Evolutionary Quantum Dynamics of a Generic Universe

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The implications of an Evolutionary Quantum Gravity are addressed in view of formulating a new dark matter candidate. We consider a Schrödinger dynamics for the gravitational field associated to a generic cosmological model and then we solve the corresponding eigenvalues problem, inferring its phenomenological issue for the actual Universe. The spectrum of the super-Hamiltonian is determined including a free inflaton field, the ultrarelativistic thermal bath and a perfect gas into the dynamics. We show that, when a Planckian cut-off is imposed in the theory and the classical limit of the ground state is taken, then a dark matter contribution can not arise because its critical parameter $\Omega_{\text{dm}}$ is negligible today when the appropriate cosmological implementation of the model is provided. Thus, we show that, from a phenomenological point of view, an Evolutionary Quantum Cosmology overlaps the Wheeler-DeWitt approach and therefore it can be inferred as appropriate to describe early stages of the Universe without significant traces on the later evolution. Finally, we provide indications that the horizon paradox can be solved in the Planck era by the morphology of the Universe wave function.

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INTRODUCTION

A long standing problem in Quantum Gravity is the so-called frozen formalism characterizing the canonical approach, i.e. the absence of a time evolution for the wavefunctional \cite{1, 2} (see also \cite{3, 4, 5} and references therein). In a recent approach \cite{6, 7}, it has been proposed that such feature disappears as soon as the impossibility of a physical slicing without frame fixing is recognized for a quantum space-time. Since the existence of a timelike direction is ensured on a quantum level only by including a fluid into the dynamics (its lightcone structure has to survive), then an equivalence between time and a frame of reference can be established \cite{8} (see also \cite{9, 10} and \cite{11, 12}).

In the light of these results, here we address a Schrödinger Quantum Cosmology relative to a generic inhomogeneous Universe and discuss the phenomenological implications of the super-Hamiltonian spectrum in view of a dark matter candidate. While previous cosmological implementations of the same quantum dynamics \cite{13, 14} were restricted to the isotropic symmetry, here the real quantum Big-Bang picture is outlined, showing how the energy of the gravitational and matter fields (vanishing in the standard Wheeler-DeWitt approach \cite{1}) provides in the classical limit, a non relativistic relic component.

To reproduce the primordial Universe we include in the dynamics a free inflaton field, the ultrarelativistic energy density of the thermal bath and a perfect gas contribution to account for Planck mass particles. To develop the phenomenological implications of the spectrum, we impose a cut-off at the Planck scale and then we show that the critical parameter of our dark matter candidate cannot fulfill unity in correspondence to an efficient inflationary scenario; in fact our dark matter candidate would work only for an e-folding of about 23, too small for a solution of the horizons paradox. In this respect our model offers quantum features of the dynamics which outline a possible explanation of such a paradox within the Planck era.

The structure of the letter is the following. In Sec.I we discuss the Schrödinger Quantum Gravity and the picture resulting from its classical limit; dualism existing in Quantum Gravity between time and matter field is stressed. In Sec.II the eigenvalue problem corresponding to the generic inhomogeneous Universe is formulated and solved in Sec.III. Sec.IV is devoted to discuss the classical limit of the model, both in the sense of a WKB approximation, and with respect to the spectrum. Finally, in Sec.V we implement the Planckian cut-off and study the implications on the spectrum cut-off; the phenomenological issues in view of a dark matter scenario are then investigated and concluding remarks outlined.

I. SCHRÖDINGER QUANTUM GRAVITY

In this section, in agreement with the point of view outlined in \cite{6, 7, 8, 9, 10, 13}, we analyze the implication of a Schrödinger formulation of the quantum dynamics for the gravitational field (A) and then we establish a dualism between time evolution and matter fields (B).
A. Evolutionary Quantum Gravity

In order to preserve the covariance on space-like hypersurfaces, we define the state functional $\Psi$ on the Wheeler Superspace of the 3-geometries $\{h_{ij}\}$ $(i,j = 1,2,3)$ (i.e. it is annihilated by the super-momentum operator $\hat{H}_s$) and we require the theory to evolve along the space-time slicing so that $\Psi = \Psi(t, \{h_{ij}\})$.

