Tree-level Amplitudes in the Nonlinear Sigma Model

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Abstract: We study in detail the general structure and further properties of the tree-level amplitudes in the $SU(N)$ nonlinear sigma model. We construct the flavor-ordered Feynman rules for various parameterizations of the $SU(N)$ fields $U(x)$, write down the Berends-Giele relations for the semi-on-shell currents and discuss their efficiency for the amplitude calculation in comparison with those of renormalizable theories. We also present an explicit form of the partial amplitudes up to ten external particles. It is well known that the standard BCFW recursive relations cannot be used for reconstruction of the the on-shell amplitudes of effective theories like the $SU(N)$ nonlinear sigma model because of the inappropriate behavior of the deformed on-shell amplitudes at infinity. We discuss possible generalization of the BCFW approach introducing “BCFW formula with subtractions” and with help of Berends-Giele relations we prove particular scaling properties of the semi-on-shell amplitudes of the $SU(N)$ nonlinear sigma model under specific shifts of the external momenta. These results allow us to define alternative deformation of the semi-on-shell amplitudes and derive BCFW-like recursion relations. These provide a systematic and effective tool for calculation of Goldstone bosons scattering amplitudes and it also shows the possible applicability of on-shell methods to effective field theories. We also use these BCFW-like relations for the investigation of the Adler zeroes and double soft limit of the semi-on-shell amplitudes.
E. Double soft limit of Goldstone boson amplitudes

E.1 Vector WI and symmetry with respect to $H$  
E.2 Soft vector current singularity  
E.3 Axial WI and Adler zero  
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1. Introduction

The chiral nonlinear sigma model is a widely used tool for description of many phenomena in theoretical particle physics. It is based on a simple Lie Group $G$ and the spontaneous symmetry breaking $G \times G \rightarrow G$ gives rise to massless excitations - Goldstone bosons. For instance, in the theory of strong interactions, the group $G$ is $SU(N_f)$ where $N_f = 2, 3$ is a number of light quark flavors and Goldstone bosons are associated with the triplet of pions (for $N_f = 2$) or octet of pseudoscalar mesons $\pi$, $K$ and $\eta$ (for $N_f = 3$). The interactions of these degrees of freedom dominate the hadronic world at low energies. In this context, the leading order nonlinear $U(3) \times U(3)$ chiral invariant effective Lagrangian, the kinetic part of which corresponds to the chiral nonlinear $U(3)$ sigma model, was constructed in the late sixties by Cronin [1] while the $SU(2)$ case was studied by Weinberg [2, 3], Brown [4] and Chang and Gürsey [5]. Further generalization lead to the invention of Chiral Perturbation Theory as a low energy effective theory of Quantum Chromodynamics by Weinberg [6] and by Gasser and Leutwyler [7], [8]. Chiral Perturbation Theory became a very useful tool for the investigation of the low energy hadron physics.

The focus of this paper is on scattering amplitudes of Goldstone bosons within the $SU(N)$ nonlinear sigma model described by the leading order Lagrangian. In principle, the standard Feynman diagram approach allows us to calculate arbitrary amplitude. Because the model is effective, and the Lagrangian contains an infinite tower of terms the calculation becomes very complicated for amplitudes of many external Goldstone bosons even at tree-level. It would be therefore desirable to find alternative non-diagrammatic methods which could save the computational effort and provide us with a tool to get the amplitudes more efficiently. In the past an attempt to formulate the calculation of the tree-level without any reference to the Lagrangian was made by Susskind and Frye [9]. They postulated recursive procedure for pion amplitudes based on certain algebraic duality assumptions supplemented with the requirement of Adler zero condition which should have to be satisfied separately for group-factor free kinematical functions recently known as the partial or stripped amplitudes. Such a condition had been proven in the special case of pion amplitudes described by the $SU(2)$ nonlinear sigma model by Osborn [10]. In [9] the authors successively calculated the amplitudes up to eight pions and showed that these results are equivalent to the diagrammatic calculation based on the $SU(2)$ nonlinear sigma model. The full equivalence for all amplitudes has been proven by Ellis and Renner in [11].

Over the past two decades there has been a huge progress in understanding scattering amplitudes using on-shell methods (for a review see e.g. [12–15]). They do not use explicitly the Lagrangian description of the theory and all on-shell quantities are calculated using on-shell data only with no access to off-shell physics (unlike virtual particles in Feynman diagrams). This has lead to many new theoretical tools (e.g. unitary methods [16,17], BCFW recursion relations for tree-level amplitudes [18, 19] and the loop integrand [20]) as well as practical applications of on-shell methods to LHC processes (for recent results of the next-to-leading order QCD corrections for $W + 4$-jets see [21]). Most of the recent theoretical developments have been driven by an intensive exploration of $\mathcal{N} = 4$ super Yang-Mills theory in the planar limit both at weak and strong couplings (see e.g. [22–33]).
There have been several attempts to extend some of these methods to other theories. The most natural starting point are the recursion relations for on-shell tree-level amplitudes, originally found by Britto, Cachazo, Feng and Witten for Yang-Mills theory [18], [19] and later also for gravity [34], [35]. The main idea is to perform a complex shift on external momenta and reconstruct the amplitude recursively using analytic properties of the S-matrix. More recently, this recursive approach was extended to Yang-Mills and gravity theories coupled to matter, as well as more general class of renormalizable theories [36].

In this paper, we find the new recursion relations for all on-shell tree-level amplitudes of Goldstone bosons within \( SU(N) \) nonlinear sigma model. This shows that on-shell methods can be applied also for effective field theories and it gives new computational tool in this model. Using these recursion relations we are also able to prove more properties of tree-level amplitudes that are invisible in the Feynman diagram approach.

The paper is organized as follows: In section 2 we discuss \( SU(N) \) nonlinear sigma model, introduce stripped amplitudes and using minimal parametrization (the convenient properties of which has been discussed in [11]) we calculate tree-level amplitudes up to 10 points. In section 3 we review BCFW recursion relations and their generalization to theories that do not vanish at infinity at large momentum shift. Section 4 is the main part of the paper, we first introduce semi-on-shell amplitudes, i.e. amplitudes with \( n - 1 \) on-shell and one off-shell external legs. Then we prove scaling properties under particular momentum shifts which allows us to construct BCFW-like recursion relations. Finally, we show explicit 6pt example. In section 5 we use previous results to prove Adler zeroes and double-soft limit formula for stripped amplitudes. Additional results and technical details are postponed to appendices: In Appendix A, we describe the general parametrization of the \( SU(N) \) nonlinear sigma model. In Appendix B we give the results of the amplitudes up to 10p. Appendix C is devoted to the counting of flavor-ordered Feynman graphs needed for the calculations of the amplitudes in nonlinear sigma models and other theories. In Appendix D we present additional scaling properties of the semi-on-shell amplitudes. In Appendix E, we study the double soft-limit for more general class of spontaneously broken theories for complete (not stripped) amplitudes.

2. Nonlinear sigma model

2.1 Leading order Lagrangian

Let us first assume a most general case of the principal chiral nonlinear sigma model based on a simple compact Lie group \( G \). Such a model corresponds to the spontaneous symmetry breaking of the chiral group \( G_L \times G_R \) where \( G_{L,R} = G \) to its diagonal subgroup \( G_V = G \), i.e. to the subgroup of the elements \( h = (g_L, g_R) \) where \( g_L = g_R \). The vacuum little group \( G_V \) is invariant with respect to the involutive automorphism \( (g_L, g_R) \rightarrow (g_R, g_L) \) and the homogeneous space \( G_L \times G_R / G_V \) is a symmetric space which is isomorphic to the group space \( G \). A canonical realization of such an isomorphism is via restriction of the mapping

\[
(g_L, g_R) \rightarrow g_R g_L^{-1} \equiv U
\]  

(2.1)
(which is constant on the right cosets of $G_V$ in $G_L \times G_R$) to $G_L \times G_R / G_V$. Provided we induce the action of the chiral group on $G_L \times G_R / G_V$ by means of the left multiplication, the transformation of $U$ under general element $(V_L, V_R)$ of the chiral group is linear

$$U \to V_R U V_L^{-1}. \tag{2.2}$$

This can be used to construct the most general chiral invariant leading order effective Lagrangian in general number $d$ of space-time dimensions describing the dynamics of the Goldstone bosons corresponding to the spontaneous symmetry breaking $G_L \times G_R \to G_V$ as

$$\mathcal{L}^{(2)} = \frac{F^2}{4} \langle \partial_\mu U \partial^\mu U^{-1} \rangle = -\frac{F^2}{4} \langle (U^{-1} \partial_\mu U)(U^{-1} \partial^\mu U) \rangle, \tag{2.3}$$

where $F$ is a constant\footnote{The decay constant of the Goldstone bosons.} with the canonical dimension $d/2 - 1$. Here and in what follows we use the notation $\langle \cdot \rangle = \text{Tr}(\cdot)$ and the trace is taken in the defining representation of $G$. The overall normalization factor is dictated by the form of the parametrization of the matrix $U$ in terms of the Goldstone boson fields $\phi^a$ which we write for the purposes of this subsection\footnote{In what follows we will use also more general parametrization of $U$.} as

$$U = \exp \left( \sqrt{2} i F \phi \right) \tag{2.4}$$

where $\phi = \phi^a t^a$ and $t^a, a = 1, \ldots, \text{dim} \ G$ are generators of $G$ satisfying

$$\langle t^a t^b \rangle = \delta^{ab}, \tag{2.5}$$

$$[t^a, t^b] = \sqrt{2} f^{abc} t^c. \tag{2.6}$$

Here $f^{abc}$ are totally antisymmetric structure constants of the group $G$. According to (2.2), the fields $\phi^a$ transform linearly under the little group $G_V$ as the vector in the adjoint representation of $G$ while the general chiral transformations of $\phi^a$ are nonlinear.

The Lagrangian $\mathcal{L}^{(2)}$ can be rewritten in terms of the Goldstone boson fields as follows. We have

$$U^{-1} \partial_\mu U = -\frac{\exp \left( -\sqrt{2} i F \text{Ad}(\phi) \right) - 1}{\text{Ad}(\phi)} \partial_\mu \phi = -\frac{1}{\sqrt{2}} \cdot \frac{\exp \left( -\sqrt{2} i F \partial_\mu \phi \right) - 1}{D_\phi} \cdot \partial \phi \tag{2.7}$$

where

$$\text{Ad}(\phi) \partial_\mu \phi = [\phi, \partial_\mu \phi] = \sqrt{2} i a D_\phi \partial_\mu \phi^b \equiv \sqrt{2} i \cdot D_\phi \cdot \partial \phi, \tag{2.8}$$

the matrix $D_\phi^{ab}$ is given as

$$D_\phi^{ab} = -i f^{cab} \phi^c \tag{2.9}$$

and the dot means contraction of the indices in the adjoint representation. Inserting this in (2.3) we get finally

$$\mathcal{L}^{(2)} = \frac{F^2}{4} \langle \partial \phi^T \cdot \frac{1 - \cos \left( \frac{2}{F} D_\phi \right)}{D_\phi^2} \cdot \partial \phi \rangle = \partial \phi^T \cdot \left( \sum_{n=1}^{\infty} \frac{1}{(2n)!} \left( \frac{2}{F} \right)^{2n-2} D_\phi^{2n-2} \right) \cdot \partial \phi. \tag{2.10}$$
2.2 General properties of the tree-level scattering amplitudes

Note that, the only group factors which enter the interaction vertices are the structure constants $f^{abc}$. In any tree Feynman diagram each $f^{abc}$ is contracted either with another structure constant within the same vertex or via propagator factor $\delta^{ab}$ with some structure constant entering next vertex. Therefore, using the standard argumentation for a general tree graph [12], i.e. expressing any $f^{abc}$ as a trace $f^{abc} = -\langle i [t^a, t^b] t^c \rangle / \sqrt{2}$ and then successively using the relations like $f^{cde}t^c = -i [t^d, t^e] / \sqrt{2}$ in order to replace the contracted structure constants with the commutators of the generators inside the single trace, we can prove that any tree level on-shell amplitude has a simple group structure, namely

$$M^{a_1a_2\ldots a_n}(p_1, p_2, \ldots, p_n) = \sum_{\sigma \in S_n/Z_n} (t^{a_{\sigma(1)}} t^{a_{\sigma(2)}} \ldots t^{a_{\sigma(n)}}) M_\sigma(p_1, \ldots, p_n).$$

(2.11)

Here all the momenta treated as incoming and the sum is taken over the permutation of the $n$ indices $1, 2, \ldots, n$ modulo cyclic permutations. As a consequence of the cyclicity of the trace we get

$$M_\sigma(p_1, p_2 \ldots, p_n) = M_\sigma(p_2, \ldots, p_n, p_1)$$

(2.12)

Due to the Bose symmetry, the kinematical factors $M_\sigma(p_1, \ldots, p_n)$ has to satisfy

$$M_{\sigma \circ \rho}(p_1, \ldots, p_n) = M_\sigma(p_{\rho(1)}, p_{\rho(2)}, \ldots, p_{\rho(n)})$$

(2.13)

(where $\sigma \circ \rho$ is a composition of permutations) and therefore

$$M_\sigma(p_1, \ldots, p_n) = M(p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(n)})$$

(2.14)

where we have denoted $M \equiv M_{id}$ (here id is identical permutation). The amplitudes $M(p_1, \ldots, p_n)$ are called the stripped or partial amplitudes. Note that the same arguments can be used also for the Feynman rules for the interaction vertices, the general form of which can be written as

$$V_n^{a_1a_2\ldots a_n}(p_1, p_2, \ldots, p_n) = \sum_{\sigma \in S_n/Z_n} (t^{a_{\sigma(1)}} t^{a_{\sigma(2)}} \ldots t^{a_{\sigma(n)}}) V_{\sigma}(p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(n)}).$$

(2.15)

After some algebra we get explicitly (see Appendix A for details) $V_{2n+1}(p_1, \ldots, p_{2n+1}) = 0$ and

$$V_{2n}(p_1, \ldots, p_{2n}) = \frac{(-1)^n}{(2n)!} \left( \frac{F^2}{2} \right)^{n-1} \frac{2n-1}{k=1} (-1)^{k-1} \binom{2n-2}{k-1} \sum_{i=1}^{2n} (p_i \cdot p_{i+k}).$$

(2.16)

Let us note that besides (2.3), (2.4) we need not to use any algebraic relations specific for the concrete group $G$ when deriving this formula and it is therefore valid for general $G$. In the general case we can therefore define the stripped amplitudes and stripped vertices, however, their relation is not straightforward and may depend on the group $G$. In what follows we will concentrate on the case $G = SU(N)$. 

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2.3 Tree-level amplitudes for $G = SU(N)$

2.3.1 Flavor ordered Feynman rules

The standard way of calculation of the tree-level amplitudes $M_{a_1 \ldots a_n}^{a_1 \ldots a_n}(p_1, \ldots, p_n)$ is to evaluate the contributions of all tree Feynman graphs with $n$ external legs build form the complete vertices (2.15) and propagators $\Delta_{ab} = i\delta_{ab}/p^2$. This includes rather tedious group algebra which is specific for each group $G$. In the special case of $G = SU(N)$ the calculations can be further simplified. Because we have the completeness relations for the generators $t^a$ in the form

$$\sum_{a=1}^{N^2-1} \langle Xt^a \rangle \langle t^a Y \rangle = \langle XY \rangle - \frac{1}{N} \langle X \rangle \langle Y \rangle,$$

(2.17)

we can simply merge the traces from the vertices of any tree Feynman graphs in one single trace preserving at the same time the order of the generators $t^a$ inside the trace. Note that the “disconnected” $1/N$ terms have to cancel in the sum in order to produce the single trace in (2.11). This enables us to formulate simple “flavor ordered Feynman rules” directly for the stripped amplitudes $M$ completely in terms of the stripped vertices $V_n$. The general recipe is exactly the same as in the more familiar case of $SU(N)$ Yang-Mills theory, i.e. the tree graphs built form the stripped vertices are decorated with cyclically ordered external momenta and the corresponding ordering of the momenta inside the stripped vertices are kept.

Let us note that such a simple way of the calculation of the stripped amplitudes might not be possible for general group $G$. For instance for $G = SO(N)$ we have the following completeness relations

$$\sum_{a=1}^{N(N-1)/2} \langle Xt^a \rangle \langle t^a Y \rangle = \frac{1}{2} \left( \langle XY \rangle - \langle XY^T \rangle \right)$$

(2.18)

the second term of which reverses the order of the generators in the merged vertex and the aforementioned simple argumentation leading to the flavor ordered Feynman rules has to be modified.

The $SU(N)$ case has also another useful feature. As a consequence of the completeness relations (2.17) for the group generators of $SU(N)$ and the analogous relation

$$\sum_{a=1}^{N^2-1} \langle Xt^a Y t^a \rangle = \langle X \rangle \langle Y \rangle - \frac{1}{N} \langle XY \rangle$$

(2.19)

it can be proved [12] that the traces $\langle t^{\sigma(1)} t^{\sigma(2)} \ldots t^{\sigma(n)} \rangle$ and $\langle t^{\rho(1)} t^{\rho(2)} \ldots t^{\rho(n)} \rangle$ are orthogonal in the leading order of $N$ in the sense that

$$\sum_{\sigma_1, \sigma_2, \ldots, \sigma_n} \langle t^{\sigma(1)} t^{\sigma(2)} \ldots t^{\sigma(n)} \rangle \langle t^{\rho(1)} t^{\rho(2)} \ldots t^{\rho(n)} \rangle^* = N^{n-2}(N^2-1) \left( \delta_{\sigma \rho} + O \left( \frac{1}{N^2} \right) \right)$$

(2.20)

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As we shall see in what follows, this fact can be understood as a consequence of the decoupling of the $U(1)$ Goldstone boson in the nonlinear $U(N)$ sigma model.
where $\delta_{\rho \sigma} = 1$ for $\rho = \sigma$ modulo cyclic permutation and zero otherwise. This relation is enough to uniquely determine the coefficients $T_\sigma$ in the general expansion of the form

$$T^{a_1 a_2 \ldots a_n} = \sum_{\sigma \in S_n/\mathbb{Z}_n} \langle t^{a_{\sigma(1)}} t^{a_{\sigma(2)}} \ldots t^{a_{\sigma(n)}} \rangle T_\sigma, \quad (2.21)$$

(provided the coefficients $T_\sigma$ are $N-$independent) as the leading in $N$ terms of the “scalar product”

$$\sum_{a_1, a_2, \ldots, a_n} T^{a_1 a_2 \ldots a_n} \langle t^{a_{\sigma(1)}} t^{a_{\sigma(2)}} \ldots t^{a_{\sigma(n)}} \rangle^* = N^{n-2}(N^2 - 1) \left( T_\sigma + O \left( \frac{1}{N^2} \right) \right) \quad (2.22)$$

Because the stripped amplitudes and vertices by construction do not depend on $N$, the coefficients at the individual traces in the representation (2.11) are unique and therefore the stripped amplitudes and vertices are unique.

### 2.3.2 Dependence on the parametrization

Up to now we have identified the Goldstone boson fields $\phi^a$ using the exponential parametrization (2.4) of the group elements $U(\phi^a)$. However, according the equivalence theorem, the amplitudes $M^{a_1 a_2 \ldots a_n}(p_1, p_2, \ldots, p_n)$ are the same for any other parametrization $U(\tilde{\phi}^a)$ where

$$\tilde{\phi}^a = \phi^a + F^a(\phi) \quad (2.23)$$

where $F^a(\phi) = O(\phi^2)$ is at least quadratic in the fields $\phi$. Therefore, according to the aforementioned uniqueness, the stripped amplitudes for the nonlinear $SU(N)$ sigma model do not depend on the parametrization. Note, however, that this is not true for the stripped vertices which do depend on the parametrization because the complete vertices $V^{a_1 a_2 \ldots a_n}_n(p_1, p_2, \ldots, p_n)$ do.

As far as the on-shell tree-level amplitudes are concerned, in various calculations we are thus free to use the most suitable parametrization and consequently the most useful form of the corresponding stripped vertices for a given purpose. We shall often take advantage of this freedom in what follows.

