dilaton. While as $|x - x_\ell| \rightarrow 0$, i.e. near the position of a 5-brane, these solutions become either $\text{AdS}_3 \times S^3 \times S^3 \times \mathbb{R}$ or $CW_6 \times S^3 \times \mathbb{R}$ with a linear dilaton. The string coupling becomes infinite at the position of the 5-brane but this position is at infinite affine distance away. String theory is again strongly coupled at the position of the 5-brane.

The solution (4.11) has the properties of a 5-brane with curved $\text{AdS}_3 \times S^3$ or $CW_6$ worldvolume. This is supported from the asymptotic conditions. The 5-brane charge per unit worldvolume is again $p = \sum_\ell Q_\ell$ for the (4.11) solution and $h$ as in (4.7). Observe that the size of the worldvolume geometry does not depend on the transverse coordinate $x$. In the $\text{AdS}_3 \times S^3$ case, one may also consider the flux (4.6) which is again quantized in some units.

**4.2. General hyper-Kähler $B_{hk}$**

**4.2.1. WZW models and compactifications.** It is clear from (4.2) that $h$ can taken to be a constant for any hyper-Kähler four-manifold $B_{hk}$. As a result, one can find the solutions

$$\mathbb{R}^{5,1} \times B_{hk}, \quad \text{AdS}_3 \times S^3 \times B_{hk}, \quad CW_6 \times B_{hk}.$$

(4.12)

Choosing $B_{hk} = K_3$ or $T^4$, $\mathbb{R}^{5,1} \times K_3$ and $\mathbb{R}^{5,1} \times T^4$ can be interpreted as the vacua of a $K_3$ and $T^4$ compactification of the heterotic string to six dimensions, respectively. Similarly $\text{AdS}_3 \times S^3 \times K_3$ and $\text{AdS}_3 \times S^3 \times T^3$ can be interpreted as the vacua of an $\text{AdS}_3$ compactification of the heterotic string on $S^3 \times K_3$ and $S^3 \times T^3$, respectively. In all the above backgrounds, the $\tilde{\nabla}$ connection is identified with the gauge one $A$ and $\text{hol}(\tilde{\nabla}) \subseteq \text{SO}(4)$. Of course $B_{hk}$ can be chosen to be non-compact. In such a case for any choice of a four-dimensional hyper-Kähler manifold, we get a new solution.
4.2.2. 5-branes with flat and curved worldvolume in hyper-Kähler manifolds. The solution (4.11) can be generalized by replacing \( \mathbb{R}^4 \) with any other four-dimensional hyper-Kähler manifold \( B_{\text{hk}} \). In such a case, \( h \) satisfies (4.2) where the harmonic condition is with respect to the hyper-Kähler metric on \( B_{\text{hk}} \). The resulting solution is
\[
d s^2 = ds^2(G) + hd s^2(B_{\text{hk}}), \quad H = \frac{1}{2} H_{\text{hk}} \lambda^a \wedge \lambda^b \wedge \lambda^c - \sum_{\text{bi}} dh, \quad e^{2\phi} = h, \tag{4.13}
\]
where \( G = \mathbb{R}^{5,1}, \text{AdS}_3 \times S^3 \) or \( CW_6 \). This solution has the interpretation as 5-branes located on a four-dimensional hyper-Kähler manifold with worldvolume geometry \( G \). However, the explicit expression for \( h \) depends on the choice of the hyper-Kähler metric, and so explicit solutions can be given only on a case-by-case basis. In all these cases, the \( \nabla \) connection can be identified with the gauge one in the context of heterotic string since \( \text{hol}(\nabla) \subseteq SO(4) \). It is straightforward to construct explicit solutions in some cases. For example, an explicit solution can be found if the hyper-Kähler metric is chosen to be the Gibbons–Hawking metric.

5. Non-trivial principal bundle solutions

5.1. Principal bundle connections

Before we proceed to construct solutions with an Abelian and a non-Abelian principal bundle connection \( \lambda \), we shall first express \( \lambda \) in terms of standard vector bundle connection \( C \). This strictly is not necessary since principal bundle connection theory is well established [25]. However, the analysis may look more familiar in terms of vector bundle connections. Indeed given \( C \) a vector bundle connection, write the local expression
\[
\lambda = - g^{-1} dg + g^{-1} C g \tag{5.1}
\]
where \( g \) is a function of the principal bundle with values on the gauge group \( G \). Since \( C \) takes values in the Lie algebra of the gauge group, it is clear that \( \lambda \) is also \( \mathfrak{Lie}G \)-algebra valued. Moreover, as required, it transforms under the adjoint representation of \( \mathfrak{Lie}G \) under the right action of \( G \) on the spacetime. Restricting \( C \) on the fibre \( G \), one recovers the left-invariant 1-forms of the group \( G \). Next, the curvature
\[
\mathcal{F} = d\lambda - \lambda \wedge \lambda \tag{5.2}
\]
of \( \lambda \) is
\[
\mathcal{F} = g^{-1} (dC - C \wedge C) g = g^{-1} F(C) g, \tag{5.3}
\]
where \( F(C) \) is the curvature of connection \( C \). Therefore the curvature of the principal bundle \( \mathcal{F} \) is related to the curvature of connection \( C \) up to a gauge transformation. Observe that if \( G \) is Abelian, then (5.1) can be simplified. For example, if \( G = U(1) \), taking \( g = e^{-i \theta} \), then \( \lambda = i (d\theta + C) \), where \( C \) is locally a real 1-form. Such principal bundle connections appear in black hole solutions with rotation or brane solutions with wrapping on a torus.

It is now clear that since the metric \( \eta \) on \( \mathfrak{Lie}G \) is bi-invariant, one can replace \( \mathcal{F} \) in equations (3.12) and (3.13) with \( F(C) \). The gauge transformation \( g \) is eliminated from gauge invariant expressions like the first Pontryagin class. So to find explicit solutions, it is sufficient to begin from an anti-self-dual connection \( C \) and then use the formula (5.1) to construct the principal bundle connection.

As a final remark on the relation between the principal bundle \( \lambda \) and vector bundle \( C \) connections, it is customary in some parts of the physics literature, instead of using \( \lambda \), to use
\[
\rho \equiv g \lambda g^{-1} = -dgg^{-1} + C, \tag{5.4}
\]
where \( dgg^{-1} \) are identified with the right-invariant 1-forms of \( G \). Moreover, typically \( dgg^{-1} \) are explicitly expressed in terms of the coordinates of \( G \). Since \( \rho \) and \( \lambda \) are related up to a
conjugation, all quantities of $\lambda$ can be expressed equally well in terms of $\rho$. However, we shall continue the analysis in terms of $\lambda$ because the geometric interpretation of both the metric of the spacetime and the 3-form field strength is more transparent in terms of $\lambda$.

5.2. Non-Abelian connections and $B_{hk} = \mathbb{R}^4$

5.2.1. Instantons. Before we proceed to explicitly construct supergravity solutions for $B_{hk} = \mathbb{R}^4$, we shall summarize some properties of $SU(2)$ instantons. In fact, the relevant instanton solutions are those for the gauge group $SL(2, \mathbb{R}) \times SU(2)$. It is not apparent though that there are non-Abelian anti-self-dual connections for the gauge group $SL(2, \mathbb{R})$, so we shall consider non-Abelian connections only for the $SU(2)$ subgroup.

