A CLASS OF SINGULARITY OF ARBITRARY PAIRS AND LOG CANONICALIZATIONS

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Abstract. We define a class of singularity on arbitrary pairs of a normal variety and an effective $\mathbb{R}$-divisor on it, which we call pseudo-lc in this paper. This is a generalization of the usual lc singularity of pairs and log canonical singularity of normal varieties introduced by de Fernex and Hacon. By giving examples of pseudo-lc pairs which are not lc or log canonical in the sense of de Fernex–Hacon’s paper, we show that pseudo-lc singularity is a strictly extended notion of those singularities. We prove that pseudo-lc pairs admit a small log canonicalization.

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1. Introduction

Throughout this paper we will work over the complex number field.

In the birational geometry, we often deal with not only algebraic varieties but also pairs of an algebraic variety and a divisor. Pairs of a variety and a divisor naturally appear, for example, a curve and marked points, or an open variety and the boundary of its compactification. Even when we study geometric properties of higher-dimensional algebraic varieties, pairs can be a very powerful tool to work induction on dimension of varieties. When we deal with pairs $(X, \Delta)$, we usually assume that the log canonical divisor $K_X + \Delta$ is $\mathbb{R}$-Cartier. Using this property, we often compare log canonical divisors of two pairs which are birationally equivalent in a sense. For example, when we are given pairs $(X, \Delta)$ and $(X', \Delta')$ with a birational map $X \dasharrow X'$, we take a common resolution $f: Y \to X$ and $f': Y \to X'$ of $X \dasharrow X'$ and compare $f^*(K_X + \Delta)$ and $f'^*(K_{X'} + \Delta')$. Some classes of pairs with $\mathbb{R}$-Cartier log canonical divisors and mild singularities, such as lc pairs, klt pairs, and so on (see [KM]), are in particular important to study higher-dimensional
algebraic varieties. In fact, a lot of important results in the birational geometry were proved in the framework of lc or klt pairs (for example, [BCHM], [F2], [B1], [HX], [HMX2], [B2]).

It is difficult to carry out similar arguments on pairs whose log canonical divisors are not $\mathbb{R}$-Cartier. In [dFH], de Fernex and Hacon defined the pullback of arbitrary $\mathbb{Q}$-divisors. Using it, they defined relative log canonical divisors, multiplier ideal sheaves and singularities on pairs $(X, \sum a_i Z_i)$ of a normal quasi-projective variety $X$ and a formal $\mathbb{R}_{\geq 0}$-linear combination $\sum a_i Z_i$ of proper closed subschemes $Z_i \subset X$. They proved that multiplier ideal sheaves, log canonical pairs and log terminal pairs in the sense of [dFH] have various properties similar to those on the usual pairs, for instance, vanishing theorem of multiplier ideal sheaves and that log terminal singularities have only rational singularities.

In this paper, we study an extension of lc singularity. The purpose of this paper is to generalize lc singularity to a singularity of pairs whose log canonical divisor is not necessarily $\mathbb{R}$-Cartier and to investigate relations between the new singularity and lc singularity or log canonical singularity introduced by [dFH].

We deal with arbitrary pairs of a normal variety $X$ and an effective $\mathbb{R}$-divisor $\Delta$ on it, which we denote $(X, \Delta)$ to distinguish them from pairs whose log canonical divisor is $\mathbb{R}$-Cartier. For any prime divisor $P$ over $X$, we define discrepancy of $P$ with respect to $(X, \Delta)$, denoted by $\alpha(P, X, \Delta)$ in this paper (Definition 4.1), and define pseudo-lc singularity by using it (Definition 4.2). We show that $\alpha(\cdot, X, \Delta)$ is a generalization of the usual discrepancy (Lemma 4.3), and therefore the class of pseudo-lc pairs contains lc pairs as a special case. We give a simple description of $\alpha(\cdot, X, \Delta)$ using notations in [dFH] (Proposition 4.7), and we prove that when $X$ is quasi-projective we can approximate $\alpha(\cdot, X, \Delta)$ by the usual discrepancy of pairs $(X, \Delta + G)$ with $G \geq 0$ (Theorem 4.8). Also, we prove that pseudo-lc pairs are closely related to log canonical singularity in the sense of [dFH] (Proposition 4.6) and they appear in generalized lc pairs introduced in [BZ] (Proposition 4.11). We give examples of pseudo-lc pairs which are not lc (Example 4.9) and pseudo-lc pairs which are not log canonical in the sense of [dFH] (Example 4.10). Thus, pseudo-lc singularity is a strictly extended notion. Furthermore, for any pair $(X, \Delta)$ with a boundary $\mathbb{R}$-divisor $\Delta$, we prove the existence of a log canonicalization which only extracts bad divisors measured by $\alpha(\cdot, X, \Delta)$. The following theorem is the main result of this paper.

**Theorem 1.1** (=Theorem 4.13). Let $(X, \Delta)$ be a pair such that $\Delta$ is a boundary $\mathbb{R}$-divisor. Then, there is a projective birational morphism $h: W \to X$ from a normal variety $W$ such that

1. any $h$-exceptional prime divisor $E_h$ satisfies $\alpha(E_h, X, \Delta) < -1$,
2. the reduced $h$-exceptional divisor $E_{\text{red}}$ is $\mathbb{Q}$-Cartier, and
3. if we put $\Delta_W = h^{-1}_* \Delta + E_{\text{red}}$, then $K_W + \Delta_W$ is $\mathbb{R}$-Cartier and the pair $(W, \Delta_W)$ is lc.

In particular, if $(X, \Delta)$ is pseudo-lc, then $h$ is small, i.e. $W$ and $X$ are isomorphic in codimension one.

We also have the following theorem:

**Theorem 1.2** (=Theorem 4.15). Let $(X, \Delta)$ be a pseudo-lc pair. Then, there is the relative log canonical model $h: (W, \Delta_W) \to X$ such that $h$ is small.

For definition of relative log canonical model, see [GK, 2.15 Definition].
By the main result, we see that pseudo-lc and lc singularities coincide on surfaces.

**Theorem 1.3** (=Corollary 4.14). Let \((X, \Delta)\) be a pair. If \(X\) is a surface, then \((X, \Delta)\) is pseudo-lc if and only if \(K_X + \Delta\) is \(\mathbb{R}\)-Cartier and \((X, \Delta)\) is lc.

We note that pseudo-lc pairs in Example 4.9 or Example 4.10 include threefolds. So a gap between pseudo-lc singularity and lc singularity or log canonical singularity in the sense of [dFH] arises when the dimension of the variety is greater than 2.

The contents of this paper are as follows: In Section 2, we collect definitions and some results on the log MMP. In Section 3, we show a special kinds of the relative log MMP, which is a generalization of [Ba2, Theorem 1.1]. In Section 4, which is the main part of this paper, we define pseudo-lc singularity, prove basic properties of pseudo-lc pairs and the main theorem.

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2. Preliminaries

In this section we collect definitions and some important theorems.

2.1. Definitions. We collect some definitions.

**Divisors.** Let \(\pi: X \to Z\) be a projective morphism of normal varieties. We use the standard definition of \(\pi\)-nef \(\mathbb{R}\)-divisor, \(\pi\)-ample \(\mathbb{R}\)-divisor, \(\pi\)-semi-ample \(\mathbb{R}\)-divisor, \(\pi\)-big \(\mathbb{R}\)-divisor and \(\pi\)-pseudo-effective \(\mathbb{R}\)-divisor.

**Singularities of pairs.** In this paper, we deal with two kinds of pairs.

We recall definition of the usual pairs. A pair \((X, \Delta)\) consists of a normal variety \(X\) and an effective \(\mathbb{R}\)-divisor \(\Delta\) such that \(K_X + \Delta\) is \(\mathbb{R}\)-Cartier. When coefficients of \(\Delta\) belong to \([0, 1]\), the divisor \(\Delta\) is called a boundary divisor. When we write \((X, \text{Supp}\Delta)\), we pay attention to \(X\) and the support of \(\Delta\). Therefore, \((X, \text{Supp}\Delta)\) simply denotes a pair of a variety and a subscheme of pure codimension one.

Let \((X, \Delta)\) be a pair and \(P\) be a prime divisor over \(X\). Then \(a(P, X, \Delta)\) denotes the discrepancy of \(P\) with respect to \((X, \Delta)\). In this paper, we use definitions of Kawamata log terminal (klt, for short) pair, log canonical (lc, for short) pair and divisorially log terminal (dlt, for short) pair as in [KM] or [BCHM]. Let \((X, \Delta)\) be an lc pair. An lc center of \((X, \Delta)\) is the image on \(X\) of a prime divisor \(P\) over \(X\) satisfying \(a(P, X, \Delta) = -1\).

We also deal with arbitrary pairs of a normal variety \(X\) and an effective \(\mathbb{R}\)-divisor \(\Delta\) on it. When we do not assume that \(K_X + \Delta\) is \(\mathbb{R}\)-Cartier, we denote the pair of \(X\) and \(\Delta\) by \((X, \Delta)\) to distinguish from the usual pairs. We note that pairs \((X, \Delta)\) only appear in Section 4.

**Models.** We use the definition of log minimal model, good minimal model and Mori fiber space as in [B1, Section 2].

**Remark 2.1.** Let \((X, \Delta)\) be an lc pair and \((X', \Delta')\) be a log minimal model of \((X, \Delta)\). Let \((X'', \Delta'')\) be an lc pair such that \(K_{X''} + \Delta''\) is nef, \(X''\) and \(X'\) are
isomorphic in codimension one, and $\Delta''$ is the birational transform of $\Delta'$ on $X''$. Then $(X'', \Delta'')$ is also a log minimal model of $(X, \Delta)$. Moreover, if $(X', \Delta')$ is a good minimal model of $(X, \Delta)$, then $(X'', \Delta'')$ is a good minimal model of $(X, \Delta)$.

**Definition 2.2** (Log canonical model). Let $X \to Z$ be a projective morphism from a normal variety to a variety, and let $(X, \Delta)$ be an lc pair. A weak log canonical model $(X', \Delta')$ of $(X, \Delta)$ over $Z$ is a log canonical model if $K_X + \Delta$ is ample over $Z$.

2.2. **Results related to the log MMP.** In this subsection, we collect three results on the log MMP.

In this paper, we use the following two results without any mention.

**Theorem 2.3** ([B1, Theorem 4.1]). Let $(X, \Delta)$ be a $\mathbb{Q}$-factorial lc pair such that $(X, 0)$ is klt. Let $\pi : X \to Z$ be a projective morphism of normal quasi-projective varieties.

If there is a log minimal model of $(X, \Delta)$ over $Z$, then any $(K_X + \Delta)$-MMP over $Z$ with scaling of an ample divisor terminates.

**Lemma 2.4** ([Ha2, Lemma 2.14]). Let $\pi : X \to Z$ be a projective morphism of normal quasi-projective varieties, and let $(X, \Delta)$ be an lc pair. Let $(Y, \Gamma)$ be an lc pair such that there is a projective birational morphism $f : Y \to X$ and we can write $K_Y + \Gamma = f^*(K_X + \Delta) + E$ with $f$-exceptional divisor $E \geq 0$.

Then, $(X, \Delta)$ has a weak lc model (resp. a log minimal model, a good minimal model) over $Z$ if and only if $(Y, \Gamma)$ has a weak lc model (resp. a log minimal model, a good minimal model) over $Z$.

We close this section with the following lemma. It plays an important role in the proof of Theorem 1.1.

**Lemma 2.5.** Let $\pi : X \to Z$ be a projective morphism of normal varieties, which are not necessarily quasi-projective. Let $(X, \Delta)$ be a $\mathbb{Q}$-factorial lc pair such that $(X, 0)$ is klt, and let $D$ be an $\mathbb{R}$-divisor on $X$ such that $(X, \Delta + D)$ is lc. Suppose that $(X, \Delta + tD)$ has the log canonical model over $Z$ for any $0 \leq t < 1$.

Then, there is a birational contraction $\phi : X \dashrightarrow Y$ over $Z$ such that for any $0 < t \ll 1$, the pair $(Y, \Delta_Y + tD_Y)$ is the log canonical model of $(X, \Delta + tD)$ over $Z$, where $\Delta_Y$ and $D_Y$ are the birational transforms of $\Delta$ and $D$ on $Y$, respectively. In particular, $D_Y$ is $\mathbb{R}$-Cartier.

**Proof.** Note that the divisor $K_X + \Delta$ is big over $Z$ by Definition 2.2.

First, we prove the lemma in the case when $Z$ is quasi-projective. Since the log canonical model is in particular a weak lc model with semi-ample log canonical divisor, $(X, \Delta + tD)$ has a good minimal model over $Z$ for any $0 \leq t < 1$. Let $(X, \Delta) \dashrightarrow (X', \Delta')$ be a sequence of steps of the $(K_X + \Delta)$-MMP over $Z$ to a good minimal model, and $X' \to Y_0$ be the contraction over $Z$ induced by $K_{X'} + \Delta'$, where $\Delta'$ is the birational transform of $\Delta$ on $X'$. Let $D'$ (resp. $\Delta_{Y_0}$) be the birational transform of $D$ (resp. $\Delta$) on $X'$ (resp. $Y_0$). By construction, $K_{Y_0} + \Delta_{Y_0}$ is ample over $Z$. We pick a sufficiently small $t' > 0$ such that the map $X \dashrightarrow X'$ is a sequence of steps of the $(K_X + \Delta + t'D)$-MMP. Since $(X, \Delta + tD)$ has a good minimal model over $Z$ for any $0 \leq t < 1$, we may assume that we can run the $(K_{X'} + \Delta' + t'D')$-MMP over $Z$ and get a good minimal model $(X', \Delta' + t'D') \dashrightarrow (X'', \Delta'' + t''D'')$. By the argument of the length of extremal rays and replacing $t'$ if necessary, we can assume
that the map $X' \to X''$ is a sequence of steps of the $(K_{X'} + \Delta' + t'D')$-MMP over $Y_0$. So we have the following diagram.

$$
\begin{array}{ccc}
X & \to & X' \\
\pi & \downarrow & \downarrow & \pi \\
Z & \leftarrow & Y_0 & \leftarrow & Z
\end{array}
$$

Since the divisor $K_{X''} + \Delta'' + t'D''$ is semi-ample over $Z$, it is semi-ample over $Y_0$. Let $X'' \to Y$ be the contraction over $Y_0$ induced by $K_{X''} + \Delta'' + t'D''$. Let $g: Y \to Y_0$ be the natural morphism, and $\Delta_Y$ and $D_Y$ be the birational transforms of $\Delta$ and $D$ on $Y$, respectively. Then, we have $K_Y + \Delta_Y = g^*(K_{Y_0} + \Delta_{Y_0})$, and the divisor $K_Y + \Delta_Y + tD_Y$ is ample over $Y_0$. Since $K_{Y_0} + \Delta_{Y_0}$ is ample over $Z$, for any $0 < t \ll t'$, the divisor

$$
K_Y + \Delta_Y + tD_Y = \frac{t}{t'}(K_Y + \Delta_Y + tD_Y) + \left(1 - \frac{t}{t'}\right) g^*(K_{Y_0} + \Delta_{Y_0})
$$

is ample over $Z$. By construction, for any $0 < t \leq t'$, the birational map $X \to X''$ is a sequence of steps of the $(K_X + \Delta + tD)$-MMP over $Z$. Since $K_{X''} + \Delta'' + tD''$ is the pullback of $K_Y + \Delta_Y + tD_Y$, we see that the pair $(Y, \Delta_Y + tD_Y)$ is the log canonical model of $(X, \Delta + tD)$ over $Z$ for any $0 < t \ll t'$. Therefore, the lemma holds true when $Z$ is quasi-projective.

