GROUPOIDS DECOMPOSITION, PROPAGATION AND OPERATOR $K$-THEORY

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Dedicated to the memory of Etienne Blanchard

Abstract. In this paper, we streamline the technique of groupoids coarse decomposition for purpose of $K$-theory computations of groupoids crossed products. This technique was first introduced by Guoliang Yu in his proof of Novikov conjecture for groups with finite asymptotic dimension. The main tool we use for these computations is controlled operator $K$-theory.

Keywords: Groupoids, Operator $K$-theory, Coarse Geometry, Baum-Connes Conjecture.

2010 Mathematics Subject Classification: 19K35, 22A22, 46L80

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0. Introduction

The concept of coarse decomposability for locally compact groupoids was introduced by several authors [2, 9, 18, 24] in order to compute $K$-theory of reduced $C^*$-algebras and of reduced crossed product algebras of locally compact groupoids. It generalizes the “cut-and-pasting” technique developed by G. Yu in [26] to prove the Novikov conjecture for groups with finite asymptotic dimension. The “cut-and-pasting” has been then extended by E. Guentner, R. Tessera and G. Yu in [6] in order to study topological rigidity of manifolds and in [7] in their approach of the Borel conjecture. In these works, they consider a class of finitely generated groups which satisfy a metric property called finite decomposition complexity. This property can be interpreted in terms of decomposition complexity of the coarse groupoid associated to the metric space, which leads naturally to extend this notion to locally compact groupoids. A first generalization was provided by E. Guentner, R. Willet and G. Yu in [8] in order to study the dynamical properties of finitely generated group actions on locally compact spaces with at each order a one step decomposition into pieces with “finite dynamic”. In [9], the same authors consider the case of finitely generated group actions on locally compact space which, given a sequence of orders, decompose in a finite number of steps into pieces with “finite dynamic”. They give a new proof of the Baum-Connes conjecture (with trivial coefficients) for these action groupoids. This approach rises a large amount of interest since it does not involve infinite dimension analysis and can be generalized to computations in non $C^*$-algebraic situations (for instance to $\ell^p$-crossed products as considered in [4]). The main tool used in this proof is quantitative $K$-theory. Quantitative $K$-theory was first introduced in [26] for obstruction algebras in order to prove the Novikov conjecture for groups with finite asymptotic dimension. It has been then extended in [17] to the setting of $C^*$-algebras equipped with a filtration arising from a length and in [3] to the general framework of $C^*$-algebras filtered by an abstract coarse structure. In [18] was stated a controlled Mayer-Vietoris exact sequence in quantitative $K$-theory associated to decomposition in “ideals at order $r$” which turned out to be tailored for $K$-theory computations under groupoid in decomposability (see [3] for the extension to general filtrations). It has been apply in [2], to the Künneth formula in $K$-theory for groupoid $C^*$-algebras and crossed product algebras. Loosely speaking, we consider decomposition of the set of elements of a given order of a groupoid as
the union of two open subgroupoids (see Definition 1.13). Following [9], we say that a locally compact groupoid $\mathcal{G}$ has finite complexity decomposition with respect to a family $\mathcal{D}$ of open subgroupoids if starting with a given sequence of orders, then iterating the above decomposition ends up with elements belonging to $\mathcal{D}$ in a finite number of steps (see Definition 1.17). The main result of this paper is the following:

Let $\mathcal{G}$ be locally compact groupoid with finite decomposition complexity with respect to a family $\mathcal{D}$ of relatively clopen subgroupoids (see Definition 1.2) and let $f : A \to B$ a homomorphism of $G$-algebras. If the morphism

$$K_*(A \rtimes_r \mathcal{H}) \to K_*(B \rtimes_r \mathcal{H})$$

induced in $K$-theory by $f$ is an isomorphism for any $\mathcal{H}$ in $\mathcal{D}$, then so is

$$K_*(A \rtimes_r \mathcal{G}) \to K_*(B \rtimes_r \mathcal{G}).$$

We then extend this result to morphisms induced by elements in $KK(G)(A, B)$ and we give an application of this result to the Baum-Connes conjecture for locally groupoids admitting a $\gamma$-element in sense of [23]. These stability results should be compare with those obtained by R. Willet in [24] using different technics.

Outline of the paper. In Section 1, we first recall some basic definitions concerning locally compact groupoids and their actions. We introduce the notion of $G$-order for a locally compact groupoid $\mathcal{G}$ which can be viewed as the generalization both of a length on a group and of a distance on a proper metric space. Following the idea of [9, Definition 3.14], we then introduce the notion of $R$-decomposition for a $G$-order $\mathcal{R}$, which leads to the concept of $\mathcal{D}$-decomposability of an open subgroupoid of $\mathcal{G}$ with respect to a family $\mathcal{D}$ of open subgroupoids and to $\mathcal{D}$-finite decomposition complexity ($\mathcal{D}$-fdc), generalizing finite dynamical complexity defined in [9].

Section 2 is devoted to some reminders on groupoid actions on $C^*$-algebras and their reduced crossed product algebras.

In Section 3, we introduce the primary tool for the proof of our main theorem, the controlled Mayer-Vietoris exact sequence in quantitative $K$-theory associated to a groupoid decomposition. We first review from [8] the main features of quantitative $K$-theory for $C^*$-algebra filtered by an abstract coarse structure. We observe that $\mathcal{G}$-orders provide such a structure on crossed products algebras of a groupoid $\mathcal{G}$. We recall the definition of a controlled Mayer-Vietoris pair and we show that groupoid decompositions of order $\mathcal{R}$ give rise to controlled Mayer-Vietoris pairs. Eventually, we recall the statement of the controlled Mayer-Vietoris exact sequence in quantitative $K$-theory associated to a controlled Mayer-Vietoris pair.

We prove in Section 4 the main result of the paper. Although the proof is tedious, the principle is quite simple as it is the extension of the
five lemma to the setting of controlled exact sequences. We then extend our main result to the case of the morphisms induced in $K$-theory by elements of $G$-equivariant $KK$-theory. This is done by noticing that every such element is up to $KK$-equivalence given by an equivariant homomorphism.

In Section 5, we give some applications to the Baum-Connes conjecture for locally compact groupoids. We first recall from [23] the statement of the Baum-Connes conjecture in the setting of locally compact groupoids and the definition of $\gamma$-elements. We end this section with heridity results of the Baum-Connes conjecture for groupoids with $D$-finite decomposition complexity which admit a $\gamma$-element in sense of [23].

I would like to thank warmly J. Renault for the very helpful discussions we had concerning relatively clopen subgroupoids. I am grateful to him for the comments and suggestions he made after carefully reading this paper. For the occasion of his recent retirement, I would like to express my deep admiration for him.

This paper is dedicated to the memory of Etienne Blanchard from whom I learned almost everything I know about $C(X)$-algebras.

1. Coarse decomposition for groupoid

Coarse decomposability for locally groupoids is the generalization of the concept of decomposability for a family of metric spaces introduced in [7]. In this section, after some reminders concerning locally groupoids and their actions, we introduced for a locally compact groupoid $G$ the notion of $G$-orders generalizing on one hand distances on metric spaces and on the other hand lengths on groups. Following ideas of [9], this allows to define decomposition of order $R$ for a subgroupoid of $G$ which leads naturally to coarse decomposability and to finite decomposition complexity with respect to a set of open subgroupoids of $G$.

1.1. Groupoids. We assume that the reader is familiar with the basic definition concerning groupoids. For more details, we refer to [19][20].

A groupoid with space of units $X$ consists of a set $G$ provided with

- two maps $s : G \to X$ and $r : G \to X$ respectively called the source map and the range map;
- a map $u : X \to G; x \mapsto u_x$ which is a section both for $s$ and $r$;
- an associative composition

$$G \times_X G \to G : (\gamma, \gamma') \mapsto \gamma \cdot \gamma',$$

with

$$G \times_X G = \{ (\gamma, \gamma') \in G \times G \text{ such that } s(\gamma) = r(\gamma') \}.$$
such that
\[ s(\gamma \cdot \gamma') = s(\gamma') \]
and
\[ r(\gamma \cdot \gamma') = r(\gamma) \]
for any \((\gamma, \gamma')\) in \(G \times_X G\) and
\[ \gamma \cdot u_{s(\gamma)} = u_{r(\gamma)} : \gamma = \gamma \]
for any \(\gamma\) in \(G\);
• an inverse map
\[ G \to G; \gamma \mapsto \gamma^{-1} \]
such that
\[ s(\gamma^{-1}) = r(\gamma), \]
\[ r(\gamma^{-1}) = s(\gamma), \]
\[ \gamma \cdot \gamma^{-1} = u_{r(\gamma)}, \]
and
\[ \gamma^{-1} \cdot \gamma = u_{s(\gamma)} \]
for any \(\gamma\) in \(G\).

Notation 1.1. Let \(G\) be a groupoid with space of units \(X\) and source and range maps \(s, r : G \to X\).
• Let \(Z\) be a subset of \(G\).
  - we set \(Z^{-1} = \{\gamma^{-1}; \gamma \in Z\}\);
  - for any \(Y \subseteq X\), we set \(Z_Y = s^{-1}(Y) \cap Z\) and \(Z_Y = r^{-1}(Y) \cap Z\);
  - for any subsets \(Y_1\) and \(Y_2\) of \(X\), we set \(Z_{Y_1}^{Y_2} = Z_{Y_2} \cap Z_{Y_1}\);
• Let \(Z_1\) and \(Z_2\) be subsets in \(G\), we set
\[ Z_1 \cdot Z_2 = \{\gamma_1 \gamma_2; \gamma_1 \in Z_1, \gamma_2 \in Z_2 \text{ and } s(\gamma_1) = r(\gamma_2)\}. \]

A locally compact groupoid is a groupoid provided with locally compact topology and such that the structure maps a continuous. In this paper, all the groupoids are assume to be locally compact and Hausdorff. An open subgroupoid of \(G\) is a subgroupoid \(H\) of \(G\) which is open as a subset and such that the space of units is open in the space of unit of \(G\). Notice that the latter condition always holds if the source map of \(G\) is open, for instance if \(G\) is provided with a Haar system [23, Lemma 6.5].

Definition 1.2. Let \(G\) be locally compact groupoid. A relatively clopen subgroupoid of \(G\) is an open subgroupoid \(H\) of \(G\) such that if \(Y\) stands for the unit space of \(H\), then \(H\) is closed in \(G_Y\).

Remark 1.3. Let \(G\) be locally compact groupoid and let \(H\) be a relatively clopen subgroupoid of \(G\) with unit space \(Y\). Then \(H\) is clopen in \(G_Y\) and in \(G_Y^Y\).

Next lemma is straightforward to prove.
Lemma 1.4. Let $\mathcal{G}$ be a locally compact groupoid and let $\mathcal{H}$ be an open subgroupoid of $\mathcal{G}$ with unit space $Y$. Then $\mathcal{H}$ is relatively clopen if and only if $K \cap \mathcal{H}$ is compact for any compact subset $K$ of $\mathcal{G}_Y$.

We recall that a locally compact groupoid with space of units $X$ is proper if the map

$$\mathcal{G} \to X \times X; \gamma \mapsto (r(\gamma), s(\gamma))$$

is proper. As a consequence of Lemma 1.4, we obtain the following corollary.

Corollary 1.5. Let $\mathcal{G}$ be a locally compact proper groupoid. Then relatively clopen subgroupoids of $\mathcal{G}$ are proper.

1.2. Groupoid actions. Let us recall first the definition of a (left) action of a groupoid. Let $\mathcal{G}$ be a groupoid with space of units $X$ and source and range maps $s$ and $r$. An action of the groupoid $\mathcal{G}$ on a set $Z$ consists of a map $p: Z \to X$ called the anchor map and a map

$$\mathcal{G} \times_X Z \to Z; (\gamma, z) \mapsto \gamma \cdot z,$$

with $\mathcal{G} \times_X Z = \{(\gamma, z) \in \mathcal{G} \times Z \text{ such that } s(\gamma) = p(z)\}$ such that

(i) for any $\gamma$ and $\gamma'$ in $\mathcal{G}$ and $z$ in $Z$ such that $(\gamma, \gamma')$ is in $\mathcal{G} \times_X \mathcal{G}$ and $(\gamma', z)$ is in $\mathcal{G} \times_X Z$, then $(\gamma, \gamma' \cdot z)$ belongs to $\mathcal{G} \times_X Z$ and $\gamma \cdot (\gamma' \cdot z) = (\gamma \cdot \gamma') \cdot z$;

(ii) $u_{p(z)} \cdot z = z$ for any $z$ in $Z$.

Notice that these conditions imply that $p(\gamma \cdot z) = r(\gamma)$ and $\gamma^{-1} \cdot \gamma \cdot z = z$ for any $(\gamma, z)$ in $\mathcal{G} \times_X Z$. If $x$ is an element in $X$, then the fiber of $Z$ at $x$ is $Z_x \overset{\text{def}}{=} f^{-1}(\{x\})$. If $\mathcal{G}$ is a locally compact groupoid and if $Z$ is a locally compact space, we require the anchor map and the action map to be continuous. In this case, $Z$ will be called a $\mathcal{G}$-space. In what follows, all $\mathcal{G}$-spaces are supposed to be Hausdorff. If $Z$ and $Z'$ are $\mathcal{G}$-space with anchor maps $p_Z$ and $p_{Z'}$, a map $f: Z \to Z'$ is called a $\mathcal{G}$-map if $f$ is continuous, $p_{Z'} \circ f = p_Z$ and $f(\gamma \cdot z) = \gamma \cdot f(z)$ for all $(\gamma, z)$ in $\mathcal{G} \times_X Z$.

Let $\mathcal{G}$ be a groupoid with space of units $X$ acting on a set $Z$ with anchor map $p: Z \to X$. Then the action groupoid corresponding to the action of $\mathcal{G}$ on $Z$ denoted by $\mathcal{G} \ltimes Z$ is the set $\mathcal{G} \times_X Z$, with $Z$ as space of units with source map

$$\mathcal{G} \ltimes Z \to Z; (\gamma, z) \mapsto z$$

and range map

$$\mathcal{G} \ltimes Z \to Z; (\gamma, z) \mapsto \gamma \cdot z,$$

unit map

$$Z \to \mathcal{G} \ltimes Z; z \mapsto (u_{p(z)}, z),$$
composition
\[(G \times Z) \times_Z (G \times Z); (\gamma, \gamma' z) \cdot (\gamma', z) \mapsto (\gamma \gamma', z)\]
and inverse
\[G \times Z \to G \times Z; (\gamma, z) \mapsto (\gamma^{-1}, \gamma \cdot z).\]
If \(G\) is a locally compact groupoid and \(Z\) is a \(G\)-space, then \(G \times Z\) is a locally compact groupoid. A \(G\)-space \(Z\) is called proper (or the action of \(G\) on \(Z\) is said to be proper) if the action groupoid \(G \times Z\) is proper.

**Remark 1.6.** Let \(G\) be a locally compact groupoid with space of units \(X\) acting on a locally compact space \(Y\).

(i) a \(G \times Y\)-space is precisely a \(G\)-space \(Z\) together with a \(G\)-map \(f : Z \to Y\);

(ii) in this case,
\[G \times Z \longrightarrow (G \times Y) \times Z; (\gamma, z) \mapsto (\gamma, f(z), z)\]
is a groupoid isomorphism;

(iii) in consequence, a \(G \times Y\)-space \(Z\) is proper if and only if it is proper as a \(G\)-space;

(iv) in particular, if \(Z\) is a proper \(G\)-space, then \(Z \times_X Y\) is a proper \(G \times Y\)-space with anchor map given by the projection on the second factor (here \(Z \times_X Y\) stands for the fiber product over the two anchor maps).

**Remark 1.7.** Let \(G\) be locally compact groupoid and let \(H\) be a relatively clopen subgroupoid of \(G\) with unit space \(Y\). For any left \(G\)-space \(Z\), then \(H \times Z\) is relatively clopen in \(G \times Z\).

1.3. **Induced actions.** We recall now from [1] the notion of induced action to a groupoid from a subgroupoid action. Let \(G\) be a locally compact groupoid with space of units \(X\) and open source and range maps, let \(H\) be a relatively clopen subgroupoid of \(G\) with space of units \(Y\) and let \(Z\) be a (left) \(H\)-space with anchor map \(p : Z \to Y\). Let us define on
\[G \times_Y Z \overset{\text{def}}{=} \{(\gamma, z) \in G \times Z \text{ such that } s(\gamma) = p(z)\}\]
the \(H\)-action with anchor map
\[G \times_Y Z \to Y; (\gamma, z) \mapsto p(z)\]
by
\[\gamma \cdot (\gamma', z) = (\gamma' \gamma^{-1}, \gamma z)\]
for any \(\gamma\) in \(H\) and \((\gamma', z)\) in \(G \times_Y Z\) such that \(s(\gamma) = p(z)\). The \(H\)-action defined in this way is proper and the quotient space
\[G \times_Y Z \overset{\text{def}}{=} (G \times_Y Z)/H\]
is Hausdorff and locally compact. Let us denote by \([\gamma, z]\) the class in \(G \times_H Z\) of an element \((\gamma, z)\) in \(G \times_H Z\). Then \(G \times_H Z\) is provided with a \(G\)-action with anchor map

\[
G \times H Z \to X : [\gamma, z] \mapsto r(\gamma)
\]
defined by

\[
\gamma : [\gamma', z] = [\gamma \gamma', z]
\]
for any \(\gamma\) in \(G\) and \([\gamma', z]\) in \(G \times_H Z\) such that \(s(\gamma) = r(\gamma')\) and is called the \(G\)-space induced by the \(H\)-space \(Z\).

**Proposition 1.8.** \([1]\) Let \(G\) be a locally compact groupoid with open source and range maps, let \(H\) be a relatively clopen subgroupoid of \(G\) and let \(Z\) be a proper \(H\)-space. Then the induced \(G\)-space \(G \times_H Z\) is proper.

1.4. \(G\)-orders.

**Definition 1.9.** Let \(G\) be locally compact groupoid with space of unit \(X\). A \(G\)-order is a subset \(R\) of \(G\) such that

- \(u(s(R)) \subseteq R\);
- \(R^{-1} = R\) (\(R\) is symmetric).
- for every compact subset \(Y\) of \(X\), then \(R \cap Y\) is compact.

**Remark 1.10.** Let \(G\) be locally compact groupoid with unit space.

(i) for any compact subset \(K\) of \(G\), then \(K \cup K^{-1} \cup r(K) \cup s(K)\) is a compact \(G\)-order and hence for any compact subset \(K\) of \(G\), there exists a compact \(G\)-order \(R\) such that \(K \subseteq R\).

(ii) if \(R_1\) and \(R_2\) are \(G\)-orders, then \(R_1 \cup R_2\) and \(R_1 \cap R_2\) are \(G\)-orders.

**Lemma 1.11.** Let \(G\) be locally compact groupoid, then any \(G\)-order is closed.

**Proof.** Let \(R\) be a \(G\)-order. Let us prove that \(R \cap K\) is compact for any compact subset \(K\) of \(G\). Let us set \(Y = s(K)\). Since \(Y\) is compact, then \(R_Y\) is compact and hence \(K \cap R = K \cap R_Y\) is compact. \(\square\)

If \(R_1\) and \(R_2\) are two \(G\)-orders, then

\[
R_1 \ast R_2 \overset{\text{def}}{=} (R_1 \cdot R_2) \cup (R_2 \cdot R_1)
\]
is a \(G\)-order. If \(R\) is a \(G\)-order and \(n\) an integer, then \(R^*\) stands for \(R \ast \cdots \ast R\) \((n\) products). Notice that according to the first point of Definition 1.9, we have that \(R \subseteq R^*\) for every integer \(n\). Let \(E_G\) be the set of \(G\)-orders. Then \(E_G\) is a poset for the inclusion and ordered semi-group for \(\ast\). Moreover \(E_G\) is a lattice with the infimum given by the intersection and the supremum given by the union. We denote by \(E_{G,c}\) the set of compact \(G\)-order. Then \(E_{G,c}\) is as well an ordered semi-group for \(\ast\) and a lattice for the partial order given by the inclusion.
1.5. \( \mathcal{R} \)-decomposition of a groupoid.

**Remark 1.12.** Let \( \mathcal{G} \) be a locally compact groupoid and let \( \mathcal{H} \) be a relatively clopen subgroupoid of \( \mathcal{G} \).

(i) let \( \mathcal{R} \) be a \( \mathcal{G} \)-order, then \( \mathcal{R} \cap \mathcal{H} \) is a \( \mathcal{H} \)-order denoted by \( \mathcal{R}/\mathcal{H} \).

(ii) \( \mathcal{E}_\mathcal{G} \to \mathcal{E}_\mathcal{H} : \mathcal{R} \mapsto \mathcal{R}/\mathcal{H} \) is a map of posets such that
\[
\mathcal{R}_1/\mathcal{H} \ast \mathcal{R}_2/\mathcal{H} \subseteq (\mathcal{R}_1 \ast \mathcal{R}_2)/\mathcal{H}
\]
for any \( \mathcal{G} \)-orders \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \).

