Explicit R-matrices for inhomogeneous 3D chiral Potts models: Integrability and the action formulation for IM

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Abstract

We construct the exact spectral parameter dependent vertex R-matrix for the classical 3D \( \mathcal{N} \)-state chiral Potts models, convenient for considering the model in context of the Bethe ansatz. The R-matrix is defined on the \( \mathcal{N}^4 \) dimensional space \( V_\mathcal{N} \otimes V_\mathcal{N} \otimes V_\mathcal{N} \otimes V_\mathcal{N} \), appropriate for consideration by means of the cube-equations defined in [14]. We present the 2D quantum spin Hamiltonians for general case and, at \( \mathcal{N} = 2 \), a fermionic lattice action representation corresponding to 3D Ising’s statistical model.

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1 Introduction

The Ising models are the ones of the most investigated statistical integrable models in one and two dimensions [1, 2, 6, 3, 11, 12, 13], and until now there are intensive investigations for elucidating the problem for three and more dimensions [28, 29, 30, 31, 35, 36, 37]. A direct integrable generalization of the 2D IM is the $\mathcal{N}$-state chiral Potts model [15, 16, 17, 18, 19, 20]. In this work we are demonstrating the 3D version of the vertex $R$-matrices for general inhomogeneous $\mathcal{N}$-state Potts models, starting from the $\mathcal{N} = 2$ case of IM. The explicit integrability conditions are not investigated yet, however as all the 2D projection matrices are the solutions to the Yang-Baxter equations, hence the investigating of the models on the surfaces would bring us to deal with (1+1)d integrable models.

The second section devotes to the investigation of the 3D Ising model by means of the technique which we have used in [26] for the investigation of 2D spin models. As it was done in [26], we define here an explicit form of the $R$-matrix, starting from the classical statistical weights of the model. Evaluating appropriate unitary local transformations of the states and operators we establish the operator form of $R$. Although the states are defined on the vertices and the interaction is considered as around a cube, however in this form the model can be considered as a "vertex" model. This is the direct analog of the 2D situation in [26], where we used vertex-like Yang-Baxter equations for the $R$-matrices defined on the faces of the lattices (with the spin-states situated on the vertexes). In the Bethe ansatz concept two neighboring R-matrices defined such way in the transfer matrices have common vertices, but no common links (for 2D cases) or faces (for 3D cases). One can try to employ here the cube equations presented in [14], which are appropriately defined for the vertex kind four-state R-matrices.

In the next section we generalize our approach to $\mathcal{N}$-state chiral Potts model. As it is known, for 2D case, the chiral Potts model is the integral generalisation of the 2DIM [15]-[21]. The algebraic structure of the corresponding 3D four-state R-matrix is presented. The corresponding 2D quantum spin Hamiltonian operators are presented also.

Then, in fourth section, we are formulating a scalar fermionic action for 3DIM model
(N = 2), representing the partition function in the coherent-state fermionic basis as a continual integral. Free fermionic conditions are presented. The fermionic interpretation of IM is not new [2, 31, 11, 6, 28, 30], for 3D case see for example the super-symmetric non-interacting strings model in [28], or in [30]. As it was stated in [28], the model is reduced to the two-dimensional supersymmetric Liouville theory, so, at critical point the 3D Ising model should be described by a conformal field theory. The information of statistical characteristics are obtained by Monte Carlo simulations, and there are numerous works using the conformal field conception [35, 37]. Approximate value of the homogeneous coupling constant at the critical point has been found here in the free fermionic limit and the difference from known approximate value of the constant presented in the mentioned works is \( \approx 0.05 \).

2 The 3D weight matrix and the corresponding R: IM

Here we investigate the one of the simplest 3D spin models - 3DIM. The statistical weight of the 3D Ising model, defined on the cell of the cubic lattice \( N \times N \times N \), can be written as follows

\[
W_{\sigma_1, \sigma_2, \sigma_3, \sigma_4} = e^{J_1(\sigma_1 \sigma_2 + \sigma_3 \sigma_4) + J_2(\sigma_1 \sigma_3 + \sigma_1 \sigma_4 + \sigma_2 \sigma_3) + J_3 \sum_i \sigma_1 \sigma_i},
\]

(2.1)

where \( \sigma_i = \pm 1 \) are the projections of the spin operator on the site indexed by \( i \). Then the statistical sum reads

\[
Z = \prod_{i_x, i_y, i_z=1}^N W_{\bar{\sigma}_{2i_x}, \bar{\sigma}_{2i_y}, \bar{\sigma}_{2i_z}, \bar{\sigma}_{2i_x} + \bar{\sigma}_{2i_y} + \bar{\sigma}_{2i_z}}.
\]

(2.2)

Here the projections of the vector \( \vec{a} = \{a_x, a_y, a_z\} \) are the spacings of the 3D cubic lattice in the corresponding spacial directions, the sites on the lattice are denoted by \( a_i = \{i_x a_x, i_y a_y, i_z a_z\} \), where \( (i_x, i_y, i_z) = 1, \cdots N \). As for the 2D case, we can perform following unitary transformation, at each site of the lattice placing the unity \( I = U^{-1} \times U \), with 

\[
U = \frac{1}{\sqrt{2}} \binom{1 - 1}{1 - 1}
\]

\[
R = U \otimes U \otimes U \otimes W \otimes U^{-1} \otimes U^{-1} \otimes U^{-1} \otimes U^{-1}.
\]
The form of $R$ contrary to $W$ has the advantage, namely it contains only the elements for which the constraint $\sum_{i=1}^{4} \alpha_i = \sum_{i=1}^{4} \beta_i + \text{mod}(2)$ does take place. We can present this $2^4 \times 2^4$-dimensional matrix in this $2^2 \times 2^2$ operator-matrix form:

$$
R = \begin{pmatrix}
R_{00}^{00} & R_{00}^{01} & R_{00}^{10} & R_{00}^{11} \\
R_{01}^{00} & R_{01}^{01} & R_{01}^{10} & R_{01}^{11} \\
R_{10}^{00} & R_{10}^{01} & R_{10}^{10} & R_{10}^{11} \\
R_{11}^{00} & R_{11}^{01} & R_{11}^{10} & R_{11}^{11}
\end{pmatrix}
$$

(2.4)

where the operators $R_{ij}^{kl}$ themselves can be presented as $2^2 \times 2^2$ matrices with corresponding statistical weights $R_{ij}^{kl}$ by shifting the values of the indexes of $R_{\tilde{\alpha}_1 \tilde{\alpha}_2 \tilde{\alpha}_3 \tilde{\alpha}_4}$ in (2.3) as $i_k = (\tilde{\alpha}_k + 1)/2, j_k = (\tilde{\beta}_k + 1)/2$.

