Dynamics of the entanglement rate in the presence of decoherence

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The dynamics of the entanglement rate are investigated in this paper for pairwise interaction and two special sets of initial states. The results show that for the given interaction and the decoherence scheme, the competitions between decohering and entangling lead to two different results—some initial states may be used to prepare entanglement while the others do not. A criterion on decohering and entangling is also presented and discussed.

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Entanglement plays an essential role in quantum information theory, the sharing of entanglement between sender and receiver allows for quantum teleportation\cite{1}, quantum superdense coding\cite{2} and the other applications to quantum information processing\cite{3}. Creating entanglement in a proper way is thus an important issue.

In general, entanglement between two systems can be generated if they interact in a controlled way. However, for a practical experiment, the production of entanglement is very difficult due to the weak interaction between the systems. Thus how to improve efficiency of the production by using those interactions become a very relevant problem. Very recently, Dürr et al.\cite{4} consider a situation that one has a given non-local Hamiltonian and ask, what is the most efficient way of entangling particles? Their answers are that (i) the initially entangled two particles can improve the efficiency of the production, and (ii) one can also improve the efficiency by using some ancillas.

In this paper, we shed light on this issue again by taking the decoherence effects into account. As you will see, the problem for mixed states is complicated, thus we choose two special sets of mixed state to study the problem. This is the limitation of this paper. The results show that there is a competition between entangling and decohering in the entanglement production process, some initial states may work very well in the absence of decoherence, but they do not provide the best way to produce entanglement in the presence of decoherence.

To begin with, we recall the definition of entanglement rate $\Gamma(t)$\cite{4}

$$\Gamma(t) = \frac{dE(t)}{dt},$$

where $E(t)$ denotes an entanglement measure of a state $\rho(t)$. In this paper, we pay our attention first to the case of two qubits, and then generalize the discussion to the case of d-level system with $d > 2$. We use the following notations throughout this paper: $|\xi_i\rangle (i = 1, 2, 3, 4)$ stand for bases of the two-qubit system, $|\xi_1\rangle = |00\rangle$, $|\xi_2\rangle = |01\rangle$, $|\xi_3\rangle = |10\rangle$, $|\xi_4\rangle = |11\rangle$. $|m\rangle_m = 0, 1$ denotes the two states of one qubit, and $\rho_{ij} = \langle \xi_i | \rho(t) | \xi_j \rangle$ represent the matrix elements of $\rho(t)$ in the space spanned by $|\xi_i\rangle (i = 1, ..., 4)$. With those notations, we choose 15 independent variables to describe a general two-qubit state $\rho(t)$, they consist of 3 independent diagonal elements $\rho_{11}, \rho_{22}, \rho_{33}$ and 6 complex off-diagonal elements $\rho_{ij}(i, j = 1, ..., 4, j > i)$ of matrix $\rho(t)$\cite{5}.

In order to calculate the entanglement rate, we have to express the entanglement measure of the state $\rho(t)$ as a function of $\rho_{ij}(i, j = 1, ..., 4, j \geq i)$. For an entangled two-qubit system, we may choose the Wootters concurrence as the entanglement measure

$$E(t) = \mathcal{E}(c(\rho)), \quad (2)$$

with

$$\mathcal{E}(c) = h(\frac{1 + \sqrt{1-c^2}}{2}),$$

$$h(x) = -xlog_2 x - (1-x)log_2 (1-x),$$

$$c(\rho) = max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\},$$

where the $\lambda$’s are the square roots of the eigenvalues of the non-Hermitian matrix $\rho \hat{\rho}$ with $\hat{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y)$ in decreasing order. In the space spanned by $\{\xi_i\}, i = 1, 2, 3, 4$, $\hat{\rho}$ reads

$$\hat{\rho} = \begin{pmatrix} \rho_{14} & -\rho_{34} & -\rho_{24} & \rho_{14} \\ -\rho_{34}^* & \rho_{33} & \rho_{23} & -\rho_{13} \\ -\rho_{24}^* & (\rho_{23})^* & \rho_{22} & -\rho_{12} \\ -\rho_{14}^* & -\rho_{13}^* & -\rho_{12}^* & \rho_{11} \end{pmatrix}.$$  \quad (3)

