On a curious bias arising when the $\sqrt{\chi^2/\nu}$ scaling prescription is first applied to a sub-sample of the individual results

Giulio D’Agostini
Universit` a “La Sapienza” and INFN, Roma, Italia
(giulio.dagostini@roma1.infn.it, http://www.roma1.infn.it/~dagos)

Abstract

As it is well known, the standard deviation of a weighted average depends only on the individual standard deviations, but not on the dispersion of the values around the mean. This property leads sometimes to the embarrassing situation in which the combined result ‘looks’ somehow at odds with the individual ones. A practical way to cure the problem is to enlarge the resulting standard deviation by the $\sqrt{\chi^2/\nu}$ scaling, a prescription employed with arbitrary criteria on when to apply it and which individual results to use in the combination. But the ‘apparent’ discrepancy between the combined result and the individual ones often remains. Moreover this rule does not affect the resulting ‘best value’, even if the pattern of the individual results is highly skewed. In addition to these reasons of dissatisfaction, shared by many practitioners, the method causes another issue, recently noted on the published measurements of the charged kaon mass. It happens in fact that, if the prescription is applied twice, i.e. first to a sub-sample of the individual results and subsequently to the entire sample, then a bias on the result of the overall combination is introduced. The reason is that the prescription does not guarantee statistical sufficiency, whose importance is reminded in this script, written with a didactic spirit, with some historical notes and with a language to which most physicists are accustomed. The conclusion contains general remarks on the effective presentation of the experimental findings and a pertinent puzzle is proposed in the Appendix.

“Observations, for example, such as are distant from each other by an interval of a few days […] are not to be used in the calculation as so many different positions, but it would be better to derive from them a single place, which would be, as it were, a mean among all, admitting, therefore, much greater accuracy than single observations considered separately.”

(F.C. Gauss, transl. by C.H. Davies)
1 Introduction

As reminded by Gauss’ opening quote, the reason why we combine “single observations” is to have an equivalent one “of greater accuracy than the single observations”. In fact, accidental errors tend to cancel if the observations are independent, and small systematic errors do too, if the measurements are performed using different devices and the data are analyzed by different methods. The simplest combination of the individual results is the arithmetic mean. But...— and let speak Gauss again [1]

“But if it seems that the same degree of accuracy cannot be attributed to the several observations, let us assume that the degree of accuracy in each may be considered proportional to the numbers \(e, e', e'', e'''\), etc. respectively, that is, that errors reciprocally proportional to these numbers could have been made in the observations with equal facility; the, according to the principles to be propounded below, the most probable mean value will no longer be the simple arithmetic mean, but

\[
\frac{ee\delta + e'e'\delta' + e''e''\delta'' + e'''e'''\delta''' + \text{etc.}}{ee + e'e' + e''e'' + e'''e''' + \text{etc.}},
\]

that is, in modern notation, \((\sum_i e_i^2 \delta_i) / (\sum_i e_i^2)\), in which we recognize the well known weighted average, provided \(e_i = 1/\sigma_i\). Then, later on (the conversion to modern notation is now straightforward),

“The degree of precision to be assigned to the mean found as above will be [...]

\[
\sqrt{(ee + e'e' + e''e'' + e'''e''' + \text{etc.)}};
\]

so that four or nine equally exact observations are required, if the mean is to possess a double or a triple accuracy.”

The advantage of having ‘distilled’ the many observations into a single, equivalent one is that further calculations are simplified, or made feasible at all, especially in the absence of powerful computers. But this idea works if there is no, or little, loss of information\(^2\). This leads us to the important concept of statistical sufficiency, which will be reminded in Sec. 2 for the well understood case of Gaussian errors.

\[^1\] “It is of the greatest importance, that the several positions of the heavenly body on which it is proposed to base the orbit, should not be taken from single observations, but, if possible, from several so combined that the accidental errors might, as far as may be, mutually destroy each other.” [1]

\[^2\] For example Gauss discusses the implications of averaging observations over days or weeks, during which the heavenly body has certainly changed position in the elapsed time. But the mean position in the mean time can be considered as an equivalent point in space and time, and a few of them, far apart, would be enough to determine the orbit parameters.
There is then the question of what to do in the case in which the individual results ‘appear’ to be in mutual disagreement. The reason of the quote marks has been discussed in Ref. [2], of which this work is a kind of appendix. In short, they are a reminder, if needed, of the fact that, rigorously speaking, we can never be absolute sure that the ‘discrepancies’ are not just due to statistical fluctuations. A way to implement our doubts has been shown in Ref. [2] and it consists in modifying the probabilistic model relating causes (the parameters of the model, first and foremost the ‘true’ value of the quantity we aim to infer, although with uncertainty) and effects (the empirical observations), adding some extra-causes (additional parameters of the model) which might affect the observations. All quantities of interest are then embedded in a kind of network, a graphical representation of which can be very helpful to grasp its features.3

Traditionally, at least in particle physics, a different approach is followed. The degree of disagreement is quantified by the $\chi^2$ of the differences between the individual results and the weighted average. Then, possibly, the standard deviation of the weighted mean is enlarged by a factor $\sqrt{\chi^2/\nu}$. The rationale of the prescription, as a fast and dirty rule to get an rough idea of the range of the possible values of the quantity of interest, is easy to understand. However, there are several reasons of dissatisfaction, as discussed later on in Sec. 4 and therefore this simplistic scaling should be used with some care, and definitely avoided in those cases in which the outcome is critical for fundamental physics issues.4 Moreover, it will be shown how the outcome can be biased if the prescription is first applied to a sub-sample of the individual results and, subsequently, the partial result is combined with the remaining ones using the same rule. The conclusions will also contain some general considerations on how an experimental result should be presented in order to use it at best a) to confront it with theory; b) to combine it with other results concerning the same physics quantity; c) to ‘propagate’ it into other quantities of interest.

