SCALAR GENERALIZED VERMA MODULES

HELGE MAAKESTAD

ABSTRACT. In this paper we study the Verma module $M(\mu)$ associated to a linear form $\mu \in \mathfrak{h}^*$ where $\mathfrak{sl}(E) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ is a triangular decomposition of $\mathfrak{sl}(E)$. The $\text{SL}(E)$-module $M(\mu)$ has a canonical simple quotient $L(\mu)$ with a canonical generator $v$. We study the left annihilator ideal $\text{ann}(v)$ in $U(\mathfrak{sl}(E))$. We also study scalar generalized Verma module $M(\rho)$ associated to a character $\rho$ of $\mathfrak{p}$ where $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{sl}(E)$. We prove $M(\rho)$ has a canonical simple quotient $L(\rho)$. This simple quotient is in some cases an infinite dimensional $\text{SL}(E)$-module. We get a class of mutually non-isomorphic irreducible $\text{SL}(E)$-modules containing the class of all finite dimensional irreducible $\text{SL}(E)$-modules. As a result we give an algebraic proof of a classical result of Smoke on the structure of the jet bundle as $P$-module on any flag-scheme $\text{SL}(E)/P$ where $P$ is any parabolic subgroup.

CONTENTS

1. Introduction 1
2. Scalar generalized Verma modules 2
3. Classical Verma modules and annihilator ideals 6
References 14

1. Introduction

We study the scalar generalized Verma module $M(\rho)$ associated to a character $\rho$ of $\mathfrak{p}$ where $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{sl}(E)$. We prove $M(\rho)$ has a canonical simple quotient $L(\rho)$. This simple quotient is in some cases an infinite dimensional $\text{SL}(E)$-module. We get a construction of a class of mutually non-isomorphic irreducible $\text{SL}(E)$-modules containing the class of finite dimensional irreducible $\text{SL}(E)$-modules. This class contains many infinite dimensional $\text{SL}(E)$-modules.

Let $V_\lambda$ be an arbitrary finite dimensional irreducible $\text{SL}(E)$-module. In this note we give a construction of $V_\lambda$ using the enveloping algebra $U(\mathfrak{sl}(E))$ and the Verma modules $M(\mu)$. We study the annihilator ideal $\text{ann}(v)$ in $U(\mathfrak{sl}(E))$ where $v$ is the highest weight vector in $V_\lambda$. We prove the class of simple modules $L(\rho)$ may be constructed using classical Verma modules $M(\mu)$ as done by Dixmier in [1]. As a

Date: March 2010.

1991 Mathematics Subject Classification. 20G15, 17B35, 17B20.

Key words and phrases. annihilator ideal, irreducible representation, highest weight vector, canonical filtration, canonical basis, generalized Verma module, semi simple algebraic group, Lie algebra, enveloping algebra.

Supported by a research scholarship from NAV, www.nav.no.
result we give an algebraic proof of a classical result of Smoke on the structure of the jet bundle as P-module on SL(E)/P (see Corollary 3.19).

2. Scalar generalized Verma modules

In this section we construct any scalar generalized Verma module $M(\rho)$. Here $\rho$ is a character of $\mathfrak{p}$ where $\mathfrak{p}$ is any parabolic subalgebra of $\mathfrak{sl}(E)$. We prove $M(\rho)$ has a canonical simple quotient $L(\rho)$. When $L(\rho)$ is finite dimensional we get a construction of all finite dimensional irreducible $\text{SL}(E)$-modules.

Let $G = \text{SL}(E)$ where $E$ is an $n$-dimensional vector space over an algebraically closed field $K$ of characteristic zero and let $\mathfrak{g} = \mathfrak{sl}(E)$. Let $\mathfrak{h}$ be the abelian subalgebra of $\mathfrak{sl}(E)$ of diagonal matrices. It follows $(\mathfrak{sl}(E), \mathfrak{h})$ is a split semi simple Lie algebra and determines a root system $R = R(\mathfrak{sl}(E), \mathfrak{h})$. Let $B$ be a basis for $R$. This determines the positive roots $R_+$ and the negative roots $R_-$. This determines a triangular decomposition $\mathfrak{sl}(E) = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$. Define the following character $\rho_\pm = \rho$

$$\rho : \mathfrak{p} \to \text{End}(L_\rho)$$

by

$$\rho(x) = \eta(x)w = \sum_{i=1}^{k} l_i(tr(A_1) + \cdots + tr(A_i))w.$$  

Lemma 2.1. The pair $(L_\rho, \rho)$ is a rank one $\mathfrak{p}$-module. All rank one $\mathfrak{p}$-modules arise in this way.

Proof. The proof is an exercise. □

Let $U(I(\mathfrak{sl}(E))) \subseteq U(\mathfrak{sl}(E))$ be the canonical filtration of $U(\mathfrak{sl}(E))$.

Definition 2.2. Let $M(\rho) = U(\mathfrak{sl}(E)) \otimes_{U(\mathfrak{p})} L_\rho$ be the generalized Verma module associated to $\rho$. Let $M_I(\rho) = U_I(\mathfrak{sl}(E)) \otimes_{U_I(\mathfrak{p})} L_\rho$ be the canonical filtration of $M(\rho)$.

Since $U_I(\mathfrak{sl}(E)) \otimes_{U_I(\mathfrak{p})} L_\rho$ is a $P$-submodule of $M(\rho)$ where $\mathfrak{p} = \text{Lie}(P)$ it follows $\{ M_I(\rho) \}_{I \geq 0}$ is a filtration of $M(\rho)$ by $P$-modules.

Since $\text{dim}_K(L_\rho) = 1$ we refer to $M(\rho)$ as a scalar generalized Verma module. By definition this construction give all scalar generalized Verma modules for $\text{SL}(E)$. 

