Linear Convergence of Variance-Reduced Projected Stochastic Gradient without Strong Convexity

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Abstract

Stochastic gradient algorithms compute the gradient based on only one sample (or just a few samples) and enjoy low computational cost per iteration. They are widely used in large-scale optimization problems. However, stochastic gradient algorithms are usually slow to converge and achieve sub-linear convergence rates, due to the inherent variance in the gradient computation. To accelerate the convergence, some variance-reduced stochastic gradient algorithms have been proposed. Under the strongly convex condition, these variance-reduced stochastic gradient algorithms achieve a linear convergence rate. However, in many machine learning problems, the objective function to be minimized is convex but not strongly convex. In this paper, we propose a Variance-Reduced Projected Stochastic Gradient (VRPSG) algorithm, which can efficiently solve a class of constrained optimization problems. As the main technical contribution of this paper, we show that the proposed VRPSG algorithm achieves a linear convergence rate without the strong convexity assumption. To the best of our knowledge, this is the first work that establishes the linear convergence rate for the variance-reduced stochastic gradient algorithm without strong convexity.

1 Introduction

Convex optimization has played an important role in machine learning as many machine learning problems can be cast into a convex optimization problem. Nowadays the emergence of big data makes the optimization problem challenging to solve and first-order stochastic gradient algorithms are often preferred due to their simplicity and low per-iteration cost. The stochastic gradient algorithms compute the gradient based on only one sample or just a few samples, and have been extensively studied in large-scale optimization problems [29, 11, 9, 20, 6, 18, 12, 21]. In general, the standard stochastic gradient algorithm randomly draws only one sample (or just a few samples) at each iteration to compute the gradient and then update the model parameter. The standard stochastic gradient algorithm computes the gradient without involving all samples and the computational cost per iteration is independent of the sample size. Thus, it is very suitable for large-scale problems. However, the standard stochastic gradient algorithms usually suffer from slow convergence. In particular, even under the strongly convex condition, the convergence rates of standard stochastic gradient algorithms are only sub-linear. In contrast, it is well-known that full (proximal) gradient descent algorithms can achieve linear convergence rates with the strongly convex condition [16]. It has been recognized that the slow convergence of the standard stochastic gradient algorithm results from the inherent variance in the gradient evaluation. To this end, some (implicit or explicit) variance-reduced stochastic gradient algorithms have been proposed; examples include Stochastic Average Gradient (SAG) [13], Stochastic Dual Coordinate Ascent (SDCA) [19, 20], Epoch Mixed Gradient Descent (EMGD) [28], Stochastic Variance Reduced Gradient (SVRG) [10], Semi-Stochastic Gradient Descent (S2GD) [11] and Proximal

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Stochastic Variance Reduced Gradient (Prox-SVRG) [27]. Under the strongly convex condition, these variance-reduced stochastic gradient algorithms achieve linear convergence rates. However, in practical problems, many objective functions to be minimized are convex but not strongly convex. For example, in machine learning, the least squares regression and logistic regression problems are extensively studied and both of objective functions are not strongly convex when the dimension $d$ is larger than the sample size $n$. Moreover, even without the strongly convex condition, linear convergence rates can also be proved for some full (proximal) gradient descent algorithms [24, 23, 8]. This inspires us to address the following question: can some variance-reduced stochastic gradient algorithms achieve a linear convergence rate without strong convexity?

In this paper, we give an affirmative answer to this question. Specifically, inspired by variance-reduced techniques adopted in SVRG [10] and Prox-SVRG [27], we propose a Variance-Reduced Projected Stochastic Gradient (VRPSG) algorithm to solve a class of constrained optimization problems. In particular, we establish a linear convergence rate for the proposed VRPSG algorithm without strong convexity. One challenge to prove the linear convergence rate for the proposed VRPSG algorithm without the strongly convex condition is that the optimization problem might have an optimal solution set which includes an infinite number of optimal solutions. Although we can establish the recursive relationship between the distance of the current feasible solution to some fixed optimal solution and the distance of the previous feasible solution to the same optimal solution, it is still very difficult to establish the linear convergence rate without strong convexity. We address this problem by establishing the recursive relationship between the distance of the current feasible solution to the optimal solution set and the distance of the previous feasible solution to the optimal solution set. Another challenge to prove the linear convergence rate for the proposed VRPSG algorithm is how to upper bound the distance of any feasible solution to the optimal solution set by the gap of the objective function value at the feasible solution and the optimal objective function value. This upper bound can be easily established under the condition that the objective function is strongly convex. However, without the strongly convex condition, it is not trivial to obtain such an upper bound. In this paper, by making suitable assumptions but without strong convexity, we can address this problem by adopting Hoffman’s bound [7, 14, 25]. To the best of our knowledge, our work establishes the first linear convergence rate for the variance-reduced stochastic gradient algorithm without strong convexity.

2 VRPSG: Variance-Reduced Projected Stochastic Gradient

We first introduce the general optimization problem, mild assumptions about the problem, and some examples that satisfy the assumptions. Then we present the proposed Variance-Reduced Projected Stochastic Gradient (VRPSG) algorithm to solve the optimization problem. Finally, we present a detailed convergence analysis for the VRPSG algorithm.

2.1 Optimization Problems, Assumptions and Examples

We consider the following constrained optimization problem:

$$\min_{w \in W} \{ f(w) = h(Xw) \}, \text{ where } w \in \mathbb{R}^d, \ X \in \mathbb{R}^{n \times d},$$

(1)

and make the following assumptions on the above problem throughout the paper:

**A1** $f(w)$ is the average of $n$ convex components $f_i(w)$, that is,

$$f(w) = \frac{1}{n} \sum_{i=1}^{n} f_i(w),$$

where $\nabla f(w)$ and $\nabla f_i(w)$ are Lipschitz continuous with constants $L$ and $L_i$, respectively.
A2 The effective domain of $h$, denoted by $\text{dom}(h)$, is open and non-empty. Moreover, $h(u)$ is continuously differentiable on $\text{dom}(h)$ and strongly convex on any convex compact subset of $\text{dom}(h)$.

A3 The constraint set, denoted by $W = \{w \in \mathbb{R}^d : Cw \leq b\}$, $C \in \mathbb{R}^{l \times d}$, $b \in \mathbb{R}^l$, is a polyhedral set which is compact.

Remark 1 Assumption A2 is the same as assumption 2.1 in [8], which indicates that $h(u)$ may not be strongly convex on $\text{dom}(h)$ but strictly convex on $\text{dom}(h)$. According to Weierstrass’s Theorem (Proposition A.8 in [2]), assumption A3 implies that the optimal solution set of the optimization problem in Eq. (1), denoted by $W^*$, is non-empty. Notice that $f(\cdot)$ is convex, so $W^*$ must be convex and the Euclidean projection of any $w \in \mathbb{R}^d$ onto $W^*$ must be unique. Moreover, for any $w, u \in W$, $Xw$ and $Xu$ must belong to a convex compact subset $U \subseteq \text{dom}(h)$. Thus, considering assumption A2, there exists a constant $\mu > 0$ such that

$$h(Xw) \geq h(Xu) + \nabla h(Xu)^T(Xw - Xu) + \frac{\mu}{2} \|Xw - Xu\|^2, \forall w, u \in W.$$ 

There are many examples that satisfy assumptions A1-A3, including two popular problems: $\ell_1$-constrained least squares (i.e., Lasso [22]) and $\ell_1$-constrained logistic regression. Specifically, for the $\ell_1$-constrained least squares: the objective function is $f(w) = \frac{1}{2n} \|Xw - y\|^2$; the convex component is $f_1(w) = \frac{1}{2}(x_i^T w - y_i)^2$, where $x_i^T$ is the $i$-th row of $X$; the strongly convex function is $h(u) = \frac{1}{2\eta} \|u - y\|^2$; the polyhedral set is $W = \{w : \|w\|_1 \leq \tau\} = \{w : Cw \leq b\}$ is compact, where each row of $C \in \mathbb{R}^{d \times d}$ is a $d$-tuples of the form $[\pm 1, \cdots, \pm 1]$, and each entry of $b \in \mathbb{R}^d$ is $\tau$. For the $\ell_1$-constrained logistic regression: the objective function is $f(w) = \frac{1}{n} \sum_{i=1}^n \log(1 + \exp(-y_i x_i^T w))$; the convex component is $f_1(w) = \log(1 + \exp(-y_i x_i^T w))$, where $X = [x_1^T, \cdots, x_n^T]^T$; the strongly convex function is $h(u) = \frac{1}{n} \sum_{i=1}^n \log(1 + \exp(-y_i u_i))$; the polyhedral set is the same as the $\ell_1$-constrained least squares. Additional constraint sets that satisfy assumption A3 include box constraint set $W = \{w : l_i \leq w_i \leq u_i\}$ with $-\infty < l_i \leq u_i < +\infty$ ($i = 1, \cdots, d$) and $\ell_{1,\infty}$-ball set $W = \{w : \sum_{i=1}^d \|w_i\|_\infty \leq \tau\}$ with $\cup_{i=1}^d G_i = \{1, \cdots, d\}$ and $G_i \cap G_j = \emptyset$ for $i \neq j$ [17].

