Villamayor-Zelinsky sequence for symmetric finite
tensor categories

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Abstract

We prove that if a finite tensor category $C$ is symmetric, then the monoidal
category of one-sided $C$-bimodule categories is symmetric. Consequently, the Pi-
card group of $C$ (the subgroup of the Brauer-Picard group introduced by Etinov-
Nikshych-Gelaki) is abelian in this case. We then introduce a cohomology over such
$C$. An important piece of tool for this construction is the computation of dual ob-
jects for bimodule categories and the fact that for invertible one-sided $C$-bimodule
categories the evaluation functor involved is an equivalence, being the coevaluation
functor its quasi-inverse, as we show. Finally, we construct an infinite exact sequence
a la Villamayor-Zelinsky for $C$. It consists of the corresponding cohomology groups
evaluated at three types of coefficients which repeat periodically in the sequence.

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monoidal category, cohomology.

1 Introduction

The classical Brauer group of a field (introduced in 1929) classifies finite dimensional divi-
sion algebras, in the way that it consists of equivalence classes of central simple algebras.
Its first generalization is over a commutative ring $R$, where one deals with algebras $A$
satisfying: $A \otimes_R A^{op} \cong \text{End}_R(A)$. The list of further generalizations culminates with the
Brauer group of a braided monoidal category $C$ in [27] in 1998, which consists of equiva-
ience classes of algebras $A \in C$ such that $A \otimes A^{op} \cong [A, A]$ and $A^{op} \otimes A \cong [A, A]^{op}$, where
$[A, A]$ denotes the inner hom (such algebras are called Azumaya algebras). In the follow-
ing decade various results have been made concerning the latter group and its subgroups.
In 2009, [14], the Brauer-Picard group of a fusion category $C$ was introduced, it con-
sists of equivalence classes of semisimple invertible $C$-bimodule categories $\mathcal{M}$ satisfying:
$\mathcal{M} \boxtimes_C \mathcal{M}^{op} \simeq C$ (and equivalently $\mathcal{M}^{op} \boxtimes_C \mathcal{M} \simeq C$). Similarly, the Brauer-Picard group
of a finite tensor category consists of equivalence classes of exact invertible $C$-bimodule
categories. This group is involved in the classification of extensions of a given tensor cat-
gory by a finite group, though it also has relations to mathematical physics, like rational
Conformal Field Theory and 3-dimensional Topological Field Theory, [18, 21]. In [14] it is also shown that \( \text{BrPic}(\text{Vec}_k) \cong \text{Br}(k) \), i.e. the Brauer-Picard group of the category of \( k \)-vector spaces is isomorphic to the Brauer group of the field \( k \). When \( \mathcal{C} \) is braided \( \text{BrPic}(\mathcal{C}) \) admits a subgroup \( \text{Pic}(\mathcal{C}) \) - the Picard group of \( \mathcal{C} \) in which the invertible categories are one-sided \( \mathcal{C} \)-module categories (and the action on the other side is induced via the braiding). From [9] it was known that \( \text{Pic}(\mathcal{C}) \) is isomorphic to the Brauer group of exact Azumaya algebras, and in [10, Proposition 3.7] it is proved that for \( \mathcal{C} = \text{Rep}_H \), the category of finite-dimensional representations of a finite-dimensional quasi-triangular Hopf algebra \((H, \mathcal{R})\), any Azumaya algebra in \( \text{Rep}_H \) is exact. This proves that the two groups, different in nature, are isomorphic: \( \text{Pic}(H \mathcal{M}) \cong \text{Br}(\text{Rep}_H) \), where \((H, \mathcal{R})\) is quasitriangular.

The Brauer group \( \text{Br}(k) \) of a field \( k \) has a nice cohomological interpretation: it is isomorphic to the second Galois cohomology group with respect to the separable closure of the field. In the root of this description lies the Crossed Product Theorem relating the relative Brauer group \( \text{Br}(l/k) \) with the second Galois cohomology group with respect to the Galois field extension \( l/k \). This cohomological interpretation is possible to transmit from the relative to the full Brauer group because every central simple algebra can be split by a Galois field extension. However, this is not the case if we consider Galois extensions of commutative rings, not every Azumaya algebra (over a ring) can be split by a Galois (ring) extension. Though, instead of the Crossed Product Theorem for the relative Brauer group we have an infinite exact sequence due to Villamayor and Zelinsky [29] involving cohomology groups for an extension of a commutative ring. These groups are evaluated at three types of coefficients and the three types of cohomology groups appear periodically in the sequence. If the ring extension is faithfully flat, the relative Brauer group embeds into the middle term on the second level of the sequence. If the extension is faithfully projective, one has an isomorphism, recovering thus the Crossed Product Theorem.

In [16] we constructed a version of the above mentioned infinite exact sequence for a commutative bialgebroid (a coring which is an algebra that fulfills certain compatibility conditions) and we interpreted the middle terms on the first three levels of the sequence. If \( R \to S \) is a commutative ring extension, then \( S \otimes_R S \) is a commutative bialgebroid and the new sequence generalizes the previous one.

In the present paper we introduce a cohomology for a symmetric finite tensor category \( \mathcal{C} \), which we still call Amitsur cohomology. We specialize it in three types of coefficients, one of which is the Picard group of \( \mathcal{C} \) (the subgroup of the Brauer-Picard group). For this construction we previously prove the following two properties. Firstly, that for a symmetric finite tensor category \( \mathcal{C} \) the monoidal category of one-sided \( \mathcal{C} \)-bimodule categories \((C^{br}_{\text{Mod}}, \otimes_C, \mathcal{C})\) is symmetric monoidal. This should not be a surprise, because, as we cited above: \( \text{Pic}(\text{Rep}_H) \cong \text{Br}(\text{Rep}_H) \) for any finite-dimensional quasi-triangular Hopf algebra \( H \). It is known that for any symmetric monoidal category the Brauer group of it is abelian. Therefore, when \( H \) is triangular, i.e. \( \text{Rep}_H \) is symmetric, we get that \( \text{Pic}(\text{Rep}_H) \) is abelian. Secondly, we prove that the dual object for an invertible one-sided \( \mathcal{C} \)-bimodule category is its opposite category and that the evaluation functor involved is an equivalence, whereas the coevaluation functor is its quasi-inverse. We then construct an infinite exact sequence a la Villamayor-Zelinsky for \( \mathcal{C} \), which consists of the corresponding cohomology groups evaluated at the mentioned three types of coefficients which repeat periodically in the sequence. The interpretation of some of its terms we develop in the
forthcoming paper [17].

It is known that any symmetric finite tensor category is equivalent to the category Rep H of finite-dimensional representations of a finite-dimensional triangular quasi-Hopf algebra H. If H is a Hopf algebra, we say that C is strong (in the style of [24]). As it was proved in [2, Theorem 5.1.1], [12, Theorem 4.3], any finite-dimensional triangular Hopf algebra over an algebraically closed field of characteristic zero is the Drinfel’d twist of a modified supergroup algebra. The corresponding representation categories are equivalent, hence we have that any symmetric finite tensor category C that is strong is the representation category of a modified supergroup algebra \( \Lambda(V) \times kG \) with a triangular structure \( \mathcal{R} \).

Collecting the above-said and the results from [8, Theorem 6.5] and [5], we obtain that the corresponding Picard group decomposes as: 

\[
\text{Pic}(\text{Rep}(H_4)) \cong \text{BW}(k) \times (k, +) \quad \text{and} \quad \text{Pic}(\text{Rep}(E(n))) \cong \text{BW}(k) \times (k, +)^{n(n+1)/2}
\]

where \( \text{BW}(k) \) denotes the Brauer-Wall group (the corresponding Azumaya algebras are \( \mathbb{Z}_2 \)-graded). For any finite group \( G \) it is \( \text{Pic}(\text{Rep}(G)) \cong H^2(G, k^\times) \), [19, Corollary 8.10].

The paper is organized as follows. In the Preliminaries we recall some definitions and prove one basic result. In the third section we study the left and right module structures over a Deligne tensor product of two finite tensor categories: \( \mathcal{C} \boxtimes \mathcal{D} \) and how they are related to a \( \mathcal{C} \)-\( \mathcal{D} \)-bimodule structure. We prove in Proposition 3.4 that the monoidal category \( (\mathcal{C}^{br}\text{-Mod}, \otimes, \mathcal{C}) \) of one-sided bimodule categories over a symmetric finite tensor category \( \mathcal{C} \) is symmetric. The fourth section is dedicated to the study of dual objects in the monoidal category of bimodule categories over any finite tensor category. We prove here that the evaluation functor is an equivalence functor for invertible bimodule categories. In Section 5 we introduce our Amitsur cohomology over symmetric finite tensor categories. Our infinite exact sequence a la Villamayor-Zelinsky is constructed in the last section in Theorem 6.2, which is the main result of this paper. Finally, we record that any symmetric finite tensor category is the representation category of a Drinfel’d twist of a modified supergroup algebra and how the corresponding Picard group decomposes into the above-mentioned direct product.

## 2 Preliminaries and notation

Throughout \( I \) will denote the unit object in a monoidal category \( \mathcal{C} \) and \( k \) an algebraically closed field of characteristic zero. When there is no confusion we will denote the identity functor on a category \( \mathcal{M} \) by \( \mathcal{M} \). We proceed to recall some definitions and basic properties.

A finite category over \( k \) is a \( k \)-linear abelian category equivalent to a category of finite-dimensional representations of a finite-dimensional \( k \)-algebra. A tensor category over \( k \) is a \( k \)-linear abelian rigid monoidal category such that the unit object is simple. An object \( X \) is said to be simple if \( \text{End}(X) = k \text{Id}_X \). A finite tensor category is a tensor category.
such that the underlying category is finite. For example, if $H$ is a finite-dimensional Hopf algebra (or, more generally, a finite-dimensional quasi-Hopf algebra) the category of its representations $\text{Rep} H$ is a finite tensor category.

All tensor categories will be assumed to be over a field $k$, all categories will be finite and all functors will be assumed to be $k$-linear.

We assume the reader is familiar with the notions of a left, right and bimodule categories over a tensor category, (bi)module functors, Deligne tensor product of finite abelian categories, tensor product of bimodule categories and exact module categories. For the respective definitions we refer to [15], [14], [19], [13]. The Deligne tensor product bifunctor $\otimes$ is a bifunctor of module categories $-\otimes -$ and the action bifunctor for module categories $(-\otimes -)$ are biexact in both variables [13 Proposition 1.46.2].

For finite tensor categories $C,D$ a $C$-$D$-bimodule category is a left $C \otimes D^{\text{rev}}$-module category and a right $C^{\text{rev}} \otimes D$-module category. Here $C^{\text{rev}}$ is the category with the tensor product reversed with respect to that of $C$: $X \otimes^{\text{rev}} Y = Y \otimes X$, and the associativity constraint $a_{X,Y,Z}^{\text{rev}} = a_{Z,Y,X}^{-1}$ for $X,Y,Z \in C$. A $(C,D)$-bimodule category is exact if it is exact as a left $C \otimes D^{\text{rev}}$-module category, [14], [19].

Let $C,D,E$ be finite tensor categories. For a $C$-$D$-bimodule category $\mathcal{M}$ and a $(D,E)$-bi-module category $\mathcal{N}$ the tensor product over $D$: $\mathcal{M} \boxtimes_D \mathcal{N}$ is a $(C,E)$-bi-module category. Given an $E$-$F$-bi-module category $\mathcal{P}$, there is a canonical equivalence of $C$-$F$-bi-module categories: $(\mathcal{M} \otimes_D \mathcal{N}) \boxtimes_E \mathcal{P} \simeq \mathcal{M} \boxtimes_{D} (\mathcal{N} \boxtimes_E \mathcal{P})$, [14] Remark 3.6.

A right $C$-module category $\mathcal{M}$ gives rise to a left $C$-module category $\mathcal{M}^{\text{op}}$ with the action given by (1) and associativity isomorphisms $m_{X,Y,M}^{\text{op}} = m_{M,Y,X}^{\text{op}}$ for all $X,Y \in C, M \in \mathcal{M}$. Similarly, a left $C$-module category $\mathcal{M}$ gives rise to a right $C$-module category $\mathcal{M}^{\text{op}}$ with the action given via (2). Here $^*X$ denotes the left dual object and $X^*$ the right dual object for $X \in C$. If $\mathcal{M}$ is a $(C,D)$-bimodule category then $\mathcal{M}^{\text{op}}$ is a $(D,C)$-bimodule category and $(\mathcal{M}^{\text{op}})^{\text{op}} \simeq \mathcal{M}$ as $(C,D)$-bimodule categories.

A $(C,D)$-bimodule category $\mathcal{M}$ is called invertible [14] if there are equivalences of bimodule categories

$$\mathcal{M}^{\text{op}} \boxtimes_C \mathcal{M} \simeq D, \quad \mathcal{M} \boxtimes_D \mathcal{M}^{\text{op}} \simeq C.$$ 

The Brauer-Picard group of a fusion category, introduced in [14], is a group of equivalence classes of semisimple invertible module categories. In a more general setting, the Brauer-Picard group $\text{BrPic}(\mathcal{C})$ of a finite tensor category $\mathcal{C}$ is a group of equivalence classes of exact invertible module categories (a module category over a fusion category is exact if and only if it is semisimple [15 Example 3.3 (iii)]).

$\textbf{C}$-$\textbf{balanced}$ functors. Let $\mathcal{M}$ be a right $C$-module and $\mathcal{N}$ a left $C$-module category. For any abelian category $A$ a bifunctor $F: \mathcal{M} \times \mathcal{N} \rightarrow A$ additive in every argument is called $C$-$balanced$ if there are natural isomorphisms $b_{M,X,N}: F(M \otimes X, N) \xrightarrow{\cong} F(M, X \otimes N)$ for all $M \in \mathcal{M}, X \in \mathcal{C}, N \in \mathcal{N}$ s.t.

$$F((M \otimes X) \otimes Y, N) \xrightarrow{F(\text{id}, m_{X,Y,N}^{\text{op}})} F(M \otimes (X \otimes Y), N) \xrightarrow{b_{M,X,Y,N}} F(M, (X \otimes Y) \otimes N) \xrightarrow{F(\text{id}, m_{X,Y,N})} F(M, X \otimes (Y \otimes N))$$

$$F((M \otimes X), Y \otimes N) \xrightarrow{b_{M,X,Y,N}} F(M, X \otimes (Y \otimes N)).$$
commutes. In $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ it holds: $(M \boxtimes X) \boxtimes_{\mathcal{C}} N = M \boxtimes_{\mathcal{C}} (X \boxtimes N)$, for all $M \in \mathcal{M}, N \in \mathcal{N}, X \in \mathcal{C}$. In particular, the identity functor induces a $\mathcal{C}$-balanced functor $\mathcal{M} \boxtimes \mathcal{N} \to \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$. If we denote by $\beta_{M,X,N} : (M \boxtimes X) \boxtimes_{\mathcal{C}} N \to M \boxtimes_{\mathcal{C}} (X \boxtimes N)$ the above identity in $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$, the coherence of the identity functor reads:

$$\beta_{M,X,Y,Z} \beta_{M,X,Y,N}(m_{r,M,X,Y} \boxtimes_{\mathcal{C}} N) = (M \boxtimes_{\mathcal{C}} m_{l,X,Y,N}) \beta_{M,X,Y,N}(m_{r,M,X,Y} \boxtimes_{\mathcal{C}} N)$$

(4)

A $\mathcal{C}$-balanced natural transformation $\Psi : F \to G$ between two $\mathcal{C}$-balanced functors $F, G : \mathcal{M} \times \mathcal{N} \to \mathcal{A}$ with their respective balancing isomorphisms $f_X$ and $g_X$, is a natural transformation such that the following diagram commutes:

$$\begin{array}{ccc}
F((M \boxtimes X), N) & \xrightarrow{f_{M,X,N}} & F(M, (X \boxtimes N)) \\
\Psi((M \boxtimes X), N) & \quad & \Psi(M, (X \boxtimes N)) \\
G((M \boxtimes X), N) & \xrightarrow{g_{M,X,N}} & G(M, (X \boxtimes N)).
\end{array}$$

(5)

**One-sided $\mathcal{C}$-bimodule categories.** When $\mathcal{C}$ is braided with a braiding $\Phi$, then every left $\mathcal{C}$-module category is a right and a $\mathcal{C}$-bimodule category: $M \boxtimes X = X \boxtimes M$ with the isomorphism functors $m_{l,M,X,Y} : (M \boxtimes (X \otimes Y)) \to (M \boxtimes X) \boxtimes Y$ defined via:

$$\begin{split}
M \boxtimes (X \otimes Y) & \xrightarrow{m_{l,M,X,Y}} (M \boxtimes X) \boxtimes Y \\
& = (X \otimes Y) \boxtimes M \xrightarrow{\Phi_{X,Y} \boxtimes M} (Y \otimes X) \boxtimes M \xrightarrow{m_{l,Y,X,M}} Y \boxtimes (X \otimes M),
\end{split}$$

(6)

see [9, Section 2.8]. Moreover, the $\mathcal{C}$-bimodule associativity constraint is given by:

$$\begin{split}
(X \boxtimes M) \boxtimes Y & \xrightarrow{\alpha_{X,Y}} X \boxtimes (M \boxtimes Y) \\
& = Y \boxtimes (X \boxtimes M) \xrightarrow{(m_{l})^{-1}_{Y,X,M}} (Y \otimes X) \boxtimes M \xrightarrow{\Phi_{Y,X} \boxtimes M} (X \otimes Y) \boxtimes M \xrightarrow{m_{l,Y,X,M}} X \boxtimes (Y \otimes M),
\end{split}$$

(7)

for all $X, Y \in \mathcal{C}, M \in \mathcal{M}$. The $\mathcal{C}$-bimodule categories obtained in this way are called one-sided $\mathcal{C}$-bimodule categories.

