Random hermitian matrices in an external field

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ABSTRACT

In this article, a model of random hermitian matrices is considered, in which the measure \( \exp(-S) \) contains a general \( U(N) \)-invariant potential and an external source term: \( S = N \text{tr}(V(M) + MA) \). The generalization of known determinant formulae leads to compact expressions for the correlation functions of the energy levels. These expressions, exact at finite \( N \), are potentially useful for asymptotic analysis.

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1. Introduction.

As first suggested by Wigner [1], random matrices can be used to simulate Hamiltonians of complex systems. In this approach, one would like to characterize the structure of the energy levels, which are represented by the eigenvalues of a large matrix that can be assumed random. It is now known that many statistical properties of spectra of true physical systems are indeed well described by those of random matrices (cf [9] for a review): it is therefore important to understand how much these spectral properties depend on the particular matrix ensemble chosen, i.e. determine universality classes of matrix ensembles.

We shall consider here ensembles of hermitian matrices only, which correspond to systems without time-reversal invariance. Let us introduce the probability distributions $\rho_n$ of the eigenvalues: If $M$ is a random hermitian $N \times N$ matrix, we define $\rho_n(\lambda_0, \ldots, \lambda_{n-1})$ to be the density of probability that $M$ has $(\lambda_0, \ldots, \lambda_{n-1})$ among its $N$ eigenvalues, with the normalization convention that: \[ \int \prod_{i=0}^{n-1} d\lambda_i \rho_n(\lambda_0, \ldots, \lambda_{n-1}) = 1. \] Following [2], we also define $R_n = \frac{N^n}{(N-n)!} \rho_n$, a different normalization which makes the connection with the correlation functions

\[ \left\langle \prod_{i=0}^{n-1} \text{tr} \delta(M - \lambda_i) \right\rangle = R_n(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}) \text{ for distinct } \lambda_i \] (1.1)

since these quantities only differ by $\delta$ functions for coinciding eigenvalues.

In the case of a $U(N)$-invariant measure, consisting of a simple potential term, which we briefly review in section 2, the theory of orthogonal polynomials allows to write down exact expressions at finite $N$ for these functions, in terms of a single kernel $K(\lambda, \mu)$ [2]:

\[ R_n(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}) = \det(K(\lambda_i, \lambda_j))_{i,j=0\ldots n-1}. \] (1.2)

The study of the distribution of eigenvalues then boils down to the analysis of this kernel; in particular the asymptotics of $K$ as $N \to \infty$ allow to compute asymptotics of correlation functions and find the different “universal” properties that arise in this limit. Therefore it seems quite interesting to find similar formulae for more general measures. In fact, in recent papers [12], Brézin and Hikami have shown that formula (1.2) can be generalized to the gaussian
ensemble with an external field. More precisely, for the (unnormalized) measure
\[ \exp \left( -\frac{N}{2} \text{tr} M^2 + N \text{tr} MA \right) d^N M, \]  
they introduced the kernel
\[ \tilde{K}(\lambda, \mu) = \frac{1}{N} \int dt \int dv \prod_{l=0}^{N-1} \left( \frac{it/N - a_l}{v - a_l} \right) \frac{1}{it/N - v} e^{-\frac{N}{2}v^2 - t^2/2N - it\lambda + Nv\mu} \]  
(the \( \sim \) is here to distinguish this kernel from a slightly different one that will be introduced later) where the \( a_l \) are the eigenvalues of the hermitian matrix \( A \) and the contour integral encircles these eigenvalues. (1.2) then holds with \( \tilde{K} \) instead of \( K \).

The aim of this paper is to define a kernel \( K \) such that (1.2) still holds in the more general case of an arbitrary potential with an external source term, which is the subject of section 3. As an example we consider in section 4 the gaussian ensemble with an external field and reproduce the kernel \( \tilde{K} \) obtained in [12].

