On the existence and stability of solitary-wave solutions to a class of evolution equations of Whitham type

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Abstract
We consider a class of pseudodifferential evolution equations of the form

\[ u_t + (n(u) + Lu)_x = 0, \]

in which \( L \) is a linear smoothing operator and \( n \) is at least quadratic near the origin; this class includes in particular the Whitham equation. A family of solitary-wave solutions is found using a constrained minimization principle and concentration-compactness methods for noncoercive functionals. The solitary waves are approximated by (scalings of) the corresponding solutions to partial differential equations arising as weakly nonlinear approximations; in the case of the Whitham equation the approximation is the Korteweg–deVries equation. We also demonstrate that the family of solitary-wave solutions is conditionally energetically stable.

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1. Introduction

In this paper we discuss solitary-wave solutions of the pseudodifferential equation

\[ u_t + (Lu + n(u))_x = 0 \]  

(1)
describing the evolution of a real-valued function $u$ of time $t \in \mathbb{R}$ and space $x \in \mathbb{R}$; here $L$ is a linear smoothing operator and $n$ is at least quadratic near the origin. A concrete example is the equation

$$u_t + 2uu_x + Lu_x = 0,$$

where $L$ is the spatial Fourier multiplier operator given by

$$\mathcal{F}(Lf)(k) = \left(\frac{\tanh(k)}{k}\right)^{1/2} \hat{f}(k).$$

This equation was proposed by Whitham [19] as an alternative to the Korteweg–deVries equation which features the same linear dispersion relation as the full water-wave problem and also allows for the breaking of waves [16, 20]. There have been several investigations of different variants of the Whitham equation (e.g.; see [11, 16, 21] and in particular [7] for a rigorous treatment of wave breaking), but it has remained unclear whether the Whitham equation admits travelling waves, that is solutions of the form $u = u(x - vt)$ representing waves moving from left to right with constant speed $v$. The existence of periodic travelling waves to the Whitham equation was recently established by Ehrnström and Kalisch [9], and in this paper we discuss solitary waves, that is travelling waves for which $u(x - vt) \to 0$ as $x - vt \to \pm \infty$.

Our mathematical task is therefore to find functions $u = u(x)$ which satisfy the travelling-wave equation

$$Lu - vu + n(u) = 0$$

with wave speed $v$ and asymptotic condition $u(x) \to 0$ as $x \to \pm \infty$. We examine equation (3) under the following conditions.

**Assumptions.**

(A1) The operator $L$ is a Fourier multiplier with classical symbol $m \in S^{m_0}_\infty(\mathbb{R})$ for some $m_0 < 0$, that is

$$\mathcal{F}(Lf)(k) = m(k) \hat{f}(k)$$

for some smooth function $m : \mathbb{R} \to \mathbb{R}$ with the property that

$$|m^{(\alpha)}(k)| \leq C_\alpha (1 + |k|)^{m_0 - \alpha}, \quad \alpha \in \mathbb{N}_0,$$

where $C_\alpha$ is a positive constant depending upon $\alpha$. In particular, one can write $L$ as a convolution with the (possibly distributional) kernel $K := \mathcal{F}^{-1}(m)$, that is

$$Lf = \frac{1}{\sqrt{2\pi}} K \ast f.$$  

(A2) The symbol $m : \mathbb{R} \to \mathbb{R}$ is even (to avoid non-real solutions) and satisfies $m(0) > 0$,

$$m(k) < m(0), \quad k \neq 0,$$

(so that it has a strict and positive global maximum at $k = 0$) and

$$m(k) = m(0) + \frac{m^{(2j_\ast)}(0)}{(2j_\ast)!} k^{2j_\ast} + r(k)$$

for some $j_\ast \in \mathbb{N}$, where $m^{(2j_\ast)}(0) < 0$ and $r(k) = O(k^{2j_\ast+2})$ as $k \to 0$. 


(A3) The nonlinearity $n$ is a twice continuously differentiable function $\mathbb{R} \to \mathbb{R}$ with

$$n(x) = n_p(x) + n_r(x),$$

in which the leading order part of the nonlinearity takes the form

$$n_p(x) = c_p |x|^p$$

for some $c_p \neq 0$ and $p \in [2, 4j^* + 1)$ or $n_p(x) = c_p x^p$ for some $c_p > 0$ and odd integer $p$ in the range $p \in [2, 4j^* + 1)$, while the higher order part of the nonlinearity satisfies the estimate

$$n_r(x) = O(|x|^p), \quad n'_r(x) = O(|x|^{p-1}),$$

for some $\delta > 0$ as $x \to 0$. (Occasionally we simply estimate $n(x) = O(|x|^p)$ and $n'(x) = O(|x|^{p-1})$.)

Proceeding formally, let us derive a long-wave approximation to equation (3) by introducing a small parameter $\mu$ equal to the momentum $\int_{\mathbb{R}} u^2 \, dx$ of the wave, writing $\nu$ as a small perturbation of the speed $m(0)$ of linear long waves, so that

$$\nu = m(0) + \mu^\gamma \nu_{bw},$$

and substituting the weakly nonlinear ansatz

$$u(x) := \mu^\alpha w(\mu^\beta x),$$

where $2\alpha - \beta = 1$ (so that $\int_{\mathbb{R}} u^2 \, dx = \mu$) into the equation. Choosing $(p - 1)\alpha = 2j^*\beta$ and $\gamma = 2j^*\beta$, we find that

$$\mu^{\alpha p} \left( \frac{(-1)^k}{(2j^*)!} m^{(2j^*)}(0) w^{(2j^*)} - \nu_{bw} w + n_p(w) \right) + \cdots = 0,$$

where the ellipsis denotes terms which are formally $o(\mu^{\alpha p})$; the constraints on $\alpha$ and $\beta$ imply the choice

$$\alpha = \frac{2j^*}{4j^* + 1 - p} \quad \text{and} \quad \beta = \frac{p - 1}{4j^* + 1 - p}. \quad (9)$$

This formal weakly nonlinear analysis suggests that solitary-wave solutions to (1) are approximated by (suitably scaled) homoclinic solutions of the ordinary differential equation

$$\frac{(-1)^k}{(2j^*)!} m^{(2j^*)}(0) w^{(2j^*)} - \nu_{bw} w + n_p(w) = 0 \quad (10)$$

for some constant $\nu_{bw}$. The following theorem gives a variational characterization of such solutions; it is established using a straightforward modification of the theory developed by Albert [2] and Zeng [22] for a slightly different class of equations (the proof that $E_{bw}$ is bounded below over $W_1$ is given in the appendix).

Theorem 1.1.

(i) The functional $E_{bw} : H^{j^*}(\mathbb{R}) \to \mathbb{R}$ given by

$$E_{bw}(w) = - \int_{\mathbb{R}} \left\{ \frac{m^{(2j^*)}(0)}{2(2j^*)!} (w^{(j^*)})^2 + N_{p+1}(w) \right\} \, dx, \quad (11)$$

where

$$N_{p+1}(x) := \begin{cases} \frac{c_p x^{p+1}}{p + 1}, & \text{if } n_p(x) = c_p x^p, \\ \frac{c_p x^{p+1}}{p + 1}, & \text{if } n_p(x) = c_p x^p, \end{cases}$$

is bounded below.

(ii) The function $w_0$ is a minimizer of $E_{bw}$ if and only if $w_0$ is a homoclinic solution of the ordinary differential equation (10).
is bounded below over the set
\[ W_1 = \{ w \in H^j(\mathbb{R}) : Q(w) = 1 \}, \]
where
\[ Q(w) = \frac{1}{2} \int_\mathbb{R} w^2 \, dx. \] (12)
The set \( D_{lw} \) of minimizers of \( E_{lw} \) over \( W_1 \) is a non-empty subset of \( H^{2j}(\mathbb{R}) \) which lies in
\[ W := \{ w \in H^{2j}(\mathbb{R}) : \|w\|_{2j} < S \} \]
for some \( S > 0 \). Each element of \( D_{lw} \) is a solution of equation (10); the constant \( \nu_{lw} \) is the Lagrange multiplier in this constrained variational principle.

(ii) Suppose that \( \{ w_n \}_{n \in \mathbb{N}_0} \) is a minimizing sequence for \( E_{lw} \) over \( \{ w \in H^j(\mathbb{R}) : Q(w) = 1 \} \). There exists a sequence \( \{ x_n \}_{n \in \mathbb{N}_0} \) of real numbers with the property that a subsequence of \( \{ w_n(\cdot + x_n) \}_{n \in \mathbb{N}_0} \) converges in \( H^j(\mathbb{R}) \) to an element of \( D_{lw} \).

For the Whitham equation (\( j^* = 1, p = 2, m''(0) = -\frac{1}{3} \)) the above derivation yields the travelling-wave version
\[ \frac{1}{6} w'' - \nu_{lw} w + w^2 = 0 \]
of the Korteweg–deVries equation, for which
\[ D_{lw} = \{ w_{KdV}(\cdot + y) : y \in \mathbb{R} \}, \quad w_{KdV}(x) = \left( \frac{3}{2} \right)^{\frac{2}{3}} \text{sech}^2 \left( \left( \frac{3}{2} \right)^{\frac{1}{3}} x \right) \]
(and there are no further homoclinic solutions). In general \( D_{lw} \) consists of all spatial translations of a (possibly infinite) family of ‘generating’ homoclinic solutions with different wave speeds (\( \nu_{lw} = \left( \frac{3}{2} \right)^{\frac{1}{3}} \) in the case of the Whitham equation).

Equation (3) also admits a variational formulation: local minimizers of the functional \( E : H^1(\mathbb{R}) \to \mathbb{R} \) given by
\[ E(u) = -\frac{1}{2} \int_\mathbb{R} u Lu \, dx - \int_\mathbb{R} N(u) \, dx, \]
where \( N \) is the primitive function of \( n \) which vanishes at the origin, so that
\[ N(x) := N_{p+1}(x) + N_{r}(x), \quad N_{r}(x) := \int_0^x n_r(s) \, ds, \]
under the constraint that \( Q \) is held fixed are solitary-wave solutions of (3). The technique employed by Albert and Zeng, which relies upon the fact that \( L \) is of positive order (so that \( E_{lw} \) is coercive), is, however, not applicable in the present situation in which \( L \) is a smoothing operator. Instead we use methods developed by Buffoni [6] and Groves and Wahlén [12]. We consider a fixed ball
\[ U = \{ u \in H^1(\mathbb{R}) : \|u\|_1 < R \}, \]
and seek small-amplitude solutions, that is solutions in the set
\[ U_\mu := \{ u \in U : Q(u) = \mu \}, \]
where \( \mu \) is a small, positive, real number. In particular, we examine minimizing sequences for \( E \) over \( U_\mu \) which do not approach the boundary of \( U \), and establish the following result with the help of the concentration-compactness principle.
Theorem 1.2 (Existence). There exists $\mu_*>0$ such that the following statements hold for each $\mu \in (0, \mu_*)$.

(i) The set $D_\mu$ of minimizers of $E$ over the set $U_\mu$ is non-empty and the estimate $\|u\|_1^2 = \mathcal{O}(\mu)$ holds uniformly over $u \in D_\mu$ and $\mu \in (0, \mu_*)$. Each element of $D_\mu$ is a solution of the travelling-wave equation (3); the wave speed $v$ is the Lagrange multiplier in this constrained variational principle. The corresponding solitary waves are supercritical, that is their speed $v$ exceeds $m(0)$.

(ii) Let $s < 1$ and suppose that $\{u_n\}_{n \in \mathbb{N}_0}$ is a minimizing sequence for $E$ over $U_\mu$ with the property that

$$\sup_{n \in \mathbb{N}_0} \|u_n\|_1 < R. \quad (14)$$

There exists a sequence $\{x_n\}_{n \in \mathbb{N}_0}$ of real numbers such that a subsequence of $\{u_n(\cdot + x_n)\}_{n \in \mathbb{N}_0}$ converges in $H^s(\mathbb{R})$ to a function in $D_\mu$.

Theorem 1.2 is proved in two steps. We begin by constructing a minimizing sequence which satisfies condition (14). To this end we consider the corresponding problem for periodic travelling waves (see section 3) and penalize the variational functional so that minimizing sequences do not approach the boundary of the corresponding domain in function space. Standard methods from the calculus of variations yield the existence of minimizers for the penalized problem, and a priori estimates confirm that the minimizers lie in the region unaffected by the penalization; in particular they are bounded (uniformly over all large periods) away from the boundary. A minimizing sequence $\{\tilde{u}_n\}_{n \in \mathbb{N}_0}$ for $E$ over $U_\mu$ is obtained by letting the period tend to infinity.

The minimizing sequence $\{\tilde{u}_n\}_{n \in \mathbb{N}_0}$ is used to show that the quantity

$$I_\mu := \inf \{E(u); u \in U_\mu\}$$

is strictly subadditive, that is

$$I_{\mu_1+\mu_2} < I_{\mu_1} + I_{\mu_2} \quad \text{whenever } 0 < \mu_1, \mu_2 < \mu_1 + \mu_2 < \mu_*.$$ 

The proof of this fact, which is presented in section 4, is accomplished by showing that the functions $\tilde{u}_n$ ‘scale’ in a fashion similar to the long-wave ansatz (8); we may therefore approximate $E$ by a scaling of $E_{lw}$ along this minimizing sequence. The corresponding strict subadditivity result for the latter functional is a straightforward matter, and a perturbation argument shows that it remains valid for $E$.

In a second step we apply the concentration-compactness principle to show that any minimizing sequence satisfying (14) converges—up to subsequences and translations—in $H^s(\mathbb{R})$, $s < 1$ to a minimizer of $E$ over $U_\mu$ (section 5). The strict subadditivity of $I_\mu$ is a key ingredient here. The proof of theorem 1.2(i) is completed by a priori estimates for the size and speed of solitary waves obtained in this fashion.

Section 6 examines some consequences of theorem 1.2. In particular, the relationship between the solutions to (10) found in theorem 1.1 and the solutions to (3) found in theorem 1.2 is rigorously clarified. Under an additional regularity hypothesis upon $n$ we show that every solution $u$ in the set $D_\mu$ lies in $H^{2,\mu}(\mathbb{R})$, ‘scales’ according to the long-wave ansatz (8) and satisfies

$$\text{dist}_{H^s(\mathbb{R})}(\mu^{-\alpha}u(\mu^{-\beta} \cdot), D_{\mu_0}) \to 0$$

as $\mu \searrow 0$; the convergence is uniform over $D_\mu$. Corresponding convergence results for the wave speeds and infima of $E$ over $U_\mu$ and $E_{lw}$ over $\{w \in H^s(\mathbb{R}); Q(w) = 1\}$ are also presented. These results may contribute towards the discussion of the validity of the Whitham equation.
as a model for water waves: they show that the Whitham solitary waves are approximated by Korteweg-deVries solitary waves, and it is known that solutions of the Korteweg-deVries equation do approximate the solutions of the full water-wave problem (see [3, 8, 17]).