For a braided finite tensor category $\mathcal{C}$ exact invertible one-sided $\mathcal{C}$-bimodule categories form a subgroup of $\text{BrPic}(\mathcal{C})$, called the Picard group of $\mathcal{C}$ and denoted by $\text{Pic}(\mathcal{C})$, [14, Section 4.4], [9, Section 2.8].

### 2.1 Deligne tensor product categories as (braided) monoidal categories

We start with the following:

**Lemma 2.1** Let $\mathcal{C}$ and $\mathcal{D}$ be finite tensor categories. Given a right $\mathcal{C}$-module category $\mathcal{M}$, a left $\mathcal{C}$-module category $\mathcal{M}'$, a right $\mathcal{D}$-module category $\mathcal{N}$ and a left $\mathcal{D}$-module category $\mathcal{N}'$, there is an isomorphism of categories:

$$\left( \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{M}' \right) \boxtimes \left( \mathcal{N} \boxtimes_{\mathcal{D}} \mathcal{N}' \right) \cong \left( \mathcal{M} \boxtimes \mathcal{N} \right) \boxtimes_{\mathcal{C} \boxtimes \mathcal{D}} \left( \mathcal{M}' \boxtimes \mathcal{N}' \right).$$
Proof. We will use \[14\] Remark 3.2] for the definition of a balanced functor. The functor $F : \mathcal{M} \boxtimes \mathcal{N} \boxtimes \mathcal{M}' \boxtimes \mathcal{N}' \to (\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{M}') \boxtimes (\mathcal{N} \boxtimes_{\mathcal{D}} \mathcal{N}')$ defined via $F(M \boxtimes N, M' \boxtimes N') := (M \boxtimes_{\mathcal{C}} M') \boxtimes (N \boxtimes_{\mathcal{D}} N')$ is $\mathcal{C} \boxtimes \mathcal{D}$-balanced. Indeed, the balancing isomorphism is identity: $F((M \boxtimes N) \boxtimes (C \boxtimes D), M' \boxtimes N') = F((M \boxtimes C) \boxtimes (N \boxtimes D), M' \boxtimes N') = ((M \boxtimes C) \boxtimes_{\mathcal{C}} M') \boxtimes ((N \boxtimes D) \boxtimes_{\mathcal{D}} N') = (M \boxtimes_{\mathcal{C}} (C \boxtimes M')) \boxtimes (N \boxtimes_{\mathcal{D}} (D \boxtimes N')) = F(M \boxtimes N, (C \boxtimes D) \boxtimes (M' \boxtimes N'))$, and it is clear that the $\mathcal{C} \boxtimes \mathcal{D}$-coherence for $F$ follows.

For the inverse functor observe the following diagram. The equivalences in the right vertical arrows are due to \[19\] Lemma 4.1. The functor $F_3$ is well-defined if $F_2$ is $\mathcal{C}$-balanced, whereas $F_2$ is well-defined if $F_1$ is $\mathcal{D}$-balanced. The check of the last two conditions is easy and we leave it to the reader.

The functor $F_3$ is given by: $F_3((M \boxtimes_{\mathcal{C}} M') \boxtimes (N \boxtimes_{\mathcal{D}} N')) = (M \boxtimes N) \boxtimes_{\mathcal{C} \boxtimes \mathcal{D}} (M' \boxtimes N')$. It is straightforward to see that $F$ and $F_3$ are inverse to each other.

\[\square\]

Let $\mathcal{C}, \mathcal{D}$ be finite tensor categories. The category $\mathcal{C} \boxtimes \mathcal{D}$ is a finite tensor category with unit object $I_{\mathcal{C}} \boxtimes I_{\mathcal{D}}$ and componentwise tensor product: $(X \boxtimes Y) \odot (X' \boxtimes Y') = (X \otimes_{\mathcal{C}} X') \boxtimes (Y \otimes_{\mathcal{D}} Y')$, where $\otimes_{\mathcal{C}}$ denotes the tensor product in $\mathcal{C}$ and similarly for $\mathcal{D}$. Suppose now that $\mathcal{C}$ and $\mathcal{D}$ are braided with the braidings $\Phi^\mathcal{C}$ and $\Phi^\mathcal{D}$, respectively. Then $\mathcal{C} \boxtimes \mathcal{D}$ is braided with the braiding

$$\tilde{\Phi}_{X \boxtimes_{\mathcal{C}} Y, X' \boxtimes_{\mathcal{D}} Y'} : (X \boxtimes Y) \odot (X' \boxtimes Y') \to (X' \boxtimes Y') \odot (X \boxtimes Y)$$

between the objects $X \boxtimes Y, X' \boxtimes Y' \in \mathcal{C} \boxtimes \mathcal{D}$ which is given by:

$$(X \boxtimes Y) \odot (X' \boxtimes Y') \xrightarrow{\Phi_{X \boxtimes_{\mathcal{C}} Y, X' \boxtimes_{\mathcal{D}} Y'}} (X' \boxtimes Y') \odot (X \boxtimes Y)$$

$$= (X \otimes_{\mathcal{C}} X') \boxtimes (Y \otimes_{\mathcal{D}} Y') \xrightarrow{\Phi^\mathcal{C}_{X, X'} \otimes \Phi^\mathcal{D}_{Y, Y'}} (X' \otimes_{\mathcal{D}} X) \boxtimes (Y' \otimes_{\mathcal{D}} Y)$$

(8)

Remark 2.2 Observe that the tensor product $\odot$ can be considered as $\boxtimes_{\mathcal{C} \boxtimes \mathcal{D}}$:

$$(X \boxtimes Y) \odot (X' \boxtimes Y') = (X \otimes_{\mathcal{C}} X') \boxtimes (Y \otimes_{\mathcal{D}} Y') = (X \boxtimes_{\mathcal{C}} X') \boxtimes (Y \boxtimes_{\mathcal{D}} Y') \cong (X \boxtimes_{\mathcal{C} \boxtimes \mathcal{D}} Y) \boxtimes_{\mathcal{C} \boxtimes \mathcal{D}} (X' \boxtimes Y')$$

where the last isomorphism is due to Lemma \[2.1\].
3 Bimodule categories over braided and symmetric tensor categories

Recall that for (non braided) finite tensor categories $\mathcal{C}, \mathcal{D}$ a $\mathcal{C}\mathcal{D}$-bimodule category is a left $\mathcal{C} \boxtimes \mathcal{D}^{rev}$-module category and a right $\mathcal{C}^{rev} \boxtimes \mathcal{D}$-module category, where $\mathcal{C}^{rev}$ is the category with the tensor product reversed with respect to that of $\mathcal{C}$. One would expect that when $\mathcal{C}$ and $\mathcal{D}$ are braided, that a $\mathcal{C}\mathcal{D}$-bimodule category is a one-sided $\mathcal{C} \boxtimes \mathcal{D}$-bimodule category. Let us investigate this.

When $\mathcal{C}$ is braided with a braiding $\Phi$, the category $\mathcal{C}^{rev}$ is braided in two ways. Its braiding is given by $X \otimes^{rev} Y = Y \otimes X \xrightarrow{\Phi} X \otimes Y = Y \otimes^{rev} X$, for $i = \pm 1$. We will denote the two braided monoidal categories by $\mathcal{C}^i$ with $i = \pm 1$.

**Proposition 3.1** Let $\mathcal{C}$ and $\mathcal{D}$ be finite tensor categories and let $\mathcal{M}$ be a $\mathcal{C}\mathcal{D}$-bimodule category. Then we have:

1. If $\mathcal{D}$ is braided, then $\mathcal{M}$ is a left $\mathcal{C} \boxtimes \mathcal{D}$-module category with the action given by:
   
   \[
   (X \boxtimes Y) \boxtimes M = X \boxtimes M \boxtimes Y, \text{ for } X \in \mathcal{C}, Y \in \mathcal{D}, M \in \mathcal{M} \text{ and the left associator } m^L \text{ defined via:}
   \]

   
   \[\begin{array}{c}
   ((X \boxtimes Y) \boxtimes (X' \boxtimes Y')) \boxtimes M \\
   \xrightarrow{m^L} (X \boxtimes Y) \boxtimes ((X' \boxtimes Y') \boxtimes M) \\
   \xrightarrow{a} Y
   \end{array}\]

   
   \[\begin{array}{c}
   ((X \otimes X') \boxtimes (Y \otimes Y')) \boxtimes M \\
   \xrightarrow{m^L} (X \boxtimes (X' \otimes Y')) \boxtimes Y
   \end{array}\]

   
   \[\begin{array}{c}
   ((X \otimes Y) \boxtimes (Y' \otimes Y)) \boxtimes M \\
   \xrightarrow{m^L} (X \boxtimes (Y' \otimes Y') \boxtimes (Y' \otimes Y)) \boxtimes M
   \end{array}\]

   
   where $i = \pm 1$.

2. If $\mathcal{C}$ is braided, then $\mathcal{M}$ is a right $\mathcal{C} \boxtimes \mathcal{D}$-module category with the action given by:
   
   \[
   M \boxtimes (X \boxtimes Y) = X \boxtimes M \boxtimes Y \text{ and the right associator } m^R.
   \]

   
   \[\begin{array}{c}
   M \boxtimes ((X \boxtimes Y) \boxtimes (X' \boxtimes Y')) \\
   \xrightarrow{m^R} M \boxtimes ((X \boxtimes Y) \boxtimes (X' \boxtimes Y')) \\
   \xrightarrow{m^R} M \boxtimes ((X \boxtimes Y) \boxtimes (X' \boxtimes Y'))
   \end{array}\]

   
   \[\begin{array}{c}
   M \boxtimes ((X \otimes X') \boxtimes (Y \otimes Y')) \\
   \xrightarrow{m^R} M \boxtimes ((X \boxtimes Y) \boxtimes (X' \boxtimes Y')) \\
   \xrightarrow{m^R} M \boxtimes ((X \boxtimes Y) \boxtimes (X' \boxtimes Y'))
   \end{array}\]

   
   where $i = \pm 1$.

3. If both $\mathcal{C}$ and $\mathcal{D}$ are braided, then $\mathcal{M}$ is a $\mathcal{C} \boxtimes \mathcal{D}$-bimodule category with the bimodule
associativity constraint \( A \) given by:

\[
\begin{align*}
((X \boxtimes Y) \boxtimes M) \boxtimes (X' \boxtimes Y') & \xrightarrow{A} (X \boxtimes Y) \boxtimes (M \boxtimes (X' \boxtimes Y')) = X \boxtimes ((X \boxtimes M) \boxtimes (X' \boxtimes Y')) \\
X \boxtimes (X \boxtimes (M \boxtimes Y)) \boxtimes Y' & \xrightarrow{a} X \boxtimes ((X \boxtimes M) \boxtimes (Y' \boxtimes Y)) \\
(m')^{-1} \boxtimes Y' & \xrightarrow{a} X \boxtimes ((X \boxtimes M) \boxtimes (Y' \boxtimes Y)) \\
\Phi^i_{X,X} \boxtimes \text{Id} & \xrightarrow{a} m \boxtimes \text{Id} \\
((X \boxtimes X') \boxtimes (M \boxtimes Y)) \boxtimes Y' & \xrightarrow{a^{-1}} X \boxtimes ((X \boxtimes M) \boxtimes (Y' \boxtimes Y')) \\
(X \boxtimes X) \boxtimes ((M \boxtimes Y) \boxtimes Y') & \xrightarrow{\text{Id} \boxtimes (m')^{-1}} (X \boxtimes X') \boxtimes (M \boxtimes (Y \boxtimes Y')) \\
\text{Id} \boxtimes (m')^{-1} & \xrightarrow{\Phi^i_{Y,Y'}} (X \boxtimes X') \boxtimes (M \boxtimes (Y' \boxtimes Y'))
\end{align*}
\]

where \( s, t = \pm 1 \).

**Proof.** The proof is direct. One uses the coherences satisfied by the associators \( m^l, m^r \) and \( a \) of the \( \mathcal{C} \otimes \mathcal{D} \)-bimodule structure of \( \mathcal{M} \), and apart from them, the computation reduces to certain conditions on the braidings. In the case of 1) it is: \((Z' \otimes \tilde{\Phi}^i_{X',Y'}) \tilde{\Phi}^i_{X,Y',Z'} = (\tilde{\Phi}^i_{Y',Z'} \otimes X') \tilde{\Phi}^i_{X',Y' \otimes Z'}\), which is fulfilled by the naturality of the braiding \( \Phi^i \) in the first coordinate. In the case of 3) one condition is:

\[
(Y \otimes \Phi^s_{Z,Y})(\Phi^s_{Z,X,Y})((\tilde{\Phi}^i_{Y',Z'} \otimes X')(Y' \otimes \tilde{\Phi}^i_{X',Z'})) \tilde{\Phi}^i_{X,Y' \otimes Z'} = \Phi^s_{Z,X,Y}(Z' \otimes \tilde{\Phi}^i_{X',Y'}) \tilde{\Phi}^i_{X,Y' \otimes Z'},
\]

which is fulfilled by one of the two axioms for the braiding \( \Phi \), on the one hand, and on the other, by the other axiom of the braiding \( \Phi^s \) and the naturality of the braiding \( \Phi^r \) in the first coordinate. The rest of the coherences to check are resolved in a similar fashion. Note that the claims hold true for arbitrary choices of indices \( i, j, s, t \in \{-1, 1\} \).

From the parts 1) and 2) in the above proposition we see that a \( \mathcal{C} \otimes \mathcal{D} \)-bimodule category is a left and a right \( \mathcal{C} \otimes \mathcal{D} \)-module category with: \((X \boxtimes Y) \boxtimes M = X \boxtimes M \boxtimes Y = M \boxtimes (X \boxtimes Y)\) for \( X \in \mathcal{C}, Y \in \mathcal{D}, M \in \mathcal{M} \). In order to investigate under which conditions a \( \mathcal{C} \otimes \mathcal{D} \)-bimodule category is a one-sided \( \mathcal{C} \otimes \mathcal{D} \)-bimodule category, we will consider constraints \( m^L, m^R \) and \( A \) from Proposition 3.1 starting with their corresponding indices \( i, j, s, t \in \{-1, 1\} \) and check the conditions (3) and (7). The braiding appearing in these two conditions will now refer to the braiding \( \Psi \) of \( \mathcal{C} \otimes \mathcal{D} \) where \( \mathcal{M} \) is assumed to be a left \( \mathcal{C} \otimes \mathcal{D} \)-module category. According to (4), \( \mathcal{D} \) is considered as a braided category \( \mathcal{D}' \), thus \( \Psi_{X \boxtimes Y, X' \boxtimes Y'} = \Phi_{X,X'} \otimes \tilde{\Phi}_{Y,Y'} \), see (8).

**Proposition 3.2** Let \( \mathcal{C} \) be a braided finite tensor category, \( \mathcal{D} \) a symmetric finite tensor category and \( \mathcal{M} \) a \( \mathcal{C} \otimes \mathcal{D} \)-bimodule category. Then \( \mathcal{M} \) is a one-sided \( \mathcal{C} \otimes \mathcal{D} \)-bimodule category with constraints \( m^L, m^R \) and \( A \) defined in Proposition 3.1 taking \( j = 1 \) and \( s = 1 \) in the definitions of \( m^R \) and \( A \), respectively.

