Hausdorff-Young type inequalities for vector-valued Dirichlet series

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Abstract

We study Hausdorff-Young inequalities for vector-valued Dirichlet series. These are inequalities that relate some norm of the coefficients $(a_n)_n$ and the norm of the Dirichlet series in the Hardy space $H_p(X)$. This leads us in a natural way to consider different type/cotype properties of the space. Restrictive properties as Fourier and Walsh type and cotype give the strongest inequalities, but we see that the much weaker notions of polynomial type and cotype also give very good inequalities. We present these inequalities and give conditions on a Banach space ensuring that it has polynomial type or cotype.

1 Introduction

The Hilbert space of Dirichlet series $H_2$ was first defined in [22] as those $\sum a_n n^{-s}$ for which $(a_n)_n \in \ell_2$. This was later extended by Bayart, who in [2] defined a whole scale of Hardy spaces of Dirichlet series $H_p$ for $1 \leq p \leq \infty$. Unlike the Hilbert space case, there is no general principle that allows to decide whether or not a Dirichlet series belongs to a given Hardy space just by looking at the size of the coefficients, but the classical Hausdorff-Young inequalities are a useful tool in this purpose. For each $1 \leq p \leq \infty$ the spaces $H_p$ and $H_p(T^\infty)$ (precise definitions are given below) are isometrically isomorphic. A rather straightforward computation (using, for example, standard interpolation arguments) shows that Hausdorff-Young inequalities also hold for these spaces and this immediately gives (here $r'$ denotes the conjugate of $1 \leq r \leq \infty$ so that $\frac{1}{r} + \frac{1}{r'} = 1$)

$$\left\| \sum a_n n^{-s} \right\|_{H_{p'}} \leq C \left( \sum_{n=1}^\infty |a_n|^p \right)^{\frac{1}{p}} \quad (1)$$

for every $1 \leq p \leq 2$ and

$$\left( \sum_{n=1}^\infty |a_n|^q \right)^{\frac{1}{q}} \leq C \left\| \sum a_n n^{-s} \right\|_{H_{q'}} \quad (2)$$

for all $2 \leq q \leq \infty$.

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Hardy spaces \( \mathcal{H}_p(X) \) of vector-valued Dirichlet series (that is, the coefficients \( a_n \) belong to some Banach space \( X \)) have been defined and studied in [9, 16]. Here the problem becomes more complicated. Once again, each one of these spaces is isometrically isomorphic to the corresponding \( \mathcal{H}_p(T^\infty, X) \), but in this case the Haussdorff-Young inequalities do not hold for an arbitrary Banach space. Fourier type and cotype are the notions to get vector-valued Haussdorff-Young inequalities, and for spaces enjoying those properties (again, see below for the definition) we easily get in Propositions 2.1 and 2.2 inequalities that are analogous to (1) and (2). The problem now is that these properties are very restrictive, in the sense that a Banach space has Fourier type or cotype with exponents \( p \) or \( q \) which are generally worse than those for the usual (Rademacher) type and cotype. Even more, the actual values of \( p \) and \( q \) are often unknown.

We aim at finding conditions weaker than Fourier type/cotype that provide satisfactory inequalities with good exponents relating the \( \mathcal{H}_p(X) \)-norm of a Dirichlet series and some norm of the coefficients. These properties happen to be the polynomial cotype (introduced in [10]) and its natural relative polynomial type, that we introduce in Section 3. We show in Theorems 3.7 and 3.8 that they are equivalent to some weighted variants of Haussdorff-Young inequalities for Dirichlet series, functions on \( T^\infty \), and Walsh functions. Although we are not interested here in a systematic study of the concepts of polynomial type and cotype, we give in Section 3 some general properties that allow us to tackle our questions. On the one hand we show the following fundamental fact: although polynomial type and cotype are defined in terms of inequalities that should hold for arbitrary homogeneous polynomials, it suffices to check them only for tetrahedral polynomials (where the variables appear with at most power 1, see Proposition 3.3 and Remark 3.4 for precise definitions and statements). Additionally, Proposition 3.3 gives a link to a weak version of Walsh type and cotype. We also show a duality result in the spirit of the classical duality properties of Rademacher type and cotype (see Proposition 3.9).

We then move one step further looking for conditions ensuring that a Banach space has polynomial type or cotype with the same exponents as for Rademacher type and cotype. It was already shown in [10] that spaces with cotype \( q \) and u.u.s.t. or with Fourier cotype \( q \) have polynomial cotype \( q \). We enlarge (mostly in Section 4) the spaces for which we can precisely establish the polynomial cotype, and extend this also to polynomial type. We see that Walsh type and cotype imply the corresponding polynomial properties. Type 2 and polynomial type 2 are equivalent, and for K-convex space, so are cotype 2 and polynomial cotype 2. For exponents other than 2, we show how other properties (Gaussian average property, Gordon-Lewis property, uniform \( C \)-convexity) relate with polynomial type and cotype. These properties include most Banach spaces we find in literature. However, in Example 4.11 we show that \( L_1/H_1 \) has polynomial cotype 2 but fails to have any of these additional conditions. The reformulation of polynomial type and cotype in terms of Walsh functions given in Theorems 3.8 and 3.7 is crucial for many results in Section 4, as well as for Example 4.11.

2 Definitions and first results

We denote by \( dz \) the normalized Lebesgue measure on the infinite dimensional polytorus \( T^\infty = \prod_{k=1}^\infty T \), i.e., the countable product measure of the normalized Lebesgue measure on \( T \). For any multi index \( \alpha = (\alpha_1, \ldots, \alpha_n, 0, \ldots) \in \mathbb{Z}^{[n]} \) (all finite sequences in \( \mathbb{Z} \)) the \( \alpha \)th Fourier coefficient \( \hat{f}(\alpha) \) of \( f \in L_1(T^\infty, X) \) is given by

\[
\hat{f}(\alpha) = \int_{T^\infty} f(z) z^{-\alpha} \, dz.
\]

Then, given \( 1 \leq p < \infty \), the \( X \)-valued Hardy space on \( T^\infty \) is the subspace of \( L_p(T^\infty, X) \) defined as

\[
\mathcal{H}_p(T^\infty, X) = \left\{ f \in L_p(T^\infty, X) \mid \hat{f}(\alpha) = 0, \quad \forall \alpha \in \mathbb{Z}^{[n]} \setminus \mathbb{N}_0^{[n]} \right\}.
\]
Regarding the equivalence of these two concepts and also with the norm as the image of \( H_{C > 0} \) there exists \( n \in N \) (where \( n \) is uniquely determined by its Fourier coefficients. With this in mind we consider the X-valued Bohr transform \( \mathcal{B}_X \) that to each \( f \) assigns the Dirichlet series \( \sum a_n n^{-s} \) where \( a_n = \hat{f}(\alpha) \) if \( n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \) is the prime number decomposition of \( n \). Then the Hardy space \( \mathcal{H}_p(X) \) of Dirichlet series in \( X \) is defined as the image of \( H_p(T^\infty, X) \) under the Bohr transform \( \mathcal{B}_X \). This vector space of Dirichlet series together with the norm

\[
\|D\|_{\mathcal{H}_p(X)} = \|\mathcal{B}_X^{-1}(D)\|_{H_p(T^\infty, X)}
\]

forms a Banach space. In other words, Bohr’s transform gives the isometric identification

\[
\mathcal{H}_p(X) = H_p(T^\infty, X) \quad \text{for } 1 \leq p < \infty.
\]

A detailed account on this identification can be found in [15] or [33].

There are many equivalent definitions of Fourier type and cotype (see [18]). Let us give the ones that are more akin to our framework. Given \( 1 \leq p \leq 2 \), we say that \( X \) has Fourier type \( p \) if there is a constant \( C > 0 \) such that for each choice of finitely many vectors \( x_1, \ldots, x_N \in X \) we have

\[
\left( \int_T \left\| \sum_{k=1}^N x_k z^k \right\|_p^q \, dz \right)^{1/q} \leq C \left( \sum_{k=1}^N \left\| x_k \right\|_p \right)^{1/p}.
\]

For \( 2 \leq q < \infty, X \) has Fourier cotype \( q \) if there is a constant \( C > 0 \) such that for each choice of finitely many vectors \( x_1, \ldots, x_N \in X \) we have

\[
\left( \sum_{k=1}^N \left\| x_k \right\|_q \right)^{1/q} \leq C \left( \int_T \left\| \sum_{k=1}^N x_k z^k \right\|_q' \, dz \right)^{1/q'}.
\]

We refer to the comments after Proposition 2.2 regarding the equivalence of these two concepts and also their connection with (4) and (5) below. It was shown in [10, Proposition 2.4] that a Banach space \( X \) has Fourier cotype \( q \geq 2 \) if and only if there exists \( C > 0 \) such that for every finite family \( \{x_\alpha\}_{\alpha \in N_0^{[N]}} \) we have

\[
\left( \sum_{\alpha} \left\| x_\alpha \right\|_q \right)^{1/q} \leq C \left( \int_T \left\| \sum_{\alpha} x_\alpha z^\alpha \right\|_q' \, dz \right)^{1/q'}.
\]

The proof of [10, Proposition 2.4]) also works to show that \( X \) has Fourier type \( 1 \leq p \leq 2 \) if and only if there exists \( C > 0 \) such that for every finite family \( \{x_\alpha\}_{\alpha \in N_0^{[N]}} \) in \( X \) we have

\[
\left( \int_T \left\| \sum_{\alpha} x_\alpha z^\alpha \right\|_p' \, dz \right)^{1/p'} \leq C \left( \sum_{\alpha} \left\| x_\alpha \right\|_p \right)^{1/p}.
\]

A straightforward argument using the Bohr transform (see (3)) allows to reformulate (4) and (5) in terms of Dirichlet series as

\[
\left( \sum_{n=1}^N \left\| a_n \right\|_X^q \right)^{1/q} \leq C \left\| \sum_{n=1}^N a_n n^{-s} \right\|_{\mathcal{H}_q(X)}
\]

and

\[
\left\| \sum_{n=1}^N a_n n^{-s} \right\|_{\mathcal{H}_p(X)} \leq C \left( \sum_{n=1}^N \left\| a_n \right\|_X^p \right)^{1/p},
\]

respectively, for every \( X \)-valued Dirichlet polynomial \( D = \sum_{n=1}^N a_n n^{-s} \). Note that (7) and the density of the finite sequences in \( \ell_p(X) \) (the space of \( p \)-summing sequences in \( X \)) show that the operator \( \ell_p(X) \to \)
$\mathcal{H}_{p'}(X)$ given by $(a_n) \sim \sum a_n n^{-s}$ is continuous. Analogously, by (6) and the density of the Dirichlet polynomials in $\mathcal{H}_{q'}(X)$ (see [15, 24.2.1e]), the operator $\mathcal{H}_{q'}(X) \to \ell_q(X)$ given by $\sum a_n n^{-s} \sim (a_n)$ is also continuous. This gives the equivalence between the first and third statements in each of the following two results. The equivalence between the second and third statements is a straightforward consequence of the definition of the Hardy spaces of Dirichlet series.

