Around the Chinese Remainder Theorem

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1 Introduction

The purpose of this text is to provide a convenient formulation of some folklore results. The five main statements are Theorem 28 page 15, Theorem 30 page 16, Theorem 31 page 17, Theorem 32 page 17, and Theorem 33 page 19.

By “polynomial” we mean “complex coefficient polynomial in the indeterminate $X$”.

We show that

- the computation of the quotient and the remainder of the euclidean division of a polynomial by a nonzero polynomial,
- the partial fraction decomposition of a rational fraction,
- the computation of an inductive sequence,
- the exponentiation of a matrix,
- the integration of an order $n$ constant coefficient linear differential equation,

follow from a unique, simple and obvious formula: Formula (3) page 7, which we call Taylor-Gauss Formula.

The field $\mathbb{C}(X)$ of rational fractions and the ring $E$ of entire functions are two important examples of differential rings containing $\mathbb{C}[X]$. The subring $\mathbb{C}[X]$ “controls” $E$ in the sense that an entire function can be euclideanly divided by a nonzero polynomial, the remainder being a polynomial of degree strictly less than that of the divisor. Among the rings having this property, one contains all the others: it is the differential ring

$$\prod_{a \in \mathbb{C}} \mathbb{C}[[X - a]].$$

Any torsion $\mathbb{C}[X]$-module is a module over the ring (1), and this ring is universal for this property. In particular any element $f$ of (1) can be evaluated on a square matrix $A,$ the matrix $f(A)$ being by definition $R(A)$, where $R$ is the remainder of the euclidean division of $f$ by a nonzero polynomial annihilating $A$. Since there is an obvious formula for this remainder (the Taylor-Gauss Formula), we are done.

An important example consists in taking for $f$ the exponential function, viewed as the element of (1) whose $a$-th component is the Taylor series of $e^X$ at $a$. We recover of course the usual notion of exponential of a matrix, but cleared form its artificial complications.
It is handy to embed \( \mathbb{C}(X) \) and \( E \) into the differential ring

\[
\prod_{a \in \mathbb{C}} \mathbb{C}((X - a)),
\]

which we use as a huge container. A side advantage of this ring is that it short-cuts the usual (particularly unilluminating) construction of the field of rational fractions (as a differential field) from the ring of polynomials.

2 Laurent Series

Let \( a \) be a complex number. A **Laurent series in** \( X - a \) is an expression of the form

\[
f = f(X) = \sum_{n \in \mathbb{Z}} f_{a,n} \ (X - a)^n,
\]

where \( (f_{a,n})_{n \in \mathbb{Z}} \) is a family of complex numbers for which there is an integer \( n_a \) such that \( n < n_a \) implies \( f_{a,n} = 0 \).

We define the operations of addition, multiplication and differentiation on the Laurent series in \( X - a \) by

\[
(f + g)_{a,n} = f_{a,n} + g_{a,n},
\]

\[
(fg)_{a,n} = \sum_{p+q=n} f_{a,p} \ g_{a,q},
\]

\[
(f')_{a,n} = (n + 1) \ f_{a,n+1},
\]

and we check that these operations have the same properties as on polynomials.

**Theorem 1.** Let \( f \) be a Laurent series in \( X - a \). If \( f \neq 0 \), then there is a unique Laurent series \( g \) in \( X - a \) such that \( fg = 1 \).

**Proof.** Exercise.

Put \( g = 1/f = \frac{1}{f} \) and \( h \ g = h/f = \frac{h}{f} \) if \( h \) is a Laurent series in \( X - a \).
3 Generalized Rational Fractions

A generalized rational fraction is a family \( f = (f_a)_{a \in \mathbb{C}} \) each of whose members \( f_a \) is a Laurent series in \( X - a \). The complex numbers \( f_{a,n} \) are called the coefficients of \( f \). The generalized rational fractions are added, multiplied and differentiated componentwise.

To each polynomial \( P \) is attached the generalized rational fraction

\[
\left( \sum_{n=0}^{\infty} \frac{P^{(n)}(a)}{n!} (X - a)^n \right)_{a \in \mathbb{C}}.
\]

As the formal sum in the parenthesis contains only a finite number of nonzero terms, it can be viewed as a polynomial. As such, it is of course equal to the polynomial \( P \). Hence the sums, products and derivatives of polynomials as polynomials coincide with their sums, products and derivatives as generalized rational fractions. These facts prompt us to designate again by \( P \) the generalized rational fraction (2).

We can now define a rational fraction as being a generalized rational fraction obtained by dividing a polynomial by a nonzero polynomial. The sums, products and derivatives of rational fractions are rational fractions. If a nonzero generalized rational fraction \( f = (f_a)_{a \in \mathbb{C}} \) is a rational fraction, then \( f_a \) is nonzero for all \( a \).

For all generalized rational fraction \( f \) and all complex number \( a \) put

\[\mu(a, f) := \inf \{ n \in \mathbb{Z} \mid f_{a,n} \neq 0 \}\]

with the convention \( \inf \emptyset = +\infty \), and say that \( \mu(a, f) \) is the multiplicity of \( a \) as zero, or root, of \( f \). We have

\[\mu(a, fg) = \mu(a, f) + \mu(a, g)\].

If \( \mu(a, f) \geq 0 \) say that \( f \) is defined at \( a \), and denote \( f_{a,0} \) by \( f(a) \). If a generalized rational fraction \( f \) is defined at a complex number \( a \), then we have

\[f_a = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (X - a)^n\].

Any sum, product or derivative of generalized rational fractions defined at \( a \) is a generalized rational fraction defined at \( a \).
For all generalized rational fraction $f$, all complex number $a$, and all integer $\mu$
set
$$J_a^\mu(f) := \sum_{n \leq \mu} f_{a,n} (X - a)^n;$$
and say that this Laurent series in $X - a$ is the order $\mu$ jet of $f$ at $a$. If $f$ and $g$
are generalized rational fractions defined at $a$, we have
$$J_a^\mu(f + g) = J_a^\mu(f) + J_a^\mu(g), \quad J_a^\mu(fg) = J_a^\mu\left( J_a^\mu(f) J_a^\mu(g) \right).$$

Let $a$ be a complex number, let $\mu$ be an integer, and let $f$ and $g$ be two general-
ized rational fractions.