**Definition 1.13.** Let \( \mathcal{G} \) be a locally compact groupoid, let \( \mathcal{H} \) be a subgroupoid of \( \mathcal{G} \) with space of units \( Y \) and let \( \mathcal{R} \) be a \( \mathcal{G} \)-order.

(i) an \( \mathcal{R} \)-decomposition of \( \mathcal{H} \) is a quadruple \( (V_1, V_2, \mathcal{H}_1, \mathcal{H}_2) \) where
- \( V_1 \) and \( V_2 \) are open subsets of \( Y \) with \( Y = V_1 \cup V_2 \) and such that there exists a partition of the unit subordinated to \( (V_1, V_2) \);
- \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are subgroupoids of \( \mathcal{H} \) which are open in \( \mathcal{G} \).
- \( \mathcal{R}_{V_i} \cap \mathcal{H} \) is contained in \( \mathcal{H}_i \) for \( i = 1, 2 \).

(ii) a coercive \( \mathcal{R} \)-decomposition of \( \mathcal{H} \) is a \( \mathcal{R} \)-decomposition \( (V_1, V_2, \mathcal{H}_1, \mathcal{H}_2) \) of \( \mathcal{H} \) such that \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are relatively clopen in \( \mathcal{G} \).

Following the route of [9], we introduce the notion of decomposability with respect to a set of open subgroupoids.

**Definition 1.14.** Let \( \mathcal{G} \) be locally compact groupoid and let \( \mathcal{D} \) be a set of open subgroupoids of \( \mathcal{G} \). A subgroupoid \( \mathcal{H} \) of \( \mathcal{G} \) is \( \mathcal{D} \)-decomposable if for every \( \mathcal{G} \)-order \( \mathcal{R} \), there exists an \( \mathcal{R} \)-decomposition \( (V_1, V_2, \mathcal{H}_1, \mathcal{H}_2) \) with \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) in \( \mathcal{D} \).

**Remark 1.15.** If the space of units of \( \mathcal{G} \) is second countable, then the existence of the partition of the unit in the first item of Definition 1.13 is guaranteed.

**Lemma 1.16.** Let \( \mathcal{G} \) be locally compact groupoid and let \( \mathcal{H} \) be a subgroupoid of \( \mathcal{G} \).

(i) if \( \mathcal{D} \) is a set of open subgroupoids of \( \mathcal{G} \) such that \( \mathcal{H} \) is \( \mathcal{D} \)-decomposable then \( \mathcal{H} \) is an open subgroupoid of \( \mathcal{G} \).

(ii) if \( \mathcal{D} \) is a set of relatively clopen subgroupoids of \( \mathcal{G} \) such that \( \mathcal{H} \) is \( \mathcal{D} \)-decomposable, then \( \mathcal{H} \) is a relatively clopen subgroupoid of \( \mathcal{G} \).

**Proof.** Let us prove the first point. Let \( \gamma \) be an element in \( \mathcal{H} \). According to point (i) of Remark 1.10, there exists a \( \mathcal{G} \)-order \( \mathcal{R} \) such that \( \gamma \) lies in \( \mathcal{R} \). Let \( (V_1, V_2, \mathcal{H}_1, \mathcal{H}_2) \) be a \( \mathcal{R} \)-decomposition of \( \mathcal{H} \) with \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) in \( \mathcal{D} \). By definition of an \( \mathcal{R} \)-decomposition, we see that \( \gamma \) belongs to \( \mathcal{H}_1 \cup \mathcal{H}_2 \) which is an open subset of \( \mathcal{G} \) contained in \( \mathcal{H} \).

For the second point, assume now that every subgroupoid in \( \mathcal{D} \) is relatively clopen and let \( \mathcal{H} \) be a \( \mathcal{D} \)-decomposable subgroupoid of \( \mathcal{G} \).
Let us prove that $\mathcal{H}$ is relatively clopen. Let $Y$ be the unit space of $\mathcal{H}$. According to Lemma 1.14, this amounts to prove that $\mathcal{H} \cap K$ is compact if $K$ is a compact subset of $\mathcal{G}_Y$. Consider then a $\mathcal{G}$-order $\mathcal{R}$ such that $K \subseteq \mathcal{R}$ (see point (i) of Remark 1.10) and let $(V_1, V_2, \mathcal{H}_1, \mathcal{H}_2)$ be a $\mathcal{R}$-decomposition for $\mathcal{H}$. The existence of a partition of the unit subordinated to $(V_1, V_2)$ ensures that there exists two closed subsets $F_1$ and $F_2$ of $Y$, respectively contained in $V_1$ and $V_2$ and such that $Y = F_1 \cup F_2$. Let us set $K_1 = K \cap \mathcal{G}_{F_1}$ and $K_2 = K \cap \mathcal{G}_{F_2}$. Then $K_1$ and $K_2$ are compact subsets respectively contained in $\mathcal{G}_{V_1}$ and $\mathcal{G}_{V_2}$ and moreover, we have $K = K_1 \cup K_2$. Furthermore, since $K_1 \subseteq \mathcal{R}_{V_1}$ and $K_2 \subseteq \mathcal{R}_{V_2}$ and using the definition of a $\mathcal{R}$-decomposition, we have $\mathcal{H} \cap K_1 = \mathcal{H}_1 \cap K_1$ and $\mathcal{H} \cap K_2 = \mathcal{H}_2 \cap K_2$. Since $\mathcal{H}_1$ and $\mathcal{H}_2$ are relatively clopen subgroupoids, then $\mathcal{H}_1 \cap K_1$ and $\mathcal{H}_2 \cap K_2$ are compact and hence $\mathcal{H} \cap K$ is compact.

Let $\mathcal{G}$ be locally compact groupoid. A set $\mathcal{D}$ of open subgroupoids of $\mathcal{G}$ is closed under coarse decompositions if every $\mathcal{D}$-decomposable subgroupoid of $\mathcal{G}$ is indeed in $\mathcal{D}$. If $\mathcal{D}$ is a set of open subgroupoid of $\mathcal{G}$, let $\hat{\mathcal{D}}$ be the smallest set of open subgroupoids of $\mathcal{G}$ closed under coarse decompositions.

**Definition 1.17.** Let $\mathcal{G}$ be a locally compact groupoid and let $\mathcal{D}$ be a family of open subgroupoids of $\mathcal{G}$. An open subgroupoid $\mathcal{H}$ of $\mathcal{G}$ has finite decomposition complexity with respect to $\mathcal{D}$ (D-fdc) if $\mathcal{H}$ belongs to $\hat{\mathcal{D}}$.

**Lemma 1.18.** Let $\mathcal{G}$ be a locally compact groupoid and let $\mathcal{D}$ be a set of open subgroupoids of $\mathcal{G}$ closed under taking open subgroupoids. Then $\hat{\mathcal{D}}$ is closed under taking subgroupoids.

**Proof.** Let $\mathcal{D}'$ be the set of open subgroupoids $\mathcal{H}$ of $\mathcal{G}$ such that every open subgroupoid of $\mathcal{H}$ lies in $\hat{\mathcal{D}}$. We have inclusions $\mathcal{D} \subseteq \mathcal{D}' \subseteq \hat{\mathcal{D}}$. Let us show that $\mathcal{D}'$ is closed under coarse decompositions. Let $\mathcal{H}$ be an open subgroupoid of $\mathcal{G}$ which is $\mathcal{D}'$-decomposable and let $\mathcal{H}'$ be an open subgroupoid of $\mathcal{H}$ with unit space $Y$. Let $\mathcal{R}$ be a $\mathcal{G}$-order and let us consider an $\mathcal{R}$-decomposition $(V_1, V_2, \mathcal{H}_1, \mathcal{H}_2)$ of $\mathcal{H}$ with $\mathcal{H}_1$ in $\mathcal{H}_2$ in $\mathcal{D}'$. Then $(V_1 \cap Y, V_2 \cap Y, \mathcal{H}_1 \cap \mathcal{H}', \mathcal{H}_2 \cap \mathcal{H}')$ is an $\mathcal{R}$-decomposition of $\mathcal{H}'$ with $\mathcal{H}_1 \cap \mathcal{H}'$ and $\mathcal{H}_2 \cap \mathcal{H}'$ in $\mathcal{D}$. In consequence $\mathcal{H}'$ is in $\hat{\mathcal{D}}$ for any open subgroupoid and hence $\mathcal{H}$ is in $\mathcal{D}'$. We conclude that $\hat{\mathcal{D}} \subseteq \mathcal{D}'$ and hence $\hat{\mathcal{D}} = \mathcal{D}'$. □

**Lemma 1.19.** Let $\mathcal{G}$ be a locally compact groupoid and let $\mathcal{D}$ be a set of relatively clopen subgroupoids of $\mathcal{G}$:

1. if $\mathcal{H}$ is in $\hat{\mathcal{D}}$, then $\mathcal{H}$ is relatively clopen;
2. If $\mathcal{D}$ is closed under taking relatively clopen subgroupoids, then so is $\hat{\mathcal{D}}$.
Proof. To prove the first point, let us consider the set $D'$ of relatively clopen subgroupoids of $\mathcal{G}$ that belongs to $\hat{D}$. Then we have inclusions $D \subseteq D' \subseteq \hat{D}$ and we deduce from Lemma 1.16 that $D'$ is closed under coarse decompositions. Hence we have $D' = \hat{D}$.

To prove the second point, we proceed as for the second point of Lemma 1.18 by considering the set of subgroupoids $H$ of $G$ for which every relatively clopen subgroupoid is in $\hat{D}$ and by noticing that the intersection of two relatively clopen subgroupoids is relatively clopen. □

Example 1.20. Let $X$ be a metric discrete space with bounded geometry and with finite complexity decomposition in the sense of [6] and consider then $\mathcal{G}_X$ the coarse groupoid of $X$ defined in [22]. Then $\mathcal{G}_X$ has finite decomposition complexity with respect to the set of its compact open subgroupoids. In particular, if $\Gamma$ is a finitely generated group with finite complexity decomposition and if we consider its action on its Stone-Čech compactification $\beta\Gamma$, then the action groupoid $\Gamma \ltimes \beta\Gamma$ has finite decomposition complexity with respect to the set of its compact open subgroupoids.

2. Reduced crossed product of a groupoid

In this section, we review the construction of the reduced crossed-product for a groupoid action on a $C^*$-algebra. Some good material for this construction can be founded in [14, 15].

2.1. $C(X)$-algebra.

Definition 2.1. Let $X$ be a locally compact space. A $C(X)$-algebra is a $C^*$-algebra $A$ together with a morphism $\Psi : C_0(X) \to Z(M(A))$, where $Z(M(A))$ stands for the center of the multiplier algebra of $A$, such that

$$\{\Psi(f) \cdot a; f \in C_0(X) \text{ and } a \in A\}$$

is dense in $A$.

From now on, for $f$ in $C_0(X)$ and $a$ in $A$, we will denote $\Psi(f) \cdot a$ by $f \cdot a$ and omit the structure map $\Phi$.

Let $A$ be an $C(X)$-algebra and let us consider for $x$ in $X$ the ideal $I_x$ defined as the closure of

$$\{f \cdot a; f \in C_0(X) \text{ and } a \in A \text{ such that } f(x) = 0\}.$$

We define the fiber of $A$ at $x$ as the quotient $C^*$-algebra $A_x \overset{\text{def}}{=} A/I_x$. For $a$ in $A$, we denote by $a(x)$ the image of $a$ under the quotient map $A \to A_x$. Then we have the following classical result [25].

Lemma 2.2. Let $X$ be a locally compact space and let $A$ be a $C(X)$-algebra. Then for any $a$ in $A$,
(i) the map $X \to \mathbb{R}; x \mapsto \|a(x)\|$ is upper semi-continuous and vanishing at infinity;
(ii) $\|a\| = \sup_{x \in X} \|a(x)\|.$

Let $X$ and $Y$ be locally compact space, let $A$ be a $C(Y)$-algebra and let $f : X \to Y$ be a continuous map. The algebra $C_0(X, A)$ of continuous functions $\xi : X \to A$ vanishing at infinity is then a $C(X \times Y)$-algebra. Consider in $C_0(X, A)$ the ideal $I_f$ defined as the closure of 

$\{ h \cdot \xi; h \in C_0(X \times Y), \xi \in C_0(X, A) \text{ such } h(x, f(x)) = 0 \forall x \in X \}.$

The pull back algebra of $A$ by $f$ is by definition $f^*A \overset{\text{def}}{=} C_0(X, A)/I_f.$ Pointwise multiplication by $C_0(X, A)$ on $C_0(X, A)$ induces then a $C(X)$-algebra structure on $f^*A.$ The fiber of $f^*A$ at an element $x$ of $X$ is canonically isomorphic to $A_{f(x)},$ this isomorphism being induced by the map

$$C_0(X, A) \to A_{f(x)}$$

$$\xi \mapsto \xi(x)(f(x)).$$

Let $A$ and $B$ be two $C(X)$algebras. A morphism of $C^*$-algebra $\Psi : A \to B$ is called a morphism of $C(X)$-algebra if it is in addition $C_0(X)$-linear. It is straightforward to check that a morphism of $C(X)$-algebra $\Psi : A \to B$ induced for every $x$ in $X$ a morphism $\Psi_x : A_x \to B_x.$ Moreover, $\Psi$ is an isomorphism (resp. injective, surjective) if $\Psi_x$ is an isomorphism (resp. injective, surjective) for any $x$ in $X.$

### 2.2. Groupoid actions on $C^*$-algebras.

Groupoid actions generalize to the setting of groupoid the notion of group actions by automorphisms on a $C^*$-algebra.

**Definition 2.3.** Let $\mathcal{G}$ be a locally compact groupoid with $X$ as space of units and let $A$ be a $C(X)$-algebra. An action of $\mathcal{G}$ on $A$ is given by a $C(\mathcal{G})$-isomorphism $\alpha : s^*A \to r^*A$ which satisfies

$$\alpha_{\gamma \gamma'} = \alpha_{\gamma} \circ \alpha_{\gamma'}$$

for any $\gamma$ and $\gamma'$ in $\mathcal{G}$ such that $s(\gamma) = r(\gamma'),$ where

$$\alpha_{\gamma} : A_{s(\gamma)} \to A_{r(\gamma)}$$

is the morphism fiberwise induced by $\alpha$ at $\gamma$ in $\mathcal{G}$ under the canonical isomorphisms $(s^*A)_\gamma \cong A_{s(\gamma)}$ and $(r^*A)_\gamma \cong A_{r(\gamma)}.$ A $C(X)$-algebra equipped with an action of $\mathcal{G}$ will be called a $\mathcal{G}$-algebra.

In what follows, for a $\mathcal{G}$-algebra $A$ with respect to an action $\alpha : s^*A \to r^*A,$ we shall denote for short the morphism induced fiberwise at $\gamma$ in $\mathcal{G}$ by

$$\gamma : A_{s(\gamma)} \mapsto A_{r(\gamma)}; a \mapsto \gamma(a).$$
Example 2.4. Let $G$ be a locally compact groupoid with space of units $X$ and let $Z$ be a $G$-space with respect to the anchor map $p_Z : Z \to X$.

(i) The anchor map provides a $C(X)$-algebra structure on $C(Z)$ which is acted upon by $G$ in the following way. Let us define

\[ s_*Z = \{(\gamma, z) \in G \times Z \text{ such that } s(\gamma) = p_Z(z)\} \]

and

\[ r_*Z = \{(\gamma, z) \in G \times Z \text{ such that } r(\gamma) = p_Z(z)\}. \]

The we have canonical isomorphisms $C_0(s_*Z) \cong s^*(C_0(Z))$ and $C_0(r_*Z) \cong r^*(C_0(Z))$ and under these identifications, the homeomorphism

\[ r_*Z \to s_*Z \]

\[ (\gamma, z) \mapsto (\gamma, \gamma^{-1}z) \]

gives rise to an $C(G)$-isomorphism

\[ \alpha : s^*(C_0(Z)) \xrightarrow{\sim} r^*(C_0(Z)). \]

Let $\gamma$ be an element in $G$. The fibers at $\gamma$ of $s^*(C_0(Z))$ and $r^*(C_0(Z))$ are under the above identifications respectively $C_0(Z_{s(\gamma)})$ and $C_0(Z_{r(\gamma)})$ and $\alpha$ induces fiberwise at $\gamma$ the isomorphism

\[ C_0(Z_{s(\gamma)}) \to C_0(Z_{r(\gamma)}) \]

\[ f \mapsto \gamma(f), \]

where $\gamma(f)(z) = f(\gamma^{-1}\cdot z)$ for any $z$ in $Z_{r(\gamma)}$ and any $f$ in $C_0(Z_{s(\gamma)})$.

(ii) If $A$ is a $C(Z)$-algebra, then an action of $G \times Z$ on $A$ is simply an action $\alpha : s^*A \to r^*A$ of $G$ on $A$ which is $C(Z)$-linear, where $A$ is viewed as a $C(X)$-algebra by using the anchor map.

Let $G$ be a locally compact groupoid with space of unit $X$ and let $A$ and $B$ be $G$-algebras. A $G$-morphism is a $C(X)$-morphism $f : A \to B$ such that

\[ \gamma \circ f_{s(\gamma)} = f_{r(\gamma)} \circ \gamma \]

for every $\gamma$ in $G$.

2.3. Reduced crossed products. Let $G$ be a locally compact groupoid with space of units $X$ and let $C_c(G)$ be the set of complex valued and compactly supported continuous function on $G$. We assume from now on that $G$ is provided with a Haar system $(\lambda^x)_{x \in X}$. Let $L^2(G)$ be the $C_0(X)$-Hilbert module obtained by completion of $C_c(G)$ with respect to the $C_0(X)$-scalar product

\[ \langle \eta, \eta' \rangle(x) = \int_G \bar{\eta}(\gamma^{-1})\eta'(\gamma^{-1})d\lambda^x(\gamma) \]

for any $\eta$ and $\eta'$ in $C_c(G)$. An element $h$ of $C_0(X)$ acts on $L^2(G)$ by multiplication by $h \circ s$. 
Let $A$ be a $G$-algebra. Recall that $r^*A \overset{\text{def}}{=} C_0(G) \otimes_s A$ is a $C_0(G)$-algebra and that for $h$ in $r^*A$ and $\gamma$, then $h(\gamma) \in A(\gamma)$ is the fiber evaluation of $h$ at $\gamma$ under the identification between $(s^*A)_\gamma$ and $A_{s(\gamma)}$. For $h$ in $r^*A$, the support of $h$, denoted by supp $h$, is the complementary of the largest open subset of $G$ on which $\gamma \mapsto h(\gamma)$ vanishes. Let us set then $C_c(X; G, r^*A)$ the set of elements of $r^*A$ with compact support. In the same way, we can define $C_c(X; G, s^*A)$ as the set of elements of $s^*A$ with compact support.

If $A$ is a $G$-algebra, we set $L^2(G, A) = L^2(G) \otimes_s A$. Notice that $C_c(X; G, s^*A)$ embeds in $L^2(G, A)$ and for any $\eta$ and $\eta'$ in $C_c(X; G, s^*A)$, the fiber evaluation of $(\eta, \eta')$ at an element $x \in X$ is the element of $A_x$ uniquely determined by

$$\langle \eta, \eta' \rangle(x) = \int_{G_x} \eta^*(\gamma^{-1}) \eta'(\gamma^{-1}) d\lambda^x(\gamma).$$

Recall that $C_c(X; G, r^*A)$ is provided with an involutive algebra structure such that

$$f \cdot g(\gamma) = \int_{G(\gamma)} f(\gamma') g(\gamma' \gamma^{-1}) d\lambda^r(\gamma')$$

and

$$f^*(\gamma) = \gamma(f(\gamma^{-1})^*)$$

for any $f$ and $g$ in $C_c(X; G, r^*A)$ and any $\gamma$ in $G$. Moreover, for any $f$ in $C_c(X; G, r^*A)$, the map

$$C_c(X; G, s^*A) \to C_c(X; G, s^*A)$$

$$\xi \mapsto f \cdot \xi$$

with

$$(f \cdot \xi)(\gamma) = \int_{G(\gamma)} \gamma^{-1}(f(\gamma')) \xi(\gamma' \gamma^{-1}) d\lambda^r(\gamma')$$

extends to an adjointable endomorphism of $L^2(G, A)$ and we obtain in this way an involutive and faithful representation of $C_c(X; G, r^*A)$. The reduced crossed product algebra $A \rtimes_r G$ is then the closure of $C_c(X; G, r^*A)$ in the algebra $\mathcal{L}(L^2(G, A))$ of adjointable endomorphisms of $L^2(G, A)$.

