CONTROLLABILITY PROBLEMS FOR THE HEAT EQUATION
ON A HALF-AXIS WITH A BOUNDED CONTROL IN THE
NEUMANN BOUNDARY CONDITION

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Abstract. In the paper, the problems of controllability and approximate controllability are studied for the control system
\[ w_t = w_{xx}, \quad w_x(0, \cdot) = u, \quad x > 0, \quad t \in (0, T), \]
where \( u \in L^\infty(0, T) \) is a control. It is proved that each initial state of the system is approximately controllable to each target state in a given time \( T \). A necessary and sufficient condition for controllability in a given time \( T \) is obtained in terms of solvability of a Markov power moment problem. It is also shown that there is no initial state which is null-controllable in a given time \( T \).

Orthogonal bases are constructed in \( H^1 \) and \( H_1 \). Using these bases, numerical solutions to the approximate controllability problem are obtained. The results are illustrated by examples.

1. Introduction. In the paper, we study controllability problems for the heat equation on a half-axis. Note that these problems for the heat equation on domains bounded with respect to spatial variables were investigated rather completely in a number of papers (see, e.g., \[3, 5, 12, 16, 15, 17, 18\] and references therein). However, controllability problems for the heat equation on domains unbounded with respect to spatial variables have not been fully studied.

Consider the heat equation on a half-axis
\[ w_t = w_{xx}, \quad x \in (0, +\infty), \quad t \in (0, T), \]
controlled by the Neumann boundary condition
\[ w_x(0, \cdot) = u, \quad t \in (0, T), \]

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under the initial condition
\[ w(\cdot, 0) = w^0, \quad x \in (0, +\infty), \tag{3} \]
and the steering condition
\[ w(\cdot, T) = w^T, \quad x \in (0, +\infty), \tag{4} \]
where \( T > 0, \ u \in L^\infty(0, T) \) is a control. This problem is considered in spaces of Sobolev type (see details in Section 2).

Control problems for the heat equation were studied in unbounded domains in \([1, 2, 7, 13, 14, 4]\). In particular, in \([14]\), the null-controllability problem for equation (1) controlled by the Dirichlet boundary condition
\[ w(0, \cdot) = u, \quad t \in (0, T), \tag{5} \]
under initial condition (3) with \( L^2 \)-control (\( u \in L^2(0, T) \)) was studied in a weighted Sobolev space of negative order. Rewriting the control system in similarity variables and developing the solutions in the Fourier series with respect to the orthonormal basis \( \{\phi_m\}_{m=1}^\infty \), the authors reduced the control problem to the moment problem
\[ \int_0^S e^{ms} \bar{u}(s) \, ds = \alpha_m, \quad m = 1, \infty, \]
where \( \phi_m(y) = C_m \mathcal{H}_{2m-1}(y/2)e^{-y^2/4}, \mathcal{H}_{2m-1} \) is the Hermite polynomial, \( C_m \in \mathbb{R} \), \( \alpha_m \) is determined by the Fourier coefficient of the initial state of reduced control problem, \( m = 1, \infty \). A solution to the control problem determines a solution to the moment problem and vice versa. Moreover, it was proved that the moment problem admits an \( L^2 \)-solution iff \( \alpha_m \) grows exponentially as \( m \to \infty \). In particular, it was proved that if \( \alpha_m = O(e^{m^\delta}) \) as \( m \to \infty \) for all \( \delta > 0 \), then the initial state associated with \( \{\alpha_m\}_{m=1}^\infty \) cannot be steered to the origin by \( L^2 \)-control. In \([14]\), it was also asserted that each initial state is approximately null-controllable in a given time \( T > 0 \) by \( L^2 \)-controls.

In \([7]\), reachability and controllability properties for control system (1),(3), (5) were studied in \( L^2(0, +\infty) \) with \( L^\infty \)-control (\( u \in L^\infty(0, T) \)). Note that \( L^\infty \)-controls allow us to consider initial states and solutions of the control system in the Sobolev space of order zero (\( L^2(0, +\infty) \)) in contrast to \([14]\), where the system was studied in a weighted Sobolev space of negative order as a result of using \( L^2 \)-controls. For a given constant \( L > 0 \), the reachability problem with controls bounded by \( L \) (\( \|u\|_{L^\infty(0, T)} \leq L \) was reduced to an infinite Markov power moment problem. Moreover, it was proved that the solutions to the finite Markov power moment problem give us controls bounded by \( L \) and solving the approximate reachability problem.

It was also proved that each end state \( w^T \in L^2(0, +\infty) \) is approximately reachable from the origin, using controls \( u \in L^\infty(0, T) \), in a given time \( T > 0 \). Using the results of \([14]\), it was proved that there is no initial state \( w^0 \in L^2(0, +\infty) \) that is null-controllable in a given time \( T > 0 \) by using controls \( u \in L^\infty(0, T) \). In addition, from the result on approximate reachability of (1),(3), (5), it was proved that each initial state \( w^0 \in L^2(0, +\infty) \) is approximately controllable to any end state \( w^T \in L^2(0, +\infty) \), using controls \( u \in L^\infty(0, T) \), in a given time \( T > 0 \).

In the present paper, we extend the results of \([7]\) to the heat equation controlled by the Neumann boundary condition (see (1), (2)). Analogs of all results of paper \([7]\) are obtained here.
In Section 2, considering the even extension with respect to \( x \) of the initial state, the end state, and the solution to system (1)–(4), we reduce this system to control system

\begin{equation}
W_t = W_{xx} - 2u\delta, \quad x \in \mathbb{R}, \ t \in (0, T),
\end{equation}

\begin{equation}
W(\cdot, 0) = W^0, \quad x \in \mathbb{R},
\end{equation}

\begin{equation}
W(\cdot, T) = W^T, \quad x \in \mathbb{R},
\end{equation}

in the subspace \( \hat{H}^{-1} \) of all even functions from \( H^{-1} \), where \( W^0 \in \hat{H}^1, W^T \in \hat{H}^1, \delta \) is the Dirac distribution with respect to \( x \). Here \( H^m \) is the Sobolev space of functions \( f \in L^2(\mathbb{R}) \) such that \( f^{(k)} \in L^2(\mathbb{R}), k = 0, m \), and \( H^{-m} = (H^m)^*, m = 0, 1 \) (see details in Section 2). Further, control system (6), (7) is considered instead of control system (1)–(3). In the same section, we formulate the problems under consideration and the main results of the paper. First, properties of the reachability set \( R_P(0) \) are studied for system (6), (7). Then, using them, controllability properties are obtained in Subsection 2.1. A necessary condition for controllability is obtained in Theorem 2.3. In Section 2, the controllability problem is reduced to an infinite Markov power moment problem (Theorems 2.4 and 2.9). In contrast to [14], we use the analyticity of the Fourier transform of the initial state of control problem and its expansion in a power series to obtain this moment problem. Moreover, it is proved that the solutions to the finite Markov power moment problem give us controls bounded by \( L > 0 \) and solving the approximate controllability problem (Theorems 2.5 and 2.13). Theorems 2.3–2.5 are proved in Section 3. The proofs of these theorems are analogous to the proofs of appropriate theorems of paper [7]. The results of Theorems 2.5 and 2.13 are illustrated by Examples 2 and 3 in Section 8. We also obtain Theorems 2.6 and 2.14 where it is asserted that each initial state \( W^0 \in \hat{H}^1 \) is approximately controllable to each target state \( W^T \in \hat{H}^1 \) in a given time \( T > 0 \), using controls \( u \in L^\infty(0, T) \). Theorem 2.6 is proved in Section 4. The proof of this theorem essentially differs from the proof of Theorem 5.2 of [7]. To prove Theorem 5.2 of [7], we use the orthogonal basis of \( L^2(\mathbb{R}) \) constructed by the Hermit polynomials multiplied by the exponents \( e^{-x^2/(4T)} \) and the basis in \( \hat{H}^0 \) where \( \hat{H}^0 \) is the subspace of all odd functions from \( H^0 = L^2(\mathbb{R}) \). To prove Theorem 2.6, first we prove the completeness of the system \( \{\sigma^n e^{-T\sigma^2}\}_{n=0}^\infty \) in the space \( H_1 \) (see Lemma 4.1) where \( H_1 \) is the space \( L^2(\mathbb{R}) \) with the weight \( 1 + \sigma^2 \). Then we approximate each function \( \sigma^{2n} e^{-T\sigma^2} \) by using piecewise constant controls from \( L^\infty(0, T) \). Finally, approximating a function by a finite sums of the form \( \sum_{n=0}^N \nu_n \sigma^{2n} e^{-T\sigma^2} \) and applying the inverse Fourier transform, we prove the theorem. In Section 5, we construct orthogonal bases in the spaces \( H^1 \) and \( H_1 \) in order to obtain numerical solution to the controllability problem. In Section 6, using the basis obtained in Section 5 and Theorem 3.1 from [14], we prove that there is no initial state \( W^0 \in \hat{H}^1 \) that is null-controllable in a given time \( T > 0 \) by using controls \( u \in L^\infty(0, T) \). In Section 7 for a given initial state \( W^0 \) and a given end state \( W^T \), we expand \( f = W^T - \frac{1}{2\sqrt{\pi}T} e^{-x^2/(4T)} \ast W^0 \) in Fourier series with respect to the basis obtained in Section 5. For each \( N = 0, \infty \), we find a sequence of piecewise constant controls \( \{u_p^n\}_{n=1}^\infty \) that solves the approximate controllability problem for the initial state \( W^0 = 0 \) and the end state \( (-1)^n (e^{-x^2/2})^{(2n)} \). Then we find the controls \( u_{N,t} = \sum_{p=0}^N U_p^n u_p^t, N \in \mathbb{N}, \) solving the approximate controllability problem (6)–(8) for the given initial state \( W^0 \) and the given end state \( W^T \), where
$U_p^N \in \mathbb{R}$ is a constant, $p = 0, N$, $l$ depends on $N$, $N \in \mathbb{N}$. The constants $U_p^N \in \mathbb{R}$, $p = 0, N$, $N \in \mathbb{N}$, are determined by the Fourier coefficients of $f$. The results of this section are illustrated by Example 4 in Section 8.

2. Problem formulation and main results. Let us give definitions of the spaces used in the paper. For $m = 0, 1$, denote

$$H^m = \left\{ \varphi \in L^2(\mathbb{R}) \mid \forall k = 0, m \varphi^{(k)} \in L^2(\mathbb{R}) \right\}$$

with the norm

$$\|\varphi\|^m = \left( \sum_{k=0}^{m} \binom{m}{k} \left( \|\varphi^{(k)}\|_{L^2(\mathbb{R})} \right)^2 \right)^{1/2}, \quad \varphi \in H^m,$$

and $H^{-m} = (H^m)^*$ with the strong norm $\|\cdot\|^{-m}$ of the adjoint space. We have $H^0 = L^2(\mathbb{R}) = (H^0)^* = H^{-0}$.