2. The \( U(N) \)-invariant case.

Let us consider an ensemble of random hermitian \( N \times N \) matrices with the measure
\[ Z^{-1} \exp \left( -N \text{tr} V(M) \right) d^N M \]  
where \( V \) is a polynomial and \( Z \) the partition function. A classical result [2] expresses the distribution law \( \rho_n \) of \( n \) eigenvalues \( 1 \leq n \leq N \) of \( M \) in terms of the kernel
\[ K(\lambda, \mu) = \sum_{k=0}^{N-1} F_k(\lambda)F_k(\mu). \]  
Here \( F_1 \) is the orthonormal function associated to the usual orthogonal polynomial \( P_1(\lambda) = \lambda^i + \cdots \):
\[ F_1(\lambda) = h_i^{-1/2} P_1(\lambda) \exp \left( -\frac{N}{2} V(\lambda) \right) \]
\[ \int d\lambda \exp(-NV(\lambda))P_1(\lambda)P_2(\lambda) = h_i\delta_{ij}. \]  
(see [11] for a review of orthogonal polynomials in matrix models). Let us briefly rederive this result in a manner that naturally generalizes. As the measure (2.1) only depends on the
eigenvalues of $M$, the integration over the angular variables is trivial and one finds:

$$\rho_N(\lambda_0, \lambda_1, \ldots, \lambda_{N-1}) = Z^{-1} \Delta^2(\lambda_i) \exp \left( -N \sum_{i=0}^{N-1} V(\lambda_i) \right).$$

(2.4)

The Van der Monde determinant $\Delta(\lambda_i) = \det(\lambda_i^j)_{i,j=0\ldots N-1}$ can be rewritten in terms of the orthogonal polynomials:

$$\rho_N(\lambda_0, \lambda_1, \ldots, \lambda_{N-1}) = Z^{-1} \det(P_k(\lambda_i))_{i,k=0\ldots N-1} \det(P_k(\lambda_j))_{j,k=0\ldots N-1} \exp \left( -N \sum_{i=0}^{N-1} V(\lambda_i) \right).$$

(2.5)

One can now easily compute $Z = N! \prod_{i=0}^{N-1} h_i$ by integrating over all $\lambda_i$. Combining the two determinants, we finally obtain:

$$\rho_N(\lambda_0, \lambda_1, \ldots, \lambda_{N-1}) = \frac{1}{N!} \det(K(\lambda_i, \lambda_j))_{i,j=0\ldots N-1}.$$

(2.6)

The kernel $K$ has the following properties:

$$\begin{align*}
K(\lambda, \mu) &= K(\mu, \lambda) \\
[K * K](\lambda, \rho) &\equiv \int d\mu K(\lambda, \mu) K(\mu, \rho) = K(\lambda, \rho)
\end{align*}$$

(2.7)

i.e. it is the orthogonal projector on the subspace spanned by the $F_k$, $0 \leq k \leq N - 1$. Using the property $K * K = K$ and noting that

$$\rho_n(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}) = \int d\lambda_n \rho_{n+1}(\lambda_0, \lambda_1, \ldots, \lambda_n),$$

(2.8)

one can then show inductively that

$$\rho_n(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}) = \frac{(N-n)!}{N!} \det(K(\lambda_i, \lambda_j))_{i,j=0\ldots n-1}.$$

(2.9)

for any $n \leq N$. This is equivalent to formula (1.2).
3. Generalization to the case of an external field.

We shall now see that in the case of a general measure with an external field, (1.2) still holds; a simple expression for a kernel \( K \) will be derived. Let us indeed consider the measure:

\[
Z^{-1} \exp \left( -N \text{ tr } V(M) + N \text{ tr } MA \right) d^N M
\]  

(3.1)

where \( V \) is an arbitrary polynomial, and \( A = \text{diag}(a_0, \ldots, a_{N-1}) \) can be assumed diagonal.

Particular matrix models of this type and their large \( N \) study appear in many papers [5,7]. Here we shall go beyond the \( 1/N \)-expansion and write exact expressions at finite \( N \).