From now on, we prove the general case. We cover $Z$ by a finitely many affine open subset $\{U_i\}$. Put $V_i = \pi^{-1}(U_i)$. By the quasi-projective case of the lemma, for any $i$, there is $t_i > 0$ and a birational contraction $U_i \to Y_i$ over $U_i$ such that for any $t \in (0, t_i]$ the pair $(Y_i, \Delta_{Y_i} + tD_{Y_i})$ is the log canonical model of $(V_i, \Delta_{V_i} + tD_{V_i})$ over $U_i$. Set $t'' = \min\{t_i\}$ and construct $Y$ by gluing all $Y_i$. By construction, for any $t \in (0, t'']$, the pair $(Y, \Delta_Y + tD_Y)$ is the log canonical model of $(X, \Delta + tD)$ over $Z$. Therefore, the birational map $X \to Y$ over $Z$ is the desired one. □

3. Spacial kinds of relative log MMP

In this section, we show a special kind of the relative log MMP (Theorem 3.5), which plays a crucial role in the proof of Theorem 1.1.

**Definition 3.1.** Let $X$ be a normal projective variety and $D$ be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $D$ on $X$.

First we define the **invariant Iitaka dimension** of $D$, denoted by $\kappa_i(X, D)$, as follows: If there is an $\mathbb{R}$-divisor $E \geq 0$ such that $D \sim_{\mathbb{R}} E$, set $\kappa_i(X, D) = \kappa(X, E)$. Here the right hand side is the usual Iitaka dimension of $E$. Otherwise, we set $\kappa_i(X, D) = -\infty$. We can check that $\kappa_i(X, D)$ is well-defined, i.e., $\kappa_i(X, D)$ does not depend on the choice of $E$. By definition, we have $\kappa_i(X, D) \geq 0$ if and only if $D$ is $\mathbb{R}$-linearly equivalent to an effective divisor.

Next we define the **numerical dimension** of $D$, denoted by $\kappa_\sigma(X, D)$, as follows: For any Cartier divisor $A$ on $X$, we set

$$
\sigma(D; A) = \max \left\{ k \in \mathbb{Z}_{\geq 0} \left| \limsup_{m \to \infty} \frac{\dim H^0(X, O_X(\lfloor mD + A \rfloor))}{m^k} > 0 \right. \right\}
$$

if $\dim H^0(X, O_X(\lfloor mD + A \rfloor)) > 0$ for infinitely many $m > 0$ and otherwise we set $\sigma(D; A) := -\infty$. Then, we define $\kappa_\sigma(X, D) := \max\{\sigma(D; A) \mid A: \text{Cartier}\}$.

It is known that $D$ is pseudo-effective if and only if $\kappa_\sigma(X, D) \geq 0$. 

Let $X \to Z$ be a projective morphism from a normal variety to a variety, and let $D$ be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on $X$. The relative numerical dimension of $D$ over $Z$ is defined by $\kappa_\sigma(F, D|_F)$, where $F$ is a sufficiently general fiber of the Stein factorization of $X \to Z$.

**Remark 3.2.** We write down basic properties of the invariant Iitaka dimension.

1. Let $D_1$ and $D_2$ be $\mathbb{R}$-Cartier $\mathbb{R}$-divisors on a normal projective variety $X$. Suppose that $D_1 \sim_\mathbb{R} D_2$. Then, we have $\kappa_i(X, D_1) = \kappa_i(X, D_2)$ and $\kappa_\sigma(X, D_1) = \kappa_\sigma(X, D_2)$.
2. Let $f : Y \to X$ be a surjective morphism of a normal projective varieties and $D$ be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on $X$.
   - The equalities $\kappa_i(X, D) = \kappa_i(Y, f^*D)$ and $\kappa_\sigma(X, D) = \kappa_\sigma(Y, f^*D)$ hold true.
   - Suppose that $f$ is birational and let $D'$ be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on $Y$ such that $D' = f^*D + E$, where $E$ is an effective $f$-exceptional divisor. Then, we have $\kappa_i(X, D) = \kappa_i(Y, D')$ and $\kappa_\sigma(X, D) = \kappa_\sigma(Y, D')$.

**Definition 3.3** (Relatively abundant and relatively log abundant divisor). Let $\pi : X \to Z$ be a projective morphism from a normal variety to a variety, and let $D$ be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on $X$. We say $D$ is $\pi$-abundant or abundant over $Z$ if the equality $\kappa_i(F, D|_F) = \kappa_\sigma(F, D|_F)$ holds, where $F$ is a sufficiently general fiber of the Stein factorization of $\pi$.

Let $\pi : X \to Z$ and $D$ be as above, and let $(X, \Delta)$ be an lc pair. We say $D$ is $\pi$-log abundant with respect to $(X, \Delta)$ if $D$ is $\pi$-abundant and the pullback of $D$ to the normalization of any lc center of $(X, \Delta)$ is abundant over $Z$.

**Lemma 3.4** (cf. [FG1, Theorem 4.12]). Let $\pi : X \to Z$ be a morphism of normal projective varieties and $(X, \Delta)$ be an lc pair such that $\Delta$ is an $\mathbb{R}$-divisor. Suppose that $K_X + \Delta$ is $\pi$-nef and $\pi$-log abundant with respect to $(X, \Delta)$.

Then $K_X + \Delta$ is $\pi$-semi-ample.

**Proof.** We can assume that $(X, \Delta)$ is not klt because otherwise the lemma follows from [GL, Theorem 4.3]. By adding the pullback of a sufficiently ample divisor on $Z$, we can assume that the divisor $K_X + \Delta$ is globally nef and log abundant with respect to $(X, \Delta)$. We show that $K_X + \Delta$ is semi-ample by induction on dim $X$. So we can assume $Z$ is a point.

By taking a dlt blow-up, we may assume that $(X, \Delta)$ is $\mathbb{Q}$-factorial dlt. Since $K_X + \Delta$ is abundant, there is $N \geq 0$ such that $K_X + \Delta \sim_\mathbb{R} N$. Let $\mathcal{L} \subset \text{WDiv}_\mathbb{R}(X)$ be the set of boundary $\mathbb{R}$-divisors $\Delta'$ such that $(X, \Delta')$ is lc, $\text{Supp} \Delta' = \text{Supp} \Delta$ and $\uplus \Delta' = \uplus \Delta$. By the argument of polytopes, we see that the set

$$
\left\{ \Delta' \in \mathcal{L} \mid \begin{array}{l}
(X, \Delta') \text{ is dlt,} \\
K_X + \Delta' \text{ is nef, and} \\
K_X + \Delta' \sim_\mathbb{R} N' \text{ for an } N' \geq 0 \text{ such that } \text{Supp} N' = \text{Supp} N.
\end{array} \right\}
$$

contains a rational polytope $\mathcal{T}(X) \subset \mathcal{L}$ in which $\Delta$ is contained. By shrinking $\mathcal{T}_X$, we can assume that lc centers of $(X, \Delta')$ coincide with those of $(X, \Delta)$ for any $\Delta' \in \mathcal{T}$.

By Remark 3.2 (1), $K_X + \Delta'$ is abundant for any $\Delta' \in \mathcal{T}(X)$.
Fix an lc center $S$ of $(X, \Delta)$. By construction, any divisor $\Delta' \in \mathcal{L}$ can be written as $\Delta + \sum_i d_i D_i$, where $0 \leq d_i < 1$ and $D_i$ are prime divisors which are components of $\Delta - \lambda \Delta$. Note that $\text{Supp} D_i \not\subseteq S$. Since $X$ is $\mathbb{Q}$-factorial, for any component $D_i$ of $\Delta - \lambda \Delta$, the restriction $D_i|_S$ is well-defined as an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $S$. By adjunction, $(K_X + \lambda \Delta)|_S$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. By the induction hypothesis, $(K_X + \Delta)|_S$ is semi-ample. Therefore, if we write $\Delta = \Delta + \sum d_i D_i$, then the divisor $(K_X + \Delta)|_S = (K_X + \lambda \Delta)|_S + \sum_i d_i (D_i|_S)$ can be written as an $\mathbb{R}_{\geq 0}$-linear combination of finitely many (not necessarily effective) semi-ample $\mathbb{Q}$-divisors $\{A_j\}$. We can write $(K_X + \Delta')|_S = (K_X + \lambda \Delta)|_S + \sum_j d'_j (D_j|_S)$ for any $\Delta' \in \mathcal{L}$. By these facts and the argument of polytopes, we see that the set

$$\{\Delta' \in \mathcal{T}(X) \mid (K_X + \Delta')|_S = \sum_j a_j A_j, \text{ where } a_j \in \mathbb{R}_{\geq 0}\}$$

contains a rational polytope $\mathcal{T}(S) \ni \Delta$.

Now we consider

$$\mathcal{T} = \bigcap_{S \text{ lc center of } (X, \Delta)} \mathcal{T}(S),$$

which is a rational polytope containing $\Delta$. Then, we can find positive real numbers $r_1, \cdots, r_m$ and $\mathbb{Q}$-divisors $\Delta(1), \cdots, \Delta(m)$ such that $\Delta(k) \in \mathcal{T}$ for any $1 \leq k \leq m$, $\sum_{k=1}^m r_k = 1$ and $\sum_{k=1}^m r_k \Delta(k) = \Delta$. By construction of $\mathcal{T}$, for any $\Delta' \in \mathcal{T}$, the divisor $K_X + \Delta'$ is nef and log abundant with respect to $(X, \Delta')$. Since $\Delta(k)$ are $\mathbb{Q}$-divisors, $K_X + \Delta(k)$ are semi-ample by [FG1, Theorem 4.12]. Then $K_X + \Delta$ is semi-ample since $K_X + \Delta = \sum_{k=1}^m r_k (K_X + \Delta(k))$. So we complete the proof. 

**Theorem 3.5.** Let $\pi: X \rightarrow Z$ be a projective morphism of normal quasi-projective varieties and $(X, \Delta)$ be an lc pair. Suppose that

- $-(K_X + \Delta)$ is pseudo-effective over $Z$, and
- for any lc center $S$ of $(X, \Delta)$ and its normalization $S' \rightarrow S$, the pullback of $-(K_X + \Delta)$ to $S'$ is pseudo-effective over $Z$.

Then, $(X, \Delta)$ has a good minimal model or a Mori fiber space over $Z$.

**Proof.** We can assume that $(X, \Delta)$ is not klt because otherwise the theorem follows from [Ha2, Theorem 1.2]. We prove Theorem 3.5 by induction on the dimension of $X$. The basic strategy is the same as [Ha2, Proof of Theorem 1.1]. We can assume that $\pi$ is a contraction and $K_X + \Delta$ is pseudo-effective over $Z$.

**Step 1.** In this step, we show that we may assume $X$ and $Z$ are projective.

Let $Z \hookrightarrow Z^c$ be an open immersion to a normal projective variety $Z^c$. Thanks to [Ha2, Corollary 1.3], there is an lc closure $(X^c, \Delta^c)$ of $(X, \Delta)$, that is, a projective lc pair $(X^c, \Delta^c)$ such that $X$ is an open subset of $X^c$ and $(X^c|_X, \Delta^c|_X) = (X, \Delta)$, and there is a projective morphism $\pi^c: X^c \rightarrow Z^c$. By construction of lc closures, we have $\pi^c|_X = \pi$ and $\pi^{-1}(Z) = X$. Furthermore, we can construct $(X^c, \Delta^c)$ so that any lc center $S^c$ of $(X^c, \Delta^c)$ intersects $X$ (see [Ha2, Corollary 1.3]). Then, the divisor $-(K_{X^c} + \Delta^c)$ is pseudo-effective over $Z^c$ and for any lc center $S^c$ of $(X^c, \Delta^c)$, the pullback of $-(K_{X^c} + \Delta^c)$ to the normalization of $S^c$ is pseudo-effective over $Z^c$ because relative numerical dimension of any $\mathbb{R}$-Cartier $\mathbb{R}$-divisor is determined on a sufficiently general fiber of the given morphism. Hence, we see that the morphism $(X^c, \Delta^c) \rightarrow Z^c$ satisfies the hypothesis of Theorem 3.5. If $(X^c, \Delta^c)$ has a good minimal model over $Z^c$, by restricting it over $Z$, we obtain a good minimal model of $(X, \Delta)$ over $Z$. 

In this way, by replacing \((X, \Delta)\) and \((X^c, \Delta^c)\) and \(Z^c\), we may assume that \(X\) and \(Z\) are projective.

**Step 2.** From this step to Step 6, we prove that \((X, \Delta)\) has a log minimal model over \(Z\). In this step, we construct a dlt blow-up with good properties.

By the hypothesis, the relative numerical dimension of \(K_X + \Delta\) over \(Z\) is 0. So there is \(E \geq 0\) on \(X\) such that \(K_X + \Delta \sim_{R, Z} E\). Since \(Z\) is projective, by adding the pullback of an ample divisor to \(E\), we may assume that \(\text{Supp}E\) contains any lc center of \((X, \Delta)\) which is vertical over \(Z\).

We take a log resolution \(f: \overline{X} \to X\) of \((X, \text{Supp}(\Delta + E))\) and a log smooth model \((\overline{X}, \Delta)\) of \((X, \Delta)\) (see [Ha1, Definition 2.9] for definition of log smooth models). As in [Ha1, Proof of Lemma 2.10], by replacing \((\overline{X}, \Delta)\) with a higher model, we may assume that we can write \(\overline{\Delta} = \overline{\Delta} + \overline{\Delta}'\) with \(\overline{\Delta}' \geq 0\) and \(\overline{\Delta}'' \geq 0\) such that \(\overline{\Delta}''\) is reduced and vertical over \(Z\), and all lc centers of \((\overline{X}, \overline{\Delta}')\) dominate \(Z\). We can decompose \(f^*E = \overline{G} + \overline{H}\) with \(\overline{G} \geq 0\) and \(\overline{H} \geq 0\) such that \(\overline{G}\) and \(\overline{H}\) have no common components, \(\text{Supp}G \subset \text{Supp}\overline{\Delta}_i\) and no component of \(\overline{H}\) is a component of \(\overline{\Delta}_i\). Since \((\overline{X}, \text{Supp}(\overline{\Delta} + \overline{H}))\) is log smooth and \(\overline{\Delta} + t\overline{H}\) is a boundary divisor for any sufficiently small \(t > 0\), the pair \((\overline{X}, \overline{\Delta} + t\overline{H})\) is dlt. We have \(\text{Supp}\overline{\Delta}'' \subset \text{Supp}G\) because \(\text{Supp}E\) contains any lc center of \((X, \Delta)\) which is vertical over \(Z\). Since all lc centers of \((\overline{X}, \overline{\Delta})\) dominate \(Z\), all lc centers of \((\overline{X}, \overline{\Delta} - tG)\) dominate \(Z\) for any sufficiently small \(t > 0\).