Theorem 1.2 also yields information about the stability of the set of solitary-wave solutions to (1) defined by $D_\mu$. Observing that $E$ and $Q$ are conserved quantities associated with equation (1), we apply a general principle that the solution set of a constrained minimization problem of this type constitutes a stable set of solutions of the corresponding initial problem (theorem 6.7): choosing dist$_{L^1(R)}(u(0), D_\mu)$ sufficiently small ensures that dist$_{L^2(R)}(u(t), D_\mu)$ remains small over the time of existence of a solution $u : [0, T] \to H^1(R)$ with sup$_{t \in [0, T]}\|u(t)\|_1 < R$. Of course the well-posedness of the initial-value problem for equation (1) is a prerequisite for discussing the stability of $D_\mu$. This discussion is, however, outside the scope of this paper; we merely assume that the initial-value problem is locally well posed in a sense made precise in section 6. Our stability result is conditional since it applies to solutions only for as long as they remain in $U$ (for example certain solutions of the Whitham equation (2) have only a finite time of existence [7, 16]), and energetic since distance is measured in $L^2(R)$ rather than $H^1(R)$ (note that the norms in $H^1(R)$ for $s \in [0, 1]$ are all metrically equivalent on $U$). Theorem 6.7 also refers to the stability of the entire set $D_\mu$; in the special case where the minimizer of $E$ over $U_\mu$ is unique up to translations it coincides with (conditional and energetic) orbital stability of this solution.

2. Preliminaries

Functional-analytic setting for the solitary-wave problem. Let $S(R)$ be the Schwartz space of rapidly decaying smooth functions, and let $F$ denote the unitary Fourier transform on $S(R)$, so that

$$F(\psi)(k) := \frac{1}{\sqrt{2\pi}} \int_R \psi(x) \exp(-ikx) \, dx, \quad \psi \in S(R),$$

and on the dual space of tempered distributions $S'(R)$, so that $(\hat{f}, \varphi) = (f, \hat{\varphi})$ for $f \in S'(R)$. By $L^p(R)$, $p \geq 1$ we denote the space of real-valued $p$-integrable functions with norm $\|f\|_{L^p(R)} := (\int_R |f(x)|^p \, dx)^{1/p}$, by $H^s(R)$, $s \in R$ the real Sobolev space consisting of those tempered distributions for which the norm

$$\|f\|_{s} := \left(\int_R |\hat{f}(k)|^2 (1 + k^2)^s \, dk \right)^{1/2}$$

is finite, and by $BC(R)$ the space of bounded and continuous real-valued functions with finite supremum norm $\|f\|_\infty := \sup_{x \in R} |f(x)|$; there is a continuous embedding $H^s(R) \hookrightarrow BC(R)$ for any $s > \frac{1}{2}$, so that $\|u\|_{H^s} \leq c_s \|u\|_s$ for all $u \in H^s(R)$. We write $L^2(R)$ for $H^0(R)$, and for all spaces the subscript 'c' denotes the subspace of compactly supported functions, so that $H^s_c(R) := \{f \in H^s(R): \text{supp}\,(f) \text{ is compact}\}$.

We now list some basic properties of the operators $L$, $n$ appearing in equation (3) and functionals $E$, $Q$ defined in equations (12) and (13).

Proposition 2.1.

(i) The linear operator $L$ belongs to $C^\infty(H^s(R), H^{s+|n|}(R)) \cap C^\infty(S(R), S(R))$ for each $s \geq 0$.

(ii) For each $j \in N$ there exists a constant $\tilde{C}_j > 0$ such that

$$|L u(x)| \leq \frac{\tilde{C}_j}{(\text{dist}(x, \text{supp}(u))^j} \|u\|, \quad x \in R \setminus \text{supp}(u)$$

for all $u \in L^2_c(R)$. 

(iii) Suppose that \( n \in C^k(\mathbb{R}, \mathbb{R}) \) for some \( k \in \mathbb{N} \). For each \( R > 0 \) the function \( n \) induces a continuous Nemitskii operator \( B_R(0) \subset H^s(\mathbb{R}) \to H^s(\mathbb{R}) \), where \( s \in (\frac{1}{2}, k] \).

**Proof.**

(i) This assertion follows directly from the definition of \( L \).

(ii) Assumption (4) implies that \( m^{(j)} \in L^2(\mathbb{R}) \) for any \( j \in \mathbb{N} \). Applying Plancherel’s theorem and Hölder’s inequality to the convolution formula (5), one finds that

\[
|Lu(x)| = \frac{1}{\sqrt{2\pi}} \left| \int_{\text{supp}(u)} \frac{(x - y)^j}{(x - y)^j} K(x - y)u(y) \, dy \right| \leq \frac{C_j\|m^{(j)}\|_0}{\sqrt{2\pi}} \frac{1}{\text{dist}(x, \text{supp}(u))^j} \|u\|_0.
\]

(iii) Construct a \( k \) times continuously differentiable function \( \tilde{n} : \mathbb{R} \to \mathbb{R} \) whose derivatives are bounded and which satisfies \( \tilde{n}(x) = n(x) \) for \( |x| \leq c_\delta R \) (for example by multiplying \( n \) by a smooth ‘cut-off’ function). The results given by Bourdaud and Sickel [4, theorem 7] (for \( s \in (0, 1) \)) and Brezis and Mironescu [5, theorem 1.1] (for \( s \geq 1 \)) show that \( \tilde{n} \) induces a continuous Nemitskii operator \( H^s(\mathbb{R}) \to H^s(\mathbb{R}) \) for \( s \in (\frac{1}{2}, k] \) and hence that \( n \) induces a continuous Nemitskii operator \( B_R(0) \subset H^s(\mathbb{R}) \to H^s(\mathbb{R}) \) for \( s \in (\frac{1}{2}, k] \). □

According to the previous proposition we may study (3) as an equation in \( H^s(\mathbb{R}) \) for \( s > \frac{1}{2} \) (provided that \( n \) is sufficiently regular). In keeping with this observation we work in the fixed ball \( U = \{ u \in H^1(\mathbb{R}) : \|u\|_1 < R \} \).

**Proposition 2.2.** Suppose that \( n \in C^1(\mathbb{R}) \).

(i) The functionals \( L, N \) and \( Q \) belong to \( C(U, \mathbb{R}) \) and their \( L^2(\mathbb{R}) \)-derivatives are given by the formulae

\[
L'(u) := -Lu, \quad N'(u) := -n(u), \quad Q'(u) = u.
\]

These formulae define functions \( L', N', Q' \in C(U, H^1(\mathbb{R})) \).

(ii) The functional \( E \) belongs to \( C(H^s(\mathbb{R}), \mathbb{R}) \) for each \( s > \frac{1}{2} \).

Finally, we note that solutions of the travelling-wave equation may inherit further regularity from \( n \).

**Lemma 2.3 (Regularity).** Suppose that \( n \in C^{k+1}(\mathbb{R}) \) for some \( k \in \mathbb{N} \). For sufficiently small values of \( R \), every solution \( u \in U \) of (3) belongs to \( H^{k+1}(\mathbb{R}) \) and satisfies

\[
\|u\|_{k+1} \leq c\|u\|_1.
\]

**Proof.** Differentiating (3), we find that

\[
u' = \frac{Lu'}{v - n'(u)}. \tag{15}
\]

There exists a positive constant \( c_\delta \) such that \( \nu - n'(u) \geq \delta > 0 \) whenever \( \|u\|_\infty < c_\delta \); the embedding \( H^s(\mathbb{R}) \hookrightarrow BC(\mathbb{R}) \) guarantees that this condition is fulfilled for each \( u \in U \) for sufficiently small values of \( R \).

Suppose that \( m \in \{1, \ldots, k\} \). For each fixed \( u \in H^m(\mathbb{R}) \) the formula

\[
\varphi_u(v) = \frac{v}{v - n'(u)}
\]
defines an operator in $B(L^2(\mathbb{R}), L^2(\mathbb{R}))$ and $B(H^m(\mathbb{R}), H^m(\mathbb{R}))$, and by interpolation it follows that $\psi_u \in B(H^s(\mathbb{R}), H^s(\mathbb{R}))$ for $s \in [0, m]$; its norm depends upon $\|u\|_m$.

Furthermore, recall that $L \in B(H^s(\mathbb{R}), H^{s+|m_0|}(\mathbb{R}))$ for all $s \in [0, \infty)$, so that

$$\psi_u := \psi_u \circ L \in B \left( H^s(\mathbb{R}), H^s(\mathbb{R}) \right), \quad s = \min(m, s + |m_0|),$$

and the norm of $\psi_u$ depends upon $\|u\|_m$.

It follows that any solution $w \in H^s(\mathbb{R})$ of the equation

$$w = \psi_u(w)$$

in fact belongs to $H^s(\mathbb{R})$, where $s = \min(m, s + |m_0|)$, and satisfies the estimate

$$\|w\|_s \leq c_{|u|_m} \|w\|_s.$$

Applying this argument recursively, one finds that any solution $w \in L^2(\mathbb{R})$ of (16) belongs to $H^m(\mathbb{R})$ and satisfies

$$\|w\|_m \leq c_{|u|_m} \|w\|_0.$$

Observe that equation (15) is equivalent to $u' = \psi_u(u')$. A bootstrap argument therefore shows that $u' \in H^k(\mathbb{R})$ with

$$\|u'\|_m \leq c_{|u|_m} \|u'\|_0, \quad m = 1, \ldots, k.$$



**Functional-analytic setting for the periodic problem.** Let $P > 0$. Let $L^2_\mathbb{Z}$ be the space of $P$-periodic, locally square-integrable functions with Fourier-series representation

$$u(x) = \frac{1}{\sqrt{P}} \sum_{k \in \mathbb{Z}} \hat{u}_k \exp(2\pi ikx/P),$$

and define

$$H^s_P := \left\{ u \in L^2_\mathbb{Z} : \|u\|_{H^s_P} := \left( \sum_{k \in \mathbb{Z}} \left( 1 + \frac{4\pi^2 k^2}{P^2} \right)^s |\hat{u}_k|^2 \right)^{\frac{1}{2}} < \infty \right\}$$

for $s \geq 0$. Just as for the Sobolev spaces $H^s(\mathbb{R})$ one has the continuous embedding $H^s_P \hookrightarrow \text{BC}(\mathbb{R})$ for all $s > \frac{1}{2}$; the embedding constant is independent of all sufficiently large values of $P$.

**Proposition 2.4.** The operator $L$ extends to an operator $S'(\mathbb{R}) \rightarrow S'(\mathbb{R})$ which maps $H^s_P$ smoothly into $H^{s+|m_0|}_P$, acting on the Fourier coefficients $\hat{u}_k$, $k \in \mathbb{Z}$, of a function $u$ by pointwise multiplication, so that

$$(Lu)_k = m(2\pi k/P) \hat{u}_k, \quad k \in \mathbb{Z}.$$  

**Proof.** The operator $L$ is symmetric on $L^2(\mathbb{R})$ and maps $S(\mathbb{R})$ into itself; it therefore extends to an operator $S'(\mathbb{R}) \rightarrow S'(\mathbb{R})$. In particular, the convolution theorem shows that $L$ maps $P$-periodic functions to $P$-periodic functions, acting on their Fourier coefficients by pointwise multiplication; it follows that $L \in C^\infty(H^s_P, H^{s+|m_0|}_P)$.

There is a natural injection from the set of functions $\hat{u}_P \in L^2_\mathbb{Z}$ with supp$(\hat{u}) \subset (-\frac{P}{2}, \frac{P}{2})$ to $L^2_\mathbb{Z}$, namely

$$\hat{u}_P \mapsto u_P := \sum_{j \in \mathbb{Z}} \hat{u}_P(-jP),$$

where the series converges in $S'(\mathbb{R}) \cap L^2_{\text{loc}}(\mathbb{R})$. The following proposition shows that this map commutes with $L$. 
Proposition 2.5. Any function $\tilde{u}_P \in L^2_c(\mathbb{R})$ with $\text{supp}(\tilde{u}_P) \subset (-\frac{P}{2}, \frac{P}{2})$ satisfies

$$L^2_c \ni \sum_{|j| \leq J} L\tilde{u}_P(\cdot + j P) \xrightarrow{J \to \infty} Lu_P \in L^2_p$$

in $S'(\mathbb{R}) \cap L^2_{loc}(\mathbb{R})$.

Proof. The convergence in $S'(\mathbb{R})$ follows from the continuity of $L : S'(\mathbb{R}) \to S'(\mathbb{R})$, while that in $L^2_{loc}(\mathbb{R})$ follows from the calculation

$$\left\| \sum_{|j| \geq J} L\tilde{u}_P(\cdot + j P) \right\|_{L^2(\mathbb{R})} \leq \sum_{|j| \geq J} \| L\tilde{u}_P(\cdot + j P) \|_{L^2(\mathbb{R})}$$

$$\leq (2M)^{\frac{1}{2}} \hat{C}_2 \| \tilde{u}_P \|_0 \sum_{|j| \geq J} \frac{1}{\text{dist}([M, M], \text{supp}(\tilde{u}(\cdot + j P)))^2}$$

$$\leq (2M)^{\frac{1}{2}} \hat{C}_2 \| \tilde{u}_P \|_0 \sum_{|j| \geq J} \frac{1}{((|j| - \frac{1}{2})P - M)^2}$$

$$\to 0$$

as $J \to \infty$. □

Define

$$U_P := \left\{ u \in H^1_P : \| u \|_{H^1_P} < R \right\}$$

and functionals $\mathcal{N}_P$, $\mathcal{L}_P$, $\mathcal{E}_P$, $\mathcal{Q}_P : U_P \to \mathbb{R}$ by replacing the domain of integration in the definitions of $\mathcal{N}$, $\mathcal{L}$, $\mathcal{E}$, $\mathcal{Q}$ by one period $(-\frac{P}{2}, \frac{P}{2})$. Observing that proposition 2.2 (with the obvious modifications) holds for the new functionals, we study $\mathcal{E}_P$, $\mathcal{Q}_P \in C^1(U_P, \mathbb{R})$. Each minimizer of $\mathcal{E}_P$ over the set

$$U_{P,\mu} := \{ u \in U_P : \mathcal{Q}_P(u) = \mu \}$$

is a $P$-periodic solution of the travelling-wave equation (3); the wave speed $v$ is the Lagrange multiplier in this constrained variational principle.

Additional notation.

- We denote the set of functions which are square integrable over an open subset $S$ of $\mathbb{R}$ by $L^2(S)$ and the subset of $L^2(S)$ consisting of those functions whose weak derivative exists and is square integrable by $H^1(S)$.

- The symbol $c$ denotes a a generic constant which is independent of $\mu \in (0, \mu_*)$ (and of course functions in a given set or sequence); its dependence upon other quantities is indicated by a subscript. All order-of-magnitude estimates are also uniform over $\mu \in (0, \mu_*)$, and in general we replace $\mu_*$ with a smaller number if necessary for the validity of our results.

3. The minimization problem for periodic functions

The penalization argument. Seeking a constrained minimizer of $\mathcal{E}_P$ in the set $U_{P,\mu}$ by the direct method of the calculus of variations, one is confronted by the difficulty that a minimizing sequence may approach the boundary of $U_P$. To overcome this difficulty we observe that $\mathcal{E}_P$ also defines a continuously differentiable functional on the set

$$V_P := \{ u \in H^1_P : \| u \|_{H^1_P} < 2R \}$$
and consider the auxiliary functional
\[ E_{P, \varphi}(u) := E_P(u) + \varphi \left( \|u\|_{H^1_P}^2 \right) \]
with constraint set
\[ V_{P, \mu} := \left\{ u \in H^1_P : \|u\|_{H^1_P} < 2R, \; Q(u) = \mu \right\}, \]
where we note the helpful estimate
\[ \|u\|_\infty \leq c \left\| u \right\|_{H^1_P} \]
\[ \left\| u \right\|_{H^1_P} \leq c \mu^{1/4}, \quad u \in V_{P, \mu}. \]  

Here \( \varphi : [0, (2R)^2] \to [0, \infty) \) is a smooth, increasing ‘penalization’ function such that
(i) \( \varphi(t) = 0 \) whenever \( 0 \leq t \leq R^2 \),
(ii) \( \varphi(t) \to \infty \) as \( t \nearrow (2R)^2 \),
(iii) for every constant \( a_1 \in (0, 1) \) there exist \( M_1, M_2 > 0 \) and \( a_2 > 1 \) such that
\[ \varphi'(t) \leq M_1(\varphi(t))^{a_1} + M_2(\varphi(t))^{a_2}; \]
an example of such a function \( \varphi \) can be obtained by scaling and translating the function
\[ t \mapsto \begin{cases} (1-t)^{-1} \exp(-1/t), & t \in (0, 1), \\ 0, & t \leq 0. \end{cases} \]

The following lemma is obtained by standard weak continuity arguments (e.g., see [18, sections I.1 and I.2]).

