We propose a general framework to describe Planckian deviations from Special Relativity (SR) compatible with a relativistic principle. They are introduced as the leading corrections in an asymptotic approach to SR going beyond the energy power expansion of effective field theories. We discuss the conditions in which these Planckian effects might be experimentally observable in the near future, together with the non-trivial limits of applicability of this asymptotic approach that such a situation would produce, both at the very high (ultraviolet) and the very low (infrared) energy regimes.

I. INTRODUCTION

Lorentz invariance is a main ingredient in our present day physical theories. However, Planck-scale departures from Lorentz symmetry are naturally expected as quantum gravity effects in many explored scenarios in relation with the structure of a quantum space-time [1]. It is interesting to realize that going beyond Special Relativity (SR) does not necessarily imply the existence of a preferred class of inertial observers. Doubly Special Relativity (DSR) was originally proposed as a generalization of SR in which the key idea is to have a universal, observer-independent, length (mass) scale besides the universal maximum speed present in SR. In this way one could give to the Planck length, made invariant by a suitable deformation of the Lorentz transformations between inertial observers, a physical meaning in connection with the structure of space-time [2–4].

DSR has been studied for almost ten years now [5]. It has been shown that there exist different versions of it [2, 6, 7], in which the universal energy scale represents typically a bound of the energy or/and the momentum. However, the consistency of a full theory implementing the DSR idea is still an open question, with some doubts coming from the emergence of apparent paradoxes, such as the so-called soccer ball problem (i.e., the incompatibility between the kinematics of macroscopic bodies and the assumption of a Planckian deformed dispersion relation) [4, 8], or nonlocal effects that might affect macroscopic physics [9].

One might wonder whether the DSR idea could be tested. In fact it may appear surprising that quantum gravity effects may be accesible at all to present day experiments, which involve energies many orders of magnitude lower than the Planck scale. However, the realization that in certain situations there are mechanisms that are able to amplify these corrections has opened a whole branch in the field, known generically as quantum gravity phenomenology [10]. It is in principle possible that this phenomenology is able to distinguish a DSR from a Lorentz breakdown (without a relativity principle) scenario. For example, by experimentally establishing the existence of an energy threshold for some particle decay the DSR framework could be ruled out, since such a threshold would not be observer independent [5]. Nevertheless, in a general case it will be difficult to distinguish between both scenarios. For example, most DSR versions give negligible corrections for processes typically considered in quantum gravity phenomenology, such as the analysis of the ultra-high energy cosmic-ray spectrum and the presence of the GZK cutoff [11], but there is always the possibility to cook some ad hoc DSR model making the corrections relevant [7]. What is more, at least in the next future, it seems likely that this kind of phenomenology will be sensitive at most to the leading order of DSR or Lorentz breakdown corrections, so that one should not insist on pursuing an “all-order DSR phenomenology” [5].

Taking into account these considerations, and with the same motivation that inspired the DSR idea, we try in this work to present a systematic, very general and coherent framework of asymptotic Planck-scale corrections to Special Relativity in a relativistic invariant way. These corrections, parametrized by the Planck scale, are a first-order approximation to whatever scenario lies beyond SR, so that their validity will be limited to an energy domain which will depend on the details of the implementation of the departures from SR kinematics. Since this domain will in any case lie far from the Planck energy, we will not impose any restriction of the kind of the DSR postulate of having the Planck mass as a second invariant besides the speed of light at this level of this asymptotic approach. This fact will also eliminate the need to address any soccer ball problem at this level since it would imply to go beyond the limits of validity of the approximation we are considering. We will refer to this framework as Asymptotic Special Relativity (ASR). It will contain of course the leading order of every DSR theory, but ASR maybe relevant even if DSR (as an
all-order theory) is not realized in Nature.

Implicit in all the discussion presented here is the assumption that the framework of effective field theories is not sufficiently general to incorporate all the low-energy/semiclassical implications of the quantum structure of space-time. If this were not the case then one already has an extension of the Standard Model in the effective field theory (EFT) framework incorporating physics beyond SR. A consequence of this assumption is that one is limited to kinematic considerations. We are still far from identifying a consistent dynamical theory including the main consequences of quantum gravity which allows us to understand how SR emerges as an approximate or exact symmetry, but we think there are enough reasons to explore kinematic situations which go beyond the EFT paradigm. In fact, EFT has shown to be a powerful tool in describing a large variety of phenomena, from the Fermi theory of weak interactions in particle physics to the quantum Hall effect in condensed matter, just to mention two important examples, but this does not mean that every physical effect can be explained in its framework. In condensed matter physics the description of many-body systems in terms of a few collective degrees of freedom (the quasi-particles) is always approximate and, in many cases, the construction of a local EFT is impossible due to some exchange with the fundamental degrees of freedom (the atoms) \[12\]. This would be relevant in the low-energy/semiclassical limit of quantum gravity if gravitons were emergent degrees of freedom \[14\]. In other approaches to the quantum gravity problem there are also hints pointing out that their low-energy/semiclassical limit is not completely contained in a local EFT. In quantum field theories with canonical noncommutative space-time coordinates there exists a mixing between ultraviolet and infrared modes that makes difficult its compatibility with the Wilsonian EFT framework \[15\]. In approaches independent of background metrics, such as Loop Quantum Gravity, where a different quantization (i.e., not unitarily equivalent to the Fock’s quantization) is carried out and radically new ingredients arise, it is not clear whether the low-energy/semiclassical limit of the theory is completely contained in a conventional local EFT \[16\].

In fact, the approach introduced in this paper under the name of ASR is not new. The possibility of considering DSR as a leading order Planckian effect independently of an extension to all orders and energies was already present in the first works of DSR \[2, 4\]. What we present in this paper is a general and systematic discussion of the kind of corrections consistent with the existence of a relativity principle and their phenomenological consequences, something we think was missing in previous works. Besides, we introduce some notions such as an infrared limit (additional to the ultraviolet one) in the domain of validity of the departures from SR and an intrinsic non-separability in the energy-momentum conservation law.