Such connections can be constructed using the t’Hooft ansatz which can be described as follows. On $\mathbb{R}^4$ there are two commuting (constant) hypercomplex structures $I_r$ and $J_r$, i.e.

$$I_r I_k = -\delta_{rs} I_s, \quad J_r J_s = -\delta_{rs} J_s + \epsilon_{rst} I_t, \quad [I_r, J_s] = 0$$  \hspace{1cm} (5.5)

which can be constructed from a basis of self-dual and anti-self-dual 2-forms, respectively. It is well known that $so(4) = so(3) \oplus so(3)$ and $I_r$ and $J_r$ span the two $so(3)$ subalgebras, respectively. Next, consider $\mathbb{R}^4$ as a hyper-Kähler manifold with respect to $I_r$ and write

$$C_r = (I_r)_{ij}/\partial_{ij} \log f,$$  \hspace{1cm} (5.6)

for some function $f$ of $\mathbb{R}^4$ which must be determined. The anti-self-duality condition on $F(C)$, which is equivalent to requiring that $F(C)$ is $(1,1)$ and traceless with respect to $I_r$, implies that

$$\frac{1}{f} \partial^2 f = 0.$$  \hspace{1cm} (5.7)

Thus a solution for $f$ is

$$f = 1 + \sum_{\ell=1}^{\nu} \frac{\rho^2_{\ell}}{|x - x_{\ell}|^2}, \quad x \in \mathbb{R}^4.$$  \hspace{1cm} (5.8)

This is the $\nu$-instanton solution with the $\ell$th instanton located at the pole $x_{\ell}$ and $\rho^2_{\ell}$ is identified with its size.

These solutions though do not have the correct asymptotic behaviour and are singular. In particular, $F(C)$ diverges at the positions of the instantons. However after a singular gauge transformation, the 1-instanton configuration can be rewritten as

$$C_r = 2(J_r)_{ij} x_i^j / |x| + \rho^2_t,$$  \hspace{1cm} (5.9)

where translation invariance has been used to locate the instanton at the origin in $\mathbb{R}^4$.

The instanton configurations are characterized by their first Pontryagin number or instanton number,

$$\nu = -\frac{1}{16\pi^2} \int \delta_{rs} F^s \wedge F^r.$$  \hspace{1cm} (5.10)

In particular, for the 1-instanton configuration $\nu = 1$.

5.2.2. Wrapped 5-branes on $S^3$. To find the solutions associated with the non-trivial $SU(2)$ principal bundle, we begin by considering the instanton solution (5.6) for $f$ given in (5.8). A direct computation reveals that

$$-\frac{1}{2} \delta_{rs} F^r_{ij} F^{ij} = -\frac{1}{2} \delta_{rs} F^r (C)_{ij} F^s (C)^{ij} = \partial^2 \partial^2 \log f.$$  \hspace{1cm} (5.11)

Substituting this into the equation for $h$ (3.13), observing that $\text{hol}(\tilde{\nabla}) \subseteq SO(7)$ and identifying the $\tilde{\nabla}$ with the connection $A$, we have that
\[ -\partial^2 h + \partial^2 \partial^2 \log f = 0. \]  
(5.12)

Clearly, if one allows for delta function sources, then \( h \) is not uniquely determined in terms of \( f \). In particular, one finds that

\[ h = h_0 + \partial^2 \log f, \quad h_0 = s + \sum_n \frac{Q_n}{|x - x_n|^2}, \]  
(5.13)

where \( s = 1 \) or \( s = 0 \).

Let us investigate the 1-instanton case with size \( \rho^2 \) in more detail. To begin, consider the special case for which

\[ h = 1 + \frac{4}{|x|^2} + \partial^2 \log f = 1 + 4 \frac{|x|^2 + 2\rho^2}{(|x|^2 + \rho^2)^2}, \quad f = 1 + \frac{\rho^2}{|x|^2}, \]  
(5.14)

receives contributions both from the instanton, \( f \) and \( h_0 \). This is precisely the expression for \( h \) that one would find if the smooth instanton solution (5.9) was used to calculate the Pontryagin class.

The resulting supergravity solution given by substituting (5.14) into (1.1) is smooth. Moreover, it has one asymptotic region \( \text{AdS}_3 \times S^3 \times \mathbb{R}^4 \) as \( |x| \to \infty \). Integrating \( H \) on the asymptotic 3-sphere \( S^3_\infty \subset \mathbb{R}^4 \), one detects 5-brane charge

\[ p = 4. \]  
(5.15)

So the smooth \( \nu = 1 \) instanton gives rise to 4 units of 5-brane change. It is not apparent why this is the case but this result arises from the standard normalization for 5-brane and instanton charges.

Furthermore, the geometry of the solution associated with (5.14) at \( |x| \to 0 \) is again \( \text{AdS}_3 \times S^3 \times \mathbb{R}^4 \) but this region is not asymptotic. This is because it can be reached in finite proper distance from any other region of spacetime apart from that for which \( |x| \to \infty \). The dilaton is bounded on the spacetime and so there are no strongly coupled regions. These asymptotic conditions are reminiscent of those of dyonic 5-brane backgrounds [28] which preserve 1/4 of the supersymmetry.

It is clear that the spacetime of the solution (1.1) for \( h \) given in (5.14) is a product \( M = \text{AdS}_3 \times X_7 \), where \( X_7 \) is a non-trivial \( SU(2) \)-principal bundle over \( \mathbb{R}^4 \). The size of the \( S^3 \) fibre is the same at every point of \( \mathbb{R}^4 \). The isometries of \( X_7 \) are \( SU(2) \times SO(4) \). So the symmetry group of the whole solution is \( SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \times SU(2) \times SO(4) \). Observe that if the instanton connection vanishes and \( X_7 = S^3 \times \mathbb{R}^4 \), the isometry group is \( SU(2) \times SU(2) \times SO(4) \). The presence of an instanton breaks some of the isometries of the background.

Adding further 5-branes to (5.14) as

\[ h = 1 + 4 \frac{|x|^2 + 2\rho^2}{(|x|^2 + \rho^2)^2} + \frac{Q}{|x|^2}, \]  
(5.16)

one finds another solution with the same asymptotic geometry at \( |x| \to \infty \). But now at the position of the 5-brane there is another asymptotic region with geometry \( \text{AdS}_3 \times S^3 \times \mathbb{R} \) and a linear dilaton. The brane charge detected at \( |x| \to \infty \) is \( p = 4 + Q \). This solution has the same symmetries as that given for \( h \) in (5.14). Of course (5.16) can be further generalized by adding additional 5-branes located at different positions in \( \mathbb{R}^4 \).