**Lemma 3.1.** The functional \( E_{P, \varphi} : V_{P, \mu} \to \mathbb{R} \) is weakly lower semicontinuous, bounded from below, and satisfies \( E_{P, \varphi}(u) \to \infty \) as \( \|u\|_{H^1_P} \nearrow 2R \). In particular, it has a minimizer \( \tilde{u}_P \in V_{P, \mu} \).

The next step is to show that \( \tilde{u}_P \) in fact minimizes \( E_P \) over \( U_{P, \mu} \). This result relies upon estimates for \( E_{P, \varphi} \) which are uniform in \( P \) and are derived in lemmata 3.2 and 3.3 and corollary 3.4 by examining the functional \( E \) and its relationship to \( E_{P, \varphi} \).

**Lemma 3.2.** For any \( w \in W \) the ‘long-wave test function’ \( S_{lw}w \), where
\[ (S_{lw}w)(x) = \mu^\alpha w(\mu^\beta x), \]
lies in \( U \) and satisfies
\[ E(S_{lw}w) = -\mu m(0) + \mu^{1+(p-1)\alpha} E_{lw}(w) + o(\mu^{1+(p-1)\alpha}), \]
where the values of \( \alpha \) and \( \beta \) are given by (9) and \( E_{lw} \) is defined in equation (11). The estimate holds uniformly over \( w \in W \), and \( w \in W_1 \) implies \( u \in U_{\mu} \).

**Proof.** Observe that
\[ Q(S_{lw}w) = \mu^{2a-\beta}, \quad F[S_{lw}w](k) = \mu^{a-\beta} \hat{w}(\mu^{-\beta} k) \]
and
\[ \|S_{lw}w\|_i^2 = \mu^{2a-\beta} \|w\|_i^2 + \mu^{2a+\beta} \|w'\|_i^2 \leq c \mu \]
for $\alpha, \beta > 0$ with $2\alpha - \beta \geq 1$. A direct calculation shows that
\[
\mathcal{E}(\tilde{S}_{lw}w) = -\frac{1}{2} \int_R \mu^2 \mathcal{M}(k) |\hat{F}[\tilde{S}_{lw}w](k)|^2 dk - \mu^\beta \int_R N(\mu^\beta w(x)) \ dx \\
= -\mu^{2\gamma}m(0) - \mu^{2\gamma+2(\gamma-1)} N(\mu^{2\gamma} w(0)) \int_R k^{2\gamma}|\hat{w}(k)|^2 dk \\
- \mu^{p+1}\mu^\beta \int_R N_{p+1}(w(x)) \ dx \\
- \mu^\beta \int_R N_{p+1}(\mu^\beta w(x)) \ dx - \frac{\mu^{2\gamma}}{2} \int_R r(\mu^\beta k)|\hat{w}(k)|^2 dk,
\]
and one can estimate
\[
\left|\frac{\mu^{2\gamma}}{2} \int_R r(\mu^\beta k)|\hat{w}(k)|^2 dk + \mu^\beta \int_R N_{p+1}(\mu^\beta w(x)) \ dx \right| \\
\leq c \left( \frac{\mu^{2\gamma+2(\gamma-1)}}{2} \int_R k^{2\gamma}|\hat{w}(k)|^2 dk + \mu^{p+1+1}|\hat{w}(k)|^2 \right).
\]
Choosing $\alpha$ and $\beta$ such that $(p-1)\alpha = 2\gamma\beta$ and $2\alpha - \beta = 1$, so that $\alpha$ and $\beta$ are given by (9), yields the desired estimate. □

**Lemma 3.3.** Let $\{\tilde{u}_P\}_P$ be a bounded family of functions in $H^1(\mathbb{R})$ with
\[
\text{supp}(\tilde{u}_P) \subset \left( -\frac{P}{2}, \frac{P}{2} \right) \quad \text{and} \quad \text{dist} \left( \pm \frac{P}{2}, \text{supp}(\tilde{u}_P) \right) \geq \frac{1}{2} P^\frac{1}{2}
\]
and define $u_P \in H^P$ by the formula
\[
u_P = \sum_{j \in \mathbb{Z}} \tilde{u}_P(\cdot + j P).
\]
(i) The function $u_P$ satisfies
\[
\lim_{P \to \infty} \| L\tilde{u}_P - Lu_P \|_{H^1((-\frac{P}{2}, \frac{P}{2}))} = 0, \quad \lim_{P \to \infty} \| L\tilde{u}_P \|_{H^1((|x| > \frac{P}{2}))} = 0.
\]
(ii) The functionals $\mathcal{E}$, $\mathcal{Q}$ and $\mathcal{E}_P$, $\mathcal{Q}_P$ have the properties that
\[
\lim_{P \to \infty} (\mathcal{E}(\tilde{u}_P) - \mathcal{E}_P(u_P)) = 0, \quad (\mathcal{Q}(\tilde{u}_P) = \mathcal{Q}_P(u_P)
\]
and
\[
\lim_{P \to \infty} \| \mathcal{E}(\tilde{u}_P) - \mathcal{E}_P(u_P) \|_{H^1((-\frac{P}{2}, \frac{P}{2}))} = 0, \quad \lim_{P \to \infty} \| \mathcal{E}_P(u_P) \|_{H^1((|x| > \frac{P}{2}))} = 0,
\]
\[
\| \mathcal{Q}(\tilde{u}_P) - \mathcal{Q}_P(u_P) \|_{H^1((-\frac{P}{2}, \frac{P}{2}))} = 0, \quad \| \mathcal{Q}_P(u_P) \|_{H^1((|x| > \frac{P}{2}))} = 0.
\]
Proof.

(i) Using proposition 2.1(ii), we find that

\[
\int_{-\xi}^{\xi} |\tilde{L}u_P - Lu_P|^2 \, dx = \int_{-\xi}^{\xi} \left| \sum_{|j| \geq 1} \tilde{C}_j \|\tilde{u}_P\|_0 \right|^2 \, dx \\
\leq \int_{-\xi}^{\xi} \left( \sum_{|j| \geq 1} \frac{\tilde{C}_j \|\tilde{u}_P\|_0}{\text{dist} (x + jP, \text{supp} (\tilde{u}_P))^3} \right)^2 \, dx \\
\leq \int_{-\xi}^{\xi} \left( 2 \sum_{j \geq 0} \frac{\tilde{C}_j \|\tilde{u}_P\|_0}{\left( jP + \frac{1}{2}P^\frac{1}{4} \right)^3} \right)^2 \, dx \\
\to 0,
\]

and therefore

\[
\lim_{P \to \infty} \|\tilde{L}u_P - Lu_P\|_{L^2(-\xi, \xi)} = 0,
\]

and

\[
\lim_{P \to \infty} \left( \int_{|x| > \xi} |\tilde{L}u_P|^2 \, dx \leq \tilde{C}_1^2 \|\tilde{u}_P\|_0^2 \int_{|x| > \xi} \frac{dx}{\text{dist} (x, \text{supp} (\tilde{u}_P))^2} \right) \\
\leq \tilde{C}_1^2 \|\tilde{u}_P\|_0^2 \int_{|x| > \xi} \left( |x| - \frac{1}{2} (P - P^\frac{1}{4}) \right)^2 = \frac{4\tilde{C}_1^2 \|\tilde{u}_P\|_0^2}{P^\frac{1}{2}} \to 0.
\]

and therefore

\[
\lim_{P \to \infty} \|\tilde{L}u_P - Lu_P\|_{L^2(|x| > \xi)} = 0,
\]

as \(P \to \infty\). The same calculation is valid with \(u_P\) and \(\tilde{u}_P\) replaced by respectively, \(u'_P\) and \(\tilde{u}'_P\), and since \(L\) commutes with differentiation this observation completes the proof.

(ii) Observe that

\[
|\mathcal{L}(\tilde{u}_P) - \mathcal{L}P(u_P)| = \frac{1}{2} \int_{-\xi}^{\xi} \tilde{u}_P \tilde{L}u_P \, dx - \frac{1}{2} \int_{-\xi}^{\xi} u_P Lu_P \, dx = \frac{1}{2} \int_{-\xi}^{\xi} \tilde{u}_P (\tilde{L}u_P - Lu_P) \, dx \\
\leq \frac{1}{2} \|\tilde{u}_P\|_0 \|\tilde{L}u_P - Lu_P\|_{L^2(-\xi, \xi)} \\
\to 0
\]

and

\[
\|\mathcal{L}'(\tilde{u}_P) - \mathcal{L}'P(u_P)\|_{H^1(-\xi, \xi)} = \|\tilde{L}u_P - Lu_P\|_{H^1(-\xi, \xi)} \to 0,
\]

\[
\|\mathcal{L}'(\tilde{u}_P)\|_{H^1(|x| > \xi)} = \|\tilde{L}u_P\|_{H^1(|x| > \xi)} \to 0
\]

as \(P \to \infty\). Furthermore,

\[
\mathcal{N}(\tilde{u}_P) = -\int_{\mathbb{R}} \mathcal{N}(\tilde{u}_P) \, dx = -\int_{-\xi}^{\xi} \mathcal{N}(\tilde{u}_P) \, dx = -\int_{-\xi}^{\xi} \mathcal{N}(u_P) \, dx = \mathcal{N}_P(u_P)
\]

and

\[
\mathcal{N}'(\tilde{u}_P(x)) = -n(\tilde{u}_P(x)) = \begin{cases} 
-n(u_P(x)) = \mathcal{N}'_P(u_P(x)), & x \in \left(-\frac{P}{2}, \frac{P}{2}\right), \\
0, & |x| \geq \frac{P}{2},
\end{cases}
\]
\( (N'(\tilde{u}_P))'(x) = -n'(u_P(x))\tilde{u}_P(x) \)

\[
\begin{align*}
\begin{cases}
-n'(u_P(x))u_P'(x) = (N'_P(u_P))'(x), & x \in \left(-\frac{P}{2}, \frac{P}{2}\right), \\
0, & |x| \geq \frac{P}{2},
\end{cases}
\end{align*}
\]

so that

\[
\|N'(\tilde{u}_P) - N'_P(u_P)\|_{H^1(\mathbb{R}^+)} = 0, \quad \|N'(\tilde{u}_P)\|_{H^1(4|<\frac{P}{2}|)} = 0.
\]

The result for \( E \), \( E_P \) follows from these calculations and the formulae \( E = L + N \), \( E_P = L_P + N_P \), and a similar calculation yields the result for \( Q \), \( Q_P \).

**Corollary 3.4.** There exist constants \( I_* > 0 \) and \( P_\mu > 0 \) such that

\[
I_\mu := \inf \{ E(u) : u \in U_\mu \} < -\mu m(0) - \mu^{1+1/\alpha} I_*
\]

and

\[
I_{P,\psi,\mu} := \inf \{ E_{P,\psi}(u) : u \in V_{P,\mu} \} < -\mu m(0) - \mu^{1+1/\alpha} I_*
\]

for each \( P \geq P_\mu \).

**Proof.** Taking \( \psi \in C_0^\infty(\mathbb{R}) \) with \( Q(\psi) = 1 \) and writing \( w(x) = \sqrt{\lambda} \psi(\lambda x) \), one finds that

\[
E_{lw}(w) = -\lambda^2 q \\frac{m^{(2j_\lambda)}}{2(2j_\lambda)} \int_\mathbb{R} (\psi(\lambda x))^2 dx - \lambda^{3/2} \int_\mathbb{R} N_{p+1}(\psi) dx < 0
\]

for sufficiently small values of \( \lambda \) provided that \( p < 4j_\lambda + 1 \) and \( N_{p+1}(\psi) > 0 \); these conditions are satisfied under assumption (A3) by choosing \( \psi > 0 \) if \( c_p > 0 \) and \( \psi < 0 \) if \( c_p < 0 \).

Noting that \( w \in W \) for sufficiently large values of \( S \), we find from lemma 3.2 that

\[
E(S_{lw}w) + \mu m(0) = \mu^{1+1/\alpha} E_{lw}(w) + o(\mu^{1+1/\alpha})
\]

\[
< \frac{1}{2} \mu^{1+1/\alpha} E_{lw}(w).
\]

(18)

Observe that \( \text{supp}(S_{lw}w) = \mu^{-\beta} \text{supp}(w) \), so that \( S_{lw}w \) satisfies the assumptions of lemma 3.3 if and only if \( \mu^{\beta} P \geq c_w \), where \( c_w \) is a positive constant independent of \( P \). For such \( P \) a combination of lemma 3.3 and (18) yields

\[
I_{P,\psi,\mu} \leq E_P(u_P)
\]

\[
\leq -\mu m(0) + \frac{1}{2} \mu^{1+1/\alpha} E_{lw}(w) + (E_P(u_P) - E(S_{lw}w))
\]

\[
= -\mu m(0) + \frac{1}{2} \mu^{1+1/\alpha} E_{lw}(w)
\]

as \( P \to \infty \), where

\[
u_P = \sum_{j \in \mathbb{Z}} (S_{lw}w)(\cdot + j P).
\]

The result follows by setting \( I_* := -\frac{1}{2} E_{lw}(w) \) and choosing \( P_\mu \) large enough so that \( \mu^{\beta} P \geq c_w \) and \( |E_P(u_P) - E(S_{lw}w)| < \frac{1}{2} \mu^{1+1/\alpha} E_{lw}(w) \) for \( P \geq P_\mu \) (see lemma 3.3(ii)).

Let us now return to our study of minimizers \( \tilde{u}_P \) of \( E_{P,\psi} \) over \( V_{P,\mu} \), which in view of corollary 3.4 satisfy

\[
E_{P,\psi}(\tilde{u}_P) < -\mu m(0) - \mu^{1+1/\alpha} I_*
\]

(19)

and of course

\[
dE_{P,\psi}[\tilde{u}_P] + v_P dQ_P[\tilde{u}_P] = 0
\]

(20)
for some constant $\nu_p \in \mathbb{R}$, that is
\[
\int_{-\xi}^{\xi} (L\bar{u}_p + n(\bar{u}_p)) v \, dx - 2\varrho' (\|\bar{u}_p\|_{H^1}) \int_{-\xi}^{\xi} (\bar{u}_p v + \bar{u}_p') \, dx = \nu_p \int_{-\xi}^{\xi} \bar{u}_p v \, dx
\]
for all $v \in H^1_0$. This equation implies that $\bar{u}_p'$ exists if $\varrho'(\|\bar{u}_p\|_{H^1}) > 0$ and that $\bar{u}_p$ satisfies the equation
\[
v_p \bar{u}_p = L\bar{u}_p + n(\bar{u}_p) - 2\varrho' (\|\bar{u}_p\|_{H^1}) (\bar{u}_p - \bar{u}_p').
\] (21)

**Lemma 3.5.** The estimate
\[
v_p - m(0) > \frac{1}{2} J_{\lambda} (p+1) \mu^{(p-1)\alpha} + O(\|\bar{u}_p\|_{\infty}^{p+\delta - 1}) - c_\varrho \mu^{1+\varepsilon}
\]
holds uniformly over the set of minimizers $\bar{u}_p$ of $E_{p,\varrho}$ over $V_{p,\mu}$ and $p \geq P_\mu$. Here $\varepsilon$ is a positive constant and $c_\varrho$ vanishes when $\varrho = 0$.

