Rare event asymptotics for exploration processes for random graphs

Shankar Bhamidi, Amarjit Budhiraja, Paul Dupuis, Ruoyu Wu

December 11, 2019

Abstract

Large deviations for random graph models has been a topic of significant recent research activity. Much of the work in this area is focused on the class of dense random graph models (number of edges in the graph scale as \(n^2\), where \(n\) is the number of vertices) where the theory of graphons has emerged as a principal tool in the study of large deviation properties. In contrast there is much less known about large deviations for random graph models in the sparse regime. The aim of this paper is to develop an approach for the study of large deviation problems in this regime by establishing Large Deviation Principles (LDP) on suitable path spaces for certain exploration processes of the associated random graph sequence. Exploration processes are an important tool in the study of sparse random graph models and have been used to understand detailed asymptotics of many functionals of sparse random graphs, such as component sizes, surplus, deviations from trees, etc. In the context of rare event asymptotics of interest here, the point of view of exploration process transforms a large deviation analysis of a static random combinatorial structure to the study of a small noise LDP for certain stochastic dynamical systems with jumps.

Our work focuses on one particular class of random graph models, namely the configuration model; however the general approach of using exploration processes for studying large deviation properties of sparse random graph models has broader applicability. The goal is to study asymptotics of probabilities of non-typical behavior in the large network limit. The first key step for this is to establish a LDP for an exploration process associated with the configuration model. A suitable exploration process here turns out to be an infinite dimensional Markov process with transition probability rates that diminish to zero in certain parts of the state space. Large deviation properties of such Markovian models is challenging due to poor regularity behavior of the associated local rate functions. Our proof of the LDP relies on a representation of the exploration process in terms of a system of stochastic differential equations driven by Poisson random measures and variational formulas for moments of nonnegative functionals of Poisson random measures. Uniqueness results for certain controlled systems of deterministic equations play a key role in the analysis. Next, using the rate function in the LDP for the exploration process we formulate a calculus of variations problem associated with the asymptotics of component degree distributions. The second key ingredient in our study is a careful analysis of the infinite dimensional Euler-Lagrange equations associated with this calculus of variations problem. Exact solutions of these systems of nonlinear differential equations are identified which then provide explicit formulas for decay rates of probabilities of non-typical component degree distributions and related quantities.

AMS 2010 subject classifications: 60F10, 60C05, 05C80, 90B15.

Keywords: large deviation principle, random graphs, sparse regime, diminishing rates, Euler-Lagrange equations, calculus of variations problems, configuration model, branching processes, variational representations, Poisson random measures, exploration process, singular dynamics, giant component.
1 Introduction

Large deviations for random graph models has been a topic of significant recent research activity (see, e.g., [3,13,16,25,29]). Much of the work in this area is focused on the class of dense random graph models (number of edges in the graph scale like \( n^2 \), where \( n \) is the number of vertices). In this regime, the theory of graphons obtained under dense graph limits [15,16,24] has emerged as a key tool in the study of large deviation asymptotics. In contrast to the above papers, the focus in the current work is on a sparse random graph setting where the average degree of a typical vertex is \( O(1) \) so that the number of edges in the graph are \( O(n) \) as \( n \to \infty \). Much less is known on rare event asymptotics for random graph models in this regime. The goal of this work is to develop an approach for the study of large deviation problems in the sparse regime by establishing large deviation principles for a class of stochastic dynamical systems, known as the exploration processes, that play a central role in the study of sparse random graphs. Our work focuses on one particular class of random graph models, namely the configuration model; however similar techniques are expected to be useful for other sparse random graph models as well where tractable dynamic constructions via exploration processes are available.

The configuration model refers to a sequence of random graphs with number of vertices approaching infinity and the degree distribution converging to a pre-specified probability distribution \( p = \{p_k\}_{k \in \mathbb{N}} \) on the set of non-negative integers [1,2,25]. This random graph model is a basic object in probabilistic combinatorics (cf. [30] and references therein) and is one of the standard workhorses in the study of networks in areas such as epidemiology [27] and community detection [18,23]. An important problem for such random graph models is to estimate probabilities of non-typical structural behaviors, particularly when the system size is large. Examples of such behavior include, graph components that are larger or smaller than that predicted by the law of large number analysis or degree distributions within components that deviate significantly from the overall degree distribution. In particular, we are interested in probabilities of events \( E^{n,\varepsilon}(q) \) associated with the configuration model random graph \( G_n \) on \( n \) vertices, described as

\[
E^{n,\varepsilon}(q) = \{ \text{there exists a component in } G_n \text{ with } m_k \text{ degree } k \text{ vertices, where } m_k \in [n(q_k - \varepsilon), n(q_k + \varepsilon)], \ k \in \mathbb{N} \},
\]

and where \( q = (q_k)_{k \in \mathbb{N}} \) is such that \( 0 \leq p_k \leq q_k \) for every \( k \). One of our main results (see Theorem 2.6) shows that, under conditions, for large \( n \) and small \( \varepsilon \)

\[
P \{ E^{n,\varepsilon}(q) \} \approx \exp \left\{ -n \left[ H(q) + H(p - q) - H(p) \right] \right\},
\]

where for a nonnegative sequence \( r = (r_k)_{k \in \mathbb{N}} \),

\[
H(r) = \sum_{k=1}^{\infty} r_k \log r_k - \left( \frac{1}{2} \sum_{k=1}^{\infty} k r_k \right) \log \left( \frac{1}{2} \sum_{k=1}^{\infty} k r_k \right).
\]

This result in particular gives asymptotics for probabilities of observing a component of a given size (see Remark 2.12) and explicit formulas for rates of decay of probabilities of observing a \( D \)-regular component of a given size in \( G_n \) (see Corollaries 2.8 and 2.11); see also Conjectures 2.9 and 2.10 on large deviation asymptotics for the size of the largest component in a \( D \)-regular graph.

The starting point of our analysis is a dynamical construction of the configuration model given through a discrete time infinite dimensional Markov chain referred to as the exploration process (cf. 20,26). As the name suggests, the exploration process is constructed by first appropriately selecting a vertex in the graph and then exploring the neighborhood of the chosen vertex until the component of that vertex is exhausted. After this one moves on to another ‘unexplored’ vertex resulting in successive exploration of components of the random graph until the entire graph has been explored. The stochastic process corresponding to one particular coordinate of this infinite dimensional Markov chain encodes the number of edges in any given component through the length of its excursions away from zero. The
remaining coordinates of this Markov chain can be used to read off the number of vertices of a given degree in any given component of the random graph. See Section 2.3 for a precise description of the state space of this Markov chain. The exploration process can be viewed as a small noise stochastic dynamical system in which the transition steps are of size $O(1/n)$ with $n$ denoting the number of vertices in the random graph. A key ingredient in the proof of Theorem 2.6 is a Large Deviation Principle (LDP) for an infinite dimensional jump-Markov process that can be viewed as a continuous time analogue of the exploration process. This result, given in Theorem 2.14, is our second main result. As other applications of this theorem, we recover a well known result on the asymptotics of the largest component in the configuration model due to Molloy and Reed [26] and Janson and Luczak [20], and also present a result (whose proof is omitted) on asymptotics of scaled number of components in a configuration model (see Remark 2.15). The rate function in the LDP given in Theorem 2.14 can be used to formulate a calculus of variations problem associated with the event $E^c_{n,\varepsilon}(q)$ described in (1.1).

This problem is at the heart of our analysis and by studying the corresponding infinite dimensional system of coupled Euler-Lagrange equations we construct an explicit minimizer in this optimization problem (see Lemma 7.6). The cost associated with the minimizer is the exponent on the right side of (1.2) and provides the exact expression for the decay rate for the probability of interest.

1.1 Proof techniques and overview of contributions

In addition to the study of the asymptotics of the configuration model, one of the main motivations for working on these sets of problems was the development of new techniques for handling large deviations for processes with “degeneracies.” We will give an overview of these contributions in this section.

The exploration process associated with the $n$-th random graph (with $n$ vertices) in the configuration model is described as an $\mathbb{R}^\infty$-valued ‘small noise’ Markov chain $\{X^n(j)\}_{j \in \mathbb{N}_0}$. Under our assumptions, there exists an $N \in \mathbb{N}$ such that for all $j \geq nN$, $X^n(j) = \emptyset$ for all $n \in \mathbb{N}$. In order to study large deviations for such a sequence, one usually considers a sequence of continuous times processes, or equivalently $\mathbb{C}([0,N] : \mathbb{R}^\infty)$-valued random variables, obtained by a linear interpolation of $\{X^n(j)\}_{j \in \mathbb{N}_0}$ over intervals of length $1/n$. A large deviations analysis of such a sequence in the current setting is challenging due to ‘diminishing rates’ feature of the transition kernel (see (2.5)) which in turn leads to poor regularity of the associated local rate function. By diminishing rates we mean the property that probabilities of certain transitions, although non-zero, can get arbitrarily close to 0 as the system becomes large. In the model we consider, the system will go through phases where some state transitions have very low probabilities, that are separated by phases of ‘regular behavior,’ many times. In terms of the underlying random graph the first type of phase corresponds to time periods in the dynamic construction that are close to the completion of exploration of one component and beginning of exploration of a new component. The poor regularity of the local rate function makes standard approximations of the near optimal trajectory that are used in proofs of large deviation principles for such small noise systems hard to implement. In order to overcome these difficulties we instead consider a different continuous time process associated with the exploration of the configuration model. This continuous time process is obtained by introducing i.i.d. exponential random times before each step in the edge exploration Markov chain. A precise description of this process is given in terms of stochastic differential equations (SDE) driven by a countable collection of Poisson random measures (PRM), where different PRMs are used to describe the different types of transitions (see Section 2.4). Although the coefficients in this SDE are discontinuous functions, their dependence on the state variable is much more tractable than the state dependence in the transition kernel of the discrete time model.

Large deviations for small noise SDE driven by Brownian motions have been studied extensively both in finite and infinite dimensions. An approach based on certain variational representations for moments of nonnegative functionals of Brownian motions and weak convergence methods [6,8] has been quite effective in studying a broad range of such systems (cf. references in [11]). A similar variational representation for functionals of a Poisson random measure has been obtained in [11]. There have been several recent papers that have used this representation for studying large deviation problems.
exploration process. Using this and the LDP in Theorem 2.14 one can provide an upper bound for the probability of the event in terms of a quantity measuring the difference between the original and perturbed process to move from the state $(0,0)$ to $(0,0)$ as the minimal cost for certain controlled analogues of the exploration process. However, the basic difficulties that arise in the proof of the large deviations lower bound can be described as follows. The proof of the lower bound based on variational representations and weak convergence methods, for systems driven by Brownian motions, requires establishing unique solvability of controlled deterministic equations of the form

$$dx(t) = b(x(t))dt + \sigma(x(t))\omega(t)dt, \quad x(0) = x_0, \quad (1.4)$$

where $u \in L^2([0,T]: \mathbb{R}^d)$ (space of square integrable functions from $[0,T]$ to $\mathbb{R}^d$) is a given control. It turns out that the conditions that are typically introduced for the well-posedness of the original small noise stochastic dynamical system of interest (e.g., Lipschitz properties of the coefficients $b$ and $\sigma$) are enough to give the well-posedness of $(1.4)$. For example when the coefficients are Lipschitz, one can use a standard argument based on Gronwall’s lemma and an application of the Cauchy-Schwarz inequality to establish the desired uniqueness property. In contrast, when studying systems driven by a PRM one instead needs to establish well-posedness of controlled equations of the form

$$x(t) = x(0) + \int_{[0,t] \times S} 1_{[0,\rho(x(s))]}(y)\varphi(s,y)ds m(dy), \quad 0 \leq t \leq T, \quad (1.5)$$

where $S$ is a locally compact metric space, $m$ a locally finite measure on $S$, $g: \mathbb{R} \to \mathbb{R}_+$ is a measurable map and the control $\varphi$ is a nonnegative measurable map on $[0,T] \times S$ which satisfies the integrability property

$$\int_{[0,T] \times S} \ell(\varphi(s,y))ds m(dy) < \infty,$$

where $\ell(x) = x \log x - x + 1$. If $\varphi$ were uniformly bounded and $g$ sufficiently regular (e.g., Lipschitz) uniqueness follows once more by a standard Gronwall argument. However, in general if $g$ is not Lipschitz or $\varphi$ is not bounded (both situations arise in the problem considered here, see e.g., (2.14), (2.15)) the problem of uniqueness becomes a challenging obstacle. One of the novel contributions of this work is to obtain uniqueness results for equations of the form (1.5) when certain structural properties are satisfied. The setting we need to consider is more complex than the one described above in that there is an infinite collection of coupled equations (one of which corresponds to the Skorokhod problem for one dimensional reflected trajectories) that describe the controlled system. However, the basic difficulties can already be seen for the simpler setting in (1.5). Although for a general $\varphi$ the unique solvability of equations of the form (1.5) may indeed be intractable, the main idea in our approach is to argue that one can perturb the original $\varphi$ slightly so that the solution $x(\cdot)$ stays the same and moreover this $x(\cdot)$ is the unique solution of the corresponding equation with the perturbed $\varphi$. Furthermore, the cost difference between the original and perturbed $\varphi$ is appropriately small. The uniqueness result given in Lemma 5.1 is a key ingredient in the proof of the lower bound given in Section 5. The proof of the upper bound, via the weak convergence based approach to large deviations relies on establishing suitable tightness and limit characterization results for certain controlled versions of the original small noise system. This proof is given in Section 4.

The rate function in the LDP for the exploration process in Theorem 2.14 is given as a variational formula on an infinite dimensional path space (see (2.12)). Getting useful information from such an abstract formula in general seems hopeless, however, as we show in this work, for the event considered in (1.1), the variational formula can be used to extract much more explicit information. We begin by observing (see (2.11)) that the event $E^{n,\tau}(q)$ of interest can be written explicitly in terms of the exploration process. Using this and the LDP in Theorem 2.14 one can provide an upper bound for the probability of the event in terms of a quantity $I^2_{\tau,n}((0,p),(0,p-q))$ which can be interpreted (see Section 4 for a precise definition) as the minimal cost for certain controlled analogues of the exploration process to move from the state $(0,p)$ to $(0,p-q)$ in $\tau$ units of time, where $\tau = \frac{1}{2} \sum_{k=1}^{\infty} kq_k$ (see Lemmas 8.2 and 8.3). We then show that this deterministic control problem, which can be reformulated as a calculus of variations problem, admits an explicit solution. This solution is given in Construction 1.
and its optimality is studied in Lemma 7.9. Using this optimality property, the complementary lower bound for the probability of interest is given in Lemma 8.3. Lemmas 7.6 and 8.4 form the technical heart of the proof of Theorem 2.6 and rely on a detailed and careful analysis of the infinite dimensional Euler-Lagrange equations associated with the calculus of variations problem.

1.2 Organization of the paper

The paper is organized as follows. In Section 2 we introduce the configuration model, our main assumptions, and our first main result, Theorem 2.6, on asymptotics of probabilities of \( E^{n,\epsilon}(q) \). We record some consequences of these results for \( D \)-regular graphs and subgraphs in Corollaries 2.8 and 2.11. Remark 2.12 discusses another application of this result to the study of asymptotics of probabilities of components of a given size. In Section 2.3 we review the edge-exploration algorithm (EEA) from [20,26] that gives a dynamical construction of the configuration model. For reasons discussed previously, the large deviation analysis of the discrete time EEA presents several technical obstacles and thus in Section 2.4 we introduce a closely related continuous time jump-Markov process \((X^n, Y^n)\) with values in \((\mathbb{R} \times \mathbb{R}^+) \times \mathbb{R}\) which is mathematically more tractable. Sections 2.5 and 2.6 present our second main result, Theorem 2.14, that gives a large deviation principle for the sequence \((X^n, Y^n)_{n \in \mathbb{N}}\) in a suitable infinite dimensional path space. In Section 2.7 we also note two side consequences of Theorem 2.14. The first, given in Section 2.6.1, is a law of large numbers (LLN) result that recovers well known results of Janson and Luczak (2009) on the asymptotics of the largest component in the configuration model. The second, discussed in Remark 2.15, gives a LDP for the scaled number of components in \( G_n \) as \( n \to \infty \).

Section 3 presents the variational representation from [11] for functionals of PRM that is the starting point of our proofs. Some tightness and characterization results that are used both in the upper and lower bound proofs are also given in this section. Next, Section 4 gives the proof of the large deviation upper bound whereas the proof of the lower bound is given in Section 5. Finally, Section 6 establishes the compactness of level sets of the function \( I_T \) defined in Section 2.6, thus proving that \( I_T \) is a rate function. Together, results of Sections 3, 4, 5 and 6 complete the proof of Theorem 2.14.

We next turn to the proof of Theorem 2.6 which is given in Sections 7-9. First in Section 7 we introduce a calculus of variations problem that is central to the proof of Theorem 2.6. We also introduce (see Construction 1) a candidate minimizer in this optimization problem and present several technical results (Lemmas 7.1, 7.6) that are needed for the proof of the optimality property of the candidate minimizer. Using results of Section 4, the proof of Theorem 2.6 is completed in Section 8. Finally, Section 9 contains the proofs of technical lemmas from Section 7 whereas Section 10 presents the proof of the LLN results from Section 2.6.1.

1.3 Notation

The following notation will be used. For a Polish space \( S \), denote the corresponding Borel \( \sigma \)-field by \( B(S) \). Denote by \( P(S) \) (resp. \( M(S) \)) the space of probability measures (resp. finite measures) on \( S \), equipped with the topology of weak convergence. Denote by \( C_b(S) \) (resp. \( M_b(S) \)) the space of real bounded and continuous functions (resp. bounded and measurable functions). For \( f : S \to \mathbb{R} \), let \( \|f\|_\infty = \sup_{x \in S} |f(x)| \). For a Polish space \( S \) and \( T > 0 \), denote by \( C([0,T] : S) \) (resp. \( D([0,T] : S) \)) the space of continuous functions (resp. right continuous functions with left limits) from \([0,T]\) to \( S \), endowed with the uniform topology (resp. Skorokhod topology). We say a collection \( \{X^n\} \) of \( S \)-valued random variables is tight if the distributions of \( X^n \) form a relatively compact collection in \( P(S) \). A sequence of \( D([0,T] : S) \)-valued random variables is said to be \( C \)-tight if it is tight in \( D([0,T] : S) \) and every weak limit point takes values in \( C([0,T] : S) \) a.s. We use the symbol ‘\( \Rightarrow \)’ to denote convergence in distribution.

We denote by \( \mathbb{R}^\infty \) the space of all real sequences which is identified with the countable product of copies of \( \mathbb{R} \). This space is equipped with the usual product topology. For \( x = (x_k)_{k \in \mathbb{N}}, y = (y_k)_{k \in \mathbb{N}} \), we
write $x \leq y$ if $x_k \leq y_k$ for each $k \in \mathbb{N}$. Let $C = C([0, T] : \mathbb{R})$, $C_\infty = C([0, T] : \mathbb{R})$, $D = D((0, T] : \mathbb{R})$, $D_\infty = D((0, T] : \mathbb{R})$. Let $x^+ = \max\{x, 0\}$ for $x \in \mathbb{R}$. Denote by $\mathbb{R}_+$ the set of all non-negative real numbers. Let $N_0 = \mathbb{N} \cup \{0\}$. Cardinality of a set $A$ is denoted by $|A|$. For $n \in \mathbb{N}$, let $[n] = \{1, 2, \ldots, n\}$. We use the following conventions: $0 \log 0 = 0$, $0 \log(x/0) = 0$ for $x \geq 0$, and $x \log(x/0) = \infty$ for $x > 0$.

## 2 Assumptions and Results

Fix $n \in \mathbb{N}$. We start by describing the construction of the configuration model of random graphs with vertex set $[n]$. Detailed description and further references for the configuration model can be found in [30 Chapter 7].

### 2.1 The configuration model and assumptions

Let $d(n) = \{d_i(n)\}_{i \in [n]}$ be a degree sequence, namely a sequence of non-negative integers such that $\sum_{i=1}^n d_i(n)$ is even. Let $2m(n) = \sum_{i=1}^n d_i(n)$. We will usually suppress the dependence of $d_i(n)$ and $m(n)$ on $n$ in the notation. Using the sequence $\{d_i\}$ we construct a random graph on $n$ labelled vertices $[n]$ as follows: (i) Associate with each vertex $i \in [n]$ $d_i$ half-edges. (ii) Perform a uniform random matching on the $2m$ half-edges to form $m$ edges so that every edge is composed of two half-edges. This procedure creates a random multigraph $G([n], d(n))$ with $m$ edges, allowing for multiple edges between two vertices and self-loops, and is called the configuration model with degree sequence $d(n)$. Since we are concerned with connectivity properties of the resulting graph, vertices with degree zero play no role in our analysis, and therefore we assume that $d_i > 0$ for all $i \in [n]$, $n \geq 1$. We make the following additional assumptions.

**Assumption 2.1.** There exists a probability distribution $p = \{p_k\}_{k \in \mathbb{N}}$ on $\mathbb{N}$ such that, writing $n_k^{(n)} = |\{i \in [n] : d_i = k\}|$ for the number of vertices with degree $k$, $n_k^{(n)}/n \rightarrow p_k$ as $n \rightarrow \infty$, for all $k \in \mathbb{N}$.

We will also usually suppress the dependence of $n_k^{(n)}$ on $n$ in the notation. We make the following assumption on moments of the degree distribution.

**Assumption 2.2.** There exists some $\varepsilon_p \in (0, \infty)$ such that $\sup_{n \in \mathbb{N}} \sum_{k=1}^{\infty} \frac{n_k}{n} k^{1+\varepsilon_p} < \infty$.

The above two assumptions will be made throughout this work.

**Remark 2.3.** (i) Note that Assumptions 2.1 and 2.2 along with Fatou’s lemma, imply that $\sum_{k=1}^{\infty} p_k k^{1+\varepsilon_p} < \infty$. Conversely, if $\sum_{k=1}^{\infty} p_k k^\lambda < \infty$ for some $\lambda \in (4, \infty)$ and $\{D_i\}_{i \in \mathbb{N}}$ is a sequence of i.i.d. $\mathbb{N}$-valued random variables with common distribution $\{p_k\}_{k \in \mathbb{N}}$, then using a Borel–Cantelli argument it can be shown that for a.e. $\omega$, Assumptions 2.1 and 2.2 are satisfied with $d_i = D_i(\omega)$, $i \in [n]$, $n \in \mathbb{N}$, and $\varepsilon_p = \frac{\lambda}{4} - 1$.

(ii) Under Assumptions 2.1 and 2.2 $\mu = \sum_{k=1}^{\infty} k p_k < \infty$ and the total number of edges $m = \frac{1}{2} \sum_{i=1}^{n} d_i$ satisfies $\frac{m}{n} \rightarrow \frac{1}{2} \sum_{k=1}^{\infty} k p_k$ as $n \rightarrow \infty$.

### 2.2 Large Deviation Asymptotics for Component Degree Distributions

We will say that a component of $G([n], d(n))$ has degree configuration $\{\tilde{n}_k\}$ if the component has $\tilde{n}_k$ vertices with degree $k$, for $k \in \mathbb{N}$. Given $0 \leq q = (q_k, k \in \mathbb{N}) \leq p$, we are interested in the asymptotic exponential rate of decay of the probability of the event $E^{n,\varepsilon}(q)$ introduced in 1.1 that corresponds to the existence of a component in $G([n], d(n))$ with degree configuration $\{\tilde{n}_k\}$ satisfying $(q_k - \varepsilon)n \leq \tilde{n}_k \leq (q_k + \varepsilon)n$, $k \in \mathbb{N}$, namely, we want to characterize $\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \log P \{E^{n,\varepsilon}(q)\}$. Note that for there
to exist a component with degree configuration \( \{ n q_k \} \) we must have \( \sum_{k=1}^{\infty} k q_k \geq 2 \left( \sum_{k=1}^{\infty} q_k - \frac{1}{n} \right) \). We will in fact assume a slightly stronger condition:

\[
\sum_{k=1}^{\infty} k q_k > 2 \sum_{k=1}^{\infty} q_k.
\] (2.1)

This condition says that there are strictly more edges than vertices in the component. Define \( \beta = \beta(q) \) as follows: \( \beta = 0 \) when \( q_1 = 0 \), and when \( q_1 > 0 \), \( \beta \in (0, 1) \) is the unique solution (see Remark 2.3 below) of the equation

\[
\sum_{k=1}^{\infty} k q_k = (1 - \beta^2) \sum_{k=1}^{\infty} \frac{k q_k}{1 - \beta^k}.
\]

Define the function \( K(q) \) by

\[
K(q) = \left( \frac{1}{2} \sum_{k=1}^{\infty} k q_k \right) \log(1 - \beta(q)^2) - \sum_{k=1}^{\infty} q_k \log(1 - \beta(q)^k)
\] (2.2)

and with \( H(\cdot) \) as in (1.2) define

\[
\hat{I}_1(q) = H(q) + H(p - q) - H(p) + K(q).
\] (2.3)

Remark 2.4. The existence and uniqueness of \( \beta(q) \) can be seen as follows. For \( \alpha \in (0, 1) \) consider

\[
\alpha F(\alpha) = \sum_{k=1}^{\infty} k q_k - (1 - \alpha^2) \sum_{k=1}^{\infty} \frac{k q_k}{1 - \alpha^k} = \alpha \left( \sum_{k=3}^{\infty} \frac{\alpha - \alpha^{k-1}}{1 - \alpha^k} k q_k - q_1 \right).
\]

For \( k \geq 3 \) and \( \alpha \in (0, 1) \) let \( F_k(\alpha) = (\alpha - \alpha^{k-1})/(1 - \alpha^k) \). It is easily verified that \( F_k(\cdot) \) is strictly increasing on \((0, 1)\). Thus for \( \alpha \in (0, 1) \), \( 0 = F_k(0+) < F_k(\alpha) < F_k(1-) = \frac{k-2}{k} \), and so

\[
-q_1 = F(0+) < F(\alpha) < F(1-) = \sum_{k=3}^{\infty} (k - 2) q_k - q_1.
\]

Since \( F \) is continuous on \((0, 1)\), \(-q_1 < 0\) and \( \sum_{k=3}^{\infty} (k - 2) q_k - q_1 = \sum_{k=1}^{\infty} k q_k - 2 \sum_{k=1}^{\infty} q_k > 0 \), we have the existence and uniqueness of \( \beta(q) \).

Remark 2.5. We note that for every \( 0 \leq q = (q_k, k \in \mathbb{N}) \leq p \), \( K(q) \) and \( H(q) \) are finite. Indeed, the finiteness of \( K(q) \) is immediate from Assumption 2.2. To see the finiteness of \( H(q) \), note that on the one hand \( \sum_{k=1}^{\infty} q_k \log k q_k \leq 0 \) while on the other hand

\[
\sum_{k=1}^{\infty} q_k \log k q_k = \sum_{k=1}^{\infty} q_k \log \frac{q_k}{2^{k+1}} - (\log 2) \sum_{k=1}^{\infty} (k+1) q_k
\]

\[
\geq - \left( 1 - \sum_{k=1}^{\infty} q_k \right) \log 1 - \sum_{k=1}^{\infty} q_k \frac{q_k}{2^{k-1}} - (\log 2) \sum_{k=1}^{\infty} (k+1) q_k > -\infty,
\]

where the first inequality follows from non-negativity of relative entropy and putting mass \( 1 - \sum_{k=1}^{\infty} q_k \) on \( k = 0 \), and the last inequality once more uses Assumption 2.2.

The following result gives asymptotics of the event \( E_{n, \varepsilon}(q) \). The proof of the theorem, which is based on Theorem 2.14, is given in Section 8.

Theorem 2.6. Suppose \( 0 \leq q \leq p \) and that (2.1) is satisfied. Then
(i) (Upper bound) when \( p_1 = 0 \), we have \( \beta(q) = 0 \), \( K(q) = 0 \) and

\[
\limsup_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log P \{ E^{n,\varepsilon}(q) \} \leq -\tilde{I}_1(q).
\]

(ii) (Lower bound)

\[
\liminf_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log P \{ E^{n,\varepsilon}(q) \} \geq -\tilde{I}_1(q).
\]

Remark 2.7. The proof of Theorem 2.6 relies on a large deviation principle for a certain exploration process (see Section 2.4) that is given in Theorem 2.13. The latter result does not require the condition \( p_1 = 0 \). Also note that the lower bound in Theorem 2.6 does not require the condition \( p_1 = 0 \) either. One can also give an upper bound (without requiring \( p_1 = 0 \)) in terms of a variational formula given by the right side of (2.4). When \( p_1 = 0 \), this variational expression can be simplified and is seen to be equal to \(-\tilde{I}_1(q)\). This is shown in Lemma 2.3 whose proof crucially relies on the property \( p_1 = 0 \). Whether the two expressions are equal in general when \( p_1 \neq 0 \) remains an open problem.

As an immediate corollary of Theorem 2.6 we have the following result for \( D \)-regular graphs, i.e., graphs such that each vertex is of degree \( D \). In the following \( \ast \) represents either \( \limsup \) or \( \liminf \).

Corollary 2.8. (\( D \)-regular graphs) Suppose that there exists some \( D \in \mathbb{N} \) with \( D \geq 3 \), such that \( p_k = 0 \), \( n_k = 0 \) for \( k \neq D \) and \( p_D = 1 \), \( n_D = n \). Fix \( q_D \in (0,1] \) and denote by \( E^{n,\varepsilon}_D(q) \) the event that there is a component of size \( n_D \in [n(q_D - \varepsilon), n(q_D + \varepsilon)] \). Then

\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log P \{ E^{n,\varepsilon}_D(q) \} = \left( 1 - \frac{D}{2} \right) \left( q_D \log q_D + (1 - q_D) \log(1 - q_D) \right). \tag{2.4}
\]

Proof. Let \( q_k = 0 \) for \( k \in \mathbb{N} \setminus \{D\} \) and let \( q = \{q_k, k \in \mathbb{N}\} \). Then since \( p_1 = 0 \), we have \( \beta(q) = 0 \) and \( K(q) = 0 \). Using (2.3) we have

\[
\tilde{I}_1(q) = H(q) + H(p - q) - H(p) + K(q) = q_D \log q_D - \frac{D q_D}{2} \log \left( \frac{D q_D}{2} \right) + (1 - q_D) \log(1 - q_D) - \frac{D - D q_D}{2} \log \left( \frac{D - D q_D}{2} \right)
\]

\[
+ \frac{D}{2} \log \left( \frac{D}{2} \right) = \left( 1 - \frac{D}{2} \right) \left( q_D \log q_D + (1 - q_D) \log(1 - q_D) \right).
\]

The result then follows from Theorem 2.6.

We note that the expression (2.4) has the same form when \( q_D \) is replaced by \( 1 - q_D \). This suggests that the most likely way of having a component of size around \( n q_D \) in \( D \)-regular graphs is to let almost all of the remaining \( n(1 - q_D) \) vertices be in one component. Indeed, conditioning on having a component of size around \( n q_D \), the remaining vertices can be viewed as a smaller configuration model of \( D \)-regular graphs with about \( n(1 - q_D) \) vertices. It then follows from the well known results for the asymptotics of the largest component in the configuration model \([20, 26]\) (and Theorem 2.18) that these remaining vertices are in one component with high probability.

Based on these observations we make the following conjecture.

Conjecture 2.9. (\( D \)-regular graphs, multiple components) Suppose that there exists some \( D \in \mathbb{N} \) with \( D \geq 3 \), such that \( p_k = 0 \), \( n_k = 0 \) for \( k \neq D \) and \( p_D = 1 \), \( n_D = n \). Fix \( M \in \mathbb{N} \) and \( q^{(i)}_D \in (0,1] \) for each \( i = 1, \ldots, M \), such that \( \sum_{i=1}^{M} q^{(i)}_D \leq 1 \). Let \( q^{(i)}_k = 0 \) for \( k \in \mathbb{N} \setminus \{D\} \) and let \( q^{(i)} = \{q^{(i)}_k, k \in \mathbb{N}\} \), for
each \( i = 1, \ldots, M \). Let \( q^{(M+1)} = p - \sum_{i=1}^{M} q^{(i)} \). Denote by \( E^{n, \varepsilon, M}_D \) the event that there are components of sizes \( N^{(i)}_D \in [n(q^{(i)}_D - \varepsilon), n(q^{(i)}_D + \varepsilon)], i = 1, \ldots, M \). Then

\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log P \left\{ E^{n, \varepsilon, M}_D \right\} = \sum_{i=1}^{M+1} H(q^{(i)}) - H(p) = \left( 1 - \frac{D}{2} \right) \sum_{i=1}^{M+1} q^{(i)} \log q^{(i)}.
\]

We also note that for each fixed \( a \in [0, 1] \), the function \([0, a] \ni x \mapsto x \log x + (a - x) \log(a - x) \in (-\infty, 0] \) is maximized at \( x = 0 \) and \( x = a \). This suggests that, the most likely way for the largest component to be of certain size, is to let as many of the remaining components as possible have such a size. Based on this we make the following conjecture on the large deviation behavior of the largest component size for \( D \)-regular graphs.

**Conjecture 2.10.** (\( D \)-regular graphs, largest component) Suppose that there exists some \( D \in \mathbb{N} \) with \( D \geq 3 \), such that \( p_k = 0, n_k = 0 \) for \( k \neq D \) and \( p_D = 1, n_D = n \). For each \( x \in [0, 1] \), let \( q^{(x)}_D = x \), \( q^{(x)}_k = 0 \) for \( k \in \mathbb{N} \setminus \{D\} \), and \( q^{(x)} = \{q^{(x)}_k \mid k \in \mathbb{N}\} \). Denote by \( M^n \) the size of the largest component. Then \( \frac{M^n_x}{n} \) satisfies a large deviation principle in \( \mathbb{R}_+ \) with rate function \( I_{\text{max}} \) defined by

\[
I_{\text{max}}(x) = k(x)H(q^{(x)}) + H(q^{(1-xk(x))}) - H(p) = \left( 1 - \frac{D}{2} \right) \left( \frac{\log (1 - xk(x))}{xk(x)} + \frac{\log (1 - xk(x))}{xk(x)} \right)
\]

for \( x \in [0, 1] \) and \( I_{\text{max}}(x) = \infty \) otherwise, where \( k(x) = \left\lfloor \frac{1}{x} \right\rfloor \) is the largest integer such that \( xk(x) \leq 1 \).

Recall that \( \mu = \sum_{k=1}^{\infty} k p_k < \infty \). The following result gives bounds on probabilities of observing a \( D \)-regular subgraph in a configuration model with a general degree sequence \((p_k)_k\).

**Corollary 2.11.** Suppose that Assumptions 2.1 and 2.2 hold. Also suppose that \( p_1 = 0 \). Fix \( D \in \mathbb{N} \) with \( D \geq 3 \) such that \( p_D > 0 \). Fix \( q_D \in (0, p_D) \). Denote by \( E^{n, \varepsilon}(q) \) the event that the graph has a component that is \( D \)-regular and has size \( N_D \in [n(q_D - \varepsilon), n(q_D + \varepsilon)] \). Then

\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log P \left\{ E^{n, \varepsilon}(q) \right\} = (q_D \log q_D + (p_D - q_D) \log(p_D - q_D) - p_D \log p_D) - \left( Dq_D \right) \log \left( \frac{Dq_D}{2} \right) + \mu - \frac{Dq_D}{2} \log \left( \frac{\mu}{2} \right).
\]

**Proof.** Let \( q_k = 0 \) for \( k \in \mathbb{N} \setminus \{D\} \) and let \( q = (q_k \mid k \in \mathbb{N}) \). As before, since \( q_1 = 0 \), we have \( \beta(q) = 0 \) and \( K(q) = 0 \). Using (2.24) we have

\[
\tilde{I}_1(q) = H(q) + H(p - q) - H(p) + K(q)
\]

\[
= q_D \log q_D - \frac{Dq_D}{2} \log \left( \frac{Dq_D}{2} \right) + (p_D - q_D) \log(p_D - q_D) + \sum_{k \neq D} p_k \log p_k - \mu - \frac{Dq_D}{2} \log \left( \frac{\mu}{2} \right)
\]

\[
- \sum_{k=1}^{\infty} p_k \log p_k + \mu - \frac{Dq_D}{2} \log \left( \frac{\mu}{2} \right)
\]

\[
= (q_D \log q_D + (p_D - q_D) \log(p_D - q_D) - p_D \log p_D) - \left( Dq_D \right) \log \left( \frac{Dq_D}{2} \right) + \mu - \frac{Dq_D}{2} \log \left( \frac{\mu}{2} \right).
\]

The result then follows from Theorem 2.6.

