Invariant and evolutionary properties of the skew-symmetric differential forms

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The present work pursues the aim to draw attention to unique possibilities of the skew-symmetric differential forms.

At present the theory of skew-symmetric exterior differential forms that possess invariant properties has been developed.

In the work the readers are introduced to the skew-symmetric differential forms that were called evolutionary ones because they possess evolutionary properties.

The combined mathematical apparatus of exterior and evolutionary skew-symmetric differential forms in essence is a new mathematical language. This apparatus can describe transitions from nonconjugated operators to conjugated ones. There are no such possibilities in any mathematical formalism.

In the present work it has been shown that the properties of exterior or evolutionary forms are explicitly or implicitly accounted for in all mathematical (and physical) formalisms.

Introduction

While studying the general equations of dynamics A. Poincare showed an existence of integral invariants. When investigating the integrability conditions of the differential equations and the sets of equations in total differentials E. Cartan discovered the invariant properties of integrands in multiple integrals and elucidated their own significance. He called them as the exterior differential forms because they obey the rules of the exterior multiplication by Grassmann (the skew-symmetry conditions). It appears that the exterior differential forms together with the operation of exterior differentiation may possess the invariant, group, tensor, structural and other properties that are of great functional and utilitarian importance. The exterior differential forms were found wide application in differential geometry and algebraic topology.

The analysis of differential equations shows that, besides of the skew-symmetric differential forms, which possess invariant properties and have been named the exterior differential forms, there are other skew-symmetrical differential forms, which possess evolutionary properties. The author has named these forms as evolutionary differential forms.

The evolutionary forms, as well as the exterior forms, are differential forms with exterior multiplication. A radical distinction between the evolutionary forms and the exterior ones consists in the fact that the exterior differential forms are defined on manifolds with closed metric forms, whereas the evolutionary differential forms are defined on manifolds with unclosed metric forms.

This leads to the fact that the evolutionary forms and exterior ones possess the opposing properties and the opposing mathematical apparatus. Whereas the mathematical apparatus of exterior differential forms involves the identical relations and nondegenerate transformations, the mathematical apparatus of
evolutionary forms involves the nonidentical relations and degenerate transformations. Thus, they complement each other and make up some unified whole. Between the exterior and evolutionary forms there exists a connection. *The evolutionary differential forms generate the exterior differential forms.*

Owing to these properties the mathematical apparatus of exterior and evolutionary differential forms allows description of discrete transitions, quantum steps, evolutionary processes, generation of various structures.

A transition from evolutionary forms to closed exterior forms describes a transition from nonconjugated operators to conjugated ones.

1 Properties and specific features of the exterior differential forms

It should be noted that this work is not intended to present the general theory of exterior differential forms. The reader can find information on this subject, for example, in [1-8]. Some initial information concerning exterior differential forms is outlined, the basic properties of the exterior differential forms and peculiarities of their mathematical apparatus are described. It is shown that the invariant properties of closed exterior forms reveal themselves in many branches of mathematics.

The exterior differential form of degree $p$ ($p$-form on the differentiable manifold) can be written as [5,7,8]

$$
\theta^p = \sum_{i_1\ldots i_p} a_{i_1\ldots i_p} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p} \quad 0 \leq p \leq n
$$

(1.1)

Here $a_{i_1\ldots i_p}$ are the functions of the variables $x^{i_1}, x^{i_2}, \ldots, x^{i_p}$, $n$ is the dimension of space, $\wedge$ is the operator of exterior multiplication, $dx^i, dx^i \wedge dx^j, dx^i \wedge dx^j \wedge dx^k, \ldots$ is the local basis which satisfies the condition of exterior multiplication:

$$
dx^i \wedge dx^i = 0
$$

$$
dx^i \wedge dx^j = -dx^j \wedge dx^i \quad i \neq j
$$

(1.2)

[From here on the symbol $\sum$ will be omitted and it will be implied that a summation is performed over double indices. Besides, the symbol of exterior multiplication will be also omitted for the sake of presentation convenience].

The differential of the (exterior) form $\theta^p$ is expressed as

$$
d\theta^p = \sum_{i_1\ldots i_p} da_{i_1\ldots i_p} dx^{i_1} dx^{i_2} \ldots dx^{i_p}
$$

(1.3)

and is the differential form of degree $(p + 1)$. (The exterior differentiating operator $d$ allows one to pass on from the $p$-fold integral over the $p$-dimension closed manifold to the $(p + 1)$-fold integral over the $(p + 1)$-dimension manifold bounded by first one [1]).
Local domains of manifold are the basis of the exterior form. In this section we will consider the domains of the Euclidean space [9] or differentiable manifolds [8]. (Manifolds, on which the exterior differential forms and the evolutionary forms may be defined, and the influence of the manifold properties on the differential forms will be discussed in more detail in section 2).

Let us consider some examples of the exterior differential form whose basis are the Euclidean space domains.

We consider the 3-dimensional space. In this case the differential forms of zero-, first- and second degree can be written as [7]

\[ \theta^0 = a, \]
\[ \theta^1 = a_1 dx^1 + a_2 dx^2 + a_3 dx^3, \]
\[ \theta^2 = a_{12} dx^1 dx^2 + a_{23} dx^2 dx^3 + a_{31} dx^3 dx^1 \]

With account for conditions (1.2) their differentials are the forms

\[ d\theta^0 = \frac{\partial a}{\partial x^1} dx^1 + \frac{\partial a}{\partial x^2} dx^2 + \frac{\partial a}{\partial x^3} dx^3, \]
\[ d\theta^1 = \left( \frac{\partial a_2}{\partial x^1} - \frac{\partial a_1}{\partial x^2} \right) dx^1 dx^2 + \left( \frac{\partial a_3}{\partial x^2} - \frac{\partial a_2}{\partial x^3} \right) dx^2 dx^3 + \left( \frac{\partial a_1}{\partial x^3} - \frac{\partial a_3}{\partial x^1} \right) dx^3 dx^1, \]
\[ d\theta^2 = \left( \frac{\partial a_{23}}{\partial x^1} + \frac{\partial a_{31}}{\partial x^2} + \frac{\partial a_{12}}{\partial x^3} \right) dx^1 dx^2 dx^3 \]

From (1.4)–(1.9) one can see the following.

a) Any function is the form of zero degree. Its basis is a surface of zero dimension, namely, a variety of points. A differential of the form of zero degree is an ordinary differential of the function.

b) The form of first degree is a differential expression. An ordinary differential of the function is an example of the first-degree form.

c) Coefficients of the differentials of the forms of the zero-, first- and second degrees give the gradient, curl, and divergence, respectively. That is, the operator \( d \), referred to as the exterior differentiation, is some abstract generalization of the ordinary operators of gradient, curl, and divergence. At this point it should be emphasized that in mathematical analysis the ordinary concepts of gradient, curl, and divergence are the operators applied to the vector, and in the theory of exterior forms gradient, curl, and divergence obtained as the results of exterior differentiating (the forms of the zero- first- and second degrees) are the operators applied to the pseudovector (the axial vector).

These examples are presented in order to give some initial illustrative presentation of the exterior differential forms.

From these examples one can assure oneself that, firstly, the differential of the exterior form is also the exterior form (but with the degree greater by one), and,
secondly, he can see that the components of the differential form commutator are coefficients of the form differential. Thus, a differential of the first-degree form $\omega = a_i dx^i$ can be written as $d\omega = K_{ij} dx^i dx^j$ where $K_{ij}$ are components of the commutator for the form $\omega$ that are defined as $K_{ij} = (\partial a_j/\partial x^i - \partial a_i/\partial x^j)$.

**Closed exterior differential forms**

Closed differential forms possess the invariant properties.

A form is called a closed one if its differential is equal to zero:

$$d\theta^p = 0 \quad (1.10)$$

From condition (1.10) one can see that the closed form $\{\text{the kernel of the operator } d\}$ is a conservative quantity. (This means that it can correspond to the conservation law, namely, to some conservative quantity).

A differential of the form is a closed form. That is,

$$dd\omega = 0 \quad (1.11)$$

where $\omega$ is an arbitrary exterior form.

The form that is a differential of some other form $\{\text{a mapping of the operator } d\}$:

$$\theta^p = d\theta^{p-1} \quad (1.12)$$

is called an *exact* form. Exact forms prove to be closed automatically [4]

$$d\theta^p = dd\theta^{p-1} = 0 \quad (1.13)$$

Here it is necessary to pay attention to the following points. In the above presented formulas it was implicitly assumed that the differential operator $d$ is the total one (that is, the operator $d$ acts everywhere in the vicinity of the point considered locally), and therefore, it acts on the manifold of the initial dimension $n$. However, a differential may be internal. Such a differential acts on some structure with the dimension being less than that of the initial manifold. The structure, on which the exterior differential form may become a closed *inexact* form, is a pseudostructure with respect to its metric properties. {Cohomology, sections of cotangent bundles, the eikonal surfaces, the characteristic and potential surfaces, and so on may be regarded as examples of pseudostructures.} The properties of pseudostructures will be considered later.

If the form is closed on pseudostructure only, the closure condition is written as

$$d_\pi \theta^p = 0 \quad (1.14)$$

And the pseudostructure $\pi$ is defined from the condition

$$d_\pi *\theta^p = 0 \quad (1.15)$$

where $*\theta^p$ is the dual form. (For the properties of dual forms see [10]).
A skew-symmetric tensor corresponds to the exterior differential form on the differentiable manifold, and the pseudotensor that is dual to the skew-symmetric tensor corresponds to the dual form.

From conditions (1.14) and (1.15) one can see that the form closed on pseudostructure (a closed inexact form) is a conservative object, namely, this quantity conserves on pseudostructure. (This can also correspond to some conservation law, i.e. to conservative object).

The exact form is, by definition, a differential (see condition (1.12)). In this case the differential is total. The closed inexact form is a differential too. The closed inexact form is an interior (on pseudostructure) differential, that is

$$\theta^p = d_\pi \theta^{p-1}$$

(1.16)

And so, any closed form is a differential of the form of a lower degree: the total one $\theta^p = d\theta^{p-1}$ if the form is exact, or the interior one $\theta^p = d_\pi \theta^{p-1}$ on pseudostructure if the form is inexact. (This may have the physical meaning: the form of lower degree may correspond to the potential, and the closed form by itself may correspond to the potential force.)

From conditions (1.12) and (1.16) it is possible to see that a relation between closed forms of different degree can exist.

