Differentiable Categories, differentiable gerbes and $G$-structures.

Abstract.

The theories of strings and $D$-branes have motivated the development of non Abelian cohomology techniques in differential geometry, on the purpose to find a geometric interpretation of characteristic classes. The spaces studied here, like orbifolds are not often smooth. In classical differential geometry, non smooth spaces appear also naturally, for example in the theory of foliations, the space of leaves can be an orbifold with singularities. The scheme to study these structures is identical: classical tools used in differential geometry, like connections and curvatures are adapted. The purpose of this paper is to present the notion of differentiable category which unifies all these points of view. This enables us to provide a geometric interpretation of 5-characteristic classes, and to interpret classical problems which appear in the theory of $G$-structures by using gerbes.

1 Introduction.

Differential geometry is the study of the analytic properties of topological spaces. Most of the main tools developed in this theory are issued from calculus, and a manifold does not have singularities. Moduli spaces in differential geometry are rarely smooth as show the space of orbits of the action of a compact Lie group on a manifold, the space of leaves of a foliation, the compactification of a space of curves,... Since these singular structures arise naturally in differential geometry, it is normal to try to study them by using methods created in the smooth case. An example of such a method is the theory of orbifolds created by Satake [37], which enables to study the action of finite groups on manifolds which may have fixed points (see Audin [4]), foliations with bundle-like metrics (see Molino and Pierrot [33]), strings theory (see Chen and Ruan [11]), quotient of compact affine manifolds, in particular quotient of flat affine spaces forms are studied by many authors (see Long and Reid [20], Ratcliffe and Tschantz [36]).... We can also quote other theories such as the theory of foliages which
studies the differential geometry of the space of leaves of a foliation (see Molino [31]), the study of homogeneous \((X, H)\)-manifolds see (Goldman [15]).

One of the main goal of this paper is to propose the theory of differentiable categories to unify the generalizations of classical differential geometry mentioned above: a differentiable category is a category whose objects are differentiable manifolds, and the morphisms between its objects are differentiable maps. This point of view enables to handle also new situations like generalized orbifolds such as the orbit space of the action of a compact Lie group, or the space of leaves of a foliation endowed with a bundle-like metric. (see Molino and Pierrot [33]).

In mathematics, the classification problem is the cornerstone on which relies every theory \(T\); this is performed by assigning to objects which occur in \(T\) simpler invariants which enable to describe them completely: for example, the genus of a closed surface. The scheme usually followed in classical differential geometry is to define objects and invariants locally, and glue them with partitions of unity. Local invariants in the theory of differentiable categories are more difficult to study, since even when there exists a topology, neighborhoods of different points are not always isomorphic, for example the notion of a frames bundle is not straightforward defined since the dimension of the objects in a differentiable category may vary. This situation is analog to algebraic geometry, and we intensively use the machinery developed by Grothendieck and his students in this context (see Giraud [13], [14]). In fact sheaves of categories and gerbes are nowadays intensively studied by differential geometers (see Brylinski [8], Brylinski and McLaughlin [9], [10]): the functional action in classical mechanic which describes the motion of a point is expressed by using a connection on a principal bundle. In the purpose to unify all existing fundamental strengths, physicists have defined strings and branes theories. The functional action which describes the motion of a string is defined by a gerbe, and one expects that a good notion of \(n\)-gerbes will enable to handle branes theories. In fact sheaves of categories in this context are examples of differentiable categories. A currently very active research topic is the adaptation of tools defined in classical geometry like connections on principal bundles to these objects.

We start this paper by studying the differential geometry of differentiable categories without using Grothendieck topologies. We define the notions of principal bundles, which are torsors whose fibers are Lie groups, the tangent space of a differentiable category and its DeRham cohomology. In this setting we introduce connections forms and distributions and study their holonomy.

In modern geometry, global objects are constructed by gluing local objects. For example, a manifold is obtained by gluing open subsets of a vector space, schemes in algebraic geometry are defined by gluing spectrums of commutative rings. Algebraic geometers have remarked that in many situations, the transition functions that are used to glue objects do not verify the Chasles relation. This has motivated descent theory which is presented in the setting of categories theory by Giraud [13]. We study differential descent; or equivalently descent
in the theory of differentiable categories. This is an adaptation of the analysis situs of Giraud; we introduce differentiable fibered principal functors and their connective structures. Recall that the notion of connective structure has been introduced by Brylinski [8] in the context of gerbes on manifolds to provide a geometric interpretation of characteristic classes.

The local analysis intensively used in differential geometry relies on the existence of neighborhoods of points. This is achieved in this context by differentiable Grothendieck topologies: examples of Grothendieck topologies are defined on orbifolds, generalized orbifolds, foliages,... the CechDeRham complex is then used to study cohomology. Chen and Ruan [11] have defined a new cohomology theory for orbifolds to understand mathematical strings theory. We adapt to generalized orbifolds this new cohomology theory.

With the notion of Grothendieck topologies defined, we can study sheaves of categories and gerbes in the theory of differentiable categories. The first example of such a construction can be obtained by gerbes defined on the Grothendieck topology associated to an orbifold. Lupercio and Uribe [21] have provided such a construction by using groupoids. One of the fundamental example of a differentiable gerbe is the canonical gerbe defined on a compact simple Lie group $H$ (see Brylinski [8]). The classifying cocycle of this gerbe is the canonical 3-cohomology class defined by the Killing form. Medina and Revo y [26], [27] have classified Lie groups endowed with a non degenerated bi-invariant scalar product which also defines a canonical 3-form. Remark that these Lie groups are not always compact and are even contractible when they are nilpotent and simply connected. The theory of lattices in Lie groups presented by Raghunathan [35] and the Leray-Serre spectral sequence enable us to construct fundamental examples of gerbes on compact manifolds which are the quotient of a nilpotent Lie group by a lattice.

The notion of Grothendieck topology of a differentiable category enables us to construct the curving, and the curvature of a connective structure on a differentiable principal gerbe. We also define the holonomy form which is used to study functional action on loop spaces.

An approach of the study of the differential geometry of a gerbe can be done by using right invariant distributions defined in the thesis of Molino [29]. We outline how to a gerbe defined on a manifold one can associate an invariant distribution which enables to construct the holonomy around curves.

In the last part of the paper, we study sequences of fibered categories. An example of such a construction has been done by Brylinski and McLaughlin [9] to provide a geometric representation of the Pontryagin class of degree 4. To a principal gerbe, we associate a 2-sequence of fibered categories which must be an example of a $U(1)$ 3-gerbe (recall that the notion of 3-gerbe is not well-understood yet). We associate to such a 2-sequence of fibered categories a 5-integral cohomology class.

This new tools for differential geometers can be used to tackle well-known problems in differential geometry. A $G$-structure is a reduction of the bundle of
jets defined on a manifold. This theory has been intensively studied in the sev-
enties (see Molino [32] and the thesis of Albert [1], Medina [25], Nguiffo-Boyom
[34]). We can associate to a manifold a sheaf of categories which represents the
geometric obstruction to the existence of a $G$-structure.

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References
2 Notations.

Let $C$ be a category which has a final object $e$, and $I$ a small set relatively to a given universe (see [3] SGA 4 p. 4). In fact the cardinality used throughout this paper are numerable. Since we are studying differentiable manifolds, we want our spaces to be at least paracompact, to insure existence of partitions of unity, one of the main tools used in differential geometry to show the existence of global objects.

Consider a small family $(X_i)_{i \in I}$ of objects of $C$. We denote by $X_{i_1 \ldots i_n}$ the fiber product (if it exists) of the finite subset $\{X_{i_1}, \ldots, X_{i_n}\}$ of $(X_i)_{i \in I}$ over $e$.

Let $P$ be a presheaf of categories defined over $C$. For every objects $e_i, e'_i \in P(X_i)$, and a map $u : e_i \to e'_i$, we denote by $e^{i_2 \ldots i_n}_i$ and by $u^{i_2 \ldots i_n}$ the respective restrictions of $e_i$ and $u$ to $U_{i_1 \ldots i_n}$.

3 Basic Definitions and examples.

The differentiable manifolds used in this paper are $C^\infty$, and finite dimensional.

Definition 3.1.
A differentiable category $C$ is a category such that:
- every element $X$ of the class of objects of $C$ is a differentiable manifold,
- every morphism of $C$ is a differentiable map.

Examples.

The category $Diff$, whose objects are finite dimensional differentiable manifolds, and such that the set of morphisms $\text{Hom}_{Diff}(M, N)$ between two differentiable manifolds $M$ and $N$ is the set of differentiable maps between $M$ and $N$, is a differentiable category. Remark that this category is not small relatively to an universe $U$ which contains the set of real numbers, but is $U$-small (see [3] SGA4 p. 5).

Let $N$ be a manifold, and $C_N$ the category whose objects are open subsets of $N$. The morphisms between objects of $C_N$ are the canonical imbeddings. The category $C_N$ is endowed with the structure of a differentiable category, for which each open subset of $N$ is endowed with the differentiable structure inherited from $N$.

3.1 Orbifolds, $(X, H)$-manifolds and differentiable categories.

The theory of orbifolds has been introduced by Satake (see Satake [37], Chen and Ruan[11]). Orbifolds appear in different branches of mathematics, like strings theory, foliations theory: the singular foliation defined by the adherence of the leaves of a foliation endowed with a bundle-like metric can define an orbifold (see Molino and Pierrot [33] p. 208).

Definition 3.1.1.
An n-dimensional orbifold $N$ (see also Chen and Ruan [11] definition 2.1), is a separated topological space $N$, such that:
- for every element $x \in N$, there exists an open subset $U_x$ of $N$,
- an open subset $V_x$ of $\mathbb{R}^n$, a finite group of diffeomorphisms $\Gamma_x$ of $V_x$, an element $\hat{x} \in V_x$

such that for every element $\gamma_x$ in $\Gamma_x$, $\gamma_x(\hat{x}) = \hat{x}$.

-There exists a continuous map $\phi_x : V_x \to U_x$, such that $\phi_x(\hat{x}) = x$, for each $y \in V_x$, and $\gamma_x$ in $\Gamma_x$, $\phi_x(\gamma_x(y)) = \phi_x(y)$, and the induced morphism $V_x/\Gamma_x \to U_x$ is an homeomorphism.

The triple $(V_x, \phi_x, \Gamma_x)$ is called an orbifold chart.

We suppose that the following condition is satisfied:

Let $(V_x, \phi_x, \Gamma_x)$ and $(V_y, \phi_y, \Gamma_y)$ be two orbifolds charts. We denote by $p_x : V_x \times_N V_y \to V_x$ the canonical projection.

We suppose that there exists an equivariant diffeomorphism in respect of $\Gamma_x$ and $\Gamma_y$:

$$\phi_{xy} : p_y(V_x \times_N V_y) \to p_x(V_x \times_N V_y)$$

such that:

$$\phi_{xy}|_{p_x(V_x \times_N V_y)} = \phi_{xy}|_{p_x(V_x \times_N V_y)} \circ \phi_{xy}.$$

The fact that the morphism $\phi_{xy}$ is equivariant is equivalent to saying that for every element $\gamma_y$ in $\Gamma_y$, there exists an element $\Phi_{xy}(\gamma_y)$ in $\Gamma_x$ such that:

$$\phi_{xy} \circ \gamma_y = \Phi_{xy}(\gamma_y) \circ \phi_{xy}.$$

Remark that the fiber product $V_x \times_N V_y$ is not necessarily a manifold. This can be illustrated by the following example: consider the quotient $N$ of the real line $\mathbb{R}$, by the map $x \to -x$, the fiber product $\mathbb{R} \times_N \mathbb{R}$ is the union of two non parallel lines in $\mathbb{R}^2$; but $p_x(V_x \times_N V_y)$ is an open subset of $V_x$.

The maps $\phi_x \circ \phi_{xy} \circ \phi_{yz}|_{p_x(V_x \times_N V_y \times_N V_z)}$ and $\phi_x \circ \phi_{xz}|_{p_x(V_x \times_N V_y \times_N V_z)}$ are equal. Since $\Gamma_x$ is finite, there exists an element $c_{xyz}$ in $\Gamma_x$ such that:

$$\phi_{xy} \circ \phi_{yz}|_{p_x(V_x \times_N V_y \times_N V_z)} = c_{xyz} \phi_{xz}|_{p_x(V_x \times_N V_y \times_N V_z)}.$$

Let $\gamma_z$ be an element of $\Gamma_z$, we have:

$$\phi_{xy} \circ \phi_{yz} \circ \gamma_z|_{p_x(V_x \times_N V_y \times_N V_z)} = \Phi_{xyz}(\gamma_z) \circ c_{xyz} \phi_{xz}|_{p_x(V_x \times_N V_y \times_N V_z)}.$$

We also have:

$$c_{xyz} \phi_{xz} \circ \gamma_z|_{p_x(V_x \times_N V_y \times_N V_z)} = c_{xyz} \circ \Phi_{xz}(\gamma_z) \circ \phi_{xz}|_{p_x(V_x \times_N V_y \times_N V_z)}.$$

This implies that:
\[ \Phi_{xy} \circ \Phi_{yz} = c_{xyz} \circ \Phi_{xz} \circ c_{xyz}^{-1}. \]

An example of an orbifold is the quotient of a manifold by a finite subgroup of its group of diffeomorphisms. We are going to characterize some orbifolds arising from the theory of affine manifolds. (See also Long and Reid [20]; Ratcliffe, and Tschantz [36]).

Let \( H \) be a Lie group which acts differentiably and transitively on the manifold \( X \), we first recall some basic facts of the theory of \((X, H)\) manifolds (See Goldman [15]).

An \((X, H)\)-manifold, is a differentiable manifold, endowed with an open covering \((U_i)_{i \in I} \) such that:
- For each \( i \in I \), there exists a differentiable map \( d_i : U_i \rightarrow X \), such that \( d_i : U_i \rightarrow d_i(U_i) \) is a diffeomorphism.
- The transition function \( d_j \circ d_i^{-1} \mid_{d_i(U_i \cap U_j)} : d_i(U_i \cap U_j) \rightarrow d_j(U_i \cap U_j) \) coincides with the restriction of the action of an element of \( H \) on \( d_i(U_i \cap U_j) \).

An \((X, H)\)-map \( f : N \rightarrow N' \) between the \((X, H)\)-manifolds \( N \) and \( N' \), is a differentiable map which preserves their \((X, H)\) structures.

Examples of \((X, H)\) structures are \( n \)-dimensional affine manifolds: here \( H \) is the group \( \text{Aff}(\mathbb{R}^n) \) of affine transformations of \( \mathbb{R}^n \), and \( X = \mathbb{R}^n \); \( n \)-dimensional projective manifolds where \( X \) is the real projective space \( \mathbb{P}^n(\mathbb{R}) \), and \( H \) is \( \text{PGL}(n, \mathbb{R}) \) the group of projective transformations. A \((\mathbb{R}^n, \text{Aff}(\mathbb{R}^n))\)-automorphism is called an affine transformation.