Finally, a puzzle is proposed in the Appendix in order to show that independent Gaussian errors are not a sufficient condition for the standard weighted average.

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3The network of all quantities involved in the model is also known as Bayesian network for two reasons: degrees of belief are assigned to all variables in the game (even to the observed ones, meant as conditional probabilities depending on the value of the others); inference and forecasting (the names are related to the purposes of our analysis – from the probabilistic point of view there is no difference) are then made by the use of probability rules, in particular the so called Bayes’ rule, without the need to invent prescriptions or ad hoc ‘principles’.

4This is not the case, at the moment, for the charged kaon mass, as commented in Ref. [2], especially footnote 19.
Figure 1: Graphical model behind the standard combination, assuming independent measurements of the same quantity, each characterized by a Gaussian error function with standard deviation $\sigma_i$.

2 From a sample of individual observations to a couple of numbers: the role of statistical sufficiency

Let us restart from the Eq. (5) of Ref. [2], based on the graphical model in Fig. 5 of the same paper, reproduced here for the reader’s convenience as Eq. (1) and Fig. 1.

$$f(\mathbf{x}, \mu, |\sigma) = \prod_i f(x_i | \mu, \sigma_i) \cdot f_0(\mu)$$ (1)

is the joint probability density function (pdf) of all the quantities of interest, with $\mathbf{x} = \{x_1, x_2, \ldots\}$. The standard deviations $\sigma = \{\sigma_1, \sigma_2, \ldots\}$ are instead considered just conditions of the problem. The pdf $f_0(\mu)$ models our prior beliefs about the ‘true’ value of the quantity of interest (see Ref. [2] for details, in particular footnote 9). The pdf of $\mu$, also conditioned on $\mathbf{x}$, is then, in virtue of a well known theorem of probability theory,

$$f(\mu | \mathbf{x}, \sigma) = \frac{f(\mathbf{x}, \mu, |\sigma)}{f(\mathbf{x} | \sigma)}.$$ (2)
Noting that, given the model and the observed values $x$, the denominator is just a number, although in general not easy to calculate, and making use of Eq. (1), we get

$$f(\mu \mid x, \sigma) \propto \left[ \prod_i f(x_i \mid \mu, \sigma_i) \right] \cdot f_0(\mu)$$

Speaking in terms of *likelihood*, and ignoring multiplicative factors\(^5\), we can rewrite the previous equation as

$$f(\mu \mid x, \sigma) \propto \left[ \prod_i \mathcal{L}(\mu \mid x_i, \sigma_i) \right] \cdot f_0(\mu),$$

that is, indeed, the particular case, valid for independent observations $x_i$, of the more general form

$$f(\mu \mid x, \sigma) \propto \mathcal{L}(\mu \mid x, \sigma) \cdot f_0(\mu),$$

since, under condition of independence, $\mathcal{L}(\mu \mid x, \sigma) = \prod_i \mathcal{L}(\mu \mid x_i, \sigma_i)$.

- The inference depends on the product of likelihood and prior (note ‘;’ instead of ‘|’ in the notation, to remind that in ‘conventional statistics’ $\mathcal{L}$ is simply a mathematical function of $\mu$, with parameters $x$ and $\sigma$);

- if the prior is ‘flat’\(^6\) then the inference is determined by the likelihood,

$$f(\mu \mid x, \sigma) \propto \mathcal{L}(\mu \mid x, \sigma).$$

In particular, the most probable value\(^7\) (‘mode’) of $\mu$ is the value which maximizes the likelihood\(^8\).

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\(^5\)This is related to the so called *Likelihood Principle*, which is consider a good feature by frequentists, though not all frequentistic methods respect it\(^4\). In practice, it says that the result of an inference should not depend on multiplicative factors of the likelihood functions. This ‘principle’ arises automatically in a probabilistic (‘Bayesian’) framework.

\(^6\)I refer again to footnote 9 of Ref.\(^2\), reminding that the *Princeps Mathematicorum* derived the ‘Gaussian’ as the error function such that the maximum of the posterior for a flat ‘prior’ (explicitly stated) had a maximum corresponding to the arithmetic average of the observations, provided they were independent and they had “the same level of accuracy”.

\(^7\)Citing again Gauss, he wrote explicitly of most probable value of ‘$\mu$’, under the hypothesis that before the experiment all its values were equally probable\(^1\).

