Semiclassical gravitational effects in de Sitter space at finite temperature

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Abstract. In the framework of finite-temperature conformal scalar field theory on de Sitter spacetime the linearized Einstein equations for the renormalized stress tensor are solved exactly. In this theory quantum field fluctuations are concentrated near two spheres of the de Sitter radius, propagating as light wavefronts. The analysis, performed for a flat expanding universe, shows exponential damping of the back-reaction effects far from these spherical objects. The solutions obtained for the semiclassical Einstein equations in a de Sitter background can also be extended straightforwardly to the anti-de-Sitter geometry.

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1. Introduction

In the modern picture of the early universe the exponential expansion period is known to be successful in solving a number of important problems occurring in standard cosmology. During this inflationary phase, described by de Sitter geometry, quantum field effects played an essential role for the subsequent formation of the present universe. In this framework, the knowledge of the corresponding quantum state, originating from the Planck epoch, is important for reconstructing the phase transition pattern of the inflation. One of the possibilities generally used [1, 2] is to consider this quantum state as a de Sitter invariant vacuum experienced by a freely moving observer as a thermal bath at the Hawking temperature $\beta^{-1}$ [3].

A wider class of quantum states, associated in static de Sitter coordinates with a thermal equilibrium at arbitrary temperature $\beta^{-1}$, has been investigated in our previous work [4], where we studied properties of some integral quantities such as a one-loop effective potential and total energy for a scalar field model. However, this analysis seems not to be quite exhaustive. For instance, it has long been known that for arbitrary $\beta^{-1}$ the renormalized stress tensor is singular on the horizon surface [5]. By itself this undesirable fact is not sufficient to exclude these quantum states as unphysical. It indicates the presence of a strong back-reaction effect in the region near the horizon, and shows the necessity of a non-
perturbative approach to this problem in the framework of quantum gravity, which would provide a natural short-distance cut-off. As a first step in this direction it is also worth analysing back-reaction effects in the semiclassical approximation, which could then be useful for investigating the physical relevance of such a quantum theory. Another motivation for this study is that the conformally invariant scalar model, which we restrict ourselves to, represents one more example where quantum corrections to the background metric can be explicitly obtained in the linearized approximation.

In this model, the finite-temperature Green function has a simple analytical form that enables one to demonstrate how to define it globally on the whole spacetime. The traceless part of the related renormalized stress tensor is singular on the horizon surface of two antipodal static coordinate systems, where thermal equilibrium is introduced.

Let us point out that the thermal properties of the system inside and outside the static regions are different. The field inside is in thermal equilibrium characterized by temperature $\beta^{-1}$, internal energy, entropy etc; whereas the system outside expands adiabatically, so that the energy density tends to its value at $\beta = \beta_H$. For different spatial sections of the de Sitter hyperboloid these static regions represent two $3D$ spherical domains or bubbles; having a certain surface energy provided by quantum fluctuations of the field near the horizon.

This paper is organized as follows. In section 2 we compute the renormalized stress tensor at finite temperature $\beta \neq \beta_H$, and discuss its properties inside and outside the static regions. In section 3 this tensor is used to evaluate, in the linear approximation, the back-reaction effects. The model of the flat expanding de Sitter universe is then considered in section 4 to give a geometrical interpretation of the results obtained. Finally, we give our conclusions and remarks.

2. The quantum state and stress tensor

Let us determine the average value of the energy-momentum tensor for conformally invariant scalar field theory $\langle \mathcal{T}^{\mu\nu} \rangle_\beta = T^{\mu\nu}_\beta(\beta)$ in static de Sitter space, where $\beta$ is the inverse temperature of the system. This tensor can be decomposed as

$$ T^{\mu\nu}_\beta = \tilde{T}^{\mu\nu}_\beta(\beta) + \frac{1}{4} \delta^{\mu\nu} T $$

(2.1)

where $\tilde{T}^{\mu\nu}_\beta(\beta)$ is the traceless part, and $T \equiv T^{\mu\mu}_\beta(\beta)$ is the local conformal anomaly which does not depend on the quantum state of the system [11]. In particular, for de Sitter space

$$ T = \frac{1}{240\pi^2 r_0^4} $$

(2.2)

($r_0$ stands for the de Sitter radius), and due to the conservation law $\tilde{T}^{\mu\nu}_\mu = 0$, the traceless part has only one independent component. Consequently, to find all its components it is sufficient to calculate only one of them.