One diagonalizes \( M \): if \( M = \Omega \Lambda \Omega^\dagger \) where \( \Lambda = \text{diag}(\lambda_0, \ldots, \lambda_{N-1}) \), the integral over \( \Omega \) is the usual Itzykson–Zuber integral [4] on the unitary group and we find:

\[
\rho_N(\lambda_0, \lambda_1, \ldots, \lambda_{N-1}) = Z^{-1} \Delta(\lambda_i) \frac{\det(\exp N \lambda_j a_l)}{\Delta(a_l)} \exp \left( -N \sum_{i=0}^{N-1} V(\lambda_i) \right) .
\]  

(3.2)

We replace as usual powers of \( \lambda \) in the Van der Monde with the orthogonal polynomials \( P_k(\lambda) \) of the measure \( \exp -NV(\lambda) d\lambda \). \( Z \) can now be computed:

\[
Z = \frac{1}{\Delta(a_l)} \int \prod_{i=0}^{N-1} d\lambda_i \det(P_k(\lambda_i)) \exp N \sum_{i=0}^{N-1} (-V(\lambda_i) + a_i \lambda_i)
\]

\[
= \frac{1}{\Delta(a_l)} \det \left( \int d\lambda P_k(\lambda) \exp N (-V(\lambda) + a_l \lambda) \right)_{k,l=0 \ldots N-1}.
\]  

(3.3)

Inserting (3.3) into (3.2) yields

\[
\rho_N(\lambda_0, \lambda_1, \ldots, \lambda_{N-1}) = \frac{1}{N!} \frac{\det(P_k(\lambda_i))_{i,k=0 \ldots N-1} \det(\exp Na_l \lambda_j)_{j,l=0 \ldots N-1}}{\det(\int d\lambda P_k(\lambda) \exp N (-V(\lambda) + a_l \lambda))_{k,l=0 \ldots N-1}} \exp \left( -N \sum_{i=0}^{N-1} V(\lambda_i) \right) .
\]  

(3.4)

This formula has a remarkably simple structure, which can be made more explicit by introducing \( F_k(\lambda) = h_k^{-1/2} P_k(\lambda) \exp -\frac{N}{2} V(\lambda) \) as before, and \( G_l(\lambda) = \exp [Na_l \lambda - \frac{N}{2} V(\lambda)] \):

\[
\rho_N(\lambda_0, \lambda_1, \ldots, \lambda_{N-1}) = \frac{1}{N!} \frac{\det(F_k(\lambda_i))_{i,k=0 \ldots N-1} \det(G_l(\lambda_j)_{j,l=0 \ldots N-1}}{\det(\int d\lambda F_k(\lambda) G_l(\lambda))_{k,l=0 \ldots N-1}}.
\]  

(3.5)

The matrix \( (\int d\lambda G_l(\lambda) F_k(\lambda))_{l,k=0 \ldots N-1} \) possesses an inverse, which we denote by \( \alpha_{kl} \); putting
together the three determinants we finally obtain:

$$\rho_N(\lambda_0, \lambda_1, \ldots, \lambda_{N-1}) = \frac{1}{N!} \det(K(\lambda_i, \lambda_j))_{i,j=0 \ldots N-1} \quad (3.6)$$

where

$$K(\lambda, \mu) = \sum_{k,l=0}^{N-1} F_k(\lambda) \alpha_{kl} G_l(\mu). \quad (3.7)$$

The kernel $K$ satisfies the property:

$$[K \ast K](\lambda, \rho) = \sum_{k,k',l,l'=0}^{N-1} \alpha_{kl} F_k(\lambda) \left[ \int d\mu G_l(\mu) F_{k'}(\mu) \right] \alpha_{k'l'} G_{l'}(\rho) \alpha_{k'l'} = K(\lambda, \rho). \quad (3.8)$$

Thus, one can follow the same line of reasoning as in the $U(N)$-invariant case to obtain the determinant formulae

$$\rho_n(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}) = \frac{(N-n)!}{N!} \det(K(\lambda_i, \lambda_j))_{i,j=0 \ldots n-1} \quad (3.9)$$

for any $n \leq N$.