For the operators $R_{ij}^{kl} = R_{ij}^{kl}$ which have the property $\sum_{k=1}^{2} i_k = \sum_{k=1}^{2} j_k + \text{mod}(2)$, the corresponding matrix also have the same structure, i.e.

$$
R_{ij}^{kl} = \begin{pmatrix}
R_{ij}^{kl} & 0 & 0 & R_{ij}^{kl} \\
0 & R_{ij}^{kl} & R_{ij}^{kl} & 0 \\
0 & R_{ij}^{kl} & R_{ij}^{kl} & 0 \\
R_{ij}^{kl} & 0 & 0 & R_{ij}^{kl}
\end{pmatrix}.
$$

(2.5)

For the remaining matrix operators, when $\sum_{k=1}^{2} i_k = \sum_{k=1}^{2} j_k + 1 + \text{mod}(2)$, correspondingly we can deduce

$$
R_{ij}^{kl} = \begin{pmatrix}
0 & R_{ij}^{kl} & R_{ij}^{kl} & 0 \\
R_{ij}^{kl} & 0 & 0 & R_{ij}^{kl} \\
R_{ij}^{kl} & 0 & 0 & R_{ij}^{kl} \\
0 & R_{ij}^{kl} & R_{ij}^{kl} & 0
\end{pmatrix}.
$$

(2.6)

This operator can be represented by means of the tensor products of the basic $2 \times 2$ matrices, in terms of the generators of the algebra $sl(2), \sigma^z = \{1, 0\, \overline{10}, -1\}, \sigma^+ = \{01, \overline{00}\}, \sigma^- = \{00, \overline{10}\}$ and the unity operator $I = \{10, \overline{01}\}$. Let us write the $R$-matrix in this operator form, where we have used following notations $\sigma_0^0 = (I + \sigma^z)/2, \sigma_1^1 = (I - \sigma^z)/2, \sigma_0^1 = \sigma^+, \sigma_1^0 = \sigma^-$:

$$
R = R_{ij}^{kl} \sigma_{i_1}^{j_1} \otimes \sigma_{i_2}^{j_2} \otimes \sigma_{i_3}^{j_3} \otimes \sigma_{i_4}^{j_4},
$$

(2.7)
and in the same time we can write also $\mathbf{R} = \sigma^{j_1}_{i_1} \otimes \sigma^{j_2}_{i_2} \mathbf{R}^{j_1 j_2}_{i_1 i_2}$.

**The matrix elements of the operator $R$.** The elements of the matrix $\mathbf{R}^{j_1 j_2}_{i_1 i_2}$ are presented explicitly below:

We can note, that for this matrix there are the following symmetry relations:

\[
R^{j_1 j_2}_{i_1 i_2} R^{j_3 j_4}_{i_3 i_4} = R^{j_1 j_2 j_3}_{i_1 i_2 i_3} R^{j_4}_{i_4},
\]

And we can represent the following matrix elements by the following expressions, where we have take $J_{1,2,3} = J_{y,x,z}$ and the remaining elements can be found just from the above relations:

\[
\begin{align*}
R^{0000}_{0000} &= 4(1 - 2 \cosh 2J_y \cosh 2J_z) + \cosh 4J_y + \cosh 4J_z - 2 \sinh 2J_y (1 + \cosh 4J_y \cosh 4J_z) - 2 \cosh 2J_y \cosh 2J_z, \\
R^{0010}_{0000} &= R^{0010}_{0010} = R^{0100}_{1000} = R^{1000}_{0000} = 2 \sinh 2J_y (\cosh 4J_y \cosh 4J_z), \\
R^{0011}_{0011} &= 4 \sinh 2J_y \cosh 2J_z (\cosh 4J_y \cosh 2J_z - \cosh 2J_z), \\
R^{0101}_{0101} &= -4 \sinh 2J_y \cosh 2J_z (\cosh 4J_y \cosh 2J_z + (1 + \cosh 4J_y \cosh 4J_z) \sinh 4J_z), \\
R^{0001}_{0001} &= -4 \sinh 2J_z (\cosh 2J_z \cosh 4J_z \cosh 4J_y \cosh 4J_z \sinh 2J_z \sinh 4J_z), \\
R^{0100}_{0100} &= 4 \sinh 2J_z \sinh 2J_y \sinh 2J_z (-1 + 2 \cosh 2J_y \cosh 2J_z), \\
R^{0010}_{0010} &= 4 \sinh 2J_z \sinh 2J_y \cosh 2J_z (-1 + 2 \cosh 2J_y \cosh 2J_z), \\
R^{0110}_{0110} &= -2 + \cosh 4J_y - \cosh 4J_y + \cosh 4J_z + \cosh 4J_y \cosh 4J_z, \\
R^{0111}_{0111} &= 4 \cosh 2J_y \cosh 2J_z \cosh 4J_y \cosh 4J_z \sinh 4J_y, \\
R^{0110}_{0110} &= -1 + \cosh 4J_y \cosh 4J_z \sinh 4J_z, \\
R^{1101}_{1101} &= 2 - \cosh 4J_y - \cosh 4J_x - \cosh 4J_x \cosh 4J_y \cosh 4J_z, \\
R^{1100}_{1100} &= (1 + \cosh 4J_y \cosh 4J_z) \sinh 4J_y, \\
R^{1111}_{1111} &= 4(1 + 2 \cosh 2J_y \cosh 2J_z) + \cosh 4J_x + \cosh 4J_y + \cosh 4J_z + \cosh 4J_y \cosh 4J_z, \\
R^{1110}_{1110} &= 4(\cosh 2J_x + \cosh 4J_x \cosh 2J_y \cosh 2J_z) \sinh 2J_y \sinh 2J_z, \\
R^{1111}_{1111} &= 4 \cosh 2J_z \sinh 2J_y \cosh 2J_x + (1 + \cosh 4J_x \cosh 4J_z) \sinh 4J_y, \\
R^{1101}_{1101} &= -2 + \cosh 4J_x + \cosh 4J_y - \cosh 4J_y \cosh 4J_y \cosh 4J_z, \\
R^{1100}_{1100} &= -\sinh 4J_y (-1 + \cosh 4J_y \cosh 4J_z), \\
R^{1001}_{1001} &= \sinh 4J_y \cosh 4J_z \cosh 4J_y \sinh 4J_y, \\
R^{0001}_{0001} &= \sinh 4J_y \cosh 4J_y \sinh 4J_y,
\end{align*}
\]
\[ R_{0010}^{10} = \sinh 4J_x \sinh 4J_y \cosh 4J_z, \]
\[ R_{0101}^{10} = \sinh 4J_x \sinh 4J_y \sinh 4J_z, \]
\[ R_{1100}^{10} = 4 \sinh 2J_x \cosh 2J_y \cosh 2J_z + (1 + \cosh 4J_y \cosh 4J_z) \sinh 4J_x, \]
\[ R_{1010}^{10} = 4 \sinh 2J_x \sinh 2J_y (\cosh 2J_y + \cosh 2J_z \cosh 4J_y \cosh 2J_z), \]
\[ R_{1101}^{10} = 4 \sinh 2J_x \sinh 2J_y \sinh 2J_z + \sinh 4J_x \sinh 4J_y \sinh 4J_z, \]
\[ R_{1011}^{10} = 4 \sinh 2J_x \sinh 2J_y (\cosh 2J_z + \cosh 2J_y \cosh 2J_y \cosh 4J_y \cosh 4J_z), \]