**Proposition 2.1.** Let $X$ be a Banach space. For $2 \leq q < \infty$ and $C \geq 1$, the following statements are equivalent:

(a) $X$ has Fourier cotype $q$ with constant $C$;

(b) every Dirichlet series $D = \sum a_n n^{-s} \in \mathcal{H}_{q'}(X)$ satisfies

$$\left( \sum_{n=1}^{\infty} \|a_n\|_X^q \right)^{1/q} \leq C \|D\|_{\mathcal{H}_{q'}(X)};$$

(c) every $f \in \mathcal{H}_{q'}(\mathbb{T}^{\infty}, X)$ satisfies

$$\left( \sum_{\alpha \in \mathbb{N}_0^{[n]}} \|\hat{f}(\alpha)\|_X^q \right)^{1/q} \leq C \|f\|_{\mathcal{H}_{q'}(\mathbb{T}^{\infty}, X)}.$$

**Proposition 2.2.** Let $X$ be a Banach space. For $1 \leq p \leq 2$ and $C \geq 1$, the following statements are equivalent:

(a) $X$ has Fourier type $p$ with constant $C$;

(b) for every $(a_n)_n \in \ell_p(X)$ the Dirichlet series $D = \sum a_n n^{-s}$ converges in $\mathcal{H}_{p'}(X)$ and

$$\|D\|_{\mathcal{H}_{p'}(X)} \leq C \left( \sum_{n=1}^{\infty} \|a_n\|_X^p \right)^{1/p};$$

(c) for every $(x_\alpha)_{\alpha \in \mathbb{N}_0^{[n]}} \in \ell_p(X)$ there is a function $f \in \mathcal{H}_{p'}(\mathbb{T}^{\infty}, X)$ and so that $\hat{f}(\alpha) = x_\alpha$ for every $\alpha$ and

$$\|f\|_{\mathcal{H}_{p'}(\mathbb{T}^{\infty}, X)} \leq C \left( \sum_{\alpha \in \mathbb{N}_0^{[n]}} \|\hat{f}(\alpha)\|_X^p \right)^{1/p}.$$

As a matter of fact, Fourier type and cotype can be seen as particular cases in the more general theory of Fourier type with respect to groups (see [18], whose notation we follow now, for an excellent survey on this and related subjects). Within this setting Fourier type $p$ (as we have defined it) is Fourier type $p$ with respect to $\mathbb{Z}$, and our Fourier cotype $q$ is Fourier type $q'$ with respect to $\mathbb{T}$. Then [18, Theorem 6.6] implies that $X$ has Fourier type $p$ if and only if it has Fourier cotype $p'$, and hence both concepts are equivalent. However, we have preferred to deal with them separately because we later work with other notions of type and cotype (which are not equivalent to each other) and in this way the relationship between these and the new ones becomes more apparent.

On the other hand, this abstract point of view allows a proof of Propositions 2.1 and 2.2 based on known results on Fourier type on groups. We only sketch here the arguments. Regarding Proposition 2.1, simply note that the statement (c) is Fourier type $q'$ with respect to $\mathbb{T}^{\infty}$. Then the equivalence between (a) and (c) follows from [18, Theorem 6.14].

The argument for Proposition 2.2 is slightly longer. First of all $X$ has Fourier type $p$ if and only if $X^*$ has Fourier type $p$ with respect to $\mathbb{T}$ [18, Theorem 6.3], and this happens if and only if $X^*$ has Fourier type...
p with respect to $\mathbb{T}^\infty$ by [18, Theorem 6.14]. Again by [18, Theorem 6.3], this is equivalent to $X$ having type $p$ with respect to the dual group of $\mathbb{T}^\infty$, which is $\mathbb{Z}^{(n)}$, and this is Proposition 2.2(c).

Following [32, Section 5.4] (see also [17, Chapter 13]), we consider $[-1,1]^\infty$ with the probability measure given by the infinite product of the uniform probability $(\delta_1 + \delta_{-1})/2$. For $\varepsilon = (\varepsilon_n)_n \in [-1,1]^\infty$ and $A \subset \mathbb{N}$ finite we denote

$$\varepsilon_A = \prod_{n \in A} \varepsilon_n,$$

and call it a Walsh function. The family of all such functions is known as the Walsh system. A finite sum $\sum_A x_A \varepsilon_A$ will be called a Walsh polynomial. Due to the probabilistic nature of the measure space, when dealing with $L_p([-1,1]^\infty, X)$, we will write $\mathbb{E}$ (expected value) rather than integrals. For $f \in L_1([-1,1]^\infty, X)$, the corresponding Walsh-Fourier coefficients are defined by

$$\hat{f}(A) = \mathbb{E}[f(\varepsilon) \varepsilon_A].$$

With this at hand we may introduce another notion of type/cotype. A Banach space $X$ has Walsh type $p$ if there is a constant $C > 0$ such that for every $n$ and every family $\{x_A : A \subseteq \{1, \ldots, n\}\} \subset X$ we have

$$\left( \mathbb{E}\left\| \sum_A x_A \varepsilon_A \right\|^p \right)^{\frac{1}{p}} \leq C \left( \sum_A \|x_A\|_p \right)^{\frac{1}{p}},$$

and has Walsh cotype $q$ if here is a constant $C > 0$ such that for every $n$ and every family $\{x_A : A \subseteq \{1, \ldots, n\}\} \subset X$ we have

$$\left( \sum_A \|x_A\|^q \right)^{\frac{1}{q}} \leq C \left( \mathbb{E}\left\| \sum_A x_A \varepsilon_A \right\|^{q'} \right)^{\frac{1}{q'}}.$$

Standard density arguments allow us to reformulate these concepts as inequalities analogous to Proposition 2.2(c) and Proposition 2.1(c). Indeed, $X$ has Walsh type $p$ if and only if there is $C \geq 1$ so that

$$\|f\|_{L_p([-1,1]^\infty, X)} \leq C \left( \sum_{A \in \mathbb{N}} \|\hat{f}(A)\|_p \right)^{\frac{1}{p}}, \quad (8)$$

Analogously, for $X$ with Walsh cotype $q$, we have

$$\left( \sum_{A \in \mathbb{N}} \|\hat{f}(A)\|_q \right)^{\frac{1}{q}} \leq C \|f\|_{L_q([-1,1]^\infty, X)}.$$

Once again, these notions of type/cotype sit in a more general framework, namely that of type/cotype with respect to an orthonormal system (we refer again to [18]). The concepts of Walsh type $p$ and Walsh cotype $p'$ coincide (see [18, Theorem 7.14]). To our best knowledge it is not known whether or not these are the same as Fourier type and cotype.

3 Polynomial type and cotype

We give now the notion of cotype that is going to play a major rôle for us. It was introduced in [10] under the name hypercontractive homogeneous cotype as an extension of the 'usual' (or, to be more accurate, Rademacher) cotype. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n, 0, 0, \ldots) \in \mathbb{N}^{(n)}_0$ we write $|\alpha| = \alpha_1 + \cdots + \alpha_n$. 

5
**Definition 3.1.** A Banach space $X$ has polynomial cotype $q$ if there exists $C > 0$ such that for every $m \in \mathbb{N}$ and every finite family $(x_{\alpha})_{|\alpha|=m}$ we have

$$
\left( \sum_{|\alpha|=m} \|x_{\alpha}\|^q \right)^{\frac{1}{q}} \leq C^m \left( \int_{\mathbb{T}^n} \left\| \sum_{|\alpha|=m} x_{\alpha} z^\alpha \right\|^2 dz \right)^{\frac{1}{2}}.
$$

(10)

Although it was not considered in [10] we may in a natural way introduce the type counterpart.

**Definition 3.2.** A Banach space $X$ has polynomial type $p$ if there exists $C > 0$ such that for every $m \in \mathbb{N}$ and every finite family $(x_{\alpha})_{|\alpha|=m}$ we have

$$
\left( \int_{\mathbb{T}^n} \left\| \sum_{|\alpha|=m} x_{\alpha} z^\alpha \right\|^2 dz \right)^{\frac{1}{2}} \leq C^m \left( \sum_{|\alpha|=m} \|x_{\alpha}\|^p \right)^{\frac{1}{p}}.
$$

(11)

It is clear from the definitions that polynomial type and cotype are local properties (loosely speaking, they depend on the finite dimensional subspaces of $X$). In particular, $X$ and its bidual $X^{**}$ have polynomial type and cotype for the same values of $p$ and $q$ by the local reflexivity principle.

Let us point out that Fourier cotype and type, as formulated in (4) and (5), give (10) and (11) with universal constants, independent of $m$. Actually, by a standard homogenization trick, we can see that (10) and (11) with universal constants are equivalent to Fourier cotype and type. As a consequence, absolute constants in (10) and (11) give the general estimates in Propositions 2.1 and 2.2. In the much weaker assumptions of polynomial type and cotype, the exponential dependence on $m$ (as $C^m$) still allows us to carry estimations from the homogeneous to the general setting at a reasonable price (the precise statements are given in Theorems 3.7 and 3.8).

Polynomial type and cotype involve inequalities that should be satisfied by every $m$-homogeneous polynomial. With the same argument as in [15, Theorem 8.10], it can be seen that if a Banach space has polynomial type or cotype, then inequalities as (10) and (11) are also satisfied (with the same constant) for polynomials of degree $\leq m$. On the other hand, a main step to work with these properties is to see that, as a matter of fact, it suffices to check those inequalities for polynomials of a very specific class, easier to handle: tetrahedral polynomials. These are polynomials where no power bigger than 1 appears or, in other words, the monomials involved consist only of products of different variables. More precisely, a tetrahedral polynomial is of the form

$$
\sum_{\alpha \in \{0,1\}^n} x_{\alpha} z^{\alpha}.
$$

Note that, given $A \subseteq \{1, \ldots, n\}$ we can define $\alpha = (\alpha_i)_i \in \{0,1\}^{|A|}$ as $\alpha_i = 1$ if $i \in A$ and 0 if $i \notin A$. With this idea, to each finite set we can associate a multi-index (and vice-versa), and each Walsh polynomial can be associated to a tetrahedral polynomial (and vice-versa). By a slight abuse of notation we get

$$
\sum_{\alpha \in \{0,1\}^n} x_{\alpha} z^{\alpha} \sim \sum_{A \subseteq \{1, \ldots, n\}} \sum_{\alpha \in \{0,1\}^{|A|}} x_{\alpha} \varepsilon_A z^A.
$$

Using Walsh polynomials as an intermediate step we may see that the behaviour of tetrahedral polynomials determines the polynomial type or cotype of a given space. The proof is rather technical, and is postponed to Section 5.

