On linearization of super sine-Gordon equation

M. Siddiq$^1$ and M. Hassan

Department of Physics,
University of the Punjab,
Quaid-e-Azam Campus,
Lahore-54590, Pakistan.

Abstract

Two sets of super Riccati equations are presented which result in two linear problems of super sine-Gordon equation. The linear problems are then shown to be related to each other by a super gauge transformation and to the super Bäcklund transformation of the equation.

PACS: 02.30.Ik
PACS: 12.60.Jv

$^1$On leave of absence from PRD, PINSTECH, Nilore Islamabad (Pakistan).
1 Introduction

During the last few decades, much progress has been made in the study of nonlinear evolution equations with soliton solutions. Some of these equations correspond to integrable models of field theory and are solvable by the inverse scattering method in the scheme of Zakharov-Shabat (ZS) and Ablowitz, Kaup, Newell and Segur (AKNS) [1]-[5]. The inverse scattering method and ZS/AKNS scheme are the general procedures which associate a linear system with the nonlinear evolution equations and lead to soliton solutions. This scheme incorporates all the well known integrable models such as the Korteweg deVries equation (KdV), modified KdV (mKdV) equation, Liouville theory, sine-Gordon theory, Schrodinger equation, sigma model etc. The associated linear system is related to the existence of infinitely many local conserved quantities in involution, which guarantee the the classical and quantum integrability of the model. Moreover, the Bäcklund transformation and its Riccati form are one of the direct methods of obtaining new soliton solutions and local conserved quantities of a given integrable model. The Bäcklund transformation can be derived by a suitable transformation from the Riccati equations [6] and through these Riccati equations the Bäcklund transformation is related to the inverse scattering method and the ZS/AKNS scheme. This suggests that the Riccati equations related to the inverse scattering method can be used to derive an infinite number of local conserved quantities.

More recently, many techniques have been developed to study the supersymmetric generalizations of integrable models, for example the supersymmetric extension of inverse scattering method, ZS/AKNS scheme, the Lax formalism, zero curvature formalism, conservation laws etc [7]-[21]. Among the several super integrable models, perhaps the super sine-Gordon equation is the physically most interesting equation. The super sine-Gordon equation is integrable as it contains infinitely many local conserved quantities [9] which survive quantization, rendering the theory quantum integrable and making it possible to compute the exact S-matrices [22]-[23]. The linear representation of the super sine-Gordon equation has been proposed in [12] and using that a rigorous proof of the existence of infinitely many local conserved quantities has been derived. So far there exists no systematic way of establishing a linear representation of super sine-Gordon equation which could be manifestly related to the Bäcklund transformation and its Riccati form. The purpose of this work is to present a systematic approach to determine the desired linear representation of the super sine-Gordon equation.
In our analysis, we start with two sets of Riccati equations whose compatibility condition is the super sine-Gordon equation. The two sets of Riccati equations are shown to be related to the Bäcklund transformation of the super sine-Gordon equation. The linearization of Riccati equations yields two linear systems associated with the super sine-Gordon equation. Both the linear systems are shown to be equivalent, related to each other by a super gauge transformation.

We follow the general procedure of writing manifestly supersymmetric sine-Gordon equation. The equation is defined in two dimensional super-Minkowski space with bosonic light-cone coordinates $x^\pm$ and fermionic coordinates $\theta^\pm$, which are Majorana spinors. The covariant superspace derivatives are defined by

$$D_\pm = \frac{\partial}{\partial \theta^\pm} - i \theta^\pm \partial_\pm, \quad D_\pm^2 = -i \partial_\pm, \quad \{D_+, D_-\} = 0,$$

where $\{,\}$ is an anti-commutator. The superspace lagrangian density of $N = 1$ super sine-Gordon theory is

$$\mathcal{L}(\Phi) = \frac{i}{2} D_+ \Phi D_- \Phi + \cos \Phi,$$

where $\Phi$ is a real scalar superfield. The superfield evolution equation follows from the lagrangian and is given by

$$D_+ D_- \Phi = i \sin \Phi.$$  \hfill (1.2)

The equation (1.2) is invariant under $N = 1$ supersymmetry transformations.

