Ferromagnetic phase transition in spinor Bose gases

Qiang Gu
Institut für Laser-Physik, Universität Hamburg,
Luruper Chaussee 149, 22761 Hamburg, Germany

Abstract
The achievement in cooling alkali atomic gases, such as $^{87}$Rb, $^{23}$Na and $^7$Li, to quantum degeneracy opens up a way to study magnetism in spinor bosons, because these constituent atoms usually have a hyperfine spin degree of freedom. This article reviews several basic problems related to the ferromagnetic phase transition in spinor atomic Bose gases from a theoretical perspective. After a brief discussion on various possible origins of the ferromagnetic interaction, the phase diagram of the ferromagnetically coupled spinor bosons is investigated. It is found that the ferromagnetic transition occurs always above Bose-Einstein condensation and the Bose condensate is fully polarized. The low-lying collective excitations of the spinor condensate, including spin and density modes, are discussed. The spectrum of the density mode is of the Bogoliubov form and the spin wave spectrum has a $k^2$-formed dispersion relation at long wavelengths. The spin-wave stiffness coefficient contains contributions from both the “normal” and the condensed part of the gas.

1 Introduction
Itinerant ferromagnetism of Fermi (electron) gases has long been a research topic in solid state physics [1]. This phenomenon has already been well understood today, although not everything is clear. Models based on a picture of almost free electrons is fairly useful in describing the ferromagnetism. It is well-known that a free electron gas exhibits paramagnetism (called Pauli paramagnetism), owing to the existence of the Fermi surface. If there is a magnetization density $M$ in the gas, the Fermi surfaces for spin-up and spin-down electrons are split and consequently the band energy would be increased. When an effective ferromagnetic (FM) exchange $I_s$ is present, electron gases can exhibit ferromagnetism. Within the framework of the Hartree-Fock-Stoner theory, $I_s$ results in a negative molecular field energy when $M$ is finite. As long as $I_s$ is large enough, the value of the molecular field energy becomes larger than that of the increase of the band energy caused by $M$. In
this case, a FM ground state is energetically favored. The critical value of $I_s$ is called the Stoner threshold.

Relatively, the magnetism of Bose gases was less studied in history. The reason why it was so may be partially owing to the fact that liquid $^4$He, the most prototypical Bose system being studied in earlier years, is a system of scalar particles and does not display magnetism at all. The first example of Bose gases with internal degrees of freedom should be the cold atomic hydrogen which received much attention in 1980s. The hydrogen has hyperfine spin $F$ and thus it had ever attracted some theoretical interest in studying its magnetic properties. However the original purpose in surveying this atomic gas is to explore gaseous Bose-Einstein condensation (BEC), which is the intrinsic phase transition in the Bose gas. Since the ultimate goal of BEC was not yet attained at that time, efforts towards other aspects, e.g. the magnetism, were quite limited.

Great changes took place in 1995, when BEC was experimentally realized in alkali atomic gases, such as $^{87}$Rb [2], $^{23}$Na [3] and later in $^7$Li [4]. Since then, research works on the physics of ultracold atomic gases have grown explosively in the communities of atomic physics, quantum optics and many-body physics [5, 6]. Meantime, research interest has already gone beyond BEC itself. Alkali atoms also have hyperfine spins, as the atomic hydrogen does. So magnetism of spinor atomic gases is now among the most active research topics in this field.

Earlier experiments leading to BEC in alkali atoms were performed in magnetic traps. A relatively strong external magnetic field $H$ was applied to confine the BEC system. Because the atomic spin direction adiabatically followed $H$, the spin degree of freedom was frozen. As a result, the atoms behaved like scalar particles although they carried spin. In 1998, Ketterle’s group at MIT succeeded in confining the atomic condensate in optical traps [7]. The condensate was first produced in an magnetic trap, then transferred into the optical trap for further study. In 2001, Barrett et al. realized all-optical formation of BEC in which the Bose condensate was created directly in the optical trap [8]. The optical trap is a radiation electric field with a intensity maximum in space created by focusing laser beams. If the frequency of light is detuned to the red, the energy of a ground-state atom has a spatial minimum, and then the atom is confined by the electric field. A far-detuned optical trap can confine all the hyperfine states equally. With the external magnetic field sufficiently low, the spin degree of freedom of optically trapped atoms remains active. So investigating magnetic properties of spinor condensates becomes experimentally possible.

Theoretical investigation of ferromagnetism in spinor bosons can be traced back to the early studies on the atomic hydrogen. Sigia and Ruckenstein proposed that the magnetically trapped spin-polarized hydrogen could exhibit a coherent ferromagnetism, based on a phenomenological model [9]. The spontaneous magnetization is perpendicular to the stabilizing magnetic field. Yamada considered an ideal spin-1 Bose gas [10] and argued that the spinor bosons could exhibit an intrinsic ferromagnetism associated with BEC. He showed that the magnetization $M(H)$ of the ideal spinor Bose gas was finite even if $H = 0$ once BEC took place, which suggested that the system is magnetized spontaneously [10, 11]. He called this phenomenon the Bose-Einstein ferromagnetism. Caramico
D’Auria et al. drew the same conclusion for interacting bosons (in which the interaction is spin-independent) [12]. These results appropriately reveal that the spinor Bose gas is rather apt to be magnetized by an external magnetic field.

The realization of optical confining of atomic condensate significantly provoked theoretical interest in magnetism of alkali atoms. Soon after the success of the MIT group, Ho [13] and Ohmi and Machida [14] studied the spinor nature of the $F = 1$ atomic condensate. They pointed out that a hyperfine spin-spin exchange interaction coming from the s-wave scattering can be the dominant interaction in these gases. The condensate can be either ferromagnetic or “polar”, depending on the sign, but regardless of the value, of the exchange interaction. They also discussed the collective excitations in these two kinds of condensates, including density and spin modes. Spectra of the density mode are of the Bogliubov form in both cases, while the spin-wave spectra have different dispersion relation.

The reason why the spinor bosons are so sensitive to the external field and the spin-dependent interaction is attributed to the following fact: without spin-dependent interactions, the ground states of spinor bosons are degenerate and both the ferromagnetic and the “polar” states are among the ground states [15]. Therefore an infinitesimal external field or spin-dependent interaction can lift the degeneracy, leading to a ferromagnetic or polar state.