We construct a dlt blow-up \((X_0, \Delta_0) \to (X, \Delta)\) by running the \((K_{\overline{X}} + \overline{\Delta})\text{-MMP}\) over \(X\). Let \(G_0\) and \(H_0\) be the birational transforms of \(\overline{G}\) and \(\overline{H}\) on \(X_0\), respectively. By arguments of the log MMP, we can check that \((X_0, \Delta_0 + tH_0)\) is dlt and all lc centers of \((X_0, \Delta_0 - tG_0)\) dominate \(Z\) for any sufficiently small \(t > 0\).

In this way, by replacing \((X, \Delta)\), we can assume that \((X, \Delta)\) is \(\mathbb{Q}\)-factorial dlt and \(K_X + \Delta \sim_{R, Z} G + H\) such that \(G\) and \(H\) satisfy

1. \(G \geq 0, H \geq 0, \) and \(G\) and \(H\) have no common components,
2. \(\text{Supp}G \subset \text{Supp}\overline{\Delta}_i,\)
3. any lc center of \((X, \Delta - tG)\) dominates \(Z\) for any \(0 < t \ll 1,\) and
4. \((X, \Delta + tH)\) is dlt for any \(0 < t \ll 1.\)

**Step 3.** Pick \(\epsilon > 0\) so that \(\Delta - \epsilon G \geq 0\) and \((X, \Delta + \epsilon H)\) is dlt. In this step, we construct a strictly decreasing infinite sequence \(\{e_i\}_{i \geq 1}\) of real numbers and a sequence of birational maps

\[
X \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \cdots \dashrightarrow X_i \dashrightarrow \cdots
\]

such that if we put \(\Delta_i\) and \(H_i\) as the birational transforms of \(\Delta\) and \(H\) on \(X_i\) respectively, then

1. \(0 < e_i < \epsilon\) and \(\lim_{i \to \infty} e_i = 0,\)
2. \(X \dashrightarrow X_1\) is a sequence of steps of the \((K_X + \Delta + e_1 H)\text{-MMP}\) over \(Z\) to a good minimal model,
3. the sequence \(X_1 \dashrightarrow \cdots \dashrightarrow X_i \dashrightarrow \cdots\) is a sequence of finitely many steps of the \((K_{X_1} + \Delta_1)\text{-MMP}\) over \(Z\) with scaling of \(e_1 H_1,\)
4. for any \(i \geq 1,\) the pair \((X_i, \Delta_i + e_i H_i)\) is a good minimal model of both \((X_1, \Delta_1 + e_i H_1)\) and \((X, \Delta + e_i H)\) over \(Z.\)

By (4), if we set \(\lambda_i = \{\mu \in \mathbb{R}_{\geq 0} | K_{X_i} + \Delta_i + \mu H_i\text{ is nef over } Z\},\) then \(\lambda_i \leq e_i.\)

Pick a strictly decreasing infinite sequence \(\{e_i\}_{i \geq 1}\) of positive real numbers such that \(e_i < \epsilon\) for any \(i \geq 1\) and \(\lim_{i \to \infty} e_i = 0.\) By conditions of Step 2, the pairs
By replacing \(X, \Delta + e_iH\) and \(X, \Delta - \frac{e_i}{1+e_i}G\) are dlt, and we have

\[K_X + \Delta + e_iH \sim_{\mathbb{R}, Z} (1 + e_i) \left(K_X + \Delta - \frac{e_i}{1+e_i}G\right).\]

Moreover, all lc centers of \((X, \Delta - \frac{e_i}{1+e_i}G)\) dominate \(Z\) and the relative numerical dimension of \(K_X + \Delta - \frac{e_i}{1+e_i}G\) over \(Z\) is 0 for any \(i\). By [Ha2, Proposition 3.4], the pair \((X, \Delta - \frac{e_i}{1+e_i}G)\) has a good minimal model over \(Z\), hence \((X, \Delta + e_iH)\) has a good minimal model over \(Z\) for any \(i\). By running the \((K_X + \Delta + e_iH)\)-MMP over \(Z\), we obtain a good minimal model \((X, \Delta + e_iH) \to (X_i, \Delta_i + e_iH_i)\) over \(Z\). Then, the log MMP only occurs in \(\text{Supp}(G + H)\), which does not depend on \(i\).

By replacing \(\{e_i\}_{i \geq 1}\) with its subsequence, we may assume that all birational maps \(X \to X_i\) contract the same divisors, which implies that all \(X_i\) are isomorphic in codimension one.

For any \(0 < t \leq e_1\), the pair \((X_1, \Delta_1 + tH_1)\) has a good minimal model over \(Z\). Indeed, we have \(K_{X_1} + \Delta_1 + tH_1 \sim_{\mathbb{R}, Z} (1 + t)(K_{X_1} + \Delta_1 - \frac{1}{1+t}G_1)\) and the relative numerical dimension of \(K_{X_1} + \Delta_1 - \frac{1}{1+t}G_1\) over \(Z\) is 0, where \(G_1\) is the birational transform of \(G\) on \(X_1\). Moreover, all lc centers of \((X_1, \Delta_1 - \frac{1}{1+t}G_1)\) dominate \(Z\). To check this, pick any prime divisor \(P\) over \(X_1\) such that \(a(P, X_1, \Delta_1 - \frac{1}{1+t}G_1) = -1\). Since \((X_1, \Delta_1)\) is lc, we have \(a(P, X_1, \Delta_1 - \frac{1}{1+t}G_1) = -1\). Since the birational map \(X \to X_1\) is also a sequence of steps of the \((K_X + \Delta - \frac{1}{1+t}G)\)-MMP, we have \(a(P, X, \Delta - \frac{1}{1+t}G) = -1\). By the third condition in Step 2, \(P\) dominates \(Z\). Thus, all lc centers of \((X_1, \Delta_1 - \frac{1}{1+t}G_1)\) dominate \(Z\). By [Ha2, Proposition 3.4], the pair \((X_1, \Delta_1 - \frac{1}{1+t}G_1)\) has a good minimal model over \(Z\), and so does \((X_1, \Delta_1 + tH_1)\).

Put \(X'_1 = X_1\) (resp. \(\Delta'_1 = \Delta_1, H'_1 = H_1\)). By [Ha2, Lemma 2.13], we get a sequence of steps of the \((K_{X'_1} + \Delta'_1)\)-MMP over \(Z\) with scaling of \(e_1H'_1\)

\[(X'_1, \Delta'_1) \to \cdots \to (X'_j, \Delta'_j) \to \cdots\]

such that if we set \(\lambda'_j = \inf\{\mu \in \mathbb{R}_{\geq 0} | K_{X'_j} + \Delta'_j + \mu H'_j\text{ is nef over }Z\}\), where \(H'_j\) is the birational transform of \(H'_j\) on \(X'_j\), then the \((K_{X'_1} + \Delta'_1)\)-MMP terminates after finitely many steps or we have \(\lim_{j \to \infty} \lambda'_j = 0\) when it does not terminate.

For any \(i \geq 1\), pick the minimum \(k_i\) such that \(K_{X'_i} + \Delta'_i + e_iH'_i\) is nef over \(Z\). Such \(k_i\) exists since \(\lim_{j \to \infty} \lambda'_j = 0\), and we have \(k_1 = 1\). By construction, the pair \((X'_{k_i}, \Delta'_{k_i} + e_iH'_{k_i})\) is a good minimal model of \((X'_1, \Delta'_1 + e_iH'_1)\) over \(Z\). We check that \((X'_{k_i}, \Delta'_{k_i} + e_iH'_{k_i})\) is a good minimal model of \((X, \Delta + e_iH)\) over \(Z\) which was constructed at the start of this step, and all \(X'_i\) are isomorphic in codimension one. Since we put \(X_1 = X'_{k_1}, X'_1\) and \(X_1\) are isomorphic in codimension one for any \(i\). Since \(\lim_{i \to \infty} e_i = 0\), the divisor \(K_{X'_1} + \Delta'_{k_1}\) is the limit of movable divisors. Then, the \((K_{X'_1} + \Delta'_{k_1})\)-MMP contains only flips, and hence \(X'_{k_i}\) and \(X_i\) are isomorphic in codimension one. By Remark 2.1, the pair \((X'_{k_i}, \Delta'_{k_i} + e_iH'_{k_i})\) is a good minimal model of \((X, \Delta + e_iH)\) over \(Z\).

By abuse of notations, we put \(X_i = X'_{k_i}\) (resp. \(\Delta_i = \Delta'_{k_i}, H_i = H'_{k_i}\)) for any \(i\). Note that after putting them, for any \(i \geq 2\), the birational map \(X \to X_i\) may not be a sequence of steps of the \((K_X + \Delta + e_iH)\)-MMP. By construction, \(\{e_i\}_{i \geq 1}\) and

\[X \to X_1 \to X_2 \to \cdots \to X_i \to \cdots\]

satisfy (1), (2), (3) and (4) stated at the start of this step. Indeed, (1) and (2) follow from the argument in the second paragraph. The condition (3) follows from
the argument in the fourth paragraph. The condition (4) follows from the argument in the fifth paragraph.

**Step 4.** Suppose that the above \((K_{X_l} + \Delta_l)\)-MMP over \(Z\) with scaling of \(e_lH_1\) terminates. Then \(X_l \simeq X_{l+1} \simeq \cdots\) for some \(l\), and hence, for any \(i \geq l\), the pair \((X_i, \Delta_i + e_iH_i)\) is a good minimal model of \((X, \Delta + e_iH)\) over \(Z\) by (4) in Step 3. Then, we have \(a(P, X, \Delta + e_iH) \leq a(P, X_l, \Delta_l + e_lH_l)\) for any prime divisor \(P\) over \(X\). By considering the limit \(i \to \infty\), we have \(a(P, X, \Delta) \leq a(P, X_l, \Delta_l)\). Therefore, the pair \((X_l, \Delta_l)\) is a weak lc model of \((X, \Delta)\) over \(Z\). In this way, we see that \((X, \Delta)\) has a log minimal model over \(Z\).

Therefore, to show the existence of log minimal model of \((X, \Delta)\), we only have to prove termination of the \((K_{X_l} + \Delta_l)\)-MMP.

**Step 5.** Since we have \(K_{X_l} + \Delta_l + e_lH_l \sim_{R, Z} (1 + e_l)(K_{X_l} + \Delta_l - \frac{G_1}{1 + e_l})\) and \(\text{Supp}\Psi_1 \subset \text{Supp}\Delta_{1,i}\), the \((K_{X_l} + \Delta_l)\)-MMP only occurs in \(\text{Supp}\Delta_{1,i}\) (see, for example, [Ha1, Step 2 in the proof of Proposition 5.4]).

Suppose that the \((K_{X_l} + \Delta_l)\)-MMP does not terminate. We get a contradiction by the argument of the special termination (cf. [F1]). We note that \((X_1, \Delta_1 + e_1H_1)\) is \(\mathbb{Q}\)-factorial dlt and any lc center of it is an lc center of \((X_1, \Delta_1)\). Therefore, for any \(i\), the pair \((X_i, \Delta_i)\) is \(\mathbb{Q}\)-factorial dlt and any lc center of it is normal. There is \(m \gg 0\) such that for any lc center \(S_m\) of \((X_m, \Delta_m)\) and any \(i \geq m\), the restriction of the birational map \(X_m \dashrightarrow X_i\) to \(S_m\) induces a birational map. Pick any lc center \(S_m\) of \((X_m, \Delta_m)\), and let \(S_i\) be the lc center of \((X_i, \Delta_i)\) birational to \(S_m\). We define an \(\mathbb{R}\)-divisor \(\Delta_{S_i}\) on \(S_i\) by \(K_{S_i} + \Delta_{S_i} = (K_{X_l} + \Delta_l)|_{S_i}\). In this step and the next step, we prove that for any \(i \geq m\) the induced birational map \((S_i, \Delta_{S_i}) \dashrightarrow (S_{i+1}, \Delta_{S_{i+1}})\) is an isomorphism. If we can prove this, then the \((K_{X_1} + \Delta_1)\)-MMP over \(Z\) must terminate (see [F1]), and we can get a contradiction. By induction on the dimension of \(S_m\) as in [F1], by replacing \(m\) if necessary, we can assume that for any \(i \geq m\) and any lc center \(Y_m\) of \((X_m, \Delta_m)\) contained in \(S_m\), the induced birational map \(Y_m \dashrightarrow Y_i\) is isomorphic in codimension one and the birational transform of \(\Delta_{Y_m}\) on \(Y_i\) is equal to \(\Delta_{Y_i}\). Here \(Y_i\) is an lc center of \((X_i, \Delta_i)\) corresponding to \(Y_m\). Let \((T_m, \Psi_m) \rightarrow (S_m, \Delta_{S_m})\) be a dlt blow-up. Set \(H_{T_m}\) as the pullback of \(H_m|_{S_m}\) to \(T_m\). As in [F1] (see also [B1, Remark 2.10]), we can obtain the following diagram

\[
\begin{array}{ccc}
(T_m, \Psi_m) & \dashrightarrow & (S_i, \Delta_{S_i}) \\
\downarrow & & \downarrow \\
(S_m, \Delta_{S_m}) & \dashrightarrow & (S_{i+1}, \Delta_{S_{i+1}}) \\
\end{array}
\]

such that

- \((T_i, \Psi_i) \rightarrow (S_i, \Delta_{S_i})\) is a dlt blow-up, and
- the upper horizontal sequence of birational maps is a sequence of steps of the \((K_{T_m} + \Psi_m)\)-MMP over \(Z\) with scaling of \(e_m H_{T_m}\).

We prove that the \((K_{T_m} + \Psi_m)\)-MMP over \(Z\) with scaling of \(e_m H_{T_m}\) must terminate. If we can prove this, then the map \((S_i, \Delta_{S_i}) \dashrightarrow (S_{i+1}, \Delta_{S_{i+1}})\) is an isomorphism. To prove this, it is sufficient that \((T_m, \Psi_m)\) has a log minimal model over \(Z\). Since the morphism \((T_m, \Psi_m) \rightarrow (S_m, \Delta_{S_m})\) is a dlt blow-up, it is sufficient to prove that \((S_m, \Delta_{S_m})\) has a log minimal model over \(Z\).

