About Knop’s action of the Weyl group on the set of orbits of a spherical subgroup in the flag manifold

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1 Introduction

Let $G$ be a complex connected reductive algebraic group. Let $B$ denote the flag variety of $G$. Let $H$ be an algebraic subgroup of $G$ which has a finite number of orbits in $B$; $H$ is said to be spherical. We denote by $H(B)$ the set of the $H$-orbits in $B$. The closures of these orbits are of importance in representation theory (see [Wol93]). Moreover, the elements of $H(B)$, viewed as orbits of a Borel subgroup of $G$ in $G/H$ play an important role in the geometry and topology of the $G$-equivariant embeddings $X$ of $G/H$.

In [Kno95], F. Knop introduced an action of a monoid (constructed from the Weyl group of $G$) on $H(B)$. This action is called “weak order” and studied by M. Brion in [Bri01]. But, the most spectacular combinatoric structure of the set $H(B)$ was discovered by F. Knop in [Kno95]: he defined an action of the Weyl group $W$ of $G$ on $H(B)$. Actually, the results of F. Knop are stated in a more general context. The proof of the existence of this action is very indirect and sophisticated. The aim of this note is to construct natural invariants separating the $W$-orbits. Note that our methods are elementary.

Let us fix a maximal torus $T^H$ of $H$. Denote by $W_H$ the Weyl group of $T^H$. Let $T$ be a maximal torus of $G$ containing $T^H$ and let $W$ denote the Weyl group of $T$.

Let $V \in H(B)$. Let $x$ be a point of $V$ whose the orbit by $T^H$ is of minimal dimension. Denote by $S$ the identity component of the stabilizer of $x$ in $T^H$. The group $W_H$ acts naturally on the set of subtori of $T^H$. The $W_H$-orbit of $S$ is called the type of $V$. It is shown in Section 3 that the type of $V$ only depends on $V$ and not on $x$.

The main result of this note is the following

**Theorem** Two elements of $H(B)$ are in the same $W$-orbit for Knop’s action if and only if they have the same type.

In Section 2, we recall some useful definitions about a graph with vertices the elements of $H(B)$, Knop’s action of $W$ on $H(B)$ and some classical invariants associated to the elements of $H(B)$. In Section 3, we show that the definition of the type of an orbit of $H$ is consistent. After, we study the fixed points of subtori of $H$ in the elements of $H(B)$. In Section 5, we state and prove our main results. In the following one, we give some consequences of our results and our proofs.
2 Definitions and notation

2.1 — Let us fix some general notation. If $\Gamma$ denotes a linear algebraic group, we denote by $\Gamma^0$ its identity component. If $\Gamma$ acts on an algebraic variety $X$ and $x$ belongs to $X$, we denote by $\Gamma_x$ the stabilizer of $x$ and by $\Gamma.x$ the orbit of $x$. The set of points of $X$ fixed by $\Gamma$ is denoted by $X^\Gamma$. If $S$ is a subgroup of $\Gamma$, we denote by $N_\Gamma(S)$ the normalizer of $S$ in $\Gamma$ and by $\Gamma^S$ the centralizer of $S$ in $\Gamma$.

2.2 — Let us recall that $G$ is a connected complex reductive group, $B$ its flag variety and $H$ a closed subgroup of $G$. We assume that $H$ is spherical; that is, $H$ has a dense orbit in $B$. In this article, we are interested in the set $H(B)$ of the orbits of $H$ in $B$. It is shown in [Bri86], [Vin86] or [Kno95] that $H(B)$ is finite.

We recall the definition of [Res04] of a graph $\Gamma(G/H)$ whose vertices are the elements of $H(B)$. The original construction of $\Gamma(G/H)$ due to M. Brion is very slightly different (see [Bri01]).

Consider the set $\Delta$ of conjugacy classes of minimal non solvable parabolic subgroups of $G$. If $\alpha$ belongs to $\Delta$, we denote by $P_\alpha$ the $G$-homogeneous space with isotropy $\alpha$. Then, there exists a unique $G$-equivariant map $\phi_\alpha : B \rightarrow P_\alpha$ which is a $\mathbb{P}^1$-bundle.

