ASSOUAD-NAGATA DIMENSION OF WREATH PRODUCTS OF GROUPS

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Abstract. Consider the wreath product $H \wr G$, where $H \neq 1$ is finite and $G$ is finitely generated. We show that the Assouad-Nagata dimension $\dim_{AN}(H \wr G)$ of $H \wr G$ depends on the growth of $G$ as follows:

If the growth of $G$ is not bounded by a linear function, then $\dim_{AN}(H \wr G) = \infty$, otherwise $\dim_{AN}(H \wr G) = \dim_{AN}(G) \leq 1$.
1. Introduction

P. Nowak [18] proved that the Assouad-Nagata dimension of some wreath products $H \wr G$ is infinite, where $H$ is finite and $G$ is a finitely generated amenable group whose Folner function grows sufficiently fast and satisfies some other conditions suitable for applying Erschler’s result [12]. That result says the Folner function $F(H \wr G)$ of $H \wr G$ is comparable to $F(H)^{F(G)}$ and the passage from it to Assouad-Nagata dimension of $H \wr G$ is fairly complicated as it includes Property A. Thus, results of [18] apply only to amenable groups $G$ and do not apply neither to lamplighter groups (as the Folner function of $\mathbb{Z}$ is linear) nor to wreath products with free non-Abelian groups (as those are not amenable).

In this paper we show that the Assouad-Nagata dimension of $H \wr G$ completely depends on the linearity of the growth of $G$. In particular, the lamplighter groups are not finitely presented and are of Assouad-Nagata dimension 1 which solves positively the following question of [10]:

**Question 1.1.** Is there a finitely generated group of Assouad-Nagata dimension 1 that is not finitely presented?

2. Assouad-Nagata dimension

Let $X$ be a metric space and $n \geq 0$. An *n-dimensional control function of $X$* is a function $D^n_X : \mathbb{R}_+ \to \mathbb{R}_+ \cup \infty$ with the following property:

For any $r > 0$ there is a cover $\{X_0, \ldots, X_n\}$ of $X$ whose Lebesque number is at least $r$ (that means every open $r$-ball $B(x, r)$ is contained in some $X_i$) and every $r$-component of $X_i$ is of diameter at most $D^n_X (r)$. Two points $x$ and $y$ belong to the same $r$-component of $X_i$ if there is a sequence $x_0 = x, x_1, \ldots, x_k = y$ in $X_i$ such that $\text{dist}(x_j, x_{j+1}) < r$ (such a sequence will be called an $r$-path).
The \emph{asymptotic dimension} $\asdim(X)$ is the smallest integer such that $X$ has an $n$-dimensional control function whose values are finite.

The \emph{Assouad-Nagata dimension} $\dim_{AN}(X)$ of a metric space $X$ is the smallest integer $n$ such that $X$ has an $n$-dimensional control function that is a dilation ($D^n_X(r) = C \cdot r$ for some $C > 0$).

The \emph{asymptotic Assouad-Nagata dimension} $\asdim_{AN}(X)$ of a metric space $X$ is the smallest integer $n$ such that $X$ has an $n$-dimensional control function that is linear ($D^n_X(r) = C \cdot r + C$ for some $C > 0$).

In case of metrically discrete spaces $X$ (that means there is $\epsilon > 0$ such that every two distinct points have the distance at least $\epsilon$) $\asdim_{AN}(X) = \dim_{AN}(X)$ (see [4]). In particular, in case of finitely generated groups we can talk about Assouad-Nagata dimension instead of asymptotic Assouad-Nagata dimension.

A countable group $G$ is called \emph{locally finite} if every finitely generated subgroup of $G$ is finite. A group $G$ has asymptotic dimension 0 if and only if it is locally finite [20].

Notice that $\dim_{AN}(X) = 0$ if there is $C > 0$ such that for every $r$-path the distance between its end-points is less than $C \cdot r$. In case of groups one has the following useful criterion of being 0-dimensional:

**Proposition 2.1.** Let $(G, d_G)$ be a group equipped with a proper left-invariant metric $d_G$ (that means bounded sets are finite). If $G$ is locally finite, then the following conditions are equivalent:

a. $\dim_{AN}(G, d_G) = 0$.

b. There is a constant $c > 0$ such that for each $r > 0$ the subgroup of $G$ generated by $B(1, r)$ is contained in $B(1, c \cdot r)$.

