Twisted Quantum Affine Superalgebra $U_q[gl(m|n)^{(2)}]$ and New $U_q[osp(m|n)]$ Invariant R-matrices

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Abstract

The minimal irreducible representations of $U_q[gl(m|n)]$, i.e. those irreducible representations that are also irreducible under $U_q[osp(m|n)]$ are investigated and shown to be affinizable to give irreducible representations of the twisted quantum affine superalgebra $U_q[gl(m|n)^{(2)}]$. The $U_q[osp(m|n)]$ invariant R-matrices corresponding to the tensor product of any two minimal representations are constructed, thus extending our twisted tensor product graph method to the supersymmetric case. These give new solutions to the spectral-dependent graded Yang-Baxter equation arising from $U_q[gl(m|n)^{(2)}]$, which exhibit novel features not previously seen in the untwisted or non-super cases.
I Introduction

The graded Yang-Baxter equation (YBE)

$$R_{12}(z)R_{13}(zw)R_{23}(w) = R_{23}(w)R_{13}(zw)R_{12}(z)$$  \hspace{1cm} (I.1)

plays a central role in the theory of supersymmetric quantum integrable systems. Its solutions $R^{ab}(z)$, usually called R-matrices, depend on a spectral parameter $z$ and act on the tensor product of two graded vector spaces $V_a$ and $V_b$. The multiplication in the tensor product is $\mathbb{Z}_2$-graded, i.e., for homogeneous elements $x$, $y$, $x'$ and $y'$,

$$(x \otimes y)(x' \otimes y') = (-1)^{|y||x'|} (x x' \otimes y y').$$  \hspace{1cm} (I.2)

In one-dimensional lattice integrable systems R-matrices give the Hamiltonians of quantum spin chains \cite{1,2}, and in statistical mechanics they define the Boltzmann weights of exactly solvable models \cite{3}. In integrable quantum field theory R-matrices give the exact factorizable scattering S-matrices \cite{4,5,6}. Therefore, the knowledge of R-matrices has had many physical applications. This is one of the reasons that the problem of constructing R-matrices has occupied a fundamental place in the study of low-dimensional integrable models.

The mathematical framework for the construction of trigonometric solutions of the (graded) YBE is given by the quantum affine (super)algebras $U_q(\mathcal{G}^{(k)})$ (see \cite{7} and references therein). Here for technical reasons we assume $k = 1$ or 2. Associated to any two finite-dimensional irreducible $U_q(\mathcal{G}^{(k)})$-modules $V(\lambda)$ and $V(\mu)$ there exists a trigonometric R-matrix $R^{\lambda,\mu}(z)$ which obeys the (graded) YBE. A systematic method, called the tensor product graph (TPG) method, for constructing these R-matrices arising from untwisted quantum affine algebras was initiated in \cite{8} (see also \cite{9} for the rational cases) and further developed in \cite{10}. This TPG approach was later generalized to untwisted quantum affine superalgebras \cite{11}. In \cite{12}, the TPG technique was extended and a twisted TPG method was developed, which enables one to construct spectral-dependent R-matrices arising from twisted quantum affine algebras. The twisted TPG method was used in \cite{13} to determine soliton S-matrices of certain quantum affine Toda theories. By means of the (twisted) TPG method, a large number of R-matrices have been constructed, leading to many new quantum integrable models, quantum spin chains, exactly solvable lattice models and exact scattering S-matrices.

It has been an open problem to generalize the TPG method to determine R-matrices arising from twisted quantum affine superalgebras. In \cite{14}, the $U_q[osp(2|2)]$ invariant R-matrix arising from $U_q[gl(2|2)^{(2)}]$ was determined by brute force. In this paper we formulate a systematic twisted TPG method, which enables us to determine R-matrices corresponding to the tensor product of any two minimal representations of $U_q[gl(m|n)^{(2)}]$, ($m \leq n, \ n > 2$).

The paper is organized as follows: In section II, we review the necessary facts about $gl(m|n)$ and its fixed point subalgebra $osp(m|n)$. In section III, we discuss the twisted
quantum affine superalgebra $U_q[gl(m|n)^{(2)}]$, its minimal representations and give the
equations which uniquely determine the R-matrices (the Jimbo equations). In sections IV and
V we explain how to solve these equations. Our technique is a supersymmetric general-
ization of the twisted TPG method introduced in [12]. We have obtained the R-matrices

\[ \text{corresponding to the tensor product } V(\lambda_a) \otimes V(\lambda_b) \text{ of any two minimal representations} \]

\[ V(\lambda_a) \text{ and } V(\lambda_b) \text{ of } U_q[gl(m|n)^{(2)}] \text{ (} m \leq n, \ n > 2 \text{). Particularly interesting is the case} \]

\[ \text{corresponding to } m = n > 2, \ a = b, \text{ where an indecomposable representation generally} \]

\[ \text{occurs in the decomposition of } V(\lambda_a) \otimes V(\lambda_a), \text{ so that the corresponding R-matrices ex-} \]

hibit new properties not observed in the untwisted case and in the bosonic case. This

necessitates an extension of our twisted TPG method. As we see below a combi-

nation of the twisted TPG method and the technique used in [14] is needed to determine the

corresponding R-matrices. Some concluding remarks are given in section VI.

II Preliminaries

Throughout this paper, we assume $n = 2r$ is even and set $h = [m/2]$ so that $m = 2h$

for even $m$ and $m = 2h + 1$ for odd $m$. For homogeneous operators $A, B$ we use the

notation $[A, B] = AB - (-1)^{|A||B|}BA$ to denote the usual graded commutator. Let $E^a_b$

be the standard generators of $gl(m|n)$ obeying the graded commutation relations

\[ [E^a_b, E^c_d] = \delta^c_b E^a_d - (-1)^{|a||b|+|c||d|} \delta^a_d E^c_b. \]  

(II.1)

In order to introduce the subalgebra $osp(m|n)$ we first need a graded symmetric metric
tensor $g_{ab} = (-1)^{|a||b|}g_{ba}$ which is assumed to be even. We shall make the convenient
choice

\[ g_{ab} = \xi_a \delta_{\bar{a}b}, \]  

(II.2)

where

\[ \bar{a} = \begin{cases} m + 1 - i, & a = i \\ n + 1 - \mu, & a = \mu, \end{cases}, \quad \xi_a = \begin{cases} 1, & a = i \\ (-1)^\mu, & a = \mu, \end{cases}. \]  

(II.3)

In the above equations, $i = 1, 2, \cdots, m$ and $\mu = 1, 2, \cdots, n$. Note that

\[ \xi_a^2 = 1, \quad \xi_a \xi_b = (-1)^{|a|}, \quad g^{ab} = \xi_b \delta_{\bar{a}b}. \]  

(II.4)

As generators of the subalgebra $osp(m|n = 2r)$ we take

\[ \sigma_{ab} = g_{ac}E^c_b - (-1)^{|a||b|}g_{ac}E^c_a = -(-1)^{|a||b|}\sigma_{ba} \]  

(II.5)

which satisfy the graded commutation relations

\[ [\sigma_{ab}, \sigma_{cd}] = g_{cb}\sigma_{ad} - (-1)^{|a||b|+|c||d|}g_{ad}\sigma_{cb} \]

\[ -(-1)^{|c||d|}(g_{bd}\sigma_{ac} - (-1)^{|a||b|+|c||d|}g_{ac}\sigma_{db}). \]  

(II.6)

We have an $osp(m|n)$-module decomposition

\[ gl(m|n) = osp(m|n) \oplus \mathcal{T}, \quad [\mathcal{T}, \mathcal{T}] \subset osp(m|n), \]  

(II.7)
where $\mathcal{T}$ is spanned by operators

$$T_{ab} = g_{ac}E_b^c + (-1)^{[a][b]}g_{bc}E_a^c = (-1)^{[a][b]}T_{ba}. \quad (\text{II.8})$$

It is convenient to introduce the Cartan-Weyl generators

$$\sigma^a_b = g^{ac}\sigma_{cb} = -(-1)^{[a][b]}\xi_{a\mathfrak{s}\mathfrak{b}}\sigma^\mathfrak{b}_{a}. \quad (\text{II.9})$$

As a Cartan subalgebra we take the diagonal operators

$$\sigma^a_a = E^a_a - E_a^a = -\sigma^a_a. \quad (\text{II.10})$$

Note that for odd $m = 2h + 1$ we have $\overline{h + 1} = h + 1$ and thus $\sigma^{h+1}_h = E^{h+1}_h - E_h^{h+1} = 0$.

