Existence of multiple cycles in a van der Pol system with hysteresis in the inductance

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Abstract. The van der Pol oscillator has interesting properties central to the study of electrical circuits and nonlinear dynamics. This paper demonstrates the effect of modelling the van der Pol oscillator with hysteresis in the inductance element, a feature often omitted, using the derivative of a Preisach nonlinearity. The system is then solved using a recently developed algorithm and it is shown that multiple limit cycles are present.

1. Introduction
Many integrated circuits contain resistors that act as normal resistors for high currents but offer negative resistance at lower currents. This characteristic was first observed in the behaviour of the van der Pol oscillator where vacuum tubes caused non-linear resistance to occur. In such systems negative resistance causes small current oscillations to increase while large current oscillations decrease due to the normal resistance. This is a famous Hopf bifurcation phenomenon [1] in which the oscillations tend towards a stable limit cycle. Fig. 1 shows a triode oscillating circuit that demonstrates this effect. Given \( J_a = S(x) \), where \( x \) represents the grid voltage, Kirchoff’s laws may be applied to the circuit to give

\[
x'' + \frac{R}{L} (1 - \frac{M}{RC} \frac{dS}{dx}) x' + \frac{1}{LC} x = 0 \quad (1)
\]

where \( x' \) is the first derivative of \( x \) with respect to time. It is commonly assumed that \( S(x) = \sigma x - \sigma_3 \frac{x^3}{3} \), where \( \sigma \) and \( \sigma_3 \) are normalizing coefficients. Substituting into eqn. (1) above results in an equation of the form

\[
x'' + \lambda (1 - \frac{d}{\lambda} \frac{x^2}{3}) x' + \omega_0^2 x = 0 \quad (2)
\]

where

\[
\lambda = \frac{R}{L} \frac{M \sigma}{LC}
\]

\[
d = \frac{M \sigma_3}{LC}
\]

and

\[
\omega_0^2 = \frac{1}{LC}
\]
This is the widely-studied classical van der Pol equation, first proposed by Balthaser van der Pol in the 1920's [3] with many later studies of its dynamical behaviour and the effect of an external forcing term including [4] and [5]. Taking $\lambda$ as the control parameter (due to a variation in $R$, for example) results in a Hopf bifurcation at $\lambda = 0$. For $\lambda > 0$ the system has one stable limit cycle of non-zero amplitude and an unstable fixed point at the origin.

It is proposed that a more accurate model of the van der Pol circuit can be achieved by the inclusion of hysteresis effects in the inductance element, $L$. In particular, a hysteretic relationship between the magnetic induction and the magnetic field is known to exist. Thus, modifying eqn. (2) above we get the following system

\begin{align}
y' &= -x + \lambda y - dx^2y \\ y &= \xi x' + v(P(x))'
\end{align}

where $(P(x))'$ is the derivative of the Preisach non-linearity (described in section 2 and [7]), the hystereron to be used in this investigation. If we let $v = 0$ then we have the same system as eqn. (2) with $\xi = LC$ and $\lambda = -\lambda$.

The relationship between system (3), (4) and the classical van der Pol system (eqn. (2)) is similar to that between the oscillation of the ferromagnetic pendulum and the standard pendulum. The problem of the ferromagnetic pendulum is studied in detail in Bertotti and Mayergoyz’s comprehensive 3-volumed study of hysteresis [2]. Here it is shown that the dynamical behaviour of the ferromagnetic pendulum is extremely rich. In particular, it is noted that the system contains multiple limit cycles. The similarities between system (3), (4) and the ferromagnetic pendulum suggest that such behaviour may also be found in the van der Pol oscillator with hysteresis. The paper is mainly devoted to experimental verification of this suggestion.
Section 2 of this paper will describe the mathematical and numerical model of the van der Pol system with hysteresis. Advancements in the study of hysteresis have resulted in an algorithm that can efficiently solve the system with hysteresis. This algorithm can be easily implemented in widely-used software such as MATLAB, thereby making the study of hysteretic behaviour accessible to many.