\(^8\)Hence the famous *Maximum likelihood* ‘principle’, which is then nothing but a simple case of the more general probabilistic approach. (But the value that maximizes the likelihood is not necessarily the best value to report, as it will be commented in the conclusions.)
in the case of independent Gaussian error functions the likelihood can be rewritten, besides multiplicative factors, as

\[ \mathcal{L}(\mu; \bar{x}, \sigma) = \prod_i \mathcal{L}(\mu; x_i, \sigma_i) \propto \prod_i \exp \left[ -\frac{(x_i - \mu)^2}{2 \sigma_i^2} \right] \]

\[ \propto \exp \left[ -\sum_i \frac{(x_i - \mu)^2}{2 \sigma_i^2} \right] \]

\[ \propto \exp \left[ -\chi^2 / 2 \right] \quad (3) \]

having recognized the sum in the exponent as \( \chi^2 = \sum_i (x_i - \mu)^2 / \sigma_i^2 \). Under the hypotheses and approximations of this model the most probable value of \( \mu \) can then also be obtained by minimizing \( \chi^2 \).\(^9\)

- going through the steps from Eqs. (7)-(12) of Ref. [2] and, under the assumptions stated in the previous items, we can further rewrite the Eq. (3) as

\[ \exp \left[ -\sum_i \frac{(x_i - \mu)^2}{2 \sigma_i^2} \right] \propto \exp \left[ -\frac{(\mu - \bar{x})^2}{2 \sigma_C^2} \right], \quad (4) \]

where

\[ \bar{x} = \frac{\sum_i x_i / \sigma_i^2}{\sum_i 1 / \sigma_i^2} \]

\[ \frac{1}{\sigma_C^2} = \sum_i \frac{1}{\sigma_i^2}, \quad (5) \]

in which we recognize Gauss’ Eqs. (G1) and (G2). In terms of likelihoods,

\[ \mathcal{L}(\mu; \bar{x}, \sigma) \propto \prod_i \mathcal{L}(\mu; x_i, \sigma_i) \propto \mathcal{L}(\mu; \bar{x}, \sigma_C), \quad (7) \]

Equation (7) is an important result, related to the concept of **statistical sufficiency**: the inference is exactly the same if, instead of using the detailed information provided by \( x \) and \( \sigma \), we just use the weighted mean \( \bar{x} \) and its standard deviation \( \sigma_C \), as if \( \bar{x} \) were a single equivalent observation of \( \mu \) with a Gaussian error function with “degree of accuracy” \( 1 / \sigma_C \) – this is exactly the result Gauss was aiming in Book 2, Section 3 of Ref. [1], reminded in the opening quote and in the introduction of Ref. [2].

\(^9\)And it is easy to recognize, in this sub-case of the general probabilistic approach, another famous ‘principle’.
Moreover we can split the sum of Eq. (3) in two contributions, from \( i = 1 \) to \( m \) (arbitrary) and from \( m + 1 \) to \( n \), thus having

\[
\exp \left[ -\sum_i (x_i - \mu)^2 / 2 \sigma_i^2 \right] = \exp \left[ -\sum_{i=1}^m (x_i - \mu)^2 / 2 \sigma_i^2 - \sum_{i=m+1}^n (x_i - \mu)^2 / 2 \sigma_i^2 \right].
\]

Going again through the steps from Eq. (7) to Eq. (12) of Ref. [2] we get

\[
\exp \left[ -\sum_i (x_i - \mu)^2 / 2 \sigma_i^2 \right] \propto \exp \left[ -2\overline{x}_A \mu + \mu^2 / 2 \sigma_{C_A}^2 \right] \cdot \exp \left[ -2\overline{x}_B \mu + \mu^2 / 2 \sigma_{C_B}^2 \right]
\]

where

\[
\overline{x}_A = \frac{\sum_{i=1}^m x_i / \sigma_i^2}{\sum_{i=1}^m 1 / \sigma_i^2},
\]

\[
\sigma_{C_A}^2 = \frac{1}{\sum_{i=1}^m 1 / \sigma_i^2}.
\]

\[
\overline{x}_B = \frac{\sum_{i=m+1}^n x_i / \sigma_i^2}{\sum_{i=m+1}^n 1 / \sigma_i^2},
\]

\[
\sigma_{C_B}^2 = \frac{1}{\sum_{i=m+1}^n 1 / \sigma_i^2}.
\]

It follows, writing the right hand side as product of exponentials and complementing each of them [2],

\[
\exp \left[ -(\overline{x} - \mu)^2 / 2 \sigma_C^2 \right] \propto \exp \left[ -(\overline{x}_A - \mu)^2 / 2 \sigma_{C_A}^2 \right] \cdot \exp \left[ -(\overline{x}_B - \mu)^2 / 2 \sigma_{C_B}^2 \right]
\]

\[
\propto \exp \left[ -(\overline{x}_A - \mu)^2 / 2 \sigma_{C_A}^2 - (\overline{x}_B - \mu)^2 / 2 \sigma_{C_B}^2 \right],
\]

that is, in terms of likelihoods,

\[
\mathcal{L}(\mu; \overline{x}, \sigma_C) \propto \mathcal{L}(\mu; \overline{x}_A, \sigma_{C_A}) \cdot \mathcal{L}(\mu; \overline{x}_B, \sigma_{C_B})
\]

\[
\propto \mathcal{L}(\mu; \overline{x}_A, \sigma_{C_A}, \overline{x}_B, \sigma_{C_B})
\]

\[
\propto \mathcal{L}(\mu; \overline{x}_A, \sigma_{C_A}, \overline{x}_B, \sigma_{C_B})
\]

