IRREDUCIBILITY OF A HOLOMORPHIC ETA QUOTIENT IS DETERMINABLE

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Abstract. We show that a holomorphic eta quotient has only finitely many factors. We also provide an algorithm for checking irreducibility of holomorphic eta quotients by constructing an upper bound for the minimum of the levels of the proper factors of a reducible holomorphic eta quotient.

1. INTRODUCTION

Imagine a situation where one knows the definition of the prime numbers and a few striking results about them without having any clue about how to tell whether a given integer is prime or not! Wouldn’t that be a plight? Well, our present situation regarding irreducible holomorphic eta quotients is very similar to that: A holomorphic eta quotient (see 1.2) is irreducible if it is not divisible by any holomorphic eta quotients except 1 and itself. We see several examples of such eta quotients in [6, 7, 14]. It is known that there are only finitely many irreducible holomorphic eta quotients of a given level (see [7]) and there are only finitely many primitive and irreducible holomorphic eta quotients of a given weight (see 1.6 or [5, 14]). Also, the complete classification of holomorphic eta quotients of weight 1/2 (see [6, 14, 21]) enabled us to check the irreducibility of any holomorphic eta quotient of weight 1 (see Corollary 3 and Algorithm 1). However, in general, it is extremely difficult to distinguish between the irreducible holomorphic eta quotients and the reducible ones. Before we explain the root cause of this difficulty, it is necessary to define a few things:

The Dedekind eta function is defined by the infinite product

\[ \eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \text{ for all } z \in \mathcal{H}, \]

where \( q^r = q^r(z) := e^{2\pi irz} \) for all \( r \) and \( \mathcal{H} := \{ \tau \in \mathbb{C} | \text{Im}(\tau) > 0 \} \). The function \( \eta \) is a modular form of weight 1/2 with a multiplier system on \( \text{SL}_2(\mathbb{Z}) \) (see [11]). An eta quotient \( f \) is a finite product of the form

\[ \prod \eta_d^{X_d}, \]

where \( d \in \mathbb{N}, \eta_d \) is the rescaling of \( \eta \) by \( d \), defined by

\[ \eta_d(z) := \eta(dz) \text{ for all } z \in \mathcal{H} \]

and the exponents \( X_d \in \mathbb{Z} \). Eta quotients naturally inherit modularity from \( \eta \); The eta quotient \( f \) in (1.2) transforms like a modular form of

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weight $\frac{1}{2} \sum_d X_d$ with a multiplier system on suitable congruence subgroups of $\text{SL}_2(\mathbb{Z})$: The largest among these subgroups is

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\},$$

where

$$N := \text{lcm}\{d \in \mathbb{N} \mid X_d \neq 0\}.$$  

We call $N$ the level of $f$. Since $\eta$ does not vanish anywhere on $\mathfrak{H}$, eta quotients can have nonzero orders of vanishing only at the cusps. An eta quotient is holomorphic if it does not have any pole at the cusps. Let $f$ and $g$ be two holomorphic eta quotients. If the eta quotient $f/g$ is holomorphic, we say that $f$ is divisible by $g$ and in particular, if $g \notin \{1, f\}$, we call $g$ a proper factor of $f$. A holomorphic eta quotient $f$ is irreducible if it has no proper factors. By a descent argument on the weights of holomorphic eta quotients, it follows that each holomorphic eta quotient is a product of irreducible holomorphic eta quotients, though such a factorization may not be unique. Irreducible holomorphic eta quotients were first considered by Zagier, who conjectured [21]:

(1.6) **There are only finitely many primitive and irreducible holomorphic eta quotients of a given weight.**

An eta quotient $f$ is primitive if no eta quotient $h$ and no integer $\nu > 1$ satisfy the equation $f = h_\nu$, where $h_\nu(z) := h(\nu z)$ for all $z \in \mathfrak{H}$. Since the weight of the product of two modular forms is the sum of their weights and since every nonconstant modular form has positive weight (see 3.11), weights of the factors of a holomorphic eta quotient $f$ are bounded above by the weight of $f$. As 1/2 is the least possible weight of a nonconstant holomorphic eta quotient, every holomorphic eta quotient of weight 1/2 is irreducible. So, the above conjecture implies the finiteness of the set of primitive holomorphic eta quotients of weight 1/2. Indeed, Zagier also conjectured [21]:

(1.7) **The only primitive holomorphic eta quotients of weight 1/2 are**

$$\eta, \eta^2, \eta^3, \eta^4, \eta^5, \eta \eta_4, \eta \eta_6, \eta \eta_5, \eta \eta_2 \eta_3, \eta \eta_6, \eta \eta_2 \eta_5, \eta \eta_2 \eta_3 \eta_6, \eta \eta_2 \eta_4 \eta_5, \eta \eta_2 \eta_3 \eta_5 \eta_6, \eta \eta_2 \eta_3 \eta_4 \eta_5 \eta_6.$$  

Both of Zagier’s conjectures were validated by Mersmann [14]. Since holomorphic eta quotients can have zeros only at the cusps, from the finiteness of the set of equivalence classes of the cusps of $\Gamma_0(N)$ (see 3.6), it follows via the valence formula (3.11) that there are only finitely many holomorphic eta quotients of a fixed pair of weight and level. Now, suppose we require to check whether a given holomorphic eta quotient $f$ of weight $k/2$ and level $N$ is irreducible. Recall that the weights of the factors of $f$ are bounded above by $k/2$. However, a priori we do not know how large the level of an arbitrary factor of $f$ might be. Because, the notion of reducibility
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of a holomorphic eta quotient allows factors of arbitrary levels. For example, we have

(1.8) \[ \frac{\eta \eta_2 \eta_6}{\eta_3} = \frac{\eta \eta_4 \eta_6^2}{\eta_2 \eta_3 \eta_2} \times \frac{\eta_2^2 \eta_12}{\eta_4 \eta_6}, \]

where a reducible holomorphic eta quotient of level 6 is factored into two holomorphic eta quotients of level 12. Scenarios like the above might make one wonder whether there could be a holomorphic eta quotient with infinitely many factors. In particular, such scenarios pose a serious challenge in determination of irreducibility of holomorphic eta quotients. We meet this challenge in two different ways: Our first reflex is to use the most powerful canon in this domain till date, viz. Mersmann’s finiteness theorem (1.6). However, the use of Mersmann’s theorem in this case yields only partial success: For a positive integer \( k \), we call the least common multiple of the levels of all primitive and irreducible holomorphic eta quotients of weights less than or equal to \( k/2 \) the \( k \)-th Mersmann level and denote it by \( M_k \). In Lemma 1, we see that the levels of the factors of a holomorphic eta quotient \( f \) of weight \( k/2 \) and level \( N \) divide \( \text{lcm}(N, M_{k-1}) \). Since every set of holomorphic eta quotients of bounded weights and levels is finite (see 3.13), we conclude that every holomorphic eta quotient has only finitely many factors. For example, from Zagier’s list (1.7), we see that \( M_1 = 12 \). Hence, the level of every factor of a holomorphic eta quotient of weight 1 and level \( N \) divides \( \text{lcm}(N, 12) \). In particular, 12 is the maximum possible level of the factors of a holomorphic eta quotient of weight 1 and level 6 (see 1.8). The only problem with this approach is that almost nothing is known about the values of \( M_k \) for \( k > 1 \), except their existence. So, at the end we shall circumvent altogether the use of Mersmann’s finiteness theorem and proceed instead via the notion of quasi-irreducibility: We call a holomorphic eta quotient \( f \) of level \( N \) quasi-irreducible if \( f \) has no proper factor whose level divides \( N \). From Theorem 1 in [7], it follows:

The weights of the quasi-irreducible holomorphic eta quotients of level \( N \) are bounded above by \( \kappa(N)/2 \), where

(1.9) \[ \kappa(N) := \varphi(\text{rad}(N)) \prod_{\substack{p \in \mathcal{P}_N \\text{ odd} \\text{ or } N \text{ prime} \\text{or } p^m \| N}} ((m - 1)(p - 1) + 2). \]

Here \( \varphi, \varphi_N \) and \( \text{rad}(N) \) denote resp. Euler’s totient function, the set of distinct prime divisors of \( N \) and the product of these primes.

The above statement together with Lemma 1 implies that each holomorphic eta quotient \( f \) of level \( N \) either has a proper factor whose level divides \( N \) or the weights of all the factors of \( f \) divide

(1.10) \[ \mathcal{L}_N := \text{lcm}\left(N, M_{\kappa(N)-1}\right) \]

(see Theorem 1). By \( \mathcal{M}_N \), we denote the minimum of the levels of the proper factors of a reducible holomorphic eta quotient of level \( N \). Clearly, \( \mathcal{M}_N \) divides \( \mathcal{L}_N \). However, in general, the bound \( \mathcal{L}_N \) is not explicit due to the lack of information about the size of \( M_{\kappa(N)-1} \). So, in our second approach, we construct an upper bound for \( \mathcal{M}_N \) recursively, using only the notion of quasi-irreducibility (see Corollary 7). In particular, this approach
works particularly well for eta quotients of prime power levels: It shows that every reducible holomorphic eta quotient of a prime power level $N$ has a proper factor whose level divides $N$ (see Theorem 2). As a consequence, it follows that if we rescale by a positive integer an irreducible holomorphic eta quotient $f$ of a prime power level or if we take an Atkin-Lehner involution* of $f$, we obtain another irreducible holomorphic eta quotient (see Corollary 4).

Also, with the support of a huge amount of numerical evidence, we speculate that similar statements as above hold for holomorphic eta quotients in general:

**Conjecture 1** (Reducibility Conjecture). *Every reducible holomorphic eta quotient of an arbitrary level $N$ has a proper factor whose level divides $N$.*

**Conjecture 2** (Irreducibility Conjecture). *The rescalings of an irreducible holomorphic eta quotient by positive integers are irreducible.*

In Corollary 18, we see that the last conjecture is equivalent to the following, which in turn, follows from Conjecture 1 (see Corollary 17):

**Conjecture 2’. The images of an irreducible holomorphic eta quotient under the Atkin-Lehner involutions are irreducible.**

An eta quotient on $\Gamma_0(N)$ is an eta quotient whose level divides $N$. If a holomorphic eta quotient $f$ has a decomposition into a product of two proper factors $g$ and $h$ such that both $g$ and $h$ are eta quotients on $\Gamma_0(N)$, then we say that $f$ is factorizable on $\Gamma_0(N)$. For example, the eta quotient of level 6 in (1.8) also has the following factorization into holomorphic eta quotients of level 6 and level 2:

\[(1.11) \quad \frac{\eta \eta_2 \eta_6}{\eta_3} = \frac{\eta^2 \eta_6}{\eta_2 \eta_3} \times \frac{\eta^2}{\eta}.
\]

So, the eta quotient $\frac{\eta \eta_2 \eta_6}{\eta_3}$ is indeed factorizable on $\Gamma_0(6)$.

The notions of irreducibility and factorizability make sense also for modular forms in general. In [7], we see that every modular form with the trivial multiplier system on $\text{SL}_2(\mathbb{Z})$ has a unique factorization of the form:

\[(1.12) \quad C_0 E_a^6 E_b^6 \prod_{t \in \mathbb{C}^*} (E_3^3 - t E_6^2)^{c_t},
\]

for some $C_0 \in \mathbb{C}$ and some nonnegative integers $a, b, c_t$, where $c_t$ is zero for all but finitely many $t$. For all $k \in 2\mathbb{N}$, by $E_k$ here we denote the normalized Eisenstein series of weight $k$:

\[(1.13) \quad E_k(z) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,
\]

where the function $\sigma_{k-1} : \mathbb{N} \to \mathbb{N}$ is given by

\[(1.14) \quad \sigma_{k-1}(n) := \sum_{d|n} d^{k-1}
\]

*We briefly recall the notion of Atkin-Lehner involutions in Section 3.
and the $k$-th Bernoulli number $B_k$ is defined by

\[(1.15) \quad \frac{t}{e^t-1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} \cdot t^k.\]

For each even integer $k > 2$, $E_k$ is a modular form of weight $k$ on $\text{SL}_2(\mathbb{Z})$ (see [21]). From [7], we also know that none of $E_4$, $E_6$ or $E_3^1 - tE_6^2$ for all $t \in \mathbb{C}^*$ is factorizable on $\text{SL}_2(\mathbb{Z})$, whereas each of them has proper factors on $\Gamma_0(4)$. So, the analog of Conjecture 1 fails for modular forms in general. In the Appendix, we discuss a little more about factorizability and irreducibility of certain modular forms.

We call a holomorphic eta quotient $f$ of level $N$ quasi-irreducible if it is not factorizable on $\Gamma_0(N)$. Clearly, the following is equivalent to Conjecture 1:

**Conjecture 1'.** Every quasi-irreducible holomorphic eta quotient is irreducible.

Define the extract of an eta quotient $f$ as the primitive eta quotient $f_0$ (see 1.6) of which $f$ is a rescaling by a positive integer. Now, Conjecture 2 implies the following restatement of Conjecture 1' :

**Conjecture 1''.** A holomorphic eta quotient is irreducible if its extract is quasi-irreducible.

Since quasi-irreducibility of holomorphic eta quotients on $\Gamma_0(N)$ is determinable (see Algorithm 1), the truth of Conjecture 1 would enable us to check irreducibility of holomorphic eta quotients in general.

