Excursions Above the Minimum for Diffusions *

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1. Introduction

Let $X = (X_t)_{t \geq 0}$ be a regular diffusion process on an interval $E \subset \mathbb{R}$. Let $H_t := \min_{0 \leq u \leq t} X_u$ denote the past minimum process of $X$ and consider the excursions of $X$ above its past minimum level: If $[a, b]$ is a maximal interval of constancy of $t \mapsto H_t$, then $(X_t : a \leq t \leq b)$ is the “excursion above the minimum” starting at time $a$ and level $y = H_a$. These excursions, when indexed by the level at which they begin, can be regarded (collectively) as a point process. The independent increments property of the first-passage process of $X$ implies that this point process is Poissonian in nature, albeit non-homogeneous in intensity. Moreover, intuition tells us that the distribution of an excursion above the minimum $(X_t : a \leq t \leq b)$ should be governed by the Itô excursion law corresponding to excursions above the fixed level $y = H_a(\omega)$.

Our first task is to render precise the ruminations of the preceding paragraph. This is accomplished in sections 2 and 3 by applying Maisonneuve’s theory of exit systems [10] to a suitable auxiliary process $(\overline{X}_t)$ associated with $X$. The basic result, stated in section 2, affirms the existence of a “Lévy system” for the point process of excursions of $X$ above its past minimum.

In sections 4, 5, and 6 we discuss several applications of the Lévy system constructed in section 3; these applications concern path decompositions of $X$ involving the minimum process $H$. Such decompositions, and related results, have been found by various authors (see [6, 9, 11, 12, 14, 15, 16, 17, 18]), most often in the special case where $X$ is Brownian motion. The possibility of using Lévy systems to give a unified treatment of path decompositions is, of course, not surprising. In an excellent synthesis [13] Pitman has shown how the existence of a Lévy system for a point process attached to a Markov process leads naturally to various path decompositions of the Markov process.

In section 4 we obtain a general version of Williams’ decomposition of a diffusion at its global minimum. A “local” version of Williams’ decomposition can be found in section 5. In section 6 we give a new proof of a result of Vervaat [17], which states that a Brownian bridge, when split at its minimum and suitably “rearranged” becomes a (scaled) Brownian

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excursion. Indeed, we produce an inversion of Vervaat’s transformation, showing how a Brownian excursion may be split and rearranged to yield Brownian bridge.

2. Notation and the basic result

Let \( X = (\Omega, \mathcal{F}, \theta_t, X_t, P^x) \) be a canonically defined regular diffusion on an interval \( E \subset \mathbb{R} \). Here \( \Omega \) denotes the space of paths \( \omega: [0,+\infty[ \to E \cup \{\Delta\} \) which are absorbed in the cemetery point \( \Delta \notin E \) at time \( \zeta(\omega) \), and which are continuous on \([0,\zeta(\omega)]\). For \( t \geq 0 \), \( X_t(\omega) = \omega(t) \), and \( \theta_t \omega \) denotes the path \( u \mapsto \omega(u + t) \). The \( \sigma \)-fields \( \mathcal{F} \) and \( \mathcal{F}_t \) \((t \geq 0)\) are the usual Markovian completions of \( \mathcal{F} = \sigma\{X_u : u \geq 0\} \) and \( \mathcal{F}_t = \sigma\{X_u : 0 \leq u \leq t\} \) respectively. The law \( P^x \) on \((\Omega, \mathcal{F}^x)\) corresponds to \( X \) started at \( x \in E \). We shall also make use of the killing operators \((k_t)\) defined for \( t \geq 0 \) by

\[
  k_t \omega(u) = \begin{cases} 
  \omega(u), & u < t, \\
  \Delta, & u \geq t.
  \end{cases}
\]

Let \( A = \inf E \), \( B = \sup E \), and write \( E^\circ = ]A, B[ \). We assume throughout the paper that \( A \notin E \), and that \( B \in E \) if and only if \( B \) is a regular boundary point which is not a trap for \( X \). In particular, these assumptions imply that the transition kernels of \( X \) are absolutely continuous with respect to the speed measure \( m \) (recalled below). See §4.11 of Itô-McKean [8].

Let \( s \) (resp. \( m \), resp. \( k \)) denote a scale function (resp. speed measure, resp. killing measure) for \( X \). Recall from [8] that the generator \( \mathcal{G} \) of \( X \) has the form

\[
(2.1) \quad \mathcal{G}u(x) \cdot m(dx) = du^+(x) - u(x) \cdot k(dx), \quad x \in E^\circ,
\]

for \( u \in D(\mathcal{G}) \), the domain of \( \mathcal{G} \). Here and elsewhere \( u^+ \) denotes the scale derivative:

\[
  u^+(x) = \lim_{y \downarrow x} \frac{u(y) - u(x)}{s(y) - s(x)}.
\]

Let \((U^\alpha : \alpha > 0)\) denote the resolvent family of \( X \). Subsequent calculations require an explicit expression for the density of \( U^\alpha(x,dy) \) with respect to \( m(dy) \). Recall from [8] that for each \( \alpha > 0 \) there are strictly positive, linearly independent solutions \( g_1^\alpha \) and \( g_2^\alpha \) of

\[
(2.2) \quad \mathcal{G}g(x) = \alpha g(x), \quad x \in E^\circ;
\]

\( g_1^\alpha \) (resp. \( g_2^\alpha \)) is an increasing (resp. decreasing) solution of \((2.2)\) which also satisfies the appropriate boundary condition at \( A \) (resp. \( B \)). Both \( g_1^\alpha \) and \( g_2^\alpha \) are uniquely determined
up to a positive multiple. We sometimes drop the superscript \( \alpha \), writing simply \( g_1 \) and \( g_2 \). Since \( g_1 \) and \( g_2 \) are linearly independent solutions of (2.2), the Wronskian \( W = g_1^+ g_2 g_2^- g_1^- \) is constant. The resolvent \( U^\alpha \) is given by

\[
U^\alpha f(x) = U^\alpha(x, f) = \int_E u^\alpha(x, y)f(y) \, m(dy),
\]

where

\[
u^\alpha(x, y) = \nu^\alpha(y, x) = g_1^\alpha(x)g_2^\alpha(y) / W, \quad x \leq y.
\]

See \( \S 4.11 \) of [8] and note that in (2.3) the mass \( m(\{B\}) \) is the “stickiness” coefficient occurring in the boundary condition at \( B \) for elements of \( D(G) \).