The result can be extended to averages of averages, that is

\[
\mathcal{L}(\mu; \overline{x}_A, \sigma_{C_A}, \overline{x}_B, \sigma_{C_B}) \propto \mathcal{L}(\mu; \overline{x}^{(G)}, \sigma_{C}^{(G)})
\]

7
where

\[ x^{(G)} = \frac{\bar{x}_A/\sigma_{C_A}^2 + \bar{x}_B/\sigma_{C_B}^2}{1/\sigma_{C_A}^2 + 1/\sigma_{C_B}^2} \]  
(17)

\[ \frac{1}{\sigma_{C}^2} = \frac{1}{\sigma_{C_A}^2} + \frac{1}{\sigma_{C_B}^2}. \]  
(18)

The property can be extended further to many partial averages, showing that the inference does not depend on whether we use the individual observations, their weighted average or even the grouped weighted averages, or the weighted average of the grouped averages. This is one of the ‘amazing’ properties of the Gaussian distribution, which simplifies our work when it is possible to use it. But there no guarantee that it works in general, and it should be then proved case by case.

3 Possibly discrepant results

Well known features of the combination of measurements by the weighted average, reminded in the last section, are that i) the combined result has a “degree of precision” \[1\] higher than each of the individual contributions or, in terms of standard deviations, \(\sigma_C < \sigma_i\); ii) the resulting standard deviation does not depend on the spread of the individual results around the mean value; iii) the error model of the ‘equivalent observation’ remains Gaussian. However, it is a matter of fact that, although from the probabilistic point of view there is no contradiction with the basic assumptions, since patterns of individual results ‘oddly’ scattered around their average have some chance to occur, we sometimes suspect that there it might be something ‘odd’ going on. That is, we tend to doubt on the validity of the simple model of Gaussian errors with the declared “degrees of precision”. (But someone might start to worry too early, sometimes even driven by wishful thinking, that is she hopes, rather than believes, that the reason of disagreement might be caused by new phenomenology or violation of fundamental laws of physics \[8, 9\].)

In the case we have serious suspicions about the presence of other effects, then we should change our model, make a new analysis and accept its outcome in the light of clearly stated hypotheses and conditions \[2, 3, 4\]. As a result, not only the overall

\footnote{It is rather well known that the human mind has problems when dealing with randomness. For example, if you ask a person to write, at random, a long list of 0’s and 1’s, she will tend to ‘regularize’ the series, which will then contain only short sequences of 0’s and 1’s, contrary to what happens rolling a coin, or using a (pseudo-)random generator. As an interesting book that discusses, among others, this experimental fact, Ref. \[12\] is recommended.}
Figure 2: Charged kaon mass from several experiments as summarized by the PDG [7]. Note that besides the ‘error’ of 0.013 MeV, obtained by a $\times 2.4$ scaling, also an ‘error’ of 0.016 MeV is provided, obtained by a $\times 2.8$ scaling. The two results are called ‘OUR AVERAGE’ and ‘OUR FIT’, respectively [7].

‘error’ should change, but also the shape of the final distribution should, since there is no strong reason to remain Gaussian. For example, the final distribution might be skewed or even multimodal [2], as it should be desirable if the pattern of individual measurements suggest so. In particular, the most probable value (mode) will differ from the average of the distribution and from the median. Instead, traditionally, only the ‘error’ is enlarged by an arbitrary factor depending on the frequentistic ‘test variable’ $\chi^2$, namely $\sqrt{\chi^2/\nu}$, where $\nu$ stands for the number of degrees of freedom. But the central value is kept unchanged and the interpretation of the result, explicitly stated or implicitly assumed so in subsequent analyses by other scientists, remains Gaussian.

11The quote marks are to remind that they refer, more precisely, to standard uncertainty, referring the nouns error and uncertainty to different concepts [10, 11].

12For example, the bimodal curve shown in the ideogram of Fig. 2 is a curious linear combination of Gaussians of $m_{K^\pm}$, although from a frequentistic point of view one should not be allowed to attribute probabilities, and hence pdf’s, to true values. And there are even frequentistic ‘gurus’ who use probability in quote marks, without explaining the reason but because they are aware
| i  | Authors                        | pub. year | central value $[d_i]$ (MeV) | uncertainty $[s_i]$ (MeV) |
|----|--------------------------------|-----------|-----------------------------|---------------------------|
| 1  | G. Backenstoss et al. [14]     | 1973      | 493.691                     | 0.040                     |
| 2  | S.C. Cheng et al. [15]         | 1975      | 493.657                     | 0.020                     |
| 3  | L.M. Barkov et al. [16]        | 1979      | 493.670                     | 0.029                     |
| 4  | G.K. Lum et al. [17]           | 1981      | 493.640                     | 0.054                     |
| 5  | K.P. Gall et al. [18]          | 1988      | 493.636                     | 0.011 (*)                 |
| 6  | A.S. Denisov et al. [19]       | 1991      | 493.696                     | [0.0059]                  |
|    | & Yu.M. Ivanov [20]            | 1992      | [same]                      | 0.007 (**)                |