In the next Section we present a generalization of SR kinematics for one and two particle systems. In Section III we explore the observability of the departures from SR with a few examples. In Section IV we summarize our results and proposals for future work.

II. GENERAL KINEMATIC FRAMEWORK

A. One-particle system

We start with the kinematic description of a one particle system. It is based on the assumption of translational invariance, which allows to assign energy-momentum variables \(E, P\) to each particle, and rotational invariance implemented as in SR (most attempts to go beyond SR concentrate on this possibility,\(^1\) although it is possible to go beyond this case). The main ingredient defining the kinematic framework is its asymptotic approach to SR kinematics parametrized by just one single additional scale (\(M_P\), the scale of transition from the classical to the quantum description of space-time). Under these conditions the most general expression for the asymptotic dispersion relation of a particle takes the form

\[
E^2 - P^2 - \frac{E^{2+\alpha}}{M_P^\alpha} \Delta(P/E) = \mu^2
\]

with \(\mu\) the particle mass, which in the region where Planckian corrections are completely negligible coincides with the SR definition, \(\mu^2 = E^2 - P^2\), an exponent \(\alpha > 0\) characterizing the leading approach to SR when \(E/M_P \ll 1\), and an arbitrary universal\(^2\), i.e. the same for every particle, function \(\Delta\) of the dimensionless combination of rotational

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\(^1\) There are several arguments to consider on a different footing deviations from SR at the level of rotation/boost transformations. One is the compact/noncompact nature of the transformations which prevents the possibility to test the invariance under all boost transformations. Another one is the observation that the presence of a new energy scale in the modified dispersion relation requires deforming the boost transformations but not necessarily the rotations.

\(^2\) As long as we are interested in an implementation of the relativity principle, we will not consider dispersion relations that depend on particle properties, apart from the mass.
invariant variables $P/E$. The Planckian modified dispersion relations which appear in many phenomenological studies of departures from SR are included as particular cases (with $\alpha = 1$ or $\alpha = 2$, and specific forms of $\Delta$, such as a constant value) of the asymptotic dispersion relation (1).

In order to make this kinematic description compatible with a relativity principle we need two things. First, identify a set of inertial frames for which some deformed Lorentz transformations specify the relation between energy-momentum variables in them in such a way that the dispersion relation (1) is kept invariant. Second, modify accordingly the usual conservation laws so that they become covariant under change of inertial frames. The last point requires going beyond the one particle system and will be studied in the next subsection. The simplest way to satisfy the first requisite is to introduce a set of auxiliary asymptotic variables

$$\epsilon = E \left(1 - \frac{E_0^\alpha}{2M_P^\alpha} g(P/E)\right) \quad \pi = P \left(1 - \frac{E_0^\alpha}{2M_P^\alpha} h(P/E)\right)$$

(2)

where the two functions $g, h$ defining the auxiliary variables are chosen such that

$$g(P/E) - \frac{P^2}{E^2} h(P/E) = \Delta(P/E).$$

(3)

Apart from this, there is no other restriction on these two functions but the trivial request that there must be some range of $P/E$ values where they are bounded. The different choices for $g$ and $h$ will determine the domain of validity of the asymptotic approach to SR, to be discussed in the next paragraphs. One can then write the asymptotic dispersion relation (1) in the simple form

$$\epsilon^2 - \pi^2 = \mu^2,$$

(4)

and the Lorentz transformation of energy-momentum variables in different inertial frames is obtained by translating the usual linear four-vector Lorentz transformations of the auxiliary asymptotic variables $\epsilon, \pi$ to the energy-momentum of the particle through the relations

$$E = \epsilon \left(1 + \frac{c^\alpha}{2M_P^\alpha} g(\pi/\epsilon)\right) \quad P = \pi \left(1 + \frac{c^\alpha}{2M_P^\alpha} h(\pi/\epsilon)\right).$$

(5)

The asymptotic auxiliary variables are the analog of the ‘Judes-Visser auxiliary variables’ [17] introduced in DSR. Is is important to remark that given a dispersion relation there are several possible choices of functions $g$ and $h$ compatible with the implementation of the relativity principle, each one corresponding to a transformation between inertial reference frames.

Then one has that the kinematic description of a one particle system in a generalization of SR is given in terms of an exponent $\alpha$ parametrizing the asymptotic approximation to SR and two functions $g, h$ of one variable specifying the transformation of energy-momentum variables among different frames and the asymptotic dispersion relation.

**Domain of validity of ASR in the one-particle system**

The asymptotic approach to SR is governed by $(E/M_P)^\alpha g(P/E)$ and $(E/M_P)^\alpha h(P/E)$ which control the departure of the auxiliary variables (in which Lorentz transformations act linearly) from the energy momentum variables (in which Lorentz boosts act non-linearly). Since all our observations are constrained to energies far below the Planck mass $M_P$ then we will not have observable signals of the departures from SR if the functions $g(P/E), h(P/E)$ are of order one. This is, in fact, the general situation that one has in DSR. On the contrary, in Lorentz breaking theories, which do not preserve the relativity principle, large observable effects are produced by very small changes in the dispersion relations. The reason of this different behaviour is due to inevitable cancellations owing to the consistency between the dispersion relation and the energy and momentum conservation laws with the relativity principle. In order to compensate them we have to consider observations corresponding to some kinematic limits where the functions $g(P/E), h(P/E)$ are much larger than one and the deviations from SR are amplified. This will affect the naive limit of applicability of ASR, $E \ll M_P$, restricting it to the domain where the difference between auxiliary and energy momentum variables is small (since ASR gives only asymptotic corrections to SR). Let us see how these amplifications might arise in two kinematic limits, the ultra-relativistic limit $P/E \rightarrow 1$ and the non-relativistic limit $P/E \rightarrow 0$.