It is worth remembering that the static de Sitter metric $g_{\mu\nu}$ reads

$$ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = (1 - r^2/r_0^2) dt^2 - (1 - r^2/r_0^2)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) $$

(2.3)

† Note that recently much attention has been paid to similar divergences appearing in statistical-mechanical computations of black-hole entropy [6-10], when a temperature different from the Hawking one is introduced to obtain the entropy from the partition function. A possible role of the superstring theory, providing such a fundamental cut-off, is discussed in [8]. Another possibility is to take into account the quantum fluctuation of the horizon [10].

† It is worth distinguishing these bubbles from those corresponding to different vacua and nucleated during phase transitions, and reminding the reader that we use this word just to indicate the regions of thermal equilibrium.
where \( x^\mu = (t, r, \theta, \varphi) \). In this space the temporal component of the stress tensor \( \tilde{T}_t^t(\beta) \) can be obtained observing that [5]

\[
(\tilde{T}_t^t)_\beta - (\tilde{T}_t^t)_{\beta=\infty} = \frac{\pi^2}{30} \frac{\beta^{-4}}{g_{tt}}.
\] (2.4)

The last equation can be also rewritten in the following form:

\[
(\tilde{T}_t^t)_\beta = (\tilde{T}_t^t)_{\beta_H} + \frac{\pi^2}{30} \left( 1 - \frac{\beta^4}{\beta_H^4} \right) \frac{\beta^{-4}}{g_{tt}}
\] (2.5)

where \( \beta_H \equiv 2\pi r_0 \) denotes the inverse Hawking temperature. Hence, due to the fact that for \( \beta = \beta_H \) the stress tensor is completely anomalous \( (\tilde{T}_\mu^\nu(\beta_H) = 0) \), we have

\[
(\tilde{T}_t^t)(\beta) = (\tilde{T}_t^t)_\beta - (\tilde{T}_t^t)_{\beta=\beta_H} = \frac{\pi^2}{30} \left( 1 - \frac{\beta^4}{\beta_H^4} \right) \frac{\beta^{-4}}{g_{tt}}.
\] (2.6)

This enables us to obtain the stress tensor in the form

\[
(\tilde{T}_\mu^\nu)_\beta = \frac{\pi^2}{30} \left( 1 - \frac{\beta^4}{\beta_H^4} \right) \frac{\beta^{-4}}{g_{tt}} \text{diag}(1, -\frac{1}{3}, -\frac{1}{3}) + \frac{1}{g_0 \pi^2 r_0^4} \delta_\mu^\nu
\] (2.7)

which coincides at zero temperature with the known result for the static conformal vacuum [12].

As one can see from (2.7) the traceless part dominates near the horizon and corresponds, for \( \beta < \beta_H \), to the energy–momentum tensor of a gas of massless scalar particles [13]. In this case, the energy density has the usual Planck form and turns out to be

\[
\tilde{T}_t^t(\beta) = \frac{\pi^2}{30} \beta^{-4}
\] (2.8)

where \( \beta^{-1} = \beta^{-1} \left( 1 - \frac{\beta^4}{\beta_H^4} \right)^{1/4} g_{tt}^{-1/2} \) plays the role of a local, red-shifted temperature. A similar structure for the energy density in finite-temperature Rindler space is given in [14].

Let us point out that (2.7) can be obtained by the standard procedure from the finite-temperature Green function given by [5]

\[
G_\beta(x, x') = i \left[ \frac{(1 - r^2/r_0^2)(1 - r'^2/r_0^2)}{2\beta_H \beta \sinh \alpha_1} \right]^{-1/2} \frac{\sinh(\alpha_1 \beta_H / \beta)}{\cosh(\alpha_1 \beta_H / \beta) - \cosh[(t - t')\beta_H / r_0 \beta]}
\] (2.9)

\[
(\cosh \alpha_1 = \left[ (r_0^2 - r^2)(r_0^2 - r'^2) \right]^{-1/2} \left[ r_0^2 - rr' \left[ \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\varphi - \varphi') \right] \right],
\]

