Dynamic Team Theory of Stochastic Differential Decision Systems with Decentralized Noisy Information Structures via Girsanov’s Measure Transformation

Charalambos D. Charalambous* and Nasir U. Ahmed†

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Abstract

In this paper, we present two methods which generalize static team theory to dynamic team theory, in the context of continuous-time stochastic nonlinear differential decentralized decision systems, with relaxed strategies, which are measurable to different noisy information structures. For both methods we apply Girsanov’s measure transformation to obtain an equivalent decision system under a reference probability measure, so that the observations and information structures available for decisions, are not affected by any of the team decisions.

The first method is based on function space integration with respect to products of Wiener measures. It generalizes Witsenhausen’s definition of equivalence between discrete-time static and dynamic team problems, and relates Girsanov’s theorem to the so-called “Common Denominator Condition and Change of Variables”.

The second method is based on stochastic Pontryagin’s maximum principle. The team optimality conditions are given by a “Hamiltonian System” consisting of forward and backward stochastic differential equations, and conditional variational Hamiltonians with respect to the information structure of each team member. Under global convexity conditions, we show that PbP optimality implies team optimality. We also obtain team and PbP optimality conditions for regular team strategies, which are measurable functions of decentralized information structures.

In addition, we also show existence of team and PbP optimal relaxed decentralized strategies (conditional distributions), in the weak* sense, without imposing convexity on the action spaces of the team members, and their realization by regular team strategies.

Key Words. Dynamic Team Theory, Stochastic, Decentralized, Existence, Path Integra-

* C.D. Charalambous is with the Department of Electrical and Computer Engineering, University of Cyprus, Nicosia 1678 (E-mail: chadcha@ucy.ac.cy).
† N.U Ahmed is with the School of Engineering and Computer Science, and Department of Mathematics, University of Ottawa, Ontario, Canada, K1N 6N5 (E-mail: ahmed@site.uottawa.ca).
1 Introduction

Static Team Theory is a mathematical formalism of decision problems with multiple Decision Makers (DMs) having access to different information, who aim at optimizing a common pay-off or reward functional. It is often used to formulate decentralized decision problems, in which the decision making authority is distributed through a collection of agents or players, and the information available to the DMs to implement their actions is different. Static team theory and decentralized decision making originated from the fields of management, organization behavior and government by Marschak and Radner [2–4]. However, its generalization to dynamic team theory has far reaching implications in all human activity including science and engineering systems, comprising of multiple components, in which information available to the decision making components is either partially communicated to each other or not communicated at all, and decisions are taken sequentially in time. Dynamic team theory and decentralized decision making can be used in large scale distributed dynamical systems, such as, transportation systems, smart grid energy systems, social network systems, surveillance systems, networked control systems, communication networks, financial markets, etc.

In general, decentralized decision making is a common feature of any system consisting of multiple local observation posts and control stations, where the acquisition of information and its processing is shared among the different observation posts, and the DM actions at the control stations are evaluated using different information, that is, the arguments in their control laws or policies are different. We call, as usual, “Information Structures or Patterns” the information available to the DMs at the control stations to implement their actions, and we call such informations “Decentralized Information Structures” if the information available to the DMs at the various control stations are not identical to all DMs. Early work discussing the importance of information structures in decision making and its applications is found in [2–7].

Since the late 1960’s several articles have been written on decentralized decision making and information structures, and their applications in communication and queuing networks, sensor networks, and networked control systems. Some of the early references are [1, 5–8, 28], while more recent are [29–41]. Among these references the most popular mathematical formalism is that of “Static Team Theory” developed by Marschak and Radner [2–4]. The most successful example is the discrete-time Linear-Quadratic-Gaussian (LQG) decision problem with two DMs having access to one step-delay sharing information pattern [10, 12], with common and private information parts, where the explicit solution is obtained via completion of squares and dynamic programming, respectively.

Due to the inherent difficulty in applying Marschak’s and Radner’s Static Team

\[\text{Static in the terminology in [2–4] means all elements of the team problem are Random Variables; some authors call such problems dynamic if the information structures depend on the decisions.} \]
Theory to stochastic discrete-time dynamic decentralized decision problems, two methods are proposed over the years. The first method is based on identifying conditions so that discrete-time stochastic dynamic team problems can be equivalently reduced to static team problems. The second method is based on applying dynamic programming, and identifying conditions so that Person-by-Person (PbP) optimality implies team optimality. The first method put forward in [8, 9], is based on using precedence diagrams to represent sequential decisions and information structures in discrete-time stochastic dynamic team problems, to aid the analysis and computation of the optimal team strategies with partially nested information structures. In our opinion, even when the conditions suggested in these papers hold, it is not clear whether this approach is tractable, or whether it will provide any insight into specific discrete-time stochastic dynamic team problems. Along the same direction, and contrary to the earlier believe at the time, Witsenhausen in [1] claimed that for a broad class of problems, discrete-time stochastic dynamic decentralized decision problems, with finite decisions (including some continuous alphabet models), are no harder than Marschak’s and Radner’s static team problems, by showing that such problems are equivalent to static problems. In Witsenhausen’s terminology a discrete-time stochastic dynamic decentralized decision problem is called “static” if it can be transformed to an equivalent problem such that the observations available for any one decision do not depend on the other decisions. The procedure is described in terms of “Common Denominator Condition” together with “Change of Variables”. However, by careful reading of [Section 2.1, [1]], Witsenhausen’s analysis is restricted to discrete-time stochastic decentralized decision problems without dynamics and hence, the conclusions obtained in [1] are only for a small class of models. Moreover, no expression is given for the common denominator condition and change of variables, which facilitate the equivalence between the two problems.

With respect to the second method, PbP optimality and dynamic programming are often used in real-time communication [16,22,30], in decentralized hypothesis testing [27,34], and networked control systems [22,32], for specific classes of discrete-time models and information structures. The procedure is based on identifying an information state or sufficient statistic, often employed in centralized stochastic control, to replace the observations available for decisions by a posteriori conditional distributions based on the observations [12,13]. However, identifying the information state and then applying dynamic programming are not easy tasks, when one is faced with general information structures and continuous alphabet spaces (see for example [22]), while the question on whether PbP optimality implies team optimality is difficult to resolve.

Following another research direction, recently, the authors invoked stochastic Pontryagin’s maximum principle to derive team and Person-by-Person (PbP) optimality conditions for stochastic differential decision systems with decentralized noiseless information structures [44], and decentralized noisy information structures [45], and computed the optimal team decentralized strategies for various communication and control applications [46]. However, the mathematical analysis in [14,45] is based on strong formulation of the probability space, which is restrictive in the sense that it is not easy to apply these optimality conditions

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2PbP optimality treats the decentralized decision problem by fixing the strategies of all DMs except one.

3We believe the proper and more precise terminology is “Memoryless Observations”, rather than “Static”, because it refers to the property of the observations only, while the unobserved state can be a random process (in [1] the unobserved state is a RV).
to noiseless feedback information structures and noisy information structures, unless certain strong assumptions are imposed on the elements of the stochastic differential decentralized decision systems.

The main objectives in this paper are the following.

(1) We present two methods, based on Girsanov’s theorem, which generalize Marschak’s and Radner’s [2, 3] static team theory to dynamic team theory. The first method is based on function space integration of Wiener functionals, which we also relate to Witsenhausen’s [1] common denominator condition and change of variables for continuous and discrete-time dynamic team problems. The second method is based on stochastic Pontryagin’s maximum principle, which allows us to derive both necessary and sufficient team optimality conditions.

(2) We show existence of relaxed team strategies (conditional distributions) under general conditions, using a weak∗ topological space;

(3) We show, under global convexity conditions, on the Hamiltonian functional and terminal pay-off, that PbP optimality implies team optimality;

(4) We show realizability of relaxed strategies by regular strategies using the Krein-Millman theorem.

Our approach is based on invoking Girsanov’s change of probability measure to define an equivalent stochastic dynamical decentralized decision system under a reference probability measure, in which the distributed observations and information structures available for decisions are not affected by any of the team decisions. Both methods do not impose any restrictions on the information structures as in [44–46], and they apply to general models and information structures, including nonclassical information structures [5, 6].

The first method is based on path integral of functionals of Brownian motion with respect to products of Wiener measures. We show that this method generalizes Witsenhausen’s [1] notion on equivalence between discrete-time stochastic dynamic team problems which can be transformed to equivalent static team problems, to continuous-time Itô stochastic nonlinear differential decentralized decision problems, to analogous discrete-time models, and in addition we identify the precise expression of the common denominator condition described in [1]. However, we point out certain limitations of this method in the context of computing the optimal team strategies, for the case of large number of decision stages.

The second methods is based on stochastic Pontryagin’s maximum principle; we derive necessary conditions for team optimality given by a stochastic Pontryagin’s maximum principle and sufficient conditions under global convexity assumptions. Moreover, we also show that under the global convexity conditions, PbP optimality implies team optimality. This method is much more general, and does not suffer from any limitations, in computing the optimal team strategies, compared to the first method.

The results listed under (1)-(4) above, are derived in the context of the following general continuous-time stochastic nonlinear differential decentralized decision systems (with relaxed and regular team member strategies, and general noisy information structures).
subject to stochastic differential dynamics with state $x(\cdot)$ and distributed noisy observations $\{y^i(\cdot) : i = 1, \ldots, N\}$ satisfying the Itô differential equations

$$
\begin{align*}
&dx(t) = f(t, x(t), u^1(t, y^1), \ldots, u^N(t, y^N))dt \\
&\quad + \sigma(t, x(t), u^1(t, y^1), \ldots, u^N(t, y^N))dW(t), \quad x(0) = x_0, \quad t \in (0, T], \\
&dy^i(t) = h^i(t, x(t), u^1(t, y^1), \ldots, u^N(t, y^N))dt + D^{i, \frac{1}{2}}(t)dB^i(t), \quad t \in [0, T], \quad i = 1, \ldots, N.
\end{align*}
$$

Here $\mathbb{E}^{\mathbb{P}_\Omega}$ denotes expectation with respect to a probability measure $\mathbb{P}_\Omega$ defined on an underlying measurable space $\left(\Omega, \mathbb{F}\right)$, while the elements of the team problem are the following:

- $J$ : the team pay-off or reward;
- $x : [0, T] \times \Omega \rightarrow \mathbb{R}^n$ : the unobserved state process;
- $W : [0, T] \times \Omega \rightarrow \mathbb{R}^m$ : the state process exogenous Brownian Motion (BM) process;
- $B^i : [0, T] \times \Omega \rightarrow \mathbb{R}^{d_i}$ : the $i$th distributed observation exogenous BM process, $i = 1, \ldots, N$;
- $y^i : [0, T] \times \Omega \rightarrow \mathbb{R}^{d_i}$ : the $i$th distributed observation process generating the $i$th information structure-the $\sigma$-algebra, $\mathcal{G}^y_{0,t} \triangleq \sigma\{y^i(s) : 0 \leq s \leq t\}$, $t \in [0, T], \ i = 1, \ldots, N$;
- $u^i : [0, T] \times \Omega \rightarrow \mathbb{R}^{d_i}$ : the $i$th decision process, $i = 1, \ldots, N$;
- $\mathbb{U}^i_{\text{reg}}[0,T]$ : admissible regular team strategies of the $i$th decision process $u^i, i = 1, \ldots, N$.

Although, in the above stochastic differential decentralized decision system we have assumed regular team strategies, in the paper we consider relaxed team strategies, which are regular conditional distributions $u^i_t(\Gamma) = q^i_t(\Gamma|\mathcal{G}^y_{0,t})$ for $t \in [0, T]$ and $\forall \Gamma \in \mathcal{B}(\mathbb{A}^i)$, $i = 1, \ldots, N$, and we obtain corresponding results for regular strategies as a special case of relaxed strategies. Moreover, we apply the first method to a discrete-time generalization of (1)-(4), and we demonstrate that both methods apply to arbitrary information structures, including non-classical information structures [?]. We illustrate these points in our discussions.

According to the formulation of (1)-(4) and definition of admissible team strategies, each distributed observation $\{y^i(t) : t \in [0, T]\}$ generates the information structure of the $i$th decision process $\{u^i_t : t \in [0, T]\}$ for $i = 1, \ldots, N$. With respect to our introductory discussion, the stochastic system (3) may be a compact representation of many interconnected subsystems, aggregated into a single state representation $x \in \mathbb{R}^n$, each $\{y^i(t) : t \in [0, T]\}$ corresponds to the observation process at the observation post $"i"$, and each $\{u^i_t : t \in [0, T]\}$ corresponds to the decision process applied at the $"i"$ th control station. Since in the current set up we have assumed $u^i \in \mathbb{U}^i_{\text{reg}}[0,T]$ then by definition, for each $t \in [0,T]$, the strategies are of the form $u^i_t \equiv \mu^i(t, \{y^i(s) : 0 \leq s \leq t\})$, for $i = 1, \ldots, N$, and hence the decision processes utilize decentralized noisy information structures.

\[\footnote{We will also describe more general decentralized noisy information structures.}\]
We call as usual, (1)-(4), a stochastic dynamic team problem, and a strategy \( u^o \triangleq (u^1, \ldots, u^N) \in \times_{i=1}^N U^i_{\text{reg}}[0, T] \) which achieves the infimum in (1) a team optimal regular strategy. A PbP optimal regular strategy \( u^o \in \times_{i=1}^N U^i_{\text{reg}}[0, T] \) is defined by

\[
J(u^1, \ldots, u^N) \leq J(u^1, \ldots, u^{i-1}, u^i, u^{i+1}, \ldots, u^N), \quad \forall u^i \in U^i_{\text{reg}}[0, T], \forall i = 1, \ldots, N.
\]

Clearly, team optimality implies PbP optimality, but the reverse is not generally true. In the context of team problems, PbP optimality is often of interest provided one can identify conditions so that PbP optimality implies team optimality. For static team problems such conditions, are derived in [3] and for exponential pay-off functionals in [20].

As we have mentioned earlier, our methodology is addressing stochastic dynamic team problems is based on Girsanov’s change of probability measure, which allows us to introduce an equivalent problem under a new reference probability space in which the distributed noisy observations are signal free and/or the unobserved state process is drift free, and hence they are not affected by any of the team decisions. Thus, we employ the powerful tools of stochastic calculus such as, martingale representation theorem, stochastic variational methods, function space integration of functionals of Brownian motion with respect to Wiener measures, to handle very general decentralized decision problems, with decision processes having access to any combination of information structures.

Indeed, we apply the first method, based on function space integration, to the continuous-time stochastic dynamic team problem (1)-(4), and we show that the so called “Common Denominator Condition” introduced in [1] to transformed, the simplified discrete-time stochastic dynamic decentralized decision problem (with unobserved state a RV) to an equivalent static one (in Witsenhausen’s terminology) is the existence, via Girsanov’s theorem [47], of a Radon-Nikodym derivative between the initial and the reference probability measure, so that under the reference probability measure the observations are not affected by any of the team decisions. Moreover, we show that the so called “Change of Variables” in [1] is an application of change of probability measure, expressing the initial pay-off under the reference probability measure. Therefore, we extend Witsenhausen’s notion of equivalence not only to general discrete-time stochastic dynamic team problems, but also to continuous-time stochastic dynamic team problems. We also show that under the reference probability measure, the pay-off of the team problem (1)-(4) is equivalently expressed via function space integration with respect to the product of Wiener measures, and that, in principle, this integration can be carried out precisely as in [48, 49], where examples of optimal, in mean-square sense, finite-dimensional filters are derived. However, contrary to an intuitive belief, we point out that this does not mean that such an equivalent problem is simpler, or that static optimization theory and/or the static team theory of Marschak and Radner [2-4] can be easily applied to the equivalent problem, even for the discrete-time analog of (1)-(4), including Witsenhausen’s simplified model (without dynamics) described in continuous-time. The reason is that the computation of the optimal team strategies can be quite intensive, and often not tractable.

Then, we proceed with the second method to derive new team and PbP optimality conditions; necessary conditions given by a stochastic Pontryagin’s maximum principle and sufficient conditions under global convexity assumptions. Firstly, we apply Girsanov’s measure transformation [47] to transform (1)-(4) to an equivalent stochastic differential team
game, under a reference probability measure, on which \( \{ y^i(t) : t \in [0,T] \} \), \( i = 1, \ldots, N \) are independent Brownian motions, and independent of any of the team decisions. Secondly, under the reference probability measure we show existence of team and PbP optimal relaxed strategies, in the weak* sense. Thirdly, we invoke stochastic variational methods and the Riesz representation theorem for Hilbert space semi martingales to derive team and PbP optimality conditions. We show that the necessary conditions for an admissible strategy to be team optimal is the existence of an adjoint process in an appropriate function space satisfying a backward stochastic differential equation, and that for each \( t \in [0,T] \), the optimal actions \( u^i(t) \) satisfy almost surely, a pointwise conditional variational Hamiltonian inequality with respect to the information structure \( \{ y^i(s) : 0 \leq s \leq t \} \), with all other actions are kept to their optimal values, for \( i = 1, \ldots, N \).

Under certain global convexity conditions we also show that the \( N \) conditional variational Hamiltonian inequalities are also sufficient for team optimality. Moreover, we also show that under the global convexity conditions, PbP optimality implies team optimality. The new optimality conditions are given both under the reference probability measure, and also under the initial probability measure via a reverse Girsanov’s measure transformation. The Hamiltonian system of equations is precisely the analog of stochastic differential decision problems optimality conditions, extended to decentralized decision problems using a dynamic team theory formulation.

One of the important aspect of our methodology is that the assumptions imposed to derive existence of team and PbP optimal strategies, and necessary and sufficient team and PbP optimality conditions are precisely the ones often imposed to derive analogous results for stochastic partially observable control problems which presuppose centralized information structures. However, “the challenge remains that of computing conditional expectations” via the conditional variational Hamiltonians, which is not an easy task even for centralized partially observable stochastic systems. Therefore, examples will be presented elsewhere as in [58], due to space limitations.

Throughout the paper, we develop the optimality conditions utilizing relaxed decentralized strategies, and then we show how to recover analogous optimality conditions for regular strategies.

Finally, we point out that one may invoke alternative methods, such as the ones described in [42,51,60], which are based on stochastic flows of diffeomorphisms, martingale representation theorem, and needle variations. Moreover, for the case of regular strategies with actions spaces which are not necessarily convex, one can derive optimality conditions by considering the generalized Hamiltonian system of equations, which includes also the second-order adjoint process [61] (see also [62]), provided the derivations are carried out under the reference probability measure. The important point to be made regarding the results of this paper is that, by invoking Girsanov’s measure transformation, the existence of team and PbP optimal strategies, and the team and PbP optimality conditions for stochastic differential decentralized decisions systems formulated using stochastic dynamic team theory, are derived similarly to stochastic optimal control or decision problems, with centralized information structures, and that only at the last step of the derivations, the issue of decentralization is accounted for, leading to the conditional variational Hamiltonians with respect to the different information structures.

We believe our lengthy introduction, together with the subsequent mathematical analysis,
and results derived in the paper will help clarify certain statements found in the literature concerning the application of Marschak’s and Radner’s Static Team Theory to stochastic dynamic team problems, and aid in addressing other types of optimality criteria such as, Nash Equilibrium, minimax games, etc. with decentralized information structures.

The paper is organized as follows. In Section 2 we introduce the stochastic differential decentralized decision problem and its equivalent re-formulations using the weak Girsanov’s measure transformation approach. In this section, we also discuss the precise connection via path integration between static team theory and dynamic team theory, thus generalizing [1] to continuous-time stochastic dynamic team problems, and discrete-time stochastic dynamic team problems with general unobserved state processes, and we also identify the exact expression of the common denominator conditions (Radon-Nikodym derivative). In Section 3 we first show existence of solutions of the stochastic differential system, their continuous dependence on u, and existence of team and PbP optimality using a weak* topological space of regular conditional distributions. In Section 4 we derive the variational equations which we invoke to derive the optimality conditions, both under the reference probability measure and under the initial probability measure. In Section 5 we show how to obtain corresponding results for regular strategies, and also how to realize relaxed strategies by regular strategies. Finally, in Section 6 we conclude the presentation with comments on possible generalizations of our results.

2 Dynamic Team Problem of Stochastic Differential Decision Systems

In this section, we introduce the team theoretic formulation of stochastic differential decentralized decision systems, the information structures available to the DMs, which are different and noisy, and the relaxed and regular strategies of the DMs. Then, we introduce appropriate assumptions, and we invoke Girsanov’s change of probability to show that under an appropriate choice of probability measure, the original stochastic differential decentralized decision problem is equivalent to a new problem with distributed observations which are independent, and independent of any of the team decisions.

Let $\mathbb{Z}_N \overset{\Delta}{=} \{1, 2, \ldots, N\}$ denote a subset of natural numbers, $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ denote Linear transformations mapping a linear vector space $\mathcal{X}$ into a vector space $\mathcal{Y}$, and $A(i)$ denote the ith column of a map $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m), i = 1, \ldots, n$.

Let $\left(\Omega, \mathcal{F}, \{\mathbb{F}_{0,t} : t \in [0, T]\}, \mathbb{P}\right)$ denote a complete filtered probability space satisfying the usual conditions, that is, $(\Omega, \mathcal{F}, \mathbb{P})$ is complete, $\mathbb{F}_{0,0}$ contains all $\mathbb{P}$-null sets in $\mathbb{F}$. Note that filtrations $\{\mathbb{F}_{0,t} : t \in [0, T]\}$ are monotone in the sense that $\mathbb{F}_{0,s} \subseteq \mathbb{F}_{0,t}, \forall 0 \leq s \leq t \leq T$. Moreover, $\{\mathbb{F}_{0,t} : t \in [0, T]\}$ is called right continuous if $\mathbb{F}_{0,t} = \mathbb{F}_{0,t+} \overset{\Delta}{=} \bigcap_{s>t} \mathbb{F}_{0,s}, \forall t \in [0, T)$ and it is called left continuous if $\mathbb{F}_{0,t} = \mathbb{F}_{0,t-} \overset{\Delta}{=} \sigma\left(\bigcup_{s<t} \mathbb{F}_{0,s}\right), \forall t \in (0, T]$. Throughout we assume that all filtrations are right continuous and complete, and defined by $\mathbb{F}_T \overset{\Delta}{=} \{\mathbb{F}_{0,t} : t \in [0, T]\}$.

Consider a random process $\{z(t) : t \in [0, T]\}$ taking values in $(\mathbb{Z}, \mathcal{B}(\mathbb{Z}))$, where $(\mathbb{Z}, d)$ is a metric space, defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathbb{F}_{0,t} : t \in [0, T]\}, \mathbb{P})$. It can be
shown that any such stochastic process which is measurable and adapted has a progressively measurable modification\footnote{3}. Unless otherwise specified, we shall say a process \( \{ z(t) : t \in [0, T] \} \) is \( \{ \mathbb{F}_{0,t} : t \in [0, T] \} \)-adapted if the processes is \( \{ \mathbb{F}_{0,t} : t \in [0, T] \} \)-progressively measurable.

\( C([0, T], \mathbb{R}^n) \) denotes the space of continuous real-valued \( n \)-dimensional functions defined on the time interval \([0, T]\).

\( L^2_{\mathbb{F}_T}([0, T], \mathbb{R}^n) \subset L^2(\Omega \times [0, T], d\mathbb{P} \times dt, \mathbb{R}^n) \equiv L^2([0, T], L^2(\Omega, \mathbb{R}^n)) \) denotes the space of \( \{ \mathbb{F}_{0,t} : t \in [0, T] \} \)-adapted random processes \( \{ z(t) : t \in [0, T] \} \) such that

\[
\mathbb{E} \int_{[0,T]} |z(t)|^2_{\mathbb{R}^n} dt < \infty,
\]

which is a sub-Hilbert space of \( L^2([0, T], L^2(\Omega, \mathbb{R}^n)) \).

