abstract. We study real rational models of the euclidean affine plane $\mathbb{R}^2$ up to isomorphisms and up to birational diffeomorphisms. The analogous study in the compact case, that is the classification of real rational models of the real projective plane $\mathbb{RP}^2$ is well known: up to birational diffeomorphisms, $\mathbb{P}^2(\mathbb{R})$ is the only model. A fake real plane is a smooth geometrically integral surface $S$ defined over $\mathbb{R}$ not isomorphic to $\mathbb{A}^2(R)$, whose real locus $S(\mathbb{R})$ is diffeomorphic to $\mathbb{R}^2$ and such that the complex surface $S(\mathbb{C})$ has the rational homology type of $\mathbb{A}^2(R)$.

we prove that fake planes exist by giving many examples and we tackle the question: does there exist fake planes $S$ such that $S(\mathbb{R})$ is not birationally diffeomorphic to $\mathbb{A}^2(R)$?

introduction

An algebraic complexification of a real smooth $C^\infty$-manifold $M$ is a smooth complex quasi-projective algebraic variety $V$ endowed with an anti-holomorphic involution $\sigma$ such that $M$ is diffeomorphic to the real locus $V^\sigma$ of $V$. Equivalently $V$ is the complex model $X \times_{\text{Spec}(\mathbb{R})} \text{Spec}(\mathbb{C})$ of a smooth quasi-projective algebraic variety $X$ defined over $\mathbb{R}$, whose set of $\mathbb{R}$-rational points is diffeomorphic to $M$ when equipped with the euclidean topology. Some manifolds such as real projective spaces $\mathbb{RP}^n$ and real euclidean affine spaces $\mathbb{R}^n$ have natural algebraic complexifications, given by the complex projective and affine spaces $\mathbb{P}^n_{\mathbb{C}}$ and $\mathbb{A}^n_{\mathbb{C}}$ respectively. But these also admit infinitely many other complexifications, and it is a natural problem to try to classify them up to appropriate notions of equivalence.

For example, the real projective plane $M = \mathbb{RP}^2$ is also the real locus of the quotient $V_1$ of a smooth quartic in $\mathbb{P}^3_{\mathbb{C}}$ by a fixed point-free involution. The latter is not isomorphic to $V_0 = \mathbb{P}^2_{\mathbb{C}}$ endowed with the usual complex conjugation, and actually, $V_1$ and $V_0$ are not even birational to each other. On the other hand, we obtain infinite families of pairwise non-isomorphic rational projective complexifications $V_k$ of $\mathbb{RP}^2$ by blowing-up sequences of pairs of non-real conjugated points of $\mathbb{P}^2_{\mathbb{C}}$. The resulting surfaces $V_k$ are not isomorphic to $\mathbb{P}^2_{\mathbb{C}}$ but the induced birational morphisms $V_k \rightarrow \mathbb{P}^2_{\mathbb{C}}$ restrict to diffeomorphisms between the real loci of $V$ and $\mathbb{P}^2_{\mathbb{C}}$. More generally, it is a striking result of Biswas and Huisman [3](see also [17]) that all rational algebraic complexifications of a given smooth compact surface $M$ are all $\mathbb{R}$-biregularly birationally equivalent: this means that for every pair $V$ and $V'$ of such complexifications, there exists birational maps $\varphi : V \rightarrow V'$ and $\varphi' : V' \rightarrow V$ inverse to each other and whose restrictions to the real loci of $V$ and $V'$ are diffeomorphisms inverse to each other. This classification result gave rise to many further discoveries, see the survey article [23] and the bibliography given there. In addition to $\mathbb{RP}^2$, the compact surfaces $M$ diffeomorphic to the real locus of smooth rational projective surface minimal over $\mathbb{R}$ are the sphere $S^2$, the torus $S^1 \times S^1$ and the Klein bottle $K = \mathbb{RP}^2 \# \mathbb{RP}^2$. Their respective minimal complexifications are $\mathbb{P}^2_{\mathbb{C}}$, the quadric hypersurface $x^2 + y^2 + z^2 - w^2 = 0$ in $\mathbb{P}^3_{\mathbb{C}}$, the Hirzebruch surfaces of even index $\mathbb{F}_{2k}$, $k \geq 0$, and the Hirzebruch surfaces of odd index $\mathbb{F}_{2k+1}$, $k \geq 1$. In all cases except that of $\mathbb{F}_1$, the minimality of the complexification $V$ endowed with the real structure is equivalent to the minimality of its topology as a compact complex manifold among all complexifications of $M$. The above description shows that complexifications with minimal topology are either unique, or all diffeomorphic to each others, belonging to a unique equivalence class of deformation.

In this paper, we lay the importance on affine complexifications and we discover that, contrary to the projective case, the easiest example $M = \mathbb{R}^2$ possesses a lot of affine complexifications $S$ with "minimal
topology” which are not biregularly isomorphic to $\mathbb{A}^2$. We call them fake real planes. In contrast with the projective case, where the notion of rational complexification with minimal topology is unambiguous due to the fact that the topology of a smooth rational complexification $V$ of a given compact surface $M$ is fully determined by its Picard rank $\rho(V) = \text{rk}(N_1(V))$, there are many possibilities to define a notion of minimality of the topology of an affine complexification, even in the rational case. For instance, there exists many rational affine complexifications of $\mathbb{R}^2$ with vanishing second homology group $H_2(S; \mathbb{Z})$ but nontrivial fundamental group. Inspired by the work of Totaro [31], who defined a good affine complexification of $M$ to be a complexification $S$ for which the inclusion of $M \hookrightarrow S$ of $M$ as the real locus of $S$ is a homotopy equivalence, a natural notion of minimality of affine complexifications $S$ of $M = \mathbb{R}^2$ is to require that $S$ is contractible as a smooth complex manifold. But we also consider weaker variants in which we require only that the inclusion $M \hookrightarrow S$ induces an isomorphism between the respective homology groups of $M$ and $S$, taken with integral or rational coefficients. So for $M = \mathbb{R}^2$, the corresponding smooth affine complexifications $S$ are respectively $\mathbb{Z}$-acyclic and $\mathbb{Q}$-acyclic complex surfaces.

Our first goal is to show that fake real planes do indeed exist by exhibiting many examples and to give elements of classification of these objects up to biregular isomorphism in terms of natural algebro-geometric invariants such as their logarithmic Kodaira dimension. It happens that any contractible affine complexification $S$ of $\mathbb{R}^2$ of non positive logarithmic Kodaira dimension is isomorphic to $\mathbb{A}^2$, in particular, there is no good affine complexification of $\mathbb{R}^2$ of logarithmic Kodaira dimension 0. In contrast, we give several families of rational good affine complexifications of $\mathbb{R}^2$ of logarithmic Kodaira dimension 1 and 2 respectively, which are therefore not biregularly isomorphic to $\mathbb{A}^2$ (see Examples 3.4, 3.7 and § 3.2.2).

A striking example is the real counterpart of the famous Ramanujam surface [28], a smooth complex contractible affine surface of logarithmic Kodaira dimension 2. Namely, let $S$ be the real algebraic surface obtained as the complement in $\mathbb{F}_1$ of the proper transform of the union of the cuspidal cubic $C = \{x^2z + y^3 = 0\} \subset \mathbb{P}^2_\mathbb{R}$ with its osculating conic $Q$ at a general $\mathbb{R}$-rational point $q \in C$ by the blow-up $\tau : \mathbb{F}_1 \to \mathbb{P}^2_\mathbb{R}$ of the remaining $\mathbb{R}$-rational intersection point $p$ of $C$ and $Q$. The real locus of $\mathbb{F}_1$ is diffeomorphic to the Klein bottle $K$ viewed as circle bundle $\theta : K \to S^1$ over $S^1$ with fibers equal to the set of $\mathbb{R}$-rational point of the lines through $p$ in $\mathbb{P}^2_\mathbb{R}$ and having the real loci of the exceptional divisor $A_0(p)$ of $\tau$ as a section. The real loci of the proper transforms of $C$ and $Q$ are homotopically equivalent respectively to a fiber and section of $\theta$, implying that the real locus of $S$ is homotopically equivalent to a disc, hence is diffeomorphic to $\mathbb{R}^2$. On the other hand $S_\mathbb{C} = S \times_{\text{Spec}(\mathbb{R})} \text{Spec}(\mathbb{C})$ is isomorphic to the usual complex Ramanujam surface.

Motivated by the above blow-up construction, we establish the real counter-parts of a series of results due to Thom Dieck and Petrie [7] describing the structure of $\mathbb{Z}$-acyclic and $\mathbb{Q}$-acyclic smooth complex affine surfaces in terms of blow-ups of arrangements of rational curves in $\mathbb{P}^2_\mathbb{R}$. As an application, we obtain a complete classification of $\mathbb{Z}$-acyclic smooth affine complexifications of $\mathbb{R}^2$ of logarithmic Kodaira dimension 1 (Theorem 3.3) and a precise description of the structure of $\mathbb{Q}$-acyclic smooth affine complexifications of $\mathbb{R}^2$ of logarithmic Kodaira dimension 2 (Theorem 3.8) formulated in terms of arrangements of $\mathbb{R}$-rational curves in $\mathbb{P}^2_\mathbb{R}$.

In a second step, we tackle the classification of fake real planes with $\mathbb{Q}$-acyclic smooth affine complexifications up to $\mathbb{R}$-biregular birational equivalence: we prove that a large class of such surfaces are $\mathbb{R}$-biregularly birationally equivalent to $\mathbb{A}^2$. More precisely, we establish that every smooth $\mathbb{Q}$-acyclic complexification
S of \( \mathbb{R}^2 \) with negative logarithmic Kodaira dimension admits a flat surjective morphism \( \pi : S \to A^1_\mathbb{R} \) defined over \( \mathbb{R} \), with general fiber isomorphic to the affine line (Theorem 4.1), and we show that every such fibered surface \( \pi : S \to A^1_\mathbb{R} \) with at most one singular fiber is \( \mathbb{R} \)-biregularly birationally equivalent to \( A^2_\mathbb{R} \) (Theorem 4.8). This holds for instance for the family of smooth affine surfaces \( S \subset A^3_\mathbb{R} = \text{Spec}(\mathbb{R}[x, y, z]) \) defined by equations of the form \( x^n z = y^r - x \) where \( n \geq 2 \) and \( r \geq 3 \) is an odd integer: the fibration \( \pi = \text{pr}_x : S \to A^1_\mathbb{R} \) has general fibers isomorphic to \( A^1_\mathbb{R} \) and \( \pi^{-1}(0) \) as a unique singular fiber, the map \( \mathbb{R}^2 \to \mathbb{R}^3, (x, z) \mapsto (x, \sqrt[n]{x^n z} + x, z) \) induces an homeomorphism between \( \mathbb{R}^2 \) and the real loci \( S \) of \( \mathbb{R}^2 \), while the complex model \( S_C \) of \( S \) is \( \mathbb{Q} \)-acyclic with \( H_1(S_C; \mathbb{Z}) \cong \mathbb{Z}/r\mathbb{Z} \) and \( H_2(S_C; \mathbb{Z}) = 0 \).

We saw above that in the projective case, there exist a unique minimal complexification or at most one family of pairwise non isomorphic but \( \mathbb{R} \)-biregularly birationally equivalent minimal complexifications. We conclude the paper with the construction of an infinite number of fake planes with moduli of arbitrary positive dimension of pairwise non isomorphic, deformation equivalent, \( \mathbb{Q} \)-acyclic euclidean planes all \( \mathbb{R} \)-biregularly birationally equivalent to \( A^2_\mathbb{R} \) (see §5.2).

In contrast with the projective case, the main difficulty to understand the notion of \( \mathbb{R} \)-biregular birational equivalence comes from the lack of natural numerical invariants to distinguish classes. In particular, neither the topology of the complexification \( S \) nor its logarithmic Kodaira dimension are invariants of its \( \mathbb{R} \)-biregular birational class. And even though Theorem 4.8 is a significant step towards a complete classification of fake real planes up to \( \mathbb{R} \)-biregular birational equivalence, the fact that its proof depends on the construction of explicit elementary \( \mathbb{R} \)-biregular birational links between appropriate projective models of the affine complexification \( S \) does not give any clear insight on possible numerical invariants of \( \mathbb{R} \)-biregular birational equivalence classes. As a consequence, the question of existence of fake real planes not \( \mathbb{R} \)-biregularly birationally equivalent to \( A^2_\mathbb{R} \) remains open, a good candidate being the Ramanujam surface above (see also §5.1 for another candidate of logarithmic Kodaira dimension 0).

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1. Preliminaries

In this article, the term \( k \)-variety will always refer to a geometrically integral scheme \( X \) of finite type over a base field \( k \) of characteristic zero. A morphism of \( k \)-varieties is a morphism of \( k \)-schemes. In the
subsequent, $k$ will be most of the time equal to either $\mathbb{R}$ or $\mathbb{C}$, and we will say that $X$ is a real, respectively complex, algebraic variety.

A complex algebraic variety $X$ will be said to be defined over $\mathbb{R}$ if there exists a real algebraic variety $X_0$ and an isomorphism of complex algebraic varieties between $X$ and the complexification $X_{0,C} = X_0 \times \text{Spec}(\mathbb{R})$. $	ext{Spec}(\mathbb{C})$ of $X_0$, where $\text{Spec}(\mathbb{C}) \rightarrow \text{Spec}(\mathbb{R})$ is the morphism induced by the usual inclusion $\mathbb{R} \hookrightarrow \mathbb{C} = \mathbb{R}[x]/(x^2 + 1)$. We will often use the shorter notation $X_{0,C} = X_0 \otimes _{\mathbb{R}} \mathbb{C}$. A complex variety of the form $X_{0,C}$ is naturally endowed with an additional action of the Galois group $\text{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}_2$ given by the anti-holomorphic involution $\sigma = \text{id}_{X_0} \times (x \mapsto -x)$, which we call the real structure on $X_{0,C}$.

1.1. Points and curves on surfaces.

**Definition 1.1.** Given a real algebraic surface $S$, we denote by $S(\mathbb{R})$ and $S(\mathbb{C})$ the sets of $\mathbb{R}$-rational and $\mathbb{C}$-rational points respectively. We always consider $S(\mathbb{R})$ as a subset of $S(\mathbb{C})$ via the map induced by the inclusion $\mathbb{R} \hookrightarrow \mathbb{C}$. In what follows, the elements of $S(\mathbb{R})$, $S(\mathbb{C})$ and $S(\mathbb{C}) \setminus S(\mathbb{R})$ will be frequently referred to as real points, complex points and non-real points of $S$ respectively.

The subset $(S_C(\mathbb{C}))^\sigma$ of $S_C(\mathbb{C})$ consisting of $\mathbb{C}$-rational points of $S_C$ that are fixed by the real structure $\sigma$ is called the real locus of $S_C$, and we identify it with $S(\mathbb{R})$ in the natural way.

**Definition 1.2.**
1) A curve on a surface $S$ defined over $k$ is a geometrically reduced closed sub-scheme $C \subset S$ of pure codimension 1 defined over $k$. We denote by $\mathbb{Z}(C)$ the free abelian group generated by the irreducible components of $C$.

2) An SNC divisor $B$ on a smooth surface $S$ defined over $k$ is a curve $B$ on $S$ with smooth irreducible components and ordinary double points only as singularities. Equivalently, for every closed point $p \in B$, the local equations of the irreducible components of $B$ passing through $p$ form a part of regular sequence in the maximal ideal $m_{S,p}$ of the local ring $\mathcal{O}_{S,p}$ of $S$ at $p$. The dual graph $\Gamma_B = (\Gamma_v B, \Gamma_e B)$ of an SNC divisor $B$ is the graph with vertex set $\Gamma_v B$ the set of irreducible components of $B$ and with edges set $\Gamma_e B$ the set of double points of $B$. An edge in $\Gamma_B$ connects the two vertices which intersect in it.

1.2. Birational maps.

Recall that a rational map $\varphi : X \dashrightarrow Y$ between two $k$-schemes $X$ and $Y$ is an equivalence class of pairs $(U, f)$ consisting of a dense open subset $U \subset X$ of $X$ and a morphism of $k$-schemes $f : U \rightarrow Y$, two such pairs $(U, f)$ and $(U', f')$ being declared equivalent if there exists a dense open subset $V \subset U \cap U'$ such that $f |_V = f' |_V$. The domain of definition of a rational map is the largest open subset $\text{dom}_\varphi$ on which $\varphi$ is represented by a morphism. We say that $\varphi$ is regular at a closed point $x \in \text{dom}_\varphi$. A rational map $\varphi : X \dashrightarrow Y$ is called birational if it admits a rational inverse $\psi : Y \dashrightarrow X$.

In the sequel, we will frequently make use of the following type of birational morphisms:

**Example 1.3.** (Subdivisorial expansion of a surface at a point). Let $S$ be a smooth surface defined over $k$ and let $p \in S$ be a closed point with residue field $k_\text{res}$. A subdivisorial expansion with center at $p$ is a birational morphism $\tau : S' \rightarrow S$ restricting to an isomorphism over $S \setminus \{p\}$ and such that $\tau^{-1}(p)$ is a chain of smooth $k_\text{res}$-rational curves, containing a unique irreducible component $A_0(p) \simeq \mathbb{P}^1_{k_\text{res}}$ with normal bundle of degree $-\deg(k_\text{res})/k$. Given an ordered sequence of regular parameters $(x_-, x_+)$ in the local ring $\mathcal{O}_{S,p}$ of $S$ at $p$, there exists a unique pair of coprime integers $1 \leq \mu_- \leq \mu_+$ such that $\tau : S' \rightarrow S$ coincides with the minimal resolution of the indeterminacies at $p$ of the rational map $x_+^{\mu_+}/x_-^{\mu_-} : S' \dashrightarrow \mathbb{P}^1$. For instance, the particular case $\mu_\pm = 1$ is nothing but the blow-up $\tau : S' = \text{Proj}_S(\bigoplus_{n \geq 0} \mathbb{I}_p^n) \rightarrow S$ of $S$ at $p$, where $\mathbb{I}_p \subset \mathcal{O}_S$ denotes the ideal sheaf of $p$.

In the sequel, we will mostly use birational morphisms in the particular case where $x_-$ and $x_+$ are the respective local equations of integral curves $C_-$ and $C_+$ on $S$ intersecting transversally at $p$. The integer $\mu_\pm$ is then equal to the coefficient of $A_0(p)$ in the total transform of $C_\pm$ and will say that $\tau : S' \rightarrow S$ is the subdivisorial expansion of $S$ at the $k_\text{res}$-rational point $(C_- \cap C_+)_p$, with multiplicities $(\mu_-, \mu_+)$.

**Definition 1.4.** A $k$-variety $X$ is called $k$-rational if there exist a birational map $\varphi : \mathbb{P}^{\dim X}_k \dashrightarrow X$. A geometrically reduced $k$-scheme of finite type $X$ is called geometrically rational if every irreducible component of $X\bar{k}$ is $\bar{k}$-rational, where $\bar{k}$ denotes an algebraic closure of $k$. 
Definition 1.5. An SNC divisor $B$ on a smooth complete surface $V$ defined over $k$ is said to be SNC-minimal over $k$ if there does not exist any projective strictly birational morphism $\tau : V \to V'$ onto a smooth surface defined over $k$ with exceptional locus contained in $B$ such that $\tau_*(B)$ is SNC.

Definition 1.6. Let $V$ be a smooth surface defined over $k$, and let $\overline{k}$ be an algebraic closure of $k$.

1) A geometrically rational tree on $V$ is an SNC divisor $B$ defined over $k$ such that every irreducible component of $B_{\overline{k}} \subset S_{\overline{k}}$ is a $\overline{k}$-rational complete curve and the dual graph $\Gamma(B_{\overline{k}})$ is a tree.

2) A geometrically rational chain on $V$ is geometrically rational tree $B$ defined over $k$ such that the dual graph $\Gamma(B_{\overline{k}})$ is a chain. The irreducible components $B_0, \ldots, B_r$ of $B_{\overline{k}}$ can be ordered in such a way that $B_i \cdot B_j = 1$ if $|i - j| = 1$ and 0 otherwise. A geometrically rational chain $B$ with such an ordering on the set of irreducible components of $B_{\overline{k}}$ is said to be oriented. The components $B_0$ and $B_r$ are called respectively the left and right boundaries of $B$, and we say by extension that an irreducible component $B_i$ of $B_{\overline{k}}$ is on the left of another one $B_j$ when $i < j$. The sequence of self-intersections $[B_0^2, \ldots, B_r^2]$ is called the type of the oriented geometrically rational chain $B$.

An oriented $k$-subchain of $B$ is a geometrically rational chain $Z$ whose support is contain in that of $B$. We say that an oriented geometrically rational chain $B$ is composed of $k$-subchains $Z_1, \ldots, Z_s$ and we write $B = Z_1 \triangleright \cdots \triangleright Z_s$ is the $Z_i$ are oriented $k$-subchains of $B$ whose union is $B$ and the irreducible components of $Z_i_{\overline{k}}$ precede those of $Z_j_{\overline{k}}$ for $i < j$.

Example 1.7. Let $V$ be a smooth projective surface defined over $\mathbb{R}$ and let $B_0$ and $\overline{B}_0$ be a pair of smooth $\mathbb{C}$-rational curves in $V_{\mathbb{C}}$ exchanged by the real structure $\sigma$ and intersecting transversally in a single point. Then $B_0 \cup \overline{B}_0$ is the complexification of a geometrical rational chain $B$ on $V$ which is SNC minimal over $\mathbb{R}$ even if $B_0^2 = \overline{B}_0^2 = -1$.

1.3. Logarithmic Kodaira dimension.

By virtue of Nagata compactification [26] and Hironaka desingularization [15] theorems, every smooth surface $S$ defined over $k$ admits an open immersion $S \hookrightarrow (V, B)$ into a smooth complete surface with SNC boundary divisor $B = V \setminus S$, both defined over $k$. The (logarithmic) Kodaira dimension $\kappa(S)$ of $S$ is then defined as the Iitaka dimension [18] of the pair $(V, \omega_V \log B)$ where $\omega_V \log B = (\det \Omega^1_{V/k}) \otimes \mathcal{O}_V(B)$.

More explicitly, letting $\mathcal{R}(V, B) = \bigoplus_{m \geq 0} H^0(V, \omega_V \log B)^{\otimes m}$ be the log-canonical ring of the pair $(V, B)$, we have $\kappa(S) = \text{tr} \deg_k \mathcal{R}(V, B) - 1$ if $H^0(V, \omega_V \log B)^{\otimes m} \neq 0$ for sufficiently large $m$ and otherwise, if $H^0(V, \omega_V \log B)^{\otimes m} = 0$ for every $m \geq 1$, then we set by convention $\kappa(S) = -\infty$ and we say for short that $\kappa(S)$ is negative.

The so-defined element $\kappa(S) \in \{-\infty, 0, 1, 2\}$ is independent of the choice of a smooth complete model $(V, B)$ [19], and it coincides with the usual notion of Kodaira dimension in the case where $S$ is already complete. Note that by definition, we have $\kappa(S) \geq \kappa(V)$. Furthermore, it is invariant under arbitrary extensions of the base field $k$, as a consequence of the flat base change theorem [13 Proposition III.9.3].
1.4. Euclidean topologies. Recall that when $k = \mathbb{R}$ or $\mathbb{C}$, the set of $k$-rational point of a $k$-variety $X$ can be endowed with the euclidean topology. Namely, every $k$-rational point $p$ admits an affine open neighborhood $U_p$ and the choice of a closed immersion $j : U_p \hookrightarrow \mathbb{A}^N_k$ enables to equip $j(U_p(k))$ with the subspace topology induced by the usual euclidean topology on $\mathbb{A}^N_k$. The so-constructed topology on $X(k)$ is well-defined and independent of the choices made \cite[Lemma 1 and Proposition 2]{29}. When $X$ is smooth, $X(k)$ equipped with the euclidean topology is a topological manifold which can be further equipped with the structure of a $C^\infty$-manifold locally inherited by the standard $C^\infty$-structure on $k^N$.

**Definition 1.8.** Given a real algebraic variety $X$, we always consider the sets $X(\mathbb{R})$ and $X_\mathbb{C}(\mathbb{C})$ as equipped with their respective euclidean topologies. The real structure $\sigma$ on $X_\mathbb{C}$ gives rise to a continuous involution of $X_\mathbb{C}(\mathbb{C})$, and we consider $X(\mathbb{R})$ as a subspace of $X_\mathbb{C}(\mathbb{C})$ via its identification with the set $X_\mathbb{C}(\mathbb{C})^\sigma$ of fixed point of $\sigma$.

Recall that given a coefficient ring $A$, a topological manifold $M$ is called $A$-acyclic if all its homology groups $H^i(M; A)$, $i \geq 1$, are trivial. The following classical topological characterization of $\mathbb{R}^2$ as a smooth manifold will be frequently used in the article as a tool to recognize real algebraic surfaces with real locus diffeomorphic to $\mathbb{R}^2$:

**Proposition 1.9.** A smooth 2-dimensional relatively compact real manifold $M \subset \overline{M}$ with connected boundary $\overline{M} \setminus M$ is diffeomorphic to $\mathbb{R}^2$ if and only if it is connected and $\mathbb{Z}_2$-acyclic.

**Proof.** The $\mathbb{Z}_2$-Betti numbers of $\mathbb{R}^2$ being known, let us assume that the $\mathbb{Z}_2$-Betti numbers of $M$ are the same and prove that $M \approx \mathbb{R}^2$. The universal coefficient theorem for homology, \cite[Theorem 3.A3, p. 264]{14} implies then that the rational Betti numbers of $M$ are those of $\mathbb{R}^2$. Indeed $H^2(M; \mathbb{Z})$ has no torsion because $M$ is a topological manifold of dimension 2 and then the $\mathbb{Z}_2$-dimension of $H^2(M; \mathbb{Z}) \otimes \mathbb{Z}_p$ is equal to $b_2(M)$ for any prime $p$. Furthermore, the same theorem gives that $H_2(M; \mathbb{Z}_2)$ is isomorphic to the direct sum of $H_2(M; \mathbb{Z}) \otimes \mathbb{Z}_2$ with the 2-torsion of $H_1(M; \mathbb{Z})$. Hence $b_2(M; \mathbb{Z}_2) = 0$ implies at the same time that $b_2(M) = 0$ and that $H_1(M; \mathbb{Z})$ has no 2-torsion. Then the $\mathbb{Z}_2$-dimension of $H_1(M; \mathbb{Z}) \otimes \mathbb{Z}_2$ is equal to $b_1(M)$ which is equal to $b_1(M; \mathbb{Z}_2)$ as $H_0(M; \mathbb{Z})$ has no torsion. It remains to conclude thanks to the classification of 2-dimensional compact smooth manifold with boundary that $M \approx \mathbb{R}^2$ (see e.g. \cite[Theorem 9.1.9]{16} for a proof of the surface classification using Morse theory).

