Special-series solution of the first-order linear vector differential equation

Xin-Bing Huang a, *

a Department of Physics, Peking University, Beijing 100871, China

Abstract

A special series is introduced in this paper to yield solution of the first-order linear vector differential equation. It is proved that if the differential equation satisfied by the first term of this series can be solved exactly, then other terms can be determined by the method of variation of parameters. We point out that the special series will be the solution of the first-order linear vector differential equation if the infinite special series converges. An illustrative example has been given to outline the procedure of our method.

Key words: special series, variation of parameters, first-order linear vector differential equation

1 Introduction

The content of linear systems constitutes a large and very important part in the theory of ordinary differential equations. From the early days of ordinary differential equations the subject of linear systems has been an area of great theoretical research and practical applications, and it continues to be so today. In physics, many dynamical systems or material fields can be treated as linear systems. For instance, the evolution equation of a spin-\(\frac{1}{2}\) fermion in Robertson-Walker space-time deduced from the covariant Dirac equation is a linear system [1]. In this paper we shall restrict our attention to linear systems of \(n\) differential equations in \(n\) unknown functions only.

* Corresponding author.

Email address: huangxb@pku.edu.cn (Xin-Bing Huang).
Almost in all books on ordinary differential equations, the authors always present a symbolic operator method for solving linear systems with constant coefficients. After introducing the so-called differential operators, the standard procedure for solving linear systems with constant coefficients will be described in any such book [2]. Therefore, we shall only focus on the linear systems whose coefficients are functions.

When the system of linear differential equations is of the type that have variable coefficients, the method of solving it depends upon the concrete form of the coefficients and the type of differential equations. One can try to solve this kind of linear systems by acquiring a higher order differential equation which can be solved [3,4], or to solve them approximately by various methods such as numerical methods, series methods, and graphical methods. In this paper We shall introduce a relatively general and systematic method for solving linear systems with variable coefficients.

The present paper is devoted to a method for obtaining solution of the first-order linear vector differential equation in special series form. We shall decompose the linear vector differential equation into a group of correlated vector differential equations. After this, we shall proceed to explain the method of special series. We shall then discuss the convergence of infinite special series, and point out the solution. Finally, we shall give an illustrative example to clarify the procedure of special-series method.

2 Decomposing the linear vector differential equation

We consider the normal form of linear system of \( n \) first-order differential equations in \( n \) unknown functions \( y_1, y_2, \ldots, y_n \). It is well known that this system is of the form

\[
y' = A(x)y + F(x),
\]

where we have introduced the notation \( y' \equiv \frac{dy}{dx} \), the vectors \( y \) and \( F(x) \) are defined respectively by

\[
y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \quad \text{and} \quad F(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix}
\]
and the matrix function $A(x)$ being defined by

$$
A(x) = \begin{pmatrix}
a_{11}(x) & a_{12}(x) & \cdots & a_{1n}(x) \\
a_{21}(x) & a_{22}(x) & \cdots & a_{2n}(x) \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1}(x) & a_{n2}(x) & \cdots & a_{nn}(x)
\end{pmatrix}
$$

(3)

We shall assume that all of the functions defined by $f_i(x), i = 1, 2, \ldots, n,$ and $a_{ij}(x), i = 1, 2, \ldots, n, j = 1, 2, \ldots, n,$ are continuous on a real interval $[a, b]$. If all $f_i(x) = 0, i = 1, 2, \ldots, n,$ for all $x$, then the system (1) is called homogeneous. Otherwise, the system is called nonhomogeneous.

Hence the normal form of linear system (1) is a nonhomogeneous linear vector differential equation, and the corresponding homogeneous linear vector differential equation reads

$$
y' = A(x)y
$$

(4)

Mathematician has developed the method of variation of parameters for finding a solution of the equation (1), assuming that one has known a fundamental matrix of the corresponding equation (4). Therefore, in this paper, we will focus on explaining the method of special series for finding the fundamental matrix of the homogeneous linear vector differential equation (4).