**Step 6.** We prove that \((S_m, \Delta_{S_m})\) has a log minimal model over \(Z\) by using the induction hypothesis of Theorem 3.5. Since \((X_m, \Delta_m)\) is \(\mathbb{Q}\)-factorial dlt, \(S_m\) and
all lc centers of $(S_m, \Delta_{S_m})$ are lc centers of $(X_m, \Delta_m)$ contained in $S_m$. Since the divisors $K_{S_m} + \Delta_{S_m} + e_H|_{S_m}$ are nef over $Z$ and since $S_m$ and $S_i$ are isomorphic in codimension one, $K_{S_m} + \Delta_{S_m}$ is pseudo-effective over $Z$. From these facts, it is sufficient to check that the divisor $-(K_{X_m} + \Delta_m)|_{Y_m}$ is pseudo-effective over $Z$ for any lc center $Y_m$ of $(X_m, \Delta_m)$ which is contained in $S_m$. We define $\Delta_{Y_m}$ on $Y_m$ by $K_{Y_m} + \Delta_{Y_m} = (K_{X_m} + \Delta_m)|_{Y_m}$.

Recall that for any $i \geq m$ and any lc center $Y_m$ of $(X_m, \Delta_m)$ contained in $S_m$, the induced birational map $Y_m \to Y_i$ is isomorphic in codimension one and the birational transform of $\Delta_{Y_m}$ on $Y_i$ is $\Delta_{Y_i}$. We set $H_{Y_i} = H_i|_{Y_i}$. Then $H_{Y_i} \geq 0$ and the birational transform of $H_{Y_m}$ on $Y_i$ is equal to $H_{Y_i}$. By construction of the map $(X, \Delta + e_iH_i) \dasharrow (X_i, \Delta_i + e_iH_i)$ (see (2) and (3) in Step 3), there is an lc center $Y_i$ of $(X, \Delta)$ such that the map $X \dasharrow X_i$ induces a birational map $Y \dasharrow Y_i$. We set $H_T = H|_Y$, and we define an $\mathbb{R}$-divisor $\Delta_T$ on $Y$ by $K_T + \Delta_T = (K_T + \Delta)|_Y$. Then $H_T \geq 0$. By (2) and (3) in Step 3, for any $i \geq m$, there is a common log resolution $Y_i \to X$ and $Y_i \to X_i$ of $X \dasharrow X_i$ and a subvariety $Y_{Y_i} \subset Y_i$ birational to $Y$ and $Y_i$ such that the induced morphisms $\tau_{Y_i} : Y_i \to Y$ and $\tau_i : Y_i \to Y_i$ is a common resolution of $Y \dasharrow Y_i$. Using (4) in Step 3 and the negativity lemma, by taking pullbacks of $K_X + \Delta + e_iH$ and $K_T + \Delta_T + e_iH_T$ to $Y_i$, and comparing coefficients, we see that $a(Q, \tau_{Y_i}, \Delta_T + e_iH_T) \leq a(Q, \tau_i, \Delta_T + e_iH_T)$ for any prime divisor $Q$ over $Y$.

Since $(X_m, \Delta_m)$ is $\mathbb{Q}$-factorial dlt, the pair $(Y_m, \Delta_{Y_m})$ is dlt. So there is a small $\mathbb{Q}$-factorization $Y' \to Y_m$. Then $Y'$ and $T_i$ are isomorphic in codimension one for any $i \geq m$ because $Y_m$ and $T_i$ are isomorphic in codimension one. We denote the pullback of $K_{Y_m} + \Delta_{Y_m}$ to $Y'$ by $K_{Y'} + \Delta_{Y'}$. Take a common resolution $\varphi : Y \to Y'$ and $\varphi' : Y_i \to Y_i$ of the birational map $Y \dasharrow Y'$ and $Y_i \to Y_i$. For any $i \geq m$, we take a common resolution $\tau : Y_i \to Y$ and $\tau_i : Y_i \to Y_i$ of the birational map $Y \dasharrow Y_i$. We have the following diagram.

$$\begin{array}{cccc}
\tau & \varphi & \tau_i \\
Y & \dasharrow & Y_i \\
Y_m & \dasharrow & \cdots & \dasharrow & Y_i
\end{array}$$

Since we have $a(Q, \tau_{Y_i}, \Delta_T + e_iH_T) \leq a(Q, \tau_i, \Delta_T + e_iH_T)$ for any prime divisor $Q$ over $Y$, we have

$$\tau^*\varphi^*(K_T + \Delta_T + e_iH_T) - \tau_i^*(K_T + \Delta_T + e_iH_T) \geq 0.$$ 

Therefore,

$$-\tau_i^*(K_T + \Delta_T) + e_i\tau^*\varphi^*H_T \geq -\tau_i^*\varphi^*(K_T + \Delta_T) + e_i\tau_i^*H_T.$$ 

By the hypothesis of Theorem 3.5, $-(K_T + \Delta_T)$ is pseudo-effective over $Z$. Thus, the divisor $-\tau_i^*(K_T + \Delta_T) + e_i\tau^*\varphi^*H_T$ is pseudo-effective over $Z$ since $H_T \geq 0$. We have $\varphi'_i\tau_i^*(K_T + \Delta_T) = K_{Y'} + \Delta_{Y'}$ since $Y'$ and $Y_i$ are isomorphic in codimension one. By taking the birational transform on $Y'$, we see that the divisor $-(K_{Y'} + \Delta_{Y'}) + e_i\varphi'_i\tau_i^*H_T$ is pseudo-effective over $Z$ for any $i$. Note that $Y'$ is $\mathbb{Q}$-factorial. Since $\lim_{i \to \infty} e_i = 0$, the divisor $-(K_{Y'} + \Delta_{Y'})$ is pseudo-effective over $Z$. So we see that $-(K_{Y_m} + \Delta_{Y_m})$ is pseudo-effective over $Z$. 

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In this way, the divisor \(-(K_{X_m} + \Delta_m)|_{Y_m}\) is pseudo-effective over \(Z\) for any lc center \(Y_m\) of \((X_m, \Delta_m)\) which is contained in \(S_m\). By the induction hypothesis of Theorem 3.5, \((S_m, \Delta_{S_m})\) has a log minimal model over \(Z\) (for details, see the first paragraph of this step). Therefore, for any \(i \gg m\) the induced birational map \((S_i, \Delta_{S_i}) \to (S_{i+1}, \Delta_{S_{i+1}})\) is an isomorphism (see Step 5). By the argument of the special termination ([F1]), the \((K_X + \Delta_1)\)-MMP over \(Z\) with scaling of \(c_1H_1\), which was constructed in Step 3, must terminates. So \((X, \Delta)\) has a log minimal model (see Step 4).

**Step 7.** By running the \((K_X + \Delta)\)-MMP over \(Z\), we can obtain a log minimal model \((X, \Delta) \to (X_{\min}, \Delta_{\min})\) over \(Z\). Then, the numerical dimension of \(K_{X_{\min}} + \Delta_{\min}\) over \(Z\) is 0, and for any lc center \(S'\) of \((X_{\min}, \Delta_{\min})\), the numerical dimension of \((K_{X_{\min}} + \Delta_{\min})|_{S'}\) over \(Z\) is 0. Since \(X\) and \(Z\) are both projective, we can apply Lemma 3.4. Therefore, the divisor \(K_{X_{\min}} + \Delta_{\min}\) is semi-ample, and \((X_{\min}, \Delta_{\min})\) is a good minimal model over \(Z\). So we are done.

The following result is not used in this paper, but it is interesting on its own.

**Corollary 3.6.** Let \(\pi : X \to Z\) be a projective morphism of normal quasi-projective varieties and \((X, \Delta)\) be an lc pair. Suppose that there is an \(\mathbb{R}\)-divisor \(B \geq 0\) on \(X\) such that

- \(-(K_X + \Delta + B)\) is nef over \(Z\), and
- \((X, \Delta + \epsilon B)\) is lc for a sufficiently small \(\epsilon > 0\).

Then \((X, \Delta)\) has a good minimal model or a Mori fiber space over \(Z\).

**Proof.** We can check that the morphism \((X, \Delta) \to Z\) satisfies the hypothesis of Theorem 3.5. Therefore, the corollary follows from Theorem 3.5. \(\square\)

### 4. Pseudo-lc pairs

In this section, a pair \((X, \Delta)\) simply denotes a pair of a normal variety \(X\) and an \(\mathbb{R}\)-divisor \(\Delta \geq 0\) on it. In particular, we do not assume \(K_X + \Delta\) to be \(\mathbb{R}\)-Cartier. When \(K_X + \Delta\) is \(\mathbb{R}\)-Cartier, we denote the pair of \(X\) and \(\Delta\) by \((X, \Delta)\) as usual.

**Definition 4.1.** Let \((X, \Delta)\) be a pair and \(P\) be a prime divisor over \(X\), that is, a prime divisor on a higher birational model \(Y \to X\). We define the discrepancy \(\alpha(P, X, \Delta)\) of \(P\) with respect to \((X, \Delta)\) as follows:

We fix \(K_X\) as a Weil divisor. We denote the image of \(P\) on \(X\) by \(c_X(P)\). Let \(f : Y \to X\) be a projective birational morphism from a normal variety \(Y\) such that \(P\) is a prime divisor on \(Y\). We fix \(K_Y\) so that \(f_*K_Y = K_X\) as Weil divisors. The divisor \(K_Y\) depends on the choice of \(K_X\). For any affine open subset \(U \subset X\) such that \(U \cap c_X(P) \neq \emptyset\), we put \(K_U = K_X|_U\), \(V = f^{-1}(U)\), \(f_V = f|_V\) and \(K_V = K_Y|_V\). For any \(\mathbb{R}\)-divisor \(B_U \geq 0\) on \(U\) such that \(K_U + \Delta|_U + B_U\) is \(\mathbb{R}\)-Cartier, we define

\[\alpha(P, X, \Delta)(P, U, B_U) = \text{coeff}_{P|_V} (K_V - f_V^*(K_U + \Delta|_U + B_U)).\]

By the standard argument, \(\alpha(P, X, \Delta)(P, U, B_U)\) does not depend on the choice of \(K_X\) and \(f : Y \to X\). We define

\[\alpha(P, X, \Delta) := \sup_{U, B_U} \{\alpha(P, X, \Delta)(P, U, B_U)\},\]

where \(U\) runs over all affine open subset of \(X\) such that \(U \cap c_X(P) \neq \emptyset\), and \(B_U\) runs over all effective \(\mathbb{R}\)-divisor on \(U\) such that \(K_U + \Delta|_U + B_U\) is \(\mathbb{R}\)-Cartier.
Definition 4.2. Let \( (X, \Delta) \) be a pair. We say the pair \( (X, \Delta) \) is pseudo-lc if the inequality \( \alpha(P, X, \Delta) \geq -1 \) holds for any prime divisor \( P \) over \( X \).

We show pseudo-lc singularity is a generalization of lc singularity.

Lemma 4.3. Let \( (X, \Delta) \) be a pair and \( P \) be a prime divisor over \( X \).

(i) If \( K_X + \Delta \) is \( \mathbb{R} \)-Cartier, then \( \alpha(P, X, \Delta) = a(P, X, \Delta) \), where the right hand side is the usual discrepancy.

(ii) If \( P \) is a divisor on \( X \), then \( \alpha(P, X, \Delta) = -\text{coeff}_P(\Delta) \).

(iii) Let \( 0 \leq \Delta' \leq \Delta \) be an \( \mathbb{R} \)-divisor. Then \( \alpha(P, X, \Delta') \leq \alpha(P, X, \Delta) \).

In particular, if \( (X, \Delta) \) is pseudo-lc and \( K_X + \Delta \) is \( \mathbb{R} \)-Cartier, then \( (X, \Delta) \) is lc.

Proof. These are proved by the standard arguments.

Firstly, we prove (i). The inequality \( \alpha(P, X, \Delta) \geq a(P, X, \Delta) \) follows from the definition of \( \alpha(P, X, \Delta) \). Thus, we show the inverse inequality. We use the notation as in Definition 4.1. Let \( f: Y \to X \) be a projective birational morphism such that \( P \) is a prime divisor on \( Y \). Let \( U \) be an affine open subset of \( X \) such that \( U \cap c_X(P) \neq \emptyset \). We set \( V = f^{-1}(U) \) and \( f_V = f|_V: V \to U \). For any \( \mathbb{R} \)-Cartier \( \mathbb{R} \)-divisor \( B_U \geq 0 \) on \( U \), \( a(P, X, \Delta) - \alpha_{(X, \Delta)}(P, U, B_U) \) is the coefficient of \( P|_V \) in
\[
(K_V - f_V^* (K_U + \Delta_U)) - (K_V - f_V^* (K_U + \Delta_U + B_U)) \geq 0.
\]
Hence, we have \( a(P, X, \Delta) \geq \alpha_{(X, \Delta)}(P, U, B_U) \) for any \( U \) and \( B_U \). By taking the supremum, we have \( \alpha(P, X, \Delta) \geq a(P, X, \Delta) \). So the equality holds.

Secondly, we show (ii). For any affine open subset \( U \subset X \) with \( P \cap U \neq \emptyset \) and any \( \mathbb{R} \)-divisor \( B_U \geq 0 \) on \( U \) such that \( K_U + \Delta_U + B_U \) is \( \mathbb{R} \)-Cartier, we have \( \alpha_{(X, \Delta)}(P, U, B_U) \leq -\text{coeff}_P(\Delta) \). Then \( \alpha(P, X, \Delta) \leq -\text{coeff}_P(\Delta) \) by Definition 4.1. We pick an affine open subset \( U \) such that \( P \cap U \neq \emptyset \) and \( U \) is contained in the smooth locus of \( X \). Such \( U \) exists since \( X \) is normal. Then, the divisor \( K_U + \Delta_U \) is \( \mathbb{R} \)-Cartier, and we have \( \alpha_{(X, \Delta)}(P, U, 0) = -\text{coeff}_P(\Delta) \). By Definition 4.1, we have \( \alpha(P, X, \Delta) \leq -\text{coeff}_P(\Delta) \). Thus, the equality of (ii) holds.

Finally, we show (iii). Put \( G = \Delta - \Delta' \geq 0 \), and pick any prime divisor \( P \) over \( X \). For any affine open subset \( U \subset X \) with \( U \cap c_X(P) \neq \emptyset \) and any \( \mathbb{R} \)-divisor \( B_U \geq 0 \) on \( U \) such that \( K_U + \Delta_U + B_U \) is \( \mathbb{R} \)-Cartier, we have
\[
\alpha_{(X, \Delta)}(P, U, B_U) = \alpha_{(X, \Delta')}\bigl(P, U, G|_U + B_U\bigr) \leq \alpha(P, X, \Delta')
\]
by Definition 4.1. By taking the supremum, we have \( \alpha(P, X, \Delta) \leq \alpha(P, X, \Delta') \). So we are done.

