Hadronic contribution to the photon vacuum polarization:
a theoretical estimate

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ABSTRACT

A simple model for the hadronic contribution to the photon vacuum polarization function \(\Pi_{\text{had}}(q^2)\), for spacelike momenta, is presented. For small momenta, the two loop contribution from the pseudoscalar meson octet is computed from the chiral Lagrangian. The light quark contribution (which at low momentum gives the \(O(q^6)\) counterterm in the chiral Lagrangian) is calculated within a relativistic constituent quark model incorporating the momentum dependence of the quark mass. The perturbative gluons of QCD are included in a standard fashion. The total result is close to an estimate of \(\Pi_{\text{had}}(q^2)\) that is obtained directly from \(e^+e^- \rightarrow \text{hadrons}\) data. We further use our results for \(\Pi_{\text{had}}(q^2)\) to calculate the \(O(e^4)\) hadronic contribution to lepton magnetic moments and to calculate \(\alpha_{\text{QED}}(M_Z^2)\). A simpler model of constituent quarks with momentum independent masses gives less favourable results.

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Introduction

The photon vacuum polarization function, $\Pi(q^2)$, where

$$i\Pi_{\mu\nu}(q) = i\epsilon^2 \Pi(q^2) [q^2 g_{\mu\nu} - q_\mu q_\nu]$$

(1)

plays an important role in any high precision QED observable, e.g. the lepton anomalous magnetic moments $a_l$ ($l = e, \mu, \tau$), atomic energy levels, $e^+e^- \rightarrow e^+e^-$ scattering. For many practical applications, one requires $\Pi(q^2)$ for spacelike momenta only ($q^2 < 0$) so we will specialize to this region, where $\Pi(q^2)$ is a smooth and real function. We define our subtraction point such that $\Pi(0) = 0$.

The largest source of uncertainty lies in the hadronic contribution, $\Pi_{\text{had}}(q^2)$. The most reliable estimates of $\Pi_{\text{had}}(q^2)$ that are available[1] use experimental data for $R(s) = \frac{\sigma(e^+e^- \rightarrow \gamma^* \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \gamma^* \rightarrow \mu^+\mu^-)}$ (2)

in conjunction with dispersion relations. A simple analytic fit to this estimate is given in [1], and we will compare our result to this fit.

From the theoretical viewpoint, $\Pi_{\text{had}}(q^2)$ is in principle calculable in terms of the (current) quark mass parameters and a QCD scale (e.g. $\Lambda_{\overline{MS}}$). Our approach is a less ambitious one, namely to extract $\Pi_{\text{had}}(q^2)$ (and other related physical observables) from a simple QCD-inspired model. We will show that a simple picture can go a long way towards providing some theoretical understanding of $\Pi_{\text{had}}(q^2)$.

The behavior of $\Pi_{\text{had}}(q^2)$ at both low and high momenta is constrained by QCD. The high momentum description is provided by perturbative QCD. The low momentum dependence receives an important contribution from pseudoscalar meson loops, and this can be calculated using standard techniques of chiral Lagrangians. The relevant quantity occurs at $O(q^6)$ in the low energy derivative expansion, and thus an associated two-loop calculation must be performed. Knowledge of an $O(q^6)$ counterterm is also necessary, and at present any estimate of this quantity is model dependent.

In this paper we shall be using a relativistic constituent quark model of the three light quarks to smoothly interpolate between the low and high energy QCD contributions to $\Pi_{\text{had}}(q^2)$. This is a gauged nonlocal constituent (GNC) quark model which incorporates the momentum dependence of the quark mass as a natural regulator. The pion decay constant and all other quantities appearing in the chiral Lagrangian to $O(q^4)$ have been expressed in terms of this mass function. In particular the standard quantities[2] $L_1, L_2, L_3, L_9$, and $L_{10}$ are well described in terms of one parameter (denoted by $A$ below) appearing in the mass function[3]. Reasonable values for the other $L_i$‘s and the current quark masses are obtained as well[4]. The model has also been successfully applied to certain other quantities, the pion electromagnetic form factor
and the vector-minus-axial two point function, beyond $O(q^4)$. There has also been a recent discussion of the advantages of our nonlocal regularization in the context of anomalous processes. These results suggest that some of the more essential aspects of nonperturbative QCD are accounted for by the momentum dependence of the quark mass.