Define the following sub Lie algebra of \( \mathfrak{s}(E) \): \( \mathfrak{n} \) is the subalgebra of matrices \( x \) on the form
\[
x = \begin{pmatrix}
A_1 & 0 & \cdots & 0 \\
* & A_2 & \cdots & 0 \\
& \ddots & \ddots & \ddots \\
& & * & \cdots & 0 \\
& & & * & \cdots & A_{k+1}
\end{pmatrix}
\]
where \( A_i \) is a \( d_i \times d_i \)-matrix with zero entries. It follows there is an isomorphism \( \mathfrak{n} \oplus \mathfrak{p} \cong \mathfrak{s}(E) \) as vector spaces. Let \( \mathfrak{s}(E) = \mathfrak{n}_{-} \oplus \mathfrak{h} \oplus \mathfrak{n}_{+} \) be the standard triangular decomposition of \( \mathfrak{s}(E) \) as defined in the previous section. It follows there is an inclusion \( \mathfrak{n}_{-} \subseteq \mathfrak{n}_{-} \) of Lie algebras. Let \( R = R(\mathfrak{s}(E), \mathfrak{h}) \) be the roots of \( \mathfrak{s}(E) \) with respect to \( \mathfrak{h} \). Let \( B' = \{ \alpha_1, \ldots, \alpha_m \} \) be a subset of \( R \) such that the set
\[
X_{-\alpha_1}, \ldots, X_{-\alpha_m}
\]
is a basis for \( \mathfrak{n} \).
Let \( P = (p_1, \ldots, p_m) \) and let \( X^P = X_{-\alpha_1}^{p_1} \cdots X_{-\alpha_m}^{p_m} \). Let \( \alpha_P = -p_1\alpha_1 - \cdots - p_m\alpha_m \).

**Lemma 2.3.** The following holds: The set
\[
\{ X^P \otimes w : p_1, \ldots, p_m \geq 0 \}
\]
is a basis for \( M(\rho) \) as \( K \)-vector space. The natural map
\[
\phi : \text{U}(\mathfrak{n}) \to M(\rho)
\]
defined by
\[
\phi(X^P) = X^P \otimes w
\]
is an isomorphism of left \( \mathfrak{n} \)-modules.

**Proof.** Since \( \mathfrak{n} \oplus \mathfrak{p} = \mathfrak{s}(E) \) there is by definition an isomorphism
\[
\text{U}(\mathfrak{s}(E)) \cong K\{ X^P : p_1, \ldots, p_m \geq 0 \} \text{U}(\mathfrak{p})
\]
of free right \( \text{U}(\mathfrak{p}) \)-modules. We get
\[
M(\rho) \cong \{ X^P : p_1, \ldots, p_m \geq 0 \} \text{U}(\mathfrak{p}) \otimes_{\text{U}(\mathfrak{p})} L_\rho \cong \}
\[
K\{ X^P \otimes w : p_1, \ldots, p_m \geq 0 \}.
\]
The first claim is proved. One checks the map \( \phi \) is a map of left \( \mathfrak{n} \)-modules and the Lemma is proved. \( \square \)

Let \( \omega_i = L_1 + \cdots + L_i \in \mathfrak{h} \) for \( i = 1, \ldots, n-1 \) be the fundamental weights for \( \text{SL}(E) \) and let \( \lambda = \sum_{i=1}^{k} l_i(\omega_i) \). It follows for all \( x \in \mathfrak{h} \subseteq \mathfrak{p} \) that \( \rho(x) = \lambda(x) \).

Let \( \rho_U : \text{U}(\mathfrak{p}) \to \text{End}(L_\rho) \) be the associated morphism of \( \rho \). Let \( L \) be the following left ideal of \( \text{U}(\mathfrak{p}) \):
\[
L = \text{U}(\mathfrak{p})\{ x - \rho(x)1_p : x \in \mathfrak{p} \}.
\]

**Proposition 2.4.** Let \( N = \text{ker}(\rho_U) \). There is an equality of ideals in \( \text{U}(\mathfrak{p}) \): \( N = L \). Hence \( L \) is a two-sided ideal in \( \text{U}(\mathfrak{p}) \).

**Proof.** The proof follows [1], Section 7. There is a short exact sequence of rings
\[
0 \to N \to \text{U}(\mathfrak{p}) \to K \to 0.
\]
Let $x_1, x_2, ..., x_n$ be a basis with $\rho(x_1) \neq 0$ and $\rho(x_2) = \cdots = \rho(x_n) = 0$. Assume $x = x_1^{v_1}x_2^{v_2} \cdots x_n^{v_n} \in U(p)$. It follows $\rho_0(x) = 0$ if and only if $v_2 + \cdots + v_n \geq 1$. We get by the PBW Theorem a direct sum decomposition
\[
U(p) = \{x_1^{v_1}x_2^{v_2} \cdots x_n^{v_n} : v_2 + \cdots + v_n \geq 1\} \oplus \{x_1^{v_1} : v_1 \geq 1\}.
\]
It follows there is an inclusion of vector spaces
\[
\{x_1^{v_1}x_2^{v_2} \cdots x_n^{v_n} : v_2 + \cdots + v_n \geq 1\} \subseteq N.
\]
One checks there is an equality of vector spaces
\[
x_1^{v_1} : v_1 \geq 1 = \{x_1^v : 1 \leq v \leq n\}.
\]
There is an inclusion
\[
\{x_1^{v_1}x_2^{v_2} \cdots x_n^{v_n} : v_2 + \cdots + v_n \geq 1\} \oplus \{x_1^{v_1} : v_1 \geq 1\} \subseteq L
\]
hence $codim(L, U(p)) \leq 1$. Similarly $codim(N, U(p)) = 1$. Since $L \subseteq N$ it follows there is an equality $L = N$ and the Proposition is proved. □

**Definition 2.5.** Let $\text{char}(\rho) = U(g)\{x - \rho(x)1_g : x \in p\}$ be the left character ideal of $\rho$ in $U(g)$. Let $\text{char}(\rho) = \text{char}(\rho) \cap U(l(g))$ be the canonical filtration of $\text{char}(\rho)$.

