A covariant entropy bound conjecture on the dynamical horizon *

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Abstract: As a compelling pattern for the holographic principle, our covariant entropy bound conjecture is proposed for more general dynamical horizons. Then we apply our conjecture to ΛCDM cosmological models, where we find it imposes a novel upper bound $10^{-90}$ on the cosmological constant for our own universe by taking into account the dominant entropy contribution from super-massive black holes, which thus provides an alternative macroscopic perspective to understand the longstanding cosmological constant problem. As an intriguing implication of this conjecture, we also discuss the possible profound relation between the present cosmological constant, the origin of mass, and the anthropic principle.

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1. Introduction and motivation

The generalized second law of thermodynamics was initially put forth for a system including black holes by Bekenstein\cite{1, 2, 3}. It states that the sum of one quarter of the area of the black hole event horizon plus the entropy of ordinary matter outside never decreases with time in all processes. Especially, for the formation or absorption of black holes, the generalized second law of thermodynamics can also be equivalently formulated as a covariant entropy bound. Namely, the entropy flux \( S \) through the event horizon between its two-dimensional space-like surfaces of area \( A_e \) and \( A'_e \) must satisfy

\[
S \leq \frac{A'_e - A_e}{4},
\]

(1.1)

where \( A'_e \geq A_e \) is assumed, and Planck units are used, i.e., \( c = G = \hbar = k = 1 \).

However, due to the global and teleological deficit of the event horizon, the notion of the black hole dynamical horizon has recently been developed quasi-locally to model growing black holes and its properties have also been extensively investigated, where, in particular, the first and second laws of black hole mechanics was generalized to the dynamical horizon\cite{4, 5, 6}. Along this line further, we have proposed a covariant entropy bound formulation of an analogous generalized second law of thermodynamics on the black hole dynamical horizon and its generalization to cosmological dynamical horizons in FRW universes has also been conjectured\cite{4, 8}. Moreover, its validity has been confirmed in both Vaidya black holes and FRW universes full of matter with a fixed state of equation \( w = \frac{\lambda}{\rho} \), regardless of the spatial geometry\cite{4, 8}. All of these
results suggest that our proposal, viewed as a covariant entropy bound conjecture on dynamical horizons, may be a universal law and there may be some deep reasons for its validity. In fact, the conjecture is motivated partly by Bousso’s covariant entropy bound conjecture\[9, 10, 11, 12\], and its strengthened form suggested by Flanagan, Marolf, and Wald\[13\]. These various entropy bound conjectures, including ours, can also be interpreted as a statement of the so called holographic principle, which is believed to be manifest in an underlying quantum gravity\[14, 15\].

Taking into account its success in many respects and justification as a possible fundamental principle, we have quite recently applied our covariant entropy bound conjecture to constrain those cosmological models with a positive cosmological constant plus the matter content satisfying the dominant energy condition\[14\]. Especially, for ΛCDM cosmological models, it is found that our conjecture implies a novel inequality as

\[
s\sqrt{\Lambda} \leq 2\sqrt{3}\pi\rho_m^0,
\]

where \(s\) represents the present entropy density, and \(\rho_m^0\) denotes the energy density of dust today. This is a remarkable result because it establishes a significant relation governing the cosmological constant, present entropy density and dust energy density.

In this essay, we further extend our covariant entropy bound conjecture to more general dynamical horizons. We then explore its intriguing physical implications after reworking out the inequality (1.2). Conclusion and discussion are presented in the end.

The signature of metric takes (−, +, +, +). Notation and conventions follow Ref.[17].

2. A covariant entropy bound conjecture on the dynamical horizon

We would first like to introduce the basic definition of our dynamical horizon in a more general sense. Roughly speaking, a dynamical horizon is just a hyper-surface which is foliated by closed apparent horizons. A more detailed definition can be presented as follows:

**Definition:** A three-dimensional sub-manifold in a spacetime \((M, g_{ab})\) is said to be a dynamical horizon if it can be foliated by a family of closed two-dimensional surfaces such that, on each leaf, the expansion \(\theta_l\) of one future-directed null normal \(l^a\) vanishes while the expansion \(\theta_n\) of the other future-directed null normal \(n^a\) is positive or negative, in addition, the Lie derivatives of \(\theta_l\) along \(l^a\) and \(n^a\) do not vanish simultaneously, where, the normalization of \(l^a\) and \(n^a\) is chosen such that \(l^a n_a = -2\), which implies the expansion of the null geodesics normal can be given by \(\theta_l = h^{ab} \nabla_a l_b (\theta_n = h^{ab} \nabla_a n_b)\) with the induced metric \(h_{ab} = g_{ab} + \frac{1}{2}(l_a n_b + n_a l_b)\) on each leaf.