## 2 Super Riccati equations and super Bäcklund transformation

The super sine-Gordon equation appears as the compatibility condition of the following set of Riccati equations

$$D_+ \mathcal{N} = \frac{1}{2} \sqrt{\frac{2}{\lambda}} g \left[ \mathcal{N} \cos \Phi + \mathcal{N}^2 \sin \Phi \right],$$

$$D_- \mathcal{N} = \frac{\sqrt{2\lambda}}{2} g \mathcal{N} + (1 + \mathcal{N}^2) D_- \Phi,$$

together with

$$\mathcal{N} D_+ g = -i \sqrt{\frac{2}{\lambda}} \left[ -\mathcal{N} \cos \Phi + \sin \Phi \right],$$

$$\mathcal{N} D_- g = -i \mathcal{N} \sqrt{2\lambda} + g D_- \Phi,$$  \hfill (2.2)
where $N$ and $g$ are scalar and spinor superfields, respectively, and $\lambda$ is the spectral parameter. The fermionic superfield $g$ is introduced because of supersymmetry and the oddness of the superspace derivatives $D_{\pm}$. The compatibility condition of equations (2.1) and (2.2) is equation (1.2). In the limit when fermions are set equal to zero, the super Riccati system reduces to pure bosonic Riccati system of the sine-Gordon equation.

The super sine-Gordon equation can also be expressed as the compatibility condition of another set of Riccati equations which is

$$
D_+ \Gamma = \sqrt{\frac{2g}{\lambda}} \left[ (\Gamma - 1) \exp(i\Phi) + \Gamma(\Gamma - 1) \exp(-i\Phi) \right],
$$

(2.3)

$$
D_- \Gamma = \sqrt{2\lambda g^4} (\Gamma^2 - 1) + 2i\Gamma D_- \Phi,
$$

together with

$$
(\Gamma - 1) D_+ g = -i\sqrt{\frac{2}{\lambda}} \left[ \exp(i\Phi) - \Gamma \exp(-i\Phi) \right],
$$

(2.4)

$$
(\Gamma - 1) D_- g = -i\sqrt{2\lambda} (\Gamma - 1) + (\Gamma + 1) g D_- \Phi,
$$

where $\Gamma$ is a scalar superfield. In what follows, we shall see that the linearization of set of Riccati equations (2.1)-(2.2) yields the linear system of [10]-[12], while the other set (2.3)-(2.4) gives its gauge equivalent form.

The given two sets of Riccati equations (2.1)-(2.2) and (2.3)-(2.4) can also be used to derive the Bäcklund transformation of the super sine-Gordon equation. In order to do so, we make transformations on superfields $N$ and $\Gamma$ as $N \rightarrow \tan \left( \frac{\Phi + \tilde{\Phi}}{2} \right)$ and $\Gamma \rightarrow \exp i(\Phi + \tilde{\Phi})$, and substitute them in (2.1)-(2.2) and (2.3)-(2.4), respectively. This yields

$$
D_+ (\Phi + \tilde{\Phi}) = \sqrt{\frac{2g}{\lambda}} \sin \left( \frac{\Phi + \tilde{\Phi}}{2} \right) \cos \left( \frac{\Phi - \tilde{\Phi}}{2} \right),
$$

(2.5)

$$
D_- (\Phi - \tilde{\Phi}) = -\sqrt{2\lambda} g \sin \left( \frac{\Phi + \tilde{\Phi}}{2} \right) \cos \left( \frac{\Phi + \tilde{\Phi}}{2} \right),
$$

$$
D_+ g = \sqrt{2g} \sin \left( \frac{\Phi - \tilde{\Phi}}{2} \right) \csc \left( \frac{\Phi - \tilde{\Phi}}{2} \right),
$$

$$
D_- g = \sqrt{2\lambda} \csc \left( \frac{\Phi + \tilde{\Phi}}{2} \right) g D_- \Phi,
$$

which is the super Bäcklund transformation for the super sine-Gordon equation. The compatibility condition of the Bäcklund transformation (2.5) is the super sine-Gordon
equation for both $\Phi$ and $\tilde{\Phi}$ separately. One can also relate the transformation (2.5) to the known super Bäcklund transformation of [11], by choosing $g = f \csc \left( \frac{\Phi + \tilde{\Phi}}{2} \right)$, to get