The quantum evolution of the gravitational field is then governed by the smeared Schrödinger equation (we works in units where $\hbar = c = 1$, apart from where we refer to the classical limit in which we restore $\hbar$)

$$i\partial_t \Psi = \hat{H}\Psi \equiv \int_{\Sigma^3_t} d^3x \left(N \hat{H}\right) \Psi \quad (1)$$

being $\hat{H}$ the super-Hamiltonian operator, $N$ the lapse function and $\Sigma^3_t$ the one-parameter family of compact boundaryless 3-hypersurfaces which fill the space-time.

Let us now take the following expansion for the wave functional

$$\Psi = \int Dc (\epsilon, \{h_{ij}\}) \exp \left\{ -i \int_{t_0}^{t} dt' \int_{\Sigma^3_t} d^3x (N \epsilon) \right\}, \quad Dc$$

being the Lebesgue measure in the space of the functions $\epsilon(x^i)$. Such an expansion reduces the Schrödinger dynamics to an eigenvalues problem of the form

$$\hat{H}\chi = \epsilon\chi, \quad \hat{H}_s\chi = 0, \quad (3)$$

which outlines the appearance of a non zero super-Hamiltonian eigenvalue.

In order to reconstruct the classical limit of the above dynamical constraints, we address the limit $\hbar \rightarrow 0$ and replace the wave functional $\chi$ by its corresponding zero-order WKB approximation $\chi \sim e^{iS/\hbar}$. Under these restrictions, the eigenvalues problem (3) reduces to the following classical counterpart

$$\hat{H}JS = \epsilon \equiv -2\sqrt{\epsilon} \hat{T}_{00}, \quad \hat{H}JS = 0 \quad (4)$$

where $\hat{H}J$ and $\hat{H}_sJ$ denote operators which, acting on the phase $S$, reproduce the super-Hamiltonian and super-momentum Hamilton-Jacobi equations respectively. We see that the classical limit of the adopted Schrödinger quantum dynamics is characterized by the appearance of a new matter contribution (associated with the non zero eigenvalue $\epsilon$) whose energy density reads

$$\rho \equiv T_{00} = \frac{\epsilon(x^i)}{2\sqrt{\epsilon}} \quad (5)$$

where by $T_{\mu}^\nu$ $(\mu, \nu = 0,1,2,3)$ we refer to the new matter energy-momentum tensor and $H \equiv \det h_{ij}$.

Since the spectrum of the super-Hamiltonian has, in general, a negative component, we can then infer that, when the gravitational field is in the ground state, this matter out-coming in the classical limit has a positive energy density. The explicit form of (5) is that of a dust fluid co-moving with the slicing 3-hypersurfaces, i.e. the field $n^\mu$ begin the 4-velocity normal to the 3-hypersurfaces (in other words, we deal with an energy-momentum tensor $T_{\mu\nu} = \rho n_\mu n_\nu$).

This matter contribution has to emerge in any system which undergoes a classical limit and therefore it must concern the history of the Universe. Indeed, a non-relativistic matter contribution is observed in the actual Universe and it corresponds to the dark matter component. Since our fluid is non relativistic in the very early stages of the Universe evolution, we are lead to regard it as a good candidate for the so-called cold dark matter [15, 16, 17, 18] (for a discussion on the correlation exiting between the observed dark matter and the baryonic one see [19] and for a theoretical interpretation of this feature see [20]). In this respect we below investigate the cosmological issue of a Schrödinger approach to Quantum Cosmology.

In view of the later analysis, it is worth noting that in a Schrödinger Quantum Gravity, it is possible to turn the solution space into Hilbert one and therefore a notion of probability density naturally arises, from the squared modulus of the wavefunctional.

