An Axiomatic Characterization of Split Cycle

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Abstract

A number of rules for resolving majority cycles in elections have been proposed in the literature. Recently, Holliday and Pacuit (Journal of Theoretical Politics 33 (2021) 475-524) axiomatically characterized the class of rules refined by one such cycle-resolving rule, dubbed Split Cycle: in each majority cycle, discard the majority preferences with the smallest majority margin. They showed that any rule satisfying five standard axioms plus a weakening of Arrow’s Independence of Irrelevant Alternatives (IIA), called Coherent IIA, is refined by Split Cycle. In this paper, we go further and show that Split Cycle is the only rule satisfying the axioms of Holliday and Pacuit together with two additional axioms, which characterize the class of rules that refine Split Cycle: Coherent Defeat and Positive Involvement in Defeat. Coherent Defeat states that any majority preference not occurring in a cycle is retained, while Positive Involvement in Defeat is closely related to the well-known axiom of Positive Involvement (as in J. Pérez, Social Choice and Welfare 18 (2001) 601-616). We characterize Split Cycle not only as a collective choice rule but also as a social choice correspondence, over both profiles of linear ballots and profiles of ballots allowing ties.

Contents

1 Introduction 2
2 Preliminaries 6
   2.1 VCCRs and VSCCs ................................................................. 6
   2.2 Split Cycle ................................................................. 7
3 Axioms on VCCRs 9
   3.1 Standard axioms ................................................................. 9
   3.2 Coherent IIA ................................................................. 9
   3.3 Coherent Defeat ............................................................. 11
   3.4 Positive Involvement in Defeat ............................................. 13
4 Characterization of the Split Cycle VCCR 17
1 Introduction

The possibility of cycles in the majority relation of an election—wherein for candidates $c_1, \ldots, c_n$, a majority of voters prefer $c_1$ to $c_2$, a majority prefer $c_2$ to $c_3$, and so on, while a majority prefer $c_n$ to $c_1$—has been taken to show that “majority rule is fatally flawed by an internal inconsistency” (Wolff 1970, p. 59). Yet voting theorists have studied many collective choice rules based on pairwise majority comparisons of candidates designed to resolve majority cycles. These collective choice rules output a binary relation between candidates, which we will call the relation of defeat, that is guaranteed to be free of cycles. One prominent family of such rules resolves cycles by paying attention to the size of majority victories, e.g., as measured by the majority margin between candidates $x$ and $y$, defined as the number of voters who prefer $x$ to $y$ minus the number who prefer $y$ to $x$. Examples include the Ranked Pairs (Tideman 1987, Zavist and Tideman 1989), River (Heitzig 2004b), Beat Path (Schulze 2011, 2022), Kemeny (Kemeny 1959),\(^1\) Weighted Covering (Dutta and Laslier 1999, Pérez-Fernández and De Baets 2018), and Split Cycle (Holliday and Pacuit 2021a, 2023a) rules.

For instance, according to Split Cycle,\(^2\) majority cycles are resolved as follows:

1. For each majority cycle, identify the pairwise majority victories with the smallest margin in that cycle.

2. A candidate $a$ defeats a candidate $b$ according to Split Cycle if and only if $a$ has a pairwise majority victory over $b$ that was not identified in step 1.

In other words, a majority victory of $a$ vs. $b$ counts as a defeat of $b$ if and only if in each majority cycle in which that majority victory appears, it does not have the smallest margin in the cycle. The resulting defeat relation contains no cycles. See Figure 1 for an example.

Faced with such a rule for resolving cycles, the question becomes: why this rule and not something else? The question can be partially answered by identifying axioms that distinguish between known rules.

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\(^1\) Though Kemeny is not usually defined in terms of pairwise margins of victory, it can be so defined as shown in Fischer et al. 2016, p. 87. Another rule usually defined in terms of individual preferences but also definable in terms of pairwise margins is the Borda count (see Zwicker 2016, p. 28).

\(^2\) Eppley’s (2000) “Beatpath Criterion Method” can be defined in the same way but using winning votes instead of margins as the measure of strength of majority preference. This distinction does not matter for our axiomatization in the context of linear ballots, but it does matter for our axiomatization in the context of ballots allowing ties (see Section 7).
A deeper answer comes from a complete axiomatic characterization of a rule as the unique rule satisfying some list of natural axioms. For example, such axiomatic characterizations exist for the collective choice rules that rank candidates by Copeland score (Rubinstein 1980) and Borda scores (Nitzan and Rubinstein 1981, Mihara 2017).

Recently Holliday and Pacuit (2021a) have characterized the class of rules refined by the Split Cycle rule using six axioms, five of which are standard (see Section 3.1 below) and the sixth of which is a weakening of Arrow’s (1963) axiom of Independence of Irrelevant Alternatives (IIA). They call their new axiom Coherent IIA. Recall that IIA states that for any two profiles $P$ and $P'$ of voter preferences, if $P$ and $P'$ are the same with respect to how each voter ranks $x$ vs. $y$, then if $x$ defeats $y$ in $P$, $x$ must also defeat $y$ in $P'$. The motivation for weakening IIA to Coherent IIA can be seen in a simple example from Holliday and Pacuit 2021a. In the profile $P$ in Figure 2, displayed alongside its corresponding margin graph, arguably any sensible collective choice rule should judge that candidate $a$ defeats candidate $b$. However, in the profile $P'$ in Figure 2, no sensible collective choice rule—technically, no collective choice rule that is anonymous, neutral, and guarantees that some undefeated candidate exists—can judge that $a$ defeats $b$, despite $a$ still beating $b$ head-to-head by a margin of $n$. This is a counterexample to IIA as a normative requirement. Holliday and Pacuit argue that the “Fallacy of IIA” is to ignore how the context of a full election can force us to suspend judgment on some relations of defeat that we could coherently accept in a different context.

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3A conjectured axiomatization of Ranked Pairs can be found in Tideman 1987.

4To say that a rule is refined by Split Cycle means that if a candidate $x$ defeats a candidate $y$ according to the rule, then $x$ also defeats $y$ according to Split Cycle.
In the context of a majority cycle in which \( x \) is majority preferred to \( y \), the electorate is *incoherent with respect to \( x \) and \( y \)* in the sense that while there is an argument that \( x \) should defeat \( y \), in virtue of the majority preference for \( x \) over \( y \), there is also an opposing argument that \( x \) should *not* defeat \( y \), in virtue of the path of majority preferences from \( y \) to \( x \); e.g., \( y \) is majority preferred to \( z \), who is majority preferred to \( x \). One natural measure of the degree of this incoherence is the strength of the opposing argument, which is some monotonic function of the *margins* of the relevant majority preferences; e.g., the larger the margin of \( y \) over \( z \) or of \( z \) over \( x \), the more incoherent with the majority preference for \( x \) over \( y \)—and the smaller those margins, the less incoherent with the majority preference for \( x \) over \( y \). On this view, the following is sufficient for \( P' \) to be *no more incoherent than \( P \) with respect to \( x \) and \( y \)*: the margin graph of \( P' \) is obtained from that of \( P \) by deleting or reducing the margins on zero or more edges not connecting \( x \) and \( y \) or by deleting zero or more candidates other than \( x \) and \( y \). Adopting this view about incoherence, Holliday and Pacuit accept the core intuition behind IIA whenever contextual incoherence does not interfere, leading to their axiom of Coherent IIA, which can be stated informally as follows:

- **Coherent IIA** (informally): if \( P \) and \( P' \) are the same with respect to how each voter ranks \( x \) vs. \( y \), \( x \) defeats \( y \) in \( P \), and \( P' \) is *not more incoherent than \( P \) with respect to \( x \) and \( y \)*, then \( x \) must also defeat \( y \) in \( P' \).

Holliday and Pacuit then prove that Split Cycle is the most resolute collective choice rule \(^5\) satisfying the five standard axioms plus Coherent IIA. Here “most resolute” means that for any other rule \( f \) that satisfies the six axioms, if \( x \) defeats \( y \) according to \( f \), then \( x \) defeats \( y \) according to Split Cycle.

Holliday and Pacuit’s theorem may be viewed as characterizing Split Cycle as the unique collective choice rule satisfying their six axioms plus a seventh axiom stating that the rule should be the most resolute rule satisfying the first six axioms. In a sense, however, this characterization using the notion of resoluteness is only half of a characterization of Split Cycle. \(^6\) We would like axioms on a collective choice rule such that for any rule \( f \) satisfying the axioms, \( x \) defeats \( y \) according to \( f \) if and only if \( x \) defeats \( y \) according to Split Cycle. Such an axiomatic characterization is the main result of the present paper.

Our new characterization of Split Cycle involves two natural axioms, which we prove are satisfied by exactly the collective choice rules that refine Split Cycle:

- **Coherent Defeat**: if a majority of voters prefer \( x \) to \( y \), and there is no majority cycle involving \( x \) and \( y \), then \( x \) defeats \( y \).

- **Positive Involvement in Defeat**: if \( y \) does *not* defeat \( x \) in profile \( P \), and \( P' \) is obtained from \( P \) by adding one new voter who ranks \( x \) above \( y \), then \( y \) still does not defeat \( x \) in \( P' \).

Coherent Defeat is a point of common ground between Split Cycle, Ranked Pairs, Beat Path, and GOCHA (Schwartz 1986): in the absence of cyclic incoherence, majority preference is sufficient for defeat. We will discuss other ways of motivating Coherent Defeat below, drawing on Heitzig 2002. \(^7\) The intuition behind Positive Involvement in Defeat, a variable-electorate axiom, is similar to the intuition behind fixed-electorate *monotonicity* axioms for collective choice rules, as stated in Blair and Pollak 1982: “additional support for a

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\(^5\) Technically, they characterize Split Cycle as what they call a *variable-election* collective choice rule, whose domain contains profiles with different sets of candidates and voters (see Section 2.1 below). Until the end of this section, we use ‘collective choice rule’ to refer to the variable-election variety.

\(^6\) As Holliday and Pacuit (2021a, p. 501) write, “A natural next step would be to obtain another axiomatic characterization of Split Cycle as the only VCCR satisfying some axioms without reference to resoluteness.”

\(^7\) Heitzig’s (2004a) definition of an “immune” candidate is the same as that of a Split Cycle winner if we replace “stronger” with “at least as strong” in his definition.
(pairwise) winning alternative leaves it winning” (p. 935). Perhaps surprisingly, many collective choice rules violate Positive Involvement in Defeat, including Ranked Pairs, River, Beat Path, and Covering (see Duggan 2013 for many versions) viewed as collective choice rules. As we will explain, the name is borrowed from the related axiom of “Positive Involvement” for functions that output winners instead of a defeat relation (see Saari 1995, Pérez 2001, Holliday and Pacuit 2021b). Our main result in this paper is that Split Cycle is the unique collective choice rule satisfying the five standard axioms, Coherent IIA, Coherent Defeat, and Positive Involvement in Defeat. The proof uses the well-known concept of minimal cuts in graph theory.

So far we have discussed Split Cycle as a collective choice rule that outputs for a given profile a binary relation of defeat on the set of candidates—or more precisely, what we call a variable-election collective choice rule (VCCR), whose domain includes profiles with different set of candidates and voters. In this paper, we also characterize the associated variable-election social choice correspondence (VSCC) that outputs for a given profile the set of undefeated candidates. This involves a translation from VSCCs to VCCRs which induces a translation in the reverse direction from IIA (resp. Coherent IIA) for VCCRs to IIA (resp. Coherent IIA) for VSCCs. The additional axioms of Coherent Defeat and Positive Involvement in Defeat also have natural analogues for VSCCs. Interestingly, the VSCC analogue of the VCCR axiom of Positive Involvement in Defeat, which we call Tolerant Positive Involvement, strengthens the axiom of Positive Involvement from the prior literature mentioned above. We prove that the Split Cycle VSCC is the unique VSCC satisfying five standard axioms, Coherent IIA, Coherent Defeat, and Tolerant Positive Involvement.

The rest of the paper is organized as follows. In Section 2, we formally define VCCRs and VSCCs in general and the Split Cycle VCCR and VSCC in particular. In Section 3, we review the axioms from Holliday and Pacuit 2021a (Sections 3.1-3.2) and discuss in more depth the additional axioms of Coherent Defeat (Section 3.3) and Positive Involvement in Defeat (Section 3.4). Using these axioms, we prove our characterization result for the Split Cycle VCCR in Section 4. We then turn to the Split Cycle VSCC. In Section 5, we define analogues for VSCCs of the axioms for VCCRs in Section 3. Using these analogous axioms, we prove our characterization result for the Split Cycle VSCC in Section 6. In Section 7, we adapt our characterization results to the setting in which ties are permitted in voters’ ballots. In Section 8, we show that the three special axioms—Coherent IIA, Positive Involvement in Defeat (or Tolerant Positive Involvement for VSCCs), and Coherent Defeat—are all necessary in our axiomatization. Finally, we conclude with some suggestions for related axiomatization problems in voting theory in Section 9.


2 Preliminaries

2.1 VCCRs and VSCCs

Fix infinite sets \( V \) and \( X \) of voters and candidates, respectively. We begin by assuming that voters submit linear orders of the candidates. For \( X \subseteq X \), let \( \mathcal{L}(X) \) be the set of all strict linear orders on \( X \). In Section 7, we will adapt our results to the setting in which voters may submit strict weak orders, allowing ties.

**Definition 2.1.** A (linear) profile is a function \( P : V \to \mathcal{L}(X) \) for some nonempty finite \( V \subseteq V \) and nonempty finite \( X \subseteq X \), which we denote by \( V(P) \) (called the set of voters in \( P \)) and \( X(P) \) (called the set of candidates in \( P \)), respectively. We call \( P(i) \) voter \( i \)'s ballot. When \( P \) is clear from context, we write ‘\( x \succ_i y \)’ for \( (x,y) \in P(i) \), and when \( i \) is also clear from context, we write \( x \succ y \) for \( x \succ_i y \).

Given a nonempty set \( Y \subseteq X(P) \) of candidates, the restriction \( P_{|Y} \) of \( P \) to \( Y \) is the profile such that for each \( i \in V(P) \), \( P_{|Y}(i) \) is the restriction of the ballot \( P(i) \) to \( Y \).

Sometimes we will display profiles in their anonymized form, as in Figure 2, meaning that we only display how many voters have each type of ballot, rather than indicating which voters have which ballots. The finiteness of the sets of voters and candidates in profiles will be used implicitly in many constructions in later proofs.

Our main objects of study are (i) functions that assign to each profile a binary relation on the set of candidates and (ii) functions that assign to each profile a subset of the candidates.

**Definition 2.2.** A variable-election collective choice rule (VCCR) is a function \( f \) on the domain of all profiles such that for any profile \( P \), \( f(P) \) is an asymmetric binary relation on \( X(P) \), which we call the defeat relation for \( P \) according to \( f \). For \( x,y \in X(P) \), we say that \( x \) defeats \( y \) in \( P \) according to \( f \) when \( (x,y) \in f(P) \).

**Definition 2.3.** A variable-election social choice correspondence (VSCC) is a function \( F \) on the domain of all profiles such that for any profile \( P \), we have \( \emptyset \neq F(P) \subseteq X(P) \).

The point of the adjective ‘variable-election’ is that different profiles in the domain of a function may have different sets of voters and candidates. By contrast, in the literature, collective choice rules (CCRs) and social choice correspondences (SCCs) are often defined such that all profiles in the domain of a given function have the same sets of voters and candidates, respectively.

Recall that a binary relation \( R \) is acyclic if there is no sequence \( (x_0,x_1,\ldots,x_{n+1}) \) (with \( n \geq 0 \)) such that \( x_0 = x_{n+1} \) and for any \( i = 0\ldots n \), \( (x_i,x_{i+1}) \in R \). A VCCR \( f \) is said to be acyclic if for any profile \( P \), the binary relation \( f(P) \) is acyclic. Such a VCCR induces a VSCC \( \tilde{f} \) that returns for a given profile \( P \) the maximal (undefeated) elements of \( f(P) \).

**Definition 2.4.** For any VCCR \( f \), let \( \tilde{f} \) be the function defined on all profiles such that for any profile \( P \),

\[
\tilde{f}(P) = \{ x \in X(P) \mid \text{there is no } y \in X(P) : y \text{ defeats } x \text{ in } P \text{ according to } f \}.
\]

If \( \tilde{f} \) is a VSCC, we say that \( \tilde{f} \) is the VSCC defeat-rationalized by \( f \).

