Induced differential forms on manifolds of functions

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Abstract

Differential forms on the Fréchet manifold \( \mathcal{F}(S, M) \) of smooth functions on a compact \( k \)-dimensional manifold \( S \) can be obtained in a natural way from pairs of differential forms on \( M \) and \( S \) by the hat pairing. Special cases are the transgression map \( \Omega^p(M) \to \Omega^{p-k}(\mathcal{F}(S, M)) \) (hat pairing with a constant function) and the bar map \( \Omega^p(M) \to \Omega^p(\mathcal{F}(S, M)) \) (hat pairing with a volume form). We develop a hat calculus similar to the tilda calculus for non-linear Grassmannians [HV04].

1 Introduction

Pairs of differential forms on the finite dimensional manifolds \( M \) and \( S \) induce differential forms on the Fréchet manifold \( \mathcal{F}(S, M) \) of smooth functions. More precisely, if \( S \) is a compact oriented \( k \)-dimensional manifold, the hat pairing is:

\[
\Omega^p(M) \times \Omega^q(S) \to \Omega^{p+q-k}(\mathcal{F}(S, M))
\]

\[
\hat{\omega} \cdot \alpha = \int_S \text{ev}^* \omega \wedge \text{pr}^* \alpha,
\]

where \( \text{ev} : S \times \mathcal{F}(S, M) \to M \) denotes the evaluation map, \( \text{pr} : S \times \mathcal{F}(S, M) \to S \) the projection and \( \int_S \) fiber integration. We show that the hat pairing is compatible with the canonical Diff(M) and Diff(S) actions on \( \mathcal{F}(S, M) \), and with the exterior derivative. As a consequence we obtain a hat pairing in cohomology.

The hat (transgression) map is the hat pairing with the constant function 1, so it associates to any form \( \omega \in \Omega^p(M) \) the form \( \hat{\omega} \cdot 1 = \hat{\omega} = \int_S \text{ev}^* \omega \in \Omega^{p-k}(\mathcal{F}(S, M)) \). Since \( \mathcal{X}(M) \) acts infinitesimally transitive on the open subset \( \text{Emb}(S, M) \subset \mathcal{F}(S, M) \) of embeddings of the \( k \)-dimensional oriented manifold \( S \) into \( M \) [H76], the expression of \( \hat{\omega} \) at \( f \in \text{Emb}(S, M) \) is

\[
\hat{\omega}(X_1 \circ f, \ldots, X_{p-k} \circ f) = \int_S f^*(i_{X_{p-k}} \ldots i_{X_1} \omega), \quad X_1, \ldots, X_{p-k} \in \mathcal{X}(M).
\]
When $S$ is the circle, then one obtains the usual transgression map with values in the space of $(p-1)$-forms on the free loop space of $M$.

Let $\text{Gr}_k(M)$ be the non-linear Grassmannian of $k$-dimensional oriented submanifolds of $M$. The tilda map associates to every $\omega \in \Omega^p(M)$ a differential $(p-k)$-form on $\text{Gr}_k(M)$ given by $[HV04]$

$$\tilde{\omega}(\tilde{Y}_N^1, \ldots, \tilde{Y}_N^{p-k}) = \int_N i_{\tilde{Y}_N^{p-k}} \cdots i_{\tilde{Y}_N^1} \omega, \quad \forall \tilde{Y}_N^1, \ldots, \tilde{Y}_N^{p-k} \in \Gamma(TN^\perp) = T_N \text{Gr}_k(M),$$

for $\tilde{Y}_N$ section of the orthogonal bundle $TN^\perp$ represented by the section $Y_N$ of $TM|_N$. The natural map $\pi : \text{Emb}(S,M) \to \text{Gr}_k(M)$ with $\pi(f) = f(S)$ provides a principal bundle with the group $\text{Diff}_+(S)$ of orientation preserving diffeomorphisms of $S$ as structure group.

The hat map on $\text{Emb}(S,M)$ and the tilda map on $\text{Gr}_k(M)$ are related by $\hat{\omega} = \pi^* \tilde{\omega}$. This is the reason why for the hat calculus one has similar properties to those for the tilda calculus. The tilda calculus was used to study the non-linear Grassmannian of co-dimension two submanifolds as symplectic manifold $[HV04]$. We apply the hat calculus to the hamiltonian formalism for $p$-branes and open $p$-branes $[AS05]$ $[BZ05]$.

The bar map $\bar{\omega} = \hat{\omega} \cdot \mu$ is the hat pairing with a fixed volume form $\mu$ on $S$, so

$$\bar{\omega}(Y_f^1, \ldots, Y_f^p) = \int_S \omega(Y_f^1, \ldots, Y_f^p) \mu, \quad \forall Y_f^1, \ldots, Y_f^p \in \Gamma(f^*TM) = T_fF(S,M).$$

We use the bar calculus to study $F(S,M)$ with symplectic form $\bar{\omega}$ induced by a symplectic form $\omega$ on $M$. The natural actions of $\text{Diff}_{ham}(M,\omega)$ and $\text{Diff}_{ex}(S,\mu)$, the group of hamiltonian diffeomorphisms of $M$ and the group of exact volume preserving diffeomorphisms of $S$, are two commuting hamiltonian actions on $F(S,M)$. Their momentum maps form the dual pair for ideal incompressible fluid flow $[MW83]$ $[GBV09]$.

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2 Hat pairing

We denote by $F(S,M)$ the set of smooth functions from a compact oriented $k$-dimensional manifold $S$ to a manifold $M$. It is a Fréchet manifold in a natural way $[KM97]$. Tangent vectors at $f \in F(S,M)$ are identified with vector fields on $M$ along $f$, i.e. sections of the pull-back vector bundle $f^*TM$.

Let $\text{ev} : S \times F(S,M) \to M$ be the evaluation map $\text{ev}(x,f) = f(x)$ and $\text{pr} : S \times F(S,M) \to S$ the projection $\text{pr}(x,f) = x$. A pair of differential forms $\omega \in \Omega^p(M)$ and $\alpha \in \Omega^p(S)$ determines a differential form $\hat{\omega} \cdot \alpha$ on $F(S,M)$ by the
fiber integral over $S$ (whose definition and properties are listed in the appendix)
of the $(p + q)$-form $\text{ev}^* \omega \wedge \text{pr}^* \alpha$ on $S \times \mathcal{F}(S, M)$:

$$\hat{\omega} \cdot \alpha = \int_S \text{ev}^* \omega \wedge \text{pr}^* \alpha$$

In this way we obtain a bilinear map called the hat pairing:

$$\Omega^p(M) \times \Omega^q(S) \to \Omega^{p+q-k}(\mathcal{F}(S, M)).$$

An explicit expression of the hat pairing avoiding fiber integration is:

$$(\hat{\omega} \cdot \alpha)_f(Y^1_f, \ldots, Y^{p+q-k}_f) = \int_S f^*(i_{Y^p_{p+q-k}} \cdots i_{Y^1_1}(\omega \circ f)) \wedge \alpha,$$

for $Y^1_f, \ldots, Y^{p+q-k}_f$ vector fields on $M$ along $f \in \mathcal{F}(S, M)$. Here we denote by

$f^* \beta_f$ the "restricted pull-back" by $f$ of a section $\beta_f$ of $f^*(\Lambda^m T^* M)$, which is a differential $m$-form on $S$ given by $f^* \beta_f : x \in S \mapsto (\Lambda^m T^*_x f)(\beta_f(x)) \in \Lambda^m T^*_x S$,

where $T^*_x f : T^*_x f(M) \to T^*_x S$ denotes the dual of $T_x f$.

