Entropy Spectrum of Black Holes of Heterotic String Theory via Adiabatic Invariance

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Abstract Using adiabatic invariance and the Bohr–Sommerfeld quantization rule we investigate the entropy spectroscopy of two black holes of heterotic string theory, the charged GMGHS and the rotating Sen solutions. It is shown that the entropy spectrum is equally spaced in both cases, identically to the spectrum obtained before for Schwarzschild, Reissner–Nordström and Kerr black holes. Since the adiabatic invariance method does not use quasinormal mode analysis, there is no need to impose the small charge or small angular momentum limits and there is no confusion on whether the real part or imaginary part of the modes is responsible for the entropy spectrum.

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1 Introduction

Bekenstein[1] showed that the black hole horizon area is an adiabatic invariant, so that it should be quantized by Ehrenfest principle. Considering the minimum change of the horizon area, ΔA, due to the absorption of a test particle into the black hole, Bekenstein argued that it can only be proportional to ℏ, so that the area spectrum should be linearly quantized. This equally spaced area spectrum let us understand the entropy of a black hole as proportional to the horizon area, $A = 8\pi\ell_p^2$, due to the absorption of a test particle into the black hole. Bekenstein argued that it can only be proportional to $\hbar$, so that the area spectrum should be linearly quantized.

The investigation by Hod[3] on quasinormal modes has shown the possibility to use Bohr’s correspondence principle to provide a method to quantize the horizon area of black holes. Here, the real part of quasinormal mode frequency is responsible for the area spectrum, giving for Schwarzschild’s black hole the spectrum

$$\Delta A = 4\ln 3\ell_p^2,$$

which is significantly different from Bekenstein’s result. Later, Maggiore[4] reexamined this idea by considering the imaginary contribution which is dominant for the highly excited quasinormal modes. This reproduced exactly Bekenstein’s spectrum, $\Delta A = 8\pi\ell_p^2$.

In 2012, Zeng et al.[5] obtained the same area spectrum for Schwarzschild and Kerr black holes by considering another proposal: the frequency of an outgoing wave performs periodic motion outside the horizon and since the gravity system is periodic with respect to Euclidean time with a period given by the inverse of the Hawking temperature, they propose that the frequency of the outgoing wave must be associated to this temperature.

In this work, we will investigate the spectroscopy of two black hole of heterotic string theory by the new approach presented by Mahjii and Vagenas[6] and extended for the rotating case by Chen and Yang.[7] Here the entropy spectrum is obtained by using the adiabaticity of black holes and the Bohr–Sommerfeld quantization rule. The method has been applied to different cases as Reissner–Nordström solution[8] and BTZ metrics.[9] The resulting black hole entropies show an equally spaced spectrum and the corresponding horizon area spectrum is identical to the result reported by Wei et al.[10] in the charged case. However, it is noteworthy that our treatment does not suppose the small angular momentum or small charge limits that are needed in other studies and there is no confusion on whether the real part or imaginary part of quasinormal modes is responsible for the entropy spectrum.

2 The GMGHS Black Hole

The low energy effective action of the heterotic string theory in four dimensions is given by

$$\mathcal{A} = \int d^4x\sqrt{-g}e^{-\psi}
\times\left(-R + \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} - G_{\mu\nu}\partial\psi\partial\psi + \frac{1}{8}F_{\mu\nu}F^{\mu\nu}\right),$$

where $R$ is the Ricci scalar, $G_{\mu\nu}$ is the metric that arises

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naturally in the σ model and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (3)$$

is the Maxwell field associated with a $U(1)$ subgroup of $E_8 \times E_8$. There is also a dilaton field $\psi$ and an antisymmetric tensor gauge field $B_{\mu\nu}$ given by $H_{\mu\rho\nu} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} - \partial_\rho B_{\mu\nu}$, with the gauge Chern-Simons term $\int [\Omega_3(A)]_{\mu\rho\nu}$, with the gauge $\psi$ and electric charge $q_i$ defined as $g_{\mu\nu} = e^{-\psi}G_{\mu\nu}$. The charged black hole solution to the corresponding field equations is known as the Gibbons–Maeda–Garfinkle–Horowitz–Strominger (GMGHS) metric,$^{[10]-[11]}$

