Evolutionarily Stable Strategies in Quantum Games

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Abstract

Evolutionarily Stable Strategy (ESS) in classical game theory is a refinement of Nash equilibrium concept. We investigate the consequences when a small group of mutants using quantum strategies try to invade a classical ESS in a population engaged in symmetric bimatrix game of Prisoner’s Dilemma. Secondly we show that in an asymmetric quantum game between two players an ESS pair can be made to appear or disappear by resorting to entangled or unentangled initial states used to play the game even when the strategy pair remains a Nash equilibrium in both forms of the game.

I. INTRODUCTION

An evolutionarily stable strategy (ESS) was originally defined by Maynard Smith and Price (1973) [1] with the motivation that a population playing the ESS can withstand a small, invading group. This concept was developed by combining ingredients from game theory and from some work on the evolution of the sex ratio. Maynard Smith considers a population in which members are matched randomly in pairs to play a bimatrix game. The players are anonymous, that is any pair of players plays the same symmetric bimatrix game and the players are identical with respect to their set of strategies and their payoff function. From the symmetry of the bimatrix game it is meant that for strategy set $S$ the payoff of the first player when he plays $A \in S$ and his opponent plays $B \in S$ is same as the payoff of the second player when the latter plays $A$ and the first player plays $B$. A population which adopts an ESS can withstand a small invading group [1–3]. An evolutionarily game means a model of strategic interaction continuing over time in which higher payoff strategies gradually displace strategies with lower payoffs. There is also some inertia involved in distinguishing between evolutionarily and revolutionary changes; here inertia means that aggregate behavior does not change too abruptly.

In a recent publication, Meyer [4] examined game theory from a quantum perspective
and showed that a player can enhance his expected payoff by implementing a quantum strategy. Shortly afterwards, Eisert et al [5] investigated the quantization of nonzero sum game, the Prisoner’s Dilemma (PD), and showed that, if quantum strategies are admitted, the dilemma no longer exists; moreover, they succeeded in constructing a particular quantum strategy which will always outmaneuver any classical strategy. Referring to Dawkins’ ‘Selfish Gene’[6], these authors also hinted that games of survival are being played already on the molecular level, where quantum mechanics dictates the rules. Coming back to the concept of an ESS we notice that it was perhaps introduced into classical game theory for two reasons:

1. Two player games can have multiple Nash equilibria (NE).
2. Population biology problems can be modelled with the help of this concept.

The reasons for claim that (1) holds for quantum as well as classical games are not far from obvious. In our opinion even the reason (2) may have a meaning in quantum setting. In section (ii) of this paper it is our purpose to take up the idea that ‘games of survival are played at molecular level’ and see what happens when ‘mutants’ of ESS theory come up with quantum strategies and try to invade classical ESS. What happens if such an invasion is successful and a new ESS is established; an ESS that is quantum in nature? Suppose afterwards another small group of ‘mutants’ appear equipped with some other quantum strategy. Would it be successful now to invade the quantum ESS? These questions have been considered in section (ii) for the pairwise symmetric game of PD.

We are trying to extend an idea originally proposed for problems in population biology to quantum domain and it needs more substantive evidence than we have provided. Our motivation is to look for the consequences when quantum strategies that Eisert et al have called one and two-parameter quantum strategies satisfy the criteria of an ESS. Later we also consider NE and ESS’s in the PD game when it is played via another scheme proposed recently by Marinatto and Weber [13].

PD is a symmetric game and the payoff to a player depends only on player’s strategy not on player’s identity. In section (iii) we consider quantization of a general $2 \times 2$ matrix game in asymmetric form played via Marinatto and Weber’s scheme. In evolutionary game
theory an ESS for such a game is a strategy pair that forms a strict Nash equilibrium (NE) [11]. We also search what should exactly be the initial entangled state to play a quantum game when a particular strategy pair is a NE in both the classical and quantum versions of the game but an ESS in only version.