For $m = 0, 1$, denote

$$\hat{H}^m_+ = \left\{ \varphi \in L^2(0, +\infty) \mid \forall k = 0, m \varphi^{(k)} \in L^2(0, +\infty) \right\}$$

with the norm

$$\|\varphi\|^m_+ = \left( \sum_{k=0}^{m} \binom{m}{k} \left( \|\varphi^{(k)}\|_{L^2(0, +\infty)} \right)^2 \right)^{1/2}, \quad \varphi \in \hat{H}^m_+,$$

and its subspace

$$H^m_\oplus = \left\{ \varphi \in L^2(0, +\infty) \mid \left( \forall k = 0, m \varphi^{(k)} \in L^2(0, +\infty) \right) \right\}$$

with the norm

$$\|\varphi\|^m_\oplus = \|\varphi\|^m_+, \quad \varphi \in H^m_\oplus,$$

and $\hat{H}^{-m}_+ = (\hat{H}^m_+)^*$, $H^{-m}_\oplus = (H^m_\oplus)^*$ with the strong norms $\|\cdot\|^{-m}_+, \|\cdot\|^{-m}_\oplus$ of the adjoint space respectively. We have

$$H^1_\oplus \subset \hat{H}^1_+ \subset H^0_\oplus = \hat{H}^0_+ = L^2(0, +\infty) = \hat{H}^{-0} = H^{-0} \subset \hat{H}^{-1}_+ \subset H^{-1}_\oplus.$$

For $n = -1, 1$, denote

$$H_n = \left\{ \psi \in L^2_{\text{loc}}(\mathbb{R}) \mid (1 + \sigma^2)^{n/2} \psi \in L^2(\mathbb{R}) \right\}$$

with the norm

$$\|\psi\|_n = \left\| (1 + \sigma^2)^{n/2} \psi \right\|_{L^2(\mathbb{R})}.$$

Evidently, $H_{-n} = (H_n)^*$.

By $\langle f, \varphi \rangle$, we denote the value of a distribution $f$ on a test function $\varphi$. In particular, if $f \in H^0 = H_0 = L^2(\mathbb{R})$ and $g \in H^0 = H_0 = L^2(\mathbb{R})$, then

$$\langle f, g \rangle = \langle f, g \rangle^0 = \langle f, g \rangle_0$$

is the inner product in $L^2(\mathbb{R})$. In $H^1$ and $H_1$, we consider the inner products

$$\langle f, g \rangle^1 = \langle f, g \rangle^0 + \langle f', g \rangle^0, \quad f, g \in H^1.$$
and

$$\langle F, G \rangle_1 = \langle F, G \rangle_0 + \langle \sigma F, \sigma G \rangle_0 = \left\langle \sqrt{1 + \sigma^2}F, \sqrt{1 + \sigma^2}G \right\rangle_0, \quad F, G \in H_1.$$

By $\mathcal{F} : H^{-1} \to H_{-1}$, denote the Fourier transform operator with the domain $H^{-1}$. This operator is an extension of the classical Fourier transform operator which is an isometric isomorphism of $L^2(\mathbb{R})$. The extension is given by the formula

$$\langle \mathcal{F}f, \varphi \rangle = \langle f, \mathcal{F}^{-1}\varphi \rangle, \quad f \in H^{-1}, \varphi \in H_1.$$

This operator is an isometric isomorphism of $H^m$ and $H_m$, $m = -1, 1$ [8, Chap. 1].

A distribution $f \in H^{-1}$ (or $H_{-1}$) is said to be odd if $\langle f, \varphi(\cdot) \rangle = -\langle f, \varphi(-\cdot) \rangle$, $\varphi \in H^1$ (or $H_1$ respectively). A distribution $f \in H^{-1}$ (or $H_{-1}$) is said to be even if $\langle f, \varphi(\cdot) \rangle = \langle f, \varphi(-\cdot) \rangle$, $\varphi \in H^1$ (or $H_1$ respectively).

By $\tilde{H}^n$ (or $\tilde{H}_n$), denote the subspace of all odd distributions in $H^n$ (or $H_n$), $n = -1, 1$. Evidently, $\tilde{H}^n$ (or $\tilde{H}_n$) is a closed subspace of $H^n$ (or $H_n$), $n = -1, 1$. By $\bar{H}^n$ (or $\bar{H}_n$), denote the subspace of all even distributions in $H^n$ (or $H_n$), $n = -1, 1$. Evidently, $\bar{H}^n$ (or $\bar{H}_n$) is a closed subspace of $H^n$ (or $H_n$), $n = -1, 1$.

Obviously, if $f_+$ is the restriction of a function $f \in \tilde{H}^m$ to $[0, +\infty)$, then $f_+ \in \bar{H}^m_+$, $m = 0, 1$. If $f$ is the even extension of $f_+ \in \bar{H}^m_+$, then $f \in \tilde{H}^m$, $m = 0, 1$. For $m = 0$, this assertion is obvious. Since $\bar{H}^1_+ \subset C[0, +\infty)$ (see, e. g., [8, Chap. 2]), we can see that it is also holds for $m = 1$.

**Remark 1.** Note that, for $\varphi \in H^m_{\mathbb{D}}$, its even extension $\hat{\varphi}(x) = \begin{cases} \varphi(x) & \text{if } x \geq 0 \\ \varphi(-x) & \text{if } x < 0 \end{cases}$ belongs to $\bar{H}^m$, $m = 0, 1$. The converse assertion is true only for $m = 0$, and it is not true for $m = 1$. That is why the even extension of a distribution $f \in \tilde{H}_1^{-1}$ may not belong to $\bar{H}^{-1}$. However, the following theorem holds.

**Theorem 2.1** ([6]). Let $f \in H_1^2$ and there exist $f'(0^+) \in \mathbb{R}$. Then $f'' \in H_1^{-1}$ can be extended to the even distribution $F$, and $F \in \bar{H}^{-1}$. This distribution is given by the formula

$$F = \hat{f}'' - 2f'(0^+)\delta,$$

where $\hat{f}(x) = f(x)$ if $x \geq 0$ and $\hat{f}(x) = f(-x)$ otherwise, $\delta$ is the Dirac distribution.

Consider control problem (1)–(4) where $(\frac{d}{dt})^m w : [0, T] \to \bar{H}^{1-2m}, m = 0, 1$, $w^0, w^T \in \bar{H}^1_+$. Let $W^0$ and $W(\cdot, t)$ be the even extensions of $w^0$ and $w(\cdot, t)$ with respect to $x$, $t \in [0, T]$. If $w$ is a solution to problem (1)–(3), then $W$ is a solution to control problem (6), (7) according to Theorem 2.1, where $W^0 \in \bar{H}^1_+, (\frac{d}{dt})^m W : [0, T] \to \bar{H}^{1-2m}, m = 0, 1$. The converse assertion is also true: if $W$ is a solution to (6), (7), then its restriction $w = W|_{(0, +\infty)}$ is a solution to (1)–(3), and

$$W_x(0^+, t) = u(t) \quad \text{a.e. on } [0, T]$$

(see below (16)). Evidently, (4) holds iff (8) holds, where $W^T$ is the even extension of $w^T$.

Consider control problem (6), (7). Denote $V^0 = \mathcal{F}W^0$ and $V(\cdot, t) = \mathcal{F}_{x \to \sigma}W(\cdot, t)$, $t \in [0, T]$. We have

$$V_t = -\sigma^2 V - \sqrt{\frac{2}{\pi}} u, \quad \sigma \in \mathbb{R}, \ t \in (0, T),$$

(11)
\[ V(\cdot,0) = V^0, \quad \sigma \in \mathbb{R}. \] (12)

Therefore,
\[ V(\sigma, t) = e^{-t\sigma^2} V^0(\sigma) - \sqrt{\frac{2}{\pi}} \int_0^t e^{-(t-\xi)\sigma^2} u(\xi) \, d\xi, \quad \sigma \in \mathbb{R}, \ t \in [0,T], \] (13)
is the unique solution to (11), (12). Since \( u \in L^\infty(0,T) \), we have
\[ |V(\sigma, t)| \leq |V^0(\sigma)| + \sqrt{\frac{2}{\pi}} \| u \|_{L^\infty(0,T)} \frac{1 - e^{-t\sigma^2}}{\sigma^2}, \quad \sigma \in \mathbb{R}, \ t \in [0,T]. \] (14)

Hence, \( V(\cdot, t) \in \tilde{H}_1, \ t \in [0,T] \). From (13), we obtain
\[ W(x,t) = e^{-\frac{x^2}{4tT}} W^0(x) - \sqrt{\frac{2}{\pi}} \int_0^t e^{\frac{x^2}{2\xi}} u(t-\xi) \sqrt{2\pi} \, d\xi, \quad x \in \mathbb{R}, \ t \in [0,T], \] (15)
and \( W(\cdot,t) \in \tilde{H}^1, \ t \in [0,T] \). Since for any \( t \in (0,T] \) the function \( e^{-\frac{x^2}{4t\sigma}} W^0(x) \) is even and belongs to \( C^\infty(\mathbb{R}) \), we obtain
\[ \frac{\partial}{\partial x} e^{-\frac{x^2}{4t\sigma}} W^0(x) \bigg|_{x=0} = 0. \]

Setting \( \mu = \frac{|x|}{2\sqrt{\xi}} \), we get
\[ \frac{\partial}{\partial x} \int_0^t e^{\frac{x^2}{2\xi}} u(t-\xi) \sqrt{2\pi} \, d\xi = -x \int_0^t e^{\frac{x^2}{2\xi}} \frac{u(t-\xi)}{(2\xi)^{3/2}} \, d\xi \]
\[ = -\sqrt{2} \text{sgn} \, x \int_{|x|/(2\sqrt{t})}^\infty e^{-\mu^2} u(t - \frac{x^2}{4\mu^2}) \, d\mu. \]

According to Lebesgue’s dominated convergence theorem, we get
\[ W_x(0^+, t) = \frac{2}{\sqrt{\pi}} u(t) \int_0^\infty e^{-\mu^2} = u(t) \quad \text{a.e. on } [0,T], \] (16)
i.e., (10) holds.

Thus, control systems (1)–(3) and (6), (7) are equivalent. Therefore, basing on this reason, we will further consider control system (6), (7) instead of original system (1)–(3).

For a state \( W^0 \in \tilde{H}^1 \) by \( \mathcal{R}_T(W^0) \), denote the set of all states \( W^T \in \tilde{H}^1 \) for which there exists a control \( u \in L^\infty(0,T) \) such that there exists a unique solution to (6)–(8). In other words, \( \mathcal{R}_T(W^0) \) is the set of states \( W^T \in \tilde{H}^1 \) that are reachable from \( W^0 \in \tilde{H}^1 \).

According to (15), we have
\[ \mathcal{R}_T(W^0) = \left\{ W^T \in \tilde{H}^1 \mid \forall v \in L^\infty(0,T) \right\} \]
\[ W^T = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} e^{\frac{x^2}{4T}} W^0 - \sqrt{\frac{2}{\pi}} \int_0^T e^{-\frac{x^2}{2\xi}} \frac{v(\xi)}{\sqrt{2\pi}} \, d\xi, \] (17)
in particular,
\[ \mathcal{R}_T(0) = \left\{ W^T \in \tilde{H}^1 \mid \forall v \in L^\infty(0,T) \right\} \]
\[ W^T = -\sqrt{\frac{2}{\pi}} \int_0^T e^{-\frac{x^2}{2\xi}} \frac{v(\xi)}{\sqrt{2\pi}} \, d\xi. \] (18)
First, we study $\mathcal{R}_T(0)$. Denote also

$$\mathcal{R}^L_T(0) = \left\{ W^T \in \hat{H}^1 \mid \exists v \in L^\infty(0, T) \left( \| v \|_{L^\infty(0, T)} \leq L \right. \right.$$ 

and $W^T = -\frac{\sqrt{2}}{\pi} \int_0^T e^{-\frac{\xi^2}{\pi T}} v(\xi) d\xi \}$. \hfill (19)

Evidently, the following theorem holds.

