A conjecture of Yves André. *

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1 Introduction.

In this article we deal with the following conjecture by Yves André and Frans Oort.

Conjecture 1.1 (André-Oort) Let \((G, X)\) be a Shimura datum. Let \(K\) be a compact open subgroup of \(G(\mathbb{A}_f)\) and let \(S\) be a set of special points in \(\text{Sh}_K(G, X)(\mathbb{C})\). Then every irreducible component of the Zariski closure of \(S\) in \(\text{Sh}_K(G, X)_\mathbb{C}\) is a subvariety of Hodge type.

The introduction to [2] (and references contained therein) contains a comprehensive exposition of terminology and notations relative to this conjecture. Since we use the same terminology and notations, we do not reproduce them here. The introduction to [2] also contains an exposition of results on this conjecture obtained before the article [3] came out.

In this article we prove the following theorem, which is actually the statement conjectured by Yves André in 1989 in his book [1] (see Problem 9). This statement (without the assumption of the GRH) is now referred to as a conjecture of Yves André.

Theorem 1.2 Assume the Generalised Riemann Hypothesis (GRH) for CM fields. Let \((G, X)\) be a Shimura datum. Let \(K\) be a compact open subgroup of \(G(\mathbb{A}_f)\). Let \(C\) be an irreducible closed algebraic curve contained in the Shimura variety \(\text{Sh}_K(G, X)\) and such that \(C\) contains an infinite set of special points. Then \(C\) is of Hodge type.

In the article [3] we considered a curve in a Shimura variety \(\text{Sh}_K(G, X)\) containing an infinite set \(S\) of special points satisfying the following condition. There is a faithful rational representa-

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tion of $G$ such that the $\mathbb{Q}$-Hodge structures corresponding to the points in $S$ via this representation lie in one isomorphism class. We proved that such a curve is of Hodge type. This was the strongest result towards the André-Oort conjecture at that time.

In the article [3] we introduced some technical tools to attack the André-Oort conjecture. In particular we obtained the following characterisation of subvarieties of Hodge type of a Shimura variety associated to a Shimura datum $(G, X)$ with $G$ semisimple of adjoint type. Let $Z$ be a Hodge generic subvariety of $\text{Sh}_K(G, X)$ contained in its image by some Hecke correspondence $T_g$ with $g$ an element of $G(\mathbb{Q}_p)$ i.e. $Z \subset T_gZ$. Suppose that $p$ is bigger than some integer depending on $G, X, K$ and $Z$ and that $g$ is such that for any simple factor $G_i$ of $G$, the image of $g$ in $G_i(\mathbb{Q}_p)$ is not contained in a compact subgroup. Then $Z$ is of Hodge type provided $Z$ contains at least one special point.

The strategy used to prove our main theorem 1.2 is the same as the one used in [3] (see Section 2 of [3] for details). We use the characterisation mentioned above. After having reduced ourselves to the case where the group $G$ is semisimple of adjoint type and where the curve $C$ is Hodge generic, we try to get $C$ to be contained in its image by a suitable Hecke correspondence. We consider intersections of $C$ with its images $T_gC$ by Hecke correspondences $T_g$ with $g$ some elements of $G(\mathbb{Q}_p)$ for various primes $p$. For suitably chosen $p$ and $g$ such intersection contains a Galois orbit of some special point of $C$. We prove that one can choose a prime $p$ and an element $g$, both satisfying the conditions mentioned above, in such a way that the Galois orbit is too large for the intersection $T_gC \cap C$ to be finite. The choice of a prime $p$ with this property is made possible by the assumption of the GRH and the use of the effective version of the Chebotarev density theorem. We conclude that $C$ is of Hodge type.

The heart of this paper is a proof of a theorem about lower bounds for Galois orbits of special points of Shimura varieties. Our theorem on Galois orbits is a partial answer to Edixhoven’s question Open Problem 14 in [4]. Using the GRH we refine lower bounds for Galois orbits given in [3] enough to be able to prove the conjecture of Yves André.

In section 2.2 we obtain precise information about Mumford-Tate groups of special points and their representations coming from special points on Shimura varieties. This information allows us to bring the following improvement to the main result of [3].

**Theorem 1.3** Let $(G, X)$ be a Shimura datum. Let $K$ be a compact open subgroup of $G(\mathbb{A}_f)$. Let $C$ be an irreducible closed algebraic curve contained in the Shimura variety $\text{Sh}_K(G, X)$ and such that $C$ contains an infinite set $S$ of special points satisfying the following condition.

For any point $s$ of $S$ we choose an element $(\tilde{s}, g)$ of $X \times G(\mathbb{A}_f)$ lying over $s$. We suppose that the Mumford-Tate groups $\text{MT}(\tilde{s})$ lie in one isomorphism class of $\mathbb{Q}$-tori as $s$ ranges through the set $S$. Then $C$ is of Hodge type.
2 Lower bounds for Galois orbits.

In this section we prove a theorem giving lower bounds for Galois orbits of special points of Shimura varieties.

**Theorem 2.1** Assume the GRH for CM fields. Let $N$ be a positive integer. Let $(G, X)$ be a Shimura datum with $G$ semi-simple of adjoint type, and let $K$ be a neat compact open subgroup of $G(A_f)$. Via a faithful representation of $G$, we view $G$ as a closed algebraic subgroup of $GL_n(\mathbb{Q})$, such that $K$ is contained in $GL_n(\hat{\mathbb{Z}})$. Let $V_{\mathbb{Z}}$ be the induced variation of $\mathbb{Z}$-Hodge structure on $Sh_K(G, X)$. For $s$ in $Sh_K(G, X)$, we let $V_s$ be the corresponding Hodge structure and $MT(V_s)$ its Mumford-Tate group (viewed as a closed algebraic subgroup of $GL_n(\mathbb{Z})$). Let $F \subset \mathbb{C}$ be a number field over which $Sh_K(G, X)$ admits a canonical model. For any special point $s$ in $Sh_K(G, X)$, let $L_s$ be the splitting field of $MT(V_s)$ and $d_{L_s}$ be the absolute value of its discriminant.

There exist real $c_1 > 0$ and $c_2 > 0$ such that for any special point $s$ in $Sh_K(G, X)_F(\overline{\mathbb{Q}})$ we have:

$$|Gal(\overline{\mathbb{Q}}/F) \cdot s| > c_1 \log(d_{L_s})^N \prod_{\{p \text{ prime } | MT(V_s)_{\mathbb{Q}_p} \text{ is not a torus}\}} c_2 p$$

2.2 Reciprocity morphisms and Mumford-Tate groups.

In this section we recall the definition of the Mumford-Tate group and reciprocity morphism attached to special elements of $X$ and prove some technical results about the Mumford-Tate groups and reciprocity morphisms to be used later on.