**Proposition 3.3.** For a Banach space $X$ and $2 \leq q < \infty$ the following statements are equivalent:

(a) $X$ has polynomial cotype $q$;

(b) inequality (10) holds for every $m$-homogeneous tetrahedral polynomial;
(c) there exists $C > 0$ such that for every $m$, every $n$ and for each family $\{x_A: A \subseteq \{1, \ldots, n\}, |A| = m\} \subset X$ we have
\[
\left( \sum_{A} \|x_A\|^q \right)^{\frac{1}{q}} \leq C^m \left( E \left\| \sum_{A} x_A \varepsilon_A \right\|^{2} \right)^{\frac{1}{2}}; 
\] (12)

(d) the same as (c) for the family $\{x_A: A \subseteq \{1, \ldots, n\}, |A| \leq m\} \subset X$.

**Remark 3.4.** With the same idea for the proof, one can show that a Banach space has polynomial type $p$ if and only if an inequality as (11) holds for $m$-homogeneous tetrahedral polynomials, and this happens if and only if, under the assumptions of Proposition 3.3–(c) or (d), we have
\[
\left( E \left\| \sum_{A} x_A \varepsilon_A \right\|^{2} \right)^{\frac{1}{2}} \leq C^m \left( \sum_{A} \|x_A\|^p \right)^{\frac{1}{p}}. 
\] (13)

Let us note that the expectations that we have in (12) and (13) are the $L_2$ norms of homogeneous polynomials, which in the context of Walsh polynomials are those indexed on families of sets having the same cardinality, and in this case all $L_1$-norms are equivalent. This is a particular case of [32, Corollary 5.5], from which we know that if we have a family $\{x_A: A \subset \mathbb{N}, |A| = m\}$ in a Banach space $X$ with only finitely many non-zero elements, then
\[
\left\| \sum_{A} x_A \varepsilon_A \right\|_{L_1([-1,1]^\infty, X)} \leq \left( \frac{r-1}{s-1} \right)^{\frac{m}{r}} \left\| \sum_{A} x_A \varepsilon_A \right\|_{L_1([-1,1]^\infty, X)} 
\] (for every $1 \leq s \leq r < \infty$). As a straightforward consequence we have the following result.

**Corollary 3.5.** Every Banach space with Walsh type $p$ has polynomial type $p$ and polynomial cotype $p'$.

**Remark 3.6.** As a matter of fact, the inequality in (14) has a counterpart for ‘usual’ polynomials. If $\{x_\alpha: \alpha \in \mathbb{N}_0^n, |\alpha| = m\}$ is a family in a Banach space $X$ with only finitely many non-zero elements, then
\[
\left\| \sum_{\alpha} x_\alpha z^\alpha \right\|_{L_1[T^\infty, X]} \leq \left( \frac{r}{s} \right)^{\frac{m}{r}} \left\| \sum_{\alpha} x_\alpha z^\alpha \right\|_{L_1[T^\infty, X]} 
\] (15)

(this is [10, Proposition 1.2]). Thus, (15) shows that the exponent of the integrals appearing in the definitions of polynomial cotype and type (see (10) and (11)) can be replaced by any other $1 \leq p < \infty$. Similarly, by (14), $p$-norms of Walsh polynomials can be interchanged at the cost of a constant that grows exponentially with the degree. We will most commonly use the exponents 1, 2 or $q$.

Closely related to the concepts of type and cotype is the notion of K-convexity. A Banach space $X$ is said to be K-convex if the Rademacher projection is bounded. More precisely, the mapping defined on the finite sums in $L_2([-1,1]^\infty, X)$ by
\[
P_1 \left( \sum_{A} x_A \varepsilon_A \right) = \sum_{|A| = 1} x_A \varepsilon_A
\]
extends to bounded linear operator $P_1 : L_2([-1,1]^\infty, X) \rightarrow L_2([-1,1]^\infty, X)$.

If $X$ is K-convex, we can also define for each $m$ the projection $P_m : L_2([-1,1]^\infty, X) \rightarrow L_2([-1,1]^\infty, X)$, which on finite sums is given by $P_m \left( \sum_{A} x_A \varepsilon_A \right) = \sum_{|A| = m} x_A \varepsilon_A$. By [31, Theorem 2.1] or [17, Theorem 13.16], there exists $K > 1$ such that
\[
\|P_m\| \leq K^m
\] (16)

for every $m$. Also, a Banach space is K-convex if and only if it has non-trivial type (see e.g. [17, Theorem 13.3]).
We are finally in the position to show how polynomial cotype gives an inequality in the spirit of (2). We obtain inequalities, not only for Dirichlet series, but also for functions defined on $\mathbb{T}^\infty$ or $\{-1,1\}^\infty$, as in Proposition 2.1 or (9). Comparing what we get now with those inequalities we gather that the $r$ factor is the price we pay for loosening the hypothesis of Fourier or Walsh to polynomial cotype. Let us recall that the number of prime divisors of $n \in \mathbb{N}$, counted with multiplicity is denoted by $\Omega(n)$.

**Theorem 3.7.** For a Banach space $X$ and $2 \leq q < \infty$ the following statements are equivalent:

(a) $X$ has polynomial cotype $q$;

(b) for some (every) $1 \leq p < \infty$, there exist constants $C \geq 1$ and $0 < r < 1$ such that every vector-valued Dirichlet series $D = \sum a_n n^{-s} \in \mathcal{H}_p(X)$ satisfies

$$\left(\sum_{n=1}^{\infty} r^{\Omega(n)} \|a_n\|_q^q\right)^{\frac{1}{q}} \leq C \|D\|_{\mathcal{H}_p(X)}.$$

(c) for some (every) $1 \leq p < \infty$, there exist constants $C \geq 1$ and $0 < r < 1$ such that every function $f \in \mathcal{H}_p(\mathbb{T}^\infty, X)$ satisfies

$$\left(\sum_{\alpha \in \mathbb{N}_0^n} r^{\alpha} \|\hat{f}(\alpha)\|_q^q\right)^{\frac{1}{q}} \leq C \|f\|_{\mathcal{H}_p(\mathbb{T}^\infty, X)}.$$

In addition, the next statement (d) implies (a),(c) and (b) and is equivalent to them whenever $X$ is $K$-convex:

(d) for some (every) $1 < p < \infty$, there exist constants $C \geq 1$ and $0 < r < 1$ such that every function $f \in L_p(\{-1,1\}^\infty, X)$ satisfies

$$\left(\sum_{A \subset \mathbb{N}} r^{|A|} \|\hat{f}(A)\|_q^q\right)^{\frac{1}{q}} \leq C \|f\|_{L_p(\{-1,1\}^\infty, X)}.$$

**Proof.** Observe that (c) and (b) are equivalent via Bohr’s transform. On the other hand, the fact that (c)$\Rightarrow$(a) follows by noticing that for an $m$-homogeneous polynomial $P = \sum x_\alpha z^\alpha$, the sum at the left-hand side becomes $r^{m/q} \left(\sum \|x_\alpha\|_q\right)^{1/q}$. The same argument proves that (d)$\Rightarrow$(a) invoking Proposition 3.3.

Next we see that (a)$\Rightarrow$(c). Set $f \in \mathcal{H}_p(\mathbb{T}^\infty, X)$ and for every $m \in \mathbb{N}$ let $f_m$ be its $m$-homogeneous projection (see [9, Proposition 2.5]). By Remark 3.6, there is a constant $c \geq 1$ such that for every (finite) $m$-homogeneous polynomial $P = \sum x_\alpha z^\alpha$ we have

$$\left(\sum_{|\alpha|=m} \|x_\alpha\|_q\right)^{\frac{1}{q}} \leq c^m \|P\|_p.$$

Since polynomials are dense in $\mathcal{H}_p(\mathbb{T}^\infty, X)$ and the $m$-homogeneous projection is a contraction, a straightforward density argument yields

$$\left(\sum_{|\alpha|=m} \|\hat{f}(\alpha)\|_q^q\right)^{\frac{1}{q}} \leq c^m \|f_m\|_p \leq c^m \|f\|_p. \quad (17)$$
Taking $r < 1/c^q$ we get
\[
\left( \sum_{\alpha \in \mathbb{N}_0^n} r^{|\alpha|} \| \hat{f}(\alpha) \|_q^q \right)^{\frac{1}{q}} = \left( \sum_{m=1}^{\infty} \sum_{|\alpha| = m} r^m \| \hat{f}(\alpha) \|_q^q \right)^{\frac{1}{q}} \leq \left( \sum_{m=1}^{\infty} (rc^q)^m \right)^{\frac{1}{q}} \| f \|_p \leq C \| f \|_1,
\]
which completes the argument.

We finally show that (a)$\Rightarrow$(d) for K-convex spaces. First assume that $p = 2$. In this case (16) gives a constant $K$ so that
\[
\| f_m \|_2 \leq K_m \| f \|_2
\]
for every $f \in L_2((-1,1)^\infty, X)$. This enables us to proceed exactly as in (17) to get the desired result. For the general case when $1 < p < \infty$, it only remains to show that an inequality analogous to (18) holds. On the one hand, if $2 \leq p < \infty$, using (14) we get
\[
\| f_m \|_p \leq (p-1)^{\frac{m}{p}} \| f_m \|_2 \leq (p-1)^{\frac{m}{p}} K_m \| f \|_2 \leq (p-1)^{\frac{m}{p}} K_m \| f \|_p.
\]
On the other hand, it is a well-known fact that if $X$ is K-convex, so is $X^*$ (see for example [17, Corollary 13.7 and Theorem 13.15]). Therefore, for $1 \leq p \leq 2,$
\[
\| f_m \|_p = \sup_{g \in L_{p'}(X^*)} \| g \|_{p',1} \| f_m \|_p \leq \sup_{g \in L_{p'}(X^*)} \| g \|_{p',1} \| f \|_p \leq \tilde{K} \| f \|_p,
\]
for some constant $\tilde{K} > 0$.

We turn now our attention to polynomial type, and get an analogous result (compare it also with Proposition 2.2 and (8)). The proof follows essentially the same lines as that of Theorem 3.7 (in fact, it is slightly simpler) so we omit it.

**Theorem 3.8.** For a Banach space $X$ and for $1 \leq p \leq 2$ the following statements are equivalent:

(a) $X$ has polynomial type $p$;

(b) for some (every) $1 \leq q < \infty$ there exist constants $R, C \geq 1$ such that every $X$-valued Dirichlet series $D = \sum a_n n^{-s}$ satisfies
\[
\| D \|_{H_q(X)} \leq C \left( \sum_{n=1}^{\infty} R^{|A(n)|} \| a_n \|_p \right)^{\frac{1}{p}};
\]

(c) for some (every) $1 \leq q < \infty$ there exist constants $C, R \geq 1$ and such that every function $f \in H_q(T^\infty, X)$ satisfies
\[
\| f \|_{H_q(T^\infty, X)} \leq C \left( \sum_{|\alpha| \in \mathbb{N}_0^n} R^{|\alpha|} \| \hat{f}(\alpha) \|_p \right)^{\frac{1}{p}};
\]

(d) for some (every) $1 \leq q < \infty$ there exist constants $C, R \geq 1$ such that every function $f \in L_q((-1,1)^\infty, X)$ satisfies
\[
\| f \|_{L_q((-1,1)^\infty, X)} \leq C \left( \sum_{|A| < \infty} R^{|A|} \| \hat{f}(A) \|_p \right)^{\frac{1}{p}}.
\]
The preceding inequalities should be understood as follows: if the sum at the right-hand side is finite, then the Dirichlet series (or the function) belongs to the corresponding space and its norm is controlled by the sum. But if the sum does not converge, then nothing can be said about the series or the function.