**Exercise 2.** Show that the following conditions are equivalent

1. $\mu(a, f - g) \geq \mu$,
2. $(X - a)^{-\mu} (f - g)$ is defined at $a$,
3. $J_a^{\mu-1}(f) = J_a^{\mu-1}(g)$.

If these conditions are satisfied and if $f$ and $g$ are defined at $a$, then we say that
$f$ and $g$ are congruent modulo $(X - a)^\mu$ and we write
$$f \equiv g \mod (X - a)^\mu.$$ We thus have
$$J_a^{\mu-1}(f) \equiv f \mod (X - a)^\mu,$$
as well as
$$\begin{align*}
f_1 &\equiv g_1 \mod (X - a)^\mu \quad \Rightarrow \quad f_1 + f_2 \equiv g_1 + g_2 \mod (X - a)^\mu \\
f_2 &\equiv g_2 \mod (X - a)^\mu \quad \Rightarrow \quad f_1 f_2 \equiv g_1 g_2 \mod (X - a)^\mu.
\end{align*}$$

**Exercise 3.** Assume $f$ is defined at $a$ and $\mu \geq 0$. Let $R$ be a degree $< \mu$
polynomial. Show that the following conditions are equivalent.

1. $R = J_a^{\mu-1}(f)$,
2. $R \equiv f \mod (X - a)^\mu$,
3. $(X - a)^{-\mu} (f - R)$ is defined at $a$.

In other words
$$\left( \frac{f - J_a^{\mu-1}(f)}{(X - a)^\mu}, J_a^{\mu-1}(f) \right)$$
is the unique pair $(q, R)$ such that
1. $q$ is a generalized rational fraction defined at $a$,

2. $R$ is a polynomial of degree $< \mu$,

3. $f(X) = (X - a)^\mu q(X) + R(X)$.

If we wish to extend the result of Exercise 3 to the division by an arbitrary polynomial $D$, it is natural to restrict to the generalized rational fractions defined at all point of $\mathbb{C}$. We thus define a generalized polynomial as being a generalized rational fraction defined at all point of $\mathbb{C}$. Any sum, product or derivative of generalized polynomials is a generalized polynomial.

**Notation 4.** Fix a nonconstant polynomial $D$ and, for all complex number $a$, let $\mu_a$ be the multiplicity of $a$ as a root of $D$.

**Exercise 5.** Let $f$ be a generalized polynomial. Show that the generalized rational fraction $f/D$ is a generalized polynomial if and only if $f \equiv 0 \mod (X - a)^{\mu_a}$ for all $a$.

We admit

**Theorem 6** (Fundamental Theorem of Algebra). *There a nonzero complex number $c$ satisfying*

$$D(X) = c \prod_{D(a)=0} (X - a)^{\mu_a}.$$  

**Exercise 7.** Let $P$ be a polynomial. Show that the following conditions are equivalent

1. $P \equiv 0 \mod (X - a)^{\mu_a}$ for all $a$,

2. $P/D$ is a generalized polynomial,

3. $P/D$ is a polynomial,

4. $P/D$ is a degree $\deg P - \deg D$ polynomial.

4 **Chinese Remainder Theorem**

**Theorem 8** (Chinese Remainder Theorem). *For all generalized polynomial $f$ and all nonconstant polynomial $D$ there is a unique couple $(q, R)$ such that*

1. $q$ is a generalized polynomial,

2. $R$ is a polynomial of degree $< \deg D$, 

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3. $f = Dq + R$.

In Notation 4 page 6 we have the Taylor-Gauss Formula (see [1])

$$R(X) = \sum_{D(a)=0} J_{a}^{\mu_{a} - 1} \left( f(X) \frac{(X - a)^{\mu_{a}}}{D(X)} \right) \frac{D(X)}{(X - a)^{\mu_{a}}}$$

(3)

Say that $R$ is the remainder of the euclidean division of $f$ by $D$, and that $q$ is its quotient.

**Proof.** Uniqueness. Assume

$$f = D q_{1} + R_{1} = D q_{2} + R_{2}$$

(obvious notation). We get

$$\frac{R_{2} - R_{1}}{D} = q_{1} - q_{2}$$

and Exercise 7 (2. $\Rightarrow$ 4.) implies $R_{1} = R_{2}$ and thus $q_{1} = q_{2}$.

Existence. Put

$$s_{(a)}(X) := J_{a}^{\mu_{a} - 1} \left( f(X) \frac{(X - a)^{\mu_{a}}}{D(X)} \right) \frac{1}{(X - a)^{\mu_{a}}} \quad s := \sum s_{(a)},$$

and note that $\frac{f}{D} - s_{(a)}$ is defined at $a$ by Exercise 3 that $\frac{f}{D} - s$ is a generalized polynomial $q$, and that $sD$ is a degree $< \deg D$ polynomial $R$. QED

**Example.** Let $a$ and $b$ be two complex numbers. For $n \geq 0$ write $s_{n}$ for the sum of the degree $n$ monomials in $a$ and $b$. The remainder of the euclidean division of the polynomial $\sum_{n \geq 0} a_{n} X^{n} \text{ by } (X - a)(X - b)$ is

$$\sum_{n \geq 1} a_{n} s_{n-1} X + a_{0} - a b \sum_{n \geq 2} a_{n} s_{n-2}.$$

**Corollary 9.** (First Serret Formula — see [1].) The principal part of $g := f/D$ at $a$ is

$$J_{a}^{\mu_{a} - 1} \left( g(X)(X - a)^{\mu_{a}} \right)(X - a)^{-\mu_{a}}$$

There are partial analogues to Theorem 8 and Corollary 9 over an arbitrary commutative ring (see Section 10), but the uniqueness of the partial fraction decomposition disappears. For instance we have, in a product of two nonzero rings,

$$\frac{1}{X - 1} - \frac{1}{X} = \frac{(1, -1)}{X - (1, 0)} + \frac{(-1, 1)}{X - (0, 1)}.$$
Theorem 10. Let $P$ be a polynomial, let $f$ be the rational fraction $P/D$, let $Q$ be the quotient, which we assume to be nonzero, of the euclidean division of $P$ by $D$, and let $q$ be the degree of $Q$. Then $Q$ is given by the Second Serret Formula (see [1]):