Similarly to the differential relation between the exterior forms of sequential degrees, there is an integral connection. The relevant integral relation has the form [10]

$$\int_{c^{p+1}} d\theta^p = \int_{\partial(c^{p+1})} \theta^p$$

(1.17)

In particular, the integral theorems by Stokes and Gauss follow from the integral relation for $p = 1, 2$ in three-dimensional space. (From this relation one can see that an integral of the closed form over the closed curve vanishes (in the case of a smooth manifold). However, in the case of a complex manifold (for example, a not simply connected manifold with the homology class being nonzero) the integral of the closed form (in this case the form is inexact) over the closed curve is nonzero. It may be equal to a scalar multiplied by $2\pi$, which in this case corresponds, for example, to such a physical quantity as the charge [10]. Just such integrals are considered in the theory of residues.)

Invariant properties of closed exterior differential forms.
Conjugacy of exterior differential forms

Since the closed form is a differential (a total one if the form is exact, or an interior one on the pseudostructure if the form is inexact), then it is obvious that the closed form proves to be invariant under all transformations that conserve a differential.

The examples of such transformations are unitary, tangent, canonical, gradient transformations.

A closure of exterior differential forms and hence their invariance result from the conjugacy of elements of exterior or dual forms.
From definition of the exterior differential form one can see that the exterior differential forms have complex structure. Specific features of the exterior form structure are homogeneity with respect to the basis, skew-symmetry, integrating terms each including two objects of different nature (the algebraic nature for form coefficients, and the geometric nature for base components). Besides, the exterior form depends on the space dimension and on the manifold topology. The closure property of the exterior form means that any objects, namely, elements of the exterior form, components of elements, elements of the form differential, exterior and dual forms and others, turn out to be conjugated. It is a conjugacy that leads to realization of invariant and covariant properties of the exterior and dual forms that have a great functional and utilitarian importance. A variety of objects of conjugacy leads to the fact that the closed forms can describe a great number of different structures, and this fact once again emphasizes great mathematical potentialities of the exterior differential forms.

Let us consider some types of conjugacy that make the exterior differential and dual forms closed, that is, they make the form differentials equal to zero.

As it was pointed out before, components of the exterior form commutator are coefficients of differential of this form. If the commutator of the form vanishes, the form differential vanishes too, and this indicates that the form is a closed one. Therefore, the closure property of the form may be recognized by finding whether or not the commutator of the form vanishes.

One of the types of conjugacy is that for the form coefficients.

Let us consider the exterior differential form of the first degree \( \omega = a_i dx^i \). In this case, as it was pointed before, the differential will be expressed as \( d\omega = K_{ij} dx^i dx^j \), where \( K_{ij} = (\partial a_j/\partial x^i - \partial a_i/\partial x^j) \) are components of the form commutator.

It is evident that the differential may vanish if the components of commutator vanish. One can see that the components of commutator \( K_{ij} \) may vanish if derivatives of the form coefficients vanish. This is a trivial case. Besides, the components \( K_{ij} \) may vanish if the coefficients \( a_i \) are derivatives of some function \( f(x^i) \), that is, \( a_i = \partial f/\partial x^i \). In this case, components of the commutator are equal to a difference of the mixed derivatives

\[
K_{ij} = \left( \frac{\partial^2 f}{\partial x^j \partial x^i} - \frac{\partial^2 f}{\partial x^i \partial x^j} \right)
\]

and, therefore, they vanish. One can see that the form coefficients \( a_i \), which satisfy these conditions, are conjugated quantities (the operators of mixed differentiation turn out to be commutative).

Let us consider the case when the exterior form is written as

\[
\theta = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy
\]

where \( f \) is the function of two independent variables \((x, y)\). It is evident that this form is closed because it is equal to the differential \( df \). And, for the dual form

\[
*\theta = -\frac{\partial f}{\partial y} dx + \frac{\partial f}{\partial x} dy
\]
to be closed also, it is necessary that its commutator be equal to zero

\[ \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \equiv \Delta f = 0 \]

where \( \Delta \) is the Laplace operator. As a result the function \( f \) has to be harmonic one.

Assume that the exterior differential form of the first degree has the form \( \theta = udx + vdy \), where \( u \) and \( v \) are functions of two variables \( (x, y) \) In this case, the closure condition of the form, that is, the condition under which the form commutator vanishes, takes the form

\[ K = \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = 0 \]

One can see that this is one of the Cauchy-Riemann conditions for complex functions. The closure condition of the relevant dual form \( *\theta = -vdx + udy \) is the second Cauchy-Riemann condition. {Here one can see a connection with functions of complex variables. If we consider the function \( w = u + iv \) of the complex variables \( z = x + iy \), \( \overline{z} = x - iy \) that obeys the Cauchy-Riemann conditions, then to this function will correspond the closed exterior and dual forms. (The Cauchy-Riemann conditions are the conditions under which the function of complex variables does not depend on the conjugated coordinate \( \overline{z} \). And to the harmonic function of complex variables there corresponds the closed exterior differential form, whose coefficients \( u \) and \( v \) are conjugated harmonic functions).}

It can exist the conjugacy which makes the interior differential on the pseudostructure equal to zero, \( d_\pi \theta = 0 \). Assume that the interior differential is the form of the first degree (the form itself is that of zero degree) and can be presented as \( d_\pi \theta = p_x dx + p_y dy \), where \( p \) is the form of zero degree (any function). Then the closure condition of the form \( (d_\pi \theta = p_x dx + p_y dy = 0) \) is

\[ \frac{dx}{dy} = -\frac{p_y}{p_x} \tag{1.18} \]

This is a conjugacy of the basis and derivatives of the form coefficients. One can see that formula (1.18) is one of the formulas of canonical relations [11]. The second formula of canonical relations follows from a condition that the dual form differential vanishes. This type of conjugacy is connected with the canonical transformation. For a differential of the first degree form (in this case the differential is the form of the second degree) a corresponding transformation is a gradient one.

At this point it should be remarked that relation (1.18) is a condition that the implicit function to exist. That is, the closed (inexact) form of zero degree is the implicit function. {This is an example of connection between an exterior form and analysis}.

A property of the exterior differential form being closed on pseudostructure points to another type of conjugacy, namely, a conjugacy of exterior and dual forms.

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From the properties of closed forms analyzed above it becomes evident that there are exterior and interior types of conjugacy. To the interior type of conjugacy there are assigned a conjugacy of the form coefficients, coefficients of the form and the basis, conjugacy of coefficients of the differential, and so on. To the exterior types of conjugacy there is assigned a conjugacy of the exterior and dual forms. Another example of the exterior conjugacy is that between the forms of sequential degrees (the closed differential form is a differential of the form of less by one degree, that is, the closed form is conjugated to the form of less by one degree).

With the conjugacy another characteristic property of the exterior differential forms, namely, their duality, is connected that has the fundamental meaning. {The conjugacy is some identical relation between two operators or mathematical objects. A duality is a concept that means that one object carries a double meaning or that two objects with different meanings (of different physical nature) are identically connected. If one knows any dual object, he can obtain the other object}. A conjugacy of objects of the exterior differential form generates a duality of the exterior forms.

The connection of the exterior and dual forms is an example of the duality. The exterior form and the dual form correspond to the objects of different nature: the exterior form corresponds to the physical (i.e. algebraic) quantity, and the dual form corresponds to some spatial (or pseudospatial) structure. At the same time, under conjugacy the duality of these objects manifests itself, that is, if one form is known, it is possible to find the other form. {It will be shown below that the duality between exterior and dual forms elucidates a connection between physical quantities and spatial structures (and together they form a physical structure). Here the duality is also revealed in the fact that, if the degree of the exterior form equals $p$, the dimension of the spatial structure equals $N - p$, where $N$ is the space dimension.}

Since the closed exterior form possesses invariant properties and the dual form corresponding to that possesses covariant properties, the invariance of the closed exterior form and the relevant covariance of the dual form is an example of the duality of exterior differential forms.

The other example of a duality of the closed form is connected with the fact that the closed form of degree $p$ is a differential of the form of degree $p - 1$. This duality is manifested in that, on the one hand, as it was pointed out before, the closed exterior form is a conservative quantity, and on the other hand, the closed form can correspond to a potential force [12,13]. (In the works [12,13] the physical meaning of this duality has been illustrated, and it has been shown in respect to what the closed form manifests itself as a potential force and with what the conservative physical quantity is connected).

Here it should be emphasized that the duality is a property of the closed forms only. In the case of unclosed forms one cannot speak about duality.

The further manifestation of duality of exterior differential forms that needs for more attention is a duality of the concepts of closure and of integrability of differential forms.

The form that can be presented as a differential can be called an integrable
one because it is possible to integrate it directly. In this context the closed form, which is a differential (total, if the form is exact, or interior, if the form is inexact), is integrable. And the closed exact form is integrable identically, whereas the closed inexact form is integrable on pseudostructure only.

The concepts of closure and integrability are not identical ones. The closure of the form is defined with respect to the form of greater by one degree, whereas the integrability is defined with respect to the form of degree less by one. Really, the form is closed one when the form differential, which is the form of greater by one degree, equals zero. And the form referred to as integrable one is a differential of some form of degree less by one. The closure and the integrability are dual concepts. Namely, the closure and the integrability are further examples of a duality of exterior differential forms.

Here it should emphasize once again that the duality is a property of closed forms only. In particular, nonclosure and nonintegrability are not dual concepts, and this will be revealed while analyzing evolutionary differential forms.

The duality of closed differential forms reveals under an availability of one or other type of conjugacy. The duality is a tool that untangles the mutual connection, the mutual changeability and the transitions between different objects.

**Differential and geometrical structure**

From the definition of a closed inexact exterior form one can see that to this form there correspond two conditions:

1) condition (1.14) is a closure condition of the exterior form itself, and
2) condition (1.15) is that of the dual form.

Conditions (1.14) and (1.15) can be regarded as equations for a binary object that combines the pseudostructure and the exterior differential form defined on this pseudostructure. Such a binary object can be named Bi-Structure. The Bi-Structure combines both an algebraic object, namely, the closed exterior form, and a geometric one, namely, the pseudostructure as well.

In its properties this differential and geometrical structure is a well-known G-Structure. Here a new term Bi-Structure was introduced to distinguish it from G-structures to which there correspond closed inexact exterior forms.

The specific feature of this structure consists in the fact that it combines objects that possess duality. The closed exterior differential form and the closed dual form are such objects. The existence of one object implies that the other one exists as well. It does not seem to make sense to combine mathematically these two binary objects. However, this combined structure constitutes a unified whole that carries a double meaning. (This statement can be understand by analyzing the role of such structures in physical applications. This has been shown in the authors' works [12,13].)

