Chemical potentials in three-dimensional higher spin anti-de Sitter gravity

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Abstract

We indicate how to introduce chemical potentials for higher spin charges in higher spin anti-de Sitter gravity in a manner that manifestly preserves the original asymptotic $W$-symmetry. This is done by switching on a non-vanishing component of the connection along the temporal (thermal) circles. We first recall the procedure in the pure gravity case (no higher spin) where the only “chemical potentials” are the temperature and the chemical potential associated with the angular momentum. We then generalize to the higher spin case. We find that there is no tension with the $W_N$ or $W_N$ asymptotic algebra, which is obviously unchanged by the introduction of the chemical potentials. Our argument is not perturbative in the chemical potentials.

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I. INTRODUCTION

Higher spin gauge theories in 3 spacetime dimensions [1–4], which provide a useful laboratory for understanding higher spin gauge theories in 4 and higher dimensions [5–7], have attracted recently a considerable amount of interest. One reason for this surge of activity is the rich asymptotic structure displayed by the theory at infinity, where the $W$-algebras, or their supersymmetric extensions in the graded case, emerge as asymptotic symmetry algebras [8–10]. This opens the door to an investigation of holography with the powerful tools of two-dimensional conformal field theory and representation theory of $W$-algebras [11–14].

As it is by now well known, anti-de Sitter gravity in 3 dimensions is described by a $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ gauge theory [15, 16]. It was recognized in [17] that the conditions expressing that the gravitational field approaches at infinity the anti-de Sitter solution give, in the Chern-Simons formulation, the conditions implementing the familiar Hamiltonian reduction of the $\mathfrak{sl}(2,\mathbb{R})$-current algebra to the Virasoro algebra [18–21]. This yields the gauge theory derivation of the asymptotic Virasoro algebra and central charge first obtained in [22] in the metric formulation.

The remarkable fact that the geometrical anti-de Sitter boundary conditions implement the algebraic Hamiltonian reduction remains valid for simple and extended supergravities [23, 24] and for higher spin gauge theories [8–10].

Recently, exact black hole solutions supporting a non trivial higher spin field have been obtained [25–27] (see also [28]). However, in spite of the simplicity of these black hole solutions, a suitable characterization of their global charges and their entropy is a subject which is not free of controversy, because there are tensions between various approaches, which give different results (see [29] for a lucid discussion). We show in this paper that this tension is somewhat artificial because it results from a non standard incorporation of the chemical potentials that obscures the asymptotics and hence the correct definition of the charges. Once the chemical potentials are properly introduced along the lines indicated below, there is no difficulty with the asymptotics. Our approach uses the Hamiltonian formalism, which provides a particularly transparent analysis. It is not perturbative.

Our paper is organized as follows. In the next section, we recall how the chemical potentials are introduced in the metric formulation of pure gravity in three dimensions and then translate the results in Chern-Simons terms. We find that the chemical potentials ap-
pear through the temporal components of the connection (along the thermal circles). This is in perfect agreement with experience from four-dimensional gravity where the chemical potential for the electric charge is well known to be associated with the zeroth component of the electromagnetic vector potential in the Reissner-Nordström solution. In Section III we extend the analysis to include higher spin charges and their chemical potentials. The approach makes it obvious that non-vanishing chemical potentials do not change the asymptotical properties because these potentials enter only the Lagrange multipliers. Finally, we give comments and conclusions in Section IV. We display a black hole solution that fulfills our conditions. In a subsequent paper [30, 31], we shall further discuss the asymptotics and the thermodynamics of this solution. It should be stressed that our method agrees with the discussion of [32, 33].

II. CHEMICAL POTENTIALS IN THE CHERN-SIMONS FORMULATION OF PURE ANTI-DE SITTER GRAVITY

To begin with, we start with a discussion on how the temperature and the chemical potential for the angular momentum enter in the \( sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R}) \) formulation of gravity. This simpler case illuminates the central points. Similar considerations may be found in [34].