$$Q(X^{-1}) = J_0^q(f(X^{-1}) X^q) X^{-q}$$

Proof. If $d$ is the degree of $D$ and if $R$ is the remainder of the euclidean division of $P$ by $D$, then the rational fraction

$$g(X) := X^{-1}\left(f(X^{-1}) - Q(X^{-1})\right) = \frac{X^{d-1} R(X^{-1})}{X^d D(X^{-1})}$$

is defined at 0, and we have

$$X^q f(X^{-1}) - X^q Q(X^{-1}) = X^{q+1} g(X) \equiv 0 \mod X^{q+1},$$

which, as $X^q Q(X^{-1})$ is a polynomial of degree at most $q$, implies

$$J_0^q\left(X^q f(X^{-1})\right) - X^q Q(X^{-1}) = 0.$$

QED

In Notation 4 page 6 fix a root $a$ of $D$, denote by $B$ the set of all other roots, and, for any map $u : B \to \mathbb{N}, b \mapsto u_b$, denote by $|u|$ the sum of the $u_b$.

Theorem 11. The coefficient $c_{a,k}$ of $(X - a)^k$ in $J_{a}^{\mu_{a} - 1}\left(\frac{(X-a)^{\mu_{a}}}{D(X)}\right)$ is

$$c_{a,k} = (-1)^k \sum_{\substack{u \in B \backslash \{a\} \mid u \mid = k}} \prod_{b \in B} \left(\frac{\mu_b - 1 + u_b}{\mu_b - 1}\right) \frac{1}{(a - b)^{\mu_b + u_b}}. \quad (4)$$

Proof. It suffices to multiply the jets

$$J_{a}^{\mu_{a} - 1}\left(\frac{1}{(X-b)^{\mu_b}}\right) = \sum_{n=0}^{\mu_{a} - 1} \left(\frac{\mu_b - 1 + n}{\mu_b - 1}\right) \frac{(-1)^n (X - a)^n}{(a - b)^{\mu_b + n}}. \quad (7)$$

QED

Corollary 12. The polynomial $R(X)$ in the Chinese Remainder Theorem (Theorem 8 page 5) is

$$R(X) = \sum_{\substack{D(a) = 0 \\text{k+n<}\mu_{a}}\atop{\text{k+n<}\mu_{a}}} c_{a,k} \frac{f(n)(a)}{n!} (X - a)^{k + n},$$

where $c_{a,k}$ is given by (7).
Corollary 13. The coefficients of the partial fraction decomposition of a rational fraction are integral coefficient polynomials in

- the coefficients of the numerator,
- the roots of the denominator,
- the inverses of the differences of the roots of the denominator.

These polynomials are homogeneous of degree one in the coefficients of the numerator and depend only on the multiplicities of the roots of the denominator. (We assume the denominator is monic.)

5 Exponential

The most important example of generalized polynomial is perhaps the exponential

\[ e^X := \left( e^a \sum_{n=0}^{\infty} \frac{(X-a)^n}{n!} \right)_{a \in \mathbb{C}}, \]

which satisfies

\[ \frac{d}{dX} e^X = e^X \]

as a generalized polynomial.

More generally we can, for any real number \( t \), define the generalized polynomial

\[ e^{tX} := \left( e^{at} \sum_{n=0}^{\infty} \frac{t^n (X-a)^n}{n!} \right)_{a \in \mathbb{C}}, \tag{5} \]

and observe the identity between generalized polynomials

\[ e^{tX} e^{uX} = e^{(t+u)X}. \]

We would like to differentiate \( e^{tX} \) not only with respect to \( X \) but also with respect to \( t \). To do this properly we must allow the coefficients of a generalized polynomial to be not only complex constants, but \( C^\infty \) functions from \( \mathbb{R} \) to \( \mathbb{C} \). Let us call variable coefficient generalized polynomial a generalized polynomial whose coefficients are \( C^\infty \) functions from \( \mathbb{R} \) to \( \mathbb{C} \). We then have the following identity between variable coefficient generalized polynomials

\[ \frac{\partial e^{tX}}{\partial t} = X e^{tX}. \]
One is often lead to compute jets of the form
\[ J_\mu^a(e^{tX} f(X)), \]
where \( f(X) \) is a rational fraction defined at \( a \). This jet is the unique degree \( \leq \mu \) polynomial satisfying
\[ J_\mu^a(e^{tX} f(X)) \equiv e^{at} J_\mu^a(f(X)) \sum_{n=0}^{\mu} t^n (X-a)^n \text{ mod } (X-a)^{\mu+1}. \] (6)

Let us sum this up by

**Theorem 14.** In Notation [4] page [6] the remainder of the euclidean division of \( e^{tX} \) by \( D(X) \) is given by the *Wedderburn Formula* (see [1])
\[ \sum_{D(a)=0} J_{\mu-1}^a \left( e^{tX} \frac{(X-a)^{\mu a}}{D(X)} \right) \frac{D(X)}{(X-a)^{\mu a}} \] (7)
the jet being given by (6).

### 6 Matrices

Let \( A \) a complex square matrix.

**Exercise 15.** Show that there is a unique monic polynomial \( D \) which annihilates \( A \) and which divides any polynomial annihilating \( A \).

Say that \( D \) is the *minimal polynomial* of \( A \).

**Exercise 16.** Let \( f \) be a generalized polynomial, \( D \) a polynomial annihilating \( A \), and \( R \) the remainder of the euclidean division of \( f \) by \( D \). Show that the matrix \( R(A) \) does not depend on the choice of the annihilating polynomial \( D \). [Suggestion: use Exercises 7 and 15.]

It is then natural to set
\[ f(A) := R(A). \]

**Exercise 17.** Let \( f, g \) be two generalized polynomials and \( D \) the minimal polynomial of \( A \). Show \( f(A) = g(A) \iff D \) divides \( f - g \).

