SEMISIMPLE HOPF ALGEBRAS OF DIMENSION 60

SONIA NATALE

Dedicated to Susan Montgomery

Abstract. We determine the isomorphism classes of semisimple Hopf algebras of dimension 60 which are simple as Hopf algebras.

1. Introduction and main results

We shall work over an algebraically closed field $k$ of characteristic zero. Let $H$ be a semisimple Hopf algebra over $k$. A Hopf subalgebra $K$ of $H$ is normal if it is stable under the left adjoint action of $H$. If $K$ is normal in $H$, then the quotient $H/HK^+$ is a Hopf algebra and there is an exact sequence $k \to K \to H \to \overline{H} \to k$. In this case, $H$ is isomorphic to a bicrossed product $K^\tau \#_{\sigma} \overline{H}$ with respect to appropriate compatible data.

The Hopf algebra $H$ is called simple if it contains no proper normal Hopf subalgebra. The notion of simplicity is self-dual, that is, $H$ is simple if and only if $H^*$ is simple.

For instance, if $G$ is a finite simple group, then the group algebra $kG$ and its dual $k^*G$ are simple Hopf algebras. Furthermore, in this case, any twisting deformation of $kG$ is simple [15]. However, there are examples of solvable groups that admit simple twisting deformations [5].

It was shown in [10] that, up to twisting deformations, there is no semisimple Hopf algebra of dimension < 60 which is simple as a Hopf algebra. The only simple example in dimension < 60 is a twisting of the group $D_3 \times D_3$ and has dimension 36.

In dimension 60 three examples are known of nontrivial semisimple Hopf algebras which are simple as Hopf algebras. The first two are the Hopf algebras $A_0$ and $A_1 \simeq A_0^*$ constructed by Nikshych [15]. We have $A_0 = (kA_5)^J$, where $J \in kA_5 \otimes kA_5$ is an invertible twist lifted from a nondegenerate 2-cocycle in a subgroup of $A_5$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

The third example is the self-dual Hopf algebra $B$ constructed in [5]. In this case $B = (kD_3 \otimes kD_5)^J$, where $J$ is an invertible twist also lifted from a nondegenerate 2-cocycle in a subgroup of $D_3 \times D_5$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. 

Date: November 25, 2009.
1991 Mathematics Subject Classification. 16W30.
This work was partially supported by CONICET, ANPCyT, SeCyT (UNC), FAMAF and Alexander von Humboldt Foundation.
As coalgebras, these examples are isomorphic to direct sums of full matric coalgebras, as follows:

(1.1) \[ A_1 \cong k \oplus M_3(k)^{(2)} \oplus M_4(k) \oplus M_5(k), \]

(1.2) \[ A_0 \cong k^{(12)} \oplus M_4(k)^{(3)}, \]

(1.3) \[ B \cong k^{(4)} \oplus M_2(k)^{(6)} \oplus M_4(k)^{(2)}. \]

As for the group-like elements, we have \( G(A_0) \cong \mathbb{A}_4 \) and \( G(B) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \).

At the level of fusion categories, it was shown in [3, Theorem 9.12] that \( \text{Rep}\, A_0 \cong \text{Rep}\, A_5 \) is the only simple fusion category of dimension 60. In this context, according to [3, Definition 9.10], a fusion category is called simple if it contains no proper fusion subcategories. In particular, there are simple Hopf algebras whose fusion category is not simple in the sense of [3].

As a consequence of [3, Corollary 9.14], it was shown in [1, Proposition 6.10] that if \( G(H) = 1 \), then \( H \) is of type (1.1), as a coalgebra, furthermore, \( H \) is simple and \( H \) is isomorphic to \( k^{\mathbb{A}_5} \) or to \( A_1 \). Also, by [1, Corollary 6.12], if \( H \) is simple and of coalgebra type (1.3), then \( H \) is isomorphic to the self-dual Hopf algebra \( B \). We shall show in Proposition 5.11 that if \( H \) is simple and has coalgebra type (1.2), then \( H \) is isomorphic to \( A_0 \cong A_1^* \).

The main result of this paper is the following theorem, that says that \( A_0, A_1 \) and \( B \) are actually the only nontrivial simple examples in dimension 60.

**Theorem 1.4.** Let \( H \) be a nontrivial semisimple Hopf algebra of dimension 60. Suppose \( H \) is simple. Then \( H \) is isomorphic to \( A_0 \) or to \( A_1 \) or to \( B \).

Theorem 1.4 will be proved in Sections 4 and 5. The proof relies on the main results of the paper [1]. In particular, we use the refinement, contained in [1, Theorem 1.1], of the result [13, Theorem 11] of Nichols and Richmond on semisimple Hopf algebras with an irreducible comodule of dimension 2.

We also make strong use of several tools developed in [10] regarding, for instance, the structure of quotient coalgebras and relations among the fusion rules for irreducible characters and Hopf subalgebras. We prove some facts on braided Hopf algebras over the dihedral groups \( D_n \) and the alternating group \( \mathbb{A}_4 \), that we apply to some cases where a reduction to the Radford-Majid biproduct situation is possible. See Propositions 3.5 and 3.6.

The contents of the paper are the following: basic facts and terminology on semisimple Hopf algebras and their irreducible characters, as well as some results from [10] and [1], are recalled in Section 2. We discuss there some properties about the structure of the left coideal subalgebra of coinvariants under a Hopf algebra map, that prove to be useful when considering the different possibilities in dimension 60. Section 3 concerns biproducts, and we consider here some special cases of braided Hopf algebras over \( D_n \) and \( \mathbb{A}_4 \).

Finally, the next two sections are devoted to the proof of our main result. In Section 4 we determine the different possible coalgebra types arising in
dimension 60. See Proposition \[4.3\]. These possibilities are then studied separately in each of the subsections of Section 5.

Acknowledgement. It is the author’s pleasure to express her recognition for the influence of Susan Montgomery in her research on semisimple Hopf algebras, through her own contributions, interesting discussions and references.

This work was started during a research stay of the author at the Mathematisches Institut der Universität München, as an Alexander von Humboldt Fellow. She thanks Prof. Hans-Jürgen Schneider for the kind hospitality.

2. Semisimple Hopf algebras

Let $H$ be a semisimple Hopf algebra over $k$. We next recall some of the terminology and conventions from \[10\] that will be used throughout this paper.

As a coalgebra, $H$ is isomorphic to a direct sum of full matrix coalgebras

\[(2.1)\quad H \simeq k^n \oplus \oplus_{d_i > 1} M_{d_i}(k)^{n_i},\]

where $n = |G(H)|$. The Nichols-Zoeller theorem \[14\] implies that $n$ divides both $\dim H$ and $n_i d_i^2$, for all $i$.

If we have an isomorphism as in (2.1), we shall say that $H$ is of type $(1, n; d_1, n_1; \ldots; d_r, n_r)$ as a coalgebra. If $H^*$ is of type $(1, n; d_1, n_1; \ldots)$ as a coalgebra, we shall say that $H$ is of type $(1, n; d_1, n_1; \ldots)$ as an algebra.

So that $H$ is of type $(1, n; d_1, n_1; \ldots; d_r, n_r)$ as a (co-)algebra if and only if $H$ has $n$ non-isomorphic one-dimensional (co-)representations, $n_1$ non-isomorphic irreducible (co-)representations of degree $d_1$, etc.

Let $V$ be an $H$-comodule. The character of $V$ is the element $\chi = \chi_V \in H$ defined by $\langle f, \chi \rangle = \text{Tr}_V(f)$, for all $f \in H^*$. For a character $\chi$, its degree is the integer $\deg \chi = \epsilon(\chi) = \dim V$.

If $\chi \in H$ is a character, then $\chi$ decomposes as $\chi = \sum \mu m(\mu, \chi)\mu$, where $\mu$ runs over the set of irreducible characters of $H$ and $m(\mu, \chi)$ are nonnegative integers. For all characters $\chi, \psi, \lambda \in H$, we have \[13\]:

\[(2.2)\quad m(\chi, \psi\lambda) = m(\psi^*, \lambda\chi^*) = m(\psi, \chi\lambda^*).\]

Let $\chi$ be an irreducible character of $H$. The stabilizer of $\chi$ under left multiplication by elements in $G(H)$ will be denoted by $G[\chi]$. So that a group-like element $g$ belongs to $G[\chi]$ if and only if $g\chi = \chi$. By the Nichols-Zoeller theorem \[14\], we have that $|G[\chi]|$ divides $(\deg \chi)^2$.

In view of \[13\] Theorem 10, $G[\chi] = \{ g \in G(H) : m(g, \chi\chi^*) > 0 \} = \{ g \in G(H) : m(g, \chi\chi^*) = 1 \}$. In particular,

$$\chi\chi^* = \sum_{g \in G[\chi]} g + \sum_{\deg \mu > 1} m(\mu, \chi\chi^*)\mu.$$

The irreducible characters in $H$ span a subalgebra of $H$, that coincides with the character algebra $R(H^*)$ of $H^*$. There is a bijective correspondence
between Hopf subalgebras of $H$ and standard subalgebras of $R(H^*)$, that is, subalgebras spanned by irreducible characters of $H$. This correspondence assigns to the Hopf subalgebra $K \subseteq H$ its character algebra $R(K^*) \subseteq R(H^*)$. See [13].

**Lemma 2.3.** Suppose $B \subseteq H$ is a Hopf subalgebra. Let $\chi, \lambda \in B$, $\psi \in H$, be irreducible characters. If $m(\chi, \psi \lambda) > 0$, then $\psi \in B$.

**Proof.** By (2.2) we have $m(\psi, \chi^* \lambda) = m(\chi, \psi \lambda) > 0$. Since $B$ is a Hopf subalgebra, all irreducible summands of $\chi^* \lambda$ belong to $B$. Hence $\psi \in B$, as claimed.

2.1. **Subalgebras of coinvariants.** Let $\pi : H \rightarrow \overline{H}$ be a surjective Hopf algebra map. Consider the subalgebra $H\,^{co\pi} \subseteq H$ of right coinvariants of $\pi$, defined as

$$H\,^{co\pi} = \{ h \in H : h_{(1)} \otimes \pi(h_{(2)}) = h \otimes 1 \}.$$ 

We shall also use the notation $H\,^{co\pi} := H\,^{co\pi}$. This is a left coideal subalgebra of $H$ stable under the left adjoint action. The map $\pi$ is normal if $H\,^{co\pi}$ is a subcoalgebra, hence a Hopf subalgebra, of $H$.