2.2 Algorithm and Main Result

A standard method for solving Eq. (1) is the projected gradient descent, which generates the sequence $\{w^k\}$ via

$$w^k = \Pi_W(w^{k-1} - \eta \nabla f(w^{k-1})) = \arg \min_{w \in W} \frac{1}{2} \|w - (w^{k-1} - \eta \nabla f(w^{k-1}))\|^2. \quad (2)$$

Assuming that $\eta < 1/L$ and assumptions A1 – A3 hold, the objective function sequence $\{f(w^k)\}$ generated by Eq. (2) has a linear convergence rate [25]. At each iteration of the projected gradient descent, a full gradient involving all samples is required. Thus, the computational burden per iteration is heavy when the sample size $n$ is large. To reduce the per-iteration cost, the projected stochastic gradient algorithm can be adopted to generate the sequence $\{w^k\}$ as follows:

$$w^k = \Pi_W(w^{k-1} - \eta_k \nabla f_i(w^{k-1})),$$

where $i_k$ is randomly drawn from $\{1, \cdots, n\}$ in uniform. At each iteration, the projected stochastic gradient algorithm computes the gradient involving only a single sample and thus is suitable for large-scale problems with large $n$. Although we have an unbiased gradient estimate at each step, i.e., $\mathbb{E} [\nabla f_i(w^{k-1})] = \nabla f(w^{k-1})$, the variance $\mathbb{E} [||\nabla f_i(w^{k-1}) - \nabla f(w^{k-1})||^2]$ introduced by sampling

\footnote{The function $h(u) = \frac{1}{n} \sum_{i=1}^n \log(1 + \exp(-y_i u_i))$ is strictly convex on $\mathbb{R}^n$ and strongly convex on any convex compact subset of $\mathbb{R}^n$.}
makes the step size $\eta_k$ diminishing to guarantee convergence, which finally results in the slow convergence. Therefore, the key for improving the convergence rate of the projected stochastic gradient algorithm is to reduce the variance by sampling. Inspired by the variance-reduce techniques \[10, 27\] and the linear convergence result of full gradient algorithms without strong convexity \[25\], we propose a Variance-Reduced Projected Stochastic Gradient (VRPSG) algorithm (in Algorithm 1) to efficiently solve Eq. (1). Specifically, we find an unbiased gradient estimate which can reduce the variance in a multi-stage manner. To see how the unbiased gradient estimation is constructed and how the variance is reduced, please refer to Algorithm 1, Lemma 2 and Remark 2. The main technical contribution of this paper lies in the linear convergence rate analysis for the proposed VRPSG algorithm without strong convexity (summarized in Theorem 1). Note that the strong convexity is required for all other variance-reduced stochastic gradient algorithms to achieve linear convergence rates.

**Algorithm 1: VRPSG: Variance-Reduced Projected Stochastic Gradient**

1. Choose the update frequency $m$ and the learning rate $\eta$;
2. Initialize $\tilde{w}^0 \in W$;
3. Choose $p_i \in (0, 1)$ for $i \in \{1, \cdots, n\}$ such that $\sum_{i=1}^{n} p_i = 1$;
4. for $k = 1, 2, \cdots$ do
5. $\xi_{k-1} = \nabla f(\tilde{w}^{k-1})$;
6. $w_k^0 = \tilde{w}^{k-1}$;
7. for $t = 1, 2, \cdots, m$ do
8. Randomly pick $i_t^k \in \{1, \cdots, n\}$ according to the probability $P = \{p_1, \cdots, p_n\}$;
9. $v_t^k = (\nabla f_{i_t^k}(w_{t-1}^k) - \nabla f_{i_t^k}(\tilde{w}^{k-1}))/((np_{i_t^k}) + \xi_{k-1})$;
10. $w_t^k = \Pi_W (w_{t-1}^k - \eta v_t^k) = \arg \min_{w \in W} \frac{1}{2} \|w - (w_{t-1}^k - \eta v_t^k)\|^2$;
11. end
12. $\tilde{w}^k = \frac{1}{m} \sum_{t=1}^{m} w_t^k$;
13. end

**Theorem 1** Let $w^* \in W^*$ be any optimal solution to Eq. (7), $f^* = f(w^*)$ be the optimal objective function value in Eq. (7) and $L_P = \max_{i \in \{1, \cdots, n\}} [L_i/(np_i)]$ with $p_i \in (0, 1), \sum_{i=1}^{n} p_i = 1$. In addition, let $0 < \eta < 1/(4L_P)$ and $m$ be sufficiently large such that

$$\rho = \frac{4L_P \eta (m + 1)}{(1 - 4L_P \eta) m} + \frac{\beta}{\mu \eta (1 - 4L_P \eta) m} < 1, \quad (4)$$

where $\mu > 0, \beta > 0$ are constant. Then under assumptions A1 – A3, the VRPSG algorithm (summarized in Algorithm 1) achieves a linear convergence rate in expectation:

$$\mathbb{E}_{\mathcal{F}_{m}} [f(\tilde{w}^k) - f^*] \leq \rho^k (f(\tilde{w}^0) - f^*),$$

where $\tilde{w}^k$ is defined in Algorithm 1 and $\mathbb{E}_{\mathcal{F}_{m}} [\cdot]$ denotes the expectation with respect to the random variable $\mathcal{F}_m^k$ with $\mathcal{F}_m^k$ \((1 \leq t \leq m)\) being defined as

$$\mathcal{F}_m^k = \{i_1^1, \cdots, i_m^1, i_1^2, \cdots, i_m^2, \cdots, i_m^{k-1}, \cdots, i_1^{k-1}, i_2^{k-1}, \cdots, i_m^{k-1}\},$$

and $\mathcal{F}_0^k = \mathcal{F}_m^k$, where $i_t^k$ is the sampling random variable in Algorithm 1.

We have the following remarks on the convergence result above:

- The linear convergence rate $\rho$ in Eq. (4) is similar to that of the Prox-SVRG \[27\]. The difference is that an additional constant $\beta > 0$ is introduced in the numerator of the second term, which is needed since our proposed algorithm does not require the strongly convex condition.
Let \( \eta = \gamma / L_P \) with \( 0 < \gamma < 1/4 \). When \( m \) is sufficiently large, we have

\[
\rho \approx {\frac {\beta L_P / \mu}{\gamma (1 - 4\gamma ) m}} + {\frac {4\gamma}{1 - 4\gamma}},
\]

where \( \beta L_P / \mu \) can be treated as a pseudo condition number of the problem in Eq. (3). If we choose \( \gamma = 0.1 \) and \( m = 100/\beta L_P / \mu \), then \( \rho \approx 5/6 \). Notice that at each outer iteration of Algorithm \( \hat{\rho} \) \( n + 2m \) gradient evaluations (computing the gradient on a single sample counts as one gradient evaluation) are required. Thus, to obtain an \( \epsilon \)-accuracy solution (i.e., \( \mathbb{E}_{x_i^n} \left[ f(\hat{w}^k) - f^* \right] \leq \epsilon \)), we need \( O(n + \beta L_P / \mu) \log(1/\epsilon) \) gradient evaluations by setting \( m = \Theta(\beta L_P / \mu) \). In particular, the complexity becomes \( O(n + \beta L_{avg} / \mu) \log(1/\epsilon) \) if we choose \( p_i = L_i / \sum_{i=1}^{n} L_i \) for all \( i \in \{1, \ldots, n\} \), and \( O(n + \beta L_{max} / \mu) \log(1/\epsilon) \) if we choose \( p_i = 1/n \) for all \( i \in \{1, \ldots, n\} \), where \( L_{avg} = \sum_{i=1}^{n} L_i / n \) and \( L_{max} = \max_{i \in \{1, \ldots, n\}} L_i \). Notice that \( L_{avg} \leq L_{max} \). Thus, sampling in proportion to the Lipschitz constant is better than sampling uniformly.