The fact that (1) and (2) provide the same differential form on $\mathcal{F}(S, M)$ can be deduced from the identity

$$(\text{ev}^* \omega)(x, f)(Y^1_f, \ldots, Y^{p-k}_f, X^1_x, \ldots, X^k_x) = f^*(i_{Y^p_{p+q-k}} \cdots i_{Y^1_1}(\omega \circ f))(X^1_x, \ldots, X^k_x)$$

for $Y^1_f, \ldots, Y^{p-k}_f \in T_f \mathcal{F}(S, M)$ and $X^1_x, \ldots, X^k_x \in T_x S$.

Since $\mathcal{X}(\hat{M})$ acts infinitesimally transitive on the open subset $\text{Emb}(S, M) \subset \mathcal{F}(S, M)$ of embeddings of the $k$-dimensional oriented manifold $S$ into $M$, we express $\hat{\omega}$ at $f \in \text{Emb}(S, M)$ as:

$$(\hat{\omega} \cdot \alpha)_f(X_1 \circ f, \ldots, X_{p+q-k} \circ f) = \int_S f^*(i_{X^p_{p+q-k}} \cdots i_{X^1_1}(\omega \circ f)) \wedge \alpha.$$

One uses the fact that the "restricted pull-back" by $f$ of $i_{X^p_{p+q-k} \circ f} \cdots i_{X^1_1}(\omega \circ f)$ is $f^*(i_{X^p_{p+q-k}} \cdots i_{X^1_1}(\omega)$.

Next we show that the hat pairing is compatible with the exterior derivative
of differential forms.

**Theorem 1.** The exterior derivative $\text{d}$ is a derivation for the hat pairing, i.e.

$$\text{d}(\hat{\omega} \cdot \alpha) = (\text{d}\omega) \cdot \alpha + (-1)^p \hat{\omega} \cdot \text{d}\alpha,$$

where $\omega \in \Omega^p(M)$ and $\alpha \in \Omega^q(S)$.

**Proof.** Differentiation and fiber integration along the boundary free manifold $S$ commute, so

$$\text{d}(\hat{\omega} \cdot \alpha) = \text{d} \int_S \text{ev}^* \omega \wedge \text{pr}^* \alpha = \int_S \text{d} (\text{ev}^* \omega \wedge \text{pr}^* \alpha)$$

$$= \int_S \text{ev}^* \text{d}\omega \wedge \text{pr}^* \alpha + (-1)^p \int_S \text{ev}^* \omega \wedge \text{pr}^* \text{d}\alpha = (\hat{\omega} \cdot \alpha) + (-1)^p \hat{\omega} \cdot \text{d}\alpha$$

for all $\omega \in \Omega^p(M)$ and $\alpha \in \Omega^q(S)$.

□
The differential form \( \hat{\omega} \cdot \hat{\alpha} \) is exact if \( \omega \) is closed and \( \alpha \) exact (or if \( \alpha \) is closed and \( \omega \) exact). In the special case \( p+q = k \) these conditions imply that the function \( \hat{\omega} \cdot \hat{\alpha} \) on \( \mathcal{F}(S,M) \) vanishes.

**Corollary 2.** The hat pairing induces a bilinear map on de Rham cohomology spaces

\[
H^p(M) \times H^q(S) \to H^{p+q-k}(\mathcal{F}(S,M)).
\]

In particular there is a bilinear map

\[
H^p(M) \times H^q(M) \to H^{p+q-k}(\text{Diff}(M)).
\]

**Remark 3.** The cohomology group \( H^q(S) \) is isomorphic to the homology group \( H_{k-q}(S) \) by Poincaré duality. With the notation \( n = k - q \), the hat pairing \([5]\) becomes

\[
H^p(M) \times H_n(S) \to H^{p-n}(\mathcal{F}(S,M)),
\]

and it is induced by the map \((\omega, \sigma) \mapsto \int_S \text{ev}^* \omega\), for differential \( p \)-forms \( \omega \) on \( M \) and \( n \)-chains \( \sigma \) on \( S \).

If \( S \) is a manifold with boundary, then formula \([4]\) receives an extra term coming from integration over the boundary. Let \( i_\partial : \partial S \to S \) be the inclusion and \( r_\partial : \mathcal{F}(S,M) \to \mathcal{F}(\partial S,M) \) the restriction map.

**Proposition 4.** The identity

\[
\text{d}(\hat{\omega} \cdot \hat{\alpha}) = (\text{d}\hat{\omega}) \cdot \hat{\alpha} + (-1)^p \hat{\omega} \cdot \text{d}\hat{\alpha} + (-1)^{p+q-k} r_\partial^*(\hat{\omega} \cdot i_\partial^* \hat{\alpha})
\]

holds for \( \omega \in \Omega^p(M) \) and \( \alpha \in \Omega^q(S) \), where the upper index \( \partial \) assigned to the hat means the pairing

\[
\Omega^p(M) \times \Omega^q(\partial S) \to \Omega^{p+q-k+1}(\mathcal{F}(\partial S,M)).
\]

**Proof.** For any differential \( n \)-form \( \beta \) on \( S \times \mathcal{F}(S,M) \), the identity

\[
\text{d} \int_S \beta - \int_S \text{d}\beta = (-1)^{n-k} \int_{\partial S} (i_\partial \times 1_{\mathcal{F}(S,M)})^* \beta
\]

holds because of the identity \([19]\) from the appendix. The obvious formulas

\[
\text{pr} \circ (i_\partial \times 1_{\mathcal{F}(S,M)}) = i_\partial \circ \text{pr}_\partial, \quad \text{ev} \circ (i_\partial \times 1_{\mathcal{F}(S,M)}) = \text{ev}_\partial,
\]

for \( \text{ev}_\partial : \partial S \times \mathcal{F}(S,M) \to M \) and \( \text{pr}_\partial : \partial S \times \mathcal{F}(S,M) \to \partial S \), are used to compute

\[
\text{d}(\hat{\omega} \cdot \hat{\alpha}) = \text{d} \int_S \text{ev}^* \omega \wedge \text{pr}^* \alpha
\]

