The maximum matching extendability and factor-criticality of 1-planar graphs

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Abstract: A graph is 1-planar if it can be drawn in the plane so that each edge is crossed by at most one other edge. Moreover, a 1-planar graph $G$ is optimal if it satisfies $|E(G)| = 4|V(G)| - 8$. J. Fujisawa et al. [3] first considered matching extension of optimal 1-planar graphs, obtained that each optimal 1-planar graph of even order is 1-extendable and characterized 2-extendable optimal 1-planar graphs and 3-matchings extendable to perfect matchings as well. In this short paper, we prove that no optimal 1-planar graph is 3-extendable. Further we mainly obtain that no 1-planar graph is 5-extendable by the discharge method and also construct a 4-extendable 1-planar graph. Finally we get that no 1-planar graph is 7-factor-critical and no optimal 1-planar graph is 6-factor-critical.

Keywords: $n$-extendable graph; $k$-factor-critical graph; 1-planar graph; optimal 1-planar graph

AMS subject classification: 05C70, 05C10

1 Introduction

All graphs considered here are finite, simple and undirected. We use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of a graph $G$ respectively. The order of a graph refers to the number of vertices. For a vertex $v$ in $G$, let $d_G(v)$ denote the degree of a vertex $v$ in $G$, the number of edges incident with $v$. Let $\delta(G)$ denote the minimum degree in $G$. A matching $M$ of a graph $G$ is a subset of $E(G)$ such that any two edges of $M$ have no end-vertices in common. A matching of $k$ edges is called a $k$-matching. A perfect matching of a graph is a matching covering all vertices of the graph.

In 1980 M.D. Plummer [4] introduced the concept of $n$-extendable graphs. For an integer $n \geq 0$, a connected graph $G$ with at least $2n + 2$ vertices is said to be $n$-extendable if it admits an $n$-matching and each $n$-matching can be a subset of a perfect matching of $G$. Matching extendability of graphs was widely investigated; see two surveys [7, 8] and a

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book [13]. Plummer proved that every $n$-extendable graph is $(n-1)$-extendable whenever $n \geq 1$ and the following basic result.

**Lemma 1.1** ([4]). If $G$ is $n$-extendable, then $G$ is $(n+1)$-connected.

Plummer studied the matching extendability of planar graphs and obtained

**Theorem 1.2** ([5]). No planar graph is 3-extendable.

**Theorem 1.3** ([6]). Every 5-connected planar graph of even order is 2-extendable.

For any surface $\Sigma$, Plummer [9] also posed the problem of determining the least integer $\mu(\Sigma)$ such that no graph $G$ embedded in $\Sigma$ is $\mu(\Sigma)$-extendable. $\mu(\Sigma)$ is called the matching extendability of the surface $\Sigma$. So by Theorems 1.2 and 1.3, $\mu(S_0) = 3$ for the sphere $S_0$. Later, N. Dean [10] completely solved this problem by obtaining the following result:

**Theorem 1.4** ([10]). If $\Sigma$ is any surface (orientable or non-orientable) other than the sphere, then $\mu(\Sigma) = 2 + \lfloor \sqrt{4 - 2\chi(\Sigma)} \rfloor$, where $\chi(\Sigma)$ is the Euler characteristic of the surface $\Sigma$.

A graph is 1-planar if it can be drawn in the plane so that each edge is crossed by at most one other edge. Unless otherwise stated, we consider that a given 1-planar graph is already drawn on the plane (or sphere). The notion of a 1-planar graph was first introduced by G. Ringel [1]. To now most researches on 1-planar graphs have focused on graph coloring and some structures. For structure of 1-planar graphs, I. Fabrici and T. Madras [2] showed that every 1-planar graph $G$ has at most $4|V(G)| - 8$ edges. A 1-planar graph $G$ is called an optimal 1-planar graph if it satisfies $|E(G)| = 4|V(G)| - 8$. They also got

**Lemma 1.5** ([2]). Every 1-planar graph contains a vertex of degree at most 7.

Motivated by Plummer’s work it is natural to consider matching extension of 1-planar graphs. However J. Fujisawa et al. [3] pointed out that an obstacle for such a research is the fact that the operation of edge contraction does not preserve the 1-planarity in general. In 2018 they eliminated this difficulty for optimal 1-planar graphs, and got a series of results on the matching extendability of optimal 1-planar graphs like the case of planar graphs.

