Periodically driven integrable systems with long-range pair potentials

Sourav Nandy¹, K Sengupta¹ and Arnab Sen¹,²

¹ Department of Theoretical Physics, Indian Association for the Cultivation of Science, Jadavpur, Kolkata 700032, India
E-mail: tpars@iacs.res.in

Received 9 April 2018, revised 19 June 2018
Accepted for publication 25 June 2018
Published 12 July 2018

Abstract
We study periodically driven closed systems with a long-ranged Hamiltonian by considering a generalized Kitaev chain with pairing terms which decay with distance as a power law characterized by exponent \( \alpha \). Starting from an initial unentangled state, we show that all local quantities synchronize with the driving frequency \( \omega \) at late times and relax to well-defined steady state values in the thermodynamic limit and after \( n \gg 1 \) drive cycles for any \( \alpha \) and \( \omega \). We introduce a distance measure, \( D_l(n) \), that characterizes the approach of the reduced density matrix of a subsystem of \( l \) sites to that of the final steady state. We chart out the \( n \) dependence of \( D_l(n) \) and identify a critical value \( \alpha = \alpha_c \) (which depends only on the time-averaged Hamiltonian) below which they generically decay to zero as \((\omega/n)^{1/2}\). For \( \alpha > \alpha_c \), in contrast, \( D_l(n) \sim (\omega/n)^{3/2}(\omega/n)^{1/2}\) for \( \omega \to \infty \) with at least one intermediate dynamical transition. An identical behavior is found for relaxation of all non-trivial correlation functions to their steady-state values. We also study the mutual information propagation to understand the nature of the entanglement spreading in space with increasing \( n \) for such periodically driven long-ranged systems. We point out existence of qualitatively new features (absent for \( \omega \gg 1 \)) in the space-time dependence of mutual information for \( \omega < \omega_c^{(1)} \), where \( \omega_c^{(1)} \) is the largest critical frequency for the dynamical transition for a given \( \alpha \) such as the presence of multiple light cone-like structures which persists even when \( \alpha \) is large. We also show that the space-time dependence of the mutual information of long-ranged Hamiltonians with \( \alpha < 2 \) differs qualitatively from those with \( \alpha > 2 \) for any drive frequency and relate this to the behavior of the Floquet group velocity of such driven system.

Keywords: long-ranged integrable systems, periodic driving and dynamical phase transitions, entanglement propagation

(Some figures may appear in colour only in the online journal)
1. Introduction and motivation

Recent experimental progress in manipulating well-isolated quantum systems such as ultracold quantum gases [1–4] and trapped ion systems [5–8] has led to a renewed interest in closed many-body systems driven by purely unitary dynamics. Even though the system is not connected to any external heat bath and is thus always in a pure quantum state, it has now been understood that the increase and spreading of quantum entanglement [9, 10] between its degrees of freedom as a function of time due to the time-dependence of some parameter of the system’s Hamiltonian leads to the necessity of a mixed density matrix description for any subsystem. This, in turn, leads to the possibility of well-defined steady states at late times for any of its subsystems [11–19] as long as the rest of the system (which we call ‘environment’ henceforth) is much bigger. Thus, the nature of entanglement propagation in these far-from-equilibrium regimes is central to their complete understanding. Systems that are continually driven by a periodic drive in time are of particular interest since these are known to lead to non-equilibrium states that have no equilibrium counterparts, e.g. Floquet time crystals in many-body localized systems [20, 21] and dynamical topological ordering [22, 23].

The propagation of quantum entanglement in non-relativistic systems with short-ranged interactions is a well-studied subject by now. The seminal work of Lieb and Robinson [24] showed the existence of a maximum velocity of propagation for correlations in translationally invariant spin systems with nearest neighbor interactions which also places a bound on the rate of entanglement propagation. In integrable systems, entanglement propagates ballistically [25] when the quantum dynamics is started from an initial unentangled state and the resulting ‘light cone effect’ (see [26] for experimental observation of this effect) is caused by the propagation of entangled quasiparticle pairs at finite velocities. Recent studies have now demonstrated that this ballistic spreading of entanglement may be more generic and is also present in non-integrable systems [27]. Global quantum quenches, where some parameter of the Hamiltonian is instantaneously changed to another value and the state is then propagated with the new Hamiltonian, provide possibly the simplest setup to study such entanglement propagation.

Less is known about entanglement propagation in long-ranged systems where it is expected that qualitatively different features should arise due to the non-locality of the interactions. The first generalization of the results of Lieb and Robinson to systems with a power-law interaction \(1/d^\alpha\) (with \(d\) being the separation) in \(D\) spatial dimensions [28] gave a bound of \(t \sim \log d\) for the casual region of a local perturbation when \(\alpha > D\), which suggests that entanglement spreading may even happen exponentially fast in long-ranged interacting systems. This bound was then significantly improved in [29] which applies for \(\alpha > 2D\) and gives the bound for entanglement spreading as \(t \sim d^\zeta\) with \(\zeta \leq 1\) and approaching 1 as \(\alpha \to \infty\) for a local perturbation. The study of quenches in different one-dimensional models where interactions decay as a power-law [30–34] \(1/d^\alpha\) shows that when \(\alpha > 2\), a sharp light cone is still present in the dynamics just like for short-range models. The light cone is significantly broadened in the regime \(1 < \alpha < 2\) which has been dubbed as the quasi long-range interaction regime in [30–32]. For \(\alpha < 1\), in contrast, the light cone effect is completely absent with correlations between distant points building up instantaneously.

In this work, we instead focus on the entanglement propagation for periodically driven long-ranged systems with local quantities being observed stroboscopically (i.e. after \(n = 0, 1, 2, \cdots\) where \(n\) denotes the number of full drive cycles). When the driving frequency \(\omega\) is large, the time-evolution at stroboscopic times can be equivalently described by a global quantum quench where the post-quench Hamiltonian equals the time-averaged Hamiltonian over one cycle of the periodic drive. It is then interesting to ask whether new features that are not
present for global quenches (or, equivalently when $\omega \gg 1$), can emerge for the spreading of entanglement at finite $\omega$. In the present work, we show that several such new features indeed exist and focus on identification and characterization of such features for periodically driven long-ranged systems.

Another quantity that characterizes the entanglement of a subsystem with its environment is its entanglement entropy $S$ [35]. It is defined through the reduced density matrix $\rho_r$ of the subsystem obtained after integrating out the environment via the following relation:

$$S = -\text{Tr}(\rho_r \ln \rho_r). \quad (1)$$

How does the entanglement entropy $S$ of the subsystem converges to the final entanglement entropy in the steady state as a function of time? This convergence also characterizes the approach of those local properties that can be defined using the lattice sites contained in the subsystem to their final steady state values since these are fully determined by $\rho_r$. It was recently found that the behavior of this quantity as a function of $n$ (the stroboscopic time) shows a dynamical phase transition [36] for a class of integrable models in one and two dimensions, that include the one-dimensional $S = 1/2$ transverse field Ising model [37] and the two-dimensional $S = 1/2$ Kitaev model [38]. When a parameter in the Hamiltonian of these models is driven periodically in time, the local properties of the system converge to the final steady state in two entirely different manners (which can be identified with two distinct dynamical phases) depending on the drive frequency $\omega$. However, the systems studied in [36] have interactions whose range do not extend beyond nearest neighbors (thus $\alpha \to \infty$).

In this work, we address various yet unanswered questions regarding entanglement generation and its spreading in periodically driven systems where the degrees of freedom are coupled by variable range pair potentials that decay as a power law of the form $1/d^\alpha$ with distance $d$. For instance, how does the presence of long-ranged terms in the Hamiltonian with the range being controlled by $\alpha$ affect the propagation of entanglement under periodic driving? How do such systems converge to their final nonequilibrium steady state and are there distinct dynamical phases which are distinguished by the nature of the relaxation of local quantities? Finally, does the light cone effect survive as a function of $\alpha$ when the entanglement propagation is considered stroboscopically, and do qualitatively new features, absent for global quantum quenches, emerge at finite $\omega$? We take a tractable model of a generalized Kitaev chain which consists of free fermions on a one-dimensional lattice with $p$-wave pairing terms that decay as $1/d^\alpha$ and drive it periodically in time starting from an initial unentangled pure state to address these issues.

The rest of the paper is organized in the following manner. In section 2, we define the generalized Kitaev chain where the pairing terms in the Hamiltonian are chosen to have a spatial power law decay characterized by an exponent $\alpha$. We introduce a pseudospin representation which allows us to express the time-dependent Hamiltonian of the system in terms of Pauli matrices. Using this representation, we obtain the corresponding Schrödinger equation and solve it numerically for a specific square-pulse periodic drive protocol characterized by a time period $T = 2\pi/\omega$ where $\omega$ is the drive frequency. In section 3, we discuss the convergence of the local properties of the system to their final steady state values as a function of the number of drive cycles $n$ which plays the role of time for stroboscopic measurement of system properties at times $t = nT$. We identify a critical value of $\alpha = \alpha_c$, where $\alpha_c$ depends only on the time-averaged Hamiltonian, above which the system exhibits two dynamical phases separated by at least one dynamical phase transition as a function of $\omega$; these phases are distinguished by the manner in which all local correlation functions (and hence the density matrix of a subsystem of the system) converge to their steady state value for $n \gg 1$. In particular, for $\omega > \omega_c^{(1)}$ (which denotes the largest frequency at which the last dynamical phase transition occurs as
the frequency is varied in \([0, \infty)\), all correlation functions shows a \(n^{-3/2}\) decay to their steady state value; this behavior changes to \(n^{-1/2}\) decay as \(\omega\) is reduced through \(\omega_c^{(1)}\). Such dynamical phases are generically independent of the periodic drive protocol and show a re-entrant behavior as a function of frequency. Below \(\alpha_c\), the high frequency dynamical phase is entirely absent and the relaxation follows \(n^{-1/2}\) behavior for any \(\omega\) (apart from some fine-tuned regions, where the behavior is \(n^{-3/2}\)). Thus, there is a dynamical phase transition even in the global quench limit as a function of \(\alpha\) where the late-time relaxation of local properties to the steady state changes from \(t^{-3/2}\) to \(t^{-1/2}\) below \(\alpha_c\). We also discuss the protocol and Hamiltonian parameter dependence of \(\alpha_c\). We thus generalize the notion of dynamical phases and phase transitions introduced in [36] for short-ranged integrable models (i.e. with \(\alpha \to \infty\)) to long-ranged integrable models with finite \(\alpha\). In section 4, we focus on the spreading of entanglement in the periodically driven long-ranged Kitaev chain as a function of space and time. We show that many features of the entanglement spreading can be understood from the behavior of the first and second derivatives of the Floquet Hamiltonian in momentum space. Importantly, if the decay exponent of the pairing terms is above \(\alpha_c\), we show that entanglement spreading is similar to that of a sudden global quantum quench as long as the drive frequency is higher than \(\omega_c^{(1)}\). In contrast, qualitatively new features emerge below \(\omega_c^{(1)}\) due to additional zeroes in the derivatives of the Floquet Hamiltonian in momentum space. These include the appearance of multiple light cone-like structures in the entanglement spreading in space-time even at large \(\alpha\) (i.e. effectively short-ranged models), something which is absent for unitary dynamics after a global quench. For \(\alpha \leq 2\), we also show that the entanglement spreading is instantaneous at any drive frequency due to the behavior of the Floquet group velocity leading to absence of light cone like structure, which is qualitatively different from the \(\alpha > 2\) case where a light cone effect exists at any drive frequency. Finally, we discuss our main results and conclude in section 5. Results for sinusoidal drive protocols are briefly discussed in appendix to stress the protocol independence of our results as long as some parameter in the Hamiltonian is periodically changed as a function of time.