There are two GNC quark loop diagrams which contribute to $\Pi_{\text{had}}(q^2)$. At high momenta the effects of the constituent quark mass become negligible and the GNC diagrams smoothly approach the naive one-quark-loop contribution to $\Pi_{\text{had}}(q^2)$. We add perturbative QCD corrections to the model description of the three light quarks. More precisely we add the hard gluonic corrections beginning at $O(\alpha_s)$ as extracted from a standard perturbative QCD calculation of $\Pi_{\text{had}}(q^2)$. The contributions to $\Pi_{\text{had}}(q^2)$ from the $c$ and $b$ quarks will be represented completely by perturbative QCD.

We will consider the implications that our calculated $\Pi_{\text{had}}(q^2)$ has for the hadronic contributions to the muon and tau magnetic moments. Of the two, the muon magnetic moment is less affected by the various uncertainties. We will also consider the QED running coupling at large momentum, $\alpha_{\text{QED}}(M_Z^2)$.

Before discussing the various components of our calculation in more detail, we present the main results in figs. 1 and 2. The effects of the meson loops and the hard gluonic corrections are shown separately. They are added to the GNC quark contribution to produce our total result for $\Pi_{\text{had}}(q^2)$. We find that the total result is quite close to the experimentally-based estimate. We note as well that for small $-q^2$, the GNC quark-loop contribution is significantly larger than a more naive constituent quark model with momentum independent masses, for example with the typical values $m_u = m_d = 330$ MeV and $m_s = 550$ MeV. This comparison is shown in fig. 3.

Although we divide our calculation into various parts, we stress that we are consistently using one model to describe $\Pi_{\text{had}}(q^2)$ for all spacelike momenta. This is the GNC model with perturbative gluonic effects added. The GNC model includes the pseudoscalar mesons, and as we have said, it nicely reproduces the standard chiral Lagrangian of low energy QCD. We will therefore describe our calculation of $\Pi_{\text{had}}(q^2)$ at low $-q^2$ using the language of chiral Lagrangians. In the model all meson dynamics, including the meson kinetic terms, are generated through quark loops. At higher energies the compositeness of these mesons will become evident via form factors, and their further contribution to $\Pi_{\text{had}}(q^2)$ will be damped out. We will enforce this feature of the model by cutting off the meson-loop contribution at a scale of order $m_\rho$. 

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Pseudoscalar mesons

For \( -q^2 \) much less than \( m_\rho^2 \), \( \Pi_{\text{had}}(q^2) \) can be evaluated from the \( SU(3)_L \times SU(3)_R \) chiral Lagrangian with explicit symmetry breaking terms. The one-meson-loop contribution corresponds to a calculation at \( \mathcal{O}(q^4) \) in the low energy expansion and the counterterm at this order is eliminated by the definition \( \Pi_{\text{had}}(0) = 0 \). The leading \( q^2 \) behavior of \( \Pi_{\text{had}}(q^2) \) requires a calculation at \( \mathcal{O}(q^6) \), and this corresponds to a two-loop calculation. It is this piece which contains the leading logarithms \( \ln(\mu/m_{\pi}) \) and \( \ln(\mu/m_K) \). We will add this two-loop piece to the finite one-loop contribution, and the result should provide a good approximation for \( \Pi_{\text{had}}(q^2) \) at low \( -q^2 \). All diagrams with more than two meson loops are of \( \mathcal{O}(q^8) \) or higher, and are therefore neglected.

