Algebraic Supersymmetry: A case study

Detlev Buchholz
Institut für Theoretische Physik,
Universität Göttingen, Friedrich-Hund-Platz 1,
D-37077 Göttingen, Germany
buchholz@theorie.physik.uni-goettingen.de
FAX: +49-551-399263

Hendrik Grundling
Department of Mathematics,
University of New South Wales,
Sydney, NSW 2052, Australia.
hendrik@maths.unsw.edu.au
FAX: +61-2-93857123

Dedicated to Daniel Kastler on the occasion of his 80th birthday

Abstract

The treatment of supersymmetry is known to cause difficulties in the C*-algebraic framework of relativistic quantum field theory; several no-go theorems indicate that super-derivations and super-KMS functionals must be quite singular objects in a C*-algebraic setting. In order to clarify the situation, a simple supersymmetric chiral field theory of a free Fermi and Bose field defined on \(\mathbb{R}\) is analyzed. It is shown that a meaningful C*-version of this model can be based on the tensor product of a CAR-algebra and a novel version of a CCR-algebra, the “resolvent algebra”. The elements of this resolvent algebra serve as mollifiers for the super-derivation. Within this model, unbounded (yet locally bounded) graded KMS-functionals are constructed and proven to be supersymmetric. From these KMS-functionals, Chern characters are obtained by generalizing formulae of Kastler and of Jaffe, Lesniewski and Osterwalder. The characters are used to define cyclic cocycles in the sense of Connes’ noncommutative geometry which are “locally entire”.

1 Introduction

Graded (super) derivations occur in many parts of physics: supersymmetry, BRS-constraint reduction and cyclic homology, to name a few. To adequately model these in a C*-algebra setting involves notorious domain problems. Kishimoto and Nakamura [16] showed, for example, that apparently natural domain assumptions on the supersymmetry graded derivations lead to an empty theory. Similarly, supersymmetric KMS-functionals underlying the construction of cyclic cocycles as in [15] [11] cannot exist in the case of infinitely extended systems [3]. These obstructions may explain why a general C*-algebraic framework for supersymmetry has not yet
emerged. It thus seems worthwhile to explore representative examples in more detail in order to
identify the pertinent structures.

In the present article we aim to develop tools to define and analyze in a C*-algebra setting
a simple but, with regard to the mathematical problems under investigation, generic supersym-
metric quantum field theory. It is the model of a chiral Fermi– and Bose–field, defined on the
light ray $\mathbb{R}$. As the construction of this model is easily accomplished in the Wightman setting
of (unbounded) quantum fields, we can concentrate here on the specific problems arising in the
passage to a C*-framework.

Although the model has formally the structure of a tensor product of a CAR–algebra and a
CCR–algebra, the adequate formulation of its C*-version requires some care. It turns out that
the standard Weyl algebra description of the CCR part is not suitable for the formulation of
supersymmetry. We therefore introduce a more viable variant of the CCR–algebra, the resolvent
algebra, which formally may be thought of as being generated by the resolvents of the underlying
Bose–field. These resolvents act as mollifiers for the super–derivation and allow one to define it on
a domain which is weakly dense in the underlying C*-algebra in all representations of interest.
The resolvents also lead to a mollified version of the fundamental relation of supersymmetry,
relating the square of the super–derivation and the generator of time translations. These rather
weak variants of supersymmetry turn out to be sufficient for the further analysis.

Having clarified the C*-algebraic formulation of supersymmetry, one has the necessary tools
for the analysis of the supersymmetric KMS–functionals in this model. Again, these functionals
are easily constructed in the Wightman setting. Yet, as follows from general arguments [3], they
cannot be extended continuously to the full underlying C*-algebra. In fact, one does not have
any a priori information on their domains of definition.

In the present model, the restrictions of the supersymmetric KMS–functionals to any local
subalgebra of the underlying C*-algebra turn out to be bounded. Thus these functionals are
densely defined, but their domain of definition does not contain any non–trivial analytic elements
with regard to the dynamics, as is required in the construction of cyclic cocycles given in \[15, \[11\].
Nevertheless, by relying on techniques from the theory of analytic functions of several complex
variables, it is possible to define cyclic cocycles in the present model as well. The restrictions of
these cocycles to any fixed local subalgebra of the underlying C*-algebra turn out to be entire
in the sense of Connes [5].

So the present field–theoretic model allows for a satisfactory C*-algebraic formulation of
supersymmetry and the analysis of its consequences. There are three observations which are of
interest going beyond the present model: First, a C*-algebraic formulation of supersymmetry
has to rely on the concept of mollifiers or, complementary, of unbounded operators affiliated with
the underlying $C^*$–algebra [9]. Second, there is growing evidence that supersymmetric KMS–
functionals, although being unbounded, are locally bounded, in accordance with the heuristic
considerations in [3]. And third, although these functionals generically do not have analytic
elements in their domain of definition, they can still be used to define local versions of Connes’
entire cyclic cocycles by relying on techniques from complex analysis. Based on these insights,
a proper $C^*$–algebraic framework for the formulation of supersymmetry and the analysis of its
consequences in quantum field theory seems within reach. We hope to return to this problem
elsewhere.

The plan of our paper is as follows. We will state our results in the body of the paper, and
defer almost all the proofs to the appendix. In Sect. 2 we present in the Wightman framework the
basic supersymmetry model which we wish to analyze; in Sect. 3 we prepare for its analysis in a
$C^*$–setting by considering algebraic mollifying relations for the quantum fields, which leads to the
study of the $C^*$–algebras generated by the resolvents of the fields. In Sect. 4 we use these tools to
present the $C^*$–algebraic framework of the model. In Sect. 5 we define (unbounded) graded KMS–
functionals on the model and prove basic properties for them, including their supersymmetry
invariance and their local boundedness. In Sect. 6 we use these KMS–functionals to define a
Chern character formula (generalizing the construction in [15, 11]), from which we obtain a
(locally) entire cyclic cocycle in the sense of Connes. This can then be taken as input to an index
theory for supersymmetric quantum field theories, of the type proposed by Longo [18].

2 The model

We begin by presenting here our model in the Wightman framework, which we would like to
model in a $C^*$-algebra setting. It is the the simplest example for supersymmetry on noncompact
spacetime, in that we have one dimension, one boson and one fermion.

We assume chiral fields, so there is only one space-time dimension, $\mathbb{R}$. The Fermi field is given
by the Clifford operators $c(f) = c(f)^*$, where $f \in \mathcal{S}(\mathbb{R}, \mathbb{R})$ and

$$\{c(f), c(g)\} = (f, g) := \int fg \, dx .$$

The boson field is $j(f) = j(f)^*$, where $f \in \mathcal{S}(\mathbb{R}, \mathbb{R})$ and

$$[j(f), j(g)] = i\sigma(f, g) := i \int fg' \, dx .$$

The $\mathbb{Z}_2$–grading automorphism $\gamma$ comes from the Fermi field by

$$\gamma(c(f)) = -c(f), \quad \gamma(j(f)) = j(f)$$
and defines even and odd parts of the polynomial field algebra by $A_\pm = (A \pm \gamma(A))/2$. The heuristic supercharge $Q := \int c(x)j(x) \, dx$ defines the supersymmetry generator $\delta$ as the graded derivation:

$$\delta(A) := [Q, A_+] + \{Q, A_-\}$$

which satisfies $\delta(AB) = \delta(A)B + \gamma(A)\delta(B)$ . Note that on the generating elements of the field algebra we have:

$$\delta(c(f)) = j(f), \quad \delta(j(f)) = ic(f').$$

(1)

Time evolution is given by translation, i.e.

$$\alpha_t(c(f)) := c(f_t) \quad \alpha_t(j(f)) := j(f_t)$$

where $f_t(x) := f(x-t), x \in \mathbb{R}$ . The generator of time evolution is the derivation:

$$\delta_0(c(f)) = ic(f'), \quad \delta_0(j(f)) = ij(f').$$

(2)

The supersymmetry relation is valid on the field algebra:

$$\delta^2 = \delta_0.$$  

(3)

Our problem is to realize this structure in a C*-algebra setting. Some problems already arise from the relation $\delta((c(f)) = j(f)$, in which $\delta$ takes a bounded operator to an unbounded one. We will deal with this issue in the next section. A deeper source of problems will come from the theorems of Kishimoto and Nakamura [16] which will make it hard to realize the supersymmetry relation 3 on a dense domain.

### 3 On Mollifiers and Resolvent Algebras

Here we develop tools to handle the unboundedness of the range elements of $\delta$. Recall that a selfadjoint operator $A$ on a Hilbert space $\mathcal{H}$ is affiliated with a C*-algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ if the resolvent $(i\lambda \mathbb{1} - A)^{-1} \in \mathcal{A}$ for some $\lambda \in \mathbb{R}\setminus\{0\}$ (hence for all $\lambda \in \mathbb{R}\setminus\{0\}$). This notion is used by Georgescu [9] e.a. (and is weaker than the one used by Woronowicz [25]) and it implies the usual one, i.e. that $A$ commutes with all unitaries commuting with $\mathcal{A}$ (but not conversely). Observe that

$$A(i\lambda \mathbb{1} - A)^{-1} = (i\lambda \mathbb{1} - A)^{-1}A = i\lambda(i\lambda \mathbb{1} - A)^{-1} - 1 \in \mathcal{A}.$$  

Thus the resolvent $(i\lambda \mathbb{1} - A)^{-1} = M$ acts as a “mollifier” for $A$, i.e. $\overline{MA}$ and $AM$ are bounded and in $\mathcal{A}$, and $M$ is invertible such that $M^{-1}\overline{MA} = A = AMM^{-1}$. This suggests that as $AM$ and $\overline{MA}$ in $\mathcal{A}$ carries the information of $A$ in bounded form, we can “forget” the original representation, and study the affiliated $A$ abstractly through these elements.
We want to apply this idea to a representation of the bosonic fields \( j(f) = j(f)^* \), \( f \in \mathcal{S}(\mathbb{R}) \) where
\[
[j(f), j(g)] = i\sigma(f, g) := i \int fg' \, dx
\]
on some common dense invariant core \( \mathcal{D} \subset \mathcal{H} \) of the selfadjoint fields \( j(f) \). It seems natural to look for mollifiers in the Weyl algebra
\[
\Delta(\mathcal{S}, \sigma) = C^* \{ \exp(\imath j(f)) \mid f \in \mathcal{S}(\mathbb{R}) \},
\]
(abstractly \( \Delta(\mathcal{S}, \sigma) \)) is the C*-algebra generated by a set of unitaries \( \{ \delta_f \mid f \in \mathcal{S}(\mathbb{R}) \} \) such that \( \delta^*_f = \delta_{-f} \) and \( \delta_f \delta_g = e^{-\imath \sigma(f, g)/2} \delta_{f+g} \). Unfortunately this is not possible because:

\[3.1 \text{ Proposition}\] The Weyl algebra \( \Delta(\mathcal{S}, \sigma) \) contains no nonzero element \( M \) such that \( j(f)M \) is bounded for some \( f \in \mathcal{S}(\mathbb{R}) \setminus \{0\} \). Thus \( \Delta(\mathcal{S}, \sigma) \) contains no mollifier for any nonzero \( j(f) \), and \( j(f) \) is not affiliated with \( \Delta(\mathcal{S}, \sigma) \).

\textbf{Proof:} Assume that \( M \in \Delta(\mathcal{S}, \sigma) \) is nonzero such that \( j(f)M \) is bounded for some nonzero \( f \in \mathcal{S}(\mathbb{R}) \). Let \( U(t) := \exp(\imath t j(f)) \), and denote the spectral resolution of \( j(f) \) by \( j(f) = \int \lambda \, dP(\lambda) \), then
\[
\begin{align*}
\|(U(t) - 1)M\| &= \left\| \int (e^{\imath t\lambda} - 1) dP(\lambda) M \right\| \\
&= |t| \left\| \int \frac{(e^{\imath t\lambda} - 1)}{t\lambda} dP(\lambda) \int \lambda' \, dP(\lambda') M \right\| \\
&\leq C |t| \|j(f)M\| \to 0
\end{align*}
\]
as \( t \to 0 \), where we used the bound \( \left| \frac{\sin t}{t} \right| < C \) for some constant \( C \). Let \( \mathcal{J} \subset \Delta(\mathcal{S}, \sigma) \) consist of all elements \( M \) such that \( \|(U(t) - 1)M\| \to 0 \) as \( t \to 0 \). This is clearly a norm-closed linear space, and by the inequality \( \|(U(t) - 1)MA\| \leq \|(U(t) - 1)M\| \|A\| \) it is also a right ideal. To see that it is a two sided ideal note that
\[
\begin{align*}
\left\| (U(t) - 1)e^{\imath j(g)} M \right\| &= \left\| (U(t)e^{\imath \sigma(f,g)} - 1)M \right\|
\end{align*}
\]
still converges to 0 as \( t \to 0 \), and use the fact that \( \Delta(\mathcal{S}, \sigma) \) is the norm closure of the span of \( \{ e^{\imath j(g)} \mid g \in \mathcal{S}(\mathbb{R}) \} \). But \( \Delta(\mathcal{S}, \sigma) \) is simple, hence \( \mathcal{J} \ni M \) must be zero.

Our solution is to abandon the Weyl algebra as the appropriate C*-algebra to model the bosonic fields \( j(f) \), and instead to choose the unital C*-algebra generated by the resolvents:
\[
C^* \{ 1, R(\lambda, f) \mid \lambda \in \mathbb{R} \setminus \{0\}, f \in \mathcal{S}(\mathbb{R}) \setminus \{0\} \}
\]
where \( R(\lambda, f) := (i\lambda 1 - j(f))^{-1} \). Then by construction all \( j(f) \) are affiliated to this C*-algebra and it contains mollifiers \( R(\lambda, f) \) for all of them.
The above discussion took place in a concrete setting, i.e. represented on a Hilbert space, and we would like to abstract this. Just as the Weyl algebra can be abstractly defined by the Weyl relations, we now want to abstractly define the C*-algebra of resolvents (of the \( j(f) \)) by generators and relations.

### 3.2 Definition

Given a symplectic space \((X, \sigma)\), we define \( \mathcal{R}_0 \) to be the universal unital \(*\)-algebra generated by the set \( \{ R(\lambda, f) \mid \lambda \in \mathbb{R}\setminus0, f \in X\setminus0 \} \) and the relations

\[
R(\lambda, f)^* = R(-\lambda, f) \tag{4}
\]
\[
R(\lambda, f) = \frac{1}{\lambda} R(1, \frac{1}{\lambda} f) \tag{5}
\]
\[
R(\lambda, f) - R(\mu, f) = i(\mu - \lambda)R(\lambda, f)R(\mu, f) \tag{6}
\]
\[
[R(\lambda, f), R(\mu, g)] = i\sigma(f, g)R(\lambda, f)R(\mu, g)^2R(\lambda, f) \tag{7}
\]
\[
R(\lambda, f)R(\mu, g) = R(\lambda + \mu, f + g)[R(\lambda, f) + R(\mu, g) + i\sigma(f, g)R(\lambda, f)^2R(\mu, g)] \tag{8}
\]

where \( \lambda, \mu \in \mathbb{R}\setminus0 \) and \( f, g \in X\setminus0 \), and for (6) we require \( \lambda + \mu \neq 0 \) and \( f + g \neq 0 \). That is, start with the free unital \(*\)-algebra generated by \( \{ R(\lambda, f) \mid \lambda \in \mathbb{R}\setminus0, f \in X\setminus0 \} \) and factor out by the ideal generated by the relations (4) to (8) to obtain the \(*\)-algebra \( \mathcal{R}_0 \).

### 3.3 Remark

(i) The \(*\)-algebra \( \mathcal{R}_0 \) is nontrivial, because it has nontrivial representations. For instance, in a Fock representation of the CCRs over \((X, \sigma)\) we have the CCR-fields \( \varphi(f) \) from which we can define \( \pi(R(\lambda, f)) = (i\lambda \mathbf{1} - \varphi(f))^{-1} \) to obtain a representation of \( \mathcal{R}_0 \).

(ii) Obviously (4) encodes the selfadjointness of \( j(f) \), (5) encodes \( j(\lambda f) = \lambda j(f) \), (6) encodes that \( R(\lambda, f) \) is a resolvent, (7) encodes the canonical commutation relations and (8) encodes additivity \( j(f + g) = j(f) + j(g) \). Moreover, the identity was added explicitly, we do not have that \( R(1,0) = -i\mathbf{1} \), in fact \( R(1,0) \) is undefined.

To define our resolvent C*-algebra, we need to decide on which C*-seminorm to define on \( \mathcal{R}_0 \). The obvious choice is the enveloping C*-norm, however for the purpose of our model, it is more convenient to use a different norm, which we now define. We will say that a state \( \omega \) on the Weyl algebra \( \Delta(X, \sigma) \) is **strongly regular** if the functions

\[
\mathbb{R}^n \ni (\lambda_1, \ldots, \lambda_n) \rightarrow \omega(\delta_{\lambda_1 f_1} \cdots \delta_{\lambda_n f_n})
\]

are smooth for all \( f_1, \ldots, f_n \in X \) and all \( n \in \mathbb{N} \). Of special importance is that the GNS-representation of a strongly regular state has a common dense invariant domain for all the generators \( j(f) \) of the one parameter groups \( \lambda \rightarrow \pi_\omega(\delta_{\lambda f}) \) (this domain is obtained by applying the polynomial algebra of the Weyl operators \( \{ \pi_\omega(\delta_f) \mid f \in X \} \) to the cyclic GNS-vector). Some
important classes of states, e.g. quasi-free states are strongly regular. Denote by $\pi_S$ the direct
sum of the GNS-representations of all strongly regular states, then as the resolvents of the fields
are in $\pi_S(\Delta(X, \sigma'))'$, we can extend $\pi_S$ to a representation of $\mathcal{R}_0$ by the Laplace transform:

$$\pi_S(R(\lambda, f)) := -i \int_0^\infty e^{-\lambda t} \pi_S(\delta_{-tf}) \, dt, \quad \lambda > 0.$$  \hspace{1cm} (9)

We define our resolvent algebra $\mathcal{R}(X, \sigma)$ as the abstract C*-algebra generated by $\pi_S(\mathcal{R}_0)$, i.e.
we factor $\mathcal{R}_0$ by Ker $\pi_S$ and complete w.r.t. the operator norm of $\pi_S$.

We state some elementary properties of $\mathcal{R}(X, \sigma)$.

3.4 Theorem Let $(X, \sigma)$ be a given nondegenerate symplectic space, and define $\mathcal{R}(X, \sigma)$ as above.

Then for all $\lambda, \mu \in \mathbb{R} \setminus 0$ and $f, g \in X \setminus 0$ we have:

(i) $[R(\lambda, f), R(\mu, f)] = 0$. Substitute $\mu = -\lambda$ to see that $R(\lambda, f)$ is normal.

(ii) $\|R(\lambda, f)\| = |\lambda|^{-1}$.

(iii) $R(\lambda, f)$ is analytic in $\lambda$. Explicitly, the series expansion:

$$R(\lambda, f) = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R(\lambda_0, f)^{n+1} i^n, \quad \lambda, \lambda_0 \neq 0 \quad \text{(Von Neumann series)}$$

converges in norm whenever $|\lambda_0 - \lambda| < |\lambda_0|$.

(iv) $R(\lambda, tf)$ is norm continuous in $t \in \mathbb{R} \setminus 0$.

(v) $R(\lambda, f)R(\mu, g)^2R(\lambda, f) = R(\mu, g)R(\lambda, f)^2R(\mu, g)$.

(vi) Let $T \in \text{Sp}(X, \sigma)$ be a symplectic transformation. then $\alpha(R(\lambda, f)) := R(\lambda, Tf)$ defines a
unique automorphism $\alpha \in \text{Aut} \mathcal{R}(X, \sigma)$.

Note that the von Neumann series for $R(\lambda, f)$ converges for any $z \in \mathbb{C}$ with $|z - \lambda_0| < |\lambda_0|$, i.e.
on a disk which stays off the real line. Using different $\lambda_0's$ we can thus define $R(z, f)$ for any
complex $z$ not on the real line and deduce the properties in the definition for these from the
series. Thus we obtain also resolvents $R(z, f)$ for complex $z$ in $\mathcal{R}(X, \sigma)$.

Any operator family $R_\lambda$ satisfying the resolvent equation (6) is called by Hille a pseudo-
resolvent (cf. p215 in [26]), and for such a family we know (cf. Theorem 1 p216 in [26]) that:

- All $R_\lambda$ have a common range and a common null space.
- A pseudo resolvent $R_\lambda$ is the resolvent for an operator $B$ iff Ker $R_\lambda = \{0\}$, and in this case
  Dom $B = \text{Ran} \ R_\lambda$ for all $\lambda$.

Thus we define:
3.5 Definition A regular representation \( \pi \in \text{Rep} \mathcal{R}(X, \sigma) \) is a Hilbert space representation such that

\[
\text{Ker} \, \pi(R(1, f)) = \{0\} \quad \forall \, f \in \mathcal{S}({\mathbb{R}}, {\mathbb{R}}) \backslash 0.
\]

We denote the collection of regular representations by \( \text{Reg} \).

Obviously many regular representations are known, e.g. \( \pi_S \) and the Fock representation. Given a \( \pi \in \text{Rep} \mathcal{R}(X, \sigma) \) with \( \text{Ker} \, \pi(R(1, f)) = \{0\} \), we can define a field operator by

\[
j_\pi(f) := i1 - \pi(R(1, f))^{-1}
\]

with domain \( \text{Dom} \, j_\pi(f) = \text{Ran} \, \pi(R(1, f)) \). Thus for \( \pi \in \text{Reg} \), all the field operators \( j_\pi(f) \), \( f \in \mathcal{S}({\mathbb{R}}, {\mathbb{R}}) \) are defined, and we have the resolvents \( \pi(R(\lambda, f)) = (i\lambda 1 - j_\pi(f))^{-1} \).

3.6 Theorem Let \( \mathcal{R}(X, \sigma) \) be as above, and let \( \pi \in \text{Rep} \mathcal{R}(X, \sigma) \) satisfy \( \text{Ker} \, \pi(R(1, f)) = \{0\} = \text{Ker} \, \pi(R(1, h)) \) for given \( f, h \in X \). Then

(i) \( j_\pi(f) \) is selfadjoint, and \( \pi(R(\lambda, f)) \text{Dom} \, j_\pi(h) \subseteq \text{Dom} \, j_\pi(h) \).

(ii) \( \lim_{\lambda \to \infty} i\lambda \pi(R(\lambda, f))\psi = \psi \) for all \( \psi \in \mathcal{H}_\pi \),

(iii) \( \lim_{s \to 0} i\pi(R(1, sf))\psi = \psi \) for all \( \psi \in \mathcal{H}_\pi \).

(iv) The space \( \mathcal{D} := \pi(R(1, f)R(1, h))\mathcal{H}_\pi \) is a joint dense domain for \( j_\pi(f) \) and \( j_\pi(h) \) and we have: \( [j_\pi(f), j_\pi(h)] = i\sigma(f, h) \) on \( \mathcal{D} \),

(v) \( j_\pi(\lambda f + h) = \lambda j_\pi(f) + j_\pi(h) \) for all \( \lambda \in {\mathbb{R}} \) on \( \mathcal{D} \),

(vi) \( j_\pi(f)\pi(R(\lambda, f)) = \pi(R(\lambda, f))j_\pi(f) = i\lambda \pi(R(\lambda, f)) - 1 \) on \( \text{Dom} \, j_\pi(f) \),

(vii) \( [j_\pi(f), \pi(R(\lambda, h))] = i\sigma(h, f)\pi(R(\lambda, h)^2) \) on \( \text{Dom} \, j_\pi(f) \),

(viii) Denote \( W(f) := \exp(ij_\pi(f)) \), then

\[
W(f)W(h) = e^{i\sigma(f, h)}W(h)W(f)
\]

\[
W(f)\pi(R(\lambda, h))W(f)^* = \pi(R(\lambda + i\sigma(f, h), h)).
\]

Moreover \( W(f)\mathcal{D} \subseteq \mathcal{D} \supseteq W(h)\mathcal{D} \), hence \( \mathcal{D} := \pi(R(1, f)R(1, h))\mathcal{H}_\pi \) is a common core for \( j_\pi(f) \) and \( j_\pi(h) \).

A distinguished regular representation of \( \mathcal{R}(X, \sigma) \) is of course the defining strongly regular representation \( \pi_S \). By definition \( \mathcal{R}(X, \sigma) \) is faithfully represented in it, and moreover, there is a common dense invariant domain \( \mathcal{D}_0 \) for all the field operators \( j_{\pi_S}(f), f \in X \). This domain can be enlarged to a dense invariant domain \( \mathcal{D}_T \) for both the resolvents and the fields simply by applying
all polynomials in $j_{\pi_S}(f)$ and $\pi_S(R(\lambda, f))$ to $D_0$, which makes sense, because from (i) above all resolvents preserve the joint domain $\bigcap \{ \text{Dom} j_{\pi_S}(f) \mid f \in X \}$. Thus we can form the *-algebra of (unbounded) operators

$$E_0 := \ast \text{-alg} \{ j_{\pi_S}(f), \pi_S(R(\lambda, f)) \mid f \in X, \lambda \in \mathbb{R} \setminus \{0\} \}$$

on $\mathcal{D}_T$. Then $E_0$ contains of course the *-algebra $\pi_S(\mathcal{R}_0)$ generated by resolvents alone, which is dense in $\mathcal{R}(X, \sigma)$. We will need these *-algebras $E_0 \supset \pi_S(\mathcal{R}_0)$ below, and will generally not indicate the faithful representation $\pi_S$ w.r.t. which they are defined. Note that for any strongly regular state $\omega$, its cyclic GNS-vector is in the domain of all $j_{\pi_\omega}(f)$, hence $\omega$ extends to define a functional on $E_0$. Thus we give a meaning to all expressions of the form

$$\omega(\delta_{f_1} \cdots \delta_{f_n}) := (\Omega, j_{\pi_\omega}(f_1) \cdots j_{\pi_\omega}(f_n) \pi_\omega(R(\lambda_1, g_1) \cdots R(\lambda_k, g_k)) \Omega) \omega$$

as above. A very important class of states on $\Delta(X, \sigma)$ are the quasifree states, which we will need below. They are given by

$$\omega(\delta_f) = \exp \left( -\frac{1}{2} \langle f|f \rangle_\omega \right), \quad f \in X,$$

where $\langle \cdot | \cdot \rangle_\omega$ is a (possibly semi–definite) scalar product on the complex linear space $X + iX$ satisfying

$$\langle f|g \rangle_\omega - \langle g|f \rangle_\omega = i\sigma(f, g), \quad f, g \in X.$$

Any quasifree state is also regular in the strong sense. By a routine computation one can represent the expectation values of products of Weyl operators in a quasifree state in the form

$$\omega(\delta_{f_1} \cdots \delta_{f_n}) = \exp \left( -\sum_{k<l} \langle f_k|f_l \rangle_\omega - \frac{1}{2} \sum_l \langle f_l|f_l \rangle_\omega \right).$$

Making use of the Laplace transform (9) for the GNS–representation of the resolvents, we have for $\lambda_1, \ldots, \lambda_n > 0$

$$\omega(R(\lambda_1, f_1) \cdots R(\lambda_n, f_n))$$

$$= (-i)^n \int_0^\infty dt_1 \cdots \int_0^\infty dt_n e^{-\sum_{k<l} t_k \lambda_k \omega(\delta_{f_1} \cdots \delta_{f_n})}$$

$$= (-i)^n \int_0^\infty dt_1 \cdots \int_0^\infty dt_n \exp \left( -\sum_k t_k \lambda_k - \sum_{k<l} t_k t_l \langle f_k|f_l \rangle_\omega - \frac{1}{2} \sum_l t_l^2 \langle f_l|f_l \rangle_\omega \right). \quad (10)$$

**Remark:** One can replace anywhere in this equation $f_k$ by $-f_k$, thus it does not impose any restriction of generality to assume that $\lambda_1, \ldots, \lambda_n > 0$. The relation (10) should be regarded as the definition of quasifree states on the resolvent algebra.