A wide class of parameterizations for the chiral nonlinear sigma model with $G = U(N)$ and $G = SU(N)$ has been discussed in [1]. The general form of such a parameterizations reads

$$U = \sum_{k=0}^{\infty} a_k \left( \sqrt{2} \frac{1}{F^a(\phi)} \right)^k \quad (2.24)$$

where $a_0 = a_1 = 1$ and the remaining real coefficients $a_k$ are constrained by the requirement $UU^+ = 1$. The exponential parametrization (2.4) corresponds to the choice $a_n = 1/n!$. In fact, as was proved in [1], for $SU(N)$ nonlinear sigma model with $N > 2$, the exponential parametrization is the only admissible choice within the above class of parameterizations (2.24) compatible with the nonlinearly realized symmetry with respect to the $SU(N)$ chiral transformations (2.2). On the other hand, for $SU(2)$ and for the extended chiral group $G = U(N)$ with arbitrary $N$, the parameterizations of the form (2.24) represent an infinite-parametric class. The more detailed discussion can be found in Appendix A.
2.3.3 Interrelation of the cases $G = U(N)$ and $G = SU(N)$

Let us note, that the $SU(N)$ and $U(N)$ chiral nonlinear sigma models are tightly related. Within the exponential parametrization we can write in the $U(N)$ case

$$ U = \exp \left( \frac{i}{F} \sqrt{\frac{2}{N}} \phi^0 \right) \hat{U} $$

(2.25)

where $\hat{U} \in SU(N)$ and $\phi^0$ is the additional $U(1)$ Goldstone boson corresponding to the $U(1)$ generator $t^0 = 1/\sqrt{N}$. We get then

$$ U^{-1} \partial_\mu U = \frac{i}{F} \sqrt{\frac{2}{N}} \partial_\mu \phi^0 + \hat{U}^* \partial_\mu \hat{U} $$

(2.26)

and as a consequence,

$$ \mathcal{L}^{(2)} = \frac{1}{2} \partial \phi^0 \cdot \partial \phi^0 + \frac{F^2}{4} (\partial_\mu \hat{U} \partial^\mu \hat{U}^{-1}). $$

(2.27)

Therefore $\phi^0$ completely decouples. This means that for the on-shell amplitudes in this model

$$ M^{a_1a_2 \ldots a_n}(p_1, p_2, \ldots, p_n) = 0 $$

(2.28)

whenever at least one $a_j = 0$. Note that this statement does not depend on the parametrization. We can therefore reproduce the on-shell amplitudes of the $SU(N)$ chiral nonlinear sigma model from that of the $U(N)$ one simply by assigning to the indices $a_i$ the values corresponding the $SU(N)$ Goldstone bosons. Keeping this in mind, in what follows we will freely switch between the $U(N)$ and $SU(N)$ case and use the general parameterizations (2.24) also in the context of the $SU(N)$ chiral nonlinear sigma model.

The fact that the $U(1)$ Goldstone boson decouples gives also a nice physical explanation why the “disconnected” $1/N$ term can be omitted in the relation (2.17) when summing over virtual states in the tree-level Feynman graphs for the $SU(N)$ nonlinear sigma model. This term can be interpreted as the subtraction of the extra $U(1)$ virtual state contained in the first “connected” part. However, because this state decouples, no such correction is in fact needed.

The decoupling of the $U(1)$ Goldstone boson is an effect analogous to the decoupling of the $U(1)$ component of the gauge field in the case of the $U(N)$ Yang-Mills theory. For the tree-level amplitudes (and the corresponding stripped amplitudes) we get as a consequence a set of identities constraining their form. For instance taking only one $a_j = 0$ (say $a_1$) in (2.28), we get the “dual Ward identity” (or the $U(1)$ decoupling identity)

$$ \mathcal{M}(p_1, p_2, p_3, \ldots, p_n) + \mathcal{M}(p_2, p_1, p_3, \ldots, p_n) + \cdots = \mathcal{M}(p_2, p_3, \ldots, p_1, p_n) = 0 $$

(2.29)

exactly as in the Yang-Mills case (see e.g. [12] and references therein).

2.4 Explicit examples of $SU(N)$ on-shell amplitudes

Using (2.11) we can reconstruct the complete amplitude $\mathcal{M}^{a_1 \ldots a_n}(p_1, \ldots, p_n)$ just from a single stripped amplitude $\mathcal{M}(p_1, \ldots, p_n)$ which is given by the sum of Feynman diagrams with ordered
external legs \{1, 2, \ldots, n\}. Though the aim of this paper is not to calculate scattering amplitudes using the Feynman diagram approach, in this section we provide explicit examples for diagrammatic calculation of the stripped 4pt and 6pt amplitudes of the chiral nonlinear \(SU(N)\) sigma model (the 8pt and 10pt amplitudes we postpone to the Appendix B) as the reference result for the recursive formula given in section 4.

We can easily see that the only poles in the stripped amplitude are of the form \(1/s_{i,j}\) where

\[
s_{i,j} = p_{i,j}^2 \quad \text{with} \quad p_{i,j} = \sum_{k=i}^{j} p_k
\]

(Obviously \(s_{i,j} = s_{j+1,i-1}\) due to momentum conservation). The variables \(s_{i,j}\) are therefore well suited for presentation of the amplitudes.

As we have discussed above, the \(SU(N)\) stripped amplitudes are essentially the same as those for the \(U(N)\) case and, as we have discussed above, they are independent on the parametrization of the unitary matrix \(U\) in (2.3). The most convenient one for diagrammatic calculation of on-shell scattering amplitudes is the \textit{minimal} parametrization [11]

\[
U = \sqrt{2} \frac{i}{F} \phi + \sqrt{1 - 2 \frac{\partial^2}{F^2}} = 1 + \sqrt{2} \frac{i}{F} \phi - 2 \sum_{k=1}^{\infty} \left(\frac{1}{2F^2}\right)^k C_{n-1} \phi^{2k}
\]

where \(C_n\) are the Catalan numbers (A.12). The stripped Feynman rules for vertices can be written in terms of \(s_{i,j}\) as follows (see Appendix A for details)

\[
V_{2n+2}(s_{i,j}) = \left(\frac{1}{2F^2}\right)^n \frac{1}{2} \sum_{k=0}^{n-1} C_k C_{n-k-1} \sum_{i=1}^{2n+2} s_{i,i+2k+1}
\]

Note that within this parametrization the stripped vertices do not depend on the off-shellness of the momenta entering the vertex and when expressed in terms of the variables \(s_{i,j}\) they are identical taken both on-shell or off-shell. This rapidly speeds up the calculation, because there are no partial cancelations between the numerators and propagator denominators within the individual Feynman graphs and it allows us to find the final expressions for the amplitudes in very compact form.

The four-point amplitude is directly given by the Feynman rule in the simple parametrization,

\[
2F^2 \mathcal{M}(1, 2, 3, 4) = s_{1,2} + s_{2,3}.
\]

Note that for \(n\)-point amplitude \(\sum_{k=1}^{n} p_k = 0\) and this can be used to systematically eliminate \(p_n\) or equivalently \(s_{n,n}\).

The six-point amplitude is given by diagrams in Fig. 1. The explicit formula reads

\[
4F^4 \mathcal{M}(1, 2, 3, 4, 5, 6) =
\begin{align*}
&= \frac{(s_{1,2} + s_{2,3})(s_{1,4} + s_{4,5})}{s_{1,3}} - \frac{(s_{1,4} + s_{2,5})(s_{2,3} + s_{3,4})}{s_{2,4}} - \frac{(s_{1,2} + s_{2,5})(s_{3,4} + s_{4,5})}{s_{3,5}} \\
&\quad + \frac{(s_{1,2} + s_{1,4} + s_{2,3} + s_{2,5} + s_{3,4} + s_{4,5})}{s_{1,3}}
\end{align*}
\]

\[2.34\]
This can be rewritten as
\[ 4F^4 \mathcal{M}(1, 2, 3, 4, 5, 6) = -\frac{1}{2} \frac{(s_{1,2} + s_{2,3})(s_{1,4} + s_{4,5})}{s_{1,3}} + s_{1,2} + \text{cycl}, \]

with ‘cycl’ defined for \( \text{n-point amplitude as} \)
\[ A[s_{i,j}, \ldots, s_{m,n}] + \text{cycl} \equiv \sum_{k=0}^{n-1} A[s_{i+k,j+k}, \ldots, s_{m+k,n+k}], \tag{2.35} \]

which will quite considerably shorten the 8- and 10-point formulae. These are postponed to Appendix B.

3. Recursive methods for scattering amplitudes

Feynman diagrams are completely universal way how to calculate scattering amplitudes in any theory (that has Lagrangian description). However, it is well-known that in many cases they are also very ineffective. Despite the expansion contains many diagrams each of them being a complicated function of external data, most terms vanish in the sum and the result is spectacularly simple. The most transparent example is Parke-Taylor formula \[37\] for all tree-level Maximal-Helicity-Violating amplitudes \(^4\). The simple structure of the result is totally invisible in the standard Feynman diagrams expansion.

Several alternative approaches and methods have been discovered in last decades, let us mention e.g. the Berends-Giele recursive relations for the currents \[38\] and the more recent BCFW (Britto, Cachazo, Feng and Witten) recursion relations for on-shell tree-level amplitudes that reconstruct the result from its poles using simple Cauchy theorem \[18\], \[19\].

3.1 BCFW recursion relations

For concreteness let us consider tree-level stripped on-shell amplitudes of \( n \) massless particles in \( SU(N) \) Yang-Mills theory (“gluodynamics”).\(^5\) The partial amplitude \( \mathcal{M}_n \) is a gauge-invariant rational function of external momenta and additional quantum numbers \( h \) (helicities in case of gluons)
\[ \mathcal{M}_n \equiv \mathcal{M}_n(p_1, p_2, \ldots p_n; h_1, h_2, \ldots h_n). \tag{3.1} \]

\(^4\)Scattering amplitudes of gluons where two of them have negative helicity and the other ones have positive helicity.

\(^5\)The recursion relations can be also formulated for more general cases and also for massive particles. See \[39\] for more details.
The external momenta are generically complex but if we are interested in physical amplitudes we can set them to be real in the end. Let us pick two arbitrary indices \( i, j \) and perform following shift.

\[
p_i \to p_i(z) = p_i + z q, \quad p_j \to p_j(z) = p_j - z q
\] (3.2)
such that the momentum \( q \) is orthogonal to both \( p_i \) and \( p_j \), ie. \( q^2 = (q \cdot p_i) = (q \cdot p_j) = 0 \) and the shifted momenta remain on-shell. Let us note that such \( q \) can be found only for the case of spacetime dimensions \( d \geq 4 \). The amplitude becomes a meromorphic function \( \mathcal{M}_n(z) \) of complex parameter \( z \) with only simple poles. The original expression corresponds to \( z = 0 \). If \( \mathcal{M}_n(z) \) vanishes for \( z \to \infty \) we can use the Cauchy theorem to reconstruct \( \mathcal{M}_n = \mathcal{M}_n(0) \),

\[
0 = \frac{1}{2\pi i} \int_{C(\infty)} \frac{dz}{z} \mathcal{M}_n(z) = \mathcal{M}_n(0) + \sum_k \text{Res} (\mathcal{M}_n, z_k) \frac{z_k}{z}
\] (3.3)
where \( C(\infty) \) is closed contour at infinity. \( \mathcal{M}_n \) can be then expressed as

\[
\mathcal{M}_n = - \sum_k \frac{\text{Res} (\mathcal{M}_n, z_k)}{z_k}
\] (3.4)
where \( k \) is sum of all residues of \( \mathcal{M}_n(z) \) in the complex \( z \)-plane. Residues of \( \mathcal{M}_n(z) \) can be straightforwardly calculated for the following reason: the only poles of \( \mathcal{M}_n \) are \( p^2_{a,b} = 0 \) where \( p_{a,b} = (p_{a} + p_{a+1} + \ldots p_{b}) \). The poles of \( \mathcal{M}_n(z) \) have still the same locations just shifted, namely \( p^2_{a,b}(z) = 0 \) where \( i \in (a, a+1, \ldots b) \) or \( j \in (a, a+1, \ldots b) \). If none of the indices \( i, j \) or both of them are in this range, the dependence on \( z \) in \( p_{a,b}(z) \) cancels and it is not pole in \( z \) anymore. It is easy to identify all locations of the corresponding poles \( z_{ab} \). Suppose that particle \( i \in (a, a+1, \ldots b) \),

\[
p^2_{a,b}(z) = (p_{a} + \ldots p_{i-1} + (p_i + z q) + p_{i+1} + \ldots p_{b})^2 = 0 \quad \Rightarrow \quad z_{a,b} = -\frac{p^2_{a,b}}{2(q \cdot p_{a,b})}
\] (3.5)

In the original amplitude \( \mathcal{M}_n \) the residue on the pole \( p^2_{a,b} = 0 \) is given by unitarity: on the factorization channel with given helicity the amplitude factorizes into two sub-amplitudes, and therefore

\[
\text{Res} (\mathcal{M}_n, z_{a,b}) = \sum_{h_{ab}} \mathcal{M}_{L}(z_{a,b})^{h_{ab}} \frac{i}{2(q \cdot p_{a,b})} \mathcal{M}_{R}^{h_{ab}}(z_{a,b})
\] (3.6)
where the summation over the helicities \( h_{ab} \) of the one-particle intermediate state is taken. The “left” and “right” sub-amplitudes \( \mathcal{M}_{L,R}^{\pm h_{ab}}(z_{a,b}) \) are

\[
\mathcal{M}_{L}^{-h_{ab}}(z_{a,b}) = \mathcal{M}_{b-a+2}(p_{a}, \ldots, p_{i}(z_{a,b}), \ldots p_{b}, -p_{a,b}(z_{a,b}); h_{a}, \ldots, -h_{ab})
\] (3.7)
\[
\mathcal{M}_{R}^{h_{ab}}(z_{a,b}) = \mathcal{M}_{n-(b-a)}(p_{a,b}(z_{a,b}), p_{b+1}, \ldots, p_{j}(z_{a,b}), \ldots, p_{a-1}; h_{ab}, \ldots, h_{a-1}).
\] (3.8)

The amplitude \( \mathcal{M}_n \) can be then written as

\[
\mathcal{M}_n = \sum_{ab,h_{ab}} \mathcal{M}_{L}^{-h_{ab}}(z_{a,b}) \frac{i}{p^2_{a,b}} \mathcal{M}_{R}^{h_{ab}}(z_{a,b})
\] (3.9)
It is convenient to choose \( i \) and \( j \) to be adjacent because it eliminates the number of factorization channels we have to consider.
3.2 Reconstruction formula with subtractions

The BCFW recursion relations discussed above are very generic and applicable for a large class of theories. The main restriction is the requirement of large $z$ behavior: $\mathcal{M}_n(z) \to 0$ for $z \to \infty$. However, this behavior is not guaranteed in general and there exist examples when it is broken no matter which pair of momenta $p_i$ and $p_j$ is chosen to be shifted. In such a case, an additional term (dubbed boundary term) is present on the right hand side of eq. (3.9). The boundary term, which is hard to obtain in general case, has been studied by various methods in the series of papers [40], [41] and [42], however no general solution is still available. Sometimes this problem can be cured by means of considering more general approach when all the external momenta $p_k$ are deformed (such an all-line shift has been introduced in [43], see also [44])

$$p_k \to p_k(z) = p_k + zq_k. \quad (3.10)$$

where $z$ is a complex parameter and $q_k$ are appropriate vectors compatible with the requirements of the momentum conservation and on-shell constraint for $p_k(z)$, i.e. $p_k \cdot q_k = q_k^2 = 0$. The on-shell amplitude

$$\mathcal{M}_n(z) \equiv \mathcal{M}_n(p_1(z), p_2(z), \ldots, p_n(z)) \quad (3.11)$$

become again meromorphic function of the variable $z$ the only singularities of which are simple poles and the residue at these poles have the simple structure (3.6) dictated by unitarity. In some cases the desired behavior $\mathcal{M}_n(z) \to 0$ for $z \to \infty$ can be achieved in this way. However, in general case the behavior of $\mathcal{M}_n(z)$ for $z \to \infty$ is power-like with non-negative power of $z$. This fact requires some modification of the reconstruction procedure.

This can be done as follows. Let us suppose that we have made any (linear) deformation of the external momenta $p_k \to p_k(z)$ in such a way that the deformed amplitude $\mathcal{M}_n(z)$ is a meromorphic function the only singularities of which are simple poles and let us assume the following asymptotic behavior

$$\mathcal{M}_n(z) \approx z^k \quad (3.12)$$

when $z \to \infty$. Let us denote the poles of $\mathcal{M}_n(z)$ as $z_i$, $i = 1, 2, \ldots, n$. Assume $a_j$, $j = 1, 2, \ldots, k + 1$ to be complex numbers satisfying $|a_j| < R$ different form the poles $z_i$. Then we can write for $z \neq a_j$ inside the disc $D(R)$ (i.e. inside the domain $|z| < R$ the boundary of which is a circle $C(R)$ of the radius $R$) the following “$k+1$ times subtracted Cauchy formula” (see Fig.2)

$$\int_{C(R)} \frac{1}{2\pi i} dw \, \mathcal{M}_n(w) \frac{1}{w-z} \prod_{j=1}^{k+1} \frac{1}{w-a_j} = \mathcal{M}_n(z) \prod_{j=1}^{k+1} \frac{1}{z-a_j} + \sum_{j=1}^{k+1} \frac{\mathcal{M}_n(a_j)}{a_j-z} \prod_{l=1, l \neq j}^{k+1} \frac{1}{a_j-a_l} + \sum_{i=1}^{n_{C(R)}} \text{Res} (\mathcal{M}_n; z_i) \prod_{j=1}^{k+1} \frac{1}{z_i-a_j}. \quad (3.13)$$

Here $z_1, z_2, \ldots, z_{n_{C(R)}}$ are the poles inside $D(R)$ and $\text{Res} (\mathcal{M}_n; z_i)$ are corresponding residues. In the limit $R \to \infty$ the integral vanishes due to (3.12) and $D(\infty)$ will contain all $n$ poles. As a result
we get a reconstruction formula with \( k + 1 \) subtractions

\[
\mathcal{M}_n(z) = \sum_{i=1}^{n} \frac{\text{Res} (\mathcal{M}_n; z_i)}{z - z_i} \prod_{j=1}^{k+1} \frac{z - a_j}{z_i - a_j} + \sum_{j=1}^{k+1} \mathcal{M}_n(a_j) \prod_{l=1,l\neq j}^{k+1} \frac{z - a_l}{a_j - a_l}. \tag{3.14}
\]

This is the desired generalization of the usual prescription. In order to reconstruct the amplitude with the asymptotic behavior (3.12) from its pole structure, we need therefore along with the residues at the poles \( z_i \) (which are fixed by unitarity) also supplementary information, namely the \( k + 1 \) values \( \mathcal{M}_n(a_j) \) of the amplitude at the points \( a_j \). Such a additional information is the weakest point of the relations (3.14): there exists no universal recipe how to get the values \( \mathcal{M}_n(a_j) \) for a general theory. This corresponds to the well known analogous situation of \( k + 1 \) subtracted dispersion relations, which allow to reconstruct a general amplitude from its discontinuities uniquely up to the \( k + 1 \) generally unknown subtraction constants. Note that, provided we choose \( a_j \) in such a way that \( \mathcal{M}_n(a_j) = 0 \) (i.e. \( a_j \) are the roots of the deformed amplitude \( \mathcal{M}_n(z) \)), we can reproduce the formula

\[
\mathcal{M}_n(z) = \sum_{i=1}^{n} \frac{\text{Res} (\mathcal{M}_n; z_i)}{z - z_i} \prod_{j=1}^{k+1} \frac{z - a_j}{z_i - a_j} \tag{3.15}
\]

first written in this context by Benincasa a Conde [45] and further discussed by Bo Feng, Yin Jia, Hui Luo a Mingxing Luo in [46].

