Global Behavior of Spherically Symmetric Navier-Stokes Equations with Degenerate Viscosity Coefficients

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Abstract

In this paper, we study a free boundary problem for compressible spherically symmetric Navier-Stokes equations with a gravitational force and degenerate viscosity coefficients. Under certain assumptions that imposed on the initial data, we obtain the global existence and uniqueness of the weak solution and give some uniform bounds (with respect to time) of the solution. Moreover, we obtain some stabilization rate estimates in $L^{\infty}$-norm and weighted $H^1$-norm of the solution. The results show that such system is stable under the small perturbations, and could be applied to the astrophysics.

Keywords: Compressible Navier-Stokes equations; density-dependent viscosity; free boundary; asymptotic behavior

AMS subject classifications: Primary: 35Q35; Secondary: 35D05, 76N10

1 Introduction.

We consider the compressible Navier-Stokes equations with density-dependent viscosity in $\mathbb{R}^n (n \geq 2)$, which can be written in Eulerian coordinates as

$$
\begin{cases}
\partial_\tau \rho + \nabla \cdot (\rho \vec{u}) = 0, \\
\partial_\tau (\rho \vec{u}) + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) + \nabla P = \text{div}(\mu(\nabla \vec{u} + \nabla \vec{u}^T)) + \nabla(\lambda \text{div} \vec{u}) - \rho \vec{f},
\end{cases}
$$

in a domain $\{(\vec{\xi}, \tau) | \vec{\xi} \in \Omega_\tau \subset \mathbb{R}^n, \tau > 0\}$, the initial conditions and boundary conditions are

$$
\begin{align*}
(\rho, \vec{u})(\vec{\xi}, 0) &= (\rho_0, u_0)(\vec{\xi}), \quad \vec{\xi} \in \Omega_0 = \{\vec{\xi} \in \mathbb{R}^n | a < |\vec{\xi}| < b\}, \\
\vec{u}|_{|\vec{\xi}|=a} &= 0, \quad \rho|_{\vec{\xi} \in \partial \Omega_\tau \setminus \{|\vec{\xi}|=a\}} = 0,
\end{align*}
$$

where $\Omega_\tau = \psi(\Omega_0, \tau)$ and $\psi$ is the flow of $\vec{u}$:

$$
\begin{cases}
\partial_\tau \psi(\vec{\xi}, \tau) = \vec{u}(\psi(\vec{\xi}, \tau), \tau), \quad \vec{\xi} \in \mathbb{R}^n, \\
\psi(\vec{\xi}, 0) = \vec{\xi}.
\end{cases}
$$

Here $\rho$, $P$, $\vec{u} = (u_1, \ldots, u_n)$ and $\vec{f}$ are the density, pressure, velocity and external force, respectively; $\lambda = \lambda(\rho)$ and $\mu = \mu(\rho)$ are the viscosity coefficients.

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*This work is supported by NSFC 10571158, Zhejiang Provincial NSF of China (Y605076) and China Postdoctoral Science Foundation 20060400335
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For the initial-boundary value problem \((1.1)-(1.3)\) with the spherically symmetric initial data and external force

\[
(\rho, \vec{u})(\xi, 0) = (\rho_0(r), u_0(r)\frac{\xi}{r}), \quad \xi \in \Omega_0,
\]
\[
\vec{f} = f(m, r, \tau)\frac{\xi}{r}, \quad m(\rho, r) = \int_a^r \rho(s, \tau)s^{n-1}ds, \quad \xi \in \Omega_\tau,
\]
where \(r = |\xi| = \sqrt{\xi_1^2 + \cdots + \xi_n^2}\), we are looking for spherically symmetric solutions \((\rho, \vec{u})\):

\[
\rho(\xi, \tau) = \rho(r, \tau), \quad \vec{u}(\xi, \tau) = u(r, \tau)\frac{\xi}{r}, \quad \xi \in \Omega_\tau,
\]
where \(\Omega_\tau = \{\xi \in \mathbb{R}^n \mid a < |\xi| < b(\tau), b(0) = b, b'(\tau) = u(b(\tau), \tau)\}\). Then \((\rho, u)(r, \tau)\) is determined by

\[
\begin{aligned}
\partial_t \rho + \partial_r (\rho u) + \frac{n-1}{r} \rho u &= 0, \\
\rho(\partial_t u + u \partial_r u) + \partial_r P &= \partial_r [\lambda + 2\mu(\partial_r u + \frac{n-1}{r} u)] - 2(n-1)\partial_r \mu \frac{u}{r} - \rho f,
\end{aligned}
\]
(1.5)
where \((r, \tau) \in (a, b(\tau)) \times (0, \infty)\), with the initial data

\[
(\rho, u)|_{\tau=0} = (\rho_0, u_0)(r), \quad a \leq r \leq b,
\]
(1.6)
the boundary conditions

\[
u|_{r=a} = 0, \quad \rho|_{r=b(\tau)} = 0,
\]
(1.7)
where \(b(0) = b, b'(\tau) = u(b(\tau), \tau), \tau > 0\).

To simplify the presentation, we only consider the famous polytropic model, i.e. \(P(\rho) = A\rho^\gamma\) with \(\gamma > 1\) and \(A > 0\) being constants. And we assume that the viscosity coefficients \(\mu\) and \(\lambda\) are proportional to \(\rho^\theta\), i.e.

\[
\mu(\rho) = c_1 \rho^\theta \text{ and } \lambda(\rho) = c_2 \rho^\theta \text{ where } c_1, c_2 \text{ and } \theta \text{ are three constants.}
\]

Additionally, we assume the external force \(f(m, r, \tau)\) satisfies

\[
f(m, r, \tau) = f_\infty(m, r) + \Delta f(m, r, \tau),
\]
(1.8)
for all \(m \geq 0, r \geq a\) and \(\tau \geq 0\), with

\[
f_\infty(m, r) = G \frac{M_0 + m}{r^{n-1}}, \quad \Delta f(m, r, \tau) \in C^1(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+)
\]
(1.9)

\[
\|\Delta f(\cdot, \cdot, \tau)\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^+)} \leq f_1(\tau), \quad \|\partial_r \Delta f, \partial_\tau \Delta f\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^+)} \leq f_2(\tau),
\]
(1.10)
\[
f_1 \in L^\infty \cap L^1(\mathbb{R}^+), \quad f_2 \in L^2(\mathbb{R}^+),
\]
(1.11)
where \(\mathbb{R}^+ = [0, \infty)\), \(G > 0\) is a constant, \(M_0 \geq 0\) is the total mass of the solid core surrounded by the gas, and the perturbation \(\Delta f\) tends to 0 as \(\tau \to \infty\) in some weak sense. If \(M_0 = 0\), we ignore the gravitational effect of the solid core. \(\Delta f\) expresses the influence of the outside gravitational force, \(f_\infty\) is the precise expression for its own gravitational force and the gravitational force of the solid core, in the astrophysical case (with spherical symmetry). We study the stabilization problem of such system, which could be applied to the astrophysics.

Now, we consider the stationary problem, namely

\[
(P(\rho_\infty))_r = -\rho_\infty f_\infty(m(\rho_\infty, r), r)
\]
(1.12)
in an interval \( r \in (a, l_\infty) \) with the end \( l_\infty \) satisfying
\[
\rho_\infty(l_\infty) = 0, \quad \int_a^{l_\infty} \rho_\infty r^{n-1} dr = M := \int_a^b \rho_0 r^{n-1} dr. \tag{1.13}
\]
The unknown quantities are the stationary density \( \rho_\infty \geq 0 \) and free boundary \( l_\infty > 0 \). If \( \gamma > \frac{2n-2}{n} \), from Proposition 2.5, we know that there exists a unique solution \((\rho_\infty, l_\infty)\) to the stationary system (1.12)-(1.13), satisfying \( \rho_\infty(r) \sim (l_\infty^m - r^n)^{\frac{1}{m-1}}, \) \((\rho_\infty)_r(r) < 0, a < r < l_\infty \) with \( l_\infty < +\infty \).

To handle the free boundary problem (1.5)-(1.7), it is convenient to reduce the problem in Eulerian coordinates \((r, \tau)\) to the problem in Lagrangian coordinates \((x, t)\) moving with the fluids, via the transformation:
\[
x = \int_a^r y^{n-1} \rho(y, \tau) dy, \quad t = \tau.
\]
Then the fixed boundary \( r = a \) and the free boundary \( r = b(\tau) \) become
\[
x = 0 \quad \text{and} \quad x = \int_a^{b(\tau)} y^{n-1} \rho(y, \tau) dy = \int_a^b y^{n-1} \rho_0(y) dy = M,
\]
where \( M \) is the total mass initially. So that the region \( \{(r, \tau) | a \leq r \leq b(\tau), \tau \geq 0\} \) under consideration is transformed into the region \( \{(x, t) | 0 \leq x \leq M, t \geq 0\} \), and the function \( m(\rho, r) \) becomes \( x \). Under the coordinate transformation (1.14), the equations (1.5)-(1.7) are transformed into
\[
\begin{align*}
\partial_t \rho(x, t) & = -\rho^2 \partial_x (r^{n-1} u), \\
\partial_t u(x, t) & = r^{n-1} \left\{ \partial_x [\rho (\lambda + 2\mu) \partial_x (r^{n-1} u) - P] - 2(n-1) \frac{\mu}{r} \partial_x \mu \right\} - f(x, r, t), \tag{1.15}
\end{align*}
\]
where \( (x, t) \in (0, M) \times (0, \infty) \), with the initial data
\[
(\rho, u)|_{t=0} = (\rho_0, u_0)(x), \quad r|_{t=0} = r_0(x) = \left(a^n + n \int_0^x \rho_0^{-1}(y) dy\right)^{\frac{1}{n}}, \tag{1.16}
\]
and the boundary conditions
\[
u(0, t) = 0, \quad \rho|_{x=M} = 0, \quad t > 0. \tag{1.17}
\]

It is standard that if we can solve the problem (1.15)-(1.17), then the free boundary problem (1.1)-(1.3) has a solution.

From (1.12), (1.13), it is easy to see that \( \rho_\infty(x) \) is the solution to the stationary system,
\[
Ar^{n-1}(\rho_\infty')_x = -f_\infty(x, r_\infty), \quad r_\infty^n(x) = a^n + n \int_0^x \rho_\infty^{-1}(y) dy, \quad x \in (0, M), \tag{1.18}
\]
\[
\rho_\infty(M) = 0. \tag{1.19}
\]

The results in [8, 20] show that the compressible Navier-Stokes system with the constant viscosity coefficient have the singularity at the vacuum. Considering the modified Navier-Stokes system in which the viscosity coefficient depends on the density, Liu-Xin-Yang[11] proved that such system is local well-posedness. It is motivated by the physical consideration that in the
derivation of the Navier-Stokes equations from the Boltzmann equation through the Chapman-Enskog expansion to the second order, cf. \cite{6}, the viscosity is a function of the temperature. If we consider the case of isentropic fluids, this dependence is reduced to the dependence on the density function.

Since \( n \geq 2 \) and the viscosity coefficient \( \mu \) depends on \( \rho \), the nonlinear term \( 2(n-1)\frac{1}{r}u\partial_x \mu \) in (1.15) makes the analysis significantly different from the one-dimensional case \cite{11,15,19,21,22}. When \( \mu \geq \mu > 0 \) and \( \rho_0 \geq \rho > 0 \), authors in \cite{11,24} obtained the existence, uniqueness and global behavior of the solution for compressible spherically symmetric Navier-Stokes equations with a external pressure and without the nonlinear term \( 2(n-1)\frac{1}{r}u\partial_x \mu \). Following the ideas in \cite{24}, we can obtain the existence and uniqueness results for the stationary problem in Section 2. In this paper, since viscosity coefficients and density will degenerate at the free boundary, we need to use the weighted function \( (M-x) \) to control the lower bound of the density in Section 3.

Considering the system (1.15)-(1.17) with a general external force, Chen-Zhang obtained the local existence and uniqueness of the solution in \cite{3}. In this paper, when the initial data \( (\rho_0, u_0, r_0) \) are close to the stationary state \( (\rho_\infty, 0, r_\infty) \), we will obtain some appropriate \textit{a priori} estimates and prove that the maximum existence time \( T^* = \infty \). The difficulty of this problem is to obtain the lower bound of the density \( \rho \). The key ideas are using the classical continuity method and the result of \textbf{Claim 1}. In \textbf{Claim 1}, we want to prove that there is a small positive constant \( \epsilon_1 \), such that, for any \( T > 0 \), if

\[
I(t) = \| g(\cdot,t) - g_\infty \|_{L^\infty} \leq 2\epsilon_1, \ \forall \ t \in [0,T],
\]

where \( g(x,t) = (M-x)^{-\frac{1}{r}}\rho(x,t) \) and \( g_\infty(x) = (M-x)^{-\frac{1}{r}}\rho_\infty(x) \), then

\[
I(t) \leq \epsilon_1, \ \forall \ t \in [0,T].
\]

Using the energy method and induction method, we can estimate the weighted \( L^2 \)-norm of \( g - g_\infty \) in Lemma 3.7. In such process (see Lemmas 3.5,3.6), we use the weight function \( (1+t)^\alpha \) (with \( \alpha = -\frac{1}{2} \)) to remedy the disadvantage of the nonlinear term \( 2(n-1)\frac{1}{r}u\partial_x \mu \), and use the induction method to increase \( \alpha \) to \( -\epsilon_2 \). Then, by the reduction to absurdity, we can finish the proof of \textbf{Claim 1} in Lemma 3.8.

Our results show that: such system is stable under the small perturbations, does not develop vacuum states or concentration states for all time, and the interface \( \partial \Omega_\tau \) propagates with finite speed.

The assumptions on \( c_1, c_2, \theta, \gamma \) and initial data can be stated as follows:

\textbf{(A1)} \( \gamma > \frac{2n-2}{n}, \ \theta \in (0, \gamma-1) \cap (0, \frac{\pi}{2}], \) \( c_1 \) and \( c_2 \) satisfy that \( c_1 > 0 \) and \( 2c_1 + nc_2 > 0 \);

\textbf{(A2)} \( N_1(M-x)^{1/\gamma} \leq \rho_0 \leq N_2(M-x)^{1/\gamma} \), with some positive constants \( 0 < N_1 < N_2 \), and

\( (M-x)^{1-\frac{2}{\gamma}}(\rho_0^\gamma)_x \in L^1([0,M]), \ (\rho_0^\gamma)_x \in L^2([0,M]); \)

\textbf{(A3)} \( u_0 \in L^2([0,M]), \ \rho_0^{\theta+1}(u_0)_x \in L^2([0,M]), \ u_0(0) = 0, \)

\[
\left( (2c_1 + c_2)\rho_0^{\theta+1}(r_0^{n-1}u_0)_x \right)_x - 2c_1(n-1)\frac{u_0}{r_0}(\rho_0^\theta)_x \in L^2([0,M]). \tag{1.22}
\]

Under the above assumptions (A1)-(A3), we will prove the existence of the global weak solution to the initial-boundary value problem (1.15)-(1.17) in the sense of the following definition.
**Definition 1.1.** A pair of functions \((\rho, u, r)(x, t)\) is called a global weak solution to the initial boundary value problem \((1.15)-(1.17)\) if for any \(T > 0\),

\[
\rho, u \in L^{\infty}([0, M] \times [0, T]) \cap C^1([0, T]; L^2([0, M])),
\]

\[
r \in C^1([0, T]; L^\infty([0, M])),
\]

\[
\rho^{-1}, (r^{n-2}u)_x, (r^{n-1})_x \in L^\infty([0, T]; L^1([0, M])),
\]

and

\[
\rho^{1+\theta}(r^{n-1}u)_x \in L^\infty([0, M] \times [0, T]) \cap C^{1/2}([0, T]; L^2([0, M])).
\]

Furthermore, the following equations hold:

\[
\rho_t + \rho^2(r^{n-1}u)_x = 0, \quad \rho(x, 0) = \rho_0(x), \quad a.e.
\]

\[
r_t = u, \quad r(x, 0) = r_0(x), \quad r^n(x, t) = a^n + n \int_0^x \rho^{-1}(y, t)dy, \quad a.e.
\]

\[
\int_0^\infty \int_0^M [u\psi_t + (P - \rho(\lambda + 2\mu)(r^{n-1}u)_x)(r^{n-1}\psi)_x + 2(n + 1)\mu(r^{n-2}\psi)_x - f(x, r, t)\psi]dxdt + \int_0^M u_0(x)\psi(x, 0)dx = 0,
\]

for any test function \(\psi(x, t) \in C^0_0(\Omega)\) with \(\Omega = \{(x, t) \mid 0 < x \leq M, \quad t \geq 0\}\), and

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^\epsilon udx = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{1-\epsilon}^1 \rho dx = 0.
\]

In what follows, we always use \(C(C_1)\) to denote a generic positive constant depending only on \(\gamma, \theta, f_1, f_2\) and the initial data, independent of the given time \(T\).

We now state the main theorems in this paper.