Let $(G, X)$ be a Shimura datum with $G$ semisimple of adjoint type, let $V$ be a faithful rational representation of $G$ and let $V_{\mathbb{Z}}$ be a lattice in $V$. Then $V_{\mathbb{Z}}$ induces a variation of $\mathbb{Z}$-Hodge structure over $X$. Let $h$ be a special element of $X$. The morphism $h : \mathbb{S} \rightarrow G_{\mathbb{R}}$, composed with the representation gives an $\mathbb{R}$-Hodge structure $h : \mathbb{S} \rightarrow GL(V_{\mathbb{R}})$. Let $z$ and $\varpi$ be the generators of the character group of $\mathbb{S}$. The morphism $h$ corresponds to the decomposition

$$V_{\mathbb{C}} = \bigoplus_{p,q} V^{p,q}$$

where $V^{p,q}$ is the $\mathbb{C}$-vector subspace on which $\mathbb{S}$ acts through the character $z^p \varpi^q$. The spaces $V^{p,q}$ satisfy the following condition $\overline{V^{p,q}} = V^{q,p}$. Let $W$ be the collection of pairs of integers $(p, q)$ that intervene in this representation. Since the $\mathbb{R}$-Hodge structures corresponding to elements of $X$ lie in one isomorphism class, the set $W$ does not depend on the element $h$ in $X$. The fact that $G$ is of adjoint type implies that for any $(p, q)$ in $W$ we have $p + q = 0$. Let $M \subset GL(V)$ be the Mumford-Tate group of $h$ and let $L$ be its splitting field. Let us recall that $M$ is the smallest algebraic subgroup $H$ of $GL(V)$ having the property that $h$ factors through $H_{\mathbb{R}}$. The
group $M$ is a $\mathbb{Q}$-torus because $h$ is special and $M$ is given a $\mathbb{Z}$-structure by taking its Zariski closure in the $\mathbb{Z}$-group scheme $GL(V_{\mathbb{Z}})$. We let $X^*(M)$ be the character group of $M$ i.e the group $\text{Hom}(M_{\mathbb{Q}}, \mathbb{G}_{m,\mathbb{Q}})$. The group $X^*(M)$ is a free $\mathbb{Z}$-module of rank equal to the dimension of $M$ with a continuous $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action.

**Lemma 2.3** The field $L$ is a Galois CM field. Furthermore, the degree of $L$ is bounded in terms of the dimension of $V$.

**Proof.** The field $L$ is Galois since it is the splitting field of a torus (the group $\text{Gal}(\overline{\mathbb{Q}}/L)$ is exactly the kernel of the morphism $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL(X^*(M))$ hence is a normal subgroup of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$). The fact that $L$ is a CM field follows from the fact that $ad(h(i))$ is a Cartan involution of $G_{\mathbb{R}}$ (this is a part of axioms imposed upon Shimura data).

Let $E$ be the centre of the endomorphism algebra of the Hodge structure $V$. The algebra $E$ is a finite product of number fields $E = E_1 \times \cdots \times E_m$. The torus $M$ is a subtorus of the torus $\prod_{i=1}^m \text{Res}_{E_i/\mathbb{Q}} \mathbb{G}_{m,E_i}$. Hence $M$ is split over the composite of the Galois closures of the $E_i$ whose degree is clearly bounded in terms of the dimension of $V$ only. $\square$

Let $T$ be the $\mathbb{Q}$-torus $\text{Res}_{L/\mathbb{Q}} \mathbb{G}_{m,L}$. Let $G_L$ be the Galois group of $L$ over $\mathbb{Q}$ and let $r: T \longrightarrow M \subset GL(V)$ be the reciprocity morphism associated to $h$. Let us recall how $r$ is defined. The morphism $h$ gives, by extending scalars from $\mathbb{R}$ to $\mathbb{C}$, the morphism $h_{\mathbb{C}}$ from $\mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$ to $M_{\mathbb{C}} \subset GL(V_{\mathbb{C}})$. Let $\mu: \mathbb{G}_{m,\mathbb{C}} \longrightarrow M_{\mathbb{C}}$ be the morphism $h_{\mathbb{C}}(z,1)$. This morphism $\mu$ is defined over $L$. Hence $\mu$ induces a morphism $\mathbb{G}_{m,L} \longrightarrow M_L$, which, by taking the restriction of scalars from $L$ to $\mathbb{Q}$ gives the morphism

$$\text{Res}_{L/\mathbb{Q}} \mu: \text{Res}_{L/\mathbb{Q}} \mathbb{G}_{m,L} \longrightarrow \text{Res}_{L/\mathbb{Q}} M_L.$$ 

This morphism $\text{Res}_{L/\mathbb{Q}} \mu$ followed the by the norm morphism $\text{Res}_{L/\mathbb{Q}} M_L \longrightarrow M$ gives $r$. The morphism $X^*(r)$ between character groups $X^*(M)$ and $X^*(T)$ is injective (because $r$ is a surjective morphism of $\mathbb{Q}$-tori). The Galois module $X^*(T)$ is naturally isomorphic to $\mathbb{Z}[G_L]$. We enumerate the elements of $G_L$ thus choosing a basis for $X^*(T)$ so that it now makes sense to talk about coordinates of elements of $X^*(T)$.

The morphism $r$, when composed with the representation, defines an action of the torus $T$ on the $\mathbb{Q}$-vector space $V$. There is a subset $\mathcal{X}$ of $X^*(T)$ such that this representation corresponds to a direct sum decomposition

$$V_{\overline{\mathbb{Q}}} = \bigoplus_{\chi \in \mathcal{X}} V_{\chi},$$

where each $V_{\chi}$ is a $\overline{\mathbb{Q}}$-subspace of $V_{\overline{\mathbb{Q}}}$ on which $T_{\overline{\mathbb{Q}}}$ acts through the character $\chi$. The spaces $V_{\chi}$ satisfy the condition that $V_{\chi}^\sigma = V_{\sigma \chi}$ (which insures that the representation is defined over $\mathbb{Q}$).
The representation $h_C(z, 1)$ of $G_{mC}$ corresponds to the decomposition $V_C = \oplus_{(p,q) \in W} V^{p,q}$ where $G_{mC}$ acts via the character $z^p$ on $V^{p,q}$. This representation is defined over $L$ hence induces a representation of $G_{mL}$. We get a decomposition $V_L = \oplus_{(p,q) \in W} V^{p,q}$ where $G_{mL}$ acts through the character $z^p$ on $V^{p,q}$. The representation $r$ of $T_L$ is obtained by taking the restriction of scalars of this representation of $G_{mL}$ followed by the norm from $L$ to $\mathbb{Q}$.

