The forgotten Potts models

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Abstract

The $q = 10$ and $q = 200$ state Potts models coupled to $2d$ gravity are investigated numerically and shown to have continuous phase transitions, contrary to their behavior on a regular lattice. Critical exponents are extracted and possible critical behavior for general $q$-state Potts models coupled to gravity is discussed.

1 Introduction

In the continuum formulation of $2d$ quantum gravity one is interested in studying the coupling of conformally invariant matter to gravity. The conformal matter is characterized by its central charge $c$ but yet it has only been possible to solve theories with $c \leq 1$ analytically.

This has triggered a lot of numerical work on discretized models of $2d$ gravity with $c > 1$ \cite{1}. One such discretization method is dynamical triangulations where the integration over space-time manifolds is replaced by a sum over piecewise linear surfaces constructed by gluing together equilateral triangles. The matter is simulated by spins, living either on triangles or vertices. The spin models usually used are the $q = 2$, 3 and 4-state Potts models and the Gaussian model. All these models have continuous phase transitions on regular lattices, with divergent correlation length, and hence can be associated with some conformal field theories in the continuum limit. The above mentioned models all correspond to conformal theories with $c \leq 1$ but taking multiple copies of them allows one to study numerically theories with $c > 1$ coupled to gravity. For unlike in the continuum formulation the discretized models are well defined for $c > 1$. 
For $q > 4$ the Potts models have a discontinuous or 1st order phase transition on regular lattices and hence no corresponding conformal field theories. So until now coupling them to gravity has been discarded as uninteresting. But a recent mean field solution of the $q = \infty$ state Potts model coupled to gravity showed the model to have a 3rd order magnetization transition from a high-temperature branched polymer phase to a low temperature pure gravity phase ($c = 0$) \[2\]. Moreover the model was shown to be equivalent to infinite copies of Ising models coupled to gravity (a $c = \infty$ model).

By expanding around the $q = \infty$ solution this analysis was extended down to finite values of $q$. There two phase transitions were found with high- and low-temperature pure gravity phases separated by a branched polymer phase. The critical behavior of those transitions was identical, both had the specific heat exponent $\alpha = -1$. As the value of $q$ is decreased the two phase transitions merge at some critical value $q_c$, and the branched polymer phase disappears. At that value of $q$ (estimated to be $q_c \approx 120$ in \[2\]) the perturbation expansion breaks down.

It is worth comparing this with the situation for multiple $q \leq 4$ state Potts models coupled to gravity. For $c \leq 1$ the interaction between gravity and matter is weak, for all values of the coupling except $\beta_c$ we have a pure gravity phase. But for larger values of $c$ numerical simulations indicate a high- and low-temperature pure gravity phase with a branched polymer phase in between, i.e. the same scenario as for high-$q$.

So the similarity between these two cases might indicate that the study of Potts models for large value of $q$ might give us informations about models with large central charge coupled to gravity. It might be possible that after coupling to gravity these model would describe some conformal matter coupled to gravity, the only problem being that we have not yet identified these conformal theories.

This has prompted us to look at the $q = 10$ and $q = 200$ state Potts models coupled to gravity applying Monte Carlo simulations \[3\]. Using micro-canonical simulations (fixed area) the models are defined by

$$Z(\beta, N) = \sum_{T \in \mathcal{T}(N)} \sum_{\{\sigma_i\}} \exp \left( \beta \sum_{(i,j)} \delta_{\sigma_i, \sigma_j} \right)$$

\[(1)\]

where $\sigma_i \in \{1, ..., q\}$ are the Potts spins, $i$ is a lattice triangle, $\{\sigma_i\}$ a spin configuration on the triangulation $T$ and $\mathcal{T}(N)$ an appropriate class of triangulations (of size $N$). In these simulations we include triangulations al-
Figure 1: (a) The specific heat $C_V$ versus $\beta$, for different lattice sizes, for the $q = 10$ state Potts model coupled to gravity. (b) The energy cumulant $V_L$ versus $\beta$ for the same model.

allowing sites joined by more than one link and sites connected to them selves as these can be shown to reduce finite size effects [3]. The triangulations were updated using the link-flip algorithm and the spin models with the Swendsen-Wang cluster algorithm. The lattice sizes used were between 250 and 8000 triangles and, as the auto-correlations increase linearly with $q$, we used $5 \times 10^6$ and $2 \times 10^7$ sweeps for each $\beta$ value for the $q = 10$ and $q = 200$ state Potts model respectively.