Similarly, \( L^2_{\mathbb{F}_T}(\Omega, \mathbb{R}^m) \subset L^2([0, T], L^2(\Omega, \mathbb{R}^m, \mathbb{R}^n)) \) denotes the space of \( \{ \mathbb{F}_{0,t} : t \in [0, T] \} \)-adapted \( n \times m \) matrix valued random processes \( \{ \Sigma(t) : t \in [0, T] \} \) such that

\[
\mathbb{E} \int_{[0,T]} |\Sigma(t)|^2_{\mathbb{R}^m} dt \leq \mathbb{E} \int_{[0,T]} tr((\Sigma^*(t)\Sigma(t))) dt < \infty.
\]

Next, we describe the set of admissible relaxed strategies. For each \( i \in \mathbb{Z}_N \), let \( \mathbb{A}^i \subset \mathbb{R}^{d_i} \) be closed and bounded (possibly nonconvex), and let \( \mathcal{B}(\mathbb{A}^i) \) denote the Borel subsets of \( \mathbb{A}^i \). Let \( C(\mathbb{A}^i) \) denote the space of continuous functions on \( \mathbb{A}^i \), endowed with the sup norm topology, which makes it a Banach space. Let \( \mathcal{M}(\mathbb{A}^i) \) denote the space of regular bounded signed Borel measures on \( \mathcal{B}(\mathbb{A}^i) \), having finite total variation. With respect to this norm topology, \( \mathcal{M}(\mathbb{A}^i) \) is also a Banach space. It is well known that the dual of \( C(\mathbb{A}^i) \) is \( \mathcal{M}(\mathbb{A}^i) \). We are interested in \( \mathcal{M}_1(\mathbb{A}^i) \subset \mathcal{M}(\mathbb{A}^i) \) the space of regular probability measures. Using this construction, the DM strategies with decentralized information structures will be described through the topological dual of the Banach space \( L^1_{\mathbb{G}_T^y}([0, T], C(\mathbb{A}^i)) \), the \( L^1 \)-space of \( \mathbb{G}_T^y \)-adapted \( C(\mathbb{A}^i) \) valued functions, for \( i \in \mathbb{Z}_N \). For each \( i \in \mathbb{Z}_N \) the dual of this space is given by \( L^\infty_{\mathbb{G}_T^y}([0, T], \mathcal{M}(\mathbb{A}^i)) \) which consists of weak* measurable \( \mathbb{G}_T^y \)-adapted \( \mathcal{M}(\mathbb{A}^i) \) valued functions. For each \( i \in \mathbb{Z}_M \) the DM strategies are drawn from \( L^\infty_{\mathbb{G}_T^y}([0, T], \mathcal{M}_1(\mathbb{A}^i)) \subset L^\infty_{\mathbb{G}_T^y}([0, T], \mathcal{M}(\mathbb{A}^i)) \), the set of probability measure valued \( \mathbb{G}_T^y \)-adapted functions. Hence, we have the following definition of relaxed strategies.

**Definition 1. (Admissible Relaxed Noisy Information Strategies)**

The admissible relaxed strategies for DM \( i \) are defined by

\[
\mathbb{U}_{rel}^i[0, T] \overset{\triangle}{=} L^\infty_{\mathbb{G}_T^y}([0, T], \mathcal{M}_1(\mathbb{A}^i)), \quad \forall i \in \mathbb{Z}_N,
\]

where \( \mathbb{A}^i \subset \mathbb{R}^{d_i}, \forall i \in \mathbb{Z}_N \) are closed and bounded (possibly nonconvex).

An \( N \) tuple of relaxed strategies is by definition

\[
\mathbb{U}_{rel}^{(N)}[0, T] \overset{\triangle}{=} \times_{i=1}^{N} \mathbb{U}_{rel}^i[0, T], \quad \mathcal{M}_1(\mathbb{A}^{(N)}) \overset{\triangle}{=} \times_{i=1}^{N} \mathcal{M}_1(\mathbb{A}^i), \quad \mathbb{A}^{(N)} \overset{\triangle}{=} \times_{i=1}^{N} \mathbb{A}^i.
\]
Thus, for any \( i \in \mathbb{Z}_N \), given the information \( \mathcal{G}_T^{i} \), \( \{ u_t^i : t \in [0, T] \} \) is a stochastic kernel (regular conditional distribution) defined by

\[
    u_t^i(\Gamma) = q_t^i(\Gamma | \mathcal{G}_0^{i}), \quad \text{for} \quad t \in [0, T], \quad \text{and} \quad \forall \Gamma \in \mathcal{B}(\mathbb{A}^i).
\]

Clearly, for each \( i \in \mathbb{Z}_N \) and for every \( \varphi \in C(\mathbb{A}^i) \) the process

\[
    \int_{\mathbb{A}^i} \varphi(\xi) u_t^i(d\xi) = \int_{\mathbb{A}^i} \varphi(\xi) q_t^i(d\xi | \mathcal{G}_0^{i}), \quad t \in [0, T],
\]

is \( \mathcal{G}_T^{i} \)– progressively measurable. For each \( i \in \mathbb{Z}_N \), the space \( L^\infty_{\mathcal{G}_T^{i}}([0, T], \mathcal{M}_1(\mathbb{A}^i)) \) is endowed with the weak\(^*\) topology, also called vague topology. A generalized sequence \( u_t^{i,\alpha} \in \mathcal{U}^{i}_{\text{rel}}[0, T] \) is said to converge (in the weak\(^*\) topology or) vaguely to \( u_t^{i,\circ} \), written \( u_t^{i,\alpha} \xrightarrow{v} u_t^{i,\circ} \), if and only if for every \( \varphi \in L^1_{\mathcal{G}_T^{i}}([0, T], C(\mathbb{A}^i)) \)

\[
    \mathbb{E} \int_{[0,T] \times \mathbb{A}^i} \varphi_t(\xi) u_t^{i,\alpha}(d\xi) dt \xrightarrow{\alpha \rightarrow \infty} \mathbb{E} \int_{[0,T] \times \mathbb{A}^i} \varphi_t(\xi) u_t^{i,\circ}(d\xi) dt \quad \forall i \in \mathbb{Z}_N.
\]

With respect to the vague (weak\(^*\)) topology the set \( \mathcal{U}^{i}_{\text{rel}}[0, T] \) is compact, and from here on we assume that \( \mathcal{U}^{i}_{\text{rel}}[0, T], \forall i \in \mathbb{Z}_N \) has been endowed with this vague topology.

For relaxed strategies \( u \in \mathcal{U}^{(N)}_{\text{rel}}[0, T] \), we use the following notation for the drift coefficient

\[
    f(t, x, u_t) \overset{\triangle}{=} \int_{\mathbb{A}^{(N)}} \left( f(t, x, \xi^1, \xi^2, \ldots, \xi^N) \right) \times_{i=1}^N u_t^i(d\xi^i), \quad t \in [0, T),
\]

and similarly for \( \{ \sigma, h, \ell \} \) in (\( \Pi^1 \cdot \mathbb{I}^3 \)).

Next, we define the set of admissible decentralized noisy information strategies of the team members, called regular strategies, (deterministic measurable functions).

**Definition 2.** (Admissible Regular Noisy Information Strategies)

The admissible regular strategies for DM \( i \) are defined by

\[
    \mathcal{U}^i_{\text{reg}}[0, T] \overset{\triangle}{=} L^\infty_{\mathcal{G}_T^{i}}([0, T] \times \Omega, \mathbb{A}^i), \quad \forall i \in \mathbb{Z}_N, \quad \text{(7)}
\]

the class of \( \mathcal{G}_T^\circ \)– adapted random processes defined on \([0, T]\) and taking values from the closed bounded set \( \mathbb{A}^i \subset \mathbb{R}^{d_i}, \forall i \in \mathbb{Z}_N \). Note that if \( \mathbb{A}^i \) is a closed, bounded and convex subset of \( \mathbb{R}^{d_i} \) then \( \mathcal{U}^i_{\text{reg}}[0, T] \) is a closed convex subset of \( L^\infty_{\mathcal{G}_T^{i}}([0, T] \times \Omega, \mathbb{A}^i), \forall i \in \mathbb{Z}_N \). An \( N \) tuple of regular strategies is by definition

\[
    (u^1, u^2, \ldots, u^N) \in \mathcal{U}^{(N)}_{\text{reg}}[0, T] \overset{\triangle}{=} \times_{i=1}^N \mathcal{U}^i_{\text{reg}}[0, T].
\]

Notice that the class of regular strategies embeds continuously into the class of relaxed decisions through the map \( u \in \mathcal{U}^{(N)}_{\text{reg}}[0, T] \xrightarrow{} \delta_{u_t(\omega)} \in \mathcal{U}^{(N)}_{\text{rel}}[0, T] \). Clearly, for every \( f \in L^1_{\mathcal{G}_T^{i}}([0, T] \times \Omega, C(\mathbb{A}^{(N)})) \) we have

\[
    \mathbb{E} \int_{[0,T] \times \mathbb{A}^{(N)}} f(t, \omega, \xi) \delta_{u_t(\omega)}(d\xi) dt = \mathbb{E} \int_{[0,T]} f(t, \omega, u^1_t(\omega), \ldots, u^N_t(\omega)) dt.
\]
There are several advantages of using relaxed strategies. For example, if optimal regular strategies exist from the admissible class $U^{rel}_{reg}[0,T] \subset U^{rel}_rel[0,T]$ then the optimality conditions of relaxed strategies can be specialized to the class of strategies which are simply Dirac measures concentrated $\{u^i_t : t \in [0,T]\} \in U^{rel}_reg[0,T]$. Thus, the necessary conditions for team and PbP optimality for regular strategies follow readily from those of relaxed strategies. Another advantage is the realizability of relaxed strategies by regular strategies, which is often established via the Krein-Millman theorem, without requiring convexity of $A^{(N)}$. Both advantages are discussed in the paper.

Next, we state the basic assumptions on the stochastic differential dynamics and decentralized observations, assuming relaxed strategies.

**Assumptions 1.** (Main assumptions)

The drift $f$, diffusion coefficients $\sigma$, and measurement functions $h^i, i = 1, \ldots, N$ in $[3], [3]$ are Borel measurable maps:

\[ f : [0, T] \times \mathbb{R}^n \times A^{(N)} \rightarrow \mathbb{R}^n, \quad \sigma : [0, T] \times \mathbb{R}^n \times A^{(N)} \rightarrow \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n), \]

\[ h^i : [0, T] \times \mathbb{R}^n \times A^{(N)} \rightarrow \mathbb{R}^{k_i}, \quad \forall i \in \mathbb{Z}_N, \]

which satisfy the following basic conditions.

(A0) $A^i \subset \mathbb{R}^{d_i}$ is closed and bounded, $\forall i \in \mathbb{Z}_N$.

There exists a $K \in L^{2+}([0, T], \mathbb{R})$ such that

(A1) $|f(t, x, \xi) - f(t, z, \xi)|_{\mathbb{R}^n} \leq K(t) |x - z|_{\mathbb{R}^n}$ uniformly in $\xi \in A^{(N)}$;

(A2) $|f(t, x, \xi)|_{\mathbb{R}^n} \leq K(t)(1 + |x|_{\mathbb{R}^n})$ uniformly in $\xi \in A^{(N)}$;

(A3) $|\sigma(t, x, \xi) - \sigma(t, z, \xi)|_{\mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)} \leq K(t)|x - z|_{\mathbb{R}^n}$ uniformly in $\xi \in A^{(N)}$;

(A4) $|\sigma(t, x, \xi)|_{\mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)} \leq K(t)(1 + |x|_{\mathbb{R}^n})$ uniformly in $\xi \in A^{(N)}$;

(A5) $|h^i(t, x, \xi)|_{\mathbb{R}^{k_i}} \leq K(t)(1 + |x|_{\mathbb{R}^n})$ uniformly in $\xi \in A^{(N)} \forall i \in \mathbb{Z}_N$;

(A6) $|h^i(t, x, \xi) - h^i(t, z, \xi)|_{\mathcal{L}(\mathbb{R}^n)} \leq K(t)|x - z|_{\mathbb{R}^n}$ uniformly in $\xi \in A^{(N)}, \forall i \in \mathbb{Z}_N$;

(A7) $f(t, x, \cdot), \sigma(t, x, \cdot), h^i(t, x, \cdot), i = 1, \ldots, N$ are continuous in $\xi \in A^{(N)}, \forall (t, x) \in [0, T] \times \mathbb{R}^n$;

(A8) $D^{\frac{1}{2}}$ is measurable in $t \in [0, T]$, uniformly bounded, the inverse $D^{\frac{1}{2}}$ exists and it is uniformly bounded, $\forall i \in \mathbb{Z}_N$.

Often, for simplicity we shall replace (A5) by the following condition. There exist a $K > 0$ such that

(A5') $|h^i(t, x, \xi)|_{\mathbb{R}^{k_i}} \leq K, \forall (t, x, \xi) \in [0, T] \times \mathbb{R}^n \times A^{(N)}, \forall i \in \mathbb{Z}_N$. 

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2.1 Team Problem Under Reference Probability Space

In this section we formulate the stochastic team problem utilizing the Girsanov change of measure approach, which is based on constructing a filtered probability space \( \left( \Omega, \mathcal{F}, \{\mathcal{F}_t : t \in [0,T]\}, \mathbb{P} \right) \) and Brownian motions \( \{W(t) : t \in [0,T]\} \) and \( \{B^u(t) : t \in [0,T]\} \) defined on it, such that \( \{x^u(t) : t \in [0,T]\} \) and \( \{y(t) : t \in [0,T]\} \) are the weak solution of (3) and (4) with respect to relaxed strategies (with \( B \) replaced by \( B^u \) and \( \mathbb{P}_\Omega \) by \( \mathbb{P}^u \)). Moreover, under the reference probability space \( \left( \Omega, \mathcal{F}, \{\mathcal{F}_t : t \in [0,T]\}, \mathbb{P} \right) \), \( \{x^u(t) : t \in [0,T]\} \) is a weak solution of (3), while the observations \( \{y^i(t) : t \in [0,T]\} \) are independent Brownian motions, which are independent of the team decisions, that is, they are fixed and unaffected by \( u^i, i = 1, 2, \ldots, N \). Consequently, the information structures of each team member is independent of \( u \).

Let \( \left( \Omega, \mathcal{F}, \mathbb{P} \right) \) be the canonical space of \( (x_0, \{W(t), \{B^i(t) : i = 1, \ldots, N\} : t \in [0,T]\}) \) which are defined by

**WP1** \( x(0) = x_0 \): an \( \mathbb{R}^n \)-valued Random Variable with distribution \( \Pi_0(dx) \);

**WP2** \( \{W(t) : t \in [0,T]\} \): an \( \mathbb{R}^m \)-valued standard Brownian motion, independent of \( x(0) \);

**WP3** \( \{B^i(t) : t \in [0,T]\}, i = 1, \ldots, N \): \( \mathbb{R}^{k_i} \)-valued, \( i = 1, \ldots, N \), mutually independent standard Brownian motions, independent of \( \{W(t) : t \in [0,T]\} \).

We introduce the Borel \( \sigma \)-algebra \( \mathcal{B}(C([0,T], \mathbb{R}^m)) \) on \( C([0,T], \mathbb{R}^m) \), the space of continuous \( m \)-dimensional functions on finite time \([0,T]\), generated by \( \{W(t) : 0 \leq t \leq T\} \) and let \( \mathbb{P}^W \) its Wiener measure on it.

Similarly, we introduce the Borel \( \sigma \)-algebra \( \mathcal{B}(C(0,T] , \mathbb{R}^{k_i}) \) on \( C([0,T], \mathbb{R}^{k_i}) \) generated by \( \{y^i(t) : 0 \leq t \leq T\} \) and let \( \mathbb{P}^{y_i} \) its Wiener measure on it, for \( i = 1, \ldots, N \). We define the Borel \( \sigma \)-algebra \( \mathcal{B}(C([0,T], \mathbb{R}^{k_i})) \) \( \triangleq \otimes_{i=1}^N \mathcal{B}(C(0,T], \mathbb{R}^{k_i})) \) on \( \otimes_{i=1}^N C([0,T], \mathbb{R}^{k_i}) \) generated by \( \{y^1(t), \ldots, y^N(t) : 0 \leq t \leq T\}, k = \sum_{i=1}^N k_i \), and its Wiener product measure \( \mathbb{P}^y \triangleq \times_{i=1}^N \mathbb{P}^{y_i} \) on it.

Further, we introduce the filtration \( \{\mathcal{F}^W_{0,t} : t \in [0,T]\} \) generated by truncations of \( W \in C([0,T], \mathbb{R}^m) \), and the filtration \( \{\mathcal{G}^y_{0,t} : t \in [0,T]\} \) generated by truncations of \( y^i \in C([0,T], \mathbb{R}^{k_i}) \), \( i = 1, \ldots, N \), and we define \( \mathcal{G}^y_{0,t} \triangleq \otimes_{i=1}^N \mathcal{G}^y_{0,t}, t \in [0,T] \). That is, for \( t \in [0,T] \), \( \mathcal{F}^W_{0,t} \) is the sub-\( \sigma \)-algebra generated by the family of sets

\[
\left\{ W \in C([0,T], \mathbb{R}^m) : w(s) \in A \right\} : 0 \leq s \leq t, \ A \in \mathcal{B}(\mathbb{R}^m), \quad t \in [0,T].
\]

Hence, \( \mathcal{F}^W_{0,t} : t \in [0,T] \) is the canonical Borel filtration and \( \mathcal{F}^W_{0,T} = \mathcal{B}(C[0,T], \mathbb{R}^m) \).

Define the canonical probability space \( \left( \Omega, \mathcal{F}, \{\mathcal{F}_t : t \in [0,T]\}, \mathbb{P} \right) \), called the reference probability space by
A typical element of $\Omega$ is $\omega$.

Under Assumptions 1, (A5), (A8), the processes $\{\Lambda^{i,u}(t) : t \in [0,T]\}$ is a supermartingale and by Itô’s differential rule it is the unique $\mathbb{F}_0,t$--adapted continuous solution of the stochastic differential equation

$$d\Lambda^{i,u}(t) = \Lambda^{i,u}(t)h^{i,*}(t, x(t), u_t)D^{i-1}(t)dy^i(t), \quad \Lambda^{i,u}(0) = 1, \quad t \in [0,T], \quad i = 1, \ldots, N.$$  

Under Assumptions (A5), (A8) the processes $\{\Lambda^{i,u}(t) : t \in [0,T]\}$ is a supermartingale and by Itô’s differential rule it is the unique $\mathbb{F}_0,t$--adapted continuous solution of the stochastic differential equation

$$d\Lambda^{i,u}(t) = \Lambda^{i,u}(t)h^{i,*}(t, x(t), u_t)D^{i-1}(t)dy^i(t), \quad \Lambda^{i,u}(0) = 1, \quad t \in [0,T], \quad i = 1, \ldots, N.$$  

On the the reference probability space $\left(\Omega, \mathbb{F}, \mathbb{P}\right)$ we define the decentralized observations by

$$y^i(t) = \int_0^t D^{i,2}(s)dB^i(s), \quad t \in [0,T], \quad i = 1, \ldots, N.$$  

Clearly, under the reference probability measure $\mathbb{P}$, the observations $\{y^i(t) : t \in [0,T]\}$, are independent Brownian motions, and hence they are fixed and unaffected by $u^i, i = 1, 2, \ldots, N$, and consequently, the information structures of each player are independent of $u$.

On the probability space $\left(\Omega, \mathbb{P}\right)$ by Assumptions (A1), (A2), (A3), (A4), for any $x(0)$ with finite second moment, and team strategy $u \in \mathbb{U}^{(N)}_{rel}[0,T]$ then $\{x^u(t) : t \in [0,T]\}$ is the pathwise unique $\mathbb{F}_0,t$--adapted continuous solution of

$$dx^u(t) = f(t, x^u(t), u_t)dt + \sigma(t, x^u(t), u_t)dW(t), \quad x(0) = x_0, \quad t \in (0,T].$$  

Notice that $u^i \equiv q^i(\mathcal{G}_{0,t}^u)$ is adapted to the family $\{\mathcal{G}_{0,t}^u : t \in [0,T]\}$ which is fixed and independent of $u$ (since $y^i$ is a Brownian motion).

Next, for any $u \in \mathbb{U}^{(N)}_{rel}[0,T]$ and for each observation process $\{y^i(t) : 0 \leq t \leq T\}$ defined on $\left(\{\mathbb{F}_0,t : t \in [0,T]\}, \mathbb{P}\right)$, we introduce the exponential functions

$$\Lambda^{i,u}(t) \triangleq \exp \left\{ \int_0^t h^{i,*}(s, x(s), u_s)D^{i,-1}(s)dy^i(s) - \frac{1}{2} \int_0^t h^{i,*}(s, x(s), u_s)D^{i,-1}(s)h^{i,*}(s, x(s), u_s)ds \right\}, \quad t \in [0,T], \quad \forall i \in \mathbb{Z}_N,$$  

and their products by

$$\Lambda^u(t) \triangleq \prod_{i=1}^N \Lambda^{i,u}(t), \quad t \in [0,T].$$  

Under Assumptions (A5), (A8) the processes $\{\Lambda^{i,u}(t) : t \in [0,T]\}$ is a supermartingale and by Itô’s differential rule it is the unique $\mathbb{F}_0,t$--adapted continuous solution of the stochastic differential equation

$$d\Lambda^{i,u}(t) = \Lambda^{i,u}(t)h^{i,*}(t, x(t), u_t)D^{i-1}(t)dy^i(t), \quad \Lambda^{i,u}(0) = 1, \quad t \in [0,T], \quad i = 1, \ldots, N.$$  

$$13$$
Then for any admissible strategy \( u \in \mathbb{U}_{rel}^{(N)}[0, T] \), \( \{\Lambda^u(t) : 0 \leq t \leq T\} \) is also an \( \left( \{\mathbb{F}_{0,t} : t \in [0, T]\}, \mathbb{P}\right) \)-super martingale and satisfies the stochastic differential equation

\[
d\Lambda^u(t) = \Lambda^u(t) \sum_{i=1}^{N} h^{i,u}(t, x(t), u_t) D^{i-1}(t) dy^i(t), \quad \Lambda^u(0) = 1, \ t \in [0, T]. \tag{17}
\]

Given a \( u \in \mathbb{U}_{rel}^{(N)}[0, T] \) we define the reward of the team game under the reference probability space \( (\Omega, \mathbb{F}, \mathbb{P}) \) by

\[
J(u) \triangleq \mathbb{E}\left\{ \int_{0}^{T} \Lambda^u(t) \ell(t, x(t), u_t) dt + \Lambda^u(T) \varphi(x(T)) \right\}, \tag{18}
\]

where \( \ell : [0, T] \times \mathbb{R}^n \times \mathbb{A}^{(N)} \rightarrow (-\infty, \infty] \), \( \varphi : \mathbb{R}^n \rightarrow (-\infty, \infty] \) are chosen so that \( \mathbb{A}^{(N)} \) is finite.

Notice that under the reference probability measure \( \mathbb{P} \), the pay-off \( \mathbb{A}^{(N)} \) with \( \Lambda^u(\cdot) \) given by \( \mathbb{A}^{(N)} \), subject to the state process satisfying \( \mathbb{A}^{(N)} \) is a transformed problem with observations which are not affected by any of the team decisions. It remains to show whether this transformed problem is equivalent to the original stochastic differential decentralized decision problem.

If we further assume that \( \mathbb{A}^{(N)} \) holds, then \( \{\Lambda^{i,u}(t) : 0 \leq t \leq T\} \) is an \( \left( \{\mathbb{F}_{0,t} : t \in [0, T]\}, \mathbb{P}\right) \)-martingale for \( i = 1, \ldots, N \), and the team reward \( \mathbb{A}^{(N)} \) subject to stochastic constraints of \( u^i(\cdot), \Lambda^u(\cdot) \) defined by \( \mathbb{A}^{(N)} \), \( \mathbb{A}^{(N)} \), respectively, is equivalent to the team game \( \mathbb{A}^{(N)} \) (with regular strategies replaced by relaxed strategies, and \( \mathbb{P}^{\Omega} \) replaced by \( \mathbb{E}^{u} \) and \( \mathbb{P}^{\Omega} \) by \( \mathbb{P}^{u} \)).