From conditions (1.14) and (1.15) one can see that Bi-Stricture constitute a conservative object, namely, a quantity that is conservative on the pseudostructure. Hence, such an object (Bi-Structure) can correspond to some conservation law. (It is from such object that the physical fields and corresponding manifolds are formed.)
The properties and specific features of Bi-Structures will be presented in section 2. The evolutionary forms allow us to describe characteristics of these structures.

Invariant properties of closed exterior differential forms lie at the basis of mathematical apparatus of exterior forms. Below it will be described some specific features of the mathematical apparatus of exterior forms.

Operators of the theory of exterior differential forms

In differential calculus the derivatives are basic elements of the mathematical apparatus. By contrast, the differential is an element of mathematical apparatus of the theory of exterior differential forms. It enables one to analyze the conjugacy of derivatives in various directions, which extends potentialities of differential calculus.

The operator of exterior differential $d$ (exterior differential) is an abstract generalization of ordinary mathematical operations of the gradient, curl, and divergence in the vector calculus [7]. If, in addition to the exterior differential, we introduce the following operators: 1) $\delta$ for transformations that convert the form of $(p+1)$ degree into the form of $p$ degree, 2) $\delta'$ for cotangent transformations, 3) $\Delta$ for the $d\delta - \delta d$ transformation, 4) $\Delta'$ for the $d\delta' - \delta'd$ transformations, then in terms of these operators that act on the exterior differential forms one can write down the operators in the field theory equations. The operator $\delta$ corresponds to Green’s operator, $\delta'$ does to the canonical transformation operator, $\Delta$ does to the d’Alembert operator in 4-dimensional space, and $\Delta'$ corresponds to the Laplace operator [8,10]. It can be seen that the operators of the exterior differential form theory are connected with many operators of mathematical physics.

Identical relations of exterior differential forms

In the theory of exterior differential forms the closed forms that possess various types of conjugacy play a principal role. Since the conjugacy is a certain connection between two operators or mathematical objects, it is evident that, to express a conjugacy mathematically, it can be used relations. Just such relations constitute the basis of mathematical apparatus of the exterior differential forms. This is an identical relation.

Identical relations for exterior differential forms reflect the closure conditions of differential forms, namely, vanishing the form differential (see formulas (1.10), (1.14), (1.15)) and the conditions connecting the forms of consequent degrees (see formulas (1.12), (1.16)).

An importance of the identical relations for exterior differential forms is manifested by the fact that practically in all branches of physics, mechanics, thermodynamics one faces such identical relations.

One can present the following examples:

a) the Poincare invariant [5] $ds = -H \, dt + p_j \, dq_j$,

b) the second principle of thermodynamics [14] $dS = (dE + p \, dV)/T$, 

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c) the vital force theorem in theoretical mechanics: $dT = X_i dx^i$ where $X_i$ are components of potential force, and $T = mV^2/2$ is the vital force,

d) the conditions on characteristics [11] in the theory of differential equations, and so on.

A requirement that the function is an antiderivative (the integrand is a differential of a certain function) can be written in terms of such identical relation.

An existence of the harmonic function is written by means of the identical relation: the harmonic function is a closed form, that is a differential (a differential on the Riemann surface).

The identical relations in differential forms express the fact that each closed exterior form is a differential of some exterior form (with the degree less by one).

In general form such an identical relation can be written as

$$d_x \phi = \theta^p$$  \hspace{1cm} (1.19)

In this relation the form in the right-hand side has to be a closed one. (As it will be shown below, the identical relations are satisfied only on pseudostructures).

In identical relation (1.19) in one side it stands the closed form and in other side does a differential of some differential form of the less by one degree, which is a closed form as well.

In addition to relations in differential forms from the closure conditions of the differential forms and the conditions connecting the forms of consequent degrees the identical relations of other types are obtained. The types of such relations are presented below.

Integral identical relations.

The formulas by Newton, Leibnitz, Green, the integral relations by Stokes, Gauss-Ostrogradskii are examples of integral identical relations.

Tensor identical relations.

From relations that connect exterior forms of consequent degrees one can obtain the vector and tensor identical relations that connect operators of the gradient, curl, divergence and so on.

From the closure conditions of exterior and dual forms one can obtain identical relations such as the gauge relations in electromagnetic field theory, the tensor relations between connectednesses and their derivatives in gravitation (the symmetry of connectednesses with respect to lower indices, the Bianchi identity, the conditions imposed on the Christoffel symbols) and so on.

Identical relations between derivatives.

The identical relations between derivatives correspond to the closure conditions of exterior and dual forms. The examples of such relations are the above presented Cauchi-Riemann conditions in the theory of complex variables, the transversality condition in the variational calculus, the canonical relations in the Hamilton formalism, the thermodynamic relations between derivatives of thermodynamic functions [14], a condition that a derivative of implicit function is subjected to, the eikonal relations [15] and so on.

The above presented examples show that the identical relations for exterior differential forms occur in various branches of mathematics and physics.
Identical relations of exterior differential forms are the mathematical expression of various kinds of conjugacy that leads to closed exterior forms. They describe a conjugacy of any objects: the form elements, components of each element, exterior and dual forms, exterior forms of various degrees, and others. The identical relations, which are connected with different kinds of conjugacy, elucidate invariant, structural and group properties of exterior forms that are of great importance in applications.

A functional significance of identical relations for exterior differential forms lies in the fact that they can describe a conjugacy of objects of different mathematical nature. This enables one to see internal connections between various branches of mathematics. Due to these possibilities the exterior differential forms have wide application in various branches of mathematics. (Below a connection of exterior differential forms with different branches of mathematics will be demonstrated).

**Nondegenerate transformations**

One of the fundamental methods in the theory of exterior differential forms is application of nondegenerate transformations (below it will be said about degenerate transformations).

As it has been already noted, in the theory of exterior differential forms the nondegenerate transformations are those that conserve the differential. This is connected with the property of closed differential forms. Since a closed form is a differential (a total one, if the form is exact, or an interior one on pseudostructure, if the form is inexact), it is evident that the closed form turns out to be invariant under all transformations that conserve a differential.

The examples of nondegenerate transformations in the theory of exterior differential forms are unitary, tangent, canonical, gradient transformations.

To the nondegenerate transformations there are assigned closed forms of given degree. To the unitary transformations it is assigned (0-form), to the tangent and canonical transformations it is assigned (1-form), to the gradient transformations it is assigned (2-form) and so on. It should be noted that these transformations are gauge transformations for spinor, scalar, vector, tensor (3-form) fields.

The above listed transformations are fundamental nondegenerated transformations occurring in various branches of mathematics.

A connection between nondegenerate transformations and closed exterior forms disclose an internal commonness of nondegenerate transformations: all these transformations are transformations that preserve a differential.

One can see that nondegenerate transformations can be classified by a degree of corresponding closed differential or dial forms.

From description of operators of exterior differential forms one can see that those are operators that execute some transformations. All these transformations are connected with the above listed nondegenerate transformations of exterior differential forms.
The possibility to apply nondegenerate transformations shows that the exterior differential forms possess group properties. This extends the utilitarian potentialities of the exterior differential forms.

A significance of the nondegenerate transformations consists in the fact that they allow one to get new closed differential forms, which gives the opportunity to obtain new structures.

Thus, one can see that the skew-symmetric closed exterior differential forms possess invariant properties. The invariant properties are a result of conjugacy between some objects, to this it points out a closure of exterior differential forms.

With these properties there are connected peculiarities of the mathematical apparatus of closed exterior forms, the principal elements of which are nondegenerate transformations and identical relations.

These properties of closed exterior differential forms and peculiarities of their mathematical apparatus lie at the basis of practically all branches of mathematics.

Below we outline connections between the exterior differential forms and various branches of mathematics in order to show what role the exterior differential forms play in mathematics and to draw attention to their great potentialities.

**Connection between exterior differential forms and various branches of mathematics**

1. **Algebraic and geometrical properties of exterior differential forms**

   The basis of Cartan’s method of exterior forms, namely, the method of analyzing the system of differential equations and manifolds, is the basis of the Grassmann algebra (the exterior algebra) [5]. The mathematical apparatus of exterior differential forms extends the algebraic mathematical apparatus. The differential forms treated as elements of algebra allow studying the manifold structure, finding the manifold invariants. Group properties of exterior differential forms (a connection with the Lie groups) enable one to study the integrability of differential equations. They can make up the basis of the invariant field theory.

   The exterior differential forms elucidate an internal connection between algebra and geometry. From the definition of the closed inexact form it follows that the closed inexact form is a quantity that is conservative on pseudostructure. That is, the closed inexact form is a conjugacy of algebraic and geometrical approaches. The set of relations, namely, the closure condition for exterior form \( (d_x^\theta p = 0) \) and the closure condition for the dual form \( (d_x^*\theta p = 0) \) allow us to describe such a conjugacy. The closed form possesses algebraic (invariant) properties, and the closed dual form has geometrical (covariant) properties.

   The mathematical apparatus of exterior differential forms allows one to study the elements of interior geometry. This is a fundamental formalism in the differential geometry [2]. It allows one to investigate the manifold structure. Cartan developed the method of exterior forms for investigation of manifolds. It is known that in the case of differentiable manifolds the metrical and differential
characteristics are consistent ones. Such manifolds may be regarded as objects of interior geometry, i.e. there is a possibility to study their characteristics as the properties of the surface itself irrelevant to imbedding into space. Cartan’s structure equation is a key tool in studying the manifold structures and fiber spaces [2,3].

2. Theory of functions of complex variables
The residue method in the theory of analytical functions of complex variables is based on the integral theorems by Stokes and Cauchy-Poincare that allow us to replace the integral of closed form along any closed loop by the integral of this form along another closed loop that is homological to the first one [5].

As it was already noted, the harmonic function (a differential on the Riemann surface) is a closed exterior form:

\[ \theta = p dx + q dy, \quad d\theta = 0 \]

to which there corresponds the dual form:

\[ *\theta = -q dx + p dy, \quad d^*\theta = 0 \]

where \( d\theta = 0, \ d^*\theta = 0 \) are the Cauchy-Riemann relations.

3. Differential equations
On the basis of the theory of exterior differential forms the methods of studying the integrability of the system of differential equations, the Pfaff equations (the Frobenius theorem), and of finding the integral surfaces have been developed. This problem was considered in many works concerning exterior differential forms [2, 4, 6].