Table 1: Experimental values of the charged kaon mass used in the numerical example, limited to those taken into account by the 2019 issue of PDG [7]. [(*) ‘error’ already scaled by a factor ×1.52 due to the $\sqrt{\chi^2/\nu}$ prescription (see text). (**) Value accepted by the PDG.]

As a practical example, let us take the results concerning the charged kaon mass of Fig. 2 and Tab. 1, as selected by the PDG [7]. From the weighted average and its standard deviation we get $493.6766 \pm 0.0055$ MeV, shown in Fig. 3 by the dashed red Gaussian (the solid blue Gaussians depict the results of the six results of Tab. 1. Comparing the individual results with the weighted average we calculate a $\chi^2$ of 22.9, and hence a scaling factors of 2.14, getting then $493.677 \pm 0.012$ MeV, reported on the same figure by the solid gray Gaussian below the dashed one. The two results are reported also in the entry A of the summary table 3.

4 Reasons of dissatisfaction with the $\chi^2$-motivated scaling prescription

As we see in Fig. 3, the Gaussian widened by the $\sqrt{\chi^2/\nu}$ prescription does not capture the picture offered by the ensemble of the individual results. In fact, the mass values preferred by the combined result are still distributed symmetrically around the weighted average.

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13 Besides rounding, the numbers slightly differ from those of Ref. [7], having applied there some ad hoc selections. But this does not change the essence of the message this note desires to convey.

14 Then the PDG adds a more intriguing curve in the reported ‘ideograms’ [7] (see e.g. Fig. 1 of Ref. [2]). But this non-Gaussian curve has no probabilistic meaning, as discussed in [2], and, anyway, it is not used to draw no quantitative results, as far as I understand (and I hope...).
Figure 3: Graphical representation of the results on the charged kaon mass of Tab. 1 (solid blue Gaussians). The dashed red Gaussian shows the result of the standard combination obtained by the weighted average. The solid gray Gaussian, centered with the dashed red one, shows the broadening due to the $\sqrt{\chi^2/\nu}$ prescription (see text).

More in general, the scaling factor is at least suspect. This is because it is well known that the $\chi^2$ distribution does not scale with $\nu$ and therefore, while a $\chi/\nu = 2$, for example, is quite in the norm for $\nu$ equal to 2, 3 or 4 (even a strict frequentist would admit that the resulting p-values of 0.14, 0.11 and 0.09, respectively, are nothing to worry), things get different for $\nu$ equal to 10, 20 or 30 (p-values of 0.029, 0.005 and 0.0009, respectively). Moreover, I am not aware of cases in which the standard deviation of the weighted average was scaled down, in the case that $\sqrt{\chi^2/\nu}$ was smaller than one.\footnote{It seems that people are worried only if the ‘errors’ appear small. But, as also stated by the ISO Guide\cite{10}, “Uncertainties in measurements” should be “realistic rather than safe”. In particular the method recommended by the ISO Guide “stands […] in contrast to certain older methods that have the following two ideas in common:}

- The first idea is that the uncertainty reported should be ‘safe’ or ‘conservative […] In fact, because the evaluation of the uncertainty of a measurement result is problematic, it was often made deliberately large. […]”
Table 2: Individual results reported by [18], together with their combination.

| Authors                   | pub. year | \([d_i] \) (MeV) | \([s_i] \) (MeV) |
|---------------------------|-----------|------------------|------------------|
| K.P. Gall et al. [18]     | 1988      | 493.675          | 0.026            |
|                           |           | 493.631          | 0.007            |
|                           |           | 493.806          | 0.095            |
|                           |           | 493.709          | 0.073            |
|                           |           | 493.636          | 0.011            |

Table 3: Combinations of the individual results of Tabs. 1 and 2. The subscript S means that the \(\sqrt{\chi^2/\nu}\) scaling prescription has been applied to standard deviation of the weighted average. In particular, note that \([1,2,3,4,5,6]_S\) is the same as \([1,2,3,4,\{5_a,5_b,5_c,5_d\}_S,6]_S\).