5.2.3. ADHM construction and 5-branes. One can generalize the previous example to multi-instanton solutions using the ADHM construction [29]. The simplest case to describe is that of \( \nu \) instantons for the gauge group \( Sp(r) \). For this, consider a \( (\nu + r) \times r \) matrix \( V \) of quaternions which satisfy the linear relation
that the dilaton is bounded everywhere on the spacetime. However, we have not been able to show of a singular gauge transformation.

\[ C = V^i dV, \]

where the differentiation is with respect to \( x \). After some computation, one finds that

\[ -\frac{1}{2} b_{rs}, F^r_{ij}, F^{ij} = -\frac{1}{2} b_{rs}, C (C)_{ij}, F^s (C)_{ij} = \partial^2 \partial^2 \text{Tr} f, \]

where \( f \) is the \( v \times v \) matrix

\[ f = \Delta^i \Delta = a^i a + a^i b x + x^i b^i a + x^i b^i a. \]

To construct new solutions (1.1) based on the \( SU(2) \) \( \nu \)-instanton connection given by the ADHM construction, substitute (5.19) into (3.13) to find

\[ h = s + \partial^2 \text{Tr} f, \quad s = 0, 1. \]

This is the multi-instanton generalization of (5.14), \( s = 1 \). The 1-instanton case for the gauge group \( SU(2) = Sp(1) \) that we have given in (5.14) can be recovered for

\[ \Delta^i = (\rho, x^i), \]

where \( \rho \) is a quaternion with a real only component. Observe that the ADHM construction gives the smooth instanton solutions with the expected asymptotic conditions without the need of a singular gauge transformation.

It is likely that the supergravity solution (1.1) associated with (5.21) is smooth, and the dilaton is bounded everywhere on the spacetime. However, we have not been able to show that \( h > 0 \) for \( v > 1 \). As \( \|x\| \to \infty \) and for \( s = 1 \), the spacetime becomes \( AdS_3 \times S^3 \times \mathbb{R}^4 \). The 5-brane charge detected at infinity, \( s = 1, p = 4v \). If instead \( s = 0, \) as \( \|x\| \to \infty \), the spacetime becomes \( AdS_3 \times S^3 \times S^3 \times \mathbb{R} \). This is a linear dilaton background.

The solution (1.1) associated with (5.21) has continuous moduli which can be identified with that of \( SU(2) \) instantons. In particular, it will depend on \( 8v - 3 \) parameters. The spacetime is again a product \( SL(2, \mathbb{R}) \times X_7 \). For a generic \( v \) instanton connection, the isometry group of \( X_7 \) is \( SU(2) \). So the symmetry group of the whole background is \( SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \times SU(2) \). Enhancement of symmetry can occur if the positions of the instantons and their relative orientations are chosen appropriately.

A more general solution can be constructed by adding 5-branes located at different points in \( \mathbb{R}^4 \). In particular, one can solve (3.13) as

\[ h = s + \partial^2 \text{Tr} f + \sum_{n=1}^{k} \frac{Q_n}{\|x - x_n\|^2}, \quad s = 0, 1, \]

where \( f \) is as given in (5.20). Apart from the asymptotic regions, we have mentioned above as \( \|x\| \to \infty \) for (5.21), this new solution has new asymptotic regions at the positions of the 5-branes \( \|x - x_n\| \to 0 \). The geometry of the spacetime near the positions of the 5-branes becomes \( AdS_3 \times S^3 \times S^3 \times \mathbb{R} \). This is a linear dilaton background. The 5-brane charge detected at infinity, \( \|x\| \to \infty, s = 1, p = 4v + \sum_n Q_n \).

The solution (1.1) associated with (5.23) has continuous moduli that of \( v SU(2) \) instantons. In addition, the positions of the 5-branes are new moduli. So it depends on \( 8v + 4k - 3 \) parameters. The symmetry group of a generic solution is again \( SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \times SU(2) \).
5.2.4. 5-brane probes. As further evidence for the interpretation of (5.14), we can perform a probe computation. In particular, consider a 5-brane probe which is embedded in spacetime by identifying the worldvolume probe coordinates with the worldvolume coordinates AdS$_3 \times S^3$ of the solution (5.14). Moreover, take the transverse coordinates $x$ of the spacetime to be constant. The induced metric on the probe worldvolume is that of AdS$_3 \times S^3$. Moreover, such a probe is supersymmetric. This follows immediately from the kappa-symmetry condition \[ \Gamma \epsilon = \epsilon, \] where $\Gamma$ is the kappa symmetry projector of the 5-brane and $\epsilon$ is a Killing spinor. For the embedding described above, the kappa-symmetry condition gives \[ \Gamma_{051627} \epsilon = \epsilon. \] All Killing spinors of the supergravity background (5.14) satisfy this kappa-symmetry condition and so the embedding is supersymmetric.

5.2.5. Abelian connections. The construction described above for non-Abelian connections can easily be extended to Abelian ones. If $B_{hk} = \mathbb{R}^4$, Abelian anti-self-dual connections can be constructed by writing \[ C_i = (I_r)^j \delta_j f^r \] and requiring that $f^r$ be harmonic functions. Such solutions typically will be singular at the positions of the harmonic functions $f^r$. One can make superposition of Abelian and non-Abelian gaugings. For example, one can consider principal bundles with a non-Abelian SU(2) connection but only gauging an Abelian subgroup of $SL(2, \mathbb{R})$. The gauging of a time-like direction in $SL(2, \mathbb{R})$ is of particular interest because it allows the construction of smooth solutions for the cases that $B_{hk}$ is a compact hyper-Kähler manifold. However, this may lead to a Lorentzian holonomy for $\nabla$.

5.2.6. General hyper-Kähler manifolds. It is clear that the construction of the solutions presented above can be generalized to any four-dimensional hyper-Kähler manifold $B_{hk}$. Let us consider the case that $B_{hk}$ is compact. To find a solution, one has to solve the equation \[ \nabla^2 h = -\frac{1}{2} \eta_{ab} F^a_{ij} F^{bij} \] for $h$. A partial integration formula easily reveals that there is no smooth solution for $h$ unless \[ \int_{B_{hk}} \eta_{ab} F^a \wedge F^b = 0. \] Assuming that we have only a non-trivial SU(2) connection, this condition cannot be met because the integral is proportional to the instanton number which is not zero.

To construct smooth solutions, the contribution of the SU(2) instanton must be cancelled. For this, we allow for rotation, i.e. for an Abelian connection along a time-like direction in $SL(2, \mathbb{R})$. Since $\eta$ is a Lorentzian metric, the consistency condition is now written as

\[ -\int_{B_{hk}} F^0 \wedge F^0 + \sum_{r=1}^{3} \int_{B_{hk}} F^r \wedge F^r = 0. \]

Because of the relative minus sign, the $U(1)$ instanton can be chosen to cancel the contribution from the SU(2) instantons. However, now the holonomy of $\nabla$ may be Lorentzian and so $\nabla$ cannot be identified with the gauge connection in the context of the heterotic string. In such
a case, the contribution of the curvature square terms in (3.13) should be taken into account. The condition for the existence of a $U(1)$ instanton connection is

$$
\int_{\mathcal{B}_k} \mathcal{F}^0 \wedge \omega_{h_k} = 0.
$$

(5.30)

This can be generalized to the $SU(2)$ instantons and involves the notion of stable vector bundles [30]. In addition of course (5.29) must be satisfied. Alternatively, one can add appropriate sources on the right-hand side of (5.27) (anti 5-branes) to cancel the contribution from the $SU(2)$ instantons.

