Semiclassical dynamics for nonrelativistic strings

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Abstract

We explore *folded* spinning string configurations over torsional Newton Cartan (TNC) geometry with $R \times S^2$ topology within the semiclassical approximation. To start with, considering zero temperature strong coupling limit, we compute the so-called Hamiltonian spectrum from spinning strings on $R \times S^2$. We further extend our analysis considering quantum fluctuations over the classical space of solutions and compute one loop string ($\alpha'$) correction to the energy spectrum in the gauge theory sector that is dual to TNC spinning strings on $R \times S^2$. Finally, we consider the large $c$ and/or nonrelativistic (NR) limit associated with the world-sheet d.o.f. and compute the one loop string correction to the energy spectrum in the dual Spin Matrix Theory (SMT) theory at strong ($g \gg 1$) coupling and low temperatures.

1 Overview and Motivation

The quest for a consistent (UV finite) low energy description of relativistic string theory had been one of the active areas of theoretical investigations for last couple of decades [1]-[7]. Apparently there seems to be existing two parallel formulations of nonrelativistic (NR) string theories over curved target space geometries. One of these formulations is based on taking $1/c$ limit of General Relativity (GR) in the first order formalism that eventually leads towards curved manifold structures (known as *string Newton-Cartan* (SNC) geometry) together with a flux-less auxiliary two form field [8]-[11]. It turns out that under such circumstances one could in fact define a NR quantum consistent 2D sigma model that is invariant under transformations generated by SNC generators [12]-[17]. In other words, NR closed string spectra on SNC background may be obtained through large $c$ expansion of the relativistic Nambu-Goto (NG) action.

The other approach is based on the formulation of 2D sigma models on curved manifolds called torsional Newton-Cartan (TNC) geometries [18]-[24]. TNC strings are obtained through target space null reduction of Poincare invariant 2D string sigma models while keeping the string momentum along the null isometry direction fixed\(^1\). Taking a zero tension limit of TNC strings finally leads towards an emerging new sector in the

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\(^1\)Recently, the authors in [25] have shown that under certain specific assumptions on the compact longitudinal spatial direction, the TNC strings (in the presence of background NS-NS fluxes) can be mapped to NR strings propagating over SNC geometry.
celebrated gauge/string duality. For example, in the case of $AdS_5 \times S^5$ (super)strings such scaling results in the so-called NR 2D sigma model/Spin Matrix Theory (SMT) correspondence $^{[18]-[19]}$. The NR 2D sigma model thus obtained has been found to possess an underlying Galilean Conformal Algebra together with a NR Weyl invariance. On the other hand, in the case of $AdS_5 \times S^5$ strings, the SMT limit corresponds to taking a NR limit of the relativistic magnon dispersion relation in $\mathcal{N} = 4$ SYM $^{[26]-[27]}$.

The central idea that lives at the heart of the gauge/string duality is the identification of the energy spectra associated with gauge invariant operators (in the dual gauge theory) to that with the stringy excitations over curved background geometry. In other words, the duality conjecture drives us towards a natural identification of heavy operators and/or gauge theory states with large quantum numbers (in the strongly coupled QFTs) to that with the solitonic excitations associated to certain 2D sigma model in the supergravity approximations $^{[28]-[29]}$. An immediate example along this line of argument is the celebrated correspondence between the energy spectra ($\sim E$) associated to $AdS_5 \times S^5$ super-strings in 10D type IIB supergravity and the operator spectrum ($\sim \Delta$) corresponding to $\mathcal{N} = 4$ SYM in 4D $^{[28]-[40]}$. The purpose of the present analysis is therefore to pursue similar questions in the context of NR strings/SMT correspondence where (considering certain specific limits $^{[27]}$) we explore gauge theory states (with large quantum numbers) in the strongly coupled SMT sector by probing solitonic (stringy) excitations associated to NR 2D sigma model on $R \times S^2$.

The set of gauge invariant operators (in the dual SMT theory at strong coupling and low temperatures) that we choose to work with are those with large R charge ($\tilde{J}_\varphi \gg 1$) and large energy quantum numbers ($\tilde{\Delta}_{NR} \gg 1$). On the string theory side, we realize these states as solitonic excitations $^{[27]}$ associated to closed NR folded spinning string configurations over $R \times S^2$. These strings are supposed to be stretched along the polar coordinate of $S^2$ and spinning around their center of mass that is happened to be coincident with the north pole of $S^2$ $^{[29]}$. A straightforward computation on the stringy side reveals $^{(87)}$,

$$\tilde{\Delta}_{NR} - \tilde{J}_\varphi = \tilde{f}(\tilde{g}) \tilde{J}_\varphi$$

(1)

where we compute the function,

$$\tilde{f}(\tilde{g}) = q_1 \tilde{g} \left( 1 + \frac{q_2}{\sqrt{\tilde{J}_\varphi}} \frac{1}{\tilde{g}^{1/4}} + \ldots \right)$$

(2)

upto one loop in the string corrections. Here, $\tilde{g} (= \frac{\tilde{g}}{\tilde{J}_\varphi}) \ll 1$ should be regarded as being the effective expansion parameter in the dual SMT in the regime of strong ($g = c^2 \lambda \gg 1$) coupling where, $\lambda \ll 1$ is the standard t’Hooft coupling in the $\mathcal{N} = 4$ SYM theory and $c(\rightarrow \infty)$ is the speed of light $^{[18]-[19]}$. The above result (1) is therefore a non-perturbative effect from the perspective of the dual SMT and is the key finding of the present analysis.

The rest of the paper is organised as follows. We start our analysis in Section 2, with a formal computation of the folded spinning TNC (closed) string spectra on $R \times S^2$ using NG formulation of the 2D sigma model $^{[24]}$. We further extend this analysis in Section 3 by incorporating one loop stringy effects on the operator spectrum in the dual gauge theory at strong couplings. Our analysis also reveals that in the presence of stringy fluctuations the spectrum is analytically tractable only in the limit of short
strings. However, the most significant part of this paper seems to be contained within Section 4 where we take $1/c$ expansion of the world-sheet d.o.f. which finally results in NR 2D sigma models over TNC geometry with $R \times S^2$ topology. From the perspective of $\mathcal{N} = 4$ SYM theory such a tensionless limit of (super)strings propagating over (subspace of) $AdS_5 \times S^5$ geometry corresponds to zooming into a specific sub-sector of the full operator spectrum near its unitarity bound [18]-[19] known as the Spin Matrix Theory (SMT). In our analysis, considering a the limit of strong ($g \gg 1$) coupling as well as low temperatures we find an effective way of expressing the energy eigenvalues (associated with the dual operator spectrum with large $R$ charge) from solitonic excitations [27] over $R \times S^2$. Going one step further, we also explore NR stringy effects beyond leading order approximation and compute one loop stringy corrections to the energy eigenvalues at strong coupling. Finally, we summarise and conclude in Section 5.

