Phonon Stability and Sound Velocity of Quantum Droplets in a Boson Mixture

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(Dated: May 29, 2020)

Quantum droplets have been realized in experiments on binary boson mixtures and dipolar Bose gases. In these systems, the mean-field energy of the Bose-Einstein condensation is attractive, and the repulsive Lee-Huang-Yang energy is crucial for stability. The Bogoliubov theory incorrectly predicts that the phonon mode is dynamically unstable in the long-wavelength limit. In this work, we go beyond the Bogoliubov theory to study how the phonon mode is stabilized in the quantum droplet of a binary boson mixture. Similar to Beliaev’s approach to a single-component Bose gas, we compute higher-order contributions to the self-energy of the boson propagator. We find that the interaction between spin and phonon excitations is the key for the phonon stability. We obtain the sound velocity which can be tested by measuring the superfluid critical velocity of the droplet in experiments. Beliaev damping of this quantum droplet is also discussed.

Introduction – Quantum droplet states of ultracold atoms are self-bound and can survive in the vacuum for considerable long time without trapping. In a seminal work [1], Petrov pointed out that attractive mean-field and repulsive Lee-Huang-Yang (LHY) energy [2] are keys to form quantum droplets in Bose gases. In recent several years, quantum droplets have been successfully realized in experiments on dipolar Bose gases of $^{164}$Dy[3–7], $^{168}$Er[8] atoms, the homonuclear mixture of $^{39}$K[9–11], and the heteronuclear $^{39}$K–$^{87}$Rb mixture [12].

The quantum droplet can be studied from many aspects, such as finite size effects[6, 9, 11], low-dimension matter waves[10, 13], and supersolid properties[14, 15]. The current theoretical approaches are mainly based on extended Gross-Pitaevskii equation[16–20] or quantum Monte Carlo techniques[21–24], but the microscopic theory is still incomplete. In the Bogoliubov theory of the quantum droplet [1], the phonon excitation is unstable in long-wavelength limit. It was postulated that the phonon excitations can be stabilized by integrating out higher-energy excitations, but never demonstrated so far. In the computation of LHY energy, this unstable mode is ignored based on the argument that its contribution is negligible compared to that from the higher-energy mode.

In this work, we study the stability of the phonon mode in the quantum droplet of a dilute binary boson mixture. We go beyond the Bogoliubov approximation and compute the contribution to the boson self-energy from higher order fluctuations. We find that in the dilute limit, the leading correction to the phonon energy comes from the interaction between phonon and spin excitations. This correction is positive and larger in magnitude than the phonon energy from the Bogoliubov theory in the long wavelength limit. Thus the phonon mode is stabilized by the interaction with the spin excitations. We obtain the sound velocity and find it is consistent with the hypothesis of superfluid hydrodynamics of the quantum droplet. This result can be tested in the experiment on the critical superfluid velocity of the quantum droplet. Beliaev damping of the phonon mode is also discussed.

Model – We consider a binary Bose mixture described by the Hamiltonian

$$H = \sum_{k} \varepsilon_{k}^{(\alpha)} (\hat{a}_{k}^{\dagger} \hat{a}_{k} + \hat{b}_{k}^{\dagger} \hat{b}_{k}) + \frac{1}{2V} \sum_{k_{1},k_{2},k_{3},k_{4}} \{g_{aa} \hat{a}_{k_{1}}^{\dagger} \hat{a}_{k_{2}}^{\dagger} \hat{a}_{k_{3}} \hat{a}_{k_{4}} + g_{bb} \hat{b}_{k_{1}}^{\dagger} \hat{b}_{k_{2}}^{\dagger} \hat{b}_{k_{3}} \hat{b}_{k_{4}} + 2g_{ab} \hat{a}_{k_{1}}^{\dagger} \hat{a}_{k_{2}}^{\dagger} \hat{b}_{k_{3}} \hat{b}_{k_{4}} \},$$