The kernel $K$ is of the form

$$K(\lambda, \mu) = \sum_{k=0}^{N-1} F_k(\lambda) \hat{F}_k(\mu) \quad (3.10)$$

with $\hat{F}_k(\mu) = \sum_l \alpha_{kl} G_l(\mu)$; but $\hat{F}_k \neq F_k$ and therefore $K$ is not symmetric. Further analysis of $K$ resides in the observation that as $a \to 0$,

$$\int d\lambda P_k(\lambda) \exp N(-V(\lambda) + a\lambda) = h_k \frac{N^k}{k!} a^k + O(a^{k+1}) \quad (3.11)$$

due to the orthogonality of the $P_k$. The first consequence is that it is only in the limit $A \to 0$ that the $G_l$ become linear combinations of the $F_k$, $0 \leq k \leq N - 1$, ensuring that $\hat{F}_k \to F_k$, as
expected. The second consequence is that the quantity

\[
a^{-N} \prod_{l=0}^{N-1} (a - a_l) \int d\lambda P_k(\lambda) \exp N(-V(\lambda) + a\lambda)
\]

is well-defined for any \( k \geq N \); we can define the inverse Laplace transform

\[
\Phi_k = \exp \frac{N}{2} V(\lambda) \partial^{-N} \prod_{l=0}^{N-1} (\partial - a_l) [P_k(\lambda) \exp -NV(\lambda)] ,
\]

(where \( \partial \equiv 1/N d/d\lambda \)) which is a regular function with fast decrease at infinity. One can see that we now have a complete set of eigenvectors of \( K \):

\[
\left\{ \begin{array}{l}
\int K(\lambda, \mu) F_k(\mu) d\mu = F_k(\lambda) \quad \forall k < N \\
\int K(\lambda, \mu) \Phi_k(\mu) d\mu = 0 \quad \forall k \geq N.
\end{array} \right.
\]

Again, it is only when \( A = 0 \) that \( \Phi_k = F_k \).

We conclude that \( K \) is a non-orthogonal projector on the space spanned by the \( F_k \), \( 0 \leq k \leq N - 1 \).

4. Example: the gaussian ensemble.

In the case where \( V(M) = \frac{1}{2}M^2 \), the kernel \( K \) defined in eq. (3.7) should be related to the kernel \( \tilde{K} \) of [12]. This is indeed what we find.

The orthogonal polynomials \( P_k(\lambda) \) are Hermite polynomials, their normalization is \( h_k = \sqrt{\frac{2}{\pi N k!}} N^k \), and one can compute explicitly the integral

\[
\int d\lambda P_k(\lambda) e^{N(-\frac{1}{2}\lambda^2 + a\lambda)} = \sqrt{\frac{2\pi}{Na^k}} e^{\frac{N}{2}a^2}
\]

in accordance with eq. (3.11). One can also directly calculate \( Z \) by changing variables from
\( M \) to \( M - A \) in the gaussian matrix integral: this immediately gives

\[ Z \sim \exp \left( \frac{N}{2} \sum_{l=0}^{N-1} a_l^2 \right). \tag{4.2} \]

up to \( A \)-independent factors. We deduce the explicit form of the kernel \( K \):

\[ K(\lambda, \mu) = \sqrt{\frac{N}{2\pi}} \sum_{k,l=0}^{N-1} P_k(\lambda) \beta_{kl} e^{N(a_l \mu - \frac{1}{4}(\lambda^2 + \mu^2) - \frac{1}{2}a_l^2)}. \tag{4.3} \]

where \( \beta_{kl} = \alpha_{kl} \sqrt{\frac{2}{N}} \exp \frac{N}{2} a_l^2 \) is the inverse of the Van der Monde matrix \( (a_l^k) \). By definition one has:

\[ \sum_{k=0}^{N-1} \beta_{kl} a^k = \prod_{l' \neq l} \frac{a - a_{l'}}{a_l - a_{l'}}. \tag{4.4} \]

which allows to express \( \beta_{kl} \) in terms of symmetric functions of the \((a_l^k)_{l' \neq l}\). We now use an inverse formula of eq. (4.1):

\[ P_k(\lambda) = \int_{-i\infty}^{+i\infty} \frac{da}{i\sqrt{2\pi N}} a^k e^{N(\frac{1}{4}a^2 - a\lambda + \frac{1}{4}\lambda^2)} \tag{4.5} \]

