TOWARDS THREE-DIMENSIONAL BETHE ANSATZ

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Abstract
We introduce a “pre-Bethe-Ansatz” system of equations for three dimensional vertex models. We bring to the light various algebraic curves of high genus and discuss some situations where these curves simplify. As a result we describe remarkable subvarieties of the space of parameters.

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1 Introduction

The purpose of this paper is to sketch how ideas introduced in the study of the sixteen vertex model in \[1\] can be generalized to higher lattice dimensions. We think that the ideas developed here are relevant tools for the analysis of lattice models in three or more dimensions, a widely unexplored area.

In this paper we first recall the basic results obtained for the two-dimensional case, with a special emphasis on the symmetries they uncover. We then introduce the simplest three-dimensional generalization of these results. In order to be more concrete, we describe a specific model which naturally generalizes the Baxter model. Finally, we show how this general construction points to a number of algebraic varieties of interest.

2 Preliminaries to the Bethe Ansatz

One of the keys to the Bethe Ansatz for two-dimensional vertex models is the existence (see for instance equations (B.10), (B.11a) in \[2\]) of vectors which are pure tensor products (of the form \(v \otimes w\)), and which the \(R\)-matrix maps onto another pure tensor product \(v' \otimes w'\) \[3\]. The “pre-Bethe Ansatz” equation is therefore:

\[ R(v \otimes w) = \mu \, v' \otimes w', \]  
(1)

A group \(G_{\text{Bethe}} \simeq sl_2 \times sl_2 \times sl_2 \times sl_2\) naturally acts on this equation, extending the weak-graph transformations \[4\]. With the following notations

\[ v = \begin{pmatrix} 1 \\ p \end{pmatrix}, \quad w = \begin{pmatrix} 1 \\ q \end{pmatrix}, \quad v' = \begin{pmatrix} 1 \\ p' \end{pmatrix}, \quad w' = \begin{pmatrix} 1 \\ q' \end{pmatrix}, \]
(2)

the elimination of \(w\) and \(w'\) from \[4\] yields the biquadratic relation:

\[ \alpha_{00} + \alpha_{10} p + \alpha_{01} q + \alpha_{20} p^2 + \alpha_{02} q^2 + 2\alpha_{11} pq + \alpha_{21} p^2 q' + \alpha_{12} q^2 p' + \alpha_{22} p q^2 p' + \alpha_{23} p^2 q^2 = 0. \]  
(3)

The \(\alpha_{ij}\)'s are quadratic polynomials in the homogeneous parameters \(a_1, \ldots, a_4\) of the \(R\)-matrix.

Equation \[4\] (and the similar one for \(q\) and \(q'\)) is a quadratic Frobenius relation \[4\] on theta functions. Let us recall that the Frobenius relations are intertwining relations of the product of two theta functions \[4\]. The “intertwining” matrix \(R\) is expressible in terms of ratio and product of theta functions depending on the difference of the arguments (spectral parameters) of the theta functions appearing in the parametrization of \(p, p'\) (resp. \(q, q'\)) \[4\]. This intertwining functional relations on theta functions enable to represent a Zamolodchikov algebra (see \[5\]) which is “almost” \[4\] a sufficient condition for the Yang-Baxter equations to be satisfied:

\[ \mathcal{A}_i(\theta_1) \mathcal{A}_j(\theta_2) = \sum_{kl} R_{ij}^{kl}(\theta_1 - \theta_2) \mathcal{A}_k(\theta_2) \mathcal{A}_l(\theta_1) \]  
(4)

In \[4\], it has been shown that equation \[4\] defines an elliptic curve in the \(p, p'\) plane. The discriminant \(\Delta\) seen as a quadratic polynomial in \(p\) is:

\[ D = \alpha p^4 + 4 \beta p^3 + 6 \gamma p^2 + 4 \beta' p + \alpha'. \]  
(5)

The transformations of \(D\) under the \(sl_2\) group acting on \(v'\) (homographic transformations of \(p')\) yield two fundamental invariants \(g_2\) and \(g_3\) \[4\] and a modular invariant \(J = g_3^2 / (g_2^3 - 27 g_2^2 g_3)\):

\[ g_2 = \alpha \alpha' - 4 \beta \beta' + 3 \gamma^2 \]  
(6)

\[ g_3 = \alpha \gamma \alpha' + 2 \beta \gamma \beta' - \alpha' \beta^2 - \alpha \beta^2 - \gamma^3 \]  
(7)

Footnote 1: The very existence of a Zamolodchikov algebra implies the Yang-Baxter equations to be satisfied provided the linear independence of the expressions \(\mathcal{A}_i(\theta_1) \mathcal{A}_j(\theta_2) \mathcal{A}_k(\theta_3)\). This independence condition is apparently not satisfied for theta functions of several parameters \[4\].
Exchanging the role of \( p \) and \( p' \) in equation (3) leads to the same \( g_2, g_3 \) and \( J \). More remarkably, the same operations applied to the equation in \( q \) and \( q' \) similar to (3), lead to the same \( g_2 \) and \( g_3 \). This is a quite non-trivial result. One can therefore associate the same Weierstrass’s canonical form (elliptic curve), \( y^2 = 4x^3 - g_2x - g_3 \), to equation (3), the similar equation in \( q \) and \( q' \) and also to the elliptic curve, intersection of quadrics in \( \mathbb{CP}_{15} \), which corresponds to the orbits of \( \Gamma \) (see Figure 1 in [1]).