We can show the following result relating orbifolds to affine manifolds, and to projective manifolds:

**Proposition 3.1.1.**

Let \( N \) be an affine manifold, and \( \Gamma \) a finite group of affine transformations of \( N \), such that the set \( C \) of elements of \( N \), such that for every element \( u \) of \( C \), there exists a non trivial element \( \gamma \) of \( \Gamma \) such that \( \gamma(u) = u \), is finite; moreover we suppose that every element of \( C \) is fixed by every element of \( \Gamma \). Let \( p : N \rightarrow N/\Gamma \) be the canonical projection. The blowing-up (not in the classical sense) of \( N/\Gamma \) at \( p(C) \) is a projective manifold.

**Proof.** First we are going to blow-up the action of \( \Gamma \).

Let \( u \) be an element of \( C \), and \( U \) an affine chart around \( u \). Thus \( U - \{u\} \) can be identified with a ball without the origin. Consider the submanifold \( P^n \) of \( \mathbb{R}^n \times \mathbb{P}^{n-1} \mathbb{R} \) defined by the equations:

\[
(x_1, ..., x_n, [X_1, ..., X_n]) \in P^n \iff x_iX_j - x_jX_i = 0.
\]

There exists a projection \( p' : P^n \rightarrow \mathbb{R}^n \), the restriction of the canonical \( \mathbb{R}^n \times \mathbb{P}^{n-1} \mathbb{R} \rightarrow \mathbb{R}^n \). The blowing up of \( N \) at \( u \) is the operation which replaces \( U \) by \( p'^{-1}(U) \) (see McDuff-Salamon [23] p. 233-235, See also Tsemo [39]). We can cover \( \mathbb{P}^{n-1} \mathbb{R} \) by two open affine subsets \( U_1 \) and \( U_2 \), which are the trivializations of the the \( \mathbb{R} \)-line bundle \( P^n \) over \( \mathbb{P}^{n-1} \mathbb{R} \). The coordinates change of these trivializations is the map:

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Thus the imbedding maps:

\[ u_i : i = 1, 2 : U_i \times \mathbb{R} \rightarrow P^{n+1} \mathbb{R} \]

defines a projective structure around \( u \) which can be glued with the affine atlas of \( N - C \) to obtain a projective structure on the blowing-up \( \hat{N} \), of \( N \).

We can identify the restriction of the action of the element \( \gamma \) of \( \Gamma \) on \( U \), to a linear map \( A_\gamma \), and extends it to a map \( A'_\gamma \) of \( P^n \) defined by

\[ A'_\gamma(x_1, ..., x_n[X_1, ..., X_n]) = (A_\gamma(x_1, ..., x_n), A_\gamma([X_1, ..., X_n])). \]

We thus obtain a free action of \( \Gamma \) on \( \hat{N} \) by projective maps. The quotient of \( \hat{N} \) by this action is a projective manifold \( N'' \), obtained from \( N/\Gamma \), by replacing a neighborhood of every element \( p(u), u \in C \) by the quotient of \( p'^{-1}(U) \) by \( \Gamma \). We also say that \( N'' \) is a blowing-up (not in the classical sense) of \( N/\Gamma \).

We associate to a orbifold \( N \) the differentiable category \( C_N \) defined as follows:

An object of \( C_N \) is a triple \((M, \phi_M, \Gamma_M)\) where \( M \) is a manifold, \( \Gamma_M \) a finite group of diffeomorphisms of \( M \), and \( \phi_M : M \rightarrow N \), a continuous map such that for every element \( \gamma_M \) in \( \Gamma_M \), for every element \( x \in M \), \( \phi_M(\gamma_M(x)) = \phi_M(x) \), and the induced map \( M/\Gamma_M \rightarrow N \) is a local homeomorphism. For every \( y = \phi_M(x) \) in \( N \), there exists a chart of the orbifold \((V_x, \phi_x, \Gamma_x)\) around \( y = \phi_x(x) \) (see definition 3.1.1) such that: if \( p^M : V_x \times_N M \rightarrow M \), and \( p_x : V_x \times_N M \rightarrow V_x \) are the natural projections, there exists an equivariant local diffeomorphism \( \phi^M_{V_x} : p^M(V_x \times_N M) \rightarrow p_x(V_x \times_N M) \) such that \( \phi_x \circ \phi^M_{V_x} = \phi^M_{\Gamma} \circ p^M_{\Gamma(V_x \times_N M)} \). In particular a chart of \( M \) is an object of \( C_N \).

A morphism between the objects \((M, \phi_M, \Gamma_M)\), and \((M', \phi_{M'}, \Gamma_{M'})\), is an equivariant differentiable map \( \phi : (M, \Gamma_M) \rightarrow (M', \Gamma_{M'}) \) such that \( \phi_M = \phi_{M'} \circ \phi \).

### 3.2 Actions of compact Lie groups and differentiable categories.

Let \( M \) be a finite dimensional manifold, and \( G \) a compact Lie group which acts effectively on \( M \). This is equivalent to saying that if an element of \( G \) fixes every element of \( M \), it is the identity. We denote by \( N \) the quotient of \( M \) by \( G \). The orbits of \( G \) are submanifolds, and when the action is free, a well-known elementary result implies that \( N \) is a manifold (see Audin [4] p. 13-19). In the general situation \( N \) is an orbifold with singularities. This can be seen by using the slice theorem of Koszul that we recall now:
Theorem 3.2.1 Koszul [18].

Let $G$ be a compact Lie group which acts effectively and differentiably on the manifold $M$. Let $u$ be an element of $M$. Denote by $G_u$ the subgroup of $G$ which fixes $u$, there exists an invariant neighborhood $U$ of $u$ which is isomorphic to a neighborhood of the zero section in the quotient of $G \times V$ by $G_u$, where $V$ is the quotient of the tangent space $T_uM$ by its subspace tangent to the orbit.

Thus, the slice theorem allows to construct an open covering of $M/G$ whose elements are quotient of open subsets of a vector space by the action of a compact Lie group (take a transversal to the zero section in theorem 3.2.1).

Let $N$ be the quotient space of $M$ by $G$. We associate to the action of $G$ on $M$ the following differentiable category $C_N$ defined as follows:

An object of $C_N$ is a triple $(P, H, \phi_P)$ where $P$ is a manifold endowed with an effective action of a compact Lie group $H$, such that there exists a local equivariant diffeomorphism $\phi_P : (P, H) \to (M, G)$ such that the induced map $P/H \to N = M/G$ is a local homeomorphism.

A morphism $f$ between the objects $(P, H, \phi_P)$ and $(P', H', \phi_{P'})$ is defined by an equivariant differentiable map $f : P \to P'$ such that $\phi_{P'} \circ f = \phi_P$.

The previous construction can be generalized in the following setting:

Definition 3.2.1.

Let $N$ be a separated topological space, A generalized orbifold on $N$ is defined by the following data:

For every element $u \in N$, there exists a manifold $M_u$, a compact Lie group $H_u$ which acts differentiably on $M_u$, and a continuous map $\phi_u : M_u \to N$ whose image contains $u$; such that for every $h_u$ in $H_u$, for every $x$ in $M_u$, $\phi_u(h_u(x)) = \phi_u(x)$, and the induced map $M_u/H_u \to N$ is a local homeomorphism. The triple $(M_u, H_u, \phi_u)$ is called a chart of the generalized orbifold.

Let $(M_u, H_u, \phi_u)$ and $(M_v, H_v, \phi_v)$ be two charts. Denote by $p_u : M_u \times N M_v \to M_u$ the canonical projection. There exists a local equivariant diffeomorphism $\phi_{uv} : p_v(M_u \times N M_v) \to p_u(M_u \times N M_v)$ such that $\phi_{uv} \circ p_v(M_u \times N M_u) = \phi_u \circ \phi_{uv}$.

Moduli spaces appear in different domains of differential geometry; many of them can be endowed with the structure of a differentiable category $C$. For example, consider the differentiable category whose class of objects is the class of isomorphism classes of hyperbolic surfaces of genus $h$, $h$ fixed. Let $[X_h]$ be the class of the surface of genus $h$, $X_h$. The differentiable structure of $[X_h]$ is the differentiable structure of one element picked in the class of $[X_h]$, for example $X_h$ itself. The set of morphisms $\text{Hom}([X_h], [X_{h'}])$ is the set of hyperbolic maps between $X_h^1$ and $X_{h'}^2$, where $X_h^1$ and $X_{h'}^2$ are the respective representants picked in the classes of $[X_h]$ and $[X_{h'}]$ to define the structure of the differentiable category.
3.3 Foliages and differentiable categories.

Let $N$ be a $n$-dimensional manifold, a foliation $\mathcal{F}$ on $N$ of codimension $q$ is defined by an atlas $(U_i, \phi_i : U_i \to \mathbb{R}^q \times \mathbb{R}^q)_{i \in I}$, such that $\phi_i \circ \phi_j^{-1}(x,y) = (u_{ij}(x), v_{ij}(y))$. This is equivalent to define a partition of $N$ by immersed manifolds of dimension $p$ called the leaves. In this situation we say that the couple $(N, \mathcal{F})$ is a foliated manifold. One of the main important problem in foliation theory is the study of the topology of the space of leaves, which is not always endowed with the structure of a manifold. For example consider the quotient $T^2$ of $\mathbb{R}^2$ by the group $\Gamma$ generated two translations $t_{e_1}$ and $t_{e_2}$ whose directions $e_1$ and $e_2$ are independent vectors. Let $\theta$ be an irrational integer; the foliation of $\mathbb{R}^2$ by affine lines parallel to $e_1 + \theta e_2$ defines on $T^2$ a foliation for which every leaf is dense. Thus the space of leaves of this foliation is not separated.

In [31] Molino has introduced the notion of foliation to study these situations which can be interpreted with differentiable categories:

**Definitions 3.3.1.**

Two foliated manifolds $(N_1, \mathcal{F}_1)$ and $(N_2, \mathcal{F}_2)$ are transversally equivalent if and only if there exists a foliated manifold $(N, \mathcal{F})$, two submersions $\pi_i, i = 1, 2 : N \to N_i$ such that the leaves of $\mathcal{F}$ are the preimages of the leaves of $\mathcal{F}_i$ by $\pi_i$.

Let $N$ be a topological space a foliage on $N$ is a differentiable category $C_N$ whose objects are quadruples $(U, V, \mathcal{F}, \pi)$, where $U$ is an open subset of $N$, $(V, \mathcal{F})$ a foliated manifold. We assume that the space of leaves of $\mathcal{F}$ is $U$ and $\pi : V \to U$ is the natural projection.

Let $(U', V', \mathcal{F}', \pi')$ another object of $C_N$, we denote by $p_{V'} : V \times_N V' \to V$ the natural projection. We assume the quadruples $(\pi(p_{V'}(V \times_N V')), p_{V'}(V \times N V'), \mathcal{F}_{p_{V'}(V \times_N V')}, \pi(p_{V'}(V \times_N V')), \pi'(p_{V'}(V \times_N V')))$ are objects of $C_N$ and transversally equivalent, where $\mathcal{F}_{p_{V'}(V \times_N V')}$ is the restriction of $\mathcal{F}$ to $p_{V'}(V \times_N V')$.

Finally we suppose that for every element $y \in N$, there exists an object $(U, V, \mathcal{F}, \pi)$ of $C_N$, such that $y \in U$.

A morphism between the objects $(V, U, \mathcal{F}, \pi)$ and $(V', U', \mathcal{F}', \pi')$ of $C_N$ is a differentiable map $\phi : V \to V'$ such that $\pi = \pi' \circ \phi$.

3.4 Projective presented manifolds.

In order to study a generalized equivalence Cartan problem, Molino (see Molino [32]) has studied projective presented manifolds which are examples of differentiable categories:

**Definition 3.4.1.**

A projective presented manifold is a small differentiable category whose class of objects is a projective system of manifolds $(V_i, \pi^*_j : V_j \to V_i)_{i \in I}$. The following conditions need to be satisfied:

The maps $\pi^*_j$ are submersions,
Let $\hat{V}$ be the topological projective limit of the family $(V_i, \pi^i_j)_{i \in I}$, and $(x_i)_{i \in I}, x_i \in V_i$ an element of $\hat{V}$. Let $P^i : \hat{V} \to V_i$ which associates to $(x_i)_{i \in I}$ the element $x_i$ in $V_i$. For each $i$, there exists an open neighborhood $U_i$ of $x_i$ in $V_i$, a map $c_i : U_i \to \hat{V}$, such that $P^i \circ c_i = Id_{U_i}$, and $P^j \circ c_i$ is differentiable.

Let $(x_i)_{i \in I}$ be an element of $\hat{V}$, there exists $i_0$, and a neighborhood $U_{i_0}$ of $x_{i_0}$ in $V_{i_0}$ such that for every $i > i_0$, for every $y \in U_{i_0}$, $\pi^{i_0-1}_i(y)$ is connected in $U_i$.

**Definition 3.4.2.**

Let $C$ and $C'$ be two differentiable categories; a differentiable morphism between $C$ and $C'$ is defined by a functor $F : C \to C'$, such that for every object $X$ of $C$, there exists a differentiable map $h^F_X : X \to F(X)$, such that for every morphism $f : X \to X'$ in $C$, the following square is commutative:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\downarrow h^F_X & & \downarrow h^F_{X'} \\
F(X) & \xrightarrow{F(f)} & F(X')
\end{array}
\]

We can suppose that the categories $C$ and $C'$ are imbedded in $Diff$ the category of differentiable manifolds. In this setting, a differentiable functor is a morphism between the identity functor of $C$, and a functor $F : C \to Diff$ whose image is contained in $C'$, moreover for every object $X$, the map $h^F_X : X \to F(X)$ which defines the morphism of functors is a differentiable map.

**Examples.**

Let $f : M \to N$ be a differentiable map, $f$ can be viewed as a differentiable functor $F : C \to C'$ where the unique object of $C$ is $M$ and the unique object of $C'$ is $N$. We suppose that the only morphisms in $C$ and $C'$ are the identities. The functor $F$ assigns $N$ to $M$, and $h^F_M = f$.

