A New Descent Algebra for Standard Parabolic Subgroups of $W(A_n)$

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Abstract

A new descent algebra $\sum_{W}(A_n)$ of Weyl groups of type $A_n$, constructed by present authors in [1], is generated by equivalence classes $[x_J]$ arising from the equivalence relation defined on the set of all $x_J$. In this paper, we introduce the structure of this new descent algebra for standard parabolic subgroups $W_J$ of Weyl groups of type $A_n$.

Key words and Phrases: Weyl groups, descent algebra.

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1 Introduction

In [1], the authors have constructed a new descent algebra of Weyl groups of type $A_n$. This new descent algebra is denoted by $\sum_{W}(A_n)$. The main objective of this paper is to introduce and study the structure of this new descent algebra when the group is taken a standard parabolic subgroup of Weyl group of type $A_n$.

In the next section, we shall give the structure of a new descent algebra for standard paraolic subgroups $W_J$ of $W(A_n)$. This algebra will be denoted by $\sum_{W_J}(A_n)$. Additionally, we shall define the induction and restriction maps between new descent algebras $\sum_{W_J}(A_n)$ and $\sum_{W_M}(A_n)$, where $W_J$ and $W_M$ are standard parabolic subgroups of $W(A_n)$.

We now give the notation, which is fairly standard and follows that given in Yağmur and Can [1], Carter [2], Solomon [3] and Bergeron et. al [4].

Let $e_1, e_2, ..., e_{n+1}$ be an orthonormal basis of a Euclidean space of dimension $n + 1$. Then the simple system and root system of type $A_n$ are given by

$$\Pi = \{e_1 - e_2, e_2 - e_3, ..., e_{n-1} - e_n, e_n - e_{n+1}\},$$

$$\Phi = \{e_i - e_j | i \neq j, \ i, j = 1, ..., n + 1\},$$

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respectively.

Let $\Pi$ be a simple system in root system $\Phi$ of type $A_n$ and $\Phi^+$ be the corresponding positive system. Then $W(A_n)$ is called the Weyl group of type $A_n$ generated by the reflections $w_r$ for all $r \in \Phi$ [2]. For simplicity, throughout this study, this group will be denoted by $W$.

Let $W_J$ be the subgroup of $W$, where $J$ is any subset of $\Pi$. This group is called a standard parabolic subgroup of $W$. Let $X_J$ be the set of representatives of the cosets of $wW_J$ in $W$. Then $X_{JK} = X_J^{-1} \cap X_K$ is a set of distinguished double coset representatives of $W_J w W_K$ in $W$.

Let $x_J = \sum_{d \in X_J} d$. The set of all $x_J$ is a basis for an algebra $\sum W$ over the field of rationals with integer structure constants $a_{JKL}$, where $a_{JKL}$ is the number of elements $w \in X_{JK}$ such that $w^{-1}(J) \cap K = L$. This algebra, discovered by Solomon [3] in 1976, is called the descent algebra (or Solomon algebra) of Weyl groups $W$. Moreover, in 1992 Bergeron et al. [4] has reconstructed the descent algebra systematically.

In [1], present authors have defined an equivalence relation on the set of all $x_J$ in order to form a basis for a new descent algebra $\sum W(A_n)$. Furthermore, the basis of this new descent algebra consists of equivalence classes $[x_J]$ arising from the equivalence relation

$$x_J \sim x_K \Leftrightarrow J \sim K \Leftrightarrow K = w(J), \quad for \quad w \in W.$$ 

Additionally, a ring multiplication for two basis elements is defined by

$$[x_J][x_K] = [x_J x_K] = \sum_{L \subseteq K} a_{JKL} [x_L], \quad (1)$$

where $a_{JKL}$’s are defined as Solomon’ descent algebra.

The following theorem is proved in [1, Theorem A].

**Theorem 1.1** Let $J \subseteq \Pi$, then

$$\sum A_n) = Sp\{[x_J]|J \subseteq \Pi\}$$

is a new descent algebra of Weyl groups of type $A_n$.

## 2 Main Results

In this section we firstly define a new descent algebra for standard parabolic subgroups $W_J$ of Weyl groups of type $A_n$. After, we shall study on this new algebra.