The algebraic variety defined by \( J = \infty \), i.e. \( g_3^2 - 27g_2^3 = 0 \), is a remarkable variety in the parameter space \( \mathbb{CP}_{15} \): the elliptic parametrization of (3) becomes a rational one. Recalling the well-known example of the symmetric eight-vertex Baxter model [11, 2, 12], this reduction to a rational parametrization corresponds to two remarkable situations in the parameter space: the disorder varieties, for which some dimensional reduction occurs [13], and the critical varieties [14, 1]. Quite surprisingly, these two varieties of a priori different nature appear here on the same footing.

It would be pointless to write down the condition \( J = \infty \), since it is a sum of several millions of monomials of degree 24 in the sixteen homogeneous parameters of the sixteen-vertex model.

Noticeably, the Frobenius relations do exist for theta functions of several variables (\( g \) variables) leading to some associated Zamolodchikov algebra [5]. However, they do not lead to Yang-Baxter equations [5]. Clearly, there is room for models satisfying a non-trivial functional relation like equation (1) (which is a key for the explicit construction of the Bethe Ansatz) but not the Yang-Baxter equations. We may for instance consider the generalization of the symmetric eight-vertex model constructed in [5] for which there exists a parametrization in terms of theta functions of \( g \) variables: this model can be seen as a coupling of \( g \) replicas of the Baxter model, the arrows of the vertex taking values in \( \mathbb{Z}_g \) and the row-to-row transfer matrix (with periodic boundary conditions) being a \( 2g^N \times 2g^N \) matrix. If the Yang-Baxter equations were satisfied (which is apparently not the case [5]), they would yield the existence of a \( (g \)-dimensional) family of commuting transfer matrices. This is certainly too demanding. The existence of transfer matrices commuting only in a subspace of the \( 2g^N \) vector space, corresponds to interesting models and may be sufficient to calculate their partition function (largest eigenvalue of the transfer matrix). One must recall the example of the disorder solutions [13, 14], for which a commutation of the (diagonal) transfer matrices is satisfied on a one-dimensional vector space, which is actually sufficient to calculate the partition function (see p. 148 of [16]).

The exchange of \( p \) and \( p' \) (resp. \( q \) and \( q' \)), as well as the symmetries described in [18], are realized as shifts of the spectral parameter \( \theta \) of the elliptic curve (see for instance [14]). This is a key point for the explicit construction of the Bethe Ansatz, since it enables to build a vector

\[
|\Psi(\theta)\rangle = \bigotimes_n e(\theta + 2\eta n),
\]

which transforms very simply by the row-to-row transfer matrix:

\[
|\Psi(\theta)\rangle \rightarrow |\Psi(\theta + 2\eta)\rangle.
\]

When acting on \( \Psi \), the translation by a lattice spacing amounts to the shift of the spectral parameter \( \theta \). Taking into account the translation invariance of the lattice with periodic boundary conditions, one can “Fourier transform” \( |\Psi(\theta)\rangle \) and get an eigenvector \( |\hat{\Psi}(\theta)\rangle \) of the transfer matrix (see p. 148 of [14]). Notice that this construction works for theta functions of \( g \) variables using the corresponding Frobenius relations [5]. One therefore has a commutation of transfer matrices on this (one dimensional!) vector space. This is just the first step for the construction of the Bethe Ansatz. The second step amounts to build eigenvectors made up of product vectors (similar to (8)) with a given number of “deviating” factors (see p. 148–150 of [16] and [2]).
3 Towards three-dimensional Bethe Ansatz

A similar analysis can be performed in higher dimensions. In this section, we show how we can generalize these results in dimension three. In a specific example, we find three curves linked to a matrix \( R \) living on a higher dimensional variety \([17]\).

We denote \( w(i, j, k, l, m, n) \) the Boltzmann weight of a given three-dimensional vertex. We shall only consider the simplest case where each of the spins \( i, j, k, l, m, \) and \( n \) can take only two values. The vertex weights may be arranged in an \( 8 \times 8 \) matrix of entries:

\[
R_{iklm} = w(i, j, k, l, m, n). \tag{10}
\]

The natural generalization of the “pre-Bethe Ansatz” equation (1) is:

\[
R(u \otimes v \otimes w) = \mu \cdot u' \otimes v' \otimes w'. \tag{11}
\]

Let us introduce the notation:

\[
u = \left( \begin{array}{c} 1 \\ p \\ \end{array} \right), \quad v = \left( \begin{array}{c} 1 \\ q \\ \end{array} \right), \quad w = \left( \begin{array}{c} 1 \\ r \\ \end{array} \right), \quad u' = \left( \begin{array}{c} 1 \\ p' \\ \end{array} \right), \quad v' = \left( \begin{array}{c} 1 \\ q' \\ \end{array} \right), \quad w' = \left( \begin{array}{c} 1 \\ r' \\ \end{array} \right). \tag{12}
\]

In the following subsections, we shall recall the symmetries of a three-dimensional vertex model, as described in \([18]\).