The positive roots of $\mathfrak{osp}(m|n)$ are given by the even positive roots (usual positive roots for $\mathfrak{o}(m)\oplus\mathfrak{sp}(n)$) together with the odd positive roots $\delta_i + \epsilon_i$, $1 \leq i \leq m$, $1 \leq \mu \leq r = n/2$, where we have adopted the useful convention $\epsilon_i = -\epsilon_i$, $i \leq h = \lfloor m/2 \rfloor$ so that $\epsilon_{h+1} = 0$ for odd $m = 2h + 1$. This is consistent with the $\mathbb{Z}$-gradation

$$\mathfrak{osp}(m|n) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2. \quad (\text{II.11})$$

Here $L_0 = \mathfrak{o}(m) \oplus \mathfrak{gl}(r)$, the $\mathfrak{gl}(r)$ generators are given by

$$\sigma^\mu_\nu = E^\mu_\nu - (-1)^{\mu+\nu}E^{\nu}_\mu, \quad 1 \leq \mu, \nu \leq r, \quad (\text{II.12})$$

and $L_{-2} \oplus L_0 \oplus L_2 = \mathfrak{o}(m) \oplus \mathfrak{sp}(n)$, where $L_2$ gives rise to an irreducible representation of $L_0$ with highest weight $(\bar{0}|2, \bar{0})$ spanned by the generators

$$\sigma^\mu_\nu = E^\mu_\nu - \xi_{\mu}\xi_{\nu}E^\nu_\mu = E^\mu_\nu + (-1)^{\mu+\nu}E^{\nu}_\mu, \quad 1 \leq \mu, \nu \leq r. \quad (\text{II.13})$$

$L_1$ is spanned by odd root space generators

$$\sigma^\mu_i = E^\mu_i + \xi_{\mu}E^\nu_\mu = E^\mu_i + (-1)^{\mu}E^\nu_\mu, \quad 1 \leq \mu \leq 1, \quad 1 \leq i \leq m \quad (\text{II.14})$$

and gives rise to an irreducible representation of $L_0$ with highest weight $(1, 0|1, \bar{0})$. $L_{-1}$, $L_{-2}$ give rise to irreducible representations of $L_0$ dual to $L_1$, $L_2$, respectively. Finally $\mathcal{T}$ transforms as an irreducible representation of $\mathfrak{osp}(m|n)$ [module the first order invariant of $\mathfrak{gl}(m|n)$] under the adjoint action with highest weight $\theta = \delta_1 + \delta_2$.

The simple roots of $\mathfrak{osp}(m|n = 2r)$ are thus given by the usual (even) simple roots of $L_0$ together with the odd simple root $\bar{\alpha}_s = \bar{\alpha}_{h+r} = \delta_r - \epsilon_1$ which is the lowest weight of $L_0$-module $L_1$. Note that the simple roots of $\mathfrak{o}(m)$ depend on whether $m$ is odd or even, and are given here for convenience: For $m = 2h$, $\bar{\alpha}_i = \epsilon_i - \epsilon_{i+1}$, $1 \leq i < h$, $\bar{\alpha}_h = \epsilon_{h-1} + \epsilon_h$. For $m = 2h + 1$, $\bar{\alpha}_i = \epsilon_i - \epsilon_{i+1}$, $1 \leq i < h$, $\bar{\alpha}_h = \epsilon_h$. The simple roots of $\mathfrak{gl}(r)$ are given by $\bar{\alpha}_{h+1} = \delta_\mu - \delta_{\mu+1}$, $1 \leq \mu < r$. Corresponding to these simple roots we have the generators for $\mathfrak{gl}(m|n)$ (in what follows $s \equiv h + r$):

$$E_i = \sigma^i_{i+1}, \quad F_i = \sigma^{i+1}_i, \quad H_i = \sigma^i_i - \sigma^{i+1}_{i+1}, \quad 1 \leq i < h,$$

$$E_h = \sigma^{h+1}_h, \quad F_h = \sigma^{h+1}_h, \quad H_h = \sigma^h_h, \quad \text{for } m = 2h + 1,$$

$$E_h = \sigma^{h+1}_h, \quad F_h = \sigma^{h+1}_h, \quad H_h = \sigma^h_h + \sigma^{h+1}_{h-1}, \quad \text{for } m = 2h,$$

$$E_{h+\mu} = \sigma^\mu_{\mu+1}, \quad F_{h+\mu} = \sigma^{\mu+1}_\mu, \quad H_{h+\mu} = \sigma^\mu_\mu - \sigma^{\mu+1}_{\mu+1}, \quad 1 \leq \mu < r,$$

$$E_s = \sigma^{\mu_{\mu+1}}_s = E^s_s + (-1)^sE^s_{s}, \quad F_s = \sigma^s_s, \quad H_s = \sigma^s_s + \sigma^s_s, \quad (\text{II.15})$$
which are the standard generators of \(osp(m|n)\), together with
\[
E_0 = T^{\mu=1}_{\nu=2} = E_2^1 + E_1^2, \quad F_0 = T^2_1, \quad H_0 = -(\sigma_1^1 + \sigma_2^2),
\]
where \(E_0\) is the minimal weight vector of \(T\). The graded half-sum of the positive roots of \(osp(m|n=2r)\) is given by
\[
\rho = \frac{1}{2} \sum_{i=1}^{h} (m - 2i) \epsilon_i + \frac{1}{2} \sum_{\mu=1}^{r} (n - m + 2 - 2\mu) \delta_{\mu}. \quad (II.17)
\]

It is convenient to work with generators
\[
T_{b}^a = g^{ac} T_{cb} = E_{b}^a + (-1)^{[a][[a] + [b]]} \xi_{a} \xi_{b} E_{a}^{b}
\]
which should be compared to our \(osp(m|n)\) generators \(\sigma_{b}^a\) above. This suggests that we introduce the automorphism
\[
\omega(E_{b}^a) = -(-1)^{[a][[a] + [b]]} \xi_{a} \xi_{b} E_{a}^{b}. \quad (II.19)
\]
Then it is easily verified that \(\omega\) so defined gives an automorphism of \(gl(m|n)\) of order 2, i.e.
\[
\omega([E_{b}^a, E_{d}^c]) = [\omega(E_{b}^a), \omega(E_{d}^c)], \quad \omega^2 = 1. \quad (II.20)
\]
Moreover, by inspection, we have
\[
\omega(\sigma_{b}^a) = \sigma_{b}^a, \quad \omega(T_{b}^a) = -T_{b}^a \quad (II.21)
\]
so that \(osp(m|n = 2r)\) is the fixed point subalgebra of \(\omega\) while \(T\) corresponds to the eigenspace corresponding to eigenvalue \(-1\) of \(\omega\).