2. Preliminaries

We focus on an exactly solvable fermionic model, the generalized Kitaev chain, with variable range \(p\)-wave pairing terms that decay as \(1/d^\alpha\) with the distance \(d = |i - j|\) between two lattice sites with coordinates \(i\) and \(j\). The Hamiltonian of the model is as follows:

\[
H = - \frac{\hbar}{\Delta} \sum_{j=1}^{L} (c_j^+ c_{j+1} + \text{H.c.}) + g(t) \sum_j (n_j - 1/2)
+ \frac{\Delta}{2} \sum_{j=1}^{L} \sum_{l=1}^{L-1} \left( \frac{c_j c_{j+l} + \text{H.c.}}{d_l^{\alpha}} \right),
\]

where \(c_j (c_j^+)\) denotes the (spinless) fermionic annihilation (creation) operator at site \(j\) and \(n_j = c_j^+ c_j\) is the corresponding fermion number operator. \(\hbar\) represents the fermionic hopping strength, \(\Delta\) denotes the pairing between fermions, and \(g(t)\) represents the time-dependent chemical potential which is varied in a periodic manner in time. Henceforth, we set \(\hbar = \Delta = 1/2\). We focus on the case of even \(L\) (where \(L\) denotes the number of sites in the lattice) with antiperiodic boundary conditions for the fermions. We accordingly define \(d_l = l\) if \(l \leq L/2\) and \(d_l = (L - l)\) otherwise.
When the pairing terms are restricted to be non-zero only for nearest neighbors on the lattice, this model can be mapped via the Jordan–Wigner transformation [37, 39] to the $S = 1/2$ transverse field Ising model. The model possess two critical points $(g = \pm 1)$ in this limit and furthermore, the phase diagram is symmetric under $g \to -g$. The correlation functions decay exponentially in space except at the critical points. For finite $\alpha$, the correlation functions decay exponential at short distances but algebraically at long range for $\alpha > 1$ and purely algebraically when $\alpha < 1$. We refer the readers to [40] for the equilibrium phase diagram and phase transitions of equation (2) for finite $\alpha$.

To study the periodically driven problem, it is convenient to adopt a pseudospin representation which we will detail below. In order to diagonalize the Hamiltonian (equation (2)), we go to the momentum space using the following transformation:

$$c_k = \frac{e^{iL/4}}{\sqrt{L}} \sum_x e^{-ilx} c_x$$

(3)

where the momenta $k$ equal $2\pi m/L$ where $m = -(L-1)/2, \ldots, -1/2, 1/2, \ldots, (L-1)/2$. Writing the Hamiltonian in terms of $c_k, c_k^\dagger$, we get

$$H = \sum_k \left[ (g(t) - \cos(k)) c_k^\dagger c_k + \Delta_{k,\alpha}(c_{-k} c_k + H.c.) - \frac{g(t)}{2} \right]$$

(4)

where

$$\Delta_{k,\alpha} = \frac{1}{(2/\pi)} \sum_{l=1}^{L-1} (\sin(kt)/d_l^\alpha).$$

(5)

When $L \to \infty$, this can be written as $\Delta_{k,\alpha} = \Im (Li_{\alpha}(e^{ik}))$, where $Li_{\alpha}(z)$ is the polylogarithm function of order $\alpha$ and argument $z$ and $\Im$ denotes the imaginary part of a complex number.

We note that $H$ connects the vacuum of the fermions $|0\rangle$ with $|k, -k\rangle = c_k^\dagger c_{-k}^\dagger |0\rangle$ and $|k\rangle = c_k |0\rangle$ with $|-k\rangle = c_{-k}^\dagger |0\rangle$. In this work, the initial pure state is taken to be the vacuum of the $c$ fermions. It is then enough to consider the states $|0\rangle, |k, -k\rangle$ at each $k > 0$ for the subsequent unitary dynamics. Furthermore, we introduce a pseudospin representation $\vec{\sigma}_k$ where $|\uparrow\rangle_k = |k, -k\rangle = c_k^\dagger c_{-k}^\dagger |0\rangle$ and $|\downarrow\rangle_k = |0\rangle$. We can then write $H = \sum_{k>0} H_k$ in this basis, where $H_k$ is given by

$$H_k = (g(t) - \cos(k)) \sigma_k^z + (\Delta_{k,\alpha}) \sigma_k^i.$$  

(6)

For drive protocols that preserve translational symmetry, each $k$ mode evolves independently as

$$i \frac{d}{dt} |\psi_k(t)\rangle = H_k(t) |\psi_k(t)\rangle$$

(7)

where

$$|\psi_k(t)\rangle = u_k(t) |\uparrow\rangle_k + v_k(t) |\downarrow\rangle_k,$$

(8)

Thus, specifying $u_k, v_k$ for $k > 0$ specifies the complete wavefunction of the system through equation (8). The initial state can be easily expressed in the pseudospin basis as $|\psi(0)\rangle = \otimes_{k>0} |\downarrow\rangle_k$.

For numerical convenience, we take the time-dependence of $g(t)$ as a square pulse that varies periodically in time with a period that equals $T$, i.e.
\[ g(t) = g_\ell, \quad (n-1)T \leq t \leq ((n-1/2)T) \\
= g_f, \quad ((n-1/2)T) \leq t \leq (nT). \tag{9} \]

Since we are interested in the stroboscopic behavior of the local quantities, it is enough to know the unitary time evolution operator \( U_k(T) \) at each \( k \) for a single period \( T \). The unitary evolution after a time \( t = nT \) where \( n = 0, 1, 2, \ldots \) can be calculated as

\[ |\psi_k(nT)\rangle = [U_k(T)]^n |\psi_k(0)\rangle. \tag{10} \]

We note here that most of our results are independent of the specific form of the periodic drive protocol and the above protocol has been taken to make the analysis tractable (some results for a sinusoidal protocol will be discussed in appendix).

### 3. Convergence to the steady state and dynamical phase transition

In this section, we discuss the convergence of the local properties of the generalized Kitaev chain (equation (2)) when \( g(t) \) is driven periodically in time. For this, we will use the formalism developed in our earlier work \cite{36} in the context of short-ranged integrable models with no interactions beyond nearest neighbors and show how it generalizes to the present case where the pairing terms decay as a power law in space.

Since we are dealing with a quadratic fermionic Hamiltonian in equation (2), it is enough to consider the behavior of the two-point correlators \( C_{ij}(n) = \langle c_i^+ c_j \rangle_n \) and \( F_{ij}(n) = \langle c_i^+ c_j^\dagger \rangle_n \) stroboscopically (i.e. at \( t = nT \)) to study the convergence to a possible final steady state as \( n \to \infty \). Other higher-point correlators can then be constructed from \( C_{ij}(n) \) and \( F_{ij}(n) \) by using Wick’s theorem. It is useful to look at this problem for a general periodic drive protocol that preserves the lattice translational symmetry first.

Equation (6) describes the motion of the pseudospin \( \vec{\sigma}_k \) at momentum \( k \) in a time-varying ‘magnetic field’ \( \Delta_{k,\alpha} \), \( 0 \leq g(t) = \cos(k) \). The time evolution operator for one time period for \( \vec{\sigma}_k \) can thus be parametrized as \( U_k(T) = \exp[-iH_{\text{FK}}T] \) where the hermitian operator \( H_{\text{FK}} \) is the Floquet Hamiltonian of the system at momentum \( k \), which can be written in general as

\[ H_{\text{FK}} = \vec{\sigma}_k \cdot \vec{\epsilon}_k = |\vec{\epsilon}_k| \vec{\sigma}_k \cdot \hat{n}_k \tag{11} \]

where \( \vec{\epsilon}_k = (\epsilon_{k1}, \epsilon_{k2}, \epsilon_{k3}) \), and \( \hat{n}_k = \epsilon_{k3}/|\vec{\epsilon}_k| \). Then, we can express \( U_k(T) \) as

\[ U_k(T) = \exp[-i(\vec{\sigma}_k \cdot \hat{n}_k)\phi_k] \tag{12} \]

where \( \phi_k = T|\vec{\epsilon}_k| \) and we restrict \( \phi_k \in [0, \pi] \) and each component of \( \vec{\epsilon}_k \in [-\pi/T, \pi/T] \) without loss of generality (i.e. we use the reduced zone scheme).

We now study the behavior of \( C_{ij} \) and \( F_{ij} \) stroboscopically when the initial pure state is taken to be the vacuum of the fermions, i.e. \( u_i(0) = 0 \) and \( v_j(0) = 1 \) for all \( k \). Using the form of \( |\psi(t)\rangle \) in equation (8), we get

\[ C_{ij}(n) = \langle c_i^+ c_j \rangle_n = \frac{2}{L} \sum_{k>0} |u_k(nT)|^2 \cos(k(i-j)) \]

\[ F_{ij}(n) = \langle c_i^+ c_j^\dagger \rangle_n = \frac{2}{L} \sum_{k>0} u_k^*(nT)v_k(nT) \sin(k(i-j)). \tag{13} \]

Using equations (10) and (12), and taking the \( L \to \infty \) limit in equation (13), we get the following expressions for \( \delta C_{ij}(n) = C_{ij}(n) - C_{ij}(\infty) \) and \( \delta F_{ij}(n) = F_{ij}(n) - F_{ij}(\infty) \), where \( C_{ij}(\infty) \) and \( F_{ij}(\infty) \) are the steady state values of the correlators \cite{36}:
\[ \delta C_{ij}(n) = \frac{1}{\pi} \int_0^\pi dk F_1(k) \cos(2n\phi_k) \]
\[ \delta F_{ij}(n) = \frac{1}{\pi} \int_0^\pi dk [F_2(k) \cos(2n\phi_k) + F_3 \sin(2n\phi_k)] \]
\[ C_{ij}(\infty) = \frac{1}{\pi} \int_0^\pi dk \cos(k(i-j)) \left( \frac{1}{2} (1 - \hat{n}_{\hat{\omega}_3}^2) \right) \]
\[ F_{ij}(\infty) = \frac{1}{\pi} \int_0^\pi dk \sin(k(i-j)) \left( -\frac{1}{2} \hat{n}_{\hat{\omega}_3}(\hat{n}_{k_1} + i\hat{n}_{k_2}) \right) \]

(14)

where

\[ F_1(k) = -\frac{1}{2} \cos(k(i-j))(1 - \hat{n}_{\hat{\omega}_3}^2), F_2(k) = -i\hat{n}_{\hat{\omega}_3}F_3(k) \]
\[ F_3(k) = \frac{1}{2} \sin(k(i-j))(\hat{n}_{k_1} + i\hat{n}_{k_2}). \]

(15)

We note that while converting the summation over \( k \) (in equation (13)) to an integral (in equation (14)), we have implicitly assumed that \(|i - j| \ll L\). Thus, the steady state is strictly reached only for such local operators (where \(|i - j| \ll L\) and \( C_{ij}(n)(F_{ij}(n)) \) continues to display undamped oscillations even when \( n \to \infty \) if \(|i - j| \sim O(L)\).

