Nonconvex Variance Reduced Optimization with Arbitrary Sampling

Samuel Horváth ∗ Peter Richtárik †

September 13, 2018

Abstract

We provide the first importance sampling variants of variance reduced algorithms for empirical risk minimization with non-convex loss functions. In particular, we analyze non-convex versions of SVRG, SAGA and SARAH. Our methods have the capacity to speed up the training process by an order of magnitude compared to the state of the art on real datasets. Moreover, we also improve upon current mini-batch analysis of these methods by proposing importance sampling for minibatches in this setting. Surprisingly, our approach can in some regimes lead to superlinear speedup with respect to the minibatch size, which is not usually present in stochastic optimization. All the above results follow from a general analysis of the methods which works with arbitrary sampling, i.e., fully general randomized strategy for the selection of subsets of examples to be sampled in each iteration. Finally, we also perform a novel importance sampling analysis of SARAH in the convex setting.

1 Introduction

Empirical risk minimization (ERM) is a key problem in machine learning as it plays a key role in training supervised learning models, including classification and regression problems, such as support vector machine, logistic regression and deep learning. A generic ERM problem has the finite-sum form

\[
\min_{x \in \mathbb{R}^d} f(x) \overset{\text{def}}{=} \frac{1}{n} \sum_{i=1}^{n} f_i(x),
\]

where \(x\) corresponds to the parameters/features defining a model, \(f_i(x)\) is the loss of the model associated with data point \(i\), and \(f\) is the average (empirical) loss across the entire training dataset. In this paper we will focus on the case when the functions \(f_i\) are \(L_i\)-smooth but non-convex. We assume the problem has a solution \(x^*\).

One of the most popular algorithms for solving (1) is stochastic gradient descent (SGD) [19, 18]. In recent years, tremendous effort was exerted to improve its performance, leading to various enhancements which use acceleration [11], momentum [14], minibatching [32], distributed implementation [16, 15], importance sampling [36, 30, 27, 14], higher-order information [26, 9], and a number of other techniques.

1.1 Variance reduced methods

A particularly important recent advance has to do with the design of variance-reduced (VR) stochastic gradient methods, such as SAG [33], SDCA [34, 32], SVRG [11], S2GD [13], SAGA [7], MISO [17] and FINITO [8] and SARAH [22], which operate by modifying the classical stochastic gradient direction in each step of the training
Table 1: Stochastic gradient evaluation complexity for achieving $E[\|\nabla f(x)\|^2] \leq \epsilon$ for two variants of SVRG, SAGA and SARAH for minimizing the average of smooth nonconvex functions. Constants: $c_1, c_2, c_3$ are universal constant, $L_{\text{max}} = \max_i L_i$; $\bar{L} = \frac{1}{n} \sum_i L_i$; $b = $ (average) minibatch size (hidden in $\alpha$); $\alpha$ can be for specific samplings smaller than 1 and decreasing with increasing $b$, which can lead to superlinear speedup in $b$. For SARAH this guarantee holds for one outer loop with minibatch size, where we assume $16\bar{L}^2(f(x^0) - f(x^*))^2/ (zb)^2 >> 0$, in other words, minibatch size is not too big comparing to the required precision.

1.2 Importance sampling, minibatching and non-convex models

In the context of problem (1), importance sampling refers to the technique of assigning carefully designed non-uniform probabilities $\{p_i\}$ to the $n$ functions $\{f_i\}$, and using these, as opposed to uniform probabilities, to sample the next data point (stochastic gradient) during the training process.

Despite the huge theoretical and practical success of VR methods, there are still considerable gaps in our understanding. For instance, an importance sampling variant of the popular SAGA method, with the “correct” convergence rate, was only designed very recently [10]; and the analysis applies to strongly convex $f$ only. A coordinate descent variant of SVRG with importance sampling, also in the strongly convex case, was analyzed in [12]. However, the method does not seem to admit a fast implementation. For dual methods based on coordinate descent, importance sampling is relatively well understood [20, 32, 24, 27, 3].

The territory is completely unmapped in the non-convex case, however. To the best of our knowledge, no importance sampling VR methods have been designed nor analyzed in the popular case when the functions $\{f_i\}$ are non-convex. An exception to this is dSAGA [3]; however, this method applies to an explicitly regularized version of (1), and while the individual functions are allowed to be non-convex, the average $f$ is assumed to be convex. Given the dominance of stochastic gradient type methods in training large non-convex models such as deep neural networks, theoretical investigation of VR methods that can benefit from importance sampling is much needed.

The situation is worse still when one asks for importance sampling of minibatches. To the best of our knowledge, there are only a handful of papers on this topic [30, 6], none of which apply to the non-convex setting considered here, nor to the methods we will analyze, and the problem is open. This is despite the fact that minibatch methods are de-facto the norm for training deep nets due to the volume of data that feeds into the training, and the necessity to fully utilize the parallel processing power of GPUs and other hardware accelerators for the task. In practice, typically relatively small ($O(1)$ in comparison with $n$) minibatch sizes are used.
1.3 Contributions

The main contributions of this paper are:

- **Arbitrary sampling.** We perform a general analysis of three popular VR methods—SVRG [11], SAGA [7] and SARAH [22]— in the arbitrary sampling paradigm [30, 24, 25, 27, 4]. That is, we prove general complexity results which hold for an arbitrary random set valued mapping (aka arbitrary sampling) generating the minibatches of examples used by the algorithms in each iteration.

- **Rates.** Our bounds improve the best current rates for these methods, for SVRG and SAGA even under the same $L_i$’s. Our importance sampling can be up faster by factor of $n$ comparing to the current state of the art (see Table 1 and Appendix C). Our methods can enjoy linear speedup or even for some specific samplings superlinear speedup in minibatch size. That is, the number of iterations needed to output a solution of a given accuracy drops by a factor equal or greater to the minibatch size used. This is of utmost relevance to the practice of training neural nets with minibatch stochastic methods as our results predict that this is to be expected. We design importance sampling and approximate importance sampling for minibatches which in our experiments vastly outperform the standard uniform minibatch strategies.

- **Best rates for SARAH under convexity.** Lastly, we also perform an analysis of importance sampling variant of SARAH in the convex and strongly convex case (Appendix I). These are the currently fastest rates for SARAH.

2 Importance Sampling for Minibatches

As mentioned in the introduction, we assume throughout that $f_i : \mathbb{R}^d \mapsto \mathbb{R}$ are smooth, but not necessarily convex. In particular, we assume that $f_i$ is $L_i$–smooth; that is,

$$\| \nabla f_i(x) - \nabla f_i(y) \| \leq L_i \| x - y \|, \quad x, y \in \mathbb{R}^d,$$

where $\| x \| \overset{\text{def}}{=} (\sum_{i} x_i^2)^{1/2}$ is the standard Euclidean norm. Let us define $L = \frac{1}{n} \sum_{i=1}^{n} L_i$. Without loss of generality assume that $L_1 \leq L_2 \leq \cdots \leq L_n$.

2.1 Samplings

Let $S$ be a random set-valued mapping (“sampling”) with values in $2^{[n]}$, where $[n] \overset{\text{def}}{=} \{1, 2, \ldots, n\}$. A sampling is uniquely defined by assigning probabilities to all $2^n$ subsets of $[n]$. With each sampling we associate a probability matrix $P \in \mathbb{R}^{n \times n}$ defined by

$$P_{ij} \overset{\text{def}}{=} \text{Prob}(\{i, j\} \subseteq S).$$

The probability vector associated with $S$ is the vector composed of the diagonal entries of $P$: $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$, where $p_i \overset{\text{def}}{=} \text{Prob}(i \in S)$. We say that $S$ is proper if $p_i > 0$ for all $i$. It is easy to show that

$$b \overset{\text{def}}{=} \mathbb{E}[|S|] = \text{Trace}(P) = \sum_{i} p_i. \quad (2)$$

From now on, we will refer to $b$ as the minibatch size of sampling $S$. It is known that $P - pp^\top$ is a symmetric positive semidefinite matrix [31].

Let us without loss of generality assume that $p_1 \leq p_2 \leq \cdots \leq p_n$ and define constant $k = |\{i \in [n] : p_i < 1\}| = \max\{i : p_i < 1\}$ to be number of $p_i$’s, which are not equal to one.

While our complexity results are general in the sense that they hold for any proper sampling, we shall now consider three special samplings; all with minibatch size $b \in (0, n)$:
(i) **Standard uniform minibatch sampling** \(S = S^a\) \(^1\) \(S\) is chosen uniformly at random from all subsets of \([n]\) of cardinality \(b\). Clearly, \(|S| = b\) with probability 1. The probability matrix is given by

\[
P_{ij} = \begin{cases} \frac{b}{n} & i = j, \\ \frac{b(b-1)}{n(n-1)} & i \neq j. \end{cases}
\]

(ii) **Independent sampling** \((S = S^*)\). Assume any proper \(p_i\)'s. For each \(i \in [n]\) we independently flip a coin, and with probability \(p_i\) include element \(i\) into \(S\). Hence, by construction, \(p_i = \text{Prob}(i \in S)\) and \(E[|S|] = \sum p_i = b\). The probability matrix of \(S\) is

\[
P_{ij} = \begin{cases} p_i & i = j, \\ p_i p_j & i \neq j. \end{cases}
\]

(iii) **Approximate independent sampling** \((S = S^a)\). Independent sampling has the disadvantage that \(k\) coin tosses need to be performed in order to generate the random set. However, we would like to sample at the cost \(\mathcal{O}(b + k - n)\) coin tosses instead. We now design a sampling which has this property and which in a certain precise sense, as we shall see later, approximates the independent sampling. In particular, given an independent sampling with parameters \(p_i\) for \(i \in [n]\), let \(a = [k \max_{i \leq k} p_i]\). Since \(\max_{i \leq k} p_i \geq \frac{b+k-n}{k}\), it follows that \(a \geq b + k - n\). On the other hand, if \(\max_{i \leq k} p_i = \mathcal{O}(b + k - n)/k\), then \(a = \mathcal{O}(b + k - n)\). We now sample a single set \(S'\) of cardinality \(a\) using the standard uniform minibatch sampling (just for \(i \leq k\)). Subsequently, we apply an independent sampling to select elements of \(S'\), with selection probabilities \(p_i/kp_i = p_i\). The resulting set is \(S\). Since \(\text{Prob}(i \in S) = \frac{b+1}{(a-1)} p_i = p_i\) and \(\text{Prob}(\{i,j\} \subseteq S) = \frac{(a-2)}{(a-1)} p_i p_j\), the probability matrix of \(S\) is given by

\[
P_{ij} = \begin{cases} p_i & i = j, \\ \frac{(a-1)k}{a(a-1)} p_i p_j & i \neq j, i,j \leq k, \\ p_i p_j & \text{otherwise}. \end{cases}
\]

Since \(\frac{(a-1)k}{a(a-1)} \approx 1\), the probability matrix of the approximate independent sampling approximates that of the independent sampling. Note that \(S\) includes both the standard uniform minibatch sampling and the independent sampling as special cases. Indeed, the former is obtained by choosing \(p_i = b/n\) for all \(i\) (whence \(a = b\) and \(p_i' = 1\) for all \(i\)), and the latter is obtained by choosing \(a = n\) instead of \(a = [k \max_{i \leq k} p_i]\).

### 2.2 Key lemma

The following lemma, which we use as upper bound for variance, plays a key role in our analysis.

**Lemma 1.** Let \(\zeta_1, \zeta_2, \ldots, \zeta_n\) be vectors in \(\mathbb{R}^d\) and let \(\overline{\zeta} \stackrel{def}{=} \frac{1}{n} \sum_{i=1}^n \zeta_i\) be their average. Let \(S\) be a proper sampling (i.e., assume that \(p_i = \text{Prob}(i \in S) > 0\) for all \(i\)). Assume that there is \(v \in \mathbb{R}^n\) such that

\[
P - pp^\top \preceq \text{Diag}(p_1 v_1, p_2 v_2, \ldots, p_n v_n) .
\]

Then

\[
E \left[ \left\| \sum_{i \in S} \frac{\zeta_i}{p_i} - \overline{\zeta} \right\|^2 \right] \leq \frac{1}{n^2} \sum_{i=1}^n \frac{v_i}{p_i} \|\zeta_i\|^2 ,
\]

\(^1\)In the literature, this is often referred to by the name \(b\)-nice sampling \([31, 29]\).

\(^2\) Note that just \(k\) not \(n\), because others are included with probability one.
where the expectation is taken over sampling $S$. Whenever (4) holds, it must be the case that
\[ v_i \geq 1 - p_i. \]
Moreover, (4) is always satisfied for $v_i = n(1 - p_i)$ for $i \leq k$ and 0 otherwise. Further, if $|S| \leq d$ with probability 1 for some $d$, then (4) holds for $v_i = d$. The standard uniform minibatch sampling admits $v_i = \frac{n^2}{bL_i}$, the independent sampling admits $v_i = 1 - p_i$, and the approximate independent sampling admits the choice $v_i = 1 - p_i(1 - \frac{k-a}{a(k-1)})$ if $i \leq k$, $v_i = 0$ otherwise.

The following quantities, which comes from Lemma 1 and smoothness assumption, play a key role in our general complexity results:
\[ K \overset{\text{def}}{=} \frac{b}{n^2} \sum_{i=1}^{n} v_i L_i^2 / p_i, \quad \alpha \overset{\text{def}}{=} \frac{K}{L^2}. \]

We can see through theory that it is a good idea to design samplings for which the value $\alpha$ is the smallest possible, which would lead to optimal sampling. The following result sheds light on how $S$ should be chosen, from samplings of a given minibatch size $b$, to minimize $\alpha$.

**Lemma 2.** Fix a minibatch size $b \in (0, n]$. Then the quantity $\alpha$, defined in (7), is minimized for the choice $S = S^*$ with the probabilities
\[ p_i \overset{\text{def}}{=} \begin{cases} 
(b + k - n) \frac{L_i}{\sum_{j=1}^{k} L_j}, & \text{if } i \leq k, \\
1, & \text{if } i > k,
\end{cases} \]
where $k$ is the largest integer satisfying $0 < b + k - n \leq \sum_{i=1}^{k} L_i / L_k$ (for instance, $k = n - b + 1$ satisfies this). Usually, if $L_i$'s are not too much different, than $k = O(n)$, for instance, if $bL_n \leq \sum_{i=1}^{n} L_i$ then $k = n$. If we choose $S = S^*$, then $\alpha$ is minimized for (8) with
\[ \alpha = \left( \frac{b}{(b + k - n)n^2} \left( \frac{k}{\sum_{i=1}^{k} L_i} \right)^2 - \frac{bs}{n^2} \frac{k}{\sum_{i=1}^{k} L_i^2} \right) / L^2, \]
where $s = 1$ for $S^*$ and $s = 1 - \frac{k-a}{a(k-1)}$ for $S^a$. Moreover, if we assume $bL_n \leq \sum_{i=1}^{n} L_i$, then $k = n$, thus
\[ \alpha_{S^*} = 1 - \frac{\sum_{i=1}^{n} L_i^2}{(\sum_{i=1}^{n} L_i)^2}, \quad \alpha_{S^a} = (n - b) \frac{n}{n - 1} \frac{\sum_{i=1}^{n} L_i^2}{(\sum_{i=1}^{n} L_i)^2}. \]

From now let $S^*, S^a$ denote *Independent Sampling* and *Approximate Independent Sampling*, respectively, with probabilities defined in (8).