B. Time-Matter Dualism

Let us now consider a gravitational system in the presence of a macroscopic matter source, described by a perfect fluid having a generic equation of state $p = (\xi - 1)\rho$ ($\rho$ being the pressure and $\xi$ the polytropic index).

The energy-momentum tensor, associated to this system reads

$$T_{\mu\nu} = \xi \rho \bar{n}_\mu \bar{n}_\nu - (\xi - 1)\rho g_{\mu\nu}. \quad (6)$$

To fix the constraints when matter is included in the dynamics, let us make use of the relations

$$G_{\mu\nu} n^\mu n^\nu = -\kappa \frac{H}{2\sqrt{\hbar}} \quad (7)$$

$$G_{\mu\nu} \partial_\nu y^\mu = \kappa \frac{H_s}{2\sqrt{\hbar}} \quad (8)$$

where $\kappa$ denotes the Einstein constant ($\kappa = 8\pi G$) and $\partial_\nu y^\mu$ are the tangent vectors to the 3-hypersurfaces $n_\mu \partial_\nu y^\mu = 0$. Equations (7) and (8), by (6) and identifying $u_\mu$ with $n_\mu$ (i.e. the physical space is filled by the fluid), rewrite

$$\rho = -\frac{H}{2\sqrt{\hbar}}, \quad H_s = 0; \quad (9)$$
Hence, equations (11) yield
\[ \rho_{ij} = \kappa (\xi - 1) p h_{ij}. \] (10)

We now observe that the conservation law \( T^\mu_{\nu,\mu} = 0 \) (semicolon denoting covariant derivative) implies the following two conditions
\[ \xi (\rho u^\mu)_{,\mu} = (\xi - 1) u^\mu \partial_\mu \rho \] (11)
\[ u^\nu u_{\mu,\nu} = \left( 1 - \frac{1}{\xi} \right) (\partial_\mu \ln \rho - u_\mu u^\nu \partial_\nu \ln \rho). \] (12)

If we now adapt the spacetime slicing, looking the dynamics into the fluid frame (i.e. \( n^\mu = \delta_0^\mu \)), then, by the relation \( n^\mu = (1/N, -N^i/N) \), we see that the co-moving constraint implies the synchronous nature of the reference frame. As it is well-known that a syn-

The above two points outline, in Quantum Gravity, a real dualism between time evolution and the presence of a dust fluid. Though the classical limit of a Schrödinger Quantum Gravity always provides (at least in principle) a cold dark matter candidate according to the discussion A, nevertheless below we analyze the generic inhomogeneous cosmological solution in the view-point B.

The quantum behavior of the Universe expectably concerns the Planck era and therefore non-relativistic (pressureless) components are not available as physical clock. However, Planck mass particles can be reliably inferred and appropriately described by a perfect gas contribution. We construct a solution in which such a term is included and we link the quantum number, associated to its energy density, to the eigenvalue \( \epsilon \). In doing that, we use the Planck mass particle as a clock for the quantum dynamics and, at the same level, we induce by them a non-relativistic matter component into the early Uni-