**Remark 2.4.** Let $\pi : H \rightarrow \overline{H}$ be a surjective Hopf algebra map. By [8, Theorem 8.2.4 and Proposition 8.4.4], the extension $H\,^{co\overline{H}} \subseteq H$ is $\overline{H}$-Galois. On the other hand, $H$ is free over $H\,^{co\overline{H}}$, by [18]. Hence, by [17, 3.2 (4)], $H\,^{co\overline{H}} \subseteq H$ is an $\overline{H}$-cleft extension. In particular, $H$ is isomorphic, as an $\overline{H}$-comodule algebra, to a crossed product $H\,^{co\overline{H}} \#_\sigma \overline{H}$.

**Lemma 2.5.** Let $F$ be a finite group. Let also $H$ be a semisimple Hopf algebra endowed with a surjective Hopf algebra map $\pi : H \rightarrow kF$. Then:

(i) For all irreducible characters $\zeta \in H$, $m(1, \pi(\zeta^*)) \geq \deg \zeta$. Equality holds if and only if $\pi(\zeta)$ is multiplicity free.

(ii) Let $c$ be the least common multiple among the dimensions of all simple $H\,^{co\pi}$-modules. Then, for every simple $H$-module $U$, $\dim U$ divides the product $c |F|$.

In particular, if $H\,^{co\pi}$ is commutative, then $\dim U$ divides the order of $F$, for all simple $H$-modules $U$.

**Proof.** (i) Let $n = \deg \zeta$. Then we may decompose $\pi(\zeta) = \sum_{i=1}^n x_i$, where $x_i \in F$. Part (i) follows easily from this.

(ii) By cleftness of $H$ as a $kF$-comodule algebra, $H$ is isomorphic as an algebra to a crossed product $H \simeq R\#_\sigma kF$, where $R = H\,^{co\pi}$. See Remark 2.4. The description of the irreducible representations of a crossed product given in [9] implies that $\dim U$ divides $c |F|$, for all simple $H$-modules $U$.

**Remark 2.6.** Let $\pi : H \rightarrow \overline{H}$ be a surjective Hopf algebra map. Identify $B^*$ with a Hopf subalgebra of $H^*$ by means of the transpose of $\pi$. Then, for each irreducible character $\zeta \in H$, the multiplicity $m(1, \pi(\zeta))$ is exactly $m(k1, \text{res}^H_{H^*} \zeta)$, where res denotes the restriction map.
By Frobenius reciprocity, we have
\[ m(1, \pi(\zeta)) = m(\text{Ind}^H_{B}\kappa 1, \zeta). \]
Consider the decomposition
\[ H^{\text{co}} = k1 \oplus V_1 \oplus \cdots \oplus V_m, \]
into irreducible left coideals \( V_i = k1, V_1, \ldots, V_m \). Let \( \zeta_i \in H \) be the character of \( V_i \), \( i = 0, \ldots, m \).

By [10, Lemma 1.7.1], \( H^{\text{co}} \cong \text{Ind}^H_{B}\kappa 1 \) as left \( H \)-comodules. Therefore the multiplicity \( m(1, \pi(\zeta_i)) \) coincides with the multiplicity of \( V_i \) as a direct summand of \( H^{\text{co}} \).

2.2. The Hopf subalgebras \( B[\chi] \). Let \( C \) be a simple subcoalgebra of \( H \), and let \( \chi \in C \) be the irreducible character of \( C \). We shall denote by \( B[\chi] := k[CS(C)] \) the Hopf subalgebra generated by \( CS(C) \) as an algebra. Note that \( G[\chi] \subseteq B[\chi] \).

The Hopf subalgebra \( B[\chi] \) is contained in the adjoint Hopf subalgebra \( H^{\text{coad}} \) of \( H \), which is generated by the irreducible components of \( XS(X) \), where \( X \) runs over all simple subcoalgebras of \( H \). Recall that there is a universal cocentral exact sequence
\[ k \to H^{\text{coad}} \to H \to kU \to k \]
where \( U \) is the universal grading group of the category \( H \)-comod of finite dimensional \( H \)-comodules. See [2, 8.5.], [6, Theorem 3.8].

The following is one of the main results of [1].

**Theorem 2.9.** [1, Theorem 1.1]. Suppose \( \deg \chi = 2 \). Then \( B[\chi] \) is a commutative Hopf subalgebra of \( H \) isomorphic to \( k\Gamma \), where \( \Gamma \) is a non cyclic finite subgroup of \( PSL_2(k) \) of even order.

Let \( G[\chi] \subseteq G(H) \) be the stabilizer of \( \chi \) with respect to left multiplication. Then we have

(i) If \( |G[\chi]| = 4 \), then \( B[\chi] \simeq k\mathbb{Z}_2 \times \mathbb{Z}_2 \).
(ii) If \( |G[\chi]| = 2 \), then \( B[\chi] \simeq kD_n \), where \( n \geq 3 \).
(iii) If \( |G[\chi]| = 1 \), then \( B[\chi] \simeq kA_4, kS_4, \) or \( kA_5 \).

Theorem 2.9 turns out to be useful when discussing low dimensional semisimple Hopf algebras since, in that case, most examples would admit irreducible comodules of dimension 2.

3. Braided Hopf algebras over \( D_n \) and \( A_4 \)

We recall for future use some facts on the Radford-Majid biproduct construction [7, 16].

Let \( A \) be a semisimple Hopf algebra and let \( \mathcal{A}YD \) denote the braided category of Yetter-Drinfeld modules over \( A \). Let \( R \) be a semisimple braided Hopf algebra in \( \mathcal{A}YD \).

Then \( R \) is both an algebra and a coalgebra in \( \mathcal{A}YD \), such that the comultiplication is an algebra map in \( \mathcal{A}YD \) and the identity map \( \text{id}_R \) has a convolution inverse, called the antipode of \( R \). We shall use the notation...
\[ \Delta_R(a) = a^{(1)} \otimes a^{(2)} \] and \( S_R \) for the comultiplication and the antipode of \( R \), respectively.

The compatibility between the multiplication and comultiplication in \( R \) is the following:

\[ \Delta_R(ab) = a^{(1)}((a^{(2)})_{-1}b^{(1)}) \otimes (a^{(2)})_0b^{(2)}, \]

for all \( a, b \in R \).

Let \( H = R \# A \) be the corresponding biproduct; so that \( H \) is a semisimple Hopf algebra with multiplication, comultiplication and antipode given by

\[ (a \# g)(b \# h) = a(g(1) \cdot b) \# g(2)h, \quad \Delta(a \# g) = a^{(1)} \# (a^{(2)})_{-1}g(1) \otimes (a^{(2)})_0 \# g(2), \]

\[ S(a \# g) = (1 \# S(a \_1 g))(S_R(a_0) \# 1), \]

for all \( g, h \in A, a, b \in R \).

**Remark 3.3.** Note that \( R = H^{co\pi} \), where \( \pi = \epsilon_R \otimes \text{id} : H \rightarrow A \). Hence \( R \) is a normal left coideal subalgebra of \( H \). Moreover, the action of \( A \) on \( R \) coincides with the restriction of the adjoint action of \( H \) to \( A \).

On the other hand, the map \( \text{id} \otimes \epsilon : H \rightarrow R \) induces a coalgebra isomorphism \( H/HA^+ \cong R \). The relations \( (3.2) \) imply that the coaction of \( A \) on \( R \) is given by \( \rho = (\epsilon_R \otimes \text{id} \otimes \text{id} \otimes \text{id})\Delta : R \rightarrow A \otimes R \).

A biproduct \( R \# A \) as described above is characterized by the following property: suppose \( H \) is a finite dimensional Hopf algebra endowed with Hopf algebra maps \( i : A \rightarrow H \) and \( \pi : H \rightarrow A \), such that \( \pi i : A \rightarrow A \) is an isomorphism. Then the subalgebra \( R := H^{co\pi} \) has a natural structure of Yetter-Drinfeld Hopf algebra over \( A \) such that the multiplication map \( R \# A \rightarrow H \) induces an isomorphism of Hopf algebras.

The following lemma gives the existence of Hopf subalgebras in a biproduct, under appropriate assumptions. Let \( R \) be a semisimple braided Hopf algebra over \( A \) and let \( H = R \# A \) be the biproduct.

Suppose that \( A = kG \), where \( G \) is a finite group. There is a \( G \)-grading on \( R : R = \bigoplus_{g \in G} R_g \), which is both an algebra and a coalgebra grading, i.e.,

\[ R_gR_h \subseteq R_{gh}, \quad \Delta_R(R_g) \subseteq \bigoplus_{st=g} R_s \otimes R_t, \]

for all \( g, h \in G \). This grading corresponds to the coaction \( \rho : R \rightarrow kG \otimes R \), \( \rho(a) = a_{-1} \otimes a_0 \), in such a way that \( R_g = \{ r \in R : \rho(a) = g \otimes a \} \).

The group \( G \) acts on \( R \) by algebra and coalgebra automorphisms, and the action satisfies, for all \( g, h \in G \), the Yetter-Drinfeld condition

\[ h.R_g = R_{gh^{-1}}. \]

The set \( \text{Supp}(R) \) of elements \( g \in G \) such that \( R_g \neq 0 \) will be called the *support* of \( R \). If \( \Gamma \) is a subgroup of \( G \) containing \( \text{Supp}(R) \), then \( R \) is a braided Hopf algebra over \( \Gamma \) and \( R \# k\Gamma \) is a Hopf subalgebra of \( R \# kG \) [11, Lemma 4.3.1].
The next two propositions will be used later on in our discussion in the dimension 60 context.

**Proposition 3.5.** Suppose $n \geq 3$ is odd. Let $R$ be a semisimple braided Hopf algebra over $G = D_n$ with $\dim R = n + 1$. Then $\text{Supp}(R) \subseteq T$, where $T \simeq \mathbb{Z}_n$ is the subgroup of rotations in $D_n$.

Therefore, $R \# kT$ is a Hopf subalgebra of $R \# kD_n$ of index 2. In particular, $R \# kT$ is normal in $R \# kD_n$.

**Proof.** Consider the $D_n$-grading $R = \oplus_{x \in D_n} R_x$. Assume on the contrary that $\text{Supp}(R) \not\subseteq T$. Then there exists a reflection $s \in D_n$ such that $R_s \neq 0$.

Since $n$ is odd, then the reflections in $D_n$ form a conjugacy class. In view of the compatibility condition (3.4), we get that $R_s \neq 0$, for all reflections $x \in D_n$. Since $D_n$ has exactly $n$ reflections, $s_1, \ldots, s_n$, then we see that $\text{Supp}(R) = \{1, s_1, \ldots, s_n\}$.

Since $\dim R = n + 1$, then $\dim R_{s_i} = 1$, for all $i = 0, \ldots, n$, where $s_0 = 1 \in D_n$. Let $u_0 = 1, u_1, \ldots, u_n$ be a basis of $R$ such that $u_i \in R_{s_i}$.