**Algorithm 1** (Checking factorizability on $\Gamma_0(N)$). If a holomorphic eta quotient $f$ is factorizable on $\Gamma_0(N)$, then there exists a holomorphic eta quotient $g \notin \{1, f\}$ on $\Gamma_0(N)$, such that

$$\text{ord}_s(g; \Gamma_0(N)) \leq \text{ord}_s(f; \Gamma_0(N))$$

at all cusps $s$ of $\Gamma_0(N)$ (see 3.8). Since eta quotients do not have zeros or poles on the upper half-plane, the valence formula (3.11) implies that the eta quotient $g$ is uniquely determined by its orders at the cusps of $\Gamma_0(N)$. Since for all cusps $s$ of $\Gamma_0(N)$, $24\text{ord}_s(g; \Gamma_0(N))$ is a nonnegative integer (see 3.7), and since $\Gamma_0(N)$ has only finitely many cusps, it follows that the search for such a $g$ halts eventually.*

We also see an irreducibility criterion for holomorphic eta quotients in [3].

## 2. The results

We show that a holomorphic eta quotients $f$ is irreducible, if there does not exist any proper factor of $f$ up to sufficiently large levels:

**Theorem 1.** The levels of all the factors of a quasi-irreducible holomorphic eta quotient of level $N$ divide $\mathcal{L}_N$, where $\mathcal{L}_N$ is as defined in (1.10).

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*This naïve algorithm could be improved by using (3.9) and by implementing standard linear algebraic techniques for finding a suitable lattice point (if any) in the relevant compact subset (see [7]) of the cone generated by the columns of the inverse of the order matrix $A_N$ (see 3.15, 3.18 and 3.19).
Corollary 1. The minimum of the levels of the proper factors of a reducible holomorphic eta quotient of level $N$ is bounded above in terms of $N$. \hfill \Box

Theorem 1 follows from the upper bound on the weights of the quasi-irreducible holomorphic eta quotients of level $N$ (see 1.9) and from the following lemma which we prove in Section 4.

Lemma 1. Let $k, N \in \mathbb{N}$ with $k \geq 2$ and let $f$ be a holomorphic eta quotient of weight $k/2$ and level $N$. Then the levels of the factors of $f$ divide \( \text{lcm}(N, M_{k-1}) \), where $M_{k-1}$ is the $(k-1)$-th Mersmann level.

Since the set of holomorphic eta quotients of a fixed weight and a fixed level is finite (see 3.13), we conclude:

Corollary 2. There are only finitely many factors of a holomorphic eta quotient. \hfill \Box

From Zagier’s list (1.7), we obtain $M_1 = 12$. So, Lemma 1 implies that

Corollary 3. The levels of the factors of a holomorphic eta quotient of weight 1 and level $N$ divide \( \text{lcm}(N, 12) \). \hfill \Box

In particular, if $12 | N$, then every weight 1 quasi-irreducible holomorphic eta quotient of level $N$ is irreducible. Since the $k$-th Mersmann level is not explicitly known for any $k > 1$, we shall circumvent altogether the use of Lemma 1 to prove a few effective results about irreducibility of holomorphic eta quotients:

Theorem 2. Every quasi-irreducible holomorphic eta quotient of level $p^n$ is irreducible for each prime $p$ and natural number $n$.

For example, $\eta^p / \eta_p$ is a quasi-irreducible holomorphic eta quotient for any prime $p$ (see Lemma 3 in [7] or Lemma 2.3 in [4]). So, by Theorem 2, it is irreducible (this follows also from Theorem 3 in [7]). In Section 7, we prove the above theorem and in Section 8, we deduce the following.

Corollary 4. Let $f$ be a quasi-irreducible holomorphic eta quotient of a prime power level. Then the rescalings of $f$ by the positive integers as well as the images of $f$ under the Atkin-Lehner involutions are irreducible.

Since $\eta^p / \eta_p$ is quasi-irreducible, Corollary 4 implies:

Corollary 5. For each prime $p$ and positive integer $m$, the holomorphic eta quotients

$$\frac{\eta^p_m}{\eta_pm} \text{ and } \frac{\eta^p_{pm}}{\eta_m}$$

are irreducible. \hfill \Box

Now, let us consider holomorphic eta quotients of a general level $N$. By $\text{rad}(N)$, we denote the product of distinct prime divisors of $N$. For a divisor $d$ of $N$, we say that $d$ exactly divides $N$ and write $d \| N$ if $\gcd(d, N/d) = 1$. In order to state the next result, we also introduce the functions $\Upsilon : \mathbb{R}_{>1} \rightarrow \mathbb{N}$ and $R_k : \mathbb{N} \rightarrow \mathbb{Q}$ defined by

$$\begin{align*}
\Upsilon(x) &:= \begin{cases} 
1 & \text{if } x < 2 \\
\prod_{j \in \mathbb{N}} (x - j)^{2j-1} & \text{otherwise}
\end{cases} \\
R_k : \mathbb{N} &\rightarrow \mathbb{Q} \text{ defined by}
\end{align*}$$

(2.1)
and
\[
R_k(N) := k \prod_{p \in \mathcal{P}_N} \left( \frac{p+1}{p-1} \right)^{\min\{2,m\}},
\]
where \(\mathcal{P}_N\) denotes the set of prime divisors of \(N\). We prove the following in Section 9.

**Theorem 3.** Let \(k, N \in \mathbb{Z}_{>1}\) and let \(f\) be a reducible holomorphic eta quotient of weight \(k/2\) and level \(N\). Let \(M\) be the least positive integer such that \(f\) is factorizable on \(\Gamma_0(M)\). Then we have
\[
M \leq (2Nk)^{2R_{k-1}(N)-1} \Upsilon(R_{k-1}(N))
\]
and \(\text{rad}(M) = \text{rad}(N)\).

**Corollary 6.** Irreducibility of a holomorphic eta quotient is determinable. \(\square\)

From the above theorem and from the upper bound on the weights of the quasi-irreducible holomorphic eta quotients of level \(N\), it follows that

**Corollary 7.** Let \(N > 1\) be an integer and let \(f\) be a reducible holomorphic eta quotient of level \(N\). Let \(M\) be the least positive integer such that \(f\) is factorizable on \(\Gamma_0(M)\). Then we have
\[
M \leq (2N\kappa(N))^{2R(N)-1} \Upsilon(R(N))
\]
and \(\text{rad}(M) = \text{rad}(N)\). Here \(\kappa(N)\) is as in (1.9) and \(R(N) := R_{\kappa(N)-1}(N)\) is defined by (2.2). \(\square\)

The following lemma, which we prove in Sections 5 and 6, would be instrumental in establishing both Theorem 2 and Theorem 3.

**Lemma 2.** Let \(N > 1\) be an integer and let \(f\) be a reducible holomorphic eta quotient on \(\Gamma_0(N)\).

(a) Then there exists a multiple \(M\) of \(N\) with \(\text{rad}(M) = \text{rad}(N)\) such that \(f\) is factorizable on \(\Gamma_0(M)\).

(b) Let \(M > N\) be as above and suppose there exists a factor \(g\) of \(f\) on \(\Gamma_0(M)\) such that the exponent of \(\eta_{M/N_0}\) in \(g\) is nonzero for some \(t|N_0\), where \(N_0\) is the greatest common exact divisor of \(M\) and \(N\). Let \(M_0 := M/N\). If
\[
M_0 \geq 2^\omega(M_0) \text{rad}(M_0) \tilde{\varphi}(N_0),
\]
then \(f\) has a factor of weight greater than or equal to \(\varphi(\text{rad}(N))/2\) on \(\Gamma_0(N)\). Moreover, if strict inequality holds in (2.5), then this factor of \(f\) is nontrivial. Here \(\omega(M_0)\) denotes the number of distinct prime divisors of \(M_0\), \(\varphi\) denotes Euler’s totient function and the function \(\tilde{\varphi} : \mathbb{N} \to \mathbb{N}\) is defined by
\[
\tilde{\varphi}(N) := N \prod_{p \in \mathcal{P}_N} \left( p - \frac{1}{p} \right),
\]
where \(\mathcal{P}_N\) denotes the set of prime divisors of \(N\).
3. Notations and the basic facts

By \( \mathbb{N} \) we denote the set of positive integers. We define the operation \( \odot : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) by

\[
(3.1) \quad d_1 \odot d_2 := \frac{\text{lcm}(d_1, d_2)}{\gcd(d_1, d_2)}.
\]

For \( N \in \mathbb{N} \), by \( \mathcal{D}_N \) (resp. \( \mathcal{E}_N \)) we denote the set of divisors (resp. exact divisors) of \( N \). It follows trivially that \( (\mathcal{E}_N, \odot) \) is a boolean group (i.e., each element of \( \mathcal{E}_N \) is the inverse of itself) and that \( \mathcal{E}_N \) acts on \( \mathcal{D}_N \) by \( \odot \). For \( X \in \mathbb{Z}^{\mathcal{D}_N} \), we define the eta quotient \( \eta^X \) by

\[
(3.2) \quad \eta^X := \prod_{d \in \mathcal{D}_N} \eta_d^{X_d},
\]

where \( X_d \) is the value of \( X \) at \( d \in \mathcal{D}_N \) whereas \( \eta_d \) denotes the rescaling of \( \eta \) by \( d \). Clearly, the level of \( \eta^X \) divides \( N \). In other words, \( \eta^X \) transforms like a modular form on \( \Gamma_0(N) \). For \( N, k \in \mathbb{Z} \), let \( \mathcal{E}_{N,k} \) (resp. \( \mathcal{E}_{N,k}^! \)) be the set of eta quotients (resp. holomorphic eta quotients) of weight \( k/2 \) on \( \Gamma_0(N) \). For \( n \in \mathcal{E}_N \), we define the Atkin-Lehner map \( \text{al}_{n,N} : \mathcal{E}_{N,k} \to \mathcal{E}_{N,k}^! \) by

\[
(3.3) \quad \text{al}_{n,N} \left( \prod_{d \in \mathcal{D}_N} \eta_d^{X_d} \right) := \prod_{d \in \mathcal{D}_N} \eta_d^{X_d \cdot a \odot d'}
\]

where \( \text{al}_{n,N} \) is the involution in \( \mathcal{E}_{N,k} \). It is easy to show that the above definition is compatible with the usual definition (see [1]) of Atkin-Lehner involutions of modular forms on \( \Gamma_0(N) \) up to multiplication by a complex number (see the Preliminaries in [4]). So in particular, if \( f \) is an eta quotient on \( \Gamma_0(N) \) and \( n \in \mathcal{E}_N \), then \( f \) is holomorphic if and only if so is \( \text{al}_{n,N}(f) \).

Recall that a holomorphic eta quotient \( f \) on \( \Gamma_0(N) \) is an eta quotient on \( \Gamma_0(N) \) that does not have any poles at the cusps. Under the action of \( \Gamma_0(N) \) on \( \mathbb{P}^1(\mathbb{Q}) \) by Möbius transformation, for \( a, b \in \mathbb{Z} \) with \( \gcd(a, b) = 1 \), we have

\[
(3.4) \quad [a : b] \sim_{\Gamma_0(N)} [a' : \gcd(N, b)]
\]

for some \( a' \in \mathbb{Z} \) which is coprime to \( \gcd(N, b) \) (see [8]). We identify \( \mathbb{P}^1(\mathbb{Q}) \) with \( \mathbb{Q} \cup \{\infty\} \) via the canonical bijection that maps \([a : \lambda]\) to \(a/\lambda\) if \(\lambda \neq 0\) and to \(\infty\) if \(\lambda = 0\). For \( s \in \mathbb{Q} \cup \{\infty\} \) and a weakly holomorphic modular form \( f \) on \( \Gamma_0(N) \), the order of \( f \) at the cusp \( s \) of \( \Gamma_0(N) \) is the exponent of \( q^{1/w_s} \) occurring with the first nonzero coefficient in the \( q \)-expansion of \( f \) at the cusp \( s \), where \( w_s \) is the width of the cusp \( s \) (see [8], [18]). Hence in particular, for \( N|N' \), we have

\[
(3.5) \quad \text{ord}_s(f; \Gamma_0(N')) = \frac{w'_s}{w_s} \cdot \text{ord}_s(f; \Gamma_0(N)),
\]

where \( w_s \) (resp. \( w'_s \)) is the width of the cusp \( s \) of \( \Gamma_0(N) \) (resp. \( \Gamma_0(N') \)). The following is the set of the equivalence classes of the cusps of \( \Gamma_0(N) \) (see [8, 13]):

\[
(3.6) \quad \mathcal{S}_N := \left\{ \frac{a}{t} \in \mathbb{Q} \mid t \in \mathcal{D}_N, a \in \mathbb{Z}, \gcd(a, t) = 1 \right\} / \sim,
\]
where \( \frac{a}{t} \sim \frac{b}{t} \) if and only if \( a \equiv b \pmod{\gcd(t, N/t)} \). For \( d \in \mathcal{D}_N \) and for \( s = \frac{a}{t} \in \mathcal{S}_N \) with \( \gcd(a, t) = 1 \), we have

\[
\text{ord}_s(\eta_d; \Gamma_0(N)) = \frac{N \cdot \gcd(d, t)^2}{24 \cdot d \cdot \gcd(t^2, N)} \in \frac{1}{24}\mathbb{Z}
\]

(see [13]). It is easy to check the above inclusion when \( N \) is a prime power. The general case follows by multiplicativity (see 3.15 and 3.18). It follows that for all \( X \in \mathbb{Z}^{D_N} \), we have

\[
\text{ord}_s(\eta^X; \Gamma_0(N)) = \frac{1}{24} \sum_{d \in \mathcal{D}_N} \frac{N \cdot \gcd(d, t)^2}{d \cdot \gcd(t^2, N)} X_d.
\]

In particular, that implies

\[
\text{ord}_{\phi/t}(\eta^X; \Gamma_0(N)) = \text{ord}_{1/t}(\eta^X; \Gamma_0(N))
\]

for all \( t \in \mathcal{D}_N \) and for all the \( \phi(\gcd(t, N/t)) \) inequivalent cusps of \( \Gamma_0(N) \) represented by rational numbers of the form \( \frac{a}{t} \in \mathcal{S}_N \) with \( \gcd(a, t) = 1 \). Let \( \psi(N) \) denote the index of \( \Gamma_0(N) \) in \( \text{SL}_2(\mathbb{Z}) \). Then \( \psi(N) \) is given by

\[
\psi(N) := N \cdot \prod_{p \mid N} \left( 1 + \frac{1}{p} \right)
\]

(see [8]). The valence formula for \( \Gamma_0(N) \) (see [2, 18]) states:

\[
\sum_{P \in \Gamma_0(N) \backslash \mathfrak{H}} \phi_P \cdot \text{ord}_P(f) + \sum_{s \in \mathcal{S}_N} \text{ord}_s(f; \Gamma_0(N)) = \frac{k \cdot \psi(N)}{24},
\]

where \( k \in \mathbb{Z}, f : \mathfrak{H} \to \mathbb{C} \) is a meromorphic function that transforms like a modular forms of weight \( k/2 \) on \( \Gamma_0(N) \) which is also meromorphic at the cusps of \( \Gamma_0(N) \) and \( n_P \) is the number of elements in the stabilizer of \( P \) in the group \( \Gamma_0(N)/\{\pm I\} \), where \( I \in \text{SL}_2(\mathbb{Z}) \) denotes the identity matrix.