One can go to the Cardy’s limit \[ 2J_x \approx J_1 \Delta t, 2J_y \approx J_2 \Delta t, e^{-2J_z} \approx h \Delta t, \] with \( \Delta t \ll 1 \), in order to organize continuous limit in third direction, which can be regarded as time. Thus, we can connect three dimensional statistical model with the quantum two dimensional problem, described by the corresponding Hamiltonian operator. As an example, the expansion of matrix element \( R_{0000}^{0000} \) gives \( 4(1 - 2 \cosh 2J_x \cosh 2J_y \cosh 2J_z) + \cosh 4J_x + \cosh 4J_y + \cosh 4J_z + \cosh 4J_x \cosh 4J_y \cosh 4J_z \approx 4(1 - (h \Delta t + \frac{1}{4\Delta t})) + 2 + ((h \Delta t)^2 + \frac{1}{16\Delta t^2}) + ((h \Delta t)^2 + \frac{1}{16\Delta t^2}) \approx 2(\frac{1}{\Delta t})^2 (1 - 2h \Delta t + O(\Delta t)). \) In the leading order the expansion of the \( R \)-matrix in its operator form is giving:

\[
R = \frac{1}{(h \Delta t)^2} (I \otimes I \otimes I \otimes I + \Delta t | I \otimes \sigma_x \otimes I \otimes I + \sigma_x \otimes I \otimes \sigma_x \otimes I \otimes I) + J_2 (I \otimes I \otimes \sigma_x \otimes \sigma_x + \sigma_x \otimes \sigma_x \otimes I \otimes I) - h (I \otimes I \otimes I \otimes \sigma_z) + I \otimes I \otimes \sigma_z \otimes I \otimes I + \sigma_z \otimes I \otimes I \otimes I) \right] \right). \tag{2.10}
\]

The operator in the parentheses coming with the coefficient \( \Delta t \) presents the cell Hamiltonian for 2D quantum spin model. Thus, the corresponding Hamiltonian defined on square lattice reads

\[
H = \sum_{i,j} (J_1[\sigma_x(2i, 2j)\sigma_x(2i, 2j + 1) + \sigma_x(2i + 1, 2j)\sigma_x(2i + 1, 2j + 1)] + J_2[\sigma_x(2i, 2j)\sigma_x(2i + 1, 2j) + \sigma_x(2i, 2j + 1)\sigma_x(2i + 1, 2j + 1)] - h[\sigma_z(2i, 2j) + \sigma_z(2i + 1, 2j) + \sigma_z(2i, 2j + 1) + \sigma_z(2i + 1, 2j + 1)]) \tag{2.11}
\]

At \( J_1 = 0 \) or \( J_2 = 0 \) this expression splits into the sum of two quantum 1D Ising model’s Hamiltonian operators defined on the parallel chains (rows) of the square lattice.
\[ R_{1234} = \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{R-matrix structure of 3D cubic lattice (i) and Cubic $R_{1234}$-matrix (ii)}
\end{figure}

3 \ $N$-state chiral Potts model: the weight matrix, four-state vertex R-matrix and 2D quantum Hamiltonian