In the ultra-relativistic limit one can consider a situation where $g(x)$ (or $h(x)$, or both) has the following behavior:

$$g(x) \rightarrow (1 - x)^{-\gamma}, \quad \gamma > 0, \quad \text{when } x \rightarrow 1.$$  

(6)
One has in this case
\[
g(P/E) \sim \frac{(1 + P/E)^\gamma}{(1 - P^2/E^2)^\gamma} \sim \left(1 + \frac{P}{E}\right)^\gamma \left(\frac{E}{\mu}\right)^{2\gamma}
\] (7)
since $\mu^2 = E^2 - P^2$ at leading order in the asymptotic correction, so that one gets an amplification factor $(E/\mu)^{2\gamma}$ for the Planckian corrections which can make them appreciable at subPlanckian energy scales. At the same time the domain of validity of the asymptotic description of the kinematics is now restricted to $E \ll M_P(\mu/M_P)^{1/(1+\alpha/2\gamma)}$. In fact the DSR scenario [7] with sizeable threshold anomalies is just a particular realization of this amplification mechanism.

In a similar way one can consider the possibility of an amplification of the Planckian kinematic corrections in the nonrelativistic limit. In this case a limiting behavior for $g(x)$ (and/or $h(x)$) such as
\[
g(x) \to x^{-\beta}, \quad \beta > 0, \quad \text{when } x \to 0,
\] (8)
leads to an amplification factor $(\mu/P)^\beta$ which can partially compensate the natural suppression factor $(\mu/M_P)^\alpha$ of Planckian kinematic corrections to the nonrelativistic limit. We will show in the next Section some examples along these lines which lead to observable effects in atomic interferometry. A remarkable property of this scenario is that the domain of validity of the asymptotic kinematic description is constrained not only in the UV but also in the IR by $(\mu/M_P)^\alpha(\mu/P)^\beta \ll 1$, i.e. $P \gg \mu(\mu/M_P)^{\alpha/\beta}$. This restriction is satisfied by any microscopic experiment up to the present time, a necessary condition to make this scenario not obviously phenomenologically inconsistent. On the other hand an amplification of kinematic Planckian corrections in the nonrelativistic limit may be a clue to look for sufficiently precise observations to be able to identify a small departure from SR.

\section*{Special case: particles of zero mass}

The above discussion is limited to the case where $\mu \neq 0$. For example, we see that Eq. (7) is problematic if $\mu = 0$. In fact, the generalization of SR to an asymptotic approximation in terms of the two functions $g, h$ of one variable has to be changed if we are dealing with a particle for which $\mu = 0$. For such a particle, Eq. (4) gives $\epsilon = \pi$, so that the quotient $\pi/\epsilon$ is always equal to one. This implies that in relations (4), $g(\pi/\epsilon)$ and $h(\pi/\epsilon)$ are not functions, but constants. Let us call them $\tilde{g}$ and $\tilde{h}$, respectively.

The expressions (2) are then rewritten as
\[
\epsilon = E \left(1 - \frac{E^\alpha}{2M_P^\gamma} \tilde{g}\right), \quad \pi = P \left(1 - \frac{E^\alpha}{2M_P^\gamma} \tilde{h}\right),
\] (9)
and $\epsilon^2 - \pi^2 = 0$ leads to the dispersion relation
\[
E^2 - P^2 - \frac{E^{2+\alpha}}{M_P^\alpha} \left(\tilde{g} - \frac{P^2}{E^2} \tilde{h}\right) \approx E^2 - P^2 - \frac{E^{2+\alpha}}{M_P^\alpha} \left(\tilde{g} - \tilde{h}\right) = 0,
\] (10)
where we have used that in the asymptotic correction $P \approx E$. That is, the dispersion relation for a zero mass particle in ASR is like Eq. (1), where $\mu = 0$ and $\Delta(P/E)$ is a constant.

In this case we do not have at our disposal a function which can produce amplification effects in certain kinematic limits, so the domain of validity of ASR for a zero mass particle will be given just by $E \ll M_P$. Therefore in particle reactions involving photons we will not consider deviations from SR for them. Only at the end of the paper we will come to a possible phenomenological consequence of the tiny deviations from SR for photons in their propagation through very large distances.

\section*{B. Conservation laws in a two particle system}

To complete the kinematic analysis one needs to identify the modification of the energy-momentum conservation laws of SR which are compatible with the generalized relativity principle. The simplest way to do that is to consider the set of auxiliary asymptotic variables $\epsilon, \pi$ for each particle. In order to have asymptotic conservation laws compatible with the relativity principle one has to identify the most general form to get a set of four variables out of the auxiliary variables of the different particles in the initial or final state transforming under Lorentz transformations like the
auxiliary asymptotic variables of a one particle system. In the case of a two particle system this amounts to get a four-vector out of two four-vectors, i.e., to consider a linear combination of the two four-vectors with scalar coefficients

\[ \epsilon_{12} = \epsilon_1 + \epsilon_2 + \frac{s^{\alpha/2}}{M_p^2} \left[ \epsilon_1 f \left( \frac{\mu_1^2}{s}, \frac{\mu_2^2}{s} \right) + \epsilon_2 f \left( \frac{\mu_1^2}{s}, \frac{\mu_2^2}{s} \right) \right] \]  

(11)

\[ \pi_{12} = \pi_1 + \pi_2 + \frac{s^{\alpha/2}}{M_p^2} \left[ \pi_1 f \left( \frac{\mu_1^2}{s}, \frac{\mu_2^2}{s} \right) + \pi_2 f \left( \frac{\mu_1^2}{s}, \frac{\mu_2^2}{s} \right) \right] \]  