**Lemma 2.5.** Given any acyclic VCCR \( f \), the function \( \tilde{f} \) defined above is a VSCC, the VSCC defeat-rationalized by \( f \), since \( \emptyset \neq \tilde{f}(P) \subseteq X(P) \) for any profile \( P \).

Our axiomatization proofs rely heavily on establishing that one VCCR (resp. VSCC) refines another in the following sense.
Definition 2.6. Let \( f \) and \( f' \) be VCCRs. We say that \( f \) refines \( f' \) if for any profile \( P \), \( f(P) \subseteq f'(P) \); that is, \( f \) outputs all the defeats that \( f' \) does and possibly more.

Let \( F \) and \( F' \) be VSCCs. We say that \( F \) refines \( F' \) if for any profile \( P \), \( F(P) \subseteq F'(P) \); that is, \( F \) always selects a subset of the candidates selected by \( F' \).

2.2 Split Cycle

In this section, we define the Split Cycle VCCR and VSCC. The definition starts with the notion of the margin graph of a profile and of majority paths and cycles in the margin graph.\(^8\)

Definition 2.7. Let \( P \) be a profile and \( x, y \in X(P) \). The margin of \( x \) over \( y \) in \( P \), written \( \text{Margin}_P(x, y) \), is

\[
|\{i \in V(P) \mid (x, y) \in P(i)\}| - |\{i \in V(P) \mid (y, x) \in P(i)\}|
\]

If \( \text{Margin}_P(x, y) > 0 \), we say that \( x \) is majority preferred to \( y \) in \( P \).

The margin graph of \( P \), denoted \( \mathcal{M}(P) \), is the directed graph with weighted edges such that its set of nodes is \( X(P) \) and there is an edge from \( x \) to \( y \) with weight \( \text{Margin}_P(x, y) > 0 \), the weight of which is \( \text{Margin}_P(x, y) \).

A majority path in \( P \) is a path in \( \mathcal{M}(P) \), i.e., a sequence \( \rho = (x_1, x_2, \ldots, x_n) \) of nodes of \( \mathcal{M}(P) \) such that for each \( i = 1 \ldots n-1 \) there is an edge from \( x_i \) to \( x_{i+1} \), i.e., \( \text{Margin}_P(x_i, x_{i+1}) > 0 \). Given such a majority path \( \rho \), we define its strength, written \( \text{Strength}_P(\rho) \), as

\[
\min\{\text{Margin}_P(x_i, x_{i+1}) \mid i = 1 \ldots n-1\}
\]

Such a \( \rho \) is also called a majority path from \( x_1 \) to \( x_n \). When \( x_n = x_1 \), we call such a \( \rho \) a majority cycle, and in this case \( \text{Strength}_P(\rho) \) is also called the splitting number of \( \rho \), denoted \( \text{Split}_P(\rho) \). A simple majority path is a majority path in which no candidate is repeated, while a simple majority cycle is a majority cycle in which no candidate is repeated except the first and the last.

There are many equivalent ways to define the Split Cycle VCCR, four of which are based on following lemma (for proofs, see Holliday and Pacuit 2023a and Holliday et al. 2021).

Lemma 2.8. For any profile \( P \) and \( x, y \in X(P) \), the following are equivalent:

1. \( \text{Margin}_P(x, y) > 0 \) and \( \text{Margin}_P(x, y) > \text{Split}_P(\rho) \) for every majority cycle \( \rho \) containing \( x \) and \( y \);
2. \( \text{Margin}_P(x, y) > 0 \) and \( \text{Margin}_P(x, y) > \text{Split}_P(\rho) \) for every simple majority cycle \( \rho \) containing \( x \) and \( y \);
3. \( \text{Margin}_P(x, y) > 0 \) and \( \text{Margin}_P(x, y) > \text{Split}_P(\rho) \) for every simple majority cycle \( \rho \) in which \( y \) directly follows \( x \);
4. \( \text{Margin}_P(x, y) > 0 \) and \( \text{Margin}_P(x, y) > \text{Strength}_P(\rho) \) for every simple majority path \( \rho \) from \( y \) to \( x \).

Definition 2.9. The Split Cycle VCCR, denoted \( \text{sc} \), is defined as follows: for any profile \( P \) and \( x, y \in X(P) \), \((x, y) \in \text{sc}(P)\) iff the condition on \( x, y \) in Lemma 2.8 holds. As is shown in Holliday and Pacuit 2023a, \( \text{sc} \) is an acyclic VCCR. Thus, we define the Split Cycle VSCC \( \text{SC} \) as \( \overline{\text{sc}} \).

The two-step algorithm in Section 1 provides one way of calculating \( \text{sc}(P) \).

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8Since Split Cycle needs only the information in the margin graph of a profile, which is an example of a weighted (weak) tournament, Split Cycle may also be regarded as a weighted tournament solution (see Fischer et al. 2016).
Example 2.10. Consider the following anonymized profile with its margin graph:

\[
\begin{array}{cccccccc}
9 & 5 & 3 & 1 & 8 & 4 & 3 & 7 \\
\hline
b & b & a & a & a & d & c & c \\
d & c & b & b & e & d & b & a \\
ce & e & e & c & b & e & a & b \\
d & e & d & e & c & c & c & d \\
a & a & c & d & d & e & a & b \\
\end{array}
\]

The simple cycles of the above margin graph are highlighted with thickened arrows below:

After removing the weakest edge in each simple cycle, what remains is the defeat graph:

The only undefeated candidate is \( b \), and thus \( b \) is the only Split Cycle winner for the profile.

Finding all cycles may be computationally costly; for more efficient algorithms, see Holliday and Pacuit 2023a, Footnote 20 or the Preferential Voting Tools library (https://pref-voting.readthedocs.io/).

The definition of Split Cycle leads directly to the following observation used in our later proofs.

Definition 2.11. Let \( \mathcal{M} \) be a margin graph (the margin graph of some profile) and \( k \in \mathbb{N} \). Then \( \mathcal{M} \restr k \) is the result of keeping all and only the edges in \( \mathcal{M} \) with weight at least \( k \).

Lemma 2.12. For any profile \( P \) and \( x, y \in X(P) \), \((x, y) \in \text{sc}(P)\) iff \( \text{Margin}_P(x, y) > 0 \) and there is no (simple) majority path from \( y \) to \( x \) in \( \mathcal{M}(P) \restr \text{Margin}_P(x, y) \).

Finally, since Split Cycle only cares about the margin graph of a profile (formally, if \( \mathcal{M}(P) = \mathcal{M}(P') \), then \( \text{sc}(P) = \text{sc}(P') \)), the only way that the assumption that voters submit linear orders can matter is by affecting the types of margin graphs that can arise. This happens with the following parity constraint.

Lemma 2.13. If \( P \) is a linear profile, then either all margins between distinct candidates are even or all margins between distinct candidates are odd.

Indeed, this parity constraint is the only consequence of assuming linear ballots.
Proposition 2.14 (Debord 1987). If $M$ is an asymmetric weighted directed graph with positive integer weights all having the same parity, and all weights are even if there are two vertices with no edge between them (representing a zero margin), then there is a linear profile $P$ such that $M = M(P)$.

In Section 7, we will drop the assumption of linear ballots and hence the parity constraint on margins.

3 Axioms on VCCRs

In this section, we present the axioms used in our characterization of the Split Cycle VCCR.

3.1 Standard axioms

First, we recall what Holliday and Pacuit (2021a) consider the “standard axioms” in their characterization.

Definition 3.1. Let $f$ be a VCCR.

1. $f$ satisfies Anonymity if for any profiles $P$ and $P'$, if $V(P) = V(P')$ and there is a bijection $\pi$ from $V(P)$ to $V(P')$ such that for any $i \in V(P')$, $P'(i) = P(\pi(i))$, then $f(P) = f(P')$; and $f$ satisfies Neutrality if for any profiles $P$ and $P'$, if $V(P) = V(P')$, $X(P) = X(P')$, and there is a bijection $\pi$ from $X(P)$ to $X(P')$ such that for any $i \in V(P)$ and $x, y \in X(P)$, $(x, y) \in P(i)$ iff $(\pi(x), \pi(y)) \in P'(i)$, then for any $x, y \in X(P)$, $(x, y) \in f(P)$ iff $(\pi(x), \pi(y)) \in f(P')$.

2. $f$ satisfies Availability if for any profile $P$, $f(P)$ is nonempty.

3. $f$ satisfies Homogeneity (resp. Upward Homogeneity) if for any profile $P$ and $2P$, where $2P$ is the result of replacing each voter in $P$ by 2 copies of that voter, $f(P) = f(2P)$ (resp. $f(P) \subseteq f(2P)$).

4. $f$ satisfies Monotonicity (resp. Monotonicity for two-candidate profiles) if for any profile (resp. two-candidate profile) $P$ and $P'$ obtained from $P$ by moving $x \in X(P)$ up one place in some voter $i$’s ballot in the sense that for all $y, z \neq x$, $(y, z) \in P'(i)$ iff $(y, z) \in P(i)$, and $|\{z \in X(P') \mid (x, z) \in P'(i)\}| = |\{z \in X(P) \mid (x, z) \in P(i)\}| + 1$, we have $(x, y) \in f(P)$ only if $(x, y) \in f(P')$.

5. $f$ satisfies Neutral Reversal if for any profile $P$ and $P'$ obtained from $P$ by adding two voters whose ballots are converses of each other, we have $f(P) = f(P')$.

Axioms 1-4 are widely satisfied by VCCRs in the literature. While axiom 5 (from Saari 2003) is violated by some prominent VCCRs (e.g., the Plurality VCCR according to which $x$ defeats $y$ iff $x$ receives more first-place votes than $y$, or the Pareto VCCR according to which $x$ defeats $y$ iff every voter strictly prefers $x$ to $y$), it is satisfied by all VCCRs that depend only on the majority margins between candidates (including, e.g., the Borda VCCR, which can be defined in terms of majority margins as in Zwicker 2016, p. 28).

3.2 Coherent IIA

The key axiom in Holliday and Pacuit 2021a is the axiom of Coherent IIA, already discussed informally in Section 1. First we recall Arrow’s (1963) IIA, of which Coherent IIA is a weakening.

Definition 3.2. A VCCR $f$ satisfies Independence of Irrelevant Alternatives (IIA) if for any profiles $P$ and $P'$, if $x$ defeats $y$ in $P$ according to $f$, and $P'\{x,y\} = P'\{x,y\}$, then $x$ defeats $y$ in $P'$ according to $f$.

\[\text{Holliday and Pacuit (2021a) call this principle variable-election IIA (VIIA), since it allows } P \text{ and } P' \text{ to have different sets of candidates, as opposed to fixed-election IIA (FIIA), which requires that } P \text{ and } P' \text{ have the same set of candidates.}\]
Coherent IIA strengthens the assumption of IIA on the relation of $P$ and $P'$ so that not only $P_{\{x,y\}} = P'_{\{x,y\}}$ but also $P'$ is related to $P$ in such a way that $P'$ cannot be more incoherent with respect to $x, y$, in terms of majority cycles and their strengths, than $P$ is with respect to $x, y$.

**Definition 3.3.** For any two profiles $P$ and $P'$ with $x, y \in X(P) \cap X(P')$, let

$$P \xrightarrow{\{x,y\}} P'$$

if $P_{\{x,y\}} = P'_{\{x,y\}}$ and $\mathcal{M}(P')$ can be obtained from $\mathcal{M}(P)$ by deleting zero or more candidates other than $x$ and $y$ and deleting or reducing the margins on zero or more edges not connecting $x$ and $y$.

Note that trivially $P \xrightarrow{\{x,y\}} P'$ only if $V(P') = V(P)$ and $X(P') \subseteq X(P)$; that is, the profiles must have the same set of voters and $P'$ cannot have additional candidates.

**Example 3.4.** Consider the following profile $P$ with three voters $i, j, k$ and four candidates $x, y, z, u$, whose margin graph is shown on the right:

If $Q$ is the restriction of $P$ to just the candidates $x$ and $y$, which is the profile where $i, j,$ and $k$ vote unanimously that $x$ is better than $y$, then trivially $P\xrightarrow{\{x,y\}} Q$. That is,

$$
\begin{array}{ccc}
i & j & k \\
x & u & x \\
y & x & z \\
z & y & u \\
u & z & y \\
\end{array}
\begin{array}{ccc}
i & j & k \\
x & x & x \\
y & y & y \\
z & y & u \\
u & z & y \\
\end{array}
$$

If we consider only changing the relative position of one pair of candidates for one voter, then the only way to produce a different profile $Q$ such that $P \xrightarrow{\{x,y\}} Q$ is to switch $x$ and $z$ in the ballot of $k$. That is, we have

$$
\begin{array}{ccc}
i & j & k \\
x & u & x \\
y & x & z \\
z & y & u \\
u & z & y \\
\end{array}
\begin{array}{ccc}
i & j & k \\
x & u & z \\
y & x & x \\
z & y & u \\
u & z & y \\
\end{array}
$$

with their corresponding margin graphs:
Note that from left to right, no new edges are created, and the only change is that the margin of the edge from \( x \) to \( z \) is reduced from 3 to 1. The reason this swap of \( x \) and \( z \) in the ballot of \( k \) is the only allowed swap if we want a profile \( Q \) such that \( P \xrightarrow{\sim_{x,y}} Q \) is that (1) obviously we cannot swap \( x \) and \( y \), and (2) for any other pair of candidates, the margin between them in the original margin graph is 1 so that a swap would flip the edge. The edges are directed, so flipping edges means creating new majority edges, which is not allowed in the definition of \( \sim_{x,y} \).

We now define Coherent IIA by replacing \( P_{\{x,y\}} = P'_{\{x,y\}} \) in the definition of IIA with \( P \xrightarrow{\sim_{x,y}} P' \).

**Definition 3.5.** A VCCR \( f \) satisfies Coherent IIA if for any profiles \( P \) and \( P' \), if \( x \) defeats \( y \) in \( P \) according to \( f \), and \( P \xrightarrow{\sim_{x,y}} P' \), then \( x \) defeats \( y \) in \( P' \) according to \( f \).

Note a simple consequence of Coherent IIA, known as Weak IIA (cf. Baigent 1987): for any profiles \( P \) and \( P' \), if \( x \) defeats \( y \) in \( P \) according to \( f \) and \( P_{\{x,y\}} = P'_{\{x,y\}} \), then it is not the case that \( y \) defeats \( x \) in \( P' \) according to \( f \). To see that Weak IIA follows from Coherent IIA, note that according to any VCCR satisfying Coherent IIA, if \( x \) defeats \( y \) in \( P \), \( P_{\{x,y\}} = P'_{\{x,y\}} = Q \), and \( y \) defeats \( x \) in \( P' \), then by Coherent IIA and the observation that \( P \xrightarrow{\sim_{x,y}} P_{\{x,y\}} = Q \) and \( P' \xrightarrow{\sim_{y,x}} P'_{\{x,y\}} = Q \), we have that both \( x \) defeats \( y \) and \( y \) defeats \( x \) in \( Q \), contradicting the fact that the defeat relation produced by the VCCR \( f \) must be asymmetric. For extensive discussion of Coherent IIA and its consequences, see Holliday and Pacuit 2021a, Section 4.3. In this paper we focus instead on the two new axioms used in our main result.

An axiom closely related to Coherent IIA that is also discussed in Holliday and Pacuit 2021a is the following axiom of Majority Defeat.

**Definition 3.6.** A VCCR \( f \) satisfies Majority Defeat iff for any profile \( P \) and \( x, y \in X(P) \), if \( (x, y) \in f(P) \), then \( \text{Margin}_P(x, y) > 0 \), i.e., \( x \) defeats \( y \) only if \( x \) is majority preferred to \( y \).

Majority Defeat will be used a few times later.

### 3.3 Coherent Defeat

We come now to the first of our two new axioms.

**Definition 3.7.** A VCCR \( f \) satisfies Coherent Defeat if for any profile \( P \) and \( x, y \in X(P) \), if \( \text{Margin}_P(x, y) > 0 \) and there is no majority cycle containing \( x \) and \( y \) in \( P \) (or equivalently, there is no majority path from \( y \) to \( x \)), then \( (x, y) \in f(P) \).