A jointly continuous version \( (L^y_t : t \geq 0, y \in E) \) of \textit{local time} for \( X \) may be chosen, and normalized to be occupation density relative to \( m \), so that

\[
P^x \int_0^\infty e^{-\alpha t} \, dL^y_t = \nu^\alpha(x, y).
\]

Fixing a level \( y \in E \), the local time \( (L^y_t : t \geq 0) \) is related to the Itô excursion law [7], for excursions from level \( y \), as follows. Let \( G(y) \) denote the (random) set of left-hand endpoints (in \( ]0, \zeta[ \) of intervals contiguous to the level set \( \{t > 0 : X_t = y\} \). Define the hitting time \( T_y \) by

\[
T_y = \inf\{t > 0 : X_t = y\} \quad (\inf \emptyset = +\infty).
\]

The Itô excursion law \( n_y \) is determined by the identity

\[
P^x \sum_{u \in G(y)} Z_u F \circ k_{T_y} \circ \theta_u = P^x \left( \int_0^\infty Z_u \, dL^y_u \right) \cdot n_y(F),
\]

where \( x \in E \), \( F \in pF^\circ \), and \( Z \geq 0 \) is an \( (\mathcal{F}_t) \)-optional process. Under \( n_y \) the coordinate process \( (X_t : t > 0) \) is strongly Markovian with semigroup \( (Q^y_t) \) given by

\[
Q^y_t(x, f) = P^x(f \circ X_t; t < T_y).
\]

The entrance law \( n_y(X_t \in dz) \) is determined by the corresponding Laplace transform

\[
W^\alpha f(y) = W^\alpha(y, f) = n_y \int_0^\zeta e^{-\alpha t} f \circ X_t \, dt.
\]

Conversely, \( n_y \) is the unique \( \sigma \)-finite measure on \( (\Omega, \mathcal{F}^\circ) \) which is carried by \( \{\zeta > 0\} \) and under which \( (X_t : t > 0) \) is Markovian with semigroup (2.7) and entrance law (2.8).
Let $V_y^\alpha$ denote the resolvent of the semigroup $(Q_t^y)$. Taking $Z_u = e^{-\alpha u}$, $F = \int_0^\zeta e^{-\alpha t} f(X_t) \, dt$ in (2.6), and using (2.5), we obtain the important identity

\[(2.9) \quad U^\alpha f(x) = V_y^\alpha f(x) + u^\alpha(x, y)[m(\{y\})f(y) + W^\alpha f(y)].\]

We also recall from §4.6 of [8] that the distribution of $T_y$ is given by

\[(2.10) \quad P^x(e^{-\alpha T_y}) = \begin{cases} \frac{g_1^\alpha(x)}{g_1^\alpha(y)}, & x \leq y, \\ \frac{g_2^\alpha(x)}{g_2^\alpha(y)}, & x \geq y. \end{cases}\]

Finally, the point process of excursions above the minimum is defined as follows. For $t \geq 0$ set

\[
H_t(\omega) = \begin{cases}
\min_{0 \leq u \leq t} X_u(\omega) & \text{if } t < \zeta(\omega), \\
-\infty & \text{if } t \geq \zeta(\omega);
\end{cases}
\]

\[
M(\omega) = \{ u > 0 : X_u(\omega) = H_u(\omega) \};
\]

\[
R_t(\omega) = \inf \{ u > 0 : u + t \in M(\omega) \};
\]

\[
G(\omega) = \{ u > 0 : u < \zeta(w), R_u(\omega) = 0 < R_u(\omega) \}.
\]

Thus $G$ is the random set of left-hand endpoints of intervals contiguous to the random set $M$. For $u \in G$ we have the excursion $e^u$ defined by

\[
e_u^u = \begin{cases} X_{u+t}, & 0 \leq t < R_u, \\ \Delta, & t \geq R_u. \end{cases}
\]

The point process $\Pi = (e^u : u \in G)$ admits a Lévy system as follows. Define a continuous increasing adapted process $C = (C_t : t \geq 0)$ by

\[
C_t = \begin{cases}
s(H_0) - s(H_t), & \text{if } t < \zeta, \\ C_\zeta, & \text{if } t \geq \zeta.
\end{cases}
\]

\[(2.11) \quad \text{Theorem.} \quad \text{For } Z \geq 0 \text{ and } (F_t)\text{-optional, and } F \in pF^\alpha,
\]

\[
P^x \sum_{u \in G} Z_u F(e^u) = P^x \int_0^\infty Z_u n_{X_u}^\uparrow (F) \, dC_u
\]

\[(2.12) = P^x \int_A Z_{T_y} 1\{T_y < +\infty\} n_{T_y}^\uparrow (F) \, ds(y),
\]

where $n_{T_y}^\uparrow$ denotes the restriction of $n_y$ to $\{ \omega : \omega(t) > y, \forall t \in ]0, \zeta(\omega)[ \}$.

\[(2.13) \quad \text{Remark.} \quad \text{The second equality in (2.12) follows from the first by the change of variable } u = T_y. \quad \text{The equality of the first and third terms in (2.12) amounts to the}
statement that the time-changed point process \((e^T_y : R_{T_y} - < R_T, A < y < x)\) is a
stopped Poisson point process under \(P^x\), with (non-homogeneous) intensity \(ds(y)n^\uparrow_y(d\omega)\),
stopped at the first level \(y\) for which \(T_y = +\infty\). See [14] for this result in the case of
Brownian motion, with or without drift. The general result (2.11) was suggested by §4.10
of [8].

3. Proof of Theorem (2.11)

Maisonneuve’s theory of exit systems [10] provides a Lévy system description of the point
process of excursions induced by a closed, optional, homogeneous random set. Unfortu-
nately the set \(M\) introduced in §2 is not \((\theta_t)\)-homogeneous; however the theory of [10]
can be brought to bear once we note that \(M\) is homogeneous as a functional of the strong
Markov process \((X_t, H_t), t \geq 0\). This key observation is due to Millar [12] and has been
formalized by Getoor in [4]. In the terminology of [4], the process \(H\) is a “min-functional”:
\(H_{t+u} = H_t \land H_u \circ \theta_t\). This property ensures that \(\Xi := (X, H)\) is Markovian, as a simple
computation shows.