The Wootters concurrence gives an explicit expression for the entanglement of formation, which quantifies the resources needed to create a given entangled state. Note

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that the Wootters concurrence is a function of \( \rho_{ij}(i, j = 1, \ldots, 4, j \geq i) \). Therefore, we can write

\[
\Gamma(t) = \sum_{i,j=1}^{4} \frac{\partial E}{\partial \rho_{ij}} \frac{\partial \rho_{ij}}{\partial t}, \quad (\rho_{ij} \neq \rho_{44}).
\] (4)

Eq.(4) shows that given a particular entanglement measure \( E(\rho_{ij}) \), we just have to determine \( \partial \rho_{ij}/\partial t \). In order to do that, we need to find the time evolution of the state \( \rho(t) \) in the presence of an environment. Generally speaking, interactions between a quantum system and its environment result in two kinds of irreversible effects: dissipation and dephasing. The first effect is due to the energy exchange between the system and its environment, whereas the second one comes from the system-environment interaction that does not change the system energy. Both dissipation and dephasing lead to decoherence. In what follows, we first drive an expression for the time derivative of \( \rho(t) \) in terms of Kraus operators, this equation may be useful in the case of the Krause operators are easily given. Then we adapt the other description of decoherence to put forward our discussion. Consider a quantum system of two qubits \( \rho \) interacting with an environment \( \rho_E = \sum_{\nu} \rho_\nu |\nu\rangle \langle \nu| \), after a finite time evolution governed by unitary evolution operator \( U(t, 0) \), the total density operator (the system plus the environment) \( \rho(t) \) is given as

\[
\rho_E(t) = U(t, 0)(\rho_E \otimes \rho)U^\dagger(t, 0).
\] (5)

Taking a partial trace over environment variables we can get the density operator of the two-qubit system in the following form[6]

\[
\rho(t) = \text{Tr}_E \rho_E(t) = \sum_{\mu, \nu} K_{\mu\nu}(t, 0) \rho(0) K^\dagger_{\mu\nu}(t, 0),
\] (6)

where \( K_{\mu\nu}(t, 0) = |\mu\rangle \sqrt{p_\mu} U(t, 0) |\nu\rangle \). The Kraus operators \( K_{\mu\nu} \) satisfy \( \sum_{\mu\nu} K^\dagger_{\mu\nu} K_{\mu\nu} = 1 \). No environment around the two-qubit system indicates that there is only one term in the sum eq.(6). In weak system-environment interaction limit, the density operator of the environment \( \rho_E \) remains unchanged in the whole time evolution process, this approximation can be found in the derivation of the master equation, which we will discuss later on. Under weak system-environment interaction, the Kraus operators can be expanded to first order of \( dt \) as

\[
K_{\mu\nu}(t + dt, t) = |\mu\rangle \sqrt{p_\mu} U(t + dt, t) |\nu\rangle \approx \sqrt{p_\mu} \delta_{\mu\nu} - idt\sqrt{p_\mu} |\mu\rangle H_1 |\nu\rangle,
\] (7)

where \( H_1 \) stands for the Hamiltonian of the total system (two-qubit plus its environment), and this equation holds only for very small \( dt \). Substituting Eq.(7) into Eq.(6), we obtain \( \rho(t + dt) \) in terms of \( \rho(t) \) (to first order of \( dt \))

\[
\rho(t + dt) = \rho(t) - idt \sum_{\mu, \nu} \sqrt{p_\mu} |\mu\rangle H_1 |\nu\rangle \rho(t)
\]

\[
+ idt \rho(t) \sum_{\mu, \nu} \sqrt{p_\mu} |\mu\rangle H_1 |\nu\rangle
\]

\[
\equiv \rho(t) + d\rho(t).
\] (8)