Let $V \in H(B)$ and $\alpha \in \Delta$. We assume that the restriction of $\phi_\alpha$ to $V$ is finite and we denote its degree by $d(V, \alpha)$. Then, there exists a unique open $H$-orbit $V'$ in $\phi_\alpha^{-1}(\phi_\alpha(V))$; in this case, we say that $\alpha$ raises $V$ to $V'$. One of the following three cases occurs.

- Type $U$: $H$ has two orbits in $\phi_\alpha^{-1}(\phi_\alpha(V))$ ($V$ and $V'$) and $d(V, \alpha) = 1$.
- Type $T$: $H$ has three orbits in $\phi_\alpha^{-1}(\phi_\alpha(V))$ and $d(V, \alpha) = 1$.
- Type $N$: $H$ has two orbits in $\phi_\alpha^{-1}(\phi_\alpha(V))$ ($V$ and $V'$) and $d(V, \alpha) = 2$.

**Definition.** Let $\Gamma(G/H)$ be the oriented graph with vertices the elements of $H(B)$ and edges labeled by $\Delta$, where $V$ is joined to $V'$ by an edge labeled by $\alpha$ if $\alpha$ raises $V$ to $V'$. One can find examples of graphs $\Gamma(G/H)$ in [Bri01, Pin01, Res04].

2.3 — Let us fix a Borel subgroup $B$ of $G$, and a maximal torus $T$ of $B$. Let $W$ denote the Weyl group of $T$. We now describe Knop’s action of $W$ on the set $H(B)$ (see also [Kno95]). Indeed, the action of simple reflexions easily reads off the graph $\Gamma(G/H)$.

Every $\alpha$ in $\Delta$ has a unique representative $P_\alpha$ which contains $B$. Moreover, there exists a unique $s_\alpha$ in $W$ such that $B s_\alpha B$ is dense in $P_\alpha$; and this $s_\alpha$ is a simple reflexion of $W$. The map, $\Delta \rightarrow W, \alpha \mapsto s_\alpha$ is a bijection from $\Delta$ onto the set of simple reflexions of $W$.

Consider the group $\tilde{W}$ generated by $\{s_\alpha : \alpha \in \Delta\}$ with the relations $s_\alpha^2 = 1$. There is a surjective homomorphism $\tilde{W} \rightarrow W$. Let $T$ denote its kernel.
One defines an action of \( \tilde{W} \) on the set \( H(B) \) by describing the action of the \( s_\alpha \), for any \( \alpha \in \Delta \):

- **Type U**: \( s_\alpha \) exchanges the two vertices of an edge of type \( U \) labeled by \( \alpha \).
- **Type T**: If \( \alpha \) raises \( V_1 \) and \( V_2 \) on \( V \), then \( s_\alpha V_1 = V_2 \) and \( s_\alpha V = V \).
- **Type N**: \( s_\alpha \) fixes the two vertices of a double edge labeled by \( \alpha \).
- \( s_\alpha \) fixes all others vertices of \( \Gamma(G/H) \).

In [Kno95], F. Knop showed that this action of \( \tilde{W} \) factors through \( W \); that is, that \( \mathcal{T} \) acts trivially on \( H(B) \). The aim of this paper is to describe the orbits of this action by a natural invariant and to give some consequences.

2.4 — Denote by \( H \) the \( G \)-homogeneous space \( G/H \). If \( V \) belongs to \( H(B) \), we set:

\[
V_H = \{ gH/H : g^{-1}B/B \in V \}.
\]

Then, \( V_H \) is a \( B \)-orbit in \( H \). Moreover, the map \( V \mapsto V_H \) is a bijection from \( H(B) \) onto the set \( B(H) \) of \( B \)-orbits in \( H \).