Following [4] we first construct a convenient realization of \(\Xi\). Let \(\Omega = \Omega \times (E \cup \{-\infty\})\),
\(E = \{(x, a) \in E \times E : a \leq x\}\), and for \((\omega, a) \in \Omega\) set
\[
\Xi_t(\omega, a) = (X_t(\omega), a \land H_t(\omega)),
\]
\[
\theta_t(\omega, a) = (\theta_t(\omega), a \land H_t(\omega)).
\]
Clearly \(\Xi_t \circ \theta_u = \Xi_{t+u}, \theta_t \circ \theta_u = \theta_{t+u}\). Moreover, \(M\) can be realized over \(\Xi\) as
(3.1)
\[
\overline{M}(\omega, a) = \{t > 0 : \Xi_t(\omega, a) \in D\},
\]
where \(D = \{(x, x) : x \in E\}\). Let \(\overline{F} = \sigma\{\Xi_u : u \geq 0\}, \overline{F}_t = \sigma\{\Xi_u : 0 \leq u \leq t\}\), and
for \((x, a) \in E\) let \(\overline{P}^{x,a} = P^x \otimes \epsilon_a\). The usual Markovian completion of the filtration \((\overline{F}_t)\)
relative to the laws \((\overline{P}^{x,a} : (x, a) \in E)\) is denoted by \((\overline{F}_t)\). Clearly \(\overline{P}^{x,a}(X_0 = (x, a)) = 1\)
so that \(\Xi\) has no branch points. Appealing to §2 of [4] we have the following

(3.2) Lemma. (i) \(\Xi = (\Omega, \overline{F}, \overline{F}_t, \overline{\theta}_t, \Xi_t, \overline{P}^{x,a})\) is a right-continuous, strong Markov pro-
cess with state space \(\overline{E}\) and cemetery \(\Delta = (\Delta, -\infty)\). The semigroup of \(\Xi\) maps Borel
functions to Borel functions, so that \(\Xi\) is even a Borel right process.

(ii) Let \(\pi : (\omega, a) \rightarrow \omega\) denote the projection of \(\overline{\Omega}\) onto \(\Omega\). If \(Z\) is an \((\overline{F}_t)\)-optional
process, then \(Z \circ \pi\) is \((\overline{F}_t)\)-optional.

Now \(\overline{M}\) is an \((\overline{F}_t)\)-optional, \((\overline{\theta}_t)\)-homogeneous set, and each section \(\overline{M}(\omega, a)\) is closed
in \(]0, \zeta(\omega, a)[\). Set \(\overline{R} = \inf \overline{M}\), so that \(\overline{R}\) is an exact terminal time of \(\Xi\) with \(\text{reg}(\overline{R}) = \)}
\{(x, a) \in E : \overline{P}^{x,a}(R = 0) = 1\} = D. This last fact follows from the regularity of \(X\) and the identity
\[\overline{P}^{x,a}(R = T_{u} \circ \pi) = 1, \quad (x, a) \in E.\]

Let \(G\) denote the set of left-hand endpoints of intervals contiguous to \(M\). The properties of the Maisonneuve exit system \((*\overline{P}^{x,a}, \overline{K})\) for \(M\) are summarized in the next proposition. In what follows, \(\overline{E}^*\) and \(\overline{F}^*\) denote the universal completions of \(\overline{E}\) (the Borel sets in \(\overline{E}\)) respectively.

**Proposition.** [Maisonneuve] There exists a continuous additive functional (CAF), \(\overline{K}\), of \(X\) with a finite 1-potential, and a kernel \(*\overline{P}^{x,a}\) from \((\overline{E}, \overline{E}^*)\) to \((\overline{\Omega}, \overline{F}^*)\) such that
\[(*3.4) \quad \overline{P}^{x,a} \sum_{u \in G} Z_u \overline{F}_u \circ \theta = \overline{P}^{x,a} \int_{0}^{\infty} Z_u \overline{P}^{x,u}(\overline{F}_u) dK_u,\]
whenever \(Z \geq 0\) is \((\overline{F}_t)\)-optional and \((u, \omega) \mapsto \overline{F}_u(\omega)\) is a \(\mathcal{B}[0, +\infty[ \otimes \overline{F}^*\)-measurable, positive function. The CAF \(\overline{K}\) is carried by \(D\). For each \((x, a) \in \overline{E}\), \(*\overline{P}^{x,a}\) is a \(\sigma\)-finite measure on \((\overline{\Omega}, \overline{F}^*)\) under which the coordinate process is strongly Markovian with the same transition semigroup as \(X\).

**Remarks.** The version of \((*\overline{P}^{x,a}, \overline{K})\) cited in (3.3) is a variant of that constructed in [10]; the difference stems from the possibility that \(\overline{P}^{x,a}(\zeta < +\infty)\) may be positive. The fact that \(\overline{K}\) is continuous (and so carried by \(D = \text{reg}(\overline{R})\)) follows from the construction in [10], since \(M = \{t > 0 : X_t \in D\}\) and \(D\) is finely perfect (with respect to \(X\)). Renormalizing the kernel \(*\overline{P}^{x,a}\) if necessary, we can and do assume that \(*\overline{P}^{y,y}(1 - e^{-R}) = 1\) for all \(y \in E\).

Our plan is to prove Theorem (2.11) by identifying \(*\overline{P}^{x,a}\) and \(\overline{K}\) explicitly, thereby deducing (2.12) from (3.4). First note that by taking \(x = y\) in (2.9) and using (2.4) we have
\[(*3.6) \quad W^\alpha f(y) = \int_{[A, y]} [g_1^\alpha(z)/g_1^\alpha(y)] f(z) m(dz) + \int_{[y, B]} [g_2^\alpha(z)/g_2^\alpha(y)] f(z) m(dz),\]
where \(y \in E, \alpha > 0, \) and \(f \geq 0\) is Borel measurable on \(E\).

To identify \(\overline{K}\) we define a second CAF of \(\overline{X}, \overline{C}\), by the formula
\[\overline{C}_t(\omega, a) = \begin{cases} s(a \wedge H_0(\omega)) - s(a \wedge H_t(\omega)) & \text{if } t < \overline{\zeta}(\omega, a), \\ \overline{C}_{\overline{\zeta}}(\omega, a) & \text{if } t \geq \overline{\zeta}(\omega, a); \end{cases}\]
and notice that \(\overline{C}_t(\omega, X_0(\omega)) = C_t(\omega)\). Clearly the fine support of \(\overline{C}\) is \(D\).

For \(x \in E\) put \(\psi(x) = W_{1|x, B)}(x)\).
(3.7) Proposition. The CAFs $\overline{K}$ and $\int_0^t \psi(X_u) dC_u$ are equivalent.