where $\hat{a}$ and $\hat{b}$ are the annihilation operators of the two components $\alpha$ and $\beta$, $\varepsilon_{k}^{(\alpha)} = \hbar^{2} k^{2}/(2m)$, $m$ is the mass of a boson, $V$ is the volume, and in the interaction term the total momentum is conserved, $k_{1} + k_{2} = k_{3} + k_{4}$. The s-wave coupling constants are given by $g_{ij} = 4\pi \hbar^{2} a_{ij}/m$, where $i,j = \alpha, \beta$, and $a_{ij}$ is the scattering length. Two characteristic length scales can be defined, $\delta a = (\varepsilon_{a} \varepsilon_{\beta} + \varepsilon_{\alpha}^{2})/2\varepsilon_{\alpha}^{0}$ and $\delta' = (\varepsilon_{\alpha}^{0} + \varepsilon_{\beta}^{0} - 2\varepsilon_{\alpha}^{0})/4$. The quantum droplets are formed in the region with $\delta a / \delta' < 1$ and the dilute condition $\sqrt{n_{\alpha}} \delta a < 1$ where $n_{\alpha}$ is the total boson density.

In the mean-field approximation, the bosons in the ground state are condensed with mean-field energy density given by $\frac{1}{2} \sum_{ii} g_{ii} n_{i} n_{i}$, which is negative for the quantum droplet. LHY energy density can be obtained in the Bogoliubov approximation, $\frac{1}{2} \sum_{\pm} \sum_{k} \{ \varepsilon_{\pm,k} - \varepsilon_{\pm,k}^{2} / (2\varepsilon_{k}^{0}) - \varepsilon_{k}^{0} / 2 \}$, where $\varepsilon_{\pm,k}$ are magnetic and phonon excitation energies [25]. For the quantum droplet, the phonon energy is imaginary in the long wavelength limit, $\varepsilon_{-k} \approx \frac{\hbar k}{\sqrt{m}} \left( \sqrt{8\alpha \alpha \beta \beta n_{\beta}^{2} n_{\alpha}^{2} - g_{aa} g_{\alpha\alpha} g_{\beta\beta} n_{\alpha}^{2} + 4n_{\alpha} n_{\beta} (g_{\alpha\beta}^{2} - g_{aa} g_{\beta\beta})} \right)$, which typically indicates dynamic instability of the system, but its contribution to LHY energy is much smaller in magnitude than that from magnetic excitations. In Ref. [1], it was postulated that the phonon mode can be stabilized by integrating out high-energy excitations and its contribution to LHY energy can be ignored. The total energy density of the quantum droplet is thus given by 

$$\frac{1}{2} \sum_{\pm} \sum_{k} \{ \varepsilon_{\pm,k} - \varepsilon_{\pm,k}^{2} / (2\varepsilon_{k}^{0}) - \varepsilon_{k}^{0} / 2 \} \right)_{\sqrt{n_{\alpha}} \delta a < 1}$$

and a positive compressibility can be obtained from its second derivative with density. The quantum droplet is self-bound in vacuum with zero pressure, leading to $

\gamma = \frac{\hbar k}{\sqrt{m}} \left( \sqrt{8\alpha \alpha \beta \beta n_{\beta}^{2} n_{\alpha}^{2} - g_{aa} g_{\alpha\alpha} g_{\beta\beta} n_{\alpha}^{2} + 4n_{\alpha} n_{\beta} (g_{\alpha\beta}^{2} - g_{aa} g_{\beta\beta})} \right)^{3/2}$

$$\frac{1}{2} \sum_{\pm} \sum_{k} \{ \varepsilon_{\pm,k} - \varepsilon_{\pm,k}^{2} / (2\varepsilon_{k}^{0}) - \varepsilon_{k}^{0} / 2 \} \right)_{\sqrt{n_{\alpha}} \delta a < 1}$$

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are much smaller than \( \Sigma \). In the following we consider higher-order fluctuations beyond Bogoliubov approximation to study how the phonon mode is stabilized in the long wavelength limit.

**Symmetric Case** – For illustration purposes, we first consider the system with symmetric intraspecies interactions, \( g_{aa} = g_{bb} \). In this case, the density of each component in the condensate is equal, \( n_a = n_g \). For simplicity, we assume the expectation values of the field operators are equal, \( \langle \phi_0 \rangle = \langle \tilde{\phi}_0 \rangle = \sqrt{nV/2} \), and introduce new operators related to density and spin fluctuations, \( \hat{c}_k = (\hat{\phi}_k + \hat{\tilde{\phi}}_k)/\sqrt{2} \) and \( \hat{d}_k = (\hat{\phi}_k - \hat{\tilde{\phi}}_k)/\sqrt{2} \), where \( \langle \hat{c}_0 \rangle = \sqrt{nV} \), \( \langle \hat{d}_0 \rangle = 0 \), and \( n \) is the total density of the condensate. The Hamiltonian in this representation is given by