In this article, some recent progress on the ferromagnetism in spinor atomic bosons is reviewed. In Section 2 we explore the origin of the ferromagnetism. Various mechanisms of the ferromagnetic couplings between atomic bosons are discussed. The phase diagram of the ferromagnetically coupled spinor bosons is studied in Section 3. Critical temperatures of both BEC and the FM transition are calculated. The Goldstone modes accompanying the two phase transitions are discussed in Section 4, with special attention to the spin wave. Our discussions are dedicated to the 3-dimensional homogeneous Bose system in the thermodynamic limit. Experimentally, atomic gases are in a different situation. Some of those experimental conditions are briefly discussed in the final section.

Bosons with internal degree of freedom could be formed in Fermi gases through spin-triplet Cooper pairing, because a pair of bound fermions as a whole behaves like a boson. The triplet Cooper pairing has been observed in superfluid $^3$He and in some solid state materials. Although the Cooper pair is not identical to a boson because Cooper pairing is not local and different pairs are strongly overlapped in space, one can expect that triplet Cooper pairs exhibit similar behaviors to those of the spinor bosons in some aspects. In this article, ferromagnetism in triplet-Cooper-paired Fermi gases is also concerned.

2 Spin-dependent interactions in atomic bosons

Interatomic forces in dilute atomic gases are rather weak. The spin-dependent interactions are even much smaller than the spin-independent one. Nevertheless, they are key ingredient in understanding magnetism of the spinor bosons. Up to now, the following spin-dependent interactions have been theoretically discussed. In some cases, they can be ferromagnetic.
2.1 Scattering between different internal states

The interaction between atoms $V_{at}(r)$ should be a function of the separation $r$ of the two atoms. However, it is impossible to evaluate the $r$-dependence of the interaction in detail, especially at short separations. When $r$ is comparable to the atomic size, the standard Born-Oppenheimer separation of the centers of mass and the internal electronic structures of involved atoms may break down. Consequently, atoms can not be regarded as point particles any longer and $V_{at}(r)$ is not even definable [6]. Fortunately, $r$ is relatively large in dilute atomic gases, typically of order $10^2$ nm, at which the Born-Oppenheimer approximation should be good enough and $V_{at}(r)$ could be well defined and approximated by the van der Waals interaction. To avoid having to calculate short range interactions in detail, the true interatomic interaction $V_{at}(r)$ may be replaced by an effective interaction [16],

$$U(r = r_1 - r_2) = \frac{4\pi \hbar^2 a}{m} \delta(r_1 - r_2), \quad (1)$$

where $a$ is the low-energy s-wave scattering length. The s-wave scattering perfectly describes the interactions of dilute alkali atomic gases, with $a$ is of the order of $100a_B$ where $a_B$ is the Bohr radius.

In case that involved atoms have internal degree of freedom, for example the hyperfine spins for alkali atoms, interatomic interactions may give rise to transitions between different sub-states. Two atoms in the initial state $|\alpha, \beta\rangle$ may be scattered by collisions to the state $|\alpha', \beta'\rangle$, where $\alpha$ and $\beta$ denote the hyperfine states of the two atoms. In this case the scattering is a multi-channel problem and the scattering length could vary according to channels. That is to say, the scattering length $a_{\alpha\beta, \alpha'\beta'}$ is “spin-dependent”, with a large number of free parameters. The number of free parameters could be reduced by considering particle exchange, time-reversal and especially spin rotational symmetries. Suppose that the collision between atoms does not change the hyperfine spin $F$ (this is usually the case for sufficiently low energy collisions if the atoms are in the lowest hyperfine multiplet) of the individual atoms, the pairwise interaction keeps rotationally invariant in the hyperfine spin space. Then the collisions between two atoms with hyperfine spin $F_1$ and $F_2$ only depend on the total spin $f = F_1 + F_2$ and $U(r_1 - r_2)$ is reduced, in the s-wave limit, to [13]

$$U(r_1 - r_2) = \frac{4\pi \hbar^2}{m} \delta(r_1 - r_2) \sum_f a_f P_f, \quad (2)$$

where $a_f$ is the scattering length for collisions between two atoms with total spin $f$, and $P_f$ is the projection operator which projects the total spin of the pair of atoms into $f$ channel. $P_f$ satisfies the condition $\sum_f P_f = 1$. For two bosons with spin $F$, $f$ takes the values $f = 2F, 2F - 2, 2F - 4, ..., 0$. The $f = 2F - 1, 2F - 3, ...$ channels are forbidden owing to the exchange symmetry of two bosons. Likewise,

$$F_1 \cdot F_2 = \sum_f \lambda_f P_f, \quad (3)$$
with $\lambda_f = [f(f+1) - 2F(F+1)]/2$. Combining Eqs. (2) and (3), one can get the effective interaction in term of the hyperfine spin operators. In particular, the effective interaction for the spin-1 bosons is given by [13, 14]

$$U(r_1 - r_2) = \frac{4\pi \hbar^2}{m} [g_0 + g_2 F(r_1) \cdot F(r_2)] \delta(r_1 - r_2),$$

(4)

where $g_0 = (a_0 + 2a_2)/3$, $g_2 = (a_2 - a_0)/3$, and $F = (F_x, F_y, F_z)$ are $3 \times 3$ spin matrices. The $g_2$ term is a Heisenberg-like exchange interaction. Usually the difference between $a_2$ and $a_0$ is very small, so $g_2$ is much smaller than $g_0$.

Why could scattering lengths be spin-dependent? To answer this question, we recall that atoms have internal electronic structures. The pairwise interaction of two alkali atoms depends on the spin configuration of the two valence electrons. For example, the interaction has an attractive contribution when two valence electrons are in the singlet state since they can occupy the same orbital. Otherwise, this contribution is absent.

The exchange interaction can be ferro- or antiferro-magnetic, depending on $a_0$ is greater or less than $a_2$. In principle, $a_0$ and $a_2$ can be calculated directly by solving Schrödinger equation. But exact solutions are not get-at-able for many electron atoms. It is possible to check the sign of $g_2$ indirectly by studying spin dynamics of the spinor Bose condensate, since behaviors of the ferro- and antiferro-magnetic spinor condensates are quite different [7, 17, 18]. Both theory and experiment suggested that $g_2 < 0$ for the gas of $F = 1 \text{ } ^{87}\text{Rb}$ [7, 19], and $g_2 > 0$ for $F = 1 \text{ } ^{23}\text{Na}$ [7, 20]. The spin exchange is antiferromagnetic for the $F = 2 \text{ } ^{85}\text{Rb}$ and $^{87}\text{Rb}$ atoms [17–19].