We finish this section by looking at the relationship between type and cotype on a given space and its dual. It is well-known that spaces with type $p$ have duals with cotype $p'$. However, the dual statement requires K-convexity, that is, K-convex spaces with cotype $q$ have duals with type $q'$. Let us see that the same holds for polynomial type and cotype.

**Proposition 3.9.** If a Banach space has polynomial type $p$, then its dual has polynomial cotype $p'$. Also, if a K-convex Banach space has polynomial cotype $q$, then its dual has polynomial type $q'$.

**Proof.** Assume that a Banach space $X$ has polynomial type $p$ with constant $C \geq 1$ and pick a finite family $\{x^*_\alpha : \|\alpha\| = m\}$ in $X^*$. We proceed exactly as in the classical duality result for type and cotype. Given $\delta > 0$, choose vectors $x_\alpha \in X$ such that

$$
\left( \sum_{|\alpha|=m} \|x^*_\alpha\|_{X^*}^{p'} \right)^{\frac{1}{p'}} \leq \sum_{|\alpha|=m} x^*_\alpha(x_\alpha) + \delta \quad \text{and} \quad \left( \sum_{|\alpha|=m} \|x_\alpha\|_X^{p} \right)^{\frac{1}{p}} = 1.
$$

Therefore, we have

$$
\left( \sum_{|\alpha|=m} \|x^*_\alpha\|_{X^*}^{p'} \right)^{\frac{1}{p'}} \leq \sum_{|\alpha|=m} x^*_\alpha(x_\alpha) + \delta = \int_{\mathbb{T}^n} \sum_{|\alpha|=m} x^*_\alpha f^\alpha \left( \sum_{|\alpha|=m} x_\alpha z^{-\alpha} \right) dz + \delta 
\leq \|P\|_{L^p(X)} \left( \int_{\mathbb{T}^n} \left( \sum_{|\alpha|=m} x_\alpha z^\alpha \right)^{p/2} \right)^{\frac{2}{p}} + \delta 
\leq C^m \|P\|_{L^p(X)} \left( \sum_{|\alpha|=m} \|x_\alpha\|_{X}^p \right)^{\frac{1}{p}} + \delta = C^m \|P\|_{L^p(X^*)} + \delta,
$$

where the last inequality is a consequence of the polynomial type of $X$. So, $X^*$ has polynomial cotype $p'$.

Now, suppose $X$ is K-convex and has polynomial cotype $q$. We use Proposition 3.3 and its polynomial type counterpart to work with Walsh polynomials. Let $P = \sum x^*_\alpha \varepsilon_A$ be an $m$-homogeneous Walsh polynomial of $n$ variables. Given $\delta > 0$, take a function $f \in L^2((-1,1)^n, X)$ such that $\|f\|_2 = 1$ and

$$
\|P\|_2 \leq E[P(\varepsilon)(f(\varepsilon))] + \delta.
$$

Therefore,

$$
\|P\|_2 \leq \sum_{|A|=m} x^*_A(\widehat{f}(A)) + \delta \leq \left( \sum_{|A|=m} \|x^*_A\|_{X^*}^{q'} \right)^{\frac{1}{q'}} \left( \sum_{|A|=m} \|\widehat{f}(A)\|_{\chi}^{q} \right)^{\frac{1}{q}} + \delta 
\leq C^n \left( E \left( \sum_{|A|=m} f(A) \varepsilon_A \right) \right)^{\frac{1}{q'}} \left( \sum_{|A|=m} \|x^*_A\|_{X^*}^{q'} \right)^{\frac{1}{q'}} + \delta.
$$

We used the polynomial cotype of $X$ in the last inequality. Notice that $\sum \widehat{f}(A) \varepsilon_A$ is the $m$-homogeneous projection of $f$. Since $X$ is K-convex we use (18) to get

$$
\|P\|_2 \leq C^m K^m \left( \sum_{|A|=m} \|x^*_A\|_{X^*}^{q'} \right)^{\frac{1}{q'}} + \delta,
$$

which concludes the proof.

\qed
4 Conditions ensuring polynomial type and cotype

In this section we present different conditions that ensure that a Banach space has polynomial type or cotype. These conditions are quite general and most Banach spaces with non-trivial type or cotype satisfy at least one of them. Also, we show an example of a Banach space with polynomial cotype 2 that does not enjoy any of these conditions.

4.1 Type and cotype 2

**Theorem 4.1.** For a Banach space $X$ the following assertions hold:

(a) if $X$ is K-convex and has cotype 2, then it has polynomial cotype 2;

(b) if $X$ has type 2, then it has polynomial type 2.

**Proof.** First observe that (b) follows from (a) by a duality argument using Proposition 3.9.

In order to prove (a), notice that Proposition 3.3 allows us to restrict our attention to the tetrahedral case. We proceed by induction in the degree of the polynomial. If $m=1$, we recover the cotype 2 inequality which holds by hypothesis. Now assume there is a constant $C > 0$ such that

$$\left( \sum_{|\alpha|=m-1} \|x_{\alpha}\|^2 \right)^{1/2} \leq C^{m-1} \left( \int_{\mathbb{T}^n} \left( \sum_{|\alpha|=m-1} x_{\alpha} z_{\alpha}^2 \right)^{1/2} \right),$$

holds for every homogeneous tetrahedral polynomial $\sum x_{\alpha} z_{\alpha}$ of degree $m-1$. Fix a homogeneous tetrahedral polynomial $P(z) = \sum x_{\alpha} z_{\alpha}$, of degree $m$ and $n$ variables. For every $1 \leq k \leq n$ define

$$Q_k(z) = m \tilde{P}(z, \ldots, z, e_k),$$

where $\tilde{P}$ is the symmetric $m$-linear mapping associated to $P$ and $e_k$ is the $k$-th canonical vector of length $n$. A straightforward computation shows that $Q_k$ is an homogeneous tetrahedral polynomial of degree $m-1$ for every $k$. Moreover, we have

$$Q_k(z) = \sum_{\alpha, \alpha_k=1} x_{\alpha} z_{\alpha_k} \ldots z_{k-1}^{\alpha_{k-1}} z_{k+1}^{\alpha_{k+1}} \ldots z_n^{\alpha_n}.$$

Therefore, applying the inductive hypothesis we get

$$\left( \sum_{|\alpha|=m} \|x_{\alpha}\|^2 \right)^{1/2} \leq \frac{1}{\sqrt{m}} \left( \sum_{|\alpha|=m} \sum_{k=1}^{n} \|x_{\alpha}\|^2 \right)^{1/2} = \frac{1}{\sqrt{m}} \left( \sum_{k=1}^{n} \sum_{\alpha_k=1} x_{\alpha_k} \right)^{1/2} \leq \frac{C^{m-1}}{\sqrt{m}} \left( \sum_{k=1}^{n} \int_{\mathbb{T}^n} \|Q_k(z)\|^2 dz \right)^{1/2} = \frac{C^{m-1}}{\sqrt{m}} \left( \int_{\mathbb{T}^n} \sum_{k=1}^{n} \|iz_k Q_k(z)\|^2 dz \right)^{1/2} = C^{m-1} \left( \int_{\mathbb{T}^n} \sum_{k=1}^{n} \left\| m B(z, \ldots, z, \frac{iz_k}{\sqrt{m}} e_k) \right\|^2 dz \right)^{1/2}.
Since $X$ has cotype 2 there is a constant $C_2 > 0$ such that

$$\left( \sum_{|\alpha| = m} \|x_\alpha\|^2 \right)^{1/2} \leq C^{m-1} C_2 \left( \int_{\mathbb{T}^n} \mathbb{E} \left\| \sum_{k=1}^n \varepsilon_k m \hat{P} \left( z, \ldots, z, \frac{i \varepsilon_k}{\sqrt{m}} \right) \right\| dz \right)^{1/2}$$

$$\leq C_2 C^{m-1} \left( \int_{\mathbb{T}^n} \mathbb{E} \left\| m \hat{P} \left( z, \ldots, z, \frac{i \varepsilon}{\sqrt{m}} \right) \right\|^2 dz \right)^{1/2}$$

Finally, the rather artificial introduction of the terms $iz_k$ becomes clearer by observing that

$$m B \left( z, \ldots, z, \frac{i}{\sqrt{m}} \varepsilon \right),$$

is the 1-homogeneous projection of the polynomial $P(z + i \varepsilon z/\sqrt{m})$ as a function in the variable $\varepsilon$. Thus, using K-convexity and rotation invariance we deduce

$$\left( \sum_{|\alpha| = m} \|x_\alpha\|^2 \right)^{1/2} \leq C_2 K C^{m-1} \left( \int_{\mathbb{T}^n} \mathbb{E} \left\| \left( 1 + \frac{\varepsilon}{\sqrt{m}} \right) z \right\| dz \right)^{1/2}$$

$$= C_2 K C^{m-1} \sqrt{1 + \frac{1}{m}} \left( \int_{\mathbb{T}^n} \mathbb{E} \left\| P(z) \right\|^2 dz \right)^{1/2} \leq \sqrt{\varepsilon} C_2 K C^{m-1} \left( \int_{\mathbb{T}^n} \mathbb{E} \left\| P(z) \right\|^2 dz \right)^{1/2},$$

where $K$ is the K-convexity constant (see (16)). Taking $C \geq \sqrt{\varepsilon} C_2 K$ we conclude the argument.

4.2 Gaussian Average Property

A Banach space $X$ has the Gaussian Average Property (GAP in short, see [11]) if there exists $G \geq 1$ such that for every finite choice $x_1, \ldots, x_N \in X$, the operator $T : X^* \to \ell_2$ defined by $T(x^*) = \sum_{k=1}^N x^*(x_k) e_k$ satisfies

$$\left( \mathbb{E} \left\| \sum_{k=1}^N x_k g_k \right\|^2 \right)^{1/2} \leq G \pi_1(T), \quad (19)$$

where $g_1, \ldots, g_N$ are i.i.d. Gaussian random variables. Spaces with GAP have non-trivial cotype [11, Theorem 1.3]. Our aim now is to show that spaces with GAP also have polynomial cotype (and with the same exponent as the usual cotype). To do this we are going to need the following case of the so-called Chevet-Person-Saphar inequalities. Given $\varphi_1, \ldots, \varphi_N \in \mathcal{L}_1(\mathbb{T}^n)$ and $x_1, \ldots, x_N \in X$, the operator $u : X^* \to \mathcal{L}_1(\mathbb{T}^n)$ defined by $u(x^*) = \sum_{k=1}^N x^*(x_k) \varphi_k$ satisfies

$$\pi_1(u) \leq \int_{\mathbb{T}^n} \left\| \sum_{k=1}^N \varphi_k(z) x_k \right\| dz \quad (20)$$

We refer the reader to [14, 15.10] and [14, 17.12] for the proof.