\[
H = \sum_k \left[ \varepsilon_k^d \hat{c}_k^\dagger \hat{c}_k + \varepsilon_k^d \hat{d}_k^\dagger \hat{d}_k + \frac{1}{2V} \sum_{k_{1,2,3,4}} \{ \delta g [\hat{c}_k^\dagger \hat{c}_k, \hat{d}_k^\dagger \hat{d}_k] + \hat{d}_k^\dagger \hat{c}_k, \hat{d}_k^\dagger \hat{d}_k + (\hat{c}_k^\dagger \hat{d}_k^\dagger \hat{c}_k \hat{d}_k, \text{h.c.})] + g' [\hat{c}_k^\dagger \hat{d}_k^\dagger \hat{c}_k \hat{d}_k, \text{h.c.}] + (\hat{c}_k^\dagger \hat{d}_k^\dagger \hat{c}_k \hat{d}_k, \text{h.c.}) \} \right],
\]

(2)

where \( \mu \) is the chemical potential, \( p = (p_0, \mathbf{p}) \), \( p_0 \) is the frequency, and the non-interacting Green’s function is given by \( G(p) = 1/(p^0 + \mu - \epsilon_p^d + i\delta) \). The proper self-energy is a block matrix,

\[
\Sigma(p) = \left( \begin{array}{ccc} \Sigma_{dd} & \Sigma_{cd} \\ \Sigma_{cd} & \Sigma_{cc} \end{array} \right),
\]

where each two-by-two block matrix \( \Sigma_{ij} \) is given by

\[
\Sigma_{ij}(p) = \left( \begin{array}{ccc} \Sigma^{ij}_{dd}(p) & \Sigma^{ij}_{dc}(p) \\ \Sigma^{ij}_{cd}(p) & \Sigma^{ij}_{cc}(p) \end{array} \right).
\]

Following Beliaev’s notation [26, 27], the superscript 11 of \( \Sigma^{ij}_{cd} \) refers to an ingoing and an outgoing external line of particle \( i \) and \( j \) in the Feynman diagram, and the superscript 20 of \( \Sigma^{ij}_{dd} \) refers to two outgoing lines, as shown in Fig. 1(A) in the dilute region, the mean-field energy is of the same order as LHY energy, i.e. \( \delta g \sim g' \phi' \), and the two gas parameters \( \delta \phi \) and \( \phi' \) are not of the same order, with \( \delta \phi \sim \phi'^2 \). Therefore to obtain the correction to the phonon spectrum, we only need to consider second-order diagrams of \( \Sigma_{cc} \) due to interaction with spin excitations which is of the order \( g' \phi'^2 \). As shown in Fig. 1(B), these diagrams consists of coupling constant \( g' \) and first-order Green’s function \( G_{dd}(p) \) or \( \hat{G}_{dd}(p) \). Physically, it indicates the correction to the phonon spectrum comes from interaction between phonon and spin excitations. All other higher-order effects are negligible. We obtain the corrected
In the long-wavelength limit, the phonon energy is linearly dispersed \( E \propto \vec{q} \cdot \vec{v} \), and much smaller than that of a spin excitation. This conclusion can be also drawn by considering the process as follows. A phonon with energy \( \epsilon_p^d \) cannot be split into two excitations with at least one spin excitation due to energy and momentum conservation, i.e. \( \epsilon_p^d < \epsilon_q^c + \epsilon_{p-q}^d < \epsilon_q^c + \epsilon_{p-q}^d \). The Beliaev damping only happens when a phonon at finite \( p \) decays into two phonons, with damping rate the order of \( p^3 \xi^3 \). In comparison, the damping of spin excitations is more complicated. In the lowest order, a spin excitation with energy \( \epsilon_p^c \approx \sqrt{g} n p \) can only decay into a spin excitation with energy \( \epsilon_q^d \) and a phonon with energy \( \epsilon_{p-q}^d \). Note that in the case with asymmetric intraspecies interactions \( a_{\alpha\alpha} \neq a_{\beta\beta} \), there is an additional on-shell process, allowing spin excitation decaying to two other spin excitations.