We shall assume that $A(x)$ is a nonsingular $n \times n$ matrix function. Then one can divide $A(x)$ into two $n \times n$ nonsingular matrices $H(x)$ and $Z(x)$. It is stressed that the elements of matrix function $Z(x)$ must be finite on the real interval $[a, b]$. That is,

$$
A(x) = H(x) + Z(x)
$$

(5)

Here $H(x)$ can also be a constant matrix. For brevity, we use the common sign $H(x)$ to denote both a matrix function defined on all real $x \in [a, b]$ and a constant matrix. Therefore, the equation (4) becomes

$$
y' = H(x)y + Z(x)y
$$

(6)
To find the fundamental matrix of the above equation, we shall assume that the vector function \( y \) is a special series written as follows

\[
y = \sum_{i=0}^{\infty} y_i .
\]  

(7)

And we shall assume that the first term \( y_0 \) of this series satisfies

\[
y_0' = H(x)y_0 .
\]  

(8)

Under the assumption that \( y \) is the sum of the first two terms, inserting \( y = y_0 + y_1 \) into the equation (6) yields

\[
y_0' + y_1' = H(x)(y_0 + y_1) + Z(x)(y_0 + y_1) .
\]  

(9)

With the help of the equation (8), and ignoring the term \( Z(x)y_1 \) in the equation (9), we obtain

\[
y_1' = H(x)y_1 + Z(x)y_0 .
\]  

(10)

Under the assumption that \( y = y_0 + y_1 + y_2 \), inserting \( y_0 + y_1 + y_2 \) into the equation (6) yields

\[
y_0' + y_1' + y_2' = H(x)(y_0 + y_1 + y_2) + Z(x)(y_0 + y_1 + y_2) .
\]  

(11)

With the help of the equations (8), (10), and ignoring the term \( Z(x)y_2 \) in the above equation, we obtain

\[
y_2' = H(x)y_2 + Z(x)y_1 .
\]  

(12)

Under the assumption that \( y = \sum_{j=0}^{m} y_j \), inserting \( \sum_{j=0}^{m} y_j \) into the equation (6) yields

\[
\sum_{j=0}^{m} y_j' = H(x)\sum_{j=0}^{m} y_j + Z(x)\sum_{j=0}^{m} y_j .
\]  

(13)

Using the same method as before, and ignoring the term \( Z(x)y_m \) in the above equation, we acquire

\[
y_m' = H(x)y_m + Z(x)y_{m-1} .
\]  

(14)
Through the foregoing logic chain, we can draw the following conclusion: in the case of homogeneous linear vector differential equation (4), after expanding the vector of functions $y$ into an infinite series, at the same time, dividing the coefficient matrix into two nonsingular matrices, we can obtain a recursion series, each term of which satisfies

$$
\begin{align*}
    y'_j &= H(x)y_j & j &= 0 \\
    y'_j &= H(x)y_j + Z(x)y_{j-1} & j &= 1, 2, 3, \ldots .
\end{align*}
$$

(15)

In the case of nonhomogeneous linear vector differential equation (1), we can similarly define a special series, each term of which satisfies

$$
\begin{align*}
    y'_j &= H(x)y_j + F(x) & j &= 0 \\
    y'_j &= H(x)y_j + Z(x)y_{j-1} & j &= 1, 2, 3, \ldots .
\end{align*}
$$

(16)

From the theoretical point of view, we can say, the above equations demonstrate that one can obtain every terms in the recursion series $y = \sum_{i=0}^{\infty} y_i$ if and only if the first term can be solved explicitly.

### 3 The solution of each term of special series

The theory of linear systems has proven that if the vector functions $\Phi_1, \Phi_2, \ldots, \Phi_i$ are $i$ solutions of (8) and $c_1, c_2, \ldots, c_i$ are $i$ numbers, then the vector function

$$
\Phi = \sum_{k=1}^{i} c_k \Phi_k ,
$$

(17)

is also a solution of (8).