6. Type II and M-theory solutions

6.1. Type IIB

All solutions that we have presented here for which $dH = 0$ can be considered as solutions of type II supergravity theories as well. Such generic solutions in type II will preserve 1/4 of the supersymmetry, i.e. eight supercharges. However, special solutions can preserve more supersymmetry.

In type IIB, the generic solutions we have constructed have the interpretation as wrapped NS5-branes on $G$ with transverse space any four-dimensional hyper-Kähler manifold. After an S-duality, the generic solution becomes a wrapped D5-brane on $G$ with transverse space any four-dimensional hyper-Kähler manifold. In particular, the S-dual of the solution given in (1.1) is

$$
\begin{align*}
&\text{ds}^2 = h^{-1/2} \eta_{ab} \text{d}x^a \text{d}x^b + h^{1/2} \text{d} s_\mathcal{B}(\mathcal{B}_k), \\
&\text{e}^{2\Phi} = h^{-1}, \\
&H_{RR} = \frac{1}{2} \eta_{ab} \lambda^a \wedge \lambda^b + \frac{3}{2} \eta_{ab} \lambda^a \wedge \mathcal{F}^b - \ast_{h_k} \text{d} h, \\
\end{align*}
$$

(6.1)

where now $H_{RR}$ is the Ramond–Ramond 3-form field strength.

6.2. Type IIA and M-theory

6.2.1. Intersecting branes. To give an M-theoretic interpretation of the solutions we have constructed, we shall first explain some aspects of intersecting branes; see [34] for the supergravity solutions. It is not apparent what the structure is of a supergravity solution which has the interpretation as a fully localized brane intersection\(^7\). A minimal expectation is that one should be able to locate the branes as well as their asymptotic regions that are involved in the intersection from data provided by the solution. For simplicity and for the application that we have in mind, suppose one wants to describe the fully localized M2-brane intersection with an M5-brane. In particular, it has been argued in [37] that M2-branes end on M5-branes on a self-dual string\(^8\). One way to fulfil the above-stated requirements is to be able to define two distance functions $\delta_{M2} \geq 0$ and $\delta_{M5} \geq 0$ on the spacetime which describe the distances of spacetime points from the locations $\delta_{M2} = 0$ and $\delta_{M5} = 0$ of the M2 and M5 branes, respectively. In addition, consider

$$
\delta_{M2 \perp M5} = \sqrt{\delta_{M2}^2 + \delta_{M5}^2}, \quad \Delta = \frac{\delta_{M2}}{\delta_{M5}}.
$$

(6.2)

One can then give the following definitions of some regions of spacetime:

\(^7\) Ansätze for the construction of supergravity solutions with an interpretation as brane intersections were initially proposed in [34]. For a more recent detailed account see [35], where indirect arguments and physical intuition, but not a systematic construction, are used to find solutions.

\(^8\) A partially localized solution with the interpretation of an M2-brane intersecting an M5-brane on a self-dual string can be found in [36].
Near horizon region of the self-dual string: $\delta_{M^{2,5}}$ small, $\Delta$ finite.

Overall transverse: $\delta_{M^{2,5}}$ large, $\Delta$ finite.

Near horizon of the M2-brane: $\delta_{M^{2}}$ small, $\Delta^{-1} \ll 1$.

Near horizon of the M5-brane: $\delta_{M^{5}}$ small, $\Delta^{-1} \ll 1$.

There are other regions that one can define like those that are asymptotic to either M2- or M5-branes. Under the assumptions we have made, one finds that apart from the near horizon geometry of the common intersection there are other geometries that appear in the fully localized M2- with M5-brane intersection like that of the near M2-brane geometry. This is simply the region of spacetime which is near the location of the M2-brane. Similarly, one can argue for a near M5-brane geometry.

### 6.2.2. Near M2-brane geometry and brane intersections

The background (1.1) viewed as a solution of IIA supergravity can easily be lifted to 11-dimensional supergravity. In particular, the 11-dimensional background associated with (1.1) for $h$ given in (5.14) is

$$\begin{align*}
  ds^2_{(11)} &= h^{-1} \eta_{ab} dx^a dx^b + h^2 [ds^2(\mathbb{R}^4) + dy^2] \\
  F &= H \wedge dy,
\end{align*}$$

(6.3)

where $y$ is the 11th coordinate. Recall that for this solution $G = \text{AdS}_3 \times S^3$. The claim is that (6.3) describes the near M2-brane geometry of an M2-brane ending on an M5-brane on the self-dual string. We choose $\delta_{M^5} = r$. As $r \ll 1$, $\delta_{M^5}$ becomes of the order of $\delta_{M^2}$, i.e. $\Delta$ is finite and $\delta_{M^{2,5}} \ll 1$. This region is the near horizon region of the self-dual intersection. The geometry of the spacetime in this region is $\text{AdS}_3 \times S^3 \times \mathbb{R}^5$ which lies in a finite affine distance from the non-asymptotic points of the spacetime. This is the expected geometry of the self-dual strings that lie at the intersection.

To continue, we take the 11th coordinate as one of the worldvolume coordinates of the M2-brane which is transverse to the M5-brane. Since this is claimed to be a near M2-brane geometry, one may expect to find some signs of the $\text{AdS}_3 \times S^7$ near horizon geometry of the M2-brane. Now the $\text{AdS}_3$ part of the spacetime is expected to include the distance function $\delta_{M^2}$ of the M2-brane as a coordinate. So the remaining three coordinates of $S^3$ and the four coordinates of $\mathbb{R}^4$ must give a description of $S^7$. This seems to be the case because $S^7$ can be viewed as a fibration $S^3 \to S^7 \to S^4$. Moreover this is the instanton bundle, i.e. the curvature of a principal bundle connection is anti-self-dual and has instaton number 1. Of course both the $\text{AdS}_3$ and $S^7$ geometries of the near horizon geometry of the M2-brane are distorted because of the presence of the M5-brane.

We do not have a proof that (6.3) is the most general solution of this type. A concern is that (6.3) is delocalized in the $y$-direction. So there may be a more general solution of 11-dimensional supergravity which describes the near M2-brane geometry of an M2-brane ending on an M5-brane. However, such a solution will have more active fields and it will not be a solution of the type II common sector alone.

### 7. Worldvolume theory

#### 7.1. Gauged sigma models with WZ term

To understand the worldvolume theory of the hol($\nabla$) $\subseteq SU(2)$ class of solutions and in particular the presence of the Pontryagin class of the principal bundle connection in the expression for $dH$, we shall summarize briefly the gauging of a sigma model with a WZ term [38], for the supersymmetric case see [39]. The gauging which is relevant for the
hol(\nabla) \subseteq SU(2)\) class of solutions is that for which the sigma model target space is the fibre of the principal bundle spacetime \(M = P(G, B; \pi)\).

To begin, we consider a sigma model with target space \(N\) equipped with metric \(g\) and a Wess–Zumino term \(W\), \(dW = 0\). Moreover, we assume that \(N\) admits a group action which leaves both \(g\) and \(W\) invariant. In the standard analysis for the gauging of such a system, the worldvolume dimension is 2. Here, we shall weaken this assumption, and we shall take the worldvolume dimension to be \(n\).