We have $u_i u_j \in R_{s_i s_j}$, for all $1 \leq i, j \leq n$. Since the product $s_i s_j$ of two reflections is a rotation, then $R_{s_i s_j} = 0$, for all $1 \leq i \neq j \leq n$, and therefore $u_i u_j = 0$, for all $1 \leq i \neq j \leq n$. In particular, $R$ is commutative.

On the other hand, for all $i = 1, \ldots, n$, we have $u_i^2 \in R_{s_i^2} = R_{s_0} = k1$.

After rescaling the basis $u_i$, we may assume that $u_i^2 = 0$ or 1, for all $i = 1, \ldots, n$.

Suppose that $u_i^2 = 1$ for some $i \geq 1$, and pick $j \neq i$, $1 \leq j \leq n$. We get

$$u_j = (u_i^2)u_j = u_i(u_i u_j) = u_i 0 = 0,$$

which is a contradiction.

Then $u_i^2 = 0$, for all $i = 1, \ldots, n$. But this is again a contradiction, since $R$ is semisimple, by assumption. Therefore such a grading is impossible, and the proposition follows. \qed

**Proposition 3.6.** Let $R$ be a semisimple braided Hopf algebra over $\mathbb{A}_4$ with $\dim R = 5$. Then $\text{Supp}(R) \subseteq \mathbb{K}$, where $\mathbb{K} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ is the Klein subgroup. Therefore $R \# k\mathbb{K}$ is a normal Hopf subalgebra of $R \# \mathbb{A}_4$ of dimension 20.

**Proof.** As in the proof of the previous proposition, consider the $\mathbb{A}_4$-grading $R = \oplus_{x \in \mathbb{A}_4} R_x$, and assume on the contrary that $\text{Supp}(R) \not\subseteq \mathbb{K}$. Then there exists a 3-cycle $c \in \mathbb{A}_4$ such that $R_c \neq 0$.

Condition (3.4) implies $R_x \neq 0$, for all 3-cycles $x \in \mathbb{A}_4$ conjugated to $c$. Since the conjugacy class of a 3-cycle $c$ in $\mathbb{A}_4$ has exactly 4 elements, $c = c_1, \ldots, c_4$, then we get $\text{Supp}(R) = \{1, c_1, \ldots, c_4\}$. Thus $\dim R_{c_i} = 1$, for all $i = 0, \ldots, 4$, where $c_0 = 1$.

Let, as before, $u_0 = 1, u_1, \ldots, u_4$ be a basis of $R$ such that $u_i \in R_{c_i}$. Then $u_i u_j \in R_{c_i c_j}$, for all $1 \leq i, j \leq 4$. Now, if $c_i, c_j$ are 3-cycles in the same conjugacy class $O$, then the product $c_i c_j$ does not belong to $O$. Hence $R_{c_i c_j} = 0$, for all $1 \leq i, j \leq 4$, and $u_i u_j = 0$, for all $1 \leq i, j \leq 4$. Since $R$ is semisimple, this is a contradiction. Then $\text{Supp}(R) \subseteq \mathbb{K}$, as claimed.
Hence $R\#\mathbb{K}$ is a Hopf subalgebra of $R\#A_4$ of dimension 20. Because $\mathbb{K}$ is normal in $A_4$ and $R$ is stable under the adjoint action of $H$, this is a normal Hopf subalgebra. The proof is complete. \qed

4. Coalgebra types in dimension 60

In what follows $H$ will be a semisimple Hopf algebra of dimension 60.

Suppose that $G(H) \neq 1$ and $H$ has an irreducible character $\chi$ of degree 2. Let $C \subseteq H$ be the simple subcoalgebra containing $\chi$, and consider the Hopf subalgebra $B := B[\chi] = k[CS(C)] \subseteq H_{\mathrm{coal}}$. Then we have $B \cong k^{A_4}, k^{Z_2 \times Z_2}$ or $k^{D_6}$, $n = 3$ or 5. See Subsection 2.2.

Lemma 4.1. Suppose $G(H) = 2$. Assume that $H$ has an irreducible character $\chi$ of degree 2. Then $G[\chi] = G(H)$ and $B[\chi] \cong k^{A_4}$ or $k^{D_6}$.

We have in addition:

(i) The sum of simple subcoalgebras of dimensions 1 and 2 is a Hopf subalgebra of $H$.

(ii) Let $B := k[B[\chi]] \deg \chi = 2$. Then $G(H) \subseteq G(B) \cap Z(B)$.

Proof. In this case $H$ cannot contain Hopf subalgebras isomorphic to $k^{A_4}$ or $k^{Z_2 \times Z_2}$. Then $|G[\chi]| = 2$ and thus $G[\chi] = G(H)$. Since this holds for all irreducible characters of degree 2, part (i) then follows from [10] Theorem 2.4.2. Since $G(H) = G[\chi]$ is contained in $B[\chi]$ for all $\chi$ of degree 2, and because $B[\chi]$ is commutative for all such $\chi$, then $G(H)$ is central in $B$. This proves (ii). \qed

Remark 4.2. Suppose $H$ is any semisimple Hopf algebra satisfying (ii), that is, the sum of simple subcoalgebras of dimensions 1 and 2 is a Hopf subalgebra of $H$. It follows from Lemma 2.3 that if $\chi, \lambda$ and $\psi$ are irreducible characters of $H$ such that $\deg \chi, \deg \lambda \leq 2$ and $m(\chi, \psi \lambda) > 0$, then $\deg \psi \leq 2$.

Proposition 4.3. Suppose $H$ is not cocommutative. Then, according to the order of $G(H)$, the coalgebra type of $H$ is one of the following:

(i) $|G(H)| = 1$: $(1, 1; 3, 2; 4, 1; 5, 5)$.

In this case, $H$ is simple and isomorphic to $k^{A_5}$ or to $A_1$.

(ii) $|G(H)| = 2$: $(1, 2; 2, 1; 3, 6)$, $(1, 2; 2, 1; 3, 2; 6, 1)$.

In this case the simple subcoalgebras of dimensions 1 and 4 form a Hopf subalgebra of $H$, isomorphic to $k^{D_3}$.

(iii) $|G(H)| = 2$: $(1, 2; 2, 2; 5, 2)$.

In this case the simple subcoalgebras of dimensions 1 and 4 form a Hopf subalgebra of $H$, isomorphic to $k^{D_6}$.

(iv) $|G(H)| = 3$: $(1, 3; 2, 12; 3, 1)$, $(1, 3; 3, 1; 4, 3)$.

(v) $|G(H)| = 4$: $(1, 4; 2, 14)$, $(1, 4; 2, 10; 4, 1)$.

(vi) $|G(H)| = 4$: $(1, 4; 2, 2; 4, 3)$.
In this case the simple subcoalgebras of dimensions 1 and 4 form a Hopf subalgebra of \(H\) of dimension 12.

(vii) \(|G(H)| = 4\): \((1, 4 ; 2, 6 ; 4, 2)\).

In this case, if \(H\) is simple, then \(H\) is isomorphic to the self-dual Hopf algebra \(B\).

(viii) \(|G(H)| = 6\): \((1, 6 ; 2, 9 ; 3, 2), (1, 6 ; 3, 3, 6), (1, 6, 3, 2 ; 6, 1)\).

(ix) \(|G(H)| = 10\): \((1, 10 ; 5, 2)\).

(x) \(|G(H)| = 12\): \((1, 12 ; 2, 12)\).

(xi) \(|G(H)| = 12\): \((1, 12 ; 4, 3)\).

(xii) \(|G(H)| = 15\): \((1, 15 ; 3, 5)\).

(xiii) \(|G(H)| = 20\): \((1, 20 ; 2, 10)\).

We shall show in Proposition 5.11 below that, if \(H\) is simple and has coalgebra type \((1, 12 ; 4, 3)\) as in (xi), then \(H\) is isomorphic to \(A_6 \cong A_3^*\).

Proof. By [1, Proposition 6.10], if \(G(H) = 1\), then \(H\) is simple and isomorphic to \(kA_5\) or to \(A_1\). Also, by [1, Corollary 6.12], if \(H\) is simple and of coalgebra type \((1, 4 ; 2, 6 ; 4, 2)\), then \(H\) is isomorphic to the self-dual Hopf algebra \(B\).

We shall next show that the prescribed ones are the only possible coalgebra types. We claim that the types \((1, 3, 2 ; 3, 3, 5)\) and \((1, 3, 2, 3, 3, 1 ; 6, 1)\) are impossible. Suppose on the contrary that \(H\) is of one of these types. Then \(H\) has a self-dual irreducible character \(\chi\) of degree 2, and necessarily \(G[\chi] = 1\). Therefore, since \(|G(H)| = 3\), it follows from Theorem 2.9 that the Hopf subalgebra \(B[\chi]\) isomorphic to \(kA_4\). In particular, \(\chi \notin B[\chi]\), and thus \(B[\chi]\) has index 2 in \(k[C]\), where \(C\) is the simple subcoalgebra containing \(\chi\). This is a contradiction since it implies that \(k[C]\) is of dimension 24, which does not divide \(\dim H\).

Also, the types \((1, 2 ; 2, 10 ; 3, 2)\) and \((1, 2 ; 2, 6 ; 3, 2 ; 4, 1)\) are impossible, by Lemma 4.1 (i), since \(H\) cannot contain Hopf subalgebras of dimensions 42 or 26.

The coalgebra type \((1, 2 ; 2, 2 ; 3, 2 ; 4, 2)\) is not possible either. Indeed, in this case, the sum of simple subcoalgebras of dimensions 1 and 4 is a Hopf subalgebra \(B\) of \(H\) of dimension 10. Let \(\zeta \neq \zeta' \in H\) be the irreducible characters of degree 4, \(C_{\zeta}, C_{\zeta'}\), the corresponding simple subcoalgebras, and \(C = C_{\zeta} \oplus C_{\zeta'}\). Consider the product \(\lambda \zeta\), where \(\lambda\) is an irreducible character of degree 2. Then \(\lambda \zeta\) does not contain irreducible summands of degree 1 or 2, since otherwise we would have \(\zeta \in B\), which is a contradiction; c.f. Remark 4.2. Taking degrees, it follows that \(\lambda \zeta\) is a sum of irreducible characters of degree 4. This implies that \(BC = C\). Therefore, \(C\) is a \((B, H)\)-Hopf module under the action of \(B\) given by left multiplication and the coaction of \(H\) given by the comultiplication. Then the Nichols-Zoeller theorem implies that \(\dim B\) divides \(\dim C = 32\). This contradiction discards this possibility.
Apart from these, other than the types listed in (i)–(xiii), we must consider the possibilities (1, 12; 2, 3; 6, 1), (1, 12; 2, 3; 3, 4), with \(|G(H)| = 12\), and (1, 4, 2, 1; 4, 1; 6, 1), (1, 4, 2, 5; 6, 1), (1, 4, 2, 1; 3, 4; 4, 1), (1, 4, 2, 5; 3, 4), with \(|G(H)| = 4\). In these cases, \(G(H)\) contains a subgroup of order 4 and the number of irreducible characters of degree 2 is odd. Hence, by [10, Proposition 2.1.3], \(H\) would contain a Hopf subalgebra of dimension 8, which is not possible, since 8 does not divide \(\dim H\). This discards these possibilities and proves that these are indeed the only possible coalgebra types.