The following lemma summarises the structure of the real locus of a geometrically rational tree $B \subset S$ in a real algebraic surface $S$.

**Lemma 1.10.** Let $B$ be a geometrically rational tree in a smooth real algebraic surface $V$. Then $B(\mathbb{R})$ is either empty, or a point, or homeomorphic to a connected union of circles $S^1$. Furthermore, in the case where $B$ is a geometrically rational chain which does not consist of $\mathbb{R}$-rational components only, then $B(\mathbb{R})$ is either empty, or a point or homeomorphic to $S^1$.

**Proof.** The geometrically rational tree $B_\mathbb{C}$ being defined over $\mathbb{R}$, its dual graph $\Gamma B_\mathbb{C} = (\Gamma, B_\mathbb{C}, \Gamma e B_\mathbb{C})$ inherits an action of the real structure $\sigma$ on $V_\mathbb{C}$. Since $\Gamma B_\mathbb{C}$ is bounded, this action has a global fixed point $x_0$. If $x_0 \in \Gamma e B_\mathbb{C}$ then $\sigma$ acts on $\Gamma$ as a reflexion with respect to $x_0$ and $B(\mathbb{R})$ consists of a single point, the intersection point of a pair of irreducible components of $B_\mathbb{C}$ exchanged by the involution $\sigma$. Otherwise, if $x_0 \in \Gamma, B_\mathbb{C}$, then it corresponds to an irreducible component $B_0$ of $B_\mathbb{C}$ defined over $\mathbb{R}$, which can be isomorphic to either $\mathbb{P}_\mathbb{R}^1$ or to a conic with empty real locus. Let $\Gamma'$ be the tree obtained from $\Gamma$ by contracting to $x_0$ the maximal connected fixed subtree containing $x_0$. The real structure $\sigma$ acts now on $\Gamma'$ as a reflexion of center $x_0$. So in any case, $B(\mathbb{R})$ is connected and the first result follows. Now if $B_\mathbb{C}$ is a chain with at least one $\mathbb{R}$-rational component and one non $\mathbb{R}$-rational component, then there is a pair $x_1, x_1'$ of vertices adjacent to $x_0$ in $\Gamma'$ corresponding to a pair $B_1, B_1'$ of irreducible components of
2. ALGEBRO-TOPOLOGICAL CHARACTERIZATIONS OF Q-HOMOLOGY EUCLIDEAN PLANES

**Definition 2.1.** A homology (resp. Q-homology) algebraic euclidean plane is a smooth real algebraic surface $S$ such that $S(\mathbb{R})$ is diffeomorphic to $\mathbb{R}^2$ and whose complexification $S_C(\mathbb{C})$ has the integral (resp. rational) homology type of the complex plane $\mathbb{A}_C^2$, when equipped with the euclidean topology. Equivalently, $S(\mathbb{R})$ is diffeomorphic to $\mathbb{R}^2$ and $S_C(\mathbb{C})$ is $\mathbb{Z}$-acyclic (resp. Q-acyclic).

**Proposition 2.2.** A Q-homology algebraic euclidean plane $S$ is affine and $\mathbb{R}$-rational.

**Proof.** By virtue of respective results of Fujita [9] and Gurjar-Pradeep-Shastri [11, 12], $S$ is affine and geometrically rational. Letting $(V, B)$ be a smooth projective completion of $S$ defined over $\mathbb{R}$, $V$ is geometrically rational and, because $S(\mathbb{R}) = \mathbb{R}^2$, has non empty connected real locus $V(\mathbb{R})$. The $\mathbb{R}$-rationality of $V$, whence of $S$, then follows from [30, Corollary VI.6.5] (see also [21, Theorem 33, p. 206]).

2.1. Criteria for Q-acyclicity and structure of the real locus. Every smooth real affine surface $S$ admits a completion $S \hookrightarrow (V, B)$ into a smooth projective surface $V$, with geometrically connected SNC boundary divisor $B = V \setminus S$. The following well-known lemma provides a characterization of the $\mathbb{Q}$-acyclicity of $S_C(\mathbb{C})$ in terms of the geometry of $B$ and the properties of the natural map $j_C : \mathbb{Z}(B_C) \to \text{Cl}(V_C)$ associating to an irreducible component of $B_C$ its class in the divisor class group $\text{Cl}(V_C)$ of $V_C$.

**Lemma 2.3.** Let $S$ be a smooth $\mathbb{R}$-rational real affine surface and let $S \hookrightarrow (V, B)$ be a smooth projective completion with SNC boundary divisor $B$. Then the surface $S_C(\mathbb{C})$ is $\mathbb{Q}$-acyclic if and only if $B$ is a geometrically rational tree and the map $j_C \otimes \text{id} : \mathbb{Z}(B_C) \otimes \mathbb{Q} \to \text{Cl}(V_C) \otimes \mathbb{Q}$ is an isomorphism. If in an addition, $j_C : \mathbb{Z}(B_C) \to \text{Cl}(V_C)$ is an isomorphism, then $S_C(\mathbb{C})$ is $\mathbb{Z}$-acyclic.

**Proof.** The long exact sequence of relative homology for the pair $(V_C(\mathbb{C}), B_C(\mathbb{C}))$ together with Poincaré duality $H_i(V_C(\mathbb{C}), B_C(\mathbb{C}); \mathbb{Z}) \simeq H^{4-i}(S_C(\mathbb{C}); \mathbb{Z})$ yields

$$0 \to H^1(S_C(\mathbb{C}); \mathbb{Z}) \to H_2(B_C(\mathbb{C}); \mathbb{Z}) \overset{j_C}{\to} H_2(V_C(\mathbb{C}); \mathbb{Z}) \to H^2(S_C(\mathbb{C}); \mathbb{Z}) \to H_1(B_C(\mathbb{C}); \mathbb{Z}) \to 0$$

where we have used that $H_1(V_C(\mathbb{C}); \mathbb{Z}) = H_3(V_C(\mathbb{C}); \mathbb{Z}) = 0$ as $V_C$ is rational and $H_0(V_C(\mathbb{C}); \mathbb{Z}) \simeq H_0(B_C(\mathbb{C}); \mathbb{Z}) \simeq \mathbb{Z}$ because $V_C$ is connected by hypothesis and $B_C$ is connected as $S_C$ is affine. The group $H_2(B_C(\mathbb{C}); \mathbb{Z})$ is a free $\mathbb{Z}$-module generated by the fundamental classes of the irreducible components of $B_C(\mathbb{C})$, and since every irreducible component of $B_C$ is a smooth projective curve, the group $H_1(B_C(\mathbb{C}); \mathbb{Z})$ is either trivial if $B_C$ is a tree of smooth proper rational curves or a free $\mathbb{Z}$-module of positive rank otherwise. Furthermore, since $S_C$ is affine, $S_C(\mathbb{C})$ has the homotopy type of a CW-complex of real dimension 2. So $H_2(S_C(\mathbb{C}); \mathbb{Z})$ is a free $\mathbb{Z}$-module and applying the universal coefficients theorem for cohomology, we obtain that $H^2(S_C(\mathbb{C}); \mathbb{Z}) \simeq H_2(S_C(\mathbb{C}); \mathbb{Z}) \oplus \text{Tors}(H_1(S_C(\mathbb{C}); \mathbb{Z}))$ while $H^1(S_C(\mathbb{C}); \mathbb{Z}) \simeq \text{Free}(H_1(S_C(\mathbb{C}); \mathbb{Z})).$ So $S_C(\mathbb{C})$ is $A$-acyclic, where $A = \mathbb{Z}$ or $\mathbb{Q}$, if and only if $B_C$ is a tree of rational curves and the map $H_2(B_C(\mathbb{C}); \mathbb{Z}) \otimes \mathbb{A} \to H_2(V_C(\mathbb{C}); \mathbb{Z}) \otimes \mathbb{A}$ is an isomorphism. To conclude, it remains to observe that since $p_B(V) = q(V) = 0$ because $V_C$ is rational, the cycle map $\text{cl} : \text{Cl}(V_C) \to H_2(V_C(\mathbb{C}); \mathbb{Z})$ which associates to every irreducible complex curve $D \subset V_C$ the fundamental class of $D(\mathbb{C})$ is an isomorphism, and that we have a commutative diagram

$$
\begin{array}{ccc}
H_2(B_C(\mathbb{C}); \mathbb{Z}) & \overset{j}{\to} & H_2(V_C(\mathbb{C}); \mathbb{Z}) \\
\uparrow & & \uparrow \\
\mathbb{Z}(B_C) & \overset{j_C}{\to} & \text{Cl}(V_C).
\end{array}
$$

$\square$

Not all smooth real algebraic surface $S$ with Q-acyclic complexification $S_C$ have their real locus diffeomorphic to $\mathbb{R}^2$. For instance, the real locus of the complement $S$ of a smooth conic $C$ in $\mathbb{P}_R^2$ is either diffeomorphic to $\mathbb{R}^2$ if $C(\mathbb{R}) = \emptyset$ of the disjoint union of $\mathbb{R}^2$ with a Möebius band otherwise. In this
context, the algebro-topological criterion of Lemma 2.3 can be refined as follows: since the pair \((V, B)\) is defined over \(\mathbb{R}\), the free abelian groups \(\mathbb{Z}(B_C)\) and \(\text{Cl}(V_C)\) both inherit a structure of \(G\)-module for the group \(G = \{1, \sigma\} \cong \mathbb{Z}_2\) generated by the real structure \(\sigma\) on \(V\). Furthermore, the complexification of divisors gives rise to a homomorphism \(\text{Cl}(V) \rightarrow \text{Cl}(V_C)\) whose image is contained in the subgroup \(\text{Cl}(V_C)|^G\).

Recall that for every \(G\)-module \(M\), the Galois cohomology groups \(H^1(G, M) = \text{Ker}(\text{id}_M + \sigma)/\text{Im}(\text{id}_M - \sigma)\) and \(H^2(G, M) = \text{Ker}(\text{id}_M - \sigma)/\text{Im}(\text{id}_M + \sigma)\) are both \(\mathbb{Z}_2\)-vector spaces. We have the following criterion:

**Proposition 2.4.** Let \(S\) be a smooth \(\mathbb{R}\)-rational real algebraic surface admitting a smooth projective completion \(S \hookrightarrow (V, B)\) defined over \(\mathbb{R}\) whose boundary \(B\) is a geometrically rational tree and let \(j_C : \mathbb{Z}(B_C) \rightarrow \text{Cl}(V_C)\) be the natural homomorphism. Then the following hold:

1) \(S(\mathbb{R})\) is diffeomorphic to \(\mathbb{R}^2\) if and only if \(B(\mathbb{R})\) is non empty and the homomorphism \(H^2(j_C) : H^2(G, \mathbb{Z}(B_C)) \rightarrow H^2(G, \text{Cl}(V_C))\) induced by \(j_C\) is an isomorphism.

2) If in addition \(\text{Cl}(V) \rightarrow \text{Cl}(V_C)\) is an isomorphism, then the second condition can be replaced by the requirement that \(j_C \otimes \text{id} : \mathbb{Z}(B_C) \otimes_{\mathbb{Z}} \mathbb{Z}_2 \rightarrow \text{Cl}(V_C) \otimes_{\mathbb{Z}} \mathbb{Z}_2\) is an isomorphism.

**Proof.** The long exact sequence of homology for the pair \((V, B)\) together with Poincaré duality \(H_i(V(\mathbb{R}), B(\mathbb{R}); \mathbb{Z}_2) \cong H^{2-i}(S(\mathbb{R}); \mathbb{Z}_2)\) yields the exact sequence

\[
0 \rightarrow H_2(V(\mathbb{R}); \mathbb{Z}_2) \rightarrow H^0(S(\mathbb{R}); \mathbb{Z}_2) \rightarrow H_1(B(\mathbb{R}); \mathbb{Z}_2) \xrightarrow{j} H_1(V(\mathbb{R}); \mathbb{Z}_2) \rightarrow H^1(S(\mathbb{R}); \mathbb{Z}_2) \rightarrow 0.
\]

Again, the \(\mathbb{R}\)-rationality of \(V\) implies that \(V(\mathbb{R})\) is non empty and connected, so that \(H_0(V(\mathbb{R}); \mathbb{Z}_2) \cong H_2(V(\mathbb{R}); \mathbb{Z}_2) = \mathbb{Z}_2\). Furthermore, by virtue of Lemma 1.10, \(B(\mathbb{R})\) is either empty, or a point or a connected union of curves homeomorphic to \(S^1\). So either \(H_0(B(\mathbb{R}); \mathbb{Z}_2) = 0\) if \(B(\mathbb{R})\) is empty, and then \(H^2(S(\mathbb{R}); \mathbb{Z}_2) = \mathbb{Z}_2\), or the map \(H_0(B(\mathbb{R}); \mathbb{Z}_2) \rightarrow H_0(V(\mathbb{R}); \mathbb{Z}_2)\) is an isomorphism. By the universal coefficient theorem for cohomology, we have \(H^0(S(\mathbb{R}); \mathbb{Z}_2) \cong H_0(S(\mathbb{R}); \mathbb{Z}_2)\), \(H^1(S(\mathbb{R}); \mathbb{Z}_2) \cong H_1(S(\mathbb{R}); \mathbb{Z}_2)\) and \(H^2(S(\mathbb{R}); \mathbb{Z}_2) \cong H_2(S(\mathbb{R}); \mathbb{Z}_2)\). Combined with Proposition 1.9, we conclude that \(S(\mathbb{R}) \cong \mathbb{R}^2\) if and only if \(B(\mathbb{R})\) is not empty and \(j_C\) is an isomorphism. The first assertion is then a consequence of the properties of the cycle map \(\text{cl} : \text{Cl}(V) \rightarrow H_1(V(\mathbb{R}); \mathbb{Z}_2)\) which is defined as follows: the image of every irreducible curve \(D \subset V\) defined over \(\mathbb{R}\), is equal either to the fundamental class of \(D(\mathbb{R})\), whose existence is proven in \([21, \S 3]\), if \(D(\mathbb{R})\) is of dimension 1 and to 0 otherwise. This construction does indeed define a homomorphism, which is surjective provided that \(p_g(V) = q(V) = 0\) \([30, \S 3.1]\). Since \(\text{Cl}(V) \rightarrow \text{Cl}(V_C)|^G\) is surjective as \(V(\mathbb{R})\) is not empty \([30, \text{Proposition I.4.5}]\), we obtain a surjective homomorphism \(\text{Cl}(V_C)|^G = \text{Ker}(\text{id}_{\text{Cl}(V_C)} - \sigma) \rightarrow H_1(V(\mathbb{R}); \mathbb{Z}_2)\). Its kernel contains the subgroup \((\text{id}_{\text{Cl}(V_C)} + \sigma)\text{Cl}(V_C)\) and hence, we obtain a surjective homomorphism \(H^2(G, \text{Cl}(V_C)) \rightarrow H_1(V(\mathbb{R}); \mathbb{Z}_2)\). The latter is in fact an isomorphism thanks to a Theorem of Krasnov \([22]\) asserting that \(b_1(V(\mathbb{R}); \mathbb{Z}_2) = \dim H^2(G, \text{Cl}(V_C))\) when \(V(\mathbb{R})\) is not empty and \(b_1(V_C(\mathbb{C}); \mathbb{Z}_2) = 0\), which is the case since \(V\) is \(\mathbb{R}\)-rational. Along the same lines, we get an isomorphism \(H^2(G, \mathbb{Z}(B_C)) \rightarrow H_1(B(\mathbb{R}); \mathbb{Z}_2)\) which makes the following diagram commutative

\[
\begin{array}{ccc}
H_1(B(\mathbb{R}); \mathbb{Z}_2) & \xrightarrow{j} & H_1(V(\mathbb{R}); \mathbb{Z}_2) \\
\uparrow \Leftrightarrow & & \uparrow \Leftrightarrow \\
H^2(G, \mathbb{Z}(B_C)) & \xrightarrow{H^2(j_C)} & H^2(G, \text{Cl}(V_C)).
\end{array}
\]

For the second assertion, it is enough to observe that if \(\text{Cl}(V) \rightarrow \text{Cl}(V_C)\) is an isomorphism then \(\text{Cl}(V_C) = \text{Cl}(V_C)|^G\), \((\text{id}_{\text{Cl}(V_C)} + \sigma)\text{Cl}(V_C) = 2\text{Cl}(V_C)\) and so \(H^2(G, \text{Cl}(V_C)) = \text{Cl}(V_C) \otimes_{\mathbb{Z}} \mathbb{Z}_2\). For the same reason, \(H^2(G, \mathbb{Z}(B_C)) = \mathbb{Z}(B_C) \otimes_{\mathbb{Z}} \mathbb{Z}_2\) and the conclusion follows. \(\Box\)

2.2. \(Q\)-homology euclidean planes obtained from rational plane curves arrangements. Here we setup a real counterpart of a general blow-up construction already used by tom Dieck and Petrie \([7]\) in the complex case which leads to a rough description of \(Q\)-homology euclidean planes in terms of a datum consisting of a suitable arrangement \(D\) of geometrically rational curves in \(\mathbb{P}_R^2\) and a geometrically rational, geometrically connected subtree \(B\) of the total transform of \(D\) in a log-resolution \(\tau : V \rightarrow \mathbb{P}_R^2\) of the pair \((\mathbb{P}_R^2, D)\). This construction will be refined later on in subsection 3.2.1 to describe in a more precise fashion the structure of homology euclidean planes of general type.
2.2.1. Let $k = \mathbb{R}$ or $\mathbb{C}$ and let $D \subset \mathbb{P}^2_k$ be a reduced curve defined over $k$, with geometrically rational irreducible components. Let $\tau : V \rightarrow \mathbb{P}^2_k$ be a log-resolution of the pair $(\mathbb{P}^2_k, D)$ such that the image of the exceptional locus of $\tau$ is contained in $D$ and let $\tau^{-1}(D)$ be the reduced total transform of $D$. By construction, $\tau^{-1}(D)$ is a geometrically connected SNC divisor with geometrically rational irreducible components, and up to performing additional blow-ups defined over $k$ and with centers on $\tau^{-1}(D)$, we may further assume that any two irreducible components of $\tau^{-1}(D)_C$ intersect each other at most one point. Now suppose that there exists a geometrically rational subtree $B \subset \tau^{-1}(D)$ defined over $k$ satisfying the following properties:

a) The support of $B$ contains the proper transform $\tau^{-1}_* D$ of $D$,

b) $\text{rk}(\mathbb{Z}\langle \tau^{-1}_*(D)_C \rangle) - \text{rk}(\mathbb{Z}\langle B_C \rangle) = \text{rk}(\mathbb{Z}\langle D_C \rangle) - 1$.

By construction, the set $\mathcal{E}_0$ of irreducible components of $\tau^{-1}(D)_C$ not contained in the support of $B_C$ is a subset of the set $\mathcal{E}$ of exceptional divisors of $\tau_C : V_C \rightarrow \mathbb{P}^2_{\mathbb{C}}$. The total transform $\tau^*_C C$ of every irreducible curve $C \subset \mathbb{P}^2_k$ can be written in the form $\tau^*_C C = \tau^{-1}_* C + \Delta$ where $\Delta$ is an element of the free abelian group $\mathbb{Z}\langle \mathcal{E} \rangle = \bigoplus_{E \in \mathcal{E}} \mathbb{Z} \cdot E$ generated by the exceptional divisors of $\tau_C$, and so, projecting onto the subgroup $\mathbb{Z}\langle \mathcal{E}_0 \rangle = \bigoplus_{E \in \mathcal{E}_0} \mathbb{Z} \cdot E$ of $\mathbb{Z}\langle \mathcal{E} \rangle$ and restricting to the kernel $R$ of the natural homomorphism $d_C : \mathbb{Z}\langle D_C \rangle \rightarrow \text{Cl}(\mathbb{P}^2_{\mathbb{C}})$, we obtain a homomorphism of $\mathbb{Z}$-module $\psi : R \rightarrow \mathbb{Z}\langle \mathcal{E}_0 \rangle$. When $k = \mathbb{R}$, the fact that $D$ and $B$ are defined over $\mathbb{R}$ guarantees that $R$ and $\mathbb{Z}\langle \mathcal{E}_0 \rangle$ have the additional structures of $G$-modules for the Galois group $G = \{1, \sigma\} \cong \mathbb{Z}_2$ generated by the real structure $\sigma$ on $V_C$ and that $\psi$ is a homomorphism of $G$-module.

**Lemma 2.5.** (See also [4, 3.6-3.9]) With the notation above, the following hold for the smooth quasi-projective surface $S = V \setminus B$:

a) $S_C(\mathbb{C})$ is $\mathbb{Q}$-acyclic if and only if $\varphi \otimes \text{id} : R \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \mathbb{Z}\langle \mathcal{E}_0 \rangle \otimes_{\mathbb{Z}} \mathbb{Q}$ is an isomorphism.

b) $S_C(\mathbb{C})$ is $\mathbb{Z}$-acyclic if and only if $d_C : \mathbb{Z}\langle D_C \rangle \rightarrow \text{Cl}(\mathbb{P}^2_{\mathbb{C}})$ is surjective and $\varphi : R \rightarrow \mathbb{Z}\langle \mathcal{E}_0 \rangle$ is an isomorphism.

c) If $k = \mathbb{R}$ then $S(\mathbb{R})$ is diffeomorphic to $\mathbb{R}^2$ if and only if $H^2(d_C) : H^2(G, \mathbb{Z}\langle D_C \rangle) \rightarrow H^2(G, \text{Cl}(\mathbb{P}^2_{\mathbb{C}})) \cong \mathbb{Z}_2$ is surjective and $H^2(\varphi) : H^2(G, R) \rightarrow H^2(G, \mathbb{Z}\langle \mathcal{E}_0 \rangle)$ is an isomorphism. Furthermore, when $d : \mathbb{Z}\langle D \rangle \rightarrow \text{Cl}(\mathbb{P}^2_{\mathbb{Z}})$ is surjective and $\text{Cl}(V) \rightarrow \text{Cl}(V_C)$ is an isomorphism, the second condition is satisfied if and only if $\varphi \otimes \text{id} : R \otimes_{\mathbb{Z}} \mathbb{Z}_2 \rightarrow \mathbb{Z}\langle \mathcal{E}_0 \rangle \otimes_{\mathbb{Z}} \mathbb{Z}_2$ is an isomorphism.

**Proof.** Both assertions follows essentially from Lemma 2.3 and Proposition 2.4 through a diagram chasing in the following commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & R \\
\uparrow & & \uparrow \\
\mathbb{Z}\langle B_C \rangle \oplus \mathbb{Z}\langle \mathcal{E}_0 \rangle & \rightarrow & \mathbb{Z}\langle V_C \rangle \simeq \text{Cl}(\mathbb{P}^2_{\mathbb{C}}) \oplus \mathbb{Z}\langle \mathcal{E} \rangle \\
\end{array}
\]

More precisely, letting $\mathcal{E}_1 = \mathcal{E} \setminus \mathcal{E}_0$, we have a natural identification $\mathbb{Z}\langle B_C \rangle = \mathbb{Z}\langle (\tau^{-1}_* D)_C \rangle \oplus \mathbb{Z}\langle \mathcal{E}_1 \rangle$. Furthermore, for every irreducible component $D_{i,C}$ of $D_C$, the following identity

$$
\tau^* D_{i,C} = \tau^{-1}_* D_{i,C} + \sum_{e \in \mathcal{E}_0} a_i(e) E_e + \sum_{e \in \mathcal{E}_1} a_i(e) E_e \sim \deg D_{i,C} \cdot \tau^* \ell_C
$$

holds in $\text{Cl}(V_C)$, where $\ell_C$ denotes the complexification of a general line $\ell \simeq \mathbb{P}^1 \subset \mathbb{P}^2$ Since by definition,

$$
\langle c \rangle = \langle c \rangle |_{\mathbb{Z}\langle (\tau^{-1}_* D)_C \rangle \oplus \mathbb{Z}\langle \mathcal{E}_1 \rangle} : \mathbb{Z}\langle B_C \rangle = \mathbb{Z}\langle (\tau^{-1}_* D)_C \rangle \oplus \mathbb{Z}\langle \mathcal{E}_1 \rangle \rightarrow \text{Cl}(V_C) \simeq (\mathbb{Z}\langle \tau^* \ell_C \rangle \oplus \mathbb{Z}\langle \mathcal{E}_0 \rangle) \oplus \mathbb{Z}\langle \mathcal{E}_1 \rangle
$$

where $\langle c \rangle |_{\mathbb{Z}\langle (\tau^{-1}_* D)_C \rangle \oplus \mathbb{Z}\langle \mathcal{E}_1 \rangle} : \mathbb{Z}\langle (\tau^{-1}_* D)_C \rangle \oplus \mathbb{Z}\langle \mathcal{E}_1 \rangle \rightarrow \deg D_{i,C} \cdot \tau^* \ell_C - \sum_{e \in \mathcal{E}_0} a_i(e) E_e - \sum_{e \in \mathcal{E}_1} a_i(e) E_e$, it follows that $j_C$ is an isomorphism if and only if is $\Phi = -\text{pr}_1 \circ j_C |_{\mathbb{Z}\langle (\tau^{-1}_* D)_C \rangle : \mathbb{Z}\langle (\tau^{-1}_* D)_C \rangle \rightarrow \mathbb{Z}\langle \tau^* \ell_C \rangle \oplus \mathbb{Z}\langle \mathcal{E}_0 \rangle}$. This holds if and only if $\text{pr}_1 \circ \Phi : \mathbb{Z}\langle (\tau^{-1}_* D)_C \rangle \rightarrow \mathbb{Z}\langle \tau^* \ell_C \rangle$ is surjective and $\text{pr}_2 \circ \Phi |_{R} : R \rightarrow \mathbb{Z}\langle \mathcal{E}_0 \rangle$ is an isomorphism. Assertions a) and b) follow since by construction, these maps coincide with $-d_C : \mathbb{Z}\langle D_C \rangle \rightarrow \text{Cl}(\mathbb{P}^2_{\mathbb{C}})$ and $\varphi : R \rightarrow \mathbb{Z}\langle \mathcal{E}_0 \rangle$ respectively.