(12) with

\[ s = \mu_1^2 + \mu_2^2 + 2(\epsilon_1 \epsilon_2 - \pi_1 \pi_2). \]  

(13)

If one does not go beyond a two particle system then the asymptotic generalized conservation laws can be expressed as an equality between \( \epsilon_{12} \) and \( \pi_{12} \) for the initial and final states, which are a combination of the auxiliary energy and momentum variables of the two particles whose general form is determined by a function \( f \) of two variables. In order to arrive to this result we have also added the assumptions that there is neither ordering ambiguity, nor initial/final state mixing. The absence of ordering ambiguity implies that the conservation laws do not depend on what particle we choose as the first or the second particle and the lack of initial/final state mixing means that the conservation laws can be expressed as a system of equations where the dependence on initial and final state variables can be separated.\(^3\)

The four combinations of variables which appear in the asymptotic generalized conservation laws when expressed in terms of the energy-momentum variables of the particles take the form

\[ \epsilon_{12} = E_1 \left( 1 - \frac{E_1^\alpha}{2M_p^2} g(P_1/E_1) \right) + E_2 \left( 1 - \frac{E_2^\alpha}{2M_p^2} g(P_2/E_2) \right) + \frac{s^{\alpha/2}}{M_p^2} \left[ E_1 f \left( \frac{\mu_1^2}{s}, \frac{\mu_2^2}{s} \right) + E_2 f \left( \frac{\mu_1^2}{s}, \frac{\mu_2^2}{s} \right) \right] \]  

(14)

\[ \pi_{12} = P_1 \left( 1 - \frac{E_1^\alpha}{2M_p^2} h(P_1/E_1) \right) + P_2 \left( 1 - \frac{E_2^\alpha}{2M_p^2} h(P_2/E_2) \right) + \frac{s^{\alpha/2}}{M_p^2} \left[ P_1 f \left( \frac{\mu_1^2}{s}, \frac{\mu_2^2}{s} \right) + P_2 f \left( \frac{\mu_1^2}{s}, \frac{\mu_2^2}{s} \right) \right], \]  

(15)

and the variable \( s \) can be approximated in the asymptotic limit by \( s \approx \mu_1^2 + \mu_2^2 + 2(E_1 E_2 - P_1 P_2). \)

The simplest choice corresponds to \( f = 0 \) leading to conservation laws which take the standard SR additive form in terms of the auxiliary variables. Then these laws can be trivially extended to a system with an arbitrary number of particles. This choice for \( f \) could naively lead us to think that we are dealing just with SR, only rewritten in some strange variables (the auxiliary energy-momentum). This criticism has already been presented in the DSR context (see, e.g., Ref. \[19\]), and answered several times (see, e.g., Ref. \[5\]). It is based on the wrong identification of the existence of a mapping between the mathematical descriptions of two theories with their physical equivalence.\(^4\) The main point is that the energy and momentum of a particle have an intrinsic meaning (for example, as the generators of time and spatial translations) so that one is not allowed to define a ‘new’ energy from a nonlinear combination of the previous variables.

Another option however is to have \( f \neq 0 \). In this case it is not guaranteed that the conservation law can be expressed as a sum of contributions each one depending on the energy-momentum of only one particle. In this last instance, Planckian effects make the two-particle system to loose any kind of separability between their elements.\(^5\) Then adding one more particle introduces new arbitrariness and the conservation laws have to be studied separately for a given number of particles. In particular the associative property is lost; it has been argued \[17\] that such kinematic generalizations should be ruled out but we do not find any reason to exclude them within the rules used to define an asymptotic approach to SR kinematics.

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\(^3\) More exotic possibilities have been considered \[18\] in the attempts to realize the DSR idea in the \( \kappa \)-Minkowski/\( \kappa \)-Poincaré framework.

\(^4\) For instance, the existence of a canonical transformation that relates the coordinates of the system of two coupled oscillators to the coordinates of a system of two uncoupled oscillators obviously does not establish their physical equivalence.

\(^5\) We will say a conservation law is separable if and only if there is an equivalent equation in which each member can be written as a sum of terms each one depending just on the energy-momentum of one particle. In the case of \( f(x, y) = 0 \) one can reexpress the conservation law as an equivalent equation in which there are terms which contain energy-momentum of several particles (it is in this way as the conservation law is often written in the DSR literature). However, it can be written in an equivalent way as a sum of contributions each one depending on only one particle and, therefore, when \( f(x, y) = 0 \) the conservation law is separable.
Domain of validity of ASR in the two-particle system

The possibility to have generalized conservation laws may give rise to the peculiar situation in which ASR is identical to Special Relativity at the one-particle level (with unmodified dispersion relation and Lorentz transformations) but different from it if one considers a two-particle system. This happens if Planckian corrections are such that $g = h = 0$ and $f \neq 0$. In this case the ultra-small factor $s^{\alpha/2}/M^2$ of Eq. (11) makes these corrections unobservable unless $f(\mu_1^2/s, \mu_2^2/s)$ takes values much larger than one. To see how this may occur, first note that, for two given particles, $\mu_1, \mu_2$ fixed, $f$ is a function of only one variable, the Lorentz invariant Mandelstam variable $s$. An amplification of $f$ compatible with experimental observations may then only arise for values of $s$ outside of the range where Special Relativity is well tested. In the center of mass reference frame, $s$ is determined by the momentum $P$ of each of the two particles. The study of two-particle scattering has experimental limitations both at arbitrary large values of $P$, what we call the ‘high-energy’ (ultraviolet, UV) regime, and at arbitrary low values of $P$ (since one cannot determine that the two particles are at rest with arbitrary precision), what we refer to as the ‘low-energy’ (infrared, IR) limit. This means that we may have an amplification effect only in these two limits: very large $s$ (but still $s \ll M_P$ in the framework of ASR), or very low $s$. In the first case, one gets an amplification effect if $f$ is such that