From equation (14), it is clear that \( \delta C_{ij}(n) \) and \( \delta F_{ij}(n) \) must vanish when \( n \to \infty \) (otherwise, the steady state value is undefined) by the Riemann-Lebesgue lemma. Moreover, the dominant contribution to this relaxation behavior to the steady state is controlled by the stationary points defined by \( |\partial \phi_n|/dk = 0 \) at late times. The contribution of such a stationary point at \( k = k_0 \) to \( \delta C_{ij}(n) \) and \( \delta F_{ij}(n) \) can be estimated using the stationary phase approximation [36]:

\[ \int \mathcal{F}_i(k) \exp(in\phi_k) dk \approx \exp(in\phi_{k_0})(n|\phi''(k_0)|)^{-1/2} \times \exp \left( \frac{\pi i \mu}{4} \left( \mathcal{F}_i(k_0) + i\frac{\mathcal{F}''_{ij}(k_0)}{2\phi''(k_0)} \frac{1}{n} + O \left( \frac{1}{n^2} \right) \right) \right) \]

(16)

where \( \mu \) is the sign of \( \phi''(k_0) \) and \( \mathcal{F}_i(k) \) is assumed to be a smooth function in the neighborhood around \( k = k_0 \).

Importantly, possible stationary points at the Brillouin zone (BZ) edges, \( k = 0 \) and \( k = \pi \), behave differently to those where \( k \in (0, \pi) \) (i.e., excluding \( k = 0 \) and \( k = \pi \)) [36]. To see this, we first note that \( \Delta_{k,0} = -\Delta_{-k,0} \) for the long-ranged Kitaev chain independent of the value of \( \alpha \). Thus, \( \Delta_{k,0} = 0 \) at the edges of the BZ. From this, it follows that \( U_\alpha(T) \) is diagonal in the \( |\uparrow\rangle_k, |\downarrow\rangle_k \) basis at \( k = 0, \pi \) and hence \( \hat{n}_{k_1} = \hat{n}_{k_2} = 0 \) and \( \hat{n}_{\hat{\omega}_3} = \pm 1 \) for any periodic drive protocol. From equation (15), it then follows that \( \mathcal{F}_{1,2,3}(k_0) = 0 \) for \( k_0 = 0, \pi \); in contrast, for \( k_0 \neq 0, \pi \), they are in general non-zero. Using this result, from equation (16), it is easy to see that the stationary points at the edges of the BZ thus lead to \( O(n^{-1/2}) \) decay of the correlation functions (equation (14)) to their steady state values; in contrast, for \( k_0 \neq 0, \pi \), the decay is \( O(n^{-1/2}) \). Since \( \mathcal{F}_{1,2,3}(k_0) = 0 \) both at \( k = 0 \) and \( k = \pi \), in the absence of any stationary points in \( k \in (0, \pi) \), \( C_{ij}(n)(F_{ij}(n)) \) would have decayed as \( O(n^{-2}) \) (and not as \( O(n^{-1}) \) which requires \( \mathcal{F}_{1,2,3}(k) \neq 0 \) at least at one of the BZ edges) which is sub-leading compared to both \( n^{-1/2} \) and \( n^{-3/2} \) when \( n \gg 1 \).

To study the relaxation of the entanglement entropy of a subsystem of \( l \) sites (which are assumed to be adjacent for concreteness for the rest of the paper) to its final steady state value
$S_{\infty}(l)$, we note that for a Hamiltonian of the form equation (2), both the reduced density matrix $\rho_t$ of the subsystem and its entanglement entropy $S_q(l)$ with the environment may be calculated from the knowledge of $C_q(n)$ and $F_q(n)$ where $i,j$ denote the sites that belong to the subsystem [42, 43]. Two $l \times l$ matrices can be constructed from $C_q(n)$ and $F_q(n)$, which we denote by $C$ and $F$ respectively. From these, we construct the following $2l \times 2l$ matrix:

$$C_n(l) = \begin{pmatrix} I & C^T \\ F & C \end{pmatrix},$$

Equation (17)

$S_q(l)$ can then be obtained from the $2l$ eigenvalues (denoted by $p_i$) of the matrix $C_n(l)$:

$$S_q(l) = -\text{Tr}[\rho_t \ln \rho_t] = -\sum_{i=1}^{2l} p_i \ln(p_i).$$

Furthermore, $\rho_t$ can be obtained by knowing the eigenvectors of $C_n(l)$ as well.

To characterize the approach of $\rho_t$ to the final reduced density matrix of the steady state, we define the following distance measure [44] $D_n(l)$:

$$D_n(l) = \text{Tr}[(C_{\infty}(l) - C_n(l))^\dagger (C_{\infty}(l) - C_n(l))]^{1/2} / (2l).$$

Equation (18)

This distance measure has the property that $0 \leq D_n(l) \leq 1$ and vanishes only when $C_n(l) = C_{\infty}(l)$, which also implies that $\rho_t$ itself has converged to the final steady state reduced density matrix for the subsystem. From the discussion on stationary points above, we thus see that if such stationary points are solely present on the edges of the BZ, then all the elements of $C_n(l)$ and hence $D_n(l)$ converge to the final steady state as $(\omega/n)^{1/2}$, while if there are any stationary points for $k_0 \in (0, \pi)$, then the relaxation instead shows a $(\omega/n)^{1/2}$ behavior. Thus, the long-time relaxation properties are again controlled by whether the number of stationary points of $|\vec{e}_k|$ (defined by $d|\vec{e}_k|/dk = 0$) inside the BZ $(0 < k < \pi)$, which we denote by $N_e$ henceforth, equals zero or not, just like in the case of short-ranged integrable models considered in [36].

3.1. High frequency limit

First, let us consider the case when $\omega \to \infty$. In this limit, $H_{\text{EF}} \sim \tilde{H}_t$, where $\tilde{H}$ denotes the time-averaged Hamiltonian over one drive cycle, by using $1/\omega$ as a perturbation parameter in the Dyson series for $U_0(T)$. $\tilde{H}_t$ can be obtained from equation (6) by replacing $g(t)$ by $g_{\text{avg}} = (1/T) \int_0^T g(t)dt$. It then follows that

$$|\vec{e}_k|_{\omega \to \infty} = \sqrt{(g_{\text{avg}} - \cos(k))^2 + \Delta_{k,\alpha}^2}. $$

Equation (19)

When $\alpha \to \infty$, we see that $\Delta_{k,\alpha} \to \sin(k)$ from equation (5), from which it is straightforward to show that the only stationary points of $|\vec{e}_k|$ are at the BZ edges ($k = 0, \pi$ for $g_{\text{avg}} \neq \pm 1$, $k = 0$ for $g_{\text{avg}} = -1$, and $k = \pi$ for $g_{\text{avg}} = +1$). Next, we consider the opposite limit where $\alpha \to 0$. It can then be shown there always exists one more stationary point in $0 < k < \pi$ by considering the behaviour of the following functions:

$$\Gamma_k(g_{\text{avg}}) = (\cos(k) - g_{\text{avg}}) \sin(k)$$

$$\Xi_k = \lim_{\alpha \to 0} \Im \left( \Im_{\alpha} (e^{ik}) \right) \Re \left( \Im_{\alpha} (e^{ik}) \right)$$

Equation (20)

where $\Im \Re$ (3) denotes the real (imaginary) part of a complex number. A stationary point in $0 < k < \pi$ implies that $\Gamma_k(g_{\text{avg}}) = \Xi_k$ for some $k_0 \in (0, \pi)$. This is always guaranteed when $\alpha \to 0$ because $\Xi_k$ is a monotonic function whose range extends from $(-\infty, 0]$ in $k \in [0, \pi]$ and $\Gamma_k(g_{\text{avg}}) = 0$ at the BZ edges and is negative for $k \to \pi^-$ independent of $g_{\text{avg}}$ (figure 1(a)). We have numerically checked that there are stationary points only at the edges of the BZ.
for all $\alpha > \alpha_c(g_{\text{avg}})$, while below $\alpha_c(g_{\text{avg}})$, additional stationary points arise in $k \in (0, \pi)$ (figure 1(b)). The behavior of $\alpha_c$ as a function of $g_{\text{avg}}$ is shown in figure 2. We find numerically that $\alpha_c$ is constant as a function of $g_{\text{avg}}$ ($\alpha_c \approx 1.05$) for all $g_{\text{avg}}$ until $g_{\text{avg}} \approx 2$ and it starts to then increase with decreasing $g_{\text{avg}}$ thereafter.

The determination of $\alpha_c$ is completely independent of any specific periodic drive protocol and only depends on $g_{\text{avg}}$ since it is fixed by the behavior at $\omega \to \infty$. We have thus unearthed a new dynamical phase transition even in the global quench limit for such long-ranged models where the (long time) approach of local quantities to their steady state values change from $t^{-3/2}$ for $\alpha > \alpha_c(g_{\text{avg}})$ to $t^{-1/2}$ for $\alpha < \alpha_c(g_{\text{avg}})$ where the post-quench Hamiltonian’s (equation (2)) chemical potential $\mu$ is fixed to be $g_{\text{avg}}$.

3.2. Low frequency limit

Next, we discuss the behavior of $|\vec{\xi}|$ at small $\omega$. For calculational purposes, we adopt the square pulse protocol given in equation (9). It can then be shown that [36]

$$|\vec{\xi}| = \arccos(M_k)/T$$

(21)

where

$$M_k = \cos(\Phi_{kz}) \cos(\Phi_{ly}) - \vec{N}_{kz} \cdot \vec{N}_{ly} \sin(\Phi_{kz}) \sin(\Phi_{ly}).$$

(22)

In the above expression, $\Phi_{kz}(f) = E_{kz}(f) T/2$ with $E_{kz}(f) = \sqrt{(g_{kz}(f) - \cos(k))^2 + \Delta_{kz}^2}$ and $\vec{N}_{kz}(f) = \left(\Delta_{kz}/E_{kz}(f), 0, (g_{kz}(f) - \cos(k))/E_{kz}(f)\right)$. For large $T = 2\pi/\omega$, $M_k$ rapidly oscillates in $[-1, 1]$ with the effective wavelength being set by $1/T$ in $k$ space. Thus, when $\omega \to 0$, the number of stationary points $N_s$ of $|\vec{\xi}|$ in $0 < k < \pi$ increases with decreasing $\omega$. In fact, we see that $N_s \to \infty$ as $\omega \to 0$ irrespective of the value of $\alpha$. A scaling of $N_s \sim 1/\omega$ at small $\omega$ was previously seen in [36] for one dimensional short-ranged integrable models. Interestingly, decreasing the value of $\alpha$ below a certain threshold ($\alpha \sim 1$) increases the number of stationary points greatly, particularly in the neighborhood of $k = 0$ (figure 3(a)); an effect which is absent for larger values of $\alpha$ as shown in figure 3(b). Thus the scaling of $N_s$ is actually faster than $1/\omega$ at small $\omega$ when $\alpha$ is small.