\[
= \int_S \text{d}(\text{ev}^* \omega \wedge \text{pr}^* \alpha) + (-1)^{p+q-k} \int_{\partial S} (i_\partial \times 1_{\mathcal{F}(S,M)})^* (\text{ev}^* \omega \wedge \text{pr}^* \alpha)
\]

\[
= \int_S \text{ev}^* \text{d}\omega \wedge \text{pr}^* \alpha + (-1)^p \int_S \text{ev}^* \omega \wedge \text{pr}^* \text{d}\alpha + (-1)^{p+q-k} \int_{\partial S} \text{ev}_\partial^* \omega \wedge \text{pr}_\partial^* i_\partial^* \alpha
\]

\[
= (\hat{\omega}) \cdot \hat{\alpha} + (-1)^p \hat{\omega} \cdot \text{d}\hat{\alpha} + (-1)^{p+q-k} r_\partial^* (\hat{\omega} \cdot i_\partial^* \hat{\alpha}),
\]

thus obtaining the requested identity. \( \square \)
Left \text{Diff}(M) \text{ action.} \quad \text{The natural left action of the group of diffeomorphisms} \text{Diff}(M) \text{ on} \mathcal{F}(S, M) \text{ is} \varphi \cdot f = \varphi \circ f. \text{ The infinitesimal action of} \ X \in \mathfrak{X}(M) \text{ is the vector field} \ X \text{ on} \mathcal{F}(S, M):

\[ \tilde{X}(f) = X \circ f, \quad \forall f \in \mathcal{F}(S, M). \]

We denote by \( \tilde{\varphi} \) the diffeomorphism of \( \mathcal{F}(S, M) \) induced by the action of \( \varphi \in \text{Diff}(M) \), so \( \tilde{\varphi}(f) = \varphi \circ f \) is the push-forward by \( \varphi \).

**Proposition 5.** Given \( \omega \in \Omega^p(M) \) and \( \alpha \in \Omega^q(S) \), the identity

\[ \tilde{\varphi}^* \hat{\omega} \cdot \alpha = (\varphi^* \omega) \cdot \alpha \]  

(7)

and its infinitesimal version

\[ L_X \hat{\omega} \cdot \alpha = (L_X \omega) \cdot \alpha \]  

(8)

hold for all \( \varphi \in \text{Diff}(M) \) and \( X \in \mathfrak{X}(M) \).

**Proof.** Using the expression (1) of the hat pairing and identity (15) from the appendix, we have:

\[
\tilde{\varphi}^* \hat{\omega} \cdot \alpha = \varphi^* \int_S \text{ev}^* \omega \wedge \text{pr}^* \alpha = \int_S (1_S \times \tilde{\varphi})^* (\text{ev}^* \omega \wedge \text{pr}^* \alpha) \\
= \int_S \text{ev}^* \varphi^* \omega \wedge \text{pr}^* \alpha = (\varphi^* \omega) \cdot \alpha,
\]

since \( \text{pr} \circ (1_S \times \tilde{\varphi}) = \text{pr} \) and \( \text{ev} \circ (1_S \times \tilde{\varphi}) = \varphi \circ \text{ev} \). \hfill \Box

A similar result is obtained for any smooth map \( \eta \in \mathcal{F}(M_1, M_2) \) and its push-forward \( \tilde{\eta} : \mathcal{F}(S, M_1) \to \mathcal{F}(S, M_2), \tilde{\eta}(f) = \eta \circ f \):

\[ \tilde{\eta}^* \hat{\omega} \cdot \alpha = \eta^* \hat{\omega} \cdot \alpha, \]

for all \( \omega \in \Omega^p(M_2) \) and \( \alpha \in \Omega^q(S) \).

**Lemma 6.** For all vector fields \( X \in \mathfrak{X}(M) \), the identity \( i_X \hat{\omega} \cdot \alpha = (i_X \omega) \cdot \alpha \) holds.

**Proof.** The vector field \( 0_S \times \tilde{X} \) on \( S \times \mathcal{F}(S, M) \) is ev-related to the vector field \( X \) on \( M \), so

\[
i_X \hat{\omega} \cdot \alpha = i_X \int_S \text{ev}^* \omega \wedge \text{pr}^* \alpha = \int_S i_{0_S \times \tilde{X}} (\text{ev}^* \omega \wedge \text{pr}^* \alpha) \\
= \int_S \text{ev}^*(i_X \omega) \wedge \text{pr}^* \alpha = (i_X \omega) \cdot \alpha.
\]

At step two we use formula (18) from the appendix. \hfill \Box
**Right Diff(S) action.** The natural right action of the diffeomorphism group Diff(S) on \( \mathcal{F}(S, M) \) can be transformed into a left action by \( \psi \cdot f = f \circ \psi^{-1} \). The infinitesimal action of \( Z \in \mathfrak{X}(S) \) is the vector field \( \hat{Z} \) on \( \mathcal{F}(S, M) \):

\[
\hat{Z}(f) = -Tf \circ Z, \quad \forall f \in \mathcal{F}(S, M).
\]

We denote by \( \hat{\psi} \) the diffeomorphism of \( \mathcal{F}(S, M) \) induced by the action of \( \psi \), so \( \hat{\psi}(f) = f \circ \psi^{-1} \) is the pull-back by \( \psi^{-1} \).

**Proposition 7.** Given \( \omega \in \Omega^p(M) \) and \( \alpha \in \Omega^q(S) \), the identity

\[
\hat{\psi}^* \omega \cdot \alpha = \omega \cdot \hat{\psi}^* \alpha
\]

and its infinitesimal version

\[
L_{\hat{Z}} \omega \cdot \alpha = \omega \cdot L_Z \alpha
\]

hold for all orientation preserving \( \psi \in \text{Diff}(S) \) and \( Z \in \mathfrak{X}(S) \).

**Proof.** The obvious identities \( \text{ev} \circ (1_S \times \hat{\psi}) = \text{ev} \circ (\psi^{-1} \times 1_F) \), \( \text{pr} \circ (1_S \times \hat{\psi}) = \text{pr} \circ \text{pr} \circ (\psi \times 1_F) = \psi \circ \text{pr} \) are used in the computation

\[
\hat{\psi}^* \omega \cdot \alpha = \hat{\psi}^* \int_S \text{ev}^* \omega \land \text{pr}^* \alpha = \int_S (1_S \times \hat{\psi})^* \text{ev}^* \omega \land \text{pr}^* \alpha
\]

\[
= \int_S ((\psi^{-1} \times 1_F)^* \text{ev}^* \omega) \land \text{pr}^* \alpha = \int_S \text{ev}^* \omega \land (\psi \times 1_F)^* \text{pr}^* \alpha
\]

\[
= \int_S \text{ev}^* \omega \land \text{pr}^* \psi^* \alpha = \omega \cdot \hat{\psi}^* \alpha,
\]

together with formula (17) from the appendix at step four.