The key idea behind Coherent Defeat is simple: when there is no incoherence due to majority cycles, majority preference is sufficient for defeat. In other words, majority cycles are the only reason we deviate from majority preference for deciding defeat. One caveat is that we understand incoherence locally: when deciding whether \( x \) defeats \( y \), only majority cycles involving \( x \) and \( y \) matter; thus, regardless of whether there are majority cycles involving other candidates, if there are no majority cycles involving \( x \) and \( y \), then a majority preference for \( x \) over \( y \) is sufficient for \( x \) to defeat \( y \). As mentioned in Section 1, a number of VCCRs (such as Ranked Pairs, Beat Path, and GOCHA, as proved below) together with Split Cycle share this core commitment and thus can all be seen as ways of resolving incoherence due to relevant majority cycles. To put the point in terms of the Advantage-Standard Model of Holliday and Kelley Forthcoming, according to which \( x \) defeats \( y \) once the **advantage** of \( x \) over \( y \) (which depends only on how voters ranks \( x \) versus \( y \)) exceeds the **standard** for \( x \) to defeat \( y \) (which may depend on other information in the profile,
including how voters rank \(x\) against non-\(y\) candidates and \(y\) against non-\(x\) candidates), Coherent Defeat follows assuming the advantage and standard functions satisfy the following weak constraints:

- the advantage of \(x\) over \(y\) is greater than 0 if \(x\) is majority preferred to \(y\);
- the standard for \(x\) to defeat \(y\) is 0 if there is no majority cycle involving \(x\) and \(y\).

The second constraint is clearly necessary if we take the standard for \(x\) to defeat \(y\) to measure in some way local incoherence due to majority cycles involving \(x\) and \(y\).

Coherent Defeat can also be viewed as a principle of “unchallenged defeat” that follows from three principles:

- for \(x\) to defeat \(y\), it is sufficient to find one reason for \(x\) to defeat \(y\) and make sure that no reasons for \(x\) to defeat \(y\) can be challenged;
- a majority preference for \(x\) over \(y\) is a reason for \(x\) to defeat \(y\);
- any challenge to any reason for \(x\) to defeat \(y\) must be based on a majority preference path from \(y\) to \(x\).

For a more concrete model, we can view the margin graph as providing arguments for and against propositions of the form “\(x\) defeats \(y\)” and their negations. There are two types of arguments. First, there is an argument for “\(x\) defeats \(y\)” whenever there is a majority preference for \(x\) over \(y\). Second, when there is a majority path from \(x\) to \(y\), there is an argument for “\(y\) does not defeat \(x\)”, since for each link \(a, b\) in the path we have an argument for “\(a\) defeats \(b\)”, but the defeat relation must be acyclic, so taken together the steps along a majority path constitute an argument for “\(y\) does not defeat \(x\)”. It should be noted here that an argument based on a majority path from \(x\) to \(y\) to the conclusion “\(x\) defeats \(y\)” is not necessarily a good argument, since defeat relations are not obviously transitive. Now if we grant that arguments for and against candidates defeating each other can only be generated in the above two ways, then once we have an \(x\) who is majority preferred to \(y\) and there is no majority path from \(y\) to \(x\), we have an argument for “\(x\) defeats \(y\)” but no counterargument to the contrary. Thus, we should accept that \(x\) defeats \(y\). Similar ideas of treating majority preferences and paths as arguments appear in Heitzig 2002, and our Coherent Defeat is essentially his Immunity to Binary Arguments (Im\(_A\)) applied to VCCRs, with \(A\) as the majority preference relation.

**Proposition 3.8.** Ranked Pairs, Beat Path, and GOCHA as VCCRs satisfy Coherent Defeat.

*Proof.* Ranked Pairs is standardly defined as a VSCC. We first define Ranked Pairs as a VSCC, and then consider a natural VCCR version for this proposition. For any profile \(P\), let \(Pairs(P) = \{(x, y) \in X(P)^2 \mid x \neq y\text{ and }\text{Margin}_x(x, y) \geq 0\}\). Say a tie-breaker for \(P\) is a linear order \(L \in \mathcal{L}(Pairs(P))\). Then we define the \(L\)-ordering \(\succ_{P, L}\) of \(Pairs(P)\) as follows: \((x, y) \succ_{P, L} (x', y')\) iff \(\text{Margin}_x(x, y) > \text{Margin}_x(x', y')\) or \(\text{Margin}_x(x, y) = \text{Margin}_x(x', y')\) and \((x, y)L(x', y')\). Then define \(rp(P, L)\), a linear order on \(X(P)\), as follows:

- List \(Pairs(P)\) according to \(\succ_{P, L}\) as \((x_1, y_1) \succ_{P, L} (x_2, y_2) \succ_{P, L} \cdots \succ_{P, L} (x_m, y_m)\).
- Let \(D_0 = \emptyset\), and for each \(i = 1, \ldots, m\), if \(D_{i-1} \cup \{(x_i, y_i)\}\) is acyclic, let \(D_i = D_{i-1} \cup \{(x_i, y_i)\}\), and otherwise let \(D_i = D_{i-1} \cup \{(y_i, x_i)\}\).
- Let \(rp(P, L) = D_m\).
Intuitively, \(rp(P, L)\) is a defeat relation relative to the tie-breaker \(L\). For any tie-breaker \(L\), \(rp(P, L)\) is a linear ordering of \(X(P)\). The Ranked Pairs VSCC is defined by letting \(RP(P)\) be the set of all \(x \in X(P)\) such that there is a tie-breaker \(L \in \mathcal{L}(Pairs(X(P)))\) and \(x\) is the top element of \(rp(P, L)\). We define the VCCR version of Ranked Pairs by letting \(rp(P) = \bigcap \{rp(P, L) \mid L \in \mathcal{L}(Pairs(X(P)))\}\). That is, \(x\) defeats \(y\) according to the Ranked Pairs iff for all tie-breakers \(L\), \(x\) defeats \(y\) relative to \(L\) according to Ranked Pairs.

Now we show that \(rp\) satisfies Coherent Defeat. Let \(P\) be a profile and \(x, y \in X(P)\) such that \(\text{Margin}_P(x, y) > 0\). Assume there is no majority path from \(y\) to \(x\). Let \(L\) be any tie-breaker. Since \(\text{Margin}_P(x, y) > 0\), for any \((x', y') \succ_{P, L} (x, y)\), \(\text{Margin}_P(x', y') > 0\). Using the notation above, let \(k\) be the position of the pair \((x, y)\) in the ordering \(\succ_{P, L}\). Then for any \(i = 1, \ldots, k\), \(\text{Margin}_P(x_i, y_i) > 0\), and by an easy induction up to \(k\), \(D_i\) is acyclic and the transitive closure of \(D_i\) must also be a subset of the transitive closure of \(\{(x_1, y_1), \ldots, (x_i, y_i)\}\). Since there is no majority path from \(y\) to \(x\), \((y, x)\) is not in the transitive closure of \(\{(x_1, y_1), \ldots, (x_{k-1}, y_{k-1})\}\). This means \(D_{k-1} \cup \{(x, y)\}\) is acyclic, and thus \((x, y) \in D_k\) and \((x, y) \in rp(P, L)\). Since \(L\) was arbitrary, \((x, y) \in rp(P)\).

For Beat Path, recall that for any majority path \(\rho\), the strength \(\text{Strength}(\rho)\) of \(\rho\) is the minimal weight in the majority edges of \(\rho\). For any \(x, y \in X(P)\), let

\[
\text{Strength}_P(x, y) = \max\{\text{Strength}_P(\rho) \mid \rho \text{ is a majority path from } x \text{ to } y \text{ in } P\}
\]

where the max of the empty set is stipulated to be 0. Then Beat Path as a VCCR is defined by setting \((x, y) \in \text{bp}(P)\) iff \(\text{Strength}_P(x, y) > \text{Strength}_P(y, x)\). Now if \(\text{Margin}_P(x, y) > 0\) while there is no majority path from \(y\) to \(x\), then \(\text{Strength}_P(x, y) = 0 = \text{Strength}_P(y, x)\). Thus, \((x, y) \in \text{bp}(P)\).

GOCHA is also usually defined as a VSCC, but the underlying idea also defines a VCCR; it can be seen as a purely qualitative version of Beat Path, not taking the weights into consideration. For any profile \(P\), the GOCHA VCCR \(gocha\) is defined by setting \((x, y) \in gocha(P)\) iff there is a majority path from \(x\) to \(y\) but not from \(y\) to \(x\) in \(P\).\(^{10}\) It is then immediate that \(gocha\) satisfies Coherent Defeat.

\[\]  

3.4 Positive Involvement in Defeat

Our second new axiom for VCCRs is based on the standard axiom of Positive Involvement for VSCCs (Saari 1995, Pérez 2001, Holliday and Pacuit 2021b).

Definition 3.9. A VSCC \(F\) satisfies Positive Involvement if for any profile \(P\), if \(y \in F(P)\) and \(P'\) is obtained from \(P\) by adding one new voter who ranks \(y\) in first place, then \(y \in F(P')\).

Thus, Positive Involvement says that a candidate \(y\)'s winning should be preserved under the addition of a voter who gives \(y\) maximum support by ranking \(y\) at the top of her ballot. As Perez (2001, p. 605) remarks, it can be seen as “the minimum to require concerning the coherence in the winning set when new voters are added.”

Remark 3.10. Another well-known axiom for VSCCs concerning the addition of voters to a profile is the Reinforcement axiom: if \(P\) and \(P'\) are profiles with the same set of candidates but disjoint sets of voters, then \(F(P) \cap F(P') \neq \emptyset\) implies \(F(P + P') = F(P) \cap F(P')\), where \(P + P'\) is the profile combining the voters from \(P\) and \(P'\). Positive Involvement clearly follows from Reinforcement\(^{11}\) together with the axiom of

---

\(^{10}\)This definition is due to Corollary 6.2.2 in Schwartz (1986). The VSCC GOCHA is simply \(gocha\).

\(^{11}\)Even in the weaker version requiring only \(F(P) \cap F'(P') \subseteq F'(P + P')\). Also note that the Homogeneity axiom of Definition 3.1 follows from Reinforcement.
Faithfulness (Young 1974): if \( P \) has only one voter who ranks \( x \) uniquely first, then \( F(P) = \{x\} \). However, Reinforcement is inconsistent with Condorcet Consistency (if some candidate \( x \) has positive margins over all other candidates in \( P \), then \( F(P) = \{x\} \) (see Zwicker 2016, Proposition 2.5) and hence is not satisfied by Split Cycle, which is Condorcet consistent. It is sometimes said that the Kemeny VSCC (Kemeny 1959) satisfies Reinforcement and Consistency, but this is not the case; it is only when Kemeny is regarded as a social preference function (SPF), which assigns to each profile a set of binary relations, that the Kemeny SPF can be regarded as satisfying Reinforcement for SPFs (Young and Levenglick 1978), which is a substantially weaker axiom than Reinforcement for VSCCs (see Zwicker 2016, p. 45). For doubts about the normative plausibility of Reinforcement for VSCCs, see Holliday and Pacuit 2023a, Remark 4.26.

To generalize Positive Involvement to VCCRs, we need to ask what is the “minimum to require concerning the coherence” in the defeat relation when new voters are added. First, there is a trivial way to generalize Positive Involvement to VCCRs.

**Definition 3.11.** An acyclic VCCR \( f \) satisfies **Positive Involvement** if the VSCC \( \overline{f} \) (recall Lemma 2.5) satisfies Positive Involvement: for any profile \( P \) and \( y \in X(P) \), if \( y \) is undefeated in \( P \) according to \( f \), and \( P' \) is obtained from \( P \) by adding one new voter who ranks \( y \) in first place, then \( y \) is still undefeated in \( P' \) according to \( f \).

The problem with this generalization is not that it is unreasonable for a VCCR to satisfy it but rather that we find it reasonable to ask for more. Given a VCCR \( f \), we have information not only about who is defeated or undefeated but also about who is defeated by whom. Thus, there is the question of how the defeat relation between two candidates \( x \) and \( y \) should react to the addition of a voter. A natural answer is this: adding a voter who ranks \( y \) above \( x \) should not lead to \( y \)'s defeat by \( x \).

**Definition 3.12.** A VCCR \( f \) satisfies **Positive Involvement in Defeat** if for any profile \( P \) and \( x, y \in X(P) \), if \( y \) is not defeated by \( x \) in \( P \) according to \( f \), and \( P' \) is obtained from \( P \) by adding one new voter who ranks \( y \) above \( x \), then \( y \) is not defeated by \( x \) in \( P' \) according to \( f \).

This is of course not the only possible answer. For example, we could require not only that \( y \) is ranked above \( x \) in the new ballot but also that \( y \) is ranked at the top.

**Definition 3.13.** A VCCR \( f \) satisfies **First-place Involvement in Defeat** if for any profile \( P \) and \( x, y \in X(P) \), if \( y \) is not defeated by \( x \) in \( P \) according to \( f \), and \( P' \) is obtained from \( P \) by adding one new voter who ranks \( y \) in first place, then \( y \) is not defeated by \( x \) in \( P' \) according to \( f \).

Since the requirement on \( x \) and \( y \) is stronger in First-place Involvement in Defeat, it is entailed by Positive Involvement in Defeat: any VCCR satisfying Positive Involvement in Defeat also satisfies First-place Involvement in Defeat. Moreover, it is easy to see that First-place Involvement in Defeat entails Positive Involvement for VCCRs.

It turns out, as we show in Section 8, that the stronger axiom of Positive Involvement in Defeat is necessary for our characterization of Split Cycle (compared to First-place Involvement in Defeat and hence Positive Involvement), fixing the other axioms we use. However, we do not see this as a problem for Split Cycle as an appealing VCCR, since we find the intuitive appeal of Positive Involvement in Defeat rather clear and perhaps even clearer than First-place Involvement in Defeat. If the only change to a profile in which \( y \) is already undefeated by \( x \) is the addition of a ballot ranking \( y \) above \( x \), thereby lending support to \( y \) against \( x \), then there is no reason to now say that \( x \) defeats \( y \).
**Proposition 3.14.** The Split Cycle VCCR satisfies Positive Involvement in Defeat.

*Proof.* Let \( P \) be a profile and \( x, y \in X(P) \). Suppose \( y \) is not defeated by \( x \). Then either (1) \( \text{Margin}_P(x, y) \leq 1 \) or (2) \( \text{Margin}_P(x, y) \geq 2 \) and there is a majority path \( \rho \) from \( y \) to \( x \) such that \( \text{Margin}_P(x, y) \leq \text{Strength}_P(\rho) \).

Now let \( P' \) be the result of adding to \( P \) one ballot where \( y \) is above \( x \). In case (1), \( \text{Margin}_P'(x, y) = \text{Margin}_P(x, y) - 1 \leq 0 \), so \( x \) still does not defeat \( y \) in \( P' \). In case (2), note that \( \text{Strength}_P(\rho) \geq 2 \), so every edge in \( \rho \) has margin at least 2, and from \( P \) to \( P' \) only one ballot is added, so the weakest edges in \( \rho \) have their margins decrease by at most 1 from \( P \) to \( P' \). Thus, \( \rho \) is still a majority path in \( P' \) from \( y \) to \( x \) and \( \text{Strength}_P(\rho) \geq \text{Strength}_P(\rho) - 1 \geq \text{Margin}_P(x, y) - 1 = \text{Margin}_P'(x, y) \), so \( x \) does not defeat \( y \) in \( P' \). \( \square \)

As another example of working with Positive Involvement in Defeat, we mention that Weighted Covering (Dutta and Laslier 1999, Pérez-Fernández and De Baets 2018) satisfies it. Weighted Covering will also be used in Section 8.

**Definition 3.15.** The Weighted Covering VCCR, denoted \( wc \), is defined as follows: for any profile \( P \) and \( x, y \in X(P), (x, y) \in wc(P) \) iff \( \text{Margin}_P(x, y) > 0 \) and for all \( z \in X(P), \text{Margin}_P(x, z) \geq \text{Margin}_P(y, z) \).