The action before gauging is

\[
S = \frac{1}{2} \int_{\Sigma} d^d\sigma \, g_{pq} \partial_{\mu} X^p \partial^q X^q + \int_{D \subseteq \Sigma} W_{pqrs} \epsilon^{\mu\nu\rho} \partial_\mu X^p \partial_\nu X^q \partial_\rho X^r,
\]

(7.1)

where \(\Sigma\) is the worldvolume and \(D\) is a three-dimensional subset of \(\Sigma\).

To begin, let us consider the minimal coupling of the \(W\) term. This gives

\[
W = \frac{1}{6} W_{pqrs} (dX^p + C^c_\mu \xi^p_\mu) \wedge (dX^q + C^c_\mu \xi^q_\mu) \wedge (dX^r + C^c_\mu \xi^r_\mu),
\]

(7.2)

where \(\xi\) are the Killing vector fields that also leave \(W\) invariant and \(C\) is a connection which gauges the isometries. Taking the exterior derivative of \(W\), one finds

\[
dW = \frac{1}{6} \xi^p \epsilon^{pqr} H_{pqr} F(C)^q \wedge (dX^q + C^c_\mu \xi^q_\mu) \wedge (dX^r + C^c_\mu \xi^r_\mu).
\]

(7.3)

To preserve closure, one has to add a non-minimal coupling as

\[
W = \frac{1}{6} W_{pqrs} (dX^p + C^c_\mu \xi^p_\mu) \wedge (dX^q + C^c_\mu \xi^q_\mu) \wedge (dX^r + C^c_\mu \xi^r_\mu) \wedge (-2u_{ap} (dX^p + C^b_\mu \xi^p_\mu) \wedge F(C)^b),
\]

(7.4)

where

\[
i_M W = du_u.
\]

(7.5)

Taking again the exterior derivative, one finds that

\[
dW = -c_{ab} F(C)^a \wedge F(C)^b, \quad c_{ab} = \frac{\xi^M}{\xi^u_{(ab)}} u_{bM}.
\]

(7.6)

It can be shown that \(c\) is constant. If \(c \neq 0\), the gauging is considered anomalous. For more details on the anomalies that arise in the gauging of the Wess–Zumino term of sigma models see [38] and for their relation to equivariant cohomology see [40].

It is clear that (7.6) is reminiscent to that of (3.12). To make a connection with the \(hol(\nabla) \subseteq SU(2)\) class of solutions precise, suppose that the sigma model target manifold \(N\) is a group manifold \(G\), \(N = G\), and \(\Sigma\) is \(\mathbb{R}^{5,1}\), \(su(2) \oplus su(2) \oplus su(2)\) or \(\mathfrak{so}_6\). Moreover, we gauge the right action of \(G\) on \(G\). The vector fields which generate this action are the left invariant vector fields and \(c_{ab} = -\eta_{ab}\).

It remains to explain the relation between the connections \(\lambda\) and \(C\), and the role of the base space \(B\) of \(M = P(G, B; \pi)\). The connection \(C\) that we have introduced to gauge the sigma model should not be thought of as a worldvolume field. Instead, it should be considered as a composite field which depends on the scalars of the base manifold \(B\). Now the base manifold has also a Wess–Zumino term \(\hat{H}\) which is invariant, \(\mathcal{L}_\xi \hat{H} = 0\), and \(i_\xi \hat{H} = 0\), so \(\hat{H}\) has trivial gauging. Now taking \(H = W + \pi^* \hat{H}\), where \(W\) is as given in (7.4), and evaluating \(dH\), we recover (3.12). Furthermore, the minimal coupling of the kinetic term of sigma model action (7.1), the identification of \(N = G\) and the addition of the metric of the base space \(B\), which has trivial gauging, reproduce the metric of the spacetime of the \(hol(\nabla) \subseteq SU(2)\) class of solutions.
7.2. Conserved currents and worldvolume supersymmetry

7.2.1. (1,1) supersymmetry. Suppose that the gauge connection $A$ is identified with $\hat{\nabla}$, and so $dH = 0$. In such a case, the associated worldvolume theory of all generic backgrounds with $\text{hol}(\hat{\nabla}) \subseteq SU(2)$ has (1,1) supersymmetry. Two-dimensional supersymmetric sigma models have extensively been investigated before [18, 41–43]. Here we follow [18]. To describe the worldvolume theory, introduce the worldvolume coordinates $(\sigma^a, \theta^a, \bar{\theta}^a)$, where $(\sigma^a, \theta^a)$ are even and $(\bar{\theta}^a)$ are odd coordinates, respectively, and the usual odd superspace derivatives $D_a$ and $D_{\bar{a}} = i \partial_{\bar{a}}$. Let $g$ be the spacetime metric of the $\text{hol}(\hat{\nabla}) \subseteq SU(2)$ backgrounds. Since $dH = 0$, $H = db$, the worldvolume action written in terms of (1,1) superfields is

$$S = \int d^2 \sigma d\theta^+ d\theta^- (g + b)_{MN} D_a X^M D_\alpha X^N,$$

(7.7)

where the spacetime coordinates $x^M$ have been promoted to worldvolume superfields $X^M (\sigma^a, \theta^a, \bar{\theta}^a)$. The components of the superfield $X$ are

$$X^M = X^M|, \quad \psi^-_a = D_- X^M|, \quad \psi^+_a = D_+ X^M|.$$  

(7.8)

The field equations are

$$\hat{\nabla}_- D_a X^M = 0.$$  

(7.9)

It is easy to see that in addition to the energy momentum tensor, and the (1,0) and (0,1) supersymmetry currents, for every $\hat{\nabla}$-parallel form there is a conserved current. In particular, the ‘elementary’ conserved currents of generic backgrounds are

$$T_{++} = g_{MN} D_a X^M D_a X^N, \quad T_{--} = g_{MN} D_- X^M D_- X^N,$$

$$J^a = i \theta^a D_a X^M, \quad U^a = \omega_{\alpha MN} D_a X^M D_\alpha X^N.$$  

(7.10)

Non-generic backgrounds may have additional currents. As we shall mention later that these may lead to the integrability of the string dynamics.

7.2.2. Ultraviolet finiteness. One of the issues that arises in the worldvolume theory is whether the associated sigma model is ultraviolet finite. Since the worldvolume theory of generic backgrounds has manifest (1,1) supersymmetry, it is not sufficient to guarantee ultraviolet finiteness. For this, the worldvolume supersymmetry has to be enhanced to (4,1) [18]. In turn (4,1) supersymmetry requires that the spacetime admit a hypercomplex structure.

As we have already explained that the spacetime $M$ of generic $\text{hol}(\hat{\nabla}) \subseteq SU(2)$ backgrounds admits a hypercomplex-type structure, $I_r$, which is $\hat{\nabla}$-parallel. However, the endomorphisms $I_r$ are degenerate and so cannot be thought of as complex structures on $M$. Moreover since $M$ is Lorentzian, it does not admit complex structures which are Hermitian with respect to the Lorentzian metric. Nevertheless, we shall provide an indirect argument that the worldvolume theory of some of our backgrounds is ultraviolet finite. First, all these models are finite up to two loops. This can be shown by direct observation using the identification of the $\hat{\nabla}$ connection with the gauge connection. Next, we restrict our analysis to the wrapped on $\text{AdS}_3 \times S^3$ 5-brane solutions. First observe that $M = P(SL(2, \mathbb{R}) \times SU(2), B_{\text{bk}}; \pi) = SL(2, \mathbb{R}) \times P(SU(2), B_{\text{bk}}; \pi)$. The fields along $SL(2, \mathbb{R})$ decouple from the rest of the theory. So the question is whether the sigma model on $P(SU(2), B_{\text{bk}}; \pi)$ is ultraviolet finite.