Section 3 will then offer results on the dynamical behaviour of the system. In particular, it will be shown that the system with hysteresis differs significantly from that without due to the presence of multiple limit cycles. It is concluded that the introduction of a hysteresis operator causes multiple limit cycles to appear in the van der Pol oscillator. Moreover, several Hopf bifurcations exist in the system with hysteresis. For some point $\lambda_1 > 0$, an additional Hopf bifurcation point results in a second stable limit cycle appearing, giving a region with multiple limit cycles. The system then returns to one stable limit cycle at the point $\lambda_2 > \lambda_1$.

2. Mathematical and Numerical model

Eqn. (3) and eqn. (4) above summarise the mathematical model of the system. It is a second order differential equation with hysteresis.

2.1. Preisach model of hysteresis

The non-ideal relay forms the basis of the Preisach model. This is a simple hysteretic transducer, the definition of which can be found in [6]. The non-ideal relay, $R_{\alpha,\beta}$, expresses the idea that, as $x$ increases, the system will not turn on until a threshold, $\beta$, is reached, while as $x$ decreases, the system will turn off only when a different threshold, $\alpha$, is reached where $\alpha < \beta$.

Say we now take a bundle of such relays where $\alpha$ is the minimum value of all thresholds and $\beta$ the maximum. We can now have multiple states for the transducer. The transducer is fully on for $x > \beta$ and fully off for $x < \alpha$. Each change of state corresponds to some relay, $R_{\alpha,\beta}$, being turned on or off. Placing an infinite number of relays in the range $[\alpha, \beta]$ results in a transducer with continually changing state. This gives us the Preisach nonlinearity (as defined in [7])

$$P(t) = \int \int_{\alpha \leq \beta} \mu(\alpha, \beta) \gamma_{\alpha,\beta} x(t) d\alpha d\beta$$

(5)

where $x(t)$ is the input to the system; $\mu(\alpha, \beta)$ is a weighting function which allows different relays or bundles of relays to have a greater or lesser effect and $\gamma_{\alpha,\beta}$ is a term containing the "local memory" of the system. $\gamma_{\alpha,\beta}$ is the set of local extrema of $x(t)$. Geometrically, $P(t)$ is represented by the area enclosed by the lines $x = \alpha, y = \beta, x = y$ and the piecewise linear curve, $L$. $L$ has a total of $\omega$ links parallel to the axes with the coordinates of the end points of links contained in the set $\eta$. The alternating horizontal and vertical links correspond to maximum and minimum local extrema of $x$, respectively. An algorithmic expression of the Preisach nonlinearity may be found in [6].

When including the Preisach nonlinearity in the van der Pol oscillator we are interested in the derivative, $(P(x))'$. Furthermore, a uniform weight function, $\mu(\alpha, \beta) = 1$, is applied, giving the following expression for the Preisach nonlinearity

$$(P(x))' = (\eta_\omega - \eta_{\omega-1})x'$$

(6)

where $\eta_\omega - \eta_{\omega-1}$ indicates the length of the last link in $\eta(\omega)$.

2.2. Numerical Model

The algorithm used for integration of eqn. (3) and eqn. (4) is based on that devised by Flynn and Rasskazov, details of which can be found on pages 53-55 of [6]. However, due to the presence
of the \( \xi \) term in eqn. (4), the algorithm can be modified. Rearranging eqn. (4) we get

\[
x' = \frac{y}{\xi + v(\eta_{w} - \eta_{w-1})}
\]  

(7)

The Flynn and Rasskazov algorithm uses a linear integration step for eqn. (7) at any time \( t \) where \( \eta_{w} - \eta_{w-1} = 0 \). Such a step is only necessary for situations in which \( \xi = 0 \). For all \( \xi \neq 0 \), the usual higher order integration step can be carried out. Therefore, when applying Flynn and Rasskazov’s algorithm to the van der Pol oscillator the "LinearStep" function may be omitted.