### 2.2 Super-exchange process

Now let us consider a Bose gas moving on a periodic optical lattice. Such a system can be approximately described by the boson Hubbard model [21, 22],

$$H_{\text{Hubbard}} = - \sum_{\langle ij \rangle} t_{ij} a^+_i a_j + \frac{1}{2} U_0 \sum_i n_i (n_i - 1).$$

(5)

Here $\langle ij \rangle$ labels two nearest neighbor sites on the lattice, $U_0$ is the on-site Hubbard repulsion. The optical lattice is produced by the interference of laser beams. Two counter-propagating laser beams form a standing wave, which acts on the atoms as a periodic potential, $V(r) = \sum_i V_i \sin^2 k_i r_i$, where $k_i$ is the wave vector of the laser. For a given optical potential, $t_{\langle ij \rangle}$ and $U_0$ are readily evaluated [23]. The Hubbard repulsion is proportional to the s-wave scattering length.

As is well-known, the electron Hubbard model is used to describe the Mott insulator in condensed matter physics. Provided the Hubbard repulsion is large enough in comparison to the hopping matrix $t_{ij}$, the energy band of electrons splits into two branches separated by a energy gap. At half filling, the lower Hubbard band is fully filled while the upper one is empty and thus electrons are in the Mott insulating state. With the charge degree of freedom being frozen, the low-energy behaviors of electrons can be described by an effective spin
model. One can derive the effective spin Hamiltonian directly from the Hubbard model through perturbation approach, taking the hopping term as the perturbation. To the second order of the perturbation, the spin Hamiltonian is described by the Heisenberg model. The spin coupling between localized electrons, called the super-exchange, is antiferromagnetic.

The boson Hubbard model is introduced to account for the superfluid-Mott insulator transition in lattice bosons [22, 23]. It is argued that bosons are insulating in the large $U_0$ limit, as the correlated electrons do. Following the suggestion of [23], the superfluid-Mott insulator transition has already observed experimentally [21]. By analogy, one can expect that the spinor boson Hubbard model is reduced to an effective spin Hamiltonian, too.

Let us first look at a two-component Bose system, which can be mapped into a pseudo-spin-$\frac{1}{2}$ boson Hubbard model. Yang et al. [24] and Duan et al. [25] have derived the effective spin Hamiltonian,

$$H = J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j,$$

which is just the Heisenberg model. However, the super-exchange interaction between localized bosons is ferromagnetic, $J = -4t^2/U_0$, as opposed to electrons.

For bosons with integer spins, the effective spin Hamiltonian is relatively complicated. Imambekov et al. has investigated the spin-1 boson Hubbard mode in detail [26]. To order of $t^2/U_0$, the spin Hamiltonian consists of two terms,

$$H = J_1 \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j + J_2 \sum_{\langle ij \rangle} (\mathbf{S}_i \cdot \mathbf{S}_j)^2,$$

with $J_1 = J_2 = -2t^2/U_0$. The first term tends to stabilize a ferromagnetic order, while the second term favors a local singlet of the two coupled spins.

In above derivations we assume that the number of bosons is the same as the number of lattice sites, thus each site is occupied by only one particle. But for bosons, any occupation is allowed. The system could be a Mott insulator as long as the boson density is commensurate. Ref. [26] presented detailed discussions on the spin exchange at various commensurate occupations. Moreover, an antiferromagnetic on-site coupling between bosons is considered in Ref. [26]. $J_2/J_1$ increases with increasing either the on-site antiferromagnetic coupling or the occupation number, and the system tends to have a nematic rather than a ferromagnetic ground state.

### 2.3 Magnetic dipolar interaction

Corresponding to the (hyperfine) spin degrees of freedom, spinor atomic bosons have a magnetic moment $\mathbf{m} = \gamma \mathbf{S}$, which induces the magnetic dipolar interaction between particles,

$$U_{md} = \frac{\mu_0}{4\pi r^3} [\mathbf{m}_1 \cdot \mathbf{m}_2 - 3(\mathbf{m}_1 \cdot \mathbf{r})(\mathbf{m}_2 \cdot \mathbf{r})],$$

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where $\mu_0$ is the vacuum permeability. The influence of the magnetic dipolar interaction on the properties of spinor Bose-Einstein condensates has recently attracted considerable interest [27–30]. Although very weak, it is expected to largely enrich the variety of phenomena in ultracold Bose gases, especially in the gases of atoms with larger magnetic moments, such as europium where $m = 7\mu_B$ ($\mu_B$ is the Bohr magneton) and Chromium where $m = 6\mu_B$ [31].

The magnetic dipolar interaction is anisotropic and its strength depends not only on the separation but also on the spin configurations of the two interacting particles. Pu et al. studied the magnetism of an ensemble of spinor condensates confined in a lattice [28, 29]. The “minicondensate” at each site behaves as a localized mesoscopic magnet and interacts with each other via the magnetic dipolar interaction. They showed that the ground state is ferromagnetic in a chain, and antiferromagnetic in a square lattice.

However it is rather difficult to treat precisely the dipolar interaction between particles in a gas, because of the anisotropic feature and the long-range character of the interaction. Many efforts have been done to study the ground state of the magnetic (and electric) dipolar gases (or fluids). It was reported that a ferromagnetic/ferroelectric nematic order is favored under certain conditions [32, 33]. Recently, the dipolar ferromagnetism has also been predicted in ensembles of randomly distributed nano-particles [34].

Let’s demonstrate why the magnetic dipolar interaction can result in ferromagnetism from a mean-field viewpoint [30]. When all the magnetic moments are arranged parallel, an effective magnetic field $B = \mu_0 M$ is created inside the polarized body where $M$ is the magnetization density, owing to the superposition of the intrinsic field of all the magnetic moments. The energy density of the effective magnetic field is $f_m = B^2/(2\mu_0) = \mu_0 M^2/2$. On the other hand, the magnetic moment of spinor bosons does respond to the internal magnetic field, so the spin direction should follow $B$, which leads to an energy decrease: $f_c = -M \cdot B = -\mu_0 M^2$. Therefore the total free energy density is negative, $f_m + f_c = -\mu_0 M^2/2$. Contrarily, in the “polar” or “equal spin” states the magnetic moments of particles are compensated: $M = 0$. At the mean-field level, an internal reference particle can not sense the moments of other particles and the magnetic free energy is zero.