**III Twisted quantum affine superalgebra \(U_q[gl(m|n)^{(2)}]\)**

Following the usual notation, we now set
\[
\hat{L} \equiv gl(m|n), \quad \hat{L}_0 \equiv osp(m|n), \quad \hat{L}_1 \equiv T. \quad (III.1)
\]

With this notation we have
\[
\hat{L} = \hat{L}_0 \oplus \hat{L}_1. \quad (III.2)
\]
The corresponding twisted affine superalgebra \(\hat{L}^{(2)}\) admits the decomposition
\[
\hat{L}^{(2)} = \bigoplus_{l \in \mathbb{Z}/2} \hat{L}_l^{(2)} \oplus Cc \oplus Cd, \quad (III.3)
\]
where
\[
\hat{L}_m^{(2)} = \begin{cases} \hat{L}_0(l), & l \in \mathbb{Z} \\ \hat{L}_1(l), & l \in \mathbb{Z} + \frac{1}{2}, \end{cases} \quad (III.4)
\]
with \( \hat{L}_t(l) = \{ x(l) \mid x \in \hat{L}_t \}, \ t = 0, 1 \). The graded commutation relations are defined by

\[
[x(l), y(l')] = [x, y](l + l') + l c \delta_{l+\nu,0}(x, y),
\]

\[
[d, x(l)] = l x(l), \quad [c, x(l)] = [c, d] = 0,
\]

(III.5)

where \((, )\) is a fixed invariant bi-linear form on \( \hat{L} \). As a Cartan subalgebra of \( \hat{L}(2) \), we take

\[
\hat{H} = H(0) \oplus Cc \oplus Cd,
\]

(III.6)

where \( H \in \hat{L}_0 \) is a Cartan subalgebra of \( \hat{L}_0 \). As a set of simple roots of \( \hat{L}(2) \), we take

\[
\alpha_0 = -\theta + \frac{1}{2} \delta, \quad \alpha_i = \bar{\alpha}_i, \quad 1 \leq i \leq s,
\]

(III.7)

where \( \theta = \delta_1 + \delta_2, \ \bar{\alpha}_i \) are the simple roots of \( \hat{L}_0 \) and \( \frac{1}{2} \delta \) is the minimal positive imaginary root.

Then \( \hat{L}(2) \) admits a standard set of simple generators \( \{ e_i, f_i, h_i \mid 0 \leq i \leq s \} \), given by

\[
e_i = E_i(0), \quad f_i = F_i(0), \quad h_i = H_i(0), \quad 1 \leq i \leq s,
\]

\[
e_0 = E_0(\frac{1}{2}), \quad f_0 = F_0(-\frac{1}{2}), \quad h_0 = H_0(0) + \frac{1}{2} c.
\]

(III.8)

They satisfy the following relations

\[
[h_i, e_j] = (\alpha_i, \alpha_j) e_j, \quad [h_i, f_j] = -(\alpha_i, \alpha_j) f_j,
\]

\[
[e_i, f_j] = \delta_{ij} h_i,
\]

\[
(\text{ad}_{e_i})^{1-a_{ij}} e_j = 0 = (\text{ad}_{f_i})^{1-a_{ij}} f_j, \quad i \neq j,
\]

(III.9)

where \( a_{ij} \) is the Cartan matrix

\[
a_{ij} = \begin{cases} \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}, & i \neq s \\ (\alpha_i, \alpha_j), & i = s. \end{cases}
\]

(III.10)

We remark that there are also higher order Serre relations [15]. Since these extra Serre relations are not important for our purpose here, we shall ignore them in this paper.

Twisted quantum affine superalgebra \( U_q(\hat{L}(2)) \) is a \( q \)-deformation of the universal enveloping algebra \( U(\hat{L}(2)) \) of \( \hat{L}(2) \). In terms of the simple generators \( \{ e_i, f_i, h_i \mid 0 \leq i \leq s \} \), the defining relations of \( U_q(\hat{L}(2)) \) are given by

\[
[h_i, e_j] = (\alpha_i, \alpha_j) e_j, \quad [h_i, f_j] = -(\alpha_i, \alpha_j) f_j,
\]

\[
[e_i, f_j] = \delta_{ij} q^{h_i} - q^{-h_i},
\]

\[
(\text{ad}_{q_i} e_i)^{1-a_{ij}} e_j = 0 = (\text{ad}_{f_i})^{1-a_{ij}} f_j, \quad i \neq j,
\]

(III.11)

where \( q_i = q^{(\alpha_i, \alpha_i)/2} \). Here again we have ignored the extra \( q \)-Serre relations.

\( U_q(\hat{L}(2)) \) is a quasi-triangular Hopf superalgebra with coproduct \( \Delta \) given by

\[
\Delta(q^{\pm h_i/2}) = q^{\pm h_i/2} \otimes q^{\pm h_i/2},
\]

\[
\Delta(e_i) = q^{-h_i/2} \otimes e_i + e_i \otimes q^{h_i/2},
\]

\[
\Delta(f_i) = q^{-h_i/2} \otimes f_i + f_i \otimes q^{h_i/2}.
\]

(III.12)
Throughout $\mathcal{R}$ denotes the universal R-matrix of $U_q(\hat{L}(2))$ which by definition satisfies

$$\mathcal{R}\Delta(a) = \Delta^T(a)\mathcal{R}, \quad \forall a \in U_q(\hat{L}(2)),$$

$$(1 \otimes \Delta)\mathcal{R} = \mathcal{R}_{12}\mathcal{R}_{13}, \quad (\Delta \otimes 1)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}$$  \hspace{1cm} (III.13)

where $\Delta^T(a)$ is the opposite coproduct. A direct consequence of the above relations is that $\mathcal{R}$ satisfies the graded YBE.

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$$  \hspace{1cm} (III.14)

Note that the generators $q^{\pm h_i/2}, e_i, f_i, \quad (1 \leq i \leq s)$ generate the quantum algebra $U_q(\hat{L}_0)$ which is a quasi-triangular Hopf subsuperalgebra of $U_q(\hat{L}(2))$. We denote by $R$ the universal R-matrix of $U_q(\hat{L}_0)$.

We now consider finite-dimensional irreducible representations of $U_q(\hat{L}(2))$ in order to give new solutions to the graded YBE. If a finite-dimensional irreducible representation of $U_q(\hat{L}(2))$ can be constructed by affinizing an irreducible representation of $U_q(\hat{L}_0)$, we say this irreducible representation of $U_q(\hat{L}_0)$ is affinizable. Not all irreducible representations of $U_q(\hat{L}_0)$ are affinizable to give representations of $U_q(\hat{L}(2))$. An important role is played by minimal irreducible representations of $\hat{L}$ which are those irreducible representations which are also irreducible under $\hat{L}_0$. We shall see below that all such irreducible representations of $\hat{L}_0$ can be quantized to give irreducible representations of $U_q(\hat{L}_0)$ which are affinizable to $U_q(\hat{L}(2))$. Using the graded fermion formalism developed in our previous paper [13], one can show that the anti-symmetric tensor irreducible representations of $\hat{L}$, with highest weights

$$\Lambda_a = \begin{cases} (0, 0|0\rangle, & a \leq m, \\ (1|a - m, 0\rangle, & a > m \end{cases}$$  \hspace{1cm} (III.15)

are minimal, and thus give rise to irreducible modules under $U_q(\hat{L}_0)$, with the corresponding $U_q(\hat{L}_0)$ highest weight

$$\lambda_a = (\hat{0}|a, \hat{0}).$$  \hspace{1cm} (III.16)

We thus denote this minimal irreducible representation by $V(\lambda_a)$ to denote its highest weight under $U_q(\hat{L}_0)$.