**Theorem 1.1.** Under the conditions \((1.8)-(1.11)\) and \([A1-A3]\), there exists a constant \(\epsilon_0 > 0\), such that if

\[
\|f_1\|_{L^\infty} + \int_0^\infty (1 + t)f_1^2(t)dt \leq \epsilon_0^2,
\]

\[
\|u_0\|_{L^2} + \|(M - x)^{-\frac{\theta}{2}}(\rho_0 - \rho_{\infty})\|_{L^\infty} \leq \epsilon_0,
\]

then the system \((1.15)-(1.17)\) has a unique global weak solution \((\rho, u, r)\) satisfying

\[
C^{-1}(M - x)^{\frac{1}{n}} \leq \rho(x, t) \leq C(M - x)^{\frac{1}{n}},
\]

\[
r(x, t) \in [a, C],
\]

\[
\int_0^M (M - x)^{1-\frac{\theta}{2}}(\rho^\theta - \rho_{\infty}^\theta)^2dx \leq C,
\]

and

\[
\|u(\cdot, t)\|_{L^\infty} + \|(\rho(r^{n-1}u)_x)(\cdot, t)\|_{L^\infty} \leq C,
\]

for all \(t \geq 0\) and \(x \in [0, M]\). Furthermore, if \((1 + t)\frac{2(\gamma + \theta)}{n + \gamma + \theta}f_1^2(t) \in L^1(\mathbb{R}_+),\) for any \(\eta > 0\), we have

\[
\int_0^M \left\{u^2 + (M - x)^{\frac{n-1}{2}}(g - g_{\infty})^2 + (r - r_{\infty})^2\right\} dx \leq C_\eta (1 + t)^{-\frac{2(\gamma + \theta)}{n + \gamma + \theta} + \eta},
\]
\[
\int_0^M \left( \rho^{\theta-1}u^2 + \rho^{\theta+1}u_x^2 \right) dx \leq C_\eta (1 + t)^{-\frac{2(\eta+\theta)}{3(\eta+\theta+1)}} \eta, \tag{1.30}
\]
\[
\int_0^M (M-x)^{2-\frac{2\theta}{\gamma}(\rho^\theta - \rho_{\infty}^\theta)^2} dx \leq C_\eta (1 + t)^{-\frac{\eta+\theta}{\eta+\theta+1}+\eta}, \tag{1.31}
\]
and
\[
\|\rho(\cdot,t) - \rho_{\infty}^2(\cdot)\|_{L^\infty} \leq C_\eta (1 + t)^{-\frac{3\eta+3\theta-1}{4(\eta+\theta+1)} + \frac{\eta}{2}}, \quad \|\rho_{\infty}^2(\cdot) - \rho_{\infty}^2(\cdot)\|_{L^\infty} \leq C_\eta (1 + t)^{-\frac{\eta+\theta}{\eta+\theta+1} + \frac{\eta}{2}}, \tag{1.32}
\]
\[
\|u(\cdot,t)\|_{L^\infty} \leq C_\eta (1 + t)^{-\frac{\eta+\theta}{\eta+\theta+1} + \frac{\eta}{2}}, \tag{1.33}
\]
for all \( t \geq 0 \), where \( C_\eta \) is a positive constant depending on \( \eta \).

Remark 1.1. The uniqueness of the solution in Theorems 1.1 or 3.1 means that: if \((\rho_1, u_1, r_1)\) and \((\rho_2, u_2, r_2)\) are two solutions to the system (1.15) - (1.17) with the same initial data \( (\rho_0, u_0, r_0) \), and satisfy regularity conditions in the theorem, then we have \((\rho_1, u_1, r_1) = (\rho_2, u_2, r_2)\).

Remark 1.2. In particular, the viscosity of the gas is proportional to the square root of the temperature for the hard sphere model (as pointed out in [15, 21]), and the relation between \( \theta \) and \( \gamma \) is
\[
\theta = \frac{\gamma - 1}{2}.
\]

Our condition (A1) covers it. Since the Navier-Stokes system with constant viscosity coefficients has the singularity at the vacuum [8, 20], we assume \( \theta > 0 \) in (A1).

Remark 1.3. There are no smallness assumptions on \( \| (M-x)^{1-\frac{\theta}{\gamma}}(\rho_0^\theta)_x \|_{L^1} \) and \( \| \rho_0^{\frac{\eta}{\gamma}} (u_0)_x \|_{L^2} \). Moreover, \( \epsilon_0 \) do not depend on \( \| (M-x)^{1-\frac{\theta}{\gamma}}(\rho_0^\theta)_x \|_{L^1} \) and \( \| \rho_0^{\frac{\eta}{\gamma}} (u_0)_x \|_{L^2} \).

Remark 1.4. Considering the no vacuum system in an exterior domain in \( \mathbb{R}^3 \), Kobayashi-Shibata [3] obtained \( \| (\rho - \rho_{\infty}, u)(\cdot,t)\|_{L^2} \lesssim (1 + t)^{-\frac{\gamma}{2}} \) and \( \| (\rho - \rho_{\infty}, u)(\cdot,t)\|_{L^\infty} \lesssim (1 + t)^{-\frac{\gamma}{2}} \) when \( \rho_{\infty} \) is a positive constant. Considering the no vacuum system in \( \mathbb{R}^n \) \((n \geq 3)\), Ukai-Yang-Zhao [18] obtained \( \| (\rho - \rho_{\infty}, u)(\cdot,t)\|_{L^2 \cap L^\infty} \leq C(1 + t)^{-\frac{\gamma}{4} + \epsilon} \) when \( \rho_{\infty} \) is close to a positive constant. Considering the one dimensional system with a degenerate viscosity coefficient, we [22] obtained \( \| (\rho(\cdot,t) - \rho_{\infty}^2)(\cdot,t)\|_{L^\infty} \leq C(1 + t)^{-\frac{\gamma}{2}} \) when the external force \( f_{\infty} \) is close to a positive constant \( N_0 \) and the stationary density \( \rho_{\infty} \) is close to \( \left( \frac{N_0(M-x)}{A} \right)^{\frac{1}{\gamma}} \). Since \( \frac{\gamma+\theta}{\gamma+\theta+1} > \frac{1}{2} \) and \( \frac{3\gamma+3\theta-1}{4(\gamma+\theta+1)} > \frac{1}{2} \) (if \( \gamma + \theta > 3 \)), it is easy to see that our results in Theorem 1.1 are better than the results in [22]. Using similar arguments as that in Theorem 1.1, we can also obtain similar results in one-dimensional cases which are better than the results in [22]. For example, the stabilization rate estimate \( \| (\rho(\cdot,t) - \rho_{\infty}^2)(\cdot,t)\|_{L^\infty} \leq C_\eta (1 + t)^{-\frac{\gamma+\theta}{\gamma+\theta+1} + \frac{\eta}{2}} \) in [22] can be replaced by \( \| (\rho(\cdot,t) - \rho_{\infty}^2)(\cdot,t)\|_{L^\infty} \leq C_\eta (1 + t)^{-\frac{\gamma+\theta}{\gamma+\theta+1} + \frac{\eta}{2}} \), \( t \geq 0 \), for any \( \eta > 0 \).

Remark 1.5. Since that the information of the dimension \( n \) mainly appear in the index of the radii \( r \), and we only consider the system with a solid core \( r \geq a > 0 \) in this paper, our results can not show the effect of the dimension. In [23], we studied the global behavior of the solution to the similar problem with a positive external pressure and without a solid core, and obtained the stabilization rate estimates for the solution of exponential type. The admissible range of the parameter \( \frac{\lambda}{\gamma} \) depends on the dimension \( n \) in [23]. We will study the system without a solid core \( r \geq 0 \) and with degenerate viscosity coefficients in the future, and guess that stabilization rate estimates of the solution can not better than the results in Theorem 1.1.
Theorem 1.2. (Continuous Dependence) For each \( i = 1, 2 \), let \((\rho_i, u_i, r_i)\) be the solution to the system (1.15)-(1.17) with the initial data \((\rho_0, u_0, r_0)\), which satisfies regularity conditions in Theorem 1.1. Then, we have

\[
\int_0^M \left( (u_1 - u_2)^2 + \rho_1^{1-\theta} \rho_2^{2\theta-4} (\rho_1 - \rho_2)^2 + \rho_1^{\theta-1} (r_1 - r_2)^2 \right) \, dx \\
\leq Ce^{Ct} \int_0^M \left( (u_01 - u_02)^2 + \rho_01^{1-\theta} \rho_02^{2\theta-4} (\rho_01 - \rho_02)^2 + \rho_01^{\theta-1} (r_01 - r_02)^2 \right) \, dx,
\]

for all \( t \geq 0 \).

Remark 1.6. Using similar arguments as that in [3], we can easily obtain such continuous dependence of the solution on the initial data, and omit the detail.

Remark 1.7. If we ignore the influence of self-gravitation, i.e. assume \( f_\infty(m, r) = G M_0 r^{n-1} \) with \( M_0 > 0 \), then we can also obtain the same results in Theorem 1.1-1.2.

We now briefly review the previous works in this direction. For the related free boundary problem of one-dimensional isentropic fluids with density-dependent viscosity (like \( \mu(\rho) = c \rho^\theta \)), see [11, 15, 19, 21, 22] and the references therein. For the related stabilization rate estimates of 1-D free boundary problem, see [5, 12, 16, 22] etc.. For the spherically symmetric solutions of the Navier-Stokes equations with a free boundary, see [2, 4, 13, 14, 23, 24] etc.. Also see Bresch-Desjardins[1], Lions[10] and Vaigant-Kazhikhov[17] for multidimensional isentropic fluids.

The rest of this paper is organized as follows. First, we obtain the existence and uniqueness of the solution to the stationary problem in Section 2. In Section 3, we will prove some \textit{a priori} estimates, and extend the local solution in [3] to the global solution in time. In Section 4, we obtain the stabilization rate estimates of the solution.

2 The stationary problem

Zlotnik-Ducomet\[24\] obtained the existence of the positive solution to the stationary problem with a positive external pressure. Using similar arguments as that in \[24\], we can obtain the following results for the stationary problem without an external pressure. We start with a proof of the existence of a non-negative solution to the Lagrangian stationary problem.

Proposition 2.1. If

\[
\gamma > \frac{2n - 2}{n} \quad (2.1)
\]
or

\[
\gamma = \frac{2n - 2}{n} \quad \text{and} \quad \left( \frac{n \gamma}{\gamma - 1} M^{\gamma - 1} \right)^{\frac{2n - 2}{n}} < \frac{G}{A} \left( \frac{M}{2} + M_0 \right), \quad (2.2)
\]
or

\[
n > 2, \ 1 < \gamma < \frac{2n - 2}{n} \quad \text{and} \quad \delta_6^n \left( a^n + \frac{n \gamma}{\delta_6 (\gamma - 1)} M^{\frac{\gamma - 1}{\gamma}} \right)^{\frac{2n - 2}{n}} \leq \frac{G}{A} \left( \frac{M}{2} + M_0 \right), \quad (2.3)
\]

where \( \delta_6 = a^n (1 - \frac{n}{2n - 2})^{\frac{2n - 2}{\gamma - 1}} M^{\frac{\gamma - 1}{\gamma}} \), then the Lagrangian stationary problem (1.18)-(1.19) has a non-negative solution \( \rho_\infty \in W^{1, \beta}([0, M]) \) satisfying \( C^{-1} (M - x)^{\frac{1}{\gamma}} \leq \rho_\infty(x) \leq C (M - x)^{\frac{1}{\gamma}} \), where \( \beta \in [1, \frac{1}{\gamma - 1}] \) is a constant.
Proof. We introduce the nonlinear operator
\[ I : K \rightarrow W^{1,\beta}([0, M]), \]
where \( K = \{ f \in C([0, M]) \mid f \geq 0, \| (M-x)^{\frac{1}{\beta}} f \|_{L^\infty} < \infty, \| \frac{f(x)}{(M-x)^{\frac{1}{\beta}}} \|_{L^\infty} < \infty \}, \) by setting
\[ I(f)(x) = \left( \frac{\int_x^M G(M_0+y)dy}{A} \right)^{\frac{1}{\beta}} \]
with \( r_n^\beta(x) = a^n + n \int_0^x f^{-1}(y)dy, \) \( x \in [0, M]. \)

We can restate the problem (1.18)-(1.19) as the fixed-point problem
\[ \rho_\infty = I(\rho_\infty). \quad (2.4) \]

For all \( f \in K_{\delta_1, \delta_2} = \{ f \in K \mid \delta_1 (M-x)^{\frac{1}{\beta}} \leq f(x) \leq \delta_2 (M-x)^{\frac{1}{\beta}} \} \) with \( 0 < \delta_1 \leq \delta_2 < \infty, \) we have
\[ a^n \leq r_n^\beta(x) \leq a^n + \frac{n \gamma}{\delta_1 (\gamma - 1)} M^{\frac{\gamma - 1}{\gamma}} := B^n \]
and
\[ \left( \frac{G(M + M_0)}{AB^{2n-2}} \right)^{\frac{1}{\gamma}} (M-x)^{\frac{1}{\gamma}} \leq I(f)(x) \leq \left( \frac{G(M + M_0)}{Aa^{2n-2}} \right)^{\frac{1}{\gamma}} (M-x)^{\frac{1}{\gamma}}, \]
\( x \in [0, M]. \)

If \( \gamma > \frac{2n-2}{n}, \) then \( I(K_{\delta_1, \delta_2}) \subset K_{\delta_3, \delta_4}, \) where \( \delta_4 = \left( \frac{G(M+M_0)}{Aa^{2n-2}} \right)^{\frac{1}{\gamma}} \) and \( \delta_3 \) is a positive constant satisfying \( \delta_3 \left( a^n + \frac{n \gamma}{\delta_4 (\gamma - 1)} M^{\frac{\gamma - 1}{\gamma}} \right)^{\frac{2n-2}{n}} \leq \frac{G(M_0 + M_0)}{A} \) and one can immediately verify that \( I \) is a compact operator on \( K_{\delta_1, \delta_2}. \) Since \( K_{\delta_3, \delta_4} \) is a convex closed bounded non-empty subset of \( C([0, M]), \) the problem (2.4) has a solution \( \rho \in K_{\delta_3, \delta_4} \) by Schauder’s fixed point theorem.

Similarly, if \( \gamma = \frac{2n-2}{n} \) and \( \left( \frac{n \gamma}{\delta_4 (\gamma - 1)} M^{\frac{\gamma - 1}{\gamma}} \right)^{\frac{2n-2}{n}} < \frac{G(M_0 + M_0)}{A}, \) then \( I(K_{\delta_3, \delta_4}) \subset K_{\delta_5, \delta_4}, \) where \( \delta_5 = a^n \left[ \left( \frac{G(M_0 + M_0)}{Aa^{2n-2}} \right)^{\frac{n-1}{\gamma}} - \frac{n \gamma}{\delta_4 (\gamma - 1)} M^{\frac{\gamma - 1}{\gamma}} \right], \) and problem (2.4) has a solution \( \rho \in K_{\delta_5, \delta_4}. \)

Similarly, if \( n > 2, 1 < \gamma < \frac{2n-2}{n} \) and \( \delta_6 \left( a^n + \frac{n \gamma}{\delta_5 (\gamma - 1)} M^{\frac{\gamma - 1}{\gamma}} \right)^{\frac{2n-2}{n}} \leq \frac{G(M_0 + M_0)}{A}, \) then \( I(K_{\delta_5, \delta_4}) \subset K_{\delta_6, \delta_4}, \) and problem (2.4) has a solution \( \rho \in K_{\delta_6, \delta_4}. \) \( \Box \)

Similar to [24], We say a stationary solution \( (\rho_\infty, r_n^\beta) \) is statically stable if
\[ J[W] := \int_0^M (\gamma A \rho_\infty^{1+\gamma} W_x^2 - (2n-2) G(M_0 + x) r_\infty^{2-3n} W^2) \ dx \geq \delta_7 \int_0^M \left( (M-x)^{\frac{1+\gamma}{\gamma}} W_x^2 + W^2 \right) \ dx, \quad (2.5) \]
for some \( \delta_7 > 0 \) and all
\[ W \in K_1 = \left\{ f \in C([0, M]) \mid f(0) = 0, \| (M-x)^{\frac{1}{\beta}} f'(x) \|_{L^\infty} < \infty, \left. \frac{1}{(M-x)^{\frac{1}{\beta}} f'(x)} \right|_{L^\infty} < \infty \right\}. \]

Now, the static potential energy takes the following form:
\[ S[V] = \int_0^M \left( \frac{A}{\gamma - 1} (V_x)^{1-\gamma} + \int_{a_n}^V G(M_0 + x)(nh)^{\frac{2-2n}{n}} \ dx \right) \ dx. \quad (2.6) \]
We call $V \in K_2 = \left\{ f \in C([0, M]) \mid f(0) = \frac{a^n}{n!}, \| (M - x)^\frac{1}{2} f'(x) \|_{L_{\infty}} < \infty, \left\| \frac{1}{(M - x)^\frac{1}{2}} f''(x) \right\|_{L_{\infty}} < \infty \right\}$ is a point of local quadratic minimum of $S$ if

$$S[V + W] - S[V] \geq \delta_8 \int_0^M \left( (M - x)^{\frac{1+\gamma}{2}} W_2^2 + W^2 \right) \, dx,$$

for all $W \in K_1$ and $\| (M - x)^{\frac{1}{2}} W_x \|_{L_{\infty}([0, M])} + \| W \|_{L_{\infty}} \leq \delta_9$, for some $\delta_8 > 0$ and $\delta_9 > 0$.

**Proposition 2.2.** If $\gamma > \frac{2n-2}{n}$ and $\rho_\infty$ is a solution of the problem (1.18)-(1.19) satisfying $\rho_\infty \in W^{1,\beta}([0, M])$ and $C^{-1}(M - x)^{\frac{1}{2}} \leq \rho_\infty(x) \leq C(M - x)^{\frac{1}{2}}$, then we have (2.5) and (2.7) hold with $V = V_\infty = \frac{r_\infty^2}{n}$.