**Proposition 3.16.** Weighted Covering satisfies Positive Involvement in Defeat.

*Proof.* Let \( P \) be a profile and \( x, y \in X(P) \). Suppose \( y \) is not defeated by \( x \). Then either (1) \( \text{Margin}_P(x, y) \leq 0 \) or (2) there is \( z \in X(P) \) such that \( \text{Margin}_P(x, z) < \text{Margin}_P(y, z) \).

Now let \( P' \) be the result of adding to \( P \) one ballot where \( y \) is above \( x \). In case (1), \( \text{Margin}_P'(x, y) < 0 \) since the new ballot decreases the margin from \( x \) to \( y \) by 1, and thus \( y \) is still undefeated by \( x \). In case (2), since in the new ballot, \( y \succ x \), there are only three possible places for \( z \) relative to \( y \) and \( x \): \( z \succ y \succ x \), \( y \succ z \succ x \), and \( y \succ x \succ z \). In the first and the third case, \( \Delta_{x,z} = \text{Margin}_P(x, z) - \text{Margin}_P(x, z) \) and \( \Delta_{y,z} = \text{Margin}_P(y, z) - \text{Margin}_P(y, z) \) are the same (−1 and 1 in the first and third cases, respectively), and in the second case, \( \Delta_{x,z} = -1 < \Delta_{y,z} = 1 \). Thus, in all three cases, since \( \Delta_{x,z} \leq \Delta_{y,z} \), we still have \( \text{Margin}_P'(x, z) < \text{Margin}_P'(y, z) \), and \( y \) is still not defeated by \( x \) in \( P' \). \( \square \)

On the other hand, the many non-weighted versions of Covering (see Gillies 1959, Fishburn 1977, Miller 1980, Duggan 2013) do not satisfy Positive Involvement in Defeat. For example, for any profile \( P \) and \( x, y \in X(P) \), we say that \( x \) defeats \( y \) according to the Right Covering VCCR when \( \text{Margin}_P(x, y) > 0 \) and for all \( z \in X(P) \), if \( \text{Margin}_P(y, z) > 0 \) then \( \text{Margin}_P(x, z) > 0 \). To see that the Right Covering VCCR fails Positive Involvement in Defeat, consider any profile \( P \) whose margin graph is

![Diagram](attachment:diagram.png)

By definition, \( x \) does not defeat \( y \) in \( P \) according to Right Covering because of \( z \). However, once we add a ballot \( y \succ x \succ z \) to \( P \), the margin graph becomes

![Diagram](attachment:diagram.png)
and now $x$ defeats $y$ according to Right Covering, since $x$ now also has a positive margin over $z$.

For another example of a VCCR violating Positive Involvement in Defeat, consider the following VCCR that defeat-rationalizes the Minimax VSCC (Simpson 1969, Kramer 1977). For any profile $P$ and $x \in X(P)$, define

$$\text{Weakness}_P(x) = \max\{\text{Margin}_P(z, x) \mid z \in X(P)\}.$$ 

Then the VCCR $mm$ is defined by setting $(x, y) \in mm(P)$ iff $\text{Weakness}_P(x) < \text{Weakness}_P(y)$. Now consider any profile $P$ whose margin graph is shown on the left below, and suppose $P'$ is obtained by adding to $P$ a ballot $b > y > x > a$, resulting in the margin graph on the right:

![Margin Graph](image)

It is clear that $\text{Weakness}_P(x) = 2 = \text{Weakness}_P(y)$, but $\text{Weakness}_{P'}(x) = 1 < 3 = \text{Weakness}_{P'}(y)$. In other words, $(x, y) \notin mm(P)$, but $(x, y) \in mm(P')$, while the only change from $P$ to $P'$ is that a new voter ranking $y$ above $x$ joined the election. What is notable about this failure of Positive Involvement in Defeat is that Minimax as a VSCC does satisfy the axiom of Tolerant Positive Involvement that we will introduce in Section 5.4 for VSCCs that corresponds to Positive Involvement in Defeat.

One may naturally wonder why we focus solely on Positive Involvement, given that there is also the axiom of Negative Involvement in the literature (again see Pérez 2001).

**Definition 3.17.** A VSCC $F$ satisfies **Negative Involvement** if for any profile $P$ and $y \in X(P)$, if $y \notin F(P)$ and $P'$ is obtained by adding one new voter who ranks $y$ in last place, then $y \notin F(P')$.

An acyclic VCCR $f$ satisfies **Negative Involvement** if $\overline{f}$ satisfies Negative Involvement. We say that a VCCR $f$ satisfies **Negative Involvement in Defeat** if for any profile $P$ and $x, y \in X(P)$, if $(x, y) \in f(P)$, and $P'$ is obtained from $P$ by adding one new voter who ranks $x$ above $y$, then $(x, y) \in f(P')$.

The reason we can focus solely on Positive Involvement is given by the following proposition.

**Proposition 3.18.** An acyclic VCCR (resp. VCCR) $f$ satisfying Neutral Reversal satisfies Positive Involvement (resp. Positive Involvement in Defeat) iff it satisfies Negative Involvement (resp. Negative Involvement in Defeat).

**Proof.** Let $f$ be an acyclic VCCR $f$ satisfying Neutral Reversal. Suppose $f$ fails Positive Involvement. Then we have a profile $P$, $x \in \overline{f}(P)$, and $P'$ that adds to $P$ a ballot $L$ ranking $x$ in first place and yet $x \notin \overline{f}(P')$. Let $P''$ be any profile that adds to $P'$ the converse of the ballot $L$, which ranks $x$ in last place. Then $P''$ is the result of adding to $P$ a ballot of full reversal. So by Neutral Reversal, $x \in \overline{f}(P'')$. But then the pair $P'$ and $P''$ witnesses the failure of Negative Involvement for $\overline{f}$ and also $f$. Clearly, the same strategy of adding the converse of $L$ works for the other direction and also works for showing that a VCCR $f$ satisfying Neutral Reversal satisfies Positive Involvement in Defeat iff it satisfies Negative Involvement in Defeat.

Thus, as long as we are focusing on VCCRs satisfying Neutral Reversal, and in particular on margin-based VCCRs such as Split Cycle, there is no loss of generality in focusing just on the positive axioms.
4 Characterization of the Split Cycle VCCR

We are now ready to prove our first main result, an axiomatic characterization of the Split Cycle VCCR.

**Theorem 4.1.** The Split Cycle VCCR is the unique VCCR satisfying Anonymity, Neutrality, Availability, (Upward) Homogeneity, Monotonicity (for two-candidate profiles), Neutral Reversal, Coherent IIA, Coherent Defeat, and Positive Involvement in Defeat.

Holliday and Pacuit have already proved half of what we need.

**Theorem 4.2** (Holliday and Pacuit 2021a). If \( f \) is a VCCR satisfying Anonymity, Neutrality, Availability, (Upward) Homogeneity, Monotonicity (for two-candidate profiles), Neutral Reversal, and Coherent IIA, then the Split Cycle VCCR refines \( f \): for any profile \( P \), \( \text{sc}(P) \supseteq f(P) \).

It remains to prove that the axioms in Theorem 4.1 force \( f \) to refine Split Cycle, in which case \( f = \text{sc} \).

In fact, we will prove that just Coherent Defeat and Positive Involvement in Defeat together do so. First we recall the following well-known extension lemma.

**Lemma 4.3.** Any acyclic binary relation can be extended to a strict linear order.

**Proof.** If \( R \) is acyclic, then the reflexive and transitive closure \( R^* \) of \( R \) is a partial order. Applying the Szpilrajn extension theorem (Szpilrajn 1930) to \( R^* \), we obtain a total order \( S \) extending \( R^* \) and hence \( R \). The asymmetric or equivalently the irreflexive part of \( S \) is a strict linear order extending \( R \). (Note that being acyclic, \( R \) is automatically asymmetric and irreflexive.) \( \square \)

Next comes the key lemma. The formulation is slightly cumbersome, and it actually proves more than what we need for Theorem 4.1. However, it is precisely what we need for characterizing Split Cycle as a VSCC in Section 6.

**Lemma 4.4.** For any profile \( P \) and \((x, y) \in \text{sc}(P)\), if \( \text{Margin}_P(x, y) > 2 \), then there is a ballot \( L \in \mathcal{L}(X(P)) \) such that

- for any \( z \in X(P) \setminus \{y\} \) with \( \text{Margin}_P(y, z) \leq 0 \), we have \( (y, z) \in L \) (in particular, \( (y, x) \in L \)), and
- \((x, y) \in \text{sc}(P + L)\).

In other words, if \( x \) defeats \( y \) according to Split Cycle by a sufficient margin (more than 2), then \( x \) can still defeat \( y \) after the addition of a specifically chosen ballot in which \( y \) is ranked very high in the sense that \( y \) is ranked above all candidates that it does not beat head-to-head, including \( x \) since \( y \) is defeated by \( x \) according to Split Cycle.

For the proof of Lemma 4.4, recall that in a graph containing vertices \( y \) and \( x \), a cut from \( y \) to \( x \) is a set of edges such that every path from \( y \) to \( x \) contains an edge from the set. Equivalently, a cut from \( y \) to \( x \) is a set of edges whose removal results in the loss of reachability of \( x \) from \( y \). The main idea of the proof is illustrated in Figure 3.

**Proof.** Let \( \mathcal{M} \) be the margin graph of \( P \) and \( k = \text{Margin}_P(x, y) \). Since \((x, y) \in \text{sc}(P)\), by Lemma 2.12 we know that \( x \) is not reachable from \( y \) in \( \mathcal{M}|k \). This means that the set of edges in \( \mathcal{M} \) of weight at most \( k - 1 \) is a cut from \( y \) to \( x \). Thus, there is a minimal cut \( C \) (minimal in the subset ordering) from \( y \) to \( x \) consisting only of edges of weight at most \( k - 1 \), since the graph is finite. By the minimality of \( C \), in the graph \( \mathcal{M} \setminus C \) resulting from removing the edges in \( C \) from \( \mathcal{M} \), adding back any edge of \( C \) reestablishes the reachability from \( y \) to \( x \). In other words,
for any edge \((u, v) \in C\), there is a path from \(y\) to \(u\) disjoint from \(C\), and there is a path from \(v\) to \(x\) disjoint from \(C\).

From this, it follows that there are no connecting pairs of edges from \(C\): there are no \(u, v, w\) such that \((u, v)\) and \((v, w)\) are both in \(C\), since otherwise there is a path from \(v\) to \(x\) and a path from \(y\) to \(v\), both disjoint from \(C\), forming a path from \(y\) to \(x\) disjoint from \(C\), contradicting that \(C\) is a cut from \(y\) to \(x\).

Now let \(B = \{(y, z) \in \{y\} \times (X \setminus \{y\}) \mid \text{Margin}_\rho(y, z) \leq 0\}\) (for the particular \(y\) given above). Note that \(B\) is also the set \(\{(y, z) \in \{y\} \times (X \setminus \{y\}) \mid (y, z) \notin \mathcal{M}(\rho)\}\). Now we show that \(C^{-1} \cup B\) is acyclic.

Clearly there is no reflexive loop, and also there cannot be any cycle completely inside \(C^{-1}\) since we have shown that \(C\) and hence also \(C^{-1}\) do not have connecting pairs of edges. Thus, if there is a cycle \(\rho\), it must contain an edge \((y, z)\) from \(B\). Let the next edge in \(\rho\) be \((z, u)\). Since \(z\) is not \(y\), this \((z, u)\) is not in \(B\) and must be in \(C^{-1}\). Now there are two cases. First, if \(u = y\), then we have \((z, y) \in C^{-1}\) and thus \((y, z) \in C\). But \((y, z)\) is also in \(B\), and by definition, \((y, z) \notin \mathcal{M}(\rho)\). All edges in \(C\) are in \(\mathcal{M}(\rho)\), however, so we have a contradiction in this case. Second, if \(u \neq y\), then consider the next edge \((u, v)\) in \(\rho\). Since \(u \neq y\), \((u, v) \notin B\), so \((u, v) \in C^{-1}\). But then we have both \((z, u)\) and \((u, v)\) in \(C^{-1}\), which is impossible since \(C\) does not have connecting pairs of edges. Thus, \(C^{-1} \cup B\) is acyclic.

Let \(L\) be any strict linear order in \(\mathcal{L}(X(\rho))\) extending \(C^{-1} \cup B\) by Lemma 4.3. This is the ballot required by the lemma. Since \(L\) extends \(B\), it satisfies the first requirement, and in particular, \((y, x) \in L\). Let \(\rho’ = \rho + L\) and \(\mathcal{M’} = \mathcal{M}(\rho’)\). Now we only need to show that \((x, y) \in \text{sc}(\rho’)\). Since \((y, x) \in L\), \(\text{Margin}_{\rho’}(x, y) = k - 1\). Since for each \((u, v) \in C\), \(\text{Margin}_\rho(u, v) \leq k - 1\) and \((v, u) \in L\), \(\text{Margin}_{\rho’}(u, v) \leq k - 2\). Let \(E_{\leq k-2}\) be the set of edges in \(\mathcal{M}\) of weight at most \(k - 2\). Then \(C \subseteq E_{\leq k-2}\). Note also that there may be edges in \(\mathcal{M’}\) but not \(\mathcal{M}\). However, if \((u, v)\) is in \(\mathcal{M’}\) but not in \(\mathcal{M}\), then since from \(\rho\) to \(\rho’\) only one ballot is added, \(\text{Margin}_{\rho’}(u, v) = 1\). Hence every edge in \(\mathcal{M’}\) but not in \(\mathcal{M}\) is also in \(E_{\leq k-2}\) since \(k > 2\) and hence \(k - 2 \geq 1\). Now it is easy to see that \(E_{\leq k-2}\) is a cut from \(y\) to \(x\) in \(\mathcal{M’}\): for any path \(\rho\) from \(y\) to \(x\) in \(\mathcal{M}\), if it uses only edges in \(\mathcal{M}\), then \(\rho\) intersects \(C\) and hence \(E_{\leq k-2}\); if \(\rho\) uses a new edge in \(\mathcal{M’}\) but not \(\mathcal{M}\), then since the new edges are all in \(E_{\leq k-2}\), \(\rho\) intersects \(E_{\leq k-2}\). Thus, there is no path from \(y\) to \(x\) in \(\mathcal{M’}\) since \(k - 1\), and so \((x, y) \in \text{sc}(\rho’)\).

Using Lemma 4.4, we can now prove that just two of our axioms force \(f\) to refine Split Cycle. The idea
is to use Lemma 4.4 repeatedly so that defeat can be decided merely by Coherent Defeat. An example is shown in Figure 4.

**Theorem 4.5.** If \( f \) is a VCCR satisfying Coherent Defeat and Positive Involvement in Defeat, then \( f \) refines the Split Cycle VCCR: for any profile \( P \), \( f(P) \supseteq sc(P) \).

**Proof.** Pick an arbitrary profile \( P \) and \( x, y \in X(P) \) such that \( (x, y) \in sc(P) \). We only need to show that \( (x, y) \in f(P) \). Let \( k = \text{Margin}_P(x, y) \) and \( M = \mathcal{M}(P) \). Now if \( k \leq 2 \), then it is easy to see that there cannot be a majority path from \( y \) to \( x \), as then by the parity constraint from Lemma 2.13, any majority path has strength at least \( k \) and hence \( x \) cannot defeat \( y \) according to Split Cycle. So if \( k \leq 2 \), we already have \( (x, y) \in f(P) \) by Coherent Defeat. If \( k > 2 \), then we inductively define \( P_0, P_1, P_2, \ldots, P_{k-2} \) and \( L_1, L_2, \ldots, L_{k-2} \) where \( P_0 = P \), \( P_{i+1} = P_i + L_{i+1} \), and \( L_{i+1} \) is obtained by applying Lemma 4.4 to \( P_i \). By the lemma, (1) \( (y, x) \) is in each \( L_i \), and (2) in each \( P_i \), \( (x, y) \in sc(P_i) \). As the margin from \( x \) to \( y \) decreases by \( k - 2 \) times in this sequence, \( (x, y) \in sc(P_{k-2}) \) but \( \text{Margin}_{P_{k-2}}(x, y) = 2 \). This means that there is no majority path from \( y \) to \( x \). Thus, by Coherent Defeat, \( (x, y) \in f(P_{k-2}) \). Finally, by Positive Involvement in Defeat in its contrapositive form, if \( (x, y) \notin f(P_{i+1}) \), then \( (x, y) \notin f(P_i) \), since \( x \) is ranked below \( y \) in \( L_{i+1} \). Thus, by an induction from \( k - 2 \) back to 0, \( (x, y) \in f(P_0) = f(P) \). \( \square \)

![Figure 4: An example of eliminating all paths from y to x by adding ballots that put y in a sufficiently high position. For each of the first two margin graphs, a minimal cut from y to x is highlighted by thickened arrows. There are many ways to eliminate all paths from y to x, and the key is to do this quickly before the positive margin from x to y runs out.](image)

Combining Theorems 4.5 and 4.2, any VCCR satisfying the axioms in Theorem 4.1 refines and is refined by Split Cycle, so it is equal to Split Cycle. This completes the proof of Theorem 4.1.