2 TNC strings on $R \times S^2$

Our analysis starts with a formal construction of the spinning string sigma model over torsional Newton-Cartan (TNC) geometry [18]-[19] with $R \times S^2$ topology [20],

$$ds^2_{TNC} = 2 \tau (du - m) + h_{\mu \nu} dx^\mu dx^\nu$$

where each of the individual entities could be formally expressed as [24],

$$\tau = \tau_\mu dx^\mu = \frac{1}{2} d\psi + dt - \frac{1}{2} \cos \theta d\varphi; \quad u = \frac{\psi}{4} - \frac{t}{2}$$

$$m = m_\mu dx^\mu = \frac{1}{4} \cos \theta d\varphi; \quad h_{\mu \nu} dx^\mu dx^\nu = \frac{1}{4} \left[d\theta^2 + \sin^2 \theta d\varphi^2\right].$$

Notice that, here $u$ and $\varphi$ are the isometry directions associated with the target space geometry [20],[24]. The relativistic action corresponding to the bosonic sector of the 2D string sigma model could be formally expressed as [2],

$$S_{NG} = \int d^2 \sigma L_{NG} = -\sqrt{\lambda} \int d\tau d\sigma \tilde{L}_{NG} ; \quad \sqrt{\lambda} = \frac{L^2}{\alpha'}.$$ (5)

where we identify the corresponding sigma model Lagrangian as [19],[24]

$$\tilde{L}_{NG} = \frac{\varepsilon_{\alpha' \beta' \gamma' \delta'}}{\varepsilon^{\alpha' \beta' \gamma' \delta'}} (\partial_\alpha' \zeta \partial_\beta' \zeta - \chi_{\alpha' \beta'}) (\partial_\alpha \theta \partial_\beta \theta + \sin^2 \theta \partial_\alpha \varphi \partial_\beta \varphi) - \varepsilon^{\alpha \beta} \cos \theta \partial_\alpha \varphi \partial_\beta \zeta. (6)$$

Notice that, here $\zeta$ is the additional compact direction associated with the target space geometry along which the string has non zero windings [18]-[19]. Moreover, here $\varepsilon^{01} = -\varepsilon_{01} = 1$ is the Levi-Civita symbol in 2D together with [24],

$$\chi_\alpha = 2 \partial_\alpha t + \partial_\alpha \psi - \cos \theta \partial_\alpha \varphi.$$

\[\text{Here } L \text{ is the radius of the 2 sphere.}\]

\[\text{This is the Lagrangian that represents relativistic strings propagating over TNC geometry in the semiclassical approximation. The world-sheet theory is nonrelativisite (NR) only from the perspective of the target space geometry given in the problem [18]-[19]. However, it is always possible to consider a large } c (\rightarrow \infty) \text{ limit [18]-[19] associated to 2D sigma models that eventually leads towards NR strings propagating over TNC geometry. This will be our subject of discussion in Section 4.}\]
2.1 The BMN limit

Before getting into the spinning string dynamics, it is customary first to consider the BMN-like limit for semiclassical strings propagating over TNC geometry with \( R \times S^2 \) topology. We start with a particular ansatz for the semiclassical string whose centre of mass is considered to be rotating within \( S^2 \) (at an angle \( \theta = \theta_0 = \text{const.} \)) with an angular velocity along the azimuthal direction (\( \varphi \)). The most natural ansatz in this case turns out to be,

\[
\begin{align*}
t &= \tau; \quad \varphi = \omega \tau; \quad \psi = \text{const}; \quad \zeta = n \sigma
\end{align*}
\]  

where, \( n \) is the corresponding winding number of the string along the additional compact dimension (\( \zeta \)) of the target space geometry.

The corresponding Lagrangian density \([6]\) turns out to be

\[
\tilde{\mathcal{L}}_{NG} = \frac{\sin^2 \theta_0 \dot{\varphi}^2}{(2 \ell - \cos \theta_0 \dot{\varphi})} - \cos \theta_0 \dot{\varphi}. \tag{9}
\]

The integrals of motion associated with the 2D sigma model \([9]\) could be formally expressed as,

\[
\begin{align*}
\mathcal{E} &= \frac{\sqrt{\lambda} \omega^2 \sin^2 \theta_0}{(2 - \omega \cos \theta_0)^2} \tag{10} \\
\mathcal{J}_\varphi &= \frac{\sqrt{\lambda}}{2} \left( \cos \theta_0 - \omega \sin^2 \theta_0 \left( \frac{4 - \omega \cos \theta_0}{(2 - \omega \cos \theta_0)^2} \right) \right). \tag{11}
\end{align*}
\]

The parameters of the theory, on the other hand, are constrained by the so-called Virasoro constraint(s) \([24]\),

\[
\begin{align*}
T_{\tau\tau} &= \frac{\omega}{4}(\omega - 2 \cos \theta_0) + (2 - \omega \cos \theta_0) \left( \frac{\omega}{4} \cos \theta_0 + \ell_-(2 - \omega \cos \theta_0) \right) \approx 0 \tag{12} \\
T_{\tau\sigma} &\approx 0 \tag{13}
\end{align*}
\]

where \( \ell_- \) is the so-called Lagrange multiplier.

Combining \((10)\) and \((11)\) we find,

\[
\Delta_{NR} - \mathcal{J}_\varphi = \mathcal{f}(\lambda) \tag{14}
\]

where \( \Delta_{NR} \sim \mathcal{E} \) is the energy eigenvalue associated with the operator (spectrum) (in the strongly correlated QFT) dual to that of the (relativistic) stringy excitations over TNC geometry. The above \((14)\) is a valid dispersion relation provided both \( \Delta_{NR} \sim |\mathcal{E}| \gg 1 \) and \( |\mathcal{J}_{\varphi}| \gg 1 \) such that the ratios like \( \frac{|\mathcal{E}|}{\sqrt{\lambda}} \) and \( \frac{|\mathcal{J}_{\varphi}|}{\sqrt{\lambda}} \) are finite in the strong coupling (\( \lambda \gg 1 \)) limit of the dual QFT. In other words, one essentially computes the spectrum associated with the so-called heavy operators in the dual gauge theory. The function,

\[
\mathcal{f}(\lambda) = \frac{\sqrt{\lambda}}{2} \zeta(\omega, \theta_0); \quad \zeta(\omega, \theta_0) = \omega \sin^2 \theta_0 \left( \frac{2\omega + 4 - \omega \cos \theta_0}{(2 - \omega \cos \theta_0)^2} \right) - \cos \theta_0 \quad \tag{15}
\]

\[4\] We have rescaled the Lagrangian by an overall factor of \( n \).
corresponds to the leading order correction to the energy eigenvalues ($\Delta_{NR}$) of the heavy (BMN) operators in the dual gauge theory.