It follows from Lemma 4.1 that the simple subcoalgebras of dimensions 1 and 4 form a Hopf subalgebra of \(H\), isomorphic to \(kD_3\), for the coalgebra types (1, 2; 2, 1; 3, 6) and (1, 2; 2, 1; 3, 2; 6, 1), or to \(kD_5\) for the coalgebra type (1, 2; 2, 2; 5, 2). Finally, the statement on type (1, 4; 2, 2; 4, 3) is easily seen.

The next two lemmas discard the existence of certain quotient Hopf algebras.

**Lemma 4.4.** Suppose \(H\) has a quotient Hopf algebra of dimension 12. Then we have:

(i) If \(H\) has coalgebra type (1, 2; 2, 1; 3, 6), (1, 2; 2, 1; 3, 2; 6, 1), (1, 6; 3, 6), or (1, 6; 3, 2; 6, 1), then \(H\) is not simple.

(ii) If \(|G(H)|\) is divisible by 5, then \(H\) is not simple.

**Proof.** Suppose \(H \rightarrow B\) is a Hopf algebra quotient with \(\dim B = 12\). Then \(\dim H^{co B} = 5\). Consider first the case (i). Here \(kG(H) \cap H^{co B} = kG(H)^{co B} = k1\), by [14]. On the other hand, \(H^{co B}\) cannot be contained in a Hopf subalgebra of type (1, 2; 2, 1), because \(\dim H^{co B}\) does not divide 6.

Decomposing \(H^{co B}\) into a direct sum of simple left coideals leads to a contradiction, in view of the assumptions on the coalgebra structure of \(H\). This proves (i).

Now suppose that 5 divides \(|G(H)|\), so that \(G(H)\) has a subgroup \(F\) of order 5. Then necessarily \(kF = H^{co B}\), by [14]. Thus \(kF\) is normal in \(H\) and thus \(H\) is not simple. This proves (ii). \(\square\)

**Lemma 4.5.** (i) Suppose \(H\) has coalgebra type (1, 15; 3, 5). Then \(H\) has no quotient Hopf algebra of dimension 6.

(ii) Suppose \(H\) has coalgebra type (1, 4; 2, 2; 4, 3). Then \(H\) has no quotient Hopf algebra isomorphic to \(kD_5\).

**Proof.** (i). Suppose on the contrary that there exists a Hopf algebra quotient \(\pi : H \rightarrow L\), with \(\dim L = 6\). We have \(\dim H^{co L} = 10\). Therefore \(G(H) \cap H^{co L}\) is of order 5 or 1. Decomposing \(H^{co L}\) into a direct sum of irreducible left coideals we see that the first is impossible, whence \(|G(H) \cap H^{co L}| = 1\). But this implies that \(\pi|_{G(H)}\) is injective, which contradicts [14], since \(|G(H)|\) does not divide \(\dim L\).
(ii). Since $H$ has 3 irreducible characters of degree 4, then one of them, say $\psi$, must be a fixed element under left multiplication by $G(H)$. Hence we have a decomposition

$$\psi\psi^* = \sum_{g \in G(H)} g + n\lambda + n'\lambda' + \mu,$$

where $\lambda \neq \lambda'$ are the irreducible characters of degree 2, $\mu$ is a sum of irreducible characters of degree 4, and $n, n'$ are nonnegative integers.

Since $H$ must contain a Hopf subalgebra of dimension 6, then left multiplication by $G(H)$ permutes transitively the set $\{\lambda, \lambda'\}$. Then $n = n'$, because $\psi$ is fixed under left multiplication by $G(H)$. Suppose $n \neq 0$. Then

$$\lambda\psi = \psi + \rho,$$

where $\rho$ is an irreducible character of degree 4; otherwise, $\rho$ would contain an irreducible character of degree $\leq 2$, implying, by Lemma 2.3 that $\psi$ belongs to the unique Hopf subalgebra of dimension 12 of $H$, which is impossible.

Since $n = n'$, we may assume that $\lambda$ belongs to a Hopf subalgebra of dimension 6 of $H$; that is, $\lambda^2 = 1 + a + \lambda$, where $1 \neq a$ is a group-like element of order 2. Multiplying (4.7) on the left by $\lambda$, we find

$$3\psi + \rho = \psi + \rho + \lambda\rho.$$

Hence $\lambda\rho = 2\psi$. Let $C_\lambda, C_\psi, C_\rho$, be the simple subcoalgebras containing $\lambda, \psi$ and $\rho$, respectively. Then we have $C_\lambda C_\psi \subseteq C_\psi + C_\rho$ and $C_\lambda C_\rho \subseteq C_\psi$, implying that $A(C_\psi + C_\rho) \subseteq C_\psi + C_\rho$, where $A = k[C_\lambda]$ is a Hopf subalgebra of dimension 6. Then $C_\psi + C_\rho$ is an $(A, H)$-Hopf module. But this contradicts [14], because $\dim A$ does not divide $32 = \dim(C_\psi + C_\rho)$.

Therefore we have $n = n' = 0$. That is,

$$\psi\psi^* = \sum_{g \in G(H^*)} g + \mu,$$

where $\mu$ is a sum of irreducible characters of degree 4.

Suppose on the contrary that there exists a quotient Hopf algebra $\pi : H \to kD_5$. We have $\dim H^{\text{co}\pi} = 6$. Then, either $H^{\text{co}\pi}$ contains a unique irreducible left coideal of dimension 4, or $H^{\text{co}\pi} = k1 \oplus ka \oplus U \oplus U'$, where $U$ and $U'$ are irreducible coideals of dimension 2 and $1 \neq a \in G(H)$.

In view of (4.9), the last possibility implies that $m(1, \pi(\psi\psi^*)) = 2$, which contradicts Lemma 2.6 (i). Therefore we may assume that

$$H^{\text{co}\pi} = k1 \oplus ka \oplus V,$$

as a left coideal of $H$, where $a \in G(H)$ is of order 2 and $V$ is an irreducible left coideal of dimension 4. Let $\zeta \in H$ be the irreducible character corresponding to $V$. We have $\zeta = \zeta^*$. Consider the decomposition (4.9). Using Lemma 2.3 (i) and the decomposition (4.10) of $H^{\text{co}\pi}$, we get $m(1, \pi(\mu)) \geq 2$. Hence $m(\zeta, \psi\psi^*) = m(\zeta, \mu) \geq$
2. This implies that $|G(\zeta)| = 4$; otherwise, since $\psi$ is stable under left multiplication by $G(H)$, we would also have $m(g\zeta, \psi\psi^*) \geq 2$, for some $g \in G(H)$ such that $g\zeta \neq \zeta$, whence the contradiction $\deg \psi\psi^* \geq 20$.

In particular, relation (4.9) holds for $\zeta$ in the place of $\psi$, and thus

\begin{equation}
\zeta^2 = \sum_{g \in G(H)} g + m\zeta + \zeta',
\end{equation}

where $m \geq 2$, and $\zeta'$ is irreducible of degree 4.

Write $\pi(\zeta) = 1 + x + y + z$, where $x, y, z \in D_5 \setminus \{1\}$. Since the dimension of an induced representation from $kD_5$ to $H^*$ is $6 = |H^* : kD_5|$, we see that the multiplicity of $\zeta$ in such representation is at most 1. By Frobenius reciprocity, $x$, $y$ and $z$ are pairwise distinct. Since $\pi(G(H))$ is a subgroup of $D_5$, then $|\pi(G(H))| = 2$. Thus $\pi(g)^2 = 1$, for all $g \in G(H)$. Suppose $g \in G(H)$ is such that $\pi(g) \neq 1$. The relation $g\zeta = \zeta$ implies that $m(\pi(g), \pi(\zeta)) > 0$. Hence we may assume $\pi(g) = x$.

Applying $\pi$ to the relation (4.11), we get

\begin{equation}
(1 + x + y + z)^2 = (2 + m)(1 + x) + my + mz + \pi(\zeta').
\end{equation}

Comparing the multiplicity of $x$ on both sides of this equality, and since $m \geq 2$, we see that $x = yz = zy$.

On the other hand, by self-duality of $\pi(\zeta)$ and because $x^2 = 1$, we must have $y = z^{-1}$ or $y^2 = z^2 = 1$. This is a contradiction since, in any case, neither $zy$ nor $yz$ can be of order 2. This shows that the decomposition (4.10) is impossible. Then $H$ cannot have quotient Hopf algebras isomorphic to $kD_5$, as claimed.

It was shown in [1, Theorem 6.4] that for every semisimple Hopf algebra $H$ such that all irreducible characters have degree at most 2, either $H$ or $H^*$ must contain a central group-like element. As a consequence, we have:

**Proposition 4.13.** Suppose $H$ has coalgebra type $(1, 4; 2, 14)$, $(1, 12; 2, 12)$ or $(1, 20; 2, 10)$. Then $H$ is not simple. □

5. PROOF OF THE MAIN RESULT

In the following subsections we shall consider the distinct possibilities for the coalgebra type of $H$, arising from Proposition 4.3.

The results in this section, combined with Proposition 4.3, imply the statement in Theorem 1.4, namely, that the only simple semisimple Hopf algebras of dimension 60 are exactly $A_0$, $A_1$ and $B$.

5.1. Type (vi). We know from Proposition 4.13 that if $H$ has coalgebra type $(1, 4; 2, 14)$, then $H$ is not simple. We shall show in this subsection that the same occurs for the type $(1, 4; 2, 10; 4, 1)$.

**Proposition 5.1.** Suppose $H$ is of type $(1, 4; 2, 10; 4, 1)$ as a coalgebra. Then $H$ is not simple.
Proof. There must exist irreducible characters \( \chi \) and \( \chi' \) of degree 2, such that \( \chi \chi' \) is irreducible of degree 4. Otherwise, the sum of simple subcoalgebras of dimensions 1 and 2 would be a Hopf subalgebra of \( H \) of dimension 44, which is impossible. By \[10\] Theorem 2.4.2, we have \( G[\chi] \cap G[\chi'] = 1 \), thus \( G[\chi], G[\chi'] \) are distinct subgroups of order 2. If \( H \) is simple, then by \[11\] Lemma 6.11, it should be \( H \cong B \), which contradicts the assumption on the coalgebra type of \( H \). Hence \( H \) is not simple, as claimed.