The operator \( d \) appears to be useful for expressing the integrability conditions for the systems of partial differential equations.

An example of applying the theory of exterior differential forms for analyzing integrability of differential equations and determining the functional properties of the solutions to these equations is presented in Appendix.

4. Connection with the differential and integral calculus
As it was already pointed out, the exterior differential forms were introduced for designation of integrand expressions that can form integral invariants [1].

The exterior differential forms are connected with multiple integrals (see, for example, [8]).

It should be mentioned that the closure property of the form \( f(x) dx \) indicates an existence of antiderivative of the function \( f(x) \).

Integral relations (1.17), from which it follows the formulas by Newton, Liebnitz, Green, Gauss-Ostrogradskii, Stokes were derived, are of great importance. Such potentials as the Newtonian one, the potentials of simple and double layers are integrals of closed inexact forms.

The exterior differential forms extend potentialities of the differential and integral calculus.

The theory of integral calculus establishes a connection between the differential form calculus and the homology of manifolds (cohomology).

5. Connection with tensors
As it is well-known, the exterior differential form is a skew-symmetric tensor field.

Information on a connection of the exterior differential forms with tensors can be found in [4,8,10,16].

The method of presentation of skew-symmetric tensors as differential forms extends possibilities of the mathematical apparatus based on these tensors. The tensors are known to be introduced as objects that are transformed according to some fixed rule under transforming the coordinates. The tensors are attached to the basis that can be transformed in an arbitrary way under transition to a new coordinate map. The exterior differential form is connected with differentials of the coordinates that vary according to the interior characteristics of the manifold under translation along manifold. Using differentials of coordinates instead of the base vectors enables one directly to make use of the integration and differentiation apparatus in physical applications. Instead of differentials of coordinates, a system of linearly-independent exterior one-forms can be chosen as the basis, and this makes the description independent of the choice of coordinate system [4, 10].

As it was pointed out, the operator of exterior differentiating \( d \) is an abstract generalization of the gradient, curl and divergence. This property elucidates a connection between the vectorial, algebraic and potential fields. A property of the exterior differential form, namely, the existence of differential and integral relations between the forms of sequential degrees, allows us to classify these fields according to the exterior form degree. This was shown for the three-dimensional Euclidean space.

Thus, even from this brief description of properties and specific features of the exterior differential forms and their mathematical apparatus one can clearly see their connection with such branches of mathematics as algebra, geometry, mathematical analysis, tensor analysis, differential geometry, differential equations, group theory, theory of transformations and so on. This is indicative of wide functional and utilitarian potentialities of exterior differential forms. One can show that, in essence, the field theory is based on invariant and structural properties of exterior differential forms that correspond to the conservation laws. The exterior differential forms allow us to see an internal connection between various branches of mathematics and physics.

The unique role of the exterior skew-symmetric differential forms in mathematics, as one can see, is connected with their invariant properties.

Below it will be shown that there exist skew-symmetric differential forms, which do not possess the invariant properties, nevertheless their mathematical apparatus occurs to be significantly wider. This is due to the fact that these differential forms, which were named evolutionary ones (since, as it will be shown below, they possess the evolutionary properties), can generate closed exterior differential forms. Such properties of the evolutionary differential forms allows a description of discrete transitions, quantum steps, generation of various structures and so on, which cannot be performed within the framework of the existing mathematical theories.
2 Evolutionary differential forms

The differential forms, which were named evolutionary ones and differ in their properties from the exterior differential forms, may be obtained while studying the problem of integrability of differential equations. This can be seen by the example of the first-order partial differential equation. Let

\[ F(x^i, u, p_i) = 0, \quad p_i = \frac{\partial u}{\partial x^i} \] (2.1)

be a partial differential equation of the first order. Let us consider the functional relation

\[ du = \theta \] (2.2)

where \( \theta = p_i \, dx^i \). Here \( \theta = p_i \, dx^i \) is the differential form of the first degree. The equation (2.1) will be integrable, if the functional relation is an identical one, namely, if the differential form \( \theta \) is a differential (a closed form). In the general case, from equation (2.1) it does not follow (explicitly) that the derivatives \( p_i = \frac{\partial u}{\partial x^i} \) that obey to the equation (and given boundary or initial conditions of the problem) make up a differential, that is, a closed exterior differential form. The form \( \theta = p_i \, dx^i \) appears to be an unclosed form and is not an identical relation. In Appendix it will be shown what is a role of such unclosed forms and nonidentical relations in qualitative investigation of functional properties of the solutions to differential equations.

The form \( \theta = p_i \, dx^i \) is an example of the evolutionary differential form. (The evolutionary properties of such a form are connected with the topological properties of this form commutator, which is nonzero).

Such differential forms originate while investigating the integrability of any differential equations that describe various processes [12,13].

The evolutionary differential forms, as well as the exterior forms, are skew-symmetric differential forms.

A radical distinction between the evolutionary forms and the exterior ones consists in the fact that the exterior differential forms are defined on manifolds with closed metric forms, whereas the evolutionary differential forms are defined on manifolds with unclosed metric forms.

Before going to description of evolutionary differential forms it should dwell on the properties of manifolds on which skew-symmetric differential forms are defined.

Some properties of manifolds

In the definition of exterior differential forms a differentiable manifold was mentioned. Differentiable manifolds are topological spaces that locally behave like Euclidean spaces [10,9]. But differentiable manifolds are not a single type of manifolds on which the exterior differential forms are defined. In the general case there are manifolds with structures of any types. The theory of exterior differential forms
was developed just for such manifolds. They may be the Hausdorff manifolds, fiber spaces, the comological, characteristic, configuration manifolds and so on. These manifolds and their properties are treated in [2,4,8] and in some other works. Since all these manifolds possess structures of any types, they have one common property, namely, locally they admit one-to-one mapping into the Euclidean subspaces and into other manifolds or submanifolds of the same dimension [9].

When describing any processes in terms of differential equations, one has to deal with manifolds, which do not allow one-to-one mapping described above. Such manifolds are, for example, manifolds formed by trajectories of elements of the system described by differential equations. The manifolds that can be called accompanying manifolds are variable deforming manifolds. The evolutionary differential forms can be defined on manifolds of this type.

What are the characteristic properties and specific features of deforming manifolds and of evolutionary differential forms connected with them?

To answer this question, let us analyze some properties of metric forms.

Assume that on the manifold one can set the coordinate system with base vectors $\mathbf{e}_\mu$ and define the metric forms of manifold [17]: $(\mathbf{e}_\mu, \mathbf{e}_\nu)$, $(\mathbf{e}_\mu dx^\mu)$, $(d\mathbf{e}_\mu)$. The metric forms and their commutators define the metric and differential characteristics of the manifold.

If metric forms are closed (the commutators are equal to zero), then the metric is defined $g_{\mu\nu} = (\mathbf{e}_\mu, \mathbf{e}_\nu)$ and the results of translation over manifold of the point $d\mathbf{M} = (\mathbf{e}_\mu dx^\mu)$ and of the unit frame $d\mathbf{A} = (d\mathbf{e}_\mu)$ prove to be independent of the curve shape (the path of integration).

The closed metric forms define the manifold structure, and the commutators of metric forms define the manifold differential characteristics that specify the manifold deformation: bending, torsion, rotation, twist.

It is evident that the manifolds, that are metric ones or possess the structure, have closed metric forms. It is with such manifolds that the exterior differential forms are connected.

If the manifolds are deforming manifolds, this means that their metric form commutators are nonzero. That is, the metric forms of such manifolds turn out to be unclosed. The accompanying manifolds and manifolds appearing to be deforming ones are examples of such manifolds.

The skew-symmetric evolutionary differential forms whose basis are deforming manifolds are defined on manifolds with unclosed metric forms.

Thus, the exterior differential forms are skew-symmetric differential forms defined on manifolds, submanifolds or on structures with closed metric forms. The evolutionary differential forms are skew-symmetric differential forms defined on manifolds with metric forms that are unclosed.

What are the characteristic properties and specific features of such manifolds and the related differential forms?

For description of the manifold differential characteristics and, correspondingly, the metric forms commutators one can use connectedness [2,4,5,17].

Let us consider the affine connectedness and their relations to commutators of metric forms.
The components of metric forms can be expressed in terms of connectedness \( \Gamma^\rho_{\mu \nu} \) [17]. The expressions \( \Gamma^\rho_{\mu \nu} \), \( (\Gamma^\rho_{\mu \nu} - \Gamma^\rho_{\nu \mu}) \), \( R^\mu_{\nu \rho \sigma} \) are components of the commutators of metric forms of zero-, first-, and third degrees. (The commutator of the second degree metric form is written down in a more complex manner [17], and therefore it is not given here).

As it is known [17], for the Euclidean manifold these commutators vanish identically. For the Riemann manifold the commutator of the third-degree metric form is nonzero:

\[
R^\mu_{\nu \rho \sigma} \neq 0.
\]

Commutators of metric form vanish in the case of manifolds that allow local one-to-one mapping into subspaces of the Euclidean space. In other words, the metric forms of such manifolds turn out to be closed.

The topological properties of manifolds are connected with the metric form commutators. The metric form commutators specify a manifold distortion. For example, the commutator of the zero degree metric form \( \Gamma^\rho_{\mu \nu} \) characterizes the bend, that of the first degree form \( (\Gamma^\rho_{\mu \nu} - \Gamma^\rho_{\nu \mu}) \) characterizes the torsion, the commutator of the third-degree metric form \( R^\mu_{\nu \rho \sigma} \) determines the curvature.

The evolutionary properties of differential forms are just connected with properties of the metric form commutators.