Suppose that a Lie group $G$ acts differentially on $M$, and $N$, we define $C$, to be the differentiable category which has $M$ has a unique object, and such that $Hom_C(M, M)$ is the image of the map $G \to Diff(M)$ which defines the action. Similarly, we define $C'$ to be the category whose unique object is $N$ and such that $Hom_{C'}(N, N)$ is the image of the map $G \to Diff(N)$. Let $\phi$ be an endomorphism of $G$, each $\phi$-equivariant map $f : M \to N$; that is a map such that for each $g \in G$, $f \circ g = \phi(g) \circ f$ defines a differentiable functor $F$ between $C$ and $C'$, such that $F$ assigns $N$ to $M$, $h^F_M = f$, and $F(g) = \phi(g)$.

## 4 Differentiable fibered categories.

To study the differentiable structure of differentiable categories, we are going to use the theory of fibered categories. On this purpose, we recall the following facts adapted to our setting:

**Definition 4.1.**
Let $F : P \to C$ be a differentiable functor, and $f : x \to y$ a map of $C$. Let $x', z'$ be two objects of the fiber of $x$, and $y'$ an object of the fiber of $y$. Denote by $\text{Hom}_F(z', y')$ the subset of the set of morphisms $\text{Hom}_P(z', y')$ such that for every element $l \in \text{Hom}_F(z', y')$, $F(l) = f$.

A morphism $f' : x' \to y'$ is Cartesian, if and only if the map $\text{Hom}_{\text{id}, X'}(z', x') \to \text{Hom}_F(z', y')$ which assigns to $h$ the map $f' \circ h$ is bijective for every $z'$ in the fiber of $x$.

**Definition 4.2.**

A differentiable bundle functor $F : P \to C$ is a Cartesian functor which satisfies the following conditions:

- The fiber of an object $x$ of $C$ has a unique element $p_x$.
- For every object $x$ of $C$, there exists a Lie group $H_x$ such that the canonical projection $p_x \to x$ defines on $p_x$ the structure of a total space of a $H_x$-principal bundle, whose base space is $x$. Morphisms between objects of $P$ are morphisms between principal differentiable bundles.

If the group $H_x$ is independent of $x$, we say that $F : P \to C$ is a $H$-principal differentiable bundle functor.

**Example.**

Let $H$ be a compact group which acts on the manifold $N$ by diffeomorphisms. We have attached a differentiable category $C_N$ to this action (see p. 8). Let $(P, H_P, \phi_P)$, an object of $C_N$, we can construct the principal $H_P$-bundle $P_{H_P}$ which is the quotient of $P \times H_P$ by the diagonal action of $H_P$. Let $f : (P, H) \to (P', H')$ a morphism in $C_N$ which is induced by a morphism $l_{H,H'} : H \to H'$ such that for every elements $h$ in $H$, and $p$ in $P$, $f(hp) = l_{H,H'}(h)f(p)$ since by definition $f$ is an equivariant map. We deduce a morphism $\psi_{H,H'}(f) : P \times H \to P' \times H'$ which sends $(p, h)$ to $(f(p), l_{H,H'}(h))$. For every $h_0 \in H$, we have:

$$\psi_{H,H'}(f)(h_0 p, h_0 h) = (f(h_0 p), l_{H,H'}(h_0 h)) = l_{H,H'}(h_0)\psi(p, h).$$

Thus the morphism $\psi_{H,H'}(f)$ induces a morphism $\psi_{H,H'}(f) : P_H \to P'_{H'}$.

We deduce the existence of a differentiable category $PC_N$ whose class of objects are the bundles $P_{H_P}$, and a differentiable bundle functor $F_N : PC_N \to C_N$ which sends the object $P_{H_P}$ to $P$. The Cartesian map above $f$ is $\psi_{H,H'}(f)$.

**Definition 4.3.**

Let $F : P \to C$ be a differentiable bundle functor, and $H : C' \to C$, a morphism between differentiable categories. The pull-back of $F$ by $H$ is the differentiable bundle functor $F' : P' \to C'$ defined as follows:

Let $X'$ be an object of $C'$, $h_X : X' \to F(X')$ the map which defines $H$; let $p_{F(X')} : X' \to F(X')$ the map which defines $F$; and $h_{p(F(X'))} : p_{F(X')}, F(X')$ the bundle map. The fiber of $X'$ is the fiber product of the maps $h_X$, and $h_{p(F(X'))}$.
4.1 Connection on differentiable bundle functors.

In this part we are going to study connections on differentiable bundles functors. First we recall the notion of connection on a principal bundle (See Lichnerowicz [19] p. 56, McDuff and Salamon [23] p. 207-209).

Let $H$ be a Lie group whose Lie algebra is denoted by $\mathfrak{h}$, and $p : P \to N$ a $H$-principal bundle over the $n$-dimensional manifold $N$, for every element $A \in \mathcal{H}$, we denote $A^*$ the vector field defined on $P$ by the formula:

$$A^*(x) = \lim_{t \to 0} \frac{d}{dt} x exp(tA), x \in P.$$ 

A connection defined on the $H$-principal bundle $p : P \to N$, is a 1-form $\theta : P \to \mathfrak{h}$ which verifies the following conditions:

Let $A^*$ be the fundamental vector field defined by $A \in \mathcal{H}, i_{A^*}\theta = A$.

For every element $h \in H$, $h^*\theta = Ad(h^{-1})\theta$.

A connection is also defined by a distribution on $P$ transverse to the fibers and invariant by $H$, whose rank is $n$ the dimension of $N$. To a connection form $\theta$, the distribution associated is: $\Theta_x = \{u \in TP_x, \theta(x) = 0\}$.

The curvature of $\theta$ is the $\mathfrak{h}$-valued 2-form on $P$ defined by: $\Omega = d\theta + \frac{1}{2}[\theta, \theta]$.

We adapt now this definition to differentiable categories:

**Definition 4.1.1.**

A connection on the principal bundle functor $p : P \to C$ is defined by a connection form $\theta_X$ on the principal $H_X$-bundle $p_X \to X$ of the fiber of $X$, such that for a map $h : p_X \to p_Y$ (necessarily Cartesian), the distribution defined by the kernel of $h^*(\theta_Y)$ is the distribution which defines the connection form of $\theta_X$.

**Example.**

Consider the interval $I = [-1, 1]$ of $\mathbb{R}$, and $N$ the orbifold which is the quotient of $I$ by the symmetry $h : x \to -x$, we associate to this orbifold the differentiable category $C_N$ whose class of objects contains only $I$, and the set of morphisms of $I$, $\text{Hom}_{C_N}(I, I) = \{\text{Id}_I, h\}$, remark that this is not the canonical differentiable category associated to an orbifold defined at p. 8. The real 1-form $\alpha = xdx$ is invariant by $h$, thus defines a connection on the trivial bundle functor $p : C_N \to C$ in circles over $C_N$ as follows: let $C^1$ be the circle, $P$ is the category which unique object is $e_1 = I \times C^1$. The unique non trivial morphism of $e_1$ is the map $h^1$ defined by $h^1(x,y) = (-x, y)$. Let $(u,v)$ be a vector tangent to $(x, y) \in I \times C^1$ we set $\theta_{I(x,y)}(u,v) = \alpha_x(u) + v = xu + v$.

**Definition:** Holonomy of a connection of a principal bundle functor 4.1.2.

Consider $C_1$, the canonical differentiable category defined on the interval by its structure of manifold, and $F : P \to C$ a $H$-principal bundle functor, endowed with a connection form $\theta$ and $L : C_1 \to C$ a differentiable functor. The pull-back of $F$ and $\theta$ by $L$ is a principal bundle over the interval endowed with a connection form whose holonomy map is the holonomy of $F : P \to C$, around $L$. 

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4.2 Differentiable tensors of a differentiable category.

In this section, we are going to associate to a differentiable category $C$, principal bundles functors which allow to define tensor fields. Such a theory is obviously known for manifolds. It has also been developed in the category of orbifolds see (Chen and Ruan [11]), and for foliages (see Molino [31]).

**Definition 4.2.1.**

Let $C$ be a differentiable category, the differentiable tangent bundle of $C$ is the differentiable category $T(C)$ defined as follows: the elements of the class of objects of $T(C)$ are tangent spaces $T(X)$, where $X$ is an object of $C$. A map between $T(X)$ and $T(Y)$, is a map $T(h) : T(X) \to T(Y)$ induced by a morphism $h : X \to Y$ in $C$.

A differentiable functor $F : C \to C'$, induces a tangent functor $T(F) : T(C) \to T(C')$ defined as follows: let $X$ be an object of $C$, the map $h_X^F : X \to F(X)$ induces the tangent map $T(h_X^F) : T(X) \to T(F(X))$ which defines the tangent functor.

The differentiable category of $p$-forms of $C$, $\Lambda^p(C)$, is the category whose class of objects is the class whose elements are $\Lambda^p T(X)$, where $X$ is an object in $C$. A morphism between the objects $\Lambda^p T(X)$ and $\Lambda^p T(Y)$ is a map of the form $\Lambda^p h$ where $h : X \to Y$ is a morphism in $C$.

A differentiable $p$-form is a functor $\alpha : \Lambda^p(C) \to C_R$, where $C_R$ is endowed with the structure of a differentiable category which as a unique object: the real line $R$, and such that the endomorphisms of $R$ are differentiable maps of $R$. We deduce from the definition of a differentiable functor that the following condition is satisfied: let $\alpha_X$ be a $p$-form, for every map $f : X \to Y$, there exists a diffeomorphism $\alpha(f)$ of $R$ such that the following square is commutative:

$$
\begin{array}{ccc}
\Lambda^p X & \xrightarrow{d^p f} & \Lambda^p Y \\
\downarrow \alpha_X & & \downarrow \alpha_Y \\
R & \xrightarrow{\alpha(f)} & R
\end{array}
$$

Let $f : X \to Y$ be a morphism in $C$, we don’t assume that $\alpha(f)$ is the identity of $R$. Thus $\alpha_X$ is not necessarily the pull-back of $\alpha_Y$ by $f$. We denote by $\Lambda^p_{id}(C)$ the set of $p$-forms such that for every map $f$ in $C$, $\alpha(f)$ is the identity.

**Definition-Proposition 4.2.2.**

Let $C$ be a differentiable category, and $\alpha$ a $p$-form defined on $C$, there exists a functor $d : \Lambda^p(C) \to \Lambda^{p+1}(C)$, the differential such that $d \circ d = 0$.

**Proof.** Let $\alpha$ be a $p$-form defined on $C$, for each object $X$ of $C$, $\alpha_X$ is a $p$-form, we can define $d\alpha_X$ the differential of $\alpha_X$. Let $f : X \to Y$ be a morphism in $\Lambda^p(C)$, we define $d(\alpha)(f) = d(\alpha(f))$.

**Examples.**

Let $N$ be a manifold, and $H$ a Lie group which acts differentially on $N$. Consider the differential category $C^H_N$ whose unique object is $N$, and such that the set of endomorphisms of $N$ in $C^H_N$ is the image of the map $H \to Diff(N)$.
which defines the action. Let \( \chi \) be a character of \( H \), we can define \( \Lambda^p_{H,\chi}(N) \) to be the set of \( p \)-forms on \( C^H_N \) such that for every element \( \alpha \in \Lambda^p_{H,\chi}(N) \), the following square is commutative:

\[
\begin{array}{ccc}
\Lambda^p N & \xrightarrow{d^p f} & \Lambda^p N \\
\downarrow \alpha_X & & \downarrow \alpha_Y \\
R & \xrightarrow{\chi(f)} & R
\end{array}
\]

In particular if \( \chi \) is the trivial character, we obtain the space of \( H \)-invariant \( p \)-forms, and the equivariant cohomology.

Let \( C_N \) be the differentiable category associated to a foliage. The set of \( \Lambda^p_{Id} C_N \) forms on \( C_N \) is the set of basic forms (see Molino [31]).

4.3 Frames bundle and differentiable categories.

Let \( C \) be a differentiable category, we cannot always define the bundle of linear frames, since two objects of \( C \) do not have necessarily the same dimension. Suppose that every objects \( C \) has dimension \( n \). We can define the set of vector frames \( V(C) \) as follows: Let \( X \) be an object of \( C \), and \( x \) be an element of \( X \), we denote by \( V(C)_x \) the set of linear maps \( u : \mathbb{R}^n \rightarrow T_x X \), where \( T_x X \) is the tangent space of \( x \). We can thus define the vector bundle \( V(C)(X) \) over \( X \) whose fiber at \( x \) is \( V(C)(X)_x \). Let \( f : X \rightarrow Y \) be a differentiable map, and \( u \in V(C)(X)_x \), the linear map \( df_x \circ u \) is a vector frame of \( V(C)(Y)_{f(x)} \). We have thus define the category of vector frames of \( C \). Since the morphisms in \( C \) are not necessarily local diffeomorphisms, we cannot assume that the elements of \( V(C)_x \) are isomorphisms.

Let \( C^H_N \) be the differentiable category associated to the action of a compact Lie group \( H \) on \( N \) (see p. 9). Let \( (P,H_P,\phi_P) \) be an object of \( C_N \). We can define the vector space \( TCN_x \), the quotient of the tangent space at \( x \), \( TP_x \) of \( P \), by the image of the infinitesimal action at \( x \) of \( H_P \). This space is called the tangent space at \( x \) of the action. Remark that the dimension \( \dim(TCN_x) \) of \( TCN_x \) depends only of \( \phi_P(x) \). But this dimension can vary if \( x \) varies in \( P \).

We can define the differentiable category of linear frames \( L(C_N) \) of \( C \). For each object \( X \) of \( C_N \), there exists fibration \( L(C_N)(X) \rightarrow X \) such that for every element \( x \in X \), \( L(C_N)(X)_x \) is the set of linear isomorphisms \( R^{\dim(TCN_x)} \rightarrow TCN_x \).

**Proposition 4.3.1.**

Suppose that the dimension of \( TCN(P)_x \) does not depend of \( P \), then there exists a connection on the frames bundle \( L(C_N) \), of the differentiable category \( C_N \) defined by the differentiable category defined above.

**Proof.** Let \( \theta \) be a connection on the frames bundle \( L(C_N)(N) \) of \( TCN(N) \) invariant by the action of \( H \). Let \( (P,H_P,\phi_P) \) be an object of \( C_N \). Since the dimension of \( TCN(P)_x \) does not depend neither of \( P \), nor of \( x \) in \( P \), the pull-back of \( \theta \) by \( \phi_P \) defines a connection form on \( L(C_N)(P) \).
**Definition 4.3.1.**

Let $C_N$ be the differentiable category $C_N$ defined by the action of the compact Lie group $H$ on $N$. We suppose that the dimension of $TCN_x$ does not depend of $x$. Let $(X,H_X,\phi_X)$ be an element of $C_N$, and $\alpha_X$ the fundamental 1-form of the bundle $L(C_N)(X)$. It is the form $R^{\dim(TCN)}$-valued form defined by:

$$\alpha_{X,u}(v) = u^{-1}(dp_X(v))$$

where $u$ is an element of $L(C_N)(X)_x$, $v$ element of the tangent space of $L(C_N)(X)$ at $u$, and $p_X : L(C_N)(X) \to X$ the canonical projection. The family of 1-forms $(\alpha_X)_{X \in C_N}$ defines an invariant form on $C_N$.