II. SYMMETRIC CASE

The PD game has classical available pure strategies Cooperation (C) and Defection (D) [7]. An interesting question is which strategies are likely to be stable and persistent in a population engaged in the pairwise version of the game. A simple analysis [8] show that D will be the pure classical strategy prevalent in the population and hence the classical ESS. In general, suppose that a strategy A is played by almost all members of the population, the rest of the population form a small group of mutants playing strategy B constitute a fraction $\epsilon$ of the total population. The strategy A is said to be evolutionarily stable (ES) [9] against B if $P[A, (1-\epsilon)A + \epsilon B] > P[B, (1-\epsilon)A + \epsilon B]$ where $P[A, B]$ is defined as the payoff to player playing A against player playing B, for all sufficiently small, positive $\epsilon$. There exists some $\epsilon_0$, such that for $\epsilon \in [0, \epsilon_0)$ the inequality is satisfied [9]. If for the given A and B the $\epsilon_0$ specified is as large as possible the $\epsilon_0$ is called the “invasion barrier”. If B comes at a frequency larger than $\epsilon_0$ it will lead to an invasion.

For a symmetric bimatrix game it follows [8,9] that A is an ESS with respect to B

1. If $P[A, A] > P[B, A]$ and

2. If $P[A, A] = P[B, A]$ then $P[A, B] > P[B, B]$ \hspace{1cm} (1)

If most of the players play A, then almost all potential opponents are A players, so if A does better against A than B does, B players will be persistent losers as the game evolves. However, if A and B do equally well against A, then how well the strategies perform against B becomes important. Therefore, for A to be ES against B the strategy A must then do better against B than B does against B. Strategy A is an ESS if A is ES against all $B \neq A$. For pure strategies A and B (classical as well as quantum) the fitnesses [8] can be defined
\[ W(A) = P(A, A)F_A + P(A, B)F_B \quad W(B) = P(B, A)F_A + P(B, B)F_B \] (2)

Where \( F_A \) and \( F_B \) are the classical frequencies of the pure strategies \( A \) and \( B \) respectively. A quantum strategy cannot be treated as a probabilistic sum of pure classical strategies (except under special conditions). Therefore for finding fitness the quantum strategies are treated as ‘new’ strategies that cannot be reduced to the pure classical strategies.

An ESS is usually considered another refinement of the NE concept. For symmetric bimatrix games the relationship is described as [11]

\[ \triangle^{ESS} \subset \triangle^{PE} \subset \triangle^{NE} \] and \( \triangle^{PE} \neq \Phi \) where \( \triangle^{NE}, \triangle^{PE} \) and \( \triangle^{ESS} \) are the sets of symmetric NE, symmetric proper equilibrium and evolutionarily stable strategies respectively. Application of quantum theory gives a new set of NE strategies \( \triangle^{NE} \) and \( \triangle^{ESS} \) may contain quantum strategies as well.

We assume the same quantum version of PD game as described by Eisert et al [5] between two players. A pair of qubits are prepared in unentangled state \( |CC\rangle \) and sent through the entangling gate \( \hat{J} \). \( \hat{J} \) is essentially a unitary operator known to both players and is symmetric with respect to the interchange of two players. The two players, call them Alice and Bob, then apply their local unitary operators \( U_A \) and \( U_B \) respectively. An inverse gate to \( \hat{J} \) is applied before the final measurement by the arbiter. Let \( s_A \) and \( s_B \) be Alice’s and Bob’s strategies respectively. The payoff matrix is the same as chosen by Eisert et al [5] and can be written as

\[
\begin{pmatrix}
(3,3) & (0,5) \\
(5,0) & (1,1)
\end{pmatrix}
\] (3)

Suppose the players apply their respective strategies \( s_A \) and \( s_B \). These strategies are unitary operators at player’s disposal i.e. \( s_A^{-1}U_A \) and \( s_A^{-1}U_B \). If initial state is maximally entangled state \( \rho \) then the final state[4] is

\[
\sigma = (U_A \otimes U_B)\rho(U_A \otimes U_B)^\dagger
\] (4)

The arbiter applies Kraus operators [4,10] on \( \sigma \)

\[
\pi_{CC} = |\psi_{CC}\rangle \langle \psi_{CC}| \quad \pi_{CD} = |\psi_{CD}\rangle \langle \psi_{CD}|
\]
The expected payoffs to Alice and Bob are [4]

\[ P_{A,B} = \text{tr}[\pi_{CC} \sigma] + \text{tr}[\pi_{CD} \sigma] + \text{tr}[\pi_{DC} \sigma] + \text{tr}[\pi_{DD} \sigma] \]  

Because the game is symmetric we define \( P(C, D) \) as the payoff to \( C \) player against \( D \) player. Similarly \( P(D, C) \) is defined. The subscripts of \( A \) and \( B \) are not required.