We will follow the definition of dimensional regularization counterterms \( (L_i^r(\mu)) \) given in [2]. We do not enforce the \( \mathcal{O}(q^2) \) equations of motion, so we must reinstate two terms [3] that are usually removed by the equations of motion. In the end our result can be expressed in terms of the \( L_i^r(\mu) \)'s of [2] as well as one new parameter, \( C^r(\mu) \), which is the \( \mathcal{O}(q^6) \) counterterm.

\[
\Pi_{\text{had}}(q^2) = \Pi_{\text{had}}^{(4)}(q^2) + \Pi_{\text{had}}^{(6)}(q^2)
\]

\[
\Pi_{\text{had}}^{(4)}(q^2) = \frac{1}{3(4\pi)^2} \left[ \frac{2}{3} - \left( 1 - \frac{4m_\pi^2}{q^2} \right) \int_0^1 dz \ln \left( 1 - z(1-z) \frac{q^2}{m_\pi^2} \right) + \frac{2}{3} - \left( 1 - \frac{4m_K^2}{q^2} \right) \int_0^1 dz \ln \left( 1 - z(1-z) \frac{q^2}{m_K^2} \right) \right]
\]

\[
\Pi_{\text{had}}^{(6)}(q^2) = \frac{3}{3(4\pi f_0)^2} \left[ C^r(\mu) + T_1(\mu) + T_2 + T_3(q^2) + T_4(q^2) \right]
\]

\( f_0 \) is the pseudoscalar decay constant in the chiral limit, and explicit expressions for the \( T_i \)'s are given in the appendix. We require that the \( \mu \) dependence of \( C^r(\mu) \) cancels that of \( T_1(\mu) \).

To obtain the numerical value of \( \Pi_{\text{had}}(q^2) \) for small \( -q^2 \), we need only the values of the coefficients \( C^r(\mu) \) and \( L_i^r(\mu) \). We will choose \( \mu = m_\rho \) and take the experimental values of the \( L_i^r(m_\rho) \)'s from [4]. We will give below the value of \( C^r(m_\rho) \) from the GNC quark model.

But to isolate a purely meson-loop contribution to \( \Pi_{\text{had}}(q^2) \) we will remove the \( C^r(m_\rho) \) from (5). This term will effectively be included in the low momentum behavior of the quark-loop graphs of the GNC quark model. The remaining terms in (5) represent the two-loop meson contribution. We will allow this plus the one-loop result to contribute to the growth of \( \Pi_{\text{had}}(q^2) \) up to \(-q^2 = m_\rho^2 \). On these scales there may be substantial error in a two loop calculation; we consider the implications of this below.

The one and two loop meson contributions are displayed in fig. [3] and compared to
the quark-loop contribution. The latter dominates even though the contributions at low \(-q^2\) are inversely proportional to the mass squares of the quark and pion. (The relative enhancement of the quark loops is due mainly to the colour and spin degrees of freedom.) But the meson-loop contribution is still significant. We find an uncertainty of 20\% in the meson-loop contribution at \(m_\mu^2\) due to the uncertainty in the \(L_r^i\)’s.

The fact that the one-loop meson contribution is smaller than the two-loop contribution is consistent with the dominance of the \(\rho\) meson. The \(\rho\) is, of course, integrated out of the chiral Lagrangian which means that its effects are incorporated into the counterterms \(L_r^i, C_r, \ldots\). Since these counterterms do not appear in (4), the one-loop meson contribution is suppressed.