**Proposition 4.2.** If the Banach space $X$ has GAP and cotype $q$, then $X$ has polynomial cotype $q$.

**Proof.** Let $P(z) = \sum_{|\alpha| = m} x_\alpha z^\alpha$ be a homogeneous polynomial on $\mathbb{C}^n$ with values in $X$. Since $X$ has cotype $q < \infty$, it has gaussian cotype $q$ [17, Corollary 12.28]. Therefore there is an universal constant $C > 0$ such that

$$\left( \sum_{|\alpha| = m} \|x_\alpha\|^q \right)^{1/q} \leq C \left( \mathbb{E} \left( \sum_{n=1}^N x_{\alpha(n)} g_n \right) \right)^{1/2},$$

where $\alpha(n)$ is the multi-index corresponding to the exponents of the prime number decomposition of $n$.  

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Using that $X$ has GAP we get
\[
\left( \sum_{|\alpha| \leq m} \|x_\alpha\|^{q} \right)^{1/q} \leq CG\pi_{1}(T), \quad (21)
\]
where $T(x^*) = (x^*(x_\alpha))_\alpha \in \ell_2$ and $G$ is the GAP constant of $X$ (see (19)).

Now we consider $\varphi_\alpha(z) = z^\alpha$ for $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = m$ (note that we have finitely many) and define $u : X^* \to L_1(T^n)$ by $u(x^*) = \sum_\alpha x^*(x_\alpha)\varphi_\alpha$. From (20) we have
\[
\pi_{1}(u) \leq \int_{T^n} \|P(z)\| dz. \quad (22)
\]

Finally, if $x_{1}^*, \ldots, x_{M}^* \in X^*$, then by the polynomial Khinchin-Steinhaus inequality [15, Theorem 25.9],
\[
\sum_{j=1}^{M} \|T(x_j^*)\|_{\ell_2} = \sum_{j=1}^{M} \left( \sum_{\alpha} |x_j^*(x_\alpha)|^2 \right)^{1/2} = \sum_{j=1}^{M} \left( \int_{T^n} \left| \sum_{|\alpha|=m} x_j^*(x_\alpha)z^\alpha \right|^2 dz \right)^{1/2} \leq 2^{\frac{m}{q}} \sum_{j=1}^{M} \int_{T^n} \left| \sum_{|\alpha|=m} x_j^*(x_\alpha)z^\alpha \right| dz = \sum_{j=1}^{M} \|u(x_j^*)\|_{L_1(T^n)}. \quad (23)
\]

This together with the definition of absolute summability implies
\[
\pi_{1}(T) \leq 2^{\frac{m}{q}} \pi_{1}(u). \quad (23)
\]

Joining (21), (22) and (23) completes the proof. \qed

There are several conditions that imply GAP for which we need some definitions. A Banach space $X$ has the Gordon-Lewis property (GL) if every absolutely summing operator from $X$ to an arbitrary Banach space $Y$ factors through $L_1$. If $X$ satisfies this property only for $Y = \ell_2$ we say $X$ has the Gordon-Lewis property for $\ell_2$ (GL$_2$).

Remark 4.3. By [11, Theorem 1.4], a Banach space $X$ has GAP if it satisfies any of the following conditions:

- has finite cotype and GL$_2$;
- has type 2;
- is a subspace of a Banach lattice of finite cotype.

In [10, Theorem 2.1] it is shown that a Banach space with local unconditional structure (l.u.st., see [17, Chapter 17] for the definitions) and cotype $q$ has polynomial cotype $q$. Proposition 4.2 above is an extension of this result: every space with l.u.st. and finite cotype has GAP, since l.u.st. implies GL$_2$ [20, Lemma 3.3].

Combining the previous comments with Theorem 4.1, Proposition 4.2 and Proposition 3.9 leads to the following.

Corollary 4.4. A Banach space with type 2 and cotype $2 \leq q \leq \infty$ has polynomial type 2 and polynomial cotype $q$. Analogously, a Banach space with type $1 < p \leq 2$ and cotype 2 has polynomial type $p$ and polynomial cotype 2.

An analogous result to Proposition 4.2 but slightly weaker holds for polynomial type. It can be deduced proceeding as in Lemma 2.2 and Proposition 2.3 from [10] (and noting the result from [30] used there only needs GL).

Proposition 4.5. If a Banach space $X$ has type $p$ and GL (in particular, if $X$ has type $p$ and l.u.st.), then it has polynomial type $p$. 

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4.3 Uniform $\mathbb{C}$-convexity

A Banach space $X$ is $q$-uniformly $\mathbb{C}$-convex [19] (for $q \geq 2$) if there exists $\lambda > 0$ such that

$$\left(\|x\|^q + \lambda\|y\|^q\right)^{1/q} \leq \max_{z \in T} \|x + zy\|,$$

for all $x, y \in X$ and $q$-uniformly PL-convex (see [12] or [32, Chapter 11]) if

$$\|x\|^q + \lambda\|y\|^q \leq \int_T \|x + zy\|^q dz,$$  \hspace{1cm}(24)

for all $x, y \in X$. In fact these two concepts are equivalent (see [27]) and provide an analytic version of the more familiar geometric property known as $q$-uniform convexity. A Banach space $X$ is $q$-uniformly convex (for $q \geq 2$) if there exists $\lambda > 0$ such that

$$\|x\|^q + \lambda\|y\|^q \leq E\|x + \varepsilon y\|^q.$$

It is easy to check that $q$-uniform convexity implies $q$-uniform PL-convexity.

In [4, Proposition 2.1] it is proven that $q$-uniform $\mathbb{C}$-convexity is equivalent to either of the following conditions:

(a) there exists $\lambda > 0$ such that for every analytic function $f : \mathbb{D} \to X$ we have

$$\|f(0)\|^q + \lambda\|f'(0)\|^q \leq \sup_{|z| < 1} \|f(z)\|^q.$$  \hspace{1cm}(25)

(b) there exists $\lambda > 0$ such that for every analytic function $f : \mathbb{D} \to X$ we have

$$\|f(0)\|^q + \lambda\|f'(0)\|^q \leq \sup_{0 < r < 1} \int_T \|f(rz)\|^q dz.$$  \hspace{1cm}(26)

Let us note that for every such function the mapping $r \in [0, 1[ \to \|f(r \cdot)\|_{H_q(T, X)}$ is increasing and, then, the supremum at the right-hand side of (26) is in fact a limit as $r \to 1^-$. With this, if $f : \mathbb{C} \to X$ is entire, then

$$\|f(0)\|^q + \lambda\|f'(0)\|^q \leq \int_T \|f(z)\|^q dz.$$  \hspace{1cm}(27)

Since taking $f(z) = x + zy$ for given $x$ and $y$ gives (24), the equivalence with $q$-uniform $\mathbb{C}$-convexity is maintained.

Using (25) Blasco proved in [3, Theorem 2.4] that $q$-uniformly $\mathbb{C}$-convex spaces have positive $q$-Bohr radius. That is, there exists $\rho > 0$ such that

$$\left(\sum_{n=0}^{\infty} \|x_n\|^q \rho^{qn}\right)^{1/q} \leq \sup_{|z| < 1} \|f(z)\|,$$  \hspace{1cm}(28)

for every analytic function $f = \sum_n x_n z^n$ on $\mathbb{D}$. Replacing (25) by (27) in his argument we deduce that for $q$-uniformly $\mathbb{C}$-convex spaces there exists $\rho > 0$ such that

$$\left(\sum_{n=0}^{\infty} \|x_n\|^q \rho^{qn}\right)^{1/q} \leq \left(\int_T \|f(z)\|^q dz\right)^{1/q},$$  \hspace{1cm}(29)

for every entire function $f = \sum_n x_n z^n$. The following theorem extends this fact to several variables.
**Theorem 4.6.** Let $X$ be a $q$-uniformly $C$-convex Banach space. Then there exists $\rho > 0$ such that for every $n$ and every polynomial $P = \sum x_\alpha z^\alpha$ of $n$ variables with values in $X$ we have

$$\left( \sum_\alpha \| x_\alpha \|^q \rho^{\alpha q} \right)^{1/q} \leq \left( \int_{T^n} \| P(z) \|^q dz \right)^{1/q}.$$ 

**Proof.** We proceed by induction on $n$, the number of variables. The case $n = 1$ follows from (29). Suppose now that the result holds for $n - 1$ and take some polynomial

$$P(z) = \sum_{\alpha \in F} x_\alpha z^\alpha,$$

for $z \in \mathbb{C}^n$ (where $F \subseteq \mathbb{N}_0^n$ is finite). Then we can write

$$\sum_\alpha \| x_\alpha \|^q \rho^{\alpha q} = \sum_{k=0}^N \rho^{qk} \sum_{\alpha \in F, \alpha_n = k} \| x_\alpha \|^q \rho^{(|\alpha| - \alpha_n)q}.$$ 

Applying the inductive hypothesis to each polynomial

$$z \in \mathbb{C}^{n-1} \mapsto \sum_{\alpha \in F, \alpha_n = k} c_\alpha z_1^{\alpha_1} \cdots z_{n-1}^{\alpha_{n-1}},$$

we have

$$\sum_\alpha \| x_\alpha \|^q \rho^{\alpha q} \leq \sum_{k=0}^N \rho^{qk} \int_{T^{n-1}} \left\| \sum_{\alpha \in F, \alpha_n = k} c_\alpha z_1^{\alpha_1} \cdots z_{n-1}^{\alpha_{n-1}} \right\|^q d(z_1, \ldots, z_{n-1})$$

$$= \int_{T^{n-1}} \sum_{k=0}^N \left\| \sum_{\alpha \in F, \alpha_n = k} c_\alpha z_1^{\alpha_1} \cdots z_{n-1}^{\alpha_{n-1}} \right\|^q \rho^{qk} d(z_1, \ldots, z_{n-1}).$$

Finally, for each fixed $(z_1, \ldots, z_{n-1}) \in T^{n-1}$ we may consider the polynomial $\mathbb{C} \to X$ given by

$$z \mapsto \sum_{k=0}^n \left( \sum_{\alpha \in F, \alpha_n = k} c_\alpha z_1^{\alpha_1} \cdots z_{n-1}^{\alpha_{n-1}} \right) z^k,$$

and then use the case $n = 1$ of the induction to conclude

$$\sum_{k=0}^N \left\| \sum_{\alpha \in F, \alpha_n = k} x_\alpha z_1^{\alpha_1} \cdots z_{n-1}^{\alpha_{n-1}} \right\|^q \rho^{qk}$$

$$\leq \int_T \left\| \sum_{k=0}^N \left( \sum_{\alpha \in F, \alpha_n = k} x_\alpha z_1^{\alpha_1} \cdots z_{n-1}^{\alpha_{n-1}} \right) z^k \right\|^q dz_n = \int_T \left\| \sum_{\alpha \in F} x_\alpha z_1^{\alpha_1} \cdots z_{n-1}^{\alpha_{n-1}} z_n \right\|^q dz_n.$$ 

Fubini’s theorem completes the proof. \qed

Let us note that Theorem 4.6 can be reformulated as

$$\left( \sum_{n \leq \chi} \| a_n \|^q \rho^{\Omega(n)} \right)^{\frac{1}{q}} \leq \left\| \sum a_n n^{-s} \right\|_{\mathcal{H}_q(X)} \leq \left\| \sum a_n n^{-s} \right\|_{\mathcal{H}_\infty(X)},$$

(30)
Corollary 4.7. Every q-uniformly \( C \)-convex Banach space has polynomial cotype \( q \). In particular, so does every q-uniformly convex Banach space.