**Exercise 18.** Show \( (fg)(A) = f(A) g(A) \).

**Theorem 19.** The matrix \( e^{tA} := f(A) \), where \( f(X) = e^{tX} \) (see [5]), depends differentiably on \( t \) and satisfies \( \frac{d}{dt} e^{tA} = A e^{tA}, e^{0A} = 1. \)
Exercise 20. Prove the above statement.

Exercise 21. Let

\[ D(X) = X^q + a_{q-1} X^{q-1} + \cdots + a_0 \]

be a monic polynomial, \((e_j)\) the canonical basis of \(\mathbb{C}^q\), and \(A\) the \(q\) by \(q\) matrix characterised by

\[ j < q \Rightarrow A e_j = e_{j+1}, \quad A e_q = -a_0 e_1 - \cdots - a_{q-1} e_q. \]

Show that \(D\) is the minimal polynomial of \(A\).

Let \(D\) be a degree \(q \geq 1\) polynomial, let \(f\) be a generalized polynomial, and \(b_{q-1} X^{q-1} + \cdots + b_0\) the remainder of the euclidean division of \(f\) by \(D\).

Exercise 22. Show \(f(A) e_1 = b_0 e_1 + \cdots + b_{q-1} e_q\).

7 Inductive Sequences

Let \(D\) be a degree \(q \geq 1\) monic polynomial; let \(\mathbb{C}^N\) be the set of complex number sequences; let \(\Delta\) be the shift operator mapping the sequence \(u\) in \(\mathbb{C}^N\) to the sequence \(\Delta u\) in \(\mathbb{C}^N\) defined by \((\Delta u)_t = u_{t+1}\); let \(f\) be in \(\mathbb{C}^N\), let \(c_0, \ldots, c_{q-1}\) be in \(\mathbb{C}\); let \(y\) be the unique element of \(\mathbb{C}^N\) satisfying

\[ D(\Delta) y = f, \quad y_n = c_n \text{ for all } n < q; \]

for \((n, t)\) in \(\mathbb{N}^2\) denote by \(g_n(t)\) the coefficient of \(X^n\) in the remainder of the euclidean division of \(X^t\) by \(D\).

Theorem 23. If \(t \geq q\) then \(y_t = \sum_{n<q} c_n g_n(t) + \sum_{k<t} g_{q-1}(t - 1 - k) f_k.\)

Proof. Introduce the sequence of vectors \(x_t := (y_t, \ldots, y_{t+q-1})\). We have

\[ x_{t+1} = B x_t + f_t e_q, \quad x_0 = c, \]

where \(e_q\) is the last vector of the canonical basis of \(\mathbb{C}^q\), and \(B\) is the transpose of the matrix \(A\) in Exercise 21. Hence

\[ x_t = B^t c + f_0 B^{t-1} e_q + f_1 B^{t-2} e_q + \cdots + f_{t-1} e_q. \]

It suffices then to take the first component of the left and right hand sides, and to invoke Exercise 22 QED.
8 Differential Equations

Let \( D \) be a degree \( q \geq 1 \) monic polynomial and \( y \) the unique solution to the differential equation

\[
D \left( \frac{d}{dt} \right) y = f(t), \quad y^{(n)}(0) = y_n \forall n < q := \deg D, \tag{8}
\]

where \( f : \mathbb{R} \to \mathbb{C} \) is a continuous function.

**Theorem 24.** We have the Collet Formula (see [1])

\[
y(t) = \sum_{n<q} y_n g_n(t) + \int_0^t g_{q-1}(t - x) f(x) \, dx
\]

where \( g_n(t) \) is the coefficient of \( X^n \) in the remainder of the euclidean division of \( e^{tX} \) by \( D \).

[For \( D(X) = X^q \) we recover the Taylor Formula with integral remainder.]

**Proof.** Putting \( v_n := y^{(n-1)} \), \( v_0 := y_{n-1} \) for \( 1 \leq n \leq q \), and denoting by \( e_q \) the last vector of the canonical basis of \( \mathbb{C}^q \), Equation (8) takes the form

\[
v'(t) - B \, v(t) = f(t) \, e_q, \quad v(0) = v_0,
\]

where \( B \) is the transpose of the matrix \( A \) in Exercise 21. Applying \( e^{-tB} \) we get

\[
\frac{d}{dt} e^{-tB} \, v(t) = e^{-tB} \, f(t) \, e_q, \quad v(0) = v_0,
\]

whence

\[
v(t) = e^{tB} \, v_0 + \int_0^t f(x) \, e^{(t-x)B} \, e_q \, dx.
\]

In view of the Wedderburn Formula (7), it suffices then to take the first component of the left and right hand sides, and to invoke Exercise 22. QED

Let \( h \) be a variable coefficient generalized polynomial, let \( f \) and \( y \) be two continuous functions from \( \mathbb{R} \) to \( \mathbb{C}^q \), let \( y_0 \) be a vector in \( \mathbb{C}^q \), and \( A \) a \( q \) by \( q \) complex matrix. If \( y \) is differentiable and satisfies

\[
y'(t) + h(t, A) \, y(t) = f(t), \quad y(0) = y_0, \tag{9}
\]

then

\[
H(t, A) := \int_0^t h(u, A) \, du \quad \Rightarrow \quad \frac{d}{dt} e^{H(t,A)} \, y(t) = e^{H(t,A)} \, f(t),
\]

whence
Theorem 25. The unique solution to (9) is given by the Euler Formula (see [1])

\[ y(t) = \exp \left( \int_0^t h(u, A) \, du \right) y_0 + \int_0^t \exp \left( \int_u^t h(u, A) \, du \right) f(v) \, dv \]

9 Euclid

By “ring” we mean in this text “commutative ring with 1”. For the reader’s convenience we prove the Chinese Remainder Theorem.

Theorem 26 (Chinese Remainder Theorem). Let \( A \) be a ring and \( I_1, \ldots, I_n \) ideals such that \( I_p + I_q = A \) for \( p \neq q \). Then the natural morphism from \( A \) to the product of the \( A/I_p \) is surjective. Moreover the intersection of the \( I_p \) coincides with their product.