At each outer iteration of VRPSG, the number of gradient evaluations is similar to that of full gradient methods. However, the overall complexity of VRPSG is superior over full gradient methods. Specifically, based on the last remark and Remark 4 if \( f \) is strongly convex with parameter \( \bar{\mu} \) and \( p_i = L_i / \sum_{i=1}^{n} L_i \), VRPSG has the same complexity as Prox-SVRG \( [27] \), that is, VRPSG needs \( O(n + L_{avg} / \bar{\mu}) \log(1/\epsilon) \) gradient evaluations to obtain an \( \epsilon \)-accuracy solution. In contrast, full gradient methods require \( O(n L / \bar{\mu}) \log(1/\epsilon) \) gradient evaluations to obtain a solution of the same accuracy. Obviously, the complexity of \( O(n + L_{avg} / \bar{\mu}) \log(1/\epsilon) \) is far superior over \( O(n L / \bar{\mu}) \log(1/\epsilon) \) when the sample size \( n \) and the condition number \( L / \bar{\mu} \) are very large.

If the Lipschitz constant \( L_i \) is unknown and difficult to compute, we can use an upper bound \( \hat{L}_i \) instead of \( L_i \) to define \( L_P = \max_{i \in \{1, \ldots, n\}} [\hat{L}_i / (n p_i)] \) and the theorem still holds.

We can obtain a convergence rate with high probability. According to Markov’s inequality with \( f(\tilde{w}^k) - f^* \geq 0 \), Theorem 1 implies that

\[
\Pr(f(\tilde{w}^k) - f^* \geq \epsilon) \leq \frac{\mathbb{E}_{x_i^n} \left[ f(\tilde{w}^k) - f^* \right]}{\epsilon} \leq \frac{\beta^k (f(\tilde{w}^0) - f^*)}{\epsilon}.
\]

Therefore, we have

\[
\Pr(f(\tilde{w}^k) - f^* \leq \epsilon) \geq 1 - \delta, \text{ if } k \geq \log \left( \frac{f(\tilde{w}^0) - f^*}{\delta \epsilon} \right) / \log(1/\rho).
\]

3 Technical Proof

In this section, we first provide several fundamental lemmas, based on which we complete the proof of Theorem 1. The key idea of the convergence proof in \( [10, 27] \) is to establish the recursive relationship between the distance of the current feasible solution to a unique optimal solution and the distance of the previous feasible solution to the same optimal solution. Different from \( [10, 27] \), we prove the linear convergence rate by establishing the recursive relationship between the distance of the current feasible solution to the optimal solution set and the distance of the previous feasible solution to the optimal solution set, due to the lack of strong convexity. Note that Lemmas 1, 2, 3 and Corollary 3 for constrained optimization problems which are adapted from Lemmas 1, 3 and Corollary 3 for regularized optimization problems in \( [27] \). Lemma 4 establishes an upper bound of the distance of any feasible solution to the optimal solution set by the gap of the objective function value at the feasible solution and the optimal objective function value, which is a key to establish the linear convergence rate for the proposed VRPSG algorithm. It is well-known that the bound in Lemma 4 holds under the strongly convex condition. However, without the strongly convex condition, it is non-trivial to obtain this bound. To address this problem, we make suitable assumptions but without strong convexity to establish this inequality by adopting Hoffman’s bound \( [23] \). Note that although Lemmas 1, 2, 3 for constrained optimization problems can be adapted from regularized optimization problems. Lemma 4...
may not be easily extended to regularized optimization problems. Besides, Lemma 4 may not be easily extended to non-polyhedral constrained optimization problems (please refer to Section 4 for more details).

The following lemma establishes a relation between the difference of gradients on components and the difference of objective functions.

**Lemma 1** Let \( w^* \in \mathcal{W}^* \) be any optimal solution to the problem in Eq. (1), \( f^* = f(w^*) \) be the optimal objective function value in Eq. (7), and \( L_P = \max_{i \in \{1, \ldots, n\}} [L_i/(np_i)] \) with \( p_i \in (0, 1), \sum_{i=1}^n p_i = 1 \). Then under assumptions A1-A3, for all \( w \in \mathcal{W} \), we have

\[
\frac{1}{n} \sum_{i=1}^n \frac{1}{np_i} \left\| \nabla f_i(w) - \nabla f_i(w^*) \right\|^2 \leq 2L_P [f(w) - f^*].
\]

**Proof** For any \( i \in \{1, \ldots, n\} \), we consider the following function

\[
\phi_i(w) = f_i(w) - f_i(w^*) - \nabla f_i(w^*)^T (w - w^*).
\]

It follows from the convexity of \( \phi_i(w) \) and \( \nabla \phi_i(w^*) = 0 \) that \( \min_{w \in \mathbb{R}^d} \phi_i(w) = \phi_i(w^*) = 0 \). Recalling that \( \nabla \phi_i(w) = \nabla f_i(w) - \nabla f_i(w^*) \) is \( L_i \)-Lipschitz continuous, we have for all \( w \in \mathcal{W} \):

\[
0 = \phi_i(w^*) \leq \min_{\eta \in \mathbb{R}} \phi_i(w - \eta \nabla \phi_i(w)) \leq \min_{\eta \in \mathbb{R}} \left\{ \phi_i(w) - \eta \| \nabla \phi_i(w) \|^2 + \frac{L_i \eta^2}{2} \| \nabla \phi_i(w) \|^2 \right\} = \phi_i(w) - \frac{1}{2L_i} \| \nabla \phi_i(w) \|^2 = \phi_i(w) - \frac{1}{2L_i} \| \nabla f_i(w) - \nabla f_i(w^*) \|^2,
\]

which implies for all \( w \in \mathcal{W} \):

\[
\| \nabla f_i(w) - \nabla f_i(w^*) \|^2 \leq 2L_i \phi_i(w) = 2L_i(f_i(w) - f_i(w^*)) - \nabla f_i(w^*)^T (w - w^*).
\]

Dividing the above inequality by \( n^2 p_i \) and summing over \( i = 1, \ldots, n \), we have

\[
\frac{1}{n} \sum_{i=1}^n \frac{1}{np_i} \left\| \nabla f_i(w) - \nabla f_i(w^*) \right\|^2 \leq 2L_P [f(w) - f(w^*) - \nabla f(w^*)^T (w - w^*)],
\]

(5)

where we use \( L_{avg} = \sum_{i=1}^n L_i/n \leq L_P = \max_{i \in \{1, \ldots, n\}} [L_i/(np_i)] \) (see Lemma 5 in the Appendix) and \( f(w) = \frac{1}{n} \sum_{i=1}^n f_i(w) \). Recalling that \( w^* \in \mathcal{W}^* \) is an optimal solution to Eq. (1) and \( w \in \mathcal{W} \), it follows from the optimality condition of Eq. (1) that

\[
\nabla f(w^*)^T (w - w^*) \geq 0,
\]

which together with Eq. (5) and \( f^* = f(w^*) \) immediately proves the lemma. \( \square \)

Based on Lemma 1 we bound the variance of \( v_i^k \) in terms of the difference of objective functions.

**Lemma 2** Let \( w^* \in \mathcal{W}^* \) be any optimal solution to the problem in Eq. (4), \( f^* = f(w^*) \) be the optimal objective function value in Eq. (4). Then under assumptions A1-A3, we have

\[
\mathbb{E} \left[ v_i^k \mid \mathcal{F}_{i-1}^k \right] = \nabla f(w_{i-1}^k),
\]

(6)

\[
\mathbb{E} \left[ \left\| v_i^k - \nabla f(w_{i-1}^k) \right\|^2 \mid \mathcal{F}_{i-1}^k \right] \leq 4L_P (f(w_{i-1}^k) - f^* + f(\tilde{w}^{k-1}) - f^*),
\]

(7)

where \( \mathcal{F}_{i}^k \) is defined in Theorem 4, \( v_i^k, w_{i-1}^k, \tilde{w}^{k-1} \) are defined in Algorithm 4, \( L_P = \max_{i \in \{1, \ldots, n\}} [L_i/(np_i)] \).
Proof Taking expectation with respect to $\mathcal{F}_t^k$ conditioned on $\mathcal{F}_{t-1}^k$ and noticing that $\mathcal{F}_t^k = \mathcal{F}_{t-1}^k \cup \{i_t^k\}$, we have

$$E_{\mathcal{F}_t^k} \left[ \frac{1}{np_t^k} \nabla f_{i_t^k}(w_{t-1}^k) \mid \mathcal{F}_{t-1}^k \right] = \sum_{i=1}^{n} \frac{p_i}{np_t^k} \nabla f_i(w_{t-1}^k) = \nabla f(w_{t-1}^k),$$

$$E_{\mathcal{F}_t^k} \left[ \frac{1}{np_t^k} \nabla f_{i_t^k}^k(w_{k-1}^k) \mid \mathcal{F}_{t-1}^k \right] = \sum_{i=1}^{n} \frac{p_i}{np_t^k} \nabla f_i(w_{k-1}^k) = \nabla f(w_{k-1}^k).$$