Proof. By [1; IV(2.13)] it suffices to check that the CAFs in question have the same finite 1-potential (over $X$). An argument of Vervaat [17] shows that $P^x(t \in M) = 0$ for all $x \in E$ and all $t > 0$. Consequently, $P^{x,a}(t \in M) = 0$ for all $(x, a) \in \overline{E}$ and all $t > 0$. By Fubini’s theorem, $\int_0^\infty e^{-t}1_M(t) dt = 0$ a.s. $P^{x,a}$ for all $(x, a) \in \overline{E}$. Thus taking $Z_u(\omega) = e^{-u}$, $F_u(\omega) = 1 - \exp(-R(\omega) \wedge \zeta(\omega))$ in (3.4), we may compute

$$P^{x,a} \int_0^\infty e^{-u} dK_u = P^{x,a} \sum_{u \in G} e^{-u} \left( \int_0^{R \wedge \zeta} e^{-t} \right) d\theta_u$$

$$= P^{x,a} \int_{R \wedge \zeta} e^{-u} du$$

$$= P^{x,a} (e^{-R \wedge \zeta} - e^{-\zeta})$$

$$= P^x (e^{-T_a \wedge \zeta} - e^{-\zeta})$$

$$= P^x \int_0^\zeta e^{-t} dt - P^x \int_0^{T_a \wedge \zeta} e^{-t} dt$$

$$= U^1(x) - V^1_a(x)$$

$$= \frac{u^1(x, a)}{u^1(a, a)} U^1(a),$$

where the last equality follows easily from (2.9). On the other hand, our hypothesis regarding the boundary $A$ implies that $g^1_1(A+) / g^1_2(A+) = 0$ (see [8; §4.6]). Thus

$$P^{x,a} \int_0^\infty e^{-t} \psi(X_t) dC_t = P^x \int_{T_a}^\infty e^{-t} \psi(X_t) dC_t$$

$$= P^x \int_0^a e^{-T_y} \psi(y) ds(y)$$

$$= \int_A [g^1_2(x) / g^1_2(y)] \psi(y) ds(y).$$

In (3.9) we have used the change of variables $t = T_y$ to obtain the second equality, and (2.1) to obtain the third. Now from the definition of the Wronskian $W$ we see that $d(g_1/g_2) = W \cdot [g_2]^{-2} ds$. Using this fact and the expression for $\psi$ provided by (3.6) we
may continue the computation begun in (3.9) with

\[
\begin{align*}
&= \int_A \left[ g_2(x)/g_2(y) \right] \int_{[y,B]} \left[ g_2(z)/g_2(y) \right] m(dz) ds(y) \\
&= \int_A \int_{[y,B]} \left[ g_2(x)g_2(z)/W \right] m(dz) d(g_1/g_2)(y) \\
&= \int_E \int_A \left[ g_1(z \wedge a)/g_2(z \wedge a) \right] \cdot \left[ g_2(x)g_2(z)/W \right] m(dz) \\
&= [u^1(x,a)/u^1(a,a)] U^1_1(a).
\end{align*}
\]

The last equality in (3.10) follows from (2.3) and (2.4). In view of (3.8)–(3.10), we see that \( K \) and \( \int_0^t \psi(X_s) c_\xi \) have the same finite 1-potential and so the proposition is proved. \( \square \)

For \( y \in E \) define a measure \( \overline{Q}^y \) on \( (\Omega, \mathcal{F}) \) by \( \overline{Q}^y(F) = \ast \overline{P}^{y,y}(F;\overline{k}_t) \), where \( \overline{k}_t \) is the killing operator on \( \overline{\Omega} \). Since \( \ast \overline{P}^{y,y}(\overline{R} \neq T_y \circ \pi) = 0 \), the first coordinate of \( \overline{X} \), namely \( (X_t : t > 0) \), is Markovian under \( \overline{Q}^y \), with \( (Q^y_t) \) as semigroup. Indeed, we claim that \( \psi(y) \pi(\overline{Q}^y) = n^1_\uparrow \), at least for \( ds \)-a.e. \( y \in E \). To verify this claim it suffices to compare the associated entrance laws.

(3.11) Lemma. Let \( f \) be a bounded positive Borel function on \( E \). Then for \( ds \)-a.e. \( y \in E \) we have

\[
\psi(y)\overline{Q}^y \int_0^\infty e^{-\alpha t} f(X_t) dt = W^\alpha f(y), \quad \forall \alpha > 0.
\]

Proof. Fix \( f \) as in the statement of the lemma and also fix \( \alpha > 0 \). For \( y \in E \) write

\[
\gamma(y) = \overline{Q}^y \int_0^\infty e^{-\alpha t} f(X_t) dt.
\]

As noted in the proof of (3.7), we have \( \int_0^\infty 1_M(t) dt = 0, \overline{P}^x.a.-a.s. \) for all \( (x,a) \in \overline{E} \). Thus,
for $x \in E$,

$$U^\alpha f(x) = \mathcal{P}^{x,x} \sum_{u \in G} e^{-\alpha u} \left( \int_0^\infty e^{-\alpha t} f(X_t) dt \right) \circ \theta_u$$

$$= \mathcal{P}^{x,x} \int_0^\infty e^{-\alpha u} \mathcal{P}^u \left( \int_0^\infty e^{-\alpha t} f(X_t) dt \right) dK_u$$

(3.13)

$$= \mathcal{P}^{x,x} \int_0^\infty e^{-\alpha u} \gamma(X_u) \psi(X_u) dG_u$$

$$= P^x \int_A e^{-\alpha T} \gamma(y) \psi(y) ds(y)$$

$$= \int_A \left[ g_2^\alpha(x)/g_2^\alpha(y) \right] \gamma(y) \psi(y) ds(y).$$

On the other hand, by (2.3) and (2.4), we have

(3.14) $$U^\alpha f(x) = \int_{[A,x]} [g_1^\alpha(x)g_2^\alpha(y)/W] f(y) m(dy) + \int_{[x,B]} [g_1^\alpha(y)g_2^\alpha(x)/W] f(y) m(dy).$$

If we equate the last line displayed in (3.13) with the right side of (3.14), divide the resulting identity by $g_2^\alpha(x)$, and then differentiate in $x$, we obtain

$$\left[ \int_{[x,B]} [g_2^\alpha(y)/g_2^\alpha(x)] f(y) m(dy) \right] ds(x) = \gamma(x) \psi(x) ds(x)$$

as measures on $E$, and the lemmas follows. \qed

(3.15) Corollary. For $ds$-a.e. $y \in E$, $\psi(y) \pi(Q^y) = n^+_y$ as measures on $(\Omega, \mathcal{F}^\circ)$.