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left(1 - \frac{Q^2 e^{-2\psi_0}}{M}\right)d\Omega^2, \quad (5)$$

$$e^{-2\psi} = e^{-2\psi_0}\left(1 - \frac{Q^2}{M}\right), \quad (6)$$

$$F = Q \sin \theta d\theta \wedge d\varphi, \quad (7)$$

where $f(r) = 1 - 2M/r$, $M$ and $Q$ are the mass and electric charge of the black hole respectively and $\psi_0$ is the asymptotic value of the dilaton. There is a spherical event horizon with radius

$$r_H = 2M, \quad (8)$$

and area

$$A = 4\pi r_H^2 - 8\pi Q^2 e^{-2\psi_0}. \quad (9)$$

Note that the GMGHS solution becomes a naked singularity if

$$M^2 \leq \frac{1}{2}Q^2 e^{-2\psi_0}. \quad (10)$$

The physical characteristics of the black hole are its Hawking temperature,

$$T = \frac{\hbar \kappa}{2\pi} = \frac{1}{8\pi M}, \quad (11)$$

which is independent of the electric charge and the electric potential at the event horizon,

$$\Phi = \frac{Q}{r_H} e^{-2\psi_0}. \quad (12)$$

Finally, the relation between area and entropy is

$$S = \frac{A}{4\hbar} = \frac{\pi r_H^2 - 2\pi Q^2 e^{-2\psi_0}}{\hbar}. \quad (13)$$

### 2.1 Entropy Spectrum

The black hole metric (5) can be “Euclideanized” by the transformation $t \rightarrow -i\tau$, giving

$$ds^2 = f(r)d\tau^2 + \frac{dr^2}{f(r)} + r^2\left(1 - \frac{Q^2 e^{-2\psi_0}}{M}\right)d\Omega^2. \quad (14)$$

The euclidean time $\tau$ is periodic with period $2\pi/\kappa$, where $\kappa$ is the surface gravity given by Eq. (11).

We will use the adiabatic invariant action

$$I = \sum_i p_i dq_i \quad (15)$$

to obtain the entropy spectrum of the GMGHS black hole. This is

$$I = \sum_i \int_O^{\tau_i} dp_i dq_i = \int_O^H \frac{dH'}{q_i} dq_i, \quad (16)$$

where $p_i$ is the conjugate momentum of the $q_i$ coordinate. Here we take $i = 0, 1$ with $q_0 = \tau$ the Euclidean time, $q_1 = r$ and dot means derivative with respect to $\tau$. Expanding the sum over $i$, gives

$$I = \int_O^H dh'd\tau + \int_O^H \frac{dh'}{r} dr, \quad (17)$$

and considering radial null paths,

$$\dot{r} = \frac{dr}{d\tau} = \pm i f(|r|) = R_{\pm}(|r|), \quad (18)$$

where the $\pm$ sign means outgoing or ingoing paths, respectively, we can write

$$\int_O^H dh'd\tau = \int_O^H dh'\frac{dr}{R_{\pm}(|r|)} = \int_O^H dh' \frac{dr}{r}. \quad (19)$$

Thus, the adiabatic invariant (17) becomes

$$I = 2\int_O^H dh'd\tau. \quad (20)$$

Since we will only consider outgoing paths, the limits in the $\tau$ integration will be 0 and $\pi/\kappa$, obtaining

$$I = 2\pi \int_O^H \frac{dh'}{r} = \hbar \int_O^H \frac{dh'}{T}. \quad (21)$$

To relate this result with the entropy of the black hole, consider the first law of thermodynamics for the GMGHS black hole

$$dM = T dS + \Phi dQ. \quad (22)$$

Following the work of Wald$^{[12]}$ and Sudarsky & Wald$^{[13]}$ to obtain the Hamiltonian of the Einstein-Maxwell system we have the relation of $H$ with the mass and electric potential,

$$H = M - \Phi Q. \quad (23)$$

Therefore, the adiabatic invariant in terms of the entropy is simply

$$I = \hbar S. \quad (24)$$

Implementing the Bohr-Sommerfeld quantization rule,

$$I = \sum_i p_i dq_i = 2\pi nh, \quad (25)$$

we obtain the equally spaced entropy spectrum $S = 2\pi n$ or $\Delta S = 2\pi$. Using the relation between entropy and horizon area (13) we obtain the area spectrum $\Delta A = 8\pi l_p^2$, where $l_p$ is Planck's length (in units with $G = c = 1$).

### 3 The Sen Black Hole

Sen$^{[14]-[15]}$ found a charged stationary solution of the field equations by the use of target space duality applied...
to the Kerr solution. The line element of this black hole is

\[ ds^2 = -(1 - \frac{2Mr}{\rho^2})dt^2 + \rho^2 \left( \frac{dr^2}{\Delta} + d\theta^2 \right) - \frac{4Mr\sin^2\theta}{\rho^2} dtd\varphi + \left( r(r + r_a) + a^2 + \frac{2Mr\sin^2\theta}{\rho^2} \right) \sin^2\theta d\varphi^2, \quad (26) \]

where

\[ \Delta = r(r + r_a) - 2Mr + a^2, \quad (27) \]
\[ \rho^2 = r(r + r_a) + a^2\cos^2\theta. \quad (28) \]

Here \( M \) is the mass of the black hole, \( a = J/M \) is the specific angular momentum of the black hole and the electric charge is given by

\[ r_a = \frac{Q^2}{M}. \quad (29) \]