**Proof.** a) $\implies$ b). Consider a constant $K > 0$ such that for each $r > 0$ all $r$-components of $G$ have diameter less than $K \cdot r$. Notice that if $g \in G$ belongs to $r$-component of 1 and $h \in B(1, r)$, then $d_G(g, gh) = d_G(1, h) < r$, so $gh$ lies in the $r$-component of 1. Therefore the subgroup generated by $B(1, r)$ is contained in $B(1, K \cdot r)$.

b) $\implies$ a). Let $G_r$ be the subgroup of $G$ generated by $B(1, r)$. Consider two different left cosets $y \cdot G_r$ and $z \cdot G_r$ of $G_r$ in $G$. If $d_G(yg, zh) < r$ for some $g, h \in G_r$, then $f = h^{-1}z^{-1}yg \in B(1, r) \subset G_r$, so $y = z(hfg^{-1})$, a contradiction. That means each $r$-component of $G$ is contained in a left coset of $G_r$ and its diameter is less than $2cr$, i.e. $\dim_{AN}(G, d_G) = 0$.

Let us generalize $r$-paths as follows:

By an $r$-\emph{cube} in a metric space $X$ we mean a function $f : \{0, 1, \ldots, k\}^n \to X$ with the property that the distance between $f(x)$ and $f(x + e_i)$ is less than $r$ for all $x \in \{0, 1, \ldots, k\}^n$ such that $x + e_i \in \{0, 1, \ldots, k\}^n$. Here $e_i$ belongs to the standard basis of $\mathbb{R}^n$. 


A sufficient condition for $\dim_A(X)$ being positive is the existence, for every $C > 0$, of an $r$-path joining points of distance at least $C \cdot r$. The purpose of the remainder of this section is to find a similar sufficient condition for $\dim_A(X) \geq n$.

**Lemma 2.2.** Consider the set $X = \{0,1,\ldots,k\}^n$ equipped with the $l_1$-metric. Suppose $X = X_1 \cup \ldots \cup X_n$. If the open $(n+1)$-ball of every point of $X$ is contained in some $X_i$, then a 2-component of some $X_i$ contains two points whose $i$-coordinates differ by $k$.

**Proof.** Let us proceed by contradiction and assume all 2-components of each $X_i$ do not contain points whose $i$-coordinates differ by $k$. Create the cover $A_i$, $1 \leq i \leq n$, of the solid cube $[0,k]^n$ by adding unit cubes to $A_i$ whenever all vertices of it are contained in $X_i$. Given $i \in \{1,\ldots,n\}$ consider the two faces $L_i$ and $R_i$ of $[0,k]^n$ consisting of points whose $i$-th coordinate is 0 and $k$ respectively. Let $B_i$ be the complement of the $\frac{1}{4}$-neighborhood of $A_i \cup L_i \cup R_i$. Notice that $B_i$ separates between $L_i$ and $R_i$. Indeed, if $L_i \cup R_i$ belongs to the same component of the $\frac{1}{4}$-neighborhood of $A_i \cup L_i \cup R_i$, then one can find a $\frac{1}{2}$-path in $A_i$ between points in $X_i$ whose $i$-coordinates differ by $k$. Picking points in $X_i$ in the same unit cubes as vertices of the path one gets a 2-path in $X_i$ between points in $X_i$ whose $i$-th coordinates differ by $k$.

Now we get a contradiction as $\bigcap_{i=1}^n B_i = \emptyset$ in violation of the well-known result in dimension theory about separation (see Theorem 1.8.1 in [13]).

**Corollary 2.3.** Suppose $X$ is a metric space with an $(n-1)$-dimensional control function $D_X^{n-1} : \mathbb{R}_+ \to \mathbb{R}_+ \cup \infty$. For any $r$-cube $f : \{0,1,\ldots,k\}^n \to X$ there exist two points $a$ and $b$ in $\{0,1,\ldots,k\}^n$ whose $i$-th coordinates differ by $k$ for some $i$ and $\text{dist}(f(a), f(b)) \leq D_X^{n-1}(n \cdot r)$.

