Convergence of the $q$-Stancu-Szász-Beta type operators

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Abstract
In this paper, we study on $q$-Stancu-Szász-Beta type operators. We give these operators convergence properties and obtain a weighted approximation theorem in the interval $[0,\infty)$.

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Keywords: $q$-Stancu type operators; Szász-Beta type operators; weighted approximation

1 Introduction
In [1], Mahmudov constructed $q$-Szász operators and obtained rate of global convergence in the frame of weighted spaces and a Voronovskaja type theorem for these operators.
In [2], Gupta and Mahmudov studied on the $q$-analog of the Szász-Beta type operators.
In [3], Yüksel and Dinlemez gave a Voronovskaja type theorem for $q$-analog of a certain family Szász-Beta type operators. In [4], Govil and Gupta introduced the $q$-analog of certain Beta-Szász-Stancu operators. They estimated the moments and established direct results in terms of modulus of continuity and an asymptotic formula for the $q$-operators. In [5–14], interesting generalization about $q$-calculus were given. Our aims are to give approximation properties and a weighted approximation theorem for $q$-Stancu-Szász-Beta type operators. We use without further explanation the basic notations and formulas, from the theory of $q$-calculus as set out in [15–19]. Let $A > 0$ and $f$ be a real valued continuous function defined on the interval $[0,\infty)$. For $0 < q \leq 1$, $q$-Stancu-Szász-Beta type operators are defined as

$$B_{n,q}^{(\alpha,\beta)}(f, x) = \sum_{k=0}^{\infty} s_{n,k}^{q}(x) \int_{0}^{\infty/A} b_{n,k}^{q}(t) f\left(\left[\frac{n}{|n|}\right]q^{t+\alpha} \left(\left|\frac{n}{|n|}\right|q^{t+\beta}\right) \right) \, dq\, dt,$$

where

$$s_{n,k}^{q}(x) = \frac{(\left[|n|\right]q^{x})^{k} e^{-\left[|n|\right]q^{x}}}{[k]_{q}},$$

and

$$b_{n,k}^{q}(x) = \frac{q^{k+2} x^{k}}{B_{q}(k+1,n)(1+x)^{yk+1}}.$$
Let us write $q = 1$ and $\alpha = \beta = 0$ in (1.1), then the operators $B_{n,q}^{(\alpha,\beta)}(f,x)$ are reduced to Szász-Beta type operators studied in [20–23].

2 Auxiliary results

For the sake of brevity, the notation $F_n^q(n) = \prod_{i=1}^{n} [n-i]_q$ and $G_n^q(n) = ([n]_q + \beta)$ will be used throughout the article. Now we are ready to give the following lemma for the Korovkin test functions.

**Lemma 1** Let $e_m(t) = t^m$, $m = 0, 1, 2$, we get

(i) $B_{n,q}^{(\alpha,\beta)}(e_0,x) = 1$,

(ii) $B_{n,q}^{(\alpha,\beta)}(e_1,x) = \frac{[n]_q x}{q^2 G_n^q(n) F_1^q(n)} + \frac{[n]_q}{q G_n^q(n) F_2^q(n)} + \frac{\alpha}{G_n^q(n)}$,

(iii) $B_{n,q}^{(\alpha,\beta)}(e_2,x) = \frac{[n]_q x^2}{q^6 G_n^q(n)^2 F_2^q(n)} + \left\{ \frac{[n]_q^3}{q^2 G_n^q(n)^2 F_2^q(n)} + \frac{(1 + [2]_q) [n]_q^3}{q^4 G_n^q(n)^2 F_2^q(n)} + \frac{2\alpha [n]_q}{q^2 G_n^q(n)^2 F_2^q(n)} \right\} x + \frac{2\alpha [n]_q}{q G_n^q(n)^2 F_2^q(n)} + \frac{\alpha^2}{G_n^q(n)^2}$.