$$f \left( \frac{\mu_1^2}{s}, \frac{\mu_2^2}{s} \right) \to \left( \frac{s}{\mu^2} \right)^{\gamma'}, \; \gamma' > 0, \; \text{when } s \gg \mu_1^2, \mu_2^2, \tag{16}$$

where $\mu^2$ is some combination of $\mu_1^2$ and $\mu_2^2$. Note that the UV regime does not include, for example, the collision process between ultra-high energy cosmic rays (of energy $E \sim 10^{21}$ eV) and cosmic microwave background photons (of energy $\omega \sim 10^{-3}$ eV), since for this process $s$ is not much larger than $\mu_1^2 \gamma$ (the squared of the proton mass):

$$s \approx \mu_1^2 N + 2E\omega(1 - \cos \theta) \Rightarrow \frac{s}{\mu_1^2} \approx 1 + 2 \frac{E}{10^{21} \text{ eV}} \frac{\omega}{10^{-3} \text{ eV}} (1 - \cos \theta), \tag{17}$$

where $\theta$ is the angle between the directions of the momenta of proton and photon in the laboratory reference frame.

The other scenario in which amplification effects for $f$ compatible with the observed phenomenology may arise is in the IR limit. In the case of two massive particles, when $s$ approaches its lower bound, $s \sim (\mu_1 + \mu_2)^2$, the arguments of $f$ take the values $(\mu_1^2 + \mu_2^2, \mu_3^2/\mu_1 + \mu_2^2)$. For a combination of two particles with given masses one could make an ad hoc choice for the function $f$ such that one had an amplification effect that would be specific to this combination of particles. We will consider the simpler and, as we will see, phenomenologically interesting case, in which one of the particles is massive with mass $\mu_1$ and the other is a zero mass particle, such as a photon. The IR limit in this case corresponds to values of the arguments of $f$ equal to 1 and 0, independently of the value of $\mu_1$, and an amplification in the infrared is obtained if

$$f(x, 0) \to (1 - x)^{-\beta'}, \; \beta' > 0, \; \text{when } x \to 1. \tag{18}$$

Consistency within the ASR framework requires that, in the case of an ultraviolet amplification taking place at $s \gg \mu_1^2, \mu_2^2$ as in Eq. (16), the modification in the conservation laws has to be small, $(s^{\alpha/2}/M^2)(s/\mu^2)^{\gamma'} \ll 1$ or $s \ll \mu_1^2/(1 + \alpha/2\gamma') M_P^{2/(1 + 2\alpha/\alpha)}$. This defines a domain of validity of ASR in the ultraviolet for a system of two particles which is smaller than the naive domain $s \ll M_P^2$.

In the case of the infrared limit of a massive particle $\mu_1$, and a photon, and with the infrared amplification Eq. (18), ASR is limited to the domain where $(s^{\alpha/2}/M^2)(1 - \mu_1^2/s)^{-\beta'} \ll 1$, or $s - \mu_1^2 \gg \mu_1^2(1/M_P)^{\alpha/\beta'}$.

III. SOME PHENOMENOLOGICAL IMPLICATIONS OF CORRECTIONS TO SR KINEMATICS

In order to illustrate the sensitivity of different observations to departures from SR we are going to consider the spectrum of ultra high energy cosmic rays (UHECR) as the best possibility to observe a departure from SR with an amplification mechanism in the UV/ultrarelativistic limit, the frequency shift in a double Raman transition, which is the most sensitive observation to an amplification in the IR/nonrelativistic limit of a departure from SR, and the effect of Planckian kinematic corrections in the energy dependence of the velocity of propagation of a particle. In all these cases we will take for definiteness a value $\alpha = 1$ for the exponent characterizing the asymptotic approach to SR so that the Planckian corrections will be proportional to $(1/M_P)$.
A. GZK cutoff

We first consider a separable conservation law with \( f = 0 \). In this case the kinematic analysis of SR applies directly to the auxiliary variables. Let us remind the derivation of the threshold for the energy loss of protons propagating in the cosmic microwave background (CMB) through pion production. The dispersion relation of the proton, in terms of auxiliary variables, is

\[ \epsilon \approx \pi + \frac{\mu_N^2}{2\pi} \quad \pi = \pi n, \tag{19} \]

with \( n \) a unit vector in the direction of propagation of the proton. The threshold can be determined by considering in the final state both the nucleon and the pion momenta in the same direction as the initial proton so that

\[ \epsilon'_1 \approx x_1 \pi + \frac{\mu_N^2}{2x_1 \pi} \quad \pi'_1 \approx x_1 \pi n \tag{20} \]

for the nucleon and

\[ \epsilon'_2 \approx x_2 \pi + \frac{\mu_N^2}{2x_2 \pi} \quad \pi'_2 \approx x_2 \pi n \tag{21} \]

for the pion. Then energy conservation leads to

\[ \pi + \frac{\mu_N^2}{2\pi} + \omega \approx x_1 \pi + \frac{\mu_N^2}{2x_1 \pi} + x_2 \pi + \frac{\mu_N^2}{2x_2 \pi} \tag{22} \]

and momentum conservation gives

\[ \pi - \omega = x_1 \pi + x_2 \pi, \tag{23} \]

where \( \omega \approx 10^{-3} \text{ eV} \) is the energy of the CMB-photon whose momentum is chosen in the opposite direction to the proton in order to determine the threshold of the reaction. Terms proportional to the proton momentum \( \pi \) cancel when we subtract Eqs. (22) and (23),

\[ \frac{\mu_N^2}{2\pi} + 2\omega \approx \frac{\mu_N^2}{2\pi} + \frac{\mu_N^2}{2(1-x)\pi}. \tag{24} \]