Considering the centre of mass of the string to be rotating along the equator of $S^2$ at a speed of light amounts of setting $\theta_0 = \frac{\pi}{2}$ and $\varpi = 1$ which finally yields,

$$\Delta_{NR} \approx J_\varphi \sqrt{1 - \frac{3}{4} \tilde{\lambda}} = J_\varphi \left(1 - \frac{3}{8} \tilde{\lambda} + ..\right) \quad (16)$$

which is the pp wave analogue of [28] where one could think of $\Delta_{NR}$ as being the dimension associated to BMN (like) operators in the near BPS sector whose energy eigenvalue gets corrected due to an effective coupling, $\tilde{\lambda} = \frac{\lambda}{J_\varphi}$ such that $|\tilde{\lambda}| < 1$ in the limit when both $|\lambda| \gg 1$ and $|J_\varphi| \gg 1$.

Comparing (16) to that with the corresponding (semi)classical stringy excitations over $AdS_5 \times S^5$ [33]-[35] we propose the following “BMN like” expression for the semi-classical stringy excitations over TNC geometry with $R \times S^2$ topology,

$$E = k_0 J_\varphi \left(1 + k_1 \tilde{\lambda} + ..\right) \quad (17)$$

where we identify $k_0 = 1$ and $k_1 = -\frac{3}{8}$ which are clearly distinct numbers as compared to that with the corresponding relativistic scenario [33]. We further notice that, like in the $AdS_5 \times S^5$ example [28], the non-perturbative (quantum) corrections (to the classical NR stringy excitations) are indeed suppressed in the limit of large $J_\varphi$.

### 2.2 More general ansatz

We now proceed towards establishing the dispersion relation corresponding to spinning string configurations over $R \times S^2$. Unlike the previous example, the centre of mass of the string does not move on $S^2$ rather the string itself is considered to be spinning around its centre of mass and is stretched along the direction of the polar coordinate ($\theta$). We choose the centre of mass of the spinning string to be coincident to that with the north pole of $S^2$. Therefore, we choose to work with an ansatz that corresponds to folded spinning strings inside an $S^2$ which is extended along the polar angle ($\theta$) with its end points rotating along the compact azimuthal ($\varphi$) direction [29],

$$t = \tau ; \ \varphi = \varpi \tau ; \ \psi = \text{const} ; \ \zeta = \sigma ; \ \theta = \theta(\sigma). \quad (18)$$

The corresponding Lagrangian density could be formally expressed as,

$$\tilde{L}_{NG} = \frac{\dot{\varphi}^2 \sin^2 \theta}{(2 \dot{t} - \dot{\varphi} \cos \theta)} - (2 \dot{t} - \dot{\varphi} \cos \theta) \theta'^2 - \dot{\varphi} \cos \theta \quad (19)$$

where prime stands for derivative w.r.t. $\sigma$ and dot corresponds to derivative w.r.t. $\tau$.

The equation of motion corresponding to $\theta$ could be found as,

$$2\theta''(2 - \varpi \cos \theta) + \varpi \sin \theta(1 + \theta'^2) + \frac{\varpi^2 \sin \theta}{(2 - \varpi \cos \theta)^2}(4 \cos \theta - \varpi(1 + \cos^2 \theta)) = 0. \quad (20)$$

On the other hand, the Virasoro constraint(s) [24] yield,

$$T_{\tau\tau} = T_{\sigma\sigma} = \theta'^2 + (2 - \varpi \cos \theta)(4 \ell_-(2 - \varpi \cos \theta) + \varpi \cos \theta) + \varpi(\varpi - 2 \cos \theta) \approx 0 \quad (21)$$

$$T_{\tau\sigma} = \ell_+ \chi_\tau \approx 0 \quad (22)$$
which thereby identically sets one of the Lagrange multipliers \( \ell_+ = 0 \). Furthermore, in order to simplify our analysis, and without loss of any generality we set \( \omega = 1 \) which thereby yields,

\[
\theta^2(\sigma) = |(2 \cos \theta - 1) - (2 - \cos \theta)(a + b \cos \theta)| \equiv \varrho(\theta)
\]

where \( a = 8\ell_- \) and \( b = 1 - 4\ell_- \) are two real constants for our problem.

In the following we note down the corresponding conserved charges \(^5\)

\[
\mathcal{E} - \frac{1}{2} J_\varphi = \frac{\sqrt{\lambda}}{2\pi} \int_0^{\theta_0} d\theta \left[ \cos \theta \left( \frac{(5 - 4 \cos \theta)(2 - \cos \theta)^2 - \varrho}{(2 - \cos \theta)^2 - \varrho} \right) + 4\varrho \right] = \frac{\sqrt{\lambda}}{2\pi} \Lambda(\theta_0) \tag{24}
\]

where we identify the (semi)classical energy of the string \(^6\)

\[
\mathcal{E} = \frac{\sqrt{\lambda}}{\pi} \int_0^{\theta_0} d\theta \left( \cos \theta \left( \frac{2 \sin^2 \theta}{(2 - \cos \theta)^2} + 2\varrho \right) \right) = \frac{\sqrt{\lambda}}{\pi} \Theta(\theta_0). \tag{25}
\]

A straightforward computation further reveals that both \( \Lambda(\theta) \) and \( \Theta(\theta) \) are pretty complicated functions of various elliptic integrals. However both of them vanish exactly at the pole of \( S^2 \) namely for \( \theta = 0 \). Therefore, the corresponding values associated with these conserved charges are solely fixed by the upper bound of the polar angle \( \theta \sim \theta_0 \).

### 2.2.1 Short strings

We first consider the limit in which the folded strings are considered to be spinning close to the north pole. The short strings are therefore defined in the limit \( |\theta_0| \ll 1 \) which finally yields,

\[
\Lambda(\theta_0) \approx \frac{\theta_0(-3a - 3b + 4)}{\sqrt{|a - b + 1|}} + \frac{\theta_0^3 (7ab + a(5a - 2) + 2b^2)}{12|a - b + 1|^{3/2}} + .. \tag{26}
\]

Using \(^24\) and \(^26\) it is now straightforward to show,

\[
\Delta_{NR} \sim \mathcal{E} \approx \frac{1}{2} J_\varphi \left( 1 + \frac{\tilde{\lambda}}{\pi^2} g(\theta_0) \right) \tag{27}
\]

where we introduce the function,

\[
g(\theta_0) = \theta_0^2 \left( \frac{1}{a + b - 1} - 3a - 3b + 1 \right) + \frac{1}{6} \theta_0^4 \left( - \frac{6}{a + b - 1} - \frac{a + 2}{(a + b - 1)^2} - a + 2b + 12 \right) + .. \tag{28}
\]

Here, \( \tilde{\lambda} = \frac{\lambda}{J_\varphi} \ll 1 \) is the effective coupling in the dual gauge theory in the semiclassical approximation, \( J_\varphi \gg 1 \).
For short strings, it is also possible to find the embedding function \( \theta(\sigma) \) to certain approximations. Using (20) and (23) and after some simple steps it is quite straightforward to find,

\[
\theta''(\sigma) + \kappa \theta \approx 0 \tag{29}
\]

which thereby yields\(^7\)

\[
\theta(\sigma) \equiv \theta_s(\sigma) = \sin \sqrt{\kappa} \sigma \approx \sqrt{\kappa} \sigma + .. \tag{30}
\]

where, \( \kappa = 2 - \frac{1}{2}(a + b) \) and \( -\theta_0 \leq \sigma \leq \theta_0 \).