Proof. Multiplying the equalities \( A = I_1 + I_p \) for \( p = 2, \ldots, n \) we get

\[ A = I_1 + I_2 \cdots I_n. \] (10)

In particular there is an \( a_1 \) in \( A \) such that

\[ a_1 \equiv 1 \mod I_1, \quad a_1 \equiv 0 \mod I_p \forall p > 1. \]

Similarly we can find elements \( a_p \) in \( A \) such that \( a_p \equiv \delta_{pq} \mod I_q \) (Kronecker delta). This proves the first claim. Let \( I \) be the intersection of the \( I_p \). Multiplying (10) by \( I \) we get

\[ I = I_1 I + I_2 \cdots I_n \subset I_1 (I_2 \cap \cdots \cap I_n) \subset I. \]

This gives the second claim, directly for \( n = 2 \), by induction for \( n > 2 \). QED

Let \( A \) be a PID (principal ideal domain). Construct the ring \( A^\wedge \) as follows. A family \((a_d)_{d \neq 0}\) of elements of \( A \) indexed by the nonzero elements \( d \) of \( A \) represents an element of \( A^\wedge \) if it satisfies

\[ d \mid e \Rightarrow a_d \equiv a_e \mod d \]

(where \( d \mid e \) means “\( d \) divides \( e \)”) for all pair \((d, e)\) nonzero elements of \( A \). Two such families \((a_d)_{d \neq 0}\) and \((b_d)_{d \neq 0}\) represent the same element of \( A^\wedge \) if and only if

\[ a_d \equiv b_d \mod d \quad \forall d \neq 0. \]

The ring structure is defined in the obvious way. Embed \( A \) into \( A^\wedge \) by mapping \( a \) in \( A \) to the constant family equal to \( a \). By abuse of notation we often designate by the same symbol an element of \( A^\wedge \) and one of its representatives. Let \( P \) be a representative system of the association classes of prime elements of \( A \).
Lemma 27. Let \( a = (a_b)_{b \neq 0} \) be in \( A^\wedge \) and \( d \) a nonzero element of \( A \) such that \( a_d \equiv 0 \mod d \). Then there is a \( q \) in \( A^\wedge \) such that \( a = dq \).

Proof. Let \( p \) be in \( P \) and \( i \) the largest integer \( j \) such that \( p^j \) divides \( d \). In other words there is an element \( d' \) of \( A \) which is prime to \( p \) and satisfies \( d = p^i d' \). For all nonnegative integer \( j \) we have

\[ a_{p^i+j} \equiv 0 \mod p^i. \]

As a result there is an element \( a'_j \) of \( A \) such that \( a_{p^i+j} = p^i a'_j \). We have

\[ p^i a'_{j+1} \equiv a_{p^i+j+1} \equiv p^i a'_j \mod p^{i+j} \]

and thus

\[ a'_{j+1} \equiv a'_j \mod p^j. \]

For all \( j \) choose \( q_{p^j} \) such that

\[ d' q_{p^j} \equiv a'_j \mod p^j \]

and thus

\[ d q_{p^j} \equiv a_{p^j} \mod p^j. \]

We get

\[ d' q_{p^j+1} \equiv a'_{j+1} \equiv a'_j \equiv d' q_{p^j} \mod p^j \]

and thus

\[ q_{p^j+1} \equiv q_{p^j} \mod p^j. \]

Let \( b \) be in \( A, b \neq 0 \), and \( P_b \) the (finite) set of those elements of \( P \) which divide \( b \). For \( p \) in \( P_b \) denote by \( i(p) \) the largest integer \( j \) such that \( p^j \) divides \( b \) and choose a solution \( q_b \) in \( A \) to the congruence system

\[ q_b \equiv q_{p^{i(p)}} \mod p^{i(p)}, \quad p \in P_b, \]

solution which exists by the Chinese Remainder Theorem (Theorem 26 page 13). One then checks that the family \( q := (q_b)_{b \neq 0} \) is in \( A^\wedge \) and that we do have \( d q = a \).

QED

See Theorem 32 page 17 for a generalization.

Let \( a \) be in \( A^\wedge \), let \( d \) be a nonzero element of \( A \), let \( \mu(p) \) be the multiplicity of \( p \) in \( P \) as a factor of \( d \), let

\[ J_p^{\mu(p)-1}(\alpha \frac{p^{\mu(p)}}{d}) \]
be an element of $A$ satisfying
\[
J_p^{\mu(p)-1}(a \frac{p^{\mu(p)}}{d}) \frac{d}{p^{\mu(p)}} \equiv a_{p^{\mu(p)}} \pmod{p^{\mu(p)}},
\]
and put
\[
\rho := \sum_{p \mid d} J_p^{\mu(p)-1}(a \frac{p^{\mu(p)}}{d}) \frac{d}{p^{\mu(p)}} \in A.
\]

Let $B$ be a ring and $A$ an integral domain contained in $B$. Say that $A$ is **principal in** $B$ if for all $b$ in $B$ and all nonzero $d$ in $A$ there is a $q$ in $B$ and an $r$ in $A$ such that $b = dq + r$.

**Theorem 28** (First Main Statement). Let $A$ be a PID, let $a$ be an element of $A^\wedge$, and let $d$ be a nonzero element of $A$. In the above notation, the elements $a$, $a_d$ and $\rho$ of $A^\wedge$ are congruent modulo $d$. Moreover, if $A$ is principal in $B$, then there is a unique $A$-algebra morphism from $B$ to $A^\wedge$.

**Proof.** Modulo $d$ we have $a \equiv a_d$ by the Lemma, and $a_d \equiv \rho$ by the Chinese Remainder Theorem (Theorem 26 page 13). This proves the first claim. Let us check the second one, starting with the uniqueness. Let $f$ be an $A$-linear map from $B$ to $A^\wedge$. If $b$ and $q$ are in $B$, and if $d \neq 0$ and $r$ are in $A$, then the equality $b = dq + r$ implies $f(b) = df(q) + r$, and thus $f(b)_d \equiv r \pmod{d}$. Consequently there is at most one such map. The existence is proved by putting $f(b)_d := r$ in the above notation, and by checking that this formula does define an $A$-linear map from $B$ to $A^\wedge$. QED

An $A$-module is **torsion** if each of its vectors is annihilated by some nonzero scalar.