In the last step we have made the approximation \( \pi - \omega \approx \pi \) in the momentum conservation equation which then leads to \( x_1 + x_2 \approx 1 \) and we have used the notation \( x_1 = x \). Then one has

\[ \pi = \frac{1}{4\omega} \left[ \frac{\mu_N^2}{x} + \frac{\mu_N^2}{1-x} - \mu_N^2 \right]. \tag{25} \]

The threshold in the auxiliary variable

\[ \pi \geq \pi_{th} = \frac{\mu_N(\mu_N + 2\mu_N)}{4\omega} \tag{26} \]

coincides with the SR-threshold (\( P^{SR}_{th} \)). The Planckian correction to the momentum threshold is

\[ \frac{P_{th}}{P^{SR}_{th}} - 1 = \frac{\epsilon_{th}}{\epsilon_{th}} \frac{\hbar}{M_P} \left( \frac{\pi_{th}}{\epsilon_{th}} \right). \tag{27} \]

It is clear that in order to have an appreciable effect of departures from SR in the cosmic ray (CR) spectrum at very high energies it is necessary to consider a generalized kinematics with \( |h(x)| \gg 1 \) when \( x \approx 1 \). As in Eq. (16) we can consider

\[ h(x) \approx \eta_\gamma (1-x)^{-\gamma} \text{ when } x \approx 1, \tag{28} \]

with \( \gamma \) a positive real number and \( \eta_\gamma \), a constant coefficient which can be used as a measure of the energy scale characterizing the kinematic corrections. A value of \( \eta_\gamma \) of order one means an energy scale of the order of magnitude of the Planck mass as expected for an effect due to quantum gravity. Inserting Eq. (28) into Eq. (27) we get

\[ \frac{P_{th}}{P^{SR}_{th}} - 1 \approx \eta_\gamma 2^\gamma \left( \frac{\mu_N}{M_P} \right) \left[ \frac{\mu_\pi}{2\omega} \left( 1 + \frac{\mu_\pi}{2\mu_N} \right)^{(2\gamma+1)} \right], \tag{29} \]
and, since the quantity between square brackets is of the order $10^{11}$, there may have observable corrections for $\eta_\gamma$ of order one (i.e. Planckian effects) and even much smaller values depending on the value of the exponent $\gamma$ which controls the amplification of the Planckian corrections. One can arrive to the same conclusions directly from the expression of the momentum in terms of auxiliary variables

$$P = \pi \left[ 1 + \eta_\gamma \frac{\epsilon}{M_P} \left( 1 - \frac{\pi}{\epsilon} \right)^{-\gamma} \right] \approx \pi \left[ 1 + 2^\gamma \eta_\gamma \frac{\mu_N}{M_P} \frac{E_{\mu_N}}{\epsilon} \right]^{2^\gamma+1}$$

(30)

where in the second step we have assumed $\epsilon \gg \mu_N$ and we have made the approximation $\epsilon \approx E$ in the Planckian correction. In this way one can see that varying $\gamma$ one changes the domain of energies where the difference between the momentum variable and the corresponding auxiliary variable is appreciable and then one can have signals of departures from SR kinematics in the UHECR spectrum.

As an alternative to the previous example where one gets an observable effect in the cosmic ray spectrum as a consequence of an appreciable difference between the momentum variable and the auxiliary variable $\pi$ at scales much lower than the Planck mass, let us consider an example were $g = h = 0$ and then there is no correction in the kinematic description of a one particle system. In this case all the departures from SR are characterized by a function $f$. But as it has already been pointed out in the previous Section, the natural suppression of Planckian kinematic corrections by a factor $\sqrt{s}/M_P$ and the cancellations due to the relativity principle can not be compensated by an amplification in the ultraviolet limit, something which would require to go to energies far beyond those explored in CR physics. An amplification in the infrared limit would require to consider energies far below the energy range where the CMB background has any appreciable effect in the CR spectrum. To summarize the UHECR spectrum is not affected by this type of departures from SR.

\[\text{Some comments on ASR compared to DSR}\]

Besides the obvious suppression in kinematic Planckian corrections due to the smallness of the highest accessible energy in microscopic systems with respect to the Planck scale (the GZK threshold is nine orders of magnitude lower than the Planck energy), in theories that preserve a relativity principle there is another source of suppression in the form of cancellations of Planckian effects owing to the consistency between the conservation laws of the energy-momentum, the dispersion relation and the relativity principle. For this reason, in general, one has no signals on the UHECR spectrum of a departure from SR compatible with a relativity principle unless one makes special choices of the deformed kinematics. One explicit example of this situation corresponds to the family of DSR proposals which produce an observable change in the prediction of the GZK cut-off (see Ref. [7]). One can ask which is the relation between these proposals and the UV amplification mechanism introduced in the ASR framework. As it has already been said in the Introduction, the ASR idea is closely related to the leading contribution of DSR at scales much lower than the Planck scale. In fact, the leading terms of all DSR proposals compatible with the treatment of conservation laws in Ref. [17] are particular cases of ASR with $f = 0$.$^6$ In the case of the DSR proposals of Ref. [7], their leading term reduces to the form of Eq. (28) at low enough energies. Of course, different DSR models can have the same behavior at low energies (that is to say, their leading-order correction to SR can be the same) and therefore they converge to the same ASR. The reason why Planckian effects become relevant in these proposals is related to the amplification mechanisms discussed in this work.

With respect to our discussion of modified conservation laws, we have not found any DSR proposal corresponding to $f \neq 0$. We think that the analysis presented in this work could be of help to introduce such a generalization in the DSR framework. One should note however that our analysis excludes cases like conservation laws where the dependence on initial and final state variables cannot be separated as it happens in certain DSR proposals based on attempts to construct a relativistic theory in $\kappa$-Minkowski.

\[\text{B. Atomic interferometry}\]

We now consider the experiment with greater sensitivity to kinematic corrections in the nonrelativistic/IR limit. The possibility to identify a deviation from SR predictions in atomic interferometry has received some attention

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$^6$ Note that, according to its definition, ASR may include non-analytical behaviour in the modified dispersion relation. In this aspect it also goes beyond the DSR theories constructed so far.
recently. A first analysis of the sensitivity of these experiments within a simple parametrization of a generalized energy-momentum relation for an atom in the nonrelativistic limit has lead to the remarkable conclusion that one could detect Planckian kinematic corrections thanks to the extreme precision of the determination of the difference between the absorbed and emitted frequencies by cold atoms and the possibility to observe these transitions with very slow atoms. It is then natural to discuss this problem in the framework introduced in this work.