**Proof.** The diagram chasing argument yields that the conditions (4) and (7), in order for \( \mathcal{M} \) to be a one-sided \( \mathcal{C} \otimes \mathcal{D} \)-bimodule category, reduce to the following:

\[
((X' \otimes X) \otimes \tilde{\Phi}^i_{Y,Y})(\Phi_{X,X'} \otimes \tilde{\Phi}^i_{Y,Y'}) = \Phi^i_{X,X'} \otimes (Y \otimes Y')
\]

and

\[
((X \otimes X') \otimes \tilde{\Phi}^i_{Y,Y})(\Phi_{X',X} \otimes \tilde{\Phi}^i_{Y,Y})(X' \otimes X) \otimes \tilde{\Phi}^i_{Y,Y'}) = ((X \otimes X') \otimes \tilde{\Phi}^i_{Y,Y})(\Phi^i_{X',X} \otimes (Y \otimes Y')).
\]
The first one implies that $j = 1$ and that $\Phi^i_{Y,Y'} \Phi^i_{X,Y'} = \text{Id}_{Y \otimes Y'}$ for every choice of $i$, hence $\mathcal{D}$ should be symmetric. The second one reduces to: $\Phi^i_{X',X} \otimes \Phi^i_{Y,Y'} = \Phi^i_{X',X} \otimes \Phi^t_{Y,Y'}$. Then $s = 1$ and the indices $i$ and $t$ are irrelevant, since $\mathcal{D}$ symmetric.

**Corollary 3.3** If $\mathcal{C}$ is a symmetric finite tensor category, then every $\mathcal{C}$-bimodule category $\mathcal{M}$ is a one-sided $\mathcal{C} \boxtimes \mathcal{C}$-bimodule category.

As a matter of fact, the converse of Proposition 3.1 parts 1) and 2) also holds and for a symmetric finite tensor category $\mathcal{C}$ we have:

$$\mathcal{M} \in \mathcal{C} \text{Bimod} \iff \mathcal{M} \in \mathcal{C} \boxtimes \mathcal{C} \text{-Mod} \iff \mathcal{M} \in \text{Mod}\mathcal{C} \boxtimes \mathcal{C}$$

and moreover $\mathcal{M}$ is a one-sided $\mathcal{C} \boxtimes \mathcal{C}$-bimodule category.

The construction of one-sided $\mathcal{C}$-bimodule categories, for a braided finite tensor category $\mathcal{C}$, induces in fact an embedding of categories (even of 2-categories):

$$\text{C-Mod} \hookrightarrow \text{C-Bimod}.$$ see [9, Section 2.8]. The module structures on the functors we will see in details in Subsection 4.2 and Subsection 4.3. The obtained subcategory (of one-sided $\mathcal{C}$-bimodule categories) we will denote by $\mathcal{C}^{\text{br-Mod}}$.

**Proposition 3.4** For a symmetric finite tensor category $\mathcal{C}$ the category $(\mathcal{C}^{\text{br-Mod}}, \boxtimes_{\mathcal{C}}, \mathcal{C})$ is symmetric monoidal.

**Proof.** Let $\mathcal{M}$ and $\mathcal{N}$ be two one-sided $\mathcal{C}$-bimodule categories. The braiding of $\mathcal{C}$ enables to define a $\mathcal{C}$-balanced functor $F : \mathcal{M} \boxtimes \mathcal{N} \to \mathcal{N} \boxtimes_{\mathcal{C}} \mathcal{M}$, $F(M \boxtimes N) = N \boxtimes_{\mathcal{C}} M$. For $M, N \in \mathcal{M}$ and $X \in \mathcal{C}$ we have: $F(M \boxtimes X, N) = N \boxtimes_{\mathcal{C}} (M \boxtimes X) = N \boxtimes_{\mathcal{C}} (X \boxtimes M) = (\overline{\boxtimes} X) \boxtimes_{\mathcal{C}} M = (\overline{\boxtimes} N) \boxtimes_{\mathcal{C}} M = F(M, X \boxtimes N)$. The isomorphism $b_X : F(M \boxtimes X, N) \to F(M, X \boxtimes N)$ we may consider identity. Then the coherence condition for $F$ to be $\mathcal{C}$-balanced, $F(M, m^{-1}_{iX,Y,N}) \boxtimes_{\mathcal{C}} m_{M \boxtimes X,Y,N} \boxtimes_{\mathcal{C}} N F(\overline{\boxtimes} m_{M,X,Y,N} \boxtimes_{\mathcal{C}} n) = b_{M,X \boxtimes Y,N}$ for $M \in \mathcal{M}$, $N \in \mathcal{N}$. $X, Y \in \mathcal{C}$ reduces to: $(\overline{\boxtimes} m_{iX,Y,N} \boxtimes_{\mathcal{C}} N M \boxtimes_{\mathcal{C}} m_{iX,Y,N} \boxtimes_{\mathcal{C}} N M) = N \boxtimes_{\mathcal{C}} (M \boxtimes (X \boxtimes Y))$. To check this identity we compose it with the isomorphism $(\overline{\boxtimes} m^{-1}_{iX,Y,N}) \boxtimes_{\mathcal{C}} M)$. Recall that the right associator $m_r$ by [9] is given via $m_r = m_{iYX,M}(\Phi_{XY} \boxtimes M)$. Now we compute:

$$((\Phi_{XY} \boxtimes M) \boxtimes_{\mathcal{C}} M) (\overline{\boxtimes} m_{iX,Y,N} \boxtimes_{\mathcal{C}} N M) = (\overline{\boxtimes} m_{iX,Y,N} \boxtimes_{\mathcal{C}} N M) (\overline{\boxtimes} \Phi_{XY} \boxtimes M).$$

In the first equation we applied the identity of the right associator $m_r$ for $\mathcal{N}$ and [11]. In the second one we used that $\mathcal{N}$ is a one-sided $\mathcal{C}$-bimodule. Since $\mathcal{C}$ is symmetric, we may cancel out the composed factor to recover $(\overline{\boxtimes} m_{iX,Y,N} \boxtimes_{\mathcal{C}} M) (\overline{\boxtimes} \Phi_{XY} \boxtimes M).$ Thus $F$ is $\mathcal{C}$-balanced and it induces a unique functor

$$\tau : \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to \mathcal{N} \boxtimes_{\mathcal{C}} \mathcal{M}, \quad M \boxtimes_{\mathcal{C}} N \mapsto N \boxtimes_{\mathcal{C}} M$$

(13)
which clearly is an isomorphism functor. It remains to see if $\tau$ is $C$-bilinear. The isomorphism $\tau(X \otimes M \boxtimes N) \rightarrow X \otimes \tau(M \boxtimes N)$ given by the composition $N \boxtimes_C (X \otimes M) = (N \otimes X) \boxtimes_C M = (X \otimes N) \boxtimes_C M$ may be considered as the identity. The coherence for left $C$-linearity of $\tau$ then reduces to $N \boxtimes_C m_{lX,Y,M} = m_{lX,Y,N} \boxtimes_C M$. The diagram chasing argument shows that this identity follows by the identity (1) and the rules for the right and the two-sided associator: (2) and (7). The coherence for the right $C$-linearity similarly reduces to: $m_{rN,X,Y} \boxtimes_C M = N \boxtimes_C m_{rM,X,Y}$. By the relation between left and right associator functors, this comes down to: $(m_{lY,X,N} \boxtimes_C M)((\Phi_{X,Y} \otimes N) \boxtimes_C M) = (N \boxtimes_C m_{lY,X,M})(N \boxtimes_C (\Phi_{X,Y} \otimes M))$, which is similarly proved as the coherence for the left side.

\[ \square \]

4 Dual objects for bimodule categories

In this section we compute the left and the right dual objects for the objects of the monoidal category $(\mathcal{C} \text{-Bimod}, \boxtimes_C, \mathcal{C})$, where $\mathcal{C}$ is a finite tensor category, proving thus that the former is a closed monoidal category. In Subsection 4.3 we conclude that if $\mathcal{C}$ is symmetric, then the left and the right dual objects in $(\mathcal{C}^{br} \text{-Mod}, \boxtimes_C, \mathcal{C})$ coincide. Actually, this is well-known in braided monoidal categories, and we saw in Proposition 3.4 that in order for $(\mathcal{C} \text{-Bimod}, \boxtimes_C, \mathcal{C})$ (or some subcategory) to be braided, it it necessary: that $\mathcal{C}$ be braided, that we restrict to one-sided $\mathcal{C}$-bimodule categories, and finally also that $\mathcal{C}$ be even symmetric.

4.1 Inner hom objects for module categories

There are following four pairs of adjoint functors $\mathcal{C} \rightarrow \mathcal{C}$:

\[(X \otimes -, *X \otimes -), \ (X^* \otimes -, X \otimes -), \ (- \otimes *X, - \otimes X), \ (- \otimes X, - \otimes X^*). \quad (14)\]

For a left $\mathcal{C}$-module category $\mathcal{M}$ let $\underline{\text{Hom}}_{\mathcal{M}}(M_1, M_2)$ be the inner hom object for $M_1, M_2 \in \mathcal{M}$. It is an object in $\mathcal{C}$ such that

\[ \text{Hom}_{\mathcal{M}}(X \boxtimes M_1, M_2) \cong \text{Hom}_{\mathcal{C}}(X, \underline{\text{Hom}}_{\mathcal{M}}(M_1, M_2)) \]

for $X \in \mathcal{C}$. Then $\text{Hom}_{\mathcal{M}}(-, M) : \mathcal{M} \rightarrow \mathcal{C}^{\text{op}}$ and $\underline{\text{Hom}}_{\mathcal{M}}(M, -) : \mathcal{M} \rightarrow \mathcal{C}$ are left $\mathcal{C}$-linear, thus they are functors in Fun$_{\mathcal{C}}(\mathcal{M}, \mathcal{C})$, [13, Corollary 2.10.15]. ($\mathcal{M}^{\text{op}}$ is the module category defined via relations (1) and (2).)

Similarly, for a right $\mathcal{C}$-module category $\mathcal{M}$ let $\underline{\text{Hom}}_{\mathcal{M}}(M_1, M_2)$ denote the corresponding inner hom object. It is such an object in $\mathcal{C}$ that:

\[ \text{Hom}_{\mathcal{M}}(M_1 \boxtimes X, M_2) \cong \text{Hom}_{\mathcal{C}}(X, \underline{\text{Hom}}_{\mathcal{M}}(M_1, M_2)). \quad (15)\]

Then we have:

\[ \text{Hom}_{\mathcal{C}}(Y, \underline{\text{Hom}}_{\mathcal{M}}(M_1 \boxtimes X, M_2)) = \text{Hom}_{\mathcal{M}}((M_1 \boxtimes X) \boxtimes Y, M_2) = \text{Hom}_{\mathcal{M}}(M_1 \boxtimes (X \otimes Y), M_2) \]

\[ \text{Hom}_{\mathcal{C}}(X \otimes Y, \underline{\text{Hom}}_{\mathcal{M}}(M_1, M_2)) \equiv \text{Hom}_{\mathcal{C}}(Y, *X \otimes \underline{\text{Hom}}_{\mathcal{M}}(M_1, M_2)). \]

This yields:

\[ \underline{\text{Hom}}_{\mathcal{M}}(M_1 \boxtimes X, M_2) = *X \otimes \underline{\text{Hom}}_{\mathcal{M}}(M_1, M_2). \quad (16)\]
Similarly, we find:

\[
\text{Hom}_C(Y, \underline{\text{Hom}}_M(M_1, M_2 \underline{\otimes} X)) = \text{Hom}_M(M_1 \underline{\otimes} Y, M_2 \underline{\otimes} X) \quad (14) \\
\text{Hom}_C(Y \otimes {}^*X, \underline{\text{Hom}}_M(M_1, M_2)) = \text{Hom}_C(Y, \underline{\text{Hom}}_M(M_1, M_2) \otimes X)
\]

which yields:

\[
\underline{\text{Hom}}_M(M_1, M_2 \underline{\otimes} X) = \underline{\text{Hom}}_M(M_1, M_2) \otimes X. \quad (17)
\]

Then (17) implies that \(\underline{\text{Hom}}_M(-, M) : \mathcal{M} \to \text{op}\mathcal{C}\) is right \(\mathcal{C}\)-linear and (17) implies that \(\underline{\text{Hom}}_M(M, -) : \mathcal{M} \to \mathcal{C}\) is right \(\mathcal{C}\)-linear. Thus both can be seen as functors in \(\text{Fun}(\mathcal{M}, \mathcal{C})\). We explain below what \(\text{op}\mathcal{M}\) means.

A right \(\mathcal{C}\)-module category \(\mathcal{M}\) gives rise to a left \(\mathcal{C}\)-module category \(\text{op}\mathcal{M}\) with the action given by (18) and associativity isomorphisms \(m_{X,Y,M}^{op} = m_{M,Y,^*X}\) for all \(X, Y \in \mathcal{C}, M \in \mathcal{M}\). Similarly, a left \(\mathcal{C}\)-module category \(\mathcal{M}\) gives rise to a right \(\mathcal{C}\)-module category \(\text{op}\mathcal{M}\) with the action given via (19). If \(\mathcal{M}\) is a \((\mathcal{C}, \mathcal{D})\)-bimodule category then \(\text{op}\mathcal{M}\) is a \((\mathcal{D}, \mathcal{C})\)-bimodule category and \(\text{op}(\text{op}\mathcal{M}) \cong \mathcal{M}\) as \((\mathcal{C}, \mathcal{D})\)-bimodule categories.

\[
X^{op \underline{\otimes} M} = M \underline{\otimes} X^* \quad (18) \\
M^{op \underline{\otimes} X} = ^*X \underline{\otimes} M \quad (19)
\]

If \(\mathcal{M}\) is a \((\mathcal{D}, \mathcal{C})\)-bimodule category, we find the following behaviour of the right \(\mathcal{C}\)-linear inner hom object:

\[
\text{Hom}_C(Y, \underline{\text{Hom}}_M(D \underline{\otimes} -, M)) = \text{Hom}_M((D \underline{\otimes} -) \underline{\otimes} Y, M) = \text{Hom}_M(D \underline{\otimes} (- \underline{\otimes} Y), M) \quad (14) \\
= \text{Hom}_M(- \underline{\otimes} Y, ^*D \underline{\otimes} M) = \text{Hom}_C(Y, \underline{\text{Hom}}_M(-, ^*D \underline{\otimes} M)).
\]

Consequently:

\[
\underline{\text{Hom}}_M(D \underline{\otimes} -, M) = \underline{\text{Hom}}_M(-, ^*D \underline{\otimes} M) \quad (20)
\]

for every \(D \in \mathcal{D}\).

### 4.2 Left dual object for a bimodule category

For a \(\mathcal{C}-\mathcal{D}\)-bimodule category \(\mathcal{M}\) and a \(\mathcal{C}-\mathcal{E}\)-bimodule category \(\mathcal{N}\) the category \(\text{Fun}_\mathcal{C}(\mathcal{M}, \mathcal{N})\) is a \(\mathcal{D}-\mathcal{E}\)-bimodule category via

\[
(X \underline{\otimes} F) = F(- \underline{\otimes} X) \quad (21)
\]

and

\[
(F \underline{\otimes} Y) = F(-) \underline{\otimes} Y \quad (22)
\]

for \(X \in \mathcal{D}, Y \in \mathcal{E}, F \in \text{Fun}_\mathcal{C}(\mathcal{M}, \mathcal{N})\) and \(M \in \mathcal{M}\).

Let \(\mathcal{M}\) be a \(\mathcal{C}\)-\(\mathcal{D}\)-bimodule category and \(\mathcal{N}\) a \(\mathcal{C}\)-\(\mathcal{E}\)-bimodule category. We denote by

\[
\theta_{M,N} : \mathcal{M}^{op} \underline{\otimes} \mathcal{C} \mathcal{N} \to \text{Fun}_\mathcal{C}(\mathcal{M}, \mathcal{N}) \quad (23)
\]

the \(\mathcal{D}-\mathcal{E}\)-bimodule equivalence from [14, Proposition 3.5] given by \(M \underline{\otimes} _\mathcal{C} N \mapsto \underline{\text{Hom}}_\mathcal{M}(-, M) \underline{\otimes} N\).

Denote by \(R : \mathcal{C} \to \text{Fun}_\mathcal{C}(\mathcal{M}, \mathcal{M})\) the functor given by \(R(X) = - \underline{\otimes} X\), where \(\mathcal{M}\) is a \(\mathcal{C}\)-bimodule category. It is \(\mathcal{C}\)-bilinear: \(R(C \otimes X) = - \underline{\otimes} (C \otimes X) \cong (- \underline{\otimes} C) \underline{\otimes} X = - \underline{\otimes} C \underline{\otimes} X\).
$R(X)(-R C) \cong C \otimes R(X)$ and $R(X \otimes C) \cong (-X) \otimes C = R(X) \otimes C$ for $X, C \in \mathcal{C}$. The coherence for the left and right $\mathcal{C}$-linearity of $R$ are precisely the coherence for the right $\mathcal{C}$-action on $\mathcal{M}$.