II. THE GENERIC QUANTUM UNIVERSE

We now apply the Evolutionary Quantum Gravity approach to the dynamics of a generic inhomogeneous Uni-
verse, i.e. a model in which any specific symmetry has been removed. There are some reliable indications [21, 22] that the early stages of the Universe evolution are characterized by such a degree of generality, but in the quantum regime, dealing with the absence of global symmetry, it is required by the indeterminism; in fact on different causal regions the geometry ‘fluctuates’ independently, so preventing global isometries. A generic cosmological model has to contain four physically independent functions of the spatial coordinates [23] and, in the ADM formalism [21, 22], its line element reads
\[ ds^2 = N^2 dt^2 - h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \] (13)
with \( N^i \) the shift vector and \( h_{ij} \) the 3-metric
\[ h_{ij} = \epsilon^{\epsilon \delta} \delta_{ab} O_i^a O_j^b \delta y^a \delta y^b, \quad a, b, c, d = 1, 2, 3 \] (14)
where \( q_a \) and \( y^a \) are dynamical variables, while \( O_i^a \) denotes three rotation matrices. By virtue of the analysis developed in [22], the dynamics of this generic model can be summarized, asymptotically to the Big-Bang, by the following variational principle
\[ \delta S = \delta \int_{\Sigma^2 \times \mathbb{R}} dt^3 x(p_a \partial_t q^a - NH) \] (15)
where adopting Misner-like variables \((R, \beta_+, \beta_-)\) [24], the super-Hamiltonian has the structure
\[ H(x^t) = \kappa \left[ - \frac{\rho_{\perp}^2}{R} + \frac{1}{R^2} (\rho_{\perp}^2 + p_{\perp}^2) \right] + \frac{3}{8\pi} \frac{p_{\parallel}^2}{R^3} + \frac{R^3}{4n_{\text{in}}} V(\beta_{\pm}) + \frac{\mu^2}{R} + \frac{\sigma^2}{R^2} \] (16)
where \( l_{\text{in}} \) denotes the comoving physical scale of inhomogeneities and the potential term accounts for the spatial curvature of the model:
\[ V(\beta_{\pm}) = e^{-8\beta_{\pm}} + e^{4(\beta_{\pm} + \sqrt{3} \beta_{-})} + e^{4(\beta_{\pm} - \sqrt{3} \beta_{-})}. \] (17)
The \( R \) variable describes the isotropic Universe expansion, while \( \beta_{\pm} \) are associated to the space anisotropies. The following three remarks about the above super-Hamiltonian are relevant, as it appears in the long-wavelength approximation [25].

i) Since the relation \( \partial_t y^a = N^i \partial_i y^a \) holds [22], in a synchronous reference \((N = 1, N^i = 0)\) the variables \( y^a \) become passive functions and the dynamics reduces point by point, to the one of a Bianchi IX model [26, 27]. As show in [22, 23, 28, 29], the classical behavior of a generic model is chaotic and, in this representation,
each space point stands for a causal region (cosmological horizon).

ii) We included in the dynamics a scalar field $\phi$ in order to account for the inflaton field which lives in the very early Universe. The corresponding potential term has been omitted because negligible at temperatures corresponding to the quantum Universe \cite{30}. Furthermore, it is well know that a minimally coupled (classical) scalar field can suppress Mixmaster oscillations in the approach to the singularity of generic cosmological spacetimes \cite{31}.

iii) We add two different contributions, an ultrarelativistic term (associated to $\mu(x)$) and a perfect gas one (associated to $\sigma^2(x)$) \cite{32} to the energy of the system. The former term describes the thermal bath of all fundamental particles (whose mass is negligible with respect to the temperature); the latter is due to particles of Planck mass that, in the quantum Universe, can be described as a perfect gas.

The canonical quantization of this model \cite{33} is reached by replacing the momenta $p_x = -i \partial_x$. The corresponding dynamics is summarized by the eigenvalue problem \cite{33}, which here reads (addressing the normal order defined in [6])

$$\hat{H} \chi \equiv \left\{ \begin{array}{ll}
\kappa \left[ \partial_R \chi - 1 \right] & - \frac{3}{8 \pi R^3} \phi \partial_\phi^2 + \\
R^3 & 4 \kappa^2 \sigma^2 \end{array} \right\} \chi = \epsilon \chi, \quad (18)
$$

Since in the Evolutionary Quantum Gravity $|\Psi|^2$ provides a probability density and in a stationary state it coincides with $|\chi|^2$, then boundary conditions appropriate to this problem take the form

$$\begin{align*}
\chi(R = 0, \beta, \phi) &< \infty \\
\chi(R \to \infty, \beta, \phi) &\to 0.
\end{align*} \quad (19)
$$

In fact, we investigate the quantum dynamics near the Big-Bang and the first boundary condition relies on the idea that the Universe is singularity-free, while the second one ensures a physical behavior at infinite radius of curvature.