The $N$-state Potts model is the generalization of the IM, when at each site there are attached $N$ spin variables: $\bar{\sigma} = \{e^{2\pi i \bar{n}}\}, \bar{n} = 0, 1, ..., N - 1 \mod N$. If to reformulate the product of the spins in the definition (2.1) by the following expression - $\bar{\sigma}_\alpha \bar{\sigma}_\gamma \Rightarrow N(\delta(\bar{\sigma}_\alpha - \bar{\sigma}_\gamma) - 1/N)$ (which is obviously an equality at $N = 2$), then we shall have the weight function of the $N$-state ordinary Potts model defined on the three dimensional cubic lattice

\[ [W^P_{\bar{\sigma}_{\alpha_1} \bar{\sigma}_{\alpha_2} \bar{\sigma}_{\alpha_3} \bar{\sigma}_{\alpha_4}]} = e^{H_1 + H_2 + H_3}, \]

\[ H_1 = J_1 N\left(\delta(\bar{\sigma}_{\alpha_1} - \bar{\sigma}_{\alpha_2}) + \delta(\bar{\sigma}_{\alpha_3} - \bar{\sigma}_{\alpha_4}) + \delta(\bar{\sigma}_{\beta_1} - \bar{\sigma}_{\beta_2}) + \delta(\bar{\sigma}_{\beta_3} - \bar{\sigma}_{\beta_4}) - \frac{4}{N}\right), \]

\[ H_2 = J_2 N\left(\delta(\bar{\sigma}_{\alpha_1} - \bar{\sigma}_{\alpha_3}) + \delta(\bar{\sigma}_{\alpha_2} - \bar{\sigma}_{\alpha_4}) + \delta(\bar{\sigma}_{\beta_1} - \bar{\sigma}_{\beta_3}) + \delta(\bar{\sigma}_{\beta_2} - \bar{\sigma}_{\beta_4}) - \frac{4}{N}\right), \]

\[ H_3 = J_3 N\sum_i^4 \left(\delta(\bar{\sigma}_{\alpha_i} - \bar{\sigma}_{\beta_i}) - \frac{1}{N}\right). \]

The statistical sum is reproduced in the same way as in (2.2). The two dimensional statistical model (if, e.g. $J_3=0$) at the self-dual point $(e^{J_1} - 1)(e^{J_2} - 1) = N$ is the $Z_N$ parafermionic Fateev-Zamolodchikov model which has second order transition and can be described by conformal field theory (with $c = 2(N - 1)/(N + 2)$) [15]. An integrable generalization of this model is the 2D chiral Potts model, for which 2D vertex $R$-matrix has been constructed and which is satisfying the ordinary Yang-Baxter equations.
3D version of chiral Potts model can be constructed in the following way. Note, that
\[
\delta(\bar{\sigma}_1^{\alpha} - \bar{\sigma}_2^{\alpha}) = \delta(e^{\frac{2\pi i n_1^{\alpha}}{N}} - e^{\frac{2\pi i n_2^{\alpha}}{N}}) \equiv \delta(\bar{n}_1^{\alpha} - \bar{n}_2^{\alpha} + \text{mod}N) = \frac{1}{N} \sum_{k=0}^{N-1} e^{\frac{2\pi ik}{N}(\bar{n}_1^{\alpha} - \bar{n}_2^{\alpha})}. \tag{3.2}
\]

In chiral models there is assumed asymmetry, which allows to attach to each summand of this sum with power \(k\) its own coupling coefficient \((J_a)^k\). E.g., for the 3D chiral Potts model the cell operators \(H_r\) can be formulated as follows

\[
H_1 = \sum_{k=1}^{N-1} J_1^k \left( e^{\frac{2\pi ik}{N}(\bar{n}_1^{\alpha} - \bar{n}_2^{\alpha})} + e^{\frac{2\pi ik}{N}(\bar{n}_3^{\alpha} - \bar{n}_4^{\alpha})} + e^{\frac{2\pi ik}{N}(\bar{n}_1^{\beta} - \bar{n}_2^{\beta})} + e^{\frac{2\pi ik}{N}(\bar{n}_3^{\beta} - \bar{n}_4^{\beta})} \right),
\]

\[
H_2 = \sum_{k=1}^{N-1} J_2^k \left( e^{\frac{2\pi ik}{N}(\bar{n}_1^{\alpha} - \bar{n}_3^{\alpha})} + e^{\frac{2\pi ik}{N}(\bar{n}_2^{\alpha} - \bar{n}_4^{\alpha})} + e^{\frac{2\pi ik}{N}(\bar{n}_1^{\beta} - \bar{n}_3^{\beta})} + e^{\frac{2\pi ik}{N}(\bar{n}_2^{\beta} - \bar{n}_4^{\beta})} \right),
\]

\[
H_3 = \sum_{k=1}^{N-1} \left( J_3^k \sum_{i=1}^{4} e^{\frac{2\pi ik}{N}(\bar{n}_1^{\alpha} - \bar{n}_i^{\beta})} \right). \tag{3.3}
\]

The vertex \(R^P\)-operator for \(N\)-state case can be obtained from the statistical weight in similarity with the Ising case (2.3) by using the generalization of unitary 2 \(\times\) 2-operator \(U\) to case of \(N \times N\)-operators \(U^N_s\) which has the matrix elements

\[
[U^N]_k^p = \frac{1}{\sqrt{N}} e^{\frac{2\pi i (k-1)(p-1)}{N}}, \quad k, p = 1, \ldots, N. \tag{3.4}
\]

In order to reproduce the 2D quantum Hamiltonian corresponding to 3D chiral Potts model, we shall follow the logic of the works [21]. We can involve \(Z^N\)-symmetry operators \(X, X^+\) and \(Z, Z^+\). The operators act on the linear space with basis vectors \(|k\rangle, k = 0, \cdots, N - 1\) as

\[
Z_k^p = \delta_k^{p-1}, \quad X_k^p = e^{\frac{2\pi i}{N} \delta_k^p}, \quad X^N = 1, \quad Z^N = 1, \quad XZ = e^{\frac{2\pi i}{N}} ZX. \tag{3.5}
\]

Once we define the weights \(W^x(n_a - n_b), W^y(n_a - n_b)\) and \(W^z(n_a - n_b)\) on the links which connect the vertexes \((a, b)\) with the state-variables \(\sigma_{a,b} = e^{\frac{2\pi i(n_a - n_b)}{N}}\) along the axes \(x, y\) and \(z\) as

\[
W^x(n_a - n_b) = e^{\sum_{h=1}^{N-1} J_1^h e^{\frac{2\pi i(n_a - n_b)h}{N}}},
\]
we can reformulate the weight matrices in a following way

\[ W^y(n_a - n_b) = e^{\sum_{k=1}^{N} J_k e^{\frac{2\pi i}{N} (n_a - n_b)k}}, \]
\[ W^x(n_a - n_b) = e^{\sum_{k=1}^{N} J_k e^{\frac{2\pi i}{N} (n_a - n_b)k}}, \]

For the reformulation of the $R$ and construct the 2D quantum lattice Hamiltonian for the chiral Potts model.