**Theorem 2.2.** Let $T > 0$. We have

(i) $\mathcal{R}_T(0) = \bigcup_{L > 0} \mathcal{R}^L_T(0)$;

(ii) $\mathcal{R}^L_T(0) \subset \mathcal{R}^{L'}_T(0)$, $L \leq L'$;

(iii) $f \in \mathcal{R}^L_T(0)$ $\Leftrightarrow$ $L f \in \mathcal{R}^L_T(0)$, $L > 0$;

(iv) $f^1 \in \mathcal{R}^L_T(f^0) \Leftrightarrow f^1 - \frac{1}{2\sqrt{\pi T}} e^{-\frac{\pi^2}{4 T^2}} * f^0 \in \mathcal{R}^L_T(0)$, $L > 0$;

(v) $f^1 \in \mathcal{R}_T(f^0) \Leftrightarrow f^1 - \frac{1}{2\sqrt{\pi T}} e^{-\frac{\pi^2}{4 T^2}} * f^0 \in \mathcal{R}_T(0)$.

We obtain the following necessary condition for $f$ to belong to $\mathcal{R}^L_T(0)$.

**Theorem 2.3.** Let $L > 0$. If $f \in \mathcal{R}^L_T(0)$, then

$$\int_0^\infty e^{\frac{-x^2}{2T}} |f(x)| dx \leq 2LT. \hfill (20)$$

In Section 8 (see Example 1) for any $T > 0$, it is constructed a function $f \in \hat{H}^1$ such that condition (20) holds for it and $f \not\in \mathcal{R}^L_T(0)$. So, condition (20) is not sufficient for $f$ to belong to $\mathcal{R}^L_T(0)$. The following necessary and sufficient condition holds.

**Theorem 2.4.** Let $L > 0$, $f \in \hat{H}^1$ and (20) hold. Let

$$\omega_n = -\frac{n!}{(2n)!} \int_0^\infty x^{2n} f(x) dx, \quad n = 0, \infty. \hfill (21)$$

Then $f \in \mathcal{R}^L_T(0)$ iff there exists $v \in L^\infty(0, T)$ such that $\| v \|_{L^\infty(0, T)} \leq L$ and

$$\int_0^T \xi^n v(\xi) d\xi = \omega_n, \quad n = 0, \infty. \hfill (22)$$

We see that $f \in \mathcal{R}^L_T(0)$ iff infinite Markov power moment problem (22) is solvable. By the following theorem, if finite Markov power moment problem (23) is solvable, then $f \in \overline{\mathcal{R}^L_T(0)}$.

**Theorem 2.5.** Let $L > 0$, $f \in \hat{H}^1$ and (20) hold. Let $\{\omega_n\}_{n=0}^\infty$ be defined by (21). If for each $N \in \mathbb{N}$ there exists $v_N \in L^\infty(0, T)$ such that $\| v_N \|_{L^\infty(0, T)} \leq L$ and

$$\int_0^T \xi^n v_N(\xi) d\xi = \omega_n, \quad n = 0, N, \hfill (23)$$

then $f \in \overline{\mathcal{R}^L_T(0)}$ (the closure is considered in $\hat{H}^1$).

We also obtain the following two theorems.

**Theorem 2.6.** Let $T > 0$. We have $\overline{\mathcal{R}_T(0)} = \hat{H}^1$.

**Theorem 2.7.** Let $T > 0$. For each $f \in \hat{H}^1 \setminus \{0\}$, we have $0 \not\in \mathcal{R}_T(f)$. 


2.1. Controllability and approximate controllability. In this subsection we consider the problems of controllability and approximate controllability for control system (6), (7).

**Definition 2.8.** For control system (6), (7), a state \( W^0 \in \hat{H}^1 \) is said to be controllable to a state \( W^T \in \hat{H}^1 \) in a given time \( T > 0 \) if \( W^T \in \mathcal{R}_T(W^0) \).

In other words, a state \( W^0 \in \hat{H}^1 \) is said to be controllable to a state \( W^T \in \hat{H}^1 \) in a given time \( T > 0 \) if there exists a control \( u \in L^\infty(0, T) \) such that there exists a unique solution to (6)–(8).

Taking into account Theorems 2.2 (iv) and 2.4, we get

**Theorem 2.9.** Let \( L > 0, T > 0, W^0 \in \hat{H}^1, W^T \in \hat{H}^1, f = W^T - e^{-\frac{x^2}{4T}} * W^0/(2\sqrt{\pi T}) \), and condition (20) hold for \( f \). Let \( \{\omega_n\}_{n=0}^\infty \) be defined by (21). Then the state \( W^0 \in \hat{H}^1 \) is controllable to the target state \( W^T \in \hat{H}^1 \) in the time \( T \) iff there exists \( v \in L^\infty(0, T) \) such that \( ||v||_{L^\infty(0, T)} \leq L \) and condition (22) holds. In addition, the control \( u(\xi) = v(T - \xi), \xi \in [0, T] \), solves the controllability problem in the time \( T \).

**Definition 2.10.** For control system (6), (7), a state \( W^0 \in \hat{H}^1 \) is said to be null-controllable in a given time \( T > 0 \) if \( 0 \in \mathcal{R}_T(W^0) \).

In [14], the lack of null controllability for (1) controlled by the Dirichlet boundary condition was established in some spaces of Sobolev type (different from those in the present paper). In contrast to the present paper and paper [7], controls solving the controllability problem belong to \( L^2(0, T) \) in [14] and have an influence on smoothness of solutions. With regard to Theorem 2.7, we obtain the following analog of the result of [14].

**Theorem 2.11.** If a state \( W^0 \in \hat{H}^1 \) is null-controllable in a time \( T > 0 \), then \( W^0 = 0 \).

**Definition 2.12.** For control system (6), (7), a state \( W^0 \in \hat{H}^1 \) is said to be approximately controllable to a target state \( W^T \in \hat{H}^1 \) in a given time \( T > 0 \) if \( W^T \in \mathcal{R}_T(W^0) \), where the closure is considered in the space \( \hat{H}^1 \).

In other words, the state \( W^0 \in \hat{H}^1 \) is approximately controllable to a target state \( W^T \in \hat{H}^1 \) in a given time \( T > 0 \) iff for each \( \varepsilon > 0 \), there exists \( u_\varepsilon \in L^\infty(0, T) \) such that there exists a unique solution \( W \) to (6), (7) with \( u = u_\varepsilon \) and \( ||W(\cdot, T) - W^T||^\varepsilon < \varepsilon \).

With regard to Theorems 2.2 (iv) and 2.5, we obtain

**Theorem 2.13.** Let \( L > 0, T > 0, W^0 \in \hat{H}^1, W^T \in \hat{H}^1, f = W^T - e^{-\frac{x^2}{4T}} * W^0/(2\sqrt{\pi T}) \), and condition (20) hold for \( f \). Let \( \{\omega_n\}_{n=0}^\infty \) be defined by (21). If for each \( N \in \mathbb{N} \) there exists \( v_N \in L^\infty(0, T) \) such that \( ||v_N||_{L^\infty(0, T)} \leq L \) and condition (23) holds, then the state \( W^0 \in \hat{H}^1 \) is approximately controllable to the target state \( W^T \in \hat{H}^1 \) in the time \( T \) by controls \( u_N(\xi) = v_N(T - \xi), \xi \in [0, T], N \in \mathbb{N} \).

This theorem is illustrated by Examples 2 and 3 (see below Section 8).

Taking into account Theorems 2.2 (v) and 2.6, we get the following theorem.

**Theorem 2.14.** Each state \( W^0 \in \hat{H}^1 \) is approximately controllable to each target state \( W^T \in \hat{H}^1 \) in a given time \( T > 0 \).
Theorem 2.14 is illustrated by Example 4 (see below Section 8).

We can see that a state \( W^0 \in \hat{H}^1 \) is controllable to a target state \( W^T \in \hat{H}^1 \) in a given time \( T > 0 \) iff they satisfy very special conditions (Theorem 2.9). In particular, any state \( 0 \neq W^0 \in \hat{H}^1 \) is not controllable to the target state \( W^T = 0 \) in a given time \( T > 0 \) (Theorem 2.11). However, each state \( W^0 \in \hat{H}^1 \) is approximately controllable to each target state \( W^T \in \hat{H}^1 \) (Theorem 2.14).

3. Proofs of Theorems 2.3–2.5. In this section we prove Theorems 2.3–2.5.

Proof of Theorem 2.3. Using (19), we have
\[
\int_0^\infty e^{\frac{\pi^2}{4\sigma}} |f(x)| \, dx \leq \sqrt{\frac{2L}{\pi}} \int_0^\infty e^{\frac{x^2}{4\sigma}} \left( \int_0^T e^{-\frac{x^2}{4\sigma}} \frac{d\xi}{\sqrt{2\xi}} \right) \, dx
\]
\[
= \sqrt{2L} \int_0^T \frac{1}{\sqrt{2\xi}} \int_0^\infty e^{-x^2(\frac{1}{4\sigma} + \frac{1}{\xi})} \, dx \, d\xi
\]
\[
= L \int_0^T \frac{1}{\sqrt{\xi}} \sqrt{\frac{T\xi}{T - \xi}} \, d\xi = 2LT.
\]
The theorem is proved. \qed

Proof of Theorem 2.4. According to (19), \( f \in \mathcal{R}_x^L(0) \) iff there exists \( v \in L^\infty(0, T) \) such that \( \|v\|_{L^\infty(0, T)} \leq L \) and
\[
f = -\sqrt{\frac{2}{\pi}} \int_0^T e^{-\frac{\pi^2}{4\sigma^2} v(\xi)} \, d\xi.
\]
Denoting \( F = \mathcal{F}f \), we have
\[
F(\sigma) = -\sqrt{\frac{2}{\pi}} \int_0^T e^{-\pi^2 \sigma^2 v(\xi)} \, d\xi.
\]
Due to the Paley–Wiener theorem, \( F(\sigma) \) is an even entire function. Therefore,
\[
\sum_{n=0}^{\infty} \frac{(F(2n))(0)}{(2n)!} \sigma^{2n} = F(\sigma) = -\sqrt{\frac{2}{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sigma^{2n} \int_0^T \xi^n v(\xi) \, d\xi.
\]
Since
\[
(F(2n))(0) = \sqrt{\frac{2}{\pi}} \int_0^\infty (-ix)^{2n} f(x) \, dx = -\sqrt{\frac{2}{\pi}} (-1)^n \frac{(2n)!}{n!} \omega_n,
\]
we conclude the assertion of the theorem. \qed

Proof of Theorem 2.5. By \( W_N \), denote the solution to problem (6), (7) with \( W^0 = 0, W^T = f, u(t) = v_N(T - t) \). Denote also \( V^T = \mathcal{F}W^T, V_N(\cdot, t) = \mathcal{F}_{x \to \sigma} W_N(\cdot, t), t \in [0, T] \). Then \( V_N \) is the unique solution to (11), (12) with \( V^0 = 0 \) and the same \( u \). Evidently,
\[
\int_0^\infty (1 + \sigma^2) |V^T(\sigma)|^2 \, d\sigma \to 0 \quad \text{as} \ a \to \infty.
\]
(24)

Put
\[
\mathcal{W}_T = \int_0^\infty e^{\frac{\pi^2}{4\sigma}} |W^T(\sigma)| \, d\sigma.
\]
For \( n = 0, \infty \), we have
\[
(V^T)^{(2n+1)}(0) = 0, \quad (V^T)^{(2n)}(\sigma) = (-1)^n \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-i\sigma x} x^{2n} W^T(x) \, dx, \quad \sigma \in \mathbb{R}.
\]
Therefore, we get
\[ \left| (V^T)^{(2n)}(\sigma) \right| \leq \sqrt{\frac{2}{\pi}} \int_0^\infty \left( x^{2n}e^{-\frac{x^2}{4\pi}} \right) \left( e^{\frac{x^2}{2\pi}} |W^T(x)| \right) dx \]
\[ \leq \sqrt{\frac{2}{\pi}} W_T \left( \frac{n}{e} \right)^n (4T)^n, \quad \sigma \in \mathbb{R}. \] 
(26)