The process to consider is the absorption of a photon by an atom, both moving in the same direction, and posterior emission of a photon in the opposite direction. In the case of a separable generalized conservation law ($f = 0$) once more one can repeat the standard kinematic analysis of this process using the auxiliary variables. The energy conservation law gives

$$M + \frac{\pi^2}{2M} + \nu \approx M + \frac{\pi'^2}{2M} + \nu'$$

and for the momentum conservation law one has

$$\pi + \nu \approx \pi' - \nu',$$

where $M$ is the mass of the atom, $\nu$ ($\nu'$) the frequency of the absorbed (emitted) photon and $\pi$ ($\pi'$) the auxiliary variable of the initial (final) atom. We have neglected the subleading effect due to kinematic corrections to the description of the absorbed and emitted photons (see Ref. [20]). One can also approximate at the level of the momentum conservation law $\nu \approx \nu' \approx \nu^*$, with $\nu^*$ the Bohr frequency associated with the intermediate excited atom. Then from the energy conservation law one has

$$\frac{\pi^2}{2M} + \nu \approx \frac{(\pi + 2\nu^*)^2}{2M} + \nu'$$

and the frequency shift is given by

$$\Delta \nu = \nu - \nu' = \frac{2\nu^*(\nu^* + \pi)}{M}.$$  

We can reexpress the frequency shift as a function of the momentum $P$ of the atom using the nonrelativistic approximation $E \approx M$ in the terms suppressed by the Planck mass of Eq. (2):

$$\Delta \nu = \Delta \nu_{SR} - \frac{\nu^* P}{M_p} h(P/M),$$

with $\Delta \nu_{SR} = 2\nu^*(\nu^* + P)/M$. The correction to the frequency shift is

$$\frac{\Delta \nu}{\Delta \nu_{SR}} - 1 = -\frac{M}{2M_p}(1 + \nu^*/P)^{-1} h(P/M).$$

The suppression factor $M/M_p$ makes this kinematic correction totally unobservable unless one has $h(x) \gg 1$ when $x \ll 1$, partially compensating the Planck mass suppression and leading to a correction of the order of magnitude of the precision in the determination of the frequency shift. A simple example with an amplification of the kinematic correction in the nonrelativistic limit is that of Eq. (5),

$$h(x) \approx \xi \beta x^{-\beta}$$

leading to

$$\frac{\Delta \nu}{\Delta \nu_{SR}} - 1 \approx -\xi \beta \frac{M}{2M_p}(1 + \nu^*/P)^{-1} \frac{M^\beta}{P^\beta}.$$  

Introducing a value $10^{-8}$ for the correction to the frequency shift (36) as an observability criteria we find that the corresponding value for $\xi \beta$ is

$$\xi \beta \approx \pm 10^{-8}\frac{2M_p}{M}(1 + \nu^*/P)\frac{P^\beta}{M^\beta},$$

which is of order one (Planckian effect) for $\beta = 1$ and $P \sim 100$ eV. In this way one reproduces, within the present general kinematic framework compatible with the relativity principle, the result that atomic interferometry experiments can be sensitive to kinematic corrections with an energy scale of the order of the Planck mass [20].
In analogy with the discussion of kinematic corrections in the UV limit, one can consider the possibility to have an appreciable effect in the IR limit in a situation where there is no modification of the energy-momentum relation for the atom and the only kinematic correction appears at the level of the generalized conservation laws. In this case one has
\[
\frac{P^2}{2M} + \nu + \frac{\sqrt{s}}{M_P} M f(M^2/s, 0) \approx \frac{(P + 2\nu^*)^2}{2M} + \nu^* + \frac{\sqrt{s'}}{M_P} M f(M^2/s', 0),
\]
which is the generalized energy conservation law Eq. (40) particularized to the case \(\alpha = 1\) (Planckian corrections proportional to \(1/M_P\)) and \(\mu_2 = 0\) (one massless particle), and where we have neglected \(E_2 f\) in comparison with \(E_1 f \approx M f\). We also have
\[
s \approx M^2 + 2M\nu, \quad s' \approx M^2 + 2M\nu'.
\]
It is remarkable that now there is a momentum (of the atom) independent contribution in the energy conservation equation which does not cancel. This is the main reason why one can easily find choices of the modified conservation laws (function \(f\)) leading to observable effects in the frequency shift.

As a first example we consider a constant function \(f = \rho_0\). In this case one finds from Eq. (40) that
\[
\Delta \nu \approx \Delta \nu_{SR} \left[ 1 - \rho_0 \frac{M}{M_P} \right]
\]
and then Planckian corrections are unobservable unless \(\rho_0 > 10^9\) (we are assuming a precision on the determination of \(\Delta \nu\) of order \(10^{-8}\)). Next we can consider \(f(x, 0) = \rho_\beta' (1-x)^{-\beta'}\) [see Eq. (18)] which gives an amplification of the Planckian corrections
\[
\Delta \nu \approx \Delta \nu_{SR} \left[ 1 + \frac{\beta' \rho_\beta'}{2^\beta'} \left( \frac{M}{\nu'^*} \right)^{1+\beta'} \frac{M}{M_P} \right]
\]
such that, for typical values \(M/\nu'^* \sim 10^{11}\), one can have an observable effect of Planckian corrections for \(\rho_\beta'\) of order one (Planckian sensitivity) and even much smaller values depending on the choice of the exponent \(\beta'\) which controls the amplification of the Planckian corrections.