4. BCFW-like relations for semi-on-shell amplitudes

The straightforward application of the BCFW reconstruction procedure is not possible for the \( SU(N) \) nonlinear sigma model because the amplitudes \( \mathcal{M}_n(z) \) do not have appropriate asymptotic behavior for \( z \to \infty \). The reason is that due to the derivative coupling of the Goldstone bosons the interaction vertices are quadratic in the momenta. Therefore after the BCFW shift the vertices
along the “hard” $z$–dependent line of the Feynman graph are in general linear in $z$ and the linear large $z$ behavior of the propagators cannot compensate for it. For instance, under the shift\(^6\) (3.2) with $i = 1$, $j = 2$ we get for the 6pt amplitude (2.34) for $z \to \infty$

$$
\mathcal{M}_6(z) = -2z \left( \frac{(q \cdot p_{2,3}) (s_{1,4} + s_{4,5} - s_{1,3})}{s_{1,3}} + \frac{(q \cdot p_{2,5}) (q \cdot p_{2,3})}{(q \cdot p_{2,4})} + \frac{(q \cdot p_{2,5}) (s_{3,4} + s_{4,5} - s_{3,5})}{s_{3,5}} \right) + O(z^0),
$$

and analogously $\mathcal{M}_n(z) = O(z)$ for general\(^7\) $n$. As discussed in the previous section, in order to reconstruct such an amplitude from its pole structure, it would be sufficient to know the values of $\mathcal{M}_n(z)$ for two fixed values of $z$. However, such an information is difficult to gain solely from the Feynman graph analysis restricted only to the amplitudes $\mathcal{M}_n$. It is therefore useful to take into account also more flexible objects, namely the semi-on-shell amplitudes, which unlike the on-shell amplitudes depend on the parametrization of the matrix $U$ and from which the on-shell amplitudes can be straightforwardly derived. As we would like to show in this section, appropriate choice of parametrization together with suitable way of BCFW-like deformation of the semi-on-shell amplitudes allows to substitute for the missing information on the amplitudes $\mathcal{M}_n$ and to construct generalized BCFW-like relations for them.

### 4.1 Semi-on-shell amplitudes and Berends-Giele relations

The semi-on-shell amplitudes $J^{a_1 a_2 \ldots a_n}_{n}(p_1, p_2, \ldots, p_n)$ (or *currents* in the terminology of the original paper [38], where they were introduced for QCD and more generally for the $SU(N)$ Yang-Mills theory) can be defined in our case as the matrix elements of the Goldstone boson field $\phi^a(0)$ between vacuum and the $n$ Goldstone boson states $|\pi^{a_1}(p_1) \ldots \pi^{a_n}(p_n)\rangle$

$$
J^{a_1 a_2 \ldots a_n}_{n}(p_1, p_2, \ldots, p_n) = \langle 0|\phi^a(0)|\pi^{a_1}(p_1) \ldots \pi^{a_n}(p_n)\rangle. \quad (4.2)
$$

Here the momentum $p_{n+1}$ attached to $\phi^a(0)$

$$
p_{n+1} = -\sum_{j=1}^{n} p_j, \quad (4.3)
$$

is off-shell. Note that $J^{a_1 a_2 \ldots a_n}_{n}(p_1, p_2, \ldots, p_n)$ has a pole for $p_{n+1}^2 = 0$.

In complete analogy with the on-shell amplitudes, at the tree level the right hand side of (4.2) can be expressed in terms of the flavor-stripped semi-on-shell amplitudes $J_n(p_1, p_2, \ldots, p_n)$ in the form

$$
\langle 0|\phi^a(0)|\pi^{a_1}(p_1) \ldots \pi^{a_n}(p_n)\rangle|_{\text{tree}} = \sum_{\sigma \in S_n} \text{Tr}(t^a t^{a_{\sigma(1)}} \ldots t^{a_{\sigma(n)}}) J_n(p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(n)}). \quad (4.4)
$$

\(^6\)Under the all-line (anti)holomorphic BCFW shift the large $z$ behavior is the same. Here we can use the general formulae derived in [44] which relate the number $n$ of external particles, the sum $H$ of their helicities and the overall dimension $c$ of the couplings to the asymptotics of the amplitude under the all-line holomorphic ($O(z^+)$) and anti-holomorphic ($O(z^-)$) shift. These formulae reads $2s = 4 - n - c + H$ and $2a = 4 - n - c - H$. In our case $H = 0$ and the only coupling constant is $F^{-1}$, therefore $c = 2 - n$, therefore in general case $a = s = 1$ independently on $n$.

\(^7\)The general statement can be derived by induction from Brends-Giele recursive relations discussed in the next subsection.
Let us note that, at higher orders in the loop expansion the group structure contains also multiple
trace terms. We normalize the one particle states according to
\[ J_1(p) = 1. \] (4.5)

In this section the above semi-on-shell flavor-stripped amplitudes \( J_n(p_1, p_2, \ldots, p_n) \) will be the main
subject of our interest. The on-shell stripped amplitudes \( M(p_1, p_2, \ldots, p_{n+1}) \) can be extracted from
them by means of the Lehmann-Symanzik-Zimmermann (LSZ) formulas
\[ M(p_1, p_2, \ldots, p_{n+1}) = -\lim_{p_{n+1} \to 0} p_{n+1}^2 J_n(p_1, p_2, \ldots, p_n). \] (4.6)

The main advantage of the semi-on-shell amplitudes \( J_n(p_1, p_2, \ldots, p_n) \) (in what follows we also use
short-hand notation \( J(1, 2, \ldots, n) \)) is that they allow to abandon the Feynman diagram approach
using appropriate recursive relation. The latter has been first formulated by Berends and Giele
in the context of QCD [38] and proved to be very efficient for the calculation of the tree-level
multi-gluon amplitudes. For the \( U(N) \) nonlinear sigma model the generalized recurrent relations
of Berends-Giele type can be written in the form (see Fig.3)
\[ J(1, 2, \ldots, n) = \frac{1}{p_{ij}^2} \sum_{m=2}^{n} \sum_{\{j_k\}} iV_{m+1}(p_{1,j_1}, p_{j_1+1,j_2}, \ldots, p_{j_{m-1}+1,n}, -p_{1,n}) \prod_{k=1}^{m} J(j_{k-1}+1, \ldots, j_k) \] (4.7)

where the sum is over all splittings of the ordered set \( \{1, 2, \ldots, n\} \) into \( m \) non-empty ordered subsets
\( \{j_{k-1}+1, j_{k-1}+2, \ldots, j_k\} \), (here \( j_0 = 0 \) and \( j_m = n \))\(^8\), \( V_{m+1} \) is the flavor-stripped Feynman rule
for vertices with \( m+1 \) external legs and \( p_{i,k} = \sum_{j=1}^{k} p_j \) as above.

Let us note that, because the Lagrangian of the nonlinear sigma model includes infinite number
of vertices with increasing number of fields, the above Berends-Giele relation for \( J_n \) have to contain

\[^8\]Explicitly
\[ \sum_{\{j_k\}} \equiv \sum_{j_1=1}^{n-m+1} \sum_{j_2=j_1+1}^{n-m+2} \cdots \sum_{j_{m-1}=j_{m-2}+1}^{n-m+(m-1)}. \]
vertices up to $n + 1$ legs, i.e. much more terms than in the case of power-counting renormalizable
theories like QCD where the number of vertices is finite\textsuperscript{9}. This fact rather reduces the efficiency
of these relation for the calculations of the amplitudes. We illustrate this in the Tab. 1, where
the number of terms on the right hand side of the Berends-Giele relation (4.7) written for $J_{2n+1}$
denoted as $t(2n + 1)$ and the total number of terms necessary for the calculation of the same
semi-on-shell amplitude using the Berends-Giele recursion (denoted as $b(2n + 1)$) is compared with
the total number $f(2n + 1)$ of the flavor ordered Feynman graphs contributing to $J_{2n+1}$ and with
the same numbers valid for the theory with only quadrilinear vertices ("$\phi^4$ theory") denoted with subscript "4". See Appendix C for more details and for derivation of the explicit formulæ for
these and other related cases.

On the other hand, as we will see in what follows, the Berends-Giele relations can be used as
a very suitable tool for the investigation of the general properties of the semi-on-shell amplitudes. Let us mention e.g. the following simple relations valid for $J(1, 2, \ldots, n)$

\begin{equation}
J(1, 2, \ldots, 2n) = 0
\end{equation}

\begin{equation}
J(1, 2, \ldots, n) = J(n, n - 1, \ldots, 2, 1).
\end{equation}

These relation are valid independently on the field redefinition. However, as we shall see in what
follows, some properties of the semi-on-shell amplitudes are not valid universally and are tightly
related to a given parametrization.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
$n$ & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
\hline
$t(2n + 1)$ & 4 & 12 & 33 & 88 & 232 & 609 & 1596 & 4180 & 10945 & 28656 \\
b$(2n + 1)$ & 5 & 17 & 50 & 138 & 370 & 979 & 2575 & 6755 & 17700 & 46356 \\
f$(2n + 1)$ & 4 & 21 & 126 & 818 & 5594 & 39693 & 2575 & 6755 & 17700 & 46356 \\
t$_4(2n + 1)$ & 3 & 6 & 10 & 15 & 21 & 28 & 36 & 45 & 55 & 66 \\
b$_4(2n + 1)$ & 4 & 10 & 20 & 35 & 56 & 84 & 120 & 165 & 220 & 286 \\
f$_4(2n + 1)$ & 3 & 12 & 55 & 273 & 1428 & 7752 & 43263 & 246675 & 1430715 & 8414640 \\
\hline
\end{tabular}
\caption{A comparison of the number $t$ of the terms on the right hand side of the Berends-Giele recursive
relation with the total number $b$ of terms needed for the Berends-Giele recursive calculation of the amplitude
$J(1, 2, \ldots, 2n + 1)$ and with the total number $f$ of flavor ordered Feynman graphs contributing to the same
amplitude. In the last three row we compare these numbers with the analogous ones for the case of "$\phi^4$
theory".}
\end{table}

\textsuperscript{9}The number of terms on the right hand side of (4.7) grows exponentially with increasing $n$ in contrast to the
polynomial growths typical for the renormalizable theories. See Appendix C for details.

\section*{4.2 Cayley parametrization}

Unlike the on-shell amplitudes $\mathcal{M}^{a_1, \ldots, a_n}(p_1, p_2, \ldots, p_n)$, which are physical observables and do not depend on the choice of the field variables provided the different choices are related by means of admissible (generally nonlinear) transformations, the concrete form of $J_n^{a, a_1, \ldots, a_n}(p_1, p_2, \ldots, p_n)$ as well as the flavor-stripped amplitudes $J_n(p_1, p_2, \ldots, p_n)$ depends on the parametrization of the
\(U(N)\) nonlinear sigma model. In what follows we will almost exclusively use the so called Cayley parameterizations

\[
U = \frac{1 + \frac{i}{\sqrt{2F}} \phi}{1 - \frac{i}{\sqrt{2F}} \phi} = 1 + 2 \sum_{n=1}^{\infty} \left( \frac{i}{\sqrt{2F}} \phi \right)^{n},
\]

where the Goldstone boson fields are arranged into the hermitian matrix \(\phi = \phi^a t^a\) with \(t^a\) being the \(U(N)\) generators. As described in Appendix A, representation (4.10) is a special member of a wide class of parameterizations suited for the construction of the flavor-stripped Feynman rules. The interrelation between the field \(\phi\) and analogous field \(\tilde{\phi}\) of the more usual exponential parametrization \(U = \exp \left( \frac{i}{\sqrt{2F}} \tilde{\phi} \right)\) is through the following admissible nonlinear field redefinition

\[
\phi = 2F \tan \left( \frac{i}{2F} \tilde{\phi} \right) = \tilde{\phi} + O \left( \tilde{\phi}^3 \right).
\]

As is shown in Appendix A, the flavor-stripped Feynman rules for vertices read in the Cayley parametrization

\[
V_{2n+1} = 0
\]

\[
V_{2n+2} = - \frac{(-1)^n}{2^{n+1}} \left( \frac{1}{F} \right)^{2n} \sum_{j=0}^{n} \sum_{i=1}^{n} (p_i \cdot p_{i+j+1}) = \frac{(-1)^n}{2^{n}} \left( \frac{1}{F} \right)^{2n} \left( \sum_{i=0}^{n} p_{2i+1} \right)^2,
\]

where we have used the momentum conservation in the last row. For the first non-trivial vertex \(V_4\) we get

\[
V_4 = - \frac{1}{2F^2} (p_1 + p_3)^2 = - \frac{1}{2F^2} (p_2 + p_4)^2
\]

and the first two non-trivial semi-on-shell amplitudes read in the Cayley parametrization

\[
J(1, 2, 3) = \frac{1}{2F^2 p_3^2} (p_1 + p_3)^2 \quad (4.14)
\]

\[
J(1, 2, 3, 4, 5) = \frac{1}{4F^4 p_6^2} \left[ \frac{(p_1 + p_2 + p_3 + p_5)(p_1 + p_3)^2}{(p_1 + p_2 + p_3)^2} + \frac{(p_1 + p_3 + p_4 + p_5)(p_3 + p_5)^2}{(p_3 + p_4 + p_5)^2} \right.
\]

\[
\left. + \frac{(p_1 + p_5)^2(p_2 + p_4)^2}{(p_2 + p_3 + p_4)^2} - (p_1 + p_3 + p_5)^2 \right].
\]

Let us illustrate explicitly the dependence of the semi-on-shell amplitudes on the parametrization. Using the exponential one we obtain different amplitude \(J(1, 2, 3)\), namely

\[
J(1, 2, 3)_{\text{exp}} = - \frac{1}{6F^2} \frac{(p_1 + p_2)^2 + (p_2 + p_3)^2 - 2(p_1 + p_3)^2}{p_3^2}.
\]

However, both \(J(1, 2, 3)\) and \(J(1, 2, 3)_{\text{exp}}\) give the same on-shell amplitude (2.33).

In the next subsection we will prove additional useful properties of the semi-on-shell amplitudes.
4.3 Scaling properties of semi-on-shell amplitudes

The Cayley parametrization is specific in the sense that the semi-on-shell amplitudes $J_n(p_1, \ldots, p_n)$ in this parametrization obey simple scaling properties when some subset of the momenta $p_i$ are scaled $p_i \to tp_i$ and the scaling parameter $t$ is then send to zero. Here we will study two important scaling limits, corresponding to the case when all odd or all even on-shell momenta are scaled. As we shall see in the following section, these two scaling limits are the key ingredients for the construction of the BCFW-like relations for semi-on-shell amplitudes in the Cayley parametrization.

We will prove that for $n > 1$ and $t \to 0$

$$J_{2n+1}(tp_1, p_2, tp_3, p_4, \ldots, p_{2r}, tp_{2r+1}, p_{2r+2}, \ldots, p_{2n}, tp_{2n+1}) = O(t^2)$$

and

$$\lim_{t \to 0} J_{2n+1}(p_1, tp_2, p_3, tp_4, \ldots, tp_{2r}, p_{2r+1}, tp_{2r+2}, \ldots, tp_{2n}, p_{2n+1}) = \frac{1}{(2F^2)^n}. \quad (4.18)$$

The general proof of (4.17) and (4.18) is by induction. Let us first verify the base cases. While the second statement holds already for $n = 1$

$$J_3(p_1, p_2, p_3) = \frac{1}{F^2} \frac{(p_1 \cdot p_3)}{(p_1 + p_2 + p_3)^2} \to \frac{1}{2F^2}, \quad (4.19)$$

the first one is not valid unless $n = 2$. Indeed

$$J_3(tp_1, p_2, tp_3) = \frac{1}{2F^2} \frac{t(p_1 \cdot p_3)}{(p_1 \cdot p_2) + (p_2 \cdot p_3) + t(p_1 \cdot p_3)} = O(t). \quad (4.20)$$

On the other hand, using the explicit form of $J_5$ (cf. (4.15)) we get

$$J_5(tp_1, p_2, tp_3, p_4, tp_5) = O(t^2); \quad (4.21)$$

we can therefore proceed by induction starting at $n = 2$.

Let us first prove the scaling property (4.17). Suppose, that (4.17, 4.18) holds for all $\bar{n}$, where $1 < \bar{n} < n$ and write for the left hand side of (4.17) the Berends-Giele relation (4.7) expressing $J_{2\bar{n}+1}$ in terms of $J_{2\bar{n}+1}$ with $\bar{n} < n$. After the scaling $p_{2k+1} \to tp_{2k+1}$, the $t \to 0$ behavior of $p_{2\bar{n}+2}^2$ and $V_{m+1}$ is $O(t^0)$ and $O(t^r)$ where $r \geq 0$ respectively. The scaling of the remaining semi-on-shell amplitudes on the right hand side of (4.7) can be deduced from the induction hypothesis. Note that it depends on the number of the external on-shell legs of $J(j_{i-1} + 1, \ldots, j_i)$ as well as on the parity of $j_{i-1} + 1$, because the semi-on-shell amplitude with scaled even or odd momenta scales differently. Namely, according to the induction hypothesis, the scaling of these building blocks of the right hand side of (4.7) is as follows (see Fig. 4)

$$J(j) = 1 = O(t^0), \quad J(2j - 1, 2j, 2j + 1) = O(t), \quad J(2j, \ldots, 2k) = O(t^0),$$

$$J(2j + 1, \ldots, 2k + 1) = O(t^2) \quad \text{for} \quad k - j > 1. \quad (4.22)$$

This implies, that those terms of Berends-Giele relations which are depicted in Fig. 5, i.e. those which contain at least one block $J(2j + 1, \ldots, 2k + 1) = O(t^2)$ with $k - j > 1$ or at least two building
Figure 4: Scaling of the building blocks on the right hand hand of the Berends-Giele recursion relation according to the induction hypothesis when the odd momenta are scaled.

Figure 5: The terms on the right hand hand of the Berends-Giele recursion relation which are automatically \( O(t^2) \) using the induction hypothesis when the odd momenta are scaled.

blocks \( J(2j-1, 2j, 2j+1) \) are automatically \( O(t^2) \). Therefore, the only dangerous terms on the right hand side of (4.7) are those without the buildings block of the type \( J(2j+1, \ldots, 2k+1) = O(t^2) \) with \( k - j > 1 \) and at the same time without (case I) or with just one (case II) building block \( J(2j-1, 2j, 2j+1) = O(t) \) (see Fig. 6). To this terms the induction hypothesis cannot be applied directly.

In the case I, the odd lines of the corresponding vertex \( V_{2m+2} \) are attached to \( J(2j_k+1) = 1 \) and such a vertex is then proportional to the squared sum of the odd momenta \( tp_{2j_k+1} \), (cf. (4.12))

\[
V_{2m+2}(tp_1, p_{2j_1}, tp_{2j_1+1}, \ldots, tp_{2n+1}) \sim (tp_1 + tp_{2j_1+1} + \cdots + tp_{2n+1})^2
\]

which means that it scales as \( O(t^2) \). This is in fact the scaling of the complete contribution of the
Figure 6: Typical terms on the right hand hand of the Berends-Giele recursion relation to which the induction hypothesis (4.17) cannot be applied directly. In both cases, to all (case I) or to all but one (case II) odd lines of the vertex the blocks $J_1$ are attached. In the case II, one building block $J_3$ is attached to remaining odd line.

**Case I**

$$V_{2m+2} \sim (tp_1 + tp_{2j_1+1} + \ldots + tp_{2n+1})^2 = O(t^2)$$

**Case II**

$$V_{2m+2} \sim (tp_{2j-1} + p_{2j} + tp_{2j+1} + \sum_k tp_{2j_k+1})^2 = O(t)$$

Therefore, the complete contribution of the dangerous terms in the case II is in fact $O(t^2)$ for $t \to 0$ because both $V_{2m+2}$ and $J_3(tp_{2j-1}, p_{2j}, tp_{2j+1})$ scale as $O(t)$ and again all the remaining building blocks are of the order $O(t^0)$ for $t \to 0$. All the other “non-dangerous” terms on the right hand side of the Berends-Giele relations scale at least as $O(t^2)$, which finishes the proof of (4.17).

Let us now prove (4.18), i.e. the case when all even momenta are scaled. Suppose validity of this relation for $\bar{n} < n$ and again write the Berends-Giele relation for the left hand side of (4.18). Thanks to the just proven statement (4.17), the terms on the right hand side of (4.7) with at least one building block $J(j_k + 1, \ldots, j_{k+1})$ with odd $j_k$ and $j_{k+1} - j_k > 1$ do not contribute in the limit $t \to 0$. Such a block can be attached only to the even line of the vertex $V_{m+1}$. Therefore, the only
terms which can contribute in the limit $t \to 0$ have the form depicted in Fig. 7, i.e. those with the building blocks $J_1$ attached to all even lines of the vertex.

\[ \frac{(-1)^k}{2^k F^{2k}} \prod_{l=1}^{k+1} \frac{1}{(2F^2)^{j_l-j_{l-1}-1}} = \frac{(-1)^k}{2^n F^{2n}} \]  

(4.25)

where we denote $j_0 = 0$ and $j_{k+1} = n + 1$. Sum of all such contributions is

\[ \sum_{k=1}^{n} \sum_{1 \leq j_1 < j_2 < \ldots < j_k \leq n} \frac{(-1)^{k-1}}{2^n F^{2n}} \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} = \frac{1}{2^n F^{2n}}. \]  

(4.26)

which finishes the proof.