GNC quarks

The GNC Lagrangian\[3\] contains the octet of pseudoscalar mesons \(\pi^a\) and a quark triplet \(\psi\) with a dynamical quark mass \(\Sigma(−q^2)\) and a current quark mass matrix \(\mathcal{M}\).

\[
\mathcal{L}_{\text{GNC}}(x, y) = \bar{\psi}(x)\delta(x − y)\left[ i\gamma^\mu(\partial_\mu − iR_\mu(y)) − \mathcal{M}\right]\psi(y) − \bar{\psi}(x)\Sigma(x − y)\xi(x)X(x, y)\xi(y)\psi(y) \tag{6}
\]

\[
X(x, y) = P\exp \left[ −i \int_x^y \Gamma_\mu(z)d\mu\right] \tag{7}
\]

\[
\Gamma_\mu(z) = \frac{i}{2} [\xi(z)(\partial_\mu − iR_\mu(z))\xi^\dagger(z) + \xi^\dagger(z)(\partial_\mu − iL_\mu(z))\xi(z)] \tag{8}
\]

\[
\xi(x) = \exp \left[ −\frac{i\gamma_5}{f_0} \sum_{a=1}^8 \lambda^a \pi^a(x) \right] \tag{9}
\]

\[
\Sigma(−q^2) = \frac{(A + 1)m_\mu^2}{Am_0^2 − q^2} \tag{10}
\]

Note that \(X(x, y)\) is a path-ordered exponential. \(L_\mu = V_\mu − A_\mu\gamma_5\) and \(R_\mu = V_\mu + A_\mu\gamma_5\) are left and right handed external gauge fields, respectively. For \(\mathcal{M} = 0\) the model has local \(SU(3)_L \times SU(3)_R\) symmetry, like QCD in the presence of external gauge fields. Numerically, we use the current quark masses \(m_u = m_d = 8\, \text{MeV}\) and \(m_s = 180\, \text{MeV}\).

The dynamical quark mass \(\Sigma(−q^2)\) in (10) is the Fourier transform of the \(\Sigma(x − y)\) appearing in (3). The parameter \(A\) specifies the value of \(m_0\) through its relation to \(f_0\).

\[
f_0^2 = \frac{N_c}{8\pi^2} \int_0^\infty ds \frac{s\Sigma(s)[2\Sigma(s) − s\Sigma'(s)]}{[s + \Sigma^2(s)]^2} \tag{11}
\]

For the most part we will use the values \(f_0 = 84\, \text{MeV}\) and \(A = 2\) which correspond to \(m_0 = 317\, \text{MeV}\).
The GNC quark contribution comes from two one-quark-loop contributions to the vacuum polarization — one with the two external photons attached at one vertex, and the other with the two photons attached at two distinct vertices. The resulting contribution to $\Pi_{\text{had}}(q^2)$ is shown in fig. 2. In this contribution we have included the small effects of the naive one-loop graphs of the $c$ and $b$ quarks. (The commonly-used ranges $1.3 < m_c < 1.7$ GeV and $4.7 < m_b < 5.3$ GeV produce an uncertainty less than 2%.) The gluonic corrections to light and heavy quark-loop graphs will be treated below. We may also consider the sensitivity of the GNC result to the quantities $f_0$ and $A$ within an allowed range $84$ MeV $< f_0 < 88$ MeV and $2 < A < 3.3$.[3] The GNC contribution is reduced by less than 15% and we find that $0.19 \lesssim C'(m_\rho) \lesssim 0.20$.

In fig. 3, we compare our $u, d, s$ GNC result to the analogous contribution from a more naive model with momentum independent masses of typical values $m_u = m_d = 330$ MeV and $m_s = 550$ MeV. Note that the shapes of the two curves are quite different, and this is true for any values of the momentum independent quark masses. The GNC curve is more consistent with the fit of Burkhardt et. al.[1]