**Lemma 2.5.** Let $G$ be a locally compact groupoid with space of units $X$ provided with a Haar system. Let $V$ be an open subset and let $\phi : X \to \mathbb{C}$ be a bounded and continuous function with support in $V$. Then there exists a bounded operator

$$\Lambda^*_\phi : A \rtimes_r G \to A \rtimes_r G$$

such that

(i) $\Lambda^*_\phi$ has operator norm bounded by $\sup_{x \in X} |\phi(x)|$;

(ii) $\Lambda^*_\phi(h) = h \cdot \phi \circ s$ for all $h$ in $C_c(X; G, A)$.
Proof. Let us set $M = \sup_{x \in X} |\phi(x)|$. The map

$$C_c(G) \longrightarrow C_c(G); f \mapsto f \cdot \phi \circ r$$

extends to an adjointable operator $T_\phi : L^2(G) \longrightarrow L^2(G)$ such that $\|T_\phi\| \leq M$. Then right multiplication by $T_\phi \otimes C_0(X) \text{Id}_A$ on $L(L^2(G, A))$ preserves the subalgebra $A \rtimes_r G$ and hence induces a bounded operator $\Lambda_\phi^r : A \rtimes_r G \rightarrow A \rtimes_r G$ which satisfies the required conditions. □

Remark 2.6. In the same way, left multiplication by $T_\phi$ on $L(L^2(G, A))$ preserve $A \rtimes_r G$ and hence induces a bounded operator $\Lambda_\phi^l : A \rtimes_r G \rightarrow A \rtimes_r G$ such that

(i) $\Lambda_\phi^l$ has operator norm bounded by $M = \sup_{x \in X} |\phi(x)|$
(ii) $\Lambda_\phi^l(h) = h \cdot \phi \circ r$ for all $h$ in $C_c(X; G, r^* A)$.
(iii) $\Lambda_\phi^l$ and $\Lambda_\phi^r$ commute for any continuous and bounded function $\phi' : X \rightarrow \mathbb{C}$ with support in $V$;
(iv) $\Lambda_\phi^l \circ \Lambda_\phi^r : A \rtimes_r G \rightarrow A \rtimes_r G$ is positive with operator norm bounded by $M^2$;
(v) $\Lambda_\phi^l \circ \Lambda_\phi^r(h) = \overline{\phi} \circ r \cdot h \cdot \phi \circ s$ for all $h$ in $C_c(X; G, r^* A)$.

For any open subgroupoid $H$ of $G$ with unit space $Y$, let $A/Y$ be the closure of

$$\{f \cdot a, f \in C_0(Y) \text{ and } a \in A\}$$

in $A$. Then $A/Y$ is a $H$-algebra and moreover, the Haar system of $G$ induced by restriction a Haar system on $H$. We will denote the crossed product $A/Y \rtimes_r H$ by $A \rtimes_r H$. Notice that since $H$ is an open subgroupoid of $G$, then $A \rtimes_r H$ can be viewed as a $C^*$-subalgebra of $A \rtimes_r G$.

3. CONTROLLED MAYER-VIETORIS EXACT SEQUENCE IN QUANTITATIVE $K$-THEORY

The concept of quantitative operator $K$-theory was first introduced in [26] for localisation algebras in order to prove the Novikov conjecture for finitely generated groups with finite asymptotic dimension. It has then been extended in [17] to the setting of $C^*$-algebras equipped with a filtration arising from a length. C. Dell’Aiera developed in [3] quantitative $K$-theory in the general framework of $C^*$-algebras filtered by abstract coarse structure.

3.1. Review on quantitative $K$-theory. In this subsection, we review from [3] the main features of quantitative $K$-theory in the framework of $C^*$-algebras filtered by an abstract coarse structure.

Definition 3.1. A coarse structure $E$ is an ordered abelian semi-group which is a lattice for the order. Recall that a lattice is a poset for which every pair $(E, E')$ admits a supremum $E \vee E'$ and an infimum $E \wedge E'$. 
Example 3.2. If $G$ is a locally compact groupoid, then the semi-group $(E_G, \ast)$ of $G$-orders partially ordered by the inclusion is a coarse structure with supremum and infimum respectively given by the union and the intersection. The same holds for the set $E_{G,c}$ of compact $G$-orders.

Definition 3.3. Let $E$ be a coarse structure. A $E$-filtered $C^*$-algebra $A$ is a $C^*$-algebra equipped with a family $(A_E)_{E \in E}$ of closed linear subspaces such that:

- $A_E \subseteq A_{E'}$ if $E \leq E'$;
- $A_E$ is stable by involution;
- $A_E \cdot A_{E'} \subseteq A_{E+E'}$;
- the subalgebra $\bigcup_{E \in E} A_E$ is dense in $A$.

Elements of $A_E$ for $E$ in $E$ are called elements with $E$-propagation (less than) $E$. If $A$ is unital, we also require that the identity $1$ is an element of $A_E$ for every $E$ in $E$.

Let $E$ be a coarse structure and let $A$ and $B$ be two $E$-filtered $C^*$-algebras. A $C^*$-algebras homomorphism $\phi : A \to B$ is called $E$-filtered if $\phi(A_E) \subseteq B_E$ for any $E$ in $E$.

Example 3.4. Let $G$ be a locally compact groupoid provided with a Haar system and let $A$ be a $G$-algebra. For any $G$-order $R$, we define $A \rtimes_r R$ as the closure in $A \rtimes_r G$ of the set of element $g$ in $C_c(X; G, r^*A)$ with support in $R$. Then

- $(A \rtimes_r R)_{R \in E_G}$ provides $A \rtimes_r G$ with a structure of $E_G$-filtered $C^*$-algebra;
- $(A \rtimes_r R)_{R \in E_{G,c}}$ provides $A \rtimes_r G$ with a structure of $E_{G,c}$-filtered $C^*$-algebra;
- if $H$ is an open subgroupoid of $G$, then $A \rtimes_r H$ is a $E_G$-filtered $C^*$-subalgebra of $A \rtimes_r G$, i.e $A \rtimes_r H$ is filtered by $(A \rtimes_r H) \cap (A \rtimes_r R)_{R \in E_G}$;
- In the same way, $A \rtimes_r H$ is an $E_{G,c}$-filtered $C^*$-subalgebra of $A \rtimes_r G$.

Notice that if $A$ and $B$ are two $G$-algebras and if $\phi : A \to B$ is a homomorphism of $G$-algebras. Then the induced homomorphism $\phi_G : A \rtimes_r G \to B \rtimes_r G$ is a $E_G$-filtered homomorphism. The same holds for $E_{G,c}$.

Let $E$ be a coarse structure and let $A$ be a $E$-filtered $C^*$-algebra. If $A$ is not unital, let us denote by $A^+$ its unitarization, i.e.,

$A^+ = \{(x, \lambda); x \in A, \lambda \in \mathbb{C}\}$

with the product

$$(x, \lambda)(x', \lambda') = (xx' + \lambda x' + \lambda' x, \lambda \lambda')$$
for all \((x, \lambda)\) and \((x', \lambda')\) in \(A^+\). Then \(A^+\) is \(\mathcal{E}\)-filtered with
\[
A^+_E = \{(x, \lambda); x \in A_E, \lambda \in \mathbb{C}\}
\]
for any \(E\) in \(\mathcal{E}\). We also define \(\rho_A : A^+ \to \mathbb{C}; (x, \lambda) \mapsto \lambda\).

Let \(\mathcal{E}\) be a coarse structure and let \(A\) be a unital \(\mathcal{E}\)-filtered \(C^*\)-algebra. For any positive number \(\varepsilon\) with \(\varepsilon < 1/4\) and any element \(E\) in \(\mathcal{E}\), we call

- an element \(u\) in \(A\) an \(\varepsilon\)-\(\mathcal{E}\)-unitary if \(u\) belongs to \(A_E\), \(\|u^* \cdot u - 1\| < \varepsilon\) and \(\|u \cdot u^* - 1\| < \varepsilon\). The set of \(\varepsilon\)-\(\mathcal{E}\)-unitaries on \(A\) will be denoted by \(\mathcal{U}^{\varepsilon, \mathcal{E}}(A)\).
- an element \(p\) in \(A\) an \(\varepsilon\)-\(\mathcal{E}\)-projection if \(p\) belongs to \(A_E\), \(p = p^*\) and \(\|p^2 - p\| < \varepsilon\). The set of \(\varepsilon\)-\(\mathcal{E}\)-projections on \(A\) will be denoted by \(\mathcal{P}^{\varepsilon, \mathcal{E}}(A)\).

Then \(\varepsilon\) is called the control and \(E\) is called the propagation of the \(\varepsilon\)-\(\mathcal{E}\)-projection or of the \(\varepsilon\)-\(\mathcal{E}\)-unitary. Notice that an \(\varepsilon\)-\(\mathcal{E}\)-unitary is invertible, and that if \(p\) is an \(\varepsilon\)-\(\mathcal{E}\)-projection in \(A\), then it has a spectral gap around \(1/2\) and then gives rise by functional calculus to a projection \(\kappa_0(p)\) in \(A\) such that \(\|p - \kappa_0(p)\| < 2\varepsilon\).

**Lemma 3.5.** Let \(\mathcal{E}\) be a coarse structure and let \(A\) be a unital \(\mathcal{E}\)-filtered \(C^*\)-algebra. Then for any \(\varepsilon\) in \((0, 1/12)\) and any \(E\) in \(\mathcal{E}\) the following holds.

(i) Let \(u\) and \(v\) be \(\varepsilon\)-\(\mathcal{E}\)-unitaries in \(A\), then \(\text{diag}(u, v)\) and \(\text{diag}(uv, 1)\) are homotopic as \(3\varepsilon\)-\(2\mathcal{E}\)-unitaries in \(M_2(A)\);
(ii) Let \(u\) be an \(\varepsilon\)-\(\mathcal{E}\)-unitary in \(A\), then \(\text{diag}(u, u^*)\) and \(I_2\) are homotopic as \(3\varepsilon\)-\(2\mathcal{E}\)-unitaries in \(M_2(A)\).

For any \(n\) integer, we set \(U_n^{\varepsilon, \mathcal{E}}(A) = \bigcup U_n^{\varepsilon, \mathcal{E}}(M_n(A))\) and \(P_n^{\varepsilon, \mathcal{E}}(A) = \bigcup P_n^{\varepsilon, \mathcal{E}}(M_n(A))\). Let us consider the inclusions
\[
P_n^{\varepsilon, \mathcal{E}}(A) \hookrightarrow P_{n+1}^{\varepsilon, \mathcal{E}}(A); p \mapsto \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}
\]
and
\[
U_n^{\varepsilon, \mathcal{E}}(A) \hookrightarrow U_{n+1}^{\varepsilon, \mathcal{E}}(A); u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}.
\]
This allows us to define
\[
U_{\infty}^{\varepsilon, \mathcal{E}}(A) = \bigcup_{n \in \mathbb{N}} U_n^{\varepsilon, \mathcal{E}}(A)
\]
and
\[
P_{\infty}^{\varepsilon, \mathcal{E}}(A) = \bigcup_{n \in \mathbb{N}} P_n^{\varepsilon, \mathcal{E}}(A).
\]
For a unital filtered \(C^*\)-algebra \(A\), we define the following equivalence relations on \(P_{\infty}^{\varepsilon, \mathcal{E}}(A) \times \mathbb{N}\) and on \(U_{\infty}^{\varepsilon, \mathcal{E}}(A)\):
• if \( p \) and \( q \) are elements of \( \mathbb{P}^{e,R}(A) \), \( l \) and \( l' \) are positive integers, 
\((p,l) \sim (q,l')\) if there exists a positive integer \( k \) and an element 
\( h \) of \( \mathbb{P}_{\infty}^{e,E}(A[0,1]) \) such that \( h(0) = \text{diag}(p, I_{k+l'}) \) and \( h(1) = \text{diag}(q, I_{k+l}) \).

• if \( u \) and \( v \) are elements of \( \mathbb{U}_{\infty}^{e,E}(A) \), \( u \sim v \) if there exists an element \( h \) of \( \mathbb{U}_{\infty}^{e,2E}(A[0,1]) \) such that \( h(0) = u \) and \( h(1) = v \).

If \( p \) is an element of \( \mathbb{P}_{\infty}^{e,E}(A) \) and \( l \) is an integer, we denote by \([p,l]_{e,E}\) the equivalence class of \((p,l)\) modulo \( \sim \) and if \( u \) is an element of \( \mathbb{U}_{\infty}^{e,E}(A) \) we denote by \([u]_{e,E}\) its equivalence class modulo \( \sim \).

**Definition 3.6.** Let \( \mathcal{E} \) be a coarse structure, let \( A \) be a \( \mathcal{E} \)-filtered \( C^* \)-algebra, let \( \mathcal{E} \) be an element of \( \mathcal{E} \) and \( \varepsilon \) be positive numbers with \( \varepsilon < 1/4 \).

We define:

(i) \( K_0^{\varepsilon,E}(A) = \mathbb{P}_{\infty}^{e,E}(A) \times \mathbb{N}/ \sim \) for \( A \) unital and
\[ K_0^{\varepsilon,E}(A) = \{ [p,l]_{\varepsilon,E} \in \mathbb{P}_{\varepsilon}^{e,E}(A^+) \times \mathbb{N}/ \sim \text{ such that } \kappa_0(\rho_A(p)) = l \} \]
for \( A \) non unital (\( \kappa_0(\rho_A(p)) \) being the spectral projection associated to \( \rho_A(p) \));

(ii) \( K_1^{\varepsilon,E}(A) = \mathbb{U}_{\infty}^{e,E}(A)/ \sim \) if \( A \) is unital and \( K_1^{\varepsilon,E}(A) = \mathbb{U}_{\infty}^{e,E}(A^+)/ \sim \) if not.

Then \( K_0^{\varepsilon,E}(A) \) turns to be an abelian group, where
\[ [p,l]_{\varepsilon,E} + [p',l']_{\varepsilon,E} = [\text{diag}(p,p'),l+l']_{\varepsilon,E} \]
for any \([p,l]_{\varepsilon,E} \) and \([p',l']_{\varepsilon,E} \) in \( K_0^{\varepsilon,E}(A) \). According to Corollary 3.5, \( K_1^{\varepsilon,E}(A) \) is equipped with a structure of abelian group such that
\[ [u]_{\varepsilon,E} + [u']_{\varepsilon,E} = [\text{diag}(u,v)]_{\varepsilon,E} \]
for any \([u]_{\varepsilon,E} \) and \([u']_{\varepsilon,E} \) in \( K_1^{\varepsilon,E}(A) \).

If \( \mathcal{E} \) is a coarse structure, we have for any \( \mathcal{E} \)-filtered \( C^* \)-algebra \( A \), any \( E, E' \) in \( \mathcal{E} \) and any positive numbers \( \varepsilon \) and \( \varepsilon' \) with \( \varepsilon \leq \varepsilon' < 1/4 \) and \( E \leq E' \) natural group homomorphisms called the structure maps:

- \( \iota^{\varepsilon,E}_0 : K_0^{\varepsilon,E}(A) \rightarrow K_0^{\varepsilon',E}(A) \), \( [p,l]_{\varepsilon,E} \mapsto [\kappa_0(p)] - [l] \) (where \( \kappa_0(p) \) is the spectral projection associated to \( p \));
- \( \iota^{\varepsilon,E}_1 : K_1^{\varepsilon,E}(A) \rightarrow K_1^{\varepsilon',E}(A) \), \( [u]_{\varepsilon,E} \mapsto [u] \); 
- \( \iota^{\varepsilon',E}_0 = \iota^{\varepsilon,0,E} \oplus \iota^{\varepsilon,0,E} \);
- \( \iota^{\varepsilon',E,E'} : K_0^{\varepsilon,E}(A) \rightarrow K_{\varepsilon',E'}(A) \), \( [p,l]_{\varepsilon,E} \mapsto [p,l]_{\varepsilon',E'} \);
- \( \iota^{\varepsilon',E,E'}_1 : K_1^{\varepsilon,E}(A) \rightarrow K_{1}^{\varepsilon',E'}(A) \), \( [u]_{\varepsilon,E} \mapsto [u]_{\varepsilon',E'} \);
- \( \iota^{\varepsilon',E,E',E'}_0 = \iota^{\varepsilon',0,E,E'} \oplus \iota^{\varepsilon',0,E,E'} \).

If some of the indices \( E, E' \) or \( \varepsilon, \varepsilon' \) are equal, we shall not repeat them in \( \iota^{\varepsilon',E,E,E'} \). In order to avoid overloading superscript in the structure maps, we shall write \( \iota^{\varepsilon',E,E'}_0 \) for \( \iota^{\varepsilon',E,E'}_0 \) when \( \varepsilon \) and \( E \) in the source are implicit, \( \iota^{\varepsilon',E,E'}_0 \) for \( \iota^{\varepsilon',E,E'}_0 \) when \( \varepsilon' \) and \( E' \) in the range are implicit and \( \iota^{\varepsilon,-} \) where \( \varepsilon \) and \( E \) in the source and \( \varepsilon' \) and \( E' \) in the range are both implicit.
There is in the setting of quantitative $K$-theory the equivalent of the standard form. We first deal with the even case.

**Lemma 3.7.** Let $\mathcal{E}$ be a coarse structure and let $A$ be a non unital $\mathcal{E}$-filtered $C^*$-algebra. Let $\varepsilon$ be in $(0, \frac{1}{16})$ and let $E$ be an element in $\mathcal{E}$. Then for any $x$ in $K_0^{\varepsilon,E}(A)$, there exist

- two integers $k$ and $n$ with $k \leq n$;
- a $9\varepsilon$-$E$-projection $q$ in $M_n(\tilde{A})$ such that $\rho_A(q) = \text{diag}(I_k, 0)$ and $x = [q, k]_{9\varepsilon,E}$ in $K_0^{9\varepsilon,E}(A)$.

We have a similar result in the odd case.

**Lemma 3.8.** Let $\mathcal{E}$ be a coarse structure and let $A$ be a non unital $\mathcal{E}$-filtered $C^*$-algebra. Let $\varepsilon$ be in $(0, \frac{1}{84})$ and let $E$ be an element in $\mathcal{E}$.

(i) for any $x$ in $K_1^{\varepsilon,E}(A)$, there exists a $21\varepsilon$-$E$-unitary $u$ in $M_n(A^+)$ such that $\rho_A(u) = I_n$ and $\iota_{\varepsilon, 21\varepsilon,E}(x) = [u]_{21\varepsilon,E}$ in $K_1^{21\varepsilon,E}(A)$; (ii) if $u$ and $v$ are two $\varepsilon$-$E$-unitaries in $M_n(A^+)$ such that $\rho_A(u) = \rho_A(v) = I_n$ and $[u]_{\varepsilon,E} = [v]_{\varepsilon,E}$ in $K_1^{\varepsilon,E}(A)$, then there exists an integer $k$ and a homotopy $(w_t)_{t \in [0, 1]}$ of $21\varepsilon$-$E$-unitaries of $M_{n+k}(A^+)$ between $\text{diag}(u, I_k)$ and $\text{diag}(v, I_k)$ such that $\rho_A(w_t) = I_{n+k}$ for every $t$ in $[0, 1]$.

Let $\mathcal{E}$ be a coarse structure and let $\phi : A \to B$ be a homomorphism of $\mathcal{E}$-filtered $C^*$-algebras. Then $\phi$ preserves $\varepsilon$-$E$-projections and $\varepsilon$-$E$-unitaries and hence $\phi$ induces for any $E$ in $\mathcal{E}$ and any $\varepsilon \in (0, 1/4)$ a group homomorphism

$$\phi_{\varepsilon,E} : K_\varepsilon^{\varepsilon,E}(A) \longrightarrow K_\varepsilon^{\varepsilon,E}(B).$$

Moreover quantitative $K$-theory is homotopy invariant with respect to homotopies which preserve $\mathcal{E}$-propagation [17, Lemma 1.26]. There is also a quantitative version of Morita equivalence [17, Proposition 1.28].

**Proposition 3.9.** Let $\mathcal{E}$ be a coarse structure, let $A$ be a $\mathcal{E}$-filtered algebra and let $\mathcal{H}$ be a separable Hilbert space, then the homomorphism

$$A \to \mathcal{K}(\mathcal{H}) \otimes A; \ a \mapsto \begin{pmatrix} a & 0 & \cdots \\ 0 & \ddots & \cdots \\ \vdots & \ddots & \ddots \end{pmatrix}$$

induces a ($\mathbb{Z}_2$-graded) group isomorphism (the Morita equivalence)

$$\mathcal{M}_{\varepsilon,E} : K_{\varepsilon,E}(A) \to K_{\varepsilon,E}(\mathcal{K}(\mathcal{H}) \otimes A)$$

for any $E$ in $\mathcal{E}$ and any $\varepsilon \in (0, 1/4)$.

The following observation establishes a connection between quantitative $K$-theory and classical $K$-theory (see [17, Remark 1.17]).
Proposition 3.10. Let $\mathcal{E}$ be a coarse structure.

(i) Let $A$ be a $\mathcal{E}$-filtered $C^*$-algebra. For any positive number $\varepsilon$ with $\varepsilon < \frac{1}{5}$ and any element $y$ of $K_*(A)$, there exists $E$ in $\mathcal{E}$ and an element $x$ of $K^{\varepsilon,E}_*(A)$ such that $\iota_{\varepsilon}^E(x) = y$.