Eisert et al [4] have used following matrix representations of the unitary operators of one and two-parameter strategies respectively.

\[ U(\theta) = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \]

\[ U(\theta, \phi) = \begin{pmatrix} e^{i\phi} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & e^{-i\phi} \cos(\theta/2) \end{pmatrix} \]  

Where \( \theta \in [0, \pi] \) and \( \phi \in [0, \pi/2] \). The classical pure strategies \( C \) and \( D \) are realized as \( C^-U(0) \), \( D^-U(\pi) \) respectively for one-parameter strategies and \( C^-U(0, 0) \), \( D^-U(\pi, 0) \) respectively for two-parameter strategies. We consider three cases:

A. A small group of mutants appear equipped with one-parameter quantum strategy \( U(\theta) \) when \( D \) exists as a classical ESS.

B. The mutants are equipped with two-parameter quantum strategy \( U(\theta, \phi) \) against classical ESS.

C. The mutants have successfully invaded and a two-parameter quantum strategy \( Q^-U(0, \pi/2) \) has established itself as a new quantum ESS. Again another small group of mutants appear using some other two-parameter quantum strategy and try to invade the quantum ESS \( Q \).

**Case A**

The expected payoffs are found as

\[ P(\theta, D) = \sin^2(\theta/2) \]

\[ P(\theta, \theta) = 2 \cos^2(\theta/2) + 5 \cos^2(\theta/2) \sin^2(\theta/2) + 1 \]

\[ P(D, \theta) = 5 \cos^2(\theta/2) + \sin^2(\theta/2) \]

\[ P(D, D) = 1 \]  

Where, for example \( P(\theta, D) \) is the payoff to mutant employing one-parameter strategy.
θ against the opponent using D. Now \( P(D, D) > P(\theta, D) \) for all \( \theta \in [0, \pi) \). Hence, the first condition for an ESS is satisfied and \( D^* U(\pi) \) is an ESS. The case \( \theta = \pi \) corresponds to the case when one-parameter mutant strategy coincides with the ESS and is ruled out. If \( D^* U(\pi) \) is played by almost all the members of the population, which correspond to high frequency \( F_D \) for \( D \), we have then \( W(D) > W(\theta) \) for all \( \theta \in [0, \pi) \). Therefore the fitness of a one-parameter quantum strategy, which also corresponds to the case of mixed (randomized) classical strategies [4], cannot be greater than that of a classical ESS. A one-parameter quantum strategy, therefore, cannot succeed to invade a classical ESS.

**Case B**

The expected payoffs are

\[
P(D, D) = 1
\]

\[
P(D, U) = 5 \cos^2(\phi) \cos^2(\theta/2) + \sin^2(\theta/2)
\]

\[
P(U, D) = 5 \sin^2(\phi) \cos^2(\theta/2) + \sin^2(\theta/2)
\]

\[
P(U, U) = 3 |\cos(2\phi) \cos^2(\theta/2)|^2 + 5 \cos^2(\theta/2) \sin^2(\theta/2) |\sin(\phi) - \cos(\phi)|^2
\]

\[+ |\sin(2\phi) \cos(\theta/2) + \sin^2(\theta/2)|^2 \tag{9}\]

Here \( P(D, D) > P(U, D) \) if \( \phi < \arcsin(1/\sqrt{5}) \) and if \( P(D, D) = P(U, D) \) then \( P(D, U) > P(U, U) \). Therefore \( D \) is an ESS if \( \phi < \arcsin(1/\sqrt{5}) \) otherwise the strategy \( U(\theta, \phi) \) will be in position to invade \( D \). Alternatively if most of the members of the population play \( D^* U(\pi, 0) \), meaning high frequency \( F_D \) for \( D \), then the fitness \( W(D) \) will remain greater than the fitness \( W[U(\theta, \phi)] \) if \( \phi < \arcsin(1/\sqrt{5}) \). For \( \phi > \arcsin(1/\sqrt{5}) \) the strategy \( U(\theta, \phi) \) can invade the strategy \( D \) which is an ESS. The possession of a richer strategy by the mutants in this case leads to an invasion of \( D \) when \( \phi > \arcsin(1/\sqrt{5}) \). Mutants having access to richer strategies may seem non-judicious but even in classical setting an advantage by the mutants leading to invasion may be seen in similar context.

