MODULI OF REFLEXIVE SHEAVES ON SMOOTH PROJECTIVE 3-FOLDS

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Abstract. We compute the expected dimension of the moduli space of torsion-free rank 2 sheaves at a point corresponding to a stable reflexive sheaf and give conditions for the existence of a perfect tangent-obstruction complex on a class of smooth projective threefolds; this class includes Fano and Calabi-Yau threefolds. We also explore both local and global relationships between moduli spaces of reflexive rank 2 sheaves and the Hilbert scheme of curves.

1. Introduction and Preliminaries

We work over an algebraically closed field of characteristic 0.

In this paper, motivated by Qin’s [26], [27] and Qin and Li’s [20] work on the relationship between the moduli of vector bundles on surfaces and Hilbert schemes of points, by Hartshorne’s work [7],[8],[9],[10] on curves in $\mathbb{P}^3$ via the Serre Correspondence, and by the theory of virtual fundamental classes and Donaldson-Thomas invariants [28], we study the moduli of reflexive rank 2 sheaves on smooth projective threefolds. Some very precise statements concerning the structure of these moduli spaces on Fano threefolds (especially smooth hypersurfaces) can be found in [1], [2], [4], [5], [12], [13], [14], [15], [16], [23], [29].

Recall that a coherent sheaf $F$ is torsion-free if the natural map of $F$ to its double-dual $h : F \to F^{**}$ is injective, and that $F$ is reflexive if $h$ is an isomorphism. We refer the reader to [8] for basic properties of reflexive sheaves. Recall the following Serre Correspondence for reflexive sheaves:

Theorem 1. [8, 4.1] Let $X$ be a smooth projective threefold, $M$ an invertible sheaf with $H^1(X, M^*) = H^2(X, M^*) = 0$. There is a one-to-one correspondence between

1. pairs $(F, s)$ where $F$ is a rank 2 reflexive sheaf on $X$ with $\det F = M$ and $s \in \Gamma(F)$ is a section whose set has codimension 2

2. pairs $(Y, \xi)$ where $Y$ is a closed Cohen-Macaulay curve in $X$, generically a local complete intersection, and $\xi \in \Gamma(Y, \omega_Y \otimes \omega^*_X \otimes M^*)$ is
a section which generates the sheaf $\omega_Y \otimes \omega_X^* \otimes M^*$ except at finitely many points.

Furthermore, $c_3(F) = 2p_a(Y) - 2 + c_1(X)c_2(F) - c_1(F)c_2(F)$.

Note that if $F$ is locally free, then the corresponding curve $Y$ is a local complete intersection, $\omega_Y \otimes \omega_X^* \otimes M^* \cong \mathcal{O}_Y$, $\xi$ is a non-zero section, and $c_3(F) = 0$. In this case we say $Y$ is subcanonical.

Note that any smooth curve in $X$ can be made subcanonical by blowing up:

**Proposition 2.** Let $X$ be a smooth projective variety of dimension $n$, $C \subset X$ a smooth curve. Then there exists a finite set $D \subset C$ and a line bundle $L \in \text{Pic}(X)$ such that if $\pi : \tilde{X} = \text{Bl}_D(X) \to X$ then

1. $\omega_{\tilde{C}} \otimes \pi^* L^* \otimes \omega_X^* = \mathcal{O}_{\tilde{C}}$, and so the proper transform of $C$ is subcanonical.
2. $H^i(\tilde{X}, \pi^* L^*) = 0$ for $i \geq 1$.

In particular, if $n = 3$ there is a rank two locally free sheaf $F$ on $\tilde{X}$ with $\wedge^2 F = \pi^* L$ and with a section $s \in H^0(\tilde{X}, F)$ whose zero scheme is $\tilde{C}$.

**Proof:** Let $L \in \text{Pic}(X)$ be such that

1. $H^i(X, L^*) = 0$ for $i \geq 1$.
2. $\omega_C \otimes L^* \otimes \omega_X^* = \mathcal{O}_C((n-1)D)$ where $D \subset C$ is a collection of distinct points.

Both conditions are satisfied by a sufficiently ample $L^*$.

Now, $H^i(\tilde{X}, \pi^* L^*) = 0$ for $i \geq 1$ follows immediately. Letting $\{E_i | i = 1, \ldots, |D|\}$ be the exceptional divisors we have

$$
\omega_{\tilde{C}} \otimes \pi^* L^* \otimes \omega_X^* = \omega_{\tilde{C}} \otimes \pi^* (L^* \otimes \omega_X^*) \otimes \mathcal{O}_{\tilde{C}}((1-n)E_i)
= \omega_C \otimes L^* \otimes \omega_X^* \otimes \mathcal{O}_C((1-n)D)
= \mathcal{O}_C
$$

Unfortunately from the point of view of moduli, in the case of a threefold the associated locally free sheaf $F$ on $\tilde{X}$ is in some sense less likely to be stable than the original reflexive sheaf on $X$ (though perhaps this could be repaired by considering the moduli of pairs). Thus we do not pursue this direction and instead in Section 2 we study general properties of the moduli spaces of reflexive sheaves on $X$. In particular, we calculate (Corollary 13) the expected dimension of the space in cases where the canonical bundle is at least somewhat non-positive. Further, we show that the space
of stable, rank 2 reflexive sheaves admits a perfect tangent-obstruction complex (Corollary 16) in a significant class of varieties not covered by Thomas’ results [28].

In Section 3, we study of the local relationship between the moduli space of reflexive sheaves, the moduli space of ideal sheaves, and the Hilbert scheme of curves (Theorem 19). Section 4 is concerned with the global relationship between these moduli spaces; we give some results (Proposition 23) in the case of vector bundles of rank 2.