For a meromorphic function \( f : \mathfrak{H} \to \mathbb{C} \) which transforms like a modular form on \( \Gamma_0(N) \), let \( \Pi_f \) denote the product of all the images of \( f \) under the operations by the elements of a minimal set of right coset representatives of \( \Gamma_0(N) \) in \( \text{SL}_2(\mathbb{Z}) \). Then \( \Pi_f \) transforms like a modular form on \( \text{SL}_2(\mathbb{Z}) \). It is easy to check that if the valence formula holds for \( \Pi_f \), then it also holds for \( f \). Thus, the case for an arbitrary \( N \) in (3.11) reduces to the case \( N = 1 \) which in turn, follows from contour integration of the logarithmic derivative of \( \Pi_f \) along the boundary of a fundamental domain for \( \text{SL}_2(\mathbb{Z}) \) (see [18]).

In particular, if \( f \) is an eta quotient, then from (3.11) we obtain

\[
\sum_{s \in \mathcal{S}_N} \text{ord}_s(f; \Gamma_0(N)) = \frac{k \cdot \psi(N)}{24},
\]

because eta quotients do not have poles or zeros on \( \mathfrak{H} \). It follows from (3.12), (3.6) and (3.9) that for an eta quotient \( f \) of weight \( k/2 \) on \( \Gamma_0(N) \), the valence formula further reduces to

\[
\sum_{1 \mid N} \phi(\gcd(t, N/t)) \cdot \text{ord}_{1/t}(f; \Gamma_0(N)) = \frac{k \cdot \psi(N)}{24},
\]
where \( \varphi \) denotes Euler’s totient function. Since \( \text{ord}_{1/t}(f: \Gamma_0(N)) \in \frac{1}{24} \mathbb{Z} \) (see 3.7), from (3.13) we conclude that the number of holomorphic eta quotients of weight \( k/2 \) on \( \Gamma_0(N) \) is at most the number of solutions of the equation \( \sum_{t|N} \varphi(\gcd(t, N/t)) \cdot x_t = k \cdot \psi(N) \) in nonnegative integers \( x_t \).

Also, the following result of Mersmann / Rouse-Webb (see [14], Theorem 2 in [19] or Corollary 1 in [5]) implies an upper bound on the number of such eta quotients:

\[
\text{For } X \in \mathbb{Z}^{D_N}, \text{ let } \eta^X \text{ be a holomorphic eta quotient of weight } k/2 \text{ on } \Gamma_0(N). \text{ Then we have } \|X\| \leq R_k(N).
\]

Here \( \|X\| := \sum_d |X_d| \) and the function \( R_k : \mathbb{N} \to \mathbb{Q} \) is as defined in (2.2).

Moreover, if the holomorphic eta quotient \( \eta^X \) is not factorizable on \( \Gamma_0(N) \), then Theorem 1 in [7] implies that \( \|X\| \) is bounded above by a function of only \( N \) (see 1.9 and 3.14).

We define the order map \( O_N : \mathbb{Z}^{D_N} \to \frac{1}{24} \mathbb{Z}^{D_N} \) of level \( N \) as the map which sends \( X \in \mathbb{Z}^{D_N} \) to the ordered set of orders of the eta quotient \( \eta^X \) at the cusps \( \{1/t\}_{t \in D_N} \) of \( \Gamma_0(N) \). Also, we define the order matrix \( A_N \in \mathbb{Z}^{D_N \times D_N} \) of level \( N \) by

\[
A_N(t, d) := 24 \cdot \text{ord}_{1/t}(\eta_d: \Gamma_0(N))
\]

for all \( t, d \in D_N \). For example, for a prime power \( p^n \), we have

\[
A_{p^n} = \begin{pmatrix}
p^n & p^{n-1} & \ldots & p & 1 \\
p^{n-2} & p^{n-1} & \ldots & p & 1 \\
p^{n-4} & p^{n-3} & \ldots & p & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & p & p^2 & \ldots & p^{n-1} & p^{n-2} \\
1 & p & p^2 & \ldots & p^{n-1} & p^n
\end{pmatrix}.
\]

By linearity of the order map, we have

\[
O_N(X) = \frac{1}{24} \cdot A_N X.
\]

For \( r \in \mathbb{N} \), if \( Y, Y' \in \mathbb{Z}^{D_N} \) is such that \( Y - Y' \) is nonnegative at each element of \( D_N \), then we write \( Y \geq Y' \). In particular, for \( X \in \mathbb{Z}^{D_N} \), the eta quotient \( \eta^X \) is holomorphic if and only if \( A_N X \geq 0 \).

From (3.15) and (3.7), we note that \( A_N(t, d) \) is multiplicative in \( N, t \) and \( d \). Hence, it follows that

\[
A_N = \bigotimes_{p^\ell|N} A_{p^\ell},
\]

where by \( \otimes \), we denote the Kronecker product of matrices.*

---

*Kronecker product of matrices is not commutative. However, since any given ordering of the primes dividing \( N \) induces a lexicographic ordering on \( D_N \) with which the entries of \( A_N \) are indexed, Equation (3.18) makes sense for all possible orderings of the primes dividing \( N \).
It is easy to verify that for a prime power $p^n$, the matrix $A_{p^n}$ is invertible with the tridiagonal inverse:

$$A_{p^n}^{-1} = \frac{1}{p^n \cdot (p - \frac{1}{p})} \begin{pmatrix} p & -p & & 0 \\ -1 & p^2 + 1 & -p^2 & \\ & -p & p \cdot (p^2 + 1) & -p^3 \\ & & \ddots & \ddots & \\ 0 & & & -p^2 & p^2 + 1 & -1 \\ & & & & -p & \\ & & & & & 1 \\ \\ & & & & & & 1 \\ & & & & & & & 1 \\ & & & & & & & & 1 \\ & & & & & & & & & 1 \\ & & & & & & & & & & 1 \\ & & & & & & & & & & & 1 \\ & & & & & & & & & & & & 1 \\ & & & & & & & & & & & & & 1 \\ & & & & & & & & & & & & & & 1 \\ & & & & & & & & & & & & & & & 1 \\ & & & & & & & & & & & & & & & & 1 \\ & & & & & & & & & & & & & & & & & 1 \\ \end{pmatrix},$$

where for each positive integer $j < n$, the nonzero entries of the column $A_{p^n}^{-1}(p^j, p^j)$ are the same as those of the column $A_{p^n}^{-1}(p^0, p^0)$ shifted down by $j - 1$ entries and multiplied with $p^{\min\{j - 1, n - j - 1\}}$. More precisely,

$$\tilde{\varphi}(p^n) \cdot A_{p^n}^{-1}(p^i, p^j) = \begin{cases} p & \text{if } i = j = 0 \text{ or } i = j = n \\ -p^{\min\{j + n - j\}} & \text{if } |i - j| = 1 \\ p^{\min\{j - 1, n - j - 1\}} \cdot (p^2 + 1) & \text{if } 0 < i = j < n \\ 0 & \text{otherwise}, \end{cases}$$

where $\tilde{\varphi}$ is as defined in (2.6). For general $N$, the invertibility of the matrix $A_N$ now follows by (3.18). Hence, any eta quotient on $\Gamma_0(N)$ is uniquely determined by its orders at the set of the cusps $\{1/t\}_{t \in D_N}$ of $\Gamma_0(N)$. In particular, for distinct $X, X' \in \mathbb{Z}^{D_N}$, we have $\eta^X \neq \eta^{X'}$. The last statement is also implied by the uniqueness of $q$-series expansion: Let $\eta^X$ and $\eta^{X'}$ be the eta products (i.e. $\hat{X}, \hat{X}' \geq 0$) obtained by multiplying $\eta^X$ and $\eta^{X'}$ with a common denominator. The claim follows by induction on the weight of $\eta^X$ (or equivalently, the weight of $\eta^{X'}$) when we compare the corresponding first two exponents of $q$ occurring in the $q$-series expansions of $\eta^X$ and $\eta^{X'}$.

We define $1_N$ and $\alpha_N \in \mathbb{Q}^{D_N}$ by

$$1_N(t) := 1 \text{ for all } t \in D_N \text{ and } \alpha_N := (A_N^{-1})^T 1_N.$$ 

Comparing (3.17) with (3.13) and recalling that for $X \in \mathbb{Z}^{D_N}$, the weight the eta quotient $\eta^X$ is $\frac{1}{2} \sum_{d \in D_N} X_d$, we get

$$\alpha_N(t) = \frac{\varphi(\gcd(t, N/T))}{\psi(N)} \text{ for all } t \in D_N.$$

Equation (3.22) also follows directly from (3.19) and (3.18). So, rather than obtaining (3.13) as a corollary to the valence formula for $\Gamma_0(N)$, one could also deduce it from (3.17) and (3.22) (see the Preliminaries in [4]).

Next, we briefly recall eta quotients with rational exponents (see [10]). We define $L : \mathcal{O} \to \mathbb{C}$ by

$$L(z) := \frac{\pi i z}{12} + \sum_{n \in \mathbb{N}} \log(1 - e^{2\pi i n z}) = \frac{\pi i z}{12} - \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} \frac{e^{2\pi i m n z}}{m}.$$
The double series above converges absolutely and uniformly on compact subsets of \( \mathcal{H} \). So, \( L \) is holomorphic on \( \mathcal{H} \). Also, \( L \) exhibits a nice transformation behaviour under the action of \( SL_2(\mathbb{Z}) \) on \( \mathcal{H} \), from which the modular transformation property \( \eta = e^L \) follows (See [17]). For \( X \in \mathbb{Q}^{D_N} \), we define

\[
\eta^X := \prod_{d \in D_N} e^{X_d L_d},
\]

where \( X_d \) is the value of \( X \) at \( d \in D_N \) whereas \( L_d \) denotes the rescaling of \( L \) by \( d \). The modularity of \( \eta^X \) under \( \Gamma_0(N) \) again follows from the transformation property of \( L \) (see the Preliminaries in [4]). In particular, holomorphic eta quotients with rational exponents provide us with examples of modular forms of arbitrary rational weights.

Let \( f \) be an eta quotient with fractional exponents on \( \Gamma_0(N) \). Then there exists an \( n \in \mathbb{N} \) such that \( g := f^n \) is an eta quotient with integer coefficients. Naturally, the order of \( f \) at a cusp \( s \) of \( \Gamma_0(N) \) is defined by

\[
\text{ord}_s(f; \Gamma_0(N)) := \frac{\text{ord}_s(g; \Gamma_0(N))}{n}.
\]

4. Proof of Theorem 1

Theorem 1 follows immediately from (1.9) and Lemma 1. We prove this lemma in the following:

**Proof of Lemma 1.** The claim holds trivially if \( f \) is irreducible. So, let us assume that \( f \) is reducible. We proceed by induction on the weight of \( f \). So, first we consider the case where \( f \) is of weight 1. Since \( f \) is reducible, there exist nonconstant holomorphic eta quotients \( g \) and \( h \) such that \( f = g \times h \). Since \( f \) is of weight 1 and since the weight of any nonconstant holomorphic eta quotient is at least 1/2, both \( g \) and \( h \) must be of weight 1/2. Let \( X \in \mathbb{Z}^{D_N} \) be such that \( f = \eta^X \) and let \( d \in D_N \) be such that \( X_d \) is nonzero. Then the exponent of \( \eta_d \) in either \( g \) or \( h \) must also be nonzero. Without loss of generality, we may assume that the exponent of \( \eta_d \) in \( g \) is nonzero. Let \( g_0 \) denote the extract of \( g \). Since the weight of any nonconstant holomorphic eta quotient is at least 1/2, each eta quotient of weight 1/2 is irreducible. In particular, so is \( g_0 \). Since \( g_0 \) is a primitive irreducible holomorphic eta quotient of weight 1/2, it follows from Mersmann’s finiteness theorem (1.6) that the level of \( g_0 \) divides the 1st Mersmann level \( M_1 \). Since \( g \) is a rescaling of \( g_0 \) by some divisor of \( d \), the level of \( g \) divides \( \text{lcm}(d, M_1) \). Since the level of \( h \) must divide the least common multiple of the levels of \( f \) and \( g \), it follows that the level of \( h \) is a divisor of \( \text{lcm}(N, M_1) \). Thus, the claim holds in the weight 1 case.