**4.4 Differentiable descent and connection in fibered categories.**

In this part we are going to study the notions of connection and holonomy on differentiable fibered categories.

Let us recall some facts on the analysis situs in differentiable categories (see Giraud [13]):

Let $F : P \to C$ be a Cartesian functor, a clivage is a family $L$ of morphisms of $P$ such that:

- every element in $L$ is cartesian,
- for every morphism $f : x \to y$ in $C$, and $y' \in P_y$, there exists a unique morphism $f' \in L$ whose target is $y'$ and such that $F(f') = f$. A clivage is a scindage if and only if it is stable by composition of maps. A clivage is the analog of a reduction in differential geometry.

Let $l : x \to y$ be a map in $C$, and $L$ a clivage. The clivage $L$ and $l$ induce a functor $l^* : P_y \to P_x$ defined as follows: The image of the object $z \in P_y$ is the source of the unique Cartesian map $q(z) : l^*(z) \to z$ in $L$ over $l$.

Consider two maps $l$ and $m$, such that the target of $l$ is the source of $m$, there exists a natural transformation:

$$c_{l,m} : (m \circ l)^* \to l^* \circ m^*$$

which satisfies the relation:

$$c_{m,l} \circ c_{l,m} = c_l \circ c_m$$

(See also Giraud [13] p.3).

Let $p : P \to C$ a Cartesian functor between differentiable categories. We assume that there exists a Lie group $H$ such that for every object $X$ of $C$, every object $e_X$ in the fiber of $X$ is endowed with the structure of an $H$-space; i.e the group $H$ acts on the right and freely on $e_X$. 

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There exists a projection \( p : e_X \to X \), such that for every \( h \in H \), and \( x \in e_X \), \( p(xh) = p(x) \).

A morphism \( f : e_X \to e_X \) in \( C \) is a differentiable map \( f \) such that for every element \( h \in H \), we have \( f \circ h = h \circ f \).

Let \( f \) be an endomorphism of \( e_X \), and \( x \in e_X \). We denote by \( u(x) \) the element of \( H \) such that \( f(x) = xu(x) \). For every element \( h \in H \), we have \( f(xh) = (xh)u(xh) = f(x)h = xu(x)h \). This implies that:

\[
u(xh) = h^{-1}u(x)h.
\]

We suppose that there exists a principal \( H \)-bundle functor \( \text{Aut}(P) \to C \), such that for every object \( e_X \) in the fiber of \( X \in C \), there exists a canonical isomorphism \( \text{Aut}(P)(X) \to \text{End}(e_X) \), which is natural in respect of morphisms between objects.

Let \( A \) be an element of \( \mathcal{H} \) the Lie algebra of \( H \), for every object \( e_X \), we can define the vector field:

\[
dt x \exp(tA).
\]

which is a fundamental vector field. This allows to identify \( \mathcal{H} \) with a subbundle of the tangent space \( Te_{e_X} \) of \( e_X \).

Let us start by a motivating example. Let:

\[
1 \longrightarrow H \to L' \to L \to 1
\]

be an exact sequence of Lie groups. Consider a principal \( L \)-bundle \( p : P \to N \) over the manifold \( N \). The obstruction to extend the structural group \( L \) to \( L' \), is defined by a sheaf of categories \( C_H \) defined as follows: for every open subset \( U \) of \( N \), \( C_H(U) \) is the category whose objects are \( L' \)-principal bundles over \( U \) whose quotient by \( H \) is the restriction of \( p \) to \( U \). Morphisms between objects of \( C_H(U) \) are morphisms of \( L' \)-bundles which induce the identity on the restriction of \( p \) to \( U \).

Let \( \mathcal{L} \) and \( \mathcal{L}' \) be the respective Lie algebras of \( L \) and \( L' \). We know the definition of a connection form \( \theta \) on \( p \), and we want to generalize this definition. A natural way is to take for each object \( e_U \in C_H(U) \) a connection \( \alpha_U \), such that the composition of \( \alpha_U \) with the natural projection \( \mathcal{L}' \to \mathcal{L} \) descends to the restriction \( \theta_U \) of \( \theta \) to \( U \). The choice of \( \alpha_U \) is not canonical since it is not necessarily preserved by every automorphism \( h \) of \( e_U \).

Remark that the form:

\[
\alpha_h = h^*(\alpha_U) - \alpha_U = \text{Ad}(h^{-1})(\alpha_U) - \alpha_U + h^{-1}dh
\]

is a \( \mathcal{H} \)-valued form.

This motivates the following definition (compare with Brylinski [8] p. 206, and with Breen and Messing [7]):

**Definition 4.4.1.**
Let \( p : P \to C \) be a \( H \)-principal fibered category, and \( H \) the Lie algebra of \( H \). A connective structure on \( C \) is a map which assigns to every object \( e_U \) of \( P_U \), an affine space \( Co(e_U) \) such that:

The vector space of \( Co(e_U) \) is the set of \( H \)-forms \( \Omega^1(U, H) \).

For every morphisms \( h' : U' \to U'' \), \( h : U \to U' \), and for every object \( e_U \) in the fiber of \( U \), \( e_{U'} \) in the fiber of \( U' \) and \( e_{U''} \) in the fiber of \( U'' \), there exists a morphism:

\[
h_* : Co(e_U) \to Co(e_{U'})
\]

which is compatible with composition: \( (h'h)_* = h'_*h_* \).

There exists a morphism:

\[
u_h : h^*(Co(e_{U'})) \longrightarrow Co(h^*(e_{U'}))\]

such that the following square is commutative:

\[
\begin{array}{ccc}
h*(h'^*Co(e_{U''})) & \xrightarrow{\nu_{h'}} & h^*Co(h'^*e_{U''}) \\
\downarrow \alpha_{h',h}^* & & \downarrow \alpha_{h',h}^* \\
(h'h)^*Co(e_{U''}) & \xrightarrow{\nu_{h'h}} & Co((h'h)^*e_{U''})
\end{array}
\]

where the morphisms \( c_{h,h'} \) is the morphism defined by a morphism in the analysis situs (see p. 16), and \( \alpha_{h',h} \) the canonical isomorphism of torsors.

Let \( u : e_U \to e_{U'} \) be a Cartesian morphism above \( h : U \to U' \), we have the compatibility diagram:

\[
\begin{array}{ccc}
h^*Co(e_U) & \xrightarrow{u_\ast} & h^*Co(e_{U'}) \\
\downarrow u_h & & \downarrow u_h \\
Co(h^*e_U) & \xrightarrow{u_\ast} & Co(h^*(e_{U'})).
\end{array}
\]

There exists an action of \( Aut_U(e_U) \) on \( Co(e_U) \) such that for every element \( h \) of \( Aut(e_U) \), and every element \( \theta \) in \( Co(e_U) \). We have the relation:

\[
h_\ast(\theta) = h.(\theta) + h^{-1}dh.
\]

And for every element \( \alpha \in \Omega^1(U, H) \), we have:

\[
h_\ast(\theta + \alpha) = h_\ast(\theta) + Ad(h^{-1})(\alpha).
\]

An alternative definition of connective structure can be done by considering torsors \( Co(e_U) \) whose vector space is the space of closed \( H \)-valued 1-forms if \( H \) is commutative.

**A fundamental relation.**

Suppose now that \( F : P \to C \) is a differentiably fibered category, consider a clivage \( L \). For each objects \( X \) of \( C \), and \( X' \) in the fiber of \( X \), consider a
morphism \( u_{XY} : Y \to X \), and its lift to a Cartesian morphism \( u_{X'Y'} : Y' \to X' \) of \( L \).

Let \( \alpha \) be a connective structure defined on this differentiable fibered bundle, we denote by \( \alpha_{Y'} \) an element of \( Co(Y') \), and by \( \alpha_{X',Y'} \), the 1-form such that \( \alpha_{X'} = \alpha_{X'Y'} + u_{X'Y'}(\alpha_{Y'}) \) we have:

\[
u_{X'Y'}(\alpha_{Y'}Z') - \alpha_{X'Z'} + \alpha_{X'Y'} =
\]

\[
= u_{X'Y'}(\alpha_{Y'} - u_{Y'Z'}(\alpha_{Z'})) - (\alpha_{X'} - u_{X'Z'}(\alpha_{Z'})) + (\alpha_{X'} - u_{X'Y'}(\alpha_{Y'}))
\]

\[
= u_{X'Z'}(\alpha_{Z'}) - u_{X'Y'}u_{Y'Z'}(\alpha_{Z'})
\]

Since \( F : P \to C \) is a fibered category, there exists a morphism \( c_{X',Y',Z'} \) such that \( u_{X'Y'}u_{Y'Z'} = u_{X'Z'}c_{X',Y',Z'} \), we deduce that:

\[
u_{X'Y'}(\alpha_{Y'}Z') - \alpha_{X'Z'} + (\alpha_{X'}Y') =
\]

\[
u_{X'Z'}(\alpha_{Z'} - c_{X',Y',Z'}(\alpha_{Z'}))
\]

\[
= u_{X'Z'}((\alpha_{Z'}) - c_{X',Y',Z'}(\alpha_{Z'}) - c_{X',Y',Z'}^{-1}dc_{X',Y',Z'})
\]

5 Grothendieck topologies in differentiable categories.

We have studied differentiable categories without emphasizing on the global topology. This can be achieved by using the notion of differentiable Grothendieck topology (see [3] S.G.A 4-1; p. 219; or Giraud [14]).

**Definitions 5.1.**

Let \( C \) be a differentiable category, a sieve \( R \) in \( C \) is a subclass \( R \) of the class of objects of \( C \) such that if \( U \) is an object of \( R \), and \( V \to U \) is a morphism in \( C \), then \( V \) is in \( R \).

A Grothendieck topology on \( C \) is defined by assigning to each object \( U \) of \( C \) a non empty family of sieves \( J(U) \) of the category over \( U, C \uparrow U \) such that the following conditions are satisfied:

- For every morphism \( h : V \to U \), and every sieve \( R \in J(U) \), the pull-back sieve \( R^h \) is in \( J(V) \).
- A sieve \( R \) of \( C \uparrow U \) is in \( J(U) \) if and only if for every map \( h : V \to U \), \( R^h \in J(V) \).

**Examples.**

An example of a Grothendieck topology can be defined as follows: Let \( N \) be a topological space, and \( C_N \) the category whose objects are open subsets,
and whose maps are canonical imbeddings between open subsets. For an open subset \( U \), an element of \( J(U) \) is a family of open subsets \((U_i)_{i \in I}\) of \( U \) such that \( \bigcup_{i \in I} U_i = U \). This topology is often called the small site.

Let \( N \) be a generalized orbifold (see definition 3.2.1). We can define on \( N \) the following Grothendieck topology:

A covering of an open subset \( U \) of \( N \), is a family of objects \((P_i, H_i, \phi_i)_{i \in I}\) which is \( U \)-jointly surjective. This equivalent to saying that \( \bigcup_{i \in I} \phi_i(P_i) = U \).

In particular if for every object \((P, H_P, \phi_P)\) in \( C_N \), the groups \( H_P \) is discrete, we obtain a Grothendieck topology on orbifolds.

Consider the space of hyperbolic surfaces of a given genus \( h \). Each of this surface can be cut in pants. The hyperbolic length of the boundaries cycles of these pants are the Fenchel-Nielsen coordinates which identify the set of isomorphic classes of hyperbolic surfaces of genus \( h \) to a cell. (See [6] X. Buff and al p.13-15).

**Definition 5.2.**

Let \((C, J)\) be a category endowed with a Grothendieck topology, we suppose that \( C \) has a final object \( e \). A global covering of \( C \) is a cover of \( e \), that is an element of \( J(e) \).

**Definition 5.3.**

A presheaf defined on the differentiable category \( C \), is a contravariant functor \( F \), from \( C \) to the category of sets.

A sheaf is a presheaf which satisfies 1-descent in respect to any sieve \( R \) in \( J(U) \). This is equivalent to saying that for every object \( U \) of \( C \), and every sieve \( R \) in \( J(U) \), the natural map:

\[
F(U) \rightarrow \lim_{V \rightarrow U \in R} F(V)
\]

is bijective.

### 5.1 Grothendieck topologies and cohomology of differentiable categories.

The cohomology of orbifolds is studied in algebraic geometry and symplectic geometry, since orbifolds arise as phase spaces in theoretical physics. Grothendieck and his collaborators (see S.G.A. 4 II, p.16) have defined Cech cohomology in Grothendieck sites. We shall apply this point of view to generalized orbifolds. We shall also generalize Chen and Ruan cohomology of orbifolds (see [11]) to generalized orbifolds.

Let \( J_N \) be the Grothendieck topology associated to the generalized orbifold \( N \). We can define the presheaf \( \Omega^p_N \), such that for each object \( e = (P, H_P, \phi_P) \) of \( C_N \), \( \Omega^p_N \) is the vector space of \( p \)-differentiable forms invariant by \( H_P \) defined on \( P \) (see also p. 14). If \( h : e \rightarrow e' \) is a morphism in \( C_N \), the restriction is defined by the pull-back of differentiable forms.
Consider a covering \((U_i, H_i, \phi_i)_{i \in I}\) of \(N\). We cannot defined the classical Cech resolution, since the differentiable category \(C_N\) associated to \(N\) is not necessarily stable fiber products. Let \((U_{i_1}, H_{i_1}, \phi_{i_1}), \ldots, (U_{i_n}, H_{i_n}, \phi_{i_n})\), be objects of \(C_N, \phi_{i_n}(U_{i_1}, \ldots, i_n)\) is a manifold. We can defined the bi-graded complex \(\Omega_N^1(\phi_{i_n}(U_{i_1}, \ldots, i_n))\) endowed with two derivations: the Cech-derivation and the canonical derivation of differentiable forms. We denote by \(H^{*,*}_{(U_i, H_i, \phi_i), i \in I}\) (N) the induced bigraded cohomology groups.

We say that the covering \((U_i', H_i', \phi_i')_{i' \in I'}\) is finer than the covering \((U_i, H_i, \phi_i)_{i \in I}\), if and only if for every \(i' \in I'\), there exists \(i \in I\) such that \(\phi_i(U_i') \subset \phi_i(U_i)\). This relation defines an inductive system on the set of coverings, the inductive limit of \(H^{*,*}_{(U_i, H_i, \phi_i), i \in I}\) (N) is the Cech-DeRham cohomology of the generalized orbifold.