**Proof.** From $r_\infty \geq a$, $(A\rho_\infty^2)_x = -G \frac{M_0 + x}{r_\infty^2}$ and $(r_\infty^2)_x = n \rho_\infty^{-1}$, using integration by parts, we have

$$J[W] = \int_0^M \left( \gamma A\rho_\infty^{1+\gamma} W_2^2 + (2n - 2) A(\rho_\infty^2)_x r_\infty^{-n} W^2 \right) \, dx$$

$$= \int_0^M \left( \gamma A\rho_\infty^{1+\gamma} W_2^2 + 2(2n - 2) A\rho_\infty^2 r_\infty^{-n} W W_x \right.$$

$$+ n(2n - 2) A\rho_\infty^{-1} r_\infty^{-2n} W^2 \big) \, dx,$$

for all $W \in K_1$.

If $\gamma > \frac{2n-2}{n}$, we have

$$J[W] \geq C^{-1} \int_0^M \left( (M - x)^{\frac{1+\gamma}{2}} W_2^2 + (M - x)^{\frac{n-1}{2}} W^2 \right) \, dx.$$

From $r_\infty \geq a$ and $(A\rho_\infty^2)_x = -G \frac{M_0 + x}{r_\infty^2}$, using integrating by parts and the Cauchy-Schwarz inequality, we have

$$\int_0^M W^2 \, dx \leq C \int_0^M (M - x)^{1+\frac{\gamma}{2}} W_2^2 \, dx + C \int_0^M G \frac{M_0 + x}{r_\infty^{2n-2}} W^2 \, dx$$

$$= C \int_0^M (M - x)^{1+\frac{\gamma}{2}} W^2 \, dx - C \int_0^M A(\rho_\infty^2)_x W^2 \, dx$$

$$\leq C \int_0^M \left( (M - x)^{\frac{1+\gamma}{2}} W_2^2 + (M - x)^{\frac{n-1}{2}} W^2 \right) \, dx,$$

then, we have (2.5) immediately.

Similarly, we obtain

$$S[V_\infty + W] - S[V_\infty]$$

$$= \frac{1}{2} \int_0^M \left\{ A[\gamma + O(|(M - x)^{\frac{1}{4}} W_x|)] \rho_\infty^{1+\gamma} W_2^2 - [2n - 2 + O(|W|)] G(M_0 + x) r_\infty^{2-3n} W^2 \right\} \, dx$$

$$= \frac{1}{2} \int_0^M \left\{ A[\gamma + O(|(M - x)^{\frac{1}{4}} W_x|)] \rho_\infty^{1+\gamma} W_2^2$$

$$- 2A[2n - 2 + O(|W|)] \rho_\infty^{-1} r_\infty^{-2n} WW_x + nA[2n - 2 + O(|W|)] \rho_\infty^{-1} r_\infty^{-2n} W^2 \right\} \, dx,$$

for all $W \in K_1$. Here, $O(d)$ means $O(d) \to 0$ as $d \to 0$. If $\gamma > \frac{2n-2}{n}$, choosing $\delta_9$ small enough, we can obtain (2.7) immediately.
Proof. From (1.18)-(1.19), we have

\[
\gamma > \frac{2n-2}{n}
\]

then we have

\[
\| (M - x)^{-\frac{1}{n}} (\rho_{\infty}(x) - \rho_2(x)) \|_{L^\infty} \leq \delta_{10}
\]

with a small enough positive constant \( \delta_{10} \), then we have \( \rho_{\infty}(x) = \rho_2(x) \), a.e. \( x \in [0, M] \).

Proposition 2.3. Let \( \rho_{\infty} \) be a solution obtained in Proposition 2.1, and \( \rho_2 \) be another solution of the problem (1.18)-(1.19) satisfying \( \rho_2 \in W^{1,\beta}([0, M]) \) and \( C^{-1}(M-x)^{\frac{1}{n}} \leq \rho_2(x) \leq C(M-x)^{\frac{1}{n}} \). If \( \gamma > \frac{2n-2}{n} \) and \( \| (M - x)^{-\frac{1}{n}} (\rho_{\infty}(x) - \rho_2(x)) \|_{L^\infty} \leq \delta_{10} \) with a small enough positive constant \( \delta_{10} \), then we have \( \rho_{\infty}(x) = \rho_2(x) \), a.e. \( x \in [0, M] \).

Proof. From (1.18)-(1.19), we have

\[
A\rho_{\infty} = \int_x^M G\frac{M_0 + y}{r_{2n-2}} dy, \quad r_{\infty}(x) = a^n + n \int_0^x \rho_{\infty}^{-1}(y) dy,
\]

\[
A\rho_2 = \int_x^M G\frac{M_0 + y}{r_2} dy, \quad r_2(x) = a^n + n \int_0^x \rho_2^{-1}(y) dy,
\]

and

\[
A(\rho_{\infty} - \rho_2) = \int_x^M \left( G(M_0 + y) \left( r_{\infty}^{2n-2} - r_2^{2n-2} \right) \right) dy.
\]

Multiplying the above equality by \( (\rho_{\infty}^{-1} - \rho_2^{-1}) \), integrating over \([0, M]\), and using the fact that

\[
\int_0^M n(\rho_{\infty}^{-1} - \rho_2^{-1})(\rho_{\infty}^{-1} - \rho_2^{-1}) \int_x^M g(y) dy dx = \int_0^M g(y) dy dx - \int_0^M (r_{\infty}^{2n} - r_2^{2n}) dx,
\]

we obtain

\[
0 = \int_0^M \left\{ A(\rho_{\infty} - \rho_2)(\rho_{\infty}^{-1} - \rho_2^{-1}) - \frac{G(M_0 + x)}{n} \left( r_{\infty}^{2n-2} - r_2^{2n-2} \right) \right\} dx
\]

\[
= \int_0^M \left\{ -A[\gamma + O((M - x)^{-\frac{1}{n}} (\rho_{\infty} - \rho_2))]|\rho_{\infty}^1 + \gamma (\rho_{\infty}^{-1} - \rho_2^{-1})|^2
\]

\[
-\frac{A}{n^2} \left[ 2n - 2 + O(|r_{\infty}^n - r_2^n|) \right] (\rho_{\infty}^{-1} - \rho_2^{-1})^2 \right\} dx
\]

\[
= -\int_0^M \left\{ A[\gamma + O((M - x)^{-\frac{1}{n}} (\rho_{\infty} - \rho_2))]|\rho_{\infty}^1 + \gamma (\rho_{\infty}^{-1} - \rho_2^{-1})|^2
\]

\[
-\frac{2A}{n} \left[ 2n - 2 + O(|r_{\infty}^n - r_2^n|) \right] \rho_{\infty}^{-1} r_{\infty}^{-n} (r_{\infty}^n - r_2^n) (\rho_{\infty}^{-1} - \rho_2^{-1})
\]

\[
+\frac{A}{n} \left[ 2n - 2 + O(|r_{\infty}^n - r_2^n|) \right] \rho_{\infty}^{-1} r_{\infty}^{-2n} (r_{\infty}^n - r_2^n)^2 \right\} dx
\]

\[
\leq -C^{-1} \int_0^M \left( (M - x)^{\frac{1}{n}} (\rho_{\infty}^{-1} - \rho_2^{-1})^2 + (M - x)^{\frac{1}{n}} (r_{\infty}^n - r_2^n)^2 \right) dx,
\]

when \( \gamma > \frac{2n-2}{n} \) and \( \delta_{10} \) is small enough. Thus, we can obtain \( \rho_{\infty} = \rho_2 \) immediately.

Now, we shall use the shooting method to prove the uniqueness of the solution \( \rho_{\infty} \in K \).

Proposition 2.4. Under the assumption (2.1), the Lagrangian stationary problem (1.18)-(1.19) has a unique solution \( \rho_{\infty} \in K \).

Proof. We consider the Cauchy problem

\[
(A\rho) = -G(M_0 + x)(nV)^{\frac{2}{2n-2}}, \quad (V)_x = \rho^{-1}, \quad x \in (0, M),
\]

\[
\rho|_{x=0} = \sigma, \quad V|_{x=0} = \frac{a^n}{n},
\]
for the unknown functions \( \rho(\sigma, x) \) and \( V(\sigma, x) \), where \( \sigma > 0 \) is the shooting parameter. Thus, for each \( \sigma > 0 \), using the classical ODE theory, there exists a unique solution to the problem (2.9)-(2.10) satisfying \( \rho(\sigma, x) > 0 \) for \( x \in [0, M_\sigma) \), where either \( \rho|_{x=M_\sigma} = 0 \) and \( M_\sigma \in (0, M) \) or \( M_\sigma = M \).

Clearly, if \( \rho_\infty \in K \) is a solution to the problem (1.18)-(1.19), then \( \rho_\infty \) satisfies (2.9)-(2.10) for some \( \sigma_0 > 0 \) and \( M_{\sigma_0} = M \). We will show it is possible only for one value of \( \sigma \). Using similar arguments as in [7] (§V.3), we obtain that \( \langle \partial_\sigma \rho^\gamma, \partial_\sigma V \rangle \) is well defined and satisfies the linear Cauchy problem

\[
A(\partial_\sigma \rho^\gamma)_x = (2n-2)G(M_0 + x)(nV)^{\frac{2-\alpha}{n}} \partial_\sigma V, \quad (\partial_\sigma V)_x = -\frac{1}{\gamma} \rho^{-\gamma-1} \partial_\sigma \rho^\gamma, \quad \sigma \in [0, M_\sigma), \quad x \in [0, M_\sigma), \quad (2.11)
\]

\[
\partial_\sigma \rho^\gamma|_{x=0} = 1, \quad \partial_\sigma V|_{x=0} = 0. \tag{2.12}
\]

It is easy to see that

\[
\partial_\sigma \rho^\gamma > 0, \quad (\partial_\sigma V)_x < 0, \quad \partial_\sigma V < 0
\]

hold on \( (0, M_4) \), where either \( \partial_\sigma \rho^\gamma|_{x=M_4} = 0 \) and \( M_4 \in (0, M_\sigma) \) or \( M_4 = M_\sigma \). We claim that only \( M_4 = M_\sigma \) can occur.

Assume that \( M_4 \in (0, M_\sigma) \). Letting \( \phi = A \rho^\gamma (\partial_\sigma V)_x + \frac{n}{2n-2} A \partial_\sigma \rho^\gamma (V)_x \), from (2.9) and (2.11), we have

\[
\int_0^{M_4} \phi dx = \left\{ A \rho^\gamma \partial_\sigma V + \frac{n}{2n-2} A \partial_\sigma \rho^\gamma V \right\} |_{0}^{M_4}.
\]

By the estimates \( \rho(\sigma, M_4) > 0, \quad \partial_\sigma \rho^\gamma|_{x=M_4} = 0, \quad \partial_\sigma V|_{x=M_4} < 0 \) and the initial condition (2.10) and (2.12), we get

\[
\int_0^{M_4} \phi dx < 0.
\]

On the other hand, from (2.9) and (2.11), we have

\[
\phi = A \rho^{-\gamma} \partial_\sigma \rho^\gamma (\frac{n}{2n-2} - \frac{1}{\gamma}) > 0, \quad x \in (0, M_4).
\]

It is a contradiction.

Thus, we obtain

\[
\rho(\sigma, x) > 0, \quad \partial_\sigma (\sigma, x) \rho > 0, \quad x \in (0, M_\sigma),
\]

and \( M_\sigma \) is non-decreasing on \( \sigma \in (0, \infty) \). Therefore, for each fixed point \( x \in [0, M_b) \), the function \( \rho(\sigma, x) \) is strictly increasing on \( \sigma \geq b \).

If there exists \( \sigma_1 \neq \sigma_0 \) such that \( M_{\sigma_1} = M_{\sigma_0} = M \) and \( \rho(\sigma_1, x) \in K \), then there exists \( \min\{\sigma_0, \sigma_1\} < \sigma_2 < \max\{\sigma_0, \sigma_1\} \) such that \( 0 < (M \gamma - \gamma)(\rho(\sigma_2, x) - \rho(\sigma_0, x)) \leq 2 \delta_0 \). From Proposition 2.3, we have \( \rho(\sigma_2, x) = \rho(\sigma_0, x) = \rho_\infty(x) \), which is a contradiction. Thus, we finish the proof of Proposition 2.4.

Using the properties of the transformation (1.14) and Propositions 2.1, 2.4 we can obtain the following proposition immediately.

**Proposition 2.5.** Under the assumption (2.1), the Eulerian stationary problem (1.12)-(1.13) has a unique solution \( (\rho_\infty, l_\infty) \), satisfying \( \rho_\infty(r) \sim (l_\infty - r^n)^{-\frac{1}{n-1}}, \quad (\rho_\infty)_r(r) < 0, \quad a < r < l_\infty \) with \( l_\infty < +\infty \).

**Remark 2.1.** The uniqueness of the solution in Proposition 2.5 means that: if \( (\rho_{\infty 1}, l_{\infty 1}) \) and \( (\rho_{\infty 2}, l_{\infty 2}) \) are two solutions to the Eulerian stationary problem (1.12)-(1.13) with the same total mass \( M \), and satisfy \( \rho_{\infty i}(r) \sim (l_{\infty i} - r^n)^{-\frac{1}{n-1}}, \quad i = 1, 2 \), then we have \( (\rho_{\infty 1}, l_{\infty 1}) = (\rho_{\infty 2}, l_{\infty 2}) \).
3 Global Existence

Using similar arguments as that in [3], we obtain the following local existence and uniqueness result and omit the proof.

**Theorem 3.1.** (Local Result) Under the assumptions in Theorem 1.1, there is a positive constant $T_1 > 0$ such that the free boundary problem (1.15)-(1.17) admits a unique weak solution $(\rho, u, r)(x, t)$ on $[0, M] \times [0, T_1]$ in the sense that

\[ \rho(x, t), u(x, t), r(x, t) \in L^\infty([0, M] \times [0, T_1]) \cap C^1([0, T_1]; L^2([0, M])), \]

\[ \rho^{\theta+1}\partial_x(r^{n-1}u) \in L^\infty([0, M] \times [0, T_1]) \cap C^{1/2}([0, T_1]; L^2([0, M])), \]

\[ \partial_x r^{n-1}, \partial_x(r^{n-2}u) \in L^\infty([0, T_1], L^1([0, M])), \]

and the following equations hold:

\[ \partial_t \rho = -\rho^2 \partial_x(r^{n-1}u), \quad \rho(x, 0) = \rho_0, \]

\[ \partial_t r(x, t) = u(x, t), \quad r^n(x, t) = a^0 + n \int_0^x \rho^{-1}(y, t)dy, \quad (3.1) \]

\[ (r^\beta(\rho^\gamma)x_t) = -\frac{\theta r^{1+\beta-n}}{2c_1 + c_2} u_t - \frac{\theta}{2c_1 + c_2} \left( Ar^{\beta}(\rho^\gamma)x + r^{1+\beta-n}f \right), \quad (3.2) \]

\[ (2c_1 + c_2)\rho^{1+\theta}(r^{n-1}u)_x = A\rho^\gamma + 2c_1(n-1)\rho^\beta \frac{u}{r} + \int_x^M \left\{ -\frac{u}{r^{n-1}} + 2c_1(n-1)\rho^{\beta \left( \frac{u}{r} \right) x} - \frac{f}{r^{n-1}} \right\} dy, \quad (3.3) \]

for almost all $x \in [0, M]$, any $t \in [0, T_1]$, where $\beta = \frac{2(n-1)c_1\theta}{2c_1 + c_2}$,

\[ \int_0^x \int_0^M \left[ u\psi_t + (P - \rho(\lambda + 2\mu)(r^{n-1}u)_x)(r^{n-1}\psi)_x \right. \]

\[ + 2(n-1)\mu(r^{n-2}u\psi)_x - f(x, r, t)\psi]dxdt + \int_0^M u_0(x)\psi(x, 0)dx = 0, \quad (3.4) \]

for any test function $\psi(x, t) \in C^\infty_0((0, M] \times [0, T_1))$. Furthermore, we have

\[ \frac{N_1}{3}(1 - x)\frac{1}{7} \leq \rho(x, t) \leq 3N_2(1 - x)^{\frac{1}{7}}, \quad (x, t) \in [0, M] \times [0, T_1], \quad (3.5) \]

\[ (M - x)^{-\frac{1}{7}}\rho(x, t) \in C([0, T_1]; L^\infty([0, M])), \quad (3.6) \]

\[ (M - x)^{\frac{\theta}{2n}\gamma}(\rho^\beta)_x, \rho_t, u_t \in L^\infty([0, T_1]; L^2([0, M])), \]

\[ \rho^{\frac{\beta+1}{2}}u_{xt} \in L^2([0, M] \times [0, T_1]), \rho \partial_x u \in L^\infty([0, M] \times [0, T_1]). \]

**Remark 3.1.** From (1.15), (3.5) and $\rho \partial_x u \in L^\infty_{t,x}$, we have $(M - x)^{-\frac{1}{7}}\partial_t \rho \in L^\infty_{t,x}$. Thus, (3.6) holds.
Assume the maximum existence time of the weak solution in Theorem 3.1 is $T^*$. In this section, under the small assumptions on the initial data, we will obtain the following a priori estimates and prove that $T^* = \infty$. In the following, we may assume that $(\rho, u, r)(x, t)$ is suitably smooth since the following estimates are valid for the solutions with the regularities indicated in Theorem 3.1 by using the Friedrichs mollifier.