We define an automorphism $D_z$ of $U_q(\hat{L}(2))$ by

$$D_z(e_i) = z^{\delta_{i0}}e_i, \quad D_z(f_i) = z^{-\delta_{i0}}f_i, \quad D_z(h_i) = h_i.$$  \hspace{1cm} (III.17)

Given any two minimal irreps $\pi_\lambda$ and $\pi_\mu$ of $U_q(\hat{L}_0)$ and their affinizations to irreducible representations of $U_q(\hat{L}(2))$, we obtain a one-parameter family of representations $\Delta_{\lambda\mu}^z$ of $U_q(\hat{L}(2))$ on $V(\lambda) \otimes V(\mu)$ defined by

$$\Delta_{\lambda\mu}^z(a) = \pi_\lambda \otimes \pi_\mu ((D_z \otimes 1)\Delta(a)), \quad \forall a \in U_q(\hat{L}(2)),$$  \hspace{1cm} (III.18)

where $z$ is the spectral parameter. We define the spectral parameter dependent R-matrix

$$R_{\lambda\mu}(z) = (\pi_\lambda \otimes \pi_\mu)((D_z \otimes 1)\mathcal{R}).$$  \hspace{1cm} (III.19)
It follows that this R-matrix gives a solution to the spectral parameter dependent YBE. From the defining property (III.13) of the universal R-matrix one derives the equations

$$R^\lambda \mu(z) \Delta^\hat{\lambda}_{\mu}(a) = (\Delta^T)^\hat{\lambda}_{\mu}(a) R^\lambda \mu(z)$$  \hspace{1cm} (III.20)

which, because the representations \(\Delta^\hat{\lambda}_{\mu}\) are irreducible for generic \(z\), uniquely determine \(R^\lambda \mu(z)\) up to a scalar function of \(z\).

We normalize \(R^\mu \mu(u)\) such that

$$\tilde{R}^\lambda \mu(z) \tilde{R}^\mu \lambda(z^{-1}) = I \quad \text{and} \quad R(0) = \pi_\lambda \otimes \pi_\mu(R),$$  \hspace{1cm} (III.21)

where \(R\) is the R-matrix of \(U_q(\hat{L}_0)\) and \(\tilde{R}^\lambda \mu(z) = PR^\mu \mu(z)\) with \(P : V(\lambda) \otimes V(\mu) \rightarrow V(\mu) \otimes V(\lambda)\) the usual graded permutation operator.

In order for the equation (III.20) to hold for all \(a \in U_q(\hat{L}^{(2)})\) it is sufficient that it holds for all \(a \in U_q(\hat{L}_0)\) and in addition for the extra generator \(e_0\). The relation for \(e_0\) reads explicitly

$$R^\lambda \mu(z) \left( z \pi_\lambda(e_0) \otimes \pi_\mu(q^{h_0/2}) + \pi_\lambda(q^{-h_0/2}) \otimes \pi_\mu(e_0) \right)$$

$$= \left( z \pi_\lambda(e_0) \otimes \pi_\mu(q^{h_0/2}) + \pi_\lambda(q^{-h_0/2}) \otimes \pi_\mu(e_0) \right) R^\lambda \mu(z),$$  \hspace{1cm} (III.22)

or equivalently

$$\tilde{R}^\lambda \mu(z) \left( z \pi_\lambda(e_0) \otimes \pi_\mu(q^{h_0/2}) + \pi_\lambda(q^{-h_0/2}) \otimes \pi_\mu(e_0) \right)$$

$$= \left( \pi_\mu(e_0) \otimes \pi_\lambda(q^{h_0/2}) + z \pi_\mu(q^{-h_0/2}) \otimes \pi_\lambda(e_0) \right) \tilde{R}^\lambda \mu(z).$$  \hspace{1cm} (III.23)

Eq.(III.23) is the Jimbo equation for the twisted quantum affine superalgebras. In the next section, we shall solve this equation for all minimal representations of \(U_q(\hat{L}^{(2)})\), for the case \(m \leq n, \ n > 2\).

**IV Twisted Tensor Product Graph**

We shall determine \(U_q[osp(m|n)]\) invariant R-matrices \(\tilde{R}^{\lambda_a \lambda_b}(z)\) on the tensor product of any two minimal irreducible representations \(V(\lambda_a) \otimes V(\lambda_b), \ a \leq b\), arising from \(U_q[gl(m|n)^{(2)}]\).

Let \(V(\lambda)\) and \(V(\mu)\) denote any two minimal irreducible representations of \(U_q(\hat{L}^{(2)})\). Except for the case \(a - b = m - n = 0\) (see below), the tensor product module \(V(\lambda) \otimes V(\mu)\) is completely reducible into irreducible \(U_q(\hat{L}_0)\)-modules as

$$V(\lambda) \otimes V(\mu) = \bigoplus_\nu V(\nu)$$  \hspace{1cm} (IV.1)

and there are no multiplicities in this decomposition. We denote by \(P^\lambda \mu_\nu\) the projection operator of \(V(\lambda) \otimes V(\mu)\) onto \(V(\nu)\) and set

$$P^\lambda \mu_\nu = \tilde{R}^\lambda \mu(1) P^\lambda \mu_\nu = P^\mu \lambda \tilde{R}^\lambda \mu(1).$$  \hspace{1cm} (IV.2)
We may thus write
\[ \tilde{R}^{\lambda \mu}(z) = \sum_\nu \rho_\nu(z) P^{\lambda \mu}_{\nu}, \quad \rho_\nu(1) = 1. \] (IV.3)

Following our previous approach [10], the coefficients \( \rho_\nu(z) \) may be determined according to the recursion relation
\[ \rho_\nu(z) = \frac{q^{C(\nu)/2} + \epsilon_\nu \epsilon_\nu' z q^{C(\nu')/2}}{z q^{C(\nu)/2} + \epsilon_\nu \epsilon_\nu' q^{C(\nu')/2} \rho_{\nu'}(z)}, \] (IV.4)

which holds for any \( \nu \neq \nu' \) for which
\[ P^{\lambda \mu}_{\nu'} \left( \pi_\lambda(e_0) \otimes \pi_\mu \left( q^{h_0/2} \right) \right) P^{\lambda \mu}_{\nu'} \neq 0. \] (IV.5)

Here \( C(\nu) \) is the eigenvalue of the universal Casimir element of \( \hat{L}_0 \) on \( V(\nu) \) and \( \epsilon_\nu \) denotes the parity of \( V(\nu) \subseteq V(\lambda) \otimes V(\mu) \).

To graphically encode the recursive relations between the different \( \rho_\nu \) we introduce the **Twisted TPG** \( \tilde{G}^{\lambda \mu} \) associated to the tensor product module \( V(\lambda) \otimes V(\mu) \). The nodes of this graph are given by the highest weights \( \nu \) of the \( U_q(\hat{L}_0) \)-modules occuring in the decomposition (IV.1) of the tensor product module. There is an edge between two nodes \( \nu \neq \nu' \) iff (IV.5) holds.