It follows that

$$E_{\mathcal{F}_t^k} \left[ v_t^k \mid \mathcal{F}_{t-1}^k \right] = E_{\mathcal{F}_t^k} \left[ \frac{1}{np_t^k} (\nabla f_{i_t^k}(w_{t-1}^k) - \nabla f_{i_t^k}(w_{k-1}^k)) + \nabla f(w_{k-1}^k) \mid \mathcal{F}_{t-1}^k \right] = \nabla f(w_{t-1}^k).$$

We next prove Eq. (7) as follows:

$$E_{\mathcal{F}_t^k} \left[ ||v_t^k - \nabla f(w_{k-1}^k)||^2 \mid \mathcal{F}_{t-1}^k \right] = E_{\mathcal{F}_t^k} \left[ \left\| \frac{1}{np_t^k} \left( \nabla f_{i_t^k}(w_{t-1}^k) - \nabla f_{i_t^k}(w_{k-1}^k) \right) \right. \right. \left. \left. \right\| \mid \mathcal{F}_{t-1}^k \right]$$

$$= E_{\mathcal{F}_t^k} \left[ \left\| \frac{1}{np_t^k} \left( \nabla f_{i_t^k}(w_{t-1}^k) - \nabla f_{i_t^k}(w_{k-1}^k) \right) \right. \right. \left. \left. \right\| \mid \mathcal{F}_{t-1}^k \right] + \left\| \frac{1}{np_t^k} \left( \nabla f_{i_t^k}(w_{k-1}^k) - \nabla f_{i_t^k}(w^*) \right) \right\| \mid \mathcal{F}_{t-1}^k \right]$$

$$\leq 2E_{\mathcal{F}_t^k} \left[ \left\| \frac{1}{np_t^k} \left( \nabla f_{i_t^k}(w_{t-1}^k) - \nabla f_{i_t^k}(w_{k-1}^k) \right) \right. \right. \left. \left. \right\| \mid \mathcal{F}_{t-1}^k \right]$$

$$+ 2E_{\mathcal{F}_t^k} \left[ \left\| \frac{1}{np_t^k} \left( \nabla f_{i_t^k}(w_{k-1}^k) - \nabla f_{i_t^k}(w^*) \right) \right\| \mid \mathcal{F}_{t-1}^k \right]$$

$$= 2 \sum_{i=1}^{n} \frac{p_i}{np_t^k} \left\| \nabla f_i(w_{t-1}^k) - \nabla f_i(w^*) \right\|^2 + 2 \sum_{i=1}^{n} \frac{p_i}{np_t^k} \left\| \nabla f_i(w_{k-1}^k) - \nabla f_i(w^*) \right\|^2$$

$$\leq 4LP \left( f(w_{t-1}^k) - f(w^*) + f(w_{k-1}^k) - f(w^*) \right)$$

$$= 4LP \left( f(w_{t-1}^k) - f^* + f(w_{k-1}^k) - f^* \right),$$

where the second equality is due to

$$E_{\mathcal{F}_t^k} \left[ \frac{1}{np_t^k} \left( \nabla f_{i_t^k}(w_{t-1}^k) - \nabla f_{i_t^k}(w_{k-1}^k) \right) \mid \mathcal{F}_{t-1}^k \right] = \nabla f(w^k) - \nabla f(w_{k-1}^k)$$

and $E \left[ \|\xi - E[\xi]\|^2 \right] = E \left[ \|\xi\|^2 \right] - E \left[ \|\xi\|^2 \right]$ for all random vector $\xi \in \mathbb{R}^d$; the second inequality is due to $\|x + y\|^2 \leq 2\|x\|^2 + 2\|y\|^2$; the third inequality is due to Lemma 11 with $w_{t-1}^k, w_{k-1}^k \in \mathcal{W}$, where $w_{t-1}^k \in \mathcal{W}$ is obvious and $w_{k-1}^k \in \mathcal{W}$ follows from the fact that $w_{k-1}^k$ is a convex combination of vectors in the convex set $\mathcal{W}$. \hfill \square

Remark 2 Eq. (8) implies that $v_t^k$ is an unbiased estimate of $\nabla f(w_{t-1}^k)$. To see that the variance is reduced, we notice that, according to Eq. (7), the variance $E_{\mathcal{F}_t^k} \left[ \left\| v_t^k - \nabla f(w_{t-1}^k) \right\|^2 \mid \mathcal{F}_{t-1}^k \right]$ approaches zero when both $\tilde{w}_{k-1}^k$ and $w_{t-1}^k$ converge to any optimal solution $w^*$. 

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The following lemma presents a bound independent of the algorithm. The terms in the left-hand side of the bound will appear in the proof of Theorem 1.

**Lemma 3** Let \( w^* \in W^* \) be any optimal solution to the problem in Eq. \((1)\), \( f^* = f(w^*) \) be the optimal objective function value in Eq. \((2)\), \( \delta^k_t = \nabla f(w^*_t - w^*_k) - v^k_t \), \( g^k_t = (w^*_t - w^*_k) / \eta \) and \( 0 < \eta \leq 1 / L \). Then we have

\[
(w^* - w^*_k)^T g^k_t + \frac{\eta}{2}\|g^k_t\|^2 \leq f^* - f(w^*_k) - (w^* - w^*_k)^T \delta^k_t.
\]

**Proof** We know that \( w^* \in W^* \subseteq W \). Thus, by the optimality condition of \( w^*_k = \Pi_W(w^*_t - \eta v^k_t) \),

\[
\arg \min \{ f(w^*_k) \mid \|w - (w^*_k - \eta v^k_t)\|^2 \}, \text{ we have}
\]

\[
(w^*_k - w^*_t + \eta v^k_t)^T (w^* - w^*_k) \geq 0,
\]

which together with \( g^k_t = (w^*_t - w^*_k) / \eta \) implies that

\[
(w^* - w^*_k)^T v^k_t \geq (w^* - w^*_k)^T g^k_t.
\]

By the convexity of \( f(\cdot) \), we have

\[
f(w^*) \geq f(w^*_k) + \nabla f(w^*_t - w^*_k)^T (w^* - w^*_k).
\]

Recalling that \( f(\cdot) \) is \( L \)-Lipschitz continuous gradient, we have

\[
f(w^*_t - w^*_k) \geq f(w^*_k) - \nabla f(w^*_k - w^*_t - w^*_k) - \frac{L}{2} \|w^*_k - w^*_t - w^*_k\|^2,
\]

which together with Eq. \((9)\) implies that

\[
f(w^*) \geq f(w^*_k) + \nabla f(w^*_k - w^*_k)^T (w^* - w^*_k) - \frac{L}{2} \|w^*_k - w^*_k\|^2 + \nabla f(w^*_t - w^*_k)^T (w^* - w^*_k)
\]

\[
= f(w^*_k) + (w^* - w^*_k)^T \delta^k_t + (w^* - w^*_k)^T v^k_t - \frac{L}{2} \|g^k_t\|^2
\]

\[
\geq f(w^*_k) + (w^* - w^*_k)^T \delta^k_t + (w^* - w^*_k)^T g^k_t - \frac{L}{2} \|g^k_t\|^2
\]

\[
= f(w^*_k) + (w^* - w^*_k)^T \delta^k_t + (w^* - w^*_k - w^*_k - w^*_k)^T g^k_t - \frac{L}{2} \|g^k_t\|^2
\]

\[
= f(w^*_k) + (w^* - w^*_k)^T \delta^k_t + (w^* - w^*_k - w^*_k)^T g^k_t + \frac{\eta}{2} \|g^k_t\|^2
\]

\[
\geq f(w^*_k) + (w^* - w^*_k)^T \delta^k_t + (w^* - w^*_k - w^*_k)^T g^k_t + \frac{\eta}{2} \|g^k_t\|^2,
\]

where the first and fourth equalities are due to \( g^k_t = (w^*_k - w^*_k) / \eta \); the second equality is due to \( \delta^k_t = \nabla f(w^*_k - w^*_k) - v^k_t \); the second inequality is due to Eq. \((8)\); the last inequality is due to \( 0 < \eta \leq 1 / L \). Rearranging the above inequality by noticing that \( f^* = f(w^*) \), we prove the lemma. \( \Box \)

The following lemma presents an upper bound of the distance of any feasible solution to the optimal solution set by the gap of the objective function value at the feasible solution and the optimal objective function value, which is the key to establish the linear convergence without strong convexity.

