Nontrivial Twisted States in Nonlocally Coupled Stuart-Landau Oscillators

Seunghae Lee† and Katharina Krischer‡
Physik-Department, Technische Universität München,
James-Franck-Straße 1, 85748 Garching, Germany
(Dated: September 20, 2022)

A twisted state is an important yet simple form of collective dynamics in an oscillatory medium. Here, we describe a nontrivial type of twisted state in a system of nonlocally coupled Stuart-Landau oscillators. The nontrivial twisted state (NTS) is a coherent traveling wave characterized by inhomogeneous profiles of amplitudes and phase gradients, which can be assigned a winding number. To further investigate its properties, several methods are employed. We perform a linear stability analysis in the continuum limit and compare the results with Lyapunov exponents obtained in a finite-size system. The determination of covariant Lyapunov vectors allows us to identify collective modes. Furthermore, we show that the NTS is robust to small heterogeneities in the natural frequencies and present a bifurcation analysis revealing that NTSs are born/annihilated in a saddle-node bifurcation and change their stability in Hopf bifurcations. We observe stable NTSs with winding number 1 and 2. The latter can lose stability in a supercritical Hopf bifurcation, leading to a modulated 2-NTS.

I. INTRODUCTION

The collective dynamics of an ensemble of coupled oscillators is key to the functioning of many systems in practically all scientific fields [1, 2]. Accordingly, the dynamics of ensembles of oscillators with different couplings has been studied intensively during the last decades. One coupling topology that has proven important for discovering new synchronization patterns and revealing the mechanisms that give rise to them is nonlocal coupling in a ring geometry, the most prominent example being a chimera state [3, 4]. When the coupling between the oscillators is weak, the dynamics can be captured by considering only the evolution equations of the phases of the oscillators [5, 7]. For somewhat stronger coupling, the amplitudes of a part of the oscillators along the ring might exhibit variations decisive for the dynamics, as found in amplitude-mediated chimera states [8, 11].

Another prominent and compared to the chimera state simpler collective dynamics found in a ring of nonlocally coupled oscillators is a so-called twisted state [12, 10]. In a ‘traditional’ twisted state, the phase difference between adjacent oscillators is always the same such that the phase winds around the ring an integer multiple of $2\pi$ whereas the amplitude of all the oscillators attains the same constant value. A twisted state has thus been seen as a typical phenomenon that is fully captured by a phase-reduced model. The phase profile evolves according to $\dot{\phi}(x, t) = 2\pi x + \Omega t$, where $\Omega$ is a collective frequency and $L$ is the length of the medium. Correspondingly, the phase gradient is everywhere given by $\partial_x \phi(x, t) = \frac{2\pi}{L} = \text{const.}$ with $q \in \mathbb{Z}$ defining a winding number, while the trivial amplitude dynamics obeys $r(x, t) = r_0 \in \mathbb{R}_{>0}$ for all $x \in [0, L]$.

In this paper, we show that in a ring of nonlocally coupled oscillators another type of a twisted state might form. This state is characterized by a non-constant gradient of the phase profile and an inhomogeneous amplitude profile which travels along the ring with a fixed shape and a constant speed. The phase still advances by a multiple of $2\pi$ when going once around the ring so that the solution can be characterized by a winding number and the state be considered a twisted state. Due to the spatio-temporal variations of the amplitude the state does not exist in the classical phase-reduced model but its description requires a priori planar oscillators. In order to contrast this novel type of twisted state from the so far known constant phase-gradient and constant amplitude twisted state, we coin a twisted state with non-uniform amplitude and phase gradient profiles a nontrivial twisted state (NTS) and a twisted state with uniform profiles of amplitude and phase gradient a trivial twisted state (TTS). Both these states are discussed in the following sections with a system of nonlocally coupled identical Stuart-Landau oscillators in a 1-d ring, with an emphasis on the dynamical and spectral properties of the NTS.

In Sec. [11] we first discuss the dynamical properties of NTSs in the original space and time coordinates. Then we perform a linear stability analysis in a moving and co-rotating reference frame where both amplitude and phase profiles become stationary. Finally, we compare the stability results with those of the TTS. In Sec. [11] we address the stability of finite-size ensembles and study how the spectral properties of the finite-size state converge to those of the continuum limit as the system size increases. The stability of the finite-size ensemble is obtained from the numerical determination of the Lyapunov exponents (LEs). In addition, covariant Lyapunov vectors (CLVs) are considered to confirm the existence of collective Lyapunov modes [17–22]. Next, in Sec. [11] we demonstrate that the NTS is robust with respect to small heterogeneities in the natural frequencies of the Stuart-Landau oscillators [13, 23]. Finally, a bifurcation analysis

\* seunghae.lee@tum.de
krischer@tum.de
is performed in Sec. [V] following the procedure described in [24]. It reveals that the NTS is stable in a large parameter range. We summarize the results in Sec. [VI].

II. NONTRIVIAL TWISTED STATE

A. Governing Equation and Observable Dynamics

We consider nonlocally coupled, identical Stuart-Landau oscillators along a 1-d ring of length \( L \). The oscillators are described by complex-valued dynamical agents \( W(x, t) = r(x, t)e^{i\phi(x, t)} \in \mathbb{C} \) where \( x \in [0, L] \). The oscillator field is governed by

\[
\frac{\partial}{\partial t} W(x, t) = \mathcal{F}(W(x, t)) + \varepsilon e^{-i\alpha H(x, t)} H(x, t)
\]

\[
= (1 + i\omega) W(x, t) - |W(x, t)|^2 W(x, t)
\]

\[
+ \varepsilon e^{-i\alpha H(x, t)} \int_0^L G(x - x')W(x', t) dx' \quad (1)
\]

where periodic boundary conditions are imposed. The uncoupled local dynamics is given by a Stuart-Landau oscillator \( \mathcal{F}(W) = (1 + i\omega) W - |W|^2 W \) and the nonlinear phase-lag function is assumed to be \( \alpha H(x, t) = \alpha_0 + \alpha_1 |H(x, t)|^2 \) with real parameters \( \alpha_0, \alpha_1 \in \mathbb{R} \) [25-27]. The coupling strength \( \varepsilon \) is a real parameter and the frequency of the identical oscillators is set to \( \omega = 0 \).