III. SPECTRUM OF THE QUANTUM UNIVERSE

In order to study the previous eigenvalue problem \cite{13} we take the following integral representation for the wavefunction $\chi(R, \beta, \phi)$ evaluated in a fixed space point

$$\chi(R, \beta, \phi) = \int \theta_K(R) F_K(R, \beta, \phi) dK. \quad (20)$$

Therefore, performing an adiabatic approximation ($|\partial_R F| \ll |\partial_R \theta|$) (allowed by the vanishing behavior that the spatial curvature has near the singularity), we obtain the following reduced problems:

$$\kappa \frac{d}{dR} \left( \frac{1}{R} \frac{d\theta}{dR} \right) + \left( \kappa \frac{K^2}{R^3} + \frac{\mu^2}{R} + \frac{\sigma^2}{R^2} - \epsilon \right) \theta = 0, \quad (21)$$

$$-(\partial_\beta^2 + \partial_\phi^2 + \frac{3}{8 \pi R^6} \partial_R^2) F + \frac{R^6}{4 \kappa^2 \sigma^2} V(\beta) F = K^2(R) F. \quad (22)$$

Because of the presence of the scalar field, near the cosmological singularity ($R \to 0$), the dynamical role of the potential term in the equation \cite{22} can be neglected as soon as the following condition holds

$$R^3 \ll \mathcal{O} \left( l_P^3 \frac{< K^2 >}{|V(\beta)|} \right) \quad (23)$$

where, instead of ideal monochromatic solutions, we deal with real wave packets which are flat over the width $\Delta \beta \approx 1/\Delta k_\beta$ ($\Delta k_\beta$ being the standard deviation in the momenta space).

Hence, the eigenvalue problem \cite{13} reduces to a system of $\infty^3$ independent eigenvalue problem (in each space point isomorphic to a Bianchi I model). The solution of \cite{22} now reads as plane waves (we replace the variables $\beta_\pm$ with the polar ones $r$ and $\alpha$, in order to outline the vanishing behavior that the wave function assumes at the infinite of the $(r, \alpha)$-plane):

$$F(r, \alpha, \phi) = e^{i \sqrt{8 \pi R^3/3 k_\phi \phi}} \cdot \sum_{m=0}^{\infty} J_m(k_\beta r) [A_m \cos(m \alpha) + B_m \sin(m \alpha)], \quad (25)$$

$J_m$ being the first type Bessel function and $k_\beta^2$ the anisotropies eigenvalue: $K^2 = k_\beta^2 + k_\gamma^2$.

To solve the equation \cite{21} we use a function of the form

$$\theta(R) = \omega(R) \exp \left[ - \frac{1}{2 R^2} \left( R + \frac{\epsilon l_P^2}{16 \pi} \right)^2 \right], \quad (26)$$

so requiring the probability density to decay with a Gaussian weight in the region where the potential term is important. Therefore we obtain the following equation for $\omega$:
\[ \kappa \left[ R^2 \omega'' + R \omega' \sum_{i=0}^{2} a_i R^i \right] + \omega \sum_{j=0}^{4} b_j R^j = 0, \]  

(27)

where

\[ a_0 = -1, \quad a_1 = -\epsilon/8\pi, \quad a_2 = -2/l_p^2, \]  

(28)

\[ b_0 = \kappa R^2, \quad b_1 = \frac{\kappa \epsilon}{16\pi} + \sigma^2, \quad b_2 = \kappa \left( \frac{e l_p}{16\pi} \right)^2 + \mu^2, \quad b_3 = 0, \quad b_4 = \frac{\kappa}{l_p}. \]  

(29)