The **character group** \( \mathcal{X}(V_H) \) of \( V \) (or \( V_H \)) is the set of all characters of \( B \) that arise as weights of eigenvectors of \( B \) in the function field \( \mathbb{C}(V_H) \). Then \( \mathcal{X}(V_H) \) is a free abelian group of finite rank \( \text{rk}(V_H) \) (or \( \text{rk}(V) \)), the rank of \( V \).

3 The type of an orbit of \( H \)

3.1 — In this section, we define the type of a \( H \)-orbit in general (not only in \( B \)). We start with two technical lemmas.

Let us fix a maximal torus \( T^H \) of \( H \). If \( V \) is a \( H \)-homogeneous space, we set:

\[
\rho_V = \min_{x \in V} \dim(T^H.x).
\]

**Lemma 3.1** Let \( V \in H(B) \). Then, for all \( x \in V \), the following are equivalent:

(i) \( \dim(T^H.x) = \rho_V \),

(ii) \( (T^H_x)^\circ \) is a maximal torus of \( H_x \).

**Proof**: Assume that \( \dim(T^H.x) = \rho_V \). Let \( S' \supseteq (T^H_x)^\circ \) be a maximal torus of \( H_x \). Then, there exists \( h \) in \( H \) such that \( h^{-1}S'h \) is contained in \( T^H \). But, \( h^{-1}S'h \) fixes \( h^{-1}x \). Therefore, \( \dim T^H - \dim T^H_x = \rho_V \leq \dim(T^H.h^{-1}x) \leq \dim T^H - \dim S' \); hence \( \dim S' \leq \dim T^H_x \). It follows that \( S' = (T^H_x)^\circ \).

The converse is obvious since \( (T^H_x)^\circ \) is always a torus of \( H_x \). \( \square \)
Lemma 3.2 Let $x$ and $y$ belong to $V$ such that $\dim(T^H.x) = \dim(T^H.y) = \rho_V$. Set $S_x = (T^H_x)^\circ$ and $S_y = (T^H_y)^\circ$.

Then, we have:

(i) There exists $h$ in $H$ such that $y = h.x$ and $S_y = hS_xh^{-1}$.

(ii) There exist $\hat{w} \in N_H(T^H)$ such that $\hat{w}^{-1}S_y\hat{w} = S_x$ and $\hat{w}^{-1}.y \in H^{S_x}.x$.

Proof: Let $h_1 \in H$ such that $y = h_1.x$. By Lemma 3.1, $h_1^{-1}S_yh_1$ and $S_x$ are maximal tori of $H_x = h_1^{-1}H_yh_1$. Therefore, (see [Hum75, 21.3]) there exists $h_2$ in $H_x$ such that $h_2^{-1}h_1^{-1}S_yh_1h_2 = S_x$. Then, $h = h_1h_2$ satisfies Assertion 1.

Notice that $H^{S_x} = h^{-1}H^{S_y}h$. Then, $T^H$ and $h^{-1}T^Hh$ are maximal tori of $H^{S_x}$; so there exists $g_1$ in $H^{S_x}$ such that $g_1^{-1}h^{-1}T^Hhg_1 = T^H$. But, we have: $g_1^{-1}h^{-1}S_yhg_1 = S_x$. Then, $\hat{w} = hg_1$ satisfies Assertion 2. □

Let $W_H = N_H(T^H)/T^H$ denote the Weyl group of $H$. The group $W_H$ acts by conjugacy on the set of subtori of $T^H$. Let $V$ be a $H$-homogeneous space. Let us fix $x$ in $V$ such that $\rho_V = \dim(T^H.x)$. Then, by Lemma 3.2, the orbit $W_H.(T^H_x)^\circ$ does not depend on $x$ but only on $V$; we call it the type of $V$.

3.2 — We have:

Proposition 3.1 Let $S$ belong to the type of $V$. Then, we have:

(i) $V^S$ is a unique orbit of $N_H(S)$.

(ii) The irreducible components of $V^S$ are orbits of $(H^S)^\circ$.