In order to calculate the entanglement rate, we have to compute \( \partial \rho_{ij}/\partial t \). Using standard perturbation theory, we find \( \partial \rho_{ij}/\partial t \) as follows

\[
\frac{\partial \rho_{ij}}{\partial t} = (\xi_i |\partial \rho(t)/\partial t| \xi_j)
\] (9)

with

\[
\frac{\partial \rho(t)}{\partial t} = -i\sum_{\mu, \nu} \sqrt{p_\mu} |\mu\rangle H_1 |\nu\rangle, \rho(t)\]

Eq.(8) and Eq.(9) show that \( \sum_{\mu, \nu} \sqrt{p_\mu} |\mu\rangle H_1 |\nu\rangle \) play a role of effective Hamiltonian for the two-qubit system, in this sense we may rewrite Eq.(8) in the following form

\[
\frac{d\rho}{dt} = [H_e, \rho],
\] (10)

where \( H_e = \sum_{\mu, \nu} \sqrt{p_\mu} |\mu\rangle H_1 |\nu\rangle \). This expression is useful and easy to handle when we know the Kraus operators.

The other tool to study the quantum dissipative system is the master equation, which can be obtained in Markovian limit [7-9]. This approximation is very useful because it is valid for many physical relevant situations and its numerical solutions can be easily found. As given by Gardiner, Walls and Millburn, Louisell in their textbook [9], the reduced density matrix \( \rho \) of the open system which is linearly coupled to its environment obeys the following master equation of Lindblad form [10]

\[
\frac{\partial \rho}{\partial t} = -i[H_0, \rho] + \frac{1}{2} \sum_{m} K_m (2X_m^+ \rho X_m^+ - X_m^+ X_m \rho - \rho X_m^+ X_m^+) + \frac{1}{2} \sum_{m} G_m (2X_m^+ \rho X_m - X_m^+ X_m \rho - \rho X_m^+ X_m^+)
\] (11)

with

\[
K_m = 2\text{Re} \left[ \int_0^\infty d\tau e^{i\omega_m \tau} \text{Tr}_{\text{env}} \{ A_m(\tau) A_m^\dagger(0) \rho_{\text{env}} \} \right],
\]

\[
G_m = 2\text{Re} \left[ \int_0^\infty d\tau e^{i\omega_m \tau} \text{Tr}_{\text{env}} \{ A_m^\dagger(\tau) A_m(0) \rho_{\text{env}} \} \right].
\]

Here, \( \rho(t) = \rho(t, K_m, G_m) \) stands for the density operator of the system and \( \rho_{\text{env}} \) denotes the density operator of the environment. \( X_m^\pm \) are eigenoperators of the system satisfying \( [H_0, X_m^\pm] = \pm i\hbar \omega_m X_m^\pm \). \( H_0 \) stands for the free Hamiltonian of the system, and \( A_m(A_m^\dagger) \) are operators of the environment through which the system and
its environment couples together. Notice from Eq.(11) that \( G_m \) should vanish at zero temperature \( T = 0 \), while \( K_n \) should not if \( A_m \) are indeed destruction operators of some kind.

The time derivative of the matrix elements \( \rho_{ij} \) in this case is

\[
\frac{\partial \rho_{ij}}{\partial t} = \frac{\partial \rho_{ji}^*}{\partial t} = \langle \xi_i | \dot{\rho} | \xi_j \rangle. \tag{12}
\]

Further more, we consider a case of a two-qubit system coupling to environments that consists of a set of harmonic oscillators. In this case, the master equation takes the following form at zero temperature

\[
\dot{\rho}(t) = -i[H_s - \rho H_s] + \frac{\gamma}{2} \sum_{i=1}^{2} (\sigma^i \rho \sigma^i + \sigma^i \sigma^i \rho - \rho \sigma^i \sigma^i), \tag{13}
\]

where \( H_s \) is the system Hamiltonian, which governs time evolution of the two-qubit system in the absence of its environment, \( \sigma^i (\sigma^i) \) are pauli matrices, and \( \gamma \) represents the damping rate. Considering a system Hamiltonian

\[
H_s = \frac{\hbar \omega \sigma^i_1}{2} + \frac{\hbar \omega \sigma^i_2}{2} + \hbar g (\sigma^1 \sigma^2_1 + \sigma^1_2 \sigma^2), \tag{14}
\]