We recall some basic terms and results:

Definition 3. Let $L$ be an ample line bundle on a smooth projective variety $X$. A torsion-free sheaf $F$ on $X$ is $L$-stable (resp. $L$-semistable) if for every coherent subsheaf $F'$ of $F$ with $0 < \text{rank } F' < \text{rank } F$, we have $\mu(F', L) < \mu(F, L)$ (resp. $\leq$), where for any coherent sheaf $G$ we define

$$\mu(G, L) = \frac{c_1(G) \cdot [L]}{\text{dim } X - \text{dim } X - 1} \cdot \frac{\text{dim } X}{\text{dim } X}$$

Note that if $\text{rank } F = 2$, it suffices to take $F'$ invertible. If $X \subset \mathbb{P}^n$ we simply say $F$ is (semi)stable to mean $F$ is $\mathcal{O}_X(1)$-(semi)stable. A coherent sheaf $F$ on a projective variety $X$ is called simple if $\dim \text{Hom}_{\mathcal{O}_X}(F, F) = 1$.

Recall ([11, 1.2.8]) that stable sheaves are simple.

Theorem 4. [6, A.5.3] Let $F$ be a coherent sheaf of rank $r$ on a smooth projective threefold $X$. The Riemann-Roch formula is

$$\chi(X, F) = \frac{1}{6} c_1^3(F) - \frac{1}{2} c_1(F) c_2(F) - \frac{1}{2} c_1(X) c_2(F) + \frac{1}{4} c_1(X)^2 c_1(F)$$

$$+ \frac{1}{12} c_2^3(X) c_1(F) + \frac{1}{12} c_2(X) c_1(F) + \frac{r}{24} c_1(X) c_2(X) + \frac{1}{2} c_3(F)$$

Note that if $F$ is a locally free sheaf of rank $r$, then

$$\chi(F) + \chi(F^*) = -c_1(X) c_2(F) + \frac{1}{2} c_1(X) c_1^2(F) + \frac{r}{12} c_1(X) c_2(X)$$

Proposition 5. [8, 2.5] Let $F$ be a reflexive sheaf on a normal projective threefold $X$, $G$ a sheaf of $\mathcal{O}_X$-modules. Then there are isomorphisms

$$H^0(X, \mathcal{E}xt^0_{\mathcal{O}_X}(F, G)) = \mathcal{E}xt^0_{\mathcal{O}_X}(F, G)$$

$$H^3(X, \mathcal{E}xt^0_{\mathcal{O}_X}(F, G)) = \mathcal{E}xt^3_{\mathcal{O}_X}(F, G)$$

and an exact sequence

$$0 \to H^1(X, \mathcal{E}xt^0_{\mathcal{O}_X}(F, G)) \to \mathcal{E}xt^1_{\mathcal{O}_X}(F, G) \to H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(F, G))$$

$$\to H^2(X, \mathcal{E}xt^0_{\mathcal{O}_X}(F, G)) \to \mathcal{E}xt^2_{\mathcal{O}_X}(F, G) \to 0$$
2. Extension Calculations

Notation 6. For a coherent sheaf $F$ of rank $r$, we denote $\Delta(F) = 2r c_2(F) \cdot (r - 1) c_1^2(F) = c_2(F \otimes F^*)$. Note that $\Delta(F) = \Delta(F \otimes L)$ for any invertible sheaf $L$. \hfill \Box

It is known \cite{24,25} that there is a coarse projective moduli space for semistable torsion-free sheaves with given Chern classes on a smooth projective threefold $X$. The tangent space at a stable sheaf $F$ is $\text{Ext}^1_{\mathcal{O}_X}(F, F)$, the obstructions lie in $\text{Ext}^2_{\mathcal{O}_X}(F, F)$, and if $\text{Ext}^2_{\mathcal{O}_X}(F, F) = 0$ then the moduli space is smooth at $F$. This motivates:

Definition 7. Let $F$ be a torsion-free sheaf on a smooth projective threefold $X$. The expected dimension of the coarse moduli space of stable torsion-free sheaves with Chern classes equal to that of $F$ is

$$\mathcal{D}(F) = \dim \text{Ext}^1_{\mathcal{O}_X}(F, F) - \dim \text{Ext}^2_{\mathcal{O}_X}(F, F)$$

\hfill \Box

We make a formal computation:

Proposition 8. Let $F$ be a rank $r$ coherent sheaf of homological dimension 1 on a smooth projective threefold $X$. Then

$$\sum_{i=0}^{3} (-1)^i \dim \text{Ext}^i_{\mathcal{O}_X}(F, F) = \frac{r^2 c_1(X) c_2(X)}{24} - \frac{c_1(X)}{2} \Delta(F)$$

PROOF: By definition, because $F$ has homological dimension 1 there is a resolution

$$0 \to E_1 \to E_0 \to F \to 0$$

with $E_0$, $E_1$ locally free of rank $k + r$ and $k$ respectively. Furthermore, as in \cite[3.4]{8} we compute the global extension groups as the hypercohomology of the complex

$$E_0^* \otimes E_1 \to (E_0^* \otimes E_0) \oplus (E_1^* \otimes E_1) \to E_1^* \otimes E_0$$

and so

$$\sum_{i=0}^{3} (-1)^i \dim \text{Ext}^i(F, F) = \chi(E_0^* \otimes E_0) + \chi(E_1^* \otimes E_1) - \chi(E_0^* \otimes E_1) - \chi(E_1^* \otimes E_0)$$
In general, if $r_1 = \text{rank}(E)$ and $r_2 = \text{rank}(F)$, one finds

$$
c_1(E \otimes F) = r_2 c_1(E) + r_1 c_1(F)
$$

$$
c_2(E \otimes F) = \binom{r_2}{2} c_1^2(E) + r_2 c_2(E) + (r_1 r_2 - 1) c_1(E) c_1(F)
$$