The forcing field is defined as an integral convolution operator, i.e.,

\[
H(x, t) = (GW)(x, t) := \int_0^L G(x - x') W(x', t) dx'. \quad (2)
\]

The nonlocal coupling kernel given by

\[
G(y) = \frac{\kappa}{2\sinh(\kappa L/2)} \cosh\left(\kappa(|y| - L/2)\right)
\]

for \(|y| \leq L/2\) so that both the normalization condition \( \int_{-L/2}^{L/2} G(y) dy = 1 \) and the Green’s function of the inhomogeneous Helmholtz equation [25, 28]

\[
(\partial_x^2 - \kappa^2) H(x, t) = -\kappa^2 W(x, t)
\]

are satisfied with the periodic boundary conditions: \( H(0, t) = H(L, t) \) and \( \partial_x H(0, t) = \partial_x H(L, t) \). Note that \( \kappa \) is a real parameter and \( \kappa^{-1} \) determines the coupling range and has the dimension of a length. Thus, \( \kappa^{-1} \) also characterizes the length of the medium [23, 25, 28]. In the limit of \( \kappa L \to \infty \), the coupling kernel becomes \( G(x) = \kappa e^{-\kappa |x|}/2 \), as used in [8]. In the following (up to Sec. [V]), we use the following parameter values: \( \varepsilon = 1 \), \( \kappa = 4.874 \), \( L = 1 \), \( \alpha_0 = -0.4\pi \) and \( \alpha_1 = -(\pi/2 + \alpha_0)/0.36 \). Note that \( \varepsilon \) is no longer small such that the amplitude variables may follow nontrivial dynamics.

For the chosen parameter values, the microscopic dynamics of the finite-size approximation with \( N = 200 \) oscillators may exhibit an NTS along the ring, as depicted in Fig. [1]. In (a,b) the amplitude dynamics \( r_j(t) \) and in (c,d) the phase dynamics \( \phi_j(t) \) is shown where \( r_j(t)e^{i\phi_j(t)} = W_j(t) \) at \( x_j = \frac{j-1}{N-1} \in [0, 1] \) such that \( \phi_j(t) \) and \( r_j(t) \) are governed by

\[
\frac{d\phi_j}{dt} = \omega + \frac{\varepsilon}{r_j} \text{Im} \left[ H_j(t) e^{-i\phi_j} e^{-i\alpha(H_j(t))} \right] \quad (4)
\]

and

\[
\frac{dr_j}{dt} = r_j - r_j^3 + \varepsilon \text{Re} \left[ H_j(t) e^{-i\phi_j} e^{-i\alpha(H_j(t))} \right] \quad (5)
\]

As apparent from the two amplitude snapshots in Fig. [1(a)], the amplitudes form a smooth time-dependent curve as a function of \( x \). The spatio-temporal evolution of the amplitude profiles shown in Fig. [1(b)] evidence that the profiles travel along the ring with a fixed shape and a constant speed. The amplitude dynamics thus constitutes a traveling wave solution. As apparent from the two snapshots of the phase profiles depicted in Fig. [1(c)], the phase appears to be smooth along \( x \) and exhibits large variations in the region where the amplitude variations are large, and shallow variations where the amplitude varies only slightly. If we define the phase difference modulo \( 2\pi \) in the interval \((-\pi, \pi)\) as \( \Delta_{i,j} := \phi_i - \phi_j \) with \( \phi_{N+1} \equiv \phi_1 \), then we can assign a winding number to the observed state according to \( q = \frac{1}{2\pi} \sum_{j=1}^{N} \Delta_{j+1,j} \in \mathbb{Z} \). In the example shown in Fig. [1], \( q = -1 \) (but note that depending on the initial condition \( q \) can also be +1). Besides, \( \Delta_{j+1,j} \neq \text{const} \). The dynamics thus constitutes an NTS as defined in the introduction. Furthermore, the phase profile is uniformly rotating with a collective frequency \( \Omega \) and, like the amplitude profile, it travels to the left with the lateral speed \( c \) (Fig. [1(c,d)]). In an appropriately rotating frame, an NTS thus constitutes a traveling...
Fig. 2. (a) Red curve: Snapshot of the amplitude profile of an NTS on the ring, i.e., $r(\frac{2\pi}{N}x)\cos(\frac{2\pi}{N}x)$ vs. $r(\frac{2\pi}{N}x)\sin(\frac{2\pi}{N}x)$ for $x \in [0, L]$ and $N = 200$; black line: trajectory of the first oscillator in the complex plane: $\text{Re}[W_1(t)]$ vs. $\text{Im}[W_1(t)]$; (b) Instantaneous phase velocity as a function of time for $N = 6$. (c) Time evolution of $\text{Re}[W(t)]$ for $N = 200$. The black line highlights the time series of one of the oscillators. (d) Poincaré map of $W_{10}$ and $W_{90}$ in the complex plane where the Poincaré section is defined by $\phi_1(t) \equiv 0 \mod 2\pi$. (e) Modulus of the global Kuramoto order parameter as a function of time for different system sizes $N$. All numerical values shown for $t \geq 10^3$. (f) Period of the modulus of the global Kuramoto order parameter as a function of the system size $N$. The numerically obtained value (red) coincides with $T/N$ (blue).

In order to validate that the NTS is in fact a coherent traveling wave, just as a TTS does, which is a coexisting solution at the same parameter values, as demonstrated below.