\[
\begin{align*}
D_+ (\Phi + \tilde{\Phi}) &= \sqrt{\frac{2}{\lambda}} f \cos \left( \frac{\Phi - \tilde{\Phi}}{2} \right), \\
D_- (\Phi - \tilde{\Phi}) &= -\sqrt{2\lambda} f \cos \left( \frac{\Phi + \tilde{\Phi}}{2} \right), \\
D_+ f &= -i \sqrt{\frac{2}{\lambda}} \sin \left( \frac{\Phi - \tilde{\Phi}}{2} \right), \\
D_- f &= -i \sqrt{2\lambda} \sin \left( \frac{\Phi + \tilde{\Phi}}{2} \right).
\end{align*}
\]

The super Bäcklund transformations (2.5) and (2.6) reduce to the Bäcklund transformation of the purely bosonic sine-Gordon equation when fermions are set equal to zero. In ref. [11] the Bäcklund transformation (2.6) has been used to derive an infinite number of conservation laws of the super sine-Gordon equation.

3 Linear systems and Lax representation in superspace

The two sets of Riccati equations (2.1)-(2.2) and (2.3)-(2.4) can be transformed to systems of linear equations for bosonic superfields $\Psi_1, \Psi_2$ and a fermionic superfield $\Psi_3$ for the set (2.1)-(2.2) by change of variables $N = \frac{\Psi_3}{\Psi_2}$, $g = \frac{\Psi_3}{\Psi_1}$ and also for bosonic superfields $\Phi_1, \Phi_2$ and a fermionic superfield $\Phi_3$ for the set (2.3)-(2.4) by change of variables $\Gamma = \frac{\Phi_3}{\Phi_2}$, $g = \frac{\Phi_3}{\Phi_1 - \Phi_2}$.

The super Riccati equations become equivalent to the following linear system of differential equations

\[
\begin{align*}
D_\pm \Psi &= \mathcal{A}_\pm \Psi, \\
D_\pm \Omega &= \mathcal{B}_\pm \Omega,
\end{align*}
\]

where

\[
\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix}, \quad \Omega = \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix},
\]

and $\mathcal{A}_\pm$ and $\mathcal{B}_\pm$ are $3 \times 3$ matrices

\[
\mathcal{A}_+ = \sqrt{\frac{1}{2\lambda}} \begin{pmatrix} 0 & 0 & \cos \Phi \\ 0 & 0 & -\sin \Phi \\ 2i \cos \Phi & -2i \sin \Phi & 0 \end{pmatrix},
\]

\[
\mathcal{B}_+ = \sqrt{\frac{1}{2\lambda}} \begin{pmatrix} 0 & 0 & \cos \Phi \\ 0 & 0 & -\sin \Phi \\ 2i \cos \Phi & -2i \sin \Phi & 0 \end{pmatrix},
\]

\[
\mathcal{B}_- = \sqrt{\frac{1}{2\lambda}} \begin{pmatrix} 0 & 0 & \cos \Phi \\ 0 & 0 & -\sin \Phi \\ 2i \cos \Phi & -2i \sin \Phi & 0 \end{pmatrix},
\]
\[
A_+ = \frac{1}{2} \begin{pmatrix}
0 & 2D_+ \Phi & \sqrt{2\lambda} \\
-2D_+ \Phi & 0 & 0 \\
-2i\sqrt{2\lambda} & 0 & 0 \\
\end{pmatrix}, \\
B_+ = \frac{1}{4}\sqrt{\frac{2}{\lambda}} \begin{pmatrix}
0 & 0 & e^{i\Phi} \\
0 & 0 & -e^{-i\Phi} \\
4i e^{-i\Phi} & -4i e^{i\Phi} & 0 \\
\end{pmatrix}, \\
B_- = \frac{1}{4} \begin{pmatrix}
4i D_+ \Phi & 0 & \sqrt{2\lambda} \\
0 & -4i D_+ \Phi & -\sqrt{2\lambda} \\
-4i\sqrt{2\lambda} & 4i\sqrt{2\lambda} & 0 \\
\end{pmatrix}.
\]