With this short string approximation (30), it is now straightforward to compute,

\[
\mathcal{E} - \frac{1}{2} J_{\phi} \approx \frac{2\sqrt{\lambda}}{3\pi} \theta_0 \left( 3 - 8\theta_0^2 \right) \tag{31}
\]

where we set \( a = 2 \) and \( b = 0 \) for the sake of simplicity.

Using (31), after few simple steps one finally arrives at the dispersion relation of the following form,

\[
\Delta_{NR} \sim \mathcal{E} \approx \frac{1}{2} J_{\phi} \left( 1 + \hat{\gamma}_S(\tilde{\lambda}) \right) \tag{32}
\]

where we identify the function,

\[
\hat{\gamma}_S(\tilde{\lambda}) \approx h_1 \tilde{\lambda} + .. \tag{33}
\]

together with the coefficient,

\[
h_1 \approx \frac{128\theta_0^4}{3\pi^2} + \mathcal{O}(\theta_0^5) \tag{34}
\]

Our goal in the next section would be to compute the one loop string correction to the above expression in (33). This would be achieved considering a specific embedding for fluctuating TNC strings over \( R \times S^2 \).

### 2.2.2 Extended strings

We now consider the extended string limit which essentially corresponds to folded spinning strings whose end points reach the equator namely, \( |\theta_0| \sim \frac{\pi}{2} \). This finally yields the dispersion relation of the following form,

\[
\mathcal{E} - \frac{1}{2} J_{\phi} \approx \frac{\sqrt{\Lambda}}{2\pi} \Lambda \left( \frac{\pi}{2} \right). \tag{35}
\]

It turns out that \( \Lambda \left( \frac{\pi}{2} \right) \) is a quite cumbersome function unless we fix the Lagrange multiplier(s) (associated to TNC string sigma models [19],[24]) which in turn fixes the constants \( a \) and \( b \) in the problem.

\(^7\)In the short string approximation one may therefore choose to work with the static gauge \( \theta = \sigma \)\(^3\). Here \( 4\theta_0 \) stands for the periodicity along \( \sigma \).
A careful analysis further reveals that for extended strings,

$$\Delta_{NR} \sim \mathcal{E} \approx \frac{1}{2} J_\varphi \left(1 + \hat{\gamma}_L(\tilde{\lambda})\right)$$  \hspace{1cm} (36)$$

where we identify,

$$\hat{\gamma}_L(\tilde{\lambda}) \approx f_1 \tilde{\lambda} + .. \hspace{1cm} (37)$$

together with the coefficient,

$$f_1 = \frac{\Lambda(\frac{\pi}{2})}{\pi^2} \left(2\Theta\left(\frac{\pi}{2}\right) - \Lambda\left(\frac{\pi}{2}\right)\right).$$  \hspace{1cm} (38)$$

As an example, setting \(a = 2\) and \(b = 0\) we note,

$$\Theta\left(\frac{\pi}{2}\right) = \frac{1}{3} \left(\sqrt{5} - 6F\left(\frac{\pi}{4}\right) - 8\right) + 14E\left(\frac{\pi}{4}\right) - 8\right)$$  \hspace{1cm} (39)$$

$$\Lambda\left(\frac{\pi}{2}\right) = \frac{1}{6} \Im \left(-156F\left(\sin^{-1}\left(\sqrt{5}\right) \left|\frac{1}{9}\right\right) + 171E\left(\sin^{-1}\left(\sqrt{5}\right) \left|\frac{1}{9}\right\right)\right) + \frac{8}{3} \Im \left(\Pi\left(-\frac{1}{3}; -\sin^{-1}\left(\sqrt{5}\right) \left|\frac{1}{9}\right\right)\right)\right) - \frac{\sqrt{5}}{3}$$  \hspace{1cm} (40)$$

where, \(\Im\) stands for the imaginary part of the complex elliptic integrals [41] above in (40).

### 3 Stringy corrections

Below we develop a general algorithm to find one loop string corrections to the energy eigenvalue \((\Delta_R)\) corresponding to TNC spinning string solitons on \(\mathcal{R} \times S^2\). This is in spirit is quite similar to that with the earlier analysis done in the context of \(AdS_5 \times S^5\) string solitons [30]. Our aim would be to find a specific solvable sector within the TNC string solitonic sigma models and calculate one loop correction to the corresponding energy eigenvalue.

#### 3.1 The formalism

We now build up a general formalism to compute quantum corrections to the dispersion relation for semiclassical strings propagating over TNC geometry. In order to do so, we consider the effects of quantum fluctuations up to leading order in the t’Hooft coupling,

$$t = \tau + \tilde{\tau}(\sigma^\alpha) \frac{1}{\lambda^{1/4}}; \quad \psi = \tilde{\psi}(\sigma^\alpha) \frac{1}{\lambda^{1/4}}; \quad \varphi = \tau + \tilde{\varphi}(\sigma^\alpha) \frac{1}{\lambda^{1/4}}; \quad \theta = \theta(\sigma) + \tilde{\theta}(\sigma^\alpha) \frac{1}{\lambda^{1/4}}; \quad \zeta = \sigma + \tilde{\zeta}(\sigma^\alpha) \frac{1}{\lambda^{1/4}}$$  \hspace{1cm} (41)$$

where, \(\sigma^\alpha = \{\tau, \sigma\}\) stands for the coordinates on the string world-sheet. The above perturbative expansion [41] naturally yields,

$$\chi_\alpha = \chi^0_\alpha + \frac{1}{\lambda^{1/4}} \chi^{(1)}_\alpha + \frac{1}{\sqrt{\lambda}} \chi^{(2)}_\alpha + .. \hspace{1cm} (42)$$
where, the individual entities could be formally expressed as,

\[
\chi^{(0)}_\alpha &= 2\partial_\alpha \tau - \cos \theta \partial_\alpha \tau \\
\chi^{(1)}_\alpha &= 2\partial_\alpha \tilde{\tau} + \partial_\alpha \tilde{\psi} - \cos \theta \partial_\alpha \tilde{\varphi} + \tilde{\theta} \partial_\alpha \tau \sin \theta \\
\chi^{(2)}_\alpha &= \tilde{\theta} \sin \theta \partial_\alpha \tilde{\varphi}.
\]