A. Metric formulation

In the usual formulation of black hole thermodynamics, the temperature and the chemical potential for the angular momentum do not enter the metric of the black hole explicitly. They appear indirectly through the identifications involving the imaginary time and the angle, which must be made to avoid a singularity at the horizon in the Euclidean section. This means that the range of the coordinates is not fixed but varies from one solution to another.

It is useful to have a description in which the range of the coordinates is fixed once and for all. This can be achieved by redefining the time coordinates \( t \rightarrow \lambda t' \) and \( \theta \rightarrow \theta' = \theta + \omega t \), where \( \lambda \) and \( \omega \) are chosen such that \( t' \) and \( \theta' \) have a constant range. This induces a non trivial lapse and shift in the three-dimensional black hole solution [35, 36], which reads
(dropping primes on coordinates),

\[ ds^2 = -(N_\infty)^2 f^2 dt^2 + f^{-2} dr^2 + r^2 \left[ \left( -\frac{J}{2r^2} N_\infty + N_\infty^\theta \right) dt + d\theta \right]^2 \]  

(1)

with

\[ f^2 = \left( \frac{r}{T} \right)^2 - M + \frac{J^2}{4r^2}. \]

(2)

If one chooses the coordinates \( t \) and \( \theta \) such that \( N_\infty = 1, N_\infty^\theta = 0 \), then the ranges of the identifications in \( t \) and \( \theta \) depend on the solution. If one wants fixed ranges, one must therefore allow for \( N_\infty \) and \( N_\infty^\theta \) to vary. We impose that on the Euclidean section \( t \sim t + 2\pi l \) and \( \theta \sim \theta + 2\pi \) (always). The variables \( N_\infty \) and \( N_\infty^\theta \) are clearly related to the temperature and the chemical potential for the angular momentum and will for this reason be called “the chemical potentials”. [We use quotation marks here because the temperature stands on a special footing but nevertheless it is convenient in what follows to include it among the standard chemical potentials.]

We shall from now on deal with the grand canonical ensemble, where the chemical potentials are held fixed to arbitrary values. The appropriate variational principle has then \( N_\infty \) and \( N_\infty^\theta \) fixed. One finds the value of the conjugate variables, namely the mass \( M \) and the angular momentum \( J \) on-shell, by requiring the absence of singularity in the Euclidean section at the horizon, which imposes in particular that \( N_\infty^\theta = -\frac{J}{2r^2} N_\infty + N_\infty^\theta \) should vanish at the horizon.

When \( N_\infty \neq 1 \) and \( N_\infty^\theta \neq 0 \), the metric does not fulfill at infinity the boundary conditions of [22], which, from the present perspective, would correspond to fixed \( \beta = \frac{1}{2\pi l} \) and zero chemical potential for the angular momentum. However, it is very easy to translate these boundary conditions to generic values of the chemical potentials, just like it is very easy to translate the asymptotic flat boundary conditions written in cartesian coordinates to spherical coordinates through the appropriate coordinate transformation. The asymptotic symmetry is of course the same. When the chemical potentials are introduced, one should not talk about a relaxation of the boundary conditions, but rather of a (straightforward in this case) extension of the formalism to cover different values of the (held fixed) chemical potentials.

The only case where the metric is not asymptotically AdS is when \( N_\infty = 0 \), which corresponds to the infinite temperature limit and to a degenerate metric (\( \det g = 0 \)). We shall not consider this case in this paper.
B. Connection formulation