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\begin{table}[h]
\begin{center}
\begin{tabular}{|c|c|c|}
\hline
$\theta_l = 0$ & $\theta_n > 0$ & $\theta_n < 0$ \\
\hline
$\mathcal{L}_n \theta_l > 0, \mathcal{L}_l \theta_l < 0$ (timelike) & expanding FRW universes with $-1 < w < \frac{1}{3}$ & time reversal \\
& time reversal & growing Vaidya-De sitter black holes \\
\hline
$\mathcal{L}_n \theta_l > 0, \mathcal{L}_l \theta_l = 0$ (null generated by $l^a$) & expanding De sitter spacetime & time reversal \\
\hline
$\mathcal{L}_n \theta_l < 0, \mathcal{L}_l \theta_l < 0$ (spacelike) & time reversal & growing Vaidya black holes \\
& expanding FRW universes with $\frac{1}{3} < w \leq 1$ & time reversal \\
\hline
$\mathcal{L}_n \theta_l < 0, \mathcal{L}_l \theta_l = 0$ (null generated by $l^a$) & time reversal & Schwarzchild black holes \\
\hline
$\mathcal{L}_n \theta_l = 0, \mathcal{L}_l \theta_l < 0$ (null generated by $n^a$) & expanding FRW universe with $w = \frac{1}{3}$ & time reversal \\
\hline
\end{tabular}
\end{center}
\caption{Classification and examples of dynamical horizons. A specific dynamical horizon may be a union of various kinds of cases.}
\end{table}

Note that there is no restriction on the signature of the dynamical horizon in our definition. In particular, if $\mathcal{L}_n \theta_l$ does not vanish, we can always choose a vector field $v^a = l^a - fn^a$ for some $f$ such that $v^a$ is tangential to the dynamical horizon. It follows from $v_\alpha v^\alpha = 4f$ that the dynamical horizon is spacelike, null, or timelike, depending on whether $f$ is positive, zero, or negative, respectively. By virtue of $\mathcal{L}_l \theta_l = 0$ and the Raychaudhuri equation, we have on the dynamical horizon

$$f \mathcal{L}_n \theta_l = \mathcal{L}_l \theta_l = -\sigma^2 - R_{ab} l^a l^b,$$

where $\sigma$ is the shear of $l^a$, and $R_{ab}$ is the Ricci tensor. With the dominant energy condition satisfied by matter, Eq. (2.1) implies that $\mathcal{L}_n \theta_l < 0$ when the dynamical horizon is spacelike, and $\mathcal{L}_n \theta_l > 0$ when the dynamical horizon is timelike. For the specific classification and corresponding examples of dynamical horizons, please see Table \[\text{[1]}\].

On the other hand, our definition appears somewhat restrictive since it rules out the degenerate cases in which $\theta_n = 0$ or/and $\mathcal{L}_l \theta_l = \mathcal{L}_n \theta_l = 0$. Nonetheless, as demonstrated in Table \[\text{[1]}\] these degenerate cases rarely occur in our interested dynamical spacetimes, thus our definition does not lose its generality.

Now a covariant entropy bound conjecture on the general dynamical horizon can be stated in a concise way: \textit{The entropy flux $S$ through the dynamical horizon between its apparent horizons of area $A_d$ and $A'_d$ must satisfy $S \leq \frac{|A_d - A'_d|}{4}$ if the dominant energy condition holds for matter.}
It is noteworthy that our general conjecture itself is manifestly time reversal invariant. So its origin must be static rather than thermodynamic, although it can be regarded as a reformulation of the generalized second law of thermodynamics on the dynamical horizon in some cases such as growing black holes and expanding universes\[7, 8\].