**Remark 1.** Lemma 2 guarantees that the sampling $S^*$ is optimal. Moreover, let $b_{\text{max}} \overset{\text{def}}{=} \max\{b | bL_n \leq \sum_{i=1}^{n} L_i\}$ then in theory, we have super linear speed up for $b$ up to $b_{\text{max}}$ for all three algorithms.

### 3 SVRG, SAGA and SARAH

In all of the results of this section we assume that $S$ is an arbitrary proper sampling. Let $b = E[|S|]$ be the (average) minibatch size. We assume that $v$ satisfies (4) and that $\alpha$ (which depends on $v$) is defined as in (7). All complexity results will depend on $\alpha$ and $b$.

We propose three methods, Algorithm 1, 2 and 3 which are generalizations of original SVRG 28, SAGA and SARAH to the arbitrary sampling setting, respectively. The original non-minibatch methods arise as special
Thus in terms of stochastic gradient evaluations to obtain $\epsilon$-accurate solution, one needs following number of iterations

$$
\max\left\{ n, \frac{\mu_2 L_{\bar{2}}(2/3)(f(x^0) - f(x^*))}{\epsilon \nu_2} \left(1 + \frac{\alpha}{3\mu_2}\right)\right\}.
$$

In the next theorem we provide a generalization of the results in [29].

**Theorem 4** (Complexity of SAGA with arbitrary sampling). There exist universal constants $\mu_3 > 0$, $0 < \nu_3 < 1$ such that the output of Alg. 2 with mini-batch size $b \leq \alpha n^{2/3}$, step size $\eta = b/(\mu_3 L_{\bar{2}} n^{2/3})$, and parameter $d = b/\alpha$ satisfies:

$$
E \left[ \|\nabla f(x_a)\|^2 \right] \leq \frac{\alpha L n^{2/3}[f(x^0) - f(x^*)]}{bT \nu_3}.
$$

3Note, that this can be always satisfied, if we uplift the smallest $L_i$'s, because if function is $L$-smooth, then it is also smooth with larger $L'$. 

---

Algorithm 1 SVRG with arbitrary sampling ($x^0, m, T, \eta, S$)

1. $x^0 = x_m^0 = x^0$, $M = [T/m]$
2. for $s = 0$ to $M - 1$
3. $x_{s+1}^0 = x_m^s$
4. $g_{s+1} = \frac{1}{n} \sum_{i=1}^m \nabla f_i(x^s)$
5. for $t = 0$ to $m - 1$
6. Draw a random subset (minibatch) $S_t \sim S$
7. $v_{t+1} = \sum_{i \in S_t} \frac{1}{n_{p_{i}}} \left( \nabla f_i(x^t) - \nabla f_i(\hat{x}^t) \right) + g_{s+1}$
8. $x_{t+1} = x_{t} + \eta_{s+1} v_{t+1}$
9. end for
10. $\hat{x}_{s+1} = x_m^s$
11. end for

**Output**: Iterate $x_{a}$ chosen uniformly random from $\{x^t_{i=0}^{m}\}_{t=0}^{M}$.

Algorithm 2 SAGA with arbitrary sampling ($x^0, d, T, \eta, S$)

1. $\alpha_0^t = x^0$ for $i \in [n]$
2. $g^0 = \frac{1}{n} \sum_{i=1}^n \nabla f_i(\alpha_0^t)$
3. for $t = 0$ to $T - 1$
4. Draw a random subset (minibatch) $S_t \sim S$
5. Pick random subset $J_t$ of $[n]$ such that $\text{Prob}(j \in J_t) = d/n$
6. $v^t = \sum_{i \in J_t} \frac{1}{n_{p_{i}}} \left( \nabla f_i(x^t) - \nabla f_i(\alpha_{t+1}) \right) + g^t$
7. $x_{t+1}^s = x^t - \eta_v^t v^t$
8. $\alpha_{t+1}^t = x^t$ for $j \in J_t$ and $\alpha_{t+1}^t = \alpha^t_j$ for $j \notin J_t$
9. $g_{t+1} = g^t - \frac{1}{n} \sum_{j \in J_t} (\nabla f_j(\alpha_j^t) - \nabla f_j(\alpha_{t+1}^t))$
10. end for
11. **Output**: Iterate $x_{a}$ chosen uniformly random from $\{x^t\}_{t=0}^{T}$.
Consider one outer loop of Alg. 3 with $\eta = \frac{2}{L(\sqrt{1 + \frac{4\alpha b^2}{n}})}$. Thus in terms of stochastic gradient evaluations to obtain $\epsilon$-accurate solution, one needs following number of iterations
\[
n + \frac{\bar{L}n(2/3)(f(x^0) - f(x^*))}{\epsilon \nu_d}(1 + \alpha).
\]

We now introduce Algorithm 3 a general form of the SARAH algorithm [23].

**Theorem 5 (Complexity of SARAH with arbitrary sampling).** Consider one outer loop of Alg. 3 with
\[
\eta \leq \frac{2}{L(\sqrt{1 + \frac{4\alpha b^2}{n}})}.
\]

Then the output $x_\alpha$ satisfies:
\[
E[\|\nabla f(x_\alpha)\|^2] \leq \frac{2}{\eta(m + 1)}[f(x^0) - f(x^*)].
\]

Thus in terms of stochastic gradient evaluations to obtain $\epsilon$-accurate solution, one needs following number of iterations
\[
n + \frac{16\alpha \bar{L}^2(f(x^0) - f(x^*))^2 + \sqrt{16\alpha^2 L^4(f(x^0) - f(x^*))^4} + 16\epsilon^2 L^2(f(x^0) - f(x^*))^2 b^2}{2\epsilon^2}.
\]

If all $L_i$’s are the same and we choose $S$ to be $S^n$, thus uniform with mini-batch size $b$, we can get back original result from [23]. Taking $b = n$, we can restore gradient descent with the correct step size.

## 4 Additional Results

### 4.1 Gradient dominated functions

**Definition 1.** Function $f$ is $\tau$-gradient dominated if $f(x) - f(x^*) \leq \tau \|\nabla f(x)\|^2$, for all $x \in \mathbb{R}^d$, where $x^*$ is optimal solution of (1).

Gradient dominance is weaker version of strong convexity due to the fact that if function is $\mu$-strongly convex then it is $\tau$-gradient dominated, where $\tau = 2/\mu$.

Any of the non-convex methods in this paper can be used as a subroutine of Algorithm 4, where $T$ is the number of steps of the subroutine and $A$ is the set of optimal parameters for the subroutine. We set $T = \alpha T^{2/3}/(b \nu_d)$ for SVRG and $T = \alpha T^{2/3}/(b \nu_d)$ for SAGA. In the case of SARAH, $T$ is obtained by solving $m + 1 = 2/\eta$ in $m$ and setting $T \leftarrow m$. Using Theorems 3, 4, 5 and the above special choice of $T$, we get
\[
E[\|\nabla f(x^k)\|^2] \leq \frac{1}{2\tau} E[f(x^{k-1})] - f(x^*)
\]
Algorithm 4 GD-Algorithm \((x^0, T, \mathbb{A})\)

| Input: \(x^0 \in \mathbb{R}^d, T, \mathbb{A}\) |
|---|
| for \(k = 0\) to \(K\) do |
| \(x^k = \text{Non-convex algorithm}(x^{k-1}, T, \mathbb{A})\) |
| end for |
| Output: \(x^K\) |

Combined with Definition 1, this guarantees linear convergence with the same constants \(\alpha, \bar{L}\) and \(b\) we had before in our analysis.

4.2 Importance sampling for SARAH under convexity

In addition to the results presented in previous sections, we also establish importance sampling results for SARAH in convex and strongly convex cases (Appendix I) with similar improvements as for the non-convex algorithm. Ours are the best current rates for SARAH in these settings.

Further, we also provide specialized non-minibatch versions of non-convex SAGA, SARAH and SVRG, which are either special cases of their minibatch versions presented in the main part, or slight modifications, with slightly improved guarantees.

5 Experiments

In this section, we perform experiments with regression for binary classification, where our loss function has form \(\frac{1}{n} \sum_{i=1}^{n} (1 - y_i \sigma(x_i^\top a))^2\), where \(\sigma(z)\) is the sigmoid function, thus is smooth but non-convex. We use four LIBSVM datasets\(^4\): covtype, ijcnn1, splice, australian.

Parameters of each algorithm are chosen as suggested by theorems in section 3 and \(x^0\) is set to be all zeros vector. The y axis shows the norm of the gradient, and the x axis shows how many times the algorithm evaluates the full gradient. Evaluation of the gradient of a single function from the empirical loss costs \(1/n\) of the full gradient evaluation. For SARAH, we chose \(m = \lceil n/b \rceil\).

5.1 Importance and uniform sampling comparison

Here we provide comparison of the methods with uniform \(S^u\) and with importance \(S^*\) sampling.

![Figure 1: Comparison of all methods with uniform and importance sampling](https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/)

From Figure 1 one can see that method with importance sampling outperform uniform sampling and can be faster by orders of magnitude. For instance, for the first graph, we can see improvement up to 4 orders of magnitude.

\(^4\)The LIBSVM dataset collection is available at [https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/](https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/)
5.2 Linear or more than linear speedup

Theory suggests that linear or even more than linear speed up can be obtained using $S^*$. Our experiment suggest that this is possible for all three algorithms.

Figure 2: *ijcnn1* dataset, # iterations means number of stochastic gradient evaluation divided by minibatch size $b$, when we assume parallelism on $b$ cores.

Figure 2 confirms that linear, even superlinear, speedup (with increasing batch size, we need the same or less full gradient evaluations) can be obtained also in practice, not just in theory. However, it is limited, and for this dataset it is up to minibatch size of 250. Upper graphs at Figure 2 visualize convergence under assumption of multi-core settings, where we assume number of cores is the same as the minibatch size.

5.3 Independent Sampling vs. Approximate Independent Sampling

In the theory, *Independent Sampling* $S^*$ is slightly better than *Approximate Independent Sampling* $S^a$. However, it is more expensive in terms of computations. The goal of the next experiment is to show that in practice $S^a$ yields comparable or faster convergence. Hence, it is more reasonable to use this sampling for datasets, where number of data points $n$ is big (if we implement $S^a$ efficiently we can almost get rid of dependence on $n$). The intuition behind why $S^a$ could work better is that $S^a$ has smaller variance in batch size than $S^*$. 

Figure 3: $S^*$ vs. $S^a$, *splice* dataset.
It can be seen from Figure 3 that $S^a$ can outperform $S^*$, thus, however, $S^*$ is optimal in theory, one should use $S^a$ in practice.

References

[1] Zeyuan Allen-Zhu. Katyusha: The first direct acceleration of stochastic gradient methods. *STOC 2017: Symposium on Theory of Computing*, 19-23, 2016.

[2] Zeyuan Allen-Zhu and Elad Hazan. Variance reduction for faster non-convex optimization. In *The 33rd International Conference on Machine Learning*, pages 699–707, 2016.

[3] Zeyuan Allen-Zhu, Zheng Qu, Peter Richtárik, and Yang Yuan. Even faster accelerated coordinate descent using non-uniform sampling. In *The 33rd International Conference on Machine Learning*, pages 1110–1119, 2016.

[4] Antonin Chambolle, Matthias J. Ehrhardt, Peter Richtárik, and Carola-Bibiane Schönlieb. Stochastic primal-dual hybrid gradient algorithm with arbitrary sampling and imaging applications. *arXiv:1706.04957*, 2017.

[5] Dominik Csiba and Peter Richtárik. Primal method for erm with flexible mini-batching schemes and non-convex losses. *arXiv:1506.02227*, 2015.

[6] Dominik Csiba and Peter Richtárik. Importance sampling for minibatches. *arXiv:1602.02283*, 2016.

[7] Aaron Defazio, Francis Bach, and Simon Lacoste-Julien. Saga: A fast incremental gradient method with support for non-strongly convex composite objectives. *Advances in Neural Information Processing Systems 27*, 2014.

[8] Aaron Defazio, Tiberio Caetano, and Justin Domke. Finito: A faster, permutable incremental gradient method for big data problems. *The 31st International Conference on Machine Learning*, 2014.

[9] Robert Mansel Gower, Donald Goldfarb, and Peter Richtárik. Stochastic block BFGS: squeezing more curvature out of data. In *The 33rd International Conference on Machine Learning*, pages 1869–1878, 2016.

[10] Robert Mansel Gower, Peter Richtárik, and Francis Bach. Stochastic quasi-gradient methods: variance reduction via Jacobian sketching. *arXiv:1805.02632*, 2018.

[11] Rie Johnson and Tong Zhang. Accelerating stochastic gradient descent using predictive variance reduction. In *Advances in Neural Information Processing Systems 26*, pages 315–323, 2013.

[12] Jakub Konečný, Zheng Qu, and Peter Richtárik. S2CD: Semi-stochastic coordinate descent. *Optimization Methods and Software*, 32(5):993–1005, 2017.

[13] Jakub Konečný and Peter Richtárik. S2GD: Semi-stochastic gradient descent methods. *Frontiers in Applied Mathematics and Statistics*, pages 1–14, 2017.

[14] Nicolas Loizou and Peter Richtárik. Momentum and stochastic momentum for stochastic gradient, newton, proximal point and subspace descent methods. *arXiv:1712.09677*, 2017.

[15] Chenxin Ma, Jakub Konečný, Martin Jaggi, Virginia Smith, Michael I Jordan, Peter Richtárik, and Martin Takáč. Distributed optimization with arbitrary local solvers. *Optimization Methods and Software*, 32(4):813–848, 2017.

[16] Chenxin Ma, Virginia Smith, Martin Jaggi, Michael I. Jordan, Peter Richtárik, and Martin Takáč. Adding vs. averaging in distributed primal-dual optimization. In *The 32nd International Conference on Machine Learning*, pages 1973–1982, 2015.
[17] Julien Mairal. Incremental majorization-minimization optimization with application to large-scale machine learning. *SIAM Journal on Optimization*, 25(2):829–855, 2015.

[18] A Nemirovski, A Juditsky, G Lan, and A Shapiro. Robust stochastic approximation approach to stochastic programming. *SIAM Journal on Optimization*, 19(4):1574–1609, 2009.

[19] Arkadi Nemirovsky and David B. Yudin. *Problem complexity and method efficiency in optimization*. Wiley, New York, 1983.

[20] Yurii Nesterov. Efficiency of coordinate descent methods on huge-scale optimization problems. *SIAM Journal on Optimization*, 22(2):341–362, 2012.

[21] Yurii Nesterov. *Introductory lectures on convex optimization: A basic course*, volume 87. Springer Science & Business Media, 2013.

[22] Lam Nguyen, Jie Liu, Katya Scheinberg, and Martin Takáč. SARAH: A novel method for machine learning problems using stochastic recursive gradient. *The 34th International Conference on Machine Learning*, 2017.