5.2. Type (iv). We show in the next two propositions that there is no simple Hopf algebra in this type.

**Proposition 5.2.** Suppose \( H \) is of type \((1,3;2,12;3,1)\) as a coalgebra. Then \( H^{\text{coad}} \) is a commutative Hopf subalgebra of dimension 12. In particular, \( H \) is not simple.

*Proof.* Since \( |G(H)| \) is odd, then for all irreducible character \( \chi \) of degree 2 we have \( \chi \chi^* = 1 + \lambda \), where \( \lambda \) is irreducible of degree 3. It follows that the irreducible subcoalgebra of dimension 9 generates a commutative Hopf subalgebra \( A \) of dimension 12, such that \( \chi \chi^* \in A \), for all irreducible characters \( \chi \in H \). See \[13\], \[1\] Remark 3.4. Then we have \( H^{\text{coad}} = A \). The proposition follows since \( H^{\text{coad}} \) is a normal Hopf subalgebra of \( H \).

**Proposition 5.3.** Suppose \( H \) is of type \((1,3;3,1;4,3)\) as a coalgebra. Then \( H \) is not simple.

*Proof.* It is easily seen that the simple subcoalgebras of dimensions 1 and 9 form a Hopf subalgebra \( K \) isomorphic to \( kA_4 \). Suppose that \( H \) is simple.

If \( \pi : H \to B \) is a Hopf algebra quotient such that \( \dim B = 12 \), then \( \dim H^{\text{co}} B = 5 \) and thus \( K \cap H^{\text{co}} B = k1 \). Then \( \pi \) restricts to an isomorphism \( K \to B \). In particular, \( B^* \cong kA_4 \subseteq kG(\pi^*) \), and \( H^* \) is a biproduct \( H^* \cong R \# kA_4 \). Hence, by Proposition 3.6, \( H \) is not simple.

By Propositions 4.3, 4.13, 5.1 and 5.2, \( H^* \) must be of type \((1,2;2,2;5,2)\) as a coalgebra. In particular, there is a Hopf algebra quotient \( \pi : H \to B \), with \( B \cong kD_5 \).

Let \( R = H^{\text{co}} B \). We have \( \dim R = 6 \) and necessarily \( G(H) \subseteq R \). In view of the coalgebra type of \( H \), this implies that, as a left coideal of \( H \),

\[ R = kG(H) \oplus V, \]

where \( V \) is an irreducible left coideal of dimension 3.

Write \( G(H) = \{1,a,a^2\} \), and let \( \psi \in K \) be the irreducible character of degree 3. Let also \( \zeta \in H \) be an irreducible character of degree 4.

We have a decomposition \( \zeta \zeta^* = 1 + n\psi + m_1 \zeta + m_2 a \zeta + m_3 a^2 \zeta \). Taking degrees, this implies \( n \neq 0 \). This gives in turn \( \psi \zeta = \zeta + a \zeta + a^2 \zeta \), since \( \psi \), and thus also \( \psi \zeta \), are stable under left multiplication by \( G(H) \). Hence \( n = 1 \).
The decomposition (5.4) implies that \( m(1, \pi(a^i \zeta)) = 0 \), for all \( i = 0, 1, 2 \), in view of Remark 2.6. On the other hand, \( m(1, \pi(\psi)) = 1 \). Thus we get \( m(1, \pi(\zeta^*)) = 2 < \deg \zeta = 4 \). This contradicts Lemma 2.5 (i). The contradiction comes from the assumption that \( H \) is not simple, hence the proposition follows. □

5.3. Type (viii). By Lemma 4.4 we may assume that \( H \) has no quotient Hopf algebra of dimension 12. In view of Lemma 4.5 (i), \( H^* \) is not of type (1, 15; 3, 5). Therefore, Proposition 4.3 and the previous results imply that there is a quotient Hopf algebra \( \pi : H \to \overline{H} \), where \( \dim \overline{H} = 10 \) or 6.

**Proposition 5.5.** Suppose \( H \) has coalgebra type (1, 6; 2, 9; 3, 2). Then \( H \) is not simple.

**Proof.** If \( G[\chi] \neq 1 \), for all irreducible characters \( \chi \) of degree 2, then the sum of simple subcoalgebras of dimensions 1 and 2 is a Hopf subalgebra of \( H \) of dimension 42, which is a contradiction. Then \( G[\chi] = 1 \) for some of these characters. Then \( B[\chi] \simeq k^{A_4} \) of coalgebra type (1, 3; 3, 1).

On the other hand, \( G(H) \) contains a unique normal subgroup \( F \simeq \mathbb{Z}_3 \). We have \( F \subseteq B[\chi] \simeq k^{A_4} \), with \( \chi \) as before. Consider the Hopf subalgebra \( K = k[G(H), B[\chi]] \). Then \( kF \) is a normal Hopf subalgebra of \( K \).

Since \( \dim K > 24 \) and \( \dim K \) is divisible by 12, then \( K = H \) and \( kF \) is normal in \( H \). Therefore \( H \) is not simple, as claimed. □

It remains to consider the types (1, 6; 3, 6) and (1, 6; 3, 2; 6, 1). Let \( F \subseteq G(H) \) be the unique subgroup of order 3. Then \( F \) is the common stabilizer of all irreducible characters of degree 3.

**Proposition 5.6.** Suppose \( H \) is of type (1, 6; 3, 2; 6, 1). Then \( H \) is not simple.

**Proof.** Let \( \chi \neq \chi' \in H \) be the irreducible characters of degree 3. Then \( g\chi = \chi = \chi g \), for all \( g \in F \), and \( a\chi = \chi' = \chi a \), where \( a \in G(H) \) is any element of order 2. Then there are decompositions

\[
(5.7) \quad \chi\chi^* = \chi' (\chi')^* = \sum_{g \in F} g + \zeta,
\]

where \( \zeta \) is the irreducible character of degree 6. Otherwise, the product of irreducible characters of degree 3 would be a sum of irreducible characters of degree 1 and 3, implying that there is a Hopf subalgebra of coalgebra type (1, 6; 3, 2), which is impossible by [14]. In particular, \( k[C] = H \) for all simple subcoalgebras \( C \) of dimension 9.

Suppose first that there is a quotient \( \pi : H \to \overline{H} \), with \( \dim \overline{H} = 10 \). Then \( H^{co}\overline{H} = kF \oplus V \), where \( V \) is an irreducible character of degree 3. Let \( C \subseteq H \) be the simple subcoalgebra containing \( V \). By [10] Corollary 3.5.2, \( kF \) is normal in \( k[C] = H \). Then \( H \) is not simple.

Consider the case where there is a quotient \( \pi : H \to \overline{H} \), with \( \dim \overline{H} = 6 \). We first claim that \( \overline{H} \) must be cocommutative. To see this, we consider the
intersection $H^{co}\pi \cap kG(H)$ and the possible decompositions of $H^{co}\pi$ as a left coideal of $H$. We have $\dim H^{co}\pi = 10$. Counting dimensions we get that $\dim H^{co}\pi \cap kG(H) \neq 2$. Also, $\dim H^{co}\pi \cap kG(H)$ is not divisible by 3, by \cite{14}. Hence $H^{co}\pi \cap kG(H) = k1$, implying that the restriction of $\pi$ induces an isomorphism $\pi : kG(H) \to \overline{H}$. Thus $\overline{H}$ is cocommutative, as claimed.

Counting dimensions, we see that the multiplicity of a simple comodule of dimension 6 in $H^{co}\overline{H}$ can be 1 or 0. Let $\zeta \in H$ be the irreducible character of such a comodule. By Remark \ref{remark:3}, $m(1, \pi(\zeta)) = 1$ or 0. Combining this with the decomposition \eqref{equation:5.7}, we get that $m(1, \pi(\chi^*)) = 2$ or 1. This contradicts Lemma \ref{lemma:2.6}. The contradiction shows that $H$ is not simple and finishes the proof of the proposition. \hfill $\square$

**Proposition 5.8.** Suppose $H$ is of type $(1,6;3,6)$ as a coalgebra. Then $H$ is not simple.

*Proof.* Assume on the contrary that $H$ is simple. We first claim that there is no quotient $\pi : H \to \overline{H}$, with $\dim \overline{H} = 10$.

Suppose on the contrary that such a quotient exists. Then $H^{co}\overline{H} = kF \oplus V$, where $V$ is an irreducible character of degree 3. Let $C \subseteq H$ be the simple subcoalgebra containing $V$.

Since $H^{co}\overline{H}$ is a subalgebra of $H$ and $V$ is the only 3-dimensional irreducible left coideal contained in $H^{co}\overline{H}$, then $gV = V = Vg$, for all $g \in F$.

By \cite{10} Corollary 3.5.2, $kF$ is normal in $k[C] = H$. Since $\dim k[C] \geq 12$ and we are assuming that $H$ is simple, then $k[C]$ is of dimension 12 and moreover, the coalgebra type of $k[C]$ is $(1,3;3,1)$. In particular, $k[C]$ is commutative. Consider the Hopf subalgebra $K = k[G(H),C]$. Since $kG(H), k[C] \subseteq K$, then $K = H$, by dimension. On the other hand, the subgroup $kF$ is normal in $kG(H)$ and also in $k[C]$, hence $kF$ is normal in $H$. This implies the claim.

In view of the above, it follows from Proposition \ref{proposition:4.3} that there is a quotient $\pi : H \to \overline{H}$, with $\dim \overline{H} = 6$. We have $kG(H) \cap H^{co}\pi = k1$, by counting dimensions. Thus $\pi$ restricts to an isomorphism $kG(H) \to \overline{H}$, implying that $H$ is a biproduct $H = R\#kG(H)$.

As a left coideal of $H$, we must have a decomposition $R = k1 \oplus V_1 \oplus V_2 \oplus V_3$, where $V_i$ is an irreducible left coideal of dimension 3, $i = 1,2,3$.

Since $F$ stabilizes all irreducible characters of degree 3, then, for all $i = 1,2,3$, we have $(V_i \#1)(1\#g) = V_ig \simeq V_i$. In particular, for each $i$, $V_i \#kF \subseteq C_i$, where $C_i$ is the simple subcoalgebra containing $V_i$. Hence, by dimension, $V_i \#kF = C_i$ is a subcoalgebra of $H$. This implies that the subalgebra $K = R\#kF \subseteq H$ is also a subcoalgebra, hence a Hopf subalgebra. Then $H$ is not simple, since $|H : K| = 2$. \hfill $\square$

5.4. **Type (xii).** In this case $H$ is of type $(1,15;3,5)$ as a coalgebra. By Lemmas \ref{lemma:4.4} and \ref{lemma:4.5} we may assume that $H$ has no quotient Hopf algebra
of dimension 12 or 6. By Proposition 4.3 and the results in the previous subsections, we may assume that $H^*$ is of type (iii), (ix) or (xii).