**Proposition 2.6.** The natural map $\phi : U(g) \to M(\rho)$ defined by $\phi(x) = x \otimes w$ defines an exact sequence of left $U(g)$-modules
\[
0 \to \text{char}(\rho) \to U(g) \to \phi M(\rho) \to 0.
\]

**Proof.** Let $X_{-\alpha_1}, ..., X_{-\alpha_m}$ be the basis for $n$ constructed above and let
\[
X^P = X_{-\alpha_1}^{p_1} \cdots X_{-\alpha_m}^{p_m}
\]
with $p_i \geq 0$ integers. There is an isomorphism of right $U(p)$-modules
\[
U(g) \cong \{X^P : p_i \geq 0\} U(p)
\]
hence we get an isomorphism of vector spaces
\[
M(\rho) \cong \{X^P \otimes w : p_i \geq 0\}.
\]
Assume $X \in U(g)$ is an element. We may write $X = \sum_P X^P x_P$ with $x_P \in U(p)$. Assume $\phi(X) = X \otimes w = 0$. We get
\[
\phi(X) = X \otimes w = \sum_P X^P x_P \otimes w = \sum_P X^P \otimes x_P w = 0
\]
It follows
\[
x_P w = 0
\]
for all $P$ hence $x_P \in ker(\rho_0) = U(p)\{y - \rho(y)1_p : y \in p\}$. It follows $X = \sum_P X^P x_P \in \text{char}(\rho)$ hence $ker(\phi) \subseteq \text{char}(\rho)$. One checks $\text{char}(\rho) \subseteq ker(\phi)$ and the Proposition is proved. □

By Lemma 2.3 it follows
\[
M(\rho) = K\{X^P \otimes w : p_i \geq 0\}.
\]
Let
\[
\alpha_P = -p_1\alpha_1 - \cdots - p_m\alpha_m \in h^*.
\]
Assume $x \in h$. We get
\[
x(X^P \otimes w) = [x, X^P] \otimes w + X^P x \otimes w = \alpha_P(x)X^P \otimes w + X^P \otimes xw =
\]
\[ \alpha_P(x)X^P \otimes w + \lambda(x)X^P \otimes w = \\
(\lambda + \alpha_P)(x)X^P \otimes w. \]

Hence
\[ M(\rho)_{\lambda+\alpha_P} = K\{X^P \otimes w\}. \]

It follows
\[ M(\rho) = \oplus_\rho M(\rho)_{\lambda+\alpha_P}. \]

Define
\[ M(\rho)_+ = \oplus_{\rho \geq 1} M(\rho)_{\lambda+\alpha_P}. \]

It follows
\[ M(\rho) = M(\rho)_{\lambda} \oplus M(\rho)_+ \]

where
\[ M(\rho)_{\lambda} = K\{1_\mathfrak{g} \otimes w\}. \]

Let the vector \(1_\mathfrak{g} \otimes v \in M(\rho)\) be the canonical generator of \(M(\rho)\).

**Theorem 2.7.** Let \(M(\rho)\) be the scalar generalized Verma module associated to the character \(\rho\). The following holds: \(M(\rho)\) contains a maximal non-trivial sub-\(\mathfrak{g}\)-module \(K\). The quotient \(L(\rho) = M(\rho)/K\) is simple and \(\dim_K(L(\rho)) \geq 2\). Let \(v = 1 \otimes w \in L(\rho)\). The ideal \(\text{ann}(v)\) is the largest non-trivial left ideal in \(U(\mathfrak{g})\) containing \(\text{char}(\rho)\). The vector \(v\) satisfies the following: \(U(\mathfrak{n}_+)v = 0\). For all \(x \in \mathfrak{h}\) it follows \(xv = \lambda(x)v\) hence \(v\) has weight \(\lambda\).

**Proof.** Consider the element \(X_{-\alpha_m} \otimes w \in M(\rho)_+\) and look at the product

\[ X_{\alpha_m}X_{-\alpha_m} \otimes w. \]

We get
\[ X_{\alpha_m}X_{-\alpha_m} \otimes w = [X_{\alpha_m}, X_{-\alpha_m}] \otimes w + X_{-\alpha_m} \otimes X_{\alpha_m}w = \\
H_{\alpha_m} \otimes w = 1_\mathfrak{g} \otimes H_{\alpha_m}w = \\
\lambda(H_{\alpha_m})(1_\mathfrak{g} \otimes w). \]

It follows
\[ X_{\alpha_m}X_{-\alpha_m} \otimes w \in M(\rho)_{\lambda} \]

hence \(M(\rho)_+\) is not \(\mathfrak{g}\)-stable.

Assume \(L \subseteq M(\rho)\) is a non-trivial \(\mathfrak{g}\)-stable module. It follows \(L \cap M(\rho)_{\lambda} = 0\) hence \(L \subseteq M(\rho)_+\). Let \(K\) be the sum of all non-trivial sub-\(\mathfrak{g}\)-modules of \(M(\rho)\). It follows \(K \subseteq M(\rho)_+\) since \(M(\rho)_+\) is not \(\mathfrak{g}\)-stable. Hence \(K \subseteq M(\rho)\) is a maximal non-trivial sub-\(\mathfrak{g}\)-module of \(M(\rho)\) and \(L(\rho) = M(\rho)/K\) is a simple quotient. It follows \(\dim_K(L(\rho)) \geq 2\).

One checks the vector \(v\) is annihilated by \(U(\mathfrak{n}_+)\) and has weight \(\lambda\).

By Proposition 2.6 there is an exact sequence of left \(U(\mathfrak{g})\)-modules
\[ 0 \rightarrow \text{char}(\rho) \rightarrow U(\mathfrak{g}) \rightarrow M(\rho) \rightarrow 0 \]

hence there is an isomorphism
\[ M(\rho) \cong U(\mathfrak{g})/\text{char}(\rho) \]
of left \(U(\mathfrak{g})\)-modules. It follows there is a bijection between the set of left sub-\(\mathfrak{g}\)-modules of \(M(\rho)\) and left sub-\(\mathfrak{g}\)-modules of \(U(\mathfrak{g})/\text{char}(\rho)\). This induce a bijection between the set of left ideals in \(U(\mathfrak{g})\) containing the ideal \(\text{char}(\rho)\) and the set of left sub-\(\mathfrak{g}\)-modules of \(M(\rho)\). It follows the submodule \(K\) corresponds to a maximal
non-trivial left ideal \( J \) in \( U(\mathfrak{g}) \) containing \( \text{char}(\rho) \). The ideal \( J \) is by definition the annihilator ideal of \( v \): There is an equality 
\[
J = \text{ann}(v).
\]

The Theorem is proved. \( \square \)

**Corollary 2.8.** The following holds:
\[(2.8.1) \quad L(\rho) \text{ is simple for all } \underline{l} \in \mathbb{Z}^k.\]
\[(2.8.2) \quad \text{If } \underline{l} \neq \underline{l}' \text{ it follows } L(\rho^{\underline{l}}) \neq L(\rho^{\underline{l}'})\]
\[(2.8.3) \quad \text{If } l_1, \ldots, l_k < 0. \text{ It follows } \dim_p(L(\rho)) = \infty\]