with the subtraction of its value corresponding to the Hawking temperature. Although expression (2.9) is only defined in the region bounded by the horizon surface, it can be extended to the total de Sitter spacetime. In terms of the conformal diagram, figure 1, where the static coordinate system (2.3) maps into region a, it is easy to prove that \( G_\beta(x, x') \) can be smoothly continued from region a into the casually connected domains c or d, by moving, one by one, its arguments through the horizon surface. Then extending this procedure from c or d to b we obtain a globally defined two-point Green function, smooth everywhere for non-coinciding arguments. Such a procedure completely fixes, for all de Sitter spacetime, the quantum state, whose properties can be derived from the structure of the associated global energy–momentum tensor. This global stress tensor, as opposed to the Green function, is singular on the horizon surface, which means that close to the horizons our semiclassical approach is not applicable and the complete theory of quantum gravity should be used for computing the quantum effects.

This energy singularity corresponds to a matter distribution over the surface of two spherical bubbles having horizon size \( r_0 \), and propagating as fronts of light waves. Regions
a and b represent their internal regions, where the stress tensor has identical forms, (2.7). In the external regions c and d the metric (2.3) changes to

$$ds^2 = (r^2/r_0^2 - 1)^{-1} dr^2 - (r^2/r_0^2 - 1) dt^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

and describes an expanding (or shrinking) universe where $r$ now plays the role of time and $t$ has the meaning of a spatial coordinate. In these cases, the horizon surfaces are removed to spatial infinities ($t = \pm \infty$), and the structure of the stress tensor can be found on the base of (2.7). It follows immediately that in these domains $\tilde{T}^{\mu\nu}(\beta)$ corresponds to a non-equilibrium thermal system. Although the energy flows from the infinities ($t = \pm \infty$) are absent, the energy density decreases due to the factor $[g_{\mu\nu}(r)]^{-1}$ to its value at $\beta = \beta_H$ in accordance with adiabatic expansion of this universe.

### 3. The semiclassical analysis

Now consider changes of the de Sitter geometry induced by the vacuum polarization in the given quantum states. Einstein equations in semiclassical approximation are known to read as [11]

$$R_{\mu\nu}(\tilde{g}) - \frac{1}{2} R(\tilde{g}) \tilde{g}_{\mu\nu} + \Lambda \tilde{g}_{\mu\nu} + \alpha \ (1) H_{\mu\nu}(\tilde{g}) + \beta \ (2) H_{\mu\nu}(\tilde{g}) = -8\pi G (\tilde{T}_{\mu\nu})_\beta .$$

Here $\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$ indicates the solution of (3.1) including quantum corrections $h_{\mu\nu}$ to the de Sitter metric $g_{\mu\nu}$, and $\Lambda = 3/r_0^2$ is the cosmological constant. As for the additional tensors $(1) H_{\mu\nu}$ and $(2) H_{\mu\nu}$ which appear due to the quantum corrections, and are defined in terms of the geometrical quantities [11], we will neglect them in further analysis assuming the unknown constants $\alpha, \beta$ in (3.1) to be equal to zero. This assumption is based on the observation that the values of $\alpha$ and $\beta$, which have to be fixed by experiment, cannot be large due to the good agreement of the Einstein equations in their classical formulation with the experimental data. Then, expanding (3.1) up to the first order in the metric perturbation...
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We have

\[ h_{\mu \nu} = h_\mu \nu^\alpha - h_\mu^\alpha \nu - h_{\nu \mu \alpha} + \frac{\delta}{r_0} h_{\mu \nu} + \left( h_\mu \nu^\alpha - h_\mu^\alpha \nu - \frac{3}{r_0^2} h_{\mu \nu} \right) g_{\mu \nu} = 16\pi G (\tilde{T}_{\mu \nu})_\beta . \]  

(3.2)

Essential progress in solving these equations can be achieved by observing that the anomalous part of the renormalized stress tensor in the RHS of (3.2) only gives rise to a redefinition of the de Sitter radius \( r_0 \). Taking into account this trivial effect instead of the total tensor we can use its traceless part \( \tilde{T}^\mu_\mu (\beta) \). Thus, after a little transformation, the initial equations take the form:

\[ h_{\mu \nu}^\alpha + \frac{2}{r_0^2} h_{\mu \nu} = -16\pi G \tilde{T}_{\mu \nu} (\beta) \]  

(3.3)

where we have imposed the gauge conditions \( h_{\mu \nu}^\nu = 0 \) and \( h_{\mu}^\nu = 0 \) (the latter is compatible with the dynamical equations and fixes the residual gauge freedom). Besides this, the property of de Sitter space for which \( R_{\mu \nu \alpha \beta} = r_0^2 (g_{\mu \nu} g_{\alpha \beta} - g_{\mu \beta} g_{\alpha \nu}) \) has been used in (3.3).