and substituting Eq.(13) into Eq.(12), we obtain(setting \( \hbar = 1 \))

\[
\begin{align*}
\rho_{11} &= \gamma (\rho_{22} + \rho_{33}), \\
\rho_{22} &= -ig\rho_{32} + ig \rho_{23} + \gamma \rho_{44} - \gamma \rho_{22}, \\
\rho_{33} &= -ig \rho_{23} + ig \rho_{32} + \gamma \rho_{44} - \gamma \rho_{33}, \\
\rho_{44} &= -2 \gamma \rho_{44}, \\
\rho_{12} &= ig \rho_{13} + \gamma \rho_{34} - 0.5 \gamma \rho_{12} + i \omega \rho_{12}, \\
\rho_{13} &= ig \rho_{12} + \gamma \rho_{24} - 0.5 \gamma \rho_{13} + i \omega \rho_{13}, \\
\rho_{14} &= -\gamma \rho_{14} + 2i \omega \rho_{14}, \\
\rho_{23} &= -ig \rho_{33} + ig \rho_{22} - \gamma \rho_{23}, \\
\rho_{24} &= -ig \rho_{34} - 1.5 \gamma \rho_{24} + i \omega \rho_{24}, \\
\rho_{34} &= -ig \rho_{24} - 1.5 \gamma \rho_{34} + i \omega \rho_{34} \\
\end{align*}
\tag{15}
\]

The Hamiltonian \( H_s(14) \) describes two two-level atoms with dipole-dipole interactions, which are a source of creating entanglement for trapped atoms in an optical lattice[11,12]. For an initial state, if the interaction terms(with coupling constant \( g \) in Eq.(15)) have no effects in the time evolution process, it(an example is given below) could not be used to create entangled state or to increase entanglement. Initial state with the form of

\[
\rho_0 = \begin{pmatrix}
\frac{c+d}{2} & 0 & 0 & \frac{d-c}{2} \\
0 & \frac{a+b}{2} & \frac{b-a}{2} & 0 \\
0 & \frac{b-a}{2} & \frac{a+b}{2} & 0 \\
\frac{d-c}{2} & 0 & 0 & \frac{c+d}{2}
\end{pmatrix}.
\tag{16}
\]

is a family of such initial states. In terms of Bell bases, eq.(16) can be written as

\[
\rho_0 = a|\psi^+\rangle\langle \psi^+ | + b|\psi^-\rangle\langle \psi^- | + c|\phi^+\rangle\langle \phi^+ | + d|\phi^-\rangle\langle \phi^- |.
\]

This is just the Werner state and any two-qubit entangled state can be expressed in this form by performing a random bilateral rotation on each shared pair[3]. We may use this kind of entangled state to demonstrate how it decoheres, although it could not be used to increase entanglement. We would like to mention that for a general mixed state characterized by 15 independent parameters, the problem becomes complicated. Hence we choose two special sets of initial states to get some insights into the formalism. In terms of Wootters concurrence, the entanglement measure for states(16) is

\[
E(t) = -x \log_2 x - (1 - x) \log_2 (1 - x), \tag{17}
\]

where \( x = (1 + \sqrt{1 - F^2})/2, \ F = a - b - c - d \) for \( a - b - c - d > 0 \), or 0 for \( a - b - c - d < 0 \), and we assume \( a > b, a > c, a > d \) without loss of generality.

It is easy to show that the entanglement rate defined by (1) takes the following form \( (a + b + c + d = 1) \)

\[
\Gamma(t) = \frac{\partial E(t)}{\partial a} \frac{\partial a}{\partial t}, \tag{18}
\]

with

\[
\frac{\partial E(t)}{\partial a} = (\log_2 x - \log_2 (1 - x)) \frac{F}{\sqrt{1 - F^2}}, \\
\frac{\partial a}{\partial t} = \frac{c + d}{2} \gamma - a \gamma. \tag{19}
\]