As result about our belief that the quantum behavior of Universe is confined near the classical singularity, we have to investigate equation (27) only in the limit \( R \to 0 \). Thus we can neglect the higher-order terms \( O(R^4) \); then the solution of the above equation reads

\[ \omega(R) = \sum_{n=0}^{\infty} c_n R^{n+\gamma}, \quad \gamma = 1 - \sqrt{1 - K^2}, \quad c_0 \neq 0, \]  

(30)

whose coefficients obey the following recurrence relations

\[ c_n = -f(n, \gamma) \left\{ -\frac{\epsilon}{8\pi} \left( n + \frac{\gamma}{2} - \frac{3}{2} \right) + \frac{\sigma^2}{8\pi l_p^2} \right\} c_{n-1} + \left[ -\frac{2}{l_p} (n + \gamma - 2) + \left( \frac{\epsilon}{16\pi} \right)^2 + \frac{\mu^2}{8\pi l_p} \right] c_{n-2} \}, \]  

(31)

with \( f(n, \gamma) = (n + \gamma)(n + \gamma - 2) + K^2)^{-1} \). Since we required the wavefunction to decay at large \( R \), where the potential term becomes relevant, then, to avoid its divergence far from the Big-Bang, the series (30) must therefore terminate, i.e. \( \omega(R) \) must be polynomial in \( R \). From the recurrence relations (31) this condition yields

\[ \epsilon_{n, \gamma} = -\frac{\sigma^2}{l_p (n + \gamma - 1/2)} \]  

(32)

\[ 2(n + \gamma) = \left( \frac{l_p \epsilon_{n, \gamma}}{16\pi} \right)^2 + \frac{\mu^2}{8\pi}. \]  

(33)

The previous relations stand for the spectrum of the Universe super-Hamiltonian and of the quantum number associated to the ultrarelativistic term. It is to be emphasized that the ground state \( n = 0 \) eigenvalue

\[ \epsilon_{0, \gamma} = -\frac{\sigma^2}{l_p (1/2 - \gamma)} \]  

(34)

for \( \gamma < 1/2 \) is negative; thus the real ground state, according to equation (33) for \( \mu^2 = 0 \), corresponds to \( \gamma \sim 1.8 \cdot 10^{-3} \), i.e.

\[ \epsilon_0 \simeq -\frac{2\sigma^2}{l_p}, \]  

(35)

is associated to a positive dust energy density.

IV. QUASI-CLASSICAL LIMIT OF THE MODEL

To complete the study of the eigenvalue problem, let us now show that the \( R \)-dependence of the wave function \( \chi \) approaches a quasi-classical limit before the potential becomes important. To this end we take the following representation

\[ \theta = \exp \left[ \frac{i}{\hbar} \left( \Sigma_0 + \frac{\hbar}{2} \Sigma_1 \right) \right] \]  

(36)

where, by the full quantum equation, we get (at zero and first order in \( \hbar \) respectively) \( \Sigma_0 = \sqrt{\frac{|\beta|}{\kappa}} R^{3/2} + O(\sqrt{R}) \) and \( \Sigma_1 \sim \ln R \). For sufficiently large values of \( R \), the logarithmic correction decays and the solution approaches the quasi-classical limit, even if the limit \( \hbar \to 0 \) is not taken. A good estimation for the achievement of such quasi-classical limit is obtained by requiring a large value of \( \Sigma_0 \)

\[ R^3 \gg O \left( \frac{l_p^2}{|\epsilon|} \right). \]  

(37)

Comparing the above relation with inequality (24), we establish the constraint which ensures that the quasi-classical limit is reached before the potential term becomes important, explicitly

\[ l_{in} \gg \sqrt{\frac{|V(\beta)|}{1 < K^2 > |\epsilon|}}. \]  

(38)

This constraint states the degree of inhomogeneity allowed by the self-consistency of our scheme and the prediction of the spectrum obtained neglecting the potential; as far as the ground state eigenvalue \( \epsilon_{0, \gamma} \) is concerned, the above relation correlates the inhomogeneous scale to the Planck length, i.e. the cosmological horizon associated to the quantum era of the Universe.