We can express the system of equations to be integrated in the vector form

\[
\frac{dx}{dt} = F(x)
\]  

(8)

Let \( \nu_{n}(x_{0}) \) denote the \( n \)-th point on the numerical trajectory obtained by numerical integration. Then, given the initial conditions \( x_{0}, \eta_{0} \) and \( \nu_{0}(x_{0}) = x_{0} \), the point \( \nu_{n+1}(x_{0}) \) can be found using the classical Runge-Kutta method with a time step \( h \)

\[
\nu_{n+1}(x_{0}) = \nu_{n}(x_{0}) + \frac{h}{6}(k_{1}(\nu_{n}(x_{0}), h) + 2k_{2}(\nu_{n}(x_{0}), h) + 2k_{3}(\nu_{n}(x_{0}), h) + k_{4}(\nu_{n}(x_{0}), h)) + \omega_{n}
\]  

(9)

where

\[
\begin{align*}
k_{1}(x, \tau) &= F(x) \\
k_{2}(x, \tau) &= F(x + \frac{\tau}{2}k_{1}(x, \tau)) \\
k_{3}(x, \tau) &= F(x + \frac{\tau}{2}k_{2}(x, \tau)) \\
k_{4}(x, \tau) &= F(x + \frac{\tau}{2}k_{3}(x, \tau))
\end{align*}
\]

and \( \omega_{n} \) is the numerical error of the method. This is a 4th order integration method. It is important to note that, although \( L = \eta_{w} - \eta_{w-1} \) is a function of \( x \), \( L \) remains constant throughout each time step. Using an adaptive \( L \) (i.e. \( L_{2} = f(x + \frac{k_{1}}{2}), L_{3} = f(x + \frac{k_{2}}{2}), L_{4} = f(x + \frac{k_{4}}{2}) \)) can cause non-convergence of the solution at \( y \approx 0 \).

2.3. Execution of the Numerical Model

The numerical model was executed using MATLAB version 7.0.1. The values assigned to the parameters are given in table 1.

The relatively large absolute values of \( \alpha \) and \( \beta \) are chosen to ensure that the Preisach nonlinearity is always present in the system. We wish to avoid the situation of \( x < \alpha \) or \( x > \beta \) as both result in \( (P(x))' = 0 \). While such cases are physically feasible we are concerned here with the more interesting case of a continuously changing Preisach nonlinearity.

The values \( \lambda \) and \( \xi \) are treated as the control parameters of the system and so are varied across the range given above. Given \( \lambda = \frac{R}{L} - \frac{M\sigma}{LC} \) and \( \xi = \frac{LC}{M} \) it is reasonable that the physical circuit can be changed to reflect changes in the control parameters. In fact, the resistance, \( R \), capacitance, \( C \), self-inductance, \( L \) and mutual inductance, \( M \), may all be altered as long as the relationship \( \frac{M}{LC} \) remains constant. In this way, \( \lambda \) and \( \xi \) may be altered but not \( d \).

A time step, \( h = 0.001 \), is used for the numerical integration of the system, while the initial conditions required for integrating the system are \( x_{0}, y_{0} \) and \( \eta_{0} \).
Table 1. Parameter values assigned to van der Pol system

| α  | β  | λ   | d  | ξ   | υ   |
|----|----|-----|----|-----|-----|
| -100 | 100 | [0, 2] | 1  | (0.0, 1] | 1   |

3. Limit cycles
The behaviour of the system is largely characterized by the number, stability and amplitude of limit cycles. Thus, the numerical model discussed in Section 2 is used to search for such cycles in the van der Pol system. Because the trajectory of the system is non-circular and not centered at the origin (see fig. 2) but oscillates about the origin the following convention is adopted as a measure of the amplitude: distance from the origin to the point where the trajectory crosses the positive x-axis.

Figure 2. Trajectory of the system with $\lambda = 0.2$, $\xi = 0.03$ and the initial conditions $x_0 = 0.001$, $y_0 = 0$, $\eta_0 = [0.1, 0.001]$. The vertical scale is an order of magnitude less than the horizontal scale. Therefore, the trajectory is roughly elliptical in shape with a major axis much greater than the minor axis. Note also how the trajectory intersects itself at two points during the first oscillation.