In order to state such results, we need the following notations. An edge in a 1-planar graph $G$ is called crossing if it crosses with another edge, and non-crossing otherwise. A cycle $C$ in a connected graph $G$ is said to be separating if $G - V(C)$ is disconnected. A
separating cycle $C$ of a 1-planar graph is called a barrier cycle if each edge of $C$ is non-crossing, $G - V(C)$ consists of two odd components which lie in the interior and exterior of $C$ respectively.

**Theorem 1.6** ([3]). *Every optimal 1-planar graph $G$ of even order is 1-extendable.*

**Theorem 1.7** ([3]). *An optimal 1-planar graph $G$ of even order is 2-extendable unless $G$ contains a barrier cycle of length 4.*

**Theorem 1.8** ([3]). *Let $G$ be a 5-connected optimal 1-planar graph of even order and $M$ be a matching of $G$ with $|M| = 3$. Then $M$ is extendable unless $G$ contains a barrier cycle $C$ of length 6 such that $V(M) = V(C)$.*

Theorem 1.7 implies that a 5-connected optimal 1-planar graph $G$ of even order is 2-extendable since it cannot contain a barrier cycle of length 4. In this paper, we get the following result for any optimal 1-planar graphs.

**Theorem 1.9.** *No optimal 1-planar graph is 3-extendable.*

By Theorems 1.8 and 1.9, we can obtain the following corollary.

**Corollary 1.10.** *Any 5-connected optimal 1-planar graph $G$ of even order contains a barrier cycle $C$ of length 6.*

Next we mainly consider the maximum matching extendability of 1-planar graphs. Since $n$-extendable graphs are $(n + 1)$-connected, Lemma 1.5 implies that no 1-planar graph is 7-extendable. However we prove that no 1-planar graph is 5-extendable by combining the discharge method and Dean’s lemma as follows. We construct a 4-extendable 1-planar graph (see Fig. 5), which shows that such result is best possible.

**Theorem 1.11.** *No 1-planar graph is 5-extendable.*

The remaining sections of this paper are organized as follows. In Section 2, we give the proof of Theorem 1.9 via Suzuki’s relation between optimal 1-planar graphs and quadrangulations on the sphere. In Section 3 we use Dean’s lemma and discharging method to prove Theorem 1.11. At the end of this section, we give a 4-extendable 1-planar graph. In Section 4, as remarks we show that no 1-planar graph is 7-factor-critical and no optimal 1-planar graph is 6-factor-critical. Some examples show such results are best possible.
2 Proof of Theorem 1.9

A quadrangulation of the sphere is a simple graph embedded on the sphere with no crossing point such that each of its faces is bounded by a cycle of length 4. If we remove all the crossing edges of an optimal 1-planar graph $G$, then the resulting graph is denoted by $Q(G)$. To prove Theorem 1.9, we first give a clear relationship between optimal 1-planar graphs and quadrangulations on the sphere as follows (see Theorem 11 in [14]).

**Theorem 2.1** ([14]). Let $H$ be a simple quadrangulation on the sphere. Then there exists a simple optimal 1-planar graph $G$ such that $H = Q(G)$ if and only if $H$ is 3-connected.

From Theorem 2.1, adding two diagonal edges within every face of a 3-connected quadrangulation on the sphere results in an optimal 1-planar graph. As an immediate consequence we have the following result.

**Lemma 2.2.** Let $G$ be an optimal 1-planar graph. Then for any $v \in V(G)$, all edges incident with $v$ in $G$ are alternately crossing and non-crossing edges in clockwise way. So the degree of each vertex in $G$ is even.