**Proof.** Consider a cover $X = X_1 \cup \ldots \cup X_n$ of $X$ of Lebesque number at least $n \cdot r$ such that $n \cdot r$-components of each $X_i$ are of diameter at most $D_X^{n-1}(n \cdot r)$. The cover $\{0,1,\ldots,k\}^n = f^{-1}(X_1) \cup \ldots \cup f^{-1}(X_n)$ has the property that the open $(n+1)$-ball of every point is contained in some $f^{-1}(X_i)$, so by Corollary 2.2 a 2-component (in the $l_1$-metric) of some $f^{-1}(X_i)$ contains two points $a$ and $b$ whose $i$-coordinates differ by $k$. Therefore $f(a)$ and $f(b)$ belong to the same $r$-component of $X_i$ and $\text{dist}(f(a), f(b)) \leq D_X^{n-1}(n \cdot r)$. We need an upper bound on the size of $r$-cubes $f$ in terms of dimension control functions and the Lipschitz constant of $f^{-1}$. One should
view the next result as a discrete analog of the fact that one cannot embed $I^n$ into an $(n - 1)$-dimensional topological space.

**Corollary 2.4.** Suppose $X$ is a metric space with an $(n - 1)$-dimensional control function $D_X^{n-1} : \mathbb{R}_+ \to \mathbb{R}_+ \cup \infty$. If $f : \{0, 1, \ldots, k\}^n \to X$ is an $r$-cube, then $k \leq D_X^{n-1}(n \cdot r) \cdot \text{Lip}(f^{-1})$.

**Proof.** By [2.3] there is an index $i \leq n$ and points $a$ and $b$ whose $i$-coordinates differ by $k$ such that $\text{dist}(f(a), f(b)) \leq D_X^{n-1}(n \cdot r)$. Since $k \leq \text{dist}(a, b) \leq \text{Lip}(f^{-1}) \cdot \text{dist}(f(a), f(b)) \leq D_X^{n-1}(n \cdot r) \cdot \text{Lip}(f^{-1})$, we are done. \hfill \blacksquare

### 3. Wreath Products

Let $A$ and $B$ be groups. Define the action of $B$ on the direct product $A^B$ (functions have finite support) by

$$bf(\gamma) := f(b^{-1} \gamma),$$

for any $f \in A^B$ and $\gamma \in B$. The **wreath product** of $A$ and $B$, denoted $A \wr B$, is the semidirect product $A^B \rtimes B$ of groups $A^B$ and $B$. That means it consists of ordered pairs $(f, b) \in A^B \times B$ and $(f_1, b_1) \cdot (f_2, b_2) = (f_1(b_1f_2), b_1b_2)$.

We will identify $(1, b)$ with $b \in B$ and $(f_a, 1)$ with $a \in A$, where $f_a$ is the function sending $1 \in B$ to $a$ and $B \setminus \{1\}$ to $1$. This way both $A$ and $B$ are subgroups of $A \wr B$ and it is generated by $B$ and elements of the form $b \cdot a \cdot b^{-1}$. That way the union of generating sets of $A$ and $B$ generates $A \wr B$.

The *lamplighter group* $L_n$ is the wreath product $\mathbb{Z}/n \wr \mathbb{Z}$ of $\mathbb{Z}/n$ and $\mathbb{Z}$.

If $g \in G$ and $a \in H \setminus \{1\}$, then $g \cdot a \cdot g^{-1} \in K$ will be called the $a$-bulb indexed by $g$ or the $(g, a)$-bulb. A bulb is a $(g, a)$-bulb for some $a \in H$ and some $g \in G$.

Consider the wreath product $H \wr G$, where $H$ is finite and $G$ is finitely generated. Let $K$ be the kernel of $H \wr G \to G$. $K$ is a locally finite group (the direct product of $|G|$ copies of $H$). In case $H$ is finite we choose as a set of generators of $H \wr G$ the union of $H \setminus \{1\}$ and a set of generators of $G$.

**Lemma 3.1.** Suppose $n > 1$. Any product of bulbs indexed by mutually different elements $g_i \in G$, $i \in \{1, \ldots, n\}$, has length at least $n$.

