Learning Correlated Stackelberg Equilibrium in General-Sum Multi-Leader-Single-Follower Games

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Abstract

Many real-world strategic games involve interactions between multiple players. We study a hierarchical multi-player game structure, where players with asymmetric roles can be separated into leaders and followers, a setting often referred to as Stackelberg game or leader-follower game. In particular, we focus on a Stackelberg game scenario where there are multiple leaders and a single follower, called the Multi-Leader-Single-Follower (MLSF) game. We propose a novel asymmetric equilibrium concept for the MLSF game called Correlated Stackelberg Equilibrium (CSE). We design online learning algorithms that enable the players to interact in a distributed manner, and prove that it can achieve no-external Stackelberg-regret learning. This further translates to the convergence to approximate CSE via a reduction from no-external regret to no-swap regret. At the core of our works, we solve the intricate problem of how to learn equilibrium in leader-follower games with noisy bandit feedback by balancing exploration and exploitation in different learning structures.

1 Introduction

Game theory studies the interactions between multiple strategic players or agents (Roughgarden, 2010; Osborne et al., 2004). Many real-world domains such as economics and policy making can be described using a hierarchical game structure among the players, where the two levels of players have asymmetric roles and can be partitioned into leaders and followers (Sherali et al., 1983). This type of games is called Stackelberg or leader-follower games. Depending on the game structures, the Stackelberg game literature can be categorized into single-leader-single-follower (SLSF) games (Conitzer and Sandholm, 2006; Blum et al., 2014), single-leader-multi-follower (SLMF) games (Ramos et al., 2016; Salas et al., 2020), multi-leader-single-follower (MLSF) games (Aussel et al., 2016; Escobar and Jofré, 2008; Hu and Ralph, 2007; Gan et al., 2018), and multi-leader-multi-follower (MLMF) games (Mallozzi and Messalli, 2017; Sherali et al., 1983). We focus on the repeated general-sum MLSF game setting, where at each round, multiple leaders first make a collective decision, and then a single follower reacts to the leaders’ decision. Such a game setting has broad implications in real-world problems, such as security games (Gan et al., 2018), deregulated electricity markets (Aussel et al., 2016), and industrial eco-parks (Ramos et al., 2016). Most of the MLSF games literature assume that the game parameters (e.g., players’ loss functions) are known a-priori, and they focus on finding equilibrium in MLSF games via optimization methods (Leyffer and Munson, 2010; Kulkarni and Shanbhag, 2014). In this paper, we present the first study of MLSF games from a learning perspective. In particular, we ask the fundamental research question:

Can we design efficient learning algorithms that provably reach equilibrium in repeated general-sum MLSF games?

An immediate follow-up question is: what is an appropriate equilibrium concept for this setting? Gan et al. (2018) propose an equilibrium concept called Nash stackelberg equilibrium (NSE) for multi-defender-single-attacker security games. But as pointed out in Gan et al. (2018), when there exist malicious defenders, an ε-NSE may not exist for every ε > 0, which means that NSE does not always exist in general-sum MLSF games. Inspired by this and the correlated equilibrium concept (Aumann, 1974, 1987), we propose a more viable and realistic equilibrium concept, which we call Correlated Stackelberg Equilibrium (CSE).

Using CSE as the equilibrium concept, we give an affirmative answer to the above question for a broad range of repeated general-sum MLSF games. We summarize our key contributions as follows: (i) We start with a simpler setting where each leader knows the loss functions of the follower and itself, and prove that classical adversarial online
learning algorithms like Hedge (Cesa-Bianchi and Lugosi, 2006; Littlestone and Warmuth, 1994) can learn to reach approximate CSE. The result also holds when it is extended to a slightly more complicated setting where the leaders do not know the loss function of the follower, but have access to an oracle that returns the best response of the follower. (ii) Building on the insights from the simpler settings, we then study the more challenging setting of MLSF games with noisy bandit feedback. We design a distributed learning algorithm (called αExp3-UCB) for leaders and the follower. We first study a degenerate scenario of MLSF games, which is the SLSF games setting with noisy bandit feedback (a special case of CSE with a single leader), and prove that it converges to the Stackelberg equilibrium using the αExp3-UCB algorithm. This result is a non-trivial improvement of Bai et al. (2021) in the online learning setting. (iii) In the ultimate setting of MLSF games, we first provide complexity analysis that shows the hardness of the αExp3-UCB algorithm in a true MLSF setting. Based on the analysis, we then devise a more efficient two-stage learning algorithm that still provably learns to converge to CSE in the MLSF setting.

2 Related Work

Leader-follower games There is a line of works that studies optimization-based methods for finding equilibrium in MLSF games (Leyffer and Munson, 2010; Kulkarni and Shanbhag, 2014; Vicente and Calamai, 1994). They center around a bilevel optimization problem structure called equilibrium program with equilibrium constraints (EPEC), and aim at devising efficient optimization methods to solve the bilevel optimization problem. As important application domains, many works study the specific MLSF scenarios of deregulated electricity markets (Allevi et al., 2018; Escobar and Jofrè, 2008), or multi-defender security game (Jiang et al., 2013; Gan et al., 2018; Basilico et al., 2017). Gan et al. (2018) propose NSE as an equilibrium concept for the multi-defender security games. They show that an exact equilibrium may fail to exist, and deciding whether it exists is NP-hard. Moreover, they show that an approximate ε-NSE may not exist for every ε > 0 in the presence of malicious defenders. This motivates us to come up with CSE as a more viable equilibrium concept.

Learning Stackelberg equilibrium in games The line of works that is close to our work is the literature on learning Stackelberg equilibrium in SLSF games. Peng et al. (2019) study the problem of learning the optimal leader strategy in Stackelberg games with a follower best response oracle, which means that the leader knows the exact follower best response to its action. Such kind of assumption is also made in other works along that line Letchford et al. (2009); Blum et al. (2014). Two player zero-sum games have been extensively studied in the broader games and learning literature (Fasoulakis et al., 2021; Rakhlin and Sridharan, 2013; Balduzzi et al., 2019), and it is known that Stackelberg equilibrium is equivalent to Nash equilibrium in convex-concave setting due to von Neumann’s minimax theorem (v. Neumann, 1928). The equilibrium in two player zero-sum game is easier to compute because the two players essentially have the same objective for any action pair. This property no longer exists for general-sum Stackelberg games, and therefore general-sum Stackelberg games are considered harder to learn. Bai et al. (2021) study learning algorithms for Stackelberg equilibrium in general-sum SLSF games with noisy bandit feedback in a batch version. Their proposed algorithm needs to query each pair of leader-follower actions for sufficient rounds to calculate the empirical mean. Therefore, it requires a centralized authority to learn the equilibrium instead of distributed player self-learning, as is in this paper. Moreover, we focus on the more challenging problem of learning CSE in general-sum MLSF games with noisy bandit feedback, which is much more computationally expensive because of the exponentially growing joint action space for multiple leaders. To the best of our knowledge, this is the first work on MLSF games of any kind from a learning perspective.

3 Preliminary

3.1 Repeated general-sum MLSF games

A general-sum MLSF game is represented as a tuple \( \{A, B, l\} \). In this setting, two levels of decision makers are considered: a set of \( m \) leaders \( 1, \ldots, m \) and one follower \( f \). \( A_i \) represents the action set of leader \( i \), and \( B \) represents the action set of follower \( f \). For each leader \( i \), \( |A_i| = n_i \) is the cardinality of its action set. We assume \( n_1 = n_2 = \cdots = n_m = n \) for clarity, and all the results can be generalized when they are not the same. We denote \( A = A_1 \times A_2 \times \cdots \times A_m \) the action set for all the leaders, and \( A \times B \) the joint action set of all the leaders and the follower. We assume all action sets are discrete and finite. For any joint action \( (a, b) \in A \times B \), the loss function for leader \( i \) is \( l_i = \{ l_i(a, b) : A \times B \rightarrow [0, 1] \} \). To distinguish the loss function and the noisy (i.e., stochastic) loss value for a given data sample, we use a different notation \( \xi_i(a, b) \in [0, 1] \) to represent the noisy form loss value for leader \( i \). By definition, \( l_i(a, b) = E[\xi_i(a, b)] \). Similarly for follower \( f \), the loss function and the noisy loss value for one data sample are respectively represented as \( l_f = \{ l_f(a, b) : A \times B \rightarrow [0, 1] \} \) and \( \xi_f(a, b) \in [0, 1] \), where \( l_f(a, b) = E[\xi_f(a, b)] \). For each leader and the follower, the goal is to minimize its own loss function.

In a repeated game, the players play iteratively at each round \( t \), with a time horizon of \( T \) rounds. Because of the asynchronous moves, the leader usually maintains a
Assumption 1. For simplicity, we assume that for every action $a_i \in A_i$, the set of follower best responses $Br(a) = \arg \min_{b \in B} l_f(a, b)$ is a singleton, i.e., $|Br(a)| = 1$. This means that for any action played by leaders, the follower has a unique best response.

Remark. There has been ambiguity in the equilibrium formulation in Stackelberg games when there may be multiple follower best responses, since there is no explicit ways of breaking ties between multiple follower best responses. A typical way for tie-breaking is to assume that the follower is either in favor of the leader utility (optimistic) or against it (pessimistic). However, we found that this tie-breaking rule incurs further ambiguity in the MLSF games setting because if each leader assumes that follower is in favor of (or against) the leader itself, then the leaders are assuming different follower best responses (one for each leader). This inconsistency makes it infeasible to converge to any equilibrium in a repeated MLSF games setting. Therefore, we follow the common practice in similar problems such as multi-armed bandit games (Auer et al., 2002b; Audibert et al., 2010) and MLSF games (Aussel and Svensson, 2020) and assume a unique best response. Since equilibrium concepts like NSE (Gan, 2020) do not generally hold under wild conditions, we propose a more viable equilibrium concept as follows:

Definition 1 ($\epsilon$-CSE). Define swap function $s : A_i \rightarrow A_i$, $\chi$ is an $\epsilon$-CSE when the following inequality holds for any swap function $s$, for any $i \in [n]$:

$$\mathbb{E} [l_i(a, Br(a))] \leq \mathbb{E} [l_i(s(a_i), a_{-i}, Br(s(a_i), a_{-i}))] + \epsilon$$

$$\sum_{t=1}^{T} L_i^f(a_i, a_{-i}) = \mathbb{E}_{a_{-i} \sim P_{-i}^*} \left[ L_i(a_i, a_{-i}, Br(a_{-i}, a_{-i})) \right].$$

The objective of each leader $i$ is to minimize the following Stackelberg-regret:

$$R_i^S(T) = \sum_{t=1}^{T} \mathbb{E}_{a_{-i} \sim P_{-i}^*} \left[ L_i^f(a_i, a_{-i}) - L_i^f(a_i, a_{-i}) \right],$$

where $a_{i,*} = \arg \min_{a_i \in A_i} \sum_{t=1}^{T} L_i^f(a_i)$ is the optimal action in hindsight. The superscript $S$ implies “Stackelberg”.