**Theorem 29.** If $A$ is principal in $B$, then any torsion $A$-module admits a unique $B$-module structure which extends its $A$-module structure. Moreover any $A$-linear map between torsion $A$-modules is $B$-linear.

**Proof.** Let us check the uniqueness. Let $b$ be in $B$ ; let $v$ be in our torsion module $V$ ; let $d \neq 0$ in $A$ satisfy $dv = 0$ ; let $q$ in $B$ and $r$ in $A$ verify $b = dq + r$. We then get $bv = rv$, which proves the uniqueness. The existence is obtained by setting $bv := rv$ in the above notation, and by checking that this formula does define a $B$-module structure on $V$ which extends the $A$-module structure. The last assertion is clear. QED

[Let $\text{Hom}_{ct}(G, H)$ be the group of continuous morphisms from the topological group $G$ to the abelian topological group $H$, and $\hat{\mathbb{Z}}$ the profinite completion of $\mathbb{Z}$.]
Z). Equip \( \mathbb{Q}/\mathbb{Z} \) with the discrete topology or with the topology induced by that of \( \mathbb{R}/\mathbb{Z} \). Equip also \( \mathbb{Q}/\mathbb{Z} \) with the \( \widehat{\mathbb{Z}} \)-module structure provided by Theorem 29. We then have

\[
\text{Hom}_{ct}(\widehat{\mathbb{Z}}, \mathbb{R}/\mathbb{Z}) = \text{Hom}_{ct}(\widehat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) = \text{Hom}_{\mathbb{Z}}(\widehat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}.
\]

We recover the well known fact that \( \mathbb{Q}/\mathbb{Z} \) is the dual of \( \widehat{\mathbb{Z}} \) in the category of locally compact abelian groups. See Weil’s *L’intégration dans les groupes topologiques*, pp. 108-109.]

Let \( L \) be a sublattice of the lattice of ideals of an arbitrary ring \( A \). Here are some examples:

1. \( L \) is the set of nonzero ideals of an integral domain,
2. \( L \) is the set of powers of a fixed ideal of an arbitrary ring,
3. let \( Y \) be a set of prime ideals of \( A \), and \( L \) the set of those ideals \( I \) of \( A \) such that any prime ideal containing \( I \) is in \( Y \),
4. \( L \) is the set of open ideals of a topological ring.

Construct the ring \( A^\wedge \) as follows. An element of \( A^\wedge \) is represented by a family \((a_I)_{I \in L}\) of elements of \( A \) satisfying

\[ I \subset J \Rightarrow a_I \equiv a_J \mod J \]

for all \( I, J \) in \( L \). Two such families \((a_I)_{I \in L}\) and \((b_I)_{I \in L}\) represent the same element of \( A^\wedge \) if and only if

\[ a_I \equiv b_I \mod I \quad \forall I \in L. \]

The ring structure is defined in the obvious way. By abuse of notation we often designate by the same symbol an element of \( A^\wedge \) and one of its representatives. Map \( A \) to \( A^\wedge \) by sending \( a \) in \( A \) to the constant family equal to \( a \).

Call **torsion module** any \( A \)-module each of whose vector is annihilated by some element of \( L \). [In the case of Example 3 above, an \( A \)-module is torsion if and only if its support is contained in \( Y \) — Bourbaki, *Alg. Com.* II.4.4.] Assume to simplify that the intersection of the elements of \( L \) reduces to zero, and consider \( A \) as a subring of \( A^\wedge \).

**Theorem 30** (Second Main Statement). Any torsion \( A \)-module \( V \) admits a unique \( A^\wedge \)-module structure such that

\[ a \in A^\wedge, v \in V, I \in L, Iv = 0 \Rightarrow av = a_I v. \]

This \( A^\wedge \)-module structure on \( V \) extends the \( A \)-module structure.
Proof. The uniqueness being obvious, lets us check the existence. For \( v \) in \( V \) and \( I, J \) in \( L \) such that \( Iv = 0 = Jv \), we have \( a_I v = a_{I+J} v \); as this vector depends only on \( a \) and \( v \), it can be denoted \( av \). Let us show \( a (v + w) = av + aw \). If \( a \) is in \( A^\wedge \), and if \( I, J \) are in \( L \) and verify \( Iv = 0 = Jw \), we have

\[
a (v + w) = a_{I \cap J} (v + w) = a_{I \cap J} v + a_{I \cap J} w = a v + a w.
\]

QED

Theorem 31 (Third Main Statement). Let \( B \) a ring containing \( A \). Assume that each torsion \( A \)-module is equipped with a \( B \)-module structure which extends its \( A \)-module structure, and that each \( A \)-linear map between torsion \( A \)-modules is \( B \)-linear. Then there is a unique \( A \)-algebra morphism \( f \) from \( B \) to \( A^\wedge \) which satisfies \( b v = f(b) v \) for all vector \( v \) of any torsion \( A \)-module, and all \( b \) in \( B \).

Proof. For all \( I \) in \( L \) denote by \( 1_I \) the element 1 of \( A/I \). Let us verify the uniqueness. Let \( b \) be in \( B \) and \( I \) in \( L \). There is an \( a \) in \( A \) satisfying

\[
a \cdot 1_I = b \cdot 1_I = f(b) \cdot 1_I = f(b)_I \cdot 1_I,
\]

and thus \( f(b)_I \equiv a \mod I \). To check the existence, we put \( f(b)_I := a \) in the above notation, and verify that this formula does define an \( A \)-linear map \( f \) from \( B \) to \( A^\wedge \) which satisfies \( b v = f(b) v \) for all vector \( v \) of any torsion \( A \)-module, and all \( b \) in \( B \). [To check that \( f(b)_I \) and \( f(b)_J \) are congruent modulo \( J \) for \( I \subset J \) in \( L \), we use the assumption that the canonical projection from \( A/I \) to \( A/J \) is \( B \)-linear.] QED

Let \( A \) be a Dedekind domain; let \( L \) be the set of nonzero ideals of \( A \); and let \( M \) be the set of maximal ideals of \( A \). For any \( A \)-module \( V \), let \( V^\wedge \) be the projective limit of the \( V/IV \) with \( I \) in \( L \); and, for any \( P \) in \( M \), let \( V_P^\wedge \) be the projective limit of the \( V/P^nV \). Then \( A^\wedge \) and \( A_P^\wedge \) are \( A \)-algebras; by the Chinese Remainder Theorem (Theorem 25 page 13) \( A^\wedge \) is the direct product of the \( A_P^\wedge \); the \( A \)-module \( V^\wedge \) is an \( A^\wedge \)-module; the \( A \)-module \( V_P^\wedge \) is an \( A_P^\wedge \)-module; \( V^\wedge \) is the direct product of the \( V_P^\wedge \). Here is a generalization of Lemma 27 page 14.