**Proposition 5.9.** $H$ is not simple.

*Proof.* There is a Hopf algebra quotient $\pi : H^* \to k^{G(H)}$, and we have $\dim(H^*)^{\co \pi} = 4$. If 2 divides $|G(H^*)|$ and $\Gamma \subseteq G(H^*)$ is a subgroup of order 2, then $\Gamma \subseteq (H^*)^{\co \pi}$, by [14]. Hence we may assume that $(H^*)^{\co \pi} = k\Gamma \oplus V$, where $V$ is an irreducible left coideal of dimension 2. This discards the possibility (ix) for the coalgebra type of $H^*$.

Thus $H^*$ is of type (iii) in this case. By Proposition 4.3, the simple subcoalgebras of dimensions 1 and 4 form a Hopf subalgebra $K$ of $H$, isomorphic to $k^{D_5}$. In view of the decomposition of $(H^*)^{\co \pi}$, we have $(H^*)^{\co \pi} \subseteq K$. But this is impossible since $\dim(H^*)^{\co \pi} = 4$ does not divide $\dim K$.

Therefore we may assume that $|G(H^*)|$ is odd, and thus that $H^*$ is of type (xii) as a coalgebra. In this case $\pi|_{k^{G(H^*)}} : kG(H^*) \to k^{G(H)}$ is an isomorphism and $H$ is a biproduct $H \simeq R \# k\mathbb{Z}_{15}$, where $R$ is a Yetter-Drinfeld Hopf algebra over $\mathbb{Z}_{15}$ of dimension 4. By [10] Proposition 4.4.6, $H$ is not simple. □

5.5. **Type (ix).** Here, $H$ is of type $(1, 10; 5, 2)$ as a coalgebra. By Lemma 3.4, we may assume that $H$ has no quotient Hopf algebra of dimension 12. By Proposition 4.3 and previous results, $|G(H^*)| = 2$ or 10.

**Proposition 5.10.** $H$ is not simple.

*Proof.* By Proposition 4.3 we may further assume there is a Hopf algebra quotient $\pi : H \to B$, where $\dim B = 6$ or 10.

If $\dim B = 6$, then $\dim H^{\co B} = 10$ and, by [14], $H^{\co B} \cap kG(H) = kF$, where $F$ is the unique subgroup of order 5 of $G(H)$. Then $H^{\co B} = kF \oplus U$, where $U$ is an irreducible left coideal of dimension 5. Then, for all $g \in F$, $gV = V = Vg$, and by [10] Corollary 3.5.2, $kF$ is a normal Hopf subalgebra in $k[C]$, where $C$ is the simple subcoalgebra containing $U$. Since $F$ is also normal in $G(H)$, then $kF$ is normal in $k[G(H), C] = H$. Hence $H$ is not simple in this case.

Finally, suppose $\dim B = 10$. Then $H^{\co B} \cap G(H) = k1$, in view of [14] and the coalgebra type of $H$. Then $\pi$ induces an isomorphism $kG(H) \simeq B$. Thus $H$ is a biproduct $H = R \# kG(H)$, where $R$ is a 6-dimensional Yetter-Drinfeld (braided) Hopf algebra over $G(H)$. Moreover, $R \simeq H/HkG(H)^+$ as coalgebras. Since the stabilizer of a simple subcoalgebra of $H$ is cyclic of order 5, then $R$ is commutative, by Remark 3.3 and [10] Remark 3.2.7 and Corollary 3.3.2.

The action of $G(H)$ permutes the 5 nontrivial group-like elements of $R$. If $G(H)$ is cyclic, then it contains a nontrivial subgroup $F$ acting trivially on $R$ (since $S_5$ does not have elements of order 10). Then $kF$ would be a normal Hopf subalgebra of $H$ [10] Lemma 4.4.4]. Therefore we may assume that $G(H) \simeq D_5$. Now the result follows from Proposition 3.5. □
5.6. Type (xi). The following proposition says that the simple Hopf algebra \( A_0 \simeq (kA_4)^4 \) is indeed characterized by its coalgebra type. This has already been shown for the other two simple examples, \( A_1 \) and \( B \), in [1].

**Proposition 5.11.** Suppose \( H \) is of type \((1,12;4,3)\) as a coalgebra. If \( H \) is simple, then \( H \simeq A_0 \).

**Proof.** Suppose that \( H \) is simple. If \( G(H^*) = 1 \), then we know from [1] Proposition 6.10 that \( H^* \simeq A_1 \). Hence \( H \simeq A_0 \). So we may assume that there is a proper Hopf algebra quotient \( \pi : H \to B \).

By Proposition 4.3, Lemma 4.4, and the previous results, we may further assume that \( \dim B = 10 \) or 12. The first possibility implies that a subgroup \( F \simeq \mathbb{Z}_3 \) of \( G(H) \) must be contained in \( H^{coB} \). Since \( \dim H^{coB} = 6 \), and \( H \) has no irreducible left coideals of dimension 3, then \( H^{coB} \subseteq kG(H) \) is a normal Hopf subalgebra of \( H \).

Then \( \dim B = 12 \) and we see, after decomposing \( H^{coB} \) into a sum of irreducible left coideals, that \( \pi|_{kG(H)} : kG(H) \to B \) must be an isomorphism.

Then \( H \) is a biproduct \( R \# kG(H) \), where \( R \) is a Yetter-Drinfeld Hopf algebra of dimension 5 over \( G(H) \). By Proposition 3.6, we may assume that \( G(H) \) is not isomorphic to \( \mathbb{A}_4 \).

If \( R \) is cocommutative, the action of \( G(H) \) on \( R \) being by coalgebra automorphisms, must permute the set \( G(R) \setminus \{1\} \) of nontrivial group-likes in \( R \). Thus, it induces a group homomorphism \( \theta : G(H) \to S_4 \). The group algebra of the kernel of \( \theta \) is a normal Hopf subalgebra of \( H \) [10] Lemma 4.4.4]. Then \( G(H) \) acts faithfully on \( G(R) \setminus \{1\} \). Therefore, \( G(H) \) is isomorphic to \( \mathbb{A}_4 \) (the only subgroup of \( S_4 \) of order 12), against our assumption. Thus we may assume that \( R \) is not cocommutative.

As a left coideal of \( H \), \( R = k1 \oplus U \), where \( U \) is an irreducible left coideal of dimension 4. Let \( C \subseteq H \) be the simple subcoalgebra containing \( U \), and let \( \Gamma \subseteq G(H) \) be the stabilizer of \( C \): that is, \( Cg = C \), for all \( g \in \Gamma \). So that \( \Gamma \) is of order 4 and \( \Gamma \) is not cyclic, otherwise \( R \) would be cocommutative, in view of [10] Remark 3.2.7 and Corollary 3.3.2).

For all \( g \in \Gamma \), we have \( U \# g = Ug \simeq U \). Hence \( U \# g \subseteq C \), and therefore, by dimension, \( C = U \# k\Gamma \). By Remark 4.3, the coaction of \( kG(H) \) on \( R \) is given by \( \rho = (\epsilon_R \otimes id)\Delta : R \to kG(H) \otimes R \). Then \( \rho(U) \subseteq k\Gamma \otimes R \), and thus \( \rho(R) \subseteq k\Gamma \otimes R \). By [10] Lemma 4.3.1, \( K = R \# k\Gamma \) is a Hopf subalgebra of \( H \) of dimension 20. The coalgebra structure of \( H \) forces \( K \) to be commutative.

If \( \Gamma \) is normal in \( G(H) \), then \( K \) is normal in \( H \), since \( K \) and \( G(H) \) generate \( H \) as an algebra. Then we can assume that \( \Gamma \) is not normal, and therefore \( G(H) \) is a semidirect product \( G(H) = T \rtimes \Gamma \), where \( T \) is a subgroup of order 3, with respect to an action \( \Gamma \to \text{Aut} T \simeq \mathbb{Z}_2 \) by group automorphisms. This implies that \( \Gamma \) has an element \( g \neq 1 \) which is central in \( G(H) \). Then \( g \) is central in \( H \), because \( g \in K \), which is commutative, and \( K \) and \( G(H) \) generate \( H \) as an algebra. This finishes the proof of the proposition. \( \Box \)
5.7. **Type (ii).** Let $B \simeq kD_3 \subseteq H$ be the unique Hopf subalgebra of dimension 6.

**Lemma 5.12.** Suppose $H$ is simple. Then $H^*$ is of type (iii) as a coalgebra.

**Proof.** In view of previous results, the possible types for $H^*$ can be (ii), (iii) and (vi). Type (vi) is discarded by Lemma 4.4. Thus it is enough to discard the possibility of $H^*$ being also of type (ii). Suppose on the contrary that this occurs. Then there is a quotient Hopf algebra $\pi : H \to kD_3$, and we have $\dim H^{\co \pi} = 10$. Counting dimensions in the possible decompositions of $H^{\co \pi}$ as a left coideal of $H$, we see that $B^{\co \pi} = B \cap H^{\co \pi} = k1$, since $\dim B^{\co \pi}$ must divide $\dim B$.

Hence $\pi$ restricts to an isomorphism $B \to kD_3$, which is a contradiction, because $B$ is not cocommutative. Thus $H$ has no quotient Hopf algebra isomorphic to $kD_3$, and the lemma is proved. \qed

**Proposition 5.13.** $H$ is not simple.

**Proof.** Let $\chi \in H$ be an irreducible character of degree 3. Then there is a decomposition $\chi \chi^* = 1 + n\lambda + \mu$, where $\lambda \in B$ is the irreducible character of degree 2, and $\mu$ is a sum of irreducible characters of degrees 3 or 6. In particular, $n \neq 0$. Moreover, since for $1 \neq a \in G(H)$ we have $a\lambda = \lambda$, then $\lambda \chi = \chi + a\chi$. Therefore, since $a\chi \neq \chi$, we find $n = m(\lambda, \chi \chi^*) = m(\chi, \lambda \chi) = 1$. Hence,

$$
\chi \chi^* = 1 + \lambda + \mu.
$$

(5.14)

This implies that $B \subseteq k[C]$, where $C$ is the simple subcoalgebra containing $\chi$. In addition, $\dim k[C] \geq \dim B + \dim C = 15$. Since 6 = $\dim B$ divides $\dim k[C]$, then $\dim k[C] = 30$ or $k[C] = H$. If $\dim k[C] = 30$, then $k[C]$ is normal in $H$ and we are done. Thus we can consider the case where $k[C] = H$, for all simple subcoalgebras $C \subseteq H$ of dimension 9.