## 5 Axioms on VSCCs

In this section, we characterize Split Cycle as a VSCC. As before, we first introduce the standard axioms (5.1). Then we define Coherent IIA for VSCCs (5.2). Care must be taken here as existing IIA-like axioms for SCCs have implicit commitments irrelevant to the spirit of IIA. Next we define Coherent Defeat for VSCCs (5.3), the obvious analogue of Coherent Defeat for VCCRs, and finally a new axiom of Tolerant Positive Involvement (5.4), a proper strengthening of Positive Involvement necessary for our characterization.

### 5.1 Standard axioms

Adapting the standard axioms for VCCRs from Section 3.1 to VSCCs, we obtain the following.

**Definition 5.1.** Let \( F \) be a VSCC.
1. $F$ satisfies **Anonymity** if for any profiles $P$ and $P'$, if $V(P) = V(P')$ and there is a bijection $\pi$ from $V(P)$ to $V(P')$ such that for any $i \in V(P)$, $P'(i) = P(\pi(i))$, then $F(P) = F(P')$; and $F$ satisfies **Neutrality** if for any profiles $P$ and $P'$, if $V(P) = V(P')$, $X(P) = X(P')$, and there is a bijection $\pi$ from $X(P)$ to $X(P')$ such that for any $i \in V(P)$ and $x, y \in X(P)$, $(x, y) \in P(i)$ iff $(\pi(x), \pi(y)) \in P'(i)$, then for any $x \in X(P)$, $x \in F(P)$ iff $\pi(x) \in F(P')$.

2. $F$ satisfies **Homogeneity** (resp. **Upward Homogeneity**) if for any profile $P$ and $2P$, where $2P$ is the result of replacing each voter in $P$ by 2 copies of that voter, we have $F(P) = F(2P)$ (resp. $F(P) \supseteq F(2P)$).

3. $F$ satisfies **Monotonicity** (resp. **Monotonicity for two-candidate profiles**) if for any profile (resp. two-candidate profile) $P$ and $P'$ obtained from $P$ by moving $x \in X(P)$ up one place in some voter’s ballot (see Definition 3.1 for a precise formulation), we have $x \in F(P)$ only if $x \in F(P')$.

4. $F$ satisfies **Neutral Reversal** if for any profile $P$ and $P'$ obtained from $P$ by adding two voters whose ballots are converses of each other, we have $F(P) = F(P')$.

Observe that we no longer have the Availability axiom for VSCCs, since by definition they must return a non-empty set of winners.

### 5.2 Coherent IIA

We now turn to the crucial question of how to formulate the analogue for VSCCs of Coherent IIA for VCCRs. First, we must ask: what is the analogue for VSCCs of Arrow’s IIA for VCCRs? One answer is the following, adopting terminology of Denicolo (2000), based on Hansson 1969.

**Definition 5.2.** A VSCC $F$ satisfies **Hansson’s Pairwise Independence** (HPI) if for any profiles $P$ and $P'$ with $x, y \in X(P)$, if $x \in F(P)$, $y \notin F(P)$, and $P'_{\{x,y\}} = P_{\{x,y\}}$, then $y \notin F(P')$.

The problem with this proposal, simply put, is this: who said $x$ is the candidate who defeats $y$ in $P$? If we knew on the basis of $x \in F(P)$ and $y \notin F(P)$ that $x$ defeats $y$ in $P$, then we could indeed conclude that $x$ still defeats $y$ in $P'$, so $y \notin F(P')$. But it does not follow (without assumptions beyond IIA) from $x \in F(P)$ and $y \notin F(P)$ that $x$ in particular is the candidate who defeats $y$ in $P$; all that follows it that $x$ is undefeated and $y$ is defeated by someone or other. HPI in effect assumes that $F$ is rationalized by a variable-election social welfare function (see Denicolo 1993, Theorem 1), i.e., a VCCR $f$ for which $f(P)$ is always a strict weak order, in which case $x \in F(P)$ and $y \notin F(P)$ do imply that $x$ defeats $y$. Thus, HPI smuggles in an additional “social rationality” assumption, which should not be part of a pure independence condition. But we can fix this problem with the following weaker definition.

**Definition 5.3.** A VSCC $F$ satisfies **Pure IIA** if for any profile $P$ and $y \in X(P)$, if $y \notin F(P)$, then there is an $x \in X(P)$ such that for any profile $P'$ with $P'_{\{x,y\}} = P_{\{x,y\}}$, we have $y \notin F(P')$.

The following proposition verifies that Pure IIA is the correct analogue of IIA for VSCCs.

**Proposition 5.4.** For any VSCC $F$, the following are equivalent:

1. $F$ satisfies Pure IIA;

2. $F$ is defeat-rationalized by a VCCR satisfying IIA.

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12 Equivalently: $X(P) \setminus F(P) \subseteq X(P) \setminus F(2P)$. In short, Upward Homogeneity for VSCCs means losing is preserved under doubling the profile.
Definition 5.9. A VSCC characterizing Split Cycle. For the Split Cycle VCCR, we used the strengthening of Positive Involvement in As we noted when we first introduced Positive Involvement in Section 3.4, a strengthening is required for preference relation, only phrased in our variable-election setting.

Translating Coherent Defeat from VCCRs to VSCCs is straightforward.

5.3 Coherent Defeat

Proposition 5.7. For any VCCR \( f \) satisfying Coherent IIA, \( \bar{f} \) also satisfies Coherent IIA.

Due to Definition 5.5 being largely a translation, we immediately have the following.

Remark 5.6. Dong (2021, Theorem 12) arrives at a similar axiom (dubbed crucial defeat) for VSCCs inspired by Coherent IIA for VCCRs, which Dong uses to characterize Split Cycle as the VSCC satisfying several axioms and refining all VSCCs satisfying those axioms, as Theorem 4.2 did for VCCRs. We will give our analogue of Theorem 4.2 for VSCCs in Theorem 6.4 below.

Definition 5.5. A VSCC \( F \) satisfies Coherent IIA if for any profile \( P \) and \( y \in X(P) \), if \( y \notin F(P) \), then there is \( x \in X(P) \) such that for any profile \( P' \) with \( P \sim_{x,y} P' \), we have \( y \notin F(P') \).

5.3 Coherent Defeat

5.4 Tolerant Positive Involvement

As we noted when we first introduced Positive Involvement in Section 3.4, a strengthening is required for characterizing Split Cycle. For the Split Cycle VCCR, we used the strengthening of Positive Involvement in Defeat. For the Split Cycle VSCC, we use the following.

Definition 5.9. A VSCC \( F \) satisfies Tolerant Positive Involvement if for any profile \( P \), if \( x \in F(P) \) and \( P' \) is obtained from \( P \) by adding one new voter who ranks \( x \) above every other candidate \( y \) such that \( x \) is not majority preferred to \( y \) in \( P \), then \( x \in F(P') \).
We use the word ‘tolerant’ since Tolerant Positive Involvement is applicable more broadly than Positive Involvement; it says that for a winner \( x \)'s winning to be preserved under the addition of a ballot \( L \), \( x \) can tolerate being ranked below some candidates as long as \( x \) is ranked high enough: \( x \) is above all those \( y \) to which \( x \) is not majority preferred before adding \( L \). In other words, if \( L \) weakens all potential weaknesses of \( x \) in the sense that \( L \) decreases all non-negative margins over \( x \), then adding \( L \) will not cause \( x \) to lose. In contrast, Positive Involvement guarantees the preservation of \( x \)'s winning only for ballots that weaken all margins over \( x \) (by putting \( x \) at the top), not just the non-negative margins.

**Example 5.10.** To illustrate the difference between Positive Involvement and Tolerant Positive Involvement, let us consider Instant Runoff Voting—in particular, the version (as in Taylor and Pacelli 2008, p. 7) where at each stage, all candidates with the fewest first-place votes are eliminated, unless all candidates would thereby be eliminated. Consider the following anonymized profile \( P \):

\[
\begin{array}{ccc}
3 & 4 & 2 \\
x & y & z \\
y & x & x \\
z & z & y \\
\end{array}
\]

According to Instant Runoff, \( x \) is the winner in \( P \). It is also not hard to see that Instant Runoff satisfies Positive Involvement, and in this particular case, if we add a voter whose ballot puts \( x \) at the top, then \( x \) will remain the sole winner as \( z \) and \( y \) will still be eliminated consecutively. Now note that the margin graph of \( P \) is

![Margin Graph](image)

This shows that \( x \) is the Condorcet winner, the candidate who is majority preferred to every other candidate. In other words, there are no other candidates to whom \( x \) is not majority preferred. This makes the precondition for Tolerant Positive Involvement trivial, and for Tolerant Positive Involvement to hold for Instant Runoff, \( x \) should remain a winner no matter what ballot is added to \( P \). But this is not the case. Let \( Q \) be the result of adding to \( P \) a new voter whose ballot is \( z \succ x \succ y \):

\[
\begin{array}{ccc}
3 & 4 & 3 \\
x & y & z \\
y & x & x \\
z & z & y \\
\end{array}
\]

According to Instant Runoff, \( x \) is eliminated in the first round and therefore does not win in \( Q \). Thus, Instant Runoff satisfies Positive Involvement but not Tolerant Positive Involvement.

Since Tolerant Positive Involvement is a strengthening of Positive Involvement, which many Condorcet methods violate (see Pérez 2001), and has a majoritarian component, which scoring rules like Borda are unlikely to respect, it is perhaps not surprising that many well-known VSCCs fail Tolerant Positive Involvement. In fact, we know of only two standard VSCCs other than Split Cycle that satisfy the axiom, namely Minimax and Weighted Covering as a VSCC.
Proposition 5.11. The Minimax VSCC $MM$, defined by

$$MM(P) = \{ x \in X(P) \mid \forall y \in X(P), \text{Weakness}_P(x) \leq \text{Weakness}_P(y) \},$$

satisfies Tolerant Positive Involvement.

Proof. The idea is similar to that in the proof showing that Weighted Covering satisfies Positive Involvement in Defeat (Proposition 3.16). Let $P$ be any profile, $x \in MM(P)$, and $P'$ the result of adding to $P$ a single voter who ranks $x$ above all $y \in X(P)$ such that $\text{Margin}_P(y, x) \geq 0$. Then observe that $\text{Weakness}_P(x)$ is either 0 or $\text{Weakness}_P(x) - 1$. The weakness of any candidate from $P$ to $P'$ can decrease by at most 1 and must be at least 0. Thus $\text{Weakness}_P(x)$ is still the smallest possible in $P'$.

The following proposition relates Tolerant Positive Involvement to Positive Involvement in Defeat, which can be used to show that Split Cycle and Weighted Covering as VSCCs satisfy Tolerant Positive Involvement. For this we recall the axiom of Majority Defeat (Definition 3.6): for any profile $P$ and $x, y \in X(P)$, if $(x, y) \in f(P)$, then $\text{Margin}_P(x, y) > 0$, i.e., $x$ defeats $y$ only if $x$ is majority preferred to $y$.

Proposition 5.12. If an acyclic VCCR $f$ satisfies Positive Involvement in Defeat and Majority Defeat, then $f$ satisfies Tolerant Positive Involvement. In particular, the Split Cycle VSCC and the Weighted Covering VSCC $WC = wc$ satisfy Tolerant Positive Involvement since both $sc$ and $wc$ satisfy Positive Involvement in Defeat and Majority Defeat.

Proof. Let $f$ be an acyclic VCCR satisfying Positive Involvement in Defeat and Majority Defeat. Let $P$ be a profile, $x \in F(P)$, and $P'$ a profile obtained by adding to $P$ a new voter with ballot $L$. Moreover, assume that for any $y \in X(P) \setminus \{x\}$, if $\text{Margin}_P(y, x) \geq 0$, then $xLy$. Our goal is to show that $x \in f(P')$, i.e., that for any $y \in X(P) \setminus \{x\}$, we have $(y, x) \notin f(P')$. Now there are two cases: either $\text{Margin}_P(y, x) \geq 0$ or $\text{Margin}_P(y, x) < 0$. In the former case, by the assumption on $L$, $xLy$. Also, since $x \in f(P)$, we have $(y, x) \notin f(P)$. So $(y, x) \notin f(P')$ since $f$ satisfies Positive Involvement in Defeat. In the latter case, since only one new voter is added, $\text{Margin}_P(y, x) \leq 0$. Then $(y, x) \notin f(P')$ by Majority Defeat.

6 Characterization of the Split Cycle VSCC

We are now ready to axiomatically characterize Split Cycle as a VSCC.

Theorem 6.1. The Split Cycle VSCC is the unique VSCC satisfying Anonymity, Neutrality, (Upward) Homogeneity, Neutral Reversal, Monotonicity (for two-candidate profiles), Coherent IIA, Coherent Defeat, and Tolerant Positive Involvement.

Unlike the situation with the Split Cycle VCCR where we can directly use the theorem in Holliday and Pacuit (2021a) to finish one direction of the theorem, we need to prove both directions specifically for the VSCC. On the face of it, this can be explained by the fact that the standard axioms, when stated for VSCCs, are weaker than their counterparts for VCCRs, since a VSCC carries less information than a VCCR, and the standard axioms involve preservation of information over profiles. For example, Upward Homogeneity for VSCCs says that losing is preserved under doubling profiles. However, one could be losing for different reasons, i.e., by being defeated by different candidates. Upward Homogeneity for VCCRs also preserves the defeaters of the losers by preserving the whole defeat relation, while Upward Homogeneity for VSCCs cannot track the defeaters since a VSCC does not carry the full information of defeat. It is an interesting question
when we can recover from a VSCC a VCCR that defeat rationalizes the VSCC while satisfying various
natural preservation axioms for the defeat relation, such as IIA, Coherent IIA, and Positive Involvement in
Defeat. We leave this for future work.

We start with the direction that uses the standard axioms. First, by a standard argument using sym-
meteries, Anonymity, Neutrality, and Monotonicity for two-candidate profiles, the loser (if there is one) is majority dispreferred. This will be used a few times later and is
already used in Holliday and Pacuit 2021a, Lemma 4.2, so we formally state it here.

Definition 6.2. A VCCR \( f \) satisfies Majority Defeat for two-candidate profiles if for any profile \( P \) with
\( X(P) = \{x, y\} \) and \( x \neq y \), if \( xf(P)y \) then \( \text{Margin}_P(x, y) > 0 \).

Let \( \text{VCCR} \) and \( \text{VSCC} \) be two VCRs. Let \( f \) be a VCCR that defeat rationalizes the
VSCC while satisfying various natural preservation axioms for the defeat relation, such as IIA, Coherent IIA, and Positive Involvement in
Defeat. We leave this for future work.

Now we prove the analogue for VSCCs of Theorem 4.2.

Theorem 6.4. Let \( SC \) be the Split Cycle VSCC and \( F \) a VSCC satisfying Anonymity, Neutrality, (Upward)
Homogeneity, Neutral Reversal, Monotonicity (for two-candidate profiles), and Coherent IIA. Then \( SC \)
refines \( F \): for any profile \( P \), \( SC(P) \subseteq F(P) \).

Proof. The proof is only slightly different than the corresponding one in Holliday and Pacuit 2021a. Let
\( P \) be any profile and \( y \in SC(P) \). Now we need to show that \( y \in F(P) \). Using Upward Homogeneity,
\( F(P) \supseteq F(2P) \). Hence we only need to show that \( y \in F(2P) \). Since \( SC(P) = SC(2P) \), for notational
convenience, let us assume from now on that \( P \) has an even number of voters and \( y \in SC(P) \).