3.1 The group of inversions \( \Gamma_{3D} \)

As in \([18]\), we first introduce the involution \( I \) changing \( R \) to its matrix inverse (we let appear an overall factor \( \lambda \) since the entry of \( R \) are taken projectively):

\[
\sum_{\alpha_1, \alpha_2, \alpha_3} (IR)^{i_1 i_2 i_3}_{\alpha_1 \alpha_2 \alpha_3} \cdot R^{i_1 i_2 i_3}_{j_1 j_2 j_3} = \lambda \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} \delta^{i_3}_{j_3}. \tag{13}
\]

Multiplying both sides of (11) by \( IR \), we get an equation of the same form as (11) with \( u \) and \( u' \), \( v \) and \( v' \) and \( w \) and \( w' \) exchanged and \( R \) replaced by \( IR \).

In \([18]\), we also introduced three partial transposition \( t_1, t_2 \) and \( t_3 \). \( t_1 \) is defined by:

\[
(t_1 R)^{i_1 i_2 i_3}_{j_1 j_2 j_3} = R^{i_1 i_2 i_3}_{t_1 j_1 t_2 j_3}. \tag{14}
\]

The definitions of \( t_2 \) and \( t_3 \) are similar.

The four involutions \( I \) and \( t_i \ (i = 1, 2, 3) \) generate an infinite discrete group \( \Gamma_{3D} \) \([18]\). The so-called inversion relations of the statistical mechanics model can be simply expressed with these building blocks. They are:

\[
I, \quad J = t_1 t_2 t_3, \quad K = t_2 t_3 t_1, \quad L = t_3 t_1 t_2. \tag{15}
\]

Considering the parameter space as a projective space (the entries of the \( R \)-matrix are homogeneous parameters), the elements of the group \( \Gamma_{3D} \) have a non-linear representation in terms of birational transformations. This group of symmetry of the parameter space of the model is very large. The number of elements of length \( l \) grows exponentially with \( l \). It is actually a hyperbolic Coxeter group \([19]\). The symmetry group of the Yang-Baxter equations in two dimensions is a mere affine Coxeter group \([15, 20]\) \([18]\).

The group \( \Gamma_{3D} \) has been shown in \([18]\) to enter the description of the group of automorphisms of the tetrahedron equations (generalization of the Yang-Baxter equations in three dimensions). We shall use this symmetry group beyond integrability, that is to say for models which do not have to verify the tetrahedron equations.
3.2 Weak-graph duality for 3D models: the gauge group \( G \)

A “gauge” group \( G = sl_2 \times sl_2 \times sl_2 \) acts linearly on the matrix \( R \) by similarity transformations (the weak-graph transformations, see [21] for details). If \( g = g_1 \times g_2 \times g_3 \), we define:

\[
g(R) = g_1 g_2 g_3 \cdot R \cdot g_1^{-1} g_2^{-1} g_3^{-1}.
\] (16)

Each of the \( g_i \)'s acts on the corresponding vector space and \( g_1 \) for example is a short hand notation for \( g_1 \otimes I \otimes I \). The action of \( G \) and \( \Gamma_{3D} \) do not commute. However, \( G \) and \( I \) do commute, and the commutation relation between the \( t_i \)'s and \( G \) gives a rather simple semi-direct product structure to the combined group:

\[
t_1 g = g^{t_1} t_1,
\] (17)

with:

\[
g^{t_1} = t_1 g_1^{-1} \times g_2 \times g_3,
\] (18)

and similar relations for \( t_2 \) and \( t_3 \). In particular, \( \Gamma_{3D} \) sends orbits of \( G \) onto orbits of \( G \). The compatibility of these two groups is described in [1] in a two-dimensional case, the sixteen-vertex model.

The effect of such a transformation on the pre-Bethe-Ansatz equation (11) is simple: \( g_1 \) act naturally on \( u \) and \( u' \), \( g_2 \) on \( v \) and \( v' \) and \( g_3 \) on \( w \) and \( w' \).

4 A three-dimensional model

The most general vertex models on a cubic lattice has a large number of parameters (sixty-four). We therefore impose some relations on the Boltzmann weights of the three-dimensional vertex. We require that these relations are invariant under the inverse \( I \) [22, 23] and the three partial transpositions \( t_1 \), \( t_2 \) and \( t_3 \) (equation (14)). They will thus be invariant under the group \( \Gamma_{3D} \). We are particularly interested in generalizations of the Baxter symmetric eight-vertex model, and define here a specific three-dimensional model, denoted in the sequel \( B_{3D} \). It is possible to “project” down a three-dimensional model onto a bidimensional one by just taking the trace of the matrix \( R \) on one of the spaces 1, 2, or 3: take for example space 3.

\[
\hat{R}^{ij}_{kl} = \sum_{\alpha_3} R^{i,j,\alpha_3}_{k,l,\alpha_3}
\] (19)

The constraints verified by \( B_{3D} \) are such that the three possible projections are symmetric Baxter models.