By Lemma 5.12, we may assume that there is a quotient $\pi : H \to kD_5$. The left coideal subalgebra $H^{\co \pi}$ has dimension 6. Hence, unless $H^{\co \pi} = B$, in which case we are done, we may assume that $H^{\co \pi} = k1 \oplus U \oplus V$ as a left coideal of $H$, where $U$ is an irreducible coideal of dimension 2 and $V$ is an irreducible coideal of dimension 3. Let $\chi \in H$ be the irreducible character corresponding to $V$, and $C$ the simple subcoalgebra containing $\chi$. Since $V$ is the only 3-dimensional irreducible left coideal contained in the self-dual left coideal $H^{\co \pi}$, we have $\chi^* = \chi$.

By Remark 2.6 $m(1, \pi(\chi)) = 1$. Also, $m(1, \pi(\chi^2)) = m(1, \pi(\chi \chi^*)) = 3$, by Lemma 2.3 (i).

Write $\pi(\chi) = 1 + x + y$, where $x, y \in D_5$, $x \neq 1 \neq y$. Then $\pi(k[C]) \subseteq (x, y)$, and therefore $(x, y) = D_5$, because $k[C] = H$. In particular, $x \neq y, y^{-1}$. Furthermore, $m(1, \pi(\chi^2)) \geq 3$, and $\pi(\chi^2) = (1 + x + y)^2$, hence $x^2 = y^2 = 1$.

Let $\pi(\lambda) = 1 + t$, with $1 \neq t \in D_5$. Since $a\lambda = \lambda$, then $\pi(a) = t$ and $t^2 = 1$. In view of the decomposition (5.14), we have $m(t, \pi(\chi^2)) = m(t, \pi(\chi \chi^*)) > 0$. Hence, $t = x$, $y$ or $xy$. In the first two cases, we find that $m(1, \pi(a\chi)) > 0$, otherwise...
which is impossible since \(a\chi \neq \chi\), and \(\chi\) is the only irreducible character of degree 3 appearing in \(H^{\text{co}\pi}\). Therefore \(t = xy\). But, since \(x\) and \(y\) are reflections in \(D_5\), then the order of \(xy\) divides 5. Thus \(t = 1\). This implies that \(a \in H^{\text{co}\pi}\), against our assumption. Hence we conclude that \(H\) is not simple.

5.8. **Type (iii).** Let \(kD_5 \simeq B \subseteq H\) be its (unique) Hopf subalgebra of dimension 10, which has coalgebra type \((1; 2; 2)\).

**Proposition 5.15.** \(H\) is not simple.

**Proof.** In view of the previous results, if \(H\) is simple, then \(H^*\) is of type (iii). But we shall show that \(H\) admits no Hopf algebra quotient \(\pi : H \rightarrow \overline{H}\), with \(\overline{H} \simeq kD_5\). This will imply the proposition. Suppose on the contrary that such quotient exists. We have \(\dim H^{\text{co}\overline{H}} = 6\). The coalgebra structure of \(H\) forces \(H^{\text{co}\overline{H}} \subseteq B\) or \(H^{\text{co}\overline{H}} \cap B = k1\). However, since 6 does not divide \(\dim B\), then \(H^{\text{co}\overline{H}} \nsubseteq B\). Also, \(H^{\text{co}\overline{H}} \cap B \neq k1\), because otherwise \(\pi\) would induce an isomorphism \(kD_5 \rightarrow kD_5\), which is impossible. \(\square\)

5.9. **Type (vi).** Suppose that \(H\) is a semisimple Hopf algebra of dimension 60 which is simple as a Hopf algebra. In view of the results in Section 4, unless \(H\) is isomorphic to \(A_0\), \(A_1\) or \(B\), then \(H\) and \(H^*\) are both of type \((1; 4; 2; 2; 2; 4, 3)\) as coalgebras.

We shall show in this subsection, c. f. Proposition 5.18, that this cannot occur, that is, such a semisimple Hopf algebra cannot be simple. This will conclude the proof of Theorem 1.4.

Let \(B \subseteq H\) be the (unique) Hopf subalgebra of dimension 12, which has coalgebra type \((1; 4; 2; 2)\).

**Lemma 5.16.** Suppose \(H\) contains a Hopf subalgebra \(K\) of dimension 20. Then \(G(H) \cap Z(H) \neq 1\).

**Proof.** In view of the coalgebra structure of \(H\), \(K\) must be commutative and \(G(H) \subseteq K\). Let \(1 \neq g \in G(H)\) be a central group-like element of \(B\). Such central group-like exists in view of the classification of semisimple Hopf algebras of dimension 20 \([11]\). Since \(G(H) \subseteq K\), then \(g\) is central in \(K\) and therefore \(g\) is central in \(k[B, K]\). On the other hand, \(k[B, K] = H\), by dimension. Hence \(H \cap Z(H) \neq 1\), as claimed. \(\square\)

**Lemma 5.17.** Assume that \(H\) is simple. Then \(H\) is a biproduct \(H \simeq R\#B\), where \(R\) is a Yetter-Drinfeld Hopf algebra over \(B\) of dimension 5.

**Proof.** Since \(H^*\) is also of type \((1; 4; 2; 2; 4, 3)\), then there is a Hopf algebra quotient \(q : H \rightarrow B'\), where \(B'\) is a Hopf algebra of dimension 12, such that \((B')^* \subseteq H^*\) is of coalgebra type \((1; 4; 2; 2)\). Since \(\dim H^{\text{co}q} = 5\), then \(\dim H^{\text{co}q} \cap B = k1\). Thus \(q|B : B \rightarrow B'\) is an isomorphism, and \(H\) is a biproduct, as claimed. \(\square\)

**Proposition 5.18.** \(H\) is not simple.
Proof. The proof will follow from Lemmas 5.19, 5.20 and 5.21 below. □

Lemma 5.19. Suppose $B$ is commutative. Then $H$ is not simple.

Proof. By Lemma 5.17, $H^*$ and $H$ have the same coalgebra type and $H \simeq R \# B$ is a biproduct, where $R$ is a Yetter-Drinfeld Hopf algebra over $B$ of dimension 5. If $B$ where commutative, then $B^* \subseteq H^*$ would be a cocommutative Hopf subalgebra of dimension 12, which is not possible. □

Combining Lemma 5.19 with the classification of semisimple Hopf algebras of dimension 12 [4], we may assume that $B \simeq A_0$ or $A_1$, where $A_0$ and $A_1$ are the nontrivial semisimple Hopf algebras of dimension 12 such that $G(A_0) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and $G(A_1) \simeq \mathbb{Z}_4$. See [10, 5.2].

Lemma 5.20. Suppose $B \simeq A_0$. Then $H$ is not simple.

Proof. By [10, Proposition 5.2.1], $B \simeq A_0$ is a twisting of the group $G = \mathbb{Z}_3 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ corresponding to the action by group automorphisms of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on $\mathbb{Z}_3$ defined by $s.a = a^2$ and $t.a = a^2$, where $Z_3 = \langle a \mid a^3 = 1 \rangle$, $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle s, t \mid s^2 = t^2 = 1 \rangle$. Therefore, there exists an invertible twist $J \in B \otimes B$ such that $B^J \simeq kG$.

Consider the twisting $H^J$ of $H$. Since $B^J \subseteq H^J$ is a Hopf subalgebra, then $G$ is isomorphic to a subgroup of $G(H^J)$ and, in particular, $|G(H^J)|$ is divisible by 12. Moreover, we may assume that $|G(H^J)| = 12$, and therefore $G(H^J) \simeq G$. Otherwise $H^J$ would be cocommutative, hence a group algebra, implying that $H$ is not simple by [4, Theorem 4.10].

By Proposition 4.13, $H^J$ is of type $(1, 12; 2, 12)$ or $(1, 12; 4, 3)$ as a coalgebra. Since $G$ is not isomorphic to $A_4$, then $H^J$ cannot be isomorphic to the simple Hopf algebra $A_0$. Therefore, by Propositions 4.13 and 5.11, $H^J$ is not simple.

Consider first the possibility $(1, 12; 2, 12)$ for the coalgebra type of $H^J$. Let $K \subseteq H^J$ be a proper normal Hopf subalgebra. Then $K \subseteq kG(H^J)$. Otherwise, dim $K = 20$ and dim $H^J / H^J K^+ = 3$. Hence $(H^J)^*$ contains a group-like element of order 3, which is impossible since $H^* \simeq H$ as coalgebras.

Thus $K = k\Gamma$ for a normal subgroup $\Gamma$ of $G(H^J) = G$, and $|\Gamma|$ is either 12, 6, 3, or 2. If $|\Gamma| = 12$, then $k\Gamma = kG$ and thus $J^{-1} \in k\Gamma \otimes k\Gamma$. Hence $B \simeq A_0 \simeq (kG)^{J^{-1}}$ is a normal Hopf subalgebra of $H$ [10, Lemma 5.4.1].

If $|\Gamma| = 6$, there is a (not necessarily normal) quotient Hopf algebra $\pi : H \rightarrow \overline{H}$ with dim $\overline{H} = 10$, implying that $H^*$ contains a Hopf subalgebra of dimension 10. This is impossible because of the coalgebra type of $H^*$. Similarly, if $|\Gamma| = 3$, then there is a Hopf subalgebra $A \subseteq H^*$ with dim $A = 20$. By Lemma 5.16, $H^*$ and thus also $H$, are not simple.

Consider next the type $(1, 12; 2, 12)$. By [1 Corollary 6.3], either $H^J$, and thus also $H$, has a nontrivial central group-like element, or there is a cocentral exact sequence $k \rightarrow K \rightarrow H^J \rightarrow kU(\mathcal{C})$, where $U(\mathcal{C})$ is the universal grading group of the category $H^J$-comod of finite dimensional $H^J$-comodules, and $K = (H^J)_{\text{coad}} \subseteq H^J$. 