It follows that the characters of $X^*(T)$ that belong to $\mathcal{X}$ have coordinates (with respect to the basis we have chosen) can be only integers $p$ or $q$ where $(p, q)$ is some element of $W$. In particular they are bounded, in absolute value, independently of the element $h$. Furthermore, the characters in $\mathcal{X}$ have the property that for any $\chi$ in $\mathcal{X}$, the character $\chi^\vee$ is the identity because the morphism $r$ satisfies the so-called Seere’s condition (the group $G$ is of adjoint type) and $p + q = 0$ for every pair $(p, q)$ in $W$. We refer to Section 2 of Chapter I of [5], in particular the Proposition 2.4 for facts about Hodge structures of CM type. We summarise what has been said in the following proposition.

Proposition 2.4 There is an integer $k > 0$ such that the following holds. Let $h$ be a special element of $X$, $M$ its Mumford-Tate group and $L$ its splitting field. Choose a basis for $X^*(T)$ by enumerating the elements of $G_L$. With respect to the basis the coordinates of the characters of $T$ that intervene in the decomposition $V^{\vee}_Q = \bigoplus_{\chi \in \mathcal{X}} V_{\chi}$ coming from the representation $r$ associated to $h$ have absolute value at most $k$. Furthermore, for any character $\chi$ in $\mathcal{X}$ the character $\chi^\vee$ is the identity.

We now apply this Proposition to prove a number of results about Mumford-Tate groups of special elements of $X$ and reciprocity morphisms attached to such elements. These results will be used later on to prove lower bounds for Galois orbits.

Proposition 2.5 There is a real $e > 0$ such that the following holds. Let $h$ be a special element of $X$. Let $M$ be the Mumford-Tate group of $h$ and $L$ be its splitting field. Let $r: T \rightarrow M$ be the reciprocity morphism attached to $h$. Let $p$ be a prime. The index of $r((\mathbb{Q}_p \otimes \mathbb{Q})^*)$ in $M(\mathbb{Q}_p)$ is finite bounded above by $e$. The index of $r((\mathbb{Z}_p \otimes \mathbb{Q}_L)^*)$ in the maximal compact open subgroup of $M(\mathbb{Q}_p)$ is finite and bounded above by $e$.

Proof. Let $P$ be the $\mathbb{Z}$-submodule of $X^*(T)$ spanned by the vectors in $\mathcal{X}$. Recall that we identify $X^*(M)$ with its image by $X^*(r)$ i.e we view it as a submodule of $X^*(T)$ and we have chosen a basis for $X^*(T)$. The module $X^*(M)$ is $P$. The group $M(\mathbb{Q}_p)$ is canonically isomorphic to the group $\text{Hom}_{G_L}(X^*(M), (\mathbb{Q}_p \otimes \mathbb{Q})^*)$ of $G_L$-invariant homomorphisms. Similarly the group $(\mathbb{Q}_p \otimes \mathbb{Q})^*$ is isomorphic to $\text{Hom}_{G_L}(X^*(T), (\mathbb{Q}_p \otimes \mathbb{Q})^*)$ and the morphism $r: T(\mathbb{Q}_p) \rightarrow M(\mathbb{Q}_p)$ is

$$r: \text{Hom}_{G_L}(X^*(T), (\mathbb{Q}_p \otimes \mathbb{Q})^*) \rightarrow \text{Hom}_{G_L}(X^*(M), (\mathbb{Q}_p \otimes \mathbb{Q})^*)$$

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which is just the restriction. The group $X^*(T)/P$ is a product of a free abelian group and a torsion group. Let $E$ be the order of this torsion subgroup. Since, by the previous Proposition, the coordinates of the vectors generating $P$ are bounded (in absolute value) by a uniform constant $k$, the number $E$ is bounded in terms of $k$ and $n_L$ only. It is straightforward to see that the order of the cokernel is bounded in terms of $E$ and $n_L$ only. The first claim follows.

The maximal compact open subgroup of $M(\mathbb{Q}_p)$ is $\text{Hom}_{G_L}(X^*(M), (\mathbb{Z}_p \otimes \mathcal{O}_L)^*)$. The second claim is proved using exactly the same arguments.

**Proposition 2.6** There is an integer $B > 0$ such that the following holds. Let $h$ be a special element of $X$ and let $M$ be its Mumford-Tate group and $L$ its splitting field. Let $p$ be a prime splitting $L$ (hence $M$). There is a $\mathbb{Z}$-basis of the character group $X^*(M)$ such that the differences of coordinates of the characters (with respect to this basis) that intervene in the representation $V_{\mathbb{Q}_p}$ of $M_{\mathbb{Q}_p}$ have absolute value at most $B$.

**Proof.** The module $X^*(M_{\mathbb{Q}_p})$ is a submodule of $X^*(T)$ (along with its given basis), generated by vectors whose coordinates are bounded in absolute value by the integer $k$ from the Proposition 2.4. This integer is independent of the point $h$. It follows that there is only finite number (depending on $k$ and $n_L$ only) of possibilities for the set $\mathcal{X}$ and hence for the submodule $X^*(M)$ of $X^*(T)$. Choose some basis for $X^*(M)$ for each of this finite number of cases. Take $B$ to be the maximum of absolute values of the differences of coordinates of characters in $\mathcal{X}$ with respect to these bases. □

**Proposition 2.7** There is a real $C > 0$ such that the following holds. Let $p$ be a prime. For any special element $h$ in $X$ with Mumford-Tate group $M$ such that $M_{\mathbb{F}_p}$ is a torus, the following holds. Let $Y$ be a subspace of $V_{\mathbb{F}_p}$. Let $T$ be the stabiliser of $Y$ in $M_{\mathbb{F}_p}$ (as defined in the Lemma 3.3.1 of [3]). The order of the group of connected components of $T_{\mathbb{F}_p}$ has order at most $C$. The order of the cokernel of the morphism $M(\mathbb{F}_p) \rightarrow (M/T)(\mathbb{F}_p)$ is at most $C$.