It is worth noting that the dipolar ferromagnetism could manifest itself more easily in a Bose gas than in a Boltzmann gas or a Fermi gas [30]. The ferromagnetically ordered state can be destroyed due to the entropy increase at finite temperatures. So a Boltzmann gas can show the dipolar ferromagnetism only at low temperatures comparable to the energy scale of the dipolar interaction. Since the Bose condensate has no entropy, the ferromagnetic state could survive in the Bose condensed particles below the BEC temperature which is relatively high in comparison to the energy scale of the dipolar interaction. As for the Fermi gas, it can hardly display the dipolar ferromagnetism, because the dipolar interaction is too weak to reach the Stoner threshold for the itinerant ferromagnetism. That’s why the dipolar interaction plays a less important role than exchange interactions in understanding magnetism in solid. The latter is much stronger.
3 Phase diagram of ferromagnetic spinor Bose gases

As suggested in the Weiss molecular field theory (for classical systems) and the Stoner theory (for Fermi systems) [1], the ferromagnetic coupling is expected to induce a ferromagnetic transition in Bose gases. On the other side, in a Bose system whose particle number is conserved there exists an intrinsic phase transition, Bose-Einstein condensation. It is natural to suppose that the FM transition temperature $T_F$ depends on the energy scale of the coupling $I$. Therefore $T_F$ is possibly smaller than $T_c$, the BEC temperature, for small values of $I$, and $T_F > T_c$ if $I$ is sufficiently large. Is that true?

In this section we study how the FM transition and BEC emerge in a spinor Bose system. We first calculate the phase transitions by dealing with a simple microscopic model, then analyze the predicted phase diagram from a phenomenological point of view. For simplicity, we consider a 3D homogeneous spinor Bose gas.

3.1 A microscopic model

In Ref. [35] we have investigated the interplay between the FM transition and BEC. The calculation starts with a simplified Hamiltonian given by

$$H = -t \sum_\sigma \langle a_i^\dagger \sigma a_j \sigma \rangle - I_H \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j. \quad (9)$$

The particles are treated as being on some kind of “lattice”. $\langle ij \rangle$ denotes two nearest-neighboring sites. The first term in the Hamiltonian represents the kinetic energy. In following calculations it is replaced by the kinetic energy of free particles with mass $m^*$, $\epsilon_k = \hbar^2 k^2 / 2m^*$.

The second term describes the FM coupling, which could be decoupled via the mean-field approximation,

$$- \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j \approx - \sum_{\langle ij \rangle} \langle \mathbf{S}_i \rangle \cdot \langle \mathbf{S}_j \rangle + \langle \mathbf{S}_i \rangle \cdot \langle \mathbf{S}_j \rangle - \langle \mathbf{S}_i \rangle \cdot \langle \mathbf{S}_j \rangle, \quad (10)$$

where $M_i = \langle \mathbf{S}_i \rangle$ serves as the ferromagnetic order parameter, and is chosen to be along the $z$ direction, $\langle \mathbf{S}_z \rangle = (0, 0, M_i)$ and $M_i = \langle S_i^z \rangle = \sum_\sigma \langle a_i^\dagger \sigma a_i \sigma \rangle$ where $a_i \sigma$ annihilates a boson with spin quantum number $\sigma$ at site $i$. We take $M_i = \hat{M}$ for a homogeneous boson gas. Following the Stoner theory for fermion gases, we call $H_m = Z I_H M$ the molecular field and $I_s = Z I_H$ the Stoner exchange, where $Z$ is the effective “coordination number” which is an irrelevant dimensionless parameter of order unity for a gas. Then the effective Hamiltonian reads

$$H - N\mu = \sum_{k\sigma} (\epsilon_k - \mu - \sigma H_m) n_{k\sigma}, \quad (11)$$

where $\mu$ is the grand canonical chemical potential.
The mean-field self-consistent equations consist of two equations,

\[ n = \frac{1}{V} \sum_{k\sigma} \langle n_{k\sigma} \rangle, \quad M = \frac{1}{V} \sum_{k\sigma} \sigma \langle n_{k\sigma} \rangle. \]

\( n = N/V \) is the density of particles, \( V \) is the volume of the system and \( N \) is the total number of particles. \( M \) is the magnetization density. Let us consider a \( F = 1 \) Bose gas with the self-consistent equations given by

\[ 1 = n_0 + t \frac{3}{2} \left[ f_{\frac{3}{2}}(a) + f_{\frac{3}{2}}(a + b) \right], \quad (12a) \]

\[ 1 = M_0 + t \frac{3}{2} \left[ f_{\frac{3}{2}}(a) - f_{\frac{3}{2}}(a + 2b) \right], \quad (12b) \]

where \( a = -(\mu + H_m)/(k_B T) \), \( b = H_m/(k_B T) \), the reduced temperature and coupling are given by \( t = k_B T m^* / (2\pi^2 n^2 / 3) \) and \( I = I_s n^1 / (2\pi^2 h^2) \), respectively, \( n_0 \) is the condensate density, \( \overline{n}_0 = n_0/n \) is the condensate fraction, \( \overline{M} = M/n \) is the normalized magnetization, and \( f_s(a) \) is the polylogarithm function defined as

\[ f_s(a) \equiv \text{Li}_s(e^{-a}) = \sum_{p=1}^{\infty} \frac{(e^{-a})^p}{p^s}. \]  \( (13) \)

We note that \( f_s(0) = \zeta(s) \), the Riemann zeta function.

