Calculating few-body resonances using an oscillator trap

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Abstract

We investigate the possibility of calculating the parameters of few-body resonances using the oscillator trap boundary conditions. We place the few-body system in an oscillator trap and calculate the energy spectrum and the strength function of a suitably chosen transition. Broader resonances are identified as Lorentzian peaks in the strength function. Narrower resonances are identified through the pattern of avoided crossings in the spectrum of the system as function of the trap size. As an example we calculate $0^+_2$ and $0^+_3$ resonances in $^{12}$C within the 3α model.

1 Introduction

Few-body resonances are often calculated using the complex scaling method where the resonances are identified as generalized complex eigenvalues of the Hamiltonian with the corresponding generalized eigenfunctions (see e.g. [1] and references therein). The method has the advantage of having a simple boundary condition: the few-body wave-function vanishes at large distances. However, it also has certain disadvantages. Complex arithmetics and algorithms are generally slower and complex matrices need more computer memory. Calculating extremely narrow resonances is difficult as it demands calculations of the eigenvalues with exceedingly high accuracy. Interpretation of the generalized eigenfunctions is also not trivial [1], especially for heavy complex scaling needed for broader resonances.

In this contribution we investigate the possibility of calculating the parameters of few-body resonances using the same simple boundary condition as in complex scaling method but working with real energies and real wave-functions.

We place the few-body system in an artificial oscillator trap of length $b$ which is significantly larger than the characteristic length of the few-body system. We then calculate the (discrete) spectrum of the system in the trap and estimate the strength function of a certain transition from a suitable chosen initial state to the positive-energy states of the system in the trap.

The broader resonances with width $\Gamma \gtrsim \hbar^2/(2mb^2)$, where $m$ is the characteristic mass of the few-body system, can be identified as Lorentzian peaks in the strength function.

The narrow resonances with $\Gamma < \hbar^2/(2mb^2)$ need an investigation of the spectrum of the system in the trap as function of the trap length. When an energy level in the trap, following its general behavior as $b^{-2}$, approaches the system's resonance level to within its width, the two levels interfere and avoid crossing. The pattern of avoided crossings in the spectrum of the system in the trap reveals the position of the narrow resonance. The width of the resonance can be estimated from the size of the region of avoided crossing, or, more precisely, from the variation of the energy levels with respect to the trap length.
The approach is similar to the box method (also called the stabilization method) \[3, 4\]. However, the difference is that we use an oscillator trap instead of a box and that we resort to strength function method for broader resonances where the avoided crossings method is less reliable. The box boundary condition is more complicated as the wave-function has to vanish identically at the box boundary which for few-body systems is a multi-dimensional surface. The oscillator trap can be potentially used in stochastic variational calculations with correlated Gaussians \[5\].

As an example we apply the approach to the 3α system in the \(J^\pi=0^+\) channel where there exist a narrow, \(0_2^+\), and a broader, \(0_3^+\), resonance. We show that the approach allows to reliably calculate the two resonances in this system.

2 The few-body system and the trap

We consider the 3α system with the total angular momentum and parity \(J^\pi=0^+\). The Ali-Bodmer type α-α potential is taken from \[6\],

\[
V_{\alpha\alpha}(r) = \left( 125 \hat{P}_l=0 + 20 \hat{P}_l=2 \right) e^{-\left( \frac{r}{1.53} \right)^2} - 30.18 e^{-\left( \frac{r}{2.85} \right)^2} + 4 \cdot 1.44 \frac{r}{2.32} \text{erf} \left( \frac{r}{2.32} \right),
\]

(1)

where all energies are in MeV, all lengths in fm, \(\hat{P}_l\) is the projection operator onto a state with relative orbital momentum \(l\), and \(r\) is the distance between α-particles. In addition a three-body force

\[
V_3(\rho) = -76 \text{MeV} \exp\left( -\rho^2/(4\text{fm})^2 \right),
\]

(2)

is employed to simulate the contribution of “compound nucleus” degrees of freedom at shorter distances where all three α-particles overlap. The three-body force is defined in terms of the hyper-radius

\[
\rho^2 = \frac{m_\alpha}{m} \sum_{i=1}^{3} r_i^2,
\]

(3)

where \(r_i\) are the c.m. coordinates of the α-particles, \(m=939\) MeV is the chosen mass scale and \(m_\alpha=3.97m\).