Our goal is to design online learning algorithms for both leaders and the follower that are able to achieve the $\epsilon$-CSE. In the following context, we consider different settings in general-sum MLSF games. Depending on the types of feedback information on a player’s loss function, we separate them into three categories, including full information,
semi-bandit information and noisy bandit feedback. In the first two simpler settings, leaders have access to the follower’s best response oracle. In the last and more challenging setting (where our main results are focused), all players can only get noisy bandit feedback.

4 Warm-up: follower best response as an oracle

We begin with two simple settings as warm up. In both cases, the leaders have access to an oracle which returns the follower’s best response given the leaders’ joint action. Therefore, we can essentially treat the follower as part of the environment that affects the leaders’ loss values (the Br(a_i, a_i^t) term in Eq.(2)) via the best response oracle.

4.1 Existence of CSE with full information

In the full information setting, each leader i knows its own loss function l_i and the follower’s loss function l_f, and can observe the joint mixed strategy χ^t at each round t. Note that although we call it “full information”, leader i does not need to know the other leaders’ loss functions, so that the process is still distributed.

Hedge is an online learning algorithm first designed for solving how to learn from expert advice, aiming to minimize the expected cumulative losses in an adversarial environment (Littlestone and Warmuth, 1994; Cesa-Bianchi and Lugosi, 2006). By applying it into full information MLSF games, we can prove that:

**Proposition 1.** If leader i uses the Hedge algorithm in full information repeated general-sum MLSF games, define R_i^S(T) as in Eq.(3), it achieves no-external Stackelberg-regret in the following sense:

\[ R_i^S(T) \leq O\left(\sqrt{\ln n T}\right). \]

Hedge can achieve no-external regret learning against adversarial losses at each round by using a classical exponential weights update rule of the policies. At each round t, after all leaders show their mixed strategy distribution, each leader i uses the Hedge algorithm to update its mixed strategy with the expected loss defined as L_i^S(a_i) in Eq. (2) for every action a_i ∈ A_i. Based on the regret analysis of the Hedge algorithm, each leader i achieves no-external Stackelberg-regret learning in this process. We provide detailed descriptions of the Hedge algorithm and the proof in Appendices A and B, respectively.

**Corollary 1.** In the full information setting, when all leaders use Hedge as the learning algorithm, together with a reduction from no-external to no-swap regret (Blum and Mansour, 2007; Ito, 2020), the time averaged joint strategy profile distribution \( \hat{\chi} = \frac{1}{T} \sum_{t=1}^{T} \chi^t \) converges to an \( \epsilon^T \)-CSE. \( \epsilon^T = O \left( \sqrt{\frac{n \ln n}{T}} \right) \to 0 \) as \( T \to \infty \), which implies \( \epsilon \)-CSE always exists in general-sum MLSF game for any \( \epsilon > 0 \).

See proof in Appendix B for the detailed proof.

4.2 Semi-bandit MLSF games

We now consider a more realistic setting with semi-bandit feedback, which is the same as the full information setting except that the leaders do not know the exact follower loss function l_f, but instead only receives a bandit feedback l_i(a_i, b_i) and the joint action (a_i, b_i) that is taken at round t. In other words, at each round t, every leader i observes the joint action (a_i, b_i) that is taken, b_i = Br(a_i), and can obtain the loss value l_i(a_i, b_i) for that round. This is opposed to full information setting where each leader i knows the expected loss value L_i^S(a_i) for any action a_i ∈ A_i defined in Eq. (2).

EXP3 (Auer et al., 2002b) is a classical algorithm modified from Hedge to suit the bandit information settings, which can be used to achieve no regret learning from partial bandit feedback. Since every leader only receives information for selected actions from the environment, we use the EXP3 algorithm combined with the concentration inequality method to bound the Stackelberg-regret R_i^S(T) in Eq.(3). Formally, we have

**Proposition 2.** If leader i uses EXP3 in semi-bandit MLSF games, it achieves no-external Stackelberg-regret learning with probability at least 1 − p, i.e.,

\[ R_i^S(T) \leq O \left( \sqrt{\ln n T + \ln \frac{1}{p}} \right). \]

Detailed descriptions of the EXP3 algorithm and the proof of Lemma 2 can be found in Appendices C and D, respectively. With a reduction from no-external to no-swap regret (Blum and Mansour, 2007; Ito, 2020), we immediately obtain that

**Corollary 2.** When all leaders use EXP3 as the underlying learning algorithm in repeated general-sum semi-bandit MLSF games, the time averaged joint strategy profile distribution \( \hat{\chi} = \frac{1}{T} \sum_{t=1}^{T} \chi^t \) converges to an approximate \( \epsilon^T \)-CSE with probability at least 1 − p, where \( \epsilon^T = O \left( \sqrt{\frac{1}{T} n^2 \ln n + \frac{1}{T} n \ln \frac{1}{p}} \right) \).

5 MLSF games with noisy bandit feedback

In this section, we present our main results on a more realistic but also more challenging scenario, where we do not assume that each leader or the follower knows the form of its own loss function, but can only get noisy (stochastic) feedback of the loss value of each round. Due to the practicality, learning equilibrium from noisy bandit feedback
has been widely studied in both the game theory and online learning literature (Helioiu et al., 2017; Bai et al., 2021).

In particular, the work of Bai et al. (2021), which is perhaps the closest to our work, considers learning equilibrium in (single) leader-follower games with noisy bandit feedback. But the analysis is based on querying batches of samples with same sizes for every action pair \((a, b)\) all at a time, as opposed to the online learning setting that we focus on. This implicitly requires a third-party authority that is able to control the sampling strategies of all the players. Our analysis does not rely on such a centralized sampling procedure, but instead allows the players to learn from self-playing on-the-go.

However—online distributed learning in leader-follower games is considered harder to solve, since it cannot be restricted to evenly query every action pair. When leaders learn without a follower best response oracle, they can only use the feedback from interactions with the follower to update their strategies. But if the algorithm does not sample every leaders’ joint action sufficient times, the follower cannot get enough information to learn a stable best response to the leaders. This in turn makes it hard for the leaders to learn its stabilized loss since the follower’s best response varies from round to round to the same leaders’ joint action. In other words, leaders are not guaranteed to achieve no Stackelberg-regret learning if they seldomly choose certain actions. Therefore, it is critical to add a stable exploration to each action \(a\) in the leaders’ learning algorithm to avoid this issue.

### 5.1 \(\alpha\)EXP3-UCB

As a reminder, our goal is to design decentralized online learning algorithms for both leaders and the follower to be able to achieve the \(\epsilon\)-CSE, which can be induced when \(R^S_i(T)\) is sublinear in \(T\) for every \(i \in [m]\).

Following the above intuition, we propose a new algorithm \(\alpha\)EXP3-UCB for learning CSE with noisy bandit feedback, as shown in Algorithm 1. On the high-level, the algorithm is run repeatedly in \(T\) rounds. At round \(t\), each leader \(i\) uses the \(\alpha\)EXP3 algorithm as the underlying learning method to sample actions (Lines 4-5) and update the strategy (Lines 8-9). For each \(a \in A\), the follower conducts a corresponding Upper Confidence Bound algorithm UCB(\(a\)) (Lai et al., 1985; Auer et al., 2002a) for the arms \(k \in [n_f]\), where \(k\) is the \(k\)-th action of the follower, and \([n_f] = \{1, \ldots, n_f\}\) is the set of all the arms of the follower (Lines 6-7).

More specifically, the leaders’ learning algorithm \(\alpha\)EXP3 is essentially the classical EXP3 algorithm (Auer et al., 2002b; Cesa-Bianchi and Lugosi, 2006; Orabona, 2019) plus an extra explicit exploration term when selecting actions. At round \(t\), each leader \(i\)’s joint action is selected as \(a_i^t \sim \tilde{P}_i^t\), where \(\tilde{P}_i^t = (1 - \alpha)P_i^t + \alpha[1/n, \ldots, 1/n]\) is a linear combination of \(P_i^t\) and a uniform probability. The parameter \(\alpha\) can be interpreted as the minimum amount of exploration that is guaranteed. It turns out that setting an appropriate \(\alpha\) is critical in balancing between sample efficiency and algorithm convergence. Lines 8-9 perform an EXP3-style update of the leaders’ strategies, where \(\tilde{P}_i^t\) is an unbiased estimate of the average loss for action \(j \in [n]\) using importance sampling, \(P^t_{i+1}(a_i, j)\) is the base strategy that is exponential w.r.t. the negative of \(\tilde{P}_i^t\). \(\mathbb{I}\{C\}\) is an indicator function with a value of 1 when condition \(C\) is met, and 0 otherwise.