We will see later that the notion of pseudo-lc singularity is closely related to log canonical singularity introduced in [dFH] and generalized lc pairs introduced in [BZ] (Proposition 4.6 and Proposition 4.11).

Before showing results on pseudo-lc pairs, we introduce log canonical and log terminal singularities in the sense of [dFH]. Our definition seems to be the most natural way to modify singularities introduced in [dFH] to singularities of pairs of a normal variety and an effective \( \mathbb{R} \)-divisor. We only write down notations and definitions used in this paper.

Notation 4.4. Let \( \Delta = \sum d_i \Delta_i \) be the decomposition to prime divisors. We set \( \mathcal{I}_\Delta = \sum d_i \cdot \mathcal{O}_X(-\Delta_i) \), a formal \( \mathbb{R}_{\geq 0} \)-linear combination of Weil divisorial sheaves. For any prime divisor \( P \) over \( X \), we denote by \( \nu_P: \mathcal{C}(X) \to \mathbb{Z} \) the corresponding divisorial valuation on the field of rational functions \( \mathcal{C}(X) \).
Firstly, for any Weil divisor $D$ on $X$, we define
\[
\nu_p^i(D) := \min \left\{ \nu_p(\phi) \ \middle| \ \phi \in \mathcal{O}_X(-D)(U), \ U \subset X \text{ is open}, \ U \cap c_X(P) \neq \emptyset \right\}.
\]
We define $\nu_p(\mathcal{I}_\Delta) := \sum d_i \cdot \nu_p^i(\Delta_i)$ ([DFH, Definition 2.1 and Definition 2.2]).

Secondly, for any birational morphism $f: Y \to X$ from a normal variety $Y$, we define
\[
f^i_*(D) := \sum_{E: \text{prime divisor on } Y} \nu_E^i(D)E
\]
([DFH, Definition 2.6]).

Thirdly, for any $\mathbb{Q}$-divisor $D$, we can define $\nu_p(D)$ by
\[
\nu_p(D) := \inf_{k \geq 1} \frac{\nu_p^k(kD)}{k} = \lim_{k \to \infty} \inf \frac{\nu_p^k(kD)}{k} = \lim_{k \to \infty} \frac{\nu_p^k(k!D)}{k!}
\]
([DFH, Lemma 2.8 and Definition 2.9]). When $D$ is $\mathbb{Q}$-Cartier, $\nu_p(D)$ coincides with the usual valuation along $P$.

Finally, we fix a birational morphism $f: Y \to X$ from a normal variety $Y$ and Weil divisors $K_X$ and $K_Y$ such that $f_*K_Y = K_X$. For any $m \geq 1$, we put
\[
K_{m,Y/X} := K_Y - \frac{1}{m}f^i_*(mK_X)
\]
([DFH, Definition 3.1]). Note that $K_{m,Y/X}$ does not depend on the choice of $K_X$ and $K_Y$.

**Definition 4.5** (cf. [DFH, Definition 7.1]). Let $\langle X, \Delta = \sum d_i \Delta_i \rangle$ be a pair. We say the pair $\langle X, \Delta \rangle$ is log canonical (resp. log terminal) in the sense of [DFH] if $X$ is quasi-projective and there is $m \geq 1$ such that
\[
\text{coeff}_P(K_{m,Y/X}) + 1 - \nu_p(\mathcal{I}_\Delta)
\]
is non-negative (resp. positive) for any projective birational morphism $f: Y \to X$ and any prime divisor $P$ on $Y$, where $\mathcal{I}_\Delta = \sum d_i \cdot \mathcal{O}_X(-\Delta_i)$ is as in Notation 4.4.

We prove that pseudo-lc singularity is a generalization of log canonical singularity in the sense of [DFH].

**Proposition 4.6.** Let $\langle X, \Delta = \sum d_i \Delta_i \rangle$ be a pair. If $\langle X, \Delta \rangle$ is log canonical in the sense of [DFH], then $\langle X, \Delta \rangle$ is pseudo-lc.

**Proof.** Let $\mathcal{I}_\Delta = \sum d_i \cdot \mathcal{O}_X(-\Delta_i)$ be as in Notation 4.4. By the hypothesis, there is $m \geq 1$ such that
\[
\text{coeff}_P(K_{m,Y/X}) - \nu_p(\mathcal{I}_\Delta) = \text{coeff}_P(K_Y) - \frac{1}{m} \cdot \nu_p^i(mK_X) - \sum d_i \cdot \nu_p^i(\Delta_i) \geq -1
\]
for any projective birational morphism $f: Y \to X$ and any prime divisor $P$ on $Y$. We pick any $P$ over $X$ and fix $f: Y \to X$ such that $P$ is a prime divisor on $Y$. By (1) in Notation 4.4, there is an open subset $U_0 \subset X$ and a rational function $\phi_0$ such that $U_0 \cap c_X(P) \neq \emptyset$, $\phi_0 \in \mathcal{O}_X(-mK_X)(U_0)$ and $\nu_p(\phi_0) = \nu_p^i(mK_X)$. Similarly, for any $i$, we can find $U_i$ and $\phi_i$ such that $U_i \cap c_X(P) \neq \emptyset$, $\phi_i \in \mathcal{O}_X(-\Delta_i)(U_i)$ and
Proposition 4.7. Let \( \langle X, \Delta \rangle \) be a pair such that \( \Delta \) is a \( \mathbb{Q} \)-divisor, and let \( P \) be a prime divisor over \( X \). Let \( f : Y \to X \) be a projective birational morphism from a normal variety \( Y \) such that \( P \) is a divisor on \( Y \). Fix \( K_X \) and \( K_Y \) such that \( f_*K_Y = K_X \). Then, 
\[
\alpha(P, X, \Delta) = \text{coeff}_P(K_Y - f^*(K_X + \Delta)) = \text{coeff}_P(K_Y) - v_P(K_X + \Delta).
\]

Proof. The second equality is obvious from the definition of \( f^*(K_X + \Delta) \). We prove the equality \( \alpha(P, X, \Delta) = \text{coeff}_P(K_Y) - v_P(K_X + \Delta) \). Pick any \( m \) such that \( m = k! \) for some integer \( k > 0 \) and \( m\Delta \) is a Weil divisor. By Notation 4.4 (1), we can find an open subset \( U \subset X \) and a rational function \( \phi \in \mathcal{O}_X(-m(K_X + \Delta))(U) \) such that \( U \cap c_X(P) \neq \emptyset \) and \( v_P(\phi) = v_P^*(m(K_X + \Delta)) \). By shrinking \( U \), we may assume

\[
v_P(\phi) = v_P^*(m(K_X + \Delta)).
\]
\( U \) is affine. Set \( B_U = \frac{1}{m}(\text{div}(\phi) - m(K_U + \Delta|_U)) \). Then \( B_U \geq 0 \) and the divisor \( K_U + \Delta|_U + B_U \) is \( \mathbb{Q} \)-Cartier. If we put \( V = f^{-1}(U) \) and \( f_V = f|_V \), then
\[
\alpha(P, X, \Delta) \geq \alpha_{(X, \Delta)}(P, U, B_U) = \text{coeff}_{P|_V}(K_V - f_V^*(K_U + \Delta|_U + B_U)) \\
= \text{coeff}_P(K_Y) - \frac{1}{m}v_P(\phi) \\
= \text{coeff}_P(K_Y) - \frac{1}{m}v_P^3(m(K_X + \Delta))
\]

By Notation 4.4 (2), we have \( v_P(K_X + \Delta) = \lim_{k \to \infty} \frac{v_P(K_X + \Delta)}{k} \). Therefore, considering the limit \( k \to \infty \), we obtain \( \alpha(P, X, \Delta) \geq \text{coeff}_P(K_Y) - v_P(K_X + \Delta) \).

On the other hand, pick an affine open subset \( U' \subset X \) and an \( \mathbb{R} \)-divisor \( C_{U'} \geq 0 \) on \( U' \) such that \( U' \cap c_X(P) \neq \emptyset \) and \( K_{U'} + \Delta|_{U'} + C_{U'} \) is \( \mathbb{R} \)-Cartier. Then, there are positive real numbers \( r_1, \ldots, r_n \) and effective \( \mathbb{Q} \)-divisors \( C_1, \ldots, C_n \) on \( U' \) such that \( \sum_{j=1}^n r_j = 1 \), \( \sum_{j=1}^n r_j C_j = C_{U'} \) and \( K_{U'} + \Delta|_{U'} + C_j \) are \( \mathbb{Q} \)-Cartier. By definition of \( \alpha_{(X, \Delta)}(U', C_{U'}) \), we have \( \alpha_{(X, \Delta)}(P, U', C_{U'}) = \sum_{j=1}^n r_j \alpha_{(X, \Delta)}(P, U', C_j) \). For some index \( j' \), we have \( \alpha_{(X, \Delta)}(P, U', C_{U'}) \leq \alpha_{(X, \Delta)}(P, U', C_{j'}) \). Pick a sufficiently large and divisible integer \( m > 0 \) such that \( m\Delta \) and \( mc_{U'} \) are both well divisors and \( m(K_{U'} + \Delta|_{U'} + C_{j'}) \) is Cartier. By shrinking \( U' \), we can write \( m(K_{U'} + \Delta|_{U'} + C_{j'}) = \text{div}(\sigma) \) for some rational function \( \sigma \). Since \( C_{j'} \geq 0 \), we have \( \sigma \in \mathcal{O}_X(-m(K_X + \Delta))(U') \) and therefore we obtain \( v_P(\sigma) = v_P^3(m(K_X + \Delta)) \).

With Notation 4.4 (1), we have
\[
\frac{1}{m}v_P(\sigma) \geq \frac{1}{m}v_P^3(m(K_X + \Delta)) \geq v_P(K_X + \Delta).
\]

We put \( V' = f^{-1}(U') \) and \( f_{V'} = f|_{V'} \). From the above facts, for any \( U' \) and \( C_{U'} \), we have
\[
\alpha_{(X, \Delta)}(P, U', C_{U'}) = \alpha_{(X, \Delta)}(P, U', C_{j'}) \\
= \text{coeff}_{P|_{V'}}(K_{V'} - f_{V'}^*(K_{U'} + \Delta|_{U'} + C_{j'})) \\
= \text{coeff}_P(K_Y) - \frac{1}{m}v_P(\sigma) \\
\leq \text{coeff}_P(K_Y) - v_P(K_X + \Delta).
\]

By taking the supremum, we have \( \alpha(P, X, \Delta) \leq \text{coeff}_P(K_Y) - v_P(K_X + \Delta) \). So we obtain the desired equality.

Theorem 4.8 below says that when \( X \) is quasi-projective the discrepancy \( \alpha(\cdot, X, \Delta) \) on any pair \( (X, \Delta) \) can be approximated by the usual discrepancies of suitable pairs.

**Theorem 4.8.** Let \( (X, \Delta) \) be a pair such that \( X \) is quasi-projective. Then, for any projective birational morphism \( f: Y \to X \) from a normal quasi-projective variety \( Y \) and any real number \( \epsilon > 0 \), there is an effective \( \mathbb{R} \)-divisor \( G \) on \( X \) such that
\[
\Delta \quad \text{and} \quad G \quad \text{have no common components}, \quad \text{and} \\
K_X + \Delta + G \quad \text{is} \quad \mathbb{R} \text{-Cartier} \quad \text{and} \quad \alpha(P, X, \Delta) - a(P, X, \Delta + G) \leq \epsilon \quad \text{for any prime divisor} \ P \quad \text{on} \ Y, \text{where} \ a(P, X, \Delta + G) \quad \text{is the usual discrepancy}.
\]

In particular, for any prime divisor \( P \) over \( X \), we have
\[
\alpha(P, X, \Delta) = \sup\{a(P, X, \Delta + G) \mid G \geq 0 \text{ such that } K_X + \Delta + G \text{ is } \mathbb{R} \text{-Cartier}\}.
\]

**Proof.** The second assertion immediately follows from the first assertion. So we only prove the first assertion. Pick \( f: Y \to X \) and \( \epsilon > 0 \) as in Theorem 4.8. By
replacing $Y$ by a higher smooth model, we can assume that $Y$ is smooth. Fix Weil divisors $K_X$ and $K_Y$ such that $f_*K_Y = K_X$. We prove Theorem 4.8 in two steps.

**Step 1.** First we prove Theorem 4.8 when $\Delta$ is a $\mathbb{Q}$-divisor. We borrow the idea of [dFH, Proof of Theorem 5.4].