For $\alpha > \alpha_c(g_{\text{avg}})$, we thus see that $N_s = 0$ when $\omega \to \infty$ and $N_s \to \infty$ for $\omega \to 0$. Hence, $D_1(n) \sim (\omega/n)^{3/2}$ for fast drives and $D_1(n) \sim (\omega/n)^{1/2}$ for slow drives as long as $\alpha > \alpha_c(g_{\text{avg}})$ irrespective of the specific details of the periodic drive protocol. As a result, there must be at least one dynamical phase transition between these two dynamical phases distinguished by the relaxation of $D_1(l)$ (as defined in equation (18)). Consequently all local quantities relax to their steady state values either as $(\omega/n)^{3/2}$ or as $(\omega/n)^{1/2}$ as the drive frequency $\omega$ is varied keeping other parameters fixed. We illustrate this in figure 4 where $\alpha$ is taken to be greater than $\alpha_c$. The two different drive frequencies $\omega$ show the different scalings of $D_1(l) \sim (\omega/n)^{3/2}$ and $D_1(l) \sim (\omega/n)^{1/2}$ respectively.

3.3. Dynamical phase transitions

Since $N_s$ is an integer, its value cannot change smoothly from $N_s = 0$ to $N_s = 1$ as $\omega$ is decreased from $1/\omega = 0$ if the limit $N_s(\omega \to \infty)$ exists, and can only turn non-zero for the first time at a finite value of $\omega^{(1)}$ for any periodic drive protocol where the range of the pairing terms is greater than $\alpha(g_{\text{avg}})$. For any $\omega \in (\omega^{(1)}, \infty)$, $D_1(n) \sim (\omega/n)^{3/2}$ from the previous discussion. To calculate $\omega^{(1)}$ for the square pulse protocol [36], we note that the new zero in $d|\vec{\xi}|/dk$ can
Figure 1. (a) The behaviour of the functions $\Gamma_k(g_{\text{avg}})$ and $\Xi_k$ as a function of $k$. (b) The behavior of $d|\vec{\epsilon}_k|/dk$ shown as a function of $k$ at $g_{\text{avg}} = 4$ for two particular values of $\alpha$. For $\alpha = 1.2$ (red curve), the stationary point is only present at $k = \pi$, while for $\alpha = 1.2$ (black curve), an additional stationary point is present in $k \in (0, \pi)$.

Figure 2. The behavior of $\alpha_c$ as a function of $g_{\text{avg}}$. 
Figure 3. (a) The behavior of $d|\vec{\epsilon}_k|/dk$ as a function of $k$ at a small $\omega$ ($\omega/\pi = 0.1$) for $\alpha = 0.9$ shows a large number of stationary points in the vicinity of $k = 0$. (b) The behavior of $d|\vec{\epsilon}_k|/dk$ as a function of $k$ at the same $\omega$ but at a larger $\alpha = 8.0$. The drive protocol used here is the square pulse protocol with $g_i = 2$ and $g_f = 0$.

Figure 4. The behavior of $D_n(l)$ as a function of $n$ for two different driving frequencies that belong to different dynamical phases. At $\omega/\pi = 4.0(0.75)$, $D_n(l) \sim (\omega/n)^{3/2}((\omega/n)^{1/2})$ both for $l = 8$ and $l = 16$, where $l$ denotes the number of consecutive sites in the subsystem. The other parameters are $g_i = 2, g_f = 0, \alpha = 2.5$ and $L = 2 \times 10^5$. 
can be calculated by expanding

\( \sin(\alpha) \).

The change in

\( \omega \)

and

\( \alpha \)

as a function of

\( \omega \)

varies as a function of

\( \alpha \),

which explains the

\( \alpha \)

and

\( \omega \)

end-points in

\( \alpha \)

case. We show the behavior of

\( N_s \)

due to the fine-tuned case of

\( \alpha \)

variation of

\( \alpha \).

We have numerically checked that irrespective of the

\( \alpha \)

value of

\( \alpha > \alpha_c(g_{avg}) \),

the new zero emerges from

\( k = \pi \)

for this specific protocol. Then, for a given

\( \alpha \),

\( \omega^{(1)} \)

can be calculated by expanding

\( d\epsilon_k/dk \)

for

\( k = \pi - \epsilon \)

and finding the value of

\( \omega \)

where the \( \mathcal{O}(\epsilon) \)

term first changes its sign. In fact,

\( \omega^{(1)} = 2\pi/T_0 \)

where

\( T_0 \)

is the smallest non-zero solution of the following equation:

\[
2 \sin(\mathcal{G}_1 T_0) [1 + g_f] [1 + g_i] T_0 \\
+ 4^{1-\alpha} (2^\alpha - 4)^2 (\zeta(\alpha - 1))^2 \\
\times \left\{ \left[ g_f - g_i \right]^2 \left( \cos(\mathcal{G}_2 T_0) - \cos(\mathcal{G}_1 T_0) \right) \right\} \\
- 2 (1 + g_f) (1 + g_i) \mathcal{G}_1 T_0 \sin(\mathcal{G}_1 T_0) = 0
\]

(23)

where

\( \mathcal{G}_1 = (g_i + g_f + 2)/2 \),

\( \mathcal{G}_2 = (g_f - g_i)/2 \)

and

\( \zeta(s) \)

denotes the Riemann zeta function.

In figure 5, we show how this

\( \omega^{(1)} \)

varies as a function of

\( \alpha \)

for different values of

\( g_f \)

and

\( g_i \).

Interestingly, one can see that for a given set of

\( g_f \)

and

\( g_i \),

\( \omega^{(1)} \)

is rather insensitive to the variation of

\( \alpha \)

(note that

\( \omega^{(1)} \)

ceases to exist below

\( \alpha(g_{avg}) \)

which explain the ‘end-points’ in figure 5).

For

\( \alpha < \alpha_c(g_{avg}) \),

the situation is qualitatively different. Here

\( N_s \neq 0 \)

even when

\( \omega \to \infty \)

and hence

\( \mathcal{D}_n(l) \sim (\omega/n)^{1/2} \)

both for fast and slow drives. There is thus no generic reason for a dynamical phase transition to occur as the frequency

\( \omega \)

is varied when

\( \alpha < \alpha_c(g_{avg}) \),

except in the fine-tuned case where

\( N_s \)

changes from

\( 2 \)

to

\( 0 \)

and then back to

\( 2 \)

as the frequency is varied. Calculations using the square pulse protocol below

\( \alpha_c(g_{avg}) \)

indeed shows that to be the case. We show the behavior of

\( \mathcal{D}_n(l) \)

for such a case both for fast and slow driving frequencies in figure 6(a) from which it is evident that

\( \mathcal{D}_n(l) \sim (\omega/n)^{1/2} \)

in both the regimes of high and low frequencies. We also show an instance where a dynamical phase transition occurs below

\( \alpha_c(g_{avg}) \)

in figure 6(b) when the relaxation is

\( \mathcal{D}_n \sim (\omega/n)^{1/2} \)

both when

\( \omega \to \infty \)

and

\( \omega \to 0 \)

due to the fine-tuned case of

\( N_s \)

changing from

\( 2 \)

to

\( 0 \)

caused by the coalescing of two stationary points in

\( k \in (0, \pi) \)

in some finite-\( \omega \)

interval.

For the case when

\( \alpha > \alpha_c(g_{avg}) \),

as

\( \omega \)

is decreased further below

\( \omega^{(1)} \),

the change in

\( N_s \)

can be non-monotonic in nature when

\( N_s \)

is small. It is then possible that in some frequency range,

\( N_s \)

may revert back to zero leading to a re-entrant behavior [36] of the dynamical phases as a function of

\( \omega \).

Such re-entrance is however ruled out when

\( \omega \to 0 \)

since

\( N_s \gg 1 \)

is this limit (figure 3). Due to this re-entrance effect, the phase diagram for the two dynamical phases has a rich structure as a function of the frequency and amplitude of the periodic drive.
We illustrate the phase diagram for the dynamical phases in figure 7 for the square pulse protocol with a fixed $g_f=0$ and varying $g_i$ and $\omega$ at different values of $\alpha$. Firstly, for $\alpha=4.5$ (figure 7(a)), the phase diagram for the dynamical phases is practically indistinguishable from the case of $\alpha \to \infty$ where the pairing terms are restricted to be between nearest neighbors only. Even when $\alpha=2.5$ (figure 7(b)), the broad features of the phase diagram remain the same though there are now clear deviations compared to the larger value of $\alpha$, especially in the region $\omega/\pi \in [0, 1]$. For $\alpha=1.5$ (figure 7(c)), we first encounter the effect that for a given amplitude $g_i$, the dynamical phase where $D_n(l) \sim (\omega/n)^{1/2}$ is completely absent upon tuning the value of $\omega$. Furthermore, the re-entrant region of $D_n(l) \sim (\omega/n)^{3/2}$ in $\omega/\pi \in [0, 1]$ present for both $\alpha=4.5$ and $\alpha=2.5$ is completely absent. The case $\alpha=1.06$ (figure 7(d)) shows even stronger departures compared to the case of $\alpha \to \infty$ especially when $g_i \in [1, 4]$. Interestingly, when $\alpha$ is reduced further, e.g. to $\alpha=1.05$, only the dynamical phase characterized by $D_n(l) \sim (\omega/n)^{1/2}$ survives for the shown parameter range of $(\omega, g_i)$. This discontinuous change in the nature of the dynamical phase diagram is because $\alpha=1.05$ is below $\alpha_c(g_{\text{avg}})$ for the parameters $(g_i, g_f)$ considered in figure 7. Furthermore, we see that $D_n(l) \sim (\omega/n)^{1/2}$

Figure 6. (a) The behavior of $D_n(l)$ as a function of $n$ for fast ($\omega/\pi = 4.0$) and slow ($\omega/\pi = 0.75$) driving frequencies when $\alpha$ is chosen to be below $\alpha_c(g_{\text{avg}})$ (here $\alpha = 0.9$, $g_f = 2$, $g_f = 0$ and $L = 2 \times 10^5$). In both the regimes, $D_n(l) \sim (\omega/n)^{1/2}$ for $l = 8, 16$. (b) Fine-tuned region below $\alpha_c(g_{\text{avg}})$ where $D_n(l) \sim (\omega/n)^{3/2}$ (dark region) (even though $D_n \sim (\omega/n)^{1/2}$ both for high and low frequencies) caused by $N_s$ changing from 2 to 0 due to the coalescing of two stationary points in $0 < k < \pi$ for a certain interval in $\omega$ ($g_f = 0$ and $\alpha = 1.06$ in (b)).
whenever $\omega \to 0$ irrespective of the value of $g_i$ and $\alpha$ and the complexity of the phase diagram (figure 7) which is consistent with $N_s \to \infty$ as $\omega \to 0$ irrespective of the value of $\alpha$. Finally, we also show the perfect agreement of the location of the last dynamical transition in frequency, $\omega_c^{(1)}$, obtained from equation (23) in figure 7 for all the different values of $\alpha$.