The system is placed in an oscillator trap

\[
V_{\text{trap}} = \frac{\hbar^2}{2m} \frac{\rho^2}{b^4},
\]

(4)

where the trap length \(b\) is varied around 30-40 fm.

The three-body problem in the trap is solved using the adiabatic hyper-spherical method (see e.g. \[2\] and references therein). First for every fixed hyper-radius \(\rho\) the eigenvalue problem for all remaining variables (denoted collectively as hyper-angles \(\Omega\)) is solved and the spectrum of hyper-angular eigenvalues \(\epsilon_i(\rho)\) together with the angular eigenfunctions \(\Phi_i(\rho,\Omega)\) are obtained. The functions \(\Phi_i(\rho,\Omega)\) are then used as a full basis in the \(\Omega\) space and the total wave-function \(\psi\) is represented as a series

\[
\psi(\rho,\Omega) = \sum_{i=1}^{\infty} f_i(\rho)\Phi_i(\rho,\Omega),
\]

(5)

where the expansion coefficients \(f_i(\rho)\) are obtained by solving the hyper-radial equations where the eigenvalues \(\epsilon_i(\rho)\) serve as effective potentials.
3 Strength function

A resonance can be identified as a peak in a reaction cross-section with approximately Lorentzian shape. The amplitude of a quantum transition, caused by an operator $F$, from some initial state $\psi_a$ into one of the discrete state $\psi_n$ of the system in the trap, is given in the Born approximation as

$$M_{n\rightarrow a} = \langle \psi_n | F | \psi_a \rangle .$$

(6)

Since a resonance per definition must be seen in any reaction channel, the particular choice of the excitation operator $F$ and the initial state $\psi_a$ should be irrelevant as soon as the matrix element does not vanish identically. We thus choose the initial state in the form of the large $\rho$ asymptotics of a bound three-body state $[7, 8]$, 

$$f_i(\rho) = \rho^{-5/2} \exp(-\rho/b_3),$$

(7)

in every hyper-radial channel $i$. The constant $b_3=4$ fm is chosen close to the size of the bound state of $3\alpha$ system. The excitation operator is taken as 

$$F = \rho^2 .$$

(8)

The cross-section of a reaction into the final states with energies $E \pm \Delta E$ is determined by the strength function, defined as

$$S(E) = \frac{1}{\Delta E} \sum_{E_n \in E \pm \Delta E} |M_{n\rightarrow a}|^2 .$$

(9)

The energy bin size $\Delta E$ has to be chosen on the one hand small enough as not to smear out the essential features of the cross-section, and on the other hand large enough to include many states. In our calculations the energy bins include four states each.

The calculated strength function is shown on Figure 1. It reveals a broader peak at 4.3 MeV and a narrow unresolved peak at $\sim 0.4$ MeV. In the region of the broader peak the strength function is well converged with respect to the trap length, and the bin size is quite appropriate for the description of the width of the peak as there are many points within the peak region.

The shape of the peak is well described by a Lorentzian

$$S(E) \approx \frac{1}{E - E_r} \frac{1}{(E - E_r)^2 + \Gamma^2} .$$

(10)
Figure 2: Left: the spectrum of the $3\alpha$ system in an oscillator trap as function of the trap size $b$ in the region of the narrow peak on Figure 1: the sequence of avoided crossings indicates a resonance at $\sim 0.38$ MeV; Right: zoom-in into the region of avoided crossings: the resonance energy is fitted with $E(b) = E_r + K b^{-4}$ with $E_r = 0.38435$ MeV and $K = 7.471$ MeV fm$^4$.

with $E_r = 4.3$ MeV and $\Gamma = 0.9$ MeV. These numbers are consistent with [2].