**Proof.** In this proof all estimates hold uniformly in $p \geq P_\mu$.

Inequality (19) asserts that
\[
- \int_{-\xi}^{\xi} N(\bar{u}_p) \, dx - \frac{1}{2} \int_{-\xi}^{\xi} \bar{u}_p L\bar{u}_p \, dx + \varrho (\|\bar{u}_p\|_{H^1}) < -m(0) \mu,
\]
for all $P \geq P_\mu$, and assumption (6) implies that
\[
\frac{1}{2} \int_{-\xi}^{\xi} \bar{u}_p L\bar{u}_p \, dx \leq \frac{m(0)}{2} \int_{-\xi}^{\xi} \bar{u}_p^2 \, dx = m(0) \mu.
\] (22)

Adding these inequalities, we find that
\[
\varrho (\|\bar{u}_p\|_{H^1}) \leq \int_{-\xi}^{\xi} N(\bar{u}_p) \, dx \leq c\|\bar{u}_p\|_{\infty}^{p-1} \int_{-\xi}^{\xi} \bar{u}_p^2 \, dx \leq c \mu^{(p+3)/4},
\]
where we have estimated $\|\bar{u}_p\|_{\infty} \leq c \mu^{1/2}$ (see (17)). Using property (iii) of the penalization function, we conclude that
\[
\varrho (\|\bar{u}_p\|_{H^1}) \leq c \mu^{1+\varepsilon}.
\] (23)

Multiplying (21) by $\bar{u}_p$ and integrating over $(-\xi, \xi)$, one finds that
\[
2v_p \mu = (p+1) \int_{-\xi}^{\xi} \left( \frac{1}{2} \bar{u}_p L\bar{u}_p + N(\bar{u}_p) \right) \, dx - \frac{p-1}{2} \int_{-\xi}^{\xi} \bar{u}_p L\bar{u}_p \, dx
\]
\[
- \int_{-\xi}^{\xi} ((p+1)N(\bar{u}_p) - \bar{u}_p n(\bar{u}_p)) \, dx - 2\varrho' (\|\bar{u}_p\|_{H^1}) \|\bar{u}_p\|_{H^1}^2
\]
\[
= - (p+1)E_{p,\varrho}(\bar{u}_p) - \frac{p-1}{2} \int_{-\xi}^{\xi} \bar{u}_p L\bar{u}_p \, dx + (p+1)\varrho (\|\bar{u}_p\|_{H^1})
\]
\[
+ O(\|\bar{u}_p\|_{\infty}^{p+\delta - 1}\|\bar{u}_p\|_{L^2}^2) - 2\varrho'(\|\bar{u}_p\|_{H^1}) \|\bar{u}_p\|_{H^1}^2
\]
because $(p+1)N(u(x)) - un(u(x)) = O(|u(x)|^{p+\delta + 1})$ uniformly over $u \in V_p$ and $x \in \mathbb{R}$. It follows that
\[
v_p > m(0) + \frac{1}{2} J_{\lambda} (p+1) \mu^{(p-1)\alpha} + O(\|\bar{u}_p\|_{\infty}^{p+\delta - 1}) - c \mu^{1+\varepsilon},
\]
where we have used inequalities (19), (22) and (23). \qed
It follows from lemma 3.5 and the estimate \( \| \tilde{u}_P \|_\infty \leq c \mu^{1/2} \) (see (17)) that \( v_P > \frac{3}{2} m(0) \) uniformly over the set of minimizers \( \tilde{u}_P \) of \( E_{P, \rho} \) over \( V_{P, \rho} \) and \( P \geq P_\mu \). This bound is used in the following estimate of the size of \( \tilde{u}_P \).

**Lemma 3.6.** The estimate

\[
\| \tilde{u}_P \|_1^2 \leq c \mu
\]

holds uniformly over the set of minimizers of \( E_{P, \rho} \) over \( V_{P, \rho} \) and \( P \geq P_\mu \).

**Proof.** In this proof all estimates again hold uniformly in \( P \geq P_\mu \).

Multiplying (21) by \( \tilde{u}_P - \tilde{u}_P^n \) if \( \varrho' (\| \tilde{u}_P \|_{H^2}^2) > 0 \) or applying the operator \( \tilde{u}_P + \tilde{u}_P \frac{d}{dt} \) if \( \varrho' (\| \tilde{u}_P \|_{H^2}^2) = 0 \), we find that

\[
v_P \| \tilde{u}_P \|_{H^1}^2 = \int_{-\xi}^{\xi} (\tilde{u}_P L \tilde{u}_P + \tilde{u}_P L \tilde{u}_P) \, dx + \int_{-\xi}^{\xi} (\tilde{u}_P n(\tilde{u}_P) + |\tilde{u}_P|^2 n'(\tilde{u}_P)) \, dx
\]

\[
- 2 \varrho' (\| \tilde{u}_P \|_{H^2}^2) \left( \| \tilde{u}_P \|_{H^2}^2 + 2 \int_{-\xi}^{\xi} |\tilde{u}_P|^2 \, dx \right)
\]

\[
\leq c \| \tilde{u}_P \|_{H^2}^2 \| \tilde{u}_P \|_{H^2}^2 + \left( \sup_{|x| \leq \| \tilde{u}_P \|_\infty} |n'(x)| \right) \| \tilde{u}_P \|_{H^1}^2.
\]

because \( m \in S_{w, \infty}^\infty (\mathbb{R}) \) and

\[
|n(\mu P(x))| \leq \left( \sup_{|x| \leq \| \tilde{u}_P \|_\infty} |n'(x)| \right) |\mu P(x)|
\]

uniformly over \( x \in \mathbb{R} \). Because \( \sup_{|x| \leq \| \tilde{u}_P \|_\infty} |n'(x)| \to 0 \) as \( \| \tilde{u}_P \|_\infty \to 0 \) and hence as \( \| \tilde{u}_P \|_{H^2} \to 0 \) this quantity is bounded by \( \frac{3}{2} m(0) \) for sufficiently small values of \( R \), so that

\[
\| \tilde{u}_P \|_{H^1}^2 \leq c \| \tilde{u}_P \|_{H^2}^2 \| \tilde{u}_P \|_{H^2}^2.
\]

Estimating

\[
\| \tilde{u}_P \|_{H^2}^2 \| \tilde{u}_P \|_{H^2}^2 = \begin{cases} \| \tilde{u}_P \|_{L^2}^2 \| \tilde{u}_P \|_{H^2}^{2|m_0|}, & |m_0| < 2, \\ \| \tilde{u}_P \|_{L^2}^2, & |m_0| \geq 2 \end{cases}
\]

shows that

\[
\| \tilde{u}_P \|_{H^2}^2 \leq c \| \tilde{u}_P \|_{L^2}^2 \leq c \mu.
\]

\[ \square \]

**Theorem 3.7 (Existence of periodic minimizers).** For each \( P \geq P_\mu \) there exists a function \( \tilde{u}_P \in U_{P, \rho} \) which minimizes \( E_P \) over \( U_{P, \rho} \), so that

\[
E_P(\tilde{u}_P) = I_{P, \rho} := \inf \left\{ E_P(u) : u \in U_{P, \rho} \right\},
\]

and satisfies the Euler–Lagrange equation

\[
E'_P(\tilde{u}_P) + v_P \mathcal{Q}'_P(\tilde{u}_P) = 0
\]

for some real number \( v_P \); it is therefore a periodic solution of the travelling-wave equation (3) with wave speed \( v_P \). Furthermore

\[
\| \tilde{u}_P \|_{H^2}^2 \leq c \mu, \quad 0 < v_P \leq c
\]

uniformly over \( P \geq P_\mu \).
Proof. Let $\tilde{u}_P$ be a minimizer of $E_{P,\rho}$ over $V_{P,\mu}$. It follows from lemma 3.6 that $\|\tilde{u}_P\|_{L^1}^2 \leq c\mu$, so that $g(\tilde{u}_P)$ and $g'(\tilde{u}_P)$ vanish. In particular, $\tilde{u}_P$ belongs to $U_{P,\mu}$, and since it minimizes $E_{P,\rho}$ over $V_{P,\mu}$ it certainly minimizes $E_{P,\rho} = E_P$ over $U_{P,\mu}$. Furthermore, equation (20) is equivalent to

$$E_P'(\tilde{u}_P) + \nu_P Q_P'(\tilde{u}_P) = 0,$$

from which it follows that

$$\nu_P = -\frac{1}{2\mu} \frac{\|\tilde{u}_P\|_{L^1}^2}{\|\tilde{u}_P\|_{H^1}} \leq c\mu.$$

□

Construction of a special minimizing sequence for $E$. We proceed by extending the minimizers $\tilde{u}_P$ of $E_P$ over $U_{P,\mu}$ found above to functions in $H^1(\mathbb{R})$ by scaling, translation and truncation in the following manner. For each sufficiently large value of $P$ there exists an open subinterval $I_P := (x_P - \frac{1}{4}P, x_P + \frac{1}{4}P)$ of $(-\frac{P}{8}, \frac{P}{8})$ such that $\|\tilde{u}_P\|_{H^1(I_P)} < P^{-1}$; we may assume that this property holds for all $P \geq P_\mu$. Let $\chi : [0, \infty) \to [0, \infty)$ be a smooth, increasing ‘cut-off’ function with

$$\chi(r) = \begin{cases} 0, & 0 \leq r \leq 1/2, \\ 1, & r \geq 1, \end{cases}$$

let $u_P$ be the $P$-periodic function defined by

$$u_P(x) := A_P v_P \left(x + \frac{P}{2}\right),$$

where

$$v_P(x)|_{[-\frac{P}{4}, \frac{P}{4}]} := \chi \left(\frac{|x|}{P^{1/4}}\right) \tilde{u}_P(x + x_P), \quad A_P = \sqrt{\frac{2\mu}{\|v_P\|_{L^2}}},$$

and finally define $\tilde{u}_P \in H^1(\mathbb{R})$ by the formula

$$\tilde{u}_P(x) := \begin{cases} u_P(x), & |x| \leq \frac{P}{2}, \\ 0, & |x| > \frac{P}{2}, \end{cases}$$

so that

$$u_P = \sum_{j \in \mathbb{Z}} \tilde{u}_P(\cdot + jP).$$

Let us examine the sequence $\{\tilde{u}_n\}_{n \in \mathbb{N}_0}$, where $\tilde{u}_n := \tilde{u}_{P_n}$ and $\{P_n\}_{n \in \mathbb{N}_0}$ is an increasing, unbounded sequence of positive real numbers with $P_0 \geq P_\mu$.

Theorem 3.8 (Special minimizing sequence for $E$). The sequence $\{\tilde{u}_n\}_{n \in \mathbb{N}_0}$ is a minimizing sequence for $E$ over $U_{\mu}$ which satisfies

$$\sup_{n \in \mathbb{N}_0} \|\tilde{u}_n\|_{L^1}^2 \leq c\mu, \quad \lim_{n \to \infty} \|E'(\tilde{u}_n) + v_n Q'(\tilde{u}_n)\|_1 = 0,$$

where $v_n = v_{P_n}, n \in \mathbb{N}_0$. 
Proof. Observe that

$$
\left\| u_p - \tilde{u}_p \left( \cdot + x_p + \frac{P}{2} \right) \right\|_{L^2_x}^2 = \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \left| A_p \chi \left( \frac{2|x|}{P^2} \right) - 1 \right|^2 |\tilde{u}_p(x + x_p)|^2 \, dx
$$

$$
= \int_{|x| < \frac{\varepsilon}{2}P^2} \left| A_p \chi \left( \frac{2|x|}{P^2} \right) - 1 \right|^2 |\tilde{u}_p(x + x_p)|^2 \, dx
$$

$$
+ \left| A_p - 1 \right|^2 \int_{|x| > \frac{\varepsilon}{2}P^2} |\tilde{u}_p(x + x_p)|^2 \, dx
$$

$$
\leq \|\tilde{u}_p\|_{L^2_x}^2 < R
$$

as $P \to \infty$; the first integral vanishes by the choice of the intervals $I_p$, while the factor $|A_p - 1|$ also vanishes because $\lim_{P \to \infty} v_p = \lim_{P \to \infty} \|\tilde{u}_P\|_{L^2_x} = \sqrt{2\mu}$. Similarly,

$$
\left\| u'_p - \tilde{u}_p \left( \cdot + x_p + \frac{P}{2} \right) \right\|_{L^2_x}^2
$$

$$
= \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \left| A_p \chi' \left( \frac{2|x|}{P^2} \right) \right|^2 |\tilde{u}_p(x + x_p)|^2 \, dx
$$

$$
\to 0
$$

as $P \to \infty$ (the above argument shows that the first integral vanishes, while the second integral is bounded). It follows that

$$
\left\| u_p - \tilde{u}_p \left( \cdot + x_p + \frac{P}{2} \right) \right\|_{H^1_x} \to 0 \quad \text{as} \quad P \to \infty,
$$

and this result shows in particular that

$$
\|\tilde{u}_p\|_1 = \|u_p\|_{H^1_x} \leq \left\| u_p - \tilde{u}_p \left( \cdot + x_p + \frac{P}{2} \right) \right\|_{H^1_x} + \|\tilde{u}_p\|_{H^1_x} \leq c\mu
$$

for $P \geq P\mu$ (where $P\mu$ is replaced with a larger constant if necessary).

Next note that

$$
\mathcal{E}_P(u_p) - \mathcal{E}_P(\tilde{u}_p) = \mathcal{E}_P(u_p) - \mathcal{E}_P \left( \tilde{u}_p \left( \cdot + x_p + \frac{P}{2} \right) \right)
$$

$$
\leq \sup_{u \in \mathcal{U}_P} \|\mathcal{E}_P'(u)\|_{L^2_x} \left\| u_p - \tilde{u}_p \left( \cdot + x_p + \frac{P}{2} \right) \right\|_{L^2_x}
$$

$$
\to 0
$$

as $P \to \infty$ (because $\|\mathcal{E}_P'(u)\|_{L^2_x}$ is bounded uniformly over $u \in U_P$ and $P > 0$) and

$$
\mathcal{E}(\tilde{u}_p) - \mathcal{E}_P(u_p) \to 0
$$

as $P \to \infty$ (lemma 3.3(ii)). Observe further that $I_{P,\mu} \to I_\mu$ as $P \to \infty$:

- Take $\tilde{w} \in C^\infty_c(\mathbb{R})$ with $Q(\tilde{w}) = \mu$, so that $w_P := \sum_{j \in \mathbb{Z}} \tilde{w}(\cdot + jP)$ satisfies $I_{P,\mu} \leq \mathcal{E}_P(w_P)$ and $\mathcal{E}_P(w_P) \to \mathcal{E}(\tilde{w})$ as $P \to \infty$ (see lemma 3.3(ii)). It follows that $\limsup_{P \to \infty} I_{P,\mu} \leq \mathcal{E}(\tilde{w})$, and hence that

$$
\limsup_{P \to \infty} I_{P,\mu} \leq \inf \left\{ \mathcal{E}(\tilde{u}) : \tilde{u} \in C^\infty_c(\mathbb{R}) \cap U_\mu \right\} = I_\mu.
$$
On the other hand,

\[ I_\mu \leq \mathcal{E}(\bar{u}_P) = (\mathcal{E}(\bar{u}_P) - \mathcal{E}_P(u_P)) + (\mathcal{E}_P(u_P) - \mathcal{E}_P(\bar{u}_P)) + I_{P,\mu}, \]

in which the first and second terms on the right-hand side vanish as \( P \to \infty \), so that

\[ I_\mu \leq \lim \inf_{P \to \infty} I_{P,\mu}. \]

We conclude that

\[ \mathcal{E}(\bar{u}_P) = (\mathcal{E}(\bar{u}_P) - \mathcal{E}_P(u_P)) + (\mathcal{E}_P(u_P) - \mathcal{E}_P(\bar{u}_P)) \to I_\mu \]

as \( P \to \infty \).