For the last assertion, we note that $\Phi$ is a homomorphism of $G$-modules and that $H^2(j_C) : H^2(G, \mathbb{Z}\langle B_C \rangle) \rightarrow H^2(G, \text{Cl}(V_C))$ is an isomorphism if and only if so is $H^2(\Phi)$. Furthermore, letting $v : \mathbb{Z}\langle (\tau^{-1}_* D)_C \rangle$ be
the injective homomorphism of \( G \)-module mapping an element \( \sum \lambda_i D_{i,C} \in R \subset \mathbb{Z}(D_C) \) to \( \sum \lambda_i \tau^{-1}_i D_{i,C} \in \mathbb{Z}(\tau^{-1}(D)_C) \), we have a commutative diagram

\[
\begin{array}{ccc}
H^2(G, \mathbb{Z}(\tau^{-1}(D)_C)) & \xrightarrow{H^2(\Phi)} & H^2(G, \mathbb{Z}(\tau^* \ell_C) \oplus \mathbb{Z}(\mathcal{E}_0)) \\
H^2(v) & \uparrow & H^2(\text{pr}_2) \\
H^2(G, R) & \xrightarrow{H^2(\varphi)} & H^2(G, \mathbb{Z}(\mathcal{E}_0)).
\end{array}
\]

The homomorphism \( H^2(\text{pr}_2) \) is clearly surjective and we claim that \( H^2(v) \) is injective. Indeed, let \( r \in \text{Ker}(\text{id}_R - \sigma) \) be such that \( v(r) \in \text{Im}(\text{id}_\mathbb{Z}(\tau^{-1}(D)_C) + \sigma) \). By assumption \( r \) can be written in the form \( \sum_{i \in I} \lambda_i D_{i,C} + \sum_{j \in I} \mu_j (D_{j,C} + \sigma(D_{j,C})) \), where \( D_{i,C} \) is \( \sigma \)-invariant while \( \sigma(D_{j,C}) \neq D_{j,C} \), and where the coefficients \( \lambda_i, \mu_j \in \mathbb{Z} \) satisfy the identity \( \deg(r) = \sum_{i \in I} \deg(D_{i,C}) + \sum_{j \in I} \mu_j \deg(D_{j,C}) = 0 \). On the other hand, the hypothesis that \( v(r) \in \text{Im}(\text{id}_\mathbb{Z}(\tau^{-1}(D)_C) + \sigma) \) means that there exists

\[
s = \sum \alpha_i \tau^{-1}_i D_{i,C} + \sum (\beta_j \tau^{-1}_j D_{j,C} + \gamma_j \sigma(\tau^{-1}_j D_{j,C})) \in \mathbb{Z}(\tau^{-1}(D)_C)
\]

such that \( v(r) = s + \sigma(s) \). This implies that \( 2 \deg(s) = 2 \sum \alpha_i \deg(D_{i,C}) + 2 \sum (\beta_j + \gamma_j) \deg(D_{j,C}) = \deg(r) = 0 \). So \( s \in \text{Im}(v) \) and \( r \in \text{Im}(\text{id}_R + \sigma) \). Now, since \( H^2(\text{pr}_1 \circ \text{pr}_2) : H^2(G, R) \to \mathbb{Z}_2(\tau^* \ell_C) \) is the trivial map, we conclude that \( H^2(\Phi) \) is an isomorphism if and only if \( H^2(d_C) : H^2(G, \mathbb{Z}(D_C)) \to H^2(G, \text{Cl}(\mathbb{P}^2_C)) \cong \mathbb{Z}_2(\ell_C) \) is surjective and \( H^2(\varphi) \) is an isomorphism, as desired.

**Remark 2.6.** In the construction given in §2.2.1 above, the initial configuration can be replaced by any pair \((V_0, D)\) consisting of a smooth \( k \)-rational projective surface \( V_0 \) and a reduced geometrically rational curve \( D \) defined over \( k \). Letting \( \tau : V \to V_0 \) be a log-resolution of the pair \((V_0, D)\) such that the image of the exceptional locus of \( \tau \) is contained in \( D \), the geometrically rational subtree \( B \subset \tau^{-1}(D) \) containing the proper transform \( \tau^{-1}_*(D) \) should then be chosen so that \( \text{rk}(\mathbb{Z}(\tau^{-1}(D)_C)) - \text{rk}(\mathbb{Z}(\mathcal{B}_C)) = \text{rk}(\mathbb{Z}(\mathcal{D}_C)) - \text{rk}(\mathbb{Z}(\mathcal{B}_C)) \). Letting \( R \) be the kernel of the homomorphism \( d_C : \mathbb{Z}(\mathcal{D}_C) \to \text{Cl}(V_0, C) \) and \( \mathcal{E}_0 \) be as in §2.2.1 the same diagram chasing argument as in the proof of the previous lemma shows that the surface \( S(C) = (V_0 \setminus B_0)(C) \) is \( \mathbb{Q} \)-acyclic if and only if the induced homomorphism \( \varphi \circ \text{id} : R \otimes \mathbb{Q} \to \mathbb{Z}(\mathcal{E}_0) \otimes \mathbb{Q} \) is an isomorphism (see [2.3.8]).

**Theorem 2.7.** Let \( k = \mathbb{R} \) or \( \mathbb{C} \) and let \( S \) be a smooth \( k \)-rational surface such that \( S(C) \) is \( \mathbb{Q} \)-acyclic. In the case where \( k = \mathbb{R} \), assume further that \( S(\mathbb{R}) \) is non compact. Then there exists an arrangement \( D \) of reduced geometrically rational curves in \( \mathbb{P}^2_k \) and a rational subtree \( B \) of the total transform of \( D \) in a log-resolution \( \tau : V \to \mathbb{P}^2_k \) of the pair \((V, D)\) satisfying properties a) and b) in §2.2.1 above such that \( S \cong V \setminus B \).

**Proof.** Let \( S \hookrightarrow (V', B') \) be a smooth projective completion of \( S \) defined over \( k \), with geometrically rational tree boundary \( B' \). Since \( V \) is \( k \)-rational, the output \( W \) of a MMP process \( \alpha : V \to W \) over \( k \) ran from \( V \) is isomorphic over \( k \) to either \( \mathbb{P}^2_k \), or to a Hirzebruch surface \( \pi_N : \mathbb{F}_n = \mathbb{P}(O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-n)) = \mathbb{P}^2_k, n \in \mathbb{Z}_{>0} \setminus \{1\} \), or to the smooth quadric \( Q = \{x^2 + y^2 + z^2 - t^2 = 0\} \subset \mathbb{P}^3_k \), the latter being isomorphic to \( \mathbb{F}_0 \) when \( k = \mathbb{C} \). Let us assume for the moment \( W \cong \mathbb{P}^2_k \). Since \( S \) is affine, the geometrically rational tree \( B' \) is the support of an ample divisor and hence it cannot be fully contained in the exceptional locus of \( \alpha \). Its image \( D = \alpha_*(B') \) is thus a reduced divisor defined over \( k \), with geometrically rational irreducible components, containing the image of the exceptional locus of \( \alpha \). Furthermore, since \( S(C) \) is \( \mathbb{Z} \)-acyclic, the map \( j_C : \mathbb{Z}(\mathcal{D}_C) \to \text{Cl}(\mathbb{P}^2_C) \) is surjective, because \( j_C : \mathbb{Z}(\mathcal{B}_C) \to \text{Cl}(V_0, C) \) is an isomorphism. Let \( \beta : V \to V' \) be a log-resolution of \( \alpha^{-1}(D) \) over \( k \) minimal with respect to the property that any two irreducible components of \( \mathcal{B}_C = ((\alpha \circ \beta)^{-1}(D))_C \) intersect each other in at most one point. By construction, \( \tau = \alpha \circ \beta : V \to \mathbb{P}^2_k \) is a log-resolution of \( \mathbb{P}^2_k \) and \( S \) is isomorphic to the complement in \( V \) of the geometrically rational subtree \( B = \beta^{-1}(B') \) of \( B \). Furthermore, since the exceptional locus of \( \alpha \) is a disjoint union of geometrically rational trees, the image in \( V' \) of the exceptional locus of \( \beta \) is contained in the support of \( B_0 \). The minimality of \( \beta \) then implies that \( \text{rk}(\mathbb{Z}(\mathcal{B}_C)) \) and \( \text{rk}(\mathbb{Z}(\mathcal{B}_C)) \) differ precisely by the number of irreducible components of the exceptional locus \( \text{Exc}(\alpha_\tau) \) of \( \alpha_\tau \) which are not contained in the support of \( B'_C \). Since \( \text{rk}(\mathbb{Z}(\mathcal{B}_C)) = \text{rk}(\mathbb{Z}(\mathcal{D}_C)) + \text{rk}\mathbb{Z}(\text{Exc}(\alpha_\tau) \cap \mathcal{B}_C) \) and since \( \text{rk}(\mathbb{Z}(\mathcal{B}_C)) = \text{rk}(\text{Cl}(\mathcal{B}_C)) = 1 + \text{rk}(\mathbb{Z}(\text{Exc}(\alpha_\tau))) \), because \( S \) is \( \mathbb{Q} \)-homology plane, we conclude that \( \text{rk}(\mathbb{Z}(\mathcal{B}_C)) - \text{rk}(\mathbb{Z}(\mathcal{B}_C)) = \text{rk}(\mathbb{Z}(\mathcal{D}_C)) - 1 \). So \( S \) is
isomorphic to a surface of the form $V \setminus B$ obtained by the procedure described in §2.2.1 above. Now it remains to show that the initial smooth completion $S' \hookrightarrow (V', B')$ and the MMP process $\alpha : V' \to W$ can be chosen so that $W \simeq \mathbb{P}^2_k$. Starting with an arbitrary smooth projective completion $S \hookrightarrow (V_0, B_0)$ with boundary geometrically rational tree $B_0$ and an arbitrary MMP process $\alpha_0 : V_0 \to W_0$, we proceed as follows.

Suppose first that $W_0$ is a Hirzebruch surface $\pi_n : \mathbb{F}_n \to \mathbb{P}^1_k$, $n \geq 2$, with exceptional section $C \simeq \mathbb{P}^1_k$ of self-intersection $-n$. First note that $D_0 = (\alpha_0)_*B_0$ cannot be equal to $C$ only, for otherwise $V \setminus \alpha_0^{-1}(D_0) \subset S = V \setminus B_0$ would contain a complete curve, for instance the inverse image of a section of $\pi_n$ disjoint from $C$, in contradiction with the affineness of $S$. Futhermore every irreducible component of $D_0$ distinct from $C$ intersects $C$ in a finite number of closed points. So if $k = \mathbb{R}$ (resp. $k = \mathbb{C}$), it follows that there exists a non real $C$-rational point $p$ of $\mathbb{P}^1_{\mathbb{R}}$ (resp. a closed points $p \in \mathbb{P}^1_{\mathbb{C}}$) such that the intersection of $\pi_n^{-1}(p)$ with $D_0$ is nonempty and disjoint from $D_0 \cap C$. Let $\varphi : \mathbb{F}_n \dashrightarrow \mathbb{F}_{n-2}$ (resp. $\varphi : \mathbb{F}_n \dashrightarrow \mathbb{F}_{n-1}$) be the elementary birational map consisting of blowing-up a point $q \in \pi_n^{-1}(p) \cap D_0$ and contracting the proper transform of $\pi_n^{-1}(p)$. We obtain a commutative diagram

$$(V_0, B_0) \xrightarrow{f} (V_1, B_1) \quad \quad \alpha_0 \downarrow \quad \quad \downarrow \alpha_1$$

$W_0 = \mathbb{F}_n \dashrightarrow W_1 = \begin{cases} \mathbb{F}_{n-2} & \text{if } k = \mathbb{R} \\ \mathbb{F}_{n-1} & \text{if } k = \mathbb{C} \end{cases}$$

where $f$ and $B_1$ are defined as follows: if $q$ belongs to the image of the exceptional locus of $\alpha_0$ then $f$ is an isomorphism of pairs, otherwise $f$ is the blow-up of the point $q \in B_0(\mathbb{C})$ and $B_1 = f^{-1}(B_0)$. In each case, $B_1$ is a geometrically rational tree defined over $k$, $V_1 \setminus B_1 \simeq S$ and the induced birational map $\alpha_1 : V_1 \to W_1$ is a process of MMP over $k$. Arguing by induction, we reach a smooth projective completion $S \hookrightarrow (V_t, B_t)$ defined over $k$ with geometrically rational tree boundary $B_t$ and a process of MMP over $k$, $\alpha_t : V_t \to \mathbb{F}_t$, where $t = 0, 1$. In the case where $t = 1$, we eventually obtain the desired birational morphism $\tau : V_1 \to \mathbb{F}_1$ defined over $k$ by contracting the negative section of $\pi_1$.

So it remains to treat the case where $W_0$ is isomorphic either to $\mathbb{F}_0 = \mathbb{P}^1_k \times \mathbb{P}^1_k$ or to the smooth quadric $Q = \{x^2 + y^2 + z^2 - t^2 = 0\} \subset \mathbb{P}^2_k$. The hypothesis implies that $D_0 = (\alpha_0)_*B_0$ has a $k$-rational point $p$. Indeed, this is clear if $k = \mathbb{C}$ and, in the case where $k = \mathbb{R}$, the emptiness of $D_0(\mathbb{R})$ would imply that of $\alpha_0^{-1}(D_0)$, and we would have

$$S(\mathbb{R}) = V_0(\mathbb{R}) \setminus B_0(\mathbb{R}) \supset V_0(\mathbb{R}) \setminus \alpha_0^{-1}(D_0)(\mathbb{R}) = W_0(\mathbb{R}) \simeq \begin{cases} \mathbb{T}^2 & \text{if } W_0 = \mathbb{P}^1_k \times \mathbb{P}^1_k \\ \mathbb{S}^2 & \text{if } W_0 = Q, \end{cases}$$

in contradiction with the non compactness of $S(\mathbb{R})$. In the case where $W_0 = \mathbb{P}^1_k \times \mathbb{P}^1_k$, we let $\varphi : \mathbb{P}^1_k \times \mathbb{P}^1_k \dashrightarrow \mathbb{F}_1$ be the blow-up of $p$ followed by the contraction of the fiber of $pr_1$ containing $p$. Similarly as in the previous case, we obtain a commutative diagram

$$(V_0, B_0) \xrightarrow{f} (V_1, B_1) \quad \quad \alpha_0 \downarrow \quad \quad \downarrow \alpha_1$$

$W_0 = \mathbb{P}^1_k \times \mathbb{P}^1_k \dashrightarrow W_1 = \mathbb{F}_1 = W_1,$

where $f$ is either an isomorphism of pairs if $p$ belongs to the image of the exceptional locus of $\alpha_0$, or the blow-up of the point $p \in B_0(k)$ in which case $B_1 = f^{-1}(B_0)$. By construction, $\alpha_1 : V_1 \to \mathbb{F}_1$ is a process of MMP over $k$. The composition of $\alpha_1$ with the contraction of the exceptional section of $\pi_1$ is the desired morphism $\tau : V_1 \to \mathbb{P}^2_k$. Finally, in the remaining case where $k = \mathbb{R}$ and $W_0 = Q$, we let $\varphi : Q \dashrightarrow \mathbb{P}^2_{\mathbb{R}}$ be the blow-up of $p$ followed by the contraction of the unique curve $\Delta \simeq \mathbb{P}^1_{\mathbb{C}}$ passing through $p$ and whose complexification
\( \Delta_C \) is of type \((1, 1)\) in \( \text{Cl}(Q_C) \). Again, we obtain a commutative diagram

\[
\begin{align*}
(V_0, B_0) & \xrightarrow{f} (V_1, B_1) \\
\alpha_0 \downarrow & \downarrow \alpha_1 \\
W_0 = Q & \xrightarrow{\varphi} W_1 = \mathbb{P}^2_{\mathbb{R}}
\end{align*}
\]

where \( f \) is the blow-up of \( p \in B_0(\mathbb{R}) \) and \( B_1 = f^{-1}(B_1) \) if \( p \) does not belong to the image of the exceptional locus of \( \alpha_0 \) and an isomorphism of pairs otherwise. By construction, \((V_1, B_1)\) is a smooth projective completion of \( S \) with geometrically rational tree boundary \( B_1 \) and \( \tau = \alpha_1 : V_1 \rightarrow \mathbb{P}^2_{\mathbb{R}} \) is the desired morphism.

3. Elements of classification of homology euclidean planes

In this section, we consider homology euclidean planes \( S \) up to biregular isomorphisms of schemes of \( \mathbb{R} \) according to their (logarithmic) Kodaira dimension. The cases where \( S \) have Kodaira dimension 0 or \(-\infty\) are easily dispensed by the following observations: first there is no smooth complex homology plane of Kodaira dimension 0 at all [see e.g. [24, Theorem 4.7.1 (1), p. 244]] and second, a complex homology plane of negative Kodaira dimension is isomorphic to \( \mathbb{A}^2_{\mathbb{R}} \) by virtue of [25]. Combined with the fact that there are no nontrivial forms of the affine 2-space over a field of characteristic zero [20], this implies that \( \mathbb{A}^2_{\mathbb{R}} \) is the only homology euclidean plane of non positive Kodaira dimension, up to isomorphisms of schemes over \( \mathbb{R} \).

So we are left with the problem of classifying homology euclidean planes of Kodaira dimension 1 and 2. A complete classification in the first case is given in the next subsection, in the form of a real counterpart of the classification of smooth complex homology planes of Kodaira dimension 1 given by Gurjar and Miyanishi [10] and tom Dieck and Petrie [5].

3.1. Homology euclidean planes of Kodaira dimension 1. Here we establish the real counterpart of the classification of smooth complex homology planes of Kodaira dimension 1 given by Gurjar and Miyanishi [10] and tom Dieck and Petrie [5].

3.1.1. A blow-up construction. Let \( k = \mathbb{R} \) or \( \mathbb{C} \). We let \( D \subseteq \mathbb{P}^2_{\mathbb{R}} \) be the union of a collection \( E_{0,0}, \ldots, E_{n,0} \cong \mathbb{P}^1_{\mathbb{R}} \) be a collection of of \( n+1 \geq 3 \) lines in \( \mathbb{P}^2_{\mathbb{R}} \) intersecting in a same \( k \)-rational point \( x \) and of a general line \( C_1 \cong \mathbb{P}^1_{\mathbb{R}} \). For every \( i = 1, \ldots, n \), we choose a pair of coprime integers \( 1 \leq \mu_i- < \mu_i+ \) in such a way that for \( v_\pm = t (\mu_1-, \ldots, \mu_n-) \in \mathcal{M}_{n,1}(\mathbb{Z}) \) and \( \Delta_+ = \text{diag}(\mu_1, \ldots, \mu_n) \in \mathcal{M}_{n,n}(\mathbb{Z}) \), the following two conditions are satisfied:

\[
(3.1) \quad \text{a)} \ \eta = n - 1 - \sum_{i=1}^{n} \frac{1}{\mu_{i,+}} > 0 \quad \text{and} \quad \text{b)} \ \text{The matrix } \mathcal{N} = \begin{pmatrix} -1 & \cdots & -1 \\ v_- & \cdots & v_+ \end{pmatrix} \quad \text{belongs to } \text{GL}_{n+1}(\mathbb{Z}).
\]

Then we let \( \tau : V \rightarrow \mathbb{P}^2_{\mathbb{R}} \) be the smooth projective surface obtained by the following blow-up procedure:

1) We first blow-up \( x \) with exceptional divisor \( C_0 \cong \mathbb{P}^1_{\mathbb{R}} \). The resulting surface is isomorphic the Hirzebruch surface \( \pi_1 : F_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1_{\mathbb{R}}} \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{R}}}(-1)) \rightarrow \mathbb{P}^1_{\mathbb{R}} \) with \( C_0 \) as the negative section of \( \pi_1 \), the proper transforms of \( E_{0,0}, \ldots, E_{n,0} \) are \( k \)-rational fibers of \( \pi_1 \) while the strict transform of \( C_1 \) is a section of \( \pi_1 \) disjoint from \( C_0 \).

2) Then for every \( i = 1, \ldots, n \), we perform the subdivisorial expansion of at the \( k \)-rational point \( p_i = C_1 \cap E_{i,0} \) with multiplicity \( (\mu_{i,-}, \mu_{i,+}) \) (see Example [13]) and exceptional divisors \( E_{i,1}, \ldots, E_{i,r_i-1}, E_{i,r_i} = A_0(p_i) \).

3) Finally, we perform a sequence of blow-ups starting with the blow-up of a \( k \)-rational point \( p_0 \in E_{0,0} \setminus (C_0 \cup C_1) \), with exceptional divisor \( E_{0,1} \cong \mathbb{P}^1_{\mathbb{R}} \) and continuing with a sequence of \( r_0 - 1 \geq 0 \) blow-ups of \( \mathbb{R} \)-rational points \( p_{0,i} \in E_{0,i} \setminus E_{0,i-1}, \ i = 1, \ldots, r_0 - 1 \), with exceptional divisors \( E_{0,i+1} \). We let \( A_0(p_0) = E_{0,r_0} \).

The union \( B \) of the proper transforms of \( C_0, C_1 \), and the divisors \( E_{i,j}, \ i = 0, \ldots, n, \ j = 0, \ldots, r_{i-1} \), is a rational subtree of the total transform \( B \) of \( D \) by the so-constructed morphism \( \tau : V \rightarrow \mathbb{P}^2_{\mathbb{R}} \). We
let $S = V \setminus B$ and we let $\rho : S \to \mathbb{P}^1_k$ be the morphism induced by the restriction of the $\mathbb{P}^1$-fibration $\pi_1 \circ \tau : V \to \mathbb{P}^1_k$.

**Proposition 3.1.** With the notation above, the surface $S$ is smooth, $k$-rational, geometrically integral. Its Kodaira dimension is equal to 1 and $S(C) = \mathbb{Z}$-acyclic. If $k = \mathbb{R}$ then $S(\mathbb{R})$ is diffeomorphic to $\mathbb{R}^2$.

Furthermore, $\rho : S \to \mathbb{P}^1_k$ is a fibration with generic fiber isomorphic to the punctured affine line $A^1_k$ over the function field of $\mathbb{P}^1_k$, which coincides with the restriction to $S$ of the Iitaka fibration associated with the log-canonical divisor $K_V + B$ on $V$.

**Proof.** The fact that $S$ is defined over $k$, smooth, $k$-rational and geometrically integral is clear from the construction. The morphism $j_C : \mathbb{Z}(D_C) \to \text{Cl}(\mathbb{P}^2_k)$ is clearly surjective and the elements $C_1 - E_{0,0}$ and $E_{0,i} - E_{0,0}$, $i = 1, \ldots, n$ form a basis of its kernel $R$. With the notation of §2.2.1, the abelian group $\mathbb{Z}(\mathcal{E}_0)$ is generated by the curves $A_0(p_i)$, $i = 0, \ldots, n$, and by construction, the matrix of the homomorphism $\varphi : R \to \mathbb{Z}(\mathcal{E}_0)$ is equal to $\mathcal{N}$. Since $\mathcal{N}$ is invertible by hypothesis, the $\mathbb{Z}$-acyclicity of $S(C)$ follows from Lemma 2.5 a). In the case where $k = \mathbb{R}$, the fact that $S(\mathbb{R}) \approx \mathbb{R}^2$ follows from c) in the same lemma and the surjectivity of $j : \mathbb{Z}(D) \to \text{Cl}(\mathbb{P}^2_\mathbb{R})$.

The generic fiber of $\pi_1 \circ \tau$ is isomorphic to the projective line $\mathbb{P}^1_k(t)$ over the function field of $\mathbb{P}^1_k$ and since $B$ contains a pair of disjoint $k$-rational sections of $\pi_1 \circ \tau$, the generic fiber of $\rho$ is isomorphic to the complement of two $k(t)$-rational points in $\mathbb{P}^1_k(t)$. By virtue of [24, Lemma 4.5.3 p. 237] and the proof of Theorem 4.6.1 p. 238 in loc. cit. we have

$$K_V + B \sim (\pi_1 \circ \tau)^*K_{\mathbb{P}^1_k} + \ell + \sum_{i=1}^n ((\pi_1 \circ \tau)^*E_{i,0} - A_0(p_i)) \sim (n - 1)\ell - \sum_{i=1}^n A_0(p_i) \sim \eta\ell + N = P + N$$

where $\ell$ is the proper transform of a fiber of $\pi_1$ over a general $k$-rational point of $\mathbb{P}^1_k$ and $N$ is an effective $\mathbb{Q}$-divisor supported on $\sum_{i=1}^n ((\pi_1 \circ \tau)^*E_{i,0})_{\text{red}} - A_0(p_i)$. So the intersection matrix of $N$ is negative definite and since $\eta > 0$ by hypothesis, it follows that $P$ is nef. The complexification of the above expression $K_V + B \sim P + N$ is thus the Zariski decomposition of $K_V + B_C$. This shows that $\kappa(S) = \kappa(V \setminus B) = 1$ and that $\pi_1 \circ \tau : V \to \mathbb{P}^1_k$ is the morphism induced by a multiple of the positive part of $K_V + B$, as claimed.