Let us assume that the claim holds for all the cases where \( f \) is of weight less than or equal to \( k_0/2 \) for some integer \( k_0 \geq 2 \). Now, consider the case where \( f \) is of weight \((k_0+1)/2\). Since \( f \) is reducible, there exist nonconstant holomorphic eta quotients \( g \) and \( h \) such that \( f = g \times h \). Let \( X \in \mathbb{Z}^{D_N} \) be such that \( f = \eta^X \) and let \( d \in D_N \) be such that \( X_d \) is nonzero. Then the exponent of \( \eta_d \) in either \( g \) or \( h \) must also be nonzero. Without loss of generality, we may assume that the exponent of \( \eta_d \) in \( g \) is nonzero. Then there exists an irreducible factor \( g' \) of \( g \) such that the exponent of \( \eta_d \) in \( g' \) is nonzero. Since \( g' \neq f \), the weight of \( g' \) is at most \( k_0/2 \). Since \( g' \) is irreducible, so is its extract
\[ g'_0. \] Since \( g'_0 \) is a primitive irreducible holomorphic eta quotient of weight at most \( k_0/2 \), it follows from Mersmann’s finiteness theorem (1.6) that the level of \( g'_0 \) divides the \( k_0 \)-th Mersmann level \( M_{k_0} \). Since \( g' \) is a rescaling of \( g'_0 \) by some divisor of \( d \), the level of \( g' \) divides \( \text{lcm}(d, M_{k_0}) \). Since the level of \( h' := f/g' \) must divide the least common multiple of the levels of \( f \) and \( g' \), it follows that the level of \( h' \) is a divisor of \( \text{lcm}(N, M_{k_0}) \). Since the weight of \( h' \) is at most \( k_0/2 \), it follows from the induction hypothesis that the level of each factor of \( h' \) is a divisor of \( \text{lcm}(N, M_{k_0}) \). In particular, since \( h' = h \times g/g' \), both the levels of \( h \) and \( g = g' \times g/g' \) divide \( \text{lcm}(N, M_{k_0}) \). Thus, the level of each factor of \( f \) divides \( \text{lcm}(N, M_{k_0}) \).

5. Level lowering map

Here we show that given a factorization of a level \( N \) holomorphic eta quotient \( f \) of the form \( f = g \times h \), we can trim the levels of \( g \) and \( h \) off the primes which does not divide \( N \) by constructing a pair of holomorphic eta quotients \( \tilde{g} \) and \( \tilde{h} \) such that \( f = \tilde{g} \times \tilde{h} \) (see Lemma 4). Thus, the assertion in Lemma 2.(a) would follow. We shall also see some applications of the level lowering map in the proof of Theorem 3 in Section 9.

For \( M \in \mathbb{N} \) and \( N \in \mathcal{D}_M \), we define the linear map \( \mathcal{P}_{M,N} : \mathbb{Z}^D \rightarrow \mathbb{Q}^D \) by

\[
(\mathcal{P}_{M,N}(X))_d := d \cdot \sum_{\substack{t \in \mathcal{D}_M \\gcd(t,N)=d}} t_d \cdot X_t \quad \text{for all } d \in \mathcal{D}_N.
\]

Here \( t_d \) is the largest divisor of \( t \) that is coprime to \( d \). Let \( \mathcal{E}^1_M := \bigcup_{k \in \mathbb{Z}} \mathcal{E}^{1}_{M,k} \) (resp. \( \mathcal{E}^1_M := \bigcup_{k \in \mathbb{Z}} \mathcal{E}^{1}_{M,k} \)) be the group (resp. monoid) of eta quotients (resp. holomorphic eta quotients) on \( \Gamma_0(M) \). Let \( \hat{\mathcal{E}}^{1}_M \) (resp. \( \hat{\mathcal{E}}^1_M \)) denote the generalization of \( \mathcal{E}^1_M \) (resp. \( \mathcal{E}^1_M \)) to eta quotients with rational exponents. Then for \( N \in \mathcal{D}_M \), the map \( \mathcal{P}_{M,N} \) induces the homomorphism \( \mathcal{P}_{M,N} : \mathcal{E}^1_M \rightarrow \hat{\mathcal{E}}^{1}_N \) given by

\[
\mathcal{P}_{M,N}(\eta^X) := \eta^{\mathcal{P}_{M,N}(X)}.
\]

**Lemma 3.** For \( M \in \mathbb{N} \) and \( N \in \mathcal{D}_M \), let \( \mathcal{P}_{M,N} : \mathcal{E}^1_M \rightarrow \hat{\mathcal{E}}^{1}_N \) be the homomorphism defined above. Then the following statements hold:

(a) If \( n \in \mathcal{D}_N \) and if \( f \) is an eta quotient of level \( n \), then \( \mathcal{P}_{M,N}(f) = f \).

(b) If \( N \| M \), then the homomorphism \( \mathcal{P}_{M,N} \) maps \( \mathcal{E}^1_M \) onto \( \mathcal{E}^1_N \) and moreover, \( \mathcal{P}_{M,N} \) preserves weight.

(c) If \( N' \in \mathcal{D}_M \) such that \( N' \| M \) and \( N \in \mathcal{D}_{N'} \), then \( \mathcal{P}_{M,N} = \mathcal{P}_{N,N'} \circ \mathcal{P}_{M,N} \).

(d) For all \( M \in \mathbb{N} \) and \( N \in \mathcal{D}_M \), the homomorphism \( \mathcal{P}_{M,N} \) preserves holomorphy.

**Proof.** (a) Let \( X \in \mathcal{D}_M \) be such that \( f = \eta^X \). Since \( f \) is of level \( n \),

\[
X_t = 0 \quad \text{for all } t \notin n.
\]
So, for \( d \in D_N \), we have
\[
(\mathcal{P}_{M,N}(X))_d = d \cdot \sum_{\substack{t \in D_M \\text{gcd}(t,N) = d}} \frac{t_d}{t} \cdot X_t = \begin{cases} X_d & \text{if } d | n \\ 0 & \text{otherwise}, \end{cases}
\]
where \( t_d \) is the largest divisor of \( t \) that is coprime to \( d \). Thus, we obtain that \( p_{M,N}(f) = f \).

(b) Let \( t \in D_M \) with \((t,N) = d\). Since \( N \| M \), \( t/d \) is the largest divisor of \( t \) that is coprime to \( d \). Hence, it follows that
\[
(\mathcal{P}_{M,N}(X))_d = \sum_{\substack{t \in D_M \\text{gcd}(t,N) = d}} X_t.
\]
So, the range of \( p_{M,N} \) is contained in \( E_N \). The surjectivity follows from (a). The fact that \( p_{M,N} \) preserves weight follows from (5.3) and from the equality:
\[
\sum_{d \in D_N, t \in D_M \\text{gcd}(t,N) = d} X_t = \sum_{t \in D_M} X_t,
\]
which holds because \( \text{gcd}(N,M/N) = 1 \).

(c) Let \( X \in \mathbb{Z}^D \) and let \( X' := \mathcal{P}_{M,N'}(X) \). Since \( N' \| M \), from (b) we get that
\[
X' = \sum_{\substack{t \in D_M \\text{gcd}(t,N) = t'}} X_t,
\]
for all \( t' \in D_{N'} \). Hence, for \( d \in D_N \), we have
\[
(\mathcal{P}_{N',N} \circ \mathcal{P}_{M,N'}(X))_d = d \cdot \sum_{\substack{t' \in D_{N'} \\text{gcd}(t',N) = d}} \frac{t'_d}{t'} \cdot \sum_{\substack{t \in D_M \\text{gcd}(t,N) = t'}} X_t
\]
\[
= d \cdot \sum_{\substack{t \in D_M \\text{gcd}(t,N) = d}} \frac{t_d}{t} \cdot X_t = (\mathcal{P}_{M,N}(X))_d,
\]
where \( t_d \) (resp. \( t'_d \)) is the largest divisor of \( t \) (resp. \( t' \)) that is coprime to \( d \). Above, the second equality follows from (5.4) below. Since \( N' \| M \), the minimum positive multiple of \( d \in D_N \subseteq D_{N'} \) that divides \( t \in D_M \cap d\mathbb{Z} \) exactly, is also an exact divisor of \( t' = \text{gcd}(t,N') \). Both sides of the following identity represents the reciprocal of this exact divisor:
\[
\frac{t_d}{t} = \frac{t'_d}{t'},
\]
where \( t_d \) and \( t'_d \) are as before.

(d) We require to show that \( \mathcal{O}_N(\mathcal{P}_{M,N}(X)) \geq 0 \) if \( \mathcal{O}_M(X) \geq 0 \). By \( P_{M,N} \in \mathbb{Q}^{D_N \times D_M} \), we denote the matrix of the linear map \( \mathcal{P}_{M,N} \) with respect to the standard bases of \( \mathbb{Q}^{D_M} \) and \( \mathbb{Q}^{D_N} \). So, for \( d \in D_N \) and \( t \in D_M \), we have
\[
P_{M,N}(d,t) = \begin{cases} \frac{d t_d}{t} & \text{if } d = \text{gcd}(t,N) \\ 0 & \text{otherwise}, \end{cases}
\]
where $t_d$ is the largest divisor of $t$ that is coprime to $d$. From (3.19), we get
\[ O_N(P_{M,N}(X)) = A_N P_{M,N} A_M^{-1} O_M(X). \]
Hence, it is enough to show that $A_N P_{M,N} A_M^{-1} \geq 0$. Let $M_1 \in \mathbb{N}$ be an exact divisor of $M$ and let $N_1 := \gcd(M_1, N)$. Let $M_2 := M/M_1$ and $N_2 := N/N_1$. From (5.5), we get that
\[ P_{M_1,N_1}(d_1, t_1) \cdot P_{M_2,N_2}(d_2, t_2) = P_{M,N}(d_1 d_2, t_1 t_2), \]
where $d_i \in D_{N_i}$ and $t_i \in D_M$, for $i \in \{1, 2\}$. That implies: $P_{M,N} = P_{M_1,N_1} \otimes P_{M_2,N_2}$ or more elaborately,
\[ P_{M,N} = \bigotimes_{p^m \| M, p^n \| N} P_{p^m,p^n}, \tag{5.6} \]
where by $\otimes$, we denote the Kronecker product of matrices. From (3.18) and (5.6) we get that
\[ A_N P_{M,N} A_M^{-1} = \bigotimes_{p^m \| M, p^n \| N} A_{p^n} P_{p^m,p^n} A_{p^m}^{-1}, \]
Hence, it suffices to show that $A_{p^n} P_{p^m,p^n} A_{p^m}^{-1} \geq 0$ for each prime $p$ and for all integers $m \geq n \geq 0$.

First we consider the case $m \geq n = 0$: We have $P_{p^n,1} = 1_{p^n}^T$. So, from (3.21) and (3.22) we get that
\[ A_1 P_{p^n,1} A_{p^n}^{-1} = \alpha_{p^n}^T \geq 0. \]
On the other hand, for $m \geq n > 0$, we have
\[ P_{p^n,p^n} = \begin{pmatrix} 1 & & & & \hline & 1 & \frac{1}{p} & \frac{1}{p^2} & \ldots & \frac{1}{p^{m-n}} & \hline & & & & \end{pmatrix}_{(n+1) \times (m+1)}. \tag{5.7} \]
From (3.19), we recall that $A_{p^n}^{-1}$ is a tridiagonal matrix with columns of the form $a \cdot (p, -1, 0, \ldots, 0)^T$, $b \cdot (0, \ldots, 0, -p^2 + 1, -p, 0, \ldots, 0)^T$ and $c \cdot (0, \ldots, 0, -1, 0)^T$, where $a, b, c$ are some positive rational numbers. Again, from the structures of $A_{p^n}$ and $P_{p^m,p^n}$ (see (3.16) and (5.7)), it is easy to note that each entry of $A_{p^n} P_{p^m,p^n}$ is of the form $p^\ell$ for some $\ell \in \mathbb{Z}$ and the exponents of $p$ in any two consecutive entries in a row of $A_{p^n} P_{p^m,p^n}$ differ by 1. It follows that each entry of $A_{p^n} P_{p^m,p^n} A_{p^m}^{-1}$ is a positive multiple of $X Y^T$ where $X \in \{(p^2, p, 1), (1, p, 1), (1, p, p^2)\}$ and $Y \in \{(p, -1, 0), (-p, p^2 + 1, -p), (0, -1, p)\}$, i.e., $X$ is a row of $A_{p^2}$ and $Y^T$ is a column of $A_{p^1}$ up to a positive rational multiple. \hfill \square

*Kronecker product of matrices is not commutative. However, since any given ordering of the primes dividing $N'$ induces a lexicographic ordering on $D_N'$ and $D_N$ with which the entries of $P_{M,N}$ are indexed, Equation (5.6) makes sense for all possible orderings of the primes dividing $N'$.\hfill
Corollary 8. Let $f$ be a holomorphic eta quotient of level $N$. If there exists an $N_1\|N$ such that $p_{N,N_1}(f)$ is irreducible, then $f$ is also irreducible.