By using the Stirling formula:
\[ \sqrt{2\pi n^{n+\frac{1}{2}}} e^{-n} \leq n! \leq en^{n+\frac{1}{2}} e^{-n}, \quad n \in \mathbb{N}, \]
(27)
we have
\[ \left| (V^T)^{(2n)}(\sigma) \right| \leq \sqrt{\frac{2}{\pi}} W_T (4T)^n \left( \frac{n}{e} \right)^n \left( \frac{e}{2n} \right)^{2n} \frac{1}{2 \sqrt{\pi n}} \]
\[ \leq \frac{W_T T^n}{\sqrt{2n\pi}} \left( \frac{e}{n} \right)^n \leq \frac{e W_T T^n}{\sqrt{2\pi n!}}, \quad \sigma \in \mathbb{R}. \] 
(28)

Taking into account the Tailor formula for \( M \in \mathbb{N} \), we get
\[ V^T(\sigma) = \sum_{n=0}^M \frac{(V^T)^{(2n)}(0)}{(2n)!} \sigma^{2n} + R^M(\sigma), \quad \sigma \in \mathbb{R}, \] 
(29)
where, according to (28),
\[ |R^M(\sigma)| \leq \frac{|\sigma|^{2M+2}}{(2M+2)!} \max_{|\xi| \leq |\sigma|} \left| (V^T)^{(2M+2)}(\xi) \right| \leq \frac{e W_T (T|\sigma|^2)^{M+1}}{\sqrt{2\pi} (M+1)!}, \quad \sigma \in \mathbb{R}. \] 
(30)

Therefore,
\[ V^T(\sigma) = \sum_{n=0}^\infty \frac{(V^T)^{(2n)}(0)}{(2n)!} \sigma^{2n}, \quad \sigma \in \mathbb{R}. \] 
(31)

Due to (14), we get
\[ |V_N(\sigma,T)| \leq \sqrt{\frac{2}{\pi}} L \frac{1 - e^{-T|\sigma|^2}}{|\sigma|^2}, \quad \sigma \in \mathbb{R}. \] 
(32)

Hence,
\[ \int_a^\infty (1 + |\sigma|^2) |V_N(\sigma,T)|^2 d\sigma \leq \frac{2L^2}{\pi} \frac{1 + a^2}{a^2} \int_a^\infty \frac{1 - e^{-T|\sigma|^2}}{|\sigma|^2} d\sigma \]
\[ \leq \frac{8L^2}{\pi} \frac{1 + a^2}{a^2} \int_a^\infty \frac{d\sigma}{\sigma^2} = \frac{8L^2}{\pi} \frac{1 + a^2}{a^2} \to 0 \quad \text{as } a \to \infty. \] 
(33)

According to (13), we get
\[ V_N(\sigma,T) = -\sqrt{\frac{2}{\pi}} \int_0^T e^{-\xi|\sigma|^2} v_N(\xi)d\xi \]
\[ = -\sqrt{\frac{2}{\pi}} \sum_{n=0}^N \frac{(-1)^n}{n!} \sigma^{2n} \int_0^T \xi^n v_N(\xi)d\xi + R^N_N(\sigma), \quad \sigma \in \mathbb{R}, \] 
(34)

where
\[ |R^N_N(\sigma)| \leq \sqrt{\frac{2}{\pi}} L \frac{|\sigma|^{2N+2}}{(N+1)!} \int_0^T \xi^{N+1} d\xi = \sqrt{\frac{2}{\pi}} L \frac{|\sigma|^{2N+2} T^{N+2}}{(N+2)!}, \quad \sigma \in \mathbb{R}. \] 
(35)
With regard to (23), (24), (29), and (34), we obtain
\[ V^T(\sigma) - V_N(\sigma, T) = R_N(\sigma) - R^N_N(\sigma), \quad \sigma \in \mathbb{R}. \]

For \( a > 0 \) taking into account (30) and (35), we conclude that
\[
s_N(a) = \sup_{\sigma \in [-a,a]} |V^T(\sigma) - V_N(\sigma, T)| \leq \frac{(Ta^2)^{N+1}}{(N+1)!} \left( e^{W_T} + \sqrt{\frac{2}{\pi}} \frac{LT}{N+2} \right) \to 0 \quad \text{as } N \to \infty.
\]

Therefore,
\[
\int_{-a}^{a} (1 + \sigma^2) |V^T(\sigma) - V_N(\sigma, T)|^2 d\sigma \leq 2a(1 + a^2) (s_N(a))^2 \to 0 \quad \text{as } N \to \infty. \tag{36}
\]

With regard to (25), (33) and (36), we obtain
\[
\|W^T(\sigma) - W_N(\sigma, T)\|_1 = \|V^T(\sigma) - V_N(\sigma, T)\|_1 \to 0 \quad \text{as } N \to \infty,
\]
i.e., \( W^T \in \mathbb{R}_k^L(0) \).

4. Proof of Theorem 2.6. In this section we prove Theorem 2.6. First, we prove the following lemma.

**Lemma 4.1.** Let \( T > 0 \),
\[
\varphi^T_m(\sigma) = \sigma^m e^{-T\sigma^2}, \quad \sigma \in \mathbb{R}, \quad m = 0, \infty. \tag{37}
\]
Then the system \( \{\varphi^T_m\}_{m=0}^\infty \) is complete in \( H_1 \).

**Proof.** Evidently, \( \varphi^T_m \in H_1, m = 0, \infty. \) Denote
\[
f^T(\sigma) = \sqrt{1 + \sigma^2} e^{-T\sigma^2}, \quad \sigma \in \mathbb{R}.
\]
We have \( \sigma^m f^T(\sigma) = \varphi^T_m \sqrt{1 + \sigma^2}, \sigma \in \mathbb{R}. \) First, let us prove that the system \( \{\sigma^m f^T\}_{m=0}^\infty \) is complete in \( H_0 = L^2(\mathbb{R}) \). Suppose the converse, i.e., suppose that this system is not complete. According to Hahn–Banach theorem, there exists a function \( h \in L^2(\mathbb{R}) \) such that \( h \neq 0 \) and
\[
\forall m = 0, \infty \int_{-\infty}^{\infty} \sigma^m f^T(\sigma) \overline{h}(\sigma) d\sigma = 0. \tag{38}
\]
Denote \( g = T^{-1} f^T \). The function \( g \) can be extended to a holomorphic function in the strip \( |\text{Im } z| < \delta \) for any \( \delta > 0 \). According to (38), \( g^{(m)}(0) = 0, m = 0, \infty. \) Therefore, \( g(z) = 0 \) in the strip \( |\text{Im } z| < \delta. \) Hence, \( f^T h = 0 \) and \( h = 0 \) in \( L^2(\mathbb{R}). \) This contradiction proves that the system \( \{\sigma^m f^T\}_{m=0}^\infty \) is complete in \( H_0 = L^2(\mathbb{R}). \)

Let \( p \in H_1. \) Then \( \sqrt{1 + \sigma^2} p \in H_0. \) For any \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) and \( \rho_0, \ldots, \rho_N \in \mathbb{C} \) such that
\[
\|p - p_N\|_1 = \left\| \sqrt{1 + \sigma^2} (p - p_N) \right\|_0 < \varepsilon,
\]
where
\[
p_N(\sigma) = \sum_{m=0}^{N} \rho_m \varphi^T_m(\sigma) = \frac{1}{\sqrt{1 + \sigma^2}} \sum_{m=0}^{N} \rho_m \sigma^m f^T(\sigma), \quad \sigma \in \mathbb{R}.
\]
Therefore, the system \( \{\varphi^T_m\}_{m=0}^\infty \) is complete in \( H_1. \) \( \square \)
Proof of Theorem 2.6. From Lemma 4.1, it follows that the system \( \{ \varphi_{2n}^T \}_{n=0}^{\infty} \) defined by (37) is complete in \( \hat{H}_1 \). Denote

\[
\varphi_{2n}^{T,l}(\sigma) = \sigma^2 e^{-T\sigma^2} \left( \frac{e^{\sigma^2/l} - 1}{\sigma^2/l} \right)^{n+1}, \quad \sigma \in \mathbb{R}, \; l > \frac{n+1}{T}, \; n \in \mathbb{N} \cup \{0\},
\]

\[
u_{n}^{l}(\xi) = \begin{cases} (-1)^{n-j} \binom{n}{j} l^{n+1}, & \xi \in \left( \frac{j}{l}, \frac{j+1}{l} \right), \; j = 0, n, \; l \in \mathbb{N}, \; n \in \mathbb{N} \cup \{0\} \end{cases},
\]

Note that \( \nu_{n}^{l} \to \delta^{(n)} \) as \( l \to \infty \) in \( H^{-1} \) for each \( n = 0, \infty \). For \( g \in L^\infty(0,T) \), denote \( \Phi_g(\sigma) = \int_0^T e^{-(T-\xi)\sigma^2} g(\xi) \, d\xi \). For \( n \in \mathbb{N} \cup \{0\} \) and \( l > \frac{n+1}{T} \), we have

\[
\Phi_{\nu_{n}^{l}}(\sigma) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} l^{n+1} \int_{j/l}^{(j+1)/l} e^{-(T-\xi)\sigma^2} \, d\xi
\]

\[
= l^{n+1} e^{-T\sigma^2} \frac{\sigma^2}{\sigma^2} \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} \left( e^{(j+1)\sigma^2/l} - e^{j\sigma^2/l} \right)
\]

\[
= l^{n+1} e^{-T\sigma^2} \frac{n+1}{\sigma^2} \sum_{j=0}^{n+1} \binom{n+1}{j} (-1)^{n-j+1} e^{j\sigma^2/l}
\]

\[
= \sigma^2 e^{-T\sigma^2} \left( \frac{e^{\sigma^2/l} - 1}{\sigma^2/l} \right)^{n+1} = \varphi_{2n}^{T,l}(\sigma).
\]

Figure 1 illustrates the functions \( \nu_{n}^{l} \).

If \( l > (2n+2)/T \), we have

\[
\left| \varphi_{2n}^{T,l}(\sigma) \right| \leq \sigma^2 e^{-T\sigma^2} e^{(n+1)\sigma^2/l} = \sigma^2 e^{-\sigma^2(T-n/2)} \leq \sigma^2 e^{-T\sigma^2/2}
\]

and \( \varphi_{2n}^{T,l} \to \varphi_{2n}^{T} \) as \( l \to \infty \) a.e. on \( \mathbb{R} \). According to Lebesgue’s dominated convergence theorem, we get

\[
\left\| \varphi_{2n}^{T} - \varphi_{2n}^{T,l} \right\|_1 = \left\| \sqrt{1 + \sigma^2} \left( \varphi_{2n}^{T} - \varphi_{2n}^{T,l} \right) \right\|_0 \to 0 \quad \text{as} \quad l \to \infty, \quad n = 0, \infty.
\]
Let $f \in \hat{H}^1$. Denote $F = \mathcal{F} f$. Then $F \in \hat{H}_1$. Since the system $\{\varphi_{2n}^T\}_{n=0}^\infty$ is complete in $\hat{H}_1$, for each $\varepsilon > 0$, we can find $N \in \mathbb{N}$ and $v_n^N \in \mathbb{R}$, $n = 0, N$, such that for

$$
F^N = \sum_{n=0}^N v_n^N \varphi_{2n}^T,
$$

we have

$$
\|F^N - F\|_1 < \varepsilon / 2.
$$

(43)

Taking into account (41), we conclude that there exist $l \in \mathbb{N}$ such that

$$
\|\varphi_{2n}^T - \varphi_{2n}^{T,l}\|_1 < \varepsilon \left(2 \sum_{m=0}^N |v_m^n|\right)^{-1}, \quad n = 0, N.
$$

Denoting

$$
F_l^N = \sum_{n=0}^N v_n^N \varphi_{2n}^{T,l},
$$

we have

$$
\|F_l^N - F^N\|_1 < \varepsilon / 2.
$$

(44)

Summarizing (43) and (44), we get

$$
\|F_l^N - F\|_1 \leq \|F_l^N - F^N\|_1 + \|F^N - F\|_1 < \varepsilon.
$$

Applying the inverse Fourier transform, we obtain

$$
\|f_l^N - f\|_1 < \varepsilon,
$$

where

$$
f_l^N = \mathcal{F}^{-1} F_l^N = \sum_{n=0}^N v_n^N \mathcal{F}^{-1} \varphi_{2n}^{T,l}.
$$

Due to (40), we have

$$
f_l^N = -\sqrt{\frac{2}{\pi}} \int_0^T e^{-\frac{\xi^2}{4}} U_l^N(T - \xi) \frac{d}{d\xi} d\xi,
$$

where $U_l^N = -\sqrt{\frac{\pi}{2}} \sum_{n=0}^N v_n^N u_l^n$.