On the other hand, we observe that the ground state probability density is a Gaussian distribution, peaked around \( R = \sigma^2/8\pi \) by the width \( l_p/\sqrt{2} \), i.e.

\[ |\theta_0(R)|^2 = \frac{1}{2\pi l_p^2} \exp \left[ -\frac{1}{l_p} \left( R - \frac{\sigma^2}{8\pi} \right)^2 \right]; \]  

(39)
such a distribution provides a singularity-free behavior of the early Universe and, in the limit \( l_P \to 0 \), it approaches, according to the classical limit request, the Dirac delta-function \( \delta(R - \sigma^2/8\pi) \).

Finally we face the question of the classical limit of the spectrum in the sense of large occupation numbers \( n \to \infty \). As the Universe approaches excited states, the eigenvalue \( \epsilon \) takes positive values and the associated dust energy density would be negative. However in the limit of very high occupation numbers, the eigenvalue approaches zero as \( 1/n \) and no significant phenomenology can come out from the negative energy density. In other words, for very large \( n \), our quantum dynamics would overlap the Wheeler-DeWitt approach (characterized just by \( \epsilon \equiv 0 \)).

Summarizing we expect that for large enough values of \( R \), the quantum dynamics reaches a quasi-classical representation and, regarding \( h \) as small, the ground state wavefunction provides a localized initial condition for the further evolution.

As a next step we want to investigate the phenomenological implications of the obtained spectrum in view of the role \( \epsilon \) can play as dark matter candidate.

V. PHENOMENOLOGY OF THE DUST FLUID: DISCUSSION AND CONCLUSIONS

In order to investigate the implications of the spectrum \([32]\) with respect to the formulation of a dark matter candidate via the non-zero eigenvalue, since the existence of a Plackian lattice for the gravitational field is a well grounded statement \([33,34]\), then we put by hands a cutoff length in our model. In the eigenvalues problem \([13]\) we deal with a perfect gas contribution and therefore we have to implement a minimal length \( l \) per particle, that is (in this section we restore \( h \) and \( \epsilon \))

\[
l^3 \equiv \frac{V}{N} = \frac{3}{2} \frac{h c l_P}{\rho_{pg} \lambda^2}, \quad (40)
\]

where \( N \) is the number of particles, \( \lambda \) the thermal length of the particles and \( \rho_{pg} \) the perfect gas energy density, i.e. \( \rho_{pg} = \sigma^2/R^2_P \leq \sigma^2/l_P^2 \) \( (R_P \) being the radius of curvature at the Planck era, to be regarded as the minimal one). Thus, by requiring \( l \geq l_P \), we get the following inequality for the quantum number \( \sigma^2 \)

\[
\sigma^2 \leq \mathcal{O}(h c l_P) \quad (41)
\]

as soon as it is recognized that the thermal length of a Planck mass particle can be suitably estimated by the Planck one (the quantum Universe has the Planck temperature as an upper limit). This constraint on the range of values available \( \sigma^2 \) implies some relevant phenomenological issues.

i)—The upper limit for \( \sigma^2 \) ensures that the spectrum is limited by below and we have \( |\epsilon| \leq \mathcal{O}(h c / l_P) \sim \mathcal{O}(10^{13} \text{GeV}) \). Since in the limit \( h \to 0 \) the quantity \( \sigma^2 \) vanishes like \( h^{3/2} \), we see that the ground state probability density behaves like \( \delta(R) \) and the singular behavior of the classical Universe is restored.