Regarding polynomial type, an analogous result holds for the dual version of uniform convexity known as uniform smoothness. A Banach space \( X \) is \( p \)-uniformly smooth (for \( 1 < p \leq 2 \)) if there exists \( C > 0 \) such that
\[
\mathbb{E}\|x + \varepsilon y\|^p \leq \|x\|^p + C\|y\|^p.
\]
Banach spaces which are \( p \)-uniformly smooth are \( K \)-convex since they have type \( p \) (see [25, Theorem 1.e.16]) and have \( p' \)-uniformly convex duals (see [1, Lemma 5]).

Corollary 4.8. Every \( p \)-uniformly smooth Banach space has polynomial type \( p \).

Proof. Let \( X \) be \( p \)-uniformly smooth. Since \( X^* \) is \( p' \)-uniformly convex, it has polynomial cotype \( p' \) by Corollary 4.7. Moreover, \( X^* \) is \( K \)-convex for \( X \) is \( K \)-convex. Applying Proposition 3.9, we deduce that \( X^{**} \) (and then \( X \)) has polynomial type \( p \).

One might wonder if there is a dual version of uniform \( C \)-convexity which is weaker than uniform smoothness and implies polynomial type. The straightforward strategy would be to reverse inequality (24), and to define Banach space \( X \) to be \( p \)-uniformly \( C \)-smooth (for \( 1 < p \leq 2 \)) if there exists \( C > 0 \) such that
\[
\int_T \|x + zy\|^p \, dz \leq \|x\|^p + C\|y\|^p. \tag{31}
\]
This, however, does not work since it is equivalent to uniform smoothness. This phenomenon was noticed by Xu in [34] where it is shown that the property known as Lusin cotype (which is equivalent to having a \( p \)-uniformly smooth renorming) has no analytic counterpart.

Remark 4.9. For a Banach space \( X \), \( p \)-uniform smoothness and \( p \)-uniform \( C \)-smoothness are equivalent. First assume \( X \) is \( p \)-uniformly smooth. Given \( x, y \in X \) and \( z \in T \) we have
\[
\mathbb{E}\|x + \varepsilon y\|^p \leq \|x\|^p + C\|y\|^p = \|x\|^p + C\|y\|^p,
\]
for some constant \( C > 0 \). Averaging in \( T \), we get
\[
\int_T \|x + zy\|^p \, dz = \int_T \mathbb{E}\|x + \varepsilon y\|^p \, dz \leq \|x\|^p + C\|y\|^p,
\]
so \( X \) is \( p \)-uniformly \( C \)-smooth.

Conversely, suppose (31) holds for some \( C > 0 \). It suffices to check that
\[
\mathbb{E}\left\|x + \frac{\varepsilon}{2} y\right\|^p \leq \int_T \|x + zy\|^p \, dz.
\]
For a fixed \( z \in T \) we have
\[
\mathbb{E}\left\|x + \frac{\varepsilon}{2} y\right\|^p = \mathbb{E}\left\|x + \frac{\varepsilon}{2} zy\right\|^p = \mathbb{E}\left\|x + \frac{1}{2}(\text{Re}(z)\varepsilon - i\text{Im}(z)\varepsilon)zy\right\|^p \leq \frac{1}{2} \left(\mathbb{E}\|x + \text{Re}(z)\varepsilon y\|^p + \mathbb{E}\|x - \text{Im}(z)\varepsilon y\|^p\right) \leq \frac{1}{2} \left(\mathbb{E}\|x + \varepsilon y\|^p + \mathbb{E}\|x - i\varepsilon y\|^p\right),
\]
and
where in the last inequality we used the Contraction Principle (see e.g. [17, Theorem 12.2]). Averaging in the torus and using the rotation invariance we get

$$\mathbb{E}\|x + \frac{\varepsilon}{2} y\|^p \leq \frac{1}{2} \left( \mathbb{E} \int_T \|x + \varepsilon y\|^p \, dz + \mathbb{E} \int_T \|x - \varepsilon y\|^p \, dz \right) = \int_T \|x + y\|^p \, dz.$$

Joining this with (31) implies $X$ is $p$-uniformly smooth with constant $2^p C$.

### 4.4 Examples

In [10, Section 2.3] it is shown that $L^p$-spaces have polynomial cotype $\max\{2, p\}$ for every $1 \leq p \leq \infty$ and Schatten classes $S_p$ have polynomial cotype $p$ for every $p \geq 2$, which coincides with their usual cotypes. Since these spaces have type $\min\{p, 2\}$, Corollaries 4.4 and 4.7 allow us to complete the picture for these families including the polynomial cotype of $S_p$ when $1 \leq p < 2$ and the polynomial type for all $p$. Just as for (Rademacher) type and cotype, we have the following.

**Example 4.10.** For $1 \leq p \leq \infty$, $L^p$-spaces and Schatten classes $S_p$ have polynomial type $\min\{2, p\}$ and polynomial cotype $\max\{2, p\}$ for every $1 \leq p \leq \infty$. These values are optimal.

This is mainly a consequence of Corollary 4.4 except for the polynomial cotype of $S_1$ where it follows from Corollary 4.7 since $S_1$ is 2-uniformly $\mathbb{C}$-convex (see [21] or [5, Theorem 3.6]).

We do not know whether the notions of polynomial type and cotype are equivalent to their usual counterparts except for the type 2 case (as shown in Corollary 4.4). Also, the conditions ensuring polynomial cotype are not necessary, as the following example shows.

**Example 4.11.** The quotient space $L_1(T)/H_1(T)$ (for which, to keep notation as simple as possible, we simply write $L_1/H_1$) has polynomial cotype 2, but fails K-convexity, GAP and $q$-uniform $\mathbb{C}$-convexity for every $q \geq 2$.

The proof of this fact will be split into two lemmas. Bourgain proved that $L_1/H_1$ has cotype 2 in [7]. Later, he and Davis gave a shortened proof (see [8, Theorem 2.1]) which provides an explicit lifting of Rademacher averages in $L_1/H_1$ to Rademacher averages in $L_1(T)$ with roughly the same norm. We can adapt their argument to lift Walsh polynomials, and then use the equivalence between polynomial cotype and the corresponding property with Walsh polynomials (Proposition 3.3) to get the desired result. Note that working directly with polynomial cotype implies lifting continuous random variables, which is far more tricky. The proof of the next lemma follows the lines of [8, Theorem 2.1].

**Lemma 4.12.** The space $L_1/H_1$ has polynomial cotype 2.

**Proof.** As mentioned above, by Proposition 3.3, it suffices to study homogeneous Walsh polynomials. Let $P \in L_1((-1,1)^\infty, L_1/H_1)$ be an $m$-homogeneous polynomial defined by

$$P(\varepsilon) = \sum_{|A|=m} x_A \varepsilon_A.$$

The key step is to find a good lifting $\tilde{P} \in L_1((-1,1)^\infty, L_1(T))$ which preserves the polynomial structure, since this allows to use the polynomial cotype of $L_1(T)$ to conclude the argument. Fix $\delta > 0$ and for each $\varepsilon \in (-1,1)^\infty$ let $F(\varepsilon) \in L_1(T)$ be a lifting of $P(\varepsilon)$ such that

$$\|F(\varepsilon)\|_{L_1(T)} \leq (1 + \delta)\|P(\varepsilon)\|_{L_1/H_1}.$$

Equivalently, regarding $F$ as a function in $L_1((-1,1)^\infty, L_1(T))$ we have

$$\|F\|_{L_1((-1,1)^\infty, L_1(T))} \leq (1 + \delta)\|P\|_{L_1((-1,1)^\infty, L_1/H_1)}.$$
For each finite $A \subseteq \mathbb{N}$, following the notation in [8] we write $\varphi_A = \hat{F}(A) \in L_1(T)$ for the Fourier-Walsh coefficients of $F$ and let $q : L_1(T) \to L_1/H_1$ be the quotient map. Notice that $q(\varphi_A) = \chi_A$ if $|A| = m$ and $q(\varphi_A) = 0$ if $|A| \neq m$ (which means that $\varphi_A \in H_1(T)$). Now, define $\tilde{h} \in L_1(T)$ by $\tilde{h}(z) = E_\xi|F(\varepsilon, z)|$. We do not take $\Phi$ the outer function on $\mathbb{D}$ with kernel $\log(\varepsilon + \delta)$ (note that $\varepsilon \geq 0$). Then, $|\Phi| = \varepsilon + \delta$ on $T$ and $\Phi$ has an holomorphic square root.

Finally, given $A$ with $|A| = m$ we consider $\Psi_A = \Phi^{1/2}R(\Phi^{-1/2}\varphi_A)$ where $R$ is the Riesz projection and set

$$\tilde{P}(\varepsilon) = \sum_{|A|=m} \Psi_A \varepsilon_A.$$

A careful calculation yields that $q(\Psi_A) = \chi_A$, so $\tilde{P}$ is a lifting of $P$.

In the following chain of inequalities we use the polynomial cotype of $q$ (see [2], Corollary 1.4] that the Hilbert transform (and thus the Riesz projection) is bounded from $L_2(T, L_1((-1, 1)^\infty))$ to $L_2(T, L_1/2((1, 1)^\infty))$. So we have:

$$\|R(\Phi^{-1/2} F)\|_{L_2(T, L_1/2((-1, 1)^\infty))} \leq C_0 \|\Phi^{-1/2} F\|_{L_2(T, L_1((-1, 1)^\infty))} \leq C_0 \|F\|_{L_1((-1, 1)^\infty, L_1(T))}^{1/2} \leq C_0 (1 + \delta) \|\tilde{P}\|_{L_1((-1, 1)^\infty, L_1/H_1)}^{1/2},$$

which completes the proof.