According to (18), we obtain $f_l^N \in \mathcal{R}_T(0)$. Since we have considered an arbitrary $\varepsilon > 0$, we conclude that $f \in \mathcal{R}_T(0)$.

5. **Orthogonal bases in $H^1$, $H_1$, and their subspaces.** Consider the Hermite polynomial [10, p. 775]

$$
\mathcal{H}_n(x) = (-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2} = n! \sum_{k=0}^{[n/2]} \frac{(-1)^k}{(n-2k)!k!} (2x)^{n-2k},
$$

where $[\cdot]$ denotes the integer part of a number, $n = 0, \infty$.

First, we consider an orthogonal basis in $H^0$. Let $\psi_n^0(x) = \mathcal{H}_n(x) e^{-x^2}$, $x \in \mathbb{R}$, $n = 0, \infty$. It is easy to see that

$$
\mathcal{F} \psi_n^0 = (-i)^n \psi_n^0, \quad n = 0, \infty.
$$

(45)

It is well known [10, p. 775] that

$$
\langle \psi_n^0, \psi_m^0 \rangle = \int_{-\infty}^{\infty} \psi_n^0(x) \psi_m^0(x) dx = \sqrt{\pi} 2^n n! \delta_{nm}, \quad m, n = 0, \infty,
$$

(46)
where $\delta_{mn}$ is the Kronecker delta, and $\{\psi_n^0\}_{n=0}^{\infty}$ is an orthogonal basis in $L^2(\mathbb{R})$. In Lemma 4.1, it has been proved that the system $\{\varphi_n^T\}_{n=0}^{\infty}$ is complete in $H_1$. Therefore, the system $\{\psi_n^0\}_{n=0}^{\infty}$ is complete in $H_1$ and $H^1$.

For $m = 0, \infty$, denote

$$
\psi_{2m}^1(x) = \sum_{k=0}^{m} \frac{(-1)^k H_{2k}(i)}{2^k(2k)!} \psi_{2k}^0(x), \quad x \in \mathbb{R},
$$

(47)

$$
\psi_{2m+1}^1(x) = \sum_{k=0}^{m} \frac{(-1)^k iH_{2k+1}(i)}{2^{k+1}(2k+1)!} \psi_{2k+1}^0(x)
$$

(48)

Evidently, $\psi_n^1(x) \in \mathbb{R}$, $x \in \mathbb{R}$, $n = 0, \infty$. In addition, $\psi_{2m}$ is even, and $\psi_{2m+1}$ is odd, $m = 0, \infty$. Denote $\hat{\psi}_n^1 = \mathcal{F}\psi_n^1$, $n = 0, \infty$. Due to (45), for $m = 0, \infty$, we obtain

$$
\hat{\psi}_{2m}^1(\sigma) = \sum_{k=0}^{m} \frac{H_{2k}(i)}{2^k(2k)!} \psi_{2k}^0(\sigma)
$$

(49)

$$
\hat{\psi}_{2m+1}^1(\sigma) = \sum_{k=0}^{m} \frac{H_{2k+1}(i)}{2^{k+1}(2k+1)!} \psi_{2k+1}^0(\sigma)
$$

(50)

By using the Christoffel–Darboux formula [11, pp. 438, 439], we get

$$
\sum_{k=0}^{n} \frac{H_k(i)H_k(\sigma)}{2^k k!} = \frac{H_{n+1}(\sigma)H_n(i) - H_n(\sigma)H_{n+1}(i)}{2^{n+1} n! (\sigma - i)} + \frac{iH_{n+1}(\sigma)H_n(i) - iH_n(\sigma)H_{n+1}(i)}{2^{n+1} n! (\sigma^2 - 1)}, \quad \sigma \in \mathbb{R}.
$$

Since $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$, $n = 1, \infty$ (see [10, p. 782]), we obtain

$$
\sum_{k=0}^{n} \frac{H_k(i)H_k(\sigma)}{2^k k!} e^{-\sigma^2} = \frac{H_{n+2}(\sigma)H_{n+1}(i) - H_{n+1}(\sigma)H_n(i)}{2^{n+2} n! (\sigma^2 + 1)} e^{-\sigma^2} + \frac{H_{n+1}(\sigma)H_{n+1}(i) - H_{n+1}(\sigma)H_{n+1}(i)}{2^{n+1} (n-1)! (\sigma^2 + 1)} e^{-\sigma^2} = \frac{H_n(i)\psi_{n+2}^0(\sigma) - H_{n+2}(i)\psi_n^0(\sigma)}{2^{n+2} n! (\sigma^2 + 1)} + \frac{H_{n-1}(i)\psi_{n+1}^0(\sigma) - H_{n+1}(i)\psi_{n-1}^0(\sigma)}{2^{n+1} (n-1)! (\sigma^2 + 1)}, \quad \sigma \in \mathbb{R}.
$$
Evidently, \( \hat{\psi}_{2m}(\sigma) \) has real values for \( \sigma \in \mathbb{R} \), and \( \hat{\psi}_{2m+1}(\sigma) \) has imaginary values for \( \sigma \in \mathbb{R} \). Therefore,

\[
\hat{\psi}_{2m}(\sigma) = \operatorname{Re} \sum_{k=0}^{2m} \frac{\mathcal{H}_k(i)\mathcal{H}_k(\sigma)}{2^k k!} e^{-\frac{\sigma^2}{2}} = \frac{\mathcal{H}_{2m+2}(\sigma)\mathcal{H}_{2m}(i) - \mathcal{H}_{2m}(\sigma)\mathcal{H}_{2m+2}(i)}{2^{2m+2}(2m)!}(\sigma^2 + 1) e^{-\frac{\sigma^2}{2}}, \quad \sigma \in \mathbb{R}, \tag{51}
\]

\[
\hat{\psi}_{2m+1}(\sigma) = i \operatorname{Im} \sum_{k=0}^{2m+1} \frac{\mathcal{H}_k(i)\mathcal{H}_k(\sigma)}{2^k k!} e^{-\frac{\sigma^2}{2}} = \frac{\mathcal{H}_{2m+3}(\sigma)\mathcal{H}_{2m+1}(i) - \mathcal{H}_{2m+1}(\sigma)\mathcal{H}_{2m+3}(i)}{2^{2m+3}(2m + 1)!(\sigma^2 + 1)} e^{-\frac{\sigma^2}{2}}, \quad \sigma \in \mathbb{R}. \tag{52}
\]

For \( m \geq n \), taking into account (46), (49), and (51), we have

\[
\langle \hat{\psi}_{2m}, \hat{\psi}_{2n} \rangle = \frac{1}{2^{2m+2}(2m)!} \sum_{k=0}^{n} \frac{\mathcal{H}_{2k}(i)}{2^{2k}(2k)!} \left( \mathcal{H}_{2m}(i) \langle \psi_{2m+2}, \psi_{2k} \rangle_0 - \mathcal{H}_{2m+2}(i) \langle \psi_{2m}, \psi_{2k} \rangle_0 \right).
\]

Analogously, for \( m \geq n \), we obtain

\[
\langle \hat{\psi}_{2m+1}, \hat{\psi}_{2n+1} \rangle = -\frac{\mathcal{H}_{2m+1}(i)\mathcal{H}_{2m+3}(i)}{2^{2m+3}(2m + 1)!} \delta_{mn}.
\]

Since \( \hat{\psi}_{2m} \) is even, and \( \hat{\psi}_{2m+1} \) is odd, \( m = 0, \infty \), we obtain

\[
\langle \hat{\psi}_{p}, \hat{\psi}_{n} \rangle = -\sqrt{\pi} \frac{\mathcal{H}_{n}(i)\mathcal{H}_{n+2}(i)}{2^n n!} \delta_{pn}, \quad p, n = 0, \infty. \tag{53}
\]

Note that, \( \mathcal{H}_n(i)\mathcal{H}_{n+2}(i) \) is real and negative, \( n = 0, \infty \).

Due to Lemma 4.1, the system \( \{\psi_{2m}^{1,\infty}\}_{m=0} \) is complete in \( H_1 \). Taking into account (53), we conclude that the system \( \{\hat{\psi}_{n}^{1,\infty}\}_{n=0} \) is an orthogonal basis in \( H_1 \). Therefore, the system \( \{\psi_{n}^{1,\infty}\}_{n=0} \) is an orthogonal basis in \( H^1 \). In addition, the system \( \{\psi_{2m}^{1,\infty}\}_{m=0} \) is an orthogonal basis in \( \hat{H}^1 \), and the system \( \{\psi_{2m+1}^{1,\infty}\}_{m=0} \) is an orthogonal basis in \( \hat{H}^1 \).

6. Proof of Theorem 2.7. In this section we prove Theorem 2.7.

Remark 2. For \( T > 0 \), \( T^* > 0 \), \( W^0 \in \hat{H}^1 \), \( W^T \in \hat{H}^1 \), denote \( \hat{W}(x,t) = W(x\sqrt{T^*/T}, tT/T^*) \), \( \hat{W}^0(x) = W^0(x\sqrt{T^*/T}) \), \( \hat{W}^T(x) = W^T(x\sqrt{T^*/T}) \), \( t \in [0, T^*] \), \( x \in \mathbb{R} \). One can see that \( W \) is the solution to (6)–(8) if \( \hat{W} \) is the solution to the same system with \( T = T^* \), \( W^0 = \hat{W}^0 \), \( W^T = \hat{W}^T \). In addition,

\[
\frac{1}{C_T} \left\| W(\cdot, tT/T^*) \right\|_1 \leq \left\| \hat{W}(\cdot, t) \right\|_1 \leq C_T^* \left\| W(\cdot, tT/T^*) \right\|_1, \quad t \in [0, T^*]. \tag{54}
\]
where \( CT^* = \frac{1+T/T^*}{\sqrt{1/T^*}} \).