**Proof.** Claim 2.8.1 This is by definition of \( L(\rho) \) since the submodule \( K \) is maximal.
Claim 2.8.2 Assume \( \phi : L(\rho^{\underline{l}}) \to L(\rho^{\underline{l}'}) \) is a map of \( \mathfrak{g} \)-modules. It follows \( \phi \) is the zero map or an isomorphism since the modules are simple. If it is an isomorphism it follows the weights are equal. This implies \( \underline{l} = \underline{l}' \) a contradiction. The claim is proved. We prove claim 2.8.3 Assume \( \dim_K(L(\rho)) < \infty \). It follows \( L(\rho) \cong V_\lambda \) where \( V_\lambda \) has a highest weight vector \( v' \) with highest weight \( \lambda' = \sum_{j=1}^{\infty} l'_j \omega_j \) with \( l'_j \geq 0 \). Since the vector \( v \) in \( L(\rho) \) has weight \( \lambda \) with \( l_i < 0 \) one gets a contradiction. The Corollary follows. \( \square \)

Assume \( p = \mathfrak{b}_+ = \mathfrak{h} \oplus \mathfrak{n}_+ \). It follows from [1], Section 7 the \( \text{SL}(E) \)-module \( L(\lambda + \delta) \) is isomorphic to \( V_\lambda \) - the finite dimensional irreducible \( \text{SL}(E) \)-module with highest weight \( \lambda = \sum_{i=1}^k l_i \omega_n_i \). Here \( l_i \geq 1 \) is an integer for \( i = 1, \ldots, k \). Hence the class
\[
\{ L(\rho^{\underline{l}}) : \rho^{\underline{l}} : p \to K, \underline{l} \in \mathbb{Z}^k \}
\]
is a class of mutually non-isomorphic \( \text{SL}(E) \)-modules parametrized by a parabolic subalgebra \( p \subseteq \mathfrak{sl}(E) \) and a character \( \rho : p \to K \) containing the class of all finite dimensional irreducible \( \text{SL}(E) \)-modules.

Let \( K_l = K \cap M_l(\rho) \subseteq M(\rho) \) be the induced filtration on \( K \). There is an exact sequence of \( P \)-modules
\[
0 \to K_1 \to M_1(\rho) \to L_1(\rho) \to 0.
\]

**Definition 2.9.** Let \( \{ L_l(\rho) \}_{l \geq 0} \) be the canonical filtration of \( L(\rho) \).

It follows \( \{ L_l(\rho) \}_{l \geq 0} \) is a filtration of \( L(\rho) \) by \( P \)-modules.

**Corollary 2.10.** Assume \( L(\rho) = V_\lambda \) is a finite dimensional irreducible \( \text{SL}(E) \)-module with highest weight vector \( v \) and highest weight \( \lambda \). It follows \( L_l(\rho) \cong U_l(\mathfrak{sl}(E)) \cdot v \).

**Proof.** The proof is obvious. \( \square \)

3. Classical Verma modules and annihilator ideals

Let \( \mathfrak{b}_+ = \mathfrak{h} \oplus \mathfrak{n}_+ \). It follows \( \mathfrak{b}_+ \) is a sub Lie algebra of \( \mathfrak{sl}(E) \). Let \((x, n), (y, m)\) be elements of \( \mathfrak{b}_+ \). The Lie product on \( \mathfrak{sl}(E) \) induce the following product on \( \mathfrak{b}_+ \): define the following action of \( \mathfrak{h} \) on \( \mathfrak{n}_+ \):
\[
ad : \mathfrak{h} \to \text{End}(\mathfrak{n}_+)
\]
\[
ad(x)(n) = [x, n].
\]
It follows \( \mathfrak{n}_+ \) is a \( \mathfrak{h} \)-module. Let \( \text{ad}(x)(h) = x(h) \).
Lemma 3.1. Define
\[ [(x, n), (y, m)] = (0, x(m) - y(n) + [n, m]) \]
where \([ ]\) is the bracket on \(n_+\). It follows the natural injection \(b_+ \rightarrow \mathfrak{sl}(E)\) is a map of Lie algebras.

Proof. The proof is obvious. \(\square\)

Let \(\mu \in \mathfrak{h}^*\) be a linear form on \(\mathfrak{h}\). Let \(L_\mu = Kw\) be the free rank one \(K\)-module on \(w\). Define the following map
\[ \tau_\mu : b_+ \rightarrow \text{End}(L_\mu) \]
by
\[ \tau_\mu(x, n)(v) = \mu(x)v. \]

Lemma 3.2. The map \(\tau_\mu\) makes \(L_\mu\) into a \(b_+\)-module.

Proof. It is clear \([\tau_\mu(x, n), \tau_\mu(y, m)] = \tau_\mu([(x, n), (y, m)])\) hence the claim is proved. \(\square\)

It follows \(L_\mu\) is a left \(b_+\)-module. By definition \(U(\mathfrak{sl}(E))\) is a right \(b_+\)-module and we may form the tensor product
\[ M(\mu) = U(\mathfrak{sl}(E)) \otimes_{U(b_+)} L_\mu. \]
It follows \(M(\mu)\) is a left \(G\)-module.

Definition 3.3. The \(G\)-module \(M(\mu)\) is the Verma module associated to the linear form \(\mu \in \mathfrak{h}^*\).

Let \(\delta = \frac{1}{2} \sum_{\alpha \in R_+} \alpha\) and let \(\lambda\) be an element in \(\mathfrak{h}\). Let \(L_{\lambda - \delta} = Kv\) be the free rank one \(K\)-vectorspace on the element \(v\). By the result above we get a character
\[ \tau_{\lambda - \delta} : b_+ \rightarrow \text{End}(L_{\lambda - \delta}) \]
defined by
\[ \tau_{\lambda - \delta}(h, n) = (\lambda - \delta)(h) \]
where \((h, n)\) is an element in \(b_+ = \mathfrak{h} \oplus n_+\). Let \(\alpha_1, \ldots, \alpha_m\) be the \(m\) distinct elements of \(R_+\). Let \(X_{-\alpha_i}\) be an element in \(\mathfrak{g}^{-\alpha_i}\). It follows the set
\[ X_{-\alpha_1}, \ldots, X_{-\alpha_m} \]
is a basis for \(n_-\) as vector space. There is a decomposition \(\mathfrak{g} = n_- \oplus b_+\) and it follows \(U(\mathfrak{g})\) is a free right \(U(b_+)-\)module as follows:
\[ U(\mathfrak{g}) = \{ X_{p_1} \cdots X_{p_m}^{\alpha_m} : p_1, \ldots, p_m \geq 0 \} U(b_+). \]