Proof. As noted earlier, both $\psi(y)\pi(Q^y)$ and $n^+_y$ make the coordinate process on $(\Omega, \mathcal{F}^\circ)$ into a Markov process with transition semigroup $(Q^y_t)$. It follows from Lemma (3.11) that these measures have the same one-dimensional distributions (and consequently the same finite dimensional distributions) for $ds$-a.e. $y$. Since $\mathcal{F}^\circ = \sigma(X_u : u \geq 0)$ is countably generated, the corollary follows. \qed

Proof of Theorem (2.11). Let $Z \geq 0$ be $(\mathcal{F}_t)$-optional and let $F \geq 0$ be $\mathcal{F}^\circ$-measurable. By (3.2)(ii), the process $Z \circ \pi$ is $(\mathcal{F}_t)$-optional. We may now use (3.4), (3.7), and (3.15) to
compute
\[
P^x \sum_{u \in G} Z_u F(e^u) = \sum_{u \in G} Z_u \circ \pi F(\pi \circ \overline{K} \circ \overline{\theta}_u)
\]
\[
= \sum_{u \in G} Z_u \circ \pi F(\pi \circ \overline{K}) d\overline{K}_u
\]
\[
= \sum_{u \in G} Z_u \circ \pi F(\pi \circ \overline{K}) \psi(X_u) d\overline{C}_u
\]
\[
= P^x \int_0^\infty Z_u \circ \pi F(\pi \circ \overline{K}) dK_u
\]
\[
= P^x \int_0^\infty Z_u \circ \pi F(\pi \circ \overline{K}) d\overline{C}_u
\]

The proof of Theorem (2.11) is complete. □

4. Williams’ decomposition

In this section we use the Lévy system (2.12) to obtain a new proof (of a general version) of Williams’ decomposition [18] of a diffusion at its global minimum. A more “computational” proof of Williams’ theorem, based on the same idea used in the present paper, may be found in [3].

For simplicity we assume that \( \gamma := H_{\zeta_-} \) satisfies \( P^x(\gamma > A) = 1 \) for all \( x \in E \). We also assume that \( \rho := \inf\{t > 0 : X_t = \gamma\} \) satisfies \( P^x(\rho < \zeta) = 1 \) for all \( x \in E \). Then \( \rho \) is the unique time at which \( X \) takes its global minimum value \( \gamma \) (cf. [17]). Note that for \( x \geq y \) (both in \( E \)),

\[
P^x(\gamma > y) = P^x(T_y = +\infty).
\]

Define a function \( r \) on \( E \) by

\[
r(x) = \begin{cases} P^x(T_{x_0} < +\infty), & x \geq x_0, \\ \frac{[P^{x_0}(T_x < +\infty)]^{-1}}{x < x_0}, & \end{cases}
\]

where \( x_0 \in E^\circ \) is fixed but arbitrary. Clearly \( r \) is strictly positive and decreasing. Arguing as in [8; pp. 128–129] one may check that \( r \) is the unique positive decreasing solution of \( G r \equiv 0 \) on \( E^\circ \) which satisfies \( r(x_0) = 1 \) and the boundary condition at \( B \). Note that

\[
P^x(T_y < +\infty) = r(x)/r(y), \quad x > y.
\]

Before proceeding to the decomposition theorem we need a preliminary result.

(4.3) Lemma. For \( y \in E^\circ \) let \( S_y = \inf\{t > 0 : X_t = y\} \). Then

\[
n^+_y(S_y = +\infty) = -\frac{r^+(y)}{r(y)}, \quad \forall y \in E^\circ.
\]
(Recall that $r^+ = d^+ r / ds^+$.)

**Proof.** Let $q$ be an increasing solution of $Gq \equiv 0$ on $E^\circ$ such that $q$ is linearly independent of $r$. We assume that $q$ is normalized so that the Wronskian $q^+ r - r^+ q$ is identically 1.

Fix $a < b$ both in $E^\circ$ and let $v_{ab}$ denote the potential density (relative to $m$) of $X$ killed at time $T_a \wedge T_b$. One checks that for $x \leq y$,

$$v_{ab}(x, y) = v_{ab}(y, x) = \frac{D(a, x)D(y, b)}{D(a, b)},$$

where $D(x, y)$ is the determinant

$$\begin{vmatrix} q(y) & q(x) \\ r(y) & r(x) \end{vmatrix}.$$  

Note that $D(x, y) > 0$ if $x < y$. Now let $y \in ]a, b[$ and use (2.6) to compute

$$P_y^y(T_b < T_a) = P_y^y \sum_{u \in G(y)} 1_{\{u < T_a \wedge T_b\}} 1_{\{T_b < T_y\}} \cdot \theta_u$$

$$= P_y^y(L_{T_a \wedge T_b}^y) n_y(T_b < \zeta).$$

But clearly $P_y^y(T_b < T_a) = D(a, y)/D(a, b)$ while $P_y^y(L_{T_a \wedge T_b}^y) = v_{ab}(y, y)$, so that

$$n_y(T_b < \zeta) = [D(a, y)/D(a, b)]/v_{ab}(y, y) = D(y, b)^{-1}.$$

Finally,

$$n_y^+ (S_y = +\infty) = \lim_{x \downarrow y} n_y^+(T_x < +\infty, S_y = +\infty)$$

$$= \lim_{x \downarrow y} n_y^+(T_x < +\infty) P_x^y(T_y = +\infty)$$

$$= \lim_{x \downarrow y} n_y(T_x < \zeta)[1 - r(x)/r(y)]$$

$$= \lim_{x \downarrow y} \left[ \frac{1}{r(y)} \cdot \frac{r(y) - r(x)}{s(x) - s(y)} \cdot \frac{s(x) - s(y)}{D(y, x)} \right]$$

$$= -r(y)^{-1} \cdot r^+(y),$$

since $\lim_{x \downarrow y}[s(x) - s(y)]/D(y, x)$ is the reciprocal of the Wronskian $q^+ r - r^+ q \equiv 1$. 

Now define probability laws on $(\Omega, F^\circ)$ by

(4.4) $$P_y^x(F) = P_x^x(F \circ k_{T_y} | T_y < +\infty),$$

(4.5) $$P_y^+ (F) = n_y^+ (F | S_y = +\infty),$$
whenever \( x > y > A \). The coordinate process is a diffusion under any of these laws: \( P_y^\downarrow \) is the law of \( X \) started at \( x \), conditioned to converge to \( A \), and then killed at \( T_y \); \( P_y^\uparrow \) is the law of \( X \) started at \( y \) and conditioned to never return to \( y \). These conditionings are accomplished by means of the appropriate \( h \)-transforms. In particular, the associated infinitesimal generators are given by

\[
G_y^\downarrow f(z) = r(z)^{-1} G(fr)(z), \quad z > y;
\]

\[
G_y^\uparrow f(z) = r_y(z)^{-1} G(fr_y)(z), \quad z > y,
\]

where \( r_y(z) = 1 - r(z)/r(y) \).