To find whether the sigma model on $P(SU(2), B_{\text{bk}}; \pi)$ is ultraviolet finite, consider instead the sigma model on $P(SU(2) \times U(1), B_{\text{bk}}; \pi) = P(SU(2), B_{\text{bk}}; \pi) \times U(1)$, i.e. add
an additional free superfield $X$ in the sigma model on $P(SU(2), B_{hk} : \pi)$. It turns out now that $P(SU(2) \times U(1), B_{hk} : \pi)$ admits a $\mathbb{V}$-parallel hypercomplex structure given by

$$\Omega' = \zeta'^{\alpha\beta} \lambda^\alpha \wedge \lambda^\beta + \omega' ,$$

(7.11)

where $\zeta'$ is the hypercomplex structure on $U(1) \times SU(2) = S^1 \times S^3$. It is well known that $S^1 \times S^3$ is an HKT manifold. Consequently, the sigma model on $P(SU(2) \times U(1), B_{hk} : \pi)$ has (4,1) worldvolume supersymmetry and so it is ultraviolet finite as a consequence of the results of [18]. Since $P(SU(2), B_{hk} : \pi)$ differs from $P(SU(2) \times U(1), B_{hk} : \pi)$ up to the addition of a free superfield, which does not affect the interactions, the sigma model on $P(SU(2), B_{hk} : \pi)$ is also ultraviolet finite. Thus all wrapped 5-brane solutions on $AdS_3 \times S^3$ are ultraviolet finite. We do not have an argument which generalizes to the rest of the solutions. However, it is likely that all backgrounds of the class $hol(\mathbb{V}) \subseteq SU(2)$ which are associated with a (1,1)-supersymmetric worldvolume theory are ultraviolet finite.

### 7.3. String integrability

As we have argued a large class of the supersymmetric heterotic backgrounds with eight supersymmetries, and possibly all, are ultraviolet finite. So, the question that arises is whether string theory can be solved on such backgrounds. With this we mean to solve the classical embedding equation of a string in such backgrounds, quantize the theory and construct the Hilbert space of string states. This does not appear to be the case for all backgrounds. For example, string theory has not been solved on the standard 5-brane background of [7].

Generic backgrounds do not appear to admit sufficient conserved charges (7.10) that may lead to integrability. Indeed although the theory has a Kac–Moody algebra as a symmetry, it does not have two copies of it as in the case of other integrable models like the WZW ones. Nevertheless, string theory is integrable on some special backgrounds which in some cases appear as limits of generic ones. These special backgrounds are

$$G \times B_{hk}, \quad G \times S^3 \times \mathbb{R}; \quad B_{hk} = \mathbb{R}^4,$$

$$G = \mathbb{R}^{5,1}, \quad AdS_3 \times S^3, \quad CW_6,$$

(7.12)

up to discrete identifications. One can take $B_{hk}$ to be another hyper-Kähler manifold instead of $\mathbb{R}^4$, or its discrete identification $T^4$, but this choice will suffice in what follows.

All backgrounds $G \times \mathbb{R}^4$, $B_{hk} = \mathbb{R}^4$, are either the ten-dimensional Minkowski space or a product of a WZW model with flat space. Moreover, the dilaton is constant. The solution of string theory on such backgrounds follows from that on flat spaces and WZW models, see e.g. [44]. The solution of string theory in Minkowski space is well known. For the $CW_6 \times \mathbb{R}^4$ background, the solution of string theory on the plane wave $CW_6$ with a non-trivial H-flux has been given in [45]. It remains to investigate the solution of string theory in the $AdS_3 \times S^3 \times \mathbb{R}^4$ background. The spectrum of string theory on $AdS_3$ has been investigated in detail in [46] and references therein. Moreover, the string spectrum on $SU(2)$ is given by that of the standard WZW model. The central charge of the bosonic sector of the background is

$$c = \frac{3k}{k-2} + \frac{3k}{k+2} + 4,$$

(7.13)

where each term corresponds to the contribution of the associated space in the product $AdS_3 \times S^3 \times \mathbb{R}^4$.

The backgrounds $G \times S^3 \times \mathbb{R}$ appear as near horizon geometries of 5-branes with worldvolume $G$. These are linear dilaton backgrounds. The dilaton depends on the coordinate of $\mathbb{R}$. The solution of string theory on $G$ has already been explained in the $G \times \mathbb{R}^4$ case above. In addition, the solution of string theory on $S^3 \times \mathbb{R}$ has already been investigated in [7] in the
context of 5-branes with worldvolume $\mathbb{R}^5$. The theory on $S^3$ is a WZW model while the theory along $\mathbb{R}$ is a Feigin–Fuks theory associated with the linear dilaton. For example, the total central charge of the bosonic sector of the $\text{AdS}_3 \times S^3 \times S^3 \times \mathbb{R}$ background is

$$c = \frac{3k}{k-2} + \frac{3k}{k+2} + \frac{3p}{p+2} + \left(1 + \frac{6}{p}\right),$$

(7.14)

where $p$ is the 5-brane charge computed at infinity. The first two contributions to the central charge are due to $\text{AdS}_3 \times S^3$ worldvolume geometry. The first of the other two contributions to the central charge is due to the WZW model associated with the near horizon 3-sphere while the last contribution is due to the Feigin–Fuks theory. Unlike the $G \times \mathbb{R}^4$ case above, the presence of a linear dilaton in these backgrounds gives rise to regions in spacetime which are strongly coupled and so string loop corrections should be taken into account.

### 7.3.1. (1,0) supersymmetry.

So far, we have investigated the worldvolume theory of $\text{hol}(\nabla) \subseteq \text{SU}(2)$ backgrounds for which the $\tilde{\nabla}$ connection has been identified with the gauge one, and so $dH = 0$. If this is no longer the case, then Bianchi identity of $H$ may receive an anomalous contribution (2.3). The worldvolume theory of such generic backgrounds admits (1,0) supersymmetry. Let $g$ be the spacetime metric. At the zeroth order in $\alpha'$, $dH = 0 + O(\alpha')$, and so $H = db + O(\alpha')$. In addition, let $(\sigma^+, \sigma^-, \theta^+)$ be the worldvolume superspace coordinates, where $(\sigma^+, \sigma^-)$ are commuting light-cone coordinates and $\theta^+$ is an odd coordinate. Moreover, introduce the superspace coordinate $D_+ \equiv i\partial_\theta$. Then the worldvolume action of string propagating in such backgrounds is

$$S = -i \int d^2\sigma d\theta^+ [(g + b)_{MN} D_+ X^M \partial_\sigma X^N + ih_{ab} \psi_a^\dagger \nabla_+ \psi_+],$$

(7.15)

where the spacetime coordinates $x^M$ have been promoted to superfields $X^M(\sigma^+, \sigma^-, \theta^+)$, $A$ is the gauge connection and $\nabla_+ \psi_a^\dagger = D_+ \psi_a^\dagger + DX^M A_M^{ab} \psi_b$. The theory has an additional fermionic superfield $\psi_-$. The components of both types of superfields are

$$X^M = X^M|, \quad X^M_+ = D_+ X^M|, \quad \psi_a^\dagger = \psi_a^\dagger|, \quad \ell^a = \nabla_+ \psi_a^\dagger|,$$

(7.16)

where $\ell$ is an auxiliary field.