In order to validate that the NTS is in fact a coherent traveling wave, as a TTS is, we determined the Kuramoto local order parameter $z(x, t)$ defined as [29, 30]

$$z(x, t) = \lim_{N \to \infty} \frac{1}{B_0^N(x)} \sum_{j \in B_0^N(x)} e^{i\phi_j(t)} \quad (6)$$

with $B_0^N(x) = \{j : 1 \leq j \leq N, |x - x_j| < \delta\}$ for small enough $0 < \delta \ll 1$. Equation (6) directly shows how to calculate the local order parameter numerically from the finite-size microscopic dynamics. In the continuum limit, Equation (6) is equivalent to the more intuitive version defined for a spatially extended 1-dimensional system, which reads

$$z(x, t) = \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} e^{i\phi(x', t)} dx' \quad (7)$$

for $0 < \delta \ll 1$ [12, 25]. Hence, the local order parameter provides a coarse-grained macroscopic observable that is continuous both in $x$ and $t$, even though $\phi(x, t)$ in general is not, and characterizes a local degree of coherence in a small neighborhood around $x$ [5, 7, 31]. In fact, we obtain $|z(x, t)| = 1$ for all $x \in [0, 1]$, ensuring that the NTS is a coherent traveling wave.

In Fig. 2(a), the trajectory of one of the oscillators (black line) is depicted together with a snapshot of the amplitude profile (red curve) in the complex plane. The trajectories encircle the origin, but exhibit a backward motion in phase when the amplitude of the oscillation exhibits a pronounced deformation from a circular structure (akin to the apparent retrograde motion of a planet from the earth’s viewpoint) (cf. also Fig. 1(d)). The reversal of the direction of phase change reflects the negative values the instantaneous phase velocity attains when the oscillation amplitude goes through the hump (Fig. 2(b)). The time evolution of $\text{Re}(W)$ of the oscillators is shown in Fig. 2(c). We notice that each individual oscillator exhibits some apparently irregular oscillation (illustrated by the black highlighted curve) while the motion of the entire ensemble displays a periodically oscillating envelope. In Fig. 2(d), the trajectories of two representative oscillators ($W_{10}$ and $W_{90}$) are depicted in a Poincaré section defined by $\phi_1(t) \equiv 0 \mod 2\pi$. Clearly, all points of the trajectory of each oscillator lie on two close curves which reveals that the oscillators exhibit in fact a quasi-periodic motion in phase space.

Further dynamical properties of the NTS can be de-
rived from the modulus of the global Kuramoto order parameter \( \Gamma \) defined as

\[
\Gamma(t) := \frac{1}{L} \int_0^L e^{i\phi(x, t)} dx
\]

which corresponds to \( \frac{1}{N} \sum_{j=1}^{N} e^{i\phi_j(t)} \) in the finite-size approximation. In Fig. 3(c), \( |\Gamma| \) exhibits a periodic motion for small system size \( N \) whereby the period and amplitude of its oscillations decrease as \( N \) increases. For large \( N \), \( |\Gamma| \) eventually attains a constant value, and \( \Gamma(t) \) rotates uniformly with \( \Omega \), i.e., \( \Gamma(t) = |\Gamma| e^{i\Omega t} \). The behavior of \( |\Gamma| \) with system size can be understood from the observation that the instantaneous phase velocities \( \{ \phi_i(t) \}_{i=1}^{N} \) of all oscillators are periodic functions with the same period \( T \approx 23 \), and identical shapes while being shifted in time by equal amounts, as can be seen in Fig. 3(b). A similar phenomenon was reported for a so-called Poisson chimera state in a two-population network \[19\]. Following the same argument as given therein, it is assumed that \( \phi_i(t - \frac{T}{N}) = \phi_{i+j}(t) \) for an arbitrary \( j \in \{1, ..., N\} \), which gives \( \phi_i(t - \frac{T}{N}) = \phi_{i+1}(t) + \Theta_0 \) for \( i = 1, ..., N \) with a common constant shift \( \Theta_0 \in \mathbb{R} \) and \( \phi_1 \equiv \phi_{N+1} \). Substituting this into Eq. 8,

\[
|\Gamma(t)| = \left| \frac{1}{N} \sum_{j=1}^{N} e^{i\phi_{j+1}(t)} \right| = \left| e^{-i\Theta_0} \sum_{j=1}^{N} e^{i\phi_j(t - \frac{T}{N})} \right|
\]

we obtain \( |\Gamma(t)| = |\Gamma(t - \frac{T}{N})| \) for all \( t \), that is, the modulus of the global order parameter is periodic with the period \( \tau = \frac{T}{N} \) decreasing with increasing \( N \). This is also numerically verified in Fig. 3(f): The period of \( |\Gamma| \) numerically obtained from Eq. 8 coincides with \( T/N \), i.e., the period \( T \) of each instantaneous phase velocity divided by the system size \( N \). Hence, the modulus of the global Kuramoto order parameter of an NTS has a non-zero constant value for a sufficiently large system size and oscillates around a non-zero mean for small system sizes. This is in contrast to a TTS which has a zero global Kuramoto order parameter \( |\Gamma(t)| = 0 \) for all \( N \in \mathbb{N} \).

B. Linear Stability of the Nontrivial Twisted States

The linear stability of an NTS can be obtained by going to a reference frame moving with a constant speed \( c \) and rotating uniformly with \( \Omega \). In this reference frame, both phase and amplitude profiles are stationary. Therefore, we make the ansatz

\[
W(x, t) = W_0(\xi) e^{i\Omega t}
\]

where \( \xi = x - ct \). The winding number of the NTS is then given by \( q = \frac{1}{2\pi} \sum_{j=1}^{N} (\Phi_{j+1} - \Phi_j) \) where \( \Phi_j = \arg[W_0(\xi_j)] \) at \( \xi_j = \frac{j-1}{N-1} \in [0, 1] \) for \( j = 1, ..., N \), and the NTS satisfies

\[
-c\partial_\xi W_0(\xi) = (1 + i\Delta)W_0(\xi) - |W_0(\xi)|^2 W_0(\xi) + \varepsilon e^{-i\sigma(\Gamma W_0(\xi))} (\Gamma W_0(\xi))
\]

where \( \partial_\xi := \frac{d}{d\xi} \) and \( \Delta = \omega - \Omega \) is a real unknown constant. Here, the integral convolution operator reads

\[
(\Gamma W_0)(\xi) = \int_0^L G(\xi - \xi') W_0(\xi') d\xi'
\]

where \( G(y) \) is defined in Eq. 3.