Another independent scaling properties of the semi-on-shell amplitudes $J_{2n+1}$ can be proven using the same strategy. For instance, when all odd momenta and one additional even momentum (say $p_{2r}$) are scaled, we get

\[ \lim_{t \to 0} J_{2n+1}(tp_1, p_2, t^2, p_4, \ldots, tp_{2r-1}, tp_{2r}, tp_{2r+1}, \ldots, p_{2n}, tp_{2n+1}) = 0 \]  

(4.27)

for $n > 1$. We postpone the proof to the Appendix D.

Let us note that due to the homogeneity of $J(1, 2, \ldots, 2n + 1)$ we can rewrite the relations (4.17) and (4.18) as a statement on the asymptotic behavior of the scaled amplitudes for $t \to \infty$, namely

\[ \lim_{t \to \infty} J_{2n+1}(tp_1, p_2, \ldots, p_{2n}, tp_{2n+1}) = \lim_{t \to \infty} J_{2n+1}(p_1, t^{-1}p_2, \ldots, t^{-1}p_{2n}, p_{2n+1}) = \frac{1}{(2F^2)^n} \]  

(4.28)

and

\[ J_{2n+1}(p_1, tp_2, \ldots, tp_{2n}, p_{2n+1}) = J_{2n+1}(t^{-1}p_1, p_2, \ldots, p_{2n}, t^{-1}p_{2n+1}) = O(t^{-2}). \]  

(4.29)
4.4 BCFW reconstruction

As we have mentioned in the previous subsection, the standard BCFW-like deformation of the external momenta $p_i$ yields deformed amplitudes which behave as a non-negative power of $z$ for $z \to \infty$. As a result, for the reconstruction of the amplitude from its pole structure we need to use the general reconstruction formula (3.14) for which additional information on the on-shell amplitude (its values at several points) is necessary. However, such an information is not at our disposal. We solve this problem by the following trick: we relax some demands placed on the usual BCFW-like deformation and allow more general ones for which either the reconstruction formula without subtractions can be applied or additional information on the deformed amplitudes is accessible.

The momentum conservation cannot be evidently avoided, what remains is the on-shell condition of all the external momenta. It seems therefore to be natural to relax this constraint and instead of the on-shell amplitudes $M_{2n+1}^2$ to use the semi-on-shell amplitudes $J_{2n+1}^2$, or the cut semi-on-shell amplitudes $M_{2n+1}^2$ defined as

$$M_{2n+1}^2(p_1, \ldots, p_{2n+1}) = p_{2n+1}^2 J_{2n+1}^2(p_1, \ldots, p_{2n+1}). \tag{4.30}$$

Motivated by the results of the previous section let us assume the following deformation of the semi-on-shell amplitude $M_{2n+1}^2$ in the Cayley parametrization

$$M_{2n+1}(z) \equiv M_{2n+1}(p_1, zp_2, zp_3, zp_4, \ldots, zp_{2r}, p_{2r+1}, zp_{2r+2}, \ldots, zp_{2n}, p_{2n+1}) \tag{4.31}$$

i.e. all even momenta are scaled by the complex parameter $z$ and the odd momenta are not deformed

$$p_{2k}(z) = zp_{2k}, \quad p_{2k+1}(z) = p_{2k+1} \tag{4.32}$$

Note that in contrast to the standard BCFW shift this deformation is possible for general number of space-time dimensions $d$. The physical amplitude corresponds to $z = 1$. For $n = 1$ we get explicitly

$$M_3(z) = \frac{1}{F^2(p_1 \cdot p_3)} \tag{4.33}$$

For general $n$ let us denote the sums of all odd (even) momenta as

$$p_- = \sum_{k=0}^{n} p_{2k+1}, \quad p_+ = \sum_{k=1}^{n} p_{2k} \tag{4.34}$$

Then in general case the function $M_{2n+1}(z)$ has the following important properties:

1. With generic fixed $p_i$ it is a meromorphic function of $z$ with simple poles.

2. The asymptotics of $M_{2n+1}(z)$ can be deduced from the known properties of $J_{2n+1}$, namely for $n > 1$ we get as a consequence of (4.29)

$$M_{2n+1}(z) = (p_+ z + p_-)^2 J_{2n+1}(p_1, zp_2, \ldots, zp_{2n}, p_{2n+1}) = O(z^0). \tag{4.35}$$
3. For $n \geq 1$ we have according to known scaling property (4.18) of $J_{2n+1}$

$$\lim_{z \to 0} M_{2n+1}(z) = \frac{1}{(2F^2)^n} p_+^2$$

(4.36)

The first two properties allows us to write for $M_{2n+1}(z)$ the reconstruction formula with one subtraction, i.e. the relation (3.14) with $k = 0$. The third property is the key one for the complete reconstruction and determines both the “subtraction point” $a_1 = 0$ and the “subtraction constant” $M_{2n+1}(a_1) = p_+^2 / (2F^2)^n$. The resulting formula reads\(^\text{10}\)

$$M_{2n+1}(z) = \frac{1}{(2F^2)^n} p_+^2 + \sum_P \text{Res} (M_{2n+1}, z_P) \frac{z}{z - z_P}$$

(4.37)

where the sum is over the poles $z_P$ of $M_{2n+1}(z)$. The position of the poles is known and the corresponding residues can be determined recursively as in usual BCFW relations, however, there are some subtleties.

The poles $z_P$ of $M_{2n+1}(z)$ correspond to the vanishing denominators of the deformed propagators $p_P^2(z) = 0$, where

$$p_P^2(z) \equiv p_{i,j}(z)^2 = 0, \quad \text{for } 2 \leq j - i < 2n$$

(4.38)

and where $j - i$ is even; in this formula $p_{i,j}(z) = z p_{i,j}^+ + p_{i,j}^-$ with

$$p_{i,j}^+ = \sum_{i \leq 2k \leq j} p_{2k}, \quad p_{i,j}^- = \sum_{i \leq 2k+1 \leq j} p_{2k+1},$$

(4.39)

i.e. $p_{i,j}^\pm$ is a sum of all even (odd) momenta from the ordered set $p_i, p_{i+1}, \ldots, p_{j-1}, p_j$. Explicitly for $j - i > 2$

$$z_{i,j}^\pm = \frac{-(p_{i,j}^+ \cdot p_{i,j}^-) \pm \left( -G(p_{i,j}^+, p_{i,j}^-) \right)^{1/2}}{p_{i,j}^2}$$

(4.40)

where $G(a, b) = a^2 b^2 - (a \cdot b)^2$ is the Gram determinant, which is nonzero for generic momenta $p_i, \ldots, p_j$. Therefore in the generic case for $j - i > 2$ we deal with doublets of single poles.

The case of three-particle poles corresponding to $j - i = 2$ has to be treated separately. In this case either $p_{i,j}^+ = 0$ or $p_{i,j}^- = 0$ (this sets in for $p_{i,j}^+ = p_{i+1}$ or for $p_{i,j}^- = p_{i+1}$ respectively; let us remind that $p_k$ are on-shell). In the first case we have only one pole

$$z_{2j-1,2j+1} = -\frac{(p_{2j-1} \cdot p_{2j+1})}{p_{2j} \cdot (p_{2j-1} + p_{2j+1})}$$

(4.41)

\(^{10}\)Let us note, that we could write analogous reconstruction formula directly for the currents $J_{2n+1}$ as we did in [49]. In such a case we do not need any subtraction. The price to pay is that we get two more poles, the residues of which cannot be determined recursively from unitarity. Fortunately, the relation (4.29) and the residue theorem can be used in order to obtain the unknown residues in terms of the remaining ones. The resulting formula is fully equivalent to (4.37), however it is a little bit less elegant.
\[
\begin{align*}
p_{i,j}(z_{i,j}^+) &= 0 \\
\frac{z_{i,j}^- - z_{i,j}^+}{p_{i,j} \cdot (p_{2j} + p_{2j+2})} 
\end{align*}
\] (4.42)

\[
\begin{align*}
z_{i,j}^+ &= -\frac{p_{2j+1} \cdot (p_{2j} + p_{2j+2})}{(p_{2j} \cdot p_{2j+2})}
\end{align*}
\] (4.43)

However \(z_{i,j}^+ = 0\) cannot be a pole according to (4.36) and the corresponding residue has to be zero.

The residues of the function \(M_{2n+1}(z)\) are dictated by unitarity and at the poles they factorize (see Fig. 8). Writing for \(j - i > 2\)

\[
(zp_{i,j}^+ + p_{i,j}^-)^2 = p_{i,j}^{+2}(z - z_{i,j}^+)(z - z_{i,j}^-) 
\] (4.44)

we get for \(j - i > 2\)

\[
\text{Res} \left( M_{2n+1}, z_{i,j}^\pm \right) = \pm \frac{M_L^{(i,j)}(z_{i,j}^\pm)M_R^{(i,j)}(z_{i,j}^\pm)}{p_{i,j}^{+2}(z_{i,j}^+ - z_{i,j}^-)} \] (4.45)

where we denoted

\[
\begin{align*}
M_L^{(i,j)}(z_{i,j}^\pm) &= M_{2n+1-(j-i)}(p_{1}(z_{i,j}^\pm), \ldots, p_{i-1}(z_{i,j}^\pm), p_{i,j}(z_{i,j}^\pm), p_{j+1}(z_{i,j}^\pm), \ldots, p_{2n+1}(z_{i,j}^\pm)) \\
M_R^{(i,j)}(z_{i,j}^\pm) &= M_{j-i+1}(p_{i}(z_{i,j}^\pm), p_{i+1}(z_{i,j}^\pm), \ldots, p_{j}(z_{i,j}^\pm)).
\end{align*}
\] (4.46, 4.47)

Note that, while the amplitude \(M_L^{(i,j)}\) remains semi-on-shell, the amplitude \(M_R^{(i,j)}\) is fully on-shell, because the deformed momentum \(p_{i,j}(z)\) is on-shell for \(z = z_{i,j}^\pm\).

The formula (4.45) is valid also for the three-particle pole \(z_{2j,2j+2}\) given by (4.43). However the pole \(z_{2j-1,2j+1}\) deserves a special remark because the corresponding residue is determined by the formula different from (4.45), namely

\[
\text{Res} \left( M_{2n+1}, z_{2j-1,2j+1} \right) = \frac{M_L^{(2j-1,2j+1)}(z_{2j-1,2j+1})M_R^{(2j-1,2j+1)}(z_{2j-1,2j+1})}{2p_{2j-1,2j+1} \cdot p_{2j-1,2j+1}} 
\] (4.48)
where $M_{L,R}^{(2j-1,2j+1)}(z_{2j-1,2j+1})$ are given by (4.46) and (4.47) with $z_{i,j}^{\pm}$ replaced by $z_{2j-1,2j+1}$.

To summarize, we have found a closed system of recursive BCFW-like relations for the tree cut semi-on-shell amplitudes $M_{2n+1}$, which consists of the reconstruction formula (4.37), the pole positions (4.40), (4.41) and (4.43) and the residue formulae (4.45) and (4.48). Note that the initial condition for the recursion (4.33) can be understood as the special case of (4.37) for $n = 1$ because then there is no pole $z_{i,j}$ with $2 \leq j - i < 2$ and the sum of the residue contributions is empty. The physical amplitude $M_{2n+1}(p_1, \ldots, p_{2n+1})$ corresponds to $z = 1$

$$M_{2n+1}(p_1, \ldots, p_{2n+1}) = \frac{1}{(2F^2)^n p^2_2} + \sum_p \text{Res} (M_{2n+1}, z_p) \frac{1}{1 - z_p}. \quad (4.49)$$

As a final result we get then using (4.45), (4.48), (4.41), (4.43) and (4.44)

$$M_{2n+1}(p_1, \ldots, p_{2n+1}) = \frac{1}{(2F^2)^n p^2_2} + \sum_p M_L^{(p)}(z_p) \frac{R_P}{p^2_2} M_R^{(p)}(z_p). \quad (4.50)$$

Note that there is an extra function $R_P$ in contrast to the standard BCFW formula (3.9), namely

$$R_P = \begin{cases} 
\frac{z_p^{-2}}{z_p^{-1}} & \text{for } z_p = z_{2j,2j+2} \\
\frac{1}{z_{i,j}^{\pm} - z_{i,j}^{\pm}} & \text{for } z_p = z_{i,j}^{\pm} 
\end{cases} \quad (4.51)$$

For further convenience, we rewrite (4.50) with help of (4.33) in the following more explicit form

$$M_{2n+1}(p_1, \ldots, p_{2n+1}) = \frac{1}{(2F^2)^n p^2_2} + 
+ \sum_{j=1}^{n-1} M_L^{(2j,2j+2)}(z_{2j,2j+2}) \frac{1}{p^2_{2j,2j+2}} \frac{p_{2j} \cdot p_{2j+2}}{F^2} 
- \sum_{j=1}^{n} M_L^{(2j-1,2j+1)}(z_{2j-1,2j+1}) \frac{1}{p^2_{2j-1,2j+1}} \frac{p^+_{2j-1,2j+1} \cdot p^-_{2j-1,2j+1}}{F^2} 
+ \sum_{2 < j - i < 2n} \frac{1}{z_{i,j}^{\pm} - z_{i,j}^{\pm}} \left( M_L^{(i,j)}(z_{i,j}^{\pm}) \frac{1 - z_{i,j}^{\pm}}{p_{i,j}^{\pm}} - M_R^{(i,j)}(z_{i,j}^{\pm}) \frac{1 - z_{i,j}^{\pm}}{p_{i,j}^{\pm}} \right). \quad (4.52)$$

The on-shell amplitude is then

$$M_{2n}(1, 2, \ldots, 2n - 1; 2n) = - \lim_{p^2_{2n-1} \to 0} M_{2n-1}(1). \quad (4.53)$$

4.5 Explicit example of application of BCFW relations: 6pt amplitude

As an illustration let us apply the BCFW-like recursive relations (4.37) to the amplitude $M_5(z) \equiv M_5(p_1, z p_2, p_3, z p_4, p_5)$. In this case we have three poles, all of them being three-particle, namely

$$z_{1,3} = 1 - \frac{s_{1,3}}{s_{1,2} + s_{2,3}}, \quad z_{2,4} = \left(1 - \frac{s_{2,4}}{s_{2,3} + s_{3,4}}\right)^{-1}, \quad z_{3,5} = 1 - \frac{s_{3,5}}{s_{3,4} + s_{4,5}} \quad (4.54)$$
where the variables \( s_{i,j} \) are given by (2.30). The residues are given by the relations (4.45) for \( z_{2,4} \) and (4.48) for \( z_{1,3} \) and \( z_{3,5} \). After simple algebra using the explicit form of the poles (4.54) we get

\[
\begin{align*}
\text{Res} \left( M_5, z_{1,3} \right)_{z_{1,3}} & = \frac{1}{4F^4} (1 - z_{1,3})(s_{2,5} - s_{2,4} + s_{3,4} - s_{3,5}) - \frac{1}{4F^4} (s_{1,5} - s_{1,4} - s_{4,5}) \\
\text{Res} \left( M_5, z_{3,5} \right)_{z_{3,5}} & = \frac{1}{4F^4} (1 - z_{3,5})(s_{1,4} - s_{1,3} + s_{2,3} - s_{2,4}) - \frac{1}{4F^4} (s_{1,5} - s_{1,2} - s_{2,5}) \\
\text{Res} \left( M_5, z_{2,4} \right)_{z_{2,4}} & = \frac{1}{4F^4} (s_{1,5} - s_{1,4} + s_{2,4} - s_{2,5}) .
\end{align*}
\]

(4.55)

Note that the potential unphysical poles \( z_{i,j}(p_k) = 0 \) have canceled completely. We have also

\[
(1 - z_{1,3})^{-1} = \frac{s_{1,2} + s_{2,3}}{s_{1,3}}, \quad (1 - z_{3,5})^{-1} = \frac{s_{3,4} + s_{4,5}}{s_{3,5}}, \quad (1 - z_{2,4})^{-1} = 1 - \frac{s_{2,3} + s_{3,4}}{s_{2,4}} \quad (4.56)
\]

These factors are responsible for setting of the physical poles in the resulting amplitude. After inserting this to the formula (4.49) we get for the individual contributions to the semi-on-shell amplitude in the Cayley parametrization

\[
\begin{align*}
\text{Res} \left( M_5, z_{1,3} \right)_{z_{1,3}(1 - z_{1,3})} & = \frac{1}{4F^2} \left( \frac{(s_{1,4} + s_{4,5} - s_{1,5})(s_{1,2} + s_{2,3})}{s_{1,3}} + s_{2,5} - s_{2,4} + s_{3,4} - s_{3,5} \right) \\
\text{Res} \left( M_5, z_{3,5} \right)_{z_{3,5}(1 - z_{3,5})} & = \frac{1}{4F^2} \left( \frac{(s_{1,2} + s_{2,5} - s_{1,5})(s_{3,4} + s_{4,5})}{s_{3,5}} + s_{1,4} - s_{1,3} + s_{2,3} - s_{2,4} \right) \\
\text{Res} \left( M_5, z_{2,4} \right)_{z_{2,4}(1 - z_{2,4})} & = \frac{1}{4F^2} \left( \frac{(s_{1,4} + s_{2,5} - s_{1,5})(s_{2,3} + s_{3,4})}{s_{2,4}} + s_{1,5} - s_{1,4} + s_{2,4} - s_{2,5} - s_{2,3} - s_{3,4} \right) \\
p^2_{-} \left( \frac{1}{4F^2} \right) & = \frac{1}{4F^2} \left[ s_{1,3} - s_{1,2} - s_{2,3} + s_{1,5} - s_{1,4} + s_{2,4} - s_{2,5} + s_{3,5} - s_{3,4} - s_{4,5} \right] .
\end{align*}
\]

(4.57)

Finally we get

\[
4F^2 M_5(1) = \\
\left( \frac{s_{1,4} + s_{4,5} - s_{1,5}}{s_{1,3}} \right) (s_{1,2} + s_{2,3}) + \left( \frac{s_{1,2} + s_{2,5} - s_{1,5}}{s_{3,5}} \right) (s_{3,4} + s_{4,5}) + \left( \frac{s_{1,4} + s_{2,5} - s_{1,5}}{s_{2,4}} \right) (s_{2,3} + s_{3,4}) \\
+ 2s_{1,5} - s_{1,2} - s_{1,4} - s_{2,3} - s_{2,5} - s_{3,4} - s_{4,5} .
\]

(4.58)

Taking this amplitude on-shell according to (4.53), i.e. setting \( s_{1,5} \to 0 \) and changing the overall sign, we reproduce the parametrization independent physical amplitude (2.34).

5. More properties of stripped semi-on-shell amplitudes

The BCFW recursive relations provides us with a Lagrangian-free formulation of the tree-level nonlinear \( SU(N) \) sigma model in the Cayley parametrization. We can use them similarly as the Berends-Giele relations as a tool for the investigation of further interesting features of the stripped semi-on-shell amplitudes \( M_{2n+1} \) and \( J_{2n+1} \). As we have already mentioned, these features are not
universal because of the parametrization dependence of \( M_{2n+1} \) and \( J_{2n+1} \), however, their implications for the fully on shell amplitudes hold universally\(^{11}\). In this section we will concentrate on the problem of single soft limits (Adler zeroes) and double soft limit of the semi-on-shell amplitudes.