**Proof.** Proceeding as in the proof of the Lemma 4.4.1 of [3], we reduce the proof of this proposition to the proof of the fact that stabilisers of lines satisfy the conclusion of the statement above. We have a decomposition $V_{\mathbb{F}_p} = \oplus_{\chi \in \mathcal{X}} V_{\chi}$. Let $v$ be an element of $V_{\mathbb{F}_p}$, write $v = \sum_{\chi} v_{\chi}$. The stabiliser of the line $kv$ is the intersection of the kernels of $\chi - \chi'$ with $\chi$ and $\chi'$ distinct characters such that $v_{\chi} \neq 0$ and $v_{\chi'} \neq 0$. Since the torsion of each $\mathbb{Z}$-module $X^*(T)/(\chi - \chi')\mathbb{Z}$ is bounded independently of $s$ and of the characters $\chi$ and $\chi'$ in $\mathcal{X}$, the order of the group of connected components of the stabiliser of $k \cdot v$ is bounded independently of $s$, $p$ and the subspace. This proves the first claim.
As for the second claim, using the Lemma 4.4.2 of [3], we see that the order of the cokernel of the map \( M(F_p) \rightarrow (M/T)(F_p) \) is bounded by the order of the group of connected components of \( T_{F_p} \), which is uniformly bounded by what has just been said. The second claim follows. \( \square \)

**Proposition 2.8** There is a real \( D > 0 \) such that the following holds. Let \( h \) be a special element of \( X \) and let \( M \) be its Mumford-Tate group. Let \( K_M \) be the maximal compact open subgroup of \( M(\mathbb{A}_f) \). The intersection \( M(\mathbb{Q}) \cap K_M \) is finite of order bounded by \( D \).

**Proof.** The group \( M(\mathbb{Q}) \cap K_M \) is finite because \( M(\mathbb{R}) \) is compact (\( M(\mathbb{R}) \) stabilises the point \( h \) of the Hermitian symmetric domain \( X \) and the group \( G_{\mathbb{R}} \) is of adjoint type) and the group \( M(\mathbb{Q}) \cap K_M \) is discrete. Let \( L \) be the splitting field of \( M \). Choose any basis for the character group \( X^*(M) \) and use this basis to embed \( M \) into a product of \( \dim(M) \) copies of \( T_L \). Then the group \( M(\mathbb{Q}) \cap K_M \) is, via this embedding, a finite subgroup of the product of \( d \) copies of \( O_L^1 \). It follows that it is contained in the product of \( \dim(M) \) copies of the group of roots of unity in \( L \) which is finite of order bounded independently of the point \( h \). The claim follows. \( \square \)

### 2.9 Getting rid of \( G \).

Choose a set of representatives \( R \) in \( G(\mathbb{A}_f) \) for the set of double classes \( G(\mathbb{Q}) \backslash G(\mathbb{A}_f) / K \). Note that \( R \) is finite. For \( s \in Sh_K(G, X) \) there exists a unique \( g_s \) in \( R \) and an element \( \tilde{s} \) in \( X \) unique up to \( \Gamma_s := G(\mathbb{Q}) \cap g_s K g_s^{-1} \), such that \( s = (\tilde{s}, g_s) \). Let \( K_s \) be the compact open subgroup of \( G(\mathbb{A}_f) \) defined by \( K_s := g_s K g_s^{-1} \). We let \( MT(\tilde{s}) \) be the Mumford-Tate group of \( \tilde{s} \) (the smallest algebraic subgroup \( H \) of \( G \) such that \( \tilde{s} \) factors through \( H_{\mathbb{R}} \)). The Mumford-Tate group \( MT(V_s) \) is the image of \( MT(\tilde{s}) \) by the representation (this follows from the explicit construction of the variation of Hodge structures over \( Sh_K(G, X) \) given in the Section 3.2 of [3]). The element \( \tilde{s} \) gives an embedding of the Shimura datum \( (MT(\tilde{s}), \{\tilde{s}\}) \) into \( (G, X) \).

In this section we reduce the problem of giving a lower bound for the Galois orbit of the point \( s \) of \( Sh_K(G, X) \) to the one of giving a lower bound for the Galois orbit of the point \( (\tilde{s}, 1) \) of \( Sh_{K \cap MT(\tilde{s})(\mathbb{A}_f)}(MT(\tilde{s})) \).

**Proposition 2.10** The morphism of Shimura varieties

\[
Sh_{K \cap MT(\tilde{s})(\mathbb{A}_f)}(MT(\tilde{s})) \rightarrow Sh_K(G, X)
\]

sending \( (\tilde{s}, t) \) to \( (\tilde{s}, t \cdot g_s) \) is injective.

**Proof.** Let \( M := MT(\tilde{s}) \). Let \( H \) be the centraliser of \( M \) in \( G \). Let \( (\tilde{s}, t) \) and \( (\tilde{s}, t') \) be two points of \( Sh_{K \cap MT(\tilde{s})(\mathbb{A}_f)}(MT(\tilde{s})) \) such that \( (\tilde{s}, t \cdot g_s) = (\tilde{s}, t' \cdot g_s) \) in \( Sh_K(G, X) \). There exists an
element \( q \) of \( H(Q) \) and an element \( k \) of \( K \) such that we have the following relation

\[
    t = qt' g_s k g_s^{-1}
\]

Since \( H(Q) \) and \( M(\mathbb{A}_\mathbb{F}) \) commute, this relation implies that \( tt'^{-1} \) belongs to \( M(\mathbb{A}_\mathbb{F}) \cap H(Q)U_s \) with \( U_s := H(\mathbb{A}_\mathbb{F}) \cap K_s \). Hence what we need to prove is that \( M(\mathbb{A}_\mathbb{F}) \cap H(Q)K_s = M(Q)(M(\mathbb{A}_\mathbb{F}) \cap U_s) \). Consider the quotient of algebraic groups \( H \longrightarrow \overline{H} = H/M \), which is well defined since \( M \) is normal in \( H \). The image of \( U_s \) of \( U_s \) in \( \overline{H}(\mathbb{A}_\mathbb{F}) \) is neat. On the other hand \( \overline{H}(\mathbb{R}) \) is compact since \( H(\mathbb{R}) \) is compact (as a stabiliser of a point in a hermitian symmetric domain and because \( G_{\mathbb{R}} \) is of adjoint type) and the map \( H(\mathbb{R}) \longrightarrow \overline{H}(\mathbb{R}) \) is surjective on identity components. It follows that \( \overline{H}(Q) \) is discrete in \( \overline{H}(\mathbb{A}_\mathbb{F}) \) and hence \( \overline{H}(Q) \cap U_s \) is trivial by neatness of \( U_s \).