Similarly, note that

\[ \|\mathcal{E}_P'(u_P) - \mathcal{E}_P'(\bar{u}_P)\|_{H^1_x} = \left\| \mathcal{E}_P'(u_P) - \mathcal{E}_P'\left( \bar{u}_P \left( \cdot + x_P + \frac{P}{2} \right) \right) \right\|_{H^1_x} \]

\[ \leq \sup_{u \in U_P} \|d\mathcal{E}_P[u]\|_{H^1_x} \left\| u_P - \bar{u}_P \left( \cdot + x_P + \frac{P}{2} \right) \right\|_{H^1_x} \to 0 \]

as \( P \to \infty \) (it follows from the calculation \( d\mathcal{E}_P[u](v) = -Lv - n'(u)v \) that

\[ \|d\mathcal{E}_P[u]\|_{H^1_x} \leq c \left( m(0) + \sup_{|x| \leq c, R} |n'(x)| + \sup_{|x| \leq c, R} |n''(x)| \right) \leq c \] (24)

uniformly over \( u \in U_P \) and \( P > 0 \), and lemma 3.3(ii) shows that

\[ \lim_{P \to \infty} \|\mathcal{E}_P'(\bar{u}_P)\|_{H^1_x} = 0, \quad \lim_{P \to \infty} \|\mathcal{E}_P(\bar{u}_P)\|_{H^1_x} = 0; \]

the same results hold for \( Q, Q_P \). We conclude that

\[ \|\mathcal{E}'(\bar{u}_P) + v_P Q'(\bar{u}_P)\|_1 \leq \|\mathcal{E}_P'(\bar{u}_P) - \mathcal{E}_P'(u_P)\|_{H^1_x} + v_P \|Q'(\bar{u}_P) - Q'_P(u_P)\|_{H^1_x} \]

\[ + \|\mathcal{E}_P'(u_P) - \mathcal{E}_P'(\bar{u}_P)\|_{H^1_x} + v_P \|Q'_P(u_P) - Q'_P(\bar{u}_P)\|_{H^1_x} \]

\[ + \|Q'_P(\bar{u}_P)\|_{H^1_x} + v_P \|Q'_P(\bar{u}_P)\|_{H^1_x} + \|\mathcal{E}_P'(\bar{u}_P)\|_{H^1_x} \]

\[ \to 0 \]

as \( P \to \infty \) because \( \{v_P\} \) is bounded. \( \square \)

4. Strict subadditivity

In this section we show that the quantity

\[ I_\mu := \inf \left\{ \mathcal{E}(u) : u \in U_\mu \right\} \]

is strictly subadditive, that is

\[ I_{\mu_1 + \mu_2} < I_{\mu_1} + I_{\mu_2} \quad \text{whenever} \quad 0 < \mu_1, \mu_2 < \mu_1 + \mu_2 < \mu_* \].

This result is needed in section 5 to exclude ‘dichotomy’ when applying the concentration-compactness principle to a minimizing sequence \( \{u_n\} \) for \( \mathcal{E} \) over \( U_\mu \). It is proved by approximating the nonlinear term \( N(u) \) by its leading-order homogeneous part
− \int_{\mathbb{R}} N_{p+1}(u_n) \, dx \) (strict subadditivity for a problem with a homogeneous nonlinearity follows by a straightforward scaling argument). However, the requisite estimate
\[
\int_{\mathbb{R}} N_t(u_n) \, dx = o(\mu^{1+(p-1)\alpha})
\]
may not hold for a general minimizing sequence; it does however hold for the special minimizing sequence \( \{\tilde{u}_n\}_{n \in \mathbb{N}_0} \) constructed in section 3.

Scaling. We now examine functions \( u \in U_\mu \) which are ‘near minimizers’ of \( \mathcal{E} \) in the sense that
\[
\mathcal{E}'(u) + v Q'(u) \parallel_1 \leq c \mu^N
\]
for some \( v \in \mathbb{R} \) and natural number \( N \geq \max \{ \frac{1}{2} (1 + 4j_\beta), 1 + (p - 1)\alpha \} \). We show that their low-wavenumber part is a long wave which ‘scales’ in a fashion similar to the ansatz (8); this result allows us to conclude in particular that \( \| u \|_\infty \leq c \mu^{1-\epsilon} \) for any \( \epsilon > 0 \) (see corollary 4.5).

Our results are obtained by studying the identity
\[
v u - L u = n(u) + \mathcal{E}'(u) + v Q'(u); \tag{26}
\]
they apply to minimizers \( u \) of \( \mathcal{E} \) over \( U_\mu \), for which \( \mathcal{E}'(u) + v Q'(u) = 0 \) for some Lagrange multiplier \( v \), and to the functions \( \tilde{u}_n \) in the minimizing sequence \( \{\tilde{u}_n\}_{n \in \mathbb{N}_0} \), which satisfy
\[
\lim_{n \to \infty} \| \mathcal{E}'(\tilde{u}_n) + v_n Q(\tilde{u}_n) \|_1 = 0. \tag{27}
\]
(Without loss of generality we may assume that \( v_n \) does not depend upon \( n \): the bounded sequence \( \{v_n\}_{n \in \mathbb{N}_0} \) has a convergent subsequence whose limit \( v \) satisfies
\[
\lim_{n \to \infty} \| \mathcal{E}'(\tilde{u}_n) + v_n Q(\tilde{u}_n) \|_1 = 0
\]
because \( \{\| Q(\tilde{u}_n) \|_1 \}_{n \in \mathbb{N}_0} \) is bounded.)

We begin with the following preliminary result, which is proved in the same fashion as lemma 3.5.

**Proposition 4.1.** The estimate
\[
v - m(0) > \frac{3}{2} L (p + 1) \mu^{(p-1)\alpha} + O(\| u \|_\infty^{p+\delta-1}) + O(\mu^{N-\frac{1}{2}})
\]
holds uniformly over the set of \( u \in U_\mu \) satisfying (25).

According to proposition 4.1 one may replace (25) by
\[
v - m(0) > O(\| u \|_\infty^{p+\delta-1}), \quad \| \mathcal{E}'(u) + v Q'(u) \|_1 \leq c \mu^N \tag{27}
\]
and most of the results in the present section apply to this more general situation. In particular, estimating
\[
\| u \|_\infty \leq c \| u \|_0 \| u \|_1^{\frac{1}{2}} \leq c \mu^\frac{1}{4}, \quad u \in U_\mu
\]
we find that \( v > \frac{3}{2} m(0) \); our next result is obtained from this bound in the same fashion as lemma 3.6.

**Proposition 4.2.** The estimate
\[
\| u \|_1^2 \leq c \mu
\]
holds uniformly over the set of \( u \in U_\mu \) satisfying (27).
The next step is to decompose a function \( u \in H^1(\mathbb{R}) \) into low- and high-wavenumber parts in the following manner. Choose \( k_0 > 0 \) so that \( m(k) \leq \frac{1}{4} m(0) \) for \( |k| \geq k_0 \), let \( \xi \) be the characteristic function of the set \([-k_0, k_0]\), and write \( u = u_1 + u_2 \), where

\[
\hat{u}_1(k) := \xi(k) \hat{u}(k), \quad \hat{u}_2(k) := (1 - \xi(k)) \hat{u}(k).
\]

We proceed by writing (26) as coupled equations for the low- and high-wavenumber parts of \( u \), namely

\[
(v - m)\hat{u}_1 = \xi \mathcal{F}[n(u) + \mathcal{E}'(u) + v\mathcal{Q}'(u)],
\]

\[
(v - m)\hat{u}_2 = (1 - \xi) \mathcal{F}[n(u) + \mathcal{E}'(u) + v\mathcal{Q}'(u)],
\]

and estimating \( u_1 \) using the weighted norm

\[
\|v\|_{r,\mu} := \left( \int_{\mathbb{R}} (v^2 + \mu^{-4j_\beta(r)}(v^{(2j))]^2 \right)^{\frac{1}{2}} dx, \quad r < 1
\]

for \( H^{2j_\beta}(\mathbb{R}) \), which is useful in estimating the \( L^\infty(\mathbb{R}) \)-norm of \( u_1 \) and its derivatives.

**Proposition 4.3.** The estimate

\[
\|v^{(j)}\|_\infty \leq c\mu^{(j+1)j_\beta} \|v\|_{r,\mu}, \quad j = 0, \ldots, 2j_\beta - 1
\]

holds for all \( v \in H^{2j_\beta}(\mathbb{R}) \).

**Proof.** Observe that

\[
\|v^{(j)}\|_{\infty} \leq \frac{1}{2\pi} \|k^{j} \tilde{v}\|_{L^1(\mathbb{R})} \leq \frac{1}{2\pi} \left( \int_{\mathbb{R}} \left| \frac{k^{2j}}{1 + \mu^{-4j_\beta(r)}k^{2j_\beta}} \right| dx \right) \|v\|_{r,\mu} \leq c\mu^{(j+1)j_\beta} \|v\|_{r,\mu}. \quad \Box
\]

**Theorem 4.4 (Scaling).** Choose \( \tau < 1 \). The estimates

\[
\|u_1\|_{r,\mu}^2 \leq c_1 \mu \quad \|u_2\|_{1}^2 \leq c_2 \mu^{\tau(p-1)+p}
\]

hold for all \( u \in U_\mu \), which satisfy (27).

**Proof.** Note that \( v - m(k) \geq \frac{1}{4} m(0) \) for \( |k| \geq k_0 \) (since \( v > \frac{3}{4} m(0) \)), so that

\[
\mathcal{F}^{-1}[(v - m)^{-1}(1 - \xi)\mathcal{F}(\cdot)] \in B(H^1(\mathbb{R})^*),
\]

where the operator norm is bounded uniformly over \( v > \frac{3}{4} m(0) \), and it follows from equation (29) that

\[
\|u_2\|_{1} \leq c\|n(u)\|_{1} + \|\mathcal{E}'(u) + v\mathcal{Q}'(u)\|_{1}
\]

\[
\leq c\mu \frac{1}{2} \|u_1\|_{\infty}^{p-1} + \mu^{\frac{1}{2}(p-1)} \|u_2\|_{1} + \mu^{N},
\]

where we have estimated

\[
\|n(u)\|_{1}^{2} = \|n(u)\|_{0}^{2} + \|n'(u)u'\|_{0}^{2}
\]

\[
\leq c\|u\|_{\infty}^{2p-2} \|u\|_{1}^{2}
\]

\[
\leq c\|u_1\|_{\infty}^{2p-2} \|u_1\|_{1}^{2} + \|u_2\|_{\infty}^{2p-2} \|u_2\|_{1}^{2} + \|u\|_{\infty}^{2p-2} \|u_2\|_{1}^{2}
\]

\[
\leq c\|u_1\|_{\infty}^{2p-2} \|u_1\|_{1}^{2} + \|u_2\|_{\infty}^{2p-2} \|u_2\|_{1}^{2} + \|u\|_{\infty}^{2p-2} \|u_2\|_{1}^{2}
\]

\[
\leq c\|u_1\|_{\infty}^{2p-2} \|u_1\|_{1}^{2} + \|u\|_{\infty}^{2p-2} \|u_2\|_{1}^{2}
\]

\[
\leq c\mu \|u_1\|_{\infty}^{2p-2} + \mu^{p-1} \|u_2\|_{1}^{2}
\]

We conclude that

\[
\|u_2\|_{1} \leq c(\mu^{\frac{1}{2}} \|u_1\|_{\infty}^{p-1} + \mu^{N}).
\]

Turning to equation (28), observe that
\[ v - m(k) > (v - m(0)) - \frac{c m^{(2j_1)}(0)}{(2j_1)!} k^{2j_1} > -\frac{c m^{(2j_1)}(0)}{(2j_1)!} k^{2j_1} + O(\|u\|^{p+\delta-1}) \]
for \( |k| < k_0 \) and uniformly over \( u \in U_\mu \), so that
\[
\int_R |u_1^{(2j_1)}|^2 \, dx \leq c \int_R (v - m(k))^2 |\hat{u}_1(k)|^2 \, dk + c \|u_1\|^2_{1,\mu} \|u_2\|^{2(p-1)}_{2,\mu}
\]
\[
\leq c \left( \|n(u)\|_0^2 + \|E^*(u) + vQ'(u)\|_0^2 + \|u\|_1^2 \|u_1\|^{2(p-1)}_{\infty} + \|u_1\|^2_{1,\mu} \|u_2\|^2_{2,\mu} \right)
\]
\[
\leq c \left( \|n(u)\|_0^2 + \|E^*(u) + vQ'(u)\|_0^2 + \|u\|_1^2 \|u_1\|^{2(p-1)}_{\infty} + \|u_1\|^2_{1,\mu} \|u_2\|^2_{2,\mu} \right)
\]
\[
\leq c (\mu \|u_1\|^{2(p-1)}_{\infty} + \|u_2\|^2_{2,\mu} + \mu^{2N})
\]
\[
\leq c (\mu \|u_1\|^{2(p-1)}_{\infty} + \|u_2\|^2_{2,\mu} + \mu^{2N})
\]
\[
\leq c \left( \mu^{1+(p-1)(\tau^\delta+1)} \left( \frac{\|u_1\|_{1,\mu}}{\mu^2} \right)^{2(p-1)} + \mu^{2N} \right),
\]
where we have estimated
\[
\|n(u)\|_0^2 \leq c (\|u_1\|^{2(p-2)}_{\infty} + \|u_2\|^2_{2,\mu}) \leq c (\mu \|u_1\|^{2p-2}_{\infty} + \|u_2\|^2_{2,\mu})
\]
and used (31) and proposition 4.3. Multiplying this estimate by \( \mu^{-4\mu\tau^\beta} \) and adding the inequality \( \int_R u_1^2 \, dx \leq 2\mu \), one finds that
\[
\|u_1\|^2_{1,\tau,\mu} \leq c \mu \left( 1 + \mu^{(1-\tau)(p-1)} \left( \frac{\|u_1\|^2_{1,\mu}}{\mu} \right)^{p-1} \right).
\]
Define \( Q = \tau \in (-\infty, 1) : \|u_1\|^2_{1,\mu} \leq c_\tau \mu \). The inequality \( \|u_1\|^2_{1,\tau,\mu} \leq \|u_1\|^2_{1,\mu} \) for \( \tau_1 \leq \tau_2 \) shows that \( (-\infty, \tau] \subset Q \) whenever \( \tau \in Q \); furthermore \( (-\infty, 0] \subset Q \) because \( \|u_1\|^2_{0,\mu} \leq \|u_1\|^2_{0,0} \leq 2\mu \). Suppose that \( \tau_* := \sup Q \) is strictly less than unity, choose \( \epsilon > 0 \) so that \( \tau_* + (1 + 8j_1\beta)\epsilon < 1 \) and observe that
\[
\frac{\|u_1\|^2_{1,\tau_*,\mu}}{\mu} \leq c \left( 1 + \mu^{1-\tau_* - 8j_1\beta - 1}\left( \frac{\|u_1\|^2_{1,\mu}}{\mu} \right)^{p-1} \right)
\]
\[
\leq c_{\tau_* + \epsilon}
\]
which leads to the contradiction that \( \tau_* + \epsilon \in Q \). It follows that \( \tau_* = 1 \) and \( \|u_1\|^2_{1,\mu} \leq c_1 \mu \) for each \( \tau < 1 \).