We shall be concerned with $n$ vector functions, and we shall use the following common notation for the $n$ vector functions in the following discussion. We
let $\Phi_1, \Phi_2, \ldots, \Phi_n$ be the $n$ vector functions defined respectively by

$$
\Phi_1(x) = \begin{pmatrix}
\phi_{11} \\
\phi_{21} \\
\vdots \\
\phi_{n1}
\end{pmatrix},
\Phi_2(x) = \begin{pmatrix}
\phi_{12} \\
\phi_{22} \\
\vdots \\
\phi_{n2}
\end{pmatrix},
\ldots,
\Phi_n(x) = \begin{pmatrix}
\phi_{1n} \\
\phi_{2n} \\
\vdots \\
\phi_{nn}
\end{pmatrix}
$$

(18)

The $n \times n$ determinant

$$
\begin{vmatrix}
\phi_{11} & \phi_{12} & \ldots & \phi_{1n} \\
\phi_{21} & \phi_{22} & \ldots & \phi_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{n1} & \phi_{n2} & \ldots & \phi_{nn}
\end{vmatrix}
$$

(19)

is called the Wronskian of the $n$ vector functions $\Phi_1, \Phi_2, \ldots, \Phi_n$ defined by (18). We will denote it by $W(\Phi_1, \Phi_2, \ldots, \Phi_n)$ and its value at the point $x$ by $W(\Phi_1, \Phi_2, \ldots, \Phi_n)(x)$.

Let the vector functions $\Phi_1, \Phi_2, \ldots, \Phi_n$ defined by (18) be $n$ solutions of the homogeneous linear vector differential equation (8) on the real interval $[a, b]$. These $n$ solutions $\Phi_1, \Phi_2, \ldots, \Phi_n$ of (8) are linearly independent on $[a, b]$ if and only if

$$
W(\Phi_1, \Phi_2, \ldots, \Phi_n)(x) \neq 0
$$

(20)

for all $x \in [a, b]$. A set of $n$ linearly independent solutions of (8) is called a fundamental set of solutions of (8). If the vector functions $\Phi_1, \Phi_2, \ldots, \Phi_n$ defined by (18) make up a fundamental set of solutions of (8), then the $n \times n$ square matrix

$$
M(x) = \begin{pmatrix}
\phi_{11}(x) & \phi_{12}(x) & \ldots & \phi_{1n}(x) \\
\phi_{21}(x) & \phi_{22}(x) & \ldots & \phi_{2n}(x) \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{n1}(x) & \phi_{n2}(x) & \ldots & \phi_{nn}(x)
\end{pmatrix}
$$

(21)
is a fundamental matrix of (8). Furthermore, let $\tilde{\Phi}_0(x)$ be an arbitrary solution of (8) on the real interval $[a, b]$. Then there exists a suitable constant vector $\mathbf{c} = (c_1, c_2, \ldots, c_n)^T$ (22) such that

$$
\tilde{\Phi}_0(x) = M(x)\mathbf{c} 
$$

(23) on $[a, b]$.

Inserting the solution (23) of $y_0$ into the equation (10) yields

$$
y'_1 = H(x)y_1 + Z(x)M(x)c . 
$$

(24)

We shall now proceed to obtain a solution of above equation by variation of parameters. It has been proven [2] that $\tilde{\Phi}_1$ defined by

$$
\tilde{\Phi}_1(x) = M(x) \int_{x_0}^x M^{-1}(t)Z(t)M(t)dtdt , 
$$

(25)

where $x_0 \in [a, b]$, is a solution of the nonhomogeneous linear vector differential equation (24) on $[a, b]$.

We have shown that $\tilde{\Phi}_1$ is a solution of differential equation (10). Directly inserting this solution into the differential equation (12) yields

$$
y'_2 = H(x)y_2 + Z(x)\tilde{\Phi}_1(x) . 
$$

(26)

We shall also proceed to obtain a solution of above equation by variation of parameters. It has been proven that $\tilde{\Phi}_2$ defined by

$$
\tilde{\Phi}_2(x) = M(x) \int_{x_0}^x M^{-1}(t_2)Z(t_2)M(t_2) \int_{x_0}^{t_2} M^{-1}(t_1)Z(t_1)M(t_1)dtdtdt , 
$$

(27)

where $x_0 \in [a, b]$, is a solution of the nonhomogeneous linear vector differential equation (26) on $[a, b]$.

