Random Matrix Theory and neutrino propagation in a turbulent medium

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It is becoming ever clearer that the neutrino signal from the next supernova in our Galaxy can reveal missing information about the neutrino as well as allowing us to probe the explosion of the star by decoding the temporal and spectral evolution of the flavor composition of the signal. But this information may be lost if turbulence in the supernova ‘depolarizes’ the neutrinos so that the observed flux for each flavor is an equal mixture of the initial - unencoded - spectra. Determining if depolarization occurs is one of the most pressing issues of this field. The most difficult aspect of studying the effect of turbulence upon the neutrinos is the lack of any theoretical models that allow us to understand the results of numerical studies. This paper makes the suggestion that Random Matrix Theory (RMT) may shine some light in this direction and presents support for this the possibility by comparing the distribution of crossing and survival probabilities obtained numerically for some ‘test case’ calculations with the distributions one expects from RMT in the calculable limit of depolarization of N neutrino flavors.

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Introduction

The subject of neutrino propagation in core-collapse supernova has been the focus of intense interest in recent years with many new effects discovered. From this ever-growing body of results we now can expect that the signal from the next Galactic supernova to reveal both unknown properties of the neutrinos - such as the ordering of the neutrino masses - as well as details of the explosion that would allow us to test the basic paradigm. This information is imprinted into the neutrino flavor composition of the flux as a function of time and energy due to neutrino collective effects and the evolving profile of the star as it explodes. For a review the reader may consult [1, 2].

But lurking in the shadows is the flavor transformation of the neutrino flavor due to the turbulence in the supernova. Turbulence in a medium creates additional Mikheev-Smirnov-Wolfenstein (MSW) resonances - locations where mixing is strongest - which, together with the weaker non-resonant fluctuations, can cause flux depolarization [3, 8]. Flux depolarization is fatal for any search for information in the supernova neutrino signal because spectral features are removed or will not be imprinted. Determining if depolarization indeed occurs is a very high priority. But the purpose of this paper is not to reconsider the potential of the supernova neutrino signal to reveal properties of the supernova and/or the neutrino. Instead we grapple with the problem that studying and quantifying the effect of turbulence in supernova lacks theoretical models that can provide insight into the problem. We make the suggestion that Random Matrix Theory (RMT) may be the doorway to progress in future studies.

At heart neutrino propagation through a supernova is a scattering problem in which we try to relate observed neutrino states to their initial states. The applicability of RMT to scattering problems has long been recognized: see, for example [9, 12]. That RMT may be useful for neutrino propagation is suggested by the similarity of the non-adiabaticity parameter in Kneller & McLaughlin [13] to a Breit-Wigner resonance and recalling Ericson’s prediction [14] from RMT that of fluctuations in nuclear cross sections from intermediate states of compound nuclei. If RMT is indeed useful then it must be able to predict the distributions of survival and crossing probabilities obtained from numerical turbulence studies. This is no trivial challenge and deciding if RMT has any shot of success is left to the future. The view that RMT may be one path to progress is supported by its success in one particular case - the depolarized limit - where one can derive the expected distribution with relative ease. The plan of this paper is thus to present the results of a small numerical study of turbulence and neutrinos for some selected test cases, to derive the expected distributions in the depolarized limit from an ensemble of N-flavor random unitary matrices [15], and then make the comparison between them pointing out the similarities but also the differences that future, more sophisticated, applications of RMT will have to predict.

Turbulence And Supernovae Neutrinos Test Cases

The quantities we are interested in calculating are the survival and crossing probabilities: i.e. \( P(\nu_j \rightarrow \nu_i) \equiv P_{ij} \) since the flux of neutrinos emerging from the supernova is given by these quantities multiplied by the appropriate initial fluxes at the proto-neutron star. Throughout this paper we chose to calculate the ‘matter’ basis probabilities because in this basis there are no confusing adiabatic MSW transitions. The reader may find the relationship between the matter and flavor bases in Kneller & McLaughlin [17]. These probabilities can be calculated from the S matrix as \( P_{ij} = |S_{ij}|^2 \). The state of the neutrino after traveling a distance \( r \) is related to the initial state at \( r = 0 \) by \( |\nu(r)\rangle = S(r, 0) |\nu(0)\rangle \) where the matrix \( S(r, 0) \) is the solution of the equation

\[
\frac{dS}{dr} = H.S
\]