$$
+ r_1 c_2(F) + \binom{r_1}{2} c_1^2(F)
$$

$$
c_3(E \otimes F) = \binom{r_2}{3} c_1^3(E) + 2 \binom{r_2}{2} c_1(E) c_2(E) + (r_1 r_2 - 2) c_1(E) c_2(F)
$$

$$
+ \frac{1}{2} (r_2 - 1)(r_1 r_2 - 2) c_1^2(E) c_1(F) + \frac{1}{2} (r_1 - 1)(r_1 r_2 - 2) c_1(E) c_2^2(F)
$$

$$
+ (r_1 r_2 - 2) c_2(E) c_1(F) + 2 \binom{r_1}{2} c_1(F) c_2(F) + \binom{r_1}{3} c_1^3(F)
$$

$$
+ r_2 (r_2^2 - 3r_2 + 3) c_3(E) + r_1 (r_1^2 - 3r_1 + 3) c_3(F)
$$

Substituting in our case, we obtain:

$$
c_1(E_i \otimes E_i^*) = c_3(E_i \otimes E_i^*) = 0
$$

$$
c_2(E_0 \otimes E_0^*) = 2(k + r) c_2(E_0) - (k + r - 1) c_1^2(E_0) = \Delta(E_0)
$$

$$
c_2(E_1 \otimes E_1^*) = 2k c_2(E_1) - (k - 1) c_1^2(E_1) = \Delta(E_1)
$$

Computing via Riemann-Roch (Theorem 4) we see

$$
\chi(E_0 \otimes E_0^*) = -\frac{1}{2} c_1(X) \left( 2(k + r) c_2(E_0) - (k + r - 1) c_1^2(E_0) - \frac{(k + r)^2}{12} c_2(X) \right)
$$

$$
\chi(E_1 \otimes E_1^*) = -\frac{1}{2} c_1(X) \left( 2k c_2(E_1) - (k - 1) c_1^2(E_1) - \frac{k^2}{12} c_2(X) \right)
$$

For the other two terms, we again substitute into the general formulae to obtain:

$$
c_1(E_0^* \otimes E_1) = (k + r) c_1(E_1) - kc_1(E_0)
$$

$$
c_2(E_0^* \otimes E_1) = \binom{k}{2} c_1^2(E_0) + k c_2(E_0) - (k^2 + rk - 1) c_1(E_0) c_1(E_1)
$$

$$
+ (k + r) c_2(E_1) + \binom{k + r}{2} c_1^2(E_1)
$$

and so by Theorem 4 we have

$$
\chi(E_0^* \otimes E_1) + \chi(E_0 \otimes E_1^*) = \frac{c_1(X)}{2} \left( c_1^2(E_0^* \otimes E_1) + \frac{k(k + r)}{6} c_2(X) - 2 c_2(E_0^* \otimes E_1) \right)
$$
Putting the four terms together:

\[
\sum_{i=0}^{3} (-1)^i \dim \text{Ext}^i(F, F) = \chi(E_0^* \otimes E_0) + \chi(E_1^* \otimes E_1) - \chi(E_0^* \otimes E_1) - \chi(E_1^* \otimes E_0)
\]

\[
= \frac{r^2 c_1(X)c_2(X)}{24} + rc_1(X)(c_2(E_1) - c_2(E_0))
\]

\[
+ \frac{c_1(X)}{2} ((r - 1)c_1^2(E_0) - (r + 1)c_1^2(E_1) + 2c_1(E_0)c_1(E_1))
\]

\[
= \frac{r^2 c_1(X)c_2(X)}{24} - \frac{c_1(X)}{2} \Delta(F)
\]

Where the last equality holds after replacing \(c_1(F), c_2(F)\) with the appropriate Chern classes of \(E_0\) and \(E_1\) as derived from the resolution of \(F\).

\[\blacksquare\]

**Corollary 9.** Let \(F\) be a rank 2 reflexive sheaf on a smooth projective threefold \(X\). Then

\[
\sum_{i=0}^{3} (-1)^i \dim \text{Ext}_{\mathcal{O}_X}^i(F, F) = \frac{c_1(X)c_2(X)}{6} - \frac{c_1(X)}{2} \Delta(F)
\]

**Proof:** This follows immediately from Proposition 8 noting that reflexive sheaves have homological dimension 1 [8, 1.2]. \(\blacksquare\)

Since stable sheaves are simple, to compute \(\mathcal{D}(F)\) for a stable sheaf we need only compute \(\dim \text{Ext}_{\mathcal{O}_X}^3(F, F)\). We give two results, one for varieties with effective anticanonical divisor (Proposition 10) and one for a class including Fano varieties (Proposition 12).

**Proposition 10.** Let \(F\) be a reflexive sheaf on a smooth projective threefold \(X\) with \(\omega_X^*\) effective.

1. If \(\omega_X = \mathcal{O}_X\) then \(\text{Ext}_{\mathcal{O}_X}^3(F, F)^* = \text{Hom}_{\mathcal{O}_X}(F, F)\).
2. If \(\omega_X \neq \mathcal{O}_X\) then \(\dim \text{Ext}_{\mathcal{O}_X}^3(F, F) < \dim \text{Hom}_{\mathcal{O}_X}(F, F)\).

**Proof:** (Cf. [28, 3.39]) We have

\[
\text{Ext}_{\mathcal{O}_X}^3(F, F) = \text{H}^3(X, \mathcal{H}om_{\mathcal{O}_X}(F, F))
\]

\[
= \text{Hom}_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(F, F), \omega_X)^*
\]
where the first equality is Proposition 5 and the second is Serre Duality. We also have
\[ \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) = H^0(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})) \]
\[ = H^0(X, \omega_X \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}))^* \]
\[ = \text{Hom}_{\mathcal{O}_X}(\omega_X \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}), \omega_X) \]
\[ = \text{Hom}_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}), \mathcal{O}_X) \]

where the first equality is Proposition 5; the second is \([8, 2.5]\); the third is Serre Duality; the fourth is \([6, \text{III.6.7}]\).