* We denote the transpose of $\mathbf{A}$ by $\mathbf{A}^T$, where $\mathbf{A}$ being any matrix.
According to above argument, we can acquire \( \Phi_3(x) \), \( \Phi_4(x) \), \( \Phi_5(x) \), \ldots etc term by term. Now we shall assume that the solutions \( \Phi_0(x) \), \( \Phi_1(x) \), \( \Phi_2(x) \), \ldots \( \Phi_{m-1}(x) \) have been acquired. Then, from the recursion formula (14), we can simplify the differential equation satisfied by \( y_m \)

\[
y'_m = H(x)y_m + Z(x)\Phi_{m-1}(x) .
\]

(28)

It is easy to prove that \( \Phi_m \) defined by

\[
\Phi_m(x) = M_x \int_{x_0}^x M_{tm}^{-1}Z_{tm}M_{tm} \int_{x_0}^{t_m} M_{tm-1}^{-1} \ldots M_{t_2} \int_{x_0}^{t_2} M_{t_1}^{-1}Z_{t_1}M_{t_1} \cdots dt_1dt_2 \cdots dt_m ,
\]

(29)

where \( M_t \equiv M(t) \) and \( x_0 \in [a, b] \), is a solution of the nonhomogeneous linear vector differential equation (14) on \([a, b]\).

Hence the above reasoning demonstrate that \( \Phi_j \) defined by

\[
\Phi_j(x) = M_x \int_{x_0}^x M_{t_j}^{-1}Z_{t_j}M_{t_j} \int_{x_0}^{t_j} M_{t_{j-1}}^{-1} \ldots M_{t_2} \int_{x_0}^{t_2} M_{t_1}^{-1}Z_{t_1}M_{t_1} \cdots dt_1dt_2 \cdots dt_j ,
\]

(30)

where \( x_0 \in [a, b] \), is a solution of the nonhomogeneous linear vector differential equation (15) on \([a, b]\).

It is well known that \( \Phi_0(x) \) defined by

\[
\Phi_0(x) = M(x) \int_{x_0}^x M^{-1}(t)F(t)dt ,
\]

(31)

where \( x_0 \in [a, b] \), is a solution of the nonhomogeneous linear vector differential equation

\[
y'_0 = H(x)y_0 + F(x)
\]

(32)
on the real interval \([a, b]\). A theorem [2] has been proven that an arbitrary solution \(\Phi_0(x)\) of the nonhomogeneous differential equation (32) is of the form
\[
\Phi_0(x) = \Phi_0(x) + M(x)c
\] (33)

for a suitable choice of \(c_1, c_2, \ldots, c_n\). According to the same logic chain as what has been explained in the study of the homogeneous linear vector differential equation (4), A solution of the nonhomogeneous linear vector differential equation (16) on \([a, b]\) can easily be acquired, that is
\[
\Phi_j(x) = M_x \int_{x_0}^x M_{t_j}^{-1} Z_{t_j} M_{t_j} \int_{x_0}^{t_j} M_{t_j-1}^{-1} \cdots M_{t_2} \int_{x_0}^{t_2} M_{t_1}^{-1} Z_{t_1} M_{t_1} \int_{x_0}^{t_1} M_{t_0}^{-1} F dt dt_1 dt_2 \cdots dt_j,
\] (34)

where \(x_0 \in [a, b]\).

4 The solution of the first-order linear vector differential equation

**THEOREM** the special series is the solution of the first-order linear vector differential equation if the infinite special series converges on the real interval \([a, b]\).

**proof.** Consider the partial sums of vectors \([5]\)

\[
S_0 = y_0,
\]
\[
S_1 = y_0 + y_1,
\]
\[
S_2 = y_0 + y_1 + y_2,
\]
\[\cdots.\]

Then \(S_l\), the \(l\)th partial sum of vectors, is given by
\[
S_l = y_0 + y_1 + y_2 + \cdots + y_l = \sum_{k=0}^{l} y_k.
\] (36)
In the case of homogeneous linear vector differential equation (4), from the equations (15), we directly acquire

$$S'_l = H(x)S_l + Z(x)S_{l-1}, \quad l \geq 1.$$  \hspace{1cm} (37)

In the case of nonhomogeneous linear vector differential equation (1), summing all the equations in (16) for the set $0 \leq j \leq l$, we obtain

$$S'_l = H(x)S_l + Z(x)S_{l-1} + F(x), \quad l \geq 1.$$  \hspace{1cm} (38)