**Lemma 8.** The identity \( i_{\hat{Z}} \omega \cdot \alpha = (-1)^p \omega \cdot i_Z \alpha \) holds for all vector fields \( Z \in \mathfrak{X}(S) \), if \( \omega \in \Omega^p(M) \).

**Proof.** The infinitesimal version of the first identity in the proof of proposition 7 is \( T \text{ev}. (0_S \times \hat{Z}) = T \text{ev}. (-Z \times 0_{\mathcal{F}(S,M)}) \), so we compute:

\[
i_{\hat{Z}} \omega \cdot \alpha = i_{\hat{Z}} \int_S \text{ev}^* \omega \land \text{pr}^* \alpha = \int_S i_{0_S \times \hat{Z}} (\text{ev}^* \omega \land \text{pr}^* \alpha)
\]

\[
= \int_S (i_{0_S \times \hat{Z}} \text{ev}^* \omega) \land \text{pr}^* \alpha = \int_S (i_{-Z \times 0_{\mathcal{F}(S,M)}} \text{ev}^* \omega) \land \text{pr}^* \alpha
\]

\[
= \int_S i_{-Z \times 0_{\mathcal{F}(S,M)}} (\text{ev}^* \omega \land \text{pr}^* \alpha) - \int_S (-1)^p \text{ev}^* \omega \land i_{-Z \times 0_{\mathcal{F}(S,M)}} \text{pr}^* \alpha
\]

\[
= (-1)^p \int_S \text{ev}^* \omega \land \text{pr}^* (i_Z \alpha) = (-1)^p \omega \cdot i_Z \alpha.
\]

At step two we use formula (18) from the appendix.
3 Tilda map and hat map

Let $\text{Gr}_k(M)$ be the non-linear Grassmannian (or differentiable Chow variety) of compact oriented $k$-dimensional submanifolds of $M$. It is a Fréchet manifold [KM97] and the tangent space at $N \in \text{Gr}_k(M)$ can be identified with the space of smooth sections of the normal bundle $TN^\perp = (TM|_N)/TN$. The tangent vector at $N$ determined by the section $Y_N \in \Gamma(TM|_N)$ is denoted by $\tilde{Y}_N \in T_N \text{Gr}_k(M)$.

The \textit{tilda map} [HV04] associates to any $p$-form $\omega$ on $M$ a $(p-k)$-form $\tilde{\omega}$ on $\text{Gr}_k(M)$ by:

$$\tilde{\omega}_N(\tilde{Y}_N^1, \ldots, \tilde{Y}_N^{p-k}) = \int_N \iota_{\gamma_N^{p-k}} \cdots \iota_{\gamma_N} \omega.$$ (9)

Here all $\tilde{Y}_N^j$ are tangent vectors at $N \in \text{Gr}_k(M)$, i.e. sections of $TN^\perp$ represented by sections $Y_N^j$ of $TM|_N$. Then $\iota_{\gamma_N^{p-k}} \cdots \iota_{\gamma_N} \omega \in \Omega^k(N)$ does not depend on representatives $Y_N^j$ of $\tilde{Y}_N^j$, and integration is well defined since $N \in \text{Gr}_k(M)$ comes with an orientation.

Let $S$ be a compact oriented $k$-dimensional manifold. The \textit{hat map} is the hat pairing with the constant function $1 \in \Omega^0(S)$. It associates to any form $\omega \in \Omega^p(M)$ the form $\hat{\omega} \in \Omega^{p-k}(\mathcal{F}(S, M))$:

$$\hat{\omega} = \tilde{\omega} \cdot 1 = \int_S \text{ev}^* \omega.$$ (10)

On the open subset $\text{Emb}(S, M) \subset \mathcal{F}(S, M)$ of embeddings, formula (2) gives

$$\hat{\omega}(X_1 \circ f, \ldots, X_{p-k} \circ f) = \int_S f^*(i_{X_{p-k}} \cdots i_{X_1} \omega).$$ (11)

Remark 9. The hat map induces a transgression on cohomology spaces

$$H^p(M) \to H^{p-k}(\mathcal{F}(S, M)).$$

When $S$ is the circle, then one obtains the usual transgression map with values in the $(p-1)$-th cohomology space of the free loop space of $M$.

Let $\pi$ denote the natural map

$$\pi : \text{Emb}(S, M) \to \text{Gr}_k(M), \quad \pi(f) = f(S),$$

where the orientation on $f(S)$ is chosen such that the diffeomorphism $f : S \to f(S)$ is orientation preserving. The image $\pi(\text{Emb}(S, M))$ is the manifold $\text{Gr}_k^S(M)$ of $k$-dimensional submanifolds of $M$ of type $S$. Then $\pi : \text{Emb}(S, M) \to \text{Gr}_k^S(M)$ is a principal bundle over $\text{Gr}_k^S(M)$ with structure group $\text{Diff}_+(S)$, the group of orientation preserving diffeomorphisms of $S$.

Note that there is a natural action of the group $\text{Diff}(M)$ on the non-linear Grassmannian $\text{Gr}_k(M)$ given by $\varphi \cdot N = \varphi(N)$. Let $\tilde{\varphi}$ be the
of $\text{Gr}_k(M)$ induced by the action of $\varphi \in \text{Diff}(M)$. Then $\tilde{\varphi} \circ \pi = \pi \circ \varphi$ for the restriction of $\tilde{\varphi}(f) = \varphi \circ f$ to a diffeomorphism of $\text{Emb}(S, M) \subset \mathcal{F}(S, M)$. As a consequence, the infinitesimal generators for the $\text{Diff}(M)$ actions on $\text{Gr}_k(M)$ and on $\text{Emb}(S, M)$ are $\pi$–related. This means that for all $X \in \mathfrak{X}(M)$, the vector fields $\tilde{X}$ on $\text{Gr}_k(M)$ given by $\tilde{X}(N) = X|_N$ and $\bar{X}$ on $\text{Emb}(S, M)$ given by $\bar{X}(f) = X \circ f$ are $\pi$–related.

**Proposition 10.** The hat map on $\text{Emb}(S, M)$ and the tilda map on $\text{Gr}_k(M)$ are related by $\hat{\omega} = \pi^* \tilde{\omega}$, for any $k$–dimensional oriented manifold $S$.

**Proof.** For the proof we use the fact that $\mathfrak{X}(M)$ acts infinitesimally transitive on $\text{Emb}(S, M)$, so $T_f \text{Emb}(S, M) = \{X \circ f : X \in \mathfrak{X}(M)\}$. With (9) and (11) we compute:

$$((\pi^* \tilde{\omega})_f(X_1 \circ f, \ldots, X_{p-k} \circ f) = \tilde{\omega}_{f(S)}(X_1|_{f(S)}, \ldots, X_{p-k}|_{f(S)}) = \int_{f(S)} i_{X_{p-k}} \ldots i_{X_1} \omega) = \int_S f^*(i_{X_{p-k}} \ldots i_{X_1} \omega) = \hat{\omega}_f(X_1 \circ f, \ldots, X_{p-k} \circ f),$$

since $\tilde{X}$ and $\bar{X}$ are $\pi$–related.