**Remark 3.2.** In the case where $k = \mathbb{R}$, we could have added to the construction given in §3.1.1 above a step 1') consisting of adding to $D$ a nonempty collection of non real $C$-rational curves $G_{1,0}, \ldots, G_{m,0} \simeq \mathbb{P}^1_C$ in $\mathbb{P}^2_\mathbb{R}$ passing through $x$ and performing sub divisorial expansions at the $C$-rational points $q_k = C_1 \cap G_{k,0}$ with with multiplicity $(\nu_{k,-}, \nu_{k,+})$, and exceptional divisors $G_{k,1}, \ldots, G_{k,s_k-1}, G_{k,s_k} = A_0(q_k) \simeq \mathbb{P}^1_C$. Letting $\tau' : V' \to \mathbb{P}^2_\mathbb{R}$ be the corresponding smooth projective surface defined over $\mathbb{R}$, the union $B'$ of the proper transforms of $C_0, C_1$, and the divisors $E_{i,j}$, $i = 0, \ldots, n$, $j = 0, \ldots, r_{i-1}$ and $G_{k,\ell}$, $k = 1, \ldots, m$, $\ell = 0, \ldots, s_k - 1$ is a rational geometrically connected subtree of the total transform of $D' = D \cup G_{1,0}, \ldots, G_{m,0}$
and the same argument as in the proof of the previous proposition shows that the surface \( S' = V' \setminus B' \) has Kodaira dimension 1 provided that
\[
\eta' = (n + 2m - 1 - \sum_{i=1}^{n} \frac{1}{\mu_{i,+}} - 2 \sum_{k=1}^{m} \frac{1}{\nu_{k,+}}) > 0.
\]

In \( V'_C \), each curve \((G_{k,\ell})_C\) consists of a pair of disjoint copies of \( \mathbb{P}^1_C \), say \( T_{k,\ell} \) and \( \overline{T}_{k,\ell} \) which are exchanged by the real structure \( \sigma \). The corresponding homomorphism \( \varphi' : R' \to \mathbb{Z}(E'_0) \) from the kernel \( R' \) of the surjective homomorphism \( j_C : \mathbb{Z}(D'_C) \to \text{Cl}(\mathbb{P}^2_C) \) is represented in appropriate basis by the matrix
\[
N' = \begin{pmatrix}
-1 & -1 \\
v_- & \Delta_+
\end{pmatrix} \in \mathcal{M}_{2m+n+1}(\mathbb{Z}) \cap \text{GL}_{2m+n+1}(\mathbb{Q}),
\]
where
\[
\begin{aligned}
v_- &= \ell(\mu_{1,-}, \ldots, \mu_{m,-}, \nu_{1,-}, \ldots, \nu_{m,-}, \nu_{1,1}, \ldots, \nu_{m,1}) \\
\Delta_+ &= \text{diag}(\mu_{1,+}, \ldots, \mu_{m,+}, \nu_{1,+}, \ldots, \nu_{m,+}).
\end{aligned}
\]

We deduce from Lemma 2.5 a) that \( S'_C \) is \( \mathbb{Q} \)-acyclic. But it is never \( \mathbb{Z} \)-acyclic as \( \nu_{k,+} \geq 2 \) for every \( k = 1, \ldots, m \). On the other hand, the induced homomorphism \( H^2(\varphi') : H^2(G; R') \to H^2(G; \mathbb{Z}(E'_0)) \) is still an isomorphism, implying that \( S'(\mathbb{R}) \cong \mathbb{R}^2 \) by virtue of c) in the same lemma.

3.1.2. Classification of homology euclidean planes of Kodaira dimension 1. By virtue of a result of Gurjar and Miyanishi [10], every smooth \( \mathbb{Z} \)-acyclic complex surface of Kodaira dimension 1 is isomorphic to one obtained from \( \mathbb{P}^2_C \) by the procedure described in § 3.1.1 above with \( k = \mathbb{C} \). We have the following counterpart for surfaces defined over \( \mathbb{R} \):

**Theorem 3.3.** Every integral homology algebraic euclidean plane \( S \) of Kodaira dimension 1 is isomorphic over \( \mathbb{R} \) to a surface constructed by the procedure described in § 3.1.1 above. In particular, every such surface admits a smooth projective model \((V, B)\) obtained from \( \mathbb{P}^2_R \) by blowing-up \( \mathbb{R} \)-rational points only and whose boundary is a tree of smooth \( \mathbb{R} \)-rational curves.

**Proof.** Let \((V_0, B_0)\) be a smooth projective completion of \( S \) with SNC boundary defined over \( \mathbb{R} \). Since \( \kappa(S) = 1 \), a multiple of the positive part of the Zariski decomposition of \( K_{V_0,C} + B_0,C \) induces a rational map \( \rho_0 : V_0 \to Z \) over a smooth positive curve \( Z \) defined over \( \mathbb{R} \) whose restriction to \( S \) is a morphism \( \rho : S' \to U \) over a Zariski open subset \( U \) of \( Z \) defined over \( \mathbb{R} \), whose generic fiber is a form of the punctured affine line over the function field of \( Z \) [24, Theorem 6.1.5]. Since \( S \), whence \( V_0 \) is \( \mathbb{R} \)-rational and contains an \( \mathbb{R} \)-rational point, it follows that \( Z \cong \mathbb{P}^1_R \). Letting \( \beta : V' \to V_0 \) be a minimal resolution of the indeterminacies of \( \rho_0 \) and \( B' = \beta^{-1}(B_0) \), the composition \( \overline{\rho}' = \rho_0 \circ \beta : V' \to Z = \mathbb{P}^1_R \) is a surjective fibration with generic fiber isomorphic to a form of the projective line over the function field \( \mathbb{R}(t) \) of \( Z \), which restricts to \( \rho \) on the open subset \( V' \setminus B' \cong S \). The generic fiber of \( \overline{\rho}' \) is in fact necessarily isomorphic to \( \mathbb{P}^1_{\mathbb{R}(t)} \) since otherwise the output of a relative MMP over \( Z \) applied to \( V' \) would be a conic bundle over \( Z \) with at least a reducible fiber, contradicting the \( \mathbb{R} \)-rationality of \( V' \). This implies in turn that \( B \) contains precisely a pair of cross-sections of \( \overline{\rho}' \), say \( C'_0, C'_1 \cong \mathbb{P}^1_R \). By replacing \((V', B')\) by the surface obtained by blowing-up the intersection points of \( C'_0 \) and \( C'_1 \), including infinitely near ones, we may assume that \( C'_0 \) and \( C'_1 \) are disjoint and, contracting if necessary all possible exceptional curves of the first kind supported simultaneously in \( B' \) and in the fibers fibers of \( \overline{\rho}' \) while keeping the property that the successive proper transforms of \( B' \) are simple normal crossing divisors, we may assume from that \((V', B')\) is minimal with this property. So \( \overline{\rho}' : V' \to Z = \mathbb{P}^1_R \) is a fibration with generic fiber isomorphic to \( \mathbb{P}^1_{\mathbb{R}(t)} \), \( B' \) is an SNC divisor defined over \( \mathbb{R} \) which contains a pair of disjoint cross-sections \( C'_0, C'_1 \cong \mathbb{P}^1_{\mathbb{R}(t)} \) of \( \overline{\rho}' \), and the only possible vertical exceptional curves of the first kind contained in \( B' \) intersect either \( C'_0 \) and \( C'_1 \) simultaneously or at least three other irreducible components of \( B' \).

After the base change to \( \text{Spec}(\mathbb{C}) \), we obtain a \( \mathbb{P}^1 \)-fibration \( \overline{\rho}^*_C : V'^*_C \to Z_C \cong \mathbb{P}^1_C \) whose restriction \( \rho_C : S_C \cong V'^*_C \setminus B'^*_C \to U_C \) coincides with the itakawa fibration associated with the log-canonical divisor \( K_{V'^*_C} + B'^*_C \). Since by hypothesis \( S_C \) is \( \mathbb{Z} \)-acylic, it follows from the aforementioned result of Gurjar and Miyanishi [10] that there exists a sequence of blow-ups \( \tau : V \to \mathbb{P}^2_C \) as in § 3.1.1 with \( k = \mathbb{C} \) and
an isomorphism \( \psi : S_C \xrightarrow{\sim} V \setminus B \) of schemes over \( \mathbb{C} \). Furthermore, by virtue of Proposition 3.1, the composition \( (\pi_1 \circ \tau) \mid_{V \setminus B} \circ \psi \) coincides with \( \rho_C \), so that \( \psi \) defines a birational map \( V'_C \longrightarrow V \) restricting to an isomorphism of fibered surfaces

\[
\begin{align*}
S_C & \xrightarrow{\psi} V \setminus B \\
\rho_C & \downarrow (\pi_1 \circ \tau) \mid_{V \setminus B} \\
U_C & \xrightarrow{\sim} \mathbb{P}^1_C.
\end{align*}
\]

We claim that \( \psi^{-1} \) is an isomorphism between \( V \) and \( V'_C \). Indeed, let \( V \xleftarrow{\alpha} Y \xrightarrow{\alpha'} V'_C \) be a minimal resolution of \( \psi^{-1} \), where \( \alpha \) consists of a possibly empty sequence of blow-ups of points supported on the successive total transforms of \( B \), and let \( B_Y = \alpha^{-1}(B) \). Then we have a commutative diagram

\[
\begin{array}{ccc}
V & \xleftarrow{\alpha} & Y \\
\pi_1 \circ \tau \downarrow & \xrightarrow{\psi} & \downarrow (\pi_1 \circ \tau) \mid_{V \setminus B} \\
\mathbb{P}^1_C & \xrightarrow{\sim} & \mathbb{P}^1_C.
\end{array}
\]

Note that the proper transforms of \( C_0 \) and \( C_1 \) in \( Y \) are cross-sections of the \( \mathbb{P}^1 \)-fibrations \( \pi_1 \circ \tau \circ \alpha \) and \( \overline{\rho}_C \circ \alpha' \). So \( \psi \) must restrict to an isomorphism in a Zariski neighborhood of them and since \( (C'_0)_C \) and \( (C'_1)_C \) are the only cross-sections of \( \overline{\rho}_C \circ \alpha' \) supported on \( B'_C \), we have \( \psi_* (C_0 + C_1) = (C'_0)_C + (C'_1)_C \). If \( \alpha' \) is not an isomorphism, then it factors through the contraction of a \((-1)\)-curve supported on the proper transform of \( B \) in \( Y \). By construction, the only possible such curves are the proper transforms of the curves \( C_1, C_0 \) and \( E_{0,0} \) of \( \mathbb{P}^1 \). Since \( \psi \) does not contract any of the first two, the only possibility would be that the proper transform of \( E_{0,0} \) in \( Y \) is a \((-1)\)-curve, and this can occur if and only if no proper base point of \( \alpha^{-1} \) is supported on the proper transform of \( E_{0,0} \) in \( V \). This implies in turn that the proper transform of \( E_{0,0} \) in \( Y \) still intersects those of \( C_0 \) and \( C_1 \). But then, after the contraction of \( E_{0,0} \), the images of \( C_0 \) and \( C_1 \) would intersect each other. This would imply in turn that \( \psi_* (C_0) \) and \( \psi_* (C_1) \) intersect each other, in contradiction with the construction of \((V, B)\). So \( \alpha' \) is an isomorphism and we may assume from now on that \((Y, B_Y) = (V'_C, B'_C)\). Now suppose that \( \alpha \) is not an isomorphism. Then at least one of its exceptional divisors is a \((-1)\)-curve supported on the boundary \( B'_C \) and since \( B \) is SNC, \( E \) intersects at most two other irreducible components of \( B'_C \). By construction of \( B \), this implies that \( E \) intersects \((C'_0)_C \) and \((C'_1)_C \) simultaneously. But since the latter coincide with the proper transforms of \( C_0 \) and \( C_1 \), this is impossible.

Summing up, we conclude that \( \psi : V'_C \longrightarrow V \) is in fact an isomorphism of \( \mathbb{P}^1 \)-fibered surfaces. This implies in turn that \( V' \) is obtained from \( \mathbb{P}^1 \) by a sequence of blow-ups \( r' : V' \rightarrow \mathbb{P}^1 \) as in §3.1 in the case \( k = \mathbb{R} \), with a possible additional Step 1') as described in Remark 3.2. But it was observed there that the existence of a nontrivial Step 1') always leads to a surface \( S'_C \) which is \( \mathbb{Q} \)-acyclic but not \( \mathbb{Z} \)-acyclic. This completes the proof. \( \square \)

**Example 3.4.** (Homotopy euclidean planes of Kodaira dimension 1). Let \( 1 < a < b \) be a pair of coprime integers and let \( \psi : \mathbb{P}^2_{\mathbb{R}} = \text{Proj}(\mathbb{R}[x, y, z]) \longrightarrow \mathbb{P}^1_{\mathbb{R}} \) be the pencil generated by the curves \( \{ x^az^{-a} = 0 \} \) and \( \{ y^b = 0 \} \). So \( \psi \) has two proper base points \( b_0 = [0 : 0 : 1] \) and \( b_1 = [1 : 0 : 0] \), a general geometrically irreducible member of \( \psi \) is an \( \mathbb{R} \)-rational cuspidal curve and \( \psi \) has precisely two degenerate members: \( \psi^{-1}([1 : 0]) \) which is the union of the lines \( L_x = \{ x = 0 \} \) and \( L_z = \{ z = 0 \} \) counted with multiplicities \( a \) and \( b - a \) respectively, and \( \psi^{-1}([0 : 1]) \) which is equal to the line \( L_y = \{ y = 0 \} \) counted with multiplicity \( b \). Up to exchanging the roles of \( x \) and \( z \), we assume from now on that \( a > b - a \).

Let \( E_{0,0} \) be a general member \( q \), for instance \( E_{0,0} = q^{-1}([1 : 1]) = \{ x^az^{-a} + y^b = 0 \} \) and let \( D = E_{0,0} \cup L_z \). Let \( p_0 \in E_{0,0}(\mathbb{R}) \setminus \{ b_1 \} \) be a smooth \( \mathbb{R} \)-rational point, for instance \( p_0 = [1 : -1 : 1] \), let \( \beta_0 : X(a, b; r_0) \rightarrow \mathbb{P}^2_{\mathbb{R}} \) be the birational morphism obtained by first blowing-up \( p_0 \) with exceptional divisor \( E_{0,1} \), and then performing a sequence of \( r_0 - 1 \geq 0 \) blow-ups of \( \mathbb{R} \)-rational points \( p_{0,i} \in E_{0,i} \setminus E_{0,i-1}, i = 1, \ldots, r_0 - 1 \) with exceptional divisor \( E_{0,i+1} \). We let \( S(a, b; r_0) = X \setminus \{ E_{0,0} \cup \cdots \cup E_{0,r_0-1} \cup L_z \} \) where we identified a curve in \( \mathbb{P}^2_{\mathbb{R}} \) with its proper transform in \( X \). A minimal resolution \( \alpha : V \rightarrow X(a, b; r_0) \) of the
induced rational pencil $\beta_{r_0} \circ q : X(a,b;r_0) \rightarrow \mathbb{P}^1_{\mathbb{R}}$ is isomorphic to a surface $\tau : V \rightarrow \mathbb{P}^2_{\mathbb{R}}$ obtained by the construction of §3.1.1 with $k = \mathbb{R}$, $n = 2$ and multiplicities $(\mu_{1,-}, a)$ and $(\mu_{2,-}, b)$, where $1 \leq \mu_{1,-} < a$ and $1 \leq \mu_{2,-} < b$ are uniquely determined in terms of $a$ and $b$ (see [5] (2.7) and [7] (5.3) for the computation).

Figure 3.2.

Via this isomorphism, the boundary $B$ coincides with the total transform of $E_{0,0} \cup \cdots \cup E_{0,r_0-1} \cup L_z$, the sections $C_0$ and $C_1$ coincide respectively with the last exceptional divisors of $\alpha$ of the points $q_0$ and $q_1$ and the curves $A_1$ and $A_2$ are the proper transforms of $L_x$ and $L_y$ respectively. The surface $S(a,b;r_0)_\mathbb{C}$ is thus $\mathbb{Z}$-acyclic with $S(a,b;r_0)(\mathbb{R}) \approx \mathbb{R}^2$. In fact, $S(a,b;r_0)_\mathbb{C}(\mathbb{C})$ is even contractible [5], and, using the same method as in the proof of Theorem 3.3 above, one can deduce from the classification of smooth complex contractible surfaces of Kodaira dimension 1 given in loc. cit. that every homology euclidean plane $S$ of Kodaira dimension 1 such that $S_\mathbb{C}(\mathbb{C})$ is contractible is isomorphic over $\mathbb{R}$ to $S(a,b;r_0)$ for some parameters $a,b,r_0$ as above.

Example 3.5. Specializing the values $(a,b,r_0)$ to $(2,3,1)$ in the previous example, $E_{0,0}$ is the cuspidal cubic $\{x^2z + y^3 = 0\} \subset \mathbb{P}^2_{\mathbb{R}}$ and the fact that the real locus of the corresponding surface $S = S(2,3,1)$ is homeomorphic to $\mathbb{R}^2$ can be seen directly as follows. Since $\beta_1 : X \rightarrow \mathbb{P}^2_{\mathbb{R}}$ consists only of the blow-up of the point $p_0 = [1:-1:1]$, $X(\mathbb{R})$ is a Klein bottle which we view as circle bundle over $\theta : X(\mathbb{R}) \rightarrow S_1$ with fibers equal to the set of $\mathbb{R}$-rational point of the lines through $p_0$ in $\mathbb{P}^2_{\mathbb{R}}$. The sets $E_{0,1}(\mathbb{R})$ and $L_z(\mathbb{R})$ are two sections of $\theta$ which do no intersect each other. On the other hand $E_{0,0}(\mathbb{R})$ is a connected closed curve which intersects $E_{0,1}(\mathbb{R})$ and $L_z(\mathbb{R})$ transversally in one point and in one point with multiplicity 3 respectively.

Figure 3.3. The initial arrangement in $\mathbb{P}^2_{\mathbb{R}}$ and the corresponding curves in the Klein bottle $X(\mathbb{R})$
The pair \((X(\mathbb{R}), E_{0,0}(\mathbb{R}) \cup L_{z}(\mathbb{R}))\) is thus homotopically equivalent to \((X(\mathbb{R}), \ell \cup E_{0,1}(\mathbb{R}))\) where \(\ell\) is a fiber of \(\theta\). So \(S(\mathbb{R})\) is homotopically equivalent to \(X(\mathbb{R})\) minus a fiber and a section of \(\theta\) whence to a disc, implying that \(S(\mathbb{R})\) is homeomorphic to \(\mathbb{R}^{2}\).

3.2. Homology euclidean planes of general type. In this subsection, we establish the real counterpart of a refined procedure to construct homology euclidean homology planes of general type due to tom Dieck and Petrie [7] in the complex case. We then study the possible real forms of certain known complex families.

3.2.1. Cycle-cutting construction. Let again \(k = \mathbb{R}\) or \(\mathbb{C}\), let \(D \subset \mathbb{P}_{k}^{2}\) be a reduced curve defined over \(k\), with geometrically rational irreducible components, and let \(\beta : V_{0} \to \mathbb{P}_{k}^{2}\) be a minimal log-resolution of the pair \((V, D)\). Given a partition \(\mathcal{E}(\beta) = \mathcal{E}_{0} \sqcup \mathcal{E}_{1}\) of the set \(\mathcal{E}(\beta)\) of irreducible exceptional divisors of \(\beta\), with associated indicator function \(\chi : \mathcal{E}(\beta) \to \{0, 1\}\), we let \(R_{0} = \sum_{E \in \mathcal{E}_{0}} E, R_{1} = \sum_{E \in \mathcal{E}_{1}} E\) and we let \(D(\chi)\) be the SNC divisor on \(V_{0}\) defined by

\[
D(\chi) = \beta_{s}^{-1}(D) + R_{1} \subset \beta^{-1}(D) = \beta_{s}^{-1}(D) + R_{1} + R_{0}.
\]

**Definition 3.6.** A cutting datum for a pair \((\mathbb{P}_{k}^{2}, D)\) as above consists of

a) A partition of \(\mathcal{E}(\beta)\) with indicator function \(\chi : \mathcal{E}(\beta) \to \{0, 1\}\) such that \(D(\chi)C\) is connected and

\[
\text{rk}\mathbb{Z}\langle R_{0}\rangle_{C} + s(D(\chi)C) = \text{rk}\mathbb{Z}\langle D_{C}\rangle - 1,
\]

where \(s(D(\chi)C)\) denote the number of independent cycles of the dual graph \(\Gamma(D(\chi)C) = (\Gamma_{0}(D(\chi)C), \Gamma_{1}(D(\chi)C))\) of \(D(\chi)C\).

b) A subset \(\Phi\) of the set of double points of \(\text{Supp}(D(\chi))\) such that the subgraph \((\Gamma_{0}(D(\chi)C), \Gamma_{1}(D(\chi)C))\setminus \Phi_{C})\) of \(\Gamma(D(\chi)C)\) is a tree.

3.2.1.1. Given a cutting datum \((\chi, \Phi)\) for a pair \((\mathbb{P}_{k}^{2}, D)\), we denote by \(B(\mathbb{P}_{k}^{2}, D, \chi, \Phi)\) the set of isomorphy classes of birational morphisms \(\alpha : V = V(\chi) \to V_{0}\) restricting to isomorphisms over \(V_{0} \setminus \Phi\) and such that for every \(p \in \Phi\), there exists an open neighborhood \(V_{0,p}\) of \(p\) over which \(\alpha\) restricts to a subervisory expansion of \(V_{0,p}\) with center at \(p\) (see § 1.3). For every \((\alpha : V(\chi) \to V_{0}) \in B(\mathbb{P}_{k}^{2}, D, \chi, \Phi)\), we let \(B(\alpha) = \alpha^{-1}(D(\chi)) - \sum_{p \in \Phi} A_{0}(p)\) and \(S(\alpha) = V(\chi) \setminus B(\alpha)\).

**Example 3.7.** (Ramanujam Surfaces [28, 7 Example 3.15]). Let \(D \subset \mathbb{P}_{\mathbb{R}}^{2} = \text{Proj}(\mathbb{R}[x, y, z])\) be the union of the cuspidal cubic \(C = \{x^{3}z + y^{3} = 0\}\) with its osculating conic \(Q\) at a general \(\mathbb{R}\)-rational point \(q \in C(\mathbb{R})\). So \(Q\) is a smooth \(\mathbb{R}\)-rational conic intersecting \(C\) at \(q\) with multiplicity 5 and transversally at a second \(\mathbb{R}\)-rational point \(p\).

**Figure 3.4.** The Ramanujam surface for the choice \((\mu_{-}, \mu_{+}) = (1, 1)\)

Let \(1 \leq \mu_{-} \leq \mu_{+}\) be a pair of integers such that \(2\mu_{-} - 3\mu_{+} = \pm 1\), let \(\gamma : V' \to \mathbb{P}_{\mathbb{R}}^{2}\) be the subervisory expansion with center at the \(\mathbb{R}\)-rational point \((C \cap Q)_{p}\) with multiplicities \((\mu_{-}, \mu_{+})\) and last exceptional divisor \(A_{0}(p)\), let \(B' = \gamma^{-1}(D) - A_{0}(p)\) and let \(S = V' \setminus B'\). Choosing \(r = 2C - 3Q\) as the generator of the kernel \(R\) of the surjective homomorphism \(d : Z(D) \to \text{Cl}(\mathbb{P}_{\mathbb{R}}^{2})\), the choice of \((\mu_{-}, \mu_{+})\) guarantees that the coefficient of \(A_{0}(p)\) in \(\gamma^{*}(r)\) is equal to \(2\mu_{-} - 3\mu_{+} = \pm 1\), whence that the induced homomorphism \(\varphi : R \to \mathbb{Z}(A_{0}(p))\) (see § 2.2.1) is an isomorphism. Since \(D(\mathbb{R})\) is not empty, we deduce from Lemma 2.5
that \( S_C(\mathbb{C}) \) is \( \mathbb{Z} \)-acyclic and that \( S(\mathbb{R}) \approx \mathbb{R}^2 \). In fact it is known that \( S_C(\mathbb{C}) \) is even contractible. The reduced total transforms of \( D \) and \( B' \) in the minimal log-resolutions \( \beta : V_0 \to \mathbb{P}_\mathbb{R}^2 \) and \( \beta' : V'_0 \to V' \) of the pairs \((\mathbb{P}_\mathbb{R}^2, D)\) and \((V', B')\) respectively have the following structures:

![Figure 3.5. Resolved boundaries](image)

So \( S \) belongs to \( B(\mathbb{P}_\mathbb{R}^2, D, \chi, \Phi) \) where \((\chi, \Phi) = (\mathcal{E}(\beta), \{q\})\). Clearly, the pair \((V'_0, (\beta')^{-1}(B'))\) cannot be birationally equivalent to either \((\mathbb{P}_\mathbb{R}^2, \text{Line})\) or a pair \((V, B)\) described in \(3.1.1\) via a birational map restricting to an isomorphism on \( S \). So Theorem 3.3 and the fact that \( A^2_\mathbb{R} \) is the only homology euclidean plane of Kodaira dimension \( \leq 0 \) imply that \( S \) is a homology euclidean plane of general type.

Theorem A in [7] admits the following real counterpart:

**Theorem 3.8.** Let \( k = \mathbb{R} \) or \( \mathbb{C} \) and let \( S \) be a smooth \( k \)-rational surface of general type such that \( S_C(\mathbb{C}) \) is \( \mathbb{Q} \)-acyclic. In the case where \( k = \mathbb{R} \), assume further that \( S(\mathbb{R}) \) is non compact. Then there exists an arrangement \( D \) of reduced geometrically rational curves in \( \mathbb{P}_k^2 \) such that \( S \) is isomorphic over \( k \) to the surface \( S(\alpha) \) associated with a birational morphism \((\alpha : V(\alpha) \to W) \in B(\mathbb{P}_k^2, D, \chi, \Phi) \) for a suitable cutting datum \((\chi, \Phi)\).