\( \square \)
Remark 2.12. Theorem 2.6 can be used to extract other asymptotic results. We give below one example without proof. Suppose that Assumptions 2.1 and 2.2 hold. Also suppose that \( p_1 = p_2 = 0 \). Let \( r \in (0, 1) \) and denote by \( E_r^{n, \varepsilon} \) the event that the graph has a component that has size \( N_r \in [n(r - \varepsilon), n(r + \varepsilon)] \). Then

\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log P \{ E_r^{n, \varepsilon} \} = \inf_{0 \leq q \leq \rho, q \neq 1} \left\{ H(p) - H(q) - H(p - q) \right\}.
\]

The proof of Theorem 2.6 is given in Section 8.

2.3 Edge-exploration algorithm (EEA)

One can construct \( G([n], d(n)) \) whilst simultaneously exploring its component structure [20, 26], which we now describe. This algorithm traverses the graph by exploring all its edges, unlike typical graph exploration algorithms, which sequentially explore vertices. At each stage of the algorithm, every vertex in \([n]\) is in one of two possible states, sleeping or awake, while each half-edge is in one of three states: sleeping (unexplored), active or dead (removed). Write \( \mathcal{A}_V(j), \mathcal{S}_V(j) \) for the set of active and sleeping vertices at step \( j \) and similarly let \( \mathcal{S}_E(j), \mathcal{A}_E(j), \mathcal{D}_E(j) \) be the set of sleeping, active and dead half-edges at step \( j \). We call a half-edge “living” if it is either sleeping or active. Initialize by setting all vertices and half-edges to be in the sleeping state. For step \( j \geq 0 \), write \( A(j) \doteq |\mathcal{A}_E(j)| \) for the number of active half-edges and \( V_k(j) \) for the number of sleeping vertices \( v \in \mathcal{S}_V(j) \) with degree \( k \). Write \( \mathcal{V}(j) \doteq (V_k(j), k \in \mathbb{N}) \) for the corresponding vector in \( \mathbb{R}_+^\infty \). At step \( j = 0 \), all vertices and half-edges are asleep hence \( A(0) = 0 \) and \( V_k(0) = n_k \) for \( k \geq 1 \). The exploration process proceeds as follows:

1. If the number of active half-edges and sleeping vertices is zero, i.e. \( A(j) = 0 \) and \( \mathcal{V}(j) = 0 \), all vertices and half-edges have been explored and we terminate the algorithm.

2. If \( A(j) = 0 \) and \( \mathcal{V}(j) \neq 0 \), so there exist sleeping vertices, pick one such vertex with probability proportional to its degree (alternatively pick a sleeping half-edge uniformly at random) and mark the vertex as awake and all its half-edges as active. Thus the transition \( (A(j), \mathcal{V}(j)) \rightarrow (A(j + 1), \mathcal{V}(j + 1)) \) at step \( j + 1 \) takes the form

\[
(0, v) \mapsto (k, v - e_k) \quad \text{with probability} \quad \frac{k v_k}{\sum_{i=1}^{\infty} i v_i}, \quad k \in \mathbb{N},
\]

where \( e_k \) is the \( k \)-th unit vector.

3. If \( A(j) > 0 \), pick an active half-edge uniformly at random, pair it with another uniformly chosen living half-edge (either active or sleeping), say \( e^* \), merge both half-edges to form a full edge and kill both half-edges. If \( e^* \) was sleeping when picked, wake the vertex corresponding to the half-edge \( e^* \), and mark all its other half-edges active. Thus in this case the transition takes the form

\[
(a, v) \mapsto (a - 2, v) \quad \text{with probability} \quad \frac{a - 1}{\sum_{i=1}^{\infty} i v_i + a - 1},
\]

\[
(a, v) \mapsto (a + k - 2, v - e_k) \quad \text{with probability} \quad \frac{k v_k}{\sum_{i=1}^{\infty} i v_i + a - 1}, \quad k \in \mathbb{N}.
\]

The statements in (2) and (3) can be combined as follows: If \( A(j) \neq 0 \) or \( \mathcal{V}(j) \neq 0 \), then the transition \( (A(j), \mathcal{V}(j)) \rightarrow (A(j + 1), \mathcal{V}(j + 1)) \) takes the form

\[
(a, v) \mapsto (a - 2, v) \quad \text{with probability} \quad \frac{(a - 1)^+}{\sum_{i=1}^{\infty} i v_i + (a - 1)^+},
\]

\[
(a, v) \mapsto (a + k - 2, v - e_k) \quad \text{with probability} \quad \frac{k v_k}{\sum_{i=1}^{\infty} i v_i + (a - 1)^+}, \quad k \in \mathbb{N}.
\]

(2.5)
The random graph $G([n], d(n))$ formed at the termination of the above algorithm has the same
distribution as the configuration model with degree sequence $d(n)$ [20,25].

We note that for $j > 0$, $A(j) = 0$ if and only if the exploration of a component in the random
graph $G([n], d(n))$ is completed at step $j$. Thus the number of edges in a component equals the length
of an excursion of $\{A(j)\}$ away from 0 and the largest excursion length gives the size of the largest
component, namely the number of edges in the component with maximal number of edges. The vertices
in each component are those that are awakened during corresponding excursions.

Note that at each step in the EEA, either a new vertex is woken up or two half-edges are killed.
Since there are a total of $n$ vertices and $2m$ half-edges, we have from Assumptions 2.1 and 2.2 that the
algorithm terminates in at most $m$

$\sum_{k=1}^{\infty} k \frac{\theta k^2}{r^2} \leq \infty$. We define $A(j) = 0$ and $V(j) = 0$ for all $j \geq j_0$ where $j_0$ is the step at which the algorithm terminates.

2.4 An equivalent continuous time exploration process

A natural way to study large deviations properties of the configuration model is through the discrete time
sequence $\{A(j), V(j)\}_{j \in \mathbb{N}}$ in EEA which can be viewed as a discrete time “small noise” Markov process.
In order to study large deviations for such a sequence, a standard approach is to consider the sequence
$C([0, L] : \mathbb{R})$-valued random variables obtained by a linear interpolation of $\{A(j), V(j)\}_{j \in \mathbb{N}}$ over
intervals of length $1/n$. As was noted in the Introduction, the ‘diminishing rates’ feature of the transition
kernel (2.5) makes the large deviations analysis of this sequence challenging. Thus we introduce below
a different continuous time process associated with the exploration of the configuration model.

We begin with some notation that will be needed to formulate the continuous time model. For
a locally compact Polish space $\mathbb{S}$, let $\mathcal{M}_{FC}(\mathbb{S})$ be the space of all measures $\nu$ on $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$ such that
$\nu(K) < \infty$ for every compact $K \subset \mathbb{S}$. We equip $\mathcal{M}_{FC}(\mathbb{S})$ with the usual vague topology. This
topology can be metrized such that $\mathcal{M}_{FC}(\mathbb{S})$ is a Polish space (see [11] for one convenient metric).
A Poisson random measure (PRM) $N$ on a locally compact Polish space $\mathbb{S}$ with intensity measure
$\nu \in \mathcal{M}_{FC}(\mathbb{S})$ is an $\mathcal{M}_{FC}(\mathbb{S})$-valued random variable such that for each $A \in \mathcal{B}(\mathbb{S})$ with $\nu(A) < \infty,
N(A)$ is Poisson distributed with mean $\nu(A)$ and for disjoint $A_1, \ldots, A_k \in \mathcal{B}(\mathbb{S}), N(A_1), \ldots, N(A_k)$ are
mutually independent random variables (cf. [19]).

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space on which are given i.i.d. PRM $\{N_k(ds \, dy \, dz)\}_{k \in \mathbb{N}_0}$ on
$\mathbb{R}_+ \times [0, 1] \times \mathbb{R}_+$ with intensity measure $ds \, dy \, dz$. Let

$$\tilde{\mathcal{F}}_t = \sigma\{N_k((0, s] \times A \times B), 0 \leq s \leq t, A \in \mathcal{B}([0, 1]), B \in \mathcal{B}(\mathbb{R}_+), k \in \mathbb{N}_0\}, t \geq 0$$

and let $\{\mathcal{F}_t\}$ be the $\mathcal{P}$-completion of this filtration. Fix $T \in (0, \infty)$. Let $\mathcal{P}$ be the $\{\mathcal{F}_t\}_{0 \leq t \leq T}$-predictable
$\sigma$-field on $\Omega \times [0, T]$. Let $\mathcal{A}_t$ be all $\mathcal{P} \otimes \mathcal{B}([0, 1]))/\mathcal{B}(\mathbb{R}_+)$-measurable maps from $\Omega \times [0, T] \times [0, 1]$ to
$\mathbb{R}_+$. For $\varphi \in \mathcal{A}_t$, define a counting process $N^\varphi_k$ on $[0, T] \times [0, 1]$ by

$$N^\varphi_k([0, t] \times A) = \int_{[0,t] \times A \times \mathbb{R}_+} 1_{\{0, \varphi(s, y)\}}(z) N_k(ds \, dy \, dz), t \in [0, T], A \in \mathcal{B}([0, 1]), k \in \mathbb{N}_0.$$ 

We think of $N^\varphi_k$ as a controlled random measure, where $\varphi$ is the control process that produces a thinning of the point process $N_k$ in a random but non-anticipative manner to produce a desired intensity. We will write $N^\varphi_k$ as $N^\theta_k$ if $\varphi \equiv \theta$ for some constant $\theta \in \mathbb{R}_+$. Note that $N^\theta_k$ is a PRM on $[0, T] \times [0, 1]$ with
intensity $\theta ds \, dy$. For $x = (x_0, x_1, x_2, \ldots) \in \mathbb{R} \times \mathbb{R}^\infty_+$, let

$$r(x) = (x_0)^+ + \sum_{k=1}^\infty k x_k, \quad r_0(x) = (x_0)^+ 1_{\{r(x) > 0\}}, \quad r_k(x) = \frac{k x_k}{r(x)} 1_{\{r(x) > 0\}}, \quad k \in \mathbb{N}. \quad (2.6)$$

Note that $\sum_{k \in \mathbb{N}_0} r_k(x) = 1$ whenever $r(x) > 0$. Recall that $e_k$ is the $k$-th unit vector in $\mathbb{R}^\infty$, $k \in \mathbb{N}_0$.
Define the state process $X^n(t) = (X^n_0(t), X^n_1(t), X^n_2(t), \ldots)$ with values in $\mathbb{R} \times \mathbb{R}^\infty_+$ as the solution to
the following SDE:

\[
X^n(t) = X^n(0) + \frac{1}{n} \int_{[0,t]} \left( 1_{\{X^n_0(s-) \geq 0\}} [-2e_0] 1_{[0,r_0(X^n(s-))]}(y) N_k^n(dy) \right. \\
\left. + \sum_{k=1}^{\infty} \frac{1}{n} \int_{[0,t] \times [0,1]} 1_{\{X^n_0(s-) \geq 0\}} [(k - 2)e_0 - e_k] 1_{[0,r_k(X^n(s-))]}(y) N_k^n(dy) \right) \\
+ \sum_{k=1}^{\infty} \frac{1}{n} \int_{[0,t] \times [0,1]} 1_{\{X^n_0(s-) < 0\}} [ke_0 - e_k] 1_{[0,r_k(X^n(s-))]}(y) N_k^n(dy), \tag{2.7}
\]

where \( X^n(0) = \frac{1}{n}(\cdots) \). The existence and uniqueness of solutions to this SDE follows from Assumption 2.2. It is not difficult to see that \( \frac{1}{n}(A(j) - 1, V_1(j), V_2(j), \ldots) \) in the discrete time EEA can be viewed as the embedded Markov chain associated with \( X^n \). In defining the state process, one could replace \( X^n_0(0) \) with the asymptotically equivalent process \( X^n_0(0) + 1/n \) which starts from 0 and is more directly comparable with the sequence \( A(j)/n \). However some of the expressions are simplified (see, e.g., the formulas for rates in (2.6) and the transition probabilities in (2.5)) when describing the state in terms of \( X^n_0(0) \) instead of \( X^n_0(0) + 1/n \). We now rewrite the evolution of \( X^n \) as follows:

\[
X^n(t) = X^n(0) + e_0 \sum_{k=0}^{\infty} \frac{k - 2}{n} \int_{[0,t] \times [0,1]} 1_{[0,r_k(X^n(s-))]}(y) N_k^n(dy) - \sum_{k=1}^{\infty} e_k \frac{1}{n} \int_{[0,t] \times [0,1]} 1_{[0,r_k(X^n(s-))]}(y) N_k^n(dy) \\
+ e_0 \sum_{k=0}^{\infty} \frac{2}{n} \int_{[0,t] \times [0,1]} 1_{\{X^n_0(s-) < 0\}} 1_{[0,r_k(X^n(s-))]}(y) N_k^n(dy).
\]

Here the first two integrands do not depend on the sign of \( X^n_0 \) and are interpreted as the main contribution to the evolution. The last sum is a ‘reflection’ term in the \( e_0 \) direction and makes a contribution of \( \frac{2}{n} e_0 \) only when \( X^n_0(s-) < 0 \). For \( t \geq 0 \) define

\[
Y^n(t) = X^n(0) + \sum_{k=0}^{\infty} \frac{k - 2}{n} \int_{[0,t] \times [0,1]} 1_{[0,r_k(X^n(s-))]}(y) N_k^n(dy), \tag{2.8}
\]

\[
\eta^n(t) = \sum_{k=0}^{\infty} \frac{2}{n} \int_{[0,t] \times [0,1]} 1_{\{X^n_0(s-) < 0\}} 1_{[0,r_k(X^n(s-))]}(y) N_k^n(dy). \tag{2.9}
\]

Using these we can write

\[
X^n_0(t) = Y^n(t) + \eta^n(t), \tag{2.10}
\]

\[
X^n_k(t) = X^n(0) - \frac{1}{n} \int_{[0,t] \times [0,1]} 1_{[0,r_k(X^n(s-))]}(y) N_k^n(dy), \quad k \in \mathbb{N}. \tag{2.11}
\]

Here \( \eta^n \) is viewed as the regulator function which ensures that \( X^n_0(t) \geq -\frac{1}{n} \). Note that for \( k \in \mathbb{N}, X^n_k(t) \) is non-increasing and non-negative. Also, from (2.7) we see that \( r(X^n_0(t)) \) is non-increasing.

### 2.5 Rate Function

The main result of this work gives a large deviation principle for \( \{(X^n, Y^n)\}_{n \in \mathbb{N}} \) in the path space \( \mathcal{D}_\infty \times \mathcal{D} \). In this section we define the associated rate function \( I_T \). Including the process \( Y^n \) in the LDP is convenient for obtaining large deviation results, for the degree distribution in giant components, of the form given in Section 10.
Recall the probability distribution \( p \equiv \{ p_k \}_{k \in \mathbb{N}} \) introduced in Assumption 2.1. Let \( \Gamma : \mathcal{C} \to \mathcal{C} \) denote the one-dimensional Skorokhod map defined by

\[
\Gamma(\psi)(t) = \psi(t) - \inf_{0 \leq s \leq t} \psi(s) \wedge 0, \quad t \in [0, T], \quad \psi \in \mathcal{C}.
\]

Let \( \mathcal{C}_T \) be the subset of \( \mathcal{C}_{\infty} \times \mathcal{C} \), consisting of those functions \( (\zeta, \psi) \) such that

(a) \( \psi(0) = 0 \), and \( \psi \) is absolutely continuous on \([0, T]\).
(b) \( \zeta_0(t) = \Gamma(\psi)(t) \) for \( t \in [0, T] \).
(c) For each \( k \in \mathbb{N} \), \( \zeta_k(0) = p_k \), \( \zeta_k \) is non-increasing and absolutely continuous and \( \zeta_k(t) \geq 0 \) for \( t \in [0, T] \).

For \( (\zeta, \psi) \in (\mathcal{D}_{\infty} \times \mathcal{D}) \setminus \mathcal{C}_T \), define \( I_T(\zeta, \psi) = \infty \). For \( (\zeta, \psi) \in \mathcal{C}_T \), define

\[
I_T(\zeta, \psi) = \inf_{\varphi \in \mathcal{S}_T(\zeta, \psi)} \left\{ \sum_{k=0}^{\infty} \int_{[0,T] \times [0,1]} \ell(\varphi_k(s, y)) \, ds \, dy \right\}.
\]

(2.12)

Here for \( x \geq 0 \),

\[
\ell(x) = x \log x - x + 1,
\]

(2.13)

and \( \mathcal{S}_T(\zeta, \psi) \) is the set of all intensities \( \varphi = (\varphi_k)_{k \in \mathbb{N}_0} \), \( \varphi_k : [0, T] \times [0, 1] \to \mathbb{R}_+ \), such that

\[
\psi(t) = \sum_{k=0}^{\infty} (k - 2) \int_{[0,t] \times [0,1]} 1_{[0, r_k(\zeta(s))]}(y) \varphi_k(s, y) \, ds \, dy, \quad t \in [0, T]
\]

(2.14)

\[
\zeta_k(t) = p_k - \int_{[0,t] \times [0,1]} 1_{[0, r_k(\zeta(s))]}(y) \varphi_k(s, y) \, ds \, dy, \quad k \in \mathbb{N}, \quad t \in [0, T].
\]

(2.15)

**Remark 2.13.** Suppose \( (\zeta, \psi) \in \mathcal{C}_T \) satisfies (2.14) and (2.15) for some \( \varphi \in \mathcal{S}_T(\zeta, \psi) \).

(a) From Assumptions 2.1 and 2.2 it follows that the following uniform integrability holds: As \( K \to \infty \),

\[
\sup_{0 \leq t \leq T} \sum_{k=K}^{\infty} k \zeta_k(t) \leq \sum_{k=K}^{\infty} k \sup_{0 \leq t \leq T} \zeta_k(t) = \sum_{k=K}^{\infty} kp_k \to 0.
\]

This in particular says that \( r(\zeta(\cdot)) \in \mathcal{C} \), where \( r(\cdot) \) is defined in (2.6).

(b) For any \( k \in \mathbb{N} \), whenever \( \zeta_k(t_k) = 0 \) for some \( t_k \in [0, T] \), we must have \( \zeta_k(t) = 0 \) for all \( t \in [t_k, T] \). This follows since \( \zeta_k \) is non-increasing and non-negative for every \( k \).

(c) Whenever \( r(\zeta(t^*)) = 0 \) for some \( t^* \in [0, T] \), we must have from part (b) that \( \zeta_k(t) = 0 \) for all \( t \in [t^*, T] \) and \( k \in \mathbb{N} \). This, together with (2.14), implies that \( \psi(\cdot) \) is non-increasing on the interval \([t^*, T] \). Hence by property (b) of \( \mathcal{C}_T \), \( \zeta_0(t) \) is non-increasing and non-negative for \( t \in [t^*, T] \). Since \( \zeta_0(t^*) = 0 \), we must then have \( \zeta_0(t) = 0 \) for \( t \in [t^*, T] \), which means that \( \zeta(t) = 0 \) for \( t \in [t^*, T] \). Thus whenever such a \( t^* \) exists, \( \zeta(t) = 0 \) after the time instant

\[
\tau_\zeta = \inf\{ t \in [0, T] : r(\zeta(t)) = 0 \} \wedge T.
\]

(2.16)
2.6 LDP and LLN for the Exploration Process

The following LDP is one of our main results and is key to the proof of Theorem 2.6.

**Theorem 2.14.** The function $I_T$ in (2.12) is a rate function on $D_\infty \times D$ and the sequence $\{(X^n, Y^n)\}_{n \in \mathbb{N}}$ satisfies a large deviation principle in $D_\infty \times D$ with rate function $I_T$.

**Outline of the proof:** Due to the equivalence between a large deviation principle and a Laplace principle, it suffices to show the following three statements (cf. [17, Section 1.2] or [9, Section 1.2]).

1. Laplace principle upper bound: For all $h \in C_b(D_\infty \times D)$,
   \[
   \limsup_{n \to \infty} \frac{1}{n} \log E e^{-nh(X^n, Y^n)} \leq - \inf_{(\zeta, \psi) \in C_\infty \times C} \{I_T(\zeta, \psi) + h(\zeta, \psi)\}.
   \] (2.17)

2. Laplace principle lower bound: For all $h \in C_b(D_\infty \times D)$,
   \[
   \liminf_{n \to \infty} \frac{1}{n} \log E e^{-nh(X^n, Y^n)} \geq - \inf_{(\zeta, \psi) \in C_\infty \times C} \{I_T(\zeta, \psi) + h(\zeta, \psi)\}.
   \] (2.18)

3. $I_T$ is a rate function on $D_\infty \times D$: For each $M \in [0, \infty)$, $\{(\zeta, \psi) \in D_\infty \times D : I_T(\zeta, \psi) \leq M\}$ is a compact subset of $D_\infty \times D$.

Statements (1), (2) and (3) will be shown in Sections 4, 5 and 6, respectively.

**Remark 2.15.** As noted above, the LDP in Theorem 2.14 is a key to the proof of Theorem 2.6. In the next subsection we will show how this LDP can be used to easily give a LLN result. The LDP can be used to establish other asymptotic results as well. We give one such example without proof below. Denote by $C_n$ the number of components in $G([n], d(n))$. Then $\eta^n$ defined in (2.9) can be used to represent $\frac{C_n}{n}$. Note that whenever the EEA starts to explore a new component, $X^n_0$ will jump from $-\frac{1}{n}$ and as a result, $\eta^n$ will increase by $\frac{2}{n}$. Therefore

\[
\frac{C_n}{n} = \sup_{t > 0} \frac{\eta^n(t)}{2} = \lim_{T \to \infty} \frac{\eta^n(T)}{2}.
\]

Observe from (2.10) that $\eta^n = X^n_0 - Y^n$, and that for large deviation asymptotics one can assume that the EEA terminates by time $N \geq 1 + \lfloor \sup_{n} \frac{1}{2} \sum_{k=1}^{\infty} k \frac{C_n^{(0)}}{n} \rfloor < \infty$ (see Lemma 8.1 and its proof for precise details). Using this fact, Theorem 2.14 and the contraction principle one can establish that $\frac{C_n}{n}$ satisfies a large deviation principle in $\mathbb{R}_+$ with rate function $\hat{I}$ defined by

\[
\hat{I}(x) = \lim_{T \to \infty} \inf_{(\zeta, \psi) \in C_\infty \times C_0} \{T(I_T(\zeta, \psi)) = 2x\}.
\]

The rate function $\hat{I}(x)$ has the following alternative representation.

\[
\hat{I}(x) = \inf_{(\zeta, \psi) \in C_\infty \times C_0} \left\{ r_0(\zeta) + \sum_{k=1}^{\infty} r_k(\zeta) \left( \frac{20(\zeta)}{9(\zeta)} \right) - \frac{1}{9}(\zeta) \right\}.
\]

2.6.1 Law of large number limits

The LDP in Theorem 2.14 can be used to identify the LLN limit $(\zeta, \psi)$ of the exploration process $(X^n, Y^n)$, which corresponds to the unique pair satisfying $I_T(\zeta, \psi) = 0$. In particular we recover well known results for the asymptotics of the largest component in the configuration model [20][26]. We assume the following strengthened version of Assumption 2.2.

**Assumption 2.16.** $\sup_{n \in \mathbb{N}} \sum_{k=1}^{\infty} \frac{1}{n^2 k^2} < \infty$. 

14
Remark 2.17. Under our standing assumptions, namely Assumption 2.1 and 2.2, one can show by following the arguments in Section 3 that \(\{X^n, Y^n\}_{n \in \mathbb{N}}\) is tight and any weak limit point \((\zeta, \psi)\) of this sequence is in \(C_T\) and satisfies 2.14 and 2.15 with \(\varphi_k = 1\) for \(k \in \mathbb{N}_0\). However it seems hard to argue the uniqueness of this limiting system of equations without additional conditions. Instead we show that if Assumption 2.2 is replaced with the stronger condition in Assumption 2.10 then there is an explicit trajectory \((\zeta, \psi)\) for which the rate function vanishes and in fact it is the unique such trajectory. This is the content of Theorem 2.18 and Proposition 2.19. From these results the LLN follows immediately. Whether the LLN holds under the weaker Assumption 2.2 is an open problem.

Recall \(\mu = \sum_{k=1}^{\infty} kp_k\) and note that \(\mu < \infty\). Define, for \(z \in [0, 1]\),
\[
G_0(z) = \sum_{k=1}^{\infty} p_k z^k \quad \text{and} \quad G_1(z) = \sum_{k=1}^{\infty} \frac{kp_k z^{k-1}}{\mu}.
\]

Define \(F_s(t) = G_0(s) - G_0(st)\) for \(s \in (0, 1]\) and \(t \in [0, 1]\). Then \(F_s: [0, 1] \to [0, G_0(s)]\) is strictly decreasing and continuous. Let \(F_s^{-1}(\cdot)\) denote the inverse of \(F_s\). Define
\[
f_s(t) = \begin{cases} F_s^{-1}(t) & \text{when } 0 \leq t \leq G_0(s), \\ 0 & \text{when } t > G_0(s). \end{cases}
\]

Then \(f_s(t)\) is strictly decreasing until it hits zero. Note that in particular, \(f_1(t) = F_1^{-1}(t)1_{[0,1]}(t)\). Define \(f_0(t) = 0\) for \(t \geq 0\).

Fix \(T \geq \frac{1}{2}\). The following theorem together with Proposition 2.19 characterizes the unique \((\zeta, \psi) \in C_T\) that minimizes the rate function \(I_T(\zeta, \psi)\). Letting
\[
\nu = \frac{\sum_{k=1}^{\infty} k(k-1)p_k}{\sum_{k=1}^{\infty} kp_k},
\]
part 1 of the theorem considers the subcritical and critical cases \(\nu \leq 1\), where the size of the largest component is \(o(n)\), while part 2 considers the supercritical case \(\nu > 1\), where the size of the largest component is \(O(n)\). Proofs of Theorem 2.18 and Proposition 2.19 are provided in Section 10.

Theorem 2.18. Suppose that Assumptions 2.1 and 2.10 hold.

1. Suppose \(\sum_{k=1}^{\infty} k(k-2)p_k \leq 0\). Define \(\zeta(t) = (\zeta_k(t))_{k \in \mathbb{N}_0}\) and \(\psi(t)\) by
\[
\zeta_0(t) = 0, \zeta_k(t) = p_k(f_1(t))^k, \quad k \in \mathbb{N},
\]
\[
\psi(t) = -2 \int_0^t r_0(\zeta(s)) \, ds + \sum_{k=1}^{\infty} (k-2)(p_k - \zeta_k(t)).
\]

Then \((\zeta, \psi) \in C_T\) and \(I_T(\zeta, \psi) = 0\).

2. Suppose \(\sum_{k=1}^{\infty} k(k-2)p_k > 0\). If \(p_1 > 0\), then there exists a unique \(\rho \in (0, 1)\) such that \(G_1(\rho) = \rho\). If \(p_1 = 0\), \(G_1(\rho) = \rho\) with \(\rho \leq 0\). Define \(\tau = \frac{1}{2}(1 - \rho^2) > 0\) and define \(\zeta(t) = (\zeta_k(t))_{k \in \mathbb{N}_0}\) and \(\psi(t)\) by
\[
\zeta_0(t) = \left[\mu - 2t - \mu \sqrt{1 - 2t/\mu} G_1(\sqrt{1 - 2t/\mu})\right]1_{[0, \tau]}(t),
\]
\[
\zeta_k(t) = \begin{cases} p_k(1 - 2t/\mu)^{k/2} & \text{when } 0 \leq t \leq \tau, \\ p_k \rho^k(f_1(t - \tau))^k & \text{when } t > \tau, \end{cases} \quad k \in \mathbb{N},
\]
\[
\psi(t) = -2 \int_0^t r_0(\zeta(s)) \, ds + \sum_{k=1}^{\infty} (k-2)(p_k - \zeta_k(t)).
\]

Then \((\zeta, \psi) \in C_T\) and \(I_T(\zeta, \psi) = 0\).
The following proposition says that there is a unique \((\zeta, \psi)\) satisfying \(I_T(\zeta, \psi) = 0\), so that this pair is the law of large numbers limit.

**Proposition 2.19.** Suppose Assumptions 2.1 and 2.16 hold. Then the pair \((\zeta, \psi)\) defined in Theorem 2.18 is the unique element of \(D_\infty \times D\) such that \(I_T(\zeta, \psi) = 0\).

### 3 Representation and Weak Convergence of Controlled Processes

We will use the following useful representation formula proved in [11]. For the second equality in the theorem see the proof of Theorem 2.4 in [7]. The representation in the cited papers is given in terms of a single Poisson random measure with points in a locally compact Polish space. However for the current work it is convenient to formulate the representation in terms of a countable sequence of independent Poisson random measures on \([0, T] \times [0, 1]\). This representation is immediate from the results in [11] and [7] by viewing the countable sequence of Poisson random measures with points in \([0, T] \times [0, 1]\) and intensity the Lebesgue measure \(\lambda_T\) on \([0, T] \times [0, 1]\) as a single PRM with points in the augmented space \([0, T] \times [0, 1] \times \mathbb{N}_0\) with intensity \(\lambda_T \otimes \varrho\), where \(\varrho\) is the counting measure on \(\mathbb{N}\). Recall that \(\mathcal{A}_+\) denotes the class of \((P \times \mathcal{B}(\mathbb{R}_+)))/\mathcal{B}(\mathbb{R}_+)\)-measurable maps from \(\mathbb{N}_0 \times [0, T] \times [0, 1]\) to \(\mathbb{R}_+\). For each \(m \in \mathbb{N}\) let

\[
\mathcal{A}_{b,m} = \{ (\varphi_k)_{k \in \mathbb{N}_0} : \varphi_k \in \mathcal{A}_+ \text{ for each } k \in \mathbb{N}_0 \text{ such that for all } (\omega, t, y) \in \mathbb{N}_0 \times [0, T] \times [0, 1], \]

\[
1/m \leq \varphi_k(\omega, t, y) \leq m \text{ for } k \leq m \text{ and } \varphi_k(\omega, t, y) = 1 \text{ for } k > m \}
\]

and let \(\mathcal{A}_b = \bigcup_{m=1}^\infty \mathcal{A}_{b,m}\). Recall the function \(\ell\) defined in (2.13).

**Theorem 3.1.** Let \(F \in \mathcal{M}_b([\mathcal{M}_F([0, T] \times [0, 1]))^\infty\). Then for \(\theta > 0\),

\[
-\log E e^{-F((N^T_k)_{k \in \mathbb{N}_0})} = \inf_{\varphi_k \in \mathcal{A}_{b+}, k \in \mathbb{N}_0} E \left[ \theta \sum_{k=0}^\infty \int_{[0, T] \times [0, 1]} \ell(\varphi_k(s, y)) ds dy + F((N^\theta \varphi_k)_{k \in \mathbb{N}_0}) \right]
\]

\[
= \inf_{\varphi = (\varphi_k)_{k \in \mathbb{N}_0} \in \mathcal{A}_b} E \left[ \theta \sum_{k=0}^\infty \int_{[0, T] \times [0, 1]} \ell(\varphi(s, y)) ds dy + F((N^\theta \varphi_k)_{k \in \mathbb{N}_0}) \right].
\]

Fix \(h \in \mathcal{C}_b(D_\infty \times D)\). Since \((X^n, Y^n)\) can be written as \(\Psi((N^n_k)_{k \in \mathbb{N}_0})\) for some measurable function \(\Psi\) from \([\mathcal{M}_F([0, T] \times [0, 1]))^\infty\) to \(D_\infty \times D\), we have from the second equality in Theorem 3.1 that with \((\theta, F) = (n, nh \circ \Psi)\),

\[
-\frac{1}{n} \log E e^{-nh(X^n, Y^n)} = \inf_{\varphi^n = (\varphi^n_k)_{k \in \mathbb{N}_0} \in \mathcal{A}_b} E \left\{ \sum_{k=0}^\infty \int_{[0, T] \times [0, 1]} \ell(\varphi^n_k(s, y)) ds dy + h(X^n, Y^n) \right\},
\]

(3.1)

Here \((\tilde{X}^n, \tilde{Y}^n) = \Psi((N^{n\varphi_n}_k)_{k \in \mathbb{N}_0})\), which solves the controlled analogue of (2.8)–(2.11), namely \(\tilde{X}^n(0) = 1/(n-1, n_1, n_2, \ldots)\), and for \(t \in [0, T]\),

\[
\tilde{Y}^n(t) = \tilde{X}^n_0(0) + \sum_{k=0}^\infty \frac{k-2}{n} \int_{[0, t] \times [0, 1]} 1_{[0, r_k(X^n(s-)))(y)N^{n\varphi_n}_k(ds\,dy)}(y) N^{n\varphi_n}_k(ds\,dy)
\]

(3.2)

\[
\tilde{X}^n_0(t) = \tilde{Y}^n(t) + \frac{2}{n} \sum_{k=0}^\infty \int_{[0, t] \times [0, 1]} 1_{[\tilde{X}^n_0(s-)<0]} 1_{[0, r_k(X^n(s-)))(y)N^{n\varphi_n}_k(ds\,dy)}(y) N^{n\varphi_n}_k(ds\,dy)
\]

(3.3)

\[
\tilde{X}^n_k(t) = \tilde{X}^n_k(0) - \frac{1}{n} \int_{[0, t] \times [0, 1]} 1_{[0, r_k(X^n(s-)))(y)N^{n\varphi_n}_k(ds\,dy)}(y) N^{n\varphi_n}_k(ds\,dy), \quad k \in \mathbb{N}.
\]

(3.4)
For 'controlled' processes, given in terms of the control sequence $\varphi^n \doteq (\varphi^n_k)_{k \in \mathbb{N}_0}$. We will occasionally suppress the dependence on $\varphi^n$ in the notation and will make this dependence explicit if there are multiple controls (e.g. as in Section 4).

In the proof of both the upper and lower bound we will show it is sufficient to consider a sequence $\{\varphi^n_k \in A_k, k \in \mathbb{N}_0\}$ that satisfies the following uniform bound for some $M_0 < \infty$:

$$\sup_{n \in \mathbb{N}} \sum_{k=0}^{\infty} \int_{[0,T] \times [0,1]} \ell(\varphi^n_k(s,y)) \, ds \, dy \leq M_0, \text{ a.s. } P. \quad (3.5)$$

In the rest of this section we study tightness and convergence properties of controlled processes $(\bar{X}^n, \bar{Y}^n)$ that are driven by controls $\{\varphi^n_k\}$ that satisfy the above a.s. bound.

From (3.2)–(3.4) we can rewrite

$$\bar{Y}^n(t) = \bar{X}^n_0(0) + \sum_{k=0}^{\infty} (k-2) \bar{B}^n_k(t), \quad (3.6)$$

$$\bar{X}^n_0(t) = \bar{Y}^n(t) + \bar{\eta}^n(t), \quad (3.7)$$

$$\bar{X}^n_k(t) = \bar{X}^n_k(0) - \bar{B}^n_k(t), k \in \mathbb{N}, \quad (3.8)$$

where

$$\bar{B}^n_k(t) = \frac{1}{n} \int_{[0,t] \times [0,1]} 1_{[0,r_k(\bar{X}^n(s-))]}(y) \, N^{n\varphi^n_k}(ds \, dy), k \in \mathbb{N}_0, \quad (3.9)$$

$$\bar{\eta}^n(t) = \sum_{k=0}^{\infty} \frac{2}{n} \int_{[0,t] \times [0,1]} 1_{\{\bar{X}^n_0(s-)<0\}} 1_{[0,r_k(\bar{X}^n(s-))]}(y) \, N^{n\varphi^n_k}(ds \, dy)$$

$$= \sum_{k=1}^{\infty} \frac{2}{n} \int_{[0,t] \times [0,1]} 1_{\{\bar{X}^n_0(s-)<0\}} 1_{[0,r_k(\bar{X}^n(s-))]}(y) \, N^{n\varphi^n_k}(ds \, dy). \quad (3.10)$$

Here the last line follows on observing that $1_{\{\bar{X}^n_0(s-)<0\}} 1_{[0,r_0(\bar{X}^n(s-))]}(y) = 0$.