[23] Lam M Nguyen, Jie Liu, Katya Scheinberg, and Martin Takáč. Stochastic recursive gradient algorithm for nonconvex optimization. *arXiv:1705.07261*, 2017.

[24] Zheng Qu and Peter Richtárik. Coordinate descent with arbitrary sampling I: algorithms and complexity. *Optimization Methods and Software*, 31(5):829–857, 2016.

[25] Zheng Qu and Peter Richtárik. Coordinate descent with arbitrary sampling II: expected separable overapproximation. *Optimization Methods and Software*, 31(5):858–884, 2016.

[26] Zheng Qu, Peter Richtárik, Martin Takáč, and Olivier Fercoq. SDNA: stochastic dual newton ascent for empirical risk minimization. In *The 33rd International Conference on Machine Learning*, pages 1823–1832, 2016.

[27] Zheng Qu, Peter Richtárik, and Tong Zhang. Quartz: Randomized dual coordinate ascent with arbitrary sampling. In *Advances in Neural Information Processing Systems 28*, pages 865–873, 2015.

[28] Sashank J Reddi, Ahmed Hefny, Suvrit Sra, Barnabas Poczos, and Alex Smola. Stochastic variance reduction for nonconvex optimization. In *The 33th International Conference on Machine Learning*, pages 314–323, 2016.

[29] Sashank J Reddi, Suvrit Sra, Barnabás Póczos, and Alex Smola. Fast incremental method for smooth nonconvex optimization. In *Decision and Control (CDC), 2016 IEEE 55th Conference on*, pages 1971–1977. IEEE, 2016.

[30] Peter Richtárik and Martin Takáč. On optimal probabilities in stochastic coordinate descent methods. *Optimization Letters*, 10(6):1233–1243, 2016.

[31] Peter Richtárik and Martin Takáč. Parallel coordinate descent methods for big data optimization. *Mathematical Programming*, 156(1-2):433–484, 2016.

[32] Peter Richtárik and Martin Takáč. Iteration complexity of randomized block-coordinate descent methods for minimizing a composite function. *Mathematical Programming*, 144(2):1–38, 2014.

[33] Nicolas Le Roux, Mark Schmidt, and Francis Bach. A stochastic gradient method with an exponential convergence rate for finite training sets. In *Advances in Neural Information Processing Systems*, pages 2663–2671, 2012.

[34] Shai Shalev-Shwartz and Tong Zhang. Stochastic dual coordinate ascent methods for regularized loss. *Journal of Machine Learning Research*, 14(1):567–599, 2013.
[35] Martin Takáč, Avleen Bijral, Peter Richtárik, and Nathan Srebro. Mini-batch primal and dual methods for SVMs. In 30th International Conference on Machine Learning, 2013.

[36] Peilin Zhao and Tong Zhang. Stochastic optimization with importance sampling for regularized loss minimization. In The 32rd International Conference on Machine Learning, pages 1–9, 2015.
Appendix

A Proof of Lemma 1

Proof. Let $1_{i \in S} = 1$ if $i \in S$ and $1_{i \not\in S} = 0$ otherwise. Likewise, let $1_{i,j \in S} = 1$ if $i,j \in S$ and $1_{i,j \not\in S} = 0$ otherwise. Note that $E[1_{i \in S}] = p_i$ and $E[1_{i,j \in S}] = p_{ij}$. Next, let us compute the mean of $X \stackrel{\text{def}}{=} \sum_{i \in S} \frac{\xi_i}{np_i}$:

$$E[X] = E\left[\sum_{i \in S} \frac{\xi_i}{np_i}\right] = E\left[\sum_{i=1}^{n} \frac{\xi_i}{np_i} 1_{i \in S}\right] = \sum_{i=1}^{n} \frac{\xi_i}{np_i} E[1_{i \in S}] = \frac{1}{n} \sum_{i=1}^{n} \xi_i = \bar{\xi}. \quad (11)$$

Let $A = [a_1, \ldots, a_n] \in \mathbb{R}^{d \times n}$, where $a_i = \frac{\xi_i}{\bar{\xi}}$, and let $e$ be the vector of all ones in $\mathbb{R}^n$. We now write the variance of $X$ in a form which will be convenient to establish a bound:

$$E\left[\|X - E[X]\|^2\right] = E\left[\|X\|^2\right] - \|E[X]\|^2$$

$$= E\left[\left\|\sum_{i \in S} \frac{\xi_i}{np_i}\right\|^2\right] - \|\bar{\xi}\|^2$$

$$= E\left[\sum_{i,j} \frac{\xi_i}{np_i} \frac{\xi_j}{np_j} 1_{i,j \in S}\right] - \|\bar{\xi}\|^2$$

$$= \sum_{i,j} p_{ij} \frac{\xi_i}{np_i} \frac{\xi_j}{np_j} - \sum_{i,j} \frac{\xi_i}{n} \frac{\xi_j}{n}$$

$$= \frac{1}{n^2} \sum_{i,j} (p_{ij} - p_{i} p_{j}) a_i^T a_j$$

$$= \frac{1}{n^2} e^T ((P - pp^T) \circ A^T A) e. \quad (12)$$

Since by assumption we have $P - pp^T \preceq \text{Diag}(p \circ v)$, we can further bound

$$e^T ((P - pp^T) \circ A^T A) e \leq e^T (\text{Diag}(p \circ v) \circ A^T A) e = \sum_{i=1}^{n} p_i v_i |a_i|^2.$$ 

To obtain (5), it remains to combine this with (12).

Inequality (6) follows by comparing the diagonal elements of the two matrices in (11). Let us now verify the formulas for $v$.

• Since $P - pp^T$ is positive semidefinite [31], we can bound $P - pp^T \preceq n \text{Diag}(P - pp^T) = \text{Diag}(p \circ v)$, where $v_i = n(1 - p_i)$.

• It was shown in [25][Theorem 4.1] that $P \preceq d \text{Diag}(p)$ provided that $|S| \leq d$ with probability 1. Hence, $P - pp^T \preceq P \preceq d \text{Diag}(p)$, which means that $v_i = d$ for all $i$.

• Consider now the independent sampling. Clearly,

$$P - pp^T = \begin{bmatrix}
    p_1(1-p_1) & 0 & \ldots & 0 \\
    0 & p_2(1-p_2) & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & p_n(1-p_n)
\end{bmatrix} = \text{Diag}(p_1 v_1, \ldots, p_n v_n),$$

where $v_i = 1 - p_i$. 

13
Consider the $b$–nice sampling (standard uniform minibatch sampling). Direct computation shows that the probability matrix is given by

$$
P = \begin{bmatrix}
\frac{b}{n} & \frac{b(b-1)}{n(n-1)} & \ldots & \frac{b(b-1)}{n(n-1)} \\
\frac{b(b-1)}{n(n-1)} & \frac{b}{n} & \ldots & \frac{b(b-1)}{n(n-1)} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{b(b-1)}{n(n-1)} & \frac{b(b-1)}{n(n-1)} & \ldots & \frac{b}{n}
\end{bmatrix},
$$

as claimed in [3]. Therefore,

$$
P - pp^\top = \begin{bmatrix}
\frac{b - b^2}{n} & \frac{b(b-1)}{n(n-1)} & \ldots & \frac{b(b-1)}{n(n-1)} \\
\frac{b(b-1)}{n(n-1)} & \frac{b}{n} & \ldots & \frac{b(b-1)}{n(n-1)} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{b(b-1)}{n(n-1)} & \frac{b(b-1)}{n(n-1)} & \ldots & \frac{b}{n}
\end{bmatrix},
$$

Letting $t = \frac{(a-1)k}{a(k-1)}$ and $s = 1 - t = \frac{k-a}{a(k-1)}$ the probability matrix of the approximate independent sampling satisfies

$$
P - pp^\top = \begin{bmatrix}
p_1(1 - p_1) & (t-1)p_1p_2 & \ldots & (t-1)p_1p_k & 0 & \ldots & 0 \\
(t-1)p_2p_1 & p_2(1 - p_2) & \ldots & (t-1)p_2p_k & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & 0 & \ldots & 0 \\
(t-1)p_np_1 & (t-1)p_n p_2 & \ldots & p_k(1-p_k) & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0
\end{bmatrix}
$$

$$
= \text{Diag}(p_1(1 - p_1(1 - s)), \ldots, p_k(1 - p_k(1 - s)), 0, \ldots, 0) - sp_kp_k^\top
\leq \text{Diag}(p_1(1 - p_1(1 - s)), \ldots, p_n(1 - p_n(1 - s)), 0, \ldots, 0),
$$

where $p_k = (p_1, \ldots, p_k, 0, \ldots, 0)^\top$. Therefore, $v_i = 1 - p_i(1 - s)$ for $i \leq k$ and $v_i = 0$ otherwise works.

Finally, as remarked in the introduction, the standard uniform minibatch sampling ($b$–nice sampling) arises as a special case of the approximate independent sampling for the choice $p_1 = b/n$. Thus $k = n$, $a = b$ and hence $s = \frac{n-b}{b(n-1)}$. Based on the previous result, $v_i = 1 - \frac{b}{n}(1 - \frac{n-b}{b(n-1)}) = \frac{n-b}{n-1}$ works.

\begin{proof}
We first establish a lemma we will need in order to prove Theorem 2.

\textbf{Lemma 6.} Let $0 < L_1 \leq L_2 \leq \cdots \leq L_n$ be positive real numbers, $0 < b \leq n$, and consider the optimization problem

$$
\begin{aligned}
\text{minimize}_{p \in \mathbb{R}^n} & \quad \Omega(p) \overset{\text{def}}{=} \sum_{i=1}^{n} \frac{L_i^2}{p_i} \\
\text{subject to} & \quad \sum_{i=1}^{n} p_i = b, \\
& \quad 0 \leq p_i \leq 1, \quad i = 1, 2, \ldots, n.
\end{aligned}
$$

(13)
\end{proof}
Let be the largest integer for which \(0 < b + k - n \leq \frac{\sum_{i=1}^{k} L_i}{L_k}\) (note that the inequality holds for \(k = n - b + 1\)). Then (13) has the following solution:

\[
p_i = \begin{cases} 
(b + k - n) \frac{L_i}{\sum_{j=1}^{k} L_j}, & \text{if } i \leq k, \\
1, & \text{if } i > k.
\end{cases}
\] (14)

**Proof.** The Lagrangian of the problem is

\[
L(p, y, \lambda_1, \ldots, \lambda_n, u_1, \ldots, u_n) = \sum_{i=1}^{n} \frac{L_i^2}{p_i} - \sum_{i=1}^{n} \lambda_ip_i - \sum_{i=1}^{n} u_i(1 - p_i) + y \left( \sum_{i=1}^{n} p_i - b \right).
\]

The constraints are linear and hence KKT conditions hold. The result can be deduced from the KKT conditions.

We can now proceed with the proof. Since \(n, b\) and \(\bar{L}\) are constants, the problem is equivalent to

\[
\text{minimize}_S \quad \psi(S) = \sum_{i=1}^{n} \frac{v_i L_i^2}{p_i}
\]

subject to \(v_i\) satisfies (4).

In view of (6),

\[
\psi(S) \geq \sum_{i=1}^{n} \frac{(1 - p_i) L_i^2}{p_i} = \sum_{i=1}^{n} \frac{L_i^2}{p_i} - \sum_{i=1}^{n} L_i^2 = \Omega(p) - \sum_{i=1}^{n} L_i^2,
\]

where function \(\Omega(p)\) was defined in Lemma 6. Since \(b = E[|S|] = \sum_i p_i\), and \(0 \leq p_i \leq 1\) for all \(i\), then in view of Lemma 6 we have

\[
\Psi(S) \geq \Omega(p^*) - \sum_{i=1}^{n} L_i^2,
\]

where \(p^*\) is defined by (8).

On the other hand, from Lemma 1 we know that the independent sampling \(S = S^*\) with probability vector \(p^*\) defined in (8) satisfies inequality (4) with \(v_i = 1 - p_i\), and hence

\[
\Psi(S^*) = \Omega(p^*) - \sum_{i=1}^{n} L_i^2.
\]

Hence, it is optimal.

**C Improvements**

Let us compute \(\alpha\) for uniform sampling.

\[
\alpha = \left( \frac{b}{n^2} \sum_{i=1}^{n} \frac{v_i L_i^2}{p_i} \right) / \bar{L}^2
\]

\[
= \left( \frac{(n - b)}{(n - 1)n} \sum_{i=1}^{n} L_i^2 \right) / \bar{L}^2
\]

\[
= \frac{n(n - b)}{(n - 1)} \frac{\sum_{i=1}^{n} L_i^2}{\left( \sum_{i=1}^{n} L_i \right)^2}
\]

It is easy to see that \(L_{\text{max}} \geq \bar{L}\). To prove that we have improved current best known rates, we need to show that \(\bar{L} \alpha \leq L_{\text{max}}\) and \(\bar{L}^2 \alpha \leq \frac{(n-b)}{(n-1)} L_{\text{max}}^2\)
"Proof.

\[
\bar{L}_\alpha = n \frac{(n-b)}{(n-1)} \frac{\sum_{i=1}^{n} L_i^2}{(\sum_{i=1}^{n} L_i)} \leq \frac{\sum_{i=1}^{n} L_i^2}{(\sum_{i=1}^{n} L_i)} = \frac{\sum_{i=1}^{n} L_{\max} L_i}{(\sum_{i=1}^{n} L_i)} = L_{\max},
\]

\[
\bar{L}^2 \alpha = n \frac{(n-b)}{(n-1)} \frac{\sum_{i=1}^{n} L_i^2}{(\sum_{i=1}^{n} L_i)} \leq \frac{(n-b)}{(n-1)} n \sum_{i=1}^{n} L_i^2 \leq (n-b) \frac{1}{n} \sum_{i=1}^{n} L_{\max} L_i \leq (n-b) \frac{1}{n} L_{\max}^2,
\]

\]

Let’s take \( b = 1 \). If \( L_\alpha \gg L_i, \forall i \in [n] \), then \( L_{\max} \approx n\bar{L} \) and \( \alpha S^* \leq 1 \) and \( \alpha S_\mu \approx n \), which essentially means, that we can have in theory speed up by factors of \( n \).

D Stochastic gradients evaluation complexity

D.1 SVRG

For SVRG, each outer loop costs \( n + mb \) evaluations of stochastic gradient. If we want to obtain \( \epsilon \)-solution, following must hold (Theorem 3)

\[
\frac{\alpha \bar{L}n^{(2/3)}(f(x^0) - f(x^*))}{bMm\nu_2} \leq \epsilon
\]

Combining these two equations with definition from Theorem 3, we get total complexity in terms of stochastic gradients evaluation

\[
\mu_2 \frac{\bar{L} n^{(2/3)}(f(x^0) - f(x^*))}{\epsilon \nu_2} (1 + \frac{\alpha}{3\mu_2})
\]

D.2 SAGA

For SAGA, each loop costs \( d + b \) evaluations of stochastic gradient. If we want to obtain \( \epsilon \)-solution, following must hold (Theorem 4)

\[
\frac{\alpha \bar{L}n^{(2/3)}(f(x^0) - f(x^*))}{b\bar{T}\nu_2} \leq \epsilon
\]

Combining these two equations with definition from Theorem 4, we get total complexity in terms of stochastic gradients evaluation

\[
n + \frac{\bar{L}n^{(2/3)}(f(x^0) - f(x^*))}{\epsilon \nu_3} (1 + \alpha),
\]

because of evaluation of full gradient on the start.