Let $\{E_i\}_i$ be the set of all $f$-exceptional prime divisors on $Y$. Since the set $\{E_i\}_i$ is a finite set, by Notation 4.4 (3), there is a sufficiently large and divisible integer $m > 0$ such that $m\Delta$ is a Weil divisor and $\frac{1}{m}v_{E_i}(m(K_X + \Delta)) \leq v_{E_i}(K_X + \Delta) + \epsilon$ for all $E_i$. We pick $m > 0$ so that $\frac{1}{m} \leq \epsilon$. By Proposition 4.7, we have

$$\alpha(E_i, X, \Delta) = \text{coeff}(K_Y) - v_{E_i}(K_X + \Delta) \leq \text{coeff}(K_Y) - \frac{1}{m}v_{E_i}(m(K_X + \Delta)) + \epsilon.$$ 

Pick a Weil divisor $D \geq 0$ on $X$ such that $m(K_X + \Delta) - D$ is Cartier, and take an ample Cartier divisor $A$ such that the sheaf $\mathcal{O}_X(A - D)$ is globally generated. We can find such $D$ and $A$ since $X$ is quasi-projective. By construction of $\mathcal{O}_X(A - D)$, we have

$$\min\{v_P(\psi) \mid \psi \in \mathcal{H}^0(X, \mathcal{O}_X(A - D))\} = v_P^2(D - A)$$

for any prime divisor $P$ on $Y$, where $v_P^2(\cdot)$ is as in Notation 4.4 (1). We define a linear system

$$\{A' \mid |A'| - D \geq 0\} = \{\text{div}(\psi) + A \mid \psi \in \mathcal{H}^0(X, \mathcal{O}_X(A - D))\}$$

and consider its pullback $f^*[A - D] := \{f^*A' \mid A' \in |A - D|\}$. Then, the fixed part $\text{Fix}(f^*[A - D])$ is

$$\text{Fix}(f^*[A - D]) = \sum_{P: \text{prime divisor on } Y} \left(\min\{\text{coeff}_P(f^*A') \mid A' \in |A - D|\}\right)P$$

$$= \sum_{P: \text{prime divisor on } Y} \left(\min\{v_P(\psi) \mid \psi \in \mathcal{H}^0(X, \mathcal{O}_X(A - D))\} + \text{coeff}_P(f^*A)\right)P$$

$$= f^*A + \sum_{P: \text{prime divisor on } Y} v_P^2(D - A) \cdot P = f^*A + f^2(D - A)$$

$$= f^2D,$$

where the final equality follows from [dFH, Lemma 2.4]. Therefore, we can find a movable Cartier divisor $M$ such that $M + f^2D \sim f^*A$. Then, $f_*M + D$ is Cartier. Thus, $m(K_X + \Delta) + f_*M = m(K_X + \Delta) - D + (D + f_*M)$ is Cartier and we have $M + f^2D = f^*(f_*M + D)$. Pick $M \geq 0$ so that $M$ is reduced and it contains no $f$-exceptional divisors or components of $f_{*}^{-1}\Delta$ in its support. Then

$$K_Y = \frac{1}{m}M - \frac{1}{m}f^2(m(K_X + \Delta))$$

$$= K_Y - \frac{1}{m}M - \frac{1}{m}f^2(m(K_X + \Delta) - D + D)$$

$$= K_Y - \frac{1}{m}M - \frac{1}{m}f^2D - \frac{1}{m}f^*(m(K_X + \Delta) - D)$$

$$= K_Y - \frac{1}{m}f^*(f_*M + D) - \frac{1}{m}f^*(m(K_X + \Delta) - D)$$

$$= K_Y - f^*(K_X + \Delta + \frac{1}{m}f_*M),$$
where the second equality follows from [dFH, Lemma 2.4] and that the divisor $m(K_X + \Delta) - D$ is Cartier. We recall that $m$ satisfies $\frac{1}{m} \leq \epsilon$, and also recall that we have $a(E_i, X, \Delta) \leq \text{coeff}_{E_i}(K_Y) - \frac{1}{m} v_{E_i}^2(m(K_X + \Delta)) + \epsilon$ for any $f$-exceptional prime divisor $E_i$ on $Y$. Pick any prime divisor $P$ on $Y$. Since $M$ contains no $f$-exceptional divisors, if $P$ is $f$-exceptional, we have

$$a(P, X, \Delta + \frac{1}{m} f_x M) = \text{coeff}_P(K_Y - \frac{1}{m} M - \frac{1}{m} f^3(m(K_X + \Delta)))$$

$$= \text{coeff}(K_Y) - \frac{1}{m} v_P^2(m(K_X + \Delta))$$

$$\geq a(P, X, \Delta) - \epsilon.$$

If $P$ is a divisor on $X$, we have

$$a(P, X, \Delta) - a(P, X, \Delta + \frac{1}{m} f_x M) = \frac{1}{m} \cdot \text{coeff}_P(f_x M) \leq \frac{1}{m} \leq \epsilon,$$

where the first equality follows from Lemma 4.3 (ii) and the second inequality follows from that $M$ is reduced. So $\frac{1}{m} f_x M$ satisfies the conditions of Theorem 4.8.

**Step 2.** From now on we prove Theorem 4.8 when $\Delta$ is an $\mathbb{R}$-divisor.

Let $\{E_i\}_i$ be the set of all $f$-exceptional prime divisors on $Y$. By Definition 4.1, there are affine open subsets $U_i \subset X$ with $c_X(E_i) \cap U_i \neq \emptyset$ and $\mathbb{R}$-divisors $B_i \geq 0$ on $U_i$ such that $K_{U_i} + \Delta|_{U_i} + B_i$ are $\mathbb{R}$-Cartier and $\alpha(E_i, X, \Delta) - \frac{1}{3} \leq \alpha(X, \Delta)(E_i, U_i, B_i)$ for all $i$. Let $E \subset \text{WDiv}_\mathbb{R}(X)$ be the set of effective $\mathbb{R}$-divisors on $X$ whose support is contained in $\text{Supp}\Delta$. For any $i$, let $\mathcal{B}_i \subset \text{WDiv}_\mathbb{R}(U_i)$ be the set of effective $\mathbb{R}$-divisors on $U_i$ whose support is contained in $\text{Supp}B_i$. We identify $E$ (resp. $\mathcal{B}_i$) with a subset of the $\mathbb{R}$-vector space whose basis is given by all components of $\Delta$ (resp. the $\mathbb{R}$-vector space whose basis is given by all components of $B_i$). Consider the set

$$\mathcal{E} = \{(\Delta', (B'_i)_i) \in E \times \prod_i \mathcal{B}_i \mid \text{for each } i, \text{ there exists an affine open subset } U_i \text{ with } c_X(E_i) \cap U_i \neq \emptyset \text{ and an } \mathbb{R} \text{-Cartier divisor } K_{U_i} + \Delta'|_{U_i} + B'_i \text{ on } X \}$$

which contains $(\Delta, (B_i)_i)$. By the argument of polytopes, we see that the set contains a rational polytope in $E \times \prod_i \mathcal{B}_i$ containing $(\Delta, (B_i)_i)$. Therefore, we can find positive real numbers $r_1, \cdots, r_n$, effective $\mathbb{Q}$-divisors $\Delta^{(1)}, \cdots, \Delta^{(n)}$ on $X$ and effective $\mathbb{Q}$-divisors $B_1^{(1)}, \cdots, B_n^{(n)}$ on $U_i$ such that $\sum_{l=1}^n r_l = 1$, $\sum_{l=1}^n r_l \Delta^{(l)} = \Delta$, $\sum_{l=1}^n r_l B_1^{(l)} = B_i$ and $K_{U_i} + \Delta^{(l)}|_{U_i} + B_i^{(l)}$ is $\mathbb{Q}$-Cartier for any $i$. By choosing those $\mathbb{Q}$-divisors sufficiently close to $\Delta$ and $B_i$, we can assume that inequality $\alpha(X, \Delta)(E_i, U_i, B_i^{(l)}) - \frac{1}{3} \leq \alpha(X, \Delta)(E_i, U_i, B_i)$ holds for any $i$ and $l$. Then

$$\alpha(E_i, X, \Delta^{(l)}) \geq \alpha(X, \Delta^{(l)})(E_i, U_i, B_i^{(l)})$$

$$\geq \alpha(X, \Delta)(E_i, U_i, B_i) - \epsilon$$

$$\geq \alpha(E_i, X, \Delta) - \frac{2}{3} \epsilon,$$

where the first inequality follows from Definition 4.1. Furthermore, we can assume that $\text{Supp}\Delta = \text{Supp}\Delta^{(l)}$ and all coefficients of $\Delta - \Delta^{(l)}$ belong to $[-\frac{1}{2} \epsilon, \frac{1}{2} \epsilon]$ for any $1 \leq l \leq n$. Then, by Lemma 4.3 (ii) and the above inequality, we obtain

$$\alpha(P, X, \Delta) - \alpha(P, X, \Delta^{(l)}) \leq \frac{2}{3} \epsilon.$$
for any prime divisor $P$ on $Y$. By the $\mathbb{Q}$-divisor case of Theorem 4.8, we can find effective $\mathbb{R}$-divisors $G^{(1)}, \cdots, G^{(n)}$ on $X$ such that

- $\Delta^{(l)}$ and $G^{(l)}$ have no common components, and
- $K_X + \Delta^{(l)} + G^{(l)}$ is $\mathbb{R}$-Cartier and $\alpha(P, X, \Delta^{(l)}) - a(P, X, \Delta^{(l)} + G^{(l)}) \leq \frac{t}{3}$ for any prime divisor $P$ on $Y$

for any $1 \leq l \leq n$. We set $G = \sum_{l=1}^{n} r_l G^{(l)}$. By construction, we have

$$K_X + \Delta = \sum_{l=1}^{n} r_l (K_X + \Delta^{(l)} + G^{(l)}),$$

and so $K_X + \Delta + G$ is $\mathbb{R}$-Cartier. Since $\text{Supp}\Delta = \text{Supp}\Delta^{(l)}$ for any $1 \leq l \leq n$ and since $\Delta^{(l)}$ and $G^{(l)}$ have no common components, we see that $\Delta$ and $G$ have no common components. We pick any prime divisor $P$ on $Y$. By construction, we have $\alpha(P, X, \Delta + G) = \sum_{l=1}^{n} r_l \cdot \alpha(P, X, \Delta^{(l)} + G^{(l)})$. Recalling $\sum_{l=1}^{n} r_l = 1$, we obtain

$$\alpha(P, X, \Delta) - a(P, X, \Delta + G)$$

$$= \sum_{l=1}^{n} r_l (\alpha(P, X, \Delta) - a(P, X, \Delta^{(l)} + G^{(l)}))$$

$$= \sum_{l=1}^{n} r_l (\alpha(P, X, \Delta) - \alpha(P, X, \Delta^{(l)}) + \alpha(P, X, \Delta^{(l)}) - a(P, X, \Delta^{(l)} + G^{(l)}))$$

$$\leq \sum_{l=1}^{n} r_l \left( \frac{2}{3} \epsilon + \frac{1}{3} \epsilon \right) = \epsilon.$$

In this way, $G$ satisfies the conditions of Theorem 4.8. So we are done.

\[\square\]

We give two examples of pseudo-lc pairs. First one is pseudo-lc pairs $\langle Z, \Delta_Z \rangle$ which are not lc.

**Example 4.9.** Let $(X, \Delta)$ be a projective $\mathbb{Q}$-factorial klt pair such that the Picard number $\rho(X)$ is greater than 1 and $-(K_X + \Delta)$ is nef but not numerically trivial. Pick a very ample Cartier divisor $A$ on $X$ such that there is no real number $r$ such that $rA \sim_{\mathbb{R}} K_X + \Delta$. Note that we only use $\rho(X) > 1$ for the existence of such $A$. We set $Y = \mathbb{P}_X(O_X \oplus O_X(-A))$, and let $f: Y \to X$ be the natural morphism. We can write

$$K_Y + 2S + f^* \Delta + f^* A = f^*(K_X + \Delta),$$

where $S$ is the unique section corresponding to $O_Y(1)$. We note that $S$ is Cartier, $S \sim X$ and the pair $(Y, S + f^* \Delta)$ is plt. We construct a cone $Z$ by contracting $S$. Let $\pi: Y \to Z$ be the natural morphism. By construction, the image $\pi(S)$ is a point. Moreover, we can write $S + f^* A \sim_{\mathbb{Q}} \pi^* H$ for an ample $\mathbb{Q}$-divisor $H$ on $Z$. We put $\Delta_Z = \pi_* f^* \Delta$.

We show that $\langle Z, \Delta_Z \rangle$ is pseudo-lc. For any real number $t > 0$, pick a general ample $\mathbb{R}$-divisor $A_t \sim_{\mathbb{R}} tA - (K_X + \Delta)$. Since we have $K_X + \Delta + A_t \sim_{\mathbb{R}} tA$, we see that $K_Z + \Delta_Z + \pi_* f^* A_t$ is $\mathbb{R}$-Cartier ([F3, Proposition 7.2.8]). Then, we can write

$$K_Y + f^*(A_t + \Delta) = \pi^*(K_Z + \Delta_Z + \pi_* f^* A_t) + aS,$$
with some \( a \in \mathbb{R} \). We restrict the equation to \( S \), and apply \( S|_S \sim_{\mathbb{Q}} -f^*A|_S \) and \((K_Y + S)|_S \sim_{\mathbb{Q}} f^*K_X|_S \). Then, we obtain \( a = -(1 + t) \), and hence we have
\[
K_Y + f^*A_t + f^*\Delta + (1 + t)S = \pi^*(K_Z + \Delta_Z + \pi_*f^*A_t).
\]
Let \( P \) be any prime divisor over \( Z \). By replacing \( A_t \) if necessary, we can assume \( c_Y(P) \not\subset \text{Supp } f^*A_t \). Then, we have \( a(P, Z, \Delta_Z + \pi_*f^*A_t) = a(P, Y, (1 + t)S + f^*\Delta) \), where both hand sides are the usual discrepancies. By definition of \( a(P, Z, \Delta_Z) \) (see Definition 4.4.1), we have \( \alpha(P, Z, \Delta_Z) \geq a(P, Z, \Delta_Z + \pi_*f^*A_t) \) for any \( t > 0 \). Thus, we obtain \( a(P, Z, \Delta_Z) \geq a(P, Y, (1 + t)S + f^*\Delta) \) for any \( t > 0 \). By the standard argument of discrepancies and since the pair \((Y, S + f^*\Delta)\) is plt, the function \( \mathbb{R} \ni t' \mapsto a(P, Y, (1 + t')S + f^*\Delta) \) is continuous and \( a(P, Y, S + f^*\Delta) \geq -1 \). Since we have \( a(P, Z, \Delta_Z) \geq a(P, Y, (1 + t)S + f^*\Delta) \) for any \( t > 0 \), by considering the limit \( t \to 0 \), we obtain \( a(P, Z, \Delta_Z) \geq -1 \). Thus, we see that \((Z, \Delta_Z)\) is pseudo-lc.

We show that \((Z, \Delta_Z)\) is not lc. It is sufficient to show that \( K_Z + \Delta_Z \) is not \( \mathbb{R} \)-Cartier. Recall that there is no real number \( r \) such that \( rA \sim_{\mathbb{R}} K_X + \Delta \). Then, \( K_Z + \Delta_Z \) is not \( \mathbb{R} \)-Cartier by [F3, Proposition 7.2.8]. Thus, \((Z, \Delta_Z)\) is not lc.

Next example shows that there is a pseudo-lc pair which is not log canonical in the sense of [dFH]

**Example 4.10.** Let \( X \) be a normal projective variety such that \((X, 0)\) is \( \mathbb{Q} \)-factorial klt, \(-K_X\) is nef and there is no effective \( \mathbb{Q} \)-divisor \( \Delta \sim_{\mathbb{Q}} -K_X \) such that \((X, \Delta)\) is lc. Such variety \( X \) exists even if \( X \) is a smooth surface (see [S, Example 1.1]). As in Example 4.9, we pick a very ample divisor \( A \) on \( X \) and set \( Y = \mathbb{P}(\mathcal{O}_X \oplus \mathcal{O}_X(-A)) \). Note that there is no real number \( r \) such that \( K_X \sim_{\mathbb{R}} rA \) by the assumption on \( K_X \). Let \( f : Y \to X \) be the natural morphism and \( \pi : Y \to Z \) be the contraction of the section \( S \) corresponding to \( \mathcal{O}_Y(1) \). We have \( K_Y + 2S + f^*A = f^*K_X \) and \( S + f^*A \sim_{\mathbb{Q}} \pi^*H \) for an ample \( H \) on \( Z \). We also have \( S \simeq X \), and \( \pi(S) \) is a point.