Based on this simple model, we have shown that an infinitesimal FM coupling can induce a FM phase transition at a finite temperature above BEC. As is well-known, the BEC critical temperature \( t_c \) can be determined by calculating the chemical potential [36]: \( a > 0 \) for \( t > t_c \) and \( a \to 0 \) as \( t \to t_c \) from above. Assume a FM phase transition is induced by \( I \) at the transition temperature \( t_F \). We first suppose \( I \) is very large so that \( t_F > t_c \). Provided that the FM transition is continuous, i.e., \( b \to 0 \) with \( t \to t_F \), the mean-field equations become

\[ 1 = 3t_F \frac{3}{2} f_{\frac{3}{2}}(a_F), \quad (14a) \]

\[ 1 = 2t_F \frac{1}{2} f_{\frac{1}{2}}(a_F). \quad (14b) \]

where \( a_F = a(t_F) \). These equations define a relation between \( t_F \) and \( I \). \( I \) is a monotonically decreasing function of \( a_F \). As \( a_F \to 0 \), \( f_{\frac{3}{2}}(a_F) \to \zeta(3/2) \approx 2.612 \), and \( f_{\frac{1}{2}}(a_F) \approx \sqrt{\pi/a_F} \). So for small values of \( I \) and \( a_F \) we have

\[ a_F \approx 4\pi t_0 I^2, \quad (15) \]

where \( t_0 = 1/[3\zeta(3/2)]^{2/3} \) is the reduced BEC critical temperature for the Bose gas with \( I = 0 \), \( a_F \) is larger than zero at \( t_F \), which means BEC does not yet take place. The Bose gas undergoes BEC at a lower temperature.
Figure 1: Reduced temperature $t$ vs. reduced FM coupling $I$ phase diagram of spin-1 Bose gases. The paramagnetic normal phase (PM), ferromagnetic normal phase (FM), and coexisting phase of ferromagnetism and Bose condensation (FM+BEC) are indicated.

By solving the mean-field equations numerically, we can obtain that both the FM transition temperature and the BEC critical temperature increase with the FM coupling. Asymptotic analysis at very small $I$ shows that

$$\frac{\delta t_F}{t_0} \sim \frac{\delta t_c}{t_0} \propto I,$$

where $\delta t_F = t_F - t_0$ and $\delta t_c = t_c - t_0$. We should note that the asymptotic analysis is not appropriate for this simple model within mean-field approximation, because numerical results show that both FM transition and BEC are discontinuous at very small $I$. Approximately, the critical value of $I$ under which the transition becomes discontinuous is 0.35 for the FM transition and 0.2 for BEC. For larger couplings, the FM transition is well described as being continuous. This point is consistent with the Weiss molecular field theory for classical particles, in which the FM transition is continuous [1], because for large couplings, the FM transition occurs at a relatively high temperature, when the Bose statistics reduces to Boltzmann statistics.

Figure 1 shows the phase diagram of spin-1 Bose gases. The curves are obtained by solving the reduced self-consistent equations supposing both FM transition and BEC are continuous. Diamonds and circles denote numerical results of Eqs. (12a) and (12b). In order to make a comparison with the Fermi gas, Fig. 2 plots the schematic phase diagram for both the Bose and Fermi gases together.

### 3.2 Phenomenological analysis

In this subsection, we examine the above-predicted phase diagram on the basis of Ginzburg-Landau (GL) phenomenological theory [30]. The advantage of the GL theory is that one
Figure 2: Schematic relations between transition temperatures and FM couplings. $T_{F}^b$ and $T_{F}^f$ represent the FM transition temperature for Bose and Fermi gases respectively. $T_c$ and $T_0$ denote the BEC critical temperature for spinor bosons with and without couplings respectively. $I_0$ is the Stoner threshold.

can describe properties of a system by analyzing the symmetry of the system in a simple way, without a detailed knowledge of the microscopic background. It is applicable to the case of interacting bosons, in which the phase transitions, both FM transition and BEC, are supposed to be of second order.

To begin with, we derive an appropriate GL free energy density functional that describes the coexistence of BEC and ferromagnetism. Generally, such a free energy density consists of three different parts [30, 37]:

$$f_t(\Psi, \Psi^{\dagger}, M) = f_b(\Psi, \Psi^{\dagger}) + f_m(M) + f_c(\Psi, \Psi^{\dagger}, M),$$

(17)
corresponding to the Bose condensed phase, the ferromagnetic phase of the normal gas and the coupling between the two phases. Here $\Psi^{\dagger} \equiv (\Psi^{T})^* = (\Psi_F^+, \Psi_F^{*+1}, ..., \Psi_F^{*+F})$ is the complex order parameter of the spinor Bose condensate. The condensed Bose gas is described by the two-fluid model. We suppose $f_m(M)$ only represents the ferromagnetic phase of the normal gas, with the order parameter $M$ proportional to the magnetization density.

Following Ginzburg and Pitaevskii [38], the free energy density of an isotropic spin-$F$ Bose-Einstein condensate is modelled as

$$f_b = \frac{\hbar^2}{2m} \nabla \Psi^{\dagger} \cdot \nabla \Psi + \alpha |\Psi|^2 + \frac{\beta_0}{2} |\Psi|^4 + \frac{\beta_s}{2} \Psi^{\dagger}_\sigma \Psi^{*}_\sigma \cdot F_{\sigma\gamma} \cdot F_{\sigma\gamma'} \Psi^{\dagger}_\gamma \Psi_\gamma,$$

(18)

where $|\Psi|^2 = \Psi^{\dagger} \Psi$ and repeated sub-indices represent summation taken over all the hyperfine states. $\alpha = \alpha' (T - T_{c0})$ and $T_{c0}$ is the BEC critical temperature. Both $\alpha'$ and $\beta_0$ are positive parameters, and $\beta_0$ contains contributions of the spin-independent interactions.
The fourth term has SO(3) symmetry, arising from the spin-exchange interactions. \( \beta_s \) can be positive or negative, depending on the exchange interaction antiferromagnetic or ferromagnetic. Due to the \( \beta_s \) term, the time-reversal symmetry in the Bose condensate should be broken spontaneously.\(^1\) Hereinafter we consider the ferromagnetic case \( (\beta_s < 0) \).

As mentioned in the last subsection, a FM transition in normal gas happens above BEC. The free energy density for this ferromagnetic phase can be expanded in powers of \( |M|^2 \):

\[
f_m = c |\nabla M|^2 + a'(T - T_f) \frac{|M|^2}{2} + b |M|^4/4, \tag{19}\]

where \( a = a'(T - T_f) \), \( c \), \( a' \) and \( b \) are positive constants, \( T_f \) is the FM transition temperature in the normal Bose gas. We suppose that the ferromagnetic normal gas couples linearly to the spinor condensate,

\[
f_c = -g M \cdot \Psi^*_\sigma \mathcal{F}_{\sigma\gamma} \Psi \gamma, \tag{20}\]

with the coupling constant \( g > 0 \). This is the simplest coupling term that satisfies the physical situation.