Indeed if \( \mathbb{A}^{(N)} \) holds, by the \( \left( \{\mathbb{F}_{0,t} : t \in [0, T]\}, \mathbb{P}\right) \)-martingale property of \( \Lambda^u(\cdot) \) defined by \( \mathbb{A}^{(N)} \), it has constant expectation and hence, \( \int_{\Omega} \Lambda^u(t, \omega) d\mathbb{P}(\omega) = 1, \forall t \in [0, T] \). Therefore, we can introduce a probability measure \( \mathbb{P}^{v} \) on \( (\Omega, \{\mathbb{F}_{0,t} : t \in [0, T]\}) \) by setting

\[
\left. \frac{d\mathbb{P}^{u}}{d\mathbb{P}} \right|_{\mathbb{F}_T} = \Lambda^u(T) \tag{19}
\]

Moreover, under the probability measure \( \left( \Omega, \mathbb{F}, \mathbb{P}^{u}\right) \), then \( \{B^{i,u} : t \in [0, T]\} \) is a standard Brownian motion defined by

\[
B^{i,u}(t) \triangleq B^{i}(t) - \int_{0}^{t} D^{i-\frac{1}{2}}(s) h^{i}(s, x(s), u_s) ds, \quad t \in [0, T], \ i = 1, 2, \ldots, N. \tag{20}
\]

Hence, by \( \mathbb{A}^{(N)} \) the observations of the team members are defined by

\[
dy^i(t) = h^i(t, x(t), u_t) dt + D^{i-\frac{1}{2}}(t) dB^{i,u}(t) \quad t \in [0, T], \ i = 1, 2, \ldots, N, \tag{21}
\]
and the state process \( \{x(t) : t \in [0, T]\} \) is defined by (13) (its distribution is unchanged). Thus, we have constructed the probability space \( (\Omega, \mathcal{F}, \{\mathbb{F}_t : t \in [0, T]\}, \mathbb{P}^u) \) and Brownian motion \( \{B^i(t), \ldots, B^N(t) : t \in [0, T]\} \) defined on it such that \( \{x(t), y^1(t), \ldots, y^N(t) : t \in [0, T]\} \) are weak solutions of (18), (13). Moreover, under the probability space \( (\Omega, \mathcal{F}, \{\mathbb{F}_t : t \in [0, T]\}, \mathbb{P}) \), by using (19) into (18) then the team problem reward is given by

\[
J(u) = \mathbb{E}^u \left\{ \int_0^T \ell(t, x(t), u_t) dt + \varphi(x(T)) \right\}. \tag{22}
\]

Therefore, we have two equivalent formulations of the stochastic differential team game. The one defined under probability space \( (\Omega, \mathcal{F}, \{\mathbb{F}_t : t \in [0, T]\}, \mathbb{P}^u) \) given by (13), (17), (18), and the one defined under the reference probability space \( (\Omega, \mathcal{F}, \{\mathbb{F}_t : t \in [0, T]\}, \mathbb{P}) \) given by (13), (17), (18), in which \( \{y^i(t) : t \in [0, T]\}, i = 1, \ldots, N \) are Brownian motions.

One of the important aspects of this weak Girsanov formulation is that the probability measure \( \mathbb{P}^u \) and the observations Brownian motions \( \{B^i(t) : t \in [0, T]\}, i = 1, \ldots, N \) depend on \( u \) but the filtrations \( \{\mathcal{G}^i_t : t \in [0, T]\}, i = 1, \ldots, N \) are fixed \( \alpha \) priori and they are independent of \( u \) (although they are correlated and not generated by Brownian motions). Girsanov’s approach is used extensively in the derivation of maximum principle for both fully observed and partially observed centralized stochastic control problems with regular strategies [42, 51, 52, 54, 55, 63], by working under the reference probability measure \( \mathbb{P} \), in which the observations are Brownian motions.

In the above formulation we have imposed condition (A5’) so that \( \Lambda^u(t) : 0 \leq t \leq T \) is an \( (\{\mathbb{F}_t : t \in [0, T]\}, \mathbb{P}) \)-martingale; a sufficient condition for \( \Lambda^u(\cdot) \) to be such a martingale is the Novikov [47] condition, \( \mathbb{E} \exp \left\{ \frac{1}{2} \int_0^T |D^{\frac{1}{2}}(s)h(s, x(s), u_s)|_{\mathbb{R}^k}^2 ds \right\} < \infty \).

**Equivalent Team Game Under Reference Probability Space—** \( (\Omega, \mathcal{F}, \mathbb{P}) \)

Next, we define the equivalent team game under the reference probability space \( (\Omega, \mathcal{F}, \{\mathbb{F}_t : t \in [0, T]\}, \mathbb{P}) \), by considering the augmented state process \( \{\Lambda, x\} \) defined by (17), (13), and reward (18).

Define the augmented vectors, drift, diffusion coefficients, and functions in the pay-off as follows.

\[
X \triangleq \text{Vector} \{\Lambda, x\} \in \mathbb{R} \times \mathbb{R}^n, \quad B \triangleq \text{Vector} \{B^1, B^2, \ldots, B^N\} \in \mathbb{R}^k \times \prod_{i=1}^{N} \mathbb{R}^{k^i},
\]

\[
y \triangleq \text{Vector} \{y^1, y^2, \ldots, y^N\} \in \mathbb{R}^k, \quad W \triangleq \text{Vector} \{D^{\frac{1}{2}}B, W\} \in \mathbb{R}^{k+m}
\]

\[
F(t, X, u) \triangleq \begin{bmatrix}
0 \\
f(t, x, u)
\end{bmatrix}, \quad G(t, X, u) \triangleq \begin{bmatrix}
\Lambda h^*(t, x, u) D^{-\frac{1}{2}}(t) \\
0 \\
\sigma(t, x, u)
\end{bmatrix},
\]

\[
h(t, x, u) \triangleq \text{Vector} \{h^1(t, x, u), \ldots, h^N(t, x, u)\}, \quad D(t) = \text{diag} \{D^1(t), \ldots, D^N(t)\},
\]

\[
L(t, X, u) \triangleq \Lambda \ell(t, x, u), \quad \Phi(X) \triangleq \Lambda \varphi(x).
\]
Then, under the reference probability space \((\Omega, \mathcal{F}, \mathbb{P})\) the augmented state satisfies the stochastic differential equation
\[
dX(t) = F(t, X(t), u_t)dt + G(t, X(t), u_t)dW(t), \quad X(0) = X_0, \quad t \in (0, T].
\]
(23)
The reward is given by
\[
J(u) = \mathbb{E}\left\{ \int_0^T L(t, X(t), u_t)dt + \Phi(X(T)) \right\}.
\]
(24)
Therefore, using the second method we shall investigate the equivalent team game under the reference measure \(\mathbb{P}\) described by (23) and (24), with augmented state \(X = \text{Vector}\{\Lambda, x\}, \{\mathcal{G}^{y_i}_{0,t} : t \in [0, T]\}\) a fixed filtration generated by Brownian motions, for \(i = 1, \ldots, N\), filtration \(\{\mathcal{F}_{0,t} : t \in [0, T]\}\) generated by Brownian motions and the initial condition, i.e. both filtrations are independent of any of the team decisions. Indeed, under the reference probability measure \(\mathbb{P}\) we use the second method to derive team and PbP optimality conditions for the stochastic differential decentralized decision problem (23), (24), by applying stochastic variational methods, the semi martingale representation theorem and the Riesz representation theorem for Hilbert space processes. This is the approach we consider in Section 4.

We are now ready to state the rigorous definitions of team optimality and PbP optimality.

**Problem 1. (Team Optimality)** Given the pay-off functional (22), state constraint (13), observations (21), and the admissible relaxed noisy information strategies, the N tuple of strategies \(u^o = (u^1,o, u^2,o, \ldots, u^N,o) \in U^N_{rel}[0, T]\) is called team optimal if it satisfies
\[
J(u^1,o, u^2,o, \ldots, u^N,o) \leq J(u^1,u^2, \ldots, u^N), \quad \forall u \buildrel \Delta \over = (u^1,u^2, \ldots, u^N) \in U^N_{rel}[0, T],
\]
and its corresponding \(x^o(\cdot) \equiv x(\cdot; u^o(\cdot))\) (satisfying (13)) is called an optimal state process. Under the reference probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_{0,t} : t \in [0, T]\}, \mathbb{P})\) this problem is equivalent to (23) and (24).

By definition, Problem 1 is a special case of stochastic dynamic games, in the sense that there is only one reward or pay-off criterion, instead of an individual pay-off criterion for each team member \(u^i, i \in \mathbb{Z}_N\). Thus, the decision making authority is distributed among the \(N\) team members and their collective actions are evaluated based on a single reward. Moreover, Problem 1 is a stochastic dynamic team problem with team members having access to different information structures. An alternative approach to handle such problems is to restrict the definition of optimality to the so-called PbP equilibrium as defined next.

**Problem 2. (PbP Optimality)** Given the pay-off functional (22), state constraint (13), observations (21), and the admissible relaxed noisy information strategies, the N tuple of strategies \(u^o \in U^N_{rel}[0, T]\) is called PbP optimal if it satisfies
\[
\tilde{J}(u^i,o, u^{-i,o}) \leq \tilde{J}(u^i,u^{-i,o}), \quad \forall u^i \in U^i_{rel}[0, T], \quad \forall i \in \mathbb{Z}_N,
\]
(26)
where

\[ \tilde{J}(v, u^{-i}) \triangleq J(u^1, u^2, \ldots, u^{i-1}, v, u^{i+1}, \ldots, u^N). \]

Under the probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_{0,t} : t \in [0, T]\}, \mathbb{P})\) this problem is equivalent to \((23)\) and \((24)\).

However, for team problems PbP optimality is often of interest provided one can identify sufficient conditions so that PbP optimality implies team optimality, much as is done in the Static Team Theory of Marschak and Radner.

**Remark 1.** In some applications it might be appropriate to consider other variations of decentralized optimality criteria such as, Nash-Equilibrium, minimax games, etc. For example, robustness of centralized control systems is often dealt with via minimax techniques, and risk-sensitive pay-off functionals \([42]\). These optimization criteria can be dealt with by using the current Girsanov measure transformation.

### 2.2 Function Space Integration: Equivalence of Static and Dynamic Team Problems

In this section, we discuss the derivation of team and PbP optimality condition via the first method, based on function space integration of functionals of Brownian motions with respect to the product of Wiener measures. As we have elaborated in Section 1, we will also generalize Witsenhausen’s equivalence between discrete-time static and dynamic team problems \([4]\), to an unobserved state which is a random processes. We will identify the exact expression of the common denominator condition (described in \([4]\)), and we will demonstrate the limitations of applying this method to compute the optimal team strategies, in terms of computational tractability, at least for problems with large number of decision stages.

**Continuous-Time Stochastic Dynamic Team Problems.**

We need the following assumption.

**Assumptions 2.** The Borel measurable diffusion coefficient \(\sigma\) in \((3)\) is replaced by \(\tilde{G}\)

(A9) \(G : [0, T] \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)\) (i.e. it is independent of \((x, \xi) \in \mathbb{R}^n \times \mathbb{A}^{(N)})\), \(G^{-1}\) exists and both are uniformly bounded;

(A10) There exists a \(K > 0\) such that \(|G^{-1}(t) f(t, x, \xi)|_{\mathbb{R}^n} \leq K, \forall (t, x, \xi) \in [0, T] \times \mathbb{R}^n \times \mathbb{A}^{(N)}\).

Under the additional Assumptions 2 by Girsanov’s theorem, we define the original stochastic differential decision problem \((22), (13), (21)\) (with \(\sigma(t, x, u) = G(t)\)) by considering two consecutive change of probability measures as follows.

We start with a probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_{0,t} : t \in [0, T]\}, \mathbb{F})\), under which \(\{(x(t), y(t)) : t \in [0, T]\}\) are defined by

\[
\begin{align*}
    x(t) &= x(0) + \int_0^t G(s)dW(s) \equiv x(0) + \hat{W}(t), \\
    y(t) &= \int_0^t D_1\hat{\pi}(s)dB(s),
\end{align*}
\]

\footnote{\(G\) can be allowed to depend on \(x\).}
where \(\{(W(t), B(t)) : t \in [0, T]\}\) are independent standard Brownian motions (independent of \(x(0)\)), and \(\left(\Omega, \mathbb{F}, \{\mathbb{F}_t : t \in [0, T]\}\right)\) are defined by [8,10].

Then we introduce the \((\{\mathbb{F}_t : t \in [0, T]\}, \mathbb{P})\)-exponential martingale defined by

\[
\Upsilon^u(t) \triangleq \exp \left\{ \int_0^t f^*(s, x(s), u_s) \left( G(s)G^*(s) \right)^{-1} dx(s) - \frac{1}{2} \int_0^t f^*(s, x(s)) \left( G(s)G^*(s) \right)^{-1} f(s, x(s)) ds \right\}, \quad t \in [0, T].
\]

(28)

Therefore, we define the reference measure \(\mathbb{P}\) on \(\left(\Omega, \{\mathbb{F}_t : t \in [0, T]\}\right)\) by setting

\[
\frac{d\mathbb{P}}{d\mathbb{P}_T} |_{\mathbb{F}_T} = \Upsilon^u(T)
\]

(29)

Then, under the reference probability space \(\left(\Omega, \mathbb{F}, \left(\{\mathbb{F}_t : t \in [0, T]\}, \mathbb{P}\right)\right)\), the process \(\{W^u(t) : t \in [0, T]\}\) is a Brownian motion defined by

\[
dW^u(t) = G^{-1}(t) \left( dx(t) - f(t, x(t), u_t) dt \right).
\]

(30)

Thus, \(\{x(t) : t \in [0, T]\}\) is a weak solution, which is pathwise unique, and \(\{\mathbb{F}_t : t \in [0, T]\}\)-adapted \(C([0, T], \mathbb{R}^n), \mathbb{P}\)-a.s., satisfying

\[
dx(t) = f(t, x(t), u_t) dt + G(t) dW^u(t), \quad x(0) = x_0, \quad t \in (0, T],
\]

and \(\{y(t) : t \in [0, T]\}\) is given by (27).

Next, we introduce the \((\{\mathbb{F}_t : t \in [0, T]\}, \mathbb{P})\)-exponential martingale \(\{\Lambda^u(t) : t \in [0, T]\}\) given by (17), and we define the original measure \(\mathbb{P}^u\) on \(\left(\Omega, \{\mathbb{F}_t : t \in [0, T]\}\right)\) by

\[
d\mathbb{P}^u(\omega) = \Lambda^u(T) d\mathbb{P}(\omega) = \Upsilon^u(T) \Upsilon^u(T) d\mathbb{P}(\omega).
\]

(31)

Thus, under the probability space \(\left(\Omega, \mathbb{F}, \{\mathbb{F}_t : t \in [0, T]\}, \mathbb{P}^u\right)\), the processes \(\{(x(t), y(t)) : t \in [0, T]\}\) are solutions of (30) and (21).

Regarding the pay-off, given a \(u \in U_r(N)[0, T]\), the team game pay-off under the probability space \(\left(\Omega, \mathbb{F}, \{\mathbb{F}_t : t \in [0, T]\}, \mathbb{P}\right)\) is defined by

\[
J(u) \triangleq \mathbb{E}\left\{ \int_0^T \Lambda^u(t) \times \Upsilon^u(t) \ell(t, x(t), u_t) dt + \Lambda^u(T) \times \Upsilon^u(T) \varphi(x(T)) \right\},
\]

(32)

where \(\mathbb{E}\{\cdot\}\) denotes expectation with respect to the product measure \(\mathbb{P}(d\xi, dw, dy) \triangleq \Pi_0(\xi) \times \mathcal{W}_W(dw) \times \mathcal{W}_y(dy)\), where \(\mathcal{W}_W(\cdot)\) is the Wiener measure on the sample paths \(\{W(t) : t \in [0, T]\} \subset C([0, T], \mathbb{R}^n)\), and \(\mathcal{W}_y(\cdot)\) is the Wiener measure of the sample paths \(\{y(t) : t \in [0, T]\} \subset C([0, T], \mathbb{R}^k)\) of Brownian motion \(y(t) = \int_0^t D \hat{\xi}(s) dB(s), t \in [0, T]\), defined by (27).
Moreover, by using (31) under the probability space \( (\Omega, \mathbb{F}, \{\mathbb{F}_{0,t} : t \in [0, T]\}, \mathbb{P}^u) \) the pay-off (32) reduced to the pay-off (22).

Hence, we have shown that the stochastic differential decision problem with \( \{(x(t), y(t)) : t \in [0, T]\} \) satisfying (30), (21), and pay-off (22), is equivalent to the decision problem with \( \{(x(t), y(t)) : t \in [0, T]\} \) independent Brownian motions, independent of any of the team decisions, satisfying (27), and pay-off (32), and hence the transformed problem is “Static” in Witsenhausen’s [1] terminology.

Now, we consider the equivalent transformed pay-off (32), and we integrate by parts the stochastic integral terms appearing in \( \Lambda^u(\cdot) \) [Theorem 6.1, [1]], which is easily verified by comparing (35) and [equation (6.4), [1]]. In fact, \( \Upsilon(x, t, y, u) = \Lambda^u(x, t, u) \phi(x) \), \( t \in [0, T], y, u \) is to write the differential equations of the class of models. One way to compute the optimal team strategies via a tractable procedure is to write the differential equations of \( \{\Lambda^u(t), \Upsilon^u(t) : t \in [0, T]\} \) and treat them as states of the pay-off (32). However, this is a stochastic dynamic optimization problem, leading to stochastic Pontryagin’s maximum principle, and hence it can not be dealt with via static team theory.

**Discrete-Time Stochastic Dynamic Team Problems.**

Next, we consider a general discrete-time stochastic dynamic system operating over the time period \( \{0, \ldots, T\} \), with observations collected at each time by \( N \) observation posts,
each serving one of the $N$ control stations. Let $\mathbb{N}_0^T \triangleq \{0, 1, 2, \ldots, T\}$, $\mathbb{N}_1^T \triangleq \{1, 2, \ldots, T\}$ denote discrete time-index set.

We start with a reference probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_{0,t} : t \in \mathbb{N}_0^T\}, \mathbb{P}\right)$, under which $\{(x(t), y^m(t)) : t \in \mathbb{Z}_0^T\}$ are sequences of independent RVs, with $x(0)$ having distribution $\Pi_0(dx)$, $\{x(t) \sim \zeta_t(\cdot) \triangleq \text{Gaussian}(0, I_{n \times n}) : t \in \mathbb{N}_1^T\}$, and $\{y^m(t) \sim \lambda^m(\cdot) \triangleq \text{Gaussian}(0, I_{k_m \times k_m}) : t \in \mathbb{N}_0^T\}$, for $m = 1, \ldots, N$. Thus, $y^m(t)$ is the observation output at the $m$th observation post at time $t \in \mathbb{N}_0^T$, for $m = 1, \ldots, N$.

Let $\{\mathcal{F}_{0,t} : t \in \mathbb{N}_0^T\}$ denote the filtration generated by the completion of the $\sigma$–algebra $\sigma\{x(\tau), y^1(\tau), \ldots, y^N(\tau) : \tau \leq t\}, t \in \mathbb{N}_1^T$, and $\{\mathcal{G}^m_{0,t} : t \in \mathbb{N}_0^T\}$ the filtration generated by completion of the $\sigma$–algebra $\sigma\{y^m(\tau) : \tau \leq t\}, t \in \mathbb{N}_0^T$, for $m = 1, \ldots, N$. Similarly, we define by $\{\mathcal{G}^m_{0,t} : t \in \mathbb{N}_0^T\}$ the filtration generated by completion of the minimum $\sigma$–algebra $\bigvee_{m=1}^N \mathcal{G}^m_{0,t}, t \in \mathbb{N}_0^T$.

Next, we specify the information structures available as arguments of the control laws, following the definitions given in [6]. For each $t \in \{0, 1, \ldots, T\}$, let $\mathcal{Y}_t \triangleq \{\tau, m) \in \{0, 1, \ldots, t\} \times \{1, 2, \ldots, N\}\}$. A data basis at time $t \in \{0, 1, \ldots, T\}$ is a subset $A \subseteq \mathcal{Y}_t$, while $\mathcal{Y}_t$ is the maximal data basis at time $t$. The array of vectors specified by $A$ is denoted by $y_A$, where $y_A \triangleq \{y^m(t) : (t, m) \in A\}$.

An information structure is the assignment to each $(t, k) \in \mathcal{Y}_t$ of a data basis at time $t$, denoted by $\mathcal{Y}_{t,k}$. The interpretation is that the control applied by the $k$th station at time $t$ is based on $\{y^m(\tau) : (\tau, \mu) \in \mathcal{Y}_{t,k}\}$. Thus, the argument of control applied by the $k$th station at time $t$ is $\{y^m(\tau) : (\tau, \mu) \in \mathcal{Y}_{t,k}\}$ (in [6] the information structure includes past controls as well).

Thus, for a given Borel measurable mapping $\gamma^k_t(\cdot)$, the control actions at the $k$th control station are generated by

$$u^k(t) = \gamma^k_t\left(y^m(\tau) : (\tau, \mu) \in \mathcal{Y}_{t,k}\right), \ t \in \{0, 1, \ldots, T\}, \ k = 1, \ldots, N. \quad (36)$$

That is, $\gamma^k_t(\cdot)$ is the control law or strategy at time $t$, which generates the control actions applied by the $i$th station, while its argument is the information structure.

Given an information structure, control station $k$ is said to have “Perfect Recall” if for $t = 0, \ldots, T - 1$, $\mathcal{Y}_{t,k} \subseteq \mathcal{Y}_{t+1,k}$. Note that perfect recall means that a station that at some time has available certain information will have available the same information at any subsequent times.

An information structure is called “Classical” if the following two conditions hold: i) all stations receive the same information (i.e. the information structures are independent of $k$), ii) all stations have perfect recall. Thus, classical information structure implies that the $\sigma$–algebras generated by the information structures at each control station over successive times are nested (i.e. these generate filtrations).

Given the information structures, we denote the set of admissible strategies at the $k$th control station at time $t \in \mathbb{N}_0^T$, by $\gamma^k_t(\cdot) \in \mathbb{U}^k[t]$, their $(T + 1)$–tuple by $\gamma^k_{[0,T]}(\cdot) \triangleq (\gamma^k_0(\cdot), \ldots, \gamma^k_T(\cdot)) \in \mathbb{U}^k[0, T] \triangleq \times_{t=0}^{T} \mathbb{U}^k[t]$, and $\gamma_{[0,T]}(\cdot) \triangleq (\gamma^1_{[0,T]}, \ldots, \gamma^N_{[0,T]}(\cdot) \in \mathbb{U}^{(N)}[0, T] \triangleq \times_{i=0}^{N} \mathbb{U}^{(N)}[0, T]$.