The presence of Adler zeroes for the on-shell Goldstone boson amplitudes \( \mathcal{M}^{a_1 \cdots a_{2n}}(p_1, \ldots, p_{2n}) \), i.e. validity of the limit

\[
\lim_{p_j \to 0} \mathcal{M}^{a_1 a_2 \cdots a_{2n}}(p_1, p_2, \ldots, p_{2n}) = 0,
\]

is a well known consequence of the nonlinearly realized chiral symmetry. More generally it is an universal (non-perturbative) feature in the theories with spontaneous breakdown of a global symmetry. In such theories the amplitudes with one extra Goldstone boson \( \pi^a \) in the out (or in) state vanishes when the Goldstone boson become soft, e.g.

\[
\lim_{p \to 0} \langle f^+ + \pi^a(p) | \text{out} | i, \text{in} \rangle = 0,
\]

provided the \( \pi^a \) cannot be emitted from the external lines corresponding to the states \( |i, \text{in} \rangle \) or \( |f, \text{out} \rangle \). In the \( SU(N) \) nonlinear sigma model the Adler zero is present also for the stripped on-shell amplitudes \( \mathcal{M}_{2n}(p_1, p_2, \ldots, p_{2n}) \) due to the leading \( N \) orthogonality relations (2.20) and corresponding uniqueness of the decomposition (2.11). However, this property is not guaranteed automatically for the semi-on-shell amplitudes \( M_{2n+1} \) and the soft Goldstone boson behavior can depend on the parametrization. For instance using the Cayley parametrization, we find for the amplitude \( M_3 = (p_1 \cdot p_3)/F^2 \) the Adler zero for soft \( p_1 \) and \( p_3 \), however there is no zero for soft \( p_2 \) in general when keeping \( p_4 \) off-shell. For the same amplitude in the exponential parametrization (cf. (4.16)) we have no Adler zero at all. As we shall show in this section, for the semi-on-shell amplitudes \( M_{2n+1} \) in the Cayley parametrization we can prove, using the BCFW-like relation, the Adler zero for half of the momenta (namely for those \( p_j \) with odd index \( j \)).

The double soft limit of the Goldstone boson on-shell amplitudes \( \mathcal{M}^{a_1 a_2 \cdots a_{2n+2}}(p_1, p_2, \ldots, p_{2n+2}) \) is more complicated and has been studied relatively recently in connection with the regularized action of the broken generators on the \( n \) Goldstone boson states [50]. Motivated by direct inspection of the six Goldstone boson amplitude in the nonlinear chiral \( SU(2) \) sigma model it was conjectured that provided the two soft momenta are sent to zero with the same rate, the following limit holds

\[
\lim_{t \to 0} \mathcal{M}^{aba_1 a_2 \cdots a_{2n}}(tp, tq, p_1, p_2, \ldots, p_{2n}) = -\frac{1}{2F^2} \sum_{i=1}^n f^{abc} f_{cd} c_{a_1}^{(d} p_i \cdot (p - q) \mathcal{M}^{a_1 \cdots a_{i-1} d a_{i+1} \cdots a_{2n}}(p_1, p_2, \ldots, p_{2n}),
\]

where \( f^{abc} \) are the structure constants. Analogous statement has been then rigorously proven for the tree-level amplitudes in the \( N = 8 \) supergravity using BCFW relations. In fact, for the on-shell amplitudes, the formula (5.3) can be proven non-perturbatively under some assumptions for the general enough case of the theory with global symmetry breaking (including the case of chiral nonlinear sigma model with general chiral group \( G \)) using the symmetry arguments only (cf. the PCAC soft-pions theorems [48]). We postpone the details to the Appendix E.

\(^{11}\)Let us remind that the on-shell amplitudes are parametrization independent.
In terms of the stripped on-shell amplitudes the relation (5.3) can be rewritten as

\[
\lim_{t \to 0} M_{2n+2}(p_1, \ldots, p_{i-1}, tp_i, \ldots, tp_j, p_{j+1}, \ldots, p_{2n+2}) = \frac{1}{4F^2} \delta_{j,i+1} \left( \frac{p_{i+2} \cdot (p_i - p_{i+1})}{p_{i+2} \cdot (p_i - p_{i+1})} - \frac{p_{i-1} \cdot (p_i - p_{i+1})}{p_{i-1} \cdot (p_i - p_{i+1})} \right) M_{2n}(p_1, \ldots, p_{i-1}, p_{i+2}, \ldots, p_{2n+2}).
\]

In this section we will prove this relation also for the tree-level semi-on-shell amplitudes \(J_{2n+1}\) (and consequently for \(M_{2n+1}\)) of the \(SU(N)\) nonlinear sigma model in the Cayley parametrization using suitable form of the generalized BCFW representation.

### 5.1 Adler zeroes

In this subsection we will use the BCFW-like relations (4.52) derived in the previous section and prove the presence an Adler zero at \(M_{2n+1}\) when one of the odd momenta, say \(p_{2l-1}\), is soft, i.e. we will prove that for \(l = 1, 2, \ldots, n + 1\)

\[
\lim_{t \to 0} M_{2n+1}(p_1, p_2, \ldots, p_{2l-2}, tp_{2l-1}, p_{2l+1}, \ldots, p_{2n+1}) = 0.
\]

For the fundamental amplitude \(M_3(p_1, p_2, p_3)\) we have explicitly

\[
M_3(tp_1, p_2, p_3) = M_3(p_1, p_2, p_3) = \frac{1}{F^2} t(p_1 \cdot p_3) \to 0.
\]

In the general case the proof of (5.5) is by induction. Let us assume validity of (5.5) for \(m < n\). This assumption also means that, taking the cut semi-on-shell amplitude \(M_{2m+1}\) on shell, i.e. for \(p_{2m+1}^2 \to 0\), the Adler zero is in fact present at \(M_{2m+1}\) on shell = \(-M_{2m+2}\) for all momenta, i.e.

\[
\lim_{t \to 0} M_{2m+1}(p_1, p_2, \ldots, tp_{j}, \ldots, p_{2m+1}) \bigg|_{\text{on shell}} = 0
\]

for all \(j = 1, \ldots, 2m + 1\) due to the cyclicity of \(M_{2m+2}\).

Let us now substitute \(p_{2l-1} \to tp_{2l-1}\) to the right hand side of (4.52). Note that, under such substitution, the position of the poles \(z_{2j,2j+2}, z_{2j-1,2j+1}\) and \(z_{i,j}^\pm\) become \(t\)-dependent. The \(t\)-dependence of the right hand side of (4.52) is therefore both explicit (due to the explicit dependence on \(p_{2l-1}\)) and implicit (due to the implicit \(t\)-dependence of the poles \(z_P\)).

We will now inspect the behavior of the individual terms under the limit \(t \to 0\). The first term gives finite limit

\[
\frac{1}{(2F^2)n} \to \frac{1}{(2F^2)n} \bigg|_{p_{2l-1} \to 0}.
\]

As far as the second term is concerned, the individual terms of the sum over \(j\) vanish in this limit unless \(j = l - 1\). The reason is as follows. For \(j \neq l - 1\) (the case A in the Figure 9), the

\[\text{Note however that for } t \to 0 \text{ according to (4.18),}
M_3(p_1, tp_2, p_3) \to \frac{1}{2F^2} (p_1 + p_3)^2
\]

and therefore the statement analogous to (5.5) for even momenta does not hold.
For $j$ and because the induction hypothesis to conclude that

$$p_{2j} \cdot p_{2j+2}/p_{2j,2j+2}^2$$

is denoted by dashed line in the case A. In the case B, $O(t)$ indicates the order of the $t$-dependent $z_{2j,2j+2}$.

kinematical factor $p_{2j} \cdot p_{2j+2}/p_{2j,2j+2}^2$ as well as the position of the pole $z_{2j,2j+2}$ are $t$-independent and because $tp_{2l-1}$ is placed on the odd position in $M_L^{(2j,2j+2)}(z_{2j,2j+2})$, we can safely\(^{13}\) use the induction hypothesis to conclude that

$$\lim_{t \to 0} M_L^{(2j,2j+2)}(z_{2j,2j+2})|_{p_{2l-1} \to 0} = 0.$$ 

For $j = l - 1$ (the case B in the Figure 9), the kinematical factor $p_{2j} \cdot p_{2j+2}/p_{2j,2j+2}^2$ becomes explicitly $t$-dependent and tends to $1/2$ for $t \to 0$, while $M_L^{(2j,2j+2)}(z_{2j,2j+2})$ has both explicit (through $p_{2j,2j+2} = z_{2j,2j+2}(p_{2j} + p_{2j+2} + tp_{2j+1})$ and implicit $t$-dependence. In this case $z_{2j,2j+2} = O(t)$, as can be seen from (4.43). Therefore, all even momenta in $M_L^{(2j,2j+2)}(z_{2j,2j+2})$ are scaled by $O(t)$ factor, in the same way as in (4.18). We can therefore conclude with help of (4.18) that

$$\lim_{t \to 0} M_L^{(2j,2j+2)}(z_{2j,2j+2}) \frac{1}{p_{2j,2j+2}^2} p_{2j} \cdot p_{2j+2} \frac{1}{F^2} = \delta_{j,l-1} \frac{1}{(2F^2)^n} p_{2l-1}^2 |_{p_{2l-1} \to 0}. \quad (5.9)$$

The third term on the right hand side of (4.52) can be treated exactly in the same way as the

\(^{13}\)Indeed, in general the momenta $p_k(z_{2j,2j+2})$ and $p_{2j,2j+2}(z_{2j,2j+2})$ are $t$-independent and nonzero.
second (see Fig. 10). Also here the individual terms of the sum over $j$ do not contribute with the only exception of $j = l$ and $j = l - 1$ by induction hypothesis applied to $M_L^{(z_{2j-1,2j+1})}$ which has for $j \neq l, l - 1$ only explicit $t$-dependence. In the remaining two cases $j = l$ and $j = l - 1$, the explicitly $t$-dependent kinematical factors $p_{2j-1,2j+1}^2 \cdot p_{2j-1,2j+1}/p_{2j-1,2j+1}$ tend again to 1/2 and within $M_L^{(z_{2j-1,2j+1})}$ the even momenta are scaled by $2z_{2j-1,2j+1} = O(t)$ (see (4.41)) and thus (4.18) can be used\(^\text{14}\) to conclude that

\[
\lim_{t \to 0} M_L^{(z_{2j-1,2j+1})} \left( z_{2j-1,2j+1} \right) \frac{1}{p_{2j-1,2j+1}^2} \frac{p_{2j-1,2j+1}^2 \cdot p_{2j-1,2j+1}}{F^2} = (\delta_{j,l} + \delta_{j,l-1}) \frac{1}{(2F^2)^n} p^2 \big|_{p_{2l-1} \to 0}.
\]  

The fourth term on the right hand side of (4.52) vanish completely in the limit $t \to 0$. This is easy to see for those terms of the sum over $(i,j)$ for which\(^\text{15}\) $\lim_{t \to 0} z_{i,j}^+ \neq 0$. In this case either $M_L^{(i,j)}(z_{i,j}^\pm)$ or $M_R^{(i,j)}(z_{i,j}^\pm)$ have explicit $t$-dependence through $tp_{2l-1}$ (which is for $M_L^{(i,j)}(z_{i,j}^\pm)$ on

\[^{14}\text{Note that, the odd momenta are } t\text{-dependent with the only exception of } p_{2j-1,2j+1}(z_{2j-1,2j+1})|_{p_{2l-1} \to p_{2l-1}} \text{ the limit of which is } p_{2j}^\pm.\]

\[^{15}\text{It is easy to realize that } \lim_{t \to 0} z_{i,j}^+ \neq \lim_{t \to 0} z_{i,j}^- \text{ for generic } p_k.\]
odd position) and thus the induction hypothesis in the form (5.5) or (5.7) can be used. By direct inspection of (4.40) we find that the only case for which the above argumentation does not apply is the case \( j - i = 4 \) with \( i \) even and \( i \leq 2l - 1 \leq j \). Here \( \lim_{t \to 0} z_{i,j}^+ \neq 0 \) and so for the “minus” part of this \((i,j)\) term we can use the induction hypothesis as above. However, the “plus” part might be problematic because

\[
z_{i,j}^+ = -\frac{(p_{2l-1} \cdot p_{2l-1} \pm 2)}{(p_{2l-1} \pm 2 \cdot p_{i,j}^+)} t + O(t^2).
\]  

(5.11)

Using this formula and (4.15) we find after some algebra

\[
M_R^{(i,j)}(z_{i,j}^+) = M_5(p_i(z_{i,j}^+), \ldots, tp_{2l-1}, \ldots, p_j(z_{i,j}^+)) = O(t^2).
\]  

(5.12)

which shows that also the “plus” part has vanishing \( t \to 0 \) limit.

Putting therefore the only nonzero contributions (5.8), (5.9) and (5.10) together we get finally

\[
\lim_{t \to 0} M_{2n+1}(p_1, p_2, \ldots, p_{2l-1}, p_{2l+1}, \ldots, p_{2n+1}) = \frac{1}{(2F^2)^n} \bigg|_{p_{2l-1} \to 0} p^2 \left( 1 + \sum_{j=1}^{n-1} \delta_{j,l-1} - \sum_{j=1}^{n} \left( \delta_{j,l} + \delta_{j,l-1} \right) \right) = 0,
\]

which finishes the proof.

### 5.2 Double-soft limit

Let us now study the behavior of the semi-on-shell amplitude \( J_{2n+1} \) in the Cayley parametrization under the double soft limit, i.e. the case when two external momenta, say \( p_i \) and \( p_j \), are scaled according to \( p_i, j \to tp_i, j \) and \( t \) is sent to zero. In this section we will prove, that for \( 1 < i < j < 2n+1 \)

\[
\lim_{t \to 0} J_{2n+1}(p_1, \ldots, p_{2n+1})|_{p_i \to tp_i, p_j \to tp_j} = \delta_{j,i+1} \frac{1}{2F^2} \left( \frac{(p_i \cdot p_{i+2})}{p_{i+2} \cdot (p_{i+1} + p_i)} - \frac{(p_i \cdot p_{i-1})}{p_{i-1} \cdot (p_{i+1} + p_i)} \right) J_{2n-1}(p_1, \ldots, p_{i-1}, p_{i+2}, \ldots, p_{2n+1})
\]  

(5.13)

which has an identical form as (5.4). The key ingredient of the proof is the generalized form of the BCFW representation mentioned in Section 3.2 written for a suitable two-parameter complex deformation of the amplitude \( J_{2n+1} \). Such a representation allows us to calculate the double soft limit with help of the known behavior of the poles and corresponding residues in this limit. Useful information on this behavior can be inferred from the statement (5.5) concerning the Adler zeroes proved in the previous subsection.

The above mentioned deformation of \( J_{2n+1} \) can be defined as the following function of two complex variables \( z \) and \( t \)

\[
S_{i,j}^n(z, t) = J(p_1, \ldots, p_{2n+1})|_{p_i \to tp_i, p_j \to tp_j},
\]  

(5.14)

---

\(^{16}\)Let us remind that \( M_R^{(i,j)}(z_{i,j}^+) \) is fully on-shell.

\(^{17}\)Indeed,

\[
\frac{(p_i \cdot p_{i+2})}{p_{i+2} \cdot (p_{i+1} + p_i)} - \frac{(p_i \cdot p_{i-1})}{p_{i-1} \cdot (p_{i+1} + p_i)} = \frac{1}{2} \left( \frac{p_{i+2} \cdot (p_i - p_{i+1})}{p_{i+2} \cdot (p_{i+1} - p_i)} - \frac{p_{i-1} \cdot (p_i - p_{i+1})}{p_{i-1} \cdot (p_{i+1} - p_i)} \right).
\]
therefore
\[ S_{i,j}^n(1,1) = J_{2n+1}(p_1, \ldots, p_{2n+1}) \] (5.15)

Various types of the double soft limit correspond then to various ways of taking the limit \((z,t) \to (0,0)\) in the double complex plane \((z,t)\); the limit (5.13) corresponds to \(\lim_{t \to 0} S_{i,j}^n(t,t) \equiv S_{i,j}^{n,0}\).

For \(z \to \infty\) and \(t > 0\) fixed the following asymptotic behavior holds
\[ S_{i,j}^n(z,t) = O(z^0), \] (5.16)
as can be easily proved e.g. by induction with help of the Berends-Giele recursive relations (4.7).

We can therefore write the generalized BCFW relation with one subtraction in the form (3.14)
\[ S_{i,j}^n(z,t) = S_{i,j}^n(a,t) + \sum_{k,l} \text{Res} \left( S_{i,j}^n(z; z_{k,l}(t)) \right) \frac{z - a}{z - z_{k,l}(t) - a}. \] (5.17)

where \(a \neq z_{k,l}(t)\) is a priori arbitrary, however, as we shall see in what follows, appropriate choice of \(a\) can simplify the calculation.

The poles \(z_{k,l}(t)\) for \(k \leq j \leq l\) correspond to the conditions \(p_{k,l}^2|_{p_i \to tp_i, p_j \to zp_j} = 0\), or explicitly
\[ z_{k,l}(t) = -\frac{p_{k,l}^2|_{p_i \to tp_i, p_j \to 0}}{2(p_j \cdot p_{k,l})|_{p_i \to tp_i}}. \] (5.18)

The residues at the poles \(z_{k,l}(t)\) factorize
\[ \text{Res} \left( S_{i,j}^n(z; z_{k,l}(t)) \right) = \frac{1}{2(p_j \cdot p_{k,l})|_{p_i \to tp_i}} [J_{2n+1-(t-k)}(p_1, \ldots, p_{k-1}, p_{k,l}, p_{l+1}, \ldots, p_{2N+1}) \times M_{l-k+1}(p_{k, \ldots, l})|_{p_i \to tp_i, p_j \to zp_j}]|_{z = z_{k,l}(t)}, \] (5.19)
where \(M_{l-k+1}\) is the cut amplitude (4.30). Namely the latter two formulae along with (5.5) contain sufficient amount of information for the calculation of the double soft limit.

Let us first assume \(i < j\) where \(i\) is odd and \(j\) arbitrary. This choice is a technical one, and as we shall see, the general case can be easily obtained using the symmetry properties of the amplitude. In what follows we set \(a = 1\) in (5.17), the double soft limit then simplifies to
\[ S_{i,j}^{n,0} \equiv \lim_{t \to 0} S_{i,j}^n(t,t) = \lim_{t \to 0} \sum_{k,l} \text{Res} \left( S_{i,j}^n(z; z_{k,l}(t)) \right) \frac{t - 1}{t - z_{k,l}(t) - 1}, \] (5.20)
where we have used the existence of the Adler zero for \(S_{i,j}^n(1,t) = J_{2n+1}(p_1, \ldots, tp_i, \ldots, p_{2n+1})\) and \(i\) odd (cf. (5.5)).