The compatibility condition of the linear systems (3.1)-(3.2) in superspace is equivalent to equation (1.2). The system (3.1) is the same as obtained earlier in [12] but here it is derived systematically from the Bäcklund transformation of super sine-Gordon equation. Both linear systems (3.1)-(3.2) are equivalent, related to each other by a gauge transformation

\[
\Psi = G\Omega, 
\]

where

\[
G = \begin{pmatrix}
-i & i & 0 \\
1 & 1 & 0 \\
0 & 0 & -i \\
\end{pmatrix},
\]

so that

\[
A_\pm = GB_\pm G^{-1}.
\]

The matrices \(A_\pm\) and \(B_\pm\) obey the zero curvature condition in superspace separately, that is

\[
D_+ A_+ + D_- A_+ - \{A_+, A_-\} = 0, \\
D_+ B_- + D_- B_+ - \{B_+, B_-\} = 0.
\]

The linear systems (3.1)-(3.2) can be reexpressed as

\[
\partial_\pm \Psi = \tilde{A}_\pm \Psi, \\
\partial_\pm \Omega = \tilde{B}_\pm \Omega,
\]

where the matrices \(\tilde{A}_\pm\) and \(\tilde{B}_\pm\) are given by

\[
\tilde{A}_+ = \begin{pmatrix}
-\frac{i}{2\lambda} \cos(2\Phi) & \frac{i}{2\lambda} \sin(2\Phi) & -\frac{i}{2} \sqrt{\frac{2}{\lambda}} \sin(\Phi) D_+ \Phi \\
\frac{i}{2\lambda} \sin(2\Phi) & \frac{i}{2\lambda} \cos(2\Phi) & -\frac{i}{2} \sqrt{\frac{2}{\lambda}} \cos(\Phi) D_+ \Phi \\
\sqrt{\frac{2}{\lambda}} \sin(\Phi) D_+ \Phi & \sqrt{\frac{2}{\lambda}} \cos(\Phi) D_+ \Phi & -\frac{1}{2\lambda} \\
\end{pmatrix},
\]

\[\tilde{B}_+ \]
\[\tilde{A}_- = \begin{pmatrix} \frac{\lambda}{2} & \partial_- \Phi & 0 \\ -\partial_- \Phi & -\frac{\lambda}{2} & \frac{i}{2} \sqrt{2\lambda D_- \Phi} \\ 0 & \frac{i}{2} \sqrt{2\lambda D_- \Phi} & \frac{i}{2} \end{pmatrix},\]

\[\tilde{B}_+ = \begin{pmatrix} 0 & \frac{1}{\lambda} e^{2i\Phi} & -\frac{i}{\lambda} \sqrt{\frac{2}{\lambda}} e^{i\Phi} D_+ \Phi \\ \frac{i}{\lambda} e^{-2i\Phi} & 0 & -\frac{i}{\lambda} \sqrt{\frac{2}{\lambda}} e^{-i\Phi} D_+ \Phi \\ \frac{i}{\lambda} \sqrt{\frac{2}{\lambda}} e^{-i\Phi} D_+ \Phi & i\sqrt{\frac{2}{\lambda}} e^{i\Phi} D_+ \Phi & -\frac{1}{2\lambda} \end{pmatrix},\]

\[\tilde{B}_- = \begin{pmatrix} i\partial_- \Phi & -\frac{\lambda}{2} & \frac{\sqrt{2\lambda}}{4} D_- \Phi \\ -\frac{\lambda}{2} & -i\partial_- \Phi & \frac{\sqrt{2\lambda}}{4} D_- \Phi \\ i\sqrt{2\lambda D_- \Phi} & i\sqrt{2\lambda D_- \Phi} & \frac{i}{2} \end{pmatrix}.\]

These matrices are also gauge equivalent to each other by the gauge transformation (3.3)

\[\tilde{A}_\pm = G \tilde{B}_\pm G^{-1}.\]

These matrices also obey the following zero curvature condition

\[\partial_+ \tilde{A}_- - \partial_- \tilde{A}_+ + [\tilde{A}_-, \tilde{A}_+] = 0,\]

\[\partial_+ \tilde{B}_- - \partial_- \tilde{B}_+ + [\tilde{B}_-, \tilde{B}_+] = 0.\]

The zero curvature condition for the given connection can be reformulated as a Lax equation with a given Lax operator. The Lax operator associated with the linear eigenvalue equation

\[L \Psi = \frac{\lambda}{2} \Psi,\]

and solves the super sine-Gordon equation in the sense of the inverse scattering method.