Asymmetric case – The intraspecies interactions of the \( ^{39} \text{K} \) quantum droplet in the experiment are asymmetric. Our above method can be readily generalized to this case. For simplicity we assume \( \langle \hat{d}_0 \rangle > 0 \) and \( \langle \hat{\beta}_0 \rangle > 0 \). We can perform the following unitary transformation to decouple the two components in the Bogoliubov Hamiltonian,

\[
\begin{pmatrix}
\hat{c}_k \\
\hat{d}_k
\end{pmatrix} =
\begin{pmatrix}
\sqrt{n_{\alpha}/n} & \sqrt{n_{\beta}/n} \\
\sqrt{n_{\beta}/n} & -\sqrt{n_{\alpha}/n}
\end{pmatrix}
\begin{pmatrix}
\hat{\alpha}_k \\
\hat{\beta}_k
\end{pmatrix}.
\]

In the new representation the expectation values of new annihilation operators are given by \( \langle \hat{\alpha}_0 \rangle = \sqrt{n} \hat{\alpha} \) and \( \langle \hat{\beta}_0 \rangle = 0 \). As the symmetric case, the Bogoliubov quasi-particle of type-\( c \) is the phonon of density fluctuation, and that of type-\( d \) corresponds to spin fluctuation which changes the density ratio of \( \alpha \) and \( \beta \) components. The phonon excitation energy given by the Bogoliubov theory is imaginary in the long-wavelength limit, and our approach is essentially the same as before. We look for the second-order contribution to the self-energy \( \Sigma_{cc} \) due to the interaction between phonon and spin excitations. We obtain the same results as given in Eq. (7)-(9) except the parameters in these equations are now redefined as follows,

\[
\begin{align*}
\delta g &= -\gamma \sqrt{a_{\alpha\alpha} b_{\beta\beta}}, \\
\gamma' &= \sqrt{a_{\alpha\alpha} b_{\beta\beta}}, \\
d' &= \sqrt{a_{\alpha\alpha} b_{\beta\beta}}.
\end{align*}
\]
In the long wavelength limit, the phonon excitation energy is given by $\varepsilon_p \approx h v p$, where the phonon velocity is

$$v = \frac{5\pi h \gamma^{3/2}}{64m\sqrt{\alpha_{aa}\alpha_{bb}}}.$$  \hspace{1cm} (12)

The ground-state energy of the quantum droplet can be also computed in this Green’s function approach and the leading correction to the mean-field energy is the same as LYH energy in Ref [1]. From this ground-state energy and zero-pressure condition, we obtain the positive compressibility given by $\kappa = -4/(\delta g n \gamma^2)$. The sound velocity obtained from thermodynamic relation $v = 1/\sqrt{m n \kappa}$ agrees with the phonon behavior in Eq. (9), which indicates that quantum droplet obeys the superfluid hydrodynamics.

**Discussion and conclusion** – The phonon excitation is associated with the propagation of density sound wave. For the $^{39}$K droplet[9], the sound velocity according to Eq. (12) is about $7.8 \times 10^{-4}$m/s. It is possible in experiments to measure sound velocity in droplet. For example, one can excite density perturbations and observe the propagation of sound waves as the single component system[28]. Another method is stirring a droplet with fixed velocity[29]. According to Landau’s criterion, the condensate is dissipationless when scan velocity is below the critical velocity, i.e. the sound velocity. Alternatively, a Bragg spectroscopy can be used to determine the excitation spectrum[30].

Measuring the sound velocity requires as large a droplet as possible. On the length scale $\xi$, the interaction energy becomes comparable with the kinetic energy[31]. The healing length of Eq. (9) is $\xi_c = 64\sqrt{\alpha_{aa}\alpha_{bb}} \gamma^{3/2}/(5\sqrt{2})$. A homogeneous droplet ball with diameter $\xi_c$ contains about $N_{\xi} = \frac{32}{15\pi} \gamma^{-5/2}$ atoms. For $^{39}$K droplet mixture[9], the number $N_{\xi}$ is about 4400. When the atom number $N \gg N_{\xi}$, the excitations of droplet are collective, which helps to observe sound velocity. In comparison, a droplet with atom number $N \lessgtr N_{\xi}$ may facilitate global excitations under a local disturbance[10].