Now clearly if \(\omega_X = \mathcal{O}_X\) then \(\text{Ext}^3_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})^* = \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})\). Otherwise, letting \(\omega_X^* = \mathcal{O}_X(D)\) we have
\[ 0 \to \omega_X \to \mathcal{O}_X \to \mathcal{O}_D \to 0 \]
Applying \(\text{Hom}_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}), \cdot)\) yields
\[ 0 \to \text{Ext}^3_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})^* \to \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) \xrightarrow{f} \text{Hom}_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}), \mathcal{O}_D) \to \cdots \]
where \(f\) is not the zero map as it preserves at least the homotheties of \(\mathcal{F}\). \(\square\)

**Corollary 11.** Let \(\mathcal{F}\) be a reflexive sheaf on a smooth projective threefold \(X\) with \(\omega_X = \mathcal{O}_X\). Then \(\mathcal{D}(\mathcal{F}) = 0\)

**Proof:** As \(c_1(X) = c_1(\mathcal{O}_X)\), this follows immediately from Propositions 8 and 10. \(\square\)

We also have the following, which applies, in particular, to Fano varieties:

**Proposition 12.** Let \(\mathcal{F}\) be a stable rank two reflexive sheaf on a smooth projective threefold \(X \subset \mathbb{P}^n\). If there exists an \(n \in \mathbb{Z}\) such that \(H^0(X, \mathcal{F} \otimes \omega_X^n) \neq 0\) and \(H^0(X, \mathcal{F} \otimes \omega_X^{n+1}) = 0\), then \(\text{Ext}^3_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) = 0\).

**Proof:** By hypothesis, for some \(n\) we have \(H^0(X, \mathcal{F} \otimes \omega_X^n) \neq 0\) and \(H^0(X, \mathcal{F} \otimes \omega_X^{n+1}) = 0\). Let \(D \subset X\) be an effective divisor such that \(\mathcal{F} \otimes \omega_X^n \otimes \mathcal{O}_X(-D)\) has a section whose zero locus is a curve \(C\); note that \(D\) may be empty and that \(H^0(X, \mathcal{F} \otimes \omega_X^{n+1} \otimes \mathcal{O}_X(-D)) = 0\). Letting \(\mathcal{G} = \mathcal{F} \otimes \omega_X^n \otimes \mathcal{O}_X(-D)\), we compute \(\text{Ext}^3_{\mathcal{O}_X}(\mathcal{G}, \mathcal{G})\).

The section of \(\mathcal{G}\) induces an exact sequence
\[ 0 \to \mathcal{O}_X \to \mathcal{G} \to \mathcal{I}_C \otimes \det \mathcal{G} \to 0 \]
Applying \(\text{Hom}_{\mathcal{O}_X}(\cdot, \mathcal{G})\) yields
\[ \cdots \to \text{Ext}^3_{\mathcal{O}_X}((\mathcal{I}_C \otimes \det \mathcal{G})^*, \mathcal{G}) \to \text{Ext}^3_{\mathcal{O}_X}(\mathcal{G}, \mathcal{G}) \to H^3(X, \mathcal{G}) \to 0 \]
We have \(H^3(X, \mathcal{G}) = H^0(X, \mathcal{G}^* \otimes \omega_X)^* = H^0(X, \mathcal{G} \otimes \omega_X \otimes \det \mathcal{G}^*)^*\) but the last group is zero by stability of \(\mathcal{G}\) and the above discussion.
Applying Hom$_{\mathcal{O}_X}(\cdot, \mathcal{G})$ to the basic sequence
\[ 0 \to \mathcal{I}_C \otimes \det \mathcal{G} \to \det \mathcal{G} \to \mathcal{O}_C \otimes \det \mathcal{G} \to 0 \]
yields
\[ \cdots \to \text{Ext}^3_{\mathcal{O}_X}(\det \mathcal{G}, \mathcal{G}) \to \text{Ext}^3_{\mathcal{O}_X}(\mathcal{I}_C \otimes \det \mathcal{G}, \mathcal{G}) \to 0 \]
We know $\text{Ext}^3_{\mathcal{O}_X}(\det \mathcal{G}, \mathcal{G}) = H^3(X, \mathcal{G}^*) = H^0(X, \mathcal{G} \otimes \omega_X) = 0$, therefore $\text{Ext}^3_{\mathcal{O}_X}(\mathcal{I}_C \otimes \det \mathcal{G}, \mathcal{G}) = 0$ and the vanishing of $\text{Ext}^3_{\mathcal{O}_X}(\mathcal{G}, \mathcal{G})$ follows.

Corollary 13. Let $\mathcal{F}$ be a stable rank two reflexive sheaf on a smooth projective threefold $X$. Assume either that $\omega_X$ is non-trivial and effective or that there exists an $n \in \mathbb{Z}$ such that $H^0(X, \mathcal{F} \otimes \omega_X^n) \neq 0$ and $H^0(X, \mathcal{F} \otimes \omega_X^{n+1}) = 0$. Then
\[ \mathcal{D}(\mathcal{F}) = 1 - \frac{c_1(X)c_2(X)}{6} + \frac{c_1(X)\Delta(\mathcal{F})}{2} \]

It is interesting to see when the expected dimension is zero in the easily computable cases of smooth Fano complete intersection threefolds. In particular, if $X$ is a hypersurface of degree $r \leq 4$ in $\mathbb{P}^4$ then a case-by-case examination shows that $\mathcal{D}(\mathcal{F})$ cannot be 0 if $r = 1, 3, 4$. Similarly, if $X$ is a smooth complete intersection $X \subseteq \mathbb{P}^5$ of type $(2,2)$ or is a smooth complete intersection $X \subseteq \mathbb{P}^6$ of type $(2,2,2)$, then $\mathcal{D}(\mathcal{F})$ cannot be 0. This leaves two cases which serve as useful examples in Section 4.