Let $K \subseteq J \subseteq \Pi$. An element $[x^J_K]$ can be defined as

$$[x^J_K] = \{x^J_L|L \sim K, \quad L \subseteq J\}.$$
Note that, if we take $J = \Pi$, then because of $x^\Pi_L = x_L$ (see Bergeron et. al [4]) we obtain

$$[x^\Pi_K] = \{x^\Pi_L | L \sim K, \ L \subseteq \Pi\}$$
$$= \{x_L | L \sim K\}$$
$$= [x_K].$$

For this reason,

$$\sum_{W_J}(A_n) = Sp\{[x^J_K]|K \subseteq J\}$$

is a new descent algebra for parabolic subgroups of Weyl groups of type $A_n$. Furthermore, the set $\{[x^J_K]|K \subseteq J\}$ is a basis for this algebra.

**Theorem 2.1** Let $a^J_{MNP}$ be the structure constants of the new descent algebra $\sum_{W_J}(A_n)$ corresponding to the $[x^J_K]$ basis. If $K, N \subseteq \Pi$, then

$$[x_K][x_N] = \sum_{P \subseteq N} \left( \sum_{M \subseteq J} a_{KJM} a^J_{MNP} \right) [x_P],$$

for all $J \subseteq \Pi$ such that $N \subseteq J$. Moreover, the structure constants satisfy the identities

$$a_{KNP} = \sum_{M \subseteq J} a_{KJM} a^J_{MNP}.$$

For the proof of this theorem, we need to state the following crucial lemma proved in [4, Lemma 2.1].

**Lemma 2.2** If $K \subseteq J \subseteq \Pi$, then $x_K = x_Jx^J_K$.

**Proof of Theorem 2.1** For $K, N \subseteq \Pi$ and $N \subseteq J$, by using Lemma 2.2 and equation (1), we have

$$[x_K][x_N] = [x_Kx_N]$$
$$= [x_Kx_Jx^J_N]$$
$$= [x_Kx_J][x^J_N]$$
$$= \sum_{M \subseteq J} a_{KJM}[x_M][x^J_N]$$
$$= \sum_{M \subseteq J} a_{KJM}[x_Jx^J_M][x^J_N]$$
$$= \sum_{M \subseteq J} a_{KJM}[x_J][x^J_Mx^J_N].$$
\[
\sum_{M \subseteq J} a_{KJM} [x_J] \sum_{P \subseteq N} a_{MNP}^J \]

\[
= \sum_{M \subseteq J, P \subseteq N} a_{KJM} a_{MNP}^J [x_J][x_P]
\]

\[
= \sum_{M \subseteq J, P \subseteq N} a_{KJM} a_{MNP}^J [x_Jx_P]
\]

\[
= \sum_{M \subseteq J, P \subseteq N} a_{KJM} a_{MNP}^J [x_P]
\]

\[
= \sum_{P \subseteq N} (\sum_{M \subseteq J} a_{KJM} a_{MNP}^J) [x_P].
\]

Moreover, by the equation (1), we get

\[
[x_K][x_N] = \sum_{P \subseteq N} a_{KNP} [x_P]
\]

So, since the set \(\{[x_P]| P \subseteq N\}\) is linear independence, we obtain

\[
a_{KNP} = \sum_{M \subseteq J} a_{KJM} a_{MNP}^J.
\]

This completes the proof. \[\Box\]

We now give the definitions of induction and restriction maps between the new descent algebras \(\sum_{WJ}(A_n)\) and \(\sum_{WM}(A_n)\), but for simple presentation, we may denote these algebras as \(\sum_{WJ}\) and \(\sum_{WM}\), respectively.