For the follower, because it maintains one UCB(\(a\)) subroutine for each leader joint action \(a \in A\), it actually conducts \(|A|\) UCB algorithms. In Line 6, the follower first observes the leaders’ joint action \(a\), and then uses the corresponding algorithm UCB(\(a\)) to obtain the response strategy:

\[
b^t = \arg\min_{k \in [n_f]} A_{a_i,k}^{t-1},
\]

where

\[
A_{a_i,k}^{t} = \left\{ \begin{array}{ll} \hat{\mu}_{a_i,k} - \sqrt{\frac{2\ln(n_f)}{T_k(n_a(t))}}, & T_k(n_a(t)) \neq 0 \\ -\infty, & \text{otherwise} \end{array} \right. 
\]

Here \(n_a(t)\) is the number of times leader plays action \(a\) in the first \(t\) rounds, \(T_k(n_a(t))\) is the number of times the follower plays its \(k\)-th action when leaders play action \(a\) in the first \(t\) rounds, and \(\hat{\mu}_{a_i,k}^t\) is the estimated average loss of arm \(k\) under leaders’ joint action \(a\). In Line 7, the follower then uses the observed noisy feedback \(\xi^t_j(a^t, b^t)\) to update

| Algorithm 1: \(\alpha\)EXP3-UCB |
|---|
| 1: Require: \(\eta > 0, \beta \geq 3\) |
| 2: \(w_i^1 = [1, \ldots, 1], T_k(n_a(0)) = 0, \tilde{\mu}_{a_i,k}^0 = 0\) for any \(a \in A, k \in [n_f]\) |
| 3: for \(t = 1 \ldots T\) do |
| 4: Each leader \(i\) sets \(\tilde{P}_i^t = (1 - \alpha)P_i^t + \alpha[1/n, \ldots, 1/n]\) |
| 5: Each leader \(i\) draws action \(a_i^t \sim \tilde{P}_i^t\) |
| 6: Follower \(f\) observes \(a^t; responses with \(b^t\) in Eq. (4) |
| 7: Follower \(f\) observes \(\xi^t_j(a^t, b^t)\) and updates \(T_k(n_a(t))\) and \(\tilde{\mu}_{a_i,k}^t\) in Eqs. (5)-(6) |
| 8: Each leader \(i\) observes \(\xi^t_j(a^t, b^t)\) and constructs the estimate \(\hat{P}_j^t = \frac{\xi^t_j(a^t, b^t)}{P^t_{i}(a^t)}\{a_i^t = a_{i,j}\} \) for \(j \in [n]\) |
| 9: Each leader \(i\) updates \(P^t_{i+1}: w_i^{t+1}(j) \leftarrow w_i^t(j) \cdot \exp(-\eta \hat{P}_j^t), P_i^{t+1}(a_{i,j}) \leftarrow \frac{w_i^{t+1}(j)}{\sum_{j=1}^{n_f} w_i^{t+1}(j)} \) for \(j \in [n]\) |
| 10: end for |
where $a_\star = \arg \min_{a \in A} \sum_{t=1}^T l(a, Br_a)$, and the $\bar{O}(\cdot)$ notation hides factors that are polynomial in $\ln T$.

Proof sketch. We present the high-level idea of our proof here and refer to Appendix E for the full proof. We decompose the noisy Stackelberg-regret of the leader into the following three terms

$$
\bar{R}^S(T) = \mathbb{E} \left[ \sum_{t=1}^T l(a^t, Br(a^t)) - l(a_\star, Br(a_\star)) \right]
$$

$$
\leq \bar{O} \left( n T \sqrt{\frac{1}{p}} + n T \sqrt{\frac{1}{p}} + n T \frac{2}{\beta} \right),
$$

where $a_\star = \arg \min_{a \in A} \sum_{t=1}^T l(a, Br(a))$, and the $\bar{O}(\cdot)$ notation hides factors that are polynomial in $\ln T$.

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$$
\bar{R}^S(T) = \mathbb{E} \left[ \sum_{t=1}^T l(a^t, Br(a^t)) - l(a_\star, Br(a_\star)) \right]
$$

$$
= \mathbb{E} \left[ \sum_{t=1}^T l(a^t, Br(a^t)) - l(a^t, b^t(a^t)) \right] + \mathbb{E} \left[ \sum_{t=1}^T l(a^t, b^t(a^t)) - l(a_\star, b^t(a_\star)) \right] + \mathbb{E} \left[ \sum_{t=1}^T l(a_\star, b^t(a_\star)) - l(a_\star, Br(a_\star)) \right].
$$

and bound each term separately. $b^t(a)$ represents the follower’s response to $a$ at round $t$, which is determined by the follower’s subroutine UCB$(a)$ at round $t$, and is not necessarily equal to the best response $Br_a$.

First, since $\xi(a,b) \in [0, 1]$ for any $(a, b) \in A \times B$, the first term (I) and the third term (III) can be respectively bounded with the following two inequalities:

$$
\sum_{t=1}^T \left| \xi^t(a^t, Br(a^t)) - \xi^t(a^t, b^t(a^t)) \right| \leq \sum_{t=1}^T \mathbb{I}[b^t(a^t) \neq Br(a^t)]
$$

$$
\sum_{t=1}^T \left| \xi^t(a^t, Br(a^t)) - \xi^t(a_\star, b^t(a_\star)) \right| \leq \sum_{t=1}^T \mathbb{I}[b^t(a_\star) \neq Br(a_\star)]
$$

To further bound these two terms to be sublinear in $T$, the idea is to guarantee that the number of times that a suboptimal arm is played is sublinear in $T$. Because the underlying algorithm of the follower is UCB, the sublinearity will be satisfied after the UCB subroutines explore sufficient rounds. Therefore, a critical step is to make sure that every $a \in A$ will be played sufficient rounds, or more specifically, $a$ needs to be played at least once in a time interval sublinear in $T$ (e.g., $O(T^{2/3})$) with high probability. This is satisfied by the extra $\alpha$-explicit exploration in the leader’s part of the algorithm. By combining the above steps with concentration inequality, term (I) and term (III) can be bounded as follows — for a sufficient small $p$,

$$
(\text{I}) \leq O \left( n f_n \sqrt{\frac{8 \beta \ln T}{\epsilon^2}} + \sqrt{T \ln \frac{1}{p}} \right),
$$

$$
(\text{III}) \leq O \left( \frac{1}{\alpha} n f_n \left( 1 - \frac{1}{p} \right)^2 \frac{8 \beta \ln T}{\epsilon^2} \right).
$$

Second, in term (II), since the follower response is consistently in the same form, it can be treated as part of the environment. Hence term (II) is essentially the regret of the $\alpha$EXP3 algorithm. Based on this observation, we bound term (II) through a regret analysis on the $\alpha$EXP3 algorithm, together with an adaptation that uses an additional analysis
on the losses for extra exploration:

\[
(\text{II}) \leq \mathcal{O}\left(\frac{\ln n}{\eta} + \frac{\eta n^2 T}{\alpha} + 2\alpha T\right).
\]

Then, we set an appropriate explicit exploration parameter \(\alpha\) and learning rate \(\eta\) as follows so that it bounds each term to be sublinear in \(T\):

\[
\alpha = \mathcal{O}\left(n^{\frac{2}{3}}(\ln n)^{\frac{1}{3}} T^{-\frac{1}{3}}\right), \eta = \mathcal{O}\left(n^{-\frac{2}{3}}(\ln n)^{\frac{1}{3}} T^{-\frac{1}{3}}\right).
\]

Last, after we bound the noisy Stackelberg-regret, we bound true Stackelberg-regret for the leader using concentration inequality:

\[
R^S(T) \leq \tilde{R}^S(T) + \mathcal{O}\left(\sqrt{T \ln \frac{1}{p}}\right).
\]

Following Theorem 1, we immediately have:

Theorem 2. Using \(\alpha\text{EXP}-\text{UCB}\) to learn in a SLSF game with noisy bandit feedback, set \(\alpha = \mathcal{O}\left(n^{\frac{2}{3}}(\ln n)^{\frac{1}{3}} T^{-\frac{1}{3}}\right), \) with probability at least \(1 - p\), the joint empirical strategy profile \(\hat{P} = \frac{1}{T} \sum_{t=1}^{T} P_t\) is an approximate Stackelberg equilibrium,

\[
\mathbb{E}_{a \sim \hat{P}}[t(a, Br(a))] \leq t(a_*, Br(a_*)) + \epsilon^T,
\]

where \(\epsilon^T = \tilde{\mathcal{O}}\left(n^{\frac{2}{3}}n_f \frac{\beta}{\epsilon^2} T^{-\frac{1}{3}}(\ln \frac{1}{p})^2 + n^{\frac{2}{3}}(\ln n)^{\frac{1}{3}} T^{-\frac{1}{3}}\right)\).

Note that the Stackelberg equilibrium is a special case of CSE in MLSF games when there is only one leader.

5.3 MLSF games with noisy bandit feedback

We now study the more complicated scenario of MLSF games with noisy bandit feedback. Compared to the above SLSF bandit games, MLSF bandit games are generally harder because the size of joint action space \(|\mathcal{A}| = |\mathcal{A}_1| \times |\mathcal{A}_2| \times \cdots \times |\mathcal{A}_m| = n^m\) increases exponentially w.r.t. \(m\). In this case the explicit exploration parameter \(\alpha\) needs to be big enough to guarantee that the expected number of times that each joint leader action \(a \in \mathcal{A}\) being played (i.e. \(\mathbb{E}\left[\sum_{t=1}^{T} I(a^t = a)\right] = \left(\frac{\alpha}{n}\right)^m T\)) is at least sublinear in \(T\).