We can now state the general version of Williams’ theorem. Recall that \( \gamma = H_{\zeta^-} \) and \( \rho = \inf\{t > 0 : X_t = \gamma\} \).

\((4.8)\) Theorem. (a) The joint law of \((\gamma, \rho, \zeta)\) is given by

\[
P^x(f(\gamma)e^{-\alpha\rho-\beta\zeta}) = \int_A^x [g_2^\alpha+\beta(x)/g_2^\alpha+\beta(y)] f(y) P_y^\uparrow(e^{-\beta\zeta}) \int r(y) ds(y).
\]

(b) For \( F, G \in bF^\circ \) and \( \psi \) bounded and Borel on \( E \),

\[
P^x(F \circ k_\rho \psi(\gamma) G \circ \theta_\rho) = P^x(P_{\gamma}^\downarrow(F) \psi(\gamma) P_{\gamma}^\uparrow(G)).
\]

\((4.11)\) Remark. The intuitive content of (4.10) is that the processes \((X_t : 0 \leq t < \rho)\) and \((X_{\rho+t} : 0 \leq t < \zeta - \rho)\) are conditionally independent under \( P^x \), given \( \gamma \); and that the conditional distributions, given that \( \gamma = y \), are \( P_y^\downarrow \) and \( P_y^\uparrow \) respectively.

Proof of (4.8). Define \( J(y, \omega) = 1_{\{S_y = +\infty\}}(\omega) \) and observe that \( \rho(\omega) = u \) if and only if \( u \in G(\omega) \) and \( J(X_u(\omega), e^u(\omega)) = 1 \). Thus, using (2.11),

\[
P^x(F \circ k_\rho \psi(\gamma) G \circ \theta_\rho) = P^x \sum_{u \in G} F \circ k_u \psi(X_u) G(e^u) J(X_u, e^u)
\]

\[
= \int_A^x \int P^x(F \circ T_{\gamma}; T_{\gamma} < +\infty) \psi(y) n_{\gamma}^\uparrow(G \cdot J(y, \cdot)) ds(y) \int P_{\gamma}^\downarrow(F) \psi(y) P_{\gamma}^\uparrow(G) P^x(T_{\gamma} < +\infty)n_{\gamma}^\uparrow(S_{\gamma} = +\infty) ds(y).
\]

Taking \( F = G = 1 \) in (4.12) we see that

\[
P^x(\gamma \in dy) = P^x(T_{\gamma} < +\infty)n_{\gamma}^\uparrow(S_{\gamma} = +\infty) ds(y).
\]
Now (4.13) substituted into the last line of (4.12) yields (4.10). To obtain (4.9) use (4.10) with 
\[ F = e^{-(\alpha + \beta)\zeta} \text{ and } G = e^{-\alpha\zeta}, \]
noting that \( F \circ k_{\rho} = e^{-(\alpha + \beta)\rho} \) and \( \zeta = \rho + \zeta_{\theta_{\rho}} (P^x\text{-a.s.)} \)
since \( \rho < \zeta \), \( P^x\text{-a.s.} \). Thus
\[
P^x(f(\gamma)e^{-\alpha\rho}e^{-\beta\zeta}) = P^x(f(\gamma)[e^{-(\alpha + \beta)\zeta} \circ k_{\rho} [e^{-\beta\zeta} \circ \theta_{\rho}])
= P^x(P^x_{\gamma}e^{-(\alpha + \beta)\zeta})f(\gamma)P^x_{\gamma}(e^{-\beta\zeta}).
\]
Formula (4.9) now follows since
\[
P^x_{\gamma}(e^{-(\alpha + \beta)\zeta}) = P^x(e^{-(\alpha + \beta)T_{\gamma}})/P^x(T_{\gamma} < +\infty)
= [g_2^{\alpha+\beta}(x)/g_2^{\alpha+\beta}(y)]/P^x(T_{\gamma} < +\infty),
\]
and since \( n^+_y(S_y = +\infty) = -r^+(y)/r(y) \) (Lemma (4.3)). \( \square \)

(4.14) Corollary. \( P^x(\rho \in dt, \gamma \in dy) = P^x(T_{\gamma} \in dt) \frac{-dr(y)}{r(y)}. \)

5. A local decomposition

Fix \( t > 0 \) and define
\[
\rho_t = \inf\{u > 0 : X_u = H_t\} \land t.
\]
Arguing as in [17] one can show that, almost surely on \( \{t < \zeta\} \), \( \rho_t \) is the unique \( u \in ]0, t[ \)
such that \( X_u = H_t \). Our purpose in this section is to describe the conditional distribution of \( \{X_u : 0 \leq u \leq t\} \) under \( P^b \), given that \( H_t = y, \rho_t = u, \) and \( X_t = x \). This conditional distribution has been computed by Imhof [6] for the Brownian motion (and closely related processes). The joint law of \( (H_t, \rho_t, X_t) \), again in the case of Brownian motion, has been found by Shepp [16]. See also [2, 9, 15] for related results.

We begin by computing the joint law of \( (H_t, \rho_t, X_t) \). Recall from [8; §4.11] that the first passage distribution \( P^x(T_y \in dv) \) has a density \( f(v; x, y) \) on \( ]0, +\infty[ \) relative to Lebesgue measure. Note that if we set \( F_{t,y}(x) = P^x(t < T_y < +\infty), \) then (see [8; p. 154])
\[
(5.1) \quad f(t; x, y) = -\frac{\partial}{\partial t} F_{t,y}(x) = GF_{t,y}(x), \quad x > y \in E, t > 0.
\]
Applying \( Q^y(z, dx) \) to both sides of (5.1) and integrating over \( x \in ]y, +\infty[ \cap E \) (making use of the relation \( Q^y_s \mathcal{G} = \mathcal{G}Q^y_s \) on \( ]y, +\infty[ \)), we obtain
\[
f(t + s; z, y) = \int_{]y, +\infty[} Q^y_s(z, dx) f(t; x, y).
\]
In other words, \((t, x) \mapsto f(t; x, y)\) is an exit law for the semigroup \((Q^y_x)\).