**Lemma 4** Let \( w \in W = \{ w : Cw \leq b \} \), \( w^* = \Pi_{W^*}(w) \) and \( f^* \) be the optimal objective function value in Eq. \((1)\). Then under assumptions \( A1-A3 \), there exist constants \( \mu > 0 \) and \( \beta > 0 \) such that

\[
f(w) - f^* \geq \frac{\mu}{2\beta} \|w - w^*\|^2, \forall w \in W.
\]

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Thus, we have

\[ \|w - w^*\| \leq \theta \|Xw - r^*\|. \]

Noticing that \( \bar{w} = \Pi_{W^*}(w) \), we have \( \|w - \bar{w}\| \leq \|w - w^*\| \) and \( X\bar{w} = r^* \). Thus,

\[ \|Xw - X\bar{w}\|^2 = \|Xw - r^*\|^2 \geq \frac{1}{\beta} \|w - w^*\|^2 \geq \frac{1}{\beta} \|w - \bar{w}\|^2, \tag{10} \]

where \( \beta = \theta^2 \). By assumption A3, we know that \( W \) is compact. Thus, for any \( w \in W \), both \( Xw \) and \( X\bar{w} \) belong to some convex compact subset \( U \subseteq \mathbb{R}^n \). By the strong convexity of \( h(\cdot) \) on the subset \( U \), there exists a constant \( \mu > 0 \) such that

\[ h(Xw) - h(X\bar{w}) \geq \nabla h(X\bar{w})^T (Xw - X\bar{w}) + \frac{\mu}{2} \|Xw - X\bar{w}\|^2, \]

which together with \( f(w) = h(Xw) \) implies that

\[ f(w) - f(\bar{w}) \geq \nabla f(\bar{w})^T (w - \bar{w}) + \frac{\mu}{2} \|Xw - X\bar{w}\|^2. \tag{11} \]

Noticing that \( w \in W \) and \( \bar{w} \in W^* \), we have

\[ \nabla f(\bar{w})^T (w - \bar{w}) \geq 0, \]

which together with Eqs. (10), (11) proves the lemma. \( \square \)

We are now ready to complete the proof of Theorem 1 as follows:

**Proof of Theorem** 1 Different from the convergence proof in [10, 27], we begin the proof by establishing the recursive relationship between the distance of the current feasible solution to the optimal solution set and the distance of the previous feasible solution to the optimal solution set. Let \( \bar{w}^k_t = \Pi_{W^*}(w^k_t) \) for all \( k, t \geq 0 \). Then we have \( \bar{w}^k_{t-1} \in W^* \), which together with the definition of \( \bar{w}^k_t \) and \( g^k_t = (w^k_{t-1} - w^k_t) / \eta \) implies that

\[ \|w^k_t - \bar{w}^k_t\|^2 \leq \|w^k_t - \bar{w}^k_{t-1}\|^2 = \|w^k_{t-1} - \eta g^k_t - \bar{w}^k_{t-1}\|^2 \]

\[ = \|w^k_{t-1} - \bar{w}^k_{t-1}\|^2 + 2\eta (\bar{w}^k_{t-1} - w^k_{t-1})^T g^k_t + \eta^2 \|g^k_t\|^2 \]

\[ \leq \|w^k_{t-1} - \bar{w}^k_{t-1}\|^2 + 2\eta (f^* - f(w^k_t) - (w^k_{t-1} - w^k_t)^T \delta^k_t) \tag{12} \]

where the last inequality is due to Lemma 3 with \( \bar{w}^k_{t-1} \in W^* \) and \( 0 < \eta < 1/(4L_P) < 1/(L_P) \leq 1/L \) (see Lemma 5). To bound the quantity \(- (\bar{w}^k_{t-1} - w^k_t)^T \delta^k_t \), we define an auxiliary vector as

\[ \tilde{w}^k_t = \Pi_W(w^k_{t-1} - \eta \nabla f(w^k_{t-1})). \]

Thus, we have

\[ - (\bar{w}^k_{t-1} - w^k_t)^T \delta^k_t = (w^k_t - \tilde{w}^k_t + \tilde{w}^k_t - \bar{w}^k_{t-1})^T \delta^k_t \]

\[ \leq \|w^k_t - \tilde{w}^k_t\| \|\delta^k_t\| + (\tilde{w}^k_t - \bar{w}^k_{t-1})^T \delta^k_t \]

\[ \leq \|w^k_t - \tilde{w}^k_t\| \|\delta^k_t\| + (\tilde{w}^k_t - \bar{w}^k_{t-1})^T \delta^k_t \]

\[ = \eta \|\delta^k_t\|^2 + (\tilde{w}^k_t - \bar{w}^k_{t-1})^T \delta^k_t, \]

where
where the second inequality is due to the non-expansive property of projection (Proposition B.11(c) in [2]). The above inequality and Eq. (12) imply that

\[
\|w^k_t - \tilde{w}^k_t\|^2 \leq \|w^k_{t-1} - \tilde{w}^k_{t-1}\|^2 - 2\eta (f(w^k_t) - f^*) + 2\eta^2 \|\delta^k_t\|^2 + 2\eta (\tilde{w}^k_t - \tilde{w}^k_{t-1})^T \delta^k_t.
\] (13)

Considering Lemma 2 with \(\delta^k_t = \nabla f(w^k_{t-1}) - w^k_t\) and noticing that \(\tilde{w}^k_t - \tilde{w}^k_{t-1}\) is independent of the random variable \(\xi^k_t\) and \(\mathcal{F}^k_{t-1} = \mathcal{F}^k_{t-1} \cup \{\xi^k_t\}\), we have \(\mathbb{E}_{\mathcal{F}^k_{t-1}}[\|\delta^k_t\|^2 | \mathcal{F}^k_{t-1}] \leq 4L_P (f(w^k_{t-1}) - f^* + f(\tilde{w}^{k-1}_t) - f^*)\) and \(\mathbb{E}_{\mathcal{F}^k_{t}}[(\tilde{w}^k_t - \tilde{w}^k_{t-1})^T \delta^k_t | \mathcal{F}^k_{t-1}] = (\tilde{w}^k_t - \tilde{w}^k_{t-1})^T \mathbb{E}_{\mathcal{F}^k_{t}}[\delta^k_t | \mathcal{F}^k_{t-1}] = 0\). Taking expectation with respect to \(\mathcal{F}^k_{t-1}\) conditioned on \(\mathcal{F}^k_{t-1}\) on both sides of Eq. (13), we have

\[
\mathbb{E}_{\mathcal{F}^k_{t-1}} \left[ \|w^k_t - \tilde{w}^k_t\|^2 | \mathcal{F}^k_{t-1} \right] \leq \|w^k_{t-1} - \tilde{w}^k_{t-1}\|^2 - 2\eta \mathbb{E}_{\mathcal{F}^k_{t-1}} [f(w^k_t) - f^* | \mathcal{F}^k_{t-1}] + 2\eta^2 \mathbb{E}_{\mathcal{F}^k_{t-1}} [\|\delta^k_t\|^2 | \mathcal{F}^k_{t-1}] + 2\eta (\tilde{w}^k_t - \tilde{w}^k_{t-1})^T \mathbb{E}_{\mathcal{F}^k_{t}}[\delta^k_t | \mathcal{F}^k_{t-1}] \leq \|w^k_{t-1} - \tilde{w}^k_{t-1}\|^2 - 2\eta \mathbb{E}_{\mathcal{F}^k_{t-1}} [f(w^k_t) - f^* | \mathcal{F}^k_{t-1}] + 8L_P \eta^2 (f(w^k_{t-1}) - f^* + f(\tilde{w}^{k-1}_t) - f^*). \]

Taking expectation with respect to \(\mathcal{F}^k_{t-1}\) on both sides of the above inequality and considering the fact that \(\mathbb{E}_{\mathcal{F}^k_{t-1}} \left[ \mathbb{E}_{\mathcal{F}^k_{t}} \left[ \|w^k_t - \tilde{w}^k_t\|^2 | \mathcal{F}^k_{t-1} \right] \right] = \mathbb{E}_{\mathcal{F}^k_{t}} \left[ \|w^k_t - \tilde{w}^k_t\|^2 \right]\), we have

\[
\mathbb{E}_{\mathcal{F}^k_{t}} \left[ \|w^k_t - \tilde{w}^k_t\|^2 \right] \leq \mathbb{E}_{\mathcal{F}^k_{t-1}} \left[ \|w^k_{t-1} - \tilde{w}^k_{t-1}\|^2 \right] - 2\eta \mathbb{E}_{\mathcal{F}^k_{t-1}} [f(w^k_t) - f^*] + 8L_P \eta^2 \mathbb{E}_{\mathcal{F}^k_{t-1}} [f(w^k_{t-1}) - f^* + f(\tilde{w}^{k-1}_t) - f^*].
\]