Given a tensor product module and its decomposition, it is not in general an easy task to determine the twisted TPG because in order to determine between which nodes of the graph relation (IV.5) holds requires detailed calculations. We therefore introduce the **Extended Twisted TPG** \( \tilde{G}^{\lambda \mu} \) which has the same set of nodes as the twisted TPG but has an edge between two vertices \( \nu \neq \nu' \) whenever
\[ V(\nu') \subseteq V(\theta) \otimes V(\nu) \] (IV.6)

and
\[ \epsilon_\nu \epsilon_\nu' = \begin{cases} +1 & \text{if } V(\nu) \text{ and } V(\nu') \text{ are in the same irreducible representation of } \hat{L} \\ -1 & \text{if } V(\nu) \text{ and } V(\nu') \text{ are in different irreducible representations of } \hat{L}. \end{cases} \] (IV.7)

The conditions (IV.6) and (IV.7) are necessary conditions for (IV.5) to hold and therefore the twisted TPG is contained in the extended twisted TPG. To see why (IV.6) is a necessary condition for (IV.5) one must realize that \( e_0 \otimes q^{h_0/2} \) is the lowest component of a tensor operator corresponding to \( V(\theta) \). The necessity of (IV.7) follows from the fact that two vertices \( \nu \neq \nu' \) connected by an edge in the twisted TPG (i.e., for which (IV.5) is satisfied) must have the same parity if \( V(\nu) \) and \( V(\nu') \) belong to the same irreducible \( \hat{L} \)-module while they must have opposite parities if they belong to different irreducible \( \hat{L} \)-modules.

While the extended twisted TPG will always include the twisted TPG, it will in general have more edges. Only if the extended twisted TPG is a tree are we guaranteed that it coincides with the twisted TPG.
We will impose a relation (IV.4) for every edge in the extended twisted TPG. Because the extended TPG will in general have more edges than the (unextended) twisted TPG, we will be imposing too many relations. These relations may be inconsistent and we are therefore not guaranteed a solution. If however a solution exists, then it must be the unique correct solution to the Jimbo’s equation.

As seen below, for the cases we are considering, the extended twisted TPG is always consistent and thus will always give rise to a solution of the graded YBE.

Throughout we adopt the convenient notation
\[
\langle a \rangle_{\pm} = \frac{1 \pm z q^a}{z \pm q^a},
\] (IV.8)
so that the relation (IV.4) may be expressed as
\[
\rho_{\nu}(z) = \left\langle \frac{C(\nu') - C(\nu)}{2} \right\rangle_{\epsilon_{\nu} \epsilon_{\nu'}} \rho_{\nu'}(z).
\] (IV.9)

V Solution to Jimbo Equation

We will now determine the R-matrices for any tensor product \(V(\lambda_a) \otimes V(\lambda_b)\) of two minimal representations \(V(\lambda_a), V(\lambda_b)\) of \(U_q(\hat{L}(2))\). Note that for our case, \(\theta = \delta_1 + \delta_2\).

V.1 The case of either \(a \leq b, \ m < n, \ n > 2\) or \(a < b, \ m = n > 2\)

We consider the decomposition of the tensor product of two minimal irreducible representations of \(U_q(\hat{L}_0)\): \(V(\lambda_a) \otimes V(\lambda_b)\). Recall that \(V(\lambda_a), V(\lambda_b)\) also carry irreducible representations of \(\hat{L}\). We denote by \(\hat{V}(a, b)\) the “two-column” irreducible representation of \(\hat{L}\) with highest weight
\[
\Lambda_{a,b} = \begin{cases} 
(\hat{2}_a, \hat{1}_b, \hat{0}|\hat{0}), & a + b \leq m \\
(\hat{2}_a, \hat{1}|a + b - m, \hat{0}), & a \leq m, \ a + b > m \\
(\hat{2}|a + b - m, a - m, \hat{0}), & a > m.
\end{cases}
\] (V.1)

Then we have the following decomposition into irreducible \(\hat{L}\)-modules
\[
V(\lambda_a) \otimes V(\lambda_b) \equiv \hat{V}(\Lambda_{a,b}) \otimes \hat{V}(\Lambda_{b})
= \bigoplus_{c=0}^{a} \hat{V}(c, a + b - 2c).
\] (V.2)

The reduction of the two-column irreducible representations of \(\hat{L}\) into irreducible representations of \(\hat{L}_0\) has been worked out in our previous paper [16] by using the quasi-spin graded-fermion formalism. We thus arrive at the following irreducible \(\hat{L}_0\), and thus \(U_q(\hat{L}_0)\) module decomposition,
\[
V(\lambda_a) \otimes V(\lambda_b) = \bigoplus_{c=0}^{a} \bigoplus_{k=0}^{c} V(k, a + b - 2c);
\] (V.3)
here and throughout $V(a, b)$ denotes an irreducible $U_q(\hat{\mathfrak{L}}_0)$ module with highest weight
\[
\lambda_{a,b} = (\hat{0}|a + b, a, \hat{0}). \quad (V.4)
\]
Note that one can only get an indecomposable in (V.3) when $m = n > 2$ and $a + b - 2c = 0$, which is the case we shall consider in the next subsection. Since $a \leq b$, $c \leq a$, this can only occur when $a = b$ and $c = a$. In that case the $U_q(\hat{\mathfrak{L}}_0)$-modules $V(k, 0)$, $k = 0, 1$, will form an indecomposable.

We now show that the minimal irreducible $U_q(\hat{\mathfrak{L}}_0)$, $V(\lambda_a)$, with highest weight $\lambda_a$, are affinizable. We first note that the vector module $V(\lambda_1)$ of $U_q(\hat{\mathfrak{L}}_0)$ with higest weight $\lambda_1 = (\hat{0}|1, \hat{0})$ is minimal. It is also affinizable since it is undeformed.

Following our previous approach [12], we consider the corresponding twisted TPG for $V(\lambda_2)$, has a quite different topology to the untwisted TPG:

\[
\begin{align*}
\lambda_0 & \quad \lambda_1 + \delta_2 & \quad \lambda_2 \\
\lambda_0 & \quad \lambda_2 & \quad \lambda_1 + \delta_2
\end{align*}
\]

which indicates that $V(\lambda_2)$ is not affinizable to carry an irreducible representation of $U_q(\hat{\mathfrak{L}}_0^{(1)})$.

More generally, we have the following twisted TPG for for $V(\lambda_2)$, has a quite different topology to the untwisted TPG:

\[
\begin{align*}
\lambda_{a-1} & \quad \lambda_a + \delta_2 & \quad \lambda_{a+1} \\
\lambda_{a-1} & \quad \lambda_a + \delta_2 & \quad \lambda_{a+1}
\end{align*}
\]

so that $\lambda_{a+1}$ is an extremal node and hence, by recursion, each of the irreducible representations $V(\lambda_a)$, is affinizable to $U_q(\hat{\mathfrak{L}}_0^{(2)})$. Thus we have shown that all minimal irreducible $U_q(\hat{\mathfrak{L}}_0)$ modules $V(\lambda_a)$ are affinizable to carry irreducible representations of $U_q(\hat{\mathfrak{L}}_0^{(2)})$.

The $U_q(\hat{\mathfrak{L}}_0)$-module decomosition of the tensor product of any two such representations is given by (V.3). Thus we have the corresponding extended twisted TPG for $V(\lambda_a) \otimes V(\lambda_b)$, given below by Figure 1.