Proof: Since $V$ is stable by $H$, $V^S$ is stable by $N_H(S)$. Let $x$ and $y$ belong to $V^S$. Let $h \in H$ such that $y = h.x$. Then, $h^{-1}Sh$ is contained in $H_x$. So by Lemma 3.1, $S$ and $h^{-1}Sh$ are maximal tori of $H_x$ and hence there exists $h_1$ in $H_x$ such that $h_1^{-1}h^{-1}Shh_1 = S$. Then, $y = hh_1.x$ belongs to $N_H(S).x$. Assertion 1 is proved.

By [Hum75, Corollary 16.3], the identity component of $N_H(S)$ is $(H^S)^\circ$. Then, Assertion 2 follows from Assertion 1. □

4 The type of an orbit of $H$ in $B$

4.1 — In the previous section, we associated to each $H$-homogeneous space $V$ a type and an integer $\rho_V$. Now, we apply these constructions to the orbits $V$ of $H$ in $B$. First, Proposition 4.1 below shows that the type of $V$ corresponds to the character group of $V$. We will deduce that $\rho_V - \text{rk}(V)$ is independent of $V$.

Let us fix a maximal torus $T$ of $G$ containing $T^H$. Let $B$ be a Borel subgroup of $G$ containing $T$.
Proposition 4.1 Let $V$ be in $\mathbf{H}(\mathcal{B})$ and $S$ be a subtorus of $T$ which belongs to the type of $V$. Let $w \in W$ such that $V$ intersects the irreducible component $G^s.wB/B$ of $\mathcal{B}^S$.

Then, $\mathcal{X}(V) \otimes \mathbb{Q}$ is equal to $\mathcal{X}(T)^w^{-1}sw \otimes \mathbb{Q}$.

Proof: Let $g \in G$ such that $gB/B$ belongs to $V \cap G^s.wB/B$. Consider $y = g^{-1}H/H$. By replacing $g$ by an element of $gB$, we may assume that $\dim(T.y) = \min_{y' \in B,y} \dim(T.y')$. But, by Lemma 3.1 $T^o_y$ is a maximal torus of $B_y$. Since the unipotent radical of $B^o_y$ is contained in $U$, it is equal to $U_y$. Then, we have: $G^o_y = T^o_y U_y$.

We have: $\mathcal{X}(V) \otimes \mathbb{Q} = \mathcal{X}(B)^{B_y} \otimes \mathbb{Q}$. Moreover, the restriction map from $\mathcal{X}(B^o_y)$ to $\mathcal{X}(T^o_y)$ is injective. Therefore, $\mathcal{X}(V) \otimes \mathbb{Q} = \mathcal{X}(T)^{T^o_y} \otimes \mathbb{Q}$.

Since $B_y = g^{-1}H_xg$, $g^{-1}Sg$ is a maximal torus of $B_y$. Therefore, there exists $b \in B^o_y$ such that $S = gbT^o_y b^{-1}g^{-1}$. By replacing $g$ by $gb$ (and keeping $x$ and $y$ unchanged), we may assume that $b$ is trivial; that is, that $S = gT^o_y g^{-1}$.

It follows that $T$ and $gTg^{-1}$ are maximal tori of $G^S$. Then, there exists $s \in G^S$ such that $sg$ normalizes $T$. Let $w_1$ be the class of $sg$ in the Weyl group of $T$. Then, $T^o_y = w_1^{-1}Sw_1$.

On the other hand, since $sg \in G^swB$, there exists $w'$ in the Weyl group of $G^S$ such that $w_1 = w'.w$. Then, $T^o_y = w^{-1}Sw$ and the proposition follows. \[ \square \]

Corollary 4.1 Let $V$ be an orbit of $H$ in $\mathcal{B}$. We have:

(i) $\rho_V - \text{rk}(V) = \text{rk}(G) - \text{rk}(H)$.

(ii) The rank of $V$ is minimal in $\mathbf{H}(\mathcal{B})$ if and only if $V$ contains points fixed by $T^H$.