Now suppose that \( h \) is in \( H(Q) \) and \( u \) in \( U_s \) such that \( hu \) is in \( M(\mathbb{A}_\mathbb{F}) \). Then, in \( \overline{H}(\mathbb{A}_\mathbb{F}) \), we have \( \overline{h} \cdot \overline{u} = 1 \), hence \( \overline{h} = \overline{u} = 1 \) in \( \overline{H}(\mathbb{A}_\mathbb{F}) \). That means that \( h \) is in \( M(Q) \) and \( u \) is in \( M(\mathbb{A}_\mathbb{F}) \cap U_s \).

The claim follows. \( \square \)

2.11 Lower bounds for Galois orbits.

We keep the notations of the preceding section. Let furthermore \( L \) be the splitting field of \( MT(\bar{s}) \). Let \( r \) be the reciprocity morphism attached to \( \bar{s} \) as explained in Section 2.2. To simplify the notation we write \( M \) for \( MT(\bar{s}) \). The morphism \( \text{Sh}_{K_s \cap MT(\bar{s})} \) \( (\text{MT}(\bar{s})) \longrightarrow \text{Sh}_{K_s}(G, X) \) is defined over \( L \). The action of \( \text{Gal}(\overline{Q}/L) \) on the Hecke orbit of \( (\bar{s}, g_s) \) is defined as follows. The group \( \text{Gal}(\overline{Q}/L) \) acts through its maximal abelian quotient, which is, by class field theory, isomorphic to a quotient of a product of a finite group of connected components of \( \mathbb{R} \times L \) (of order bounded in terms of the degree of \( L \) only) and of \( (\mathbb{A}_\mathbb{F} \times L)^*/(\hat{\mathbb{Z}} \otimes O_L)^* \). Let \( \sigma \) be an element of \( \text{Gal}(\overline{Q}/L) \) and \( t \) be an element of \( (\mathbb{A}_\mathbb{F} \times L)^* \) such that some element in the preimage of \( \sigma \) in \( (\mathbb{A} \times L)^* \) followed by the projection to \( (\mathbb{A}_\mathbb{F} \times L)^* \) is \( t \). Then

\[
    \sigma(\bar{s}, g_s) = (\bar{s}, r(\bar{t}) \cdot g_s)
\]

It follows that the size of the Galois orbit is, up to a uniformly bounded factor, the size of the set \( (\bar{s}, r((\mathbb{A}_\mathbb{F} \times L)^*) \cdot g_s) \). From the last lemma it follows that to prove the Theorem 2.1 it suffices to give a lower bound for the size of the image of the set \( (\bar{s}, r((\mathbb{A}_\mathbb{F} \times L)^*)) \) in \( \text{Sh}_{K_s \cap M(\mathbb{A}_\mathbb{F})}(M) \). Since the set \( R \) of elements \( g_s \) is finite, the index \( [K : K_s \cap K] \) is bounded independently of \( s \) and it suffices to give a lower bound for the Galois orbit of the point \( (\bar{s}, 1) \) of the Shimura variety \( \text{Sh}_{\text{GL}_n(\mathbb{Z}) \cap M(\mathbb{A}_\mathbb{F})}(M) \).

Lemma 2.12 There is an element \( q \) of \( \text{GL}_n(Q) \) such that the torus \( M' := qMq^{-1} \) satisfies the condition that \( M'_p \) is a torus for any prime \( p \) not dividing the discriminant of \( L \).
Proof. Let $S$ be the finite set of primes $p$ such that $M(\mathbb{Z}_p)$ is not the maximal compact subgroup $\text{Hom}_{GL_1}(X^*(M), (\mathbb{Z}_p \otimes O_L)^*)$ of $M(\mathbb{Q}_p)$. For every prime $p$ in $S$, choose a lattice $L_p$ in $\mathbb{Q}_p^n$ invariant under the maximal compact subgroup of $M(\mathbb{Q}_p)$. Let $g = (g_p)$ be an element of $\text{GL}_n(\mathbb{A}_f)$ such that each $g_p$ is an element of $\text{GL}_n(\mathbb{Q}_p)$ such that $L_p = g_p Z_p^n$. As $\text{GL}_n(\mathbb{A}_f) = \text{GL}_n(\mathbb{Q})\text{GL}_n(\hat{\mathbb{Z}})$, we get an element $q$ of $\text{GL}_n(\mathbb{Q})$ such that $q = qk$ for some $k$ in $\text{GL}_n(\hat{\mathbb{Z}})$. By the Lemma 3.3.1 of [3], the torus $M' := qMq^{-1}$ is a torus for every $p$ unramified in $L$. \hfill \Box

The morphism $\text{inn}_q$ induces an isomorphism between $M(\mathbb{Q}) \setminus M(\mathbb{A}_f)/M(\mathbb{A}_f) \cap \text{GL}_n(\hat{\mathbb{Z}})$ and $M'(\mathbb{Q}) \setminus M'(\mathbb{A}_f)/M'(\mathbb{A}_f) \cap q\text{GL}_n(\hat{\mathbb{Z}})q^{-1}$. We let $r'$ denote the morphism $\text{inn}_q \circ r$. To give a lower bound for the Galois orbit of the point $(\tilde{s}, 1)$ of $\text{Sh}_{M'(\mathbb{A}_f) \cap q\text{GL}_n(\hat{\mathbb{Z}})q^{-1}}(M')$ it suffices to give a lower bound for the image of $r'((\mathbb{A}_f \otimes L)^*)$ in $M'(\mathbb{Q}) \setminus M'(\mathbb{A}_f)/M'(\mathbb{A}_f) \cap q\text{GL}_n(\hat{\mathbb{Z}})q^{-1}$.

**Proposition 2.13** The size of the image of $r'((\mathbb{A}_f \otimes L)^*)$ in $\text{Sh}_{M'(\mathbb{A}_f) \cap q\text{GL}_n(\hat{\mathbb{Z}})q^{-1}}(M')$ is, up to a uniform (i.e depending only on the Shimura variety, not on $s$) constant, the size of the image of $r'((\mathbb{A}_f \otimes L)^*) \cap M'(\hat{\mathbb{Z}})$ in $M'(\mathbb{Z})/M'(\hat{\mathbb{Z}}) \cap q\text{GL}_n(\hat{\mathbb{Z}})q^{-1}$.