Since in the reference frame defined above the NTS is a stationary solution, we can obtain its linear stability by linearizing the evolution equation around the stationary wave profile and determining the eigenvalues of the linearized equations. To do so, we first consider the coordinate transformation

\[
W_0(\xi) = X_0(\xi) + iY_0(\xi)
\]

where \( \text{Re}W_0 = X_0 \) and \( \text{Im}W_0 = Y_0 \) are real-valued functions that are periodic in \( \xi \): \( X_0(\xi + L) = X_0(\xi) \) and \( Y_0(\xi + L) = Y_0(\xi) \). Then, we rewrite Eq. 11 as follows

\[
-c\partial_\xi \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} = \begin{pmatrix} 1 - \Delta \\ \Delta \end{pmatrix} - (X_0^2 + Y_0^2) I_2 \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} + \varepsilon \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} (G X_0)(\xi) \\ (G Y_0)(\xi) \end{pmatrix}
\]

\[
\alpha = \alpha_0 + \alpha_1 |H_0(\xi)|^2
\]

where \( I_2 \) is a \( 2 \times 2 \) identity matrix. Next, we consider a small deviation from \( W_0(\xi) \): \( v_1 = X(\xi, t) - X_0(\xi) \) and \( v_2 = Y(\xi, t) - Y_0(\xi) \) with \( |v_i| \ll 1 \) for \( i = 1, 2 \). Note that we treat \( \xi \) here as a time-independent spatial variable. Then, the linearized equation is given by

\[
\frac{dV}{dt} = \mathcal{L}V
\]

where \( V = (v_1, v_2)^T \) and \( \mathcal{L} := \mathcal{M} + \mathcal{K} \) is a time-independent linear operator that governs the tangent space dynamics of the perturbation whose point and continuous spectra \( \sigma(\mathcal{L}) = \sigma_{pl}(\mathcal{L}) \cup \sigma_{cont}(\mathcal{L}) \) determine the linear stability of the profiles of the NTS. To numerically investigate the spectral properties of an NTS profile, we consider uniformly discretized operators that are calculated at each \( \xi = \xi_j \) for \( j = 1, ..., M \) with \( M > 1 \) \[28\]. The operator given by a multiplication then reads

\[
(\mathcal{M}V)(\xi) = \mathcal{M}(\xi)V(\xi) = \begin{pmatrix} (cD + 2Y_0^2) & -\Delta \\ \Delta & (cD + 2X_0^2) \end{pmatrix} V(\xi)
\]

\[
+ \begin{pmatrix} \text{Re} \eta(\xi) & \text{Im} \eta(\xi) \\ \text{Im} \eta(\xi) & \text{Re} \eta(\xi) \end{pmatrix} V(\xi),
\]

\[
\eta(\xi) = 1 - 3(X_0^2 + Y_0^2) - i2X_0Y_0
\]
where $\mathcal{D} \equiv \partial_t$ is a differential operator which in our numerical approach we evaluate spectrally following Ref. [22]. It is approximately treated as a constant matrix operator. From the numerical evaluation of the eigenvalues of $\mathcal{M}$ it follows that we obtain only a discretization of continuous eigenvalue branches so that $\sigma(\mathcal{M}) = \sigma_{\text{cont}}(\mathcal{M})$ holds [5]. On the other hand, the compact integral operator $\mathcal{K}$ is given by

\[(\mathcal{K}V)(\xi) = \varepsilon \left( A(\xi) + 2\alpha_1 B(\xi) \right) \left( \begin{array}{c} (Gv_1)(\xi) \\ (Gv_2)(\xi) \end{array} \right), \]

where $A(\xi) = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}$, $\alpha = \alpha_0 + \alpha_1 |H_0(\xi)|^2$, $B(\xi) = \begin{pmatrix} -\sin \alpha & \cos \alpha \\ \cos \alpha & -\sin \alpha \end{pmatrix}$, $\sigma \left( \begin{array}{c} \text{Re}^2 H_0 \\ 0 \\ 0 & \text{Im}^2 H_0 \end{array} \right) + \text{Im}H_0 \text{Re}H_0 \begin{pmatrix} \cos \alpha & -\sin \alpha \\ -\sin \alpha & -\cos \alpha \end{pmatrix}$, and $(Gv_i)(\xi) = \int_0^L G(\xi - \xi')v_i(\xi')d\xi'$, $i = 1, 2$

where $\text{Re}^2 H_0 = (\text{Re}H_0)^2$ and $\text{Im}^2 H_0 = (\text{Im}H_0)^2$ are also discretized with the same method [23].

In Fig. 3 (a), the eigenvalues of the linear operator $L$ are shown in the complex plane. Figure 3 (b) shows a magnification of (a) highlighting the eigenvalues with small imaginary part. The blue dots indicate the eigenvalues which do not coincide with the eigenvalues of the multiplication operator, i.e., $\sigma(\mathcal{L}) \setminus \sigma(\mathcal{M})$, and the black dots display a discretization of the continuous spectrum. These eigenvalues are obtained from the discretization of the linear operator with $\sigma = 2^{10}$. (c) Some eigenvalues of the unstable, coexisting TTS near the origin of the complex plane. The parameters are specified in Sec. IIA.

FIG. 3. (a) Eigenvalues of the linearized system around the NTS solution in a reference frame moving with $c$ and rotating with $\Omega$. (b) Magnification of (a) to highlight the eigenvalues with small imaginary part. The blue dots indicate the eigenvalues which do not coincide with the eigenvalues of the multiplication operator, i.e., $\sigma(\mathcal{L}) \setminus \sigma(\mathcal{M})$. (c) Some eigenvalues of the unstable, coexisting TTS near the origin of the complex plane. The parameters are specified in Sec. IIA.