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Write \( \lambda_t(\cdot) = \prod_{m=1}^N \lambda_t^m(\cdot), t \in \mathbb{N}_0^T \), and \( y \triangleq \text{Vector}\{y^1, \ldots, y^N\}, h \triangleq \text{Vector}\{h^1, \ldots, h^M\}, D^\frac{1}{2} \triangleq \text{diag}\{D^1, \ldots, D^N\}. \) Consider the following measurable functions.

\[
f(t, \cdot, \cdot) : \times_{\tau=0}^T (\mathbb{R}^n) \times_{t=1}^T (\mathbb{R}^k_t) \rightarrow \mathbb{R}^n, \ t \in \mathbb{N}_0^{T-1}
\]

\[
h^m(t, \cdot, \cdot) : \mathbb{R}^n \times_{t=1}^T (\mathbb{R}^k_t) \rightarrow \mathbb{R}^m, \ t \in \mathbb{N}_0^T, \ m = 1, \ldots, N.
\]

For any admissible strategy \( u^k(t) \equiv \gamma^k_t(\{y^\mu(\tau) : (\tau, \mu) \in \mathcal{Y}_{t,k}\}, k = 1, \ldots, N, t \in \mathbb{N}_0^T \), we define the following quantity \( (u \equiv (u^1, \ldots, u^N)). \)

\[
\Theta^u_t = \prod_{\tau=1}^T \frac{\zeta_t(G^{-1}(\tau - 1)(x(\tau) - f(\tau - 1, x(0), \ldots, x(\tau - 1), u^1(\tau - 1), \ldots, u^N(\tau - 1))))}{|G(\tau - 1)|\zeta_t(x(\tau))} \cdot \frac{\lambda_t(D^\frac{1}{2}(\tau - 1)(y(\tau) - h(\tau, x(\tau), u^1(\tau), \ldots, u^N(\tau))))}{|D^\frac{1}{2}(\tau)|\lambda_t(y(\tau))}, \ t \in \mathbb{N}_1^T.
\]

\[
\Theta^u = \prod_{t=0}^T \frac{\lambda_0(\gamma^0_t(0)(y(0) - h(0, x(0), u^1(0), \ldots, u^N(0)))))}{|D^\frac{1}{2}(0)|\lambda_0(y(0))}.
\]

Under the reference probability space \( (\Omega, \mathcal{F}, \{\mathbb{F}_t : t \in \mathbb{N}_0^T\}, \mathbb{P}) \) the discrete-time pay-off is defined by

\[
J(u^1, \ldots, u^N) \equiv \mathbb{E}\left\{ \Theta^u_t(x(0), u^1(0), \ldots, u^N(0), y(0), \ldots, x(T), u^1(T), \ldots, u^N(T), y(T)) \right\}
\]

\[
= \left\{ \prod_{t=1}^T \zeta_t(x(t)) \cdot \lambda_t(y(t)) \right\}
\]

\[
\Pi_0(dx(0)) \cdot \lambda_0(y(0)) \prod_{t=1}^T \zeta_t(x(t)) \cdot \lambda_t(y(t)) dx(t). dy(t).
\]

The team problem is defined by

\[
\inf \left\{ J(\gamma_{[0,T]}^0) : \gamma_{[0,T]}^0 \in \mathcal{U}^{(N)}[0,T] \right\}.
\]
It can be shown that \( \{ \Theta_{0,t}^n : t \in \mathbb{N}_0^T \} \) is an \( \left( \Omega, \mathbb{F}, \{ \mathbb{F}_{0,t} : t \in \mathbb{N}_0^T \}, \mathbb{P} \right) \)-martingale, and hence \( \int \Theta_{0,t}^n(\omega)d\bar{\mathbb{P}}(\omega) = 1 \). Therefore, we can define a probability measure \( \mathbb{P}^u \) on \( \left( \Omega, \{ \mathbb{F}_{0,t} : t \in \mathbb{N}_0^T \} \right) \) by setting
\[
\left. \frac{d\mathbb{P}^u}{d\mathbb{P}} \right|_{\mathbb{F}_{0,t}} = \Theta_{0,t}, \quad t \in \mathbb{N}_0^T
\] (42)
Then under this probability measure \( \mathbb{P}^u \), the processes defined by
\[
w^u(t) \triangleq G^{-1}(t-1) \left( x(t) - f(t-1, x(0), x(1), \ldots, x(t-1), u^1(t-1), \ldots, u^N(t-1)) \right), \quad t \in \mathbb{N}_1^T,
\]
\[
\vartheta^m,u(t) \triangleq D_{m-\frac{1}{2}}(t) \left( y^m(t) - h^m(t, x(t), u^1(t), \ldots, u^N(t)) \right), \quad t \in \mathbb{N}_0^T, \quad m = 1, \ldots, N,
\] (44)
are two sequences of independent normally distributed RVs with densities, \( \zeta_t(\cdot), t \in \mathbb{N}_1^T, \) and \( \lambda^m(\cdot), t \in \mathbb{N}_0^T, m = 1, \ldots, N, \) respectively.
Therefore, under the probability space \( \left( \Omega, \mathbb{F}, \{ \mathbb{F}_{0,t} : t \in \mathbb{N}_0^T \}, \mathbb{P}^u \right) \), the discrete-time stochastic team problem has state and observations given by
\[
x(t + 1) = f(t, x(0), x(1), \ldots, x(t), u^1(t), \ldots, u^N(t)) + G(t)w^u(t + 1), \quad x(0) = x_0, \quad t \in \mathbb{N}_1^{T-1},
\]
\[
y^m(t) = h^m(t, x(t), u^1(t), \ldots, u^N(t)) + D_{m-\frac{1}{2}}(t)\vartheta^m,u(t), \quad t \in \mathbb{N}_0^T, \quad m = 1, \ldots, N.
\] (46)
Thus, for any admissible team strategy \( u \equiv \gamma_{[0,T-1]} \in \mathcal{U}(\mathcal{N})[0,T] \), under measure \( \mathbb{P}^u \) the team pay-off is
\[
J(u) = \mathbb{E}^u \left\{ \sum_{t=0}^{T-1} \ell(t, x(t), u^1(t), \ldots, u^N(t)) + \varphi(x(T)) \right\}, \quad \inf \left\{ J(\gamma_{[0,T]}): \gamma_{[0,T]} \in \mathcal{U}(\mathcal{N})[0,T] \right\}.
\] (47)
Therefore, we have shown that the dynamic team problem (45), (46) with pay-off (47) can be transformed to the equivalent problem (static problem in the terminology of Witsenhausen [11]) with pay-off (39), where the sequences \( \{ x(t), y(t) : t \in \mathbb{N}_0^T \} \) are independent, sequences, distributed according to \( x(0) \sim \Pi_0(dx), x(t) \sim \zeta_t(\cdot), t \in \mathbb{N}_1^T, y^m(t) \sim \lambda^m_t(\cdot), t \in \mathbb{N}_0^T, m = 1, \ldots, N. \)
Consequently, we conclude that under appropriate conditions (for example those in [3, 20]), we can apply static team optimality conditions. Indeed, we can invoke any of Theorems found in [20]. We illustrate this point by stating one such theorem from [20], when all control stations do not have perfect recall (the rest are also applicable).

**Theorem 1.** Suppose none of the control stations have perfect recall. Assume the following conditions hold.

\((S1)\) \( L : \times_{t=0}^{T,N} (\mathcal{A}_t^k) \times \times_{t=0}^{T} (\mathbb{R}^m) \longrightarrow \mathbb{R} \) is Borel measurable;
(S2) $L(\cdot, x(0), \ldots, x(T), y(0), \ldots, y(T))$ is convex and differentiable uniformly in $(x(0), y(0), \ldots, x(T), y(T)) \in \times_{i=0}^{T} (\mathbb{R}^{n+k})$;

(S3) $\inf \left\{ J(\gamma_{[0,T]}): \gamma_{[0,T]} \in \mathbb{U}^{(N)}[0,T] \right\} > -\infty$;

(S4) There exists a $\gamma_{[0,T]}^o \in \mathbb{U}^{(N)}[0,T]$ such that $J(\gamma_{[0,T]}^o) < \infty$;

(S5) For all $\gamma_{[0,T]} \in \mathbb{U}^{(N)}[0,T]$ such that $J(\gamma_{[0,T]}) < \infty$, the following holds

$$E \{ \sum_{k=1,t=0}^{N,T} L_{u_k(t)}(\gamma_{[0,T]}^o,x(0),\ldots,x(T),y(0),\ldots,y(T))$$

$\cdot \left( \gamma^k_t - \gamma^{k,o}_t \right) \} \geq 0, \ \forall \gamma_{[0,T]} \in \mathbb{U}^{(N)}[0,T].$ (48)

Then $\gamma_{[0,T]}^o \in \mathbb{U}^{(N)}[0,T]$ is a team optimal strategy. If in addition, $L(\cdot, x(0), \ldots, x(T), y(0), \ldots, y(T))$ is strictly convex, $\mathbb{P} - a.s.,$ then $\gamma_{[0,T]}^o \in \mathbb{U}^{(N)}[0,T]$ is unique $\mathbb{P} - a.s.$

Proof. Since none of the control stations have perfect recall, and $(t,k) \in \{0,\ldots,T\} \times \{1,\ldots,N\}$, there are $(T+1)N$ policies. Since, under the reference measure $\mathbb{P}$ the information structures are independent of any of the team decisions, then under the reference measure the problem is equivalent to a static team problem. By the above construction, we can apply static team theory to the equivalent pay-off, i.e. [Theorem 2, [20]] is applicable. □

Notice that by using the fact that Radon-Nykodym derivative $\Theta_{0,t}^u(\cdot), \forall t \in \mathbb{N}_0^T$ is an exponential function then we can expressed (48) under the initial probability measure $\mathbb{P}^u$. Moreover, it we can show that (48) implies the following component wise conditional variational inequalities.

$$E \{ L_{u_k(t)}(u^o(0),\ldots,u^o(T),x(0),\ldots,x(T),y(0),\ldots,y(T)) | \{ y^m(\tau): (\tau,\mu) \in \mathcal{Y}_{t,k} \}$$

$\cdot \left( u^k(t) - u^{k,o}(t) \right) \} \geq 0, \ \forall u^k(t) \in A_k^t, \ \mathbb{P} - a.s., k = 1,\ldots,N, t = 0,\ldots,T.$ (49)

Furthermore, (49) can be expressed under the original probability measure $\mathbb{P}^{u^o}$ as well.

Therefore, for discrete-time team problems we have extended Witsenshausen’s equivalence of static and dynamic team problems, to observations which are functions of an unobserved process with memory (rather than a RV), and we have identified the “Common Denominator Condition and Change of Variables”. Although, the above method appears feasible, to compute the optimal team strategies using this procedure is expected to be computational intensive for large number of decision stages $T$ and large number of control stations $N$. This is the reason why we pursue the second method based on stochastic Pontryagin’s maximum principle.

Next, we apply the static team optimality conditions to the Witsenhausen counter example described in [5]. This is a two-stage stochastic control problem is described by the
Admissible Strategies:

Under the reference probability measure

\[ p_0(\cdot) \]

Paper have been written on it.

Pay-Off:

\[ \gamma \]

Output Equations:

\[ x_1 = x_0 + u_1, \quad x_2 = x_1 - u_2, \quad x_0 \sim p_0(\cdot) \] (50)

State Equations:

\[ y_0 = x_0, \quad y_1 = x_1 + v, \quad v \sim \lambda_v(\cdot) \] (51)

Admissible Strategies:

\[ u_1 = \gamma_1(y_0), \quad u_2 = \gamma_2(y_1) \] (52)

Pay-Off:

\[ J(\gamma_1^*, \gamma_2^*) \triangleq \inf_{(\gamma_1(y_0), \gamma_2(y_1))} J(\gamma_1, \gamma_2), \quad J(u_1, u_2) \triangleq \mathbb{E}^{u_1, u_2}\{k^2(u_1)^2 + (x_2)^2\}, \] (53)

where \((\gamma_1, \gamma_2)\) are measurable functions, \(x_0\) is a Random Variable with known probability density function \(p_0(\cdot)\), and \(v\) is a Random noise term with a known probability density function \(\lambda_v(\cdot)\), both having zero mean and finite second moments, and \(x_0\) independent of \(v\). The objective is to find \((\gamma_1^*, \gamma_2^*)\) which minimizes \(J(\gamma_1, \gamma_2)\). The information pattern is nonclassical since \(y_0\) is known to the control \(u_1\) at the first stage but it is not known to the control \(u_2\) at the second stage. This problem remains unsolved since 1968, although several papers have been written on it.

Under the reference probability measure \(\mathbb{P}\), in which \(y_1\) is distributed according to \(\lambda_v(\cdot)\), the equivalent pay-off is

\[ J(u_0, u_1) = \mathbb{E}\{\frac{\lambda_v(y_1 - x_0 - u_1)}{\lambda_v(y_1)} \left( k^2(u_1)^2 + (x_0 + u_1 - u_2)^2 \right) \}. \] (54)

Applying static team theory, we have

\[ \frac{\partial}{\partial u_1} \mathbb{E}\{\frac{\lambda_v(y_1 - x_0 - u_1)}{\lambda_v(y_1)} \left( k^2(u_1)^2 + (x_0 + u_1 - u_2^*)^2 \right) \}|_{u_1 = u_1^*} = 0, \] (55)

\[ \frac{\partial}{\partial u_2} \mathbb{E}\{\frac{\lambda_v(y_1 - x_0 - u_1^*)}{\lambda_v(y_1)} \left( k^2(u_1^*)^2 + (x_0 + u_1^* - u_2^2)^2 \right) \}|_{u_2 = u_2^*} = 0. \] (56)

Suppose \(v\) is Gaussian distributed \(N(0, 1)\). Then, we deduce, after elementary calculations, that under the initial probability measure \(\mathbb{P}^{u_1, u_2}\), the optimal control strategies \((\gamma_1^*, \gamma_2^*)\) are given by the following expressions.

\[ \gamma_1^*(y_0) = -\frac{1}{2k^2} \mathbb{E}_{\gamma_1^*} \gamma_2^* \left\{ (y_1 - x_1) \left(x_1 - \gamma_2^*(y_1)\right)^2 | y_0 \right\} - \frac{1}{2k^2} \mathbb{E}_{\gamma_1^*} \gamma_2^* \left\{ x_1 - \gamma_2^*(y_1) | y_0 \right\} \] (57)

\[ = -\frac{1}{2k^2} \mathbb{E}_{\gamma_1^*} \gamma_2^* \left\{ (y_1 - x_0 - \gamma_1^*(y_0)) \left(x_0 + \gamma_1^*(y_0) - \gamma_2^*(y_1)\right)^2 | y_0 \right\} \]

\[ - \frac{1}{k^2} \mathbb{E}_{\gamma_1^*} \gamma_2^* \left\{ x_0 + \gamma_1^*(y_0) - \gamma_2(y_1) | y_0 \right\}, \] (58)

\[ \gamma_2^*(y_1) = \mathbb{E}_{\gamma_1^*} \left\{ x_1 | y_1 \right\} \] (59)

\[ = \mathbb{E}_{\gamma_1^*} \left\{ x_0 | y_1 \right\} + \mathbb{E}_{\gamma_1^*} \left\{ \gamma_1^*(y_0) | y_1 \right\}. \] (60)
These are the equations which give the optimal two-stage decision strategies. One may proceed further to compute the conditional density $p_{x_1|y_1}(x_1|y_1)$, and then substitute Eq. (62) into Eq. (59) to obtain a nonlinear equation in terms of $\gamma^*_1(y_0)$.

## 3 Existence of Relaxed Team Optimal Strategies

In this section we consider the augmented systems of previous section, and we show continuous dependent of the solutions on $u$. This property is required when we invoke the semi martingale representation for Hilbert space processes to obtain the variational equation of the augmented system. Moreover, we introduce additional assumptions on $\ell, \varphi$ and we also show existence of team and PbP optimal strategies.

Let $B^\infty_{\mathcal{F}_T}([0,T], L^2(\Omega, \mathbb{R}^{n+1}))$ denote the space of $\{\mathbb{F}_t : t \in [0,T]\}$-adapted $\mathbb{R}^{n+1}$—valued second order random processes endowed with the norm topology $\| \cdot \|$ defined by

$$
\| X \|_2 \triangleq \sup_{t \in [0,T]} \mathbb{E}[X(t)^2].
$$

Next, we show existence of solutions and their continuous dependence on $u$.

**Lemma 1. (Existence and Continuous Dependence of Solutions)**

Suppose Assumptions 1, (A0)-(A4), (A5'), (A6)-(A8) hold. Then for any $\mathbb{F}_{0,t}$-measurable initial state $x_0$ having finite second moment, and any $u \in \U_{\text{rel}}([0,T])$, the following hold.

1. System (23) has a unique solution $X \in B^\infty_{\mathcal{F}_T}([0,T], L^2(\Omega, \mathbb{R}^{n+1}))$ having a continuous modification, that is, $X \in C([0,T], \mathbb{R}^{n+1})$, $\mathbb{P}$–a.s. Moreover, $\Lambda^u(t) \in L^p(\Omega, \mathbb{F}_{0,t}, \mathbb{P}; \mathbb{R})$, $\forall t \in [0,T]$ for any finite $p$, and also $\Lambda^u \in L^n(\Omega, \mathbb{F}_{0,t}, \mathbb{P}; \mathbb{R})$, $\forall t \in [0,T]$;

2. The solution of system (23) is continuously dependent on the control, in the sense that, as $u^{i,\alpha} \xrightarrow{\mathcal{V}} u^{i,\circ}$ in $\U_{\text{rel}}([0,T])$, $\forall i \in \mathbb{Z}_N$, then $X^\alpha \xrightarrow{\mathcal{S}} X^\circ$ in $B^\infty_{\mathcal{F}_T}([0,T], L^2(\Omega, \mathbb{R}^{n+1}))$.

**Proof.** (1). We consider $X = (\Lambda, x)$ and we show the statements component wise. The proof regarding $x$ is standard and can be shown using the Banach fixed point theorem, hence it is omitted. Regarding $\Lambda^u$ which satisfies (17), by the conditions (A5'), (A8) we deduce that

$$
\mathbb{P}\{\int_0^T |D^{-1}(t)h(t, x(t), u_t)|_{\mathbb{R}^k}^2 dt < \infty\} = 1,
$$

hence by (17) there exists a unique non-negative continuous solution $\Lambda^u(t) = \prod_{i=1}^N \Lambda^i(u)(t)$, where $\Lambda^i(u)(\cdot)$ is given by (14). Next, we show that $\Lambda^u \in B^\infty_{\mathcal{F}_T}([0,T], L^2(\Omega, \mathbb{R}))$. Define

$$
\tau_n \triangleq \inf \big\{ t \in [0,T] : \int_0^t h^*(s, x(s), u_s)D^{-1}(s)dy(s) \geq n \big\}.
$$

By the integral solution of (17) (obtained by Itô’s formulae), and taking $t = \tau_n$ we have

$$
\Lambda^u(\tau_n) = 1 + \int_0^{\tau_n} \Lambda^u(s)h^*(s, x(s), u_s)D^{-1}(s)dy(s). \quad (61)
$$

Taking the expectation of both sides we obtain

$$
\mathbb{E}\{ \Lambda^u(\tau_n) \} = 1. \quad (62)
$$
But \( \tau_n \rightarrow T \) as \( n \rightarrow \infty \), \( \mathbb{P} - a.s. \). Since \( \Lambda^u(\cdot) \) is nonnegative, letting \( n \rightarrow \infty \) and using Fatou’s Lemma we have

\[
\mathbb{E} \liminf_{n \rightarrow \infty} \Lambda^u(\tau_n) \leq \liminf_{n \rightarrow \infty} \mathbb{E}\Lambda^u(\tau_n) = 1.
\]  

(63)

Hence,

\[
\mathbb{E}\{ \Lambda^u(T) \} \leq 1.
\]

(64)

Now,

\[
|\Lambda^u(t)|^2 = \exp \left\{ \frac{2}{2} \int_0^t h^*(s, x(s), u_s)D^{-1}(s)dy(s) - \frac{2}{2} \int_0^t |D^{-\frac{1}{2}}(s)h(s, x(s), u_s)|^2_{\mathbb{R}^k} ds \right\}
\]

\[
= \exp \left\{ \frac{2}{2} \int_0^t h^*(s, x(s), u_s)D^{-1}(s)dy(s) - \frac{4}{2} \int_0^t |D^{-\frac{1}{2}}(s)h(s, x(s), u_s)|^2_{\mathbb{R}^k} ds \right\}
\]

\[
\times \exp \left\{ \int_0^t |D^{-\frac{1}{2}}(s)h(s, x(s), u_s)|^2_{\mathbb{R}^k} ds \right\}.
\]

(65)

Using \((A5')\), \((A8)\) then there exists a \( K > 0 \) such that

\[
|\Lambda^u(t)|^2 \leq \exp\{K^2T\} \exp \left\{ \frac{2}{2} \int_0^t h^*(s, x(s), u_s)D^{-1}(s)dy(s) - \frac{4}{2} \int_0^t |D^{-\frac{1}{2}}(s)h(s, x(s), u_s)|^2_{\mathbb{R}^k} ds \right\}
\]

(66)

By \((64)\) which also holds for \( t \in [0, T] \) instead of \( T \) and by replacing \( h^*(t, x, u) \) by \( 2h^*(t, x, u) \) then we obtain

\[
\mathbb{E}|\Lambda^u(t)|^2 \leq \exp\{K^2T\} \mathbb{E}\left\{ \exp \left\{ \frac{2}{2} \int_0^t h^*(s, x(s), u_s)D^{-1}(s)dy(s)
\right.ight.
\]

\[
- \frac{4}{2} \int_0^t |D^{-\frac{1}{2}}(s)h(s, x(s), u_s)|^2_{\mathbb{R}^k} ds \right\} \right\} \leq \exp\{K^2.T\}.1
\]

(67)

This shows that \( \Lambda^u \in B_{\mathbb{F}^T}([0, T], L^2(\Omega, \mathbb{R})) \). The same procedure can be used to show that \( \Lambda^u(\cdot) \in L^p(\Omega, \mathbb{F}_t, \mathbb{F}^T; \mathbb{R}), \forall t \in [0, T] \) for any finite \( p \).

Since \( \Lambda^u(\cdot) \) is an \( \mathbb{F}_T \)-martingale (with right continuous trajectories) then \( |\Lambda^u(\cdot)|^2 \) is an \( \mathbb{F}_T \)-submartingale with right continuous trajectories. Therefore, we can apply Doob’s \( L^p \)-inequality for \( p = 2 \) to obtain

\[
\mathbb{P}\left\{ \sup_{t \in [0, T]} |\Lambda^u(t)|^2 > n \right\} \leq \frac{\mathbb{E}|\Lambda^u(T)|^4}{n^2} \leq \frac{1}{n^2} K, \quad K > 0
\]

where the last inequality follows from \( \Lambda^u(t) \in L^p(\Omega, \mathbb{F}_t, \mathbb{F}^T; \mathbb{R}), \forall t \in [0, T] \), and hence for \( p = 4 \). Let \( \Delta_n \triangleq \left\{ \omega \in \Omega : \sup_{t \in [0, T]} |\Lambda^u(t, \omega)|^2 > n \right\} \), then

\[
\sum_{n=1}^{\infty} \mathbb{P}\{\Delta_n\} \leq K \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.
\]

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Define $\Delta^* \triangleq \lim\Delta_n = \bigcap_{n=1}^{\infty} \bigcup_{n=1}^{\infty} \Delta_n \equiv \{\Delta_n \text{ i.o.}\}$. By the Borel-Cantelli lemma (first part) we have, $P\{\Delta_n \text{ i.o.}\} = 0$. This means that $P - \text{ess sup}_{\omega \in \Omega} \sup_{t \in [0, T]} |\Lambda^u(t, \omega)|^2 < M$ for some finite $M > 0$.