Substituting (41) into (5) and expanding the action up to quadratic order in fluctuations we find,

\[
S_{NG} = S_{NG}^{(0)} + S_{NG}^{(1)} + S_{NG}^{(2)} + O(1/\sqrt{\lambda})
\]

where, the effective Lagrangian density in the quadratic fluctuations could be formally expressed as\(^8\)

\[
\mathcal{L}^{(2)}_{NG} = \frac{\mathcal{M}^{(0)}}{\chi^{(0)}_\tau} \left( \left( \tilde{\zeta} + \frac{\chi^{(1)}_\chi}{\chi^{(0)}_\tau} \right)^2 - \Phi(\sigma^\alpha) \right) - \frac{\mathcal{M}^{(1)}}{\chi^{(0)}_\tau} \left( \tilde{\zeta} + \frac{\chi^{(1)}_\chi}{\chi^{(0)}_\tau} \right) + \frac{\mathcal{M}^{(2)}}{\chi^{(0)}_\tau} - \mathcal{K}^{(2)}
\]

where the individual entities above in (47) are given by,

\[
\Phi(\sigma^\alpha) &= \varepsilon^{\alpha\beta} \chi^{(1)}_\alpha \partial_\beta \tilde{\zeta} + \chi^{(2)}_\alpha \\
\mathcal{M}^{(0)} &= \sin^2 \theta - (\chi^{(0)}_\tau)^2 \theta^2 \\
\mathcal{M}^{(1)} &= -2\chi^{(0)}_\tau \theta (\chi^{(1)}_\tau \theta' + \chi^{(0)}_\tau \theta'') + 2(\dot{\tilde{\varphi}} + \tilde{\zeta}') \sin^2 \theta + 2\tilde{\theta} \sin \theta \cos \theta
\]

\[
\mathcal{M}^{(2)} &= \tilde{\theta}^2 - 2\theta' \dot{\tilde{\varphi}} + \tilde{\zeta}' \theta^2 - \tilde{\theta}^2 (\chi^{(0)}_\tau)^2 - 2\chi^{(0)}_\tau \chi^{(1)}_\tau \theta' \theta'' - 2\theta^2 \chi^{(0)}_\tau \chi^{(2)}_\tau - \theta^2 (\chi^{(1)}_\tau)^2 \\
&+ \tilde{\theta}^2 \cos^2 \theta + 4\tilde{\theta} (\dot{\tilde{\varphi}} + \tilde{\zeta}') \sin \theta \cos \theta + (\dot{\tilde{\varphi}}^2 + \tilde{\zeta}'^2 + 2\tilde{\varphi}' \tilde{\zeta}') + 2\varepsilon^{\alpha\alpha'} \partial_\alpha \tilde{\varphi} \partial_\alpha \tilde{\zeta} \sin^2 \theta \\
&+ 2\chi^{(0)}_\tau \theta' \varepsilon^{\alpha\alpha'} \partial_\alpha \tilde{\varphi} \chi^{(1)}_\alpha \chi^{(1)}_\tau \tilde{\varphi}' - (\chi^{(0)}_\tau)^2 \tilde{\varphi}'^2 - (\chi^{(1)}_\tau)^2 \sin^2 \theta
\]

and,

\[
\mathcal{K}^{(2)} = -\varepsilon^{\alpha\beta} \cos \theta \partial_\alpha \tilde{\varphi} \partial_\beta \tilde{\zeta} + \tilde{\theta} \sin \theta (\tilde{\zeta}' + \dot{\tilde{\varphi}}).
\]

Next, we note down the equations of motion corresponding to different fluctuations which we present categorically here in the following. These equations should be understood as being solved order by order as a perturbation in \(\frac{1}{\sqrt{\lambda}}\) in the semiclassical approximation. However, in our analysis, we would be interested in solving these equations only at the level of leading and/or zeroth order in \(\frac{1}{\sqrt{\lambda}}\) which are thereby obtained by varying the action quadratic in fluctuations namely, \(S_{NG}^{(2)} = \int d^2 \sigma \mathcal{L}_{NG}^{(2)}\).

### 3.2 An example: Short strings

#### 3.2.1 Solving fluctuations

The dynamics of stringy fluctuations is indeed quite difficult to deal with until and unless one takes some special limits. In order to solve the dynamics of stringy fluctuations, we

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\(^8\)We rescale the Lagrangian by an overall factor of \(-\frac{1}{4\pi}\).
therefore choose to work with the so called short string limit namely, $|\theta_0| \ll 1$ together
with a particular string embedding of the following form,

$$\bar{\tau}(\tau, \sigma) = \xi \tau ; \bar{\theta} = \bar{\theta}(\sigma) ; \bar{\psi}(\tau, \sigma) = \gamma_2 \tau ; \bar{\varphi}(\tau, \sigma) = \gamma_1 \tau ; \bar{\zeta}(\tau, \sigma) = \eta \sigma.$$  \hspace{1cm} (53)

The resulting Lagrangian could be approximated as,

$$\mathcal{L}_{NG}^{(2)} \approx \left((\eta + \nu)^2 - \eta \nu\right) (\theta_s^2 - \theta_s'^2) - \left(\eta + \frac{\nu}{2}\right) \theta_s \bar{\theta} - (\eta + \nu) \left(2\theta_s \bar{\theta} + 2(\gamma_1 + \eta)\theta_s^2 - 2\nu \theta_s^2 - 2\theta_s' \bar{\theta}'\right) \hspace{1cm} (54)$$

where $\theta(\sigma) \equiv \theta_s(\sigma)$ (with $\kappa = 1$) is the classical background solution \hspace{1cm} (30) in the limit of
short strings together with, $\nu = 2\xi + \gamma_2 - \gamma_1$ and $\epsilon_1 = \gamma_1^2 + \eta^2 + 4\gamma_1 \eta$. Here we drop all
those terms and their derivatives which are $\sim \mathcal{O}(\theta^3)$ and higher order in the fluctuations.

The resulting equation of motion could be formally expressed as,

$$\bar{\theta}''(\sigma) + \bar{\theta}(\sigma) + j \sin \sigma \approx 0$$  \hspace{1cm} (55)

where, $j = \eta - 9\nu - \frac{3\gamma_1}{2}$. The corresponding solution turns out to be,

$$\bar{\theta}(\sigma) = \cos \sigma \left(\epsilon_1 + \frac{3\sigma}{2}\right) + \epsilon_2 \sin \sigma$$

$$\approx \epsilon_1 + \sigma \left(\epsilon_2 + \frac{3}{2}\right) - \frac{1}{2} \epsilon_1 \sigma^2 + .. \hspace{1cm} (56)$$

Finally, we note down the Virasoro constraints which for the present example yields,

$$T_{\tau\tau} = T_{\sigma \sigma} \approx 1 + \theta_s'^2 + \frac{1}{\lambda^{1/4}} \left(\xi + \frac{\gamma_2}{2} - \frac{\gamma_1}{2} + \frac{\theta_s' \bar{\theta}'}{2}\right) + .. \approx 0 \hspace{1cm} (57)$$

$$T_{\tau \sigma} \approx 0. \hspace{1cm} (58)$$

### 3.2.2 Conserved charges

With the above machinery in hand, we are now in a position to compute the one loop ($\alpha' \sim \frac{1}{\sqrt{\lambda}}$) quantum corrections to the energy (E) and R charge (J_r) of NR strings propagating
over TNC geometry. To start with, the formula we derive below is valid for generic TNC
strings spinning over $R \times S^2$. A straightforward computation reveals,

$$\mathcal{E} = \sqrt{\lambda} \frac{\sqrt{2}}{2\pi} \int_0^{4\theta_0} d\sigma \left[\frac{\zeta' \phi^2 \sin^2 \theta}{(2\bar{t} + \psi - \cos \theta \bar{\varphi})^2} + \frac{\theta^2}{\zeta'}\right] \hspace{1cm} (59)$$

$$|J_r| = \sqrt{\lambda} \frac{\sqrt{2}}{4\pi} \int_0^{4\theta_0} d\sigma \left[\zeta' \sin^2 \theta \bar{\varphi} \frac{(4i + 2\psi - \cos \theta \bar{\varphi})}{(2\bar{t} + \psi - \cos \theta \bar{\varphi})^2} + \frac{\cos \theta}{\zeta'} \left(\theta'^2 - \zeta'^2\right)\right]. \hspace{1cm} (60)$$