Example 14. Let $\mathcal{F}$ be a stable rank two reflexive sheaf on a smooth quadric hypersurface $X \subseteq \mathbb{P}^4$ and suppose that $\wedge^2 \mathcal{F} = \mathcal{O}_X(k)$. Then $\mathcal{D}(\mathcal{F}) = 0$ if and only if $k^2 + 1 = 2c_1(\mathcal{O}_X(1))c_2(\mathcal{F})$.

Let $C \subset X$ be a line. We associate to $C$ a rank 2 vector bundle $\mathcal{F}$ with $\wedge^2 \mathcal{F} = \mathcal{O}_X(1)$. One can show that $\mathcal{F}$ is stable and that $\text{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) = 0$, hence the moduli space is smooth of dimension 0. Note, however, that $h^0(C, N_{C/X}) = 3$ and $H^1(C, N_{C/X}) = 0$, hence the Hilbert scheme is smooth of dimension three at $C$.

Example 15. Let $\mathcal{F}$ be a stable rank two reflexive sheaf on a smooth complete intersection $X \subseteq \mathbb{P}^5$ of type $(2,3)$. Then $c_1(X) = c_1(\mathcal{O}_X(1))$ and $c_2(X) = 4c_2^2(\mathcal{O}_X(1))$. If $\wedge^2 \mathcal{F} = \mathcal{O}_X(k)$ then $\mathcal{D}(\mathcal{F}) = 0$ exactly when $2c_1(\mathcal{O}_X(1))c_2(\mathcal{F}) = 3(1 + k^2)$.

Let $C$ be a smooth plane cubic and $X$ a smooth complete intersection of type $(2,3)$ which contains it. We associate to $C$ a rank 2 vector bundle $\mathcal{F}$ with $\wedge^2 \mathcal{F} = \mathcal{O}_X(1)$. One can show that $\mathcal{F}$ is stable and that $\text{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) = 0$, hence the moduli space is again smooth of dimension 0. Note, however,
that $h^0(C, N_{C/X}) = 3$ and $H^1(C, N_{C/X}) = 0$, hence the Hilbert scheme is again smooth of dimension three at $C$. \qed

By [28, 3.30] and [19, 3.7], Proposition 12 also immediately implies:

**Corollary 16.** Let $X$ be a smooth projective threefold and let $\mathcal{M}$ be the moduli space of rank 2 semistable sheaves with Chern classes $c_i$ and determinant $L$. Suppose that all such sheaves are stable and reflexive and suppose that for each $\mathcal{F} \in \mathcal{M}$ there exists an $n \in \mathbb{Z}$ such that $H^0(X, \mathcal{F} \otimes \omega_X^n) \neq 0$ and $H^0(X, \mathcal{F} \otimes \omega_X^{n+1}) = 0$ (e.g. if $X$ is Fano). Then $\mathcal{M}$ admits a perfect tangent-obstruction complex; further, there is a virtual cycle $Z_0 \subset \mathcal{M}$ of dimension $D(F)$ defined by the tangent-obstruction functors. \qed

### 3. The Local Structure of Moduli of Reflexive Sheaves

Several results in this section contain the hypothesis that $H^2(X, \mathcal{F}) = 0$. To show this is not very restrictive (especially when $\mathcal{F}$ is stable and $X$ is Fano) we have:

**Lemma 17.** Let $X$ be a smooth projective threefold with $H^2(X, \mathcal{O}_X) = 0$, $\mathcal{F}$ a rank 2 reflexive sheaf with $H^2(X, \det \mathcal{F}) = 0$. Suppose that $\mathcal{F}$ has a section whose zero scheme is a curve $C$ and that $\det \mathcal{F} \otimes \mathcal{O}_C$ is non-special. Then $H^2(X, \mathcal{F}) = 0$.

**Proof:** This follows immediately from the sequence $0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{F} \rightarrow I_C \otimes \det \mathcal{F} \rightarrow 0$. \qed

We compare the local structure of moduli of reflexive sheaves to moduli of ideal sheaves (see [30] and the remarks after Proposition 23 for relations between this and the Hilbert scheme)

**Proposition 18.** Let $X$ be a smooth projective threefold, $\mathcal{F}$ a rank 2 reflexive sheaf. Suppose that $\mathcal{F}$ has a section whose zero scheme is a curve $C$.

1. If $H^2(X, \mathcal{F}) = 0$ and $H^1(X, I_C \otimes \det \mathcal{F} \otimes \omega_X) = 0$, then the vanishing of $\text{Ext}^2_{\mathcal{O}_X}(I_C, I_C)$ implies the vanishing of $\text{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$.
2. If $H^0(X, \mathcal{F} \otimes \omega_X) = 0$ and $H^1(X, I_C \otimes \det \mathcal{F}) = 0$, then the vanishing of $\text{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$ implies the vanishing of $\text{Ext}^2_{\mathcal{O}_X}(I_C, I_C)$.

**Proof:**

Part (1):

Applying $\text{Hom}_{\mathcal{O}_X}(\cdot, \mathcal{F})$ to the sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{F} \rightarrow I_C \otimes \det \mathcal{F} \rightarrow 0$$

we have

$$\cdots \rightarrow \text{Ext}^2_{\mathcal{O}_X}(I_C \otimes \det \mathcal{F}, \mathcal{F}) \rightarrow \text{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) \rightarrow \text{Ext}^2_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) \rightarrow \cdots$$
where $\text{Ext}^2_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) = H^2(X, \mathcal{F}) = 0$ by hypothesis. Therefore, it suffices to show that $\text{Ext}^2_{\mathcal{O}_X}(\mathcal{I}_C \otimes \det \mathcal{F}, \mathcal{F}) = \text{Ext}^2_{\mathcal{O}_X}(\mathcal{I}_C, \mathcal{F}^*) = 0$.