To see that this graph is consistent we have to consider the closed loops of the form

\[
\begin{align*}
(c, k - 1) & \quad (c, k) \\
(c + 1, k) & \quad (c + 1, k + 1)
\end{align*}
\]
Figure 1: The extended twisted TPG for $U_q[gl(m|n)^{(2)}]$ for the tensor product $V(\lambda_a) \otimes V(\lambda_b)$, for the cases we are considering. The vertex labelled by the pair $(c, k)$ corresponds to the irreducible $U_q(\hat{L}_0)$ module $V(k, a + b - 2c)$. The rows and columns are labelled by $c$ and $k$, respectively. The rows labelled by $c$ correspond to irreducible representations of $\hat{L}_0$ in the same irreducible representation of $\hat{L}$ and so all vertices in a row have the same parity. The vertices along the diagonal edges of the graph have alternating parities.

where we have indicated the relative parities of the vertices. We denote the eigenvalue of the universal Casimir element of $\hat{L}_0$ on the irreducible representation labelled by $(c, k)$ by $C_{c,k}$. Then it is easily seen that

\begin{align}
C_{c,k} - C_{c,k-1} &= C_{c+1,k+1} - C_{c+1,k} = 2(\rho, \delta_1 + \delta_2) - 2(a + b - 1) + 4(c - k), \\
C_{c,k} - C_{c+1,k+1} &= C_{c,k-1} - C_{c+1,k} = 2(\rho, \delta_1 - \delta_2) - 2(a + b - 1) + 2c.
\end{align}

This implies that the extended twisted TPG is consistent, i.e. that the recursion relations (V.4) give the same result independent of the path along which one recurses.

We can now read off the R-matrix from the extended twisted TPG

\begin{align}
\hat{R}_{\lambda_a, \lambda_b}^\rho(z) &= \sum_{c=0}^{a} \sum_{k=0}^{c-k} \prod_{j=1}^{c-k} (m - n + 2j - a - b)_+ \\
&\prod_{i=1}^{c} (i - a - b - 1) \cdot P_{(a+b-2c+k)\delta_1+k\delta_2}^{\lambda_a, \lambda_b}.
\end{align}

V.2 The case of $a = b, \ m = n > 2$

As indicated in the last subsection, for the case at hand the $U_q[osp(n|n)]$-modules $V(k, 0)$, $k = 0, 1$, appearing in the right hand side of the tensor product decomposition (V.3),
form an indecomposable representation of \(U_q[osp(n|n)]\). From now on we denote by \(V\) this indecomposable module. We thus have an \(U_q[osp(n|n)]\) module decomposition

\[
V(\lambda_a) \otimes V(\lambda_a) = \bigoplus_\nu V(\nu) \bigoplus V,
\]

where the sum on \(\nu\) is over the irreducible highest weights and \(V\) is the unique indecomposable. Note that \(V\) contains a unique submodule \(\bar{V}(\delta_1 + \delta_2)\) which is maximal, indecomposable and cyclically generated by a maximal vector of weight \(\delta_1 + \delta_2\) such that \(V/\bar{V}(\delta_1 + \delta_2) \cong V(\hat{0}|\hat{0})\) (the trivial \(U_q[osp(n|n)]\)-module). Moreover \(V\) contains a unique irreducible submodule \(\bar{V}(\hat{0}|\hat{0}) \subset \bar{V}(\delta_1 + \delta_2)\). The usual form of Schur’s lemma applies to \(\bar{V}(\delta_1 + \delta_2)\) and so the space of \(U_q[osp(n|n)]\) invariants in \(\text{End}(V)\) has dimension 2 (see Appendix A). It is spanned by the identity operator \(I\) together with an invariant \(N\) (unique up to scalar multiples) satisfying

\[
NV = V(\hat{0}|\hat{0}) \subset \bar{V}(\delta_1 + \delta_2), \quad N\bar{V}(\delta_1 + \delta_2) = (0).
\]

It follows that \(N\) is nilpotent, i.e.

\[
N^2 = 0.
\]

We now determine the extended twisted TPG for the decomposition given by (V.9). We note that \(V\) can only be connected to two nodes corresponding to highest weights

\[
\nu = \begin{cases} 
2\delta_1 & \text{(opposite parity)}, \\
2(\delta_1 + \delta_2) & \text{(same parity)},
\end{cases} \quad (c, k) = (a - 1, 0) \quad \text{or} \quad (c, k) = (a, 2).
\]

We thus arrive at the (consistent) extended twisted TPG for (V.9), given by Figure 2.

Let \(P_V \equiv P_V^{\lambda_0, \lambda_a}\) be the projector onto \(V(\nu)\) for \(\nu \neq 0\) and \(P_0 \equiv P_V^{0, \lambda_a}\) the projector onto \(V(\nu)\). Then the R-matrix \(\hat{R}(z) \equiv \hat{R}^{\lambda_0, \lambda_a}(z)\) from the extended twisted TPG can be expanded in terms of the operators \(N\), \(P_V\) and \(P_0\):

\[
\hat{R}(z) = \rho_N(z)N + \rho_V(z)P_V + \sum_\nu \rho_\nu(z)P_\nu.
\]

The coefficients \(\rho_N(z)\) can be obtained recursively as in the last subsection from the extended twisted TPG. However, the coefficients \(\rho_N(z)\) and \(\rho_V(z)\) can not be read off from the extended twisted TPG since the corresponding vertex refers to an indecomposable module. So it remains to determine these two coefficients. Following our previous approach \[14\] to \(U_q[gl(2|2)^{(2)}]\), we proceed as follows.

We write the R-matrix \(\hat{R} \equiv \hat{R}(0) = PR, \ R \equiv R^{\lambda_0, \lambda_a}\) of \(U_q[osp(n|n)]\) in the form

\[
\hat{R} = \rho_N(0)N + \rho_V(0)P_V + \sum_\nu \rho_\nu(0)P_\nu,
\]

where the coefficients \(\rho_N(0)\), \(\rho_V(0)\), \(\rho_\nu(0)\) are all known from the representation theory of \(U_q[osp(n|n)]\), as is the nilpotent operator satisfying

\[
P_\nu N = NP_\nu = 0, \quad P_N N = NP_N = N, \quad N^2 = 0.
\]
Figure 2: The extended twisted TPG for $U_q[gl(n|n)^{(2)}]$ ($n > 2$) for the tensor product $V(\lambda_a) \otimes V(\lambda_a)$. The vertex labelled by the pair $(c, k)$ corresponds to the irreducible $U_q[osp(n|n)]$ module $V(k, 2a - 2c)$ except for the vertex corresponding to $c = a, k = 1$, which has been circled to indicate that it is an indecomposable $U_q[osp(n|n)]$-module.

Explicitly,

$$N = \frac{1}{2\rho_V(0)}(R^TR - \rho_V(0)^2)P_V,$$

where $T$ is the usual graded twist map,

$$\rho_N(0) = 1, \quad \rho_V(0) = \epsilon_V q^{-C_{\lambda_a}}$$

with $C_{\lambda_a}$ being the eigenvalue of the universal Casimir of $U_q[osp(n|n)]$ on $V(\lambda_a)$ and $\epsilon_V$ being the parity of $V$ which is given by $(-1)^a$ (i.e. +1 for $a$ even and -1 for $a$ odd).

Multiplying the Jimbo equation from the right by $P_\nu$ and from the left by $P_V$, utilising (V.13) and (V.15), one gets

$$(\rho_V(z)P_V + \rho_N(z)N)\left(z e_0 \otimes q^{h_0/2} + q^{-h_0/2} \otimes e_0\right)P_\nu = \rho_\nu(z)P_V\left(e_0 \otimes q^{h_0/2} + q^{-h_0/2} \otimes e_0\right)P_V.$$  

Similarly multiplying the Jimbo equation from the left by $P_\nu$ and from the right by $P_V$ gives

$$\rho_\nu(z)P_\nu\left(z e_0 \otimes q^{h_0/2} + q^{-h_0/2} \otimes e_0\right)P_V = \rho_V(z)P_\nu\left(e_0 \otimes q^{h_0/2} + q^{-h_0/2} \otimes e_0\right)P_V,$$

where we have employed

$$P_\nu \left(e_0 \otimes q^{h_0/2}\right) N = P_\nu \left(q^{-h_0/2} \otimes e_0\right) N = 0.$$  

(V.16)
This is seen as follows. We have $N \left( V(\lambda_a) \otimes V(\lambda_b) \right) = V(\hat{0}|\hat{0}) \subseteq \bar{V}(\delta_1 + \delta_2)$. Thus

$$\left( e_0 \otimes q^{h_0/2} \right) N, \left( q^{-h_0/2} \otimes e_0 \right) N \subseteq \bar{V}(\delta_1 + \delta_2) \quad (V.21)$$

from which \((V.20)\) follows.