We have not constructed explicit solutions of this class of backgrounds. However, it is straightforward to see that one can easily generalize the wrapped 5-brane solutions on $G$ to this case. We shall not give the details here. We shall focus instead on the issue of ultraviolet finiteness for these models. In particular, consider the case of wrapped 5-brane solutions on $\text{AdS}_3 \times S^3$ for which $S^3$ is gauged but the $\tilde{\nabla}$ connection is not identified with the gauge one. Generalizing the argument we have presented for the backgrounds with (1,1) worldvolume supersymmetry, we can show that such (1,0)-supersymmetric backgrounds are also ultraviolet finite. However in this case, the proof for ultraviolet finiteness is based on that of (4,0)-supersymmetric sigma models [18, 19] instead of that of (4,1) supersymmetric ones. It is known that the couplings of (4,0)-supersymmetric sigma models receive corrections from changing the renormalization scheme from a (1,0)-supersymmetric to a manifestly (4,0)-supersymmetric one [19]. Such corrections are consistent with the contribution from the curvature square terms in (3.13).
8. Holonomy $\mathbb{R}^8$ solutions

8.1. Geometry $\mathbb{R}^8$ solutions

The backgrounds with eight supersymmetries and $\text{hol}(\hat{V}) \subseteq \mathbb{R}^8$ [14] admit a light-cone orthonormal frame $e^A = (e^-, e^+, e^i)$, $i = 1, 6, 2, 7, 3, 8, 4, 9$. In this frame, the conditions that arise from the Killing spinor equations can be written as

$$\text{hol}(\hat{V}) \subseteq \mathbb{R}^8, \quad \hat{\nabla}e^- = 0, \quad \delta_a \Phi = 0, \quad de^- \in \mathbb{R}^8, \quad H_{ijk} = 0, \quad 2\partial_i \Phi - H_{-ai} = 0. \quad (8.1)$$

Moreover, the spacetime metric and $H$ are

$$ds^2 = 2e^- e^+ + \delta_{ij} e^i e^j, \quad H = e^+ \wedge de^- + \Omega_{-ij} e^- \wedge e^i \wedge e^j,$$  

where $\Omega$ is the spin connection. To solve the conditions that arise from the Killing spinor equations, first observe that $(8.1)$ implies that the vector field $X$ associated with the null 1-form $e^-$ is Killing. Adapting a coordinate to $X$ as $X = \partial_u$. The metric is independent of $u$ since $X$ is an isometry. Next, the condition $de^- \in \mathbb{R}^8$ can also be written as

$$e^- \wedge de^- = 0. \quad (8.3)$$

This implies that there is a coordinate $v$ such that

$$e^- = h^{-1}(y, v) dv,$$  

for some function $h$ which depends on the coordinates $(v, y^i)$, where $y^i$ are the coordinates along the transverse directions to the light cone.

In terms of the coordinates $(u, v, y^i)$, the frame can be written as

$$e^- = h^{-1}(y, v) dv, \quad e^+ = du + V dv + n_I dy^I, \quad e^i = e^i_J dy^j + t^I dv. \quad (8.5)$$

This frame is not unique. There is a residuum $\mathbb{R}^8$-gauge symmetry which leaves $e^-$ and the Killing spinors invariant which can be used to set $t = 0$. So one has

$$e^- = h^{-1} dv, \quad e^i = e_i^j dy^j, \quad e^+ = du + V dv + n_I dy^I, \quad (8.6)$$

where $h, e_i^j, V$ and $n_I$ depend on both $v$ and $y^i$ coordinates. The metric and 3-form field strength can be written as

$$ds^2 = 2h^{-1} dv(du + V dv + n_I dy^I) + \delta_{ij} e^i_J dy^j, \quad H = e^+ \wedge de^- - (h\delta_{ij} e^i_J \partial_j e^e_J + \partial_i n_J e^i_J) e^- \wedge e^i \wedge e^j. \quad (8.7)$$

Using the first condition in $(8.1)$, $H_{ijk} = 0$, and the torsion free conditions for the frame $(e^-, e^+, e^i)$, one finds that

$$\partial_i (2\Phi + \log h) = 0, \quad \partial_i e^i_J = 0, \quad \Omega_{-ij} = e^i_J (-\partial_i V + \partial_a n_I), \quad \Omega_{-ai} = -h\partial_i (e^i_J) \partial_a e^e_J - \partial_i n_J e^i_J.$$

The first condition relates the dilaton to $h$. The second condition implies that there are functions $x^i(y^j, v)$ such that

$$e^i_J = \partial_i x^i.$$  

Performing the coordinate transformation $x^i = x^i(y^j, v)$ and after an appropriate redefinition of $V$ and $n$, the fields can be rewritten as

$$ds^2 = 2e^- e^+ + \delta_{ij} dx^i dx^j, \quad e^- = h^{-1} dv, \quad e^+ = du + V dv + n_I dx^I, \quad H = d(e^- \wedge e^+), \quad e^2 dv = h^{-1} g(v), \quad (8.10)$$

21
where $h, V, n$ are functions of $v$ and $x^i$, and $g$ is an arbitrary function of $v$. This concludes the analysis of the gravitino and dilatino Killing spinor equations.

The gaugino Killing spinor equation implies

$$F_{ui} = F_{ij} = F_{vu} = 0,$$  \hspace{1cm} (8.11)

where we have suppressed the gauge indices. In the coordinates of (8.10), these can be written as

$$F_{ui} = F_{ij} = F_{vu} = 0.$$  \hspace{1cm} (8.12)

Choosing the gauge $A_u = 0$, we have that the above conditions imply

$$A_i = A_i(v, y), \quad A_v = A_v(v, x^i), \quad A_i = U^{-1} \partial_i U.$$  \hspace{1cm} (8.13)

After performing a gauge transformation using $U$, the only non-vanishing component of $A$ is $A_v$ and

$$F_{vi} = \partial_i A_v.$$  \hspace{1cm} (8.14)

Clearly the one-loop contribution to the Bianchi identity of $H$ due to the anomaly cancellation mechanism for both $A$ and $\nabla$ connections vanishes. So the solutions are at least two-loop exact.

It is clear from the analysis so far that to find explicit solutions in this case, one has to determine the unknown functions that appear in (8.10) and (8.14). These are $h(x, v)$, $V(x, v)$, $n_i(x, v)$ and $A_v(x, v)$. This is done by solving the field equations.