ii)—The contribution of our dark matter candidate to the actual critical parameter is provided by

\[
\Omega_{dm} \sim \frac{\rho_{dm}}{\rho_{Today}} \sim \mathcal{O} \left( \frac{\epsilon}{R^2_{Today}} \frac{d^2_{P}}{h c} \right), \quad (42)
\]

where \( \rho_{Today} \equiv 1.88 h^2 \times 10^{-29} g/cm^3 \) \( (h \in [0.4,1]) \) denoting the present critical density, \( R_{Today} \sim \mathcal{O}(10^{28} cm) \) the present radius of curvature and \( d_H \sim \mathcal{O}(10^{27} cm) \) is the Hubble size of the Universe. Using the relation \( \epsilon_0 \sim h c / l_P \), the critical parameter \( \Omega_{dm} \) rewrites

\[
\Omega_{dm} \sim \frac{d^2_{P}}{R^2_{Today}} \sim \mathcal{O}(10^{-60}) \quad (43)
\]

Such a parameter is much less than unity and it can not be recognized as the dark matter contribution (the dark matter critical parameter being estimated \( \mathcal{O}(0.3)) \).

Thus, our analysis has shown that the quantum energy of the gravitational field, described by an evolutionary theory, is not a suitable candidate for cold dark matter; from a phenomenological point of view the evolutionary and the Wheeler-DeWitt scheme overlap.

iii)—It is worth stressing that to have, for our candidate, a critical parameter order of unity, it is necessary a ground state energy of \( \mathcal{O}(10^{60}) \) times \( \epsilon_0 \) and, as discussed in \([14]\) for the case of the Friedmann-Robertson-Walker model, this would simply corresponds to fix the Planck radius of curvature \( R_P \) about to \( \mathcal{O}(10^{-21} cm) \), instead of the Planck length taken above. Despite the idea of a Planckian Universe much larger than the Planck volume is acceptable, in the context of our model, two facts are against it.

a) The Planckian radius associated to the cut-off \( R_P^{cut} \), the classical one corresponding to \( \epsilon_0 R_P^{class} \) and the mean value of \( R \) in the ground state \( < R > \), respectively read

\[
R_P^{cut} \sim l_P \sqrt{\frac{l_P \epsilon_0}{h c}}, \quad R_P^{class} \sim l_P \sqrt{\frac{l_P \epsilon_0}{h c}}, \quad < R > \sim l_P \left( \frac{l_P \epsilon_0}{h c} \right); \quad (44)
\]

these three values are of the same order \( (l_P) \), as they must, only for \( \epsilon_0 \sim h c / l_P \) (like above) and when \( \Omega_{dm} \sim 1 \), i.e. for \( \epsilon_0 \sim \mathcal{O}(10^{60} h c / l_P) \), they deeply differ.

b) For the inflationary scenario to take place in its standard formulation (at the energy scale \( \Sigma \sim \mathcal{O}(10^{14} \text{GeV}) \)),
we have to require the dust energy density to be smaller than the vacuum energy and this leads to the following value for the radius of curvature $R_i$ when inflation starts

$$R_i = \left( \frac{\epsilon_0}{\Sigma} \right)^{4/3} \frac{hc}{\epsilon_0}; \quad (45)$$

If $\epsilon_0 \sim hc/l_P$, then inflation starts, as in the standard case, at $R_i \sim O(10^{6})l_P$, while for the value of $\epsilon_0 \sim O(10^{66}hc/l_P)$ we get an initial radius of curvature 20 orders of magnitude greater. In this second case inflation would start when the Universe has a radius of curvature too large for implementing a suitable de-Sitter phase; in fact, the corresponding $e$-folding would be here about 23 and this value would be not enough to solve the horizon paradox.

Thus, we can conclude that, from a phenomenological point of view, the Evolutionary Quantum Cosmology overlaps, in the generic inhomogeneous case, the Wheeler-Dewitt description and no evidence appears of the non-zero eigenvalue in the Universe critical parameter. As a final point, we want to emphasize how the Planckian dimension of the quantum Universe, predicted by our model, provides good indications on the solution of the horizon paradox within a Quantum Mechanics endowed with a cut-off. In fact, if (like here) the mean size of the primordial Universe is comparable to the classical horizon at the Planck time, no real puzzle arises about its later strong uniformity.

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