**Case C**

Eisert et al [4] showed that the quantum strategy \( Q^* U(0, \pi/2) \) played by both the players is the unique NE and one player cannot gain without lessening the other player’s expected payoff. The expected payoffs are
\[ P(Q,Q) = 3 \]
\[ P(U,Q) = [3 - 2\cos^2(\phi)]\cos^2(\theta/2) \]
\[ P(Q,U) = [3 - 2\cos^2(\phi)]\cos^2(\theta/2) + 5\sin^2(\theta/2) \quad (10) \]

Now \( P(Q,Q) > P(U,Q) \) holds true for all \( \theta \in [0, \pi] \) and \( \phi \in [0, \pi/2] \) except when \( \theta = 0 \) and \( \phi = \pi/2 \) which is the case when the mutant strategy \( U(\theta, \phi) \) is the same as \( Q \) and is ruled out. Therefore the first condition for \( Q \) to be an ESS is satisfied. The condition \( P(Q,Q) = P(U,Q) \) implies \( \theta = 0 \) and \( \phi = \pi/2 \). We have again the situation of the mutant strategy to be same as \( Q \) and we neglect it. If \( Q \) is played by most of the players, meaning high frequency \( F_Q \) for \( Q \), then it is seen that \( W(Q) > W[U(\theta, \phi)] \) for all \( \theta \in (0, \pi) \) and \( \phi \in [0, \pi/2] \). Therefore a two parameter quantum strategy \( U(\theta, \phi) \) cannot invade the quantum ESS i.e. the strategy \( Q^{-1}U(0, \pi/2) \) for this particular game. The mutants having access to richer strategy space remains an advantage not any more now. For the population as well as the mutants \( Q \) is the unique NE and ESS of the game.

The invasion of the mutants in case B does not seem so unusual given the richer structure of strategy space they exploit and they are unable to invade when it doesn’t remain an advantage and most of the population have access to it.

We now see what happens to PD game when played via Marinatto’s scheme[13]. In this scheme the players apply their ‘tactics’ by restricting themselves to a probabilistic choice between the identity operator \( \hat{I} \) and the Pauli spin-flip operator \( \hat{\sigma}_x \). The purpose [12] for such a choice as described by the authors is to have the smallest set of operations able to reproduce, when applied to a factorizable couple of strategies, the results of the classical theory of games. However new results come out from the richer structure of the strategic space, i.e. from the entangled couple of strategies[13]. S.C.Benjamin in his comment [14] have considered it a severe restriction on the full range of quantum mechanically possible manipulations but Marinatto and Weber have replied [12] by describing it a ‘minimal’ choice enough to reproduce the classical results.

For the initial entangled state
\[ |\psi_{in}\rangle = a |CC\rangle + b |DD\rangle \quad |a|^2 + |b|^2 = 1 \quad (11) \]
when $\hat{I}$ and $\hat{\sigma}_x$ correspond to strategies $C$ and $D$ respectively with the payoff matrix (3).

Payoffs to Alice and Bob are:

$$P_A(p, q) = 3 \{pq |a|^2 + (1 - p)(1 - q) |b|^2\} + 5 \{p(1 - q) |b|^2 + q(1 - p) |a|^2\}$$
$$+ \{pq |b|^2 + (1 - p)(1 - q) |a|^2\}$$

$$P_B(p, q) = 3 \{pq |a|^2 + (1 - p)(1 - q) |b|^2\} + 5 \{p(1 - q) |a|^2 + q(1 - p) |b|^2\}$$
$$+ \{pq |a|^2 + (1 - p)(1 - q) |b|^2\}$$

(12)

where $p$ and $q$ are the probabilities of Alice and Bob respectively to act with the operator $\hat{I}$. PD is symmetric game and remains symmetric after quantizing it. For a symmetric bimatrix games an ESS is recognized as a symmetric NE with an additional property usually called ‘the stability property’[8].

We search for symmetric NE from the inequalities using only the parameter $b$ of the initial state $|\psi_{in}\rangle$ because for the state $|\psi_{in}\rangle = a |CC\rangle + b |DD\rangle$ the game reduces to classical when $|b|^2 = 0$ i.e. when the initial state becomes unentangled. NE inequalities are then

$$P_A(\hat{p}, \hat{q}) - P_A(p, \hat{q}) = (\hat{p} - p)\{3 |b|^2 - (\hat{q} + 1)\} \geq 0$$

$$P_B(\hat{p}, \hat{q}) - P_B(\hat{p}, q) = (\hat{q} - q)\{3 |b|^2 - (\hat{p} + 1)\} \geq 0$$

(13)

The parameters of the initial entangled state $a$ and $b$ may decide some of the possible NE. Three symmetric NE are