To proceed with our analysis, it is worth remembering that in the given quantum state \( (\beta \neq \beta_H) \) the total de Sitter symmetry \( SO(1,4) \) is broken to the subgroup \( T_1 \times SO(3) \), where \( T_1 \) stands for de Sitter boosts associated with translations along the coordinate \( t \). This enables us to restrict the degrees of freedom of \( h_{\mu \nu} \) to the only non-zero components \( h_{\mu \alpha}^\nu, h^\alpha_{\nu \alpha}, h_{\alpha}^{\mu \alpha} \), and \( h^\alpha_{\nu \mu} = -a(r) r^2 \gamma_{i k} \), where the indices \( i \) and \( k \) refer to the \( \theta \) and \( \phi \) coordinates, and \( \gamma_{i k} \) is the metric on \( S^2 \) (in accordance with (2.3) \( g_{i k} = -r^2 \gamma_{i k} \)).

Due to the residual symmetry and gauge conditions imposed, there is only one independent equation among (3.3). After some algebra one can show that in static coordinates it reduces to a second-order differential equation for \( h_{rr} \):

\[ \left[ 4 (1 - y) \frac{d^2}{dy^2} - \frac{26y - 16}{y} \frac{d}{dy} - \frac{28y - 8}{y^2} \right] h_{rr}(y) = -\frac{K(\beta)}{y^4} \]  

(3.4)

written in terms of the variable \( y(r) = 1 - r^2/r_0^2 \) and a constant

\[ K(\beta) = \frac{G}{90\pi r_0^2} \left( 1 - \frac{\beta_H^4}{\beta^4} \right) . \]  

(3.5)

Remarkably, the homogeneous equation associated with (3.4) admits two simple solutions: \( y^{-2} \), and \( [y^2(1 - y)^{3/2}]^{-1} \). Thus the general integral of (3.4) can be given in the following form:

\[ h_{rr}(r) = \frac{K(\beta)}{4y^2(r)} \left\{ 2 \left[ \frac{1}{3} + \frac{2}{1 - y(r)} + \frac{1}{[1 - y(r)]^{3/2}} \log \left[ 1 - \sqrt{1 - y(r)} \right] \right] \right\} \right. \] 

\[ + \frac{A}{y^2(r)} + \frac{B}{y^2(r)[1 - y(r)]^{3/2}} . \]  

(3.6)

It is worth mentioning that coordinate systems in de Sitter spaces, (2.3) and (2.10), can be treated on an equal footing if one adds a small imaginary part to the de Sitter radius, \( r_0 \rightarrow r_0 + i\varepsilon \). This regularization preserves the structure of all the equations in both quantum and classical theory. It removes the singularity at the horizons in such a way that (2.10) and (2.3) can be unified in a single expression for the metric valid for \( 0 \leq r < \infty \). Then the solution (3.6) can be defined in the same region. However, the integration constants \( A \) and \( B \) cannot be chosen to be equal in the whole of this interval. Indeed, if \( A \) and \( B \) are fixed in some way in the inner domain (a), then after passing in the external region, say (c), one obtains, due to the logarithm, complex values for the function \( h_{rr}(r) \). It means that
integration constants for external and internal problems in the given semiclassical approach should be found independently, on the basis of additional physical motivations.

In the static regions $A$ and $B$ can be chosen from the condition $h_{\mu\nu}(r) = 0$ at $r = 0$, which corresponds to the natural assumption for the vacuum effects to disappear in the flat limit $r/r_0 \ll 1$, when the background space converts into a Minkowski one. As for the external domains, one can require for the perturbed metric to approach the de Sitter metric in the limit $r \to \infty$, where local quantum effects disappear (see (2.7)). The components $h_{tt}$ and $h_{tt}$ can be obtained from (3.6) using constraints. On the other hand, the non-diagonal element $h_{tr}$, fixed by the condition $h_{tr}^2 = 0$, turns out to be completely independent of the stress tensor and other components. For this reason, we put it to be zero everywhere, which inside the horizon can also be justified by claiming its regularity at $r = 0$.