Since $m_1 = \sup_{n \in \mathbb{N}} \sum_{k=1}^{\infty} k \frac{2}{n} < \infty$ by Assumption 2.2 we have that for $t \in [0,T]$, $-\frac{1}{n} \leq \bar{X}^n_0(t) \leq m_1$, $0 \leq r(\bar{X}^n(t)) \leq m_1$ and $0 \leq \bar{X}^n_k(t) \leq \frac{2m_1}{n}$. Also note that both $r(\bar{X}^n(\cdot))$ and $\bar{X}^n_k(\cdot)$ for $k \in \mathbb{N}$ are non-increasing.

The following lemma summarizes some elementary properties of $\ell$. For part (a) we refer to [10, Lemma 3.1], and part (b) is an easy calculation that is omitted.

**Lemma 3.2.** (a) For each $\beta > 0$, there exists $\gamma(\beta) \in (0, \infty)$ such that $\gamma(\beta) \to 0$ as $\beta \to \infty$ and $x \leq \gamma(\beta) \ell(x)$, for $x \geq \beta > 1$.

(b) For $x \geq 0$, $x \leq \ell(x) + 2$.

The next lemma gives some uniform integrability properties for the control sequence $\varphi^n$.

**Lemma 3.3.** For $K \in \mathbb{N}$ define

$$\bar{U}_K = \sup_{n \in \mathbb{N}} \int_{[0,T] \times [0,1]} k \varphi^n_k(s,y) 1_{[0,r_k(\bar{X}^n(s-))]}(y) \, ds \, dy. \quad (3.11)$$

Then $\bar{U}_K < \infty$ for each $K \in \mathbb{N}$, and $\lim_{K \to \infty} \bar{U}_K = 0$. \hfill 17
Proof. From (3.9) and (3.8) it follows that

\[ U_K = \sup_{n \in \mathbb{N}} E \left\{ \sum_{k=K}^{\infty} k \tilde{B}_n^k(T) \right\} = \sup_{n \in \mathbb{N}} E \left\{ \sum_{k=K}^{\infty} k [\tilde{X}_k^n(0) - \tilde{X}_k^n(T)] \right\} \leq \sup_{n \in \mathbb{N}} \sum_{k=K}^{\infty} \frac{kn_k}{n}. \]

Recalling \( \varepsilon_p \in (0, \infty) \) introduced in Assumption 2.2 we have

\[ \sup_{n \in \mathbb{N}} \sum_{k=K}^{\infty} \frac{kn_k}{n} \leq K^{-\varepsilon_p} \sup_{n \in \mathbb{N}} \sum_{k=1}^{\infty} \frac{n_k^{1+\varepsilon_p}}{n} \to 0 \]

as \( K \to \infty \). The result follows. \( \square \)

The following lemma proves some key tightness properties. Write \( \tilde{B}^n = \{ \tilde{B}_n^k \}_{k \in \mathbb{N}_0} \), for where \( k \in \mathbb{N}_0 \),

\[ \tilde{v}_n^k([0, t] \times A) = \int_{[0,t] \times A} \varphi_n^k(s, y) \, ds \, dy, \quad t \in [0, T], \, A \in \mathcal{B}([0, 1]). \] (3.12)

Lemma 3.4. Suppose that the bound in (3.3) is satisfied. Then the sequence of random variables \( \{ (\tilde{v}^n, \tilde{X}^n, \tilde{Y}^n, \tilde{B}^n, \tilde{\eta}^n) \} \) is tight in \( [\mathcal{M}([0, T] \times [0, 1]) ]^\infty \times \mathcal{D}_\infty \times \mathcal{D} \times \mathcal{D}_\infty \times \mathcal{D} \).

Proof. We will argue the tightness of \( \{ \tilde{v}^n \} \) in \( [\mathcal{M}([0, T] \times [0, 1]) ]^\infty \) and the \( \mathcal{C} \)-tightness of \( \{ \tilde{X}^n \}, \{ \tilde{Y}^n \}, \{ \tilde{B}^n \}, \) and \( \{ \tilde{\eta}^n \} \) in \( \mathcal{D}_\infty \), \( \mathcal{D} \), \( \mathcal{D}_\infty \), and \( \mathcal{D} \) respectively. This will complete the proof.

Consider first \( \{ \tilde{v}^n \} \). Note that \([0, T] \times [0, 1]\) is a compact metric space. Also from Lemma 3.2(b) and (3.5) we have a.s. for each \( k \in \mathbb{N}_0 \),

\[ \tilde{v}_n^k([0, T] \times [0, 1]) = \int_{[0,T] \times [0,1]} \varphi_n^k(s, y) \, ds \, dy \leq \int_{[0,T] \times [0,1]} (\ell(\varphi_n^k(s, y)) + 2) \, ds \, dy \leq M_0 + 2T. \]

Hence \( \{ \tilde{v}_n^k \} \) is tight in \( \mathcal{M}([0, T] \times [0, 1]) \).

Next, since \( \tilde{X}^n_k(0) \in [0, 1] \) for \( k \in \mathbb{N} \) a.s., we see from (3.7) and (3.8) that \( \mathcal{C} \)-tightness of \( \{ \tilde{X}^n \} \) in \( \mathcal{D}_\infty \) follows once we show \( \mathcal{C} \)-tightness of \( \{ \tilde{Y}^n \}, \{ \tilde{B}^n \} \) and \( \{ \tilde{\eta}^n \} \).

We now show that \( \{ (\tilde{Y}^n(t), \tilde{B}^n(t), \tilde{\eta}^n(t))) \} \) is tight for each \( t \). From (3.6), (3.9) and (3.10),
We now consider fluctuations of \((\bar{Y}^n, \bar{B}^n, \bar{\eta}^n)\). Recall the filtration \(\{\mathcal{F}_t\}_{0 \leq t \leq T}\). For \(\delta \in [0, T]\), let \(\mathcal{T}_\delta\) be the collection of all \([0, T - \delta]\)-valued stopping times \(\tau\). Note that for \(\tau \in \mathcal{T}_\delta\),

\[
E|\bar{Y}^n(\tau + \delta) - \bar{Y}^n(\tau)| \leq E \left[ \sum_{k=0}^{\infty} (k + 2) \left| \bar{B}^n_k(\tau + \delta) - \bar{B}^n_k(\tau) \right| \right].
\]

Thus in order to argue tightness of \(\{(\bar{Y}^n, \bar{B}^n, \bar{\eta}^n)\}\), by the Aldous–Kurtz tightness criterion (cf. [22] Theorem 2.7) it suffices to show that

\[
\limsup_{\delta \to 0} \limsup_{n \to \infty} \sup_{\tau \in \mathcal{T}_\delta} E \left[ \sum_{k=0}^{\infty} (k + 2) \left| \bar{B}^n_k(\tau + \delta) - \bar{B}^n_k(\tau) \right| + |\bar{\eta}^n(\tau + \delta) - \bar{\eta}^n(\tau)| \right] = 0. \tag{3.13}
\]

From (3.9) and (3.11) it follows that for every \(K \in \mathbb{N}\) and \(M \in (0, \infty)\),

\[
E \left[ \sum_{k=0}^{\infty} (k + 2) \left| \bar{B}^n_k(\tau + \delta) - \bar{B}^n_k(\tau) \right| + |\bar{\eta}^n(\tau + \delta) - \bar{\eta}^n(\tau)| \right]
\leq E \sum_{k=0}^{\infty} \int_{(\tau, \tau + \delta] \times [0, 1]} (k + 4) \varphi^n_k(s, y) 1_{[0, r_k(\bar{X}^n(s))]}(y) \, ds \, dy
\leq E \sum_{k=0}^{K-1} \int_{(\tau, \tau + \delta] \times [0, 1]} (k + 4) \varphi^n_k(s, y) 1_{\{\varphi^n_k(s, y) > M\}} \, ds \, dy
\quad + \int_{(\tau, \tau + \delta] \times [0, 1]} (k + 4) \varphi^n_k(s, y) 1_{\{\varphi^n_k(s, y) \leq M\}} \, ds \, dy
\quad + 5\bar{U}_K.
\]

Using Lemma 3.2 (a) and 3.3, we can bound the last display by

\[
E \sum_{k=0}^{K-1} \int_{(\tau, \tau + \delta] \times [0, 1]} (K + 3) \gamma(M) \ell(\varphi^n_k(s, y)) \, ds \, dy + K(K + 3)M\delta + 5\bar{U}_K
\leq (K + 3)\gamma(M)M_0 + K(K + 3)M\delta + 5\bar{U}_K.
\]

Therefore

\[
\limsup_{\delta \to 0} \limsup_{n \to \infty} \sup_{\tau \in \mathcal{T}_\delta} E \left[ \sum_{k=0}^{\infty} (k + 2) \left| \bar{B}^n_k(\tau + \delta) - \bar{B}^n_k(\tau) \right| + |\bar{\eta}^n(\tau + \delta) - \bar{\eta}^n(\tau)| \right]
\leq (K + 3)\gamma(M)M_0 + 5\bar{U}_K.
\]

Taking \(M \to \infty\) and then \(K \to \infty\), we have from Lemmas 3.2 (a) and 3.3 that (3.13) holds. Finally \(C\)-tightness is immediate from the following a.s. bounds and Assumption 2.2 for any \(k \in \mathbb{N}_0, K \in \mathbb{N}\) and \(t \in (0, T)\),

\[
|\bar{B}^n_k(t) - \bar{B}^n_k(t^-)| \leq \frac{1}{n}, \quad |\bar{\eta}^n(t) - \bar{\eta}^n(t^-)| \leq \frac{2}{n}, \quad |\bar{Y}^n_k(t) - \bar{Y}^n_k(t^-)| \leq \frac{K}{n} + \sum_{j=K+1}^{\infty} \frac{jn_j}{n}.
\]

This completes the proof. \(\square\)

Next we will characterize weak limit points of \(\{(\nu^n, X^n, \bar{Y}^n, \bar{B}^n, \bar{\eta}^n)\}\). For that, we need the following notation. For \(k \in \mathbb{N}_0\) define the compensated process

\[
\tilde{N}^n_{\nu^n_k}(ds \, dy) \equiv N^n_{\nu^n_k}(ds \, dy) - n\nu^n_k(s, y) \, ds \, dy.
\]

---

19
Then $\tilde{N}_k^n([0, t] \times A)$ is an $\{\mathcal{F}_t\}$-martingale for $A \in \mathcal{B}([0, 1])$ and $k \in \mathbb{N}_0$. Let

$$B_k^n(t) = \tilde{B}_k^n(t) - \tilde{B}_k^n(t), \quad t \in [0, T], \quad k \in \mathbb{N}_0.$$  \hfill (3.14)

where

$$\tilde{B}_k^n(t) \doteq \frac{1}{n} \int_{[0,t] \times [0,1]} 1_{[0, r_k(\tilde{X}^n(s-))]}(y) \tilde{N}^{n^n}_k(ds \, dy)$$

is an $\{\mathcal{F}_t\}$-martingale and

$$\tilde{B}_k^n(t) \doteq \int_{[0,t] \times [0,1]} 1_{[0, r_k(\tilde{X}^n(s))]}(y) \tilde{N}^{n^n}_k(s, y) \, ds \, dy.$$  \hfill (3.15)

Write $\tilde{B}^n \doteq (\tilde{B}_k^n)_{k \in \mathbb{N}_0}$ and $\hat{B}^n \doteq (\hat{B}_k^n)_{k \in \mathbb{N}_0}$. Let $\lambda_t$ be Lebesgue measure on $[0, t] \times [0, 1]$.

We have the following characterization of the weak limit points. Recall $S_T(\zeta, \psi)$ defined in (2.14) and (2.15).

**Lemma 3.3.** Suppose Assumptions 2.1 and 2.2 hold. Also assume that the bound (3.5) is satisfied and suppose that $(\nu^n, X^n, \bar{Y}^n, B^n, \bar{\eta}^n)$ converges along a subsequence, in distribution, to $(\nu, \bar{X}, \bar{Y}, \bar{B}, \bar{\eta}) \in [\mathcal{M}([0, T] \times [0, 1])]^\infty \times \mathcal{D}_\infty \times \mathcal{D} \times \mathcal{D}_\infty \times \mathcal{D}$ given on some probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$. Then the following holds $\mathbb{P}^*$-a.s.

(a) For each $k \in \mathbb{N}_0$, $\nu_k \ll \lambda_T$.

(b) $(\bar{X}, \bar{Y}, \bar{B}, \bar{\eta}) \in C_\infty \times C \times C_\infty \times C$, and for $t \in [0, T]$

$$\bar{X}_k(t) = p_k - \bar{B}_k(t) \geq 0, \quad k \in \mathbb{N},$$  \hfill (3.16)

$$\bar{Y}(t) = \sum_{k=0}^{\infty} (k - 2) \bar{B}_k(t),$$  \hfill (3.17)

$$\bar{X}_0(t) = \bar{Y}(t) + \bar{\eta}(t) \geq 0.$$  \hfill (3.18)

(c) For $k \in \mathbb{N}_0$ let $\varphi_k(s, y) \doteq \frac{d\nu_k}{d\lambda_T}(s, y), \quad (s, y) \in [0, T] \times [0, 1]$. Then for $t \in [0, T]$ and $k \in \mathbb{N}_0$

$$\bar{B}_k(t) = \int_{[0,t] \times [0,1]} 1_{[0, r_k(\bar{X}(s))]}(y) \varphi_k(s, y) \, ds \, dy.$$  \hfill (3.19)

(d) $\bar{X}_0 = \Gamma(\bar{Y})$. In particular, $(\bar{X}, \bar{Y}) \in C_T$ and $\varphi \in S_T(\bar{X}, \bar{Y})$.

**Proof.** Assume without loss of generality that $(\nu^n, X^n, \bar{Y}^n, B^n, \bar{\eta}^n) \Rightarrow (\nu, \bar{X}, \bar{Y}, \bar{B}, \bar{\eta})$ along the whole sequence as $n \rightarrow \infty$.

(a) This is an immediate consequence of the bound in (3.3) and Lemma A.1 of [7].

(b) The first statement is an immediate consequence of the $C$-tightness argued in the proof of Lemma 3.4. Then using (3.8), Assumption 2.1 and the fact that $X^n_k(t) \geq 0$ a.s., we have (3.15). Next, note that by Assumption 2.2 as $K \rightarrow \infty$

$$\sup_{n \in \mathbb{N}} \sup_{0 \leq t \leq T} \left| \sum_{k=K}^{\infty} (k - 2) B_k^n(t) \right| \leq \sup_{n \in \mathbb{N}} \sum_{k=K}^{\infty} \frac{kn_k}{n} \leq K^{-\varepsilon_p} \sup_{n \in \mathbb{N}} \sum_{k=K}^{\infty} \frac{kn_k}{n} \rightarrow 0.$$  \hfill (3.19)

Hence $\sum_{k=0}^{\infty} (k - 2) B_k^n \Rightarrow \sum_{k=0}^{\infty} (k - 2) \bar{B}_k \in C$. From this and (3.6) we see that (3.16) holds. Next, since $(\bar{Y}^n, \bar{\eta}^n) \Rightarrow (\bar{Y}, \bar{\eta}) \in C^2$ and $X^n_0(t) \geq -\frac{1}{n}$ a.s., we have from (3.7) that (3.17) holds.

20
(c) By Doob's inequality, as $n \to \infty$

$$E^{\infty} \sup_{k=0}^{\infty} \int_{0 \leq t \leq T} |\tilde{B}^n_k(t)|^2 \leq \frac{4}{n} E^{\infty} \sum_{k=0}^{\infty} \int_{0,T} \varphi^n_k(s,y) \mathbb{1}_{[0,r_k(\bar{X}^n(s))]}(y) \, ds \, dy$$

$$\leq \frac{4}{n} E^{\infty} \sum_{k=0}^{\infty} \int_{0,T} \left[ \ell(\varphi^n_k(s,y)) + 2 \right] \mathbb{1}_{[0,r_k(\bar{X}^n(s))]}(y) \, ds \, dy$$

$$\leq \frac{4}{n} (M_0 + 2T) \to 0,$$

where the second inequality follows from Lemma \ref{lem3.2}(b) and the third inequality follows from \ref{lem3.5}. Therefore as $n \to \infty$

$$\tilde{B}^n \Rightarrow 0. \quad (3.20)$$

By appealing to the Skorokhod representation theorem, we can assume without loss of generality that $(\tilde{\nu}^n, \bar{\bar{X}}^n, \bar{\bar{Y}}^n, \bar{B}^n, \bar{\bar{\eta}}^n, \bar{\bar{B}}^n) \to (\nu, \bar{X}, \bar{Y}, \bar{B}, \bar{\eta}, 0)$ a.s. on $(\Omega^*, \mathcal{F}^*, P^*)$, namely there exists some event $F \in \mathcal{F}^*$ such that $P^*(F^c) = 0$ and

$$(\tilde{\nu}^n, \bar{\bar{X}}^n, \bar{\bar{Y}}^n, \bar{B}^n, \bar{\bar{\eta}}^n, \bar{\bar{B}}^n) \to (\nu, \bar{X}, \bar{Y}, \bar{B}, \bar{\eta}, 0) \quad \text{on } F.$$

Fix $\bar{\bar{\omega}} \in F$. The rest of the argument will be made for such an $\bar{\bar{\omega}}$ which will be suppressed from the notation. From \ref{lem3.14} we have that as $n \to \infty$

$$r(\bar{X}^n(t)) = (\bar{X}^n_0(t))^+ + \sum_{k=1}^{\infty} k \bar{X}^n_k(t) \to (\bar{X}_0(t))^+ + \sum_{k=1}^{\infty} k \bar{X}_k(t) = r(\bar{X}(t))$$

uniformly in $t \in [0,T]$, and $r(\bar{X}(.))$ is continuous. Let $\bar{\bar{\tau}} = \tau_{\bar{X}}$, where $\tau_{\bar{X}}$ is defined through \ref{eq:tau}, namely $\bar{\bar{\tau}} = \inf\{t \in [0,T] : r(\bar{X}(t)) = 0\} \wedge T$. We will argue that \ref{eq:3.18} holds for all $t < \bar{\bar{\tau}}, t = \bar{\bar{\tau}}$ and $t > \bar{\bar{\tau}}$.

For $t < \bar{\bar{\tau}}$, we have $r(\bar{X}(t)) > 0$. Hence for each $k \in \mathbb{N}_0,$

$$1_{[0,r_k(\bar{X}^n(s))]}(y) \to 1_{[0,r_k(\bar{X}(s))]}(y) \quad (3.21)$$

as $n \to \infty$ for $\lambda_t$-a.e. $(s,y) \in [0,t] \times [0,1]$ since $\lambda_t\{(y,s) : y = r_k(\bar{X}(s))\} = 0$. From \ref{eq:3.21} and the uniform integrability of $(s,y) \mapsto 1_{[0,r_k(\bar{X}^n(s))]}(y) - 1_{[0,r_k(\bar{X}(s))]}(y) \varphi^n_k(s,y)$ (with respect to the normalized Lebesgue measure on $[0,T] \times [0,1]$) which follows from \ref{lem3.5} and the superlinearity of $\ell$, we have that

$$\int_{[0,t] \times [0,1]} 1_{[0,r_k(\bar{X}(s))]}(y) \varphi^n_k(s,y) \, ds \, dy \to 0.$$

Also, from the bound in \ref{lem3.5} it follows that

$$\int_{[0,t] \times [0,1]} 1_{[0,r_k(\bar{X}(s))]}(y) \varphi^n_k(s,y) \, ds \, dy \to \int_{[0,t] \times [0,1]} 1_{[0,r_k(\bar{X}(s))]}(y) \varphi_k(s,y) \, ds \, dy.$$

Combining the two convergence statements we have

$$\tilde{B}^n_k(t) \to \int_{[0,t] \times [0,1]} 1_{[0,r_k(\bar{X}(s))]}(y) \varphi_k(s,y) \, ds \, dy. \quad (3.22)$$

The above convergence along with \ref{eq:3.14} and \ref{eq:3.20} gives \ref{eq:3.18} for $t < \bar{\bar{\tau}}$. Since \ref{eq:3.18} holds for $t < \bar{\bar{\tau}}$, it also holds for $t = \bar{\bar{\tau}}$ by continuity of $\tilde{B}$ and of the right side in \ref{eq:3.18}.

Now suppose $T \geq t > \bar{\bar{\tau}}$. Since $r(\bar{X}(\cdot))$ is continuous, we see from the definition of $\bar{\bar{\tau}}$ that $r(\bar{X}(\bar{\bar{\tau}})) = 0$. Noting that $r(\bar{X}^n(\cdot))$ is non-negative and non-increasing, so is $r(\bar{X}(\cdot))$. Therefore $r(\bar{X}(t)) = 0$ and
\( \dot{X}(t) = 0 \) for \( \tilde{\tau} \leq t \leq T \). From this we see that the right hand side of (3.18) remains constant for \( \tilde{\tau} \leq t \leq T \) and it suffices to show that \( B(t) = B(\tilde{\tau}) \) for \( \tilde{\tau} < t \leq T \). From (3.9) it follows that, for \( k \in \mathbb{N} \), \( \sup_{\tilde{\tau} \leq t \leq T} |\tilde{B}_k^n(t) - \tilde{B}_k^n(\tilde{\tau})| = \tilde{B}_k^n(T) - \tilde{B}_k^n(\tilde{\tau}) = X_k^n(\tilde{\tau}) - \tilde{X}_k^n(T) \leq \tilde{X}_k^n(\tilde{\tau}), \quad (3.23) \)

which converges to \( \bar{X}_k(\tilde{\tau}) = 0 \) as \( n \to \infty \). Hence \( \tilde{B}_k^n(t) = \tilde{B}_k^n(\tilde{\tau}) \) for \( \tilde{\tau} < t \leq T \) and this gives (3.18) for each \( k \in \mathbb{N} \). Next we show \( \tilde{B}_0(t) = \tilde{B}_0(\tilde{\tau}) \) for \( \tilde{\tau} < t \leq T \). From (3.9) and (3.14),

\[
\sup_{\tilde{\tau} < t \leq T} |\tilde{B}_0^n(t) - \tilde{B}_0^n(\tilde{\tau})| \\
\leq \sup_{\tilde{\tau} < t \leq T} |\tilde{X}_0^n(t) - \tilde{X}_0^n(\tilde{\tau})| + \sup_{\tilde{\tau} < t \leq T} |\bar{\eta}^n(t) - \bar{\eta}^n(\tilde{\tau})| + \sum_{k=1}^{\infty} |k - 2| \sup_{\tilde{\tau} < t \leq T} |\tilde{B}_k^n(t) - \tilde{B}_k^n(\tilde{\tau})|. \quad (3.24)
\]

Since \( \tilde{X}_0^n(t) \geq -\frac{1}{n} \), we have

\[
\sup_{\tilde{\tau} < t \leq T} |\tilde{X}_0^n(t) - \tilde{X}_0^n(\tilde{\tau})| \leq \sup_{\tilde{\tau} < t \leq T} |\tilde{X}_0^n(t)| + |\tilde{X}_0^n(\tilde{\tau})| \leq \sup_{\tilde{\tau} < t \leq T} (\tilde{X}_0^n(t))^+ + \frac{1}{n} + (\tilde{X}_0^n(\tilde{\tau}))^+ + \frac{1}{n} \\
\leq \sup_{\tilde{\tau} < t \leq T} r(\tilde{X}^n(\tilde{\tau})) + r(\tilde{X}^n(t)) + \frac{2}{n} \leq 2r(\tilde{X}^n(\tilde{\tau})) + \frac{2}{n},
\]

where the last line follows from the fact that \( r(\tilde{X}^n(t)) \) is non-increasing for \( t \in [0, T] \). From (3.10) and (3.9) it follows that

\[
\sup_{\tilde{\tau} < t \leq T} |\bar{\eta}^n(t) - \bar{\eta}^n(\tilde{\tau})| \\
= \sup_{\tilde{\tau} < t \leq T} 2 \sum_{k=1}^{\infty} \frac{1}{n} \int_{(\tilde{\tau}, t] \times [0, 1]} 1_{\{\tilde{X}_k^n(\tilde{\tau}) < 0\}} 1_{[0, r_k(\tilde{X}^n(\tilde{\tau}))]}(y) N_k^{n\tilde{\eta}^n}(du, dy) \\
\leq \sup_{\tilde{\tau} < t \leq T} 2 \sum_{k=1}^{\infty} \frac{1}{n} \int_{(\tilde{\tau}, t] \times [0, 1]} 1_{[0, r_k(\tilde{X}^n(\tilde{\tau}))]}(y) N_k^{n\tilde{\eta}^n}(du, dy) \\
= \sup_{\tilde{\tau} < t \leq T} 2 \sum_{k=1}^{\infty} |\tilde{B}_k^n(t) - \tilde{B}_k^n(\tilde{\tau})|.
\]

Combining above two estimates with (3.24), we see that as \( n \to \infty \),

\[
\sup_{\tilde{\tau} < t \leq T} |\tilde{B}_0^n(t) - \tilde{B}_0^n(\tilde{\tau})| \leq 2r(\tilde{X}^n(\tilde{\tau})) + \frac{2}{n} + \sup_{\tilde{\tau} < t \leq T} \sum_{k=1}^{\infty} (k + 4)|\tilde{B}_k^n(t) - \tilde{B}_k^n(\tilde{\tau})| \\
\leq 2r(\tilde{X}^n(\tilde{\tau})) + \frac{2}{n} + \sum_{k=1}^{\infty} (k + 4)\tilde{X}_k^n(\tilde{\tau}) \leq 7r(\tilde{X}^n(\tilde{\tau})) + \frac{2}{n} \quad (3.25)
\]

\[
\rightarrow 7r(\tilde{X}(\tilde{\tau})) = 0,
\]

where the second inequality follows from (3.23). Since we have proved (3.18) for all \( t < \tilde{\tau}, t = \tilde{\tau} \) and \( t > \tilde{\tau} \), part (c) follows.

(d) From (3.17) and a well known characterization of the solution of the Skorohod problem (see, e.g., [21 Section 3.6.C]), it suffices to show that \( \tilde{\eta}(0) = 0, \tilde{\eta}(t) \geq 0, \tilde{\eta}(t) \) is non-decreasing for \( t \in [0, T] \) and \( \int_0^T \tilde{X}_0(t) \tilde{\eta}(dt) = 0 \). Since \( \tilde{\eta}^n(0) = 0, \tilde{\eta}^n(t) \geq 0 \) and \( \tilde{\eta}^n(t) \) is non-decreasing for \( t \in [0, T] \), so is \( \tilde{\eta} \). It remains to show \( \int_0^T \tilde{X}_0(t) \tilde{\eta}(dt) = 0 \). Note that \( \tilde{\eta}^n(t) \) increases only when \( \tilde{X}_0^n(t-) < 0 \), namely \( \tilde{X}_0^n(t-) = -\frac{1}{n} \). Therefore

\[
\int_0^T \left( \tilde{X}_0^n(t-) + \frac{1}{n} \right) \tilde{\eta}^n(dt) = 0.
\]

22
From this we have
\[
\left| \int_0^T \tilde{X}_0(t) \tilde{\eta}(dt) \right| = \left| \int_0^T \tilde{X}_0(t) \tilde{\eta}(dt) - \int_0^T \left( \tilde{X}_0^n(t) + \frac{1}{n} \right) \tilde{\eta}^n(dt) \right| \\
\leq \left| \int_0^T \tilde{X}_0(t) \tilde{\eta}(dt) - \int_0^T \tilde{X}_0^n(t) \tilde{\eta}^n(dt) \right| + \int_0^T |\tilde{X}_0(t) - \tilde{X}_0^n(t)| \tilde{\eta}^n(dt) + \frac{\tilde{\eta}^n(T)}{n},
\]
(3.26)
Since both \(\tilde{\eta}^n\) and \(\tilde{\eta}\) are non-decreasing, we see that \(\tilde{\eta}^n \rightarrow \tilde{\eta}\) as measures on \([0, T]\). Combining this with the fact that \(\tilde{X}_0 \in C_b([0, T] : \mathbb{R})\), we get
\[
\left| \int_0^T \tilde{X}_0(t) \tilde{\eta}(dt) - \int_0^T \tilde{X}_0^n(t) \tilde{\eta}^n(dt) \right| \rightarrow 0
\]
as \(n \rightarrow \infty\). Also from continuity of \(\tilde{X}_0\), we have uniform convergence of \(\tilde{X}_0^n\) to \(\tilde{X}_0\) and hence
\[
\int_0^T |\tilde{X}_0(t) - \tilde{X}_0^n(t)| \tilde{\eta}^n(T) \leq \left( \sup_{0 \leq t \leq T} |\tilde{X}_0^n(t) - \tilde{X}_0(t)| + \frac{1}{n} \right) \tilde{\eta}^n(T) \rightarrow 0
\]
as \(n \rightarrow \infty\). Combining these two convergence results with (3.26), we see that \(\int_0^T \tilde{X}_0(t) \tilde{\eta}(dt) = 0\). This proves part (d) and completes the proof.

4 Laplace upper bound

In this section we prove the Laplace upper bound (2.17).

From (3.1), for every \(n \in \mathbb{N}\), we can choose \(\hat{\varphi}^n \equiv (\hat{\varphi}^n_k)_{k \in \mathbb{N}_0} \in \mathcal{A}_b\) such that
\[
-\frac{1}{n} \log Ee^{-h(X^n,Y^n)} \geq E \left\{ \sum_{k=0}^{\infty} \int_{[0,T] \times [0,1]} \ell(\hat{\varphi}^n_k(s,y)) ds dy + h(X^n,\hat{\varphi}^n, Y^n,\hat{\varphi}^n) \right\} - \frac{1}{n},
\]
where \((X^n,\hat{\varphi}^n, Y^n,\hat{\varphi}^n)\) are defined by (3.3)–(3.4) by replacing \(\varphi^n\) with \(\hat{\varphi}^n\). Since \(\|h\|_{\infty} < \infty\),
\[
\sup_{n \in \mathbb{N}} \sum_{k=0}^{\infty} \int_{[0,T] \times [0,1]} \ell(\hat{\varphi}^n_k(s,y)) ds dy \leq 2\|h\|_{\infty} + 1 = M_h.
\]
Now we modify \(\hat{\varphi}^n\) so that the last inequality holds not in the sense of expectation, but rather almost surely, for a possibly larger constant [see (3.5)]. Fix \(\sigma \in (0, 1)\) and define
\[
\hat{r}^n \equiv \inf \left\{ t \in [0, T] : \sum_{k=0}^{\infty} \int_{[0,t] \times [0,1]} \ell(\hat{\varphi}^n_k(s,y)) ds dy > 2M_h\|h\|_{\infty}/\sigma \right\} \wedge T.
\]
For \(k \in \mathbb{N}_0\), letting \(\varphi^*_{n,k}(s,y) \equiv \hat{\varphi}^*_{n,k}(s,y)1_{\{s \leq \hat{r}^n\}} + 1_{\{s > \hat{r}^n\}}, \ (s,y) \in [0, T] \times [0,1]\), we have \(\varphi^n \equiv (\varphi^*_{n,k})_{k \in \mathbb{N}_0} \in \mathcal{A}_b\) since \(\hat{r}^n\) is an \(\mathcal{F}_t\)-stopping time. Also
\[
E \sum_{k=0}^{\infty} \int_{[0,T] \times [0,1]} \ell(\varphi^*_{n,k}(s,y)) ds dy \leq E \sum_{k=0}^{\infty} \int_{[0,T] \times [0,1]} \ell(\hat{\varphi}^n_k(s,y)) ds dy
\]
and
\[
P(\varphi^n \neq \hat{\varphi}^n) \leq P\left( \sum_{k=0}^{\infty} \int_{[0,T] \times [0,1]} \ell(\hat{\varphi}^n_k(s,y)) ds dy > 2M_h\|h\|_{\infty}/\sigma \right)
\leq \frac{\sigma}{2M_h\|h\|_{\infty}} E \sum_{k=0}^{\infty} \int_{[0,T] \times [0,1]} \ell(\hat{\varphi}^n_k(s,y)) ds dy \leq \frac{\sigma}{2\|h\|_{\infty}}.
\]
Letting $(\bar{X}^n, \phi^n, \bar{Y}^n, \psi^n)$ be defined through (3.2)–(3.11) using $\phi^n$, we have
\[
\left| Eh(\bar{X}^n, \phi^n, \bar{Y}^n, \psi^n) - Eh(\bar{X}^n, \phi^n, \bar{Y}^n, \phi^n) \right| \leq 2\|h\|_\infty P(\phi^n \neq \phi^n) \leq \sigma.
\]
Hence we have
\[
-\frac{1}{n} \log E e^{-nh(\bar{X}^n, \bar{Y}^n)} \geq E \left\{ \sum_{k=0}^\infty \int_{[0,T] \times [0,1]} \ell(\phi_k^n(s, y)) \, ds \, dy + h(\bar{X}^n, \bar{Y}^n) \right\} - \frac{1}{n} - \sigma
\]
and
\[
\sup_n \sum_{k=0}^\infty \int_{[0,T] \times [0,1]} \ell(\phi_k^n(s, y)) \, ds \, dy \leq 2Mh/\sigma \leq K_0, \text{ a.s. } P.
\]

Now we can complete the proof of the Laplace upper bound. Recall that $h \in C_0(D_\infty \times D)$. Write $(\bar{p}^n, \bar{X}^n, \bar{Y}^n) \equiv (\bar{p}^n, \phi^n, \bar{X}^n, \psi^n, \bar{Y}^n, \psi^n)$, where $\bar{p}^n$ is as defined in (3.12) using $\phi^n$. Noting from (1.1) that (3.9) is satisfied with $M_0 = K_0$, we have from Lemma 3.1 that $(\bar{p}^n, \bar{X}^n, \bar{Y}^n)$ is tight. Assume without loss of generality that $(\bar{p}^n, \bar{X}^n, \bar{Y}^n)$ converges along the whole sequence weakly to $(\bar{p}, \bar{X}, \bar{Y})$, given on some probability space $(\Omega^*, \mathcal{F}^*, \bar{P}^*)$. By Lemma 3.8 we have $(\bar{X}, \bar{Y}) \in C_T$ and $\bar{p} = \bar{p}^\phi$ for some $\phi \in S_T(\bar{X}, \bar{Y})$ a.s. $\bar{P}^*$, where $\bar{p}^\phi$ is as defined in (3.12) using $\phi$. Owing to the topology used for the measure component and the relation (3.12), Lemma A.1 in [7] (see also [9, Appendix A.4.3, Lemma A.11]) implies the lower semicontinuity of the cost that is needed for the second inequality below. Using Fatou’s lemma and the definition of $I_T$ in (2.12)
\[
\liminf_{n \to \infty} -\frac{1}{n} \log E e^{-nh(\bar{X}^n, \bar{Y}^n)} \geq \liminf_{n \to \infty} E \left\{ \sum_{k=0}^\infty \int_{[0,T] \times [0,1]} \ell(\phi_k^n(s, y)) \, ds \, dy + h(\bar{X}^n, \bar{Y}^n) - \frac{1}{n} - \sigma \right\}
\geq E^* \left\{ \sum_{k=0}^\infty \int_{[0,T] \times [0,1]} \ell(\phi_k(s, y)) \, ds \, dy + h(\bar{X}, \bar{Y}) \right\} - \sigma
\geq \inf_{(\zeta, \psi) \in D_\infty \times D} \{ I_T(\zeta, \psi) + h(\zeta, \psi) \} - \sigma.
\]

Since $\sigma \in (0, 1)$ is arbitrary, this completes the proof of the Laplace upper bound.

## 5 Laplace lower bound

In this section we prove the Laplace lower bound (2.14).

The following lemma, which shows unique solvability of the ODE (2.14) and (2.15) for controls $\phi$ in a suitable class, is key in the proof.

**Lemma 5.1.** Fix $\sigma \in (0, 1)$. Given $(\zeta, \psi) \in C_T$ with $I_T(\zeta, \psi) < \infty$, there exists $\phi^* \in S_T(\zeta, \psi)$ such that

(a) $\sum_{k=0}^\infty \int_{[0,T] \times [0,1]} \ell(\phi_k(s, y)) \, ds \, dy \leq I_T(\zeta, \psi) + \sigma$.

(b) If $(\tilde{\zeta}, \tilde{\psi})$ is another pair in $C_T$ such that $\phi^* \in S_T(\tilde{\zeta}, \tilde{\psi})$, then $(\tilde{\zeta}, \tilde{\psi}) = (\zeta, \psi)$.