III. LYAPUNOV EXPONENTS AND COLLECTIVE MODES

In a finite-size system, an NTS cannot be represented as a stationary solution in an appropriate reference frame. Rather, we have to treat an NTS as a time-evolving reference trajectory in phase space. Then, we can obtain its spectral properties from a Lyapunov analysis, which yield information about its stability. Therefore, we consider the Jacobian matrix evaluated along a reference trajectory in phase space

\[(J)_{ij} = \left( \frac{\partial \phi_j}{\partial \phi_i}, \frac{\partial \phi_j}{\partial r_i} \right) \in \mathbb{R}^{2N \times 2N}, \quad i, j = 1, ..., N. \quad (14)\]

Defining the tangent linear propagator $M(t, t_0) = O(t)O^{-1}(t_0)$ where $O(t)$ is the fundamental matrix solution of $O(t) = J(t)O(t)$ with the identity matrix $O(0) = I_{2N} [19] [21]$, we obtain the Lyapunov exponents $\Lambda_i$ as an exponential growth rate

\[\Lambda_i = \lim_{t \to \infty} \frac{1}{t} \log \left| \frac{||M(t, t_0)u(t_0)||}{||u(t_0)||} \right| \quad (15)\]
along the perturbation vector in the tangent space $T_{x^{\text{NTS}}(t)}(\mathbb{R}^{2N})$ where $x^{\text{NTS}}(t)$ is a given NTS reference trajectory in phase space, and $u(\theta_0)$ is perturbation vector belonging to each Oseledets’ splitting for $i = 1, ..., 2N$ [18, 22, 33].

In Fig. 4 (a), we show numerically obtained Lyapunov spectra $\Lambda(\nu)$ for different system sizes $N$ as a function of the re-scaled index $\nu = \frac{i-1}{2N-1}$ (black and gray tone) together with the real part of the point and continuous eigenvalues from the continuum limit analysis (red points). All Lyapunov spectra have two zero Lyapunov exponents which arise from the two continuous eigenvalues from the continuum limit analysis. The inset depicts the first half of the re-scaled index $\nu$ for $N = 40$. (b) and (c) The IPR as a function of system size $N$ corresponding to the discrete Lyapunov exponents around $\nu = 0.0$ and $\nu = 1.0$, respectively.

In general, one intuitively expects any propagating wave to be dominated by collective modes since all elements behave in the same way and their entirety forms a propagating structure. Lyapunov analysis also allows one to measure the ‘collectivity’ of the different Lyapunov modes. The covariant Lyapunov vectors (CLV), which are the spanning set of the Oseledets’ splittings, directly indicate the perturbation directions in which the Lyapunov exponents exhibit an exponential growth rate in phase space [20, 21, 33]. From the CLVs, we can derive the time-averaged inverse participation ratios (IPRs) for various system sizes, which in turn make it possible to identify collective modes [19, 22, 33]:

$$\text{IPR}^{(i)}(N) = \left\langle \exp \left( \frac{1}{q-1} \log \sum_{j=1}^{2N} |v^{(i)}_j(t)|^{2q} \right) \right\rangle_t$$ (16)

where $q = 2$ and $\text{IPR}^{(i)} \in [(2N)^{-1}, 1]$, and $v^{(i)}_j$ is the $j^{th}$ component of the CLV $v^{(i)} \in T_{x^{\text{NTS}}(t)}(\mathbb{R}^{2N})$ corresponding to a certain Lyapunov exponent $\Lambda_i(N)$ in Eq. (15) for $i = 1, ..., 2N$. When the components of a CLV spread out through all the oscillators, IPR$^{(i)}(N) \sim \frac{1}{N}$ as $N \to \infty$ and the CLV is a collective Lyapunov mode [33]. In contrast, when IPR$^{(i)}(N) \sim \text{const.}$ as $N$ increases, the vector is well localized.

In Fig. 4 (b-c), we show the IPRs as a function of $N$ for the first and the last few Lyapunov modes, which correspond to the discrete LEs near $\nu \gtrsim 0.0$ and $\nu \lesssim 1.0$, respectively. Their IPR decreases as $N$ increases with IPR(N) $\sim \frac{1}{N}$, classifying these modes as collective Lyapunov modes. Also the discrete LEs around $\nu = \frac{2N-1}{2N-1} \approx 0.5$ show the same scaling, so that they too are collective Lyapunov modes. From this, we conclude that an NTS trajectory is indeed governed by collective modes that can be captured by Lyapunov analysis.

IV. HETEROGENEOUS NATURAL FREQUENCIES

In this section, we demonstrate the robustness of the NTS by adding a small heterogeneity to the natural frequencies of the up to now identical Stuart-Landau oscillators. Therefore, we consider the Cauchy-Lorentz distribution

$$g(\omega) = \frac{\gamma}{\pi} \frac{1}{\omega^2 + \gamma^2}$$
no longer a set of nearly identical values; rather, the values tend to decrease as a function of the re-scaled index. This is reflected in the standard deviation of the LEs between \( \nu = 0.6 \) and \( 0.9 \), which is approximately \( 10^{-4} \) (considered zero within our numerical accuracy) for the system of identical oscillators, and approximately 0.028 for the heterogeneous system. Furthermore, we cannot observe collective modes based on the Lyapunov analysis, as apparent from the inset in Fig. 6 (c): There is no Lyapunov mode whose IPR decreases with \( 1/N \). The incoherent motion at the microscopic level caused by the heterogeneity of the natural frequencies apparently overshadows the collective response of the oscillators that causes the propagation of the profiles.

V. BIFURCATION SCENARIO

A. Trivial twisted states and nontrivial twisted state with winding number \( q = 1 \)

In this section, we perform a bifurcation analysis of the TTS and the NTS, based on pseudo-arclength continuation combined with the Newton-Raphson method. The algorithm is described in detail in Ref. 24 and applications can be found in Refs. 35, 36.