4. Propagation of mutual information

In this section, we study the spread of entanglement in the system described by equation (2) as a function of space and time when $g(t)$ is a periodic function in time. For this purpose, we monitor the mutual information $I_n(A, B)$ between two disjoint spatial regions $A$ and $B$ to measure the total amount of correlations present between $A$ and $B$ [45]. $I_n(A, B)$ is defined in the following manner:

$$I_n(A, B) = S_n(A) + S_n(B) - S_n(A \cup B).$$

\[24\]
For this study, we take both the regions $A$ and $B$ to contain $l$ adjacent sites each with $l_s$ sites separating these non-overlapping regions (shown schematically in figure 8). $A \cup B$ represents the $2l$ sites of these two subsystems together, and $S_\alpha(R)$ is the entanglement entropy of the subsystem $R$ after $n$ drive cycles using equation (1). Henceforth, we will denote the mutual information between two disjoint subsystems by $I_n(l, l_s)$. $I_n(l, l_s)$ has the property that it is positive and can only vanish if $\rho_r(A \cup B) = \rho_r(A) \otimes \rho_r(B)$. Therefore, starting from an unentangled state at $n = 0$, $I_n(l, l_s)$ provides an unbiased measure of when the two regions $A$ and $B$ get entangled with each other as $n$ is progressively increased.

The behavior of $I_n(l, l_s)$ is shown in figure 9 for the power-law decay exponent $\alpha = 8.0$ (figure 9(a)), $\alpha = 2.5$ (figure 9(b)), $\alpha = 1.8$ (figure 9(c)) and $\alpha = 0.9$ (figure 9(d)) respectively for a square pulse protocol (equation (9)) with the parameters being $g_l = 2$, $g_s = 0$, and $\omega/\pi = 10$. The pure state at $n = 0$ is the vacuum state of the fermions. We take a fixed size of $l = 10$ adjacent sites for both the regions $A$ and $B$ and show the results for $I_n(l, l_s)$ for a separation of $l_s = 100, 200$ and 400 sites as a function of the stroboscopic time $n$ in figure 9.

For $\alpha = 8.0$ (figure 9(a)), we have checked that the behavior of the mutual information is practically indistinguishable from the short-ranged case where the pairing terms are restricted to be between nearest neighbors (i.e. $\alpha \to \infty$). $I_n(l, l_s)$ becomes non-zero only after a finite $n$, the value of which increases linearly with the distance between the disjoint blocks ($l_s$) (figure 9(a)), thus clearly showing the light cone effect with a well-defined velocity. For a fixed $l_s$, $I_n(l, l_s)$ shows a strong peak at a value of $n$ close to where it first becomes non-zero (inset of figure 9(a)). For $\alpha = 2.5$ (figure 9(b)), there are already significant deviations compared to $\alpha \to \infty$. For example, the peak in $I_n(l, l_s)$ for a fixed $l_s$ as a function of $n$ does not appear soon after it first turns non-zero (inset of figure 9(b)) but only at a much later value of $n$ unlike when $\alpha = 8.0$. However, the mutual information again first turns non-zero only after a finite $n$ that scales linearly with the distance between the blocks $l_s$. Moreover, the position of the peak in the mutual information that emerges only at a much later $n$ also scales linearly with increasing $l_s$ with a different velocity that is distinct from the light cone velocity. In figures 9(c) and (d), we display the effect of lowering $\alpha$ further on the propagation of mutual information. Both for $\alpha = 1.8$ (figure 9(c)) and for $\alpha = 0.9$ (figure 9(d)), the mutual information behaves completely differently from the cases shown in figures 9(a) and (b) in that no matter how large the separation between the blocks ($l_s$), the mutual information is always non-zero for any $n > 0$ which implies that the blocks become entangled with each other instantaneously showing the absence of a strict light cone effect. The immediate growth of the mutual information for any $n > 0$ is demonstrated more clearly in the insets of the corresponding figures in figures 9(c) and (d). However, in spite of the absence of a light cone effect, there are still clear features in terms of local peaks of the mutual information as a function of $n$ where the peak positions in $n$ increase linearly with $l_s$ (main panels of figures 9(c) and (d)). This means that one can associate the notion of a well-defined velocity for such features even at small $\alpha$ where there is an instantaneous propagation of the entanglement.
The results displayed in figure 9 for $\omega/\pi = 10$ can be qualitatively understood by using results from previous studies of quantum quenches in such long-ranged models. We note that at large $\omega$, the Floquet Hamiltonian that describes the stroboscopic time evolution equals the time-averaged Hamiltonian over one drive cycle $\bar{H}$ as $\omega \to \infty$ and the problem can be formally mapped to a global quantum quench with the post-quench Hamiltonian being equal to $\bar{H}$. We can then directly apply the results obtained in [30–32] which we summarize below. The group velocity of the quasiparticles at momentum $k$ can be obtained from $v_g(k) = d|\tilde{\epsilon}_k|/dk$ where $|\tilde{\epsilon}_k|$ is given in equation (19) when $\omega \to \infty$. The maximum of the magnitude of the group velocity $v_{g}^{\text{max}}(k)$ as a function of $k$, which we denote by $v_{g}^{\text{max}}$, is finite [30–32] when $\alpha > 2$, which justifies the presence of the light cone effect for global quenches even in such long-ranged systems. However, $v_{g}^{\text{max}} \to \infty$ when $\alpha \to 2^+$. Near $k = 0$, the dispersion relation of the quasiparticle energy behaves as [30–32]

$$|\tilde{\epsilon}_k|_{\omega \to \infty} \sim \epsilon_0 + Ak^{\alpha - 1}. \quad (25)$$

Figure 9. The propagation of mutual information $I_n(l, l_s)$ where $l = 10$ and $l_s = 100$ (black), $l_s = 200$ (red), $l_s = 400$ (blue). The periodic drive parameters are $g_i = 2$, $g_f = 0$, and $\omega/\pi = 10$ with system size $L = 2 \times 10^5$. The state at $n = 0$ is the vacuum state of the fermions. The four panels show data for (a) $\alpha = 8.0$, (b) $\alpha = 2.5$, (c) $\alpha = 1.8$ and (d) $\alpha = 0.9$. The insets of the panels in (a) and (b) (data for $l_s = 400$) show that mutual information becomes non-zero only after a finite $n$. Dotted vertical lines in the insets are at $n = l_s/(2T(v_{g}^{\text{max}}))$ with $l_s = 400$. The insets of panels in (c) and (d) (data for $l_s = 400$) show that mutual information becomes non-zero immediately for any $n > 0$. The dotted lines in the main panels of (a)–(d) are at $n = l_s/(2T|v_{g}^{\text{max}}(k^*)|)$ (with $l_s = 400$ here) such that $D_F^l(k \to k^*) \to \infty$. 

The results displayed in figure 9 for $\omega/\pi = 10$ can be qualitatively understood by using results from previous studies of quantum quenches in such long-ranged models. We note that at large $\omega$, the Floquet Hamiltonian that describes the stroboscopic time evolution equals the time-averaged Hamiltonian over one drive cycle $\bar{H}$ as $\omega \to \infty$ and the problem can be formally mapped to a global quantum quench with the post-quench Hamiltonian being equal to $\bar{H}$. We can then directly apply the results obtained in [30–32] which we summarize below. The group velocity of the quasiparticles at momentum $k$ can be obtained from $v_g(k) = d|\tilde{\epsilon}_k|/dk$ where $|\tilde{\epsilon}_k|$ is given in equation (19) when $\omega \to \infty$. The maximum of the magnitude of the group velocity $v_g(k)$ as a function of $k$, which we denote by $v_g^{\text{max}}$, is finite [30–32] when $\alpha > 2$, which justifies the presence of the light cone effect for global quenches even in such long-ranged systems. However, $v_g^{\text{max}} \to \infty$ when $\alpha \to 2^+$. Near $k = 0$, the dispersion relation of the quasiparticle energy behaves as [30–32]

$$|\tilde{\epsilon}_k|_{\omega \to \infty} \sim \epsilon_0 + Ak^{\alpha - 1}. \quad (25)$$
Thus the group velocity near \( k = 0 \) diverges as \( k^{\alpha - 2} \) for any \( \alpha < 2 \). The spectrum is also unbounded as \( k \to 0 \) when \( \alpha < 1 \). Thus, there is no sharp light cone for a quantum quench when \( \alpha < 2 \), consistent with the behavior displayed in figures 9(c) and (d) for a large drive frequency.

At any finite \( \omega \), the spreading of the mutual information deviates from the global quantum quench. Then, a natural question that arises is that when do qualitatively new features appear in the entanglement propagation as the drive frequency of the periodic protocol is decreased? In figure 10, we show the mutual information propagation for the same combination of \( \alpha, g_i \) and \( g_f \) as in figures 9(a) and (c) but at a lower drive frequency of \( \omega/\pi = 0.5 \). The mutual information profile is now completely different compared to the case where \( \omega/\pi = 10 \) (which was similar to that of a global quench) and has much more structure. Crucially, there is still a well-defined light cone effect for \( \alpha > 2 \) (as shown for \( \alpha = 8.0 \) in figure 10(a), inset) while the entanglement builds up immediately when \( \alpha < 2 \) (as shown for \( \alpha = 1.8 \) in figure 10(b), inset) even at low \( \omega \). In particular, for large \( \alpha \), the space-time propagation of the mutual information shows a simple behavior with a single sharp light cone front when the driving protocol frequency is large (figure 9(a)), but clear multiple light cone fronts with distinct velocities for lower \( \omega \) (as can be seen in figure 10(a)).

The presence (absence) of light cone like features in the spreading of mutual information in space-time for \( \alpha > 2 \) (\( \alpha < 2 \)) at any drive frequency \( \omega \) can be easily seen by plotting \( T_s(l, l) \) as a function of both the subsystem separation \( (l) \) and the stroboscopic time \( (nT) \) as shown in figure 11. For \( \alpha = 8.0 \), we see a single light cone feature for a large drive frequency \( \omega/\pi = 10.0 \) (figure 11(a)). For the same \( \alpha \), we see the presence of multiple light cone features in the mutual information propagation for a lower drive frequency of \( \omega/\pi = 0.5 \) (figure 11(b)). For a low \( \alpha (=1.8) \), we can see that there is no sharp light cone effect irrespective of whether the drive frequency \( \omega \) is large (figure 11(c)) or small (figure 11(d)). Also, we can clearly see that the mutual information propagation in space-time for low \( \alpha \) is qualitatively different at \( \omega/\pi = 0.5 \) compared to the high-frequency drive frequency case \( \omega/\pi = 10.0 \).

In figure 12, we see that the appearance of new features in the propagation of the mutual information in space-time is intimately tied to the last dynamical phase transition in frequency for any \( \alpha > \alpha_c \) as \( \omega \) is varied in the range \([0, \infty]\) (discussed in section 3). More precisely, for a drive frequency \( \omega \in (\omega_c^{(1)}, \infty) \), the mutual information spreading shows no new features compared to the \( \omega \to \infty \) limit irrespective of whether \( \alpha > 2 \) (figure 12(a)) (where there is a strict light cone effect present at any \( \omega \)) or \( \alpha < 2 \) (figure 12(b)) (where there is no light cone effect at any \( \omega \)). When \( \omega < \omega_c^{(1)} \), qualitatively new features emerge both when \( \alpha > 2 \) (figure 12(a)) and \( \alpha < 2 \) (figure 12(b)).