**Proof.** Since $I_T(\zeta, \psi) < \infty$, we can choose some $\phi \in S_T(\zeta, \psi)$ such that
\[
\sum_{k=0}^\infty \int_{[0,T] \times [0,1]} \ell(\phi_k(s, y)) \, ds \, dy \leq I_T(\zeta, \psi) + \frac{\sigma}{2}.
\]
Next we will modify $\phi$ to get the desired $\phi^*$. Since $\ell$ is convex and nonnegative and $\ell(1) = 0$, we can assume without loss of generality that for each $k \in \mathbb{N}_0$ and $(t, y) \in [0, T] \times [0, 1]$, $\phi_k(t, y)$ takes the form
\( \varphi_k(t, y) = \rho_k(t)1_{[0, r_k(\zeta(t))])(y) + 1_{[r_k(\zeta(t)), 1]}(y) \) for some \( \rho_k(t) \in [0, \infty) \). Fix \( \epsilon \in (0, 1) \). We will shrink the support of \( \varphi \) to get the desired \( \varphi^\epsilon \) for sufficiently small \( \epsilon \).

For all \( t \in [0, T] \), let

\[
\varphi^\epsilon_k(t, y) = \frac{\rho_k(t)}{1 - \epsilon}1_{[0, (1 - \epsilon)r_k(\zeta(t))])(y) + 1_{[(1 + \epsilon)r_k(\zeta(t)), 1]}(y).
\]

Clearly \( \varphi^\epsilon \in \mathcal{S}_T(\zeta, \psi) \). Note that \( \varphi^\epsilon_k(t, y) = 0 \) for \((1 - \epsilon)r_k(\zeta(t)) < y < (1 + \epsilon)r_k(\zeta(t)) \), which will be a key when we prove uniqueness in part (b). Recall \( \tau_\epsilon \) introduced in (2.10). Then

\[
\sum_{k=0}^{\infty} \int_{[0, T] \times [0, 1]} \epsilon(\varphi^\epsilon_k(t, y)) \, dt \, dy - \sum_{k=0}^{\infty} \int_{[0, T] \times [0, 1]} \epsilon(\varphi_k(t, y)) \, dt \, dy
\]

\[
= \sum_{k=0}^{\infty} \int_{0}^{\tau_\epsilon} \left[ (1 - \epsilon)r_k(\zeta(t)) \epsilon \left( \frac{\rho_k(t)}{1 - \epsilon} \right) + 2\epsilon r_k(\zeta(t)) \epsilon(0) - r_k(\zeta(t)) \epsilon(\rho_k(t)) \right] \, dt
\]

\[
= \sum_{k=0}^{\infty} \int_{0}^{\tau_\epsilon} r_k(\zeta(t)) \left[ \left( \frac{\rho_k(t)}{1 - \epsilon} \right) - \rho_k(t) + 1 - \epsilon \right] + 2\epsilon
\]

\[
= \sum_{k=0}^{\infty} \int_{0}^{\tau_\epsilon} r_k(\zeta(t)) \left[ \rho_k(t) \log \left( \frac{1}{1 - \epsilon} \right) + \epsilon \right] \, dt.
\]

From Lemma 3.2 (b) we have

\[
\sum_{k=0}^{\infty} \int_{[0, T] \times [0, 1]} \epsilon(\varphi_k(t, y)) \, dt \, dy - \sum_{k=0}^{\infty} \int_{[0, T] \times [0, 1]} \epsilon(\varphi_k(t, y)) \, dt \, dy
\]

\[
\leq \sum_{k=0}^{\infty} \int_{0}^{\tau_\epsilon} r_k(\zeta(t)) \left[ \left( \rho_k(t) + 2 \right) \log \left( \frac{1}{1 - \epsilon} \right) + \epsilon \right] \, dt
\]

\[
= \log \left( \frac{1}{1 - \epsilon} \right) \sum_{k=0}^{\infty} \int_{[0, T] \times [0, 1]} \epsilon(\varphi_k(t, y)) \, dt \, dy + 2\tau_\epsilon \log \left( \frac{1}{1 - \epsilon} \right) + T \epsilon
\]

Choosing \( \epsilon \) small enough so that the last display is no larger than \( \frac{\sigma}{2} \), we have

\[
\sum_{k=0}^{\infty} \int_{[0, T] \times [0, 1]} \epsilon(\varphi_k(s, y)) \, ds \, dy \leq \sum_{k=0}^{\infty} \int_{[0, T] \times [0, 1]} \epsilon(\varphi_k(s, y)) \, ds \, dy + \frac{\sigma}{2} \leq I_T(\zeta, \psi) + \sigma.
\]

Part (a) then holds with \( \varphi^* = \varphi^\epsilon \) for such an \( \epsilon \).

We now show that part (b) is satisfied with such a \( \varphi^* \). Suppose that, in addition to \( (\zeta, \psi) \), there is another pair of \( (\tilde{\zeta}, \tilde{\psi}) \) such that \( (\tilde{\zeta}, \tilde{\psi}) \in \mathcal{C}_T \) and \( \varphi^* \in \mathcal{S}_T(\tilde{\zeta}, \tilde{\psi}) \). Let \( \tau = \inf \{ t \in [0, T] : \zeta(t) \neq \tilde{\zeta}(t) \} \wedge T \). We claim that \( \tau = T \). Once the claim is verified, it follows from continuity of \( \zeta \) and \( \tilde{\zeta} \) that \( \zeta(t) = \tilde{\zeta}(t) \) for all \( t \in [0, T] \). Then from (2.14) \( \psi = \tilde{\psi} \) proving part (b).

Now we prove the claim that \( \tau = T \). We will argue via contradiction. Suppose that \( \tau < T \). To complete the proof, it suffices to reach the following contradiction

\[
\zeta(t) = \tilde{\zeta}(t), t \in [\tau, \tau + \delta] \text{ for some } \delta > 0.
\]

From definition of \( \tau \) and (2.14) it follows that \( (\zeta(t), r(\zeta(t)), \psi(t)) = (\tilde{\zeta}(t), r(\tilde{\zeta}(t)), \tilde{\psi}(t)) \) for all \( t < \tau \). From Remark 2.13 (a) we have that \( r(\zeta(\cdot)), r(\tilde{\zeta}(\cdot)) \in \mathcal{C} \). Then by continuity, \( (\zeta(t), r(\zeta(t)), \psi(t)) = (\tilde{\zeta}(t), r(\tilde{\zeta}(t)), \tilde{\psi}(t)) \) for all \( t < \tau \). Therefore, \( \zeta(t) = \tilde{\zeta}(t) \) for all \( t \in [\tau, \tau + \delta] \). This contradicts the definition of \( \tau \). Hence, \( \tau = T \).
\((\tilde{\zeta}(t), r(\tilde{\zeta}(t)), \tilde{\psi}(t))\) for all \(t \leq \tau\). If \(r(\zeta(\tau)) = r(\tilde{\zeta}(\tau)) = 0\), then from Remark 2.13(c) we have \(\zeta(t) = \tilde{\zeta}(t) = 0\) for all \(t \geq \tau\), which gives (5.1). Now we show (5.1) for the remaining case: \(r(\zeta(\tau)) = r(\tilde{\zeta}(\tau)) > 0\). For this, note that by continuity of \(r(\zeta)\) and \(r(\tilde{\zeta})\), there exists some \(\delta > 0\) such that for all \(t \in [\tau, \tau + \delta]\),
\[
  r(\zeta(t)) > 0, r(\tilde{\zeta}(t)) > 0, \quad \left| \frac{r(\zeta(t))}{r(\zeta(t))} - 1 \right| < \varepsilon,
\]
where \(\varepsilon\) is as in part (a) and recall that \(\varphi^* = \varphi^\varepsilon\). We will argue in two steps.

Step 1: We will prove that
\[
  \zeta_k(t) = \tilde{\zeta}_k(t) \quad \text{for all} \quad t \in [\tau, \tau + \delta], k \in \mathbb{N}.
\]  
Suppose not, namely there exists \(k \in \mathbb{N}\) such that \(\tau_k = \inf\{t \in [\tau, \tau + \delta] : \zeta_k(t) \neq \tilde{\zeta}_k(t)\} \land T\) satisfies \(\tau \leq \tau_k < \tau + \delta\). By continuity, we have \(\zeta_k(t) = \tilde{\zeta}_k(t)\) for \(t \leq \tau_k\). We must have \(\zeta_k(\tau_k) = \tilde{\zeta}_k(\tau_k) > 0\), since otherwise \(\zeta_k(\tau_k) = \tilde{\zeta}_k(\tau_k) = 0\) and so from Remark 2.13(b) \(\zeta_k(t) = \tilde{\zeta}_k(t) = 0\) for all \(t \geq \tau_k\), which contradicts the definition of \(\tau_k\). From (5.2) it then follows that
\[
  r_k(\zeta(\tau_k)) = \frac{k\zeta_k(\tau_k)}{r(\zeta(\tau_k))} > 0,
\]
\[
  \left| r_k(\zeta(\tau_k)) - r_k(\tilde{\zeta}(\tau_k)) \right| = \left| \frac{k\zeta_k(\tau_k)}{r(\zeta(\tau_k))} - \frac{k\tilde{\zeta}_k(\tau_k)}{r(\tilde{\zeta}(\tau_k))} \right| = \left| \frac{k\zeta_k(\tau_k)}{r(\zeta(\tau_k))} - \frac{k\tilde{\zeta}_k(\tau_k)}{r(\tilde{\zeta}(\tau_k))} \right| < \varepsilon r_k(\zeta(\tau_k)).
\]
Once more by continuity, there exists some \(\delta_k > 0\) such that last two inequalities hold for \(t \in [\tau_k, \tau_k + \delta_k]\), namely
\[
  r_k(\zeta(t)) > 0, \quad (1 - \varepsilon) r_k(\zeta(t)) < r_k(\tilde{\zeta}(t)) < (1 + \varepsilon) r_k(\zeta(t)).
\]
From construction of \(\varphi^\varepsilon\), we see that for \(t \in [\tau_k, \tau_k + \delta_k]\),
\[
  \int_{(\tau_k, t) \times [0,1]} 1_{[0, r_k(\zeta(s))]}(y) \varphi^\varepsilon_k(s, y) \, ds \, dy = \int_{(\tau_k, t) \times [0,1]} 1_{[0, r_k(\tilde{\zeta}(s))]}(y) \varphi^\varepsilon_k(s, y) \, ds \, dy.
\]
It then follows from (5.2) that \(\zeta_k(t) = \tilde{\zeta}_k(t)\) for all \(t \leq \tau_k + \delta_k\). This contradicts the definition of \(\tau_k\). Therefore (5.3) must hold.

Step 2: We will prove that
\[
  \zeta_0(t) = \tilde{\zeta}_0(t) \quad \text{for all} \quad t \in [\tau, \tau + \delta].
\]
Let \(\eta(t) = \zeta_0(t) - \psi(t)\) and \(\tilde{\eta}(t) = \tilde{\zeta}_0(t) - \tilde{\psi}(t)\). From properties of the Skorokhod map \(\Gamma\) (see, e.g., [21], Section 3.6.C), we have that
\[
  \eta(0) = 0, \eta(t) \text{ is non-decreasing and} \int_0^T \zeta_0(t) \, \eta(dt) = 0,
\]
\[
  \tilde{\eta}(0) = 0, \tilde{\eta}(t) \text{ is non-decreasing and} \int_0^T \tilde{\zeta}_0(t) \, \tilde{\eta}(dt) = 0.
\]
Consider \([\zeta_0(t) - \tilde{\zeta}_0(t)]^2\). Since \(\zeta_0, \psi, \tilde{\zeta}_0, \tilde{\psi}\) are absolutely continuous, we have for \(t \in [\tau, \tau + \delta]\),
\[
  (\zeta_0(t) - \tilde{\zeta}_0(t))^2 = (\zeta_0(\tau) - \tilde{\zeta}_0(\tau))^2 + 2 \int_\tau^t (\zeta_0(s) - \tilde{\zeta}_0(s))(\zeta'_0(s) - \tilde{\zeta}'_0(s)) \, ds
\]
\[
  = 2 \int_\tau^t (\zeta_0(s) - \tilde{\zeta}_0(s))(\psi'(s) - \tilde{\psi}'(s)) \, ds + 2 \int_\tau^t (\zeta_0(s) - \tilde{\zeta}_0(s))(\tilde{\eta}'_0(s) - \tilde{\eta}'_0(s)) \, ds.
\]
From (2.14) and (2.15) we see that for $t \in [\tau, \tau + \delta]$,

$$\psi(t) = \sum_{k=1}^{\infty} (k - 2)(p_k - \zeta_k(t)) - \int_{[0,t] \times [0,1]} 1_{[0, r_0(\zeta(s))]}(y) \varphi_0^*(s, y) ds dy,$$

$${\hat{\psi}}(t) = \sum_{k=1}^{\infty} (k - 2)(p_k - {\hat{\zeta}}_k(t)) - \int_{[0,t] \times [0,1]} 1_{[0, r_0(\zeta(s))]}(y) \varphi_0^*(s, y) ds dy.$$

Taking the difference of these two displays and using (5.3), we have that for $t \in [\tau, \tau + \delta]$,

$$\psi(t) - {\hat{\psi}}(t) = -2 \int_{[0,t] \times [0,1]} \left( 1_{[0, r_0(\zeta(s))]}(y) - 1_{[0, r_0({\hat{\zeta}}(s))]}(y) \right) \varphi_0^*(s, y) ds dy.$$  

(5.8)

Since for each fixed $y \geq 0$ the function $x \mapsto \frac{r}{x+y}$ is non-decreasing on $(-y, \infty)$, we have from (5.3) and (5.2) that if for some $t \in [\tau, \tau + \delta]$, $\zeta_0(t) \geq \zeta_0(t)$, then

$$r_0(\zeta(t)) = \frac{r_0(t)}{\zeta(t) + \sum_{k=1}^{\infty} k\zeta_k(t)} = \frac{r_0(t)}{\zeta(t) + \sum_{k=1}^{\infty} k\zeta_k(t)} = r_0({\hat{\zeta}}(t)).$$

Therefore for $t \in [\tau, \tau + \delta]$,

$$1_{[0, r_0(\zeta(s))]}(y) \geq 1_{[0, r_0({\hat{\zeta}}(s))]}(y) \quad \text{when} \quad \zeta_0(t) \geq \zeta_0(t)$$

and similarly

$$1_{[0, r_0(\zeta(s))]}(y) \leq 1_{[0, r_0({\hat{\zeta}}(s))]}(y) \quad \text{when} \quad \zeta_0(t) \leq \zeta_0(t).$$

Combining these two inequalities with (5.8), we see that

$$(\zeta_0(s) - \zeta_0(s))(\psi'(s) - {\hat{\psi}}'(s)) \leq 0, \quad \text{a.e.} \quad s \in [\tau, \tau + \delta].$$  

(5.9)

Next from (5.5) and (5.6) we see that for $t \in [\tau, \tau + \delta]$,

$$\int_{\tau}^{t} 1_{\{\zeta_0(s) > \zeta_0(s)\}}(\zeta_0(s) - \zeta_0(s))(\eta'_0(s) - \eta'_{\hat{0}}(s)) ds \leq \int_{\tau}^{t} 1_{\{\zeta_0(s) > \zeta_0(s)\}}(\zeta_0(s) - \zeta_0(s))\eta'_0(s) ds$$

$$\leq \int_{\tau}^{t} 1_{\{\zeta_0(s) > 0\}}(\zeta_0(s) \eta_0(ds) = 0,$$

and similarly

$$\int_{\tau}^{t} 1_{\{\zeta_0(s) < \zeta_0(s)\}}(\zeta_0(s) - \zeta_0(s))(\eta'_0(s) - \eta'_{\hat{0}}(s)) ds \leq 0.$$

Combining these two inequalities with (5.9) and (5.7), we have for $t \in [\tau, \tau + \delta]$, $|\zeta_0(t) - \zeta_0(t)|^2 \leq 0$, proving (5.4). Combining (5.3) and (5.4) gives (5.1) and completes the proof.

We can now complete the proof of the Laplace lower bound. Fix $h \in C_b(D_\infty \times D)$ and $\sigma \in (0, 1)$. Fix some $\sigma$-optimal $(\zeta^*, \psi^*) \in C_T$ with $I_T(\zeta^*, \psi^*) < \infty$, namely

$$I_T(\zeta^*, \psi^*) + h(\zeta^*, \psi^*) \leq \inf_{(\zeta, \psi) \in D_\infty \times D} \{ I_T(\zeta, \psi) + h(\zeta, \psi) \} + \sigma.$$

Let $\varphi^* \in S_T(\zeta^*, \psi^*)$ be as in Lemma [5.1] (with $(\zeta, \psi)$ there replaced by $(\zeta^*, \psi^*)$). For each $n \in \mathbb{N}$ and $(s, y) \in [0, T] \times [0, 1]$, consider the deterministic control

$$\varphi^*_k(s, y) = \frac{1}{n} \varphi^*_k(s, y) 1_{\varphi^*_k(s, y) \leq \frac{1}{n}} + \varphi^*_k(s, y) 1_{\frac{1}{n} < \varphi^*_k(s, y) < n} + n \varphi^*_k(s, y) 1_{\varphi^*_k(s, y) \geq n}, \quad k \leq n,$$

$$\varphi^*_k(s, y) = 1, \quad k > n.$$
Then \( \varphi^n \equiv (\varphi^n_k) \in \mathcal{A}_b \), and from (3.1) we have

\[
- \frac{1}{n} \log E \exp \left\{ -n h(X^n, Y^n) \right\} \leq E \left\{ \sum_{k=0}^{\infty} \int_{[0,T] \times [0,1]} \ell(\varphi^n_k(s, y)) \, ds \, dy + h(X^n, Y^n) \right\},
\]

where \((X^n, Y^n)\) are given as in (3.2) - (3.4). Noting that for all \( n \in \mathbb{N}, k \in \mathbb{N}_0 \) and \((s, y) \in [0, T] \times [0, 1], \ell(\varphi^n_k(s, y)) \leq \ell(\varphi^n_k(s, y)) \), we have from Lemma 5.1(a) that (3.3) holds with \( M_0 \) replaced by \( I_T(\zeta^*, \psi^*) + 1 \). Define \( \{\tilde{\nu}^n\} \) as in (3.12) with controls \( \varphi^n \). From Lemma 3.4 it follows that \( \{\tilde{\nu}^n, X^n, Y^n\} \) is tight. Assume without loss of generality that \((\bar{\nu}^n, X^n, Y^n)\) converges along the whole sequence weakly to \((\tilde{\nu}, X, Y)\), given on some probability space \((\Omega^*, \mathcal{F}^*, P^*)\). From the construction of \( \varphi^n \) we must have \( \tilde{\nu} = \nu^* \) a.s. \( P^* \), where \( \nu^* \) is as defined in (3.12) using \( \varphi \). By Lemma 3.5 we have \((X, Y) \in \mathcal{C}_T \) and \( \varphi^* \in \mathcal{S}_T(X, Y) \) a.s. \( P^* \). From Lemma 5.1(b) it now follows that \((X, Y) = (\zeta^*, \psi^*) \) a.s. \( P^* \). Finally, from Lemma 5.1(a),

\[
\limsup_{n \to \infty} - \frac{1}{n} \log E \exp \left\{ -n h(X^n, Y^n) \right\} \leq \limsup_{n \to \infty} E \left\{ \sum_{k=0}^{\infty} \int_{[0,T] \times [0,1]} \ell(\varphi^n_k(s, y)) \, ds \, dy + h(X^n, Y^n) \right\}
\]

\[
\leq \sum_{k=0}^{\infty} \int_{[0,T] \times [0,1]} \ell(\varphi^n_k(s, y)) \, ds \, dy + E^* h(X, Y)
\]

\[
= \sum_{k=0}^{\infty} \int_{[0,T] \times [0,1]} \ell(\varphi^n_k(s, y)) \, ds \, dy + h(\zeta^*, \psi^*)
\]

\[
\leq I_T(\zeta^*, \psi^*) + h(\zeta^*, \psi^*) + \sigma
\]

\[
\leq \inf_{(\zeta, \psi) \in \mathcal{D}_\infty \times \mathcal{D}} \{I_T(\zeta, \psi) + h(\zeta, \psi)\} + 2\sigma.
\]

Since \( \sigma \in (0, 1) \) is arbitrary, this completes the proof of the Laplace lower bound.

### 6 Compact Sub-level Sets

In this section we prove that the function \( I_T \) defined in (2.12) is a rate function, namely the set \( \Gamma_N = \{(\zeta, \psi) \in \mathcal{D}_\infty \times \mathcal{D} : I_T(\zeta, \psi) \leq N\} \) is compact for each fixed \( N \in [0, \infty) \). Since the proof (as is usual) is very similar to the proof of the Laplace upper bound we will only provide details on steps that are significantly different.

Take any sequence \( \{(\zeta^n, \psi^n)_{n\in\mathbb{N}}\} \subset \Gamma_N \). Then \((\zeta^n, \psi^n) \in \mathcal{C}_T \) and there exists some \( \frac{1}{n} \)-optimal \( \varphi^n \in \mathcal{S}_T(\zeta^n, \psi^n), \) namely

\[
\sum_{k=0}^{\infty} \int_{[0,T] \times [0,1]} \ell(\varphi^n_k(s, y)) \, ds \, dy \leq I_T(\zeta^n, \psi^n) + \frac{1}{n} \leq N + \frac{1}{n}, \quad (6.1)
\]

Recalling (2.14) and (2.15) and letting \( \eta^n(t) = \zeta^n_0(t) - \psi^n(t) \), we can write for \( t \in [0, T] \),

\[
\zeta^n_0(t) = \Gamma(\psi^n(t)) = \psi^n(t) + \eta^n(t) = \sum_{k=0}^{\infty} (k - 2) B^n_k(t) + \eta^n(t), \quad (6.2)
\]

where

\[
B^n_k(t) \equiv \int_{[0,T] \times [0,1]} 1_{[0,r_k(\zeta^n(s))]}(y) \varphi^n_k(s, y) \, ds \, dy, \quad k \in \mathbb{N}_0. \quad (6.3)
\]

From standard properties of the one-dimensional Skorokhod Problem we have

\[
\eta^n(0) = 0, \quad \eta^n(t) \text{ is non-decreasing and } \int_0^T 1_{\{\zeta^n_0(t) > 0\}} \eta^n(dt) = 0. \quad (6.4)
\]
Write $B^n = (B^n_k)_{n \in \mathbb{N}_0}$ and let $\nu^n$ be defined as in (3.12) with deterministic controls $\varphi^n$. The following lemma shows that $\{(\nu^n, \zeta^n, \psi^n, B^n, \eta^n)\}$ is pre-compact. The proof is similar to that of Lemma 3.1 and is therefore omitted.

**Lemma 6.1.** $\{(\nu^n, \zeta^n, \psi^n, B^n, \eta^n)\}$ is pre-compact in $[\mathcal{M}_{FC}([0, T] \times [0, 1])]^\infty \times \mathcal{C}_{\infty} \times \mathcal{C} \times \mathcal{C}_{\infty} \times \mathcal{C}$.

The following lemma characterizes limit points of $(\nu^n, \zeta^n, \psi^n, B^n, \eta^n)$.

**Lemma 6.2.** Suppose $(\nu^n, \zeta^n, \psi^n, B^n, \eta^n)$ converges along a subsequence to $(\nu, \zeta, \psi, B, \eta) \in [\mathcal{M}([0, T] \times [0, 1])]^\infty \times \mathcal{C}_{\infty} \times \mathcal{C} \times \mathcal{C}_{\infty} \times \mathcal{C}$. Then the following hold.

(a) For each $k \in \mathbb{N}_0$, $\nu_k \ll \lambda_T$, and letting $\varphi_k \equiv \frac{dm}{d\lambda_T}$, $\sum_{k=0}^\infty \int_{[0, T] \times [0, 1]} f(\varphi_k(s, y)) ds dy \leq N$.

(b) For each $t \in [0, T]$,

$$\zeta_0(t) = \Gamma(\psi)(t) = \psi(t) + \eta(t), \quad \psi(t) = \sum_{k=0}^\infty (k - 2)B_k(t)$$

$$\zeta_k(t) = p_k - B_k(t), \quad k \in \mathbb{N}.$$  

(c) For each $t \in [0, T]$,

$$B_k(t) = \int_{[0, t] \times [0, 1]} 1_{[0, r_k(\zeta(s))]}(y) \varphi_k(s, y) ds dy, \quad k \in \mathbb{N}_0,$$  

and in particular $(\zeta, \psi) \in \mathcal{C}_T$ and $\varphi \in \mathcal{S}_T(\zeta, \psi)$.

**Proof.** Assume without loss of generality that

$$(\nu^n, \zeta^n, \psi^n, B^n, \eta^n) \to (\nu, \zeta, \psi, B, \eta)$$  

as $n \to \infty$ along the whole sequence. Much of the proof is similar to that of Lemma 3.5 except the proof of (6.5) for $k = 0$. Thus we only give details for the latter statement.

From (6.6) and arguments similar to Lemma 3.3, it follows that

$$r(\zeta^n(t)) = (\zeta^n_0(t))^+ + \sum_{k=1}^\infty k\zeta^n_k(t) \to (\zeta_0(t))^+ + \sum_{k=1}^\infty k\zeta_k(t) = r(\zeta(t))$$  

uniformly in $t \in [0, T]$ as $n \to \infty$. Therefore $r(\zeta(\cdot))$ is continuous. Let $\tau = \inf\{t \in [0, T] : r(\zeta(t)) = 0\} \wedge T$. We will argue that (6.5), for $k = 0$, holds for all $t < \tau$, $t = \tau$ and $t > \tau$. The proof of the cases $t < \tau$ and $t = \tau$ is similar to that of (3.18) and is omitted.

Now consider $T \geq t > \tau$. From (6.4) and (6.2), for $\tau < t \leq T$,

$$|\eta^n(t) - \eta^n(\tau)| = \int_{\tau}^{t} \eta^n(s) - \int_{\tau}^{t} 1_{\{\zeta^n_0(s) = 0\}} \eta^n(s) = \int_{\tau}^{t} 1_{\{\zeta^n_0(s) = 0\}} \left( d\zeta^n_0(s) - \sum_{k=0}^\infty (k - 2) dB^n_k(s) \right).$$

From (6.3) we see that $\int_{\tau}^{t} 1_{\{\zeta^n_0(s) = 0\}} dB^n_0(s) = 0$. Also since $\zeta^n_0$ is non-negative and absolutely continuous, we have $1_{\{\zeta^n_0(s) = 0\}}(\zeta^n_0)'(s) = 0$ for a.e. $s \in [0, T]$. Therefore

$$|\eta^n(t) - \eta^n(\tau)| \leq \sum_{k=1}^\infty |k - 2||B^n_k(t) - B^n_k(\tau)|.$$  

29
Applying the triangle inequality to (6.2) and using this estimate, we see that
\[
\sup_{\tau < t \leq T} |B^n_0(t) - B^n_0(\tau)| \leq \sup_{\tau < t \leq T} |\zeta^n_0(t) - \zeta^n_0(\tau)| + 2 \sum_{k=1}^\infty |k - 2| \sup_{\tau < t \leq T} |B^n_k(t) - B^n_k(\tau)|.
\]
Now as in the proof of (3.25) we have \(\sup_{\tau < t \leq T} |B^n_0(t) - B^n_0(\tau)| \leq 4r(\zeta^n(\tau))\), which converges to \(4r(\zeta(\tau)) = 0\) as \(n \to \infty\). Hence \(B^n_0(t) = B_0(\tau)\) for \(\tau < t \leq T\) and this gives (6.5) for \(k = 0\).
Since we have proved (6.6) for \(k = 0\) and all \(t < \tau\), \(t = \tau\) and \(t > \tau\), the proof is complete. \(\square\)

**Proof of compact sub-level sets** \(\Gamma_M\): Now we are ready to prove that \(\Gamma_M\) is compact for each fixed \(M \in [0, \infty)\). Recall \((\zeta^n, \psi^n)\) introduced above (6.1) and \(\nu^n\) introduced above Lemma 6.1. From Lemma [0.1] we have pre-compactness of \(\{(\nu^n, \zeta^n, \psi^n)\}\) in \([M([0, T] \times [0, 1])]^{\infty} \times C_\infty \times C\). Assume without loss of generality that \((\nu^n, \zeta^n, \psi^n)\) converges along the whole sequence to some \((\nu, \zeta, \psi)\). By Lemma 6.2 \((\zeta, \psi) \in C_T\) and \(\nu = \nu^k\), where for \(k \in \mathbb{N}_0\), \(\nu^k\) is as defined by the right side of (3.12) replacing \(\varphi^k\) with \(\varphi_k\), and
\[
I_T(\zeta, \psi) \leq \sum_{k=0}^\infty \int_{[0,T] \times [0,1]} \ell(\varphi_k(s,y)) \, ds \, dy \leq M.
\]
Therefore \((\zeta, \psi) \in \Gamma_M\) which proves that \(\Gamma_M\) is compact.

**Remark 6.3.** Suppose that for all \(n \in \mathbb{N}\), \((\zeta^n, \psi^n) = (\zeta, \psi)\) for some \((\zeta, \psi) \in C_T\) with \(I_T(\zeta, \psi) < \infty\) and \(M = I_T(\zeta, \psi)\). Then taking \(\varphi^k\) satisfying (6.1) (with \((\zeta^n, \psi^n)\) replaced with \((\zeta, \psi)\)), we see from the above argument that there exists some \(\varphi \in S_T(\zeta, \psi)\) such that
\[
I_T(\zeta, \psi) \leq \sum_{k=0}^\infty \int_{[0,T] \times [0,1]} \ell(\varphi_k(s,y)) \, ds \, dy \leq I_T(\zeta, \psi),
\]

namely \(I_T(\zeta, \psi)\) is achieved at some \(\varphi \in S_T(\zeta, \psi)\).

### 7 Calculus of Variations Problem

In this section we study a calculus of variations problem that is key in proof of Theorem 2.6. We begin by giving an overview of the proof strategy. Let \(0 \leq q \leq p\). First note that the set \(E^{n,q}(q)\) can be written as
\[
E^{n,q}(q) = \{\exists t_1, t_2 \in [0, \infty) \text{ such that } X^n_0(t_1) = X^n_0(t_2) = -1/n, X^n_0(t) > -1/n \text{ for } t \in [t_1, t_2), |X^n_k(t_1) - X^n_k(t_2) - q_k| \leq \varepsilon \text{ for all } k \in \mathbb{N}\}.
\]
Here \(t_1\) (resp. \(t_2\)) corresponds to the time instant the first vertex (resp. the last edge) in a component is woken up (resp. is formed).

For \(t_2 \geq t_1 \geq 0\) and \((\zeta, \psi) \in C_{t_2}\), define
\[
I_{t_1, t_2}(\zeta, \psi) = \inf_{\varphi \in S_{t_2}(\zeta, \psi)} \sum_{k=0}^\infty \int_{[t_1, t_2] \times [0,1]} \ell(\varphi_k(s,y)) \, ds \, dy.
\]
Further for \(x^{(1)}, x^{(2)} \in \mathbb{R}^\infty_+\), define
\[
J_{t_1, t_2}^0(x^{(1)}, x^{(2)}) = \{(\zeta, \psi) \in C_{t_2} : \zeta(t_1) = x^{(1)}(t_1), \zeta(t_2) = x^{(2)}(t_2)\},
\]
\[
J_{t_1, t_2}^1(x^{(1)}, x^{(2)}) = \{(\zeta, \psi) \in J_{t_1, t_2}^0(x^{(1)}, x^{(2)}) : \psi(t) \geq 
\]
\[
J_{t_1, t_2}^2(x^{(1)}, x^{(2)}) = \{(\zeta, \psi) \in J_{t_1, t_2}^1(x^{(1)}, x^{(2)}) : \frac{dr(\zeta(t))}{dt} = -2 \text{ for a.e. } t \in (t_1, t_2)\},
\]

30
and
\[ I_j^{I_1,I_2}(x^{(1)},x^{(2)}) \doteq \inf_{(\zeta,\psi) \in \mathcal{J}_{I_1,I_2}(x^{(1)},x^{(2)})} I_{I_1,I_2}(\zeta,\psi), \quad j = 0,1,2. \] (7.3)

Here as usual, the infimum over an empty set is infinity.

The proof of Theorem 2.6 proceeds through the following steps. Let \( \tau \doteq \tfrac{1}{3} \sum_{k=1}^{\infty} kq_k \) and assume \( \sum_{k=1}^{\infty} kq_k > 2 \sum_{k=1}^{\infty} q_k \).

- Lemma 8.4 shows that when \( p_1 \) is 0 the upper and lower bounds coincide.

- Finally Proposition 7.4 shows that
\[ I_{0,\tau}^{2}((0,p),(0,p-q)) = H(q) + H(p-q) - H(p) + K(q) \]

completing the proof of Theorem 2.6.

Note that for \( (\zeta,\psi) \in \mathcal{J}_{I_1,I_2}(x^{(1)},x^{(2)}), \zeta_0(t) = x^{(1)}_0 + \psi(t) - \psi(t_1) \) for \( t \in [t_1,t_2] \). Intuitively, on the event \( \{ (X^n,Y^n) \in \mathcal{J}_{I_1,I_2}(x^{(1)},x^{(2)}) \} \) the exploration remains in the same component over \([t_1,t_2]\), and on the smaller event \( \{ (X^n,Y^n) \in \mathcal{J}_{I_1,I_2}(x^{(1)},x^{(2)}) \} \) the exploration pace matches that for the discrete-time exploration process (with time steps of length \( 1/n \)), in which at each step \( 2 \) half-edges are killed. The main idea in the proof of the theorem is that in characterizing the asymptotics of the probability of interest one can restrict to \( \mathcal{J}_{I_1,I_2}(x^{(1)},x^{(2)}) \), which roughly means that one can restrict to trajectories that avoid the boundary and whose evolution matches that of the original discrete time process of interest removing the artificial “continuous time” aspect of the evolution.

Define for \( x = (x_k)_{k \in \mathbb{N}_0} \in \mathbb{R}_+^{\infty} \) and \( \beta = (\beta_k)_{k \in \mathbb{N}_0} \in \mathbb{R} \times [-1,0]^{\infty} \) with \( \sum_{k=1}^{\infty} \beta_k \geq -1 \),
\[ L(x,\beta) \doteq R(\nu(\cdot|\beta)\|\mu(\cdot|x)) = \sum_{k=0}^{\infty} \nu(k|\beta) \log \left( \frac{\nu(k|\beta)}{\mu(k|x)} \right) \doteq \sum_{k=0}^{\infty} L_k(x,\beta), \] (7.6)
where
\[ \nu(0|\beta) \doteq 1 + \sum_{k=1}^{\infty} \beta_k, \quad \nu(k|\beta) \doteq -\beta_k, \quad k \in \mathbb{N}, \] (7.7)
\[ \mu(k|x) \doteq r_k(x), x \neq 0, \quad \mu(k|x) \doteq 1_{\{k=0\}}, x = 0, \quad k \in \mathbb{N}_0. \] (7.8)

We set \( L(x,\beta) = \infty \), if \( \beta \in \mathbb{R} \times [-1,0]^{\infty} \) and \( \sum_{k=1}^{\infty} \beta_k < -1 \). Note that \( \beta_0 \) actually does not play a role in the definition of \( L(x,\beta) \) or \( \nu(\cdot|\beta) \). Later on \( (x,\beta) \) will be usually replaced by \( (\zeta(t),\zeta'(t)) \) for some absolutely continuous path \( \zeta \) and \( t \geq 0 \).