From the properties of the hat pairing presented in proposition 5, lemma 6 and theorem 1, a hat calculus follows easily:

**Proposition 11.** For any $\omega \in \Omega^p(M)$, $\varphi \in \text{Diff}(M)$, $X \in \mathfrak{X}(M)$, and $\eta \in \mathcal{F}(M', M)$ with push-forward $\bar{\eta} : \mathcal{F}(S, M') \rightarrow \mathcal{F}(S, M)$, the following identities hold:

1. $\varphi^* \hat{\omega} = \tilde{\varphi}^* \omega$ and $\bar{\eta}^* \hat{\omega} = \tilde{\eta}^* \omega$
2. $L_X \hat{\omega} = \hat{L}_X \omega$
3. $i_X \hat{\omega} = \tilde{i}_X \omega$
4. $d\hat{\omega} = \hat{d}\omega$.

**Remark 12.** If $S$ is a manifold with boundary, then the formula 4. above receives an extra term coming from integration over the boundary $\partial S$ as in proposition 4:

$$d\hat{\omega} = \hat{d}\omega + (-1)^{p-k} r_{\partial}^* \tilde{\omega}^{\partial}$$

(12)

for $\omega \in \Omega^p(M)$. As before, $r_{\partial} : \mathcal{F}(S, M) \rightarrow \mathcal{F}(\partial S, M)$ denotes the restriction map on functions and $\omega \in \Omega^p(M) \mapsto \tilde{\omega}^{\partial} \in \Omega^{p-k+1}(\mathcal{F}(\partial S, M))$.

Now the properties of the tilda calculus follow immediately from proposition 11.

**Proposition 13.** [HV04] For any $\omega \in \Omega^p(M)$, $\varphi \in \text{Diff}(M)$ and $X \in \mathfrak{X}(M)$, the following identities hold:
1. \( \tilde{\varphi} \ast \tilde{\omega} = \varphi \ast \omega \)
2. \( L \tilde{X} \tilde{\omega} = L_X \omega \)
3. \( i \tilde{X} \tilde{\omega} = i_X \omega \)
4. \( d \tilde{\omega} = \tilde{d} \omega \)

Proof. We verify the identities 1. and 4. From relation 1. from proposition [11] we get that

\[
\pi \ast \tilde{\varphi} \ast \tilde{\omega} = \bar{\varphi} \ast \pi \ast \tilde{\omega} = \bar{\varphi} \ast \bar{\omega} = \hat{\omega} = \pi \ast \tilde{\varphi} \ast \omega,
\]

and this implies the first identity. Using identity 4. from proposition [11] we compute

\[
\pi \ast d \tilde{\omega} = d \pi \ast \tilde{\omega} = d \tilde{\omega} = \tilde{d} \omega = \pi \ast \tilde{d} \omega,
\]

which shows the last identity. \(\Box\)

**Hamiltonian formalism for \( p \)-branes**

In this section we show how the hat calculus appears in the hamiltonian formalism for \( p \)-branes and open \( p \)-branes [AS05] [BZ05].

Let \( S \) be a compact oriented \( p \)-dimensional manifold. The phase space for the \( p \)-brane world volume \( S \times \mathbb{R} \) is the cotangent bundle \( T^* \mathcal{F}(S, M) \), where the canonical symplectic form is twisted. The twisting consists in adding a magnetic term, namely the pull-back of a closed 2-form on the base manifold, to the canonical symplectic form on a cotangent bundle [MR99]. These twisted symplectic forms appear also in cotangent bundle reduction.

We consider a closed differential form \( \tilde{H} \in \Omega^{p+2}(M) \). Since \( \dim S = p \), the hat map (10) provides a closed 2-form \( \tilde{H} \) on \( \mathcal{F}(S, M) \). If \( \pi_X : T^* \mathcal{F}(S, M) \rightarrow \mathcal{F}(S, M) \) denotes the canonical projection, the twisted symplectic form on \( T^* \mathcal{F}(S, M) \) is

\[
\Omega_H = -d\Theta_X + \frac{1}{2} \pi_X \tilde{H},
\]

where \( \Theta_X \) is the canonical 1-form on \( T^* \mathcal{F}(S, M) \).

For the description of open branes one considers a compact oriented \( p \)-dimensional manifold \( S \) with boundary \( \partial S \) and a submanifold \( D \) of \( M \). The phase space is in this case the cotangent bundle \( T^* \mathcal{F}_D(S, M) \) over the manifold [MS0]

\[
\mathcal{F}_D(S, M) = \{ f : S \rightarrow M | f(\partial S) \subset D \}.
\]

The twisting of the canonical symplectic form is done with a closed differential form \( H \in \Omega^{p+2}(M) \) with \( i^* H = dB \) for some \( B \in \Omega^{p+1}(D) \), where \( i : D \rightarrow M \) denotes the inclusion. The twisted symplectic form on \( T^* \mathcal{F}_D(S, M) \) is

\[
\Omega_{(H,B)} = -d\Theta_D + \frac{1}{2} \pi_D^* (\tilde{H} - \partial^* \tilde{B}^\partial)
\]
with $\partial : \mathcal{F}_D(S, M) \to \mathcal{F}(\partial S, D)$ the restriction map and $\pi_{\mathcal{F}_D} : T^*\mathcal{F}_D(S, M) \to \mathcal{F}_D(S, M)$. To distinguish between the hat calculus for $\mathcal{F}(S, M)$ and the hat calculus for $\mathcal{F}(\partial S, M)$, we denote $\sim^\partial : \Omega^n(M) \to \Omega^{n-p+1}(\mathcal{F}(\partial S, M))$.

The only thing we have to verify is the closedness of $\hat{H} - \partial^*\hat{B}^\partial$. We first notice that (12) implies $d\hat{H} = \hat{d}H + r^\partial*\hat{H}^\partial$, where $r^\partial : \mathcal{F}(S, M) \to \mathcal{F}(\partial S, M)$ denotes the restriction map, and identity 4 from proposition 11 implies $\hat{d}B^\partial = d\hat{B}^\partial$. On the other hand identity 1 from proposition 11 ensures that $\hat{i}^*H^\partial = \hat{\bar{i}}^*H^\partial$, with $\bar{i} : \mathcal{F}(\partial S, D) \to \mathcal{F}(\partial S, M)$ denoting the push-forward by $i : D \to M$. Knowing that $r^\partial = \bar{i} \circ \partial$, we compute:

$$d\hat{H} = \hat{d}H + r^\partial*\hat{H}^\partial = \partial^*\hat{i}^*H^\partial = \partial^*\hat{\bar{i}}^*H^\partial = \partial^*\hat{d}B^\partial = d\partial^*\hat{B}^\partial,$$

so the closed 2–form $\hat{H} - \partial^*\hat{B}^\partial$ provides a twist for the canonical symplectic form on the cotangent bundle $T^*\mathcal{F}_D(S, M)$.