**Specific features of the evolutionary differential form**

Let us point out some properties of evolutionary forms and show in what their difference from exterior differential forms is manifested. The evolutionary differential form of degree \( p \) (\( p \)-form), as well as the exterior differential form, can be written down as

\[
\omega^p = \sum_{\alpha_1 \ldots \alpha_p} a_{\alpha_1 \ldots \alpha_p} dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \ldots \wedge dx^{\alpha_p} \quad 0 \leq p \leq n \quad (2.3)
\]

where the local basis obeys the condition of exterior multiplication

\[
\begin{align*}
dx^\alpha \wedge dx^\alpha &= 0 \\
dx^\alpha \wedge dx^\beta &= -dx^\beta \wedge dx^\alpha \quad \alpha \neq \beta
\end{align*}
\]

But the evolutionary form differential cannot be written similarly to that presented for exterior differential forms (see formula (1.3)). In the evolutionary form differential there appears an additional term connected with the fact that the basis of the form changes. For differential forms defined on the manifold with unclosed metric form one has \( d(dx^{\alpha_1} dx^{\alpha_2} \ldots dx^{\alpha_p}) \neq 0 \) (it should be noted that for differentiable manifold the following is valid: \( d(dx^{\alpha_1} dx^{\alpha_2} \ldots dx^{\alpha_p}) = 0 \)). For this reason a differential of the evolutionary form \( \omega^p \) can be written as

\[
d\omega^p = \sum_{\alpha_1 \ldots \alpha_p} da_{\alpha_1 \ldots \alpha_p} dx^{\alpha_1} dx^{\alpha_2} \ldots dx^{\alpha_p} + \sum_{\alpha_1 \ldots \alpha_p} a_{\alpha_1 \ldots \alpha_p} d(dx^{\alpha_1} dx^{\alpha_2} \ldots dx^{\alpha_p})
\]

(2.4)

where the second term is connected with a differential of the basis. The second term is expressed in terms of the metric form commutator. For the manifold with a closed metric form this term vanishes.
For example, we again inspect the first-degree form $\omega = a_\alpha dx^\alpha$. The differential of this form can be written as $d\omega = K_{\alpha \beta} dx^\alpha dx^\beta$, where $K_{\alpha \beta} = a_{\beta;\alpha} - a_{\alpha;\beta}$ are components of the commutator of the form $\omega$, and $a_{\beta;\alpha}$, $a_{\alpha;\beta}$ are the covariant derivatives. If we express the covariant derivatives in terms of the connectedness (if it is possible), then they can be written as $a_{\beta;\alpha} = \partial a_{\beta}/\partial x^\alpha + \Gamma^\sigma_{\beta\alpha} a_\sigma$, where the first term results from differentiating the form coefficients, and the second term results from differentiating the basis. (In the Euclidean space covariant derivatives coincide with ordinary ones since in this case derivatives of the basis vanish). If we substitute the expressions for covariant derivatives into the formula for the commutator components, we obtain the following expression for the commutator components of the form $\omega$:

$$K_{\alpha \beta} = \left( \frac{\partial a_{\beta}}{\partial x^\alpha} - \frac{\partial a_{\alpha}}{\partial x^\beta} \right) + (\Gamma^\sigma_{\beta\alpha} - \Gamma^\sigma_{\alpha\beta}) a_\sigma \quad (2.5)$$

Here the expressions $(\Gamma^\sigma_{\beta\alpha} - \Gamma^\sigma_{\alpha\beta})$ entered into the second term are just the components of commutator of the first-degree metric form.

That is, the corresponding metric form commutator will enter into the differential form commutator.

Thus, differentials and, correspondingly, the commutators of exterior and evolutionary forms are of different types.

**Unclosure of evolutionary differential forms**

The evolutionary differential form commutator, in contrast to that of the exterior one, cannot be equal to zero because it involves the metric form commutator being nonzero. This means that the evolutionary form differential is nonzero. Hence, the evolutionary differential form, in contrast to the case of the exterior form, cannot be closed.

The commutators of evolutionary forms depend not only on the evolutionary form coefficients but on the characteristics of manifolds, on which this form is defined, as well. As a result, such a dependence of the evolutionary form commutator produces the topological and evolutionary properties of both the commutator and the evolutionary form itself (this will be demonstrated below).

Since the evolutionary differential forms are unclosed, the mathematical apparatus of evolutionary differential forms does not seem to possess any possibilities connected with the algebraic, group, invariant and other properties of closed exterior differential forms. However, the mathematical apparatus of evolutionary forms includes some new unconventional elements.

Specific features of the mathematical apparatus of evolutionary differential forms are shown below.

**Nonidentical relations of evolutionary differential forms**

Above it was shown that the identical relations lie at the basis of the mathematical apparatus of exterior differential forms.
In contrast to this, nonidentical relations lie at the basis of the mathematical apparatus of evolutionary differential forms.

The identical relations of closed exterior differential forms reflect a conjugacy of any objects. The evolutionary forms, being unclosed, cannot directly describe a conjugacy of any objects. But they allow description of the process in which the conjugacy may appear (the process when closed exterior differential forms are generated). Such a process is described by nonidentical relations.

The concept of “nonidentical relation” may appear to be inconsistent. However, it has a deep meaning.

The identical relations establish exact correspondence between the quantities (or objects) involved into this relation. It is possible in the case when the quantities involved into the relation are measurable ones. [A quantity is called a measurable quantity if its value does not change under transition to another, equivalent, coordinate system. In other words, this quantity is invariant one.] In the nonidentical relations one of the quantities is unmeasurable. (Nonidentical relations with two unmeasurable quantities are meaningless). If this relation is evolutionary one, it turns out to be a selfvarying relation, namely, a variation of some object leads to variation of other one, and in turn a variation of the second object leads to variation of the first and so on. Since in the nonidentical relation one of the objects is a unmeasurable quantity, the other object cannot be compared with the first, and therefore, the process cannot stop. Here the specific feature is that in the process of such selfvarying it may be realized the additional conditions under which the identical relation can be obtained from nonidentical relation. The additional condition may be realized spontaneously while selfvarying the nonidentical relation if the system possesses any symmetry. When such additional relations are realized the exact correspondence between the quantities involved in the relation is established. Under the additional condition a unmeasurable quantity becomes a measurable quantity as well, and the exact correspondence between the objects involved in the relation is established. That is, an identical relation can be obtained from a nonidentical relation.

The nonidentical relation is a relation between a closed exterior differential form, which is a differential and is a measurable quantity, and an evolutionary form, which is an unmeasurable quantity.

Nonidentical relations of such type appear in descriptions of any processes (see, for example, functional relation (2.2) built of derivatives of differential equation). These relations may be written as

\[ d\psi = \omega^p \]  

Here \( \omega^p \) is the \( p \)-degree evolutionary form that is nonintegrable, \( \psi \) is some form of degree \( (p - 1) \), and the differential \( d\psi \) is a closed form of degree \( p \).

Nonidentical relations obtained while describing any processes are evolutionary ones.

In the left-hand side of this relation it stands the form differential, i.e. a closed form that is an invariant object. In the right-hand side it stands the nonintegrable unclosed form that is not an invariant object. Such a relation cannot be identical.
One can see a difference of relations for exterior forms and evolutionary ones. In the right-hand side of identical relation (see relation (1.19)) it stands a closed form, whereas the form in the right-hand side of nonidentical relation (2.6) is an unclosed one.

How is this relation obtained?

Let us consider this by the example of the first degree differential forms. A differential of the function of more than one variables can be an example of the first degree form. In this case the function itself is the exterior form of zero degree. The state function that specifies a state of material system can serve as an example of such function. When the physical processes in a material system are being described, the state function may be unknown, but its derivatives may be known. The values of the function derivatives may be equal to some expressions that are obtained from the description of a real physical process. And it is necessary to find the state function.

Assume that $\psi$ is the desired state function that depends on the variables $x^\alpha$ and also assume that its derivatives in various directions are known and equal to some quantities $a_\alpha$, namely:

$$\frac{\partial \psi}{\partial x^\alpha} = a_\alpha \quad (2.7)$$

Let us set up the differential expression $(\partial \psi/\partial x^\alpha)dx^\alpha$. This differential expression is equal to

$$\frac{\partial \psi}{\partial x^\alpha}dx^\alpha = a_\alpha dx^\alpha \quad (2.8)$$

Here the left-hand side of the expression is a differential of the function $d\psi$, and in the right-hand side it stands the differential form $\omega = a_\alpha dx^\alpha$. Relation (2.8) can be written as

$$d\psi = \omega \quad (2.9)$$

It is evident that relation (2.9) is of the same type as (2.6) under the condition that the differential form degrees are equal to 1.

This relation is nonidentical because the differential form $\omega$ is an unclosed differential form. The commutator of this form is nonzero since the expressions $a_\alpha$ for the derivatives $(\partial \psi/\partial x^\alpha)$ are nonconjugated quantities. They are obtained from the description of an actual physical process and are unmeasurable quantities. (While finding the state function it is commonly assumed that its derivatives are conjugated quantities, that is, their mixed derivatives are commutative. But in physical processes the expressions for these derivatives are usually obtained independently of one another. And they appear to be unmeasurable quantities, and hence they are not conjugated).

One can come to relation (2.9) by means of analyzing the integrability of the partial differential equation. An equation is integrable if it can be reduced to the form $d\psi = dU$. However it appears that, if the equation is not subjected to an additional condition (the integrability condition), it is reduced to the form (2.9), where $\omega$ is an unclosed form and it cannot be expressed as a differential. The first principle of thermodynamics is an example of nonidentical relation.
It arises a question of \textit{how to work with nonidentical relation?}

Two different approaches are possible.

The first, evident, approach is to find a condition under which the nonidentical relation becomes identical and to obtain a closed form under this condition. In other words, the nonidentical relation is subjected the condition, under which this relation is transformed into an identical relation (if it is allowed).

Such an approach is traditional and is always used implicitly. It may be shown that additional conditions are imposed on the mathematical physics equations obtained in description of the physical processes so that these equations should be invariant (integrable) or should have invariant solutions.

\begin{quote}
Here a psychological point should be noted. While investigating real physical processes one often faces the relations that are nonidentical. But it is commonly believed that only identical relations can have any physical meaning. For this reason one immediately attempts to impose a condition onto the nonidentical relation under which this relation becomes identical, and it is considered only that this relation can satisfy the additional conditions. And all remaining is rejected. It is not taken into account that a nonidentical relation is often obtained from a description of some physical process and it has physical meaning at every stage of the physical process rather than at the stage when the additional conditions are satisfied. In essence, the physical process does not considered completely. At this point it should be emphasized that the nonidentity of the evolutionary relation does not mean the imperfect accuracy of the mathematical description of a physical process. The nonidentical relations are indicative of specific features of the physical process development.\end{quote}

This approach does not solve the evolutionary problem.

Below we present the evolutionary approach to investigation of the nonidentical relation. At the basis of this approach it lies a peculiarity of the relation that is a nonidentical evolutionary relation. Such a relation is a selfvarying relation.

\textbf{Selfvariation of the evolutionary nonidentical relation}

The evolutionary nonidentical relation is selfvarying, because, firstly, it is nonidentical, namely, it contains two objects one of which appears to be unmeasurable, and, secondly, it is an evolutionary relation, namely, a variation of any object of the relation in some process leads to variation of another object and, in turn, a variation of the latter leads to variation of the former. Since one of the objects is an unmeasurable quantity, the other cannot be compared with the first one, and hence, the process of mutual variation cannot stop.