Unlike a single component BEC with attractive interaction[31], a mixture BEC with negative mean-field energy can be long lived even at large atom number. This indicates the unstable-miscible boundary is modified due to interaction between two quasi-particle branches. The actual boundary is closer to the zero-pressure condition $\gamma = \frac{64}{5\sqrt{\pi}} \sqrt{n(a_{aa}\alpha_{bb}^2)}^{3/2}$, rather than $\sqrt{\alpha_{aa}\alpha_{bb}} + a_{ab} = 0$.

Thus our results are readily applicable to trapped BEC mixture with small positive mean-field energy and fixed density ratio $n_1/n_2 \approx \sqrt{a_{22}/a_{11}}$ in the miscible region.

In conclusion, we go beyond the Bogoliubov theory to study excitations in a binary quantum droplet. Different from the predictions of the Bogoliubov theory, the phonon excitation is found to be stable with a positive sound velocity, which can be readily tested in experiments. The Beliaev damping of phonon is greatly suppressed in a quantum droplet. Our work theoretically confirms the quantum droplet is in superfluid hydrodynamic region.

We would like to thank Z.-Q. Yu and B. Liu for helpful discussions. This work is supported by the National Basic Research Program of China under Grant No. 2016YFA0301501.

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SUPPLEMENTAL MATERIALS

The supplemental materials provides the derivation of phonon velocity’s correction in asymmetric case, i.e. $a_{aa} \neq a_{bb}$.

Hamiltonian in atomic basis is

$$H = \sum_{\mathbf{k}} \left[ (e_{k}^{0} - \mu_{e}) \hat{c}_{\mathbf{k}}^{\dagger} \hat{c}_{\mathbf{k}} + (e_{k}^{0} - \mu_{h}) \hat{b}_{\mathbf{k}}^{\dagger} \hat{b}_{\mathbf{k}} \right] + \frac{1}{2V} \sum_{\mathbf{k},1,2,3,4} \left[ g_{aa} \hat{c}_{\mathbf{k}_1}^{\dagger} \hat{c}_{\mathbf{k}_2}^{\dagger} \hat{c}_{\mathbf{k}_3} \hat{c}_{\mathbf{k}_4} + g_{bb} \hat{b}_{\mathbf{k}_1}^{\dagger} \hat{b}_{\mathbf{k}_2}^{\dagger} \hat{b}_{\mathbf{k}_3} \hat{b}_{\mathbf{k}_4} + 2g_{ab} \hat{c}_{\mathbf{k}_1}^{\dagger} \hat{b}_{\mathbf{k}_2}^{\dagger} \hat{b}_{\mathbf{k}_3} \hat{c}_{\mathbf{k}_4} \right].$$

(S1)

The coupling parameters $g_{ij}$ is renormalized as $\frac{1}{g_{ij}} = \frac{1}{\lambda_{ij}} - \Lambda$, where $f_{ij} = \frac{e^{2} \sigma_{ij}}{m} a_{ij}$ and $\Lambda = \int \frac{1}{V} d\mathbf{q}$. In this article we adopt the convention $\int d\mathbf{q} = \frac{1}{(2\pi)^{3}} \int d^{4}q$ and $\int d\mathbf{q} = \frac{1}{(2\pi)} \int d^{3}q$.

In low-density limit, one can sum up the ladder diagrams to obtain the T-matrices,

$$\Gamma^{ij}_{ij}(p) = \frac{1}{(\lambda_{ij} + \Lambda) - (\chi_{p} + \Lambda)},$$

(S2)

where $\chi_{p} = \int d\mathbf{q} \frac{1}{p^{0} - (p^{2}/4 + \mathbf{q}^{2}) + i0^{+}}$. Usually Eq.(S2) approximated to the leading term $f_{ij}$ is enough for studying problems in the Bogoliubov level. Considering $p \sim \sqrt{n_{0} f_{ij}}$, the leading imaginary parts of T-matrices are of order $f_{ij}^{2}(\chi_{p} + \Lambda) \sim i f_{ij} \sqrt{n_{0} f_{ij}} \sim i\beta f_{ij}$. Therefore, we must retain this term to investigate the second order effect,

$$\Gamma^{ij}_{ij}(p) \approx f_{ij} + f_{ij}^{2}(\chi_{p} + \Lambda),$$

(S3)