Next, recall from §4.11 that the semigroup \((Q^y_t)\) has a density \(q^y(t; x, z)\) (for \(x \wedge z > y\)) relative to the speed measure \(m(dz)\); we have \(q^y > 0\) on \([0, +\infty) \times [|y, B]|^2\) and \(q^y(t; x, z) = q^y(t; z, x)\). The entrance law for \(n^+_y\) can now be expressed as

\[
(5.2) \quad n^+_y(X_t \in dx) = q^+_y(t; x) \, m(dx),
\]

where

\[
(5.3) \quad q^+_y(t; x) = \int_{[y, B]} n^+_y(X_{t-u} \in dz) q^y(u; z, x).
\]

Substituting (5.2) into (5.3) and using the symmetry of \(q^y\), we see that

\[
(5.4) \quad q^+_y(t + u; x) = \int_{[y, B]} Q^y_u(x, dz) q^+_y(t; z).
\]

But (5.4) means that \((t, x) \mapsto q^+_y(t; x)\) is also an exit law for \((Q^y_t)\). Finally, using (3.6), if \(\alpha > 0\) and \(h\) is positive, measurable, and vanishes off \([y, B]\), we may compute

\[
\int_0^\infty e^{-\alpha t} \left( \int_E q^+_y(t; x) h(x) \, m(dx) \right) dt = W^\alpha h(y)
\]

\[
= \int_{[y, B]} \left[ g^0_2(x) / g^0_2(y) \right] h(x) \, m(dx)
\]

\[
= \int_{[y, B]} P^x(e^{-\alpha T^y}) h(x) \, m(dx)
\]

\[
= \int_0^\infty e^{-\alpha t} \left( \int_E f(t; x, y) h(x) \, m(dx) \right) dt.
\]

By Laplace inversion,

\[
(5.5) \quad q^+_y(t; x) = f(t; x, y)
\]

for \(dt \otimes dm\)-a.e. \((t, x)\) in \([0, +\infty) \times |y, B]\). Since both sides of (5.5) are exit laws (and so excessive functions in time-space), it follows that (5.5) holds identically for \(t > 0\), \(y \in E^\circ\) and \(E \ni x > y\). See §3 of [5], and especially (3.17) therein.

(5.6) Proposition. For \(b \in E, x \in E, u \in [0, t[,\) and \(y \in [A, b \wedge x[\),

\[
(5.7) \quad P^b_H(t \in dy, \rho_t \in du, X_t \in dx) = f(u; b, y) f(t - u; x, y) ds(y) \, du \, m(dx).
\]
Proof. Let \( g, h, \) and \( \phi \) be bounded positive Borel functions on \( \mathbb{R} \), vanishing off \( E, E, \) and \( ]0, t[ \), respectively. Put \( J(v, y, \omega) = 1_{\{S_\omega > v\}}(\omega) \). Using (2.11) we have, since \( u = \rho_t(\omega) \) if and only if \( u \in G(\omega) \) and \( J(t - u, X_u(\omega), e^u(\omega)) = 1, \)

\[
P^b(g(H_t)\phi(\rho_t)h(X_t)) = P^b \sum_{u \in G} g(X_u)\phi(u)h(X_{t-u} \circ \theta_u)J(t - u, X_u, e^u)
= P^b \sum_{u \in G} g(X_u)\phi(u)h(e^u_{t-u})J(t - u, X_u, e^u)
= \int_A \int_{\Omega} g(y)\phi(T_y(\omega))n^\uparrow_y(h(X_{t-T_y(\omega)})P^b(d\omega) ds(y)
= \int_A \int_{]0, t]} g(y)\phi(u)n^\uparrow_y(h(X_{t-u})f(u; b, y) du ds(y).
\]

The proposition now follows from (5.2) and (5.5).

Our local decomposition of \( X \) will be expressed in terms of certain “bridges” of \( X \). First, let \( \hat{K}^{y, \ell, x} \) denote the \( h \)-transform of \( P^\uparrow_y \) by means of the time-space harmonic function

\[
h_{\ell, x}(t, z) = q^y(\ell - t; z, x) \left[ \frac{r(y) - r(x)}{r(y) - r(z)} \right] 1_{]0, \ell[}(t),
\]

where \( \ell > 0 \) and \( x > y \). Straightforward computations show that the absolute probabilities and transition probabilities under \( \hat{K}^{y, \ell, x} \) are given by

\[
\hat{K}^{y, \ell, x}(X_t \in dz) = \frac{q^y(\ell - t; z, x)f(t; y, z)}{f(\ell; y, x)} m(dz),
\]

and

\[
\hat{K}^{y, \ell, x}(X_{t+v} \in dw|X_t = z) = \frac{q^y(v; z, w)q^y(\ell - t - v; w, x)}{q^y(\ell - t; z, x)} m(dw).
\]

Moreover (cf. [15])

\[
(5.8) \quad \hat{K}^{y, \ell, x}(\zeta = \ell, X_{\zeta-} = x) = 1,
\]

\[
\int_{[y, B]} \hat{K}^{y, \ell, x}(F) P^\uparrow_y(X_\ell \in dx) = P^\uparrow_y(F \circ k_\ell).
\]

Thus, \( \{\hat{K}^{y, \ell, x} : x \in [y, B]\} \) is a regular version of the conditional probabilities \( F \mapsto P^\uparrow_y(F \circ k_\ell|X_\ell = x) \).

Now let \( K^{x, \ell, y} \) denote the image of \( \hat{K}^{y, \ell, x} \) under the time-reversal mapping, taking \( \omega \) to the path \( \gamma_\ell \omega \) defined by

\[
(\gamma_\ell \omega)(t) = \begin{cases} 
\omega(\ell - t), & 0 < t < \ell \\
\omega(\ell -), & t = 0 \\
\Delta, & t \geq \ell.
\end{cases}
\]
Like $\hat{K}^{y,\ell,x}$, $K^{x,\ell,y}$ is the law of a non-homogeneous Markov diffusion; from (5.8) we see that

$$K^{x,\ell,y}(X_0 = x, \zeta = \ell, X_{\zeta -} = y) = 1.$$ 

Moreover, computation of finite dimensional distributions shows that the transition probabilities for $K^{x,\ell,y}$ are given by

$$(5.9) \quad K^{x,\ell,y}(X_{t+u} \in dw|X_t = z) = \frac{q^y(v; z, w)f(\ell - t - v; w, y)}{f(\ell - t; z, y)}.$$ 

It follows that $\{K^{x,\ell,y}: \ell > 0\}$ is a regular version of the conditional probabilities

$$P^{x\downarrow}_y(\cdot|\zeta = \ell).$$

(5.10) **Theorem.** Let $b \in E$. Then under $P^b$ the path fragments $(X_t: 0 \leq t < \rho_t)$ and $(X_{\rho_t+u}: 0 \leq u < t - \rho_t)$ are conditionally independent given $(H_t, \rho_t, X_t)$ on $\{X_t \in E\} = \{t < \zeta\}$. Moreover, given that $H_t = y$, $\rho_t = u$, and $X_t = x$ ($0 < u < t$, $y > x$), the above processes have conditional laws $K^{b,u,y}$ and $\hat{K}^{y,t-u,x}$ respectively.