In our calculations below, we will frequently need the following differentiability of quasifree states:
3.7 Proposition  Let $\omega$ be a quasifree state as above, and let $x_k \in \mathbb{R} \mapsto f_k(x_k) \in X$, $k = 1, \ldots, n$ be paths in $X$ for which the functions $x_k, x_l \mapsto \langle f_k(x_k) | f_l(x_l) \rangle_\omega$, $k, l = 1, \ldots, n$, are smooth. Then

$$\frac{\partial}{\partial x_r} \omega \left( R(\lambda_1, f_1(x_1)) \cdots R(\lambda_n, f_1(x_n)) \right)$$

$$= - \sum_{k=1}^{r-1} \frac{\partial}{\partial x_r} \left( \langle f_k(x_k) | f_r(x_r) \rangle_\omega \right) \frac{\partial^2}{\partial \lambda_r \partial \lambda_k} \omega \left( R(\lambda_1, f_1(x_1)) \cdots R(\lambda_n, f_1(x_n)) \right)$$

$$- \frac{1}{2} \frac{\partial}{\partial x_r} \left( \langle f_r(x_r) | f_r(x_r) \rangle_\omega \right) \frac{\partial^2}{\partial \lambda_r^2} \omega \left( R(\lambda_1, f_1(x_1)) \cdots R(\lambda_n, f_1(x_n)) \right)$$

$$- \sum_{k=r+1}^{n} \frac{\partial}{\partial x_r} \left( \langle f_r(x_r) | f_k(x_k) \rangle_\omega \right) \frac{\partial^2}{\partial \lambda_r \partial \lambda_k} \omega \left( R(\lambda_1, f_1(x_1)) \cdots R(\lambda_n, f_1(x_n)) \right)$$

and all the partial derivatives involved in this formula exist.

4 C*-algebra formulation of Supersymmetry.

Here we want to write our model of Section 2 in a C*-algebra framework. However, to motivate our choices made below, let us recall a theorem of Kishimoto and Nakamura [16]:

4.1 Theorem  Let $\mathcal{A}$ be a C*-algebra with $\mathbb{Z}_2$-grading $\gamma$, let $\alpha : \mathbb{R} \to \Aut \mathcal{A}$ be a pointwise continuous action with generator $\delta_0$ having a smooth domain $C^\infty(\delta_0) := \bigcap_{n=1}^{\infty} \Dom(\delta_0^n)$. Let $\delta$ be a closable graded derivation with $\Dom(\delta) \supset C^\infty(\delta_0)$, $\delta \circ \alpha_t = \alpha_t \circ \delta$ for all $t$, and $\delta^2 = \delta_0$ on $C^\infty(\delta_0)$. Then $\delta$ is bounded.

Thus it will be hard to obtain the supersymmetry relation on natural dense domains.

For the fermion field, let $\mathcal{H} = L^2(\mathbb{R})$ and define $\text{CAR}(\mathcal{H})$ in Araki’s self-dual form (cf. [11]) as follows. On $\mathcal{K} := \mathcal{H} \oplus \mathcal{H}$ define an antiunitary involution $\Gamma$ by $\Gamma(h_1 \oplus h_2) := \overline{h_2} \oplus h_1$. Then $\text{CAR}(\mathcal{H})$ is the unique simple C*-algebra with generators $\{ \Phi(k) \mid k \in \mathcal{K} \}$ such that $k \to \Phi(k)$ is antilinear, $\Phi(k)^* = \Phi(\Gamma k)$, and

$$\{ \Phi(k_1), \Phi(k_2)^* \} = (k_1, k_2) \mathbb{1}, \quad k_i \in \mathcal{K}.$$ 

The correspondence with the heuristic creators and annihilators of fermions is given by $\Phi(h_1 \oplus h_2) = a(h_1) + a^*(\overline{h_2})$, where

$$a(h) = \int a(x) \overline{h(x)} \, dx, \quad a^*(h) = \int a^*(x) h(x) \, dx.$$ 

To obtain the Clifford operators $c(f) = c(f)^*$, $f \in \mathcal{S}(\mathbb{R}, \mathbb{R})$ we take $c(f) := \Phi(f \oplus f)/\sqrt{2}$, in which case we have $c(f) = c(f)^*$ and $\{c(f), c(g]\} = \langle f, g \rangle = \int fg \, dx$. Let $\text{Cliff}(\mathcal{S}(\mathbb{R})) := C^\star \{ c(f) \mid f \in \mathcal{S}(\mathbb{R}) \}$ and notice that $\overline{c}(f) := i\Phi(f \oplus -f)/\sqrt{2}$ also satisfies the Clifford relations.
hence generates another copy of $\text{Cliff}(S(\mathbb{R}))$ in $\text{CAR}(\mathcal{H})$, and together these two Clifford algebras generate all of $\text{CAR}(\mathcal{H})$. In fact, since the $c(f)$ and $\bar{c}(g)$ anticommute, we have that $\text{CAR}(\mathcal{H}) \cong \text{Cliff}(S(\mathbb{R}) \oplus S(\mathbb{R}))$. Conversely, if we are given a real pre-Hilbert space $X$ with complexified completion $Y$ and a projection $P$ and antiunitary involution $\Gamma$ such that $\Gamma PT = 1 - P$ and these preserve $X$, then we have an isomorphism $\text{Cliff}(X) \cong \text{CAR}(Y)$ given by $\Phi(x) = (c(Px) - ic(\Gamma Px) + c(PTx) + ic((1 - P)x))/\sqrt{2}$.

For the bosonic part we take the resolvent algebra $\mathcal{R}(S(\mathbb{R}), \sigma)$ where $\sigma(f, g) := \int f g' \, dx$, and so the full $\text{C}^*$-algebra in which we want to define our model is

$$\mathcal{A} := \text{Cliff}(S(\mathbb{R})) \otimes \mathcal{R}(S(\mathbb{R}), \sigma)$$

where the tensor norm is unique because the CAR-algebra is nuclear. The grading automorphism $\gamma$ is the identity on $\mathcal{R}(S(\mathbb{R}), \sigma)$, and $\gamma(\Phi(k)) = -\Phi(k)$ for all $k$ on the CAR-part.

Next, we want to define on some suitable domain in $\mathcal{A}$ the supersymmetry graded derivation $\delta$ corresponding to the relations $[\Gamma]$. First, considering $\delta(j(f)) = ic(f')$, since $\delta$ is a derivation on the bosonic part, it is natural to define

$$\delta(R(\lambda, f)) := ic(f') R(\lambda, f)^2 \in \mathcal{A}. \quad (11)$$

However due to the unbounded rhs of $\delta(c(f)) = j(f)$ we cannot define $\delta$ directly on the $c(f)$, so we need to multiply by mollifiers. Define

$$\zeta(f) := c(f)R(1, f), \quad (12)$$

then

$$\delta(\zeta(f)) := iR(1, f) - 1 + ic(f')R(1, f)^2 \in \mathcal{A}. \quad (13)$$

where we made use of the graded derivation property, the relations $[\Gamma]$ and $j(f)R(\lambda, f) = i\lambda R(\lambda, f) - 1$. Next, we would like to extend $\delta$ as a graded derivation to the $\ast$-algebra generated by these basic objects:

$$\mathcal{D}_S := \ast\text{-alg} \{ 1, R(\lambda, f), \zeta(f) \mid \lambda \in \mathbb{R}\setminus 0, f \in S(\mathbb{R})\setminus 0 \} \subset \mathcal{A}. \quad (14)$$

Observe that $\mathcal{D}_S$ is not norm-dense in $\mathcal{A}$, however due to Theorem 3.6(ii) applied to $R(\lambda, f)c(f) = \zeta(f/\lambda)$, it will be strong operator dense in $\mathcal{A}$ in any regular representation. Note that $\delta$ does not preserve $\mathcal{D}_S$, it takes its image in the norm dense $\ast$-algebra

$$\mathcal{A}_0 := \ast\text{-alg} \{ 1, R(\lambda, f), c(f) \mid \lambda \in \mathbb{R}\setminus 0, f \in S(\mathbb{R})\setminus 0 \} \subset \mathcal{A}. \quad (15)$$

To see that $\delta$ extends as a graded derivation to $\mathcal{D}_S$, we proceed as follows. Let $\pi_0$ be any representation of $\text{Cliff}(S(\mathbb{R}))$ then $\pi_0 \otimes \pi_S$ is a faithful representation of $\mathcal{A}$ and there is a common dense invariant domain $\mathcal{D} := \mathcal{H}_{\pi_0} \otimes \mathcal{D}_T$ for all $\pi_0(c(f)) \otimes 1, 1 \otimes j_{\pi_S}(f)$ and $1 \otimes \pi_S(R(\lambda, f))$ where
Denotes the domain of \( \mathcal{E}_0 \) defined at the end of Section 3 (Henceforth we will not indicate tensoring by 1 nor the representations \( \pi_0, \pi_0 \) when the context makes clear what is meant). Let

\[
\mathcal{E} := \ast\text{-algebra}\{ c(f), j(f), R(\lambda, f) \mid f \in \mathcal{S}(\mathbb{R}), \lambda \in \mathbb{R}\setminus 0 \} \supset \mathcal{A}_0
\]

so we have the \( \ast \)-algebras \( \mathcal{R}_0 \subset \mathcal{E}_0 \subset \mathcal{E} \) on \( \mathcal{D} \). Define on the generating elements of \( \mathcal{E} \) a map \( \overline{\delta} \), setting

\[
\overline{\delta}(j(f)) = ic(f'), \\
\overline{\delta}(R(\lambda, f)) = ic(f')R(\lambda, f)^2, \\
\overline{\delta}(c(f)) = j(f).
\]

We will see that this map extends to a graded derivation on \( \mathcal{E} \). For the proof it suffices to show that \( \overline{\delta} \) is linear and satisfies the graded Leibniz rule on any finite polynomial involving operators \( j(f), R(\lambda, f) \) and \( c(f) \), i.e. in each instance only a finite number of test functions \( f \) and real parameters \( \lambda \) are involved. We will take advantage of this fact as follows.

Let \( X_s \subset \mathcal{S}(\mathbb{R}) \) be any finite–dimensional subspace and consider the subalgebra \( \mathcal{E}(X_s) \subset \mathcal{E} \) generated by the elements \( j(f), R(\lambda, f) \) and \( c(f) \) with \( \lambda \in \mathbb{R}\setminus 0, f \in X_s \). We extend \( X_s \) to a space \( X_s' \subset \mathcal{S}(\mathbb{R}) \) by adding to the elements of \( X_s \) also their first derivatives. Picking in \( X_s' \) some (finite) orthonormal basis \( \{h_n\} \) with regard to the scalar product \( (\cdot, \cdot) \), we have for any \( f \in X_s \) the “completeness relations” \( \sum_n (h_n, f) h_n = f, \sum_n (h_n, f') h_n = f' \).

Next, we define an operator \( Q_s \in \mathcal{E} \), setting

\[
Q_s = \sum_n c(h_n)j(h_n).
\]

As \( Q_s \) is of fermionic (odd) type, we can consistently define with the help of it a graded derivation \( \overline{\delta}_s \) on \( \mathcal{E} \), setting for even and odd elements \( E_\pm \in \mathcal{E} \), respectively,

\[
\overline{\delta}_s(E_+) = [Q_s, E_+] = Q_s E_+ - E_+ Q_s, \quad \overline{\delta}_s(E_-) = \{Q_s, E_-\} = Q_s E_- + E_- Q_s.
\]

Computing the action of \( \overline{\delta}_s \) on the even elements \( j(f), R(\lambda, f) \) and odd elements \( c(f) \), where \( \lambda \in \mathbb{R}\setminus 0, f \in X_s \), we obtain from the basic relations in \( \mathcal{E} \) by some elementary algebraic manipulations.

\[
\overline{\delta}_s(j(f)) = i\sum_n c(h_n) (h_n, f') = ic(f'), \\
\overline{\delta}_s(R(\lambda, f)) = i\sum_n c(h_n) (h_n, f') R(\lambda, f)^2 = ic(f')R(\lambda, f)^2, \\
\overline{\delta}_s(c(f)) = \sum_n (h_n, f) j(h_n) = j(f).
\]

Thus we conclude that the action of \( \overline{\delta} \) on the generating elements of \( \mathcal{E}(X_s) \) coincides with the action of the graded derivation \( \overline{\delta}_s \). As the choice of the subspace \( X_s \) was arbitrary, it follows that \( \overline{\delta} \) extends to a graded derivation on the whole polynomial algebra \( \mathcal{E} \).
The final step consists in showing that the action of $\delta$ on the generating elements $R(\lambda, f), \zeta(f)$ of $\mathcal{D}_S$ coincides with the action of the graded derivation $\bar{\delta}$. But this follows immediately from the relations given above. Thus $\delta$ extends to a graded derivation with domain $\mathcal{D}_S$ and range in $\mathcal{A}_0$. Uniqueness is clear from the graded derivation property, so we have proven:

4.2 Theorem There is a unique graded derivation $\delta : \mathcal{D}_S \to \mathcal{A}$ satisfying relations (11) and (13).

Next we need to define the time evolution derivation $\delta_0$ in this C*-setting. From the equations (2) this suggest that we define on $\mathcal{E}$ a *-derivation $\bar{\delta}_0$ satisfying:

$$\bar{\delta}_0(j(f)) = ij(f'),$$
$$\bar{\delta}_0(R(\lambda, f)) = iR(\lambda, f)j(f')R(\lambda, f),$$
$$\bar{\delta}_0(c(f)) = ic(f')$$

and then proceed to the corresponding mollified relations in $\mathcal{A}$. For the proof that $\bar{\delta}_0$ extends to a *-derivation on $\mathcal{E}$, we proceed as in the discussion of the superderivation: We pick any finite dimensional subspace $X_s \subset \mathcal{S}(\mathbb{R})$, consider the corresponding subalgebra $\mathcal{E}(X_s) \subset \mathcal{E}$ and choose in the extended space $X_s' \subset \mathcal{S}(\mathbb{R})$, containing the elements of $X_s$ and their first derivatives, some orthonormal basis $\{h_n\}$. In addition to the completeness relations mentioned above we will also make use of $\sum_n (h_n, f) h_n' = f'$ for $f \in X_s$.

We consider now the symmetric operator in $\mathcal{E}$

$$H_s = \frac{1}{2} \sum_n \{ic(h_n')c(h_n) + j(h_n)j(h_n)\}.$$ 

Putting

$$\bar{\delta}_0 s(\cdot) = [H_s, \cdot],$$

it induces a *-derivation on $\mathcal{E}$. Its action on the generating elements of $\mathcal{E}(X_s)$ can easily be computed:

$$\bar{\delta}_0 s(j(f)) = \sum_n j(h_n) i(h_n, f') = i j(f'),$$
$$\bar{\delta}_0 s(R(\lambda, f)) = \sum_n i(h_n, f') R(\lambda, f) j(h_n) R(\lambda, f) = i R(\lambda, f) j(f') R(\lambda, f),$$
$$\bar{\delta}_0 s(c(f)) = \frac{1}{2} \sum_n i\{(h_n, f) c(h_n') + (h_n, f') c(h_n)\} = i c(f').$$

Thus we conclude as in the preceding discussion that the action of $\bar{\delta}_0$ on the generating elements of $\mathcal{E}(X_s)$ coincides with the action of the derivation $\bar{\delta}_0 s$. As $X_s$ was arbitrary, it follows that $\bar{\delta}_0$ extends to a derivation on the whole polynomial algebra $\mathcal{E}$.

The supersymmetry relation $\bar{\delta}^2 = \bar{\delta}_0$ can now be verified on the generating elements of $\mathcal{E}$ and thus holds on the whole algebra $\mathcal{E}$.
The question now is how one should define the time evolution \( \delta_0 \) and the square \( \delta^2 \) on the C*-algebra \( \mathcal{A} \) from the unbounded versions in \( \mathcal{E} \). Since \( \delta : \mathcal{D}_S \to \mathcal{A}_0 \), its square \( \delta^2 \) does not make sense on \( \mathcal{D}_S \). Note however that for every \( A \in \mathcal{A}_0 \) there is a monomial \( M \in \mathcal{D}_S \) of resolvents \( R(\lambda, f) \) such that
\[
AM \in \mathcal{D}_S \ni MA.
\]
(By Theorem 3.6(ii) we know that in regular representations we can let these mollifiers \( M \) go to \( 1 \) in the strong operator topology.)

4.3 Definition

For each \( A \in \mathcal{D}_S \) let \( M_A \in \mathcal{D}_S \) be a monomial of resolvents \( R(\lambda, f) \) such that
\[
M_A \delta(A) \in \mathcal{D}_S.
\]
Define
\[
M_A \delta^2(A) := \delta(M_A \delta(A)) - \delta(M_A) \delta(A) \in \mathcal{A}_0.
\]
Note that this definition coincides with \( M_A \delta^2(A) \) in \( \mathcal{E} \), however the definition above involves only bounded quantities, so it can be defined independently in the C*-setting on \( \mathcal{D}_S \). Of course we then have the mollified SUSY–relations \( M_A \delta^2(A) = M_A \tilde{\delta}_0(A) \) for all \( A \in \mathcal{D}_S \) from the unbounded SUSY relation in \( \mathcal{E} \). This is not however acceptable for a bounded SUSY–relation until we have demonstrated the connection of \( M_A \tilde{\delta}_0(A) \) with the time evolution. The time evolution \( \alpha : \mathbb{R} \to \text{Aut} \mathcal{A} \) is just translation, as this is a chiral theory
\[
\alpha_t(c(f)) := c(f_t), \quad \alpha_t(R(\lambda, f)) = R(\lambda, f_t).
\]
The desired connection
\[
M_A \tilde{\delta}_0(A) = -i \frac{d}{dt} M_A \alpha_t(A)
\]
exists only in specific regular representations on suitable domains, and for these one will then have supersymmetry. In many applications, one only needs the supersymmetry weakly, i.e.
\[
\omega(BM_A \tilde{\delta}_0(A)C) = -i \frac{d}{dt} \omega(BM_A \alpha_t(A)C)
\]
for \( A, B, C \) in a suitable domain and \( \omega \) a distinguished functional. We will verify this relation explicitly below for the functionals used in our constructions.

5 Graded KMS–functionals.

Graded KMS–functionals are used in supersymmetric theories to calculate cyclic cocycles [11, 15], and here we want to develop this theory in the current context for our simple supersymmetric model as a first application of it.
5.1 Definition Let $A$ be a unital $C^*$-algebra with a grading automorphism $\gamma \in \text{Aut} A$, $\gamma^2 = 1$, and a (pointwise continuous) action $\alpha : \mathbb{R} \to \text{Aut} A$ such that $\alpha_t \circ \gamma = \gamma \circ \alpha_t$ for all $t$. Then a graded KMS–functional is a (possibly unbounded) functional $\varphi$ on $A$ such that

(i) $\text{Dom} \varphi$ is a unital dense $^*$–subalgebra of $A$ such that

$$\gamma(\text{Dom} \varphi) \subseteq \text{Dom} \varphi \supseteq \alpha_t(\text{Dom} \varphi) \quad \forall t,$$

(ii) For all $A, B \in \text{Dom} \varphi$ there is a continuous complex function $F_{A,B} : S \to \mathbb{C}$ on the strip $S := \mathbb{R} + i[0,1]$ which is analytic on the interior of $S$ and satisfying on the boundary:

$$F_{A,B}(t) = \varphi(A \alpha_t(B)) \quad \forall t$$

$$F_{A,B}(t + i) = \varphi(\alpha_t(B)\gamma(A)) \quad \forall t \in \mathbb{R}.$$ 

(iii) For $A, B \in \text{Dom} \varphi$ we have

$$|F_{A,B}(t + is)| < C(1 + |t|)^N \quad \forall t \in \mathbb{R}, \ s \in (0, 1)$$

and some $C \in \mathbb{R}^+$ and $N \in \mathbb{N}$ depending on $A$ and $B$.

5.2 Example Below for our model, we will define on

$$A = \text{Cliff}(\mathcal{S}(\mathbb{R})) \otimes \mathcal{R}(\mathcal{S}(\mathbb{R}), \sigma)$$

a functional $\varphi = \psi \otimes \omega$, with $\text{Dom} \varphi = A_0$ where $\psi$ and $\omega$ are quasi-free with two-point functions

$$\omega(j^2(f)) = \int \frac{p}{1 - e^{-p}} |\hat{f}(p)|^2 dp,$$

$$\psi(c(f)c(g)) = \lim_{\varepsilon \to 0^+} \int \frac{p}{1 - e^{-p}} \frac{e^{-p} - e^{-p - \varepsilon}}{p^2 + \varepsilon^2} \hat{f}(p) \hat{g}(p) dp$$

and we will verify that $\varphi = \psi \otimes \omega$ is a graded KMS–functional. Note that $\omega$ is a state on $\mathcal{R}(\mathcal{S}(\mathbb{R}), \sigma)$, but $\psi$ is unbounded and nonpositive. It does however satisfy supersymmetry, in that $\varphi \circ \delta = 0$, and equation (14) holds weakly.

The motivation for using graded KMS–functionals come from several sources:

- Physicists used graded KMS–functionals to construct supersymmetric field theories in a thermal background [8, 10].

- Jaffe e.a. [11] and Kastler [15] used graded KMS–functionals to construct cyclic cocycles in Connes’ cyclic cohomology.
The reasons why one has to use nonpositive unbounded KMS–functionals for field theories on noncompact spacetime are as follows. First, there is the theorem of Buchholz and Ojima [4] that supersymmetry breaks down in spatially homogeneous KMS–states, and second there is the theorem of Buchholz and Longo [3] that if \( \varphi \) is a bounded graded KMS–functional of \( \mathcal{A} \) with time evolution \( \alpha : \mathbb{R} \to \text{Aut} \mathcal{A} \), and if there are \( \beta_n \in \text{Aut} \mathcal{A} \) such that

\[
\lim_{n \to \infty} \varphi(\mathcal{C}[A, \beta_n(B)]) = 0 \quad \forall A, B, C \in \mathcal{A}
\]

then \( \alpha = \iota \). On noncompact spaces, the translations will produce the \( \beta_n \) in a local field theory. Thus in local field theories on noncompact spaces, we are inevitably led to unbounded graded KMS–functionals for supersymmetry.

First, we would like to establish a few general properties of graded KMS–functionals.

5.3 Proposition Given a graded KMS-functional \( \varphi \) defined w.r.t. the data \((\mathcal{A}, \gamma, \alpha)\), then

(i) \( \varphi \) is \( \alpha \)–invariant.

(ii) \( \varphi \) is \( \gamma \)–invariant.

Moreover if a functional \( \varphi \) satisfies the graded KMS–property on a subset \( Y \subset \text{Dom} \varphi \subset \mathcal{A} \), then it also satisfies the graded KMS–property on \( \text{Span} Y \).

5.4 Proposition Let \( \mathcal{A} = \mathcal{C} \otimes \mathcal{B} \) where \( \mathcal{C} \) and \( \mathcal{B} \) are unital C*-algebras with \( \mathcal{C} \) nuclear. Let \( \sigma : \mathbb{R} \to \text{Aut} \mathcal{C} \) and \( \beta : \mathbb{R} \to \text{Aut} \mathcal{B} \) be dynamical systems, and let \( \gamma \in \text{Aut} \mathcal{C} \) be a grading automorphism, \( \gamma^2 = \iota \). Let \( \omega \in \mathcal{S}(\mathcal{B}) \) be a KMS–state on \( \mathcal{B} \) w.r.t. \( \beta \), and let \( \psi \) be a graded KMS–functional on \( \mathcal{C} \) w.r.t. \( \sigma \). Define a functional \( \varphi := \psi \otimes \omega \) with \( \text{Dom} \varphi := \text{Span} \{ C \otimes B \mid C \in \text{Dom} \psi, B \in \mathcal{B} \} \) by \( \varphi(C \otimes B) := \psi(C)\omega(B) \). Then \( \varphi \) is a graded KMS–functional w.r.t. the grading \( \gamma \otimes \iota \), and the C*-dynamical system \( \sigma \otimes \beta : \mathbb{R} \to \text{Aut} (\mathcal{C} \otimes \mathcal{B}) \).

Thus for our model, as \( \mathcal{A} = \text{Cliff}(\mathcal{S}(\mathbb{R})) \otimes \mathcal{R}(\mathcal{S}(\mathbb{R}), \sigma) \), it suffices to define a graded KMS–functional \( \psi \) on \( \text{Cliff}(\mathcal{S}(\mathbb{R})) \) and a KMS–state \( \omega \) on \( \mathcal{R}(\mathcal{S}(\mathbb{R}), \sigma) \) from which we can then construct the graded KMS–functional \( \varphi := \psi \otimes \omega \). We start by defining the KMS–state \( \omega \) on \( \mathcal{R}(\mathcal{S}(\mathbb{R}), \sigma) \).