We define \( B_{3D} \) by imposing the following restrictions on the entries [18]:

\[
R^{i_1 i_2 i_3}_{j_1 j_2 j_3} = R^{-i_1, -i_2, -i_3}_{-j_1, -j_2, -j_3}
\] (20)

\[
R^{i_1 i_2 i_3}_{j_1 j_2 j_3} = 0 \text{ if } i_1 i_2 i_3 j_1 j_2 j_3 = -1
\] (21)

These constraints imply that the \( 8 \times 8 \) matrix \( R \) is the direct product of two times the same \( 4 \times 4 \) submatrix [17]. It is further possible to impose that this \( 4 \times 4 \) matrix is symmetric, since such a symmetry is preserved by the partial transpositions \( t_1 \), \( t_2 \), \( t_3 \) [13, 17], that is:

\[
R^{i_1 i_2 i_3}_{j_1 j_2 j_3} = R^{j_1 j_2 j_3}_{i_1 i_2 i_3}
\] (22)

We shall use the following notations for the entries of this \( 4 \times 4 \) submatrix:

\[
\begin{pmatrix}
a & d_1 & d_2 & d_3 \\
d_1 & b_1 & c_3 & e_2 \\
d_2 & c_3 & b_2 & c_1 \\
d_3 & c_2 & c_1 & b_3
\end{pmatrix}.
\] (23)
The four rows and columns of this matrix correspond to the states \((+,-,+), (-,+,+), (-,-,-), (+,+,-)\) and \((-,-,+)\) of the triplets \((i_1,i_2,i_3)\) or \((j_1,j_2,j_3)\). The matrix \(R\) can be completed by spin reversal, according to (20). Note that \(t_1\) (resp. \(t_2, t_3\)) simply exchanges \(c_2\) with \(d_2\) and \(c_3\) with \(d_3\) (resp. circular permutations). \(I\) acts as the inversion of this \(4 \times 4\) matrix.

It is quite remarkable that there exist four quantities which are covariant by all the four generating involutions \(I, t_1, t_2, t_3\), and therefore the whole group \(\Gamma_{3D}\). Let us introduce:

\[ p_3 = ab_3 + b_1b_2 - c_3^2 - d_3^2, \quad q_3 = c_1d_1 - c_2d_2, \]

and the polynomials obtained by permutations of 1, 2 and 3. They form a five dimensional space of polynomials. Any ratio of these polynomials is invariant under all the four generating involutions \(I, t_1, t_2, t_3, \mathbb{CP}_9\) is thus foliated by five dimensional algebraic varieties invariant under the whole group \(\Gamma_{3D}\). We can also express it by saying that the polynomials (24) define a map from the parameter space \(\mathbb{CP}_9\) to \(\mathbb{CP}_4\) invariant under \(\Gamma_{3D}\).

If we consider a subgroup \(\Gamma_2\) generated by only two involutions, say \(I\) and \(L\) or equivalently \(I\) and \(t_3\), one gets three more independent covariant polynomials leading to algebraic surfaces (see Fig. 1). They read:

\[ r_3 = ab_3 - b_1b_2 - c_3^2 + d_3^2, \quad s_3 = (a + b_3)c_3 - d_1d_2 - c_1c_2, \]

\[ t_3 = (b_1 + b_2)d_3 - d_1c_2 - c_1d_2. \]

From the projection (19) we get a Baxter model. If we denote by \(a_B, b_B, c_B, d_B\) the non zero entries of the \(R\)-matrix of this model, we have:

\[ a_B = a + b, \quad b_B = b_1 + b_2, \quad c_B = 2c, \quad d_B = 2d. \]

5 Study of the three-dimensional "pre-Bethe" equations.

5.1 A first attempt.

In the study of the equation (11), we can start by eliminating the variables \(p\) and \(p'\). We obtain the following system of five equations on the remaining variables \(q, q', r, r'\):

\[
0 = c_3d_3 - b_2b_3\ q^2 - c_1^2\ r^2 + (d_2d_3 - b_2c_2 + c_2c_3 - b_3c_1)\ r q + c_2d_2\ q^2 r^2 - ab_1\ q^2 \\
+ (ab_3 - c_2^2 + b_1b_2 - d_3^2)\ q q' + (ac_1 - c_2c_3 + b_1c_1 - d_2d_3)\ r q' \\
+ (b_2d_1 - c_3d_2 - c_2d_3 + b_3d_1)\ r q^2 q' + 2(c_1d_1 - c_2d_2)\ r^2 q q' \\
- (a d_1 + b_1d_1 - c_2d_3 - c_3d_2)\ r q q'^2 + c_3d_3\ q^2 q'^2 + c_2d_2\ r^2 q'^2 - d_1^2\ q^2 r^2 q'^2, \tag{27}
\]

\[
0 = (c_1c_3 - b_2c_2)\ q + (b_3c_3 - c_1c_2)\ r + (c_1d_2 - b_2d_3)\ q^2 r + (b_3d_2 - c_1d_3)\ r q^2 \\
+ ac_1\ q'^2 - ac_3\ r' - c_1d_3\ q^2 q' - b_3d_2\ r^2 q' \\
+ b_2d_3\ q^2 r' + c_1d_2\ r^2 q' + d_1d_3\ q^2 r'^2 - b_3d_2\ q^2 r'^2 - d_1d_2\ q^2 r^2 q'^2, \\
+ (a d_3 + c_2d_1 - b_3d_3 - c_1d_2)\ q r q' - (a d_2 + c_3d_1 - b_2d_2 - c_1d_3)\ q r r', \tag{28}
\]