Theorem 32 (Fourth Main Statement). The ring \( A^\wedge \) is \( A \)-flat. If \( V \) is a finitely generated \( A \)-module, then the natural morphism from \( A^\wedge \otimes_A V \) to \( V^\wedge \) is an isomorphism. In particular, if \( I \) is in \( L \), then \( IA^\wedge \) is the kernel of the canonical projection of \( A^\wedge \) onto \( A/I \).

Proof. The first claim follows from Proposition VII.4.2 of Homological Algebra by Cartan and Eilenberg (Princeton University Press, 1956). The second claim follows from Exercise II.2 of the same book in conjunction with Proposition 10.13 of Introduction to Commutative Algebra by Atiyah and Macdonald.
(Addison-Wesley, 1969). The third claim is obtained by tensoring the exact sequence
\[ 0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0 \]
with \( A^\wedge \) over \( A \). QED

10 The Case of an Arbitrary Ring

(Reminder: by “ring” we mean “commutative ring with 1”.) Let \( A \) be a ring, \( X \)
an indeterminate, and \( a \) an element of \( A \). For any formal power series
\[ f = \sum_{n \geq 0} a_n (X - a)^n \in A[[X - a]] \]
and any nonnegative integer \( k \) put
\[ \frac{f^{(k)}}{k!} := \sum_{n \geq k} \binom{n}{k} a_n (X - a)^{n-k} \in A[[X - a]]. \]

We then get the Taylor Formula
\[ f = \sum_{n \geq 0} \frac{f^{(n)}(a)}{n!} (X - a)^n. \]

We regard the \( A \)-algebra \( A[[X - a]] \) as an \( A[[X]] \)-algebra via the morphism from \( A[[X]] \) to \( A[[X - a]] \) which maps \( P \) to
\[ \sum_{n \geq 0} \frac{P^{(n)}(a)}{n!} (X - a)^n. \]

For any \( f \) in \( A[[X - a]] \) and any nonnegative integer \( k \) we call \( k \)-jet of \( f \) at \( a \)
the polynomial
\[ J^k_a(f) := \sum_{n \leq k} \frac{f^{(n)}(a)}{n!} (X - a)^n. \]

Then \( J^k_a \) induces a ring morphism
\[ J^k_a : A[[X - a]] \rightarrow \frac{A[X]}{(X - a)^k}. \]

Let \( a_1, \ldots, a_r \) be in \( A \); let \( m_1, \ldots, m_r \) be positive integers; let \( D \) be the product of the \((X - a_i)^{m_i}\); and let \( u \) be the natural morphism from \( A[X]/(D) \) to the product \( B \) of the \( A[X]/(X - a_i)^{m_i} \):
\[ u : \frac{A[X]}{(D)} \rightarrow B := \prod_{i=1}^r \frac{A[X]}{(X - a_i)^{m_i}}. \]
Assume that $a_i - a_j$ is invertible for all $i \neq j$. In particular

$$D_i(X) := \frac{D(X)}{(X - a_i)^{m_i}}$$

is invertible in $A[[X - a_i]]$. Let $v$ be the morphism

$$v : \left( P_i \mod (X - a_i)^{m_i} \right)_{i=1}^r \mapsto \sum_{i=1}^r J_{a_i}^{m_i-1} \left( \frac{P_i}{D_i} \right) D_i \mod D$$

from $B$ to $A[X]/(D)$.

**Theorem 33** (Fifth Main Statement). The maps $u$ and $v$ are inverse ring isomorphisms. Moreover Corollary 12 page 8 remains true (with the obvious changes of notation).

**Proof** The composition $u \circ v$ is obviously the identity of $B$. Since both rings are rank $\deg D$ free $A$-modules, this proves the statement. QED

## 11 Universal Remainder

In this Section we work over some unnamed ring (reminder: by “ring” we mean “commutative ring with 1”). The remainder of the euclidean division of $X^r$ by

$$(X - a_1) \cdots (X - a_k)$$

is a universal polynomial in $a_1, \ldots, a_k, X$ with integral coefficients. To compute this polynomial first recall Newton’s interpolation.

As a general notation set

$$f(u, v) := \frac{f(v) - f(u)}{v - u},$$

define the **Newton interpolation polynomial**

$$N(f(X); a_1, \ldots, a_k; X)$$

of an arbitrary polynomial $f(X)$ at $a_1, \ldots, a_k$ by

$$N(f(X); a_1, \ldots, a_k; X) := f(a_1) + f(a_1, a_2)(X - a_1) + f(a_1, a_2, a_3)(X - a_1)(X - a_2) + \cdots + f(a_1, \ldots, a_k)(X - a_1) \cdots (X - a_{k-1}),$$
and put
\[ g(X) := N(f(X); a_1, \ldots, a_k; X), \]
\[ h(X) := N(f(a_1, X); a_2, \ldots, a_k; X). \]

**Theorem 34** (Newton Interpolation Theorem). We have \( g(a_i) = f(a_i) \) for \( i = 1, \ldots, k \). In particular \( g(X) \) is the remainder of the euclidean division of \( f(X) \) by \((X - a_1) \cdots (X - a_k)\). Moreover \( f(a_1, \ldots, a_i) \) is symmetric in \( a_1, \ldots, a_i \).

**Proof.** Argue by induction on \( k \), the case \( k = 1 \) being easy. Note
\[ g(X) = f(a_1) + (X - a_1) h(X). \]
The equality \( g(a_1) = f(a_1) \) is clear. Assume \( 2 \leq i \leq k \). By induction hypothesis we have \( h(a_i) = f(a_1, a_i) \) and thus
\[ g(a_i) = f(a_1) + (a_i - a_1) f(a_1, a_i) = f(a_i). \]
Since \( f(a_1, \ldots, a_k) \) is the leading coefficient of \( g(X) \), it is symmetric in \( a_1, \ldots, a_k \).