1. $\hat{p} = \hat{q} = 0$ when $3 |b|^2 \leq 1$
2. $\hat{p} = \hat{q} = 1$ when $3 |b|^2 \geq 2$
3. $\hat{p} = \hat{q} = 3 |b|^2 - 1$ when $1 < 3 |b|^2 < 2$  

(14)

The first two NE are independent of the parameters $a$ and $b$ of the initial state. However, the third NE depends on these. We now ask which of these NE can be ESS’s assuming that a particular NE exists with reference to a particular set of initial states $|\psi_{in}\rangle$ for which it can be found. The payoff to a player using $\hat{I}$ with probability $p$ when the opponent uses $\hat{I}$ with probability $q$ is

$$P(p, q) = 3 \{pq |a|^2 + (1 - p)(1 - q) |b|^2\} + 5 \{p(1 - q) |b|^2 + q(1 - p) |a|^2\}$$
$$+ \{pq |b|^2 + (1 - p)(1 - q) |a|^2\}$$

(15)
For the first case \( p^* = q^* = 0 \). The payoff \( P(0, 0) > P(p, 0) \) when \( 3 |b|^2 < 1 \) and \( P(0, 0) = P(p, 0) \) imply \( 3 |b|^2 = 1 \). Also \( P(q, q) = -q^2 + \frac{5}{3}(q + 1) \) and \( P(0, q) = \frac{5}{3}(q + 1) \). Now \( P(0, q) > P(q, q) \) when \( q \neq 0 \). Therefore, \( p^* = q^* = 0 \) is an ESS when \( 3 |b|^2 \leq 1 \).

Consider \( p^* = q^* = 1 \) now. \( P(1, 1) > P(p, 1) \) means \( 3 |b|^2 > 2 \) if \( p \neq 1 \). And \( P(1, 1) = P(p, 1) \) means for \( p \neq 1 \) we have \( 3 |b|^2 = 2 \). In such case \( P(q, q) = -q^2 + \frac{1}{3}(q + 7) \) and \( P(1, q) = \frac{5}{3}(2 - q) \). Now \( P(1, q) > P(q, q) \) because \((1 - q)^2 > 0 \) for \( q \neq 1 \). Therefore \( p^* = q^* = 1 \) is an ESS when \( 3 |b|^2 \geq 2 \).

The third case \( p^* = q^* = 3 |b|^2 - 1 \). Here \( P(3 |b|^2 - 1, 3 |b|^2 - 1) = -36 |b|^6 + 36 |b|^4 - 5 |b|^2 + 6 \). Also we find \( P(p, 3 |b|^2 - 1) = -21 |b|^4 + 21 |b|^2 - 3 \). Therefore, the condition \( P(3 |b|^2 - 1, 3 |b|^2 - 1) > P(p, 3 |b|^2 - 1) \) holds and \( p^* = q^* = 3 |b|^2 - 1 \) is an ESS too for \( 1 < 3 |b|^2 < 2 \).

All three possible symmetric NE definable for different ranges of \( |b|^2 \) turn out ESS’s. Each of the three sets of initial states \( |\psi_{in}\rangle \) give a unique NE that is an ESS too. Switching from one to the other sets of initial states also changes the NE and ESS accordingly. A question rises here: is it possible that a particular NE switches over between ‘ESS’ and ‘not ESS’ when the initial state changes between certain possible choices?. The transition between classical and quantum game is also controlled by a change in the initial state. For example classical payoffs can be obtained when the initial state is unentangled. It implies that it may be possible to switch over between ‘ESS’ and ‘not ESS’ by a change between ‘classical’ and ‘quantum’ forms of a game i.e. when the initial state is unentangled and entangled respectively. This possibility makes ESS interesting for the quantum game theory as well. Because PD does not allow such a possibility we now investigate asymmetric games to look for an answer for our question.

### III. ASYMMETRIC CASE

The players are anonymous in a symmetric bimatrix game. Such a game is written as \( G = (M, M^T) \) where \( M \) is a square matrix and \( M^T \) is its transpose. An ESS for an asymmetric bimatrix game i.e. \( G = (M, N) \) when \( N \neq M^T \) is defined as a strict NE [11]. A
strategy pair \((\hat{x}, \hat{y}) \in S\), is an Evolutionarily Stable Strategy Pair of the asymmetric bimatrix game \(G = (M, N)\) if it satisfies the NE conditions with strict inequality i.e.