(ii) There exists a positive number $\lambda_0 > 1$ such that for any $\mathcal{E}$-filtered $C^*$-algebra $A$, any $E$ in $\mathcal{E}$, any $\varepsilon$ in $(0, \frac{1}{5\lambda_0})$ and any element $x$ of $K^{\varepsilon,E}_*(A)$ for which $\iota_{\varepsilon}^E(x) = 0$ in $K_*(A)$, then there exists $E'$ in $\mathcal{E}$ with $E' \geq E$ such that $\iota_{\varepsilon}^{\lambda_0,\varepsilon,E',E'}(x) = 0$ in $K^{\lambda_0,\varepsilon,E',E'}_*(A)$.

Apply to $\mathcal{G}$-orders of a locally compact groupoid provided with a Haar system, we deduce the following result.

Lemma 3.11. Let $\mathcal{G}$ be a locally compact groupoid and let $A$ be a $\mathcal{G}$-algebra.

(i) for every $\varepsilon$ in $(0,1/4)$ and any $y$ in $K_* (A \rtimes \mathcal{G})$, there exist a compact $\mathcal{G}$-order $R$ such that $\iota_{\varepsilon}^R(x) = y$.

(ii) there exists $\lambda_0 \geq 1$ such that for any $\varepsilon$ in $(0, \frac{1}{5\lambda_0})$, any $\mathcal{G}$-order $R$ and any $x$ in $K^{\varepsilon,R}_*(A \rtimes \mathcal{G})$ satisfying $\iota_{\varepsilon}^R(x) = 0$ in $K_*(A \rtimes \mathcal{G})$, there exist a $\mathcal{G}$-order $R'$ with $R \subseteq R'$ such that $\iota_{\varepsilon}^{\lambda_0,\varepsilon,R,R'}(x) = 0$ in $K^{\lambda_0,\varepsilon,R,R'}_*(A \rtimes \mathcal{G})$. The constant $\lambda_0$ depends neither on $A$ nor on $\mathcal{G}$. Moreover, if $R$ is compact, then $R'$ can be chosen compact.

Quantitative $K$-theory inherits many features from $K$-theory. In particular there is a quantitative version of Bott periodicity and of the six-term exact sequence.

3.2. Controlled Mayer-Vietoris pair. The concept of controlled Mayer-Vietoris pair was introduced in [18] to streamline the "cut-and-pasting" technology developed by G. Yu in [25] to prove the Novikov conjecture for groups with finite asymptotic dimension. It was then extended in [2] to the general setting of $C^*$-algebras filtered by a coarse structure. It gives rise to a controlled exact sequence that allows to compute the $K$-theory by letting the propagation go to infinity.

Definition 3.12. Let $\mathcal{E}$ be a coarse structure, let $A$ be a $\mathcal{E}$-filtered $C^*$-algebra, let $E$ be an element of $\mathcal{E}$ and let $\Delta$ be a closed linear subspace of $A_E$. Then a sub-$C^*$-algebra $B$ of $A$ is called an $E$-controlled $\Delta$-neighborhood-$C^*$-algebra if

- $B$ is filtered by $(B \cap A_{E'})_{E' \in \mathcal{E}}$;
- $\Delta + A_{5E} \cdot \Delta + \Delta \cdot A_{5E} + A_{5E} \cdot \Delta \cdot A_{5E} \subseteq B$.

Definition 3.13. Let $\mathcal{E}$ be a coarse structure, let $A$ be a $\mathcal{E}$-filtered $C^*$-algebra, let $E$ be an element of $\mathcal{E}$ and let $c$ be a positive number. A complete coercive decomposition pair of order $E$ for $A$ is a pair $(\Delta_1, \Delta_2)$ of closed linear subspaces of $A_E$ such that for any $E'$ in $\mathcal{E}$ with $E' \leq\leq$
Definition 3.14. Let $S_1$ and $S_2$ be two subsets of a $C^*$-algebra $A$. The pair $(S_1, S_2)$ is said to have complete intersection approximation property (CIA) if there exists $c > 0$ such that for any positive number $\varepsilon$, any integer $n$, any $x \in M_n(S_1)$ and any $y \in M_n(S_2)$ with $||x - y|| < \varepsilon$, there exists $z \in M_n(S_1 \cap S_2)$ satisfying

$$||z - x|| < c\varepsilon, \quad ||z - y|| < c\varepsilon.$$ 

The positive number $c$ is called the **coercivity** of the pair $(S_1, S_2)$.

Definition 3.15. Let $E$ be a coarse structure, let $A$ be an $E$-filtered $C^*$-algebra and let $E$ be an element in $E$. An $E$-controlled Mayer-Vietoris pair for $A$ is a quadruple $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ such that for some positive number $c$.

(i) $(\Delta_1, \Delta_2)$ is a completely coercive decomposition pair for $A$ of order $E$ with coercivity $c$.

(ii) $A_{\Delta_i}$ is an $E$-controlled $\Delta_i$-neighborhood-$C^*$-algebra for $i = 1, 2$;

(iii) the pair $(A_{\Delta_1', E'}, A_{\Delta_2, E'})$ has the CIA property with coercivity $c$ for any $E'$ in $E$.

The positive number $c$ is called the **coercivity** of the $E$-controlled Mayer-Vietoris pair $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$.

Remark 3.16. In the above definition,

(i) $(\Delta_1 \cap A_{E'}, \Delta_2 \cap A_{E'}, A_{\Delta_1}, A_{\Delta_2})$ is an $E'$-controlled Mayer-Vietoris pair for any $E'$ in $E$ with $E' \leq E$ with same coercivity as $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$.

(ii) $A_{\Delta_1} \cap A_{\Delta_2}$ is $E$-filtered by $(A_{\Delta_1, E} \cap A_{\Delta_2, E})_{E \in E}$.

(iii) If $A$ is a unital, we will view $A_{\Delta_1}^\times$ the unitarization of $A_{\Delta_1}$ as $A_{\Delta_1} + \mathbb{C} \cdot 1 \subseteq A$ and similarly for $A_{\Delta_2}$ and $A_{\Delta_1} \cap A_{\Delta_2}$.

For purpose of rescaling the control and the propagation of an $\varepsilon$-$E$-projection or of an $\varepsilon$-$E$-unitary, we introduce the following concept of $E$-control pair.

Definition 3.17. A control pair is a pair $(\lambda, h)$, where

- $\lambda$ is a positive number with $\lambda > 1$;
- $h : (0, \frac{1}{\lambda}) \rightarrow \mathbb{N} \setminus \{0\}; \varepsilon \mapsto h_\varepsilon$ is a non-increasing map.

The set of control pairs is equipped with a partial order: $(\lambda, h) \leq (\lambda', h')$ if $\lambda \leq \lambda'$ and $h_\varepsilon \leq h'_\varepsilon$ for all $\varepsilon$ in $(0, \frac{1}{\lambda})$.

Proposition 3.18. For every positive number $c$, there exists a control pair $(\alpha, l)$ such that the following holds.
Let $\mathcal{E}$ be a coarse structure, let $A$ be a unital $\mathcal{E}$-filtered $C^*$-algebra, let $E$ be an element in $\mathcal{E}$, let $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ be controlled Mayer-Vietoris pair for $A$ of order $E$ and coercivity $c$.

Then for any $\varepsilon \in (0, \frac{1}{4c})$ and any $\varepsilon$-$E$-unitary $u$ in $A$ homotopic to 1, there exist a positive integer $k$ and $w_1$ and $w_2$ two $\alpha \varepsilon$-$E$-unitaries in $M_k(A)$ such that

1. $w_i - I_k$ is an element in the matrix algebra $M_k(A_{\Delta_i})$ for $i = 1, 2$;
2. for $i = 1, 2$, there exists a homotopy $(w_{i,t})_{t \in [0,1]}$ of $\alpha \varepsilon$-$E$-unitaries between 1 and $w_i$ such that $w_{i,t} - I_k \in M_k(A_{\Delta_i})$ for all $t$ in $[0,1]$;
3. $\|\text{diag}(u, I_{k-1}) - w_1 w_2\| < \alpha \varepsilon$.

If $A$ is a non unital $\mathcal{E}$-filtered $C^*$-algebra, then the same result holds for $u$ in $A^+$ such that $u - 1$ is in $A$ and $u$ is homotopic to 1 as an $\varepsilon$-$E$-unitary in $A^+$.

### 3.3. Applications to coercive decompositions of groupoids.

In this subsection, we show that coercive decompositions of groupoids give rise to controlled Mayer-Vietoris pairs.

For any $\mathcal{G}$-order $\mathcal{R}$ and any open subset $V$ of the unit space of $\mathcal{G}$, we define $A \ltimes_r \mathcal{R}_V$ as the closure of set of element $h$ in $C_c(\mathcal{G}, A)$ with support in $\mathcal{R}_V$.

**Lemma 3.19.** Let $\mathcal{G}$ be a locally compact groupoid with unit space $X$ provided with a Haar system and let $A$ be a $\mathcal{G}$-algebra. Let $V_1$ and $V_2$ be open subsets of $X$ with $X = V_1 \cup V_2$ and such that there exists a partition of the unit subordinated to $(V_1, V_2)$. Then $(A \ltimes_r \mathcal{R}_{V_1}, A \ltimes_r \mathcal{R}_{V_2})$ is for any $\mathcal{G}$-order a coercive $\mathcal{R}$-decomposition pair for $A \ltimes_r \mathcal{G}$ with coercivity 1.

**Proof.** Let $(\phi_1, \phi_2)$ be a partition of the unit for $X$ subordinated to $(V_1, V_2)$. Let us consider for $i = 1, 2$ the bounded operator

$$
\Lambda_{\phi_i}^s : A \ltimes_r \mathcal{G} \to A \ltimes_r \mathcal{G}
$$

of Lemma 2.5 and for any $x$ in $A \ltimes_r \mathcal{R}$, let us set $x_i = \Lambda_{\phi_i}^s(x)$. According to Lemma 2.3, we have $x = x_1 + x_2$, $\|x_i\| \leq 1$ and $x_i$ lies in $A \ltimes_r \mathcal{R}_{V_i}$ for $i = 1, 2$.

Notice that if $\mathcal{R}'$ is a $\mathcal{G}$-order with $\mathcal{R} \subseteq \mathcal{R}'$ and $V$ is an open subset of the unit space of $\mathcal{G}$, then $A \ltimes_r \mathcal{R}_V \subseteq A \ltimes_r \mathcal{R}'_V$.

**Lemma 3.20.** Let $\mathcal{G}$ be a locally compact groupoid with unit space $X$ provided with a Haar system. Let $\mathcal{R}$ and $\mathcal{R}'$ be $\mathcal{G}$-orders such that $\mathcal{R}' \subseteq \mathcal{R}$ and let $V$ be an open subset of $X$. Then $A \ltimes_r \mathcal{R}'_V \subseteq A \ltimes_r \mathcal{R}_V \cap A \ltimes_r \mathcal{R}'$.

**Proof.** We clearly have $A \ltimes_r \mathcal{R}'_V \subseteq A \ltimes_r \mathcal{R}_V \cap A \ltimes_r \mathcal{R}'$. Conversely, let $x$ be an element in $A \ltimes_r \mathcal{R}'_V \cap A \ltimes_r \mathcal{R}'$. Then there exist two sequences $(h_n)_{n \in \mathbb{N}}$ and $(h'_n)_{n \in \mathbb{N}}$ in $C_c(X; \mathcal{G}, r^*A)$ with support respectively in $\mathcal{R}_V$ and in $\mathcal{R}'$ converging to $x$. Let us set $K_n = s(\text{supp} h_n)$ for any integer $n$ and let $\phi_n : X \to [0,1]$ be a continuous compactly supported in $V$.
and such that \( \phi_n(x) = 1 \) for any \( x \) in \( K_n \). According to Lemma 2.23, we see that
\[
(h_n - h'_n \cdot \phi_n \circ s)_{n \in \mathbb{N}} = (A^*_{\phi_n}(h_n - h'_n))_{n \in \mathbb{N}}
\]
converges to zero in \( A \rtimes G \) and hence \( (h'_n \cdot \phi_n \circ s)_{n \in \mathbb{N}} \) is a sequence of elements in \( C_c(X; G, r^*A) \) with support in \( \mathcal{R}_V \) converging to \( x \). \( \square \)

As a consequence we obtain:

**Corollary 3.21.** Under the assumption of Lemma 3.19, then \( (A \rtimes_r \mathcal{R}_{V_1}, A \rtimes_r \mathcal{R}_{V_2}) \) is for any \( G \)-order \( \mathcal{R} \) a complete coercive \( \mathcal{R} \)-decomposition pair for \( A \rtimes_r G \) with coercivity 1.

**Proof.** Since for every integer \( n \) and for \( G \)-order \( \mathcal{R}' \) such that \( \mathcal{R}' \subseteq \mathcal{R} \), we have
\[
(A \rtimes_r \mathcal{R}') \otimes M_n(\mathbb{C}) \cap (A \rtimes_r \mathcal{R}_{V_1}) \otimes M_n(\mathbb{C}) = (A \otimes M_n(\mathbb{C})) \rtimes_r \mathcal{R}_{V_i}'
\]
for \( i = 1, 2 \), the result is a consequence of Lemma 3.19. \( \square \)

**Lemma 3.22.** Let \( G \) be a locally compact groupoid provided with a Haar system and let \( \mathcal{H} \) be a relatively clopen subgroupoid of \( G \) with unit space \( Y \). Then for any compactly supported continuous function \( \phi : Y \to \mathbb{C} \) and for any \( G \)-algebra \( A \), there exists a positive continuous linear map \( \Upsilon_\phi : A \rtimes_r G \to A \rtimes_r \mathcal{H} \) such that

(i) \( \Upsilon_\phi(f) = \phi \circ r \cdot f_{|\mathcal{H}} \circ \tilde{\phi} \circ s \), for any \( f \) in \( C_c(X; G, r^*A) \), where \( f_{|\mathcal{H}} : \mathcal{H} \to \mathbb{C} \) is the restriction of \( f \) to \( \mathcal{H} \);

(ii) \( \Upsilon_\phi \) is bounded in norm by \( \sup_{y \in Y} |\phi(y)|^2 \).

(iii) \( \Upsilon_\phi \) maps \( A \rtimes_r \mathcal{R} \) to \( A \rtimes_r \mathcal{R}_{/\mathcal{H}} \) for any \( G \)-order \( \mathcal{R} \).

**Proof.** Let us denote by \( \lambda = (\lambda^x)_{x \in X} \) the Haar system for \( G \). Then the restriction of \( \lambda \) to \( \mathcal{H} \) is a Haar system for \( \mathcal{H} \) that we shall denote by \( \lambda_{/\mathcal{H}} = (\lambda^\gamma)_{\gamma \in Y} \). Using the inclusion \( C_c(\mathcal{H}) \hookrightarrow C_c(\mathcal{G}) \), we see that \( L^2(\mathcal{H}) \) can be viewed as a \( C_0(X) \)-Hilbert submodule of \( L^2(\mathcal{G}) \) and therefore \( L^2(\mathcal{H}, A) \) is a right \( A \)-Hilbert submodule of \( L^2(\mathcal{G}, A) \). Since \( \mathcal{H} \) is clopen in \( \mathcal{G}^Y \), we get that \( \phi \circ r : \mathcal{H} \to \mathbb{C} \) extends to a continuous function \( \psi : \mathcal{G} \to \mathbb{C} \) defined by \( \psi(\gamma) = \phi \circ r(\gamma) \) if \( \gamma \) is in \( \mathcal{H} \) and \( \psi(\gamma) = 0 \) else. We have \( \sup \psi \leq \mathcal{H} \) and \( |\psi(\gamma)| \leq M \) for any \( \gamma \) in \( \mathcal{H} \) with \( M = \sup_{y \in Y} |\phi(y)| \). Define \( T_\psi : L^2(\mathcal{G}) \to L^2(\mathcal{G}) \) as the unique bounded operator extending the map
\[
C_c(\mathcal{G}) \to C_c(\mathcal{G}); \xi \mapsto \psi \xi.
\]
Then \( T_\psi \) has operator norm bounded by \( M \) and \( \text{Im} T_\psi \subseteq L^2(\mathcal{H}) \). In consequence \( T_\psi \otimes 1 \) maps \( L^2(\mathcal{G}, A) \) to \( L^2(\mathcal{H}, A) \). Consider the map
\[
\Upsilon_\phi : A \rtimes_r \mathcal{G} \to \mathcal{L}(L^2(\mathcal{H}, A)) \quad x \mapsto T_\psi \cdot x \cdot T_\psi^*
\]
Since \( T_\psi^* = T_\psi \), we deduce that \( \Upsilon_\phi \) is a positive operator with norm bounded by \( M^2 \). Moreover, for any \( f \) in \( C_c(X; G, r^*A) \), any \( \xi \) in
\[ C_c(Y; \mathcal{H}, s^*A/Y) \] and any \( \gamma \) in \( \mathcal{H} \), we have
\[
(\Upsilon_{\phi}(f) \cdot \xi)(\gamma) = \bar{\psi}(\gamma) \int_{\mathcal{H}^\gamma} \gamma^{-1}(f(\gamma'))\psi(\gamma^{-1}\gamma)\xi(\gamma^{-1}\gamma)d\lambda^r(\gamma)(\gamma')
\]
\[
= \bar{\psi}(\gamma) \int_{\mathcal{H}^\gamma} \gamma^{-1}(f(\gamma'))\psi(\gamma^{-1}\gamma)\xi(\gamma^{-1}\gamma)d\lambda^r(\gamma)(\gamma')
\]
\[
= \bar{\psi}(\gamma) \int_{\mathcal{H}^\gamma} \gamma^{-1}(f(\gamma'))\psi(\gamma^{-1}\gamma)\xi(\gamma^{-1}\gamma)d\lambda^r(\gamma)(\gamma')
\]
\[
= \int_{\mathcal{H}^\gamma} \gamma^{-1}(g(\gamma'))\xi(\gamma^{-1}\gamma)d\lambda^r(\gamma)(\gamma')
\]

With
\[
g(\gamma) = \bar{\psi}(\gamma) \cdot \psi^{-1}(\gamma) \cdot f(\gamma) = \bar{\psi}(\gamma) \cdot \psi^{-1}(\gamma) \cdot f(\gamma)
\]
(1)
for any \( \gamma \) in \( \mathcal{H} \). Moreover \( g \) has support in \( \text{supp} \psi \cap \text{supp} f \cap \text{supp}^{-1} \psi \).
Since \( \text{supp} \psi \) is closed in \( \mathcal{G} \) and contained in \( \mathcal{H} \), we deduce that \( g \) is in \( C_c(\mathcal{H}, r^*A) \). Hence \( \Upsilon_{\phi} \) maps \( C_c(X; \mathcal{G}, r^*A) \) to \( A \rtimes_r \mathcal{H} \) and by continuity maps \( A \rtimes_r G \) to \( A \rtimes_r \mathcal{H} \). It is then clear that \( \Upsilon \) satisfies the required conditions.

**Remark 3.23.** According to Equation (1) and since \( \mathcal{H} \) is open in \( \mathcal{G} \), we see that if \( f \) is in \( C_c(X; \mathcal{G}, r^*A) \), then \( \Upsilon_{\phi}(f) \) is supported in \( \mathcal{H} \cap \text{supp} f \).

**Corollary 3.24.** Let \( \mathcal{H} \) be a relatively clopen subgroupoid of a locally compact groupoid \( \mathcal{G} \), let \( \mathcal{R} \) be a \( \mathcal{G} \)-order and let \( V \) be an open subset of \( X \). Then we have
\[
(A \rtimes_r \mathcal{H}) \cap (A \rtimes_r \mathcal{R}) = A \rtimes_r \mathcal{R}_{/\mathcal{H}}
\]
for any \( \mathcal{G} \)-algebra \( A \).

**Proof.** We clearly have \( A \rtimes_r \mathcal{R}_{/\mathcal{H}} \subseteq (A \rtimes_r \mathcal{H}) \cap (A \rtimes_r \mathcal{R}) \). Conversely, let \( x \) be an element in \( (A \rtimes_r \mathcal{H}) \cap (A \rtimes_r \mathcal{R}) \). Then there exist two sequences \( (h_n)_{n \in \mathbb{N}} \) and \( (h'_n)_{n \in \mathbb{N}} \) in \( C_c(X; \mathcal{G}, r^*A) \) with support respectively in \( \mathcal{H} \) and in \( \mathcal{R} \) converging to \( x \). Let us set \( K_n = s(\text{supp} h_n) \cup r(\text{supp} h_n) \) for any integer \( n \) and let \( \phi_n : X \rightarrow [0, 1] \) be a continuous function compactly supported in the unit space of \( \mathcal{H} \) and such that \( \phi_n(x) = 1 \) for any \( x \) in \( K_n \). According to Lemma 3.22 we see that
\[
(h_n - \Upsilon_{\phi_n}(h'_n))_{n \in \mathbb{N}} = (\Upsilon_{\phi_n}(h_n - h'_n))_{n \in \mathbb{N}}
\]
converges to zero in \( A \rtimes_r \mathcal{G} \). In view of Remark 3.23 we deduce that \( \Upsilon_{\phi_n}(h'_n) \) has compact support in \( \mathcal{R} \cap \mathcal{H} = \mathcal{R}_{/\mathcal{H}} \), and thus \( (\Upsilon_{\phi_n}(h'_n))_{n \in \mathbb{N}} \) is a sequence in \( C_c(X; \mathcal{G}, r^*A) \) with support in \( \mathcal{R}_{/\mathcal{H}} \) converging to \( x \) and hence \( x \) belongs to \( A \rtimes_r \mathcal{R}_{/\mathcal{H}} \).