(2) Since the class of policies $U_{rel}^i[0, T], \forall i \in \mathbb{Z}_N$ is compact in the vague topology, then $\times_{i=1}^{N} U_{rel}^i[0, T]$ is also compact in this topology. Utilizing this observation the proof regarding $x$ is identical to that of [50], Lemma 3.1. Next, we consider the second part asserting the continuity of $u$ to solution map $u \rightarrow \Lambda$. Let $\{u^{i, \alpha} : i = 1, 2, \ldots, N\}$, $u^0$ be any pair of strategies from $\mathbb{U}^{(N)}_i[0, T] \times \mathbb{U}^{(N)}_i[0, T]$ and $\{x^\alpha, x^0\}$, $\{\Lambda^\alpha, \Lambda^0\}$ denote the corresponding pair of solutions of the system [23]. Let $u^{i, \alpha} \overset{v}{\rightarrow} u^{i, \alpha}$ in $\mathbb{U}^{i}_r[0, T], i = 1, 2, \ldots, N$. We must show that $\Lambda^\alpha \overset{\delta}{\rightarrow} \Lambda^0$ in $B_{F_T}^{\infty}([0, T], L^2(\Omega, \mathbb{R}))$. By the definition of solution to [17], then

$$
\Lambda^\alpha(t) - \Lambda^0(t) = \int_0^t \{\Lambda^\alpha(s) - \Lambda^0(s)\} h^*(s, x^\alpha(s), u^\alpha_s) D^{-1}(s) dy(s) \\
+ \int_0^t \Lambda^\alpha(s) \left\{ h^*(s, x^\alpha(s), u^\alpha_s) - h^*(s, x^0(s), u^0_s) \right\} D^{-1}(s) dy(s), \quad t \in [0, T] \\
= \int_0^t \{\Lambda^\alpha(s) - \Lambda^0(s)\} h^*(s, x^\alpha(s), u^\alpha_s) D^{-1}(s) dy(s) + e_1^\alpha(t) + e_2^\alpha(t),
$$

(68)

where

$$
e_1^\alpha(t) \triangleq \int_0^t \Lambda^\alpha(s) \left\{ h^*(s, x^\alpha(s), u^\alpha_s) - h^*(s, x^0(s), u^0_s) \right\} D^{-1}(s) dy(s),$$

$$
e_2^\alpha(t) \triangleq \int_0^t \Lambda^\alpha(s) \left\{ h^*(s, x^\alpha(s), u^\alpha_s) - h(s, x^0(s), u^0_s) \right\} D^{-1}(s) dy(s).$$

From (68) using Doob's martingale inequality, it follows that there exists constants $C_1, C_2 > 0$ such that

$$E|\Lambda^\alpha(t) - \Lambda^0(t)|^2 \leq 4C_1 E \int_0^t |\Lambda^\alpha(s) - \Lambda^0(s)|^2 |D^{-\frac{1}{2}}(s) h^*(s, x^\alpha(s), u^\alpha_s)|_{L^2}^2 ds \\
+ C_2 (E|e_1^\alpha(t)|_{L^2}^2 + E|e_2^\alpha(t)|_{L^2}^2).$$

(69)

Clearly, by Assumptions H (A6), (A7) we also obtain

$$E|e_1^\alpha(t)|_{L^2}^2 \leq 4E \int_0^t |\Lambda^\alpha(t)|^2 |D^{-\frac{1}{2}}(s) \left( h^*(s, x^\alpha(s), u^\alpha_s) - h(s, x^\alpha(s), u^\alpha_s) \right)|_{L^2}^2 ds,$$

(70)

$$E|e_2^\alpha(t)|_{L^2}^2 \leq 4E \int_0^t |\Lambda^\alpha(t)|^2 |D^{-\frac{1}{2}}(s) \left( h^*(s, x^\alpha(s), u^\alpha_s) - h(s, x^\alpha(s), u^\alpha_s) \right)|_{L^2}^2 ds.$$

(71)

Define $\tau_n \triangleq \inf \left\{ t \in [0, T] : |D^{-\frac{1}{2}}(t) h(t, x^\alpha(t), u^\alpha_t)|_{L^2}^2 > n \right\}$. Using this stopping time in [69]
we have
\[
\mathbb{E}[\Lambda^\alpha(t \land \tau_n) - \Lambda^\alpha(t \land \tau_n)]^2 \leq 4C_1 \mathbb{E} \int_0^{t \land \tau_n} \left| \Lambda^\alpha(s) - \Lambda^\alpha(s) \right|^2 D^{-\frac{1}{2}}(s) h(s, x^\alpha(s), u^\alpha_s) ds \\
+ 4C_2 \mathbb{E} \int_0^{t \land \tau_n} K^2(s) \left| \Lambda^\alpha(s) \right|^2 \left| x^\alpha(s) - x^\alpha(s) \right|_{\mathbb{R}^n}^2 ds \\
+ 4C_2 \mathbb{E} \int_0^{t \land \tau_n} \left| \Lambda^\alpha(t) \right|^2 D^{-\frac{1}{2}}(s) \left( h(s, x^\alpha(s), u^\alpha_s) - h(s, x^\alpha(s), u^\alpha_s) \right)_{\mathbb{R}^k}^2 ds, \quad t \in [0, T].
\]

Hence,
\[
\mathbb{E}[\Lambda^\alpha(t \land \tau_n) - \Lambda^\alpha(t \land \tau_n)]^2 \leq 4nC_1 \mathbb{E} \int_0^{t \land \tau_n} \left| \Lambda^\alpha(s) - \Lambda^\alpha(s) \right|^2 ds \\
+ 4C_2 \mathbb{E} \int_0^{t \land \tau_n} K^2(s) \left| \Lambda^\alpha(s) \right|^2 \left| x^\alpha(s) - x^\alpha(s) \right|_{\mathbb{R}^n}^2 ds, \\
+ 4C_2 \mathbb{E} \int_0^{t \land \tau_n} \left| \Lambda^\alpha(t) \right|^2 D^{-\frac{1}{2}}(s) \left( h(s, x^\alpha(s), u^\alpha_s) - h(s, x^\alpha(s), u^\alpha_s) \right)_{\mathbb{R}^k}^2 ds, \quad t \in [0, T].
\]

It is easy to see that this inequality is the same as the following one
\[
\mathbb{E}\left\{ \left| \Lambda^\alpha(t \land \tau_n) - \Lambda^\alpha(t \land \tau_n) \right|^2 \right\} \leq nC_1 \int_0^{t \land \tau_n} \mathbb{E}\left\{ \left| \Lambda^\alpha(s \land \tau_n) - \Lambda^\alpha(s \land \tau_n) \right|^2 \right\} ds \\
+ 4C_2 \mathbb{E} \int_0^{t \land \tau_n} K^2(s) \left| \Lambda^\alpha(s) \right|^2 \left| x^\alpha(s) - x^\alpha(s) \right|_{\mathbb{R}^n}^2 ds, \\
+ 4C_2 \mathbb{E} \int_0^{t \land \tau_n} \left| \Lambda^\alpha(t) \right|^2 D^{-\frac{1}{2}}(s) \left( h(s, x^\alpha(s), u^\alpha_s) - h(s, x^\alpha(s), u^\alpha_s) \right)_{\mathbb{R}^k}^2 ds, \quad t \in [0, T].
\]

Applying Gronwall Lemma to (4) we obtain
\[
\mathbb{E}\left\{ \left| \Lambda^\alpha(t \land \tau_n) - \Lambda^\alpha(t \land \tau_n) \right|^2 \right\} \leq 4nC_1 \int_0^{t \land \tau_n} \mathbb{E}\left\{ \left| \Lambda^\alpha(s \land \tau_n) - \Lambda^\alpha(s \land \tau_n) \right|^2 \right\} ds \\
+ 4C_2 \mathbb{E} \int_0^{t \land \tau_n} K^2(s) \left| \Lambda^\alpha(s) \right|^2 \left| x^\alpha(s) - x^\alpha(s) \right|_{\mathbb{R}^n}^2 ds \\
+ 4C_2 \mathbb{E} \int_0^{t \land \tau_n} \left| \Lambda^\alpha(t) \right|^2 D^{-\frac{1}{2}}(s) \left( h(s, x^\alpha(s), u^\alpha_s) - h(s, x^\alpha(s), u^\alpha_s) \right)_{\mathbb{R}^k}^2 ds, \quad t \in [0, T].
\]

giving
\[
\sup_{t \in [0, T]} \mathbb{E}\left| \Lambda^\alpha(t \land \tau_n) - \Lambda^\alpha(t \land \tau_n) \right|^2 \leq 4C_2 \exp\{4nC_1 T\} \mathbb{E} \int_0^{T} \left| \Lambda^\alpha(s) \right|^2 \left( K^2(s) \left| x^\alpha(s) - x^\alpha(s) \right|_{\mathbb{R}^n}^2 \\
+ D^{-\frac{1}{2}}(s) \left( h(s, x^\alpha(s), u^\alpha_s) - h(s, x^\alpha(s), u^\alpha_s) \right)_{\mathbb{R}^k}^2 \right) ds.
\]
Since by part (1), $\mathbb{P} - \text{ess sup}_{\omega \in \Omega} \text{ sup}_{t \in [0,T]} |A^\alpha(t,\omega)|^2 < M$ for some finite $M > 0$, applying this in \((76)\) we deduce the following bound.

$$\sup_{t \in [0,T]} E|A^\alpha(t \wedge \tau_n) - A^\alpha(t \wedge \tau_n)|^2 \leq 4C_2 \exp\{4nC_1T\} \cdot \mathbb{P} - \text{ess sup}_{\omega \in \Omega} \text{ sup}_{t \in [0,T]} |A^\alpha(t,\omega)|^2$$

$$\leq 4C_2 \exp\{4nC_1T\} \cdot M \left\{ \int_0^T K^2(s) \mathbb{E}|x^\alpha(s) - x^\alpha(s)|_2^2 ds + \mathbb{E} \int_0^T |D^{-\frac{1}{2}}(s)(h(s,x^\alpha(s),u^\alpha_s) - h(s,x^\alpha(s),u^\alpha_s))|_2^2 ds \right\}$$

$$+ \mathbb{E} \int_0^T |D^{-\frac{1}{2}}(s)(h(s,x^\alpha(s),u^\alpha_s) - h(s,x^\alpha(s),u^\alpha_s))|_2^2 ds \right\}, \quad (77)$$

Now, utilizing (1), letting $\alpha \to \infty$ and recalling that $x^\alpha$ converges to $x^\alpha$ in $B^\infty_{\mathbb{P}}([0,T], L^2(\Omega, \mathbb{R}^n))$, the first integrand in the right hand side of (77) converge to zero for almost all $s \in [0,T], \mathbb{P}-a.s.$ Also, by virtue of vague convergence $u^{i,\alpha} \overset{v}{\to} u^{i,o}$, and the uniform continuity assumption (A7) on $h(t,x,\cdot)$, the second integrand in the right hand side of (77) converge to zero for almost all $s \in [0,T], \mathbb{P}-a.s.$ Since by our assumptions the integrands are dominated by integrable functions, by Lebesgue dominated convergence theorem we obtain

$$\lim_{\alpha \to \infty} \sup_{t \in [0,T]} E|A^\alpha(t \wedge \tau_n) - A^\alpha(t \wedge \tau_n)|^2 = 0, \text{ for every } n \in N. \quad (78)$$

Since

$$\mathbb{P}\left\{ \sup_{t \in [0,T]} |D^{-\frac{1}{2}}(t)h(t,x^\alpha(t),u^\alpha_t)|_2^2 > n \right\} \to 0 \text{ as } n \to \infty,$$

it is clear that $\lim_{n \to \infty} \tau_n = T$. Hence, $A^\alpha$ converges in the mean square sense on $[0,T]$.

Next, we address the question of existence of team and PbP optimal strategies. We need the following assumptions.

**Assumptions 3.** *The functions $\ell$ and $\varphi$ associated with the pay-off (24) are Borel measurable maps:*

$$\ell : [0,T] \times \mathbb{R}^n \times A^{(N)} \to (-\infty, +\infty], \quad \varphi : \mathbb{R}^n \to (-\infty, +\infty].$$

*satisfying the following basic conditions:*

(B1) $x \to \ell(t,x,\xi)$ is continuous on $\mathbb{R}^n$ for each $t \in [0,T]$, uniformly with respect to $\xi \in A^{(N)}$;

(B2) $\exists h \in L^1_1([0,T],\mathbb{R})$ such that for each $t \in [0,T]$, $|\ell(t,x,\xi)| \leq h(t)(1 + |x|_2^2)$;

(B3) $x \to \varphi(x)$ is continuous on $\mathbb{R}^n$ and $\exists c_0, c_1 \geq 0$ such that $|\varphi(x)| \leq c_0 + c_1|x|_2^2$.

Using the results of Lemma 1 in the next theorem we establish existence of team and PbP optimal strategies $u^\alpha \in \mathbb{U}^{rel}_{r}(0,T)$ for Problem 1 [2].
Theorem 2. (Existence of Team Optimal Strategies) Consider Problem 1 and suppose Assumptions hold. Then there exists a team decision \( u^o \triangleq (u^{1,o}, u^{2,o}, \ldots, u^{N,o}) \) \( \in \mathbb{U}_{rel}^N [0, T] \) at which \( J(u^1, u^2, \ldots, u^N) \) attains its infimum.

Existence also holds for PbP decisions of Problem 2.

\[
\text{Proof.} \quad \text{Since the class of control policies } \mathbb{U}_{rel}^j [0, T] \text{ is compact in the vague topology, it suffices to prove that } J(\cdot) \text{ given by (24) is lower semicontinuous with respect to this topology. Suppose } u^{i,o} \xrightarrow{v} u^{i,o} \text{ in } \mathbb{U}_{rel}^j [0, T] \text{ for } i = 1, 2, \ldots, N \text{ and let } \{X^\alpha, X^o\} \subset B_{\mathbb{F}_T}^\infty([0, T], L^2(\Omega, \mathbb{R}^{n+1})) \text{ denote the solutions of equation (23) corresponding to the sequence } \{\{u^{i,o} : i = 1, 2, \ldots, N\}, u^o\} \subset \mathbb{U}_{rel}^N [0, T]. \text{ Then by Lemma 1 along a subsequence if necessary, } x^\alpha \xrightarrow{\mathbb{P}} x^o \text{ in } B_{\mathbb{F}_T}^\infty([0, T], L^2(\Omega, \mathbb{R}^n)) \text{ and } \Lambda^\alpha \xrightarrow{\mathbb{P}} \Lambda^o \text{ in } B_{\mathbb{F}_o,T}^\infty([0, T], L^2(\Omega, \mathbb{R})) \text{ Firstly, in view of the strong convergence, along a subsequence if necessary, } x^\alpha(T) \xrightarrow{\mathbb{P}} x^o(T), \Lambda^\alpha(T) \xrightarrow{\mathbb{P}} \Lambda^o(T), \mathbb{P}-\text{a.s.} \text{ By the continuity of } \varphi, \text{ Assumptions } 3, \text{ (B3), we have } \varphi(x^o(T)) = \lim \inf_n \varphi(x^n(T)) \text{, } \mathbb{P}-\text{a.s. and also get } \Lambda^\alpha(T) \varphi(x^\alpha(T)) \leq \lim \inf_n \Lambda^\alpha(T) \varphi(x^n(T)), \mathbb{P}-\text{a.s.} \text{ Hence, } \mathbb{E}\{\Lambda^\alpha(T) \varphi(x^\alpha(T))\} \leq \lim \inf_n \mathbb{E}\{\Lambda^\alpha(T) \varphi(x^n(T))\}. \text{ Thus, it follows from Assumptions 3, (B3), by applying Fatou’s lemma that }
\]

\[
\mathbb{E}\{\Lambda^\alpha(T) \varphi(x^\alpha(T))\} \leq \lim \inf_n \mathbb{E}\{\Lambda^\alpha(T) \varphi(x^n(T))\}. \quad (79)
\]

Next, consider the integral pay-off; it can be shown that

\[
\mathbb{E}\int_{[0,T]} \Lambda^\alpha(t) \ell(t, x^\alpha(t), u^o_t) dt = \mathbb{E}\int_{[0,T]} \Lambda^\alpha(t) \ell(t, x^\alpha(t), u^o_t - u^n_t) dt + \mathbb{E}\int_{[0,T]} \Lambda^\alpha(t) \ell(t, x^o(t), u^o_t) dt + \mathbb{E}\int_{[0,T]} \left( \Lambda^\alpha(t) \ell(t, x^\alpha(t), u^o_t) - \Lambda^\alpha(t) \ell(t, x^o(t), u^o_t) \right) dt. \quad (80)
\]

By virtue of vague convergence (as in the derivation of Lemma 1) of the product measure \( \times_{i=1}^N u^{i,o} \) to \( \times_{i=1}^N u^{i,o} \), and Doobs inequality together with Borel-Cantelli lemma, we have \( \mathbb{P} - \text{ess sup}_{\omega \in \Omega} \sup_{t \in [0,T]} |\Lambda^\alpha(t, \omega)| < M \) for some finite \( M > 0 \), hence it is evident that for every \( \varepsilon > 0 \) there exists an integer \( \alpha_{1,\varepsilon} \) sufficiently large, such that the absolute value of the first term on the right hand side of equation (80) is less than \( \varepsilon/3 \) for all \( \alpha \geq \alpha_{1,\varepsilon} \). Expressing the last right hand side term in (80) as

\[
\mathbb{E}\int_{[0,T]} \left( \Lambda^\alpha(t) \ell(t, x^\alpha(t), u^o_t) - \Lambda^\alpha(t) \ell(t, x^o(t), u^o_t) \right) dt
\]

\[
= \mathbb{E}\int_{[0,T]} \Lambda^\alpha(t) \left( \ell(t, x^\alpha(t), u^o_t) - \ell(t, x^o(t), u^o_t) \right) dt + \mathbb{E}\int_{[0,T]} \left( \Lambda^\alpha(t) - \Lambda^\alpha(t) \right) \ell(t, x^o(t), u^o_t) dt,
\]

by Assumptions 3, (B1), (B2), in particular the continuity of \( \ell \) in \( x \) uniformly in \( \Lambda^{(N)} \), it is easy to verify that there exists an integer \( \alpha_{2,\varepsilon} \) such that for all \( \alpha \geq \alpha_{2,\varepsilon} \), the absolute value of the first term on the right hand side of (81) is less than \( \varepsilon/3 \), and that by Lebesgue Dominated convergence theorem, and Lemma 1 that there exists an integer \( \alpha_{3,\varepsilon} \) such that
for all $\alpha \geq \alpha_{3,\varepsilon}$, the absolute value of the second term on the right hand side of (81) is less than $\varepsilon/3$. By combining these observations we obtain the following inequality

$$
\mathbb{E} \int_{[0,T]} \ell(t, x^\alpha(t), u_t^\alpha)dt \leq \varepsilon + \int_{[0,T]} \ell(t, x^\alpha(t), u_t^\alpha)dt, \quad \forall \alpha \geq \alpha_{1,\varepsilon} \lor \alpha_{2,\varepsilon} \lor \alpha_{3,\varepsilon}.
$$

Since $\varepsilon > 0$ is arbitrary, it follows from the above inequality that

$$
\mathbb{E} \int_{[0,T]} \ell(t, x^\alpha(t), u_t^\alpha)dt \leq \liminf_{\alpha} \mathbb{E} \int_{[0,T]} \ell(t, x^\alpha(t), u_t^\alpha)dt.
$$

Combining (79) and (82) we arrive at the conclusion that

$$
J(u^1,\alpha, u^2,\alpha, \ldots, u^N,\alpha) \leq \liminf_{\alpha} J(u^1,\alpha, u^2,\alpha, \ldots, u^N,\alpha),
$$

establishing lower semicontinuity of $J(\cdot)$ in the vague topology. Since $U_{rel}^{(N)}[0,T]$ is compact in the product (vague) topology, $J(\cdot)$ attains its minimum on it. This proves the existence of an optimal team decision from $U_{rel}^{(N)}[0,T]$. Existence of an optimal PbP decision is shown similarly.

Since our derivation of stochastic minimum principle (necessary conditions of optimality) or stochastic Pontryagin’s minimum principle will be based on the martingale representation approach, we state a version of Hilbert space semi martingale representation which can be found in many references.

**Definition 3.** An $\mathbb{R}^n$-valued random process $\{m(t) : t \in [0,T]\}$ is said to be a square integrable continuous $\mathbb{F}_T$-semi martingale if and only if it has a representation

$$
m(t) = m(0) + \int_0^t v(s)ds + \int_0^t \Sigma(s)dW(s), \quad t \in [0,T],
$$

for some $v \in L^2_{\mathbb{F}_T}([0,T],\mathbb{R}^n)$ and $\Sigma \in L^2_{\mathbb{F}_T}([0,T],\mathcal{L}(\mathbb{R}^m,\mathbb{R}^n))$ and for some $\mathbb{R}^n$-valued $\mathbb{F}_{0,0}$-measurable random variable $m(0)$ having finite second moment. The set of all such semi martingales is denoted by $\mathcal{SM}^2([0,T],\mathbb{R}^n)$.

Introduce the following class of $\mathbb{F}_T$-semi martingales:

$$
\mathcal{SM}^2_0([0,T],\mathbb{R}^n) \triangleq \left\{ m : m(t) = \int_0^t v(s)ds + \int_0^t \Sigma(s)dW(s), \quad t \in [0,T], \right. \\
\left. \text{for } v \in L^2_{\mathbb{F}_T}([0,T],\mathbb{R}^n) \text{ and } \Sigma \in L^2_{\mathbb{F}_T}([0,T],\mathcal{L}(\mathbb{R}^m,\mathbb{R}^n)) \right\}.
$$

Now we present the fundamental result which is utilized in the maximum principle derivation.
4 Team and PbP Optimality Conditions

In this section, we consider relaxed DM strategies and we derive necessary and sufficient optimality conditions for team optimality (see Problem 1), and we deduce analogous results for PbP optimality (see Problem 2).

For the derivation of stochastic optimality conditions we shall require stronger regularity conditions on the coefficients of the state and observation equations \( \{f, \sigma, h\} \), and coefficients in the reward function \( \{\xi, \varphi\} \). These are given below.

**Assumptions 4.** \( \mathbb{E}|x(0)|^2 < \infty \), \( (A^i, d), \forall i \in \mathbb{Z}_N \) are compact, the maps \( f, \sigma, \xi, \{h^i, D^{i, \frac{1}{2}}, \forall i \in \mathbb{Z}_N\} \) are measurable in \( t \in [0, T] \), \( \varphi \) is Borel measurable, defined by

\[
\begin{align*}
  f &: [0, T] \times \mathbb{R}^n \times A^{(N)} \rightarrow \mathbb{R}^n, \quad \sigma &: [0, T] \times \mathbb{R}^n \times A^{(N)} \rightarrow \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n), \quad \varphi &: \mathbb{R}^n \rightarrow \mathbb{R}, \\
  \xi &: [0, T] \times \mathbb{R}^n \times A^{(N)} \rightarrow \mathbb{R}, \quad h^i &: [0, T] \times \mathbb{R}^n \times A^{(N)} \rightarrow \mathbb{R}^k, \quad D^{i, \frac{1}{2}} &: [0, T] \rightarrow \mathcal{L}(\mathbb{R}^{k_1}, \mathbb{R}^{k_1}),
\end{align*}
\]

for \( i = 1, \ldots, N \), and they satisfy the following conditions.

**C1** The maps \( \{f, \sigma\} \) are once continuously differentiable with respect to \( x \in \mathbb{R}^n \);

**C2** The first derivatives \( \{f_x, \sigma_x\} \) are bounded uniformly on \( [0, T] \times \mathbb{R}^n \times A^{(N)} \);

**C3** The maps \( \{\xi, \varphi\} \) are once continuously differentiable with respect to \( x \in \mathbb{R}^n \), and there exists a \( K > 0 \) such that

\[
\begin{align*}
  \left(1 + |x|^2 \right)^{-1} |\xi(t, x, u)|_R + \left(1 + |x|_R \right)^{-1} |\xi_x(t, x, u)|_R^n \\
  \left(1 + |x|^2 \right)^{-1} |\varphi(x)|_R + \left(1 + |x|_R \right)^{-1} |\varphi_x(x)|_R^n &\leq K;
\end{align*}
\]

**C4** The map \( h^i \) is once continuously differentiable with respect to \( x \in \mathbb{R}^n \), and \( \{h^i, h^i_x\} \) are bounded uniformly on \( [0, T] \times \mathbb{R}^n \times A^{(N)} \), for \( i = 1, 2, \ldots, N \);

**C5** \( D^{i, \frac{1}{2}} \) is uniformly bounded, the inverse \( D^{i, -\frac{1}{2}} \) exists and it is uniformly bounded, for \( i = 1, 2, \ldots, N \).
4.1 Optimality Conditions Under Reference Probability Space 
\( (\Omega, F, \mathbb{P}) \)

Next, we state and prove the optimality conditions by utilizing the augmented system (23) and reward (24), under the reference probability space \( (\Omega, F, \{\mathbb{P}_t : t \in [0, T]\}) \).