The Hamiltonian will have similar to the 1D quantum chain chiral Potts Hamiltonian view

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For the reformulation of the $R^P$-operator in terms of $Z$, $X$ matrices, let us define the following link transfer operators: horizontal $S^{1,2}$ transfer operator along the axes $x, y$ and the vertical transfer operator $T$ along the axis $z$.

\[ S^{1,2} = \sum W^{x,y}(k) (X \otimes X^+)^k, \quad W^{x,y}(k) = \frac{1}{N} W^{x,y}(n) e^{\frac{2\pi i}{N} (nk)}, \]
\[ W^{x,y}(\bar{n} - \tilde{n}) = \langle \bar{n} | \otimes \bar{n}' | S^{1,2} | \bar{n}' \rangle \otimes | \tilde{n} \rangle, \]
\[ T = \sum W^z(k) Z^k, \quad W^z(\bar{n} - \tilde{n}) = \langle \bar{n} | T | \tilde{n}' \rangle. \]

Then

\[ [R^P]_{1234} = \left[ [S^1]_{12}([S^2]_{13} \otimes [S^2]_{24})[S^1]_{34} \right] (T_1 \otimes T_2 \otimes T_3 \otimes T_4) \left[ [S^1]_{12}([S^2]_{13} \otimes [S^2]_{24})[S^1]_{34} \right]. \]

After implementation of the unitary transformations by means of the mentioned operators, which is actually Fourier transformation of the vector basis, the matrix forms of the operators $X$ and $Z$ are interchanging their view, and now the operator $Z$ is diagonal.

As in the case of 3DIM we can take the Cardy’s limit at $\Delta t \ll 1$ for this generalized case and construct the 2D quantum lattice Hamiltonian for the chiral Potts model.

\[ R^P = I \otimes I \otimes I \otimes I + \Delta t H^P. \]

The Hamiltonian will have similar to the 1D quantum chain chiral Potts Hamiltonian view [21]
\[ H^P = \sum_{i,j}^{N-1} \left( J_{zk} \left[ X^k(2i, 2j)X^{+k}(2i, 2j + 1) + X^k(2i + 1, 2j)X^{+k}(2i + 1, 2j + 1) \right] \right) + J_{yk} \left[ X^k(2i, 2j)X^{+k}(2i + 1, 2j) + X^k(2i + 1, 2j + 1) \right] \]

\[ + Z^k(2i + 1, 2j) + Z^k(2i, 2j + 1) + Z^k(2i, 2j + 1) \] \hspace{1cm} (3.13)

The Hamiltonian of this basis has the symmetry \([H^P, Z^P] = 0\) with charge \(Z^P = \prod_{i,j}^{N,N} Z(i, j)\).

4 \( \mathcal{N} = 2 \): The fermionic representation: Free fermionic conditions

In the article [26] we have represented the 2d model in terms of the graded basis [23], associating with each site of the lattice a pair of the creation and annihilation fermionic operators (so called "0"-spin fermions), \(c_\alpha^+, c_\alpha, c_\alpha^+c_\alpha' + c_\alpha'c_\alpha^+ = \delta_{\alpha\alpha'}\). Then the operators defined above \(\sigma_j^i\) in the Fock space with basis \(|0\rangle, |1\rangle = c^+|0\rangle\) can be represented in terms of the fermionic operators, \(\sigma_1^0 = |0\rangle\langle 1| = c, \quad \sigma_1^1 = |1\rangle\langle 0| = c^+, \quad \sigma_0^0 = |0\rangle\langle 0| = 1 - c^+c, \quad \sigma_0^1 = |1\rangle\langle 1| = c^+c\). (4.1)

This is the reflection of the spin-fermion correspondence (Jordan-Wigner transformation) [13]. In the lattices with definite arrangement of the sites at which the spin operators \(\sigma_\alpha^{\pm, z}\) are attached, the Jordan-Wigner transformation is non-local, in order to ensure the anti-commutation behavior of the fermionic operators at different sites:

\(\sigma_\alpha^+ = \prod_{\gamma=1}^{\alpha-1} \left[ 1 - 2c_\gamma^+c_\gamma \right] c_\alpha^+, \quad \sigma_\alpha^- = \prod_{\gamma=1}^{\alpha-1} \left[ 1 - 2c_\gamma^+c_\gamma \right] c_\alpha, \quad \sigma_\alpha^z = 2c_\alpha^+c_\alpha - 1\). (4.2)

For the three dimensional cubic lattice the variable \(\alpha\) denotes the vertices labelled with the integers \(\{i, j, k\}\) corresponding to the coordinates \(\{x, y, z\} = \{ia, ja, ka\}\) - where \(a\) is the lattice spacing.
The 3D R-matrix (2.7) under consideration can be expressed by fermionic operators in accordance with approach developed in articles [22, 23, 24, 25] and adapted for evaluating the partition functions (2D IM, XY cases) in [26]. As a result we shall have the following graded formulae for the operator (2.7) defined on the space \( V_1 \otimes V_2 \otimes V_3 \otimes V_4, V_k = \{0\}_k |1\}_k \},

\[
R_{1234} = \sum_{i_k=1,j_k=1[k=1,2,3,4]} R_{i_1i_2i_3i_4}^{j_1j_2j_3j_4} |j_1\>_1 |j_2\>_2 |j_3\>_3 |j_4\>_4 \langle i_1\>_1 \langle i_2\>_2 \langle i_3\>_3 \langle i_4\>_4 = \quad (4.3)
\]