**Proof.** By virtue of Theorem 2.7 there exists an arrangement \( D \) of reduced geometrically rational curves in \( \mathbb{P}_k^2 \) and a birational morphism \( \alpha : V \to \mathbb{P}_k^2 \) with the property that \( \tau^{-1}(D) \) is SNC and that \( B \) is a subtree of \( \tau^{-1}(D) \) containing the proper transform \( \tau^{-1}D \) of \( D \). Furthermore, the image of the exceptional locus of \( \tau \) is supported on \( D \) and, because \( S_Q \) is \( \mathbb{Z} \)-acyclic, we have \( \text{rk}(\mathbb{Z}(\tau^{-1}(D)_C)) - \text{rk}(\mathbb{Z}(B_C)) = \text{rk}(\mathbb{Z}(D_C)) - 1 \). Since \( \tau : V \to \mathbb{P}_k^2 \) is a log-resolution of the pair \((\mathbb{P}_k^2, D)\), there exists a unique birational morphism \( \alpha : V \to V_0 \) such that \( \alpha_*\tau^{-1}(D) = \beta^{-1}(D) \). The function \( \chi : \mathcal{E}(\beta) \to \{0, 1\} \) defined by \( \chi(E) = 1 \) if and only if \( E \in \alpha_*B \) defines a partition of \( \mathcal{E}(\beta) \) and letting \( \Phi \) be the image of the exceptional locus of \( \alpha \), it is enough to check that \((\chi, \Phi)\) is a cutting datum for the pair \((\mathbb{P}_k^2, D)\) for which \( (\alpha : V \to V_0) \) belongs to \( B(\mathbb{P}_k^2, D, \chi, \Phi) \). The proof is a verbatim of that of Proposition 2.3 in [7].

**3.2.2. Real forms of homology euclidean planes of general type.** It follows from Theorem 3.3 that a homology euclidean plane \( S \) of Kodaira dimension 1 does admit non trivial real forms. Furthermore, it always admits a smooth projective completion \( S \hookrightarrow (V, B) \) obtained from \( \mathbb{P}_k^2 \) by blowing-up \( \mathbb{R} \)-rational points only and whose rational boundary tree consists of \( \mathbb{R} \)-rational curves only. Here we construct examples of homology euclidean plane of general type for which both properties fail.

**3.2.2.1.** Consider the nodal cubic curves \( C_1, C_2 \subset \mathbb{P}_R^2 \) with respective equations \((x - y)(x^2 + y^2) - xyz = 0 \) and \((x - y)(x^2 - 4y^2) - xyz = 0 \). Both have the \( \mathbb{R} \)-rational point \([1 : 1 : 0]\) as a flex but \( C_1 \) has a second \( \mathbb{C} \)-rational flex \( C_1 \cap \{x^2 + y^2 = 0\} \) while \( C_2 \) possesses two other \( \mathbb{R} \)-rational flexes \([2 : 1 : 0]\) and \([-2 : 1 : 0]\). So \( C_1 \) and \( C_2 \) are not \( \mathbb{R} \)-isomorphic but their complexifications are both projectively equivalent over \( \mathbb{C} \) to the curve with equation \( x^3 + y^2 - xyz = 0 \). The projective duals \( \Gamma_1 \) and \( \Gamma_2 \) of \( C_1 \) and \( C_2 \) respectively are cuspidal quartics, which are nontrivial \( \mathbb{R} \)-forms of each other, with projectively...
equivalent complexifications. They both have an ordinary \( \mathbb{R} \)-rational cusp \( p_0 \) corresponding to the common \( \mathbb{R} \)-rational flex of \( C_1 \) and \( C_2 \), and either a second \( \mathbb{C} \)-rational ordinary cusp \( q \) for \( \Gamma_1 \), or a pair of additional \( \mathbb{R} \)-rational ordinary cusps \( q_1 \) and \( q_2 \) for \( \Gamma_2 \). The tangent \( L \) to \( \Gamma_1 \) (resp. \( \Gamma_2 \)) at \( p_0 \) intersects \( \Gamma_1 \) (resp. \( \Gamma_2 \)) transversally in a second \( \mathbb{R} \)-rational point \( p \).

**Figure 3.6.** Real forms of a tricuspidal quartic

Let \( D_i = \Gamma_i \cup L, i = 1,2 \) and let \( 1 \leq \mu_- \leq \mu_+ \) be a pair of integers such that \( \mu_- - 4\mu_+ = \pm 1 \). Then let \( \gamma_i : V'_i \to \mathbb{P}^2, i = 1,2 \), be the projective surface obtained from \( \mathbb{P}^2 \) by the subdivisorial expansion with center at the \( \mathbb{R} \)-rational point \( (\Gamma_i \cap L)_p \) with multiplicities \( (\mu_-,\mu_+) \) and last exceptional divisor \( A_0(p) \), let \( B'_i = \gamma_i^{-1}(D_i) - A_0(p) \) and let \( S_i = V'_i \setminus B'_i \). Letting \( W_i \to V'_i \) be a minimal resolution of the pair \((V'_i, B'_i)\), the reduced total transform \( B_2 \) of \( B'_2 \) consists of \( \mathbb{R} \)-rational curves only, while the reduced total transform \( B_1 \) of \( B'_1 \) contains a chain of \( \mathbb{C} \)-rational curves arising from the resolution of the \( \mathbb{C} \)-rational cuspidal point \( q \).

**Figure 3.7.** Total transforms of the boundaries in the minimal resolution

The complexifications \( B_{i,\mathbb{C}} \) of \( B_i \) have the same structure: every irreducible component of \( B_{2,\mathbb{C}} \) is invariant under the real structure \( \sigma \), while \( \sigma \) acts on \( B_{1,\mathbb{C}} \) by permuting the two “cuspidal branches” \([-3,-1,-2]\) and leaving all other irreducible component invariant.

**Proposition 3.9.** With the notation above, the following hold:

1) The surfaces \( S_1 \) and \( S_2 \) are non isomorphic homology euclidean planes of general type, with isomorphic complexifications \( (S_1)_{\mathbb{C}} \) and \( (S_2)_{\mathbb{C}} \).

2) The surface \( S_1 \) does not admit any smooth SNC-minimal completion \( S_1 \hookrightarrow (V,B) \) defined over \( \mathbb{R} \) for which \( B \) consists of \( \mathbb{R} \)-rational curves only.
Proof. Choosing $r_i = \Gamma_i - 4L$ as the generator of the kernel $R_i$ of the surjective homomorphism $\mathbb{Z}((D_i)) \to \text{Cl}(\mathbb{P}^2_\mathbb{R})$, the choice of $(\mu_-, \mu_+)$ guarantees that the coefficient of $A_0(p)$ in $\gamma_i^*(r_i)$ is equal to $\mu_- - 4\mu_+ = \pm 1$, whence that the induced homomorphism $\varphi_i : R_i \to \mathbb{Z}(A_0(p))$ (see §2.2.1) is an isomorphism. Since $D_i(\mathbb{R})$ is not empty, we deduce from Lemma 2.5 that $S_{i,C}(\mathbb{C})$ is $\mathbb{Z}$-acyclic and that $S_i(\mathbb{R}) \approx \mathbb{R}^2$. The fact that $S_i$ is of general type follows from the same argument as in Example 3.7 by comparing the structure of the minimal rational boundary tree $B_i$ in Figure 3.7 above with those described in §3.1.1. Since $\Gamma_{1,C}$ and $\Gamma_{2,C}$ are projectively equivalent, $S_{1,C}$ and $S_{2,C}$ are isomorphic by construction. Now suppose that $S_1$ admits smooth SNC-minimal completion $S_1 \hookrightarrow (V, B)$ defined over $\mathbb{R}$ for which $B$ consists of $\mathbb{R}$-rational curves only. Then there would exists a birational map of pairs $\varphi : (V_{1,C}, B_{1,C}) \dasharrow (V_C, B_C)$ defined over $\mathbb{R}$ and restricting to an isomorphism $V_{1,C} \setminus B_{1,C} \approx V_C \setminus B_C$. Since every irreducible component of $B$ is $\mathbb{R}$-rational, the real structure $\sigma$ on $V_C$ acts trivially on the set of irreducible components $B_C$. So $\varphi$ cannot be an isomorphism of pairs because, as observed before, the real structure $\sigma$ on $V_{1,C}$ acts non trivially on the set of irreducible components of $B_{1,C}$. So $\varphi$ must be strictly birational and, letting $V_{1,C} \xleftarrow{\sigma} X \xrightarrow{\sigma^1} V_C$ be a minimal resolution of $\varphi$ defined over $\mathbb{R}$, the morphism $\alpha' : X \to V_C$ would consists of a sequence of blow-downs of either $\sigma$-invariant $(-1)$-curves or pairs of disjoint $(-1)$-curves exchanged by $\sigma$ supported on the strict transform of $B_{1,C}$ by $\alpha$. The structure of $B_{1,C}$ depicted in Figure 3.7 above implies that the only possible such curves are the proper transforms of the last exceptional divisors of the minimal log-resolution of the pair $(V_{1,C}', B_{1,C}')$ over the three singular points of $\Gamma_{1,C}$. But the image of $\alpha'^{-1}(B_{1,C})$ after their contraction would no longer be SNC, which is excluded since $B_C$ is an SNC divisor by hypothesis. So the boundary $B$ of every smooth SNC-minimal completion $S_1 \hookrightarrow (V, B)$ must contain at least one non $\mathbb{R}$-rational component, which shows 2). Since the pair $(V_2, B_2)$ constructed in §3.2.2.1 is a smooth SNC-minimal completion of $S_2$ for which $B_2$ consists of $\mathbb{R}$-rational curves only, we deduce in turn that $S_1$ and $S_2$ are not isomorphic as schemes over $\mathbb{R}$. \hfill $\Box$

Remark 3.10. The surfaces $S_1$ and $S_2$ above can also be obtained by a cutting-cycle construction from arrangements $\Delta_1$ and $\Delta_2$ in $\mathbb{P}^2_\mathbb{R}$ consisting of lines and conics and whose complexifications are both projectively equivalent to the following arrangement $\Delta$ of 7 lines with 3 double points and 6 triple points (see [6]).

![Figure 3.8. Arrangement of lines associated to tricuspidal quartics](image)

In the case of $S_1$ the real structure $\sigma$ on $\mathbb{P}^2_\mathbb{C}$ acts on $\Delta$ by exchanging the lines $pq$ and $pr$ and fixing the others while in the case of $S_2$, the real structure acts trivially on $\Delta$.

4. $\mathbb{Q}$-acyclic euclidean planes of negative Kodaira dimension

This section is devoted to the study of $\mathbb{Q}$-acyclic euclidean planes $S$ of negative Kodaira dimension. We first give a complete classification of these up to isomorphisms of schemes over $\mathbb{R}$. More precisely, we show that they all admit an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_\mathbb{R}$ defined over $\mathbb{R}$, that is, a surjective flat morphism with generic fiber isomorphic to the affine line of the function field $\mathbb{R}(t)$ of $\mathbb{A}^1_\mathbb{R}$ and we characterize the $\mathbb{Q}$-acylicity of $S_C(\mathbb{C})$ and the property that $S(\mathbb{R}) \approx \mathbb{R}^2$ in terms of the degenerate fibers of $\pi$. 


4.1. Structure of \( \mathbb{Q} \)-acyclic euclidean planes of negative Kodaira dimension. This subsection is devoted to the proof of the following characterization

**Theorem 4.1.** For a smooth geometrically integral surface \( S \) defined over \( \mathbb{R} \) the following are equivalent:

1. The Kodaira dimension of \( S \) is negative, the surface \( S_\mathbb{C}(\mathbb{C}) \) is \( \mathbb{Q} \)-acyclic and \( S(\mathbb{R}) \approx \mathbb{R}^2 \).

2. The surface \( S \) admits an \( \mathbb{A}^1 \)-fibration \( \pi : S \to \mathbb{A}^1_\mathbb{R} \) defined over \( \mathbb{R} \), whose closed degenerate fibers are all isomorphic to the affine line over the corresponding residue fields when equipped with their reduced structure and whose degenerate fibers over \( \mathbb{R} \)-rational points of \( \mathbb{A}^1_\mathbb{R} \) all have odd multiplicities.

4.1.1. We begin with a recollection on \( \mathbb{A}^1 \)-fibrations over curves and their smooth SNC completions into \( \mathbb{P}^1 \)-fibrations on projective surfaces. Let \( k = \mathbb{R} \) or \( \mathbb{C} \), let \( S \) be a smooth geometrically integral surface defined over \( k \) such that \( S(k) \) is empty and let \( \pi : S \to C \) be an \( \mathbb{A}^1 \)-fibration over a smooth curve, either affine or projective, defined over \( k \). Letting \( V \) be any smooth projective completion of \( S \), the fibration \( \pi : S \to C \) extends to a rational map \( \overline{\pi} : V \dashrightarrow \overline{C} \) over the smooth projective model \( \overline{C} \) or \( C \) over \( k \), and, after taking a log-resolution of the indeterminacies of \( \overline{\pi} \) and of the boundary divisor \( V \setminus S \), we may assume that \( V \) is a smooth projective completion of \( S \) with SNC boundary \( B \) on which \( \pi \) extends to a morphism \( \overline{\pi} : V \to \overline{C} \). Up to performing a sequence of blow-downs defined over \( k \), we may further assume that \( B \) does not contain any irreducible component \( B_i \cong \mathbb{P}^1_\kappa \) where \( \kappa = \mathbb{R} \) or \( \mathbb{C} \), with self-intersection \( -\deg(\kappa/k) \) contained in a fiber of \( \overline{\pi} \) and intersecting at most two other irreducible components of \( B \). Since \( V \) is smooth and the generic fiber of \( \pi \) is isomorphic to the affine line over the function field of \( C \), the generic fiber of \( \overline{\pi} \) is isomorphic to the projective line over the function field of \( \overline{C} \), and the boundary divisor \( B \) contains precisely one irreducible component, say \( \overline{C}_0 \), which is a section of \( \overline{\pi} \). So there exists a birational morphism \( \tau : V \to \mathbb{P}(E) \) defined over \( k \) to a ruled surface \( \mathbb{P}(E) \to \overline{C} \) for a certain rank 2 vector bundle \( E \) over \( \overline{C} \). Up to changing \( \tau \) for a different birational morphism, we may further assume that \( \tau \) restricts to an isomorphism in a open neighborhood of \( \overline{C}_0 \). As a consequence, every fiber of \( \overline{\pi} \) over a closed point \( c \in \overline{C} \) is a geometrically rational tree defined over of the residue field \( \kappa(c) \) of \( c \), containing at least one irreducible component isomorphic to \( \mathbb{P}^1_{\kappa(c)} \).

With our minimality assumptions, it follows that if \( S \) is affine, then the boundary \( B = V \setminus S \) is a tree which can be written in the form \( B = \bigcup_{c \in \overline{C} \setminus C} F_c \cup \overline{C}_0 \cup \bigcup_{p \in C} H_p \), where \( F_c = \overline{\pi}^{-1}(c) \cong \mathbb{P}^1_{\kappa(c)} \) and \( H_p \) is a strict rational subtree of \( \overline{\pi}^{-1}(p) \) either empty or containing a \( \kappa(p) \)-rational component intersecting \( \overline{C}_0 \), the full fiber \( \overline{\pi}^{-1}(p) \) being equal to the union of \( H_p \) and of the closure in \( V \) of \( \pi^{-1}(p) \). The closure in \( V \) of every irreducible component of \( \pi^{-1}(p) \) is isomorphic to the projective line over a finite extension \( \kappa' \) of \( \kappa(p) \) and it intersects \( H_p \) transversally in a unique \( \kappa' \)-rational point.

One of the assertions of Theorem 4.1 now follows from the next proposition:

**Proposition 4.2.** Let \( S \) be a smooth geometrically integral surface defined over \( k = \mathbb{R} \) or \( \mathbb{C} \) and let \( \pi : S \to C \) be an \( \mathbb{A}^1 \)-fibration defined over \( k \).

1. The surface \( S_\mathbb{C}(\mathbb{C}) \) is \( \mathbb{Q} \)-acyclic if and only if \( C \cong \mathbb{A}^1_k \) and every closed fiber of \( \pi \) is isomorphic to the affine line when equipped with its reduced structure.

2. If \( k = \mathbb{R} \), \( S \) is affine and \( C \cong \mathbb{A}^1_\mathbb{R} \), then \( S(\mathbb{R}) \approx \mathbb{R}^2 \) if and only if the scheme theoretic fiber of \( \pi \) over every \( \mathbb{R} \)-rational point is of the form \( mR + R' \), where \( R \cong \mathbb{A}^1_\mathbb{R} \), \( m \geq 1 \) is odd, and \( R' \) is an effective divisor whose support is disjoint from \( R \) and consists of a disjoint union of affine lines defined over \( \mathbb{C} \).

**Proof.** In the first case, the affineness of \( S \) follows from the \( \mathbb{Q} \)-acyclicity of \( S_\mathbb{C}(\mathbb{C}) \). Furthermore, since \( S_\mathbb{C} \) is also rational, \( C \) is geometrically rational, and so, the complexification of a smooth projective completion
Figure 4.1. Structure of degenerate fibers in a minimal completion

(V, B) of S as in § above is obtained from a Hirzerbruch surface ρn : Fn,∞ → P1 by a sequence of blow-ups τ : Vn → Fn mapping the irreducible component C0 of B isomorphically onto a section of ρn. The divisor class group of Fn,∞ being generated by τ of C0 and a fiber of ρn, B must contain exactly one full fiber π : V → C over a k-rational point of C. Indeed otherwise, the homomorphism jC : Z(B) → Cl(V) would be either non surjective if B contains no full fiber of π or non injective if B contains more than one full fiber of π. Thus C is isomorphic to k and so C ∼ k.

In both cases, up to making elementary transformation with center on the fiber F∞ of π over ∞ = P1 \ A1, we may assume further from now on that V is obtained from F1 by a birational morphism τ : V → F1 defined over k and mapping C0 isomorphically onto the exceptional section of p1. Suppose that the closed scheme theoretic fibers of π are not all isomorphic to the affine line over the corresponding residue fields, and let p1, ..., ps ∈ C(k) and, in the case k = R, let q1, ..., qr be the closed C-rational points, over which the fiber of : S → k is degenerate. Let Fj and Gj be the fibers of p at over these points. The images of F∞, Fj and Gj in P2 k by the contraction of τ of C0 form an arrangement D of lines and geometrically reducible conics meeting each other in a unique k-rational point. This provides a presentation of (V, B) as the blow-up of an arrangement D of s + 2r + 1 lines meeting each other in a unique point. The image of F∞ ∼ P1 k in P2 k is a generator of Cl(P2) k, and basis of the kernel R of the homomorphism Z(D) → Cl(P2) consists for instance of the images in P2 C of the divisors Fj - F∞, C, C - F∞, C and C - F∞, C, where Tj and Tj denote the two connected components of Gj, exchanged by the real structure σ. On the other hand, with the notation of § 2.1, the set E0 C of the complexifications of the closures Aij C(pj), ℓ = 1, ..., ni and Aij C(qj), ℓ = 1, ..., mj in V of the fibers of π over the points pj and qj. We let μij,ℓ ≥ 1 and νij,ℓ ≥ 1 be the multiplicities Aij C(pj) and Aij C(qj) in the fibers π(pj) and π(qj) respectively.

By virtue of a) in Lemma 2.5, S C is Q-acyclic if and only the homomorphism φ ⊗ id : R ⊗ Z Q → Z(E0) ⊗Z Q is an isomorphism, it must be that R and Z(E0) have the same rank s + 2r. This is possible only if j = 1 for every j = 1, ..., r, ni = 1 for every i = 1, ..., s and Aij(pj) is k-rational, isomorphic to P1 k. Conversely, assuming that these conditions are satisfied, the matrix of φ : R → Z(E0) is equivalent to the one diag(μ1, ..., μs, ν1, ..., νr, ν1, ..., νr). Summing up, S C is Q-acyclic if and only if C ∼ k and the fiber of : S → C over every closed point c ∈ C is isomorphic to P1 k when equipped with its reduced structure.

In the second case, since B(R) is not empty, it follows from c) in Lemma 2.5 that S(R) ≈ R2 if and only if the map H2(φ) : H2(Z, R) → H2(Z, Z(E0)) is an isomorphism. The Galois cohomology groups H2(Z, R) and H2(Z, Z(E0)) have bases consisting respectively of the classes of the σ-invariant curves Fj - F∞ and of the classes of the complexifications of the R-rational curves among the Aij C(pj). By construction, H2(φ)(Fj - F∞) = ∑ ni=1 μij Aij C(pj) C where the sum is taken over these R-rational irreducible components. So H2(φ) is an isomorphism if and only if for every i = 1, ..., s, there exists exactly one R-rational curve among the Aij C(pj), say Aij C(pj), and the residue class of μij modulo 2 is nonzero. This proves 2).
Example 4.3. (Q-acyclic surfaces completable by a chain of rational curves). Let $S$ be smooth affine complex surface admitting an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_C$ and such that $S_C(C)$ is $\mathbb{Q}$-acyclic and let $(V,B)$ be an SNC-minimal smooth projective completion of $S$ on which $\pi$ extends to a $\mathbb{P}^1$-fibration $\tilde{\pi} : \tilde{V} \to \mathbb{P}^1_C$. By virtue of Proposition 4.2 and the description given in §4.1, the boundary $B$ has the form $F_\infty \cup C_0 \cup \bigcup_{p \in \mathbb{A}^1_C} H_p$ where $F_\infty \simeq \mathbb{P}^1_C$ is the fiber of $\tilde{\pi}$ over $\infty = \mathbb{P}^1_C \setminus \mathbb{A}^1_C$, $C_0$ is a section of $\tilde{\pi}$ which, up to performing elementary birational transformations with center on $F_\infty$, can be assumed to have self-intersection $-1$, and $H_p$ is a strict rational subtree of $\tilde{\pi}^{-1}(p)$ containing a irreducible component intersecting $C_0$. Now suppose that there exists a unique $p \in \mathbb{A}^1_C$, say $p = \{0\}$, such that $H_p$ is not empty and that $E = H_{\{0\}}$ consists of a chain of rational curves. By the minimality assumption, every irreducible component of $E$ has self-intersection $\leq -2$ and since $S_C(C)$ is $\mathbb{Q}$-acyclic, $\tilde{\pi}^{-1}\{\{0\}\}$ is the union of $E$ with the closure $D$ in $V$ of the unique irreducible component of $\tilde{\pi}^{-1}\{\{0\}\}$. The latter has multiplicity $\mu \geq 2$ as a component of $\tilde{\pi}^{-1}\{\{0\}\}$ and is the unique $(-1)$-curve contained in $\tilde{\pi}^{-1}\{\{0\}\}$. It follows that there exists a unique birational morphism $\tau : V \to \mathbb{F}_1$ to the Hirzebruch surface $\rho_1 : \mathbb{F}_1 \to \mathbb{P}^1_\mathbb{C}$ restricting to an isomorphism outside $(E \setminus C_0) \cup D$ and mapping $\tilde{\pi}^{-1}\{\{0\}\}$ and $D$ to $F_0 = \rho_1^{-1}\{\{0\}\}$ and a point on $\mathbb{F}_0 \setminus \tau_*(C_0)$ respectively.

Note in particular that the left and right boundary curves of the chain $E$ have distinct self-intersections, except if $E$ consists of three irreducible components with self-intersection $-2$. In this case, the contraction of the chain $C_0 \owns E$ defines a birational morphism $\beta : V \to \mathbb{P}^2_\mathbb{C}$ which maps $S$ isomorphically onto the complement of the image $\beta_*(F_\infty)$ of $F_\infty$, which is a smooth conic.

The next proposition completes the proof of Theorem 4.1.

Proposition 4.4. Let $S$ be a smooth geometrically integral surface defined over $\mathbb{R}$ of negative Kodaira dimension such that $S_C(C)$ is $\mathbb{Q}$-acyclic and $S(\mathbb{R})$ is non compact. Then $S$ admits an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_\mathbb{R}$ defined over $\mathbb{R}$.