For the special case of two leaders \((m = 2)\), when all players use \(\alpha\text{EXP}-\text{UCB}\), we can still get a sub-linear no-external Stackelberg regret \(R^S_i(T) \leq \tilde{\mathcal{O}}\left(n_f n^{\frac{2}{3}} T^{\frac{1}{3}} \ln \frac{1}{p} + n_f n \frac{\beta}{\epsilon^2}\right)\) for \(i = 1, 2\) with probability at least \(1 - p\) by setting the exploration parameter \(\alpha = \mathcal{O}\left(n^{\frac{2}{3}}(\ln n)^{\frac{1}{3}} T^{-\frac{1}{3}}\right)\) (see proof in Appendix G). However, if we still use Algorithm 1 for cases when \(m > 2\), we need to set a much larger \(\alpha\) to ensure a more aggressive exploration. This results in an extremely slow convergence. Formally:

Theorem 3. Using \(\alpha\text{EXP}-\text{UCB}\) for MLSF games with noisy bandit feedback, with \(\alpha = \mathcal{O}\left(n^{\frac{2}{3}} T^{-\frac{1}{3}}\right), T \geq \mathcal{O}(n^{m+1})\), define \(L^*_i(a_i)\) in Eq.(2), it achieves no-external regret learning with probability at least \(1 - p\)

\[
R^S_i(T) = \mathbb{E}\left[\sum_{t=1}^{T} L^*_i(a_i) - L^*_i(a_{i,*})\right] \leq \tilde{\mathcal{O}}\left(n_f n \frac{\beta}{\epsilon^2} (\ln \frac{1}{p})^2 \frac{T}{\mathcal{O}(n^m)} + n_f n \frac{\beta}{\beta - 2}\right).
\]

See proof in Appendix G. Because \(\alpha \leq 1\) and \(\alpha = \mathcal{O}\left(n^{\frac{2}{3}} T^{-\frac{1}{3}}\right),\) we require that \(T \geq \mathcal{O}(n^{m+1})\). Intuitively, the big exploration parameter \(\alpha\) incurs very low sample efficiency and more regret because it suffers from losses by unnecessarily exploring many “bad” actions. Although the term \(n^m\) can not be avoided in general (because we need to enumerate every leaders-follower action pair \((a, b)\) sufficient times to learn a reasonable estimated loss function), we can still find a more efficient algorithm to improve the regret w.r.t. \(T\).

5.3.1 A sample-efficient two-stage learning algorithm

To overcome the above issue, our intuition is that the players should initially use more aggressive exploration to obtain exact best response with high probability, and then reduce exploration for the sake of algorithm convergence. Based on this intuition, we propose a two-stage learning algorithm (see Algorithm 2) which is provably more efficient than Algorithm 1.

In the first stage (when round \(t \leq t_0;\) Lines 2-7), the leaders perform a pure random exploration without updating their strategies (Lines 3&5), while the follower uses the highly explorative type algorithm Upper Confidence Bound Exploration (UCB-E) (Audibert et al., 2010) to select its best response (Line 4) and update its strategy (Line 6). In Line 4, the best response is chosen as

\[
b^t = \arg\min_{k \in [n_f]} B^t_{a^t,k}, \quad (7)
\]

where

\[
B^t_{a,k} = \left\{ \begin{array}{ll}
\hat{\mu}^t_{a,k} - \sqrt{\frac{T}{2T_k(n_a(t))}}, & T_k(n_a(t)) \neq 0 \\
-\infty, & \text{otherwise}
\end{array} \right.
\]

Here \(e\) is a parameter that specifies the extent of exploration. Because of the way that an arm is selected, UCB-E is a highly explorative type algorithm designed for solving the best arm identification problem in multi-armed bandit games. In Line 6, the strategy update is following the same UCB-style as in Algorithm 1, where \(T_k(n_a(t))\) and \(\hat{\mu}^t_{a,k}\) are respectively updated by Eq.(5) and Eq.(6).

As the end of the first stage (Line 9), the follower learns the best response predictors \(\{\hat{Br}(a), a \in \mathcal{A}\}\) that has the
minimal estimated average loss up to round $t_0$:

$$\left\{ \hat{Br}(a), a \in \mathcal{A} \right\} : \hat{Br}(a) = \arg \min_{k \in \{n_f\}} \mu_{a,k}^{t_0}. \quad (8)$$

Since leaders conduct pure and explicit exploration in the first stage, the follower’s best response predictor is found by each sub-routine UCB-E$(a)$ with a high probability due to the sufficient exploration.

In the second stage (when $t > t_0$; Lines 9-13), the leaders then switch to EXP3 to update their strategies, while the follower commits to the strategy learned from the first-stage exploration and stops updating it. Because the follower stops updating its strategy in this stage, it actually reduces to the semi-bandit setting in Section 4.2.

Overall, for the leaders’ algorithm, it is equivalent to the $\alpha^t$EXP3 algorithm by setting the explicit exploration parameter $\alpha_t$ to be $\alpha_t = \begin{cases} 1, & t \leq t_0 \\ 0, & t > t_0 \end{cases}$, and not performing strategy update in the first stage. For the follower’s algorithm, it is essentially a learn-to-commit procedure.

**Algorithm 2:** Two-stage bandit algorithm

1: $T_k(n_a(0)) = 0$, $\mu_{a,k}^0 = 0$ for any $a \in \mathcal{A}, k \in \{n_f\}$
2: for $t = 1, 2, \ldots, t_0$ do
3: Each leader $i$ selects $a_i^t \sim [1/n, \ldots, 1/n]$
4: Follower $f$ observes $a^t$; responds with $b^t$ in Eq.(7)
5: Each leader $i$ receives $\xi_i^t(a^t, b^t)$; no strategy update
6: Follower $f$ receives $\xi_f^t(a^t, b^t)$ and updates $T_k(n_a(t))$ and $\hat{\mu}_{a,k}^t$ by Eq. (5) and Eq. (6)
7: end for
8: Follower $f$ learns best response predictors $\left\{ \hat{Br}(a), a \in \mathcal{A} \right\}$ in Eq.(8)
9: for $t = t_0 + 1, t_0 + 2, \ldots, T$ do
10: Each leader $i$ selects $a_i^t$ with EXP3
11: Follower $f$ observes $a^t$ and selects $b^t = \hat{Br}(a^t)$
12: Each leader $i$ receives $\xi_i^t(a^t, b^t)$ and updates strategy with EXP3
13: end for

**5.3.2 Two stage learning results**

Before presenting our final results, we first have the following Lemma as a prerequisite:

**Lemma 1.** (Audibert et al., 2010) If UCB-E is run with parameter $0 < \epsilon \leq \frac{25}{36} \frac{T_0 - n_f}{H_a}$, let $T_a$ be the number of times leaders choose $a$ in the first $t_0$ rounds, $H_a = \sum_{k=1}^{n_f} \frac{1}{\lambda k}$, then it satisfies

$$\mathbb{P}\left( \hat{Br}(a) \neq Br(a) \right) \leq 2T_a n_f \exp \left( -\frac{2\epsilon}{25} \right).$$

In particular, when we set $\epsilon = \frac{25}{36} \frac{T_0 - n_f}{H_a}$, we have

$$\mathbb{P}\left( \hat{Br}(a) \neq Br(a) \right) \leq 2T_a n_f \exp \left( -\frac{T_0 - n_f}{18H_a} \right).$$

Lemma 1 guarantees that the follower learns a best response predictor with a high probability for any $a \in \mathcal{A}$ through UCB-E in the first stage of the algorithm. Building on top of that, we have

**Theorem 4.** For a MLSF game with noisy bandit feedback, if every leader $i$ uses Algorithm 2, let $q \geq 18H_a \left( \ln \frac{2n_a}{p} + \ln n \right) + n_f, \epsilon = \frac{25}{36} \frac{q - n_f}{H_a}$, $t_0 = O\left( n^m q \right)$, and define $R_i^f(T)$ as in Eq. (3), with probability at least $1 - 2p$, we have

$$R_i^f(T) \leq O \left( t_0 + \sqrt{Tn \ln n + \frac{T \ln \frac{1}{p}}{p}} \right)$$

and use a reduction from no-external to no-swap regret (Blum and Mansour, 2007; Ito, 2020), we get an $\epsilon^T$-CSE for leaders in MLSF bandit game, where $\epsilon^T = O\left( \frac{1}{T} + \frac{1}{T} \ln^2 \ln n + \frac{1}{T} \ln \frac{1}{p} \right)$. And the follower learns best response predictor with a high probability for any $a \in \mathcal{A}$

$$\mathbb{P}\left( \hat{Br}(a) \neq Br(a) \right) \leq \frac{n}{m}.$$ 

We refer to the full proof in Appendix H. It is worth mentioning that the term $O\left( n^m \right)$ is inevitable, since in the bandit noisy feedback setting we need to go through every action in the action space $\mathcal{A}(\{|A| = n^m\})$ sufficient times to find the follower best response $Br(a)$ for every $a$.

**Based on Lemma 1,** the follower can get the exact best response predictors $\hat{Br}(a) = Br(a)$ for every $a \in \mathcal{A}$ with probability at least $1 - p$ using the union bound. After the follower commits its identification, the follower will play a fixed best response action for each $a \in \mathcal{A}$ in the second stage. With the reliable best response predictors learned by the follower, i.e., with a follower best response oracle that holds with high probability, the second-stage game is reduced to a semi-bandit MLSF game for leaders in Section 4.2 and therefore the proof of Lemma 2 can be re-used here with a slight adaptation.

**6 Conclusion**

This paper is the first to take a learning perspective of general-sum multi-leader-single-follower games. We first propose a new viable equilibrium concept called correlated Stackelberg equilibrium and prove its existence in full information and semi-bandit settings. We then study the more challenging setting where leaders and the follower can only obtain noisy bandit feedback, and prove convergence results of our proposed learning algorithms. Our work opens up many potential future directions at the intersection of learning and MLSF games. For example, it is interesting to see: 1) Can we relax the type of learning algorithms for the players? 2) Can we find computationally more efficient algorithms to reach the CSE? 3) Can our results be generalized to the SLMF and MLMF settings?
Societal impact

General-sum MLSF games have broad applications in many real-world problems, such as security games, wildlife conservation, deregulated electricity markets, and industrial eco-parks. Our work studies the learning aspects of this set of problems, and takes a key step by allowing decentralized learning among the players. Our proposed algorithms, together with the fundamental theoretical analysis, lay the foundation of applying realistic learning algorithms towards the set of practical problems, and therefore can create huge societal impact to the above domains.

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A The Hedge algorithm

Lemma 2. Let \( l_t^2 \) denote the \( n \)-dimensional vector of square losses, i.e., \( l_t^2(i) = (l_t(i))^2 \), let \( \eta > 0 \), and assume all losses to be non-negative. The Hedge algorithm satisfies for any expert \( i^* \in [n] \)

\[
\sum_{t=1}^{T} x_t^T l_t - \sum_{t=1}^{T} l_t(i^*) \leq \frac{\ln n}{\eta} + \eta \sum_{t=1}^{T} x_t^T l_t^2.
\]

Proof. See Theorem 1.5 of Hazan (2019) for a detailed proof.