**Perturbative gluons**

The final contribution that must be considered is due to radiative gluons. We have already included one-quark-loop diagrams with the correct non-zero masses (momentum dependent masses for the light quarks), so we wish to extract the gluonic corrections given by terms containing $\alpha_s$. We use the following dispersion relation

$$\left[\Pi_{\text{had}}(q^2) - \Pi_{\text{had}}(0)\right]_{\text{pert}} = -\frac{q^2}{12\pi^2} \int_{m_c^2}^{\infty} ds \frac{R(s) - R_0(s)}{s(s-q^2)}$$

where $q^2 < 0$. $R(s)$ has been calculated using the $\overline{MS}$ scheme to $\mathcal{O}((\alpha_s/\pi)^3)$ in [3] for $N_f$ quark flavours with charges $Q_i$.

$$R(s) = 3 \left(\sum Q_i^2\right) \left[1 + \frac{\alpha_s}{\pi} + r_1 \left(\frac{\alpha_s}{\pi}\right)^2 + r_2 \left(\frac{\alpha_s}{\pi}\right)^3\right] + \mathcal{O}\left(\frac{\alpha_s}{\pi}\right)^4$$

$$r_1 = 1.9857 - 0.1153 N_f$$

$$r_2 = -6.6368 - 1.2001 N_f - 0.0052 N_f^2 - 1.2395 \left[\frac{(\sum Q_i^2)^2}{3\sum(Q_i^2)}\right]$$

$R_0(s)$ is the value of $R(s)$ when $\alpha_s$ is set to zero, and its appearance in [12] removes the naive one-quark-loop result.

$\alpha_s(s)$ is obtained by solving the QCD $\beta$ function.[4]

$$\frac{d\alpha_s}{d\mu} = -\frac{\beta_0}{2\pi} \alpha_s^2 - \frac{\beta_1}{8\pi^2} \alpha_s^3 - \frac{\beta_2}{32\pi^3} \alpha_s^4 + \mathcal{O}(\alpha_s^5)$$
\[
\begin{align*}
\beta_0 &= 11 - \frac{2}{3} N_f \\
\beta_1 &= 102 - \frac{38}{3} N_f \\
\beta_2 &= \frac{2857}{2} - \frac{5033}{18} N_f + \frac{325}{54} N_f^2
\end{align*}
\] (17) (18) (19)

We have cut off the integral in (12) at \( s = m_c^2 \). The result is an estimate of hard gluonic corrections missing in the GNC model, and it is shown separately in figs. 1 and 2. The gluonic contribution contains an uncertainty of roughly 23\%(16\%) at \(-q^2 = m_s^2(M_Z^2)\) corresponding to the range \(.11 < \alpha_s(M_Z^2) < .13\) and an uncertainty of 26\%(6\%) at \(-q^2 = m_c^2(M_Z^2)\) for \(1.3 < m_c < 1.7\) GeV.

By our choice of the cutoff in (12) we have added to the GNC quark model only those perturbative corrections which can be reliably calculated. Our intent is to see how the model does at describing the contributions that cannot be calculated perturbatively.

**Lepton anomalous magnetic moments**

To lowest order in \( \alpha_{QED} \) the hadronic contribution to a lepton’s magnetic moment has the following algebraic form [10].

\[
\begin{align*}
\alpha_{l,had}^{(4)} &\equiv \left(\frac{g-2}{2}\right)^{(4)}_{l,had} = \frac{1}{\pi} \int_{4 m_l^2}^{\infty} \frac{dt}{t} X_l(t) \\
X_l(t) &= \frac{e^4 m_l^2}{4 \pi \sqrt{t(t-4 m_l^2)}} \int_{-1}^{1} dz \left(\frac{1 + 3 z}{2}\right) \Pi_{had}(f_l^2) \\
f_l^2 &= -\frac{1}{2} (t-4 m_l^2)(1-z)
\end{align*}
\] (20) (21) (22)

The superscript “\((4)\)” reminds us that this is all at \( O(e^4) \). Notice that \( f_l^2 \leq 0 \) for the entire range of integration, so we only need the value of \( \Pi_{had}(q^2) \) for spacelike \( q^2 \).