D.3 SARAH

For SARAH with one outer loot, each inner loop costs \( 2b \) evaluations of stochastic gradient. If we want to obtain \( \epsilon \)-solution, following must hold (Theorem 5)

\[
2\bar{L}(f(x^0) - f(x^*)) \left( \frac{1 + 4\alpha}{\bar{b}} \right) \leq \epsilon
\]

Solving this equation for \( m \), we get

\[
m \leq \frac{16\alpha \bar{L}^2(f(x^0) - f(x^*))^2 + \sqrt{16\alpha^2 \bar{L}^4(f(x^0) - f(x^*))^4 + 16\alpha^2 \bar{L}^2(f(x^0) - f(x^*))^2 b^2}}{2b}\]

16
Combining this equation with complexity off each inner loop we obtain total complexity in terms of stochastic gradients evaluation:

\[
\frac{16\alpha L^2 (f(x^0) - f(x^*))^2 + \sqrt{16^2 \alpha^2 L^4 (f(x^0) - f(x^*))^4 + 16\epsilon^2 L^2 (f(x^0) - f(x^*))^2 b^4}}{2\epsilon^2}.
\]
E  Proofs for SVRG

Lemma 7. For $c_t,c_{t+1},\beta > 0$, suppose we have
\[ c_t = c_{t+1}(1 + \eta \beta + 2\eta^2 K) + K\eta^2 L. \]

Let $\eta$, $\beta$ and $c_{t+1}$ be chosen such that $\Gamma_t > 0$ (in Theorem (17)). The iterate $x_{t+1}^*$ in Algorithm 3 satisfy the bound:
\[
E \left[ \|\nabla f(x_{t+1}^*)\|^2 \right] \leq \frac{R_t^{s+1} - R_{t+1}^{s+1}}{\Gamma_t},
\]
where $R_{t+1}^{s+1} \overset{\text{def}}{=} E \left[ f(x_{t+1}^*) + c_t \|x_t^* - \hat{x}^*\|^2 \right]$ for $0 \leq s \leq S - 1$.

Proof. Since $f_i$ is $L_i$-smooth we have
\[
E [f_i(x_{t+1}^{s+1})] \leq E [f_i(x_t^{s+1}) + \langle \nabla f_i(x_{t+1}^{s+1}), x_t^{s+1} - x_t^{s+1} \rangle + \frac{L_i}{2} \|x_t^{s+1} - x_t^{s+1}\|^2].
\]
Summing through all $i$ and dividing by $n$ we obtain
\[
E \left[ f(x_{t+1}^{s+1}) \right] \leq E \left[ f(x_t^{s+1}) + \langle \nabla f(x_{t+1}^{s+1}), x_t^{s+1} - x_t^{s+1} \rangle + \frac{L}{2} \|x_t^{s+1} - x_t^{s+1}\|^2 \right].
\]
Using the IS-SVRG update in Algorithm 5 and its unbiasedness ($E [v_t^*] = \nabla f(x_t^*)$), the right hand side above is further upper bounded by
\[
E \left[ f(x_t^{s+1}) - \eta \|\nabla f(x_t^{s+1})\|^2 + \frac{L\eta^2}{2} \|v_t^{s+1}\|^2 \right].
\]
(15)
Consider now the Lyapunov function
\[
R_{t+1}^{s+1} \overset{\text{def}}{=} E \left[ f(x_{t+1}^{s+1}) + c_t \|x_t^* - \hat{x}^*\|^2 \right].
\]
For bounding it we will require the following:
\[
E \left[ \|x_{t+1}^{s+1} - \hat{x}^*\|^2 \right] = E \left[ \|x_{t+1}^{s+1} - x_t^{s+1} + x_t^{s+1} - \hat{x}^*\|^2 \right] = E \left[ \|x_{t+1}^{s+1} - x_t^{s+1}\|^2 + \|x_t^{s+1} - \hat{x}^*\|^2 \right] + 2 \langle x_{t+1}^{s+1} - x_t^{s+1}, x_t^{s+1} - \hat{x}^* \rangle = E \left[ \eta^2 \|v_t^{s+1}\|^2 + \|x_t^{s+1} - \hat{x}^*\|^2 \right] - 2\eta E \left[ \langle \nabla f(x_t^{s+1}), x_t^{s+1} - \hat{x}^* \rangle \right] \overset{(54),(55)}{=} E \left[ \eta^2 \|v_t^{s+1}\|^2 + \|x_t^{s+1} - \hat{x}^*\|^2 \right] + 2\eta E \left[ \frac{1}{2\beta} \|\nabla f(x_t^{s+1})\|^2 + \frac{1}{2} \beta \|x_t^{s+1} - \hat{x}^*\|^2 \right].
\]
(16)
The second equality follows from the unbiasedness of the update of IS-SVRG. Plugging Equation (15) and Equation (16) into $R_{t+1}^{s+1}$, we obtain the following bound:
\[
R_{t+1}^{s+1} \leq E \left[ f(x_t^{s+1}) - \eta \|\nabla f(x_t^{s+1})\|^2 + \frac{L\eta^2}{2} \|v_t^{s+1}\|^2 \right] + E \left[ c_{t+1} \eta^2 \|v_t^{s+1}\|^2 + c_{t+1} \|x_t^{s+1} - \hat{x}^*\|^2 \right] + 2c_{t+1} \eta E \left[ \frac{1}{2\beta} \|\nabla f(x_t^{s+1})\|^2 + \frac{1}{2} \beta \|x_t^{s+1} - \hat{x}^*\|^2 \right] \leq E \left[ f(x_t^{s+1}) - \left( \eta - \frac{c_{t+1}}{c_{t+1} + c_{t+1} \eta \beta} \right) \|\nabla f(x_t^{s+1})\|^2 \right] + \left( \frac{L\eta^2}{2} + c_{t+1} \eta^2 \right) E \left[ \|v_t^{s+1}\|^2 \right] + (c_{t+1} + c_{t+1} \eta \beta) E \left[ \|x_t^{s+1} - \hat{x}^*\|^2 \right].
\]
(17)
To further bound this quantity, we use Lemma 10 to bound \( E \left[ \|v_t^{s+1}\|^2 \right] \), so that upon substituting it in Equation (17), we see that
\[
R_{t+1}^{s+1} \leq E \left[ f(x_t^{s+1}) \right] - \left( \eta - \frac{c_{t+1}}{\beta} - \eta^2 L - 2c_{t+1} \eta^2 \right) E \left[ \|\nabla f(x_t^{s+1})\|^2 \right] + \left[ c_{t+1} (1 + \eta \beta + 2\eta^2 K) + \eta^2 K L \right] E \left[ \|x_t^{s+1} - \tilde{x}^s\|^2 \right]
\]
\[
\leq R_{t+1} - (\eta - \frac{c_{t+1}}{\beta} - \eta^2 L - 2c_{t+1} \eta^2) E \left[ \|\nabla f(x_t^{s+1})\|^2 \right]. \quad (18)
\]
The second inequality follows from the definition of \( c_t \) and \( R_t^{s+1} \), thus concluding the proof.

**Proof of Lemma 7 and Theorem 17**

*Proof.* Since \( \eta = \eta \) for \( t \in \{0, \ldots, m - 1\} \), using Lemma 7 and telescoping the sum, we obtain
\[
\sum_{t=0}^{m-1} E \left[ \|\nabla f(x_t^{s+1})\|^2 \right] \leq \frac{R_0^{s+1} - R_m^{s+1}}{\gamma_n}. \quad (19)
\]
This inequality in turn implies that
\[
\sum_{t=0}^{m-1} E \left[ \|\nabla f(x_t^{s+1})\|^2 \right] \leq \frac{E \left[ f(\tilde{x}^s) - f(\tilde{x}^{s+1}) \right]}{\gamma_n}, \quad (20)
\]
where we used that \( R_0^{s+1} = E \left[ f(x_m^{s+1}) \right] = E \left[ f(\tilde{x}^{s+1}) \right] \) (since \( c_m = 0 \)), and that \( R_0^{s+1} = E \left[ f(\tilde{x}^s) \right] \) (since \( x_0^{s+1} = \tilde{x}^s \)). Now sum over all epochs to obtain
\[
\frac{1}{T} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} E \left[ \|\nabla f(x_t^{s+1})\|^2 \right] \leq \frac{f(x_0) - f(x^*)}{T \gamma_n}. \quad (21)
\]
The above inequality used the fact that \( \tilde{x}^0 = x^0 \). Using the above inequality and the definition of \( x_a \) in Algorithm 5, we obtain the desired result.

**Proof of Theorem 18**

*Proof.* For our analysis, we will require an upper bound on \( c_0 \). Let \( m = \lfloor Kn/(3L^2)_{\mu_0} \rfloor, \eta = \mu_0 \bar{L}/(Kn^{2/3}) \).

We observe that \( c_0 = \frac{\mu_0 \bar{L}^2}{Kn^{2/3}} (1 + \theta)^m \) where \( \theta = 2K \eta^2 + \eta \beta \). This is obtained using the relation \( c_t = c_{t+1} (1 + \eta \beta + 2K \eta^2) + \eta^2 K \bar{L} \) and the fact that \( c_m = 0 \). Using the specified values of \( \beta \) and \( \eta \) we have
\[
\theta = 2K \eta^2 + \eta \beta = \frac{2 \mu_0 \bar{L}^2}{Kn^{2/3}} + \frac{\mu_0 \bar{L}^2}{Kn} \leq \frac{3 \mu_0 \bar{L}^2}{Kn}. \quad (22)
\]
The above inequality follows since \( \mu_0 \leq 1 \) and \( n \geq 1 \). Using the above bound on \( \theta \), we get
\[
c_0 = \frac{\mu_0^3 \bar{L}^3 (1 + \theta)^m - 1}{n^2 K} = \frac{\mu_0 \bar{L} ((1 + \theta)^m - 1)}{2 \mu_0 + \frac{1}{3}} \leq \frac{\mu_0 \bar{L} ((1 + \frac{3 \mu_0 \bar{L}^2}{Kn^{2/3}}) \lfloor Kn/3 \mu_0 \bar{L}^2 \rfloor - 1)}{2 \mu_0 + \frac{1}{3}} \leq n^{-\frac{1}{3}} (\mu_0 \bar{L} (e - 1)), \quad (23)
\]
wherein the second inequality follows upon noting that \( (1 + \frac{1}{t})^t \) is increasing for \( t > 0 \) and \( \lim_{t \to \infty} (1 + \frac{1}{t})^t = e \) (here \( e \) is the Euler’s number). Now we can lower bound \( \gamma_n \), as
\[
\gamma_n = \min \left( \eta - \frac{c_{t+1}}{\beta} - \eta^2 \bar{L} - 2c_{t+1} \eta^2 \right) \geq \left( \eta - \frac{c_0}{\beta} - \eta^2 \bar{L} - 2c_0 \eta^2 \right) \geq \frac{\mu_0 \bar{L}}{Kn^{2/3}}, \quad (24)
\]

19
where $\nu$ is a constant independent of $n$. The first inequality holds since $c_t$ decreases with $t$. The second inequality holds since (a) $c_0/\beta$ is upper bounded by a constant independent of $n$ as $c_0/\beta \leq \mu_0(e-1)$ (follows from Equation (23)), (b) $\eta^2L \leq \mu_0\eta$ and (c) $2c_0\eta^2 \leq 2\mu_0^2(e-1)\eta$ (follows from Equation (23)). By choosing $\mu_0$ (independent of $n$) appropriately, one can ensure that $\gamma_n \geq \nu \bar{L}/(Kn^2)$ for some universal constant $\nu$. For example, choosing $\mu_0 = 1/4$, we have $\gamma_n \geq \nu \bar{L}/(Kn^2)$ with $\nu = 1/40$. Substituting the above lower bound in Equation (21), we obtain the desired result.

\[\square\]

F Minibatch SVRG

Proof of Theorem 3

The proofs essentially follow along the lines of Lemma 7, Theorem 17 and Theorem 18 with the added complexity of mini-batch. We first prove few intermediate results before proceeding to the proof of Theorem 3.

Lemma 8. Suppose we have

\[
\begin{align*}
\mathcal{R}_{t+1}^{s+1} & \overset{\text{def}}{=} \mathbb{E} \left[ f(x_t^{s+1}) + \nu_t \| x_t^{s+1} - \bar{x}^s \|^2 \right], \\
\nu_t & = \nu_{t+1}(1 + \eta\beta + \frac{2K\eta^2}{b} + \frac{K\eta^2\bar{L}}{b}).
\end{align*}
\]

for $0 \leq s \leq S - 1$ and $0 \leq t \leq m - 1$ and the parameters $\eta, \beta$ and $\nu_{t+1}$ are chosen such that

\[
\left( \eta - \frac{\nu_{t+1}\eta}{\beta} - \eta^2\bar{L} - 2\nu_{t+1}\eta^2 \right) \geq 0.
\]

Then the iterates $x_t^{s+1}$ in the mini-batch version of Algorithm 3 i.e., Algorithm 1 with expected mini-batch size $b$ satisfy the bound:

\[
\mathbb{E}\left[ \| \nabla f(x_t^{s+1}) \|^2 \right] \leq \frac{\mathcal{R}_{t+1}^{s+1} - \mathcal{R}_{t+1}^{s+1}}{\left( \eta - \frac{\nu_{t+1}\eta}{\beta} - \eta^2\bar{L} - 2\nu_{t+1}\eta^2 \right)}.
\]

Proof. Using essentially the same argument as the proof of Lemma 7 until Equation (17), we have

\[
\begin{align*}
\mathcal{R}_{t+1}^{s+1} & \leq \mathbb{E} \left[ (x_t^{s+1}) - \left( \eta - \frac{\nu_{t+1}\eta}{\beta} \right) \| \nabla f(x_t^{s+1}) \|^2 + \left( \frac{L\eta^2}{2} + \nu_{t+1}\eta^2 \right) \mathbb{E}\left[ \| v_t^{s+1} \|^2 \right] \\
& \quad + (\nu_{t+1} + \nu_{t+1}\eta\beta) \mathbb{E}\left[ \| x_t^{s+1} - \bar{x}^s \|^2 \right].
\end{align*}
\]

We use Lemma 11 in order to bound $\mathbb{E}\left[ \| v_t^{s+1} \|^2 \right]$ in the above inequality. Substituting it in Equation (26), we see that

\[
\begin{align*}
\mathcal{R}_{t+1}^{s+1} & \overset{(30)}{=} \mathbb{E} \left[ f(x_t^{s+1}) \right] - \left( \eta - \frac{\nu_{t+1}\eta}{\beta} - \eta^2\bar{L} - 2\nu_{t+1}\eta^2 \right) \mathbb{E}\left[ \| \nabla f(x_t^{s+1}) \|^2 \right] \\
& \quad + \mathbb{E} \left[ \| x_t^{s+1} - \bar{x}^s \|^2 \right] \\
& \overset{(25)}{\leq} \mathcal{R}_{t}^{s+1} - \left( \eta - \frac{\nu_{t+1}\eta}{\beta} - \eta^2\bar{L} - 2\nu_{t+1}\eta^2 \right) \mathbb{E}\left[ \| \nabla f(x_t^{s+1}) \|^2 \right].
\end{align*}
\]

The second inequality follows from the definition of $\nu_t$ and $\mathcal{R}_{t+1}^{s+1}$, thus concluding the proof.