We now define $\text{coev} : \mathcal{C} \to \mathcal{M}^{\text{op}} \boxtimes \mathcal{M}$ such that the triangle $\langle 1 \rangle$ in the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\text{coev}} & \mathcal{M}^{\text{op}} \boxtimes \mathcal{M} \\
R & \xrightarrow{\theta_{\mathcal{M},\mathcal{M}}} & \mathcal{M}^{\text{op}} \boxtimes \mathcal{M} \\
\text{Fun}_C(M, M) & \xrightarrow{\text{db}} & \text{Fun}_C(M, M) \boxtimes \mathcal{M} \\
\end{array}$$

The functor $\text{db} : \text{Fun}_C(M, \mathcal{C}) \boxtimes \mathcal{M} \to \text{Fun}_C(M, M)$ is defined so that $\text{db}(F \boxtimes M) = F(-) \boxtimes M$, for $F \in \text{Fun}_C(M, M), M \in \mathcal{M}$. Then the triangle $\langle 2 \rangle$ above commutes. It is clear that $\text{db}$ is a $\mathcal{C}$-bimodule equivalence functor, and $\text{coev}$ is a $\mathcal{C}$-bimodule functor. From the definition of $\text{coev}$ we see that $\text{coev}(I) = \oplus \text{Hom}_M(-, W_i) \boxtimes V_i$. That is, for every $M \in \mathcal{M}$ one has:

$$\oplus \text{Hom}_M(M, W_i) \boxtimes V_i = M. \tag{25}$$

In [19, Corollary 3.22] it is proved that a $\mathcal{C}$-$\mathcal{D}$-bimodule category $\mathcal{M}$ induces an adjoint pair of functors $\mathcal{M} \boxtimes \mathcal{D} \dashv : \mathcal{D}$-$\mathcal{E}$-Bimod $\to \mathcal{C}$-$\mathcal{E}$-Bimod : $\text{Fun}_C(\mathcal{M}, -)$. The unit of the adjunction evaluated at a $\mathcal{D}$-$\mathcal{E}$-bimodule category $\mathcal{N}$ is a $\mathcal{D}$-$\mathcal{E}$-bilinear functor $\alpha(\mathcal{N}) : \mathcal{N} \to \text{Fun}_C(\mathcal{M}, \mathcal{M} \boxtimes \mathcal{D}) \mathcal{N}$ given by $N \mapsto - \boxtimes \mathcal{N}$. If $\mathcal{D} = \mathcal{E} = \mathcal{C} = \mathcal{N}$, we get a $\mathcal{C}$-bilinear functor $\alpha : \mathcal{C} \to \text{Fun}_C(\mathcal{M}, \mathcal{M} \boxtimes \mathcal{C}) = \text{Fun}_C(\mathcal{M}, \mathcal{M})$ such that $\alpha(X) = - \boxtimes \mathcal{C} X$. The counit of the adjunction in this case is a $\mathcal{C}$-bilinear functor $\beta : \mathcal{M} \boxtimes \mathcal{C} \text{Fun}_C(\mathcal{M}, \mathcal{C}) \to \mathcal{C}$ which is the evaluation functor.

**Remark 4.1** When $\mathcal{M}$ is an invertible $\mathcal{C}$-bimodule category we have that the unit of the adjunction $R : \mathcal{C} \to \text{Fun}_C(\mathcal{M}, \mathcal{M})$, and hence also the counit $\text{Ev} : \mathcal{M} \boxtimes \text{Fun}_C(\mathcal{M}, \mathcal{C}) \to \mathcal{C}$, is an equivalence [14, Proposition 4.2]. In this case the functor $\text{coev}$ from (24) is an equivalence, too.

Finally, define the functor $\text{ev} : \mathcal{M} \boxtimes \mathcal{C} \mathcal{M}^{\text{op}} \to \mathcal{C}$ through the commuting diagram:

$$\begin{array}{ccc}
\mathcal{M} \boxtimes \mathcal{C} \mathcal{M}^{\text{op}} & \xrightarrow{\text{ev}} & \mathcal{C} \\
\mathcal{M} \boxtimes \mathcal{C} \theta_{\mathcal{M},\mathcal{C}} & \xrightarrow{\text{Ev}} & \mathcal{C} \\
\mathcal{M} \boxtimes \mathcal{C} \text{Fun}_C(\mathcal{M}, \mathcal{C}) & \xrightarrow{\text{ev}} & \mathcal{C} \\
\end{array}$$

Then $\text{ev}$ is $\mathcal{C}$-bilinear and it is $\text{ev}(M \boxtimes C N) = \text{Hom}_M(M, N)$.

**Proposition 4.2** Let $\mathcal{M}$ be a $\mathcal{C}$-bimodule category. The object $\mathcal{M}^{\text{op}}$ together with the functors

$$\text{ev} : \mathcal{M} \boxtimes \mathcal{C} \mathcal{M}^{\text{op}} \to \mathcal{C} \quad \text{and} \quad \text{coev} : \mathcal{C} \to \mathcal{M}^{\text{op}} \boxtimes \mathcal{C} \mathcal{M}$$

is a left dual object for $\mathcal{M}$ in the monoidal category ($\mathcal{C}$-Bimod, $\mathcal{C}$, $\boxtimes$).

Consequently, $(\text{ev} \boxtimes \mathcal{C} \mathcal{M})(\mathcal{M} \boxtimes \mathcal{C} \mathcal{M}^{\text{op}}) \cong \text{Id}_\mathcal{M}$ and $(\mathcal{M}^{\text{op}} \boxtimes \mathcal{C} \mathcal{M}^{\text{op}})(\mathcal{M}^{\text{op}} \boxtimes \mathcal{C} \mathcal{M}) \cong \text{Id}_\mathcal{M}^{\text{op}}$.

If $\mathcal{M}$ is an invertible $\mathcal{C}$-bimodule category, the functors $\text{ev}$ and $\text{coev}$ are $\mathcal{C}$-bimodule equivalence functors.
Proof. Take \( M \in \mathcal{M} \), we find: \((\text{ev} \otimes \mathcal{C} \mathcal{M})(\mathcal{M} \otimes \mathcal{C} \text{coev})(\mathcal{M}) \cong (\text{ev} \otimes \mathcal{C} \mathcal{M})(\oplus_{i \in J} \mathcal{M} \otimes \mathcal{C} W_i \otimes \mathcal{C} V_i) = \oplus_{i \in J} \text{Hom}_{\mathcal{M}}(M, W_i) \otimes \mathcal{C} V_i = M \) by (23). To check the other identity, observe that \( \text{ev}[\mathcal{M} \otimes \mathcal{C} (\mathcal{M}^{op} \otimes \mathcal{C} \text{ev})(\text{coev} \otimes \mathcal{C} \mathcal{M}^{op})] \cong \text{ev}[(\mathcal{M} \otimes \mathcal{C} \mathcal{M} \otimes \mathcal{C} \mathcal{M}^{op})] \cong \text{ev} \), as the first axiom for dual objects is satisfied. Now by the universal property of the evaluation functor (which is the counit of the adjunction) it follows that the second axiom also holds.

The last part follows by Remark [11] and (26).

### 4.3 Right dual object for a bimodule category

Let \( \mathcal{M} \) be an \( \mathcal{E} \)-\( \mathcal{C} \)-bimodule category and \( \mathcal{N} \) a \( \mathcal{D} \)-\( \mathcal{C} \)-bimodule categories. The functor category \( \text{Fun}(\mathcal{M}, \mathcal{N})_c \) is a \( \mathcal{D} \)-\( \mathcal{E} \)-bimodule category via

\[
(X \otimes F)(M) = X \otimes F(M) \quad (27)
\]

\[
(F \otimes Y)(M) = F(Y \otimes M) \quad (28)
\]

for \( X \in \mathcal{D}, Y \in \mathcal{E}, F \in \text{Fun}(\mathcal{M}, \mathcal{N})_c \) and \( M \in \mathcal{M} \).

#### Lemma 4.3

Let \( \mathcal{C}, \mathcal{D}, \mathcal{E} \) be finite tensor categories, let \( \mathcal{M} \) be an \( \mathcal{E} \)-\( \mathcal{C} \)-bimodule category and \( \mathcal{N} \) a \( \mathcal{D} \)-\( \mathcal{C} \)-bimodule category. The functor

\[
\sigma_{\mathcal{M}, \mathcal{N}} : \mathcal{M} \otimes \mathcal{C} \mathcal{N} \to \text{Fun}(\mathcal{N}, \mathcal{M})_c
\]

given by \( \sigma(M \otimes \mathcal{C} N) = M \otimes \text{Hom}_{\mathcal{N}}(-, N) \) is an \( \mathcal{E} \)-\( \mathcal{D} \)-bimodule equivalence.

**Proof.** Observe that the functor \( \sigma_1 : \mathcal{M} \otimes \mathcal{C} \mathcal{N} \to \text{Fun}(\mathcal{N}, \mathcal{M}) \) given by \( \sigma_1(M \otimes \mathcal{C} N) = M \otimes \text{Hom}_{\mathcal{N}}(-, N) \) is \( \mathcal{C} \)-balanced:

\[
\sigma_1(M \otimes X, N) = (M \otimes X) \otimes \text{Hom}_{\mathcal{N}}(-, N) \cong M \otimes (X \otimes \text{Hom}_{\mathcal{N}}(-, N))
\]

\[
= M \otimes (\text{Hom}_{\mathcal{N}}(-, N) \otimes X^*) \quad (17)
\]

\[
M \otimes \text{Hom}_{\mathcal{N}}(-, N) \otimes X^* = \sigma_1(M, X \otimes \mathcal{C} N).
\]

The isomorphism \( \sigma_1(M \otimes X, N) \to \sigma_1(M, X \otimes \mathcal{C} N) \) is thus given by the composition of the identity (17) and the right associativity functor \( m_e \) for \( \mathcal{M} \); it satisfies the coherence for a \( \mathcal{C} \)-balanced functor because of the coherence for the right \( \mathcal{C} \)-action on \( \mathcal{M} \) and the coherence of the successive application of (17) for objects \( X, Y \in \mathcal{C} \). Thus \( \sigma_1 \) induces \( \sigma_2 : \mathcal{M} \otimes \mathcal{C} \mathcal{N} \to \text{Fun}(\mathcal{N}, \mathcal{M}) \). The functor \( \sigma_2(M \otimes \mathcal{C} N) = M \otimes \text{Hom}_{\mathcal{N}}(-, N) \) is right \( \mathcal{C} \)-linear, as so is \( \text{Hom}_{\mathcal{N}}(-, N) : \mathcal{N} \to \mathcal{C} \). Thus \( \sigma_{\mathcal{M}, \mathcal{N}} \) is well-defined. It is clearly left \( \mathcal{C} \)-linear. For right \( \mathcal{D} \)-linearity we find:

\[
\sigma_{\mathcal{M}, \mathcal{N}}(M \otimes \mathcal{C} N^{op} \otimes D) = M \otimes \text{Hom}_{\mathcal{N}}(-, N^{op} \otimes D) \quad (18)
\]

\[
M \otimes \text{Hom}_{\mathcal{N}}(-, D \otimes -, N) \quad (20)
\]

\[
= M \otimes \text{Hom}_{\mathcal{N}}(-, N) \otimes D = \sigma_{\mathcal{M}, \mathcal{N}}(M \otimes \mathcal{C} N) \otimes D.
\]

The proof that \( \sigma_{\mathcal{M}, \mathcal{N}} \) is an equivalence is analogous to the proof of [19] Theorem 3.20, Lemma 3.21]. The inverse of \( \sigma_{\mathcal{M}, \mathcal{N}} \) is induced by the functor \( J : \text{Fun}(\mathcal{N}, \mathcal{M})_c \to \mathcal{M} \otimes \mathcal{C} \mathcal{N} \) such that \( J(F) \) is the representing object of the functor \( M \otimes \mathcal{C} N \to \text{Hom}(M, F(N)) \), for \( F \in \text{Fun}(\mathcal{N}, \mathcal{M})_c \). That is, there is an equivalence \( \text{Hom}_{\mathcal{M} \otimes \mathcal{C} \mathcal{N}}(M \otimes \mathcal{C} N, J(F)) = \text{Hom}_{\mathcal{M}}(M, F(N)) \). \( \square \)
(In [11, Corollary 3.4.11] the category equivalence from above lemma appears with a different, non-explicit proof.)

Let \( L : C \to \text{Fun}(\mathcal{M}, \mathcal{M})_\mathcal{C} \) be the functor given by \( L(X) = X \otimes - \). Then \( L \) is \( \mathcal{C} \)-bilinear. For left \( \mathcal{C} \)-linearity we have: \( L(X \otimes C) = (X \otimes C) \otimes - \cong X \otimes (C \otimes -) = X \otimes L(C) \) for \( X, C \in \mathcal{C} \), and for the right one: \( L(C \otimes X) \cong C \otimes (X \otimes -) \cong L(C) \otimes X \).

Now let \( \mathcal{M} \) be a \( \mathcal{C} \)-bimodule category. From Lemma 4.3 we have that \( \sigma := \sigma_{\mathcal{C}, \mathcal{M}} : \mathcal{M} \otimes \mathcal{C} \xrightarrow{\sim} \text{Fun}(\mathcal{M}, \mathcal{C})_\mathcal{C} \) as \( \mathcal{C} \)-bimodule categories. We define \( \text{coev} : \mathcal{C} \to \mathcal{M} \otimes \mathcal{C} \) \( \mathcal{M} \) such that the triangle (1) in the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\text{coev}} & \mathcal{M} \otimes \mathcal{C} \\
L \downarrow & \sigma_{\mathcal{M}, \mathcal{C}} & \downarrow \text{db} \\
\text{Fun}(\mathcal{M}, \mathcal{M})_\mathcal{C} & \xrightarrow{\text{Fun}(\mathcal{M}, \mathcal{C})_\mathcal{C}} & \mathcal{M} \otimes \text{Fun}(\mathcal{M}, \mathcal{C})_\mathcal{C}
\end{array}
\]

Then \( \text{coev} \) too is a \( \mathcal{C} \)-bimodule functor. The functor \( \text{db} : \mathcal{M} \otimes \mathcal{C} \text{Fun}(\mathcal{M}, \mathcal{C})_\mathcal{C} \to \text{Fun}(\mathcal{M}, \mathcal{M})_\mathcal{C} \) is defined so that \( \text{db}(M \otimes_c F) = M \otimes F(-) \), for \( F \in \text{Fun}(\mathcal{M}, \mathcal{M})_\mathcal{C}, M \in \mathcal{M} \). Then the triangle (2) above commutes, and \( \text{db} \) is a \( \mathcal{C} \)-bimodule equivalence functor. From the definition of \( \text{coev} \) we see that \( \text{coev}(I) = \bigoplus_{i \in J} V_i \otimes_c W_i \) is such an object in \( \mathcal{M} \otimes \mathcal{C} \) \( \mathcal{M} \) such that \( \text{Id}_\mathcal{M} = \bigoplus_{i \in J} V_i \otimes \text{Hom}_\mathcal{M}(-, W_i) \). That is, for every \( M \in \mathcal{M} \) one has:

\[
\bigoplus_{i \in J} V_i \otimes \text{Hom}_\mathcal{M}(M, W_i) = M.
\]

The next properties are easy to deduce:

**Lemma 4.4**

1) \( \text{op}(\text{Fun}(\mathcal{M}, \mathcal{N})_\mathcal{C}) \cong \text{Fun}(\mathcal{N}, \mathcal{M})_\mathcal{C} \) as \( \mathcal{D} \text{-}\mathcal{E} \)-bimodule categories, for \( \mathcal{D} \mathcal{M}_\mathcal{C}, \mathcal{E} \mathcal{N}_\mathcal{C} \);

2) \( \text{op}(\mathcal{M} \otimes \mathcal{N})_\mathcal{C} \cong \text{op} \mathcal{N} \otimes \mathcal{C} \text{op} \mathcal{M} \) as \( \mathcal{E} \cdot \mathcal{D} \)-bimodule categories, for \( \mathcal{D} \mathcal{M}_\mathcal{C}, \mathcal{E} \mathcal{N}_\mathcal{C} \).