**Proof** Using the $q$-Gamma and $q$-Beta functions in [15, 24], we obtain the following equality:

\[
q^2 \int_0^{\ln^1/\lambda} \frac{1}{B(k+1,n)} \frac{t^{k+m}}{(1+t)_q^{n+1}} dt = \frac{[n+k]_q [n-m-1]_q q^{[2k-(k+m)(k+m+1)]/2}}{[k]_q [n-1]_q^2}.
\] (2.1)

Then, using (2.1), for $m = 0$, we get

\[
B_{n,q}^{(\alpha,\beta)}(e_0,x) = e^{-[n]_q x} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{[k]_q!} q^{k(k-1)/2} = e^{-[n]_q x} E_q^{[n]_q x} = 1,
\]
and the proof of (i) is finished. With a direct computation, we obtain (ii) as follows:

\[
B_{n,q}^{(\alpha,\beta)}(e_1,x) = \frac{[n]_q}{G_n^q(n) F_1^q(n)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{[k]_q!} q^{k(k-3)/2} e^{-[n]_q x} + \frac{[n]_q}{G_n^q(n) F_1^q(n)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{[k]_q!} q^{k(k-1)/2} e^{-[n]_q x} + \frac{\alpha}{G_n^q(n)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{[k]_q!} q^{k(k-1)/2} e^{-[n]_q x}.
\]
Using the equality

\[ [n]_q = [s]_q + q^s \cdot [n-s]_q, \quad 0 \leq s \leq n, \]  

we get

\begin{align*}
B_{n,q}^{(\alpha,\beta)}(e_2, x) &= \frac{[n]_q^2 x^2}{q^2 G_\beta(n)^2 F_1^2(n)} \\
& \quad + \left\{ \frac{[n]_q^2}{q^2 G_\beta(n)^2 F_1^2(n)} + \frac{(1 + [2]_q)[n]_q^3}{q^2 G_\beta(n)^2 F_1^2(n)} + \frac{2\alpha[n]_q^2}{q^2 G_\beta(n)^2 F_1^2(n)} \right\} x \\
& \quad + \frac{[2]_q[n]_q^2}{q^2 G_\beta(n)^2 F_1^2(n)} + \frac{2\alpha[n]_q}{q^2 G_\beta(n)^2 F_1^2(n)} + \frac{\alpha^2}{G_\beta(n)^2},
\end{align*}

and so we have the proof of (iii). \qed

To obtain our main results we need to compute the second moment.

**Lemma 2** Let \( q \in (0,1) \) and \( n > 2 \). Then we have the following inequality:

\[ B_{n,q}^{(\alpha,\beta)}((t-x)^2, x) \leq \left( \frac{2(1-q^4)}{q^6} + \frac{164(\alpha + \beta + 1)^2[n]_q^3}{q^4 F_1^2(n)} \right) x(x+1) + \frac{6(\alpha + 1)^2}{q^3 G_\beta(n)^2}. \]

**Proof** From the linearity of the \( B_{n,q}^{(\alpha,\beta)} \) operators and Lemma 1, we write the second moment as

\begin{align*}
B_{n,q}^{(\alpha,\beta)}((t-x)^2, x) &= \left\{ \frac{[n]_q^4}{q^6 G_\beta(n)^2 F_1^2(n)} - \frac{2[n]_q^2}{q^2 G_\beta(n)^2 F_1^2(n)} + 1 \right\} x^2 \\
& \quad + \left\{ \frac{(1 + [2]_q)[n]_q^3}{q^2 G_\beta(n)^2 F_1^2(n)} + \frac{2\alpha[n]_q^2}{q^2 G_\beta(n)^2 F_1^2(n)} - \frac{2[n]_q}{q G_\beta(n)^2 F_1(n)} - \frac{2\alpha}{G_\beta(n)^2} \right\} x \\
& \quad + \frac{[2]_q[n]_q^2}{q^2 G_\beta(n)^2 F_1^2(n)} + \frac{2\alpha[n]_q}{q^2 G_\beta(n)^2 F_1^2(n)} + \frac{\alpha^2}{G_\beta(n)^2} \\
& \leq \left\{ \frac{[n]_q^4}{q^4 G_\beta(n)^2 F_1^2(n)} - \frac{2[n]_q^2}{q^2 G_\beta(n)^2 F_1^2(n)} + 1 + \frac{(1 + [2]_q)[n]_q^3}{q^2 G_\beta(n)^2 F_1^2(n)} \\
& \quad + \frac{2\alpha[n]_q}{q^2 G_\beta(n)^2 F_1^2(n)} + \frac{\alpha^2}{G_\beta(n)^2} \right\} x(x+1) + \frac{[2]_q[n]_q^2}{q^2 G_\beta(n)^2 F_1^2(n)} + \frac{2\alpha[n]_q}{q G_\beta(n)^2 F_1(n)} + \frac{\alpha^2}{G_\beta(n)^2}.
\end{align*}
\[
\begin{align*}
&\leq \left\{ \frac{[n]_q^4(1 + q^6) - 2q^4[n - 2]_q^4 + 2\beta q^6[n]_q[n - 1]_q[n - 2]_q}{q^6 G_2(n)^2 F_2(n)} \\
&\quad + \frac{(q + q^2 + [2]_q^2 q^2)[n]_q^2}{q^6 G_2(n)^2 F_2(n)} + \frac{q^6 \beta^2 [n - 1]_q[n - 2]_q}{q^6 G_2(n)^2 F_2(n)} + \frac{2\alpha q^4 [n]_q^2 [n - 2]_q}{q^6 G_2(n)^2 F_2(n)} \right\} x(x + 1) \\
&\quad + \frac{([2]_q + 2\alpha q^2 + \alpha^2 q^3)[n]_q}{q^3 G_2(n) F_2(n)}. \\
\end{align*}
\]