| data set                        | \(m_{K^+}/\text{MeV}\)  |
|---------------------------------|-------------------------|
| A) \([1,2,3,4,5,6]\) \(\{1,2,3,4,5,6\}_S\) | 493.6766 ± 0.0055       |
|                                 | \textbf{493.677} ± 0.12 |
| B) \([5_a,5_b,5_c,5_d]\) \(\{5_a,5_b,5_c,5_d\}_S\) | 493.6355 ± 0.0067       |
|                                 | 493.636 ± 0.010         |
| C) \([1,2,3,4,5_a,5_b,5_c,5_d,6]\) \(\{1,2,3,4,5_a,5_b,5_c,5_d,6\}_S\) | 493.6644 ± 0.0046       |
|                                 | \textbf{493.664} ± 0.011 |
| D) \([1,2,3,4,5_b,5_c,5_d]\) \(\{1,2,3,4,5_a,5_b,5_c,5_d\}_S\) | 493.6404 ± 0.0061       |
|                                 | 493.6404 ± 0.0076       |
| E) \([\{1,2,3,4,5_a,5_b,5_c,5_d\}_S,6\] \(\{1,2,3,4,5_a,5_b,5_c,5_d\}_S,6\) | 493.6705 ± 0.0051       |
|                                 | 493.671 ± 0.028         |

But there is another subtle issue with the method, which I have realized only very recently, going through the details of the charged kaon mass measurements: if the prescription is applied to a sub-sample of results and then to all them (taking for the sub-sample weighted average and scaled standard deviation), then a bias is introduced in the final result with respect to when all results were taken individually. This is because the summary provided by such a prescription is not a sufficient statistics.

The lowest, high precision mass value of 493.636 ± 0.011 (see Tab. 1 and Fig. 3) come in fact from the combination, done directly by the experimental team [18] applying the \(\sqrt{\chi^2/\nu}\) prescription. Without this scaling, the four individual results, reported in Tab. 2, had given a weighted average of 493.6355 ± 0.0067 MeV, with a \(\chi^2\) of 7.0. Now it is true that \(\chi^2/\nu\) is equal to 2.32, but this is not a reason to worry, being \(\nu = 3\). In fact the \(p\)-value, calculated as \(P(\chi^2|\nu = 3) > 7.0\), is 0.073, that is even above the (in-)famous 0.05 threshold [9].

Nevertheless, if we apply to the standard deviation a scaling factor of \(\sqrt{2.32} = \)
1.52, then we get 493.636 ± 0.010 MeV (the difference between this value of 0.010 MeV and 0.011 MeV of Tabs. 1 and 2 could be just due to rounding of the individual values). The result is shown in Fig. 4 together with the individual results that enter the analysis (see also entry B of the summary table 3).

It is interesting to see what we get if we use the nine individual points, i.e. 1, 2, 3, 4 and 6 of Tab. 1, together with 5\textsubscript{a}, 5\textsubscript{b}, 5\textsubscript{c} and 5\textsubscript{d} of Tab. 2. The combined weighted average, shown in Fig. 5, comes out right in the middle of the two most precise results, with little overlap with them. The average is 493.6644 MeV, with standard deviation 0.0046 MeV, which becomes 0.011 MeV after the $\chi^2$ motivated scaling\textsuperscript{16} of

\begin{align*}
\text{Fig. 4: Individual results of Ref. [18] (cyan solid lines), with the weighted average with and without } \sqrt{\chi^2/\nu} \text{ scaling factor (same graphic notation of Fig. 3).}
\end{align*}

\textsuperscript{16}The shift of the average almost in the middle of the two most precise results makes the contributions to the $\chi^2$ huge. Here are the differences of the individual measurements in unit of their standard deviations: 0.67, −0.37, 0.19, −0.45, 0.41, −4.77, 1.49, 0.61, 4.52, resulting in a $\chi^2$ of 46.7 and a consequent p-value of $1.7 \times 10^{-7}$. Therefore, also to whom who are critical against p-values [8, 9], in a case of this kind, an alarm bell should sound [21]. But to all persons of good sense a similar alarm, concerning the frequentist solution of the problem, should sound too (see
Figure 5: Combination of the individual results of Ref. [18], together with the other results (i.e. excluding nr. 5) of Tab. 1. For details of the graphic notation see the previous figures.

$\times 2.42$. As we can see, the central value differs by $-12$ keV with respect from the one obtained above [see also section. 3 and entry C of the summary table 3]: the use of the pre-combined result of Ref. [18] produces a bias of $+12$ keV in the final result, that is comparable with the quoted ‘error’. The reason is due to the fact that the $\sqrt{\chi^2/\nu}$ prescription used to enlarge the standard deviation does not hold sufficiency. As a consequence, the relevance of the ensemble of results of Ref. [18] gets reduced.

As a further example to show this effect on the same data, let us make the academic exercise of grouping the data in a different way. For example we first combine all results published before year 1990 (1-4,5a-5d, with references to Tabs. 1 and 2, and include the most recent one (6 of Tab. 1) in a second step. The outcome of the exercise is reported in Fig. 6 and in the entries D and E of the summary table 3. The weighted average of the eight results before year 1990 (upper plot of Fig. 6)

Ref. [2] for a sceptical alternative.)
Figure 6: Example of arbitrary grouping of the results in before and after year 1990 (see text).
and entry D in Tab. 3) gives \( m_{K^\pm} = 493.6404 \pm 0.0061 \text{ MeV} \) (dashed red line). The \( \chi^2 \) is equal to 10.8, producing a scaling factor of 1.24 and thus a modified result of \( m_{K^\pm} = 493.6404 \pm 0.0076 \text{ MeV} \) (solid brown line of Fig. 6 and entry D in Tab. 3).