5.5 Theorem (i) There is a quasi–free state on \( \overline{\Delta(\mathcal{S}, \sigma)} \) defined by \( \omega(\delta_f) := \exp[-s(f, f)/2] \), \( f \in \mathcal{S}(\mathbb{R}, \mathbb{R}) \) where

\[
s(f, g) := \int\frac{p}{1-e^{-p}}\hat{f}(p)\overline{\hat{g}(p)}\,dp = \langle f|g\rangle_{\omega}.
\]

(ii) This quasi–free state \( \omega \) on \( \overline{\Delta(\mathcal{S}, \sigma)} \) extends to a KMS–functional on \( \pi(\overline{\Delta(\mathcal{S}, \sigma)})'' \), hence on \( \mathcal{R}(\mathcal{S}(\mathbb{R}), \sigma) \), where it is defined by Equation (10). The time evolution used for the KMS–condition is translation of test functions \( f \).
Next, we would like to define a graded KMS–functional $\psi$ on $\text{Cliff}(S(\mathbb{R}))$ with $\text{Dom} \psi = *$-alg \{ $c(f) \mid f \in S(\mathbb{R})$ \}. By the last part of Proposition 5.3 it suffices to define $\psi$ and check its KMS-properties on the monomials $c(f_1) \cdots c(f_n)$. Recall that a quasi–free functional on the Clifford algebra is uniquely defined by its two point functional and the relations:

$$\psi(c(f_1) \cdots c(f_{2k+1})) = 0$$

$$\psi(c(f_1) \cdots c(f_{2k})) = (-1)^{\frac{k}{2}} \sum_{P} (-1)^{p} \prod_{j=1}^{k} \psi(c(f_{P(j)})c(f_{P(k+j)}))$$

where $k \in \mathbb{N}$ and $P$ is any permutation of $\{1, 2, \ldots, 2k\}$ such that $P(1) < \cdots < P(k)$ and $P(j) < P(k+j)$ for $j = 1, \ldots, k$ (cf. p89 in [21]). Using this formula, we define a quasi–free functional $\psi$ with two point function

$$\theta(f, g) := \psi(c(f)c(g)) = \lim_{\epsilon \rightarrow 0^+} \int \frac{p}{1 - e^{-p}} \frac{p}{p^2 + \epsilon^2} \hat{f}(p) \overline{g(p)} \, dp$$

$$= \lim_{\epsilon \rightarrow 0^+} \left( \int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right) \frac{1}{1 - e^{-p}} \hat{f}(p) \overline{g(p)} \, dp$$

$$= \lim_{\epsilon \rightarrow 0^+} \left( \int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right) \left( \frac{1}{p} \right) \frac{p}{1 - e^{-p}} \hat{f}(p) \overline{g(p)} \, dp$$

$$= \mathcal{P} \left( \frac{1}{p} \right) (G)$$

where $\mathcal{P} \left( \frac{1}{p} \right)$ denotes the distribution consisting of the Cauchy Principal Part integral of $1/p$, and $G(p) := \frac{p}{1 - e^{-p}} \hat{f}(p) \overline{g(p)}$ which is differentiable everywhere and of fast decay since $\frac{p}{1 - e^{-p}}$ is differentiable and of linear growth, and $f, g$ are real-valued Schwartz functions. Thus $\theta(f, g)$ is well-defined for all $f, g \in S(\mathbb{R})$.

We mention as an aside that the quasifreeness of a graded KMS–functional $\psi$ on the Clifford algebra and the formula for its two–point function are a consequence of the graded KMS–condition, as can be shown by similar arguments as in [23].

This quadratic form $\theta$ is unbounded and not positive definite, because $(1 - e^{-p})^{-1}$ is unbounded and not positive. $\theta$ has the following useful properties.

5.6 Theorem Let $f, g \in S(\mathbb{R})$, then

(i) $\theta(f, g) = (g, (P + T)f)$ where $P = 2\pi \times \text{projection onto positive spectrum of } D := id/dx$, and $T$ is an unbounded operator given explicitly by

$$(g, Tf) = 2i \int dx \int dy \, f(x)g(y)(x - y) \int_{0}^{\infty} dp \ln(1 - e^{-p}) \cos(p(x - y)).$$

Moreover $P_TTP_T$ is trace–class and selfadjoint for all compact intervals $J \subset \mathbb{R}$ where $P_T$ is the projection onto $L^2(J) \subset L^2(\mathbb{R})$.
(ii) For $z \in S = \mathbb{R} + i[0,1]$ define

$$G(z) := \theta(f, g_z) := \lim_{\varepsilon \to 0^+} \left( \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \frac{e^{ipz}}{1 - e^{-p}} \hat{f}(p) \overline{\hat{g}(p)} \, dp.$$ 

Then $G$ is continuous on $S$, analytic on its interior, and satisfies

$$|G(t + is)| \leq A + B|t| \quad \text{for} \ t \in \mathbb{R}, \ s \in [0,1]$$

and constants $A, B$.

Using these properties, one can now establish that:

\[5.7 \text{ Theorem}\]

The quasi–free functional $\psi$ with two point functional $\theta$ and domain $\text{Dom} \psi = \ast\text{-alg} \{ c(f) \mid f \in S(\mathbb{R}) \}$, is a graded KMS–functional on $\text{Cliff}(S(\mathbb{R}))$ where time evolution is given by translation.

Thus by Proposition 5.4 we have a KMS–functional $\varphi = \psi \otimes \omega$ on $\mathcal{A}$ with domain

$$\text{Dom} \varphi = \text{Span} \{ C \otimes R \mid C \in \text{Dom} \psi, R \in \mathcal{R}(S(\mathbb{R}), \sigma) \}.$$ 

What makes this KMS–functional interesting, is that it satisfies supersymmetry, i.e.

\[5.8 \text{ Theorem}\]

For the quasifree functional $\varphi$ above, we have that

(i) $\mathcal{A}_0 \subset \text{Dom} \varphi$,

(ii) $\varphi(\delta(A)) = 0$ for all $A \in \mathcal{D}_S$

(iii) $\varphi(BM_A \delta_0(A)C) = -i \frac{d}{dA} \varphi(BM_A \alpha_t(A)C) \bigg|_0 = \varphi(BM_A \delta^2(A)C)$ for all $A \in \mathcal{D}_S$ and $B, C \in \mathcal{A}_0$ where $M_A$ is a monomial of resolvents $R(\lambda, f)$ such that $M_A \delta(A) \in \mathcal{D}_S$ (as in Definition 4.3).

Whilst the functional $\varphi$ is unbounded, it is locally bounded in the sense of the theorem below.

For the local algebras, let $J \subset \mathbb{R}$ be a bounded interval and define

$$\mathcal{A}(J) := C^* \{ c(f), R(\lambda, f) \mid \text{supp } f \subset J, f \in S(\mathbb{R}), \lambda \in \mathbb{R}\setminus\{0\} \}$$

$$\mathcal{A}_0(J) := \ast\text{-alg} \{ c(f), R(\lambda, f) \mid \text{supp } f \subset J, f \in S(\mathbb{R}), \lambda \in \mathbb{R}\setminus\{0\} \}$$

hence $\mathcal{A}_0(J)$ is a dense $\ast$-algebra of $\mathcal{A}(J)$, and $\mathcal{A}_0(J) \subset \text{Dom} \varphi$. Then:

\[5.9 \text{ Theorem}\]

For the quasifree functional $\varphi$ above, and a bounded interval $J \subset \mathbb{R}$ we have that

$$\|\varphi^{\uparrow} \mathcal{A}_0(J)\| \leq \exp(K|J|^2)$$

where $K$ is a constant (independent of $J$), and $|J|$ is the length of $J$.

Thus $\varphi$ is bounded on all the local algebras $\mathcal{A}_0(J)$. 

18
6 The JLO–cocycle.

From an assumed supersymmetry structure on a C*-algebra and a KMS-functional, Jaffe, Lesniewski and Osterwalder [11] and Kastler [15] constructed with a Chern character formula an entire cyclic cocycle in the sense of Connes [5]. Their assumed supersymmetry assumptions are too restrictive to include quantum field theories on noncompact spacetimes. Here we want to show that we can adapt the JLO cocycle formula to produce a well-defined entire cyclic cocycle for our model, using the KMS-functional in the preceding section. We first need to make sense of the Chern character formula:

$$\tau_n(a_0, \ldots, a_n) := i^n \int_{\sigma_n} \varphi(a_0 \alpha_{i_1}(\delta \gamma(a_1)) \alpha_{i_2}(\delta(a_2)) \alpha_{i_3}(\delta \gamma(a_3)) \cdots \alpha_{i_n}(\delta \gamma(a_n))) \; ds_1 \cdots ds_n, \quad a_i \in D_S, \quad (19)$$

where $\epsilon_n := n \mod 2$, $\varphi$ is the graded KMS–functional above w.r.t. the data $\gamma$, $\alpha$, $\delta$ above, and

$$\sigma_n := \{ s \in \mathbb{R}^n \mid 0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \leq 1 \}.$$ 

Since only $\alpha : \mathbb{R} \to \text{Aut} \; \mathcal{A}$ is given, the expressions $\alpha_{i_s}, \; s \in \mathbb{R}$ in the formula are undefined, and need to be interpreted. Let $b_0, \ldots, b_n \in \mathcal{A}_0 \subset \text{Dom} \; \varphi$, then using the KMS–property of $\varphi$, the function

$$Q(t_1, \ldots, t_n) := \varphi(b_0 \alpha_{t_1}(b_1 \alpha_{t_2}(b_2 \cdots \alpha_{t_n}(b_n))) \cdots) = \varphi(b_0 \alpha_{t_1}(b_1) \cdots \alpha_{t_1+\cdots+t_n}(b_n))$$

can be analytically extended in each variable $t_j$ into the strip $\{ z_j \mid 0 \leq \text{Im} \; z_j \leq 1 \}$ keeping the other variables real. This produces $n$ functions $Q^j : T_j \to \mathbb{C}$ where $T_j := \{ z \in \mathbb{C}^n \mid z_j \in \mathbb{R} + i[0,1], \; z_k \in \mathbb{R} \text{ for } k \neq j \}$ such that $Q^j|_{\mathbb{R}^n} = Q^j|_{\mathbb{R}^n}$ for all $i, j$. The sets $T_j$ are flat tubes, i.e. of the form $T_B = \mathbb{R}^n + iB$ where the basis $B \subset \mathbb{R}^n$ is of dimension less than $n$. To continue, we now need the Flat Tube Theorem [2]:

6.1 Theorem Let $T_{B_1}, T_{B_2} \subset \mathbb{C}^n$ be two flat tubes whose bases $B_1, B_2$ are convex, with closures which contain 0 and are star-shaped w.r.t. 0. Let $F_1, F_2$ be any two functions analytic in $T_{B_1}, T_{B_2}$ respectively, with continuous boundary values on $\mathbb{R}^n$ and such that $F_1|_{\mathbb{R}^n} = F_2|_{\mathbb{R}^n}$. Then there is a unique function $F$ extending $F_1$ and $F_2$ analytically into the tube $\overline{T_{B_1 \cup B_2}}$ (where $B_1 \cup B_2$ is the convex hull of $B_1 \cup B_2$) and with continuous boundary values on $\mathbb{R}^n$. We have that $T_{B_1 \cup B_2} = \bigcup_{0 \leq \lambda \leq 1} T_{B_\lambda}$ where $B_\lambda := (1 - \lambda)B_1 + \lambda B_2$.

Using this inductively, we can extend $Q$ by analytic continuation into the tube $T_n := \mathbb{R}^n + i\Sigma_n$ where $\Sigma_n$ is the convex hull of the unit intervals on the axes, i.e. the simplex $\Sigma_n := \{ s \in \mathbb{R}^n \mid 0 \leq s_i \forall i, \; s_1 + \cdots + s_n \leq 1 \}$. So we will interpret

$$\varphi(b_0 \alpha_{z_1}(b_1) \cdots \alpha_{z_1+\cdots+z_n}(b_n)) := Q(z_1, \ldots, z_n)$$
for \( z \in \mathcal{T}_n \) to be this unique analytic continuation. The change of variables \( w_1 := z_1, w_2 := z_1 + z_2, \ldots, w_n := z_1 + \cdots + z_n \) defines an invertible complex linear map \( W : \mathbb{R}^n + i\Sigma_n \to \mathbb{R}^n + i\sigma_n \), so both \( W \) and \( W^{-1} \) are analytic, and hence

\[
(Q \circ W^{-1})(w_1, \ldots, w_n) =: \varphi(b_0 \alpha_{w_1}(b_1) \cdots \alpha_{w_n}(b_n))
\]

is analytic on \( \mathbb{R}^n + i\sigma_n \). In particular

\[
\int_{\Sigma_n} \varphi(b_0 \alpha_{i_1}(b_1) \cdots \alpha_{i_1+\cdots+i_n}(b_n)) \, dr_1 \cdots dr_n = \int_{\sigma_n} \varphi(b_0 \alpha_{i_1}(b_1) \cdots \alpha_{i_n}(b_n)) \, ds_1 \cdots ds_n
\]

by the change of variables \( s_1 = r_1, s_2 = r_1 + r_2, \ldots, s_n = r_1 + \cdots + r_n \). By the substitutions \( a_0 = b_0, b_1 = \delta \gamma(a_1), \ldots, b_n = \delta \gamma^n(a_n) \) into this formula, we arrive at a consistent interpretation of the Chern character formula \([19]\).

Let us recall from \([11, 12, 5]\) the definition of an entire cyclic cocycle.

**6.2 Definition** Equip \( \mathcal{D}_S \) with the Sobolev norm \( \| a \|_* = \| a \| + \| \delta a \| \), and for any \(*\)-algebra \( \mathcal{D} \subseteq \mathcal{D}_S \) let \( C^n(\mathcal{D}) \) denote the space of \((n+1)\)-linear functionals on \( \mathcal{D} \) which are continuous w.r.t. the norm \( \| \cdot \|_* \), and let \( \| \cdot \|_* \) denote also the norm on \( C^n(\mathcal{D}) \) w.r.t. the norm \( \| \cdot \|_* \) on \( \mathcal{D} \). Define the space of cochains \( C(\mathcal{D}) \) to be the space of sequences \( \rho = (\rho_0, \rho_1, \ldots) \) where \( \rho_n \in C^n(\mathcal{D}) \) which satisfy the entire analyticity condition:

\[
\lim_{n \to \infty} n^{1/2} \| \rho_n \|_*^{1/n} = 0 .
\]

The entire cyclic cohomology is defined by a coboundary operator \( \partial = b + B \) on \( C(\mathcal{D}) \) for operators

\[
\begin{align*}
\partial : C^n(\mathcal{D}) &\to C^{n+1}(\mathcal{D}), \\
B : C^{n+1}(\mathcal{D}) &\to C^n(\mathcal{D})
\end{align*}
\]

i.e.

\[
(\partial \rho)(a_0, \ldots, a_n) = (b \rho_{n-1})(a_0, \ldots, a_n) + (B \rho_{n+1})(a_0, \ldots, a_n)
\]

where \( b \) and \( B \) are given by:

\[
(b \rho_n)(a_0, \ldots, a_{n+1}) = \sum_{j=0}^{n} (-1)^j \rho_n(a_0, \ldots, a_j a_{j+1}, \ldots, a_{n+1})
\]

\[
+(-1)^{n+1} \rho_n(\tilde{a}_{n+1} \cdot a_0, a_1, \ldots, a_n)
\]

\[
(B \rho_n)(a_0, \ldots, a_{n-1}) = \rho_n(1, a_0, \ldots, a_{n-1}) + (-1)^{n-1} \rho_n(a_0, \ldots, a_{n-1}, 1)
\]

\[
+\sum_{j=1}^{n-1} (-1)^{(n-1)j} \left[ \rho_n(1, \gamma(a_{n-j}), \ldots, \gamma(a_{n-1}), a_0, \ldots, a_{n-j-1})
\right]
\]

\[
+(-1)^{n-1} \rho_n(\gamma(a_{n-j}), \ldots, \gamma(a_{n-1}), a_0, \ldots, a_{n-j-1}, 1)
\]

where \( \tilde{a} := \left\{ \begin{array}{ll} \gamma(a) & \text{if } \rho_n \in C^n_+(\mathcal{D}) \\ a & \text{if } \rho_n \in C^n_-(\mathcal{D}) \end{array} \right. \)

where \( C^n_+(\mathcal{D}) \) denotes the even part under \( \gamma \), and \( C^n_-(\mathcal{D}) \) the odd part. The entire cyclic cocycles are those \( \rho \in C(\mathcal{D}) \) for which \( \partial \rho = 0 \), i.e.

\[
(b \rho_{n-1})(a_0, \ldots, a_n) = -(B \rho_{n+1})(a_0, \ldots, a_n), \quad n = 1, 2, \ldots
\]
Below we will use the even part of $\tau$ to define an entire cyclic cocycle.

### 6.3 Theorem

For $\tau_n$ defined in Equation (19) we have that

$$|\tau_n(a_0, \ldots, a_n)| \leq \frac{A}{n!} e^{Bn} \|a_0\|_* \cdots \|a_n\|_*$$

for all $a_i \in \mathcal{D}_k := \ast\text{-alg} \{1, R(1, f), \zeta(f) \mid \text{supp}(f) \subseteq [-k, k]\} \subset \mathcal{D}_S$ and where $\|a\|_* := \|a\| + \|\delta a\|$ as above, and for some constants $A$ and $B$ which depend on $k > 0$ but are independent of $n$. Thus condition (20) holds for $\tau$, i.e.

$$\lim_{n \to \infty} n^{1/2} \|\tau_n\|_1^{1/n} = 0.$$  

Using this, we can now prove that:

### 6.4 Theorem

The sequence $\tilde{\tau} := (\tau_0, 0, -\tau_2, 0, \tau_4, 0, \ldots) \in \mathcal{C}(\mathcal{D}_k)$ defines an entire cyclic cocycle for each $k > 0$, i.e.

$$(b\tau_{n-1})(a_0, \ldots, a_n) = (B\tau_{n+1})(a_0, \ldots, a_n), \quad n = 1, 3, 5, \ldots$$

and the entire analyticity condition holds.

It is possible to have taken the choice $\partial = b - B$ for the cyclic coboundary operator above; this is in fact done in [13], and would have led to the cyclic cocycle $(\tau_0, 0, \tau_2, 0, \tau_4, 0, \ldots)$ instead of $\tilde{\tau}$ above.

Note that whilst we have obtained entire cyclic cocycles on each compact set $[-k, k]$ these do not define an entire cyclic cocycle on $\mathcal{D}_{\text{comp}} := \ast\text{-alg} \{1, R(1, f), \zeta(f) \mid \text{supp}(f) \text{ is compact}\} \subset \mathcal{D}_S$ because one can choose a sequence $\{a_0, a_1, \ldots\}$ with $a_j \in \mathcal{D}_{k_j}$ where $k_j$ grows sufficiently fast so that through the dependencies of the constants $A$ and $B$ in Theorem 6.3 on $k_j$ the entire analytic condition fails. One expects to use an inductive limit argument, to define an index on $\mathcal{D}_{\text{comp}}$ from the indices on the $\mathcal{D}_k$.

From this cyclic cocycle we can calculate an index for this quantum field theory, but its physical significance is presently unclear, though one would expect it to remain stable under deformations. This type of index is discussed in more detail in Longo [18].

### 7 Conclusions

In this paper we have explored how supersymmetric quantum fields can be treated in a C*-algebra setting, avoiding the obstructions found by Kishimoto and Nakamura [16] and by Buchholz and Longo [3]. We did this in detail for a simple one-dimensional model.
In order to establish a reasonable domain of definition for the super-derivation, we found it necessary to analyze a notion of “mollifiers” for the quantum fields and to introduce a corresponding C*-algebra, the resolvent algebra. The full algebra $\mathcal{A}$ defining the model can then be taken as the tensor product of this resolvent algebra and the familiar CAR-algebra.

The super-derivation is defined on a subalgebra which is weakly dense in $\mathcal{A}$ in all representations of physical interest; alternatively, one can define it on a norm dense subalgebra of $\mathcal{A}$ with range in a *-algebra $\mathcal{E}$ of bounded and unbounded operators which are affiliated with $\mathcal{A}$ in the sense of [9]. Similarly, the basic supersymmetry relation can either be formulated in a mollified form on some weakly dense domain or, alternatively, as a relation between maps which have been extended to the *-algebra $\mathcal{E}$. These findings reveal some basic features of supersymmetric quantum field theories which have to be taken into account in a general C*-framework covering such theories. The tools developed here should also be useful in other areas of quantum field theory where one needs to use graded derivations, e.g. in BRS-constraint theory.

We also exhibited in the present model graded KMS-functionals for arbitrary positive temperatures which are supersymmetric. In accordance with the general results in [3], these functionals are unbounded. Yet their restrictions to any local subalgebra of the underlying C*-algebra $\mathcal{A}$ are bounded. It is an interesting question whether these functionals are also locally normal with respect to the vacuum representation of the theory, as one would heuristically expect.

The KMS-functionals were then employed to define cyclic cocycles. In view of the fact that the domain of definition of these functionals does not contain analytic elements with regard to the time evolution, the strategy outlined in [15, 11] could not be applied here. That these cocycles can be constructed, nevertheless, is due to the fact that the functionals inherit sufficiently strong analyticity properties from the KMS-condition which allow one to perform the necessary complex integrations. Moreover, the resulting cocycles are entire on all local algebras.

These functionals may thus be taken as an input for a quantum index theory as suggested by Longo [18]. Such an index should be stable under deformations, and one can easily think of possible deformations of our model, e.g. deform the supersymmetry generator $Q$ by an appropriate function $M$:

$$Q_M := \int M(x) j(x) c(x) \, dx$$

so

$$\delta_M(c(f)) = j(Mf), \quad \delta_M(j(f)) = c(Mf')$$

thus:

$$\delta^2_M =: \delta_0 M$$

defines the new generator for time evolution.
8 Appendix

In this appendix we give proofs of the statements in the main body of the text.

Proof of Theorem 3.4

(i) By (6) we have that \( i(\mu - \lambda)R(\lambda, f)R(\mu, f) = R(\lambda, f) - R(\mu, f) = -(R(\mu, f) - R(\lambda, f)) = i(\mu - \lambda)R(\mu, f)R(\lambda, f) \), i.e. \([R(\lambda, f), R(\mu, f)] = 0\).

(ii) The Fock representation is a subrepresentation of \( \pi_S \), and the resolvents of the fields \( \varphi(f) \) give the Fock representation induced on \( R(X, \sigma) \), i.e. \( \pi(R(\lambda, f)) = (i\lambda - \varphi(f))^{-1} \). Since this is nonzero, \( R(\lambda, f) \neq 0 \) for all nonzero \( f \) and \( \lambda \). Now by \( R(\lambda, f)^* = R(-\lambda, f) \) we get

\[
2|\lambda||R(\lambda, f)||^2 = \|2\lambda R(\lambda, f)R(\lambda, f)^*\| = \|R(\lambda, f) - R(\lambda, f)^*\| \leq 2\|R(\lambda, f)\|. 
\]

Thus by \( R(\lambda, f) \neq 0 \) we find \( \|R(\lambda, f)\| \leq 1/|\lambda| \). Now

\[
\|R(\lambda, f)\| \geq \|\pi(R(\lambda, f))\| = \|(i\lambda - \varphi(f))^{-1}\| = \sup_{t \in \sigma(\varphi(f))} \left| \frac{1}{i\lambda - t} \right| = \frac{1}{|\lambda|}
\]

using the fact that the spectrum \( \sigma(\varphi(f)) = \mathbb{R} \). Thus \( \|R(\lambda, f)\| = 1/|\lambda| \).

(iii) Rearrange equation (6) to get:

\[
R(\lambda, f)\left(1 - i(\lambda_0 - \lambda)R(\lambda_0, f)\right) = R(\lambda_0, f).
\]

Now by (ii) above, if \(|\lambda_0 - \lambda| < |\lambda_0| \) then \( \|i(\lambda_0 - \lambda)R(\lambda_0, f)\| < 1 \), and hence \( (1 - i(\lambda_0 - \lambda)R(\lambda_0, f))^{-1} \) exists, and is given by a norm convergent power series in \( i(\lambda_0 - \lambda)R(\lambda_0, f) \). That is, we have that

\[
R(\lambda, f) = R(\lambda_0, f)(1 - i(\lambda_0 - \lambda)R(\lambda_0, f))^{-1} = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R(\lambda_0, f)^{n+1} i^n 
\]

when \(|\lambda_0 - \lambda| < |\lambda_0| \), as claimed.

(iv) From equation (6) we get \( R(\lambda, tf) = \frac{1}{t} R(1, \frac{1}{t} f) = \frac{1}{t} R(\frac{1}{t}, f) \) so

\[
R(\lambda, sf) - R(\lambda, tf) = R(\frac{1}{s} \lambda, f) - R(\frac{1}{t} \lambda, f) 
= \frac{1}{s} \left( R(\frac{1}{s} \lambda, f) - R(\frac{1}{t} \lambda, f) \right) + \left( \frac{1}{s} - \frac{1}{t} \right) R(\frac{1}{t} \lambda, f) 
= \frac{i\lambda}{s} \left( \frac{1}{s} - \frac{1}{t} \right) R(\frac{1}{s} \lambda, f) R(\frac{1}{t} \lambda, f) + \left( \frac{1}{s} - \frac{1}{t} \right) R(\frac{1}{t} \lambda, f). 
\]

Thus \( \|R(\lambda, sf) - R(\lambda, tf)\| \leq \left| \frac{1}{s} - \frac{1}{t} \right| 2 \left| \frac{1}{t} \lambda \right| \) from which continuity away from zero is clear.

(v) This follows directly from equation (7) by interchanging \( \lambda \) and \( f \) with \( \mu \) and \( g \) resp.

(vi) Recall that we have a faithful (strongly regular) representation \( \pi_S \) of \( R(X, \sigma) \) which is an extension of a regular representation of the Weyl algebra \( \Delta(X, \sigma) \) such that \( \pi_S(\Delta(X, \sigma))'' \supset \)
\( \pi_S(\mathcal{R}(X, \sigma)) \) and such that each \( \pi_S(R(\lambda, f)) \) is the resolvent of the generator \( j_{\pi_S}(f) \) of the one-parameter group \( t \to \pi_S(\delta_{t^f}) \). Let \( T \in \text{Sp}(X, \sigma) \) then it defines an automorphism of \( \Delta(X, \sigma) \) by \( \alpha_T(\delta_f) = \delta_{Tf} \) which preserves the set of strongly regular states, and in fact defines a bijection on the set of strongly regular states by \( \omega \to \omega \circ \alpha_T \). Now \( \pi_S \) is the direct sum of the GNS-representations of all the strongly regular states, and hence \( \pi_S \circ \alpha_T \) is just \( \pi_S \) where its direct summands have been permuted. Such a permutation of direct summands can be done by conjugation of a unitary, thus \( \pi_S \) is unitarily equivalent to \( \pi_S \circ \alpha_T \), and so we can extend \( \alpha_T \) by unitary conjugation to \( \pi_S (\Delta(X, \sigma))'' \). By equation (9) we get that \( \alpha_T (\pi_S (R(\lambda, f))) = \pi_S (R(\lambda, Tf)) \), and hence \( \alpha_T \) preserves \( \mathcal{R}(X, \sigma) \), so defines an automorphism on it.