We define the Gateaux derivative of \( G \) with respect to the variable at the point \((t, z, \nu) \in [0, T] \times \mathbb{R}^{n+1} \times \mathcal{M}_1(\mathbb{A})\) in the direction \( \eta_1 \in \mathbb{R}^{n+1} \) by

\[
G_X(t, z, \nu; \eta_1) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ G(t, z + \varepsilon \eta_1, \nu) - G(t, z, \nu) \right\}, \quad t \in [0, T].
\]

Clearly, for each column of \( G \) denoted by \( G^{(j)}, j = 1, \ldots, m + k \), the Gateaux derivative of \( G^{(j)} \) component wise is given by \( G^{(j)}(t, z, \nu; \eta_1) = G^{(j)}(t, z, \nu) \eta_1, t \in [0, T] \).

In order to present the necessary conditions of optimality we need the so called variational equation. Suppose \( u^o \triangleq (u^{1,o}, u^{2,o}, \ldots, u^{N,o}) \in \mathcal{U}_{rel}[0, T] \) denotes the optimal decision and \( u \triangleq (u^1, u^2, \ldots, u^N) \in \mathcal{U}_{rel}[0, T] \) any other decision. Since \( \mathcal{U}_{rel}[0, T] \) is convex \( \forall \varepsilon \in \mathbb{Z}_N \), it is clear that for any \( \varepsilon \in [0, 1] \),

\[
u_{t}^i \varepsilon \triangleq u_{t}^{i,o} + \varepsilon (u_{t}^{i} - u_{t}^{i,o}) \in \mathcal{U}_{rel}[0, T], \quad \forall i \in \mathbb{Z}_N.
\]

Let \( X^\varepsilon(\cdot) \equiv X^\varepsilon(\cdot; u^\varepsilon(\cdot)) \) and \( X^o(\cdot) \equiv X^o(\cdot; u^o(\cdot)) \in B^\infty_{\mathcal{F}_T}([0, T], L^2(\Omega, \mathbb{R}^{n+1})) \) denote the solutions of the system equation (23) corresponding to \( u^\varepsilon(\cdot) \) and \( u^o(\cdot) \), respectively. Consider the limit

\[
Z(t) \triangleq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ X^\varepsilon(t) - X^o(t) \right\}, \quad t \in [0, T].
\]

We have the following result characterizing the variational process \( \{Z(t) : t \in [0, T]\} \).

**Lemma 2.** Suppose Assumptions \( \square \) hold. The process \( \{Z(t) : t \in [0, T]\} \) is an element of the Banach space \( B^\infty_{\mathcal{F}_T}([0, T], L^2(\Omega, \mathbb{R}^{n+1})) \) and it is the unique solution of the variational stochastic differential equation

\[
dZ(t) = F_X(t, X^o(t), u^o(t))Z(t)dt + G_X(t, X^o(t), u^o(t); Z(t))d\mathcal{W}(t)
+ F(t, X^o(t), u_t - u^o_t)dt + G(t, X^o(t), u_t - u^o_t)d\mathcal{W}(t), \quad Z(0) = 0. \tag{86}
\]

where

\[
F(t, X, u - u^o) \triangleq \sum_{i=1}^{N} F(t, X^o, u^{-i,o} - u^i - u^{i,o}), \quad G(t, X^o, u - u^o) \triangleq \sum_{i=1}^{N} G(t, X^o, u^{-i,o}, u^i - u^{i,o}),
\]

having a continuous modification.

**Proof.** The derivation utilizes the statements of Lemma \( \square \). Note that for \( t \in (0, T] \), (86) is a linear stochastic differential equation. Define the component vectors of (86) by

\[
Z \triangleq \text{Vector}\{Z^1, Z^2\}.
\]
Then

\[dZ^1(t) = Z^1(t)h^*(t, x^o(t), u^o_t)D^{-1}(t)dy(t) + \Lambda^o(t)h^o(t, x^o(t), u^o_t, Z^2(t))D^{-1}(t)dy(t) + \Lambda^o(t)h^o(t, x^o(t), u_t - u^o_t)D^{-1}(t)dy(t), \quad Z^1(0) = 0.\] (87)

\[dZ^2(t) = f_x(t, x^o(t), u^o_t)Z^2(t)dt + \sigma_x(t, x^o(t), u^o_t; Z^2(t))dW(t) + f(t, x^o(t), u_t - u^o_t)dt + \sigma(t, x^o(t), u_t - u^o_t)dW(t), \quad Z^2(0) = 0.\] (88)

Let \(\{Z_h^2(t) : t \in [0, T]\}\) denote the homogenous part of (88) given by

\[dZ_h^2(t) = f_x(t, x^o(t), u^o_t)Z_h^2(t)dt + \sigma_x(t, x^o(t), u^o_t; Z^2(t))dW(t), \quad Z_h^2(s) = \zeta^2, \quad t \in [s, T].\] (89)

By Assumptions 4 and Lemma 1 there is a unique solution \(\{Z_h^2(t) : t \in [s, T]\}\) given by

\[Z_h^2(t) = \Psi^2(t, s)\zeta^2, \quad t \in [s, T],\]

where \(\Psi^2(t, s), t \in [s, T]\) is the random \(\{\mathbb{P}_0,t : t \in [0, T]\}\)–adapted transition operator for the homogenous system. Since the derivatives of \(f\) and \(\sigma\) with respect to the state are uniformly bounded, the transition operator \(\Psi^2(t, s), t \in [s, T]\) is uniformly \(\mathbb{P}\)–a.s. bounded (with values in the space of \(n \times n\) matrices).

Consider now the non homogenous stochastic differential equation (88), then its solution is given by

\[Z^2(t) = \int_0^t \Psi^2(t, s)d\eta^2(s), \quad t \in [0, T],\] (90)

where \(\{\eta^2(t) : t \in [0, T]\}\) is the semi martingale given by the following stochastic differential equation

\[d\eta^2(t) = f(t, x^o(t), u_t - u^o_t)dt + \sigma(t, x^o(t), u_t - u^o_t)dW(t), \quad \eta^2(0) = 0, \quad t \in (0, T].\] (91)

Note that \(\{\eta^2(t) : t \in [0, T]\}\) is a continuous square integrable \(\{\mathbb{P}_0,t : t \in [0, T]\}\)–adapted semi martingale. The fact that it has continuous modification follows directly from the representation (90) and the continuity of the semi martingale \(\{\eta^2(t) : t \in [0, T]\}\).

Similarly, let \(\{Z_h^1(t) : t \in [0, T]\}\) denote the homogenous part of (87) given by

\[dZ_h^1(t) = Z_h^1(t)h^*(t, x^o(t), u^o_t)D^{-1}(t)dy(t), \quad Z_h^1(s) = \zeta^1, \quad t \in [s, T].\] (92)

By Assumptions 4 and Lemma 1 there is a unique solution \(\{Z_h^1(t) : t \in [s, T]\}\) given by

\[Z_h^1(t) = \Psi^1(t, s)\zeta, \quad t \in [s, T],\]

where \(\Psi^1(t, s), t \in [s, T]\) is the random \(\{\mathbb{P}_0,t : t \in [0, T]\}\)–adapted transition operator for the homogenous system. Since \(h\) is uniformly bounded, the transition operator \(\Psi^1(t, s), t \in [s, T]\) is uniformly \(\mathbb{P}\)–a.s. bounded.
Consider now the non homogenous stochastic differential equation \((87)\), then its solution is given by

\[
Z^1(t) = \int_0^t \Psi^1(t, s) d\eta^1(s), \quad t \in [0, T],
\]  

(93)

where \(\{\eta^1(t) : t \in [0, T]\}\) is the martingale given by the following stochastic differential equation

\[
d\eta^1(t) = \Lambda^o(t) h^*_x(t, x^o(t), u^o; Z^2(t)) D^{-1}(t) dy(t)
+ \Lambda^o(t) h^*_x(t, x^o(t), u_t - u^o_t) D^{-1}(t) dy(t), \quad \eta^1(0) = 0, \quad t \in (0, T].
\]  

(94)

Note that

\[
E[\eta^1(t)]^2 \leq 2 E \int_0^t |\Lambda^o(t)|^2 |D^{-1/2}(s) h_x(s, x^o(s), u^o_s)^2_{\mathbb{R}^n}| Z^2(s)^2_{\mathbb{R}^n} ds
+ 2 E \int_0^t |\Lambda^o(t)|^2 |D^{-1/2}(s) h(s, x^o(s), u_t - u^o_t)^2_{\mathbb{R}^n} ds,
\]

\[
\leq 2C_1 E \int_0^t |\Lambda^o(t)|^2 |Z^2(s)|_{\mathbb{R}^n}^2 ds + 2C_2 E \int_0^t |\Lambda^o(t)|^2 ds,
\]  

(95)

where the last inequality follows from Assumptions (C4), (C5). Since by Lemma (1) \(\Lambda^o \in L^\infty(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})\), and by the previous calculations \(Z^2 \in B^\infty_{\mathcal{F}_T}([0, T], L^2(\Omega, \mathbb{R}^n))\), then from (95) we deduce that \(\eta^1 \in B^\infty_{\mathcal{F}_T}([0, T], L^2(\Omega, \mathbb{R}))\), and \(\{\eta^1(t) : t \in [0, T]\}\) is a continuous square integrable \(\{\mathcal{F}_{0,t} : t \in [0, T]\}\)—adapted martingale. By invoking (93) we also obtain that \(Z^1 \in B^\infty_{\mathcal{F}_T}([0, T], L^2(\Omega, \mathbb{R}))\).

Putting together \(Z^1 = Vector\{Z^1, Z^2\}, \eta^1 = Vector\{\eta^1, \eta^2\}\) then for \(t \in (0, T]\), the homogenous part of \((86)\) is a linear stochastic differential equation satisfying

\[
dZ^1(t) = F_X(t, X^o(t), u^o_t) Z^1(t) dt + G_X(t, X^o(t), u^o_t) d\overline{W}(t), \quad Z^1(s) = \zeta, \quad t \in [s, T],
\]  

(96)

and by the properties of \(\{Z_1^1(t), Z_2^2(t) : t \in [s, T]\}\) there is a unique solution \(\{Z(t) : t \in [s, T]\}\) given by

\[
Z(t) = \Psi(t, s) \zeta, \quad t \in [s, T],
\]

where \(\Psi(t, s)\) constructed from \(\{\Psi^1(t, s), \Psi^2(t, s)\}, t \in [s, T]\) is the random \(\{\mathcal{F}_{0,t} : t \in [0, T]\}\)—adapted transition operator for the homogenous system of \((86)\).

The solution of the non homogenous stochastic differential equation \((86)\), is given by

\[
Z(t) = \int_0^t \Psi(t, s) d\eta(s), \quad t \in [0, T],
\]  

(97)

where \(\{\eta(t) : t \in [0, T]\}\) is the semi martingale given by the following stochastic differential equation

\[
d\eta(t) = F(t, X^o(t), u_t - u^o_t) dt
+ G(t, X^o(t), u_t - u^o_t) d\overline{W}(t), \quad \eta(0) = 0, \quad t \in (0, T].
\]  

(98)
By the properties of \( \{\eta^1, \eta^2\} \) then \( \{\eta(t) \equiv Vector\{\eta^1(t), \eta^2(t)\} : t \in [0, T]\} \) is a continuous square integrable \( \{\mathbb{F}_{0,t} : t \in [0, T]\} \)-adapted semi martingale. The fact that it has continuous modification follows directly from the representation (9) and the continuity of martingale and semi martingale \( \{\eta^1(t), \eta^2(t) : t \in [0, T]\} \).

By constructing the difference \( \tilde{Z}^\epsilon(t) \triangleq \frac{1}{\epsilon} \left( X^{\epsilon}(t) - X^\alpha(t) \right) - Z(t) \) and then utilizing Assumptions [3] and Lemma [11] it can be shown that in the limit, as \( \epsilon \rightarrow 0 \), \( \tilde{Z}^\epsilon \) converges to zero in \( B_{\mathbb{F}_{T}}^\infty([0, T], L^2(\Omega, \mathbb{R}^{n+1})) \).

\[ \square \]

**Remark 2.** Note that

\[
G_X(t, X, u; Z) \, d\overline{W} \equiv \sum_{j=1}^{k+m} G_X^{(j)}(t, X, u) Z d\overline{W}_j, \\
G(t, X, u - \bar{u}) \, d\overline{W} \equiv \sum_{j=1}^{k+m} G^{(j)}(t, X, u - \bar{u}) d\overline{W}_j.
\]

where \( G^{(j)} \) is the \( j \)-th column of \( G \), and \( G_X^{(j)} \) is the derivatives of \( G^{(j)} \) with respect to \( X \), \( j = 1, 2, \ldots, k + m \).

Before we give the main theorems we introduce the augmented Hamiltonian system of equations corresponding to (23) and (24).

Define the Hamiltonian of the augmented system

\[
\mathcal{H}_{rel} : [0, T] \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathcal{L}(\mathbb{R}^{m+k}, \mathbb{R}^{n+1}) \times \mathcal{M}_1(A^{(N)}) \rightarrow \mathbb{R}
\]

\[
\mathcal{H}_{rel}(t, X, \Psi, Q, u) \triangleq \langle F(t, X, u, \Psi) + tr\left(Q^*G(t, X, u)\right) + L(t, X, u), \quad t \in [0, T].
\]

(99)

For any \( u \in \mathcal{U}_{rel}^{(N)}[0, T] \), the adjoint process \( (\Psi, Q) \in L_{\mathbb{F}_{T}}^2([0, T], \mathbb{R}^{n+1}) \times L_{\mathbb{F}_{T}}^2([0, T], \mathcal{L}(\mathbb{R}^{m+k}, \mathbb{R}^{n+1})) \) satisfies the following backward stochastic differential equations

\[
d\Psi(t) = -F_X(t, X(t), u_t) \Psi(t) dt - V_Q(t) dt - L_X(t, X(t), u_t) dt + Q(t) d\overline{W}(t), \quad t \in [0, T), \\
= -\mathcal{H}_{rel}^c(t, X(t), \Psi(t), Q(t), u_t) dt + Q(t) d\overline{W}(t), \quad \Psi(T) = \Phi_X(X(T)),
\]

(100)

where \( V_Q \in L_{\mathbb{F}_{T}}^2([0, T], \mathbb{R}^{n+1}) \) is given by

\[
\langle V_Q(t), \zeta \rangle = tr\left(Q^*(t)G_X(t, X(t), u_t; \zeta)\right), \quad t \in [0, T].
\]

Clearly,

\[
V_Q(t) = \sum_{j=1}^{k+m} \left(G_X^{(j)}(t, X(t), u_t)\right)^* Q^{(j)}(t), \quad t \in [0, T].
\]

(101)
The state process satisfies the stochastic differential equation \( \text{(23)} \) expressed in terms of the Hamiltonian as follows:

\[
\frac{dX(t)}{dt} = \mathcal{H}^{rel}_\Psi(t, X(t), \Psi(t), Q(t), u_t)dt + G(t, X(t), u_t)d\overline{W}(t), \quad X(0) = X_0, \quad t \in (0, T].
\]

Next, we state the first set of optimality conditions for an element \( u^o \in \mathcal{U}^{(N)}_{rel}[0, T] \) with the corresponding augmented solution \( X^o \equiv (\Lambda^o, x^o) \) to be team optimal.

**Theorem 4. (Team Optimality Necessary Conditions under Reference Measure \( \mathbb{P} \))**

**Necessary Conditions.** For an element \( u^o \in \mathcal{U}^{(N)}_{rel}[0, T] \) with the corresponding solution \( X^o \in \mathcal{B}^\infty_{\mathbb{P}}([0, T], L^2(\Omega, \mathbb{R}^{n+1})) \) to be team optimal, it is necessary that the following hold.

1. There exists a semi-martingale \( m^o \in \mathcal{SM}_0^2([0, T], \mathbb{R}^{n+1}) \) with the intensity process \( (\Psi^o, Q^o) \in \mathcal{L}_0^2([0, T], \mathbb{R}^{n+1}) \times \mathcal{L}_0^2([0, T], \mathcal{L}(\mathbb{R}^{m+k}, \mathbb{R}^{n+1})) \).

2. The variational inequality is satisfied:

\[
\sum_{i=1}^N \mathbb{E}\left\{ \int_0^T \mathcal{H}^{rel}(t, X^o(t), \Psi^o(t), Q^o(t), u_t^{i-o}, u_t^{i-o})dt \right\} \geq 0, \quad \forall u \in \mathcal{U}^{(N)}_{rel}[0, T].
\]

Moreover, \( \text{(103)} \) holds if and only if the following variation inequality holds.

\[
\mathbb{E}\left\{ \int_0^T \mathcal{H}^{rel}(t, X^o(t), \Psi^o(t), Q^o(t), u_t^{i-o}, u_t^{i-o})dt \right\} \geq 0, \quad \forall u^i \in \mathcal{U}^i_{rel}[0, T], i = 1, 2, \ldots, N.
\]

(3) The process \( (\Psi^o, Q^o) \in \mathcal{L}_0^2([0, T], \mathbb{R}^{n+1}) \times \mathcal{L}_0^2([0, T], \mathcal{L}(\mathbb{R}^{m+k}, \mathbb{R}^{n+1})) \) is a unique solution of the backward stochastic differential equation \( \text{(100)} \) such that \( u^o \in \mathcal{U}^{(N)}_{rel}[0, T] \) satisfies the point wise almost sure inequalities with respect to the \( \mathcal{P} \)-measurables \( \mathcal{Q}_{t,N}^\Psi \subset \mathcal{P} \), \( t \in [0, T], i = 1, 2, \ldots, N : \)

\[
\mathbb{E}\left\{ \mathcal{H}^{rel}(t, X^o(t), \Psi^o(t), Q^o(t), u_t^{i-o}, \nu^i)|\mathcal{Q}_{0,t}^\Psi \right\} \geq \mathbb{E}\left\{ \mathcal{H}^{rel}(t, X^o(t), \Psi^o(t), Q^o(t), u_t^{i-o})|\mathcal{Q}_{0,t}^\Psi \right\},
\]

\[\forall \nu^i \in \mathcal{M}_1(\mathcal{P}^i), a.e.t \in [0, T], \mathbb{P}|_{\mathcal{Q}_{0,t}^\Psi} - a.s., i = 1, 2, \ldots, N \]

**Proof.** (1). The derivation utilizes the variational equation of Lemma \( \mathbb{L} \) and the semi-martingale representation stated under Theorem \( \mathbb{3} \). We describe the initial steps. Suppose \( u^o \in \mathcal{U}^{(N)}_{rel}[0, T] \) is an optimal team decision and \( u \in \mathcal{U}^{(N)}_{rel}[0, T] \) any other admissible decision. Since \( \mathcal{U}^{(N)}_{rel}[0, T] \) is convex \( \forall i \in \mathbb{Z}_N \), we have, for any \( \varepsilon \in [0, 1] \), \( u_t^{i,\varepsilon} \triangleq u_t^{i-o} + \varepsilon(u_t^{i-o} - u_t^{i-o}) \in \mathcal{U}^{(N)}_{rel}[0, T], \forall i \in \mathbb{Z}_N \). Let \( X^{\varepsilon}(\cdot) \equiv X^{\varepsilon}(\cdot; u^{\varepsilon}(\cdot)), X^o(\cdot) \equiv X^o(\cdot; u^o(\cdot)) \in \mathcal{B}^\infty_{\mathbb{P}}([0, T], L^2(\Omega, \mathbb{R}^{n+1})) \)
denote the solutions of the stochastic system \([23]\) corresponding to \(u^r(\cdot)\) and \(u^o(\cdot)\), respectively. Since \(u^o(\cdot) \in \mathbb{U}_{rel}^{(N)}[0, T]\) is optimal it is clear that
\[
J(u^o) - J(u^o) \geq 0, \quad \forall \varepsilon \in [0, 1], \quad \forall u \in \mathbb{U}_{rel}^{(N)}[0, T].
\] (106)
Define the Gateaux differential of \(J\) at \(u^o\) in the direction \(u - u^o\) by
\[
dJ(u^o, u - u^o) \triangleq \lim_{\varepsilon \to 0} \frac{J(u^o) - J(u^o)}{\varepsilon} \equiv \frac{d}{d\varepsilon} J(u^o)|_{\varepsilon=0}.
\]

It can be shown that
\[
dJ(u^o, u - u^o) = \mathcal{L}(Z) + \sum_{i=1}^{N} \mathbb{E} \int_0^T L(t, X^o(t), u_t^{-i,o}, u_t^i - u_t^{i,o}) dt \geq 0, \quad \forall u \in \mathbb{U}_{rel}^{(N)}[0, T]. \tag{107}
\]
where \(\mathcal{L}(Z)\) is given by the functional
\[
\mathcal{L}(Z) = \mathbb{E} \left\{ \int_0^T L_X(t, X^o(t), u^o_t, Z(t)) dt + \langle \Phi_X(X^o(T)), Z(T) \rangle \right\}. \tag{108}
\]
Under measure \(\mathbb{P}\), the filtration \(\{\mathbb{F}_{0,t} : t \in [0, T]\}\) is generated by the initial state \(X(0)\), and the augmented Brownian motion vector \(\{(W(t), B(t)) : t \in [0, T]\}\), and by Lemma 2, the process \(Z(\cdot) \in B_{\mathbb{F}_T}^\infty([0, T], L^2(\Omega, \mathbb{R}^{n+1}))\) and it is continuous \(\mathbb{P}\)-a.s.

Hence, by Assumptions \(\text{H}\) it follows that \(Z \to \mathcal{L}(Z)\) is a continuous linear functional. Further, by Lemma 2 \(\eta \to Z\) is a continuous linear map from the Hilbert space \(\mathcal{S}M_2^2(0, T)\) to the space \(B_{\mathbb{F}_T}^\infty([0, T], L^2(\Omega, \mathbb{R}^{n+1}))\) given by the expression \([97]\). Thus the composition map \(\eta \to Z \to \mathcal{L}(Z) \equiv \tilde{\mathcal{L}}(\eta)\) is a continuous linear functional on \(\mathcal{S}M_2^2(0, T)\). Then by virtue of Riesz representation theorem for Hilbert spaces, there exists a semi martingale \(m^o \in \mathcal{S}M_0^2([0, T], \mathbb{R}^{n+1})\) with intensity \((\psi^o, Q^o) \in L_{\mathbb{F}_T}^2([0, T], \mathbb{R}^{n+1}) \times L_{\mathbb{F}_T}^2([0, T], \mathcal{L}(\mathbb{R}^{m+k}, \mathbb{R}^{n+1}))\) such that
\[
\mathcal{L}(Z) \triangleq \tilde{\mathcal{L}}(\eta) = (m^o, \eta)_{\mathcal{S}M_0^2([0, T], \mathbb{R}^{n+1})} = \sum_{i=1}^{N} \mathbb{E} \int_0^T \langle \Psi^o(t), F(t, X^o(t), u_t^{-i,o}, u_t^i - u_t^{i,o}) \rangle dt
\]
\[+
\sum_{i=1}^{N} \mathbb{E} \int_0^T \text{tr} \left( Q^{o,s}(t) G(t, X^o(t), u_t^{-i,o}, u_t^i - u_t^{i,o}) \right) dt. \tag{109}
\]
This proves (1).