Setting $z = 0$ into \((V.19)\) gives

$$\rho_\nu(0) P_\nu \left( q^{-h_0/2} \otimes e_0 \right) P_\nu = \rho_\nu(0) P_\nu \left( e_0 \otimes q^{h_0/2} \right) P_\nu. \quad (V.22)$$

Substituting this equation into \((V.19)\) we arrive at

$$\left( z + \frac{\rho_\nu(0)}{\rho_\nu(0)} \right) \rho_\nu(z) P_\nu \left( q^{-h_0/2} \otimes e_0 \right) P_\nu = \left( 1 + z \frac{\rho_\nu(0)}{\rho_\nu(0)} \right) \rho_\nu(z) P_\nu \left( e_0 \otimes q^{h_0/2} \right) P_\nu. \quad (V.23)$$

Since $P_\nu(e_0 \otimes q^{h_0/2}) P_\nu \neq 0$, it follows that

$$\rho_\nu(z) = \frac{z + \rho_\nu(0)/\rho_\nu(0)}{1 + z \rho_\nu(0)/\rho_\nu(0)} \rho_\nu(z) \quad (V.24)$$

with $\nu$ as in \((V.12)\). Note that since the extended twisted TPG is consistent, it does not matter which $\nu$ in \((V.12)\) is used.

Before proceeding to the evaluation of $\rho_N(z)$, it is worth noting that

$$N \left( e_0 \otimes q^{h_0/2} \right) N = N \left( q^{-h_0/2} \otimes e_0 \right) N = 0. \quad (V.25)$$

Now multiplying the Jimbo equation from the left and the right by $P_\nu$ gives rise to

$$(\rho_N(z) N + \rho_\nu(z) P_\nu) \left( z e_0 \otimes q^{h_0/2} + q^{-h_0/2} \otimes e_0 \right) P_\nu = P_\nu \left( e_0 \otimes q^{h_0/2} + z q^{-h_0/2} \otimes e_0 \right) (\rho_N(z) N + \rho_\nu(z) P_\nu), \quad (V.26)$$

which gives, on multiplying from left by $N$ and using \((V.19)\) and \((V.23)\),

$$N \left( z e_0 \otimes q^{h_0/2} + q^{-h_0/2} \otimes e_0 \right) P_\nu = N \left( e_0 \otimes q^{h_0/2} + z q^{-h_0/2} \otimes e_0 \right) P_\nu. \quad (V.27)$$

Setting $z = 0$ one gets

$$N \left( q^{-h_0/2} \otimes e_0 \right) P_\nu = N \left( e_0 \otimes q^{h_0/2} \right) P_\nu. \quad (V.28)$$

Multiplying the Jimbo equation from the right by $N$, utilising \((V.15)\) and \((V.23)\), one has

$$P_\nu \left( z e_0 \otimes q^{h_0/2} + q^{-h_0/2} \otimes e_0 \right) N = P_\nu \left( e_0 \otimes q^{h_0/2} + z q^{-h_0/2} \otimes e_0 \right) N, \quad (V.29)$$

which leads to, on setting $z = 0$,

$$P_\nu \left( q^{-h_0/2} \otimes e_0 \right) N = P_\nu \left( e_0 \otimes q^{h_0/2} \right) N. \quad (V.30)$$
Also setting $z = 0$ into $(V.20)$ and using $(V.28)$ and $(V.30)$, one gets

$$\rho_V(0) P_V \left( e_0 \otimes q^{h_0/2} - q^{-h_0/2} \otimes e_0 \right) P_V = N \left( e_0 \otimes q^{h_0/2} \right) P_V - P_V \left( e_0 \otimes q^{h_0/2} \right) N.$$  \hspace{1cm} (V.31)

Finally, substituting $(V.28)$, $(V.30)$ and $(V.31)$ into $(V.26)$ gives

$$\left( (1 + z) \rho_N(z) + (z - 1) \frac{\rho_V(z)}{\rho_V(0)} \right) N(e_0 \otimes q^{h_0/2}) P_V$$

$$= \left( (1 + z) \rho_N(z) + (z - 1) \frac{\rho_V(z)}{\rho_V(0)} \right) P_V(e_0 \otimes q^{h_0/2}) N,$$  \hspace{1cm} (V.32)

which can only be satisfied if

$$\rho_N(z) = \frac{1 - z}{1 + z} \cdot \frac{\rho_V(z)}{\rho_V(0)}.$$  \hspace{1cm} (V.33)

Indeed, applying $(V.32)$ to $\bar{V}(\delta_1 + \delta_2) \subset V$ we obtain, using $N \bar{V}(\delta_1 + \delta_2) = (0)$,

$$\left( (1 + z) \rho_N(z) + (z - 1) \frac{\rho_V(z)}{\rho_V(0)} \right) N(e_0 \otimes q^{h_0/2}) \bar{V}(\delta_1 + \delta_2) = (0).$$  \hspace{1cm} (V.34)

Since $N(e_0 \otimes q^{h_0/2}) \bar{V}(\delta_1 + \delta_2) \neq (0)$, one obtains $(V.33)$.

Summarizing, the R-matrix corresponding to $(V.9)$ reads

$$\hat{R}(z) = \rho_N(z) N + \rho_V(z) P_V + \sum_{c=0}^{a} \sum_{k=0}^{c} \prod_{j=1}^{c-k} \langle 2j - 2a \rangle_+ \prod_{i=1}^{c} \langle i - 2a - 1 \rangle_- P_{(2a-2c+k)\delta_1 + k\delta_2},$$  \hspace{1cm} (V.35)

where the primes in the sums signify that terms corresponding to $c = a$ and $k = 0, 1$ are ommitted from the sums, and $\rho_V(z)$, $\rho_N(z)$ are given by

$$\rho_V(z) = \frac{z - q^2}{1 - zq^2} \prod_{j=1}^{a-1} \langle 2j - 2a \rangle_+ \prod_{i=1}^{a-1} \langle i - 2a - 1 \rangle_-,$$

$$\rho_N(z) = (-1)^a q^{-a^2} \frac{1 - z}{1 + z} \rho_V(z),$$  \hspace{1cm} (V.36)

where we have used $\rho_{2\delta_1}(0) = (-1)^{a-1} q^{\frac{1}{2} C_{2\delta_1} - C_{\lambda a}}$ and $\rho_V(0) = (-1)^a q^{-C_{\lambda a}}$.

### VI Conclusions

We have shown how to construct infinite families of new R-matrices with $U_q[\hat{L}_0 = osp(m|n)]$ invariance, arising from the minimal finite dimensional irreducible representations of the twisted quantum affine superalgebra $U_q[gl(m|n)^{(2)}]$.

These R-matrices are the only ones so far constructed with $U_q[osp(m|n)]$ invariance apart from the following special exceptions: (i) Those arising from $\hat{L}_0 = osp(2|2) \cong$
sl(2|1), whose R-matrices are already known from the $gl(m|n)$ case [11]. (ii) Those arising from the vector module of $U_q(\hat{L}_0)$, which is known to be affinizable to an irreducible representation of the untwisted quantum affine superalgebra $U_q[osp(m|n)]^{(1)}$. However the remaining minimal irreducible representations of $U_q(\hat{L}_0)$ are not affinizable in the untwisted sense, as noted in the paper. Moreover even in the case of vector representation, the R-matrices constructed above are different to those arising from the untwisted case.

