Dual Connections and Holonomy

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Dual affine connections on Riemannian manifolds have played a central role in the field of information geometry since their introduction in [1].

Here I would like to extend the notion of dual connections to general vector bundles with an inner product, in the same way as a unitary connection generalizes a metric affine connection, using Cartan decompositions of Lie algebras. This gives a natural geometric interpretation for the Amari tensor, as a “connection form term” which generates dilations, and which is reversed for the dual connections.

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1 Cartan Decomposition

Let $\mathfrak{gl}(n, \mathbb{R})$ be the Lie algebra of $\text{GL}(n, \mathbb{R})$. If $X^*$ denotes the transpose in $\mathfrak{gl}$, then the map $\theta : \mathfrak{gl} \to \mathfrak{gl}$ given by $\theta(x) := -x^*$ has the following properties.

1. It is a (linear) isomorphism of $\mathfrak{gl}$.
2. It respects Lie brackets: $\theta[x, y] = [\theta x, \theta y]$.
3. It is an involution: $\theta \circ \theta = id$.
4. If $B$ is the Killing form, then $B_\theta(x, y) := -B(x, \theta y)$ is a positive definite symmetric bilinear form.

**Definition 1.1.** A homomorphic involution of a Lie algebra $\mathfrak{g}$ which satisfies the properties above is called a *Cartan involution*.

Any real semisimple Lie algebra admits a Cartan involution, which is unique up to inner isomorphisms. A Cartan involution $\theta$ divides $\mathfrak{g}$ into two eigenspaces:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where the eigenvalues are respectively $+1$ and $-1$. For $\mathfrak{gl}$, this corresponds intuitively to the "direction of rotations", and "direction of dilations". We will see that dualizing a connection reverses precisely only the second subspace.

We have that:

1. $[\mathfrak{k}, \mathfrak{k}] = \mathfrak{k}$,
2. $[\mathfrak{k}, \mathfrak{p}] = \mathfrak{p}$,
3. $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$,

which imply that:

- $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$.
- $\mathfrak{p}$ in general is not, and its Lie subalgebras are all commutative (and 1-dimensional).
- $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal for the Killing form and for $B_\theta$. 

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For $\mathfrak{gl}$, the subalgebra $\mathfrak{k}$ is precisely $\mathfrak{so}$, on which indeed $\theta$ is the identity. In other words, we can decompose any element $x$ of $\mathfrak{g}$ into:

$$x = x^+ + x^-,$$

(2)

where:

$$x^+ := \frac{x + \theta x}{2}, \quad x^- := \frac{x - \theta x}{2}.$$

(3)

There is a corresponding involution for Lie groups, which for $\text{GL}$ corresponds to $M \mapsto (M^*)^{-1}$. This map is called $\Theta : G \to G$ and it has the property that for $x \in \mathfrak{g}$:

$$\Theta e^x = e^{\theta x}.$$

(4)

We call $K$ the subgroup of $G$ which is fixed by $\Theta$. For $\text{GL}$ it is precisely $\text{SO}$. In general it is generated by exponentials of $\mathfrak{k}$. Every element of $G$ can be expressed as a product:

$$M = e^k e^p,$$

(5)

where $k \in \mathfrak{k}$ and $p \in \mathfrak{p}$. Note that in general this is not equal to $e^{k+p}$, as $k$ and $p$ may not commute. What is true, though, is the following.

**Proposition 1.1.** Let $e^{k+p} = e^{k'} e^{p'}$, with $k, k' \in \mathfrak{k}$ and $p, p' \in \mathfrak{p}$ possibly different. Then:

$$e^{k-p} = e^{k'} e^{-p'}.$$

(6)

To see this, it is sufficient to notice that the quantity above is precisely:

$$\Theta e^{k+p} = e^{\theta(k+p)}.$$

(7)

For $\text{GL}$, this implies that any element $M$ can be written as $M = OP$, where $O$ is orthogonal, and $P$ is positive definite (and given by $P = M^* M$).

For all the details on Cartan involutions and decompositions, see [2], Chapter VI.

## 2 Holonomy

Let $V$ be a vector bundle over $X$ of rank $n$. Let $p \in X$ and let $L_p$ be the set of loops pointed at $p \in X$ equipped with the usual composition of loops (see [3]).

We can view a connection on $V$ as a smooth mapping $\nabla : L_p \to \text{Aut}(V_p)$, where $V_p$ is the fiber at $p$, such that $\nabla(l)$ is the transformation that a vector at $p$ undergoes after parallel transport along $l$. It has the following properties:
1. $\nabla$ maps the trivial loop to the identity.

2. $\nabla$ preserves composition: $\nabla(ll') = \nabla(l') \circ \nabla(l)$.