3. Constraint $\Lambda$CDM cosmological models by the covariant entropy bound conjecture

In terms of the conformal time and comoving coordinates, the flat FRW metric takes the form as
\[
ds^2 = a^2(\eta) [-d\eta^2 + dr^2 + r^2 (dr^2 + \sin^2 \theta d\phi^2)],
\]
(3.1)

Next, let us first compute the initial expansion of the future-directed null congruences orthogonal to an arbitrary sphere characterized by some value of $(\eta, r)$. Accordingly we obtain
\[
\theta_{\pm} = \frac{\dot{a}}{a} \pm \frac{1}{r},
\]
(3.2)

where the dot denotes the derivative with respect to $\eta$, and the sign $+(-)$ represents the null congruence is directed at larger(smaller) values of $r$. Thus the dynamical horizon here is identified as
\[
r_c(\eta) = \pm \frac{1}{h},
\]
(3.3)

where $h \equiv \frac{\dot{a}}{a}$.

If as usual the matter content of FRW universes is assumed to be described by the perfect fluid, with energy momentum tensor
\[
T_{ab} = a^2(\eta) \{\rho(\eta)(d\eta)_a(d\eta)_b + p(\eta)[(dr)_a d(r)_b

+ r^2 ((d\theta)_a d(\theta)_b + \sin^2 \theta (d\phi)_a (d\phi)_b)]\},
\]
(3.4)

then by the Einstein equation with a positive cosmological constant $\Lambda$, we have
\[
3h^2 = 8\pi \rho a^2 + \Lambda a^2,
\]
(3.5)
\[
-(h^2 + 2\dot{h}) = 8\pi p a^2 - \Lambda a^2,
\]
(3.6)

From here, we can further obtain
\[
\dot{h} = -\frac{4\pi}{3} [(1 + 3w)\rho - \lambda] a^2,
\]
(3.7)
\[
h^2 - \dot{h} = 4\pi (1 + w) \rho a^2,
\]
(3.8)

where $-1 \leq w \leq 1$ due to the dominant energy condition, and $\lambda \equiv \frac{\Lambda}{4\pi}$. 

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To proceed, we further assume that the evolution of FRW universes is adiabatical, which implies the conservation of the entropy current associated with the matter, i.e., \( \nabla a s^a = 0 \). Whence the entropy current can be formulated as

\[
s^a = \frac{s}{a^4}(\frac{\partial}{\partial \eta})^a, \tag{3.9}
\]

where \( s \) is actually the ordinary comoving entropy density, constant in space and time.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{dynamical_horizon.png}
\caption{The dynamical horizon with the entropy current flowing through it for \( \Lambda \)CDM cosmological models in the conformal coordinates. When \( \dot{r}_c \leq 0 \), the entropy current flows across the dynamical horizon from the interior region to the exterior one while it flows from the exterior region to the interior one for \( \dot{r}_c \geq 0 \).}
\end{figure}

On the other hand, according to Eq.(3.3), we have

\[
\dot{r}_c = \mp \frac{h}{h^2}. \tag{3.10}
\]

Note that at any moment the area of the dynamical horizon is give by \( A = 4\pi a^2 r_c^2 \).
Accordingly we can work out its time derivative, i.e.,
\[
\dot{A} = 8\pi a^2 r_c^2 (h + \frac{\dot{r}_c}{r_c}) = \frac{8\pi a^2 (h^2 - \dot{h})}{h^3}.
\] (3.11)

Obviously, by Eq.(3.8), the increase or decrease of area with time only depends on whether the universe is expanding or contracting. In what follows we shall only focus on the expanding universes, i.e., \( h \geq 0 \), where the corresponding area monotonically increases with the evolution of time.