**Proof.** Consider $x = (g_1a_1g_1^{-1}) \cdot \ldots \cdot (g_na_ng_n^{-1}) \in K$. If its length is smaller than $n$, then $x = x_1 \cdot b_1 \cdot x_2 \cdot b_2 \cdot \ldots \cdot x_k \cdot b_k \cdot x_{k+1}$, where $k < n$ and $b_i \in H$, $x_i \in G$ for all $i$. We can rewrite $x$ as $(y_1 \cdot b_1 \cdot y_1^{-1}) \cdot (y_2 \cdot b_2 \cdot y_2^{-1}) \cdot \ldots \cdot (y_k \cdot b_k \cdot y_k^{-1}) \cdot y$, where $y_1 = x_1$. Since $x \in K$, $y = 1$. 

Now we arrive at a contradiction by looking at projections of $K$ onto its summands. □

**Lemma 3.2.** Suppose $r > 1$. Any element of $K$ of length less than $r$ is a product of bulbs indexed by elements of $G$ of length less than $r$.

**Proof.** Any element of $K$ of length less than $r$ has $x_1a_1x_2a_2\ldots x_ka_kz$ as a minimal representation, so it can be rewritten as

$$(x_1a_1x_1^{-1})(x_1x_2a_2x_2^{-1}x_1^{-1})\ldots(x_1\ldots x_ka_kx_k^{-1}\ldots x_1^{-1}).$$

Therefore the bulbs involved are indexed by elements of $G$ of length less than $r$. □

In case of the lamplighter group $L_2$ there is a precise calculation of length of its elements in [6]. We need a generalization of those calculations.

**Lemma 3.3.** Let $H$ be finite. Suppose the subgroup $\mathbb{Z}$ generated by $t \in G$ is of finite index $n$ and there are generators $\{t, g_1, \ldots, g_n\}$ of $G$ such that every element $g$ of $G$ can be expressed as $g_i \cdot t^{e(g)}$ for some $i$.

1. Every element of $K$ can be expressed as a product of $(h_i, a_i)$-bulbs, $i = 1, \ldots, k$, such that $h_i \neq h_j$ for $i \neq j$.
2. The length of such product is at most $n(k + 2 + 4 \max\{|e(h_i)|\})$.

**Proof.** Observe the product of the $(g, a)$-bulb and the $(g, b)$-bulb is the $(g, a \cdot b)$-bulb, so every product of bulbs can be represented as a product of $(h_i, a_i)$-bulbs, $i = 1, \ldots, k$, such that $h_i \neq h_j$ for $i \neq j$. We will divide those bulbs in groups determined by $h_i \cdot t^{-e(h_i)}$. Since there are at most $n$ groups, it suffices to show that if $h_i \cdot t^{-e(h_i)} = g$ for all $i$, then the length of the product $x$ of $(h_i, a_i)$-bulbs is at most $k + 2 + 4 \max\{|e(h_i)|\}$. We may order $h_i$ so that the function $i \to e(h_i)$ is strictly increasing. Now,

$$g^{-1} \cdot x \cdot g = \prod_{i=1}^{k} t^{e(h_i)} \cdot a_i \cdot t^{-e(h_i)} = t^{e(h_1)} \cdot a_1 \cdot t^{-e(h_1) + e(h_2)} \cdot a_2 \cdot \ldots \cdot a_k \cdot t^{-e(h_k)}$$

and its length is at most $k + |e(h_1)| + e(h_k) - e(h_1) + |e(h_k)| \leq k + 4 \max\{|e(h_i)|\}$. Therefore the length of $x$ is at most $k + 2 + 4 \max\{|e(h_i)|\}$. □

4. Dimension control functions of wreath products

Recall that the *growth* $\gamma$ of $G$ is the function counting the number of points in the open ball $B(1, r)$ of $G$ for all $r > 0$. Notice that $\gamma$ being bounded by a linear function is independent on the choice of generators of $G$. 
The next result relates the growth function of $G$ to dimension control functions of the kernel of the projection $H \wr G \to G$.

**Theorem 4.1.** Suppose $G$ and $H$ are finitely generated and $K$ is the kernel of the projection $H \wr G \to G$ equipped with the metric induced from $H \wr G$. If $\gamma$ is the growth function of $G$ and $D^{n-1}_K$ is an $(n-1)$-dimensional control function of $K$, then the integer part of $\frac{\gamma(r)}{n}$ is at most $D^{n-1}_K(3nr)$.