In the next six lemmas \( x^{(1)} \doteq (x^{(1)}_0,p^{(1)}) \) and \( x^{(2)} \doteq (x^{(2)}_0,p^{(2)}) \) where \( x^{(1)}_0, x^{(2)}_0 \in \mathbb{R}_+ \) and \( 0 \leq p^{(2)} \leq p^{(1)} \leq p \). Let \( z \doteq x^{(1)} - x^{(2)} \). Define
\[ \tau \equiv \tau(x^{(1)},x^{(2)}) \doteq \frac{1}{2} (r(x^{(1)}) - r(x^{(2)})) = \frac{1}{2} \left( (x^{(1)}_0 - x^{(2)}_0) + \sum_{k=1}^{\infty} k(p^{(1)}_k - p^{(2)}_k) \right), \] (7.9)
The following lemma relates \( I^1,I^2 \) and \( L \).
Lemma 7.1. Fix $t_1 \in [0, \infty)$. Suppose $\tau \geq 0$. Let $x^{(0)} \equiv (0, p)$. Suppose there exists some $(\zeta^*, \psi^*) \in J_{t_1, \tau}^{0}(x^{(0)}, x^{(1)})$ such that $I_{0, t_1}(\zeta^*, \psi^*) < \infty$. Then
\begin{equation}
\inf_{t_2 \geq t_1} I_{t_1, t_2}^{1}(x^{(1)}, x^{(2)}) = I_{t_1, t_2}^{2}(x^{(1)}, x^{(2)}).
\end{equation}
Furthermore, for $(\zeta, \psi) \in J_{t_1, t_2}^{1}(x^{(1)}, x^{(2)}),$ 
\begin{equation}
I_{t_1, t_2}(\zeta, \psi) = \int_{t_1}^{t_2} L(\zeta(s), \zeta'(s)) \, ds,
\end{equation}
and if $I_{t_1, t_2}(\zeta, \psi) < \infty,$ then $\sum_{k=1}^{\infty} \zeta_k(t) \geq -1$ for a.e. $t \in [t_1, t_2]$. In particular,
\begin{equation}
\inf_{t_2 \geq t_1} I_{t_1, t_2}^{1}(x^{(1)}, x^{(2)}) = I_{t_1, t_2}^{2}(x^{(1)}, x^{(2)}) = \inf_{(\zeta, \psi) \in J_{t_1, t_2}(x^{(1)}, x^{(2)})} \int_{t_1}^{t_2} L(\zeta(s), \zeta'(s)) \, ds.
\end{equation}

Lemma 7.2. Suppose that $\sum_{k=1}^{\infty} k z_k + z_0 > 2 \sum_{k=1}^{\infty} z_k$ and, $x_0^{(2)} > 0$ or $z_1 > 0$. Then there is a unique $\beta \equiv \beta(x^{(1)}, x^{(2)}) \in (0, 1)$ such that
\begin{equation}
\sum_{k=1}^{\infty} k z_k = (1 - \beta^2) \sum_{k=1}^{\infty} \frac{k z_k}{1 - \beta^k} + x_0^{(2)} - \beta^2 x_0^{(1)}.
\end{equation}

The construction given below will be used to give an explicit representation for the minimizer for the right side of (7.14).

Construction 1. Suppose that either (i) or (ii) holds, where
(i) $x_0^{(2)} = 0$ and $z_1 = 0$.
(ii) $\sum_{k=1}^{\infty} k z_k + z_0 > 2 \sum_{k=1}^{\infty} z_k$ and, $x_0^{(2)} > 0$ or $z_1 > 0$.
Let $\beta \equiv \beta(x^{(1)}, x^{(2)}) \in [0, 1]$ be 0 in case (i) and be the unique solution in $(0, 1)$ of (7.14) in case (ii) (as ensured by Lemma 7.2). Note that $\beta$ satisfies (7.13) in both cases (i) and (ii).
Define $\tau$ as in (7.10) and suppose that $\tau \geq 0$. Let $\bar{\tau} = \tau/(1 - \beta^2)$ and $\tilde{z}_k = z_k/(1 - \beta^k)$ for $k \in \mathbb{N}$. Fix $t_1 \geq 0$ and let $x^{(0)}, (\zeta^*, \psi^*)$ be as in Lemma 7.1. Define $\tilde{\zeta}, \tilde{\psi}$ by $(\tilde{\zeta}(t), \tilde{\psi}(t)) = (\zeta^*(t), \psi^*(t))$ for $t \in [0, t_1]$ and for $t \in [t_1, \tau]$ 
\begin{equation}
\tilde{\zeta}(t) = p_k^{(1)}(t) - \tilde{z}_k \left[ 1 - \left( \frac{1 - t - t_1}{\bar{\tau}} \right)^{k/2} \right], \quad k \in \mathbb{N},
\end{equation}
\begin{equation}
\tilde{\psi}(t) = \tilde{\psi}(t_1) + \sum_{k=1}^{\infty} k (p_k^{(1)}(t) - \tilde{\zeta}(t)) - 2(t - t_1),
\end{equation}
\begin{equation}
\tilde{\zeta}(t) = p_k^{(2)}(t) - \tilde{\psi}(t_1) + \sum_{k=1}^{\infty} k (p_k^{(2)}(t) - \tilde{\zeta}(t)) - 2(t - t_1).
\end{equation}

The next two lemmas give some properties of the various quantities in the above construction. Let 
\[ \Xi = \left\{ (x^{(1)}, x^{(2)}) : \text{for } i = 1, 2, x^{(i)} = (x_0^{(i)}, p^{(i)}), x_0^{(i)} \in \mathbb{R}_+, \right\} \]
\[ 0 \leq p^{(2)} \leq p^{(1)} \leq p \text{ and } \sum_{k=1}^{\infty} k (p_k^{(1)}(t) - p_k^{(2)}(t)) + (x_0^{(1)} - x_0^{(2)}) > 2 \sum_{k=1}^{\infty} (p_k^{(1)} - p_k^{(2)}) \right\}.
\]
We will equip $\Xi$ with the topology corresponding to coordinatewise convergence.
Lemma 7.3. Both \( \beta \) and \( x_0^{(2)} \log \beta \) are continuous on \( \Xi \): for \((x^{(1),n}, x^{(2),n}) \in \Xi\) with \((x^{(1),n}, x^{(2),n}) \rightarrow (x^{(1), x^{(2)})} \in \Xi, \beta^n \equiv \beta(x^{(1),n}, x^{(2),n}) \rightarrow \beta(x^{(1), x^{(2)})} = \beta \) and \( x_0^{(2)} \log \beta^n \rightarrow x_0^{(2)} \log \beta \).

Lemma 7.4. Suppose that \( \tau \geq 0 \). Also suppose that \( \sum_{k=1}^{\infty} k z_k + z_0 > 2 \sum_{k=1}^{\infty} z_k \). Fix \( t_1 \geq 0 \). Let \((\zeta^*, \psi^*)\) be as in Lemma 7.6 and \((\tilde{\zeta}, \tilde{\psi})\) be as introduced in Construction 1. Then

\[
(a) \quad \tau \leq \tilde{\tau} = \frac{1}{2} \left( x_0^{(1)} + \sum_{k=1}^{\infty} k \tilde{z}_k \right).
\]

\[(b) \quad (\tilde{\zeta}, \tilde{\psi}) \in J_{t_1, t_1 + \tau}(x^{(1), x^{(2)})} \).
\]

\[(c) \quad \tilde{\zeta}_0(t) > 0 \text{ for } t \in (t_1, t_1 + \tau).\]

The next lemma calculates \( \int_{t_1}^{t_1 + \tau} L(\zeta(s), \zeta'(s)) \, ds \) for \((\tilde{\zeta}, \tilde{\psi})\) introduced in Construction 1. Recall that \( z = x^{(1)} - x^{(2)} \).

Lemma 7.5. Suppose that \( \tau \geq 0 \). Suppose that either (i) or (ii) in Construction 1 is satisfied. Also, let \((\zeta^*, \psi^*)\) be as in Lemma 7.6 with some \( t_1 \geq 0 \) and let \((\tilde{\zeta}, \tilde{\psi})\) be given as in Construction 1. Define the function \( K(x^{(1)}, x^{(2)}) \) by

\[
K(x^{(1)}, x^{(2)}) = \frac{z_0 + \sum_{k=1}^{\infty} k z_k}{2} \log(1 - \beta^2) - \sum_{k=1}^{\infty} z_k \log(1 - \beta^k) + x_0^{(2)} \log \beta.
\]

For \( x \in \mathbb{R} \times \mathbb{R}^\infty \) such that \( x_0 + \sum_{k=1}^{\infty} k x_k \geq 0 \), define \( \tilde{H}(x) \) by

\[
\tilde{H}(x) = \sum_{k=1}^{\infty} x_k \log x_k - \frac{x_0 + \sum_{k=1}^{\infty} k x_k}{2} \log \frac{x_0 + \sum_{k=1}^{\infty} k x_k}{2}.
\]

Then

\[
\int_{t_1}^{t_1 + \tau} L(\tilde{\zeta}(s), \tilde{\zeta}'(s)) \, ds = \tilde{H}(x) - \tilde{H}(x') + \tilde{K}(x^{(1)}, x^{(2)}) < \infty.
\]

Moreover, the right hand side is lower semicontinuous in \((x^{(1)}, x^{(2)}) \in \Xi\), namely for \((x^{(1),n}, x^{(2),n}) \in \Xi\) with \((x^{(1),n}, x^{(2),n}) \rightarrow (x^{(1), x^{(2)})} \in \Xi,

\[
\liminf_{n \rightarrow \infty} \left( \tilde{H}(z^n) + \tilde{H}(x^{(2),n}) - \tilde{H}(x^{(1),n}) + \tilde{K}(x^{(1),n}, x^{(2),n}) \right) \geq \tilde{H}(z) + \tilde{H}(x^{(2)}) - \tilde{H}(x^{(1)}) + \tilde{K}(x^{(1)}, x^{(2)}),
\]

where \( z^n = x^{(1),n} - x^{(2),n}, z = x^{(1)} - x^{(2)} \).

Recall the functions \( H \) and \( K \) from (2.3) and (2.1) respectively. We note that with \( \tilde{K} \) and \( \tilde{H} \) as introduced in the above lemma, for \( 0 \leq q \leq \bar{q} \leq p \)

\[
H(q) = \tilde{H}(0, q), \quad K(q) = \tilde{K}((0, q), (0, \bar{q} - q)).
\]

(7.17)

The next lemma shows that \((\tilde{\zeta}, \tilde{\psi})\) in Construction 1 is a minimizer for the right side of (7.12).

Lemma 7.6. Suppose that \( \tau \geq 0 \). Suppose that \( \sum_{k=1}^{\infty} k z_k + z_0 > 2 \sum_{k=1}^{\infty} z_k \). Fix \( t_1 \geq 0 \) and let \((\zeta^*, \psi^*)\) be as in Lemma 7.3 and \((\tilde{\zeta}, \tilde{\psi})\) as introduced in Construction 1. Then

\[
I^2_{t_1, t_1 + \tau}(x^{(1), x^{(2)})} = \inf_{(\zeta, \psi) \in J_{t_1, t_1 + \tau}(x^{(1), x^{(2)})} \int_{t_1}^{t_1 + \tau} L(\zeta(s), \zeta'(s)) \, ds = \int_{t_1}^{t_1 + \tau} L(\tilde{\zeta}(s), \tilde{\zeta}'(s)) \, ds.
\]

(7.18)
Proofs of Lemmas 7.1–7.6 are given in Section 9. The following proposition summarizes an important consequence of the above lemmas for the case when $x_0^{(1)} = x_0^{(2)} = 0$.

**Proposition 7.7.**

(a) Suppose $0 \leq q \leq \bar{q} \leq p$ and that either $\sum_{k=1}^{\infty} kq_k > 2\sum_{k=1}^{\infty} q_k$ or $\sum_{k=1}^{\infty} kq_k = 2\sum_{k=1}^{\infty} q_k$ but $p_1 = 0$. Given $t_1 \geq 0$, and with $x^{(0)} = (0, p)$, $x^{(1)} = (0, \bar{q})$, suppose there exists some $(\zeta^*, \psi^*) \in \mathcal{R}_{0,t_1}^0(x^{(0)}, x^{(1)})$ such that $I_0,t_1(\zeta^*, \psi^*) < \infty$. Then

$$
\inf_{t_2 \geq t_1} I_{t_1,t_2}^1((0, \bar{q}), (0, q - \bar{q})) = I_{t_1,t_1+\tau}^2((0, \bar{q}), (0, q - \bar{q})) = H(q) + H(\bar{q} - q) - H(q) + K(q),
$$

where $\tau = \frac{1}{2} \sum_{k=1}^{\infty} kq_k$.

(b) Suppose $p_1 = 0$, $q \geq 0$, $\bar{q} \geq 0$, $q + \bar{q} \leq p$, $\sum_{k=1}^{\infty} kq_k \geq 2\sum_{k=1}^{\infty} q_k$, and $\sum_{k=1}^{\infty} k\bar{q}_k \geq 2\sum_{k=1}^{\infty} q_k$. Let $\tau = \frac{1}{2} \sum_{k=1}^{\infty} kq_k$ and $\bar{\tau} = \frac{1}{2} \sum_{k=1}^{\infty} k\bar{q}_k$. Then

$$
I_{0,\bar{\tau}}^2((0, p), (0, p - \bar{q})) + I_{\bar{\tau},\bar{\tau}+\tau}^2((0, p - \bar{q}), (0, p - q - \bar{q}))
$$

$$
= I_{0,\bar{\tau}}^2((0, p), (0, p - q)) + I_{\bar{\tau},\bar{\tau}+\tau}^2((0, p - q), (0, p - q - \bar{q})).
$$

**Proof.** (a) The first equality in part (a) is a consequence of Lemma 7.1. For the second equality, consider first the case $\sum_{k=1}^{\infty} kq_k > 2\sum_{k=1}^{\infty} q_k$. From (7.17) we have

$$
H(q) + H(\bar{q} - q) - H(\bar{q}) + K(q) = \bar{H}(0, q) + \bar{H}(0, \bar{q} - q) - \bar{H}(0, q) - \bar{K}((0, q), (0, q - q)).
$$

Applying Lemma 7.5 with $x^{(1)} = (0, \bar{q})$, $x^{(2)} = (0, q - \bar{q})$, the above expression equals $\int_{t_1}^{t_1+\tau} L(\bar{\zeta}(s), \bar{\zeta}'(s)) \, ds$ where $\bar{\zeta}$ is defined by (7.31)–(7.33). Now from Lemma 7.6

$$
I_{t_1,t_1+\tau}^2((0, \bar{q}), (0, q - \bar{q})) = H(q) + H(\bar{q} - q) - H(\bar{q}) + K(q)
$$

which proves the second equality in part (a) for the considered case.

Now consider the case $\sum_{k=1}^{\infty} kq_k = 2\sum_{k=1}^{\infty} q_k$ and $p_1 = 0$. Since $p_1 = 0$, we must have $q_k = 0$ for each $k \neq 2$. Then for any $(\zeta, \psi) \in \mathcal{R}_{t_1,t_1+\tau}^2((0, \bar{q}), (0, q - \bar{q}))$ with $I_{t_1,t_1+\tau}(\zeta, \psi) < \infty$, we must have (see (2.13)) and the definition of $\mathcal{R}_{t_1,t_2}^2$ $\zeta_k(t) = -1$ and $\zeta'_k(t) = \psi'(t) = 0$, $k \neq 2$ for $t \in [t_1, t_1+\tau]$. Also, in this case $q_1 = 0$ and so we are in case (i) of Construction 1 with $x^{(1)} = (0, \bar{q})$ and $x^{(2)} = (0, q - \bar{q})$. It is easily checked that any $(\zeta, \psi)$ with the above properties is same as the minimizer $(\bar{\zeta}, \bar{\psi})$ over $[t_1, t_1+\tau]$. Thus using Lemma 7.1 and Lemma 7.3 we get

$$
I_{t_1,t_1+\tau}^2((0, \bar{q}), (0, q - \bar{q})) = \inf_{(\zeta, \psi) \in \mathcal{R}_{t_1,t_1+\tau}^2((x^{(1)}, x^{(2)}))} \int_{t_1}^{t_1+\tau} L(\zeta(s), \zeta'(s)) \, ds
$$

$$
= \int_{t_1}^{t_1+\tau} L(\bar{\zeta}(s), \bar{\zeta}'(s)) \, ds
$$

$$
= \bar{H}(0, q) + \bar{H}(0, \bar{q} - q) - \bar{H}(0, q) + \bar{K}((0, q), (0, q - q))
$$

$$
= H(q) + H(\bar{q} - q) - H(\bar{q}) + K(q).
$$

This proves part (a) in this case.

(b) From part (a),

$$
I_{0,\bar{\tau}}^2((0, p), (0, p - \bar{q})) = H(q) + H(\bar{q} - q) - H(p) + K(q)
$$

and since the right side is finite, again from part (a),

$$
I_{\bar{\tau},\bar{\tau}+\tau}^2((0, p - \bar{q}), (0, p - q - \bar{q})) = H(q) + H(\bar{q} - q) - H(p - q) + K(q).
$$
Therefore,
\[
I_{0,\tau}^2((0, p), (0, p - q)) + I_{r, r + \tau}^2((0, p - q), (0, p - q - q))
= [H(q) + H(p - q) - H(p) + K(q)] + [H(q) + H(p - q) - H(p - q) + K(q)]
= [H(q) + H(p - q) - H(p) + K(q)] + [H(q) + H(p - q) - H(p - q) + K(q)]
= I_{0,\tau}^2((0, p), (0, p - q)) + I_{r, r + \tau}^2((0, p - q), (0, p - q - q)),
\]
where the last line follows, once more, from (a). This proves (b) and completes the proof.

8 Proof of Theorem 2.6

In this section we will use Theorem 2.14 and results in Section 7 to prove Theorem 2.6. Let \( 0 \leq q \leq p \). Recall from Section 2.3 that the discrete-time EEA terminates in at most \( m \) steps where \( m = \lfloor \sum_{k=1}^{n} k \frac{n_k}{n} \rfloor + 1 < \infty \). So since the discrete time EEA is the embedded chain associated with the continuous time EEA (see Section 2.4), \( X^n \) will have at most \( nN \) jumps before arriving at the absorbing state \( (-\frac{1}{n}, 0) \). Since the total jump rate for \( X^n(t) \) at any instant before getting absorbed is \( n \sum_{k=0}^{\infty} r_k(X^n(t)) = n \), we have

\[
P(F^{n,T}) \geq P \left( \sum_{i=1}^{nN} \xi_i \leq T \right) = P \left( \frac{1}{n} \sum_{i=1}^{nN} \xi_i \leq T \right),
\]

where \( \xi_i \) are i.i.d. \( \exp(n) \) and \( \xi_i^\prime \) are i.i.d. \( \exp(1) \). Therefore

\[
\limsup_{n \to \infty} \frac{1}{n} \log P((F^{n,T})^c) \leq N \limsup_{n \to \infty} \frac{1}{nN} \log \mathbb{P} \left( \frac{1}{nN} \sum_{i=1}^{nN} \xi_i > \frac{T}{N} \right)
= -NL_1 \left( \frac{T}{N} \right) \to -\infty
\]
as \( T \to \infty \), where the second equality is from Cramér’s theorem and where \( L_1(x) \doteq x - 1 - \log x \) for \( x \geq 0 \) is the Legendre transform of the log-moment generating function of \( \xi_1 \).
The following lemma gives an upper bound for the exponential rate of decay of \( P(E^{n,\varphi}(q)) \).

**Lemma 8.2.** Suppose \( \sum_{k=1}^{\infty} kq_k > 2 \sum_{k=1}^{\infty} q_k \). Let \( \tau = \frac{1}{2} \sum_{k=1}^{\infty} kq_k \). Then the upper bound in (8.4) holds.

\[
\lim_{n \to \infty} \limsup_{\varepsilon \to 0} \frac{1}{n} \log P(E^{n,\varepsilon}(q)) \leq - \inf_{q \leq \bar{p} \leq \bar{p}, t_1 \geq 0} \left[ I^{0}_{0,\tau}((0, \bar{p}), (0, \bar{p})) + I^{2}_{t_1, t_1+\tau}((0, \bar{p}), (0, \bar{p} - q)) \right]. \tag{8.4}
\]

**Proof.** From (8.3) we have

\[
P(E^{n,\varepsilon}(q)) \leq P(E^{n,\varepsilon,T}(q)) + P((F^{n,T})^c)
\]

and hence

\[
\limsup_{n \to \infty} \frac{1}{n} \log P(E^{n,\varepsilon}(q)) \leq \max \left\{ \limsup_{n \to \infty} \frac{1}{n} \log P(E^{n,\varepsilon,T}(q)), \limsup_{n \to \infty} \frac{1}{n} \log P((F^{n,T})^c) \right\}.
\]

In view of Lemma 8.1 it suffices to show that for all sufficiently large \( T \)

\[
\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log P(E^{n,\varepsilon,T}(q)) \leq - \inf_{q \leq \bar{p} \leq \bar{p}, t_1 \geq 0} \left[ I^{0}_{0,\tau}((0, \bar{p}), (0, \bar{p})) + I^{2}_{t_1, t_1+\tau}((0, \bar{p}), (0, \bar{p} - q)) \right].
\]

Let \( \mathcal{Q}_T = \mathbb{D}([0, T] : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}) \) and consider

\[
\tilde{E}^{\varepsilon,T}(q) = \{(\zeta, \psi) \in \mathcal{Q}_T : \exists t_1, t_2 \in [0, T] \text{ such that } \zeta_0(t_1 -) = \zeta_0(t_2) \leq 0, \psi(t) \geq \psi(t_1) - \varepsilon \text{ for } t \in [t_1, t_2], |\zeta_k(t_1) - \zeta_k(t_2) - q_k| \leq \varepsilon \text{ for all } k \in \mathbb{N}\}.
\]

Denote the closure of \( \tilde{E}^{\varepsilon,T}(q) \) by \( \text{cl}\tilde{E}^{\varepsilon,T}(q) \). From the definition in (8.1), when \( n > 2\varepsilon^{-1} \)

\[
E^{n,\varepsilon,T}(q) \subset \{(X^n, Y^n) \in \tilde{E}^{\varepsilon,T}(q) \} \subset \{(X^n, Y^n) \in \text{cl}\tilde{E}^{\varepsilon,T}(q) \}.
\]

From this and Theorem 2.14 we have

\[
\limsup_{n \to \infty} \frac{1}{n} \log P(E^{n,\varepsilon,T}(q)) \leq \limsup_{n \to \infty} \frac{1}{n} \log P((X^n, Y^n) \in \text{cl}\tilde{E}^{\varepsilon,T}(q)) \leq - \inf_{(\zeta, \psi) \in \text{cl}\tilde{E}^{\varepsilon,T}(q)} I_T(\zeta, \psi).
\]

Since \( I_T(\zeta, \psi) < \infty \) only when \( (\zeta, \psi) \in \mathcal{C}_T \), we have

\[
\limsup_{n \to \infty} \frac{1}{n} \log P(E^{n,\varepsilon,T}(q)) \leq - \inf_{(\zeta, \psi) \in \tilde{E}^{\varepsilon,T}(q) \cap \mathcal{C}_T} I_T(\zeta, \psi).
\]

It is easy to see that \( \text{cl}\tilde{E}^{\varepsilon,T}(q) \cap \mathcal{C}_T = \tilde{E}^{\varepsilon,T}(q) \cap \mathcal{C}_T \). Thus we have

\[
\limsup_{n \to \infty} \frac{1}{n} \log P(E^{n,\varepsilon,T}(q)) \leq - \inf_{(\zeta, \psi) \in \tilde{E}^{\varepsilon,T}(q) \cap \mathcal{C}_T} I_T(\zeta, \psi).
\]

Letting

\[
\tilde{E}^T(q) = \{(\zeta, \psi) \in \mathcal{C}_T : \exists t_1, t_2 \in [0, T] \text{ such that } \zeta_0(t_1) = \zeta_0(t_2) \leq 0, \psi(t) \geq \psi(t_1) \text{ for } t \in [t_1, t_2], \zeta_k(t_1) - \zeta_k(t_2) = q_k \text{ for all } k \in \mathbb{N}\},
\]

we have \( \tilde{E}^T(q) = \bigcap_{\varepsilon > 0} \left( \tilde{E}^{\varepsilon,T}(q) \cap \mathcal{C}_T \right) \). From this, the lower semi-continuity and compactness of level sets of \( I_T(\zeta, \psi) \) (since \( I_T \) is a rate function; see Theorem 2.14), it follows

\[
\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log P(E^{n,\varepsilon,T}(q)) \leq - \lim_{\varepsilon \to 0} \inf_{(\zeta, \psi) \in \tilde{E}^{\varepsilon,T}(q) \cap \mathcal{C}_T} I_T(\zeta, \psi) = - \inf_{(\zeta, \psi) \in \tilde{E}^T(q)} I_T(\zeta, \psi).
\]
Breaking $(\zeta, \psi) \in \hat{E}^T(q)$ for $t \in [0, T]$ according to $t \leq t_1$, $t_1 \leq t \leq t_2$ and $t \geq t_2$,

$$\inf_{(\zeta, \psi) \in \hat{E}^T(q)} I_T(\zeta, \psi) = \inf_{q \leq p \leq p, 0 \leq t_1 \leq t \leq t_2 \leq T} \left[ I^0_{0, t_1}((0, p), (0, \bar{p}), (0, \bar{p} - q)) + I^1_{t_1, t_2}((0, p), (0, \bar{p}), (0, \bar{p} - q)) \right]$$

$$\geq \inf_{q \leq p \leq p, 0 \leq t_1 \leq t < \infty} \left[ I^0_{0, t_1}((0, p), (0, \bar{p})) + I^1_{t_1, t_2}((0, p), (0, \bar{p} - q)) \right]$$

$$= \inf_{q \leq p \leq p, t_1 \geq 0} \left[ I^0_{0, t_1}((0, p), (0, \bar{p})) + I^2_{t_1, t_1 + \tau}((0, \bar{p}), (0, \bar{p} - q)) \right],$$

where the last line follows from Lemma 8.2. The result follows.

The following lemma improves the upper bound (8.4) in Lemma 8.2 when $p_1 = 0$.

**Lemma 8.3.** Suppose $p_1 = 0$ and $0 \leq q \leq p$. Then

(a) $I^0_{0, t_1}((0, p), (0, q)) = I^1_{0, t_1}((0, p), (0, q))$ for each $t_1 \geq 0$.

(b) Let $\tau = \frac{1}{2} \sum_{k=1}^{\infty} k \bar{q}_k$ as in Lemma 8.2. The infimum on the right side of (8.4) is achieved at $t_1 = 0$:

$$\inf_{q \leq p \leq p, t_1 \geq 0} \left[ I^0_{0, t_1}((0, p), (0, \bar{p})) + I^2_{t_1, t_1 + \tau}((0, \bar{p}), (0, \bar{p} - q)) \right] = I^2_{0, \tau}((0, p), (0, p - q)).$$

**Proof.** (a) Fix $t_1 \geq 0$. It suffices to show that if $(\zeta, \psi) \in J^0_{0, t_1}((0, p), (0, q))$ satisfies $I_{0, t_1}(\zeta, \psi) < \infty$ then $(\zeta, \psi) \in J^0_{0, t_1}((0, p), (0, q))$. For such a pair $(\zeta, \psi)$, let $\varphi \in S_{t_1}(\zeta, \psi)$ be such that the associated cost is finite. In particular, $\psi$, and consequently $\zeta_0$, is absolutely continuous.

Since $\zeta_0(t) = \Gamma(\psi)(t) \geq 0$ for $t \in [0, t_1]$, we have

$$1_{\{\zeta(t) > 0\}} \zeta_0'(t) = 1_{\{\zeta(t) > 0\}} \psi'(t), \quad 1_{\{\zeta(t) = 0\}} \zeta_0'(t) = 0, \text{ a.e. } t \in [0, t_1].$$

(8.5)

Since $p_1 = 0$, we see from (2.14) that $1_{\{\zeta(t) = 0\}} \psi'(t) \geq 0$ for a.e. $t \in [0, t_1]$. Indeed, when $p_1 = 0$ the term for $k = 1$ in the sum on the right side of (2.14) is zero. Also, the term for $k = 2$ is always zero and the integrand for $k = 0$ is zero on the set $\{\zeta(t) = 0\}$. This shows that, on this set, the derivative of the sum on the right side of (2.14) is nonnegative. Combining this with (8.5), we have for $t \in [0, t_1]$,

$$\psi(t) = \int_0^t 1_{\{\zeta(s) > 0\}} \psi'(s) ds + \int_0^t 1_{\{\zeta(s) = 0\}} \psi'(s) ds \geq \int_0^t 1_{\{\zeta(s) > 0\}} \zeta_0'(s) ds$$

$$= \int_0^t 1_{\{\zeta(s) > 0\}} \zeta_0'(s) ds + \int_0^t 1_{\{\zeta(s) = 0\}} \zeta_0'(s) ds = \zeta_0(t) \geq 0.$$

This implies $(\zeta, \psi) \in J^0_{0, t_1}((0, p), (0, q))$ and part (a) follows.

(b) For $0 \leq \bar{p} \leq p$, let $\bar{q} \eqdef p - \bar{p}$ and $\bar{\tau} = \frac{1}{2} \sum_{k=1}^{\infty} k \bar{q}_k$. Since $p_1 = 0$, we always have $\sum_{k=1}^{\infty} k \bar{q}_k \geq 2 \sum_{k=1}^{\infty} \bar{q}_k$. Therefore

$$\inf_{q \leq p \leq p, t_1 \geq 0} \left[ I^0_{0, t_1}((0, p), (0, \bar{p})) + I^2_{t_1, t_1 + \tau}((0, \bar{p}), (0, \bar{p} - q)) \right]$$

$$= \inf_{q \leq p \leq p, t_1 \geq 0} \left[ I^0_{0, t_1}((0, p), (0, \bar{p})) + I^2_{t_1, t_1 + \tau}((0, \bar{p}), (0, \bar{p} - q)) \right]$$

$$= \inf_{q \leq p \leq p} \left[ I^2_{0, \bar{\tau}}((0, p), (0, \bar{q})) + I^2_{\bar{\tau}, \bar{\tau} + \bar{\tau}}((0, p - \bar{q}), (0, p - \bar{q} - q)) \right]$$

where the first equality uses part (a) with $q = \bar{p}$ and the second equality follows from Lemma 8.1 and the observation that $I^2_{t, t'}((0, \bar{p}), (0, \bar{p} - q)) = I^2_{t, t'}((0, \bar{p}), (0, \bar{p} - q))$ for all $t, t'$ as long as
$$I^0_{0,s}((0,p),(0,p)) < \infty$$ for $$s = t, t'$$ Using Proposition 7.11(b), the right side on the last line equals

$$\inf_{0 \leq q \leq p} \left[ I^2_{0,t}((0,p),(0,p-q)) + I^2_{t,t}((0,p-q),(0,p-q)) \right]$$

$$= I^2_{0,t}((0,p),(0,p-q)) + \inf_{0 \leq q \leq p-q} I^2_{t,t}((0,p-q),(0,p-q))$$

$$= I^2_{0,t}((0,p),(0,p-q)),$$

where the last equality follows by considering $$q = 0$$. This completes the proof.

Next we will prove the lower bound.

**Lemma 8.4.** Suppose $$0 \leq q \leq p$$ and $$\sum_{k=1}^{\infty} kq_k > 2 \sum_{k=1}^{\infty} q_k$$. Let $$\tau = \frac{1}{2} \sum_{k=1}^{\infty} kq_k$$. Then the lower bound in (8.4) holds.

**Proof.** Let $$(\xi, \psi)$$ be as introduced in Construction 11 with $$t_1 = 0$$, $$x^{(1)} = (0,p)$$ and $$x^{(2)} = (0,p-q)$$. We define $$(\xi, \psi)$$ for $$t > \tau$$ through (2.14)-(2.16) by setting $$\varphi_k(t,y) = 1$$ for all $$k, y$$ and $$t > \tau$$. Then $$I_t(\xi, \psi) = I_t(\xi, \psi)$$ for all $$t > \tau$$. So by Lemmas 7.1 and 7.6, for

$$I_t(\xi, \psi) = \int_0^\tau L(\xi(s), \psi(s)) ds = I_{0,\tau}^2((0,p),(0,p-q)). \quad (8.6)$$

For $$\delta \in (0,1)$$ consider the set

$$\tilde{G}_\delta(\xi, \psi) = \{ (\xi, \psi) \in D([0,\infty) : \mathbb{R} \times \mathbb{R}_\infty) : \sup_{t \in [0,\tau]} |k(t) - \xi_k(t)| < \delta, \text{ for all } k = 0, 1, 2, \ldots, |\delta^{-1}| \}. \quad (8.7)$$

Let $$\tau^n = \inf \{ t \geq \tau : X^n_0(t) = -\frac{1}{n} \}$$. Then $$\tau^n < \infty$$ a.s. Define for odd integer $$j \geq -1$$,

$$G_j^n \doteq \left\{ X^n_0(\tau) = \frac{j}{n}, X^n_k(\tau^n) = X^n_k(\tau), k \in \mathbb{N} \right\},$$

and for even integer $$j \geq -1$$,

$$G_j^n \doteq \left\{ X^n_0(\tau) = \frac{j}{n} \sum_{k=1}^{\infty} (X^n_k(\tau^n) - X^n_k(\tau)) = -\frac{1}{n} \right\}.$$ 

Intuitively, $$G_j^n$$ describes the event that from time instant $$\tau$$ to the time $$\tau^n$$ at which the current component is fully explored, the continuous-time EEA does not wake up any sleeping vertices, with the exception that if the number of active half-edges at time $$\tau$$ is odd (namely $$X^n_0(\tau) = \frac{j}{n}$$ for some even integer $$j \geq -1$$), in which case exactly one sleeping vertex (necessarily with odd degree) will be woken up. Consider the event

$$A^n_\delta(\xi, \psi) \doteq \left\{ (X^n, Y^n) \in \tilde{G}_\delta(\xi, \psi) \right\} \bigcap \left( \bigcup_{j=-1}^{\infty} G_j^n \right).$$

Fix $$\varepsilon \in (0,1)$$. We claim that there exist $$\delta_0 > 0$$ and $$n_0 > 0$$ such that

$$A^n_\delta(\xi, \psi) \subset E^{n,\varepsilon}(q) \text{ for all } \delta < \delta_0 \text{ and } n > n_0. \quad (8.8)$$

To see this first note that by Assumption 2.2 there exists $$M \in \mathbb{N}$$ such that

$$\sup_{n \in \mathbb{N}} \sum_{k=M}^{\infty} k \frac{n_k}{n} < \frac{\varepsilon}{2}, \sum_{k=M}^{\infty} k p_k < \frac{\varepsilon}{2}, \quad (8.9)$$

$$38$$
By continuity of \( \tilde{\zeta} \), there exists \( \varepsilon_0 > 0 \) such that
\[
|\tilde{\zeta}_k(t) - \tilde{\zeta}_k(0)| < \frac{\varepsilon}{4} \text{ for all } t \in [0, \varepsilon_0], k = 0, 1, \ldots, M,
\]
(8.10)
\[
|\tilde{\zeta}_k(t) - \tilde{\zeta}_k(\tau)| < \frac{\varepsilon}{4} \text{ for all } t \in [\tau - \varepsilon_0, \tau], k = 0, 1, \ldots, M.
\]
(8.11)

From Lemma (8.3) we have \( \tilde{\zeta}_0(t) > 0 \) for all \( t \in (0, \tau) \). Since \( \tilde{\zeta}_0(t) \) is continuous,
\[
\delta_0 = \left( \inf_{t \in [\varepsilon_0, \tau - \varepsilon_0]} \tilde{\zeta}_0(t) \right) \wedge \frac{\varepsilon}{4} \wedge \frac{1}{M} > 0.
\]

Take \( n_0 > \frac{4}{\varepsilon} \). We now show (8.3) with this choice of \( n_0 \) and \( \delta_0 \). Fix \( \delta < \delta_0 \) and \( n > n_0 \) and consider \( \omega \in A^n_0(\tilde{\zeta}, \psi) \). For \( t \in [\varepsilon_0, \tau - \varepsilon_0] \), since \( |X^n_0(t) - \tilde{\zeta}_0(t)| < \delta < \delta_0 \leq \tilde{\zeta}_0(t) \), we have \( \inf_{t \in [\varepsilon_0, \tau - \varepsilon_0]} X^n_0(t) > 0 \). So there exist \( t^n_1 \in [0, \varepsilon_0] \) and \( t^n_2 \in [\tau - \varepsilon_0, \tau^n] \) such that
\[
X^n_0(t^n_1) = X^n_0(t^n_2) = -\frac{1}{n}, \quad X^n_0(t) > -\frac{1}{n} \text{ for } t \in [t^n_1, t^n_2],
\]
(8.12)
where by convention \( X^n_0(0) = X^n_0(0) = -1/n \). For \( k \geq M \), it follows from (8.4) that
\[
|X^n_k(t^n_1) - X^n_k(t^n_2) - q_k| \leq |X^n_k(t^n_1) - X^n_k(t^n_2)| + \frac{n_k}{n} + p_k < \varepsilon.
\]
(8.13)

For \( 1 \leq k \leq M \leq \lfloor \delta^{-1} \rfloor \),
\[
|X^n_k(t^n_1) - X^n_k(t^n_2) - q_k| = |(X^n_k(t^n_1) - X^n_k(t^n_2)) - (\tilde{\zeta}_k(0) - \tilde{\zeta}_k(\tau))| \leq |X^n_k(t^n_1) - \tilde{\zeta}_k(0)| + |X^n_k(t^n_2) - \tilde{\zeta}_k(\tau)|.
\]
(8.14)

From (8.7) and (8.10) we have the following bound for the first term in (8.14).
\[
|X^n_k(t^n_1) - \tilde{\zeta}_k(0)| \leq |X^n_k(t^n_1) - \tilde{\zeta}_k(t^n_1)| + |\tilde{\zeta}_k(t^n_1) - \tilde{\zeta}_k(0)| < \delta + \frac{\varepsilon}{4}.
\]

For the second term in (8.14), if \( t^n_2 \leq \tau \), then using (8.7) and (8.11) we have
\[
|X^n_k(t^n_2) - \tilde{\zeta}_k(\tau)| \leq |X^n_k(t^n_2) - \tilde{\zeta}_k(t^n_2)| + |\tilde{\zeta}_k(t^n_2) - \tilde{\zeta}_k(\tau)| < \delta + \frac{\varepsilon}{4}.
\]

If \( t^n_2 > \tau \), then \( t^n_2 = \tau^n \) and from the definition of \( G^n_j \) and (8.4) we have
\[
|X^n_k(t^n_2) - \tilde{\zeta}_k(\tau)| \leq |X^n_k(\tau^n) - X^n_k(\tau)| + |X^n_k(\tau) - \tilde{\zeta}_k(\tau)| \leq \frac{1}{n} + \delta \leq \frac{\varepsilon}{4} + \delta.
\]

Combining these three displays with (8.14) gives
\[
|X^n_k(t^n_1) - X^n_k(t^n_2) - q_k| < 2 \left( \delta + \frac{\varepsilon}{4} \right) < \varepsilon, \quad k \in \mathbb{N}.
\]

From this, and (8.1) (8.2) (8.3) we have \( \omega \in E^n, \tilde{\zeta}(q) \). Since \( \delta < \delta_0 \) and \( n > n_0 \) is arbitrary, the claim (8.5) holds.