We perform the double integration by fitting our numerical result for \( \Pi_{had}(q^2) \) to a piecewise-analytic function of \( q^2 < 0 \). Since \( m_\mu < m_c \), our estimated \( \alpha_s \) corrections to \( \Pi_{had}(q^2) \) have essentially no effect on \( \alpha_{\mu,had}^{(4)} \). For the case of \( \alpha_{r,had}^{(4)} \) the uncertainties in the meson-loop contribution and the \( \alpha_s \) corrections are both larger, with the former dominating. We take a 40\% error for the meson-loop contribution in this case, double the naive error due to the uncertainty in the \( L_i^r \)’s as noted above, to include possible corrections from higher order effects in the chiral Lagrangian. The results are

\[
\begin{align*}
\alpha_{\mu,had}^{(4)} &= (6.3 \pm 0.5) \times 10^{-8} \\
\alpha_{r,had}^{(4)} &= (3.2 \pm 0.1) \times 10^{-6}
\end{align*}
\] (23) (24)
Any other choice of \((f_0, A)\) in the range \((84 \text{ MeV} < f_0 < 88 \text{ MeV}, 2 < A < 3)\) would reduce our result for \(a^{(4)}_{\mu,\text{had}}(a^{(4)}_{\tau,\text{had}})\) by less than 10\% (6\%).

These results should be compared to the calculation from integrating an experimental determination of \(R(q^2)\). \((R(q^2)\) is defined in (2). The precise form of the integral is given by Kinoshita et. al.\[11\]

\[
\text{experiment} \Rightarrow \begin{cases} 
    a^{(4)}_{\mu,\text{had}} = (7.05 \pm 0.08) \times 10^{-8} & \text{[11]} \\
    a^{(4)}_{\tau,\text{had}} = (3.6 \pm 0.3) \times 10^{-6} & \text{[12]} 
\end{cases}
\] (25)

On the other hand the parametrization of the experimentally-determined \(\Pi_{\text{had}}(q^2)\) due to Burkhardt et. al.\[\] may be used directly in eqs. (20-22). This gives

\[
\text{experimental fit} \Rightarrow \begin{cases} 
    a^{(4)}_{\mu,\text{had}} = 6.63 \times 10^{-8} \\
    a^{(4)}_{\tau,\text{had}} = 3.45 \times 10^{-6} 
\end{cases}
\] (26)

The authors of \[\] seem to claim that the uncertainty should be less than 5\%, making this result for \(a^{(4)}_{\mu,\text{had}}\) noticeably smaller than the preceding result of (25).

Finally, we consider quarks with momentum independent masses of \(m_u = m_d = 330\) MeV and \(m_s = 550\) MeV, and add the pseudoscalar mesons, heavy quarks and \(\alpha_s\) corrections exactly as discussed above.

\[
m_u = m_d = 330\text{MeV}, m_s = 550\text{MeV} \Rightarrow \begin{cases} 
    a^{(4)}_{\mu,\text{had}} = (4.3 \pm 0.5) \times 10^{-8} \\
    a^{(4)}_{\tau,\text{had}} = (2.2 \pm 0.1) \times 10^{-6} 
\end{cases}
\] (27)

Clearly the GNC model is a significant improvement. In order to reproduce the GNC results for both lepton magnetic moments in (23) we would require different sets of masses, for example \(m_{u,d,s} = 243\) MeV and \(m_{u,d,s} = 201\) MeV for \(a^{(4)}_{\mu,\text{had}}\) and \(a^{(4)}_{\tau,\text{had}}\) respectively. This illustrates the fact mentioned previously that momentum independent quark masses cannot reproduce the shape of our \(\Pi_{\text{had}}(q^2)\) for the entire range \(-q^2 < 0\).