Here the factor \((-1)^{p(R)}\) indicates, that in fermionic representation the Fock space is graded, and the states \(|0\>_i, |1\>_j\) have different gradings: the states \(|0\>_i\) with different \(i\) are commutative with one another and with the states \(|1\>_j\), and they have the parity \(p(0) = 0\), meanwhile the states \(|1\>_i\) with different \(i\) are anti-commute and have the parity \(p(1) = 1\). This means \(|a_i\>_i|a_j\>_j = |a_j\>_j|a_i\>_i(-1)^{p(a_i)p(a_j)}\). Thus we can check the parity \(p(R)\) of the \(R\)-operator in the relation (4.3),

\[
p(R) = \sum_{t=1}^{3} p(i_t) \sum_{k=t+1}^{4} (p(i_k) + p(j_k)). \quad (4.4)
\]

The fermionic representation of the discussed \(R\)-matrix is a local operator, as it is even operator in terms of the fermionic operators, and it means that the non-local term of the Jordan-Wigner transformation must be counted even times, and thus must be reduced, as \((1 - 2n)(1 - 2n) = 1\). What is the advantage of the fermionic representation - it gives an opportunity to represent the statistical sum (partition function) and the other statistical quantities as integrals with respect to the fermionic variables. This can be achieved by means of the coherent basis of the fermionic operators formulated via the Grassmann variables \(\psi_i, \bar{\psi}_i\) (5.1), which fulfill orthonormality and completeness relations, see Appendix A(5.2).

So we can represent the R-matrices in the form of \(R = A_0 : e^{A(\bar{c}c)} :\), and in the general case the fermionic action for the elementary cell of the cubic lattice can have interaction terms up to the 8th degree \(A = \sum_{i=1}^{4} A_i[c'c]^i\), where \(c', c\) are the fermionic operators from the set \(c_i, \bar{c}_j\) situated on the sites of the cube. It happens, that for the case of 2D Ising
model the fermionic action contains only quadratic terms [26] and so describes free fermions [5].

Then one can represent the partition function of the model defined on the 3D cubic lattice as integral representation over the fermionic lattice action

$$Z = \prod_{3D} R = (A_0)^{N^2} \int D\psi D\bar{\psi} e^{\sum_{3D} A((\psi),\bar{\psi})-\sum \bar{\psi}\psi}. \quad (4.5)$$

This can be achieved by writing in the partition function all the R-matrices in terms of the coherent basis, situating the unity operators in the operator form (5.5) at each vertex of the 3D cubic lattice, and then taking the trace (5.4).

In the coherent basis the cell action for the cube R-matrix which acts on the vector spaces on the square (see the figure 1) with the vertices noted by \{1, 2, 3, 4\}, has the following form

$$\langle \bar{\psi}_4|\langle \bar{\psi}_3|\langle \bar{\psi}_2|\langle \bar{\psi}_1|R|\psi_1\rangle|\psi_2\rangle|\psi_3\rangle|\psi_4\rangle = A_0 e^{A(\bar{\psi},\psi)}+\sum_i \bar{\psi}_i\psi_i, \quad (4.6)$$

where

$$A = A_2 + A_4 + A_6 + A_8, \quad (4.7)$$

$$A_2 = \sum_{i,j=1}^{4} a_{ij}^2 \bar{\psi}_i\psi_j + \sum_{i<j}^{4} a_{ij}\bar{\psi}_i\psi_j + \sum_{i<j=1}^{4} a_{ij}^2 \bar{\psi}_i\psi_j, \quad (4.8)$$

$$A_4 = \sum_{i,j,k,r=1}^{4} \left( a_{kr}^{ij} \bar{\psi}_i\psi_j\bar{\psi}_k\psi_r + a_{r}^{ij} \bar{\psi}_i\psi_j\bar{\psi}_k\psi_r + a_{jkr}^{i} \bar{\psi}_i\psi_j\bar{\psi}_k\psi_r \right) + a_{1234}^{1234} \bar{\psi}_1\bar{\psi}_2\bar{\psi}_3\bar{\psi}_4 + a_{1234}^{1234} \bar{\psi}_1\bar{\psi}_2\bar{\psi}_3\bar{\psi}_4, \quad (4.9)$$

$$A_6 = \sum_{i,j,k,r,p,t=1}^{4} \left( a_{1234}^{ij} \bar{\psi}_i\psi_j\bar{\psi}_1\bar{\psi}_2\bar{\psi}_3\bar{\psi}_4 + a_{pt}^{1234} \bar{\psi}_1\bar{\psi}_2\bar{\psi}_3\bar{\psi}_4\bar{\psi}_p\psi_t + a_{p}^{ij} \bar{\psi}_i\bar{\psi}_j\bar{\psi}_1\bar{\psi}_2\bar{\psi}_3\bar{\psi}_4 \right), \quad (4.10)$$

$$A_8 = a_{1234}^{1234} \bar{\psi}_1\bar{\psi}_2\bar{\psi}_3\bar{\psi}_4 \psi_1\psi_2\psi_3\psi_4. \quad (4.11)$$

Comparing the expressions of two realizations of the R-matrix we can easily find the relations between the coefficients $a_{-\ldots-}$ and the matrix elements $R_{-\ldots-}$. Particularly:

$$A_0 = R_{0000}^{0000}, \quad a_{ij} = R_{-i-j-}^{0000}(-1)^p/A_0, \quad a_{ij} = R_{0000}^{i-j-}(-1)^p/A_0,$$