where the integration is along the imaginary axis. Combining eq. (4.3), (4.4), and (4.5) we obtain:

\[ K(\lambda, \mu) = \sum_{l=0}^{N-1} \int_{-i\infty}^{+i\infty} \frac{da}{2\pi i} \prod_{l' \neq l} \frac{a - a_{l'}}{a_l - a_{l'}} e^{N(\frac{1}{4}a^2 - a\lambda + \frac{1}{4}\lambda^2 - \frac{1}{4}\mu^2 - \frac{1}{2}a_l^2 + a_{l'} \mu)}. \tag{4.6} \]

This can in turn be represented by a contour integral in the complex plane which picks up poles at \( v = a_l \):

\[ K(\lambda, \mu) = \int_{-i\infty}^{+i\infty} \frac{da}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{dv}{2\pi i} \prod_{l=0}^{N-1} \left( \frac{a - a_l}{v - a_l} \right) \frac{1}{a - v} e^{N(\frac{1}{4}a^2 - a\lambda + \frac{1}{4}\lambda^2 - \frac{1}{4}\mu^2 - \frac{1}{2}a_l^2 + a_{l'} \mu)}. \tag{4.7} \]

The redefinition

\[ \tilde{K}(\lambda, \mu) \equiv e^{-\frac{N}{2}\lambda^2} K(\lambda, \mu) e^{+\frac{N}{2}\mu^2}, \tag{4.8} \]
gives
\[
\tilde{K}(\lambda, \mu) = \int_{-i\infty}^{+i\infty} \frac{da}{2\pi i} \int dv \frac{dv}{2\pi i} \prod_{l=0}^{N-1} \left( \frac{a-a_l}{v-a_l} \right) \frac{1}{a-v} e^{N\left(\frac{1}{2}a^2-a\lambda-\frac{1}{2}v^2+\nu\mu\right)}.
\] (4.9)

which coincides with eq. (1.4) if one sets \(a = it/N\). Notice that transformation (4.8) does not affect the value of determinants of the type (1.2).

5. Conclusion.

We have derived determinant formulae for the correlation functions of eigenvalues in terms of a kernel \(K\) which has very simple algebraic properties: it is a projector on a space of dimension \(N\); its eigenvectors are known, even though they are not as simple as in the \(U(N)\)-invariant case, since \(K\) is no longer symmetric.

For the simple gaussian ensemble, the asymptotic behavior of the kernel \(K(\lambda, \mu)\) when \(\lambda - \mu\) is of order \(1/N, N \to \infty\), has been known for a long time [2,3]:
\[
K(\lambda, \mu) \sim \frac{\sin s}{s}, \quad s \sim N(\lambda - \mu).
\] (5.1)

It is remarkable that in the general \(U(N)\)-invariant case [8] as well as in the gaussian case with an external source [12], this behavior remains identical; combined with the determinant formulae, it implies universal properties for the correlation functions, as well as for the level spacing distribution. The latter quantity, first introduced by Wigner [1], is empirically known to be universal for a wide range of models. Here, it can be expressed in terms of \(K\) using the determinant formulae, and the relation takes a particularly simple form in the domain of universality, that is when one considers intervals of order \(1/N\): if \(p(s)\) is the level spacing distribution, the spacing variable \(\theta\) being defined by \(s = N\theta\), one has
\[
p(s) = \frac{d^2}{ds^2} \det(1 - \hat{K})
\] (5.2)

where \(\hat{K}\) is the asymptotic form of \(K\) in the interval \([-\theta/2, \theta/2]\) as \(N \to \infty, \theta \to 0, s\) fixed (cf eq. (5.1)), and \(\det\) is the usual Fredholm determinant. Thus, the universality of \(K\) implies the short distance universality of the level spacing. As opposed to the universal behavior of
the correlation function $\rho_2(\lambda, \mu)$ when $\lambda - \mu$ is of order 1, which can be obtained by various standard large $N$ techniques [6,8,10], the short distance universal behavior is more difficult to find, and its generalization to an arbitrary potential with an external source (using the newly found kernel $K$) is currently under study.

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