4 Conclusion

In summary, we have presented two sets of Riccati equations for the super sine-Gordon equation. These equations are then used to derive the super Bäcklund transformation, the
linear system and the linear eigenvalue problem. The linear systems (3.1)-(3.2) obtained by the two super Riccati equations are shown to be related to each other by a gauge transformation. By writing the linear system in superspace we have shown how the zero curvature formulation can be obtained for fermionic as well as bosonic superfields. This work can be further extended to investigate the Darboux transformation of the super sine-Gordon theory. In fact, the Darboux transformation generates multi-soliton solutions and can be related to the Hirota’s bilinear formalism of the super sine-Gordon equation. These investigations shall be presented in a separate work.

Acknowledgment

We acknowledge the enabling role of the Higher Education Commission Pakistan and appreciate its financial support through “Merit Scholarship Scheme for PhD studies in Science & Technology (200 Scholarships)”. We also acknowledge CERN scientific information Service (publication requests).

References

[1] M. J. Ablowitz and P. A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press (2003).

S. Novikov, S. Manakov, L. B. Pitaevskii, V. E. Zakharov, Theory of Solitons, Plenum (1984).

G. L. Lamb, Jr., Elements of Soliton Theory, John Wiley & Sons (1980).

[2] C. S. Gardner, J. M. Greene, M. D. Kruskal, R. M. Miura, Phys. Rev. Lett. 19 (1976) 1095.

[3] V. E. Zakharov and A. B. Shabat, Sov. Phys. JETP, 37 (1973) 823, Funct. Anal. Appl. 13 (1979) 166.

[4] M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, Phys. Rev. Lett. 30 (1973) 1262, Phys. Rev. Lett. 31 (1973) 125.

[5] P. D. Lax, Commun. Pure and Appl. Math. 21 (1968) 467.

[6] H. H. Chen, Phys. Rev. Lett. 33 (1974) 925.

[7] B. Kupershmidt, Elements of Super Integrable Systems, Kluwer Academic Publisher (1987).
[8] H. Aratyn, T. D. Imbo, W-Y. Keung and U. Sukhatme (Eds.), *Supersymmetry and Integrable Models*, Springer (1998).

[9] S. Ferrara, L. Girardello and S. Sciuto, Phys. Lett. **B76** (1978) 303.

[10] L. Girardello and S. Sciuto, Phys. Lett. **B77** (1978) 267.

[11] M. Chaichian and P. P. Kulish, Phys. Lett. **B78** (1978) 413.

[12] S. Sciuto, Phys. Lett. **B90** (1980) 75.

[13] M. Gürses and O. Oğuz, Phys. Lett. **A108** (1985) 437.

[14] Y. I. Manin, A. Radul, Commun. Math. Phys. **98** (1985) 65.

[15] P. Mathieu, Phys. Lett. **B203** (1988) 287, J. Math. Phys. **29** (1988) 2499.

[16] H. Aratyn, A. K. Das and C. Rasinarik, Mod. Phys. Lett. **A12** (1997) 2623.

[17] H. Aratyn and A. Das, “*The sAKNS Hierarchy*”, solv-int/9710026.

[18] C. Morosi, L. Pizzocchero, Commun. Math. Phys. **176** (1996) 353.

[19] Q. P. Liu and M. Manas, Phys. Lett. **B485** (2000) 293.

[20] J. M. Evans, M. Hassan, N. J. MacKay, A. J. Mountain, Nucl. Phys. **B580** (2000) 605.

[21] U. Saleem and M. Hassan, Eur. Phys. J. **C38**, (2005) 521.

[22] D. Bernard and A. Leclair, Phys. Lett. **B247** (1990) 309.

[23] C. Ahn, Nucl. Phys. **B354** (1991) 57.