Using (53), one could further simplify the above expressions (59)-(60) for short strings,

$$\mathcal{E} \approx \sqrt{\lambda} \frac{\sqrt{2}}{2\pi} \left(4\theta_0 + 4\theta_0 \left(2c_2 - \eta + j\right) - \frac{64}{3} (-4\gamma_1 + 2\gamma_2 - 2\eta + j + 4\zeta) \tilde{\theta}_0 \theta_0^2\right) + .. \hspace{1cm} (61)$$
where, we introduce the entity \( \tilde{\theta}_0 = \frac{\theta_0}{\lambda^{1/4}} \ll 1 \). Using (61) and (62), it is quite straightforward to recover (31) in the strict classical limit namely, \( \tilde{\theta}_0 \to 0 \). However, for our present purpose it is desired to find the dispersion relation of the following form,

\[
E - J_\varphi \approx \sqrt{\lambda} \left( \frac{128 \theta_0^3}{3} + 4 \tilde{\theta}_0 (2c_2 - 2\eta + j) + 32 \epsilon_1 \tilde{\theta}_0 \theta_0 \right) + 32 \sqrt{\lambda} \left( 16 \gamma_1 - 8 \gamma_2 + j + 10 \eta - 16 \xi + 6c_2 \right) \tilde{\theta}_0 \theta_0^2 + ..
\]  

(62)

where we use the following shorthand notation

\[
\tilde{a} = 4(2c_2 - \eta + j) ; \quad \tilde{b} = \frac{64}{3} (-4\gamma_1 + 2\gamma_2 - 2\eta + j + 4\xi)
\]

\[
\tilde{c} = 4(2c_2 - 2\eta + j) ; \quad \tilde{d} = 32c_1 ; \quad \tilde{e} = \frac{32}{3} (16\gamma_1 - 8\gamma_2 + j + 10\eta - 16\xi + 6c_2).
\]  

(64)

The above equation (63) could be messaged further to obtain,

\[
\Delta_{NR} \sim \mathcal{E} = J_\varphi \left( 1 + f(\tilde{\lambda}) \right)
\]  

(65)

where we identify,

\[
f(\tilde{\lambda}) \approx \frac{\lambda \theta_0^2}{2 \pi^2} \left( \frac{128 \theta_0^2}{3} + \sqrt{\alpha'} \left( \tilde{c} + \theta_0 \tilde{d} + \theta_0^2 \left( \frac{32}{3} (\tilde{a} - \tilde{c}) + \tilde{e} \right) \right) \right) + ..
\]  

(66)

as being the one loop \( (\alpha') \) corrected anomalous dimension associated to folded (short) spinning TNC strings over \( R \times S^2 \).

### 4 NR spinning strings on \( R \times S^2 \)

We conclude our analysis with a detailed discussion on the Hamiltonian spectrum corresponding to NR folded strings spinning over TNC geometry with \( R \times S^2 \) topology\(^9\). This is achieved by taking the so called scaling limit\(^{18-19}\) corresponding to 2D sigma model. The resulting sigma model action could be formally expressed as\(^{10}\)

\[
S_{NR} = \frac{\sqrt{g}}{4\pi} \int d^2 \sigma L_{NR}
\]  

(67)

where we identify the corresponding Lagrangian density as\(^{24}\),

\[
L_{NR} = \frac{\varepsilon^{\alpha\alpha'}\varepsilon^{\beta\beta'}}{t} \partial_\alpha t \partial_\beta \partial(t \partial_\alpha \theta \partial_\beta \varphi + \sin^2 \theta \partial_\alpha \varphi \partial_\beta \varphi) + \varepsilon^{\alpha\beta} \cos \theta \partial_\alpha \varphi \partial_\beta \zeta + \mathcal{O}(1/c^2)
\]  

(68)

where, \( c (\to \infty) \) is the speed of light.

\(^9\)Unlike the earlier example, here we perform an exact evaluation of \( 0 \leq \sigma \leq 2\pi \) integral without any approximation on the string length and compute the corresponding energy eigenvalue \( (\Delta_{NR}) \) and its one loop stringy correction. Following the original arguments of\(^{27}\), one should remember that the present computation is strictly valid in the limit of strong \( (g \gg 1) \) coupling and low temperatures.

\(^{10}\)Here, \( g = c^2 \lambda \) is the string tension associated to TNC strings in the large \( c (\to \infty) \) limit.
4.1 Classical solutions

We start with the folded spinning string ansatz of the following form,

\[ t = \tilde{\kappa} \tau \; ; \; \varphi = \tilde{\omega} \tau \; ; \; \zeta = \tilde{\ell} \sigma \; ; \; \theta = \theta(\sigma). \tag{69} \]

Substituting (69) into (68) we find,

\[ \mathcal{L}_{NR} = \tilde{\kappa} \theta'' + \tilde{\omega} \tilde{\ell} \cos \theta. \tag{70} \]

The corresponding equation of motion turns out to be,

\[ \theta''(\sigma) + \frac{\tilde{\omega} \tilde{\ell}}{2 \tilde{\kappa}} \sin \theta = 0 \tag{71} \]

which has the most general solution of the following form\(^{11}\)

\[ \theta(\sigma) \equiv \theta_{cl} = 2C_1 \text{am} \left( \frac{\sqrt{\tilde{\omega} \tilde{\ell} + \tilde{\kappa} c_1 (\sigma + c_2)}}{2 \sqrt{\tilde{\kappa}}} \bigg| \frac{2 \tilde{\omega} \tilde{\ell}}{\tilde{\omega} \tilde{\ell} + \tilde{\kappa} c_1} \right) \]

\[ -2C_2 \text{am} \left( \frac{\sqrt{\tilde{\omega} \tilde{\ell} + \tilde{\kappa} c_1 (\sigma + c_2)}}{2 \sqrt{\tilde{\kappa}}} \bigg| \frac{2 \tilde{\omega} \tilde{\ell}}{\tilde{\omega} \tilde{\ell} + \tilde{\kappa} c_1} \right) \]

\[ \equiv \frac{\sqrt{g}}{\pi} \text{E} \]

and\(^{12}\)

\[ \tilde{J}_\varphi = \frac{\sqrt{g} \tilde{\ell}}{4\pi} \int_0^{2\pi} \cos \theta d\sigma = \frac{\sqrt{g} \tilde{\ell}}{4\pi} \int_0^{2\pi} \sqrt{1 - \frac{4\tilde{\kappa}^2}{\tilde{\omega}^2 \tilde{\ell}^2} \theta''^2} d\sigma \]

\[ = \frac{\sqrt{g} \tilde{\ell} \sqrt{\tilde{\kappa}} N}{4\pi \tilde{f}(g)}. \tag{74} \]