QED

Assume that
\[ b_i := \prod_{j \neq i} (a_i - a_j) \]
is invertible for all \( i \). By Lagrange interpolation we have, for \( f(X) = X^r = f_r(X) \),
\[ f_r(a_1, \ldots, a_k) = \sum_i \frac{a_i^r}{b_i}. \]
The generating function of the sum \( s_n = s_n(a_1, \ldots, a_k) \) of the degree \( n \) monomials in \( a_1, \ldots, a_k \) being
\[ \frac{1}{(1 - a_1 X) \cdots (1 - a_k X)} = \sum_i a_i^{k-1} b_i^{-1} \frac{1}{1 - a_i X} = \sum_{n,i} \frac{a_i^{k-1+n}}{b_i} X^n, \]
we get
\[ s_n = \sum_i \frac{a_i^{k-1+n}}{b_i}, \quad f_r(a_1, \ldots, a_k) = s_{r-k+1} \]
with the convention \( s_n = 0 \) for \( n < 0 \), and the remainder of the euclidean division of \( X^r \) by \((X - a_1) \cdots (X - a_k)\) is
\[ \sum_{i=1}^k s_{r-i+1}(a_1, \ldots, a_i) (X - a_1) \cdots (X - a_{i-1}) \]
(even when some of the \( b_i \) are not invertible).
12 An Adjunction

Recall that a $\mathbb{Q}$-algebra is a ring containing $\mathbb{Q}$ ("ring" meaning "commutative ring with 1"), and that a derivation on a $\mathbb{Q}$-algebra $A$ is a $\mathbb{Q}$-vector space endomorphism $a \mapsto a'$ satisfying $(ab)' = a'b + ab'$ for all $a, b$ in $A$. A differential $\mathbb{Q}$-algebra is a $\mathbb{Q}$-algebra equipped with a derivation. We leave it to the reader to define the notions of $\mathbb{Q}$-algebra and of differential $\mathbb{Q}$-algebra morphism. To each $\mathbb{Q}$-algebra $A$ is attached the $\mathbb{Q}$-algebra $A[[X]]$ (where $X$ is an indeterminate) equipped with the derivation $\frac{d}{dX}$. Let $A$ be a $\mathbb{Q}$-algebra and $B$ a differential $\mathbb{Q}$-algebra. Denote respectively by

$$A(B, A) \quad \text{and} \quad D(B, A[[X]])$$

the $\mathbb{Q}$-vector space of $\mathbb{Q}$-algebra morphisms from $B$ to $A$ and of differential $\mathbb{Q}$-algebra morphisms from $B$ to $A[[X]]$. The formulas

$$f(b) = F(b)_0, \quad F(b) = \sum_{n=0}^{\infty} \frac{f(b^{(n)})}{n!} X^n,$$

where $F(b)_0$ designates the constant term of $F(b)$, set up a bijective $\mathbb{Q}$-linear correspondence between vectors $f$ of $A(B, A)$ and vectors $F$ of $D(B, A[[X]])$.

Let us check for instance the following point. Let $F$ be in $D(B, A[[X]])$, let $f$ in $A(B, A)$ be the constant term of $F$, and let $G$ in $D(B, A[[X]])$ be the extension of $f$. Let us show $G = F$. We have

$$G(b) = \sum_{n=0}^{\infty} \frac{G(b^{(n)}(0))}{n!} X^n \quad \text{by Taylor}$$

$$= \sum_{n=0}^{\infty} \frac{G(b^{(n)}(0))}{n!} X^n \quad \text{because } G(x)' = G(x')$$

$$= \sum_{n=0}^{\infty} \frac{f(b^{(n)})}{n!} X^n \quad \text{by definition of } G$$

$$= \sum_{n=0}^{\infty} \frac{F(b^{(n)}(0))}{n!} X^n \quad \text{by definition of } f$$

$$= \sum_{n=0}^{\infty} \frac{F(b^{(n)}(0))}{n!} X^n \quad \text{because } F(x') = F(x)'$$

$$= F(b) \quad \text{by Taylor.}$$
13 Wronski

Here is a brief summary of Section II.2 in H. Cartan’s book *Calcul différentiel*. If $E$ and $F$ are Banach spaces, let $\mathcal{L}(E, F)$ be the Banach space of continuous linear maps from $E$ to $F$. Let $E$ be a Banach space, $I$ a nonempty open interval, and $A$ a continuous map from $I$ to $\mathcal{L}(E, E)$. Consider the ODE’s

$$x' = A(t) x,$$

where the unknown is a map $x$ from $I$ to $E$, and

$$X' = A(t) X,$$

where the unknown is a map $X$ from $I$ to $F := \mathcal{L}(E, E)$. Let $t_0$ be in $I$. The solution to (12) satisfying $X(t_0) = Id_E$ is called the **resolvent** of (11) and is denoted $t \mapsto R(t, t_0)$.

Let $B$ be a continuous map from $I$ to $E$ and $x_0$ a vector of $E$. The solution to

$$x' = A(t) x + B(t), \quad x(t_0) = x_0$$

is

$$x(t) := R(t, t_0) x_0 + \int_{t_0}^t R(t, \tau) B(\tau) \, d\tau.$$

Here is an additional comment. For $t$ in $I$ let $A(t)$ be the left multiplication by $A(t)$. In particular $A$ is a continuous map from $I$ to $\mathcal{L}(F, F)$. Let $R$ be its resolvent. Then $R(t, t_0)$ is the left multiplication by $R(t, t_0)$.

**Corollary.** If $W$ is a differentiable map from $I$ to $F$ satisfying $W' = AW$, then

$$W(t) = R(t, t_0) W(t_0).$$

**Corollary to the Corollary.** If in addition $W(t_0)$ is invertible, then

$$R(t, t_0) = W(t) W(t_0)^{-1}.$$

* [1] For terminological justifications, type didrygaillard on Google.