### 8.2. The solution of the field equations

The Killing spinor equations imply some of the field equations. It is known [14, 15] that the remaining independent field equations that have to be solved are the $E_{--} = 0$ component of the Einstein equations, the field equations $L H_{AB} = 0$ of the 2-form gauge potential and the $L F_{-} = 0$ component of the 1-form gauge potential. The field equation $L H_{AB} = 0$ gives

$$\partial_j n_i = 0, \quad \partial_i (dn)_j + g \partial_v (g^{-1} \partial_i h) = 0.$$  \hspace{1cm} (8.15)

Using $\partial^2 h = 0$, the $E_{--} = 0$ component of the Einstein equations implies that

$$h^{-1} \partial^2 V - h^{-1} \partial_i n_i - \partial^2 V - h \partial_v h^{-1} \partial_v \log(h^{-1} g) = 0.$$  \hspace{1cm} (8.16)

Moreover, $L F_{-} = 0$ gives

$$\partial^2 A_v = 0.$$  \hspace{1cm} (8.17)

and so $A_v$ is a harmonic function on $\mathbb{R}^8$.

The above equations of motion can be simplified somewhat provided we take $g = 1$ and choose

$$\partial^i n_i = \partial_v h.$$  \hspace{1cm} (8.18)

In such a case, they can be written as

$$\partial^2 h = 0, \quad \partial^i n_i = 0, \quad \partial_i V = 0, \quad \partial^2 A_v = 0.$$  \hspace{1cm} (8.19)

Therefore $h, n, V$ and $A_v$ are $v$-dependent harmonic functions of $\mathbb{R}^8$. Moreover, it is required that $n$ and $h$ satisfy (8.18). Solutions of these equations that depend on both $v$ and $x$ coordinates have been given in [47].

Alternatively, one can assume that $h, V, n$ and $A_v$ depend only on $x$ and $g = 1$. In such a case, one finds that

$$\partial^2 h = 0, \quad \partial^i (dn)_i = 0, \quad \partial_i V = 0, \quad \partial^2 A_v = 0.$$  \hspace{1cm} (8.20)

So again $h, V$ and $A_v$ are harmonic functions of $\mathbb{R}^8$ and $n$ satisfies the Maxwell equations on $\mathbb{R}^8$. It is clear that the most general solution of this class is the superposition of a fundamental string with a pp-wave and with a null rotation.
8.3. Worldvolume theory

As we have already mentioned there is no anomalous contribution to the Bianchi identity of \( H \) in the \( \text{hol} (\hat{\nabla}) \subseteq \mathbb{R}^8 \) class of backgrounds, and so \( dH = 0 \). However, the gauge connection \( A \) need not be identified with \( \hat{\nabla}. \) In fact in the context of the heterotic string \( A \) cannot be identified with \( \hat{\nabla} \) because the holonomy of the former is compact while the holonomy of the latter is Lorentzian for generic backgrounds [20]. As a result the worldvolume theory of generic backgrounds is (1,0)-supersymmetric. The action is given as in (7.15) and we shall not repeat the analysis.

To describe the ultraviolet properties of \( \text{hol} (\hat{\nabla}) \subseteq \mathbb{R}^8 \) backgrounds, we shall consider two cases. If the solutions are taken as backgrounds of the common sector of type II strings, i.e. the gauge connection is identified with \( \hat{\nabla}, \) and so the worldvolume supersymmetry is enhanced to (1,1), then it has been shown in [20] that worldvolume theory is ultraviolet finite provided \( \partial \nu h = 0. \) Next consider the \( \text{hol} (\hat{\nabla}) \subseteq \mathbb{R}^8 \) backgrounds as solutions of the heterotic string. As has already been mentioned that \( \text{hol} (\hat{\nabla}) \) cannot be identified with \( A \) for generic backgrounds. So in general the worldvolume theory has strictly (1,0) worldvolume supersymmetry. In addition, it is known that in many cases, the 2-loop beta function does not vanish. An exhaustive analysis of the ultraviolet properties of these backgrounds in the context of heterotic string can be found in [20].

9. Conclusions

We have described all backgrounds of heterotic strings that preserve half the supersymmetry, i.e. eight supercharges. There are three classes of backgrounds distinguished by the holonomy of the connection with torsion \( \hat{\nabla}. \) The backgrounds of the \( \text{hol} (\hat{\nabla}) \subseteq SU(2) \) class can be constructed starting from any four-dimensional hyper-Kähler manifold, \( B_{hk}, \) and an anti-self-dual connection on it with gauge group \( G \) such that \( \text{Lie}(G) = \mathbb{R}^{5,1}, sl(2, \mathbb{R}) \oplus su(2), \) \( \text{sl}_{6} \) is a self-dual Lorentzian Lie algebra. In particular, the spacetime is a principal bundle with base space \( B_{hk}, \) fibre group \( G \) and equipped with a compatible anti-self-dual connection \( \lambda. \) We demonstrate that the generic solutions of this type have the interpretation of wrapped 5-branes on \( G. \)

We have constructed new solutions which are characterized by three integers and depend on a continuous modulus which includes that of \( SU(2) \) instantons. There may be a relation of some of these backgrounds to the ADHM sigma models of [48]. We also explore their interpretation in M-theory in terms of M2- and M5-brane intersections.

We have investigated the worldvolume theory of \( \text{hol} (\hat{\nabla}) \subseteq SU(2) \) backgrounds, and show that it is closely related to that of gauged WZW models. However, the worldvolume gauge field in this case is not independent but rather it is a composite that depends on the scalars associated with \( B_{hk}. \) We demonstrate that a large class of the worldvolume theories based on these backgrounds are ultraviolet finite. This may be extended to all \( \text{hol} (\hat{\nabla}) \subseteq SU(2) \) backgrounds.

We also show that the backgrounds of the \( \text{hol} (\hat{\nabla}) \subseteq \mathbb{R}^8 \) class are superpositions of the fundamental string with a pp-wave which may also include a null rotation.

For the third class of backgrounds that preserve eight supersymmetries, \( \text{hol} (\hat{\nabla}) = \{1\}. \) The spacetime is a Lorentzian group manifold. These backgrounds have been classified in [21].

The analysis we have presented is complete for the \( \text{hol} (\hat{\nabla}) \subseteq \mathbb{R}^8 \) class of backgrounds. The description of all \( \text{hol} (\hat{\nabla}) \subseteq SU(2) \) solutions requires the classification of all four-dimensional hyper-Kähler manifolds and their anti-self-dual instantons which have gauge
group $G$ with Lie algebra $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{su}(2)$ or $\mathfrak{so}_6$. There are explicit constructions of these data in many special cases but the description is not complete. Even in the case that $B_{hk} = \mathbb{R}^4$ to construct all solutions, one has to find the (non-Abelian) instantons with a Lorentzian gauge group $G$; $\Sigma_G = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{su}(2)$, $\mathfrak{so}_6$. Unlike for instantons on $\mathbb{R}^4$ with an Euclidean gauge group, little is known about the Lorentzian gauge group case. Though related configurations have been investigated in e.g. [49]. Since any four-dimensional hyper-Kähler manifold can be used to construct all solutions, one has to find the (non-Abelian) instantons with a Lorentzian gauge group $G$; $\text{Lie } G = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{su}(2)$, $\mathfrak{so}_6$.

It is clear that there are continuous families of $\text{hol}(\hat{\nabla}) \subseteq \text{SU}(2)$ backgrounds and their moduli are very large. Nevertheless all such backgrounds can be understood in terms of four-dimensional hyper-Kähler geometry and the theory of instantons on hyper-Kähler manifolds.

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