For example, $h := \frac{\eta^5_2\eta^5_3\eta^5_{12}\eta^{25}_{20}\eta^{25}_{60}}{\eta_4\eta^5_6\eta^5_{10}\eta^5_{15}\eta^{15}_{60}}$ is a holomorphic eta quotient of level 60 and we have

$$p_{60,5}(h) = \frac{\eta^5_5}{\eta}.$$

It is easy to show that $\eta^5_5/\eta$ is quasi-irreducible (see [4]). Hence, by Theorem 2, it follows that $\eta^5_5/\eta$ is irreducible. So, by Corollary 8, we get that $h$ is irreducible.

Lemma 4 (Level Lowering Lemma). Let $f$ be a holomorphic eta quotient on of level $N$ which is factorizable on $\Gamma_0(M)$ for some $M \in N \cdot \mathbb{N}$. Let $M = mn$ with $\text{rad}(n)|N$ and $(m,N) = 1$. Let $g$ be a factor of $f$ on $\Gamma_0(M)$. Then

$$f = p_{M,n}(g) \times p_{M,n}(f/g)$$

is a factorization of $f$ on $\Gamma_0(n)$.

Proof. Clearly, we have $n\|M$. So, Lemma 3 implies that the homomorphism $p_{M,n} : E^l_M \to E^l_n$ preserves both weight and holomorphy of eta quotients. Since the level of $f$ divides $n$, again from Lemma 3, we obtain

$$f = p_{M,n}(f) = p_{M,n}(g) \times p_{M,n}(h).$$

For example, the holomorphic eta quotient $\frac{\eta^5_2\eta^5_3\eta^5_{12}}{\eta^5_2\eta^5_4\eta^5_6}$ of level 12 is a factor of the holomorphic eta quotient $\frac{\eta^5_2}{\eta^5_2\eta^5_4\eta^5_6}$ of level 4. So by Lemma 4, the later is factorizable on $\Gamma_0(4)$:

$$\frac{\eta^5_2}{\eta^5_2\eta^5_4\eta^5_6} = \frac{\eta^5_2\eta^5_3\eta^5_{12}}{\eta^5_2\eta^5_4\eta^5_6} \times \frac{\eta^5_2\eta^5_{60}}{\eta^5_3\eta^5_{12}} = p_{12,4} \left( \frac{\eta^5_2\eta^5_3\eta^5_{12}}{\eta^5_2\eta^5_4\eta^5_6} \right) \times p_{12,4} \left( \frac{\eta^5_2\eta^5_{60}}{\eta^5_3\eta^5_{12}} \right) = \frac{\eta^5_2}{\eta^5_4} \times \frac{\eta^5_2}{\eta^5_4}.$$

It is easy to check that all the eta quotients above are holomorphic. In particular, from Lemma 4 we conclude:

Corollary 9. The assertion in part (a) of Lemma 2 holds.

Corollary 10. Let $M \in \mathbb{N}$ and let $N\|M$. If a holomorphic eta quotient $f$ of level $N$ is factorizable on $\Gamma_0(M)$, then $f$ is also factorizable on $\Gamma_0(N)$.

Corollary 11. If a holomorphic eta quotient $f$ of level $N$ has a proper factor of a squarefree level, then $f$ is factorizable on $\Gamma_0(N)$.

6. Construction of new factors

We already saw a partial result (Corollary 9) towards Lemma 2. Here we complete the proof of this lemma by constructing a suitable factor of a holomorphic eta quotient $f$ of level $N$, which satisfies the assumptions in its part (b). To describe this factor explicitly, first we require to implement a certain normalization of the columns of the inverse of the order matrix $A_N$ of level $N$ (see [7]). Since all the entries of $A_N^{-1}$ are rational with their denominators dividing $\phi(N)$ (see 3.18, 3.20 and 2.6), for each $t \in D_N$,
there exists a smallest positive integer $m_{t,N} \mid \hat{\varphi}(N)$ such that $m_{t,N} \cdot A_{N}^{-1}(\_ , t)$ has integer entries, where $A_{N}^{-1}(\_ , t)$ denotes the column of $A_{N}$ indexed by $t \in D_{N}$.

**Lemma 5.** For $N \in \mathbb{N}$ and $t \in D_{N}$, let $m_{t,N}$ denote the least positive integer such that $m_{t,N} \cdot A_{N}^{-1}(0 , t) \in \mathbb{Z}^{D_{N}}$. Then we have

\[
m_{t,N} = \hat{\varphi}\left(N, t'angle\right) \hat{\varphi}\left(t'' \gcd(t', t''/t')^\rho\right),
\]

where $t'$ is the quotient of $t$ by the greatest common exact divisor of $N$ and $t$, $t''$ is the least exact divisor of $N$ such that $t' | t''$ and the function $\hat{\varphi} : \mathbb{N} \to \mathbb{N}$ is as defined in (2.6).

**Proof.** If $N$ is a prime power, the claim follows from (3.20). The general case then follows by multiplicativity (see 3.18). $\square$

We define $B_{N} \in \mathbb{Z}^{D_{N} \times D_{N}}$ by

\[
B_{N}(\_, t) := m_{t,N} \cdot A_{N}^{-1}(\_, t) \quad \text{for all} \quad t \in D_{N}.
\]

Clearly, $B_{N}$ is invertible over $\mathbb{Q}$. Since $m_{t,N}$ is multiplicative in $N$ and $t$, from the multiplicativity of $A_{N}^{-1}(d, t)$ in $N$, $d$ and $t$ (see 3.18), it follows that $B_{N}(d, t)$ is also multiplicative in $N$, $d$ and $t$. That implies:

\[
B_{N} = \bigotimes_{p \in P_{N}} B_{p^{n}},
\]

where $\varphi_{N}$ denotes the set of prime divisors of $N$. For a prime $p$, from (6.2) and (3.19), we have

\[
B_{p^{n}} = \begin{pmatrix}
p & -p & & 0 \\
-1 & p^2 + 1 & -p & \\
-p & p^2 + 1 & -p & \\
& & \ddots & \ddots & \ddots \\
0 & -p & p^2 + 1 & -1 & \\
& & & -p & p
\end{pmatrix}.
\]

From (6.3) and (6.4), it follows that for all $t \in D_{N}$, the weight of an eta quotient of the form

\[
\eta^{B_{N}(\_, t)} := \prod_{d \mid N} \eta^{B_{N}(d, t)}
\]

is at least $\varphi(\text{rad}(N))/2$ (see (4.8) in [7]). So, the assertion in part (b) of Lemma 2 is implied by the following:

**Lemma 2.**(b)’. Let $N > 1$ be an integer and let $f$ be a holomorphic eta quotient on $\Gamma_{0}(N)$. Let $M > N$ be a multiple of $N$ with $\text{rad}(M) = \text{rad}(N)$ and let $g$ be a factor of $f$ on $\Gamma_{0}(M)$ such that the exponent of $\eta_{t,M/\text{rad}(N)}$ in $g$ is nonzero for some $t | N_{0}$, where $N_{0}$ is the greatest common exact divisor of $M$ and $N$. Let $M_{0} := M/N$. If

\[
M_{0} \geq 2^{\varphi(\text{rad}(M_{0}))} \varphi(N_{0}),
\]

then...
then there exists $r \in \mathcal{D}_{N_0}$ such that $\eta^{B_N(-rN_1)}$ is a factor of $f$, where $N_1 := N/N_0$. Moreover, if strict inequality holds in (6.6), then $\eta^{B_N(-rN_1)}$ is a nontrivial factor of $f$. Here $\omega(M_0)$ denotes the number of distinct prime divisors of $M_0$ and the function $\hat{\varphi} : \mathbb{N} \to \mathbb{N}$ is as defined in (2.6).

To prove the above lemma, we need some intermediate results. To state these results, first we define composition of eta quotients of coprime levels: Let $M, N \in \mathbb{N}$ with $\gcd(M, N) = 1$. Let $X \in \mathbb{Z}^{D_M}, Y \in \mathbb{Z}^{D_N}$ and let $f = \eta^X$ and $g = \eta^Y$. We define the eta quotient $f \otimes g$ on $\Gamma_0(MN)$ by

$$f \otimes g := \eta^{X \otimes Y} = \prod_{d \in D_M} \prod_{d' \in D_N} \eta^{X_dY_{d'}}.$$  

The following lemma relates the orders of $f \otimes g$ with the orders of $f$ and $g$ at the cusps:

**Lemma 6.** Let $M, N \in \mathbb{N}$ be mutually coprime. Let $f$ (resp. $g$) be an eta quotient of $\Gamma_0(M)$ (resp. $\Gamma_0(N)$). Then for $t \in \mathcal{D}_M$ and $t' \in \mathcal{D}_N$, we have

$$\text{ord}_{\frac{1}{p}}(f \otimes g; \Gamma_0(MN)) = 24 \cdot \text{ord}_{\frac{1}{p}}(f; \Gamma_0(M)) \cdot \text{ord}_{\frac{1}{p}}(g; \Gamma_0(N)).$$

**Proof.** Let $X \in \mathbb{Z}^{D_M}$ (resp. $Y \in \mathbb{Z}^{D_N}$) be such that $f = \eta^X$ (resp. $g = \eta^Y$). Since $\gcd(M, N) = 1$, it follows from (3.18) that

$$A_{MN}(X \otimes Y) = (A_M X) \otimes (A_N Y).$$

Now, the claim follows via (3.17). \hfill \square

Next we provide an essential ingredient of the proof of Lemma 2.1(b):

**Lemma 7** (New Factor Lemma). Let $N > 1$ be an integer and let $f$ be a holomorphic eta quotient on $\Gamma_0(N)$. Let $M > N$ be a multiple of $N$ with $\text{rad}(M) = \text{rad}(N)$ and let $g$ be a factor of $f$ on $\Gamma_0(M)$ such that there exist (not necessarily holomorphic) eta quotients $g' \neq 1$ (resp. $g''$) on $\Gamma_0(N_0)$ (resp. on $\Gamma_0(M/\text{rad}(M_0))$) with

$$g = g'_M \times g''_N,$$

where $N_0$ is the greatest common exact divisor of $M$ and $N$, $M_0 := M/N$, $M_1 := M/N_0$ and $g'_M$ denotes the rescaling of $g'$ by $M_1$. Suppose, there exists a nonconstant holomorphic eta quotient $h$ on $\Gamma_0(N_0)$ such that for all $t \in \mathcal{D}_{N_0}$, we have

$$2^{\omega(M_0) - 1} \cdot \text{rad}(M_0) \cdot \text{ord}_{\frac{1}{t}}(h; \Gamma_0(N_0)) \leq M_0 \cdot \text{ord}_{\frac{1}{t}}(g'; \Gamma_0(N_0)),$$

where $\omega(M_0)$ denotes the number of distinct prime divisors of $M_0$. Let $N_1 := N/N_0$.

1. Then $h \otimes \eta^{B_N(-rN_1)}$ is a factor of $f$ on $\Gamma_0(N)$.
2. The above one is a proper factor of $f$ if for at least one $t \in \mathcal{D}_{N_0}$, strict inequality holds in (6.9).

**Proof.** (1) Let $M_0 = \prod_{p \in \mathcal{P}_{M_0}} p^{m_p}$, where by $\mathcal{P}_{M_0}$ we denote the set of prime divisors of $M_0$. We shall proceed by induction on $m_p$ for all $p \in \mathcal{P}_{M_0}$. So, first we consider the case where $m_p = 1$ for all such $p$: Let $a \in \mathbb{Z}^{D_M}$ with $a_r = \text{ord}_{\frac{1}{r}}(g; \Gamma_0(M))$ for $r \in \mathcal{D}_M$. Since $M = M_0M_1$ with $N_0$ and $M_1$ mutually coprime, there is a canonical bijection between
\[ Z^{D_M} \text{ and } Z^{D_{M_0}} \times Z^{D_{M_1}}. \] Let \( \tilde{a} \in \frac{1}{24}(Z^{D_{M_0}} \times Z^{D_{M_1}}) \) denote the image of \( a \in \frac{1}{24}Z^{D_M} \) under this bijection. Since \( g \) is a factor of \( f \), we have

\[ 0 \leq a_r \leq \ord_{i/r}(f; \Gamma_0(M)) \text{ for all } r \in D_M. \]

Let \( X \in Z^{D_M} \) (resp. \( X' \in Z^{D_{M_0}} \)) be such that \( g = \eta X \) (resp. \( g' = \eta X' \)). Then from (3.17), we get:

\[ X = 24 \cdot A^{-1}_M a = 24 \cdot A^{-1}_{M_0} \bigotimes_{p \in p_{M_0}^N} A^{-1}_{p^{n+1}} a, \]

where the second equality follows from (3.18). That implies (see Lemma 4.3.1 in [9]):

\[ \tilde{X} = 24 \cdot A^{-1}_{M_0} \tilde{a} \left( \bigotimes_{p \in p_{M_0}^N} A^{-1}_{p^{n+1}} \right)^T, \]

where \( \tilde{X} \) is the image of \( X \) under the canonical bijection from \( Z^{D_M} \) to \( Z^{D_{M_0}} \times Z^{D_{M_1}} \). Since for all \( d \in D_{M_0} \), we have

\[ X_d' = X_{dM_1} = X_d \prod_{p \in p_{M_0}^N} p^{n+1}, \]

from (6.12) we obtain

\[ X' = 24 \cdot A^{-1}_{M_0} a \left( \bigotimes_{p \in p_{M_0}^N} A^{-1}_{p^{n+1}} (p^{n+1}, -) \right)^T \]

(6.13)

where \( A^{-1}_{p^{n+1}} (p^{n+1}, -) \) denotes the last row of \( A_{p^{n+1}}^{-1} \) and where \( a', a'' \in Z^{D_M} \) are defined by

\[ a'_t := \sum_{S \subseteq p_{M_0}^N \#S \text{ even}} a_{t|N_1} \prod_{p \in p_{M_0}^N \#S} p \quad \text{and} \quad a''_t := \sum_{S \subseteq p_{M_0}^N \#S \text{ odd}} a_{t|N_1} \prod_{p \in p_{M_0}^N \#S} p \]

for all \( t \in D_{M_0} \). Here by \( \#S \), we denote the number of elements in \( S \). Above, (6.13) holds since \( \tilde{\varphi} \) is multiplicative and since for any prime \( p \), we have

\[ A_{p^{n+1}}^{-1} (p^{n+1}, p^n) = \frac{1}{\tilde{\varphi}(p^n)}, \quad A_{p^{n+1}}^{-1} (p^{n+1}, p^{n+1}) = \frac{1}{\tilde{\varphi}(p^n)} \]

whereas all other entries of \( A_{p^{n+1}}^{-1} (p^{n+1}, -) \) are zero (see 3.19).