3.1. Searching for Limit Cycles
Identifying and characterising limit cycles for a given $\lambda$ and $\xi$ is achieved by carrying out a two-step process:

Step 1. The range $[0, x_n]$ along the x-axis is discretised to a medium resolution, typically $\Delta x = 0.01$. Each node in this set is then selected in turn as the initial voltage of the system
The system is then integrated for five full oscillations with the amplitude of each oscillation, $\psi$, being measured. The fourth and fifth amplitudes are compared and a value assigned

$$\theta = \begin{cases} 
1 & \text{if } \psi_5 \geq \psi_4 \\
-1 & \text{if } \psi_5 < \psi_4
\end{cases}$$

(10)

The case $\theta = 1$ indicates a trajectory with increasing amplitude while $\theta = -1$ indicates a trajectory with decreasing amplitude. In this way, each of the grid points, $x_n$, has a corresponding $\theta_n$ and a corresponding amplitude $\psi_{5n}$. We can let $x_n = \psi_{5n}$. If we have $\theta_{p-1} = 1$ and $\theta_p = -1$, we are in the region of a stable limit cycle since $\psi_{p-1} < \psi_p$. Conversely, say we have $\theta_{q-1} = -1$ and $\theta_q = 1$, we are in the region of an unstable limit cycle since $\psi_{q-1} < \psi_q$. Fig. 3 shows typical results of this step in which the two stable and one unstable limit cycles have been located to within an accuracy of 0.01.

![Figure 3](image_url)

**Figure 3.** Plot of $\theta$ versus $x$ for $\lambda = 0.7$ and $\xi = 0.05$. The grid resolution, $\Delta x = 0.01$. Where the curve jumps from +1 to -1 indicates the presence of a stable limit cycle while from -1 to +1 indicates an unstable limit cycle.

**Step 2.** Identifying the amplitude of a stable limit cycle given $x_{p-1}$ and $x_p$: The system is solved with $x_0 = x_{p-1}$. The change in amplitude, $\Delta \psi$, is measured for each oscillation

$$\Delta \psi = \psi_n - \psi_{n-1}$$

(11)

Integration continues until $\Delta \psi$ becomes less than a specified tolerance, in our case $\Delta \psi < 10^{-5}$. Similarly, the system is solved with $x_0 = x_p$ until $\Delta \psi < 10^{-5}$. This gives two approximations
for the stable limit cycle, a lower and an upper bound. Assuming the upper and lower bounds are sufficiently close, the final approximation for the location of the stable limit cycle can be found by averaging the lower and upper bounds.

Identifying the amplitude of an unstable limit cycle given $x_{q-1}$ and $x_q$: It has been confirmed in step 1 that $x_{q-1}$ corresponds to a region where oscillations have decreasing amplitude while $x_q$ corresponds to a region of increasing amplitude. Therefore, a bisection method is used whereby the midpoint, $x_{q-\frac{1}{2}}$, of $[x_{q-1}, x_q]$ is found and the direction of its amplitude, $\theta$, established as in step 1. If $\theta = 1$, indicating an increasing amplitude, then the interval of interest becomes $[x_{q-1}, x_{q-\frac{1}{2}}]$. Otherwise, the amplitude is decreasing and the interval of interest is $[x_{q-\frac{1}{2}}, x_q]$. In this way, the interval $[x_{q-1}, x_q]$ on which the unstable limit cycle lies can be reduced until its length is less than a specific tolerance.

This method of locating limit cycles is based on an important assumption that should be discussed. It assumes that, as the system is being solved, $\psi$ is monotonically increasing or decreasing. This allows us to predict the locations to which the trajectory is being attracted or from which it is being repelled. However, as is clear from fig. 2, this assumption is incorrect. It is possible for the trajectory of the system to intersect itself.

Observation has shown that intersection is limited to two particular times in the evolution of the trajectory. The first is shown in fig. 2. The first oscillation of the system may intersect later oscillations. Such intersections do not occur after the first oscillation is complete. It is proposed that the initial conditions of the system may give rise to such a trajectory. Secondly, intersection of the trajectory is observed to occur as the system approaches the stable limit cycle, where $\Delta \psi$ becomes extremely small. The trajectory tends to correct such intersections so that the location of the limit cycle is not altered. It is believed that this second type of intersection appears as a result of errors in the numerical integration.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{The classical van der Pol oscillator has 1 stable limit cycle. The plot shows the amplitude of this limit cycle for $\lambda \in (0, 2.02]$ and $\xi = 0.1$.}
\end{figure}
Figure 5. The amplitude of the limit cycles as a function of $\lambda$ for $\xi = 0.005$.

Figure 6. The amplitude of the limit cycles as a function of $\lambda$ for $\xi = 0.05$. 