(2) Substituting (109) into (107) we obtain the following variational equation.
\[
dJ(u^o, u - u^o) = \sum_{i=1}^{N} \mathbb{E} \int_0^T \langle \Psi^o(t), F(t, X^o(t), u_t^{-i,o}, u_t^i - u_t^{i,o}) \rangle dt
\]
\[+
\sum_{i=1}^{N} \mathbb{E} \int_0^T \text{tr} \left( Q^{o,s}(t) G(t, X^o(t), u_t^{-i,o}, u_t^i - u_t^{i,o}) \right) dt
\]
\[+
\sum_{i=1}^{N} \mathbb{E} \int_0^T L(t, X^o(t), u_t^{-i,o}, u_t^i - u_t^{i,o}) dt \geq 0, \quad \forall u \in \mathbb{U}_{rel}^{(N)}[0, T]. \tag{110}
\]
By the definition of the Hamiltonian \((99)\), it follows that inequality \((110)\) is precisely \((103)\) along with the pair \(\{(\psi^o(t), Q^o(t)) : t \in [0, T]\}\). Next, we show that \((103)\) implies \((104)\). Define

\[
g^i(T, \omega) \triangleq \int_0^T \mathcal{H}^{\text{rel}}(t, X^o(t), \Psi^o(t), Q^o(t), u_t^{-i, o}, u_t^i - u_t^{i, o}) dt, \quad \forall u^i \in \mathcal{U}_i^{\text{rel}}[0, T], i = 0, 1, \ldots, N.
\]

(111)

Suppose for some \(i \in \mathbb{Z}_N\), \((104)\) does not hold, and let \(A^i(T) \triangleq \{\omega : g^i(T, \omega) < 0\}\). Let \(\nu^i\) be any vaguely \(\{G_{0,t}^i : t \in [0, T]\}\)–adapted \(\nu^i \in \mathcal{M}_i(A^i)\). We can choose \(u^i\) in \((103)\) as

\[
u^i \begin{cases}
\nu^i \text{ on } A^i(T) \\
u^i \text{ outside } A^i(T)
\end{cases}
\]

(112)

together with \(u^j = u_t^{j, o}, j \neq i, j \in \mathbb{Z}_N\). Substituting \((112)\) in \((103)\) we arrive at \(\int_{A^i(T)} g^i(T, \omega) d\mathbb{P}(\omega) \geq 0\), which contradicts the definition of \(A^i(T)\), unless \(A^i(T)\) has measure zero. Hence, \((103)\) implies \((104)\). On the other hand, it is clear that \((104)\) implies \((103)\). Thus, \((103)\) and \((104)\) are equivalent.

(3). Following the same steps in derivation found in \([44]\) for noiseless information structures, we verify that the process \((\Psi^o, Q^o) \in L^2_{\mathbb{P}}([0, T], \mathbb{R}^{n+1}) \times L^2_{\mathbb{P}}([0, T], \mathcal{L}(\mathbb{R}^{m+k} \times \mathbb{R}^{n+1}))\) is a unique solution of the backward stochastic differential equation \((100)\).

Next, we show \((103)\). By using the property of conditional expectation then

\[
\mathbb{E} \left\{ \int_0^T \mathbb{E} \left\{ \mathcal{H}^{\text{rel}}(t, X^o(t), \Psi^o(t), Q^o(t), u_t^{-i, o}, u_t^i - u_t^{i, o})|G_{0,t}^{g^i}\right\} dt \right\} \geq 0, \quad \forall u^i \in \mathcal{U}_i^{\text{rel}}[0, T], i \in \mathbb{Z}_N.
\]

(113)

Let \(t \in (0, T), \omega \in \Omega\), and \(\varepsilon > 0\), and consider the sets \(I^i_\varepsilon \equiv [t, t + \varepsilon] \subset [0, T]\) and \(\Omega^i_\varepsilon \subset \Omega\) in \(G_{0,t}^{g^i}\) containing \(\omega\) such that \(|I^i_\varepsilon| \to 0\) and \(\mathbb{P}(\Omega^i_\varepsilon) \to 0\) as \(\varepsilon \to 0\), for \(i = 1, 2, \ldots, N\). For any sub-sigma algebra \(G \subset \mathcal{F}\), let \(\mathbb{P}|_G\) denote the restriction of the probability measure \(\mathbb{P}\) on to the \(\sigma\)-algebra \(G\). For any (vaguely) \(G_{0,t}^{g^i}\)–adapted \(\nu^i \in \mathcal{M}_i(A^i)\), construct

\[
u^i = \begin{cases}
\nu^i & \text{for } (t, \omega) \in I^i_\varepsilon \times \Omega^i_\varepsilon \\
u^{i, o} & \text{otherwise}
\end{cases} \quad i = 1, 2, \ldots, N.
\]

(114)

Clearly, it follows from the above construction that \(u^i \in \mathcal{U}_i^{\text{rel}}[0, T]\). Substituting \((114)\) in \((113)\) we obtain the following inequality

\[
\int_{\Omega^i_\varepsilon \times I^i_\varepsilon} \mathbb{E} \left\{ \mathcal{H}^{\text{rel}}(t, X^o(t), \Psi^o(t), Q^o(t), u_t^{-i, o}, \nu^i - u_t^{i, o})|G_{0,t}^{g^i}\right\} dt \geq 0,
\]

\[
\forall \nu^i \in \mathcal{M}_i(A^i), \text{a.e.t } \in [0, T], \mathbb{P}|_{G_{0,t}^{g^i}} \text{ a.s.}, \quad i = 1, 2, \ldots, N.
\]

(115)
Letting $|I^i_\varepsilon|$ denote the Lebesgue measure of the set $I^i_\varepsilon$ and dividing the above expression by the product measure $\mathbb{P}(\Omega^i_\varepsilon)|I^i_\varepsilon|$ and letting $\varepsilon \to 0$ we arrive at the following inequality.

\[
\mathbb{E}\left\{ \mathcal{H}^{rel}(t, X^{o}(t), \Psi^{o}(t), Q^{o}(t), u_t^{-i,o}, \nu^i)|G_{0,t}^{\varepsilon} \right\} \geq \mathbb{E}\left\{ \mathcal{H}^{rel}(t, X^{o}(t), \Psi^{o}(t), Q^{o}(t), u_t^{-i,o}, u_t^{i,o})|G_{0,t}^{\varepsilon} \right\},
\]

$\forall \nu^i \in \mathcal{M}_1(\mathbb{R}^q)$, a.e. $t \in [0,T]$, $\mathbb{P}|_{G_{0,t}^{\varepsilon}}$-a.s., $i = 1, 2, \ldots, N$. (116)

This completes the proof of (3). \hfill \square

The following remark can be used to identify the martingale part of the adjoint process.

**Remark 3.** Suppose the additional assumptions hold: $f, \sigma, h, \ell, \varphi$ are twice continuously differentiable and uniformly bounded. By an application of the Riesz representation theorem for Hilbert space martingales, we identify, the martingale term of the adjoint process $M^i_t = \int^t_0 \Psi^{o}_X(s)G(s, X^{o}(s))dW(s)$, dual to the first martingale term in the variational equation (86). Hence, $Q$ in the adjoint equation, is identified by $Q(t) \equiv \Psi_X(t)G(t, X(t), u_t)$.

**Alternative Hamiltonian System on** $\left(\Omega, \mathbb{F}, \{\mathbb{F}_t : t \in [0,T]\}, \mathbb{P}\right)$

An alternative representation of the Hamiltonian system of equations is obtained as follows. Define the following quantities:

\[
\Psi \triangleq \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}, \quad Q \triangleq \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix},
\]

$\Psi_1 \in \mathbb{R}$, $\Psi_2 \in \mathbb{R}^n$, $q_{11} \in \mathcal{L}(\mathbb{R}^k, \mathbb{R})$, $q_{12} \in \mathcal{L}(\mathbb{R}^m, \mathbb{R})$, $q_{21} \in \mathcal{L}(\mathbb{R}^k, \mathbb{R}^n)$, $q_{22} \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$.

Then an equivalent Hamiltonian system of equations under the reference measure $\mathbb{P}$ is given below. The Hamiltonian is equivalently expressed as

\[
\mathcal{H}^{rel}(t, X(t), \Psi(t), Q(t), u_t) = \langle f(t, x(t), u_t), \Psi_2(t) \rangle + \Lambda(t) \ell(t, x(t), u_t)
\]

\[
+ \text{tr}\left(q_{22}^{*}(t)\sigma(t, x(t), u_t)\right) + \Lambda(t) \text{tr}\left(q_{11}^{*}h^{*}(t, x(t), u_t)\right)\right}\} \quad \text{(117)}
\]

\[
\equiv \mathcal{H}^{rel}(t, x(t), \Lambda(t), \Psi_2(t), q_{11}(t), q_{22}(t), u_t), \quad \text{(118)}
\]

the adjoint processes $\{\Psi_1, \Psi_2, q_{11}, q_{21}, q_{12}, q_{22}\}$ satisfy the equations

\[
d\Psi_1(t) = -\ell(t, x(t), u_t)dt + q_{12}(t)dW(t)
\]

\[
+ q_{11}(t)\left(dy(t) - h(t, x(t), u_t)dt\right), \quad \Psi_1(T) = \varphi(x(t)), \quad t \in [0,T), \quad \text{(119)}
\]

\[
d\Psi_2(t) = -f_x^{*}(t, x(t), u_t)\Psi_2(t)dt - \Lambda(t)\ell_x(t, x(t), u_t)dt + q_{22}(t)dW(t)
\]

\[
- \Lambda(t)V_{q_{21}}(t)dt - V_{q_{22}}(t)dt + q_{21}(t)dy(t), \quad \Psi_2(T) = \Lambda(T)\varphi_x(x(T)), \quad t \in [0,T), \quad \text{(120)}
\]

\footnote{We redefine $q_{11} \equiv q_{11}D^{-\frac{1}{2}}, q_{21} \equiv q_{21}D^{-\frac{1}{2}}$ without changing the notation.}
where $V_{q_{11}}, V_{q_{22}}$ are given by

\begin{align}
\langle V_{q_{11}}(t), \zeta \rangle &= \text{tr} (q_{11}^*(t) h_x(t, x(t), u_t; \zeta)), \quad t \in [0, T], \quad (121) \\
\langle V_{q_{22}}(t), \zeta \rangle &= \text{tr} (q_{22}^*(t) \sigma(t, x(t), u_t; \zeta)), \quad t \in [0, T], \quad (122)
\end{align}

the state equations are given by

\begin{align}
&d \Lambda(t) = \Lambda(t) h_x^*(t, x(t), u_t) D^{-1}(t) dy(t), \quad \Lambda(0) = 1, \quad t \in (0, T], \quad (123) \\
&dx(t) = f(t, x(t), u_t) dt + \sigma(t, x(t), u_t) dW(t), \quad x(0) = x_0, \quad t \in (0, T], \quad (124)
\end{align}

and the conditional variational Hamiltonian (equivalent of (105)) is given by

\[
E\left\{ H^{rel}(t, x^o(t), \Lambda^o(t), \Psi^o(t), Q^o(t), u_t^{i, o}, u_t^{i, o} - u_t^{i, o}) \mid G_{0,t} \right\} \geq 0, \\
\forall u^i \in \mathcal{M}_1(\mathbb{A}^1), \quad a.e. t \in [0, T], \quad P|_{G_{0,t}} - a.s., \quad i = 1, 2, \ldots, N. \quad (125)
\]

The previous system of equations (117)-(125) describe the maximum principle under the reference probability measure $P$.

Next, we deduce as expected that the optimality conditions (necessary) for a $u^o \in U_{rel}^{(N)}[0, T]$ to be a PbP optimal can be derived following the procedure of Theorem $4$ and that these conditions are the same to the conditions of team optimality, with a minor difference. These results are stated as a Corollary.

**Corollary 1. (PbP Necessary Conditions Under Reference Measure $P$)**

Consider Problem $2$ under Assumptions $4$.

**Necessary Conditions.** For an element $u^o \in U_{rel}^{(N)}[0, T]$ with the corresponding solution $x^o \in B_{\mathcal{F}_T}([0, T], L^2(\Omega, \mathbb{R}^{n+1}))$ to be a PbP optimal strategy, it is necessary that the statements of Theorem $4$ (1), (3) hold and statement (2) is replaced by

\[(2') \quad \text{The variational inequalities are satisfied:}
\]

\[
E \int_0^T H^{rel}(t, X^o(t), \Psi^o(t), Q^o(t), u_t^{i, o}, u_t^{i, o} - u_t^{i, o}) dt \geq 0, \quad \forall u^i \in U_{rel}[0, T], \quad \forall i \in \mathbb{Z}_N. \quad (126)
\]

**Proof.** The derivation is based on the procedure of Theorem $4$, but we only vary in the direction $u_t^i - u_t^{i, o}$, while the rest of the strategies are optimal, $u_t^j = u_t^{j, o}$.

Clearly, every team optimal strategy for Problem $1$ is a PbP optimal strategy for Problem $2$, hence PbP optimality is a weaker notion than team optimality as expected. By comparing the statements of Theorem $4$ and Corollary $4$, it is clear that the necessary conditions for team optimality and PbP optimality are equivalent.
Remark 4. (Centralized Information Structures)
From Theorem 4 one can deduce the optimality conditions of the classical centralized partially
observable control problems, that is, when at each \( t \in [0, T] \), \( u_t \) is a stochastic kernel measurable
with respect to the centralized noisy information \( G_{0,t}^I \subseteq G_{0,t}^y \). The necessary conditions
for such a \( u^* \) to be optimal are

\[
\mathbb{E}\left\{ \mathcal{H}^{rel}(t, X^o(t), \Psi^o(t), Q^o(t), u^o) \big| G_{0,t}^I \right\} \geq \mathbb{E}\left\{ \mathcal{H}(t, X^o(t), \Psi(t), Q(t), u^o) \big| G_{0,t}^I \right\},
\]

\[
\forall u \in \mathcal{M}_1(\mathbb{A}(N)), a.e.t \in [0, T], \mathbb{P}|_{G_{0,t}^I} - a.s.,
\]

(127)

where \( \{X^o(t), \Psi^o(t), Q^o(t) : t \in [0, T]\} \) are the solutions of (100), (102).
For centralized information structure \( G_{0,t}^I = G_{0,t}^y \) a maximum principle is derive in [54,55,57], and for risk-sensitive pay-offs in [42].

4.2 Optimality Conditions Under Original Probability Space- \((\Omega, \mathcal{F}, \mathbb{P}^u)\)

Next, we prepare to express the optimality conditions of Theorem 11 and Corollary 11 with
respect to the original probability space \((\Omega, \mathcal{F}, \{\mathbb{F}_t : t \in [0, T]\}, \mathbb{P}^u)\), starting with the
explicit representation of the optimality conditions under the reference measure \( \mathbb{P} \), described by (117)-(125).

Define

\[
\psi \overset{\Delta}{=} \Lambda^{-1}\Psi_2, \quad \bar{q}_{21} \overset{\Delta}{=} \Lambda^{-1}\left(q_{21} - \Psi_2 h^* D^{-1}\right), \quad \bar{q}_{21} \overset{\Delta}{=} \Lambda^{-1}q_{21}, \quad \bar{q}_{22} = \Lambda^{-1}q_{22}.
\]

(128)

Utilizing (128) the Hamiltonian (117) is given by

\[
\mathcal{H}^{rel}(t, X(t), \Psi(t), Q(t), u_t) = \Lambda^u(t)\left\{ \langle f(t, x(t), u_t), \psi(t) \rangle + \ell(t, x(t), u_t)
+ tr\left(\bar{q}_{22}(t)\sigma(t, x(t), u_t)\right) + tr\left(q_{11}^* h^*(t, x(t), u_t)\right) \right\}.
\]

(129)

Since the right hand side of (129) is multiplied by the Radon-Nikodym derivative \( \Lambda^u = \frac{d\mathbb{P}^u}{d\mathbb{P}} \),
then as we have done in the previous subsection, we can express the Hamiltonian system of
equations under the original probability \( \mathbb{P}^u \), using the fact that under the original probability
measure \( \mathbb{P}^u \), the process \( B^u(t) = \int_0^t D^{-\frac{1}{2}}(s)\left(dy(s) - h(s, x(s))\right)ds \) is an \( \{\mathbb{F}_t : t \in [0, T]\} \)
standard Brownian motion.

Define the alternative Hamiltonian

\[
\mathbb{H}^{rel} : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{L}(\mathbb{R}^k, \mathbb{R}) \times \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) \times \mathcal{M}_1(\mathbb{A}(N)) \longrightarrow \mathbb{R}
\]

by

\[
\mathbb{H}^{rel}(t, x, \psi, q_{11}, \bar{q}_{22}, u) \overset{\Delta}{=} \langle f(t, x, u), \psi \rangle + \ell(t, x, u) + tr\left(\bar{q}_{22}^* \sigma(t, x, u)\right) + tr\left(q_{11}^* h^*(t, x, u)\right).
\]

(130)
By invoking the Itô differential rule to $\psi(\cdot) \triangleq \Lambda^{-1}(\cdot)\Psi_2(\cdot)$ we can derive the backward stochastic differential equation for $\psi(\cdot)$. Under the original probability measure $\left(\Omega, \mathcal{F}, \{\mathbb{F}_0,t : t \in [0,T]\}, \mathbb{P}^u\right)$, the minimum principle is given is given in terms of the new Hamiltonian (130) as follows.

The adjoint processes $\{\psi, q_{11}, q_{12}, q_{21}, \bar{q}_{22}\}$ ($\bar{k}_{21}$ is not included because it is redundant) satisfy the Backward stochastic differential equations

$$
\begin{align*}
\text{dx}(t) &= -\ell(t, x(t), u_t)dt + q_{12}(t)dW(t) + q_{11}(t)D^2\psi(t)dB^u(t), \quad \Psi_1(T) = \varphi(x(T)), \quad t \in [0, T),
\end{align*}
$$

the state process satisfies the forward stochastic differential equation

$$
\begin{align*}
\text{dx}(t) &= -f_x(t, x(t), u_t)\psi(t)dt - \ell_x(t, x(t), u_t)dt - V_{q_{22}}(t)dt - V_{q_{11}}(t)dt \\
&\quad+ \bar{q}_{22}(t)dW(t) + \bar{k}_{21}(t)D^2\psi(t)dB^u(t),
\end{align*}
$$

Hence, under the original probability measure $\mathbb{P}^u$ the conditional variational Hamiltonian is given by

$$
\begin{align*}
\mathbb{E}^{u^o}\left\{ \mathbb{H}^{rel}(t, x^o(t), \psi^o(t), q_{11}^o(t), q_{22}^o(t), u_t^{i,o}, \nu^i)\mid \mathcal{G}_{0,t}^{y^i} \right\} &\geq \mathbb{E}^{u^o}\left\{ \mathbb{H}^{rel}(t, x^o(t), \psi^o(t), q_{11}^o(t), q_{22}^o(t), u_t^i)\mid \mathcal{G}_{0,t}^{y^i} \right\}, \\
\forall \nu^i \in \mathcal{M}_1(A^t), \ a.e.t \in [0, T], \mathbb{P}^u\left|_{\mathcal{G}_{0,t}^{y^i}} \right. &- a.s., \ i = 1, 2, \ldots, N.
\end{align*}
$$

Thus, under the original probability measure $\left(\Omega, \mathcal{F}, \{\mathbb{F}_0,t : t \in [0,T]\}, \mathbb{P}^u\right)$, the adjoint processes are weakly defined with respect to the filtration $\{\mathbb{F}_0,t \equiv \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{F}_{0,t}^{W} \otimes \mathcal{F}_{0,t}^{y} : t \in [0,T]\}$. Consequently, we can choose to apply the optimality conditions either under measure $\mathbb{P}$ or $\mathbb{P}^u$.

We can now state the following necessary and sufficient conditions of optimality under the original probability space $\left(\Omega, \mathcal{F}, \{\mathbb{F}_0,t : t \in [0,T]\}, \mathbb{P}^u\right)$.

**Theorem 5.** (Team Optimality Conditions under Original Measure $\mathbb{P}^u$)

**Consider Problem 2 under Assumptions 4.**

**Necessary Conditions.** For an element $u^o \in U^{[N]}_{rel}[0,T]$ with the corresponding solution $x^o \in B^{\infty}_{\mathcal{F}_T}([0,T], L^2(\Omega, \mathbb{R}^n))$ to be team optimal, it is necessary that the following hold.

1. There exists a semi martingale $m^o \in \mathcal{SM}_0^2([0,T], \mathbb{R}^{n+1}) (n + 1)$ with the intensity process $\{(\Psi_0, \psi^o), \left[ q_{11}^0 \ q_{12}^0 \right] \} \in L^2_{\mathcal{F}_T}([0,T], \mathbb{R}^{n+1}) \times L^2_{\mathcal{F}_T}([0,T], \mathcal{L}(\mathbb{R}^{m+k}, \mathbb{R}^{n+1}))$.

2. The variational inequality is satisfied:
\[
\sum_{i=1}^{N} \mathbb{E}^{u^o} \left\{ \int_0^T \mathbb{H}^{rel}(t, x^o(t), \psi^o(t), q_{11}(t), \tilde{q}_{22}, u_{t}^{i,o} - u_{t}^{i,o}) dt \right\} \geq 0, \; \forall u \in \mathbb{U}^{(N)}_{rel}[0, T].
\]

Moreover, (135) holds if and only if the following variational inequality holds.

\[
\mathbb{E}^{u^o} \left\{ \int_0^T \mathbb{H}^{rel}(t, x^o(t), \psi^o(t), q_{11}(t), \tilde{q}_{22}, u_{t}^{i,o} - u_{t}^{i,o}) dt \right\} \geq 0, \; \forall u^i \in \mathbb{U}^{i}_{rel}[0, T], \; \forall i \in \mathbb{Z}_N.
\]

(3) The process \( \{(\Psi^i_t, \psi^o), \left[ \frac{q_{11}^i}{\tilde{q}_{22}^i} \right] \} \in L^2_{\mathbb{F}_T}([0, T], \mathbb{R}^{n+1}) \times L^2_{\mathbb{F}_T}([0, T], \mathcal{L}(\mathbb{R}^{m+k}\mathbb{R}^{n+1})) \) is a unique solution of the backward stochastic differential equation (137), (132) such that \( u^o \in \mathbb{U}^{(N)}_{rel}[0, T] \) satisfies the point wise almost sure inequalities with respect to the \( \sigma \)-algebras \( \mathcal{G}_t^o \subset \mathbb{F}_t, \; t \in [0, T] \):

\[
\mathbb{E}^{u^o} \left\{ \mathbb{H}^{rel}(t, x^o(t), \psi^o(t), q_{11}(t), \tilde{q}_{22}, \psi^o(t)) | \mathcal{G}_0^o \right\} \geq \mathbb{E}^{u^o} \left\{ \mathbb{H}^{rel}(t, x^o(t), \psi^o(t), q_{11}(t), \tilde{q}_{22}, u^i_t) | \mathcal{G}_0^o \right\},
\]

\[
\forall \nu^i \in \mathcal{M}_1(\mathbb{A}^i), \; a.e.t \in [0, T], \mathbb{P}^{u^o} | \mathcal{G}_0^o \text{ a.s.}, i = 1, 2, \ldots, N
\]

\[
(137)
\]

**Sufficient Conditions.** Let \( (x^o(\cdot), u^o(\cdot)) \) denote an admissible state and decision pair and let \( \psi^o(\cdot) \) the corresponding adjoint processes. Suppose the following conditions hold.

- **(C6)** \( \mathbb{H}^{rel}(t, \cdot, \psi, q_{11}, \tilde{q}_{22}, \psi) \), \( t \in [0, T] \) is convex in \( x \in \mathbb{R}^n \);
- **(C7)** \( \varphi(\cdot) \) is convex in \( x \in \mathbb{R}^n \).

Then \( (x^o(\cdot), u^o(\cdot)) \) is a relaxed team optimal if it satisfies (137).