Now, from (6.13) and (3.17), we get:

\[ a'_t - a''_t = \tilde{\varphi}(N_1) \cdot \ord_{1/t}(g'; \Gamma_0(N_0)) \]

(6.15)

for all \( t \in D_{M_0} \). As both \( a'_t \) and \( a''_t \) are nonnegative (see 6.10), it follows that

\[ \max(a'_t, a''_t) \geq \tilde{\varphi}(N_1) \cdot \ord_{1/t}(g'; \Gamma_0(N_0)) \]

(6.16)

for all \( t \in D_{M_0} \). Let \( h \) be a nonconstant holomorphic eta quotient on \( \Gamma_0(N_0) \) which satisfies (6.9) for all \( t \in D_{M_0} \). Since both the number of the subsets of \( \varphi_{M_0} \) of odd cardinality and the number of the subsets of \( \varphi_{M_0} \) of even
cardinality are equal to $2^{\omega(M_0)-1}$ and since $\text{rad}(M_0) = M_0$ in the case under consideration, from (6.9), (6.16) and (6.14), it follows that for each $t \in D_{N_0}$, there exists a subset $S_t \subseteq \mathcal{O}_{M_0}$ such that

$$
\hat{\varphi}(N_1) \cdot \text{ord}_{1/t}(h; \Gamma_0(N_0)) \leq a_{tN_1} \prod_{p \in S_t} p
$$

(6.17)

$$
\leq \text{ord}_{1/(tN_1 \prod_{p \in S_t} p)}(f; \Gamma_0(M)),
$$

where the second inequality follows from (6.10). Again, it follows from (3.4) and (3.9) that for an arbitrary subset $S \subseteq \mathcal{O}_{M_0}$, we have

$$
\text{ord}_{1/(tN_1 \prod_{p \in S} p)}(f; \Gamma_0(N)) = \text{ord}_{1/(tN_1)}(f; \Gamma_0(N))
$$

(6.18)

for all $t \in D_{N_0}$. For $\alpha, \lambda \in \mathbb{Z}$ with $\gcd(\alpha, \lambda) = 1$ and $s = [\alpha : \lambda] \in \mathbb{P}^1(\mathbb{Q})$, the width of the cusp $s$ of $\Gamma_0(N)$ is $N/\gcd(\lambda^2, N)$ (see [8, 12]). Hence, it follows from (6.5) and (6.18) that for each subset $S \subseteq \mathcal{O}_{M_0}$, we have

$$
\text{ord}_{1/(tN_1 \prod_{p \in S} p)}(f; \Gamma_0(M)) = \text{ord}_{1/(tN_1)}(f; \Gamma_0(M))
$$

(6.19)

for all $t \in D_{N_0}$. Now, (6.17) and (6.19) together imply that

$$
\hat{\varphi}(N_1) \cdot \text{ord}_{1/t}(h; \Gamma_0(N_0)) \leq \text{ord}_{1/(tN_1 \prod_{p \in S} p)}(f; \Gamma_0(M))
$$

(6.20)

for all $t \in D_{N_0}$ and for all subsets $S \subseteq \mathcal{O}_{M_0}$.

From (3.7), we obtain that for a prime $p$ and $n \in \mathbb{N}$, the orders of $\eta_{p^n}/\eta_{p^{n-1}}$ at the cusps $\{1/p^\alpha\}_{0 \leq \alpha \leq n+1}$ of $\Gamma_0(p^{n+1})$ are as follows:

$$
\text{ord}_{1/p}(\frac{\eta_{p^n}}{\eta_{p^{n-1}}}; \Gamma_0(p^{n+1})) = \begin{cases} 0 & \text{if } 0 \leq \alpha < n, \\ \varphi(p^n) & \text{otherwise}. \end{cases}
$$

(6.21)

From (6.3) and (6.4), it follows that

$$
\eta_{B_{N_1}}(-N_1) = \bigoplus_{p \in \mathcal{O}_{M_0}} \eta_{p^n}
$$

(6.22)

Let $h' := h \otimes \eta_{B_{N_1}}(-N_1)$. Since $\hat{\varphi}$ is multiplicative, from (6.21), (6.22) and Lemma 6, we conclude that for $r \in D_M$, we have

$$
\text{ord}_{1/r}(h'; \Gamma_0(M)) = \begin{cases} \hat{\varphi}(N_1) \cdot \text{ord}_{1/t}(h; \Gamma_0(N_0)) & \text{if } N_1|r \\ 0 & \text{otherwise}, \end{cases}
$$

(6.23)

where $t = \gcd(r, N_0)$. It follows from (6.20) and (6.23) that at each cusp $s$ of $\Gamma_0(M)$, we have

$$
\text{ord}_s(h'; \Gamma_0(M)) \leq \text{ord}_s(f; \Gamma_0(M)).
$$

Hence, $h'$ is indeed a factor of $f$.

Thus, we see that for $M_0 = \prod_{p \in \mathcal{O}_{M_0}} p^{m_p}$, the claim holds if $m_p = 1$ for all $p \in \mathcal{O}_{M_0}$. Now, for all such $p$, let us assume that the claim holds for $m_p = n_p$ for some $n_p \in \mathbb{N}$. Hence, to complete the induction, it is enough to show that the claim also holds for $m_{p_0} = n_{p_0} + 1$ for an arbitrary prime $p_0 \in \mathcal{O}_{M_0}$.
Let $N' := Np_0$ and let $M = Np_0^{n_{p_0} + 1} \prod_{p \in \mathcal{D}_{N_0}} p^{n_p} = N' \prod_{p \in \mathcal{D}_{N_0}} p^{n_p}$. Let $f$ be a holomorphic eta quotient on $\Gamma_0(N)$ and let $g$ be a factor of $f$ on $\Gamma_0(M)$ such that there exist (not necessarily holomorphic) eta quotients $g' \neq 1$ (resp. $g''$) on $\Gamma_0(N_0)$ (resp. on $\Gamma_0(M/\text{rad}(M_0))$) with

$$g = g'_{M_1} \times g'',$$

where $N_0$ is the greatest common exact divisor of $M$ and $N$, $M_1 := M/N_0$ and $g'_{M_1}$ denotes the rescaling of $g'$ by $M_1$. Suppose, there exists a non-constant holomorphic eta quotient $h$ on $\Gamma_0(N_0)$ such that for all $t \in \mathcal{D}_{N_0}$, Inequality (6.9) is satisfied, where $\omega(M_0)$ denotes the number of distinct prime divisors of $M_0 := M/N$. Let $M'_0 := M_0/p_0$. Inequality (6.9) is equivalent to the following:

$$(6.24) \ 2^{(M_0) - 1} \text{rad}(M_0) \cdot \text{ord}_1(h; \Gamma_0(N_0)) \leq M'_0 \cdot \text{ord}_1(g''; \Gamma_0(N_0)),$$

Let $N'_0 := N'/N_0$. Then by induction hypothesis, we have

$$(6.25) \ \tilde{h} := h \otimes \eta^{B_{N'_0}(-, N'_0)}$$

is a factor of $f_{p_0}$ on $\Gamma_0(N')$. Let $N'_0$ be the largest divisor of $N$ which is not divisible by $p_0$. Let $h'$ (resp. $h''$) be the eta quotient on $\Gamma_0(N'_0)$ (resp. $\Gamma_0(N)$) such that

$$(6.26) \ \tilde{h} = h'_{M'_1} \times h'',$$

where $h'_{M'_1}$ denotes the rescaling of $h'$ by $M'_1 := p_0^{n_{p_0} + 1}$. From (6.3) and (6.4), it follows that

$$(6.27) \ \eta^{B_{N'_0}(-, N'_0)} = \eta^{B_{N'_0}(-, N'_0)} \otimes \eta^{p_0}_{N_{p_0}^{p_0 + 1}}.$$ 

Since $\otimes$ is associative, from (6.25), (6.26) and (6.27), we obtain:

$$(6.28) \ \tilde{h}' = h \otimes \eta^{B_{N'_0}(-, N'_0)} \otimes \eta^{p_0} = \left(h \otimes \eta^{B_{N'_0}(-, N'_0)}\right)^{p_0}$$

and

$$(6.29) \ \tilde{h}'' = h \otimes \eta^{B_{N'_0}(-, N'_0)} \otimes \eta^{p_0}.$$ 

Let $N_1 := N/N_0$. Since $\tilde{h}'$ is holomorphic, it follows again by induction hypothesis that

$$\tilde{h}' \otimes \eta^{p_0}_{N_{p_0}^{p_0 + 1}} = \left(h \otimes \eta^{B_{N'_0}(-, N'_0)}\right)^{p_0} \otimes \eta^{p_0}_{N_{p_0}^{p_0 + 1}}$$

is a factor of $f_{p_0}$ on $\Gamma_0(N)$, where the last equality follows again from (6.3) and (6.4). Therefore, $h \otimes \eta^{B_{N_1}(-, N_1)}$ is a factor of $f$ on $\Gamma_0(N)$. \hfill $\Box$

(2) Follows easily from the proof of (1). \hfill $\Box$
Corollary 12. Let $N > 1$ be an integer and let $f$ be a holomorphic eta quotient on $\Gamma_0(N)$. Let $M > N$ be a multiple of $N$ with $\text{rad}(M) = \text{rad}(N)$ and let $g$ be a factor of $f$ on $\Gamma_0(M)$ such that there exist (not necessarily holomorphic) eta quotients $g' \neq 1$ (resp. $g''$) on $\Gamma_0(N_0)$ (resp. on $\Gamma_0(M/\text{rad}(M_0))$) with

$$g = g'M \times g'',$$

where $N_0$ is the greatest common exact divisor of $M$ and $N$, $M_0 := M/N$, $M_1 := M/N_0$ and $g'M_1$ denotes the rescaling of $g'$ by $M_1$. Suppose, there exists a nonconstant holomorphic eta quotient $h$ on $\Gamma_0(N_0)$ such that $f = \prod_{t \in D_{N_0}} h^\omega_t$, where $\omega_t$ denotes Euler’s totient function and $\omega(M_0)$ denotes the number of distinct prime divisors of $M_0$. \hfill $\square$

Corollary 13. Let all the assumptions of Corollary 12 hold for a pair of integers $N, M$ with $M > N > 1$, $\text{rad}(M) = \text{rad}(N)$ and a holomorphic eta quotient $f$ on $\Gamma_0(N)$. Let $k \in \mathbb{N}$ be such that the weight of $f$ is $k/2$. If the greatest common exact divisor of $M$ and $N$ is 1, then we have

$$\frac{\varphi(M_0)}{2\omega(M_0)-1} \leq k,$$

where $\varphi$ denotes Euler’s totient function and $\omega(M_0)$ denotes the number of distinct prime divisors of $M_0 := M/N$.

Proof. Let $g$ be a factor of $f$ on $\Gamma_0(M)$ such that there exist (not necessarily holomorphic) eta quotients $g' \neq 1$ (resp. $g''$) on $\text{SL}_2(\mathbb{Z})$ (resp. on $\Gamma_0(M/\text{rad}(M_0))$) with

$$g = g'M \times g'',$$

where $g'M$ denotes the rescaling of $g'$ by $M$. The existence of such a factor $g$ of $f$ follows from the assumptions of Corollary 12 and from the fact that the greatest common exact divisor of $M$ and $N$ is 1. Since an eta quotient on $\text{SL}_2(\mathbb{Z})$ is only an integer power of $\eta$, there exists a nonzero integer $m$ such that $g' = \eta^m$. Since $\text{SL}_2(\mathbb{Z})$ has only one cusp and since

$$|\text{ord}_\infty(g'; \text{SL}_2(\mathbb{Z}))| = \frac{|m|}{24} \geq \text{ord}_\infty(\eta; \text{SL}_2(\mathbb{Z})),$$

from Corollary 12, it follows that the order of the holomorphic eta quotient with rational exponents

$$f_1 := (\eta^{B_N}(-N)) \frac{M_0}{2\omega(M_0) - 1},$$

at each the cusp of $\Gamma_0(N)$ is less than or equal to the order of $f$ at that cusp. Hence, from the valence formula (3.11), it follows that the weight of $f_1$ is less than the weight of $f$. Again, (6.3) and (6.4) together imply that
that the weight of $\eta^{B_N}(\cdot, N)$ is $\varphi(\text{rad}(N))/2$ (see also (4.8) in [7]). So, we obtain:

$$M_0\varphi(\text{rad}(N)) \leq k.$$  

(6.30)

Since the set of prime divisors of $M$ and $N$ are the same and since the greatest common exact divisor of $M$ and $N$ is 1, we have $\text{rad}(N) = \text{rad}(M_0)$. Hence, the claim follows from (6.30).