**Proof of Theorem 3.6**

(i) Observe that by Theorem 1 p216 of Yosida [20], we deduce from \( \text{Ker} \pi(R(1, f)) = \{0\} \) that \( \pi(R(\lambda, f)) \) is the resolvent of \( j_{\pi}(f) \), i.e. we have now for all \( \lambda \neq 0 \) that \( j_{\pi}(f) = i\lambda 1 - \pi(R(\lambda, f))^{-1} \). Then

\[
j_{\pi}(tf) = i1 - \pi(R(1, tf))^{-1} = i1 - t\pi(R(1/f, f))^{-1} = t j_{\pi}(f).
\]

Thus

\[
j_{\pi}(f)^* = (i1 - \pi(R(1, f))^{-1})^* \supseteq -i1 - (\pi(R(1, f))^{-1})^* = -i1 - \pi(R(1, f))^{-1}
\]

\[
= -i1 + \pi(R(1, -f))^{-1} = -j_{\pi}(-f) = j_{\pi}(f)
\]

and hence \( j_{\pi}(f) \) is symmetric. To see that it is selfadjoint note that:

\[
\text{Ran} (j_{\pi}(f) \pm i1) = \text{Ran} (-\pi(R(\pm1, f))^{-1}) = \text{Dom} (\pi(R(\pm1, f))) = \mathcal{H}_{\pi}
\]

hence the deficiency spaces \( (\text{Ran} (j_{\pi}(f) \pm i1))^\perp = \{0\} \) and so \( j_{\pi}(f) \) is selfadjoint.

For the domain claim, recall that \( \text{Dom} j_{\pi}(f) = \text{Ran} \pi(R(1, f)) \). So

\[
\pi(R(\lambda, f))\text{Dom} j_{\pi}(h) = \pi(R(\lambda, f))\pi(R(1, h))\mathcal{H}_{\pi}
\]

\[
= \pi \left( R(1, h)R(\lambda, f) + i\sigma(f, h)R(1, h)R(\lambda, f)^2 R(1, h) \right) \mathcal{H}_{\pi}
\]

\[
\subseteq \pi \left( R(1, h) \right) \mathcal{H}_{\pi} = \text{Dom} j_{\pi}(h).
\]

(ii) Let \( j_{\pi}(f) = \int \lambda dP(\lambda) \) be the spectral resolution of \( j_{\pi}(f) \). Then \( \pi(R(\mu, f)) = \int \frac{i\mu}{i\mu - \lambda} dP(\lambda) \) hence

\[
i\mu \pi(R(\mu, f))\psi = \int \frac{i\mu}{i\mu - \lambda} dP(\lambda)\psi \quad \forall \psi \in \mathcal{H}_{\pi}.
\]
Since \( \left| \frac{i\mu}{i\mu - \lambda} \right| < 1 \) for \( \mu \in \mathbb{R}\setminus\{0\} \) the integrand is dominated by 1 which is an \( L^1 \)-function with respect to \( dP(\lambda) \), and as we have pointwise that \( \lim_{\mu \to \infty} \frac{i\mu}{i\mu - \lambda} = 1 \), we can apply the dominated convergence theorem to get that

\[
\lim_{\mu \to \infty} i\mu \pi(R(\mu, f))\psi = \int dP(\lambda)\psi = \psi.
\]

(iii) \( i\pi(\lambda, f) = \int \frac{i}{\pi} dP(\lambda) \psi \to \psi \) as \( s \to 0 \) by the same argument as in (ii).

(iv) Let \( D := \pi(R(1, f))\pi(1, h)H_\pi \), then by definition \( D \subseteq \text{Ran} \pi(R(1, f)) = \text{Dom} j_\pi(f) \). Moreover \( \pi(R(1, f)R(1, h))H_\pi = \pi(R(1, h)[R(1, f) + i\sigma(f, h)R(1, f)^2R(1, h)]\pi(1, f) = \text{Dom} j_\pi(h) \), i.e. \( D \subseteq \text{Dom} j_\pi(f) \cap \text{Dom} j_\pi(h) \). That \( D \) is dense, follows from (iii) of this theorem, using

\[
\lim_{s \to 0} \text{lim}_{\mu \to \infty} \pi(R(1, s)R(1, th) \psi = -\psi
\]

for all \( \psi \in H_\pi \), as well as \( sR(1, sf) = R(1/s, f) \) and the fact mentioned before (cf. Theorem 1 p216 in [20]) that all \( \pi(R(\lambda, f)) \) have the same range for \( f \) fixed.

Let \( \psi \in D \), i.e. \( \psi = \pi(R(1, f))\pi(1, h)\varphi \) for some \( \varphi \in H_\pi \). Then

\[
\pi(R(1, h)R(1, f))[j_\pi(f), j_\pi(h)]\psi
\]

\[
= \pi(R(1, h)R(1, f))[\pi(R(1, f))^{-1}, \pi(R(1, h))^{-1}]\pi(R(1, f)R(1, h))\varphi
\]

\[
= \pi(R(1, f)R(1, h) - R(1, h)R(1, f))\varphi = i\sigma(f, h)\pi(R(1, h)R(1, f)^2R(1, h))\varphi
\]

\[
= i\sigma(f, h)\pi(R(1, h)R(1, f))\psi.
\]

Since \( \text{Ker} \pi(R(1, h)R(1, f)) = \{0\} \) it follows that \( [j_\pi(f), j_\pi(h)] = i\sigma(f, h) \) on \( D \).

(v) From Equation (5) we have that

\[
\pi(R(\lambda, f)) = (1i\lambda - j_\pi(f))^{-1} = \frac{1}{\lambda} \pi(R(1, \frac{1}{\lambda} f)) = \frac{1}{\lambda} \cdot \left( i\lambda - j_\pi(1 f) \right)^{-1}
\]

and hence that \( j_\pi(f) = \lambda j_\pi(\frac{1}{\lambda} f) \), i.e. \( j_\pi(\lambda f) = \lambda j_\pi(f) \) for all \( \lambda \in \mathbb{R}\setminus\{0\} \). In equation 8:

\[
\pi(R(\lambda, f) R(\mu, g)) = \pi \left( R(\lambda + \mu, f + g)[R(\lambda, f) + R(\mu, g) + i\sigma(f, g)R(\lambda, f)^2R(\mu, g)] \right)
\]

multiply on the left by \( i(\lambda + \mu)1 - j_\pi(f + g) \) and apply to \((i\mu1 - j_\pi(g))(i\lambda1 - j_\pi(f))\psi, \psi \in D\) to get

\[
(i(\lambda + \mu)1 - j_\pi(f + g))\psi = ((i\mu1 - j_\pi(g)) + (i\lambda1 - j_\pi(f)))\psi
\]

making use of \([(i\mu1 - j_\pi(g)), (i\lambda1 - j_\pi(f))] \psi = i\sigma(g, f) \psi \). Thus \( j_\pi(f + g) = j_\pi(f) + j_\pi(g) \) on \( D \).

(vi) From the spectral resolution for \( j_\pi(f) \) we have trivially that on \( \text{Dom} j_\pi(f) \)

\[
j_\pi(f) \pi(R(\mu, f)) = \pi(R(\mu, f)) j_\pi(f) = \int \frac{\lambda}{i\mu1 - \lambda} dP(\lambda) = i\mu \pi(R(\mu, f)) - 1.
\]
(vii) Let $\psi \in \text{Dom } j_\pi(f) = \text{Ran } \pi(R(\lambda, f))$, i.e. $\psi = \pi(R(\lambda, f))\varphi$ for some $\varphi \in \mathcal{H}_\pi$. Then
\[
\pi(R(\lambda, f))[j_\pi(f), \pi(R(\lambda, g))]\psi = \pi(R(\lambda, f))[j_\pi(f), \pi(R(\lambda, g))]\pi(R(\lambda, f))\varphi = \pi([R(\lambda, g), R(\lambda, f)])\varphi = i\sigma(g, f)\pi \left(R(\lambda, f)R(\lambda, g)^2 R(\lambda, f)\right) \varphi = i\sigma(g, f)\pi \left(R(\lambda, f)R(\lambda, g)^2 \right) \psi.
\]
Since $\text{Ker } \pi(R(\lambda, f)) = \{0\}$, it follows that
\[
[j_\pi(f), \pi(R(\lambda, g))] = i\sigma(g, f)\pi \left(R(\lambda, g)^2 \right)
\]
on $\text{Dom } j_\pi(f)$.

(viii) We first prove the second equality. Let $\psi, \phi \in \tilde{D} := \text{Span } \{\chi_{[-a,a]}(j_\pi(f))\mathcal{H}_\pi \mid a > 0\}$ where $\chi_{[-a,a]}$ indicates the characteristic function of $[-a, a]$, and note that $\tilde{D}$ is a dense subspace. Since $\|j_\pi(f)^n|\chi_{[-a,a]}(j_\pi(f))\mathcal{H}_\pi\| \leq a^n$, $n \in \mathbb{N}$, we can use the exponential series, i.e.
\[
W(f)\psi := \exp(i j_\pi(f))\psi = \sum_{n=0}^{\infty} \frac{(i j_\pi(f))^n}{n!}\psi \quad \forall \psi \in \tilde{D}.
\]
By the usual rearrangement of series we then have
\[
(\varphi, W(f)\pi(R(\lambda, h))W(f)^*\psi) = \sum_{n=0}^{\infty} \frac{1}{n!} (\varphi, (\text{ad } i j_\pi(f))^n(\pi(R(\lambda, h)))\psi)
\]
for all $\varphi, \psi \in \tilde{D}$. Using part (vii) we have
\[
(\text{ad } i j_\pi(f))(\pi(R(\lambda, h)^k)) = k \sigma(f, h)\pi(R(\lambda, h)^{k+1})
\]
thus
\[
(\text{ad } i j_\pi(f))^n(\pi(R(\lambda, h))) = n! \sigma(f, h)^n \pi(R(\lambda, h)^{n+1})
\]
so
\[
(\varphi, W(tf)\pi(R(\lambda, h))W(tf)^*\psi) = \sum_{n=0}^{\infty} t^n \sigma(f, h)^n \left(\varphi, \pi(R(\lambda, h)^{n+1})\psi\right) = (\varphi, \pi(R(\lambda + it\sigma(f, h), h))\psi)
\]
whenever $|t\sigma(f, h)| < |\lambda|$ and where we made use of the Von Neumann series (Theorem 3.3(iii)) in the last step. Since the operators involved are bounded and $\tilde{D}$ is dense, it follows that $W(tf)\pi(R(\lambda, h))W(tf)^* = \pi(R(\lambda + it\sigma(f, h), h))$ for $|t\sigma(f, h)| < |\lambda|$. By analyticity in $\lambda$ this can be extended to complex $\lambda$ such that $\lambda \notin i\mathbb{R}$. Using the group property of $t \mapsto W(tf)$ we then obtain for $\lambda \in \mathbb{R}\setminus\{0\}$ that
\[
W(f)\pi(R(\lambda, h))W(f)^* = \pi(R(\lambda + i\sigma(f, h), h)).
\] (25)
To prove the first equation, let us write $W(h)$ in terms of resolvents. Note that $\lim_{n \to \infty} (1-it/n)^{-n} = e^{it}$, $t \in \mathbb{R}$ and so by the bound: $\sup_{t \in \mathbb{R}} |(1-it/n)^{-n}| = \sup_{t \in \mathbb{R}} (1 + t^2/n^2)^{-n} = 1$, it follows from spectral theory (cf. Theorem VIII.5(d), p262 in [22]) that
\[
W(h) = e^{ij_\pi(h)} = \lim_{n \to \infty} (1 - ij_\pi(h)/n)^{-n} = \lim_{n \to \infty} \pi \left(i R(1, -h/n)\right)^n
\]
in strong operator topology. Apply equation (25) to this to get

\[ W(f)W(h)W(f)^* = \lim_{n \to \infty} \pi \left( iR(1 + i\sigma(f, -\frac{h}{n}) - \frac{h}{n}) \right)^n \]

\[ = \lim_{n \to \infty} (1 - i(\sigma(f, h) + j_\pi(h))/n)^{-n} \]

\[ = \exp[i\sigma(f, h) + ij_\pi(h)] = e^{i\sigma(f, h)}W(h) \]

as required. Now

\[ W(f)D = W(f)\pi(R(\lambda, f)R(\mu, h))H_\pi = \pi(R(\lambda, f)R(\mu + i\sigma(f, h), h))H_\pi = D \]

hence we conclude that \( D \) is a core for \( j_\pi(f) \) (cf. Theorem VIII.11, p269 in [22]).

**Proof of Proposition 3.7**

Recall Equation (10)

\[ \omega(R(\lambda_1, f_1) \cdots R(\lambda_n, f_n)) \]

\[ = (-i)^n \int_0^\infty dt_1 \cdots \int_0^\infty dt_n \exp \left( - \sum_k t_k \lambda_k - \sum_{k<l} t_k t_l \langle f_k | f_l \rangle \omega - \frac{1}{2} \sum_l t_l^2 \langle f_l | f_l \rangle \omega \right) \]

then the integrand \( F(x) \) is differentiable as a function of \( x \in \mathbb{R}^n \), because by assumption all \( x_k, x_l \mapsto \langle f_k(x_k)|f_l(x_l)\rangle\omega \) are continuously differentiable. Thus by the chain rule we obtain for its partial derivatives that

\[ \frac{\partial}{\partial x_r} F(x) = \frac{\partial}{\partial x_r} \left[ - \sum_k t_k \lambda_k - \sum_{k<l} t_k t_l \langle f_k | f_l \rangle \omega - \frac{1}{2} \sum_l t_l^2 \langle f_l | f_l \rangle \omega \right] F(x) \]

\[ = \left[ - \sum_{k=1}^{r-1} t_r t_k \frac{\partial}{\partial x_r} \langle f_k(x_k)|f_r(x_r)\rangle\omega - \frac{1}{2} t_r^2 \frac{\partial}{\partial x_r} \langle f_r(x_r)|f_r(x_r)\rangle\omega \right. \]

\[ - \sum_{k=r+1}^n t_r t_k \frac{\partial}{\partial x_r} \langle f_r(x_r)|f_k(x_k)\rangle\omega \] \quad (26)

\[ = - \sum_{k=1}^{r-1} \frac{\partial}{\partial x_r} \langle f_k(x_k)|f_r(x_r)\rangle\omega \frac{\partial^2}{\partial \lambda_r \partial \lambda_k} F(x) - \frac{1}{2} \frac{\partial}{\partial x_r} \langle f_r(x_r)|f_r(x_r)\rangle\omega \frac{\partial^2}{\partial \lambda_r^2} F(x) \]

\[ - \sum_{k=r+1}^n \frac{\partial}{\partial x_r} \langle f_r(x_r)|f_k(x_k)\rangle\omega \frac{\partial^2}{\partial \lambda_r \partial \lambda_k} F(x) \] \quad (27)

making use of \( t \exp(-t\lambda) = -\frac{\partial}{\partial \lambda} \exp(-t\lambda) \). In the middle step (26), all the terms are integrable functions in the \( t_i \)-variables, and bounded by an integrable function (uniformly in \( x_r \)). To see this, recall that \( \langle \cdot | \cdot \rangle \omega \) is positive semidefinite and hence \( |t_i t_j F(x)| \leq t_i t_j \exp \left(- \sum_k t_k \lambda_k \right) \) and this is integrable in the (positive) variables \( t_1, \ldots, t_n \) because \( \lambda_k > 0 \) for all \( k \). Thus we can use the dominated convergence theorem for derivatives (cf. Theorem 2.7 in [7]) to conclude that
\( \omega(R(\lambda_1, f_1) \ldots R(\lambda_n, f_n)) \) is differentiable in all \( x_r \)-variables, and the partial derivatives can be taken into the integral to give via (27):

\[
\frac{\partial}{\partial x_r} \omega \left( R(\lambda_1, f_1(x_1)) \ldots R(\lambda_n, f_1(x_n)) \right) = - \sum_{k=1}^{r-1} \frac{\partial}{\partial x_r} \left( \langle f_k(x_k) | f_r(x_r) \rangle_r \right) \int_0^\infty dt_1 \ldots \int_0^\infty dt_n \frac{\partial^2}{\partial \lambda_r \partial \lambda_k} F(x)
\]

\[- \frac{1}{2} \frac{\partial}{\partial x_r} \left( \langle f_r(x_r) | f_r(x_r) \rangle_r \right) \int_0^\infty dt_1 \ldots \int_0^\infty dt_n \frac{\partial^2}{\partial \lambda_r^2} F(x)\]

\[- \sum_{k=r+1}^{n} \frac{\partial}{\partial x_r} \left( \langle f_r(x_r) | f_k(x_k) \rangle_r \right) \int_0^\infty dt_1 \ldots \int_0^\infty dt_n \frac{\partial^2}{\partial \lambda_r \partial \lambda_k} F(x).\]

We need to argue that we can take the partial derivatives w.r.t. the \( \lambda_i \)'s through the integrals above. Now if we have that \( \langle f_r(x_r) | f_r(x_r) \rangle_r = 0 \), then each integrand factorises into a \( t_r \)-dependent and a \( t_r \)-independent part. Then the integrand w.r.t. the \( t_r \)-variable is of the form \( t_r^k \exp(-t_r \lambda_r) \) for \( k = 0, 1, 2 \), and so we get explicitly from the Laplace transforms that we can take \( \partial / \partial \lambda_r \) through the \( t_r \)-integral. This takes care of the part of the integral corresponding to those variables \( t_r \) for which \( \langle f_r(x_r) | f_r(x_r) \rangle_r = 0 \). The remaining factor \( \tilde{F}(x) \) of \( F(x) \) depends only on variables \( t_r \) for which we have that \( \langle f_r(x_r) | f_r(x_r) \rangle_r > 0 \). Then

\[
\left| \frac{\partial^2}{\partial \lambda_r \partial \lambda_k} \tilde{F}(x) \right| \leq t_r t_k \exp \left( - \frac{1}{2} \sum_{l=1}^n t_l^2 \langle f_l | f_l \rangle_r \right)
\]

which is integrable w.r.t. the remaining variables, and likewise we also get a dominating function for the first derivatives. Thus by dominated convergence (uniformly in the \( \lambda \)-variables) we can take the partial derivatives in \( \lambda_i \) through the integral in in the remaining variables. Thus we get

\[
\frac{\partial}{\partial x_r} \omega \left( R(\lambda_1, f_1(x_1)) \ldots R(\lambda_n, f_1(x_n)) \right)
\]

\[- \sum_{k=1}^{r-1} \frac{\partial}{\partial x_r} \left( \langle f_k(x_k) | f_r(x_r) \rangle_r \right) \frac{\partial^2}{\partial \lambda_r \partial \lambda_k} \omega \left( R(\lambda_1, f_1(x_1)) \ldots R(\lambda_n, f_1(x_n)) \right)\]

\[- \frac{1}{2} \frac{\partial}{\partial x_r} \left( \langle f_r(x_r) | f_r(x_r) \rangle_r \right) \frac{\partial^2}{\partial \lambda_r^2} \omega \left( R(\lambda_1, f_1(x_1)) \ldots R(\lambda_n, f_1(x_n)) \right)\]

\[- \sum_{k=r+1}^{n} \frac{\partial}{\partial x_r} \left( \langle f_r(x_r) | f_k(x_k) \rangle_r \right) \frac{\partial^2}{\partial \lambda_r \partial \lambda_k} \omega \left( R(\lambda_1, f_1(x_1)) \ldots R(\lambda_n, f_1(x_n)) \right)\]

**Proof of Proposition 5.3**

To prove this theorem, we first need to establish the following lemma.

**8.1 Lemma** For the strip \( S := \mathbb{R} + i[0, 1[ \subset \mathbb{C}, \) let \( F : S \to \mathbb{C} \) be a continuous function, analytic on the interior of \( S \), which satisfies for some \( C > 0 \) and \( \lambda \in \mathbb{C}, |\lambda| = 1 \) the conditions:

\[
|F(t + is)| \leq C(1 + |t|)^N \quad \forall t \in \mathbb{R}, s \in (0, 1)
\]

and

\[
F(t + i) = \lambda F(t) \quad \forall t \in \mathbb{R}.
\]
Then $F = 0$ if $\lambda \neq 1$ and $F$ is constant if $\lambda = 1$.

**Proof:** Note that $\mathbb{C}$ is covered by the strips $S_n := S + in$. We define $G : \mathbb{C} \to \mathbb{C}$ by $G(z) := \lambda^n F(z - in)$ whenever $z \in S_n$. This is consistent, because on the joining lines $\mathbb{R} + in = S_n \cap S_{n-1}$ we have that $G(t + in) = \lambda^n F(t) = \lambda^{n-1} F(t + i)$. By the continuity of $F$ on $S_0 = S$, $G$ is continuous.

Now $G$ is analytic on the interior of each $S_n$ and continuous on the boundary, i.e. continuous on the lines $\mathbb{R} + in$ and analytic on either side of them. So it follows from a well-known theorem of analytic continuation that $G$ is analytic on the lines $\mathbb{R} + in$ (cf. [20] p183) hence entire. Moreover $G(z + i) = \lambda G(z)$ for all $z$. Let $\Gamma$ be a closed anticlockwise circle of radius $R$ centered at the fixed point $z_0$. If $z \in \Gamma \cap S_n$ then

$$|G(z)| = |\lambda^n F(z - in)| = |F(z - in)| \leq C(1 + |\text{Re}(z - in)|)^N = C(1 + |\text{Re}(z)|)^N \leq C(1 + R + |z_0|)^N$$

which is independent of $n$, i.e. $|G(z)| \leq C(1 + R + |z_0|)^N$ for all $z \in \Gamma$. Applying this to the Cauchy integral formula:

$$G^{(k)}(z_0) = \frac{k!}{2\pi i} \int_{\gamma} \frac{G(z)}{(z - z_0)^{k+1}} \, dz$$

we find:

$$|G^{(k)}(z_0)| \leq k! R \sup_{z \in \Gamma} \frac{|G(z)|}{R^{k+1}} \leq k! C \frac{(1 + R + |z_0|)^N}{R^k}.$$

When $k > N + 1$ this goes to zero as $R \to \infty$, hence $G^{(k)}(z_0) = 0$ for all $k > N + 1$. This is true for all $z_0 \in \mathbb{C}$, so $G$ is a polynomial. However only a constant polynomial can satisfy $G(z + i) = \lambda G(z)$ (or else it has infinitely many zeroes), hence $G$ is a constant, and if $\lambda \neq 1$, the only possible constant is zero.

(i) The KMS-condition for $A = 1$ reads $F_{A,B}(t) = \varphi(\alpha_t(B)) = F_{A,B}(t + i)$. Thus by lemma S.1 it follows that $F_{A,B}$ is constant, hence that $\varphi(\alpha_t(B))$ is independent of $t$.

(ii) Let $\gamma(A) = -A \in \text{Dom } \varphi$, then $F_{A,\gamma}(t + i) = \varphi(\gamma(A)) = -\varphi(A) = -F_{A,\gamma}(t)$ so by lemma S.1 we have $0 = F_{A,\gamma} = \varphi(A)$. For any $B \in \text{Dom } \varphi$ decompose $B = B_+ + B_-$ into $\gamma$-even and odd parts then we get $\varphi(B) = \varphi(B_+) = \varphi(\gamma(B))$.

Finally, let a functional $\varphi$ satisfy the graded KMS-condition on a set $Y \subset \text{Dom } \varphi$. Let $A, B \in \text{Span } Y$, i.e. $A = \sum_i \lambda_i A_i, B = \sum_j \mu_j B_j$ for $A_i, B_j \in Y$, and $\lambda_i, \mu_j \in \mathbb{C}$. Then

$$F_{A,B}(t) := \varphi(A \alpha_t(B)) = \sum_{i,j} \lambda_i \mu_j \varphi(A_i \alpha_t(B_j)) = \sum_{i,j} \lambda_i \mu_j F_{A_i,B_j}(t).$$
Since \( \varphi \) is \( \gamma \)–KMS on \( Y \) the \( F_{A_i,B_j} \) are \( \gamma \)–KMS functions. Thus \( F_{A,B} \) is continuous on \( S \), analytic on its interior, and

\[
F_{A,B}(t + i) = \sum_{i,j} \lambda_i \mu_j F_{A_i,B_j}(t + i) = \sum_{i,j} \lambda_i \mu_j \varphi(\alpha_t(B_j)\gamma(A_i)) = \varphi(\alpha_t(B)\gamma(A)) ,
\]

\[
|F_{A,B}(t + is)| \leq \sum_{i,j} |\lambda_i \mu_j| |F_{A_i,B_j}(t + is)| \leq \sum_{i,j} |\lambda_i \mu_j| C_{ij}(1 + |t|)^{N_{ij}} \leq C(1 + |t|)^N
\]

where \( C := \sum_{i,j} |\lambda_i \mu_j| C_{ij} \) and \( N := \max_{ij}(N_{ij}) \). So \( \varphi \) is \( \gamma \)–KMS on \( \text{Span } Y \).