After the transformation given in Eq. (10) of the main text, the Hamiltonian can be rewritten as

$$H = \sum_{\mathbf{k}} \left[ (e_{k}^{0} - \mu_{e}) \hat{c}_{\mathbf{k}}^{\dagger} \hat{c}_{\mathbf{k}} + (e_{k}^{0} - \mu_{h}) \hat{d}_{\mathbf{k}}^{\dagger} \hat{d}_{\mathbf{k}} - \mu_{s}(\hat{c}_{\mathbf{k}}^{\dagger} \hat{d}_{\mathbf{k}} + \text{h.c.}) \right] + \frac{1}{2V} \sum_{\mathbf{k},1,2,3,4} \left\{ 
\begin{array}{c}
+ \Gamma_{cc}^{cc}(p) \hat{c}_{\mathbf{k}_1}^{\dagger} \hat{c}_{\mathbf{k}_2}^{\dagger} \hat{c}_{\mathbf{k}_3} \hat{c}_{\mathbf{k}_4} + \Gamma_{dd}^{dd}(p) \hat{d}_{\mathbf{k}_1}^{\dagger} \hat{d}_{\mathbf{k}_2}^{\dagger} \hat{d}_{\mathbf{k}_3} \hat{d}_{\mathbf{k}_4} \\
+ \Gamma_{cd}^{cd}(p) \hat{c}_{\mathbf{k}_1}^{\dagger} \hat{d}_{\mathbf{k}_2}^{\dagger} \hat{c}_{\mathbf{k}_3} \hat{d}_{\mathbf{k}_4} + \Gamma_{dc}^{cd}(p) \hat{d}_{\mathbf{k}_1}^{\dagger} \hat{c}_{\mathbf{k}_2}^{\dagger} \hat{d}_{\mathbf{k}_3} \hat{c}_{\mathbf{k}_4} \\
+ \Gamma_{cc}^{cd}(p) \hat{c}_{\mathbf{k}_1}^{\dagger} \hat{d}_{\mathbf{k}_2}^{\dagger} \hat{c}_{\mathbf{k}_3} \hat{d}_{\mathbf{k}_4} + \Gamma_{cd}^{cd}(p) \hat{d}_{\mathbf{k}_1}^{\dagger} \hat{c}_{\mathbf{k}_2}^{\dagger} \hat{d}_{\mathbf{k}_3} \hat{c}_{\mathbf{k}_4} \\
+ \Gamma_{dd}^{cd}(p) \hat{c}_{\mathbf{k}_1}^{\dagger} \hat{d}_{\mathbf{k}_2}^{\dagger} \hat{c}_{\mathbf{k}_3} \hat{d}_{\mathbf{k}_4} + \Gamma_{dd}^{cd}(p) \hat{d}_{\mathbf{k}_1}^{\dagger} \hat{c}_{\mathbf{k}_2}^{\dagger} \hat{d}_{\mathbf{k}_3} \hat{c}_{\mathbf{k}_4} \\
\end{array} \right\},$$

where chemical potentials are $\mu_{e} = \frac{n_{a}}{n} \mu_{a} + \frac{n_{b}}{n} \mu_{b}$, $\mu_{d} = \frac{n_{p}}{n} \mu_{a} + \frac{n_{a}}{n} \mu_{b}$ and $\mu_{s} = \frac{1}{n} \sqrt{\mu_{a} \mu_{b}} (\mu_{a} - \mu_{b})$.

The T-matrices approximated to the first order are

$$\Gamma_{cc}^{cc}(p) = \delta f + f' \tau^{2}(\chi_{p} + \Lambda),$$

$$\Gamma_{dd}^{dd}(p) = f a_{a} + f b_{b} - 2 f' + \delta f + (f a_{a} + f b_{b} - f')^{2}(\chi_{p} + \Lambda),$$

$$\Gamma_{cd}^{cd}(p) = \Gamma_{dc}^{cd}(p) = f a_{a} + f b_{b} + \delta f + f' \tau^{3}(\chi_{p} + \Lambda),$$

$$\Gamma_{cc}^{cd}(p) = - \frac{1}{2} \sqrt{f^{3}(\sqrt{f a_{a}} - \sqrt{f b_{b}}) + \sqrt{f^{3}(\sqrt{f a_{a}} - \sqrt{f b_{b}})(\chi_{p} + \Lambda),}$$

$$\Gamma_{dd}^{dd}(p) = \frac{1}{2} \sqrt{f^{3}(\sqrt{f a_{a}} - \sqrt{f b_{b}})(2 - \gamma) + \sqrt{f^{3}(f a_{a} + f b_{b} - f')^{2}(\sqrt{f a_{a}} - \sqrt{f b_{b}})(\chi_{p} + \Lambda),}$$