The following proposition can be proved directly, or alternatively, by using successive applications of the equivalence \( \sigma \) from [29] and the properties from the above lemma:

**Proposition 4.5** Let \( \mathcal{N} \) be a \( \mathcal{D} \)-\( \mathcal{C} \)-bimodule category, \( \mathcal{M} \) be a \( \mathcal{C} \)-\( \mathcal{E} \)-bimodule category and \( \mathcal{A} \) an \( \mathcal{F} \)-\( \mathcal{E} \)-bimodule category then there is an isomorphism of \( \mathcal{F} \cdot \mathcal{D} \)-bimodule categories:

\[
\text{Fun}(\mathcal{N} \otimes \mathcal{C} \mathcal{M}, \mathcal{A})_\mathcal{E} \cong \text{Fun}(\mathcal{N}, \text{Fun}(\mathcal{M}, \mathcal{A})_\mathcal{E})_\mathcal{C}.
\]

From Proposition 4.5 we see that given a \( \mathcal{C} \)-\( \mathcal{E} \)-bimodule category \( \mathcal{M} \) there is an adjoint pair of functors: \( - \otimes \mathcal{C} \mathcal{M} : \mathcal{D} \mathcal{C} \text{-Bimod} \to \mathcal{E} \text{-Bimod} : \text{Fun}(\mathcal{M}, -)_\mathcal{E} \). The unit of the adjunction evaluated at a \( \mathcal{C} \)-\( \mathcal{D} \)-bimodule category \( \mathcal{N} \) is a \( \mathcal{C} \)-\( \mathcal{D} \)-bilinear functor \( \alpha(\mathcal{N}) : \mathcal{N} \to \text{Fun}(\mathcal{M}, \mathcal{N} \otimes \mathcal{C} \mathcal{M})_\mathcal{E} \) given by \( N \mapsto N \otimes \mathcal{C} \mathcal{M} \). If \( \mathcal{D} = \mathcal{C} = \mathcal{N} \), we get a \( \mathcal{C} \)-bilinear functor \( \alpha : \mathcal{C} \to \text{Fun}(\mathcal{M}, \mathcal{C} \otimes \mathcal{C} \mathcal{M})_\mathcal{E} = \text{Fun}(\mathcal{M}, \mathcal{M})_\mathcal{E} \) such that \( \alpha(X) = X \otimes \mathcal{C} \mathcal{M} \). The counit of the adjunction in this case is a \( \mathcal{C} \)-bilinear functor \( \beta : \text{Fun}(\mathcal{M}, \mathcal{C})_\mathcal{E} \otimes \mathcal{C} \mathcal{M} \to \mathcal{C} \) which is the evaluation functor.

**Remark 4.6** When \( \mathcal{E} = \mathcal{C} \) and \( \mathcal{M} \) is an invertible \( \mathcal{B} \)-bimodule we have that the unit of the adjunction \( L : \mathcal{C} \to \text{Fun}(\mathcal{M}, \mathcal{C} \otimes \mathcal{C} \mathcal{M})_\mathcal{C} \), and hence also the counit \( \text{Ev} : \text{Fun}(\mathcal{M}, \mathcal{C})_\mathcal{C} \otimes \mathcal{C} \mathcal{M} \to \mathcal{C} \), is an equivalence [14, Proposition 4.2]. In this case the functor \( \text{coev} \) from (30) is an equivalence, too.
For a $C$-bimodule category $\mathcal{M}$ we define the functor $\text{ev} : \text{op} \mathcal{M} \boxtimes C \mathcal{M} \to C$ through the commuting diagram:

$$
\begin{align*}
\text{op} \mathcal{M} \boxtimes C \mathcal{M} & \xrightarrow{\text{ev}} C \\
\sigma \boxtimes C \mathcal{M} & \xrightarrow{\text{ev}} \text{Fun}(\mathcal{M}, C) \boxtimes C \mathcal{M}
\end{align*}
$$

(32)

Then $\text{ev}$ is $C$-bilinear and it is $\text{ev}(M \boxtimes_c N) = \text{Hom}_\mathcal{M}(N, M)$.

**Proposition 4.7** Let $C$ be a braided finite tensor category and let $\mathcal{M}$ be a $C$-bimodule category. The object $\text{op} \mathcal{M}$ together with the functors $\text{ev} : \text{op} \mathcal{M} \boxtimes C \mathcal{M} \to C$ and $\text{coev} : C \to M \boxtimes_c \text{op} \mathcal{M}$ is a right dual object for $\mathcal{M}$ in the monoidal category $(C\text{-Bimod}, C, \boxtimes C)$. Consequently, $(\text{ev}(\text{op} \mathcal{M} \boxtimes C \mathcal{M})) = \text{Id}_{\mathcal{M}}$ and $(\text{coev} \boxtimes \text{op} \mathcal{M})(\text{op} \mathcal{M} \boxtimes_c \text{coev} \mathcal{M}) = \text{Id}_{\mathcal{M}}$.

**Proof.** Take $M \in \mathcal{M}$, we find: $(\mathcal{M} \boxtimes_c \text{ev})(\text{coev} \boxtimes_c \mathcal{M})(M) = (\mathcal{M} \boxtimes_c \text{ev})(\oplus_{i \in I} V_i \boxtimes_c W_i \boxtimes_c \mathcal{M}) = \oplus_{i \in I} V_i \boxtimes_c \text{Hom}_\mathcal{M}(M, W_i)$ by (31). To check the other identity, observe that $\text{ev}((\text{ev}(\text{op} \mathcal{M} \boxtimes_c \mathcal{M})))(\text{op} \mathcal{M} \boxtimes_c \text{coev} \mathcal{M}) = \text{ev}((\text{ev}(\text{op} \mathcal{M} \boxtimes_c \mathcal{M}))(\text{op} \mathcal{M} \boxtimes_c \text{coev} \mathcal{M})) = \text{ev}(\text{op} \mathcal{M} \boxtimes_c \mathcal{M} \boxtimes_c \text{ev} \mathcal{M})(\text{op} \mathcal{M} \boxtimes_c \text{coev} \mathcal{M} \boxtimes_c \mathcal{M}) = \text{ev}$, as the first axiom for dual objects is satisfied. Now by the universal property of the evaluation functor (which is the counit of the adjunction) it follows that the second axiom also holds.

The last part follows by Remark 4.6 and (32).

**Remark 4.8** In [26, Proposition 4.22] the left $\mathcal{M}^\sharp$ and the right dual $\sharp \mathcal{M}$ of a $C$-bimodule category are constructed. They are versions of our $\text{op} \mathcal{M}$ and $\text{op} \mathcal{M}$.

### 4.4 Some applications

We will prove here some claims that will be useful in our future computations.

**Lemma 4.9** Let $\mathcal{F}, \mathcal{G} : \mathcal{M} \to \mathcal{N}$ be $\mathcal{D}$-$\mathcal{C}$-bimodule functors and let $\mathcal{P}$ be an invertible $\mathcal{C}$-bimodule category.

1. It is $\mathcal{F} \boxtimes_c \text{Id}_\mathcal{P} = \mathcal{G} \boxtimes_c \text{Id}_\mathcal{P}$ if and only if $\mathcal{F} = \mathcal{G}$.

2. If $\mathcal{H} : \mathcal{P} \to \mathcal{L}$ is a $\mathcal{C}$-bimodule equivalence functor, it is $\mathcal{F} \boxtimes_c \mathcal{H} = \mathcal{G} \boxtimes_c \mathcal{H}$ if and only if $\mathcal{F} = \mathcal{G}$.

**Proof.**

1. We tensor the identity $\mathcal{F} \boxtimes_c \text{Id}_\mathcal{P} = \mathcal{G} \boxtimes_c \text{Id}_\mathcal{P}$ from the right by $\text{Id}_{\mathcal{P}^{\text{op}}}$ and then “conjugate” by $\text{Id}_{\mathcal{N} \boxtimes_c \text{ev}_\mathcal{P}}$. We get: $(\text{Id}_{\mathcal{N} \boxtimes_c \text{ev}_\mathcal{P}})(\mathcal{F} \boxtimes_c \text{Id}_\mathcal{P} \boxtimes_c \text{Id}_{\mathcal{P}^{\text{op}}})(\text{Id}_\mathcal{M} \boxtimes_c \text{ev}_\mathcal{P}^{-1}) = (\text{Id}_{\mathcal{N} \boxtimes_c \text{ev}_\mathcal{P}})(\mathcal{G} \boxtimes_c \text{Id}_{\mathcal{P}^{\text{op}}} \boxtimes_c \text{Id}_\mathcal{P})(\text{Id}_\mathcal{M} \boxtimes_c \text{ev}_\mathcal{P}^{-1})$, which is the same as saying that $\mathcal{F} = \mathcal{F} \boxtimes_c \text{Id}_\mathcal{C} = \mathcal{G} \boxtimes_c \text{Id}_\mathcal{C} = \mathcal{G}$. 
2. We compose the identity $F \boxtimes C H = G \boxtimes C H$ from the left by $\text{Id}_C \boxtimes C H^{-1}$ and apply the part 1).

\[\square\]

For a $C$-$D$-bimodule functor $F : \mathcal{M} \to \mathcal{N}$ the $D$-$C$-bimodule functor $\text{op} F : \text{op} \mathcal{N} \to \text{op} \mathcal{M}$ is given by $\text{op} F = \sigma_{M,C}^1 \circ F^* \circ \sigma_{N,C}$. Here $F^* : \text{Fun}_C(\mathcal{N}, C) \to \text{Fun}_C(\mathcal{M}, C)$ is given by $F^*(G) = G \circ F$ and $\sigma$ is the equivalence from (29). If $F$ is an equivalence, it is $(F^*)^{-1} = (F^{-1})^* = - \circ F^{-1}$ and consequently: $(\text{op} F)^{-1} = (\text{op} F^*)^{-1}$.

**Lemma 4.10** Let $\mathcal{C}$ be a finite tensor category and $\mathcal{M}$ an exact $\mathcal{C}$-bimodule category and let $\alpha : \mathcal{M} \to \mathcal{C}$ be a $\mathcal{C}$-bimodule equivalence. Then: $\overline{\alpha^*} = (\text{op} \alpha)^{-1} \boxtimes \mathcal{C} \alpha$ as right $\mathcal{C}$-linear functors.

**Proof.** Take $M \in \text{op} \mathcal{M}$ and $N \in \mathcal{M}$. We have that $\sigma_{\mathcal{C},\mathcal{M}}(M) = \overline{\text{Hom}}_\mathcal{M}(\alpha^{-1}(-), M)$ is a right $\mathcal{C}$-linear functor $\mathcal{M} \to \mathcal{C}$. Observe that $(\text{op} \alpha)^{-1}(M) = \sigma_{\mathcal{C},\mathcal{M}}^{-1}(\alpha^{-1}(M)) = \sigma_{\mathcal{C},\mathcal{C}}^{-1}(\overline{\text{Hom}}_\mathcal{M}(\alpha^{-1}(\alpha(\alpha^{-1}(M))))).$ Then:

\[
((\text{op} \alpha)^{-1} \boxtimes \mathcal{C} \alpha)(M \boxtimes \mathcal{C} N) = (\text{op} \alpha)^{-1}(M) \boxtimes \mathcal{C} \alpha(N)
= \sigma_{\mathcal{C},\mathcal{C}}^{-1}(\overline{\text{Hom}}_\mathcal{M}(\alpha^{-1}(\alpha^{-1}(M)))) \boxtimes \mathcal{C} \alpha(N)
= \sigma_{\mathcal{C},\mathcal{C}}^{-1}(\overline{\text{Hom}}_\mathcal{M}(\alpha^{-1}(\alpha^{-1}(M) \boxtimes \alpha(N)))
\begin{align*}
\overset{28}{=} \sigma_{\mathcal{C},\mathcal{C}}^{-1}(\overline{\text{Hom}}_\mathcal{M}(\alpha^{-1}(\alpha(N) \boxtimes -), M))
= \sigma_{\mathcal{C},\mathcal{C}}^{-1}(\overline{\text{Hom}}_\mathcal{M}(\overline{\mathcal{N} \boxtimes M}).
\end{align*}
\]

The third equality holds since $\sigma$ is right $\mathcal{C}$-linear and in the last equality we used the fact that $\alpha^{-1}$ is right $\mathcal{C}$-linear: $\overline{\text{Hom}}_\mathcal{M}(\alpha^{-1}(\alpha(N) \boxtimes X), M) = \overline{\text{Hom}}_\mathcal{M}(\alpha^{-1}(\alpha(N) \boxtimes X), M) = \text{Hom}_\mathcal{M}(\overline{\mathcal{N} \boxtimes X}, M)$. On the other hand, we have: $\overline{\alpha^*}(M \boxtimes \mathcal{C} M) = \text{Hom}_\mathcal{M}(\mathcal{N} \boxtimes \mathcal{N}, M)$ and $\sigma_{\mathcal{C},\mathcal{C}}(\overline{\alpha^*}(M \boxtimes \mathcal{C} N)) = \text{Hom}_\mathcal{C}(\mathcal{N} \boxtimes \mathcal{C}, N, M)).$ We should prove that $\text{Hom}_\mathcal{M}(\mathcal{N} \boxtimes -), M) \cong \text{Hom}_\mathcal{C}(\mathcal{N} \boxtimes \mathcal{C}, N, M)$. For arbitrary $X, Y \in \mathcal{C}$ we find:

\[
\text{Hom}_\mathcal{C}(X, \text{Hom}_\mathcal{C}(Y, \text{Hom}_\mathcal{M}(\mathcal{N}, M))) = \text{Hom}_\mathcal{C}(Y \boxtimes \mathcal{C}, \text{Hom}_\mathcal{M}(\mathcal{N}, M))
= \text{Hom}_\mathcal{M}(\mathcal{N} \boxtimes (Y \boxtimes \mathcal{C}), M) = \text{Hom}_\mathcal{M}((\mathcal{N} \boxtimes Y) \boxtimes X, M)
= \text{Hom}_\mathcal{C}(X, \text{Hom}_\mathcal{M}(\mathcal{N} \boxtimes Y, M))
\]

and the first claim follows. Note that $\overline{\alpha^*} : \text{op} \mathcal{M} \boxtimes \mathcal{C} \mathcal{M} \to \mathcal{C}$, while $(\text{op} \alpha)^{-1} \boxtimes \mathcal{C} \alpha : \text{op} \mathcal{M} \boxtimes \mathcal{C} \mathcal{M} \to \mathcal{C} \text{op} \mathcal{C} \mathcal{C}$, this is why the equality is of right $\mathcal{C}$-module functors (as in 2)).

\[\square\]

### 4.5 Dual objects for one-sided bimodules over symmetric finite tensor categories

Let $\mathcal{C}$ be a braided finite tensor category. We denote by $\mathbf{Pic}(\mathcal{C})$ the category of exact invertible one-sided $\mathcal{C}$-bimodule categories and their equivalences. They were studied in [14] Section 4.4, [9] Section 2.8. From Proposition 3.4 it follows that when $\mathcal{C}$ is symmetric, so is $\mathbf{Pic}(\mathcal{C})$ (as a monoidal subcategory of $(\mathcal{C}^{br} \text{-} \mathbf{Mod}, \boxtimes, \mathcal{C})$).

For one-sided $\mathcal{C}$-bimodules we have: $\text{Fun}_\mathcal{C}(\mathcal{M}, \mathcal{C}) = \text{Fun}(\mathcal{M}, \mathcal{C})_\mathcal{C}$, their $\mathcal{C}$-module structures (21), (28), (21), and (22) coincide. Then $\text{op} \mathcal{M} = \mathcal{M}^{op}$. We also have that the functors $\text{Hom}_\mathcal{M}(\mathcal{M}, -)$ and $\text{Hom}_\mathcal{M}(\mathcal{M}, -)$ coincide. (Namely, they are right adjoint functors
to $-\otimes M$ and $M\otimes -$ from $\mathcal{C}$ to $\mathcal{M}$, respectively, for every $M \in \mathcal{M}$, and the latter are equal as functors in $\text{Pic}(\mathcal{C})$. Then $\text{Hom}_{\mathcal{M}}(M, -)$ and $\text{Hom}_{\mathcal{M}}(-, M)$ are equal for every $M \in \mathcal{M}$, hence the bifunctors $\text{Hom}_{\mathcal{C}, \mathcal{M}}(-, -)$ and $\text{Hom}_{\mathcal{M}}(-, -)$ are equal, in particular so are $\text{Hom}_{\mathcal{M}}(-, M)$ and $\text{Hom}_{\mathcal{M}}(M, -)$.

If $\mathcal{C}$ is symmetric, we then have: $\sigma \circ \tau = \theta$, where $\tau$ is from (13), and also: $\text{ev} \circ \tau = \text{ev}$ and $\tau \circ \text{coev} = \text{coev}$. Consequently, the left and the right dual object are isomorphic in this setting.