Lemma 3.4. Assume \(X^P = X_{p_1}^{\alpha_1} \cdots X_{p_m}^{\alpha_m}\) with \(p_1, \ldots, p_m \geq 0\) integers. Let \(x \in \mathfrak{h}\). It follows
\[ x(X^P) = (-p_1 \alpha_1 - \cdots - p_m \alpha_m)(x)X^P = \alpha_P(x)X^P. \]

Proof. The proof is an easy calculation. \(\square\)

Let \(P = (p_1, \ldots, p_m)\) with \(p_i\) integers and let \(\alpha_P = p_1 \alpha_1 + \cdots + p_m \alpha_m\). Let \(X^P = X_{-\alpha_1}^{p_1} \cdots X_{-\alpha_m}^{p_m}\).
Proposition 3.5. Assume $x \in \mathfrak{h}$. The following holds:

\begin{align*}
(3.5.1) & \quad M(\lambda) = K\{X^P \otimes v : p_1, \ldots, p_m \geq 0\} \\
(3.5.2) & \quad x(X^P \otimes v) = (\lambda - \delta - \alpha_P)(x)X^P \otimes v \\
(3.5.3) & \quad M(\lambda) = \oplus_{p_1, \ldots, p_m \geq 0} M(\lambda)_{\lambda - \delta - \alpha_P} \\
(3.5.4) & \quad M(\lambda)_{\lambda - \delta} = K(1 \otimes v) \\
(3.5.5) & \quad M(\mu) = K\{X^P \otimes v : \lambda - \delta - \alpha_P = \mu\} \\
(3.5.6) & \quad U(n_-)(1 \otimes v) = M(\lambda)
\end{align*}

Proof. We prove Claim 3.5.1. There is a direct sum decomposition

$$\mathfrak{g} = n_- \oplus \mathfrak{b}_+$$

and the set

$$X_{-\alpha_1}, \ldots, X_{-\alpha_m}$$

is a basis for $n_-$ as $K$-vector space. It follows by the PBW-theorem that the set

$$\{X^P : p_1, \ldots, p_m \geq 0\}$$

is a basis for $U(n_-)$ as $K$-vector space. It follows $U(\mathfrak{g})$ is isomorphic to

$$\{X^P : p_1, \ldots, p_m \geq 0\} U(\mathfrak{b}_+)$$

as free left $U(\mathfrak{b}_+)$-module. We get

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}_+)} L_{\lambda - \delta} \cong$$

$$\{X^P : p_1, \ldots, p_m \geq 0\} U(\mathfrak{b}_+) \otimes_{U(\mathfrak{b}_+)} L_{\lambda - \delta} \cong$$

$$K\{X^P \otimes v : p_1, \ldots, p_m \geq 0\}$$

and Claim 3.5.1 is proved.

We prove Claim 3.5.2. By the previous Lemma it follows for all $x \in \mathfrak{h}$

$$[x, X^P] = \alpha_P(x)X^P.$$ 

We get

$$x(X^P \otimes v) = [x, X^P] \otimes v + X^P x \otimes v =$$

$$\alpha_P(x)X^P \otimes v + X^P \otimes (\lambda - \delta)(x)v =$$

$$(\lambda - \delta - \alpha_P)(x)X^P \otimes v$$

and Claim 3.5.2 is proved. We prove Claim 3.5.3. By Claim 3.5.1 it follows $M(\lambda)$ has a basis given as follows:

$$\{X^P \otimes v : p_1, \ldots, p_m \geq 0\}.$$ 

Since for all $x \in \mathfrak{h}$ it follows

$$x(X^P \otimes v) = (\lambda - \delta - \alpha_P)(x)X^P \otimes v$$

Claim 3.5.3 follows.

We prove Claim 3.5.4. Let $X^P \otimes v \in M(\lambda)$ It follows $x(X^P \otimes v) = (\lambda - \delta - \alpha_P)(x)X^P \otimes v$. Hence $X^P \otimes v$ is in $M(\lambda)_{\lambda - \delta}$ if and only if

$$\lambda - \delta - \alpha_P = \lambda - \delta.$$ 

Since $p_1, \ldots, p_m \geq 0$ this can only occur when $p_1 = \cdots p_m = 0$ hence $\alpha_P = 0$. The Claim is proved.

Claim 3.5.5 is by definition.
We prove Claim 3.5.6. Since the map
\[ \phi : U(n-) \rightarrow M(\lambda) \]
defined by
\[ \phi(X^P) = X^P \otimes v \]
is an isomorphism of \( n_- \)-modules Claim 3.5.6 follows. The Proposition is proved.

**Definition 3.6.** The element \( 1 \otimes v \in M(\lambda) \) is called the **canonical generator** of \( M(\lambda) \).

**Proposition 3.7.** There is an isomorphism of \( n_- \)-modules
\[ \phi : U(n-) \cong M(\lambda) \]
defined by
\[ \phi(X^P) = X^P \otimes v. \]

**Proof.** Let \( n_- \) have basis \( X_{-\alpha_1}, \ldots, X_{-\alpha_m} \) and let \( X^P = X_{-\alpha_1}^{p_1} \cdots X_{-\alpha_m}^{p_m} \). It follows \( M(\lambda) \) has a basis consisting of elements \( X^P \otimes v \). Hence the map \( \phi \) is an isomorphism of vector spaces. One checks it is \( n_- \)-linear and the Proposition follows.