**Proof of Theorem 1.9** Suppose to the contrary that there exists a 3-extendable optimal 1-planar graph $G$. By Lemma 1.1 $G$ is 4-connected. So $G$ has a vertex $v$ such that $4 \leq d_G(v) \leq 7$ by Lemma 1.5.

By Lemma 2.2 $d_G(v)$ is even, so $d_G(v) = 4$ or 6. If $d_G(v) = 4$, then by Lemma 2.2, $v$ has exactly two neighbors $v_1$ and $v_3$ in $Q(G)$ so that $G$ has one diagonal edge between $v_1$ and $v_3$ in each 4-face on two sides of 3-path $v_1v_3$. Two edges joining $v_1$ and $v_3$ clearly would become multiple edges, contradicting that $G$ is simple.

If $d_G(v) = 6$, let $vv_i$, $1 \leq i \leq 6$, be the consecutive edges incident with $v$ in counterclockwise order. Then by Lemma 2.2, without loss of generality suppose that $vv_1$, $vv_3$ and $vv_5$ are non-crossing edges, and $vv_2$, $vv_4$ and $vv_6$ are crossing edges (see Fig 1). Further, $v$ is incident with exactly three 4-faces of $Q(G)$ so that their face cycles are $vv_1v_2v_3v$ and $vv_3v_4v_5v$, and $vv_5v_6v_1v$, which implies that $Q(G)$ has a 6-cycle $v_1v_2v_3v_4v_5v_6v_1$. Hence $G$ has a 3-matching $\{v_1v_2, v_3v_4, v_5v_6\}$ covering all neighbors of $v$, which is not extendable to a perfect matching of $G$, contradicting Lemma 3.1 or the definition of 3-extendable graphs. Therefore, no optimal 1-planar graph is 3-extendable.

$\square$
3 Proof of Theorem 1.11

To obtain our main result in this section we first give some preliminaries. The following lemma due to N. Dean [10] will play a crucial role in the proof of Theorem 1.11. For a graph $G$, we use $N(v)$ to denote the neighborhood of a vertex $v$ and $G[N(v)]$ for the induced subgraph of $G$ by $N(v)$.

**Lemma 3.1 ([10]).** Let $v$ be a vertex of degree $n+t$ in an $n$-extendable graph $G$. Then $G[N(v)]$ does not contain a matching of size $t$.

We now describe some terminologies and notations of a graph $G$. A vertex is a $t$-vertex (resp., $t^+$-vertex, $t^-$-vertex) if $d_G(v) = t$ (resp., $d_G(v) \geq t$, $d_G(v) \leq t$). For a face $f$ of a connected plane graph $G$, the face degree of $f$, denoted by $d_G(f)$, is the length of the closed walk along the boundary of $f$. Such closed walk is called a face walk, and a face cycle whenever it is cycle. Similarly a $t$-face (resp., $t^+$-face, and $t^-$-face) refers to a face with degree $t$ (resp., at least $t$, and at most $t$).

The associated plane graph $G^x$ of a 1-planar graph $G$ is the plane graph that is obtained from $G$ by turning all crossings of $G$ into new vertices of degree four. These new vertices in $G^x$ are called false vertices, and the vertices of $G$ are called true vertices. A face in $G^x$ is false if it is incident with at least one false vertex; otherwise, it is true.

Next we give a lemma which is used in proof of Theorem 1.11. This lemma shows that there exists a 1-matching in $G[N(v)]$ if a true vertex $v$ of the associated plane graph $G^x$ of a 1-planar graph $G$ is incident with three consecutive false 3-faces.

**Lemma 3.2.** Let $G^x$ be the associated plane graph of a 1-planar graph $G$. If a true vertex $v$ is incident with three consecutive false 3-faces in $G^x$, then $G[N(v)]$ contains an edge of $G$.