Since the integer $l$ is unrelated with variable $x$, the limit with respect to $l$ and the differential with respect to $x$ are commutative

$$\lim_{l \to \infty} S'_l = (\lim_{l \to \infty} S_l)' ,$$ \hspace{1cm} (39)

With the help of equation (39), seeking for the limit of the partial sum of vectors, for the homogeneous linear systems, from the equation (37), we obtain

$$\lim_{l \to \infty} S'_l = H(x) \lim_{l \to \infty} S_l + Z(x) \lim_{l \to \infty} S_{l-1} ,$$ \hspace{1cm} (40)

and for the nonhomogeneous linear systems, seeking for the limit of the partial sum of vectors in the equation (38) yields

$$\lim_{l \to \infty} S'_l = H(x) \lim_{l \to \infty} S_l + Z(x) \lim_{l \to \infty} S_{l-1} + F(x).$$ \hspace{1cm} (41)

Assuming that the infinite special series converges on the real interval $[a, b]$, then

$$\lim_{l \to \infty} S_l = \lim_{l \to \infty} S_{l-1} = S,$$ \hspace{1cm} (42)

where $S$ is a finite function on $[a, b]$. Therefore, inserting the above expression into the equation (40) yields

$$S' = H(x)S + Z(x)S = A(x)S ,$$ \hspace{1cm} (43)

that is, the limit $S$ defined by (15) and (42) satisfies the homogeneous linear vector differential equation (4). Furthermore, inserting the formula (42) into the equation (41) yields

$$S' = H(x)S + Z(x)S + F(x) = A(x)S + F(x).$$ \hspace{1cm} (44)
The above equation demonstrates that the limit $S$ defined by (16) and (42) satisfies the nonhomogeneous linear vector differential equation (1). Since the infinite series converges and has sum $S$ if the sequence of partial sums converges to $S$, the special series converges to the solution of the first-order linear vector differential equation.

Thus we have shown the validity of theorem.

**CONCLUSION** If the special series converges, then:

(1) an arbitrary solution $\Phi(x)$ of the homogeneous linear vector differential equation (4) on $[a, b]$ can be expressed as

$$\Phi(x) = \sum_{j=0}^{\infty} \Phi_j(x),$$

(45)

where $\Phi_j(x)$ is defined by (30); and

(2) an arbitrary solution $\Phi'(x)$ of the nonhomogeneous linear vector differential equation (1) on $[a, b]$ is of the form

$$\Phi'(x) = \sum_{j=0}^{\infty} \Phi_j(x),$$

(46)

where $\Phi_j(x)$ is defined by (34).

**Example**

Solve the system

$$\begin{cases} y_1' = (e^{-2x} - 3e^{2x} + 2)y_1 + (e^{-2x} - 9e^{2x} + 3)y_2 \\ y_2' = (e^{2x} - e^{-2x} - 1)y_1 + (3e^{2x} - e^{-2x} - 2)y_2 \end{cases}$$

(47)

on the real interval $[0,1]$.

Clearly this is of the form (6), where

$$H(x) = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}, \quad Z(x) = \begin{pmatrix} e^{-2x} - 3e^{2x} & e^{-2x} - 9e^{2x} \\ e^{2x} - e^{-2x} & 3e^{2x} - e^{-2x} \end{pmatrix}.$$
The corresponding homogeneous differential equation satisfied by $y_0$ is

$$\frac{dy_0}{dx} = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} y_0 .$$  \hspace{1cm} (49)

we find that

$$\phi_1(x) = \begin{pmatrix} 3e^x \\ -e^x \end{pmatrix} \quad \text{and} \quad \phi_2(x) = \begin{pmatrix} e^{-x} \\ -e^{-x} \end{pmatrix}$$  \hspace{1cm} (50)

constitute a fundamental set (pair of linearly independent solutions) of (49). Thus a fundamental matrix of (49) is given by

$$M(x) = \begin{pmatrix} 3e^x & e^{-x} \\ -e^x & -e^{-x} \end{pmatrix}.$$  \hspace{1cm} (51)