Proof. Since the Kodaira dimension of $S$ is negative and $S_C$ is affine (see 2.2), it follows from [23] that $S_C$ admits an $\mathbb{A}^1$-fibration $p : S_C \to D$ over a smooth complex curve $D$. By virtue of 1) in Proposition 4.2 we have $D \simeq \mathbb{A}^1_C$ as $S_C(C)$ is $\mathbb{Q}$-acyclic. If $p : S_C \to D$ is the unique such $\mathbb{A}^1$-fibration on $S_C$ up to composition by automorphisms of $D$, then the real structure $\sigma$ on $S_C$ descends to an action of $D$ by Galois automorphisms, and so, since $D$ and the morphism $p : S_C \to D$ are affine, there exists a curve $C$ and a morphism $\pi : S \to C$ both defined over $\mathbb{R}$ from which $p : S_C \to D$ is obtained by the base change $\text{Spec}(\mathbb{C}) \to \text{Spec}(\mathbb{R})$. Since the affine line does not have nontrivial forms over fields of characteristic zero,
we conclude that $C \simeq \mathbb{A}^1_{\mathbb{R}}$ and that the generic fiber of $\pi$ is isomorphic to the affine line over the function field of $C$. So $\pi: S \to C \simeq \mathbb{A}^1_{\mathbb{R}}$ is the desired $\mathbb{A}^1$-fibration defined over $\mathbb{R}$. So it remains to consider the case where $S_C$ admits at least two $\mathbb{A}^1$-fibrations over $\mathbb{A}^1_{\mathbb{C}}$ with distinct general fibers. By virtue of [8] this holds if and only if $S_C$ is not isomorphic to $\mathbb{A}^1_{\mathbb{C}} \times (\mathbb{A}^1_{\mathbb{C}} \setminus \{0\})$ and admits a smooth SNC completion whose boundary divisor consists of a chain of smooth proper rational curves. Furthermore, the boundary in any other smooth SNC-minimal completion of $S_C$ is then again a chain of rational curves. Now let $(V, B)$ be a smooth SNC completion of $S$ defined over $\mathbb{R}$ and such that $B$ is a geometrically rational tree minimal over $\mathbb{R}$. The tree $B_C$ is then minimal over $\mathbb{C}$ unless it contains pairs of non-disjoint $(-1)$-curves exchanged by the real structure $\sigma$, in case a minimal SNC completion $(V_0, B_0)$ over $\mathbb{C}$ is obtained from $(V_C, B_C)$ by blowing-down all possible successive $(-1)$-curves in $B_C$. Since $B_0$ must be a chain, it follows that $B_C$ is a chain too, and so $B$ is a geometrically rational chain. Indeed, if $B_C$ contains an irreducible component $R$ intersecting at least three other components, then because $B_0$ is a chain, at least one the connected component of $B_C \setminus R$, say $D$, is contracted to a smooth point by the above sequence of blow-downs. So it must contain at least a $(-1)$-curve $E$ intersecting at most two other irreducible components of $B_C$, one of which being, by the minimality assumption, its image $\bar{E}$ by the real structure $\sigma$. But then $D$ contains a pair of non-disjoint $(-1)$-curves, contradicting the negative definiteness of its intersection matrix.

a) If $B$ consists of $\mathbb{R}$-rational components only then by virtue of 1) in Lemma 4.5 below, there exists a smooth SNC completion $(V', B')$ of $S$ defined over $\mathbb{R}$ whose boundary $B'$ is a chain of $\mathbb{R}$-rational curves of the form $B' = F \triangleright C \triangleright T$ where $F^2 = 0$, $C^2 = -1$ and $T$ is either empty, or an $\mathbb{R}$-rational curve with self-intersection 0, or a chain of $\mathbb{R}$-rational curves with self-intersections $\leq -2$. Since $V'$ is $\mathbb{R}$-rational, the complete linear system $|F|$ defines a $\mathbb{P}^1$-fibration $V' \to \mathbb{P}^1_{\mathbb{R}}$ having $F$ as a fiber and $C$ as a section and whose restriction to $S$ is an $\mathbb{A}^1$-fibration $\pi: S \to \mathbb{A}^1_{\mathbb{R}}$ if $T$ does not consists of a unique curve, or an $\mathbb{A}^1$-fibration $\pi: S \to \mathbb{A}^1_{\mathbb{R}} \setminus \{0\}$ otherwise, the second case being excluded by the fact that $S_C(\mathbb{C})$ is $\mathbb{Q}$-acyclic by hypothesis.

b) If $B$ contains at least one non $\mathbb{R}$-rational component then by virtue of Corollary 4.6 there exists a smooth SNC completion $(V', B')$ of $S_C$ whose boundary $B'$ is a chain of $\mathbb{C}$-rational curves $B = F \triangleright C \triangleright E$ where $F^2 = 0$, $C^2 = -1$ and where $E$ is either empty, or an irreducible curve with self-intersection 0 or a chain of curves with negative self-intersections listed in the corollary. In the first two cases, it follows that $S_C$ is either isomorphic to $\mathbb{A}^2_{\mathbb{R}}$, in which case $S \simeq \mathbb{A}^2_{\mathbb{R}}$ and we are done, or it admits an $\mathbb{A}^1$-fibration over $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$. But the second possibility is again excluded by the hypothesis that $S_C(\mathbb{C})$ is $\mathbb{Q}$-acyclic. In the remaining cases, it follows from the description given in Corollary 4.6 that the sequences of self-intersections of the irreducible components of $E$ are all symmetric. By virtue of [1] Corollary 2 (see also [11 Corollary 3.2.3]), such sequences up to reversion of the ordering are invariants of the isomorphism type of $S_C$. Comparing with the sequences obtained in Example 4.3 above for $\mathbb{Q}$-acyclic complex surfaces admitting a smooth SNC completion whose boundary is a chain, we conclude that the only possibility is that $E$ consists of a chain of three curves with self-intersection $-2$, whence that $S_C$ is isomorphic to the complement of a smooth conic in $\mathbb{P}^2_{\mathbb{C}}$. This implies in turn that $S$ is isomorphic to the complement of a smooth conic $D$ in $\mathbb{P}^2_{\mathbb{R}}$, and since $S(\mathbb{R})$ is not compact, $D$ has an $\mathbb{R}$-rational point $p$. It follows that $S$ admits an $\mathbb{A}^1$-fibration $\pi: S \to \mathbb{A}^1_{\mathbb{R}}$ defined over $\mathbb{R}$, induced by the restriction of pencil of conics osculating $D$ at $p$, i.e the pencil generated by $D$ and twice its tangent line at $p$. $\square$

In the proof of Proposition 4.4 we used the following lemma describing the structure of geometrically rational chains $B$ arising as boundary divisors in minimal completions of smooth affine surfaces.

**Lemma 4.5.** Let $k = \mathbb{R}$ or $\mathbb{C}$, let $(V, B)$ be a pair defined over $k$ consisting of a smooth projective surface $V$ and a geometrically rational chain $B$ supporting a effective ample divisor on $V$ and let $S = V \setminus B$. Then the following holds:

1) If every irreducible component of $B$ is $k$-rational, then there exists a smooth SNC completion $(V_1, B_1)$ of $S$ defined over $k$ whose boundary $B_1$ is a chain of $k$-rational curves of the form $B_1 = F \triangleright C \triangleright E$ where $F^2 = 0$, $C^2 = -1$ and $E$ is either empty, or an irreducible curve with self-intersection 0, or a chain of rational curves with self-intersections $\leq -2$. 

Let $k = \mathbb{R}$ or $\mathbb{C}$, let $(V, B)$ be a pair defined over $k$ consisting of a smooth projective surface $V$ and a geometrically rational chain $B$ supporting a effective ample divisor on $V$ and let $S = V \setminus B$. Then the following holds:

1) If every irreducible component of $B$ is $k$-rational, then there exists a smooth SNC completion $(V_1, B_1)$ of $S$ defined over $k$ whose boundary $B_1$ is a chain of $k$-rational curves of the form $B_1 = F \triangleright C \triangleright E$ where $F^2 = 0$, $C^2 = -1$ and $E$ is either empty, or an irreducible curve with self-intersection 0, or a chain of rational curves with self-intersections $\leq -2$.
2) If $B$ has a non $k$-rational irreducible component then there exists a smooth SNC completion $(V_1,B_1)$ of $S$ defined over $k$ whose boundary $B_1$ is a geometrically rational chain such that $B_{1,C}$ has one of the following forms:

a) $B_{1,C} = H \triangleright \overline{\Pi}$ where $H$ is irreducible with self-intersection 1 and $\Pi$ is its image by the real structure $\sigma$ on $V_{1,C}$.

b) $B_{1,C} = H \triangleright \overline{H}$ where $H = E \triangleright G$ is a chain consisting of an irreducible curve $G$ with self-intersection 0 and a possible empty chain $E$ of curves with self-intersections $\leq -2$ and $\overline{H}$ is the image of $H$ by the real structure $\sigma$ on $V_{1,C}$.

b') $B_{1,C} = H \triangleright \overline{\Pi}$ where $H = E \triangleright G$ is a chain consisting of an irreducible curve $G$ with self-intersection $-1$ and a nonempty chain of curves with self-intersections $\leq -2$, except maybe its right boundary which is a $(-1)$-curve, and $\overline{\Pi}$ is the image of $H$ by the real structure $\sigma$ on $V_{1,C}$.

c) $B_{1,C} = H \triangleright C \triangleright \overline{\Pi}$ where $C$ is an irreducible curve of self-intersection 0 invariant by the real structure $\sigma$ on $V_{1,C}$, $H$ is either an irreducible curve with self-intersection 0 or a chain of curves with self-intersections $\leq -2$, except maybe its right boundary which is a $(-1)$-curve, and $\overline{\Pi}$ is the image of $H$ by the real structure $\sigma$ on $V_{1,C}$.

Proof. We may assume from the very beginning that $B$ is minimal over $k$, i.e. that it does not contain any irreducible component $B_i \cong \mathbb{P}^1_k$, where $k = \mathbb{R}$ or $\mathbb{C}$ with self-intersection $-\deg(k/k)$. Being the support of an effective ample divisor on $V_{1,C}$, $B_{1,C}$ must then contain an irreducible component with self-intersection $\geq -1$.

The first assertion is well-known, so we only sketch the argument, referring the reader for instance to [4] for the detail. If every irreducible component of $B$ is $k$-rational, then the previous observation together implies that $B$ contains at least one irreducible component of nonnegative self-intersection. Fixing an orientation on $E$, we let $D$ be the leftmost irreducible component of $B$ with this property and we let $D_\leq \subset B$ be the subchain of $B$ consisting of $D$ and the components on the left of it. Then by a sequence of birational transformations, consisting of blow-ups of double points of the support of the successive total transforms of $D_\leq$ and contractions of irreducible components of them, we can transform $(V,B)$ into a smooth projective completion $(V',B')$ of $S$ defined over $k$, whose boundary $B'$ consists of $k$-rational curves, in such a way that the left boundary $F$ of $B'$ has self-intersection 0. (e.g. see [5] Lemma 2.7 for a description of such kind of birational transformations). The surface $V' \setminus B' \cong V \setminus B$ being affine, $B'$ has at least a second irreducible component and up to making additional elementary transformations with centers at $k$-rational points of $F$, we may assume that the irreducible component of $B'$ intersecting $F$, say $C$, has self-intersection $-1$. It follows that the complete linear system $[F]$ generates a $\mathbb{P}^1$-fibration $\pi : V' \to \mathbb{P}^1_k$ having $F$ as a full fiber, $C$ as a section, and the remaining irreducible components of $B'$ form a possibly empty chain $T$ contained in a single other fiber of $\pi$. If $T$ is not empty then after contracting all successive $(-1)$-curves contained in its support and performing additional elementary transformations with centers at $k$-rational points of $F$ if necessary, we reach a smooth projective completion $(V_1,B_1)$ defined over $k$, whose boundary is a chain $B_1 = F \triangleright C \triangleright T$ of $k$-rational curves with the desired properties. Note for further use that the initial boundary $B$ contained two disjoint irreducible components with non negative self-intersections if and only it consisted of a chain of three irreducible components $D \triangleright C \triangleright E$ with $D^2 = (E')^2 = 0$ and $C^2$ arbitrary. Indeed, since the birational transformations involved in the construction the pair $(V',B')$ restricts to isomorphisms outside $D_\leq$ and its successive total transforms, the proper transform $E'$ in $V'$ of an irreducible component $E \subset B$ disjoint from $D$ is disjoint from $F$ and has self-intersection $(E')^2 = E^2$. The only possibility is thus that $E'$ is a fiber of $\pi : V' \to \mathbb{P}^1_k$. Thus $E$ has self-intersection 0 and is necessarily the right boundary of $B$, and the same argument implies that $B = D \triangleright C \triangleright E$ as desired.

Now suppose that at least one of the irreducible component of $B$ is not $k$-rational. So $k = \mathbb{R}$ and according to the proof of Lemma 1.10, we have the following alternative: either $B(\mathbb{R})$ consists of a unique point $p$ and then $B_C$ is a chain of the form $H \triangleright \overline{H}$ where $H$ and $\overline{H}$ are chains intersecting each others in $p$ and exchanged by the real structure $\sigma$ on $V_C$, or $B$ contains a unique geometrically irreducible component with empty real locus, or a unique $\mathbb{R}$-rational irreducible component, say $C_0$, and $B_C$ is a chain of the
form $H \triangleright C_{0,C} \triangleright \overline{H}$ where $C_{0,C} \simeq \mathbb{P}^1$, and $H$ and $\overline{H}$ are possibly empty chains exchanged by the real structure $\sigma$ on $V_C$. We consider two sub-cases:

1) If $B_C$ is SNC-minimal, then by the observation at the beginning of the proof, there exists an irreducible component $D_0$ of $B_C$ with non negative self-intersection. If $B(\mathbb{R}) = \{p\}$ then $B_C = H \cup \overline{H}$ and since $B$ defined over $\mathbb{R}$, it follows that $B_C$ contains at least two irreducible components with non negative self-intersection, $D_0$ and its image $\overline{D}_0$ by the real structure $\sigma$ on $V_C$. If $D_0$ and $\overline{D}_0$ are disjoint then $B_C = D_0 \cup C \cup \overline{D}_0$ where $C \simeq \mathbb{P}^1$. But then it would follow that $B(\mathbb{R})$ is either empty or homeomorphic to $S^1$, a contradiction. So up to the choice of an ordering of $B_C$ and the exchange of $D_0$ and $\overline{D}_0$, we may assume that $B_C = G \triangleright D_0 \triangleright \overline{D}_0 \triangleright \overline{G}$ where $D_0$ and $\overline{D}_0$ intersect in $\{p\}$ and $G$ and $\overline{G}$ are possibly empty chains of rational curves with self-intersection $\leq -2$ exchanged by the real structure $\sigma$. Furthermore $D_0^2 \leq 1$ for otherwise, by blowing-up $\{p\}$ with exceptional $E$, we would obtain a new smooth projective completion $(V', B')$ of $S$ defined over $\mathbb{R}$ whose boundary chain $B'$ would have the property that $B'_C = G \triangleright D_0 \triangleright E \triangleright \overline{D}_0 \triangleright \overline{G}$ contains two disjoint irreducible components with positive self-intersection.

For the same reason we conclude that either $D_0^2 = 1$ and then $B = D_0 \triangleright \overline{D}_0$ or $D_0^2 = \overline{D}_0^2 = 0$ and then $B_C = G \triangleright D_0 \triangleright \overline{D}_0 \triangleright \overline{G}$ where $G$ and $\overline{G}$ are possibly empty chains of rational curves with self-intersection $\leq -2$ exchanged by the real structure $\sigma$. This corresponds to cases a) and b) respectively.

Otherwise, if $B(\mathbb{R}) = \emptyset$ or $S^1$ then $B_C = H \triangleright C_{0,C} \triangleright \overline{H}$. If $H$ contains an irreducible component with non negative self-intersection then by the same argument as above, we conclude that $B = D_0 \triangleright C_{0,C} \triangleright D_0^2$ with $D_0^2 = \overline{D}_0^2 = 0$ and by elementary transformations defined over $\mathbb{R}$ with centers on $D_0 \cup \overline{D}_0$, we may assume that $C_{0,C}^2 = 0$ or $-1$, the second case being then reduced further to the one $B = D_0 \cup \overline{D}_0$ with $D_0^2 = \overline{D}_0^2 = 1$ by contracting $C_{0,C}$. The only other possibility is that $C_{0,C}^2 \geq 0$ and that $H$ and $\overline{H}$ consists of chains of curves with self-intersections $\leq -2$ exchanged by the real structure $\sigma$. By blowing-up pairs of double points on $C_{0,C}$ exchanged by the real structure $\sigma$, we may reduce to either case c), that is, $B_C = H \triangleright C_{0,C} \triangleright \overline{H}$ where $C_{0,C}^2 = 0$, $H$ consists of a chain of curves with self-intersection $\leq -2$ except maybe its right boundary which is a $(-1)$-curve, and $\overline{H}$ is the image of $H$ by the real structure $\sigma$, or to the situation that $C_{0,C}^2 = -1$ from which we reach case b) by contracting $C_0$.

2) If $B_C$ is not minimal over $\mathbb{C}$, then the hypothesis that $B$ is minimal over $\mathbb{R}$ implies that $B_C = E \triangleright D_0 \triangleright \overline{D}_0 \triangleright \overline{E}$ where $D_0$ and $\overline{D}_0$ are irreducible with self-intersection $-1$ and $E$ is a chain of curves with self-intersection different from $-1$. Since $B_C$ is the support of an ample divisor on $V_C$, $E$ cannot be empty. Furthermore, it cannot contain any irreducible component with nonnegative self-intersection for otherwise $B_C$ would contain two disjoint such components. This yields case b3).

\begin{corollary}
Let $(V,B)$ be a pair defined over $\mathbb{R}$ consisting of a smooth projective surface $V$ and a geometrically rational chain $B$ supporting a effective ample divisor on $V$. If the irreducible component of $B$ are not all $\mathbb{R}$-rational then there exists a smooth SNC completion $(V_2, B_2)$ of $S_C = V_C \setminus B_C$ defined over $\mathbb{C}$ whose boundary $B_2$ is a chain of $\mathbb{C}$-rational curves of the form $B_2 = F \triangleright C \triangleright E$ where $F^2 = 0$, $C^2 = -1$ and where $E$ is either empty, or an irreducible curve with self-intersection 0, or a chain of the one of the following types

i) $[-e_1, -e_2, \cdots, -e_n, -e_n, \ldots, -e_1]$, where $n \geq 1$ and $e_i \geq 2$ for every $i = 1, \ldots, n$.

ii) $[-e_1, \ldots, -e_{n-1}, -2e_n, -e_{n-1}, \ldots, -e_1]$ where $n \geq 2$ and $e_i \geq 2$ for every $i = 1, \ldots, n$.

iii) $[-e_1, \ldots, -e_{n-1}, -2e_n + 1, -e_{n-1}, \ldots, -e_1]$ where $n \geq 2$ and $e_i \geq 2$ for every $i = 1, \ldots, n$.

\end{corollary}

\begin{proof}
Let $(V_1, B_1)$ be the smooth SNC completion $(V_1, B_1)$ of $S$ defined over $\mathbb{R}$ with geometrically rational chain boundary $B_1$ constructed in the previous lemma. In case a) we reach a chain of three irreducible components with self-intersection $(0, -1, 0)$ by blowing-up the intersection point of $H \triangleright \overline{H}$. In case b) and $E$ is empty, then $B_{1,C}$ consists of a pair of irreducible curves $G$ and $\overline{G}$ with self-intersection 0 which can be transformed into a pair of curves with self-intersection 0 and $-1$ by performing an elementary transformation at their intersection point. Otherwise, if $E = E_1 \triangleright \cdots \triangleright E_n$ is not empty, we let $e_i = -E_i^2 = -(E_i')^2 \geq 2$ for every $i = 1, \ldots, n$. We desired smooth completion of $S_C$ is obtained from $(V_1, B_1, C)$ by performing the following sequence of birational transformations with centers and exceptional curves all supported on the successive total transforms of $B_{1,C}$. We first blow-up the point $G \cap \overline{G}$ and contract the proper transform of $G$. The self-intersection of $E_n$ increased by 1, the self-intersection of $\overline{G}$.

\end{proof}
decreased by one and the proper transform of the exceptional divisor of the blow-up has self-intersection 0. We repeat the same operation again \( e_n - 2 \) times until the proper transform of \( E_n \) has self-intersection \(-1\). Then we blow-up again the intersection point of the proper transform of the last exceptional divisor \( E \) with \( \mathcal{G} \) to get a chain of the form \( E_1 > E_2 > \cdots > E_n > E > E' > \mathcal{G} > E_n > \cdots > E_1 \) with \( E_n^2 = E^2 = -1 \) and \( \mathcal{G}^2 = -e_n \). Then we contract \( E_n \) to get a chain of the form \( E_1 > E_2 > \cdots > E_{n-1} > E > E' > \mathcal{G} > E_n \cup \cdots \cup E_1 \) where \( E_{n-1} = -e_{n-1} + 1, E^2 = 0 \) and \((E')^2 = -1\). We continue by induction until we reach the desired smooth completion with boundary chain of type \([e_2, \ldots, -e_n, -e_n, \ldots, -e_1]\). By contracting \( E_1 \), we eventually reach the desired smooth completion with boundary chain of type \( i)\).

The remaining two cases, corresponding respectively to smooth SNC completion \((V_1, B_1)\) of the form \( c)\) and \( b^1)\) in Lemma 4.5 follow from similar arguments. We leave the detail to the reader.

Remark 4.7. The proof of Proposition 4.4 shows more generally that a smooth affine geometrically integral surface \( S \) defined over \( \mathbb{R} \) and of negative Kodaira dimension up to \( -1 \) with at most one degenerate fiber is \( \mathbb{R}\)-acyclic euclidean plane of negative Kodaira dimension.

The \( \mathbb{Q}\)-acyclicity of \( S \) and the non compactness of \( S(\mathbb{R}) \) play a crucial role in the characterization obtained in this proposition. Indeed, as already observed in the proof of Proposition 4.4, the complexification of \( S' = \{x^2 + y^2 = z^3 - 1\} \subset \mathbb{A}^3_{\mathbb{R}} \) of the surface \( S' = \{uv = z^3 - 1\} \subset \mathbb{A}^3_{\mathbb{R}} \), whose complexification has \( H_2(S(\mathbb{C}); \mathbb{Z}) \cong \mathbb{Z}^2 \). The real loci of \( S \) and \( S' \) are both homeomorphic to \( \mathbb{R}^2 \) via the maps \( \mathbb{R}^2 \to S(\mathbb{R}), (x, y) \mapsto (x, y, \sqrt{x^2 + y^2 + 1}) \) and \( \mathbb{R}^2 \to S'(\mathbb{R}), (u, v) \mapsto (u, v, \sqrt{uv + 1}) \) respectively. The surface \( S' \) admits an \( A^1 \)-fiber \( \pi' : S' \to \mathbb{A}^1_{\mathbb{R}} \) defined over \( \mathbb{R} \) whose unique degenerate fiber \( (\pi')^{-1}(0) \) consists of the disjoint union of the curves \( \mathbb{A}^1_{\mathbb{R}} = \text{Spec}(\mathbb{R}[v]) \) and \( \mathbb{A}^1_{\mathbb{C}} = \text{Spec}(\mathbb{R}[z]/(z^2 + z + 1)[v]) \). So \( \kappa(S) = \kappa(S') = \kappa(S') = -\infty \). Furthermore, \( S' \) admits a smooth SNC completion \((V', B')\) defined over \( \mathbb{R} \) whose boundary \( B \) is a chain of three \( \mathbb{R}\)-rational curves \( F > C > E \) as above where \( E \) is palindrome consisting of a unique curve with self-intersection \(-3\) (see e.g. \([\mathbb{R} \S 5.4]\)).

4.2. \( \mathbb{R}\)-biregular birational rectification of \( \mathbb{Q}\)-acyclic euclidean planes of negative Kodaira dimension. In this subsection, we consider the question of classification of \( \mathbb{Q}\)-acyclic euclidean planes of negative Kodaira dimension up to \( \mathbb{R}\)-biregular birational equivalence. We say for short that such an euclidean plane \( S \) is \( \mathbb{R}\)-biregularly birationally rectifiable if there exists an \( \mathbb{R}\)-biregular birational map \( \varphi : S \to \mathbb{A}^2_{\mathbb{R}}, \) i.e. a birational map defined over \( \mathbb{R} \), containing the real locus of \( S \) in its domain of definition and inducing a diffeomorphism \( \varphi(\mathbb{R}) : S(\mathbb{R}) \to \mathbb{R}^2 = \mathbb{A}^2_{\mathbb{R}}(\mathbb{R}) \). The following theorem implies in particular that a large class of \( \mathbb{Q}\)-acyclic euclidean planes of negative Kodaira are indeed \( \mathbb{R}\)-biregularly birationally rectifiable.

Theorem 4.8. Let \( S \) be a smooth affine geometrically integral surface defined over \( \mathbb{R} \) with an \( A^1 \)-fiber \( \pi : S \to \mathbb{A}^2_{\mathbb{R}}, \) Suppose that \( S(\mathbb{R}) \approx \mathbb{R}^2 \) and that all but at most one fibers of \( \pi \) over \( \mathbb{R}\)-rational points of \( \mathbb{A}^1_{\mathbb{R}} \) contain a reduced \( \mathbb{R}\)-rational irreducible component. Then \( S \) is \( \mathbb{R}\)-biregularly birationally rectifiable.

In particular, every \( \mathbb{Q}\)-acyclic euclidean plane of negative Kodaira dimension \( S \) with \( S(\mathbb{R}) \approx \mathbb{R}^2 \) admitting an \( A^1 \)-fiber \( \pi : S \to \mathbb{A}^1_{\mathbb{R}} \) with at most one degenerate fiber is \( \mathbb{R}\)-biregularly birationally rectifiable.
The proof of Theorem 4.8 consists of two steps, given in §4.2.1 below. We first reduce via suitable \( \mathbb{R} \)-biregular birational maps to the case of surfaces \( S \) equipped with an \( A^1_R \)-fibration \( \pi : S \to A^1_R \) defined over \( \mathbb{R} \) with irreducible fibers and at most one degenerate \( \mathbb{R} \)-rational fiber. Then we show by induction on the number of irreducible components in the boundary \( B \) of a smooth SNC-minimal completion \( S \hookrightarrow (V, B) \) defined over \( \mathbb{R} \) that every such surface is \( \mathbb{R} \)-biregularly birationally rectifiable.

4.2.1. Standard \( r \)-models. The simplest surfaces \( S \hookrightarrow A^1_R \) satisfying the hypotheses of Theorem 4.8 are those for which \( \pi : S \to A^1_R \) restricts to a trivial \( A^1 \)-bundle over \( A^1_R \setminus \{0\} \) and \( \pi^*\{0\} \) is geometrically irreducible, of odd multiplicity \( m \geq 1 \). Indeed, the fact that \( S(\mathbb{R}) \approx \mathbb{R}^2 \) is then guaranteed by 2) in Proposition 4.2.

4.2.1.1. When specialized to such surfaces, the general description given in §4.1.1 provides a smooth SNC completion \( S \hookrightarrow (V, B) \) defined over \( \mathbb{R} \) into a surface obtained from \( \rho_1 : F_1 \to \mathbb{P}^1_\mathbb{R} \) with exceptional section \( C_0 \simeq \mathbb{P}^1_\mathbb{R} \) and a pair of fixed \( \mathbb{R} \)-rational fibers \( E_{-1} = \rho_1^{-1}\{0\} \) and \( F_\infty = \rho_1^{-1}(\mathbb{P}^1_\mathbb{R} \setminus A^1_R) \) via a birational morphism \( \tau : V \to \mathbb{P}^1_\mathbb{R} \) defined over \( \mathbb{R} \) of the following form:

- If \( m = 1 \) then \( \pi : S \hookrightarrow A^1_R \) is isomorphic to the trivial \( A^1 \)-bundle \( \text{pr}_1 : S \simeq A^2_R \to A^1_R \), and we have an isomorphism \( (V, B) = (F_1, F_\infty \cup C_0) \).