**QED running coupling**

It is straightforward to evaluate the hadronic contribution to the running of \(\alpha_{QED}(-q^2)\) from our results.\[13\] Using \(f_0 = 84\) MeV and \(A = 2\) we find

\[
\Delta \equiv \left[ \frac{1}{\alpha_{QED}(0)} - \frac{1}{\alpha_{QED}(M_Z^2)} \right]_{\text{had}} = 4\pi [\Pi_{\text{had}}(-M_Z^2) - \Pi_{\text{had}}(0)] = 3.68 \pm 0.07
\] (28)
The error is dominated by the error in the perturbative QCD contribution. This may be compared to the result in [1]:

$$\Delta = 3.94 \pm 0.12$$  \hspace{1cm} (29)

When combined with the well-known non-hadronic effects, our result (28) implies

$$\alpha^{-1}_{QED}(M^2_\pi) = 129.05 \pm 0.07$$  \hspace{1cm} (30)

We note that this is also consistent with an independent determination using recent LEP data,[14] (In that analysis additional corrections are included to define an effective coupling, $\alpha_{\text{eff}}(M_\pi)$). Any other choice of $(f_0,A)$ in the range $(84 \text{ MeV} < f_0 < 88 \text{ MeV}, 2 < A < 3)$ would reduce the value of $\Delta$ by less than 2%.

**Comments**

In this paper we have used a nonlocal constituent quark model for the description of $\Pi_{\text{had}}(q^2)$ at spacelike momenta. The success of our quark level description could be considered to be a manifestation of “duality”, and it clearly relies on being far removed from the resonance structure of QCD appearing for timelike momenta. The various uncertainties we have quoted occur within the model itself, and they are not intended as an *a priori* estimate of how well the model should resemble QCD. It is only after a comparison with experimental data that our model is able to shed light on some of the essential features of QCD dynamics.

After completion of this work, we received a preprint[15] containing an independent theoretical estimate of the hadronic vacuum polarization and the muon magnetic moment.

**Appendix**

The complete result for $\Pi_{\text{had}}(q^2)$ for small $-q^2$ can be derived from the chiral Lagrangian, and is given in (8). Here we provide the explicit form of the $T_i$ parameters, using the notation of [2] at the renormalization scale $\mu$. (The $\mu$ and $q^2$ dependence of all parameters is implicit in this appendix to simplify the notation.) The parameters $\tilde{m}_\pi$, $\tilde{m}_K$ and $\tilde{m}_\eta$ are the lowest order mass values. The relation of these parameters to the physical masses can be taken from eq. (10.7) of [2]. The $T_i$’s are well-behaved functions of spacelike $q^2$ such that $q^2 T_i$ vanishes when $q^2 = 0$. The apparent $\mu$ dependence in $T_2$, $T_3$, and $T_4$ cancels out.