Based on the regret analysis of algorithm Hedge (Hazan, 2019), noticed that \( L_t^i(a_{i,j}) \in [0, 1] \) for all \( j \in [n] \), we can bound the Stackelberg-regret for leader \( i \)

\[
\sum_{t=1}^{T} \mathbb{E}_{a_t^i \sim P_t^i} [L_t^i(a_t^i) - L_t^i(a_{i,*})] \leq \frac{\ln n}{\eta} + \eta \sum_{t=1}^{T} \sum_{j=1}^{n} P_t^i(a_{i,j}) \cdot (L_t^i(a_{i,j}))^2
\]

\[
\leq \frac{\ln n}{\eta} + \eta \sum_{t=1}^{T} \sum_{j=1}^{n} P_t^i(a_{i,j}) \quad \left( P_t^i \text{ is a distribution, so } \sum_{j=1}^{n} P_t^i(a_{i,j}) = 1 \right)
\]

\[
= \eta T + \frac{\ln n}{\eta}.
\]

We choose \( \eta = \mathcal{O} \left( \frac{\ln n}{T} \right) \), then we have

\[
R_i^{S} (T) = \sum_{t=1}^{T} \mathbb{E}_{a_t^i \sim P_t^i} [L_t^i(a_t^i) - L_t^i(a_{i,*})] \leq \eta T + \frac{\ln n}{\eta} \leq \mathcal{O} \left( \sqrt{T \ln n} \right).
\]

Using reduction from no-external to no-swap regret, which is Theorem 2 of Ito (2020), for any swap function \( s : \mathcal{A}_i \rightarrow \mathcal{A}_i \), we have

\[
R_{i,swap}^{S} (T) = \sum_{t=1}^{T} \mathbb{E}_{a_t^i \sim P_t^i} [L_t^i(a_t^i) - L_t^i(s(a_t^i))] \leq \mathcal{O} \left( \sqrt{nT \ln n} \right).
\]

Let \( \epsilon^T = R_{i,swap}^{S} (T) / T \), for any \( i \in [n] \), we have,

\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_{a_t \sim \chi^t} \left[ l_t(a_t^i, Br(a_t^i)) \right] \leq \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_{a_t \sim \chi^t} \left[ l_t(s(a_t^i), Br(s(a_t^i), a_{i,-i})) \right] + \epsilon^T.
\]

Since expectations are linear, we rewrite the inequality using the time averaged joint action profile distribution \( \bar{\chi} = \frac{1}{T} \sum_{t=1}^{T} \chi^t \) as follows
We can use the following process to sample a joint action from distribution $\tilde{\chi}$: we first sample $t$ uniformly from $[T]$, we then sample a joint action $a$ from the distribution $\chi^t$. $\epsilon^T = O\left(\sqrt{\frac{n \ln n}{T}}\right) \to 0$ as $T \to \infty$, which implies that $\epsilon$-CSE exists in general-sum MLSF game for any $\epsilon > 0$.

C The Exp3 algorithm

**Algorithm 4: EXP3 for Leader $i$**

Require: $\eta, T \in \mathbb{N}$

\[P^1_i = [1/n, \ldots, 1/n]\]

for $t = 1, \ldots, T$ do

    Leader $i$ draws $a^t_i$ according to $P^t_i$ and selects $a^t_i$

    Follower observes $a^t$ and plays $b^t_i = Br(a^t)$

    Leader $i$ observes $l_i(a^t, b^t)$ and constructs the estimate $\tilde{l}^t_{i,j} = \frac{\ell(a^t_i, b^t_j)}{P^t_{t_i}(a^t_i)} \tilde{I}_t(a^t_i = a^t_i, j)$ for $j \in [n]$

    Leader $i$ updates $P^{t+1}_i$: $P^{t+1}_i(a^t_i, j) \propto P^t_{t_i}(a^t_i) \cdot \exp(-\eta \tilde{l}^t_{i,j})$

end for

D Proof of Proposition 2

In this section, for simplicity, we let $\tilde{L}^t_i(a_i) = l_i(a_i, a^t_{-i}, Br(a_i, a^t_{-i}), L^t_i(a_i)$ be defined by Eq.2, $a_i \in \mathcal{A}_i$. We decompose the Stackelberg-regret for leader $i$ as follows,

\[R^S_i(T) = \sum_{t=1}^{T} \mathbb{E}_{a^t_i \sim P^t_i} [L^t_i(a^t_i) - L^t_i(a_i, \star)] \]

\[= \sum_{t=1}^{T} \mathbb{E}_{a^t_i \sim P^t_i} [L^t_i(a^t_i) - \tilde{L}^t_i(a^t_i)] + \sum_{t=1}^{T} \mathbb{E}_{a^t_i \sim P^t_i} [\tilde{L}^t_i(a^t_i) - \tilde{L}^t_i(a_i, \star)] + \sum_{t=1}^{T} \tilde{L}^t_i(a_i, \star) - L^t_i(a_i, \star).\]

Term I and Term III are caused by the randomness of leaders choosing their actions from their mixed strategy distribution profile at each round. Term II is the regret caused by losses generated by the other leaders’ selected actions at each round. First, we bound Term II based on the regret bound of the EXP3 algorithm. Then we bound Term I and Term III by concentration inequality methods.

Based on the regret analysis of the EXP3 algorithm, which can be found in Theorem 10.2 of Orabona (2019), by setting $\eta = O\left(\frac{\ln n}{T}\right)$, we can bound Term II as

\[\text{Term II} = \sum_{t=1}^{T} \mathbb{E}_{a^t_i \sim P^t_i} [\tilde{L}^t_i(a^t_i) - \tilde{L}^t_i(a_i, \star)] \leq O\left(\sqrt{nT \ln n}\right).\]

For any $a_i \in \mathcal{A}_i, t \in [T]$

\[\mathbb{E}_{a^t_i \sim P^t_i} [\tilde{L}^t_i(a^t_i)] = L^t_i(a_i), -1 \leq \tilde{L}^t_i(a_i) - L^t_i(a_i) \leq 1.\]

Using concentration inequality, we have

\[\mathbb{P}\left[\sum_{t=1}^{T} \tilde{L}^t_i(a_i) - L^t_i(a_i) > \epsilon\right] \leq 2 \exp\left(-\frac{2\epsilon^2}{4T}\right) = 2 \exp\left(-\frac{\epsilon^2}{2T}\right).\]
Then, for any $\delta > 0$, the following inequality holds with probability at least $1 - \delta$ for any $a_i \in A_i$:

$$\left| \sum_{t=1}^{T} \bar{L}_t^i(a_i) - L_t^i(a_i) \right| \leq \sqrt{2T \ln \frac{2}{\delta}}.$$  

So we can bound Term III with probability at least $1 - \delta$

$$\text{Term III} \leq \left| \sum_{t=1}^{T} \bar{L}_t^i(a_i, \ast) - L_t^i(a_i, \ast) \right| \leq \sqrt{2T \ln \frac{2}{\delta}}.$$  

Similarly, using concentration inequality, the following inequality holds with probability at least $1 - \delta$

$$\left| \sum_{t=1}^{T} L_t^i(a_i^t) - \bar{L}_t^i(a_i^t) - \mathbb{E}_{a_i^t \sim P_t} [L_t^i(a_i^t) - \bar{L}_t^i(a_i^t)] \right| \leq 2\sqrt{2T \ln \frac{2}{\delta}}.$$  

So we can bound Term I with probability at least $1 - 2\delta$

$$\text{Term I} = \sum_{t=1}^{T} \mathbb{E}_{a_i^t \sim P_t} [L_t^i(a_i^t) - \bar{L}_t^i(a_i^t)]$$

$$= \sum_{t=1}^{T} \mathbb{E}_{a_i^t \sim P_t} [L_t^i(a_i^t) - \bar{L}_t^i(a_i^t)] - \sum_{t=1}^{T} \mathbb{E}_{a_i^t \sim P_t} [L_t^i(a_i^t) - \bar{L}_t^i(a_i^t)] + \sum_{t=1}^{T} L_t^i(a_i^t) - \bar{L}_t^i(a_i^t)$$

$$\leq 2\sqrt{2T \ln \frac{2}{\delta}} + 2\sqrt{T \ln \frac{2}{\delta}} = 3\sqrt{2T \ln \frac{2}{\delta}}.$$  

With probability at least $1 - p$, $p = 3\delta$, we have the following inequality

$$R^S_i(T) = \sum_{t=1}^{T} \mathbb{E}_{a_i^t \sim P_t} [L_t^i(a_i^t) - L_t^i(a_i, \ast)]$$

$$= \sum_{t=1}^{T} \mathbb{E}_{a_i^t \sim P_t} [L_t^i(a_i^t) - \bar{L}_t^i(a_i^t)] + \sum_{t=1}^{T} \mathbb{E}_{a_i^t \sim P_t} [\bar{L}_t^i(a_i^t) - \bar{L}_t^i(a_i, \ast)] + \sum_{t=1}^{T} \bar{L}_t^i(a_i, \ast) - L_t^i(a_i, \ast)$$

$$\leq \mathcal{O} \left( \sqrt{T \ln \frac{1}{p}} \right) + \mathcal{O} \left( \sqrt{\ln n \ln n} \right) + \mathcal{O} \left( \sqrt{T \ln \frac{1}{p}} \right).$$  

Using reduction from no-external to no-swap regret, which is Theorem 2 of Ito (2020), we have

$$R^S_i,\text{swap}(T) \leq \mathcal{O} \left( \sqrt{n^2 T \ln n} + \sqrt{n T \ln \frac{1}{p}} \right).$$  

Let $\epsilon^T = R^S_i,\text{swap}(T) / T$, for any $i \in [n]$, we have,

$$\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_{a_i \sim \chi^t} [l_i(a^t, Br(a^t))] \leq \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_{a_i \sim \chi^t} [l_i(s(a_i), Br(s(a_i), a_i, \ast))] + \epsilon^T.$$  