From (1.9), (1.18) and Proposition 2.1, we could obtain the following lemma easily.

**Lemma 3.1.** Under the assumptions of Theorem 1.1, we have

$$A\rho \gamma(x) = \int_x^M G\frac{M_0 + y}{r^{2n-2}} dy,$$  \hspace{1cm} \text{(3.7)}

$$C^{-1}(M - x)^{\frac{1}{\gamma}} \leq \rho \leq C(M - x)^{\frac{1}{\gamma}}, \quad r(x) \in [a, C],$$  \hspace{1cm} \text{(3.8)}

$$\frac{d}{dx}(A\rho \gamma(x)) = -G\frac{M_0 + x}{r^{2n-2}}, \quad C^{-1} \leq (M - x)^{1 - \frac{1}{\gamma}} \frac{d}{dx}\rho \leq C,$$  \hspace{1cm} \text{(3.9)}

for all $x \in [0, M]$.

**Lemma 3.2.** Under the assumptions of Theorem 1.1, we have

$$\frac{d}{dt} \int_0^M \left( \frac{1}{2} u^2 + \frac{A\rho \gamma}{\gamma - 1} + \int_a^r G\frac{M_0 + x}{s^{n-1}} ds \right) dx$$

$$+ \int_0^M \left\{ \left( \frac{2}{n} c_1 + c_2 \right) \rho^{1+\theta} [\partial_x(r^{n-1}u)]^2 + \frac{2(n-1)}{n} c_1 \rho^{1+\theta} (r^{n-1}u_x - \frac{u}{r\rho})^2 \right\} dx$$

$$= -\int_0^M \Delta f u dx, \quad t \in [0, T^*).$$ \hspace{1cm} \text{(3.10)}

**Proof.** Multiplying (1.15) by $u$, integrating the resulting equation over $[0, M]$, using integration by parts and the boundary conditions (1.17), we obtain

$$\frac{d}{dt} \int_0^M \frac{1}{2} u^2 dx + \int_0^M \left\{ (2c_1 + c_2) \rho^{1+\theta} [\partial_x(r^{n-1}u)]^2 - 2c_1(n-1)\rho^{\theta} \partial_x(r^{n-2}u^2) \right\} dx$$

$$= \int_0^M A\rho \gamma \partial_x(r^{n-1}u) dx - \int_0^M f u dx.$$  \hspace{1cm} \text{(3.11)}

From (1.15), we have

$$\int_0^M A\rho \gamma \partial_x(r^{n-1}u) dx = -\frac{d}{dt} \int_0^M A \frac{\rho^{\gamma-1}}{\gamma - 1} dx,$$  \hspace{1cm} \text{(3.12)}

$$- \int_0^M f u dx = -\frac{d}{dt} \int_0^M \int_a^r G\frac{M_0 + x}{s^{n-1}} ds dx - \int_0^M \Delta f u dx,$$  \hspace{1cm} \text{(3.13)}

and

$$\left(2c_1 + c_2\right) \rho^{1+\theta} [\partial_x(r^{n-1}u)]^2 - 2c_1(n-1)\rho^{\theta} \partial_x(r^{n-2}u^2)$$

$$= \left( \frac{2}{n} c_1 + c_2 \right) \rho^{1+\theta} [\partial_x(r^{n-1}u)]^2 + \frac{2(n-1)}{n} c_1 \rho^{1+\theta} (r^{n-1}u_x - \frac{u}{r\rho})^2.$$  \hspace{1cm} \text{(3.14)}

From (3.11)-(3.14), we obtain (3.10) immediately. \qed
Now, using the classical continuity method, we will obtain the estimate of \(\|(M-x)^{-\frac{1}{\gamma}}(\rho - \rho_{\infty})\|_{L^\infty}\).

**Claim 1:** Under the assumptions of Theorem 1.1, there is a small positive constant \(\epsilon_1 > \epsilon_0\), such that, for any \(T \in (0,T^*)\), if
\[
I(t) = \|g(\cdot,t) - g_{\infty}\|_{L^\infty} \leq 2\epsilon_1, \quad \forall \ t \in [0, T],
\]
where \(g(x,t) = (M - x)^{-\frac{1}{\gamma}}\rho(x,t)\) and \(g_{\infty}(x) = (M - x)^{-\frac{1}{\gamma}}\rho_{\infty}(x)\), then
\[
I(t) \leq \epsilon_1, \quad \forall \ t \in [0, T].
\]

Using the results in Lemmas 3.3-3.8, we can give the definition of \(\epsilon_1\) in (3.15) and finish the proof of **Claim 1**.

**Lemma 3.3.** Under the assumptions of Theorem 1.1 and (3.15), if \(\epsilon_1\) is small enough, we obtain
\[
C_1^{-1}(M-x)^{\frac{1}{\gamma}} \leq \rho(x,t) \leq C_1(M-x)^{\frac{1}{\gamma}},
\]
\[
r(x,t) \in [a,C_1],
\]
for all \(t \in [0, T]\) and \(x \in [0, M]\).

**Proof.** From (1.15), (3.15) and Lemma 3.1, we can easily obtain the estimate (3.17) and (3.18) when \(4\epsilon_1 < \min_{x \in [0,M]} g_{\infty}\).

**Lemma 3.4.** Under the assumptions of Lemma 3.3, if \(\epsilon_1\) is small enough, we obtain
\[
\int_0^M \left\{ u^2 + (M-x)^{\frac{1}{\gamma}}(g - g_{\infty})^2 + (r-r_{\infty})^2 \right\} dx \leq C_2\epsilon_0^2,
\]
\[
\int_0^t \|u(\cdot,s)\|_{L^\infty}^2 ds + \int_0^t \int_0^M \left( \rho^{\theta+1}u_x^2 + \rho^{\theta-1}u_x^2 \right)(x,s) dx ds \leq C_2\epsilon_0^2,
\]
for all \(t \in [0, T]\).

**Proof.** From (2.6), (3.7) and (3.10), we have
\[
\frac{d}{dt} \left( \int_0^M \frac{1}{2}u^2 dx + S[V] - S[V_{\infty}] \right)
\]
\[
= \int_0^M \left\{ \left( \frac{2}{n}c_1 + c_2 \right) \rho^{1+\theta}[\partial_x(r^{n-1}u)]^2 + \frac{2(n-1)}{n}c_1\rho^{1+\theta}(r^{n-1}u_x - \frac{u}{r\rho})^2 \right\} dx
\]
\[
= - \int_0^M \Delta f u dx
\]
where \(V_{\infty} = \frac{r_{\infty}^n}{n}\) and \(V = \frac{r^n}{n}\). From (2.7), (3.17)-(3.18) and Proposition 2.2 we have
\[
C^{-1} \int_0^M (M-x)^{\frac{1}{\gamma}}(g - g_{\infty})^2 + (r-r_{\infty})^2 dx
\]
\[
\leq \ S[V] - S[V_{\infty}] \leq C \int_0^M (M-x)^{\frac{1}{\gamma}}(g - g_{\infty})^2 + (r-r_{\infty})^2 dx,
\]
when \(\|M-x\|_{L^\infty}(\rho^{-1}-\rho_\infty^{-1})\|_{L^\infty} + \|\frac{1}{\theta}(r^n - r_\infty^n)\|_{L^\infty}\leq C_3 \epsilon_1 \leq \delta_9.\) From (1.24), (3.17)-(3.18) and (3.21), we obtain

\[
\int_0^M \left\{ u^2 + (M - x) \frac{\gamma - 1}{\theta} (g - g_\infty)^2 + (r - r_\infty)^2 \right\} dx + \int_0^t \int_0^M \left\{ \rho^{\theta+1} u_x^2 + \rho^{\theta-1} u_x^2 \right\} dxds \\
\leq C \epsilon_1^2 + C \int_0^t f_1(s) \|u(\cdot, s)\|_{L^\infty} ds. \tag{3.23}
\]

Since \(\theta \in (0, \gamma - 1),\) we obtain

\[
|u(x, t)| = \left| \int_0^x u_x dy \right| \leq C \left( \int_0^x \rho^{\theta+1} u_x^2 dy \right)^{\frac{1}{2}} \left( \int_0^x \rho^{\theta-1} dy \right)^{\frac{1}{2}} \\
\leq C \left( \int_0^x \rho^{\theta+1} u_x^2 dy \right)^{\frac{1}{2}} \left( \int_0^x (M - y) \frac{\theta+1}{\theta} dy \right)^{\frac{1}{2}} \leq C \left( \int_0^x \rho^{\theta+1} u_x^2 dy \right)^{\frac{1}{2}} \tag{3.24}
\]

and

\[
C \int_0^t f_1(s) \|u(\cdot, s)\|_{L^\infty} ds \leq \frac{1}{2} \int_0^t \int_0^M \rho^{\theta+1} u_x^2 dyds + C^2 \int_0^t f_2^2 dt. \tag{3.25}
\]

From (1.23) and (3.23)-(3.25), we can obtain (3.19)-(3.20) immediately. \(\square\)

**Lemma 3.5.** Under the assumptions of Lemma 3.3, if \(\epsilon_1\) is small enough, we obtain

\[
(1+t) \alpha \int_0^M \rho_\infty^{-1} (g - g_\infty)^2 dx + \int_0^t \int_0^M (1+s)^\alpha \left[ \rho_\infty^{-2} (g - g_\infty)^2 + (r - r_\infty)^2 \right] dxds \leq C_4 \epsilon_0, \tag{3.26}
\]

for all \(t \in [0, T],\) where \(\alpha = -\frac{\gamma}{2} .\)

**Proof.** Multiplying (1.15)2 by \(1+t) \alpha r^{-1-n} (\frac{r^n}{n} - \frac{r_\infty^n}{n})\), integrating over \([0, M],\) using integration by parts and the boundary conditions (1.17), we obtain

\[
(1+t) \alpha \int_0^M \left[ A(\rho_\infty^{-1} - \rho^{-1}) + G(M_0 + x) (r^{2-2n} - r_\infty^{2-2n})(\frac{r^n}{n} - \frac{r_\infty^n}{n}) \right] dx \\
= -(1+t) \alpha \int_0^M \frac{u_t}{r^{-1-n}} (\frac{r^n}{n} - \frac{r_\infty^n}{n}) dx - (1+t) \alpha \int_0^M \Delta r^{-1-n} (\frac{r^n}{n} - \frac{r_\infty^n}{n}) dx \\
+ (1+t) \alpha \int_0^M (2c_1 + c_2) \rho^{1+\theta} \partial_x (r^{-1-n} u) (\rho^{-1} - \rho_\infty^{-1}) dx \\
+ (1+t) \alpha \int_0^M 2c_1 (n-1) \rho^\gamma \left( \frac{u_t}{r} \frac{r^n}{n} - \frac{r_\infty^n}{n} \right) x dx := \sum_{i=1}^4 B_i. \tag{3.27}
\]

We can rewrite the left hand side of (3.27) as follows

\[
\text{L.H.S of (3.27)} = (1+t) \alpha \int_0^M \left[ A(\gamma + O(\epsilon_1)) \rho_\infty^{\gamma+1} (\rho^{-1} - \rho_\infty^{-1})^2 \\
- (2n - 2 + O(\epsilon_1)) G(M_0 + x) r_\infty^{2-3n} (\frac{r^n}{n} - \frac{r_\infty^n}{n}) \right] dx.
\]

Similar to (2.5), we have

\[
\text{L.H.S of (3.27)} \geq C_9 (1+t) \alpha \int_0^M \left[ \rho_\infty^{-2} (g - g_\infty)^2 + (r - r_\infty)^2 \right] dx, \tag{3.28}
\]

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when $\epsilon_1 \leq \delta_{10}$ is small enough.

Using (3.17)-(3.19), integration by parts and Hölder’s inequality, we can estimate $B_i$ as follows.

\[
B_1 = -\frac{d}{dt} \int_0^M (1 + t)^\alpha \frac{u}{r^{n-1}} \left( \frac{r^n}{n} - \frac{r^n}{n} \right) dx + \alpha (1 + t)^{\alpha - 1} \int_0^M \frac{u}{r^{n-1}} \left( \frac{r^n}{n} - \frac{r^n}{n} \right) dx \\
+ (1 + t)^\alpha \int_0^M u^2 \left( \frac{1}{n} + \frac{(n-1)r_n}{nr_n} \right) dx \\
\leq -\frac{d}{dt} \int_0^M (1 + t)^\alpha \frac{u}{r^{n-1}} \left( \frac{r^n}{n} - \frac{r^n}{n} \right) dx + C \int_0^M u^2 dx + C\epsilon_0^2 (1 + t)^{\alpha - 1}
\]  

(3.29)

\[
B_2 \leq C\epsilon_0 (1 + t)^{\alpha} f_1,
\]  

(3.30)

\[
B_3 = -\frac{2c_1 + c_2}{\theta} \int_0^M (\rho^\theta)_t (1 + t)^\alpha \left( \frac{1}{\rho_\infty} - \frac{1}{\rho} \right) dy \\
= -\frac{2c_1 + c_2}{\theta} \int_0^M \frac{\partial h}{\rho_\infty} (1 + t)^\alpha dx \\
= -\frac{2c_1 + c_2}{\theta} \frac{d}{dt} \int_0^M h(\rho, \rho_\infty) (1 + t)^\alpha dx + \frac{\alpha (2c_1 + c_2)}{\theta} \int_0^M h(\rho, \rho_\infty) (1 + t)^{\alpha - 1} dx
\]  

(3.31)

where $h(\rho, \rho_\infty) = \int_{r_\infty}^\rho \theta s^{\theta-1} (\frac{1}{\rho_\infty} - \frac{1}{\rho}) ds \sim \rho_\infty^{\theta-1} (g - g_\infty)^2$, and

\[
B_4 \leq C(1 + t)^\alpha \int_0^M \left[ \rho^\theta |u_x| + \rho^{\theta-1} |u| \right] dx \\
\leq C(1 + t)^\alpha \left[ \int_0^M \left( \rho^{\theta+1} u_x^2 + \rho^{\theta-1} u^2 \right) dx \right]^\frac{1}{2} \left[ \int_0^M (M - x)^{\frac{\theta-1}{2}} \right]^\frac{1}{2} \\
\leq C(1 + t)^\alpha \left[ \int_0^M \left( \rho^{\theta+1} u_x^2 + \rho^{\theta-1} u^2 \right) dx \right]^\frac{1}{2}
\]  

(3.32)

since $\gamma + \theta - 1 > 0$. From (3.27)-(3.32), we get

\[
\frac{d}{dt} \int_0^M (1 + t)^\alpha \left\{ \frac{u}{r^{n-1}} \left( \frac{r^n}{n} - \frac{r^n}{n} \right) + \frac{2c_1 + c_2}{\theta} h(\rho, \rho_\infty) \right\} dx \\
+ C^{-1} \int_0^M (1 + t)^{\alpha - 1} \rho^{\theta-1} (g - g_\infty)^2 + (1 + t)^\alpha \left\{ \rho_\infty^{\theta-1} [(g - g_\infty)^2 + (r - r_\infty)^2] \right\} dx \\
\leq C(1 + t)^\alpha \left[ \int_0^M \rho^{\theta+1} u_x^2 dx + \|u(\cdot, t)\|_{L^\infty}^2 \right]^\frac{1}{2} + C\epsilon_0 (1 + t)^{\alpha} f_1 + C \int_0^M u^2 dx + C\epsilon_0^2 (1 + t)^{\alpha - 1}.
\]

And using (1.23), (3.17)-(3.20) and Hölder’s inequality, we obtain (3.26) immediately. \[\square\]

Let $\epsilon_2 \in \left(0, \min\left\{ \frac{1}{4}, \frac{\gamma - \theta - 1}{\gamma - \theta}, \frac{\gamma - 1}{2(\gamma - 1)} \right\} \right)$ be a constant. Define $\{\beta_j\}$ and $\{\alpha_j\}$ by $\beta_{j+1} = \frac{\beta_j}{2} + \frac{1}{2} - \frac{\epsilon_2}{4}$, $\alpha_j = \frac{\beta_j}{2} - \frac{1}{2} - \frac{\epsilon_2}{4}$ and $\alpha_0 = \alpha = -\frac{5}{8}$, $j = 0, 1, \ldots$. Let $N_4$ be an integer satisfying $\beta_{N_4} \in [1 - \epsilon_2, 1 - \frac{3\epsilon_2}{8})$ and $\alpha_{N_4} \in (-\epsilon_2, -\frac{\epsilon_2}{8})$. It is easy to see that $\beta_0 = -\frac{1}{4} + \frac{\epsilon_2}{2} < 0$, $\alpha_j \in \left[ -\frac{5}{8}, -\frac{\epsilon_2}{8} \right)$ and $\beta_j \in \left( -\frac{1}{4}, 1 - \frac{3\epsilon_2}{16} \right)$, $j = 0, 1, \ldots, N_4$. Then, the following lemma can be proved by induction.
Lemma 3.6. Under the assumptions of Lemma 7.8, if $\epsilon_1$ is small enough, we obtain

\[
\int_0^t \left\{ u^2 + (M - x)^{\frac{\gamma - 1}{\gamma}} (g - g_\infty)^2 + (r - r_\infty)^2 \right\} dx \leq C_7 \epsilon_0^{2 - N_4} (1 + t)^{\theta - 1},
\]

(3.33)

\[
\int_0^t (1 + s)^{1 - \epsilon_2} \|u(\cdot, s)\|_{L^\infty}^2 ds + \int_0^t \int_0^M (1 + s)^{1 - \epsilon_2} \left( \rho^{\theta + 1} u_x^2 + \rho^{\theta - 1} u^2 \right) (x, s) dx ds \leq C_7 \epsilon_0^{2 - N_4},
\]

(3.34)

\[
\int_0^t \frac{M - x}{(1 + t)^{\epsilon_2}} (g - g_\infty)^2 dx + \int_0^t \int_0^M \rho_\infty^{-1} (g - g_\infty)^2 + (r - r_\infty)^2 \frac{dx ds}{(1 + s)^{\epsilon_2}} \leq C_7 \epsilon_0^{2 - N_4},
\]

(3.35)

for all $t \in [0, T]$, where $C_7$ is a constant depending on $\epsilon_2$.