**Proof.** If a true vertex $v$ is incident with a false 3-face, then $v$ is adjacent to one true vertex and one false vertex on the false 3-face by 1-planarity of $G$. If $v$ is incident with
three consecutive false 3-faces in \(G^x\), let \(vv_i, 1 \leq i \leq 4\), denote four consecutive edges in \(G^x\) so that the \(vv_i v_{i+1}v\) are false 3-faces for each \(i = 1, 2, 3\). Then \(v_1, v_2, v_3\) and \(v_4\) are false and true vertices in an alternative way. If \(v_1\) is a false vertex, then \(v_3\) is a false vertex, \(v_2\) and \(v_4\) are true vertices and \(v_2v_4\) is an edge of \(G\) passing through \(v_3\). Similarly, If \(v_1\) is a true vertex, then \(v_3\) is a true vertex and \(v_1v_3\) is an edge of \(G\) passing through \(v_2\). Hence either \(v_1v_3\) or \(v_2v_4\) is an edge of \(G\) (see Fig. 2). That is, \(G[N(v)]\) contains an edge. 

Fig. 2. A true vertex \(v\) in \(G^x\) which is incident with three consecutive false 3-faces.

**Proof of Theorem 1.11** Suppose to the contrary that there is a 5-extendable 1-planar graph \(G\). By Lemma 1.1, \(G\) is 6-connected and \(\delta(G) \geq 6\). In the following, we will apply the discharging method on the associated plane graph \(G^x\).

By Euler’s formula and degree-sum formulas:

\[
|V(G^x)| - |E(G^x)| + |F(G^x)| = 2 \quad (3.1)
\]

\[
\sum_{v \in V(G^x)} d_{G^x}(v) = 2|E(G^x)|, \text{ and} \quad (3.2)
\]

\[
\sum_{f \in F(G^x)} d_{G^x}(f) = 2|E(G^x)|, \quad (3.3)
\]

we have

\[
\sum_{v \in V(G^x)} (3d_{G^x}(v) - 10) + \sum_{f \in F(G^x)} (2d_{G^x}(f) - 10) = -20 \quad (3.4)
\]

First, we give an initial charge function:

- \(w(v) = 3d_{G^x}(v) - 10\), for each \(v \in V(G^x)\), and
- \(w(f) = 2d_{G^x}(f) - 10\), for each \(f \in F(G^x)\).
Next, we will design some discharging rules. Let $w'$ be the new charge after the discharging process. It suffices to show that $w'(x) \geq 0$ for each $x \in V(G^x) \cup F(G^x)$, which leads to a contradiction with Eq. (3.4).

To get the target we now present four claims as follows.

**Claim 1.** Every 6-vertex $v$ in $G^x$ is not incident with a true 3-face, and is incident with at most four false 3-faces.

**Proof.** If $v$ is incident with a true 3-face $vv_1v_2$, then $v_1$ and $v_2$ are both true vertices and $v_1v_2$ is an edge of $G$. Here $v_1v_2$ is a 1-matching in $G[N(v)]$, contradicting Lemma 3.1. Thus every 6-vertex $v$ in $G^x$ is not incident with a true 3-face.

If $v$ is incident with at least five false 3-faces, then these five false 3-faces are consecutive. By Lemma 3.2, we can find a 1-matching in $G[N(v)]$, contradicting Lemma 3.1. Hence every 6-vertex $v$ in $G^x$ is incident with at most four false 3-faces.

**Claim 2.** Every 7-vertex $v$ in $G^x$ is incident with at most six false 3-faces. Moreover, if $v$ is incident with exactly six false 3-faces, then the other face incident with $v$ is a 4$^+$-face.

**Proof.** If $v$ is incident with seven false 3-faces, let $vv_i$, $1 \leq i \leq 7$, denote the seven consecutive edges in $G^x$. By the proof of Lemma 3.2 we have that false vertices and true vertices alternate in the cyclic ordering $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_1$, which is obviously impossible. Hence every 7-vertex $v$ in $G^x$ is incident with at most six false 3-faces.