Since \(-K_X\) is nef, as in the argument in the second paragraph of Example 4.9, we see that \((Z, 0)\) is pseudo-lc. We show that \((Z, 0)\) is not log canonical in the sense of [dFH]. If \((Z, 0)\) is log canonical in the sense of [dFH], by [dFH, Proposition 7.2], there is an \( \mathbb{R} \)-divisor \( B \geq 0 \) on \( Z \) such that \( K_Z + B \) is \( \mathbb{R} \)-Cartier and \((Z, B)\) is lc. Then, we can write \( K_Y + aS + \pi_*^{-1}B = \pi^*(K_Z + B) \) with an \( a \leq 1 \), and the pair \((Y, aS + \pi_*^{-1}B)\) is sub-lc. If \( a < 1 \), by using \( S + f^*A \sim_{\mathbb{Q}} \pi^*H \), we obtain
\[
K_Y + S + \pi_*^{-1}B + (1 - a)f^*A \sim_{\mathbb{Q}} \pi^* \pi^*(K_Z + B + (1 - a)H).
\]
By restricting to \( S \), we obtain \( K_S \sim_{\mathbb{Q}} -\pi_*^{-1}B|_S - (1 - a)f^*A|_S \). We recall \( S \simeq X \). Since \( \pi_*^{-1}B|_S \geq 0 \) and \( 1 - a > 0 \), we see that \(-K_X\) is big. Because \(-K_X\) is nef and \((X, 0)\) is \( \mathbb{Q} \)-factorial klt by the hypothesis, we can find a \( \mathbb{Q} \)-divisor \( \Delta \sim_{\mathbb{Q}} -K_X \) such that \((X, \Delta)\) is klt. But it contradicts the hypothesis of \( X \). Thus, we see that \( a = 1 \). Then \( K_Y + S + \pi_*^{-1}B = \pi^*(K_Z + B) \) and the pair \((Y, \pi_*^{-1}B)\) is lc. By restricting to \( S \), we obtain \( K_S \sim_{\mathbb{R}} -\pi_*^{-1}B|_S \), and if we set \( \Delta_S = \pi_*^{-1}B|_S \), then the pair \((S, \Delta_S)\) is lc by adjunction. Since \( S \simeq X \), there is an \( \mathbb{R} \)-divisor \( \Delta_X \sim_{\mathbb{R}} -K_X \) such that \((X, \Delta_X)\) is lc. By the argument of Shokurov polytopes, we can find a \( \mathbb{Q} \)-divisor \( \Delta \sim_{\mathbb{Q}} -K_X \) such that \((X, \Delta)\) is lc. But it contradicts the hypothesis of \( X \). Therefore, \((Z, 0)\) is not log canonical in the sense of [dFH].

The following proposition says that pseudo-lc pairs appear in generalized lc pairs.

**Proposition 4.11.** Let \((X', \Delta' + M')\) be a generalized lc pair which comes with a data \( X' \to X' \to Z \) and \( M \). Then, the pair \((X', \Delta')\) is pseudo-lc.
Proof. By definition of pseudo-lc pairs, we can shrink $X'$ and $Z$. Therefore, we may assume that $Z$ is affine and there is an ample divisor on $X'$. We fix a prime divisor $P$ over $X'$ and show $\alpha(P,X',\Delta') \geq -1$. We denote $X \to X'$ by $f$. By replacing $X$, we may assume that $f$ is a log resolution of $(X',\text{Supp} \Delta')$ such that $P$ is a divisor on $X$. We can write $K_X + \Delta + M = f^*(K_{X'} + \Delta' + M')$, where $(X,\Delta)$ is sub-lc. Pick an ample divisor $A'$ on $X'$ and write $f^*A' = H + G$, where $H$ is ample and $G \geq 0$. For any $t > 0$, pick general $H_t \sim_{\mathbb{R}} tH + M$ such that $H_t \geq 0$ and $\text{Supp} H_t \not\subset P$. Then $K_{X'} + \Delta' + f_*(H_t + tG) \sim_{\mathbb{R}} K_X + \Delta + H_t + tG = f^*(K_{X'} + \Delta' + f_*(H_t + tG))$ for any $t > 0$. Since $(X,\Delta)$ is sub-lc and since $f_*(H_t + tG) \geq 0$, by definition of $\alpha(P,X',\Delta')$, we have

$$\alpha(P,X',\Delta') \geq \text{coeff}_P(-\Delta - tG) \geq -1 - t \cdot \text{coeff}_P(G)$$

for any $t > 0$. Thus $\alpha(P,X',\Delta') \geq -1$ for any prime divisor $P$ over $X'$, and we see that $(X',\Delta')$ is pseudo-lc. □

Remark 4.12. There is a generalized lc pair with zero boundary part $(Z,M_Z)$ such that there is no divisor $B$ with which the pair $(Z,B)$ is lc. Indeed, pick a smooth variety $X$ satisfying the properties stated in Example 4.10. For example, pick $X$ as in [S, Example 1.1]. Let $A, f: Y \to X$, $S$ and $\pi: Y \to Z$ be as in Example 4.10. We have $S + f^*A \sim_{\mathbb{Q}} \pi^*H$ for some ample $H$ on $Z$. Put $N = -f^*K_X + \pi^*H$, which is nef by construction of $X$. Then we have $K_Y + S + N \sim_{\mathbb{Q}} 0$. Since $(Y,S)$ is plt, $(Z,M_Z := g_*N)$ is a generalized lc pair which comes with the data $\pi: Y \to Z$ and $N$. But, as we have seen in Example 4.10, the pair $(Z,0)$ is not log canonical in the sense of [dFH]. So there is no boundary divisor $B$ such that the pair $(Z,B)$ is lc (see [dFH, Proposition 7.2]).

From now on, we prove the main result of this paper.

Theorem 4.13. Let $(X,\Delta)$ be a pair such that $\Delta$ is a boundary $\mathbb{R}$-divisor. Then, there is a projective birational morphism $h: W \to X$ from a normal variety $W$ such that

- any $h$-exceptional prime divisor $E_h$ satisfies $\alpha(E_h,X,\Delta) < -1$,
- the reduced $h$-exceptional divisor $E_{\text{red}}$ is $\mathbb{Q}$-Cartier, and
- if we put $\Delta_W = h^{-1}_*\Delta + E_{\text{red}}$, then $K_W + \Delta_W$ is $\mathbb{R}$-Cartier and the pair $(W,\Delta_W)$ is lc.

In particular, if $(X,\Delta)$ is pseudo-lc, then $h$ is small, i.e. $W$ and $X$ are isomorphic in codimension one.

Proof. We prove it with several steps.

Step 1. In this step, we construct a special log resolution of $(X,\text{Supp} \Delta)$ used in this proof.

Let $f: Y \to X$ be a log resolution of $(X,\text{Supp} \Delta)$. Let $\Gamma$ be the sum of $f_\star^{-1}\Delta$ and the reduced $f$-exceptional divisor, that is, the sum of all $f$-exceptional prime divisors with coefficients 1. Let $G$ be the reduced divisor on $Y$ which is the sum of all $f$-exceptional prime divisors whose discrepancy $\alpha(\cdot,X,\Delta)$ is less than $-1$. We have $\Gamma - G \geq 0$ and $\alpha(D,X,\Delta) \geq -1$ for any component $D$ of $\cup (\Gamma - G)$ (see Lemma 4.3 (ii)). Suppose that there exists an lc center $S_0$ of $(Y,\Gamma - G)$ such that for any prime divisor $P_0$ over $Y$ with $c_Y(P_0) = S_0$ and $\alpha(P_0,Y,\Gamma - G) = -1$, we have $\alpha(P_0,X,\Delta) < -1$. We take the blow-up $f_1: Y_1 \to Y$ along $S_0$, and set $\Gamma_1 = f_1^{-1}\Gamma + E_1$ and $G_1 = f_1^{-1}G + E_1$, where $E_1$ is the unique $f_1$-exceptional divisor.
Note that \( \alpha(E_1, X, \Delta) < -1 \) since we have \( c_Y(E_1) = S_0 \) and \( \alpha(E_1, Y, \Gamma - G) = -1 \) by construction. We also see that \( G_1 \) is the sum of all \((f \circ f_1)\)-exceptional prime divisors on \( Y_1 \) whose discrepancy \( \alpha(\cdot, X, \Delta) \) is less than \(-1\). Suppose that there exists an lc center \( S_1 \) of \((Y_1, \Gamma_1 - G_1)\) such that for any prime divisor \( P_1 \) over \( Y_1 \) with \( c_{Y_1}(P_1) = S_1 \) and \( \alpha(P_1, Y_1, \Gamma_1 - G_1) = -1 \), we have \( \alpha(P_1, X, \Delta) < -1 \). We take the blow-up \( f_2: Y_2 \to Y_1 \) along \( S_1 \), and set \( \Gamma_2 = f_2^{*}\Gamma_1 + E_2 \) and \( G_2 = f_2^{*}G_1 + E_2 \), where \( E_2 \) is the unique \( f_2\)-exceptional divisor. Note that we have \( \alpha(E_2, X, \Delta) < -1 \) like above. By the standard argument, this process eventually stops.

In this way, by replacing \((Y, \Gamma)\) if necessary, we can assume that there exists an \( f\)-exceptional divisor \( G \geq 0 \) on \( Y \) such that

- \( G \) is reduced,
- any \( f\)-exceptional prime divisor \( E_f \) on \( Y \) is a component of \( G \) if and only if \( \alpha(E_f, X, \Delta) < -1 \), and
- for any lc center \( S \) of \((Y, \Gamma - G)\), there is a prime divisor \( Q \) over \( X \) such that \( c_Y(Q) = S \), \( a(Q, Y, \Gamma - G) = -1 \) and \( \alpha(Q, X, \Delta) \geq -1 \).

**Step 2.** From this step to Step 4, we prove that for any \( 0 < t \leq 1 \) there is the log canonical model \((W_t, \Gamma_{W_t} - tG_{W_t})\) of \((Y, \Gamma - tG)\) over \( X \) such that any exceptional prime divisor \( P \) of the morphism \( W_t \to X \) satisfies \( \alpha(P, X, \Delta) < -1 \). We fix \( t \in (0, 1] \). Note that conditions on \( \Gamma \) and \( G \) stated in Step 2 hold even if we restrict \( f: Y \to X \) over an affine open subset in \( X \). Since the log canonical model can be constructed locally, from this step to Step 4, we may assume that \( X \) is affine.

We run the \((K_Y + \Gamma - tG)\)-MMP over \( X \) with scaling of an ample divisor. After finitely many steps, we obtain a model \( f': (Y', \Gamma' - tG') \to X \) such that \( K_{Y'} + \Gamma' - tG' \) is the limit of movable divisors over \( X \), where \( \Gamma' \) and \( G' \) are the birational transforms of \( \Gamma \) and \( G \) on \( Y' \), respectively. Then, for any \( f'\)-exceptional prime divisor \( E' \) on \( Y' \), we have \( \alpha(E', X, \Delta) \leq -1 \). Indeed, if \( \alpha(E', X, \Delta) > -1 \) for an \( f'\)-exceptional prime divisor \( E' \), by Theorem 4.8, there is an \( \mathbb{R} \)-divisor \( B \geq 0 \) on \( X \) such that \( K_X + \Delta + B \) is \( \mathbb{R} \)-Cartier and \( a(E', X, \Delta + B) > -1 \). Then,

\[
K_{Y'} + \Gamma' - tG' = f'^*(K_X + \Delta + B) + M' - f'^{-1}B - tG'
\]

where \( M' \) is an \( f'\)-exceptional divisor on \( Y' \). Since \( a(E', X, \Delta + B) > -1 \) and \( \Gamma' \) contains the reduced \( f'\)-exceptional divisor, the effective part of \( M' \) contains \( E' \neq 0 \) in its support. By construction of \( G \) (the second condition of Step 1 in this proof) and since \( \alpha(E', X, \Delta) > -1 \), we see that \( E' \) is not a component of \( G' \). Therefore, the divisor \( M' - f'^{-1}B - tG' \) has non-zero effective \( f'\)-exceptional part. But it contradicts [B1, Lemma 3.3] because \( K_{Y'} + \Gamma' - tG' \) is the limit of movable divisors over \( X \). So we have \( \alpha(E', X, \Delta) \leq -1 \) for any \( f'\)-exceptional prime divisor \( E' \).

By the above argument, for any \( \mathbb{R} \)-divisor \( C \geq 0 \) on \( X \) such that \( K_X + \Delta + C \) is \( \mathbb{R} \)-Cartier, we can write

\[
K_{Y'} + \Gamma' - tG' = f'^*(K_X + \Delta + C) - N
\]

with an \( N \geq 0 \). Then \( a(P', Y', \Gamma' - tG') \geq a(P', X, \Delta + C) \) for any prime divisor \( P' \) over \( X \), where both hand sides are the usual discrepancies. By Theorem 4.8, we have \( a(P', Y', \Gamma' - tG') \geq a(P', X, \Delta) \) for any prime divisor \( P' \) over \( X \).

**Step 3.** We check with Theorem 3.5 that \((Y', \Gamma' - tG')\) has a good minimal model over \( X \). Note that in this step we assume that \( X \) is affine.

It is clear that \(-(K_{Y'} + \Gamma' - tG')\) is pseudo-effective over \( X \). Pick any lc center \( S' \) of \((Y', \Gamma' - tG')\). Then \( S' \) is normal since \((Y', \Gamma' - tG')\) is \( \mathbb{Q} \)-factorial dlt. There is an
lc center $S$ of $(Y, \Gamma - tG)$ such that the birational map $Y \to Y'$ induces a birational map $S \to S'$ and $Y \to Y'$ is isomorphic near the generic point of $S$. Since $(Y, \Gamma)$ is lc, $S$ is an lc center of $(Y, \Gamma - G)$. By recalling construction of the pair $(Y, \Gamma - G)$, we can find a prime divisor $Q$ over $X$ such that $c_{\Gamma'}(Q) = S$, $a(Q, Y, \Gamma - G) = -1$ and $a(Q, X, \Delta) \geq 1$ (see the third condition of Step 1 in this proof). Since $(Y, \Gamma)$ is lc, we have $a(Q, Y, \Gamma - tG) = -1$. Since the birational map $Y \to Y'$ is isomorphic near the generic point of $S$, we see that $c_{\Gamma'}(Q) = S'$ and $a(Q, Y', \Gamma' - tG') = -1$. By [K, Corollary 1.39], we can construct the extraction $\overline{f}: (Y', \Psi) \to (Y', \Gamma' - tG')$ of $Q$, that is, the pair $(\overline{Y}, \Psi)$ is $Q$-factorial lc, $K_{\overline{Y}} + \Psi = \overline{f}(K_{Y'}, + \Gamma' - tG')$ and $Q$ appears as the unique $\overline{f}$-exceptional divisor. Then $\text{coeff}_{Q}(\overline{f}) = 1$ since we have $a(Q, Y', \Gamma' - tG') = -1$.