1. \(P_A(\hat{x}, \hat{y}) > P_A(x, \hat{y})\) for all \(x \neq \hat{x}\)
2. \(P_B(\hat{x}, \hat{y}) > P_B(\hat{x}, y)\) for all \(y \neq \hat{y}\) \(\quad (16)\)

The game of Battle of Sexes has the following matrix

\[
\begin{pmatrix}
(\alpha, \beta) & (\gamma, \gamma) \\
(\gamma, \gamma) & (\beta, \alpha)
\end{pmatrix}
\]

where \(\alpha > \beta > \gamma\) \(\quad (17)\)

is an asymmetric bimatrix game with three classical NE \(\quad [13]\)

1. \(p_1 = q_1 = 0\)
2. \(p_2 = q_2 = 1\)
3. \(p_3 = \frac{\alpha - \gamma}{\alpha + \beta - 2\gamma}, \quad q_3 = \frac{\beta - \gamma}{\alpha + \beta - 2\gamma}\) \(\quad (18)\)

Here (1) and (2) are ESS’s as well but (3) is not because it is not a strict NE. The asymmetric quantum game played via the entangled state \(|\psi_{in}\rangle = a |OO\rangle + b |TT\rangle\), where \(O\) and \(T\) denote ‘Opera’ and ‘Television’ respectively, has following three NE \(\quad [13]\)

1. \(p_1 = q_1 = 1\)
2. \(p_2 = q_2 = 0\)
3. \(p_3 = \frac{(\alpha - \gamma)|a|^2 + (\beta - \gamma)|b|^2}{\alpha + \beta - 2\gamma}, \quad q_3 = \frac{(\alpha - \gamma)|b|^2 + (\beta - \gamma)|a|^2}{\alpha + \beta - 2\gamma}\) \(\quad (19)\)

Similar to the classical case, (1) and (2) are ESS’s while (3) is not. These two ESS’s do not depend on the parameters \(a\) and \(b\) of the initial state, however, the third NE does so. We show later that for other games an ESS may also be dependent on the parameters \(a\) and \(b\). Interestingly playing the Battle of Sexes game via another entangled state \(|\psi_{in}\rangle = a |OT\rangle + b |TO\rangle\) changes the scene. The payoffs to Alice and Bob now are:

\[
P_A(p, q) = \\
p \left\{ -q(\alpha + \beta - 2\gamma) + \alpha |a|^2 + \beta |b|^2 - \gamma \right\} + q \left\{ \alpha |b|^2 + \beta |a|^2 - \gamma \right\} + \gamma
\]

\[
P_B(p, q) = \\
q \left\{ -p(\alpha + \beta - 2\gamma) + \beta |a|^2 + \alpha |b|^2 - \gamma \right\} + p \left\{ \beta |b|^2 + \alpha |a|^2 - \gamma \right\} + \gamma
\]

and there is only one NE that is not an ESS i.e.

\[
p = \frac{\beta |a|^2 + \alpha |b|^2 - \gamma}{\alpha + \beta - 2\gamma}, \quad q_3 = \frac{\alpha |a|^2 + \beta |b|^2 - \gamma}{\alpha + \beta - 2\gamma}\]

\(\quad (21)\)
and Battle of Sexes playing via the state $|\psi_{in}\rangle = a |OT\rangle + b |TO\rangle$ gives no ESS at all.

An essential requirement on a quantum version of a game is that the corresponding classical game must be its subset. Suppose for a quantum game corresponding to an asymmetric bimatrix classical game a particular strategy pair $(x, y) \in S$ is an ESS independent of an initial state $|\psi_{in}\rangle$ in its possible choices i.e. $(x, y)$ is an ESS for all $a$ and $b$. Classical game being a subset of the quantum game the strategy pair $(x, y)$ must be an ESS in the classical game as well. However, a strategy pair $(x, y)$ being an ESS in the classical game may not remain an ESS in quantum version. The quantization of an asymmetric classical game can make disappear the classical ESS’s but cannot make appear new ESS’s, provided an ESS in quantum version remains so for every possible choice of $a$ and $b$. However, when an ESS is defined as a strict NE existing only for a set of initial states for which that NE exists the statement that quantization can only make disappear classically available ESS’s may not remain valid. In such a case quantization may make appear new ESS’s definable for certain ranges of the parameters $a$ and $b$. To find games with the property that ‘a particular NE switches over between ‘ESS’ and ‘not ESS’ when the initial state changes between its possible choices’ we write down the payoffs to Alice and Bob playing an asymmetric quantum game via the method [13] of probabilistic choice of the operator $\hat{I}$ by the players. For the matrix

$$
\begin{pmatrix}
(\alpha_1, \alpha_2) & (\beta_1, \beta_2) \\
(\gamma_1, \gamma_2) & (\sigma_1, \sigma_2)
\end{pmatrix}
$$