To understand generic features of the spread of entanglement in space as a function of the stroboscopic time and its dependence on the drive frequency for the generalized Kitaev chain, it is sufficient to look at the behavior of \( \delta C_{ij}(n) \) and \( \delta F_{ij}(n) \) (equation (14)). For brevity, we only analyze \( \delta C_{ij}(n) \) (since \( \delta F_{ij}(n) \) leads to similar conclusions) and focus on the ‘space-time scaling limit’ [46] where both \( l_s = (i - j) \to \infty \) and \( n \to \infty \), with \( l_s/n = u_s \) fixed. Expressing the integrand in terms of \( u_s \) and \( n \), we get

\[
\delta C_{ij}(n) = \int_0^{\pi} \frac{dk}{8\pi} (\hat{n}_k^2 - 1) \left( e^{i\Phi_+(k)n} + e^{i\Phi_-(k)n} + c.c. \right)
\]

\[
\Phi_{\pm}(k) = (ku_s \pm 2|\xi|/T). \tag{26}
\]

Thus, along the line \( l_s/n = u_s \), the integral in equation (26) is dominated by the stationary points of \( \Phi_{\pm}(k) \) given by the \( k \) values (denote by \( k' \)) where \( d\Phi_{\pm}(k)/dk = 0 \) which gives
where we have defined the ‘Floquet group velocity’ of the quasiparticles at momentum \(k\) as
\[
\nu_F(k) = \frac{d|\epsilon_k|}{dk}
\]
and we stress again that we are working in the reduced zone scheme as explained below equation (12)). We numerically see from figure 13 that the maximum magnitude of \(\nu_F(k)\) in the BZ, which we denote by \((\nu_F)_{\text{max}}\), is finite for \(\alpha > 2\) and diverges for \(\alpha < 2\) irrespective of the value of \(\omega\) using the square pulse protocol (equation (9)), and not just when \(\omega \to \infty\) where the problem reduces to that of a global quench. Furthermore, the divergence in \(\nu_F(k)\) arises when \(k \to 0\) and is of the form \(k^{\alpha-2}\) for \(\alpha < 2\) irrespective of the value of \(\omega\) (as shown in the inset of figure 13(b)). This explains the build up of the mutual information immediately for any \(n > 0\) as shown in the inset of figures 9(c) and 10(b) when \(\alpha = 1.8\), unlike the case shown in (inset of) figures 9(a) and 10(a) where \(\alpha = 8.0\).
We now consider the behavior of mutual information for a fixed $l_s$ as a function of $n$ (as shown in figures 9 and 10). At large $n$, $\delta C_j(n)$ will receive a contribution from a stationary point $k^*$ whenever equation (27) is satisfied for a $k^* \in [0, \pi]$. From this, it is immediately clear that if

$$n < n_c \left( = \frac{l_s}{2T(\nu^*_f)^{\max}} \right)$$

where $n_c$ is defined in equation (28).
then equation (27) does not have any solution, and $\delta C_{ij}(n)_{(n)}$ is vanishingly small. This explains the resulting light cone effect whenever $(v_{\text{fg}})_{\text{max}}$ is finite, since otherwise $n_c \to 0$. In fact, the mutual information decays exponentially as $\exp(-((n_c - n)/\xi(\alpha, \omega)))$ for $n < n_c$ when $l_s$ is large (insets of figures 9(a), (b) and 10(a)), where $\xi(\alpha, \omega) \to \infty$ as $\alpha \to 2^+$ since $(v_{\text{fg}})_{\text{max}}$ diverges below $\alpha = 2$ (figure 13) for any $\omega$.

For $n > n_c$, there may be a solution at some $k^*$ where equation (27) is satisfied at a particular $n$. Apart from an oscillatory sinusoidal factor, this contribution from $k^*$ will scale as (the form of the stationary point contribution may be read off from equation (16))

$$\sqrt{D_F^g(k^*)/nT}, \text{ with } D_F^g(k) = \frac{1}{\pi^2} |dv_{\text{fg}}(k)/dk|^{-1},$$

(29)

where $D_F^g(k)$ can be interpreted as a density of states in velocity as a function of $k$ since it can be written as $D_F^g(k) = (1/\pi) \int_0^\infty dk \delta(v - v_{\text{fg}}(k))$. Thus at a fixed $l_s$, mutual information will then show strong features in the neighborhood of $n = l_s/(2T|v_{\text{fg}}(k^*)|)$ (the stationary point condition of equation (16)) when $D_F^g(k^*) \to \infty$. In figures 9(a)-(d) and 10(a) and (b),
Let us first consider the case when $\omega \gg 1$. For $\alpha = 8.0$ with $g_i = 2$ and $g_f = 0$, $D_F(k)$ has a single divergence at $k = 0$ for $\omega/\pi = 10.0$ which is also the momentum $k$ at which the Floquet group velocity $v_F(k)$ attains its maximum magnitude (figure 14(a)). This explains the simple behavior of $I_{\text{in}}(l, l_s)$ as shown in figure 9(a) where there is a single sharp mutual information front soon after it turns non-zero as a function of $n$. Lowering the value of $\alpha$ to 2.5 (keeping the other parameters the same as before) already leads to an interesting difference. $D_F(k)$ now has two divergences, both at non-zero values of $k$, but the maximum of $v_F(k)$ is still at $k = 0$ (figure 14(b)), where $D_F(k)$ goes to zero. This explains the marked difference of $I_{\text{in}}(l, l_s)$ for $\alpha = 2.5$ (figure 9(b)) compared to $\alpha \gg 1$. The mutual information is suppressed in the neighborhood of $n = n_c$ (equation (28)) because of the low density of quasiparticles that have velocities close to $(v_F)_{\text{max}}$. Instead, the peak feature in the mutual information in figure 9(b) is from the contribution of the quasiparticles in the neighborhood of $k^*$ for which $D_F(k) \to \infty$ here (figure 14(b)) and therefore, has a velocity $v_F(k^*)$, which is completely different from $(v_F)_{\text{max}}$.

Figure 13. We show here that $(v_F)_{\text{max}}$ is finite for (a) $\alpha > 2$ (data for $\alpha = 2.5$) and diverges for (b) $\alpha < 2$ (data for $\alpha = 1.8$) irrespective of the value of the drive frequency $\omega$. Here we use the square pulse protocol and $g_i = 2.0, g_f = 0.0$ for finite $\omega$. The inset of (b) shows that the divergence near $k = 0$ is of the form $k^{\alpha-2}$ for $\alpha < 2$ even at finite $\omega$. 

$n = l_s/(2T|v_F(k^*)|)$ are marked by vertical dotted lines at $l_s = 200$ in the main panels for the $k^*$ where $D_F(k)$ diverges, and we indeed see that the local peaks of the mutual information are in their neighborhood.
Importantly, $D_{gF}(k)$ is strongly sensitive to the drive frequency $\omega$. When $\omega \to 0$, $v_{gF}(k)$ crosses zero a large number of times ($\sim 1/\omega$ or larger) in the BZ as can be seen from figure 3. Since $v_{gF}(k)$ is continuous in $k$, this implies that the number of divergences in $D_{gF}(k)$ also scales in the same manner at small $\omega$, which is qualitatively different from the behavior of $D_{gF}(k)$ at large $\omega$. We show the behavior of $v_{gF}(k)$ and $D_{gF}(k)$ at a drive frequency of $\omega/\pi = 0.5$ for $\alpha = 8.0$ (figure 15(a)) and for $\alpha = 1.8$ (figure 15(b)) where the other parameters are $g_i = 2$ and $g_f = 0$. The multiple light cones in figure 10(a) for $\alpha = 8.0$ can now be seen as the direct consequence of extra divergences in $D_{gF}(k)$ apart from at $k=0$ when $\omega$ is decreased. The first light cone front as a function of $n$ arises from the quasiparticles around $k = 0$ where $v_{gF}(k)$ attains its maximum. However, the other two pronounced light cone fronts in $\mathcal{I}_n(l_l)$ (as shown in figure 10(a)) are because of the quasiparticles around $k_1^*$ and $k_2^*$, that propagate with the corresponding $v_{gF}(k)$ (figure 15(a)), which are the other momenta where $D_{gF}(k)$ diverges. Similarly, the difference in the behavior of $\mathcal{I}_n(l_l)$ for $\alpha = 1.8$ at the drive frequencies of $\omega/\pi = 10.0$ (figure 9(c)) and $\omega/\pi = 0.5$ (figure 10(b)) can again be attributed to the presence of extra divergences in $D_{gF}(k)$ as the drive frequency is varied (figure 15(b)). Thus, extra divergences in $D_{gF}(k)$ as the frequency is reduced from $1/\omega = 0$ causes the appearance of qualitatively new features that are absent in the global quench case (or equivalently, at high

![Figure 14. The behavior of $v_{gF}(k)$ (shown in red) and $D_{gF}(k)$ (shown in black) at high frequency ($\omega/\pi = 10.0$) for (a) $\alpha = 8.0$ and (b) $\alpha = 2.5$. The other drive parameters are $g_i = 2$ and $g_f = 0$. The locations of the divergences of $D_{gF}(k)$ are shown as dotted (blue) lines.](image-url)
We also note here the presence of additional local extrema in the mutual information $I_n(l, l_s)$ for both large $\alpha$ (figure 10(a)) and for small $\alpha$ (figure 10(b)) which cannot be simply explained by the divergences in $D_F^g(k)$ when the drive frequency is small. It will be useful to understand this full structure in detail in future work.

When $\alpha > \alpha_c$, we see that no new divergence develops in $D_F^g(k)$ compared to the global quench case ($\omega \to \infty$) for any $\omega \in (\omega_c^{(1)}, \infty)$ and an extra divergence is immediately generated for $\omega \to \omega_c^{(1)}$ from below irrespective of whether $\alpha > 2$ (figure 16(a)) or $\alpha < 2$ (figure 16(b)). The number of zeroes of both the functions, $v_F^g(k)$ and $dv_F^g(k)/dk$ in $k \in [0, \pi]$ stay unchanged when $\omega$ is above $\omega_c^{(1)}$. Just below $\omega_c^{(1)}$, an additional zero in $v_F^g(k)$ first enters from one of the BZ edges which causes $v_F^g(k)$ to change sign in that $k$ neighborhood (either around $k = 0$ or $k = \pi$ depending on where the new zero enters from). Moreover, it also causes $v_F^g(k)$ to develop an additional extremum between its new zero and the zero at the BZ edge. Hence, an additional divergence is immediately produced in $D_F^g(k)$ when $\omega$ goes infinitesimally below $\omega_c^{(1)}$. As $\omega$ is lowered further, additional divergences get generated in $D_F^g(k)$ at other specific values of $\omega$ (because the quantity is integer-valued) since ultimately the number of these divergences diverges as $\omega \to 0$ as discussed before. Thus, the mutual information propagation can attain a qualitatively different profile in space-time due to additional divergences in the function $D_F^g(k)$ when $\omega$ is outside the range $(\omega_c^{(1)}, \infty)$, whereas inside this frequency range, there

---

**Figure 15.** The behavior of $v_F^g(k)$ (shown in red) and $D_F^g(k)$ (shown in black) at a drive frequency of $\omega/\pi = 0.5$ for (a) $\alpha = 8.0$ and (b) $\alpha = 1.8$. The other drive parameters are $g_l = 2$ and $g_f = 0$. The locations of the divergences of $D_F^g(k)$ are shown as dotted (blue) lines.
is no qualitative distinction compared to the case of a global quantum quench. This establishes the presence of a sudden change in mutual information $I_{m(n)}(l,s)$ as a function of $\omega$ at the largest dynamical transition frequency $\omega_c^{(1)}$ for $\alpha > \alpha_c$.