**Proof.** Given $k \geq 1$ we will construct a $3r$-cube $f: \{0,k\}^n \to K$ similarly to the way paths in the Cayley graph of $K$ are constructed. There, it suffices to label the beginning vertex and all the edges and that induces labeling of all the vertices. In case of our $3r$-cube we label the origin by $1 \in K$ and each edge from $x$ to $x + e_i$, $e_i$ being an element of the standard basis of $\mathbb{R}^n$, will be labeled by $x(j,i)$, where $j$ is the $i$-th coordinate of $x$. It remains to choose $x(j,i)$, $1 \leq i \leq n$ and $0 \leq j \leq k-1$. Given $r > 0$ consider mutually different elements $g(j,i)$, $1 \leq i \leq n$ and $0 \leq j \leq k-1$ of $G$ whose length is smaller than $r$, where $k$ is the integer part of $\frac{\gamma(r)}{n}$. Pick $u \in H \setminus \{1\}$ and put $x(j,i) = g(j,i) \cdot u \cdot g(j,i)^{-1}$. By 3.1 one has $\text{Lip}(f^{-1}) \leq 1$, so $k \leq D^{n-1}_K(3nr)$ by 2.4.

If $H$ is finite, then the kernel $K$ of the projection $H \wr G \to G$ is locally finite and it has a 0-dimensional control function $D^0_K$ attaining finite values ($K$ is equipped with the metric induced from $H \wr G$). Let us relate $D^0_K$ to the growth of $G$.

**Theorem 4.2.** Suppose $G$ is finitely generated and $H \neq \{1\}$ is finite. Let $K$ be the kernel of the projection $H \wr G \to G$ equipped with the metric induced from $H \wr G$. If $\gamma$ is the growth function of $G$, then $D^0_K(r) := (2r+1)\gamma(r)$ is a 0-dimensional control function of $K$.

**Proof.** It suffices to show that $r$-component of 1 in $K$ is of diameter at most $(2r+1)\gamma(r)$ as any $r$-component of $K$ is a shift of the $r$-component containing 1. By 3.2 any element of $B(1,r)$ in $K$ is a product of bulbs indexed by elements of $G$ of length less than $r$. Therefore any product of elements in $B(1,r)$ is a product of bulbs indexed by elements of $G$ of length less than $r$ and such product can be reduced to a product of at most $\gamma(r)$ such bulbs. Each of them is of length at most $2r+1$, so the length of the product is at most $(2r+1) \cdot \gamma(r)$. ■

**Theorem 4.3** (cf. [7 Proposition 4.2]). Suppose $G$ is finitely generated and $\pi: G \to I$ is a retraction onto its subgroup $I$ with kernel $K$. $K$ is equipped with the metric induced from a word metric on $G$ so that generators of $I$ are included in the set of generators of $G$. If $D^n_I$ is an
n-dimensional control function of $I$ and $D_K^n$ is a 0-dimensional control function of $K$, then
\[ D_I^n(r) + D_K^n(r + 2D_I^n(r)) \]
is an $n$-dimensional control function of $G$.

**Proof.** Given $r > 0$ express $I$ as $I_0 \cup \ldots \cup I_n$ so that $r$-components of $I_i$ have diameter at most $D_I^n(r)$. Consider $G_i = \pi^{-1}(I_i)$. If $g_1 \cdot 1, \ldots, g_i \cdot x_m$ is an $r$-path in $G_i$, then $h_1 = \pi(g_1) \cdot 1, \ldots, h_m = \pi(g_1) \cdot y_m$ form an $r$-path in $I_i$ (here $y_j = \pi(x_j)$), so $l(y_j) \leq D_I^n(r)$ for all $j$. Consider $z_j = x_j \cdot y_j^{-1} \in K$. Notice $\text{dist}(z_j, z_{j+1}) < r + 2D_I^n(r)$. Therefore, $\text{dist}(1, z_m) \leq D_K^n(r + 2D_I^n(r))$ resulting in $l(x_m) \leq D_K^n(r + 2D_I^n(r)) + D_I^n(r)$ and $\text{dist}(g_1, g_1 \cdot x_m) \leq D_I^n(r) + D_K^n(r + 2D_I^n(r))$ which completes the proof. \[\blacksquare\]

**Definition 4.4** (cf. [15, Section VI.B]). Let $f$ and $g$ be functions from $R_+$ to $R_+$. We say that $f$ weakly dominates $g$ if there exist constants $\lambda \geq 1$ and $C \geq 0$ such that $g(t) \leq \lambda f(\lambda t + C) + C$ for all $t \in R_+$.