Combining this outcome with the 1991 result [19, 20] we get (lower plot of Fig. 6 and entry E in Tab. 3) a weighted average of \( m_{K^\pm} = 493.6705 \pm 0.0051 \text{ MeV} \), but with the very large \( \chi^2 \) of 29 (p-value \( 0.74 \times 10^{-7} \)), thus yielding a \( \times 5.4 \) scaling factor and then a widened standard deviation of 28 keV. At least, contrary to the previous cases, this time the scaled standard deviation is able to cover both individual results, although an experienced physicist would suspect that most likely only one of the two is correct. (In situations of this kind a ‘sceptical analysis’ would result in a bimodal distribution, as shown in Fig. 4 of Ref. [3].)

5 Conclusions

It is self-evident that the prescription of enlarging the standard deviation of a weighted average by a \( \chi^2 \) motivated scaling factor is often not able to capture the pattern of individual results, in the case they show a sizable skewed distribution or they tend to cluster in two different regions. Instead, I was not aware of the fact that the \( \sqrt{\chi^2/\nu} \) scaling might produce a bias, with respect to the average of all individual results, if the scaling is first applied to a sub-sample of results, and then to all of them [2]. This is due to the fact that the procedure does not hold statistical sufficiency and, therefore, individual results should be used without pre-grouping.

However, this does not imply that the latter is the correct way to proceed in the case the pattern of the individual results is at odds with the weighted average applied to all points. A more pondered analysis should rather be performed in order to model our doubts, as done e.g. in Ref. [2]. (In the case of the charged kaon mass, there is however a curious compensation, such that the biased result comes out to agree, at least in terms of central value and ‘error’, with that of the ‘sceptical analysis’ [2].)

I would like to conclude with some remarks concerning how to report an experimental result, in perspective of its further uses. In fact, a result is not an end in itself, as Physics and all Sciences are not just collections of facts (and even an experimental result is not a mere ‘fact’, since it is derived from many empirical observations through models relying on a web of beliefs\(^{17}\)).

Focusing on pure science, results are finally confronted with theoretical evaluations (not strictly ‘predictions’) in order to classify in degree of belief the possible models describing how ‘the World works’ (note that the acclaimed Popperian falsification is

\(^{17}\) As the historian of science Peter Galison puts it, “Experiments begin and end in a matrix of beliefs. . . . beliefs in instrument type, in programs of experiment inquiry, in the trained, individual judgments about every local behavior of pieces of apparatus.” [22].
an idealistic scheme that however seldom applies in practice\textsuperscript{[23, 24]). But in order to achieve the best selective power, individual results are combined together, as we have seen in this note. Moreover a result could be propagated into other evaluations, as it is, itself, practically always based on other results, since it depends on quantities which enter the theoretical model(s) on which it relies (‘principles of measurement’\textsuperscript{[10]), including those who govern the “pieces of apparatus”, as reminded in footnote\textsuperscript{17}).

Therefore, it is important to provide, as outcome of an experimental investigation, something that can be used at best, even after years, for comparison, combination and propagation. Fortunately there is something on which there is universal consensus: the most complete information resulting from the the empirical findings, concerning a quantity that can assume values with continuity, is the so called likelihood function\textsuperscript{18}. In fact, in the case of independent experiments reporting evidence on the same physics quantity the rule of the combination is straightforward, as it results from probability theory, without the need of ad hoc prescriptions: just multiply the individual likelihoods. It follows then that the likelihood (or its negative log) should be described at best in a publication, as for example done in Ref.\textsuperscript{[27], in which several negative log-likelihoods were shown in figures and parameterized around their minimum by suitable polynomials.

Reducing the detailed information provided by the likelihood in a couple of numbers does not provide, in general, an effective and unbiased way to report the result of the findings, unless the likelihood is, with some degree of approximation, Gaussian. Instead, if the likelihood is not Gaussian [or the $\chi^2$ is not parabolic, in those cases in which the likelihood can be rewritten as $\exp(-\chi^2/2)$], then reporting the value that maximize it, with an ‘error’ related to the curvature of its negative log at the minimum, or ‘asymmetric errors’ derived from a prescription that is only justified for a Gaussian likelihood, is also an inappropriate way of reporting the information contained in the findings. This is because, when a result is given in terms of $\hat{x}^{+\Delta+}_{-\Delta-}$, then $\hat{x}$ is often used in further calculations, and the $(\Delta+, \Delta-)$’s are ‘propagated’ into further uncertainties in ‘creative’ ways, forgetting that the well know formulae for propagations in linear combinations (or in linearized forms) rely on probabilistic properties of means and variances (and the Central Limit Theorem makes the result Gaussian if ‘several’ contributions are considered). There are, instead, no similar

\textsuperscript{18}In the case there are only some possibilities, or we are interested on how the data support an hypothesis over others, the quantity to report is (are) the likelihood ratio(s), as recently recognized also by the European Network of Forensic Science Institutes (ENFSI)\textsuperscript{[25]. Note that a likelihood function, or likelihood ratios (also known as Bayes factors), cannot be considered as ‘objective numbers’, because they depend on the judgments of experts. This is also recognized by the ENFSI Guidelines\textsuperscript{[25]. which appears then rather advanced with respect to naive ideals of objectivism that often are instead just a defense of established procedures and methods supported by inertia and authority.
theorems that apply to the ‘best values’ obtaining minimizing the $-\ln L$ or the $\chi^2$ and to the (possibly asymmetric) ‘errors’ obtained by the $\Delta(-\ln L)$ and the $\Delta\chi^2$ rules, still commonly used to evaluate ‘errors’. Therefore these rules might produce biased results, directly and/or in propagations [26].