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\[ a^j_i = R^j_i - (-1)^p / A_0 - \delta^j_i, \quad a^{kr}_ij = R^{-k-r}_{i-j} - (-1)^p / A_0 + \delta^k_i \delta^r_j + R^{-k-r}_{i-j} \delta^r_j (-1)^p / A_0 + \]

\[ + R^{-k-r}_{i-j} \delta^k_j (-1)^p / A_0 + R^{-k-r}_{j-i} \delta^r_i (-1)^p / A_0 - (a^r_ia^k_j (-1)^p + a^r_ja^k_i (-1)^p) - a^r_ia^k_j (-1)^p, \]

\[ a_{1234} = R^{0000}_{1111} / A_0 - a_{12}a_{34} + a_{13}a_{24} - a_{14}a_{23}, \]

\[ a^{1234} = R^{1111}_{0000} / A_0 - a^{12}a^{34} + a^{13}a^{24} - a^{14}a^{23}, \]

\[ (4.12) \]

\[ a^{ij}_r = R^{-r}_{i-j-k}(-1)^p / A_0 - (a^{i,k}_r a^{j}_k (-1)^p + a^{ij}_r a^{k}_k (-1)^p) - a^{ij}_r a^{k}_k (-1)^p, \]

\[ a^{ijk}_r = R^{-r}_{i-j-k}(-1)^p / A_0 - (a^{i,k}_r a^{j}_k (-1)^p + a^{ij}_r a^{k}_k (-1)^p) - a^{ij}_r a^{k}_k (-1)^p, \]

\[ a^{ijk}_r = R^{-r}_{i-j-k}(-1)^p / A_0 - (a^{i,k}_r a^{j}_k (-1)^p + a^{ij}_r a^{k}_k (-1)^p) - a^{ij}_r a^{k}_k (-1)^p, \]

The expressions for the elements \( A_6 \) and \( A_8 \) can be easily deduced in the same manner. Here by \( R^{-j-}_{-i-} \), \( R^{0000}_{-i-j-} \), \( R^{0000}_{-i-j-} \), ... we denote the matrix elements, for which all the indexes are 0, besides of those, which are at the positions \( i, j, ... \), e.g. \( R^{-4-}_{-1-} = R^{0001}_{1000} \). The sign \((-1)^p\) is taking into account the grading. The parity of each summand in the expressions must be checked separately. Here we can write explicitly some of the \( a \)-coefficients. At first, the coefficients of \( A_2 \) easily can be derived from (2.9). It is clear that the \( a^{ij}_r \) have the same symmetries, as the matrix elements \( R^{-j-}_{-i-} \) in (2.9), e.g. \( a^1_1 = a^2_2 = a^3_3 = a^4_4, a^2_1 = a^4_2, \) and so on. The interaction terms \( A_k, k > 2 \) also can be exactly calculated. Particularly, some of the expressions in \( A_4 \) are identically null, such as the terms \( a^{12}_{12} = 0, a^{34}_{34} = 0, a^{13}_{13} = 0, a^{24}_{24} = 0, a_{1234} = a^{1234} = 0, \) meanwhile the expressions of the following terms are

\[ a^{23}_{23} = -a^{14}_{14} = -a^{23}_{14} = 16(\sinh 2J_1 \sinh 2J_2 \sinh 2J_3)^2 / A_0^2 \] (4.13)

\[ a^{34}_{1} = -16(\sinh 2J_1 \sinh 2J_2 \sinh 2J_3)(\cosh 2J_1 \cosh 2J_2 - \cosh 2J_3) / A_0^2 \] (4.14)

From the checking all the expressions in the sets \( A_4, A_6, A_8 \) it follows that free fermionic condition (i.e. \( A_{k>2} = 0 \)) means

\[ \sinh J_1 \sinh J_2 \sinh J_3 = 0, \] (4.15)

as all the functions in \( A_4 \) are proportional to \( \sinh J_1 \sinh J_2 \sinh J_3 \). This is the case of 2DIM. The statistical sum of any free model can be evaluated simply by direct calculations, particularly using the Fourier transformation in the Grassmann variable’s space.
For being precise, before performing the fermionic transformation of the $R$-matrix, one should define at first the non-check graded $R$ matrix, as

$$R^{j_1 j_2 j_3 j_4}_{i_1 i_2 i_3 i_4} = (-1)^{\sum_{k=1}^4 (p_{i_k} \sum_{l=-k}^1 p_{j_l})} R^{j_4 j_3 j_2 j_1}_{i_1 i_2 i_3 i_4},$$  \hspace{1cm} (4.16)$$

which means, that in the formulas for $A_k$ we must take into account the following transformations $j_i \rightarrow j_{5-i}$ for the upper indexes, and the corresponding changes in the signs.

Critical point of the model in the free fermionic limit, for small coupling constants

As it is known, the critical point of the three-dimensional Ising model is described by a conformal field theory [28], and the conformal field theory is under active investigation using the method of the conformal bootstrap [35, 37]. By means of this method and by Monte Carlo simulations there are obtained rather precise information about the critical exponents. For the homogeneous 3D IM the best known critical value the of coupling constant is 0.22165455.

We also can try to find the critical points in the free fermionic limit. This means, that we must take only the quadratic part of the action in (4.7), which is justified at small $J_i$-s, as we can see from the exact values of the coefficients in the terms $A_4$, $A_6$ and $A_8$. Then, following to the steps in [26], where an exact calculations has been done for 2DIM, we can perform a Fourier transformation of the fermionic basis in 3D lattice with antiperiodic boundary conditions. After redefining the Grassmann fields at the half of the momenta space as

$$\tilde{\psi}_i (\pi - p_x, \pi - p_y, \pi - p_z) = \psi_{i+4} (p_x, p_y, p_z) \text{ and } \tilde{\psi}_i (\pi - p_x, \pi - p_y, \pi - p_z) = -\psi_{i+4} (p_x, p_y, p_z)$$

$(i = 1, 2, 3, 4)$, we can represent the partition function as a product of the determinants of $8 \times 8$ matrices. The zeroes of the partition function give the approximated value of the coupling constant, which is $J^f_c = 0.270325$ in the free fermionic limit.

Summary and Acknowledgements

In this work we have presented the 3D generalizations of the 2D integrable models - IM and $\mathcal{N}$-state chiral Potts model in the vertex four-state $R$-matrix formulation. This will give an
advantage in the theoretical (in the framework of 3D ABA) and numerical investigations of these models or their modifications.