**Proof of Proposition 5.4**

Since \( \text{Dom } \psi \) and \( \mathcal{B} \) are dense \(*\)-algebras, it follows that \( \text{Dom } \varphi := \text{Span } \{ C \otimes B \mid C \in \text{Dom } \psi, B \in \mathcal{B} \} \) is a dense \(*\)-algebra of \( \mathcal{A} = C \otimes \mathcal{B} \), and that it is invariant w.r.t. \( \gamma \otimes \iota \) and \( \sigma \otimes \beta \). Thus by the last part of Proposition 5.3 it suffices to verify the KMS–property on \( \{ C \otimes B \mid C \in \text{Dom } \psi, B \in \mathcal{B} \} \). Let \( A_i := C_i \otimes B_i \), \( i = 1, 2 \) for \( C_i \in \text{Dom } \psi \subseteq C \), and \( B_i \in \mathcal{B} \). Consider for \( t \in \mathbb{R} \) the function

\[
F_{A_1,A_2}(t) := \varphi(A_1(\sigma \otimes \beta)\iota(A_2)) = \varphi((C_1 \otimes B_1)(\sigma_t(C_2) \otimes \beta_t(B_2))) = \psi(C_1 \sigma_t(C_2))\omega(B_1 \beta_t(B_2)) = F_{C_1,C_2}^\psi(t) F_{B_1,B_2}^\omega(t)
\]

where \( F_{C_1,C_2}^\psi \) and \( F_{B_1,B_2}^\omega \) are the KMS–functions of \( \psi \) and \( \omega \) resp.. Thus, using their analytic properties, it follows that \( F_{A_1,A_2} \) extends to an function on the strip \( S = \mathbb{R} + i[0,1] \), analytic on its interior and continuous on the boundary, given by \( F_{A_1,A_2}(z) = F_{C_1,C_2}^\psi(z) F_{B_1,B_2}^\omega(z) \). Moreover

\[
F_{A_1,A_2}(t + i) = F_{C_1,C_2}^\psi(t + i) F_{B_1,B_2}^\omega(t + i) = \psi(\sigma_t(C_2)\gamma(C_1))\omega(\beta_t(B_2)B_1) = \varphi(\sigma(\sigma \otimes \iota)\iota(A_2)(\gamma \otimes \iota)(A_1))
\]

and thus \( F_{A_1,A_2} \) will be a \((\gamma \otimes \iota)\)–KMS function if the tempered growth property also holds. We have \( |F_{C_1,C_2}^\psi(t + is)| \leq K(1 + |t|)^N \) for a constant \( K \) and \( N \in \mathbb{N} \) depending on \( C_i \). Now as \( \omega \) is a state we have from the KMS–property that \( |F_{B_1,B_2}^\omega(t + is)| \leq \|B_1\| \|B_2\| \) for \( s = 0, 1 \). So by the maximum modulus principle (apply it after first mapping \( S \) to the unit disk by the Schwartz mapping principle) it follows that \( |F_{B_1,B_2}^\omega(z)| \leq \|B_1\| \|B_2\| \) for all \( z \in S \). Thus for \( t \in \mathbb{R} \), \( s \in [0,1] \) we have

\[
|F_{A_1,A_2}(t + is)| \leq \|B_1\| \|B_2\| K(1 + |t|)^N
\]

and so the tempered growth property holds for \( F_{A_1,A_2} \). Thus \( \varphi \) is a \((\gamma \otimes \iota)\)–KMS functional.
Proof of Theorem 5.5

(i) To prove there is a quasi–free state on \( \Delta(S, \sigma) \) defined by \( \omega(\delta_f) := \exp[-s(f, f)/2], \ f \in S(\mathbb{R}, \mathbb{R}) \) where \( s(f, f) := \int \frac{p}{1-e^{-p}} |\hat{f}(p)|^2 dp \), it suffices to show that \( |\sigma(f, h)|^2 \leq 4s(f, f) s(h, h) \) by [19].

\[
|\sigma(f, h)|^2 = \left| \int ip \frac{\hat{f}(p) \overline{\hat{h}(p)}}{p} dp \right|^2 \leq \left( \int |p \hat{f}(p) \overline{\hat{h}(p)}| dp \right)^2
\]

\[
= \left( 2 \int_0^\infty p |\hat{f}(p) \overline{\hat{h}(p)}| dp \right)^2 \quad \text{as } |p \hat{f}(p) \overline{\hat{h}(p)}| \text{ is even by } \hat{f}(-p) = \hat{f}(p)
\]

\[
\leq 4 \int_0^\infty p |\hat{f}(p)|^2 dp \int_k |\hat{h}(k)|^2 dk \quad \text{by Cauchy–Schwartz}
\]

\[
< 4 \int_{-\infty}^\infty \frac{p}{1-e^{-p}} |\hat{f}(p)|^2 dp \int_0^\infty \frac{k}{1-e^{-k}} |\hat{h}(k)|^2 dk \quad \text{as } p < \frac{p}{1-e^{-p}} \text{ for } p > 0
\]

\[
< 4 \int_{-\infty}^\infty \frac{p}{1-e^{-p}} |\hat{f}(p)|^2 dp \int_0^\infty \frac{k}{1-e^{-k}} |\hat{h}(k)|^2 dk \quad \text{as } 0 < \frac{p}{1-e^{-p}} \text{ for all } p
\]

\[
= 4s(f, f) s(h, h)
\]

(ii) Next we need to show that this quasi–free state \( \omega \) on \( \Delta(S, \sigma) \) extends to a KMS–functional on \( \pi_\omega(\Delta(S, \sigma))'' \), and hence on \( R(S(\mathbb{R}), \sigma) \). We first prove that \( \omega \) is KMS on the *–algebra \( \Delta_c := \text{Span} \left\{ \delta_f \mid f \in S(\mathbb{R}), \text{ supp } \hat{f} \text{ compact} \right\} \). Let \( \delta_f, \delta_h \in \Delta_c \) and consider

\[
F_1(t) := \omega(\delta_f \alpha_t(\delta_h)) = e^{-i\sigma(f, h_t)/2} \omega(\delta_{f+h_t})
\]

\[
= \exp \left[ -\frac{i}{2} \sigma(f, h_t) - \frac{1}{2} s(f + h_t, f + h_t) \right]
\]

\[
= \exp \left[ \frac{i}{2} \int \left( -p \hat{f} \hat{h} - \frac{p}{1-e^{-p}} |\hat{f} + \hat{h}|^2 \right) dp \right]
\]

\[
\quad = \exp \left[ \frac{i}{2} \int p \left( \frac{|\hat{f}|^2 + |\hat{h}|^2}{e^{-p} - 1} - \frac{2 - e^{-p}}{1 - e^{-p}} e^{ipt} \hat{f} \hat{h} - \frac{e^{-ipt}}{1 - e^{-p}} \hat{f} \hat{h} \right) dp \right]
\]

where we used \( \hat{h}_t(p) = e^{-ipt} \hat{h}(p) \). Now put \( K := \exp \left[ \frac{i}{2} \int p \left( \frac{|\hat{f}|^2 + |\hat{h}|^2}{e^{-p} - 1} \right) dp \right] \), and substitute \( p \to -p \) in the last integral, using \( \hat{f} \hat{h}(-p) = \hat{f} \hat{h}(p) \) to get:

\[
F_1(t) = K \exp \left[ -\int \frac{pe^{ipt}}{1-e^{-p}} \hat{f} \hat{h}(p) dp \right].
\]

By a similar calculation we find \( \omega(\alpha_t(\delta_h)\delta_f) = F_1(t + i) \) and this suggests that we define the KMS–function for \( z \in S = \mathbb{R} + i[0, 1] \) by:

\[
F(z) := K \exp \left[ -\int \frac{pe^{ipz}}{1-e^{-p}} \hat{f} \hat{h}(p) dp \right]. \quad (28)
\]

Let \( z = x + iy \in S = \mathbb{R} + i[0, 1] \), then we know that the integral exists for \( y = 0, 1 \). For \( y \in (0, 1) \) the function \( \frac{pe^{ipz}}{1-e^{-p}} \) is bounded, so the integral exists for all \( z \in S \), hence the definition (28) makes sense for \( F \). It is however not clear that \( F \) is analytic. However, recall that by assumption \( \text{supp } \hat{f} \) and \( \text{supp } \hat{h} \) are compact, then it follows from the dominated convergence
theorem that \( F \) is continuous on \( S \) and as the differential w.r.t. \( z \) of the integrand is continuous in \( p \), it also implies that \( F \) is analytic on the interior of \( S \). Thus \( F \) is the KMS–function for \( \omega \), hence by the last part of Proposition 5.3.4 \( \omega \) is a KMS–state on \( \Delta_c \).

Next, we prove that \( \pi_\omega(\Delta_c) \) is strong operator dense in \( \pi_\omega(\overline{\Delta(S, \sigma)})'' \). It suffices to show that for each \( f \in \mathcal{S}(\mathbb{R}) \) there is a sequence \( \{f_n\} \subset \mathcal{S}(\mathbb{R}) \) with Fourier transforms of compact support, such that \( \omega(\delta_g \delta_f \delta_h) \to \omega(\delta_g \delta_f \delta_h) \) for all \( g, h \in \mathcal{S}(\mathbb{R}) \) as \( n \to \infty \). Fix an \( f \in \mathcal{S}(\mathbb{R}) \) and define \( f_n \in \mathcal{S}(\mathbb{R}) \) by its Fourier transform \( \hat{f}_n = K_n \cdot \hat{f} \) where each \( K_n : \mathbb{R} \to [0, 1] \) is a smooth bump function of compact support which is 1 on \([-n, n]\) and zero on \( \mathbb{R} \setminus [-n - 1, n + 1] \). Then

\[
\omega(\delta_g \delta_f \delta_h) = e^{-i\sigma(g+f_n,f_n+h)/2} \omega(\delta_g+f_n+h) \\
= \exp \left[ - \frac{i}{2} \sigma(g+f_n,f_n+h) - \frac{1}{2} s(g+f_n+h,g+f_n+h) \right] \\
= \exp \left[ \frac{i}{2} \int \left( - p(\hat{g}+\hat{f}_n)(\overline{\hat{f}_n}+\overline{\hat{h}}) - \frac{p}{1-e^{-p}} |\hat{g}+\hat{f}_n+\hat{h}|^2 \right) dp \right] \\
\xrightarrow{n \to \infty} \omega(\delta_g \delta_f \delta_h)
\]

using dominated convergence as \( f \) is \( L^1 \).

Next, we want to use the strong operator denseness of \( \pi_\omega(\Delta_c) \) in \( \pi_\omega(\overline{\Delta(S, \sigma)})'' \) to show that \( \omega \) is KMS on all of \( \pi_\omega(\overline{\Delta(S, \sigma)})'' \). Let \( A, B \in \pi_\omega(\overline{\Delta(S, \sigma)})'' \) be selfadjoint, then by the Kaplansky density theorem (cf. Theorem 5.3.5 p329 in [14]) it follows that there are sequences \( \{A_n\}, \{B_n\} \subset \pi_\omega(\Delta_c) \) of selfadjoint elements such that \( \|A_n\| \leq \|A\|, \|B_n\| \leq \|B\| \) and \( A_n \to A, B_n \to B \), in strong operator topology. Now \( \omega \circ \alpha_t = \omega \) because \( s(f_t, f_t) = s(f, f) \), and thus there is an implementing unitary \( U_t \in B(\mathcal{H}_\omega) \) such that \( U_t \pi_\omega(D) U_t^* = \pi_\omega(\alpha_t(D)) \) for \( D \in \overline{\Delta(S, \sigma)} \) and \( U_t \Omega_\omega = \Omega_\omega \).

Abbreviate the KMS–functions \( F_n(z) := F_{A_n, B_n}(z) \) of \( \omega \), then for all \( t \in \mathbb{R} \):

\[
F_n(t) - F_k(t) = \omega(A_n \alpha_t(B_n)) - \omega(A_k \alpha_t(B_k)) \\
= (A_n \Omega_\omega, U_t B_n \Omega_\omega) - (A_k \Omega_\omega, U_t B_k \Omega_\omega) \\
= (U_t^* A_n \Omega_\omega, (B_n - B_k) \Omega_\omega) + (U_t^* (A_n - A_k) \Omega_\omega, B_k \Omega_\omega)
\]

Thus
\[
|F_n(t) - F_k(t)| \leq \|(B_n - B_k) \Omega_\omega\| \cdot \|A_n\| + \|(A_n - A_k) \Omega_\omega\| \cdot \|B_k\| \\
\leq \|(B_n - B_k) \Omega_\omega\| \cdot \|A\| + \|(A_n - A_k) \Omega_\omega\| \cdot \|B\|
\]

and this is independent of \( t \). Similarly

\[
|F_n(t + i) - F_k(t + i)| \leq \|(A_n - A_k) \Omega_\omega\| \cdot \|B\| + \|(B_n - B_k) \Omega_\omega\| \cdot \|A\|
\]

Now \( F_n - F_k \) is an analytic function on the strip \( S \), so by combining The Riemann mapping theorem with the maximum modulus principle we have that \( |F_n - F_k| \) takes its maximum on
the boundary of $S$. Thus by the inequalities above, $F_n - F_k$ converges uniformly to zero, hence the uniform limit $F := \lim F_n$ exists and is analytic and bounded on $S$. Since $\omega(A_n \alpha_t(B_n)) = (A_n \Omega_\omega, U_t B_n \Omega_\omega) \to (A \Omega_\omega, U_t B \Omega_\omega) = \omega(A \alpha_t(B))$ it follows that $F$ is the KMS–function $F_{A,B}$ of $\omega$. Thus $\omega$ is KMS on the selfadjoint part, hence on all of $\pi_\omega(\Delta(S, \sigma))$.

Since $\omega$ is quasifree, it is strongly regular, hence the resolvents of the generators of the one-parameter groups $t \to \pi_\omega(\delta_t)$ will provide a representation of $\mathcal{R}(S(\mathbb{R}), \sigma)$, where it is defined by Equation (10) via spectral theory.

**Proof of Theorem 5.16**

For $z \in S = \mathbb{R} + i[0, 1]$ consider

$$ G(z) := \lim_{\varepsilon \to 0^+} \left( \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \frac{e^{ipz}}{1 - e^{-p}} \hat{f}(p) \overline{\hat{g}(p)} \, dp $$

$$ = \lim_{\varepsilon \to 0^+} \int_{-\varepsilon}^{\infty} \left( \frac{e^{ipz}}{1 - e^{-p}} \hat{f}(p) \overline{\hat{g}(p)} + \frac{e^{-ipz}}{1 - e^p} \hat{f}(p) \overline{\hat{g}(p)} \right) \, dp \quad \text{by } \hat{f}(-p) = \overline{\hat{f}(p)} $$

$$ = \int_0^\infty \left( \frac{e^{ipz}}{1 - e^{-p}} \hat{f}(p) \overline{\hat{g}(p)} - \frac{e^{-ipz}}{1 - e^p} \hat{f}(p) \overline{\hat{g}(p)} \right) \, dp $$

$$ = (g, F(D)f) + \int_0^\infty dp \int dx \int dy f(x) g(y) \frac{e^{-p}}{1 - e^{-p}} \left( e^{ipz} e^{ip(y-x)} - e^{-ipz} e^{ip(x-y)} \right) $$

$$ = (g, F(D)f) + 2i \int_0^\infty dp \int dx \int dy f(x) g(y) \frac{e^{-p}}{1 - e^{-p}} \sin p(z + y - x) \quad (29) $$

where $D = id/dx$ and $F(p) := e^{ipz} \chi_{[0,\infty)}(p)$, and we used the fact that the Fourier transform diagonalises $D$. To use Fubini to rearrange these integrals, we need to show that the integrand is integrable. We need to separate the low $p$ from the high $p$ behaviour in the last integral. For the low $p$ behaviour, consider the integral

$$ \int_0^1 dp \int dx \int dy f(x) g(y) \frac{e^{-p}}{1 - e^{-p}} \sin p(z + y - x) . $$

Rearrange the integrand to $f(x) g(y) \frac{e^{-p}}{1 - e^{-p}} p(z + y - x) \frac{\sin p(z + y - x)}{p(z + y - x)}$ and observe that $p(z + y - x) \in S = \mathbb{R} + i[0, 1]$ since $p \in [0, 1]$. Now $H(z) := \sin(z)/z$ is analytic in $S$, so $|H(z)|$ takes its maximum on the boundary. On $\mathbb{R}$, $|H(x)| \leq 1$, and for $\mathbb{R} + i$ we have

$$ |H(x + i)|^2 = \left| \frac{\sin(x + i)}{x + i} \right|^2 = \frac{\cosh^2 1 - \cos^2 x}{x^2 + 1} \leq \cosh^2 1 $$

and thus for the integrand

$$ \left| f(x) g(y) \frac{e^{-p}}{1 - e^{-p}} p(z + y - x) \frac{\sin p(z + y - x)}{p(z + y - x)} \right| \leq |f(x) g(y)| \frac{e^{-p}}{1 - e^{-p}} p(z + y - x) \cosh 1 \quad (30) $$

which is clearly integrable because $f$ and $g$ are Schwartz functions so take care of the linear factor in $x$ and $y$. For the high $p$ behaviour, consider the remaining part of the integral (29), i.e.

$$ \int_1^\infty dp \int dx \int dy f(x) g(y) \frac{e^{-p}}{1 - e^{-p}} \sin p(z + y - x) . $$
Now for $z = t + is \in S, t \in \mathbb{R}, s \in [0, 1]$ we have

$$
\left| f(x)g(y) \frac{e^{-p}}{1 - e^{-p}} \sin p(z + y - x) \right| \leq \frac{1}{2} \left| f(x)g(y) \right| \frac{e^{-p}}{1 - e^{-p}} \left| e^{ip(z+y-x)} - e^{-ip(z+y-x)} \right|
$$

$$
\leq \frac{1}{2} \left| f(x)g(y) \right| \frac{e^{-p}}{1 - e^{-p}} |e^{-ps} + e^{ps}| \leq \left| f(x)g(y) \right| \frac{e^{p(s-1)}}{1 - e^{-p}}
$$

This is integrable for $s \in [0, 1)$, but not for $s = 1$. However, we will see below that $G$ is continuous on $S$, and this will be enough. Thus the integrand in (29) is integrable for $z \in \mathbb{R} + i[0, 1)$, and we can apply the Fubini theorem to rearrange the order of integrals, and we get:

$$
G(z) = (g, F(D)f) + 2i \int dx \int dy f(x)g(y) \int_0^\infty dp \frac{e^{-p}}{1 - e^{-p}} \sin p(z + y - x). \quad (31)
$$

To prove (i), let $z = 0$, so $G(0) = \partial f, g)$ and $F(p) = \chi_{[0, \infty)}(p)$ hence $P := F(D) = 2\pi \times \text{projection onto positive spectrum of } D$. So

$$
G(0) = \partial f, g) = (g, Pf) + 2i \int dx \int dy f(x)g(y) \int_0^\infty dp \frac{e^{-p}}{1 - e^{-p}} \sin p(y - x)
$$

$$
= (g, Pf) - 2i \int dx \int dy f(x)g(y) (y - x) \int_0^\infty dp \ln(1 - e^{-p}) \cos p(y - x)
$$

$$
= (g, (P + T)f)
$$

through an integration by parts. Consider the operator

$$
(Tf)(x) := 2i \int dy f(y)(x - y) \int_0^\infty dp \ln(1 - e^{-p}) \cos p(x - y) =: \int dy f(y) K(x - y)
$$

which is obviously a kernel operator with kernel $K$. Due to the factor $(x - y)$ in $K$, $T$ is unbounded. Note however that

$$
\left| \frac{\partial^p}{\partial t^n} \ln(1 - e^{-p}) \cos pt \right| \leq |p^n \ln(1 - e^{-p})|
$$

which is integrable in $p$. [To see this, note that for $0 < p \leq 1$, $|p^n \ln(1 - e^{-p})| \leq |\ln(1 - e^{-p})| \leq \ln p + |\ln(1 - e^{-p})| \text{ which is integrable, and for } e^p > 2 \text{ we have } |p^n \ln(1 - e^{-p})| = p^n(e^{-p} + e^{-2p}/2 + e^{-3p}/3 + \cdots) \leq p^n e^{-p}(1 + e^{-p}/2 + e^{-2p}/3 + \cdots) \leq p^n e^{-p} \sum_{k=0}^{\infty} (\frac{1}{2})^k \leq 2p^n e^{-p} \text{ which is integrable.}]

Thus by dominated convergence the function $t \to \int_0^\infty dp \ln(1 - e^{-p}) \cos pt$ is is smooth. Thus the kernel $K$ of $T$ is smooth. If $J$ is a compact interval then $P_JTP_J$ has kernel $\chi_J(x)K(x - y)\chi_J(y)$ which is smooth and bounded on $J$. Thus $P_JTP_J$ is trace–class by Theorem 1 p128 in Lang 17.

We also have an explicit proof that $P_JTP_J$ is trace–class below in the proof of Theorem 5.9. Selfadjointness now follows from the fact that $\theta(f, f)$ is real by its formula. This proves (i).

To prove (ii), Note that we already proved above that $G(z)$ is well-defined for $z \in \mathbb{R} + i[0, 1)$. To prove that it is well defined on all of $S$, it is only necessary to prove integrability for the high $p$ part of the integral. For this

$$
|2i \int_1^\infty dp \int dx \int dy f(x)g(y) \frac{e^{-p}}{1 - e^{-p}} \sin p(z + y - x)|
$$
where \( z = t + is \in S \), and so \( G(z) \) is well-defined for \( z \in S \).

To establish the stated inequality for \( z = s + it \in S \), consider Equation (31). Now \( |F(p)| = e^{-sp}\chi_{[0,\infty)}(p) \) implies \( \|F\| = 1 \), so \( \|F(D)\| = 1 \) and hence \( |(g, F(D)f)| \leq \|f\| \|g\| \). The high \( p \) part of the integral in (31) has an estimate (32), so for the low \( p \) integral we have for its integrand the inequality (30) above, so that

\[
\left| \int_0^1 dp \int dx \int dy \ f(x) \ g(y) \ \frac{e^{-p}}{1 - e^{-p}} \ \sin p(z + y - x) \right| \\
\leq \ \cosh 1 \cdot \ \int dx \int dy \ |f(x)g(y)| \ (1 + |t| + |x - y|) \int_0^1 dp \ \frac{pe^{-p}}{1 - e^{-p}} \\
= \ C + |t|E
\]

for some finite constants \( C \) and \( E \). Combining this with (32) and the estimate for \( |(g, F(D)f)| \), we obtain \( |G(t + is)| \leq A + B|t| \) for constants \( A, B \) as desired. It remains to prove that \( G \) is continuous on \( S \) and analytic in its interior. Now

\[
\left| \frac{d}{dz} f(x) \ g(y) \ \frac{e^{-p}}{1 - e^{-p}} \ \sin p(z + y - x) \right| \leq |f(x)g(y)| \ \frac{pe^{-p}}{1 - e^{-p}} \ \frac{1}{2} (e^{-sp} + e^{sp})
\]

which is \( L^1 \) for all \( s \in (0,1) \). Thus the last integral in (31) is analytic on the interior of \( S \). That \( (g, F(D)f) \) is analytic in \( z \) follows from spectral theory, hence \( G \) is analytic on the interior of \( S \). For continuity on \( S \), we already have that \( (g, F(D)f) \) is continuous in \( z \), and by inequalities (32) and (30) we can get \( L^1 \) estimates to ensure that \( G \) is continuous on \( S \).

**Proof of Theorem 5.7**

We now show that quasi–free functional \( \psi \) with two point functional \( \theta \) is a graded KMS–functional on \( \text{Cliff}(\mathcal{S}(\mathbb{R})) \). Its domain \( \text{Dom} \psi = *\text{-alg} \{ c(f) \mid f \in \mathcal{S}(\mathbb{R}) \} \) is clearly a unital dense *-algebra of \( \text{Cliff}(\mathcal{S}(\mathbb{R})) \) which is invariant w.r.t. both the grading \( \gamma \) and the time evolution \( \alpha_t \), so part (i) of Definition 5.1 is satisfied. For the KMS-condition (ii), it suffices to check it for the monomials \( c(f_1) \cdots c(f_k) \). Let \( A = c(f_1) \cdots c(f_k) \) and \( B = c(g_1) \cdots c(g_m) \) where \( k + m = 2n \). Then from (16) and (17) we get

\[
F_{A,B}(t) := \psi(A \alpha_t(B)) = \psi(c(f_1) \cdots c(f_k)c(T_1g_1) \cdots c(T_ng_m)) \\
= (-1)^{\binom{k}{2}} \sum_{P} (-1)^P \prod_{j=1}^{n} \theta(h_{P(j)}, h_{P(n+j)})
\]  

(33)
where \( h_1 = f_1, \ldots, h_k = f_k, h_{k+1} = T_tf_1, \ldots, h_{2n} = T_tf_m \) and \( T_tf := f_t \) is translation by \( t \). Since \( P(j) < P(n + j) \) always, the terms \( \theta(h_{P(j)}, h_{P(n+j)}) \) can only be one of the types

\[
\theta(f_t, T_tg_j) \quad \text{or} \quad \theta(T_tf_t, T_tg_j) \quad \text{if } k < n, \quad \text{or} \quad \theta(f_t, g_j) \quad \text{if } k > n.
\]

Since by the formula for \( \theta \) we have \( \theta(T_tf, T_tg) = \theta(f, g) \) the last two types are the same and constant in \( t \). For the first type, we get by definition functions \( G(t) = \theta(f_t, T_tg) \) as in Theorem 5.6(ii), which we therefore know extend analytically to the strip \( S \). Thus since Equation (83) expresses \( F_{A,B}(t) \) as a polynomial of constant functions and functions of the form \( G \), it follows that \( F_{A,B}(t) \) extends to a continuous function on \( S \) which is analytic on its interior. Now

\[
G(t + i) = \lim_{\varepsilon \to 0^+} \left( \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \frac{e^{ipt} + e^{-ipt}}{1 - e^{-p}} \hat{f}(p) \overline{\hat{g}(p)} \, dp
\]

which is the graded KMS-condition for \( F_{c(f), c(g)}(t) = \psi(c(f)\alpha_t(c(g))) \). The terms \( \theta(T_tg, f) \) are exactly the ones which occur in the corresponding expression (83) for \( \psi(\alpha_t(B)\gamma(A)) \) so the graded KMS-condition for \( F_{A,B} \) follows from the one for \( G \), Equation (34).

It remains to prove the growth condition (iii) of Definition 5.1. We already have

\[
|G(t + is)| \leq a + b|t| \quad \text{for } t \in \mathbb{R}, \ s \in [0,1]
\]

by Theorem 5.6(ii). So from formula (34) we get that for \( t + is \in S \):

\[
|F_{A,B}(t + is)| \leq (a_1 + b_1|t|) \cdots (a_n + b_n|t|) \leq C (1 + |t|)^n
\]

for suitable constants \( a_i, b_i \) and \( C \). Thus \( \psi \) is a graded KMS-functional.

**Proof of Theorem 5.8**

By construction \( \text{Dom} \varphi \) contains the *-algebra generated by all \( c(f), \ f \in \mathcal{S}(\mathbb{R}) \) as well as \( \mathcal{R}(\mathcal{S}(\mathbb{R}), \sigma) \) and so it will certainly contain the *-algebra generated by \( 1 \) and all \( c(f), \ R(\lambda, f) \), which is \( \mathcal{A}_0 \). So (i) is trivially true.

Next, for (ii), we need to prove the SUSY-invariance of \( \varphi \), and for this, we need the following lemma.
8.2 Lemma For all \( g, f_i \in S(\mathbb{R}) \setminus 0 \) and \( \lambda_i \in \mathbb{R} \setminus 0 \) we have

\[
\varphi\left(R(\lambda_1, f_1) \cdots R(\lambda_n, f_n) j(g) R(\lambda_{n+1}, f_{n+1}) \cdots R(\lambda_m, f_m)\right)
\]

\[
= \sum_{k=1}^{n} s(f_k, g) \varphi\left(R(\lambda_1, f_1) \cdots R(\lambda_k, f_k)^2 \cdots R(\lambda_n, f_n) R(\lambda_{n+1}, f_{n+1}) \cdots R(\lambda_m, f_m)\right)
\]

\[
+ \sum_{k=n+1}^{m} s(g, f_k) \varphi\left(R(\lambda_1, f_1) \cdots R(\lambda_n, f_n) R(\lambda_{n+1}, f_{n+1}) \cdots R(\lambda_k, f_k^2) \cdots R(\lambda_m, f_m)\right)
\]

where \( \varphi \) is a strongly regular state on \( R(S(\mathbb{R}), \sigma) \) so these expressions make sense on \( \mathcal{E}_0 \).