From (2.2), we have

\[
B^{(\alpha, \beta)}_{n,q}(t - x)^2, x
\]

\[
\leq \left\{ \frac{[n - 2]_q^4 (q^{14} + q^8 - 2q^4)}{q^6 G_2(n)^2 F_2(n)} \\
+ \frac{(1 + q^6)[4[2]_q q^6 [n - 2]_q^4 + 6[2]_q^2 q^4 [n - 2]_q^2 + 4[2]_q^3 q^2 [n - 2]_q + [2]_q^4]}{q^6 G_2(n)^2 F_2(n)} \\
+ \frac{(q + q^2 + [2]_q^2 q^2 + 2\beta q^6 + 2\alpha q^4)[n]_q^3 + \beta^2 q^6 [n]_q^2}{q^6 G_2(n)^2 F_2(n)} \\
+ \frac{([2]_q + q^2)([2]_q + 2\alpha q^2 + \alpha^2 q^3)}{q^3 G_2(n) F_2(n)} \right\} x(x + 1) \\
\leq \left( \frac{2(1 - q^4)}{q^6} + \frac{164(\alpha + \beta + 1)^2 [n]_q}{q^6 G_2(n) F_2(n)} \right) x(x + 1) + \frac{6(\alpha + 1)^2}{q^3 G_2(n)}.
\]

And the proof of Lemma 2 is now finished. \(\square\)

### 3 Direct estimates

Now in our considerations, \(C_B[0, \infty)\) denotes the set of all bounded-continuous functions from \([0, \infty)\) to \(\mathbb{R}\). \(C_B[0, \infty)\) is a normed space with the norm \(\|f\|_B = \sup\{|f(x)|: x \in [0, \infty]\}\).

We denote the first modulus of continuity on the finite interval \([0, b]\), \(b > 0\),

\[
\omega_{[0,b]}(f, \delta) = \sup_{0 < h < \delta, x \in [0,b]} |f(x + h) - f(x)|. \tag{3.1}
\]

The Peetre \(K\)-functional is defined by

\[
K_2(f, \delta) = \inf \{ \|f - g\|_B + \delta \|g''\|_B : g \in W_\infty^2 \}, \quad \delta > 0,
\]

where \(W_\infty^2 = \{g \in C_B[0, \infty) : g', g'' \in C_B[0, \infty)\}\). By Theorem 2.4 in [25], p.177, there exists a positive constant \(C\) such that

\[
K_2(f, \delta) \leq C\omega_2(f, \sqrt{\delta}), \tag{3.2}
\]

where

\[
\omega_2(f, \sqrt{\delta}) = \sup_{0 < h < \sqrt{\delta}, x \in [0, \infty)} |f(x + 2h) - 2f(x + h) - f(x)|.
\]

Gadjzhi proved the weighted Korovkin-type theorems in [26]. We give the Gadjzhi results in weighted spaces. Let \(\rho(x) = 1 + x^2\) and the weighted spaces \(C_\rho[0, \infty)\) denote
the space of all continuous functions $f$, satisfying $|f(x)| \leq M_f \rho(x)$, where $M_f$ is a constant depending only on $f$. $C_\rho[0, \infty)$ is a normed space with the norm $\|f\|_\rho = \sup\{\frac{|f(x)|}{\rho(x)} : x \in \mathbb{R}^+ \cup \{0\}\}$ and $C_\rho^0[0, \infty)$ denotes the subspace of all functions $f \in C_\rho[0, \infty)$ for which $\lim_{x \to \infty} \frac{|f(x)|}{\rho(x)}$ exists finitely.