Proof. The following estimates can be proved by induction:

\[
\int_0^t (1 + s)^{\beta_j} \|u(\cdot, s)\|_{L^\infty}^2 ds + \int_0^t \int_0^M (1 + s)^{\beta_j} \left( \rho^{\theta + 1} u_x^2 + \rho^{\theta - 1} u^2 \right) (x, s) dx ds \leq C_{j, \epsilon_2} \epsilon_0^{2 - j},
\]

(3.36)

\[
(1 + t)^{\beta_j} \int_0^M \rho_\infty^{-1} (g - g_\infty)^2 dx + \int_0^t \int_0^M \rho_\infty^{-1} (g - g_\infty)^2 + (r - r_\infty)^2 \frac{dx ds}{(1 + s)^{\epsilon_2}} \leq C_{j, \epsilon_2} \epsilon_0^{2 - j},
\]

(3.37)

(3.38)

for all $t \geq 0$, where $C_{j, \epsilon_2}$ is a constant depending on $j$ and $\epsilon_2$, $j = 0, 1, \ldots, N_4$.

From (3.19), (3.20) and (3.26), we obtain (3.36)-(3.38) hold with $j = 0$. Now, suppose that (3.36)-(3.38) hold with $j = k \geq 0$. To show (3.36)-(3.38) hold with $j = k + 1$, from (3.21), we have

\[
\frac{d}{dt} \left\{ (1 + t)^{\beta_{k+1}} \left( \int_0^M \frac{1}{2} u^2 dx + S[V] - S[V_\infty] \right) \right\}
\]

\[
+ (1 + t)^{\beta_{k+1}} \int_0^M \left\{ \frac{2}{n} c_1 + c_2 \rho^{1 + \theta} [\rho(x^{n-1} u)]^2 + \frac{2(n - 1)}{n} c_1 \rho^{1 + \theta} (r^{n-1} u_x - u \rho) \right\} dx 
\]

\[
= \beta_{k+1} (1 + t)^{\alpha_k} \int_0^M \frac{1}{2} u^2 (x, t) dx + S[V] - S[V_\infty] - (1 + t)^{\beta_{k+1}} \int_0^M \Delta f u dx,
\]

where $V_\infty = \frac{r^n}{n}$ and $V = \frac{r^n}{n}$. Integrating the above equality in $[0, t]$, using (1.24), (3.17)-(3.20), (3.22), (3.38) with $j = k$ and the fact that $\alpha_k < 0$, we obtain

\[
(1 + t)^{\beta_{k+1}} \int_0^M \left\{ u^2 (x, t) + (M - x)^{\frac{\gamma - 1}{\gamma}} (g - g_\infty)^2 + (r - r_\infty)^2 \right\} dx
\]

\[
+ \int_0^t \int_0^M (1 + s)^{\beta_{k+1}} \left\{ \rho^{\theta + 1} u_x^2 + \rho^{\theta - 1} u^2 \right\} dx ds
\]

\[
\leq C_6 \epsilon_0^{2 - k} + C \int_0^t (1 + s)^{\beta_{k+1}} \|u(\cdot, s)\|_{L^\infty} ds.
\]

(3.39)

From (1.23) and (3.25), we can obtain (3.36)-(3.37) with $j = k + 1$ immediately.

To show (3.38) with $j = k + 1$, from (3.27)-(3.28), we have

\[
(1 + t)^{\alpha_k + 1} \int_0^M \left[ A(\rho_\infty^{\gamma} - \rho^\gamma)(\rho^{-1} - \rho_\infty^{-1}) + G(M_0 + x)(r^{2 - 2n} - r_\infty^{2 - 2n})(\frac{r^n}{n} - \frac{r^n}{n}) \right] dx
\]

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Lemma 3.7. Using (1.23), (3.40)-(3.45), (3.36)-(3.37) with equality and the fact that Similar to (3.29)-(3.32), applying the estimates (3.17)-(3.19), integration by parts, Hölder’s inequality and the fact that \( \alpha < 0 \), we can estimate \( E \) as follows.

\[
E_1 \leq - \frac{d}{dt} \int_0^M (1 + t)^{\alpha_k+1} \frac{u}{r^{n-1}} \left( \frac{r^n}{n} - \frac{r^n}{n} \right) dx + C\|u\|_{L^\infty}^2 + C_\xi^2(1 + t)^{\alpha_k+1-1},
\]

(3.42)

\[
E_2 \leq C_\xi f_1(1 + t)^{\alpha_k+1},
\]

(3.43)

\[
E_3 = -\frac{2c_1 + c_2}{\theta} \frac{d}{dt} \int_0^M h(\rho, \rho_\infty)(1 + t)^{\alpha_k+1} \frac{\alpha_k+2(2c_1 + c_2)}{\theta} \int_0^M h(\rho, \rho_\infty)(1 + t)^{\alpha_k+1} \frac{dx}{t},
\]

and

\[
E_4 \leq C(1 + t)^{-\frac{3}{2} - \frac{1}{4}} \left[ (1 + t)^{\beta_k+1} \int_0^M \rho^{\theta+1} u^2 dx + \|u(\cdot, t)\|_{L^\infty}^2 \right]^{\frac{1}{2}}.
\]

(3.45)

Using (1.23), (3.40)-(3.45), (3.36)-(3.37) with \( j = k + 1 \) and Hölder’s inequality, we get

\[
\int_0^M (1 + t)^{\alpha_k+1} \rho_\infty^{\theta-1}(g - g_\infty)^2 dx + \int_0^t \int_0^M (1 + s)^{\alpha_k+1}\rho_\infty^{\theta-1}(g - g_\infty)^2 dx ds
\]

\[
+ \int_0^t \int_0^M (1 + s)^{\alpha_k+1} \{ \rho_\infty^{\theta-1} [(g - g_\infty)^2 + (r - r_\infty)^2] \} dx ds
\]

\[
\leq C(1 + t)^{\alpha_k+1} \int_0^M |u||r - r_\infty| dx + C \int_0^M |u_0||r_0 - r_\infty| dx
\]

\[
+ C \int_0^t \left[ f_1(1 + s)^{\alpha_k+1} + C_\xi^2(1 + s)^{\alpha_k+1-1} + \|u\|_{L^\infty}^2 \right] ds
\]

\[
+ C \int_0^t (1 + s)^{-\frac{1}{2} - \frac{1}{2}} \left[ (1 + t)^{\beta_k+1} \left( \int_0^M \rho^{\theta+1} u^2 dx + \|u(\cdot, t)\|_{L^\infty}^2 \right) \right]^{\frac{1}{2}} ds
\]

\[
\leq C\xi_0^{2-(k+1)},
\]

(3.46)

and finish the proof of (3.35) with \( j = k + 1 \). Thus, we show that (3.36)-(3.38) hold for \( j = 0,1,\ldots,N_4 \), and obtain (3.33)-(3.35) immediately. \( \square \)

From Lemma 3.6, we can obtain the following estimate of the weighted \( L^2 \)-norm of \( g - g_\infty \).

**Lemma 3.7.** Under the assumptions of Lemma 3.3, we obtain

\[
\int_0^M (M - x)^{\theta-1+\gamma-\theta} (g - g_\infty)^2 dx \leq C_{60}^{2-N_4}, \quad t \in [0, T].
\]

(3.47)
Proof. Using (3.33), (3.35) and Hölder’s inequality, we have

\[
\int_0^M (M - x)^{\frac{s-1+(\gamma-\theta)c_0}{\gamma}} (g - g_\infty)^2 \, dx \\
\leq C \left[ \int_0^M (1 + t)^{1-\epsilon_2} (M - x)^{\frac{s-1}{\gamma}} (g - g_\infty)^2 \, dx \right]^{\epsilon_2} \left[ \int_0^M (M - x)^{\frac{s-1}{\gamma}} (g - g_\infty)^2 \, dx \right]^{1-\epsilon_2} \\
\leq C \epsilon_0^{2-N_4}. 
\]

Then, using the similar argument as that in [22], we can finish the proof of Claim 1 in the following lemma.

Lemma 3.8. Under the assumptions of Lemma 3.3, if \( \epsilon_0 \) is small enough, we obtain

\[
|r(t) - \rho_\infty(t)| \leq C_{10} \epsilon_0^{\frac{\theta-1}{\gamma}} 2^{-N_4}, \quad (3.48)
\]

for all \( x \in [0, M] \) and \( t \in [0, T] \).

Proof. From (3.2), for any fixed \( x \in [0, M] \), we have

\[
I_1(x,t) + \frac{\theta}{2c_1 + c_2} \int_0^t A r^\beta(x,\tau)(\rho^\gamma(x,\tau) - \rho_\infty^\gamma(x)) \, d\tau = r_0^\beta(x) \rho_0^\theta(x) + I_2(x,t), \quad x \in [0, M], \quad t \in [0, T], \quad (3.49)
\]

where

\[
I_1(x,t) \\
= r_\infty^\beta(x) \rho^\theta(x,t) - (r_\infty^\beta(x) - r^{\beta}(x,t)) \rho^\theta(x,t) + \int_x^M \beta [(r^{\beta-n} \rho^{\theta-1})(y,t) - (r_0^{\beta-n} \rho_0^{\theta-1})(y)] \, dy \\
- \frac{\theta}{2c_1 + c_2} \int_x^M [(r^{\beta-n+1} u)(y,t) - (r_0^{\beta-n+1} u_0)(y)] \, dy + \frac{\theta(\beta - n + 1)}{2c_1 + c_2} \int_0^t \int_x^M r^{\beta-n} u^2 \, dyd\tau,
\]

and

\[
I_2(x,t) \\
= - \frac{\theta A \beta}{2c_1 + c_2} \int_0^t \int_x^M r^{\beta-n} \rho^{\gamma-1} - \rho_\infty^{\gamma} \, dyd\tau \\
+ \frac{\theta}{2c_1 + c_2} \int_0^t \int_x^M \left\{ r^{\beta} G(M_0 + y)(r^{2-2n} - r_\infty^{2-2n}) + r^{\beta-n+1} \Delta f \right\} \, dyd\tau.
\]

Using (3.17)-(3.18), (3.47), Hölder’s inequality and the condition \( \epsilon_2 < \frac{\gamma - \theta - 1}{\gamma - \theta} \), i.e., \( \frac{\theta + 1 + (\gamma - \theta) \epsilon_2}{\gamma} < 1 \), we have

\[
|(r - r_\infty)(x)| \leq C |r^n - r_\infty^n| \leq C \int_0^x |r^{\gamma-1} - r_\infty^{\gamma-1}| \, dy \leq C \int_0^x (M - y)^{\frac{1}{\gamma}} |g - g_\infty| \, dy \\
\leq C \left( \int_0^x (M - y)^{\frac{\theta + 1 + (\gamma - \theta) \epsilon_2}{\gamma}} (g - g_\infty)^2 \, dy \right)^{\frac{1}{2}} \left( \int_0^x (M - y)^{\frac{\theta + 1 + (\gamma - \theta) \epsilon_2}{\gamma}} dy \right)^{\frac{1}{2}}
\]

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Claim 1.1.

Proof of Claim 1.1. If not, there exists \( \theta \in (0, \frac{1}{2}] \) such that \( \theta \leq \frac{2\theta}{C_{1,3}} \leq \frac{2\theta}{C_{1,3} + 2c_2} \).

Thus, from (3.13)-(3.20) and (3.50)-(3.52), we obtain

\[
|I_1(x, t) - r_\infty^\theta| \leq C_{1,1}(M - x) \theta \epsilon_0^{-N_4-1},
\]

and

\[
|I_2(x, t_1) - I_2(x, t_2)| \leq C_{1,2}(M - x) \theta \epsilon_0^{-N_4-1} |t_2 - t_1|, \quad x \in [0, M].
\]

Claim 1.1. For any fixed \( x \in [0, M] \), we have

\[
I_1(x, t) \geq \min \left\{ I_1(x, 0), r_\infty^\beta \left( \rho_\infty^\gamma - \frac{C_{1,2}}{C_{1,3}} \epsilon_0^{-N_4-1} (M - x) \right)^{\frac{\theta}{\gamma}} - C_{1,1} \epsilon_0^{-N_4-1} (M - x)^{\frac{\theta}{\gamma}} \right\} := M_{1,1},
\]

where \( C_{1,3} := \frac{\theta}{C_{1,3} + 2c_2} \leq \frac{\theta}{C_{1,3} + 2c_2} \).

Proof of Claim 1.1. If not, there exists \( t_{1,1} > 0 \) such that \( I_1(x, t_{1,1}) < M_{1,1} \), then we can find \( t_{1,2} \in (0, t_{1,1}) \) such that \( I_1(x, t_{1,2}) = M_{1,1} \) and \( I_1(x, t) < M_{1,1} \) for all \( t \in (t_{1,2}, t_{1,1}) \). From (3.54) we have

\[
I_1(x, t_{1,1}) - I_1(x, t_{1,2}) + \frac{\theta}{2c_1 + c_2} \int_{t_{1,2}}^{t_{1,1}} r^\beta (\rho_\infty^\gamma - \rho_\infty^\gamma) \geq -C_{1,2} \epsilon_0^{-N_4-1} (M - x) (t_{1,1} - t_{1,2}).
\]

From (3.53), we have

\[
\rho_\infty^\theta(x, t) = r_\infty^{-\beta} (I_1(x, t) - (I_1(x, t) - r_\infty^\theta)) \leq r_\infty^{-\beta} (M_{1,1} + C_{1,1} \epsilon_0^{-N_4-1} (M - x)^{\frac{\theta}{\gamma}}) \leq \left( \rho_\infty^\gamma - \frac{C_{1,2}}{C_{1,3}} \epsilon_0^{-N_4-1} (M - x) \right)^{\frac{\theta}{\gamma}},
\]

and

\[
\rho_\infty^\gamma \leq \rho_\infty^\gamma - \frac{C_{1,2}}{C_{1,3}} \epsilon_0^{-N_4-1} (M - x),
\]

then \( I_1(x, t_{1,1}) \geq I_1(x, t_{1,2}) \). It is a contradiction. Thus, Claim 1.1 holds.

Similarly, we can obtain the following Claim.
Claim 1.2. For any fixed \( x \in [0, M] \), we have

\[
I_1(x, t) \leq \max \left\{ I_1(x, 0), r_{\infty}^\beta \left( \rho_0^\gamma + \frac{C_{1,2}}{C_{1,4}} (M - x) \right)^{\frac{\gamma}{\gamma + 1}} + C_{1,1} (M - x)^{\frac{\gamma}{\gamma + 1}} \right\} := M_{1,2},
\]

where \( C_{1,4} \) is a positive constant satisfying \( C_{1,4} \geq \frac{r_{\infty}^\beta}{2c_1 + c_2} \).

From Claim 1.1 and 1.2, we have

\[
|g(x, t) - g_\infty(x)| \leq C_{1,5} \epsilon_0^{\frac{\gamma}{\gamma + 1}},
\]

where \( x \in [0, M] \) and \( t \in [0, T] \), when \( \epsilon_0 \leq \delta_{11} \) is small enough.

Now, we can let

\[
\epsilon_1 = \epsilon_0 + C_{10} \epsilon_0^{\frac{\gamma}{\gamma + 1}}.
\]

If \( 4\epsilon_1 < \min_{x \in [0, M]} g_\infty \), \( C_3 \epsilon_1 \leq \delta_9 \), \( \epsilon_1 \leq \delta_{10} \) and \( \epsilon_0 \leq \delta_{11} \), using the results in Lemmas 3.3-3.8, we finish the proof of the Claim 1. From (3.6) and Claim 1, using the classical continuity method, we can obtain the following lemma easily.

Lemma 3.9. Under the assumptions of Theorem 1.1, we obtain (3.17)-(3.18), (3.33)-(3.35), (3.48) and (3.56) hold for all \( x \in [0, M] \) and \( t \in [0, T^\ast] \).

Proof. Let \( \mathcal{A} = \{ T \in [0, T^\ast) \mid I(t) \leq \epsilon_1 \text{ for all } t \in [0, T] \} \). Since \( I(0) \leq \epsilon_0 < \epsilon_1 \) and \( I(t) \in C([0, T^\ast)) \), then there exists a constant \( T_0 > 0 \) such that \( I(t) \leq \epsilon_1 \) for all \( t \in [0, T_0] \). Thus, \( \mathcal{A} \) is not empty and relatively closed in \( [0, T^\ast) \). To show that \( \mathcal{A} \) is also relatively open in \( [0, T^\ast) \cap [0, T] \), and hence the entire interval, it therefore suffices to show that the weaker bound

\[
I(t) \leq 2\epsilon_1, \text{ for all } t \in [0, T^\ast) \cap [0, T],
\]

implies \( I(t) \leq \epsilon_1 \) for all \( t \in [0, T^\ast] \). From Claim 1, we have \( \mathcal{A} = [0, T^\ast) \).