How do the chemical potentials enter the Chern-Simons connection? We claim that they appear as additional contributions to the thermal circles around the horizon (dt contributions to the connection), explicitly (after the $r$-dependent gauge transformation of [17] has been performed to eliminate the $r$-dependence to leading order):

$$a^\pm = \pm \left( L^\pm_{\pm 1} - \frac{2\pi}{k} \mathcal{L}_\pm L^\pm_{\mp 1} \right) dx^\pm \pm \frac{1}{l} \Lambda^\pm (\nu_\pm) dt , \quad \text{(3)}$$

$$\Lambda^\pm (\nu_\pm) = \nu_\pm L^\pm_{\pm 1} - \frac{2\pi}{k} \nu_\pm \mathcal{L}_\pm L^\pm_{\mp 1} ,$$

(asymptotically) where $\nu^\pm$ are constants and called the chemical potentials of the Chern-Simons formulation\(^1\). Indeed, with constant $\mathcal{L}_\pm$’s, the metric corresponding to (3) is (1) with

$$\frac{(N_\infty)^2}{4} = \left( \nu^+ + \nu^- + 2 \right)^2 \quad \text{(4)}$$

$$N_\infty^\theta = \frac{\nu^+ - \nu^-}{2l} \quad \text{(5)}$$

and

$$M = \frac{2\pi}{l} (\mathcal{L}_+ + \mathcal{L}_-) \quad J = 2\pi (\mathcal{L}_+ - \mathcal{L}_-) . \quad \text{(6)}$$

C. Asymptotic Analysis

We now show that the introduction of the chemical potentials does not modify the Virasoro asymptotics. This is in fact direct, and physically mandatory, but we provide an explicit argument since some confusion arose in the spin-3 case.

The discussion is most transparent in the Hamiltonian formalism. On a slice $t = \text{const}$, say the initial slice $t = 0$, the connection is asymptotically given by

$$a^\pm (t = 0) = \left( L^\pm_{\pm 1} - \frac{2\pi}{k} \mathcal{L}_\pm L^\pm_{\mp 1} \right) d\theta \quad \text{(7)}$$

The gauge transformations that preserve this form of the connection are asymptotically parametrized by a gauge parameter that takes the form

$$\Lambda^\pm (\varepsilon_\pm) = \varepsilon_\pm L^\pm_{\pm 1} + \frac{1}{2} \left( \varepsilon'^\prime - \frac{4\pi}{k} \varepsilon_\pm \mathcal{L}_\pm \right) L^\pm_{\mp 1} \quad \text{(8)}$$

\(^1\) In our conventions the level is given by $k = \frac{4}{l^2} = 2l$. 

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where $\varepsilon_\pm$ are at this stage arbitrary functions of $\theta$ and also of the slice under consideration, i.e., $t$, since one can make independent gauge transformations that preserve (7) on each slice. Here, prime denotes the derivative with respect to $\theta$.

The motion from one slice to the next is a gauge transformation parametrized by the Lagrange multiplier $a_0^\pm$ associated with the Chern-Simons Gauss constraint. To preserve the asymptotic form (7), $a_0^\pm$ should be of the form (8). The choice of the Lagrange multiplier which is made when the chemical potentials are not switched on is simply $\varepsilon_\pm = 1$, so that

$$a_0^\pm = \pm a_0^\pm.$$

The equations of motion imply that the fields are chiral with $\pm$ chiralities, and asymptotically given by (3) with $\nu_\pm = 0$.

The choice of Lagrange multipliers which is made when the chemical potentials are switched on is $\varepsilon_\pm = 1 + \nu_\pm$ yielding now (3) with $\nu_\pm$ non zero when one integrates the equations. It is immediate, by very construction, that:

- The asymptotic symmetry algebra is the conformal algebra since the connection obeys (7) on all slices (the Lagrange multipliers are taken in the allowed class of gauge parameters).
- The $L_\pm$ fulfill Virasoro algebra with the same central charge independently of $\nu_\pm$ since they depend only on the canonical variables and not on the Lagrange multipliers.