\[
0 = c_3d_2 - b_2c_1\ (q^2 + r^2) + (c_1^2 - b_1^2 + d_2^2 - c_2^2)\ r q + (ab_3 - d_2^2)\ q q' + (ac_1 - d_2d_3)\ q q' \\
+ (b_1b_2 - c_3^2)\ q r q' - ab_1\ q' r + c_3d_2\ q^2 r^2 + (b_2d_1 - c_3d_2)\ r^2 q q' + (c_1d_1 - c_3d_3)\ r q^2 q' \\
+ (c_1d_1 - c_2d_2)\ r^2 q r' - d_1^2\ q^2 r' q'^2 + c_3d_3\ q^2 r'^2 - a d_1 - b_1d_1)\ q r q' \\
+ (b_1c_1 - c_2c_3)\ r r' + c_2d_2\ r^2 q' r' + (b_2d_1 - c_3d_2)\ r' r^2 + c_3d_3\ q^2 q' r', \tag{29}
\]
\[
0 = c_3d_1 - b_2c_2 q^2 - c_1c_3 r^2 + (d_1d_2 - c_1c_2 - b_2c_3 + b_1c_3) rq + b_1d_2 q^2r^2 \\
+ (ac_2 - d_1d_3) qq' + (ac_3 - d_1d_2) rq' - ac_3 q'r' \\
+ (c_3d_1 - b_1d_2) r^2qq' + (c_2d_1 - b_1d_4) rqq'q' + c_1d_2 r^2q'q' + b_2d_3 q^2q'r' \\
- (ad_2 + c_3d_1 - c_1d_3 - b_2d_2) qrr'q' - d_1d_2 q^2r^2q'r',
\]

(30)

\[
0 = (c_3d_2 - b_2d_1) q + (c_3d_3 - c_1d_1) r - (ab_2 - d_2^2) q^2r - (ac_1 - d_2d_3) qr^2 + ad_1 q' \\
+ (a^2 + d_1^2 - d_1^2 - d_2^2) rqq' - d_2d_3 (q^2 + r^2)q' + ad_1 q^2r^2q' + (b_1b_2 - c_3^2) qq'r' \\
+ (b_1c_1 - c_2c_3) rrq'r' - (ad_1 + b_1d_1 - c_2d_3 - c_3d_2) rqq'r'^2 + c_2d_2 r^2q'^2 r' - ab_1 q^2r' \\
+ (c_1d_1 - c_2d_2) r^2q'rq'q' + (b_2d_1 - c_3d_2) rq^2r'q' - d_1^2 q^2r^2q'^2 r' + c_3d_3 q^2q'^2 r'.
\]

(31)

Two similar system of equations can be obtained by the elimination of the pair of variables \(q\) and \(q'\) or \(r\) and \(r'\). The equations are of degree two in each of the variables, that is a overall maximum degree of eight. In fact only one is of degree seven, three are of degree six and one is of degree five.

The only apparent property of this system is the invariance by changing the sign of each of the variables \(q\), \(q'\), \(r\) and \(r'\), which is linked to the zeroes of the \(R\) matrix for \(i j k l m n = -1\).

The spin reversal symmetry of the \(U\) matrix and the change of \(R\) into its inverse \(IR\) have no visible consequences. This is due to the particular choice made in the elimination of \(p\) and \(p'\). The equations we have written are just five out of a system of thirty-six equations with a number of relations among them. On this full system, the symmetries should be more manifest.

### 5.2 Necessary conditions for \(B_{3D}\).

Since this direct attempt to find a full solution to \((11)\) leads to such a confuse result, we shall study some necessary conditions. We replace the tensor product \(u \otimes v\) and \(u' \otimes v'\) by general vectors in the tensor product of space \(1\) and \(2\) and \(U\) and \(U'\). \((11)\) becomes:

\[
R (U \otimes w) = \mu U' \otimes w'.
\]

This can be written:

\[
\begin{pmatrix}
R_1 & R_2 \\
R_3 & R_4
\end{pmatrix}
\begin{pmatrix}
U \\
rU
\end{pmatrix} = \mu
\begin{pmatrix}
U' \\
r'U'
\end{pmatrix},
\]

(33)

with \(R_1, \ldots, R_4\) the \(4 \times 4\) blocs of \(R\).