We do not consider fluctuations in this subsection, so all the parameters are supposed to be given by their average: \( \langle \Psi_\sigma \rangle = \Phi_\sigma \) and \( \langle M \rangle = (0, 0, M_0) \) where the FM order parameter is chosen to be along the \( z \) direction for convenience. \( \Phi^\dagger_\sigma \Phi_\sigma = n_0 \) is just the density of condensed bosons. Hence the gradient terms in Eqs. (18) and (19) can be dropped for the homogeneous system.

Minimizing the total free energy \( f_t \) with respect to \( \Phi^\dagger \), one gets

\[
[a'(T - T_c^0) - g M_0 \sigma] \Phi_\sigma + \beta_0 |\Phi|^2 \Phi_\sigma + \beta_s \Phi^*_\sigma \mathcal{F}_{\sigma\gamma} \cdot \mathcal{F}_{\sigma'\gamma'} \Phi_\gamma \Phi_{\gamma'} = 0. \tag{21}\]

Then the stable solution reads

\[
|\Phi_0|^2 = |\Phi_{-1}|^2 = 0, \quad |\Phi_1|^2 = \frac{a'}{\beta_0 + \beta_s} \left( T - T_c^0 - \frac{g}{a'} M_0 \right). \tag{22}\]

Only the spin-1 bosons condense, occurring at an enhanced BEC transition temperature,

\[
T_c = T_c^0 + \frac{g}{a'} M_0. \tag{23}\]

Obviously, the magnetization in the normal gas promotes the BEC critical temperature. At \( T = T_c \), the order parameter of the condensate is zero, and we can derive the value of \( M_0 \) by minimizing \( f_m(M_0) \) with respect to \( M_0 \),

\[
M_0 = \sqrt{\frac{a'}{b} [T_f - T_c]}. \tag{24}\]

---

\(^1\)Using the notation \( \langle \Psi_1, \Psi_0, \Psi_{-1} \rangle = (\phi_1 e^{i\theta_1}, \phi_0, \phi_{-1} e^{i\theta_{-1}}) \), the \( \beta_s \) term becomes \( \beta_s \phi_0^2 (\phi_1^2 + \phi_{-1}^2 + 2 \phi_1 \phi_{-1} \cos(\theta_1 + \theta_{-1})) + \beta_s (\phi_1^2 - \phi_{-1}^2)^2/2 \). Since \( \theta_1 + \theta_{-1} = \pi \) for \( \beta_s > 0 \) and \( \theta_1 + \theta_{-1} = 0 \) for \( \beta_s < 0 \), it therefore determines the ground state of the condensate: it is ferromagnetic for \( \beta_s < 0 \) and “polar” for \( \beta_s > 0 \) [13, 14].
We now derive the phenomenological relations between the two transition temperatures. At very small ferromagnetic couplings, \( I \to 0 \), both \( \delta T_c = T_c - T^0_c \) and \( \delta T_f = T_f - T^0_c \) tend to zero. Substituting Eq. (24) into (23), one finds

\[
\delta T_c = \sqrt{T^* (\delta T_f - \delta T_c)},
\]

with \( T^* = (g/\alpha')^2 a'/b \). Under the condition that \( \delta T_f << T^* \) we have

\[
(\delta T_c) \approx \delta T_f \left( 1 - \frac{\delta T_f}{T^*} \right).
\]

Suppose \( \delta T_f \) increases linearly with the ferromagnetic coupling \( I \), \( \delta T_f = CI \), when \( I << 1 \), the coupling dependence of \( \delta T_c \) is given by the formula

\[
\delta T_c = CI \left( 1 - \frac{CI}{T^*} \right).
\]

Besides a linear term, \( \delta T_c \) also depends on \( I \) quadratically. So far, the phase diagram predicted in the previous subsection is roughly reproduced. It is noteworthy that the linear dependence of \( \delta T_f \) on \( I \) is only an assumption in phenomenological theory. To decide the precise relation between \( \delta T_f \) and \( I \) is definitely of theoretical interest and is still an open question.

In analogy to the ferromagnetically coupled spinor bosons, a similar system has been found in condensed matters, say, the ferromagnetic superconductors [39,40]. In these materials, such as \( \text{UGe}_2 \) [39] and \( \text{ZrZn}_2 \) [40], coexistence of ferromagnetism and superconductivity has been observed. The phase diagram, see Fig. 3, indicates that these materials first undergo a FM transition, then go into the superconducting state. Considering the strong magnetization in these materials, it is suggested that the Cooper pairing should be \( p \)-wave triplet, thus behaves like spin-1 bosons in some sense. Phenomenological theory has been performed to explain the phase diagram of FM superconductors [41,42]. This subsection is just an extension of the phenomenological theory for FM superconductors to spinor bosons.

### 4 Spin waves in spinor Bose condensates

Once a continuous symmetry is spontaneously broken, gapless Goldstone modes are expected. Spontaneous magnetization breaks the spin rotational symmetry and the corresponding Goldstone modes are referred as spin waves. In a ferromagnet, the dispersion relation of the spin wave takes the form \( \omega_s = c_s k^2 \) where \( k \) is the wave number, and the spin-wave stiffness coefficient \( c_s \) should depend on the ferromagnetic couplings. For example, the long wave-length spectrum of spin waves in a Heisenberg ferromagnet is \( \hbar \omega \sim |J|k^2 \) with the simple cubic structure, where \( J \) is the Heisenberg exchange.

The \( k^2 \)-formed spectrum can be derived qualitatively from the GL free energy density functional for FM transitions given by Eq. (19). Within GL theory, the Goldstone mode is
interpreted as transverse fluctuations on the average value of the order parameter. The full order parameter of the FM phase takes the form $\mathbf{M} = \langle \mathbf{M} \rangle + \delta \mathbf{M} = (\delta M_x, \delta M_y, M_0 + \delta M_z)$ where $\delta M_z$ is the longitudinal fluctuation and $\delta M_x, \delta M_y$ are transverse components. To linear order in $\delta \mathbf{M}$, the free energy is given by $f_m = f_m^0 + f_m^1$, with
\[
 f_m^0 = \frac{1}{2}aM_0^2 + \frac{1}{4}bM_0^4, \quad (27a)
\]
\[
 f_m^1 = \frac{1}{2}c\nabla \delta M_z \nabla \delta M_z + \frac{1}{2}(a + 3bM_0^2) \delta M_z \delta M_z 
+ \frac{1}{2}(a + bM_0^2) \delta M_+ \delta M_- . \quad (27b)
\]
Here $\delta M_+ = \delta M_x + i\delta M_y$ and $\delta M_- = \delta M_x - i\delta M_y$. Below FM transition point, $\partial f_m^0 / \partial M_0 = 0$ and we have $a + bM_0^2 = 0$. Under this condition, the dispersion relation of the transverse mode becomes gapless,
\[
 \hbar \omega_{\pm} = ck^2, \quad (28)
\]
while the longitudinal mode is gapped, $\hbar \omega_z = ck^2 + bM_0^2$ and thus this mode can be neglected at low energy. Here $c_s = c$ is a phenomenological parameter.