Varying the evolutionary form coefficients leads to varying the first term of the evolutionary form commutator (see (2.5)). In accordance with this variation it varies the second term, that is, the metric form of manifold varies. Since the metric form commutators specifies the manifold differential characteristics, which are connected with the manifold deformation (as it has been pointed out, the commutator of the zero degree metric form specifies the bend, that of second degree specifies various types of rotation, that of the third degree specifies the
curvature), this points to the manifold deformation. This means that it varies the evolutionary form basis. In turn, this leads to variation of the evolutionary form, and the process of intervariation of the evolutionary form and the basis is repeated. Processes of variation of the evolutionary form and the basis are governed by the evolutionary form commutator and it is realized according to the evolutionary relation.

Selfvariation of the evolutionary relation is executed by exchange between the evolutionary form coefficients and the manifold characteristics. (This may be, for example, a mutual exchange between physical quantities and space-time characteristics, or between algebraic and geometrical characteristics.) This is an exchange between quantities of different nature.

The process of the evolutionary relation selfvariation cannot come to the end. This is indicated by the fact that both the evolutionary form commutator and the evolutionary relation involve unmeasurable quantities.

The question arises whether the identical relation can be obtained from this nonidentical selfvarying relation, which would allow determination of the desired differential form under the differential sign.

It appears that it is possible under the degenerate transformation.

**Degenerate transformations.**

To obtain the identical relation from the evolutionary nonidentical relation, it is necessary that a closed exterior differential form should be derived from the evolutionary differential form that is included into evolutionary relation.

However, as it was shown above, the evolutionary form cannot be a closed form. For this reason a transition from the evolutionary form is possible only to an *inexact* closed exterior form that is defined on pseudostructure.

To the pseudostructure there corresponds a closed dual form (whose differential vanishes). For this reason a transition from the evolutionary form to a closed inexact exterior form proceeds only when the conditions of vanishing the dual form differential are realized, in other words, when the metric form differential or commutator becomes equal to zero.

Conditions of vanishing the dual form differential (additional conditions) determine the closed metric form and thereby specify the pseudostructure (the dual form). In this case the closed exterior (*inexact*) form is formed.

Since the evolutionary form differential is nonzero, whereas the closed exterior form differential is zero, a passage from the evolutionary form to the closed exterior form is allowed only under *degenerate transformation*. The conditions of vanishing the dual form differential (an additional condition) are the conditions of degenerate transformation.

At this point it should be emphasized that differential, which equals zero, is an interior one. The evolutionary form commutator becomes to be zero only on the pseudostructure. The total evolutionary form commutator is nonzero. That is, under degenerate transformation the evolutionary form differential vanishes only *on pseudostructure*. The total differential of the evolutionary form is nonzero. The evolutionary form remains to be unclosed.

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The conditions of degenerate transformation (additional conditions) can be realized, for example, if it will appear any symmetries of the evolutionary form coefficients or its commutator. This can happen under selfvariation of the nonidentical relation. (While describing material system such additional conditions are related, for example, to degrees of freedom of the material system). Mathematically to the conditions of degenerate transformation there corresponds a requirement that some functional expressions become equal to zero. Such functional expressions are Jacobians, determinants, the Poisson brackets, residues, and others.

**Obtaining identical relation from nonidentical one**

Let us consider nonidentical evolutionary relation (2.6).

As it has been already mentioned, the evolutionary differential form $\omega^p$, involved into this relation is an unclosed one. The commutator, and hence the differential, of this form is nonzero. That is,

$$d\omega^p \neq 0$$  \hspace{1cm} (2.10)

If the transformation is degenerate, from the unclosed evolutionary form it can be obtained a differential form closed on pseudostructure. The differential of this form equals zero. That is, it is realized the transition

$$d\omega^p \neq 0 \rightarrow (\text{degenerate transformation}) \rightarrow d_\pi \omega^p = 0, \ d_\pi^* \omega^p = 0$$

On the pseudostructure $\pi$ evolutionary relation (2.6) transforms into the relation

$$d_\pi \psi = \omega_\pi^p$$  \hspace{1cm} (2.11)

which proves to be the identical relation. Indeed, since the form $\omega_\pi^p$ is a closed one, on the pseudostructure it turns out to be a differential of some differential form. In other words, this form can be written as $\omega_\pi^p = d_\pi \theta$. Relation (2.11) is now written as

$$d_\pi \psi = d_\pi \theta$$

There are differentials in the left-hand and right-hand sides of this relation. This means that the relation is an identical one.

The evolutionary relation (2.6) becomes identical on the pseudostructure. From this relation one can find the desired differential $d_\pi \psi$ that is equal to the closed exterior differential form derived. One can obtain the desired form $\psi$ from this differential.

It can be shown that all identical relations of the exterior differential form theory are obtained from nonidentical relations (that contain the evolutionary forms) by applying degenerate transformations.

Transition from nonidentical relation (2.6) to identical relation (2.11) means the following. Firstly, it is from such a relation that one can obtain the differential $d_\pi \psi$ and find the desired function $\psi_\pi$ (the potential). And, secondly, an emergence of the closed (on pseudostructure) inexact exterior form $\omega_\pi^p$ (right-hand side of relation (2.11)) points to an origination of the conservative object.
This object is a conservative quantity (the closed exterior form $\omega^p$) on the pseudostructure (the dual form $\ast \omega^p$, which defines the pseudostructure). This object is an example of the differential and geometrical structure (G-Structure). Such objects were already described in section 1 and were named as Bi-Structure.

This complex is a new conjugated object. Below it will be shown a relation between characteristics of these objects and characteristics of the evolutionary differential forms.

Thus, the mathematical apparatus of evolutionary differential forms describes a process of generation of the closed inexact exterior differential forms, and this discloses a process of origination of a new conjugated object.

The evolutionary process of obtaining the identical relation from the nonidentical one and obtaining a closed (inexact) exterior form from the unclosed evolutionary form describes a process of conjugating any objects.

**Transition from nonconjugated operators to conjugated operators**

In section 1 it has been shown that the condition of the closure of exterior differential forms is a result of the conjugacy of any constituents of the exterior or dual forms (the form elements, components of each element, exterior and dual forms, exterior forms of various degrees, and others). Conversely, if there is any closed form, this points to a presence of one or other type of conjugacy.

The conjugacy of any objects is a conjugacy of one or other type connected with the exterior differential forms. To the conjugated objects, including operators, there are assigned some closed exterior or dual form.

Since the identical relations of exterior differential forms is a mathematical record of the closure conditions of exterior differential forms and, correspondingly, of conjugacy of any objects, the process of obtaining the identical relation from nonidentical one (selfmodification of the nonidentical evolutionary relation and degenerate transformation) is a process of conjugating any objects.

It can be seen that the process of conjugating the objects is a mutual exchange between the quantities of different nature (for example, between the algebraic and geometric quantities, between the physical and spatial quantities) and vanishing some functional expressions (Jacobians, determinates and so on). This follows from the fact that a selfvariation of the nonidentical evolutionary relation and a transition from the nonidentical evolutionary relation to identical one develop as a result of mutual variations of the evolutionary form coefficients (which have the algebraic nature) and the manifold characteristics (which have the geometric nature), and a realization of the degenerate transformation with obeying additional conditions.

The evolutionary differential form is an unclosed form, that is, it is the form whose differential is not equal to zero. The differential of the exterior differential form equals zero. To the closed exterior form there correspond conjugated operators, whereas to the evolutionary form there correspond nonconjugated operators. A transition from the evolutionary form to the closed exterior form is that from nonconjugated operators to conjugated ones. This is expressed
mathematically as a transition from a nonzero differential (the evolutionary form differential is nonzero) to a differential that equals zero (the closed exterior form differential equals zero). This is effected as a transition from one coordinate system to another (nonequivalent) coordinate system.

Since the conjugated objects, to which there correspond the closed exterior forms and the identical relations, are obtained from the evolutionary differential forms and the nonidentical relations, it is evident that the characteristics of conjugated objects are defined by characteristics of the evolutionary differential form commutators, which control the evolutionary process. The evolutionary forms enable us to describe the characteristics of Bi-Structure originated.

**Characteristics of Bi-Structure**

Since the closed exterior differential form, which corresponds to the Bi-Structure arisen, was obtained from the nonidentical relation that involves the evolutionary form, it is evident that the Bi-Structure characteristics must be connected with those of the evolutionary form and of the manifold on which this form is defined, with the conditions of degenerate transformation and with the values of commutators of the evolutionary form and the manifold metric form.

While describing the mechanism of Bi-Structure origination one can see that at the instant when the Bi-Structure originates there appear the following typical functional expressions and quantities: (1) the condition of degenerate transformation, i.e. vanishing of the interior commutator of the metric form; (2) vanishing of the interior commutator of the evolutionary form; (3) the value of the nonzero total commutator of the evolutionary form that involves two terms, namely, the first term is composed of the derivatives of the evolutionary form coefficients, and the second term is composed of the derivatives of the coefficients of the dual form that is connected with the manifold (here we deal with a value that the evolutionary form commutator assumes at the instant of Bi-Structure origination). They determine the following characteristics of the Bi-Structure. The conditions of degenerate transformation, as it was said before, determine the pseudostructures. The first term of the evolutionary form commutator determines the value of the discrete change (the quantum), which the quantity conserved on the pseudostructure undergoes at the transition from one pseudostructure to another. The second term of the evolutionary form commutator specifies a characteristics that fixes the character of the initial manifold deformation, which took place before the Bi-Structure arose. (Spin is such an example).

A discrete (quantum) change of a quantity proceeds in the direction that is normal (more exactly, transverse) to the pseudostructure. Jumps of the derivatives normal to the potential surfaces are examples of such changes.

Bi-Structure may carry a physical meaning. Such binary objects are the physical structures from which the physical fields are formed. This has been shown by the author in the works [12,13].

A connection of Bi-Structure with the skew-symmetric differential forms allows to introduce a classification of Bi-structure in dependence on parameters...
that specify the skew-symmetric differential forms and enter into nonidentical and identical relation of the skew-symmetric differential forms. To determine these parameters one has to consider the problem of integration of the nonidentical evolutionary relation.

Integration of the nonidentical evolutionary relation

Under degenerate transformation from the nonidentical evolutionary relation one obtains a relation being identical on pseudostructure. Since the right-hand side of such a relation can be expressed in terms of differential (as well as the left-hand side), one obtains a relation that can be integrated, and as a result he obtains a relation with the differential forms of less by one degree.