Let \( M \) be the largest margin of \( P \). Since \( P \) is a linear profile, \( M \) is even. Let \( P' \) be the result of adding
to \( P \) enough pairs of voters with converse ballots so that \( |V(P')| \geq M \cdot |X(P)| \). Since adding these pairs
of voters does not change the margins or the number of candidates, \( M \) is also the largest margin of \( P' \) and
\( |V(P')| \geq M \cdot |X(P')| \).

By Neutral Reversal, it is enough to show that \( y \in F(P') \). Toward a contradiction, suppose that
\( y \notin F(P') \). Then by Coherent IIA, there is an \( x \in X(P') = X(P) \) such that for any profile \( Q' \) with
\( P' \sim_{x,y} Q' \) \( y \notin F(Q') \). Consider the two-candidate profile \( Q' = P'(\{x,y\}) \). Clearly \( P' \sim_{x,y} Q' \), and by
assumption \( y \notin F(Q') \). Thus, by Lemma 6.3, \( \text{Margin}_{P'}(x, y) = \text{Margin}_{Q}(x, y) > 0 \).

Let \( m = \text{Margin}_{P'}(x, y) > 0 \), which is even since \( |V(P')| \) is even. Since \( y \in SC(P') \), there must be a
majority cycle \((x, y, z_1, z_2, \ldots, z_n, x)\) such that all the margins of the consecutive edges are at least \( m \). Now
construct \( Q \) and \( Q' \) as in Holliday and Pacuit 2021a:

\begin{itemize}
  \item First, take \( m \) voters in \( P' \) who rank \( x \) above \( y \), and let half of them vote \( x \succ y \succ z_1 \succ z_2 \succ \cdots \succ z_n \)
and the other half vote \( z_n \succ z_{n-1} \succ \cdots \succ z_1 \succ x \succ y \). Since \( m = \text{Margin}_{P'}(x, y) = \text{Margin}_{P}(x, y) \),
the required \( m \) voters can be found.

  \item The rest of the voters can then be split evenly according to whether they rank \( x \) above \( y \) or \( y \) above
\( x \). So take \( m/2 \) fresh voters in \( P' \) ranking \( y \) above \( x \) and let them vote \( y \succ z_1 \succ z_2 \succ \cdots \succ z_n \succ x \succ y \succ z_1 \). Then take \( m/2 \) fresh voters in \( P' \)
ranking \( x \) above \( y \) and let them vote \( x \succ z_n \cdots \succ z_2 \succ y \succ z_1 \succ x \).

  \item For any \( i = 1, 2, \ldots, n-1 \), first take \( m/2 \) fresh voters in \( P' \) ranking \( x \) above \( y \) and let them vote
\( z_i \succ z_{i+1} \succ z_{i+2} \succ \cdots \succ z_n \succ x \succ y \succ z_1 \cdots \succ z_{i-1} \succ z_1 \). Then take \( m/2 \) fresh voters in \( P' \)
ranking \( y \) above \( x \) and let them vote \( z_{i-1} \succ \cdots \succ z_1 \succ y \succ x \succ z_n \cdots \succ z_{i+2} \succ z_i \succ z_{i+1} \).
\end{itemize}
Let $\Pi A, y \in m$ consecutive edge's margin being $P$. Observe now that $\Sigma$ of candidates with $\sigma(x) = y, \sigma(y) = z_1, \sigma(z_i) = z_{i+1},$ and $\sigma(z_n) = x,$ then applying $\sigma$ to $Q$ results in the same anonymized profile. See Figure 6. This contradicts the assumption that $F$ is a VSCC.

Now we prove the other direction.

**Theorem 6.5.** Let $SC$ be the Split Cycle VSCC and $F$ any VSCC satisfying Coherent Defeat and Tolerant Positive Involvement. Then $F$ refines $SC$: for any profile $P, F(P) \subseteq SC(P)$. 

\[\begin{array}{cccc|cccc|c}
\hline
 m/2 & m/2 & m/2 & m/2 & m/2 & m/2 & m/2 & m/2 \\
\hline
 x & z_n & y & x & z_1 & y & \cdots & z_n & z_{n-1} \\
y & \vdots & z_1 & z_n & z_2 & x & \cdots & x & \vdots \\
z_1 & \vdots & z_2 & \vdots & \vdots & z_n & \cdots & y & z_1 \\
\vdots & z_1 & \vdots & z_2 & z_n & \vdots & \cdots & z_1 & y \\
x & z_n & y & x & z_1 & \cdots & \cdots & z_n & x \\
z_n & y & x & z_1 & y & z_2 & \cdots & z_{n-1} & x \\
\hline
\end{array}\]

Figure 5: the profile $Q$.

\[\begin{array}{cccc|cccc|c}
\hline
 m/2 & m/2 & m/2 & m/2 & m/2 & m/2 & m/2 & m/2 \\
\hline
 y & x & z_1 & y & z_2 & z_1 & \cdots & x & z_n \\
z_1 & z_n & z_2 & x & z_3 & y & \cdots & y & \vdots \\
z_2 & \vdots & \vdots & z_n & \vdots & x & \cdots & z_1 & \vdots \\
\vdots & z_2 & z_n & \vdots & z_n & \vdots & \cdots & z_1 & \vdots \\
z_n & y & x & z_1 & y & z_2 & \cdots & \cdots & \cdots & x \\
x & z_1 & y & z_2 & z_1 & z_3 & \cdots & z_n & y \\
\hline
\end{array}\]

Figure 6: the profile $\sigma(Q)$.

- Take $m/2$ fresh voters in $P'$ ranking $x$ above $y$ and let them vote $z_n \succ x \succ y \succ z_1 \succ \cdots \succ z_{n-1}$. Then take $m/2$ fresh voters in $P'$ ranking $y$ above $x$ and let them vote $z_{n-1} \succ \cdots \succ z_1 \succ y \succ z_n \succ x$.
- The above uses $(n+2) \cdot m$ many voters in $P'$. Since $n+2 \leq X(P'), m \leq M,$ and $V(P') \geq M \cdot |X(P')|,$ we have enough voters for the above construction. Thus, let $Q$ be the profile using the voters used above with their ballots also specified as above. Figure 5 gives the anonymized summary of profile $Q$. There may be unused voters in $P'$. That is, $V(P') \setminus V(Q)$ may be non-empty. However, they must come in pairs with respect to whether they rank $x$ above $y$ or $y$ above $x,$ since $\text{Margin}_Q(x, y) = \text{Margin}_P(x, y)$. Thus pick a ballot $L \in L\{(x, y, z_1, \cdots, z_n)\}$ with $xLy,$ and construct $Q'$ by adding to $Q$ the voters in $X(P') \setminus X(Q)$ and let them vote $L$ if they rank $x$ above $y$ in $P'$ and $L^{-1}$ otherwise.

Observe now that $P' \sim_{x,y} Q'$ since the margin graph of $Q'$ is a pure cycle with $x, y, z_1, z_2, \cdots, z_n$ with each consecutive edge's margin being $m,$ which is no greater than their original margin in $P'$. Thus, by Coherent II A, $y \notin F(Q')$. Since $Q'$ is the result of adding (possibly zero) reversal pairs to $Q,$ $y \notin F(Q)$. However, using the symmetries of the profile $Q,$ by Anonymity and Neutrality, $x$ and the $z_i$'s are also not in $F(Q)$, making $F(Q) = \varnothing$. (This is the same argument as in Holliday and Pacuit 2021a; if we consider the rotation $\sigma$ of candidates with $\sigma(x) = y, \sigma(y) = z_1, \sigma(z_i) = z_{i+1},$ and $\sigma(z_n) = x,$ then applying $\sigma$ to $Q$ results in the same anonymized profile. See Figure 6.) This contradicts the assumption that $F$ is a VSCC. \[\square\]
Proof. We prove the contrapositive. Suppose that \( y \not\in SC(P) \). Then there is an \( x \in X(P) \) such that \((x, y) \in sc(P)\), where \( sc \) is the Split Cycle VCCR. Let \( k = \text{Margin}_P(x, y) \) and \( M = M(P) \). If \( k \leq 2 \), then since \( P \) is a linear profile, \( k = 2 \) and there cannot be a majority path from \( y \) to \( x \). Then by Coherent Defeat, \( y \not\in F(P) \), and we are done. If \( k > 2 \), then we use Lemma 4.4 again. Inductively construct \( P_0, P_1, \ldots, P_{k-2} \) and \( L_1, L_2, \ldots, L_{k-2} \) where \( P_{i+1} = P_i + L_{i+1} \) and \( L_{i+1} \) is obtained by Lemma 4.4 applied to \( P_i \). Then according to the lemma and a simple induction:

- \( y \) is ranked by \( L_i \) above any candidate whom \( y \) does not beat head-to-head in \( P_{i-1} \);
- then by Tolerant Positive Involvement, if \( y \not\in F(P_{i+1}) \), then \( y \not\in F(P_i) \) for each \( i = 0, \ldots, k - 3 \);
- \((x, y)\) is in each \( sc(P_i) \), and in particular, \((x, y) \in sc(P_{k-2})\);
- \( \text{Margin}_{P_{k-2}}(x, y) = 2 \).

Since \( \text{Margin}_{P_{k-2}}(x, y) = 2 \) and \((x, y) \in sc(P_{k-2})\), there cannot be a majority path from \( y \) to \( x \). Then by Coherent Defeat, \( y \not\in F(P_{k-2}) \). Then, by backward induction from \( k - 2 \) to 0, \( y \not\in F(P_0) = F(P) \).

Combining Theorems 6.4 and 6.5, the main theorem Theorem 6.1 follows.

7 Ballots with Ties

In many applications of voting methods, it is unreasonable to expect voters to submit a linear ordering of all candidates, since the voter may lack information or willingness to make strict comparison between all pairs of candidates. Mathematically, this can be accommodated by using weak orders instead of linear orders, where candidates are first put into equivalence classes of tied candidates, and then the equivalence classes are linearly ordered. In this section, we show that a slight modification of the above axioms also axiomatizes Split Cycle in this generalized framework.

For any \( X \subseteq X \), let \( W(X) \) be the set of all strict weak orders on \( X \). Recall that \( \succ \) is a strict weak order on \( X \) iff \( \succ \) is irreflexive, transitive, and negatively transitive: for all \( x, y, z \in X \), if \( x \not\succ y \) and \( y \not\succ z \), then \( x \not\succ z \). Given a strict weak order \( \succ \) on \( X \), we say that \( x, y \in X \) are in a tie, written \( x \sim y \), if \( x \not\succ y \) and \( y \not\succ x \). The relation \( \succ \) being a strict weak order guarantees that being in a tie is an equivalence relation, and note that the empty set \( \emptyset \) is a strict weak order where all candidates are in a big tie. Now a profile is a function \( P : V \rightarrow W(X) \) for some nonempty finite \( V \subseteq V \) and \( X \subseteq X \). Given a nonempty finite set \( X \), we typically specify a strict weak order \( \succ \in W(X) \) by the notation

\[
P_1 \succ P_2 \succ \cdots \succ P_m
\]

where \( \{P_1, P_2, \cdots, P_m\} \) is a partition of \( X \); e.g., where \( X = \{a, b, c, d, e\} \), we may write \( \{c, e\} \succ \{b\} \succ \{a, d\} \). This means that \( \{P_1, P_2, \cdots, P_m\} \) is the set of all \( \sim \)-equivalence classes (groups that are tied internally) and for any \( i < j \), \( x \in P_i \), and \( y \in P_j \), \( x \succ y \); these two conditions uniquely determine a strict weak ordering of \( X \). When some \( P_i \) is a singleton \( \{x\} \), we may write ‘\( \cdots \succ x \succ \cdots \)’ instead of ‘\( \cdots \succ \{x\} \succ \cdots \)’.

In this section, all notions previously defined on linear profiles are now defined relative to all profiles, including the definitions of VCCR and VSCC. Thus, for example, a VCCR must return an asymmetric relation for any profile of strict weak orders. In some previous definitions, we used the word ‘above’ applied to places in a ballot. Those uses are intended for strict preference. Thus, when understanding previous
definitions in the context of profiles of strict weak orders, ‘$x$ is above $y$ in a ballot $\succ'$ is understood as $x \succ y$, even though in this section we will typically say ‘strictly above’. Moreover, the concept of margin is now relative to strict weak orders and is still counting and comparing only strict preferences; the margin of $x$ over $y$ does not depend directly on how many voters put $x$ or $y$ in a tie. As before, we say ‘$x$ is majority preferred to $y'$ when the margin of $x$ over $y$ is greater than 0, but because of the possibility of ties, the number of voters putting $x$ strictly above $y$ may not really be a majority among all voters. But we take this as only a minor verbal inconvenience, as it can be understood that ‘majority’ here is relative to those voters who express a strict preference between $x$ and $y$.

When defining the axiom of Monotonicity, we used the notion of ‘moving a candidate up one place in a ballot’. To make this precise for strict weak orders, we use the following definition.

**Definition 7.1.** Let $X$ be a finite set, $x \in X$, and $\succ, \succ'$ two strict weak orders on $X$ where $x$ is not the greatest element in $\succ$, i.e., there is $x' \in X \setminus \{x\}$ such that $x \not\succ x'$. Since $X$ is finite, we can find a minimal (relative to $\succ$) $x' \neq x$ such that $x \not\succ x'$. We say that $\succ'$ is the result of **moving $x$ up one place in $\succ$** if

- for any $y, z \in X \setminus \{x\}$, $y \succ z$ iff $y \succ' z$;
- if $x$ is in a tie with $x'$ in $\succ$, then in $\succ'$, $x$ is not in a tie with any other element, $x \succ x'$, and for any $y \in X \setminus \{x\}$, if $y \succ x$ then $y \succ' x$;
- if $x$ is not in a tie (relative to $\succ$) with any other element, then $x$ is in a tie with $x'$ relative to $\succ'$.

Thus, intuitively, to move $x$ up one place, we either break $x$ from a tie so that $x$ is immediately above those with whom $x$ previously tied, or we merge $x$ into the tied group that was immediately above $x$ if $x$ was not in a tie. Then Lemma 6.3 holds under this definition of ‘moving up one place’.

Still, the generalization to strict weak orders requires a new axiom. As we noted, the empty set that puts every candidate in a single tie is a legal ballot, and intuitively such a ballot should not affect the outcome of an election. We codify this requirement as an axiom.

**Definition 7.2.** A VCCR or VSCC $f$ satisfies **Neutral Indifference** if for any profile $P$ and $P'$ obtained from $P$ by adding a voter whose ballot is the empty set, we have $f(P) = f(P')$.

While Neutral Indifference is similar to Neutral Reversal, it does not follow from Neutral Reversal without enough help from other axioms. For example, we can easily devise VSCCs satisfying Anonymity, Neutrality, and Neutral Reversal by running different rules depending on the parity of the number of voters. Full Homogeneity (requiring precisely the same output for $P$ and $2P$, in contrast to Upward Homogeneity) is one axiom that can reduce Neutral Indifference to Neutral Reversal.

**Proposition 7.3.** Any VCCR or VSCC $f$ satisfying Neutral Reversal and Homogeneity satisfies Neutral Indifference.

*Proof.* Let $P : V \rightarrow W(X)$ be any profile, $P + \emptyset$ a profile obtained by adding a voter $v$ with $\emptyset$ as her ballot to $P$, and $f$ a VCCR or VSCC satisfying Neutral Reversal and Homogeneity. Note that we can first double $P$ to $2P$ (not using $v$) and obtain $2P + \emptyset + \emptyset$ by adding $v$ and another $v'$ with $\emptyset$ as both of their ballots. Now by Homogeneity, $f(P) = f(2P)$. By Neutral Reversal and noting that $\emptyset = \emptyset^{-1}$, $f(2P) = f(2P + \emptyset + \emptyset)$. But $2P + \emptyset + \emptyset$ is a doubling of $P + \emptyset$. So by Homogeneity and chaining the previous equalities, $f(P) = f(P + \emptyset)$. \qed
Let \( f \) be a VCCR satisfying Anonymity, Neutrality, Availability, Neutral Indifference, Monotonicity (for two-candidate profiles), Neutral Reversal, and Coherent IIA. Then Split Cycle refines \( f \): for any profile \( P \), \( sc(P) \supseteq f(P) \).