**Proof of Theorem 2.7.** According to Remark 2, we may suppose \( T = 1/4 \), without loss of generality. Find \( u \in L^\infty(0, T) \) such that there exists a unique solution to (6), (7) and \( W(\cdot, T) = 0 \). Denote \( V^0 = \mathcal{F}W^0 \), \( V(\cdot, t) = \mathcal{F}_{x\to \sigma} W(\cdot, t), \ t \in [0, T] \). Taking into account (13), we obtain

\[
V^0(\sigma) = \sqrt{\frac{2}{\pi}} \int_0^{1/4} e^{\xi \sigma^2} u(\xi) \, d\xi, \quad \sigma \in \mathbb{R}.
\]  

(55)

We have

\[
\sum_{m=0}^{\infty} \nu_m \left( \psi_{2m}^1 \right)^2 = \sum_{m=0}^{\infty} \int_0^{1/4} \mu_m(\xi) u(\xi) \, d\xi \left( \psi_{2m}^1 \right)^2,
\]

where

\[
\begin{align*}
\nu_m &= 2 \int_0^\infty V^0(\sigma) \psi_{2m}^1(\sigma) \, d\sigma, \\
\mu_m(\xi) &= 2 \sqrt{\frac{2}{\pi}} \int_0^\infty e^{\xi \sigma^2} \psi_{2m}^1(\sigma) \, d\sigma, \quad \xi \in \left[ \frac{1}{4}, \frac{1}{4} \right].
\end{align*}
\]

Therefore,

\[
\int_0^{1/4} \mu_m(\xi) u(\xi) \, d\xi = \nu_m, \quad m = 0, \infty.
\]  

(56)

Let \( m = 0, \infty \) be fixed. We have (see (47))

\[
\begin{align*}
\mu_m(\xi) &= 2 \sqrt{\frac{2}{\pi}} \sum_{k=0}^{m} \frac{\mathcal{H}_{2k}(i)}{2^{2k} k!} \sum_{p=0}^{k} \frac{(-1)^{k-p} 2p}{(k-p)! (2p)!} \int_0^{\infty} \sigma^{2p} e^{-(1/2-\xi)\sigma^2} \, d\sigma \\
&= 2 \sqrt{\frac{2}{\pi}} \sum_{k=0}^{m} \frac{\mathcal{H}_{2k}(i)}{2^{2k} k!} \sum_{p=0}^{k} \frac{(k)! (1/2-\xi)^p}{(k-p)! (2p)!} \\
&= 2 \sqrt{\frac{2}{\pi}} \sum_{k=0}^{m} \frac{\mathcal{H}_{2k}(i)}{2^{2k} k!} \left( \frac{1+2\xi}{1-2\xi} \right)^k, \quad \xi \in \left[ \frac{1}{4}, \frac{1}{4} \right].
\end{align*}
\]  

(57)

Replacing \( 1/2+k \xi \) by \( e^s \) and taking into account (56), we get

\[
\nu_m = 2 \sum_{k=0}^{m} \frac{\mathcal{H}_{2k}(i)}{2^{2k} k!} \int_0^{1/4} \left( 1 + 2 \xi \right)^k \frac{u(\xi)}{\sqrt{1-2\xi}} \, d\xi = \sum_{k=0}^{m} \frac{\mathcal{H}_{2k}(i)}{2^{2k} k!} \int_0^{\ln 3} e^{ks} U(s) \, ds,
\]

where \( U(s) = \frac{\sqrt{2 e^s}}{(e^s+1)^{1/2}} \), \( s \in [0, \ln 3] \). Therefore,

\[
\int_0^{\ln 3} e^{ms} U(s) \, ds = \nu_m^*, \quad m = 0, \infty,
\]

(58)

where \( \nu_0^* = \nu_0 \) and \( \nu_m^* = \frac{2^{2m} m!}{\mathcal{H}_{2m}(i)} (\nu_m - \nu_{m-1}), \ m = 1, \infty \). With regard to (53), for \( m = 0, \infty \), we obtain

\[
|\nu_m| \leq \| V^0 \|_1 \left\| \psi_{2m}^1 \right\|_1 \leq \| V^0 \|_1 \sqrt{\frac{\mathcal{H}_{2m}(i) \mathcal{H}_{2m+2}(i)}{2^{2m+2} (2m)!}}\]

\[
\leq \| V^0 \|_1 \sqrt{\frac{\mathcal{H}_{2m+2}(i)}{2^{m+1} \sqrt{2} (2m)!}},
\]

because \( |\mathcal{H}_{2m}(i)| \leq \frac{1}{2} |\mathcal{H}_{2m+2}(i)|, \ m = 0, \infty.\)
Hence,
\[
\frac{\|\nu^*_m\|}{\|V^0\|_1} \leq \pi^{1/4} \frac{2^{m-1/2}(m+1)!}{\sqrt{2(2m)!}} \frac{|H_{2m} (i)|}{|H_{2m} (i)|}, \quad m = 0, \infty.
\]
Due to (60) (see below Proposition 1 in Section 7) with \(a = 3/4\) and \(b = 3/2\), we have
\[
\sqrt{\pi} \frac{2^{3m+3/2}}{e^5 3^{m+1/2} \sqrt{m}} \leq |H_{2m} (i)| \leq \frac{e^{4} 2^{4m+1/2}}{\pi 3^{m+1/2} \sqrt{m}}.
\]
Therefore, we obtain
\[
\frac{\|\nu^*_m\|}{\|V^0\|_1} \leq \frac{e^9 2^{2m+5/2} \sqrt{m(m+1)(m+1)!}}{3^{5/4} \sqrt{(2m)!}}, \quad m = 0, \infty.
\]
Applying the Stirling formula (27), we get
\[
\frac{\|\nu^*_m\|}{\|V^0\|_1} \leq \frac{e^{9} 2^{7/4} 2^{m+7/2}}{3^{2/3} \sqrt{m}} \frac{(m+1)}{m} m^{1/4} (m+1)^{2} \leq \frac{e^{9} 2^{7/4} 2^{m+7/2}}{3^{2/3} \sqrt{m}} m^{1/4} (m+1)^{2}, \quad m = 0, \infty.
\]
Therefore, for all \(\delta > 0\) there exists \(C_{\delta} > 0\) such that
\[
|\nu^*_m| \leq C_{\delta} e^{\delta \bar{m}}, \quad m = 0, \infty.
\] (59)
We have
\[
\int_0^{\ln 3} |U(s)|^2 \, ds = \int_0^{1/4} |u(\xi)|^2 (1 + 2 \xi) \, d\xi \leq \frac{5}{16} \left( ||u||_{L^2(0,1/4)} \right)^2.
\]
Thus, \(U \in L^2(0, \ln 3)\) and (58), (59) hold. Due to [14, Theorem 3.1, b)], we obtain \(\nu^*_m = 0, m = 0, \infty\), i.e., \(V^0 = W^0 = 0\).

7. Numerical solutions to the approximate controllability problem.

**Proposition 1.** Let \(b > 1, 0 < a < 1\). Then for \(m \in \mathbb{N}\)
\[
\sqrt{\pi} \frac{2^{2m+1}}{e^{2+b/(b-1)} b^{m+1/2} \sqrt{m}} \leq |H_{2m} (i)| \leq \frac{e^{1+a/(1-a)} 2^{m-1/2}}{\pi a^{m+1/2} \sqrt{m}} m!.
\] (60)

**Proof.** Using the following formula from [11, p. 447]:
\[
H_n (x) = \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} (x + iy)^n e^{-y^2} \, dy, \quad n = 0, \infty,
\]
we obtain
\[
H_{2m} (i) = \frac{(-1)^m 2^{2m}}{\sqrt{\pi}} \int_{-\infty}^{\infty} (1 + z)^{2m} e^{-z^2} \, dz, \quad m = 0, \infty.
\]
Since \(z^2 \geq a(z+1)^2 \frac{a}{1-a} \) for \(0 < a < 1\) and \(z \in \mathbb{R}\), we have
\[
|H_{2m} (i)| \leq \frac{2^{2m}}{\sqrt{\pi}} \int_{-\infty}^{\infty} (1 + z)^{2m} e^{-a(z+1)^2} \, dz \leq \frac{2^{2m}}{\sqrt{\pi}} e^{a/(1-a)} (-1)^m \left( \frac{d}{da} \right)^m \int_{-\infty}^{\infty} e^{-a y^2} \, dy
\]
\[
= (-1)^m 2^{2m} e^{a/(1-a)} \left( \frac{d}{da} \right)^m \frac{1}{\sqrt{a}}.
\]
Due to (53), for each $\varepsilon$ the reasoning below.

Using (27), we get

$$|\mathcal{H}_{2m}(i)| \leq \frac{e^{1+a/(1-a)}}{a^{m+1/2}} \frac{e^{2m}}{\sqrt{2\pi}} \left( \frac{2m}{m} \right)^m \frac{\sqrt{2m}}{\sqrt{m}}$$

$$= \frac{2^{2m} e^{1+a/(1-a)}}{\sqrt{\pi} a^{m+1/2}} \left( \frac{m}{e} \right)^m \frac{\sqrt{m}}{\sqrt{m}} \leq \frac{2^{2m} e^{1+a/(1-a)}}{\sqrt{\pi} a^{m+1/2}} \frac{m!}{\sqrt{2\pi} \sqrt{m}}$$

$$= \frac{e^{1+a/(1-a)}}{a^{m+1/2}} \frac{2^{2m-1/2} m!}{\sqrt{2\pi} \sqrt{m}}, \quad m = 0, \infty.$$  

Thus, the right-hand inequality in (60) is proved.

Since $z^2 \leq b(z+1)^2 + \frac{b}{b-1}$ for $b > 1$ and $z \in \mathbb{R}$, then

$$|\mathcal{H}_{2m}(i)| \geq \frac{2^{2m} e^{-b/(b-1)}}{\sqrt{\pi}} \int_{-\infty}^{\infty} (1+z)^{2m} e^{-b(z+1)^2-b/(b-1)} dz$$

$$= \frac{2^{2m} e^{-b/(b-1)}}{\sqrt{\pi}} (-1)^m \left( \frac{d}{db} \right)^m \int_{-\infty}^{\infty} e^{-b y^2} dy$$

$$= e^{-b/(b-1)} \frac{(2m)!}{m! b^{(2m+1)/2}}, \quad m = 0, \infty.$$  

Using again (27), we obtain

$$|\mathcal{H}_{2m}(i)| \geq \frac{2 \sqrt{\pi} e^{-b/(b-1)}}{b^{m+1/2}} \frac{2^{2m} m!}{e} \frac{\sqrt{m}}{\sqrt{m}} \frac{1}{\sqrt{e^{2+b/(b-1)}}} \frac{2^{2m+1} m!}{\sqrt{2\pi} \sqrt{m}}, \quad m = 0, \infty,$$

and the left-hand inequality in (60) is proved.

According to Remark 2, we may suppose $T = 1/2$ without loss of generality in the reasoning below.