(1)

with initial condition that \( S(0, 0) = 1 \). The Hamiltonian
$H$ has two components: the vacuum mass term $K$ and the potential $V$ that accounts for the effect of matter. We shall not consider the case where the neutrino density is so high that collective neutrino effects need to be included. The vacuum mass term in the flavor basis, $K^{(f)}$, is set by the mass square differences and the parameters $\theta_{12}, \theta_{13}, \theta_{23}$ of the Maki-Nakagawa-Sakata-Pontecorvo (MNSP) mixing matrix. The two Majorana phases of the MNSP matrix are irrelevant and we set its CP phase $\delta$ to zero. For this paper we adopt $\delta m^2_{32} = 8 \times 10^{-3} \text{eV}^2$, $|\delta m^2_{21}| = 3 \times 10^{-3} \text{eV}^2$, $\sin^2 2\theta_{13} = 0.83$ and $\sin^2 2\theta_{23} = 1$ [18]. All values of the unknown mixing angle $\theta_{13}$ we shall consider will be ‘large’ i.e. they will be above the Dighe & Smirnov [19] threshold of $\sin^2(2\theta_{13}) \sim 10^{-5}$. If the sign of $\delta m^2_{23}$ is positive then the ‘hierarchy’ - the ordering of the masses - is known as ‘normal’; if the sign negative then the hierarchy is ‘inverted’. The potential affects only the electron neutrinos and antineutrinos and, in the flavor basis, the only non-verted’. The potential affects only the electron neutrinos and antineutrinos and, in the flavor basis, the only non-verted’.

The density profile of the 2D model described there. We do not distribute the underlying profile is the same for every instantiation so that collective neutrino effects need to be included. The potential affects only the electron neutrinos and antineutrinos and, in the flavor basis, the only non-verted’.

This particular simulation was chosen because it most closely resembles the profiles of the 2D model described there. We do not distribute the turbulence throughout the entire profile because the profile is dissected into distinct regions by the forward and reverse shocks. Within each region one would expect the turbulence, if any, to be quite different. For this paper we shall only insert turbulence between the reverse shock at $r_r$ and the forward shock at $r_s$ which is the region with largest amplitude fluctuations as indicated by multi-dimensional supernova simulations [21, 23]. The density profile $\rho(r)$ is thus constructed as $\rho(r) = \langle \rho(r) \rangle + \delta \rho(r)$ where $\langle \rho(r) \rangle$ is the adopted one-dimensional profile from the simulation. As is common, we model $\delta \rho(r)$ as $\delta \rho(r) = F(r) \langle \rho(r) \rangle$ with $F(r)$ a random field with vanishing expectation value. For this paper, $F(r)$ is proportional to: a parameter $C_s$ which sets the amplitude of the fluctuations, the factor $\tanh((r - r_f)/\lambda) \tanh((r_s - r)/\lambda)$ whose purpose is to avoid introducing additional discontinuities at $r_r$ and $r_s$ and where $\lambda$ is a scale set to $\lambda = 100 \text{ km}$, and to

$E(k) = (\alpha - 1) \left( \frac{k_s}{k} \right)^\alpha$.

Now that we have our density profiles we can calculate how a neutrino propagates through it for some fixed set of oscillation parameters and neutrino energy. The method used to solve equation (1) is described in Kneller & McLaughlin [17]. One finds that the probabilities $P_{ij}$ for the most past are constant as a function of the distance $r$ and change only in the vicinity of the ‘resonances’ which are the locations where the separations of the eigenvalues of $H$ are minimal. Resonances occur at two different densities: the resonance at high density is known as the H resonance, the one at low density is the L resonance. The L resonance always occurs between states $\nu_1$ and $\nu_2$ because the sign of $\delta m^2_{13}$ is known. For a normal hierarchy the H resonance occurs between states $\nu_2$ and $\nu_3$: for an inverted hierarchy it is antineutrino states $\bar{\nu}_1$ and $\bar{\nu}_3$ that mix.
Numerical Results

Inverted Hierarchy

With the setup complete we present numerical results; we consider first the case of an inverted hierarchy. In figure (1) we show the evolution of the survival probability \( P_{11} = P(\bar{\nu}_1 \to \bar{\nu}_1) \) as a function of the radius \( r \) through one instantiation of the potential. Without fluctuations the neutrino experiences three H resonances; when turbulence is added the number of MSW resonances increases dramatically. Since the value \( \theta_{13} \) is ‘large’ we observe that the semi-adiabaticity of the additional MSW resonances kicks the state of the neutrino at it passes through them. But the effect of turbulence is not just the addition of new H resonances because we also observe that non-resonant fluctuations also cause the probability \( P_{11} \) to evolve. Once the neutrino has passed through the entire turbulence region we find there is a significant difference between the final states of the neutrino through the profiles with and without turbulence. This difference is not fixed: if we use a new instantiation of \( F \) we obtain a different result. By repeating the calculation with 1000 different instantiations we generate figure (2) which is a frequency distribution of the final state i.e. the state at the edge of the star. The figure indicates that the distribution of the final states is consistent with uniform and the final state is completely uncorrelated with the initial state. So for this energy, \( \theta_{13} \) and turbulence amplitude we have a case where two flavor depolarization occurs.