Two functions are weakly equivalent if each weakly dominates the other.

**Theorem 4.5.** Suppose $G$ is finitely generated infinite group and $H \neq \{1\}$ is finite. Let $\gamma$ be the growth function of $G$ and $D_G^n$ be an $n$-dimensional control function of $G$. Then for any $k \geq n$ there is a $k$-dimensional control function of $H \wr G$ which is weakly dominated by $(D_G^n(t) + t) \cdot \gamma(D_G^n(t) + t)$. Also, for any $k \geq n$ every $k$-dimensional control function of $H \wr G$ weakly dominates the function $\gamma$.

**Proof.** Notice that $\gamma$ dominates a linear function and combine 4.2 and 4.3 To get the estimate from below, notice that a $k$-dimensional control function of $H \wr G$ works as a $k$-dimensional control function of the kernel $K$, and apply 4.1 \[\blacksquare\]

Our next result gives a better solution to Question 2 in [18].

**Corollary 4.6.** Suppose $G$ is a finitely generated group of exponential growth and $H \neq \{1\}$ is finite. If $\dim_{AN}(G) \leq n$ then for any $k \geq n$ the $k$-dimensional control function of $H \wr G$ is weakly equivalent to the function $2^t$ (i.e. there is a $k$-dimensional control function of $H \wr G$ weakly dominated by $2^t$ and every such control function weakly dominates $2^t$).

**Corollary 4.7.** Let $F_2$ be the free non-Abelian group of two generators. For every $n \geq 1$ the $n$-dimensional control function of $Z/2tF_2$ is weakly equivalent to the function $2^t$ (i.e. there is an $n$-dimensional control function of $Z/2tF_2$ weakly dominated by $2^t$ and every such control function weakly dominates $2^t$).
Proof. Notice that the function \( f(t) = 2^t \) is weakly equivalent to the growth function of \( F_2 \) and \( \dim_{AN}(F_2) = 1 \).

5. Assouad-Nagata dimension of wreath products

Suppose \( G \) is finitely generated and \( H \neq 1 \) is finite. If \( \dim_{AN}(G) = 0 \), then \( G \) is finite and so is \( H \wr G \). In such case \( \dim_{AN}(H \wr G) = 0 = \dim_{AN}(G) \). Therefore it remains to consider the case of infinite groups \( G \).

Theorem 5.1. Suppose \( G \) is an infinite finitely generated group and \( H \) is a finite group. If the growth of \( G \) is bounded by a linear function, then \( \dim_{AN}(K) = 0 \) and \( \dim_{AN}(H \wr G) = \dim_{AN}(G) = 1 \).

Proof. Notice that 4.2 does provide a 0-dimensional control function for \( K \). However, it may not be bounded by a linear function, so we have to do more precise calculations.

\( G \) is a virtually nilpotent group by Gromov’s Theorem (see [13] or Theorem 97 in [17]). Let \( F \) be a nilpotent subgroup of \( G \) of finite index. Pick elements \( a_i, i = 1, \ldots, k \), of \( G \) such that \( G = \bigcup_{i=1}^{k} a_i \cdot F \) and pick a natural \( n \) satisfying \( |a_i| \leq n \) for all \( i \leq k \). Every two elements of \( F \) can be connected in \( G \) by a 2-path. From each point of the path (other than initial and terminal points) one can move to \( F \) by distance at most \( n \) (by representing that point as \( a_i \cdot x \) for some \( x \in F \)). Therefore we can create a \((2n + 2)\)-path in \( F \) joining the original points. That means \( F \) is generated by its elements of length at most \( 2n + 1 \).