Proof: Recall by Theorem 5.5 that \( \varphi \) is a quasi–free state on \( \Delta(S, \sigma) \) defined by \( \varphi(\delta \varphi) := \exp[-s(f, f)/2], \ f \in S(\mathbb{R}, \mathbb{R}) \) where \( s \) is given in Equation (15). Since the maps \( t, r \to s(rf, tg) \) are smooth, we can apply Proposition 3.7 to \( \varphi \) w.r.t. the maps \( t \to rf, tg \). From the two relations

\[
\frac{d}{dt} R(\lambda, tf) = j(f) R(\lambda, tf)^2 \quad \text{and} \quad \lim_{t \to 0} \lambda t \pi = \psi \Rightarrow \psi,
\]

we get

\[
\varphi\left(R(\lambda_1, f_1) \cdots R(\lambda_n, f_n) j(g) R(\lambda_{n+1}, f_{n+1}) \cdots R(\lambda_m, f_m)\right)
\]

\[
= -\mu^2 \lim_{t \to 0} \frac{\partial}{\partial t} \varphi\left(R(\lambda_1, f_1) \cdots R(\lambda_n, f_n) R(\mu, tf) R(\lambda_{n+1}, f_{n+1}) \cdots R(\lambda_m, f_m)\right)
\]

\[
= -\mu^2 \lim_{t \to 0} \left\{ -\sum_{k=1}^{n} \frac{d}{dt} s(f_k, tg) \frac{\partial^2}{\partial \mu \partial \lambda_k} \varphi\left(R(\lambda_1, f_1) \cdots R(\lambda_n, f_n) R(\mu, tf) R(\lambda_{n+1}, f_{n+1}) \cdots R(\lambda_m, f_m)\right)
\]

\[
- \frac{1}{2} \frac{d}{dt} s(tg, tg) \frac{\partial^2}{\partial \mu^2} \varphi\left(R(\lambda_1, f_1) \cdots R(\lambda_n, f_n) R(\mu, tf) R(\lambda_{n+1}, f_{n+1}) \cdots R(\lambda_m, f_m)\right)
\]

\[
- \sum_{k=n+1}^{m} \frac{d}{dt} s(tg, f_k) \frac{\partial^2}{\partial \mu \partial \lambda_k} \varphi\left(R(\lambda_1, f_1) \cdots R(\lambda_n, f_n) R(\mu, tf) R(\lambda_{n+1}, f_{n+1}) \cdots R(\lambda_m, f_m)\right) \right\}
\]

(By Proposition 3.7)

\[
= \sum_{k=1}^{n} s(f_k, g) \varphi\left(R(\lambda_1, f_1) \cdots R(\lambda_k, f_k)^2 \cdots R(\lambda_n, f_n) R(\lambda_{n+1}, f_{n+1}) \cdots R(\lambda_m, f_m)\right)
\]

\[
+ \sum_{k=n+1}^{m} s(g, f_k) \varphi\left(R(\lambda_1, f_1) \cdots R(\lambda_n, f_n) R(\lambda_{n+1}, f_{n+1}) \cdots R(\lambda_k, f_k^2) \cdots R(\lambda_m, f_m)\right)
\]

where we used \( \frac{d}{dx} R(\lambda, f) = -iR(\lambda, f)^2 \).

Next we need to show that \( \varphi \circ \delta \) is zero on \( \mathcal{D}_S \), i.e. that it vanishes on all the monomials:

\[
\zeta(f_1) \cdots \zeta(f_n) R(\lambda_1, g_1) \cdots R(\lambda_m, g_m) = c(f_1) \cdots c(f_n) R(1, f_1) \cdots R(1, f_n) R(\lambda_1, g_1) \cdots R(\lambda_m, g_m).
\]

Recall that \( \delta \) is a restriction to \( \mathcal{D}_S \) of a graded derivation on \( \mathcal{E} \) defined by

\[
\delta(j(f)) = ic(f'), \quad \delta(R(\lambda, f)) = ic(f') R(\lambda, f)^2, \quad \text{and} \quad \delta(c(f)) = j(f).
\]

So we calculate:

\[
\varphi \circ \delta (c(f_1) \cdots c(f_n) R(1, f_1) \cdots R(1, f_n) R(\lambda_1, g_1) \cdots R(\lambda_m, g_m))
\]
where we made use of the lemma 8.2. Note that as \( \varphi \) is quasifree, \( n \) must be odd for the last expression to be nonzero, and also:

\[
\varphi(c(f_1) \cdots c(f_n)) = \sum_{k=1}^{n-1} (-1)^{k+1} \varphi(c(f_k) \ c(f_n)) \varphi(c(f_1) \cdots \hat{c}(f_k) \cdots c(f_n))
\]

So we get:

\[
\varphi \circ \delta (c(f_1) \cdots c(f_n) R(1, f_1) \cdots R(1, f_n) R(\lambda_1, g_1) \cdots R(\lambda_m, g_m))
\]

\[
= \sum_{k=1}^{n} (-1)^{k+1} \varphi(c(f_1) \cdots \hat{c}(f_k) \cdots c(f_n)) \left\{ \sum_{r=1}^{n} [s(f_k, f_r) - i\varphi(c(f_k) c(f'_r))] \right\} \times \varphi \left( R(1, f_1) \cdots R(1, f_r)^2 \cdots R(1, f_n) R(\lambda_1, g_1) \cdots R(\lambda_m, g_m) \right)
\]

\[
+ \sum_{p=1}^{m} \left[ s(f_k, g_p) - i\varphi(c(f_k) c(g'_p)) \right]
\]

\[
\times \varphi \left( R(1, f_1) \cdots R(1, f_n) R(\lambda_1, g_1) \cdots R(\lambda_p, g_p)^2 \cdots R(\lambda_m, g_m) \right) \right\}.
\]

However for the two-point functions we have:

\[
\varphi(c(f) c(g')) = \lim_{\epsilon \to 0^+} \left( \int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right) \frac{-ip}{1 - e^{-p}} \hat{f}(p) \hat{g}(p) \ dp
\]

\[
= -i \int_{-\infty}^{\infty} \frac{p}{1 - e^{-p}} \hat{f}(p) \hat{g}(p) \ dp = -is(f, g)
\]

and so we get \( \varphi \circ \delta = 0 \) as desired.

Finally, for (iii) we need to prove that

\[
\varphi\left( BM_A \delta_0(A) C \right) = \left. \frac{d}{dt} \varphi\left( BM_A \alpha_t(A) C \right) \right|_0
\]

(35)
for all \( A \in D_S \) and \( B, C \in A_0 \). Since \( \alpha_t \) and \( \delta_0 \) do not mix \( \text{Cliff}(S(\mathbb{R})) \) and \( \mathcal{R}(S(\mathbb{R}), \sigma) \) and \( \varphi \) has a product structure, it suffices to verify (35) on the CAR and CCR parts separately. First, on the Clifford algebra we have \( \overline{\varphi}_0(c(f)) = ic(f') \), and by the derivative property we only need to check (35) for \( A = c(f) \). However, \( \varphi \) is quasifree so it suffices to check for the two-point functions that

\[
\frac{d}{dt} \varphi(c(g) \alpha_t(c(f)))\bigg|_0 = i \varphi(c(g)\overline{\delta}_0(c(f))) = - \varphi(c(g) c(f'))
\]

for all \( f, g \in S(\mathbb{R}) \). The differentiability of \( G(t) = \varphi(c(g) \alpha_t(c(f))) \) was proven above in Theorem 57. So:

\[
\frac{d}{dt} \varphi(c(f) \alpha_t(c(g)))\bigg|_0 = \frac{d}{dt} \lim_{\varepsilon \to 0^+} \left( \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \frac{e^{ipt}}{1 - e^{-p}} \hat{f}(p) \overline{g(p)} dp \bigg|_0
\]

\[
= \lim_{\varepsilon \to 0^+} \left( \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \frac{ip}{1 - e^{-p}} \hat{f}(p) \overline{g(p)} dp
\]

\[
= - \varphi(c(f) c(g'))
\]

as required. Next, we need to check (35) on the resolvent algebra. By the derivative property, it suffices to do this for \( A = R(\lambda, f) \), and by linearity for the remaining terms being monomials of resolvents. That is, we need to prove that

\[
\frac{d}{dt} \varphi(R(\lambda_1, f_1) \cdots R(\lambda_n, f_n) R(\mu, T_T g) R(\lambda_{n+1}, f_{n+1}) \cdots R(\lambda_m, f_m))\bigg|_0
\]

\[
= i \varphi(R(\lambda_1, f_1) \cdots R(\lambda_n, f_n) R(\lambda_{n+1}, f_{n+1}) \cdots R(\lambda_m, f_m)) \tag{36}
\]

where \( \overline{\delta}_0(R(\mu, g)) = i R(\mu, g) j(g') R(\mu, g) \). This is an expression of the form of Proposition 3.7, so to apply this, we need to check that the functions \( t \to s(f, T_T g) \) are smooth (note that \( s(T_f, T_T g) = s(f, g) \)), and this is an easy verification. In fact, \( s(\cdot, \cdot) \) is clearly a distribution in each entry as it is an expectation value \( (f, Ag) \) where \( A \) is multiplication by a smooth function which is polynomially bounded. Applying Proposition 3.7 we get:

\[
\frac{d}{dt} \varphi(R(\lambda_1, f_1) \cdots R(\lambda_n, f_n) R(\mu, T_T g) R(\lambda_{n+1}, f_{n+1}) \cdots R(\lambda_m, f_m))\bigg|_{t=0}
\]

\[
= - \sum_{k=1}^n \frac{d}{dt} s(f_k, T_T g) \frac{\partial^2}{\partial \mu \partial \lambda_k} \varphi(R(\lambda_1, f_1) \cdots R(\lambda_n, f_n) R(\mu, T_T g) R(\lambda_{n+1}, f_{n+1}) \cdots R(\lambda_m, f_m))\bigg|_{t=0}
\]

\[
- \frac{1}{2} \frac{d}{dt} s(T_T g, T_T g) \frac{\partial^2}{\partial \mu^2} \varphi(R(\lambda_1, f_1) \cdots R(\lambda_n, f_n) R(\mu, T_T g) R(\lambda_{n+1}, f_{n+1}) \cdots R(\lambda_m, f_m))\bigg|_{t=0}
\]

\[
- \sum_{k=n+1}^m \frac{d}{dt} s(T_T g, f_k) \frac{\partial^2}{\partial \mu \partial \lambda_k} \varphi(R(\lambda_1, f_1) \cdots R(\lambda_n, f_n) R(\mu, T_T g) R(\lambda_{n+1}, f_{n+1}) \cdots R(\lambda_m, f_m))\bigg|_{t=0}
\]

\[
= - \sum_{k=1}^n s(f_k, g') \varphi(R(\lambda_1, f_1) \cdots R(\lambda_k, f_k)^2 \cdots R(\lambda_n, f_n) R(\mu, g)^2 R(\lambda_{n+1}, f_{n+1}) \cdots R(\lambda_m, f_m))
\]

\[
- \sum_{k=n+1}^m s(g', f_k) \varphi(R(\lambda_1, f_1) \cdots R(\lambda_n, f_n) R(\mu, g)^2 R(\lambda_{n+1}, f_{n+1}) \cdots R(\lambda_k, f_k)^2 \cdots R(\lambda_m, f_m))
\]
However, recall that

\[ \text{Proof of Theorem 5.9} \]

\[ \text{formula for } T. \]

Recall from Theorem 5.6 that:

\[ \text{from Lemma 8.2, that:} \]

\[ \text{By requiring complex linearity for } \Phi(\Gamma x) \text{ is symmetrical about the origin, we can define } \Gamma : R \rightarrow R \text{ such that} \]

\[ \text{with a self–dual CAR–algebra explicitly. First observe that} \]

\[ \text{On the other hand, for the right hand side of Equation (36) we have} \]

\[ \text{Therefore (} \Gamma f(\cdot,t) \text{) = } f(-t) \text{. Then } (\Gamma f)(x) = f(-x) \text{ and} \]

\[ \text{Define for } f \in L^2(J,R) \text{ by} \]

\[ \Phi(f) := \frac{1}{\sqrt{2}} \left( c(Pf) - ic(\Gamma Pf) + c(PTf) + ic((1-P)f) \right) \]

\[ \text{and observe that } \Phi(\Gamma f) = \Phi(f)^* \text{ and } \{ \Phi(f), \Phi(g) \} = (\Gamma f, g) \mathbb{1}, \text{ which establishes the isomorphism.} \]

\[ \text{By requiring complex linearity for } \Phi(f), \text{ we get } \Phi(f) + i\Phi(g) =: \Phi(f + ig) \text{ hence get in fact} \]

\[ \text{However, } s(g,g') = -s(g',g) \text{ and so the last two terms cancel and hence we have proven} \]

\[ \text{Proof of Theorem 5.9} \]

Recall that

\[ \mathcal{A}_0(J) := \ast\text{-alg} \{ c(f), R(\lambda, f) \mid \text{supp } f \subseteq J, f \in S(\mathbb{R}), \lambda \in \mathbb{R} \setminus 0 \} = C(J) \otimes R_0(J) \]

\[ \text{where } \mathcal{C}(J) := \ast\text{-alg} \{ c(f) \mid \text{supp } f \subseteq J, f \in S(\mathbb{R}) \} \]

\[ \text{and } R_0(J) := \ast\text{-alg} \{ R(\lambda, f) \mid \text{supp } f \subseteq J, f \in S(\mathbb{R}), \lambda \in \mathbb{R} \setminus 0 \}. \]

\[ \text{Since } \varphi = \psi \otimes \omega \text{ is a product functional, and } \omega \text{ is a state, we have that } \| \varphi \mathcal{A}_0(J) \| = \| \psi \mathcal{C}(J) \|, \]

\[ \text{and so this is what we need to estimate. Without loss of generality we may assume } J \text{ to be a closed interval, and also symmetrical about the origin (since } \varphi \text{ is invariant w.r.t. translations).} \]

Recall from Theorem 5.6 that \( \psi(c(f)c(g)) = \theta(f,g) = (g, (P+T)f) \) where \( P \) is a projection (after a normalisation) and \( P_TTP_T \) is trace-class and selfadjoint for all compact intervals \( J \), and so this is the case for \( c(f), c(g) \in C(J) \). We will need to use the isomorphism of the Clifford algebra \( \overline{C(J)} \) with a self–dual CAR–algebra explicitly. First observe that \( \overline{C(J)} = \text{Cliff}(L^2(J,R)) \) by continuity of \( c(f) \). Since \( J \) is symmetrical about the origin, we can define \( \Gamma : L^2(J,R) \rightarrow L^2(J,R) \) by \( (\Gamma f)(p) := \hat{f}(-p) \). Then \( (\Gamma f)(x) = f(-x) \) and \( \Gamma P = (1-P)\Gamma \), and \( \Gamma TT = -T \) by the explicit formula for \( T \). Define for \( f \in L^2(J,R) \)

\[ \Phi(f) := \frac{1}{\sqrt{2}} \left( c(Pf) - ic(\Gamma Pf) + c(PTf) + ic((1-P)f) \right) \]

and observe that \( \Phi(\Gamma f) = \Phi(f)^* \) and \( \{ \Phi(f), \Phi(g) \} = (\Gamma f, g) \mathbb{1}, \) which establishes the isomorphism.
isomorphism of $\overline{C(J)}$ with $\text{CAR}(L^2(J, \mathbb{C}))$. Note that by $\Phi(\Gamma f) = \Phi(f)^*$ the involution $\Gamma$ has to extend to $L^2(J, \mathbb{C}) =: \mathcal{K}$ in a conjugate linear way. The image of $\mathcal{C}(J)$ under the isomorphism, is the dense *-algebra $\text{CAR}_0$ generated by all $\Phi(f)$, $f \in \mathcal{K}$, and this is the domain of $\psi$.

With respect to the decomposition $\mathcal{K} = P\mathcal{K} \oplus (1 - P)\mathcal{K} \ni f \oplus g$ we decompose $T = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right)$ so by $T = T^*$ we get $A = A^*$, $D = D^*$ and $B = C^*$, and these operators are trace-class because $T$ is. From the relation $\Gamma TT = -T$ we then find that $A = -D$ and $B^* = -B$, hence $T = \left( \begin{array}{cc} A & B \\ -B & -A \end{array} \right)$. Since $T$ preserves the original real space $L^2(J, \mathbb{R})$ we have $\overline{Af} = A\overline{f}$ and $\overline{Bf} = B\overline{f}$. Then the two-point function of $\psi$ on $\text{CAR}_0$ is for $f \oplus g$, $h \oplus k \in L^2(J, \mathbb{R})$:

$$\psi(\Phi(f \oplus g) \Phi(h \oplus k))$$

$$= \frac{1}{2} \psi \left( (c(f \oplus 0) - i c(0 \oplus f) + c(g \oplus 0) + i c(0 \oplus g)) \cdot (c(h \oplus 0) - i c(0 \oplus h) + c(k \oplus 0) + i c(0 \oplus k)) \right)$$

$$= \frac{1}{2} \left( |(h, f) + (k, f) + (h, g) + (k, g)| + (h, Af) - i(k, Bf) + i(h, Bg) + (k, Ag) \right)$$

$$= \left( \Gamma(h \oplus k), \left[ \frac{1}{2} \begin{array}{cc} I & I \\ I & I \end{array} \right] + \left( -iB \begin{array}{cc} A \\ A \end{array} iB \right) \right) (f \oplus g).$$

This expression is complex linear in both entries, so we can extend it by linearity to $\mathcal{K}$ to get for all $f, g \in \mathcal{K}$ that

$$\psi(\Phi(f) \Phi(g)) = (\Gamma(g), (R + Q)f) \quad \text{where:}$$

$$R := \frac{1}{2} \begin{array}{cc} I & I \\ I & I \end{array} \quad \text{and} \quad Q := \left( -iB \begin{array}{cc} A \\ A \end{array} iB \right)$$

Define an operator $S$ by $\frac{1}{2} + S := R + Q$, then $S$ is a bounded selfadjoint operator which satisfies $\Gamma ST = S$. As $S_0 := \frac{1}{2} \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}$ has eigenvalues $\pm \frac{1}{2}$ and $Q$ is trace class, $S = S_0 + Q$ has discrete spectrum with the only possible accumulation points $\pm \frac{1}{2}$ (cf. Theorem 9.6 [21]). We now need:

**8.3 Lemma** 

(i) $E([\frac{1}{2}, \frac{1}{2}]^c) \cdot (2|S| - 1)$ on $L^2(J)$ is trace class where $E$ is the spectral resolution of $S$. Moreover we have for the trace-norms $\| \cdot \|_1$ that

$$\left\| E([\frac{1}{2}, \frac{1}{2}]^c) \cdot (2|S| - 1) \right\|_1 \leq b \| PT_P \|_1$$

for a positive constant $b$ (independent of $J$).

(ii) Let $\{e_j \mid j \in \mathbb{J}\}$ be an orthonormal system of eigenvectors of $S$ corresponding to the eigenvalues $s_j \in [-\frac{1}{2}, \frac{1}{2}]^c$, and exhausting these eigenspaces. For each $j \in \mathbb{J}$, let $\mathcal{C}_j$ be the two–dimensional abelian *-algebras generated by $\Phi(e_j)^* \Phi(e_j)$, and let $\mathcal{C}_0 := \ast\text{-alg} \left\{ \Phi(f) \mid f \in E([\frac{1}{2}, \frac{1}{2}]^c) \mathcal{K} \right\}$. Then

$$\| \psi \mathcal{C}(J) \| = \prod_{j \in [0] \cup \mathbb{J}} \| \psi_j \| = \prod_{j \in \mathbb{J}} 2|s_j| < \infty$$

where $\psi_j$ denotes the restriction of $\psi$ to $\mathcal{C}_j$. 

41
Proof: (i) Let $E_0(\cdot)$ be the spectral resolution corresponding to $S_0$. For any $s \in (0, \frac{1}{2})$ which is not in the spectrum $\sigma(S)$ of $S$, since $S$ and $S_0$ are bounded, we obtain by spectral calculus that

$$E((s, \infty)) - E_0((s, \infty)) = (2\pi i)^{-1} \int_C dz \left( (z - S)^{-1} - (z - S_0)^{-1} \right)$$

$$\quad = (2\pi i)^{-1} \int_C dz (z - S_0)^{-1}Q(z - S)^{-1}$$  \quad (39)$$

where $C$ is a suitable closed path in $\mathbb{C}$, e.g. a large anticlockwise simple contour with $\sigma(S) \cap [s, \infty)$ and $\sigma(S_0) \cap [s, \infty)$ in its interior, and crossing the real axis only at $s$ and some $t > s$. So from (39) we conclude that $E((s, \infty)) - E_0((s, \infty))$ is a trace class operator. Taking into account that $E_0((s, \infty))(2S_0 - 1) = 0$ we obtain

$$E((s, \infty))(2S - 1) = (E((s, \infty)) - E_0((s, \infty)))(2S - 1) + E_0((s, \infty))2Q,$$  \quad (40)$$

showing that $E((s, \infty))(2S - 1)$ and hence a fortiori $E((\frac{1}{2}, \infty))(2S - 1)$ is trace class. A similar argument establishes that $E((0, \frac{1}{2})) (2S + 1)$ is trace class, and hence $E([-\frac{1}{2}, \frac{1}{2}]) \cdot (2|S| - 1)$ is trace class. From Equation (40) we get that

$$\|E((s, \infty))(2S - 1)\|_1 \leq \|(E((s, \infty)) - E_0((s, \infty)))(2S - 1)\|_1 + \|E_0((s, \infty))2Q\|_1$$

since all the terms in (40) are trace class, and $\|AQ\|_1 \leq \|A\| \cdot \|Q\|_1$ for $A$ bounded. Let $P_0$ be the projection onto the eigenspace of $S_0$ with eigenvalue $\frac{1}{2}$ (since $S_0 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ this is just the space of even functions w.r.t. the decomposition associated with $P$). Then by substituting

$$(z - S_0)^{-1} = (z - \frac{1}{2})^{-1}P_0 + (z + \frac{1}{2})^{-1}(1 - P_0) \quad \text{and} \quad (z - S)^{-1}(2S - 1) = (2z - 1)(z - S)^{-1} - 21$$

into (39) $(2S - 1)$ we get that

$$[E((s, \infty)) - E_0((s, \infty))](2S - 1)$$

$$\quad = \frac{1}{2\pi i} \left[ P_0Q \int_C dz \left( (z - \frac{1}{2})^{-1} \left( \frac{2z - 1}{z - S} - 2 \right) + (1 - P_0)Q \int_C dz (z + \frac{1}{2})^{-1} \left( \frac{2z - 1}{z - S} - 2 \right) \right) \right]$$

$$\quad = \frac{1}{2\pi i} P_0Q \int_C dz 2(z - S)^{-1} - 2P_0Q + \frac{1}{2\pi i}(1 - P_0)Q \int_C dz \left( \frac{2z - 1}{z - S} \right)$$

$$\quad = P_0Q 2E((s, \infty)) - 2P_0Q + (1 - P_0)Q f(S)$$

where $f(z) := \frac{2z - 1}{z + 1/2} \chi_H(z)$ and $H := \{ z \in \mathbb{C} \mid \Re(z) > s \}$. Now $\|f(S)\| \leq \|f\|(s, \infty)\|_\infty = 2$, and so

$$\|E((s, \infty)) - E_0((s, \infty))(2S - 1)\|_1 \leq 6\|Q\|_1 \leq a\|P_TTP_T\|_1$$

42
for a constant $a > 0$, where we obtain the last inequality $\|Q\|_1 \leq \text{const.}\|P_T\|_1$ from the decomposition in (32) from which $PQP = -iB = -iP(P_T)(1-P)(1-P)QP = P(P_T)P$ etc. Thus by (31) we get

$$\|E((\frac{1}{2}, \infty))(2S - 1)\|_1 = \|E((\frac{1}{2}, \infty)E(s, \infty)(2S - 1)\|_1 \leq \|E(s, \infty)(2S - 1)\|_1 \leq a\|P_T\|_1.$$  

A similar argument establishes that $\|E(-\infty, -\frac{1}{2}](2S + 1)\|_1 \leq a\|P_T\|_1$ and hence that $\|E([\frac{1}{2}, \infty] \cdot (2|S| - 1))\|_1 \leq b\|P_T\|_1$ for a positive constant $b$.