Then, from Lemmas 3.3-3.8, we obtain (3.17)-(3.18), (3.33)-(3.35), (3.48) and (3.56) hold for all \( x \in [0, M] \) and \( t \in [0, T^\ast] \).

We will prove an estimate in weighted \( L^2([0, M] \times [0, T^\ast]) \) norm of the function \( g - g_\infty \).

Lemma 3.10. Under the assumptions of Theorem 1.1, we obtain

\[
\int_0^t \int_0^M (1 + s)^{-c_2} (g - g_\infty)^2 dx ds \leq C,
\]

where \( t \in [0, T^\ast] \).

Proof. From (1.15), we have

\[
A(\rho^\gamma - \rho_{\infty}^\gamma) = \int_x^M \left( \frac{u_t}{r_{n-1}} + \frac{\Delta f}{r_{n-1}} \right) dy + \int_x^M G(M_0 + g)(r^{2-2n} - r_{\infty}^{2-2n})dy + (2c_1 + c_2)\rho^{\theta + 1}(r_{n-1}u)_x - 2c_1(n - 1)\rho^{\theta} \frac{u}{r} - 2c_1(n - 1) \int_x^M \rho^{\theta} \left( \frac{u}{r} \right)_x dy
\]
Multiplying the above equality by \((1+t)^{-\epsilon_2}(M-x)^{-2}(\rho^\gamma - \rho_\infty^\gamma)\), integrating the resulting equation over \([0, M] \times [0, t]\), we obtain

\[
\int_0^t \int_0^M A(1+s)^{-\epsilon_2}(M-x)^{-2}(\rho^\gamma - \rho_\infty^\gamma)^2 dx ds
= \int_0^t \int_0^M (1+s)^{-\epsilon_2}(M-x)^{-2}(\rho^\gamma - \rho_\infty^\gamma) \int_x^M \frac{u_t}{r^{n-1}} dy dx ds
+ \int_0^t \int_0^M (1+s)^{-\epsilon_2}(M-x)^{-2}(\rho^\gamma - \rho_\infty^\gamma) \int_x^M \frac{\Delta f}{r^{n-1}} dy dx ds
+ \int_0^t \int_0^M (1+s)^{-\epsilon_2}(M-x)^{-2}(\rho^\gamma - \rho_\infty^\gamma) \int_x^M G(M_0 + y)(r^{2-2n} - r_\infty^{2-2n}) dy dx ds
+ \int_0^t \int_0^M (2c_1 + c_2)(1+s)^{-\epsilon_2}(M-x)^{-2}(\rho^\gamma - \rho_\infty^\gamma)\rho^\theta + 1(r^{n-1}u)_x dx ds
- \int_0^t \int_0^M 2c_1(n-1)(1+s)^{-\epsilon_2}(M-x)^{-2}(\rho^\gamma - \rho_\infty^\gamma)\rho^\theta + 1(r^{n-1}u)_x dx ds
- \int_0^t \int_0^M 2c_1(n-1)(1+s)^{-\epsilon_2}(M-x)^{-2}(\rho^\gamma - \rho_\infty^\gamma) \int_x^M \rho^\theta \left(\frac{u}{r}\right)_x dy dx ds
\]

\[
:= \sum_{i=1}^6 F_i, \quad (3.58)
\]

Using \((1.15), (1.23), (3.17)-(3.18), (3.33)-(3.35), (3.48), Lemma 3.9,\) integration by parts and the Cauchy-Schwarz inequality, we can estimate \(F_i\) as follows.

\[
F_1 = \left\{ \int_0^M (1+s)^{-\epsilon_2}(M-x)^{-2}(\rho^\gamma - \rho_\infty^\gamma) \int_x^M \frac{u}{r^{n-1}} dy dx \right\}_0^t
+ \int_0^t \int_0^M \epsilon_2(1+s)^{-\epsilon_2-1}(M-x)^{-2}(\rho^\gamma - \rho_\infty^\gamma) \int_x^M \frac{u}{r^{n-1}} dy dx ds
+ \int_0^t \int_0^M \gamma(1+s)^{-\epsilon_2}(M-x)^{-2}\rho^\gamma + 1(r^{n-1}u)_x \int_x^M \frac{u}{r^{n-1}} dy dx ds
+ \int_0^t \int_0^M (n-1)(1+s)^{-\epsilon_2}(M-x)^{-2}(\rho^\gamma - \rho_\infty^\gamma) \int_x^M \frac{u^2}{r^{n-1}} dy dx ds
\leq C\|g - g_\infty\|_{L^\infty_{2t}} \left( \int_0^M |u|^2 dx \right)^{\frac{1}{2}} \left( \int_0^M (M-x)^{-\frac{n-2}{2}} dx \right)^{\frac{1}{2}} + C
+ C \left( \int_0^t \int_x^M \frac{\rho_\infty^{\gamma - 1}}{(1+s)^{\epsilon_2}} (g - g_\infty)^2 dx ds \right)^{\frac{1}{2}} \left( \int_0^t \|u(\cdot, s)\|_{L^\infty}^2 ds \int_0^M (M-x)^{-\frac{n-1}{\gamma}} dx \right)^{\frac{1}{2}}
+ C \left( \int_0^t \int_x^M \rho^\theta + 1(r^{n-1}u)^2 dx ds \right)^{\frac{1}{2}} \left( \int_0^t \|u(\cdot, s)\|_{L^\infty}^2 ds \int_0^M (M-x)^{-\frac{\theta-1}{\gamma}} dx \right)^{\frac{1}{2}}
+ C\|g - g_\infty\|_{L^\infty_{2t}} \int_0^t \|u(\cdot, s)\|_{L^\infty_{2t}}^2 ds
\leq C, \quad (3.59)
\]

\[
F_2 \leq C \left( \int_0^t (1+t)^{-1-\epsilon_2} \right)^{\frac{1}{2}} \left( \int_0^t (1+t)^{f_1^2} dt \right)^{\frac{1}{2}} \leq C, \quad (3.60)
\]
\[ |r(x, t) - r_\infty(x)| \leq C \int_0^x \rho_x^{-1} |g - g_\infty| \, dy \leq C \left( \int_0^x \rho_x^{-1} |g - g_\infty|^2 \, dy \right)^{1/2} \left( \int_0^x \rho_x^{-1} \, dy \right)^{1/2} \]
\[ \leq C (M - x)^{-\frac{3}{2} \frac{1}{2}} \left( \int_0^M \rho_x^{-1} |g - g_\infty|^2 \, dx \right)^{1/2}, \] (3.61)

\[ F_3 \leq C \int_0^t \int_0^M (1 + s)^{-\epsilon_2} (M - x)^{-2} |\rho^\gamma - \rho_x^\gamma| \left( \int_0^M \rho_x^{-1} |g - g_\infty|^2 \, dx \right)^{1/2} \int_0^M (M - y)^{-\frac{1}{2}} \, dy \, dx \]
\[ \leq A \int_0^t \int_0^M (1 + s)^{-\epsilon_2} (M - x)^{-2} (\rho^\gamma - \rho_x^\gamma)^2 \, dx \, ds \]
\[ + C \int_0^t \int_0^M (1 + s)^{-\epsilon_2} \rho_x^{-1} |g - g_\infty|^2 \, dx \, dz \int_0^M (M - z)^{-\frac{1}{2}} \, dz \]
\[ \leq A \int_0^t \int_0^M (1 + s)^{-\epsilon_2} (M - x)^{-2} (\rho^\gamma - \rho_x^\gamma)^2 \, dx \, ds + C, \] (3.62)

\[ F_4 = -\frac{2c_1 + c_2}{\theta} \int_0^t \int_0^M (1 + s)^{-\epsilon_2} (M - x)^{-2} (\rho^\gamma - \rho_x^\gamma) (\rho^\theta) \, dx \, ds \]
\[ = -\frac{2c_1 + c_2}{\theta} \left\{ (1 + s)^{-\epsilon_2} \int_0^M (M - x)^{-2} \left( \frac{\theta}{\gamma + \theta} \rho^\gamma + \theta - \rho_x^\gamma \rho^\theta \right) \, dx \right\}^t_0 \]
\[ - \epsilon_2 (2c_1 + c_2) \int_0^t \int_0^M (1 + s)^{-\epsilon_2-1} (M - x)^{-2} \left( \frac{\theta}{\gamma + \theta} \rho^\gamma + \theta - \rho_x^\gamma \rho^\theta \right) \, dx \]
\[ \leq C \|g - g_\infty\|_{L_\infty} \int_0^M (M - x)^\gamma^{-1} \, dx \left( 1 + \int_0^t (1 + s)^{-1-\epsilon_2} \, ds \right) + C \]
\[ \leq C, \] (3.63)

\[ F_5 \leq C \int_0^t \int_0^M (1 + s)^{-\epsilon_2} \|u(\cdot, t)\|_{L_\infty} (M - x)^{\theta-1} \, dx \, ds \]
\[ \leq C \left\{ \int_0^t (1 + s)^{1-\epsilon_2} \, ds \right\}^{1/2} \left\{ \int_0^t (1 + s)^{1-\epsilon_2} \|u(\cdot, t)\|_{L_\infty}^2 \, ds \right\}^{1/2} \leq C \] (3.64)

and

\[ F_6 \leq C \int_0^t \int_0^M (1 + s)^{-\epsilon_2} (M - x)^{-1} \int_x^M \left( |\rho^\theta u_x| + |\rho^\theta-1 u| \right) \, dy \, dx \, ds \]
\[ \leq C \int_0^t \int_0^M (1 + s)^{-\epsilon_2} (M - x)^{-1} \left[ \int_x^M \left( \rho^{\theta+1} u_x^2 + \rho^{\theta-1} u^2 \right) \, dy \right]^{1/2} \left[ \int_x^M \rho^{\theta-1} \, dy \right]^{1/2} \, dx \, ds \]
\[ \leq C \int_0^t \int_0^M (1 + s)^{-\epsilon_2} (M - x)^{\theta-2/\gamma} \left[ \int_x^M \left( \rho^{\theta+1} u_x^2 + \rho^{\theta-1} u^2 \right) \, dy \right]^{1/2} \, dx \, ds \]
\[ \leq C \left[ \int_0^t \int_0^M (1 + s)^{1-\epsilon_2} \left( \rho^{\theta+1} u_x^2 + \rho^{\theta-1} u^2 \right) \, dx \, ds \right]^{1/2} \left[ \int_0^t (1 + s)^{1-\epsilon_2} \, ds \right]^{1/2} \]
\[ \leq C. \] (3.65)

From (3.58)-(3.65), we get (3.57) immediately. \[\square\]
Lemma 3.11. Under the assumptions of Theorem 1.1, we obtain
\[(1 + t)^{-\epsilon_2} \int_0^M (M - x)^{1 - \frac{\theta}{2}} (\rho^\theta - \rho_\infty^\theta)_x^2 dx + \int_0^t \int_0^M (1 + s)^{-\epsilon_2} (M - x)^{2 - \frac{2\theta}{\gamma}} (\rho^\theta - \rho_\infty^\theta)_x^2 dx ds \leq C, \tag{3.66}\]
for all \(t \in [0, T^*).\)

Proof. From (3.2), we have
\[
\frac{\partial}{\partial t} \left[ \frac{\theta}{2c_1 + c_2} r^{1+\beta-n} u + r^\beta (\rho^\theta)_x - r^\beta_\infty (\rho^\theta)_x \right] + \frac{A_\gamma \rho^\gamma - \theta}{2c_1 + c_2} \left[ \frac{\theta}{2c_1 + c_2} r^{1+\beta-n} u + r^\beta (\rho^\theta)_x - r^\beta_\infty (\rho^\theta)_x \right] = \frac{-\theta}{2c_1 + c_2} \left[ r^\beta G (M_0 + x) - r^\beta_\infty (A \rho^\gamma_\infty)_x \right]. \tag{3.67}
\]
Let \(H = \frac{\theta}{2c_1 + c_2} r^{1+\beta-n} u + r^\beta (\rho^\theta)_x - r^\beta_\infty (\rho^\theta)_x,\) multiplying (3.67) by \((1 + t)^{-\epsilon_2} (M - x)^{1 - \frac{\theta}{2}} H^2,\)
integrating the resulting equation over \([0, M],\) and using the Cauchy-Schwarz inequality, we obtain
\[
\frac{d}{dt} \int_0^M (1 + t)^{-\epsilon_2} (M - x)^{1 - \frac{\theta}{2}} H^2(x,t) dx + C_{13} \int_0^M (1 + t)^{-\epsilon_2} (M - x)^{2 - \frac{2\theta}{\gamma}} H^2(x,t) dx \leq C \int_0^M (1 + t)^{-\epsilon_2} (M - x)^{1 - \frac{\theta}{2}} \left( |Hu| + |u^2 H| + |\Delta f H| \right) dx + C \int_0^M (1 + t)^{-\epsilon_2} (M - x)^{1 - \frac{\theta}{2}} \left| r^\beta G (M_0 + x) + \frac{\rho^\gamma - \theta}{\rho^\gamma_\infty} (A \rho^\gamma_\infty)_x \right| |H| dx \leq C_{13} \int_0^M (1 + t)^{-\epsilon_2} (M - x)^{2 - \frac{2\theta}{\gamma}} H^2(x,t) dx + C \int_0^M (1 + t)^{-\epsilon_2} \left| r^\beta G (M_0 + x) + \frac{\rho^\gamma - \theta}{\rho^\gamma_\infty} (A \rho^\gamma_\infty)_x \right|^2 dx. \tag{3.68}
\]
Here, we use the estimates (3.17)-(3.18) and the condition \(\theta \in (0, \gamma - 1).\) From (3.39) and (3.17)-(3.18), we have
\[
\int_0^M (1 + t)^{-\epsilon_2} \left| r^\beta G (M_0 + x) + \frac{\rho^\gamma - \theta}{\rho^\gamma_\infty} (A \rho^\gamma_\infty)_x \right|^2 dx = \int_0^M (1 + t)^{-\epsilon_2} \left| r^\beta G (M_0 + x) - G (\rho^\gamma_\infty \frac{\rho^\beta}{\rho^\gamma_\infty} (M_0 + x)) \right|^2 dx \leq C \int_0^M (1 + t)^{-\epsilon_2} \left[ (r - r_\infty)^2 + (g - g_\infty)^2 \right] dx. \tag{3.69}
\]
From (3.33) and (3.68)-(3.69), we obtain
\[
\frac{d}{dt} \int_0^M (1 + t)^{-\epsilon_2} (M - x)^{1 - \frac{\theta}{2}} H^2(x,t) dx + C_{13} \int_0^M (1 + t)^{-\epsilon_2} (M - x)^{2 - \frac{2\theta}{\gamma}} H^2(x,t) dx \leq C \int_0^M (1 + t)^{-\epsilon_2} (M - x)^{2 - \frac{2\theta}{\gamma}} H^2(x,t) dx.
\]
Using (A3), (1.23), (3.17)-(3.18), (3.33)-(3.34) and the Cauchy-Schwarz inequality, we obtain

From (A2), (1.23), (3.33)-(3.35), (3.56)-(3.57) and (3.70), we obtain (3.66) immediately.

Lemma 3.12. Under the assumptions of Theorem 1.1, we obtain

\[(1 + t)^{1-\epsilon_2} \int_0^M \left( \rho^{\theta-1} u^2 + \rho^{\theta+1} u_x^2 \right) (x,t) dx + \int_0^t \int_0^M (1 + s)^{1-\epsilon_2} u_t^2 (x,s) dx ds \leq C, \tag{3.71}\]

\[\|u(\cdot, t)\|_{L^{\infty}} \leq C (1 + t)^{-\frac{1}{2} + \epsilon_2}, \tag{3.72}\]

for all \(t \in [0,T^*)\).