2 The order of the phase transitions

The first interesting question is the order of the phase transitions, are they continuous or discontinuous after coupling to gravity. This we determine by looking at the scaling of the maximum of the specific heat with lattice size. If a phase transition is 1st order $C_V(\text{max})$ should scale linearly with the volume. From fig. 1a, where we plot the specific heat versus $\beta$ for the $q = 10$ state Potts model, we see that the maximum increases slowly with lattice size. Looking at the quantity $C_V(\text{max})/N$ it approaches zero in the infinity volume limit, which is a strong indication for a continuous phase transition.

Another quantity sensitive to the order of the phase transition is the
Table 1: Measured critical exponents for the $q = 10$ and $q = 200$ state Potts models coupled to gravity. For $\nu d_H$ we have values from (a) the scaling of $\max\{\partial BC/\partial \beta\}$ and (b) the scaling of $M$ and $\partial M/\partial \beta$. In a same way for $\alpha$ (a) comes from $C_V(\max)$ and (b) from $C_V(\beta_c)$.

\[
V_L = 1 - \frac{\langle E^4 \rangle}{\langle E^2 \rangle^2}.
\]

If the transition is 1st order $V_L$ should reach a non-zero minimum in the critical point, otherwise it is zero for all values of $\beta$. In fig. 1b we show that for the $q = 10$ state Potts model no indication of a non-zero minimum can be seen.

Exactly the same is seen for the $q = 200$ state Potts model, and from these measurements we conclude that both models have continuous magnetization transitions after coupling to gravity. It it might at first seem strange that the order of the phase transition can change that drastically with coupling to gravity, but we should remember that even for one Ising model (which can be solved explicitly) there is a change from a 2nd to a 3rd order transition with coupling to gravity.

### 3 Critical exponents

To locate the critical temperatures for the models we used the finite size scaling behavior of two quantities: (a) location of the peak in the specific heat and (b) the intersection of Binder’s cumulant for different lattice sizes. Both quantities are expected to approach $\beta_c$ as $N^{-1/\nu d_H}$. The fits to the scaling behavior are made easier as $\nu d_H$ can be determined directly from Binder’s cumulant, as the maximum of its slope scales as $N^{\nu d_H}$. The results
are shown in table 1. We then applied standard finite size scaling in $\beta_c$ to extract the critical exponents $\beta$ (from the magnetization $M$), $\nu d_H$ (from $\partial M/\partial \beta$), $\gamma$ (from the magnetic susceptibility $\chi$) and $\alpha$ (both from the scaling of $C_V(\text{max})$ and $C_V(\beta_c)$). The results are shown in table 1. For $q = 10$ we tested that the scaling of the specific heat fits equally well to logarithmic divergence ($C_V \sim \log(N)$).

4 Discussion

It is an interesting observation that the critical exponents for the $q = 10$ state Potts model are very close to that of the $q = 4$ state Potts model coupled to gravity (which has $\beta = 1/2$, $\gamma = 1$, $\alpha = 0$ and $\nu d_H = 2$). As the value $q = 10$ is chosen arbitrary this might indicate that there exists an whole range of $q$ values, starting at $q = 4$ and ending at some $q_c$, where we have the same critical behavior after coupling to gravity.

On the other hand the $q = 200$ state Potts model has $\alpha$ close to $-1$, so we might be in the high-$q$ region were the calculation in [2] is valid. Then we would expect a branched polymer phase and two phase transitions. Unfortunately we have only been able to locate one, but as the other one is not a magnetization transition it is not clear how to identify it. But for an interval of $\beta$, before $\beta_c$, we see some evidence of a branched polymer phase, supporting the picture in [2].

From this the following critical behavior of the $q$-state Potts models coupled to gravity sounds reasonable: For $q = 4$ up to some $q_{c1}$ we have the same critical behavior, i.e. a 2nd order phase transition ($\alpha = 0$) where the details of the Potts models are less important than the coupling to gravity. And for $q > q_{c2}$ we also have the same critical behavior, this time with (presumably two) 3rd order phase transitions ($\alpha = -1$) and a branched polymer phase. Whether $q_{c1} = q_{c2}$ or if we have some different behavior at intermediary values of $q$ is not determined in these simulations. But it is clear that we need some theoretical understanding of what kind of conformal matter (if any) the $q > 4$ state Potts model coupled to gravity describes.

References

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