Thus we are ready to give direct results. The following lemma is routine and its proof is omitted.

**Lemma 3** Let

$$
\overline{B}_{n,q}^{(\alpha, \beta)}(f, x) = B_{n,q}^{(\alpha, \beta)}(f, x) - f(D_{n,q}^{(\alpha, \beta)}(x)) + f(x). \tag{3.3}
$$

Then the following assertions hold for the operators (3.3):

(i) $\overline{B}_{n,q}^{(\alpha, \beta)}(1, x) = 1$,

(ii) $\overline{B}_{n,q}^{(\alpha, \beta)}(t, x) = x$,

(iii) $\overline{B}_{n,q}^{(\alpha, \beta)}(t - x, x) = 0$,

where $D_{n,q}^{(\alpha, \beta)}(x) = \left[\frac{\alpha_\rho^2 x}{q^2 C_\rho^0(n) F_1(n)} + \frac{\beta_\rho}{q^2 C_\rho^0(n) F_1(n)} \right]$.

**Lemma 4** Let $q \in (0, 1)$ and $n > 2$. Then for every $x \in [0, \infty)$ and $f'' \in C_\rho[0, \infty)$, we have the inequality

$$
\|\overline{B}_{n,q}^{(\alpha, \beta)}(f, x) - f(x)\| \leq \gamma_{n,q}^{(\alpha, \beta)}(x) \|f''\|_B,
$$

where $\gamma_{n,q}^{(\alpha, \beta)}(x) = \left(\frac{2(1-q^2)}{q^2} + \frac{263(\alpha + \beta + 1)^2}{q^2 F_1(n)}\right)x(x + 1) + \frac{5(\alpha + 1)^2}{q^4 C_\rho^0(n)}$.

**Proof** Using Taylor’s expansion

$$
f(t) = f(x) + (t - x)f'(x) + \int_x^t (t - u) f''(t) du
$$

and Lemma 3, we obtain

$$
\overline{B}_{n,q}^{(\alpha, \beta)}(f, x) - f(x) = \overline{B}_{n,q}^{(\alpha, \beta)}\left(\int_x^t (t - u) f''(u) du, x\right).
$$

Then, using Lemma 1 and the inequality

$$
\left|\int_x^t (t - u)f''(u) du\right| \leq \|f''\|_B \frac{(t - x)^2}{2},
$$

we get

$$
\|\overline{B}_{n,q}^{(\alpha, \beta)}(f, x) - f(x)\|
\leq \left|B_{n,q}^{(\alpha, \beta)}\left(\int_x^t (t - u)f''(u) du, x\right) - \int_x^t D_{n,q}^{(\alpha, \beta)}(x) - u f''(u) du \right|
$$
Theorem 1 Let \( (q_n) \subset (0,1) \) a sequence such that \( q_n \to 1 \) as \( n \to \infty \). Then for every \( n > 2 \), \( x \in [0,\infty) \) and \( f \in C_{\rho}[0,\infty) \), we have the inequality

\[
|B^{(\alpha,\beta)}_{n,q_n}(f,x) - f(x)| \leq 2M\omega_2\left(f, \sqrt{\delta_{n,q_n}(x)}\right) + w(f, \eta_{n,q_n}(x)),
\]

where \( \eta_{n,q_n}(x) = \left(\frac{[n]_{q_n}}{q_nG_{\rho}^{\alpha}(n)F_1^{\alpha}(n)} - 1\right)x + \frac{[n]_{q_n}}{q_nG_{\rho}^{\alpha}(n)F_1^{\alpha}(n)} + \frac{\alpha}{G_{\rho}^{\alpha}(n)}. \)