Reporting the likelihood is also very important in the case of ‘negative searches’, in which a lower/upper bound is usually reported. In fact, although there is no way to combine the bounds (and so people often rely on the most stringent one, which could be just due a larger fluctuation of the background with respect to its expectation) there are little doubts on how to ‘merge’ the individual (independent) likelihoods in a single combined likelihood, from which conventional bounds can be evaluated (see Ref. [28] and chapter 13 of Ref. [29]).

Finally, a puzzle is proposed in the Appendix, as a warning on the use of the weighted average to combine results, even if they are believed to be independent and affected by Gaussian errors.

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Appendix – Independent Gaussian errors are not a sufficient condition to rely on the standard weighted average: an instructive puzzle

Imagine we have a sample of \( n \) observations, characterized by independent Gaussian errors with unknown \( \mu \), associated to the true value of interest, and also unknown \( \sigma \). Our main interest is to infer \( \mu \), but in this case also \( \sigma \) needs to be estimated from the same sample. The ‘estimators’ (to use frequentistic vocabulary, to which most readers are most likely familiar) of \( \mu \) and \( \sigma \) are the arithmetic mean \( \overline{x} \) and the standard deviation \( s \) calculated from the sample, respectively.\(^{19}\) In particular the ‘error’ on \( \mu \) is calculated as \( s/\sqrt{n} \) (hereafter we focus on the determination of \( \mu \), although a similar reasoning and a related puzzle concerns the determination of \( \sigma \)).

Now the question is what happens if we divide the samples in sub-samples, ‘determine’ \( \mu \) from each sub-sample and then combine the partial results. In order to avoid abstract speculations, let us concentrate on the following simulated sample:\(^{20}\)

\[
\begin{align*}
\text{2.691952} & \quad \text{2.805799} & \quad \text{3.826049} & \quad \text{1.908438} & \quad \text{3.844093} & \quad \text{2.406228} & \quad \text{5.176920} & \quad \text{1.925284} \\
\text{1.688440} & \quad \text{2.309165} & \quad \text{3.046256} & \quad \text{3.211285} & \quad \text{2.302760} & \quad \text{2.966700} & \quad \text{2.301784} & \quad \text{2.232128}
\end{align*}
\]

From the arithmetic average (2.7902) and the ‘empirical’ standard deviation (0.8970), we get
\[
\mu^{(\text{All})} = 2.790 \pm 0.224
\]
(the exaggerate number of decimal digits, with respect to reasonable standards, is only to make comparisons easier).

Let us now split the values into two sub-samples (first and second row, respectively). The ‘determinations’ of \( \mu \) are now
\[
\begin{align*}
\mu^{(A)} &= 3.073 \pm 0.399 \\
\mu^{(B)} &= 2.507 \pm 0.183.
\end{align*}
\]

\(^{19}\)See e.g. Sec. 39.2.1 of Ref [7], \hspace{1cm}^{20}\)The sample has been obtained with the Gaussian random number generator \texttt{rnorm()}\ of the R language\[30], with the following commands

\begin{verbatim}
set.seed(20200102)
n = 16; mu = 3; sigma = 1
x = rnorm(n, mu, sigma)
\end{verbatim}
so that we know the ‘true \( \mu \)’. (The random seed, used to make the numbers reproducible, was set to the date in which the sample was generated.) Mean and standard deviation are then calculated using R functions:

\begin{verbatim}
m = mean(x)
s = sd(x)
\end{verbatim}
Combining then the two results calculating the weighted average and its standard deviation, we get

\[
\mu^{(A&B)} = 2.605 \pm 0.1660, \\
\]
sensibly different from \(\mu^{(All)}\) calculated above.

We can then split again the two samples (first four values and second four values of each row), thus getting

\[
\mu^{(A_1)} = 2.808 \pm 0.394 \\
\mu^{(A_2)} = 3.338 \pm 0.736 \\
\mu^{(B_1)} = 2.564 \pm 0.352 \\
\mu^{(B_2)} = 2.451 \pm 0.173.
\]

Combining the four partial results we get then

\[
\mu^{(A_1\&A_2\&B_1\&B_2)} = 2.548 \pm 0.142,
\]
different from \(\mu^{(All)}\) and from \(\mu^{(A&B)}\).

**What is going on?** Or, more precisely, what should be the combination rule such that \(\mu^{(All)}\), \(\mu^{(A&B)}\) and \(\mu^{(A_1\&A_2\&B_1\&B_2)}\) would be the same? (Sufficient hints are in the paper and a note could possibly follow with a detailed treatement of the case.)