(22)

with the condition

$$
\begin{pmatrix}
\alpha_1 \\
\beta_1 \\
\gamma_1 \\
\sigma_1
\end{pmatrix} \neq \begin{pmatrix}
\alpha_2 \\
\beta_2 \\
\gamma_2 \\
\sigma_2
\end{pmatrix}^T
$$

the payoffs for the initial state $|\psi_{in}\rangle = a |OO\rangle + b |TT\rangle$ are as follows, given that $O$ and $T$ are again the two pure classical strategies that no more represent ‘Opera’ and ‘Television’ only.

$$
P_A(p, q) = \alpha_1 \left\{ p q |a|^2 + (1 - p)(1 - q) |b|^2 \right\} + \beta_1 \left\{ p(1 - q) |a|^2 + q(1 - p) |b|^2 \right\}
+ \gamma_1 \left\{ p(1 - q) |b|^2 + q(1 - p) |a|^2 \right\} + \sigma_1 \left\{ p q |b|^2 + (1 - p)(1 - q) |a|^2 \right\}$$

$$
P_B(p, q) = \alpha_2 \left\{ p q |a|^2 + (1 - p)(1 - q) |b|^2 \right\} + \beta_2 \left\{ p(1 - q) |a|^2 + q(1 - p) |b|^2 \right\}
+ \gamma_2 \left\{ p(1 - q) |b|^2 + q(1 - p) |a|^2 \right\} + \sigma_2 \left\{ p q |b|^2 + (1 - p)(1 - q) |a|^2 \right\}
$$

(23)
The NE conditions are then

\[ P_A(\hat{p}, \hat{q}) - P_A(p, \underline{q}) = \]
\[ (\hat{p} - p) \left[ |a|^2 (\beta_1 - \sigma_1) + |b|^2 (\gamma_1 - \alpha_1) - \underline{q} \left\{ (\beta_1 - \sigma_1) + (\gamma_1 - \alpha_1) \right\} \right] \geq 0 \]
\[ P_B(\hat{p}, \hat{q}) - P_B(p, q) = \]
\[ (\hat{q} - q) \left[ |a|^2 (\gamma_2 - \sigma_2) + |b|^2 (\beta_2 - \alpha_2) - \hat{p} \left\{ (\gamma_2 - \sigma_2) + (\beta_2 - \alpha_2) \right\} \right] \geq 0 \] (24)

Let now \( \hat{p} = \hat{q} = 0 \) be a NE i.e.

\[ P_A(0, 0) - P_A(p, 0) = -p \left[ (\beta_1 - \sigma_1) + |b|^2 \left\{ (\gamma_1 - \alpha_1) + (\beta_1 - \sigma_1) \right\} \right] \geq 0 \]
\[ P_B(0, 0) - P_B(0, q) = -q \left[ (\gamma_2 - \sigma_2) + |b|^2 \left\{ (\gamma_2 - \sigma_2) + (\gamma_2 - \sigma_2) \right\} \right] \geq 0 \] (25)

When the strategy pair \((0, 0)\) is an ESS in the classical game \(|b|^2 = 0\) we should have

\[ P_A(0, 0) - P_A(p, 0) = -p(\beta_1 - \sigma_1) > 0 \text{ for all } p \neq 0 \]
\[ P_B(0, 0) - P_B(0, q) = -q(\gamma_2 - \sigma_2) > 0 \text{ for all } q \neq 0 \] (26)

It implies \((\beta_1 - \sigma_1) < 0\) and \((\gamma_2 - \sigma_2) < 0\).