Eliminating \(U\) and \(U'\) gives the necessary condition:

\[
det(R_1 r' + R_2 r r' - R_3 - R_4 r) = 0.
\]

(34)

For the model introduced in section \((1)\) (eqs. \((24), (21), (22)\)), equation \((34)\) reads in terms of \(r\) and \(r'\):

\[
A_3 \cdot (r^4 r'^4 + 1) + B_3 \cdot (r^4 r'^2 + r^2 r'^4 + r^2 + r'^2) + C_3 \cdot (r^4 + r'^4) \\
+ D_3 \cdot (r^3 r'^3 + rr') + E_3 \cdot (r^3 r' + r r^3) + F_3 \cdot r^2 r'^2 = 0.
\]

(35)

Here \(A_3, \ldots, F_3\) are polynomial expressions of degree four in the homogeneous entries of the \(R\)-matrix \((23)\) \((a, \ldots, d_3)\). The simplest expressions are:

\[
A_3 = (c_1d_1 - c_2d_2)^2 = q_3^2, \\
C_3 = (b_1b_2 - c_3^2)(ab_3 - d_3^2) = \frac{1}{4}(p_3 + r_3)(p_3 - r_3).
\]

In fact, all the coefficients \(A_3, \ldots, F_3\) can be expressed as quadratic expressions in the polynomials invariant by the subgroup \(\Gamma_2\) of \(\Gamma_{3D}\) listed in \((24)\) and \((25)\). This shows that equation \((35)\) is invariant by this infinite group. Moreover, they verify the relation:

\[
4A_3 \cdot (F_3 - 2A_3 + 2C_3 + 2E_3) = (D_3 + 2B_3)^2.
\]

(36)
We recognize \( q_3 \) in the coefficient of \( r^4 + 1 \) and \( p_2 - p_1 \) in the coefficient of \( r^2 \). These coefficients are thus \( \Gamma_{3D} \)-covariant polynomials.

Of course two similar eliminations can be performed on \( (1) \) yielding constraints between \( p \) and \( p' \) (resp. \( q \) and \( q' \)).

To take into account the symmetry of \( (35) \) by the exchanges \( r \leftrightarrow r' \), \( r \leftrightarrow 1/r \) and \( r' \leftrightarrow 1/r' \), one may introduce the variables \( X = rr' + 1/rr' \) and \( Y = r/r' \). Equation \( (35) \) then becomes a conic:

\[
A_3 \ X^2 + B_3 \ XY + C_3 \ Y^2 + D_3 \ X + E_3 \ Y + \tilde{F}_3 = 0, \tag{38}
\]

with \( \tilde{F}_3 = F_3 - 2A_3 - 2C_3 \).

An invariant \( I_{\text{proj}} \) is naturally associated to the conic \( (38) \), it is the determinant of the \( 3 \times 3 \) matrix \( M \):

\[
M = \begin{pmatrix}
A_3 & B_3/2 & D_3/2 \\
B_3/2 & C_3 & E_3/2 \\
D_3/2 & E_3/2 & \tilde{F}_3
\end{pmatrix} \tag{39}
\]

The value of this invariant is (taking into account the relations between the entries of \( M \)):

\[
I_{\text{proj}} = -I^2/4A_3, \quad \text{with}
\]

\[
I = 2B_3^2 + D_3B_3 - 2E_3A_3 - 8A_3C_3. \tag{40}
\]

\( I_{\text{proj}} = 0 \) is a projectively invariant condition for the conic, meaning that it is the union of two lines. Remark that this does not imply the existence of a rational uniformization of \( (38) \). A similar phenomenon happens in the Baxter model, for which \( (3) \) becomes linear in \( X \) and \( Y \), but this does not provide the elliptic parametrization.

In order to obtain a parametrization of \( (38) \), we look at it as a polynomial of degree four in \( r \), the coefficients being polynomials in \( r' \):

\[
\alpha(r')r^4 + 4\beta(r')r^3 + 6\gamma(r')r^2 + 4\beta'(r')r + \alpha'(r') = 0 \tag{41}
\]

Its discriminant reads \( (3) \): \( \Delta(r') \) is a genus eleven curve.

\[
\Delta(r') = g_2(r')^3 - 27g_3(r')^2, \tag{42}
\]

with:

\[
g_2(r') = \alpha(r')\alpha'(r') - 4\beta(r')\beta'(r') + 3\gamma(r')^2, \tag{43}
\]

and:

\[
g_3(r') = \alpha(r')\gamma(r')\alpha'(r') + 2\beta(r')\gamma(r')\beta'(r') - \alpha(r')\beta'(r')^2 - \beta(r')^2\alpha'(r') - \gamma(r')^3. \tag{44}
\]

For a general vertex model in three dimensions, equation \( (34) \) leads to an equation like \( (11) \) where \( \alpha(r') \), \( \beta(r') \), \( \gamma(r') \), \( \beta'(r') \), \( \alpha'(r') \) are polynomials of degree four in \( r' \). The polynomials \( g_2(r'), g_3(r'), \Delta(r') \) are polynomials of degree 8, 12 and 24 respectively. For the general vertex model the hyperelliptic curve \( y^2 = \Delta(r') \) is a genus eleven curve.

However, one verifies easily that, for the model \( B_{3D} \), \( \Delta(r') \) is a polynomial of degree twelve in \( r'^2 \). Moreover the polynomial \( \Delta(r')/r'^{12} \) is symmetric under the inversion \( r' \leftrightarrow 1/r' \). Hence, introducing the variable \( s' = r'^2 + r'^{-2} \), \( \Delta(r')/r'^{12} \) becomes a degree six polynomial in \( s' \), denoted \( P_6(s') \).