The following theorem provides convergence rate of mini-batch IS-SVRG.

Theorem 9. Let $\gamma_n$ denote the following quantity:

\[
\gamma_n \overset{\text{def}}{=} \min_{0 \leq t \leq m - 1} \left( \eta - \frac{\nu_{t+1}\eta}{\beta} - \eta^2\bar{L} - 2\nu_{t+1}\eta^2 \right).
\]
Suppose \( \tau_m = 0 \), \( \tau_t = \tau_{t+1}(1 + \eta \beta + 2K\eta^2) + \frac{K\eta^2L}{b} \) for \( t \in \{0, \ldots, m-1\} \) and \( \tau_n > 0 \). Then for the output \( x_a \) of mini-batch version of Algorithm \( \text{S} \) with mini-batch size \( b \), we have

\[
E \left[ \|\nabla f(x_a)\|^2 \right] \leq \frac{f(x^0) - f(x^*)}{\tau_n},
\]

where \( x^* \) is an optimal solution to (1).

**Proof.** Since \( \eta = \eta \) for \( t \in \{0, \ldots, m-1\} \), using Lemma 8 and telescoping the sum, we obtain

\[
\sum_{t=0}^{m-1} E \left[ \|\nabla f(x_{t+1}^*)\|^2 \right] \leq \frac{R_{0}^{s+1} - R_{m}^{s+1}}{\gamma_n}.
\]

This inequality in turn implies that

\[
\sum_{t=0}^{m-1} E \left[ \|\nabla f(x_{t+1}^*)\|^2 \right] \leq E \left[ f(\tilde{x}^*) - f(\tilde{x}^{s+1}) \right],
\]

where we used that \( R_{m}^{s+1} = E \left[ f(x_{m}^{s+1}) \right] = E \left[ f(x_{s+1}^*) \right] \) (since \( \tau_m = 0 \)), and that \( R_{0}^{s+1} = E \left[ f(\tilde{x}^*) \right] \). Now sum over all epochs and using the fact that \( \tilde{x}^0 = x^0 \), we get the desired result. \( \square \)

We now present the proof of Theorem 3 using the above results.

**Proof of Theorem 3.** We first observe that using the specified values of \( \beta = \bar{L}/n^{1/3}, \eta = \mu_2 b \bar{L}/(Kn^{2/3}) \) and \( \eta = \lfloor nK/(bL^2 \mu_2) \rfloor \), we obtain

\[
\eta = \frac{2K\eta^2}{b} + \eta \beta = \frac{2\mu_2^2 b \bar{L}^2}{Kn^{4/3}} + \frac{\bar{L}^2 \mu_2 b}{Kn} \leq \frac{3\mu_2 \bar{L}^2 b}{Kn}.
\]

The above inequality follows since \( \mu_2 \leq 1 \) and \( n \geq 1 \). For our analysis, we will require the following bound on \( \gamma_n \):

\[
\gamma_n = \min \left\{ \eta - \frac{\tau_{t+1} \eta}{\beta}, -\eta^2 \bar{L} - 2 \tau_{t+1} \eta^2 \right\} \geq \frac{b L \nu_2}{K n^{2/3}},
\]

where \( \nu_2 \) is a constant independent of \( n \). The first inequality holds since \( \gamma_n \) decreases with \( t \). The second one holds since (a) \( \gamma_n/\beta \) is upper bounded by a constant independent of \( n \) as \( \gamma_n/\beta \leq \mu_2(e - 1) \) (due to Equation (28)), (b) \( \eta^2 \bar{L} \leq \mu_2 \eta \) (as \( b \leq K\bar{L}^2 \eta^{2/3} \)) and (c) \( 2 \tau_{t+1} \eta^2 \leq 2 \mu_2^2 (e - 1) \eta \) (again due to Equation (28) and the fact \( b \leq K\bar{L}^2 \eta^{2/3} \)). By choosing an appropriately small constant \( \mu_2 \) (independent of \( n \)), one can ensure that \( \gamma_n \geq \frac{b L \nu_2}{(Kn^{2/3})} \) for some universal constant \( \nu_2 \). For example, choosing \( \mu_2 = 1/4 \), we have \( \gamma_n \geq \frac{b L \nu_2}{(Kn^{2/3})} \) with \( \nu_2 = 1/40 \). Substituting the above lower bound in Theorem 3, we obtain the desired result. \( \square \)
Lemmas

Lemma 10. For the intermediate iterates $v_{t+1}^s$ computed by Algorithm 3, we have the following:

$$E [∥v_{t+1}^s∥^2] \leq 2E [∥\nabla f(x^s_{t+1})∥^2] + 2KE [∥x^s_{t+1} - \hat{x}^s∥^2].$$  \hspace{1cm} (29)

Proof. The proof simply follows from the proof of Lemma 11 with $S_t = \{i_t\}$.

We now present a result to bound the variance of mini-batch IS-SVRG.

Lemma 11. Let $v_{t+1}^s$ be computed by the mini-batch version of Algorithm 3 i.e., Algorithm 2 with sampling $S$. Then,

$$E [∥v_{t+1}^s∥^2] \leq 2E [∥\nabla f(x^s_{t+1})∥^2] + \frac{2K}{b} E [∥x^s_{t+1} - \hat{x}^s∥^2].$$  \hspace{1cm} (30)

Proof. For the simplification, we use the following notation:

$$\zeta_{t+1} = \sum_{i_t \in S_t} \frac{1}{np_{i_t}} (\nabla f_{i_t}(x^s_{t+1}) - \nabla f_{i_t}(\hat{x}^s)).$$

We use the definition of $v_{t+1}^s$ to get

$$E [∥v_{t+1}^s∥^2] = E [∥\zeta_{t+1} + \nabla f(\hat{x}^s)∥^2]$$
$$= E [∥\zeta_{t+1} + \nabla f(\hat{x}^s) - \nabla f(x^s_{t+1}) + \nabla f(x^s_{t+1})∥^2]$$
$$\leq 2E [∥\nabla f(x^s_{t+1})∥^2] + 2E [∥\zeta_{t+1} - E [\zeta_{t+1}]∥^2]$$
$$= 2E [∥\nabla f(x^s_{t+1})∥^2]$$
$$+ 2E \left[ \left\| \sum_{i_t \in S_t} \left( \frac{1}{np_{i_t}} (\nabla f_{i_t}(x^s_{t+1}) - \nabla f_{i_t}(\hat{x}^s)) - E [\zeta_{t+1}] \right) \right\|^2 \right].$$

The first inequality follows from fact that $∥x + y∥^2 \leq 2∥x∥^2 + 2∥y∥^2$ and the fact that $E [\zeta_{t+1}] = \nabla f(x^s_{t+1}) - \nabla f(\hat{x}^s)$. From the above inequality, we get

$$E [∥v_{t+1}^s∥^2] \leq 2E [∥\nabla f(x^s_{t+1})∥^2] + 2 \sum_{i=1}^n \frac{v_ip_i}{np_i^2} ∥(\nabla f_i(x^s_{t+1}) - \nabla f_i(\hat{x}^s))∥^2$$
$$\leq 2E [∥\nabla f(x^s_{t+1})∥^2] + \frac{2K}{b} E [∥x^s_{t+1} - \hat{x}^s∥^2].$$  \hspace{1cm} (31)

\hspace{1cm} (52), (7)

\hspace{1cm} (52), (7)
G \hspace{1em} \textbf{Proofs for SAGA}

Lemma 12. For $c_t, c_{t+1}, \beta > 0$, suppose we have
\[
    c_t = c_{t+1}(1 - \frac{d}{n} + \eta \beta + 2 \frac{K\eta^2}{b}) + \frac{K\eta^2L}{b}.
\]

Also let $\eta, \beta$ and $c_{t+1}$ be chosen such that $\Gamma_t > 0$. Then, the iterates $\{x^t\}$ of Algorithm \ref{alg:saga} satisfy the bound
\[
    \mathbb{E} \left[ \|\nabla f(x^t)\|^2 \right] \leq \frac{R^t - R^{t+1}}{\Gamma_t},
\]
where $R^t \triangleq \mathbb{E} [f(x^t)] + c_t \max_{i \in [n]} \mathbb{E} \left[ \|x^t - \alpha^t_i\|^2 \right]$.

\textbf{Proof.} Since $f$ is $\bar{L}$-smooth we have
\[
    \mathbb{E} \left[ f(x^{t+1}) \right] \leq \mathbb{E} \left[ f(x^t) + \langle \nabla f(x^t), x^{t+1} - x^t \rangle + \frac{\bar{L}}{2} \|x^{t+1} - x^t\|^2 \right].
\]

We first note that the update in Algorithm \ref{alg:saga} is unbiased i.e., $\mathbb{E}[v^t] = \nabla f(x^t)$. By using this property of the update on the right hand side of the inequality above, we get the following:
\[
    \mathbb{E} \left[ f(x^{t+1}) \right] \leq \mathbb{E} \left[ f(x^t) - \eta \|\nabla f(x^t)\|^2 + \frac{L\eta^2}{2} \|v^t\|^2 \right]. \tag{31}
\]

Here we used the fact that $x^{t+1} - x^t = -\eta v^t$ (see Algorithm \ref{alg:saga}). Consider now the Lyapunov function
\[
    R^t \triangleq \mathbb{E} \left[ f(x^t) \right] + c_t \max_{i \in [n]} \mathbb{E} \left[ \|x^t - \alpha^t_i\|^2 \right] .
\]

For bounding $R^t$ we need the following:
\[
    \mathbb{E} \left[ \|x^{t+1} - \alpha^{t+1}_i\|^2 \right] = \frac{d}{n} \mathbb{E} \left[ \|x^{t+1} - x^t\|^2 \right] + \frac{n - d}{n} \mathbb{E} \left[ \|x^{t+1} - \alpha^{t}_i\|^2 \right]. \tag{32}
\]

The above equality follows from the definition of $\alpha^{t+1}_i$ and the definition of randomness of index $j_t$ in Algorithm \ref{alg:saga} and Algorithm \ref{alg:is-saga}. The term $T_1$ in (32) can be bounded as follows
\[
    T_1 = \mathbb{E} \left[ \|x^{t+1} - x^t + x^{t} - \alpha^{t}_i\|^2 \right] = \mathbb{E} \left[ \|x^{t+1} - x^t\|^2 + \|x^{t} - \alpha^{t}_i\|^2 \right] + 2 \langle x^{t+1} - x^{t}, x^{t} - \alpha^{t}_i \rangle = \mathbb{E} \left[ \|x^{t+1} - x^t\|^2 + \|x^{t} - \alpha^{t}_i\|^2 \right] - 2\eta \mathbb{E} \left[ \langle \nabla f(x^t), x^{t} - \alpha^{t}_i \rangle \right] \tag{34}
\]

The second equality again follows from the unbiasedness of the update of IS-SAGA. The last inequality follows from a simple application of Cauchy-Schwarz and Young's inequality. Plugging (31) and (33) into $R^t$, we obtain the following bound:
\[
    R^t \leq \mathbb{E} \left[ f(x^t) - \eta \|\nabla f(x^t)\|^2 + \frac{L\eta^2}{2} \|v^t\|^2 \right] + \mathbb{E} \left[ c_{t+1} \|x^{t+1} - x^{t}\|^2 \right] + \frac{n - d}{n} \mathbb{E} \left[ \|x^{t+1} - \alpha^{t}_i\|^2 \right] \leq \mathbb{E} \left[ f(x^t) - \left( \eta - \frac{c_{t+1} \eta}{n} \right) \|\nabla f(x^t)\|^2 \right] + \left( \frac{L\eta^2}{2} + c_{t+1} \eta^2 \right) \mathbb{E} \left[ \|v^t\|^2 \right] + \left( \frac{n - d}{n} c_{t+1} + c_{t+1} \eta \beta \right) \max_{i \in [n]} \mathbb{E} \left[ \|x^{t} - \alpha^{t}_i\|^2 \right], \tag{34}
\]

23
where we use that \( \|x^t - \alpha^t_{i_{\max}}\| \leq \max_{i \in [n]} \|x^t - \alpha^t_i\| \). To further bound the quantity in (34), we use Lemma 13 to bound \( \mathbb{E}[\|v^t\|^2] \), so that upon substituting it into (34), we obtain

\[
R^{t+1} \leq \mathbb{E}[f(x^t)] - \left( \eta - \frac{c_{t+1} \eta}{\beta} - \eta^2 \bar{L} - 2 \epsilon_{t+1} \eta^2 \right) \mathbb{E}[\|\nabla f(x^t)\|^2] + \left( \epsilon_{t+1} \left( 1 - \frac{d}{n} + \eta \beta + 2 \frac{K \eta^2}{b} \right) + \frac{K \eta^2 \bar{L}}{b} \right) \max_{i \in [n]} \mathbb{E}[\|x^t - \alpha^t_i\|^2]
\]

\[
\leq R^t - \left( \eta - \frac{c_{t+1} \eta}{\beta} - \eta^2 \bar{L} - 2 \epsilon_{t+1} \eta^2 \right) \mathbb{E}[\|\nabla f(x^t)\|^2]
\]

(35)

The second inequality follows from the definition of \( \epsilon_t \) i.e., \( \epsilon_t = \epsilon_{t+1} \left( 1 - \frac{d}{n} + \eta \beta + 2 \frac{K \eta^2}{b} \right) + \frac{K \eta^2 \bar{L}}{b} \) and \( R^t \) specified in the statement, thus concluding the proof.

The following lemma provides a bound on the variance of the update used in Mini-batch IS-SAGA algorithm. More specifically, it bounds the quantity \( \mathbb{E}[\|v^t\|^2] \).