Proof: The proposition shows that the rank of $V$ is the dimension of $T$ minus the dimension of $S$. On the other hand, $\rho_V$ is the difference between the rank of $H$ and the dimension of $S$. Assertion 1 follows.

Since $T^H$ has fixed points in $\mathcal{B}$, the rank of $V$ is minimal if and only if $\rho_V = 0$; that is, if and only if $V$ contains points fixed by $T^H$. \[ \square \]

4.2 — Let $V$ be in $\mathbf{H}(\mathcal{B})$ and $S$ belong to the type of $V$. We are now interested in the set $V^S$. We can make Proposition 3.1 more precise:

Proposition 4.2 (i) The intersection of $V^S$ and an irreducible component of $\mathcal{B}^S$ is a unique orbit of $H^S$.

(ii) If $H$ is connected, the intersection of $V$ and one irreducible component of $\mathcal{B}^S$ is irreducible.

Proof: Let $x$ and $y$ be two points of $V^S$ in the same irreducible component of $\mathcal{B}^S$. Since the irreducible components of $\mathcal{B}^S$ are orbits of $G^S$, there exists $g \in G^S$ such that $y = g.x$. By Assertion (i) there exists $h \in N_H(S)$ such that $y = h.x$. Then, $g^{-1}h$ belongs to $G_x$ which is a Borel subgroup of $G$ which contains $S$. Moreover, $g^{-1}h$ normalizes $S$. But, by [Hum75, Proposition 19.4], we have: $N_{G_x}(S) = G_x^S$. So, $g^{-1}h$ and $h$ centralize $S$. Assertion (iii) follows.
If $H$ is connected, Theorem 22.3 of [Hum75] shows that $H^S$ is connected. Now, Assertion (iv) follows from Assertion (iii).

4.3 — We are now interested in the set of irreducible components of $B^S$ which intersect $V$. By Proposition 4.2, if $H$ is connected, this set is in bijection with the set of the irreducible components of $V^S$.

Since the irreducible components of $B^S$ are the $G^SwB/B$ for $w$ in $W$, we set:

$$\mathcal{C}(V, S) = \{w \in W : V \cap G^SwB/B \neq \emptyset\}. $$

To describe $\mathcal{C}(V, S)$, we need two technical lemmas.

**Lemma 4.1** Set $N_H(S)G^S = \{hg : h \in N_H(S) \text{ and } g \in G^S\}$.

Then, $N_H(S)G^S$ is a closed subgroup of $N_G(S)$ whose identity component is $G^S$. Moreover, the group $(N_H(S)G^S)/G^S$ is isomorphic to $N_H(S)/H^S$ (the Weyl group of $S$ in $H$, denoted by $W(H, S)$).

**Proof:** Notice that, $N_H(S)$ normalizes $G^S$. Now, one easily checks that $N_H(S)G^S$ is a subgroup of $G$. Moreover, $N_H(S)G^S$ is clearly contained in $N_G(S)$ and contains $G^S$. But by [Hum75, Corollary 16.3], $G^S$ is the identity component of $N_G(S)$. It follows that the index of $G^S$ in $N_H(S)G^S$ is finite. Then, $N_H(S)G^S$ is closed in $N_G(S)$ and its identity component is $G^S$. The last assertion is obvious.

Notice that $T$ is contained in $N_H(S)G^S$. Set $W_{N_H(S)G^S} = N_{N_H(S)G^S}(T)/T$. Then, the inclusion of $N_{N_H(S)G^S}(T)$ in $N_G(T)$ induces an embedding of $W_{N_H(S)G^S}$ in $W$. Let $W_G^S$ denote the Weyl group of $(G^S, T)$.