From the fundamental matrix $M(x)$, we find that

$$M^{-1}(x) = \frac{1}{2} \begin{pmatrix} e^{-x} & e^{-x} \\ -e^x & -3e^x \end{pmatrix}.$$  \hspace{1cm} (52)

Thus the term $M^{-1}(x)Z(x)M(x)$ in the formula (30) becomes

$$M^{-1}(x)Z(x)M(x) = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$  \hspace{1cm} (53)

We have indicated that an arbitrary solution of (49) can be presented as

$$\tilde{\Phi}_0(x) = M(x) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$  \hspace{1cm} (54)
where \(c_1\) and \(c_2\) are constants. Inserting the equation (53) into the formula (25) yields

\[
\tilde{\Phi}_1(x) = M(x) \begin{pmatrix} 2x c_2 \\ 2x c_1 \end{pmatrix}.
\]

(55)

One can acquire \(\tilde{\Phi}_2(x), \tilde{\Phi}_3(x), \ldots\) from the formula (30) term by term, that is

\[
\tilde{\Phi}_2(x) = M(x) \begin{pmatrix} \frac{(2x)^2}{2} c_1 \\ \frac{(2x)^2}{2} c_2 \end{pmatrix}, \quad \tilde{\Phi}_3(x) = M(x) \begin{pmatrix} \frac{(2x)^3}{3!} c_2 \\ \frac{(2x)^3}{3!} c_1 \end{pmatrix}, \ldots
\]

(56)

Obviously, the special series given by (54), (55), and (56) is convergent. By theorem, we know that

\[
\tilde{\Phi}(x) = M(x) \begin{pmatrix} c_1 + 2x c_2 + \frac{(2x)^2}{2!} c_1 + \frac{(2x)^3}{3!} c_2 + \frac{(2x)^4}{4!} c_1 + \cdots \\ c_2 + 2x c_1 + \frac{(2x)^2}{2!} c_2 + \frac{(2x)^3}{3!} c_1 + \frac{(2x)^4}{4!} c_2 + \cdots \end{pmatrix}
\]

\[
= \frac{1}{2} M(x) \begin{pmatrix} e^{2x}(c_1 + c_2) + e^{-2x}(c_1 - c_2) \\ e^{2x}(c_2 + c_1) + e^{-2x}(c_2 - c_1) \end{pmatrix}
\]

is a solution of the linear system (47) for every real number \(x \in [0, 1]\). Furthermore, we simplify the solution of (47) as follows

\[
\tilde{\Phi}(x) = \begin{pmatrix} (-e^{-3x} + 3e^{-x} + e^x + 3e^{3x}) \tilde{c}_1 + (e^{-3x} - 3e^{-x} + e^x + 3e^{3x}) \tilde{c}_2 \\ (e^{-3x} - e^{-x} - e^x - e^{3x}) \tilde{c}_1 + (-e^{-3x} + e^{-x} - e^x - e^{3x}) \tilde{c}_2 \end{pmatrix},
\]

where \(2\tilde{c}_1 = c_1\) and \(2\tilde{c}_2 = c_2\). Inserting the above vector into the linear vector differential equation (47) really produces two identities.

An important fundamental property of a normal linear system (1) is its relationship to a single \(n\)th-order linear differential equation in one unknown function. The so-called normalized \(n\)th-order linear differential equation is of the form

\[
\frac{d^n y}{dx^n} + a_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x) \frac{dy}{dx} + a_n(x) y = F(x).
\]

(57)

\[\dagger\] In this example, we have selected \(x_0 = 0\) for simplicity.
It is well known that the single $n$th-order equation (57) can be transformed into a special case of the normal linear system (1) of $n$ equations in $n$ unknown functions. Hence the method of special series can be used for solving the normalized $n$th-order linear differential equation.

5 Conclusions

A special series has been introduced to yield the solution of the linear vector differential equation. A group of recursion linear vector differential equations satisfied by the terms of special series has been obtained and solved. We proved that the special series is the solution of the first-order linear vector differential equation if the infinite special series converges. We have given an example to outline the procedure of our method. In principle, the method of special series can be used for solving any first-order linear vector differential equation if and only if the considered special series converges.

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