Since expectations are linear, we can rewrite the inequality using the time averaged joint action profile distribution $\bar{\chi} = \frac{1}{T} \sum_{t=1}^{T} \chi^t$ as follows

$$\mathbb{E}_{a_i \sim \bar{\chi}} [l_i(a, Br(a))] \leq \mathbb{E}_{a_i \sim \bar{\chi}} [l_i(s(a_i), Br(s(a_i), a_i, \ast))] + \epsilon^T.$$  

We can use the following process to sample a joint action from distribution $\bar{\chi}$: first, we sample $t$ uniformly from $[T]$. Then, we sample a joint action $a$ from the distribution $\bar{\chi}$. So $\bar{\chi}$ is an $\epsilon^T$-CSE.
Lemma 3. Assume that the losses of the follower’s actions minus their expectations are 1-subgaussian and has a mean of zero and $\xi$. Since $\mathbb{E} [\xi] = 0$, we decompose the noisy Stackelberg-regret of the leader into the following three terms

$$R^S(T) = \mathbb{E} \left[ \sum_{t=1}^T \xi^t(a^t, \text{Br}(a^t)) - \xi^t(a^t, \text{Br}(a^t)) \right]$$

$$= \mathbb{E} \left[ \sum_{t=1}^T \xi^t(a^t, \text{Br}(a^t)) - \xi^t(a^t, b^t(a^t)) \right]$$

$$+ \mathbb{E} \left[ \sum_{t=1}^T \xi^t(a^t, b^t(a^t)) - \xi^t(a^t, b^t(a^t)) \right] + \mathbb{E} \left[ \sum_{t=1}^T \xi^t(a^t, b^t(a^t)) - \xi^t(a^t, \text{Br}(a^t)) \right].$$

and bound each term separately. $b^t(a)$ represents the the follower’s response to $a$ at round $t$, which is determined by the follower’s subroutine UCB(a) at round $t$, and is not necessarily equal to the best response Br(a).

Intuitively, term (I) means the cumulative gap between the noisy losses of the leaders’ action $a^t$ when the follower respectively uses any response $b^t(a)$ or the best response $\text{Br}(a)$. Term (II) represents the cumulative gap between the noisy losses of the leaders’ any action $a^t$ and optimal action $a^t$, when the follower plays response $b^t(a)$. Term (III) is the cumulative gap between noisy losses of the leaders’ optimal action $a^t$ when the follower uses any response $b^t(a)$ or best response $\text{Br}(a)$.

E.1 Bound $T_k(n_a(T))$ for suboptimal response $k \neq \text{Br}(a)$

Since $\xi_f(a, b) \in [0, 1], \xi_f(a, b) = \mathbb{E} [\xi_f(a, b)]$ is 1-subgaussian variable. This is based on the fact that if a random variable $X$ has a mean of zero and $X \in [a, b]$ almost surely, then $X$ is $(b-a)/2$-subgaussian.

Lemma 3. Assume that the losses of the follower’s actions minus their expectations are 1-subgaussian and $\beta \geq 3$. Then, using UCB algorithm guarantees that for any $a \in A$, if $k \neq \text{Br}(a)$, then

$$\mathbb{E} [T_k(n_a(T))] \leq \frac{8\beta \log(n_a(T))}{\Delta_{ak}^2} + \frac{\beta}{\beta - 2}.$$

And if at time $t$, $T_k(n_a(t-1)) > \frac{8\beta \log(n_a(T))}{\Delta_{ak}^2}$, then for $k \neq \text{Br}(a)$

$$\mathbb{P} \{a^t = a\} \mathbb{P} \{b^t = k\} \leq 2(n_a(t-1)) n_a(t)^{-\beta}.$$

Proof. See Theorem 10.14 of Orabona (2019) for a detailed proof.

Assume that action $a$ is chosen at round $t_1, t_2, \cdots, t_{n_a(T)}$. For any suboptimal $k$ for leader’s action $a$, if $T_k(n_a(T-1)) \leq \frac{8\beta \log(n_a(T))}{\Delta_{ak}^2}$, then we have $T_k(n_a(T)) \leq \frac{8\beta \log(T)}{\Delta_{ak}^2} + 1$. We next assume that $T_k(n_a(T-1)) > \frac{8\beta \log(n_a(T))}{\Delta_{ak}^2}$. Let $j$, $j \in [n_a(T)]$ be the biggest index such that $T_k(n_a(t_j-1)) \leq \frac{8\beta \log(n_a(T))}{\Delta_{ak}^2}$. Then at any round $t_i > t_j$, $i \in [n_a(T)]$, using the above lemma, we have

$$\mathbb{P} \{a^{t_i} = a \land b^{t_i} = k\} \leq 2(i-1) \beta^{-\beta}.$$

We let $p_i = 2(i-1) \beta^{-\beta}$. When $\beta \geq 3$,

$$\sum_{i=1}^{n_a(T)} 2(i-1) \beta^{-\beta} \leq \sum_{i=2}^{n_a(T)} 2i \beta^{-\beta} \leq 2 \int_1^{+\infty} x^{-\beta} \, dx = \frac{2}{\beta - 2}.$$

So

$$\sum_{i=1}^{n_a(T)} p_i \leq \frac{2}{\beta - 2}.$$
We construct the following filtration. For any $a \in A$, $\beta \geq 3$, $k \in [n_f]$, $k \neq Br(a)$, with probability at least $1 - \delta$

$$T_k(n_a(T)) \leq T_k(n_a(t_{j-1})) + 1 + \sum_{i=j+1}^{n_a(T)} \mathbb{I}\{a^i = a \land b^i = k\} \leq T_k(n_a(t_{j-1})) + 1 + \sum_{i=1}^{n_a(T)} p_i + O\left(\frac{1}{\delta} + \ln n_f n\right) \leq T_k(n_a(t_{j-1})) + 1 + \frac{2}{\beta - 2} + O\left(\frac{1}{\delta} + \ln n_f n\right) \leq \frac{8\beta \ln(n_a(T))}{\Delta_{\beta}^2} + \frac{\beta}{\beta - 2} + O\left(\frac{1}{\delta} + \ln n_f n\right).$$

E.2 Bound for Term I

We can see that when $b'(a^t) = Br(a^t)$, then $\xi^t(a^t, Br(a^t)) - \xi^t(a^t, b'(a^t)) = 0$. Since $\xi^t(a^t, Br(a^t)) \in [0, 1]$, so $|\xi^t(a^t, Br(a^t)) - \xi^t(a^t, b'(a^t))| \leq 1$. Based on these two arguments, we have

$$\sum_{t=1}^{T} \mathbb{I}\{\xi^t(a^t, Br(a^t)) - \xi^t(a^t, b'(a^t))\} \leq \sum_{t=1}^{T} \mathbb{I}\{b'(a^t) \neq Br(a^t)\} = \sum_{a \in A} \sum_{k \neq Br(a)} T_k(n_a(T)).$$

We define

$$M_t = \mathbb{E}_{a^t \sim \tilde{p}_t} \left[\xi^t(a^t, Br(a^t)) - \xi^t(a^t, b'(a^t))\right] - \left[\xi^t(a^t, Br(a^t)) - \xi^t(a^t, b'(a^t))\right],$$

$$-2 \leq M_t \leq 2.$$

We construct the following filtration. For any $t \in [T]$, we define following $\sigma$-algebra as follows

$$\mathcal{F}^t = \sigma\left\{\{a^i, \xi^i(a^i, b'(a^i)), \xi^i(a^i, b'(a^i))\}_{i \in [t]}\right\}.$$

$M_1, M_2, \ldots, M_T$ is a martingale difference sequence with respect to filtration $\{\mathcal{F}^t\}_{t \in [T]}$, which means

$$\mathbb{E}[M_{t+1}|\mathcal{F}^t] = 0.$$

By applying Azuma’s inequality to the martingale difference sequence, we have

$$\mathbb{P}\left[\left|\sum_{t=1}^{T} M_t\right| > \varepsilon\right] \leq 2 \exp\left(\frac{-2\varepsilon^2}{16T}\right) = 2 \exp\left(\frac{-\varepsilon^2}{8T}\right).$$
So the following inequality holds with probability at least $1 - \delta$

$$
\text{Term I} - \left[ \sum_{t=1}^{T} \xi^t(a^t, \text{Br}(a^t)) - \xi^t(a^t, b^t(a^t)) \right]
\leq \sum_{i=1}^{T} E_{a_t \sim \tilde{P}_t} \left[ \xi^t(a^t, \text{Br}(a^t)) - \xi^t(a^t, b^t(a^t)) \right] - \left[ \sum_{i=1}^{T} \xi^t(a^t, \text{Br}(a^t)) - \xi^t(a^t, b^t(a^t)) \right]
\leq 2 \sqrt{2T \ln \frac{2}{\delta}}.
$$

Using the results above, we have the following inequality holds with probability at least $1 - \delta$

$$
\text{Term I} = \left[ \sum_{t=1}^{T} \xi^t(a^t, \text{Br}(a^t)) - \xi^t(a^t, b^t(a^t)) \right] + \text{Term I} - \left[ \sum_{t=1}^{T} \xi^t(a^t, \text{Br}(a^t)) - \xi^t(a^t, b^t(a^t)) \right]
\leq \sum_{a \in A} \sum_{k \neq \text{Br}(a)} T_k(n_a(T)) + 2 \sqrt{2T \ln \frac{2}{\delta}}
\leq O \left( n_j \left( \frac{8 \beta \log(T)}{\varepsilon^2} + \frac{\beta}{\beta - 2} + \ln \frac{1}{\delta} + \ln n_j n \right) + \sqrt{T \ln \frac{1}{\delta}} \right).
$$

E.3 Bound for Term II

In this part, we focus on the influence of the additional explicit exploration parameter $\alpha$ to the regret bound. For simplicity, since we only have one leader, we let $L^t(a) = \xi^t(a, b^t(a)) \in [0, 1]$, $a \in A$, and let $\tilde{L}^t(a) = \frac{\xi^t(a, b^t(a))}{P^t(a)} \mathbb{1}\{a^t = a\}$. We denote $a_j$ is the $j$-th action of the leader.