Summing the above inequality over \(t = 1, 2, \cdots, m\) by noticing that \(\mathcal{F}^k_0 = \mathcal{F}^k_{m-1}\), we have

\[
\mathbb{E}_{\mathcal{F}^k_{m}} \left[ \|w^k_m - \tilde{w}^k_m\|^2 \right] + 2\eta \sum_{t=1}^{m} \mathbb{E}_{\mathcal{F}^k_{t}} [f(w^k_t) - f^*] \leq \mathbb{E}_{\mathcal{F}^k_{m-1}} \left[ \|w^k_0 - \tilde{w}^k_0\|^2 \right] + 8L_P \eta^2 \sum_{t=1}^{m} \mathbb{E}_{\mathcal{F}^k_{t-1}} [f(w^k_{t-1}) - f^*] + 8L_P \eta^2 m \mathbb{E}_{\mathcal{F}^k_{m-1}} [f(w^{k-1}) - f^*],
\]

Thus, we have

\[
\mathbb{E}_{\mathcal{F}^k_{m}} \left[ \|w^k_m - \tilde{w}^k_m\|^2 \right] + 2\eta \mathbb{E}_{\mathcal{F}^k_{m}} [f(w^k_m) - f^*] + 2\eta (1 - 4L_P \eta) \sum_{t=1}^{m-1} \mathbb{E}_{\mathcal{F}^k_{t}} [f(w^k_t) - f^*] \leq \mathbb{E}_{\mathcal{F}^k_{m-1}} \left[ \|w^k_0 - \tilde{w}^k_0\|^2 \right] + 8L_P \eta^2 \mathbb{E}_{\mathcal{F}^k_{m-1}} [f(w^k_0) - f^* + m(f(\tilde{w}^{k-1}_t) - f^*)],
\]

which together with \(\mathbb{E}_{\mathcal{F}^k_{m}} \left[ \|w^k_m - \tilde{w}^k_m\|^2 \right] \geq 0, 2\eta \mathbb{E}_{\mathcal{F}^k_{m}} [f(w^k_m) - f^*] \geq 0, 2\eta > 2\eta (1 - 4L_P \eta) > 0\) and \(w^k_0 = \tilde{w}^k_{m-1}\) implies that

\[
2\eta (1 - 4L_P \eta) \sum_{t=1}^{m} \mathbb{E}_{\mathcal{F}^k_{t}} [f(w^k_t) - f^*] \leq \mathbb{E}_{\mathcal{F}^k_{m-1}} \left[ \|w^k_0 - \tilde{w}^k_0\|^2 \right] + 8L_P \eta^2 (m + 1) \mathbb{E}_{\mathcal{F}^k_{m-1}} [f(w^{k-1}) - f^*],
\]

(14)

where we use the fact that \(\mathbb{E}_{\mathcal{F}^k_{t-1}} [f(w^k_0) - f^*] = \mathbb{E}_{\mathcal{F}^k_{t-1}} [f(w^{k-1}) - f^*] = \mathbb{E}_{\mathcal{F}^k_{m-1}} [f(w^{k-1}) - f^*].\) By the convexity of \(f(\cdot)\), we have

\[
f(w^k) = f \left( \frac{1}{m} \sum_{t=1}^{m} w^k_t \right) \leq \frac{1}{m} \sum_{t=1}^{m} f(w^k_t).
\]
Thus, we have

\[ m \left( f(\hat{w}^k) - f^* \right) \leq \sum_{i=1}^{m} \left( f(w^k_i) - f^* \right), \]  

(15)

Considering Lemma 4 with \( \hat{w}^{-1} = w^0 \in \mathcal{W} \) and \( \hat{w}^k = \Pi_{\mathcal{W}}(w^k) \), we have

\[ f(\hat{w}^{k-1}) - f^* = f(w^k_0) - f^* \geq \frac{\mu}{2\beta} \|w^k_0 - \hat{w}^k_0\|^2, \]

which together with Eqs. (14), (15) implies that

\[ 2\eta(1 - 4L_p\eta)m E_{x_m^k} \left[ f(\hat{w}^k) - f^* \right] \leq \frac{4L_p\eta(m + 1)}{(1 - 4L_p\eta)m} + \frac{\beta}{\mu\eta(1 - 4L_p\eta)m} E_{x_m^k} \left[ f(\hat{w}^{k-1}) - f^* \right], \]

Thus, we have

\[ E_{x_m^k} \left[ f(\hat{w}^k) - f^* \right] \leq \left( \frac{4L_p\eta(m + 1)}{(1 - 4L_p\eta)m} + \frac{1}{\mu\eta(1 - 4L_p\eta)m} \right) E_{x_m^k} \left[ f(\hat{w}^{k-1}) - f^* \right]. \]

Using the above recursive relation and considering the definition of \( \rho \) in Eq. (4), we complete the proof of the theorem. \( \square \)

**Remark 3** If \( f \) is strongly convex with parameter \( \bar{\mu} \), then the inequality in Lemma 4 holds with \( \beta = 1 \) and \( \mu = \bar{\mu} \). Therefore, we can easily obtain from the proof of Theorem 4 that

\[ E_{x_m^k} \left[ f(\hat{w}^k) - f^* \right] \leq \left( \frac{4L_p\eta(m + 1)}{(1 - 4L_p\eta)m} + \frac{1}{\mu\eta(1 - 4L_p\eta)m} \right) \left( f(\hat{w}^0) - f^* \right), \]

which has the same convergence rate as [27].

**4 Discussion**

Recall that one of the assumptions for the convergence analysis in Theorem 1 is that the constraint set is polyhedral. An interesting question is whether we can extend the linear convergence rate in Theorem 1 to constrained optimization problems beyond polyhedral sets. Let us consider the following sphere constrained optimization problem:

\[ \min_{w \in \mathbb{R}^d} \{ f(w) \quad s.t. \quad w \in \mathcal{W} = \{ w : \|w\| \leq \tau \} \}, \]  

(16)

where \( f(w) = h(Xw) \) satisfies assumptions A1, A2 and \( \tau > 0 \). Obviously the sphere constrained set does not satisfy assumption A3. Let \( f(w) = (w_1 + w_2 - \sqrt{2}) \) and \( \tau = 1 \). Then the optimal solution set of Eq. (16) is

\[ \mathcal{W}^* = \left\{ \left[ \sqrt{2}/2, \sqrt{2}/2 \right]^T \right\}. \]

Let \( w = [\cos(\omega), \sin(\omega)]^T \). It is easy to obtain that \( w \in \mathcal{W} \), \( \|w - \bar{w}\|^2 = (\cos(\omega) - \sqrt{2}/2)^2 + (\sin(\omega) - \sqrt{2}/2)^2 = -\sqrt{2}(\cos(\omega) + \sin(\omega) - \sqrt{2}) \) and \( f(w) - f^* = (\cos(\omega) + \sin(\omega) - \sqrt{2})^2 \). Thus, we have

\[ \lim_{\omega \rightarrow \pi/4} \frac{f(w) - f^*}{\|w - \bar{w}\|^2} = 0, \]
which implies that Lemma 4 does not hold for the sphere constrained optimization problem in Eq. (16). Notice that Lemma 4 may not be a necessary condition of the linear convergence analysis in Theorem 1. So we cannot conclude that it is impossible to extend the linear convergence rate in Theorem 1 to the sphere constrained optimization problem in Eq. (16). However, this simple example illustrates that the extension of the linear convergence analysis to the non-polyhedral constrained optimization problem may not be easy.

It is well-known that the constrained optimization problem in Eq. (1) is equivalent to some regularized optimization problems. However, such an example illustrates that the analysis may be highly non-trivial even if the extension is possible.