- Otherwise, if \( m \geq 2 \) then \( \tau = \tau_0 \circ \cdots \circ \tau_n \) is a sequence of blow-ups of \( \mathbb{R} \)-rational points, starting with the blow-up \( \tau_0 : V_1 \to V_0 = F_1 \) of a point \( p_0 \in E_{-1} \setminus C_0 \), say with exceptional divisor \( E_0 \), followed by the blow-up \( \tau_1 : V_2 \to V_1 \) of the intersection point \( p_1 \) of \( E_0 \) with the proper transform of \( E_{-1} \), with exceptional divisor \( E_1 \), and continuing with a sequence of blow-ups \( \tau_i : V_{i+1} \to V_i \) of \( \mathbb{R} \)-rational points \( p_i \in E_{i-1} \) with exceptional divisor \( E_i \). The last step \( \tau_n : V = V_{n+1} \to V_n \) is the blow-up of an \( \mathbb{R} \)-rational point \( p_n \in E_{n-1} \) with exceptional divisor \( A_0 \). The surface \( S \) is then isomorphic to the complement in \( V \) of the SNC divisor \( B = F_\infty \cup C_0 \cup E \), where \( E = \bigcup_{i=0}^n E_{i-1} \) is a tree of \( \mathbb{R} \)-rational curves, the \( A^1 \)-fibration \( \pi : S \to A^1_R \) coincides with the restriction to \( S \) of the \( \mathbb{P}^1 \)-fibration \( \pi : V \to \mathbb{P}^1_\mathbb{R} \) and \( \pi^{-1}\{0\} = A_0 \cap S \). Note that by construction \( E_i^2 \leq -2 \) for every \( i = -1, \ldots, n - 1 \).

![Figure 4.3. Structure of the divisor \( B \cup A_0 \)](image)

Since \( m \geq 2 \) is odd, the sequence \( \tau_i \) is not completely arbitrary: for instance, the branching components of the tree \( E \), i.e. the irreducible components of \( E \) intersecting at least two other irreducible components, must have odd multiplicities as irreducible components of the degenerate fiber \( E_\ast^\ast\{0\} \) of the \( \mathbb{P}^1 \)-fibration \( \pi : V \to \mathbb{P}^1_\mathbb{R} \). More precisely, we have the following description

**Lemma 4.9.** For a pair \( (V, B = F_\infty \cup C_0 \cup E) \) as above, the following holds:

1) Every branching component of \( E \) has odd multiplicity as an irreducible component of the degenerate fiber \( E_\ast^\ast\{0\} \).

2) Let \( L \subset C_0 \cup E \) be the unique minimal subchain containing \( C_0 \) and \( E_0 \) and let \( E_i \) be the unique branching component of \( E \) contained in \( L \). If \( E_i \cap E_0 \neq \emptyset \) then \( i = 2p \) for some \( p \geq 1 \) and \( L = C_0 \supset E_{-1} \supset \cdots \supset E_{2p} \supset E_0 \) is a chain of type \((-1, -2, \ldots, -2, E_{2p}, -(2p + 1))\).
Proof. If \( E \) has a branching component with even multiplicity, say \( E_i \) for some \( i \geq 2 \), then since \( \tau_{i+1} : V_{i+2} \rightarrow V_{i+1} \) necessarily consists of the blow-up of a simple point of \( E_{i-1} \cup \cdots \cup E_i \) supported on \( E_i \), the proper transform in \( V \) of its exceptional divisor \( E_{i+1} \) has the same multiplicity as \( E_i \) in \( \pi^*(\{0\}) \). Since the center of the next blow-up \( \tau_{i+2} \) is either the point \( E_i \cap E_{i+1} \) or a simple point supported on \( E_{i+1} \), it follows by induction that the proper transform in \( V \) of every divisor \( E_j, j \geq i \) arises with even multiplicity in \( \pi^*(\{0\}) \). As a consequence, \( A_0 \) would have even multiplicity as an irreducible component of \( \pi^*(\{0\}) \), in contradiction with the fact that \( A_0 \cap S \) has odd multiplicity \( m \) as a scheme theoretic fiber of \( \pi \). The second assertion follows immediately from the observation that if \( E_i \cap E_0 \neq \emptyset \) then the multiplicity of \( E_i \) as a component of \( \pi^*(\{0\}) \) is equal to \( -E_0^2 \geq 2 \).

Definition 4.10. An \( r \)-standard \( A^1 \)-fibered surface is a smooth geometrically integral affine surface \( S \) with an \( A^1 \)-fibration \( \pi : S \rightarrow A^1_R \) defined over \( \mathbb{R} \), restricting to a trivial \( A^1 \)-bundle over \( A^1_R \backslash \{0\} \) and such \( \pi^{-1}(\{0\}) \) is geometrically irreducible, of odd multiplicity \( m \geq 1 \). An \( r \)-standard pair is a pair \((V,B)\) consisting of a smooth geometrically integral projective surface \( V \) and a geometrically rational tree \( B \) both defined over \( \mathbb{R} \) isomorphic to the completion of an \( r \)-standard \( A^1 \)-fibered surface constructed in §4.2.1.1.

The next proposition reduces the study of the \( \mathbb{R} \)-biregular birational rectifiability of surfaces considered in Theorem 4.3 to the case of \( r \)-standard surfaces:

Proposition 4.11. Let \( S \) be a smooth affine geometrically integral surface defined over \( \mathbb{R} \) with an \( A^1 \)-fibration \( \pi : S \rightarrow A^1_R \). Suppose that \( S(\mathbb{R}) \approx \mathbb{R}^2 \) and that all but at most one fiber of \( \pi \) over \( \mathbb{R} \)-rational points of \( A^1_R \) contain a reduced \( \mathbb{R} \)-rational irreducible component. Then there exists an \( r \)-standard affine \( A^1 \)-fibered surface \( \pi_0 : S_0 \rightarrow A^1_R \) and an \( \mathbb{R} \)-biregular birational map \( \varphi : S \rightarrow S_0 \) such that the following diagram commutes:

\[
\begin{array}{ccc}
S & \xrightarrow{\varphi} & S_0 \\
\downarrow \pi & & \downarrow \pi_0 \\
A^1_R & \xrightarrow{\sim} & A^1_R.
\end{array}
\]

Proof. The strategy is of course to eliminate on the one hand all degenerate fibers of \( \pi \) over \( \mathbb{C} \)-rational points of \( A^1_R \) and on the other hand all non \( \mathbb{R} \)-rational irreducible components of the degenerate fibers of \( \pi \) over \( \mathbb{R} \)-rational points. Since \( S(\mathbb{R}) \approx \mathbb{R}^2 \) it follows from 2) in Proposition 4.2 that for every \( \mathbb{R} \)-rational point \( p \in A^1_R \), the fiber \( \pi^*(p) \) has the form \( mr + R' \), where \( R \approx A^1_R : m \geq 1 \) is odd, and \( R' \) is an effective divisor whose support is disjoint from \( R \) and consists of a disjoint union of affine lines defined over \( \mathbb{C} \).

The complement \( S' \) of all non \( \mathbb{R} \)-rational irreducible components of \( \pi^{-1}(p) \) where \( p \) runs through the finitely many \( \mathbb{R} \)-rational points of \( A^1_R \) over which the fiber of \( \pi \) is degenerate is an affine open subset of \( S \) on which \( \pi \) restricts to an \( A^1 \)-fibration \( \pi' : S' \rightarrow A^1_R \) whose fibers over \( \mathbb{R} \)-rational point of \( A^1_R \) are all isomorphic to \( A^1_R \) when equipped with their reduced structure. Furthermore, the hypotheses imply that there exists at most one \( \mathbb{R} \)-rational point of \( A^1_R \) over which the scheme theoretic fiber of \( \pi' \), say \( (\pi')^{-1}(\{0\}) \), is degenerate. By construction, the inclusion \( S' \hookrightarrow S \) is an \( \mathbb{R} \)-biregular birational map. Now let \( S' \hookrightarrow (V',B') \) be a smooth completion of \( S' \) defined over \( \mathbb{R} \) with geometrically rational boundary tree \( B' = F'_\infty \cup C'_0 \cup \bigcup_{p \in A^1_R} H'_p \) as in §4.1.1, where \( F'_\infty \approx \mathbb{P}^1_R \) and \( C'_0 \approx \mathbb{P}^1_R \) are respectively the fiber over \( \infty = \mathbb{P}^1_R \backslash A^1_R \) and a section of the \( \mathbb{P}^1 \)-fibration \( \pi' : V' \rightarrow \mathbb{P}^1_R \) extending \( \pi' \). Since \( (\pi')^{-1}(\{0\}) \) is the unique possibly degenerate fiber of \( \pi' \) over an \( \mathbb{R} \)-rational point of \( A^1_R \), the divisor \( \bigcup_{p \in A^1_R} H'_p \) can be decomposed into the disjoint union \( \bigcup_{q \in A^1_R(\mathbb{C})} H_q \) where \( H_0 \) is a possibly empty tree of \( \mathbb{R} \)-rational curve supported on \( (\pi')^{-1}(\{0\}) \). For every \( \mathbb{C} \)-rational point \( q \) of \( A^1_R \) for which \( H_q \) is not empty, equivalently, for every \( \mathbb{C} \)-rational point \( q \) of \( A^1_R \) over which the fiber \( (\pi')^*(q) \) is degenerate, there exists a sequence of contractions \( \beta_q : V' \rightarrow V'_q \) of curves defined over \( \mathbb{R} \) supported on \( (\pi')^{-1}(q) \) such that \( \beta_q((\pi')^*(q)) \approx \mathbb{P}^1_R \) is a smooth fiber of the \( \mathbb{P}^1 \)-fibration \( \pi'_q = \pi' \circ \beta_q^{-1} : V'_q \rightarrow \mathbb{P}^1_R \). Let be the \( \mathbb{P}^1 \)-fibered surface obtained from \( V' \) by performing such sequences of contractions for every \( \mathbb{C} \)-rational point \( q \in A^1_R \) such that \( (\pi')^*(q) \) is degenerate, let \( \beta : V' \rightarrow V_0 \) be the corresponding birational morphism and let \( B_0 \) be the image of \( F'_\infty \cup C'_0 \cup H_0 \) by \( \beta \). By construction, \( \pi'_0(\{0\}) \) is the unique degenerate fiber of \( \pi_0 : V_0 \rightarrow \mathbb{P}^1_R \) and the restriction of \( \beta \) to \( S \) induces an \( \mathbb{R} \)-biregular birational map \( S \rightarrow S_0 = V_0 \setminus B_0 \) which commutes with the \( A^1 \)-fibrations \( \pi \) and \( \pi_0 \).
\[ \pi_0 \] induced by \( \pi \) and \( \pi_0 \) respectively. So \( \pi_0 : S_0 \rightarrow \mathbb{A}^1 \) is a standard \( \mathbb{A}^1 \)-fibered surface, provided that \( S_0 \) is affine. First note that \( S_0 \) does not contain any complete algebraic curve. Indeed, otherwise such an irreducible curve \( D \) would not intersect \( F_0 \) in \( V_0 \), whence would be contained in a fiber of \( \pi_0 \). Since every fiber of \( \pi_0 \) but \( \pi_0^{-1}(0) \) is smooth and \( C_0 \) is a section of \( \pi_0 \), it would follow that \( D \) is contained in \( \pi_0^{-1}(0) \). But since \( \beta \) restricts to an isomorphism in a neighborhood of \( (\pi')^{-1}(0) \), it would follow that \( \beta^{-1}(D) \) is a complete curve in \( S' \), which is absurd since \( S' \) is affine. It remains to observe that \( B_0 \) is the support of an effective \( \mathbb{Q} \)-divisor \( \Delta \) on \( V_0 \) whose intersection with every irreducible component of \( B_0 \) is strictly positive: since \( F_0^2 = 0 \) and the dual graph of \( B_0 \) is a tree, such a \( \Delta \) is obtained by assigning a positive coefficient \( a_0 \in \mathbb{Q} \) to \( F_0 \) and assigning to the other irreducible components of \( B_0 \) a sequence of positive rational coefficients decreasing rapidly with respect to the distance to \( F_0 \) in the dual graph of \( B_0 \). The so constructed \( \Delta \) is ample by virtue of the Nakai-Moishezon criterion and so, \( S_0 \) is affine as desired. \( \square \)

4.2.2. Elementary birational links between \( r \)-standard pairs. Let \((V, B = F_\infty \cup C_0 \cup E)\) be an \( r \)-standard pair with non empty tree \( E \) and let \( \tau = \tau_0 \circ \cdots \circ \tau_m : V \rightarrow \mathbb{P}^1 \) be the birational morphism constructed in \( \S\) 4.2.1.1 Since the proper base point \( p_0 \) of \( \tau^{-1} \) belongs to \( E_{-1} \setminus C_0 \), the pencil \( \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \) lifting the projection \( \mathbb{P}^2_R \longrightarrow \mathbb{P}^1 \) from \( p_0 \) via the contraction \( \xi : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \) of \( C_0 \) lifts to a \( \mathbb{P}^1 \)-fibration \( \xi : V \rightarrow \mathbb{P}^1 \) with a unique degenerate fiber supported by the closure in \( V \) of \((C_0 \cup E \setminus E_0) \cup A_0 \) and having the proper transforms of \( F_\infty \) and \( E_0 \) as cross-sections. The restriction of \( \xi : S = V \setminus B \) is thus a surjective fibration \( S \rightarrow \mathbb{P}^1 \) defined over \( \mathbb{R} \) whose generic fiber is isomorphic to the 1-punctured affine line over the function field of \( \mathbb{P}^1_R \).

**Definition 4.12.** With the notation above, we call *elementary links* the birational transformations of pairs \( \eta : (V, B) \longrightarrow (V^{(1)}, B^{(1)}) \) defined as follows: :

a) If \( E_0^2 = -2s \) is even, we choose \( s \) distinct smooth fibers \( \ell_i \simeq \mathbb{P}^1_C \) of \( \xi_0 : V \rightarrow \mathbb{P}^1 \) over \( \mathbb{C} \)-rational points of \( \mathbb{P}^1_R \) and we let \( \eta' : V \longrightarrow V' \) be the birational map defined over \( \mathbb{R} \) consisting of the blow-up of the \( \mathbb{C} \)-rational points \( \ell_i \cap F_\infty \), with respective exceptional divisors \( \tilde{\ell}_i \simeq \mathbb{P}^1 \), followed by the contraction of the proper transforms of the \( \tilde{\ell}_i \), \( i = 1, \ldots, s \). The proper transforms of \( E_0 \) and \( F_\infty \) in \( V' \) have self-intersections 0 and \(-2s\) respectively, while the self-intersections of the remaining irreducible components of \( B \) are left unchanged.

![Elementary link: even case](image)

Since \( V' \) is rational, the complete linear system \(|E_0|\) generates a \( \mathbb{P}^1 \)-fibration \( \pi' : V' \rightarrow \mathbb{P}^1 \) having the unique irreducible component \( E_{i_1} \) of \( E \) intersecting \( E_0 \) as a section. Furthermore, since \( E_0^2 \) is even, the multiplicity of \( E_{i_1} \) as an irreducible component of the degenerate fiber of \( \xi_0 \) is even and so, it follows from Lemma 4.9 that \( E_{i_1} \) is not a branching component of \( E \). This implies that the closure \( T \) in \( V' \) of \( B \setminus (E_{i_1} \cup E_0) \) is contained in a unique degenerate fiber of \( \pi' \). After making a sequence of birational transformations consisting on the one hand of elementary transformations with \( \mathbb{R} \)-rational centers (including infinitely near ones) on \( E_0 \setminus E_{i_1} \) and contracting the proper transform of \( E_0 \), and on the other hand of contracting all successive \((-1)\)-curves supported on the proper transform of \( T \) starting from that of \( C_0 \), the total transform of \( E_0 \cup E_{i_1} \cup T \) can be re-written in the form \( B^{(1)} = F_\infty^{(1)} \cup C_0^{(1)} \cup E^{(1)} \), where \( F_\infty^{(1)} \simeq \mathbb{P}^1 \) is the last exceptional divisor of the sequence of elementary transformations, \( C_0^{(1)} \) is the
proper transform of $E_{i_1}$ and $E^{(1)}$ is the image of $T$. We let $V^{(1)}$ be the so constructed smooth projective surface and we let $\eta : (V, B) \dasharrow (V^{(1)}, B^{(1)})$ be the corresponding birational map. By construction, $E^{(1)}$ is either empty or has strictly less irreducible components than $E$.

b) If $E_0^2 = -(2s + 1)$ is odd, we choose $s + 1$ distinct smooth fibers $\ell_i \simeq \mathbb{P}^1_{\mathbb{C}}$ of $\xi_{p_0} : \mathbb{P} \to \mathbb{P}^1_{\mathbb{R}}$ over $\mathbb{C}$-rationals point of $\mathbb{P}^1_{\mathbb{R}}$ and we let $\eta' : V \dasharrow V'$ be the birational map defined over $\mathbb{R}$ consisting of the blow-up of the $\mathbb{C}$-rational points $\ell_i \cap F_\infty$, with respective exceptional divisors $\tilde{\ell}_i \simeq \mathbb{P}^1_{\mathbb{R}}$, followed by the contraction of the proper transforms of the $\ell_i$, $i = 1, \ldots, s + 1$. The proper transforms of $E_0$ and $F_\infty$ in $V''$ have self-intersections $1$ and $-2s - 2$ respectively, while the self-intersections of the remaining irreducible components of $B$ are left unchanged.

**Figure 4.5. Elementary link: odd case**

Let $V'' \to V'$ be the surface obtained by blowing-up the intersection point of $E_0$ with the closure $T$ of the strict transform of $B \setminus E_0$ and let $C_0^{(1)}$ be the exceptional divisor. Then $T$ is contained in a unique degenerate fiber of the $\mathbb{P}^1$-fibration $\pi'' : V'' \to \mathbb{P}^1_{\mathbb{R}}$ generated by the complete linear system $|E_0|$ on $V''$ while $C_0^{(1)}$ is a section of this fibration. After contracting all successive $(-1)$-curves supported in $T$, starting from that of $C_0$ and continuing with that of the successive proper transforms of the irreducible components of the nonempty chain of $(-2)$-curves joining $C_0$ to the branching component of $E$ contained in the chain $L$ as in Lemma 4.9, the total transform of $E_0 \cup C_0^{(1)} \cup T$ can be rewritten in the form $B^{(1)} = F^{(1)} \cup C_0^{(1)} \cup E^{(1)}$ where $F^{(1)}$ and $E^{(1)}$ are the proper transforms of $E_0$ and $T$ respectively. We let $V^{(1)}$ be the so constructed smooth projective surface and we let $\eta : (V, B) \dasharrow (V^{(1)}, B^{(1)})$ be the corresponding birational map defined over $\mathbb{R}$. The description given in Lemma 4.9 implies again that if not empty, $E^{(1)}$ has strictly less irreducible components than $E$.

**Proposition 4.13.** Let $(V, B = F_\infty \cup C_0 \cup E)$ be an $r$-standard pair such that $E$ is not empty and let $\eta : (V, B) \dasharrow (V^{(1)}, B^{(1)})$ be an elementary link as in Definition 4.12 above. Then $(V^{(1)}, B^{(1)})$ is an $r$-standard pair and the induced birational map $S = V \setminus B \dasharrow S^{(1)} = V^{(1)} \setminus B^{(1)}$ is $\mathbb{R}$-biregular.

**Proof.** By construction, the birational map $S \dasharrow S^{(1)}$ induces a diffeomorphism $S(\mathbb{R}) \cong S^{(1)}(\mathbb{R})$. So it is enough to show that $S^{(1)}(\mathbb{C})$ is $\mathbb{Q}$-acyclic. Indeed, if so, then $S^{(1)}$ is affine whence in particular does not contain any complete curve. As a consequence, the $\mathbb{A}^1$-fibration $\pi^{(1)} : S^{(1)} \to \mathbb{A}^1_{\mathbb{R}}$ induced by the restriction of the $\mathbb{P}^1$-fibration $\pi^{(1)} : V^{(1)} \to \mathbb{P}^1_{\mathbb{R}}$ defined by the complete linear system $|E_\infty^{(1)}|$ has at most one degenerate fiber, whose closure is contained in the fiber of $\pi^{(1)}$ over the $\mathbb{R}$-rational point $\pi^{(1)}(E^{(1)}_0) \in \mathbb{A}^1_{\mathbb{R}}$. Together with Proposition 4.2, the $\mathbb{Q}$-acyclicity of $S^{(1)}(\mathbb{C})$ and the fact that $S^{(1)}(\mathbb{R}) \approx \mathbb{R}^2$ imply that the unique possible degenerate fiber of $\pi^{(1)}$ is isomorphic to $\mathbb{A}^1_{\mathbb{R}}$ when equipped with its reduced structure and has odd multiplicity. So $\pi^{(1)} : S^{(1)} \to \mathbb{A}^1_{\mathbb{R}}$ is an $r$-standard $\mathbb{A}^1$-fibred surface.

Note that since $(V, B)$ is an $r$-standard pair, $S_{\mathbb{C}}(\mathbb{C})$ is $\mathbb{Q}$-acyclic by virtue of Proposition 4.2. Furthermore, it follows from the proof of this proposition that $H_1(S_{\mathbb{C}}(\mathbb{C}); \mathbb{Z}) \simeq \mathbb{Z}/m\mathbb{Z}$, where $m$ is the multiplicity of the unique degenerate fiber of the $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$. By the description given in §4.2.1.1 and Figure 4.3, we may view the pair $(V, B)$ as being constructed from the surface $\tau_0 : V_1 \to \mathbb{F}_1$ obtained by
blowing-up the point $p_0 \in E_{-1} \setminus C_0$ by blowing-up further a sequence of $\mathbb{R}$-rational points supported on the successive total transforms on $E_{-1}$. Contracting $C_0$ from $V_1$, we may also view $(V, B)$ as being obtained from another Hirzebruch surface $\rho'_1 : \mathbb{F}_1 \to \mathbb{P}^1_{\mathbb{R}}$ having $E_0$ and $F_\infty$ as sections with self-intersections $-1$ and $+1$ respectively by a sequence $\alpha : V \to \mathbb{F}_1$ of blow-ups of $\mathbb{R}$-rational points in such a way that the $\mathbb{P}^1$-fibration $\xi_{p_0} : V \to \mathbb{P}^1_{\mathbb{R}}$ coincides with $\rho'_1 \circ \tau'$.

The image $D = \alpha_*(B)$ of $B$ consists of the union of $E_0$, $F_\infty$ and $E_{-1}$, which is a fiber of $\tau'$. With the notation of Definition 2.2.1 the kernel $R$ of the surjective map $j_C : \mathbb{Z}(D_C) \to \text{Cl}(\mathbb{F}_1)$ is generated by $F_\infty - E_0 - E_{-1}$ while $\mathbb{Z}(E_0)$ is the free abelian group generated by $A_0, C$. Letting $f_\infty, e_0$ and $e_{-1}$ be the coefficients of $A_0$ in the total transforms in $V$ of $F_\infty, E_0$ and $E_{-1}$ respectively, the homomorphism $\varphi : R \to \mathbb{Z}(E_0)$ with respect to the chosen bases is simply the multiplication by $f_\infty - e_0 - e_{-1}$, and since the diagram chasing in the proof of Lemma 2.5 (see also Remark 2.6) provides an isomorphism $H_1(S\mathbb{C}(C); \mathbb{Z}) \cong \mathbb{Z}(E_0)/\text{Im}\varphi$, we have $f_\infty - e_0 - e_{-1} = \pm m$.

On the other hand, with the notation of Definition 4.12 $S^{(1)} = V^{(1)} \setminus B^{(1)}$ is isomorphic to the complement in the projective surface $V'$ of the proper transform $B'$ of $B$. Since by construction $V'$ is obtained from $V$ by performing $r = -E_0^2$ elementary birational transformations along $\mathbb{C}$-rational smooth fibers of $\xi_{p_0} : V \to \mathbb{P}^1_{\mathbb{R}}$ with centers on $F_\infty$, we have a commutative diagram

$$
\begin{array}{ccc}
V & \xrightarrow{\eta'} & V' \\
\alpha \downarrow & & \downarrow \alpha' \\
\mathbb{F}_1 & \longrightarrow & \mathbb{F}_{1-2r}
\end{array}
$$

where $\mathbb{F}_1 \longrightarrow \mathbb{F}_{1-2r}$ consists of $r$ elementary birational transformations along $\mathbb{C}$-rational smooth fibers of $\rho'_1 : \mathbb{F}_1 \to \mathbb{P}^1_{\mathbb{R}}$ with centers on $F_\infty$, $\eta'$ restricts to an isomorphism in a open neighborhood of $B \setminus (E_0 \cup F_\infty)$ and $\alpha' : V' \to \mathbb{F}_{1-2r}$ is a sequence of blow-ups of $\mathbb{R}$-rational points. It follows in particular that the coefficients of $A_0$ in the total transforms in $V'$ of $F_\infty, E_0$ and $E_{-1}$ are again equal to $f_\infty, e_0$ and $e_{-1}$ respectively. On the other hand, the proper transforms of $E_{-1}$, $E_0$ and $F_\infty$ in $\mathbb{F}_{1-2r}$ are respectively a fiber of the $\mathbb{P}^1$-bundle structure and a pair of sections with self-intersections $-1 + 2r$ and $1 - 2r$. Thus $E_0$ is linearly equivalent in $\mathbb{F}_{1-2r}$ to $F_\infty + (2r - 1)E_{-1}$ and, letting $D' = \alpha'_*B'$, a basis of the kernel $R'$ of the surjective map $j'_{C'} : \mathbb{Z}(D'_{C'}) \to \text{Cl}(\mathbb{F}_1)$ is generated by $F_\infty + (2r - 1)E_{-1} - E_0$. As a consequence, the homomorphism $\varphi' : R' \to \mathbb{Z}(E_0)$ coincides with the multiplication by $f_\infty + (2r - 1)e_{-1} - e_0$ and we deduce from the generalization of Lemma 2.5 given in Remark 2.6 that $S^{(1)}_\mathbb{C}(C)$ is $\mathbb{Q}$-acyclic unless $f_\infty + (2r - 1)e_{-1} - e_0 = 0$. But this second possibility never occurs because $f_\infty - e_0 - e_{-1} = \pm m$ is odd by hypothesis. This completes the proof.