Proof of Lemma 2.(b)" Let $M_1 := M/N_0$. Since the exponent of $\eta_{N_0}$ in $g$ is nonzero, there exist (not necessarily holomorphic) eta quotients $g' \neq 1$ (resp. $g''$) on $\Gamma_0(N_0)$ (resp. on $\Gamma_0(M/\text{rad}(M_0))$) with

$$g = g'_M \times g''_M.$$  

Since $g' \neq 1$, there exists $r \in \mathcal{D}_{N_0}$ such that $\text{ord}_{1/r}(g'; \Gamma_0(N_0)) \neq 0$. So, (3.7) implies that

$$|\text{ord}_{1/r}(g'; \Gamma_0(N_0))| \geq \frac{1}{24}.$$  

(6.31)

Again, from (6.2) and (3.17), it follows that for $t \in \mathcal{D}_{N_0}$, we have

$$\text{ord}_{1/t}(\eta^{B_{N_0}}(\cdot, r); \Gamma_0(N_0)) = \begin{cases} \frac{m_{r,N_0}}{24} & \text{if } t = r \\ 0 & \text{otherwise,} \end{cases}$$  

(6.32)

where $m_{r,N_0}$ is the same as in (6.1) after we replace $t$ with $r$ and $N$ with $N_0$. Recall from the discussion preceding Lemma 5 that $m_{r,N_0} | \hat{\varphi}(N_0)$. So, from (6.31), (6.31) and (6.6), we conclude that

$$\frac{2^a(M_0)-1 \text{rad}(M_0) \cdot \text{ord}_{1/t}(\eta^{B_{N_0}}(\cdot, r); \Gamma_0(N_0)) \leq M_0 \cdot |\text{ord}_{1/t}(g'; \Gamma_0(N_0))|}$$  

for all $t \in \mathcal{D}_{N_0}$. Since (6.3) implies that

$$\eta^{B_{N_0}}(\cdot, r) \otimes \eta^{B_{N_1}}(\cdot, N_1) = \eta^{B_N}(\cdot, r N_1),$$

the claim follows from Lemma 7.

\[\square\]

7. Proof of Theorem 2

Let $n \in \mathbb{N}$ and a let $p$ be a prime. Let $f$ be a quasi-irreducible holomorphic eta quotient of level $p^n$. Suppose, $f$ is reducible. Then from Lemma 2.(a), it follows that there exists an integer $m > n$ such that $f$ has a factor of level $p^m$. Now, Lemma 2.(b) implies that $g := \eta^{p^m}/\eta^{p^{m-1}}$ is a factor of $f$ and the same theorem also implies that if $m > n + 1$, then $g$ is a proper factor of $f$. So, Theorem 2 would follow if we can prove that also in the case $m = n + 1$, we have $g \neq f$. We shall do so by showing that unlike $f$, $g$ is not factorizable on $\Gamma_0(p^{n+1})$:

Lemma 8. For $n \in \mathbb{N}$ and a prime $p$, the holomorphic eta quotient $\eta^{p_n}/\eta^{p^{n-1}}$ is not factorizable on $\Gamma_0(p^{n+1})$.

\[\ast\]See also Theorem 2.31 and Corollary 2.33 in [4].
Proof. Suppose, there exist nonconstant holomorphic eta quotients $f$ and $g$ on $\Gamma_0(p^{n+1})$ such that

$$fg = \eta_{p^n}/\eta_{p^{n-1}}.$$ 

Let $X, Y \in \mathbb{Z}^D_{p^{n+1}}$ be such that $f = \eta^X$ and $g = \eta^Y$. From Lemma 3 in [7], we recall that $\eta_{p^n}/\eta_{p^{n-1}}$ is not factorizable on $\Gamma_0(p^n)$. Hence, both $f$ and $g$ must be of level $p^{n+1}$. In other words, neither $X_{p^{n+1}}$ nor $Y_{p^{n+1}}$ is zero. From (6.13), after replacing $N_0$ with 1 and $N_1$ with $p^n$, we obtain:

$$(7.1) \quad X_{p^{n+1}} = \frac{24(a' - a'')}{\hat{\varphi}(p^n)} \quad \text{and} \quad Y_{p^{n+1}} = \frac{24(b' - b'')}{\hat{\varphi}(p^n)},$$

where $a' := \text{ord}_{1/p^{n+1}}(f; \Gamma_0(p^{n+1}))$, $a'' := \text{ord}_{1/p^n}(f; \Gamma_0(p^{n+1}))$, $b' := \text{ord}_{1/p^{n+1}}(g; \Gamma_0(p^{n+1}))$ and $b'' := \text{ord}_{1/p^n}(g; \Gamma_0(p^{n+1}))$. Since $f$ and $g$ are holomorphic, we have

$$(7.2) \quad a', a'', b', b'' \geq 0.$$ 

Though both $X_{p^{n+1}}$ and $Y_{p^{n+1}}$ are nonzero, but from the fact that $fg$ is of level $p^n$, it follows that

$$(7.3) \quad X_{p^{n+1}} + Y_{p^{n+1}} = 0.$$ 

So, without loss of generality, we may assume that $X_{p^{n+1}} > 0$. Then (7.1), (7.2) and (7.3) together imply that

$$(7.4) \quad a' + b' = a'' + b'' \geq b'' = b' + \frac{\hat{\varphi}(p^n)}{24} X_{p^{n+1}} \geq \frac{\hat{\varphi}(p^n)}{24}.$$ 

Since $fg = \eta_{p^n}/\eta_{p^{n-1}}$, the sum of the orders of $f$ and $g$ at the cusps $\{1/p^j\}_{0 \leq j \leq n+1}$ of $\Gamma_0(p^{n+1})$ are then given by (6.21). Since both $f$ and $g$ are holomorphic, they have nonnegative orders at all the cusps. So, from (6.21), it follows that for $0 \leq j < n$, we have

$$(7.5) \quad \text{ord}_{1/p^j}(f; \Gamma_0(p^{n+1})) = \text{ord}_{1/p^j}(g; \Gamma_0(p^{n+1}))$$

$$= \text{ord}_{1/p^j}(\eta_{p^n}/\eta_{p^{n-1}}; \Gamma_0(p^{n+1})) = 0.$$ 

Also, from (6.21), we obtain:

$$a'' + b'' = \text{ord}_{1/p^n}(\eta_{p^n}/\eta_{p^{n-1}}; \Gamma_0(p^{n+1})) = \frac{\hat{\varphi}(p^n)}{24}$$

and

$$a' + b' = \text{ord}_{1/p^n}(\eta_{p^{n+1}}/\eta_{p^{n-1}}; \Gamma_0(p^{n+1})) = \frac{\hat{\varphi}(p^n)}{24}.$$ 

In other words, all the equalities in (7.4) hold. Hence, we have $a'' = b' = 0$ and $a' = b'' = \hat{\varphi}(p^n)/24$, i.e.

$$(7.6) \quad \text{ord}_{1/p^n}(f; \Gamma_0(p^{n+1})) = \text{ord}_{1/p^{n+1}}(g; \Gamma_0(p^{n+1})) = 0$$

and

$$(7.7) \quad \text{ord}_{1/p^{n+1}}(f; \Gamma_0(p^{n+1})) = \text{ord}_{1/p^n}(g; \Gamma_0(p^{n+1})) = \frac{\hat{\varphi}(p^n)}{24}.$$ 

Now, from (3.17), (7.5), (7.6), (7.7) and (3.20), it follows that

$$X_{p^n} = 24 \cdot A_{p^{n+1}}^{-1} (p^n, p^{n+1}) \cdot \text{ord}_{1/p^{n+1}}(f; \Gamma_0(p^{n+1})) = -1/p \notin \mathbb{Z}.$$
Thus, we get a contradiction! □

8. An Implication of the Reducibility Conjecture

Here we show that Conjecture 2 follows from the Reducibility Conjecture and we deduce Corollary 4 from Theorem 2. But at first, we require a few results on Atkin-Lehner involutions:

Lemma 9. (a) Let $N \in \mathbb{N}$ and let $n$ be an exact divisor of $N$. Let $f, g$ and $h$ be eta quotients on $\Gamma_0(N)$ such that $f = gh$. Then we have

$$a_{n,N}(f) = a_{n,N}(g) \cdot a_{n,N}(h),$$

where the map $a_{n,N}$ is as defined in (3.3).
(b) Let $f$ be an eta quotient on $\Gamma_0(N)$ and let $M \in \mathbb{N}$ be a multiple of $N$. Let $m$ be an exact divisor of $M$. Then we have

$$a_{m,M}(f) = (a_{n,N}(f))_{\nu},$$

where $n = (m, N)$, $\nu = m/n$ and $(a_{n,N}(f))_{\nu}$ denotes the rescaling of the eta quotient $a_{n,N}(f)$ by $\nu$.
(c) Let $f$ be an eta quotient on $\Gamma_0(N)$ and let $M \in \mathbb{N}$ be a multiple of $N$. Let $m$ (resp. $n$) be an exact divisor of $M$ (resp. $N$) such that $n|m$. Let $\nu := m/n$. Then we have

$$a_{m,M}(f_{\nu}) = a_{n,N}(f).$$

Proof. (a) Follows trivially from (3.3) whereas (b) from the fact that $\nu n \odot d = \nu (n \odot d)$ for all $d \in \mathbb{D}_N$ and (c) from the fact that $(\nu n) \odot (\nu d) = n \odot d$ for all $d \in \mathbb{D}_N$. □

Corollary 14. Let $f$ be a holomorphic eta quotient on $\Gamma_0(N)$ and let $n\|N$. Then $f$ is factorizable on $\Gamma_0(N)$ if and only if so is $a_{n,N}(f)$. □

Corollary 15. Let $f$ be an eta quotient on $\Gamma_0(N)$ and let $f_{\nu}$ be the rescaling of $f$ by $\nu \in \mathbb{N}$. Then we have

$$a_{\nu,N,N}(f_{\nu}) = a_{N,N}(f).$$

Corollary 16. Let $f$ be an irreducible holomorphic eta quotient on $\Gamma_0(N)$ and let $n$ be an exact divisor of $N$. Then $a_{n,N}(f)$ is not factorizable on $\Gamma_0(N)$. □

From Corollary 16, it immediately follows that Conjecture 2$'$ is a consequence of the Reducibility Conjecture:

Corollary 17. Conjecture 1 implies Conjecture 2$'$. □

Next, we show that the two forms of the Irreducibility Conjecture are equivalent:

Corollary 18. Conjecture 2 and Conjecture 2$'$ are equivalent.

Proof. It follows from Corollary 15 that Conjecture 2$'$ implies Conjecture 2, whereas the reverse implication follows from Lemma 9.(c) and Corollary 14. □
In order to show that Theorem 2 implies Corollary 4, we require a few more results on Atkin-Lehner involutions and the level lowering map which we defined in Section 5. For \( M \in \mathbb{N} \) and \( N \| M \), the following lemma lets us interchange the order of the operation of \( p_{M,N} \) (see 5.2) with that of the Atkin-Lehner involutions on the set of eta quotients on \( \Gamma_0(M) \):

**Lemma 10.** (a) Let \( M \in \mathbb{N} \) and let \( m, N \) be exact divisors of \( M \). Let \( n := \gcd(m, N) \). Then we have

\[
p_{M,N} \circ \text{al}_{m,M} = \text{al}_{n,N} \circ p_{M,N}.
\]

(b) Let \( M \in \mathbb{N} \) and let \( M' \in D_M \). Let \( N \) be an exact divisor of \( M' \) and let \( N' := \gcd(N, M') \). Let \( f \) be an eta quotient on \( \Gamma_0(M') \). Then we have

\[
p_{M,N}(f) = p_{M',N'}(f).
\]

**Proof.** (a) Follows easily from fact that for any \( M \in \mathbb{N} \), each exact divisor \( m \) of \( M \) induces an involution on \( D_M \) defined by \( d \mapsto m \circ d \) for \( d \in D_M \), where \( \circ \) is as defined in (3.1).

(b) Follows trivially from (5.1) and (5.2).

**Proposition 1.** For \( n \in \mathbb{N} \), if the Reducibility Conjecture holds for the holomorphic eta quotients whose levels have at most \( n \) distinct prime divisors, then the image of an irreducible holomorphic eta quotient of such a level under an Atkin-Lehner involution is irreducible.