**Non-linear Grassmannians as symplectic manifolds**

In this subsection we recall properties of the co-dimension two non-linear Grassmannian as a symplectic manifold.

**Proposition 14.** [I96] Let $M$ be a closed $m$–dimensional manifold with volume form $\nu$. The tilda map provides a symplectic form $\tilde{\nu}$ on $Gr_{m-2}(M)$:

$$\tilde{\nu}_N(\tilde{X}_N, \tilde{Y}_N) = \int_N i_{Y_N} i_{X_N} \nu,$$

for $\tilde{X}_N$ and $\tilde{Y}_N$ sections of $TN^\perp$ determined by sections $X_N$ and $Y_N$ of $TM|_N$.

**Proof.** The 2–form $\tilde{\nu}$ is closed since $d\tilde{\nu} = \tilde{\nu}$ by the tilda calculus. To verify that it is also (weakly) non-degenerate, let $X_N$ be an arbitrary vector field along $N$ such that $\int_N i_{Y_N} i_{X_N} \nu = 0$ for all vector fields $Y_N$ along $N$. Then $X_N$ must be tangent to $N$, so $\tilde{X}_N = 0$. \qed

In dimension $m = 3$ the symplectic form $\tilde{\nu}$ is known as the Marsden–Weinstein symplectic form on the space of unparameterized oriented links, see [MW83] [B93].

**Hamiltonian Diff$_{ex}(M, \nu)$ action.** The action of the group Diff$(M, \nu)$ of volume preserving diffeomorphisms of $M$ on $Gr_{m-2}(M)$ preserves the symplectic form $\tilde{\nu}$:

$$\varphi^*\tilde{\nu} = \hat{\varphi}^*\nu = \tilde{\nu}, \quad \forall \varphi \in \text{Diff}(M, \nu).$$

The subgroup Diff$_{ex}(M, \nu)$ of exact volume preserving diffeomorphisms acts in a hamiltonian way on the symplectic manifold $(Gr_{m-2}(M), \tilde{\nu})$. Its Lie algebra is $\mathfrak{x}_{ex}(M, \nu)$, the Lie algebra of exact divergence free vector fields, i.e. vector fields
\(X_\alpha\) such that \(i_{X_\alpha}\nu = \text{d}\alpha\) for a potential form \(\alpha \in \Omega^{m-2}(M)\). The infinitesimal action of \(X_\alpha\) is the vector field \(\tilde{X}_\alpha\). By the tilda calculus \(\tilde{\alpha} \in \mathcal{F}(\text{Gr}_{m-2}(M))\) is a hamiltonian function for the hamiltonian vector field \(\tilde{X}_\alpha\):

\[
i_{\tilde{X}_\alpha}\tilde{\nu} = i_{X_\alpha}\nu = \tilde{\text{d}}\alpha = \text{d}\tilde{\alpha}.
\]

It depends on the particular choice of the potential \(\alpha\) of \(X_\alpha\). A fixed continuous right inverse \(b: \Omega^{m-2}(M) \to \Omega^{m-2}(M)\) to the differential \(\text{d}\) picks up a potential \(b(\text{d}\alpha)\) of \(X_\alpha\). The corresponding momentum map is:

\[
J: \mathcal{M} \to \mathfrak{x}_{ex}(M, \nu)^*, \quad \langle J(N), X_\alpha \rangle = b(\text{d}\alpha)(N) = \int_N b(\text{d}\alpha).
\]

On the connected component \(\mathcal{M}\) of \(N \in \text{Gr}_{m-2}(M)\), the non-equivariance of \(J\) is measured by the Lie algebra 2–cocycle on \(\mathfrak{x}_{ex}(M, \nu)\)

\[
\sigma_N(X,Y) = \langle J(N), [X,Y]^{op} \rangle - \tilde{\nu}(\tilde{X}, \tilde{Y})(N) = (\text{bidi}_{Y} i_X\nu)(N) - (i_Y i_X\nu)(N)
\]

\[
= (\tilde{P}i_X i_Y \nu)(N) = \int_N Pi_X i_Y \nu.
\]

Here \(P = 1_{\Omega^{m-2}(M)} - b \circ \text{d}\) is a continuous linear projection on the subspace of closed \((m-2)\)-forms and \((X,Y) \mapsto [Pi_Y i_X \nu] \in H^{m-2}(M)\) is the universal Lie algebra 2–cocycle on \(\mathfrak{x}_{ex}(M, \nu)\) \(\text{[R95]}\). The cocycle \(\sigma_N\) is cohomologous to the Lichnerowicz cocycle

\[
\sigma_\eta(X,Y) = \int_M \eta(X,Y) \nu,
\]

where \(\eta\) is a closed 2-form Poincaré dual to \(N\) \(\text{[V09]}\).

If \(\nu\) is an integral volume form, then \(\sigma_N\) is integrable \(\text{[196]}\). The connected component \(\mathcal{M}\) of \(\text{Gr}_{m-2}(M)\) is a coadjoint orbit of a 1–dimensional central Lie group extension of \(\text{Diff}_{ex}(M, \nu)\) integrating \(\sigma_N\), and \(\tilde{\nu}\) is the Kostant-Kirillov-Souriau symplectic form. \(\text{[HV04]}\).

4 Bar map

When a volume form \(\mu\) on the compact \(k\)–dimensional manifold \(S\) is given, one can associate to each differential \(p\)-form on \(M\) a differential \(p\)-form on \(\mathcal{F}(S, M)\)

\[
\bar{\omega}(Y^i_f, \ldots, Y^p_f) = \int_S \omega(Y^i_f, \ldots, Y^p_f) \mu, \quad \forall Y^i_f \in T_f \mathcal{F}(S, M),
\]

where \(\omega(Y^i_f, \ldots, Y^p_f) : x \mapsto \omega_{\mathcal{F}(x)}(Y^i_f(x), \ldots, Y^p_f(x))\) defines a smooth function on \(S\). In this way a bar map is defined. Formula \(\text{[2]}\) assures that this bar map is just the hat pairing of differential forms on \(M\) with the volume form \(\mu\)

\[
\bar{\omega} = \hat{\omega} \cdot \mu = \int_S \text{ev}^* \omega \wedge \text{pr}^* \mu.
\]
From the properties of the hat pairing presented in proposition [prop], lemma [lem] and theorem [thm], one can develop a bar calculus.