**Lemma 13.** Let \( v^t \) be computed by Algorithm 2. Then,

\[
\mathbb{E}[\|v^t\|^2] \leq 2 \mathbb{E}[\|\nabla f(x^t)\|^2] + \frac{2K}{b} \max_{i \in [n]} \mathbb{E}[\|x^t - \alpha^t_i\|^2].
\]

(36)

**Proof.** For ease of exposition, we use the notation

\[
\zeta_i \overset{\text{def}}{=} \frac{1}{np_i} \left( \nabla f_i(x^t) - \nabla f_i(\alpha^t_i) \right).
\]

Using the convexity of \( \|\cdot\|^2 \) and the definition of \( v^t \) we get

\[
\mathbb{E}[\|v^t\|^2] = \mathbb{E}\left[ \left\| \sum_{i \in S_t} \zeta_i^t + \frac{1}{n} \sum_{i=1}^n \nabla f(\alpha^t_i) \right\|^2 \right]
\]

\[
= \mathbb{E}\left[ \left\| \sum_{i \in S_t} \zeta_i^t + \frac{1}{n} \sum_{i=1}^n \nabla f_i(\alpha^t_i) - \nabla f(x^t) + \nabla f(x^t) \right\|^2 \right]
\]

\[
\leq 2 \mathbb{E}[\|\nabla f(x^t)\|^2] + 2 \mathbb{E}\left[ \left\| \sum_{i \in S_t} \zeta_i^t - \mathbb{E}[\zeta^t] \right\|^2 \right]
\]

\[
\leq 2 \mathbb{E}[\|\nabla f(x^t)\|^2] + 2 \sum_{i=1}^n \mathbb{E}[p_i \|\zeta_i^t\|^2].
\]

The first inequality follows from the fact that \( \|a + b\|^2 \leq 2(\|a\|^2 + \|b\|^2) \) and that \( \mathbb{E}[\zeta^t] = \nabla f(x^t) - \frac{1}{n} \sum_{i=1}^n \nabla f(\alpha^t_i) \).

\[
\mathbb{E}[\|v^t\|^2] \leq 2 \mathbb{E}[\|\nabla f(x^t)\|^2] + 2 \sum_{i=1}^n \mathbb{E}\left[ \frac{p_i}{n^2 p_i^2} \|\nabla f_i(x^t) - \nabla f_i(\alpha^t_i)\|^2 \right]
\]

\[
\leq 2 \mathbb{E}[\|\nabla f(x^t)\|^2] + 2 \sum_{i=1}^n \mathbb{E}\left[ \eta_i L^2 \|x^t - \alpha^t_i\|^2 \right]
\]

\[
\leq 2 \mathbb{E}[\|\nabla f(x^t)\|^2] + \frac{2K}{b} \max_{i \in [n]} \mathbb{E}[\|x^t - \alpha^t_i\|^2].
\]

(37)

The last inequality follows from \( L_i \)-smoothness of \( f_i \) and using properties of \( S \) sampling, thus concluding the proof. \( \square \)
Proof of Theorem 19

Proof. We apply telescoping sums to the result of Lemma 12 to obtain
\[ \gamma_n \sum_{t=0}^{T-1} \mathbb{E} \left[ \| \nabla f(x^t) \|^2 \right] \leq \sum_{t=0}^{T-1} \Gamma_t \mathbb{E} \left[ \| \nabla f(x^t) \|^2 \right] \leq R^0 - R^T. \]  
(38)

The first inequality follows from the definition of \( \gamma_n \). This inequality in turn implies the bound
\[ \sum_{t=0}^{T-1} \mathbb{E} \left[ \| \nabla f(x^t) \|^2 \right] \leq \frac{\mathbb{E} \left[ f(x^0) - f(x^T) \right]}{\gamma_n}, \]  
(39)

where we used that \( R^T = \mathbb{E} \left[ f(x^T) \right] \) (since \( c_T = 0 \)), and that \( R^0 = \mathbb{E} \left[ f(x^0) \right] \) (since \( a^t_0 = x^0 \) for \( i \in [n] \)). Using inequality (39), the optimality of \( x^* \), and the definition of \( x_a \) in Algorithm 6, we obtain the desired result. \( \square \)

Proof of Theorem 20 and Theorem 4

Proof. With the values of \( \mu_3 = 1/3, \nu_3 = 12 \), \( \eta = b\bar{L}/(3Kn^2/3), d = b\bar{L}^2/K \) and \( \beta = \bar{L}/n^{1/3} \), let us first establish an upper bound on \( c_t \). Let \( \theta \) denote \( \frac{L^2}{Kn^2} - \eta \beta - 2K\eta^2/b \). Observe that \( \theta < 1 \) and \( \theta \geq 4\bar{L}^2b/(9Kn) \). This is due to the specific values of \( \eta \) and \( \beta \) and lower bound of \( K \). Also, we have \( c_t = c_{t+1} (1 - \theta) + K\eta^2\bar{L}/b \).

Using this relationship, it is easy to see that \( c_t = K\eta^2\bar{L}\frac{1-(1-\theta)^{T-t}}{b\theta} \). Therefore, we obtain the bound
\[ c_t = K\eta^2\bar{L}\frac{1-(1-\theta)^{T-t}}{b\theta} \leq \frac{K\eta^2\bar{L}}{b\theta} \leq \frac{\bar{L}}{4n^{1/3}}, \]  
(40)

for all \( 0 \leq t \leq T \), where the inequality follows from the definition of \( \eta \) and the fact that \( \theta \geq 4\bar{L}^2b/(9Kn) \). Using the above upper bound on \( c_t \) we can conclude that
\[ \gamma_n = \min_t \left( \eta - \frac{c_{t+1}\eta}{\beta} - \eta^2\bar{L} - 2c_{t+1}\eta^2 \right) \geq \frac{\bar{L}b}{12Kn^{2/3}}, \]

upon using the following inequalities: (i) \( c_{t+1}\eta/\beta \leq \eta/4 \), (ii) \( \eta^2\bar{L} \leq \eta/3 \) and (iii) \( 2c_{t+1}\eta^2 \leq \eta/6 \), which hold due to the upper bound on \( c_t \) in (40) and if \( b \leq K/\bar{L}^2n^{2/3} \). Substituting this bound on \( \gamma_n \) in Theorem 19 we obtain the desired result. \( \square \)

Theorem 20 is special case with \( b = 1 \) and \( d = 1 \).

SARAH-non-convex

This lemmas are modification of lemmas appeared in [23] for importance sampling with mini-batch.

Lemma 14. Consider SARAH, then we have
\[ \sum_{t=0}^{m} \mathbb{E} \left[ \| \nabla f(x^t) \|^2 \right] \leq \frac{2}{\eta} [f(x^0) - f(x^*)] + \sum_{t=0}^{m} \mathbb{E} \left[ \| \nabla f(x^t) - v^t \|^2 \right] \]
\[ - (1 - \bar{L}\eta) \sum_{t=0}^{m} \mathbb{E} \left[ \| v^t \|^2 \right], \]  
(41)

where \( x^* \) is an optimal solution of (1).

25
Proof. By $\tilde{L}$-smoothness of $f$ and $x^{t+1} = x^t - \eta v^t$, we have

$$E\left[f(x^{t+1})\right] \leq E\left[f(x^t)\right] - \eta E\left[\nabla f(x^t)^\top v^t\right] + \frac{\tilde{L}\eta^2}{2} E\left[\|v^t\|^2\right]$$

$$= E\left[f(x^t)\right] - \frac{\eta}{2} E\left[\|\nabla f(x^t)\|^2\right] + \frac{\eta}{2} E\left[\|\nabla f(x^t) - v^t\|^2\right]$$

$$- \left(\frac{\eta}{2} - \frac{L\eta^2}{2}\right) E\left[\|v^t\|^2\right],$$

where the last equality follows from the fact $r^\top q = \frac{1}{2} \left[\|r\|^2 + \|q\|^2 - \|r - q\|^2\right]$, for any $r, q \in \mathbb{R}^d$.

By summing over $t = 0, \ldots, m$, we have

$$E\left[f(x^{m+1})\right] \leq E\left[f(x^0)\right] - \frac{\eta}{2} \sum_{t=0}^m E\left[\|\nabla f(x^t)\|^2\right] + \frac{\eta}{2} \sum_{t=0}^m E\left[\|\nabla f(x^t) - v^t\|^2\right]$$

$$- \left(1 - \tilde{L}\eta\right) \sum_{t=0}^m E\left[\|v^t\|^2\right]$$

$$\leq \frac{2}{\eta} |f(x^0) - f(x^*)| + \sum_{t=0}^m E\left[\|\nabla f(x^t) - v^t\|^2\right]$$

$$- \left(1 - \tilde{L}\eta\right) \sum_{t=0}^m E\left[\|v^t\|^2\right],$$

where the last inequality follows since $x^*$ is an optimal solution of (1). (Note that $x^0$ is given.)

Lemma 15. Consider $v^t$ defined in SARAH, then for any $t \geq 1$,

$$E\left[\|\nabla f(x^t) - v^t\|^2\right] = \sum_{j=1}^t E\left[\|v^j - v^{j-1}\|^2\right] - \sum_{j=1}^t E\left[\|\nabla f(x^j) - \nabla f(x^{j-1})\|^2\right].$$

Proof. Let $\mathcal{F}_j = \sigma(x^0, i_1, i_2, \ldots, i_{j-1})$ be the $\sigma$-algebra generated by $x^0, i_1, i_2, \ldots, i_{j-1}$; $\mathcal{F}_0 = \mathcal{F}_1 = \sigma(x^0)$. Note that $\mathcal{F}_j$ also contains all the information of $x^0, \ldots, x^j$ as well as $v^0, \ldots, v^{j-1}$. For $j \geq 1$, we have

$$E\left[\|\nabla f(x^j) - v^j\|^2 | \mathcal{F}_j\right] = E\left[\left\|\nabla f(x^{j-1}) - v^{j-1}\right\|^2 + \left\|\nabla f(x^j) - \nabla f(x^{j-1})\right\|^2 - \left\|v^j - v^{j-1}\right\|^2 | \mathcal{F}_j\right]$$

$$= \left\|\nabla f(x^{j-1}) - v^{j-1}\right\|^2 + \left\|\nabla f(x^j) - \nabla f(x^{j-1})\right\|^2$$

$$+ E\left[\left\|v^j - v^{j-1}\right\|^2 | \mathcal{F}_j\right]$$

$$+ 2\left\langle \nabla f(x^{j-1}) - v^{j-1}, \nabla f(x^j) - \nabla f(x^{j-1}) \right\rangle$$

$$- 2\left\langle \nabla f(x^{j-1}) - v^{j-1}, v^j - v^{j-1} \right\rangle$$

$$- 2\left\langle \nabla f(x^j) - v^j, v^j - v^{j-1} \right\rangle$$

$$= \left\|\nabla f(x^{j-1}) - v^{j-1}\right\|^2 - \left\|\nabla f(x^j) - \nabla f(x^{j-1})\right\|^2$$

$$+ E\left[\left\|v^j - v^{j-1}\right\|^2 | \mathcal{F}_j\right].$$

26
where the last equality follows from
\[
E [v^j - v^{j-1} | F_j] = E \left[ \sum_{i \in I_j} \frac{1}{np_i} \nabla f_i(x^j) - \nabla f_i(x^{j-1}) \right] | F_j \\
= \sum_{i=1}^n \frac{p_i}{np_i} (\nabla f_i(x^j) - \nabla f_i(x^{j-1})) = \nabla f(x^j) - \nabla f(x^{j-1}).
\]
By taking expectation for the above equation, we have
\[
E \| \nabla f(x^j) - v^j \|^2 = E \| \nabla f(x^{j-1}) - v^{j-1} \|^2 - E \| \nabla f(x^j) - \nabla f(x^{j-1}) \|^2 + E \| v^j - v^{j-1} \|^2.
\]
Note that \( \| \nabla f(x^0) - v^0 \|^2 = 0 \). By summing over \( j = 1, \ldots, t \) \((t \geq 1)\), we have
\[
E \| \nabla f(x^t) - v^t \|^2 = \sum_{j=1}^t E \| \nabla f(x^j) - v^{j-1} \|^2 - \sum_{j=1}^t E \| \nabla f(x^j) - \nabla f(x^{j-1}) \|^2.
\]

With the above Lemmas, we can derive the following upper bound for \( E \| \nabla f(x^t) - v^t \|^2 \).

**Lemma 16.** Consider \( v^t \) defined in SARAH. Then for any \( t \geq 1 \),
\[
E \| \nabla f(x^t) - v^t \|^2 \leq \frac{1}{b} K \eta^2 \sum_{j=1}^t E \| v^{j-1} \|^2.
\]

**Proof.** Let
\[
\xi_i = \frac{1}{np_i} (\nabla f_i(x^j) - \nabla f_i(x^{j-1})) \quad (42)
\]
We have
\[
E \| v^j - v^{j-1} \|^2 | F_j | - \| \nabla f(x^j) - \nabla f(x^{j-1}) \|^2 \\
= E \left[ \left\| \sum_{i \in I_j} \frac{1}{np_i} [\nabla f_i(x^j) - \nabla f_i(x^{j-1})] \right\|^2 | F_j \right] - \left\| \frac{1}{n} \sum_{i=1}^n (\nabla f_i(x^j) - \nabla f_i(x^{j-1})) \right\|^2 \\
= E \left[ \sum_{i \in I_j} \| \xi_i \|^2 | F_j \right] - \left\| \frac{1}{n} \sum_{i=1}^n \xi_i \right\|^2 \\
\leq \sum_{i=1}^n \| \xi_i \|^2 \\
= \frac{1}{K} \eta^2 \sum_{i=1}^n \| \nabla f_i(x^j) - \nabla f_i(x^{j-1}) \|^2 \\
\leq \frac{1}{b} K \eta^2 \| v^{j-1} \|^2.
\]
Hence, by taking expectation, we have
\[
E \| v^j - v^{j-1} \|^2 - E \| \nabla f(x^j) - \nabla f(x^{j-1}) \|^2 \leq \frac{1}{b} K \eta^2 E \| v^{j-1} \|^2.
\]
By Lemma 15 for \( t \geq 1 \),
\[
\mathbb{E} \left[ \| \nabla f(x^t) - v^t \|^2 \right] = \sum_{j=1}^{t} \mathbb{E} \left[ \| v^j - v^{j-1} \|^2 \right] - \sum_{j=1}^{t} \mathbb{E} \left[ \| \nabla f(x^j) - \nabla f(x^{j-1}) \|^2 \right] \\
\leq \frac{1}{b} K \eta^2 \sum_{j=1}^{t} \mathbb{E} \left[ \| v^{j-1} \|^2 \right].
\]

This completes the proof. \( \square \)

**Proof of Theorem 5**

**Proof.** By Lemma 16, we have
\[
\mathbb{E} \left[ \| \nabla f(x^t) - v^t \|^2 \right] \leq \frac{1}{b} K \eta^2 \sum_{j=1}^{t} \mathbb{E} \left[ \| v^{j-1} \|^2 \right].
\]

Note that \( \| \nabla f(x^0) - v^0 \|^2 = 0 \). Hence, by summing over \( t = 0, \ldots, m \) \((m \geq 1)\), we have
\[
\sum_{t=0}^{m} \mathbb{E} \left[ \| v^t - \nabla f(x^t) \|^2 \right] \leq \frac{1}{b} K \eta^2 \left[ m \mathbb{E} \left[ \| v^0 \|^2 \right] + (m - 1) \mathbb{E} \left[ \| v^1 \|^2 \right] + \cdots + \mathbb{E} \left[ \| v^{m-1} \|^2 \right] \right].
\]