Figure 7. The range of $\lambda$ with multiple limit cycles for a given $\xi$. Only 1 stable limit cycle exists for any point below the 1st bifurcation line or above the 2nd bifurcation line. $\xi$ is on the x axis with $\lambda$ on the y axis.

Empirically, it has been shown that $\psi$ monotonically increases (or decreases) towards stable limit cycles and away from unstable limit cycles for all oscillations apart from the first oscillation and oscillations for which $\Lambda\psi$ becomes extremely small. Thus, using a minimum of five oscillations before measuring direction and ensuring that tolerance values for $\Delta\psi$ are greater than those values at which the second type of intersection occur appears accurate for measuring the direction of limit cycles.

3.2. Limit Cycles of the Classical van der Pol System
Say we let $\upsilon = 0$ in eqn. (4). This gives the classical van der Pol system with no hysteresis. Taking $\lambda$ as the control parameter and $\xi = 0$ produces the results shown in fig. 4. There is one stable limit cycle in the classical van der Pol system. This remains true for variations in all other parameters of the system.

3.3. Limit Cycles of the van der Pol System with Hysteresis
Including hysteresis in our model, i.e. by letting $\upsilon = 1$, causes the behaviour of the system to differ from the classical van der Pol oscillator. Similar to before, we let $\xi = a$ constant and identify the magnitude of the limit cycles over a range of $\lambda$. The resulting situation is shown in fig. 5 and fig. 6 for $\xi = 0.005$ and 0.05, respectively. In both cases two stable and one unstable limit cycles are identified. As $\lambda$ is increased from zero, there is initially one stable limit cycle of relatively small amplitude. When the first bifurcation point is reached a second stable limit cycle of relatively large amplitude appears, as well as an unstable limit cycle separating the two. When the second bifurcation point is reached the first stable limit cycle and the unstable limit cycle disappear. For all values of $\lambda$ greater than the second bifurcation point the system has
Figure 8. Trajectory approaching the first stable limit cycle for $\lambda = 0.7$ and $\xi = 0.05$.

Figure 9. Trajectory approaching the second stable limit cycle for $\lambda = 0.7$ and $\xi = 0.05$.

one stable limit cycle of ever increasing amplitude.

Comparing fig. 5 and fig. 6 it will be noted that for greater values of $\xi$ the value, $\lambda$, of the both the first and second bifurcation point has decreased. The magnitude of the decrease of the
second bifurcation point is much larger than that of the first. In effect, as $\xi$ increases the region in which we have multiple limit cycles is decreasing. This behaviour is verified in fig. 7 in which the range of $\lambda$ containing multiple limit cycles for a given $\xi$ is plotted. For $\xi = 0.005$, the range of $\lambda$ with multiple limit cycles is 1.42, while for $\xi = 0.1$ the range has been reduced to 0.02. As $\xi$ increases the relative effect of the Preisach nonlinearity in eqn. (4) decreases. The system is tending towards the classical van der Pol system with one stable limit cycle.

Clearly, there is a large difference in the amplitude of the first and second stable limit cycle. Further differences may also be observed, as can be seen from fig. 8 and fig. 9. The typical trajectory of the system near the first stable limit cycle is shown in fig. 8. It is elliptical in shape with a major axis in the $x$ direction and minor axis in the $y$ direction. However, the trajectory of the second stable limit cycle, as shown in fig. 9, is quite different. In this case the amplitude measured along the $y$ axis is almost equivalent to that measured along the $x$ axis, while the point of maximum amplitude of the trajectory lies at approximately $60^\circ$ to the $x$ axis. Thus, we can say that the rate of increase of the amplitude of the stable limit cycle is greatest in the direction at $60^\circ$ to the $x$ axis and lowest in the direction at $-30^\circ$ to the $x$ axis.

4. Conclusion

This paper has demonstrated the feasibility of modelling the van der Pol oscillator system with hysteresis which can then be solved using an algorithm based on standard higher order numerical integration methods. Investigations using this algorithm have shown that the classical van der Pol system is characterised by one stable limit cycle while the system with hysteresis has multiple limit cycles. In particular, it has been shown that a system with two stable and one unstable limit cycles is common for a range of resistance values of the circuit.

References

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