**Definition 2.3** Let \(J \subseteq M \subseteq \Pi\). The induction map between \(\sum_{WJ}\) and \(\sum_{WM}\) is defined by

\[
\text{ind}_J^M : \sum_{WJ} \rightarrow \sum_{WM} ; \quad \text{ind}_J^M ([x^J_K]) = [x^M_J][x^K_J].
\]

**Definition 2.4** Let \(J \subseteq M \subseteq \Pi\). The restriction map between \(\sum_{WM}\) and \(\sum_{WJ}\) is defined by

\[
\text{res}_J^M : \sum_{WM} \rightarrow \sum_{WJ} ;
\]

\[
\text{res}_J^M ([x^M_K]) = \sum_{d \in W_M \cap X_{JK}} [x^J_{d^{-1}(K)}] = \sum_{d \in W_M \cap X_{JK}} [x^J_{d(K)}].
\]

In [1, Proposition 2.10], the authors have proved that there is an isomorphism between \(\sum_{W}(A_n)\) and \(PB(W)\), where \(PB(W)\) is the parabolic Burnside ring of
associated Weyl group. For the case of induction, the permutation representation \( W_J/W_K \) in \( PB(W_J) \) induced to \( PB(W) \) is simply \( W/W_K \) [4]. The following theorem is arising from this discussion.

**Theorem 2.5** Given \( J \subseteq M \subseteq \Pi \), let

\[
\Theta_J : \sum_{W_J} \to \sum_{W_M} \quad \text{and} \quad \Theta_M : PB(W_J) \to PB(W_M)
\]

be the canonical homomorphisms. Then,

\[
\Theta_J \circ \text{res}_J^M = \text{res}_{W_J}^W \Theta_M
\]

and

\[
\Theta_M \circ \text{ind}_J^M = \text{ind}_{W_J}^W \Theta_J,
\]

where

\[
\text{res}_{W_J}^W : PB(W_M) \to PB(W_J);
\]

\[
\text{res}_{W_J}^W (W_M/W_K) = \sum_{d \in W_M \cap X_{JK}} W_J/W_{J \cap d^{-1}(K)}
\]

and

\[
\text{ind}_{W_J}^W : PB(W_J) \to PB(W_M); \quad \text{ind}_{W_J}^W (W_J/W_K) = W_M/W_K.
\]

**Proof.** For all \([x_K^M] \in \sum_{W_M} \), by using Definition 2.4 and considering the existence of isomorphism between \( \sum_{W_M} \) and \( PB(W_M) \) (see Yağmur and Can [1]), we get

\[
\Theta_J \circ \text{res}_J^M ([x_K^M]) = \Theta_J (\text{res}_J^M ([x_K^M]))
\]

\[
= \Theta_J \left( \sum_{d \in W_M \cap X_{JK}} [x_K^M] \right)
\]

\[
= \sum_{d \in W_M \cap X_{JK}} \Theta_J ([x_K^M])
\]

\[
= \sum_{d \in W_M \cap X_{JK}} W_J/W_{J \cap d^{-1}(K)}.
\]

On the other hand, we obtain

\[
\text{res}_J^M \Theta_J ([x_K^M]) = \text{res}_J^M (\Theta_J [x_K^M])
\]

\[
= \text{res}_J^M (W_M/W_K)
\]

\[
= \sum_{d \in W_M \cap X_{JK}} W_J/W_{J \cap d^{-1}(K)}
\]

\[
= \sum_{d \in W_M \cap X_{JK}} W_J/W_{J \cap d(K)}.
\]
This proves the first assertion of theorem.

Now, for all \([x^K_J] \in \sum_{W_J} \), by using Definition 2.3 and considering the existence of isomorphism between \(\sum_{W_J} \) and \(PB(W_J)\) (see Yağmur and Can [1]), we have

\[
\Theta_M \text{ind}_J^M ([x^K_J]) = \Theta_M (\text{ind}_J^M ([x^K_J]))
\]
\[
= \Theta_M ([x_J^M][x^K_J])
\]
\[
= \Theta_M ([x^K_M])
\]
\[
= W_M/W_K.
\]

Besides, we obtain

\[
\text{ind}_J^M \Theta_M ([x^K_J]) = \text{ind}_J^M (\Theta_M ([x^K_J]))
\]
\[
= \text{ind}_J^M (W_J/W_K)
\]
\[
= W_M/W_K.
\]

This completes the proof. \(\Box\)

We would also like to point out that these results are obtained from conformation of Bergeron et. al[4]'s study to new descent algebra for parabolic subgroups of \(W(A_n)\).

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