5.2 Chen-Ruan cohomology for generalized orbifolds.

Suppose that the generalized orbifold \(N\) is compact. We are going to adapt the cohomology theory defined by Chen and Ruan [11] for orbifolds. Firstly we recall the following construction in Chen and Ruan (page 6-7): let \(N\) be an orbifold, \((U_x, H_x, \phi_x)\) a local chart at \(x\), define \(\hat{N}\) to be the set whose elements are \((x, (h_x))\), where \((h_x)\) is the conjugacy class of the element \(h_x\) of \(H_x\). Remark that \(\hat{N}\) is well-defined despite the use of local charts. The orbifold \(\hat{N}\) is not necessarily connected. Its connected components are called twisted sectors (Chen and Ruan p.8). There exists a natural surjection \(p : \hat{N} \rightarrow N\), the connected components of elements of \(p^{-1}(U_x)\) can be parameterized by the set of conjugacy classes \((h_x), h_x \in H_x\). Suppose that the orbifold is endowed with a pseudo-complex structure, which defines a representation \(\rho_{H_x} : H_x \rightarrow GL(n, \mathbb{C}) (n = dim_{\mathbb{C}}(N))\). For every element \(h_x\) in \(H_x\), \(\rho_{H_x}(h_x)\) depends only of the conjugacy class \((h_x)\) of \(h_x\) in \(H_x\), they define \(i_{x, h_x} = -\frac{1}{2\pi i} Log(det(\rho_{H_x}(h_x)))\). This enables Chen and Ruan to define the orbifold \(d\)-cohomology group:

\[ H^d(X) = \oplus H^{d-2i(h)}(X(h)). \]

Let \(N\) be a generalized compact orbifold, we can find a finite cover \((U_i, H_i, \phi_i)\) for the Grothendieck topology, such that each open subset \(U_i\) is defined by the slice theorem (see theorem 3.2.1), this is equivalent to saying that \(U_i\) is the quotient \(H_i \times_{H_i} V_i\) by \(H_i\) where \(H_i'\) is the stabilizer of an element \(x_i\) of \(U_i\), \(V_i\) is the quotient of the tangent space \(TU_i\) at \(x_i\), by the image of the infinitesimal action of \(H_i\) at \(x_i\) (see Audin [4] p. 15). Let \(C_i\) be an open subset of \(V_i\) invariant by \(H_i\). Then \((C_i, H_i, \phi_i')\) is a chart of the generalized orbifold, where \(\phi_i' : C_i \rightarrow C_i/H_i\) is the canonical projection. Thus for every element \(x\) in \(N\), there exists a chart \((U_x, H_x, \phi_x)\), \(x'\) in \(U_x\) such that \(\phi_x(x') = x\), and \(H_x(x') = x'\). We are going to consider only this type of charts in the sequel. The existence of such charts is related to the definition of holonomy of singular foliations. See Molino and Pierrot [33] p. 208, for the definition of the holonomy a foliation defined by the action of a compact Lie group, or Debord [12].
Let $H$ be a closed subgroup of $H_x$, we denote by $(H)$ the conjugacy class of $H$ in $H_x$. Let $y$ be an element of $U_x$, and $H^y_x$ the subgroup of $H_x$ which fixes $y$. We say that $y$ and $y'$ have the same type if and only if $(H^y) = (H_x^y)$. Let $(U_y, H_y, \phi_y)$ be a chart such that $H_y(y) = y$, denote by $\lambda_y : H_y \rightarrow H_x$ the morphism induced by the transition function $\phi_{xy}$. We suppose that the stabilizer of $\phi_{xy}(y)$ in $U_x$ is $\lambda_y(H_y)$. We can define:

$$\tilde{N} = \{(x,(H)), H \subset H_x, (H) = (H^y_x)\}$$

where $x \in N$, $(U_x, H_x, \phi_x)$ is a local chart at $x$. We denote by $H''_x$ the set of subgroups of $H_x$ which are type of an orbit, and by $H'_x$ the set of conjugacy classes of these subgroups. Remark that the argument in Audin [4] p. 17 proposition 2.2.3 implies that we can assume that the number of types of orbits contained in every chart is finite. The reunion $D_H$ of the orbits whose type is $(H)$ is a submanifold. The following proposition is shown for orbifolds by Chen and Ruan [11] p.7.

**Proposition 5.2.1.**

There exists a generalized orbifold structure on $\tilde{N}$. Let $(U_x, H_x, p_{H_x})$ be a chart of $N$, and $(H) \in H'_x$. We denote by $U^H$ the fixed point subset of $U$ by the action of $H$, and by $C(H)$ the normalizer of $H$ in $H_x$, then $((U^H, C(H)), C(H), \phi_H)$ is a chart of the generalized orbifold $\tilde{N}$, where $\phi_H : U^H \rightarrow U^H / C(H)$ is the natural projection.

**Proof.** Consider $(U_x, H_x, \phi_x)$ a chart at $x$. Let $y$ be an element of $\phi_x(U_x)$. Consider a chart $(U_y, H_y, \phi_y)$, such that $U_y$ contains an element $y'$ such that $\phi_y(y') = y$. The equivariant transition function $\phi_{xy} : p_y(U_x \times N U_y) \rightarrow p_x(U_x \times N U_y)$ is the canonical projection induces a morphism $\lambda_{y'} : H_y \rightarrow H_x$. Let $H = H^y_x$ and $h \in H$, the element $\lambda_{y'}(h)$ fixes $\phi_{xy}(c)$. We deduce a map $\Phi$ which associates to $(y, (H))$ the projection of $\phi_{xy}(y')$ in $\bigcup_{H = H''_x \in H'_x} U^H_x / H_x$, where an element $h$ of $H_x$ acts on $U^H_x / H_x$ by sending the element $c \in U^H_x$ to $b(c) \in U^H_x$. If instead of taking $H$, we take the element $H'$ in $(H)$, $H' = aHa^{-1}$, $\phi_{xy}(az) \in U^{\lambda_{y'}(H')} / H_x$, and $\Phi(y, (aHa^{-1}))$ is the projection to $\bigcup_{H = H''_x \in H'_x} U^H_x / H_x$ of $\phi_{xy}(y')$ in $U^H_x / H_x$.

If we take $y''$ such that $\phi_x(y'') = \phi_x(y')$, $y'' = by'$, $b \in H_x$, and $\phi_{xy}(bz) \in U^{b\lambda_{y'}(H')b^{-1}} / H_x$, and $\Phi(y, (H))$ is the projection of $y'' \in U^{b\lambda_{y'}(H')b^{-1}} / H_x$ to $\bigcup_{H = H''_x \in H'_x} U^H_x / H_x$. Thus the map $\Phi$ is well defined. This map is surjective; this can be shown by the fact that we can linearize the action of compact Lie group. It is injective: If $\phi(y, (H)) = \phi(y_1, (H_1))$, and $\Phi(y, (H))$ and $\Phi(y_1, (H_1))$ are the projections of $y'$ and $y'_1$ in $\bigcup_{H = H''_x \in H'_x} U^H_x / H_x$, there exists $a \in H_x$ such that $y'_1 = ay'$. This implies that $y = y_1$. The definition of $\Phi$ implies then that $(H) = (H')$. Remark that the image of the previous map is in bijection with $\bigcup_{(H) \in H'_x} U^H_x / C(H)$.

We endow $\tilde{N}$ with the topology topology generated by the image of the maps $U^H \rightarrow N$. The triples $(U^H_x, C(H), \phi_H)$ defines a covering atlas of the generalized orbifold where $\phi_H : U^H \rightarrow U^H / C(H)$ is the projection map $\bullet$. 

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Let \( \mathcal{H} = \mathcal{H}_z \), \( U^H/H \) is an open subset of a suborbifold of \( \hat{N} \) completely determined by \( (\mathcal{H}) \) if \( N \) is connected that we denote \( \mathcal{N}_H \).

Consider a pseudo-complex structure defined on \( \mathcal{C}_N \), this is equivalent to suppose that each chart is endowed with a pseudo-complex structure, and morphisms in \( \mathcal{C}_N \) preserve pseudo-complex structures. Consider a chart \( (U_x, H_x, \phi_x) \). For every and \( (H) \) in \( \mathcal{H}'_z \), we define \( 2i(H) = \text{dim}_C(U_x) - \text{dim}_C(N_H) \).

We can define:

\[
H^d(N) = \oplus H^{d-2i(H)}(N_H).
\]

6 Sheaf of categories and gerbes in differentiable categories.

Recall that if \( C \) is a differentiable category endowed with a topology, \( U \) an object of \( C \) and \( R \) a sieve in \( J(U) \). The forgetful functor from \( R \) to \( C \) which sends a map \( V \to U \) to \( V \) is Cartesian.

**Definition 6.1.**

Let \( F : \mathcal{P} \to C \) be a differentiable fibered functor, where the category \( C \) is equipped with a Grothendieck topology, we say that \( F \) is a sheaf of categories, if for every object \( U \) of \( C \), and for every sieve \( R \in J(U) \), the natural restriction map:

\[
\text{Cart}_C(C \uparrow U, F) \to \text{Cart}_C(R, F)
\]

is a 2-descent map, otherwise said, an equivalence of categories. (See Giraud [14])

The sheaf of categories is called a gerbe bounded by the sheaf \( H \) if the following conditions are satisfied:

- \( F \) is locally connected: this is equivalent to saying that for every object \( U \) of \( C \), there exists a sieve \( R \in J(U) \) such that for every map \( V \to U \in R \), the objects of the fiber \( P_V \) of \( V \) are isomorphic each other.

There exists a sheaf in groups \( H \) defined on \( (C, J) \) such that for every object \( U \in C \), and \( e_U \in P_U \) the group \( \text{Aut}_U(e_U) \) of automorphisms of \( e_U \) over the identity of \( U \) is isomorphic to \( H(U) \), and these family of isomorphisms commute with morphisms between objects and restrictions. The sheaf \( H \) is called the band of the gerbe.

Let \( (C, J) \) be a site, two fibered categories \( F_i, i = 1, 2 : C_i \to C \) are equivalent, if there exists a Cartesian isomorphism between \( C_1 \) and \( C_2 \).

An equivalence between the gerbes \( F_i, i = 1, 2 : P_i \to C \) is a Cartesian isomorphism which commutes with their bands. Let \( H \) be a sheaf defined on the differentiable site \( (C, J) \), we denote by \( H^2(C, H) \) the set of equivalences classes of \( H \)-gerbes. This set is often called the non-abelian 2-cohomology group of the sheaf \( H \).
6.1 The classifying cocycle of a gerbe.

Suppose that the differentiable category $C$ has inductive limits, finite projective limits, a final and initial object.

Let $R$ be a covering of the final object $e$. We suppose that $R$ is a good covering, that is every gerbe defined on an object $X_i$ of $C$ such that there exists a map $X_i \to e$ in $R$ is trivial and connected.

Let $F : P \to C$ be a gerbe, and $e_i$ an object of the fiber $P_{X_i}$. There exists an isomorphism:

$$u_{ij} : e_j^i \to e_i^j$$

We denote by $c_{ijl}$ the isomorphism $u_{ij} \circ u_{jl} \circ u_{li}^j$.

We have the relation:

$$c_{i2}^{i1} c_{i1}^{i3} c_{i3}^{i4} u_{i1}^{i2} u_{i3}^{i4} = c_{i1}^{i2} c_{i2}^{i3} c_{i3}^{i4} u_{i1}^{i2} u_{i3}^{i4}$$

The family of 2-chains $c_{i1,i2,i3}$ which satisfies the relation above is called a non-abelian 2-cocycle. Giraud [14] has shown that there exists a 1 to 1 correspondence between the set of gerbes bounded by $H$ and non abelian $H$ 2-cocycles (see also the proof in Brylinski [8] p. 200-203 for commutative gerbes).

7 Examples of sheaf of categories and gerbes.

The differentiable category $C_H$ which represents the geometric obstruction to extend the structural group of a principal bundle is a gerbe (see page 17).

Recently, Lupercio and Uribe [21] have introduced Abelian gerbes on orbifolds. For an orbifold $N$, we can define a gerbe on $N$ to be a gerbe defined on the Grothendieck site $J_N$ (see p. 20).

7.1 Gerbes and $G$-structures.

We are going to define a fundamental example of a gerbe, that we are going to apply to the study of $G$-structures. Let $G$ be a Lie group, and $H$ a closed subgroup of $G$. Consider a principal $G$-bundle $p : P \to N$ over the manifold $N$. A natural question is to ask wether the bundle has an $H$-reduction, that is wether there exists coordinates change which take their values in $H$. This problem is equivalent to the following question: Consider the bundle $p' : P' \to N$ whose typical fiber is the homogeneous space $G/H$ obtained by making the quotient of each fiber of $p$ by $H$. Is there exists a global section of $p'$?(see Albert and Molino [2] p. 64). We have the following:

**Proposition 7.1.1.**

The correspondence defined on the category of open subsets of $N$, which assigns to every open subset $U$ the category $C_H(U)$, whose objects are $H$-reductions of the restriction of $p$ to $U$, and whose morphisms, are morphisms of $H$-bundles is a sheaf of categories.
**Proof.** Gluing condition for objects.

Let \((U_i)_{i \in I}\) be an open covering of \(U\), \(e_i\) an object of \(C_H(U_i)\) such that there exists a morphism \(u_{ij} : e_j \to e_i\) such that \(u_{ij} u_{ij} = u_{ji}\). Then there exists an object \(e\) in \(C_H(U)\) whose restriction to \(U_i\) is \(e_i\), since we can glue \(H\)-bundles.

Gluing conditions for arrows:

Let \(e\) and \(e'\) be two objects of \(C_H(U)\), the correspondence which assigns to every open subset \(V\) of \(U\), \(\text{Hom}(C_H(U)(e|_V, e'|_V))\) is a sheaf, since it is the sheaf of morphisms between two bundles.

This sheaf of categories can be applied to the following situation: suppose that \(N\) is a \(n\)-dimensional manifold. Let \(R_p(N)\) be the bundle of \(p\)-linear frames of \(N\), and \(G\) a subgroup of \(Gl_p(n, \mathbb{R})\) the group of invertible \(p\)-jets of \(\mathbb{R}^n\). The geometric obstruction of the existence of a \(G\)-structure on \(N\) is defined by the sheaf of categories that we have just defined.

Let \(U\) be a contractible open subset of \(N\), \(C_G(U)\) is not empty, since the restriction of \(P\) to \(U\) is a trivial bundle. But the objects of \(C_G(U)\) are not always isomorphic: suppose that \(N\) is a \(n\)-dimensional manifold, and take \(G = O(n, \mathbb{R})\); the \(G\)-reductions of the bundle of linear frames \(R(N)\) of \(N\) define the differentiable metrics. It is well-known that two differentiable metrics are not locally isomorphic if their curvatures are distinct.