Proof. Multiplying (1.15) by \((1 + t)^{1-\epsilon_2} u_t\), integrating the resulting equation over \([0, M] \times [0, t]\), using integration by parts and the boundary conditions (1.17), we obtain

\[\int_0^t \int_0^M (1 + s)^{1-\epsilon_2} u_t^2 (x,s) dx ds \]

\[= \int_0^t \int_0^M A(1 + s)^{1-\epsilon_2} \rho^\gamma \partial_x (r^{n-1} u) dx ds \]

\[= - \int_0^t \int_0^M (2c_1 + c_2)(1 + s)^{1-\epsilon_2} \rho^1 \partial_x (r^{n-1} u) \partial_x (r^{n-1} u) dx ds \]

\[+ \int_0^t \int_0^M 2c_1(n-1)(1 + s)^{1-\epsilon_2} \rho^\theta \partial_x (r^{n-2} uu_t) dx ds - \int_0^t \int_0^M (1 + s)^{1-\epsilon_2} fu_t dx ds \]

\[\leq \sum_{i=1}^4 H_i. \tag{3.73}\]

Using (A3), (1.23), (3.17)-(3.18), (3.33)-(3.34) and the Cauchy-Schwarz inequality, we obtain

\[H_2 + H_3 \]

\[= \left\{ (1 + s)^{1-\epsilon_2} \int_0^M \left[ \frac{-2c_1 + c_2}{2} \rho^{1+\theta} \partial_x (r^{n-1} u)^2 + c_1(n-1) \rho^\theta \partial_x (r^{n-2} u^2) \right] dx \right\}_0^t \]

\[+ \int_0^t \int_0^M \frac{2c_1 + c_2}{2} (1 - \epsilon_2) (1 + s)^{-\epsilon_2} \rho^{1+\theta} (r^{n-1} u)_x^2 dx ds \]

\[= \int_0^t \int_0^M c_1(n-1)(1 - \epsilon_2) (1 + s)^{-\epsilon_2} \rho^\theta \partial_x (r^{n-2} u^2) dx ds \]

\[+ \int_0^t \int_0^M (1 + s)^{1-\epsilon_2} \left\{ (2c_1 + c_2)(n-1) \rho^{1+\theta} \partial_x (r^{n-1} u) \partial_x (r^{n-2} u^2) \right\} \]

\[\leq C + C \int_0^t (\|u\|_{L^\infty} + \|\rho(r^{n-1} u)_x\|_{L^\infty}) \int_0^M \left[ \rho^{1+\theta} (r^{n-1} u)_x^2 + \rho^{\theta-1} u^2 \right] dx ds \]
Using (3.17)-(3.18), (3.24), (3.33), integration by parts and the Cauchy-Schwarz inequality, we obtain

\[ -C_{14}(1 + t)^{1-\epsilon_2} \int_0^M \left[ \rho^{1+\theta}(r^{n-1}u)_x^2 + \rho^{\theta-1}u^2 \right] dx, \quad (3.74) \]

\[
H_1 = \left\{ (1 + s)^{1-\epsilon_2} \int_0^M A\rho^\gamma \partial_x(r^{n-1}u)dx \right\}^t_0 + \int_0^t \int_0^M A\gamma(1 + s)^{1-\epsilon_2} \rho^{\gamma+1} [\partial_x(r^{n-1}u)]^2 dx ds \]
\[ - \int_0^t \int_0^M 2A(n - 1)(1 + s)^{1-\epsilon_2} \rho^{\gamma-1} u r \partial_x(r^{n-1}u) dx ds \]
\[ + \int_0^t \int_0^M A(n - 1)(1 + s)^{1-\epsilon_2} \rho^{\gamma-1} u^2 r^2 dx ds \]
\[ - \int_0^t \int_0^M A(1 - \epsilon_2)(1 + s)^{-\epsilon_2} \rho^\gamma \partial_x(r^{n-1}u) dx ds \]
\[ \leq (1 + s)^{1-\epsilon_2} \int_0^M A\rho^\gamma \partial_x(r^{n-1}u)dx + C, \quad (3.75) \]

\[
H_4 = - \left\{ (1 + s)^{1-\epsilon_2} \int_0^M G\frac{u(M_0 + x)}{r^{n-1}} dx \right\}^t_0 + (1 - \epsilon_2) \int_0^t \int_0^M (1 + s)^{-\epsilon_2} G\rho(M_0 + x) dx ds \]
\[ + \int_0^t \int_0^M (1 - n)(1 + s)^{1-\epsilon_2} G(M_0 + x)r^{n-1}u^2 dx ds - \int_0^t \int_0^M (1 + s)^{1-\epsilon_2} \Delta u u_t dx ds \]
\[ \leq -(1 + s)^{1-\epsilon_2} \int_0^M G\frac{u(M_0 + x)}{r^{n-1}} dx + C + \frac{1}{2} \int_0^t \int_0^M (1 + s)^{1-\epsilon_2} u_t^2 dx ds. \quad (3.76) \]

Using (3.17), (3.18), (3.24), (3.33), integration by parts and the Cauchy-Schwarz inequality, we obtain

\[
(1 + s)^{1-\epsilon_2} \int_0^M A\rho^\gamma \partial_x(r^{n-1}u)dx - (1 + s)^{1-\epsilon_2} \int_0^M G\frac{u(M_0 + x)}{r^{n-1}} dx \]
\[ = (1 + s)^{1-\epsilon_2} \int_0^M (A\rho^\gamma \partial_x(r^{n-1}u) + r^{n-1}u(A\rho^\gamma)_x - Gr^{n-1}u(M_0 + x)(r^{2-2n} - r^{2-2n}_\infty)) dx \]
\[ = (1 + s)^{1-\epsilon_2} \int_0^M (A\rho^\gamma - \rho^\gamma_\infty) \partial_x(r^{n-1}u) - Gr^{n-1}u(M_0 + x)(r^{2-2n} - r^{2-2n}_\infty)) dx \]
\[ \leq \frac{C_{14}}{4} (1 + t)^{1-\epsilon_2} \int_0^M \rho^{1+\theta}(r^{n-1}u)_x^2 dx + C \int_0^M (1 + t)^{1-\epsilon_2} [\rho^{\gamma-1}(g - g_\infty)^2 + (r - r_\infty)^2] dx \]
\[ \leq \frac{C_{14}}{4} (1 + t)^{1-\epsilon_2} \int_0^M \rho^{1+\theta}(r^{n-1}u)_x^2 dx + C. \quad (3.77) \]

From (3.73)-(3.77), we can obtain

\[
(1 + t)^{1-\epsilon_2} \int_0^M \left[ \rho^{1+\theta}(r^{n-1}u)_x^2 + \rho^{\theta-1}u^2 \right] dx + \int_0^t \int_0^M (1 + s)^{1-\epsilon_2} u_t^2 dx ds \]
\[ \leq C + C \int_0^t \left( \|u\|_{L^\infty} + \|\rho(r^{n-1}u)_x\|_{L^2} \right) (1 + s)^{1-\epsilon_2} \]
\[ \times \int_0^M \left[ \rho^{1+\theta}(r^{n-1}u)_x^2 + \rho^{\theta-1}u^2 \right] dx ds. \quad (3.78) \]

From (3.3), we have

\[ \rho(r^{n-1}u)_x \]
Using conditions $\theta \in (0, \frac{2}{\gamma}] \cap (0, \gamma - 1)$, (1.23), estimates (3.17)-(3.20) and Hölder’s inequality, we conclude that

$$\|\rho \partial_x(r^{n-1}u)\|_{L^\infty_x} \leq C + C \left(\|u(\cdot, t)\|_{L^\infty_x}^2 + \int_0^M \left[\rho^{1+\theta}(r^{n-1}u)^2_x + u_t^2\right] dx\right)^{\frac{1}{2}}. \quad (3.79)$$

Using (3.24), (3.34), (3.78)-(3.79) and the Cauchy-Schwarz inequality, we can obtain

$$(1 + t)^{1-\epsilon_2} \int_0^M \rho^{1+\theta}(r^{n-1}u)^2_x + \rho^\theta u^2 \right] dx + \int_0^t \int_0^M (1 + s)^{1-\epsilon_2} u_t^2 dx ds \leq C + C \int_0^t \left(\int_0^M (1 + s)^{1-\epsilon_2} \left[\rho^{1+\theta}(r^{n-1}u)^2_x + \rho^\theta u^2\right] dx\right)^2 ds. $$

Using Gronwall’s inequality and the estimate (3.34), we obtain (3.71) immediately. From (3.17), (3.71) and the fact $\theta \in (0, \gamma - 1)$, we can obtain

$$|u(x, t)| \leq \left|\int_0^x u_x dy\right| \leq C \left(\int_0^x \rho^{\theta+1} u_x^2 dy\right)^{\frac{1}{2}} \left(\int_0^x (M - y)^{-\frac{\theta+1}{\gamma}} dy\right)^{\frac{1}{2}} \leq C(1 + t)^{-\frac{1}{\gamma}+\frac{1}{\gamma}^2}, \quad (x, t) \in [0, M] \times [0, T^*).$$

Lemma 3.13. Under the assumptions of Theorem 1.1 we obtain

$$\int_0^M u_t^2(x, t) dx + \int_0^t \int_0^M \left[\rho^{1+\theta} u_{xt}^2 + \rho^\theta u_t^2\right] dx ds \leq C_{11}, \quad (3.80)$$

$$\|\rho(r^{n-1}u)\|_{L^\infty} \leq C_{11}, \quad (3.81)$$

for all $t \in [0, T^*)$.

Proof. We differentiate the equation (1.15) with respect to $t$, multiply it by $u_t$ and integrate it over $[0, M] \times [0, t]$, using the boundary conditions (1.17), then derive

$$\int_0^M \frac{1}{2} u_t^2 dx = \int_0^M \frac{1}{2} u_t^2(x, 0) dx - \int_0^t \int_0^M \left[(2c_1 + c_2)\rho^{1+\theta} \partial_x(r^{n-1}u) - A\rho^\gamma - 2c_1(n-1)\rho^{\theta \frac{u}{r}}\right]$$

$$\times \partial_x((n-1)r^{n-2}u_t) dx ds - \int_0^t \int_0^M \partial_t \left[(2c_1 + c_2)\rho^{1+\theta} \partial_x(r^{n-1}u) - A\rho^\gammaight]$$

$$-2c_1(n-1)\rho^{\theta \frac{u}{r}} \partial_x(r^{n-1}u_t) dx ds + \int_0^t \int_0^M 2c_1(n-1)\partial_t(r^{n-1}\rho^{\theta \partial_x\left(\frac{u}{r}\right)})u_t dx ds$$

$$- \int_0^t \int_0^M f_t u_t dx ds,$$

$$:= \sum_{i=1}^5 J_i. \quad (3.82)$$
From (A2)-(A3), we have
\[
J_1 \leq C \left( \left\| \left( (2c_1 + c_2)\rho_0^{\theta+1}(r_0^{-n}u_0)x \right)_x - 2c_1(n-1)\frac{u_0}{r_0}\rho_0^{\theta}x \right\|_{L^2} + \|\rho_0^{\gamma}x\|_{L^2} + \|f(x, r_0, 0)\|_{L^2} \right)^2 \leq C.
\] (3.83)

From (3.17)-(3.20) and the Cauchy-Schwarz inequality, we get
\[
J_3 + J_4 = -\int_0^t \int_0^M \left[ (2c_1 + c_2)\rho^{1+\theta}(r^{-n}u_t)^2 - 2c_1(n-1)\rho^{\theta}(r^{-n-2}u_t^2)x \right] dxds
+ \int_0^t \int_0^M \left\{ (2c_1 + c_2)(1+\theta)\rho^{\theta+2}[\partial_x(r^{-n}u)]^2 - (n-1)(2c_1 + c_2)\rho^{1+\theta}\partial_x(r^{-n-2}u^2)
- \gamma\rho^{\gamma+1}\partial_x(r^{-n}u) - 2c_1(n-1)\theta\rho^{\theta+1}\partial_x(r^{-n}u)\frac{u}{r} - 2c_1(n-1)\rho\frac{u^2}{r^2} \right\} u_t
\times \left[ (n-1)\frac{u_t}{r}\rho + r^{-n}u_t|dxds + 2c_1(n-1)\int_0^t \int_0^M \left\{ (n-1)r^{-n-2}u\rho^{\theta} \left( \frac{u}{r} \right)_x \right\} u_t
- \theta r^{-1}\rho^{\theta+1}(r^{-n}u)_x \left( \frac{u}{r} \right)_x u_t - r^{-n-1}\rho^{\theta} \left( \frac{u^2}{r^2} \right)_x u_t \right] dxds
\leq -C_{15} \int_0^t \int_0^M \left( \rho^{\theta+1}u^2 + \rho^{\theta-1}u^2 \right) dxds + C
+ C \int_0^t \left( \|u\|_{L^\infty}^2 + \|\rho(r^{-n}u)_x\|_{L^\infty}^2 \right) \int_0^M \left[ \rho^{\theta+1}u^2 + \rho^{\theta-1}u^2 \right] dxds,
\] (3.84)

From (1.11), (3.17)-(3.20) and the Cauchy-Schwarz inequality, we obtain
\[
J_2 \leq \frac{C_{15}}{8} \int_0^t \int_0^M \left( \rho^{\theta-1}u_t^2 + \rho^{\theta+1}u^2_t \right) dxds + C
+ C \int_0^t \left( \|u\|_{L^\infty}^2 + \|\rho(r^{-n}u)_x\|_{L^\infty}^2 \right) \int_0^M \left[ \rho^{\theta+1}u^2 + \rho^{\theta-1}u^2 \right] dxds
\] (3.85)

and
\[
J_5 \leq \frac{C_{15}}{8} \int_0^t \int_0^M \rho^{-1}u_t^2 dxds
+ C \int_0^t \int_0^M (G(M_0 + x)r^{-n}|u| + |\partial_x u| + |\partial_t u|)^2 \rho^{1-\theta} dxds
\leq \frac{C_{15}}{8} \int_0^M r^{-2}u_t^2 dxds + C.
\] (3.86)

From (3.52)-(3.56), we have
\[
\int_0^M u_t^2(x, t) dx + \int_0^t \int_0^M \left[ \rho^{\theta+1}u^2 + \rho^{\theta-1}u^2 \right] dxds
\leq C + C \int_0^t \left( \|u\|_{L^\infty}^2 + \|\rho(r^{-n}u)_x\|_{L^\infty}^2 \right) \int_0^M \left[ \rho^{\theta+1}u^2 + \rho^{\theta-1}u^2 \right] dxds.
\] (3.87)
From (3.71)-(3.72) and (3.79), we have
\[ \|\rho \partial_x (r^{1-1} u)\|_{L^\infty} \leq C + C\| u_t \|_{L^2}. \] (3.88)

From (3.20), (3.71)-(3.72) and (3.87)-(3.88), we obtain
\[
\int_0^M u_t^2(x,t) dx + \int_0^t \int_0^M \left[ \rho^{\theta+1} u_{x1}^2 + \rho^{\theta-1} u_t^2 \right] dx ds \\
\leq C + C \int_0^t \int_0^M \left[ \rho^{\theta+1} u_x^2 + \rho^{\theta-1} u^2 \right] dx \| u_t \|_{L^2}^2 ds.
\]

Using Gronwall’s inequality and the estimate (3.20), we obtain (3.80)-(3.81) immediately. \( \square \)

**Proof of existence and uniqueness**

If \( T^* < \infty \), from Lemmas 3.9-3.13 we have, for all \( t \in [0,T^*) \),
\[ C^{-1}(M - x)^{1/\gamma} \leq \rho(x,t) \leq C(M - x)^{1/\gamma}, \ x \in [0,M], \]
\[ \int_0^M (M - x)^{1-\frac{\theta}{2}} (\rho^\theta)_x^2 dx \leq C, \]
\[ \int_0^M (\rho^\gamma)_x^2 dx \leq C, \]
\[ \int_0^M u^2 + (M - x)^{\theta+1} u_x^2 dx \leq C, \]
\[ u(0,t) = 0, \]
\[ \int_0^M \left\{ \left( (2c_1 + c_2) \rho^{\theta+1} (r^{n-1} u)_x \right)_x - 2c_1(n-1) \frac{u}{r} \partial_x \rho^\theta \right\}^2 dx \leq C. \]

Thus, from Theorem 3.1 there exists \( T^*_2 > 0 \) such that the free boundary problem (1.15)-(1.17) admits a unique weak solution \((\rho, u, r)(x,t)\) on \([0,M] \times [T^*_2, T^*_2 + T^*_2] \), with initial data \((\rho, u, r)(x, T^*_2 - T^*_2)\). Using the uniqueness result in Theorem 3.1, we obtain that
\[ (\tilde{\rho}, \tilde{u}, \tilde{r})(x,t) = \begin{cases} (\rho, u, r)(x,t), & t \in [0, T^*_2 - T^*_2] \\ (\rho_2, u_2, r_2)(x,t), & t \in [T^*_2 - T^*_2, T^*_2 + T^*_2] \end{cases} \]

is a solution of the system (1.15)-(1.17), which is contradiction with the definition of \( T^* \). Thus, we have \( T^* = \infty \). From Lemma 3.9-3.13 we can show that the global weak solution satisfies the regularity conditions (1.25)-(1.26) and (1.26) in Theorem 1.1.

**Remark 3.2.** The uniqueness of the solution of Theorem 3.1 is obtained by the energy method. Let \((u_i, \rho_i, r_i), \ i = 1, 2\), be two solutions of the system (1.15)-(1.17) satisfying the regularity conditions in Theorem 1.1. Using similar arguments as that in the uniqueness part in 3, we can obtain, for all \( T > 0 \),
\[
\frac{d}{dt} \int_0^M \left( w^2 + \rho_1^{1-\theta} \rho_2^{2\theta-4} \rho^2 + \rho_1^\theta \rho_2^{-1} \mathcal{R}^2 \right) dx \\
+ C^{-1} \int_0^M \rho_1^{1+\theta} \left( \rho_1 r_1^{2n-2} (\partial_x w)^2 + \frac{w^2}{r_1^2} \right) dx \\
\leq C \int_0^M \left( w^2 + \rho_1^{1-\theta} \rho_2^{2\theta-4} \rho^2 + \rho_1^\theta \rho_2^{-1} \mathcal{R}^2 \right) dx, \ t \in [0,T],
\]
where \((w, \rho, \mathcal{R}) = (u_1 - u_2, \rho_1 - \rho_2, r_1 - r_2)\). Using Gronwall’s inequality, we could obtain \((u_1, \rho_1, r_1) = (u_2, \rho_2, r_2)\), a.e. \((x,t) \in [0,M] \times [0,T] \).
4 Further decay result

Lemma 4.1. Let $\nu$ be a positive constant satisfying $\nu < \min\{1, \frac{2\gamma - 2}{\gamma + \theta}\}$. Under the assumptions of Theorem 1.1, we obtain

\[
\left\| \rho^{\frac{\nu + \theta}{2}}(\cdot, t) - \rho_{\infty}^{\frac{\nu + \theta}{2}}(\cdot) \right\|_{L^{\infty}} \leq C(1 + t)^{\frac{1}{\gamma} + \frac{\nu}{2}},
\]  

for all $t \geq 0$, where $C_\nu$ is a positive constant depending on $\nu$.