For generic \(p_r\) there exist a finite limit
\[ z_{k,l}(0) = \lim_{t \to 0} z_{k,l}(t) \neq 1 \] (5.21)
In fact the only nonzero contributions to the right hand side of (5.20) stem from the cases for which
\( z_{k,l}(0) = 0 \). Indeed, for \( z_{k,l}(0) \neq 0 \) we get for the corresponding contribution
\[
\frac{1}{z_{k,l}(0)(z_{k,l}(0) - 1)} \lim_{t \to 0} \text{Res} \left( S^{m}_{i,j}; z_{k,l}(t) \right),
\]
and, according to (5.5), on the right hand side of (5.19) we get either
\[
\lim_{t \to 0} [M_{l-k+1}(p_{k}, \ldots, p_{l})|_{p_{i} \to t p_{i}, p_{j} \to z_{p_{j}}}]|_{z \to z_{k,l}(t)} = 0
\]
for \( k \leq i < j \leq l \) or
\[
\lim_{t \to 0} J_{2n+1-(l-k)}(p_{1}, \ldots, p_{i}, \ldots, p(k, l)(t), p_{k+1}, \ldots, p_{2n+1}) = 0
\]
for \( i < k < j \leq l \). In both cases the complementary factor has finite limit and therefore
\[
\lim_{t \to 0} \text{Res} \left( S^{m}_{i,j}; z_{k,l}(t) \right) = 0.
\]
Let us therefore discuss the contributions form the poles for which \( z_{k,l}(0) = 0 \). Note that, for
generic \( p_{r} \) such a pole does not exist provided \( j > i + 2 \). We can therefore immediately conclude
\[
S^{m,0}_{i,j} = 0 \quad \text{for} \quad j > i + 2.
\]
What remains are the following two alternatives for which the three-particle poles \( z_{k,l}(t) \) with
\( l = k + 2 \) can vanish in the limit \( t \to 0 \) (see Fig. 11)

1. \( j = i + 1 \) and either \( k = i \) or \( k = i - 1 \). In this case either
\[
p_{i-1,i+1}^{2}|_{p_{i} \to t p_{i}, p_{j} \to 0} \to p_{i-1}^{2} = 0
\]
\[
\text{or}
\]
\[
p_{i,i+2}^{2}|_{p_{i} \to t p_{i}, p_{j} \to 0} \to p_{i+2}^{2} = 0
\]
2. \( j = i + 2 \) and \( k = i \), in this case
\[
p_{i,i+2}^{2}|_{p_{i} \to t p_{i}, p_{j} \to 0} = p_{i+1}^{2} = 0.
\]
In what follows we will discuss separately the cases \( j = i + 1 \) and \( j = i + 2 \). Let us first study the
double soft limit of two adjacent momenta, i.e. \( j = i + 1 \) where \( i \) is odd. We will investigate the
contributions of individual poles \( z_{k,l}(t) \) on the right hand side of (5.20) separately. In this case we
get for \( i > 1 \) only two potentially nonzero contributions (i.e. (5.28) and (5.27)) to the right hand
side of (5.20), namely
\[
S^{m,0}_{i,i+1} = \lim_{t \to 0} \frac{\text{Res} \left( S^{m}_{i,i+1}; z_{i-1,i+1}(t) \right)}{t - z_{i-1,i+1}(t) - 1} + \lim_{t \to 0} \frac{\text{Res} \left( S^{m}_{i,i+1}; z_{i,i+2}(t) \right)}{t - z_{i,i+2}(t) - 1}.
\]
We get for the poles $z_{i-1,i+1}(t)$ and $z_{i,i+2}(t)$

\[ z_{k,k+2}(t) = \frac{-p^2_{k,k+2}|p_i\to p_{k+1}, p_j\to 0}{2(p_j \cdot p_{k,k+2})|p_i\to t_{p_i}} = -t \frac{(p_i \cdot p_r)}{(p_j \cdot p_r)} + O(t^2), \]

where either $r = i+2$ (for $k = i$) or $r = i-1$ (for $k = i-1$), and as a consequence,

\[ \frac{1}{t - z_{k,k+2}(t)} \frac{t - 1}{z_{k,k+2}(t) - 1} = \frac{1}{t \ p_r \cdot (p_j + p_i)} (1 + O(t)). \]

We have further

\[ p_{k,k+2}(t) = tp_i + z_{k,k+2}(t)p_j + p_r \to p_r \neq 0 \]

and therefore in both cases

\[ \lim_{t \to 0} J_{2n-1}(p_1, \ldots, p_{k-1}, p_{k,k+2}(t), p_{k+3}, \ldots, p_{2n+1}) = J_{2n-1}(p_1, \ldots, p_{i-2}, p_{i-1}, p_{i+2}, \ldots, p_{2n+1}). \]
For the remaining ingredients of the formula (5.19) we get

\[ M_3(tp_i, z_{i,i+2}(t)p_{i+1}, p_{i+2}) = \frac{1}{F^2} t(p_i \cdot p_{i+2}) \]  
(5.35)

\[ M_3(p_{i-1}, tp_i, z_{i-1,i+1}(t)p_{i+1}) = \frac{1}{F^2} z_{i-1,i+1}(t)(p_{i-1} \cdot p_{i+1}) = -t \frac{1}{F^2}(p_i \cdot p_{i-1})(1 + O(t)). \]  
(5.36)

Inserting this into the formulae (5.19) and (5.30) get finally for \( i > 1 \)

\[ S_{n,i+1}^{0,0} = \frac{1}{2F^2} \left( \frac{(p_i \cdot p_{i+2})}{p_{i+2} \cdot (p_{i+1} + p_i)} - \frac{(p_i \cdot p_{i-1})}{p_{i-1} \cdot (p_{i+1} + p_i)} \right) J_{2n-1}(p_1, \ldots, p_{i-2}, p_{i-1}, p_{i+2}, \ldots, p_{2n+1}). \]  
(5.37)

In the same way, for \( i = 1 \) only the first term on the right hand side of (5.37) contributes.

Let us proceed to the case 2. when \( j = i + 2 \) and \( z_{i,i+2}(t) \to 0 \) for \( t \to 0 \) is the only pole which can give nonzero contribution to (5.20). In this case we have

\[ M_3(tp_i, p_{i+1}, z_{i,i+2}(t)p_{i+2}) = \frac{1}{F^2} t z_{i,i+2}(t)(p_i \cdot p_{i+2}) = O(t^2). \]  
(5.39)

which implies \( S_{n,i+2}^{0,0} = 0. \)

To summarize, we have for \( k > 0 \)

\[ \lim_{t \to 0} J_{2n+1}(p_1, \ldots, p_{2k}, tp_{2k+1}, \ldots, tp_j, \ldots, p_{2n+1}) = \delta_{j,2k+2} \frac{1}{2F^2} J_{2n-1}(p_1, \ldots, p_{2k}, p_{2k+3}, \ldots, p_{2n+1}) \left( \frac{(p_{2k+1} \cdot p_{2k+3})}{p_{2k+3} \cdot (p_{2k+2} + p_{2k+1})} - \frac{(p_{2k+1} \cdot p_{2k})}{p_{2k} \cdot (p_{2k+2} + p_{2k+1})} \right) \]  
(5.40)

and for \( k = 0 \)

\[ \lim_{t \to 0} J_{2n+1}(tp_1, \ldots, tp_j, \ldots, p_{2n+1}) = \delta_{j,2} \frac{1}{2F^2} \frac{(p_1 \cdot p_3)}{(p_2 \cdot p_3) + (p_1 \cdot p_3)} J_{2n-1}(p_3, \ldots, p_{2n+1}). \]

As it is clear from the above discussion, the “asymmetry” of the latter result stems from the fact that \( p_{2n+2} \) is off-shell and therefore the three-particle pole corresponding to \((p_3 + p_4 + \ldots + p_{2n+1})^2 = (p_{2n+2} - p_1 - p_2)^2 \to p_{2n+2}^2 \neq 0 \) does not contribute.

Because

\[ J(1, 2, \ldots, 2n + 1) = J(2n + 1, 2n, \ldots, 2, 1), \]  
(5.41)

we get for \( j < 2k + 1 \)

\[ J_{2n+1}(p_1, \ldots, tp_j, \ldots, p_{2k}, tp_{2k+1}, \ldots, p_{2n+1}) = J_{2n+1}(p_{2n+1}, \ldots, tp_{2k+1}, p_{2k}, \ldots, tp_j, \ldots, p_1). \]  
(5.42)
On the right hand side of this identity the momentum $p_{2k+1}$ stays on the odd position and thus

$$
\lim_{t\to 0} J_{2n+1}(p_1, \ldots, tp_j \ldots p_{2k}, tp_{2k+1}, \ldots, p_{2n+1}) = \delta_{j,2k} \frac{1}{2F^2} J_{2n-1}(p_1, \ldots, p_{2k-1}, p_{2k+2}, \ldots, p_{2n+1}) \left( -\frac{(p_{2k} \cdot p_{2k-1})}{p_{2k-1} \cdot (p_{2k} + p_{2k+1})} + \frac{(p_{2k} \cdot p_{2k+2})}{p_{2k+2} \cdot (p_{2k} + p_{2k+1})} \right) 
$$

Putting (5.40) and (5.43) together the final result (5.13) follows.

6. Summary and conclusion

We have studied various aspects of the $SU(N)$ chiral nonlinear sigma model which describes the low-energy dynamics of the Goldstone bosons corresponding to the spontaneous chiral symmetry breaking $SU(N) \times SU(N) \rightarrow SU(N)$. As we have shown, the tree-level scattering amplitudes of the Goldstone bosons can be constructed from the stripped amplitudes, which are identical as those of the $U(N)$ chiral nonlinear sigma model. It is therefore possible to use this correspondence and to investigate both the $SU(N)$ and $U(N)$ cases on the same footing. Especially we are allowed to choose any parametrization (field redefinition) of the chiral unitary matrix $U(x)$ entering the Lagrangian from the wide class of parametrizations admissible for the extended $U(N)$ case, because the fully on-shell stripped amplitudes do not depend on the parametrization. For the direct calculation of the flavor ordered Feynman graphs, the most convenient choice proved to be the minimal parametrization (2.31), which we have chosen in order to calculate the on-shell amplitudes up to 10 Goldstone bosons.

The proliferation of the Feynman graphs with increasing number of the Goldstone bosons call for alternative methods of calculation. The more efficient method is based on the Berends-Giele recursive relations for the semi-on-shell amplitudes, but due to the infinite number of the interaction vertices in the Lagrangian of the nonlinear sigma model, the number of terms necessary to evaluate the $n$–point amplitude grows much faster (exponentially) with $n$ than for the case of the power-counting renormalizable theories (where the growth is polynomial).

The BCFW recursive relations could make the calculation of the on-shell stripped amplitude as effective as for the renormalizable theories at least as far as the number of terms (which is in both cases related to the number of factorization channels) is concerned. However, the standard way of the BCFW reconstruction is not directly applicable for the nonlinear sigma model because of the bad behavior of the BCFW deformed amplitudes at infinity. We have therefore proposed an alternative deformation of the semi-on-shell amplitudes based on the scaling of all odd or all even momenta, for which we were able to prove exact results concerning the behavior of the semi-on-shell amplitudes when the scaling parameter tended to zero. Using the Berends-Giele recursive relations we were able to prove this scaling properties for general $n$–point amplitude. An essential ingredient of the proof was the fact that the semi-on-shell amplitudes (unlike the on-shell ones) are parametrization dependent and we could therefore make an appropriate choice of the parametrization (the Cayley one). We have then used these exact scaling properties for a generalized BCFW reconstruction formula (with one subtraction) which determines fully all the semi-on-shell amplitudes in the Cayley
parametrization including the basic four-point one. Putting then the semi-on-shell amplitudes on-shell we reconstruct simply the parametrization independent on-shell amplitudes. In contrast to the standard BCFW relations our procedure is not restricted to \( d \geq 4 \) space-time dimensions.

The BCFW recursive relation are also a suitable tool for investigation of the properties of the amplitudes. We have illustrated this in two cases, namely we have proved the presence of the Adler zero and established the general form of the double soft limit for the semi-on-shell amplitudes in the Cayley parametrization.

The existence of BCFW recursion relations for power-counting non-renormalizable effective theory as the \( SU(N) \) chiral nonlinear sigma model gives an evidence that the on-shell methods can be used for much larger classes of theories than has been considered so far. It also indicates that the \( SU(N) \) chiral nonlinear sigma model is rather special and deeper understanding of all its properties is desirable. For future directions, it would be interesting to see whether the construction can be re-formulated purely in terms of on-shell scattering amplitudes not using the semi-on-shell ones. Next possibility is to focus on loop amplitudes. As was shown in [20] the loop integrand can be also in certain cases constructed using BCFW recursion relations, it would be spectacular if the similar construction can be applied for effective field theories.

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A. General parametrization

In this Appendix we will discuss a very general class of parameterizations of the \( U(N) \) sigma model originally studied in [1], which is suited for a derivation of the stripped Feynman rules. Within this class the field \( U(x) \in U(N) \) is expressed in the form

\[
U = \sum_{k=0}^{\infty} a_k \left( \sqrt{\frac{1}{2F}} \phi \right)^k
\]  

where \( \phi = t^a \phi^a \), \( \phi^a \) are the Goldstone boson fields, \( t^a \) are the \( U(N) \) generators normalized according to \( \langle t^a t^b \rangle = \delta^{ab} \) and \( a_k \) are real coefficients. These coefficients are not completely arbitrary, because the unitarity condition \( U^+U = 1 \) implies the following constraint

\[
\sum_{k=0}^{n} a_k a_{n-k} (-1)^k = \delta_{n,0}.
\]

For \( n = 0 \) we get \( a_0^2 = 1 \) and without lose of generality we can set \( a_0 = 1 \). In order to preserve the correct normalization of the kinetic term and to keep the interpretation of \( F \) as the decay constant for the fields \( \phi^a \) we have to fix also \( a_1 = 1 \).
For \( n \) odd the relations (A.2) are satisfied automatically while for \( n = 2k \) we can solve them for \( a_{2k} \) and get a recurrent formula for the even coefficients expressed in terms of the odd ones

\[
a_{2k} = -\frac{(-1)^k}{2} a_k^2 - \sum_{j=1}^{k-1} (-1)^j a_{2j} a_{2k-j}. \tag{A.3}
\]

This gives up to \( k = 3 \)

\[
\begin{align*}
a_2 &= \frac{1}{2} a_1^2 = \frac{1}{2} \\
a_4 &= -\frac{1}{2} a_2^2 + a_1 a_3 = -\frac{1}{8} + a_3 \\
a_6 &= \frac{1}{2} a_3^2 + a_1 a_5 - a_2 a_4 = \frac{1}{16} - \frac{1}{2} a_3 + \frac{1}{2} a_5^2 + a_5
\end{align*} \tag{A.4}
\]

The explicit solution of the recurrent relations (A.3) to all orders can be easily found by means of the following trick. Let us introduce the generating function \( f(x) \) of the above coefficients \( a_k \)

\[
f(x) = \sum_{k=0}^{\infty} a_k x^k. \tag{A.5}
\]

The relations of unitarity with the initial conditions \( a_0 = a_1 = 1 \) are then equivalent to

\[
f(-x)f(x) = 1, \quad f(0) = 1, \quad f'(0) = 1 \tag{A.6}
\]

which represents a functional equations for the generating functions \( f(x) \). Let us define \( f_{\pm}(x) \) to be the even and odd part of \( f(x) \), i.e. \( f_{\pm}(x) = (f(x) \pm f(-x))/2 \). From (A.6) we get then

\[
f_+(x)^2 - f_-(x)^2 = 1 \tag{A.7}
\]

or finally

\[
f_+(x) = \sqrt{1 + f_-(x)^2}. \tag{A.8}
\]

The formal series expansion of both sides of the last equation at \( x = 0 \) gives the solution of the recurrent relations (A.3), i.e. the explicit expressions for \( a_{2k} \) in terms of an infinite number of free parameters \( a_{2k+1} \). The general solution of the functional equation (A.6) is then

\[
f(x) = f_-(x) + \sqrt{1 + f_-(x)^2} \tag{A.9}
\]

where \( f_-(x) \) is arbitrary odd real function analytic for \( x = 0 \) satisfying \( f'(0) = 1 \). The minimal parameter-free solution corresponds to the choice \( a_{2k+1} = 0 \) for \( k > 0 \), i.e. \( f_{\text{min}}(x) = x \) and

\[
f_{\text{min}}(x) = x + \sqrt{1 + x^2} \tag{A.10}
\]

i.e. for \( k \geq 1 \)

\[
a_{2k}^{\text{min}} = \frac{(-1)^{k+1}}{2^{2k-1}} C_k, \tag{A.11}
\]
where
\[ C_n = \frac{1}{n+1} \binom{2n}{n} \]  
(A.12)
are the Catalan numbers.

Another frequently used choices are the exponential and Cayley parameterizations corresponding to \( f_{\text{exp}}(x) \) and \( f_{\text{Cayley}}(x) \) respectively, where
\[
f_{\text{exp}}(x) = e^x \quad \text{(A.13)}
\]
\[
f_{\text{Cayley}}(x) = \frac{1 + (x/2)}{1 - (x/2)} \quad \text{(A.14)}
\]

or in terms of the coefficients \( a_k \)
\[
a_k^{\text{exp}} = \frac{1}{k!} \quad \text{(A.15)}
\]
\[
a_k^{\text{Cayley}} = \frac{1}{1 + \delta_{k,0}} \frac{1}{2^k} \quad \text{(A.16)}
\]

These two parameterizations can be understood as minimal parameter-free variants with respect to other two possible forms of the general solutions of the functional equation (A.6), namely
\[
f(x) = \exp g(x) \quad \text{(A.17)}
\]
and
\[
f(x) = \frac{h(x)}{h(-x)} \quad \text{(A.18)}
\]
where \( g(x) \) and \( h(x) \) are arbitrary real functions analytic for \( x = 0 \) for which
\[
g(x) = -g(-x) \quad \text{(A.19)}
\]
\[
g(0) = 0, \quad g'(0) = 1 \quad \text{(A.20)}
\]
and
\[
h'(0) = \frac{1}{2} h(0) \neq 0 \quad \text{(A.21)}
\]

As was proved in [1], for \( N > 2 \) the only parametrization from the class (A.1) admissible also for \( SU(N) \) sigma model is the exponential one. The reason is that, under the general axial \( SU(N) \) transformation
\[
U(x)' = \sum_{k=0}^{\infty} a_k \left( \sqrt{2} \frac{i}{F} \phi' \right)^k = U_A \sum_{k=0}^{\infty} a_k \left( \sqrt{2} \frac{i}{F} \phi \right)^k U_A \quad \text{(A.22)}
\]
which defines corresponding nonlinear transformation of the matrix of the Goldstone boson fields \( \phi = \sum_{a=1}^{N^2-1} \phi^a t^a \) the \( SU(N) \) condition for the trace \( \langle \phi' \rangle = 0 \) is not preserved unless \( a_k = 1/k! \). Of course, in the case \( N > 2 \) we can use different admissible parameterizations of \( SU(N) \) which, however, do not belong to the class (A.1) (see e.g. [47]).

Let us now find the stripped Feynman rules. Using the general parametrization (A.1) we can write the Lagrangian of the nonlinear \( U(N) \) sigma model in the expanded form
\[ \mathcal{L}^{(2)} = \frac{F^2}{4} \langle \partial U \cdot \partial U^+ \rangle = \sum_{n,m=0}^{\infty} v_{n,m} \langle \partial \phi \phi^n \cdot \partial \phi \phi^m \rangle. \]  
(A.23)

where we get for \( v_{n,m} \) after some algebra (and using the unitarity condition (A.2))

\[ v_{n,m} = (1 + (-1)^{n+m})(-i)^{n+m} \frac{1}{4F^{n+m}} \sum_{k=0}^{m} a_k a_{m+n+2-k} (-1)^{k+1} (k - 1 - m) \]  
(A.24)

Therefore only the terms with even number of fields survive, explicitly

\[ \mathcal{L}^{(2)} = \sum_{n=0}^{\infty} \mathcal{L}^{(2)}_{2n+2} \]  
(A.25)

where

\[ \mathcal{L}^{(2)}_{2n+2} = \sum_{k=0}^{2n} v_{k,2n-k} \langle \partial \phi \phi^k \cdot \partial \phi \phi^{2n-k} \rangle \]  
(A.26)

The usual Feynman rules for the vertices can be easily obtained as a sum over permutations

\[ V^{a_1, \ldots, a_{2n+2}}(p_1, p_2, \ldots, p_{2n+1}; p_{2n+2}) = -2^{n+1} \sum_{\sigma \in S_{2n+2}} \langle t^{a_{\sigma(1)}} \ldots t^{a_{\sigma(2n+2)}} \rangle \]  
\times \sum_{i=1}^{2n} v_{k,2n-k} \langle p_{i} \cdot p_{i+k+1} \rangle \]  
(A.27)

The stripped Feynman rule then follows in the form

\[ V_{2n+2}(p_1, p_2, \ldots, p_{2n+1}; p_{2n+2}) = -2^{n+1} \sum_{k=0}^{2n+2} \sum_{i=1}^{2n} v_{k,2n-k} \langle p_{i} \cdot p_{i+k+1} \rangle \]  
(A.28)

Inserting (A.15) into (A.24) we get after some algebra for the exponential parametrization

\[ v_{k,2n-k}^{\exp} = \left( -1 \right)^n \frac{(-1)^k}{2F^{2n}} \frac{(2n+2)!}{2n!} \frac{2n}{k} \]  
(A.29)

while for the Cayley parametrization we have \( v_{2k+1,2n-2k-1}^{\text{Cayley}} = 0 \) and

\[ v_{2k,2n-2k}^{\text{Cayley}} = \left( -1 \right)^n \frac{1}{2F^{2n}} \frac{2n+1}{2^{2n+1}}. \]  
(A.30)

Similar calculations can be made also for the minimal parametrization, but the result is much more lengthy and we will not need it explicitly. Instead we will rewrite the Feynman rules for the vertex \( V_{2n+2} \) with \( 2n + 2 \) external legs in terms of the variables

\[ s_{i,j} = p_{i,j}^2 \]  
(A.31)
where $1 \leq i < j \leq 2n + 1$ and

$$p_{i,j} = \sum_{k=i}^{j} p_k$$

(A.32)

Here we identify

$$s_{2n+2,2n+2+k} = s_{k+1,2n+1}$$

(A.33)

$$s_{i,2n+2+k} = s_{k+1,i-1}.$$  

(A.34)

The scalar products $(p_i \cdot p_j)$ can be then expressed as

$$(p_i \cdot p_i) = s_{i,i}$$

(A.35)

$$(p_i \cdot p_{i+1}) = \frac{1}{2}(s_{i,i+1} - s_{i,i} - s_{i+1,i+1})$$

(A.36)

and for $k \geq 2$

$$(p_i \cdot p_{i+k}) = \frac{1}{2}(s_{i,i+k} - s_{i,i+k-1} + s_{i+1,i+k-1} - s_{i+1,i+k} ).$$