A particular situation is the example of flat \(G\)-structures like symplectic structures (see Albert and Molino [2] p. 177). For every elements \(x\) and \(y\) in \(N\), there exists neighborhoods \(U_x\) and \(U_y\) of \(x\) and \(y\) in \(N\), and a diffeomorphism \(h : U_x \to U_y\) which preserves the \(G\)-structures induced by \(N\) on \(U_x\) and \(U_y\). If \(U\) is contractible open subset of \(N\), two elements of \(C_G(U)\) are connected. The theory of gerbes and \(G\)-structures, will be intensively studied in [42].

### 7.2 Gerbes and invariant scalar product on Lie groups.

Another example of gerbes can be described as follows: consider a Lie group \(H\) which is not commutative, and \(L\) a lattice in \(H\). Consider the manifold \(H/L\), and let suppose that \(H\) is endowed with an orthogonal bi-invariant metric: this is equivalent to the existence of a scalar product \(<,>\) (i.e a non-degenerated real valued bilinear form not necessarily positive definite) on the Lie algebra \(\mathcal{H}\) of \(H\) such that for every elements \(x, y, z \in \mathcal{H}\):

\[
< [x, y], z > + < y, [x, z] > = 0.
\]

The 3-invariant form \(\nu\) defined on the Lie algebra \(\mathcal{H}\) of \(H\) by:

\[
\nu(x, y, z) = < [x, y], z >
\]

defines on \(H/L\) a closed 3-form \(\nu_L\).

The space of bilinear symmetric forms on \(\mathcal{H}\) corresponds to real 3-cocycles as shows Koszul [17] p. 95. Medina [26] has shown that the dimension of this space is either 1 or 2.
Let $H$ be a $n$-dimensional nilpotent Lie group, and $L$ a lattice of $H$. Recall that there exists a basis $e_1, ..., e_n$ of the Lie algebra $\mathcal{H}$ of $H$, such $[e_i, e_j] = \sum_{ijl} c_{ijl} e_l, c_{ijl} \in \mathbb{Q}$ (see Raghunathan [35] p. 34). We say in this situation that the constants of structure are rational. A lattice $L$ is the image of $\mathbb{Z}e_1 \oplus ... \oplus \mathbb{Z}e_n$ by the exponential map.

**Proposition 7.2.1.**

Under the notations above, if there exists an invariant scalar product $<,>$, such that $<e_i, e_j> \in \mathbb{Q}$, then the 3-form $\nu$ on $H/L$ induces canonically a rational 3-form $\nu_L$ on $H/L$.

**Proof.** The proof uses a theorem of Nomizu quoted in [35] Raghunathan p. 123 in the real case. Let $H_0$ be the center of $H$, the intersection $L_0 = L \cap H_0$ is a lattice in $H_0$. (If $H$ is commutative, we take $H_0$ to be a non trivial subgroup different of $H$. See Raghunathan [35] p. 40). Thus the foliation of $H/L$ whose leaves are orbits of $H_0$, has compact leaves. The space of leaves of this foliation is the quotient $M$ of $H/H_0$ by $L/L_0$. The natural projection $H/L \rightarrow M$ is a fibration whose fibers are $n$-dimensional torus $T^n$, where $n$ is the dimension of $H_0$.

We can apply the Leray-Serre spectral sequence to this fibration for the rational cohomology we obtain:

$$E_2^{p,q} = H^p(M, H^q(T^n, \mathbb{Q})) \simeq H^p(M, \Lambda \mathbb{Q}^q),$$

$$E_\infty^{p,q} \Rightarrow H^{p+q}(N, \mathbb{Q}).$$

Consider $\hat{E}_2^{p,q}$ the Leray-Serre spectral sequence associated to the space of $H$-invariant forms on $N$ and $M$, we have:

$$\hat{E}_2^{p,q} = H^p(\mathcal{H}/\mathcal{H}_0, \Lambda \mathbb{Q}^q),$$

$$\hat{E}_\infty^{p,q} \Rightarrow H^{p+q}(\mathcal{H}, \mathbb{Q}).$$

The recursive hypothesis implies that $H^*(M, \mathbb{Q}) = H^*(\mathcal{H}/\mathcal{H}_0, \mathbb{Q})$. This implies that $H^*(N, \mathbb{Q}) = H^*(\mathcal{H}, \mathbb{Q})$. The image of $\nu$ by the isomorphism $H^3(\mathcal{H}, \mathbb{Q}) \rightarrow H^3(N, \mathbb{Q})$ is the form $\nu_L$. The result of Koszul [17] p. 95 shows that we can realize this form by using an invariant bilinear form $\bullet$.

The classification theorem of Giraud [14] implies the existence of a gerbe over $H/L$ whose classifying class is the cohomology class of $p\nu_L$, where $p$ is an integer. We call such a gerbe, a Medina-Revoy gerbe.

**Examples of Medina-Revoy gerbes.**

Lie groups endowed with bi-invariant scalar product have been intensively studied by Aubert, Dardie, Diatta, Medina and Revoy. Medina and Revoy [27] have shown that they can be constructed from simple Lie groups and the 1-dimensional Lie group by the processus of double extension. Here is an example
of a Medina Revoy gerbe constructed from the double extension of the two dimensional Euclidean space, endowed with its commutative structure of a Lie algebra.

Consider the nilpotent Lie algebra constructed as follows: Let \((e_1, e_2)\) be an orthogonal basis of the 2-dimensional Euclidean space \((U, \langle, \rangle)\), and \(h : U \to U\) the linear endomorphism such that \(h(e_1) = e_2, h(e_2) = 0\) considered also as a derivation of the trivial underlying Lie algebra of \(U\). Let \(V\) be the 1-dimensional commutative Lie algebra, and \(V^*\) its dual. For every elements \(u_1, u_2 \in U\), we denote by \(w(u_1, u_2) : V \to V^*\) the map which assigns to \(v \in V = R\) the scalar \(\langle vh(u_1), u_2 \rangle\). The double extension of \((U, \langle, \rangle)\) by \(V\) and \(h\) is the nilpotent Lie algebra \(\mathcal{L} = V^* \oplus U \oplus V\) whose bracket is defined by the formula:

\[
[(v'_1, u_1, v_1); (v'_2, u_2, v_2)] = (w(u_1, u_2), v_1 h(u_2) - v_2 h(u_1), 0)
\]

The Lie algebra \(V^* \oplus U \oplus V\) is endowed with the scalar product:

\[
\langle (v'_1, u_1, v_1); (v'_2, u_2, v_2) \rangle' = \langle u_1, u_2 \rangle + v_1 v_2 + v'_1(v_2) + v'_2(v_1)
\]

The constant of structures of \(\mathcal{L}\) are integral in its canonical basis. The 3-form \(\nu_\mathcal{L}\) defined on \(L\) by \((u, v, w) \to \langle [u, v], w \rangle'\) is rational.

Let \(\Gamma\) be the lattice of the 1-connected Lie group \(L\) associated to \(\mathcal{L}\) which is generated by the image of an integral basis of \(\mathcal{L}\). The classification theorem of Giraud implies the existence of a gerbe on \(L/\Gamma\) whose classifying cocycle is \(p \nu_\mathcal{L}\), where \(p \in N\) is such that \(p \nu_\mathcal{L}\) is integral.

### 7.3 A sheaf of categories on an orbifold with singularities.

Let \(H\) be a compact Lie group which acts on a manifold, the quotient space \(N/H\) is an example of a generalized orbifold \(C_N\) (see definition 3.2.1).

**Proposition 7.3.1.**

Let \(N\) be a generalized compact orbifold. The correspondence defined on the category of open subsets of \(N\) which assigns to \(U\) the category \(C_N(U)\), whose objects are elements \((P, H_P, \phi_P)\) of \(C_N\), such that the image of \(\phi_P\) is \(U\) is a sheaf of categories.

**Proof.** Gluing conditions of objects.

Let \(U_i\) be an open subset of \(N\), and \((U_i)_{i \in I}\) an open covering of \(U\). Consider for each \(i \in I\), an object \(e_i = (P_i, H_i, \phi_i)\) in \(C_N(U_i)\), and a morphism \(u_{ij} : e_j^i \to e_i^j\) such that \(u_{ij}^* u_{ij} = u_{ij}^*\). Since the morphisms \(u_{ij}\) are local diffeomorphisms, there exists a manifold \(P\) obtained by gluing the family of manifolds \(P_i\) with \(u_{ij}\). We can glue the Lie groups \(H_i\) and their actions to define a Lie group \(H\) which acts on \(P\), and such that the map \(P/H \to N\) is a local homeomorphism.

Let \(l_i\) be the restriction of the action of \(H_i\) to \(p_i(P_i \times N P_j)\). We can identify \(p_i(P_i \times N P_j)\) with \(p_j(P_i \times N P_j)\) with \(u_{ij}\). We denote \(H_{ij}\) the limit of the maps \(l_i\) and \(l_j\). The Lie group \(H_{ij}\) acts on the gluing of \(P_i\) and \(P_j\) by \(u_{ij}\). Without restricting the generality, we can suppose that \(I\) is a numerable set, construct
$H_{01}, H_{01..n}$ obtained by gluing recursively the action of $H_0, ..., H_n$. The Lie group $H$ is the limit of the groups $H_{01..n}$.

Gluing condition of arrows.

Let $U$ be an open subset of $N$, and $P$, and $P'$ two objects of $C_N(U)$. The correspondence defined on the category of open subsets of $U$, which assigns to $V$ the set $Hom_{C_N(V)}(P|_V, P'|_V)$, where $P|_V$ is $\phi^{-1}_P(V)$ is a sheaf since we can glue differentiable maps.

Another example of sheaf of categories is defined by the theory of foliages (see Molino [31], or definition 3.3.1). Let $N$ be a topological manifold endowed with a structure of a foliage. A natural problem is to determine whether this foliage is induced by a foliation on a manifold. The following proposition provides the obstruction which solves this problem.

**Proposition 7.3.2.**

Let $N$, be a topological space endowed with the structure of a foliation, for every open subset $U$ of $N$, denote by $C_N(U)$ the class of objects $(U, V, F, \pi)$ of $C_N$. The correspondence which assigns $C_N(U)$ to $U$ is a sheaf of categories which is the geometric obstruction of the existence of a manifold $\hat{N}$, endowed with a foliation $F_N$, such that $N$ is the space of leaves of $F_N$.

**Proof.** Gluing conditions of objects.

Let $U$ be an open subset of $N$, $(U_i)_{i\in I}$ an open covering of $U$ such that for each element $i$ of $I$, there exists an object $e_i = (U_i, V_i, F_i, \pi_i)$ in $C_N(U_i)$, morphisms $u_{ij} : e_i^j \to e_i^j$ such that $u_{ij}^l u_{ij}^l = u_{ij}^l$. The morphisms $u_{ij}$ allow to glue the family of manifolds $V_i$ to obtain a manifold $V$, on which is defined a foliation $F$ whose restriction to $U_i$ is $F_i$.

Gluing condition for arrows.

Let $e = (U, V, F, \pi)$ and $e' = (U', V', F', \pi')$ two objects of $C_N(U)$. The correspondence defined on the category of open subsets of $U$, which assigns to $U'$ the set $Hom_{C_N}(e|_U, e'|_U)$, is a sheaf, since we can glue differentiable foliated maps.

There exists a sheaf $L$ on $N$ which assigns to every open subset $U$ of $N$, the set of isomorphisms of an object $e_U$ of $C_N(U)$ (the foliated isomorphisms), and the sheaf of categories that we have just defined is a gerbe bounded by $L$.

**8 Differential geometry of sheaves of categories.**

In this part, we are going to analyze the tools defined in the general context of differentiable categories to study their geometry by using the underlying topology.

Let $C$ be a differentiable category endowed with a topology. We suppose that $C$ has a final object and a good global covering $(U_i)_{i \in I}$ (see p. 24). Let $P \to C$ be a gerbe, we suppose that there exists a Lie group $H$, a principal $H$-torsor $A : Aut(P) \to C$, such that every object $e_U \in P_U, U \in C$ is a bundle $p_{e_U} : e_U \to U$ endowed with a free right action of $H$. The set of morphisms
between two objects of $P_U$ are morphisms between bundles which project to the identity of $U$, and the set of automorphisms of $e_U$ can be identified with gauge transformations of $\text{Aut}(P)(U)$ by a map which commutes with morphisms between objects and with restrictions. We denote by $\text{aut}(P)$ the vector $\mathcal{H}$-bundle on $C$ associated to $\text{Aut}(P)$: If the coordinate changes of $\text{Aut}(P)$ are defined by the maps $(u_{ij})_{i,j \in I}$, the coordinate changes of $\text{aut}(P)$ are defined by the map $(\text{Ad}(u_{ij}))_{i,j \in I}$. Such a gerbe is called a $H$-gerbe.

8.1 Induced gerbes.

Let $p : P \to C$ be an $H$-principal gerbe, that is: for every object $U$ of $C$, the map $p_{UV} : e_U \to U$, endows $e_U$ with the structure of a $H$-principal bundle.

Consider a morphism of Lie groups $h : H \to H'$, we can construct a principal $H'$-gerbe $p' : P' \to C$ as follows:

Let $a : A \to U, U \in C$ be an object of $P_U$, it is a $H$-principal torsor defined by the trivialization $(U, u_{ij} \in H)_{i,j \in I}$. We can define the image of $a$ by $h$. It is the torsor whose coordinates change are defined by: $(U, h(u_{ij}))_{i,j \in I}$. The family of these images is the induced gerbe.

An example is the situation when $H$ is $SU(n)$ or $O(n)$, and $h$ is the determinant morphism.

**Proposition 8.1.**

There exists a connective structure on each $H$-gerbe $p : P \to C_N$, where $N$ is a manifold, and $C_N$ the differentiable category associated to $N$ defined at p.5.

**Proof.** Let $(U_i)_{i \in I}$ be an open cover of $N$, and $e_i$ an object of $P_{U_i}$, we denote by $e'_i$ the quotient of $e_i$ by the action of $H$.

Consider an isomorphism $u_{ij} : e'_j \to e'_i$. We can construct from $u_{ij}$ a morphism $u'_{ij} : e'_{ij} \to e'_{ij}$ which is its quotient. With these morphisms, we can glue the family of quotients $(e'_i)_{i \in I}$ to define a fiber bundle $p' : P' \to N$.