Recall from (26) that $\text{ev} : M \boxtimes_{\mathcal{C}} \mathcal{M}^{\text{op}} \to \mathcal{C}$ is given by $\text{ev}(M \boxtimes_{\mathcal{C}} N) = \text{Hom}_{\mathcal{M}}(M, N)$ for $M \boxtimes_{\mathcal{C}} N \in M \boxtimes_{\mathcal{C}} N$ and set $\text{coev}(I) = \bigoplus_{i \in J} W_i \boxtimes_{\mathcal{C}} V_i \in M^{\text{op}} \boxtimes_{\mathcal{C}} M$. Then $\text{ev} \circ \tau \circ \text{coev}(I) = \bigoplus_{i \in J} \text{Hom}_{\mathcal{M}}(V_i, W_i)$.

**Corollary 4.11** For a symmetric finite tensor category $\mathcal{C}$ and $M \in \text{Pic}(\mathcal{C})$, it is $\text{ev}^{-1} = \tau \circ \text{coev} \simeq \text{coev}$. In particular, it is $\bigoplus_{i \in J} \text{Hom}_{\mathcal{M}}(V_i, W_i) \simeq I$.

**Proof.** By Proposition 4.2 the functors $\text{ev}$ and $\text{coev}$ are equivalences. The identity $(\text{ev} \boxtimes_{\mathcal{C}} \mathcal{M})(\mathcal{M} \boxtimes_{\mathcal{C}} \text{coev}) \simeq \text{id}_\mathcal{M}$ yields $\mathcal{M} \boxtimes_{\mathcal{C}} \text{coev} \simeq \text{ev}^{-1} \boxtimes_{\mathcal{C}} \mathcal{M}$. Compose this with the isomorphism $\tau$ and apply Lemma 4.9, 1) to get the claim. \qed

## 5 Amitsur cohomology over symmetric finite tensor categories

Amitsur cohomology was first introduced in [1] for commutative algebras over fields, it can be viewed as an affine version of Čech cohomology. It was further developed in [7, 22]. For more details see [4]. We construct an analogous cohomology for symmetric finite tensor categories.

Given a tensor functor $\eta : \mathcal{C} \to \mathcal{E}$, then $\mathcal{E}$ is a left (and similarly a right) $\mathcal{C}$-module category. The action bifunctor $\mathcal{C} \times \mathcal{E} \to \mathcal{E}$ is given by $\otimes(\eta \times \text{Id}_\mathcal{E})$, where $\otimes$ is the tensor product in $\mathcal{E}$, and the associator functor is $m_{X,Y,F} = \alpha_{\eta(X), \eta(Y), E}(\xi_{X,Y} \otimes E)$ (33)

for every $X, Y \in \mathcal{C}$ and $E \in \mathcal{E}$, where $\xi_{X,Y} : \eta(X \otimes Y) \to \eta(X) \otimes \eta(Y)$ determines the monoidal structure of the functor $\eta$ and $\alpha$ is the associativity constraint for $\mathcal{E}$. The constraint for the action of the unit is defined in the obvious manner. Moreover, $\mathcal{E}$ is a $\mathcal{C}$-$\mathcal{E}$-bimodule category with the bimodule constraint $\gamma_{X,E,F} : (X \otimes E) \otimes F \to X \otimes (E \otimes F)$ for $X \in \mathcal{C}, E, F \in \mathcal{E}$, given via $\gamma_{X,E,F} = \alpha_{\eta(X), E,F}$.

**Definition and Lemma 5.1** Let $F : \mathcal{D} \to \mathcal{C}$ be a tensor functor.

- Given another tensor functor $G : \mathcal{D}' \to \mathcal{C}$ and a $\mathcal{C}$-bimodule category $\mathcal{M}$, the category $\mathcal{F}\mathcal{M}_G$ equal to $\mathcal{M}$ as an abelian category with actions:

$$X \boxtimes M \boxtimes X' = F(X) \boxtimes M \boxtimes G(X')$$

for all $X \in \mathcal{D}, X' \in \mathcal{D}'$ and $M \in \mathcal{M}$ is a $\mathcal{D}$-$\mathcal{D}'$-bimodule category.
• If \( \mathcal{H} : \mathcal{C} \to \mathcal{E} \) is another tensor functor and \( \mathcal{M} \) is a left \( \mathcal{E} \)-module category, then there is an obvious equivalence of left \( D \)-modules categories:

\[
\mathcal{F} \mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{H} \mathcal{M} \simeq \mathcal{H}_F \mathcal{M}.
\]

**Proof.** For the second part, we have that the functor \( \beta : \mathcal{F} \mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{H} \mathcal{M} \to \mathcal{H}_F \mathcal{M} \), given by \( C \boxtimes M \mapsto H(C) \otimes M \), is \( \mathcal{C} \)-balanced with the natural isomorphism \( b_{C,X,M} = m_{C,X,M} = \alpha_{H(C),H(X),M}(\xi_{X,Y} \otimes M) \), similarly as in (33). The coherence (3) for this balance is precisely the coherence for \( \mathcal{M} \) to be a left \( \mathcal{C} \)-module category (through \( \mathcal{H} \)).

Fix \( \mathcal{C} \) a finite tensor category. In order to simplify the notation we will often write: \( X_1 \cdots X_n \) for the tensor product \( X_1 \otimes \cdots \otimes X_n \) in \( \mathcal{C} \). An object \( X \in \mathcal{C}^{\otimes n} \) we will write as \( X = X^1 \boxtimes X^2 \boxtimes \cdots \boxtimes X^n \), where the direct summation is understood implicitly. Recall that the category \( \mathcal{C}^{\otimes n} \) is a finite tensor category with unit object \( I^{\otimes n} \) and the componentwise tensor product: \( (X^1 \boxtimes X^2 \boxtimes \cdots \boxtimes X^n) \otimes (Y^1 \boxtimes Y^2 \boxtimes \cdots \boxtimes Y^n) = X_1 Y_1 \boxtimes \cdots \boxtimes X_n Y_n \).

We consider the functors \( e^n_i : \mathcal{C}^{\otimes n} \to \mathcal{C}^{\otimes (n+1)} \) with \( i = 1, \cdots, n+1 \), given by

\[
e^n_i(X^1 \boxtimes \cdots \boxtimes X^n) = X^1 \boxtimes \cdots \boxtimes I \boxtimes X^i \boxtimes \cdots \boxtimes X^n
\]

They are clearly tensor functors. We have:

**Lemma 5.2** For \( i \geq j \in \{1, \cdots, n+1\} \) it is

\[
e^{n+1}_j \circ e^n_i = e^{n+1}_{i+1} \circ e^n_j.
\]

Let \( P \) be an additive covariant functor from a full subcategory of the category of (symmetric) tensor categories that contains all Deligne tensor powers \( \mathcal{C}^{\otimes n} \) of \( \mathcal{C} \) to abelian groups. We define \( \mathcal{C}^{\otimes 0} = k \). Then we consider

\[
\delta_n = \sum_{i=1}^{n+1} (-1)^{i-1} P(e^n_i) : P(\mathcal{C}^{\otimes n}) \to P(\mathcal{C}^{\otimes (n+1)}).
\]

It is straightforward to show, using Lemma 5.2, that \( \delta_{n+1} \circ \delta_n = 0 \), so we obtain a complex:

\[
\begin{align*}
0 \longrightarrow & \quad P(\mathcal{C}) \xrightarrow{\delta_1} P(\mathcal{C}^{\otimes 2}) \xrightarrow{\delta_2} P(\mathcal{C}^{\otimes 3}) \xrightarrow{\delta_3} \cdots \\
\end{align*}
\]

We will call it *Amitsur complex* \( C(\mathcal{C}/\text{vec}, P) \). We have:

\[
Z^n(\mathcal{C}, P) = \text{Ker} \ \delta_n, \quad B^n(\mathcal{C}, P) = \text{Im} \ \delta_n, \quad \text{and} \quad H^n(\mathcal{C}, P) = Z^n(\mathcal{C}, P)/B^n(\mathcal{C}, P).
\]

We will call \( H^n(\mathcal{C}, P) \) the \( n \)-th Amitsur cohomology group of \( \mathcal{C} \) with values in \( P \). Elements in \( Z^n(\mathcal{C}, P) \) are called \( n \)-cocycles, and elements in \( B^n(\mathcal{C}, P) \) are called \( n \)-coboundaries.

We will consider the cases: \( P = \text{Pic} \), where \( \text{Pic}(\mathcal{C}) \) is the Picard group of a braided monoidal category \( \mathcal{C} \), consisting of equivalence classes of exact invertible one-sided \( \mathcal{C} \)-bimodule categories ([14, Section 4.4], [9, Section 2.8]), and \( P = \text{Inv} \), where \( \text{Inv}(\mathcal{C}) \) is the group of invertible objects of \( \mathcal{C} \), which we define below. Since we need \( P \) to map to abelian groups, we need to work only with symmetric categories \( \mathcal{C} \).
Lemma 5.3 For a braided (symmetric) finite tensor category $C$ the category $C^\otimes n$ is a braided (symmetric) finite tensor category for every $n \in \mathbb{N}$.

If $C$ is braided, every left $C^\otimes n$-module category is a $C^\otimes n$-bimodule category so we may consider the Picard group $\text{Pic}(C^\otimes n)$. If $C$ is symmetric by Proposition 3.4 the monoidal category $((C^\otimes n)^{br}, \text{Bimod}, \boxtimes_{C^\otimes n}, C^\otimes n)$ is symmetric and we have:

Corollary 5.4 For a symmetric finite tensor category $C$ the Picard group $\text{Pic}(C^\otimes n)$ is abelian for every $n \in \mathbb{N}$.

An object $X \in C$ is called invertible if there exists an object $Y \in C$ such that $X \otimes Y \cong I \cong Y \otimes X$. For such an object $Y$ we will write $Y = X^{-1}$. Isomorphism classes of invertible objects in $C$ form a group $\text{Inv}(C)$ with the product induced by the tensor product in $C$. If $C$ is braided $\text{Inv}(C)$ is an abelian group (in two ways).

Set $\tilde{\partial}_n(X) := \prod_{i=1}^{n+1} \text{Inv}(e^n_i)(X)(-1)^i \frac{2.2}{C_i} \circ \prod_{i=1}^{n-1} e^n_i(X)(-1)^i$.

The following lemma we will use in the proof of Theorem 6.2:

Lemma 5.5 Let $C$ be braided and let $M$ be a one-sided $C^\otimes n$-bimodule category. For $X \in \text{Inv}(C^\otimes n)$ we denote by $m(X)$ the $C^\otimes n$-bilinear autoequivalence of $M$ given by acting by $X$, that is $m(X)(M) = M \boxtimes X = M \boxtimes_{C^\otimes n} X$ for all $M \in M$. Then

$$\partial_n(m(X)) = m(\tilde{\partial}_n(X)). \quad (37)$$

Proof. That $m(X)$ is $C^\otimes n$-bilinear it follows from the fact that $C$ is braided. The left linearity is clear. The natural isomorphism $s_{MY} : m(X)(M \boxtimes Y) \to m(X)(M) \boxtimes Y$ for any $M \in M$ and $Y \in C^\otimes n$ is given by $M \boxtimes \Phi_{Y,X} : M \boxtimes Y \boxtimes X \to M \boxtimes X \boxtimes Y$ (we omit the parentheses, since the associativity for the action holds). Then the coherence diagram for $s_{MY}$ holds by one of the braiding axioms. Set $X = X^1 \boxtimes X^2 \boxtimes \cdots \boxtimes X^n$. First note that $\tilde{\partial}_n(X) = X_1 \boxtimes_{C^\otimes (n+1)} X_2^{-1} \boxtimes_{C^\otimes (n+1)} \cdots \boxtimes_{C^\otimes (n+1)} X_n^{-1}$, where we applied Remark 2.2 for the tensor product in $C^\otimes (n+1)$. Hence we have:

$$m(\tilde{\partial}_n(X)) = m(I \boxtimes X) \boxtimes_{C^\otimes (n+1)} \cdots \boxtimes_{C^\otimes (n+1)} m(X^1 \boxtimes \cdots \boxtimes X^{i-1} \boxtimes I \boxtimes X^i \boxtimes \cdots \boxtimes X^n)(-1)^{-1} \boxtimes_{C^\otimes (n+1)} \cdots \boxtimes_{C^\otimes (n+1)} m((X \boxtimes I)^{-1})^{-1}$$

$$= m(X \boxtimes \boxtimes_{C^\otimes (n+1)} \cdots \boxtimes_{C^\otimes (n+1)} m(X^1 \boxtimes \cdots \boxtimes X^{i-1}) \boxtimes_{C^\otimes (n+1)} \cdots \boxtimes_{C^\otimes (n+1)} m(X^n)^{-1} \boxtimes_{C^\otimes (n+1)} \cdots \boxtimes_{C^\otimes (n+1)} m((X \boxtimes I)^{-1})^{-1}$$

$$= m(X) \boxtimes_{C^\otimes (n+1)} \cdots \boxtimes_{C^\otimes (n+1)} m(X)^{-1} \boxtimes_{C^\otimes (n+1)} \cdots \boxtimes_{C^\otimes (n+1)} m(X)^{-1}$$

$$= \delta_n(m(X)). \quad \square$$

5.1 The Picard category of a symmetric finite tensor category and the Amitsur cohomology

Throughout this subsection let $C$ be a symmetric finite tensor category. Then $C^\otimes n$ is a symmetric finite tensor category and the category $\overline{\text{Pic}(C^\otimes n)}$ is symmetric monoidal with
the tensor product $\boxtimes_{C^\otimes_n}$. The objects of $\underline{\text{Pic}}(C^\otimes_n)$ are exact invertible one-sided $C^\otimes_n$-bimodule categories $\mathcal{M}$ so that there are equivalence functors $\mathcal{M} \boxtimes_{C^\otimes_n} \mathcal{M}^{op} \simeq C^\otimes_n$ (and $\mathcal{M}^{op} \boxtimes_{C^\otimes_n} \mathcal{M} \simeq C^\otimes_n$).

Let us consider the functors $E^n_i : \underline{\text{Pic}}(C^\otimes_n) \to \underline{\text{Pic}}(C^\otimes(n+1))$ for $i = 1, \ldots, n + 1$ given by
\[
E^n_i(M) = M_i = \mathcal{M} \boxtimes_{C^\otimes_n} e_i^\ast C^\otimes(n+1)
\]
and
\[
E^n_i(F) = F_i = F \boxtimes_{C^\otimes_n} e_i^\ast C^\otimes(n+1)
\]
for every object $M$ and every functor $F$ in $\underline{\text{Pic}}(C^\otimes_n)$, with $e_i^\ast$'s from (35).

**Lemma 5.6** For $i \geq j \in \{1, \ldots, n + 1\}$ and $M \in \underline{\text{Pic}}(C^\otimes_n)$, we have a natural isomorphism:
\[
\mathcal{M}_{ij} \cong \mathcal{M}_{j(i+1)}.
\]

**Proof.**
\[
\mathcal{M}_{ij} = (\mathcal{M} \boxtimes_{C^\otimes_n} e_i^\ast C^\otimes(n+1)) \boxtimes_{C^\otimes(n+1)} e_{i+1}^\ast C^\otimes(n+2)
\]
\[
\cong \mathcal{M} \boxtimes_{C^\otimes_n} (e_{i+1}^\ast e_i^\ast) C^\otimes(n+2) \boxtimes_{C^\otimes(n+1)} \mathcal{M} \boxtimes_{C^\otimes_n} (e_{i+1}^\ast e_i^\ast) C^\otimes(n+2)
\]
\[
\cong (\mathcal{M} \boxtimes_{C^\otimes_n} e_i^\ast C^\otimes(n+1)) \boxtimes_{C^\otimes(n+1)} e_{i+1}^\ast C^\otimes(n+2) = \mathcal{M}_{j(i+1)}.
\]

Now for every non-zero $n \in \mathbb{N}$, we define a functor
\[
\delta_n : \underline{\text{Pic}}(C^\otimes_n) \to \underline{\text{Pic}}(C^\otimes(n+1)),
\]
by
\[
\delta_n(M) = M_1 \boxtimes_{C^\otimes(n+1)} M_2^{op} \boxtimes_{C^\otimes(n+1)} \cdots \boxtimes_{C^\otimes(n+1)} \mathcal{N}_{n+1},
\]
\[
\delta_n(F) = F_1 \boxtimes_{C^\otimes(n+1)} (F_2^{op})^{-1} \boxtimes_{C^\otimes(n+1)} \cdots \boxtimes_{C^\otimes(n+1)} (G_{n+1})^{\pm 1},
\]
with $\mathcal{N} = \mathcal{M}$ or $\mathcal{M}^{op}$ and $G = F$ or $F^{op}$ depending on whether $n$ is even or odd. Up to the permutation of the factors in the Deligne tensor product - we use the fact that $\underline{\text{Pic}}(C^\otimes_n)$ is symmetric - it is clear that the functor $\delta_n$ is monoidal.