3. If $-l$ it the inverse loop of $l$, then $\nabla(-l) = (\nabla(l))^{-1}$.

These properties, which remind of a group homomorphism, can indeed define a homomorphism provided that a suitable group structure is defined on the space of loops, through quotienting. But we will not need it here. Even without defining a group structure for loops, the properties above imply that:

**Proposition 2.1.** The image of $\nabla$ is a Lie subgroup of $\text{Aut}(V_p)$.

We call such image the holonomy group of $\nabla$ at $p$, and we denote it by $\text{Hol}_p(\nabla)$. If $X$ is path connected, all holonomy groups are isomorphic. In that case we drop the reference to the base point, and simply write $\text{Hol}(\nabla)$. Local coordinates around $p$ give an isomorphism between $\text{Aut}(V_p)$ and $GL(n, \mathbb{R})$, so that $\text{Hol}(\nabla)$ is isomorphic to a subgroup of $GL$. For example:

- A trivial connection has trivial holonomy group.
- A general affine connection may have the whole $GL$ as holonomy group.
- A metric connection has holonomy group isomorphic to (a subgroup of) $O(n)$. Different metrics (or different coordinates) yield different isomorphisms.
- A connection on an oriented bundle has holonomy group isomorphic to (a subgroup of) $SL(n)$. A metric connection here will yield $SO(n)$.
- Special subgroups of $O(n)$, like for example $SU(n/2)$ for $n$ even, are the holonomy groups of the so-called manifolds of special holonomies.

Let $\text{hol}(\nabla)$ be the Lie algebra of the holonomy group. Then we can express the connection $\nabla$ locally, in suitable coordinates, as a 1-form $\omega$ with values in $\text{hol}(\nabla)$, i.e., an element of $T^*X \otimes \text{hol}(\nabla)$, mapping linearly a tangent vector $v$ to an element $x$ of the Lie algebra. If we also choose coordinates on the fiber, we have a mapping from tangent vectors to a subalgebra of $\mathfrak{gl}(n)$.

For details about holonomy, the reader is referred to [3].
3 Dual Connection

We can put together the results of the previous two sections, and define dual connections. Let $\nabla : L_p \to \text{Aut}(V_p)$ be a connection on $V$. Since $\text{Aut}(V_p)$ is isomorphic to $\mathfrak{gl}$, it admits Cartan involutions. Let $\Theta$ be such a Cartan involution. Then $\nabla^* : L_p \to \text{Aut}(V_p)$ defined by $\Theta \circ \nabla$ is called the dual connection with respect to the Cartan involution $\Theta$.

Different Cartan involutions will yield different dual connections, and this is equivalent to choosing a different inner product on $V$ (if $V$ is the tangent bundle, a Riemannian metric). This is done by indentifying the metric adjoint $M \mapsto \Theta(M^{-1})$.

Let now be $\langle , \rangle$ an inner product on $V$. Let $l$ be a loop at $p$. Then we can decompose the parallel transport according to Cartan as:

$$\nabla(l) = e^k e^p.$$ (8)

The dual connection will instead yield (see Proposition 1.1):

$$\nabla^*(l) = e^k e^{-p},$$ (9)

so that if $v, w$ are vectors of $V_p$:

$$\langle \nabla^*(l)v, \nabla(l)w \rangle = \langle e^k e^{-p}v, e^k e^p w \rangle = \langle v, (e^k e^{-p})e^k e^p w \rangle = \langle v, \Theta(e^k e^{-p})^{-1}e^k e^p w \rangle = \langle v, \Theta(e^{-p})^{-1}\Theta(e^k)^{-1}e^k e^p w \rangle = \langle v, \Theta(e^p)e^p w \rangle = \langle v, e^{-p}e^p w \rangle = \langle v, w \rangle,$$ (10)

which is the property traditionally defining dual connections (see [4]).

At the Lie algebra level, the connection form $\omega^*$ of the dual connection $\nabla^*$ is obtained by $\omega$ as the mapping $\omega^* : TX \to \text{hol}(\nabla)$ given by $\theta \circ \omega$.

Applying the decomposition (2) to $\omega$, we get that:

$$\omega = \omega^+ + \omega^-,$$ (16)

$$\omega^* = \omega^+ - \omega^-.$$ (17)

Equivalently:

$$\omega^* = \omega - 2\omega^-.$$ (18)
If $\omega^- = 0$ we have a metric connection (see the Ambrose-Singer holonomy theorem in [3]). In general $\omega^-$ measure how much our connection tends to change the length of the vectors, and the dual connection does the opposite.

On Riemannian manifolds, if $\nabla$ is torsion free, then $\omega^- = 0$ gives precisely the Levi-Civita connection. In general $\omega^-$ is precisely proportional to the Amari tensor (see [4]). This suggests that we can generalize the concept of $\alpha$-connections to general vector bundles, by taking:

$$\omega^\alpha = \omega^+ + \alpha \omega^-,$$
$$\omega^{-\alpha} = \omega^+ - \alpha \omega^-.$$

Moreover, this way we have a very natural geometric interpretation of the Amari tensor: it can be written as a 1-form with values in $p$, i.e. a subspace of $\text{hol}(\nabla)$ orthogonal to $\mathfrak{k} \equiv \mathfrak{so}$. Intuitively, it is the part of the connection which generates dilations. Since every subalgebra of $p$ is 1-dimensional, the parameter $\alpha$ spans it completely.

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References

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