By virtue of (3.6) and conditions chosen we can represent all the non-zero components of the metric perturbations in the following form:

$$h_{tt}(r) = \frac{K(\beta)}{4} \left[ 2 - \frac{4r^2}{3r_0^2} + \frac{r_0}{2r} \log \left( \frac{r_0 - r}{r_0 + r} \right)^2 \right] + A \left( 3 - \frac{2r^2}{r_0^2} \right) + B \frac{r}{r_0} \tag{3.7}$$

$$h_{rr}(r) = \frac{K(\beta)}{4y^2(r)} \left[ \frac{2}{3} - \frac{2r_0^2}{r^2} + \frac{r_0^3}{2r^3} \log \left( \frac{r_0 - r}{r_0 + r} \right)^2 \right] + \frac{A}{y^2(r)} + B \frac{r_0^3}{2y^2(r)r^3} \tag{3.8}$$

$$a(r) = \frac{K(\beta)}{4} \left[ - \frac{2}{3} + \frac{r_0^2}{r^2 y(r)} + \frac{r_0^3}{4r^3} \log \left( \frac{r_0 - r}{r_0 + r} \right)^2 \right] - A + B \frac{r_0^3}{2r^3}. \tag{3.9}$$

Following the previously mentioned requirements, the integration constants $A$ and $B$ have to be put equal to zero for $0 \leq r < r_0$, whereas for $r_0 < r < \infty$ we have $A = -K(\beta)/6$. As far as the value of $B$ in the external region is concerned, it is left undetermined. This fact can be understood remembering that in the approximate approach of semiclassical theory two regions are separated by an infinite barrier, whereas in the correct theory both solutions should be matched at the horizon, thus fixing the remaining constant. Note also that solutions (3.7)-(3.9) are only valid in the region where the perturbation expansion is reliable. In terms of the Planck length $l_{\text{Pl}} = \sqrt{G}$, this region results to be $|r - r_0| \gg 1 - \beta_0^2/\beta_0^4/2l_{\text{Pl}}$. Therefore, if the de Sitter radius $r_0 \approx l_{\text{Pl}}$ the present analysis is not applicable to the bubble interior.

Finally, we observe that solutions (3.7)-(3.9) of the semiclassical Einstein equations in the de Sitter background can be immediately extended to anti-de-Sitter space just by substituting $r_0$ by $ir_0$. This is connected to the fact that the renormalized stress tensor at finite temperature in anti-de-Sitter space has the same structure as the de Sitter one, once a complex scalar field with Neumann and Dirichlet boundary conditions for its two independent components is chosen [16].

4. Gravitational effects

We now discuss the properties of the space taking into account quantum corrections (3.7)-(3.9) to the de Sitter metric. It can be done by considering the different models of de Sitter space corresponding to flat, open or closed expanding universes. For the sake of simplicity in this paper we restrict our analysis to the flat model, which is generally used for cosmological applications.

The corresponding metric has the familiar form

$$ds^2 = dt^2 - e^{2\xi/r_0}(d\xi^2 + \xi^2 d\Omega^2) \quad 0 \leq \xi < \infty \tag{4.1}$$

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\[ ds^2 = d\theta^2 + \sin^2 \theta \, d\phi^2 \] is the line element on a sphere. This system of coordinates maps only half of the de Sitter hyperboloid. For a more clear representation we assume it to cover regions a and c in figure 1. In this case, one of the singularities of the global stress tensor is placed inside the universe (4.1) in such a way that the origin of one of the static regions and the centre of the corresponding bubble is at the point \( \xi = 0 \). At the same time, the boundary of the other static domain is at spatial infinity \( (\xi \to \infty) \). The map between (4.1) and (2.3) or (2.10) can be written as follows:

\[ t = \tau - \frac{1}{q} r_0 \log \left( 1 - \frac{\xi^2}{r_0^2} e^{2\xi/r_0} \right)^2 \quad r = \xi e^{\xi/r_0}. \] (4.2)