To close this subsection, we note that the introduction of the chemical potentials through the temporal components of the connection (i.e., the components along the thermal circles) is in fact familiar from the thermodynamics of Reissner-Nordström black holes, where the chemical potential for the electric charge is introduced through the temporal component $A_0$ of the electromagnetic vector potential. $A_0$ is the Lagrange multiplier for Gauss’ law. This procedure actually guarantees the interpretation of the Lagrange multipliers as chemical potentials. Indeed, quite generally, the Lagrange multipliers $\lambda^A$ enter the action as $-\int d^4 x \lambda^A \mathcal{G}_A - \lambda^A_\infty q_A$ where $\mathcal{G}_A$ are the constraints (which vanish on-shell) and $q_A$ the charges, which are given by the surface terms at infinity that must accompany the bulk constraints to make the variational principle well defined [37]. On shell, this sum reduces to $-\lambda^A_\infty q_A$. 
D. Some comments

We thus see that it is rather straightforward to handle the chemical potentials in the Chern-Simons formulation. It has been proposed in the literature to include them not along the thermal circle, but along the conjugate null directions, as

\[ a^\pm = \pm \left( L^\pm_{\pm 1} - \frac{2\pi}{k} L^\pm_{\pm 1} \right) dx^\pm \pm \Lambda^\pm(\nu_\pm) dx^\mp. \]  

(9)

Now, this is more than just a choice of the Lagrange multiplier. One sees that this choice has also the effect of modifying the phase space variables \( a^\pm_0 \) as

\[ a^\pm(t = \text{const}) = \left( (1 - \nu_\pm)L^\pm_{\pm 1} - \frac{2\pi(1 - \nu_\pm)}{k} L^\pm_{\pm 1} \right) d\theta, \]

which is not any more of the requested asymptotic form (7). However, even though not of the requested asymptotic form, one can bring the spatial connection to it by redefinitions, so that the “penalty” paid is not very high: the formulas need only direct, although somewhat awkward, adjustments. As we shall see, this is not the case for the higher spin charges, where this different approach destroys the asymptotics.

III. HIGHER SPIN CHEMICAL POTENTIALS

We now turn to the higher spin case. For definite, we consider the theory based on \( sl(3, \mathbb{R}) \oplus sl(3, \mathbb{R}) \), which contains a spin 3 field in addition to the spin 2 one, and which illustrates the main points.

A. Direct approach

The asymptotic form of the connections when the chemical potentials are not included in them read \[8, 9\], after the \( r \)-dependent gauge transformation of \[17\] has been performed,

\[ a^\pm = \pm \left( L^\pm_{\pm 1} - \frac{2\pi}{k} L^\pm_{\pm 1} \right) dx^\pm. \]  

(10)

So, on a slice \( t = \text{const} \), the spatial connection (which contains the reduced phase space variables) takes the asymptotic form

\[ a^\pm(t = \text{const}) = \left( L^\pm_{\pm 1} - \frac{2\pi}{k} L^\pm_{\pm 1} - \frac{\pi}{2k} W^\pm_{\pm 1} \right) d\theta. \]  

(11)
The gauge transformations that leave this asymptotic form invariant are \([8, 9]\):

\[
\Lambda^\pm (\varepsilon_\pm, \chi_\pm) = \varepsilon_\pm L^\pm_{\pm 1} + \chi_\pm W^\pm_{\pm 2} \mp \varepsilon_\pm' L^\pm_0 \mp \chi_\pm' W^\pm_{\pm 1} + \frac{1}{2} \left( \varepsilon^\prime_\pm - \frac{4\pi}{k} \varepsilon_\pm \Lambda^\pm + \frac{8\pi}{k} W^\pm \chi_\pm \right) L^\pm_{\mp 1}
\]

\[
- \left( \frac{\pi}{2k} W^\pm \varepsilon_\pm + \frac{7\pi}{6k} \mathcal{L}^\prime_\pm \chi_\pm + \frac{\pi}{3k} \chi_\pm \mathcal{L}^\prime_\pm \chi_\pm + \frac{4\pi}{3k} \mathcal{L}^\prime_\pm \chi_\pm' - \frac{4\pi^2}{k^2} \mathcal{L}^2_\pm \chi_\pm - \frac{1}{24k} \mathcal{L}^\prime_\pm \right) W^\pm_{\mp 2}
\]