Let \( \{F_i\} \) be the lower central series of \( F \) and let \( d_i \) be the rank of \( F_i/F_{i+1}, i \geq 0 \). Since the growth of \( F \) is also linear, Bass’ Theorem (see [2] or Theorem 103 in [17]) stating that the growth of \( F \) is polynomial of degree \( d = \sum_{i=0}^{\infty} (i + 1) \cdot d_i \) implies that \( d_0 = 1 \) and all the other ranks \( d_i \) are 0. Hence the abelianization of \( F \) is of the form \( \mathbb{Z} \times A \), \( A \) being a finite group, and the commutator group of \( F \) is finite. Therefore \( F \) is virtually \( \mathbb{Z} \) and that means \( G \) is virtually \( \mathbb{Z} \) as well.

Let \( n \) be the index of \( \mathbb{Z} \) in \( G \) and pick elements \( g_1, \ldots, g_n \) of \( G \) such that any element of \( G \) can be expressed as \( g_i \cdot t^k \) for some \( i \leq n \) and some \( k \), where \( t \) is the generator of \( \mathbb{Z} \subset G \). Without loss of generality we may assume that the set of generators of \( G \) chosen to compute the word length \( l(w) \) of elements \( w \in H \wr G \) is \( t, g_1, \ldots, g_n \). For \( H \) we choose all of \( H \setminus \{1\} \) as the set of generators.

We need existence of \( C > 0 \) such that \( \frac{|w|}{C} \leq l(t^k) \leq |k| \) for all \( k \). It suffices to consider \( k > 0 \). Since the number of points in \( B(1_G, 4) \) is
finite, there is $C > 0$ such that $t^u \in B(1_G, 4)$ implies $|u| \leq C$. Now, if 
$l(t^k) = m$ and $t^k = x_1 \cdot \ldots \cdot x_m$, where $l(x_i) = 1$, then there are $u(i)$ such that dist$(x_1 \cdot \ldots \cdot x_i, t^{u(i)}) \leq 1$ for all $i \leq k$ (we choose $u(m) = k$ obviously). Therefore dist$(t^{u(i)}, t^{u(i+1)}) \leq 3$ and $u(i + 1) - u(i) \leq C$. 
Now $k = u(m) = (u(m) - u(m-1)) + \ldots + (u(2) - u(1)) + u(1) \leq C \cdot m$ implying $l(t^k) = m \geq \frac{k}{C}$.

By \underline{3.2} any element of $K$ of length less than $r$ is a product of bulbs 
indexed by elements of $G$ of length less than $r > 1$. If $l(g_i \cdot t^k) < r$, 
then $l(t^k) < r + 1 < 2r$ and $|k| \leq C \cdot l(t^k) \leq 2Cr$. Therefore there are 
at most $n \cdot 4Cr$ such words and any product of such bulbs is of length 
at most $n(4Crn + 2 + 2Cr) \leq r(4Crn^2 + 2n + 2Cn)$ by \underline{3.3}.

Therefore the group generated by $B(1, r)$ in $K$ is contained in $B(1, Lr)$, 
where $L = 4Crn^2 + 2n + 2Cn$, and dim$_{AN}(K) = 0$ by \underline{2.1}. Using 
the Hurewicz Theorem for Assouad-Nagata dimension of $K$, we get 
dim$_{AN}(H \wr G) \leq \dim_{AN}(G) = 1$ (one can also use \underline{1.3}). Since $H \wr G$ 
is infinite, its Assouad-Nagata dimension is positive and dim$_{AN}(H \wr G) = 
\dim_{AN}(G) = 1$.

**Corollary 5.2.** If the growth of $G$ is not bounded by a linear function and $H \neq 1$, then dim$_{AN}(H \wr G) = \infty$.

**Proof.** Let $\gamma$ be the growth of $G$ in some set of generators. Suppose 
dim$_{AN}(K) < n < \infty$, so it has an $(n - 1)$-dimensional function of the form 
$D^n_{K^{-1}}(r) = C \cdot r$ for some $C > 0$. By \underline{4.1} one has $\gamma(r)/n \leq C \cdot 3nr + 1$. Thus $\gamma(r) \leq n \cdot (3Cn + 1)$ and the growth of $G$ is bounded 
by a linear function, a contradiction. ■

**Problem 5.3.** Suppose $G$ is a locally finite group equipped with a 
proper left-invariant metric $d_G$. If dim$_{AN}(G, d_G) > 0$, is dim$_{AN}(G, d_G)$ 
infinite?

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