**Proof.** We are interested in the size of the set

$$r'((\mathbb{A}_f \otimes L)^*)/r'((\mathbb{A}_f \otimes L)^*) \cap (M'(\mathbb{Q})(q\text{GL}_n(\hat{\mathbb{Z}})q^{-1} \cap M'(\mathbb{A}_f))).$$

Since $M'(\hat{\mathbb{Z}})$ is the maximal compact subgroup of $M'(\mathbb{A}_f)$, we have an inclusion

$$M'(\mathbb{A}_f) \cap q\text{GL}_n(\hat{\mathbb{Z}})q^{-1} \subset M'(\hat{\mathbb{Z}}).$$

Hence the size of the set we are interested in is the size of

$$r'((\mathbb{A}_f \otimes L)^*)/r'((\mathbb{A}_f \otimes L)^*) \cap M'(\mathbb{Q})M'(\hat{\mathbb{Z}})$$

times that of

$$r'((\mathbb{A}_f \otimes L)^*) \cap M'(\mathbb{Q})M'(\hat{\mathbb{Z}})/r'((\mathbb{A}_f \otimes L)^*) \cap (M'(\mathbb{Q})M'(\mathbb{A}_f) \cap q\text{GL}_n(\hat{\mathbb{Z}})q^{-1}).$$

The order of $M'(\mathbb{Q}) \cap M'(\hat{\mathbb{Z}})$ is bounded independently of the point $s$ by the Proposition 2.8, hence the size of

$$r'((\mathbb{A}_f \otimes L)^*) \cap M'(\mathbb{Q})M'(\hat{\mathbb{Z}})/r'((\mathbb{A}_f \otimes L)^*) \cap (M'(\mathbb{Q})q\text{GL}_n(\hat{\mathbb{Z}})q^{-1} \cap M'(\mathbb{A}_f))$$

is, up to a uniformly bounded constant, that of the of the image of $r'((\mathbb{A}_f \otimes L)^*) \cap M'(\hat{\mathbb{Z}})$ in $M'(\mathbb{Z})/M'(\hat{\mathbb{Z}}) \cap q\text{GL}_n(\hat{\mathbb{Z}})q^{-1}$. \hfill \Box
Theorem 2.15 Assume the GRH for CM fields. Let $N$ be a positive integer. There is a real constant $c > 0$ independent of the choice of $s$ and $M'$ (but depending on $N$) such that the size of the set $r'((\mathbb{A}_f \otimes L)^*) / r'((\mathbb{A}_f \otimes L)^*) \cap M'(\mathbb{Q}) M'(\mathbb{Z})$ is at least $c \log(d_L)^N$.

Proof. In what follows, we write $M$ for $M'$ and $r$ for $r'$ to simplify the notations. Let $n_L$ be the degree of $L$ over $\mathbb{Q}$. Let $m > 0$ be an integer at most $\frac{\log(d_L)^5}{15 n_L \log\log(d_L)}$ and let $p_1, \ldots, p_m$ be $m$ distinct primes split in $L$ and smaller than $\log(d_L)^5$. Their existence is provided by the effective Chebotarev theorem (under GRH), provided $d_L$ is bigger than some absolute constant, which we assume. We refer to the Proposition 8.2 of [2] for the exact statement of the effective Chebotarev theorem that we use. For each $i = 1, \ldots, m$, we choose a place $v_i$ of $L$ lying over $p_i$. We let $P_i$ be the uniformiser at the place $v_i$. Let $n_1, \ldots, n_m$ be integers satisfying $|n_i| < N$. Let $I$ be the...
element of \((A_f \otimes L)^*\) that equals \(P_i^{m_i}\) at the place \(v_i\) for \(i = 1, \ldots, m\) and 1 everywhere. Suppose that \(r(I)\) belongs to \(M(\mathbb{Q})M(\hat{\mathbb{Z}})\). Let \(\pi\) be a corresponding element of \(M(\mathbb{Q})\) (this element is defined up to an element of \(M(\mathbb{Q}) \cap M(\hat{\mathbb{Z}})\) which is, by the Proposition 2.8 a finite group of uniformly bounded order). Let, as before, \(\text{Lemma 2.16}\) \(G\) be a corresponding element of \(\text{Gal}(\mathbb{Q})\) \(\cap M(\hat{\mathbb{Z}})\) which is Galois invariant. Let \(\pi\) be the cardinality of \(\mathcal{X}\) and let \((\pi_1, \ldots, \pi_d)\) be the \(d\) elements of \(L^*\) which are images of \(\pi\) by the \(d\) characters in \(\mathcal{X}\). The field \(\mathbb{Q}(\pi_1, \ldots, \pi_d)\) is Galois because the set \(\mathcal{X}\) is Galois invariant.

**Lemma 2.16** Suppose that not all \(n_i\) are zero. Then the field \(\mathbb{Q}(\pi_1, \ldots, \pi_d)\) is \(L\).

**Proof.** It suffices to prove that the group \(\text{Gal}(L/\mathbb{Q}(\pi_1, \ldots, \pi_d))\) acts trivially on \(\mathbb{Q} \otimes X^*(M)\) (alternatively on \(\mathbb{Q} \otimes X_*(M)\), \(X_*(M)\) being the group of cocharacters).

To simplify the exposition we suppose that \(m = 1\) (the general case is done using exactly the same arguments). Let \(\sigma\) be an element of \(\text{Gal}(L/\mathbb{Q}(\pi_1, \ldots, \pi_d))\). So we have a prime \(p\) splitting \(L\), we choose a place \(v\) of \(L\) lying over \(p\) and a uniformiser \(P\) at \(v\). We consider the idele \(I = P^n\) with \(n > 1\) some integer. We suppose that \(r(I)\) belongs to \(M(\mathbb{Q})M(\hat{\mathbb{Z}})\). As \(p\) splits \(M\), we have \(M(\mathbb{Q}_p) = \text{Hom}(X^*(M), \mathbb{Q}_p^*) = X_*(M) \otimes \mathbb{Q}_p^*\). It follows that the evaluation map \(\nu_p: \mathbb{Q}_p^* \rightarrow \mathbb{Z}\) induces an isomorphism between \(M(\mathbb{Q}_p)/M(\mathbb{Z}_p)\) and the group of cocharacters \(X_*(M)\) of \(M\). Let \(K\) be the kernel of \(r\), then we have an exact sequence of \(\mathbb{Q}\)-vector spaces with \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\)-action

\[
0 \rightarrow \mathbb{Q} \otimes X_*(K) \rightarrow \mathbb{Q} \otimes X_*(T) \rightarrow \mathbb{Q} \otimes X_*(M) \rightarrow 0.
\]