The bound for \( \|u_2\|^2_{1,\mu} \) follows from inequality (31), proposition 4.3 and the bound for \( \|u_1\|^2_{1,\mu} \).

\[ \square \]

**Corollary 4.5.** Choose \( \tau < 1 \). The estimate
\[ \|u\|_{\infty} \leq c_\tau \mu^a \mu^{(1-\tau)(\frac{1}{2} - \delta)} \]
holds for all \( u \in U_\mu \) which satisfy (27).
Proof. Using proposition 4.3, theorem 4.4 and the relation \( \beta = 2 \alpha - 1 \), one finds that
\[
\| u \|_{\infty} \leq c (\mu^{1/2} \| u_1 \|_{\infty} + \| u_2 \|_{1}) \\
\leq c \mu^{1/2} \mu^{1/2} \| u \|_{\infty} + \mu^{1/2} \| u \|_{1} \\
= c \mu^{1/2} \mu^{1/2} \mu^{(1-\tau)(1/2-\alpha)}.
\]
\[\square\]

Corollary 4.6. Any function \( u \in U_\mu \) satisfying (25) has the property that \( v - m(0) > 0 \).

Proof. Using corollary 4.5, we find that
\[
\| u \|_{p+1} \leq c \| u \|_{p+1} \| u \|_{\infty}^{-1} \leq c \mu^{1/2} \mu^{(1-\tau)(1/2-\alpha)} \mu^{(p-1)(p-1)\alpha} \\
= c \mu^{(1-\tau)(1/2-\alpha)} \mu^{(p-1)\alpha} + o(\mu^{(p-1)\alpha}) \\
\leq o(\mu^{(p-1)\alpha})
\]
uniformly over \( u \in D_\mu \) for \( \tau \) sufficiently close to 1, whereby proposition 4.1 shows that
\[ v - m(0) > \frac{1}{2} I_\mu^{(p+1)(p-1)\alpha} + o(\mu^{(p-1)\alpha}) > 0 \].
\[\square\]

Strict subhomogeneity. A function \( \mu \mapsto I_\mu \) is said to be strictly subhomogeneous on an interval \( (0, \mu^*) \) if
\[ I_{a\mu} < a I_\mu \]
whenever \( 0 < \mu < a \mu < \mu^* \); a straightforward argument shows that strict subhomogeneity implies strict subadditivity on the same interval (see [6, p 48]).

Proposition 4.7.
(i) Any function \( u \in U_\mu \) with the property
\[ E(u) > -\mu m(0) - I_\mu \mu^{1+(p-1)\alpha} \]
(32)
satisfies
\[ N(u) \leq -c \mu^{1+(p-1)\alpha}. \]
This result holds in particular for any minimizing sequence \( \{u_n\}_{n \in \mathbb{N}_0} \) for \( E \) over \( U_\mu \).
(ii) Any function \( u \in U_\mu \) with the property (32) satisfies
\[ \int_\mathbb{R} N_{p+1}(u) \, dx \geq c \mu^{1+(p-1)\alpha}. \]
This result holds in particular for the sequence \( \{\tilde{u}_n\}_{n \in \mathbb{N}_0} \).

Proof. The first result is a consequence of the equation
\[ N(u) = E(u) - L(u) \]
and the estimates (32) and
\[ -L(u) = \frac{1}{2} \int_\mathbb{R} u Lu \, dx \leq \frac{m(0)}{2} \int_\mathbb{R} u^2 \, dx \equiv \mu m(0) \].
while the second is obtained from the first using the estimate
\[
\left| \int_\mathbb{R} N_t(u) \, dx \right| \leq c \|u\|_1^{2+\delta} \|u\|_\infty^{p-1} \\
\leq c \mu^{\frac{1+\delta}{2}} \mu^{(1-\tau)(\frac{1}{2}-\omega)(p-1)} \\
= c \mu^{\frac{1+\delta}{2}(1-\tau)(\frac{1}{2}-\omega)(p-1)} \mu^{1+\omega} \\
= o(\mu^{1+\omega})
\] (33)
for \( \tau \) sufficiently close to 1 (see corollary 4.5).

\[ \Box \]

**Lemma 4.8.** The map \( \mu \mapsto I_\mu \) is strictly subhomogeneous for \( \mu \in (0, \mu^*) \).

**Proof.** Fix \( a > 1 \) and note that \( \|a^\tau \tilde{u}_n\|^2 \leq c a \mu < R \). We have that
\[
I_{a\mu} \leq \mathcal{E}(a^\tau \tilde{u}_n) = \mathcal{L}(a^\tau \tilde{u}_n) - \int_\mathbb{R} N_{p+1}(a^\tau \tilde{u}_n) \, dx - \int_\mathbb{R} N_t(a^\tau \tilde{u}_n) \, dx \\
= a \mathcal{L}(\tilde{u}_n) - a^{\tau(p+1)} \int_\mathbb{R} N_{p+1}(\tilde{u}_n) \, dx + o(\mu^{1+p-1}) \\
= a \mathcal{E}(\tilde{u}_n) - (a^{\tau(p+1)} - a) \int_\mathbb{R} N_{p+1}(\tilde{u}_n) \, dx + o(\mu^{1+p-1}) \\
\leq a \mathcal{E}(\tilde{u}_n) - c(a^{\tau(p+1)} - a) \mu^{1+p-1} + o(\mu^{1+p-1}),
\] (34)
in which we have used proposition 4.7(ii) and the estimate
\[
\left| \int_\mathbb{R} N_t(a^\tau \tilde{u}_n) \, dx \right| \leq ca^{\tau(p+1)} \|\tilde{u}_n\|^2_1 \|\tilde{u}_n\|_\infty^{p-1} = o(\mu^{1+p-1})
\]
(see calculation (33)). In the limit \( n \to \infty \) inequality (34) yields
\[
I_{a\mu} \leq a I_\mu - c(a^{\tau(p+1)} - a) \mu^{1+p-1} + o(\mu^{1+p-1}),
\]
from which it follows that \( I_{a\mu} < a I_\mu \). \[ \Box \]

5. Concentration-compactness

In this section we present the proof of theorem 1.2 with the help of the concentration-compactness principle [15], which we now recall in a form suitable for our purposes.

**Theorem 5.1 (Concentration-compactness).** Any sequence \( \{e_n\}_{n \in \mathbb{N}_0} \subset L^1(\mathbb{R}) \) of non-negative functions with the property that
\[
\int_\mathbb{R} e_n \, dx \equiv l > 0
\]
admits a subsequence, denoted again by \( \{e_n\}_{n \in \mathbb{N}_0} \), for which one of the following phenomena occurs.

Vanishing: For each \( r > 0 \) one has that
\[
\lim_{n \to \infty} \sup_{x_0 \in \mathbb{R}} \int_{B_r(x_0)} e_n \, dx = 0.
\] (35)
Concentration: There is a sequence \( \{x_n\}_{n \in \mathbb{N}_0} \subset \mathbb{R} \) with the property that for each \( \varepsilon > 0 \) there exists \( r > 0 \) with
\[
\int_{B_r(x_n)} e_n \, dx \geq 1 - \varepsilon, \tag{36}
\]
for all \( n \in \mathbb{N}_0 \).

Dichotomy: There are sequences \( \{x_n\}_{n \in \mathbb{N}_0}, \{M_n\}_{n \in \mathbb{N}_0}, \{N_n\}_{n \in \mathbb{N}_0} \subset \mathbb{R} \) and a real number \( \lambda \in (0,1) \) with the properties that \( M_n, N_n \to \infty, M_n/N_n \to 0 \),
\[
\int_{B_{N_n}(x_n)} e_n \, dx \to \lambda \quad \text{and} \quad \int_{B_{M_n}(x_n)} e_n \, dx \to \lambda, \tag{37}
\]
as \( n \to \infty \).

We proceed by applying theorem 5.1 to the functions \( e_n = u_n^2, n \in \mathbb{N}_0 \), where \( \{u_n\}_{n \in \mathbb{N}_0} \) is a minimizing sequence for \( \mathcal{E} \) over \( U_\mu \) with the property that \( \sup_{n \in \mathbb{N}_0} \|u_n\|_1 < R \), so that \( \ell = 2\mu \).

It is a straightforward matter to exclude ‘vanishing’.

**Lemma 5.2.** No subsequence of \( \{e_n\}_{n \in \mathbb{N}_0} \) has the ‘vanishing’ property.

**Proof.** Suppose that \( \{e_n\}_{n \in \mathbb{N}_0} \) satisfies (35), and observe that
\[
\|N(u_n)\| \leq \int \|N(u_n)\| \, dx \leq c \sum_{j \in Z} \int_{2j-1}^{2j+1} |u_n|^{p+1} \, dx \leq c \|u\|_\infty^{p-1} \sum_{j \in Z} \int_{2j-1}^{2j+1} |u_n|^2 \, dx
\]
\[
\leq c \|u\|_1^{p-1} \left( \sup_{x_0 \in \mathbb{R}} \int_{B_1(x_0)} e_n \, dx \right) \leq c \sup_{x_0 \in \mathbb{R}} \int_{B_1(x_0)} e_n \, dx
\]
\[
\to 0
\]
as \( n \to \infty \), which contradicts proposition 4.7(i). \( \square \)

**Lemma 5.3.** Choose \( s \in (0,1) \) and suppose that a subsequence of \( \{e_n\}_{n \in \mathbb{N}_0} \) ‘concentrates’. There exists a subsequence of \( \{u_n(\cdot + x_0)\}_{n \in \mathbb{N}_0} \) which converges in \( H^s(\mathbb{R}) \) to a minimizer of \( \mathcal{E} \) over \( U_\mu \).

**Proof.** Write \( v_n := u_n(\cdot + x_0) \), so that \( \sup_{n \in \mathbb{N}_0} \|v_n\|_1 < R \). Equation (36) implies that for any \( \varepsilon > 0 \) there exists \( r > 0 \) such that
\[
\|v_n\|_{L^2(\{1 < r\})} < \varepsilon.
\]
On the other hand \( \{v_n\}_{n \in \mathbb{N}_0} \) converges weakly in \( H^1(\mathbb{R}) \) and strongly in \( L^2(\mathbb{R}) \) to a function \( v \) with \( \|v\|_1 < R \); it follows that \( v_n \to v \) in \( L^2(\mathbb{R}) \) as \( n \to \infty \). In view of the interpolation inequality \( \|v_n - v\|_1 \leq \|v_n - v\|_0^{1-s}\|v_n - v\|_s \) we conclude that \( v_n \to v \) in \( H^s(\mathbb{R}) \) as \( n \to \infty \), so that \( \mathcal{E}(v_n) \to \mathcal{E}(v) \) as \( n \to \infty \) (proposition 2.2(ii)) with \( \mathcal{E}(v) = I_\mu \) (by uniqueness of limits). \( \square \)

Suppose now that ‘dichotomy’ occurs, and that \( \{e_n\}_{n \in \mathbb{N}_0} \) satisfies (37); note in particular that the sequence \( \{v_n\}_{n \in \mathbb{N}_0} \) with \( v_n = u_n(\cdot + x_0) \) satisfies
\[
\|v_n\|_{L^2(|M_n < \cdot < N_n|)} = \int_{-N_n}^{N_n} e_n \, dx - \int_{-M_n}^{M_n} e_n \, dx \to 0 \tag{38}
\]
as \( n \to \infty \). Let \( \zeta \) be a smooth, decreasing ‘cut-off’ function with
\[
\zeta(r) := \begin{cases} 1, & 0 \leq r \leq 1, \\ 0, & r \geq 2, \end{cases}
\]
and define
\[
\begin{align*}
u_n^{(1)}(x) &:= v_n(x)\zeta \left( \frac{|x|}{M_n} \right), \\
u_n^{(2)}(x) &:= v_n(x) \left( 1 - \zeta \left( \frac{2|x|}{N_n} \right) \right),
\end{align*}
\]

so that
\[
\text{supp}(\nu_n^{(1)}) \subseteq [-2M_n, 2M_n], \quad \text{supp}(\nu_n^{(2)}) \subseteq \mathbb{R} \setminus \left( -\frac{N_n}{2}, \frac{N_n}{2} \right),
\]
which in view of the properties of \( M_n \) and \( N_n \) are disjoint sets for large values of \( n \).

**Proposition 5.4.** The sequences \( \{\nu_n^{(1)}\}_{n \in \mathbb{N}_0} \) and \( \{\nu_n^{(2)}\}_{n \in \mathbb{N}_0} \) satisfy
\[
\|\nu_n^{(1)}\|_{2, \lambda}^2 \to 0, \quad j = 1, 2, \quad \text{as } n \to \infty.
\]

**Proof.** The limits (39) are a direct consequence of (38) since \( |\nu_n^{(1)}|, |\nu_n^{(2)}| \leq |v_n| \). It follows that
\[
\|\nu_n^{(1)}\|_{2, \lambda}^2 \to 0
\]
and
\[
\|\nu_n^{(2)}\|_{2, \lambda}^2 \to 0
\]
as \( n \to \infty \). Using these results, we find that
\[
\|\nu_n^{(1)}\|_0^2 = \|\nu_n^{(1)}\|_{\xi, \lambda}^2 + \|\nu_n^{(1)}\|_{\xi, \lambda}^2
\]
\[
= \int_{-M_n}^{M_n} v_n \, dx + \|\nu_n^{(1)}\|_{\xi, \lambda}^2
\]
\[
\to \lambda
\]
and
\[
\|\nu_n^{(2)}\|_0^2 = \|\nu_n^{(2)}\|_{\xi, \lambda}^2 + \|\nu_n^{(2)}\|_{\xi, \lambda}^2
\]
\[
= \|\nu_n\|_0^2 - \int_{-N_n}^{N_n} v_n \, dx + \|\nu_n^{(2)}\|_{\xi, \lambda}^2
\]
\[
= 2\mu
\]
\[
\to 2\mu - \lambda
\]
as \( n \to \infty \). \( \square \)

Define
\[
\begin{align*}
u_n^{(1)} := \frac{\sqrt{\lambda}}{\|\nu_n^{(1)}\|_0} \nu_n^{(1)}, \quad \nu_n^{(2)} := \frac{\sqrt{2\mu - \lambda}}{\|\nu_n^{(2)}\|_0} \nu_n^{(2)},
\end{align*}
\]
so that
\[
\|\nu_n^{(1)}\|_0^2 = \lambda, \quad \|\nu_n^{(2)}\|_0^2 = 2\mu - \lambda,
\]
for all \( n \in \mathbb{N}_0 \). According to the next proposition we can assume without loss of generality that \( \{\nu_n^{(1)}\} \subset U_\frac{\lambda}{2} \) and \( \{\nu_n^{(2)}\} \subset U_{\frac{2\mu - \lambda}{2}} \).
Proposition 5.5. The sequences \( \{u_n^{(1)}\}_{n \in \mathbb{N}_0} \) and \( \{u_n^{(2)}\}_{n \in \mathbb{N}_0} \) satisfy

(i) \( \lim_{n \to \infty} \|v_n - u_n^{(1)} - u_n^{(2)}\|_0^2 = 0 \).