Using (73) and (74) it is now straightforward to obtain\(^{13}\)

\[ \tilde{\Delta}_{NR} \sim \tilde{\xi} = \tilde{J}_\varphi \left( 1 + \tilde{f}(g) \right) \]

\(^{11}\)The entity introduced on the r.h.s. of (72) is known as the Jacobi amplitude \( \phi(u,k) \equiv \text{am}(u|k) = \int_0^u \text{dn}(u',k)du' \) where, \( \text{dn}(u',k) \) is known as Jacobi elliptic function with elliptic modulus) which is defined as being the inverse of the elliptic integral function of first kind. For more details on different types of Jacobi elliptic functions the enthusiastic reader is encouraged to see [41].

\(^{12}\)See Appendix for details.

\(^{13}\)Here \( \tilde{\Delta}_{NR} \) is the energy eigenvalue associated with the corresponding dual operator spectrum in the SMT theory at strong coupling and low temperatures.
where we identify,

\[ \tilde{f}(\tilde{g}) = \frac{\tilde{g}}{4\pi^2} \tilde{\ell} \sqrt{\tilde{\kappa} N} \left( E - \frac{\tilde{\ell} \sqrt{\tilde{\kappa} N}}{4} \right) \equiv q_1 \tilde{g} \]  

(76)

as being the leading order correction to the Hamiltonian spectrum in the limit of strong \((g \gg 1)\) coupling. Here we identify, \(\bar{g} = \frac{g}{\sqrt{\eta}}\) as an effective coupling constant in the dual SMT theory at low temperatures.

4.2 One loop corrections

Our aim now is to compute quantum \((\sqrt{g})^{-1}\) corrections to the above function in (76). In order to do so, we choose to work with the string embedding of the following form,

\[
t = \tilde{\kappa} \tau + \hat{t}(\sigma^\alpha) \quad ; \quad \varphi = \tilde{\omega} \tau + \hat{\varphi}(\sigma^\alpha) \quad ; \quad \zeta = \tilde{\ell} \sigma + \hat{\zeta}(\sigma^\alpha) \quad ; \quad \theta = \theta_{cl}(\sigma) + \hat{\theta}(\sigma)
\]  

(77)

Substituting (77) into (68) we arrive at the quadratic Lagrangian of the following form,

\[
L^{(2)}_{NR} = \tilde{\kappa} \hat{\theta}'' + 2 \hat{\theta}' \theta_{cl}' + \bar{w} \cos \theta_{cl} - (\tilde{\omega} l + \tilde{\ell} w) \hat{\theta} \sin \theta_{cl}
\]  

(78)

where we choose to work with the following ansatz,

\[
\hat{t}(\sigma^\alpha) = \tau \quad ; \quad \hat{\varphi}(\sigma^\alpha) = w \tau \quad ; \quad \hat{\zeta}(\sigma^\alpha) = l \sigma
\]  

(79)

together with the fact that \(w\) and \(l\) are some real positive integers.

The equation of motion corresponding to \(\hat{\theta}(\sigma)\) could be formally expressed as,

\[
2 \tilde{\kappa} \hat{\theta}''(\sigma) + 2 \hat{\theta}'(\sigma) + (\tilde{\omega} l + \tilde{\ell} w) \sin \theta_{cl} = 0.
\]  

(80)

The above equation (80) has a remarkably simple solution once we set,

\[
\tilde{\kappa} = \tilde{\omega} = \tilde{\ell} = w = l = 1
\]  

(81)

which by means of (71) yields,

\[
\hat{\theta}(\sigma) = \theta_{cl}(\sigma) + s_1 \sigma + s_2
\]  

(82)

where \(s_{1,2}\) are integration constants.

Using (82), it is now straightforward to compute one loop stringy correction to the energy spectrum\[14\] \[15\]

\[
\tilde{E} = \frac{\sqrt{\tilde{g}}}{\pi} \left( E + \frac{\sqrt{\delta}}{L} \Delta E \right)
\]  

(83)

where we identify one loop correction to the NR string spectrum as\[15\]

\[
\Delta E = -gd \left( \frac{1}{\sqrt{2}} \right) + gd \left( \frac{1 + 2\pi}{\sqrt{2}} \right) + \sqrt{2} \sinh \left( \frac{\sqrt{2} \pi}{\sqrt{2}} \right) \sech \left( \frac{1}{\sqrt{2}} \right) \sech \left( \frac{1 + 2\pi}{\sqrt{2}} \right).
\]  

(84)

\[14\]For the sake of computational simplicity we set, \(c_1 = c_2 = s_1 = 1\) and \(s_2 = 0\).

\[15\]Notice that here, \(gd(x) = \int_0^x \frac{dt}{\cosh t}\) is the so called Gudermannian function.\[14\]
Finally, we compute one loop correction to the R charge,
\[ \tilde{J}_{\rho} = \frac{\sqrt{g}}{4\pi} \left( \frac{N}{D} + \sqrt{\alpha'} \frac{L}{\Delta J} \right) \]  
(85)

where,
\[ \Delta J = 4\sqrt{2} \left( 2e^{\frac{1+2\pi}{\sqrt{2}}} \left( gd \left( \frac{1+2\pi}{\sqrt{2}} \right) + \pi \right) + 3 \right) - \frac{4\sqrt{2} \left( 2e^{\frac{1}{\sqrt{2}}} \left( \frac{1}{\sqrt{2}} \right) + 1 \right)}{1 + e^{\sqrt{2}(1+2\pi)}} 
- 4\sqrt{2} + 2\pi + 6 + 8\tan^{-1} \left( e^{\frac{1}{\sqrt{2}}} \right) - 8\tan^{-1} \left( e^{\frac{1+2\pi}{\sqrt{2}}} \right) - \sqrt{2} \cosh^{-1}(3). \]  
(86)

Combining (83) and (85) we finally obtain,
\[ \Delta_{NR} \sim \tilde{E} = \tilde{J}_{\rho}(1 + \tilde{f}(\bar{g})) \]  
(87)

where we identify the function,
\[ \tilde{f}(\bar{g}) = \tilde{f}(\bar{g})(1 + \Delta \tilde{f}) \]  
(88)

that include one loop stringy correction of the form,
\[ \Delta \tilde{f} = \frac{\sqrt{\alpha'}}{L} \left( \frac{\Delta E + \Delta \bar{S}^2}{\bar{S} \frac{E}{g} - \frac{N}{2D}} \right) + O(\alpha'/L^2) \]
\[ = \frac{q_2}{\sqrt{\tilde{J}_{\rho}}} + O(1/(\tilde{J}_{\rho} \sqrt{\bar{g}})) \]  
(89)

where, \[ q_2 = \left( \frac{\Delta E + \Delta \bar{S}^2}{\bar{S} \left( \frac{E}{g} - \frac{N}{2D} \right)} \right). \]