**Corollary 3.25.** Let \( \mathcal{G} \) be a locally compact groupoid with unit space \( X \) provided with a Haar system. Let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be relatively clopen subgroupoids of \( \mathcal{G} \). Then the following holds:

(i) \( A \rtimes_r (\mathcal{H}_1 \cap \mathcal{H}_2) = (A \rtimes_r \mathcal{H}_1) \cap (A \rtimes_r \mathcal{H}_2) \);
Proof. Let us prove the first point. We clearly have
\[ A \times_r R / \mathcal{H}_1 \cap \mathcal{H}_2 = (A \times_r R / \mathcal{H}_1) \cap (A \times_r R / \mathcal{H}_2) \] for any \( \mathcal{G} \)-order \( R \).

Conversely, let \( x \) be an element in \((A \times_r \mathcal{H}_1) \cap (A \times_r \mathcal{H}_2)\). Then there exist two sequences \((h_n)_n \in \mathbb{N}\) and \((h'_n)_n \in \mathbb{N}\) in \(C_c(X; \mathcal{G}, r^*A)\) with support respectively in \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) converging to \( x \). Let us set
\[ K_n = s(\text{supp } h_n) \cup r(\text{supp } h_n) \]
for any integer \( n \) and let \( \phi_n : X \to [0, 1] \) be a continuous function compactly supported in the unit space of \( \mathcal{H}_1 \) and such that \( \phi_n(x) = 1 \) for any \( x \) in \( K_n \). According to Lemma 3.22, we see that
\[(h_n - \Upsilon(h'_n))_{n \in \mathbb{N}} = (\Upsilon_n(h_n - h'_n))_{n \in \mathbb{N}}\]
converges to zero in \( A \times_r \mathcal{G} \). In view of Remark 3.23, we deduce that \( \Upsilon(h'_n) \) has compact support in \( \mathcal{H}_1 \cap \mathcal{H}_2 \), and thus that \( (\Upsilon_n(h'_n))_{n \in \mathbb{N}} \) is a sequence in \(C_c(X; \mathcal{G}, r^*A)\) with support in \( \mathcal{H}_1 \cap \mathcal{H}_2 \) converging to \( x \) and hence \( x \) belongs to \( A \times_r (\mathcal{H}_1 \cap \mathcal{H}_2) \). To prove the second point, let us observe that according to Corollary 3.24 we have
\[ A \times_r R / \mathcal{H}_1 \cap \mathcal{H}_2 = A \times_r R \cap A \times_r (\mathcal{H}_1 \cap \mathcal{H}_2) \]
The result is then a consequence of the first point. \( \square \)

Remark 3.26. The proof of the first point only requires \( \mathcal{H}_1 \) to be relatively clopen.

Theorem 3.27. Let \( \mathcal{G} \) be a locally compact groupoid provided with a Haar system, let \( A \) be a \( \mathcal{G} \)-algebra and let \( R \) and \( R' \) be \( \mathcal{G} \)-orders such that \( R^{\ast 6} \subseteq R' \). Assume that \((V_1, V_2, \mathcal{H}_1, \mathcal{H}_2)\) is a coercive \( R' \)-decomposition for \( \mathcal{G} \). Then
\[ (A \times_r R V_1, A \times_r R V_2, A \times_r \mathcal{H}_1, A \times_r \mathcal{H}_2) \]
is a coercive Mayer-Vietoris pair of order \( R \) with coercivity 2.

Proof. According to Corollary 3.21 \((A \times_r R V_1, A \times_r R V_2)\) is a complete coercive \( R \)-decomposition pair for \( A \times_r \mathcal{G} \) with coercivity 1. Let us prove that \( A \times_r \mathcal{H}_i \) is for \( i = 1, 2 \) an \( R \)-controlled \( A \times_r R V_i \)-neighborhood-\( C^* \)-algebra. By Lemma 3.24 we see that the \( C^* \)-algebra \( A \times_r \mathcal{H}_i \) is filtered by
\[ ((A \times_r \mathcal{H}_i) \cap (A \times_r R))_{R \in \mathcal{E}_R} = (A \times_r R / \mathcal{H}_i)_{R \in \mathcal{E}_R}. \]
Since \( R^{\ast 6} \subseteq R' \), \( R^{\ast 5} V_i \subseteq \mathcal{H}_i \) and \( \mathcal{H}_i \) is a subgroupoid of \( \mathcal{G} \), we see that
- \( \mathcal{R}_V \subseteq \mathcal{H}_i; \)
- \( \mathcal{R}^{\ast 5} \cdot \mathcal{R}_V \subseteq \mathcal{H}_i; \)
- \( \mathcal{R}_V \cdot \mathcal{R}^{\ast 5} \subseteq \mathcal{H}_i; \)
- \( \mathcal{R}^{\ast 5} \cdot \mathcal{R}_V \cdot \mathcal{R}^{\ast 5} \subseteq \mathcal{H}_i; \)
and hence
- \( A \times_r R V_i \subseteq A \times_r \mathcal{H}_i; \)
\[ A \times_r R_{V_1} \cdot A \times_r R^{s_5} \subseteq A \times_r \mathcal{H}_i; \]
\[ A \times_r R^{s_5} \cdot A \times_r \mathcal{R}_{V_1} \subseteq A \times_r \mathcal{H}_i; \]
\[ A \times_r R^{s_5} \cdot A \times_r \mathcal{R}_{V_1} \cdot A \times_r R^{s_5} \subseteq A \times_r \mathcal{H}_i. \]

This proves that \( A \times_r \mathcal{H}_i \) is an \( \mathcal{R} \)-controlled \( A \times_r \mathcal{R}_{V_1} \)-neighborhood-\( C^* \)-algebra.

Let us prove that \(( A \times_r \mathcal{H}_1, A \times_r \mathcal{H}_2)\) satisfies the CIA property with coercivity \(2\). Up to replace \( A \) by \( A \otimes M_n(\mathbb{C}) \), it is enough to show that for every positive number and any \( x_1 \) in \( A \times_r \mathcal{R}_{/\mathcal{H}_1} \) and \( x_2 \) in \( A \times_r \mathcal{R}_{/\mathcal{H}_2} \) such that \( \|x_1 - x_2\| < \varepsilon \) then there exists \( z \) in \(( A \times_r \mathcal{R}_{/\mathcal{H}_1}) \cap ( A \times_r \mathcal{R}_{/\mathcal{H}_2})\) such that \( \|z - x_1\| < \varepsilon \). Notice that in view of Corollary \(3.25\) we have

\[ (A \times_r \mathcal{R}_{/\mathcal{H}_1}) \cap (A \times_r \mathcal{R}_{/\mathcal{H}_2}) = A \times_r \mathcal{R}_{/(\mathcal{H}_1 \cap \mathcal{H}_2)}. \]

Set \( \alpha = \varepsilon - \|x_1 - x_2\| \) and let \( h \) be an element in \( C_c(X; \mathcal{G}, r^*A) \) with support included in \( \mathcal{R}_{/\mathcal{H}_1} \) and such that \( |x_1 - h| < \alpha/2 \). Let \( \phi : \mathcal{G} \to [0,1] \) be a continuous function compactly supported in the space of unit of \( \mathcal{H}_1 \) and such that \( \phi(x) = 1 \) for all \( x \) in \( r(\text{supp} \ h) \cup s(\text{supp} \ h) \).

According to Lemma \(3.22\) we see that \( \Upsilon_\phi(h) = h, \ \Upsilon_\phi(x_2) \) belongs to \( A \times_r \mathcal{R}_{/\mathcal{H}_1} \) and

\[
\|x_1 - \Upsilon_\phi(x_2)\| < \|x_1 - h\| + \|h - \Upsilon_\phi(x_2)\| < \frac{\alpha}{2} + |\Upsilon_\phi(h) - \Upsilon_\phi(x_2)| < \frac{\alpha}{2} + \|h - x_2\| < \frac{\alpha}{2} + \|h - x_1\| + \|x_1 - x_2\| < \frac{\alpha}{2} + \frac{\alpha}{2} + \|x_1 - x_2\| < \varepsilon.
\]

But \( x_2 \) is a limit of elements of \( C_c(X; \mathcal{G}, r^*A) \) with support in \( \mathcal{R}_{/\mathcal{H}_2} \) and hence according to Remark \(3.23\) \( \Upsilon_\phi(x_2) \) is also a limit of element of \( C_c(\mathcal{G}, r^*A) \) with support in \( \mathcal{R}_{/\mathcal{H}_2} \) and therefore \( \Upsilon_\phi(x_2) \) belongs to \( A \times_r \mathcal{R}_{/\mathcal{H}_2} \).

\[ \square \]

**Remark 3.28.** In view of \cite{13} Theorem 6.1, we can show in the same way that under assumptions of Lemma \(3.19\) if \( A \) is an exact and separable, then \(( A \times_r \mathcal{R}_{V_1}, A \times_r \mathcal{R}_{V_2}, A \times_r \mathcal{H}_1, A \times_r \mathcal{H}_2)\) is a \( \mathcal{R} \)-controlled nuclear Mayer-Vietoris pair in sense of \cite{18} Definition 4.8 for \( A \times_r \mathcal{G} \) with coercivity \(2\).

### 3.4. The Mayer-Vietoris controlled exact sequence.

A coercive Mayer-Vietoris pair gives rise to a controlled six term exact sequence that computes the quantitative \( K \)-theory up to the order of the pair and up to rescaling by a control pair. In view of Theorem \(3.27\), it turns out that this controlled Mayer-Vietoris six term exact sequence was shaped out for \( K \)-theory computations in the setting of coercive decompositions for groupoids.
**Notation 3.29.** Let $\mathcal{E}$ be a coarse structure, let $A$ be a $\mathcal{E}$-filtered $C^*$-algebra, let $E$ be an element in $\mathcal{E}$ and let $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ be a $E$-controlled Mayer-Vietoris pair for $A$. We denote by

- $j_{\Delta_1} : A_{\Delta_1} \to A$;
- $j_{\Delta_2} : A_{\Delta_2} \to A$;
- $j_{\Delta_1, \Delta_2} : A_{\Delta_1} \cap A_{\Delta_2} \to A_{\Delta_1}$;
- $j_{\Delta_2, \Delta_1} : A_{\Delta_1} \cap A_{\Delta_2} \to A_{\Delta_2}$

the obvious inclusion maps.

**Proposition 3.30.** For every positive number $c$, there exists a control pair $(\alpha, l)$ such that for any coarse structure $\mathcal{E}$, any $\mathcal{E}$-filtered $C^*$-algebra $A$, any $E$ in $\mathcal{E}$ and any $E$-controlled Mayer-Vietoris pair

$$(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$$

for $A$ with coercitivity $c$, then the following holds:

- for any $\varepsilon$ in $(0, \frac{1}{4c})$, for any $E'$ in $\mathcal{E}$ such that $l \cdot E' \leq E$, for any $y_1$ in $K^E_{\alpha, l, E'}(A_{\Delta_1})$ and any $y_2$ in $K^E_{\alpha, l, E'}(A_{\Delta_2})$ such that
  $$j_{\Delta_1, \Delta_2,*}^E(y_1) = j_{\Delta_2, \Delta_1,*}^E(y_2)$$
  in $K^E_{\alpha, l, E'}(A)$, then there exists an element $x$ in $K^E_{\alpha, l, E'}(A_{\Delta_1} \cap A_{\Delta_2})$ such that
  $$j_{\Delta_1, \Delta_2,*}^{\alpha, E'}(x) = i_{\varepsilon}^{-\alpha, E'}(y_1)$$
  in $K^E_{\alpha, l, E'}(A_{\Delta_1})$ and
  $$j_{\Delta_2, \Delta_1,*}^{\alpha, E'}(x) = i_{\varepsilon}^{-\alpha, E'}(y_2)$$
  in $K^E_{\alpha, l, E'}(A_{\Delta_2})$.

In other word, this means that the composition

$$K_{\alpha, l, E'}(A_{\Delta_1} \cap A_{\Delta_2}) \xrightarrow{(j_{\Delta_1, \Delta_2,*}^{\alpha, E'}, j_{\Delta_2, \Delta_1,*}^{\alpha, E'})} K_{\alpha, l, E'}(A_{\Delta_1}) \oplus K_{\alpha, l, E'}(A_{\Delta_2}) \xrightarrow{(j_{\Delta_1,*}^{\alpha, E'}, j_{\Delta_2,*}^{\alpha, E'})} K_{\alpha, l, E'}(A)$$

is exact at order $E$, up to rescaling by $(\alpha, l)$. We shall see later on that this composition can be completed at order $E$, up to rescaling by a control pair, in a six term exact sequence (called in [2, Theorem 8.4] $E$-controlled Mayer-Vietoris exact sequence).

We introduce first the quantitative boundary map that fits into the controlled Mayer-Vietoris exact.

**Lemma 3.31.** For every positive number $c$, there exists a control pair $(\lambda, k)$ such that the following holds:

- Let $\mathcal{E}$ be a coarse structure, let $A$ be a unital $\mathcal{E}$-filtered $C^*$-algebra, let $E$ be an element in $\mathcal{E}$ and let $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ be a $E$-controlled Mayer-Vietoris pair for $A$ with coercitivity $c$. Let $E'$ be an element in $\mathcal{E}$ such that $2 \cdot E' \leq E$, let $\varepsilon$ be in $(0, \frac{1}{4c})$, let $m$ and $n$ be integers and let $u$ be in $U^E_{m}(A)$, let $v$ be in $U^E_{m}(A)$ and let $w_1, w_2$ be $\varepsilon$-$E'$-unitaries in $M_{n+m}(A)$ such that
• $w_i - I_{n+m}$ is an element in the matrix algebra $M_{n+m}(A_{\Delta_1})$ for $i = 1, 2$;
• $\| \text{diag}(u, v) - w_1 w_2 \| < \varepsilon$.

Then,

(i) there exists a $\lambda \varepsilon$-homotopy $q$ in $M_{n+m}(A)$ such that

• $q - \text{diag}(I_n, 0)$ is an element in the matrix algebra $M_{n+m}(A_{\Delta_1} \cap A_{\Delta_2})$;
• $\| q - w_1^{*} \text{diag}(I_n, 0) w_1 \| < \lambda \varepsilon$;
• $\| q - w_2^{*} \text{diag}(I_n, 0) w_2 \| < \lambda \varepsilon$.

(ii) if $q$ and $q'$ are two $\lambda \varepsilon$-homotopies in $M_{n+m}(A)$ that satisfy the first point, then

$$[q, n]_{\lambda \varepsilon, k, E'} = [q', n]_{\lambda \varepsilon, k, E'}$$

in $K_0^{\lambda \varepsilon, k, E'}(A_{\Delta_1} \cap A_{\Delta_2})$.

(iii) Let $(w_1, w_2)$ and $(w'_1, w'_2)$ be two pairs of $\varepsilon$-unitaries in $M_{n+m}(A)$ satisfying the assumption of the lemma and let $q$ and $q'$ be $\lambda \varepsilon$-homotopies in $M_{n+m}(A)$ that satisfy the first point relatively to respectively $(w_1, w_2)$ and $(w'_1, w'_2)$, then

$$[q, n]_{\lambda \varepsilon, 2k, E'} = [q', n]_{\lambda \varepsilon, 2k, E'}$$

in $K_0^{\lambda \varepsilon, 2k, E'}(A_{\Delta_1} \cap A_{\Delta_2})$.

Remark 3.32. We have a similar statement in the non-unital case with $u$ in $U_n^{E'}(A^+)$ and $v$ in $U_m^{E'}(A^+)$ such that $u - I_n$ and $v - I_m$ have coefficients in $A$.

We recall now the definition of the quantitative boundary map associated to a controlled Mayer-Vietoris pair. Let $E$ be a coarse structure, let $A$ be a $E$-filtered $C^*$-algebra and let $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ be a $E$-controlled Mayer-Vietoris pair for $A$ with coercitivity $c$. Assume first that $A$ is unital.

Let $(\alpha, l)$ be a control pair as is Proposition 3.18. For any $\varepsilon \in (0, \frac{1}{4w})$, any $E'$ in $E$ such that $2E' \subseteq E$ and any $\varepsilon$-unitary $u$ in $M_n(A)$, let $v$ be an $\varepsilon$-unitary in some $M_m(A)$ such that $\text{diag}(u, v)$ is homotopic to $I_{n+m}$ as a $3\varepsilon \cdot 2E'$-unitary in $M_{n+m}(A)$, we can take for instance $v = u^*$ (see Lemma 3.5). According to Proposition 3.18 and up to replacing $v$ by $\text{diag}(v, I_k)$ for some integer $k$, there exists $w_1$ and $w_2$ two $3\varepsilon \cdot 2E'$-unitaries in $M_{n+m}(A)$ such that

• $w_i - I_{n+m}$ is an element in the matrix algebra $M_{n+m}(A_{\Delta_1})$ for $i = 1, 2$;
• for $i = 1, 2$, there exists a homotopy $(w_{i,t})_{t \in [0, 1]}$ of $3\varepsilon \cdot 2E'$-unitaries between 1 and $w_1$ such that $w_{1,t} - I_{n+m}$ is an element in the matrix algebra $M_{n+m}(A_{\Delta_1})$ for all $t$ in $[0, 1]$.
• $\| \text{diag}(u, v) - w_1 w_2 \| < 3\varepsilon$. 


Let \((\lambda, k)\) be the control pair of Lemma [\ref{lem:control}](recall that \((\lambda, k)\) depends only on the coercivity \(c\)). Then if \(\varepsilon\) is in \((0, \frac{1}{2\alpha c})\), there exists a \(3\alpha\varepsilon\)-projection \(q\) in \(M_{n+m}(A)\) such that

- \(q - \text{diag}(I_\varepsilon, 0)\) is an element in the matrix algebra 
  \[M_{n+m}(A_{\Delta_1 \cap \Delta_2});\]
- \(\|q - w_1^* \text{diag}(I_\varepsilon, 0)w_1\| < 3\alpha \varepsilon;\)
- \(\|q - w_2^* \text{diag}(I_\varepsilon, 0)w_2\| < 3\alpha \varepsilon.\)

In view of the second point of Lemma [\ref{lem:control}] the class \([q, n]_{3\alpha\lambda\varepsilon, 4l_{3\varepsilon}k_{3\alpha\varepsilon}E'}\) in

\[K_0^{3\alpha\lambda\varepsilon, 4l_{3\varepsilon}k_{3\alpha\varepsilon}E'}(A_{\Delta_1 \cap \Delta_2}) \]

does not depend on the choice of \(w_1, w_2\) or \(q\). Set then \(\alpha_c = 3\alpha\lambda^3\) and

\[k_{c,\varepsilon} : \left(0, \frac{1}{4\alpha_c}\right) \to \mathbb{N} \setminus \{0\}, \varepsilon \mapsto 4l_{3\varepsilon}k_{3\alpha\varepsilon}\]

and define \(\partial^{c,E'}_{\Delta_1,\Delta_2,1}([u], E) = [q, n]_{\alpha_c, k_c, E'}.\) Then for any \(\varepsilon\) in \((0, \frac{1}{4\alpha_c})\) and any \(E'\) in \(\mathcal{E}\) such that \(k_{c,\varepsilon} E' \leq E\), the morphism

\[\partial^{c,E'}_{\Delta_1,\Delta_2,1} : K_{0}^{c, E'}(A) \to K_0^{\alpha_c, k_c E'}(A_{\Delta_1 \cap \Delta_2})\]

is well defined.

In the non unital case \(\partial^{c,s}_{\Delta_1,\Delta_2,1}\) is defined similarly by noticing that in view of Lemma [\ref{lem:quantitative}] and up to rescaling \(\varepsilon\), every element \(x\) in \(K_1^{c, E}(A)\) is the class of an \(\varepsilon\)-unitary \(u\) in some \(M_n(A^+_{\varepsilon})\) such that \(u - I_n\) has coefficients in \(A\). It is straightforward to check that \(\partial^{c,s}_{\Delta_1,\Delta_2,1}\) is compatible with the structure morphisms, i.e

\[t_{**}^{c, E'} \circ \partial^{c,E'}_{\Delta_1,\Delta_2,1} = \partial^{c', E''}_{\Delta_1,\Delta_2,1} \circ t_{**}^{c', E'},\]

for any \(\varepsilon'\) and \(\varepsilon''\) in \((0, \frac{1}{4\alpha_c})\) and any \(E'\) and \(E''\) in \(\mathcal{E}\) with \(E' \leq E''\) and \(k_{c,\varepsilon'} E' \leq k_{c,\varepsilon''} E'' \leq E\).