If \( P_6(s') \) were a generic polynomial of degree six, the hyperelliptic curve \( y^2 = P_6(s') \) would be a genus two curve, meaning that, as far as parametrization is concerned, one is obliged to deal with theta functions of two variables (the Jacobian associated to the genus two curve \( 3 \)) or automorphic functions (see \( 24 \) and page 455 of \( 23 \)). One can envisage handy parametrization when the
hyperelliptic curves degenerate into elliptic ones, that is, when two roots of \( P_6(s') \) coincide, or equivalently when the discriminant of \( P_6(s') \) vanishes\(^1\).

It is important to note that the model of section 4, corresponds to such a situation where \( P_6(s') \) can be written as:

\[
P_6(s') = (s' - s_0)^2 \cdot P_4(s')
\]

where \( P_4(s') \) is a polynomial of the fourth degree in \( s' \), containing 289 monomials of degree eight in the coefficients \( A_3, \ldots, E_3 \). Noticeably, \( s_0 \) is a quite simple expression:

\[
s_0 = -\frac{D_3 + 2B_3}{2A_3}
\]

which reads in terms of the entries \( a, \ldots, d_3 \) of \( R \):

\[
s_0 = \frac{(c_1^2 - c_2^2 + d_1^2 - d_2^2) - (a - b_3)(b_1 - b_2)}{c_1d_1 - c_2d_2}
\]

Expression (47) is invariant under the group \( \Gamma_{3D} \) (see equations (24)). In equation (46), index 3, and the equations similar to (53) relating \( p \) and \( p' \), or \( q \) and \( q' \), lead to equations like (47), where 1,2 and 3 are permuted. It would be interesting to look for conditions on the entries of the \( R \) such that these three elliptic curves identify.

Clearly, a particular variety plays a special role: the subvariety in the space of models where these three elliptic curves identify.

To sum up, the three-dimensional vertex model \( B_{3D} \) yields a generalization of the intertwining quadratic Frobenius relations in the form of an intertwining of three different elliptic curves by an \( R \)-matrix living on an algebraic variety of dimension five given by the intersection of five quadrics (equations (24)).

5.3 Further analysis

The polynomial \( P_4(s') \) appearing in equation (45) is worth analyzing. We see that \( g_2 \) and \( \Delta = g_2^3 - 27g_3^2 \) factorize:

\[
g_2 = A_3^3 \cdot g_2^{(1)} \cdot g_2^{(2)}, \quad g_3 = A_3^3 \cdot g_3^{(1)}, \quad \Delta = A_3^8 \cdot \Delta_1 \Delta_2 \Delta_3 \Delta_4 \Delta_5^3
\]

with:

\[
g_2^{(1)} = 3E_3^2A_3 + 8C_3E_3A_3 + 16A_3^2C_3 - D_3^2C_3 - 16C_3^2A_3 - E_3B_3D_3 - 2B_3^2E_3 - 4B_3^2A_3 + 4C_3B_3^2,
\]

\[
\Delta_1 = 2E_3A_3 - D_3B_3, \quad \Delta_2 = 2B_3^2 + D_3B_3 - 2E_3A_3 - 8A_3C_3,
\]

\[
\Delta_3 = 4B_3^2A_3 + 16C_3^2A_3 - 4C_3B_3^2 - D_3^2C_3 + E_3^2A_3 + 8C_3E_3A_3 - 16A_3^2C_3 - 4C_3B_3D_3 + 8C_3D_3A_3 - 4B_3E_3A_3,
\]

\[
\Delta_4 = 4B_3^2A_3 + 16C_3^2A_3 - 4C_3B_3^2 - D_3^2C_3 + E_3^2A_3 + 8C_3E_3A_3 - 16A_3^2C_3 - 4C_3B_3D_3 - 8C_3D_3A_3 + 4B_3E_3A_3.
\]

The expressions \( g_2^{(2)} \) and \( \Delta_5 \) are polynomials of degree ten in the variables \( A_3, B_3, C_3, D_3 \) and \( E_3 \) (for instance, \( g_2^{(2)} \) contains 147 monomials). \( g_3^{(1)} \) is a polynomial of degree 20 in the same variables. Their explicit expressions involve too many terms to be reproduced here.

\(^2\) One should note that this is just an auxiliary parametrization and not a uniformization of eq. (44).
Expressing the coefficients $A_3, \cdots, E_3$ in terms of the entries of the $R$-matrix, one discovers further factorizations:

\[ g_2^{(1)} = (c_1 d_1 - c_2 d_2) \cdot G_2^{(1)}, \quad \Delta_2 = (c_1 d_1 - c_2 d_2) \cdot \delta_2, \]
\[ \Delta_3 = (c_1 d_1 - c_2 d_2)^2 \cdot \delta_3, \quad \Delta_4 = (c_1 d_1 - c_2 d_2)^2 \cdot \delta_4 \]

where $G_2^{(1)}$ is the sum of 1570 monomials of degree ten, $\delta_2$ is the sum of 104 monomials of degree six, $\delta_3$ and $\delta_4$ are the sum of 780 monomials of degree eight, and $\Delta_1$ is the sum of 256 monomials of degree eight in the entries of the three-dimensional $R$-matrix, i.e. $a, \cdots, d_3$.