Normal Hierarchy

When we switch to a normal hierarchy the H resonance now affects the neutrinos. For modest turbulence, \( C^2_s \lesssim 0.1 \), we obtain uniform distributions of final states for \( \nu_2 \) and \( \nu_3 \). But as \( C_s \) increases we find something new. The reason is that for the normal hierarchy the L and H resonances both occur for the neutrinos so large amplitude fluctuations break HL factorization i.e. the L resonances no longer occur after all the H resonances but rather the two are mixed together. An example of a case with broken HL factorization can be seen in figure (3). For a normal hierarchy the H resonance mixes states \( \nu_2 \) and \( \nu_3 \) and state \( \nu_1 \) is unaffected but for this particular calculation the density drops very close to the L resonance at around \( r \sim 50,000 \) km whereupon the matter state \( \nu_1 \) mixes with matter state \( \nu_2 \). The density thereafter returns to the H resonance where further mixing between states \( \nu_2 \) and \( \nu_3 \) occurs. This mixing between states \( \nu_1 \) and \( \nu_2 \) means that the distributions for the final state probabilities are now quite different. The distributions of the final states for the neutrinos from 1563 calculations for this hierarchy, \( C^2_s = 0.3 \) and \( E = 60 \) MeV are shown in figure (4). Without turbulence \( P_{11} \) is always unity for the mixing parameters used and one notices that this is still a common result. But we also see that there appears to be a triangular component in both the distributions of \( P_{11} \) and \( P_{22} \); if state \( \nu_3 \) has a triangular component it is not apparent. We thus complete our rudimentary study of turbulence effects upon supernovae neutrinos; in the next section we derive the expected distributions in the limit of N flavor depolarization and then compare them with these numerical results.

Predictions For Random Unitary S Matrices

We make the conjecture that in the depolarized limit the \( S(r, 0) \) matrices have a block structure such that each block is a random N-flavor unitary matrix [15]. From this ansatz we can derive the distribution of the probabilities \( P_{ij} = |S_{ij}|^2 \) in this limit. The N real components, \( x_{ij} \), plus the N imaginary components, \( y_{ij} \), of the elements in a row or column of the block form a 2N Euclidean space. The requirement of unitarity defines a unit sphere in this space and since these 2N quantities are identically distributed the probability of a particular set of the elements must be uniform over the surface of the sphere. That is, the probability

\[
P(\bar{\nu}_1 \to \bar{\nu}_1) = \frac{1}{10^4} \times \frac{1}{10^4} = \frac{1}{10^8}
\]
We change variables so that the simply proportional to the area element for a column
that we are located at \( \{x_{1j}, y_{1j}, x_{2j}, y_{2j}, \ldots \} \) where, say for a column \( S_{1j} = x_{1j} + iy_{1j}, S_{2j} = x_{2j} + iy_{2j}, \ldots \), is simply proportional to the area element \( dA \)
\[
P(x_{1j}, y_{1j}, x_{2j}, y_{2j}, \ldots) \propto d^N x d^N y \propto dA
\] (3)
\[
P(x_{1j}, y_{1j}, x_{2j}, y_{2j}, \ldots) \propto \delta \left( 1 - \sum_{i=1}^{N} x_{ij}^2 - \sum_{i=1}^{N} y_{ij}^2 \right) \prod_{i=1}^{N} dx_{ij} \prod_{i=1}^{N} dy_{ij}.
\] (4)
We change variables so that the \( N \) independent pairs \( x_{ij}, y_{ij} \) are expressed as
\[
x_{1j} = \sqrt{P_{1j}} \cos \theta_{1j}, \quad y_{1j} = \sqrt{P_{1j}} \sin \theta_{1j},
\] (5)
\[
x_{2j} = \sqrt{P_{2j}} \cos \theta_{2j}, \quad y_{2j} = \sqrt{P_{2j}} \sin \theta_{2j},
\] (6)
and so on. After inserting these new variables into equation (4) and integrating over the angles we find that
\[
P(P_{1j}, \ldots, P_{Nj}) \propto \delta \left( 1 - \sum_{i=1}^{N} P_{ij} \right) \prod_{i=1}^{N} dP_{ij}.
\] (7)
Thus, the set \( \{P_{1j}, \ldots, P_{Nj}\} \) are uniformly distributed on the surface of a standard \( N - 1 \) simplex. By integrating equation (7) over \( N - 1 \) of the \( P \)'s and normalizing one derives our final result that element \( P_{ij} \) must be distributed according to
\[
P(P_{ij}) = (N-1) \left( 1 - P_{ij} \right)^{N-2}
\] (8)
For \( N = 2 \) the distribution is uniform with mean 1/2 and variance 1/12; for \( N = 3 \) the distribution is triangular with variance 1/18.

**Discussion**

We find that the expected distributions in the depolarized limit possess some resemblance with the numerical results. For the test case results shown in figure (3) we have achieved two flavor depolarization since the distribution is uniform; the results shown in figure (4) appear to be a transitional stage between the \( N=2 \) and \( N=3 \) depolarized limits. This resemblance is encouraging and supports our suggestion that better RMT calculations - obviously not based on Dyson's ensemble - may allow to predict the distributions for all cases. This suggestion is the principal message of this paper though we are fully aware that deriving the distributions for general cases will certainly be a challenge.

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