5. Conclusions and outlook

In this work, we have analyzed a driven generalized Kitaev chain where the degrees of freedom are spinless fermions with a nearest neighbor hopping, an onsite chemical potential and long-ranged $p$-wave pairing terms whose decay in space is characterized by an exponent $\alpha$ (described by equation (2)). The system is driven by a purely unitary dynamics generated from the time-dependence of the chemical potential ($g(t)$) that is periodically varied in time with a frequency $\omega$. Short-ranged integrable models with free fermion representations are known to asymptotically synchronize with the drive frequency such that when local (in space) properties are observed stroboscopically in time (i.e. when the time intervals are separated by an integer multiple of the time period ($T$) of the drive such that $t = nT$), the late time properties reach a steady state that can be described by a periodic generalized Gibbs ensemble which has a volume law scaling of entanglement instead of the well-known area law scaling for ground states and unentangled pure states. The motivation for this work is two-fold:
whether and how such a long-ranged system reaches its steady state (locally) as a function of time when driven periodically in time, and

- how does the entanglement propagate in space and time when the system is started from an initial unentangled pure state (the vacuum of fermions in this study)?

Regarding the former point, we show that the local properties of such a long-ranged integrable system always reaches an asymptotic steady state irrespective of the value of $\alpha$ and the drive frequency $\omega$ in the thermodynamic limit. We address how the local properties relax to their final values as a function of the stroboscopic time $nT$ by defining an appropriate distance measure, $D_n(l)(c \in [0,1])$, which is zero iff all non-trivial correlation functions that can be defined by using any subset of $l$ adjacent sites in the system coincide with their corresponding values in the final steady state. We show that there are only two possible dynamical phases when the drive frequency is varied for any value of $\alpha$ which are characterized by either a $D_n(l) \sim (\omega/n)^{-3/2}$ or a $D_n(l) \sim (\omega/n)^{-1/2}$ behavior when $n \gg 1$ for any finite $l$ in the thermodynamic limit. We also show that there exists a critical range $\alpha_c$ that only depends on the time-averaged value of $g(t)$ over one full drive cycle, denoted by $g_{\text{avg}}$, such that above $\alpha_c(g_{\text{avg}})$, $D_n(l) \sim (\omega/n)^{3/2}$ whereas below $\alpha_c(g_{\text{avg}})$, $D_n(l) \sim (\omega/n)^{1/2}$ both for high and low frequency driving. Since the problem maps on to a global quantum quench with the post-quenched Hamiltonian equal to the time-averaged one (over one full period of the drive) when $\omega \gg 1$, this implies that there is a dynamical phase transition at $\alpha_c$ (keeping other parameters fixed) with a global quench protocol. We also map out the rich phase diagram for these dynamical phases as a function of the drive frequency and amplitude for different values of $\alpha$ and point out the distinctions between (effectively) short-ranged ($\alpha \gg 1$) and long-ranged ($\alpha \sim 1$) pairing terms.

Regarding the latter point, we study the mutual information $I_n(l,l_s)$ which is a reliable measure of entanglement generation as a function of $n$, $\omega$ and $\alpha$. Our study finds qualitatively different features in $I_n(l,l_s)$ as a function of $\omega$ and $\alpha$ which can be quantitatively understood from the properties of the Floquet group velocity $v_F^i(k)$ and the corresponding density of states $D_F^i(k)$. We find that for $\alpha > 2$, $\alpha_c$, where at least one dynamical transition exists at $\omega = \omega^{(1)}_c$, $I_n(l,l_s)$ exhibits a single light-cone like feature analogous to the one obtained for quantum quenches [30–32] for $\omega > \omega^{(1)}_c$. In contrast, for $\omega < \omega^{(1)}_c$, it shows multiple light-cone like features which can be shown to be the consequence of appearance of new zeroes in $v_F^i(k)$. The first of such additional zeroes appear at the dynamic transition with the highest frequency ($\omega = \omega^{(1)}_c$); the behavior of $I_n(l,l_s)$ as a function of $n$ changes suddenly at this point relating the dynamic transition to the behavior of $I_n(l,l_s)$. We also find that the behavior of $I_n(l,l_s)$ for $\alpha \leq 2$ is fundamentally different from its counterpart for $\alpha > 2$ at least in two major ways. First, $I_n(l,l_s)$ do not exhibit a light cone structure for any $\omega$ and second the propagation of entanglement between two subsystems is instantaneous for $\alpha \leq 2$ making $I_n(l,l_s)$ finite for any $n > 0$ in contrast to its counterpart for $\alpha > 2$ which is finite for $n > n_c$ (equation (28)). These differences may be understood from the fact that for $\alpha \leq 2$, $v_F^i(k)$ diverges at $k = 0$; thus equation (28) has a solution for any $n > 0$ which ensure instant propagation of entanglement. In contrast, for $\alpha > 2$, $v_F^i(k)$ and hence $n_c$ is finite for all $k$, leading to single or multiple light cone like features along with finite entanglement propagation time. We note that the fact that $n_c$ is zero for all $\alpha < 2$ indicates that the spread of mutual information can not clearly distinguish between quasi long-range ($1 < \alpha < 2$) and long range ($\alpha < 1$) interaction regimes [30–32] in the sense that it propagates instantaneously for any $\alpha < 2$. Our work
therefore points out that the spread of entanglement in a closed quantum system depends on both the drive frequency and the long/short-range nature of its Hamiltonian.

To conclude, we have studied a periodically driven Kitaev chain whose pair-potential decays in space with an exponent $\alpha$. For $\alpha > \alpha_c$, we have found the existence of at least one dynamic transition in this model separating two dynamical phases in which all correlator of the system decay to their steady state values as $(\omega/n)^{3/2}$ for high frequencies and as $(\omega/n)^{1/2}$ for low frequencies. For $\alpha < \alpha_c$, no such transition exists and all correlator exhibit $n^{-1/2}$ decay at all frequencies (except for fine-tuned regions); this allows for a change in the phase of the driven system at high frequencies by tuning $\alpha$ through $\alpha_c$. We have also shown that the behavior of the entanglement entropy exhibits a sudden change at the dynamic transition; at high frequencies, the space-time behavior of the mutual information exhibits a single light cone when $\alpha > 2$ while at low frequencies, multiple light cones exist. This change can be understood from an analysis of the Floquet Hamiltonian of the system. For $\alpha_c < \alpha < 2$, even though the entanglement propagation is instantaneous and no light cone like features exist at any $\omega$, the behavior of the mutual information again shows no new features when $\omega \in (\omega_c, \infty)$ while qualitatively new features appear when $\omega < \omega_c$. Finally, our work suggests that it will be interesting to explore the presence of such dynamical phases in Bethe-integrable systems [47] and in the pre-thermal regime of non-integrable models [48], which are close to integrable points, and to understand the dynamics of entanglement spreading in aperiodically driven (both random and quasiperiodic) integrable systems [49].

Acknowledgments

The work of AS is partly supported through the Partner Group program between the Indian Association for the Cultivation of Science (Kolkata) and the Max Planck Institute for the Physics of Complex Systems (Dresden). The authors thank R Ghosh and T Kuwahara for useful discussions.

Appendix. Protocol independence

We have used a square pulse protocol (equation (9)) for the calculations shown in the main text. The main advantage of using this protocol is that most of the important quantities (for instance, the Floquet spectrum) can be calculated in a closed analytical form. However, we note that most of the results discussed in the main text are independent of the exact functional form of the periodic drive protocol. Here, we illustrate this point with some results using a sinusoidal protocol.

For a sinusoidal protocol with time period denoted by $T$, the variation of $g(t)$ takes the form:

$$g(t) = g_{\text{avg}} + A \cos \left( \frac{2\pi t}{T} \right). \quad (A.1)$$

Here, unlike for the square pulse protocol, the unitary matrix for one drive cycle, $U_k(T)$, cannot be expressed analytically. $U_k(T)$ may then be numerically calculated by using the following approach. We note that the evolution operator $U_k(T)$ that evolves the system for a drive cycle can be written in the form

\[^3\text{See for example [47].}\]
The parametrization of $U_k$ essentially follows from the unitary nature of the evolution and $\theta_k$, $\alpha_k$, $\gamma_k$ are real valued function of $k$. We note that the for any arbitrary initial state $|\psi_k(0)\rangle$, the parameters $\theta_k$, $\alpha_k$ and $\gamma_k$ can be expressed in terms of the initial and the final wavefunctions $|\psi_k(T)\rangle$ after one drive cycle. Here we consider the initial state to be $|\psi_k(0)\rangle = (0, 1)^T$ for all the $k$-modes. Then it can be shown with few lines of simple algebra that $\sin(\theta_k) = |u_{kF}|$, $\alpha_k = -\text{Arg}(v_{kF})$ and $\gamma_k = \text{Arg}(u_{kF})$, where $|\psi_k(t = T)\rangle = (u_{kF}, v_{kF})^T$. To compute $u_{kF}$ and $v_{kF}$, we use the standard fourth order Runge–Kutta method and solve the time-dependent Schrödinger equation. Thus, having obtained the unitary matrix, we can compute $H_k F$ as follows. We note that the unitary structure of the evolution matrix $U_k$ guarantees that $H_k F$ can be expressed in terms of the Pauli matrices. We have,

$$
H_k F = \vec{\sigma} \cdot \vec{n}_k = |\vec{e}_k| |\vec{\sigma} \cdot \vec{n}_k|
$$

where $\vec{e}_k = (\epsilon_{k1}, \epsilon_{k2}, \epsilon_{k3})$ and $\vec{n}_k = \vec{e}_k / |\vec{e}_k|$. Now, it can be shown that
\[
\hat{n}_{k1} = - \sin(\theta_k) \sin(\gamma_k) / D_k \\
\hat{n}_{k2} = - \sin(\theta_k) \cos(\gamma_k) / D_k \\
\hat{n}_{k3} = - \sin(\alpha_k) \cos(\theta_k) / D_k \\
D_k = \sqrt{1 - \cos^2(\theta_k) \cos^2(\alpha_k)} \\
|\vec{\epsilon}_k| = \arccos(\cos(\theta_k) \cos(\alpha_k))/T. \tag{A.4}
\]

Once the Floquet Hamiltonian is obtained following the above procedure then the wavefunction after \(n\) drive cycles can be computed easily using the following:

\[
|\psi_k(t = nT)\rangle = |U_k(T)|n|\psi_k(t = 0)\rangle \\
= \exp[-iH_{kF}T]|\psi_k(t = 0)\rangle. \tag{A.5}
\]

Furthermore, \(v_F^k(k)\) and \(D_F^k(k)\) can be computed by taking appropriate derivates of \(|\vec{\epsilon}_k|\).