By definitions, \( F > 0 \) and \( x > \frac{1}{2} \), then \( \frac{\partial E}{\partial a} > 0 \). So, wether the entanglement increase or decrease for the initial states (16) only depends on \( \frac{\partial a}{\partial t} \). And \( \frac{\partial a}{\partial t} \) is always below zero. Therefore, we could not increase entanglement starting from the initial states (16). Because \( \frac{\partial E}{\partial a} \) does not depend on \( c \) and \( d \), the entanglement rate depends on \( c \) and \( d \) linearly, and with \( c + d \) increase, the entanglement decreases. Whereas the entanglement rate is inversely proportional to \( \gamma \). The dependence of \( \Gamma(t) \) on parameter \( a \) is shown in figure 1.

![FIG. 1. Dependence of the entanglement rate \( \Gamma \) on \( a \) with \( \gamma = 0.01 \) and \( c + d = 0.1 \).](image_url)
This figure shows that the larger the parameter \( a \), the smaller the entanglement rate. For a limit case \( a = 1 \), and \( c = d = b = 0 \) that corresponds to the maximally entangled state \( |\psi^+\rangle \), the entanglement rate takes its minimum over states (16).

In contrast to example eq.(16), we present here another kind of states

\[
\rho_0 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & p & q & 0 \\
0 & q^* & 1 - p & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\tag{20}
\]

This kind of state is of interest because entanglement contained in those states range from zero to one(maximally entangled state). And it is a typical family of states for a ring of entangled state. And it is a typical family of states for a ring of N qubits in a translation invariant quantum state[13].

The entanglement measure for this family of states has the same expression as Eq.(17) except for replacing \( F \) by \( G \)

\[
G = 2|q|.
\tag{21}
\]

The positivity of the state \( \rho_0 \) Eq.(20) require

\[
R = p^2 - p + |q|^2 \leq 0,
\tag{22}
\]

this indicates that \( |q| \leq \frac{1}{2} \) for all family of states (20). And for \( |q| = \frac{1}{2} \), there is only one value available for \( p \), i.e. \( p = \frac{1}{2} \). This is shown in Fig.2 which gives the dependence of \( R \) on \( p \) and \( |q| \).

The entanglement rate in this example is

\[
\Gamma(t) = \frac{\partial E(t)}{\partial p} \frac{\partial p}{\partial t} + \frac{\partial E(t)}{\partial q^R} \frac{\partial q^R}{\partial t} + \frac{\partial E(t)}{\partial q^I} \frac{\partial q^I}{\partial t},
\tag{23}
\]

where \( q^{R(I)} \) represents the real (imaginary) part of \( q \). Eq.(15) and (17) together give

\[
\frac{\partial E(t)}{\partial p} = 0,
\frac{\partial p}{\partial t} = -2gq^I + \frac{1}{2}\gamma(1 - 2p),
\frac{\partial E(t)}{\partial q^I} = (\log_{2y} - \log_{2(1 - y)}) \frac{G}{\sqrt{1 - G^2}} \frac{\partial G}{\partial q^I},
\frac{\partial q^I}{\partial t} = g(2p - 1) - \gamma q^I,
\frac{\partial E(t)}{\partial q^R} = (\log_{2y} - \log_{2(1 - y)}) \frac{G}{\sqrt{1 - G^2}} \frac{\partial G}{\partial q^R},
\frac{\partial q^R}{\partial t} = -\gamma q^R,
\frac{\partial G}{\partial q^{R(I)}} = \frac{4q^{R(I)}}{|q|},
\tag{24}
\]

where \( y = \frac{1}{2}(1 + \sqrt{1 - G^2}) \). Equations (23) (24) together give

\[
\Gamma(t) = \log_{2y} \frac{y}{1 - y} \frac{G}{\sqrt{1 - G^2}} \frac{4gq^I(2p - 1) - 4\gamma|q|^2}{|q|}
\]

This shows that there are competitions between entangling and decohering. If \( g/\gamma > |q|^2/(q^I(2p-1)) \), \( \Gamma(t) > 0 \), entanglement increases. Otherwise the entanglement decreases. In other words, in order to get a positive entanglement rate, the decoherence rate \( \gamma \) and the coupling constant \( g \) should satisfy condition \( g > \gamma|q|^2/(q^I(2p-1)) \)

For parameters \( p = 0.6, g = 0.2, \gamma = 0.01 \), the entanglement rate \( \Gamma \) versus \( q^I \) and \( q^R \) is illustrated in Fig.3.