\[
+ \frac{1}{2} \left( \chi_\pm'' - \frac{8\pi}{k} \mathcal{L} \chi_\pm \right) W^\pm_0 \mp \frac{1}{6} \left( \chi^\prime_\pm - \frac{8\pi}{k} \chi_\pm \mathcal{L}^\prime_\pm \chi_\pm - \frac{20\pi}{k} \mathcal{L}^\prime_\pm \chi_\pm' \right) W^\pm_{\mp 1},
\]

where \(\varepsilon_\pm, \chi_\pm\) are arbitrary functions of \(\theta\) (on each slide). Furthermore, the functions \(\mathcal{L}_\pm\) and \(\mathcal{W}_\pm\) of the canonical variables appearing in the asymptotic form of the connection transform as \([8, 9]\):

\[
\delta \mathcal{L}_\pm = \varepsilon_\pm \mathcal{L}^\prime_\pm + 2 \mathcal{L}_\pm \varepsilon_\pm' - \frac{k}{4\pi} \varepsilon_\pm'' - 2 \chi_\pm \mathcal{W}^\prime_\pm - 3 \mathcal{W}_\pm \chi_\pm',
\]

\[
\delta \mathcal{W}_\pm = \varepsilon_\pm \mathcal{W}^\prime_\pm + 3 \mathcal{W}_\pm \varepsilon_\pm' - \frac{64\pi}{3k} \mathcal{L}^2_\pm \chi_\pm + 3 \chi_\pm \mathcal{L}^\prime_\pm \chi_\pm'' + 5 \mathcal{L}^\prime_\pm \chi_\pm'' + 2 \chi_\pm \mathcal{L}^\prime_\pm - \frac{k}{12\pi} \chi_\pm'''
\]

\[
- \frac{64\pi}{3k} \left( \chi_\pm \mathcal{L}^\prime_\pm - \frac{5k}{32\pi} \chi_\pm''' \right) \mathcal{L}_\pm,
\]

leading to the \(W_3\) algebra (with central charge).

As we explained above, the Lagrange multipliers \(a^\pm_0\) must preserve the boundary conditions and so must be of the form \([12]\). Now, in the case where the chemical potentials are not incorporated in the connection \([8, 9]\), one simply takes \(\varepsilon_\pm = 1, \chi_\pm = 0\) and one finds that on-shell, the connections are chiral and take the asymptotic form \([11]\).

If one wants to include the chemical potentials in the connection, one should, as explained above, allow for extra terms in \(a^\pm_0\). The temporal component of the connection reads now

\[
a^\pm_0 = \pm \left( L^\pm_{\pm 1} - \frac{2\pi}{k} \mathcal{L}^\pm_{\pm 1} \right) \frac{dt}{l} \pm \frac{1}{l} \Lambda^\pm(\nu_\pm, \mu_\pm) dt,
\]

where \(\nu_\pm\) and \(\mu_\pm\) are constants and are the respective chemical potentials for the spin-2 part and the spin-3 part. The connection is therefore

\[
a^\pm = \pm \left( L^\pm_{\pm 1} - \frac{2\pi}{k} \mathcal{L}^\pm_{\pm 1} \right) dx^\pm \pm \frac{1}{l} \Lambda^\pm(\nu_\pm, \mu_\pm) dt.
\]

Note that this is not really a relaxed set of boundary conditions since the spatial part of the connection is unchanged. Only the Lagrange parameters are modified. This is an extension of the formalism that incorporates the chemical potentials.