In the even case, the quantitative boundary map associated to a controlled Mayer-Vietoris pair is defined by using controlled Bott periodicity [\ref{bott} Section 2]. Up to rescale the control pair \((\alpha_c, k_c)\), we obtain for any \(\varepsilon\) in \((0, \frac{1}{4\alpha_c})\) and any \(E'\) in \(\mathcal{E}\) such that \(k_{c,\varepsilon} E' \leq E\), the morphism

\[\partial^{c,E'}_{\Delta_1,\Delta_2,0} : K_0^{c, E'}(A) \to K_1^{\alpha_c, k_c E'}(A_{\Delta_1 \cap \Delta_2}).\]

We set then

\[\partial^{c,E'}_{\Delta_1,\Delta_2,*} = \partial^{c,E'}_{\Delta_1,\Delta_2,0} \oplus \partial^{c,E'}_{\Delta_1,\Delta_2,1}.\]

Then

\[\partial^{c,E'}_{\Delta_1,\Delta_2,*} : K_0^{c, E'}(A) \to K_0^{\alpha_c, k_c E'}(A_{\Delta_1 \cap \Delta_2})\]

is a morphism of degree 1 compatible with the structure morphisms called the \(\varepsilon\)-\(E'\)-quantitative Mayer-Vietoris boundary map.
Notice that the quantitative boundary map associated to an $E$-controlled Mayer-Vietoris pair is natural in the following sense: let $E$ be a coarse structure, let $A$ and $B$ be $E$-filtered $C^*$-algebras, let $E$ be an element in $E$, let $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ and $(\Delta'_1, \Delta'_2, B_{\Delta'_1}, B_{\Delta'_2})$ be respectively $E$-controlled Mayer-Vietoris pairs for $A$ and $B$ with coercivity $c$ and let $f : A \to B$ be a homomorphism of $E$-filtered $C^*$-algebras such that $f(\Delta_1) \subseteq \Delta'_1$, $f(\Delta_2) \subseteq \Delta'_2$, $f(A_{\Delta_1}) \subseteq B_{\Delta'_1}$ and $f(A_{\Delta_2}) \subseteq B_{\Delta'_2}$. Then we have

\[ f_{A_{\Delta_1} \cap A_{\Delta_2}} \circ \partial^{E'}_{\Delta_1, \Delta_2, *} = \partial^{E'}_{\Delta'_1, \Delta'_2, *} \circ f^{E'}_*, \tag{2} \]

for any $\varepsilon$ in $(0, \frac{1}{4c})$ and any $E'$ in $E$ with $k_{E, E'} \leq E$, where $f_{A_{\Delta_1} \cap A_{\Delta_2}} : A_{\Delta_1} \cap A_{\Delta_2} \to B_{\Delta'_1} \cap B_{\Delta'_2}$ is the restriction of $f$ to $A_{\Delta_1} \cap A_{\Delta_2}$.

We now investigate controlled exactness at the source for the quantitative boundary map associated to a controlled Mayer-Vietoris pair. We start with the following lemma which will play a key role in the proof of the main theorem.

**Lemma 3.33.** There exists a control pair $(\lambda, l)$ such that

- for any coarse structure $\mathcal{E}$, any unital $\mathcal{E}$-filtered $C^*$-algebra $A$ and any subalgebras $A_1$ and $A_2$ of $A$ such that $A_1$, $A_2$ and $A_1 \cap A_2$ are respectively filtered by $(A_1 \cap A_2)_E \in \mathcal{E}$, $(A_2 \cap A_2)_E \in \mathcal{E}$ and $(A_1 \cap A_2)_E \in \mathcal{E}$;

- for any positive number $\varepsilon$ with $\varepsilon < \frac{1}{4\lambda}$, any $E$ in $\mathcal{E}$, any integers $n$ and $m$ and any $\varepsilon$-unitaries $u_1$ in $M_n(A)$ and $u_2$ in $M_m(A)$;

- for any $\varepsilon$-unitaries $v_1$ and $v_2$ respectively in $M_{n+m}(A_1^+)$ and $M_{n+m}(A_2^+)$ such that

  - $\| \text{diag}(u_1, u_2) - v_1 v_2 \| < \varepsilon$;

  - there exists an $\varepsilon$-projection $p$ in $M_{n+m}(A)$ such that $q = \text{diag}(I_n, 0)$ is in $M_{n+m}(A_1 \cap A_2)$, $\| q - v_1^* \text{diag}(I_n, 0) v_1 \| < \varepsilon$ and $[q, n]_{E, E} = 0$ in $K_0^E(A_1 \cap A_2)$.

Then there exists an integer $k$ and $E$-unitaries $w_1$ and $w_2$ respectively in $M_{n+k}(A_1^+)$ and $M_{n+k}(A_2^+)$ such that

- $\| \text{diag}(u_1, I_k) - \text{diag}(w_1, w_2) \| < \lambda \varepsilon$.

Moreover, if $v_i - I_{n+k}$ lies in $M_{n+k}(A_1)$ for $i = 1, 2$ then $w_1$ and $w_2$ can be chosen such that $w_i - I_{n+k}$ lies in $M_{n+m}(A_i)$ for $i = 1, 2$.

As a consequence, we deduce controlled exactness at the source for the quantitative boundary map associated to a controlled Mayer-Vietoris pair. Moreover, this controlled exactness is persistent at any order.

**Corollary 3.34.** For any positive number $c$, there exists a control pair $(\lambda, l)$ such that

- for any coarse structure $\mathcal{E}$ and any $\mathcal{E}$-filtered $C^*$-algebra $A$;
for any \( E \) in \( \mathcal{E} \) and any \( E \)-controlled Mayer-Vietoris pair \((\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})\) for \( A \) with coercitivity \( c \).

for any positive numbers \( \varepsilon' \) and \( \varepsilon'' \) with \( 0 < \alpha \varepsilon' \leq \varepsilon'' < \frac{1}{\lambda} \) and any \( E' \) and \( E'' \) in \( \mathcal{E} \) with \( k_{\varepsilon,\varepsilon'} E' \leq E \) and \( k_{\varepsilon,\varepsilon''} E' \leq E'' \)

then for any \( y \) in \( K_{*}^{E',E''}(A) \) such that
\[
\iota_{*}^{E'',E''} \circ \partial_{\Delta_1,\Delta_2,*}^{E',E''}(y) = 0
\]
in \( K_{*+1}^{E'',E''}(A_{\Delta_1} \cap A_{\Delta_2}) \), there exist \( x_1 \) in \( K_{*}^{E'',E''}(A_{\Delta_1}) \) and \( x_2 \) in \( K_{*}^{E'',E''}(A_{\Delta_2}) \) such that
\[
\iota_{*}^{E'',E''}(y) = J_{\Delta_1,*}^{E''}(x_1) - J_{\Delta_2,*}^{E''}(x_2).
\]

We now investigate the controlled exactness at the range for the quantitative boundary map associated to a controlled Mayer-Vietoris pair. We start with the following lemma which will play also a key role in the proof of the main theorem.

**Lemma 3.35.** There exists a control pair \((\lambda, h)\) such that the following holds:

- Let \( \mathcal{E} \) be a coarse structure, let \( A \) be a unital \( \mathcal{E} \)-filtered \( C^* \)-algebra and let \( A_1 \) and \( A_2 \) be subalgebras of \( A \) such that \( A_1 \) and \( A_2 \) and \( A_1 \cap A_2 \) are respectively filtered by \((A_1 \cap A_E)_{E \in \mathcal{E}}, (A_2 \cap A_E)_{E \in \mathcal{E}} \) and \((A_1 \cap A_2 \cap A_E)_{E \in \mathcal{E}} \);
- let \( \varepsilon \) be in \((0, \frac{1}{\lambda})\) and let \( E \) be in \( \mathcal{E} \);
- let \( n \) and \( N \) be positive integers with \( n \leq N \) and let \( p \) an \( \varepsilon \)-projection in \( M_N((A_1 \cap A_2)^+) \) such that \( \rho_{A_1 \cap A_2}(p) = \text{diag}(I_n, 0) \).

Assume that

- \( p \) is homotopic to \( \text{diag}(I_n, 0) \) as an \( \varepsilon \)-projection in \( M_N(A_1^+) \);
- \( p \) is homotopic to \( \text{diag}(I_n, 0) \) as an \( \varepsilon \)-projection in \( M_N(A_2^+) \).

Then there exist an integer \( N' \) with \( N' \geq N \), and four \( \lambda \varepsilon \)-unitaries \( w_1 \) and \( w_2 \) in \( M_{N'}(A) \), \( u \) in \( M_n(A) \) and \( v \) in \( M_{N'-n}(A) \) such that

- \( w_i - I_{N'} \) is an element in \( M_{N'}(A_i) \) for \( i = 1, 2 \);
- \[
\|w_i^* \text{diag}(I_n, 0) w_i - \text{diag}(p, 0)\| < \lambda \varepsilon
\]
and
\[
\|w_2 \text{diag}(I_n, 0) w_2^* - \text{diag}(p, 0)\| < \lambda \varepsilon.
\]

- for \( i = 1, 2 \), then \( w_i \) is connected to \( I_{N'} \) by a homotopy of \( \lambda \varepsilon \)-\( h \)-\( E \)-unitaries \( (w_i,t)_{t \in [0,1]} \) in \( M_{N'}(A) \) such that \( w_i(t) - I_{N'} \) is in \( M_{N'}(A_i) \) for all \( t \) in \([0,1]\).

As a consequence, we deduce controlled exactness at the range for the quantitative boundary map associated to a controlled Mayer-Vietoris pair.
Proposition 3.36. For every positive number \( c \), there exists a control pair \( (\alpha, \lambda) \) such that for any coarse structure \( \mathcal{E} \), any \( \mathcal{E} \)-filtered \( C^* \)-algebra \( A \), any \( E \) in \( \mathcal{E} \) and any \( E \)-controlled Mayer-Vietoris pair \( (\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2}) \) for \( A \) with coercitivity \( c \), then the following holds:

for any \( \varepsilon \) in \( (0, \frac{1}{4\lambda c}) \) and any \( E' \) in \( \mathcal{E} \) with \( k_{c,\lambda \varepsilon} E' \leq E \), any \( y \) in \( K_*^{\varepsilon, E'}(A_{\Delta_1} \cap A_{\Delta_2}) \) such that

\[
J_{\Delta_1, \Delta_2, \ast}^{\varepsilon, E'}(y) = 0
\]
in \( K_*^{\varepsilon, E'}(A_{\Delta_1}) \) and

\[
J_{\Delta_2, \Delta_1, \ast}^{\varepsilon, E'}(y) = 0
\]
in \( K_*^{\varepsilon, E'}(A_{\Delta_2}) \), there exists an element \( x \) in \( K_*^{\lambda \varepsilon, E'}(A) \) such that

\[
\partial x_{\Delta_1, \Delta_2, \ast}(x) = \iota_{\varepsilon, E'}(y)
\]
in \( K_*^{\alpha, \lambda \varepsilon, k_{c,\lambda \varepsilon}} E'(A_{\Delta_1} \cap A_{\Delta_2}) \).

Example 3.37. With notations of Theorem 3.27 we denote by

\[
\begin{align*}
J_{1,2, A} : A \times_{r}(H_1 \cap H_2) & \hookrightarrow A \times_{r} H_1 \\
J_{2,1, A} : A \times_{r}(H_1 \cap H_2) & \hookrightarrow A \times_{r} H_2 \\
J_{1, A} : A \times_{r} H_1 & \hookrightarrow A \times_{r} H \\
J_{2, A} : A \times_{r} H_2 & \hookrightarrow A \times_{r} H
\end{align*}
\]

the obvious inclusions and for any \( \varepsilon \) in \( (0, 1) \) and any \( R_0 \) in \( \mathcal{E} \) such that \( k_{c,\varepsilon} R_0 \subseteq R \), we denote by

\[
\partial x_{\varepsilon, R_0} = \partial x_{A \times_{r} (H_1 \cap H_2), A \times_{r} H, \ast} : K_*^{\varepsilon, R_0}(A \times_{r} H) \to K_*^{\alpha, \lambda \varepsilon, k_{c,\varepsilon} R_0}(A \times_{r} (H_1 \cap H_2))
\]
\( \varepsilon \)-\( R_0 \)-quantitative Mayer-Vietoris boundary map associated to the \( R_0 \)-controlled Mayer-Vietoris pair

\[
(A \times_{r} R_{V_1}, A \times_{r} R_{V_2}, A \times_{r} H_1, A \times_{r} H_2).
\]

4. Statement of the main result

Theorem 4.1. Let \( \mathcal{G} \) be a locally compact groupoid provided with a Haar system and let \( f : A \to B \) be a homomorphism of \( \mathcal{G} \)-algebras. Let us assume that there exists a subset \( \mathcal{D} \) of relatively clopen groupoids of \( \mathcal{G} \), closed under taking relatively clopen subgroupoids and such that

(i) \( f_{\mathcal{H}, \ast} : K_*(A \times_{r} \mathcal{H}) \to K_*(B \times_{r} \mathcal{H}) \) is an isomorphism for any \( \mathcal{H} \) in \( \mathcal{D} \)

(ii) \( \mathcal{G} \) has finite \( \mathcal{D} \)-complexity.

Then

\[
f_{\mathcal{G}, \ast} : K_*(A \times_{r} \mathcal{G}) \to K_*(B \times_{r} \mathcal{G})
\]
is an isomorphism.

Theorem 4.1 is a consequence of the following result:
Lemma 4.2. Let $\mathcal{G}$ be a locally groupoid provided with a Haar system, let $\mathcal{H}$ be a relatively clopen subgroupoid of $\mathcal{G}$, let $f : A \to B$ be a homomorphism of $\mathcal{G}$-algebras. Let us assume that there exists a subset $\mathcal{D}$ of relatively clopen groupoids of $\mathcal{G}$, closed under taking relatively clopen subgroupoids and such that

(i) $f_{H, *} : K_*(A \rtimes_r H') \to K_*(B \rtimes_r H')$ is an isomorphism for any $H'$ in $\mathcal{D}$.

(ii) $\mathcal{H}$ is $\mathcal{D}$-decomposable.

Then

$f_{H, *} : K_*(A \rtimes_r H) \to K_*(B \rtimes_r H)$

is an isomorphism.

Proof. By Bott periodicity, this amounts to prove that

$f_{H, *} : K_1(A \rtimes_r H) \to K_1(B \rtimes_r H)$

is an isomorphism. Let

$f_{\tilde{H}} : \tilde{A} \rtimes_r H \to \tilde{B} \rtimes_r H$

be the unitarisation of $f_{H}$ with $f_{\tilde{H}}$, $\tilde{A} \rtimes_r H$ and $\tilde{B} \rtimes_r H$ respectively equal to $f_{H}$, $A \rtimes_r H$ and $B \rtimes_r H$ if $f_{H}$ is already a morphism of unital $C^*$-algebras and $f_{\tilde{H}}$, $(A \rtimes_r H)^+$ and $(B \rtimes_r H)^+$ else. Let us fix a control pair $(\lambda, l)$ such that

- $(\lambda, l)$ is larger than
  - the control pair corresponding to the quantitative boundary map associated to a coarse Mayer-Vietoris pair with coercitivity $c = 2$ (see Section 3.4),
  - the control pairs of Proposition 3.18 and of Lemmas 3.33 and 3.35.

We proceed by using a quantitative version of the five lemma.

4.1. Injectivity part. Let $x$ be an element in $K_1(A \rtimes_r H)$ such that $f_{H,*}(x) = 0$ in $K_1(B \rtimes_r H)$. Let us show then that $x = 0$. We divide the proof in five steps.

Step 1. Let us fix a positive number $\varepsilon$ in $(0, \frac{1}{20\lambda_0})$. According to Lemma 3.11, there exist up to stabilisation a $\mathcal{G}$-order $\mathcal{R}_0$ and an $\varepsilon$-$\mathcal{R}_0$-unitary in $1 + A \rtimes_r H$ such that

- $\iota^g_{\varepsilon, \mathcal{R}_0}(\varepsilon, \mathcal{R}_0) = x$;
- $[f_{\tilde{H}}(u)]_{\varepsilon, \mathcal{R}_0} = 0$ in $K_1(\varepsilon, \mathcal{R}_0)$.

Let $(V_1, V_2, H_1, H_2)$ be a $(6\varepsilon \cdot \mathcal{R}_0)$-decomposition of $\mathcal{H}$ with $H_1$ and $H_2$ in $\mathcal{D}$. In view of Theorem 3.27, we see that

$$(A \rtimes_r \mathcal{R}_1, A \rtimes_r \mathcal{R}_2, A \rtimes_r H_1, A \rtimes_r H_2)$$
is a \( l \epsilon \cdot \mathcal{R}_0 \)-controlled Mayer-Vietoris pair relatively to \( A \times_r \mathcal{H} \)
\[
(B \times_r \mathcal{R}_{V_1}, B \times_r \mathcal{R}_{V_2}, B \times_r \mathcal{H}_1, B \times_r \mathcal{H}_2)
\]
is quantitative Mayer-Vietoris pair of order \( l \epsilon \cdot \mathcal{R}_0 \) relatively to \( B \times_r \mathcal{H} \).

For seek of simplicity, we rescale \((\alpha \epsilon, k \epsilon)\) to be equal to \((\lambda, l)\).

According to Proposition 3.18 applied with coercity \( c = 2 \), there exists up to stabilisation two \( \lambda \epsilon-(l \epsilon \cdot \mathcal{R}_0)\)-unitaries \( v_1 \) and \( v_2 \) in \( B \times_r \mathcal{H} \) such that

- \( v_i - 1 \) is in \( B \times_r \mathcal{H}_i \) for \( i = 1, 2 \);
- \( v_i \) is homotopic to 1 as an \( \lambda \epsilon-(l \epsilon \cdot \mathcal{R}_0)\)-unitaries in \( B \times_r \mathcal{H}_i \) for \( i = 1, 2 \);
- \( \|f_i(u) - v_1 v_2\| < \lambda \epsilon \).

**Step II.** By naturality of the quantitative Mayer-Vietoris boundary map (see Equation (2) of Section 3.4), we have
\[
f_{H_1 \cap H_2,*}^{\lambda \epsilon, l \epsilon \cdot \mathcal{R}_0} \circ \partial_{H_1,H_2,*}^{\mathcal{R}_0}([u]_{\mathcal{R}_0}) = \partial_{H_1,H_2,*}^{\mathcal{R}_0}([f(u)]_{\mathcal{R}_0}) = 0.
\]

In particular,
\[
f_{H_1 \cap H_2,*}^{\lambda \epsilon, l \epsilon \cdot \mathcal{R}_0} \circ \partial_{H_1,H_2,*}^{\mathcal{R}_0}([u]_{\mathcal{R}_0}) = \lambda \epsilon \cdot l \epsilon \cdot \mathcal{R}_0 \circ f_{H_1,H_2,*}^{\lambda \epsilon, l \epsilon \cdot \mathcal{R}_0}([u]_{\mathcal{R}_0}) = 0,
\]
and since \( f_{H_1 \cap H_2,*} \) is injective, we deduce from Lemma 3.11 that there exists a compact \( \mathcal{G} \)-order \( \mathcal{R} \) containing \( l \epsilon \cdot \mathcal{R}_0 \) such that
\[
l_\ast \lambda^2 \epsilon \cdot \mathcal{R} \circ \partial_{H_1,H_2,*}^{\mathcal{R}_0}([u]_{\mathcal{R}_0}) = 0.
\]

**Step III.** According to Lemma 3.33 up to stabilisation and up to replacing \( \mathcal{R} \) by \( l \lambda \epsilon \cdot \mathcal{R} \), there exists two \( \lambda^2 \epsilon \cdot \mathcal{R}\)-unitaries \( w_1 \) and \( w_2 \) in \( A \times_r \mathcal{H} \) such that

- \( w_i - 1 \) is in \( A \times_r \mathcal{H}_i \) for \( i = 1, 2 \);
- \( \|u - w_1 w_2\| < \lambda^3 \epsilon \).