Taking \( a = 0 \), Eq. (26) coincides with the GMGHS solution (5) while taking \( r_a = 0 \) it reconstructs the Kerr metric. The Sen space has an event horizon, with radius defined by the largest root of the equation \( \Delta = 0 \),

\[ r_H = M - \frac{Q^2}{2M} + \sqrt{\left( M - \frac{Q^2}{2M} \right)^2 - \frac{J^2}{M^2}}, \quad (30) \]

and with area

\[ A = 8\pi \left( M - \frac{Q^2}{2M} \right) - \left[ M - \frac{Q^2}{2M} + \sqrt{\left( M - \frac{Q^2}{2M} \right)^2 - \frac{J^2}{M^2}} \right]. \quad (31) \]

Equation (30) shows that the horizon disappears unless \( |J| \leq M^2 - Q^2/2 \), where the equal sign defines the extremal black hole which has \( A = 8\pi J^2 \). The physical characteristics of the black hole are the angular velocity at the horizon,

\[ \Omega = \frac{J}{2M^2 - M - Q^2/2M + \sqrt{(M - Q^2/2M)^2 - J^2/M^2}}, \quad (32) \]

the electrostatic potential at the horizon,

\[ V = \frac{Q}{2M}, \quad (33) \]

and the Hawking temperature,

\[ T_H = \frac{\kappa_{\mu}h}{2\pi} = \frac{h\sqrt{(2M^2 - Q^2)^2 - 4J^2}}{4\pi M(2M^2 - Q^2 + \sqrt{(2M^2 - Q^2)^2 - 4J^2})}. \quad (34) \]

### 3.1 Entropy Spectrum

In order to calculate the entropy spectrum for the Sen black hole we consider again the adiabatic invariant action in Eq. (17). However, in this case there is an ergosphere between the outer horizon and the infinite redshift surface and to avoid the dragging effect, one should perform the coordinate transformation,

\[ \phi = \varphi - \Omega t, \quad (35) \]

where the dragging angular velocity \( \Omega \) is given by Eq. (17). The line element becomes

\[ ds^2 = -F(r)dt^2 + \frac{1}{G(r)}dr^2 + \rho^2 d\theta^2 + H^2(r)d\phi^2, \quad (36) \]

where

\[ F(r) = \frac{\Delta \rho^2}{(r + r_a)(a^2)^2 - \Delta a^2 \sin^2\theta}; \quad (37) \]
\[ G(r) = \frac{\Delta}{\rho^2}; \quad (38) \]
\[ H^2(r) = \frac{\sin^2\theta}{\rho^2} \left[ (r(r + r_a) + a^2)^2 - \Delta a^2 \sin^2\theta \right]. \quad (39) \]

This line element is “Euclideanized” by making \( t \to -i\tau \),

\[ ds^2 = F(r)d\tau^2 + \frac{1}{G(r)}dr^2 + \rho^2 d\theta^2 + H^2(r)d\phi^2, \quad (40) \]

so radial null outgoing or ingoing paths are

\[ \dot{\tau} = \frac{dr}{d\tau} = \pm i\sqrt{F(r)G(r)} = R_{\pm}(r). \quad (41) \]

This gives

\[ \int_o^H dH'\dot{\tau} = \int_o^H dH'\frac{dr}{R_{\pm}(r)} = \int_o^H dH'\frac{dr}{r}. \quad (42) \]

and Eq. (17) becomes

\[ I = 2\pi \int_o^H \frac{dH'}{\kappa} = h \int_o^H \frac{dH'}{T}. \quad (43) \]

where we have considered the euclidean time and only outgoing paths. This time, the first law of thermodynamics is

\[ dM = TdS + Jd\Omega + \Phi dQ, \quad (44) \]

while the Hamiltonian relates with the electric potential and angular momentum by

\[ H = M - \Phi Q - 2\Omega J. \quad (45) \]

Implementing the Bohr–Sommerfeld quantization(25), we obtain the equally spaced spectrum for the entropy,

\[ \Delta S = 2\pi. \quad (46) \]

### 4 Conclusions

Although the quantum gravity theory has not been found, it is meaningful to investigate quantum corrections to the entropy spectrum. Here we have investigated the entropy of two stringy black holes, GMGHS and Sen solutions, with the help of Bohr–Sommerfeld quantization rule and the adiabatic invariance. The results show an equally spaced entropy spectrum in both cases, which is the same as the obtained for Schwarzschild, Reissner–Nordström and Kerr black holes. This fact confirms the proposal of Bekenstein that the area spectrum of a black hole is independent of its parameters. Even more, our result is identical to the spectrum reported by Wei et al.[10] using the quasinormal modes analysis for the charged case.
However, the adiabatic invariance calculation does not need the quasinormal mode frequency, so there is no confusion on whether the real part or imaginary part is responsible for the entropy spectrum and the small charge and angular momentum limits, which are necessary in the quasinormal mode analysis, were not imposed.

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