For fixed \( \delta < \delta_0 \) and \( n > n_0 \) consider the following two probabilities
\[
P(A^n_0(\tilde{\zeta}, \psi), \quad P((X^n, Y^n) \in \tilde{G}_s(\tilde{\zeta}, \psi)).
\]

Write
\[
P(A^n_0(\tilde{\zeta}, \psi)) = \sum_{j=-1}^{\infty} P \left( \{(X^n, Y^n) \in \tilde{G}_s(\tilde{\zeta}, \psi) \} \cap G^n_j \right) = \sum_{j=-1}^{\lfloor \delta n \rfloor} E \left[ 1_{\{(X^n, Y^n) \in \tilde{G}_s(\tilde{\zeta}, \psi) \}} P \left( G^n_j | \mathcal{F}_\tau \right) \right]
\]
(8.15)
where we only have to sum up $|\delta n|$ in the last line when $(X^n, Y^n) \in \tilde{G}_s(\tilde{\zeta}, \tilde{\psi})$ since $\tilde{\zeta}_0(\tau) = 0$. Since $\tau^n = \tau$ on $\{X^n_0(\tau) = -\frac{1}{n}\}$, we have

$$P(G^n_{-1} | F_\tau) = 1_{\{X^n_0(\tau) = -\frac{1}{n}\}}. \quad (8.16)$$

From Assumption 2.2 and the fact that $r(X^n(t))$ is non-increasing it follows

$$\sup_{n \in \mathbb{N}} \sup_{t \geq 0} r(X^n(t)) \leq \sup_{n \in \mathbb{N}} \sum_{k=1}^{\infty} \frac{k n_k}{n} \leq C_0 < \infty.$$ 

Hence for odd integer $1 \leq j \leq |\delta n|$ and $\delta < \frac{C_0}{2}$,

$$1_{\{(x^n, y^n) \in \tilde{G}_s(\tilde{\zeta}, \tilde{\psi})\}} P(G^n_j | F_\tau)
= 1_{\{(x^n, y^n) \in \tilde{G}_s(\tilde{\zeta}, \tilde{\psi})\}} \frac{j}{n r(X^n(\tau))} \cdot \frac{j-2}{n r(X^n(\tau)) - 2} \cdots \frac{1}{n r(X^n(\tau)) - (j-1)} 1_{\{X^n_0(\tau) = \frac{1}{n}\}}
\geq 1_{\{(x^n, y^n) \in \tilde{G}_s(\tilde{\zeta}, \tilde{\psi})\}} \frac{j}{C_0 n} \cdot \frac{j-2}{C_0 n} \cdots \frac{1}{C_0 n} 1_{\{X^n_0(\tau) = \frac{1}{n}\}}
\geq 1_{\{(x^n, y^n) \in \tilde{G}_s(\tilde{\zeta}, \tilde{\psi})\}} \frac{|2\delta n|!}{(C_0 n)!} 1_{\{X^n_0(\tau) = \frac{1}{n}\}}, \quad (8.17)$$

where the last inequality follows since the term on the last line includes more fractions that are less than 1 than the one on the previous line. For the first equality we have used the fact that on the event $G^n_j$ all the active $j + 1$ half edges (an even number) at time instant $\tau$ should merge among themselves (without waking any sleeping vertices) by the time instant $\tau^n$, whereas the total number of available half edges (either awake or sleeping) at time instant $\tau$ equals $n r(X^n(\tau)) + 1$.

For even integer $0 \leq j \leq |\delta n|$, we consider three different cases for values of $p$ and $q$.

**Case 1:** There exists some odd $m \in \mathbb{N}$ such that $p_m > q_m \geq 0$. Let $C_m = \frac{1}{2}(p_m - q_m) > 0$. For $\delta < \frac{1}{m} \land \delta_0 \land C_m$ and $(X^n, Y^n) \in \tilde{G}_s(\tilde{\zeta}, \tilde{\psi})$, we have from (8.7),

$$X^n_m(\tau) = \tilde{X}^n_m(\tau) - (\tilde{X}^n_m(\tau) - X^n_m(\tau)) = (p_m - q_m) - \delta > C_m,$$

which implies $X^n_m(\tau) \geq 1/n$ for $n \geq \delta^{-1}$. So for even integer $0 \leq j \leq |\delta n|$ and $n > \frac{m}{\delta} \vee n_0$,

$$1_{\{(x^n, y^n) \in \tilde{G}_s(\tilde{\zeta}, \tilde{\psi})\}} P(G^n_j | F_\tau)
\geq 1_{\{(x^n, y^n) \in \tilde{G}_s(\tilde{\zeta}, \tilde{\psi})\}} \frac{m n X^n_m(\tau)}{n r(X^n(\tau))} \cdot \frac{j + m - 2}{n r(X^n(\tau)) - 2} \cdots \frac{j + m - 4}{n r(X^n(\tau)) - 4} \frac{1}{n r(X^n(\tau)) - (j + m - 1)} 1_{\{X^n_0(\tau) = \frac{1}{n}\}}
\geq 1_{\{(x^n, y^n) \in \tilde{G}_s(\tilde{\zeta}, \tilde{\psi})\}} \frac{m n}{C_0 n} \cdot \frac{j + m - 2}{C_0 n} \cdots \frac{j + m - 4}{C_0 n} \frac{1}{C_0 n} 1_{\{X^n_0(\tau) = \frac{1}{n}\}}
\geq 1_{\{(x^n, y^n) \in \tilde{G}_s(\tilde{\zeta}, \tilde{\psi})\}} \frac{|2\delta n|!}{(C_0 n)!} \frac{1}{(2\delta n)!} 1_{\{X^n_0(\tau) = \frac{1}{n}\}}, \quad (8.18)$$

where the last inequality follows once again as in (8.17). Combining (8.13) and (8.18) implies that, for
$\delta < \frac{C_\mu}{2} \land \frac{1}{m} \land \delta_0 \land C_m$ and $n > \frac{\pi}{\delta} \lor n_0$,

$$
P(A^n_\delta(\tilde{\zeta}, \tilde{\psi})) \geq \sum_{j=1}^{\lfloor \delta n \rfloor} E \left[ 1_{\{(X^n,Y^n) \in \tilde{G}_\delta(\tilde{\zeta}, \tilde{\psi})\}} \frac{2|\delta n|!}{(C_0 n)^{|2\delta n|}} 1_{\{X^n_\delta(\tau) = \frac{j}{n}\}} \right]
$$

$$
= \frac{2|\delta n|!}{(C_0 n)^{|2\delta n|}} P((X^n, Y^n) \in \tilde{G}_\delta(\tilde{\zeta}, \tilde{\psi})) 
$$

$$
\geq \sqrt{2\pi |2\delta n|} \left( \frac{|2\delta n|!}{C_0^{|2\delta n|}} \right)^{\frac{1}{2\delta n}} P((X^n, Y^n) \in \tilde{G}_\delta(\tilde{\zeta}, \tilde{\psi})),
$$

where the last line uses Stirling’s approximation $n! \geq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. From this and (8.19) we have

$$
\liminf_{n \to \infty} \frac{1}{n} \log P(E^{n,\varepsilon}(q)) 
\geq \liminf_{n \to \infty} \frac{1}{n} \log P(A^n_\delta(\tilde{\zeta}, \tilde{\psi})) 
\geq \liminf_{n \to \infty} \left[ \frac{1}{2n} \log (2\pi |2\delta n|) + \frac{|2\delta n|}{n} \log \left( \frac{|2\delta n|!}{C_0^{|2\delta n|}} \right) + \frac{1}{n} \log P((X^n, Y^n) \in \tilde{G}_\delta(\tilde{\zeta}, \tilde{\psi})) \right] 
= 2\delta \log \left( \frac{C_0^{|2\delta n|}}{|2\delta n|!} \right) + \liminf_{n \to \infty} \frac{1}{n} \log P((X^n, Y^n) \in \tilde{G}_\delta(\tilde{\zeta}, \tilde{\psi})).
$$

(8.19)

Define the open set

$$
G_{\delta,\tau}(\tilde{\zeta}, \tilde{\psi}) = \{ (\zeta, \psi) \in D([0, \tau] : \mathbb{R} \times \mathbb{R}^\infty \times \mathbb{R}) : \sup_{t \in [0, \tau]} |\zeta_k(t) - \tilde{\zeta}_k(t)| < \delta \text{ for all } k = 0, 1, 2, \ldots, |\delta^{-1}| \}.
$$

It follows from Theorem 2.14 that

$$
\liminf_{n \to \infty} \frac{1}{n} \log P((X^n, Y^n) \in \tilde{G}_\delta(\tilde{\zeta}, \tilde{\psi})) = \liminf_{n \to \infty} \frac{1}{n} \log P((X^n, Y^n) \in G_{\delta,\tau}(\tilde{\zeta}, \tilde{\psi})) 
\geq - \inf_{(\zeta, \psi) \in G_{\delta,\tau}(\tilde{\zeta}, \tilde{\psi})} I_\tau((\zeta, \psi)) \geq -I_\tau(\tilde{\zeta}, \tilde{\psi}).
$$

Combining this with (8.19), (8.6) and sending $\delta \to 0$ gives

$$
\liminf_{n \to \infty} \frac{1}{n} \log P(E^{n,\varepsilon}(q)) \geq -I_\tau(\tilde{\zeta}, \tilde{\psi}) = -I_{\delta,\tau}^2((0,p), (0,p-q)).
$$

(8.20)

The lower bound in Case 1 now follows on sending $\varepsilon \to 0$.

Case 2: $p_m = 0$ for all odd $m \in \mathbb{N}$. It suffices to establish a similar estimate as in (8.18), the lower bound in Case 2 will then follow as in Case 1. From Assumptions 2.3 and 2.2

$$
\sum_{k=0}^{\infty} (2k+1) \frac{n_{2k+1}}{n} \rightarrow \sum_{k=0}^{\infty} (2k+1) = \frac{1}{2}.
$$

Therefore for each $\kappa \in (0, 1)$, there exists some $\tilde{n}_\kappa \in \mathbb{N}$ such that $0 \leq \sum_{k=0}^{\tilde{n}_\kappa} (2k+1) \frac{n_{2k+1}}{n} < \kappa$ for $n > \tilde{n}_\kappa$, which implies $n_m = 0$ for all odd $m \geq \kappa n$. Consider now an even integer $0 \leq j \leq |\delta n|$ and $n > \tilde{n}_\kappa$. Denote by $M^n$ the largest odd degree for which there is a sleeping vertex at time instant $\tau$ in the continuous time EEA. Note that $M^n \leq \kappa n$ a.s. Therefore for $\kappa < \delta < \frac{C_\mu}{2} \land \delta_0$ and $n > \tilde{n}_\kappa$,

$$
1_{\{(X^n,Y^n) \in \tilde{G}_\delta(\tilde{\zeta}, \tilde{\psi})\}} P(G_j^n | \mathcal{F}_\tau) 
\geq 1_{\{(X^n,Y^n) \in \tilde{G}_\delta(\tilde{\zeta}, \tilde{\psi})\}} \sum_{1 \leq m \leq \lfloor \kappa n \rfloor, m \text{ is odd}} \left[ 1_{\{M^n=m\}} \frac{m n X^n_\delta(\tau)}{n r(X^n_\delta(\tau))} \cdot \frac{j+m-2}{n r(X^n_\delta(\tau)) - (j+m-1)} \right] 1_{\{X^n_\delta(\tau) = \frac{j}{n}\}}
$$

41
Since \( nX^m_n(\tau) \geq 1 \) on the set \( \{M^n = \omega\} \) the right side can be bounded below by
\[
1_{\{(X^n,Y^n)\in G_d(\xi,\psi)\}} \sum_{1 \leq m \leq \lfloor \kappa n \rfloor, m \text{ is odd}} \left[ 1_{\{M^n = \omega\}} \left( \frac{j + m - 2}{m} \right)^{\lfloor \kappa n \rfloor} \right] 1_{\{X^n_\omega(\tau) = \frac{1}{m}\}}
\geq 1_{\{(X^n,Y^n)\in G_d(\xi,\psi)\}} \sum_{1 \leq m \leq \lfloor \kappa n \rfloor, m \text{ is odd}} \left[ 1_{\{M^n = \omega\}} \left( \frac{2\delta_n!}{(C_0\kappa n)^{2\delta_n}} \right) \right] 1_{\{X^n_\omega(\tau) = \frac{1}{m}\}}
\]
\[
= 1_{\{(X^n,Y^n)\in G_d(\xi,\psi)\}} \frac{2\delta_n!}{(C_0\kappa n)^{2\delta_n}} 1_{\{X^n_\omega(\tau) = \frac{1}{m}\}},
\]
where the last inequality follows once again as in (8.18). Therefore we have the same inequality as in (8.18) for \( n > n_0 \vee \tilde{n}_x \) and \( \delta < \frac{C_0}{\kappa} \wedge \delta_\omega \), and so the lower bound in Case 2 follows.

Case 3: There exists an odd \( m \in \mathbb{N} \) such that \( p_m > 0 \) but \( p_m = q_m \). For \( i \in \mathbb{N} \), consider the vector \( q^i = (q^i_k)_{k \in \mathbb{N}} \), where \( q^i_k = q_k \) for \( k \neq m \) and \( q^i_m = q_m - \frac{1}{x} \). Fix \( \varepsilon \in (0,1) \). Choose \( i \) so that \( p_m > q^i_m > 0 \), \( \varepsilon > \frac{1}{x} = q_m - q^i_m \) and \( \sum_{k=1}^{\infty} kq^i_k > 2 \sum_{k=1}^{\infty} q^i_k \). When \( q \) is replaced by \( q^i \) we are in Case 1 and thus for \( \varepsilon^i < (\varepsilon - \frac{1}{x}) \), from the lower bound (8.20) for Case 1,
\[
\liminf_{n \to \infty} \frac{1}{n} \log P(E^{\varepsilon^i}(q)) \geq \liminf_{n \to \infty} \frac{1}{n} \log P(E^{\varepsilon}(q^i)) \geq -I_{0,\rho,\varepsilon}^2(\{(0,p),(0,p-q^i)\}).
\] (8.21)

From Proposition (7.21) we have
\[
I_{0,\rho,\varepsilon}^2(\{(0,p),(0,p-q^i)\}) = H(q^i) + H(p-q^i) - H(p) + K(q^i).
\] (8.22)

Since \( \{(0,p),(0,p-q^i)\} \) and \( \{(0,p),(0,p-q)\} \) are in \( \Xi \) and \( q^i \to q \), from Lemma (7.3) we have \( K(q^i) \to K(q) \). Also, clearly
\[
H(q^i) + H(p-q^i) - H(p) \to H(q) + H(p-q) - H(p).
\]
Thus, as \( i \to \infty \), the right side of (8.22) converges to
\[
H(q) + H(p-q) - H(p) + K(q) = I_{0,\rho,\varepsilon}^2(\{(0,p),(0,p-q)\}),
\]
where the equality follows again by Proposition (7.21). The desired result now follows on sending \( i \to \infty \) and then \( \varepsilon \to 0 \) in (8.21).

The above three cases cover all possible values of \( p \) and \( q \). This completes the proof.

\section{Completing the proof of Theorem 2.6}

The upper bound of Theorem 2.6 follows from Lemma (8.2) Lemma (8.3) b) and Proposition (7.21) a). The lower bound of Theorem 2.6 follows from Lemma (8.4) and Proposition (7.21) a).

\section{Proofs of Auxiliary Lemmas}

In this section we prove the lemmas in Section 7. Specifically, in Section 9.1 we prove Lemma (7.1) in Section 9.2, we prove Lemmas (7.2) and (7.3) in Section 9.3, we prove Lemma (7.4) in Section 9.4, we prove Lemma (7.5), and finally in Section 9.5 we prove Lemma (7.6).

We start with the following remark.

\textbf{Remark 9.1.} Fix \( 0 \leq t_1 < t_2 < \infty \), \( \alpha^{(1)}, \alpha^{(2)} \in \mathbb{R}^\infty \) and \( (\zeta, \psi) \in \mathcal{J}_{t_1,t_2}^{\alpha^{(1)}, \alpha^{(2)}} \) such that \( I_{t_1,t_2}(\zeta, \psi) < \infty \). Fix \( \varepsilon \in (0,1) \). Then there exists \( \varphi \in \mathcal{S}_{t_2}(\zeta, \psi) \) such that
\[
\sum_{k=0}^{\infty} \int_{[t_1,t_2] \times [0,1]} \ell(\varphi_k(s,y)) \, ds \, dy \leq I_{t_1,t_2}(\zeta, \psi) + \varepsilon.
\]
Using convexity of ℓ, we can assume without loss of generality that \( \varphi_k(t, y) = \rho_k(t)1_{[0, r_k(\zeta(t))]}(y) + 1_{[r_k(\zeta(t)), 1]}(y) \) for \( t \in [t_1, t_2] \), where \( \rho_k \) is some nonnegative function. From (2.11) and (2.14) we see that for a.e. \( t \in [t_1, t_2] \),

\[
\zeta'_k(t) = -\rho_k(t)r_k(\zeta(t)), \quad k \in \mathbb{N}, \\
\psi'(t) = \sum_{k=0}^{\infty} (k - 2)\rho_k(t)r_k(\zeta(t)).
\] (9.1) (9.2)

Since \( (\zeta, \psi) \in J_{t_1, t_2}(x(1), x(2)) \), \( \zeta_0(t) = \zeta_0(t_1) + \psi(t) - \psi(t_1) \) over \( (t_1, t_2) \), namely there is no reflection over this interval. Therefore for a.e. \( t \in [t_1, t_2] \), we have \( \zeta'_0(t) = \psi'(t) \) and

\[
-\frac{1}{2} \frac{d}{dt}r(\zeta(t)) = -\frac{1}{2} \left( \psi'(t) + \sum_{k=1}^{\infty} k\zeta'_k(t) \right) = \sum_{k=0}^{\infty} \rho_k(t)r_k(\zeta(t)), \quad t \in [t_1, t_2].
\] (9.3)

Now we prove the lemmas in Section 7.

9.1 Proof of Lemma 7.1

We first prove (7.11). Since

\[
\inf_{t_2 \geq t_1} I_{t_1, t_2}(x(1), x(2)) \leq I_{t_1, t_1+\tau}(x(1), x(2)) \leq I_{t_1, t_1+\tau}(x(1), x(2)),
\]

it suffices to show

\[
\inf_{t_2 \geq t_1} I_{t_1, t_2}(x(1), x(2)) \geq I_{t_1, t_1+\tau}(x(1), x(2))
\] (9.4)

when \( \inf_{t_2 \geq t_1} I_{t_1, t_2}(x(1), x(2)) < \infty \). Fix \( \varepsilon \in (0, 1) \). There exist \( t_2^* \geq t_1 \), \( (\zeta, \psi) \in J_{t_1, t_2}^{1}(x(1), x(2)) \) and \( \varphi \in S_{t_2^*}(\zeta, \psi) \) such that

\[
\sum_{k=0}^{\infty} \int_{[t_1, t_2^*] \times [0, 1]} \ell(\varphi_k(s, y)) ds dy \leq I_{t_1, t_2^*}(\zeta, \psi) + \varepsilon \leq \inf_{t_2 \geq t_1} I_{t_1, t_2}(x(1), x(2)) + 2\varepsilon.
\] (9.5)

Recall that \( t \mapsto r(\zeta(t)) \) is a non-increasing function (see (9.3)). We claim that in fact we can assume without loss of generality that \( t \mapsto r(\zeta(t)) \) is strictly decreasing for \( t \in [t_1, t_2] \). Indeed, if this function is not strictly decreasing, we can modify \( (\zeta, \psi) \) such that for the modified trajectory strict monotonicity holds and the associated cost is not any higher. Such a modification can be constructed via a limiting argument as follows. Consider \( (\zeta^n, \psi^n) \) defined recursively as: \( (\zeta^n, \psi^n) = (\zeta, \psi, \varphi^n) \) on \( [0, t_1^n] \) where \( t_1^n = t_2^n \). For \( n \in \mathbb{N} \), having defined \( (\zeta^n, \psi^n) \) and \( t_2^n \leq t_2^n \), where \( (\zeta^n, \psi^n) \in J_{t_1, t_2^n}^{1}(x(1), x(2)) \) and \( \varphi^n \in S_{t_2^n}(\zeta^n, \psi^n) \) such that

\[
\sum_{k=0}^{\infty} \int_{[t_1, t_2^n] \times [0, 1]} \ell(\varphi_k^n(s, y)) ds dy \leq \sum_{k=0}^{\infty} \int_{[t_1, t_2^n] \times [0, 1]} \ell(\varphi_k(s, y)) ds dy.
\] (9.6)

we modify \( (\zeta^n, \psi^n) \), in case \( r(\zeta(\cdot)) \) is not strictly decreasing on \( [t_1, t_2^n] \) as follows. Let \( [s_1^n, s_2^n] \subset [t_1, t_2^n] \) be the largest constant piece of \( r(\zeta^n(\cdot)) \), namely \( r(\zeta^n(t)) \) is constant on \( [s_1^n, s_2^n] \) and \( s_2^n - s_1^n \) is maximized among all such possible pieces. Let \( t_2^{n+1} = t_2^n - (s_2^n - s_1^n) \) and define \( (\zeta^{n+1}, \psi^{n+1}) \) by shrinking \( (\zeta^n, \psi^n) \) over \( [s_1^n, s_2^n] \), namely let \( (\zeta^{n+1}(t), \psi^{n+1}(t)) = (\zeta^n(t), \psi^n(t)) \) for \( t \leq s_1^n \) and \( (\zeta^{n+1}(t), \psi^{n+1}(t)) = (\zeta^n(t + s_2^n - s_1^n), \psi^n(t + s_2^n - s_1^n)) \) for \( s_1^n < t \leq t_2^{n+1} \). Clearly \( (\zeta^{n+1}, \psi^{n+1}) \in J_{t_1, t_2^{n+1}}^{1}(x(1), x(2)) \) and the associated control \( \varphi^{n+1} \) satisfies (9.6) with \( n \) replaced with \( n + 1 \). If \( r(\zeta(t)) \) only has \( N \) constant pieces over \( [t_1, t_2^n] \), then \( (\zeta^N, \psi^N) \) is the desired modification of \( (\zeta, \psi) \). If \( r(\zeta(t)) \) has countably many constant pieces over \( [t_1, t_2^n] \), then the sequence \( (\zeta^n, \psi^n) \) is well defined and (9.6) holds for every \( n \). Since the
sequence $t_2^*$ is non-increasing, it converges to some point $\bar{t}_2$. Since $I(t)$ has compact sub-level sets, this sequence (of paths over the time interval $[t_1, t_2]$) has a limit point $(\zeta, \psi)$. It is easy to check that this limit point must belong to $J_{t_1, t_2}^1(\zeta^{(1)}, \zeta^{(2)})$ and $I_{t_1, t_2}^1(\zeta, \psi) \leq \liminf_{n \to \infty} I_{t_1, t_2}^n(\zeta^n, \psi^n) \leq I_{t_1, t_2}^1(\zeta, \psi)$.

From the construction one can show that for fixed $\delta$ in $(0, 1)$, $\inf_{s \in [t_1, t_2-\delta]} |r(\zeta^n(s)) - r(\zeta^n(s+\delta))|$ is non-decreasing in $n$ and eventually positive. Therefore $|r(\zeta(s)) - r(\zeta(s+\delta))| > 0$ for each $s \in [t_1, t_2-\delta]$. As $\delta$ is arbitrary, $(\zeta, \psi)$ is the desired modification of $(\zeta, \psi)$ verifying the claim.

From Remark 9.1 we can assume without loss of generality that $\varphi_k(t, y) = \rho_k(t)1_{[0, r_k(\zeta(t))]}(y) + 1_{[r_k(\zeta(t)), 1]}(y)$ for $t \in [t_1, t_2]$ and (9.1) - (9.3) hold for $t \in [t_1, t_2]$. We now introduce a time transformation. Consider the non-decreasing function $f$ defined as: $f(0) = 0,$

$$f'(t) = \begin{cases} 1, \\ -\frac{1}{2} \frac{dr}{dt}(\zeta(t)) = \sum_{k=0}^{\infty} \rho_k(t)r_k(\zeta(t)), \end{cases}$$

where the equality in the second line follows from (9.3). Since $r(\zeta(t))$ is strictly decreasing for $t \in [t_1, t_2]$, $f(t)$ must be strictly increasing for $t \in [t_1, t_2]$. So $g = f^{-1}$ is well-defined and absolutely continuous on $[0, f(t_2^*)]$. Note that $f(t_2^*) = f(t_1) + \int_{t_1}^{t_2^*} f'(t) \, dt = t_1 - \frac{1}{2}(r(\zeta(t_2^*)) - r(\zeta(t_1))) = t_1 + \tau$, where the last equality is from (9.1). Define $\tilde{\varphi_k}(t, \tilde{\psi}(t)) = \varphi_k(\zeta(t), \psi(g(t)))$ for $t \in [0, t_1 + \tau]$. Then it is easy to see that $(\tilde{\zeta}, \tilde{\psi}) \in J_{t_1, t_1+\tau}(\zeta^{(1)}, \zeta^{(2)})$. Since $f(g(t)) = t$, $f'(g(t))g'(t) = 1$ and so

$$\frac{d}{dt} r(\zeta(t)) = -2f'(g(t))g'(t) = -2 \text{ for a.e. } t \in [t_1, t_1 + \tau].$$

Therefore $(\tilde{\zeta}, \tilde{\psi}) \in J_{t_1, t_1+\tau}(\zeta^{(1)}, \zeta^{(2)})$. Define

$$\tilde{\varphi_k}(t, y) = \begin{cases} \varphi_k(t, y), \\ \tilde{\rho_k}(t)1_{[0, r_k(\tilde{\zeta}(t))]}(y) + 1_{[r_k(\tilde{\zeta}(t)), 1]}(y), \end{cases} \quad t \in [0, t_1],$$

where $\tilde{\rho_k}(t) = \rho_k(g(t))g'(t)$ for $t \in [t_1, t_1 + \tau]$. From (9.1) and (9.2), for $t \in [t_1, t_1 + \tau],$

$$\phi_k'(t) = \phi_k'(g(t))g'(t) = -\rho_k(g(t))r_k(\zeta(g(t)))g'(t) = -\tilde{\rho_k}(t)r_k(\tilde{\zeta}(t)), k \in \mathbb{N},$$

$$\psi'(t) = \psi'(g(t))g'(t) = \sum_{k=0}^{\infty} (k-2)\rho_k(g(t))r_k(\zeta(g(t)))g'(t) = \sum_{k=0}^{\infty} (k-2)\tilde{\rho_k}(t)r_k(\tilde{\zeta}(t)).$$

So $\tilde{\phi} \in S_{t_1+\tau}(\tilde{\zeta}, \tilde{\psi})$. We claim that

$$\sum_{k=0}^{\infty} \int_{[t_1, t_2^*] \times [0, 1]} \ell(\varphi_k(t, y)) \, dy \geq \sum_{k=0}^{\infty} \int_{[t_1, t_1+\tau] \times [0, 1]} \ell(\tilde{\varphi_k}(t, y)) \, dy. \quad (9.7)$$

To see the claim, first note that the left hand side of (9.7) equals

$$\sum_{k=0}^{\infty} \int_{t_1}^{t_2^*} r_k(\zeta(t))\ell(\rho_k(t)) \, dt.$$

Since $g(f(t)) = t$, we have $g'(f(t))f'(t) = 1$ and hence the right hand side of (9.7) is

$$\sum_{k=0}^{\infty} \int_{t_1}^{t_1+\tau} r_k(\tilde{\zeta}(t))\ell(\tilde{\rho_k}(t)) \, dt = \sum_{k=0}^{\infty} \int_{t_1}^{t_2^*} r_k(\tilde{\zeta}(f(t)))\ell(\tilde{\rho_k}(f(t)))f'(t) \, dt$$

$$= \sum_{k=0}^{\infty} \int_{t_1}^{t_2^*} r_k(\zeta(t))\ell \left( \frac{\rho_k(t)}{f'(t)} \right) f'(t) \, dt.$$
We first show that (9.2) and the fact that \( \varphi \) find some \( \psi \), Assume without loss of generality that \( I \) for a suitable sequence of non-negative functions \( \varepsilon \). Since \( \varepsilon \) for fixed \( (0, 1) \) is arbitrary, (9.4) follows, which, as argued previously, gives (7.10).

Combining the above two facts, we have
\[
\sum_{k=0}^{\infty} \int_{[t_1, t_1+\tau] \times [0,1]} \ell(\varphi_k(t, y)) \, dt \, dy - \sum_{k=0}^{\infty} \int_{[t_1, t_1+\tau] \times [0,1]} \ell(\tilde{\varphi}_k(t, y)) \, dt \, dy = \sum_{k=0}^{\infty} \int_{t_1}^{t_1+\tau} r_k(\zeta(t)) \big| \rho_k(t) \log f'(t) - f'(t) + 1 \big| \, dt
\]
\[
= \int_{t_1}^{t_1+\tau} \left[ \sum_{k=0}^{\infty} r_k(\zeta(t)) \rho_k(t) \right] \log f'(t) - f'(t) + 1 \big| \, dt,
\]
\[
= \int_{t_1}^{t_1+\tau} \ell(f'(t)) \, dt \geq 0,
\]
where the next to last equality uses the fact that \( \sum_{k=0}^{\infty} r_k(\zeta(t)) = 1 \) for all \( t \in [t_1, t_1+\tau] \) and the last equality uses the definition of \( f'(t) \). This proves the claim in (9.7). Combining (9.3) and (9.7) with the fact that \( \varphi \in S_{t_1, t_1+\tau}(\zeta, \psi) \) and \( (\zeta, \psi) \in J_{t_1, t_1+\tau}^2(x^{(1)}, x^{(2)}) \) gives

\[
I_{t_1, t_1+\tau}^2(x^{(1)}, x^{(2)}) \leq \sum_{k=0}^{\infty} \int_{[t_1, t_1+\tau] \times [0,1]} \ell(\tilde{\varphi}_k(t, y)) \, dt \, dy \leq \inf_{t_2 \geq t_1} I_{t_1, t_2}^1(x^{(1)}, x^{(2)}) + 2\varepsilon.
\]

Since \( \varepsilon \in (0, 1) \) is arbitrary, (9.8) follows, which, as argued previously, gives (7.10).

Next consider (7.11) and the third statement in the lemma for fixed \( (\zeta, \psi) \in J_{t_1, t_1+\tau}^2(x^{(1)}, x^{(2)}) \). We first show that
\[
I_{t_1, t_1+\tau}(\zeta, \psi) \geq \int_{t_1}^{t_1+\tau} L(\zeta(s), \zeta'(s)) \, ds.
\]

Assume without loss of generality that \( I_{t_1, t_1+\tau}(\zeta, \psi) < \infty \). Fix \( \varepsilon \in (0, 1) \). From Remark 9.4 we can find some \( \varphi \in S_{t_1, t_1+\tau}(\zeta, \psi) \) such that
\[
\sum_{k=0}^{\infty} \int_{[t_1, t_1+\tau] \times [0,1]} \ell(\varphi_k(s, y)) \, ds \, dy \leq I_{t_1, t_1+\tau}(\zeta, \psi) + \varepsilon,
\]
\[
\varphi_k(t, y) = \rho_k(t) 1_{[0, r_k(\zeta(t))](y) + 1_{[r_k(\zeta(t)), 1]}(y)}, \quad t \in [t_1, t_1+\tau], k \in \mathbb{N}_0,
\]
for a suitable sequence of non-negative functions \( \rho_k \), and (9.11), (9.3) hold for \( t \in [t_1, t_1+\tau] \). Using (9.1), (9.2) and the fact that \( \frac{d}{dt} r(\zeta(t)) = -2 \) for a.e. \( t \in [t_1, t_1+\tau] \) we have
\[
\rho_0(t) r_0(\zeta(t)) = -\psi'(t) + \sum_{k=1}^{\infty} (k-2) \xi'_k(t) = 1 + \sum_{k=1}^{\infty} \xi'_k(t),
\]
which also implies \( \sum_{k=1}^{\infty} \xi'_k(t) \geq -1 \) for a.e. \( t \in [t_1, t_1+\tau] \), proving the third statement in the lemma. Furthermore we have
\[
\sum_{k=0}^{\infty} \int_{[t_1, t_1+\tau] \times [0,1]} \ell(\varphi_k(s, y)) \, ds \, dy \leq \sum_{k=0}^{\infty} \int_{t_1}^{t_1+\tau} r_k(\zeta(t)) \, dt \, dy + \sum_{k=1}^{\infty} r_k(\zeta(t)) \ell \left( \frac{1 + \sum_{k=1}^{\infty} \xi'_k(t)}{r_0(\zeta(t))} \right) \, dt \geq \int_{t_1}^{t_1+\tau} L(\zeta(t), \zeta'(t)) \, dt.
\]

(9.9)
where the last equality uses the definition of $\ell$ in (2.13) and $L$ in (7.4) and we use the convention that $0 \ell (x/0) = 0$ for $x \geq 0$. Therefore

$$
\int_{t_1}^{t_1 + \tau} L(\zeta(t), \zeta'(t)) \, dt = \sum_{k=0}^{\infty} \int_{[t_1, t_1 + \tau] \times [0, 1]} \ell(\varphi_k(s, y)) \, ds \, dy \leq I_{t_1, t_1 + \tau}(\zeta, \psi) + \varepsilon. \tag{9.10}
$$

Since $\varepsilon \in (0, 1)$ is arbitrary, we have (9.8).