Note that by definition, the shell of the bubble coincides with the de Sitter horizon, and thus its size remains unchanged during inflation, but due to the expansion every point, once inside it, will move away. The metric (4.1) changes to

\[ ds^2 = (1 + h_{\eta \eta}) d\eta^2 + 2 h_{\eta \xi} d\eta \, d\xi - \left( e^{2n/r_0} - h_{\xi \xi} \right) d\xi^2 - e^{2n/r_0} \xi^2 (1 + a) \, d\Omega^2 \] (4.3)

where we introduced \( \eta \) instead of \( \tau \) to indicate that now it is not the proper time. The metric perturbations \( h_{\eta \eta}, h_{\eta \xi} \) and \( h_{\xi \xi} \) follow from (3.7) and (3.8), and map (4.2), where \( r \) is replaced by \( \eta \):

\[ h_{\eta \eta}(\eta, \xi) = \frac{1}{y^2} h_{\eta \eta} + \frac{\xi^2}{r_0^2} e^{2n/r_0} h_{rr}, \] (4.4)

\[ h_{\eta \xi}(\eta, \xi) = \frac{\xi}{r_0} e^{2n/r_0} \left( \frac{1}{y^2} h_{\eta \eta} + h_{rr} \right), \] (4.5)

\[ h_{\xi \xi}(\eta, \xi) = e^{2n/r_0} \left( \frac{\xi^2}{r_0^2} y^2 e^{2n/r_0} h_{\eta \eta} + h_{rr} \right). \] (4.6)

In equations (4.3)-(4.6) all the functions \( y, a, h_{\eta \eta} \) and \( h_{rr} \) are supposed to be expressed in terms of \( \eta \) and \( \xi \). So long as \( \eta \) does not represent the proper time we need an additional transformation to pass to co-moving coordinates \( (\xi, x, \theta, \phi) \), similar to (4.1):

\[ ds^2 = dt^2 - e^{2r/r_0} \left[ (1 + h_{xx}(x, \tau)) \, dx^2 + x^2 (1 + h_{\Omega \Omega}(x, \tau)) \, d\Omega^2 \right] \equiv dt^2 - \gamma_{ij} \, dx^i \, dx^j \] (4.7)

with \( i, j = 1, 2, 3 \) corresponding to \( x, \theta, \) and \( \phi \), respectively. The above transformation reads

\[ \eta = \tau + \eta_1(\tau, x) \quad \xi = x + \xi_1(\tau, x) \] (4.8)

and the functions \( \eta_1 \) and \( \xi_1 \) obey to the following differential equations:

\[ 2 \frac{\partial \eta_1}{\partial \tau} + h_{\eta \eta} = 0 \quad e^{2r/r_0} \frac{\partial \xi_1}{\partial \tau} - h_{\eta \xi} - \frac{\partial \eta_1}{\partial x} = 0. \] (4.9)

In terms of these functions the metric corrections \( h_{xx} \) and \( h_{\Omega \Omega} \) result in

\[ h_{xx} = \frac{\eta_1}{r_0} + \frac{\partial \xi_1}{\partial x} - e^{-2r/r_0} h_{\xi \xi}, \quad h_{\Omega \Omega} = \frac{\eta_1}{r_0} + \frac{\xi_1}{x} + a. \] (4.10)

It is worth observing that the integration constant for \( \xi_1 \), obtained from (4.9), is not relevant because it can always be changed by redefinition of the \( x \)-coordinate in (4.7). As far as \( \eta_1 \) is concerned, its value is fixed, up to an inessential numerical constant, by requiring that

\[^\dag \text{A discussion about this point is given in section 5.}\]
metric components (4.7) depend only on the distance \( r = x \exp(\tau/r_0) \) from the centre of the perturbation.