Now we see that the space $L_1/H_1$ lacks every property we found to be sufficient for having polynomial cotype. First, $L_1/H_1$ is not $K$-convex since it contains a copy of $L_1$ (see [6]). As a consequence, it cannot have nontrivial Fourier or Walsh cotype: otherwise, it would have nontrivial Fourier/Walsh type and, in particular, nontrivial (Rademacher) type, which is equivalent to $K$-convexity. Regarding uniform $C$-convexity, in [12] (see also [32, Corollary 11.55]) it is shown that $L_1/H_1$ cannot be renormed to be $q$-uniformly $C$-convex for any $q \geq 2$. Finally, it only remains to check that $L_1/H_1$ does not have GAP.

**Lemma 4.13.** The space $L_1/H_1$ does not have GAP.

**Proof.** Let $A$ be the disk algebra and $M_s(T)$ be the Banach space of singular measures on $T$. As a consequence of the F. and M. Riesz theorem (see for example [28, Chapter 1]) we have

$$A^* \simeq L_1/H_1 \oplus_1 M_s(T).$$

If $L_1/H_1$ has GAP, then it also has $GL_2$ since it is of cotype 2 (see [11, Theorem 1.4]). Recall that $M_s(T)$ is complemented in $M(T)$ by Lebesgue’s decomposition theorem. In addition, $M(T)$ has l.u.s.t.

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and therefore $GL_2$ since it is an $L_1$-space. Thus, $M_k(\mathbb{T})$ enjoys $GL_2$ and so does $A^*$. As $GL_2$ is a self-dual property (see [17, Proposition 17.9]), $A$ should have $GL_2$. However, this is not the case. Let us sketch the proof.

In [28, Chapter 4] it is shown that $A$ fails $GL$. Introducing minor changes to the argument yields it also fails $GL_2$. Consider the projection $P : A \to \ell_2$ given by $P(f) = (f(2^n))_{n \in \mathbb{N}}$. This map is absolutely summing by Paley’s inequality (see [26] or [28, Chapter 2]). Proceeding exactly as in [28, Theorem 4.2 (i)], we get that $P$ does not factor through $L_1$ and, therefore, $A$ does not have $GL_2$.

\[\square\]

### 5 Proof of Proposition 3.3

We need several lemmas in order to prove Proposition 3.3. The first one estimates the norm of the homogeneous projection of a Walsh polynomial and can be found in [24, Lemma 2] (see also [13, Lemma 3.2.4]). A proof is included since the constant is not explicitly computed there, and we need it to grow exponentially on the degree of the polynomial (i.e., of the form $B^m$ for some $B > 0$).

**Lemma 5.1.** For every Banach space $X$ there is a constant $B > 0$ such that for every $X$-valued Walsh polynomial $P$ of degree $m$ its $k$-homogeneous projection $P_k$ satisfies

\[
\left(\mathbb{E}\|P_k(\varepsilon)\|_X^q\right)^{1/q} \leq B^m \left(\mathbb{E}\|P(\varepsilon)\|_X^q\right)^{1/q}.
\]

**Proof.** For each $m$ we consider the functions $\{1, t, \ldots, t^m\}$ in $L_2(0,1)$. We show that there are polynomials $\{p_1^{(m)}, \ldots, p_{m+1}^{(m)}\}$ of degree at most $m$ such that

\[
\int_0^1 t^{i-1} p_j^{(m)}(t)\, dt = \delta_{ij},
\]

for every $1 \leq i, j \leq m + 1$. Indeed, writing $p_j^{(m)}(t) = \sum_{k=1}^{m+1} a_{kj}^{(m)} t^{k-1}$ we get

\[
\delta_{ij} = \int_0^1 t^{i-1} p_j^{(m)}(t)\, dt = \sum_{k=1}^{m+1} a_{kj}^{(m)} \int_0^1 t^{i+k-2} dt = \sum_{k=1}^{m+1} \frac{1}{i+k-1} a_{kj}^{(m)},
\]

for every $1 \leq i, j \leq m + 1$. In other words, we obtain the matrix identity

\[I = HA,\]

where $H$ is the well-known Hilbert matrix and $A$ is the matrix defined by the coefficients $a_{ij}^{(m)}$. Thus, we have $A = H^{-1}$, which provides a specific formula for the polynomials $p_j^{(m)}$. It is easy to check that there is a constant $C > 0$ so that $\sup_{0 < i < 1} |p_j^{(m)}(t)| \leq C^m$ and therefore

\[
\sup_{0 < t < 1} |p_j^{(m)}(t)| \leq B^m
\]

for some universal constant $B$. Notice that, if $Q$ is a tetrahedral polynomial of degree $m$, then

\[P_k(\varepsilon) = \int_0^1 P(\varepsilon) p_{k+1}^{(m)}(t)\, dt,\]

for every $0 \leq k \leq m$. So we get

\[
\left(\mathbb{E}\|P_k(\varepsilon)\|_X^q\right)^{1/q} \leq \int_0^1 \left(\mathbb{E}\|P(\varepsilon) p_{k+1}^{(m)}(t)\|_X^q\right)^{1/q} dt \leq B^m \int_0^1 \left(\mathbb{E}\|P(\varepsilon)\|_X^q\right)^{1/q} dt.
\]
Now, [13, Lemma 3.2.3] gives
\[
\left( \mathbb{E} \left\| P(t \varepsilon) \right\|_X^q \right)^{1/q} \leq \left( \mathbb{E} \left\| P(\varepsilon) \right\|_X^q \right)^{1/q}
\]
for every \( 0 \leq t \leq 1 \), and this completes the proof.

The following lemma shows that tetrahedral Steinhaus polynomials and their Walsh counterparts have equivalent norms up to exponential constants.

**Lemma 5.2.** Let \( X \) be a Banach space and set \( 1 \leq p < \infty \). For every tetrahedral polynomial \( P \) of degree \( m \) and \( n \) variables we have
\[
(1 + \sqrt{2})^{-m} \left( \mathbb{E} \left\| P(\varepsilon) \right\|_X^q \right)^{1/p} \leq \left( \int_{\mathbb{R}^n} \left\| P(z) \right\|_X^q \, dz \right)^{1/p} \leq (1 + \sqrt{2})^m \left( \mathbb{E} \left\| P(\varepsilon) \right\|_X^q \right)^{1/p}.
\]

**Proof of Lemma 5.2.** In [23, p. 2764] it is shown that for every polynomial \( Q : \mathbb{C}^n \to \mathbb{C} \) of degree \( m \), we have
\[
\sup_{z \in \mathbb{T}^n} |Q(z)| \leq (1 + \sqrt{2})^m \sup_{x \in [-1,1]^n} |Q(x)|.
\]
If we assume \( Q \) to be tetrahedral, we observe that
\[
\sup_{x \in [-1,1]^n} |Q(x)| = \sup_{\varepsilon \in [-1,1]^n} |Q(\varepsilon)|,
\]
since \( Q \) is affine in every coordinate. Thus,
\[
\sup_{\varepsilon \in [-1,1]^n} |Q(\varepsilon)| \leq \sup_{z \in \mathbb{T}^n} |Q(z)| \leq (1 + \sqrt{2})^m \sup_{\varepsilon \in [-1,1]^n} |Q(\varepsilon)|.
\]

Equivalently, for every finite choice of scalars \( \{c_A\}_{|A| \leq m} \subseteq \mathbb{C} \) we have
\[
\sup_{\varepsilon \in [-1,1]^n} \left| \sum_{|A| \leq m} c_A \varepsilon^A \right| \leq \sup_{z \in \mathbb{T}^n} \left| \sum_{|A| \leq m} c_A z^A \right| \leq (1 + \sqrt{2})^m \sup_{\varepsilon \in [-1,1]^n} \left| \sum_{|A| \leq m} c_A \varepsilon^A \right|,
\]
where for simplicity we also used Walsh notation for the variable \( z \). Consider the sets of characters \( \{\varepsilon_A\}_{|A| \leq m} \) and \( \{z_A\}_{|A| \leq m} \) of the compact abelian groups \( (-1,1)^n \) and \( \mathbb{T}^n \) respectively. Since these sets satisfy (33), the conditions of [29, Theorem 1] are met. So we get
\[
(1 + \sqrt{2})^{-m} \left\| \sum_{|A| \leq m} x_A \varepsilon^A \right\|_{L^p([-1,1]^n,X)} \leq \left\| \sum_{|A| \leq m} x_A z^A \right\|_{L^p(\mathbb{T}^n,X)} \leq (1 + \sqrt{2})^m \left\| \sum_{|A| \leq m} x_A \varepsilon^A \right\|_{L^p([-1,1]^n,X)},
\]
for every choice of vectors \( \{x_A\}_{|A| \leq m} \subseteq X \). This concludes the proof since it is equivalent to (32).

We also need a rather convoluted description of a polynomial in terms of the parity of the exponents of the variables. Fix an even \( m \in \mathbb{N} \). Given \( A \subseteq \{1, \ldots, n\} \) we define
\[
\Lambda_A = \{ \alpha \in \mathbb{N}_0^n : |\alpha| = m, \alpha_i \text{ is odd if and only if } i \in A \}.
\]
Since \( m \) is even, it is clear that \( A \) has even cardinality whenever \( \Lambda_A \neq \emptyset \). In the rest of this discussion we only consider \( A \) with \( \Lambda_A \neq \emptyset \). Note that for any \( \varepsilon \in (-1,1)^n \) and \( z \in \mathbb{T}^n \), we have
\[
(\varepsilon z)^\alpha = \varepsilon_A z^\alpha
\]
for any choice of \( \{x_A\}_{|A| \leq m} \subseteq X \). This concludes the proof since it is equivalent to (32).
for every $\alpha \in \Lambda_A$, where, as always, $\varepsilon_A = \prod_{i \in A} \varepsilon_i$.

Now, for an $m$-homogeneous polynomial of $n$ variables $P(z) = \sum_{|\alpha| = m} x_\alpha z^\alpha$ we write

$$P_A(z) = \sum_{\alpha \in \Lambda_A} x_\alpha z^\alpha.$$

With this notation, we clearly have

$$P(\varepsilon z) = \sum_{A \subseteq \{1, \ldots, n\}} \varepsilon_A P_A(z).$$

As we can see from the expression above, $P(\varepsilon z)$ regarded as a polynomial on $\varepsilon$ is tetrahedral. Also, we may write $P(\varepsilon z)$ as the sum of its homogeneous components (as a function of $\varepsilon$). As we have already mentioned, each $A$ considered has even cardinality. So, if we define

$$A_k = \{A \subseteq \{1, \ldots, n\} : |A| = 2k\},$$

we can write

$$P(\varepsilon z) = \sum_{k=0}^{m/2} \sum_{A \in A_k} \varepsilon_A P_A(z). \quad (34)$$

Note that, whenever $i$ belongs to some $A$, the exponents of $z_i$ are odd for every monomial in $P_A(z)$. Also, since $m$ is even, given $\alpha \in \Lambda_A$, we have that $\sum_{i \in A} \alpha_i$ must be even. We then define

$$\Lambda_{A,l} = \{\alpha \in \Lambda_A : \sum_{i \in A} \alpha_i = 2l\},$$

which allows us to write, for $A \in A_k$,

$$P_A(z) = \sum_{l=k}^{m/2} \sum_{\alpha \in \Lambda_{A,l}} x_\alpha z^\alpha = \sum_{l=k}^{m/2} P_{A,l}(z). \quad (35)$$

Note that $P_{A,l}(z)$ is the $2l$-homogeneous component of the polynomial $P_A(z)$ regarded as a function of the variables $z_i$ with $i \in A$ (that is, the variables with odd exponents). In other words, the polynomial $P_{A,l}(z)$ consists of the monomials $x_\alpha z^\alpha$ of $P_A(z)$ where the sum of the odd exponents equals $2l$.