Applying $\text{Hom}_{\mathcal{O}_X}(\mathcal{I}_C, \cdot)$ to the sequence

$$0 \to \det \mathcal{F}^* \to \mathcal{F}^* \to \mathcal{I}_C \to 0$$

we have

$$\cdots \to \text{Ext}^2_{\mathcal{O}_X}(\mathcal{I}_C, \det \mathcal{F}^*) \to \text{Ext}^2_{\mathcal{O}_X}(\mathcal{I}_C, \mathcal{F}^*) \to \text{Ext}^2_{\mathcal{O}_X}(\mathcal{I}_C, \mathcal{I}_C) \to \cdots$$

where $\text{Ext}^2_{\mathcal{O}_X}(\mathcal{I}_C, \det \mathcal{F}^*) = H^1(X, \mathcal{I}_C \otimes \det \mathcal{F} \otimes \omega_X) = 0$ by hypothesis. Therefore, if $\text{Ext}^2_{\mathcal{O}_X}(\mathcal{I}_C, \mathcal{I}_C) = 0$ then $\text{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) = 0$.

Part (2):

Applying $\text{Hom}_{\mathcal{O}_X}(\cdot, \mathcal{I}_C)$ to the sequence

$$0 \to \det \mathcal{F}^* \to \mathcal{F}^* \to \mathcal{I}_C \to 0$$

we have

$$\cdots \to \text{Ext}^1_{\mathcal{O}_X}(\det \mathcal{F}^*, \mathcal{I}_C) \to \text{Ext}^2_{\mathcal{O}_X}(\mathcal{I}_C, \mathcal{I}_C) \to \text{Ext}^2_{\mathcal{O}_X}(\mathcal{F}^*, \mathcal{I}_C) \to \cdots$$

where $\text{Ext}^1_{\mathcal{O}_X}(\det \mathcal{F}^*, \mathcal{I}_C) = H^1(X, \det \mathcal{F} \otimes \mathcal{I}_C) = 0$ by hypothesis.

Applying $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}^*, \cdot)$ to the sequence

$$0 \to \det \mathcal{F}^* \to \mathcal{F}^* \to \mathcal{I}_C \to 0$$

we have

$$\cdots \to \text{Ext}^2_{\mathcal{O}_X}(\mathcal{F}^*, \mathcal{F}^*) \to \text{Ext}^2_{\mathcal{O}_X}(\mathcal{F}^*, \mathcal{I}_C) \to \text{Ext}^3_{\mathcal{O}_X}(\mathcal{F}^*, \det \mathcal{F}^*) \to \cdots$$

where $\text{Ext}^3_{\mathcal{O}_X}(\mathcal{F}^*, \det \mathcal{F}^*) = H^3(X, \mathcal{F}^*) = 0$ by hypothesis. Therefore, the vanishing of $\text{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$ implies $\text{Ext}^2_{\mathcal{O}_X}(\mathcal{I}_C, \mathcal{I}_C) = 0$. \hfill \Box

Especially when $X$ is Fano, part (2) of Proposition 18 is at least consistent with the idea that the Hilbert scheme of curves should fiber over the moduli of reflexive sheaves. This will be discussed further in Section 4.

If our primary interest is the moduli of reflexive sheaves, then combining Corollary 17 with part (1) of Proposition 18 yields the following local result:

**Theorem 19.** Let $\mathcal{F}$ be a stable reflexive sheaf of rank 2 on a smooth projective threefold $X$ with $H^2(X, \mathcal{F}) = 0$ and assume that either $\omega_X^*$ is effective or that there exists an $n \in \mathbb{Z}$ such that $H^0(X, \mathcal{F} \otimes \omega_X^n) \neq 0$ and $H^0(X, \mathcal{F} \otimes \omega_X^{n+1}) = 0$. Suppose further that $\mathcal{F}$ has a section whose zero scheme is a curve $C$, and that

1. $H^1(X, \mathcal{I}_C \otimes \det \mathcal{F} \otimes \omega_X) = 0$
2. $\text{Ext}^2_{\mathcal{O}_X}(\mathcal{I}_C, \mathcal{I}_C) = 0$
Then the (coarse) projective moduli space of semi-stable coherent rank 2 torsion-free sheaves at the point corresponding to $\mathcal{F}$ is smooth of dimension $D(\mathcal{F})$. $\square$

A particularly nice application of Theorem 19 is the case $C$ is a rational curve on a Fano threefold.

**Corollary 20.** Let $X$ be a smooth projective Fano threefold, $\mathcal{F}$ a stable rank 2 reflexive sheaf with $\det \mathcal{F}$ big and nef. Suppose that $\mathcal{F}$ has a section whose zero scheme is a rational curve $C$. If $\text{Ext}^1_{\mathcal{O}_X}(\mathcal{I}_C, \mathcal{O}_C) = 0$ then the (coarse) projective moduli space of semi-stable coherent rank 2 torsion-free sheaves is smooth of dimension

$$1 - \frac{c_1(X)c_2(X)}{6} + \frac{c_1(X)\Delta(\mathcal{F})}{2}$$

at the point corresponding to $\mathcal{F}$.