Next we show that

$$
I_{t_1, t_1 + \tau}(\zeta, \psi) \leq \int_{t_1}^{t_1 + \tau} L(\zeta(s), \zeta'(s)) \, ds. \tag{9.11}
$$

Assume without loss of generality that $\int_{t_1}^{t_1 + \tau} L(\zeta(s), \zeta'(s)) \, ds < \infty$. Since there exists some $(\zeta^*, \psi^*) \in J_{0, t_1}^0(x^{(0)}, x^{(1)})$ such that $I_{0, t_1}(\zeta^*, \psi^*) < \infty$, we can further assume without loss of generality that $I_{0, t_1}(\zeta, \psi) < \infty$. Then there exists some $\varphi^* \in S_{t_1}(\zeta, \psi)$. Let $\varphi(t, y) \equiv \varphi^*(t, y)$ for $t \in [0, t_1)$, and for $t \in [t_1, t_1 + \tau]$ define

$$
\rho_k(t) \equiv - \frac{\zeta_k'(t)}{r_k(\zeta(t))} \mathbf{1}_{\{r_k(\zeta(t)) \neq 0\}}, \quad k \in \mathbb{N},
$$

$$
\rho_0(t) \equiv \sum_{k=1}^{\infty} (k - 2) \rho_k(t) r_k(\zeta(t)) - \psi'(t) \mathbf{1}_{\{r_0(\zeta(t)) \neq 0\}},
$$

$$
\varphi_k(t, y) \equiv \rho_k(t) \mathbf{1}_{\{r_k(\zeta(t)) \neq 0\}}(y) + 1_{[0, 1]}(y), \quad y \in [0, 1], \quad k \in \mathbb{N}_0.
$$

Clearly (9.11) and (9.2) hold for $t \in [t_1, t_1 + \tau]$ and hence $\varphi \in S_{t_1 + \tau}(\zeta, \psi)$. Also one can check that (9.9) still holds. Therefore

$$
I_{t_1, t_1 + \tau}(\zeta, \psi) \leq \sum_{k=0}^{\infty} \int_{[t_1, t_1 + \tau] \times [0, 1]} \ell(\varphi_k(s, y)) \, ds \, dy = \int_{t_1}^{t_1 + \tau} L(\zeta(t), \zeta'(t)) \, dt.
$$

This gives (9.11) and completes the proof of (7.11).

Finally, (7.12) follows on combining (7.10), (7.11) and (7.11). This completes the proof of the lemma. \hfill \square

### 9.2 Proofs of Lemmas 7.2 and 7.3

**Proof of Lemma 7.2** Consider for $(x^{(1)}, x^{(2)}) \in \Xi$, the function $\alpha \mapsto B(\alpha)$ on $(0, 1)$, defined by

$$
B(\alpha) \equiv B(\alpha; x^{(1)}, x^{(2)}) \equiv \frac{1}{\alpha} \left(1 - \alpha^2 \sum_{k=1}^{\infty} \frac{k z_k}{1 - \alpha^k} - \sum_{k=1}^{\infty} k z_k + x_0^{(2)} - \alpha^2 x_0^{(1)}\right)
$$

$$
= z_1 - \sum_{k=3}^{\infty} k z_k \frac{\alpha - \alpha^{k-1}}{1 - \alpha^k} + \frac{x_0^{(2)}}{\alpha} - \alpha x_0^{(1)}
$$

$$
= z_1 - \sum_{k=3}^{\infty} k z_k B_k(\alpha) + \frac{x_0^{(2)}}{\alpha} - \alpha x_0^{(1)}. \tag{9.12}
$$

where $B_k(\alpha) \equiv (\alpha - \alpha^{k-1})/(1 - \alpha^k)$ for $k \geq 3$ and $z = x^{(1)} - x^{(2)}$ as before. For each $k \geq 3$ and $\alpha \in (0, 1)$, using the inequality of arithmetic and geometric means one can verify that

$$
B'_k(\alpha) = \left(1 - \frac{\alpha^2}{1 - \alpha^k}\right) \left(1 + \frac{\alpha^2 + \alpha^4 + \cdots + \alpha^{2k-4}}{k - 1} - \alpha^{k-2}\right) > 0, \tag{9.13}
$$

and

$$
0 = B_k(0+) \leq B_k(\alpha) \leq B_k(1-) = (k - 2)/k. \tag{9.14}
$$
So \( B(1-) = z_1 - \sum_{k=3}^{\infty} (k-2)z_k + x_0^{(2)} - x_0^{(1)} = \left(\sum_{k=1}^{\infty} (k-2)z_k + \alpha\right) < 0 \) by assumption and \( B(\alpha) \) is decreasing in \( \alpha \in (0, 1) \). Also note that the assumption \( \sum_{k=1}^{\infty} k\beta + \alpha > 2 \sum_{k=1}^{\infty} z_k \) can be rewritten as

\[
\sum_{k=3}^{\infty} (k-2)z_k + x_0^{(1)} > z_1 + x_0^{(2)},
\]

which implies

\[
either x_0^{(1)} > 0 \text{ or } z_k > 0 \text{ for some } k \geq 3. \tag{9.15}
\]

From this and \( \Box_1 \) we see that \( B(\alpha) \) is actually strictly decreasing in \( \alpha \in (0, 1) \). Since each \( B_k(\alpha) \) is continuous on \((0, 1)\), \( B(\alpha) \) is also continuous by \( \Box_2 \) and the dominated convergence theorem. Finally, since \( B(0+) = z_1 + \alpha \cdot 1_{\{x_0^{(0)} > 0\}} > 0 \) and \( B(1-) < 0 \), there must exist a unique \( \beta \in (0, 1) \) such that \( B(\beta) = 0 \). This completes the proof of the lemma.

\[\Box\]

**Proof of Lemma 7.2**  Suppose \((x_0^{(1, n)}, x_0^{(2, n)}) \rightarrow (x_0^{(1)}, x_0^{(2)})\) as \( n \rightarrow \infty \), where \((x_0^{(1, n)}, x_0^{(2, n)}), (x_0^{(1)}, x_0^{(2)}) \in \Xi\). Recall the function \( B(\cdot) \) defined above \( \Box_3 \) and the definition of \( \beta(\cdot) \) from Section 2.2. We consider two possible cases for the values of \( x_0^{(2)} \) and \( z_1 \).

**Case 1:** \( x_0^{(2)} = 0 \) and \( z_1 = 0 \). In this case \( \beta = \beta(x_0^{(1)}, x_0^{(2)}) = 0 \) by definition and \( x_0^{(2)} \log x_0^{(2)} = 0 \) by our convention. Since \( B(0+) = z_1 + \alpha \cdot 1_{\{x_0^{(0)} > 0\}} = 0 \) and \( B(\alpha) \) is strictly decreasing in \( \alpha \in (0, 1) \), we have \( B(\alpha) < 0 \) for every \( \alpha \in (0, 1) \). Fixing \( \alpha \in (0, 1) \), from \( \Box_3 \) and the dominated convergence theorem one has

\[
B^{(n)}(\alpha) = B(\alpha; x_0^{(1, n)}, x_0^{(2, n)}) \rightarrow B(\alpha) = B(\alpha; x_0^{(1)}, x_0^{(2)}) \tag{9.16}
\]

as \( n \rightarrow \infty \). Therefore \( B^{(n)}(\alpha) < 0 \) for sufficiently large \( n \). Since \( B^{(n)} \) is decreasing, we must have \( \beta^{(n)} \geq \beta(x_0^{(1, n)}, x_0^{(2, n)}) \leq \alpha \) for all such \( n \). Since \( \alpha \in (0, 1) \) is arbitrary, this implies that as \( n \rightarrow \infty \), \( \beta^{(n)} \rightarrow 0 = \beta \). Next note that the convergence of \( x_0^{(2, n)} \log x_0^{(n)} \rightarrow x_0^{(2)} \log x_0^{(2)} = 0 \) holds trivially if \( x_0^{(2, n)} = 0 \) for all sufficiently large \( n \). Suppose now that \( x_0^{(2, n)} > 0 \) for every \( n \). Also take \( n \) to be sufficiently large, so that \( x_0^{(2, n)} < 1 \). From \( \Box_3 \) and since \( kB_k^{(n)}(\alpha) \leq (k-2) \) from \( \Box_4 \) [applied with \( (x_1, x_2) \) replaced by \( (x_0^{(1, n)}, x_0^{(2, n)}) \)] we have

\[
B^{(n)}((x_0^{(2, n)})^2) \geq 0 - \sum_{k=3}^{\infty} (k-2)z_k^2 + \frac{1}{x_0^{(2, n)}} - (x_0^{(2, n)})^2 x_0^{(1, n)} > 0
\]

for \( x_0^{(2, n)} \) sufficiently small. So \( \beta^{(n)} \geq (x_0^{(2, n)})^2 \) for such \( n \) and so \( x_0^{(2, n)} \log x_0^{(n)} \rightarrow x_0^{(2)} \log x_0^{(2)} \).

**Case 2:** \( x_0^{(2)} > 0 \) or \( z_1 > 0 \). In this case, for \( n \) sufficiently large, we must have \( x_0^{(2, n)} > 0 \) or \( z_1^2 > 0 \). So \( \beta^{(n)} \) satisfies \( B^{(n)}(\beta^{(n)}) = 0 \) for all such \( n \). Since \( \beta > 0 \), and by proof of Lemma 7.2 \( B(\beta) = 0 \), \( B(0+) > 0 \) and \( B(\cdot) \) is strictly decreasing, we have \( B(\beta/2) > 0 \). As in the proof of \( \Box_4 \), we see that \( B^{(n)}(\beta/2) \rightarrow B(\beta/2) \) as \( n \rightarrow \infty \), and so \( B^{(n)}(\beta/2) > 0 \) for all sufficiently large \( n \). Since \( B^{(n)} \) is decreasing, we must have \( \beta^{(n)} \geq \beta/2 > 0 \) for all sufficiently large \( n \). From this, \( \Box_4 \) and the dominated convergence theorem one can show that along any convergent subsequence of \( \beta^{(n)} \), \( B^{(n)}(\beta^{(n)}) \rightarrow B(\lim \beta^{(n)}) \). So any limit point of \( \beta^{(n)} \) is a solution to \( B(\alpha) = 0 \) defined on \((0, 1)\). But \( \beta \) is the unique solution to this equation. So \( \beta^{(n)} \rightarrow \beta \) and also \( x_0^{(2, n)} \log x_0^{(n)} \rightarrow x_0^{(2)} \log x_0^{(2)} \). This completes the proof of the lemma.

\[\Box\]

### 9.3 Proof of Lemma 7.4

(a) Recall the definition of \( \tau \) in \( \Box_3 \) and \( \hat{\tau}, \hat{z}_k \) from Construction \( \Box_4 \). From \( \Box_3 \) we have

\[
\hat{\tau} = \frac{\tau}{1 - \beta^2} = \frac{x_0^{(1)} - x_0^{(2)} + \sum_{k=1}^{\infty} k\hat{z}_k}{2(1 - \beta^2)} = \frac{1}{2} \left(x_0^{(1)} + \sum_{k=1}^{\infty} k\hat{z}_k\right). \tag{9.17}
\]
Since \( \beta \in [0, 1) \), we have \( \tau \leq \tau/(1 - \beta^2) = \tilde{\tau} \). This proves part (a).

(b) We first show that \( (\tilde{\zeta}, \tilde{\psi}) \in \mathcal{J}_{t_1, t_1 + \tau}^{(1)}(x^{(1)}, x^{(2)}) \). For this, it suffices to check

\[
\tilde{\zeta}_k(t_1 + \tau) = p_k^{(1)} - \tilde{z}_k(1 - \tilde{\tau})^{k/2} = p_k^{(2)} - \tilde{z}_k(1 - \tilde{\tau})^{k/2} = p_k^{(1)} - \tilde{z}_k = p_k^{(2)},
\]

which gives the first statement in (9.18). From this, (7.15) and (7.19) it follows that

\[
\tilde{\zeta}_0(t_1 + \tau) = x^{(1)}_0 + \sum_{k=1}^{\infty} k(p_k^{(1)} - p_k^{(2)}) - 2\tilde{\tau} = x^{(2)}_0,
\]

which gives the second statement in (9.18).

For (9.19), applying the change of variable \( t - t_1 = \tilde{\tau}(1 - \alpha_t^2) \), namely \( \alpha_t = (1 - \frac{t - t_1}{\tilde{\tau}}) \) for \( t \in [t_1, t_1 + \tau] \), we have

\[
\begin{align*}
\sum_{k=1}^{\infty} k(p_k^{(1)} - \tilde{\zeta}_k(t)) - 2(t - t_1) &= \sum_{k=1}^{\infty} k\tilde{\zeta}_k(1 - \tilde{\tau})^{k/2} - 2(t - t_1) \\
&= \sum_{k=1}^{\infty} k\tilde{\zeta}_k(1 - \alpha_t^k) - 2\tilde{\tau}(1 - \alpha_t^2) \\
&= \sum_{k=1}^{\infty} k\tilde{\zeta}_k(1 - \alpha_t^k) - \sum_{k=1}^{\infty} k\tilde{\zeta}_k(1 - \alpha_t^2) - x^{(1)}_0(1 - \alpha_t^2) = F(\alpha_t),
\end{align*}
\]

where the third equality follows from part (a). Using this we can write, for \( t \in [t_1, t_1 + \tau] \),

\[
\tilde{\zeta}_0(t) = x^{(1)}_0 + F(\alpha_t) = \tilde{\psi}(t) - \tilde{\psi}(t_1) + x^{(1)}_0,
\]

Note that, for \( t \in [t_1, t_1 + \tau] \),

\[
x^{(1)}_0 + F(\alpha_t) = \alpha_t x^{(1)}_0 + \sum_{k=1}^{\infty} k\tilde{\zeta}_k(\alpha_t^2 - \alpha_t) = \alpha_t(1 - \alpha_t) \left( \frac{\alpha_t x^{(1)}_0}{1 - \alpha_t} - \tilde{z}_1 + \sum_{k=3}^{\infty} k\tilde{\zeta}_k \tilde{B}_k(\alpha_t) \right)
\]

\[
= \alpha_t(1 - \alpha_t) \left( \frac{\alpha_t x^{(1)}_0}{1 - \alpha_t} - \tilde{z}_1 + \sum_{k=3}^{\infty} k\tilde{\zeta}_k \tilde{B}_k(\alpha_t) \right) = \alpha_t(1 - \alpha_t) \tilde{B}(\alpha_t),
\]

where \( \tilde{B}_k(\alpha) = (\alpha - \alpha^{k-1})/(1 - \alpha) \) for \( k \geq 3 \) and \( \tilde{B}(\alpha) = \sum_{k=3}^{\infty} k\tilde{\zeta}_k \tilde{B}_k(\alpha) \). One can verify (e.g. using Young’s inequality) that

\[
\tilde{B}_k(\alpha) = \frac{(k - 2)\alpha^{k-1} - (k - 1)\alpha^{k-2} + 1}{(\alpha - 1)^2} > 0, \quad k \geq 3, \quad \alpha \in [0, 1).
\]

So \( \tilde{B}(\alpha) \) is increasing. Using (7.13) one can verify that \( \tilde{B}(\beta) = \tilde{x}^{(2)}_{t_1 - \beta t}1_{\beta > 0} \geq 0. \) Since for \( t \in (t_1, t_1 + \tau), \alpha_t \in [\beta, 1), \) for all such \( t, x^{(1)}_0 + F(\alpha_t) \geq 0. \) This along with (9.20) gives (9.19).

So far we have verified that \( (\tilde{\zeta}, \tilde{\psi}) \in \mathcal{J}_{t_1, t_1 + \tau}^{(1)}(x^{(1)}, x^{(2)}) \). From (7.19) we also have \( \frac{d}{dt} r(\tilde{\zeta}(t)) = -2 \) for \( t \in [t_1, t_1 + \tau] \). Thus actually \( (\tilde{\zeta}, \tilde{\psi}) \in \mathcal{J}_{t_1, t_1 + \tau}^{(1)}(x^{(1)}, x^{(2)}) \), completing the proof of (b).

(c) Since for \( t \in (t_1, t_1 + \tau), \tilde{\zeta}_0(t) = x^{(1)}_0 + F(\alpha_t) = \alpha_t(1 - \alpha_t) \tilde{B}(\alpha_t) \) and \( \tilde{B}(\beta) \geq 0 \), it suffices to show that \( \tilde{B}(\alpha) \) is strictly increasing in \( \alpha \in [\beta, 1) \). But thanks to (9.19), this is immediate from the fact that \( \tilde{B}_k(\alpha) > 0 \) and \( \frac{x^{(1)}_0}{1 - \alpha} \) is strictly increasing when \( x^{(1)}_0 > 0 \). This gives part (c) and completes the proof of the lemma. \( \square \)
9.4 Proof of Lemma 7.5

Let \( \mu_0 = \frac{1}{2} \left( \sigma_0^{(1)} + \sum_{k=0}^{\infty} k \sigma_k^{(1)} \right) \). Recall \( \tilde{z}_k = \frac{z_k}{1 - \beta \tau} \). It then follows from (7.14), (7.15) and Lemma 7.4] a) that for \( t \in [t_1, t_1 + \tau] \),

\[
\tilde{z}_k(t) = \frac{-k \tilde{z}_k}{2 \tau - 2(t - t_1)} \left( 1 - \frac{t - t_1}{\tau} \right)^{k/2} = -\frac{k}{2 \tau - 2(t - t_1)}(\tilde{z}_k(t) - p_k^{(1)} + \tilde{z}_k),
\]

(9.21)

\[
1 + \sum_{k=1}^{\infty} \tilde{z}_k(t) = \frac{x_0^{(1)} - 2(t - t_1) + \sum_{k=1}^{\infty} kp_k^{(1)} - \tilde{z}_k(t)}{2 \tau - 2(t - t_1)} = \frac{\tilde{z}_0(t)}{2 \tau - 2(t - t_1)},
\]

(9.22)

\[
r(\tilde{z}(t)) = 2\mu_0 - 2(t - t_1).
\]

From these we have

\[
\int_{t_1}^{t_1 + \tau} L(\tilde{\zeta}(s), \tilde{\zeta}'(s)) \, ds
\]

\[
= \int_{t_1}^{t_1 + \tau} \left[ \left( 1 + \sum_{k=1}^{\infty} \tilde{z}_k(t) \right) \log \left( \frac{1 + \sum_{k=1}^{\infty} \tilde{z}_k(t)}{\tilde{\zeta}_0(t)/r(\tilde{\zeta}(t))} \right) + \sum_{k=1}^{\infty} (-\tilde{z}_k(t)) \log \left( \frac{-\tilde{\zeta}_k(t)}{k \tilde{\zeta}_k(t)/r(\tilde{\zeta}(t))} \right) \right] \, dt
\]

\[
= \int_{t_1}^{t_1 + \tau} \left[ (2\mu_0 - 2(t - t_1)) - (2\tau - 2(t - t_1)) - \sum_{k=1}^{\infty} \tilde{z}_k(t) \log \left( \frac{\tilde{\zeta}_k(t) - p_k^{(1)} + \tilde{z}_k}{\tilde{\zeta}_k(t)} \right) \right] \, dt.
\]

(9.23)

We claim that we can interchange the integration and summation in the last line. To see this, first note that there exists some \( M \in \mathbb{N} \) such that \( z_k \leq \tilde{z}_k \leq 2z_k \leq 2p_k^{(1)} < 1 \) for \( k \geq M \). Since \( \tilde{\zeta}_k(t) \) is non-increasing, we have

\[
\int_{t_1}^{t_1 + \tau} \sum_{k=M}^{\infty} \tilde{z}_k(t) \log \left( \frac{\tilde{\zeta}_k(t) - p_k^{(1)} + \tilde{z}_k}{\tilde{\zeta}_k(t)} \right) \, dt
\]

\[
= \sum_{k=M}^{\infty} \int_{t_1}^{t_1 + \tau} \log \left( \frac{\tilde{\zeta}_k(t) - p_k^{(1)} + \tilde{z}_k}{\tilde{\zeta}_k(t)} \right) \, d(p_k^{(1)} - \tilde{\zeta}_k(t)) = \sum_{k=M}^{\infty} \int_0^{z_k} \log \left( \frac{\tilde{z}_k - u}{p_k^{(1)} - u} \right) \, du
\]

\[
\leq \sum_{k=M}^{\infty} \int_0^{z_k} \left( - \log(\tilde{z}_k - u) - \log(p_k^{(1)} - u) \right) \, du \leq -2 \sum_{k=M}^{\infty} \int_0^{p_k^{(1)}} \log u \, du.
\]

Using \( \tilde{\ell}(x) = x \log x - x = \int \log x \, dx \), the last expression equals

\[
-2 \sum_{k=M}^{\infty} (\tilde{\ell}(2p_k^{(1)}) - \tilde{\ell}(0)) = 4 \sum_{k=M}^{\infty} p_k^{(1)} \log \left( \frac{1}{p_k^{(1)}} \right) - 4(2 - 1) \sum_{k=M}^{\infty} p_k^{(1)}.
\]

Here the last term is clearly finite. Letting \( \tilde{M} = \sum_{k=M}^{\infty} p_k^{(1)} \in (0, 1] \), we have

\[
\sum_{k=M}^{\infty} p_k^{(1)} \log \left( \frac{1}{p_k^{(1)}} \right) = \tilde{M} \sum_{k=M}^{\infty} \frac{p_k^{(1)}}{k^2} \log k^2
\]

\[
\leq \tilde{M} \log \left( \sum_{k=M}^{\infty} \frac{1}{k^2} \right) + 2 \sum_{k=M}^{\infty} kp_k^{(1)} < \infty,
\]

where the inequality holds since \( \log x \) is concave and \( \log x \leq x \). Therefore

\[
\sum_{k=M}^{\infty} \int_{t_1}^{t_1 + \tau} \tilde{z}_k(t) \log \left( \frac{\tilde{\zeta}_k(t) - p_k^{(1)} + \tilde{z}_k}{\tilde{\zeta}_k(t)} \right) \, dt < \infty,
\]

49
One can easily verify that for \(1 \leq k \leq M\),

\[
\int_{t_1}^{t_1+\tau} \tilde{\zeta}(t) \log \left( \frac{\tilde{\zeta}(t) - p_k^{(1)} + \tilde{z}_k}{\tilde{\zeta}(t)} \right) \, dt = \int_0^{\tilde{z}_k} \log \left( \frac{\tilde{z}_k - u}{p_k^{(1)} - u} \right) \, du < \infty.
\]

So the claim holds. Actually we have also shown that \(\int_{t_1}^{t_1+\tau} L(\zeta(s), \zeta'(s)) \, ds < \infty\).

From (9.23) it then follows that

\[
\int_{t_1}^{t_1+\tau} L(\tilde{\zeta}(s), \tilde{\zeta'}(s)) \, ds
\]

\[
= \frac{\beta^2 \tau}{1 - \beta^2} \log \frac{\beta^2 \tau}{1 - \beta^2} - \frac{\tau}{1 - \beta^2} \log \frac{\tau}{1 - \beta^2}
\]

\[
= -\tau \log \tau + \tau \log(1 - \beta^2) + \frac{2\beta^2 \tau}{1 - \beta^2} \log \beta.
\]

Using \(\tilde{\zeta}(t_1) = x^{(1)}\) and \(\tilde{\zeta}(t_1 + \tau) = x^{(2)}\). Using \(\tilde{\tau} = \tau/(1 - \beta^2)\),

\[
(\tilde{\tau} - \tau) \log(\tilde{\tau} - \tau) - \tilde{\tau} \log \tilde{\tau} = \frac{\beta^2 \tau}{1 - \beta^2} \log \frac{\beta^2 \tau}{1 - \beta^2} - \frac{\tau}{1 - \beta^2} \log \frac{\tau}{1 - \beta^2}
\]

where the last line follows from \(\tilde{\zeta}(t_1) = x^{(1)}\) and \(\tilde{\zeta}(t_1 + \tau) = x^{(2)}\). Using \(\tilde{\tau} = \tau/(1 - \beta^2)\),

\[
(\tilde{\tau} - \tau) \log(\tilde{\tau} - \tau) - \tilde{\tau} \log \tilde{\tau} = \frac{\beta^2 \tau}{1 - \beta^2} \log \frac{\beta^2 \tau}{1 - \beta^2} - \frac{\tau}{1 - \beta^2} \log \frac{\tau}{1 - \beta^2}
\]

\[
= -\tau \log \tau + \tau \log(1 - \beta^2) + \frac{2\beta^2 \tau}{1 - \beta^2} \log \beta.
\]

Since \(\tilde{z}_k = z_k/(1 - \beta^k)\), we have

\[
\sum_{k=1}^{\infty} [-(\tilde{z}_k - z_k) \log(\tilde{z}_k - z_k) + \tilde{z}_k \log \tilde{z}_k]
\]

\[
= \sum_{k=1}^{\infty} \left[ z_k \log z_k - z_k \log(1 - \beta^k) - k\beta^k z_k \log \beta \right]
\]

\[
= \sum_{k=1}^{\infty} \left[ z_k \log z_k - z_k \log(1 - \beta^k) \right] + \sum_{k=1}^{\infty} \left( k z_k \frac{z_k}{1 - \beta^k} \log \beta \right)
\]

\[
= \sum_{k=1}^{\infty} \left[ z_k \log z_k - z_k \log(1 - \beta^k) \right] + x_0^{(2)} \left( \frac{2\beta^2 \tau}{1 - \beta^2} \log \beta \right),
\]

where the last line is from (7.13) and (9.17). The last two displays along with (9.24) give

\[
\int_{t_1}^{t_1+\tau} L(\tilde{\zeta}(s), \tilde{\zeta'}(s)) \, ds
\]

\[
= -\mu_0 \log \mu_0 - \tau \log \tau + \mu_0 \log \mu_0 + \tau \log(1 - \beta^2)
\]

\[
+ \sum_{k=1}^{\infty} \left[ z_k \log z_k + p_k^{(2)} \log p_k^{(2)} - p_k^{(1)} \log p_k^{(1)} \right] - \sum_{k=1}^{\infty} z_k \log(1 - \beta^k) + x_0^{(2)} \log \beta
\]

\[
= \hat{H}(z) + \hat{H}(x^{(2)}) - \hat{H}(x^{(1)}) + \hat{K}(x^{(1)}, x^{(2)}).
\]

50
Finiteness of the above follows as in Remark 2.5. This gives the first statement in the lemma.

For the lower semicontinuity, first note that \(-((\mu_0 - \tau) \log(\mu_0 - \tau) - \tau \log \mu + \mu_0 \log \mu_0 + \tau \log(1 - \beta^2)) - \sum_{k=1}^{\infty} z_k \log(1 - \beta^k) + \chi_0^{(2)} \log \beta\) is continuous from Lemma 7.3 and Assumption 2.2. The remaining terms in (9.22) can be written as

\[
\sum_{k=1}^{\infty} \left[ z_k \log z_k + p_k^{(2)} \log p_k^{(2)} - p_k^{(1)} \log p_k^{(1)} \right] = \sum_{k=0}^{\infty} \left[ z_k \log z_k + p_k^{(2)} \log p_k^{(2)} - p_k^{(1)} \log p_k^{(1)} \right] - \left[ z_0 \log z_0 + p_0^{(2)} \log p_0^{(2)} - p_0^{(1)} \log p_0^{(1)} \right] = \sum_{k=0}^{\infty} \left[ z_k \log z_k \right] + \sum_{k=0}^{\infty} \left[ p_k^{(2)} \log p_k^{(2)} \right] - \left[ z_0 \log z_0 + p_0^{(2)} \log p_0^{(2)} - p_0^{(1)} \log p_0^{(1)} \right],
\]

where \(z_0 = 1 - \sum_{k=1}^{\infty} z_k\), and \(p_0^{(i)} = 1 - \sum_{k=1}^{\infty} p_k^{(i)}\) for \(i = 1, 2\). The last term in the above display is clearly a lower semicontinuous function of \((x^{(1)}, x^{(2)}) \in \Xi\). The lemma follows. \(\square\)

### 9.5 Proof of Lemma 7.6

We begin with a lemma that gives the statement in Lemma 7.6 under a stronger assumption.

**Lemma 9.2.** Suppose the same setting as in Lemma 7.6. Suppose in addition that: (i) \(x_0^{(1)}, x_0^{(2)} > 0\), and (ii) for every \(k \in \mathbb{N}\), if \(p_k^{(1)} > 0\) then \(p_k^{(2)} > 0\). Then (7.18) is satisfied.

**Proof.** The first equality in (7.18) is proved in Lemma 7.1. For the second equality, we need to show that \((\tilde{\zeta}, \psi)\) is the minimizer of the function

\[
\tilde{G}(\zeta, \psi) \doteq \int_{t_1}^{t_1 + \tau} L(\zeta(s), \zeta'(s)) \, ds, \quad (\zeta, \psi) \in J_{t_1, t_1 + \tau}^2(x^{(1)}, x^{(2)}).
\]

We will prove this via contradiction. First note that \(J_{t_1, t_1 + \tau}^2(x^{(1)}, x^{(2)})\) is a convex set. Also using the definition of \(L\), one can verify that \(\tilde{G}(\zeta, \psi)\) is a convex function in \((\zeta, \psi) \in J_{t_1, t_1 + \tau}^2(x^{(1)}, x^{(2)})\). Now suppose there exists some \((\zeta, \psi) \in J_{t_1, t_1 + \tau}^2(x^{(1)}, x^{(2)})\) such that \(\tilde{G}(\zeta, \psi) < \tilde{G}(\tilde{\zeta}, \tilde{\psi})\). From Lemma 7.5 we have \(\tilde{G}(\tilde{\zeta}, \tilde{\psi}) < \infty\). For \(\varepsilon \in [0, 1]\), construct the family of paths \((\zeta^\varepsilon, \psi^\varepsilon) \doteq (1 - \varepsilon)(\tilde{\zeta}, \tilde{\psi}) + \varepsilon(\zeta, \psi)\). Letting \(g(\varepsilon) \doteq \tilde{G}(\zeta^\varepsilon, \psi^\varepsilon)\), we have \(g(1) = \tilde{G}(\zeta, \psi) < \tilde{G}(\tilde{\zeta}, \tilde{\psi}) = g(0)\). It follows from the convexity that \(g\) is left and right differentiable wherever it is finite. We will show that \(g'(0) = 0\), where \(g'\) is the right derivative of \(g\). The convexity of \(g\) will then give the desired contradiction.

By convexity of \(g\), we have \(g(\varepsilon) < g(0)\) for every \(\varepsilon \in (0, 1]\). From Lemma 7.4(c), assumption (i) and continuity of \(\tilde{\zeta}_0\) we have

\[
\delta \doteq \inf_{t \in [t_1, t_1 + \tau]} \tilde{\zeta}_0(t) > 0.
\] (9.26)

From (9.22) we see

\[
1 + \sum_{k=1}^{\infty} \tilde{\zeta}_k(t) = \frac{\tilde{\zeta}_0(t)}{2\tau - 2(t - t_1)} \geq \frac{\delta}{2\tau} > 0, \quad t \in [t_1, t_1 + \tau].
\] (9.27)

Now fix \(0 < \varepsilon < \frac{1}{4} \land \delta \land \frac{\delta}{2\tau}\). Then \(\zeta_0(t) > \frac{\delta}{2\tau}\) for all \(t \in [t_1, t_1 + \tau]\).

We next argue that one can assume without loss of generality that

\[
\zeta_k(t) = \tilde{\zeta}_k(t) \text{ for all } t \in [t_1, t_1 + \tau] \text{ and } k \geq n_0
\] (9.28)
for some large enough \( n_0 \in \mathbb{N} \). To show this, we define \((\zeta^n, \psi^n)\) for \( n \in \mathbb{N} \) as follows: For \( t \in [0, t_1) \), \((\zeta^n(t), \psi^n(t)) = (\zeta^n(t), \psi^n(t))\), and for \( t \in [t_1, t_1 + \tau] \),
\[
\zeta_k^n(t) = \tilde{\zeta}_k(t), \quad k \geq n, \\
\zeta_k^n(t) = \zeta_k^n(t), \quad 1 \leq k < n, \\
\zeta_0^n(t) = \zeta_0(t) + \sum_{k=1}^{\infty} k(p_k^{(1)} - \zeta_k^n(t)) - 2(t - t_1), \\
\psi^n(t) = \psi(t) + \sum_{k=1}^{\infty} k(p_k^{(1)} - \zeta_k^n(t)) - 2(t - t_1).
\]

From this definition we have \((\zeta_k^n)_{k \in \mathbb{N}} \to (\zeta_k^\infty)_{k \in \mathbb{N}}\) in \( C([0, t_1 + \tau] : \mathbb{R}_+^\infty) \) as \( n \to \infty \). So \((\zeta_0^n, \psi^n) \to (\zeta_0^\infty, \psi^\infty)\) in \( C([0, t_1 + \tau] : \mathbb{R}^2) \) as \( n \to \infty \). From this we see \( \psi^n(t) - \psi^n(t_1) + \zeta_0^{(1)} = \zeta_0^n(t) \) is uniformly bounded away from 0 in \( t \in [t_1, t_1 + \tau] \) for sufficiently large \( n \). So \( \zeta^n \in \mathcal{J}_{t_1, t_1 + \tau}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \) for all such \( n \). Recall \( L_k \) and \( L \) defined in (7.3). Using the definition of \( \zeta_k^n \) for \( 1 \leq k < n \)
\[
\tilde{G}(\zeta^n, \psi^n) - \tilde{G}(\zeta^\infty, \psi^\infty) = \int_{t_1}^{t_1 + \tau} \left[ L(\zeta^n(s), (\zeta^n(s))') - L(\zeta^\infty(s), (\zeta^\infty(s))') \right] ds \\
= \int_{t_1}^{t_1 + \tau} \left[ L_0(\zeta^n(s), (\zeta^n(s))') - L_0(\zeta^\infty(s), (\zeta^\infty(s))') \right] ds \\
+ \int_{t_1}^{t_1 + \tau} \sum_{k=n}^{\infty} \left[ L_k(\zeta^n(s), (\zeta^n(s))') - L_k(\zeta^\infty(s), (\zeta^\infty(s))') \right] ds.
\]

We claim that both terms on the right side converge to 0 as \( n \to \infty \). To see this, note that
\[
L_0(\zeta^n(s), (\zeta^n(s))') = (1 + \sum_{k=1}^{\infty} (\zeta_k^n)'(s)) \log \left[ (1 + \sum_{k=1}^{\infty} (\zeta_k^n)'(s)) \right] \to L_0(\zeta^\infty(s), (\zeta^\infty(s))')
\]
as \( n \to \infty \), for each \( s \in [t_1, t_1 + \tau] \). From (9.27) and the choice of \( \varepsilon \) we have that
\[
1 \geq 1 + \sum_{k=1}^{\infty} (\zeta_k^n)'(s) \geq \frac{\delta}{2\tau} - \varepsilon > 0.
\]
Since \( \zeta_0^n(s) \) and \( \tilde{\zeta}_0(s) \) are both bounded from above and away from 0 for all \( s \in [t_1, t_1 + \tau] \),
\[
\sup_{n \in \mathbb{N}} \sup_{s \in [t_1, t_1 + \tau]} |L_0(\zeta^n(s), (\zeta^n(s))')| < \infty.
\]
The first term on the right side of (9.29) then converges to 0 as \( n \to \infty \) by the dominated convergence theorem. For the second term, note that
\[
L_k(\zeta^n(s), (\zeta^n(s))') = -(\zeta_k^n)'(s) \log \left[ \frac{\zeta_k^n(s)}{r(\zeta_k^n(s))} \right] \to L_k(\zeta^\infty(s), (\zeta^\infty(s))')
\]
as \( n \to \infty \), for each \( s \in [t_1, t_1 + \tau] \). Since \( \zeta^n \in \mathcal{J}_{t_1, t_1 + \tau}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \), we have \( r(\zeta^n(s)) = r(\tilde{\zeta}(s)) = r(\zeta^\infty(s)) \) for each \( s \in [t_1, t_1 + \tau] \), and hence
\[
|L_k(\zeta^n(s), (\zeta^n(s))')| \leq |L_k(\tilde{\zeta}(s), (\zeta^\infty(s))')| + |L_k(\zeta^\infty(s), (\zeta^\infty(s))')|.
\]
Since \( \tilde{G}(\zeta^\infty, \psi^\infty) < \infty \) and \( \tilde{G}(\zeta, \psi) < \infty \), we see that the last expression is summable over \( k \in \mathbb{N} \) and integrable over \( s \in [t_1, t_1 + \tau] \). Therefore the second term in the claim converges to 0 as \( n \to \infty \) by the dominated convergence theorem. From the above claim we then have that \( \tilde{G}(\zeta^n, \psi^n) < \tilde{G}(\zeta, \psi) \) for
sufficiently large \( n_0 \). We now fix such a \( n_0 \) and, abusing notation, denote \( (\zeta, \psi) = (\zeta^{n_0}, \psi^{n_0}) \) and define \( (\tilde{\zeta}, \tilde{\psi}) \) as before, by so that \[ 9.28 \] holds.