The narrow peak at in the strength function $\sim 0.4$ MeV is represented by only one point. The position of the point reveals the resonance energy but not the width. To resolve the width at least several points are needed within the peak region. For resonances width exceedingly narrow width $\Gamma$ this would demand unreasonably large trap lengths of the order $b \sim \sqrt{\hbar^2/(2m\Gamma)}$.

However instead of the strength function the avoided crossings method can be used to calculate narrow resonances using reasonably sized traps.

### 4 Avoided crossings

For large trap lengths the energy levels in the trap scale with the trap size as $\hbar \omega \propto b^{-2}$. Varying the trap size a level in the trap can be moved close to the resonance level of the system. If the resonance were behind a completely impenetrable barrier (thus having a vanishing width) there would be no interference through the barrier between the resonance and the state in the external trap. The resonance would then be insensitive to the trap size. The spectrum of the system in the trap, as function of the trap size, would thus show the trap levels scaling as $b^{-2}$ and crossing the resonance energy represented by a horizontal line.

If the barrier has small but finite penetrability the trap level approaching the resonance to within its width becomes perturbed by the resonance resulting in the “repulsion” of the two interfering levels. This shows up as a sequence of avoided crossings in the spectrum of the system in the trap as function of the trap size.

Indeed the spectrum of the $3\alpha$ system in the trap reveals such a sequence of avoided crossings in the vicinity of the narrow resonance, see Figure 2 (left).

The resonance state gets a contribution from the oscillator potential [4] which at large $b$ is proportional to $b^{-4}$. This contribution can be determined by a fit $E_r + K b^{-4}$ through the resonance energies as shown on Figure 2 (right). The fit also provides the asymptotic estimate of the resonance energy $E_r = 0.38435$ MeV.

The figure 3 shows the reduced energy $E_3' \equiv E_3 - K b^{-4}$ where the oscillator contribution is subtracted. It is possible to estimate the width $\Gamma$ of the resonance from the plot assuming that the energy region where the avoided crossing takes place is determined by the width of the resonance,

$$\frac{\Gamma}{2} = \Delta b \left. \frac{\partial E_n'}{\partial b} \right|_{E_n = E_r},$$

(11)
Figure 3: The reduced energy $E'_3 \equiv E_3 - Kb^{-4}$ of the third level of the $3\alpha$ system in an oscillator trap as function of the trap length $b$ in the region of the resonance $E_r = 0.38435$ MeV (indicated as a horizontal line) where the parameters $K$ and $E_r$ are from the fit of the resonance energy on Figure 2. The energy $E'_3$ is fitted with the curve \( \frac{b - b_3}{\Delta b_3} = \arctan \frac{\Gamma / 2}{E' - E_r} \) where $b_3$, $\Delta b_3$, and $\Gamma$ are fitting parameters. The fit gives $\Gamma = 77$ eV.

where $\Delta b$ is the distance between the neighboring avoided crossings.

However, instead of numerical differentiation it is better to estimate the width by fitting the calculated energies with the curve

\[
\frac{b - b_n}{\Delta b_n} = \arctan \frac{\Gamma / 2}{E'_n - E_r},
\]

where $b_n$, $\Delta b_n$, and $\Gamma$ are fitting parameters. Figure 3 (right) shows such a fit for $E_3$. The fit gives $\Gamma = 77$ eV. This value is consistent with the estimates of 10-30 eV in [6, 9] taking into account that our three-body potential provides a slightly higher $E_r$.

5 Conclusion

Using the two lowest $J^p=0^+$ resonances in the $3\alpha$ system as an example, we have investigated the possibility of calculating the energies and widths of few-body resonances by placing the few-body system in an artificial oscillator trap. The oscillator trap has particularly simple boundary condition and can be potentially used in stochastic variational calculations with correlated Gaussians.

We have shown that broader resonances with the width $\Gamma \gtrsim \hbar^2/(2mb^2)$, where $b$ is the trap size, can be identified as Lorentzian peaks in the strength function of a suitably chosen “gedanken” transition. Narrower resonances can be identified through the pattern of avoided crossings in the spectrum of the system in the trap as function of the trap size.

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