In this section, we validate the effectiveness of VRPSG by solving the following logistic regression problem:

\[
\min_{\mathbf{w} \in \mathbb{R}^d} \left\{ f(\mathbf{w}) \right\}, \quad \text{s.t.} \quad \|\mathbf{w}\|_1 \leq \tau, \tag{17}
\]

where \( f(\mathbf{w}) = h(\mathbf{X} \mathbf{w}) \) satisfies assumptions A1, A2. It is well-known that Eq. (17) and Eq. (18) have the same optimal solution set when \( \tau \) and \( \lambda \) choose appropriate values. In the following, we focus on the case where Eq. (17) and Eq. (18) have the same optimal solution set. It is easy to verify that the \( \ell_1 \)-constrained problem in Eq. (17) satisfies assumptions A1-A3 and thus Theorem 1 is applicable to Eq. (17). It is known that we can use Algorithm 1 to solve the \( \ell_1 \)-constrained problem in Eq. (18) by replacing the projection step in Algorithm 1 (Line 10) by the proximal step. But the question is if we can extend the convergence analysis in Theorem 1 with respect to \( F(\cdot) \). One key building block to establish a similar linear convergence rate as in Theorem 1 is to prove a bound similar to Lemma 4. Specifically, is there a constant \( \theta > 0 \) such that

\[
F(\mathbf{w}) - F^s \geq \theta \|\mathbf{w} - \bar{\mathbf{w}}\|^2 \tag{19}
\]

holds for all \( \mathbf{w} \in \mathbb{R}^d \) where \( F^s \) is the optimal objective function value in Eq. (18)? Let us consider the following example by setting \( f(\mathbf{w}) = (w_1 + w_2 - 1)^2, \tau = 0.5 \) and \( \lambda = 1 \). It is easy to verify that Eq. (17) and Eq. (18) have the same optimal solution set

\[
\mathcal{W}^* = \{w_1 + w_2 = 0.5, w_1 \geq 0, w_2 \geq 0\}.
\]

Let \( \mathbf{w} = [w_1, w_2]^T \) with \( w_1 + w_2 = 0.5 \) and \( w_1 > 0, w_2 < 0 \). It is easy to obtain that \( \bar{\mathbf{w}} = \Pi_{\mathcal{W}^*}(\mathbf{w}) = [0.5, 0]^T \), \( \|\mathbf{w} - \bar{\mathbf{w}}\|^2 = (w_1 - 0.5)^2 + w_2^2 = 2w_2^2 > 0 \) and \( F(\mathbf{w}) - F^s = w_1 - w_2 - 0.5 = -2w_2 > 0 \). Thus, we have

\[
\lim_{w_2 \to -\infty} \frac{F(\mathbf{w}) - F^s}{\|\mathbf{w} - \bar{\mathbf{w}}\|^2} = 0,
\]

which implies that there does not exist a constant \( \theta > 0 \) such that Eq. (19) holds for all \( \mathbf{w} \in \mathbb{R}^d \). Since Eq. (19) may not be a necessary condition of the linear convergence rate for solving Eq. (18), the example above only shows that the convergence analysis in Theorem 1 may not be extended to regularized optimization problems. However, such an example illustrates that the analysis may be highly non-trivial even if the extension is possible.

5 Experiments

In this section, we validate the effectiveness of VRPSG by solving the following \( \ell_1 \)-constrained logistic regression problem:

\[
\min_{\mathbf{w} \in \mathbb{R}^d} \left\{ f(\mathbf{w}) = \frac{1}{n} \sum_{i=1}^n \log(1 + \exp(-y_i \mathbf{x}_i^T \mathbf{w})) \right\}, \quad \text{s.t.} \quad \|\mathbf{w}\|_1 \leq \tau,
\]
where \( n \) is the number of samples; \( \tau > 0 \) is the constrained parameter; \( \mathbf{x}_i \in \mathbb{R}^d \) is the \( i \)-th sample; \( y_i \in \{1, -1\} \) is the label of the sample \( \mathbf{x}_i \). For the above problem, it is easy to obtain that the convex component is \( f_i(\mathbf{w}) = \log(1 + \exp(-y_i \mathbf{x}_i^T \mathbf{w})) \) and the Lipschitz constant of \( \nabla f_i(\mathbf{w}) \) is \( \|\mathbf{x}_i\|^2/4 \).

We conduct experiments on three real-world data sets: classic (\( n = 7094, d = 41681 \)), reviews (\( n = 4069, d = 18482 \)) and sports (\( n = 8580, d = 14866 \)). The three data sets are multi-class sparse text data and can be downloaded from [http://www.shi-zhong.com/software/docdata.zip](http://www.shi-zhong.com/software/docdata.zip). To adapt the data to the two-class logistic regression problem, we transform the multi-class data into a two-class by labeling the first half of all classes as positive class, and the remaining classes as the negative class.

### 5.1 Sensitivity Studies for VRPSG

We conduct sensitivity studies for VRPSG on the sampling distribution parameter \( \mathbf{p} = [p_1, \cdots, p_n]^T \), the inner iterative number \( m \) and the step size \( \eta \) by varying one parameter and keeping the other two parameters fixed. Notice that the projection (line 10 in Algorithm 1) onto the \( \ell_1 \)-ball is easy to solve \footnote{Computing the gradient on a single sample counts as one gradient evaluation.} \footnote{We do not include SAG \cite{Jenatton13} and SDCA \cite{Shalev-Shwartz14} in comparison, since SAG is only applicable to unconstrained optimization problems and SDCA is adopted to solve regularized optimization problems.} \footnote{Computing the gradient on a single sample counts as one gradient evaluation.} \footnote{We do not include SAG \cite{Jenatton13} and SDCA \cite{Shalev-Shwartz14} in comparison, since SAG is only applicable to unconstrained optimization problems and SDCA is adopted to solve regularized optimization problems.} \footnote{Computing the gradient on a single sample counts as one gradient evaluation.} \footnote{Computing the gradient on a single sample counts as one gradient evaluation.} \footnote{Computing the gradient on a single sample counts as one gradient evaluation.} and thus the dominant computational cost is to compute the gradient. To provide an implementation independent result, we report the objective function value \( f(\mathbf{w}^k) \) vs. the number of gradient evaluations \( \#\text{grad}/n \) plots in Figure 1, Figure 2 and Figure 3. From these results, we have the following observations: (a) The VRPSG algorithm with non-uniform sampling (i.e., \( p_i = L_i / \sum_{i=1}^n L_i \)) is much more efficient than that with uniform sampling (i.e., \( p_i = 1/n \)), which is consistent with the analysis in the remarks of Theorem 1. (b) In general, the VRPSG algorithm by setting \( m = 0.5n, n \) has the most stable performance, which indicates that a small or large \( m \) will degrade the performance of the VRPSG algorithm. (c) The optimal step sizes of the VRPSG algorithm on different data sets are slightly different. Moreover, the VRPSG algorithm with step sizes \( \eta = 1/L_P \) and \( \eta = 5/L_P \) converges quickly, which demonstrates that the VRPSG algorithm still performs well even if the step size is much larger than that required in the theoretical analysis (\( \eta < 0.25/L_P \) is required in Theorem 1). This shows the robustness of the VRPSG algorithm.

![Figure 1: Sensitivity study of VRPSG on the parameter \( \mathbf{p} = [p_1, \cdots, p_n]^T \): the objective function value \( f(\mathbf{w}^k) \) vs. the number of gradient evaluations \( \#\text{grad}/n \) plots (averaged on 10 runs). “Uniform” and “Non-uniform” indicate that \( p_i = 1/n \) and \( p_i = L_i / \sum_{i=1}^n L_i \), respectively. Other parameters are set as \( \tau = 10, m = n, \eta = 1/L_P \).](image)

### 5.2 Comparison with Other Algorithms

We conduct comparison by including the following algorithms:

- AFG: the accelerated full gradient algorithm proposed in \cite{Beck13}, where the adaptive line search scheme is used.
our best knowledge, this is the first linear convergence result of variance-reduced stochastic gradient.

In this paper, we propose a Variance-Reduced Projected Stochastic Gradient (VRPSG) algorithm to efficiently solve a class of constrained optimization problems. Our main technical contribution is to establish a linear convergence rate for the VRPSG algorithm without strong convexity. To our best knowledge, this is the first linear convergence result of variance-reduced stochastic gradient.

Figure 2: Sensitivity study of VRPSG on the parameter $m$: the objective function value $f(\tilde{w}^k)$ vs. the number of gradient evaluations ($\#\text{grad}/n$) plots (averaged on 10 runs). Other parameters are set as $\tau = 10$, $p_i = L_i/\sum_{i=1}^{n} L_i$, $\eta = 1/L_P$.

Figure 3: Sensitivity study of VRPSG on the parameter $\eta$: the objective function value $f(\tilde{w}^k)$ vs. the number of gradient evaluations ($\#\text{grad}/n$) plots (averaged on 10 runs). Other parameters are set as $\tau = 10$, $m = n$, $p_i = L_i/\sum_{i=1}^{n} L_i$.