(ii) Recall that $S$ has a purely discrete spectrum in $[-\frac{1}{2}, \frac{1}{2}]$, so choose an orthonormal system \{ $e_j \in \mathcal{K}$ | $j \in J \subseteq \mathbb{N}$\} of eigenvectors of $S$, corresponding to eigenvalues $s_j \in [-\frac{1}{2}, \frac{1}{2}]$ and exhausting these eigenspaces (some $s_j$ will coincide for higher multiplicities). Let $E_j$ be the one-dimensional orthogonal projection onto $e_j$, $j \in J$, and let $\lambda = (\lambda_1, \lambda_2, \ldots) \in \bigoplus_{j \in J} \{-1, 1\}$. We define on $\mathcal{K}$ the unitaries

$$V(\lambda) := E([-\frac{1}{2}, \frac{1}{2}]) + \sum_{j \in J} \lambda_j E_j.$$  

Since $V(\lambda)$ commutes with $\Gamma$, these unitaries induce an action $\gamma : \bigoplus_{j \in J} \{-1, 1\} \to \text{Aut CAR}(\mathcal{K})$ given by

$$\gamma_\lambda(\Phi(f)) := \Phi(V(\lambda)f).$$  

Since $V(\lambda)^2 = 1$, we can decompose CAR($\mathcal{K}$) into odd and even parts w.r.t. each $\gamma_\lambda$. Moreover, since $V(\lambda)$ commutes with $S$ we have that $\psi \circ \gamma_\lambda = \psi$ and so $\psi$ must vanish on the odd part of CAR$_0$ with respect to each $\gamma_\lambda$. Let $\mathcal{C} \subset$ CAR$_0$ be the *-algebra generated by $\{ \Phi(e_j) | j \in J \} \cup \{ \Phi(f) | f \in E([-\frac{1}{2}, \frac{1}{2}])\mathcal{K} \}$, then $\mathcal{C}$ is mapped to itself by all $\gamma_\lambda$. Since the two-point functional $\theta$ is bounded on $L^2(J)$, it suffices to calculate the norm of $\psi$ on $\mathcal{C}$, and in fact on the intersection of all the even parts of $\mathcal{C}$ with respect to $\gamma_\lambda$, and we denote this *-algebra by $\varepsilon(\mathcal{C})$. It is produced by a projection $\varepsilon$ which we can consider as the projection defined on $\mathcal{C}$ by averaging over the action of $\gamma_\lambda$ on $\mathcal{C}$. Since for each $A \in \mathcal{C}$ only a finite number of $j$’s are involved, these averages will again be in the *-algebra $\mathcal{C}$. Now we only need to consider monomials in the $\Phi(e_j)$ and $\Phi(e_j)^*$ which are even in each index $j$. In a given monomial $\Phi(e_{j_1}) \cdots \Phi(e_{j_n}) \in \varepsilon(\mathcal{C})$ if we collect all (even number of) terms with the same $j$ together, we can then simplify it with the relations $2 \Phi(e_j)^2 = (\Gamma e_j e_j)\1$ and $[\Phi(e_j)^* \Phi(e_j)]^2 = \Phi(e_j)^* \Phi(e_j) - \frac{1}{4}[(\Gamma e_j, e_j)]^2 \1$. Thus $\varepsilon(\mathcal{C})$ is generated by the two-dimensional abelian *-algebras $\mathcal{C}_j := \text{*–alg}\{\Phi(e_j)^* \Phi(e_j)\}$ and $\mathcal{C}_0 := \text{*–alg}\{\Phi(f) | f \in E([-\frac{1}{2}, \frac{1}{2}])\mathcal{K}\}$. Since for $i \neq j$ we have

$$\varepsilon(\{\Phi(e_i)^* \Phi(e_i), \Phi(e_j)^* \Phi(e_j)\}) = \varepsilon(\Gamma e_i, e_j)\Phi(e_j)^* \Phi(e_i) + (\Gamma e_i, e_j)\Phi(e_j)^* \Phi(e_i)^* = 0$$  

it follows that all the $\mathcal{C}_i$ commute, and in fact we have the (incomplete) tensor product decomposition

$$\varepsilon(\mathcal{C}) = \bigotimes_{j \in \{0\} \cup J} \mathcal{C}_j.$$  

43
Moreover, \( \psi \) is a product functional on this tensor product. Hence its norm, if it exists, is given by

\[
\| \psi \| = \prod_{j \in (0) \cup J} \| \psi_j \|
\]

where \( \psi_j \) denotes the restriction of \( \psi \) to \( C_j \). Now \( \psi_0 \) is by construction a state on \( C_0 \) because \( \frac{1}{2} + S \) is positive on \( E(\{ -\frac{1}{2}, \frac{1}{2} \}) \mathcal{K} \), hence the two-point function is positive and so \( \| \psi_0 \| = 1 \). Since for \( a, b \in \mathbb{C} \)

\[
\psi(a_1 + b \Phi(e_j)^* \Phi(e_j)) = a + b(\frac{1}{2} + s_j)
\]

and

\[
\| a_1 + b \Phi(e_j)^* \Phi(e_j) \| = \max\{|a|, |a + b|\}
\]

one obtains \( \| \psi_j \| = 2|s_j|, j \in J \). Thus since \( 2|s_j| > 1 \), a necessary and sufficient condition for the existence of \( \| \psi \| \) is \( \sum_{j \in J} (2|s_j| - 1) < \infty \). However, this is guaranteed by part (i) \( \square \)

Using this lemma, we can now prove:

**8.4 Lemma** We have

\[
\| \varphi | A_0(J) \| \leq \exp (b \| P_J TP_J \|_1)
\]

where \( b \) is a positive constant (independent of \( J \)) and \( \| \cdot \|_1 \) denotes the trace–norm.

**Proof:** recall from Equation (38) that

\[
\| \varphi | A_0(J) \| = \prod_{j \in (0) \cup J} \| \psi_j \| = \prod_{j \in J} 2|s_j| = \prod_{j \in J} (1 + t_j) < \infty
\]

where \( t_j := 2|s_j| - 1 \). Now \( \ln(1 + x) \leq x \) for \( x \geq 0 \), so

\[
\ln \prod_{j=1}^{N} (1 + t_j) = \sum_{j=1}^{N} \ln (1 + t_j) \leq \sum_{j=1}^{N} t_j \quad \text{hence} \quad \prod_{j \in J} (1 + t_j) \leq \exp \left( \sum_{j \in J} t_j \right)
\]

Now \( \sum_{j \in J} t_j = \sum_{j \in J} (2|s_j| - 1) = \| E([-\frac{1}{2}, \frac{1}{2}]) \cdot (2|S| - 1) \|_1 \leq b \| P_J TP_J \|_1 \) for a constant \( b > 0 \) by Lemma 8.3(i). Combining these claims prove the lemma \( \square \)

To conclude the proof of the theorem, we need to estimate \( \| P_J TP_J f \|_1 \). Recall from Theorem 5.6 that

\[
(P_J TP_J f)(x) := 2i \chi_J(x) \int dy f(y) (x - y) \int_0^\infty dp \ln(1 - e^{-p}) \cos p(x - y) \chi_J(y)
= i \chi_J(x) \int dy f(y) (x - y) \int_{-\infty}^\infty dp \ln(1 - e^{-|p|}) e^{ip(x-y)} \chi_J(y)
= \text{const.} \left[ \chi_J(X) \left[ X, \ln(1 - e^{-|P|}) \right] \chi_J(X) f \right](x)
\]

(42)

where \( X \) is the multiplication operator \( (Xf)(x) = xf(x) \), and \( P \) is as usual \( i \frac{d}{dx} \), and the constant incorporates the \( 2\pi \) factors from the Fourier transforms. Now \( D := -\ln(1 - e^{-|P|}) \) is a positive
operator, so write trivially \(\chi_j(X) X D \chi_j(X) = (\chi_j(X) X D^{1/2})(D^{1/2} \chi_j(X))\), then we show that both factors are Hilbert–Schmidt. Now

\[
(\chi_j(X) X D^{1/2} f)(x) = \int dy \ K(x, y) \quad \text{where:}
\]

\[
K(x, y) = \chi_j(x) x \int dp \ [ - \ln(1 - e^{-|p|}) ]^{1/2} e^{ip(x-y)}
\]

so

\[
\|\chi_j(X) X D^{1/2}\|_2^2 = \int dx \ dy \ |K(x, y)|^2 = \int dx \ x^2 \ \int dp \ [ - \ln(1 - e^{-|p|}) ] \leq \text{const.} |J|^3,
\]

using the integrability of \(\ln(1 - e^{-|p|})\). Likewise, we get

\[
\|(D^{1/2} \chi_j(X))\|_2^2 = \int dx \ \int dp \ [ - \ln(1 - e^{-|p|}) ] \leq \text{const.} |J|.
\]

Thus \(\chi_j(X) X D \chi_j(X)\) is trace class, and as \(P_j TP_j\) is, so is \(\chi_j(X) X D \chi_j(X)\). For their trace norms we find

\[
\|\chi_j(X) X D \chi_j(X)\|_1 \leq \|\chi_j(X) X D^{1/2}\|_2 \cdot \|(D^{1/2} \chi_j(X))\|_2 \leq \text{const.} |J|
\]

and likewise \(\|\chi_j(X) X D \chi_j(X)\|_1 \leq \text{const.} |J|^2\). Thus by (12) we get that

\[
\|P_j TP_j\|_1 \leq \text{const.} \|\chi_j(X) X D \chi_j(X)\|_1 + \text{const.} \|(\chi_j(X) X D \chi_j(X))\|_1 \leq K|J|^2
\]

for a constant \(K\) (independent of \(J\)). Now from Lemma 8.4 we get the claim of the theorem, i.e. that

\[
\|\varphi[A_0(J)]\| \leq \exp(K |J|^2).
\]

**Proof of Theorem 6.3**

Fix a compact interval \(J = [-k, k]\) and let \(a_i \in A_0(J)\) for all \(i\), then as \(\alpha_t(a_i) \in A_0(J + t)\) we have \(\alpha_{t_0}(a_0) \cdots \alpha_{t_n}(a_n) \in A_0([-M, M])\) where \(M := k + \sup|t_i|\). Thus by Theorem 5.9 we get

\[
|\varphi(\alpha_{t_0}(a_0) \cdots \alpha_{t_n}(a_n))| \leq e^{4KM^2} \|a_0\| \cdots \|a_n\|.
\]

Now

\[
M^2 = k^2 + \sup \sum ti^2 + 2k \sup |t_i| \leq k^2 + \sum ti^2 + 2k \sup(1 + ti^2)
\]

\[
\leq k^2 + \sum ti^2 + 2k(1 + \sum ti^2) = k^2 + 2k + (1 + 2k) \sum ti^2
\]

hence:

\[
|\varphi(\alpha_{t_0}(a_0) \cdots \alpha_{t_n}(a_n))| \leq A \exp(B \sum ti^2) \|a_0\| \cdots \|a_n\|.
\]

for suitable constants \(A\) and \(B\) depending only on \(k\) (but not on \(n\)). Now let \(t_0 = 0, t_1 = s_1, t_2 = s_1 + s_2, \ldots, t_n = s_1 + \cdots + s_n\), and define for all \(s_i \in \mathbb{R}\):

\[
F_{a_0, \ldots, a_n}(s_1, \ldots, s_n) := \exp \left( - B \sum_{k=1}^n (s_1 + \cdots + s_k)^2 \right) \cdot \varphi(a_0 \alpha_{s_1}(a_1) \cdots \alpha_{s_1+\cdots+s_n}(a_n)).
\]
Then we have $|F_{a_0,\ldots,a_n}(s_1,\ldots, s_n)| \leq A \|a_0\| \cdots \|a_n\|$, and by the KMS–property of $\varphi$, the
function $F_{a_0,\ldots,a_n}$ can be analytically continued in each variable $s_j$ into the strip $S_j := \{z_j \in \mathbb{C}^n \mid \text{Im } z_j \in [0, 1]\}$, keeping the other variables real. This produces functions $F^{(j)}_{a_0,\ldots,a_n}$ analytic in the flat tubes $T^j := \mathbb{R}^{n-1} \times S_j$, and hence by using the Flat Tube Theorem 6.1 inductively, we obtain an analytic continuation of $F_{a_0,\ldots,a_n}$ in the tube $T_n := \mathbb{R}^n + i\Sigma_n$ where $\Sigma_n := \{s \in \mathbb{R}^n \mid 0 \leq s_i \forall i, s_1 + \ldots + s_n \leq 1\}$, coinciding with all $F^{(j)}_{a_0,\ldots,a_n}$ on $T^j$. We want to obtain a bound for this analytic function $F$. We start by finding bounds for the $F^{(j)}$. Let $G(s_1,\ldots, s_n) := \varphi(a_0 \alpha_{s_1}(a_1) \cdots \alpha_{s_1+\ldots+s_n}(a_n))$ which has analytic extensions to each $T^j$, and by the definition of KMS–functionals we know that $|G(s_1,\ldots, s_j + ir_j,\ldots, s_n)| \leq C(1 + |s_j|)^N$ where $C$ and $N$ are independent of $s_j$ and $r_j \in [0, 1]$. Now

$$F_{a_0,\ldots,a_n}(s_1,\ldots, s_j + ir_j,\ldots, s_n) = G(s_1,\ldots, s_j + ir_j,\ldots, s_n) \exp[Br^2_j(n + 1 - j) - B \sum_{k=1}^n (s_1 + \ldots + s_k)^2 + i\theta]$$

where $\theta$ is real. Thus from the exponential damping factor in $s_j$ we conclude that $F^{(j)}_{a_0,\ldots,a_n}$ is bounded. By the maximum modulus principle (applied after first mapping $S_j$ to a unit disk by the Schwartz mapping principle), the bound of $|F^{(j)}_{a_0,\ldots,a_n}|$ is attained on the boundary of $S_j$ (this also follows from the Phragmen Lindelöf theorem, cf. p138 in [6]). So on the real part of the boundary of $S_j$ we have already from above that $|F^{(j)}_{a_0,\ldots,a_n}(s_1,\ldots, s_n)| \leq A \|a_0\| \cdots \|a_n\|$ and by the KMS–condition and translation invariance of $\varphi$ we have on the other part

$$|G(s_1,\ldots, s_j + i, \ldots, s_n)| = |\varphi(\alpha_{s_1+\ldots+s_j}(a_j) \cdots \alpha_{s_1+\ldots+s_{j-1}}(a_{j-1}) a_0 \alpha_{s_j}(a_1) \cdots \alpha_{s_1+\ldots+s_{j-1}}(a_{j-1}))| \leq A \exp\left(B \sum_{k=1}^n (s_1 + \ldots + s_k)^2\right) \|a_0\| \cdots \|a_n\|$$

hence by (43):

$$|F^{(j)}_{a_0,\ldots,a_n}(s_1,\ldots, s_j + i,\ldots, s_n)| \leq A e^{Bn} \|a_0\| \cdots \|a_n\|$$

and thus as $e^{Bn} > 1$,

$$|F^{(j)}_{a_0,\ldots,a_n}(s_1,\ldots, z_j,\ldots, s_n)| \leq A e^{Bn} \|a_0\| \cdots \|a_n\| =: C$$

for all $z_j \in S_j$.

Now define $H_\alpha(z_1,\ldots, z_n) := [F_{a_0,\ldots,a_n}(z_1,\ldots, z_n) - e^{i\alpha} C]^{-1}$ where $\alpha \in [0, 2\pi]$ for $(z_1,\ldots, z_n) \in T_n$. Then by the estimates above for $|F^{(j)}_{a_0,\ldots,a_n}|$, each map $z_j \to H_\alpha(s_1,\ldots, z_j,\ldots, s_n)$ is analytic on the strip $S_j$, and thus by the Flat Tube Theorem 6.1 $H_\alpha$ has a unique extension as an analytic function to $T_n$, and hence cannot have any singularities in $T_n$, i.e. $F_{a_0,\ldots,a_n}(z_1,\ldots, z_n) \neq e^{i\alpha} C$ for all $\alpha$. By continuity of $F$, the image set $F_{a_0,\ldots,a_n}(T_n)$ must be connected. By assumption, this set has some points inside the circle $|z| = C$, hence the entire image set is inside the circle $|z| = C$, i.e.

$$|F_{a_0,\ldots,a_n}(z_1,\ldots, z_n)| \leq A e^{Bn} \|a_0\| \cdots \|a_n\| \quad \forall (z_1,\ldots, z_n) \in T_n \quad \text{and } a_i \in A_0(J).$$

Consider now the Chern character formula (19): 

$$\tau_n(a_0,\ldots, a_n) := i^n \int_{\sigma_n} \varphi(a_0 \alpha_{is_1}(\delta\gamma(a_1)) \alpha_{is_2}(\delta(a_2)) \alpha_{is_3}(\delta\gamma(a_3)) \cdots$$
where we used first, that the volume of 

\[ i^n \int_{\Sigma_n} \varphi(b_0 \alpha_{i_1}(b_1) \cdots \alpha_{i_r} \cdots + \alpha_{i_r} \cdots (b_n)) \, dr_1 \cdots dr_n \]

where we made a change of variables \( s_1 = r_1, \ s_2 = r_1 + r_2, \ldots, s_n = r_1 + \cdots + r_n \) and substitutions 
\( a_0 = b_0, \ b_1 = \delta \gamma(a_1), \ldots, b_n = \delta \gamma^n(a_n) \) as in Section 6.3 making use of the Flat Tube Theorem. (Note that \( b_i \in \mathcal{A}_0(J) \ \forall \ i \) for some \( J \).) In fact, from the uniqueness part of the extensions to \( T_n \) we have that on \( T_n \)

\[ \varphi(b_0 \alpha_{z_1}(b_1) \cdots \alpha_{z_1+\cdots+z_n}(b_n)) = \exp \left[ B \sum_{k=1}^{n} (z_1 + \cdots + z_n)^2 \right] \cdot F_{b_0,\ldots,b_n}(z_1, \ldots, z_n) \]

and so for \((z_1, \ldots, z_n) = i(r_1, \ldots, r_n) \in i\Sigma_n\) we have

\[
\left| \varphi(b_0 \alpha_{ir_1}(b_1) \cdots \alpha_{ir_1+\cdots+ir_n}(b_n)) \right| \leq \exp \left[ - B \sum_{k=1}^{n} (r_1 + \cdots + r_n)^2 + Bn \right] A \|b_0\| \cdots \|b_n\| \\
\text{hence: } |\tau_n(a_0, \ldots, a_n)| \leq \frac{A}{n!} e^{Bn} \|b_0\| \cdots \|b_n\| \leq \frac{A}{n!} e^{Bn} \|a_0\|^{*} \cdots \|a_n\|^{*}
\]

where we used first, that the volume of \( |\Sigma_n| = 1/n! \), and second, that \( \|b_j\| \leq \|a_j\|^{*} \) because \( b_j = \delta \gamma(a_j) = -\gamma \delta(a_j) \) for \( j > 0 \). Thus \( \|\tau_n\|^{*} \leq A e^{Bn}/n! \) and hence it is clear that 
\( \lim_{n \to \infty} n^{1/2} \|\tau\|^{1/n} \leq e^{B} \lim_{n \to \infty} n^{1/2}(A/n!)^{1/n} = 0 \) by Stirling’s formula, which concludes the proof.

**Proof of Theorem 6.4**

By Theorem 6.3 we already have the entireness condition for \( \bar{\tau} \) so it is only necessary to prove the cocycle condition for \( a_i \in \mathcal{D}_c : \)

\[ (b_{\tau_{n-1}})(a_0, \ldots, a_n) = (B_{\tau_{n+1}})(a_0, \ldots, a_n), \quad n = 1, 3, 5, \ldots \]

with \( b \) and \( B \) given by Equations (22) and (23). We will roughly follow the technique used in (12), but due to the different analytic properties of our model, we will need to go explicitly through the steps. In order to manipulate the expressions involved with Equation (24), we need the results in the following Lemma.

**8.5 Lemma** Let \( b_i \in \mathcal{A}_0 \), then:

\( i \) \( \varphi(b_0 \alpha_{i s_1}(b_1) \cdots \alpha_{i s_n}(b_n)) = \varphi(\gamma(b_0) \alpha_{i(1-s_1)}(b_0) \alpha_{i(1-s_2)+s_1}(b_1) \cdots \alpha_{i(1-s_n+s_{n-1})}(b_n-1)) \)

for all \( (s_1, \ldots, s_n) \in \sigma_n. \)

\( ii \) \( \int_{\sigma_n} \varphi(b_0 \alpha_{i s_1}(b_1) \cdots \alpha_{i s_n}(b_n)) \, ds_1 \cdots ds_n = \int_{\sigma_n} \varphi(\gamma(b_0) \alpha_{i s_1}(b_0) \cdots \alpha_{i s_n}(b_n-1)) \, ds_1 \cdots ds_n \)

\( iii \) The functions \( (t_1, \ldots, t_n) \to \varphi(b_0 \alpha_{t_1}(b_1) \cdots \delta(\alpha_{t_k}(b_k)) \cdots \alpha_{t_n}(b_n)) \) and

\( (t_1, \ldots, t_n) \to \varphi(b_0 \alpha_{t_1}(b_1) \cdots \gamma(\alpha_{t_k}(b_k)) \cdots \alpha_{t_n}(b_n)) \) both have analytic continuations to \( \mathbb{R}^n + i\sigma_n \), and for these we have
\[
\varphi\left(b_0 \alpha_{is_1}(b_1) \cdots \delta(\alpha_{is_k}(b_k)) \cdots \alpha_{is_n}(b_n)\right) = \varphi\left(b_0 \alpha_{is_1}(b_1) \cdots \delta(b_k) \cdots \alpha_{is_n}(b_n)\right) \quad \text{and}
\]
\[
\varphi\left(b_0 \alpha_{is_1}(b_1) \cdots \gamma(\alpha_{is_k}(b_k)) \cdots \alpha_{is_n}(b_n)\right) = \varphi\left(b_0 \alpha_{is_1}(b_1) \cdots \alpha_{is_k}(\gamma(b_k)) \cdots \alpha_{is_n}(b_n)\right).
\]

(iv) For \(j = 2, \ldots, n\) we have:
\[
\int_{\sigma_{n+1}} \frac{\partial}{\partial s_j} \varphi\left(b_0 \alpha_{is_1}(b_1) \cdots \alpha_{is_{n+1}}(b_{n+1})\right) ds_1 \cdots ds_{n+1}
\]
\[
= \int_{\sigma_n} \left[ \varphi\left(b_0 \alpha_{is_1}(b_1) \cdots \alpha_{is_{j-1}}(b_{j-1} b_j) \cdots \alpha_{is_n}(b_n)\right) - \varphi\left(b_0 \alpha_{is_1}(b_1) \cdots \alpha_{is_{j-1}}(b_{j-1} b_j) \cdots \alpha_{is_n}(b_{n+1})\right) \right] ds_1 \cdots ds_n
\]

(44)
\[
\int_{\sigma_{n+1}} \frac{\partial}{\partial s_1} \varphi\left(b_0 \alpha_{is_1}(b_1) \cdots \alpha_{is_{n+1}}(b_{n+1})\right) ds_1 \cdots ds_{n+1}
\]
\[
= \int_{\sigma_n} \left[ \varphi\left(b_0 \alpha_{is_1}(b_1 b_2) \alpha_{is_2}(b_3) \cdots \alpha_{is_n}(b_n)\right) - \varphi\left(b_0 \alpha_{is_2}(b_2) \cdots \alpha_{is_n}(b_n)\right) \right] ds_1 \cdots ds_n
\]

(45)
\[
\int_{\sigma_{n+1}} \frac{\partial}{\partial s_{n+1}} \varphi\left(b_0 \alpha_{is_1}(b_1) \cdots \alpha_{is_{n+1}}(b_{n+1})\right) ds_1 \cdots ds_{n+1}
\]
\[
= \int_{\sigma_n} \left[ \varphi\left(\gamma(b_{n+1}) b_0 \alpha_{is_1}(b_1) \cdots \alpha_{is_n}(b_n)\right) - \varphi\left(b_0 \alpha_{is_1}(b_1) \cdots \alpha_{is_n}(b_{n+1})\right) \right] ds_1 \cdots ds_n
\]

(46)

**Proof:** (i) Recall that the left hand side is defined by the analytic extension of the function
\[
F_{t_1, \ldots, t_n}(b_0, \ldots, b_n) := \varphi(b_0 \alpha_{t_1}(b_1) \cdots \alpha_{t_n}(b_n))
\]
to the tube \(\mathbb{R}^n + i\sigma_n\) by the KMS-condition and Flat Tube theorem, so \(\varphi\left(b_0 \alpha_{is_1}(b_1) \cdots \alpha_{is_n}(b_n)\right) := F_{is_1, \ldots, is_n}(b_0, \ldots, b_n)\). By the invariance \(\varphi \circ \alpha_t = \varphi\) we have
\[
F_{t_1, \ldots, t_n}(b_0, \ldots, b_n) = \varphi(\alpha_t(b_0) \alpha_{t_1+t_1}(b_1) \cdots \alpha_{t+t_n}(b_n))
\]
\[
= F_{t, t_1, \ldots, t_1, t_2, \ldots, t_2, \ldots, t_n}(1, b_0, \ldots, b_n) = F_{t_1, \ldots, t_{n+1}}(\alpha_t(b_0), b_1, \ldots, b_n).
\]
The latter function has an analytic continuation in the variables \((t + t_1, \ldots, t + t_n)\) to \(\mathbb{R}^n + i\sigma_n\) and from the former function it also has an analytic extension in \(t\) to the strip \(\mathbb{R} + i[0,1]\). Thus by the flat tube theorem we get a unique analytic extension to all of \(\mathbb{R}^{n+1} + i\sigma_n\). Put \(t_j = is_j\) where \(s \in \sigma_n\) and \(t = i(1-s_n)\), then
\[
F_{is_1, \ldots, is_n}(b_0, \ldots, b_n) = F_{i(1-s_n), i(1-s_n+s_1), \ldots, i(1-s_n+s_n-1), 1}(1, b_0, \ldots, b_n)
\]
which is justified because we have that the variables
\[
r = (r_1, \ldots, r_n) := (1 - s_n, 1 - s_n + s_1, \ldots, 1 - s_n + s_n - 1) \in \sigma_n.
\]
Now the function \(F_{ir_1, \ldots, ir_n, 1}(1, b_0, \ldots, b_n)\) is obtained from the analytic extension of \(F_{t_1, \ldots, t_n, v}(1, b_0, \ldots, b_n), t_i, v \in \mathbb{R}\) to \(\mathbb{R}^{n+1} + i\sigma_n\). By the KMS-condition:
\[
F_{t_1, \ldots, t_n, v}(1, b_0, \ldots, b_n) = \varphi\left(b_n \gamma(\alpha_{t_1}(b_0) \cdots \alpha_{t_n}(b_{n-1}))\right)
\]
\[
= \varphi\left(\gamma(b_n) \alpha_{t_1}(b_0) \cdots \alpha_{t_n}(b_{n-1})\right) = F_{t_1, \ldots, t_n}(\gamma(b_n), b_0, \ldots, b_{n-1})
\]
Thus by uniqueness of the analytic continuations we have

\[ F_{i_{s_1}, \ldots, i_{s_n}}(b_0, \ldots, b_n) = F_{i_{s_1}, \ldots, i_{s_n}}(1, b_0, \ldots, b_n) = F_{i_{s_1}, \ldots, i_{s_n}}(\gamma(b_n), b_0, \ldots, b_{n-1}) \]

which is the statement (i) of the lemma.

(ii) By part (i) we have:

\[
\int_{\sigma_n} \varphi\left(b_0 \alpha_{i_{s_1}}(b_1) \cdots \alpha_{i_{s_n}}(b_n)\right) ds_1 \cdots ds_n \\
= \int_{\sigma_n} \varphi\left(\gamma(b_n) \alpha_{i_{1-s_n}}(b_0) \alpha_{i_{1-s_n+s_1}}(b_1) \cdots \alpha_{i_{1-s_n+s_n-1}}(b_{n-1})\right) ds_1 \cdots ds_n \\
= \int_{\sigma_n} \varphi\left(\gamma(b_n) \alpha_{i_{r_1}}(b_0) \cdots \alpha_{i_{r_n}}(b_{n-1})\right) dr_1 \cdots dr_n
\]

making use of the change of variables \( s \rightarrow r \) above (with Jacobian = 1), and the fact that \( r \in \sigma_n \) iff \( s \in \sigma_n \).

(iii) Since \( \varphi\left(b_0 \alpha_{t_1}(b_1) \cdots \delta(\alpha_{t_k}(b_k)) \cdots \alpha_{t_n}(b_n)\right) = \varphi\left(b_0 \alpha_{t_1}(b_1) \cdots \alpha_{t_k}(b_k) \cdots \alpha_{t_n}(b_n)\right) \) and the latter obviously has an analytic extension to \( \mathbb{R}^n + \sigma_n \) the claim follows. Likewise for the other one.