If $v$ is incident with exactly six false 3-faces, let $vv_1v_2, vv_2v_3, vv_3v_4, vv_4v_5, vv_5v_6, vv_6v_7$ denote these six consecutive false 3-faces. Then $v_1, v_2, v_3, v_4, v_5, v_6, v_7$ are all true vertices and $v_2, v_4$ and $v_6$ are all false vertices. So $v_1v_3$ is an edge of $G$ passing through $v_2$, and $v_5v_7$ is an edge of $G$ passing through $v_6$. Then we find a 2-matching \{ $v_1v_3, v_5v_7$ \} in $G[N(v)]$, contradicting Lemma 3.1. Thus $v_1$ and $v_7$ are both false vertices, and $v_2$ and $v_6$ are true vertices (see Fig. 3(b)). By the 1-planarity the edges with endvertices $v_2$ and $v_6$ in $G$ passing through $v_1$ and $v_7$ respectively are different. The other endvertices (may be the same) of them and vertices $v, v_1$ and $v_7$ all lie on the last one face incident with $v$, which must be a false 4$^+$-face.

**Claim 3.** Every 7-vertex $v$ in $G^x$ is incident with at most two true 3-faces. Moreover, if $v$ is incident with two true 3-faces, then such two true 3-faces are adjacent.

**Proof.** Obviously each true 3-face incident with $v$ has an edge $G[N(v)]$. If $v$ is incident with two nonconsecutive true 3-faces, then we can find a 2-matching in $G[N(v)]$ consisting of one edge in each such true 3-face, contradicting Lemma 3.1. Hence the assertion holds.
Claim 4. Let $v$ be a 7-vertex of $G^\times$. Then the following two statements hold.

(i) if $v$ is incident with one true 3-face, then $v$ is incident with at most four false 3-faces;

(ii) if $v$ is incident with two true 3-faces, then $v$ is incident with at most three false 3-faces.

Proof. (i) Suppose that $v$ is incident with one true 3-face, say $f = vv_1v_2$. Suppose to the contrary that $v$ is incident with at least five false 3-faces.

If $v$ is incident with three consecutive false 3-faces such that they have no common edges with the true 3-face $vv_1v_2$. By Lemma 3.2, we can find an edge of $G[N(v)]$ among such three false 3-faces, which together with edge $v_1v_2$ form a 2-matching in $G[N(v)]$, contradicting Lemma 3.1.

Otherwise, there are three consecutive false 3-faces, say $vv_2v_4, vv_4v_6, vv_6v_7$, incident with $v$ having a common edge $vv_2$ with $vv_1v_2$ and the other two consecutive false 3-faces, say $vv_1v_3$ and $vv_3v_5$, having a common edge $vv_1$ with $vv_1v_2$ (see Fig. 4(a)). Since $v_1$ and $v_2$ are true vertices, $v_5$ and $v_6$ are both true vertices, and $v_1v_5$ is an edge of $G$ passing through $v_3$ and $v_2v_6$ is an edge of $G$ passing through $v_4$. Now \(\{v_1v_5, v_2v_6\}\) is a 2-matching in $G[N(v)]$, contradicting Lemma 3.1. This shows that Statement (i) holds.

(ii) Suppose that $v$ is incident with two true 3-faces. Then such two true 3-faces are consecutive by Claim 3, which are denoted by $vv_1v_2$ and $vv_2v_3$. Suppose to the contrary that $v$ is incident with at least four false 3-faces.

If there are three consecutive false 3-faces incident with $v$, then they have no common edges with at least one of the two true 3-face $vv_1v_2$ and $vv_2v_3$. Similar to (i) we can find a 2-matching in $G[N(v)]$, contradicting Lemma 3.1.