C. Velocities

To end up this exploratory analysis of possible ways to detect implications of ASR we can consider observations which are sensitive to corrections in the energy dependence of the velocity of propagation. In fact, the strongest constraints to departures from SR come in many cases \[21\] from limits to an energy dependence for the light propagation velocity and its consequences in cosmic gamma rays. In order to discuss these limits it is required to go beyond the framework we have considered until now. The previous analysis was based exclusively on the kinematic description of particles with energy-momentum variables and the relations among these variables (conservation laws) for processes with initial and final states with up to two particles. The simplest way to extend the discussion considering the propagation of a particle in space-time is to add the assumption\[7\] that the energy-momentum variables used in the kinematic description of the one particle systems are the generators of translations in space-time. In this case one has
\[
\frac{dx}{dt} = \frac{\partial E}{\partial p} = \frac{p}{E} \left[ 1 + \frac{3E}{2M_P} \Delta(p/E) + \frac{E}{2M_P} \left( \frac{E}{p} - \frac{p}{E} \right) \Delta'(p/E) \right],
\]
where we have used the asymptotic dispersion relation \([1]\) with \(\alpha = 1\). If one considers an amplification in the nonrelativistic and/or ultrarelativistic limit in order to have some appreciable effect of the kinematic Planckian corrections, then one can use the corresponding choices for the functions \(g, h\) to calculate the function \(\Delta\) which fixes the modification in the dispersion relation and the velocity of propagation of particles. A systematic analysis of the modification of velocities in different limits and the possibility to observe them will be presented elsewhere.

\[7\] In fact the structure of space-time in different attempts to go beyond SR is an open issue.
In the case of photons, the dispersion relation \( \frac{dE}{dt} \) leads to

\[
c(E) = \frac{dE}{dt} = 1 + \frac{E}{M_P}(\tilde{g} - \tilde{h}),
\]

which is the standard energy dependence for the velocity of light as expected on pure dimensional grounds. From different observations of cosmic gamma rays one has very stringent limits on \( \tilde{g} \) (which is the standard energy dependence for the velocity of light as expected on pure dimensional grounds. From possible signs of the space-time structure of a theory of quantum gravity.

We have introduced a convenient parametrization for the asymptotic approach to SR kinematics compatible with the relativity principle. In the simplest case of a separable generalized energy-momentum conservation law it is given in terms of two functions of one variable which fix at the same time the modification of the dispersion relation and the conservation laws. It is also possible to have nonseparable contributions in the generalized conservation laws; these contributions require then the introduction of an additional function of two variables. In fact it is possible to have a generalized conservation law with the standard SR dispersion relation (setting the two functions of one variable equal to zero) while maintaining the relativity principle. This provides a general framework within which to explore possible signs of the space-time structure of a theory of quantum gravity.

We have shown how it is possible to escape from the difficulty to observe these signs at energies much smaller than the Planck scale. The cancellation of corrections in an approach compatible with the relativity principle can be (at least partially) compensated by an amplification mechanism leading to observable effects at energies much lower than the Planck mass. Two limits where one can introduce an amplification are the UV/IR limits (particles with momentum much higher/lower than the mass). An amplification in the UV limit implies a reduction in the maximum energy-momentum of the domain of validity of the asymptotic kinematic description and an amplification in the IR limit leads to the striking conclusion that the Planckian corrected kinematics cannot be applied to arbitrarily small momenta.

We have applied the proposal for a generalization of SR kinematics to the simplest reaction responsible for the energy loss of protons propagating in the CMB. We have found simple examples of an amplification in the UV limit leading to observable effects in the spectrum of very high energy cosmic rays due to the modification of the dispersion relation for protons. The generalized kinematics has also been used to calculate the frequency shift in a double-Raman atomic transition. In this case we have identified simple examples of an amplification in the IR limit with appreciable corrections to the SR result due either to a modification in the momentum dependence of the kinetic energy of the atom or to a non-separable generalization of the energy-momentum conservation law for a photon-atom system. Once the very simple analysis based on the results for the momentum threshold of the photopion production from protons and the frequency shift in a double Raman transition has lead us to the conclusion that there can be appreciable effects of Planckian kinematic corrections both at high and low momenta one can go further to try to extract information of the details of these corrections from present and/or future data. This requires to go beyond the simple analysis presented in this work.

In this direction, at high momenta one should try to extend the usual analysis of the high energy cosmic ray spectrum based on SR kinematics incorporating the Planckian corrections. It would also be interesting to explore other reactions where the amplification introduced in the UV limit could lead to appreciable effects; one case which seems specially promising is the scattering of very high energy (TeV) neutrinos where the very small value of the neutrino masses makes the amplification more important.

At low momenta it could be interesting to extend the analysis of cold atom experiments exploring the possibility to get complementary information on the Planckian corrections by considering different setups corresponding to different directions for the absorbed and emitted photons. The analysis of properties in particle reactions which are sensitive to IR divergences is another way to look for data which can be sensitive to kinematic Planckian corrections. Another possibility to go towards the limits of validity of the generalized kinematic framework and then to the onset of appreciable corrections is to consider systems at extremely low temperatures like Bose-Einstein condensates. When comparing the different strategies to look for signs of kinematic Planckian corrections one should keep in mind that the most straightforward way, the approach to the limits of validity of the simple asymptotic kinematic description, is limited by the increase of the unknown corrections to the asymptotic limit. Alternatively, high precision data can be sensitive to a small correction well within the domain of validity of the asymptotic description and then such kind of experiments are better candidates to determine the details of the asymptotic approach to SR. Of course the primordial objective is the identification of a departure from SR and the best strategy to get this result depends on the amplification mechanisms accompanying these departures.
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