In what follows, we focus on the mutual information propagation and show few results obtained by using the sinusoidal protocol. First, in figure A1, we show that the appearance of new features in the propagation of the mutual information in space-time is connected to the
last dynamical phase transition in frequency for any \( \alpha \geq \alpha_c \) as \( \omega \) is varied in the range \([0, \infty)\).

This aspect of mutual information propagation was already demonstrated in the main text in figure 12 for the square pulse protocol. Figure A1 illustrates that for drive frequencies in the range \( \omega \in (\omega_c^{(1)}, \infty) \), there is no new feature in the mutual information propagation compared to a quench i.e \( \omega \to \infty \). Furthermore, as already shown in the main text for the square pulse protocol, this happens irrespective of whether \( \alpha > 2 \) or \( \alpha \leq 2 \). Below \( \omega_c^{(1)} \), new features emerge in the mutual information propagation because of the appearance of new stationary points in the Floquet spectrum where \( D_g^F(k) \) diverges.

In figure A1, we have shown two cases for \( \alpha = 8.0 \) (figures A1(a) and (b)) and \( \alpha = 1.8 \) (figures A1(c) and (d)). By comparing figures A1(a) and (b), we clearly see that below \( \omega_c^{(1)} \), new features arise in \( I_n(l, I_s) \). As discussed in the main text, \( I_n(l, I_s) \) show sharp features in the neighbourhood of \( n = \frac{h}{2 [v^F(k)]} \) where \( k' \) are the momenta where \( D_g^F(k) \) diverges. These special values of \( n \), computed from the Floquet group velocity, are marked with dotted lines in figure A1. For completeness, we show \( v^F(k) \) and \( D_g^F(k) \) as a function of \( k \) for two values of \( \omega \) for \( \alpha = 8.0 \) in figures A2(a) and (b) respectively. These two figures show the appearance of new divergences in \( D_g^F(k) \) below \( \omega_c^{(1)} \) that leads to the emergence of new features in \( I_n(l, I_s) \).

We witness the same phenomenon for \( \alpha \leq 2 \) as well. As concrete illustration, we show the behaviour of \( I_n(l, I_s) \) for \( \alpha = 1.8 \) (figures A1(c) and (d)). The corresponding \( v^F(k) \) and \( D_g^F(k) \) as a function of \( k \) are shown in figures A2(c) and (d) respectively.

**ORCID iDs**

Arnab Sen  
https://orcid.org/0000-0001-5342-8332

**References**

[1] Bloch I 2005 Ultracold quantum gases in optical lattices Nat. Phys. 1 23
[2] Goldman N and Dalibard J 2014 Periodically driven quantum systems: effective Hamiltonians and engineered gauge fields Phys. Rev. X 4 031027
[3] Langen T, Geiger R and Schmiedmayer J 2015 Ultracold atoms out of equilibrium Annu. Rev. Condens. Matter Phys. 6 201
[4] Eckardt A 2017 Colloquium: atomic quantum gases in periodically driven optical lattices Rev. Mod. Phys. 89 011004
[5] Leibfried D, Blatt R, Monroe C and Wineland D 2003 Quantum dynamics of single trapped ions Rev. Mod. Phys. 75 281
[6] Moehring D L, Maunz P, Olmschenk S, Younge K C, Matsukevich D N, Duan L-M and Monroe C 2007 Entanglement of single-atom quantum bits at a distance Nature 449 68
[7] Kim K, Chang M-S, Korenblit S, Islam R, Edwards E E, Freericks J K, Lin G-D, Duan L-M and Monroe C 2010 Quantum simulations of frustrated Ising spins with trapped ions Nature 465 590
[8] Duan L-M and Monroe C 2010 Colloquium: quantum networks with trapped ions Rev. Mod. Phys. 82 1209
[9] Popescu S, Short A J and Winter A 2006 Entanglement and the foundations of statistical mechanics Nat. Phys. 2 754
[10] Nandkishore R and Huse D A 2015 Many body localization and thermalization in quantum statistical mechanics Annu. Rev. Condens. Matter Phys. 6 15
[11] Calabrese P and Cardy J 2006 Time dependence of correlation functions following a quantum quench Phys. Rev. Lett. 96 136801
[12] Kollath C, Läuflch A and Altman E 2007 Quench dynamics and non equilibrium phase diagram of the Bose–Hubbard model Phys. Rev. Lett. 98 180601
[13] Rigol M, Dunjko V and Olshanii M 2008 Thermalization and its mechanism for generic isolated quantum systems Nature 452 854
[14] Moessl M and Kehrein S 2008 Interaction quench in the Hubbard model Phys. Rev. Lett. 100 175702
[15] Polkovnikov A, Sengupta K, Silva A and Vengalattore M 2011 Non-equilibrium dynamics of closed interacting quantum systems Rev. Mod. Phys. 83 863
[16] Lazarides A, Das A and Moessner R 2014 Periodic thermodynamics of isolated systems Phys. Rev. Lett. 112 150401
[17] Lazarides A, Das A and Moessner R 2014 Equilibrium states of generic quantum systems subject to periodic driving Phys. Rev. E 90 012110
[18] D’Alessio L and Rigol M 2014 Long-time behaviour of isolated periodically driven interacting lattice systems Phys. Rev. X 4 041048
[19] Ponte P, Chandran A, Papic Z and Abanin D A 2015 Periodically driven ergodic and many-body localized quantum systems Ann. Phys. 353 196
[20] Else D V, Bauer B and Nayak C 2016 Floquet time crystals Phys. Rev. Lett. 117 090402
[21] Khemani V, Lazarides A, Moessner R and Sondhi S L 2016 On the phase structure of driven quantum systems Phys. Rev. Lett. 116 250401
[22] Kitagawa T, Berg E, Rudner M and Demler E 2010 Topological characterization of periodically driven systems Phys. Rev. B 82 235114
[23] Nathan F and Rudner M S 2015 Topological singularities and the general classification of Floquet–Bloch systems New. J. Phys. 17 125014
[24] Lieb E H and Robinson D 1972 The finite group velocity of quantum spin systems Phys. Rev. Lett. 28 251
[25] Calabrese P and Cardy J 2005 Evolution of entanglement entropy in one-dimensional systems J. Stat. Mech. P04010
[26] Chenau M, Barmettler P, Poletti D, Enders M, Schau P, Fukuhara T, Gross C, Bloch I, Kollath C and Kuhr S 2012 Light-cone-like spreading of correlations in a quantum many-body system Nature 481 484
[27] Kim H and Huse D A 2013 Ballistic spreading of entanglement in a diffusive nonintegrable system Phys. Rev. Lett. 111 127205
[28] Hastings M and Koma T 2006 Spectral gap and exponential decay of correlations Commun. Math. Phys. 265 781
[29] Foss-Feig M, Gong Z-X, Clark C W and Gorshkov A V 2015 Nearly linear light cones in long-range interacting quantum systems Phys. Rev. Lett. 114 157201
[30] Haque P and Tagliacozzo L 2013 Spread of correlations in long-range interacting quantum systems Phys. Rev. Lett. 111 207202
[31] Regemortel M V, Sels D and Wouters M 2016 Information propagation and equilibration in long-range Kitaev chains Phys. Rev. A 93 032311
[32] Buyskikh A S, Fagotti M, Schachenmayer J, Essler F and Daley A J 2016 Entanglement growth and correlation spreading with variable-range interactions in spin and fermion tunneling models Phys. Rev. A 93 053620
[33] Dutta A and Dutta A 2017 Probing the role of long-range interactions in the dynamics of a long-range Kitaev Chain Phys. Rev. B 96 125113
[34] Cevolani L, Despres J, Carleo G, Tagliacozzo L and Sanchez-Palencia L 2017 Universal scaling laws for correlation spreading in quantum systems with short- and long-range interactions Phys. Rev. B 98 024302
[35] Eisert J, Cramer M and Plenio M B 2010 Colloquium: area laws for the entanglement entropy Rev. Mod. Phys. 82 277
[36] Sen A, Nandy S and Sengupta K 2016 Entanglement generation in periodically driven integrable systems: dynamical phase transitions and steady state Phys. Rev. B 94 214301
[37] Sachdev S 2011 Quantum Phase Transitions (Cambridge: Cambridge University Press)
[38] Kitaev A 2006 Anyons in an exactly solved model and beyond Ann. Phys. 321 2
[39] Lieb E, Schlutze T and Mattis D 1961 Two solvable models of an antiferromagnetic chain Ann. Phys., NY 16 407
[40] Vodola D, Lepori L, Ercolelli E, Gorshkov A V and Pupillo G 2014 Kitaev chains with long-range pairing Phys. Rev. Lett. 113 156402
[41] Kolodrubetz M, Clark B K and Huse D A 2012 Nonequilibrium dynamic critical scaling of the quantum ising chain Phys. Rev. Lett. 109 015701
[42] Vidal G, Latorre J I, Rico E and Kitaev A 2003 Entanglement in quantum critical phenomena Phys. Rev. Lett. 90 227902
[43] Peschel I 2003 Calculation of reduced density matrices from correlation functions J. Phys. A: Math. Gen. 36 L205
[44] Fagotti M and Essler F H L 2013 Reduced density matrix after a quantum quench Phys. Rev. B 87 245107
[45] Wolf M M, Verstraete F, Hastings M B and Cirac J I 2008 Area laws in quantum systems: mutual information and correlations Phys. Rev. Lett. 100 070502
[46] Calabrese P, Essler F H L and Fagotti M 2011 Quantum quench in the transverse-field Ising chain Phys. Rev. Lett. 106 227203
[47] Andrei N 1994 Integrable models in condensed matter physics (arXiv:cond-mat/9408101)
Andrei N 2016 Quench dynamics in integrable systems Strongly Interacting Quantum Systems out of Equilibrium (Lecture Notes of the Les Houches Summer School vol 99) (August 2012) (Oxford: Oxford University Press)
[48] Abanin D A, De Roeck W and Huveneers F 2015 Exponentially slow heating in periodically driven many-body systems Phys. Rev. Lett. 115 256803
Abanin D A, De Roeck W and Ho W W 2017 Effective Hamiltonians, prethermalization and slow energy absorption in periodically driven many-body systems Phys. Rev. B 95 014112
Bukov M, Gopalakrishnan S, Knap M and Demler E 2015 Prethermal floquet steady states and instabilities in the periodically driven, weakly interacting Bose–Hubbard model Phys. Rev. Lett. 115 205301
Canovi E, Kollar M and Eckstein M 2016 Stroboscopic prethermalization in weakly interacting periodically driven systems Phys. Rev. E 93 012130
Bukov M, Heyl M, Huse D A and Polkovnikov A 2016 Heating and many-body resonances in a periodically driven two-band system Phys. Rev. B 93 155132
Kuwahara T, Mori T and Saito K 2016 Floquet–Magnus theory and generic transient dynamics in periodically driven many-body quantum systems Ann. Phys. 367 96
[49] Nandy S, Sen A and Sen D 2017 Aperiodically driven integrable systems and their emergent steady states Phys. Rev. X 7 031034