To proceed, we consider Goldstone modes in a spinor Bose condensate. This problem has received much attention recently [13, 14, 43–46]. Since both the conservation of particle numbers and the spin rotational symmetry are spontaneously broken in the ferromagnetic condensate, an interesting question arises: how the spin wave manifests itself therein. Ho [13] and Ohmi and Machida [14] studied the zero-temperature collective excitations in spinor condensates based on the Bogliubov approximation using an equation of motion approach. As they pointed out, the density, spin and “quadrupolar” spin fluctuations are related to $\delta \Psi_1, \delta \Psi_0, \text{and} \delta \Psi_{-1}$, respectively, since $\delta n = \sqrt{n_0}(\delta \Psi_1 + \delta \Psi_1^*)$. 

Figure 3: Schematic Phase diagram of the FM superconductor $\text{ZrZn}_2$. $T_F$ and $T_S$ are the FM and superconducting transition temperatures. $P$ is the pressure.
\[ \delta M_\perp = \sqrt{n_0} \delta \Psi^*_0 \text{ and } \delta M^2_\perp = 2 \sqrt{n_0} \delta \Psi^*_1. \]

A Bogliubov spectrum for density fluctuations and a \( k^2 \)-formed dispersion for spin fluctuations were derived,

\[
\hbar \omega_1 = \sqrt{\epsilon_k^2 + \frac{4\pi \hbar^2}{m} (g_0 + g_s) n_0 \epsilon_k}, \quad \hbar \omega_0 = \epsilon_k = \frac{\hbar^2 k^2}{2m}. \tag{29}
\]

consistent with available theories of BEC and ferromagnetism. Ueda obtained the same results through a many-body mean-field approach [43]. These results hold for a dilute weak-interacting atomic gas, in which the condensate fraction is almost equal to one at sufficiently low temperatures.

In the following, we generalize the equation of motion approach to finite temperature cases, especially near the BEC point, making use of GL theory [37]. The full order parameters for a condensate are written as \( \Psi^\dagger = (\Phi^*_1 + \delta \Psi^*_1, \delta \Psi^*_0, \delta \Psi^*_1) \). Adopting the results of Section 3.2 that only \( F = 1 \) bosons condense, the free energy density for a ferromagnetic condensate reads

\[
f_b = f^0_b + f^I_b, \tag{30a}
\]

\[
f^0_b = \alpha \Phi^2_1 + \frac{1}{2} (\beta_0 + \beta_s) \Phi^4_1, \tag{30b}
\]

\[
f^I_b = \frac{\hbar^2}{2m} \nabla \delta \Psi^*_\sigma \nabla \delta \Psi_\sigma + [\alpha + (\beta_0 + \beta_s) \Phi^2_1] \delta \Psi^*_\sigma \delta \Psi_\sigma + \frac{1}{2} (\beta_0 + \beta_s) \Phi^2_1 (\delta \Psi^*_1 + \delta \Psi_1)^2 - 2 \beta_s \Phi^2_1 \delta \Psi^*_1 \delta \Psi_1. \tag{30c}
\]

Here \( \beta = \beta_0 + \beta_s \). The equations of motion are obtained from \( f^I_b \) according to \( i \hbar \partial_t \delta \Psi_\sigma = (\partial f^I_b)/(\partial \delta \Psi^*_\sigma) \). We have

\[
i \hbar \partial_t \begin{pmatrix} \delta \Psi^*_1 \\ \delta \Psi_1 \\ \delta \Psi^*_0 \\ \delta \Psi_{-1} \end{pmatrix} = \begin{pmatrix} (\epsilon_k + \alpha + \beta \Phi^2_1) \delta \Psi^*_1 \\ \beta \Phi^2_1 (\delta \Psi^*_1 + \delta \Psi^*_0) \\ (\epsilon_k + \alpha + \beta \Phi^2_1) \delta \Psi_0 \\ (\epsilon_k + \alpha + \beta \Phi^2_1 - 2 \beta_s \Phi^2_1) \delta \Psi_{-1} \end{pmatrix}. \tag{31}
\]

It is easy to get the frequency of \( \delta \Psi^*_1 \):

\[
\hbar \omega_1 = \sqrt{(\epsilon_k + \alpha + \beta \Phi^2_1)^2 + 2 \beta \Phi^2_1 (\epsilon_k + \alpha + \beta \Phi^2_1)}. \tag{33}
\]

Under BEC, \( f^0_b \) satisfies the relation \( \partial f^0_b / \partial \Phi_1 = 0 \), which yields

\[
\alpha + \beta \Phi^2_1 = 0. \tag{34}
\]

This condition guarantees a gapless Goldstone mode in the Bogliubov form. We can also derive frequencies of \( \delta \Psi_0 \) and \( \delta \Psi_{-1} \):

\[
\hbar \omega_0 = \epsilon_k + \alpha + \beta \Phi^2_1, \quad \hbar \omega_{-1} = \epsilon_k + \alpha + \beta \Phi^2_1 - 2 \beta_s \Phi^2_1. \tag{35}
\]

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Under condition of Eq. (34), $\omega_0$ also becomes gapless. Up to now, the results of Refs. [13] and [14] are reproduced, except that the meaning of parameters is different. One advantage of the phenomenological approach is that it could reveal more evidently the relation between the Goldstone mode and the phase transition.