We shall now explore how our covariant entropy bound conjecture provides an intriguing constraint on \( \Lambda \text{CDM} \) cosmological models, where \( \rho = \frac{\rho_m}{a^4} + \frac{\rho_\Lambda}{a^3} \). However, as demonstrated in Figure 1, it is noteworthy that there is an obvious difference between \( \dot{r}_c \leq 0 \) and \( \dot{r}_c \geq 0 \). Thus employing the conservation of the entropy current and Gauss theorem, our conjecture can be equivalently expressed as

\[
\frac{\dot{A}}{4} + \dot{S} \geq 0 
\] (3.12)

for \( \dot{r}_c \leq 0(\dot{h} \geq 0) \), and

\[
\frac{\dot{A}}{4} - \dot{S} \geq 0 
\] (3.13)

for \( \dot{r}_c \geq 0(\dot{h} \leq 0) \). Here \( S \) denotes the entropy flux through the interior region \( r \leq r_c \), given by \( S = \frac{4\pi}{3} s r_c^3 \), whereby we have

\[
\dot{S} = 4\pi s r_c^2 \dot{r}_c = -\frac{4\pi s \dot{h}}{h^4}.
\] (3.14)

So by Eqs.(3.5), (3.7), and (3.8), our conjecture gives

\[
s \leq \frac{\sqrt{3(8\pi \rho a^2 + \Lambda a^2)(1 + w) \rho a^2}}{2[(1 + 3w) \rho - \lambda]} 
\] (3.15)

when \((1 + 3w) \rho > \lambda\), and

\[
s \leq \frac{\sqrt{3(8\pi \rho a^2 + \Lambda a^2)(1 + w) \rho a^2}}{2[\lambda - (1 + 3w) \rho]} 
\] (3.16)

when \((1 + 3w) \rho < \lambda\), which apparently corresponds to the later stages of expanding universes. In particular, to guarantee that the bound (3.16) holds at the very remote future, we thus obtain the novel inequality (1.2) by setting the present scale factor \( a_0 = 1 \).

For our own universe, as is well known, \( \Lambda \sim 10^{-120} \) and \( \rho_m^0 \sim \frac{1}{3} \Lambda \), so our conjecture follows that the present entropy density should be less than \( 10^{-60} \), which is satisfied with
a wide safety margin, since the realistic entropy density, dominated by super-massive black holes, is around of order $10^{-75}$ today. That is to say, our conjecture supports the existence of our own universe as it should do. On the other hand, if we take the present entropy density and dust energy density as input data, our conjecture gives a novel upper bound on the cosmological constant, i.e., $\Lambda < 10^{-90}$, which obviously much alleviates the cosmological constant problem why the cosmological constant is so small in Planck units. Last but not least, the presence of cosmological constant, albeit small, appears to be in favor of the anthropic principle: To make our conjecture satisfied, there should be the dust matter in our universe, which is assumed to be a very basic condition for the creation of life. Furthermore, due to the fact that the dust matter must be massive, our conjecture seems to indicate a new close tie between our large scale fiducial $\Lambda$CDM cosmological model and our small scale standard model or beyond for elementary particles, namely, the origin of mass for the ordinary matter and cold dark matter is determined to be intertwined with the present cosmological constant somehow or others.

4. Conclusion and discussion

Advance in fundamental physics has often been driven by the recognition of a new principle, a key insight to guide the search towards a successful theory. In the ongoing search for a complete theory of quantum gravity, the holographic principle stands out as such a principle, which, in essence, relates geometric aspects of spacetime to the number of quantum states of matter in a surprisingly strong way, believed to be a law of physics that captures one of the most crucial aspects of quantum gravity. As a compelling pattern for the holographic principle, our covariant entropy bound conjecture has been addressed for more general dynamical horizons.

On the other hand, without knowledge of its microscopic make up and specific dynamics, the use of general principles to investigate a system can be very rewarding. Thus we have also applied our proposed covariant entropy bound conjecture to $\Lambda$CDM cosmological models. As a result, it is shown that our conjecture implies a remarkable upper bound $10^{-90}$ on the cosmological constant for our own universe, which thus opens an alternative macroscopic perspective to shed light on the longstanding cosmological constant problem. In addition, our conjecture also indicates that there may be a certain profound connection among the presence of the cosmological constant, the origin of mass, and the anthropic principle.

We conclude with an honest caveat. Although the results obtained so far are particularly attractive as well as consistent with our observational data, there remains a possibility that our starting conjecture proves incorrect. It may be quite successful in
many respects only as a coincidence, but one should regard it as as a warning, showing that our covariant entropy bound conjecture may require a certain reformulation where it is violated, rather than as a criterion or tool to constrain various models. Therefore it is clear that both to provide more indirect or peripheral justifications and to signify a deeper origin of our conjecture in an underlying quantum theory of gravity are needed.

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