**Theorem 7.4.** Let \( f \) be a VCCR satisfying the stated axioms. Toward a contradiction, let \( P \) be a profile and \( x, y \in X(P) \) such that \((x, y) \in f(P) \setminus sc(P)\). By Lemma 6.3 applied to \( P \setminus \{x, y\} \) and Coherent IIA, \( m \) := \( Margin_{P}(x, y) = Margin_{P \setminus \{x, y\}}(x, y) > 0 \). By the definition of Split Cycle, there are \( z_1, z_2, \ldots, z_n \in X(P) \) such that \((x, y, z_1, z_2, \ldots, z_n, x)\) is a majority cycle and \( Margin_{P}(x, y) \) is smallest among the margins of the consecutive edges.

Now we add voters to \( P \) without affecting the result of \( f \). Let \( N_P(x \succ y) \) be the number of voters in \( P \) who rank \( x \) above \( y \), \( N_P(x \prec y) \) the number of voters who rank \( x \) below \( y \), and \( N_P(x \sim y) \) the number of voters who put \( x \) and \( y \) in a tie. Fix an arbitrary strict weak order \( L \) on \( X(P) \) in which \( x \succ y \). Then let \( P' \) be the result of adding to \( P \)

- \( \max(3m - N_P(x \succ y), 0) \) many pairs of voters who vote \( L \) and \( L^{-1} \), respectively, and
- \( \max((2n - 1)m - N_P(x \sim y), 0) \) many voters whose ballots have only ties.

Note first that by Neutral Reversal and Neutral Indifference, \((x, y)\) is still in \( f(P')\). Also, by the number of pairs of voters and voters added, \( Margin_{P'}(x, y) \) is still \( m \), and we have \( N_{P'}(x \succ y) \geq 3m \), \( N_{P'}(x \prec y) = N_{P'}(x \succ y) - m \geq 2m \), and \( N_{P'}(x \sim y) \geq (2n - 1)m \).

Next we construct a profile \( Q \), depicted in Figure 7, using some voters in \( P' \) such that \( M(Q) \) consists of precisely a majority cycle \((x, y, z_1, \ldots, z_n, x)\) with each edge’s weight being precisely \( m \); the tallying at the end of the last paragraph ensures that we have enough voters with desired types.

- Take \( m \) voters in \( P' \) ranking \( x \) above \( y \), and let their ballots in \( Q \) be \( x \succ y \succ \{z_1, \ldots, z_n\} \). Then take \( m \) voters in \( P' \) tying \( x \) with \( y \), and let their ballots in \( Q \) be \( \{z_1, \ldots, z_n\} \succ \{x, y\} \).

- Take another \( m \) voters ranking \( y \) above \( x \) in \( P' \), and let their ballots in \( Q \) be \( y \succ z_1 \succ \{x, z_2, \ldots, z_n\} \). Then take another \( m \) voters with \( x \) above \( y \) in \( P' \), and let their ballots in \( Q \) be \( \{z_2, \ldots, z_n, x\} \succ \{y, z_1\} \).

![Figure 7: the profile Q for Theorem 7.4. Candidates inside the same gray area are tied.](image-url)
• For each \( i = 1, \ldots, n - 1 \), first take another \( m \) voters tying \( x \) with \( y \) in \( P' \) and let their ballots in \( Q \) be \( z_i \succ z_{i+1} \succ \{z_1, \ldots, z_{i-1}, z_{i+2}, \ldots, z_n, x, y\} \). Then take another \( m \) voters tying \( x \) with \( y \) in \( P' \) and let their ballots in \( Q \) be \( \{z_1, \ldots, z_{i-1}, z_{i+2}, \ldots, z_n, x, y\} \succ \{z_i, z_{i+1}\} \).

• Finally, take another \( m \) voters ranking \( x \) above \( y \) in \( P' \) and let their ballots in \( Q \) be \( z_n \succ x \succ \{y, z_1, \ldots, z_{n-1}\} \). Then take another \( m \) voters ranking \( y \) above \( x \) in \( P' \) and let their ballots in \( Q \) be \( \{y, z_1, \ldots, z_{n-1}\} \succ \{x, z_n\} \).

By a standard argument with permutations, using Anonymity, Neutrality, and Availability, \( (x, y) \notin f(Q) \).

Now let \( L \) be an arbitrary strict weak order on \( x, y, z_1, \ldots, z_n \) with \( xLy \). Then let \( Q' \) be the result of adding the remaining voters in \( V(P') \setminus V(Q) \) to \( Q \) with their ballots given by:

- if the voter ranks \( x \) above \( y \) in \( P' \), her ballot in \( Q' \) is \( L \);
- if the voter ranks \( y \) above \( x \) in \( P' \), her ballot in \( Q' \) is \( L^{-1} \);
- if the voter ties \( x \) with \( y \), then her ballot in \( Q' \) is \( \emptyset \).

Two immediate observations follow. First, it is easy to check that \( P'(x, y) = Q'(x, y) \). Second, since \( \text{Margin}_Q(x, y) \) is easily seen to be \( m \), the number of voters from \( V(P') \setminus V(Q) \) ranking \( x \) above \( y \) in \( P' \) must be equal to the number of voters from \( V(P') \setminus V(Q) \) ranking \( y \) above \( x \). Thus, \( Q' \) is the result of adding to \( Q \) some (possibly zero) pairs of voters with converse ballots and some (possibly zero) voters with the fully tied ballot. Thus:

- \( \mathcal{M}(Q') = \mathcal{M}(Q) \). Since the latter is just a majority cycle with each edge’s weight being \( m \), \( \mathcal{M}(Q') \) can be obtained from \( \mathcal{M}(P') = \mathcal{M}(P) \) by deleting candidates and edges and lowering weights not involving the edge between \( x \) and \( y \) (the weights of the edges in the cycle \( x, y, z_1, \ldots, z_n, x \) are originally all at least \( m \)). So Coherent IIA applies, and \( (x, y) \notin f(Q') \).

- Since \( (x, y) \notin f(Q) \), and \( Q' \) is the result adding pairs of converse ballots and ballots with one big tie, by Neutral Reversal and Neutral Indifference, \( (x, y) \notin f(Q') \).

So we have a contradiction. \( \square \)

Appeal to Neutral Indifference cannot be eliminated in the proof since we may have to add an odd number of big tie ballots at some point. In particular, \( \max((2n-1)M-Np(x \sim y), 0) \) may be odd.

We now prove the analogous theorem for the Split Cycle VSCC over profiles of strict weak orders.

**Theorem 7.5.** Let \( F \) be a VSCC satisfying Anonymity, Neutrality, Neutral Indifference, Monotonicity (for two-candidate profiles), Neutral Reversal, and Coherent IIA. Then \( SC \) refines \( F \): for any profile \( P \), \( SC(P) \subseteq F(P) \).

**Proof.** The proof essentially combines the strategies used for Theorems 6.4 and 7.4. Let \( F \) be a VSCC satisfying the axioms and \( P \) a profile (allowing ties in ballots). Since we are dealing with a VSCC, as in the proof for Proposition 6.4, we need to first add voters to prepare for a later construction. Let \( M \) be the largest margin in the margin graph of \( P \). Then fix a linear order \( L^* \) on \( X(P) \), and let \( P' \) be the result of adding to \( P \)

- 3\( M \) pairs of voters in which one voter’s ballot is \( L^* \) and the other voter’s ballot is \( (L^*)^{-1} \) and
Clearly $\mathbf{P}$ and $\mathbf{P}'$ have the same margin graph, and by the assumed axioms, $F(\mathbf{P}') = F(\mathbf{P})$. So to show that $SC(\mathbf{P}) \subseteq F(\mathbf{P})$, it is enough to show that $SC(\mathbf{P}') \subseteq F(\mathbf{P}')$.

To apply Lemma 7.6 directly to our axiomatization for all profiles allowing ties, one may hope to relax the assumption that Margin $\mathbf{P}(x, y) > 1$ to Margin $\mathbf{P}(x, y) > 0$. But this cannot be done as shown by the following example:

\[ \begin{array}{c}
\text{a} \\
\Downarrow 2 \\
2 \\
\Updownarrow 1 \\
\text{x} \\
\text{y} \\
\Downarrow 2 \\
\Updownarrow 1 \\
\text{b} \\
\end{array} \]

- For any profile $\mathbf{P} : V \to \mathcal{W}(X)$ and any $(x, y) \in \text{sc}(\mathbf{P})$, if Margin$_\mathbf{P}(x, y) > 2$, then there is a linear ballot $L \in \mathcal{L}(X)$ such that
  - for any $z \in X \setminus \{y\}$ with Margin$_\mathbf{P}(y, z) \leq 0$, we have $(y, z) \in L$ (in particular, $(y, x) \in L$), and
  - $(x, y) \in \text{sc}(\mathbf{P} + L)$.

To apply Lemma 7.6 directly to our axiomatization for all profiles allowing ties, one may hope to relax the assumption that Margin$_\mathbf{P}(x, y) > 1$ to Margin$_\mathbf{P}(x, y) > 0$. But this cannot be done as shown by the following example:
For any ballot $L$ in which $y$ is ranked above $x$ and $a$, adding $L$ to any profile $P$ generating the above margin graph would render $(x, y)$ no longer a defeat edge according to Split Cycle, since there will be a majority cycle $(x, y, a, x)$ in which $(x, y)$ is a weakest edge with margin 1.

For this reason, we resort to the axiom of Downward Homogeneity to restore the parity constraint.

**Definition 7.7.** A VCCR $f$ (resp. VSCC $F$) satisfies **Downward Homogeneity** if for any profile $P$ and $2P$, where $2P$ is the result of replacing each voter in $P$ by 2 copies of that voter, $f(P) \supseteq f(2P)$ (resp. $F(P) \subseteq F(2P)$).

**Theorem 7.8.** Let $f$ be a VCCR satisfying Downward Homogeneity, Coherent Defeat, and Positive Involvement in Defeat. Then $f$ refines the Split Cycle VCCR: for any profile $P$, $f(P) \supseteq sc(P)$.

**Proof.** Pick an arbitrary profile $P : V \rightarrow W(X)$ and $x, y \in X$ such that $(x, y) \in sc(P)$. Using Downward Homogeneity, to show that $(x, y) \in f(P)$, it is enough to show that $(x, y) \in f(2P)$. Let $k = \text{Margin}_{2P}(x, y)$ (which must be at least 2) and inductively define $P_0, \ldots, P_{k-2}$ where $P_0 = 2P$ and $P_{i+1} = P_i + L_i$ with $L_i$ obtained by applying Lemma 7.6 to $P_i$. An easy induction shows that $(x, y) \in sc(P_{k-2})$ and $\text{Margin}_{P_{k-2}}(x, y) = 2$. But more importantly, since each $L_i$ is linear, the parity constraint is preserved, and in each $P_i$ inductively constructed, the margins in $P_i$’s margin graph share the same parity. This means that there is no majority edge in $P_{k-2}$ with margin 1. Thus, there is no majority path from $y$ back to $x$ in $P_{k-2}$. By Coherent Defeat, $(x, y) \in f(P_{k-2})$, and by backward induction using Positive Involvement in Defeat, $(x, y) \in f(2P)$. \qed

**Theorem 7.9.** Let $F$ be a VSCC satisfying Downward Homogeneity, Coherent Defeat, and Tolerant Positive Involvement. Then $F$ refines the Split Cycle VSCC: for any profile $P$, $F(P) \subseteq SC(P)$.

**Proof.** By combining the parity argument in the proof of Theorem 7.8 and the inductive argument using Tolerant Positive Involvement in the proof of Theorem 6.5. \qed

To sum up, now we have the following axiomatizations of Split Cycle when ties are allowed in ballots.

**Theorem 7.10.** Allowing ties in ballots, the Split Cycle VCCR is the unique VCCR satisfying Anonymity, Neutrality, Availability, Downward Homogeneity, Neutral Indifference, Neutral Reversal, Monotonicity (for two-candidate profiles), Coherent IIA, Coherent Defeat, and Positive Involvement in Defeat. Downward Homogeneity and Neutral Indifference can be replaced by Homogeneity.

**Theorem 7.11.** Allowing ties in ballots, the Split Cycle VSCC is the unique VSCC satisfying Anonymity, Neutrality, Downward Homogeneity, Neutral Indifference, Neutral Reversal, Monotonicity (for two-candidate profiles), Coherent IIA, Coherent Defeat, and Tolerant Positive Involvement. Downward Homogeneity and Neutral Indifference can be replaced by Homogeneity.

### 8 The Necessity of Three Special Axioms

In this section, we show the necessity of the three special axioms in the axiomatization of Split Cycle: Coherent IIA, Coherent Defeat, and Positive Involvement in Defeat (and Tolerant Positive Involvement in the context of VSCCs). For simplicity, we go back to the linear ballot setting.
8.1 Coherent IIA

We first exhibit a VCCR satisfying all the standard axioms, Coherent Defeat, and Positive Involvement in Defeat, but not Coherent IIA. By Theorem 4.5, this VCCR must refine Split Cycle.

Definition 8.1. Recall from Definition 3.15 the weighted covering VCCR \( wc \) defined by \((x, y) \in wc(P)\) iff \( \text{Margin}_p(x, y) > 0 \) and for all \( z \in X(P) \), \( \text{Margin}_p(x, z) \geq \text{Margin}_p(y, z) \). Define the VCCR \( scwc \) as the profile-wise union of \( sc \) and \( wc \).\(^{13}\) That is, \( scwc(P) = sc(P) \cup wc(P) \). In other words, \( x \) defeats \( y \) according to \( scwc \) iff either \( x \) defeats \( y \) according to \( sc \) or \( x \) defeats \( y \) according to \( wc \).

It requires a delicate argument to show that \( scwc \) satisfies Availability.

Proposition 8.2. The VCCR \( scwc \) satisfies Availability. In fact, \( scwc(P) \) is always acyclic.

Proof. Pick any profile \( P \). We claim that for any \( x, y, z \in X(P) \), if \((x, y) \in sc(P) \) and \((y, z) \in wc(P) \), then \((x, z) \in sc(P) \). Suppose \((x, y) \in sc(P) \) and \((y, z) \in wc(P) \). From \((y, z) \in wc(P) \), we have \( \text{Margin}_p(y, x) \geq \text{Margin}_p(z, x) \), which implies \( \text{Margin}_p(x, z) \geq \text{Margin}_p(x, y) \). Now pick any majority path \( \rho \) from \( z \) to \( x \), and our goal is to show that \( \text{Margin}_p(x, z) \geq \text{Strength}_p(\rho) \). Let \( a \) be the candidate immediately after \( z \) in the majority path \( \rho \), and let \( \rho' \) be the majority path that starts with \( y \), goes to \( a \) immediately, and then continues to \( x \) following \( \rho \). Since \((y, z) \in wc(P) \), \( \text{Margin}_p(y, a) \geq \text{Margin}_p(z, a) \). This means \( \text{Strength}_p(\rho') \geq \text{Strength}_p(\rho) \). Since \((x, y) \in sc(P) \), \( \text{Margin}_p(x, y) > \text{Strength}_p(\rho') \). Connecting this with inequalities (1) and (2), \( \text{Margin}_p(x, z) > \text{Strength}_p(\rho) \), which shows that \((x, z) \in sc(P) \).

Now we note that \( wc(P) \) is transitive and irreflexive and hence acyclic. So if there is a cycle in \( scwc(P) \), the cycle must contain at least one edge in \( sc(P) \). But then we can repeatedly use the above claim and obtain a cycle in \( sc(P) \), contradicting the acyclicity of \( sc \).

Proposition 8.3. The VCCR \( scwc \) satisfies all the standard axioms, Coherent Defeat, and Positive Involvement in Defeat. Moreover, it satisfies Majority Defeat. Thus, using Proposition 5.12, \( scwc \) satisfies all the standard axioms, Coherent Defeat, and Tolerant Positive Involvement.

Proof. Since \( scwc(P) \) only depends on the margin graph of \( P \)—and indeed only on the ordering of margins by size, not the numerical values—it follows that Anonymity, Neutrality, Homogeneity, and Neutral Reversal are satisfied. Monotonicity is also easy to see as both \( sc \) and \( wc \) satisfy Monotonicity. We showed in Proposition 8.2 that \( scwc \) satisfies Availability. Next, \( scwc \) satisfies Coherent Defeat since it refines \( sc \), and \( wc \) satisfies Coherent Defeat. Since both \( sc \) and \( wc \) satisfy Positive Involvement in Defeat by Propositions 3.14 and 3.16, reasoning by cases we see that \( scwc \) satisfies Positive Involvement in Defeat. Thus \( scwc \) satisfies all the stated axioms. Since \( sc \) and \( wc \) both satisfy Majority Defeat, \( scwc \) also satisfies Majority Defeat, which implies that \( scwc \) satisfies Tolerant Positive Involvement by Proposition 5.12. That \( scwc \) satisfies the other stated axioms is easy to check.