Just as in the pure gravity case, it is obvious that this choice is such that:
• The asymptotic symmetry algebra is the conformal $W_3$ algebra since the connection obeys (11) on all slices (the Lagrange multipliers are taken in the allowed class of gauge parameters).

• The $L_\pm, W_\pm$ fulfill in the Poisson-Dirac bracket the $W_3$ algebra of \[8, 9\] with the same central charge independently of the chemical potentials since these generators depend only on the canonical variables and not on the Lagrange multipliers.

B. Comments

If one were to introduce the chemical potentials through extra non-vanishing components of the connection not along the thermal circles but along the conjugate timelike directions, one would run into serious difficulties. Indeed, if one were to impose asymptotically

$$a_\pm = \pm \left( L_\pm^\pm - \frac{2\pi}{k} \mathcal{L}_{L_\pm^\pm} - \frac{\pi}{2k} \mathcal{W}_{W_\pm^\pm} \right) dx^\pm \pm \Lambda_\pm^\pm (\nu_\pm, \mu_\pm) dx^\mp,$$

one would modify the spatial connection in a way incompatible with the $W_3$ symmetry since the terms proportional to the chemical potentials $\mu_\pm$ for the spin-3 charge enter $a_\theta^\pm$ multiplied by Lie algebra generators that are not highest (lowest) weight states and hence are not compatible with the asymptotic conditions (11) implementing the Hamiltonian reduction of the $sl(3)$ current algebra to the $W_3$ algebra,

$$a_\pm (t = \text{const}) = \left( L_\pm^\pm - \frac{2\pi}{k} \mathcal{L}_{L_\pm^\pm} - \frac{\pi}{2k} \mathcal{W}_{W_\pm^\pm} \right) d\theta + (\nu_\pm L_\pm^\pm + \mu_\pm W_\pm^\pm) d\theta$$

$$+ \left[ \frac{1}{2} \left( - \frac{4\pi}{k} \nu_\pm \mathcal{L}_{\pm} + \frac{8\pi}{k} \mathcal{W}_{\pm} \mu_\pm \right) L_\pm^\pm - \left( \frac{\pi}{2k} \mathcal{W}_{\pm} \nu_\pm - \frac{4\pi^2}{k^2} \mathcal{L}_{\pm}^2 \mu_\pm \right) W_\pm^\pm \right] d\theta$$

$$- \frac{4\pi}{k} \mathcal{L}_{\pm} \mu_\pm W_0^\pm d\theta.$$   

The “offending terms” are absent when the chemical potentials $\mu_\pm$ are zero (although rescalings are still needed in that case), but present otherwise.

In fact, when the chemical potentials $\mu_\pm$ are non zero, the full asymptotic asymptotic symmetry at infinity, i.e., the set of all gauge transformations preserving the asymptotic form of $a_\theta$ is the algebra $W_3^\pm$ corresponding to the other non trivial embedding of $sl(2, \mathbb{R})$ into $sl(3, \mathbb{R})$ \[30, 31\]. In the enveloping algebra of $W_3^\pm$, one may try to pick out a $W_3$ algebra, for instance by requiring analyticity in $\mu$. Perturbative efforts in that direction may be found in \[38\] where the equations have been analyzed to finite order $O(\mu^4)$. 


IV. CONCLUSIONS

We have shown in this paper how to incorporate the chemical potentials associated with higher spin charges in higher spin three-dimensional gravity. Although we considered only the case of $sl(3, \mathbb{R})$, it is clear that our method extends straightforward to any $sl(N, \mathbb{R})$ (with any non-trivial embedding of $sl(2, \mathbb{R})$ and even to infinite-dimensional higher-spin algebras). As here, the asymptotic symmetry algebra is obviously unchanged when the chemical potentials are switched on.