Let in the following \( E = K\{e_1, \ldots, e_n\} \) be an \( n \)-dimensional vector space over \( K \) and let \( g = sl(E) \) with \( h \) in \( g \) the abelian sub-algebra of diagonal matrices. Let \( E_i = K\{e_1, \ldots, e_i\} \) for \( i = 1, \ldots, n-1 \). It follows we get a complete flag
\[ E_\bullet : 0 \neq E_1 \subseteq E_2 \subseteq \cdots \subseteq E_{n-1} \subseteq E \]
in \( E \) with \( \dim(E_i) = i \). Let \( l = (l_1, \ldots, l_{n-1}) \) with \( l_i \geq 0 \) integers. Let
\[ W(l) = \text{Sym}^{l_1}(E) \otimes \text{Sym}^{l_2}(\wedge^2 E) \otimes \cdots \otimes \text{Sym}^{l_{n-1}}(\wedge^{n-1} E). \]
Let \( v_i = \wedge^i E_i \) and let \( v = v_1 l_1 \otimes \cdots \otimes v_{n-1} l_{n-1} \) be a line in \( W(l) \). Let \( P \) in \( SL(E) \) be the subgroup of elements fixing the flag \( E_\bullet \). It follows \( P \) consists of upper triangular matrices with determinant one. Let \( p = \text{Lie}(P) \). It follows \( p \) consists of upper triangular matrices with trace zero. Let
\[ \omega_i = L_1 + \cdots + L_i \]
where
\[ h^* = K\{L_1, \ldots, L_n\}/L_1 + \cdots + L_n. \]
It follows \( \omega_1, \ldots, \omega_{n-1} \) are the fundamental weights for \( g \). Let \( \lambda = \sum_{i=1}^{n-1} l_i \omega_i \).

**Proposition 3.8.** Let \( x \in p \) be an element. The following formula holds:
\[ x(v) = \sum_{i=1}^{n-1} l_i (a_{11} + \cdots + a_{ii}) v \]
where \( a_{ii} \) is the \( i \)’th diagonal element of \( x \). It follows the vector \( v \) has weight \( \lambda \). The \( SL(E) \)-module \( V_\lambda \) generated by \( v \) is an irreducible finite dimensional \( SL(E) \)-module with highest weight vector \( v \) and highest weight \( \lambda \).

**Proof.** The Proposition follows from [1], Section 7 and an explicit calculation.
Let $L_v$ in $V_\lambda$ be the line spanned by the vector $v$. It follows the subgroup of SL($E$) of elements fixing $L_v$ equals the group $P$. We get a character

$$\rho : p \to \text{End}(L_v)$$

defined by

$$\rho(x)v = x(v) = \sum_{i=1}^{n-1} l_i (a_{1i} + \cdots + a_{ii})v.$$ 

We get an exact sequence of Lie algebras

$$0 \to p_v \to p \to \text{End}(L_v) \to 0$$

where $p_v = \ker(\rho)$.

**Definition 3.9.** Let

$$\text{char}(\rho) = \mathcal{U}(\mathfrak{g}) \{x - \rho(x)1_\mathfrak{g} : x \in p\}$$

be the left character ideal of $\rho$. Let

$$\text{ann}(v) = \{x \in \mathcal{U}(\mathfrak{g}) : x(v) = 0\}$$

be the left annihilator ideal of $v$. Let

$$\text{char}_l(\rho) = \text{char}(\rho) \cap \mathcal{U}_l(\mathfrak{g})$$

and

$$\text{ann}_l(v) = \text{ann}(v) \cap \mathcal{U}_l(\mathfrak{g})$$

for all $l \geq 1$.

If $x \in \mathfrak{h}$ it follows $\rho(x) = \lambda(x)$ where $\lambda$ is the highest weight of $V_\lambda$. By [1], Section 7 the following holds: Let for $\alpha \in B$ the integer $m_\alpha$ be defined as follows:

$$m_\alpha = \lambda(H_\alpha) + 1$$

where

$$0 \neq H_\alpha \in [X_\alpha, X_{-\alpha}].$$

If $\alpha_i = L_i - L_{i+1}$ it follows $X_{\alpha_i} = E_{i,i+1}$. It also follows $X_{-\alpha_i} = E_{i+1,i}$. We get

$$H_{\alpha_i} = E_{ii} - E_{i+1,i+1}.$$ 

**Lemma 3.10.** The following holds for all $i = 1, \ldots, n - 1$:

$$m_{\alpha_i} = l_i + 1.$$

**Proof.** The proof is an easy calculation. \hfill \Box

We get by [1], Proposition 7.2.7, Section 7 the following description of the ideal $\text{ann}(v)$ in $\mathcal{U}(\mathfrak{g})$. Note that $\mathcal{U}(\mathfrak{g})$ is a noetherian associative algebra and the ideal $\text{ann}(v)$ is a left ideal in $\mathcal{U}(\mathfrak{g})$. It follows $\text{ann}(v)$ has a finite set of generators. The following holds: Let

$$I(v) = \mathcal{U}(\mathfrak{g})n_+ + \sum_{x \in \mathfrak{h}} \mathcal{U}(\mathfrak{g})(x - \lambda(x)1_\mathfrak{g}).$$

It follows $I(v) \subseteq \mathcal{U}(\mathfrak{g})$ is a left ideal. It follows

$$\text{ann}(v) = I(v) + \sum_{\alpha \in B} \mathcal{U}(\mathfrak{g})X_{-\alpha}^{m_\alpha}.$$
Let \( I_l(v) = I(v) \cap U_l(\mathfrak{g}) \) for all \( l \geq 1 \) integers. Let for any element \( x \in U(\mathfrak{g}) \)
\[
fil(x) = \min \{ l : x \in U_l(\mathfrak{g}) \}
\]
be the filtration of the element \( x \).

**Lemma 3.11.** There is for every integer \( l \geq 1 \) an equality
\[
\text{ann}_l(v) = I_l(v) + \sum_{\alpha \in B} U_{l-m_\alpha}(\mathfrak{g}) X_{-\alpha}^{m_\alpha}
\]
of vector spaces.

**Proof.** The inclusion
\[
I_l(v) + \sum_{\alpha \in B} U_{l-m_\alpha}(\mathfrak{g}) X_{-\alpha}^{m_\alpha} \subseteq \text{ann}_l(v)
\]
is obvious. Assume
\[
x = u + v \in \text{ann}_l(v) = (I(v) + \sum_{\alpha \in B} U(\mathfrak{g}) X_{-\alpha}^{m_\alpha}) \cap U_l(\mathfrak{g})
\]
with
\[
u \in I(v)
\]
and
\[
v \in \sum_{\alpha \in B} U(\mathfrak{g}) X_{-\alpha}^{m_\alpha}.
\]
It follows \( fil(u), fil(v) \leq fil(x) \). Hence \( u, v \in U_l(\mathfrak{g}) \). It follows \( u \in I_l(v) \) and
\[
v \in \sum_{\alpha \in B} U_{l-m_\alpha}(\mathfrak{g}) X_{-\alpha}^{m_\alpha}.
\]
The Lemma follows. \( \square \)