The R-matrices of this paper will thus give rise to new integrable models with $U_q[osp(m|n)]$ invariance, which will be investigated elsewhere. It is particularly interesting in the case $a = b$, $m = n > 2$, that the R-matrices admit a $U_q[osp(n|n)]$ invariant nilpotent component, a feature not seen previously in the untwisted or non-super cases.

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A Appendix

Throughout $L_0 = \mathfrak{o}(n = 2r) \oplus gl(r)$ denotes the zeroth $\mathbb{Z}$-graded component of $\hat{L}_0 = osp(m = n|n)$. From [16], $V$ in (V.18) admits a composition series of length 3,

$$V \supseteq \tilde{V}(\delta_1 + \delta_2) \supseteq V(\hat{0}|\hat{0}).$$

Here $\tilde{V}(\delta_1 + \delta_2)$ is indecomposable and cyclically generated by a $U_q(\hat{L}_0)$ highest weight vector of weight $\delta_1 + \delta_2$ and is the unique maximal submodule of $V$. $V(\hat{0}|\hat{0})$ is the trivial one dimensional $U_q(\hat{L}_0)$ module which is the unique submodule of $\tilde{V}(\delta_1 + \delta_2)$, while the factor module $V/\tilde{V}(\delta_1 + \delta_2)$ is isomorphic, as a $U_q(\hat{L}_0)$ module, to $V(\hat{0}|\hat{0})$. It is our aim here to prove

**Proposition 1**: The space of $U_q(\hat{L}_0)$-invariant operator on $V$ is spanned by the identity operator $I$ on $V$ and a nilpotent invariant operator $N$, unique up to a scalar multiples, satisfying

$$NV = V(\hat{0}|\hat{0}), \quad N^2 = (0).$$

**Proof.** First we note that $V$ is completely reducible as a $U_q(L_0)$ module, so we have a $U_q(L_0)$ module decomposition

$$V = W \oplus V(\hat{0}|\hat{0})$$

for some $U_q(L_0)$ submodule $W$ of co-dimension 1 in $V$. Let $v_0^+$ be the maximal vector of the irreducible $U_q(L_0)$ module $V(\hat{0}|\hat{0})$ and $\xi \in V$ the canonical generator of the factor
module $V/\tilde{V}(\delta_1 + \delta_2)$, so that

$$(a - \epsilon(a))\xi \in \tilde{V}(\delta_1 + \delta_2), \quad \forall a \in U_q(\hat{L}_0)$$

(A.2)

with $\epsilon$ being the co-unit.

We define an operator $N$ on $V$ by

$$N\xi = v_0^+, \quad N\tilde{V}(\delta_1 + \delta_2) = (0).$$

(A.3)

Since $\xi$ is uniquely determined modulo $\tilde{V}(\delta_1 + \delta_2)$, and $N$ vanishes on this subspace, $N$ is unique, up to scalar multiples. Moreover $N$ is $U_q(\hat{L}_0)$-invariant since, for all $a \in U_q(\hat{L}_0)$,

$$aN\xi = \epsilon(a)v_0^+ = \epsilon(a)N\xi = Na\xi,$$

by (A.2, A.3),

while it is obvious from (A.3) that

$$(aN - Na)\tilde{V}(\delta_1 + \delta_2) = (0) = (aN - Na)V, \quad \forall a \in U_q(\hat{L}_0),$$

so $N$ on $V$ is invariant as stated.

Now let $A \in \text{End}(V)$ be any $U_q(\hat{L}_0)$-invariant and note that the usual form of Schur’s lemma applies to $\tilde{V}(\delta_1 + \delta_2)$. Thus there exists $\alpha \in \mathbb{C}$ such that

$$(A - \alpha I)\tilde{V}(\delta_1 + \delta_2) = (0),$$

and in particular, from (A.2),

$$(A - \alpha I)(a - \epsilon(a))\xi = (a - \epsilon(a))(A - \alpha I)\xi = (0), \quad \forall a \in U_q(\hat{L}_0).$$

(A.4)

In view of the decomposition (A.1) we may write

$$(A - \alpha I)\xi = w + \beta v_0^+$$

for some $w \in W$, $\beta \in \mathbb{C}$. Hence, $\forall a \in U_q(\hat{L}_0),$

$$\epsilon(a)w + \epsilon(a)\beta v_0^+ = \epsilon(a)(A - \alpha I)\xi = a(A - \alpha I)\xi, \quad \text{by (A.4)}$$

$$= aw + \beta av_0^+ = aw + \epsilon(a)\beta v_0^+$$

$$\implies aw = \epsilon(a)w, \quad \forall a \in U_q(\hat{L}_0),$$

so

$$w \in W \cap V(\hat{0}|\hat{0}) = (0).$$

Hence we must have

$$(A - \alpha I)\xi = \beta v_0^+, \quad (A - \alpha I)\tilde{V}(\delta_1 + \delta_2) = (0)$$

so that, from (A.3),

$$A - \alpha I = \beta N \implies A = \alpha I + \beta N,$$

which is sufficient to prove the result.

Finally we note that $N$ acting on $V \subset V(\lambda_a) \otimes V(\lambda_a)$ satisfies the requirements of proposition 1, as desired.
References

[1] E.K. Sklyanin, L.A. Takhtadzhyan, L.D. Faddeev, Theor. Math. Phys. 40 (1979) 194.

[2] V.E. Korepin, N.M. Bogoliubov, A.G. Izergin, Quantum inverse scattering method and correlations functions, Cambridge University Press (1993).

[3] R.J. Baxter, Exactly Solved Models in Statistical Mechanics, Academic Press, London (1982).

[4] A. B. Zamolodchikov, Al. B. Zamolodchikov, Ann. Phys. 120 (1979) 253.

[5] T.J. Hollowood, Int. J. Mod. Phys. A8 (1993) 947.

[6] G.W. Delius, Nucl. Phys. B451 (1995) 445.

[7] M. Jimbo, Introduction to the Yang-Baxter Equation, Int. J. Mod. Phys. A4 (1989) 3759.

[8] R.B. Zhang, M.D. Gould, A.J. Bracken, Nucl. Phys. B354 (1991) 625.

[9] N.J. MacKay, J. Phys. A25 (1992) L1343.

[10] G.W. Delius, M.D. Gould, Y.-Z. Zhang, Nucl. Phys. B432 (1994) 377.

[11] G.W. Delius, M.D. Gould, J.R. Links, Y.-Z. Zhang, Int. J. Mod. Phys. A10 (1995) 3259; J. Phys. A28 (1995) 6203.

[12] G.W. Delius, M.D. Gould, Y.-Z. Zhang, Int. J. Mod. Phys. A11 (1996) 3415.

[13] G.M. Gandenberger, N.J. MacKay, G.M.T. Watts, Nucl. Phys. B465 (1996) 329.

[14] M.D. Gould, J.R. Links, I. Tsohantjis, Y.-Z. Zhang, J. Phys. A30 (1997) 4313.

[15] H. Yamane, On defining relations of the affine Lie superalgebras and their quantized universal enveloping superalgebras, e-print q-alg/9603015.

[16] M.D. Gould, Y.-Z. Zhang, Quasi-spin graded-fermion formalism and \( gl(m|n) \downarrow osp(m|n) \) branching rules, e-print math-ph/9905002.

[17] G.W. Delius, Y.-Z. Zhang, J. Phys. A28 (1995) 1915.