Proof. From (3.48), (3.56) and (3.66), we have

\[
\int_{0}^{M} \left( \rho^{\frac{\nu + \theta}{2}} - \rho_{\infty}^{\frac{\nu + \theta}{2}} \right)^{2} \, dx \leq C(1 + t)^{2}, \quad t \geq 0.
\]  

Combining (3.33) and the Gagliardo-Nirenberg inequality $\|\phi\|_{L^{\infty}} \leq \|\phi\|_{L^{\nu}}^{\frac{1}{\nu}} \|\phi'\|_{L^{2}}^{\frac{1}{2}}$, we obtain

\[
\left\| \rho^{\frac{\nu + \theta}{2}} - \rho_{\infty}^{\frac{\nu + \theta}{2}} \right\|_{L^{\infty}} \leq C(1 + t)^{-\frac{\nu}{2} + \frac{\nu}{2}}, \quad t \geq 0.
\]

Using similar arguments as that in Lemmas 3.6, 3.11-3.12 and 4.1 with $\nu = \frac{\gamma - 1}{2\gamma}$, we can obtain the following lemma and omit the proof.

Lemma 4.2. Under the assumptions of Theorem 1.1, we have

\[
\int_{0}^{M} \left( M - x \right)^{\frac{\nu - 1}{\gamma}} (g - g_{\infty})^2 \, dx + \int_{0}^{t} \int_{0}^{M} \left[ \rho_{\infty}^{\gamma - 1}(g - g_{\infty})^2 + (r - r_{\infty})^2 \right] \, dx \, ds \leq C,
\]

\[
\int_{0}^{M} \left\{ u^2(x, t) + (M - x)^{\frac{\nu - 1}{\gamma}} (g - g_{\infty})^2 + (r - r_{\infty})^2 \right\} \, dx \leq C(1 + t)^{-1},
\]

\[
\int_{0}^{t} (1 + s)\|u(\cdot, s)\|_{L^{\infty}}^{2} \, ds + \int_{0}^{t} \int_{0}^{M} \left( \rho^{\theta + 1}u_x^2 + \rho^{\theta - 1}u^2 \right) (x, s) \, dx \, ds \leq C,
\]

\[
\int_{0}^{M} (M - x)^{1 - \frac{\nu}{2}} (\rho - \rho_{\infty})^{2}(x, t) \, dx + \int_{0}^{t} \int_{0}^{M} (M - x)^{2 - \frac{2\nu}{\gamma}} (\rho - \rho_{\infty})^{2}(x, s) \, dx \, ds \leq C,
\]

\[
(1 + t) \int_{0}^{M} \left( \rho^{\theta - 1}u_x^2 + \rho^{\theta + 1}u^2 \right) (x, t) \, dx + \int_{0}^{t} \int_{0}^{M} (1 + s)u^2(x, s) \, dx \, ds \leq C,
\]

for all $t \geq 0$. 

Remark 4.1. The key point is: similar to (4.32), using the estimates (4.18), (4.34), (4.12) and the condition $\epsilon_2 < \frac{\gamma - 1}{3(\gamma + 1)}$, we have

\[
\int_0^t \int_0^M 2c_1(n - 1)\rho^\theta \left( \frac{u)n - \rho^\theta}{n} \right) dxds \\
\leq C \int_0^1 \int_0^M \left\{ |r - r_\infty| \left( |\rho^\theta u| + |\rho^\theta u - \rho^\theta - u| \right) + \rho^\theta |u(r - \rho^\theta - u)| \right\} dxds \\
\leq C \int_0^1 \int_0^M (1 + s)^{1 - \epsilon_2} \left( \rho^\theta u^2 + \rho^\theta u^2 \right) (x, s)dxs \\
+ C \int_0^1 \int_0^M (1 + s)^{1 - \epsilon_2} \|r(\cdot, t) - r_\infty(\cdot)\|^2_{L^\infty} \int_0^M (M - x) \frac{\theta + 1}{\gamma} dxds \\
+ C \int_0^1 \int_0^M (1 + s)^{1 - \epsilon_2} \left| \rho^\theta r(\cdot, t) - \rho^\theta r_\infty(\cdot) \right|^2_{L^\infty} \int_0^M (M - x) \frac{\theta + 1}{\gamma} dxds \\
\leq C + C \int_0^1 (1 + s)^{1 - \epsilon_2 + \epsilon_2} ds \leq C, \text{ } t \geq 0.
\]

Without loss of generality, we assume $\eta \in (0, \frac{2(\gamma + 1)}{\gamma + \theta})$. Let $\epsilon_4 \in (0, \frac{\gamma + \theta - 1}{\gamma + \theta + 1})$ be a constant satisfying $2\frac{1 - \epsilon_4}{2(\gamma + \theta) + 1} > \frac{2(\gamma + \theta)}{\gamma + \theta + 1} - \eta$. Define $\{\kappa_j\}$ and $\{\eta_j\}$ by $\eta_j + 1 = 1 + \kappa_j$, $\kappa_j = \frac{2 + \theta - 1}{\gamma + \theta} \eta_j - \frac{\epsilon_4}{\gamma} \eta_j - \frac{1}{2} - \frac{\epsilon_4}{2}$ and $\eta_0 = 1$. Let $N_5$ be a positive integer satisfying $\eta_{N_5} > \frac{2(\gamma + \theta)}{\gamma + \theta + 1} - \eta$. It is easy to see that $\eta < 2$ and $\kappa_j < 1$, $j = 0, 1, \ldots, N_5$. Using similar arguments as that in Lemma 4.2, applying the induction method, we can obtain the following lemma and omit the proof.

**Lemma 4.3.** Under the assumptions of Theorem 1.1, we have

\[
\int_0^M \left\{ u^2(x, t) + (M - x)^{\frac{\gamma - 1}{\gamma}} \left( g - g_\infty \right)^2 + (r - r_\infty)^2 \right\} dx \leq C_{\eta,j}(1 + t)^{-\eta_j}, \tag{4.9}
\]

\[
\int_0^1 (1 + s)^{\eta_j} \|u(\cdot, s)\|^2_{L^\infty} ds + \int_0^1 (1 + s)^{\eta_j} \left( \rho^\theta u^2 + \rho^\theta u^2 \right) (x, s)dxds \leq C_{\eta,j}, \tag{4.10}
\]

\[
\left\| \rho^\theta r(\cdot, t) - \rho^\theta r_\infty(\cdot) \right\|_{L^\infty} \leq C_{\eta,j}(1 + t)^{-\eta_j}, \tag{4.11}
\]

\[
\int_0^1 (1 + s)^{\eta_j} \left[ \rho^\theta_r(g - g_\infty)^2 + (r - r_\infty)^2 \right] dxds \leq C_{\eta,j}, \tag{4.12}
\]

\[
(1 + t)^{\eta_j} \int_0^M \left( \rho^\theta u^2 + \rho^\theta u^2 \right) (x, t) dx + \int_0^1 (1 + s)^{\eta_j} u^2(x, s) dxds \leq C_{\eta,j}, \tag{4.13}
\]

\[
\|u(\cdot, t)\|_{L^\infty} \leq C_{\eta,j}(1 + t)^{-\eta_j}, \tag{4.14}
\]

for all $t \geq 0$, $j = 0, \ldots, N_5$, where $C_{\eta,j}$ is a positive constant depending on $\eta$ and $j$.

**Remark 4.2.** The main difficulty is to show (4.12) with $j = k$, when (4.9)-(4.11) hold with $j = k$. From (3.27)-(3.28), we have

\[
\int_0^T \int_0^M (1 + t)^{k_k} \left[ A(\rho^\gamma \rho^\gamma + \rho^\gamma - \rho^\gamma) + G(M_0 + x)(r^2 - 2n - r^2 - 2n) \right] \frac{n}{n} \frac{n}{n} dxdt \\
= - \int_0^T \int_0^M (1 + t)^{k_k} \frac{u_t}{r_1 - n} \left( \frac{n}{n} - \frac{n}{n} \right) dxdt - \int_0^T \int_0^M (1 + t)^{k_k} \Delta f^1 - n \left( \frac{n}{n} - \frac{n}{n} \right) dxdt
\]
\[ + \int_0^T \int_0^M (1 + t)^{\kappa_k} (2c_1 + c_2) \rho^{1 + \theta} \rho_x (r^{n-1} u)(\rho_{\infty}^{-1} - \rho^{-1}) \, dx \, dt \]
\[ + \int_0^T \int_0^M (1 + t)^{\kappa_k} 2c_1(n - 1) \rho^\theta \left( \frac{u^2}{r} - \frac{r_{\infty}^2}{n} \right) \, dx \, dt := \sum_{i=1}^4 Q_i, \quad T > 0 \] 

(4.15)

and

L.H.S of (4.15) \[ \geq C_{12} \int_0^T \int_0^M (1 + t)^{\kappa_k} \left[ \rho_{\infty}^{\gamma - 1} (g - g_{\infty})^2 + (r - r_{\infty})^2 \right] \, dx \, dt. \] 

Similar to (3.29)-(3.32), applying the estimates (3.17)-(3.20), (4.4), integration by parts, the Cauchy-Schwarz inequality and the fact that \( \kappa_j = \frac{3\gamma + 3\theta - 1}{4(\gamma + \theta)} \eta_j - \frac{\epsilon}{2} = \frac{1}{2} - \frac{\epsilon}{4} < \eta_j \), we can estimate \( Q_1 \) as follows.

\[ Q_1 \leq - \int_0^M (1 + t)^{\kappa_k} \frac{u}{r^{n-1}} \left( \frac{r^n}{n} - \frac{r_{\infty}^n}{n} \right) \, dx \bigg|_0^T + C \int_0^T (1 + t)^{\kappa_k} \| u \|_{L_\infty}^2 \, dt \]
\[ + C \int_0^T \int_0^M (1 + t)^{\kappa_k - 1} \| u \| \| r - r_{\infty} \| \, dx \, dt \]
\[ \leq C \] 

(4.17)

\[ Q_2 \leq \frac{C_{12}}{6} \int_0^T \int_0^M (1 + t)^{\kappa_k} (r - r_{\infty})^2 \, dx \, dt + C \int_0^T f_1^2 (1 + t)^{\kappa_k} \, dt \]
\[ \leq \frac{C_{12}}{6} \int_0^T \int_0^M (1 + t)^{\kappa_k} (r - r_{\infty})^2 \, dx \, dt + C, \] 

(4.18)

\[ Q_3 \leq C \int_0^T \int_0^M (1 + t)^{\eta_k} \rho^{1 + \theta} (r^{n-1} u_x)^2 \, dx \, dt \]
\[ + C \int_0^T \int_0^M (1 + t)^{2\kappa_k - \eta_k} \frac{\eta_k \nu_1}{2} (M - x) \frac{\rho^{\theta - 1}}{\gamma} \, dx \, dt \]
\[ \leq C, \] 

(4.19)

where \( \nu_1 = \frac{\gamma + \theta - 1}{\gamma + \theta} - \epsilon_4 \),

\[ \| r(\cdot, t) - r_{\infty}(\cdot) \|_{L_\infty} \leq C (1 + t)^{\frac{\nu_1 \eta_4}{4}}, \]

and

\[ Q_4 \leq C \int_0^T \int_0^M (1 + t)^{\eta_k} \left( \rho^{1 + \theta} (r^{n-1} u_x)^2 + \rho^{\theta - 1} u^2 \right) \, dx \, dt \]
\[ + C \int_0^T \int_0^M (1 + t)^{2\kappa_k - \eta_k} \frac{\eta_k \nu_1}{2} (M - x) \frac{\rho^{\theta - 1}}{\gamma} \, dx \, dt \]
\[ \leq C. \] 

(4.20)

From (4.15)-(4.20), we finish the proof of (4.12) with \( j = k \).

From (4.12) with \( j = N_5 \), using similar arguments as that in Lemma 3.11 we can obtain the following lemma and omit the proof.

**Lemma 4.4.** Under the assumptions of Theorem 1.1, we have

\[ \int_0^M (M - x)^{2 - \frac{2\theta}{\gamma}} \rho^{\theta - \rho_{\infty}^2} \, dx \leq C_s (1 + t)^{\frac{\theta}{4}(\gamma + \theta) - 1}, \] 

(4.21)

\[ \| \rho^\gamma (\cdot, t) - \rho_{\infty}^\gamma (\cdot) \|_{L_\infty} \leq C (1 + t)^{\frac{\gamma}{2} - \frac{3\gamma + 3\theta - 1}{4(\gamma + \theta)}}, \quad t \geq 0. \] 

(4.22)

Thus, we finish the proof of Theorem 1.1.
References

[1] D. Bresch and B. Desjardins, On the construction of approximate solutions for the 2D viscous shallow water model and for compressible Navier-Stokes models, J. Math. Pure Appl., 86(2006), pp. 362–368.

[2] G.Q. Chen and M. Kratka, Global solutions to the Navier-Stokes equations for compressible heat-conducting flow with symmetry and free boundary, Comm. Partial Differential Equations, 27(5-6)(2002), pp. 907–943.

[3] P. Chen and T. Zhang, A vacuum problem for multidimensional compressible Navier-Stokes equations with degenerate viscosity coefficients, preprint, arXiv:math.AP/0701150.

[4] B. Ducomet and A.A. Zlotnik, Viscous compressible barotropic symmetric flows with free boundary under general mass force. I. uniform-in-time bounds and stabilization, Math. Methods Appl. Sci., 28(7)(2005), pp. 827–863.

[5] B. Ducomet and A.A. Zlotnik, Lyapunov functional method for 1D radiative and reactive viscous gas dynamics, Arch. Rational Mech. Anal., 177(2005), pp. 185–229.

[6] H. Grad, Asymptotic theory of the Boltzmann equation II, In: Rarefied Gas Dynamics, 1. J. Laurmann, ed., New York: Academic Press, 1963, pp. 26–59.

[7] P. Hartman, Ordinary differential equations, Wiley, New York, 1964.

[8] D. Hoff and D. Serre, The failure of continuous dependence on initial data for the Navier-Stokes equations of compressible flow, SIAM J. Appl. Math., 51(4)(1991), pp. 887–898.

[9] T. Kobayashi and Y. Shibata, Decay estimates of solutions for the equations of motion of compressible viscous and heat-conductive gases in an exterior domain in $\mathbb{R}^3$, Comm. Math. Phys., 200(3)(1999), pp. 621–659.

[10] P.L. Lions, Mathematical topics in fluid mechanics, Vol. 1-2, Oxford University Press: New York, 1996, 1998.

[11] T.P. Liu, Z.P. Xin and T. Yang, Vacuum states of compressible flow, Discrete Contin. Dynam. Systems, 4(1)(1998), pp. 1–32.

[12] A. Matsumura and S. Yanag, Uniform boundedness of the solutions for a one-dimensional isentropic model system of a compressible viscous gas, Comm. Math. Phys., 175(1996), pp. 259–274.

[13] Š. Matušů-Nečasová, M. Okada and T. Makino, Free boundary problem for the equation of spherically symmetric motion of viscous gas (II)-(III), Japan J. Indust. Appl. Math., 12(1995) pp. 195–203; 14(1997), pp. 199–213.

[14] M. Okada and T. Makino, Free boundary value problems for the equation of spherically symmetrical motion of viscous gas, Japan J. Appl. Math., 10(1993), pp. 219–235.

[15] M. Okada, Š. Matušů-Nečasová and T. Makino, Free boundary problem for the equation of one-dimensional motion of compressible gas with density-dependent viscosity, Ann. Univ. Ferrara Sez. VII (N.S.), 48(2002), pp. 1–20.
[16] I. Straškraba and A.A. Zlotnik, Global behavior of 1d-viscous compressible barotropic fluid with a free boundary and large data, J. Math. Fluid Mech., 5(2003), pp. 119–143.

[17] V.A. Vaigant and A.V. Kazhikhov, On existence of global solutions to the two-dimensional Navier-Stokes equations for a compressible viscosity fluid, Siberian Math. J., 36(1995), pp. 1108–1141.

[18] S. Ukai, T. Yang and H.J. Zhao, Convergence rate for the compressible Navier-Stokes equations with external force, J. Hyperbolic Differ. Equ., 3(3)(2006), pp. 561–574.

[19] S.W. Vong, T. Yang and C.J. Zhu, Compressible Navier-Stokes equations with degenerate viscosity coefficient and vacuum(II), J. Differential Equations, 192(2)(2003), pp. 475–501.

[20] Z.P. Xin, Blow-up of smooth solution to the compressible Navier-Stokes equations with compact density, Comm. Pure Appl. Math., 51(3)(1998), pp. 229–240.

[21] T. Yang and C.J. Zhu, Compressible Navier-Stokes equations with degenerate viscosity coefficient and vacuum, Comm. Math. Phys., 230(2)(2002), pp. 329–363.

[22] T. Zhang and D.Y. Fang, Global behavior of compressible Navier-Stokes equations with a degenerate viscosity coefficient, Arch. Rational Mech. Anal., 182(2)(2006), pp. 223–253.

[23] T. Zhang and D.Y. Fang, Global behavior of spherically symmetric Navier-Stokes equations with density-dependent viscosity, preprint, arXiv:math.AP/0701216.

[24] A.A. Zlotnik and B. Ducomet, The stabilization rate and stability of viscous compressible barotropic symmetric flows with a free boundary for a general mass force, Sb. Math., 196(11-12)(2005), pp. 1745–1799.