**Proof.** Let \( N \in \mathbb{N} \) have at most \( n \) distinct prime divisors and let \( f \) be an irreducible holomorphic eta quotient of level \( N \). Suppose, there exists a multiple \( M \) of \( N \) and an exact divisor \( m \) of \( M \) such that \( \text{al}_{m,M}(f) \) is reducible. Then there exists a multiple \( M' \) of \( M \) such that \( \text{al}_{m,M}(f) \) is factorizable on \( \Gamma_0(M') \). Let \( N' \) be the exact divisor of \( M' \) with \( \text{rad}(N') = \text{rad}(N) \). Let \( N_1 := \gcd(N', M) \). Then \( N \) divides \( N_1 \). Let \( n_1 := \gcd(m, N_1) \). Then we have

\[
p_{M',N'} \circ \text{al}_{m,M}(f) = p_{M,N_1} \circ \text{al}_{m,M}(f) = \text{al}_{n_1,N_1} \circ p_{M,N_1}(f) = \text{al}_{n_1,N_1}(f),
\]

where the first two equalities follow from Lemma 10.(b) and Lemma 10.(a), whereas the last one follows from Lemma 3.(a). Since \( \text{al}_{m,M}(f) \) is factorizable on \( \Gamma_0(M') \), from (8.1) and from Lemma 3.(b), it follows that \( \text{al}_{n_1,N_1}(f) \) is factorizable on \( \Gamma_0(N') \). Again, since \( N_1 \) has at most \( n \) distinct prime divisors, according to our assumption, the Reducibility Conjecture holds for eta quotients of level \( N_1 \). Therefore, \( \text{al}_{n_1,N_1}(f) \) is factorizable on \( \Gamma_0(N_1) \). Hence, Corollary 14 implies that \( f \) is reducible. Thus, we get a contradiction!

The last proposition and Corollary 15 together imply:

**Corollary 19.** For \( n \in \mathbb{N} \), if the Reducibility Conjecture holds for the holomorphic eta quotients whose levels have at most \( n \) distinct prime divisors, then the rescaling by any positive integer of an irreducible holomorphic eta quotient of such a level is irreducible.

From Proposition 1 and Corollary 19, it follows that Theorem 2 implies Corollary 4.
9. Proof of Theorem 3

Let $f$ be a reducible holomorphic eta quotient of weight $k/2$ and level $N$ and let $M \in \mathbb{N}$ be the least positive multiple of $N$ such that $f$ is factorizable on $\Gamma_0(M)$. Then from Lemma 2.(a), it follows that $\text{rad}(M) = \text{rad}(N)$. Let $g, h \notin \{1, f\}$ be two holomorphic eta quotients on $\Gamma_0(M)$ such that $f = g \times h$. Our strategy for proving Theorem 3 is as follows:

For $M$ sufficiently large, we provide a construction of two nonconstant holomorphic eta quotients $\tilde{g}$ and $\tilde{h}$ on $\Gamma_0(M)$ for some positive multiple $\tilde{M}$ of $N$ with $\tilde{M} < M$ such that $f = \tilde{g} \times \tilde{h}$, using the level lowering map which we provided in Section 5. This would contradict the minimality of $M$. Henceforth, we shall assume that $g \neq h$.

Now, define the sequence

\[ M(9.1) \]

$$\\frac{\varphi(M_0)}{2^{\omega(M_0) - 1}} \leq k$$

which implies:

\[ (9.1) \quad M' = NM_0 \leq 2Nk. \]

Now, define the sequence $\{d_0, d_1, d_2, \ldots\}$ in $\mathcal{D}_M$ as follows: $d_0 := M'$ and for $j \geq 1$, let $d_j \in \mathcal{D}_M$ be such that

1. $d_j$ does not divide $L_j := \text{lcm}(d_0, d_1, \ldots, d_{j-1})$.
2. $X_{d_j}$ is nonzero.
3. $\frac{d_{d_j}}{\gcd(d_j, L_j)} \leq \frac{d}{\gcd(d, L_j)}$ for all $d \in \mathcal{D}_M$ which satisfies (1) and (2) above.
Since $d_j \not\mid N$ for all $j$ and since the level of $g \times h = f$ is $N$, it follows that the exponents of $\eta_{d_j}$ is zero in $f$ for all $j$. In other words, we have
\begin{equation}
Y_{d_j} = -X_{d_j}
\end{equation}
for all $j$. So, we would have obtained the same sequence $\{d_0, d_1, d_2, \ldots\}$ if for $j \geq 1$, we would have defined $d_j \in \mathcal{D}_M$ instead by replacing $X_{d_j}$ with $Y_{d_j}$ in (2) above.

Since $g$ is a proper factor of $f$, the weight of $g$ is less than or equal to $(k - 1)/2$. So, (3.14) implies that
\begin{equation}
\|X\| = R_{k-1}(M) = R_{k-1}(N),
\end{equation}
where the last equality follows from (2.2), since the set of prime divisors of $M$ and $N$ are the same. Recall from (3.14) that $\|X\| = \sum_d |X_d|$.

In particular, (9.3) implies that the number of $d \in \mathcal{D}_M$ with $X_d \neq 0$ is bounded above by $R_{k-1}(N)$. So, the sequence $\{d_0, d_1, d_2, \ldots\}$ terminates at some $n \in \mathcal{D}_M$ such that
\begin{equation}
n \leq R_{k-1}(N) - 1.
\end{equation}
It follows that each $d \in \mathcal{D}_M$ with nonzero $X_d$ (or equivalently, nonzero $Y_d$) must divide $L_{n+1}$. In other words, we have
\begin{equation}
M = L_{n+1}.
\end{equation}
Since $d_0$ is a multiple of the level of $f$, from Lemma 3.(a) it follows that
\begin{equation}
f = \mathcal{P}_{M,L_j}(g) \times \mathcal{P}_{M,L_j}(h)
\end{equation}
for all $j$. Clearly, for all $j \in \{1, \ldots, n\}$, there exist nonconstant (but not necessarily holomorphic) eta quotients $g'_j$, $g''_j$, $h'_j$ and $h''_j$ on $\Gamma_0(M)$, where the levels of $g'_j$ and $h'_j$ divide $L_j$ and the exponents of $\eta_d$ are zero in $g''_j$ and $h''_j$ for all $d | L_j$ such that
\begin{equation}
g = g'_j \times g''_j \quad \text{and} \quad h = h'_j \times h''_j.
\end{equation}
From the same reasoning by which we obtained (9.2), it follows that
\begin{equation}
g''_j \times h''_j = 1.
\end{equation}
Since $\mathcal{P}_{M,L_j}$ is a homomorphism, we have
\begin{equation}
\mathcal{P}_{M,L_j}(g''_j) \times \mathcal{P}_{M,L_j}(h''_j) = 1.
\end{equation}
Since the levels of $g'_j$ and $h'_j$ divide $L_j$, from Lemma 3.(a) it follows that $g'_j := \mathcal{P}_{M,L_j}(g'_j)$ and $h'_j := \mathcal{P}_{M,L_j}(h'_j)$. Hence, (9.6), (9.7) and (9.9) together imply that
\begin{equation}
f = g'_j \times h'_j.
\end{equation}
Since $g$ (resp. $h$) is holomorphic, Lemma 3.(d) implies that $\mathcal{P}_{M,L_j}(g) = g'_j \times \mathcal{P}_{M,L_j}(g'_j)$ (resp. $\mathcal{P}_{M,L_j}(h) = h'_j \times \mathcal{P}_{M,L_j}(h'_j)$) is holomorphic. Since the exponents in $g'_j$ and $h'_j$ are all integers, it follows from (3.7) that for each cusp $s$ of $\Gamma_0(L_j)$, both $24 \cdot \text{ord}_s(g'_j; \Gamma_0(L_j))$ and $24 \cdot \text{ord}_s(h'_j; \Gamma_0(L_j))$ are integers.

Hence, if at each cusp $s$ of $\Gamma_0(L_j)$, we have
\begin{equation}
\max\{|\text{ord}_s(\mathcal{P}_{M,L_j}(g''_j); \Gamma_0(L_j))|, |\text{ord}_s(\mathcal{P}_{M,L_j}(h''_j); \Gamma_0(L_j))|\} < \frac{1}{24},
\end{equation}
then \( g'_j \) (resp. \( h'_j \)) must be holomorphic. But the fact that \( g'_j \) and \( h'_j \) are nonconstant holomorphic eta quotients on \( \Gamma_0(L_j) \) with \( L_j < M \) such that their product is \( f \) (see (9.10) contradicts the minimality of \( M \).

Let \( X', X'' \in \mathbb{Z}^{D_M} \) be such that \( g'_j = \eta^{X'} \) and \( g''_j = \eta^{X''} \). Then from Conditions (1) and (2) in the definition of the finite sequence \( \{d_0, d_1, \ldots, d_n\} \), it follows that \( \|X'\| \geq j \). Since \( X = X' + X'' \), (9.3) implies that

\[
(9.12) \quad \|X''\| \leq R_{k-1}(N) - j.
\]

From (5.1), (5.2) and from Condition (3) in the definition of the finite sequence \( \{d_0, d_1, \ldots, d_n\} \), it follows that the absolute value of each exponent in \( p_{M,L_j}(g''_j) \) is less than or equal to

\[
\|X''\| \frac{\gcd(d_j, L_j)}{d_j}.
\]

Hence, (3.17), (3.15), (3.7) and (9.12) together imply that at each cusp \( s \) of \( \Gamma_0(L_j) \), the absolute value of the order of \( p_{M,L_j}(g''_j) \) is less than or equal to

\[
\mathcal{M}_j := L_j(R_{k-1}(N) - j) \frac{\gcd(d_j, L_j)}{24 \cdot d_j}.
\]

Similarly, we obtain that the absolute value of the order of \( p_{M,L_j}(h''_j) \) is less than or equal to \( \mathcal{M}_j \) at each cusp \( s \) of \( \Gamma_0(L_j) \). So, from (9.11) and from the discussion following it, we conclude that the inequality: \( \mathcal{M}_j < 1/24 \) leads to a contradiction to the minimality of \( M \) with respect to the fact that \( f \) factorizes on \( \Gamma_0(M) \). Therefore, for all \( j \), we must have \( \mathcal{M}_j \geq 1/24 \), i.e.

\[
\frac{d_j}{\gcd(d_j, L_j)} \leq L_j(R_{k-1}(N) - j).
\]

That implies:

\[
(9.13) \quad L_{j+1} = \text{lcm}(d_j, L_j) \leq L_j^2(R_{k-1}(N) - j)
\]

for all \( j \in \{1, \ldots, n\} \). By induction, from the recurrence inequality above, we obtain:

\[
M = L_{n+1} \leq L_1^2(R_{k-1}(N) - n)^{2^{n-1}} \cdots (R_{k-1}(N) - 1) \leq (2nk)^{2^{R_{k-1}(N)-1}} \Upsilon(R_{k-1}(N)),
\]

where the function \( \Upsilon : \mathbb{N} \to \mathbb{N} \) is as defined in (2.1). Since \( L_1 = M' \), The last inequality follows from (9.1) and (9.4).

**APPENDIX: IRREDUCIBILITY OF MODULAR FORMS IN GENERAL**

Recall that the notions of irreducibility and factorizability also makes perfect sense for modular forms in general. In [7], we see that the modular forms \( E_4, E_6 \) and \( E_4^t - tE_6^t \) for all \( t \in \mathbb{C} \) are not factorizable on \( \text{SL}_2(\mathbb{Z}) \). In the following, we provide examples of modular forms with the trivial multiplier system on \( \Gamma_0(N) \) which are not factorizable on \( \Gamma_0(N) \) for an arbitrary integer \( N > 1 \).

By \( X_0(N) \), we denote the compact modular curve \( \Gamma_0(N) \backslash \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q}) \). Given \( x_0 \in X_0(N) \), if there exists a modular form \( f_{x_0,N} \) on \( \Gamma_0(N) \) which vanishes nowhere on \( X_0(N) \) except at \( x_0 \) such that \( f_{x_0,N} \) has the least order of vanishing among all the modular forms on \( \Gamma_0(N) \) which vanishes only
at $x_0$, then clearly, $f_{x_0,N}$ is not factorizable on $\Gamma_0(N)$. In particular, from the invertibility of the order matrix (see 3.15), it follows that both $f_{0,N}$ and $f_{\infty,N}$ are given by eta quotients for all $N \in \mathbb{N}$. More precisely, $f_{0,N}$ (resp. $f_{\infty,N}$) is the least positive integer power of $\eta^{B_N(-,1)}$ (resp. $\eta^{B_N(-,N)}$) (see 6.5) which satisfies Newman’s criteria (see [15], [16] or [19]). Proceeding in this way or by an easy generalization a result from [20], we obtain that for all $n \in \mathbb{N}$ and for all primes $p \geq 5$, we have

$$f_{0,p^n} = \left(\frac{\eta^p}{\eta_p}\right)^2 \text{ and } f_{\infty,p^n} = \left(\frac{\eta_p^{p^n}}{\eta_p^{p^n-1}}\right)^2.$$  

(9.15)

In particular, for each prime $p \geq 5$, the modular form $(\eta^p/\eta_p)^2$ of weight $p - 1$ is not factorizable on $\Gamma_0(p^n)$ for all $n \in \mathbb{N}$. It follows by a similar argument that for all $N \in \mathbb{N}$ which have at least two distinct odd prime divisors and for $B_N$ as defined in (6.3), the modular form

$$\eta^{B_N(-,1)} = \prod_{d|N} \eta^{B_N(d,1)}$$

has trivial multiplier system on $\Gamma_0(N)$ and it is not factorizable on $\Gamma_0(M)$ for any multiple $M$ of $N$, whose set of prime divisors is the same as that of $N$. However, the lack of an analog of the level lowering map (see Section 5) which preserves weight for modular forms in general, prevents us from concluding that the above modular form is irreducible. In fact, instance of occurrence of any irreducible modular form of weight greater than 2 is yet undiscovered and we doubt whether any such thing exists at all:

**Open question 1.** Does there exist an irreducible modular form with the trivial multiplier system of some weight greater than 2?

On the contrary, it follows from Corollary 5 (or from Theorem 3 in [7], Corollary 1 in [3]) that there exist irreducible holomorphic eta quotients* of arbitrarily large weights.

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