- SGD: the stochastic gradient descent algorithm in Eq. (3). As suggested by [4], we set the step size as $\eta_k = \eta_0/\sqrt{k}$, where $\eta_0$ is an initial step size.
- VRPSG: the variance-reduced projected stochastic gradient algorithm proposed in this paper.
- VRPSG2: a hybrid algorithm by executing SGD for one pass over the data and then switching to the VRPSG algorithm (similar schemes are also adopted in [10, 27]).

Notice that SGD is sensitive to the initial step size $\eta_0$ [4]. To have a fair comparison of different algorithms, we set different values of $\eta_0$ for SGD to obtain the best performance ($\eta_0 = 5, 1, 0.2, 0.04$). To comprehensively show the convergence behaviors of different algorithms, we report the objective function value $f(\tilde{w}^k)$ and the objective function value gap $f(\tilde{w}^k) - f^*$ vs. the number of gradient evaluations ($\#\text{grad}/n$) plots in Figure 4, from which we have the following observations: (a) Both stochastic algorithms (VRPSG and SGD with a proper initial step size) outperform the full gradient algorithm (AFG). (b) SGD quickly decreases the objective function value in the beginning and gradually slows down in the proceeding iterations. In contrast, VRPSG decreases the objective function value quickly. This phenomenon is commonly expected due to the sub-linear convergence rate of SGD and the linear convergence rate of VRPSG. (c) VRPSG2 performs slightly better than VRPSG, which demonstrates that the hybrid scheme can empirically improve the performance (similar results are also reported in [10, 27]).

6 Conclusion

In this paper, we propose a Variance-Reduced Projected Stochastic Gradient (VRPSG) algorithm to efficiently solve a class of constrained optimization problems. Our main technical contribution is to establish a linear convergence rate for the VRPSG algorithm without strong convexity. To our best knowledge, this is the first linear convergence result of variance-reduced stochastic gradient.
Figure 4: Comparison of different algorithms: the objective function value $f(\mathbf{w}^k)$ (first row) and the objective function value gap $f(\mathbf{w}^k) - f^*$ vs. the number of gradient evaluations ($\#\text{grad}/n$) plots (averaged on 10 runs). The parameter of VRPSG are set as $\tau = 10$, $\eta = 1/LP$, $m = n$, $p_i = L_i/\sum_{i=1}^n L_i$; the step size of SGD is set as $\eta_k = \eta_0/\sqrt{k}$.

algorithms without the strongly convex condition. In the future work, we will try to develop a more general convergence analysis for a wider range of problems including both non-polyhedral constrained optimization problems and regularized optimization problems.

Appendix

Lemma 5 Let $L$ and $L_i$ be the Lipschitz constants of $\nabla f(\mathbf{w})$ and $\nabla f_i(\mathbf{w})$, respectively. Moreover, let $L_{\text{avg}} = \sum_{i=1}^n L_i/n$, $L_{\text{max}} = \max_{i \in \{1, \ldots, n\}} L_i$ and $LP = \max_{i \in \{1, \ldots, n\}} [L_i/(np_i)]$ with $p_i \in (0, 1), \sum_{i=1}^n p_i = 1$. Then we have

$$L \leq L_{\text{avg}} \leq LP \text{ and } L_{\text{avg}} \leq L_{\text{max}}.$$ 

Proof Based on the definition of Lipschitz continuity, we obtain that $L$ and $L_i$ are the smallest positive constants such that for all $\mathbf{w}, \mathbf{u} \in \mathbb{R}^d$:

$$\|\nabla f(\mathbf{w}) - \nabla f(\mathbf{u})\| \leq L\|\mathbf{w} - \mathbf{u}\|, \quad (20)$$

$$\|\nabla f_i(\mathbf{w}) - \nabla f_i(\mathbf{u})\| \leq L_i\|\mathbf{w} - \mathbf{u}\|. \quad (21)$$

Dividing Eq. (21) by $n$ and summing over $i = 1, \ldots, n$, we have

$$\frac{1}{n} \sum_{i=1}^n \|\nabla f_i(\mathbf{w}) - \nabla f_i(\mathbf{u})\| \leq \frac{1}{n} \sum_{i=1}^n L_i\|\mathbf{w} - \mathbf{u}\|. \quad (22)$$

Based on the triangle inequality and $\nabla f(\mathbf{w}) = \frac{1}{n} \sum_{i=1}^n \nabla f_i(\mathbf{w})$ we have

$$\|\nabla f(\mathbf{w}) - \nabla f(\mathbf{u})\| \leq \frac{1}{n} \sum_{i=1}^n \|\nabla f_i(\mathbf{w}) - \nabla f_i(\mathbf{u})\|,$$

which together with $L_{\text{avg}} = \sum_{i=1}^n L_i/n$ and Eqs. (20), (22) implies that $L \leq L_{\text{avg}}$.

Define $\mathbf{s} = [L_1/p_1, \ldots, L_n/p_n]^T$. Noticing that $LP = \max_{i \in \{1, \ldots, n\}} [L_i/(np_i)]$ with $p_i \in (0, 1), \sum_{i=1}^n p_i = 1$ and considering the definition of the dual norm, we have

$$nLP = \max_{i \in \{1, \ldots, n\}} \frac{L_i}{p_i} = \|\mathbf{s}\|_{\infty} = \sup_{\|\mathbf{t}\| \leq 1} \mathbf{t}^T \mathbf{s} \geq \sum_{i=1}^n p_i \frac{L_i}{p_i},$$

$$15$$
which together with $L_{\text{avg}} = \sum_{i=1}^{n} L_i/n$ immediately implies that $L_{\text{avg}} \leq L_P$. $L_{\text{avg}} \leq L_{\text{max}}$ is obvious by the definition of $L_{\text{avg}} = \sum_{i=1}^{n} L_i/n$ and $L_{\text{max}} = \max_{i \in \{1, \ldots, n\}} L_i$.

\[ \Box \]

**Lemma 6** Under assumptions A1-A3, for all $w^* \in W^*$, there exists a unique $r^*$ such that $Xw^* = r^*$. Moreover, $W^* = \{ w^* : Cw^* \leq b, \ Xw^* = r^* \}$.

**Proof** By assumption A3, we know that $W^*$ is not empty. Assume that there are $w^*_1, w^*_2 \in W^*$ such that $Xw^*_1 \neq Xw^*_2$. Then, the optimal objective function value is $f^* = f(w^*_1) = f(w^*_2)$. Due to $w^*_1, w^*_2 \in W^*$ and the convexity of $W^*$, we have $(w^*_1 + w^*_2)/2 \in W^*$. Therefore,

$$f^* = f \left( \frac{1}{2}(w^*_1 + w^*_2) \right) = h \left( \frac{1}{2}Xw^*_1 + \frac{1}{2}Xw^*_2 \right). \tag{23}$$

On the other hand, the strong convexity of $h(\cdot)$ implies that

$$h \left( \frac{1}{2}Xw^*_1 + \frac{1}{2}Xw^*_2 \right) < \frac{1}{2}h(Xw^*_1) + \frac{1}{2}h(Xw^*_2) = \frac{1}{2}(f(w^*_1) + f(w^*_2)) = f^*,$$

leading to a contradiction with Eq. (23). Thus, there exists a unique $r^*$ such that for all $w^* \in W^*$, $Xw^* = r^*$ and $f^* = h(r^*)$.

If $w^* \in W^*$, then $w^* \in W$ and $Xw^* = r^*$, that is, $w^* \in \{ w^* : Cw^* \leq b, \ Xw^* = r^* \}$ and hence $W^* \subseteq \{ w^* : Cw^* \leq b, \ Xw^* = r^* \}$. If $w^* \in \{ w^* : Cw^* \leq b, \ Xw^* = r^* \}$, then $w^*$ is a feasible solution and $f(w^*) = h(Xw^*) = h(r^*) = f^*$, that is, $w^* \in W^*$ and hence $W^* = \{ w^* : Cw^* \leq b, \ Xw^* = r^* \} \subseteq W^*$. Therefore, we have $W^* = \{ w^* : Cw^* \leq b, \ Xw^* = r^* \}$.

\[ \Box \]

**Lemma 7** (Hoffman’s bound, Lemma 4.3 [25]) Let $V = \{ w : Cw \leq b, \ Xw = r \}$ be a non-empty polyhedron. Then for any $w \in \mathbb{R}^d$, there exist a feasible point $w^*$ of $V$ and a constant $\theta > 0$ such that

$$\| w - w^* \| \leq \theta \left\| \frac{Cw - b}{Xw - r} \right\|,$$

where $|Cw - b|^+_{Xw - r}$ denotes the Euclidean projection of $Cw - b$ onto the non-negative orthant and $\theta$ only depends on $C$ and $X$.

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