Let us note that $\Delta_2 = \mathcal{I}$ (see equation (40)), and is thus related to the projective invariant $\mathcal{I}_{\text{proj}}$ of the conic (38).

### 5.4 Subcases of $B_{3D}$

The three-dimensional model $B_{3D}$ was built in such a way that it “projects” down to the two dimensional Baxter model, as defined in section 4. It is natural to consider the conditions on $B_{3D}$ obtained by writing that the three projections lie on the critical or disorder varieties of the Baxter model.

For example, writing the three disorder conditions $a_B + d_B = b_B + c_B$ (4) of the Baxter model for the three projections $(i = 1, 2, 3)$, yields a codimension-three subvariety of the three-dimensional model parametrized as follows:

\[ a = b_1 + b_2 + b_3 - 2z, \quad c_1 = b_1 + d_1 - z \]
\[ c_2 = b_2 + d_2 - z, \quad c_3 = b_3 + d_3 - z \quad (49) \]

On the subvariety (49), the discriminant of $P_4(s')$ vanishes, $\Delta_2 = \Delta_4 = 0$, the conic (38) degenerates since $\mathcal{I}_{\text{proj}} = \Delta_2 = 0$ and even more remarkably, $P_4(s')$ gets proportional to $(s' - 2)^4$.

Similarly, the three criticality conditions $a_B = b_B + c_B + d_B$ (4) yield a codimension three subvariety:

\[ 2d_1 = a + b_1 - b_2 - b_3 - 2c_1, \quad 2d_2 = a + b_2 - b_3 - b_1 - 2c_2, \]
\[ 2d_3 = a + b_3 - b_1 - b_2 - 2c_3 \quad (50) \]

On the subvariety (50), the discriminant of $P_4(s')$ vanishes, $\Delta_2 = \Delta_3 = 0$, and now $P_4(s')$ gets proportional to $(s' + 2)^4$. This last codimension-three subvariety is particularly interesting, since it is $\Gamma_{3D}$ invariant.

### 5.5 A solvable case.

Another interesting model is obtained by setting $d_1 = d_2 = d_3 = 0$. The projections yield six-vertex models (25). The biquartic equation (33) becomes a homogeneous equation of degree 4 and the solution is the union of four lines $r' = \lambda r$. Remarkably, for this model the left-hand side of (33) factorizes for $r' = -r$:

\[
r^4(ab_2 + b_1 b_3 - c_2^2 + ab_1 + b_2 b_3 - c_1^2 - 2ac_3 - 2b_3 c_3 + 2b_1 c_1) \\
(35) \quad (ab_2 + b_1 b_3 - c_2^2 + ab_1 + b_2 b_3 - c_1^2 + 2ac_3 + 2b_3 c_3 - 2b_1 c_1). \quad (51)
\]

What is more interesting is that we can get the conditions for the existence of solutions to (31) in this case. In (11), there are two equations which fix uniquely some scale factors:

\[ a = \mu, \quad a \ pqr = \mu \ p' q' r'. \quad (52)\]
Using these informations, the six others components of (11) fall into three pairs of equations like:

\[
\begin{align*}
ap' &= b_1p + c_2q + c_2r, \\
\frac{a}{p'} &= b_1\frac{1}{p} + c_2\frac{1}{q},
\end{align*}
\]  

(53)

In writing these equations, we discarded the trivial solution \( p = q = r = 0 \) which always exists in this case. Multiplying pairwise these equations, we obtain three linear equations for the variables \( X_p = \frac{q}{r} + r/q \), \( X_q = r/p + p/r \) and \( X_r = p/q + q/p \). This system of equation can be easily solved. These three variables are not independent since they depend only on the ratios of \( p, q \) and \( r \). They satisfy the relation:

\[
X_pX_qX_r - (X_p^2 + X_q^2 + X_r^2) + 4 = 0.
\]  

(54)

Rewritten in the homogeneous variables \( a, b_1, b_2, b_3, c_1, c_2, c_3 \), equation (54) is a necessary condition for equation (11) to have non-trivial solutions. This is however not the end of the story since the relation \( pqr = p'q'r' \) yields another condition on \( R \) once we have solved for \( r/p \) and \( q/p \). Note that the normalization of the variables \( p, q, r \) and \( p', q', r' \) remains free.

A complete analysis therefore yields the existence of non trivial solutions to (11) when \( R \) is on some codimension-two subvariety in the parameter space.

6 Conclusion.

We have shown how to associate algebraic curves with three-dimensional vertex models. We have described a specific model for which the analysis of these curves is handable. We have introduced a generalization of the quadratic Frobenius relations (associated to elliptic functions). It corresponds to new intertwining relations of products of more than two algebraic curves by \( R \)-matrices living on algebraic varieties which are no longer curves. In the example detailed in this paper, one has an intertwining of three curves by an \( R \)-matrix living on a higher-dimensional algebraic variety. We think that these equations are a key ingredient for the construction of the generalization of the Bethe Ansatz in higher dimensions, the quest of solutions of the tetrahedron equations and more generally any exact calculation (inversion trick [27], quest of critical manifolds [28]) performed on higher dimensional models.

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