FIG. 2. Dependence of the quantity \( R \) on \( p \) and \( |q| \), that shows which region of \( |q| \) and \( p \) are available for the state(20).

FIG. 3. Entanglement rate for the family of state (20) versus \( q^I \) and \( q^R \). The parameters chosen are \( p = 0.6, g = 0.2, \gamma = 0.01 \).

The maximum of the entanglement rate is 0.4 corresponding \( q^I = 0.5, q^R = 0.0 \), while the minimum of \( \Gamma \) is -0.1 at about \( q^I = 0, q^R = 0.5 \). Similarly, the maximum and minimum of the entanglement rate change with \( g \) and \( \gamma \), but \( q^{I(R)}_{\text{max(min)}} \) corresponding to \( \Gamma_{\text{max(min)}} \) do not.
In the end of this paper, we generalize the formulas derived above to the case of multilevel systems, we denote by \( \sigma_i \) the generators of the group \( SU(N) \) with \( N \) being the dimension of the Hilbert space of the system \( A \), and \( \tau_i \) the generators corresponding to system \( B \) with dimension \( M \). With this notation, we may write any density matrix of the composite system \( A \) and \( B \) in the following general form

\[
\rho_{AB} = \frac{1}{MN}(1 + \sum_i \alpha_i \sigma_i + \sum_j \beta_j \tau_j + \sum_{ij} \gamma_{ij} \sigma_i \otimes \tau_j).
\]

(25)

Here we choose \( \alpha_i \), \( \beta_i \) and \( \gamma_{ij} \) as the independent variables to characterize the state of system \( A \) plus \( B \). The entanglement measure \( E(\alpha_i, \beta_j, \gamma_{ij}) \) is thus a function of \( \alpha_i \), \( \beta_j \), and \( \gamma_{ij} \). It is natural to express the entanglement rate in the following way

\[
\Gamma(t) = \sum_i \frac{\partial E}{\partial \alpha_i} \frac{\partial \alpha_i}{\partial t} + \sum_i \frac{\partial E}{\partial \beta_i} \frac{\partial \beta_i}{\partial t} + \sum_{ij} \frac{\partial E}{\partial \gamma_{ij}} \frac{\partial \gamma_{ij}}{\partial t}.
\]

(26)

Given an entanglement measure \( E(\alpha_i, \beta_j, \gamma_{ij}) \), the derivatives \( \partial E/\partial \alpha_i \), \( \partial E/\partial \beta_j \) and \( \partial E/\partial \gamma_{ij} \) can be easily calculated. The remained task is only to determine the time derivative of \( \alpha_i \), \( \beta_j \), and \( \gamma_{ij} \). Proceeding as before, we obtain

\[
\frac{\partial \alpha_i}{\partial t} = \text{Tr}_A(\sigma_i \frac{\partial \rho_{AB}(t)}{\partial t}), \quad \frac{\partial \beta_j}{\partial t} = \text{Tr}_B(\tau_j \frac{\partial \rho_{AB}(t)}{\partial t}), \quad \frac{\partial \gamma_{ij}}{\partial t} = \text{Tr}_{AB}(\sigma_i \frac{\partial \rho_{AB}(t)}{\partial t} \tau_j).
\]

(27)

Where \( \partial \rho_{AB}/\partial t \) has a similar expression with Eq.(10) or Eq.(11), depending on what formalism you choose to describe the time evolution of the system.

In summary, taking the decoherence effects into account, we study dynamics of the entanglement rate for two special sets of initial states. The interaction under consideration is of pairwise. The results show that there are competitions between decohering and entangling, those competitions lead to (1). For a specific interaction and a decoherence scheme, some initial state could not be used to prepare entanglement. (2). Some initial states can be used to prepare or increase entanglement under a proper choice of the parameters.

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