Proposition 8.4. Neither the VCCR \( scwc \) nor the VSCC \( scwc \) satisfies Coherent IIA.

Proof. Let \( P \) be the following profile with its margin graph displayed on the right:

\[ \text{Margin}_p(x, z) > \text{Margin}_p(y, z) \]
The column under $v_{i,i+1}$ means that the voters $v_i$ and $v_{i+1}$ submitted the same ballot. Observe that $sc(P) = \{(x,b)\}$ and $wc(P) = \{(x,y)\}$. (A quick method to check: if there is a 3-cycle, then none of the three edges are in $wc$.) Thus $scwc(P) = \{(x,b),(x,y)\}$ and $isc(P) = \{a,x\}$. Now consider profile $Q$ and its margin graph:

Observe that $P \not\sim x,y Q$ (only $v_2$ and $v_3$ changed ballots). Now $(x,y) \notin scwc(Q) = \emptyset$. Thus $P,Q$ witness $scwc$ failing Coherent IIA. To show that $isc$ fails Coherent IIA, we must be more careful. Starting with the fact that $y \notin scwc(P)$, we must show that for any candidate $u \neq y$, there is $P'$ such that $P \not\sim y,a P'$ and $y \in scwc(P')$. If $u = b$, we can use $P' = P_{\{(y,b)\}}$. If $u = a$, we can use $P' = P_{\{(y,a)\}}$. And finally if $u = x$, we use $Q$.

Thus, for either the VCCR $sc$ or the VSCC $SC$, Coherent IIA cannot be dropped from our axioms.

8.2 Positive Involvement in Defeat and Tolerant Positive Involvement

Now we turn to the necessity of Positive Involvement in Defeat and Tolerant Positive Involvement. For the following definition, as a convention we define the minimum of the empty set to be $\infty$ (an object greater than every natural number) and the maximum of the empty set to be 0.

**Definition 8.5.** For any positively weighted directed graph $M$ and simple path $\rho$ from $x$ to $y$ in $M$, define the *ignore-source strength* $\text{Strength}^{is}_M(\rho)$ of $\rho$ to be the minimum of the weights of the consecutive edges in $\rho$ that do not start from $x$. The *ignore-source strength* $\text{Strength}^{is}_M(x,y)$ from $x$ to $y$ is the maximum of $\text{Strength}^{is}_M(\rho)$ for all majority paths $\rho$ from $x$ to $y$. Then define the VCCR *Ignore-source Split Cycle isc* by: for any profile $P$ and $x,y \in X(P)$, $(x,y) \in isc(P)$ iff $\text{Margin}_P(x,y) > \text{Strength}^{is}_M(x,y)$. The Ignore-source Split Cycle VSCC $ISC$ is defined as $isc$ (recall Lemma 2.5).

From the definition, the following observations are immediate.

**Lemma 8.6.** Let $M$ be a margin graph and $x,y$ vertices in $M$.

1. $\text{Strength}^{is}_M(x,y) = \infty$ iff $(x,y)$ is an edge in $M$. $\text{Strength}^{is}_M(x,y) = 0$ iff $y$ is not reachable from $x$. 

2. $\text{Strength}^{is}_M(x,y) = \text{Strength}_{M'}(x,y)$ where $M'$ is the weighted directed graph obtained from $M$ by increasing to $\infty$ the weight of each edge whose source is $x$. Here $\text{Strength}_{M'}$ is calculated in the standard way: it is the maximum of the strengths of all paths from $x$ to $y$, where the strength of a path is the minimum of the weights of the consecutive edges in that path. Thus ‘ignore source’ can also be viewed as ‘infinity source’.
3. Suppose $\text{Strength}^i_{\mathcal{M}}(x, y) < \infty$. Then if we delete all edges in $\mathcal{M}$ that (1) do not start from $x$ and (2) have weights no greater than $\text{Strength}^i_{\mathcal{M}}(x, y)$, then $y$ is no longer reachable from $x$. In other words, when $\text{Strength}^i_{\mathcal{M}}(x, y) < \infty$, the set of all edges not starting from $x$ and with weights no greater than $\text{Strength}^i_{\mathcal{M}}(x, y)$ is a cut from $x$ to $y$.

4. For any cut $C$ from $x$ to $y$ that does not use any edge starting from $x$, $\text{Strength}^i_{\mathcal{M}}(x, y)$ is at most the maximal weight of the edges in $C$.

**Proposition 8.7.** $isc$ satisfies Anonymity, Neutrality, Availability, Homogeneity, Monotonicity, Neutral Reversal, Coherent IIA, Coherent Defeat, and First-place Involvement in Defeat.

**Proof.** We only verify First-place Involvement in Defeat here; the rest can be verified in the same way as they are verified for Split Cycle. Suppose $x$ does not defeat $y$ according to $isc$ in $P$, and let $P'$ be the result of adding a ballot that ranks $y$ at the top. Then $\text{Margin}_{P}(x, y) \leq \text{Strength}^i_{\mathcal{M}(P)}(y, x)$, and we want to show the same inequality for $P'$. Since the new ballot ranks $y$ at the top, $\text{Margin}_{P'}(x, y) = \text{Margin}_{P}(x, y) - 1$. So we only need to show that $\text{Strength}^i_{\mathcal{M}(P')} (y, x) \geq \text{Strength}^i_{\mathcal{M}(P)} (y, x) - 1$. For the standard strength, this is trivial since we only added a single ballot, so the margin changes can be at most 1. However, by the example in the proof of the next proposition, extra care must be taken for $\text{Strength}^i$. Since the added ballot ranks $y$ at the top, all the edges starting from $y$ in $\mathcal{M}(P)$ are strengthened in weight, and in particular every edge in $\mathcal{M}(P)$ starting from $y$ is still in $\mathcal{M}(P')$. Thus, letting $\mathcal{M}(P)^+$ and $\mathcal{M}(P')^+$ be the results of raising the weights of all edges from $y$ to $\infty$ in $\mathcal{M}(P)$ and $\mathcal{M}(P')$, respectively, the weight decrease of the edges from $\mathcal{M}(P)^+$ to $\mathcal{M}(P')^+$ is at most 1 (including eliminating edges with weight 1); the weight increase could be infinite due to new edges starting from $y$, but this is irrelevant for the direction of the inequality we are trying to show. Hence $\text{Strength}_{\mathcal{M}(P')^+} (y, x) \geq \text{Strength}_{\mathcal{M}(P)^+} (y, x) - 1$. Then using Lemma 8.6.2, we are done.

As we have seen that First-place Involvement in Defeat entails Positive Involvement, $isc$ satisfies Positive Involvement as well.

**Proposition 8.8.** $isc$ does not refine Split Cycle, and it fails Positive Involvement in Defeat. Moreover, $ISC$ fails Tolerant Positive Involvement.

**Proof.** Consider a profile $P$ whose margin graph is

![Diagram](attachment:diagram.png)

According to Split Cycle, $b$ defeats $a$. But according to $isc$, $b$ does not defeat $a$, since $\text{Strength}^{isc}(a, b) = 5$ as we need to ignore the edge from $a$ to $c$. Since clearly $c$ does not defeat $a$ either according to $isc$, $a$ is undefeated and a winner according to $ISC$. To see the failure of Positive Involvement in Defeat and Tolerant Positive Involvement, let $P'$ be the result of adding a ballot $\succ$ such that $c \succ a \succ b$ (where $a$ is above $b$ and thus above all candidates to whom $a$ is not majority preferred) to $P$. Then $\mathcal{M}(P')$ is
Now $b$ defeats $a$ according to $isc$ since there is no path from $a$ to $b$ and the ignore-source strength from $a$ to $b$ decreased from 5 above to 0.

Thus, when axiomatizing Split Cycle, we cannot replace Positive Involvement in Defeat by First-place Involvement in Defeat for the VCCR or replace Tolerant Positive Involvement by Positive Involvement for the VSCC. On the other hand, we can show that Coherent Defeat and First-place Involvement in Defeat provide a one-sided axiomatization for $isc$ in the direction opposite to what Holliday and Pacuit (2021a) did for the Split Cycle VCCR.

**Theorem 8.9.** For any VCCR $f$ that satisfies Coherent Defeat and First-place Involvement in Defeat, $f$ refines $isc$.

**Proof.** We repeat the same cut argument in Lemma 4.4 and the inductive argument in Theorem 4.5. The counterpart of Lemma 4.4 for $isc$ takes the following form: for any profile $P: V \to \mathcal{L}(X)$ and $(x, y) \in isc(P)$, if $\text{Margin}_P(x, y) > 2$, then there is a ballot $L \in \mathcal{L}(X)$ such that

- $y$ is at the top of $L$, and
- $(x, y) \in isc(P + L)$.

The proof is almost the same as that of Lemma 4.4: since we are using $\text{Strength}^{isc}_{M_t}$, we can guarantee that the cut does not include any edge starting from the defeated candidate according to $isc$. Hence the union of the converse of the cut and the set of all pairs starting from $y$ is acyclic, and the rest of the proof is exactly the same. Then we repeatedly use this counterpart of Lemma 4.4 to show that if $(x, y) \in isc(P)$, then there is a sequence of ballots $L_1, \cdots, L_n$, all putting $y$ at the top, such that $(x, y) \in isc(P + L_1 + \cdots + L_n)$, and in $P + L_1 + \cdots + L_n$, $x$ defeats $y$ merely by Coherent Defeat. Then by First-place Involvement in Defeat, $x$ defeats $y$ in $P$ according to $f$.

### 8.3 Coherent Defeat

The example showing the necessity of Coherent Defeat is the following VCCR.

**Definition 8.10.** Let $oca$ (one-covers-all) be the VCCR such that for any profile $P$ and $x, y \in X(P)$, $(x, y) \in oca(P)$ iff for all $z \in X(P)$, $\text{Margin}_P(x, y) > \text{Margin}_P(y, z)$.

**Proposition 8.11.** The VCCR $oca$ satisfies all the standard axioms, and thus the VSCC $\overline{oca}$ also satisfies all the standard axioms. Moreover, $oca$ satisfies Coherent IIA and Positive Involvement in Defeat, while $\overline{oca}$ satisfies Coherent IIA and Tolerant Positive Involvement.

**Proof.** For the standard axioms, the only one worth commenting on is Availability for $oca$. Indeed, we can show that $oca$ is acyclic, just as we show that $sc$ is acyclic. First, note that $oca$ satisfies Majority Defeat, since if $(x, y) \in oca(P)$, then $\text{Margin}_P(x, y) > \text{Margin}_P(y, x)$. Also, the weakest edges of any majority cycle cannot be defeats according to $oca$. So there cannot be a defeat cycle according to $oca$. 

35
To see that $oca$ satisfies Coherent IIA, suppose $(x, y) \in oca(P)$ and $P \sim_{x,y} Q$. Then, $\text{Margin}_P(x, y) = \text{Margin}_Q(x, y) > 0$, while for any $z \in X(Q)$, $\text{Margin}_Q(y, z)$ is either $< 0$ or $\leq \text{Margin}_P(y, z)$. In either case, $\text{Margin}_Q(x, y) > \text{Margin}_Q(y, z)$. So $(x, y) \in oca(Q)$. By Proposition 5.7, $\overline{oca}$ also satisfies Coherent IIA.

To see that $oca$ satisfies Positive Involvement in Defeat, say $(x, y) \notin oca(P)$ and $Q$ is obtained by adding one ballot ranking $y$ above $x$. This means that there is $z \in X(P)$ such that $\text{Margin}_P(x, y) \leq \text{Margin}_P(y, z)$. Now $\text{Margin}_Q(x, y) = \text{Margin}_P(x, y) - 1$, while $|\text{Margin}_P(y, z) - \text{Margin}_Q(y, z)| \leq 1$. Thus $\text{Margin}_Q(x, y) \leq \text{Margin}_Q(y, z)$, and thus $(x, y) \notin Q$. Using Proposition 5.12, $\overline{oca}$ satisfies Tolerant Positive Involvement.

Proposition 8.12. Both $oca$ and $\overline{oca}$ fail Coherent Defeat.

Proof. Any profile $P$ whose margin graph is

![Diagram](attachment:image.png)

will do. Here $oca(P) = \{(y, z)\}$ and $\overline{oca}(P) = \{x, y\}$, even though $y$ is coherently defeated by $x$ as $\text{Margin}_P(x, y) > 0$ and there is no majority path from $y$ to $x$.

9 Conclusion

We have provided axiomatizations for Split Cycle both as a VCCR and as a VSCC, both over profiles prohibiting ties and over profiles allowing ties. Most of the axioms are largely uncontroversial, while the special axioms—Coherent IIA, Coherent Defeat, Positive Involvement in Defeat, and Tolerant Positive Involvement—have direct intuitive appeal. The axiomatizations also show where and how Split Cycle diverges from similar margin-based voting methods such as Ranked Pairs and Beat Path: Coherent Defeat is a shared starting point; then Positive Involvement in Defeat (resp. Tolerant Positive Involvement for VSCCs) forces the method to refine Split Cycle; and Coherent IIA forces the method to be refined by Split Cycle.

We conclude with several directions for possible future investigation. First, while our method for the axiomatization of Split Cycle cannot be applied directly to Beat Path and Ranked Pairs or other unaxiomatized margin-based voting methods, our results at least suggest that the axiomatization question for margin-based voting methods could be amenable to analysis of graphs by well-known graph theoretical concepts. In particular, given the similarity between the definitions of Beat Path and Split Cycle, we believe an axiomatization of Beat Path is within reach.

Second, as we mentioned at the beginning of Section 6, in general a VSCC carries less information than a VCCR, but when we focus on VCCRs and VSCCs satisfying certain axioms, recovering canonically a rationalizing VCCR from a VSCC may be possible. In fact, the axiom of Coherent IIA itself suggests a method of recovery: given a VSCC $F$ that is rationalized by some VCCR $f$ satisfying Coherent IIA and a losing candidate $y$ in a profile $P$, any candidate $x$ that defeats $y$ according to $f$ should be such that for any $P'$ with $P \sim_{x,y} P'$, $x$ should still defeat $y$ in $P'$ according to $f$, making $y$ still lose according to $F$ in $P'$. Viewing this statement about $x$ as a property of a potential defeater of $y$, we can take as the defeaters of $y$ according to the VSCC $F$ all candidates $x \in X(P)$ such that for any $P'$ with $P \sim_{x,y} P'$, $y \notin F(P')$. Some of these candidates may not defeat $y$ according to $f$, but we can still use this method to recover a VCCR $f'$ from $F$. It remains to be seen what axioms can be preserved by this method of constructing a VCCR from a VSCC and what other interesting properties this method may exhibit.
Third, as we showed in Section 8, within the set of all the margin-based VCCRs satisfying Monotonicity, Coherent IIA, Coherent Defeat, and First-place Involvement in Defeat, what we call *Ignore-source Split Cycle* is the least refined while Split Cycle is the most refined. It seems to be an interesting problem to classify all VCCRs in this set and also study their ordering by refinement. This could help us better understand the axioms (especially First-place Involvement in Defeat).

Finally, we believe it is important to systematically study the class of refinements of Split Cycle that are, unlike Split Cycle (see Holliday and Pacuit 2023a, § 5.4.2.2), *asymptotically resolvable*: for any number of candidates, the probability\(^{14}\) of selecting multiple winners goes to zero as the number of voters goes to infinity. Methods in this class, which include Ranked Pairs (Tideman 1987), River (Heitzig 2004b), Beat Path (Schulze 2011, 2022), and Stable Voting (Holliday and Pacuit 2023b), can be viewed as ways of deterministically breaking ties among Split Cycle winners. No known method in this class satisfies Positive Involvement or its variations studied in this paper (see Holliday 2024 for a related impossibility theorem), but no known impossibility result yet rules out the existence of such a refinement of Split Cycle.

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