The metric components can be readily obtained from (4.9) on the total space \((0 \leq r < r_0\) and \(r_0 < r < \infty\)). For the sake of brevity we only present the expressions for the outside region to investigate the metric asymptotic at infinity. They read

\[
\begin{align*}
h_{\Omega \Omega} &= K(\beta) \left[ -\frac{5}{4} \log \left( \frac{r^2}{r^2 - r_0^2} \right) + \frac{r_0^3}{8r^3} \log \left( \frac{r - r_0}{r + r_0} \right) + \frac{r_0^2}{4r^2} \frac{2r_0^2 - 3r^2}{r_0^2 - r^2} \right] \\
h_{xx} &= K(\beta) \left[ -\frac{5}{4} \log \left( \frac{r^2}{r^2 - r_0^2} \right) + \frac{r_0^2}{4r^2} \frac{r_0^2 - 3r^2}{r_0^2 - r^2} \right]
\end{align*}
\] (4.11) (4.12)

where the undefined constant \( B \) has been neglected, because it does not affect the long-range behaviour \((r \gg r_0)\) of perturbations. By using (4.11) and (4.12) one can see that the gravitational field far from the bubbles decreases like \( r^{-2} \). It corresponds, due to the expansion, to an exponential falling with time \( \tau \) for a point of given \( x \).

The effects of the quantum corrections to the metric (4.1) can be expressed in another way by computing the spatial curvature \( R^{(3)}(\gamma) \) of the universe (4.7):

\[
R^{(3)}(\gamma) = h_{\mu j}^{; \nu;j} - h_{\mu j}^{; \nu} - \frac{2}{\kappa_0^2} e^{-2\kappa_0 / \kappa_0} \frac{d}{dx}(xf(x, \tau))
\] (4.13)

\[
f(x, \tau) = \frac{d}{dx} (x h_{\Omega \Omega}) - h_{xx} = 3a + x \frac{d}{dx} a.
\] (4.14)

Remarkably, due to (4.14), it turns out to be completely independent of the constant \( B \) and takes the simple analytical forms

\[
R^{(3)}(\gamma)|_{r < r_0} = K(\beta) \frac{3r_0^2 - r^2}{(r^2 - r_0^2)^2} \quad R^{(3)}(\gamma)|_{r > r_0} = K(\beta) \frac{r_0^2}{r^2} \frac{r^2 + r_0^2}{(r^2 - r_0^2)^2}
\] (4.15)

for inside and outside regions, respectively. By definition (3.5) of \( K(\beta) \) it follows that curvature is negative if the bubble temperature \( \beta^{-1} \) is higher then the Hawking one \( \beta^{-1}_H \), and changes sign when \( \beta^{-1} < \beta^{-1}_H \). Moreover, its value at \( r = 0 \) does not depend on time \( R^{(3)}(\gamma)|_{r = 0} = 3K(\beta)r_0^2 \), whereas in the external region \( R^{(3)} \) decreases as \( r^{-4} \). This long-range behaviour of the gravitational field is connected to the massless nature of the chosen scalar field. For massive matter fields one can expect that the gravitational effects will be exponentially smeared on a length of the order of the inverse mass outside the bubble shell.

5. Conclusions and remarks

In this paper the local properties of thermal states for a quantum field in de Sitter spacetime have been investigated. In such a theory quantum fluctuations are concentrated on two spherical surfaces (boundaries of bubbles) moving far away from one another. Such domains, characterized by temperature, entropy, etc give rise to a number of semiclassical effects.

To find the gravitational field produced by these perturbations, the linearized Einstein equations on de Sitter background have been solved exactly for the stress tensor of a conformal scalar field. The computations show that the gravitational effects are weak far from the bubbles, and in the expanding universe they are exponentially damped.

However, other aspects of this problem are worth being formulated as open questions. It is hardly probable for the considered thermal states to appear as a result of classical processes. It is more reasonable to investigate if they could be produced during the Planck epoch, at the moment when classical de Sitter geometry occurs, and whether they are stable.
or decay immediately. In any case, in realistic cosmological models both the radius and the thermodynamical properties of the bubbles might change during the evolution of the universe. Note that the external region of the static coordinates (2.3) can be considered as a 'black hole' absorbing 'information' from the inner domain due to decays of unstable field configurations of the Higgs scalars. As a result of this process, the bubble size, associated with the horizon, grows and achieves cosmological values determined by the inverse of the Hubble constant ($H^{-1} \approx 10^{28}$ cm at present). However, due to the exponential damping, the presence of these inhomogeneous regions does not seem to affect the observable part of the universe, provided it is far enough from them.

Finally, it is worth observing that the solutions obtained for the semiclassical Einstein equations in the de Sitter background can be extended straightforwardly to the anti-de-Sitter geometry. In this case, in fact, the stress tensor has the same structure as the de Sitter one, once a complex scalar field with appropriate boundary conditions is chosen [16].

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