We recall that $a(Q, Y', \Gamma' - tG') \geq a(Q, X, \Delta)$ (see the last sentence of Step 2 in this proof). Thus, we have

$$-1 = a(Q, Y', \Gamma' - tG') \geq a(Q, X, \Delta) \geq 1$$

and therefore we see that $a(Q, X, \Delta) = -1$. By Theorem 4.8, for any $k \in \mathbb{Z}_{\geq 0}$, we can find an $\mathbb{R}$-divisor $C_k \geq 0$ on $X$ such that $K_X + \Delta + C_k$ is $\mathbb{R}$-Cartier and $a(Q, X, \Delta + C_k) \geq 1 - \frac{1}{k}$. We set $\beta_k = 1 + a(Q, X, \Delta + C_k)$. Then $-\frac{1}{k} \leq \beta_k \leq 0$ because we have $a(Q, X, \Delta + C) \leq a(Q, X, \Delta) = -1$ by Definition 4.1. We recall that for any $\mathbb{R}$-divisor $C \geq 0$ on $X$ such that $K_X + \Delta + C$ is $\mathbb{R}$-Cartier, we can write $K_{\overline{Y}} + \Gamma' - tG' = f^*(K_X + \Delta + C) - N$ with an $N \geq 0$. This fact is stated in the last paragraph of Step 2 in this proof. Therefore, with an effective $\mathbb{R}$-divisor $N_k$ on $Y'$, we can write $K_{\overline{Y}} + \Gamma' - tG' = f^*(K_X + \Delta + C_k) - N_k$. By a simple calculation of discrepancies, we have

$$\text{coeff}_{Q}(-\overline{f} N_k) = a(Q, X, \Delta + C_k) - a(Q, Y', \Gamma' - tG') = a(Q, X, \Delta + C_k) + 1 = \beta_k.$$ 

Since $Q$ is the unique $\overline{f}$-exceptional prime divisor, if we put $\overline{N}_k = \overline{f}^{-1}N_k$, we have $-\overline{f} N_k = \beta_k Q - \overline{N}_k$. Since we have $K_{\overline{Y}} + \Psi = \overline{f}(K_{Y'}, + \Gamma' - tG')$ by construction, we can write

$$K_{\overline{Y}} + \Psi = \overline{f}(K_{Y'} + \Gamma' - tG') = \overline{f}^*(K_X + \Delta + C_k) - \overline{f} N_k \sim_{\mathbb{R}, X} \beta_k Q - \overline{N}_k.$$ 

Note that $\overline{N}_k \geq 0$ since $N_k \geq 0$, and $\text{Supp}(\overline{N}_k \not\subset Q$ for any $k$.

We consider the divisor $-(K_{\overline{Y}} + \Psi)\mid_Q$. By the above relation, we have

$$-(K_{\overline{Y}} + \Psi)\mid_Q + \beta_k Q\mid_Q \sim_{\mathbb{R}, X} \overline{N}_k\mid_Q \geq 0.$$ 

Since $\lim_{k \to \infty} \beta_k = 0$, we see that $-(K_{\overline{Y}} + \Psi)\mid_Q$ is pseudo-effective over $X$. We recall that the morphism $\overline{f}: (\overline{Y}, \Psi) \to (Y', \Gamma' - tG')$ is the extraction of $Q$ and $\overline{f}(Q) = c_{\Gamma'}(Q) = S'$, where $S'$ is an lc center of $(Y', \Gamma' - tG')$. By construction, $-(K_{\overline{Y}} + \Psi)\mid_Q$ is the pullback of the divisor $-(K_{Y'} + \Gamma' - tG')\mid_{S'}$ to $Q$. From these facts, we see that $-(K_{Y'} + \Gamma' - tG')\mid_{S'}$ is pseudo-effective over $X$. Since $S'$ is any lc center of $(Y', \Gamma' - tG')$, the morphism $(Y', \Gamma' - tG') \to X$ satisfies the hypothesis of Theorem 3.5. Recall that in this step $X$ is assumed to be affine. Hence, the pair $(Y', \Gamma' - tG')$ has a good minimal model over $X$.

**Step 4.** We successively assume that $X$ is affine. We run the $(K_{Y'} + \Gamma' - tG')$-MMP over $X$, and we can get a good minimal model $(Y', \Gamma' - tG') \to (Y'', \Gamma'' - tG'')$ over $X$. Let $Y'' \to W_t$ be the contraction over $X$ induced by $K_{Y''} + \Gamma'' - tG''$. Because the birational map $Y \to Y''$ is a sequence of steps of the $(K_{Y} + \Gamma - tG)$-MMP over $X$, the pair $(W_t, \Gamma_{W_t} - tG_{W_t})$ is the log canonical model of $(Y, \Gamma - tG)$ over $X$, A class of singularity and log canonicalizations 23
where $\Gamma_{W_i}$ and $G_{W_i}$ are the birational transforms of $\Gamma$ and $G$ on $W_i$, respectively. We denote the morphism $Y'' \to X$ by $f''$. Now we have the following diagram:

\[
\begin{array}{c}
(Y, \Gamma - tG) \twoheadrightarrow (Y', \Gamma' - tG') \twoheadrightarrow (Y'', \Gamma - tG'') \\
\downarrow f \hspace{1cm} \downarrow f' \hspace{1cm} \downarrow f'' \\
(W, \Gamma_{W_i} - tG_{W_i}) \quad (W_i, \Gamma_{W_i} - tG_{W_i}) \quad (W_i, \Gamma_{W_i} - tG_{W_i})
\end{array}
\]

We prove that any exceptional prime divisor $P$ of the morphism $W_i \to X$ satisfies $\alpha(P, X, \Delta) < -1$. To prove this, we prove that the morphism $Y'' \to W_i$ contracts all $f''$-exceptional prime divisors $E''$ satisfying $\alpha(E'', X, \Delta) \geq -1$. By construction, $\Gamma''$ is the sum of $f''_*^{-1}\Delta$ and the reduced $f''$-exceptional divisor. We recall the second condition on $\Gamma$ and $G$ stated in Step 1 in this proof. From the condition, $E''$ is not a component of $\Gamma''$, and hence $E''$ is an lc center of $(Y'', \Gamma'' - tG'')$. We also recall that the restriction $-(K_{Y'''} + \Gamma'' - tG'')|_{E''}$ is pseudo-effective over $X$ for any lc center $S'$ of $(Y', \Gamma' - tG')$, which is proved in Step 3. Therefore, by taking a common resolution of the map $Y' \to Y''$ and applying [F1, Lemma 4.2.10], we see that the divisor $-(K_{Y'''} + \Gamma'' - tG'')|_{E''}$ on $E''$ is pseudo-effective over $X$. On the other hand, since $(Y'', \Gamma'' - tG'')$ is a good minimal model over $X$, the divisor $K_{Y'''} + \Gamma'' - tG''$ is semi-ample over $X$. From these facts, we see that the restriction of $(K_{Y'''} + \Gamma'' - tG'')|_{E''}$ to any sufficiently general fiber of the morphism $E'' \to X$ is numerically trivial. This implies that the morphism $Y'' \to W_i$ contracts all sufficiently general fibers of $E'' \to X$. In particular, $E''$ is contracted by $Y'' \to W_i$. In this way, we see that the morphism $Y'' \to W_i$ contracts all $f''$-exceptional prime divisors $E''$ on $Y''$ satisfying $\alpha(E'', X, \Delta) \geq -1$.

**Step 5.** In this step, $X$ is not necessarily affine. Let $f: (Y, \Gamma) \to X$ and $G$ be as in Step 1. By steps 2, 3 and 4, for any $t \in [0, 1]$, there exists the log canonical model $(W_t, \Gamma_{W_t} - tG_{W_t})$ of $(Y, \Gamma - tG)$ over $X$ such that any exceptional prime divisor $P$ of the morphism $W_t \to X$ satisfies $\alpha(P, X, \Delta) < -1$. Since $G_{W_t}$ is the birational transform of $G$ on $W_t$, it is the reduced exceptional divisor of $W_t \to X$ (see the second condition of Step 1 in this proof).

Let $\{e_n\}_{n \geq 1}$ be a strictly decreasing sequence of positive real numbers such that $e_n \leq 1$ and $\lim_{n \to \infty} e_n = 0$. We apply Lemma 2.5 to $(Y, \Gamma - e_nG) \to X$ and $e_nG$. For each $n$, we can find $t_n \in (0, e_n)$ and a birational contraction $Y \to W_{t_n}$ such that $(W_{t_n}, \Gamma_{W_{t_n}} - t_nG_{W_{t_n}})$ is the log canonical model of $(Y, \Gamma - t_nG)$ over $X$ and $G_{W_{t_n}}$ is $\mathbb{Q}$-Cartier. By construction, the pair $(W_{t_n}, \Gamma_{W_{t_n}} - G_{W_{t_n}})$ is lc, $\lim_{n \to \infty} t_n = 0$ and the log canonical threshold $\text{lt}(W_{t_n}, \Gamma_{W_{t_n}} - G_{W_{t_n}})$ is not less than $1 - t_n$. By [HIMX1, Theorem 1.1], we can find $n$ such that $\text{lt}(W_{t_n}, \Gamma_{W_{t_n}} - G_{W_{t_n}}) = 1$. For this $n$, put $W = W_{t_n}$, $\Delta_W = \Gamma_{W_{t_n}}$ and $G_W = G_{W_{t_n}}$. We denote the morphism $W \to X$ by $h$.

We check that $h: (W, \Delta_W) \to X$ satisfies all the conditions of Theorem 4.13. The first condition of Theorem 4.13 follows from construction of $h: W \to X$ (see steps 2, 3 and 4, or the third sentence of this step). Recall that $G_W$ is the reduced $h$-exceptional divisor (see the last sentence in the first paragraph of this step). Put $E_{\text{red}} = G_W$, which is $\mathbb{Q}$-Cartier. Therefore, the second condition of Theorem 4.13 is satisfied. We have $\Delta_W = h^{-1}_*\Delta + E_{\text{red}}$ by construction of $\Gamma$ in Step 1 in this proof. Since $K_W + \Delta_W - E_{\text{red}}$ is $\mathbb{R}$-Cartier and $\text{lt}(W, \Delta_W - E_{\text{red}}; E_{\text{red}}) = 1$, the third condition of Theorem 4.13 is satisfied. So we complete the proof.
Corollary 4.14. Let $\langle X, \Delta \rangle$ be a pair. If $X$ is a surface, then $\langle X, \Delta \rangle$ is pseudo-lc if and only if $K_X + \Delta$ is $\mathbb{R}$-Cartier and $(X, \Delta)$ is lc.

Note that $(W, \Delta_W)$ in Theorem 4.13 is not the relative log canonical model of $\langle X, \Delta \rangle$ (for definition of relative log canonical model, see [GK, 2.15 Definition]). If there is the relative log canonical model $\langle X', \Delta' \rangle$ of $\langle X, \Delta \rangle$, then $\langle X', \Delta' \rangle$ satisfies the first and third condition of Theorem 4.13.

Notations as in the proof of Theorem 4.13, the pair $(W, \Gamma_W, -tG_W)$ is the log canonical model of $(Y, \Delta - tG)$ over $X$ for any $0 < t \leq 1$, and $G$ is the sum of $f$-exceptional prime divisor $E$ over $X$ satisfying $\alpha(E, X, \Delta) < -1$. Therefore, when the pair $(X, \Delta)$ is pseudo-lc, we have $G = 0$. So we obtain the following theorem.

Theorem 4.15. Let $\langle X, \Delta \rangle$ be a pseudo-lc pair. Then, there is the relative log canonical model $h : (W, \Delta_W) \to X$ such that $h$ is small.

Here we would like to remark about the relative log canonical model of $\langle X, \Delta \rangle$. If $\Delta$ is a $\mathbb{Q}$-divisor and $K_X + \Delta$ is $\mathbb{Q}$-Cartier, then the relative log canonical model of $\langle X, \Delta \rangle$ exists ([OX, Theorem 1.1]). But, as we see in the following example, the existence of relative log canonical models for non-$\mathbb{Q}$-Cartier pairs is in general a very difficult problem.

Example 4.16 (cf. [FG2, Proof of Lemma 3.2]). Let $X$ be a smooth projective variety such that $K_X$ is pseudo-effective. Let $A, f : Y \to X, S$ and $\pi : Y \to Z$ be as in Example 4.9. By construction, we have $K_Y + 2S + f^*A \sim_{\mathbb{Q}, Z} K_Y + S$. Since $S \simeq X$, we have $\kappa(S) = \kappa(X)$, where both hand sides are Kodaira dimensions.

Suppose that the pair $(Z, 0)$ has the relative log canonical model $(Z', \Delta_{Z'}) \to Z$. Then, $\Delta_{Z'}$ is the reduced exceptional divisor over $Z$, $K_{Z'} + \Delta_{Z'}$ is $\mathbb{Q}$-Cartier and ample over $Z$, and $(Z', \Delta_{Z'})$ is lc. We show that $(Y, S)$ has a good minimal model over $Z$. Indeed, since $(Z', \Delta_{Z'})$ is lc, we have $a(S, Z', \Delta_{Z'}) \geq -1 = a(S, Y, S)$. By taking a common resolution of the birational map $Y \dashrightarrow Z'$ and by the negativity lemma, we see that $(Z', \Delta_{Z'})$ is a weak lc model of $(Y, S)$ over $X$ with relatively ample log canonical divisor. Note that $S$ is the unique exceptional divisor of the map $Y \dashrightarrow Z'$ because it is the unique exceptional divisor of the morphism $\pi : Y \to Z$.

By [Ha2, Lemma 2.10], we see that $(Y, S)$ has a good minimal model over $Z$.

Let $(Y, S) \dashrightarrow (Y', S')$ be a sequence of steps of the $(K_Y + S)$-MMP over $Z$ to a good minimal model. Then, $S$ is not contracted by the log MMP because $K_S$ is pseudo-effective. Furthermore, we have $(K_{Y'} + S')|_{S'} = K_{S'}$ and $\kappa(S) \geq \kappa(S')$ by construction. Since $K_{S'}$ is semi-ample, we have $\kappa(S') \geq 0$. Then $\kappa(X) \geq 0$.

In this way, the existence of the relative log canonical model of $\langle Z, 0 \rangle$ implies the non-vanishing theorem for $X$.

Finally, we introduce a klt analogue of pseudo-lc singularity.

Definition 4.17. Let $\langle X, \Delta \rangle$ be a pair. We say the pair $\langle X, \Delta \rangle$ is pseudo-klt if the inequality $\alpha(P, X, \Delta) > -1$ holds for any prime divisor $P$ over $X$.

As in the proof of Proposition 4.6, we see that $\langle X, \Delta \rangle$ is pseudo-klt if it is log terminal in the sense of [dFH]. It is not known that whether pseudo-klt singularity is a strictly extended notion of log terminal singularity in the sense of [dFH].

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