(S5)

As we can see, $\Gamma_{cc}^{cc}(p), \Gamma_{dc}^{cd}(p)$ and $\Gamma_{cd}^{cd}(p)$ are of order $\delta f$, while the others are of order $f'$. In the Bogoliubov approximation the Hamiltonian is decoupled after ignoring such terms that of the order $\delta f$. The first-order self-energies $\Sigma_{dd}$ and chemical potential $\mu_{d}$ of $G_{d}$ are given by

$$\Sigma_{dd}^{1d} = n_{0} \Gamma_{dd}^{dd}(0) = f' n_{0},$$

$$\Sigma_{dd}^{2d} = n_{0} \Gamma_{cc}^{cc}(0) = f' n_{0},$$

$$\mu_{d} = 0.$$  

(S6)
Then we can obtain the first-order Green’s functions,

\[
G_d(p + \mu) = \frac{(p^0 + \varepsilon_p^0 + f' n_0)}{(p^0 - \varepsilon_p^d + i\delta)},
\]
\[
\hat{G}_d(p + \mu) = \hat{G}_d(p + \mu) = -f' n_0 / (p^0 - \varepsilon_p^d + i\delta),
\]

where \(\varepsilon_p^d = \sqrt{\varepsilon_p^0(\varepsilon_p^0 + 2f' n_0)}\). And the first-order correction to \(G_c\) is still absent.

As for the second-order correction, the necessary \(T\)-matrices are \(\Gamma_{cc}^{\omega}(p)\), \(\Gamma_{cd}^{\omega}(0)\) and \(\Gamma_{cc}^{dd}(0)\). As shown in Fig.1(B) in the main text, diagrams \(F\) are

\[
F_a(p'_1, p'_2; p_1, p_2) = if^{f2} \int dq \, G_d(q + \mu_d) \hat{G}_d(p'_1 + q - p_1 + \mu_d),
\]
\[
F_b(p'_1, p'_2; p_1, p_2) = if^{f2} \int dq \, \hat{G}_d(q + \mu_d) \hat{G}_d(p_1 + q - p'_1 + \mu_d),
\]
\[
F_c(p'_1, p'_2; p_1, p_2) = if^{f2} \int dq \, [G_d(q + \mu_d)G_d(p_1 + p_2 - q + \mu_d) - G^0_d(q + \mu_d)G^0_d(p_1 + p_2 - q + \mu_d)],
\]
\[
F_d(p'_1, p'_2, p'_i; p_1) = if^{f2} \int dq \, \hat{G}_d(q + \mu_d)G_d(p'_i + p'_2 - q + \mu_d),
\]
\[
F_e(p'_1, p'_2, p'_2; p_1, p_2) = if^{f2} \int dq \, \hat{G}_d(q + \mu_d)\hat{G}_d(p_1 + p_2 - q + \mu_d),
\]
\[
F_j(p_1, p_2, p_3; p_4) = if^{f2} \int dq \, \hat{G}_d(q + \mu_d)\hat{G}_d(p_2 + p_3 - q + \mu_d),
\]
\[
F_k(p'_1, p'_2, p'_2; p_1, p_2) = if^{f2} \int dq \, \hat{G}_d(q + \mu_d)\hat{G}_d(p'_1 + p'_2 - q + \mu_d),
\]
\[
F_h(p'_1; p_1) = if^{f} \int dq \, G_d(q + \mu_d),
\]
\[
F_i(p_1, p_2) = if^{f} \int dq \, [\hat{G}_d(q + \mu_d) - f' n_0 G^0_d(q + \mu_d)G^0_d(-q + \mu_d)],
\]
\[
F_k(p_1, p_2) = if^{f} \int dq \, [\hat{G}_d(q + \mu_d) - f' n_0 G^0_d(q + \mu_d)G^0_d(-q + \mu_d)].
\]