Let $T = 1/2$, $W^T \in \tilde{H}^1$, $W^0 \in \tilde{H}^1$,

$$f = W^T - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} * W^0.$$  

For the state $W^0$, let us construct controls approximately targeting the state $W^T$. Denote $F = \mathcal{F}f$. Then

$$f = \sum_{n=0}^{\infty} \omega_n \psi^1_{2n}, \quad F = \sum_{n=0}^{\infty} \omega_n \tilde{\psi}^1_{2n}.$$  

Due to (53), for each $\varepsilon^1 > 0$, there exists $N \in \mathbb{N}$ such that

$$\left( \left\| \sum_{n=N+1}^{\infty} \omega_n \tilde{\psi}^1_{2n} \right\|_1^2 \right)^2 = S_N = \sqrt{\pi} \sum_{n=N+1}^{\infty} |\omega_n| \frac{|\mathcal{H}_{2n}(i)\mathcal{H}_{2n+2}(i)|}{2^{2n+1}(2n)!} < (\varepsilon^1)^2. \quad (61)$$

Put

$$\Omega^N_k = \sum_{n=k}^{N} \omega_n, \quad k = 0, N. \quad (62)$$
Then we have
\[
\sum_{n=0}^{N} \omega_n \hat{\psi}_{2n}^1 = \sum_{n=0}^{N} \omega_n \sum_{k=0}^{n} \mathcal{H}_{2k}(i) \psi_{2k} = \sum_{k=0}^{N} \Omega_k \mathcal{H}_{2k}(i) \psi_{2k} = \sum_{k=0}^{N} \frac{2k!}{2^{2k}k!(2k)!} \psi_{2k} = \sum_{k=0}^{N} \frac{2^{2k}p}{p!(k-p)!} \psi_{2k} = \sum_{p=0}^{2^{p}} \frac{2^{2p}p}{p!(k-p)!} h_p^N \psi_{2p},
\]
(63)
where
\[
h_p^N = \sum_{k=p}^{N} \frac{2^{2k}p}{p!(k-p)!} \Omega_k \mathcal{H}_{2k}(i) \psi_{2k}.
\]
(64)

Let us estimate \( \| \varphi_{2p}^{1,l} \| - \| \varphi_{2p}^{1,2} \| \). We have
\[
\left( \left\| \varphi_{2p}^{1,l} \right\| - \left\| \varphi_{2p}^{1,2} \right\| \right)^2 = \int_{-\infty}^{\infty} \left| \sqrt{1 + \sigma^2} \sigma^2 e^{-\sigma^2/2} \left( \left( \frac{e^{\sigma^2/2} - 1}{\sigma^2/2} \right)^{p+1} - 1 \right) \right|^2 d\sigma.
\]
Since
\[
|y+1|^{p+1} - 1 \leq (p+1)(y+1)^p y, \quad y > 0, \\
e^{\sigma^2/2} - 1 \leq \sigma^2, \quad \sigma > 0,
\]
then
\[
\left| \left( \frac{e^{\sigma^2/2} - 1}{\sigma^2/2} \right)^{p+1} - 1 \right| \leq (p+1) \left( \frac{e^{\sigma^2/2} - 1}{\sigma^2/2} \right)^p \left( \frac{e^{\sigma^2/2} - 1}{\sigma^2/2} - 1 \right) \leq \frac{p+1}{2l} \sigma^2 e^{(p+1)/2}.
\]
Thus,
\[
\left( \left\| \varphi_{2p}^{1,l} \right\| - \left\| \varphi_{2p}^{1,2} \right\| \right)^2 \leq \left( \frac{p+1}{2l} \right)^2 \int_{-\infty}^{\infty} \left( 1 + \sigma^2 \right)^{4p+4} e^{-\sigma^2(1-2p/2)} d\sigma.
\]
if \( l \geq 2(p+1) \). It is easy to see that
\[
\int_{-\infty}^{\infty} \sigma^{2m} e^{-\sigma^2} d\sigma = \sqrt{\pi} \frac{(2m-1)!!}{2^m} \alpha^{-m+1/2}, \quad \alpha > 0, \quad m \in \mathbb{N}.
\]
Therefore,
\[
\left\| \varphi_{2p}^{1,l} \right\| - \left\| \varphi_{2p}^{1,2} \right\| \leq \sqrt{\pi} \frac{p+1}{2l} \left( \frac{(4p+3)!!}{2^{2p+2}} \left( \frac{l}{l-2p-2} \right)^{2p+3/2} \right. \\
+ \left. \frac{(4p+5)!!}{2^{2p+3}} \left( \frac{l}{l-2p-2} \right)^{2p+5/2} \right)^{1/2}. \quad (65)
\]
For any \( N \in \mathbb{N} \) and any \( l > 2N + 1 \), denote
\[
F_N^l = \sum_{p=0}^{N} \frac{2^{2p}p}{p!(2p)!} h_p^N \varphi_{2p}^{1,l}, \quad (66)
\]
where \( h_p^N \) is defined by (64). Due to (61), (63), and (65), we have
\[
\left\| W^T - W_N^l \right\|_1 = \left\| F - F_N^l \right\|_1 \leq \varepsilon^1 + \varepsilon^2, \quad (67)
\]
where
\[ W_N' = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} * W_0 + \scr{F}^{-1} F_N', \]
\[ \varepsilon^2 \geq \frac{\sqrt{\pi}}{2l} \sum_{p=0}^{N} \frac{2^{2p}(p+1)}{(2p)!} |h_N^p| E_p, \]
\[ E_p = \sqrt{\frac{(4p+3)!}{2^{2p+2}} \left( \frac{l}{l-2p-2} \right)^{2p+3/2} + \frac{(4p+5)!}{2^{2p+3}} \left( \frac{l}{l-2p-2} \right)^{2p+5/2}}. \]

Put
\[ u_{N,l} = -\sqrt{\frac{\pi}{2}} \sum_{p=0}^{N} \frac{2^{2p}}{(2p)!} h_N^p u_N^p, \]
where \( u_N^p \) is defined by (39), \( h_N^p \) is defined by (64). With regard to (40) and (66), we get
\[ F_N' = -\sqrt{\frac{2}{\pi}} \Phi_{u_{N,l}} \quad \text{and} \quad W_N' = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} * W_0 - \sqrt{\frac{2}{\pi}} \int_{0}^{T} e^{-\frac{x^2}{2}} u_{N,l}(T-\xi) \frac{d\xi}{\sqrt{2\xi}}. \]

Taking into account (67), we conclude that the state \( W_0 \) is approximately controllable to the state \( W^T \) in the time \( T = 1/2 \) by the controls (68).

8. Examples. The following example illustrates the results of Theorem 2.3.

Example 1. Let \( T > 0, f(x) = \frac{1}{\sqrt{T}} e^{-\frac{x^2}{T}}, x \in \mathbb{R}. \) Evidently, \( f \in \hat{H}^1 \), and condition (20) holds for it. Let us prove that \( f \notin \scr{R}_L^T(0) \) for all \( L > 0 \). Suppose the converse, i.e., suppose \( f \in \scr{R}_L^T(0) \) for some \( L > 0 \). Then there exists \( u \in L^\infty(0,T) \) such that \( \|u\|_{L^\infty(0,T)} \leq L \) and
\[ e^{-T\sigma^2/2} = F(\sigma) = -\sqrt{\frac{2}{\pi}} \int_{0}^{T} e^{-(T-\xi)\sigma^2} u(\xi) d\xi = (\scr{F}u)(-i\sigma^2), \quad \sigma \in \mathbb{R}, \]
where \( F = \scr{F}f, \ U(\xi) = -2u(T-\xi)(H(\xi) - H(\xi-T)). \) Due to the Paley–Wiener theorem, \( \scr{F}u \) is an entire function. Therefore, \( (\scr{F}u)(\mu) = e^{-i\mu T \sigma^2/2}. \) Hence, \( \mu(\xi) = \sqrt{2\pi} \delta(\xi - T/2) \) and \( U \notin L^\infty(0,T). \) Thus, \( f \notin \scr{R}_L^T(0). \) This contradiction proves that \( f \notin \scr{R}_L^T(0). \)

The following two examples illustrate the results of Theorems 2.5 and 2.13.

Example 2. Let \( T = 1, W^T(x) = \frac{1}{\sqrt{2\pi}} \int_{0}^{T} e^{-\frac{x^2}{8} - \frac{2\xi}{\sqrt{T}}} d\xi, \) \( W_0(x) = 0. \) Evidently, condition (20) holds for \( f = W^T. \) We find the controls \( u_N(\xi) = v_N(T-\xi), \xi \in [0,T], \) where \( v_N \) is the solution to (23) for \( N = 2P - 1, P \in \mathbb{N}. \) We use the algorithm given in [9] to find \( v_N \) in the form
\[ v_N(\xi) = \begin{cases} 1 & \text{if } \xi \in [\nu_{2p-1}, \nu_{2p}] \text{ for } p = \frac{1}{2}P, \\ -1 & \text{if } \xi \in [\nu_{2p}, \nu_{2p+1}] \text{ for } p = \frac{0}{2}P, \end{cases} \]
where \( 0 = \nu_0 \leq \nu_1 \leq \nu_2 \leq \cdots \leq \nu_{2P-1} \leq \nu_{2P} \leq \nu_{2P+1} = T. \) Thus the controls \( u_N \) are of the form
\[ u_N(\xi) = \begin{cases} 1 & \text{if } \xi \in [\xi_{2p-1}, \xi_{2p}] \text{ for } p = \frac{1}{2}P, \\ -1 & \text{if } \xi \in [\xi_{2p}, \xi_{2p+1}] \text{ for } p = \frac{0}{2}P, \end{cases} \]
By $W_N$, we denote the value at $t = T$ of the solution to (6), (7) with the control $u = u_N$. The controls $u_N$, $N = 3, 5, 7, 15$, and the influence of these controls on the end states of solutions $W_N$ are given in Figure 2.

(A) $N = 3$, $\xi_0 = 0$, $\xi_1 \approx 0.01769$, $\xi_2 \approx 0.37776$, $\xi_3 \approx 0.61231$, $\xi_4 \approx 0.75224$, $\xi_5 = 1$

(b) $N = 5$, $\xi_0 = 0$, $\xi_1 \approx 0.005922$, $\xi_2 \approx 0.259772$, $\xi_3 \approx 0.367787$, $\xi_4 \approx 0.559854$, $\xi_5 \approx 0.783907$, $\xi_6 \approx 0.837990$, $\xi_7 = 1$

(c) $N = 7$, $\xi_0 = 0$, $\xi_1 \approx 0.00253$, $\xi_2 \approx 0.18727$, $\xi_3 \approx 0.24105$, $\xi_4 \approx 0.42568$, $\xi_5 \approx 0.56972$, $\xi_6 \approx 0.67551$, $\xi_7 \approx 0.86200$, $\xi_8 \approx 0.88684$, $\xi_9 = 1$

(d) $N = 15$, $\xi_0 = 0$, $\xi_1 \approx 0.00026$, $\xi_2 \approx 0.069975$, $\xi_3 \approx 0.07683$, $\xi_4 \approx 0.17566$, $\xi_5 \approx 0.20061$, $\xi_6 \approx 0.30915$, $\xi_7 \approx 0.36190$, $\xi_8 \approx 0.457982$, $\xi_9 \approx 0.540445$, $\xi_{10} \approx 0.60981$, $\xi_{11} \approx 0.71317$, $\xi_{12} \approx 0.75240$, $\xi_{13} \approx 0.857902$, $\xi_{14} \approx 0.873279$, $\xi_{15} \approx 0.957004$, $\xi_{16} \approx 0.959855$, $\xi_{17} = 1$.

(e) The influence of the control $u_N$ on the end state $W_N$ in the cases: (1) $u = 0$, (2) $N = 3$, (3) $N = 5$, (4) $N = 7$, (5) $N = 15$.

(f) The difference $W_T - W_N$ in the cases: (1) $N = 3$, (2) $N = 5$, (3) $N = 7$, (4) $N = 15$.

Figure 2. (A)–(D): The controls $u_N$ defined by (70). (E), (F): The influence of these controls on the end state of the solution to (6), (7) with $W_T(x) = \frac{1}{\sqrt{\pi}} \int_0^T e^{-\frac{2t^2}{\xi}} d\xi$ and $u = u_N$. 
Example 3. Let $T = 1$, $W^T(x) = \frac{1}{\sqrt{T}} \int_0^T e^{-\frac{x^2}{4T}} \frac{2(\xi - \frac{1}{2})^2 - 1}{\sqrt{\xi}} d\xi$, $W^0(x) = 0$. Evidently, condition (20) holds for $f = W^T$. We find the controls $u_N(\xi) = v_N(T-\xi)$, $\xi \in [0, T]$, where $v_N$ is the solution to (23) for $N = 2P - 1$, $P \in \mathbb{N}$. We use the algorithm given in [9] to find $v_N$ in the form (69). By $W_N$, we denote the value at $t = T$ of the solution to (6), (7) with the control $u = u_N$ defined by (70). The controls $u_N$, $N = 3, 5, 7, 15$, and the influence of these controls on the end states of solutions $W_N$ are given in Figure 3.