For a generic information structure \( \{I^i(t) : t \in [0, T]\} \) available to each DM \( i \), generating a \( \sigma \)-algebra \( \mathcal{G}^{I(t)} \cong \mathcal{G}^{f(t)} \), \( t \in [0, T] \), which is not necessarily nested (nonclassical), in the sense that, \( \mathcal{G}^{f(t)} \not\subset \mathcal{G}^{f(t)}, t > t \) (see Remark [A] the conditioning in (137) is taken with respect to \( \mathcal{G}^{f(t)} \), for \( i = 1, 2, \ldots, N \).

**Proof. Necessary Conditions.** This follows from the discussion prior to the statement of the Theorem.

**Sufficient Conditions.** Let \( u^o \in \mathbb{U}^{(N)}_{rel}[0, T] \) denote a candidate for the optimal team decision and \( u \in \mathbb{U}^{(N)}_{rel}[0, T] \) any other decision. Then

\[
J(u^o) - J(u) = \mathbb{E}^{u^o} \left\{ \int_0^T \left( \ell(t, x^o(t), u^o_t) - \ell(t, x(t), u_t) \right) dt + \left( \varphi(x^o(T)) - \varphi(x(T)) \right) \right\}.
\]

(138)
By the convexity of $\varphi(\cdot)$ then
\[ \varphi(x(T)) - \varphi(x^o(T)) \geq \langle \varphi_x(x^o(T)), x(T) - x^o(T) \rangle. \] (139)
Substituting (139) into (138) yields
\[ J(u^o) - J(u) \leq \mathbb{E}^u \left\{ \langle \varphi_x(x^o(T)), x^o(T) - x(T) \rangle \right\} \]
\[ + \mathbb{E}^u \left\{ \int_0^T (\ell(t, x^o(t), u^o_t) - \ell(t, x(t), u_t)) dt \right\}. \] (140)
Applying the Ito differential rule to $\langle \psi^o, x - x^o \rangle$ on the interval $[0, T]$ and then taking expectation we obtain the following equation.
\[ \mathbb{E}^u \left\{ \langle \psi^o(T), x(T) - x^o(T) \rangle \right\} = \mathbb{E}^u \left\{ \langle \psi^o(0), x(0) - x^o(0) \rangle \right\} \]
\[ + \mathbb{E}^u \left\{ \int_0^T (\langle -f^*_x(t, x^o(t), u^o_t)\psi^o(t) dt - V^*_q(t) - V^*_q(t) - \ell_x(t, x^o(t), u^o_t), x(t) - x^o(t) \rangle dt \right\} \]
\[ + \mathbb{E}^u \left\{ \int_0^T \langle \psi^o(t), f(t, x(t), u_t) - f(t, x^o(t), u^o_t) \rangle dt \right\} \]
\[ + \mathbb{E}^u \left\{ \int_0^T tr \left( \tilde{q}^*_2(t) \sigma(t, x(t), u_t) - \tilde{q}^*_2(t) \sigma(t, x^o(t), u^o_t) \right) \right\} \]
\[ = -\mathbb{E}^u \left\{ \int_0^T \langle H^q_{x^o}(t, x^o(t), \psi^o(t), q^o_{11}(t), q^o_{22}(t), u^o_t), x(t) - x^o(t) \rangle dt \right\} \]
\[ + \mathbb{E}^u \left\{ \int_0^T tr \left( \tilde{q}^*_2(t) \sigma(t, x(t), u_t) - \tilde{q}^*_2(t) \sigma(t, x^o(t), u^o_t) \right) \right\} \] (141)
Note that $\psi^o(T) = \varphi_x(x^o(T))$. Substituting (141) into (140) we obtain
\[ J(u^o) - J(u) \leq \mathbb{E}^u \left\{ \int_0^T \left[ H^{rel}(t, x^o(t), \psi^o(t), q^o_{11}(t), q^o_{22}(t), u^o_t) - H^{rel}(t, x(t), \psi^o(t), q^o_{11}(t), q^o_{22}(t), u^o_t) \right] dt \right\} \]
\[ - \mathbb{E}^u \left\{ \int_0^T \langle H^q_{u^o}(t, x^o(t), \psi^o(t), q^o_{11}(t), q^o_{22}(t), u^o_t), x^o(t) - x(t) \rangle dt \right\}. \] (142)
Since by hypothesis $H^{rel}$ is convex in $x \in \mathbb{R}^n$ and linear in $\nu \in \mathcal{M}_1(A^{(N)})$, then
\[ H^{rel}(t, x^o(t), \psi^o(t), q^o_{11}(t), q^o_{22}(t), u^o_t) \geq \sum_{i=1}^N H^{rel}(t, x^o(t), \psi^o(t), q^o_{11}(t), q^o_{22}(t), u^o_t) \]
\[ \geq \sum_{i=1}^N H^{rel}(t, x^o(t), \psi^o(t), q^o_{11}(t), q^o_{22}(t), u^o_t, u^{i,o} - u^{i,o}_t) \]
\[ + \langle H^q_{u^o}(t, x^o(t), \psi^o(t), q^o_{11}(t), q^o_{22}(t), u^o_t), x(t) - x^o(t) \rangle, \quad t \in [0, T] \] (143)
Substituting (143) into (142) yields
\[ J(u^o) - J(u) \leq -\mathbb{E}^u \left\{ \sum_{i=1}^N \int_0^T H^{rel}(t, x^o(t), \psi^o(t), q^o_{11}(t), q^o_{22}(t), u^o_t, u^{i,o} - u^{i,o}_t) dt \right\}. \] (144)
By (137) and the definition of conditional expectation we have
\[
\mathbb{E}^{\omega}\left\{ I_{A_1}(\omega)\mathbb{H}^{rel}(t, x^o(t), \psi^o(t), q_{11}^o(t), g_{22}^o(t), u_{i}^{-i,o}, u_{i}^{-i,o}) \right\} \\
= \mathbb{E}^{\omega}\left\{ I_{A_1}(\omega)\mathbb{E}^{\omega}\left\{ \mathbb{H}^{rel}(t, x^o(t), \psi^o(t), q_{11}^o(t), g_{22}^o(t), u_{i}^{-i,o}, u_{i}^{-i,o})|\mathcal{G}_{0,t}^i \right\} \right\} \geq 0, \forall A_1^i \in \mathcal{G}_{0,t}^i, i \in \mathbb{Z}_N.
\tag{145}
\]
Hence, \( \mathbb{H}^{rel}(t, x^o(t), \psi^o(t), q_{11}^o(t), g_{22}^o(t), u_{i}^{-i,o}, u_{i}^{-i,o}) \geq 0, \forall u_{i} \in \mathcal{M}_1(A_i), a.e.t \in [0, T], \mathbb{P}^u - a.s., i = 1, 2, \ldots, N. \) Substituting this inequality into (144) gives
\[
J(u^o) \leq J(u), \forall u \in \mathbb{U}^{(N)}[0, T]。
\]
Hence, sufficiency of (137) is shown.

The last statement is obvious.

Note that the global convexity conditions (C6), (C7) are precisely the conditions often used to show sufficiency of the necessary conditions for centralized information structures.

For PbP optimality we have the following Corollary.

**Corollary 2.** (PbP Optimality Conditions under \( \mathbb{P}^u \))

**Consider Problem 2 under Assumptions 4.**

**Necessary Conditions.** For an element \( u^o \in \mathbb{U}^{(N)}[0, T] \) with the corresponding solution \( x^o \in B_{\mathbb{C}}^\infty([0, T], L^2(\Omega, \mathbb{R}^n)) \) to be a relaxed PbP optimal strategy, it is necessary that the statements of Theorem 5, (1), (3) hold and statement (2) is replaced by
\[
\mathbb{E}^{\omega}\int_0^T \mathbb{H}^{rel}(t, x^o(t), \psi^o(t), q_{11}^o(t), g_{22}^o(t), u_{i}^{-i,o}, u_{i}^{-i,o})dt \geq 0, \forall u_{i} \in \mathbb{U}^{i}_{rel}[0, T], \forall i \in \mathbb{Z}_N.
\tag{146}
\]

**Sufficient Conditions.** Let \( (x^o(\cdot), u^o(\cdot)) \) denote an admissible state and control pair and let \( \psi^o(\cdot) \) the corresponding adjoint processes. Suppose the conditions of the conditions of Theorem 5 (C6), (C7) hold. Then \( (x^o(\cdot), u^o(\cdot)) \) is PbP optimal if it satisfies (137).

**Proof.** **Necessary Conditions.** The methodology is similar to that of Theorem 5 with the exception that we only vary in the direction \( u_{i}^o - u_{i}^{i,o} \) while all other strategies assume their optimal values.

**Sufficient Conditions.** This is precisely as done in Theorem 5.
5 Team Optimality Conditions for Regular Strategies

In this section, we illustrate how the optimality conditions derived based on relaxed strategies can be reduced to optimality conditions based regular strategies.

Suppose optimal regular team and PbP strategies exist from the admissible class $U^{(N)}_{reg}[0,T] \subset U^{(N)}_{rel}[0,T]$. Then the necessary and sufficient conditions presented in the previous section can be specialized to the class of decision strategies which are simply Dirac measures concentrated $\{u^o \in U^{(N)}_{reg}[0,T] \subset U^{(N)}_{rel}[0,T] \}$. Specifically, consider the class of regular decisions $U^{(N)}_{reg}[0,T]$, where the sets $\mathbb{A}^i$ are compact subsets of $\mathbb{R}^d$, $i = 1, 2, \ldots, N$. This class of regular decisions embeds continuously into the class of relaxed decisions through the map $u \in U^{(N)}_{reg}[0,T] \rightarrow \delta_{u^o}(\omega) \in U^{(N)}_{rel}[0,T]$. Clearly, for every $g \in L^1_{\mathcal{B}}([0,T] \times \Omega, C(\mathbb{A}^i))$ we have

$$
\mathbb{E} \int_{[0,T] \times \mathbb{A}^i} g(t, \omega, t, \psi) \delta_{u^o(t)(\omega)}(d\xi) dt = \mathbb{E} \int_{[0,T]} g(t, \omega, u^o(t)) dt, \quad \forall i \in \mathbb{Z}_N.
$$

The important advantage of the theory of relaxed strategies is that the necessary conditions of optimality for regular strategies follow readily from those of relaxed strategies, without having to repeat the derivations.

Thus, by simply replacing the relaxed strategies by Dirac measures, from previous section we obtain the following Hamiltonian (corresponding to regular strategies)

$$
\mathbb{H}^{reg} : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{L}(\mathbb{R}^k, \mathbb{R}) \times \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) \times \mathbb{A}^i \rightarrow \mathbb{R},
$$

where

$$
\mathbb{H}^{reg}(t, x, \psi, q_{11}, \tilde{q}_{22}, u) \triangleq \langle f(t, x, u), \psi \rangle + \ell(t, x, u) + tr(\tilde{q}_{22}^{*} \sigma(t, x, u)) + tr(q_{11}^{*} h^{*}(t, x, u)),
$$

and the Hamiltonian system of equations is given by (131)-(133), with relaxed strategies replaced by regular strategies $u \in U^{(N)}_{reg}[0,T]$.

**Theorem 6.** (Regular team optimality conditions) Consider Problem 7 under the Assumptions of Theorem 4 with admissible decisions from the regular class taking values in $\mathbb{A}^i$, a closed, bounded and convex subset of $\mathbb{R}^d$, $\forall i \in \mathbb{Z}_N$.

**Necessary Conditions.** For an element $u^o \in U^{(N)}_{reg}[0,T]$ with the corresponding solution $x^o \in B_{\mathbb{A}^i}^{22}([0,T], L^2(\Omega, \mathbb{R}^n))$ to be team optimal, it is necessary that the following hold.

1. **Statement (1) of Theorem 4** holds.
2. The variational inequality is satisfied:

$$
\frac{d}{dt} \left( \sum_{i=1}^{N} \mathbb{E}^{u^o} \int_{0}^{T} \mathbb{H}^{reg}(t, x^o(t), \psi^o(t), q_{11}^o(t), \tilde{q}_{22}^o(t), u_i^{-1,o}, u_i^o) dt \right) \geq \frac{d}{dt} \left( \sum_{i=1}^{N} \mathbb{E}^{u^o} \int_{0}^{T} \mathbb{H}^{reg}(t, x^o(t), \psi^o(t), q_{11}^o(t), \tilde{q}_{22}^o(t), u_i^o) dt \right), \quad \forall u \in U^{(N)}_{reg}[0,T].
$$

(149)
(3) The process \( \{(\Psi^o, \psi^o), (q^o_{11}, q^o_{12}, \tilde{q}^o_{21}, \tilde{q}^o_{22}) \} \) is a unique solution of the backward stochastic differential equation (131), (132), with \( H^{rel} \) replaced by \( H^{reg} \) such that \( u^o \in U^{(N)}_\text{reg}[0, T] \) satisfies the point wise almost sure inequalities with respect to the \( \sigma \)-algebras \( \mathcal{G}_{0,t}^i, t \in [0, T] \):

\[
\mathbb{E}^{u^o} \left\{ \left( H^{reg}(t, x^o(t), \psi^o(t), q^o_{11}(t), \tilde{q}^o_{22}(t), u_{t-i}^o, u^o_t) - H^{reg}(t, x^o(t), \psi^o(t), q^o_{11}(t), \tilde{q}^o_{22}(t), u^o_t) \right) \right| \mathcal{G}_{0,t}^i \right\} \geq 0, \\
\forall u^i \in A^i, \text{a.e.} t \in [0, T], \mathbb{P}^{u^o}_{\mathcal{G}_{0,t}^i} \text{-a.s.}, i = 1, 2, \ldots, N. \quad (150)
\]

**Sufficient Conditions.** Let \((u^o(\cdot), x^o(\cdot))\) denote an admissible decision and state pair and \( \psi^o(\cdot) \) the corresponding adjoint processes.

Suppose the conditions (C7) holds and in addition

(C8) \( f, \sigma, h, \ell \) are continuously differentiable in \( u \in A^{(N)} \) and uniformly bounded;

(C9) \( H^{reg}(t, \cdot, \psi, q_{11}, \tilde{q}_{22}, \cdot), t \in [0, T], \) is convex in \( (x, u) \in \mathbb{R}^n \times A^{(N)} \);

Then \((x^o(\cdot), u^o(\cdot))\) is optimal if it satisfies (150).

**Proof.** The necessary conditions follow from Theorem 5 by simply replacing relaxed strategies by Dirac measures concentrated at \( u^o \in U^{(N)}_{\text{reg}}[0, T] \). The derivation of sufficient conditions is done precisely as in Theorem 5 by using the additional condition (C8).

\( \square \)

Person-by-person optimality conditions for regular decision strategies follow from their relaxed counterparts, as discussed above.

### 5.1 Realization of Relaxed by Regular Strategies

The existence of optimal relaxed strategies shown in Theorem 2 does not assume convexity of the actions spaces \( A^i, i = 1, \ldots, N \). Since is some applications of stochastic dynamic team theory, it is often desirable and easier to construct regular team strategies, next we address the question of whether a regular team strategy with corresponding team pay-off is close to that realized by an optimal relaxed team strategy. To this end, we have the following result.

**Theorem 7.** Consider the regular team strategies \( U^{(N)}_{\text{reg}}[0, T] \), where \( A^{(N)} \) is closed and bounded, but not necessarily convex. Suppose the assumptions of Lemma 4 and Theorem 2 hold, and consider the team problem stated in Problem 4.

Let \( u^o \in U^{(N)}_{\text{reg}} \) be the optimal relaxed team strategy. Then, for every \( \varepsilon > 0 \) there exists a regular team strategy control \( u_r \in U^{(N)}_{\text{reg}}[0, T] \) such that

\[
J(u_r) \leq \varepsilon + J(u^o).
\]
Proof. Since $\mathbb{U}_{rel}^{(N)}[0,T] \equiv L_{G_{T}}^{\infty}([0,T] \times \Omega, \mathcal{M}(\mathbb{R}^{d}))$ is compact in the vague topology (that is weak star topology) and convex (by the convexity of $\mathcal{M}(\mathbb{R}^{d}))$, it follows from the well known Krein-Millman theorem that

$$\mathbb{U}_{rel}^{(N)}[0,T] = cl_{v}conv(\text{ext}(\mathbb{U}_{rel}^{(N)}[0,T])),\tag{B3}$$

i.e. $\mathbb{U}_{rel}^{(N)}[0,T]$ is the weak star closed convex hull of its extreme points. By considering the embedding $\mathbb{U}_{reg}^{(N)}[0,T] \hookrightarrow \mathbb{U}_{rel}^{(N)}[0,T]$, it can be verified that the extreme points of $\mathbb{U}_{rel}^{(N)}[0,T]$ are precisely the set of regular strategies $\mathbb{U}_{reg}^{(N)}[0,T]$ through the map $u \in \mathbb{U}_{reg}^{(N)}[0,T] \longrightarrow \delta_{u} \in \mathbb{U}_{rel}^{(N)}[0,T]$. Thus, if $u^{o} \in \mathbb{U}_{reg}^{(N)}[0,T]$ is the optimal (relaxed) strategy there exists a sequence $\{u^{n}\}$ of the form

$$u^{n} \equiv \sum_{i=1}^{n} \alpha_{i}^{n}u_{i}, \quad u_{i} \in \mathbb{U}_{reg}^{(N)}[0,T], \quad \alpha_{i}^{n} \geq 0, \quad \sum_{i=1}^{n} \alpha_{i}^{n} = 1, \quad n \in N$$

such that $u^{n} \mathop{\rightarrow}^{v} u^{o}$. Let $\{X^{n}, X^{o}\} \subset B_{F_{L}}^{\infty}([0,T], L_{2}(\Omega, \mathbb{R}^{d+1}))$ denote the solutions of the augmented system (23) corresponding to $\{u^{n}, u^{o}\}$ respectively. By Lemma 1 along a subsequence if necessary, it follows that $X^{n} \mathop{\rightarrow}^{a} X^{o}$ in $B_{F_{L}}^{\infty}([0,T], L_{2}(\Omega, \mathbb{R}^{d+1}))$. Consequently, it follows from continuity of $L$ and $\Phi$ in the augmented state variable $X$, Assumptions 3, Assumptions (B1)-(B3), and Lebesgue dominated convergence theorem that $\lim_{n \rightarrow \infty} J(u^{n}) = J(u^{o})$. Note that for every $n \in N$, $u^{n} \in \mathbb{U}_{reg}^{(N)}[0,T]$, and so, for every $\varepsilon > 0$, there exists an $n_{\varepsilon} \in N$ such that $|J(u^{n}) - J(u^{o})| < \varepsilon$ for all $n \geq n_{\varepsilon}$. Taking $u_{r} = u^{n_{\varepsilon}}$ we have $J(u_{r}) \leq \varepsilon + J(u^{o})$. This completes the derivation.

Remark 5. By the previous theorem an $\varepsilon$-optimal team strategy can be found from the class of regular strategies (measurable functions with values in $\mathbb{A}^{(N)}$), though the limit of such strategies may be a relaxed. More specifically, if $\mathbb{A}^{i} \subset \mathbb{R}^{d_{i}}$ consists of a finite set of points, it is clearly non-convex, and optimal team strategies may not exist from the class of regular strategies $\mathbb{U}_{reg}^{(N)}[0,T]$ based on the set $\mathbb{A}^{i}$. However, optimal relaxed team strategies do exist. In this case the sequence of regular strategies approximating the optimal relaxed strategies may oscillate violently between the finite set of points of $\mathbb{A}^{i}$ with increasing frequency (converging to infinity). This is known as chattering.

Remark 6. (General Information Structures) The optimality conditions apply to many other forms information structures. We describe two such generalizations.

Nested Information Structures. Suppose each team members actions $u_{i}^{t}$ at time $t \in [0,T]$ is a nonanticipative measurable function of the noisy observations $\{y^{i}(s) : 0 \leq s \leq t\}$, and delayed noisy observations $\{y^{i}(s-\epsilon_{j}) : \epsilon_{j} > 0, j \in \mathcal{O}(i), 0 \leq s \leq t\}$, of any subset $\mathcal{O}(i) \subset \{1, 2, \ldots, i-1, i+1, \ldots, N\}$ of the rest of observations which are communicated to member $i$, for $i = 1, \ldots, N$. Let $\mathcal{G}_{0,t}^{i} \overset{\Delta}{=} \sigma\{I^{i}(s) : 0 \leq s \leq t\}$ denote the minimum $\sigma$-algebra generated by $\{I^{i}(s) \overset{\Delta}{=} \{y^{i}(s), y^{i}(s-\epsilon_{j}) : \epsilon_{j} > 0, j \in \mathcal{O}(i)\} : 0 \leq s \leq t\}$, $t \in [0,T]$, the information available to $u_{i}^{t}$, at $t \in [0,T]$, for $i = 1, \ldots, N$. Clearly, $\mathcal{G}_{0,t}^{i}$ is a nested information structure since $\mathcal{G}_{0,t}^{i} \subseteq \mathcal{G}_{0,\tau}^{i}, \forall \tau > t$. 

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Nonnested Information Structures. Suppose each team member's actions $u^t_i$ at time $t \in [0, T]$ is a measurable function of $I^i(t) \triangleq \{y^i(t), y^j(t - \epsilon_j) : \epsilon_j > 0, j \in \mathcal{O}(i)\}$. Let $\mathcal{G}^I(t) \triangleq \sigma\{I^i(t)\}$ denote the minimum $\sigma-$algebra generated by $I^i(t)$, $t \in [0, T]$, the information available to $u^t_i$, at $t \in [0, T]$, for $i = 1, \ldots, N$.

Clearly, $\mathcal{G}^I(t)$ is a nonnested (nonclassical) information structure since $\mathcal{G}^I(t) \not\subset \mathcal{G}^I(\tau)$, $\forall \tau > t$. For such information structures the conditioning of the Hamiltonian is replaced by the conditioning with respect to $\mathcal{G}^I(\tau)$, $t \in [0, T]$.

This completes our analysis on team and PbP optimality conditions for decision systems with decentralized noisy information structures. We point out that the challenge is in the implementation of the new variational Hamiltonians and the computation the optimal decentralized strategies for specific examples.

In future work we will investigate applications of the results of this part to specific linear and nonlinear distributed stochastic differential decision systems from Communication and Control areas as in [45].

6 Conclusions and Future Work

In this paper we presented two methods which generalize static team theory to dynamic team theory, in the context of continuous-time stochastic differential decentralized decision problems, with relaxed and regular decentralized team strategies. Both methods utilize Girsanov’s measure transformation to transform the original problem to an equivalent problem under a reference probability space, in which the observations and/or the unobserved state are not affected by any of the team decisions. The first generalizes and makes precise Witsenhausen’s [1] notion of equivalence between static and dynamic team problems. The second method is based on Pontryagin’s minimum principle and consists of forward and backward stochastic differential equations, and a conditional Hamiltonian with respect to the information structure available to each team player. We also show existence of team and PbP optimality among the class of relaxed decentralized strategies.

The methodology is very general; it is applicable to variety of examples, including nonlinear stochastic differential team problems, and it is easily generalized to other systems and games. Below, we provide a short list of additional generalizations and issues which can be further investigated.

(F1) For team problems with regular decentralized strategies with non-convex action spaces $A^i, i = 1, 2, \ldots, N$, and diffusion coefficients which depend on the decision variables it is necessary to derive optimality conditions based on second-order variations. If one considers our function spaces, then the extra equation coming from second order variations, then the modified conditional Hamiltonian can be easily obtained as in [62].

(F2) The derivation of optimality conditions can be used in other type of games such as Nash-equilibrium strategies with different information structures for each player, and minimax games.
(F3) The derivation of optimality conditions can be extended to differential systems driven by both continuous Brownian motion processes and jump processes, such as Lévy or Poisson jump processes. If one invokes our spaces then this generalization follows directly from [50].

(F4) The Pontryagin’s optimality conditions obtained for continuous systems can be easily transformed to analogous optimality conditions for discrete-time systems, by invoking the discrete-time Girsanov’s measure transformation, the semi martingale and Riesz representation theorems for discrete-time Hilbert processes. This direction is worth pursuing to gain additional insight into decentralized decision making.

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