To conclude our description of $P$, for $\alpha \in \Lambda_A$ define exponents $\beta$, $\gamma$ and $l_A$ by

$$\beta_i = \begin{cases} 0 & \text{if } i \in A \\ \frac{\alpha_i}{2} & \text{if } i \in A^c \end{cases}, \quad \gamma_i = \begin{cases} \frac{\alpha_i - 1}{2} & \text{if } i \in A \\ 0 & \text{if } i \in A^c \end{cases}, \quad l_A,i = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{if } i \in A^c \end{cases},$$

for every $1 \leq i \leq n$. Note that $\alpha = 2\beta + 2\gamma + l_A$ where $\beta \in \mathbb{N}^n_0$ is supported in $A^c$ and $\gamma, l_A \in \mathbb{N}^n_0$ are supported in $A$. Moreover, for $\alpha \in \Lambda_{A,l}$ we have

$$|\beta| = \sum_{i=1}^{n} \beta_i = \sum_{i \in A^c} \frac{\alpha_i}{2} = \frac{|\alpha|}{2} - \sum_{i \in A} \frac{\alpha_i}{2} = \frac{m}{2} - l,$$

and

$$|\gamma| = \sum_{i=1}^{n} \gamma_i = \sum_{i \in A} \frac{\alpha_i - 1}{2} = \frac{2l - |A|}{2} = l - k.$$
Denote the set of all the exponents \( \beta \) supported in \( A^c \) with \( |\beta| = m/2 - 1 \) by \( B_{A,1} \) and the set of all the exponents \( \gamma \) supported in \( A \) with \( |\gamma| = 1 - k \) by \( \Gamma_{A,k} \). We get

\[
P_{A,l}(z) = \sum_{\alpha \in A_{A,1}} x_{\alpha} z^\alpha = \sum_{\gamma \in \Gamma_{A,k}} \sum_{\beta \in B_{A,1}} x_{2\beta + 2\gamma + 1A} z^{2\beta + 2\gamma + 1A_l} = \sum_{\gamma \in \Gamma_{A,1}} \left( \sum_{\beta \in B_{A,1}} x_{2\beta + 2\gamma + 1A} z^{2\beta} \right)^{z^{2\gamma + 1A}}.
\]  

Gathering (34), (35) and (36) we get the full description of \( P(\varepsilon z) \) proving the following lemma.

**Lemma 5.3.** For an even \( m \in \mathbb{N} \), an \( m \)-homogeneous polynomial in \( n \) variables

\[
P(z) = \sum_{|\alpha| = m} x_{\alpha} z^\alpha,
\]

and \( \varepsilon \in \{-1, 1\}^n \) we have

\[
P(\varepsilon z) = \sum_{k=0}^{m/2} \sum_{A \in A_k} \sum_{\gamma \in \Gamma_{A,1}} \sum_{\beta \in B_{A,1}} x_{2\beta + 2\gamma + 1A} \varepsilon A z^{2\beta + 2\gamma + 1A_l}.
\]

A similar formula can be deduced for \( m \) odd. We are now in position to prove Proposition 3.3.

**Proof of Proposition 3.3.** Lemma 5.2 implies the equivalence (b) \( \iff \) (c). Notice that (a) \( \implies \) (b) and (d) \( \implies \) (c) are obvious. On the other hand, to prove (c) \( \implies \) (d), first write

\[
\sum_A \|x_{A}\|^q = \sum_{k=0}^{m} \sum_{|A|=k} \|x_{A}\|^q
\]

and apply (c) to each of the \( m \) inner sums. Now, (d) follows from Lemma 5.1. It only remains to prove that (c) \( \implies \) (a). We prove this by induction on the degree \( m \).

In view of Remark 3.6 we may replace all exponents involved by \( q \). Assume that for a constant \( C > 0 \) and every \( (x_{\alpha})_{|\alpha|=k} \subseteq X \) with \( \alpha \in \mathbb{N}_0^n \) and \( k < m \) we have

\[
\left( \sum_{|\alpha|=k} \|x_{\alpha}\|^q \right)^{1/q} \leq C^k \left( \int_{\mathbb{T}^n} \left( \sum_{|\alpha|=k} x_{\alpha} z^\alpha \right)^q |dz| \right)^{1/q}.
\]

We show only the case when \( m \) is even since the odd case is completely analogous. Fix an \( m \)-homogeneous polynomial in \( n \) variables

\[
P(z) = \sum_{|\alpha|=m} x_{\alpha} z^\alpha.
\]

Since our goal involves estimating an integral of \( P(z) \), we take advantage of the rotation invariance and work with \( P(\varepsilon z) \). For a fixed \( 1 \leq k \leq m/2 \) and \( A \subseteq \{1, \ldots, n\} \) with \( |A| = 2k \), take \( k \leq l \leq m/2 \) and define \( P_A \) and \( P_{A,l} \) as before. Intuitively, \( P_{A,l} \) detaches the \( z_i \)'s with odd exponent from the \( z_i \)'s with even exponent. This enables us to use inductive hypothesis twice (once for the odd and once for the even part) to assemble the polynomials \( P_{A,l} \). Let \( \mathbb{T}^{A^c} \) denote \(|A^c|\) copies of the torus indexed in \( A^c \). We get

\[
\sum_{\gamma \in \Gamma_{A,1}} \sum_{\beta \in B_{A,1}} \|x_{2\beta + 2\gamma + 1A_l}\|^q \leq \sum_{\gamma \in \Gamma_{A,1}} C^q(m/2-1) \int_{\mathbb{T}^{A^c}} \left( \sum_{\beta \in B_{A,1}} x_{2\beta + 2\gamma + 1A} z^{2\beta} \right)^q |dz|
\]

\[
\leq C^q(m/2-1) \int_{\mathbb{T}^{A^c}} \sum_{\gamma \in \Gamma_{A,1}} \sum_{\beta \in B_{A,1}} x_{2\beta + 2\gamma + 1A} z^{2\beta} |dz|
\]

\[
= C^q(m/2-1) \int_{\mathbb{T}^{A^c}} \sum_{\gamma \in \Gamma_{A,1}} \sum_{\beta \in B_{A,1}} x_{2\beta + 2\gamma + 1A} z^{2\beta} |dz|.
\]
where the last step follows by a change of variables. Since $\beta$ is supported in $A^c$, the variables $z_i$ with $i \in A$ do not appear in the expression above. So, we are still able to introduce them by applying the inductive hypothesis again. We obtain

$$
\sum_{\gamma \in I_{A,l}} \sum_{\beta \in B_{A,l}} \|x_{2\beta + 2\gamma + 1_A}\|^q \\
\leq C q(m/2-1) C^{l-k} \int_{T^n} \sum_{\gamma \in I_{A,l}} \left( \sum_{\beta \in B_{A,l}} x_{2\beta + 2\gamma + 1_A} z^{2\beta} \right) z^{\gamma} \|q\| dz
$$

$$
= C q(m/2-k) \int_{T^n} \sum_{\gamma \in I_{A,l}} \sum_{\beta \in B_{A,l}} x_{2\beta + 2\gamma + 1_A} z^{2\beta} z^{\gamma} \|q\| dz
$$

$$
= C q(m/2-k) \int_{T^n} \sum_{\gamma \in I_{A,l}} \sum_{\beta \in B_{A,l}} x_{2\beta + 2\gamma + 1_A} z^{2\beta} z^{\gamma} \|q\| dz
$$

$$
= C q(m/2-k) \int_{T^n} \|P_{A,l}(z)\|^q dz,
$$

where in the last step we used (36). Since $P_{A,l}$ is the 2l-homogeneous component of $P_A$ regarded as a function depending only on the variables $z_i$ with $i \in A$, we have

$$
\int_{T^n} \|P_{A,l}(z)\|^q dz \leq \int_{T^n} \|P_A(z)\|^q dz.
$$

From (37) and (38), we deduce

$$
\sum_{l=k}^{m/2} \sum_{\gamma \in I_{A,l}} \sum_{\beta \in B_{A,l}} \|x_{2\beta + 2\gamma + 1_A}\|^q \leq \left( \frac{m}{2} - k \right) C q(m/2-k) \int_{T^n} \|P_{A,l}(z)\|^q dz
$$

$$
\leq m C q(m/2-k) \int_{T^n} \|P_A(z)\|^q dz.
$$

Finally we construct $P(\varepsilon z)$ using (c) and (34). Define $C_\varepsilon$ to be the constant provided by (c) and assume $C > C_\varepsilon^2$. Taking Lemma 5.3 into consideration we get

$$
\sum_{|\alpha|=m} \|x_{\alpha}\|^q = \sum_{k=1}^{m/2} \sum_{A \in A_k} \sum_{l=k}^{m/2} \sum_{\gamma \in I_{A,l}} \sum_{\beta \in B_{A,l}} \|x_{2\beta + 2\gamma + 1_A}\|^q
$$

$$
\leq m \int_{T^n} \sum_{k=1}^{m/2} C q(m/2-k) \sum_{A \in A_k} \|P_A(z)\|^q dz
$$

$$
\leq m \int_{T^n} \sum_{k=1}^{m/2} C q(m/2-k) C_\varepsilon^{2qk} \varepsilon^2 \sum_{A \in A_k} \|P_A(z)\|^q dz
$$

$$
\leq m C q^{m/2} \int_{T^n} \sum_{k=1}^{m/2} \varepsilon^2 \sum_{A \in A_k} \|P_A(z)\|^q dz.
$$

If $B$ is the constant from Lemma 5.1, we have

$$
\sum_{|\alpha|=m} \|x_{\alpha}\|^q \leq m C q^{m/2} B^{q^m} \int_{T^n} \sum_{k=1}^{m/2} \varepsilon^2 \|P(\varepsilon z)\|^q dz
$$

$$
\leq m^2 C q^{m/2} B^{q^m} \int_{T^n} \|P(\varepsilon z)\|^q dz \leq C q^m \int_{T^n} \|P(z)\|^q dz,
$$

if we take $C > B^2$ and $m$ sufficiently large. □
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