(A.37)

On-shell we get $s_{i,i} = 0$ and $s_{1,2n+1} = 0$. The stripped Feynman rule in these variables can be written in the form valid for $n \geq 1$

$$V_{2n+2}(s_{i,j}) = (-1)^n \left( \frac{2}{F^2} \right)^n \sum_{k=0}^{n} w_{k,n} \sum_{i=1}^{2n+2} s_{i,i+k}$$

(A.38)

where

$$w_{0,n} = (-1)^n 2 F^{2n} (2v_{0,2n} - v_{1,2n-1})$$

(A.39)

$$w_{k,n} = (-1)^n 2 F^{2n} (2v_{k,2n-k} - v_{k-1,2n+1-k} - v_{k+1,2n-1-k}) \text{ for } k < n$$

(A.40)

$$w_{n,n} = (-1)^n 2 F^{2n} (v_{n,n} - v_{n-1,n+1}).$$

(A.41)

Within the general parametrization we get from (A.24) and (A.2) after some algebra

$$w_{k,n} = \frac{(-1)^k}{1 + \delta_{kn}} a_{k+1} a_{2n+1-k}.$$  

(A.42)

For the above special cases this reads for $N \geq 1$

$$w_{0,n}^{\exp} = \frac{(-1)^k}{1 + \delta_{kn}} \frac{1}{2n+2!} \binom{2n+2}{k+1}$$

(A.43)

$$w_{k,n}^{\text{Cayley}} = \frac{(-1)^k}{1 + \delta_{kn}} \frac{1}{2n}$$

(A.44)

$$w_{0,n}^{\min} = w_{2k,n}^{\min} = 0$$

(A.45)

$$w_{2k+1,n}^{\min} = \frac{1}{1 + \delta_{2k+1,n}} \frac{(-1)^n}{2n} C_k C_{n-k-1}.$$  

(A.46)

Note that, for the minimal parametrization the coefficients $w_{0,n}^{\min}$ at $s_{i,i} = p_i^2$ vanish, therefore the stripped Feynman rules for vertices do not depend on the off-shellness of the momenta in this case. This fact has been observed already in [11] without calculating the explicit Feynman rules.
B. More examples of amplitudes

The eight-point amplitude is

\[
8F^6 M(1, 2, 3, 4, 5, 6, 7, 8) = \frac{1}{2} \left( s_{1,2} + s_{2,3} \right) \left( s_{1,4} + s_{4,7} \right) \left( s_{5,6} + s_{6,7} \right) + \frac{1}{2} \left( s_{1,2} + s_{2,3} \right) \left( s_{1,4} + s_{4,5} \right) \left( s_{6,7} + s_{7,8} \right) s_{1,3} s_{6,8} s_{1,3} - \frac{1}{2} \left( s_{1,2} + s_{2,3} \right) \left( s_{4,5} + s_{4,7} + s_{5,6} + s_{6,7} + s_{7,8} \right) + 2s_{1,2} + \frac{1}{2} s_{1,4} + \text{cycl} \quad (B.1)
\]

and graphically in Fig. 12. Finally the ten-point amplitude is given by

\[
16F^8 M(1, 2, 3, 4, 5, 6, 7, 8, 9, 10) = -\frac{s_{1,2} + s_{2,3}}{s_{1,3}} \left\{ \right.
\frac{1}{2} \left( s_{1,4} + s_{4,9} \right) \left( s_{5,8} + s_{6,9} \right) \left( s_{6,7} + s_{7,8} \right) s_{5,9 s_{6,8}} + \frac{1}{2} \left( s_{1,4} + s_{4,5} \right) \left( s_{1,8} + s_{6,9} \right) \left( s_{6,7} + s_{7,8} \right) + \frac{1}{2} \left( s_{1,4} + s_{4,5} \right) \left( s_{1,6} + s_{6,9} \right) \left( s_{1,8} + s_{8,9} \right) s_{1,5 s_{6,8}} + \frac{1}{2} \left( s_{1,4} + s_{4,5} \right) \left( s_{4,7} + s_{5,8} \right) \left( s_{5,6} + s_{6,7} \right) + \frac{1}{2} \left( s_{1,4} + s_{4,9} \right) \left( s_{4,5} + s_{5,6} \right) \left( s_{7,8} + s_{8,9} \right) s_{4,6 s_{7,9}} - \frac{1}{2} \left( s_{1,4} + s_{4,9} \right) \left( s_{4,5} + s_{4,7} + s_{5,6} + s_{5,8} + s_{6,7} + s_{7,8} \right) s_{4,8} - \frac{1}{2} \left( s_{1,4} + s_{1,6} + s_{4,5} + s_{4,7} + s_{5,6} + s_{6,7} \right) s_{1,7} - \frac{1}{2} \left( s_{1,4} + s_{1,6} + s_{4,5} + s_{4,9} + s_{5,6} + s_{6,9} \right) s_{7,9} - \frac{1}{2} \left( s_{1,4} + s_{4,5} \right) \left( s_{1,6} + s_{1,8} + s_{6,7} + s_{6,9} + s_{7,8} + s_{8,9} \right) s_{1,5} - \frac{1}{2} \left( s_{1,4} + s_{4,9} \right) \left( s_{5,6} + s_{5,8} + s_{6,7} + s_{6,9} + s_{7,8} + s_{8,9} \right) s_{5,9} + 2s_{1,4} + 2s_{1,6} + 2s_{4,5} + 2s_{4,7} + 2s_{4,9} + 2s_{5,6} + 2s_{5,8} + 2s_{6,7} + 2s_{6,9} + 2s_{7,8} + 2s_{8,9} \left\} \right.
\]

Figure 12: Graphical representation of the 8-point amplitude (B.1) with cycling tacitly assumed.
Figure 13: Graphical representation of the 10-point amplitude (B.2) with cycling tacitly assumed.

\[-\frac{1}{2}(s_{1,2} + s_{1,4} + s_{2,3} + s_{2,5} + s_{3,4} + s_{4,5})(s_{1,6} + s_{1,8} + s_{6,7} + s_{6,9} + s_{7,8} + s_{8,9}) \]
\[+ 5s_{1,2} + 2s_{1,4} + \text{cycl} \]  
with one-to-one correspondence with Fig. 13

C. Relative efficiency of Feynman diagrams and Berends-Giele relations

In this appendix we review the solution of several types of recursive relations which count the number of ordered Feynman graphs needed for the semi-on-shell amplitude \(J(1, 2, \ldots, n)\) in the nonlinear sigma model and related toy models.

C.1 Number of the Feynman graphs

Let us start with the case of nonlinear sigma model, i.e. with the case with infinite number of vertices in the interaction Lagrangian. The above recursive relations, which determine the number \(f(2n + 1)\) of the (flavor ordered) Feynman graphs which contribute to \(J(1, 2, \ldots, 2n + 1)\), are tightly related to the Berends-Giele relations (4.7). Indeed, after making the following substitution to (4.7)

\[J(1, 2, \ldots, 2n + 1) \rightarrow f(2n + 1), \quad \frac{i}{p_{2n+2}} \rightarrow 1, \quad iV_{2k+1} \rightarrow 0, \quad iV_{2k+2} = 1, \]  
(C.1)
the individual terms on the right hand side just count the number of Feynman graphs generated from these terms by the iterations of the recursive procedure. As a result we get for $f(2n + 1)$ the following recursive relation

$$f(2n + 1) = \sum_{k=1}^{n} \sum_{n_i} 2^{k+1} \prod_{i=1}^{2k+1} f(2n_i + 1), \quad (C.2)$$

with the initial condition $f(1) = 1$. In the above formula the sum over $\{n_i\}$ is constrained by the requirement

$$\sum_{i=1}^{2k+1} (2n_i + 1) = 2n + 1 \iff \sum_{i=1}^{2k+1} n_i = n - k \quad (C.3)$$

i.e. it corresponds to the sum over all possible decompositions of ordered set of $2n + 1$ momenta to non-empty clusters with odd number of momenta in each cluster (cf. (4.8) and Fig. 3), i.e. more explicitly

$$f(2n + 1) = \sum_{k=1}^{n} \sum_{n_i=n-k} 2^{k+1} \prod_{i=1}^{2k+1} f(2n_i + 1), \quad f(1) = 1. \quad (C.4)$$

Standard method for solution of this type of recursive relation is based on the generating function defined as

$$A(x) = \sum_{n=0}^{\infty} f(2n + 1)x^n. \quad (C.5)$$

The recursive formula (C.4) implies the following equation for $A(x)$

$$A = 1 + \sum_{k=1}^{\infty} x^k A^{2k+1} = 1 + \frac{xA^3}{1 - xA^2} \quad (C.6)$$

or

$$x = \frac{B}{(B + 1)^2 (2B + 1)} \equiv \frac{B}{g(B)} \quad (C.7)$$

where $B = A - 1$ and $g(z) = (z + 1)^2(2z + 1)$. In this form, the problem is prepared for the application of the Lagrange–Bürmann inversion formula

$$B(x) = \sum_{n=0}^{\infty} x^n \frac{d^{n-1}}{dz^{n-1}} g(z)^n |_{z=0} = \sum_{n=1}^{\infty} x^n \frac{d^{n-1}}{dz^{n-1}} (z + 1)^2n(2z + 1)^n |_{z=0}. \quad (C.8)$$

After straightforward algebra with help of Leibnitz rule we get for $n \geq 1$

$$f(2n + 1) = 2^{n-1} \sum_{k=0}^{n-1} \binom{n}{k+1} \binom{2n}{k} 2^{-k} = 2^{n-1} \, _2F_1 \left( 1 - n, -2n, 2; \frac{1}{2} \right), \quad (C.9)$$

where $_2F_1(\alpha, \beta, \gamma; z)$ is the hypergeometric function. In the same way one can solve the recurrence relations for the number of ordered Feynman graphs for the semi-on-shell amplitudes $J(1, 2, \ldots, n)$ in the cases when only quadrilinear vertices (”$\phi^4$ theory”), only trilinear vertices (”$\phi^3$ theory”) or
Table 2: Number of flavor ordered Feynman graphs for $J(1, \ldots, n)$ and $J(1, \ldots, 2n + 1)$ in the models of the type $\phi^3$, $\phi^3 + \phi^4$, $\phi^4$ and nonlinear sigma model.

| $n$ | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $f_3(n)$ | 1   | 2   | 5   | 14  | 42  | 132 | 429 | 1430| 4862| 16796|
| $f_{3+4}(n)$ | 1   | 3   | 10  | 38  | 154 | 654 | 2871| 12925| 59345| 276835|
| $f_4(2n + 1)$ | 3   | 12  | 55  | 273 | 1428| 7752| 43263| 246675| 1430715| 8414640|
| $f(2n + 1)$ | 4   | 21  | 126 | 818 | 5594| 39693| 289510| 2157150| 16348960| 125642146|

both trilinear and quadrilinear vertices ("$\phi^3 + \phi^4$ theory") are present in the Lagrangian. In the first case, similarly to the nonlinear sigma model, only $J(1, 2, \ldots, n)$ with $n$ odd can be nonzero, while in the remaining two cases $J(1, 2, \ldots, n)$ both parities of $n$ are generally allowed. Let us denote the number of the Feynman graphs for $J(1, 2, \ldots, n)$ as $f_4(n)$, $f_2(n)$ and $f_{3+4}(n)$ respectively. We get the following recurrence relations

$$f_4(2n + 1) = \sum_{n_1 + n_2 + n_3 = n - 1, \ n_i \geq 0} f_4(2n_1 + 1)f_4(2n_2 + 1)f_4(2n_3 + 1) \tag{C.10}$$

$$f_3(n) = \sum_{n_1 + n_2 = n, \ n_i \geq 1} f_3(n_1)f_3(n_2) \tag{C.11}$$

$$f_{3+4}(n) = \sum_{n_1 + n_2 = n, \ n_i \geq 1} f_{3+4}(n_1)f_{3+4}(n_2)$$

$$+ \sum_{n_1 + n_2 + n_3 = n, \ n_i \geq 1} f_{3+4}(n_1)f_{3+4}(n_2)f_{3+4}(n_3) \tag{C.13}$$

with initial conditions $f_j(1) = 1$, $j = 3, 4, 3 + 4$. The corresponding generating functions

$$A_4(x) = \sum_{n=0}^{\infty} f_4(2n + 1)x^n, \quad A_{3,3+4}(x) = \sum_{n=1}^{\infty} f_{3,3+4}(n)x^n \tag{C.14}$$

then satisfy

$$A_4 = 1 + xA_4^3, \quad A_3 = x + A_3^2, \quad A_{3+4} = x + A_{3+4}^2 + A_{3+4}^3. \tag{C.15}$$

In the second case we get

$$A_3(x) = \frac{1 - \sqrt{1 - 4x}}{2} = \frac{1}{2} \left( 1 - \sum_{n=0}^{\infty} \binom{1/2}{n} (-4x)^n \right) \tag{C.16}$$

and therefore

$$f_3(n) = \frac{1}{n} \binom{2(n - 1)}{n - 1} = C_{n-1} \tag{C.17}$$

where $C_n$ are the Catalan numbers. In the first case, writing

$$x = \frac{A_4 - 1}{A_4^3} = \frac{B_4}{(B_4 + 1)^3} \tag{C.18}$$
and using the Lagrange–Bürmann inversion formula we get for \( n > 0 \)
\[
f_4(2n + 1) = \frac{1}{n!} \frac{d^{n-1}}{dz^{n-1}}(z + 1)^{3n}|_{z=0} = \frac{1}{2n + 1} \left( \begin{array}{c} 3n \\ n \end{array} \right). \tag{C.19}
\]

In the third case, we get from
\[
x = A_{3+4} \left( 1 - A_{3+4} - A_3^2 \right)
\]
and using the Lagrange–Bürmann inversion formula
\[
f_{3+4}(n) = \frac{1}{n!} \frac{d^{n-1}}{dz^{n-1}}\left( \frac{1}{1 - z - z^2} \right)^n \bigg|_{z=0} = \frac{(-1)^n}{n!} \frac{d^{n-1}}{dz^{n-1}}\left( \frac{1}{z_1 - z} \right)^n \left( \frac{1}{z_2 - z} \right)^n |_{z=0} \tag{C.20}
\]
(where \( z_1 = -\phi, z_2 = \phi - 1 \) and \( \phi = \frac{1 + \sqrt{5}}{2} \) is the Golden ratio) the result
\[
f_{3+4}(n) = (-1)^{n+1} \phi^{1-n} \sum_{k=0}^{n-1} \frac{n - 1 + k}{n} \binom{2(n-1) - k}{n-1} \left( \frac{\phi}{1 - \phi} \right)^k
\]
\[
= \left( -\frac{4}{\phi} \right)^{n-1} \Gamma \left( n - \frac{1}{2} \right) 2F_1 \left( 1 - n, n, 2 - 2n; \frac{\phi}{1 - \phi} \right). \tag{C.22}
\]

The first twelve members of the above sequences are illustrated in the Table 2.

### C.2 Efficiency of the Berends-Giele relations

We can compare this with the number of terms generated by Berends-Giele recursion. For the nonlinear sigma model, the number of terms on the right hand side of (4.7) is just
\[
t(2n + 1) = \sum_{k=1}^{n} \sum_{\{n_k\}} 1 = \sum_{k=1}^{n} \binom{n + k}{n - k} = F_{2n+1} - 1 \tag{C.23}
\]
where
\[
F_n = \frac{1}{\sqrt{5}} \left( \phi^n - (\phi - 1)^n \right) \tag{C.24}
\]
are the Fibonacci numbers and \( \phi = \frac{1 + \sqrt{5}}{2} \) is the Golden ratio. Therefore, using the known results for \( J(1, 2, \ldots, 2m + 1) \) with \( m < n \) at each step, we need to evaluate altogether
\[
b(2n + 1) = \sum_{m=1}^{n} t(2m + 1) = \frac{1}{\sqrt{5}} \left( \phi^3 \phi^{2n} - \phi + 1 \right)^3 - n \tag{C.25}
\]
terms in order to calculate \( J(1, 2, \ldots, 2n + 1) \) using the Berends-Giele recursion. We show the sequences \( t(2n + 1) \) and \( b(2n + 1) \) in the first and second row of Tab.1 respectively.

In the same way we can calculate analogous numbers \( t_j(n) \) and \( b_j(n) \) for \( j = 3, 4, 3+4 \), i.e. for \( \phi^3 \) theory, \( \phi^4 \) theory or \( \phi^3 + \phi^4 \) theory. For instance, for \( t_4(2n + 1) \) we have (see Tab. 1 for numerical values)
\[
t_4(2n + 1) = \binom{n + 1}{2}, \quad b_4(2n + 1) = \sum_{m=1}^{n} t_4(2m + 1) = \frac{1}{6} n(n + 1)(n + 2) \tag{C.26}
\]

Note the exponential growth of \( t(2n + 1) \) and \( b(2n + 1) \) with increasing \( n \) in contrast to the only polynomial growth of \( t_4(2n + 1) \) and \( b_4(2n + 1) \).
D. Other example of scaling properties of the semi-on-shell amplitudes

In this appendix we prove the following scaling limit

$$\lim_{t \to 0} J_{2n+1}(tp_1, tp_2, tp_3, p_4, \ldots, tp_{2r-1}, tp_{2r}, tp_{2r+1}, \ldots, p_{2N}, tp_{2n+1}) = 0 \quad (D.1)$$

which is valid for for $n > 1$. Let us note, however, that

$$J_3(tp_1, tp_2, tp_3) = J_3(p_1, p_2, p_3) \neq 0. \quad (D.2)$$

On the other hand, for $N = 2$ we get by direct calculation

$$\lim_{t \to 0} J_5(tp_1, tp_2, tp_3, p_4, tp_5) = \lim_{t \to 0} J_5(tp_1, p_2, tp_3, tp_4, tp_5) = 0 \quad (D.3)$$

and we can therefore proceed by induction based on Berends-Giele relations almost exactly as in the case of the proof of (4.17). The only modification here is that, along with the “dangerous” contributions without blocks $J(jk+1, \ldots, jk+1)$ where $jk$ is even and $jk+1 - jk > 1$ attached to the odd line of the vertex $V_{m+1}$ (provided at least one such a block is present, the contribution vanish either by the induction hypothesis or by (4.17) ) we have to discuss separately new type of “dangerous” terms with building block $J(p_{2r-1}, p_{2r}, p_{2r+1})$ (this block does not vanish due to (D.2)). The “old” dangerous terms do not in fact contribute as was already discussed within the proof of (4.17). The “new” dangerous terms have the following general form form

$$\frac{i}{P_{2N+2}} iV_{2k+2}(p_1, p_2, j_1, p_{2j_1+1}, \ldots, p_{2j_1+2}, 2r-2, P_{2r-12r+1}, p_{2r+2j_1+1}, \ldots, p_{2j_1-1}, 2n, p_{2n+1}, -p_{1,2n+1})$$

$$\times J(p_1)J(2, \ldots, 2j_1)J(p_{2j_1+1}) \ldots J(2j_1 + 2, \ldots, 2r - 2)$$

$$\times J(p_{2r-1}, p_{2r}, p_{2r+1})J(2r + 2, \ldots, 2j_{1}+1) \ldots J(2j_{k-1}, \ldots, 2n)J(p_{2n+1}). \quad (D.4)$$

Note that, $p_{2r-1,2r+1}$ is attached to the odd line of the vertex $V_{2k+2}$ and scales as

$$p_{2r-1,2r+1} \to tp_{2r-1,2r+1} \quad (D.5)$$

i.e. in the same way as the remaining momenta attached to the odd lines of the vertex. The vertex being proportional the squared sum of the odd line momenta scales therefore as $O(t^2)$, and the contribution of the “new” dangerous terms vanish. This finishes the proof.