In particular, according to the first point of Lemma 3.5 we have
\[
[u]_{3 \lambda^3 \epsilon, 2 \mathcal{R}} = f_{1, A,*}^{3 \lambda^3 \epsilon, 2 \mathcal{R}}([w_1]_{3 \lambda^3 \epsilon, 2 \mathcal{R}}) + f_{2, A,*}^{3 \lambda^3 \epsilon, 2 \mathcal{R}}([w_2]_{3 \lambda^3 \epsilon, 2 \mathcal{R}})
\]
in \( K_1^{3 \lambda^3 \epsilon, 2 \mathcal{R}}(A \times_r \mathcal{H}) \). Moreover, we have
\[
\|v_1 v_2 - \tilde{f}(w_1)\tilde{f}(w_2)\| < 2 \lambda^3 \epsilon
\]
and in consequence,
\[
\|v_1^* \tilde{f}(w_1) - v_2 \tilde{f}(w_2)\| < 8 \lambda^3 \epsilon.
\]
The CIA-condition with coercivity \( c = 2 \) implies that up to replace \( \mathcal{R} \) by \( 2 \cdot \mathcal{R} \), there exists an element \( v \) in \( 1 + B \times_r \mathcal{R} / H_1 \cap H_2 \) such that
\[
\|v - v_1 \tilde{f}(w_1)\| < 16 \lambda^3 \epsilon.
\]
and
\[ \| v - v_2 \tilde{f}_H(w_2) \| < 16 \lambda^3 \varepsilon. \]
In particular, \( v \) is a \( \lambda' \varepsilon \)-unitary with \( \lambda' = 64 \lambda^3 \). Moreover, \( v \) is homotopic to \( v_1 \tilde{f}_H(w_1) \) as a \( \lambda' \varepsilon \)-\( R \)-unitary in \( B \times_r H_1 \) and homotopic to \( v_2 \tilde{f}_H(w_2) \) as a \( \lambda' \varepsilon \)-\( R \)-unitary in \( B \times_r H_2 \). By surjectivity of \( \tilde{f}_{H_1 \cap H_2.*} \) and in view of Lemma 3.11 there exists a compact \( G \)-order \( \mathcal{R}' \) containing \( \mathcal{R} \) and an element \( z \) in \( R_{1\lambda' \varepsilon, \mathcal{R}'}(A \times_r (H_1 \cap H_2)) \) such that
\[ \tilde{f}_{H_1 \cap H_2.*}^{\lambda' \varepsilon, \mathcal{R}'}(z) = [v]_{\lambda' \varepsilon, \mathcal{R}'} \]
in \( R_{1\lambda' \varepsilon, \mathcal{R}'}(B \times_r (H_1 \cap H_2)) \).

\textbf{Step IV.} Let us set
\[ z_1 = l_{1,2,A,*}^{\lambda' \varepsilon, \mathcal{R}'}(z) \]
and
\[ z_2 = j_{2,1,A,*}^{\lambda' \varepsilon, \mathcal{R}'}(z). \]
We deduce from the discussion at the end of the previous step that
\[
\tilde{f}_{H_1.*} \circ l_{s}^{\lambda' \varepsilon, \mathcal{R}'}(z_1) = l_{s}^{\lambda' \varepsilon, \mathcal{R}'} \circ \tilde{f}_{H_1.*}^{\lambda' \varepsilon, \mathcal{R}'}(z_1) = l_{s}^{\lambda' \varepsilon, \mathcal{R}'}([v_1 \tilde{f}_{H_1}(w_1)]_{\lambda' \varepsilon, \mathcal{R}'}) = l_{s}^{\lambda' \varepsilon, \mathcal{R}'}([f_{H_1}(w_1)]_{\lambda' \varepsilon, \mathcal{R}'}) = f_{H_1.*} \circ l_{s}^{\lambda' \varepsilon, \mathcal{R}'}([w_1]_{\lambda' \varepsilon, \mathcal{R}'}),
\]
where the third equality holds because \( v_1 \) is homotopic to 1 as a \( \lambda' \varepsilon \)-\( l_\varepsilon \cdot \mathcal{R}_0 \)-unitary in \( B \times_r H_1 \). Since \( f_{H_1.*} \) is one-to-one, we get that
\[ l_{s}^{\lambda' \varepsilon, \mathcal{R}'}(z_1) = l_{s}^{\lambda' \varepsilon, \mathcal{R}'}([w_1]_{\lambda' \varepsilon, \mathcal{R}'}) \]
and similarly,
\[
l_{s}^{\lambda' \varepsilon, \mathcal{R}'}(z_2) = -l_{s}^{\lambda' \varepsilon, \mathcal{R}'}([w_2]_{\lambda' \varepsilon, \mathcal{R}'}). \]

\textbf{Step V.} According to Lemma 3.11 there exists a compact \( G \)-order \( \mathcal{R}'' \) containing \( \mathcal{R}' \) and such that
\[ l_{s}^{-\lambda' \varepsilon, \mathcal{R}''}(z_1) = [w_1]_{\lambda' \varepsilon, \mathcal{R}''} \]
and
\[ l_{s}^{-\lambda' \varepsilon, \mathcal{R}''}(z_2) = -[w_2]_{\lambda' \varepsilon, \mathcal{R}''} \]
From Equation 3.3 we deduce
\[
[u]_{\lambda' \varepsilon, \mathcal{R}''} = f_{1,2,A,*}^{\lambda' \varepsilon, \mathcal{R}''} \circ l_{s}^{-\varepsilon}(z_1) - f_{2,1,A,*}^{\lambda' \varepsilon, \mathcal{R}''} \circ l_{s}^{-\varepsilon}(z_2) = l_{s}^{-\lambda' \varepsilon, \mathcal{R}''} \circ f_{1,2,A,*}^{\lambda' \varepsilon, \mathcal{R}'}(z_1) - l_{s}^{-\lambda' \varepsilon, \mathcal{R}''} \circ f_{2,1,A,*}^{\lambda' \varepsilon, \mathcal{R}'}(z_2) = l_{s}^{-\lambda' \varepsilon, \mathcal{R}''} \circ (f_{1,2,A,*}^{\lambda' \varepsilon, \mathcal{R}'} \circ f_{1,2,A,*}^{\lambda' \varepsilon, \mathcal{R}'} - f_{2,1,A,*}^{\lambda' \varepsilon, \mathcal{R}''} \circ f_{2,1,A,*}^{\lambda' \varepsilon, \mathcal{R}'})(z) = 0
\]
and hence
\[ x = l_{s}^{\lambda' \varepsilon, \mathcal{R}''}[u]_{\lambda' \varepsilon, \mathcal{R}''} = 0. \]
4.2. **Surjectivity part.** Let us set \( \alpha_0 = 9\lambda^5 \). In view of Lemma 3.11 let us prove that for every \( \varepsilon \in (0, \frac{1}{4\omega}) \), any \( \mathcal{G} \)-order \( \mathcal{R}_0 \) and any \( y \) in \( K_1^{\varepsilon, \mathcal{R}_0}(B \times_r \mathcal{G}) \), there exists a compact \( \mathcal{G} \)-order \( \mathcal{R}_1 \) containing \( \mathcal{R}_0 \) and \( x \) an element in \( K_1^{\varepsilon, \mathcal{R}_1}(A \times_r \mathcal{G}) \) such that \( f_{\mathcal{H}, \mathcal{R}_1}(x) = \ell_*^{\varepsilon, \mathcal{R}_0, \mathcal{R}_1}(y) \) in \( K_1^{\varepsilon, \mathcal{R}_1}(B \times_r \mathcal{G}) \). We divide this proof in four steps.

**Step I.** Up to replacing \( \mathcal{R}_0 \) by \( 21\varepsilon \) and in view the first point of Lemma 3.8 we can assume that there exists an \( \varepsilon \)-\( \mathcal{R}_0 \)-unitary \( u' \) in \( I_{N'} + M_{N'}(B \times_r \mathcal{H}) \) for some integer \( N' \) such that \( y = [u']_{\varepsilon, \mathcal{R}_0} \). Let \( (V_1, V_2, H_1, H_2) \) be a \((6\varepsilon \cdot \mathcal{R}_0)\)-decomposition for \( \mathcal{H} \) with \( H_1 \) and \( H_2 \) in \( \mathcal{D} \). Since \( f_{\mathcal{H}_1 \cap \mathcal{H}_2, \varepsilon} \) is onto and according to Lemma 3.11 there exists a compact \( \mathcal{G} \)-order \( \mathcal{R} \) containing \( \ell_* \cdot \mathcal{R}_0 \) and an element \( x_{1,2} \) in \( K_0^{\lambda_2, \mathcal{R}}(A \times_r (H_1 \cap H_2)) \) such that

\[
\begin{align*}
\ell_*^{\lambda_2, \mathcal{R}}(x_{1,2}) &= \ell_*^{\lambda_2, \mathcal{R}} \circ f_{\mathcal{H}, \mathcal{R}}(x_{1,2}) \\
&= \ell_*^{\lambda_2, \mathcal{R}} \circ f_{\mathcal{H}_1, \mathcal{R}}(x_{1,2}) \\
&= \ell_*^{\lambda_2, \mathcal{R}} \circ f_{\mathcal{H}_2, \mathcal{R}}(x_{1,2}) \\
&= 0
\end{align*}
\]

Let us set \( x_1 = f_{1,2, A, \varepsilon}(x_{1,2}) \) in \( K_0^{\lambda_2, \mathcal{R}}(A \times_r H_1) \) and \( x_2 = f_{1,2, A, \varepsilon}(x_{1,2}) \) in \( K_0^{\lambda_2, \mathcal{R}}(A \times_r H_2) \). Then we have

\[
\ell_*^{\lambda_2, \mathcal{R}}(x_1) = 0
\]

and in the same way \( \ell_*^{\lambda_2, \mathcal{R}}(x_2) = 0 \). Since \( f_{\mathcal{H}_1, \varepsilon} \) and \( f_{\mathcal{H}_2, \varepsilon} \) are one-to-one, we deduce that \( \ell_*^{\lambda_2, \mathcal{R}}(x_1) = 0 \) and \( \ell_*^{\lambda_2, \mathcal{R}}(x_2) = 0 \). According to Lemma 3.11 there exists a compact \( \mathcal{G} \)-order \( \mathcal{R}' \) containing \( \mathcal{R} \) and such that

\[
\ell_*^{\lambda_2, \mathcal{R}'}(x_{1,2}) = 0
\]

and

\[
\ell_*^{\lambda_2, \mathcal{R}'}(x_{1,2}) = 0.
\]

**Step II.** In view of Lemma 3.7 there exists for some positive integers \( n \) and \( N \) with \( n \leq N \) a \( 9\lambda_2, \mathcal{R} \)-projection in \( \text{diag}(I_n, 0) + M_N(A \times_r \mathcal{H}) \) such that

\[
\ell_*^{9\lambda_2, \mathcal{R}}(x_{1,2}) = [p, n]_{9\lambda_2, \mathcal{R}}
\]

in \( K_0^{9\lambda_2, \mathcal{R}}(A \times_r (H_1 \cap H_2)) \). According to Lemma 3.35 and up to stabilisation, there exists four \( 9\lambda_3 \)-\( \mathcal{R} \)-unitaries, \( v_1 \) and \( v_2 \) in \( I_{N'} + M_N(A \times_r \mathcal{H}) \), \( u_1 \) in \( M_n(A \times_r \mathcal{H}) \) and \( u_2 \) in \( M_{N-n}(A \times_r \mathcal{H}) \) such that

\[
\begin{align*}
&\|v_1 \text{ diag}(I_n, 0)v_1 - p\| < 9\lambda_3 \varepsilon; \\
&\|v_2 \text{ diag}(I_n, 0)v_2 - p\| < 9\lambda_3 \varepsilon;
\end{align*}
\]
• \[ \| \text{diag}(u_1, u_2) - v_1v_2 \| < 9\lambda^3 \varepsilon; \]
• for \( i = 1, 2 \), then \( v_i \) is connected to \( I_N \) by a homotopy of \( 9\lambda^3 \varepsilon\)-
\( l_{9\lambda^2 \varepsilon} \mathcal{R} \)-unitaries in \( I_N + M_N(A \rtimes_r \mathcal{H}_i) \).

**Step III.** Let \( u' \) be an \( \varepsilon \)-\( \mathcal{R}_0 \)-unitary in \( I_{N'} + M_{N'}(B \rtimes_r \mathcal{H}) \) for some integer \( N' \) such that \( y = [u']_{\varepsilon, \mathcal{R}_0} \). By construction of the controlled
Mayer-Vietoris boundary applied to \( -y = [u^*]_{\varepsilon, \mathcal{R}_0} \), there exists two
\( \lambda \varepsilon \)-\( \mathcal{L} \)-unitaries \( w_1 \) and \( w_2 \) in \( M_{2N'}(B \rtimes_r \mathcal{H}) \) and \( q \) an \( \lambda \varepsilon \)-projection in \( \text{diag}(I_{N'}, 0) + M_{2N'}(B \rtimes_r \mathcal{H}) \) such that
• \( w_i - I_{2N'} \) lies in \( M_{2N'}(B \rtimes_r \mathcal{H}_i) \) for \( i = 1, 2 \);
• \( \| \text{diag}(u^*, u') - w_1w_2 \| < \lambda \varepsilon; \)
• \( \| w_i^* \text{diag}(I_{N'}, 0) - q \| < \lambda \varepsilon \) and \( \| w_2 \text{diag}(I_{N'}, 0)w_2 - q \| < \lambda \varepsilon; \)
• \( -\theta_{\mathcal{H}_i, \mathcal{H}_2, B, \mathcal{R}_0, \varepsilon}(z) = [q, N']_{\lambda \varepsilon, \mathcal{L}, \mathcal{R}_0} \).

**Step IV.** If we apply Lemma 3.33 to \( \text{diag}(\tilde{f}_H(u_1), u^*), \text{diag}(\tilde{f}_H(u_2), u') \)
and to the matrices obtained from \( \text{diag}(\tilde{f}_H(v_1), w_1), \text{diag}(\tilde{f}_H(v_2), w_2) \)
and \( \text{diag}(\tilde{f}_H(p), q) \) by flipping the coordinates \( n + 1, \ldots, N \) and \( N +
1, \ldots, N + N' \), we see that up to replacing \( \mathcal{R}' \) by \( l_{9\lambda^5 \varepsilon} \mathcal{R}' \), there exist for some integer and for \( i = 1, 2 \) a \( 9\lambda^5 \varepsilon \)-\( \mathcal{R}' \)-
unitary \( v'_i \) in \( I_{N''} + M_{N''}(B \rtimes_r \mathcal{H}_i) \) s.t. \( v'_i \)

\[ [v'_1]_{9\lambda^5 \varepsilon, \mathcal{R}'} + [v'_2]_{9\lambda^5 \varepsilon, \mathcal{R}'} = [\tilde{f}_H(u_1)]_{9\lambda^5 \varepsilon, \mathcal{R}'} - \iota_{v'_i}^{9\lambda^5 \varepsilon, \mathcal{R}_0, \mathcal{R}_1}(y). \]

Since \( f_{H_{1, \varepsilon}} \) and \( f_{H_{2, \varepsilon}} \) are onto and according to Lemma 3.11 there exist a compact \( \mathcal{G} \)-order \( \mathcal{R}_1 \) containing \( \mathcal{R}' \) and for \( i = 1, 2 \) an element \( x_i \) in \( K_{9\lambda^5 \varepsilon, \mathcal{R}_1}^1(A \rtimes_r \mathcal{H}_i) \) such that \( f_{H_{1, \varepsilon}}(x_i) = -[v'_1]_{9\lambda^5 \varepsilon, \mathcal{R}_1} \) in \( K_{9\lambda^5 \varepsilon, \mathcal{R}_1}^1(B \rtimes_r \mathcal{H}_i) \). Then we have
\[ \iota_{v'_i}^{9\lambda^5 \varepsilon, \mathcal{R}_0, \mathcal{R}_1}(y) = f_{H_{1, \varepsilon}}(j_{1, A, \varepsilon}^1(x_1) + j_{2, A, \varepsilon}^1(x_2) + [u_1]_{5\lambda^5 \varepsilon, \mathcal{R}_1}) \]
and hence \( f_{H_{1, \varepsilon}} \) is onto.

**4.3. Extension to Kasparov product.** The aim of this section is

to extend Theorem 4.1 to morphisms induced in \( K \)-theory by right
Kasparov product. Indeed, this a consequence of the following standard
fact which says that up to \( KK \)-equivalence, a Kasparov element is
equivalent to \( C^* \)-algebra homomorphism (see [16] for an approach via
triangulated categories).

**Lemma 4.3.** Let \( \mathcal{G} \) be a locally groupoid provided with a Haar system,
let \( A \) and \( B \) be \( \mathcal{G} \)-algebras and let \( z \) be an element in \( KK_{\mathcal{G}}^*(A, B) \). Then
there exists
• \( A' \) and \( B' \) two \( \mathcal{G} \)-algebras;
• \( f : A' \to B' \) a homomorphism of \( \mathcal{G} \)-algebras;
• \( \alpha \) in \( KK_{\mathcal{G}}^*(A, A') \) and \( \beta \) in \( KK_{\mathcal{G}}^*(B', B) \) invertible elements,
such that
\[ z = f_\ast(\alpha) \otimes_{B'} \beta. \]
Proof. Let us first prove the result for \( z \) in \( KK^q_1(A, B) \). The imprimitivity \( \mathcal{H}(\mathcal{L}^2(G, A))-A \)-bimodule \( \mathcal{L}^2(G, A) \) gives rise an invertible element \([M]\) in \( KK^q_1(\mathcal{H}(\mathcal{L}^2(G, A)), A) \) and hence, this amounts to prove the result for \([M] \otimes_A z\). According to Lemma 3.5 of the appendix of \([12]\) (see also \([1]\) Section 5), this amounts to prove the result for any element \( z \) in \( KK^q_1(A, B) \) that can be represented by a Kasparov \( K \)-cycle \((\mathcal{E}, \pi, F)\) such that \( F : \mathcal{E} \to \mathcal{E} \) is \( \mathcal{G} \)-equivariant. Up to adding a degenerated Kasparov \( K \)-cycle, we can assume without loss of generality that the linear space generated by \( \{\langle \xi, \nu \rangle, \xi \text{ and } \nu \in \mathcal{E}\} \) is dense in \( B \). Let us set \( P = \frac{1}{2}(F + Id_\mathcal{E}) \) and let us consider the \( \mathcal{G} \)-algebra \( E_P = \{(a, f) \in A \oplus \mathcal{L}(\mathcal{E}) \text{ such that } T - P \cdot \pi(a) \cdot P \text{ belongs to } \mathcal{K}(\mathcal{E}) \} \).

Then the projection on the first factor of \( E_P \) gives rise to an extension of \( \mathcal{G} \)-algebra

\[
0 \longrightarrow \mathcal{K}(\mathcal{E}) \longrightarrow E_P \longrightarrow A \longrightarrow 0
\]

semi-splited by

\[
A \longrightarrow E_P; a \mapsto (a, P \cdot \pi(a) \cdot P).
\]

Let \([M]\) be the element in \( KK^q_1(\mathcal{H}(\mathcal{E}), B) \) corresponding to the \( \mathcal{H}(\mathcal{E})-B \)-imprimitivity bimodule \( \mathcal{E} \). Then \([M]\) is invertible and \( z \otimes_B [M]^{-1} \) is the class in \( KK^q_1(A, \mathcal{K}(\mathcal{E})) \) of the semi-split extension \((4,3)\). Hence this amounts to prove that the lemma holds for the class \([\partial_{I,A}]\) in \( KK^q_1(A/I, I) \) of any semi-split extension \( 0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0 \). We proceed by using the mapping cone. For \( B \) a \( \mathcal{G} \)-algebra let us set

\[
B(0, 1) = \{f : [0, 1] \to \mathbb{C} \text{ continuous such that } f(0) = 0 \},
\]

\[
B(0, 1) = \{f : [0, 1] \to \mathbb{C} \text{ continuous such that } f(0) = f(1) = 0 \},
\]

and let us consider the class \([\partial_B]\) in \( KK^q_1(B, B(0, 1)) \) of the semi-split extension of \( \mathcal{G} \)-algebras

\[
0 \longrightarrow B(0, 1) \longrightarrow B(0, 1) \xrightarrow{ev_1} B \longrightarrow 0,
\]

where \( ev_1 : B(0, 1) \to B \) is the evaluation at 1. Recall that \([\partial_B]\) is invertible. For a semi-split extension

\[
0 \longrightarrow I \longrightarrow A \longrightarrow A/I \xrightarrow{q} 0,
\]

we define the mapping cone algebra of \( A \) by

\[
C_q = \{(x, f) \in A \oplus A/I(0, 1] \text{ such that } f(1) = q(x) \}.
\]

Let us consider the morphisms of \( \mathcal{G} \)-algebras

\[
e_q : I \longrightarrow C_q; x \mapsto (x, 0).
\]

and

\[
\phi_q : A/I(0, 1) \longrightarrow C_q; f \mapsto (0, f).
\]

According to \([21]\), the element \([e_q]\) in \( KK^q_1(I, C_q) \) induced by \( e_q \) is invertible and moreover,

\[
e_{q,*}[\partial_{I,A}] = \phi_{q,*}[\partial_{A/I}] = \phi_{q,*}[\partial_{A/I}],
\]

where \( \phi_{q,*} \) denotes the pull-back of the morphism \( \phi_q \).
Hence we have

\[[\partial_{I,A}] = \phi_{q,*}[\partial_{A/I}] \otimes C_q[e_q]^{-1}\]

Since \([\partial_{A/I}]\) is an invertible element in \(KK^G(A/I, A/I(0, 1))\), we deduce that the conclusion of the lemma holds for \([\partial_{I,A}]\).