The following example illustrates the result of Theorems 2.6, 2.14, and numerical method of Section 7.

Example 4. Let $T = 1/2$, $W^T(x) = \cosh x e^{-\frac{x^2}{4} - \frac{1}{4}}$, $W^0(x) = 0$. Consider the approximate controllability problem for system (6), (7). Denote $f = W^T$, $F = \mathcal{F} f$. Then $F(\sigma) = e^{-\sigma^2/2 + 3/4} \cos \sigma$. Since $F \in \tilde{H}_1 \subset \tilde{H}_0$, then

$$V^T = \sum_{m=0}^{\infty} \tilde{\omega}_m \psi_2^0,$$

where $\tilde{\omega}_m = \frac{(-1)^m}{2^{2m}(2m)!}$. Using (51) and (46), we have

$$\left\langle F, \psi_2^1 \right\rangle_1 = \sum_{m=0}^{\infty} \tilde{\omega}_m \left\langle \psi_2^0, \psi_2^1 \right\rangle_1$$

$$= \sum_{m=0}^{\infty} \frac{\tilde{\omega}_m}{2^{2m+2}(2m)!} \left( H_{2m}(i) \left\langle \psi_2^0, \psi_2^{0+2} \right\rangle_0 - H_{2m+2}(i) \left\langle \psi_2^0, \psi_2^{0+2} \right\rangle_0 \right)$$

$$= \frac{\tilde{\omega}_{n+1}}{2^{2n+2}(2n)!} H_{2n}(i) \left\langle \psi_2^{0+2}, \psi_2^{0+2} \right\rangle_0 - \frac{\tilde{\omega}_n}{2^{2n+2}(2n)!} H_{2n+2}(i) \left\langle \psi_2^{0+2}, \psi_2^{0+2} \right\rangle_0$$

$$= (-1)^n \frac{H_{2n}(i)}{2^{2n+2}(2n)!} \sqrt{\pi} 2^{2n+2}(2n+2)!$$

$$- (-1)^n \frac{H_{2n+2}(i)}{2^{2n+2}(2n)!} \sqrt{\pi} 2^{2n}(2n)!$$

$$= (-1)^{n+1} \sqrt{\pi} \frac{H_{2n}(i) + H_{2n+2}(i)}{2^{2n+2}(2n)!}.$$

Therefore,

$$\omega_n = \frac{\left\langle F, \psi_2^1 \right\rangle_1}{\left\langle \psi_2^1, \psi_2^1 \right\rangle_1} = (-1)^n \frac{H_{2n}(i) + H_{2n+2}(i)}{H_{2n}(i) H_{2n+2}(i)}, \quad \omega_n = \rho_n \psi_2^1.$$
\begin{itemize}
  \item[(A)] \( N = 3 \), \( \xi_0 = 0 \), \( \xi_1 \approx 0.38575 \), \( \xi_2 \approx 0.46602 \), \( \xi_3 \approx 0.72627 \), \( \xi_4 \approx 0.97933 \), \( \xi_5 = 1 \)
  \item[(B)] \( N = 5 \), \( \xi_0 = 0 \), \( \xi_1 \approx 0.26349 \), \( \xi_2 \approx 0.28768 \), \( \xi_3 \approx 0.55421 \), \( \xi_4 \approx 0.67565 \), \( \xi_5 \approx 0.80465 \), \( \xi_6 \approx 0.99236 \), \( \xi_5 = 1 \)
  \item[(C)] \( N = 7 \), \( \xi_0 = 0 \), \( \xi_1 \approx 0.18768 \), \( \xi_2 \approx 0.19625 \), \( \xi_3 \approx 0.42618 \), \( \xi_4 \approx 0.48263 \), \( \xi_5 \approx 0.65917 \), \( \xi_6 \approx 0.78484 \), \( \xi_7 \approx 0.85387 \), \( \xi_8 \approx 0.99651 \), \( \xi_5 = 1 \)
  \item[(D)] \( N = 15 \), \( \xi_0 = 0 \), \( \xi_1 \approx 0.06836 \), \( \xi_2 \approx 0.06878 \), \( \xi_3 \approx 0.17460 \), \( \xi_4 \approx 0.17853 \), \( \xi_5 \approx 0.30885 \), \( \xi_6 \approx 0.32391 \), \( \xi_7 \approx 0.45536 \), \( \xi_8 \approx 0.49091 \), \( \xi_9 \approx 0.60057 \), \( \xi_{10} \approx 0.66095 \), \( \xi_{11} \approx 0.73455 \), \( \xi_{12} \approx 0.81336 \), \( \xi_{13} \approx 0.84990 \), \( \xi_{14} \approx 0.92928 \), \( \xi_{15} \approx 0.93978 \), \( \xi_{16} \approx 0.99958 \), \( \xi_{17} = 1 \).
\end{itemize}

\begin{figure}
\centering
\begin{subfigure}{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{(A) The influence of the control \( u_N \) on the end state \( W_N \) in the cases: \( \circ \) \( a = 0 \), \( \bullet \) \( a = 3 \), \( \square \) \( a = 5 \), \( \triangle \) \( a = 7 \), \( \square \) \( a = 15 \).}
\end{subfigure} \quad
\begin{subfigure}{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{(B) The difference \( W^T - W_N \) in the cases: \( \circ \) \( N = 3 \), \( \square \) \( N = 5 \), \( \triangle \) \( N = 7 \), \( \square \) \( N = 15 \).}
\end{subfigure}
\end{figure}

\textbf{Figure 3.} (A)-(D): The controls \( u_N \) defined by (70). (E), (F): The influence of these controls on the end state of the solution to (6), (7) with \( W^T(x) = \frac{1}{\sqrt{2\pi}} \int_0^T e^{-\frac{x^2}{2\sigma^2}} d\xi \) and \( u = u_N \).

Using Proposition 1 and taking \( a = 3/4 \) and \( b = 3/2 \) in (60), we obtain

\[
\frac{|H_{2m+2}(i)|}{|H_{2m}(i)|} \leq \sqrt{2} \frac{e^{3+a/(1-a)+b/(b-1)}\eta^{m+1/2}}{\pi^{3/2}a^{m+3/2}} \sqrt{m(m+1)} \leq \frac{e^{9/2}m}{3\pi^{3/2}(m+1)}.
\]
Continuing (72), we get
\[ S_N \leq \sqrt{\pi} \frac{9}{4} \sum_{m=N+1}^{\infty} \frac{\varepsilon^9 \beta^m}{3^{3/2} \pi^{m+2} (2m)!} \frac{m+1}{2^{m+1/2} (2m+1)!} \]
\[ = 3\varepsilon^9 \frac{2^{3/2}}{2\pi} \left( \sum_{m=N+1}^{\infty} \frac{1}{2^{m+1/2} (2m+1)!} + \sum_{m=N+1}^{\infty} \frac{1}{2^{m+1} (2m)!} \right) \]
\[ \leq 3\varepsilon^9 \frac{2^{3/2}}{2\pi} \left( \frac{1}{\sqrt{2}} \frac{\cosh \left( \frac{1}{\sqrt{2}} \right)}{(2N+1)!} \left( \frac{1}{\sqrt{2}} \right)^{2N+1} + \frac{\sinh \left( \frac{1}{\sqrt{2}} \right)}{(2N+2)!} \left( \frac{1}{\sqrt{2}} \right)^{2N+2} \right) \]
\[ = 3\varepsilon^9 \frac{2^{N+2}}{2^{N+2} \pi} \left( \frac{1}{2^{N+1}} + \frac{\sinh \left( \frac{1}{\sqrt{2}} \right)}{(2N+2)!} \right). \] (73)

Put
\[ u_{N,l} = -\sqrt{\frac{\pi}{2}} \sum_{p=0}^{N} \frac{2^{2p}}{(2p)!} h_p^N u_p^l, \]
where \( u_p^l \) is defined by (39), \( h_p^N \) is defined by (64). Due to (67) for \( l \geq 2(N + 1) \), we get
\[ \| W^\top - W_N^l \| \leq \varepsilon_N^1 + \varepsilon_{N,l}^2, \]
where
\[ W_N^l = -\sqrt{\frac{2}{\pi}} \int_0^T e^{-\frac{\pi}{4} u_{N,l}(T - \xi)} d\xi, \]
\[ \varepsilon_N^1 = \sqrt{3\varepsilon^9 \frac{2^{N+2}}{2\pi} \left( \frac{1}{2^{N+1}} + \frac{\sinh \left( \frac{1}{\sqrt{2}} \right)}{(2N+2)!} \right)}, \]
\[ \varepsilon_{N,l}^2 = \sqrt{\frac{\pi}{2l}} \sum_{p=0}^{N} \frac{2^{2p}(p+1)}{(2p)!} |h_p^N| E_p, \]
\[ E_p = \sqrt{(4p + 3)!!} \left( \frac{l}{l - 2p - 2} \right)^{2p+3/2} + \frac{(4p + 5)!!}{(l - 2p - 2)^{2p+3}}. \]

Thus, the controls \( u_{N,l}, l = 2(N + 1), \infty, N = 1, \infty \), solve the approximate controllability problem for the given system.

Some estimates for \( \| W^\top - W_N^l \| \) are given in Table 1 and the influence of the control \( u_{N,l} \) on the end state \( W_N^l \) of solution to (6), (7) with the control \( u = u_{N,l} \) and the target state \( W^\top \) is shown in Figure 4. The shape of the control in the case of \( N = 3, l = 200 \) and the case of \( N = 4, l = 150 \) are similar to the shape of the control in the case of \( N = 3, l = 100 \) and the case of \( N = 4, l = 400 \), respectively.

In this section two methods of solving to approximate controllability problems for system (6), (7) are illustrated. The first of them deals with initial and target states of special form (see Theorem 2.13) but gives us controls \( \{u_N\}_{N=1}^\infty \subset L^\infty[0,T] \) that
Table 1. The estimates for $\|W_T - W_N^l\|_1$.

| $N, l$   | $\varepsilon_N^1$ | $\varepsilon_N^2$ | $\varepsilon_N^1 + \varepsilon_N^2$ |
|----------|--------------------|--------------------|--------------------------------------|
| $N = 3, l = 100$ | 0.18666          | 0.12756         | 0.31422                              |
| $N = 3, l = 200$ | 0.18666          | 0.05927         | 0.24593                              |
| $N = 4, l = 150$ | 0.01535          | 0.08648         | 0.10183                              |
| $N = 4, l = 400$ | 0.01535          | 0.03038         | 0.04573                              |

Figure 4. (A), (B): The controls $u_{N,l}$ defined by (70). (C), (D): The influence of these controls on the end state $W_N^l$ of the solution to (6), (7) with $W_T(x) = \cosh xe^{-\frac{x^2}{4}}$ and $u = u_{N,l}$. 

(c) The given $W_T(x)$.

(d) The difference $W_T - W_N^l$ in the cases: 1) $N = 3, l = 100$; 2) $N = 3, l = 200$; 3) $N = 4, l = 150$; 4) $N = 4, l = 400$. 

are uniformly bounded with a given constant $L > 0$: $\|u_N\|_{L^\infty[0, T]} \leq L$, $N = 1, \infty$.

The second of them has no restrictions on initial and target states (Theorem 2.14), but controls $\{u_N\}_{N=1}^\infty \subset L^\infty[0, T]$ solving this problem are not uniformly bounded: $\|u_N\|_{L^\infty[0, T]} \to \infty$ as $N \to \infty$.

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