First, we look at a continuation of the TTS with the coupling strength \( \varepsilon \) as a bifurcation parameter. In Fig. 6 (a) the TTS is depicted by the black and gray lines, where black indicates unstable TTSs and gray stable ones. The TTS is unstable for low values of \( \varepsilon \) and becomes stabilized in a Hopf bifurcation at the point HB, i.e. at a comparatively high value of \( \varepsilon \). Our numerical results strongly suggest that the Hopf bifurcation is subcritical. Furthermore, the bifurcation analysis predicts that the velocity \( c \) of the TTS depends linearly on \( \varepsilon \). This can be easily understood from the properties of the TTS together with Eq. (4): Since all amplitudes \( r_j \approx 1 \) and all phase differences of adjacent oscillators are the same, the coupling term in Eq. (4) is identical for all oscillators for a given \( \varepsilon \) and scales linearly with \( \varepsilon \), resulting in a linear increase of \( |c| \) with \( \varepsilon \). The branch of unstable TTS continues actually up to \( \varepsilon = 0 \), while also \( c = 0 \).

Let us now focus on the continuation of the NTS with winding number \( q = \pm 1 \) as a function of \( \varepsilon \). The corresponding bifurcation diagram is depicted by the red and blue lines in Fig. 6 (a). Stable states are shown in red, unstable ones in blue. Coming from small values of \( \varepsilon \), the unstable NTS is stabilized in a Hopf bifurcation at point A. As judged from the course of the eigenvalues with \( \varepsilon \) in the complex plane (see the left panel in the middle row of Fig. 6), Beyond the Hopf bifurcation, the stable NTS exists in a large \( \varepsilon \) interval, ensuring that its existence is not restricted to a practically inaccessible parameter range. Along this stable curve, the difference between the maximum and minimum values of the amplitude hump increases with increasing \( \varepsilon \). The fully synchronized state is stable for \( \varepsilon \gtrsim 0.773 \). Thus, the NTS coexists with the
uniform oscillation in most of its existence range. However, in 30 simulations with random initial conditions, 28 and 29 trajectories approached the NTS state at $\varepsilon = 1.0$ and $\varepsilon = 1.2$, respectively. This suggests that the basin of attraction of the NTS is considerably larger than the one of the synchronized oscillation.

At the high-$\varepsilon$ end of its existence interval, the NTS solution is annihilated in a saddle-node bifurcation (SN) at point B (cf. the middle panel in the middle row). Continuing the unstable NTS branch that is born in the SN bifurcation B, we observe that it is further destabilized in a series of Hopf bifurcations starting at C. At the same time, the amplitude profile becomes flatter and flatter approaching the uniform profile as $c$ approaches zero.

Obviously, stable NTS and stable TTS do not coexist. Their existence range is separated by the $\varepsilon$ interval between points B and HB where none of them is stable. In this interval, more precisely, in 100 numerical integration with $\varepsilon = 1.6$ as well as $\varepsilon = 1.8$ and random initial conditions, the trajectory always approached the fully synchronized oscillation. In comparison, when doing the same numerical experiment for $\varepsilon = 2.0$ and $\varepsilon = 2.2$, 95% of the initial conditions end up on the TTS and only 5% on the uniform oscillation. For $\varepsilon = 1.0$ and $\varepsilon = 1.2$, where the NTS is stable, none and 2 out of 30 simulations with random initial conditions approach the uniform state, respectively. Furthermore, in the shown range of $\varepsilon$ the bifurcation diagrams for TTS and NTS remain well separated, suggesting that the two solutions do not interact directly at any bifurcation point. We note that we were not able to continue the NTS for values of $\varepsilon$ smaller than the ones shown in Fig. 6 due to convergence problems.

Besides the region between B and HB, there is a second region in which neither NTS nor TTS are stable, namely for $\varepsilon < \varepsilon (A)$. In this parameter interval we observed states with discontinuous amplitude dynamics. Thus, here, the amplitudes do not form a smooth curve. Examples include states where the phase dynamics seems similar to the one of the irregular inhomogeneous states reported in Ref. 22, amplitude-mediated chimera states [8], as well as different kinds of NTS solutions, as discussed in the next subsection.

Finally, a continuation of the NTS with the parameter $\kappa$ (the inverse of the interaction range) reveals that the solution is restricted to a certain interaction range, or certain length of the system (Fig. 6 (b)).

**B. Nontrivial twisted states with winding number $q = 2$**

So far, we only discussed an NTS with a winding number $|q| = 1$. However, as apparent from the definition, just as a TTS, an NTS may also have a winding number $|q| > 1$. In this section, we address an NTS solution of Eq. (1) with winding number $|q| = 2$, and call it a 2-NTS. In Fig. 7 (a-b), such a 2-NTS solution is depicted. The amplitude profile features two humps and the phase profile winds twice along the ring, changing in total by $4\pi$. From the Lyapunov exponents, we can conjecture that the 2-NTS, too, is a stable solution. However, it is less stable than the 1-NTS since the first half of the Lyapunov exponents are closer to zero than in the case of the 1-NTS (compare Fig. 7 (c), (h) to Fig. 3 (a)). This can be also verified by determining the eigenvalues of the linearized equation in the continuum limit analysis (Fig. 7 (e)): One of the continuous parts of the spectrum of the 2-NTS is closer to the imaginary axis than that of the 1-NTS.

A bifurcation analysis reveals that the stable 2-NTS solution can be observed in some parameter region, ensuring it is a robust solution (Fig. 7 (d)). However, the parameter interval is smaller than the one in which the stable 1-NTS solution exists. Moreover, it is found at lower values of the coupling strength $\varepsilon \approx 0.6$, and its speed tends to be lower than the one of the stable 1-NTS. At both ends of its existence interval the stable 2-NTS becomes destabilized through Hopf bifurcations (cf. Fig. 7 (f) for HB1). Furthermore, we numerically
verified that one of them (HB1) is indeed a supercritical Hopf bifurcation. Before HB1, the 2-NTS solution exhibits a stationary amplitude profile in a moving reference frame (Fig. 7(b-2)). Beyond HB1, e.g. at $\epsilon = 0.67$, a 2-NTS state is still observed, but now its amplitude profile oscillates periodically, rendering the state a modulated traveling wave (Fig. 7(g)). The modulated NTS features three zero Lyapunov exponents, two of which arise from the continuous symmetries, the third originating from the modulation frequency (Fig. 7(i)).