**Remark 5.7** Throughout we will use similar switch functor isomorphisms in the identities.

Computations similar to the computations in the proof of the previous lemma show that:
\[
\delta_{n+1} \delta_n(M) = (\boxtimes_{C^\otimes(n+2)})^{n+2}_{j=2} (\boxtimes_{C^\otimes(n+2)})^{-1}_{j=1} (\mathcal{M}_{ij} \boxtimes_{C^\otimes(n+2)} \mathcal{M}_{ij}^{op}),
\]
\[
\delta_{n+1} \delta_n(F) = (\boxtimes_{C^\otimes(n+2)})^{n+2}_{j=2} (\boxtimes_{C^\otimes(n+2)})^{-1}_{j=1} (F_{ij} \boxtimes_{C^\otimes(n+2)} (F_{ij}^{op})^{-1}),
\]
so we have a natural equivalence:
\[
\lambda_M = (\boxtimes_{C^\otimes(n+2)})^{n+2}_{j=2} (\boxtimes_{C^\otimes(n+2)})^{-1}_{j=1} e_{ij} \mathcal{M}_{ij} : \delta_{n+1} \delta_n(M) \to C^\otimes(n+2).
\]

(Here $\lambda_M$ is an equivalence by Proposition 4.7.) Similarly, one gets:
\[
\delta_{n+2}(\lambda_M) = \lambda_{\delta_n(M)}.
\]
Remark 5.8 For $\mathcal{M}, \mathcal{N} \in \text{Pic}(\mathcal{C}^{2n})$ it is: $\lambda_{\mathcal{M} \otimes \mathcal{N}^{\mathcal{G}_{2n}}} \simeq \lambda_{\mathcal{M}} \otimes \mathcal{C}^{\mathcal{G}_{2(n+2)}} \lambda_{\mathcal{N}}$, or which is the same:

$$
(\mathcal{C}^{\mathcal{G}_{2n}})^{n+2} \mathcal{M} \otimes^{\mathcal{G}_{2(n+2)}} \mathcal{N} = (\mathcal{C}^{\mathcal{G}_{2(n+2)}})^{n+2} \mathcal{M} \otimes^{\mathcal{G}_{2(n+2)}} \mathcal{N}.
$$

To see this it is equivalent to prove that $ev_{\mathcal{M} \otimes \mathcal{N}} \simeq ev_{\mathcal{M}} \otimes ev_{\mathcal{N}}$ for $\mathcal{M}, \mathcal{N} \in \text{Pic}(\mathcal{C})$. The functor $ev_{\mathcal{M} \otimes \mathcal{N}}$ is defined through the commutative diagram (3) below:

![Diagram](image)

The diagram (1) commutes by naturality of the braiding $\tau$ in $(\mathcal{C}^{br}, \text{Mod}, \mathcal{C}, \mathcal{C})$, and (2) commutes obviously. Then the commutativity of the outer diagram yields: $ev_{\mathcal{M} \otimes \mathcal{N}} = ev_{\mathcal{M} \otimes \mathcal{N}^{\mathcal{G}_{2n}}} (\mathcal{M} \otimes \mathcal{N})^{\mathcal{G}_{2(n+2)}}$, or $ev_{\mathcal{M} \otimes \mathcal{N}} = ev_{\mathcal{M}} \otimes ev_{\mathcal{N}}$ up to the switch isomorphism functor.

Observe that we also have:

$$
\delta_{n+1} \delta_n (ev_{\mathcal{M}}) = (\mathcal{C}^{\mathcal{G}_{2n}})^{n+2} (\mathcal{C}^{\mathcal{G}_{2n}})^{n+2} \mathcal{M} \otimes^{\mathcal{G}_{2(n+2)}} \mathcal{M} \otimes^{\mathcal{G}_{2(n+2)}} \mathcal{N} = (\mathcal{C}^{\mathcal{G}_{2(n+2)}})^{n+2} \mathcal{M} \otimes^{\mathcal{G}_{2(n+2)}} \mathcal{N}.
$$

For a consequence of Lemma 4.10 1), we have:

**Corollary 5.9** For any $\mathcal{M} \in \text{Pic}(\mathcal{C}^{2n})$ and a $\mathcal{C}^{2n}$-bimodule equivalence $\alpha : \mathcal{M} \rightarrow \mathcal{C}^{2n}$ it is $\lambda_{\mathcal{M}} = \delta_{n+1} \delta_n (\alpha)$.

**Proof.** We compute:

$$
\lambda_{\mathcal{M}} = (\mathcal{C}^{\mathcal{G}_{2n}})^{n+2} (\mathcal{C}^{\mathcal{G}_{2n}})^{n+2} \mathcal{M} = (\mathcal{C}^{\mathcal{G}_{2(n+2)}})^{n+2} \mathcal{M} = \delta_{n+1} \delta_n (\alpha).
$$

When $\mathcal{M} = \mathcal{C}^{2n}$ and $\alpha = \text{Id}$, one gets:

$$
\lambda_{\mathcal{C}^{2n}} = \mathcal{C}^{\mathcal{G}_{2(n+2)}}.
$$

We define $\mathcal{C}^{\mathcal{G}_{2n}}(\mathcal{C}, \text{Pic})$ to be the category with objects $(\mathcal{M}, \alpha)$, with $\mathcal{M} \in \text{Pic}(\mathcal{C}^{2n})$, and $\alpha : \delta_n (\mathcal{M}) \rightarrow \mathcal{C}^{\mathcal{G}_{(n+1)}}$ an equivalence of $\mathcal{C}^{\mathcal{G}_{(n+1)}}$-module categories so that $\delta_{n+1} (\alpha) = \lambda_{\mathcal{M}}$. A morphism $(\mathcal{M}, \alpha) \rightarrow (\mathcal{N}, \beta)$ is an equivalence of $\mathcal{C}^{2n}$-module categories $F : \mathcal{M} \rightarrow \mathcal{N}$.
such that $\beta \circ \delta_1(F) = \alpha$. Then $Z^n(C, \text{Pic})$ is a symmetric monoidal category, with tensor product $(\mathcal{M}, \alpha) \otimes (\mathcal{N}, \beta) = (\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}, \alpha \boxtimes_{\mathcal{C}} \mathcal{N} + 1, \beta)$ and unit object $(\mathcal{C}, \mathcal{C}(n+1))$. Note that every object in this category is invertible, so we can consider the Grothendieck group:

$$K_0 Z^n(C, \text{Pic}) = Z^n(C, \text{Pic}).$$

There is a strongly monoidal functor

$$d_{n-1} : \text{Pic}(C^{2n(n-1)}) \to Z^n(C, \text{Pic}),$$

$$d_{n-1}(\mathcal{N}) = (\delta_{n-1}(\mathcal{N}), \lambda_\mathcal{N}).$$

Let $B^n(C, \text{Pic})$ be the subgroup of $Z^n(C, \text{Pic})$, consisting of elements represented by $d_{n-1}(\mathcal{N})$, where $\mathcal{N} \in \text{Pic}(C^{2n(n-1)})$. We then define:

$$H^n(C, \text{Pic}) = Z^n(C, \text{Pic})/B^n(C, \text{Pic}).$$

**Remark 5.10** Observe that for $(\mathcal{L}, \alpha), (\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}, \beta) \in Z^1(C, \text{Pic})$ we have:

$$(\mathcal{L}, \alpha) \cong (\mathcal{L} \boxtimes_{\mathcal{C}} \mathcal{C}, \alpha \boxtimes_{\mathcal{C}} \mathcal{C}) \quad \text{and} \quad (\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}, \beta) \cong (\mathcal{N} \boxtimes_{\mathcal{C}} \mathcal{M}, \beta \circ \delta_1(\tau))$$

where the equivalences inducing the corresponding isomorphisms are the obvious ones. This clearly extends to the similar properties in $Z^n(C, \text{Pic})$ for all $n \in \mathbb{N}$. Thus the omitting of these equivalence functors, which we applied so far when computing in $\text{Pic}(C)$, is justified also when passing to to the category $Z^n(C, \text{Pic})$.

### 6 Categorical Villamayor-Zelinsky sequence

This section is dedicated to the construction of the infinite exact sequence of the type Villamayor-Zelinsky. Originally, it involved Amitsur cohomology groups for commutative algebras over fields. We construct a version of it for symmetric finite tensor categories. In the proof of Theorem 6.2 we will frequently use the following result.

**Proposition 6.1** Given a finite tensor category $C$ and an invertible $C$-bimodule category $\mathcal{M}$, every $C$-module autoequivalence $F \in \text{Fun}_C(\mathcal{M}, \mathcal{M})$ is of the form $F \cong - \otimes X$ for some $X \in \text{Inv}(C)$.

**Proof.** Since $\mathcal{M}$ is invertible, by [14, Proposition 4.2] any $F \in \text{Fun}_C(\mathcal{M}, \mathcal{M})$ is of the form $F \cong - \otimes X$ for some $X \in C$. If $F$ is an equivalence, then clearly $X \in \text{Inv}(C)$. \qed

Observe that in the way the notation is fixed in the following result the morphisms $\alpha_1$ and $\beta_1$ are trivial maps, as $H^0(C, \text{Pic})$ is the trivial group ($\text{Pic}(k) = 0$).

**Theorem 6.2** Let $C$ be a symmetric finite tensor category. There is a long exact sequence

$$1 \longrightarrow H^2(C, \text{Inv}) \overset{\alpha_2}{\longrightarrow} H^1(C, \text{Pic}) \overset{\beta_2}{\longrightarrow} H^1(C, \text{Pic}) \longrightarrow$$

$$\overset{\gamma_2}{\longrightarrow} H^3(C, \text{Inv}) \overset{\alpha_3}{\longrightarrow} H^2(C, \text{Pic}) \overset{\beta_3}{\longrightarrow} H^2(C, \text{Pic}) \longrightarrow$$

$$\overset{\gamma_3}{\longrightarrow} H^4(C, \text{Inv}) \overset{\alpha_4}{\longrightarrow} H^3(C, \text{Pic}) \overset{\beta_4}{\longrightarrow} H^3(C, \text{Pic}) \longrightarrow \cdots$$

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Proof. Definition of $\alpha_n$. Let $X \in Z^n(C, \text{Inv})$. Then $(C^\otimes(n-1), m(X)) \in Z^{n-1}(C, \text{Pic})$, since

$$\delta_n(m(X)) = m(\delta_n(X)) = m(I^\otimes(n+1)) = C^\otimes(n+1) \overset{15}{\cong} \lambda_{C^\otimes(n-1)}.$$ 

In the case that $X$ is a coboundary: $X = \delta_{n-1}(Y)$ for some $Y \in \text{Inv}(C^\otimes(n-1))$, we have that $m(Y) : C^\otimes(n-1) \to C^\otimes(n-1)$ is an isomorphism between $(C^\otimes(n-1), m(\delta_{n-1}(Y)))$ and $(C^\otimes(n-1), C^\otimes(n))$ because of (13). Then the map

$$\alpha_n([X]) = [(C^\otimes(n-1), m(X))]$$

is well-defined ($\alpha_n$ is clearly a group map).

Definition of $\beta_n$. We define $\beta_n([M, \alpha]) = [M]$. 

Definition of $\gamma_n$. Let $[M] \in Z^{n-1}(C, \text{Pic})$. Then there exists a $C^\otimes n$-module equivalence $\alpha : \delta_{n-1}(M) \to C^\otimes n$. The composition $\lambda_M \circ \delta_n(\alpha)^{-1} : C^\otimes(n+1) \to C^\otimes(n+1)$ is an equivalence of $C^\otimes(n+1)$-module categories, so it is equal to $m(X)$ for some $X \in \text{Inv}(C^\otimes(n+1))$. Now:

$$\begin{align*}
m(\delta_{n+1}(X)) &\overset{13, \text{Cor.} 5.9}{=} \delta_{n+1}(m(X)) = \delta_{n+1}(\lambda_M) \circ ((\delta_{n+1} \circ \delta_n)(\alpha))^{-1} \\
&= \lambda_{\delta_{n+1}(M)} \circ \lambda_{\delta_{n-1}(M)}^{-1} = C^\otimes(n+2),
\end{align*}$$

so $\delta_{n+1}(X) = I^\otimes(n+2)$, and $X \in Z^{n+1}(C, \text{Inv})$.

If $\alpha' : \delta_{n-1}(M) \to C^\otimes n$ is another $C^\otimes n$-module equivalence, then we have $\lambda_M \circ \delta_n(\alpha)^{-1}(X') \to C^\otimes(n+1)$. Then $\alpha' \circ \alpha^{-1}$ is a $C^\otimes n$-module autoequivalence of $C^\otimes n$, so $\alpha' \circ \alpha^{-1} = m(Z^{-1})$, for some $Z \in \text{Inv}(C^\otimes n)$. Now we have

$$m(X') = \lambda_M \circ \delta_n(\alpha')^{-1} = \lambda_M \circ \delta_n(\alpha)^{-1} \circ \delta_n(m(Z)) \overset{13}{=} m(X) \circ m(\delta_{n-1}(Z)) = m(X\delta_n(Z)),$$

yielding $X' = X\delta_n(Z)$, so $[X] = [X']$ in $H^{n+1}(C, \text{Inv})$. Thus we have a well-defined map $Z^{n-1}(C, \text{Pic}) \to H^{n+1}(C, \text{Inv})$ given by $[M] \mapsto [X]$ (such that $m(X) = \lambda_M \circ \delta_n(\alpha)^{-1}$ for some $C^\otimes n$-module equivalence $\alpha : \delta_{n-1}(M) \to C^\otimes n$).

This map induces a map $\gamma_n : H^{n-1}(C, \text{Pic}) \to H^{n+1}(C, \text{Inv})$. Indeed, if $[M] \in B^{n-1}(C, \text{Pic})$ so that $M = \delta_{n-1}(N)$ for some $N \in \text{Pic}(C^\otimes(n-2))$, then $\lambda_M : \delta_{n-1}(M) \to C^\otimes n$ is a $C^\otimes n$-module equivalence and as above we have for some $Y \in \text{Inv}(C^\otimes(n+1))$:

$$m(Y) = \lambda_M \circ \delta_n(\lambda_N)^{-1} = \lambda_M \circ \lambda_{\delta_{n-2}(N)}^{-1} = m(I^\otimes(n+1)),$$

hence $Y = I^\otimes(n+1)$.

Exactness at $H^{n-1}(C, \text{Pic})$. It is clear that $\beta_n \circ \alpha_n = 1$.

Take $[(M, \alpha)] \in H^{n-1}(C, \text{Pic})$ such that $\beta_n([(M, \alpha)] = [M] = 1$ in $H^{n-1}(C, \text{Pic})$. Then $M \simeq \delta_{n-2}(N)$ for some $N \in \text{Pic}(C^\otimes(n-2))$. The composition

$$\lambda_N^{-1} \circ \alpha : (\delta_{n-1} \circ \delta_{n-2})(N) \to (\delta_{n-1} \circ \delta_{n-2})(N)$$

is a $C^\otimes n$-module equivalence, so it is given by $m(X)$ for some $X \in \text{Inv}(C^\otimes n)$. Then it follows:

$$m(\delta_n(X)) = \delta_n(\lambda_N)^{-1} \circ \delta_n(\alpha) = \lambda_{\delta_{n-2}(N)}^{-1} \circ \lambda_M = C^\otimes(n-1)$$

meaning that $\delta_n(X) = I^\otimes(n+1)$ and $X \in H^n(C, \text{Inv})$.

To prove that $[(M, \alpha)]$ is in the image of $\alpha_n$, we will need the following:
Claim 6.3 Let $\mathcal{M}, \alpha$ and $X$ be as above. Then $m(X) \circ \delta_{n-1}(\overline{\nu}_M) = \lambda_{\mathcal{N}^\mathrm{op}} \boxtimes \mathbb{C}^{\otimes n} \alpha$. Consequently, $\overline{\nu}_M : \mathcal{M}^{\mathrm{op}} \boxtimes \mathbb{C}^{\otimes (n-1)} \mathcal{M} \to \mathcal{C}^{\otimes (n-1)}$ is an isomorphism

$$(\mathcal{M}^{\mathrm{op}} \boxtimes \mathbb{C}^{\otimes (n-1)} \mathcal{M}, \lambda_{\mathcal{N}^\mathrm{op}} \boxtimes \mathbb{C}^{\otimes n} \alpha) \to (\mathbb{C}^{\otimes (n-1)}, m(X))$$

in $\mathbb{Z}^{n-1}(\mathcal{C}, \mathrm{Pic})$.