$$T_1(\mu) = 8l_\pi + 8l_K - \frac{2}{3(4\pi)^2} (l_\pi^2 + l_K^2 + l_\pi l_K)$$  \hspace{1cm} (31)
\[ T_2 = \frac{8}{3} \left[ L_9^r - \frac{1}{8(4\pi)^2} \left( \frac{1}{6} + l_\pi + l_K \right) \right] \]  
\[ T_3(q^2) = Y_1\phi_\pi + Y_2\phi_K + \frac{8\tilde{m}_\pi^2}{q^2} \left[ 2L_4^r + L_5^r - 4L_6^r - 2L_8^r - \frac{1}{24(4\pi)^2} \left( 3l_\pi - \frac{1}{3}l_\eta \right) \right] + \frac{8\tilde{m}_K^2}{q^2} \left[ 4L_4^r + L_5^r - 8L_6^r - 2L_8^r - \frac{1}{18(4\pi)^2} l_\eta \right] \]  
\[ T_4(q^2) = Y_3\phi_\pi + Y_4\phi_\pi\phi_K + Y_5\phi_K^2 \]  
\[ Y_1 = 4 \left( 1 - 4\frac{\tilde{m}_\pi^2}{q^2} \right) \left[ L_9^r - \frac{1}{12(4\pi)^2} \left( \frac{1}{2} + 2l_\pi + l_K \right) \right] + 48\frac{\tilde{m}_\pi^2}{q^4} \left[ L_4^r + L_5^r - 2L_6^r - 2L_8^r - \frac{1}{24(4\pi)^2} \left( 3l_\pi + \frac{1}{3}l_\eta \right) \right] + 96\frac{\tilde{m}_\pi^2\tilde{m}_K^2}{q^4} \left[ L_4^r - 2L_6^r + \frac{1}{36(4\pi)^2} l_\eta \right] \]  
\[ Y_2 = 4 \left( 1 - 4\frac{\tilde{m}_\pi^2}{q^2} \right) \left[ L_9^r - \frac{1}{12(4\pi)^2} \left( \frac{1}{2} + l_\pi + 2l_K \right) \right] + 48\frac{\tilde{m}_K^2}{q^4} \left[ 2L_4^r + L_5^r - 4L_6^r - 2L_8^r - \frac{1}{9(4\pi)^2} l_\eta \right] + 48\frac{\tilde{m}_\pi^2\tilde{m}_K^2}{q^4} \left[ L_4^r - 2L_6^r + \frac{1}{36(4\pi)^2} l_\eta \right] \]  
\[ Y_3 = \frac{-1}{6(4\pi)^2} \left( 1 - 4\frac{\tilde{m}_\pi^2}{q^2} \right)^2 \]  
\[ Y_4 = \frac{-1}{6(4\pi)^2} \left( 1 - 4\frac{\tilde{m}_\pi^2}{q^2} \right) \left( 1 - 4\frac{\tilde{m}_K^2}{q^2} \right) \]  
\[ Y_5 = \frac{-1}{6(4\pi)^2} \left( 1 - 4\frac{\tilde{m}_K^2}{q^2} \right)^2 \]  
\[ l_P = \ln \left( \frac{\tilde{m}_P}{\mu} \right) \]  
\[ \phi_P = \int_0^1 dz \ln \left( 1 - z(1-z)\frac{q^2}{\tilde{m}_P^2} \right) \]  
\[ P = (\pi, K, \eta) \]
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**Figure Captions**

Figure 1: All contributions to our total result for $\Pi_{had}(q^2)$ for $|q| < 800$MeV. The fit from [1] is also shown.

Figure 2: All contributions to our total result for $\Pi_{had}(q^2)$ for $|q| < 3$GeV. The fit from [1] is also shown.

Figure 3: Comparison of the contribution to $\Pi_{had}(q^2)$ from the $u$, $d$ and $s$ quarks in two distinct approaches: the GNC model (as used in the present analysis) and a simple model with momentum independent masses $m_u = m_d = 330$ MeV and $m_s = 550$ MeV.
This figure "fig2-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9304264v2
This figure "fig3-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9304264v2
This figure "fig4-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9304264v2
Figure 1

- Burkhardt et. al.
- Total model result
- u,d,s (GNC) + c,b
- 1 & 2 meson loops
- 1 meson loop
- Perturbative gluons

\[ \Pi_{\text{had}}(q^2) \]

\[ |q| \text{ [MeV]} \]
Burkhardt et al. meson loops

u,d,s (GNC) + c,b

perturbative gluons

Figure 2

\( \Pi_{\text{had}}(q^2) \)

\( |q| \) [GeV]

\( 0 \)

\( 0.5 \)

\( 1 \)

\( 1.5 \)

\( 2 \)

\( 2.5 \)

\( 3 \)

0

0.02

0.04

0.06

0.08

0.1

Burkhardt et. al.

total model result

u,d,s (GNC) + c,b

1 & 2 meson loops

perturbative gluons
Figure 3

\[ \Pi_{\text{had}}(q^2) \]

- u,d,s (GNC model)
- \( m_u = m_d = 330\text{MeV}, m_s = 550\text{MeV} \)