As a reminder, $\tilde{P}_i^t = (1 - \alpha)P_i^t + \alpha[1/n, ..., 1/n]$. To facilitate our proof, we denote $\tilde{P}_j^t = \tilde{P}_j^t(a_j)$, $P_j^t = P_j^t(a_j)$ for $j \in [n]$.

We calculate the expectation of $\tilde{L}_j^t(a), j \in [n]$:

$$
\mathbb{E}\left[ \tilde{L}_j^t(a) \right] = \mathbb{E}_{a_t \sim \tilde{P}_t} \left[ \tilde{L}_j^t(a) \right] = \mathbb{E}_{a_t \sim \tilde{P}_t} \left[ \frac{\xi^t(a_j, b^t(a_j))}{P^t(a_j)} \mathbb{1}\{a^t = a_j\} \right]
= \frac{\xi^t(a_j, b^t(a_j))}{P^t(a_j)} \mathbb{E}_{a_t \sim \tilde{P}_t} \left[ \mathbb{1}\{a^t = a_j\} \right]
= \xi^t(a_j, b^t(a_j)) = L^t(a_j).
$$

Similarly, the variance of $\tilde{L}_j^t(a), j \in [n]$:

$$
\mathbb{E}\left[ (\tilde{L}_j^t(a))^2 \right] = \mathbb{E}_{a_t \sim \tilde{P}_t} \left[ (\tilde{L}_j^t(a))^2 \right] = \mathbb{E}_{a_t \sim \tilde{P}_t} \left[ \frac{\xi^t(a_j, b^t(a_j))^2}{P^t(a_j)^2} \mathbb{1}\{a^t = a_j\} \right] = \frac{(L^t_1(a_j))^2}{P_j^t}.
$$
So we have $L_t = \sum_{t=1}^T \xi_t(a^t, b^t(a^t)) - \xi_t(a_*, b^t(a_*))$, so based on the regret bound of Hedge algorithm, and $\sum_{t=1}^T E_{a_j \sim \tilde{p}_t}^t [L^t(a^t) - L^t(a_*)] = 0$,

\[
\sum_{t=1}^T E_{a_j \sim \tilde{p}_t}^t \left[ \sum_{j=1}^n \tilde{L}^t(a_j) \tilde{P}_j^t - L^t(a_*) \right] = (1 - \alpha) E \left[ \sum_{t=1}^T \sum_{j=1}^n \tilde{L}^t(a_j) P_j^t - L^t(a_*) \right] + \frac{\alpha}{n} \sum_{t=1}^T E_{a_j \sim \tilde{p}_t}^t \left[ \sum_{j=1}^n \tilde{L}^t(a_j) \right] + \alpha T.
\]

We can see that $P^t$ is produced by the Hedge algorithm’s updating rule with loss $\tilde{L}^t(a_j)$ for action $a_j \in A$ at each round $t$, so based on the regret bound of Hedge algorithm, and $\tilde{P}_j^t \geq \frac{\alpha}{n}$ we have,

\[
\sum_{t=1}^T E_{a_j \sim \tilde{p}_t}^t \left[ \sum_{j=1}^n \tilde{L}^t(a_j) \tilde{P}_j^t - L^t(a_*) \right] \leq \ln n + \frac{\eta}{\alpha} \ln n + \sum_{t=1}^T \sum_{j=1}^n P_j^t E \left[ (\tilde{L}^t(a_j))^2 \right]
\]

\[
= \ln n + \eta \sum_{t=1}^T \sum_{j=1}^n P_j^t E \left[ (\tilde{L}^t(a_j))^2 \right]
\]

\[
\leq \ln n + \eta \sum_{t=1}^T \sum_{j=1}^n P_j^t \frac{l^t(a_j)^2}{P_j^t} = \ln n + \eta \sum_{t=1}^T \sum_{j=1}^n P_j^t \frac{n}{\alpha} = \ln n + \frac{m^2T}{\alpha}.
\]

So we have

\[
\text{Term II} \leq (1 - \alpha) \left[ \sum_{t=1}^T \sum_{j=1}^n \tilde{L}^t(a_j) P_j^t - L^t(a_*) \right] + \alpha T + \alpha T
\]

\[
\leq (1 - \alpha) \left[ \frac{\ln n}{\eta} + \frac{\eta m^2T}{\alpha} \right] + 2\alpha T
\]

\[
\leq \frac{\ln n}{\eta} + \frac{\eta m^2T}{\alpha} + 2\alpha T.
\]

Set $\alpha = O \left( n^{\frac{1}{2}} (\ln n)^{\frac{1}{4}} T^{-\frac{1}{2}} \right)$, $\eta = O \left( n^{-\frac{1}{2}} (\ln n)^{\frac{1}{4}} T^{-\frac{1}{2}} \right)$, we have

\[
\text{Term II} \leq \frac{\ln n}{\eta} + \frac{\eta m^2T}{\alpha} + 2\alpha T = O \left( n^{\frac{1}{2}} (\ln n)^{\frac{1}{4}} T^{\frac{1}{2}} \right).
\]
E.4 Bound for Term III

For any action \( a \in A \), we assume that action \( a \) is chosen at round \( t_1, t_2, \ldots, t_{n_a(T)} \). For any \( a \in A \), let \( s_i = t_i - t_{i-1} \) represents the length of interval leader chooses action \( a \) between the \( i \)-th time and the \( (i+1) \)-th time, \( i \in [n_a(T)] \), and let \( s_{n_a(T)+1} = T - n_a(T), t_0 = 0 \).

We notice that when \( t_{i-1} < t \leq t_i, t \in [T], b^i(t) = b^i(a) \), so we have the following inequality

\[
\text{Term III} = \mathbb{E} \left[ \sum_{t=1}^{T} \xi^t(a, b^t(a)) - \xi^t(a, Br(a)) \right]
= \sum_{t=1}^{T} \xi^t(a, b^t(a)) - \xi^t(a, Br(a))
\leq \sum_{t=1}^{T} \mathbb{I}\{b^t(a) \neq Br(a)\}
\leq s_{n_a(T)+1} + \sum_{i=1}^{n_a(T)} s_i \mathbb{I}\{b^{i}(a) \neq Br(a)\}
\leq \max_{i \in [n_a(T)+1]} \{s_i\} \left( 1 + \sum_{i=1}^{n_a(T)} \mathbb{I}\{b^{i}(a) \neq Br(a)\} \right)
= s_{\text{max}} \left( 1 + \sum_{k \neq Br(a)} T_k(n_a(T)) \right)
\leq 2s_{\text{max}} \sum_{k \neq Br(a)} T_k(n_a(T)).
\]

Next we need to bound \( s_{\text{max}} \). Since we add an explicit exploration parameter \( \alpha \) in the algorithm, the probability of leader choosing any action \( a \) is at least \( \frac{\alpha}{n} \) at each round, so we have

\[
\mathbb{P}\{s_{\text{max}} \geq k\} \leq \left( 1 - \frac{\alpha}{n} \right)^k, \text{ Let } \delta = \left( 1 - \frac{\alpha}{n} \right)^k.
\]

Noticed that \( \ln(1-x) \leq -x \) when \( x \in [0,1] \), so we have

\[
\ln \delta = k \ln \left( 1 - \frac{\alpha}{n} \right) \leq k \left( -\frac{\alpha}{n} \right), k \leq \frac{\ln \frac{1}{\delta}}{\frac{\alpha}{n}} = \frac{n \ln \frac{1}{\delta}}{\alpha}.
\]

So with probability at least \( 1 - \delta \),

\[
s_{\text{max}} \leq \frac{n \ln \frac{1}{\delta}}{\alpha}.
\]

We set \( \alpha = \mathcal{O}\left( n^{\frac{7}{2}}(\ln n)^{\frac{5}{2}}T^{-\frac{1}{2}} \right) \). So with probability at least \( 1 - \delta \), \( s_{\text{max}} \leq \mathcal{O}\left( n^{\frac{7}{2}}(\ln n)^{-\frac{1}{2}}T^{\frac{1}{2}} \ln \frac{1}{\delta} \right) \).

\[
\text{Term III} \leq s_{\text{max}} \sum_{k \neq Br(a)} T_k(n_a(T))
\leq \mathcal{O}\left( n^{\frac{7}{2}}T^{\frac{1}{2}} \ln \frac{1}{\delta} n_f \left( \frac{8\beta \log(T)}{\varepsilon^2} + \frac{\beta}{\beta - 2} + \ln \frac{1}{\delta} + \ln n_f n \right) \right).
\]

E.5 Bound for \( R^S(T) \)

Using concentration inequality, the following inequality holds with probability at least \( 1 - \delta \),
\[ |R^S(T) - \tilde{R}^S(T)| \leq \sqrt{2T \ln \frac{2}{\delta}}. \]

For a sufficient small \( p, \ln \frac{1}{p} \leq (\ln \frac{1}{p})^2 \). Set \( \alpha = O \left( n^\frac{2}{3} (\ln n)^{\frac{4}{3}} T^{\frac{1}{3}} \right), \eta = O \left( n^{-\frac{2}{3}} (\ln n)^{\frac{4}{3}} T^{-\frac{2}{3}} \right) \), by the union bound, the following inequality holds with probability at least \( 1 - p, p = 4\delta \),

\[ R^S(T) = \tilde{R}^S(T) + R^S(T) - \tilde{R}^S(T) \leq R^S(T) + O \left( \sqrt{T \ln \frac{1}{p}} \right) \]

\[ = \text{Term I} + \text{Term II} + \text{Term III} + O \left( \sqrt{T \ln \frac{1}{p}} \right) \]

\[ \leq O \left( n \ln \left( \frac{8 \log(T)}{\epsilon^2} + \frac{\beta}{\beta - 2} + \ln \frac{1}{p} + \ln n \right) + \sqrt{T \ln \frac{1}{p}} \right) \]

\[ + O \left( n^\frac{2}{3} T^{\frac{1}{3}} \ln \frac{1}{p} \left( \frac{8 \log(T)}{\epsilon^2} + \frac{\beta}{\beta - 2} + \ln \frac{1}{p} + \ln n \right) \right) \]

\[ + O \left( n^\frac{2}{3} (\ln n)^{\frac{4}{3}} T^{\frac{1}{3}} \right) + O \left( T \ln \frac{1}{p} \right) \]

\[ \leq \tilde{O} \left( n^\frac{2}{3} n \beta \frac{\epsilon}{2} T^{\frac{1}{3}} (\ln \frac{1}{p})^2 + n^\frac{2}{3} (\ln n)^{\frac{4}{3}} T^{\frac{1}{3}} + n \frac{\beta}{\beta - 2} \right). \]

\[ \sum_{t=1}^{T} \mathbb{E}_{a \sim \tilde{P}^t} \left[ l_i(a^t, Br(a^t)) \right] \leq l(a_*, Br(a_*)) + \epsilon^T. \]