The method is straightforward because the chemical potentials enter only the temporal components of the connection, which are Lagrange multipliers. The canonical variables - and hence the canonical generators of the symmetry at infinity - are unaffected. In that sense, the boundary conditions with chemical potentials included are not true relaxations of the original boundary conditions. True relaxations would be relaxations on the behavior of the canonical variables and not just the Lagrange multipliers. The Hamiltonian formalism makes the analysis particularly transparent and direct. The argument is non-perturbative and exact in the chemical potentials.

As we pointed out, the introduction of the chemical potentials through the temporal (i.e., along the thermal circles) components of the connection is in fact familiar from the thermodynamics of Reissner-Nordström black holes. The thermodynamical significance of the Lagrange multipliers as chemical potentials is immediate given the structure of the action, where the constraints and the accompanying charges (given by surface integrals at infinity) are multiplied by the Lagrange multipliers. It is very satisfying that what works in four dimensions also works in three.

In a future paper [30, 31], we shall provide more insight on the analysis by investigating the static black hole solution endowed with a spin-3-field

$$a^\pm = \left( \pm L_{\pm 1}^\pm + \frac{2\pi}{k} L^\pm_{\mp 1} - \frac{\pi}{2k} \mathcal{W} W^\pm_{\mp 2} \right) dx^\pm + \left( \mu W_{\pm 2}^\pm - \frac{4\pi}{k} \mu \mathcal{W}^\pm_0 \pm \nu L^\pm_{\pm 1} \pm \frac{2\pi}{k} \left[ 2\mu \mathcal{W} - \nu \mathcal{L} \right] L^\pm_{\mp 1} + \frac{\pi}{2k} \left[ 8\pi \mu \mathcal{L}^2 - \nu \mathcal{W} \right] W^\pm_{\mp 2} \right) dt$$

where $\mathcal{L}$, $\mathcal{W}$, $\mu$, $\nu$ are integration constants. We shall confirm through the study of this specific “$W_3$-black hole” that there is no tension between the holographic and canonical approaches. The analysis of the thermodynamics and the conformal properties is direct, because the coefficients $\mathcal{L}$ and $\mathcal{W}$ in the connection really mean what they are, namely, the
generators of the $W_3$ algebra, without needing translation through a dictionary.

We shall also investigate the rotating solution, which is given by (16) with $L_\pm, W_\pm, \nu_\pm$ and $\mu_\pm$ fixed to constants, which we write by displaying explicitly the spatial and temporal components

$$a^\pm = \left( L_\pm^2 - \frac{2\pi}{k} L_\pm L_\mp + \frac{\pi}{2k} W_\pm W_\mp \right) \frac{d\theta}{\pm} \pm \frac{1}{2} (1 + \nu_\pm) L_\pm^2 + \mu_\pm W_\mp^2 - \frac{4\pi}{k} \frac{1}{2} \mu_\pm L_\pm W_\mp^2$$

$$- \frac{2\pi}{k} ((1 + \nu_\pm) L_\pm - 2\mu_\pm W_\pm) L_\mp^2 - \frac{\pi}{2k} \left( (1 + \nu_\pm) W_\pm - \frac{8\pi}{k} \mu_\pm L_\mp^2 \right) W_\mp^2 \right) dt ,$$

as well as the analog “$W_3^{(2)}$-black holes”.

Perhaps the following question can be asked as a final note. We have seen that the chemical potentials $\nu_\pm$ of the gravitational sector can be absorbed through a definition $t \rightarrow \alpha t$, $\theta \rightarrow \theta + \omega t$ of the coordinates, at the price of having new ranges for the new coordinates. It would be interesting to see if analogous absorptions of the higher spin chemical potentials could take place by suitable redefinitions in the yet-to-be-found geometry incorporating the higher spin fields.

Note added: In the interesting paper [39], devoted to the higher spin black hole solutions of [25], it is advocated that the chemical potentials should be defined in that case in terms of the components of the connection along the thermal circles. This work relies on the approach of [38], where the $W_3$-symmetry was proved to be present (perturbatively, to order $O(\mu^4)$).

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