Proof Using (3.3) for any \( g \in W^2_\infty \), we obtain the following inequality:

\[
|B^{(\alpha,\beta)}_{n,q_n}(f,x) - f(x)| \leq \left|B^{(\alpha,\beta)}_{n,q_n}(f,g,x) - (f-g)(x) + B^{(\alpha,\beta)}_{n,q_n}(g,x) - g(x)\right|
\]

\[
+ \left|f\left(\frac{[n]_{q_n}}{q_nG_{\rho}^{\alpha}(n)F_1^{\alpha}(n)}x + \frac{[n]_{q_n}}{q_nG_{\rho}^{\alpha}(n)F_1^{\alpha}(n)} + \frac{\alpha}{G_{\rho}^{\alpha}(n)}\right) - f(x)\right|.
\]

From Lemma 4, we get

\[
|B^{(\alpha,\beta)}_{n,q_n}(f,x) - f(x)| \leq 2\|f - g\|_B + \delta_{n,q_n}(x)\|g''\|
\]

\[
+ \left|f\left(\frac{[n]_{q_n}}{q_nG_{\rho}^{\alpha}(n)F_1^{\alpha}(n)}x + \frac{[n]_{q_n}}{q_nG_{\rho}^{\alpha}(n)F_1^{\alpha}(n)} + \frac{\alpha}{G_{\rho}^{\alpha}(n)}\right) - f(x)\right|.
\]

By using equality (3.1) we have

\[
|B^{(\alpha,\beta)}_{n,q_n}(f,x) - f(x)| \leq 2\|f - g\|_B + \delta_{n,q_n}(x)\|g''\|_B + w(f, \eta_{n,q_n}(x)).
\]

Taking the infimum over \( g \in W^2_\infty \) on the right-hand side of the above inequality and using the inequality (3.2), we get the desired result. \( \square \)

Theorem 2 Let \( (q_n) \subset (0,1) \) a sequence such that \( q_n \to 1 \) as \( n \to \infty \). Then \( f \in C^*_\rho[0,\infty) \), and we have

\[
\lim_{n \to \infty} \|B^{(\alpha,\beta)}_{n,q_n}(f) - f\|_\rho = 0.
\]

Proof From Lemma 1, it is obvious that \( B^{(\alpha,\beta)}_{n,q_n}(e_0) - e_0\|_\rho = 0 \). Since \( \frac{[n]_{q_n}}{q_nG_{\rho}^{\alpha}(n)F_1^{\alpha}(n)}x + \frac{\alpha}{G_{\rho}^{\alpha}(n)} - x\right| \leq (x + 1)\sigma(1) \) and \( \frac{1}{1+e^2} \) is positive and bounded from above for
each $x \geq 0$, we obtain
\[
\|B^{(\alpha,\beta)}_{n,q}(e_1) - e_1\|_p \leq \frac{x + 1}{1 + x^2} o(1).
\]

And then $\lim_{n \to \infty} \|B^{(\alpha,\beta)}_{n,q}(e_1) - e_1\|_p = 0$.

Similarly for every $n > 2$, we write
\[
\|B^{(\alpha,\beta)}_{n,q}(e_2) - e_2\|_p = \sup_{x \in (0,\infty)} \left\{ \frac{\left( \frac{1}{q_nC^m_p(\alpha)^2F_2(n)} - 1 \right)x^2}{1 + x^2} \right. \\
\frac{\left( 1 + q_n(\frac{2}{q_n}n_{\alpha q})^2 + 2q_n^2n_{\beta q}\frac{1}{q_nC^m_p(\alpha)^2F_2(n)} \right)x + \frac{2}{q_nC^m_p(\alpha)^2F_2(n)}}{1 + x^2} \\
+ \frac{\frac{2}{q_nC^m_p(\alpha)^2F_2(n)} + \frac{q_n^2}{C^m_p(\alpha)^2}}{1 + x^2} \left\} \right. \\
\leq \sup_{x \in (0,\infty)} \frac{1 + x + x^2}{1 + x^2} - o(1),
\]

we get $\lim_{n \to \infty} \|B^{(\alpha,\beta)}_{n,q}(e_2) - e_2\|_p = 0$. Thus, from AD Gadjiev’s theorem in [26], we obtain the desired result of Theorem 2. □

Competing interests
The author declares to have no competing interests.

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