The relation obtained after integration proves to be nonidentical as well.

The resulting nonidentical relation of degree \((p-1)\) (relation that contains the forms of the degree \((p-1)\)) can be integrated once again if the corresponding degenerate transformation has been realized and the identical relation has been formed.

By sequential integrating the evolutionary relation of degree \(p\) (in the case of realization of the corresponding degenerate transformations and forming the identical relation), one can get closed (on the pseudostructure) exterior forms of degree \(k\), where \(k\) ranges from \(p\) to 0.

In this case one can see that under such integration closed (on the pseudostructure) exterior forms, which depend on two parameters, are obtained. These parameters are the degree of evolutionary form \(p\) (in the evolutionary relation) and the degree of created closed forms \(k\).

In addition to these parameters, another parameter appears, namely, the dimension of space. If the evolutionary relation generates the closed forms of degrees \(k = p, k = p-1, \ldots, k = 0\), to them there correspond the pseudostructures of dimensions \((N-k)\), where \(N\) is the space dimension. \{It is known that to the closed exterior differential forms of degree \(k\) there correspond skew-symmetric tensors of rank \(k\) and to corresponding dual forms there do the pseudotensors of rank \((N-k)\), where \(N\) is the space dimensionality. The pseudostructures correspond to such tensors, but only on the space formed.\}

The properties of pseudostructures and closed exterior forms.
Forming fields and manifolds

As mentioned before, the additional conditions, namely, the conditions of degenerate transformation, specify the pseudostructure. But at every stage of the evolutionary process it is realized only one element of pseudostructure, namely, a certain minipseudostructure. The additional conditions determine a direction (a derivative of the function that specifies the pseudostructure) on which the evolutionary form differential vanishes. (However, in this case the total differential of the evolutionary form is nonzero). The closed exterior form is formed along this direction.
While varying the evolutionary variable the minipseudostructures form the pseudostructure.

The example of minipseudostructure is the wave front. The wave front is the eikonal surface (the level surface), i.e. the surface with a conservative quantity. A direction that specifies the pseudostructure is a connection between the evolutionary and spatial variables. It gives the rate of changing the spatial variables. Such a rate is a velocity of the wave front translation. While its translation the wave front forms the pseudostructure.

Manifolds with closed metric forms are formed by pseudostructures. They are obtained from manifolds with unclosed metric forms. In this case the initial manifold (on which the evolutionary form is defined) and the formed manifold with closed metric forms (on which the closed exterior form is defined) are different spatial objects.

It takes place a transition from the initial manifold with unclosed metric form to the pseudostructure, namely, to the created manifold with closed metric forms. Mathematically this transition (degenerate transformation) proceeds as a transition from one frame of reference to another, nonequivalent, frame of reference.

The pseudostructures, on which the closed *inexact* forms are defined, form the pseudomanifolds. (Integral surfaces, pseudo-Riemann and pseudo-Euclidean spaces are the examples of such manifolds). In this process dimensions of formed manifolds are connected with the evolutionary form degree.

To transition from pseudomanifolds to metric manifolds there corresponds a transition from closed *inexact* differential forms to *exact* exterior differential forms. (Euclidean and Riemann spaces are examples of metric manifolds).

Here it is to be noted that the examples of pseudometric spaces are potential surfaces (surfaces of a simple layer, a double layer and so on). In these cases the type of potential surfaces is connected with the above listed parameters.

Since the closed metric form is dual with respect to some closed exterior differential form, the metric forms cannot become closed by themselves, independently of the exterior differential form. This proves that manifolds with closed metric forms are connected with the closed exterior differential forms. This indicates that the fields of conservative quantities are formed from closed exterior forms at the same time when the manifolds are created from the pseudostructures. (The specific feature of the manifolds with closed metric forms that have been formed is that they can carry some information.) That is, the closed exterior differential forms and manifolds, on which they are defined, are mutually connected objects. On the one hand, this shows duality of these two objects (the pseudostructure and the closed inexact exterior form), and, on the other hand, this means that these objects constitute a unified whole. This whole is a new conjugated object (Bi-Structure).

**Summary.**

In section 1 it had been described invariant properties of skew-symmetric closed exterior differential forms and was shown that due to their properties the
closed exterior forms play a fundamental role in various branches of mathematics.

In section 2 we showed that besides the exterior skew-symmetric differential forms, which possess invariant properties, there are skew-symmetric differential forms, which possess the evolutionary properties (which, for this reason, have been named the evolutionary forms).

By comparing the exterior and evolutionary forms one can see that they possess the opposite properties. At the same time the exterior and evolutionary forms constitute a unified whole: evolutionary differential forms generate the closed exterior differential forms.

The mathematical apparatus of exterior and evolutionary skew-symmetric differential forms constitute a new closed mathematical apparatus that possesses the unique properties. It includes new, unconventional, elements: "nonidentical relation", "degenerate transformation", "transition from one frame of reference to another, nonequivalent, frame of reference". This allows to create the mathematical language that has radically new abilities.

Due to their properties, the skew-symmetric differential forms enable one to see the internal connection between various branches of mathematics. Many foundations of the mathematical apparatus of skew-symmetric differential forms presented in this work may turn out to be of great importance for various sections of mathematics. Identical and nonidentical relations, nondegenerate and degenerate transformations, transitions from nonidentical relations to identical ones, transition from nonconjugated operators to conjugated operators, a possibility to describe an origination of structure and formation of fields and manifolds, and other foundations and potentialities of the mathematical apparatus of skew-symmetric differential forms presented may find many applications in such branches of mathematics as the qualitative theory of differential and integral equations, differential geometry and topology, theory of functions, theory of series, theory of numbers and others.

Due to their properties skew-symmetric differential forms have a great significance in applications. In the works by the author [12,13] it has been shown that the apparatus of skew-symmetrical differential forms discloses a mechanism of originating the physical structures and forming the physical fields, and also explains the causality of these processes.

Below we present the example of application of the skew-symmetric differential forms for qualitative studying solutions of differential equations.

Appendix

Application of the mathematical apparatus of the skew-symmetric differential forms to qualitative investigation of functional properties of the solutions to differential equations

The presented method of investigating the solutions to differential equations is not new. Such an approach was developed by Cartan [2] in his analysis of the integrability of differential equations. Here this approach is presented to demonstrate a role of exterior and evolutionary differential forms.
A role of exterior differential forms in the qualitative investigation of the solutions to differential equations is conditioned by the fact that the mathematical apparatus of these forms enables one to determine the conditions of consistency for various elements of differential equations or for the system of differential equations. This enables one, for example, to define the consistence of the partial derivatives in the partial differential equations, the consistence of the differential equations in the system of differential equations, the conjugacy of the function derivatives and of the initial data derivatives in ordinary differential equations and so on. The functional properties of the solutions to differential equations are just depend on whether or not the conjugacy conditions are satisfied.

The basic idea of the qualitative investigation of the solutions to differential equations can be clarified by the example of the first-order partial differential equation.

Let
\[ F(x^i, u, p_i) = 0, \quad p_i = \frac{\partial u}{\partial x^i} \] (A.1)

be the partial differential equation of the first order. Let us consider the functional relation
\[ du = \theta \] (A.2)

where \( \theta = p_i dx^i \) (the summation over repeated indices is implied). Here \( \theta = p_i dx^i \) is the differential form of the first degree.

The specific feature of functional relation (A.2) is that in the general case this relation turns out to be nonidentical.

The left-hand side of this relation involves a differential, and the right-hand side includes the differential form \( \theta = p_i dx^i \). For this relation to be identical, the differential form \( \theta = p_i dx^i \) must be a differential as well (like the left-hand side of relation (A.2)), that is, it has to be a closed exterior differential form. To do this it requires the commutator \( K_{ij} = \frac{\partial p_j}{\partial x^i} - \frac{\partial p_i}{\partial x^j} \) of the differential form \( \theta \) has to vanish.

In the general case, from equation (A.1) it does not follow (explicitly) that the derivatives \( p_i = \frac{\partial u}{\partial x^i} \) that obey to the equation (and given boundary or initial conditions of the problem) make up a differential. Without any supplementary conditions the commutator of the differential form \( \theta \) defined as \( K_{ij} = \frac{\partial p_j}{\partial x^i} - \frac{\partial p_i}{\partial x^j} \) is not equal to zero. The form \( \theta = p_i dx^i \) proves to be unclosed and is not a differential like the left-hand side of relation (A.2). The functional relation (A.2) appears to be nonidentical: the left-hand side of this relation is a differential, but the right-hand side is not a differential.

Functional relation (A.2) is an example of nonidentical evolutionary relation.

The nonidentity of functional relation (A.2) points to a fact that without additional conditions derivatives of the initial equation do not make up a differential. This means that the corresponding solution to the differential equation \( u \) will not be a function of \( x^i \). It will depend on the commutator of the form \( \theta \), that is, it will be a functional.

To obtain the solution that is the function (i.e., derivatives of this solution form a differential), it is necessary to add the closure condition for the form
\[ \theta = p_i dx^i \] and for the dual form (in the present case the functional \( F \) plays a role of the form dual to \( \theta \)) [2]:

\[
\begin{align*}
\{ dF(x^i, u, p_i) &= 0 \\
 d(p_i dx^i) &= 0 \quad \text{(A.3)}
\end{align*}
\]

If we expand the differentials, we get a set of homogeneous equations with respect to \( dx^i \) and \( dp_i \) (in the \( 2n \)-dimensional space – initial and tangential):

\[
\begin{align*}
&\left( \frac{\partial F}{\partial x^i} + \frac{\partial F}{\partial u} p_i \right) dx^i + \frac{\partial F}{\partial p_i} dp_i = 0 \\
&dp_i dx^i - dx^i dp_i = 0 \quad \text{(A.4)}
\end{align*}
\]

The solvability conditions for this set (vanishing of the determinant composed of coefficients at \( dx^i, dp_i \)) have the form:

\[
\frac{dx^i}{\partial F/\partial p_i} = \frac{-dp_i}{\partial F/\partial x^i + p_i \partial F/\partial u} \quad \text{(A.5)}
\]

These conditions determine an integrating direction, namely, a pseudostructure, on which the form \( \theta = p_i dx^i \) turns out to be closed one, i.e. it becomes a differential, and from relation (A.2) the identical relation is produced. If conditions (A.5), that may be called the integrability conditions, are satisfied, the derivatives constitute a differential \( \delta u = p_i dx^i = du \) (on the pseudostructure), and the solution becomes a function. Just such solutions, namely, functions on the pseudostructures formed by the integrating directions, are the so-called generalized solutions [19]. The derivatives of the generalized solution constitute the exterior form that is closed on the pseudostructure.