For the pair \((0, 0)\) to be ‘not ESS’ for some \(|b|^2 \neq 0\) let take \(\gamma_1 = \alpha_1\) and \(\beta_2 = \alpha_2\) we have

\[ P_A(0, 0) - P_A(p, 0) = -p(\beta_1 - \sigma_1) \{1 - |b|^2\} \]
\[ P_B(0, 0) - P_B(0, q) = -q(\gamma_2 - \sigma_2) \{1 - |b|^2\} \] (27)

And the pair \((0, 0)\) doesn’t remain an ESS when \(|b|^2 = 1\). A game with these properties is given by the matrix

\[
\begin{pmatrix}
(1, 1) & (1, 2) \\
(2, 1) & (3, 2)
\end{pmatrix}
\] (28)

For this game the pair \((0, 0)\) is an ESS when \(|b|^2 = 0\) (classical game) but it is not when for example \(|b|^2 = \frac{1}{2}\), though it remains a NE in both the cases. Therefore, a NE can be switched between ESS and ‘not ESS’ by adjusting the parameters \(a\) and \(b\). An ESS may also appear when unentangled strategies become entangled opposite to the previous case.

An example of a game for which it happens is

\[
\begin{pmatrix}
(2, 1) & (1, 0) \\
(1, 0) & (1, 0)
\end{pmatrix}
\] (29)

Playing this game again via \(|\psi_{in}\rangle = a \langle OO\rangle + b \langle TT\rangle\) gives following payoff differences for
the strategy pair \((0, 0)\) for Alice and Bob respectively

\[ P_A(0, 0) - P_A(p, 0) = p|b|^2 \quad \text{and} \quad P_B(0, 0) - P_B(0, q) = q|b|^2 \quad (30) \]

Therefore (29) is an example of a game for which \((0, 0)\) is not an ESS when initial state in unentangled but \((0, 0)\) is an ESS for entangled initial states i.e. \(0 < |b|^2 < 1\).

**IV. CONCLUDING REMARKS**

We have shown that in a population engaged in symmetric bimatrix classical game of Prisoner’s Dilemma an invasion of classical ESS is possible rather easily by the mutants exploiting two-parameter set of quantum strategies. However, the mutants cannot invade when they are deprived of using entanglement or when entanglement doesn’t remain an advantage. For an asymmetric quantum game between two players we have shown that a strategy pair can be made an ESS for either classical (using unentangled \(|\psi_{in}\rangle\)) or quantum (using entangled \(|\psi_{in}\rangle\)) version of the game even when the strategy pair remains a Nash equilibrium in both the versions. It shows that in certain types of games entanglement can be used to make appear or disappear ESS’s while retaining corresponding Nash equilibria.

The notion of an ESS in multiplayer classical games have been used in classical game theory. Recently S.C.Benjamin [15] have shown that coherent equilibria of mostly cooperative nature can exist in multiplayer quantum games. We think that ESS can be useful refinement concept in multiplayer quantum games having multiple NE and entanglement can also be related to ESS’s in these games as well.

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VI. REFERENCES

[1] Maynard Smith, J. and Price, G.R. (1973) The logic of animal conflict. Nature, 246, 15-18

[2] Maynard Smith, J. (1982) Evolution and the theory of games. CUP.

[3] Grafen, A (1990) Biological signals as handicaps. J. Theor. Bio. 144, 517-546

[4] D.Meyer, Phys. Rev. Lett. 82, 1052 (1999) and quant-ph/0004092

[5] J.Eisert, M. Wilkens, and M. Lewenstein, Phys. Rev. Lett. 83, 3077 (1999) and quant-ph/0004076

[6] R. Dawkins, The Selfish Gene (Oxford University Press, Oxford, 1976).

[7] R.B. Myerson, Game Theory: An Analysis of Conflict (MIT Press, Cambridge, MA, 1991).

[8] Game Theory, A report by K. Prestwich, Department of Biology, College of the Holy Cross, Worcester, MA, USA 01610. 1999.

[9] Evolution in knockout conflicts. A report by M.Broom, Center for Statistics and Stochastic Modeling, School of Mathematical Sciences, University of Sussex, U.K. September 17, 1997.

[10] States, Effects, and Operations: Fundamental Notions of Quantum Theory, by K.Kraus. Lecture Notes in Physics, Vol. 190. (Springer-Verlag, Berlin, 1983).

[11] Evolutionary game theory and the Modelling of Economic Behaviour, by Gerard van der Laan and Xander Tieman. November 6, 1996. Research Program ”Competition and Cooperation” of the Faculty of Economics and Econometrics, Free University, Amsterdam.

[12] Reply to “Comment on: A Quantum Approach to Static Games of Complete Information” quant-ph/0009103

[13] A Quantum Approach to Static Games of Complete Information. Phys. Lett, 272, 291 (2000);

[14] Comment on: “A quantum approach to static games of complete information” quant-ph/0008127
[15] Multiplayer Quantum Games. quant-ph/0007038