Let us prove now that the result holds in the even case. Let \(z\) be an element in \(KK^G(A,B)\). Noticing that \([\partial_A]\) is invertible in \(KK^G(A,A(0,1))\) and applying the odd case to \([\partial_A]^{-1} \otimes_A z\), we deduce the result for \(z\).

\[\square\]

As a consequence, we can extend Theorem 4.1 to \(KK\)-elements. Recall that for any locally groupoid \(G\) provided with a Haar system, then

\[J_G: KK^G_* (\bullet, \bullet) \rightarrow KK_* (\bullet \rtimes_r G, \bullet \rtimes_r G)\]

stands for the Kasparov transformation. For any \(G\)-algebras \(A\) and \(B\) and for any element \(z\) in \(KK^G_* (A,B)\), we denote by

\[\otimes J_G(z): K_* (A \rtimes_r G) \rightarrow K_* (B \rtimes_r G)\]

the morphism induced by Kasparov right multiplication by \(J_G(z)\).

**Corollary 4.4.** Let \(G\) be a locally groupoid provided with a Haar system, let \(A\) and \(B\) be \(G\)-algebras and let \(z\) be an element in \(KK^G_* (A,B)\). Let us assume that there exists a subset \(D\) of relatively clopen groupoids of \(G\), closed under taking relatively clopen subgroupoids and such that

(i) \(G\) has finite \(D\)-complexity.

(ii) for any subgroupoid \(H\), the morphism

\[\otimes J_H(z/H): K_* (A \rtimes_r H) \rightarrow K_* (B \rtimes_r H)\]

is an isomorphism.

Then

\[\otimes J_G(z): K_* (A \rtimes_r G) \rightarrow K_* (B \rtimes_r G)\]

is an isomorphism.

**Proof.** Let \(A'\) and \(B'\) be \(G\)-algebras, let \(f: A' \rightarrow B'\) be a morphism of \(G\)-algebras and let \(\alpha\) in \(KK^G_* (A,A')\) and \(\beta\) in \(KK^G_* (B',B)\) be invertible elements as in Lemma 4.3. Then

\[\otimes J_G(z): K_* (A \rtimes_r G) \rightarrow K_* (B \rtimes_r G)\]

is an isomorphism if and only if

\[f_{G,*}: K_* (A' \rtimes_r G) \rightarrow K_* (B' \rtimes_r G)\]

is an isomorphism and in the same way

\[\otimes J_H(z/H): K_* (A \rtimes_r H) \rightarrow K_* (B \rtimes_r H)\]

is an isomorphism if and only if

\[f_{H,*}: K_* (A' \rtimes_r H) \rightarrow K_* (B' \rtimes_r H)\].

The corollary is then consequence of Theorem 4.1 applied to \(f: A' \rightarrow B'\) for any \(H\) in \(D\).
5. Application the Baum-Connes conjecture

In this section, we show that for groupoids which admit a $\gamma$-element in sense of [23], then the Baum-Connes conjecture is closed under closed decomposition.

5.1. The Baum-Connes conjecture for groupoids. Let us recall first the statement of the Baum-Connes conjecture for groupoids. Let $G$ be a locally compact groupoid provided with the Haar system, let $A$ be a $G$-algebra and let $A \rtimes_r G$ be the reduced crossed product of $A$ by $G$ (with respect to the given Haar system). Then the Baum-Connes conjecture for the pair $(A, G)$ is the claim that the assembly map

$$\mu_{A,G} : K^G_{\text{top}}(G, A) \rightarrow K_*^G(A \rtimes_r G)$$

is an isomorphism, the left-hand side being the topological $K$-theory for the groupoid $G$ with coefficient, defined as the inductive limit

$$\lim_X KK^G_*\left(C_0(X), A\right),$$

where $X$ runs through $G$-compact subsets of the universal example for proper actions of $G$ (see [23] for a complete description of the Baum-Connes conjecture in the setting of groupoids). Although the conjecture holds for a large class of pair $(A, G)$, (e.g if $G$ is an amenable groupoid), counterexamples have been exhibited by N. Higson, V. Lafforgues and G. Skandalis in [5].

5.2. The case of groupoids admitting a $\gamma$-elements. The concept of $\gamma$-element was introduced by G. Kasparov in [10] in order to prove the Novikov conjecture for discrete subgroups of almost connected groups. He showed that for an almost connected group $G$ acting on a $C^*$-algebra $A$ strongly continuously by automorphisms, then the image of the Baum-Connes assembly map is the range of $\gamma$ acting on $K_*^G(A \rtimes_r G)$ as an idempotent. The notion of $\gamma$-element was extended to groupoid by J. L. Tu in [23], where he developed an abstract setting for such an element.

**Definition 5.1.** A locally compact groupoid admits a $\gamma$-element if there exists an element $\gamma$ in $KK_*^G(C_0(X), C_0(X))$, a proper $G$-space $Z$, a $Z \rtimes G$-algebra $A$, an element $\eta$ in $KK_*^G(C_0(X), A)$ and an element $D$ in $KK_*^G(C_0(X), A)$ such that

- $\gamma = \eta \otimes_A D$;
- $p_{Z'}^* \gamma = 1$ in $KK_*^{Z \rtimes G}(C_0(Z'), C_0(Z'))$ for every proper $G$-space $Z'$, where $p_{Z'} : Z' \rtimes G \rightarrow G$ is the forgetful map.

Such an element if it exists is unique and is called a $\gamma$-element. As in the case of the $\gamma$-element of Kasparov, a $\gamma$-element is an idempotent of $KK_*^G(C_0(X), C_0(X))$ and as such acts as an idempotent on $K_*^G(A \rtimes_r G)$. This idempotent is given by right Kasparov product by $J_G(\tau_A(\gamma))$, where $\tau_A(\gamma) \in KK_*^G(A, A)$ is obtained by tensorization over
Proof. Let us denote respectively by \( \gamma \) to Corollary 1.5, the proper action of \( G \) and \( \eta \) let \( H \)\(^{\bot}\) a relatively clopen subgroupoid of \( G \) and which satisfies the Baum-Connes conjecture with coefficients, then any relatively clopen subgroupoid of \( G \) satisfies the Baum-Connes conjecture with coefficients.

**Proposition 5.2.** Let \( \mathcal{G} \) be a locally compact groupoid provided with a Haar system admitting a \( \gamma \)-element and let \( A \) be a \( \mathcal{G} \)-algebra. Then the following assertions are equivalent:

(i) \( \mu_{A,\mathcal{G}}:K_{*}^{\text{top}}(\mathcal{G},A)\to K_{*}(A\rtimes_{r}\mathcal{G}) \) is an isomorphism;

(ii) \( J_{\mathcal{G}}(\tau_{A}(\gamma)) \) acts as the identity by right Kasparov product on \( K_{*}(A\rtimes_{r}\mathcal{G}) \).

**Remark 5.3.** Since \( J_{\mathcal{G}}(\tau_{A}(\gamma)) \) is an idempotent, it acts as the identity by right Kasparov product on \( K_{*}(A\rtimes_{r}\mathcal{G}) \) if and only if it acts as an isomorphism.

The restriction of a \( \gamma \)-element to a relatively clopen subgroupoid is a \( \gamma \)-element.

**Lemma 5.4.** Let \( \mathcal{G} \) be a locally compact groupoid and let \( \mathcal{H} \) be a relatively clopen subgroupoid of \( \mathcal{G} \). If \( \mathcal{G} \) admits a \( \gamma \)-element, then the restriction of \( \gamma \) to \( \mathcal{H} \) is a \( \gamma \)-element for \( \mathcal{H} \).

**Proof.** Let us denote respectively by \( X \) and \( Y \) the space of units of \( \mathcal{G} \) and \( \mathcal{H} \). Let \( Z \) be a proper \( \mathcal{G} \)-space, let \( A \) be a \( \mathcal{G} \times Z \)-algebra, let \( \eta \) be an element in \( KK_{*}^{\mathcal{G}}(C_{0}(X),A) \) and let \( D \) be an element in \( KK_{*}^{\mathcal{G}}(A,C_{0}(X)) \) as in Definition 5.1. According to Remark 1.7 and to Corollary 1.5 the proper action of \( \mathcal{G} \) on \( Z \) restricts to a proper action of \( \mathcal{H} \) on \( Z_{Y} \). Let \( A_{Z,Y} \) be the restriction of \( A \) to \( Z_{Y} \). According to the second point of Example 2.4 then \( A_{Z,Y} \) is a \( \mathcal{H} \times Z_{Y} \) algebra. Let \( \gamma/\mathcal{H} \) in \( KK_{*}^{\mathcal{H}}(C_{0}(Y),C_{0}(Y)) \), \( \eta/\mathcal{H} \) in \( KK_{*}^{\mathcal{H}}(C_{0}(Y),A_{Z,Y}) \) and \( D/\mathcal{H} \) in \( KK_{*}^{\mathcal{H}}(A_{Z,Y},C_{0}(Y)) \) be respectively the restriction of \( \gamma, \eta \) and \( D \) to \( \mathcal{H} \) (i.e induced by functoriality in the groupoids by the inclusion \( \mathcal{H} \hookrightarrow \mathcal{G} \)). Since the restriction respects Kasparov products, we deduce that \( \gamma/\mathcal{H} = \eta/\mathcal{H} \otimes_{A_{Z,Y}} D/\mathcal{H} \). Let us check the second point of the definition of a \( \gamma \)-element. Let \( Z' \) be a proper \( \mathcal{H} \)-space, let \( Z'' = \mathcal{G} \times_{\mathcal{H}} Z' \) be the proper induced \( \mathcal{G} \)-space (see Section 1.3) and let us recall that \( p_{Z''}: \mathcal{G} \times Z'' \to \mathcal{G} \) stands for the forgetful map. We have by definition of a \( \gamma \)-element that \( p_{Z''}^{*}\gamma = 1 \) in \( K_{*}^{\mathcal{G} \times_{\mathcal{H}} Z''}(C_{0}(Z''),C_{0}(Z'')) \). We have an obvious inclusion of groupoids

\[ \mathcal{H} \times Z' \hookrightarrow \mathcal{G} \times Z''; \quad (\gamma, z) \mapsto (\gamma, [u_{p_{Z''}}(z), z]) \]

which pulls back \( p_{Z''}^{*}\gamma \) to \( p_{Z'}^{*}\gamma \) and hence \( p_{Z'}^{*}\gamma = 1 \) in \( K_{*}^{\mathcal{H} \times Z'}(C_{0}(Z'),C_{0}(Z')) \). We conclude that \( \gamma/\mathcal{H} \) is a \( \gamma \)-element for \( \mathcal{H} \). \( \square \)

**Remark 5.5.** As a consequence and using induced algebras [1], we can prove that if \( \mathcal{G} \) is a locally compact groupoid which admits a \( \gamma \)-element and which satisfies the Baum-Connes conjecture with coefficients, then any relatively clopen subgroupoid of \( \mathcal{G} \) satisfies the Baum-Connes conjecture with coefficients.
An action groupoid of a groupoid with a $\gamma$-element has a $\gamma$-element.

**Lemma 5.6.** Let $\mathcal{G}$ be a locally compact groupoid and let $Y$ be a locally compact (left) $\mathcal{G}$-space. If $\mathcal{G}$ admits a $\gamma$-element, then the action groupoid $\mathcal{G} \times Y$ admits a $\gamma$-element.

**Proof.** Let us denote by $X$ the space of units of $\mathcal{G}$ and let $q_Y : Y \to X$ be the anchor map for the $\mathcal{G}$-action on $Y$. Let $Z$ be a proper $\mathcal{G}$-space, let $A$ be a $\mathcal{G} \times Z$-algebra, let $\eta$ be an element in $KK^Y_*(A, C_0(X))$ and let $D$ be an element in $KK^Y_*(A, C_0(X))$ as in Definition 5.1. Let $p_Y : \mathcal{G} \times Y \to \mathcal{G}$ be the forgetful map with respect to the $\mathcal{G}$-action on $Y$. According to the fourth point of Remark 1.6, we see that $Z \times_X Y$ is a proper $\mathcal{G} \times Y$-space with anchor map $q_{Z \times_X Y} : Z \times_X Y \to Y$ given by the projection on the second factor. Consider then the elements $\gamma_Y = p_Y^* \gamma$ in $KK^Y_{*+1}(C_0(Y), C_0(Y))$, $\eta_Y = p_Y^* \eta$ in $KK^Y_*(A, C_0(Y), q_Y^* A)$ and $D_Y = p_Y^* D$ in $KK^Y_*(A, C_0(Y))$. Using the second point of Example 2.3, we see that $q_Y^* A = A \otimes C_0(Y)$ is a $(\mathcal{G} \times Y) \times (Z \times_X Y)$-algebra and since $p_Y^*$ preserves Kasparov products, we have

$$\gamma_Y = \eta_Y \otimes q_Y^* A D_Y.$$ 

Let us check now the second condition of Definition 5.1. Let $Z'$ be a proper $\mathcal{G} \times Y$-space. According to the first and the third point of Remark 1.6, we see that $Z'$ is a proper $\mathcal{G}$-space equipped with a $\mathcal{G}$-map $Z' \to Y$. Let

$$p_{Z'} : (\mathcal{G} \times Y) \times Z' \to \mathcal{G} \times Y$$

be the forgetful map with respect to the $\mathcal{G} \times Y$-action on $Z'$. Then we have

$$p_{Z'}^* \gamma_Y = p_{Z'}^* (p_Y^* \gamma) = (p_Y \circ p_{Z'})^* \gamma_Y$$

But under the identification between $(\mathcal{G} \times Y) \times Z'$ and $\mathcal{G} \times Z'$ of the second point of Remark 1.6, then

$$p_Y \circ p_{Z'} : (\mathcal{G} \times Y) \times Z' \to \mathcal{G}$$

corresponds to the forgetful map $\mathcal{G} \times Z' \to \mathcal{G}$ with respect to the proper $\mathcal{G}$-action on $Z'$. From this we deduce that $p_{Z'}^* \gamma_Y = 1$ in $KK^Y_{*+1}(C_0(Z'), C_0(Z'))$ and hence $\gamma_Y$ is a $\gamma$-element for $\mathcal{G} \times Y$.

\[\square\]

**Example 5.7.** The following examples of locally compact group $G$ are known to have a $\gamma$-element

(i) if $G$ acts properly on a simply connected manifold with non-positive sectional curvature \[10\];

(ii) if $G$ is (a closed subgroup of) an almost connected group \[10\];

(iii) if $G$ is groups acting properly on an Euclidean buildings \[11\].

For any action of such a group $G$ on a locally compact space $X$, then the action groupoid $G \times X$ has a $\gamma$-element.
5.3. Baum-Connes conjecture and coarse decomposability.

Theorem 5.8. Let $\mathcal{G}$ be a locally groupoid provided with a Haar system which moreover admits a $\gamma$-element in sense of [23] and let $A$ be a $\mathcal{G}$-algebra. Assume that there exists a subset $\mathcal{D}$ of relatively clopen subgroupoids of $\mathcal{G}$, closed under taking relatively clopen subgroupoids such that

(i) every groupoid in $\mathcal{D}$ satisfies the Baum-Connes conjecture with coefficients in $A$;
(ii) $\mathcal{G}$ has finite $\mathcal{D}$-complexity.

Then $\mathcal{G}$ satisfies the Baum-Connes conjecture with coefficients in $A$.

Proof. This is a consequence of Corollary 4.4 applied to $\tau_{\mathcal{A}}(\gamma)$ and of Lemma 5.4, by noticing that $\tau_{\mathcal{A}}(\gamma)/\mathcal{H} = \tau_{\mathcal{A}}(\gamma/\mathcal{H})$ for any relatively clopen subgroupoids $\mathcal{H}$ in $\mathcal{D}$.

We end this paper with an application to the Baum-Connes conjecture with coefficients.

Corollary 5.9. Let $\mathcal{G}$ be a locally groupoid provided with a Haar system and which moreover admits a $\gamma$-element in sense of [23]. Assume that there exists a subset $\mathcal{D}$ of relatively clopen subgroupoids of $\mathcal{G}$ such that

(i) every groupoid in $\mathcal{D}$ satisfies the Baum-Connes conjecture with coefficients;
(ii) $\mathcal{G}$ has finite $\mathcal{D}$-complexity.

Then $\mathcal{G}$ satisfies the Baum-Connes conjecture with coefficients.

Proof. Let $\mathcal{D}'$ be the set of all relatively clopen subgroupoids of elements of $\mathcal{D}$. According to Remark 5.5, any groupoid $\mathcal{H}$ in $\mathcal{D}'$ satisfies the Baum-Connes conjecture with coefficients. Since $\mathcal{G}$ has finite $\mathcal{D}$-complexity, it has $\mathcal{D}'$-complexity. The result is then a consequence of Theorem 5.8.

References

[1] Christian Bönicke. A Going-Down principle for ample groupoids and the Baum-Connes conjecture, preprint.
[2] Christian Bönicke and Clément Dell’Aiera. Going-down functors and the Künneth formula for crossed products by étale groupoids. Trans. Amer. Math. Soc. 372 (11):8159–8194, 2019.
[3] Clément Dell’Aiera. Controlled $K$-theory for groupoids & applications to coarse geometry. J. Funct. Anal. 275 (7):175–1807, 2018.
[4] Yeong Chyuan Chung. Quantitative $K$-theory for Banach algebras. J. Funct. Anal. 274 (1):278–340, 2018.
[5] Nigel Higson, Vincent Lafforgue and Georges Skandalis. Counterexamples to the Baum-Connes conjecture. Geom. Funct. Anal. 12 (2):330–354, 2002.
[6] Erik Guentner, Romain Tessera and Guoliang Yu. A notion of geometric complexity and its application to topological rigidity, Invent. Math. 189 (2):315–357, 2012.
[7] Erik Guentner, Romain Tessera and Guoliang Yu. Discrete groups with finite decomposition complexity, Groups Geom. Dyn. 7 (2):377–402, 2013.
[8] Erik Guentner, Rufus Wilett and Guoliang Yu. Dynamic asymptotic dimension: relation to dynamics, topology, coarse geometry, and $C^*$-algebras, Mathematische Annalen, 367:785-829, 2017.
[9] Erik Guentner, Rufus Wilett and Guoliang Yu. Dynamical complexity and controlled operator $K$-theory. preprint.
[10] Gennadi Kasparov. Equivariant $KK$-theory and the Novikov conjecture. Invent. Math., 91(1):147–201, 1988.
[11] Gennadi Kasparov and Georges Skandalis. Groups acting on buildings, operator $K$-theory, and Novikov’s conjecture. $K$-Theory, 4 (4):303–337, 1991.
[12] Vincent Lafforgue. $K$-théorie bivariante pour les algèbres de Banach, groupoîdes et conjecture de Baum-Connes. Avec un appendice d’Hervé Oyono-Oyono. J. Inst. Math. Jussieu, 6(3):415–451, 2007.
[13] Scott M. LaLonde. Nuclearity and exactness for groupoid crossed products. J. Operator Theory, 74 (1):213–245, 2015.
[14] Pierre-Yves Le Gall. Théorie de Kasparov équivariante et groupoîdes. Thèse de Doctorat, Université Paris VII, 1994.
[15] Pierre-Yves Le Gall. Théorie de Kasparov équivariante et groupoîdes I. $K$-Theory 16(4), 361–390, 1999.
[16] Ralf Meyer and Ryszard Nest. The Baum-Connes conjecture via localisation of categories. Topology 45(2):209–259, 2006.
[17] Hervé Oyono-Oyono and Guoliang Yu. On a quantitative operator $K$-theory. Annales de l’Institut Fourier, 65(2):605–674, 2015.
[18] Hervé Oyono-Oyono and Guoliang Yu. Quantitative $K$-theory and the Künneth formula for operator algebras, J. Funct. Anal. 277 (7):2003—-2091, 2019.
[19] Alan Paterson. Groupoids, inverse semigroups, and their operator algebras. Progress in Mathematics, 170, Birkhäuser Boston, Inc., Boston, MA, 1999.
[20] Jean Renault. A groupoid approach to $C^*$-algebras. Lecture Notes in Mathematics, 793. Springer, Berlin, 1980.
[21] Georges Skandalis. Exact sequences for the Kasparov groups of graded algebras. Can. J. Math. 37(2):324–343, 1980.
[22] Georges Skandalis, Jean-Louis Tu and Guoliang Yu. The coarse Baum-Connes conjecture and groupoids. Topology, 41(4):807–834, 2002.
[23] Jean-Louis Tu. La conjecture de Novikov pour les feuilletages hyperboliques $K$-Theory, 16(3):129–184, 1999.
[24] Rufus Wilett. Decompositions and Künneth formula, preprint.
[25] Dana P. Williams. Crossed products of $C^*$-algebras. Mathematical Surveys and Monographs, 134. American Mathematical Society, Providence, RI, 2007.
[26] Guoliang Yu. The Novikov conjecture for groups with finite asymptotic dimension. Ann. of Math. (2), 147(2):325–355, 1998.

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