Finally, substituting (89) into (87) we find,
\[ \Delta_{NR} = \left( 1 + q_1 \bar{g} + \frac{q_1 q_2}{\sqrt{\tilde{J}_{\rho}}} \bar{g}^{3/4} + .. \right) \tilde{J}_{\rho}. \]  
(90)

5 Summary and final remarks

The present paper is an attempt towards understanding the NR sigma model/SMT correspondence through some explicit computations of the energy spectrum in the dual gauge theory at strong (\( g \gg 1 \)) coupling. In our analysis, we consider specific example of NR folded closed string configurations on \( R \times S^2 \) spinning around the north pole of \( S^2 \). By exploring solitonic excitations associated with these NR 2D sigma models we finally compute the Hamiltonian spectrum (\( \Delta_{NR} \)) corresponding to heavy states (\( \tilde{J}_{\rho} > \sqrt{g} \gg 1 \)) in the dual gauge theory at strong coupling. In other words, on the gauge theory side we consider states with large length (\( L \sim \tilde{J}_{\rho} \)) which thereby implies an effective planar limit at low temperatures. On the dual stringy sector this limit corresponds to free strings propagating over \( S^2 \) [27]. The key observation of our analysis turns out to be
the identification of an effective expansion parameter $\bar{g}(= \frac{\tilde{g}}{g}) \ll 1$ for the gauge theory which allows us to compute quantum corrections associated with the Hamiltonian spectra (around its classical value) in the limit of strong ($g \gg 1$) coupling and low temperatures. An interesting question along this line of arguments turns out to be how does the spectrum changes (at strong coupling) as temperature increases above the critical transition temperature ($T_c$) where the classical stringy picture breaks down and one therefore needs to take into account the so called *non planar effects* [27]. The other question that might be of worth exploring is whether it is possible to match the spectrum at *finite* (or, *small*) coupling ($g$). In other words, to check the spin chain (or, near planar $\mathcal{N} = 4$ SYM)/ NR sigma model correspondence using the notion of effective coupling constant ($\bar{g}$) discussed in this paper. We hope to address some of these questions in the near future.

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Appendix: Detailed expressions for $N$ and $D$
Here, in the Appendix, we provide detailed expressions for the functions $N$ and $D$ those which appear in the expression for the R charge (74) associated to NR TNC strings spinning over $R \times S^2$. Below we enumerate their individual expressions as,

$$N = -\left(c_1 c_2 \sqrt{\kappa} \sqrt{c_1 \kappa + \widehat{\omega} \ell} - 2 \left(c_1 \kappa + \widehat{\omega} \ell\right) E \left(\text{am} \left(\frac{\sqrt{\widehat{\omega} \ell + \kappa c_2} (\sqrt{c_1 + \kappa c_2})}{2 \sqrt{\kappa}} \right) \left| \frac{2 \widehat{\omega} \ell}{\widehat{\omega} \ell + \kappa c_1} \right| \left| \frac{2 \widehat{\omega} \ell}{\widehat{\omega} \ell + \kappa c_1} \right)\right)\right)$$

$$\times \text{cn} \left(\frac{\sqrt{\widehat{\omega} \ell + \kappa c_2} (\sqrt{c_1 + \kappa c_2})}{2 \sqrt{\kappa}} \right)^2 \text{dn} \left(\frac{\sqrt{\widehat{\omega} \ell + \kappa c_2} (\sqrt{c_1 + \kappa c_2})}{2 \sqrt{\kappa}} \right) - \text{sn} \left(\frac{\sqrt{\widehat{\omega} \ell + \kappa c_2} (\sqrt{c_1 + \kappa c_2})}{2 \sqrt{\kappa}} \right)^2$$

$$\times \left(-2 \widehat{\omega} \ell \text{sn} \left(\frac{\sqrt{\widehat{\omega} \ell + \kappa c_2} (\sqrt{c_1 + \kappa c_2})}{2 \sqrt{\kappa}} \right)^2 + c_1 \kappa + \widehat{\omega} \ell\right)$$

$$+ \left(c_1 (c_2 + 2 \pi) \sqrt{\kappa} \sqrt{c_1 \kappa + \widehat{\omega} \ell} - 2 \left(c_1 \kappa + \widehat{\omega} \ell\right) E \left(\text{am} \left(\frac{\sqrt{\widehat{\omega} \ell + \kappa c_2} (\sqrt{c_1 + \kappa c_2})}{2 \sqrt{\kappa}} \right) \left| \frac{2 \widehat{\omega} \ell}{\widehat{\omega} \ell + \kappa c_1} \right| \left| \frac{2 \widehat{\omega} \ell}{\widehat{\omega} \ell + \kappa c_1} \right)\right)\right)$$

$$\times \text{dn} \left(\frac{c_2 \sqrt{c_1 \kappa + \widehat{\omega} \ell}}{2 \sqrt{\kappa}} \right) \left(\text{cn} \left(\frac{\sqrt{\widehat{\omega} \ell + \kappa c_2} (\sqrt{c_1 + \kappa c_2})}{2 \sqrt{\kappa}} \right)^2 - \text{sn} \left(\frac{\sqrt{\widehat{\omega} \ell + \kappa c_2} (\sqrt{c_1 + \kappa c_2})}{2 \sqrt{\kappa}} \right)^2\right)$$

$$\times \left(-2 \widehat{\omega} \ell \text{sn} \left(\frac{\sqrt{\widehat{\omega} \ell + \kappa c_2} (\sqrt{c_1 + \kappa c_2})}{2 \sqrt{\kappa}} \right)^2 + c_1 \kappa + \widehat{\omega} \ell\right).$$
and,

\[ D = \tilde{\omega} \sqrt{c_1 \tilde{\kappa}} + \tilde{\omega} \tilde{\ell} \text{dn} \left( \frac{\sqrt{\tilde{\omega} \tilde{\ell} + \tilde{\kappa} c_1 c_2}}{2 \sqrt{\tilde{\kappa}}} \right) \text{dn} \left( \frac{\sqrt{\tilde{\omega} \tilde{\ell} + \tilde{\kappa} c_1 (c_2 + 2 \pi)}}{2 \sqrt{\tilde{\kappa}}} \right) \]

\[ \times \left( \text{cn} \left( \frac{\sqrt{\tilde{\omega} \tilde{\ell} + \tilde{\kappa} c_1 c_2}}{2 \sqrt{\tilde{\kappa}}} \right) \right)^2 - \text{sn} \left( \frac{\sqrt{\tilde{\omega} \tilde{\ell} + \tilde{\kappa} c_1}}{2 \sqrt{\tilde{\kappa}}} \right)^2 \]
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