(ii) \( \limsup_{n \to \infty} \|u_n^{(1)} + u_n^{(2)}\|_1 < R \) and \( \limsup_{n \to \infty} \|u_n^{(j)}\|_1 < R, \ j = 1, 2. \)

Proof.

(i) Clearly

\[
\|v_n - v_n^{(1)} - v_n^{(2)}\|_0^2 = \|v_N - v_N^{(1)} - v_N^{(2)}\|_{L^2(M_n, \mu)}^2 \to 0
\]

as \( n \to \infty \) in view of the triangle inequality and the limits (38) and (39). On the other hand

\[
\|u_n^{(1)} + u_n^{(2)} - v_n^{(1)} - v_n^{(2)}\|_0^2 = \|u_n^{(1)} - v_n^{(1)}\|_0^2 + \|u_n^{(2)} - v_n^{(2)}\|_0^2
\]

\[
= \left( \frac{\sqrt{\lambda}}{\|v_n^{(1)}\|_0^2} - 1 \right)^2 \|v_n^{(1)}\|_0^2 + \left( \frac{\sqrt{2\mu - \lambda}}{\|v_n^{(2)}\|_0^2} - 1 \right)^2 \|v_n^{(2)}\|_0^2
\]

\[
\to 0
\]

as \( n \to \infty \) (proposition 5.4).

(ii) Note that \( \|v_n'\|_0^2 \leq R \) and

\[
\left| \frac{d}{dx} \left( \frac{|x|}{M_n} \right) \right| + 1 - \frac{2|x|}{N_n} \leq cM_n^{-1},
\]

uniformly over \( x \in \mathbb{R} \), whence

\[
\|(v_n^{(1)} + v_n^{(2)})'\|_0^2 \leq \|v_n'\|_0^2 + O(M_n^{-1}),
\]

and (41) shows that

\[
\|v_n^{(1)} + v_n^{(2)}\|_0^2 = \|v_n\|_0^2 + o(1)
\]

as \( n \to \infty \). Combining these estimates, one finds that

\[
\|v_n^{(1)} + v_n^{(2)}\|_1^2 \leq \|v_n\|_1^2 + o(1),
\]

which in the light of (42) implies that

\[
\|u_n^{(1)} + u_n^{(2)}\|_1^2 \leq \|v_n\|_1^2 + o(1)
\]

as \( n \to \infty \).

The previous inequality shows that

\[
\limsup_{n \to \infty} \|u_n^{(1)} + u_n^{(2)}\|_1 \leq \sup_{n \in \mathbb{N}_0} \|v_n\|_1 < R,
\]

and the results for \( \limsup_{n \to \infty} \|u_n^{(j)}\|_1, \ j = 1, 2 \) follow from the estimates

\[
\|u_n^{(j)}\|_1 \leq \|u_n^{(1)} + u_n^{(2)}\|_1, \quad j = 1, 2.
\]

Our next result shows that \( \{\mathcal{E}(v_n)\}_{n \in \mathbb{N}_0} \) decomposes into two parts for large values of \( n \).

Proposition 5.6. The sequences \( \{u_n^{(1)}\}_{n \in \mathbb{N}_0} \) and \( \{u_n^{(2)}\}_{n \in \mathbb{N}_0} \) satisfy

\[
\lim_{n \to \infty} \left( \mathcal{E}(v_n) - \mathcal{E}(u_n^{(1)}) - \mathcal{E}(u_n^{(2)}) \right) = 0.
\]
Proof. First note that
\[ |E(v_n) - E(u_n^{(1)} + u_n^{(2)})| \leq \sup_{u \in U} \|E'(u)\|_0 \|v_n - u_n^{(1)} - u_n^{(2)}\|_0 \to 0 \] (43)
as \( n \to \infty \) since \( \|E'(u)\|_0 \) is bounded on \( U \).
Furthermore,
\[ \mathcal{L}(u_n^{(1)} + u_n^{(2)}) = \mathcal{L}(u_n^{(1)}) + \mathcal{L}(u_n^{(2)}) - \int_{\mathbb{R}} u_n^{(2)} Lu_n^{(1)} \, dx, \]
and
\[ \left| \int_{\mathbb{R}} u_n^{(2)} Lu_n^{(1)} \, dx \right| \leq \tilde{C}_1 \|u_n^{(1)}\|_0 \int_{\mathbb{R}} \frac{|u_n^{(2)}(x)|}{\text{dist}(x, \text{supp}(u_n^{(1)}))} \, dx \]
\[ \leq \tilde{C}_1 R \int_{|x| > \frac{N_n}{2}} \frac{|u_n^{(2)}(x)|}{\text{dist}(x, [-2M_n, 2M_n])} \, dx \]
\[ \leq \tilde{C}_1 R^2 \left( 2 \int_{N_n/2}^{\infty} \frac{dx}{(x - 2M_n)^2} \right)^{\frac{1}{2}} \]
\[ = \tilde{C}_1 R^2 \left( \frac{4}{N_n \left(1 - \frac{4M_n}{N_n}\right)} \right)^{\frac{1}{2}} \]
\[ \to 0 \]
as \( n \to \infty \), so that
\[ \lim_{n \to \infty} (\mathcal{L}(u_n^{(1)} + u_n^{(2)}) - \mathcal{L}(u_n^{(1)}) - \mathcal{L}(u_n^{(2)})) = 0. \]
Combining this result with the equation
\[ \mathcal{N}(u_n^{(1)} + u_n^{(2)}) = \mathcal{N}(u_n^{(1)}) + \mathcal{N}(u_n^{(2)}) \]
(the supports of \( u_n^{(1)} \) and \( u_n^{(2)} \) are disjoint), one finds that
\[ \lim_{n \to \infty} (E(u_n^{(1)} + u_n^{(2)}) - E(u_n^{(1)}) - E(u_n^{(2)})) = 0. \] (44)
The stated result follows from (43) and (44).

Lemma 5.7. No subsequence of \( \{v_n\}_{n \in \mathbb{N}} \) has the 'dichotomy' property.

Proof. Recall that \( \{v_n\}_{n \in \mathbb{N}} \) is a minimizing sequence for \( E \) over \( U_\mu \) and that \( E(u_n^{(1)}) \geq I_{\mu_1}^*, \quad E(u_n^{(2)}) \geq I_{\mu_2} \). Using lemma 5.6 and the strict subadditivity of \( \mu \mapsto I_\mu \) on \( (0, \mu^*) \), we arrive at the contradiction
\[ I_\mu < I_{\mu_1} + I_{\mu_2} \]
\[ \leq \lim_{n \to \infty} (E(u_n^{(1)}) + E(u_n^{(2)})) \]
\[ = \lim_{n \to \infty} E(v_n) \]
\[ = I_\mu. \] \( \square \)
According to theorem 5.1 and lemmata 5.2 and 5.7 a subsequence of \( \{e_n\}_{n \in \mathbb{N}_0} \) concentrates, so that the hypotheses of lemma 5.3 are satisfied. It follows that \( D_\mu \) is non-empty and theorem 1.2(ii) holds. The remaining assertions in theorem 1.2(i) are proved by applying proposition 4.1 and corollary 4.6 to \( u \in D_\mu \).
6. Consequences of the existence theory

An a priori result for supercritical solitary waves. We now record an a priori estimate for supercritical solutions \( u \in U_\mu \) of (3). The result states that such solutions are long waves which ‘scale’ in a fashion similar to the ansatz (8). More precisely, we show that \( \|u\|_{r,\mu}^2 \leq c_r \mu \) for \( r < 1 \), where \( \| \cdot \|_{r,\mu} \) is the weighted norm for \( H^{2r}(\mathbb{R}) \) defined by formula (30), so that \( \|u^{(j)}\|_\mu \leq c_\mu \mu^{\frac{\delta+j}{2}} \) for \( j = 1, \ldots, 2 j_* \). We make the following additional assumption on the nonlinearity \( n \), which ensures that \( u \in H^{2r}(\mathbb{R}) \) with \( \|u\|_{2r} = O(\mu^2) \) (see lemma 2.3).

(A4) The nonlinearity \( n \) belongs to \( C^{2, r}(\mathbb{R}) \) with

\[
n^{(j)}_\mu(x) = O(|x|^{\nu + \delta - j}), \quad j = 0, \ldots, 2 j_
\]

for some \( \delta > 0 \) as \( x \to 0 \).

Lemma 6.1. Suppose that the additional regularity assumption (A4) holds. Every supercritical solution \( u \in U_\mu \) of (3) satisfies \( \|u\|_{r,\mu}^2 \leq c \mu \) for all \( r < 1 \).

Proof. Write (3) as

\[
(v - m)\hat{u}_1 = \xi F[n(u)], \quad \hat{u}_2 = (v - m)^{-1}(1 - \xi) F[n(u)].
\]

Observing that

\[
v - m(k) > (v - m(0)) - \frac{cm^{(2, j_\mu)(0)}}{j_\mu!} k^{2 j_\mu} \quad > - \frac{cm^{(2, j_\mu)(0)}}{j_\mu!} k^{2 j_*}
\]

for \( |k| < k_0 \), we find that

\[
\int_\mathbb{R} |u_2^{(j_\mu)}|^2 \, dx \leq c \int_\mathbb{R} (v - m(k))^2 |\hat{u}_1(k)|^2 \, dk \leq c \|n(u)\|_0 \leq c \|u\|_{p-1,\mu}^2 \|u\|_{0,\mu}^2.
\]

On the other hand

\[
F^{-1}[(v - m)^{-1}(1 - \xi) F(\cdot)] \in B(L^2(\mathbb{R}), L^2(\mathbb{R}))
\]

where the operator norm is bounded uniformly over \( v > m(0) \), so that

\[
\|u_2^{(2, j_\mu)}\|_0 \leq c \|n(u)\|_{2,\mu} \leq c \sum_{i=1}^{2 j_*} \|n^{(i)}(u) B_{2 j_*}(u', \ldots, u^{2 j_* - i + 1})\|_0
\]

\[
\leq c \sum_{i=1}^{2 j_*} \||u||_{p-1,\mu} \|u'\|_{L^2(R)} \cdots \|u^{2 j_* - i + 1}\|_{L^2([0,\mu])} B_{2 j_*}(u, \ldots, u^{2 j_* - i + 1})\|_0
\]

\[
\leq c \sum_{i=1}^{2 j_*} \||u||_{p-1,\mu} \|u^{2 j_* - i + 1}\|_0 \left(\|u\|_{0,\mu} \|u^{2 j_* - i + 1}\|_0 \right)^j \cdots \left(\|u\|_{0,\mu} \|u^{2 j_* - i + 1}\|_0 \right)^{j_\mu - i + 1}\|u^{2 j_* - i + 1}\|_0
\]

where \( B_{j_\mu} \) denote the Bell polynomials,

\[
J_i = \{(j_1, \ldots, j_{2 j_* - i + 1}) : j_1 + \ldots + j_{2 j_* - i + 1} = i, \quad j_1 + 2 j_2 + \ldots + (2 j_* - i + 1) j_{2 j_* - i + 1} = 2 j_* \}.
\]
and the generalized Hölder and Gagliardo–Nirenberg inequalities have been used (see [13, theorem 8.8] and [10, theorem 9.3]).

It follows that

$$
\int_{\mathbb{R}} |u|^{2(j^\ast)} \, dx \leq c \|u\|_\infty^{2(p-1)} \leq c \mu \|u\|_\infty^{2(p-1)}
$$

$$
\leq c (\mu^{1+(p-1)\tau\beta} \|u\|_{\tau,\mu}^{2(p-1)}) \leq c \left( \mu^{1+(p-1)\tau\beta} \left( \frac{\|u\|_{\tau,\mu}}{\mu^\tau} \right)^{2(p-1)} \right),
$$

(45)

and multiplying this estimate by $\mu^{-4j^\ast\tau\beta}$ and adding $\int_{\mathbb{R}} u^2 \, dx = 2\mu$ yields

$$
\|u\|^2_{\tau,\mu} \leq c \mu \left( 1 + \mu^{(1-\tau)(p-1)} \left( \frac{\|u\|^2_{\tau,\mu}}{\mu} \right)^{p-1} \right).
$$

The stated estimate is obtained from this inequality using the argument given at the end of the proof of theorem 4.4.

Convergence to long waves. In this section we work under the additional regularity condition (A4) and examine the relationship between $D_\mu$ and $D_{lw}$, beginning with that between the quantities

$$
I^\mu := \inf \left\{ \mathcal{E}(u) : u \in U^\mu \right\}
$$

and

$$
I_{lw} := \inf \{ \mathcal{E}_{lw}(w) : w \in W_1 \}.
$$

Lemma 6.2.

(i) The quantity $I^\mu$ satisfies

$$
I^\mu = -m(0)\mu + \mathcal{E}_{lw}(u) + o(\mu^{1+(p-1)\alpha})
$$

uniformly over $u \in D^\mu$.

(ii) The quantities $I^\mu$ and $I_{lw}$ satisfy

$$
I^\mu = -m(0)\mu + \mu^{1+(p-1)\alpha} I_{lw} + o(\mu^{1+(p-1)\alpha}).
$$

Proof.

(i) Using the identity

$$
\mathcal{E}(u) = -m(0)\mu + \mathcal{E}_{lw}(u) - \frac{1}{2} \int_{\mathbb{R}} r(k)|\hat{u}|^2 \, dk - \int_{\mathbb{R}} N_t(u) \, dx
$$

for $u \in U^\mu \cap H^\nu(\mathbb{R})$, we find that

$$
I^\mu = \mathcal{E}(u) = -m(0)\mu + \mathcal{E}_{lw}(u) - \frac{1}{2} \int_{\mathbb{R}} r(k)|\hat{u}|^2 \, dk - \int_{\mathbb{R}} N_t(u) \, dx.
$$

for each $u \in D^\mu$, where

$$
\left| \frac{1}{2} \int_{\mathbb{R}} r(k)|\hat{u}|^2 \, dk + \int_{\mathbb{R}} N_t(u) \, dx \right| \leq c \left( \int_{\mathbb{R}} k^{2j^\ast+2}|\hat{u}|^2 \, dk + \|u\|_p^{p+\delta-1} \right)
$$

$$
\leq c \left( \mu^{2j^\ast+2+\tau\beta} \|u\|_{\tau,\mu}^2 + \mu^{1+(p-1)(\tau\beta+1)} \right)
$$

$$
\leq c \tau \left( \mu^{1+(p+1)(\tau\beta+1)} \right) = o(\mu^{1+(p-1)\alpha})
$$

uniformly over $u \in D^\mu$. 

(ii) Choosing \( u \in D_\mu \) and applying (i), one finds that
\[
I_\mu = -m(0)\mu + \mathcal{E}_{\mu}(u) + o(\mu^{1+\sigma(1-\alpha)})
\]
\[
= -m(0)\mu + \mu^{1+\sigma(1-\alpha)}\mathcal{E}_{\mu}(\hat{u}) + o(\mu^{1+\sigma(1-\alpha)})
\]
\[
\geq -m(0)\mu + \mu^{1+\sigma(1-\alpha)}I_{\mu} + o(\mu^{1+\sigma(1-\alpha)}).
\]

On the other hand, choosing \( w \in D_{\mu} \) and applying lemma 3.2, one finds that
\[
I_\mu \leq \mathcal{E}(S_{\mu}w)
\]
\[
= -m(0)\mu + \mu^{1+\sigma(1-\alpha)}\mathcal{E}_{\mu}(w) + o(\mu^{1+\sigma(1-\alpha)})
\]
\[
= -m(0)\mu + \mu^{1+\sigma(1-\alpha)}I_{\mu} + o(\mu^{1+\sigma(1-\alpha)}).
\]

Our main result shows how a scaling of \( D_\mu \) converges to \( D_{\mu} \) as \( \mu \searrow 0 \).