**Proposition 15.** For any $\omega \in \Omega^p(M)$, $\varphi \in \text{Diff}(M)$ and $X \in \mathfrak{X}(M)$, the following identities hold:

1. $\bar{\varphi}^*\bar{\omega} = \varphi^*\omega$
2. $L_{\bar{\xi}}\bar{\omega} = \bar{L}_X\omega$
3. $i_{\bar{\xi}}\bar{\omega} = i_X\omega$
4. $d\bar{\omega} = \overline{d\omega}$.

**$\mathcal{F}(S, M)$ as symplectic manifold**

Let $(M, \omega)$ be a connected symplectic manifold and $S$ a compact $k$–dimensional manifold with a fixed volume form $\mu$, normalized such that $\int_S \mu = 1$. The following fact is well known:

**Proposition 16.** The bar map provides a symplectic form $\bar{\omega}$ on $\mathcal{F}(S, M)$:

$$\bar{\omega}_f(X_f, Y_f) = \int_S \omega(X_f, Y_f)\mu.$$  

*Proof.* That $\bar{\omega}$ is closed follows from the bar calculus: $d\bar{\omega} = \overline{d\omega} = 0$. The (weakly) non-degeneracy of $\bar{\omega}$ can be verified as follows. If the vector field $X_f$ on $M$ along $S$ is non-zero, then $X_f(x) \neq 0$ for some $x \in S$. Because $\omega$ is non-degenerate, one can find another vector field $Y_f$ along $f$ such that $\omega(X_f, Y_f)$ is a bump function on $S$. Then $\bar{\omega}(X_f, Y_f) = \int_S \omega(X_f, Y_f)\mu \neq 0$, so $X_f$ does not belong to the kernel of $\bar{\omega}$, thus showing that the kernel of $\bar{\omega}$ is trivial. \hfill $\square$

**Hamiltonian action on $M$.** Let $G$ be a Lie group acting in a hamiltonian way on $M$ with momentum map $J : M \to \mathfrak{g}^*$. Then $\mathcal{F}(S, M)$ inherits a $G$-action: $(g \cdot f)(x) = g \cdot (f(x))$ for any $x \in S$. The infinitesimal generator is $\xi_f = \bar{\xi}_M$ for any $\xi \in \mathfrak{g}$, where $\xi_M$ denotes the infinitesimal generator for the $G$-action on $M$. The bar calculus shows quickly that $G$ acts in a hamiltonian way on $\mathcal{F}(S, M)$ with momentum map

$$J = \bar{J} : \mathcal{F}(S, M) \to \mathfrak{g}^*, \quad \bar{J}(f) = \int_S (J \circ f)\mu, \quad \forall f \in \mathcal{F}(S, M).$$

Indeed, for all $\xi \in \mathfrak{g}$

$$i_{\xi_f}\bar{\omega} = i_{\bar{\xi}_M}\bar{\omega} = \overline{i_{\xi_M}\omega} = \overline{d(J, \xi)} = \overline{d(J, \xi)} = \overline{d(J, \xi)} = \overline{d(J, \xi)} = \overline{d(J, \xi)}.$$
Let $M$ be connected and let $\sigma$ be the $\mathbb{R}$-valued Lie algebra 2–cocycle on $\mathfrak{g}$ measuring the non-equivariance of $J$, i.e.

$$\sigma(\xi, \eta) = \langle J(x), [\xi, \eta] \rangle - \omega(\xi_M, \eta_M)(x), \quad x \in M,$$

(both terms are hamiltonian function for the vector field $[\xi, \eta]_M = -[\xi_M, \eta_M]$).

Then the non-equivariance of $\bar{J} = \bar{\partial}$ is also measured by $\sigma$: for all $f \in \mathcal{F}(S, M)$

$$\langle \bar{J}(f), [\xi, \eta] \rangle - \bar{\omega}(\xi_F, \eta_F)(f) = \langle J, [\xi, \eta] \rangle(f) - \bar{\omega}(\xi_M, \eta_M)(f) = \sigma(\xi, \eta).$$

**Hamiltonian $\text{Diff}_{\text{ham}}(M, \omega)$ action.** The action of the group $\text{Diff}(M, \omega)$ of symplectic diffeomorphisms preserves the symplectic form $\bar{\omega}$:

$$\varphi^*\bar{\omega} = \bar{\varphi^*}\omega = \bar{\omega}, \quad \forall \varphi \in \text{Diff}(M, \omega).$$

The subgroup $\text{Diff}_{\text{ham}}(M, \omega)$ of hamiltonian diffeomorphisms of $M$ acts in a hamiltonian way on the symplectic manifold $\mathcal{F}(S, M)$. The infinitesimal action of $X_h \in \mathfrak{x}_{\text{ham}}(M, \omega)$, $h \in \mathcal{F}(M)$, is the hamiltonian vector field $\bar{X}_h$ on $\mathcal{F}(S, M)$ with hamiltonian function $\bar{h}$. This follows by the bar calculus:

$$d\bar{h} = \bar{d}h = \bar{i}_{X_h}\bar{\omega} = i_{X_h}\bar{\omega}.$$

The hamiltonian function $\bar{h}$ of $\bar{X}_h$ depends on the particular choice of the hamiltonian function $h$. To solve this problem we fix a point $x_0 \in M$ and we choose the unique hamiltonian function $h$ with $h(x_0) = 0$, since $M$ is connected. The corresponding momentum map is

$$\mathbf{J} : \mathcal{F}(S, M) \rightarrow \mathfrak{x}_{\text{ham}}(M, \omega)^*, \quad \langle \mathbf{J}(f), X_h \rangle = \bar{h}(f) = \int_S (h \circ f)\mu.$$

The Lie algebra 2–cocycle on $\mathfrak{x}_{\text{ham}}(M, \omega)$ measuring the non-equivariance of the momentum map is

$$\sigma(X, Y) = -\omega(X, Y)(x_0),$$

by the bar calculus

$$\sigma(X, Y)(f) = \langle \mathbf{J}(f), [X, Y]^\varphi \rangle - \bar{\omega}(X_F, Y_F)(f)
= \bar{\omega}(X, Y) - \omega(X, Y)(x_0)(f) - \bar{\omega}(\bar{X}, \bar{Y})(f)
= -\omega(X, Y)(x_0).$$

This is a Lie algebra cocycle describing the central extension

$$0 \rightarrow \mathbb{R} \rightarrow \mathcal{F}(M) \rightarrow \mathfrak{x}_{\text{ham}}(M, \omega) \rightarrow 0$$

where $\mathcal{F}(M)$ is enowed with the canonical Poisson bracket. A group cocycle on $\text{Diff}_{\text{ham}}(M, \omega)$ integrating the Lie algebra cocycle $\sigma$ if $\omega$ exact is studied in [ILM06].
**Hamiltonian** $\text{Diff}_{ex}(S,\mu)$ action. The (left) action of the group $\text{Diff}(S,\mu)$ of volume preserving diffeomorphisms preserves the symplectic form $\bar{\omega}$:

$$\bar{\psi}^*\bar{\omega} = \bar{\psi}^*\bar{\omega} \cdot \mu = \bar{\omega} \cdot \bar{\psi}^*\mu = \bar{\omega} \cdot \mu = \bar{\omega}, \quad \forall \psi \in \text{Diff}(S,\mu).$$