It suffices to prove that \(\sigma\) acts trivially on \(\mathbb{Q} \otimes (X_*(T)/X_*(K))\). Since \(\sigma\) fixes each \(\pi_i\) and the set of \(\pi_i\) is \(G_L\)-invariant, \(\sigma\) fixes all the elements \(r(\tau I)\) of \(M(\mathbb{Q}_p)/M(\mathbb{Z}_p)\) with \(\tau\) ranging through \(G_L\). The Galois action on \(M(\mathbb{Q}_p)/M(\mathbb{Z}_p)\) is being given by identifying it with \(X_*(M)\) which has a Galois action. Since the morphism \(X_*(T) \rightarrow X_*(M)\) is surjective, for any \(\tau\) in \(G_L\) we have \(\sigma \tau I = \tau I\) in \(\mathbb{Q} \otimes (X_*(T)/X_*(K))\). Let \(e_1, \ldots, e_{n_k}\) be the basis of \(\mathbb{Q} \otimes X_*(T)\) given by the \(n\)th powers of uniformisers at the places lying over \(p\). Their images in \(\mathbb{Q} \otimes (X_*(T)/X_*(K))\) generate this vector space. Since \(\sigma\) fixes these elements, \(\sigma\) acts trivially on \(\mathbb{Q} \otimes (X_*(T)/X_*(K))\).

The claim follows. \(\square\)

Let \(x\) be the integer \((p_1 \cdots p_m)^{N_k}\) with \(k\) the integer from the Proposition 2.4. Let \(\chi\) be a character in \(\mathcal{X}\). The element \(x\chi(I)\) of \((A_f \otimes L)^*\) belongs to \(\hat{\mathbb{Z}} \otimes O_L\). On the other hand this element is of the form \(x\pi_i\) (for some \(i\)) times some element of \((\hat{\mathbb{Z}} \otimes O_L)^*\). It follows that \(x\pi_i\) is in \(O_L\). We replace \(\pi_i\) with \(x\pi_i\). The fact that \(\chi\chi\) is the identity shows that \(\pi_i\pi_i^\star\) is \(x^2 = (p_1 \cdots p_m)^{2N_k}\). The
field $\mathbb{Q}(\pi_1, \ldots, \pi_d)$ is $L$. Let us choose a basis $b_1, \ldots, b_{n_L}$ of $L$ over $\mathbb{Q}$ consisting of monomials in $\pi_1, \ldots, \pi_d$ of degree bounded by a constant depending on $n_L$ only. The discriminant of the ring $\mathbb{Z}[b_1, \ldots, b_{n_L}]$ is the discriminant of the matrix $A$ whose entries $A_{ij} = \text{Tr}_{L/\mathbb{Q}}(b_i b_j)$. The absolute values of the $A_{ij}$ are bounded by a uniform power of $(p_1 \cdots p_m)^N$. We see that the discriminant of $A$ is the sum of $n_L!$ terms whose absolute values are bounded by a uniform power of $(p_1 \cdots p_m)^N$ hence there is a uniform constant $t$ such that

$$|\text{disc} \mathbb{Z}[b_1, \ldots, b_{n_L}]| \leq (p_1 \cdots p_m)^{Nt}.$$ 

Since $\mathbb{Z}[b_1, \ldots, b_{n_L}]$ is an order in $O_L$, we have

$$|\text{disc} \mathbb{Z}[b_1, \ldots, b_{n_L}]| \geq d_L$$

Replacing $t$ with $5t$, we get the following inequality

$$\log(d_L)^{Nm} > d_L$$

Hence, if $I$ is such that $r(I)$ belongs to $M(\mathbb{Q})M(\hat{\mathbb{Z}})$, then

$$Nm > \frac{\log(d_L)}{t \log \log(d_L)}$$

Let us now consider elements of $(A_f \otimes L)^*$ that equal $P_i^{n_i}$ at the place $v_i$ and 1 outside of the places $v_i$ and where $|n_i| < N/2$ with $N$ and $m$ are such that $Nm \leq \frac{\log(d_L)}{\tau \log \log(d_L)}$. From the above inequality it follows that these elements have distinct non-trivial images in $M(A_f)/M(\mathbb{Q})M(\hat{\mathbb{Z}})$ by $r$. It follows that the set $r((A_f \otimes L)^*)/r((A_f \otimes L)^*) \cap M(\mathbb{Q})M(\hat{\mathbb{Z}})$ contains at least $(N/2)^m$ elements if $N$ and $m$ being such that $Nm \leq \frac{\log(d_L)}{\tau \log \log(d_L)}$. Taking $m = \frac{\log(d_L)}{2Nn_L \log \log(d_L)}$ (which is possible by the effective Chebotarev), we easily see that $(N/2)^m$ is at least $c \log(d_L)^N$ elements where $c$ is some real positive constant not depending on $s$ (but of course depending on $N$). □

3 Proof of main results.

In this section we prove the Theorems 1.2 and 1.3. Let $(G, X)$ be a Shimura datum and $K$ a compact open subgroup in $G(A_f)$. Let $C$ be an irreducible closed algebraic curve in $\text{Sh}_K(G, X)$ containing an infinite set $\Sigma$ of special points. For any special point $s$ of $C$ we let $L_s$ be the splitting field of the Mumford-Tate group of some element $\tilde{s}$ lying over $s$ and we let $d_s$ be the absolute value of the discriminant of $L_s$.