**Proposition 3.12.** There is an equality
\[
I(v) = \text{char}(\rho)
\]
of left ideals in \( U(\mathfrak{g}) \).

**Proof.** Recall there is a triangular decomposition \( \mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \). It follows there is an equality \( \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{n}_+ = \mathfrak{b}_+ \) of Lie algebras. Assume \( x \in \mathfrak{n}_+ \subseteq \mathfrak{p} \). It follows \( \rho(x) = 0 \), hence \( x(v) = \rho(x)v = 0 \). Assume \( x \in \mathfrak{h} \subseteq \mathfrak{p} \). It follows \( \rho(x) = \lambda(x) \) where \( \lambda \) is the highest weigt of \( V_\lambda \) with
\[
\lambda = \sum_{i=1}^{n-1} l_i \omega_i.
\]
It follows there is an element \( x \in \mathfrak{h} \) with \( \lambda(x) \neq 0 \). Moreover we may choose elements \( h_1, \ldots, h_{n-2} \) in \( \mathfrak{h} \) with \( \rho(h_i) = \lambda(h_i) = 0 \) and
\[
\{ x, h_1, \ldots, h_{n-2} \}
\]
is a basis for \( \mathfrak{h} \). Assume \( y(x - \rho(x)1_\mathfrak{g}) \in \text{char}(\rho) \). It follows
\[
x \in \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{n}_+.
\]
It follows \( x = x_1 + x_2 \) with \( x_1 \in \mathfrak{h} \) and \( x_2 \in \mathfrak{n}_+ \). It follows
\[
y(x - \rho(x)1_\mathfrak{g}) = y(x_1 + x_2 - \rho(x_1 + x_2)1_\mathfrak{g}) = y(x_1 - \rho(x_1)1_\mathfrak{g}) + y(x_2 - \rho(x_2)1_\mathfrak{g}).
\]
Since $x_1 \in \mathfrak{h}$ it follows
\[ y(x_1 - \rho(x_1)1_\mathfrak{g}) = y(x_1 - \lambda(x_1)1_\mathfrak{g}) \subseteq \sum_{x \in \mathfrak{h}} U(\mathfrak{g})(x - \lambda(x)1_\mathfrak{g}) \subseteq I(v). \]

Since $x_2 \in \mathfrak{n}_+$ it follows $\rho(x_2) = 0$. We get
\[ y(x_2 - \rho(x_2)1_\mathfrak{g}) = yx_2 \in U(\mathfrak{g})\mathfrak{n}_+. \]

It follows $y(x - \rho(x)1_\mathfrak{g}) \in I(v)$ and we have proved the inclusion $\text{char}(\rho) \subseteq I(v)$.

We prove the reverse inclusion: Assume $x = x_1 + x_2 \in I(v)$ with $x_1 = yn \in U(\mathfrak{g})\mathfrak{n}_+$ and $y \in U(\mathfrak{g})$ and $n \in \mathfrak{n}_+$. Assume also $x_2 = z(u - \lambda(u)1_\mathfrak{g})$ with $z \in U(\mathfrak{g})$ and $u \in \mathfrak{h}$. It follows $\rho(n) = 0$ and $x_1 = y(n - \rho(n)1_\mathfrak{g}) \in \text{char}(\rho)$. Also $x_2 = z(u - \lambda(u)1_\mathfrak{g}) = z(u - \rho(u)1_\mathfrak{g}) \in \text{char}(\rho)$. It follows $x = x_1 + x_2 \in \text{char}(\rho)$ and we have proved the reverse inclusion $I(v) \subseteq \text{char}(\rho)$. The Proposition is proved.

\[ \square \]

**Corollary 3.13.** There is for every $l \geq 1$ an equality
\[ I_l(v) = \text{char}_l(\rho) \]

of filtrations.

**Proof.** The Corollary follows from Proposition 3.12.

Recall $\lambda = \sum_{i=1}^{n-1} i \omega_i$. Define $m(\lambda) = \min_{i=1, \ldots, n-1} \{l_i\}$.

**Theorem 3.14.** For all $1 \leq l \leq m(\lambda)$ there is an equality
\[ \text{ann}_l(v) = \text{char}_l(\rho) \]

of filtrations.

**Proof.** By Lemma 3.11 there is a equality
\[ \text{ann}_l(v) = I_l(v) + \sum_{\alpha_i \in B} U_{l-m_{\alpha_i}}(\mathfrak{g})X_{-\alpha_i}^{m_{\alpha_i} + 1} \]

of vector spaces. We get
\[ X_{-\alpha_i}^{m_{\alpha_i} + 1} = E_{i+1,i}^{l_i+1} \]

for $i = 1, \ldots, n-1$. We get from Corollary 3.13
\[ \text{ann}_l(v) = I_l(v) + \sum_{i=1}^{n-1} U_{l-l_i-1}(\mathfrak{g})E_{i+1,i}^{l_i+1} = \text{char}_l(\rho) + \sum_{i=1}^{n-1} U_{l-l_i-1}(\mathfrak{g})E_{i+1,i}^{l_i+1}. \]

If $1 \leq l \leq m(\lambda)$ it follows $l \leq l_i$ for all $i = 1, \ldots, n-1$. It follows $l - l_i - 1 < 0$ hence we get an equality
\[ \text{ann}_l(v) = \text{char}_l(\rho) \]

and the claim of the Theorem follows. \[ \square \]

Let $\mathfrak{p} \subseteq \mathfrak{sl}(E)$ be a parabolic subalgebra and let $\mathfrak{n}$ be the complementary Lie algebra as defined in 3. It follows there is a direct sum decomposition as vector spaces $\mathfrak{p} \oplus \mathfrak{n} \cong \mathfrak{sl}(E)$. We get inclusions $\mathfrak{n} \subseteq \mathfrak{n}_-$ and $\mathfrak{b}_+ \subseteq \mathfrak{p}$ of Lie algebras. Let $\rho : \mathfrak{p} \to \text{End}(L_\rho)$ with induced character
\[ \tilde{\rho} : \mathfrak{b}_+ \to \text{End}(L_{\tilde{\rho}}). \]
Lemma 3.15. There is a surjective map of left $U(\mathfrak{sl}(E))$-modules 
\[ \phi : M(\tilde{\rho}) \to M(\rho). \]
It follows $\ker(\phi) = \text{char}(\rho)/\text{char}(\tilde{\rho})$.