Since \((H^J)^* = H^*\) as coalgebras, then \(\widehat{U(C)} = G(kU(C))\) is of order 2 or 4. Therefore \((H^J)^*\) has a central group-like element of order 2. Hence, there is a normal Hopf subalgebra \(L \subseteq H^J\) with \(\dim L = 30\). By \cite{10} Theorem 2, \(L\) is necessarily commutative. Since \(H^J\) is a bicrossed product \(H^J \simeq L' \# k\mathbb{Z}_2\), in view of the description of the irreducible modules in the proof of \cite{9} Theorem 2.1, \((H^J)^*\) must be of type \((1, n; 2, m)\) as a coalgebra, which is a contradiction. This discards the type \((1, 12; 2, 12)\) for the coalgebra structure of \(H^J\) and finishes the proof of the lemma. □

**Lemma 5.21.** Suppose \(B \simeq A_1\). Then \(H\) is not simple.

**Proof.** We may assume that \(H\) is a biproduct \(H = R \# A_1\). As a left coideal of \(H\), we must have a decomposition \(R = k1 \oplus V\), where \(V\) is an irreducible left coideal of dimension 4. Let \(\zeta \in H\) be the character of \(V\). Then \(\zeta = \zeta^*\).

Suppose that \(G(H)\) stabilizes \(\zeta\). Then \((V \# 1)(1 \# g) = Vg \simeq V\), for all \(g \in G(H)\). In particular, \(V \# kG(H) \subseteq C\), where \(C\) is the simple subcoalgebra containing \(V\). Hence, by dimension, \(V \# kG(H) = C\) is a subalgebra of \(H\). This implies that the subalgebra \(K = R \# kG(H) \subseteq H\) is also a subcoalgebra, hence a Hopf subalgebra of \(H\). By Lemma \[5.16\] \(H\) is not simple in this case.

Therefore we may assume that \(\zeta\) is not stable under left multiplication by \(G(H)\). Note that \(H\) does contain a unique irreducible character \(\psi\) of degree 4 which is stable under left multiplication by \(G(H)\). Let \(G(H)\zeta = \{\zeta, \zeta^*\}\).

**Claim 5.22.** We have

\[
(5.23) \quad g\psi g^{-1} = \psi, \text{ for all } g \in G(H),
\]

\[
(5.24) \quad \psi \psi^* = \sum_{g \in G(H)} g + \zeta + \zeta^* + \psi.
\]

**Proof of the claim.** We have \(|G[g\psi g^{-1}]| = 4\), for all \(g \in G(H)\). Then \(g\psi g^{-1} = \psi\), for all \(g \in G(H)\), since this is the only stable irreducible character of degree 4. This proves \(5.23\).

On the other hand, we have \(\psi \psi^* = \sum_{g \in G(H)} g + n\lambda + n'\lambda' + m\zeta + m'\zeta' + r\psi\), where \(\lambda \neq \lambda'\) are the irreducible characters of degree 2 and \(n, n', m, m', r\) are nonnegative integers. Since \(\psi\) is stable and \(G(H)\lambda = \{\lambda, \lambda'\}\), we find that \(n = n'\). Suppose \(n \neq 0\). Then \(m(\psi, \lambda\psi) = m(\lambda, \psi\psi^*) = n \neq 0\). Hence \(\lambda\psi = \psi + \rho\), where \(\rho\) is irreducible of degree 4; indeed, \(\lambda\psi\) cannot have irreducible summands of degrees 1 or 2, since otherwise \(\psi \in B\), which is impossible; c. f. Lemma \[2.3\].

It follows from \(5.23\) that \(\psi\) is stable also under right multiplication by \(g \in G(H)\), and it is, moreover, the only irreducible character with this property. Then \(\rho\) is stable under right multiplication by \(G(H)\), implying that \(\rho = \psi\). Then \(n = 2\) and we have a relation \(\lambda\psi = 2\psi\). Let \(A \subseteq B\) be the smallest Hopf subalgebra of \(H\) containing \(\lambda\); then \(\dim A = 6\) or 12. By the above, \(AC_\psi = C_\psi\), where \(C_\psi\) is the simple subcoalgebra containing \(\psi\). This contradicts \[13\], since \(\dim C_\psi = 16\) is not divisible by \(\dim A\).
Therefore $n = n' = 0$, and $\psi \psi^* = \sum_{g \in G(H)} g + m\zeta + m'\zeta' + r\psi$. Since $\psi$ is stable, we have $m = m'$. Moreover, $m + m' \neq 0$, because otherwise $\psi$ and $G(H)$ would span a standard subring of the character ring of $H$ corresponding to a Hopf subalgebra of dimension 20, implying that $H$ is not simple, by Lemma 5.16. Taking degrees we find $m = m' = 1$, hence also $r = 1$. This proves (5.24).

Consider the projection $\gamma : H \to k[G(H')]$. Then $\dim H^{\co \gamma} = 15$, implying that $H^{\co \gamma} \cap G(H) = 1$ and $H^{\co \gamma} \cap A_1 = k1 \oplus U$, where $U$ is an irreducible left coideal of dimension 2. In particular, $H$ is a biproduct $\tilde{R} \# kG(H)$, with $\tilde{R} = H^{\co \gamma}$, a braided Hopf algebra over $G(H)$.

On the other hand, $R \subseteq \tilde{R}$. Hence, as a left coideal of $H$, $\tilde{R} = k1 \oplus U \oplus V \oplus V_1 \oplus V_2$, where $V_1$ are irreducible left coideals of dimension 4.

Let $C$ be the simple subcoalgebra containing $V$. If $V_1$ and $V_2$ are both contained in $CG(H)$, then $\tilde{R} \subseteq A_1 \oplus CG(H)$, and thus $H = \tilde{R} \# kG(H) \subseteq A_1 \oplus CG(H)$. This is not possible because $\dim A_1 \oplus CG(H) < \dim H$. Hence we may assume that $V_1$ is the only, up to isomorphisms, stable irreducible left coideal. Therefore $V_1 \# kG(H) \subseteq C_1$, where $C_1$ is the simple subcoalgebra of $H$ containing $V_1$. By dimension, we have $V_1 \# kG(H) = C_1$. This implies that $V_1 = (\id \otimes \epsilon)(C)$ is a subcoalgebra of $R$.

**Claim 5.25.** The multiplicity of $V_1$ as a direct summand of $\tilde{R}$ equals 1.

**Proof of the claim.** We know that $V_1$ appears in $\tilde{R}$ with positive multiplicity. Since $V_1$ is not isomorphic to $V$, it will be enough to show that the multiplicity of $V_1$ in $\tilde{R}$ is not equal to 2. Suppose on the contrary that this is the case. Then $H = \tilde{R} \# kG(H) \subseteq A_1 \oplus V \# kG(H) \oplus C_1$, where $C_1$ is the simple subcoalgebra containing $V_1$. Counting dimensions, we see that this is not possible. Hence the multiplicity of $V_1$ is 1, as claimed.

It follows from Claim 5.25 and (5.23) that $V_1 \subseteq \tilde{R}$ is also a submodule under the (adjoint) action of $G(H)$. Let $R_0 = k[V_1]$ be the subalgebra of $\tilde{R}$ generated by $V_1$. Since $V_1$ is also a subcoalgebra of $\tilde{R}$, then $R_0 \# kG(H) \subseteq \tilde{R} \# kG(H)$ is a Hopf subalgebra. If $\dim R_0 \# kG(H) = 20$, then we are done by Lemma 5.16. Otherwise $R_0 \# kG(H) = H$. By [12, Corollary 1.3.2], since $G(H)$ is cyclic, $\tilde{R}$ is a comultiplicative coalgebra.

We claim that the (adjoint) action of $G(H)$ permutates the set $G(V_1)$ transitively. Indeed, if this were not the case, the centralizer $G(H)_x$ of $x$ in $G(H)$ would be of order 2, for all $x \in G(V_1)$, because $|G(V_1)| = 4$. Then the only subgroup of order 2 of $G(H)$ would centralize $V_1$ and a fortiori all of $\tilde{R} = k[V_1]$. Then this subgroup would be central in $H$, contradicting the simplicity of $H$. This proves the claim.
In particular, $\dim V_1^{G(H)} = 1$. We may now apply [12, Proposition 1.4.3] to conclude that $\tilde{R}$ contains a nontrivial group-like element of $H$, and arriving thus to a contradiction. This shows that $H$ is not simple and finishes the proof of the lemma.

\[\square\]

References

[1] J. Bichon and S. Natale, Hopf algebra deformations of binary polyhedral groups, preprint arXiv:0907.1879v1 [math.QA].
[2] P. Etingof, D. Nikshych and V. Ostrik, On fusion categories, Ann. Math. (2) 162 (2005), 581–642.
[3] P. Etingof, D. Nikshych and V. Ostrik, Weakly group-theoretical and solvable fusion categories, preprint arXiv:0809.3031v1[math.QA].
[4] N. Fukuda, Semisimple Hopf algebras of dimension 12, Tsukuba J. Math. 21 (1997), 43–54.
[5] C. Galindo and S. Natale, Simple Hopf algebras and deformations of finite groups, Math. Res. Lett. 14 (2007), 943–954.
[6] S. Gelaki and D. Nikshych, Nilpotent fusion categories, Adv. Math. 217 (2008), 1053–1071.
[7] S. Majid, Crossed products by braided groups and bosonization, J. Algebra 163 (1994), 165–190.
[8] S. Montgomery, Hopf Algebras and their Actions on Rings, CMBS Reg. Conf. Ser. in Math. 82, Amer. Math. Soc., 1993.
[9] S. Montgomery and S. Whiteman, Irreducible representations of crossed products, J. Pure Appl. Algebra 129 (1998), 315–326.
[10] S. Natale, Semisolvability of semisimple Hopf algebras of dimension 60, Memoirs Amer. Math. Soc. 186, 123 pp. (2007).
[11] S. Natale, On semisimple Hopf algebras of dimension $pq^2$, Algebr. Represent. Theory 7 (2004), 173–188.
[12] S. Natale, On semisimple Hopf algebras of dimension $pq^2$, II, Algebr. Represent. Theory 4 (2001), 277–291.
[13] W. Nichols and M. Richmond, The Grothendieck group of a Hopf algebra, J. Pure Appl. Algebra 106 (1996), 297–306.
[14] W. Nichols and M. Zoeller, A Hopf algebra freeness theorem, Amer. J. Math. 111 (1989), 381–385.
[15] D. Nikshych, $K_0$-rings and twisting of finite-dimensional semisimple Hopf algebras, Commun. Algebra 26 (1998), 321–342.
[16] D. Radford, The structure of Hopf algebras with a projection, J. Algebra 92 (1985), 322–347.
[17] H.-J. Schneider, Representation theory of Hopf Galois extensions, Israel J. Math. 72 (1990), 196–231.
[18] S. Skryabin, Projectivity and freeness over comodule algebras Trans. Am. Math. Soc. 359 (2007), 2597–2623.

Facultad de Matemática, Astronomía y Física. Universidad Nacional de Córdoba. CIEM – CONICET. (5000) Ciudad Universitaria. Córdoba, Argentina

E-mail address: natale@mate.uncor.edu
URL: http://www.mate.uncor.edu/natale