**Proposition 3.1** Suppose that the discriminant of $L_s$ is bounded as $s$ ranges through $\Sigma$. Then $C$ is of Hodge type.
Proof. We can assume that $G$ is semisimple of adjoint type (passing to the adjoint group does not change the property of $C$ being of Hodge type by the Proposition 2.2 of [3] and does not change the property that $d_{L_s}$ is bounded). Let us choose some faithful representation $V$ of $G$. Since the discriminant of $L_s$ is bounded as $s$ ranges through $\Sigma$, there are only finitely many possibilities for $L_s$. Hence we can assume that for all points $s$ in $\Sigma$, the field $L_s$ is the same field $L$. For any $s$ in $\Sigma$, we let $\tilde{s}$ be an element of $X$ such that $s = (\tilde{s}, g)$ for some $g$ in $G(\mathbb{A}_f)$. The reciprocity morphism $r_{\tilde{s}}$ gives a rational representation of the torus $T := \text{Res}_{L/\mathbb{Q}} G_{mL}$. This representation corresponds to a direct sum decomposition $V_L = \bigoplus_{\chi \in X} V_\chi$ for some subset $X$ of $X^*(T)$. As before, we identify the $G_L$-module $X_*$ with $\mathbb{Z}[G_L]$ and enumerate elements of $G_L$ thus getting a basis for $X^*(T)$. Using the fact that coordinates of the characters in the set $X$ are bounded in absolute value by $k$ which does not depend on $s$ (Proposition 2.4), we see that there are only finitely many possibilities for the set $X$ as $s$ ranges through $\Sigma$. Hence, possibly replacing $\Sigma$ by an infinite subset, we can and do assume that the set $X$ is constant as $s$ ranges through $\Sigma$. We can further assume that the dimensions of the $V_\chi$ are constant. We now see that the $\mathbb{Q}$-Hodge structures $V_{\tilde{s}}$ are isomorphic as $s$ ranges through $\Sigma$. Hence $C$ is of Hodge type by the main theorem of [3].

From the proof of this Proposition the Theorem 1.3 follows. Indeed, let $C$ be a curve in $\text{Sh}_K(G, X)$ that contains an infinite set $\Sigma$ of points such that the corresponding Mumford-Tate groups are isomorphic as $\mathbb{Q}$-tori. Since the Mumford-Tate groups of points of $\Sigma$ are isomorphic, they have the same splitting field. From the proof of the above proposition, it follows that $C$ contains an infinite set of special points such that the $\mathbb{Q}$-Hodge structures corresponding to these points via some faithful representation of $G$ lie in one isomorphism class. By the main result of [3], $C$ is of Hodge type.

In what follows we assume that $d_{L_s}$ is unbounded as $s$ ranges through $\Sigma$. From Propositions 2.1 and 2.2 of [3], it follows that we can assume $G$ to be semisimple of adjoint type and $C$ to be Hodge generic. Write $G = G_1 \times \cdots \times G_r$ where $G_i$ are simple. We can and do assume that $K$ is the product of compact open subgroups $K_p$ of the $G(\mathbb{Q}_p)$ and that $K$ is neat. Choose a faithful representation $V$ of $G$ through which we view $G$ as a closed subgroup of $\text{GL}_n(\mathbb{Q})$ such that $K$ is in $\text{GL}_n(\hat{\mathbb{Z}})$. Also choose a $K$-invariant lattice in $V_{\mathbb{A}_f}$. This gives a variation of $\mathbb{Z}$-Hodge structure on $\text{Sh}_K(G, X)$ (Section 3.2 of [3]). Let $X^+$ be a connected component of $X$. After possibly having replaced $C$ by an irreducible component of its image under a suitable Hecke correspondence we can and do assume that $C$ is contained in the image $S$ of $X^+ \times \{1\}$ in $\text{Sh}_K(G, X)$. Since $C$ contains an infinite set of special points which are in $\text{Sh}_K(G, X)(\mathbb{Q})$, $C$ is defined over a Galois number field $F$ containing the reflex field of $(G, X)$ (as an absolutely irreducible closed subscheme $Z_F$ of $\text{Sh}_K(G, X)_{\overline{F}}$).
Proposition 3.2 Assume the GRH for CM fields. There is a prime \( p \) and a point \( s \) in \( \Sigma \) which have the following properties

1. \( p \) splits \( \MT(V_s) \).
2. \( \MT(V_s)_{F_p} \) is a torus.
3. Let \( k \) be an integer as in the Corollary 7.4.4 of [3]. Then \( |\Gal(\overline{\Q}/F) \cdot s| > p^k \).

Proof. Let, as in the section 7 of [3], define the function \( i: \Sigma \rightarrow \Z \) as follows. For \( s \) in \( \Sigma \), let \( i(s) \) be the number of prime numbers \( p \) such that \( \MT(V_s)_{F_p} \) is not a torus. Then, by the Theorem 2.1 there exist real \( c_1 > 0 \) and \( c_2 > 0 \) such that for any \( s \) in \( \Sigma \) we have

\[
|\Gal(\overline{\Q}/F) \cdot s| > c_1 \log(d_{L_s})^{5k} c_2 i(s)!
\]

where \( k \) is the integer from the Corollary 7.4.4 of [3]. Using this inequality and the effective Chebotarev theorem (in the form stated in the Proposition 8.2 of [2]) we see that the number of primes split in \( L_s \) and smaller than \( |\Gal(\overline{\Q}/F) \cdot s|^{1/k} \) is bigger than \( i(s) \) when \( d_{L_s} \) is large enough. This finishes the proof of the proposition. \( \square \)

Take a prime \( p \) and a point \( s \) given by the previous proposition. Let \( m \) be an element of \( G(\Q_p) \) given by the Corollary 7.4.4 of [3] (this Corollary can be applied because of our Proposition 2.6). Then some Galois conjugate of \( s \) is in \( C \cap T_mC \) and since \( C \cap T_mC \) is defined over \( F \) the whole Galois orbit of \( s \) is contained in \( C \cap T_mC \). If the intersection \( C \cap T_mC \) was finite, its cardinality would be smaller than \( p^k \). By the choice of \( p \) and \( s \), this intersection can not be finite hence \( C \) is contained in \( T_mC \). We conclude that \( C \) is of Hodge type using the Theorem 7.1 of [3].

References

[1] Y. André G-functions and geometry Aspects of Mathematics, E13. Friedr. Vieweg and Sohn, Braunschweig, 1989.

[2] S.J. Edixhoven On the André-Oort conjecture for Hilbert modular surfaces. Progress in Mathematics 195 (2001), 133-155, Birkhauser. Available on Edixhoven’s homepage.

[3] S.J. Edixhoven, A. Yafaev Subvarieties of Shimura varieties. Annals of Mathematics, 157 (2003), 1-25.

[4] S.J. Edixhoven, B. Moonen, F. Oort Open problems in algebraic geometry. Bull. Sci. Math. 125 (2001), n. 1, 1-22.
[5] J.S. Milne *Canonical models of (mixed) Shimura varieties and automorphic vector bundles*. Automorphic forms, Shimura varieties and $L$-functions. Vol I. Proceedings of the conference held at university of Michigan. Ann Arbor, Michigan, 1988.

[6] B.J.J. Moonen. *Linearity properties of Shimura varieties. II*. Compositio Math. 114 (1998), no. 1, 3–35.

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