4.2.3. Proof of Theorem 4.8

By hypothesis $S$ is a smooth affine geometrically integral surface defined over $\mathbb{R}$, with $S(\mathbb{R}) \cong \mathbb{R}^2$ and equipped with an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$ such that all but at most one fibers of $\pi$ over $\mathbb{R}$-rational points of $\mathbb{A}^1_{\mathbb{R}}$ contain a reduced $\mathbb{R}$-rational irreducible component. By virtue of Proposition 4.11 there exists an $\mathbb{R}$-biregular birational mapping $\varphi_0 : S \to S^{(0)}$ onto an $r$-standard surface $\pi^{(0)} : S^{(0)} \to \mathbb{A}^1_{\mathbb{R}}$ such that $\pi = \pi^{(0)} \circ \varphi_0$. Letting $(V^{(0)}, B^{(0)}) = F^{(0)}_\infty \cup C^{(0)}_0 \cup E^{(0)}$ be a smooth completion of $S^{(0)}$ defined over $\mathbb{R}$ as in § 4.2.1.1 we have the following alternative: either $E^{(0)}$ is empty and then $V^{(0)} \cong \mathbb{F}_1$ and $S^{(0)} = V^{(0)} \setminus B^{(0)} \cong \mathbb{A}^2_{\mathbb{R}}$ or, by virtue of Proposition 4.13 there exists an elementary link $\eta_0 : (V^{(0)}, B^{(0)}) \longrightarrow (V^{(1)}, B^{(1)}) = F^{(1)}_\infty \cup C^{(1)}_0 \cup E^{(1)}$ to an $r$-standard pair $(V^{(1)}, B^{(1)})$ restricting to an $\mathbb{R}$-biregular birational mapping $\eta_0 : S^{(0)} \longrightarrow S^{(1)} = V^{(1)} \setminus B^{(1)}$ between $r$-standard surfaces. Since $E^{(1)}$ is either empty or has strictly less irreducible component then $E^{(0)}$ we conclude by induction that there exists a finite sequence of elementary links

$$(V^{(0)}, B^{(0)}) \xrightarrow{\eta_0} (V^{(1)}, B^{(1)}) \xrightarrow{\eta_1} \cdots \xrightarrow{\eta_{m-1}} (V^{(m)}, B^{(m)}) \xrightarrow{\eta_m} (V^{(m+1)}, B^{(m+1)})$$

terminating with an $r$-standard pair $(V^{(m+1)}, B^{(m+1)})$ for which $E^{(m+1)}$ is empty and such that the composition $\eta_m \circ \cdots \circ \eta_0 : S^{(0)} \longrightarrow S^{(m+1)} = V^{(m+1)} \setminus B^{(m+1)} \cong \mathbb{A}^2_{\mathbb{R}}$ is an $\mathbb{R}$-biregular birational map. This completes the proof of Theorem 4.8.

Example 4.14. Let $D \subset \mathbb{P}^2_\mathbb{R}$ be a geometrically integral $\mathbb{R}$-rational curve of degree $d = 2n + 1 \geq 1$ with a unique singular point $p$ of multiplicity $2n$, such that $D_C$ has a unique analytic branch at $p$, and let
Proposition 5.1. Let \( S = \mathbb{P}^2_\mathbb{R} \setminus D \). Then \( D(\mathbb{R}) \) is connected and since \( d \) is odd, \( S(\mathbb{R}) \) is connected, hence homeomorphic to \( \mathbb{R}^2 \). On the other hand, \( S_\mathbb{C}(\mathbb{C}) \) is \( \mathbb{Q} \)-acyclic with \( H_1(S_\mathbb{C}(\mathbb{C}); \mathbb{Z}) \cong \text{Cl}(S_\mathbb{C}) \cong \mathbb{Z}_d \) and so, \( S \) is not isomorphic to \( \mathbb{A}^2_\mathbb{R} \) as a scheme over \( \mathbb{R} \). But it follows from Theorem 4.8 that \( S \) is \( \mathbb{R} \)-biregularly birationally isomorphic to \( \mathbb{A}^2_\mathbb{R} \). Indeed, the pencil \( \mathbb{P}^2_\mathbb{R} \rightarrow \mathbb{P}^2_\mathbb{R} \) generated by \( D \) and \( d \) times its tangent line \( L \cong \mathbb{P}^1_\mathbb{R} \) at \( p \) restricts on \( S \) to an \( \mathbb{A}^1 \)-fibration \( \pi : S \rightarrow \mathbb{A}^1_\mathbb{R} \) with a unique degenerate fiber isomorphic to \( \mathbb{A}^1_\mathbb{R} \), of multiplicity \( d \).

5. Complements

5.1. Exceptional \( \mathbb{Q} \)-homology euclidean planes of Kodaira dimension 0. By virtue of [19] (8.64)] (see also [24] Lemma 4.4.2), a smooth affine complex \( \mathbb{Q} \)-acyclic surface of Kodaira dimension 0 is either \( \mathbb{A}^1_\mathbb{R} \)-ruled over a base curve isomorphic to \( \mathbb{A}^1_\mathbb{C} \) or \( \mathbb{P}^1_\mathbb{C} \), or is isomorphic to one of the so-called exceptional surfaces \( Y(3, 3, 3), Y(2, 4, 4) \) or \( Y(2, 3, 6) \). Hereafter, we construct all real models of these exceptional surfaces and characterize \( \mathbb{Q} \)-acyclic euclidean planes among them. We show in particular that \( Y(2, 4, 4) \) admits two real forms of very different nature: one whose real locus is not diffeomorphic to \( \mathbb{A}^3_\mathbb{R} \). Indeed, the pencil \( \mathbb{P}^2_\mathbb{R} \rightarrow \mathbb{P}^2_\mathbb{R} \) is isomorphic to \( \mathbb{P}^2_\mathbb{R} \) generated by \( D \). Indeed, the pencil \( \mathbb{P}^2_\mathbb{R} \rightarrow \mathbb{P}^2_\mathbb{R} \) is isomorphic to \( \mathbb{P}^2_\mathbb{R} \) generated by \( D \) and \( d \) times its tangent line \( L \cong \mathbb{P}^1_\mathbb{R} \) at \( p \) restricts on \( S \) to an \( \mathbb{A}^1 \)-fibration \( \pi : S \rightarrow \mathbb{A}^1_\mathbb{R} \) with a unique degenerate fiber isomorphic to \( \mathbb{A}^1_\mathbb{R} \), of multiplicity \( d \).

5.1.1. Real model of \( Y(3, 3, 3) \). Let \( D \) be the union of four general lines \( \ell_i \simeq \mathbb{P}^1_\mathbb{R}, \ i = 0, 1, 2, 3 \) in \( \mathbb{P}^2_\mathbb{R} \) and let \( \pi : V \rightarrow \mathbb{P}^2_\mathbb{R} \) be the projective surface obtained by first blowing-up the points \( p_{ij} = \ell_i \cap \ell_j \) with exceptional divisors \( E_{ij}, \ i, j = 1, 2, 3, i \neq j \) and then blowing-up the points \( \ell_i \cap E_{12}, \ \ell_2 \cap E_{23} \) and \( \ell_3 \cap E_{13} \) with respective exceptional divisors \( E_1, E_2 \) and \( E_3 \). We let \( B = \ell_0 \cup \ell_1 \cup \ell_2 \cup \ell_3 \cup E_{12} \cup E_{23} \cup E_{31} \). The dual graphs of \( D \), its total transform \( \tau^{-1}(D) \) in \( V \) and \( B \) are depicted in Figure 5.1.

![Figure 5.1: Construction of Y(3, 3, 3)](image)

We let \( Y(3, 3, 3) = V \setminus B \). With the notation of [2, 2.7], the kernel \( R \) of the surjective map \( j_C : \mathbb{Z}(D_C) \rightarrow \text{Cl}(\mathbb{P}^2_\mathbb{C}) \) is generated by the classes \( \ell_i.C - \ell_0.C, \ i = 1, 2, 3 \) while \( \mathbb{Z}(E_0) \) is the free abelian group generated by \( E_{1.C}, E_{2.C} \) and \( E_{3.C} \). With this choice of basis, the induced homomorphism \( \varphi : R \rightarrow \mathbb{Z}(E_0) \) is represented by the matrix

\[
A = \begin{pmatrix}
2 & 1 & 0 \\
0 & 2 & 1 \\
1 & 0 & 2
\end{pmatrix}
\]

which has determinant \( \det A = 9 \). So by virtue of Lemma 2.5, \( Y(3, 3, 3)_C \) is \( \mathbb{Q} \)-acyclic with \( H_2(Y(3, 3, 3)_C; \mathbb{Z}) = 0 \) and \( H_1(Y(3, 3, 3)_C; \mathbb{Z}) \cong \mathbb{Z}_9 \). Furthermore, since \( j : \mathbb{Z}(D) \rightarrow \text{Cl}(\mathbb{P}^2_\mathbb{R}) \) is surjective and \( V \) is obtained from \( \mathbb{P}^2_\mathbb{R} \) by blowing-up \( \mathbb{R} \)-rational points only, we deduce from c) in the same Lemma that \( Y(3, 3, 3)(\mathbb{R}) \approx \mathbb{R}^2 \).

Proposition 5.1. Let \( S \) be a smooth surface defined over \( \mathbb{R} \) such that \( S(\mathbb{R}) \approx \mathbb{R}^2 \) and \( S_\mathbb{C} \cong Y(3, 3, 3)_C \). Then \( S \) is isomorphic to \( Y(3, 3, 3) \) as a scheme over \( \mathbb{R} \).
Proof. Since the automorphism group of $Y(3,3,3)_C$ is isomorphic to $\mathbb{Z}_3$, it follows that $Y(3,3,3)$ has no nontrivial $\mathbb{R}$-form: indeed isomorphy classes of $\mathbb{R}$-forms of $Y(3,3,3)$ are in one-to-one correspondence with elements of the cohomology group $H^1(\mathbb{Z}_2, \text{Aut}(Y(3,3,3)_C)) \simeq H^1(\mathbb{Z}_2, \mathbb{Z}_3) = 0$, as every element in $\mathbb{Z}_3$ is a multiple of 2. \qed

**Question 5.2.** Is $Y(3,3,3)$ $\mathbb{R}$-biregularly birationally equivalent to $\mathbb{A}^2_\mathbb{R}$?

5.1.2. *Real forms of $Y(2,4,4)$.* Starting from $\mathbb{P}^1_\mathbb{R} \times \mathbb{P}^1_\mathbb{R}$, we can construct two non-isomorphic real forms $Y_r(2,4,4)$ and $Y_c(2,4,4)$ of the same complex surface as follows:

a) The surface $Y_r(2,4,4)$. We let $D_r$ be the union of three distinct $\mathbb{R}$-rational fibers $\ell_j \simeq \mathbb{P}^1_\mathbb{R}$, $j = 1, 2, 3$, of the first projection and of three distinct $\mathbb{R}$-rational fibers $M_i \simeq \mathbb{P}^1_\mathbb{R}$, $i = 1, 2, 3$, of the second projection. We let $\pi_r : V_r \to \mathbb{P}^1_\mathbb{R} \times \mathbb{P}^1_\mathbb{R}$ be the surface obtained by first blowing-up the $\mathbb{R}$-rational points $p_{12} = M_1 \cap \ell_2$, $p_{13} = M_1 \cap \ell_3$, $p_{23} = M_2 \cap \ell_3$ and $p_{32} = M_3 \cap \ell_2$ with respective exceptional divisors $E_{12}, E_{13}, E_{23}$ and $E_{32}$, and then blowing-up the $\mathbb{R}$-rational points $M_2 \cap E_{23}$ and $M_3 \cap E_{32}$ with respective exceptional divisors $F_{23}$ and $F_{32}$. We let $B_r = M_1 \cup M_2 \cup M_3 \cup \ell_1 \cup \ell_2 \cup \ell_3 \cup E_{23} \cup E_{32}$ and we let $Y_r(2,4,4) = V_r \setminus B_r$.

b) The surface $Y_c(2,4,4)$. We let $D_c$ be the union of a $\mathbb{R}$-rational fibers $\ell_1 \simeq \mathbb{P}^1_\mathbb{R}$ and $M_1 \simeq \mathbb{P}^1_\mathbb{R}$ of the first and second projection and of a pair of conjugate $\mathbb{C}$-rational fibers $\ell \simeq \mathbb{P}^1_\mathbb{C}$ and $M \simeq \mathbb{P}^1_\mathbb{C}$ of the first and second projection respectively. We let $\pi_c : V_c \to \mathbb{P}^1_\mathbb{R} \times \mathbb{P}^1_\mathbb{R}$ be the surface defined over $\mathbb{R}$ obtained by blowing-up the $\mathbb{R}$-rational points $M_1 \cap \ell$ and $M \cap \ell$ with respective exceptional divisors $E_1$ and $E$ and then blowing-up the $\mathbb{C}$-rational point $M \cap E$ with exceptional divisor $F$. We let $B_c = M_1 \cup M \cup \ell_1 \cup \ell \cup E$ and we let $Y_c(2,4,4) = V_c \setminus B_c$.

By construction, the surfaces $Y_r(2,4,4)$ and $Y_c(2,4,4)$ are not isomorphic over $\mathbb{R}$, but their complexifications $Y_r(2,4,4)_C$ and $Y_c(2,4,4)_C$ are isomorphic over $\mathbb{C}$. With the notation of [2.2.1] the kernel $R$ of the homomorphism $j_C : Z(D_c, C) \to \text{Cl}(\mathbb{P}^1_\mathbb{R} \times \mathbb{P}^1_\mathbb{R})$ is generated by the classes $\ell_2, C - \ell_1, C, \ell_3, C - \ell_1, C, M_2, C - M_1, C$ and $M_3, C - M_1, C$ and letting $Z(\mathbb{E}_0)$ be the free abelian group generated by $E_{12}, E_{13}, E_{23}, F_{23}, C$ and $F_{32}, C$, the induced homomorphism $\varphi : R \to Z(\mathbb{E}_0)$ is represented by the matrix

$$A = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$$

which has determinant $\det A = 8$. A similar argument as in the proof of Lemma [2.5] see also Remark [2.6] shows that $Y_r(2,4,4)_C$ is $\mathbb{Q}$-acyclic with $H_2(Y_r(2,4,4)_C; \mathbb{Z}) = 0$ and $H_1(Y_r(2,4,4)_C; \mathbb{Z}) \simeq \mathbb{Z}_6$.

**Proposition 5.3.** Let $S$ be a smooth surface defined over $\mathbb{R}$ such that $S(\mathbb{R}) \approx \mathbb{R}^2$ and $S_C \simeq Y_r(2,4,4)_C \simeq Y_c(2,4,4)_C$. Then $S$ is isomorphic to $Y_c(2,4,4)$ as a scheme over $\mathbb{R}$ and is $\mathbb{R}$-regularly birationally equivalent to $\mathbb{A}^2_\mathbb{R}$.

Proof. The automorphism group of $Y_r(2,4,4)_C$ being isomorphic to $\mathbb{Z}_2$, $Y_r(2,4,4)$ and $Y_c(2,4,4)$ are the only two $\mathbb{R}$-forms of $Y_r(2,4,4)$. Since $j : Z(D_r) \to \text{Cl}(\mathbb{P}^1_\mathbb{R} \times \mathbb{P}^1_\mathbb{R})$ is surjective and $\pi_r : V_r \to \mathbb{P}^1_\mathbb{R} \times \mathbb{P}^1_\mathbb{R}$ consists of blow-up of $\mathbb{R}$-rational points only, we infer similarly as in the proof of c) in Lemma [2.5] that $Y_r(2,4,4)(\mathbb{R}) \approx \mathbb{R}^2$ if and only if $\varphi \otimes \text{id} : R \otimes \mathbb{Z}_2 \to Z(\mathbb{E}_0) \otimes \mathbb{Z}_2$ is an isomorphism, which is not the case. Since the reduction of $A$ modulo 2 is not invertible, we conclude that $Y_r(2,4,4)(\mathbb{R}) \neq \mathbb{R}^2$. On the other hand, since $\pi_c : V_c \to \mathbb{P}^1_\mathbb{R} \times \mathbb{P}^1_\mathbb{R}$ consists only of blow-ups of $\mathbb{C}$-rational points, the pair $(V_c, B_c)$ is $\mathbb{R}$-regularly birationally equivalent to $(\mathbb{P}^1_\mathbb{R} \times \mathbb{P}^1_\mathbb{R}) \setminus (M_1 \cup \ell_1)$ and so $Y_c(2,4,4)$ is $\mathbb{R}$-regularly birationally equivalent to $\mathbb{P}^1_\mathbb{R} \times \mathbb{P}^1_\mathbb{R} \setminus (M_1 \cup \ell_1) \simeq \mathbb{A}^2_\mathbb{R}$. \qed

5.1.3. *Real model of $Y(2,3,6)$.* Staring again, with two triples of $\mathbb{R}$-rational fibers $M_i$ and $\ell_j$, $i = 1, 2, 3$ in $\mathbb{P}^1_\mathbb{R} \times \mathbb{P}^1_\mathbb{R}$ of the second and first projection respectively, we let $\pi : V \to \mathbb{P}^1_\mathbb{R} \times \mathbb{P}^1_\mathbb{R}$ be the smooth projective surface obtained by first blowing-up the points $p_{13} = M_1 \cap \ell_3$, $p_{31} = M_3 \cap \ell_1$, $p_{22} = M_2 \cap \ell_2$ and $p_{32} = M_3 \cap \ell_2$ with respective exceptional divisors $E_{13}, E_{31}, E_{22}$ and $E_{32}$ and then blowing-up the points $p_{22} = E_{22} \cap \ell_2$, $p_{13} = M_1 \cap E_{13}$ and $p_{31} = E_{31} = M_3 \cap E_{31}$ with respective exceptional divisors $F_{22}, F_{13}$ and $F_{31}$. We let $B = M_1 \cup M_2 \cup M_3 \cup \ell_1 \cup \ell_2 \cup \ell_3 \cup E_{22} \cup E_{13} \cup E_{31}$ and we let $Y(2,3,6) = V \setminus B$. Then $Y(2,3,6)$ is a smooth affine surface defined over $\mathbb{R}$ such that $H_2(Y(2,3,6)_C; \mathbb{Z}) \simeq \mathbb{Z}_6$ and $H_1(Y(2,3,6)_C; \mathbb{Z}) = 0$. Since $V$ is obtained from $\mathbb{P}^1_\mathbb{R} \times \mathbb{P}^1_\mathbb{R}$ by blowing-up $\mathbb{R}$-rational points only, we deduce in a similar way as in the
previous case that $Y(2,3,6)(\mathbb{R}) \not \cong \mathbb{R}^2$. Furthermore, since the automorphism group of $Y(2,3,6)_\mathbb{C}$ is trivial, there is no nontrivial $\mathbb{R}$-form of $Y(2,3,6)$. Summing up, there is no smooth affine surface $S$ defined over $\mathbb{R}$ with $S(\mathbb{R}) \cong \mathbb{R}^2$ and $S_{\mathbb{C}} \cong Y(2,3,6)_\mathbb{C}$.

5.2. Moduli of $\mathbb{R}$-biregularly birationally rectifiable surfaces of negative Kodaira dimension. As seen in the introduction, in the rational projective case, there is a unique minimal complexification or at most one family of pairwise non isomorphic but $\mathbb{R}$-biregularly birationally and deformation equivalent minimal complexifications. Non minimal complexifications are obtained from these models by blowing-up sequences of pairs of non-real conjugate points. It is natural to expect an affine counterpart of this type of results in the form of continuous moduli of $\mathbb{Q}$-acyclic euclidean planes of negative Kodaira dimension all $\mathbb{R}$-biregularly birationally equivalent to each other. For instance, starting with the standard open embedding of $\mathbb{A}^2_{\mathbb{R}}$ in $\mathbb{P}^2_{\mathbb{R}}$ as the complement of a line $L_\infty \cong \mathbb{P}^1_{\mathbb{R}}$ and performing a sequence of blow-ups $\tau : V \to \mathbb{P}^2_{\mathbb{R}}$ defined over $\mathbb{R}$ whose centers lie over $L_\infty$, one obtains open embeddings $\mathbb{A}^2_{\mathbb{R}} \to V$ into various smooth projective surfaces defined over $\mathbb{R}$, which, in restriction to the real loci correspond to smooth open embeddings of $\mathbb{R}^2$ into smooth compact non-orientable surfaces of arbitrary genus $g \geq 1$. For a fixed number $g = 1 \geq 0$ of $\mathbb{R}$-rational points blown-up, the isomorphy type as real algebraic varieties of the so-constructed surfaces $V$ with $g(V(\mathbb{R})) = g$ depend on the choice of the points, giving rise in general to a continuous moduli of such algebraic surfaces. In contrast, it follows from [3] that their equivalence classes up to $\mathbb{R}$-biregular biratical isomorphisms depend only on $g$, which in this particular case coincides simply with the number of $\mathbb{R}$-rational irreducible components of the boundary $B = V \setminus \mathbb{A}^2_{\mathbb{R}}$.

The next proposition illustrates the existence of infinitely many deformation equivalence classes of pairwise non isomorphic $\mathbb{Q}$-acyclic euclidean planes all $\mathbb{R}$-biregularly birationally equivalent to $\mathbb{A}^2_{\mathbb{R}}$, each deformation equivalence class having further a moduli of arbitrary positive dimension $n \geq 3$.

**Proposition 5.4.** Let $Y = \text{Spec}(\mathbb{R}[a_1,\ldots,a_n])$, $n \geq 3$, let $r \geq 3$ be an odd integer, and let $\mathcal{X} \subset Y \times \mathbb{A}^3_{\mathbb{R}}$ be the subvariety with equation $x^{n+1}z = y^r + \sum_{i=2}^{n} a_i x^{i+1} + x^2 + x$. Then the following hold:

1) The projection $\text{pr}_Y : \mathcal{X} \to Y$ is smooth and $\text{pr}_Y(\mathbb{R}) : \mathcal{X}(\mathbb{R}) \to Y(\mathbb{R})$ is a trivial $\mathbb{R}^2$-bundle over $Y(\mathbb{R}) \cong \mathbb{R}^n$.

2) For every $\mathbb{R}$-rational point $p \in Y$, the scheme theoretic fiber $S = \mathcal{X}_p$ is a smooth connected affine surface defined over $\mathbb{R}$, of negative Kodaira dimension with $S(\mathbb{R}) \cong \mathbb{R}^2$ and $H_1(S(\mathbb{C});\mathbb{Z}) \cong \mathbb{Z}_r$, $H_2(S(\mathbb{C});\mathbb{Z}) = 0$. The restriction of $S$ of the projection $\text{pr}_x$ is an $\mathbb{A}^1$-fibration $q : S \to \mathbb{A}^1_{\mathbb{R}}$ with $q^{-1}(0)$ as a unique geometrically irreducible degenerate fiber, of multiplicity $r$. In particular, $S$ is $\mathbb{R}$-biregularly birationally equivalent to $\mathbb{A}^2_{\mathbb{R}}$.

3) Every $S = \mathcal{X}_p$ is deformation equivalent to $\mathcal{X}_0$ via the retraction $Y \to \{0\}$, $(a_2,\ldots,a_n) \in \mathbb{R}^2 \mapsto (ta_2,\ldots,ta_n)$, $t \in \mathbb{R}$.

4) Let $p = (a_1,\ldots,a_n) \in Y(\mathbb{R})$ and $p' = (a'_1,\ldots,a'_n) \in Y(\mathbb{R})$. Then $\mathcal{X}_p$ is isomorphic to $\mathcal{X}_{p'}$ if and only if $p = p'$.

**Proof.** The first assertion follows from the Jacobian criterion and the observation that the map

$$\psi : Y(\mathbb{R}) \times \mathbb{R}^2 \to \mathcal{X}(\mathbb{R}) \quad (a_1,\ldots,a_n,x,z) \mapsto (a_1,\ldots,a_n,x,\sqrt[n]{x^{n+1}z - \sum_{i=2}^{n} a_i x^{i+1} - x^2 - x},z)$$

is an homeomorphism. For every $p \in Y(\mathbb{R})$, $q : S = \mathcal{X}_p \to \mathbb{A}^1_{\mathbb{R}}$ is an $r$-standard $\mathbb{A}^1$-fibered surface with $q^{-1}(\mathbb{A}^1_{\mathbb{R}} \setminus \{0\}) \cong \text{Spec}(\mathbb{R}[x^{1},y]) \cong \mathbb{A}^1_{\mathbb{R}} \setminus \{0\} \times \mathbb{A}^1_{\mathbb{R}}$ and $q^*(\{0\}) \cong \text{Spec}(\mathbb{R}[g]/(y^r)[z])$. So 2) follows from Proposition 1.2 and Theorem 1.3. The third assertion is clear. For the last assertion, let $S = \mathcal{X}_p$ and $S' = \mathcal{X}_{p'}$, it follows from Theorem 6.1 in [27] that $S_{\mathbb{C}}$ and $S'_{\mathbb{C}}$ are isomorphic if and only if there exists $\lambda, \alpha, \mu \in \mathbb{C}^*$ and $\beta \in \mathbb{C}$ such that

$$(\alpha y + \beta)^r + \sum_{i=2}^{n} a_i (\lambda x)^{i+1} + (\lambda x)^2 + \lambda x = \mu(y^r + \sum_{i=2}^{n} a'_i x^{i+1} + x^2 + x).$$

The previous identity implies that $\beta = 0$, $\mu = \alpha^r = \lambda^2 = \lambda$ and $a_i \lambda^{i+1} = \mu a'_i$ for $i = 2,\ldots,n$. Thus $\lambda = \mu = 1$ necessarily and so $(a_1,\ldots,a_n) = (a'_1,\ldots,a'_n)$. □
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