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5 Appendix

A: Coherent Basis and Grassmann variables

\[ c_i | \psi_i \rangle = | \psi_i \rangle c_i, \quad \langle \bar{\psi}_i | c_i^+ = \langle \bar{\psi}_i | \bar{\psi}_i, \]

\[ \langle \bar{\psi}_i | \psi_j \rangle = \delta_{ij}, \quad \int d\bar{\psi}_i d\psi_i e^{-\bar{\psi}_i \psi_i} \langle \bar{\psi}_i | = I. \] (5.1)

Any operator \( K(\{c_i^+, c_j\}) \) in the fermionic coherent basis reads as

\[ K(\{c_i^+, c_j\}) = \prod \bar{\psi}_i K(\{c_i^+, c_j\}) \prod \psi_j = e^{\sum_i \bar{\psi}_i \psi_i} K(\{\bar{\psi}_i, \psi_j\}). \] (5.2)

The trace of the operator \( K(\{c_i^+, c_j\}) \) in coherent-states is an integral over the Grassmann variables,

\[ \text{tr} K(\{c_i^+, c_j\}) = \int D\psi D\bar{\psi} e^{\sum \bar{\psi}_i \psi_i} K(\{\bar{\psi}_i, \psi_j\}), \quad D\psi D\bar{\psi} = \prod_i d\psi_id\bar{\psi}_i. \] (5.3)

The following integral representation takes place for the identity operator.

\[ I = \int d\bar{\psi}(i, j) d\psi(i, j) e^{-\bar{\psi}(i, j) \psi(i, j)} |\psi(i, j)\rangle \langle \bar{\psi}(i, j)|. \] (5.4)

B: Local integrability equations for R-matrices defined on the cube

One can use the following simplest extension of the Baxter’s transformations \([6]\) for proper parametrization of the \( R(u, w) \)-matrix elements in order to check the integrability properties in the context of the Bethe Ansatz.

\[ e^{\pm 2J_1} = \text{cn}[i, u, k] \mp i \text{sn}[i, u, k], \]

\[ e^{\pm 2J_2} = \text{cn}[i, w, k] \mp i \text{sn}[i, w, k], \]

\[ e^{\pm 2J_3} = i(\text{dn}[i (u + w), k] \pm 1)/(k \text{sn}[i (u + w), k]). \] (5.5)
When \( J_1 = 0 \) or \( J_2 = 0 \) these relations are equivalent to the corresponding formulas for the 2d case. Another possible variation of the Baxters transformation could have such kind expression for \( J_3 \):

\[
e^{\pm 2J_3} = i(\text{dn}(u, k) + \text{dn}(w, k) \pm 1)/(k \text{sn}(u + w, k))
\]

(5.7)

By these transformations for 2d case one ensures the form of the \( R(u) \) matrix satisfying Yang-Baxter equations (YBE) with additive spectral parameter -

\[
R(u_{12})R(u_{13})R(u_{23}) = R(u_{23})R(u_{13})R(u_{12}),
\]

where \( u_{ij} = u_i - u_j \).

The symbolic extension of this relation for 3D case with \( R(u, v, u + v) \)-matrix has the form

\[
R(u_{12}, u_{51}, u_{52})R(u_{34}, u_{53}, u_{54})R(u_{36}, u_{13}, u_{16})R(u_{46}, u_{24}, u_{26}) = R(u_{46}, u_{24}, u_{26})R(u_{36}, u_{13}, u_{16})R(u_{34}, u_{53}, u_{54})R(u_{12}u_{51}, u_{52}).
\]

(5.8)

The spectral parameter dependence here is taken as for the standard 3D vertex R-matrix, defined on the tensor product of three vector spaces \( V_i \otimes V_j \otimes V_k \), for which the local integrability conditions are the vertex version of ZTE or Semi-Tetrahedral equations (see [8], [27]). The spectral parameters are attached to the three lines orthogonal to the faces of the cubes. If to check the cube equations taking the constructed \( R \)-matrices, of course, there are solutions to these equations, which correspond to the situations equivalent to the 2D case - \( J_{1,2} = 0 \), when the cube equations transform to the set of YB equations. However for general case one can take the suggested parametrization as a starting point, and look for the solutions after modifications both of the R-matrices and local equations.

Note, that for the cube equations also one can suggest restricted variant of the equations - simplified cube equations, where as intertwiner matrices one can take two-particle \( R_{ij} \) matrices, as in [27].
C: Integrable 3D model with general R matrix of non-symmetric free-fermionic structure: Commutativity of the transfer matrices. One can note, that if the given R-matrix of any dimensional statistical model has a such structure that it lets possible to represent the statistical sum as a generating functional with free particle action, then the model is integrable. Such model is an integrable model, but however it does not mean that in Bethe anzats framework such R-matrix satisfies to a local integrability condition, or transfer matrices with different spectral parameters commute and there is an intertwiner matrix ensuring it. For the known free-fermionic cases (XX, XY or 2D IM) the R-matrix itself has a similar structure \( R_f \equiv e^{A_{c \bar{c}}} :. \) For example, for the most and entirely investigated 2D case, the most general form \( R_f = e^{\sum_{i,j=1}^{2} a_i \bar{c}_i c_j + \sum_{i>j} a_{ij} \bar{c}_i c_j} : \) has arbitrary coefficients, meanwhile YBE solutions put definite restrictions on them, see e.g [6, 26] for homogeneous YBE, and [32, 33] for inhomogeneous YBE. The periodic quadratic operators can be easily diagonalized in the Fourier transformation basis.

The free-fermionic conditions for 3D matrix with standard vertex structure \( R_{ijk} \) (the vector states are situated on the six links) is presented in [27], and a solution to semi-tetrahedral equations is presented therein. The free fermionic 3D models are considered also in [10]. As we have seen in this article the free-fermionic condition of the 3D IM brings to the relation (4.15). In general case the free-fermionic conditions for the cube \( R_{ijk\bar{k}} \)-matrix (the vector states are on the eight vertices) can be defined in similar manner, expanding the corresponding exponent in the normal ordered form and comparing the matrix elements. As example we can present a relation

\[
P_{0000}^{0010} P_{1010}^{1010} = P_{1000}^{1000} P_{0010}^{0010} + P_{0100}^{0100} P_{1000}^{1000} - P_{0010}^{0010} P_{1010}^{1010}.
\]  

In fact, such kind equations, as in [27], mean the equalities between the appropriate matrix-minors in the R-matrix.
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