**Proof:** By the nef hypothesis, $H^1(C, \det \mathcal{F} \otimes \mathcal{O}_C) = 0$.

From the sequence

$$0 \to \det \mathcal{F}^* \to \mathcal{F}^* \to \mathcal{I}_C \to 0$$

we have $H^2(X, \det \mathcal{F}^*) = 0$ by hypothesis and $H^2(X, \mathcal{I}_C) = 0$ because $C$ is rational; hence $H^2(X, \mathcal{F}^*) = 0$. By Proposition 5, this gives $\text{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) = 0$, but $\text{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)^* = H^1(X, \mathcal{F} \otimes \omega_X)$. From the sequence

$$0 \to \omega_X \to \mathcal{F} \otimes \omega_X \to \mathcal{I}_C \otimes \det \mathcal{F} \otimes \omega_X \to 0$$

we have $H^1(X, \mathcal{I}_C \otimes \det \mathcal{F} \otimes \omega_X) = 0$.

From the long exact sequence

$$\cdots \to \text{Ext}^1_{\mathcal{O}_X}(\mathcal{I}_C, \mathcal{O}_C) \to \text{Ext}^2_{\mathcal{O}_X}(\mathcal{I}_C, \mathcal{I}_C) \to \text{Ext}^2_{\mathcal{O}_X}(\mathcal{I}_C, \mathcal{O}_X) \to \cdots$$

and the fact that $\text{Ext}^2_{\mathcal{O}_X}(\mathcal{I}_C, \mathcal{O}_X)^* = H^1(X, \mathcal{I}_C \otimes \omega_X) = 0$ we have that $\text{Ext}^2_{\mathcal{O}_X}(\mathcal{I}_C, \mathcal{I}_C) = 0$. $\square$

**Remark 21.** In the case of a canonically trivial threefold, a stable reflexive sheaf cannot have a section whose zero scheme is a rational curve. However, one can still show that for any reflexive sheaf $\mathcal{F}$ with a section whose zero scheme is an irreducible rational curve $C$, if $H^1(C, N_{C/X}) = 0$ then $\text{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) = 0$. $\square$
4. The Global Structure of Moduli of Locally Free Sheaves and Donaldson-Thomas Invariants

Note that in Theorem 19 we have the condition that the moduli space of ideal sheaves is smooth at the point $[\mathcal{I}_C]$. From the point of view of the Serre Correspondence some relationship between the moduli of reflexive sheaves and the moduli of ideal sheaves is not surprising; similarly, we expect a relationship with the Hilbert scheme. Note, however, that a single reflexive sheaf may have many independent sections whose zero schemes may be different curves within the same component of the Hilbert scheme. We therefore expect that components of the Hilbert scheme of curves fiber over moduli spaces of reflexive sheaves, with fibers projectivized spaces of sections (since scalar multiples of sections have the same zero scheme). This phenomenon was already described in [14] (see Example 29 below). It should also be noted that it is possible, a priori, that $\mathcal{F}$ could have many independent sections vanishing along the same scheme, though it is proved in [7, 1.3] (the result is attributed to Wever) that this is generally not the case for vector bundles on $\mathbb{P}^3$. In general, we have

**Proposition 22.** Let $X$ be a smooth projective threefold, $\mathcal{F}$ a simple rank 2 locally free sheaf with $H^1(X, \det \mathcal{F}^*) = 0$. If $\mathcal{F}$ has a section whose zero scheme is a curve $C$ and if $H^1(X, \mathcal{I}_C) = 0$, then $h^0(X, \mathcal{I}_C \otimes \mathcal{F}) = 1$.

**Proof:** Tensoring the exact sequence

$$0 \to \det \mathcal{F}^* \to \mathcal{F}^* \to \mathcal{I}_C \to 0$$

by $\mathcal{F}$, by simplicity it is enough to show $H^1(X, \mathcal{F}^*) = 0$. This follows immediately from the hypotheses. \qed

As evidence supporting these observations (in addition to Examples 14 and 15) we have the following elementary result.

**Proposition 23.** Let $X$ be a smooth projective threefold, $\mathcal{F}$ a rank 2 locally free sheaf. Suppose that $\mathcal{F}$ has a section whose zero scheme is a curve $C$ and suppose that $H^1(X, \mathcal{F}) = H^1(X, \mathcal{F}^*) = H^2(X, \mathcal{F}) = H^2(X, \mathcal{F}^*) = 0$. Then $H^0(C, N_{C/X}) \to \text{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$ and $\text{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) \hookrightarrow H^1(C, N_{C/X})$. In particular, we have

$$\dim \text{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) = h^0(C, N_{C/X}) - h^0(X, \mathcal{F}) + h^0(X, \mathcal{I}_C \otimes \mathcal{F}).$$

If, in addition, we have $H^1(X, \mathcal{I}_C) = 0$ then

$$\dim \text{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) = h^0(C, N_{C/X}) - (h^0(X, \mathcal{F}) - 1).$$

This gives precisely the naive dimension count of the above discussion. Compare this with Proposition 18 and with [30, 6.6] where, in particular, it is
shown that if $C \subset X$ is a local complete intersection curve in a smooth threefold (which is the case in Proposition 23) with $H^1(X, O_X) = 0$ then $H^0(C, N_{C/X}) \cong \text{Ext}^1_{O_X}(I_C, I_C)$ and $H^1(C, N_{C/X}) \hookrightarrow \text{Ext}^2_{O_X}(I_C, I_C)$.

**Proof:** (of Proposition 23) From the section of $F$ we have

$$0 \to \text{det} F^* \to F^* \to I_C \to 0$$

Tensoring with $F$ and taking global sections gives

$$\cdots \to H^i(X, F^*) \to \text{Ext}^i_{O_X}(F, F) \to H^i(X, F \otimes I_C) \to \cdots$$

Tensoring the standard sequence

$$0 \to I_C \to O_X \to O_C \to 0$$

with $F$ and taking global sections gives

$$\cdots \to H^i(X, F \otimes I_C) \to H^i(X, F) \to H^i(C, N_{C/X}) \to \cdots$$

The results follow from simple diagram chasing.