The second order self-energy of \(G_c\) are

\[
\Sigma^{11}_{cc}(p) = 2n_0 \Gamma_{cc}^{\omega}(p) + n_0 \left\{ F_a(p, 0; 0, 0) + F_b(p, 0; 0, 0) + F_c(p, 0; 0, 0) + F_d(p, 0; 0, 0) + F_e(p, 0; 0, 0) \right\} + F_h(p; p),
\]
\[
\Sigma^{20}_{cc}(p) = n_0 \Gamma_{cc}^{\omega}(0) + n_0 \left\{ F_a(p, -p; 0, 0) + F_b(p, -p; 0, 0) + F_c(p, 0, -p; 0) + F_d(p, 0, -p; 0) \right\} + F_h(p, -p),
\]
\[
\Sigma^{20}_{cc}(p) = n_0 \Gamma_{cc}^{\omega}(0) + F_b(0, 0; -p, 0) + F_d(0, 0; -p, 0) + F_e(0, 0; -p, 0) + F_h(0, 0; -p, 0) + F_i(p, -p),
\]
\[
\mu_c = n_0 \Gamma_{cc}^{\omega}(0) + F_b(0, 0; 0) + F_i(0, 0).
\]

By using Eq.(S5)-(S7) and after some straightforward calculations, we can obtain

\[
\Sigma^{11}_{cc}(p) = 2n_0 \delta f + \frac{56}{3\sqrt{\pi}} n_0 f' \sqrt{n_0 a^2} + \frac{4 n_0 f'^2}{4} \int \frac{d^3q}{\varepsilon_q^d \varepsilon_k^d} \left( \frac{Q_-(q, k)}{p^0 - \varepsilon_q^d - \varepsilon_k^d + i\delta^+} - \frac{Q_-(q, k)}{p^0 + \varepsilon_q^d + \varepsilon_k^d + i\delta^-} - 2\varepsilon_q^d + 2\varepsilon_k^d \right),
\]
\[
\Sigma^{20}_{cc}(p) = n_0 \delta f + \frac{8}{\sqrt{\pi}} n_0 f' \sqrt{n_0 a^2} + \frac{4 n_0 f'^2}{4} \int \frac{d^3q}{\varepsilon_q^d \varepsilon_k^d} R(q, k) \left( \frac{1}{p^0 - \varepsilon_q^d - \varepsilon_k^d + i\delta^+} - \frac{1}{p^0 + \varepsilon_q^d + \varepsilon_k^d - i\delta^-} \right),
\]
\[
\mu_c = n_0 \delta f + \frac{32}{3\sqrt{\pi}} n_0 f' \sqrt{n_0 a^2},
\]

where \(\mathbf{k} = \mathbf{q} - \mathbf{p}, R(q, k) = -\varepsilon_q^d + \varepsilon_k^d + n_0 f' (\varepsilon_q^0 + \varepsilon_k^0)\), and \(Q_+(q, k) = \varepsilon_q^d + \varepsilon_k^d + 3\varepsilon_q^0 \varepsilon_k^0 + n_0 f' (\varepsilon_q^0 + \varepsilon_k^0) \pm 2(\varepsilon_q^0 \varepsilon_k^0 + \varepsilon_q^0 \varepsilon_k^0)\).

The Green’s function \(G_c\) is given by

\[
G_c(p + \mu_c) = \frac{p^0 + \varepsilon_p^0 + \Sigma_{cc}^{11}(p)}{(p^0 - \Sigma^A)^2 - (\varepsilon_p^0 + \Sigma^S - \mu_c + \Sigma_{cc}^{20}(p))(\varepsilon_p^0 + \Sigma^S - \mu_c - \Sigma_{cc}^{20}(p))},
\]
where $\Sigma^A = [\Sigma_{11}^{31}(p) - \Sigma_{11}^{11}(-p)]/2 = \bar{A}p^0$ and $\Sigma^S = [\Sigma_{11}^{31}(p) + \Sigma_{11}^{11}(-p)]/2$. After dropping higher order terms, the pole equation becomes

$$\left(p^0\right)^2 - \left[\left(\varepsilon_p^0\right)^2(1 + 2\bar{A}) + 2\varepsilon_p^0(\Sigma^S - \mu_c)\right] = 0.$$  \hspace{1cm} (S12)

At small momentum, it is reduced to

$$p^0 = \sqrt{\varepsilon_p^0(\varepsilon_p^0 + 2\Sigma_{\varepsilon p}^2(0))} = \sqrt{\varepsilon_p^0(\varepsilon_p^0 + 2n_0\delta f + 32n_0\delta f^*\sqrt{n_0d^3}/\sqrt{\pi})}.$$  \hspace{1cm} (S13)