The proof of (5.10) is similar to that of (4.8) and is left to the interested reader as an exercise.

6. **A result of W. Vervaat**

In this last section we use the decomposition of §5 to give a new proof of a result of Vervaat [17] which concerns a path transformation carrying Brownian bridge into Brownian excursion.

In this section we take the basic process $(X_t, P^x)$ to be standard Brownian motion on $\mathbb{R}$. Let $P_0$ denote the law of Brownian bridge; namely,

$$P_0(F) = P^0(F|X_1 = 0), \quad F \in \mathcal{F}_1.$$ 

Under $P_0$ the coordinate process is centered Gaussian with continuous paths, $X_0 = 0$, and covariance $P_0(X_uX_t) = u(1 - t)$ for $0 \leq u \leq t \leq 1$.

Next, Let $P_+$ denote the law of scaled Brownian excursion. Under $P_+$ the coordinate process $(X_t: 0 \leq t \leq 1)$ is a non-homogeneous Markov diffusion with absolute probabilities

$$(6.1)(a) \quad P_+(X_t \in dx) = \frac{2x^2}{\sqrt{2\pi t^3(1-t)^3}}e^{-x^2/2t(1-t)}.$$ 

16
and transition probabilities

\[(6.1)(b) \quad P_+(X_{t+v} \in dy | X_t = x) = p(v; y-x) \left( \frac{1-t}{1-t-v} \right)^{3/2} \frac{y \exp(-y^2/2(1-t-v))}{x \exp(-x^2/2(1-v))}, \]

where \(p(v; x) = (2\pi v)^{-1/2}e^{-x^2/2v}\) is the Gauss kernel, and \(0 < t < t+v < 1, 0 < x, y.\) Also, \(P_+ (\zeta = 1) = P_+ (X_t > 0, \forall t \in ]0,1[) = P_+ (X_0 = X_1- = 0) = 1.\)

Computation of finite dimensional distributions now shows that the following identities hold:

\[k_u(P_+(\cdot|x_u = y)) = \hat{K}^{0,u,y},\]
\[\theta_u(P_+(\cdot|X_u = y)) = K^{y,1-u,0},\]

where \(\hat{K}^{0,u,y}\) and \(K^{y,1-u,0}\) are as defined in the last section, the basic process being standard Brownian motion.

Now let \(\Omega_0 = \{\omega \in \Omega : \omega(0) = \omega(1-) = 0, \zeta (\omega) = 1\}\) and \(\overline{\Omega} = \Omega_0 \times ]0,1[.\) Define a map \(\Phi: \overline{\Omega} \to \Omega_0\) by

\[\Phi(\omega, u)(t) = \Phi_u(\omega)(t) = \begin{cases} \omega(u + t) - \omega(u), & 0 \leq t < 1 - u, \\ \omega(u + t - 1) - \omega(u), & 1 - u \leq t < 1. \end{cases}\]

In the following we regard \(P_+\) and \(P_0\) as measures on \(\Omega_0\). Define \(\mathcal{P}\) on \(\Omega_0\) by \(\mathcal{P} = P_+ \otimes \lambda\), where \(\lambda\) is Lebesgue measure on \([0,1[.\) Set \(U(\omega, u) = u\) and \(V = 1 - U\) on \(\Omega_0\).

**Proposition.** The joint law of \((\Phi, V, X_U)\) under \(\mathcal{P}\) is the same as the joint law of \((\omega, \rho_1, -H_1)\) under \(P_0\).

**Proof.** For paths \(\omega\) and \(\omega'\), and \(t \in ]0,1[\) let \(\omega/t/\omega'\) denote the spliced path

\[(\omega/t/\omega')(u) = \begin{cases} \omega(u), & 0 \leq u < t, \\ \omega'(u-t), & t \leq u < 1, \end{cases}\]

and let \(\tau_y \omega(t) = \omega(t) - y.\) Let \(p_+(u, y) = P_+(X_u \in dy)/dy.\) Note that if \(\omega(u) = \omega'(0),\) then

\[\Phi(\omega/u/\omega', u) = (\tau_y \omega'/1 - u/\tau_y \omega),\]

where \(0 < u < 1\) and \(y = \omega(u)\). Thus,

\[\mathcal{P}(F \circ \Phi \psi(V, X_U)) = \int_0^1 P_+(F \circ \Phi_u \psi(1-u, X_u)) \, du\]

\[= \int_0^1 \int_0^\infty \int_\Omega \int F(\Phi_u(\omega/u/\omega')) \psi(1-u, y) \hat{K}^{0,u,y}(dw) K^{y,1-u,0}(d\omega') p_+(u, y) \, dy \, du\]

\[= \int_0^1 \int_0^\infty \int_\Omega \int F(\tau_y \omega'/1 - u/\tau_y \omega) \psi(1-u, y) \hat{K}^{0,u,y}(dw) K^{y,1-u,0}(d\omega') p_+(u, y) \, dy \, du\]

\[= \int_0^1 \int_0^\infty \int_\Omega \int F(\omega'/1 - u/\omega) \psi(1-u, y) K^{0,1-u,-y}(d\omega') K^{-y,1-u,0}(d\omega) p_+(u, y) \, dy \, du\]

\[= P_0(F \cdot \psi(\rho_1, -H_1)).\]
(6.3) Corollary. (Vervaat): Define a transformation $\Psi : \Omega_0 \to \Omega_0$ by
\[
(\Psi \omega)(t) = \begin{cases} 
\omega(\rho_1(\omega) + t) - H_1(\omega), & 0 \leq t < 1 - \rho_1(\omega), \\
\omega(\rho_1(\omega) + t + 1) - H_1(\omega), & 1 - \rho_1(\omega) \leq t < 1.
\end{cases}
\]
Then $\Psi(P_0) = P_+$. That is, the $P_0$-law of $(X_t \circ \Psi : 0 \leq t < 1)$ is $P_+$.

Proof. It is easy to check that $\Psi \circ \Phi(\omega, u) = \omega$ for all $(\omega, u) \in \Omega$. Using Proposition (6.2),
\[
P_0(F \circ \Psi) = \overline{P}(F \circ \Pi)
= \overline{P}(F \circ \pi_1)
= P_+(F),
\]
where $\pi_1 : (\omega, u) \to \omega$. □

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