Proof. First of all, observe that $(\mathcal{M}^{\mathrm{op}} \boxtimes \mathbb{C}^{\otimes (n-1)} \mathcal{M}, \lambda_{\mathcal{N}^\mathrm{op}} \boxtimes \mathbb{C}^{\otimes n} \alpha) = (\delta_{n-2}(\mathcal{N}^\mathrm{op}), \lambda_{\mathcal{N}^\mathrm{op}}) \otimes (\mathcal{M}, \alpha) \in \mathbb{Z}^{n-1}(\mathcal{C}, \mathrm{Pic})$. Now from (14) and Remark 5.8 it follows that:

$$\delta_{n-1}(\overline{\nu}_M) = 
\left((\mathbb{C}^{\otimes n})_{j=2}^n ev_{\mathcal{N}^\mathrm{op}} \right) \boxtimes \mathbb{C}^{\otimes n} \left((\mathbb{C}^{\otimes n})_{j=2}^n (\mathbb{C}^{\otimes n})_{j=1}^{j-1} ev_{\mathcal{N}^\mathrm{op}} \right).$$

Thus:

$$\delta_{n-1}(\overline{\nu}_M)(id_{\delta_{n-1}(\mathcal{M}^{\mathrm{op}})} \boxtimes \lambda_{\mathcal{N}^\mathrm{op}}) = \left((\mathbb{C}^{\otimes n})_{j=2}^n (\mathbb{C}^{\otimes n})_{j=1}^{j-1} ev_{\mathcal{N}^\mathrm{op}} \right) \boxtimes \mathbb{C}^{\otimes n} \left((\mathbb{C}^{\otimes n})_{j=2}^n (\mathbb{C}^{\otimes n})_{j=1}^{j-1} ev_{\mathcal{N}^\mathrm{op}} \right) \circ (id_{\delta_{n-1}(\mathcal{M}^{\mathrm{op}})} \boxtimes \lambda_{\mathcal{N}^\mathrm{op}})$$

$$= \left((\mathbb{C}^{\otimes n})_{j=2}^n (\mathbb{C}^{\otimes n})_{j=1}^{j-1} ev_{\mathcal{N}^\mathrm{op}} \right) \boxtimes \mathbb{C}^{\otimes n} \mathcal{C}^{\otimes n} = \lambda_{\mathcal{N}^\mathrm{op}} \boxtimes \mathbb{C}^{\otimes n} \mathcal{C}^{\otimes n}.$$

On the other hand, we have that $\delta_{n-1}(ev_M)$ is right $\mathcal{C}^{\otimes n}$-linear, since so is $ev_M$. This implies the first identity in the following computation:

$m(X) \circ \delta_{n-1}(ev_M) = \delta_{n-1}(ev_M)(Id_{\delta_{n-1}(\mathcal{M}^{\mathrm{op}})} \boxtimes \mathbb{C}^{\otimes n} m(X))$

$$= \delta_{n-1}(ev_M)(Id_{\delta_{n-1}(\mathcal{M}^{\mathrm{op}})} \boxtimes \mathbb{C}^{\otimes n} \lambda_{\mathcal{N}^\mathrm{op}})(Id_{\delta_{n-1}(\mathcal{M}^{\mathrm{op}})} \boxtimes \mathbb{C}^{\otimes n} \alpha)$

$$= (\lambda_{\mathcal{N}^\mathrm{op}} \boxtimes \mathbb{C}^{\otimes n} \mathcal{C}^{\otimes n})(Id_{\delta_{n-1}(\mathcal{M}^{\mathrm{op}})} \boxtimes \mathbb{C}^{\otimes n} \alpha)$

$$= \lambda_{\mathcal{N}^\mathrm{op}} \boxtimes \mathbb{C}^{\otimes n} \alpha.$$

Observe that $[(\delta_{n-2}(\mathcal{N}^\mathrm{op}), \lambda_{\mathcal{N}^\mathrm{op}})] = 1$ in $H^{n-1}(\mathcal{C}, \mathrm{Pic})$. Now we have:

$$[(\mathcal{M}, \alpha)] = [(\delta_{n-2}(\mathcal{N}^\mathrm{op}), \lambda_{\mathcal{N}^\mathrm{op}})][(\mathcal{M}, \alpha)]$$

$$= [(\mathcal{M}^{\mathrm{op}} \boxtimes \mathbb{C}^{\otimes (n-1)} \mathcal{M}, \lambda_{\mathcal{N}^\mathrm{op}} \boxtimes \mathbb{C}^{\otimes n} \alpha)] = [(\mathbb{C}^{\otimes (n-1)}, m(X))] = \alpha_n([X]).$$

Exactness at $H^{n-1}(\mathcal{C}, \mathrm{Pic})$. Let $[(\mathcal{M}, \alpha)] \in H^{n-1}(\mathcal{C}, \mathrm{Pic})$, then $\beta_n[(\mathcal{M}, \alpha)] = [\mathcal{M}]$. In order to compute $\gamma_n([\mathcal{M}])$, we choose the $\mathbb{C}^{\otimes n}$-module equivalence $\alpha : \delta_{n-1}(\mathcal{M}) \to \mathbb{C}^{\otimes n}$. We have $\delta_n(\alpha) = \lambda_{\mathcal{M}}$, so:

$$\delta_n(\alpha) \circ \lambda_{\mathcal{M}}^{-1} = m(I^{\otimes (n+1)}),$$

and thus $\gamma_n \circ \beta_n = 1$. Now, assume for $[\mathcal{M}] \in H^{n-1}(\mathcal{C}, \mathrm{Pic})$ that $\gamma_n([\mathcal{M}]) = 1$. Then there is a $\mathbb{C}^{\otimes n}$-module equivalence $\alpha : \delta_{n-1}(\mathcal{M}) \to \mathbb{C}^{\otimes n}$ such that $\delta_n(\alpha) \circ \lambda_{\mathcal{M}}^{-1} = m(X)$ for some $X \in \mathrm{Inv}(\mathbb{C}^{\otimes (n+1)})$, and we know that $\gamma_n([\mathcal{M}]) = [X]$, so $X \in B^{n+1}(\mathcal{C}, \mathrm{Inv})$. Then $X = \delta_n(Y)$, for some $Y \in \mathrm{Inv}(\mathbb{C}^{\otimes n})$. Consider the $\mathbb{C}^{\otimes n}$-module equivalence

$$\alpha' = m(Y^{-1}) \circ \alpha : \delta_{n-1}(\mathcal{M}) \to \mathbb{C}^{\otimes n}.$$

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Then: 
\[ \delta_n(\alpha') \circ \lambda^{-1}_M = \delta_n(m(Y^{-1})) \circ m(X) = m(\delta_n(Y^{-1})) \circ m(X) = C^{(n+1)} \]
therefore: \([\mathcal{M}, \alpha'] \in Z^{n-1}(\mathcal{C}, \text{Pic}), \text{ and } [\mathcal{M}] = \beta_n([\mathcal{M}, \alpha']).\]

Exactness at \(H^{n+1}(\mathcal{C}, \text{Inv})\). Take \([\mathcal{M}] \in H^{n-1}(\mathcal{C}, \text{Pic})\), and choose a \(C^{\mathfrak{g}}\)-module equivalence \(\alpha : \delta_{n-1}(\mathcal{M}) \to C^{\mathfrak{g}n}\). Then \(\gamma_n([\mathcal{M}]) = [X]\) for some \(X \in \text{Inv}(C^{\mathfrak{g}(n+1)})\), where \(m(X) = \lambda_\mathcal{M} \circ \delta_n(\alpha)^{-1}\), and we have: \((\alpha_{n+1} \circ \gamma_n)([\mathcal{M}]) = [(C^{\mathfrak{g}n}, m(X))].\) It is immediate that \(\alpha\) defines an isomorphism
\[ (\delta_{n-1}(\mathcal{M}), \lambda_\mathcal{M}) \to (C^{\mathfrak{g}n}, m(X)) \]
in \(Z^n(\mathcal{C}, \text{Pic}).\) It follows that \([(C^{\mathfrak{g}n}, m(X))] \in B^n(\mathcal{C}, \text{Pic}), \) and thus \(\alpha_{n+1} \circ \gamma_n = 1.\)

Take \(X \in Z^{n+1}(\mathcal{C}, \text{Inv}),\) and assume that
\[ (C^{\mathfrak{g}n}, m(X)) \cong d_{n-1}(\mathcal{N}) = (\delta_{n-1}(\mathcal{N}), \lambda_\mathcal{N}) \]
in \(Z^n(\mathcal{C}, \text{Pic})\) for some \(\mathcal{N} \in \text{Pic}(C^{\mathfrak{g}(n-1)}).\) This means that there is a \(C^{\mathfrak{g}n}\)-module equivalence \(\alpha : \delta_{n-1}(\mathcal{N}) \to C^{\mathfrak{g}n}\) such that \(\lambda_\mathcal{N} = m(X) \circ \delta_n(\alpha).\) Thus \(\gamma_n([\mathcal{N}]) = [X].\)

\(\alpha_{2}\) is injective. Take \(X \in Z^2(\mathcal{C}, \text{Inv}),\) and suppose that \(\alpha_2([X]) = [(\mathcal{C}, m(X))] = [(\mathcal{C}, m(I \boxtimes I))].\) Then there exists a \(\mathcal{C}\)-module autoequivalence \(\alpha : \mathcal{C} \to \mathcal{C}\) such that \(m(X) = m(I \boxtimes I) \circ \delta_1(\alpha).\) Moreover, \(\alpha\) is given by tensoring by some \(Y \in \text{Inv}(\mathcal{C}),\) so: \(\delta_1(\alpha) = \delta_1(m(Y)) = m(\delta_1(Y)).\) For a consequence we get: \(X = \delta_1(Y) \in B^2(\mathcal{C}, \text{Inv}).\)

\[\square\]

### 6.1 Examples of symmetric finite tensor categories and their Picard groups

Since any finite tensor category is equivalent to a category of finite-dimensional representations of a finite-dimensional weak quasi Hopf algebra, \([15, \text{Proposition 2.7}]\), then every symmetric finite tensor category is equivalent to a category \(\text{Rep} H\) of finite-dimensional representations of a finite-dimensional triangular weak quasi-Hopf algebra \(H.\) We will say that a finite tensor category is **strong** if it is equivalent to \(\text{Rep} H,\) where \(H\) is a Hopf algebra (in the style of \([24]\)).

It is known that every finite-dimensional triangular Hopf algebra over an algebraically closed field of characteristic zero is the Drinfel’d twist of a modified supergroup algebra, \([2, \text{Theorem 5.1.1}], [12, \text{Theorem 4.3}]\). A finite-dimensional triangular Hopf algebra \(H\) with an \(R\)-matrix \(\mathcal{R}\) is called a **modified supergroup algebra** if there exist:

1. a finite group \(G;\)
2. a central element \(u \in G\) with \(u^2 = 1;\)
3. a linear representation of \(G\) on a finite-dimensional vector space \(V\) on which \(u\) acts as \(-1\).
such that $H \cong \Lambda(V) \times kG$ as the Radford biproduct Hopf algebra, where the elements in $G$ are group-like and the elements in $V$ are $(u, 1)$-primitive. Namely, the action of $G$ on $V$ makes the exterior algebra $\Lambda(V)$ into a $kG$-module algebra and we can construct the smash product $\Lambda(V) \# kG$. The element of $kG \otimes kG$:

$$\mathcal{R} = \mathcal{R}_u = \frac{1}{2}(1 \otimes 1 + u \otimes 1 + 1 \otimes u - u \otimes u)$$

is a triangular structure on $kG$ and $\Lambda(V)$ is a Hopf algebra in $kG\mathcal{M}$, by defining

$$\Delta(v) = 1 \otimes v + v \otimes 1, \quad \varepsilon(v) = 0 \quad \text{and} \quad S(v) = -v.$$ 

As a matter of fact, $\Lambda(V)$ is a Yetter Drinfel’d module algebra over $kG$ with the coaction induced by $\Lambda(v) = u \otimes v$, so that $\Lambda(V)$ is indeed a Hopf algebra in $H^*_H \mathcal{YD}$. The triangular structure $\mathcal{R}_u$ extends to the triangular structure of $\Lambda(V) \times kG$.

The Hopf subalgebra of the Radford biproduct $\Lambda(V) \times kG$ which is generated by $u$ and by the $(u, 1)$-primitive elements of $V$ is isomorphic, as a triangular Hopf algebra, to Nichols Hopf algebra $E(n) \cong \Lambda(n) \times k\mathbb{Z}_2$, where $n = \dim(V)$, with the triangular structure $\mathcal{R}_u$. The Nichols Hopf algebra $E(n)$ is a modified supergroup algebra whose representation category is the most general non-semisimple symmetric finite tensor category whithout non-trivial Tannakian subcategories. (A symmetric fusion category is called Tannakian if it is equivalent to Rep$G$, the category of representations of a finite group $G$.) For $n = 1$ we obtain the Sweedler Hopf algebra $H_4$.

The Brauer-Picard group is computed for a number of categories. In [14] it is done for the representation category of any finite abelian group, and in [24] this result is extended to a number of finite groups; in [26] the Brauer-Picard group is computed for the representation category of a modified supergroup algebra, whereas in [3] it is computed for the Nichols Hopf algebra $E(n)$; we also mention [20].

The subgroup of the Brauer-Picard group of the representation category of a modified supergroup algebra $\Lambda(V) \times kG$ determined by the one-sided bimodule categories over $\mathcal{C} = \text{Rep}(\Lambda(V) \times kG)$ is the Picard group of $\mathcal{C}$. In view of the above said, this subgroup is one of the central protagonists of this article, as such $\mathcal{C}$ is the most general symmetric finite tensor category that is strong. It was computed in [19, Corollary 8.10] that $\text{Pic}(\text{Rep}G) \cong H^2(G, k^\times)$ for any finite group $G$.

In [10, Proposition 3.7] it is proved that for a finite-dimensional quasi-triangular Hopf algebra $H$ every $H$-Azumaya algebra in the Brauer group $\text{BM}(k, H, \mathcal{R})$ is exact. This allows one to conclude that $\text{Pic}(\text{Rep}H) = \text{BM}(k, H, \mathcal{R})$, in view of [9, Section 3.2]. The latter group has extensively been studied, in particular for the Radford biproduct Hopf algebras $H = B \times L$. A deep insight about the decomposition $\text{BM}(k, B \times L, \mathcal{R}) \cong \text{BM}(k, L, \mathcal{R}) \times \text{Gal}(B; L\mathcal{M})$, where $L$ is a quasi-triangular Hopf algebra whose quasi-triangular structure $\mathcal{R}$ extends to that on $H$ (denoted by $\mathcal{R}$) and $B$ is a commutative (and cocommutative) Hopf algebra in $L\mathcal{M}$, is given in [3, Theorem 6.5]. Here $\text{Gal}(B; L\mathcal{M})$ is the group of $B$-Galois objects in $L\mathcal{M}$, which are one-sided comodules over $B$ and $B$ is cocommutative in $L\mathcal{M}$. Observe that this is precisely the case in the modified supergroup algebras. Henceforth, we may write:

$$\text{Pic}(\mathcal{C}) \cong \text{Pic}(\text{Rep}(\Lambda(V) \times kG)) \cong \text{BM}(k, kG, \mathcal{R}_u) \times \text{Gal}(\Lambda(V); kG\mathcal{M})$$

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for every symmetric finite tensor category $\mathcal{C}$ that is strong. Here $\text{BM}(k, kG, \mathcal{R}_u)$ is the Brauer group of $G$-graded vector spaces with respect to the braiding induced by $\mathcal{R}_u$. Moreover, from the direct sum decomposition proved in [5] we also have:

$$\text{Pic}(\mathcal{C}) \cong \text{BM}(k, kG, \mathcal{R}_u) \times S^2(V^*)^G$$

where $S^2(V^*)^G$ is the group of symmetric matrices over $V^*$ invariant under the conjugation by elements of $G$.

In particular, for the Picard group of the representation categories of triangular Hopf algebras $H_4$ and $E(n)$ mentioned above we obtain from [6] and [28]:

$$\text{Pic}(\text{Rep}(H_4)) \cong \text{BW}(k) \times (k, +) \quad \text{and} \quad \text{Pic}(\text{Rep}(E(n)) \cong \text{BW}(k) \times (k, +)^{n(n+1)/2}$$

where $\text{BW}(k)$ denotes the Brauer-Wall group (the corresponding Azumaya algebras are $\mathbb{Z}_2$-graded). It is known that $\text{BW}(\mathbb{C}) \cong \mathbb{Z}/2\mathbb{Z}$ and $\text{BW}(\mathbb{R}) \cong \mathbb{Z}/8\mathbb{Z}$. The factor $(k, +)^{n(n+1)/2}$ corresponds to $\text{Sym}_n(k)$, the group of $n \times n$ symmetric matrices under addition.

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