(If conditions (A.5) are not satisfied, that is, the derivatives do not form a differential, the solution that corresponds to such derivatives will depend on the differential form commutator constructed of derivatives. That means that the solution is a functional rather than a function.)

Since the functions that are the generalized solutions are defined only on the pseudostructures, they have discontinuities in derivatives in the directions that are transverse to the pseudostructures. The order of derivatives with discontinuities is equal to the exterior form degree. If the form of zero degree is involved in the functional relation, the function itself, being a generalized solution, will have discontinuities.

If we find the characteristics of equation (A.1), it appears that conditions (A.5) are the equations for characteristics [11]. That is, the characteristics are examples of the pseudostructures on which derivatives of the differential equation constitute the closed forms and the solutions prove to be the functions (generalized solutions). (The characteristic manifolds of equation (A.1) are the pseudostructures \( \pi \) on which the form \( \theta = p_i dx^i \) becomes a closed form: \( \theta_\pi = du_\pi \)).

Here it is worth noting that coordinates of the equations for characteristics are not identical to independent coordinates of initial space on which equation
(A.1) is defined. A transition from the initial space to the characteristic manifold appears to be a degenerate transformation, namely, the determinant of the set of equations (A.4) becomes zero. The derivatives of equation (A.1) are transformed from the tangent space to the cotangent one. The transition from the tangent space, where the commutator of the form $\theta$ is nonzero (the form is unclosed, the derivatives do not form a differential), to the characteristic manifold, namely, the cotangent space, where the commutator becomes equal to zero (the closed exterior form is formed, i.e. the derivatives form a differential), is the example of the degenerate transformation.

A partial differential equation of the first order has been analyzed, and the functional relation with the form of the first degree analogous to the evolutionary form has been considered.

Similar functional properties have the solutions to all differential equations. And, if the order of the differential equation is $k$, the functional relation with the $k$-degree form corresponds to this equation. For ordinary differential equations the commutator is produced at the expense of the conjugacy of derivatives of the functions desired and those of the initial data (the dependence of the solution on the initial data is governed by the commutator).

In a similar manner one can also investigate the solutions to a set of partial differential equations and the solutions to ordinary differential equations (for which the nonconjugacy of desired functions and initial conditions is examined).

It can be shown that the solutions to equations of mathematical physics, on which no additional external conditions are imposed, are functionals. The solutions prove to be exact only under realization of the additional requirements, namely, the conditions of degenerate transformations: vanishing determinants, Jacobians and so on, that define the integral surfaces. The characteristic manifolds, the envelopes of characteristics, singular points, potentials of simple and double layers, residues and others are the examples of such surfaces.

Here the mention should be made of the generalized Cauchy problem when the initial conditions are given on some surface. The so called “unique” solution to the Cauchy problem, when the output derivatives can be determined (that is, when the determinant built of the expressions at these derivatives is nonzero), is a functional since the derivatives obtained in such a way prove to be nonconjugated, that is, their mixed derivatives form a commutator with nonzero value, and the solution depends on this commutator.

The dependence of the solution on the commutator may lead to instability of the solution. Equations that do not possess the integrability conditions (the conditions such as, for example, the characteristics, singular points, integrating factors and others) may have the unstable solutions. Unstable solutions appear in the case when the additional conditions are not realized and no exact solutions (their derivatives form a differential) are formed. Thus, the solutions to the equations of the elliptic type may be unstable.

Investigation of nonidentical functional relations lies at the basis of the qualitative theory of differential equations. It is well known that the qualitative theory of differential equations is based on the analysis of unstable solutions and integrability conditions. From the functional relation it follows that the
dependence of the solution on the commutator leads to instability, and the closure conditions of the forms constructed by derivatives are the integrability conditions. One can see that the problem of unstable solutions and integrability conditions appears, in fact, to be reduced to the question under what conditions the identical relation for the closed form is produced from the nonidentical relation that corresponds to the relevant differential equation (the relation such as (A.2)), the identical relation for the closed form is produced. In other words, whether or not the solutions are functionals? This is to the same question that the analysis of the correctness of setting the problems of mathematical physics is reduced.

Here the following should be emphasized. When the degenerate transformation from the initial nonidentical functional relation is fulfilled, an integrable identical relation is obtained. As a result of integrating, one obtains a relation that contains exterior forms of less by one degree and which once again proves to be (in the general case without additional conditions) nonidentical. By integrating the functional relations sequentially obtained (it is possible only under realization of the degenerate transformations) from the initial functional relation of degree \( k \) one can obtain \((k + 1)\) functional relations each involving exterior forms of one of degrees: \( k, k - 1, \ldots, 0 \). In particular, for the first-order partial differential equation it is also necessary to analyze the functional relation of zero degree.

Thus, application of the exterior differential forms allows one to reveal the functional properties of the solutions to differential equations.

It is evident that, for solutions to the differential equation be generalized solutions (i.e. solutions whose derivatives form a differential, namely, the closed form), the differential equation has to be subject the additional conditions. Clearly, only generalized solutions can correspond to various structures including physical structures. Let us consider what equations are obtained in this case.

Return to equation (A.1).

Assume that it does not explicitly depend on \( u \) and is solved with respect to some variable, for example \( t \), that is, it has the form of

\[
\frac{\partial u}{\partial t} + E(t, x^j, p_j) = 0, \quad p_j = \frac{\partial u}{\partial x^j} \tag{A.6}
\]

Then integrability conditions (A.5) (the closure conditions of the differential form \( \theta = p_idx^i \) and the corresponding dual form) can be written as (in this case \( \partial F/\partial p_1 = 1 \))

\[
\frac{dx^j}{dt} = \frac{\partial E}{\partial p_j}, \quad \frac{dp_j}{dt} = -\frac{\partial E}{\partial x^j} \tag{A.7}
\]

These are the characteristic relations for equation (A.6). As it is well known, the canonical relations have just such a form.

As a result we conclude that the canonical relations are the characteristics of equation (A.6) and the integrability conditions for this equation.

The canonical relations that are obtained from the closure condition of the differential form \( \theta = p_idx^i \) and the corresponding dual form, are the examples of the identical relation of the theory of exterior differential forms.
The conjugated coordinates \( x^j, p_j \) that obey the canonical relations can be put into correspondence with another equation, namely, the equation for the canonical relation integral \( V(t, x^j, p_j) \)

\[
\frac{\partial V}{\partial t} + [E, V] = 0 \quad (A.7')
\]

where \([E, V]\) is the Poisson bracket.

Equation (A.6) provided with the supplementary conditions, namely, the canonical relations (A.7), is called the Hamilton-Jacobi equation [11]. In other words, the equation whose derivatives obey the canonical relation is referred to as the Hamilton-Jacobi equation. The derivatives of this equation form the differential, i.e. the closed exterior differential form:

\[
\delta u = \left( \frac{\partial u}{\partial t} \right) dt + p_j dx^j = -E dt + p_j dx^j = du.
\]

The equations of field theory belong to this type.

\[
\frac{\partial s}{\partial t} + H\left( t, q_j, \frac{\partial s}{\partial q_j} \right) = 0, \quad \frac{\partial s}{\partial q_j} = p_j \quad (A.8)
\]

where \( s \) is the field function for the action functional \( S = \int L dt \). Here \( L \) is the Lagrange function, \( H \) is the Hamilton function: \( H(t, q_j, p_j) = p_j \dot{q}_j - L \), \( p_j = \partial L/\partial \dot{q}_j \). The closed form \( ds = H dt + p_j dq_j \) (the Poincare invariant) corresponds to equation (A.8).

The coordinates \( q_j, p_j \) in equation (A.8) are the conjugated ones. They obey the canonical relations. The equation for the canonical relation integral that is similar to equation (A.7') can be assigned to equation (A.8).

In quantum mechanics (where to the coordinates \( q_j, p_j \) the operators are assigned) the Schrödinger equation [18] serves as an analog to equation (A.8), and the Heisenberg equation serves as an analog to the relevant equation for the canonical relation integral. Whereas the closed exterior differential form of zero degree (the analog to the Poincare invariant) corresponds to the Schrödinger equation, the closed dual form corresponds to the Heisenberg equation.

A peculiarity of the degenerate transformation can be considered by the example of the field equation. In section 2 it was said that a transition from the unclosed differential form (which is included into the functional relation) to the closed form is the degenerate transformation. Under degenerate transformation a transition from the initial manifold (on which the differential equation is defined) to the characteristic (integral) manifold goes on.

Here the degenerate transformation is a transition from the Lagrange function to the Hamilton function. The equation for the Lagrange function, that is the Euler variational equation, was obtained from the condition \( \delta S = 0 \), where \( S \) is the action functional. In the real case, when forces are nonpotential or couplings are nonholonomic, the quantity \( \delta S \) is not a closed form, that is, \( d \delta S \neq 0 \). But the Hamilton function is obtained from the condition \( d \delta S = 0 \) which is the closure condition for the form \( \delta S \). A transition from the Lagrange function \( L \) to the Hamilton function \( H \) (a transition from variables \( q_j, \dot{q}_j \) to variables \( q_j, p_j = \partial L/\partial \dot{q}_j \)) is a transition from the tangent space, where the
form is unclosed, to the cotangent space with a closed form. One can see that this transition is a degenerate one.

The invariant field theories used only nondegenerate transformations that conserve the differential. By the example of the canonical relations it is possible to show that nondegenerate and degenerate transformations are connected. The canonical relations in the invariant field theory correspond to nondegenerate tangent transformations. At the same time, the canonical relations coincide with the characteristic relation for equation (A.8), which the degenerate transformations correspond to. The degenerate transformation is a transition from the tangent space \((q_j, \dot{q}_j)\) to the cotangent (characteristic) manifold \((q_j, p_j)\). (This is a transition from the manifold that corresponds to the material system, to physical fields. Such a transition is connected with emergence of a physical structure.) On the other hand, the nondegenerate transformation is a transition from one characteristic manifold \((q_j, p_j)\) to the other characteristic manifold \((Q_j, P_j)\). (Physically, this describes a transition from one physical structure to another physical structure.) (The formula of canonical transformation can be written as \(p_j dq_j = P_j dQ_j + dW\), where \(W\) is the generating function.)

It may be easily shown that such a property of duality is also a specific feature of transformations such as tangent, gradient, contact, gauge, conform mapping, and others.

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