E. Double soft limit of Goldstone boson amplitudes

In this appendix we will discuss the properties of the on-shell scattering amplitudes of the Goldstone bosons, which are dictated by the symmetry, namely the limits of the amplitudes for soft external momenta. Some of these properties have been obtained in the special case of pions by PCAC methods in the late sixties (see e.g. [48]). Here we enlarge and reformulate them in a more general form appropriate for our purposes with stress on the proof of the double soft limit discussed recently for pions and $N = 8$ supergravity in [50].
Let us assume a general theory with spontaneous symmetry breaking according to the pattern $G \to H$ where the homogeneous space $G/H$ is a symmetric space, i.e. the vacuum little group $H$ is the maximal subgroup invariant with respect to some involutive automorphism of $G$ ("parity"). This implies the following structure of the Lie algebra of $G$

$$[T^a, T^b] = i f^{abc}_T T^c$$
$$[T^a, X^b] = i f^{abc}_X X^c$$
$$[X^a, X^b] = i F^{abc} T^c.$$  \hspace{1cm} (E.1)

Here $T^a$ and $X^a$ are the unbroken and broken generators respectively and $f^{abc}_T$, $f^{abc}_X$ and $F^{abc}$ are the structure constants. The chiral nonlinear sigma model is a special case for which $f^{abc}_T = f^{abc}_X = F^{abc} = f^{abc}$.

The invariance of the theory with respect to the group $G$ can be expressed in terms of the Ward identities for the correlators in the general form

$$i p^\mu \langle \bar{V}_\mu^a(p) \bar{O}_1(p_1) \cdots \bar{O}_n(p_n) \rangle = - \sum_{i=1}^n i \langle \bar{O}_1(p_1) \cdots \delta_T^a \bar{O}_i(p_i + p) \cdots \bar{O}_n(p_n) \rangle \quad (E.2)$$

$$i p^\mu \langle \bar{A}_\mu^a(p) \bar{O}_1(p_1) \cdots \bar{O}_n(p_n) \rangle = - \sum_{i=1}^n i \langle \bar{O}_1(p_1) \cdots \delta_X^a \bar{O}_i(p_i + p) \cdots \bar{O}_n(p_n) \rangle. \quad (E.3)$$

Here $V_\mu^a(x)$ and $A_\mu^a(x)$ are the Noether currents corresponding to the generators $T^a$ and $X^a$ respectively (in analogy with the chiral theories we will call them vector and axial currents in what follows and to the Ward identities (E.2) and (E.3) we will refer to the vector and axial WI), $O_i(x)$ are (generally composite) local operators, $\delta_T^a O_i(x)$ and $\delta_X^a O_i(x)$ are their infinitesimal transforms with respect to the generators $T^a$ and $X^a$. The tilde means the Fourier transform

$$\bar{O}_i(p) = \int d^4 x e^{ip\cdot x} O_i(x). \quad (E.4)$$

According to the Goldstone theorem the spectrum of the theory contains as many Goldstone bosons $\pi^a$ as the broken generators $X^a$ for which the currents $A_\mu^a(x)$ play the role of the interpolating fields, i.e.

$$\langle 0 | A_\mu^a(0) | \pi^b(p) \rangle = i p_\mu F \delta^{ab}. \quad (E.5)$$

where $F$ is the Goldstone boson decay constant. Let as denote $M^{a_1 \ldots a_n}(p_1, \ldots, p_n)$ the on-shell scattering amplitude of the Goldstone bosons $\pi^{a_1}(p_1), \ldots, \pi^{a_n}(p_n)$. In what follows we will concentrate on the properties of $M^{a_1 \ldots a_n}(p_1, \ldots, p_n)$ dictated by the symmetry, i.e. those which are encoded in the WI (E.2) and (E.3).

**E.1 Vector WI and symmetry with respect to $H$**

The invariance with respect to the unbroken subgroup $H$ implies

$$\sum_{i=1}^n f^{a_1 \ldots a_n \mu}_{n} M^{a_1 \ldots a_{i-1} b a_{i+1} \ldots a_n}(p_1, \ldots, p_n) = 0. \quad (E.6)$$
This can be understood as the consequence of the vector WI of the form

\[-i\mu^\mu (\bar{V}_\mu^a (p) \bar{A}_{\mu_1}^a (p_1) \ldots \bar{A}_{\mu_n}^a (p_n)) = -\sum_{i=1}^n i\langle \bar{A}_{\mu_1}^a (p_1) \ldots \delta_\mu^a \bar{A}_{\mu_n}^a (p + p_i) \ldots \bar{A}_{\mu_n}^a (p_n) \rangle \]  

(E.7)

Note that the infinitesimal transformations \( \delta_\mu^a V^b_\nu \) and \( \delta_\mu^a A^b_{\mu_1} \) of these currents with respect to the generator \( T^a \) of the unbroken subgroup \( H \) are as follows

\[ \delta_\mu^a A^b_\nu = -\xi_{abc} A^c_\nu \]  

(E.8)

\[ \delta_\mu^a V^b_\nu = -f_{abc} V^c_\nu. \]  

(E.9)

Because there is no pole for \( p \to 0 \) in the correlator on the left hand side of (E.7), we get in this limit

\[ \sum_{i=1}^n f^{aa_1b_1} \langle \bar{A}_{\mu_1}^a (p_1) \ldots \bar{A}_{\mu_i}^b (p_i) \ldots \bar{A}_{\mu_n}^a (p_n) \rangle = 0. \]  

(E.10)

Using the LSZ formula we get according to (E.5)

\[ \langle \bar{A}_{\mu_1}^a (p_1) \ldots \bar{A}_{\mu_n}^a (p_n) \rangle = \left( \prod_{i=1}^n \frac{i}{p_i^2} Z_{\mu_i} \right) M^{a_1 \ldots a_n} (p_1, \ldots, p_n) + R_{\mu_1}^{a_1 \ldots} \]  

(E.11)

where \( Z_{\mu_i} = iF_{p_{\mu_i}} \) and the remnant \( R_{\mu_1}^{a_1 \ldots} \) is regular on shell in the sense that

\[ \lim_{p_i^2 \to 0} \left( \prod_{i=1}^n \frac{p_i^2}{p_i^2} \right) R_{\mu_1}^{a_1 \ldots} = 0. \]  

(E.12)

which implies (E.6) for the on-shell amplitude \( M^{a_1 \ldots a_n} (p_1, \ldots, p_n) \).

### E.2 Soft vector current singularity

Let us assume now the following matrix element

\[ \langle \bar{V}_\mu^a (p) | \pi^{a_1} (p_1) \ldots \pi^{a_i} (p_i) \ldots \pi^{a_n} (p_n) \rangle. \]  

(E.13)

In what follows we will discuss the behavior of this object in the limit \( p \to 0 \). On the level of the Feynman graphs, the only singularities in the soft limit \( p \to 0 \) are those which stem from the one-Goldstone-boson-reducible graphs for which the vector current \( \bar{V}_\mu^a (p) \) is attached to the external Goldstone boson line. The potential singularities are therefore of the form (see Fig. 14)

\[ \langle \bar{V}_\mu^a (p) \phi^{a_i} (0) | \pi^{a_1} (p_1) \rangle \prod_{i} \Delta_{a_i a_k} ((p - p_i)^2) \langle \phi^{a_k} (0) | \pi^{a_1} (p_1) \ldots \pi^{a_i} (p_i) \ldots \pi^{a_n} (p_n) \rangle \]  

(E.14)

where the subscript 1PI means one-Goldstone-boson-irreducible block, the hat means omitting of the corresponding particle, \( \phi^{a} (x) \) is the Goldstone boson interpolating field normalized as

\[ \langle 0 | \phi^{a} (0) | \pi^{b} (p) \rangle = \delta^{ab} \]  

(E.15)
and $\Delta^{a_ia_k}(q^2)$ is a Goldstone boson propagator. For $q^2 \to 0$ we have

$$\Delta^{a_ia_k}(q^2) = \frac{\delta^{a_ia_k}}{q^2} \left(1 + O(q^2)\right). \quad (E.16)$$

As a consequence of the Lorentz invariance, invariance with respect to $H$ and LSZ formulæ we have

$$\langle \bar{V}_\mu^a(p)\phi^{a_i}(0)|\pi^{a_1}(p_1)\rangle_{1PI} = if_X^{a_ia_j} F_V(p^2)(2p_i - p)_\mu + O((p - p_i)^2) \quad (E.17)$$

where $F_V(p^2)$ is the on-shell vector form-factor defined as\(^18\)

$$\langle \pi^{a_j}(p - p_i)|\bar{V}_\mu^a(p)|\pi^{a_1}(p_i)\rangle = if_X^{a_ia_j} F_V(p^2)(2p_i - p)_\mu. \quad (E.18)$$

We can fix the normalization of the vector currents $V^a_\mu$ in such a way that

$$F_V(p^2) = 1 + O(p^2). \quad (E.19)$$

Analogously we have

$$\langle \phi^{a_1}(0)|\pi^{a_1}(p_1)\ldots\pi^{a_i}(p_i)\ldots\pi^{a_n}(p_n)\rangle_{1PI} = M^{a_1\ldots a_i\ldots a_k a_{i+1} \ldots a_n}(p_1, \ldots, p_n) + O((p - p_i)^2). \quad (E.20)$$

Using $(p - p_i)^2 = -2(p\cdot p_i) + p^2$ and putting all the ingredients together we get for $p \to 0$

$$\langle \bar{V}_\mu^a(p)|\pi^{a_1}(p_1)\ldots\pi^{a_i}(p_i)\ldots\pi^{a_n}(p_n)\rangle = \sum_{i=1}^n f_X^{a_ia_i}(2p_i - p)_\mu M^{a_1\ldots a_i\ldots a_k a_{i+1} \ldots a_n}(p_1, \ldots, p_n) + O(1) \quad (E.21)$$

### E.3 Axial WI and Adler zero

To illustrate the method which we will use in the next subsection, let us briefly recapitulate the textbook example of the derivation of the Adler zero for the amplitude $M^{a_1\ldots a_n}(p_1, \ldots, p_n)$ (see e.g. [51]). Let us start with the axial WI in the form

$$-ip^\mu \langle \bar{A}_{\mu}^a(p)\bar{A}_{\mu_1}^{a_1}(p_1)\ldots\bar{A}_{\mu_n}^{a_n}(p_n)\rangle = \sum_{i=1}^n i\langle \bar{A}_{\mu_1}^{a_1}(p_1)\ldots\delta_X^a\bar{A}_{\mu_i}^{a_i}(p + p_i)\ldots\bar{A}_{\mu_n}^{a_n}(p_n)\rangle \quad (E.22)$$

\(^{18}\)The form of the right hand side is dictated by $H$-invariance, Bose and crossing symmetry.
where now
\[\delta^a_X A^b_\nu = -F^{abc}_\nu V^c_\nu,\]
\[\delta^a_X V^b_\nu = -f^{abc}_X A^c_\nu.\]

(E.23)

Applying on both sides of (E.22) the LSZ reduction to all but one axial currents, we get the conservation of the axial current in terms of the transversality of the matrix element of \(A^a_\mu\) between the initial and final states \(|i\rangle\) and \(|f\rangle\)
\[\langle f| \tilde{V}_c^{\mu}(p + p_i)\ldots \tilde{A}^c_{\mu_n}(p_n)\rangle = 0.\]

(E.24)

On the other hand from (E.11) we get the Goldstone boson pole dominance for \(p^2 \to 0\)
\[\langle f + \pi^a(p)|i\rangle = \frac{1}{F} p^\mu R^a_{\mu,fi} + \ldots\]

(E.25)

where \(Z_{\mu} = iFp_{\mu}\) and the remnant \(R^a_{\mu,fi}\) is regular in this limit
\[\lim_{p^2 \to 0} p^2 R^a_{\mu,fi} = 0.\]

(E.26)

Putting (E.24) and (E.25) together we get for the amplitude with emission of the Goldstone boson \(\pi^a(p)\) in the final state
\[\langle f + \pi^a(p)|i\rangle = \frac{1}{F} p^\mu R^a_{\mu,fi} + \ldots\]

(E.27)

Provided the following stronger regularity condition holds
\[\lim_{p \to 0} p^\mu R^a_{\mu,fi} = 0,\]

(E.28)

we get
\[\langle f + \pi^a(0)|i\rangle = 0,\]

(E.29)

i.e. the Adler zero for \(p \to 0\).

An useful off-shell generalization of the formula (E.25) reads
\[\langle f | A^a_\mu(p)\tilde{A}^a_{\mu_1}(p_1)\ldots\tilde{A}^a_{\mu_n}(p_n)\rangle = iF\langle \pi^a(p)|\tilde{A}^a_{\mu_1}(p_1)\ldots\tilde{A}^a_{\mu_n}(p_n)\rangle - \frac{1}{F} p^\mu R^a_{\mu,\mu_1\ldots} + \ldots\]

(E.30)

where
\[\lim_{p^2 \to 0} p^2 R^a_{\mu,\mu_1\ldots} = 0.\]

(E.31)

and using the Ward identity (E.22) and (E.23) we get
\[F\langle \pi^a(p)|\tilde{A}^a_{\mu_1}(p_1)\ldots\tilde{A}^a_{\mu_n}(p_n)\rangle = p^\mu R^a_{\mu,\mu_1\ldots} + \sum_{i=1}^{n} F^{a,a_1,c}_{\mu_1}(p_1)\ldots V^c_{\mu_i}(p + p_i)\ldots\tilde{A}^a_{\mu_n}(p_n).\]

(E.32)
\section*{E.4 Double soft limit}

Our starting point is the axial WI (E.22) rewritten in the form

\begin{equation}
- \text{i} p^\mu \langle \bar{A}_\mu^a (p) \bar{A}_\nu^b (q) \bar{A}_\mu^{a_1} (p_1) \ldots \bar{A}_\mu^{a_n} (p_n) \rangle = - \text{i} (\delta^a_2 \bar{A}_\nu^b (p+q) \bar{A}_\mu^{a_1} (p_1) \ldots \bar{A}_\mu^{a_n} (p_n))
\end{equation}

\begin{equation}
- \sum_{i=1}^n \text{i} \langle \bar{A}_\nu^b (q) \bar{A}_\mu^{a_1} (p_1) \ldots \delta^a_2 \bar{A}_\mu^{a_n} (p+i) \ldots \bar{A}_\mu^{a_n} (p_n) \rangle
\end{equation}

Multiplying then both sides by $-\text{i} q^\nu$ and using the axial WI (E.22) once again we get

\begin{equation}
- \text{i} q^\nu \langle \bar{A}_\mu^a (p) \bar{A}_\nu^b (q) \bar{A}_\mu^{a_1} (p_1) \ldots \bar{A}_\mu^{a_n} (p_n) \rangle
\end{equation}

\begin{equation}
= q^\nu F^{abc} \langle \bar{V}_\nu^c (p+q) \bar{A}_\mu^{a_1} (p_1) \ldots \bar{A}_\mu^{a_n} (p_n) \rangle
\end{equation}

\begin{equation}
+ \sum_{i \neq j; i,j=1}^n F^{a_1 c} F^{b a_i d} \langle \bar{A}_\mu^{a_1} (p_1) \ldots \bar{V}_\mu^d (p_1 + q) \ldots \bar{V}_\mu^c (p_1 + p_j) \ldots \bar{A}_\mu^{a_n} (p_n) \rangle
\end{equation}

\begin{equation}
+ \sum_{i=1}^n F^{a_1 c} F^{bcd} \langle \bar{X}_\mu^{a_1} (p_1) \ldots \bar{A}_\mu^d (p+q+p_i) \ldots \bar{A}_\mu^{a_n} (p_n) \rangle.
\end{equation}

The left hand side of (E.34) is symmetric with respect to the interchange of $(p, a) \leftrightarrow (q, b)$; its right hand side can be therefore rewritten in the manifest symmetric form

\begin{equation}
- \text{i} p^\mu q^\nu \langle \bar{A}_\mu^a (p) \bar{A}_\nu^b (q) \bar{A}_\mu^{a_1} (p_1) \ldots \bar{A}_\mu^{a_n} (p_n) \rangle
\end{equation}

\begin{equation}
= \frac{1}{2} (p-q)^\nu F^{abc} \langle \bar{V}_\nu^c (p+q) \bar{A}_\mu^{a_1} (p_1) \ldots \bar{A}_\mu^{a_n} (p_n) \rangle
\end{equation}

\begin{equation}
+ \sum_{i \neq j; i,j=1}^n F^{a_1 c} F^{b a_i d} \langle \bar{A}_\mu^{a_1} (p_1) \ldots \bar{V}_\mu^d (p_1 + q) \ldots \bar{V}_\mu^c (p_1 + p_j) \ldots \bar{A}_\mu^{a_n} (p_n) \rangle
\end{equation}

\begin{equation}
+ \frac{1}{2} \sum_{i=1}^n \left( F^{a_1 c} f^{bcd} + F^{b a_i c} f^{cad} \right) \langle \bar{A}_\mu^{a_1} (p_1) \ldots \bar{A}_\mu^d (p+q+p_i) \ldots \bar{A}_\mu^{a_n} (p_n) \rangle.
\end{equation}

On the other hand, the LSZ formula gives for $p^2, q^2 \to 0$

\begin{equation}
\langle \bar{A}_\mu^a (p) \bar{A}_\nu^b (q) \bar{A}_\mu^{a_1} (p_1) \ldots \bar{A}_\mu^{a_n} (p_n) \rangle = \sum_{c,d} \frac{i}{p^2} \langle 0 | A_\mu^a | \pi^c (p) \rangle \frac{i}{q^2} \langle 0 | A_\nu^b | \pi^d (q) \rangle
\end{equation}

\begin{equation}
\times \langle \pi^c (p) \pi^d (q) | \bar{A}_\mu^{a_1} (p_1) \ldots \bar{A}_\mu^{a_n} (p_n) + R^{ab} \ldots \rangle
\end{equation}

where the regular remnant satisfies

\begin{equation}
\lim_{p^2,q^2 \to 0} p^2 q^2 R^{ab} \ldots = 0.
\end{equation}

Therefore, using (E.5) we get

\begin{equation}
- \text{i} p^\mu q^\nu \langle \bar{A}_\mu^a (p) \bar{A}_\nu^b (q) \bar{A}_\mu^{a_1} (p_1) \ldots \bar{A}_\mu^{a_n} (p_n) \rangle
\end{equation}

\begin{equation}
= F^2 \langle \pi^a (p) \pi^b (q) | \bar{A}_\mu^{a_1} (p_1) \ldots \bar{A}_\mu^{a_n} (p_n) \rangle - p^\mu q^\nu R^{ab} \ldots
\end{equation}
On the other hand applying the LSZ reduction to (E.34, E.35) (let us note that only the first terms on the right hand side has the appropriate poles at $p^2, q^2 \to 0$) we get

$$\langle \pi^a(p) | \pi^b(q) \rangle = q^\nu F^{abc} \langle \bar{V}_c(p+q) | \pi^a(p_1) \ldots \pi^d(p_i) \ldots \pi^a(p_n) \rangle$$

and as a consequence of LSZ reduction of (E.38)

$$= -\frac{1}{2} F^{abc}(p-q) \mu \langle \bar{V}_c(p+q) | \pi^a(p_1) \ldots \pi^d(p_i) \ldots \pi^a(p_n) \rangle$$

According to (E.21) we have for $p, q \to 0$

$$\langle \pi^a(p) | \pi^b(q) \rangle = \frac{1}{2} \sum_{i=1}^n F^{abc} f_X^{ca,d} \frac{2p_i - p - q}{2(p+q) \cdot p_i} \langle \pi^a(p_1) \ldots \pi^d(p_i) \ldots \pi^a(p_n) \rangle + O(p - q)$$

$$+ O\left( p - q, \frac{p^2 - q^2}{p_i \cdot (p + q)} \right)$$

For $p^2 = q^2 = 0$ we finally get

$$\langle \pi^a(0) | \pi^b(0) \rangle = \frac{1}{2} \sum_{i=1}^n F^{abc} f_X^{ca,d} \frac{p_i \cdot (p - q)}{p_i \cdot (p + q)} \langle \pi^a(p_1) \ldots \pi^d(p_i) \ldots \pi^a(p_n) \rangle$$

Provided condition stronger than (E.37) holds, namely $\lim_{p, q \to 0} p^\mu q^\nu F^{\mu \nu}_{\pi^a(p) \pi^b(q)} = 0$ (cf. (E.28)), we get as a result

$$\lim_{t \to 0} F^2 \langle \pi^a(tp) \pi^b(tq) | \pi^a(p_1) \ldots \pi^d(p_i) \ldots \pi^a(p_n) \rangle$$

For the chiral nonlinear sigma model corresponding to the symmetry breaking $G \times G \to G$, we have $F^{abc} = f_X^{abc} = f_T^{abc}$ and we get the formula (5.3) as a special case.
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