Consider a connection on $p'$: this is a distribution $\mathcal{D}'$ of $P'$ whose rank is the dimension of $N$, and which is transverse to the fiber of $p'$ (compare with McDuff and Salamon [23] p. 210). The distribution $\mathcal{D}'$ can be also defined by a 1-form $\theta$ on $P'$ which takes its values in the canonical bundle over $P'$, such that for each $x \in P'$, the fiber of this bundle is the tangent space of $P'_{x}$, the fiber at $p'(x)$. We suppose that if $v \in T_xP'$, $x \in U, \theta(v) = v$. Such a distribution can be constructed by using a differentiable metric on $P'$, and by taking the orthogonal of the fiber.

Let $U$ be an open subset of $N$, and $e_U$ an object of $P_U$. We denote by $C(e_U)$ the set of 1-forms defined on the bundle $e_U \to U$ which take their values in the canonical vector bundle over $U$ whose fiber at $x$ is the tangent space of the fiber of $e_U \to U$ at $x$. We suppose that for every $\alpha \in C(e_U)$, and every element $A \in \mathcal{H}$ which generates the fundamental vector field $A^*$, we have:

$$\alpha(A^*) = A.$$
We suppose also that the each element of \( Co(e_U) \) descends to the restriction of \( \theta \) to \( U \).

Such a connection can be constructed as follows: Let \( (U_i)_{i \in I} \) be a trivialization of \( e_U \rightarrow e_U/H \). We can define on \( U_i \times H \) a distribution invariant by \( H \) whose projection to \( U_i \) is the restriction of \( D' \). Such a distribution is defined by a 1-form \( \alpha_i \) whose projects to the restriction of \( \theta \) to \( U_i \). By using a partition of unity, we deduce that \( Co(e_U) \) is not empty.

### 8.2 Reduction to a situation similar to the motivating example.

Now we reduce the study of the differential geometry of a \( H \)-gerbe to a situation similar to the motivating example (see p. 17) by using the following construction. Consider a \( H \)-gerbe \( p : P \rightarrow N \). We have seen that there exists a fiber bundle \( p' : P' \rightarrow N \) such that for a good covering \( (U_i)_{i \in I} \) of \( N \), the restriction of \( P' \) to \( U_i \) is the quotient of every object \( e_i \) of \( P_{U_i} \) by \( H \). Let \( F \) be the fiber of this bundle. We can define the bundle \( L(p') : L(F) \rightarrow N \) such that for every element \( u \in N \) the fiber of \( L(F) \) at \( u \) is the set of linear frames of its fiber, \( F_u \). Let \( U \) be an open subset of \( U \), and \( e_U \) an object of \( P_U \). We can define the pullback of the maps \( L(F)|_U \rightarrow e_U/H \) and \( e_U \rightarrow e_U/H \) which is an \( H \)-principal bundle \( e'_U \) over the restriction \( L(F)|_U \). The class \( P'_U \) whose elements are the \( e'_U \) just constructed defines a gerbe \( L(P) \rightarrow N \). This allows to deal only with to a situation similar to the motivating example in supposing that the bundle \( P' \rightarrow N \) is principal. In fact in the sequel, we deal only with the motivating example. Moreover we suppose that connective structures are constructed by using connections of \( P' \) like at the proposition 8.2.1.

Another variant of the previous construction is the following: let \( p : P \rightarrow N \) be a \( H \)-principal bundle defined on the manifold \( N \), to each cocycle \( c \in H^2_{Cech}(N,H) \) we can associate a gerbe \( C \) whose classifying cocycle is \( c \) (see Giraud [14]). The previous construction allows to construct a gerbe \( R(C) \) above the frames bundle of \( N \), such that an object of \( R(C)(U) \) is the pull back of an object of \( C(U) \) by the projection map \( R(U) \rightarrow U \), where \( R(U) \) is the bundle of linear frames of \( U \). Thus we can define connective structures above connections of \( R(N) \).

**Definition 8.2.1.**

A curving (see also Brylinski [8] p. 211) defined on the connective structure \( Co \) of the \( H \)-gerbe \( p : P \rightarrow C \), is a map which assigns to every object \( e_U \) of \( P_U \), and every element \( \theta \in Co(e_U) \), a 2-form \( L(e_U, \theta) \) which takes its values in \( H \) such that the following properties are satisfied:

Let \( h : e_U \rightarrow e_U' \) be a morphism, for every \( \theta \in Co(e_U') \), we have:

\[
L((e_U), h^*(\theta)) = h^*(L(e_U, \theta)).
\]

If \( h \) is an automorphism of \( e_U \), and \( \theta \) an element of \( Co(e_U) \), we have:

\[
L(e_U, h^*(\theta)) = Ad(h^{-1})(L(e_U, \theta))
\]
Let $\alpha$ be an element of $\Omega^1(U, \text{aut}(U))$, we have:

$$L(e_U, \theta + \alpha) = L(e_U, \theta) + d(\alpha) + \frac{1}{2}([\alpha, \alpha] + [\theta, \alpha] + [\alpha, \theta]).$$

The correspondence $(e_U, \theta) \mapsto L(e_U, \theta)$ is natural in respect to restrictions and morphisms between objects.

Remark that since the gerbe considered here is associated to the motivating example, the 2-form $L(e_U, \theta + \alpha) - L(e_U, \theta)$ is $\mathcal{H}$-valued.

**Proposition 8.2.1.**

Let $Co$ be a connective structure defined on the $H$-principal gerbe $p : P \to C$, there exists a curving.

**Proof.** Compare the following proof with Brylyinski [8] p. 212). We are going to assume that there exists a $L'$-bundle $p' : P' \to N$ such that for every open subset $U$ of $N$, $e_U$ an object of $P_U$, the elements of $Co(e_U)$ are connections which projects to $\alpha$.

Let $\mathcal{H}$, $L$, and $L'$ be the respective Lie algebras of $H$, $L$ and $L'$. Let $u$ be a linear section of the canonical map $L \to L'$. We can define the form $\alpha_0 = u \circ \alpha$ on $P'$. Let $\theta$ be an element of $Co(e_U)$, consider the form $\alpha e_U$, the pull-back of the restriction of $\alpha_0$ by the canonical projection $e_U \to e_U/H = P'_U$. We set:

$$L(e_U, \theta) = d\theta + \frac{1}{2}[\theta, \theta] - (d\alpha e_U + \frac{1}{2}[\alpha e_U, \alpha e_U]).$$

•

**Definition 8.2.2.**

Let $L$ be a curving of the connective structure $Co$ defined on the $H$-gerbe $P \to C_N$, where $N$ is a manifold, and $C_N$ the canonical differentiable category associated to $N$ (see page. 5). Let $(U_i)_{i \in I}$ be a good covering of $N$, and $e_i$ and object of $P_{U_i}$, and $\alpha_i$ an element of $Co(e_i)$ and $u_{ij} : e_j' \to e_i'$ a morphism.

Let $\Omega_i$ be the curvature of $\alpha_i$. On $U_i \cap U_j$, the difference $\Omega_j^{ij} - u_{ij}^* \Omega_i^j$ defines a 1-Cech $\text{aut}(P)$-cocycle. The cohomology class does not depend of the elements $e_i$ in $P_{U_i}$ and of the elements $\alpha_i$ used to define it, since $Co(e_i)$ is a torsor whose vector space is a space of $\mathcal{H}$-valued 1-forms.

The DeRham-Cech isomorphism allows to identify this cocycle to a $3 - \text{aut}(P)$ form $D$ called a curvature of the connective structure.

**Definition 8.2.3: Holonomy.**

We are going to reduce this definition to the commutative case, all the groups are compact and complex, as well as the vector bundles. Let $p : P \to C$ be a differentiable $H$-gerbe, endowed with a connective structure and a curving $L$. 

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Suppose that $P$ is the geometric obstruction to lift a $G$-bundle $P'$ over $N$ to a $G'$-bundle given the exact sequence of Lie groups $1 → H → G' → G → 1$. We assume also that in fact, the $G$-bundle $P'$ is the frames bundle of a vector bundle $V$ over $N$, and the objects of $P$ are also associated to vector bundles.

Let $N$ be a surface. Consider a morphism $h : N → C$. We can pull-back $p$ by $h$ and obtain a gerbe $p_N : P_N → N$, endowed with a connective structure which has a curving. The quotient of $p_N$ by $H$ is the pull-back $p'_G : P'^G_N → N$ of $P'$ by $h$, which is a reduction of the frames bundle of the pull-back $V_N$ of $V$ by $h$. A well-known result implies that $V_N$ is isomorphic to the summand of complex line bundles (see McDuff and Salamon [23] p. 80). Thus we can assume that the structural group of $P_N$ is a commutative subgroup $G_N$ of $G$.

The gerbe $p_N$ is also the geometric obstruction to extend the structural group $G_N$ to $G'_N$ given an exact sequence of Lie groups $1 → H_N → G'_N → G_N → 1$. This implies that we can assume also that $H_N$ and $G'_N$ are commutative.

The pull-back of the connective structure and the curving of $p$, induces a connective structure $CoN$ of $pN$ and a curving $L_N$, moreover the connection $α$ used to construct the connective structure of $p$ (see proposition 8.2.1) is supposed to be Hermitian, as well as the elements of $Co(e_U)$, where $e_U$ an object of the gerbe. Thus the connection $α_N$ on $P'^G_N$, which induces the connective structure $CoN$ preserves every Hermitian reduction. We just have to recall the definition of the holonomy in the commutative case.

Let $(U_i)_{i ∈ I}$ be a good cover of $N$. Let $e_i$ be an object of $P_{NU_i}$, and let $θ_i$ be an element of $CoN(e_i)$, we can suppose that this connection takes its values in the Lie algebra of $G'_N$. We denote by $L(e_i, θ_i)$ the curving associated to the element $θ_i ∈ Co(e_i)$. Denote by $θ_{ij}$ the form $θ_i^j − u_{ij}^i(θ_i^1)$. Since $N$ is 2-dimensional, there exists a 1-form $h_i$ such that $dh_i = L(e_i, θ_i)$. We have:

$$θ_{ij} = h_j − h_i + da_{ij}$$

We can set

$$d_{ij} = c_{i,j}a_{ij}^{-1}a_{i}a_{ij}^{-1}$$

where $c_{ij}$ is the classifying cocycle of $p_N$. The chain $d_{ij}$ is the holonomy cocycle of the gerbe $p_N$ endowed with its connective structure.

The Čech-DeRham isomorphism allows to identifies this form with a 2-form $Ω$ on $N$. (See also Mackaay and Picken [22] p. 27).

### 8.3 Holonomy and functor on loops space.

Consider the category $C^2$ whose objects are maps: $h : C^1 + .. + C^1 → N$, where $C^1 + .. + C^1$ is a finite disjoint union of circles. A morphism between the objects $h$ and $h'$, is a map from a surface $l : N → C$ such that the restriction of $l$ to the boundary of $N$ is the sum of the maps $h$ and $h'$. The holonomy defines a functor $D$ on $C^2$ which associates to $h$ the complex line $C$. Let $l : N → C$ be a morphism between $h$ and $h'$. The real holonomy around $N$ is the image $D(l)$ of $l$ by $D$. 

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We suppose here that the structural group $H$ of the $H$-gerbe $P \to N$ defined over the manifold $N$ and endowed with a connective structure is contained in $\text{Gl}(n, \mathbb{C})$. We can relate this gerbe which is associated determinant gerbe as follows:

**Proposition 8.3.1.**

Suppose that $H$ is included in $\text{Gl}(n, \mathbb{C})$, then the trace of the curvature of a principal $H$-gerbe $P \to N$ is the curvature of the associated determinant gerbe $\text{det}(P) \to N$; that is the gerbe induced by the determinant morphism $H \to \mathbb{C}$. (See 8.1).

**Proof.** Let $p : P' \to N$ be the quotient of the gerbe by $H$. For every open subset $U$ of $N$, the objects of $P_U$ are $H$-principal bundles over $P'_U$, the restriction of $P'$ to $U$.

Let $\text{det}_{e_U} : e_U \to \text{det}(e_U)$ the determinant morphism which associates to the object $e_U$ of $P_U$, the corresponding object $\text{det}(e_U)$ in $\text{det}(P)$, and $\alpha \in \text{Co}(e_U)$ whose kernel defines the distribution $C_\alpha$ on $e_U$. Since $\text{det}_{e_U}$ is an equivariant morphism between the $H$-bundle $e_U \to P'_U$ and $\text{det}(e_U)$, the image of $C_\alpha$ is a distribution $\text{det}(C_\alpha)$ invariant by the action of $U(1)$ on $\text{det}(e_U)$ and transverse to the fiber. Such a construction thus defines a connective structure on the determinant gerbe.

Suppose that the form $\alpha$ is $L \oplus \text{gl}(n, \mathbb{C})$-valued, where $L$ is the Lie algebra of the structural group of $P'$. Then the connection form which defines the distribution $\text{det}(C_\alpha)$ is the composition of $\alpha$ and $(I_L, \text{trace}_{\text{gl}(n, \mathbb{C})})$. This implies that the curving of $(\text{det}(e_U), C_{e_U})$ is the trace of the curving $L$. This implies the result.

### 8.4 Canonical relations associated to a connective structure on a gerbe.

In this part we are going to determine canonical relations associated to a connective structure. (compare with Breen and Messing [7] p. 58) The morphisms $u^*$ are pull-back, and the morphisms the $u^*$ are inverse of pull-back. Let $e_i$ be an object of $P_U$, and $\alpha_i \in \text{Co}(e_i)$. Consider the restriction $e_j^i$ of $e_j$ to $U_{ij}$ and $u_{ij} : e_j^i \to e_i^j$ an arrow. The 1-form $u_{ij,\iota}(\alpha_j)$ is an element of $\text{Co}(e_i^j)$, since $\text{Co}(e_i^j)$ is a torsor, there exists a 1-form $\alpha_{ij}$ such that:

$$\alpha_i^j = u_{ij,*}(\alpha_i^j) + \alpha_{ij}$$

We have seen that the family of forms $\alpha_{ij}$ verifies the equations:

$$u_{ij,*}\alpha_{jl} - \alpha_{il} + \alpha_{ij} = u_{it,*}(\alpha_i^{lj} - \text{Ad}(c_{ijl}^{-1})(\alpha_i^{lj})) - c^{-1}_{ijl} dc_{ijl}$$

where $c_{ijl}$ is the map $u_{ij*}u_{ij*}u_{ij*}$.

Let $L$ be a curving of the connective structure. Denote by $L_{ij}$ the 2-form $L_{ij}(e_i^j, \alpha_j) - L_{ij}(e_i^j, u_{ij,*}(\alpha_i))$. We have:
Another treatment of the differentiable structure on gerbes can be done as follows:

8.5 Uniform distributions and gerbes.

Another treatment of the differentiable structure on gerbes can be done as follows: Let \( p : P \to N \) be a \( H \)-gerbe defined on a manifold. We reduce the study to the motivating example. The natural way to study the differential geometry of a principal bundle is to use the theory of connections. Unfortunately, connections defined on a principal bundle are not necessarily invariant by the gauge group. This motivates the definition of a torsor of connections, which is invariant by the automorphisms group.