(iv) For \( 2 \leq j \leq n \) we have

\[
\int_{s_{j-1}}^{s_{j+1}} \frac{\partial}{\partial s_j} \varphi\left(b_0 \alpha_{i_{s_1}}(b_1) \cdots \alpha_{i_{s_
+1}}(b_{n+1})\right) ds_j \\
= \varphi\left(b_0 \alpha_{i_{s_1}}(b_1) \cdots \alpha_{i_{s_{j-1}}}(b_{j-1}) \alpha_{i_{s_{j+1}}}(b_j) \alpha_{i_{s_{j+1}}}(b_{j+1}) \cdots \alpha_{i_{s_{n+1}}}(b_{n+1})\right) \\
- \varphi\left(b_0 \alpha_{i_{s_1}}(b_1) \cdots \alpha_{i_{s_{j-1}}}(b_{j-1}) \alpha_{i_{s_{j+1}}}(b_j) \alpha_{i_{s_{j+1}}}(b_{j+1}) \cdots \alpha_{i_{s_{n+1}}}(b_{n+1})\right) \\
= \varphi\left(b_0 \alpha_{i_{s_1}}(b_1) \cdots \alpha_{i_{s_{j-1}}}(b_{j-1}) \alpha_{i_{s_{j+1}}}(b_j) \alpha_{i_{s_{j+1}}}(b_{j+1}) \cdots \alpha_{i_{s_{n+1}}}(b_{n+1})\right) \\
- \varphi\left(b_0 \alpha_{i_{s_1}}(b_1) \cdots \alpha_{i_{s_{j-1}}}(b_{j-1}) \alpha_{i_{s_{j+1}}}(b_j) \alpha_{i_{s_{j+1}}}(b_{j+1}) \cdots \alpha_{i_{s_{n+1}}}(b_{n+1})\right)
\]

from which equation (11) follows by a change of label of the integration variables. For equation (15) we substitute \( j = 1, s_{j-1} = 0 \) into the last equation. For equation (16) we substitute \( j = n + 1, s_{j+1} = 1 \) into the last equation to get

\[
\int_{\sigma_{n+1}} \frac{\partial}{\partial s_{n+1}} \varphi\left(b_0 \alpha_{i_{s_1}}(b_1) \cdots \alpha_{i_{s_{n+1}}}(b_{n+1})\right) ds_1 \cdots ds_{n+1} \\
= \int_{\sigma_n} \left[ \varphi\left(b_0 \alpha_{i_{s_1}}(b_1) \cdots \alpha_{i_{s_n}}(b_n) \alpha_{i_{s_n}}(b_{n+1})\right) - \varphi\left(b_0 \alpha_{i_{s_1}}(b_1) \cdots \alpha_{i_{s_n}}(b_{n+1})\right) \right] ds_1 \cdots ds_n \\
= \int_{\sigma_n} \left[ \varphi\left(\gamma(b_{n+1}) b_0 \alpha_{i_{s_1}}(b_1) \cdots \alpha_{i_{s_n}}(b_n)\right) - \varphi\left(b_0 \alpha_{i_{s_1}}(b_1) \cdots \alpha_{i_{s_n}}(b_{n+1})\right) \right] ds_1 \cdots ds_n
\]

making use of part (i) for the KMS–condition.

Let us begin with the right hand side of our desired Equation (21). From the definition (23) we have for \( a_i \in D_c \) via \( \delta(1) = 0 \) that:

\[
(B_{\gamma_{n+1}})(a_0, \ldots, a_n) = i^{\gamma_{n+1}} \int_{\sigma_{n+1}} ds_1 \cdots ds_{n+1} \left[ \varphi\left(\alpha_{i_{s_1}}(\delta\gamma a_0) \cdots \alpha_{i_{s_{n+1}}}(\delta\gamma a_n)\right) \right]
\]
\[ + \sum_{j=1}^{n} (-1)^{nj} \varphi \left( \alpha_{is_1} (\delta \gamma a_{n+1-j}) \cdots \alpha_{is_j} (\delta \gamma j+1 a_{n-j}) \alpha_{is_{j+1}} (\delta \gamma j+1 a_{n}) \cdots \alpha_{is_{n+1}} (\delta \gamma n+1 a_{n-j}) \right). \]

We can now use Lemma 8.5(ii) in all the terms on the right hand side to bring the factor with \(a_0\) to the front:

\[ (B\tau_{n+1})(a_0, \ldots, a_n) = z^{n+1} \int_{\sigma_{n+1}} ds_1 \cdots ds_{n+1} \left[ \varphi \left( \delta \gamma (a_0) \alpha_{is_1} (\delta \gamma a_1) \cdots \alpha_{is_{n+1}} (\delta \gamma n a_{n+1-j}) \right) \right. \]

\[ \cdots \alpha_{is_{n+1}} (\delta \gamma n+1 a_{n-j}) \alpha_{is_{n+1}} (\delta \gamma n a_{n+1-j}) \alpha_{is_{n+2}} (\delta \gamma n+1 a_{n+1-j}) \cdots \alpha_{is_{n+1}} (\delta \gamma n+1 a_{n-j}) \right]. \]

Now substitute \( \varphi \to \varphi \circ \delta = -\delta \circ \gamma \) and Lemma 8.5(iii):

\[ = z^{n+1} \int_{\sigma_{n+1}} ds_1 \cdots ds_{n+1} \left[ \varphi \left( \delta (a_0) \alpha_{is_1} (\delta \gamma a_1) \cdots \alpha_{is_{n+1}} (\delta \gamma n a_{n+1-j}) \right) \right. \]

\[ \cdots \alpha_{is_{n+1}} (\delta \gamma n+1 a_{n-j}) \alpha_{is_{n+1}} (\delta \gamma n a_{n+1-j}) \alpha_{is_{n+2}} (\delta \gamma n+1 a_{n+1-j}) \cdots \alpha_{is_{n+1}} (\delta \gamma n+1 a_{n-j}) \right]. \]

Now recall that \( \tau := (\tau_0, 0, -\tau_2, 0, \tau_4, \ldots) \in C(D_c) \), and hence we may assume that \( n \) is odd in the preceding expression (if \( n \) is even, \( B\tau_{n+1} = 0 \)). Thus

\[ (B\tau_{n+1})(a_0, \ldots, a_n) = \int_{\sigma_{n+1}} ds_1 \cdots ds_{n+1} \left[ \sum_{j=0}^{n} \varphi \left( \delta (a_0) \alpha_{is_1} (\delta \gamma a_1) \cdots \alpha_{is_{n+1}} (\delta \gamma n a_{n+1-j}) \right) \right. \]

\[ \cdots \alpha_{is_{n+1}} (\delta \gamma n+1 a_{n-j}) \alpha_{is_{n+1}} (\delta \gamma n a_{n+1-j}) \alpha_{is_{n+2}} (\delta \gamma n+1 a_{n+1-j}) \cdots \alpha_{is_{n+1}} (\delta \gamma n+1 a_{n-j}) \right]. \]

Since \( \alpha_{is_{n+1}} (1) = 1 \), we can do the integrals w.r.t. \( s_{n-j+1} \), and so using \( 0 \leq s_1 \leq s_2 \leq \cdots \leq s_{n+1} \leq 1 \) and a relabelling of variables, we get

\[ (B\tau_{n+1})(a_0, \ldots, a_n) = \int_{\sigma_n} ds_1 \cdots ds_n \left[ \sum_{j=1}^{n-1} (s_{n-j+1} - s_{n-j}) \varphi \left( \delta (a_0) \alpha_{is_1} (\delta \gamma a_1) \cdots \alpha_{is_{n+1}} (\delta \gamma n a_{n+1-j}) \right) \right. \]

\[ \cdots \alpha_{is_{n-j}} (\delta \gamma n a_{n-j}) \alpha_{is_{n-j+1}} (\delta \gamma n+1 a_{n+1-j}) \cdots \alpha_{is_{n-j}} (\delta \gamma n+1 a_{n-j}) \right] \]

\[ + (s_1 + 1 - s_n) \varphi \left( \delta (a_0) \alpha_{is_1} (\delta \gamma a_1) \cdots \alpha_{is_{n+1}} (\delta \gamma n a_{n+1-j}) \right). \]

Next, we turn our attention to the left hand side of our desired Equation (24). Observe first that we have

\[ \tau_n (\gamma a_0, \ldots, \gamma a_n) = (-1)^{n} \tau_n (a_0, \ldots, a_n) \quad \text{because:} \]

50
\[ \varphi(\gamma(a_0)\alpha_{i_{s1}}(\delta\gamma(\gamma a_1))\cdots\alpha_{i_{s_n}}(\delta\gamma^n(\gamma a_1))) = (-1)^n \varphi(a_0\alpha_{i_{s1}}(\delta\gamma a_1)\cdots\alpha_{i_{s_n}}(\delta\gamma^n a_n)) \]

since \( \delta \circ \gamma = -\gamma \circ \delta \), \( \varphi \circ \gamma = \varphi \) and by Lemma 8.5 (iii). Thus \( \tau \circ \gamma = \tau \), and so we have \( \tilde{a} = \gamma a \) in definition 22. An application of definition 22 to the left hand side of Equation 24 yields:

\[
(b_{\tau_{n-1}}(a_0, \ldots, a_n) = t^{r_{n-1}} \int_{\sigma_{n-1}} ds_1 \cdots ds_{n-1} \left[ \sum_{j=0}^{n-1} (-1)^j \varphi(a_0\alpha_{i_{s1}}(\delta\gamma a_1)\cdots\alpha_{i_{s_j}}(\delta\gamma^j(a_j a_{j+1}))\cdots \alpha_{i_{s_{n-1}}}(\delta\gamma^{n-1} a_n)) + (-1)^n \varphi((\gamma a_n) a_0\alpha_{i_{s1}}(\delta\gamma a_1)\cdots\alpha_{i_{s_{n-1}}}(\delta\gamma^{n-1} a_{n-1})) \right].
\]

(48)

We examine the terms in this sum more closely:

\[ j = 0: \int_{\sigma_{n-1}} ds_1 \cdots ds_{n-1} \varphi(a_0 a_1\alpha_{i_{s1}}(\delta\gamma a_2)\cdots\alpha_{i_{s_{n-1}}}(\delta\gamma^{n-1} a_n)) \]

\[ j = 1: -\int_{\sigma_{n-1}} ds_1 \cdots ds_{n-1} \varphi(a_0\alpha_{i_{s1}}(\delta\gamma(a_1 a_2))\alpha_{i_{s_{2}}} (\delta a_3)\cdots\alpha_{i_{s_{n-1}}}(\delta\gamma^{n-1} a_n)) \]

\[ = -\int_{\sigma_{n-1}} ds_1 \cdots ds_{n-1} \varphi(a_0\alpha_{i_{s1}}(\delta\gamma(a_1) a_2 + a_1 \delta\gamma a_2)\alpha_{i_{s_{2}}} (\delta a_3)\cdots\alpha_{i_{s_{n-1}}}(\delta\gamma^{n-1} a_n)) \]

\[ = -\int_{\sigma_{n-1}} ds_1 \cdots ds_{n-1} \left[ \varphi(a_0\alpha_{i_{s1}}(\delta\gamma(a_1) a_2)\alpha_{i_{s_{2}}} (\delta a_3)\cdots\alpha_{i_{s_{n-1}}}(\delta\gamma^{n-1} a_n)) + \varphi(a_0 a_1\alpha_{i_{s1}}(\delta\gamma a_2)\alpha_{i_{s_{2}}} (\delta a_3)\cdots\alpha_{i_{s_{n-1}}}(\delta\gamma^{n-1} a_n)) \right] \]

\[ - \int_{\sigma_{n}} ds_1 \cdots ds_{n} \frac{\partial}{\partial s_j} \varphi(a_0\alpha_{i_{s1}}(\delta\gamma a_1)\cdots\alpha_{i_{s_{j}}} (\delta\gamma j+1 a_j)\cdots\alpha_{i_{s_{n}}}(\delta\gamma^{n-1} a_n)) \]

where we made use of Equation 144 in the last step. Notice that we get a cancellation between the middle term and the \( j = 0 \) term in the sum. For \( 1 < j \leq n - 1 \) we have the terms:

\[ (-1)^j \int_{\sigma_{n-1}} ds_1 \cdots ds_{n-1} \varphi(a_0\alpha_{i_{s1}}(\delta\gamma a_1)\cdots\alpha_{i_{s_{j}}} (\delta\gamma j(a_j a_{j+1}))\cdots\alpha_{i_{s_{n-1}}}(\delta\gamma^{n-1} a_n)) \]

\[ = (-1)^j \int_{\sigma_{n-1}} ds_1 \cdots ds_{n-1} \varphi(a_0\alpha_{i_{s1}}(\delta\gamma a_1)\cdots\alpha_{i_{s_{j}}} ((\delta\gamma j a_j)^j a_{j+1} + \gamma^{j+1}(a_j)\delta\gamma j+1 a_{j+1})\cdots\alpha_{i_{s_{n-1}}}(\delta\gamma^{n-1} a_n)) \]

\[ = (-1)^j \int_{\sigma_{n-1}} ds_1 \cdots ds_{n-1} \left[ \varphi(a_0\alpha_{i_{s1}}(\delta\gamma a_1)\cdots\alpha_{i_{s_{j}}} ((\delta\gamma j a_j)^j a_{j+1})\cdots\alpha_{i_{s_{n-1}}}(\delta\gamma^{n-1} a_n)) + \varphi(a_0\alpha_{i_{s1}}(\delta\gamma a_1)\cdots\alpha_{i_{s_{j-1}}} ((\delta\gamma j-1 a_{j-1})\gamma^{j+1} a_{j+1})\cdots\alpha_{i_{s_{n-1}}}(\delta\gamma^{n-1} a_n)) \right] \]

\[ + (-1)^j \int_{\sigma_{n}} ds_1 \cdots ds_{n} \frac{\partial}{\partial s_j} \varphi(a_0\alpha_{i_{s1}}(\delta\gamma a_1)\cdots\alpha_{i_{s_{j}}} (\delta\gamma j+1 a_j)\cdots\alpha_{i_{s_{n}}}(\delta\gamma^{n-1} a_n)) \]

where we made use of Equation 144. Thus we get for Equation 145, taking into account cancellations between subsequent terms in the sum, that

\[
(b_{\tau_{n-1}}(a_0, \ldots, a_n) = t^{r_{n-1}}(-1)^{n-1} \int_{\sigma_{n-1}} ds_1 \cdots ds_{n-1} \varphi(a_0\alpha_{i_{s1}}(\delta\gamma a_1)\cdots\alpha_{i_{s_{n-1}}}(\delta\gamma^{n-1} a_{n-1})\gamma^{n-1} a_n)) \]

\[ + \int_{\sigma_{n}}^{n-1} \sum_{j=1}^{n-1} (-1)^j \int_{\sigma_{n}} ds_1 \cdots ds_{n} \frac{\partial}{\partial s_j} \varphi(a_0\alpha_{i_{s1}}(\delta\gamma a_1)\cdots\alpha_{i_{s_{j}}} (\delta\gamma a_{j+1})\cdots\alpha_{i_{s_{n}}}(\delta\gamma^{n-1} a_n)) \]

\[ + \int_{\sigma_{n-1}}^{n-1}(-1)^n \int_{\sigma_{n-1}} ds_1 \cdots ds_{n-1} \varphi((\gamma a_n) a_0\alpha_{i_{s1}}(\delta\gamma a_1)\cdots\alpha_{i_{s_{n-1}}}(\delta\gamma^{n-1} a_{n-1})) \]

\[ = \sum_{j=1}^{n} (-1)^j \int_{\sigma_{n}} ds_1 \cdots ds_{n} \frac{\partial}{\partial s_j} \varphi(a_0\alpha_{i_{s1}}(\delta\gamma a_1)\cdots\alpha_{i_{s_{j}}} (\delta\gamma a_{j+1})\cdots\alpha_{i_{s_{n}}}(\delta\gamma^{n-1} a_n)) \]

51
where we made use of Equation (40) and used the fact that since \( \overline{\tau} = (\tau_0, 0, -\tau_2, 0, \tau_4, \ldots) \), we may take \( n \) to be odd. Then

\[
(b \tau_{n-1})(a_0, \ldots, a_n) = \sum_{j=1}^{n} (-1)^{j+n} \int_{\sigma_n} ds_1 \cdots ds_n \frac{\partial}{\partial s_j} \varphi\left( (\gamma a_0) \alpha_{i(s_1)} (\delta a_1) \cdots \alpha_{i(s_j)} (\delta \gamma^j a_j) \cdots \alpha_{i(s_n)} (\delta \gamma^n a_n) \right)
\]

\[
= \int_{\sigma_n} ds_1 \cdots ds_n \sum_{j=1}^{n} \frac{\partial}{\partial s_j} \varphi\left( (\gamma a_0) \alpha_{i(s_1)} (\delta a_1) \cdots \alpha_{i(s_j-1)} (\delta \gamma^{j-1} a_{j-1}) \alpha_{i(s_j)} (\delta \gamma^j a_j) \cdots \alpha_{i(s_n)} (\delta \gamma^n a_n) \right). \tag{49}
\]

To make further progress, we need the following lemma.

**8.6 Lemma** Let \( a_i \in \mathcal{D}_S \) and \( (s_1, \ldots, s_n) \in \sigma_n \), then

\[
\varphi\left( \delta(a_0) \alpha_{i(s_1)} (\delta a_1) \cdots \alpha_{i(s_n)} (\delta a_n) \right) = \sum_{j=1}^{n} \frac{\partial}{\partial s_j} \varphi\left( (\gamma a_0) \alpha_{i(s_1)} (\delta a_1) \cdots \alpha_{i(s_j-1)} (\delta \gamma^{j-1} a_{j-1}) \alpha_{i(s_j)} (\delta \gamma^j a_j) \cdots \alpha_{i(s_n)} (\delta a_n) \right).
\]

**Proof:** A close examination of the proof of Theorem 5.8 shows that we actually proved that

\[
\frac{d}{dt} \varphi(B \alpha_t(A)C) \bigg|_0 = i \varphi(B \overline{\delta}_0(A)C) = i \varphi(B \overline{\delta}^2(A)C)
\]

for all \( A \in \mathcal{D}_S \) and \( B, C \in A_0 \) where the right hand side makes sense because \( \varphi \) is strongly regular on \( \mathcal{R}(S(\mathbb{R}), \sigma) \) hence is well defined on \( \mathcal{E}_0 \). Now from the graded product rule for \( \overline{\delta} \) on \( \mathcal{E}_0 \) we get

\[
\overline{\delta}(b_0) B_1 \cdots B_n = \overline{\delta}(b_0 B_1 \cdots B_n) - \sum_{j=1}^{n} \gamma(b_0 B_1 \cdots B_{j-1}) \overline{\delta}(B_j) B_{j+1} \cdots B_n
\]

for \( b_0 \in \mathcal{D}_S, B_1, \ldots, B_n \in A_0 \). Let \( B_i = \overline{\delta}(b_i) \) for \( b_i \in \mathcal{D}_S \), then

\[
\overline{\delta}(b_0) \overline{\delta}(b_1) \cdots \overline{\delta}(b_n) = \overline{\delta}(b_0 \overline{\delta}(b_1) \cdots \overline{\delta}(b_n)) - \sum_{j=1}^{n} \gamma(b_0 \overline{\delta}(b_1) \cdots \overline{\delta}(b_{j-1})) \overline{\delta}^2(b_j) \overline{\delta}(b_{j+1}) \cdots \overline{\delta}(b_n).
\]

Hence, using \( \varphi \circ \overline{\delta} = 0 \) we get:

\[
\varphi\left( \overline{\delta}(b_0) \overline{\delta}(b_1) \cdots \overline{\delta}(b_n) \right) = - \sum_{j=1}^{n} \varphi\left( \gamma(b_0 \overline{\delta}(b_1) \cdots \overline{\delta}(b_{j-1})) \overline{\delta}^2(b_j) \overline{\delta}(b_{j+1}) \cdots \overline{\delta}(b_n) \right)
\]

\[
= i \frac{d}{dt} \sum_{j=1}^{n} \varphi\left( \gamma(b_0 \overline{\delta}(b_1) \cdots \overline{\delta}(b_{j-1})) \alpha_t(b_j) \overline{\delta}(b_{j+1}) \cdots \overline{\delta}(b_n) \right) \bigg|_0.
\]

Now make the replacements \( b_0 \to a_0, b_i \to \alpha_{t_i}(a_i), i = 1, \ldots, n \) for \( a_i \in \mathcal{D}_S \) and use the fact that \( \alpha_t \circ \overline{\delta} = \overline{\delta} \circ \alpha_t \) to find that:

\[
\varphi\left( \delta(a_0) \alpha_{t_1}(\delta a_1) \cdots \alpha_{t_n}(\delta a_n) \right) = i \sum_{j=1}^{n} \frac{\partial}{\partial t_j} \varphi\left( \gamma(a_0) \alpha_{t_1}(\delta a_1) \cdots \alpha_{t_{j-1}}(\delta a_{j-1}) \alpha_{t_j}(a_j) \alpha_{t_{j+1}}(\delta a_{j+1}) \cdots \alpha_{t_n}(\delta a_n) \right)
\]

52
where we replaced $\overline{\tau}$ by $\delta$ because it is now evaluated on $D_S$ only. Now by the KMS–condition, analyticity, flat tube theorem and a complex linear change of variables, we find as in Section 6 that the functions

$$
(t_1, \ldots, t_n) \rightarrow \varphi\left(\delta(a_0) \alpha t_1 (\delta a_1) \cdots \alpha t_n (\delta a_n)\right)
$$

$$
(t_1, \ldots, t_n) \rightarrow \varphi\left(\gamma(a_0) \alpha t_1 (\gamma \delta a_1) \cdots \alpha t_j-1 (\gamma \delta a_j-1) \alpha t_j (a_j) \alpha t_{j+1} (\delta a_{j+1}) \cdots \alpha t_n (\delta a_n)\right)
$$

extend analytically to the flat tube $T_n := \mathbb{R}^n + i\sigma^n$ such that

$$
\varphi\left(\delta(a_0) \alpha z_1 (\delta a_1) \cdots \alpha z_n (\delta a_n)\right) = i \sum_{j=1}^n \frac{\partial}{\partial z_j} \varphi\left(\gamma(a_0) \alpha z_1 (\gamma \delta a_1) \cdots \alpha z_j-1 (\gamma \delta a_j-1) \alpha z_j (a_j) \alpha z_{j+1} (\delta a_{j+1}) \cdots \alpha z_n (\delta a_n)\right).
$$

In the case that $z_k = is_k$ where $(s_1, \ldots, s_n) \in \sigma^n$ we can use $\partial/\partial z_k = -i \partial/\partial s_k$ to obtain from the last equation the statement of the Lemma.

Application of the Lemma to Equation \ref{49} then produces

$$
(b \tau_{n-1})(a_0, \ldots, a_n) = \int ds_1 \cdots ds_n \varphi\left(\delta(a_0) \alpha is_1 (\delta \gamma a_1) \cdots \alpha is_n (\delta \gamma^n a_n)\right)
$$

$$
= (B \tau_{n+1})(a_0, \ldots, a_n)
$$

by Equation \ref{47} and hence $\overline{\tau}$ is a cyclic cocycle.

**Acknowledgements.**

We gratefully acknowledge discussions with Arthur Jaffe, Roberto Longo and Hajime Moriya on various aspects of supersymmetry. DB wishes to thank the Department of Mathematics of the University of New South Wales and HG wishes to thank the Institute for Theoretical Physics of the University of Göttingen for hospitality and financial support which facilitated this research. The work was also supported in part by the FRG grant PS01583.

**References**

[1] Araki, H: Bogoliubov automorphisms and Fock representations of canonical anticommutation relations. Contemp. Math. 62, 23–141 (1987)

[2] Bros, J., Buchholz, D.: Towards a relativistic KMS-condition. Nucl. Phys. B429, 291–318 (1994). Available on the web at [http://xxx.lanl.gov/abs/hep-th/9807099](http://xxx.lanl.gov/abs/hep-th/9807099)
[3] Buchholz, D., Longo R.: Graded KMS-functionals and the breakdown of supersymmetry. Adv. Theor. Math. Phys. 3, 615–626 (2000) [Addendum: ibid. 6, 1909–1910 (2000)]

[4] Buchholz D., Ojima I.: Spontaneous collapse of supersymmetry. Nucl. Phys. B498, 228–242 (1997)

[5] Connes, A.: Entire cyclic cohomology of Banach algebras and characters of $\theta$–summable Fredholm modules. K-theory 1, 519–548 (1988)

[6] Conway, J.W.: Functions of one complex variable. Springer 1978

[7] Folland, G.: Real Analysis. John Wiley & Sons, New York 2000

[8] Fuchs, J.: Thermal and superthermal properties of supersymmetric field theories. Nucl. Phys. B246, 279–301 (1984)

[9] Damak, M., Georgescu, V.: Self-adjoint operators affiliated to C*-algebras. Rev. Math. Phys. 16 257–280 (2004)

[10] Van Hove, L.: Supersymmetry and positive temperature for simple systems. Nucl.Phys.B207 15–28 (1982)

[11] Jaffe, A., Lesniewski, A., Osterwalder, K., On super-KMS functionals and entire cyclic cohomology. K-theory 2 (no. 6), 675–682 (1989)

[12] Jaffe, A., Lesniewski, A., Osterwalder, K., Quantum K-theory. Commun. Math. Phys. 118, 1–14 (1988)

[13] Jaffe, A., Osterwalder, K., Ward identities for non–commutative geometry. Commun. Math. Phys. 132, 119–130 (1990)

[14] Kadison, R.V., Ringrose, J.R.: Fundamentals of the Theory of Operator Algebras I. New York: Academic Press 1983

[15] Kastler, D.: Cyclic cocycles from graded KMS functionals. Comm. Math. Phys. 121 345–350 (1989)

[16] Kishimoto, A., Nakamura, H.: Super-derivations. Commun. Math. Phys. 159, 15–27 (1994)

[17] Lang, S.: SL2(ℝ). Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1975

[18] Longo, R.: Notes for a quantum index theorem. Commun. Math. Phys. 222, 45–96 (2001)
[19] Manuceau, J.; Verbeure, A. Quasi-free states of the CCR-algebra and Bogoliubov transformations. Comm. Math. Phys. 9, 293–302 (1968)

[20] Nehari, Z: Conformal mapping. Reprinting of the 1952 edition. Dover Publications, Inc., New York, 1975

[21] Plymen, R., Robinson, P.: Spinors in Hilbert space. Cambridge University Press 1994.

[22] Reed, M., Simon, B.: Methods of mathematical physics I: Functional analysis. Academic Press, New York, London, Sydney 1980.

[23] Rocca, F., Sirugue, M., Testard, D.: On a class of equilibrium states under the Kubo-Martin-Schwinger boundary condition. I. Fermions. Comm. Math. Phys. 13 317–334 (1969)

[24] Weidmann, J.: Linear operators in Hilbert spaces. Springer-Verlag, New York, Heidelberg, Berlin 1980.

[25] Woronowicz, S.: C*-algebras generated by unbounded elements. Rev. Math. Phys. 7, 481–521 (1995)

[26] Yosida, K.: Functional Analysis. Springer-Verlag, Berlin, Heidelberg, New York 1980.