However, we should note that the above derivation is not self-consistent, because it neglects the magnetization in the normal part of the gas. According to discussions in Section 3, the FM transition takes place above BEC. So near the BEC temperature, the magnetization in normal gas is much larger than in the condensate. Apparently, this FM phase in the thermal cloud (described by $f_m$) should be taken into account. In this case, the coupling $f_c$ between the two phases plays an important role. $f_c$ consists of two terms,

$$f_c^0 = -gM_0\Phi_1^2,$$

$$f_c^I = -gM_0(\delta\Psi_1^*\delta\Psi_1 - \delta\Psi_{-1}^*\delta\Psi_{-1}) - \frac{\sqrt{2}}{2}g\Phi_1(\delta\Psi_0^*\delta M_+ + \delta\Psi_0\delta M_-).$$

The self-consistent solution to phase transitions and Goldstone modes in spinor condensates should be acquired by treating the total free energy $f_t = f_m + f_b + f_c$ as a whole [37]. The obtained results suggested that the Bogliubov mode remains unchanged, but spin waves in the thermal cloud and in the condensate are coupled together,

$$f_{sf}^I = (\delta M_+ \delta\Psi_0^*, 
\begin{pmatrix}
ck^2 + \frac{g\Phi_1^2}{2M_0} & -\frac{\sqrt{2}}{2}g\Phi_1 \\
-\frac{\sqrt{2}}{2}g\Phi_1 & \epsilon_k + gM_0
\end{pmatrix}
\delta M_- + \delta\Psi_0).$$

This equation indicates that the transverse spin fluctuations in both phases are gapped solo. But after considering the coupling, the Goldstone theorem is recovered. The spectrum for this coupled mode at long wave length is given by

$$\hbar\omega_0 \approx \frac{gM_0ck^2 + \frac{g\Phi_1^2}{2M_0}ck}{gM_0 - \frac{g\Phi_1^2}{2M_0}} \approx ck^2 + \frac{1}{2}M_0^2\epsilon_k.$$ 

Once again, we obtain a $k^2$-formed spectrum for spin waves.

Many attempts have been made to evaluate excitation spectra at finite temperatures microscopically, on the basis of perturbation theory [44, 45] and generalized self-consistent Hartree-Fock theory [46]. The obtained results show that the dispersion relations are in the same forms as at zero temperature. The phonon velocity of the Bogliubov mode decreases while the spin-wave stiffness $c_s$ increases as the temperature is increasing. But these theories are not very much applicable near the BEC temperature, not only because of the invalidity of the calculating method itself at the temperature regime under discussion, but also because of the neglect of the normal FM phase.

As a comparison to the ferromagnetic condensate, let us take a glance at the spin waves in antiferromagnetic Bose gases in which the Bose-Einstein condensate is in the “polar” state: only spin-0 bosons condense. In this case the spin and density waves are described
by \( \delta n = \sqrt{n_0}(\delta \Psi_0 + \delta \Psi_0^*) \), \( \delta M_- = \sqrt{n_0}(\delta \Psi_{-1} + \delta \Psi_{-1}^*) \) and \( \delta M_+ = \delta M_+^\dagger \) [13, 14]. The spectra are
\[
\hbar \omega_0 = \sqrt{\epsilon_k^2 + 2 \frac{4\pi \hbar^2}{m} g_0 n_0 \epsilon_k}, \quad \hbar \omega_\pm = \sqrt{\epsilon_k^2 + 2 \frac{4\pi \hbar^2}{m} g_s n_0 \epsilon_k}. \tag{39}
\]

At very small \( k \), the spin-wave spectrum \( \hbar \omega_\pm \sim \sqrt{g_s k} \). The dispersion relation is linear in \( k \) which coincides with that in a Heisenberg antiferromagnet in which \( \hbar \omega \sim |J| k \). But, the dependence of the spin wave velocities on the couplings are different.

5 Conclusions and Further discussions

In summary, some theoretical results concerning ferromagnetic transitions in spinor Bose gases are reviewed. The subject covers the origin of the coupling (which induces the transition), the transition itself (including the critical point and its relation with BEC), and the Goldstone mode (as a result of the transition). A unique feature of the FM transition in bosonic systems is that it takes place always above Bose-Einstein condensation, regardless of the magnitude of the coupling. The spectrum of the spin wave takes the same form as in a conventional ferromagnet, \( \omega \sim k^2 \), while the spin-wave stiffness coefficient consists of two different parts: one coming from the thermal cloud, the other from the Bose condensate, which embodies the “two-fluid” feature of the system.

The above theoretical results are devoted mainly to a homogeneous system in the thermodynamic limit and the actual experimental situations for ultracold atomic gases are not well taken into account. In the following we discuss briefly some experimental facts which may cause significant differences.

1) The trapping potential. Atomic gases are experimentally confined by a trapping potential which can be approximated as being harmonic. One direct consequence of the confinement is that it changes the density of state of the gas and thus the critical behaviors [47]. So how the trapping potential affects on the phase diagram of the FM spinor boson deserves further study. More recently, Huang et al. [48] studied BEC in trapped \( F = 1 \) spinor bosons with FM couplings, but the FM transition was not considered.

2) The number of confined atoms. At present, the number of confined atoms is typically in the range of \( 10^6 - 10^8 \), far away from the thermodynamic limit. So questions arise: do theories derived in the thermodynamic limit hold in this case? Does the concept of spontaneous symmetry breaking apply? And to what extent does it apply? The observation of BEC suggests that the spontaneous symmetry breaking is still a valid concept, and the observed critical points agree with theories quite well [49]. But it is still an open question whether it is so for the FM transition.

3) The spin conservation rule. It is observed that the total spin approximately conserves through the evaluation of the Bose-Einstein condensate [7, 17, 18]. From Ref. [17, 18], this conservation dominates the spin dynamics and the final states of the system. This is in conflict with the picture of spontaneous symmetry breaking which says the total spin.
does not conserve in the ground state. It might be a consequence of the departure from the thermodynamic limit.

Isoshima et al. studied phase transitions in $F = 1$ spinor bosons under the spin conservation rule [50]. They argued that the system may have two spatially phase-separated Bose condensates with $m_F = 1$ and $-1$ respectively.

Although many work has been done, the research on ferromagnetism in spinor bosons is still in a very primary stage. Many questions remain open. Direct comparison between theories and experiments is far from being reached.

Acknowledgements

The author gratefully acknowledges collaboration with K. Sengstock, K. Bongs, R. A. Klemm, helpful discussions with K. Scharnberg and D.I. Uzunov, and support from the Deutsche Forschungsgemeinschaft through the Graduiertenkolleg “Spectroscopy of localized atomic systems”, No. 463.

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