We have
\[
\sum_{t=0}^{m} \mathbb{E} \left[ \| \nabla f(x^t) - v^t \|^2 \right] - (1 - \bar{L} \eta) \sum_{t=0}^{m} \mathbb{E} \left[ \| v^t \|^2 \right] \\
\leq \frac{1}{b} K \eta^2 \left[ m \mathbb{E} \left[ \| v^0 \|^2 \right] + (m - 1) \mathbb{E} \left[ \| v^1 \|^2 \right] + \cdots + \mathbb{E} \left[ \| v^{m-1} \|^2 \right] \right] \\
- (1 - \bar{L} \eta) \left[ \mathbb{E} \left[ \| v^0 \|^2 \right] + \mathbb{E} \left[ \| v^1 \|^2 \right] + \cdots + \mathbb{E} \left[ \| v^m \|^2 \right] \right] \\
\leq \left[ \frac{1}{b} K \eta^2 m - (1 - \bar{L} \eta) \right] \sum_{t=1}^{m} \mathbb{E} \left[ \| v^{t-1} \|^2 \right] \overset{(10)}{\leq} 0
\]

since
\[
\eta = \frac{2}{\bar{L} \left( \sqrt{1 + \frac{4Km}{L^2b}} + 1 \right)}
\]
is a root of equation
\[
\frac{1}{b} K \eta^2 m - (1 - \bar{L} \eta) = 0.
\]

Therefore, by Lemma 14, we have
\[
\sum_{t=0}^{m} \mathbb{E} \left[ \| \nabla f(x^t) \|^2 \right] \leq \frac{2}{\eta} [f(x^0) - f(x^*)] + \sum_{t=0}^{m} \mathbb{E} \left[ \| \nabla f(x^t) - v^t \|^2 \right] \\
- (1 - \bar{L} \eta) \sum_{t=0}^{m} \mathbb{E} \left[ \| v^t \|^2 \right] \overset{(44)}{\leq} \frac{2}{\eta} [f(x^0) - f(x^*)].
\]

28
If \( x_a \) is chosen uniformly at random from \( \{x^t\}_{t=0}^m \), then
\[
E \left[ \| \nabla f(x_a) \|^2 \right] = \frac{1}{m+1} \sum_{t=0}^m E \left[ \| \nabla f(x^t) \|^2 \right] \leq \frac{2}{\eta(m+1)} \left( f(x^0) - f(x^*) \right).
\]
This concludes the proof. \( \square \)

H One Sample Importance Sampling

H.1 SVRG

Algorithm 5 IS-SVRG\( (x^0, T, m, \{p_i\}_{i=0}^n, \eta) \)

1: Input: \( x^0 = x^0_m = x^0 \in \mathbb{R}^d \), epoch length \( m \), step sizes \( \{\eta_i > 0\}_{i=0}^{m-1} \), \( S = \lceil T/m \rceil \)
2: for \( s = 0 \) to \( S - 1 \) do
3: \( x^s_0 = x^s_m \)
4: \( g^{s+1} = \frac{1}{2} \sum_{i=1}^n \nabla f_i(\tilde{x}^s) \)
5: for \( t = 0 \) to \( m - 1 \) do
6: With \( \{p_i\}_{i=0}^n \) randomly pick \( i_t \) from \( \{1, \ldots, n\} \)
7: \( v^{s+1}_t = \frac{1}{np_{i_t}}(\nabla f_{i_t}(x^t_{s+1}) - \nabla f_{i_t}(\tilde{x}^s)) + g^{s+1} \)
8: \( x^{s+1}_t = x^{s+1}_t - \eta v^{s+1}_t \)
9: end for
10: \( \tilde{x}^{s+1} = x^{s+1}_m \)
11: end for
12: Output: Iterate \( x_a \) chosen uniformly random from \( \{x^{s+1}_t\}_{t=0}^m \).

In this section, we introduce SVRG algorithm for importance sampling,

Theorem 17. Let \( c_m = 0 \), \( \eta = \eta > 0 \), \( \beta = \beta > 0 \), and \( c_t = c_{t+1}(1 + \eta \beta + 2K\eta^2) + K\eta^2 \bar{L} \) such that \( \Gamma_t > 0 \)
for \( 0 \leq t \leq m - 1 \). Define the quantity \( \gamma_n \defeq \min \Gamma_t \). Further, let \( T \) be a multiple of \( m \). Then for the output \( x_a \) of Algorithm 5 we have
\[
E \left[ \| \nabla f(x_a) \|^2 \right] \leq \frac{f(x^0) - f(x^*)}{T\gamma_n},
\]
where \( x^* \) is an optimal solution to (1) and \( \Gamma_t = (\eta - \frac{c_t + \eta \beta}{\eta} - \eta^2 \bar{L} - 2c_t + \eta^2) \).

Theorem 18. Let \( \eta = \bar{L}\mu_0/(Kn^2) \) (\( 0 < \mu_0 < 1 \)), \( \beta = \bar{L}/n^4 \), \( m = \lceil Kn/(3\bar{L}^2\mu_0) \rceil \) and \( T \) is some multiple of \( m \). Then there exists universal constants \( \mu_0, \nu > 0 \) such that we have the following: \( \gamma_n \geq \frac{\mu_0}{Kn^2} \) in Theorem 17 and
\[
E \left[ \| \nabla f(x_a) \|^2 \right] \leq \frac{Kn^2[f(x^0) - f(x^*)]}{LT\nu},
\]
where \( x^* \) is an optimal solution to the problem in (1) and \( x_a \) is the output of Algorithm 5.

Comparing Theorem 17 to the previous result in [28], we can see improvement in constant, if we assume different \( L_i \)-smooth constants for different functions. If the all \( L_i \)'s are the same then our result is the same as previous result for uniform sampling, because then \( \alpha = \frac{n-1}{n-1} = 1 \).

H.2 SAGA

Here, we provide similar analysis as for SVRG with the same result. We provide more generalized improved form of theorems which appeared in [29].
Algorithm 6 IS-SAGA \( (x^0, T, \{p_i\}_{i=0}^{\eta}, \eta) \)

1: **Input**: \( x^0 \in \mathbb{R}^d \), \( \alpha_i^0 = x^0 \) for \( i \in [n] \), number of iterations \( T \), step size \( \eta > 0 \)
2: \( g^0 = \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(\alpha_i^0) \)
3: for \( t = 0 \) to \( T - 1 \) do
4: Randomly pick \( t \) from \([n]\) with \( \{p_i\}_{i=0}^{\eta} \)
5: Randomly uniformly pick \( i_t \) from \([n]\)
6: \( v^t = \frac{1}{np_t}(\nabla f_{i_t}(x^t) - \nabla f_{i_t}(\alpha_{i_t}^t)) + g^t \)
7: \( x^{t+1} = x^t - \eta v^t \)
8: \( \alpha_{j_t}^{t+1} = x^t \) and \( \alpha_{j_t}^{t+1} = \alpha_{j_t}^t \) for \( j \neq j_t \)
9: \( g^{t+1} = g^t - \frac{1}{n}(\nabla f_{i_t}(\alpha_{i_t}^t) - \nabla f_{i_t}(\alpha_{i_t}^{t+1})) \)
10: end for
11: **Output**: Iterate \( x_a \) chosen uniformly random from \( \{x^t\}_{t=0}^{T} \).

**Theorem 19.** Let \( c_T = 0, \beta > 0, \) and \( c_t = c_{t+1}(1 - \frac{1}{n} + \eta \beta + 2K\eta^2) + K\eta^2 L \) be such that \( \Gamma_t > 0 \) for \( 0 \leq t \leq T - 1 \). Define the quantity \( \gamma_n \overset{\text{def}}{=} \min_{0 \leq t \leq T - 1} \Gamma_t \). Then the output \( x_a \) of Algorithm 6 satisfies the bound
\[
E \left[ \|\nabla f(x_a)\|^2 \right] \leq \frac{f(x^0) - f(x^*)}{T \gamma_n},
\]
where \( x^* \) is an optimal solution to (1) and \( \Gamma_t = \left( \eta - \frac{c_t + 1}{n} \right) - \eta^2 L - 2c_{t+1} \eta^2 \).

**Theorem 20.** Let \( \eta = L/(3Kn^{2/3}) \) and \( \beta = L/n^{1/3} \). Then, \( \gamma_n \geq \frac{L}{12Kn^{7/3}} \) and we have the bound
\[
E \left[ \|\nabla f(x_a)\|^2 \right] \leq \frac{12Kn^{2/3}f(x^0) - f(x^*)}{LT},
\]
where \( x^* \) is an optimal solution to the problem in (1) and \( x_a \) is the output of Algorithm 6.

We can see that exactly same conclusions apply here as for SVRG and results can be interpreted in the same way.

I SARAH: Convex Case

I.1 Main result

Consider Algorithm 7 which is an importance sampling variant of the SARAH method.

Note, that only 10-th and 11-th row are changed comparing to classic SARAH algorithm presented in [22]. We do not sample uniformly anymore and also in the 11-th row of Algorithm 7 where we use factor \( \frac{1}{p_i} \) in order to stay unbiased in outer cycle.

Then using similar analysis used in [22] and additional lemmas we can prove following theorems with \( p_i \) in Algorithm 7 to be \( \frac{L}{\sum_{i=1}^{n} L_i} \).

**Theorem 21.** Suppose that \( f_i(x) \) are \( L_i \)-smooth and convex, \( f(x) \) is \( \mu \) strongly convex. Consider \( v^t \) defined in SARAH-ISc (Algorithm 7) with \( \eta < 2/L \), where \( \bar{L} = \frac{1}{n} \sum_{i=1}^{n} L_i \). Then, for any \( t \geq 1 \),
\[
E \left[ \|v^t\|^2 \right] \leq \left[ 1 - \left( \frac{2}{\eta} - 1 \right) \mu^2 \eta^2 \right] E \left[ \|v^{t-1}\|^2 \right] \leq \left[ 1 - \left( \frac{2}{\eta} - 1 \right) \mu^2 \eta^2 \right] E \left[ \|\nabla f(x^0)\|^2 \right].
\]

By choosing \( \eta = \mathcal{O}(1/L) \), we obtain the linear convergence of \( \|v^t\|^2 \) in expectation with the rate \((1 - 1/\kappa^2)\), where \( \kappa = \frac{L}{\mu} \) is condition number, This is improvement over previous result in [22], because of \( \frac{L}{\mu} \leq \frac{L_{\text{max}}}{\mu} \).

Below we show that a better convergence rate could be obtained under a stronger convexity assumption for each single \( f_i(x) \).
Algorithm 7 SARAH-ISC
1: Parameters: the learning rate $\eta > 0$ and the inner loop size $m$.
2: Initialize: $\tilde{x}_0$
3: Iterate:
4: for $s = 1, 2, \ldots$ do
5: \hspace{1em} $x_0 = \tilde{x}_{s-1}$
6: \hspace{1em} $v^0 = 1/n \sum_{i=1}^n \nabla f_i(x^0)$
7: \hspace{1em} $x_1 = x_0 - \eta v^0$
8: \hspace{1em} Iterate:
9: \hspace{2em} for $t = 1, \ldots, m-1$ do
10: \hspace{3em} Sample $i_t$ at random from $[n]$ with probability $p_i$
11: \hspace{3em} $v^t = 1/np_i (\nabla f_{i_t}(x^t) - \nabla f_{i_t}(x^{t-1})) + v^{t-1}$
12: \hspace{3em} $x_{t+1} = x^t - \eta v^t$
13: \hspace{2em} end for
14: Set $\tilde{x}_s = x^t$ with $t$ chosen uniformly at random from $\{0, 1, \ldots, m\}$
15: end for

Theorem 22. Suppose that $f_i(x)$ are $L_i$-smooth and $\mu$ strongly convex. Consider $v^t$ defined by in SARAH-ISC (Algorithm 7) with $\eta \leq 2/(\mu + \tilde{L})$. Then the following bound holds, $\forall \, t \geq 1$,
\[
E \left[ \|v^t\|^2 \right] \leq \left( 1 - \frac{2\tilde{L}n\eta}{\mu + \tilde{L}} \right) E \left[ \|v^{t-1}\|^2 \right]
\leq \left( 1 - \frac{2\tilde{L}n\eta}{\mu + \tilde{L}} \right)^t E \left[ \|\nabla f(x^0)\|^2 \right].
\]

By setting $\eta = \mathcal{O}(1/\tilde{L})$, we derive the linear convergence with the rate of $(1 - 1/\kappa)$, where $\kappa = \frac{L}{\mu}$ which is an improvement over the previous result of [22], because if we take the optimal stepsize $\nu = \frac{2}{\mu + \tilde{L}}$ than we can easily prove that $\frac{2\tilde{L}n\eta}{\mu + \tilde{L}}$ is greater than $\frac{2nL_{\max}}{\mu + L_{\max}}$, with optimal step size, where $L_{\max} = \max_i \{L_i\}$.

I.2 Lemmas

We start with modification of lemmas in [22], which we later use in the proofs of Theorem 22 and Theorem 21. The first Lemma [23] bounds the sum of expected values of $\|\nabla f(x^t)\|^2$. The second, Lemma 24, bounds $E \left[ \|\nabla f(x^t) - v^t\|^2 \right]$.

Lemma 23. Suppose that $f_i(x)$'s are $L_i$-smooth. Consider SARAH-ISC (Algorithm 7). Then, we have
\[
\sum_{t=0}^m E \left[ \|\nabla f(x^t)\|^2 \right] \leq \frac{2}{\eta} E \left[ f(x^0) - f(x^*) \right]
+ \sum_{t=0}^m E \left[ \|\nabla f(x^t) - v^t\|^2 \right] - (1 - L\eta) \sum_{t=0}^m E \left[ \|v^t\|^2 \right]. \quad (47)
\]

Lemma 24. Suppose that $f_i(x)$'s are $L_i$-smooth. Consider SARAH-ISC (Algorithm 7). Then for any $t \geq 1$,
\[
E \left[ \|\nabla f(x^t) - v^t\|^2 \right] = \sum_{j=1}^t E \left[ \|v^j - v^{j-1}\|^2 \right] - \sum_{j=1}^t E \left[ \|\nabla f(x^j) - \nabla f(x^{j-1})\|^2 \right].
\]
Lemma 25. Suppose that $f_i(x)$’s are $L_i$-smooth and convex. Consider SARAH (Algorithm 7) with $\eta < 2/\bar{L}$. Then we have that for any $t \geq 1$,

\[
\mathbb{E} \left[ \|\nabla f(x^t) - v^t\|^2 \right] \leq \frac{\eta\bar{L}}{2 - \eta L} \mathbb{E} \left[ \|v^0\|^2 \right] - \mathbb{E} \left[ \|v^t\|^2 \right]
\]

where $\bar{L} = \frac{1}{n} \sum_{i=1}^{n} L_i$.

**Proof of Lemma 23**

Proof. By Lemma 26 and $x^{t+1} = x^t - \eta v^t$, we have

\[
\mathbb{E} \left[ f(x^{t+1}) \right] \leq \mathbb{E} \left[ f(x^t) \right] - \eta \mathbb{E} \left[ \nabla f(x^t)^\top v^t \right] + \frac{\bar{L} \eta^2}{2} \mathbb{E} \left[ \|v^t\|^2 \right]
\]

\[
= \mathbb{E} \left[ f(x^t) \right] - \eta \mathbb{E} \left[ \|\nabla f(x^t)\|^2 \right] + \frac{\bar{L} \eta^2}{2} \mathbb{E} \left[ \|\nabla f(x^t) - v^t\|^2 \right]
\]

\[
- \left( \frac{\eta}{2} - \frac{\bar{L} \eta^2}{2} \right) \mathbb{E} \left[ \|v^t\|^2 \right],
\]

where the last equality follows from the fact $a^\top b = \frac{1}{2} \left[ \|a\|^2 + \|b\|^2 - \|a - b\|^2 \right]$.