Since \( (\zeta, \psi) \in \mathcal{J}_{t_1, t_1 + \tau}(\mathbf{1}, \mathbf{2}) \), we have \( r(\zeta(t)) = x_0^{(1)} + \sum_{k=1}^{\infty} kp_k^{(1)} - 2(t - t_1) \) and \( \zeta_0(t) = r(\zeta(t)) - \sum_{k=1}^{\infty} k\zeta_k(t) = x_0^{(1)} + \sum_{k=1}^{\infty} kp_k^{(1)} - \zeta_k(t) - 2(t - t_1) \) for \( t \in [t_1, t_1 + \tau] \). Using the definition of \( \mathcal{L} \), one can write

\[
\tilde{G}(\zeta, \psi) = \int_{t_1}^{t_1 + \tau} \left\{ \left( 1 + \sum_{k=1}^{\infty} \zeta_k'(t) \right) \log \left[ \left( 1 + \sum_{k=1}^{\infty} \zeta_k(t) \right) / \left( \zeta_0(t) / r(\zeta(t)) \right) \right] \right. \\
- \sum_{k=1}^{\infty} \zeta_k'(t) \log \left[ (-\zeta_k'(t)) / \left( \zeta_0(t) / r(\zeta(t)) \right) \right] \right\} dt \\
= \int_{t_1}^{t_1 + \tau} \left\{ \left( 1 + \sum_{k=1}^{\infty} \zeta_k'(t) \right) \log \left( x_0^{(1)} + \sum_{k=1}^{\infty} kp_k^{(1)} - \zeta(t) / r(\zeta(t)) \right) \\
- \sum_{k=1}^{\infty} \zeta_k'(t) \log \left( \zeta_0(t) / k\zeta_k(t) \right) \right\} dt + \int_{t_1}^{t_1 + \tau} \log \left( x_0^{(1)} + \sum_{k=1}^{\infty} kp_k^{(1)} - 2(t - t_1) \right) dt,
\]

and the analogous expression holds for \( \tilde{G}(\tilde{\zeta}, \tilde{\psi}) \). Let \( \theta \doteq \zeta - \tilde{\zeta} \). From \[ 9.28 \] we have \( \theta_k = 0 \) for \( k > n_0 \) and hence

\[
g(\varepsilon) = \int_{t_1}^{t_1 + \tau} \left\{ \left( 1 + \sum_{k=1}^{n_0} \zeta_k'(t) \right) \log \left( x_0^{(1)} + \sum_{k=1}^{\infty} \zeta_k'(t) \right) \\
- \sum_{k=1}^{n_0} \zeta_k'(t) \log \left( \zeta_0(t) / k\zeta_k(t) \right) \right\} dt + C_0 \\
= \int_{t_1}^{t_1 + \tau} \eta(t, \zeta_k(t) + \varepsilon \theta_k(t), \zeta_k(t) + \varepsilon \theta_k'(t)) dt + C_0,
\]

for some constant \( C_0 \), where

\[
\eta(t, (u_k, v_k)) = \left( 1 + \sum_{k=1}^{n_0} v_k + \alpha \right) \log \left( \frac{1 + \sum_{k=1}^{n_0} v_k + \alpha}{x_0^{(1)} + \sum_{k=1}^{\infty} kp_k^{(1)} - u_k + \gamma - 2(t - t_1)} \right) - \sum_{k=1}^{n_0} v_k \log \left( \frac{v_k}{k u_k} \right),
\]

with \( \alpha = \sum_{k=n_0+1}^{\infty} \zeta_k'(t) \) and \( \gamma = \sum_{k=n_0+1}^{\infty} kp_k^{(1)} - \zeta_k(t) \). We wish to show that differentiation under the integral over \( t \) with respect to \( \varepsilon \) is valid in a neighborhood of 0. For this, we now establish an integrable bound on the partial derivative of \( \tilde{\eta} \) with respect to \( \varepsilon \). To obtain such a bound, note that we only need to consider the contribution from \( \varepsilon \theta_k(t) \) for \( 1 \leq k \leq n_0 \) such that \( p_k^{(2)} > 0 \), since when \( p_k^{(2)} = 0 \), one has that \( p_k^{(1)} = 0 \) by assumption (ii), which implies \( \theta_k(t) \equiv 0 \). Therefore assume without loss of generality that \( p_k^{(2)} > 0 \) for every \( 1 \leq k \leq n_0 \). Further note that we can assume \( p_k^{(1)} > p_k^{(2)} \), since otherwise, once more, \( \theta_k(t) \equiv 0 \). Therefore we assume without loss of generality that

\[
p_k^{(1)} > p_k^{(2)}, \quad 1 \leq k \leq n_0.
\]

Denote by \( \frac{\partial \eta}{\partial v_k} \) and \( \frac{\partial \eta}{\partial u_k} \) the corresponding partial derivatives for the function \( \eta(t, (u_k, v_k)) \). Then one can verify that

\[
\frac{\partial \tilde{\eta}(t, \varepsilon)}{\partial \varepsilon} = \sum_{k=1}^{n_0} \frac{\partial \eta}{\partial u_k} |_{t, (\zeta_k(t), \zeta_k'(t))_{k=1}^{n_0}} \theta_k(t) + \sum_{k=1}^{n_0} \frac{\partial \eta}{\partial v_k} |_{t, (\zeta_k(t), \zeta_k'(t))_{k=1}^{n_0}} \theta_k'(t).
\]
The partial derivatives of $\eta$ are
\[
\frac{\partial \eta_t}{\partial u_k} = \frac{k(\sum_{j=1}^{n_0} v_j + \sum_{j=n_0+1}^{\infty} \tilde{\zeta}'_j(t))}{x_0^{(1)}(t) + \sum_{j=1}^{n_0} j(p_j^{(1)} - \tilde{\zeta}_j(t)) + \sum_{j=n_0+1}^{\infty} j(p_j^{(1)} - \tilde{\zeta}_j(t)) - 2(t - t_1)} + \frac{v_k}{u_k},
\]
for $1 \leq k \leq n_0$. For all $0 \leq \varepsilon < \frac{1}{4} \wedge \delta \wedge \frac{\delta}{2\pi}$ and $t \in [t_1, t_1 + \tau]$, from (3.30) and (3.32),
\[
0 < \frac{\delta}{4} \leq (1 - \varepsilon) \tilde{\zeta}_0(t) \leq \zeta_0(t) \leq \sum_{k=1}^{\infty} k p_k < \infty,
\]
where the last line uses (2.24) and Lemma 3.1. Furthermore, using (3.13) we get
\[
(\zeta^{(k)}_0(t) \leq (1 - \varepsilon) \tilde{\zeta}^k_0(t) \leq \frac{3k \tilde{z}_k}{\delta / 2} \left(1 - \frac{t - t_1}{\tau}\right)^{k/2 - 1} = \frac{3k \tilde{z}_k}{\delta / 2} (\tilde{\tau} - (t - t_1))^{k/2 - 1}.
\]
Combining these bounds we have
\[
\left| \frac{\partial \eta_t}{\partial u_k} (t, (\zeta^{(k)}_0(t), (\zeta^{(k)}_0(t)))_{k=1}^{n_0}) \right| \leq \frac{k}{\delta / 2} + \frac{1}{p_k^{(2)}},
\]
\[
\left| \frac{\partial \eta_t}{\partial v_k} (t, (\zeta^{(k)}_0(t), (\zeta^{(k)}_0(t)))_{k=1}^{n_0}) \right| \leq \max \left\{ \left| \frac{\log 1 / \delta / 2}{\log \left(\frac{\delta / 4 \tilde{\tau}}{x_0^{(1)}(t) + \sum_{j=1}^{\infty} j p_j^{(1)} - \tilde{\zeta}_j(t)}\right)} \right|, \left| \frac{\log 1 / \delta / 2}{k p_k^{(2)}} \right| \right\}
\]
\[
+ \max \left\{ \left| \frac{\log 1 / \delta / 2}{1 / k p_k^{(2)}} \right|, \left| \frac{3k \tilde{z}_k}{\delta / 2} \left(\tilde{\tau} - (t - t_1))^{k/2 - 1} \right) \right| \right\}
\]
for all $\varepsilon \in [0, 1/4]$, $t \in [t_1, t_1 + \tau]$, and $k = 1, \ldots, n_0$. Therefore one can find some $\tilde{C}_0 \in (0, \infty)$ such that
\[
\left| \frac{\partial \eta_t}{\partial \varepsilon} (t, \varepsilon) \right| \leq \tilde{C}_0 + \tilde{C}_0 (\tilde{\tau} - (t - t_1))^{1/2 - 1}, \quad \varepsilon \in [0, 1/4], \quad t \in [t_1, t_1 + \tau].
\]
Since $|\log (\tilde{\tau} - (t - t_1))|$ is integrable in $t \in [t_1, t_1 + \tau]$, we have obtained an integrable bound on $|\frac{\partial \eta_t}{\partial \varepsilon}|$ that is uniform in $\varepsilon \in [0, 1/4]$. Thus we can differentiate under the integral sign to get
\[
g'(\varepsilon) = \int_{t_1}^{t_1 + \tau} \frac{\partial \eta_t}{\partial \varepsilon} (t, \varepsilon) \, dt
\]
for all $0 < \varepsilon < \frac{1}{4} \wedge \delta \wedge \frac{\delta}{2\pi}$. Next we claim that the following Euler-Lagrange equations are satisfied.
\[
\frac{\partial \eta_t}{\partial u_n} (t, (\tilde{\zeta}_k(t), (\tilde{\zeta}'_k(t)))_{k=1}^{n_0}) = \frac{d}{dt} \partial \eta_t (t, (\tilde{\zeta}_k(t), (\tilde{\zeta}'_k(t)))_{k=1}^{n_0})
\]
for $1 \leq n \leq n_0, t \in [t_1, t_1 + \tau]$. (9.33)
Once this claim is verified, we have
\[
g'_+(0) = \sum_{k=1}^{n_0} \int_{t_1}^{t_1 + \tau} \left[ \frac{\partial \eta_t}{\partial u_k} (t, (\tilde{\zeta}_k(t), (\tilde{\zeta}'_k(t)))_{k=1}^{n_0}) \theta_k(t) + \frac{\partial \eta_t}{\partial v_k} (t, (\tilde{\zeta}_k(t), (\tilde{\zeta}'_k(t)))_{k=1}^{n_0}) \theta'_k(t) \right] dt
\]
\[
= \sum_{k=1}^{n_0} \int_{t_1}^{t_1 + \tau} \theta_k(t) \left[ - \int_{t_1}^{t_1 + \tau} \frac{\partial \eta_t}{\partial u_k} (s, (\tilde{\zeta}_k(s), (\tilde{\zeta}'_k(s)))_{k=1}^{n_0}) ds + \frac{\partial \eta_t}{\partial v_k} (t, (\tilde{\zeta}_k(t), (\tilde{\zeta}'_k(t)))_{k=1}^{n_0}) \right] dt
\]
\[
= \sum_{k=1}^{n_0} \int_{t_1}^{t_1 + \tau} \tilde{c}_k \theta'_k(t) \, dt = \sum_{k=1}^{n_0} \int_{t_1}^{t_1 + \tau} \tilde{c}_k (\theta_k(t_1 + \tau) - \theta_k(t_1)) = 0,\]
54
where the second equality follows from integration by parts, the third is a consequence of (9.33) with some suitable constants \( \tilde{c}_k \) and the last equality holds since \( \theta(t_1) = 0 = \theta(t_1 + \tau) \). This gives the desired contradiction and shows that \( (\zeta, \psi) \) is the minimizer.

Finally we prove the claim (9.33). Fix \( 1 \leq n \leq n_0 \). Using (9.31) and (9.32) one can verify that
\[
\frac{\partial n}{\partial u_n}(t, (\tilde{z}_k(t), \tilde{z}_k'(t))_{k=1}^{n_0}) = \frac{n(1 + \sum_{k=1}^{\infty} \tilde{z}_k(t))}{\zeta_0(t)} + \frac{\tilde{z}_k'(t)}{\zeta_n(t)}
\]
\[
= \frac{n}{2\tau - 2(t-t_1)} + \frac{\tilde{z}_k'(t)}{\zeta_n(t)}
\]
\[
= \frac{d}{dt} \left( -\frac{n}{2} \log(\tilde{z} - (t-t_1)) + \log(\tilde{z}_n(t)) \right).
\]
Therefore it suffices to show
\[
-\frac{n}{2} \log(\tilde{z} - (t-t_1)) + \log(\tilde{z}_n(t)) = \frac{\partial n}{\partial u_n}(t, (\tilde{z}_k(t), \tilde{z}_k'(t))_{k=1}^{n_0}) + \tilde{c}_n
\]
for some constant \( \tilde{c}_n \). From (9.32) one has that
\[
\frac{\partial n}{\partial u_n}(t, (\tilde{z}_k(t), \tilde{z}_k'(t))_{k=1}^{n_0}) = \log(n\tilde{z}_n(t)) - \log(-\tilde{z}_n'(t)) + \log \left( \frac{1 + \sum_{k=1}^{\infty} \tilde{z}_k(t)}{\zeta_0(t)} \right)
\]
\[
= \log(n\tilde{z}_n(t)) - \log(-\tilde{z}_n'(t)) - \log(2\tau - 2(t-t_1))
\]
where the last line follows from (9.22). From this we have
\[
-\frac{n}{2} \log(\tilde{z} - (t-t_1)) + \log(\tilde{z}_n(t)) - \frac{\partial n}{\partial u_n}(t, (\tilde{z}_k(t), \tilde{z}_k'(t))_{k=1}^{n_0})
\]
\[
= -\left( \frac{n}{2} - 1 \right) \log(\tilde{z} - (t-t_1)) - \log \left( \frac{n}{2} \right) + \log(-\tilde{z}_n'(t))
\]
\[
= \log \tilde{z}_n - \frac{n}{2} \log \tilde{z}.
\]
where the last line follows from (9.21) and (7.14). Therefore (9.34) holds with \( \tilde{c}_n = \log \tilde{z}_n - \frac{n}{2} \log \tilde{z} \) which proves (9.36). This completes the proof.

Proof of Lemma (7.6) The first equality in (7.18) follows as before from Lemma (7.1). Lemma (9.2) shows that the second equality holds if additional two assumptions in Lemma (9.2) are satisfied. Let \( (\xi, \psi) \in \mathcal{J}_{t_1,t_1+\tau}(x^{(1)}, x^{(2)}) \) be a trajectory such that \( \int_{t_1}^{t_1+\tau} L(\xi(s), \xi'(s))\, ds \leq \int_{t_1}^{t_1+\tau} L(\tilde{\xi}(s), \tilde{\xi}'(s))\, ds \). It suffices to show
\[
\int_{t_1}^{t_1+\tau} L(\xi(s), \xi'(s))\, ds \geq \int_{t_1}^{t_1+\tau} L(\tilde{\xi}(s), \tilde{\xi}'(s))\, ds.
\]
(9.35)
We claim that we can assume
- \( \zeta_0(t) > 0 \) for all \( t \in (t_1, t_1+\tau) \),
- if \( z_k > 0 \) for some \( k \in \mathbb{N} \), then \( \zeta_k(t) > 0 \) for all \( t \in (t_1, t_1+\tau) \).

For this, note that \( \tilde{\xi} \) satisfies these two properties. Letting \( (\xi', \psi') \doteq \varepsilon(\xi, \psi) + (1-\varepsilon)(\tilde{\xi}, \tilde{\psi}) \) for \( \varepsilon \in (0,1) \) we have that \( (\xi', \psi') \in \mathcal{J}_{t_1,t_1+\tau}(x^{(1)}, x^{(2)}) \) and it satisfies the two claimed properties. Also, from the convexity of \( L \) we see that, it suffices to prove (9.35) with \( (\zeta, \psi) \) replaced with \( (\xi', \psi') \). Therefore the claim.

Fix two sequences of time instants \( t_1^{(1)} \doteq t_1 + \frac{1}{n} \) and \( t_2^{(1)} \doteq t_1 + \tau - \frac{1}{n} \). Note that \( t_2^{(n)} = t_1^{(1)} + \tau^{(n)} \) where \( \tau^{(n)} \) is defined by (7.2) by replacing \( (x^{(1)}, x^{(2)}) \) with \( (x^{(1)}, x^{(2)^{(n)}}) = (\xi(t_1^{(n)}), \xi(t_2^{(n)})) \). Consider now the optimization problem in (9.35) associated with \( L^{2}(t_1^{(n)}, t_2^{(n)}; x^{(1)}, x^{(2)^{(n)}}) \). Note that for this problem the two additional assumptions in Lemma (9.2) are satisfied. Furthermore, the assumption \( \sum_{k=1}^{\infty} k z_k + z_0 > 2 \sum_{k=1}^{\infty} z_k \) in Lemma (7.6) also holds with \( z \) replaced by \( \bar{z} \).
large \( n \). Therefore Lemma 9.2 can be applied with \((x^{(1)}, x^{(2)})\) replaced with \((x^{(1)}, n, x^{(2)}, n)\). Let 
\((\zeta^{(n)}, \psi^{(n)}) \in \mathcal{J}^2_{t_1^{(n)}, t_1^{(n)} + r^{(n)}}(x^{(1)}, n, x^{(2)}, n)\) be the corresponding minimizer and \(\beta^{(n)} = \beta(x^{(1)}, n, x^{(2)}, n)\). Then

\[
\int_{t_1}^{t_1 + \tau} L(\zeta(s), \zeta'(s)) \, ds = \lim_{n \to \infty} \int_{t_1}^{t_1^{(n)}} L(\zeta(s), \zeta'(s)) \, ds
\]

\[
\geq \liminf_{n \to \infty} \int_{t_1}^{t_1^{(n)}} L(\tilde{\zeta}^{(n)}(s), (\tilde{\zeta}^{(n)})'(s)) \, ds
\]

\[
= \liminf_{n \to \infty} [\tilde{H}(z^n) + \tilde{H}(x^{(2)}, n) - \tilde{H}(x^{(1)}, n) + \tilde{K}(x^{(1)}, n, x^{(2)}, n)]
\]

\[
\geq \tilde{H}(z) + \tilde{H}(x^{(2)}) - \tilde{H}(x^{(1)}) + \tilde{K}(x^{(1)}, x^{(2)})
\]

\[
= \int_{t_1}^{t_1 + \tau} L(\tilde{\zeta}(s), \tilde{\zeta}'(s)) \, ds.
\]

Here the first inequality follows from Lemma 9.2 and the last three lines use Lemma 7.5.

\[\square\]

10 Proof of LLN

In this section we give the proofs of Theorem 2.18 and Proposition 2.19.

Proof of Theorem 2.18

(1) Assume without loss of generality that \( T \geq 1 \). Since \( f_1(t) \leq 1 \), we see from Assumption 2.16 that \( r(\zeta(\cdot)) \) with \( r \) from (2.14) and \( \psi \) are well-defined. Let \( \varphi_k(s, y) = 1 \) for all \( k \in \mathbb{N}_0 \) and \((s, y) \in [0, T] \times [0, 1] \). It suffices to show \( \varphi \in \mathcal{S}_T(\zeta, \psi) \) and \( (\zeta, \psi) \in \mathcal{C}_T \). Since \( f_1(t) = F_1^{-1}(t) 1_{[0,1]}(t) \), we have \( \tau_\zeta = 1 \), where \( \tau_\zeta \) was defined in (2.19). Since \( F_1(f_1(t)) = t \) for \( t \in [0, 1] \),

\[
f_1'(t) = -\frac{1}{\sum_{k=1}^{\infty} k p_k(f_1(t))^{k-1}} \quad \text{for } 0 < t < \tau_\zeta \quad \text{and} \quad f_1'(t) = 0 \quad \text{for } \tau_\zeta < t < T.
\]

Using this it follows that for \( k \in \mathbb{N} \),

\[
\zeta_k'(t) = -\frac{k \zeta_k(t)}{\sum_{j=1}^{\infty} J_{s_j}(t)} = -r_k(\zeta(t)) \quad \text{for } 0 < t < \tau_\zeta \quad \text{and} \quad \zeta_k'(t) = 0 \quad \text{for } \tau_\zeta < t < T.
\]

From this we see that (2.15) holds and we can write

\[
\psi(t) = \sum_{k=0}^{\infty} (k - 2) \int_{0}^{t} r_k(\zeta(s)) \, ds.
\]

This gives (2.14) and verifies that \( \varphi \in \mathcal{S}_T(\zeta, \psi) \).

Next we argue that \( (\zeta, \psi) \in \mathcal{C}_T \). From Assumption 2.16 for \( t < \tau_\zeta \), as \( K \to \infty \),

\[
\sum_{k=K}^{\infty} |k - 2| |\zeta_k'(t)| \leq \sum_{k=K}^{\infty} k r_k(\zeta(t)) \leq \frac{\sum_{k=K}^{\infty} k^2 p_k}{r(\zeta(t))} \to 0.
\]

In particular, \( \psi \) is absolutely continuous and thus property (a) of \( \mathcal{C}_T \) holds. Also, for \( t < \tau_\zeta \),

\[
\psi'(t) = \sum_{k=1}^{\infty} (k - 2) r_k(\zeta(t)) = \sum_{k=1}^{\infty} k(k - 2) p_k(f_1(t))^{k-1} \leq \frac{f_1(t) \sum_{k=1}^{\infty} k(k - 2) p_k}{r(\zeta(t))} \leq 0.
\]

56
Therefore $\Gamma(\psi)(t) = 0 = \zeta_0(t)$ for $t < \tau_\zeta$. For $\tau_\zeta \leq t \leq T$, clearly $\Gamma(\psi)(t) = 0 = \zeta_0(t)$. So we have checked property (b) of $\mathcal{C}_T$. Property (c) of $\mathcal{C}_T$ follows from the definition of $\zeta_k$, $k \in \mathbb{N}$. Therefore $(\zeta, \psi) \in \mathcal{C}_T$ and part (1) follows.

(2) The fact that when $p_1 > 0$ there is a unique $\rho \in (0,1)$ such that $G_1(\rho) = \rho$ is proved in [25]. Since $f_\rho(t) \leq 1$, we see from Assumption 2.10 that $r(\zeta(t))$ and $\psi$ are well-defined. Let $\varphi_k(s, y) = 1$ for all $k \in \mathbb{N}_0$ and $(s, y) \in [0, T] \times [0, 1]$. It suffices to show $\varphi \in \mathcal{S}_T(\zeta, \psi)$ and $(\zeta, \psi) \in \mathcal{C}_T$. First consider times $t < \tau$. Using the definitions of $r$, $G_1$ and $\tau$, for $t < \tau$

$$r(\zeta(t)) = \mu - 2t - \mu \sqrt{1 - 2t/\mu} G_1(\sqrt{1 - 2t/\mu}) + \sum_{k=1}^{\infty} k \rho \left(1 - \frac{2t}{\mu}\right)^{k/2} = \mu - 2t > \mu \rho^2 \geq 0.$$ 

From this one can verify that for $t < \tau$

$$\zeta_k(t) = \frac{-k \zeta_k(t)}{\mu - 2t} = -r_k(\zeta(t)).$$

Using this we see that 2.13 holds for $t < \tau$ and hence as before 2.14 holds as well. To show that $(\zeta, \psi) \in \mathcal{C}_t$ for $t < \tau$, it suffices to show that $\psi(t)$ is absolutely continuous and $\zeta_0(t) = \psi(t)$ for $t \in [0, \tau)$. Note that for $t < \tau$, $\sum_{k=1}^{\infty} |k - 2| r_k(\zeta(t))| \leq \sum_{k=1}^{\infty} \frac{k \rho}{\mu - 2t}$. So from Assumption 2.10 $\psi$ is absolutely continuous over $[0, \tau]$. Also, one can verify that for $t < \tau$

$$\zeta'_0(t) = \frac{d}{dt} r(\zeta(t)) - \sum_{k=1}^{\infty} k \zeta'_k(t) = -2 + \sum_{k=1}^{\infty} k r_k(\zeta(t)) = \psi'(t).$$

So $\zeta_0(t) = \psi(t)$ for $t < \tau$. Thus we have that $\varphi \in \mathcal{S}_t(\zeta, \psi)$ and $(\zeta, \psi) \in \mathcal{C}_t$ for each $t < \tau$.

We now consider $t \in [\tau, \tau_\zeta]$. Since $\rho \in [0, 1)$ and $G_1(\rho) = \rho$, we have

$$0 = \frac{\mu(G_1(\rho) - \rho)}{\rho - 1} = \frac{1}{\rho - 1} \sum_{k=1}^{\infty} k \rho \left(\rho^{k-1} - \rho\right) = -p_1 + \rho \sum_{k=3}^{\infty} k \rho^{k-3} - \frac{1}{\rho - 1}$$

$$= -p_1 + \rho \sum_{k=3}^{\infty} k \rho^{k-3} + \rho^{k-4} + \cdots + 1$$

$$\geq -p_1 + \rho \sum_{k=3}^{\infty} k \rho^{k-3} + \rho^{k-4} + \cdots + 1$$

and therefore $0 \geq \sum_{k=1}^{\infty} k \rho^{k-3} + \rho^{k-4} + \cdots + 1$. Namely, the assumption in part (1) is satisfied with $p$ replaced by $\zeta(\tau)$. Thus the proof for the case $t \in [\tau, \tau_\zeta]$ is very similar to that in part (1), with $f_1(t)$ replaced by $f_\rho(t - \tau)$ and $p_k$ replaced with $\zeta_k(\tau)$, and we would like to omit the detail. This completes the proof of (2).

**Proof of Proposition 2.19** Suppose for $i = 1, 2$, $(\zeta^{(i)}, \psi^{(i)})$ are two pairs such that $I_T(\zeta^{(i)}, \psi^{(i)}) = 0$. By the definition of $I_T(\cdot)$, $(\zeta^{(i)}, \psi^{(i)}) \in \mathcal{C}_T$. It will be seen in Remark 6.3 that there exists some $\varphi^{(i)} \in \mathcal{S}_T(\zeta^{(i)}, \psi^{(i)})$ such that

$$\sum_{k=0}^{\infty} \int_{[0,T] \times [0,1]} \ell(\varphi_k^{(i)}(s, y)) ds dy = I_T(\zeta^{(i)}, \psi^{(i)}) = 0.$$ 

Since $\ell(x) = 0$ if and only if $x = 1$, we must have $\varphi_k^{(i)}(s, y) = 1$ for a.e. $(s, y) \in [0, T] \times [0, 1]$ and $k \in \mathbb{N}_0$. 

57
Using such $\varphi^{(i)}$ with (2.14) and (2.15), we see that

$$\zeta_k^{(i)}(t) = p_k - \int_0^t r_k(\zeta^{(i)}(s)) \, ds, \, k \in \mathbb{N}, \tag{10.1}$$

$$\psi^{(i)}(t) = \sum_{k=0}^\infty (k-2) \int_0^t r_k(\zeta^{(i)}(s)) \, ds. \tag{10.2}$$

Since $\zeta_0^{(i)} = \Gamma(\psi^{(i)})$, for a.e. $t$, $(\zeta_0^{(i)})' \geq (\psi^{(i)})' = (k-2)r_k(\zeta^{(i)}(t))$, and by (2.3)

$$\frac{d}{dt} r(\zeta^{(i)}(t)) = (\zeta_0^{(i)})'(t) + \sum_{k=1}^\infty k(\zeta_k^{(i)})'(t) \geq \sum_{k=0}^\infty (k-2)r_k(\zeta^{(i)}(t)) - \sum_{k=1}^\infty kr_k(\zeta^{(i)}(t))$$

$$= -2 \cdot 1_{\{r(\zeta^{(i)}(t)) > 0\}} \geq -2.$$

Consider the strictly increasing function $g^{(i)}(t)$ defined by

$$g^{(i)}(0) = 0, \ (g^{(i)})'(t) = r(\zeta^{(i)}(g^{(i)}(t)))1_{\{g^{(i)}(t) < \tau_{\zeta^{(i)}}\}} + 1_{\{g^{(i)}(t) \geq \tau_{\zeta^{(i)}}\}}, \tag{10.3}$$

where $\tau_{\zeta^{(i)}}$ is as in (2.10). Since $\frac{d}{dt} r(\zeta^{(i)}(t)) \in [-2, 0]$ and $0 \leq r(\zeta^{(i)}(\cdot)) \leq r(\zeta^{(i)}(0)) = \sum_{k=1}^\infty kp_k < \infty$, we see that $r(\zeta^{(i)}(\cdot))$ is bounded and Lipschitz. Also $r(\zeta^{(i)}(t)) > 0$ for $t < \tau_{\zeta^{(i)}}$. So we have existence and uniqueness of the strictly increasing function $g^{(i)}(t)$ before it reaches $\tau_{\zeta^{(i)}}$. The existence, uniqueness and monotonicity of $g^{(i)}(t)$ after $\tau_{\zeta^{(i)}}$ is straightforward.

Define $(\tilde{\zeta}^{(i)}(t), \tilde{\psi}^{(i)}(t)) \doteq (\zeta^{(i)}(g^{(i)}(t)), \psi^{(i)}(g^{(i)}(t)))$. From (10.1) and (10.2) it follows that

$$\tilde{\zeta}^{(i)}_k(t) = p_k - \int_0^t k\tilde{\zeta}^{(i)}_k(s) \, ds, \, k \in \mathbb{N},$$

$$\tilde{\psi}^{(i)}(t) = \sum_{k=0}^\infty (k-2) \int_0^t k\tilde{\zeta}^{(i)}_k(s) \, ds - 2 \int_0^t \tilde{\psi}^{(i)}(s) \, ds - \sum_{k=1}^\infty kr_k(\zeta^{(i)}(t)) - 2 \int_0^t \Gamma(\tilde{\psi}^{(i)}(s)) \, ds.$$

Clearly $\tilde{\zeta}^{(i)}_k = \tilde{\zeta}^{(i)}_k$ for each $k \in \mathbb{N}$. Also, since $\Gamma$ is Lipschitz on path space, Gronwall’s inequality implies $\tilde{\psi}^{(i)} = \psi^{(i)}$, and hence $\tilde{\zeta}^{(i)}_0 = \zeta_0^{(i)}$. Noting that (10.3) can be written as

$$g^{(i)}(0) = 0, \ (g^{(i)})'(t) = r(\tilde{\zeta}^{(i)}(t))1_{\{r(\tilde{\zeta}^{(i)}(t)) \geq 0\}} + 1_{\{r(\tilde{\zeta}^{(i)}(t)) = 0\}},$$

we have $g^{(i)} = g^{(2)}$. Since $g^{(i)}$ is strictly increasing, its inverse function is well-defined and we must have that $(\zeta^{(1)}, \psi^{(1)}) = (\zeta^{(2)}, \psi^{(2)})$. This completes the proof. \qed

**Acknowledgement:** The research of SB was supported in part by the NSF (DMS-1606839, DMS-1613072) and the Army Research Office (W911NF-17-1-0010). The research of AB was supported in part by the NSF (DMS-1305120, DMS-1814894, DMS-1853968). The research of PD was supported in part by the NSF (DMS-1904992) and DARPA (W911NF-15-2-0122). The research of RW was supported in part by DARPA (W911NF-15-2-0122).

**References**

[1] E. A. Bender and E. R. Canfield, *The asymptotic number of labeled graphs with given degree sequences*, Journal of Combinatorial Theory, Series A 24 (1978), no. 3, 296–307.

[2] B. Bollobás, *A probabilistic proof of an asymptotic formula for the number of labelled regular graphs*, European Journal of Combinatorics 1 (1980), no. 4, 311–316.
[3] C. Bordenave and P. Caputo, *Large deviations of empirical neighborhood distribution in sparse random graphs*, Probability Theory and Related Fields **163** (2015), no. 1-2, 149–222.

[4] C. Borgs, J. T Chayes, L. Lovász, V. T Sós, and K. Vesztergombi, *Convergent sequences of dense graphs I: Subgraph frequencies, metric properties and testing*, Advances in Mathematics **219** (2008), no. 6, 1801–1851.

[5] C. Borgs, J. T Chayes, L. Lovász, V. T Sós, and K. Vesztergombi, *Convergent sequences of dense graphs II. multway cuts and statistical physics*, Annals of Mathematics **176** (2012), no. 1, 151–219.

[6] M. Boué and P. Dupuis, *A variational representation for certain functionals of Brownian motion*, The Annals of Probability **26** (1998), no. 4, 1641–1659.

[7] A. Budhiraja, J. Chen, and P. Dupuis, *Large deviations for stochastic partial differential equations driven by a Poisson random measure*, Stochastic Processes and their Applications **123** (2013), no. 2, 523–560.

[8] A. Budhiraja and P. Dupuis, *A variational representation for positive functionals of infinite dimensional Brownian motion*, Probability and Mathematical Statistics **20** (2000), no. 1, 39–61.

[9] A. Budhiraja and P. Dupuis, *Analysis and Approximation of Rare Events. Representations and Weak Convergence Methods*, Vol. 94, Series Prob. Theory and Stoch. Modelling, Springer, 2019.

[10] A. Budhiraja, P. Dupuis, and A. Ganguly, *Moderate deviation principles for stochastic differential equations with jumps*, The Annals of Probability **44** (2016), no. 3, 1723–1775.

[11] A. Budhiraja, P. Dupuis, and V. Maroulas, *Variational representations for continuous time processes*, Annales de l’Institut Henri Poincaré(B), Probabilités et Statistiques **47** (2011), no. 3, 725–747.

[12] A. Budhiraja and R. Wu, *Moderate deviation principles for weakly interacting particle systems*, Probability Theory and Related Fields **168** (2017), no. 3, 721–771.

[13] S. Chatterjee, *An introduction to large deviations for random graphs*, Bulletin of the American Mathematical Society **53** (2016), no. 4, 617–642.

[14] S. Chatterjee and S. S. Varadhan, *The large deviation principle for the Erdős-Rényi random graph*, European Journal of Combinatorics **32** (2011), no. 7, 1000–1017.

[15] J. Choi and S. Sethuraman, *Large deviations for the degree structure in preferential attachment schemes*, The Annals of Applied Probability **23** (2013), no. 2, 722–763.

[16] S. Dhara and S. Sen, *Large deviation for uniform graphs with given degrees*, arXiv preprint [arXiv:1904.07666] (2019).

[17] P. Dupuis and R. S. Ellis, *A Weak Convergence Approach to the Theory of Large Deviations*, Wiley series in probability and mathematical statistics: Probability and statistics, vol. 902, John Wiley & Sons, New York, 1997.

[18] S. Fortunato, *Community detection in graphs*, Physics Reports **486** (2010), no. 3, 75–174.

[19] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, North-Holland Mathematical Library, vol. 24, Elsevier, 1981.

[20] S. Janson and M. J Luczak, *A new approach to the giant component problem*, Random Structures & Algorithms **34** (2009), no. 2, 197–216.

[21] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, Graduate Texts in Mathematics, vol. 113, Springer New York, 1991.

[22] T. G. Kurtz, *Approximation of Population Processes*, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 36, SIAM, 1981.

[23] A. Lancichinetti, F. Radicchi, J. J Ramasco, and S. Fortunato, *Finding statistically significant communities in networks*, PloS One **6** (2011), no. 4, e18961.

[24] L. Lovász, *Large networks and graph limits*, Vol. 60, American Mathematical Society Providence, 2012.

[25] M. Molloy and B. Reed, *A critical point for random graphs with a given degree sequence*, Random Structures & Algorithms **6** (1995), no. 2-3, 161–180.

[26] M. Molloy and B. Reed, *The size of the giant component of a random graph with a given degree sequence*, Combinatorics, Probability and Computing **7** (1998), no. 3, 293-305.

[27] M. E. Newman, *Spread of epidemic disease on networks*, Physical Review E **66** (2002), no. 1, 016128.

[28] N. O'Connell, *Some large deviation results for sparse random graphs*, Probability Theory and Related Fields **110** (1998), no. 3, 277–285.

[29] A. A Puhalskii, *Stochastic processes in random graphs*, The Annals of Probability **33** (2005), no. 1, 337–412.

[30] R. Van Der Hofstad, *Random Graphs and Complex Networks*, Vol. 1, Cambridge University Press, 2016.
