The $\eta'$ and Cooling with Staggered Fermions

G. Kilcup, J. Grandy and L. Venkataraman

*Department of Physics, The Ohio State University, 174 West 18th Ave, Columbus, Ohio 43210

We present a calculation of the mass of the $\eta'$ meson using quenched and dynamical staggered fermions. We also discuss the effects of “cooling”, and suggest its use a quantitative tool.

1. $\eta'$ Lore

One of the more curious inhabitants of the low energy spectrum of QCD is the $\eta'$ meson. Deriving part of its mass from its kinship with the Goldstone boson pseudoscalars, the $\eta'$ owes most of its anomalous weight to its connection to the topological susceptibility. Specifically, in an SU(3) symmetric world, one would write

$$m_{\eta'}^2 = m_8^2 + m_0^2$$

where $m_8$ is the mass common to all of the octet meson, and $m_0$ is peculiar to the $\eta'$. In the chiral limit, the octet mass $m_8$ vanishes like $\sqrt{m_q}$, while $m_0$ remains finite. Ignoring mixing and identifying $m_8^2$ with the average mass-squared of the physical $\pi$’s, $K$’s and the $\eta$, one derives the “experimental” number of $m_0 = 860$ MeV for $N_f = 3$. To complete the calculation of the ground state hadron spectrum, one would like to compute this number from a first principles lattice simulation, free of assumptions such as the validity of the large $N_c$ expansion.

In the language of quarks, the $\eta'$ is understood to gain its mass from “hairpin” diagrams, where the quarks annihilate into glue and then reconstitute themselves. In the language of mesons, this phenomenon can be described by ascribing a strength $m_0^2$ to this interaction of flavor singlet mesons. Iteration of this process leads to a geometric series which shifts the pole in the propagator from $m_8^2$ to $m_8^2 + m_0^2$. In the quenched approximation, the series terminates with the second term. The quenched $\eta'$ propagator then has the unphysical form of a sum of a pole and a double pole, the infrared divergences of which lead to extra quenched chiral logs. It is therefore of interest to see if one can distinguish pole and double pole behavior in a lattice simulation.

The quantity of interest is the ratio of the disconnected and connected diagrams.

$$R(t) = \frac{\langle \eta'(0) \eta'(t) \rangle_{2\text{-loop}}}{\langle \eta'(0) \eta'(t) \rangle_{1\text{-loop}}}$$

If we compute the ratio using $N_{\text{val}}$ valence quarks and $N_{\text{dyn}}$ dynamical quarks, with the same quark mass for valence and dynamical, then $R(t)$ should asymptote to

$$R(t) \rightarrow \frac{N_{\text{val}}}{N_{\text{dyn}}} \left[1 - \frac{Z'}{Z} \exp(-t\Delta m)\right]$$

where $\Delta m = m_{\eta'} - m_8$, and $Z'$ and $Z$ are the residues for the creation of the singlet and octet particles, respectively. For $N_{\text{dyn}} = 0$, i.e. in the quenched approximation, $R(t)$ should never asymptote, but instead rises linearly with slope $2m_0^2/m_8$.

This approach has been taken with quenched configurations and valence Wilson fermions in ref. [1]. Here we extend the analysis to staggered fermions, and use quenched and dynamical configurations.
2. Implementation

This calculation used three ensembles of lattices of size $16^3 \times 32$, one quenched and two including dynamical quarks. The dynamical ensembles were “borrowed” from the Columbia group, and have been used previously in our $B_K$ analysis \cite{4}, and the calculation of $f_B$ by the MILC collaboration \cite{5}. These were generated using molecular dynamics with $N_f = 2$ staggered fermions with quark masses $m_{\text{dyn}} = .01$ and .025. The quenched lattices were generated on the Cray-T3D at the Ohio Supercomputer Center (OSC), using 3-subgroup $SU(2)$ updates, with 4 to 1 mixture of over-relaxed and heatbath. We saved configurations with 2000 sweeps, and obtained an average update time of 5 microseconds per link on 16 processors.

| Table 1 | The Statistical Ensemble |
|---------|-------------------------|
| $\beta$: | 6.0 5.7 5.7 |
| $N_f$:   | 0 2 2 |
| $m_{\text{dyn}}$: | $\infty$ .025 .01 |
| $N_{\text{samp}}$: | 32 35 49 |

We computed staggered propagators on the 32 node T3D at OSC and on the 256 node T3D at Los Alamos’s Advanced Computer Laboratory. We ran our fixed size problem on sets of processors ranging from 16 nodes to 128 nodes, and found the per node performance to be essentially independent of the partition size over this range. By hand coding the inner loops in CAM, the Cray T3D assembly language, we obtained a sustained performance of 45 Mflops per node, including I/O time.

To create and destroy the $\eta'$ we used the staggered flavor singlet operator $\bar{Q}(\gamma_5 \otimes I)Q$. This is a distance 4 operator, which we made gauge invariant by putting in explicit links, and averaging over the 24 paths across the edges of the hypercube.

To obtain the one-loop contractions of this operator (the denominator of fig. \cite{4}), we computed two types of propagators. One type used a noisy source which was nonzero on two timeslices $t = 0$ and $t = 1$. This source was a random phase $\eta_x = e^{i\theta(x)}$ for each color and each site, such that in the average over noise samples, $\langle \eta_x \eta^\dagger_y \rangle = \delta_{xy}$. The second type of propagator used a source obtained by transporting the noise $\eta_x$ across the diagonal of the hypercube and putting the phase appropriate for the $\eta'$ operator. Thus the operator which created the $\eta'$ was the spatial sum over a double timeslice of the hypercube operator, plus noise terms which vanish on average. We took two noise samples per configuration.