By summing over $t = 0, \ldots, m$, we have

\[
\mathbb{E} \left[ f(x^{m+1}) \right] \leq \mathbb{E} \left[ f(x^0) \right] - \frac{\eta}{2} \sum_{t=0}^{m} \mathbb{E} \left[ \|\nabla f(x^t)\|^2 \right] + \frac{\bar{L} \eta^2}{2} \sum_{t=0}^{m} \mathbb{E} \left[ \|\nabla f(x^t) - v^t\|^2 \right]
\]

\[
- \left( \frac{\eta}{2} - \frac{\bar{L} \eta^2}{2} \right) \sum_{t=0}^{m} \mathbb{E} \left[ \|v^t\|^2 \right],
\]

which is equivalent to ($\eta > 0$):

\[
\sum_{t=0}^{m} \mathbb{E} \left[ \|\nabla f(x^t)\|^2 \right] \leq \frac{2}{\eta} \mathbb{E} \left[ f(x^0) - f(x_{m+1}) \right] + \frac{\bar{L} \eta^2}{2} \sum_{t=0}^{m} \mathbb{E} \left[ \|\nabla f(x^t) - v^t\|^2 \right] - \left( 1 - \bar{L} \eta \right) \sum_{t=0}^{m} \mathbb{E} \left[ \|v^t\|^2 \right]
\]

\[
\leq \frac{2}{\eta} \mathbb{E} \left[ f(x^0) - f(x^*) \right] + \sum_{t=0}^{m} \mathbb{E} \left[ \|\nabla f(x^t) - v^t\|^2 \right]
\]

\[
- \left( 1 - \bar{L} \eta \right) \sum_{t=0}^{m} \mathbb{E} \left[ \|v^t\|^2 \right],
\]

where the last inequality follows since $x^*$ is a global minimizer of $f$. \qed
Proof of Lemma 24

Proof. Let $\mathcal{F}_j$ be $\sigma$ algebra that contains all the information of $x^0, \ldots, x^j$ as well as $v^0, \ldots, v^{j-1}$. For $j \geq 1$, we have

$$\begin{align*}
E \left[ \| \nabla f(x^j) - v^j \|^2 | \mathcal{F}_j \right] = \\
E \left[ \| \nabla f(x^{j-1}) - v^{j-1} \| + \| \nabla f(x^j) - \nabla f(x^{j-1}) \| - \| v^j - v^{j-1} \|^2 | \mathcal{F}_j \right] \\
= \| \nabla f(x^{j-1}) - v^{j-1} \|^2 + \| \nabla f(x^j) - \nabla f(x^{j-1}) \|^2 \\
+ E \left[ \| v^j - v^{j-1} \|^2 | \mathcal{F}_j \right] \\
+ 2(\nabla f(x^{j-1}) - v^{j-1})^T(\nabla f(x^j) - \nabla f(x^{j-1}) ) \\
- 2(\nabla f(x^{j-1}) - v^{j-1})^T E \left[ v^j - v^{j-1} | \mathcal{F}_j \right] \\
- 2(\nabla f(x^j) - \nabla f(x^{j-1}) )^T E \left[ v^j - v^{j-1} | \mathcal{F}_j \right] \\
= \| \nabla f(x^{j-1}) - v^{j-1} \|^2 - \| \nabla f(x^j) - \nabla f(x^{j-1}) \|^2 \\
+ E \left[ \| v^j - v^{j-1} \|^2 | \mathcal{F}_j \right],
\end{align*}$$

where the last equality follows from

$$E \left[ v^j - v^{j-1} | \mathcal{F}_j \right] = E \left[ \frac{1}{n\eta_p_j} (\nabla f_{x_j}(x^j) - \nabla f_{x_j}(x^{j-1})) | \mathcal{F}_j \right] = \nabla f(x^j) - \nabla f(x^{j-1}).$$

By taking expectation for the above equation, we have

$$E \left[ \| \nabla f(x^j) - v^j \|^2 \right] = E \left[ \| \nabla f(x^{j-1}) - v^{j-1} \|^2 \right] - E \left[ \| \nabla f(x^j) - \nabla f(x^{j-1}) \|^2 \right] \\
+ E \left[ \| v^j - v^{j-1} \|^2 \right].$$

Note that $\| \nabla f(x^0) - v^0 \|^2 = 0$. By summing over $j = 1, \ldots, t$ ($t \geq 1$), we have

$$E \left[ \| \nabla f(x^t) - v^t \|^2 \right] = \sum_{j=1}^t E \left[ \| v^j - v^{j-1} \|^2 \right] - \sum_{j=1}^t E \left[ \| \nabla f(x^j) - \nabla f(x^{j-1}) \|^2 \right].$$

Proof of Lemma 25

Proof. For $j \geq 1$, we have

$$\begin{align*}
E \left[ \| v^j \|^2 | \mathcal{F}_j \right] = \\
E \left[ \| v^{j-1} \|^2 + \frac{1}{n\eta_p_j} (\nabla f_{x_j}(x^{j-1}) - \nabla f_{x_j}(x^j)) | \mathcal{F}_j \right] \\
= \| v^{j-1} \|^2 + E \left[ \frac{1}{n\eta_p_j} \| \nabla f_{x_j}(x^{j-1}) - \nabla f_{x_j}(x^j) \|^2 | \mathcal{F}_j \right] \\
- E \left[ \frac{2}{n\eta_p_j} (\nabla f_{x_j}(x^{j-1}) - \nabla f_{x_j}(x^j))^T (x^{j-1} - x^j) | \mathcal{F}_j \right] \\
\overset{\cdot{\cdot}\cdot{\cdot}}{\leq} \| v^{j-1} \|^2 + E \left[ \frac{1}{n\eta_p_j} \| \nabla f_{x_j}(x^{j-1}) - \nabla f_{x_j}(x^j) \|^2 | \mathcal{F}_j \right] \\
- E \left[ \frac{2}{L \eta \eta_p_j} \| \nabla f_{x_j}(x^{j-1}) - \nabla f_{x_j}(x^j) \|^2 | \mathcal{F}_j \right] \\
= \| v^{j-1} \|^2 + \left( 1 - \frac{2}{\eta L} \right) E \left[ \frac{1}{n\eta_p_j} (\nabla f_{x_j}(x^{j-1}) - \nabla f_{x_j}(x^j))^2 | \mathcal{F}_j \right] \\
= \| v^{j-1} \|^2 + \left( 1 - \frac{2}{\eta L} \right) E \left[ \| v^j - v^{j-1} \|^2 | \mathcal{F}_j \right]
\end{align*}$$

33
The consequent equality follows from definition of $p_i$'s and the last equality follows from definition of SARAH-ISC. Taking expectation, we get
\[
E \left[ \|v^j - v^{j-1}\|^2 \right] \leq \frac{\eta \bar{L}}{2 - \eta \bar{L}} \left[ E \left[ \|v^{j-1}\|^2 \right] - E \left[ \|v^j\|^2 \right] \right],
\]
when $\eta < 2/\bar{L}$.

By summing the above inequality over $j = 1, \ldots, t$ ($t \geq 1$), we have
\[
\sum_{j=1}^{t} E \left[ \|v^j - v^{j-1}\|^2 \right] \leq \frac{\eta \bar{L}}{2 - \eta \bar{L}} \left[ E \left[ \|v^0\|^2 \right] - E \left[ \|v^t\|^2 \right] \right].
\]

By Lemma 24, we have
\[
E \left[ \|\nabla f(x') - v'\|^2 \right] \leq \sum_{j=1}^{t} E \left[ \|v^j - v^{j-1}\|^2 \right] \leq \frac{\eta \bar{L}}{2 - \eta \bar{L}} \left[ E \left[ \|v^0\|^2 \right] - E \left[ \|v^t\|^2 \right] \right].
\]

**Proof of Theorem 21**

*Proof.* For $t \geq 1$, we have
\[
\|\nabla f(x') - \nabla f(x'^{-1})\|^2 = \left\| \frac{1}{n} \sum_{i=1}^{n} \left[ \nabla f_i(x') - \nabla f_i(x'^{-1}) \right] \right\|^2
\]
\[
= \left\| \sum_{i=1}^{n} p_i \frac{1}{np_i} \left[ \nabla f_i(x') - \nabla f_i(x'^{-1}) \right] \right\|^2
\]
\[
\leq \sum_{i=1}^{n} p_i \left\| \frac{1}{np_i} \left[ \nabla f_i(x') - \nabla f_i(x'^{-1}) \right] \right\|^2
\]
\[
= \text{E} \left[ \left\| \frac{1}{np_i} \left( \nabla f_i(x') - \nabla f_i(x'^{-1}) \right) \right\|^2 | F_i \right].
\]

Using the proof of Lemma 25 for $t \geq 1$, we have
\[
E \left[ \|v'^{t}\|^2 | F_i \right] \leq \|v'^{-1}\|^2 + \left( 1 - \frac{2}{n \bar{L}} \right) E \left[ \|\nabla f_i(x'^{-1}) - \nabla f_i(x'^{-1})\|^2 | F_i \right]
\]
\[
\leq \|v'^{-1}\|^2 + \left( 1 - \frac{2}{n \bar{L}} \right) \|\nabla f(x') - \nabla f(x'^{-1})\|^2
\]
\[
\leq \|v'^{-1}\|^2 + \left( 1 - \frac{2}{n \bar{L}} \right) \mu^2 \eta^2 \|v'^{-1}\|^2.
\]

Note that $1 - \frac{2}{n \bar{L}} < 0$ since $\eta < 2/\bar{L}$. The last inequality follows by the strong convexity of $f$, that is, $\mu \|x'^{t} - x'^{-1}\| \leq \|\nabla f(x'^{t}) - \nabla f(x'^{-1})\|$ and the fact that $x'^{t} = x'^{-1} - \eta v'^{-1}$. By taking the expectation and applying recursively, we have
\[
E \left[ \|v'^{t}\|^2 \right] \leq \left[ 1 - \left( \frac{2}{n \bar{L}} - 1 \right) \mu^2 \eta^2 \right] E \left[ \|v'^{-1}\|^2 \right]
\]
\[
\leq \left[ 1 - \left( \frac{2}{n \bar{L}} - 1 \right) \mu^2 \eta^2 \right] E \left[ \|v'^{0}\|^2 \right]
\]
\[
= \left[ 1 - \left( \frac{2}{n \bar{L}} - 1 \right) \mu^2 \eta^2 \right] E \left[ \|\nabla f(x'^{0})\|^2 \right].
\]
Proof of Theorem 22

Proof. We obviously have $E [ \|v^0\|^2 | F_0] = \|\nabla f(x_0)\|^2$. For $t \geq 1$, we have

$$E [\|v^t\|^2 | F_t] = E \left[ \|v^{t-1} - \frac{1}{n \eta t^4} (\nabla f_{t_i}(x^{t-1}) - \nabla f_{t_i}(x^t)) \|^2 | F_t \right]$$

$$= \|v^{t-1}\|^2 + E \left[ \frac{1}{n \eta t^4} \|\nabla f_{t_i}(x^{t-1}) - \nabla f_{t_i}(x^t)\|^2 | F_t \right]$$

$$- E \left[ \frac{2}{n \eta^2} (\nabla f_{t_i}(x^{t-1}) - \nabla f_{t_i}(x^t))^\top (x^{t-1} - x^t) | F_t \right]$$

$$= \|v^{t-1}\|^2 + E \left[ \frac{1}{n \eta t^4} \|\nabla f_{t_i}(x^{t-1}) - \nabla f_{t_i}(x^t)\|^2 | F_t \right]$$

$$- \frac{2}{n \eta} (\nabla f(x^{t-1}) - \nabla f(x^t))^\top (x^{t-1} - x^t)$$

$$\leq \left( 1 - \frac{2 \mu L}{n \eta} \right) \|v^{t-1}\|^2 + E \left[ \frac{1}{n \eta t^4} \|\nabla f_{t_i}(x^{t-1}) - \nabla f_{t_i}(x^t)\|^2 | F_t \right]$$

$$- \frac{2 \mu L}{n \eta} \|v^{t-1}\|^2$$

$$= \left( 1 - \frac{2 \mu L}{n \eta} \right) \|v^{t-1}\|^2$$

$$- E \left[ \frac{1}{n \eta t^4} \|\nabla f_{t_i}(x^{t-1}) - \nabla f_{t_i}(x^t)\|^2 | F_t \right] - \|\nabla f(x^{t-1}) - \nabla f(x^t)\|^2 | F_t \right]$$

$$\leq \left( 1 - \frac{2 \mu L}{n \eta} \right) \|v^{t-1}\|^2,$$ (50)

where in the first two equalities, we used definition of SARAH-ISC. The first inequality follows from fact that $f(x)$ is $\bar{L}$-smooth and $\mu$ strongly convex, thus following inequality holds (inequality from [21])

$$(\nabla f(x) - \nabla f(x'))^\top (x - x') \geq \frac{\mu \bar{L}}{\mu + \bar{L}} \|x - x'\|^2 + \frac{1}{\mu + \bar{L}} \|\nabla f(x) - \nabla f(x')\|^2,$$ (51)

The second one uses assumption that $\eta \leq \frac{2}{\mu + \bar{L}}$, thus $\eta = \frac{2}{\mu + \bar{L}}$ is optimal step size under this analysis. By taking the expectation and applying recursively, the desired result is achieved.

J Technical Lemmas

Lemma 26. Let $f_i$’s be function, which are $L_i$-smooth, then $f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x)$ is $\bar{L}$-smooth, where $\bar{L} = \frac{1}{n} \sum_{i=1}^{n} L_i$.

Proof. For each function $f_i$ we have by definition of $L_i$-smoothness, $\forall x, y \in \mathbb{R}^d$

$$f_i(x) \leq f_i(y) + \nabla f_i(y)^\top (x - y) + \frac{L_i}{2} \|x - y\|^2$$ (52)

Summing through all $i$’s and dividing by $n$, we get

$$f(x) \leq f(y) + \nabla f(y)^\top (x - y) + \frac{\bar{L}}{2} \|x - y\|^2$$ (53)

Lemma 27 (Cauchy-Schwarz inequality). For all $x, y \in \mathbb{R}^d$ we have

$$|\langle x, y \rangle| \leq ||x|| ||y||.$$ (54)
Lemma 28 (Young’s inequality). For $a, b \in \mathbb{R}$ and $\beta > 0$ we have
\[
ab \leq \frac{a^2 \beta}{2} + \frac{b^2}{2\beta}.
\] (55)

Lemma 29 (Jensen’s inequality). Let $X$ be a random variable and $g(x)$ be a convex function. Then
\[
g(\mathbb{E}[X]) \leq \mathbb{E}[g(X)].
\] (56)