**Theorem 6.3.** The sets \( D_\mu \) and \( D_{\mu} \) satisfy
\[
\sup_{u \in D_\mu} \text{dist}_{H^s(\mathbb{R})}(S_{\mu}^{-1}u, D_{\mu}) \to 0
\]
as \( \mu \searrow 0 \).

**Proof.** Assume that the result is false. There exist \( \varepsilon > 0 \) and sequences \( \{\mu_n\}_{n \in \mathbb{N}_0} \subset (0, \mu_*) \), \( \{u_n\}_{n \in \mathbb{N}_0} \subset H^{2s}(\mathbb{R}) \) with \( u_n \in D_{\mu_n} \) such that \( \lim_{n \to \infty} \mu_n = 0 \) and
\[
\inf_{w \in D_{\mu_n}} \|w_n - w\|_\mu \geq \varepsilon,
\]
where \( w_n(x) := \mu_n^{-\beta}u_n(\mu_n^{-\beta}x) \). Using lemma 6.2(i), one finds that
\[
I_{\mu_n} = -m(0)\mu_n + \mathcal{E}_{\mu_n}(u_n) + o(\mu_n^{1+\sigma(1-\alpha)})
\]
\[
= -m(0)\mu_n + \mu_n^{1+\sigma(1-\alpha)}\mathcal{E}_{\mu_n}(w_n) + o(\mu_n^{1+\sigma(1-\alpha)})
\]
as \( n \to \infty \), and because
\[
I_{\mu_n} = -m(0)\mu_n + \mu_n^{1+\sigma(1-\alpha)}I_{\mu_n} + o(\mu_n^{1+\sigma(1-\alpha)})
\]
(lemma 6.2(ii)), it follows that
\[
\mathcal{E}_{\mu_n}(w_n) = I_{\mu_n} + o(1)
\]
as \( n \to \infty \), so that \( \{w_n\}_{n \in \mathbb{N}_0} \) is a minimizing sequence for \( \mathcal{E}_{\mu_n} \) over \( \{w \in H^{s}(\mathbb{R}) : \mathcal{Q}(w) = 1\} \). According to theorem 1.1 there exists a sequence \( \{x_n\}_{n \in \mathbb{N}_0} \) of real numbers with the property that a subsequence of \( \{w_n(\cdot + x_n)\}_{n \in \mathbb{N}_0} \) converges in \( H^{s}(\mathbb{R}) \) to an element of \( D_{\mu} \). This fact contradicts (46).

**Remark 6.4.** The previous theorem implies that \( \{S_{\mu}^{-1}u\}_{u \in D_\mu} \) is a bounded set in \( H^{s}(\mathbb{R}) \). For all \( u \in D_\mu \) we therefore find that
\[
\|u\|_{s,\infty}^2 \leq \frac{1}{2\pi} \left\| \hat{u} \right\|_{L^1(\mathbb{R})}^2 \leq \frac{1}{2\pi} \left( \int_{\mathbb{R}} \frac{1}{1 + \mu^{-2\beta}k^2} \, dk \right) \left( \int_{\mathbb{R}} (1 + \mu^{-2\beta}k^2) \left| \hat{u} \right|^2 \, dk \right)
\]
\[
= \frac{1}{2\pi} \mu^{2\alpha} \left( \int_{\mathbb{R}} \frac{1}{1 + k^{2\alpha}} \, dk \right) \left( \int_{\mathbb{R}} (1 + k^{2\alpha}) \left| \mathcal{F}[S_{\mu}^{-1}u] \right|^2 \, dk \right)
\]
\[
\leq c \mu^{2\alpha},
\]
and inequality (45) implies that
\[
\mu^{-4\beta} \int_{\mathbb{R}} |u(t,2\beta)|^2 \, dx \leq c \mu^{1+2\alpha(1-\beta)} = c \mu,
\]
whence \( \|u\|_{1,\mu}^2 \leq c \mu \). For \( u \in D_\mu \) lemma 6.1 therefore also holds with \( \tau = 1 \) (the result predicted by the long-wave ansatz (8)), and in particular \( \{S_{\mu}^{-1}u\}_{u \in D_\mu} \) lies in \( W \) for sufficiently large values of \( S \).
Finally, we relate the wave speeds \( v(u) \) and \( v_{lw}(w) \) associated with, respectively, \( u \in D_\mu \) and \( w \in D_{lw} \).

**Lemma 6.5.** There exists a family \( \{w_u\}_{u \in D_\mu} \) of functions in \( D_{lw} \) such that

\[
v(u) = m(0) + \mu^{(p-1)\alpha} v_{lw}(w_u) + o(\mu^{(p-1)\alpha})
\]

uniformly over \( u \in D_\mu \).

**Proof.** Using the identity

\[
\langle E'(u), u \rangle_0 = -2m(0) Q(u) + \langle E_{lw}'(u), u \rangle_0 - \int_\mathbb{R} r(k) |\hat{u}|^2 \, dk - \int_\mathbb{R} n_t(u) \, dx
\]

for \( u \in U \), we find that

\[
\langle E'(u), u \rangle_0 = -2m(0) \mu + \mu^{1+(p-1)\alpha} \langle E_{lw}'(S_{lw}^{-1} u), S_{lw}^{-1} u \rangle_0
\]

\[
- \mu 2^{\alpha - \beta} \int_\mathbb{R} r(\mu^\beta k) [F(S_{lw}^{-1} u)]^2 \, dk - \mu^{2^{\alpha - \beta}} \int_\mathbb{R} S_{lw}^{-1} u n_t(\mu^\beta S_{lw}^{-1} u) \, dx
\]

\[
= -2m(0) \mu + \mu^{1+(p-1)\alpha} \langle E_{lw}'(S_{lw}^{-1} u), S_{lw}^{-1} u \rangle_0 + o(\mu^{1+(p-1)\alpha})
\]

(47)

uniformly over \( u \in D_\mu \), where the second line follows from the observation that

\[
\left| \mu 2^{\alpha - \beta} \int_\mathbb{R} r(\mu^\beta k) |\hat{u}|^2 \, dk + \mu^{2^{\alpha - \beta}} \int_\mathbb{R} n_t(\mu^\beta w) \, dx \right|
\]

\[
\leq c \left( \mu 2^{\alpha + (2j^\star) - \beta} \int_\mathbb{R} k^{2j^\star + 2} |\hat{u}|^2 \, dk + \mu^{(p+1)\alpha - \beta} \int_\mathbb{R} |w|^{p+1} \, dx \right)
\]

\[
= o(\mu^{1+(p-1)\alpha})
\]

uniformly over \( w \in W \).

Theorem 6.3 asserts in particular the existence of \( w_u \in D_{lw} \) such that

\[
\|S_{lw}^{-1} u - w_u\|_{j^\star} = o(1)
\]

and therefore

\[
\langle E_{lw}'(S_{lw}^{-1} u), S_{lw}^{-1} u \rangle_0 - \langle E_{lw}'(w_u), w_u \rangle_0 = \sup_{w \in W} \|G'(w)\|_0 \|S_{lw}^{-1} u - w_u\|_0 = o(1)
\]

(48)

uniformly over \( u \in D_\mu \), where

\[
G(w) = \langle E_{lw}'(w), w \rangle_0 = -\int_\mathbb{R} \left\{ \frac{m(2j^\star)(0)}{(2j^\star)! (w^{(j^\star)})^2 + (p + 1)N_{p+1}(w)} \right\} \, dx.
\]

Furthermore, it follows from the equations

\[
E'(u) + v(u) Q'(u) = 0, \quad E_{lw}'(w_u) + v_{lw}(w_u) Q_{lw}'(w_u) = 0
\]

that

\[
2v(u) \mu = -\langle E'(u), u \rangle_0, \quad 2v_{lw}(w_u) = -(E_{lw}'(w_u), w_u)_0
\]

(49)

for each \( u \in U_\mu \). Combining (47)–(49), one finds that

\[
v(u) = m(0) + \mu^{(p-1)\alpha} v_{lw}(w_u) + o(\mu^{(p-1)\alpha})
\]

uniformly over \( u \in U_\mu \). \[\square\]
Remark 6.6. For the Whitham equation theorem 6.3 and lemma 6.5 yield the convergence results

\[
\sup_{u \in D_\mu} \inf_{y \in \mathbb{R}} \| \mu^{-\frac{1}{3}} u (\mu^{-\frac{1}{3}} (\cdot + y)) - u_{KdV} \|_1 \to 0
\]

and

\[
\sup_{u \in D_\mu} \left| v(u) - 1 - \mu^{\frac{1}{3}} \left( \frac{2}{3} \right)^{\frac{1}{3}} \right| = o(\mu^{\frac{1}{3}})
\]
as \( \mu \searrow 0 \), which show how Whitham solitary waves are approximated by a scaling of the classical Korteweg–deVries solitary wave.

Stability. In this section we explain how theorem 1.2(ii) implies that the set of solitary-wave solutions to (1) defined by \( D_\mu \) enjoys a certain type of stability, working with the following local well-posedness assumption. (Although consideration of the initial-value problem is outside the scope of this paper we note that a local well-posedness result in \( H^s(\mathbb{R}) \) for \( s > \frac{3}{2} \) may be obtained using Kato’s method [14]; see also [1].)

Well-posedness assumption. There exists a subset \( M \subset U \) with the following properties.

(i) The closure of \( M \setminus D_\mu \) in \( H^1(\mathbb{R}) \) has a non-empty intersection with \( D_\mu \).

(ii) For each initial datum \( u_0 \in M \) there exists a positive time \( T \) and a function \( u \in C([0, T], U) \) such that \( u(0) = u_0 \),

\[
E(u(t)) = E(u_0), \quad Q(u(t)) = Q(u_0)
\]

for all \( t \in [0, T] \) and

\[
\sup_{t \in [0, T]} \| u(t) \|_1 < R.
\]

Theorem 6.7 (Conditional energetic stability). Choose \( s \in [0, 1) \). For each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[
\text{dist}_{H^s(\mathbb{R})}(u(t), D_\mu) < \varepsilon,
\]

for all \( t \in [0, T] \) whenever

\[
u \in M, \quad \text{dist}_{H^s(\mathbb{R})}(u_0, D_\mu) < \delta.
\]

Proof. Assume that the result is false. There exist \( \varepsilon > 0 \) and sequences \( \{u_{0,n}\}_{n \in \mathbb{N}} \subset M, \{T_n\}_{n \in \mathbb{N}} \subset (0, \infty), \{t_n\}_{n \in \mathbb{N}} \subset [0, T_n] \) and \( \{u_n\}_{n \in \mathbb{N}} \subset C([0, T_n], U) \) such that \( u_n(0) = u_{0,n} \),

\[
E(u_n(t)) = E(u_{0,n}), \quad Q(u_n(t)) = Q(u_{0,n}), \quad t \in [0, T_n]
\]

and

\[
\text{dist}_{H^s(\mathbb{R})}(u_n(t_n), D_\mu) \geq \varepsilon, \quad \text{dist}_{H^s(\mathbb{R})}(u_{0,n}, D_\mu) < \frac{1}{n}. \quad (50)
\]

According to the last inequality there is a sequence \( \{\tilde{u}_n\}_{n \in \mathbb{N}} \subset D_\mu \) such that

\[
\lim_{n \to \infty} \| u_{0,n} - \tilde{u}_n \|_1 = 0. \quad (51)
\]

The sequence \( \{\tilde{u}_n\}_{n \in \mathbb{N}} \) is clearly a minimizing sequence for \( E \) over \( U_\mu \) with \( \sup_{n \in \mathbb{N}} \| \tilde{u}_n \|_1 < R \). It follows from theorem 1.2(ii) that there is a sequence \( \{x_n\}_{n \in \mathbb{N}} \subset \mathbb{R} \) with
the property that (a subsequence of) \( \{\bar{u}_n(\cdot + x_n)\} \) converges in \( H^s(\mathbb{R}) \) to a function \( \bar{u} \in D_\mu \). Equation (51) shows that the same is true of \( \{u_{0,n}(\cdot + x_n)\} \), and using proposition 2.2 we find that
\[ E(u_{0,n}) \to E(\bar{u}), \quad \mu_n := Q(u_{0,n}) \to Q(\bar{u}) = \mu \]
as \( n \to \infty \). Defining \( v_n := (\mu/\mu_n)^2 u_n(t_n) \), observe that
\[ Q(v_n) = \frac{\mu}{\mu_n} Q(u_n(t_n)) = \frac{\mu}{\mu_n} Q(u_{0,n}) = \mu \]
and
\[ E(v_n) \to E(u_{0,n}) = E(v_n) - E(u_n(t_n)) \leq \sup_{u \in U} \|E'(u)\|_0 \|v_n - u_n(t_n)\|_0 \]
\[ = \sqrt{2} \sup_{u \in U} \|E(u)\|_0 |\mu - \mu_n| \]
\[ \to 0 \]
as \( n \to \infty \), so that \( \{v_n\}_{n \in \mathbb{N}_0} \) is also a minimizing sequence for \( E \) over \( U_\mu \) with \( \sup_{n \in \mathbb{N}_0} \|v_n\|_1 < R \). Theorem 1.2 (ii) implies that (a subsequence of) \( \{v_n\}_{n \in \mathbb{N}_0} \) satisfies
\[ \text{dist}_{H^s(\mathbb{R})}(v_n, D_\mu) \to 0 \]
as \( n \to \infty \), and since
\[ \|v_n - u_n(t_n)\|_2^2 = \left( \frac{\mu}{\mu_n} - 1 \right) \|u_n(t_n)\|_2^2 \leq R^2 \left( \frac{\mu}{\mu_n} - 1 \right) \to 0 \]
as \( n \to \infty \), we conclude that \( \text{dist}_{H^s(\mathbb{R})}(u_n(t_n), D_\mu) \to 0 \) as \( n \to \infty \). This fact contradicts (50).

Appendix

Here we present a short argument demonstrating that \( E_{lw} \) is bounded below over \( W_1 \).

Using the Gagliardo–Nirenberg and Young inequalities, we find that
\[ \left| \int_{\mathbb{R}} N_{p+1}(w) \ dx \right| \leq c \|w\|^{p+1}_{L^{p+1}(\mathbb{R})} \]
\[ \leq c \|w\|^{(1-\theta)(p+1)}_0 \|w\|^{\theta(p+1)}_{J_{\theta}} \]
\[ \leq c \|w\|^{\frac{p+1}{j_{\theta}}}_{J_{\theta}} \]
\[ \leq c_\varepsilon + c\varepsilon \|w\|^{2}_{J_{\theta}}, \]
where \( \varepsilon \) is a small positive number and
\[ \theta = \frac{p - 1}{2j_{\theta}(p + 1)} \]
(note that \( (p - 1)/2j_{\theta} < 2 \) by assumption (A3)). It follows that
\[ E_{lw}(w) = E_{lw}(w) + Q(w) - Q(w) \]
\[ \geq c \|w\|^2_{J_{\theta}} - \left| \int_{\mathbb{R}} N_{p+1}(w) \ dx \right| - Q(w) \]
\[ \geq c \|w\|^2_{J_{\theta}} - c_\varepsilon \]
\[ \geq -c_\varepsilon \]
for sufficiently small values of \( \varepsilon \).
Acknowledgments

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