VI. DISCUSSION AND CONCLUSION

In this work, we reported a new type of collective behavior in a ring of nonlocally coupled Stuart-Landau oscillators, which we name a nontrivial twisted state. It is characterized by non-uniform profiles of amplitude and phase gradient as well as a winding number $q$. The latter characterizes it as a twisted state and implies that the structure is a coherent traveling wave. From a macroscopic point of view, the modulus of the local order parameter of an NTS (see Eq. (6)) $|z(x, t)| = 1$ for all $x \in [0, L]$ and all $t$ while the global order parameter (see Eq. (8)) $0 < |\Gamma(t)| < 1$ for all $t$. In contrast, the well-known trivial twisted state has a constant phase gradient and uniform amplitude profile, which renders the global order parameter zero $|\Gamma(t)| = 0$ while the modulus of the local order parameter remains $1, |z(x, t)| = 1$ for all $x$ and $t$. Linear stability analysis, Lyapunov analysis and bifurcation analysis revealed that NTS solutions with winding number $|q| = 1, 2$ are attracting states which exist in wide ranges of parameter sets and for many initial conditions.

In the literature, there are some examples of coherent spatio-temporal patterns of coupled oscillators in a ring geometry that resemble an NTS in some respect. Most of them were observed in studies of phase models. A coherent traveling wave solution in a model of coupled phase oscillators with a non-constant phase gradient profile was reported in [57] (see Fig. 24). However, it is not a twisted state since the difference of phases does not add up to an integer multiple of $2\pi$ but rather to 0. In other words, the phase does not form complete cycles when going once around the ring so that one cannot assign a winding number. In the Kuramoto-Sakaguchi phase model with nonlinear phase-lag function, initial conditions close to a TTS led transiently to an evolution of the phase profile that resembles the one in our NTS solutions before set-
ting down to a chimera state. However, this NTS-like dynamics was not obtained as a stable state (see Fig. 2 (c) in [12]). In the Kuramoto-Sakaguchi phase-reduced model with a trigonometric nonlocal coupling kernel, coherent wave solutions with non-constant phase gradient profile coming closest to the ones discussed here are reported (Fig. 1 (c) in [7]). However, these traveling waves are again not stable solutions but long-lived transients with many neutrally stable directions [39]. Finally, in a model of Stuart-Landau oscillators with time-delay, a traveling wave solution possessing a winding number and a slightly varying phase gradient was found (Fig. 2 (d) in [39]), so that the modulus of the global order parameter was close to zero, yet finite.

As we indicated in Sec. [V] besides the NTS solutions, the system of nonlocally coupled Stuart-Landau oscillators on a ring seems to possess various kinds of further coherent traveling wave solutions. Many of them represent apparently novel types of collective behaviors such as a solitary state presenting discontinuities at some locations in an otherwise smooth profile. In the future, further studies of nonlocally coupled Stuart-Landau oscillators may therefore reveal other novel types of collective behaviors.

Finally, it is worthwhile also to compare the NTS to the partial synchrony observed in globally coupled oscillators [40] as a splay state in the globally coupled system is similar to the TTS in the spatially extended system. The amplitude profiles of some partially synchronized states in globally coupled oscillators form a smooth closed curve as a function of phase $\phi \in [-\pi, \pi]$ in the complex plane [43]. Similarly, the amplitude of the NTS forms a smooth closed curve in the complex plane as a function of spatial variable $x \in [0, L]$. In both cases, the individual oscillators behave quasi-periodically while the collective dynamics is periodic. However, a prominent difference between them seems to be that the partial synchrony in globally coupled systems bifurcates from the splay state whereas the NTS does not bifurcate from the TTS but rather emerges in a saddle-node bifurcation.

ACKNOWLEDGMENTS

The authors would like to thank O. E. Omel’chenko and C. R. Laing for fruitful discussions. This work has been supported by the Deutsche Forschungsgemeinschaft (project KR1189/18 ‘Chimera States and Beyond’)

[1] A. Pikovsky, M. Rosenblum, and J. Kurths, Synchronization: A Universal Concept in Nonlinear Sciences (Cambridge University Press, Cambridge, 2001).
[2] S. H. Strogatz, Sync (Hyperion, New York, 2003).
[3] Y. Kuramoto and D. Battogtokh, Coexistence of coherence and incoherence in nonlocally coupled phase oscillators, Nonlinear Phenom. Complex Syst. 5, 380 (2002).
[4] D. M. Abrams and S. H. Strogatz, Chimera States for Coupled Oscillators, Phys. Rev. Lett. 93, 174102 (2004).
[5] O. E. Omel’chenko, Coherence-incoherence patterns in a ring of non-locally coupled phase oscillators, Nonlinearity 26, 2469 (2013).
[6] M. J. Panaggio and D. M. Abrams, Chimera states: coexistence of coherence and incoherence in networks of coupled oscillators, Nonlinearity 28, R67 (2015).
[7] O. E. Omel’chenko. The mathematics behind chimera states, Nonlinearity 31, R121 (2018).
[8] G. C. Sethia, A. Sen, and G. L. Johnston, Amplitude-mediated chimera states, Phys. Rev. E 88, 042917 (2013).
[9] K. Sathiyadevi, V. K. Chandrasekar, D. V. Senthilkumar, and M. Lakshmanan, Imperfect Amplitude Mediated Chimera States in a Nonlocally Coupled Network, Frontiers in Applied Mathematics and Statistics 4, 10.3389/fams.2018.00058 (2018).
[10] T. Banerjee, D. Biswas, D. Ghosh, E. Schöll, and A. Zakharova, Networks of coupled oscillators: From phase to amplitude chimeras, Chaos: An Interdisciplinary Journal of Nonlinear Science 28, 113124 (2018).
[11] K. Sathiyadevi, V. K. Chandrasekar, and D. V. Senthilkumar, Stable amplitude chimera in a network of coupled Stuart-Landau oscillators, Phys. Rev. E 98, 032301 (2018).
[12] D. Bolotov, M. I. Bolotov, L. A. Smirnov, G. V. Osipov, and A. S. Pikovsky, Twisted States in a System of Nonlinearly Coupled Phase Oscillators, Regular and Chaotic Dynamics 24, 717 (2019).
[13] O. E. Omel’chenko, M. Wolfrum, and C. R. Laing, Partially coherent twisted states in arrays of coupled phase oscillators, Chaos: An Interdisciplinary Journal of Nonlinear Science 24, 023102 (2014).
[14] S. Lee, Y. S. Cho, and H. Hong, Twisted states in low-dimensional hypercubic lattices, Phys. Rev. E 98, 062221 (2018).
[15] D. A. Wiley, S. H. Strogatz, and M. Girvan, The size of the sync basin, Chaos: An Interdisciplinary Journal of Nonlinear Science 16, 015103 (2006).
[16] T. Ginnyk, M. Hasler, and Y. Maistrenko, Multistability of twisted states in non-locally coupled Kuramoto-type models, Chaos: An Interdisciplinary Journal of Nonlinear Science 22, 013114 (2012).
[17] A. Pikovsky and A. Politi, Lyapunov Exponents: A Tool to Explore Complex Dynamics (Cambridge University Press, Cambridge, 2016).
[18] J. P. Eckmann and D. Ruelle, Ergodic theory of chaos and strange attractors, Rev. Mod. Phys. 57, 617 (1985).
[19] S. Lee and K. Krischer, Attracting Poisson chimeras in two-population networks, Chaos: An Interdisciplinary Journal of Nonlinear Science 31, 113101 (2021).
[20] P. Ginelli, H. Chaté, R. Livi, and A. Politi, Covariant lyapunov vectors, Journal of Physics A: Mathematical and Theoretical 46, 254005 (2013).
[21] P. V. Kuptsov and U. Parlitz, Theory and computation of covariant lyapunov vectors, Journal of Nonlinear Science 22, 727 (2012).
[22] K. Hohlein, F. P. Kemeth, and K. Krischer, Lyapunov spectra and collective modes of chimera states in globally coupled Stuart-Landau oscillators, Phys. Rev. E 100, 022217 (2019).