Since expectations are linear, we can rewrite the inequality using the time averaged action profile distribution \( \tilde{P} = \frac{1}{T} \sum_{t=1}^{T} \tilde{P}^t \) as follows

\[ \mathbb{E}_{a \sim \tilde{P}} \left[ l(a, Br(a)) \right] \leq l(a_*, Br(a_*)) + \epsilon^T. \]

We can use the following process to sample an action from distribution \( \tilde{P} \): first we sample \( t \) uniformly from \([T]\), then we sample an action \( a \) from the distribution \( \tilde{P}^t \).

\[ \tilde{R}^S(T) \leq \tilde{O} \left( n^2 \ln \frac{1}{\alpha^2} n \frac{\beta}{\epsilon^2} + \frac{\beta}{\beta - 2} + \ln \frac{1}{p} \right) + \frac{\ln n}{\eta} + \frac{\eta m^2 T}{\alpha} + 2\alpha T + n \frac{\beta}{\beta - 2} \]

\[ \leq \tilde{O} \left( n n^\frac{2}{3} \beta \frac{\epsilon}{2} T^{\frac{1}{3}} (\ln \frac{1}{p})^2 + n \frac{\beta}{\beta - 2} \right). \]

For the general \( m \), we define \( \tilde{\chi}^t = \tilde{P}_1^t \times \cdots \times \tilde{P}_m^t \), and we decompose the noisy regret as follows
Using Bernstein inequality, we have

\[ R_i^S(T) = \sum_{t=1}^{T} \mathbb{E}_{a_t \sim \hat{\mathcal{X}}} \left[ \xi_i(a_t, Br(a_t)) - \xi_i(a_t, a_{t-1} Br(a_t, a_{t-1})) \right] \]

\[ = \sum_{t=1}^{T} \mathbb{E}_{a_t \sim \hat{\mathcal{X}}} \left[ \xi_i(a_t, Br(a_t)) \right] - \mathbb{E}_{a_t \sim \hat{\mathcal{X}}} \left[ \xi_i(a_t, Br(a_t)) \right] \]

\[ + \sum_{t=1}^{T} \mathbb{E}_{a_t \sim \hat{\mathcal{X}}} \left[ \xi_i(a_t, Br(a_t)) - \xi_i(a_t, b'(a_t)) \right] \quad \text{(Term I)} \]

\[ + \sum_{t=1}^{T} \mathbb{E}_{a_t \sim \hat{\mathcal{X}}} \left[ \xi_i(a_t, b'(a_t)) - \xi_i(a_t, a_{t-1}, b'(a_t, a_{t-1})) \right] \quad \text{(Term II)} \]

\[ + \sum_{t=1}^{T} \mathbb{E}_{a_t \sim \hat{\mathcal{X}}} \left[ \xi_i(a_t, a_{t-1}, b'(a_t, a_{t-1})) - \xi_i(a_t, a_{t-1}, Br(a_t, a_{t-1})) \right] \quad \text{(Term III)} \]

\[ + \sum_{t=1}^{T} \xi_i(a_t, a_{t-1}, Br(a_t, a_{t-1})) - \mathbb{E}_{a_t \sim \hat{\mathcal{X}}} \left[ \xi_i(a_t, a_{t-1}, Br(a_t, a_{t-1})) \right] \quad \text{(Term IV)} \]

\[ + \sum_{t=1}^{T} \xi_i(a_t, a_{t-1}, Br(a_t, a_{t-1})) - \mathbb{E}_{a_t \sim \hat{\mathcal{X}}} \left[ \xi_i(a_t, a_{t-1}, Br(a_t, a_{t-1})) \right] \quad \text{(Term V)} \]

Term I and Term V can be bounded using similar techniques of the proof of Proposition 2. Term II, Term III and Term IV can be bounded using similar techniques of the proof of Theorem 1. We set \( \alpha = \mathcal{O} \left( \frac{nT}{m^2} \right) \), \( \eta = \mathcal{O} \left( \sqrt{T - \frac{m^2}{\beta}} \right) \), and \( T \geq \mathcal{O} \left( n^{m+1} \right) \). Combined with the techniques used in the proof of Proposition 2 and Theorem 1, with probability at least \( 1 - p \), we have

\[ R_i^S(T) \leq \tilde{\mathcal{O}} \left( \frac{n^m \ln \frac{1}{\alpha m} (nT^2 + nT \ln \frac{1}{\beta} \eta) + \ln \frac{n}{\eta} + \frac{n^m T}{\alpha} + 2n \frac{\beta}{n} \frac{\beta}{\beta} \right) \]

\[ \leq \tilde{\mathcal{O}} \left( n + n \frac{\beta}{\epsilon^2} \left( \ln \frac{1}{\beta} \right)^2 \frac{T}{\beta} \right). \]

**H Proof of Theorem 4**

Based on leaders’ pure exploration strategy, at every round \( t \leq t_0 \), for every \( a \in \mathcal{A} \): \( \mathbb{P} \left( \{a^t = a\} \right) = \frac{1}{n^m} \). Let \( p_0 = \frac{1}{n^m} \). Using Bernstein inequality, we have

\[ \mathbb{P} \left( \frac{1}{t_0} \sum_{t=1}^{t_0} \mathbb{I}\{a^t = a\} - p_0 < -\epsilon \right) \leq \exp \left( -\frac{t_0 \epsilon^2}{2 \sigma^2 + \frac{2}{3} \frac{\epsilon^2}{p_0^2}} \right), \]

where \( \sigma^2 = p_0(1 - p_0) < p_0 \). Let \( \epsilon = \frac{p_0}{2} \), we have

\[ \mathbb{P} \left( \sum_{t=1}^{t_0} \mathbb{I}\{a^t = a\} < t_0 \frac{p_0}{2} \right) \leq \exp \left( -\frac{t_0 \frac{p_0^2}{2} \epsilon^2}{2 \sigma^2 + \frac{2}{3} \frac{\epsilon^2}{p_0^2}} \right) \leq \exp \left( -\frac{t_0 \frac{p_0^2}{2} \epsilon^2}{2 \sigma^2 + \frac{2}{3} \frac{\epsilon^2}{p_0^2}} \right) \leq \exp \left( -\frac{3}{28} t_0 p_0 \right). \]

Let \( t_0 \geq \max \left\{ t_0, \frac{28}{p_0^2}, \frac{28}{p_0^2} \ln \frac{2}{p_0^2} \right\} \), with probability at least \( 1 - \frac{m^m}{n^m} \), we have

\[ T_a = \sum_{t=1}^{t_0} \mathbb{I}\{a^t = a\} \geq 2q. \]
Lemma 4. ([Audibert et al., 2010]) If UCB-E is run with parameter $0 < e \leq \frac{25}{36} T_a - n_f$, let $T_a$ be the number of times leaders choose $a$ in the first $t_0$ rounds, $H_a = \sum_{k=1}^{n_f} \frac{1}{n_k}$, then it satisfies

$$P\left(\hat{B}(a) \neq B(a)\right) \leq 2T_a n_f \exp\left(-\frac{2e}{25}\right).$$

In particular, when we set $e = \frac{25}{36} T_a - n_f$, we have

$$P\left(\hat{B}(a) \neq B(a)\right) \leq 2T_a n_f \exp\left(-\frac{T_a - n_f}{18H_a}\right).$$

Proof. See Theorem 1 of Audibert et al. (2010) for a detailed proof. 

We set $q \geq 18H_a \left(\ln \frac{2n_f}{p} + m \ln n\right) + n_f$. So $q > \ln \frac{2}{p0p}$. Since $t_0 \geq \max\left\{\frac{4q}{p0}, \frac{28}{3p0} \ln \frac{2}{p0p}\right\}$, so we can set $t_0 \geq \frac{28q}{3p0} = \frac{28q m^m}{3} = O(\ln q)$.

Base on this lemma, when $T_a \geq 2q \geq 2 \left(18H_a \left(\ln \frac{2n_f}{p0p}\right) + n_f\right) \geq 18H_a \left(\ln \frac{4n_f}{p0p}\right) + n_f$,

$$P\left(\hat{B}(a) \neq B(a)\right) \leq \frac{p0p}{2}.$$

By the union bound, with probability at least $1 - p0p$, we have $\hat{B}(a) = B(a)$.

Similarly, with probability at least $1 - n^m p0p = 1 - p$, for any $a \in A$, we have $\hat{B}(a) = B(a)$.

After the follower commits its best response predictors, the rest $T - t_0$ rounds game for the leader is actually a semi-bandit MLSF game, so we use the result of Proposition 2. So by the union bound, with probability at least $1 - 2p$,

$$R_S^L(T) \leq O\left(t_0 + \sqrt{n(T - t_0) \ln n} + \sqrt{T - t_0 \ln \frac{1}{p}}\right) \leq O\left(t_0 + \sqrt{nT \ln n} + \sqrt{T \ln \frac{1}{p}}\right).$$

Using reduction from no-external to no-swap regret, which is Theorem 2 of Ito (2020), we get an $\epsilon^T$-CSE for leaders in MLSF bandit game, where $\epsilon^T = O\left(\frac{1}{T} + \sqrt{\frac{1}{T} n^2 \ln n} + \sqrt{\frac{1}{T} n \ln \frac{1}{p}}\right)$. 