Proof. There is a natural map
\[ f : U(\mathfrak{sl}(E)) \times L_\rho \to U(\mathfrak{sl}(E)) \otimes U(p) L_\rho \]
defined by
\[ f(z, w) = z \otimes w. \]
This map induce a surjection
\[ \phi : U(\mathfrak{sl}(E)) \otimes U(b_+) L_\rho \to U(\mathfrak{sl}(E)) \otimes U(p) L_\rho \]
of left $U(\mathfrak{sl}(E))$-modules and the first claim is proved. The second claim follows easily and the Lemma is proved. □

Let $K \subseteq M(\rho)$ be the unique maximal $\mathfrak{sl}(E)$-stable submodule and let $K' \subseteq M(\tilde{\rho})$ be the unique $\mathfrak{sl}(E)$-stable submodule. It follows $\phi(K') \subseteq K$ since the image of a $\mathfrak{sl}(E)$-stable module is $\mathfrak{sl}(E)$-stable. We get a commutative diagram of maps of $\mathfrak{sl}(E)$-modules where the middle vertical arrow is surjective:

\[
\begin{array}{ccccccccc}
0 & \to & K' & \to & M(\tilde{\rho}) & \overset{\phi'}{\to} & L(\tilde{\rho}) & \to & 0 \\
& & \downarrow{\phi} & & \downarrow{\phi'} & & \downarrow{\phi'} & & \\
0 & \to & K & \to & M(\rho) & \overset{\rho}{\to} & L(\rho) & \to & 0 \\
\end{array}
\]

Lemma 3.16. The map $\phi'$ is an isomorphism of $\mathfrak{sl}(E)$-modules.

Proof. Since the map $\phi$ is a surjection of $\mathfrak{sl}(E)$-modules it follows $\phi'$ is a surjection. Since the modules $L(\rho)$ and $L(\tilde{\rho})$ are simple it follows $\ker(\phi')$ is zero. The Lemma is proved. □

Hence we get no new simple $\text{SL}(E)$-modules when we consider simple quotients $L(\rho)$ where $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{sl}(E)$ with $\mathfrak{p} \neq \mathfrak{b}_+$. The modules $L(\rho)$ may be constructed using simple quotients of classical Verma modules as done in [1].

Let $u = 1 \otimes w \in M(\rho)$ be the canonical generator and let $v$ be its image in $L(\rho)$ under the canonical projection map $M(\rho) \to L(\rho)$. Let $\text{ann}(v)$ be the left annihilator ideal in $U(\mathfrak{sl}(E))$ of the vector $v$.

Recall the exact sequence
\[ 0 \to K_l \to M_l(\rho) \to L_l(\rho) \to 0 \]
from the previous section.

Proposition 3.17. The following holds:

\[ K_l = 0 \text{ if and only if } \text{ann}(v) = \text{char}(\rho). \]
Proof. There is a commutative diagram of exact sequences of left $\mathfrak{sl}(E)$-modules

\[
\begin{array}{cccccc}
0 & \longrightarrow & \text{char}(\rho) & \overset{\phi}{\longrightarrow} & U(\mathfrak{sl}(E)) & \longrightarrow & M(\rho) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{ann}(v) & \overset{\tilde{\phi}}{\longrightarrow} & U(\mathfrak{sl}(E)) & \longrightarrow & L(\rho) & \longrightarrow & 0 \\
\end{array}
\]

One checks that $\phi(\text{ann}(v)) = K$ and $\phi(\text{ann}_l(v)) = K_l$ for all $l \geq 1$. Assume $\text{ann}_l(v) = \text{char}_l(\rho)$. It follows

\[K_l = \phi(\text{ann}_l(v)) = \phi(\text{char}_l(\rho)) = 0\]

since the diagram above is exact. Assume $K_l = 0$ and let $x \in \text{ann}_l(v)$. It follows

\[\phi(x) = x \otimes w \in K_l\]

hence $\phi(x) = 0$. It follows $x \in \text{char}_l(\rho)$ since the diagram has exact rows. It follows $\text{ann}_l(v) = \text{char}_l(\rho)$ and the Proposition follows. $\square$

Assume $p = b_+$ and $\rho|_h = \lambda$ with $\lambda = \sum_i l_i \omega_i$ and $l_i \geq 0$ for all $i$.

Corollary 3.18. If $1 \leq l \leq m(\lambda)$ it follows there is an isomorphism

\[M_l(\rho) \cong L_l(\rho)\]

of left $\text{SL}(E)$-modules.

Proof. The Corollary follows from Proposition 3.17 and Theorem 3.14. $\square$

Let $P \subseteq \text{SL}(E)$ be the parabolic subgroup with $\text{Lie}(p) = P$ and let $\mathcal{L} \in \text{Pic}^\text{SL}(E)(\text{SL}(E)/P)$ be the linebundle with $H^0(\text{SL}(E)/P, \mathcal{L})^* = L(\rho) = V_\lambda$. Let $X = \text{SL}(E)/P$ and let $\mathcal{P}_X^l(\mathcal{L})$ be the $l$'th order jetbundle of $\mathcal{L}$ as defined in [2].

Corollary 3.19. For all $1 \leq l \leq m(\lambda)$ it follows there is an isomorphism

\[\mathcal{P}_X^l(\mathcal{L})(e)^* \cong U_l(\mathfrak{sl}(E)) \otimes_{U(p)} L_\rho\]

of $P$-modules.

Proof. By [2], Theorem 3.10 there is an isomorphism

\[\mathcal{P}_X^l(\mathcal{L})(e)^* \cong U_l(\mathfrak{sl}(E)) \otimes_{U(p)} L_\rho\]

of $P$-modules when $1 \leq l \leq m(\lambda)$. The Corollary now follows from Corollary 3.18. $\square$

Corollary 3.19 gives an algebraic proof of a result proved in [4] over the complex numbers.

References

[1] J. Dixmier, Enveloping Algebras, Graduate Studies in Mathematics, American Math. Society (1996)
[2] H. Maakestad, On jetbundles and generalized Verma modules II, Preprint math arXiv 0903.3291
[3] H. Maakestad, On the annihilator ideal of a highest weight vector, Preprint math arXiv:1003.3522
[4] W. Smoke, Invariant differential operators, Trans. Am. Math. Soc. no. 127 (1967)

Institut Fourier, Grenoble
E-mail address: h_maakestad@hotmail.com