Otherwise, there are two consecutive false 3-faces, say $vv_1v_4$ and $vv_4v_6$, that has one common edge $vv_1$ with the true 3-face $vv_1v_2$, and the other two consecutive false 3-faces,

![Fig. 3. 7-vertex $v$ in $G^\times$ which is incident with exactly six false 3-faces.](image)
say \( vv_3v_5 \) and \( vv_5v_7 \), that has one common edge \( vv_3 \) with the true 3-face \( vv_2v_3 \) (see Fig. 4(b)). Since \( v_1 \) and \( v_3 \) are true vertices, \( v_6 \) and \( v_7 \) are both true vertices, and \( v_1v_6 \) is an edge of \( G \) passing through \( v_4 \) and \( v_3v_7 \) is an edge of \( G \) passing through \( v_5 \). Now \( \{v_1v_6, v_3v_7\} \) is a 2-matching in \( G[N(v)] \), contradicting Lemma 3.1. So Statement (ii) holds.

\[ \square \]

Fig. 4. 7-vertex \( v \) in \( G^x \) which is incident with at least one true 3-face.

The following are the discharging rules.

**R1** Every false vertex in \( G^x \) gives \( \frac{1}{2} \) to each incident 4\(^-\)-face.

**R2** Every \( 6^+\)-vertex in \( G^x \) gives \( \frac{4}{3} \) to each incident true 3-face, and gives \( \frac{7}{4} \) to each incident false 3-face, and gives \( \frac{1}{2} \) to each incident 4-face.

Next we verify that the new charge of each member in \( V(G^x) \cup F(G^x) \) is nonnegative. Firstly we consider any face \( f \in F(G^x) \). There are three cases according to the degree of a face.

**Case 1.** \( d_{G^x}(f) = 3 \). If \( f \) is a true 3-face, then all vertices incident with \( f \) are \( 6^+\)-vertices since \( \delta(G) \geq 6 \). By R2, \( w'(f) = 2d_{G^x}(f) - 10 + \frac{1}{2} \times 3 = 2 \times 3 - 10 + 4 = 0 \). If \( f \) is a false 3-face, then all vertices incident with \( f \) are one false vertex and two \( 6^+\)-vertices because \( \delta(G) \geq 6 \) and \( G \) is a 1-planar graph. By R1 and R2, \( w'(f) = 2d_{G^x}(f) - 10 + \frac{1}{2} + \frac{7}{4} \times 2 = 2 \times 3 - 10 + \frac{1}{2} + \frac{7}{2} = 0 \).

**Case 2.** \( d_{G^x}(f) = 4 \). Whether \( f \) is a true 4-face or a false 4-face, by R1 and R2, \( w'(f) = 2d_{G^x}(f) - 10 + \frac{1}{2} \times 4 = 2 \times 4 - 10 + 2 = 0 \).

**Case 3.** \( d_{G^x}(f) \geq 5 \). By R1 and R2, \( f \) has neither lost charge nor gained charge, so \( w'(f) = 2d_{G^x}(f) - 10 \geq 0 \).

Now we consider any vertex \( v \in V(G^x) \). There are four cases according to the degree of a vertex.

**Case 4.** \( d_{G^x}(v) = 4 \). Because \( v \) is incident with at most four 4\(^-\)-faces, \( w'(v) \geq 3d_{G^x}(v) - 10 - \frac{1}{2} \times 4 = 3 \times 4 - 10 - 2 = 0 \) by R1.
Case 5. $d_{G^x}(v) = 6$. By Claim 1 and R2, $w'(v) \geq 3d_{G^x}(v) - 10 - \frac{7}{4} \times 4 - \frac{1}{2} \times 2 = 3 \times 6 - 10 - 7 - 1 = 0$.

Case 6. $d_{G^x}(v) = 7$. If $v$ is not incident with a true 3-face, then by Claim 2 and R2, $w'(v) \geq 3d_{G^x}(v) - 10 - \frac{7}{4} \times 6 - \frac{1}{2} \times 1 = 3 \times 7 - 10 - \frac{21}{2} - \frac{1}{2} = 0$; Otherwise, $v$ is incident with one or two true 3-faces by Claim 3. Further, by Claim 4 and R2, $w'(v) \geq 3d_{G^x}(v) - 10 - \frac{7}{4} \times 4 - \frac{4}{3} \times 1 - \frac{1}{2} \times 2 = 3 \times 7 - 10 - 7 - \frac{4}{3} - 1 = \frac{5}{3} > 0$.