**Remark 24.** If Pic $X = \mathbb{Z}$, then Proposition 23 applies in particular to aCM vector bundles [3] (these are rank 2 bundles $F$ such that $h^i(X, F(n)) = 0$ for $i = 1, 2$ and $n \in \mathbb{Z}$). See Example 29 below.

**Remark 25.** Part (2) of Proposition 18 should also be evidence of this correspondence; i.e. if the Hilbert scheme of curves does, in fact, fiber over the moduli of reflexive sheaves then one may expect that smoothness of the base, together with some extra conditions, would give smoothness of the total space.

We do not review Donaldson-Thomas invariants here (nice introductions can, for example, be found in [17],[18],[21],[22]) but they are a motivation for studying the global relationship between the moduli space of reflexive sheaves and the Hilbert scheme of curves.

For example, fix a subcanonical curve $C$ on a Calabi-Yau threefold. The expected dimension, and hence the dimension of the virtual fundamental class, of both the component of the Hilbert scheme and the relevant moduli space of torsion-free sheaves is zero. However, Proposition 23 suggests that the moduli space of torsion-free sheaves may have strictly smaller dimension and so the geometry of that space may be more directly related to invariants coming from virtual fundamental classes.

**Example 26.** Let $C \subset \mathbb{P}^3$ be the twisted cubic, let $X = \text{Bl}_C \mathbb{P}^3$ be the blow up along $C$, and let $L \subset X$ be the proper transform of a secant line to $C$. Then $\omega_L \otimes \omega_L^* = O_L$, hence $L$ is subcanonical. As $H^1(X, O(-2H + E)) =$
H^2(X, O(-2H+E)) = 0 (Kodaira Vanishing) we may associate to L a unique vector bundle F with det F = O_X(2H - E) (note that O_L(2H - E) = O_L).

The linear system |2H - E| gives a morphism f : X → P^2. It turns out ([31]) that this is a P^1-bundle; in fact X = ℙ(f_*O_X(H)). Given this, it is not hard to see that F = f^*f_*O_X(H) and that H^1(X, F) = H^1(X, F^*) = H^2(X, F) = H^2(X, F^*) = 0. Further, as L is a fiber of f, h^0(L, N_{L/X}) = 2 and h^1(L, N_{L/X}) = 0. Finally, the exact sequence

0 → O_X → F → I_L(2H - E) → 0

implies that h^0(X, F) = 3. By Proposition 23, we have Ext^1_{O_X}(F, F) = Ext^2_{O_X}(F, F) = 0.

Example 27. Let C ⊂ P^3 be a smooth canonical curve of genus 4. Every such C lies (degenerately) on a smooth quintic Q ⊂ P^4. C is subcanonical since ω_C = O_C(1) and ω_Q = O_Q, and so by the Serre correspondence there is a rank 2 vector bundle F on Q with det F = O_Q(1) and with a section whose zero scheme is C (it is easy to see F is stable, Cf. [7, 3.1]). The corresponding extension

0 → O_Q → F → I_C/Q(1) → 0

immediately gives h^0(Q, F) = 2 and h^i(Q, F) = 0 for i > 0, and dually gives h^3(Q, F^*) = 2 and h^i(Q, F^*) = 0 for i < 3. Therefore, Proposition 23 applies and we see that

\dim \text{Ext}^1_{O_Q}(F, F) = h^0(C, N_{C/Q}) - 1.

Proposition 28. Notation as above, assume that C ⊂ X is a connected, subcanonical curve corresponding to a semistable vector bundle F. Then \( \pi_2 \circ \pi_1^{-1}(C) \) is a single point [F]; therefore, the induced rational map \( \Sigma : H \dashrightarrow M \) is defined at C.
Suppose additionally that $H^0(X, F(-D)) = 0$ for every non-trivial effective divisor $D$. Then as a set $\Sigma^{-1}(F) = \mathbb{P} \Gamma(X, F)$.

**Proof:** Suppose $L \otimes \omega_X \otimes O_C \cong \omega_C$. Referring to the proof of [7, 1.1], the main point is that since $\text{ext}^1_{\mathcal{O}_X}(I_C, L^*) = 1$, there is only one vector bundle $F$ such that there is a pair $(F, s)$ with $Z(s) = C$.

The second part guarantees that every section of $F$ vanishes along a curve.

\[\square\]

**Example 29.** In [14, 2.4] it is shown that if $C$ is an ACM half-canonical curve of genus 15 on a quartic threefold not contained in a quadric, then the Serre Correspondence gives a morphism from a subscheme of the Hilbert scheme to an open set of the appropriate moduli space of rank 2 vector bundles. Further, the fibers are identified with projectivized spaces of sections as in Proposition 28.

\[\square\]

To illustrate Proposition 28, we give a simple corollary.

**Corollary 30.** Let $X \subset \mathbb{P}^4$ be a smooth projective hypersurface of degree $d$, $C \subset X$ a connected, nondegenerate, subcanonical curve with $\mathcal{O}_C(d-4) \cong \omega_C$. Let $F$ be the corresponding (necessarily stable) vector bundle with $\det F = \mathcal{O}_X(1)$. Then the reduced component of the Hilbert scheme containing $[C]$ is birationally equivalent to the reduced component of the moduli space of rank 2 vector bundles containing $F$.

\[\square\]

**Example 31.** Some examples of Corollary 30 include:

1. $C$ is an elliptic normal of degree 5 curve lying on a quartic.
2. $C$ is a canonical curve of genus 5 lying on a quintic.
3. $C$ is a nondegenerate half-canonical curve lying on a sextic (e.g. of genus 15 [14]).

\[\square\]

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