For the two-loop contractions we used source $\eta$ with $U(1)$ noise at every site on the lattice. Then solving $(\not{D} + m)\phi = \eta$ we estimate the propagator as $G_{xy} = \langle m\phi_x\phi^\dagger_y \rangle$. We note that this estimator is orders of magnitude better than the alternative $G_{xy} = \langle \phi_x\eta^\dagger_y \rangle$, but it only available when the sites $x$ and $y$ are separated by an even distance. For this reason we restricted our attention to the pseudoscalar operator, and neglected the axial vector (which is distance 3). For each color we used 16 noise samples on a doubled lattice, or an effective number of 96 noise vectors per mass.

3. Results

Figure 2. Quenched $R(\tau)$.

Figure 2 shows the measured ratio $R(\tau)$ in the quenched ensemble at a valence quark mass $m_q = .01$. In principle, the sickness of the quenched approximation should manifest itself in an unending linear trend in the data. By contrast, the dotted curve is the exponential one would expect if there were $N_f = 4$ active dynamical flavors. Clearly the data rule out $N_f = 4$, and are consistent with the quenched form, but from the data alone one couldn’t rule out a nonpathological behavior with
$N_f = 2$ or fewer flavors. Extracting the slope from such curves for several quark masses, we obtain the values for $m_0$ plotted in figure 3.

![Figure 3. Quenched $m^2_0$.](image)

Extrapolating linearly to $m_q = 0$ and rescaling to the most relevant case of $N_f = 3$ degenerate flavors, we obtain

$$m^2_0(N_f = 3) = (1050 \pm 170 \text{ MeV})^2 \left( \frac{a^{-1}}{2 \text{ GeV}} \right)^2,$$

which is compatible with the “experimental” number quoted above.

In the presence of $N_f = 2$ dynamical fermions, we expect $R(\tau)$ to asymptote to the constant 2. Figure 4 shows the result for $m_{\text{dyn}} = m_{\text{val}} = .01$. To fit it to the exponential form may be putting more weight on the data than it should bear, but its behavior does stand in clear contrast to the quenched data. If we ignore our theoretical prejudice and simply press ahead to repeat the linearized analysis as in the quenched case, we find the result

$$m^2_0(N_f = 3) = (780 \pm 50 \text{ MeV})^2 \left( \frac{a^{-1}}{2 \text{ GeV}} \right)^2,$$

On the other hand, if we do the correct job and fit to the exponential form, we can extract $m_{\eta'}s$ at two points, $m_{\text{dyn}} = m_{\text{val}} = .01$ and $m_{\text{dyn}} = m_{\text{val}} = .025$, finding $488 \pm .030$ and $.676 \pm .040$ for the bare lattice numbers respectively. Extrapolating to $m_q = 0$ and rescaling to $N_f = 3$ we end up with

$$m^2_0(N_f = 3) = (730 \pm 250 \text{ MeV})^2 \left( \frac{a^{-1}}{2 \text{ GeV}} \right)^2.$$
method which could speed up this and other calculations. Any observable we care to compute can be written as

\[ \langle O \rangle = \langle O \rangle_{\text{cool}} \times \left( \frac{\langle O \rangle}{\langle O \rangle_{\text{cool}}} \right), \]

where \( O_{\text{cool}} \) indicates observables computed on cooled configurations. In general \( O_{\text{cool}} \) are much cheaper to compute, and have smaller statistical fluctuations. If the hot and cool observables are strongly correlated, it may then be the case that for a fixed amount of computing one can obtain the RHS of eqn. 7 to better precision than the LHS. The method amounts to a type of renormalization group transformation, where \( O_{\text{cool}} \) is supposed to retain the long distance physics.

In a pilot study we tried cooling by only two steps. Figure 6 is a scatterplot of \( m_\pi \) and \( m_\rho \) for a variety of quark masses in both the hot and cold ensembles. Extrapolating to zero quark mass, the hot and cool ensembles give compatible rho masses, but evidently the cool data are much more precise. Further, if we compare at fixed pion mass, the cool data are a factor of 3 to 4 cheaper in terms of CG iterations.

Figure 6. \( \rho \) and \( \pi \) masses.

Picking quark masses which are close to each other in figure 6 (\( m_q = .01 \) hot, and \( m_q = .04 \) cool), we can look for the correlations in more complicated observables, such as the disconnected \( \langle \eta'(0)\eta'(t) \rangle \) two-point function. Figure 7 shows a scatterplot of the data for \( t = 10 \). As the dispersions show, both variables are statistically consistent with zero, i.e. we are well into the noise. Still, the two observables are strongly correlated, and we can determine their ratio more precisely than we can determine either variable alone.

Figure 7. Hot and cool observables.

5. Conclusions and Future Directions

We have shown that the \( \eta' \) mass can be computed using staggered fermions. While the data are too imprecise for definitive conclusions, it is at least plausible that the quenched \( \eta' \) propagator is pathological, while the dynamical one is well behaved. To compute the study we want to repeat the calculation with smeared sources to reduce the effect of higher excitations. We have also suggested a style of calculation which may be of some use when computing expensive, propagator-intensive observables. Our initial choice of two large cooling steps is unlikely to be optimal; to fully test the idea one should try better smoothing algorithms, e.g. using an improved action.

Acknowledgment

These calculations were performed on the Cray-T3D’s at the Ohio Supercomputer Center and at the Advanced Computer Laboratory. We thank OSC and ACL for early access to these machines.

REFERENCES

1. Kuramashi et al., Phys. Rev. Lett. 72 (1994) 3448.
2. Kuramashi et al., Phys. Rev. D (1995).
3. F. Brown et al., Phys. Rev. Lett. 67 (1991) 1062.
4. G. Kilcup, Phys. Rev. Lett. 71 (1994) 1677.
5. C. Bernard, these proceedings