Case 7. $d_{G^x}(v) \geq 8$. By R2, $w'(v) \geq 3d_{G^x}(v) - 10 - \frac{7}{4} d_{G^x}(v) = \frac{5}{4} d_{G^x}(v) - 10 \geq 0$.

In summary we get that for each $x \in V(G^x) \cup F(G^x)$, $w'(x) \geq 0$, which is a contradiction. This completes the proof of Theorem 1.11.

We remark that the non-5-extendability of 1-planar graphs in Theorem 1.11 is best possible by presenting a 4-extendable 1-planar graph drawn in Fig. 5. We use a computer program to check the validation of the example: we find that the 1-planar graph has exactly 967469 4-matchings and each 4-matching can be contained in a perfect matching. Further, the graph has 1116948 perfect matchings and a 5-matching (see bold edges in Fig 5) not extendable to a perfect matching.

![Fig. 5. A 4-extendable 1-planar graph.](image)

4 Remarks on factor-criticality of 1-planar graphs

We conclude with some remarks on the maximum factor-criticality of (optimal) 1-planar graphs. Yu [12] and Favaron [11] independently formulated the definition of a $k$-factor-critical graph. A graph of order $n$ is $k$-factor-critical, where $k$ is an integer with $0 \leq k < n$ and $n + k$ is even, if $G - S$ admits a perfect matching for every set $S$ of
$k$ vertices of $G$. Favaron in [11] obtained following basic properties of $k$-factor-critical graphs.

**Theorem 4.1** ([11]). For $k \geq 2$, any $k$-factor-critical graph of order $n > k$ is $(k - 2)$-factor-critical.

**Theorem 4.2** ([11]). For $k \geq 1$, any $k$-factor-critical graph of order $n > k$ is $k$-connected and $(k + 1)$-edge-connected.

**Theorem 4.3.** No 1-planar graph is 7-factor-critical.

*Proof.* Suppose to the contrary that there exists a 7-factor-critical 1-planar graph $G$. Then by Theorem 4.2, $G$ is 8-edge-connected. Then $\delta(G) \geq 8$, contradicting Lemma 1.5.

![Fig. 6. A 6-factor-critical 1-planar graph.](image)

![Fig. 7. A 5-factor-critical 1-planar graph.](image)

In other word, each 1-planar graph is not 7-factor-critical. The non-7-factor-criticality of 1-planar graphs is best possible by providing a 6-factor-critical 1-planar graph in Fig. 6 that is a 7-regular 1-planar graph taken from [2] and a 5-factor-critical 1-planar graph of
odd order in Fig. 7. We also present a computer check to the validation of both examples: we find that the former has exactly 340361 perfect matchings and the removal of any 6 vertices results in a graph with a perfect matching, and the removal of any 5 vertices of the latter results in a graph with a perfect matching.

Next we turn to factor-criticality of optimal 1-planar graphs. The following theorem can be obtained from Theorem 1.9. Here we give a direct proof.

**Theorem 4.4.** No optimal 1-planar graph is 6-factor-critical.

*Proof.,* Suppose to the contrary that there exists a 6-factor-critical optimal 1-planar graph $G$. Then by Theorem 4.2, $G$ is 7-edge-connected and $\delta(G) \geq 7$. From Lemma 2.2, each vertex of $G$ has even degree, so $\delta(G) \geq 8$, contradicting Lemma 1.5.

Here the non-6-factor-criticality of optimal 1-planar graphs is best possible by presenting 4- and 5-factor-critical optimal 1-planar graphs shown in Fig. 8 (a) and (b) respectively. Their validation has also been confirmed by a computer program.

![Fig. 8. 4- and 5-factor-critical optimal 1-planar graphs.](image)

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