There exists another point of view used by Molino in his thesis (see [29]). Molino has studied the notion of invariant distributions on principal bundles. An invariant distribution on a principal bundle is a right invariant distribution. We do not request here that the dimension of the distribution is the dimension of the basis space of the bundle. The invariant distribution is transitive, if its summand with the tangent space of the fiber, generates the tangent space of the bundle, (see Molino [29] p. 180), transitive distributions are nothing but equivalence classes of connections. We focus on the motivating example of a \( H \)-gerbe \( P \to N \); there exists an exact sequence of Lie groups \( 1 \to H \to L' \to L \to 1 \), a \( L \)-principal bundle \( P' \to N \), such that the gerbe \( P \to N \) is the geometric obstruction to lift the structural group of the previous principal bundle to \( L' \). The objects of \( P_U \) are principal \( H \)-bundles over the restriction of \( P' \) to \( U \). For a connection \( \theta \) defined on \( P' \), we can define on each object \( e_U \) of \( P_U \), the transitive distribution which is the kernel of the pull-back of \( \theta_{P'\mid U} \) to \( e_U \). This transitive distribution is invariant by the automorphisms of \( e_U \).

Suppose now that the extension which defines the lifting problem is central. Let \( \mathcal{H}, \mathcal{L}' \) and \( \mathcal{L} \) be the respective Lie algebras of \( H, L' \) and \( L \). The coordinate changes \((u_{ij})_{i,j \in \mathbb{I}}\) of the principal \( L \)-bundle \( P' \to N \), define a \( L \)-bundle \( P_\mathcal{L} \) over \( N \) whose coordinates changes are \((Ad(u_{ij}))_{i,j \in \mathbb{I}}\). We can lift \( u_{ij} \) to an element \( u'_{ij} \) of \( L' \), since the extension is central, we can define a \( \mathcal{L}' \)-bundle \( P_{\mathcal{L}'} \) over \( N \) whose coordinates change are \((Ad(u'_{ij}))_{i,j \in \mathbb{I}}\). There exists a canonical projection \( p_0 : P_{\mathcal{L}'} \to P_\mathcal{L} \).

A connection structure defined on the gerbe \( P \), is defined as follows:

\[
L_{jl} - L_{il} + u_{jl}^* L_{jl} =
\]

\[
= L_l(e_{ij}^l, \alpha_{ij}^l) - L_l(e_{il}^l, u_{ij}^* (\alpha_{ij}^l)) - (L_l(e_{ij}^l, \alpha_{ij}^l) - L_l(e_{il}^l, u_{ij}^* (\alpha_{ij}^l))) + u_{jl}^* (L_j(e_{jl}^j, \alpha_{jl}^j) - L_j(e_{ij}^j, u_{ij}^* (\alpha_{ij}^j)))
\]

\[
= L_l(e_{ij}^l, u_{il}^* (\alpha_{il}^l)) - L_l(e_{il}^l, u_{ij}^* (\alpha_{ij}^l))
\]

\[
= u_{il}^* (L_i(e_{il}^j, \alpha_{il}^j) - Ad(c_{ij}^{-1})L(e_{jl}^j, \alpha_{jl}^j))
\]

where \( c_{ij} = u_{ij}^l u_{jl} u_{il}^l \).
Let $U$ be an open subset of $N$, and $e_U$ an object of $P_U$, there exists a transitive distribution $D_{e_U}$ of $e_U$ which is right invariant.

Let $h : e_U \rightarrow e_{U'}$ be a morphism in $P$, we assume that the pull-back of $D_{e_{U'}}$ by $h$ is $D_{e_U}$.

The distribution $D_{e_U}$ is not assume to be uniform, see (Molino [29] p. 184) when it is uniform, the connection structure can be defined by a family of 1-forms $\theta_{e_U} : e_U \rightarrow P^{e_{U'}}$ which verify the following conditions:

if $x$ is an element of $e_U$, and $v$ an element of $Te_U x$, the tangent space of $e_U$ at $x$, $\theta e_U (v)$ is an element of the fiber of $p e_U (x)$, where $p e_U : e_U \rightarrow U$ is the canonical projection.

Let $A$ be an element of $L'$, and $\tilde{A}$ the projection of $A$ in $L$ by the canonical map $\tilde{p} : L' \rightarrow L'/\mathcal{H} = L$. Denote by $A^*$ the fundamental vector field generated by $A$ on $e_U$. We assume that $\tilde{p}(\theta_{e_U}(A^*)) = \tilde{A}$.

Let $\mathcal{H}_{e_U}^*$ be the vector space of fundamental vectors generated by elements of $\mathcal{H}$. We assume that $\theta_{e_U}$ preserves $\mathcal{H}_{e_U}^*$, and its restriction to it is a projection.

Let $h : e_U \rightarrow e_{U'}$ be a morphism in $P$, we assume that $h^*(\theta_{e_{U'}}) = \theta_{e_U}$.

An horizontal path in $e_U$ is a differentiable path $c : I \rightarrow e_U$ such that for each $t$ in $I$, the tangent vector to the curve $c'(t)$ at $t$ is an element of $D_{e_U}$.

The holonomy of a transitive distribution can be defined as is defined the holonomy of a connection (see Lichnerowicz [19] p. 62, Molino [29] p. 181): Let $x$ be an element of $e_U$, the holonomy group $H_x$ at $x$, is the set of elements $l'$ in $L'$ such that there exists an horizontal path between $x$ and $x l'^{-1}$, of course if we replace $x$ by $h x$, $H_{h x} = Ad(h^{-1}) H_x$.

Let $x$ be an element of $U$, and $c : I \rightarrow U$ a differentiable path such that $c(0) = c(1) = x$. Consider $y$ an element of the fiber of $x$. Since the distribution is transitive, there exists an horizontal path over $c$ in $e_U$, $d : I \rightarrow e_U$ such that $d(0) = y$. This is implied by the fact that a transitive invariant distribution contains always a connection. (See Molino [29] p. 181). The element $y d(1)^{-1}$ does not depends of $y$ (compare with Lichnerowicz p. 94). It is called the holonomy around $c$. The holonomy group $H_x^U$ at $x$ is the set whose elements are holonomy around loops at $x$. The holonomy group depends of the object since two objects of $P_U$ are not always isomorphic.

Suppose that $U$ is contractible, then the holonomy group does not depends of the object since it is invariant by the gauge transformations which preserve the connection since the extension is central, and all the objects of $P_U$ are isomorphic. This last group can be computed by the Ambrose-Singer theorem. (See Molino [29] p. [183]).

9 Sequences of fibered categories in differentiable categories.

One of the main motivation of the introduction of gerbes theory in differential geometry is the geometric interpretation of characteristic classes. Let $N$ be a manifold. A well-known result identifies the 2-dimensional integral cohomology
space $H^2(M, \mathbb{Z})$ of $N$, with the space of isomorphic classes of $U(1)$-bundles defined on $N$. This identification is one of the main tools used in quantization in physics. String theory has created the need of finding such an interpretation for higher cohomology classes. The space of 3-dimensional integral cohomology classes is the classifying space of $U(1)$-gerbes (See Brylinski [8] p. 200). The second Pontryagin class which is an element of $H^4(N, \mathbb{Z})$ has been interpreted with 2-gerbes by Brylinski and McLaughlin (see [9] p. 625). In this part, we are going define and apply the theory and sequences of fibered categories analog to the theory defined in Tsemo [40]) to study characteristic classes in differentiable categories.

**Definition 9.1.**

A 2-sequence of fibered categories is defined by the following data:

A fibered category $p : P \to C$ over the Grothendieck site $C$, such that:

Let $U$ be an object of $C$, and $e_U$ an object of $P_U$. Recall that $e_U$ is a differentiable manifold. There exists a correspondence which assigns to $e_U$ a 2-category $Q_{e_U}$ (see Benabou for the definition of a 2-category) whose objects are gerbes defined on $e_U$.

Let $c : U \to U'$ a morphism of $C$, the restriction functor is the pull-back of gerbes.

There exists a covering $(U_i)_{i \in I}$, such that for every objects $e_i$ and $e'_i$ of $P_{U_i}$, there exists an isomorphism between the 2-categories $Q_{e_i}$ and $Q_{e'_i}$.

The set of automorphisms of an object of $Q_{e_U}$ can be identified with sections of a sheaf $L$ defined on $C$, and this identification is natural in respect to morphisms between objects and restrictions.

Let $P''$ be the category whose objects are objects of $Q_{e_U}$, $e_U$ in $P_U$, and $P'$ the category whose objects are open subsets of the manifolds $e_U$. If $e$ in $Q_{e_U}$ and $e'$ in $Q_{e_{U'}}$ are objects of $P''$, there exists an open subset $V$ of $e_U$, (resp. $V'$ of $e_{U'}$) a Lie group $H$ such that $e \to V$ is a $H$-bundle (resp. $e' \to V'$ is a $H$-bundle). A morphism $h' : e \to e'$ in $P''$ is a morphism of $H$-bundles such that there exists $h : e_U \to e_{U'}$ in $P$ such that $h(V) \subset V'$, and the following square is commutative:

\[
\begin{array}{ccc}
e & \xrightarrow{h'} & e' \\
\downarrow & & \downarrow \\
V & \xrightarrow{h} & V'
\end{array}
\]

Our descent condition is expressed by the fact that we assume that the correspondence $P'' \to P'$ which assigns to $e$, the open subset $V$ of $e_U$ is a fibered category.

**9.1 Classification 4-cocycles and sequences of 2-fibered categories.**

Before to attach to a 2-sequence of fibered categories a cocycle, we describe an automorphism $h$ above the identity of a $H$-gerbe $p : P \to N$ over a manifold $N$. 
The automorphism $h$ is defined by a family of functors $h_U$ of $P_U$ above the identity, where $U$ is an open subset in $N$, such that if $V$ is a subset of $U$ the following square commutes:

$$
\begin{array}{ccc}
P_U & \xrightarrow{h_U} & P_U \\
\downarrow r_{V,U} & & \downarrow r_{V,U} \\
P_V & \xrightarrow{h_V} & P_V
\end{array}
$$

where $r_{V,U}: P_U \to P_V$ is the restriction map.

Let $(U_i)_{i \in I}$ be a good covering of $N$, we assume that an object $e_i$ of $P_{U_i}$ is a trivial $H$-bundle. Since the square:

$$
\begin{array}{ccc}
P_{U_i} & \xrightarrow{h_{U_i}} & P_{U_i} \\
\downarrow r_{U_i \cap U_j, U_i} & & \downarrow r_{U_i \cap U_j, U_i} \\
P_{U_i \cap U_j} & \xrightarrow{h_{U_i \cap U_j}} & P_{U_i \cap U_j}
\end{array}
$$

The automorphism $h$ is described by a family of morphisms $u_{ij}: U_i \cap U_j \times H \to U_i \cap U_j \times H$ such that $u_{ij}^l u_{ij}^m = u_{ij}^m u_{ij}^l$. Thus by a $H$-bundle.

Now can describe the classifying 4-cocycle:

We assume that $L$ is commutative. Let $(U_i)_{i \in I}$ be a good cover of the site $(C, J)$, and $e_i$ and object of $P_{U_i}$, we choose a gerbe $d_i$ in $Q_{e_i}$. Since $P_{U_i \times C U_j}$ is connected, there exists a morphism: $u_{ij}: e_i \to e_j$, and a map $u_{ij}^*: d_i \to d_j$.

The automorphism $c_{ijl} = u_{ij}^* u_{ij}^* u_{ij}^*: d_i \to d_j$ is not above the identity. But

$$
c_{ijlm} = u_{ij}^* u_{ij}^* u_{ij}^* \circ c_{ijm}^{-1} \circ c_{ilm} \circ c_{jlm}^{-1}
$$

is a morphism above the identity that we identifies with a 1-form defined on $U_{ijlm}$ which takes its values in $L(U_{ijlm})$. The Cech-DeRham isomorphism identifies this with a 4-cocycle which takes its values in $L$ if $C$ is a manifold, since $L$ is assumed to be commutative.

Before to give examples, we are going to describe a weak version of a 2-sequence of fibered categories.

**Definition 9.1.2.**

A 2-sequence of torsor/fibered categories, is a 2-sequence of fibered categories where the sheaf of categories $P \to C$ is in fact a torsor.

We can associate to a 2-sequence of torsor/fibered categories a 3-cocycle as follows:

Let $(U_i)_{i \in I}$ be a good cover of $C$, and $e_i$ the object of $P_{U_i}$, and $d_i$ an object of $Q_{e_i}$, and $u_{ij}: e_i \to e_j$. There exists a morphism $u_{ij}^*: d_i \to d_j$. The morphism

$$
c_{ijl} = u_{ij}^* u_{ij}^* u_{ij}^*
$$

is above the identity. It is a 1-form defined on $U_{ijl}$ which is $L(U_{ijl})$ valued. The Cech Derham isomorphism identifies this cocycle with a 3-form defined on $C$ which is $L$-valued.
Examples.

Let $H$ be a compact simple Lie group. Consider a $H$-principal bundle $p : P \to C$ over a differentiable category $C$. We can define the following 2-sequence of torsor/fibered categories:

Let $U$ be an object of $C$, the object of $Q_{P_U}$ are $U(1)$-gerbes which induces on each fiber of $e_U \to U$ a gerbe isomorphic to the canonical $U(1)$-gerbe on $H$. This example is defined when $N$ is a manifold by Brylinski and McLaughlin [9] p. 625). In this situation the classifying cocycle is an element of $H^3(N,U(1)) = H^4(N,Z)$.

This construction can also be applied to a subcategory $C'_N$ of differentiable category $C_N$ associated to a generalized orbifold (see definition 3.2.1) where there exists a simple and compact Lie group $H$ such that every object of $C'_N$ is of the form $(P,H,\phi_P)$. We can also define a 4-integral class on the space whose elements are closure of leaves of a foliation endowed with a bundle like metric.

Let $p : P \to N$ be a $H$-principal gerbe over the manifold $N$. Without restricting the generality, we suppose that for every open subset $U$ of $N$, and for every element $e_U$ in $P_U$, $e_U$ is a principal $H$-bundle. We can construct the following 2-sequence of fibered categories:

The objects of $Q_{e_U}$ are $U(1)$-gerbes which induces on the fiber of $e_U \to U$ the canonical $U(1)$-gerbe defined on $H$.

The classifying cocycle of this 2-sequence of fibered categories is a 4-class in $H^4(N,U(1))$ which can be identified with a 5-class in $H^5(N,Z)$.

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