The subgroup $\text{Diff}_{ex}(S,\mu)$ of exact volume preserving diffeomorphisms acts in a hamiltonian way on the symplectic manifold $\mathcal{F}(S,M)$. The infinitesimal action of the exact divergence free vector field $X_\alpha \in \mathfrak{x}_{ex}(S,\mu)$ with potential form $\alpha \in \Omega^{k-2}(S)$ is the hamiltonian vector field $\hat{X}_\alpha$ on $\mathcal{F}(S,M)$ with hamiltonian function $\bar{\omega} \cdot \bar{\alpha}$. Indeed, from $i_{X_\alpha}\mu = d\alpha$ follows by the hat calculus that

$$d(\bar{\omega} \cdot \bar{\alpha}) = d\bar{\omega} \cdot \alpha + \omega \cdot d\alpha = \omega \cdot i_{X_\alpha} \mu = i_{\hat{X}_\alpha} \bar{\omega} \cdot \mu = i_{\hat{X}_\alpha} \bar{\omega}.$$

If the symplectic form $\omega$ is exact, then the corresponding momentum map is

$$\mathbf{J} : \mathcal{F}(S,M) \to \mathfrak{x}_{ex}(S,\mu)^*, \quad \langle \mathbf{J}(f), X_\alpha \rangle = (\bar{\omega} \cdot \bar{\alpha})(f) = \int_S f^*\omega \wedge \alpha.$$

It takes values in the regular part of $\mathfrak{x}_{ex}(S,\mu)^*$, which can be identified with $d\Omega^1(S)$, so we can write $\mathbf{J}(f) = f^*\omega$ under this identification.

In general the hamiltonian function $\bar{\omega} \cdot \bar{\alpha}$ of $\hat{X}_\alpha$ depends on the particular choice of the potential form $\alpha$ of $X_\alpha$. To fix this problem we consider as in Section 3 a continuous right inverse $b : d\Omega^m(S) \to \Omega^{m-2}(M)$ to the differential $d$, so $b(d\alpha)$ is a potential for $X_\alpha$. The corresponding momentum map is

$$\mathbf{J} : \mathcal{F}(S,M) \to \mathfrak{x}_{ex}(S,\mu)^*, \quad \langle \mathbf{J}(f), X_\alpha \rangle = (\omega \cdot b d\alpha)(f) = \int_S f^*\omega \wedge b(d\alpha).$$

On a connected component $\mathcal{F}$ of $\mathcal{F}(S,M)$, the non-equivariance of $\mathbf{J}$ is measured by the Lie algebra 2–cocycle

$$\sigma_\mathcal{F}(X, Y) = \langle \mathbf{J}(f), [X, Y] \rangle = \omega(\hat{X}, \hat{Y})(f) = (\omega \cdot b d i_Y i_X \mu)^*(f) = (\omega \cdot i_Y i_X \mu)^*(f)$$

$$= (\omega \cdot P i_X i_Y \mu)^*(f) = \int_S f^*\omega \wedge P i_X i_Y \mu$$

on the Lie algebra of exact divergence free vector fields, for $P = 1 - b d$ the projection on the subspace of closed $(m-2)$-forms. It does not depend on $f \in \mathcal{F}$, because the cohomology class $[f^*\omega] \in H^2(S)$ does not depend on the choice of $f$.

The cocycle $\sigma_\mathcal{F}$ is cohomologous to the Lichnerowicz cocycle $\sigma_{f^*\omega}$ defined in [13]. Since $\int_S \mu = 1$, the cocycle $\sigma_\mathcal{F}$ is integrable if and only if the cohomology class of $f^*\omega$ is integral [196].

**Remark 17.** The two equivariant momentum maps on the symplectic manifold $\mathcal{F}(S,M)$, for suitable central extensions of the hamiltonian group $\text{Diff}_{ham}(M,\omega)$ and of the group $\text{Diff}_{ex}(S,\mu)$ of exact volume preserving diffeomorphisms, form the dual pair for ideal incompressible fluid flow [MW83] [GBV09].
5 Appendix: Fiber integration

Chapter VII in [GHV72] is devoted to the concept of integration over the fiber in locally trivial bundles. We particularize this fiber integration to the case of trivial bundles $S \times M \to M$, listing its main properties without proofs.

Let $S$ be a compact $k$–dimensional manifold. Fiber integration over $S$ assigns to $\omega \in \Omega^n(S \times M)$ the differential form $\int_S \omega \in \Omega^{n-k}(M)$ defined by

$$(\int_S \omega)(x) = \int_S \omega_x \in \Lambda^{n-k}T^*_x M, \quad \forall x \in M,$$

where $\omega_x \in \Omega^k(S, \Lambda^{n-k}T^*_x M)$ is the restriction of $\omega$ to the fiber over $x$:

$$\langle \omega_x(Z^1_s, \ldots, Z^{n-k}_s), X^1_x \wedge \cdots \wedge X^k_x \rangle = \omega_{(s,x)}(X^1_x, \ldots, X^k_x, Z^1_s, \ldots, Z^{n-k}_s)$$

for all $X^i_x \in T_x M$ and $Z^j_s \in T_s S$.

The properties of the fiber integration used in the text are special cases of the propositions (VIII) and (X) in [GHV72]:

1. Pull-back of fiber integrals:

$$f^* \int_S \omega = \int_S (1_S \times f)^* \omega, \quad \forall f \in \mathcal{F}(M', M),$$

with infinitesimal version

$$L_X \int_S \omega = \int_S L_{0_S \times X} \omega, \quad \forall X \in \mathfrak{X}(M).$$

2. Invariance under pull-back by orientation preserving diffeomorphisms of $S$:

$$\int_S (\varphi \times 1_M)^* \omega = \int_S \omega, \quad \forall \varphi \in \text{Diff}_+(S),$$

with infinitesimal version $\int_S L_{Z \times 0_M} \omega = 0, \quad \forall Z \in \mathfrak{X}(S)$.

3. Insertion of vector fields into fiber integrals:

$$i_X \int_S \omega = \int_S i_{0_S \times X} \omega, \quad \forall X \in \mathfrak{X}(M).$$

4. Integration along boundary free manifolds commutes with differentiation.

When $\partial S$ denotes the boundary of the $k$–dimensional compact manifold $S$ and $i_\partial : \partial S \to S$ the inclusion,

$$d \int_S \beta - \int_S d\beta = (-1)^{n-k} \int_{\partial S} (i_\partial \times 1_M)^* \beta$$

holds for any differential $n$–form $\beta$ on $S \times M$. 

15
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