[23] M. I. Bolotov, L. A. Smirnov, E. S. Bubnova, G. V. Osipov, and A. S. Pikovsky, Spatiotemporal Regimes in the Kuramoto-Battogtokh System of Nonidentical Oscillators, Journal of Experimental and Theoretical Physics 132, 127 (2021).

[24] C. R. Laing, Numerical bifurcation theory for high-dimensional neural models, The Journal of Mathematical Neuroscience 4, 13 (2014).

[25] M. Bolotov, L. Smirnov, G. Osipov, and A. Pikovsky, Simple and complex chimera states in a non-linearly coupled oscillatory medium, Chaos: An Interdisciplinary Journal of Nonlinear Science 28, 045101 (2018).

[26] M. I. Bolotov, L. A. Smirnov, G. V. Osipov, and A. S. Pikovsky, Breathing chimera in a system of phase oscillators, JETP Letters 106, 393 (2017).

[27] G. Bordyugov, A. Pikovsky, and M. Rosenblum, Self-emerging and turbulent chimeras in oscillator chains, Phys. Rev. E 82, 035205 (2010).

[28] L. Smirnov, G. Osipov, and A. Pikovsky, Chimera patterns in the Kuramoto-Battogtokh model, Journal of Physics A: Mathematical and Theoretical 50, 08LT01 (2017).

[29] M. Wolfrum, O. E. Omel’chenko, S. Yanchuk, and Y. L. Maistrenko, Spectral properties of chimera states, Chaos: An Interdisciplinary Journal of Nonlinear Science 21, 013112 (2011).

[30] O. E. Omel’chenko, Mathematical framework for breathing chimera states, Journal of Nonlinear Science 32, 22 (2022).

[31] A. Pikovsky and M. Rosenblum, Dynamics of heterogeneous oscillator ensembles in terms of collective variables, Physica D: Nonlinear Phenomena 240, 872 (2011).

[32] L. Trefethen, Spectral Methods in MATLAB (SIAM, Philadelphia, 2000).

[33] V. Oseledec, A multiplicative ergodic theorem. Characteristic Liapunov, exponents of dynamical systems, Trans. Mosc. Math. Soc. 19, 197 (1968).

[34] K. A. Takeuchi and H. Chaté, Collective lyapunov modes, Journal of Physics A: Mathematical and Theoretical 46, 254007 (2013).

[35] C. R. Laing, Interpolating between bumps and chimeras, Chaos: An Interdisciplinary Journal of Nonlinear Science 31, 113116 (2021).

[36] C. R. Laing and O. Omel’chenko, Moving bumps in theta neuron networks, Chaos: An Interdisciplinary Journal of Nonlinear Science 30, 043117 (2020).

[37] J. Xie, E. Knobloch, and H.-C. Kao, Multicluster and traveling chimera states in non-local phase-coupled oscillators, Phys. Rev. E 90, 022919 (2014).

[38] O. E. Omel’chenko, Personal Communication, May 19, 2022.

[39] A. Gjurchinovski, E. Schöll, and A. Zakharova, Control of amplitude chimeras by time delay in oscillator networks, Physical Review E 95, 042218 (2017).

[40] C. v. Vreeswijk, Partial synchronization in populations of pulse-coupled oscillators, Phys. Rev. E 54, 5522 (1996).

[41] M. Rosenblum and A. Pikovsky, Two types of quasiperiodic partial synchrony in oscillator ensembles, Phys. Rev. E 92, 012919 (2015).

[42] P. Clusella, A. Politi, and M. Rosenblum, A minimal model of self-consistent partial synchrony, New Journal of Physics 18, 093037 (2016).

[43] N. Nakagawa and Y. Kuramoto, From collective oscillations to collective chaos in a globally coupled oscillator system, Physica D: Nonlinear Phenomena 75, 74 (1994).

[44] N. Nakagawa and Y. Kuramoto, Anomalous lyapunov spectrum in globally coupled oscillators, Physica D: Nonlinear Phenomena 80, 307 (1995).

[45] P. Clusella and A. Politi, Between phase and amplitude oscillators, Phys. Rev. E 99, 062201 (2019).