**Lemma 4.2** We have an exact sequence:

$$1 \rightarrow W_G^S \rightarrow W_{N_H(S)G^S} \rightarrow W(H, S) \rightarrow 1.$$ 

**Proof:** Let us start with the exact sequence given by Lemma 4.1:

$$1 \rightarrow G^S \rightarrow N_H(S)G^S \rightarrow W(H, S) \rightarrow 1.$$ 

By intersecting with $N_{N_H(S)G^S}(T)$, we obtain an exact sequence:

$$1 \rightarrow N_G^S(T) \rightarrow N_{N_H(S)G^S}(T) \rightarrow W(H, S),$$

and it is sufficient to prove that the last map is surjective. Let $h$ in $N_H(S)$ and $g$ in $G^S$. Since, $ghT(gh)^{-1}$ and $T$ are maximal tori of $G^S$, there exists $g' \in G^S$ such that $g'ghT(gh)^{-1}g'^{-1} = T$. The lemma follows.

If $E$ is a finite set, let $|E|$ denote its cardinality. Now, we can describe $\mathcal{C}(V, S)$:

**Proposition 4.3** (i) The set $\mathcal{C}(V, S)$ is an orbit of $W_{N_H(S)G^S}$ for its action on $W$ by left multiplication.
(ii) If \( H \) is connected, \( V^S \) has \( |W_{NH(S)GS}| \) irreducible components.

**Proof:** Let \( \sigma \) be an element of \( C(V,S) \) and let \( x \) belong to \( V \cap G^S \sigma B/B \). By Proposition 3.1, \( V^S = N_H(S).x \). Therefore \( G^S.V^S = G^S.N_H(S).x = (N_H(S)G^S)\sigma B/B \). But \( G^S.V^S \) is the union of the \( G^S, wB/B \) for \( w \in C(V,S) \). The first assertion follows.

By Proposition 4.2, each irreducible component of \( V^S \) is the intersection of \( V \) and one irreducible component of \( B^S = \bigcup_{w \in w_{GS}} w G^S wB/B \). Therefore, by the first assertion \( V^S \) has \( \frac{|W_{NH(S)GS}|}{|W_{GS}|} \) irreducible components. Now, the second assertion follows from Lemma 4.2. \( \square \)

**4.4** — Each irreducible component of \( B^S \) is isomorphic to the flag variety \( B_{GS} \) of \( G^S \). Moreover, by Proposition 4.2, \( V \) intersects any such irreducible component in one orbit of \( H^S \). We will now describe the orbits of \( H^S \) in \( B_G \) which appear in that way.

Let \( \tau \) be a \( W_H \)-orbit of subtori of \( T^H \). Let \( H(B)_\tau \) denote the set of \( H \)-orbits in \( B \) of type \( \tau \).

**Proposition 4.4** Assume that \( H(B)_\tau \) is not empty. Let us fix an element \( S \) in \( \tau \). Then,

(i) The subgroup \( H^S \) of \( G^S \) is spherical.

(ii) The rank of \( G^S/H^S \) is equal to the rank of the free abelian group \( \chi(T)^S \).

(iii) Let \( V \in H(B)_\tau \) and \( x \in V^S \). Then, \( \rho_{H^S,x} = \text{rk}(H) - \text{rk}(S) \). In particular, \( \text{rk}(H^S.x) = \text{rk}(G^S/H^S) \).

(iv) Conversely, let \( y \in B^S \) such that \( \rho_{H^S,y} = \text{rk}(H) - \text{rk}(S) \). Then, the type of \( H.y \) is \( \tau \).

**Proof:** We first prove Assertions 3 and 4. Let \( V \in H(B)_\tau \) and \( x \in V^S \).

Let \( y \in H^S.x \). Since \( y \) belongs to \( V \) and the type of \( V \) is \( \tau \), we have \( \dim(T^H.y) \leq \dim S \). Then, \( \rho_{H^S,y} \leq \text{rk}(H) - \text{rk}(S) \).

But \( \rho_{H^S,x} \geq \rho_{H,x} = \text{rk}(H) - \text{rk}(S) \). So \( \rho_{H^S,x} = \text{rk}(H) - \text{rk}(S) \). This proves Assertion 3.

Set \( \Omega = \{ y \in G^S.x : \rho_{H^S,y} \leq \text{rk}(H) - \text{rk}(S) \} \). The set \( \Omega \) is open in \( G^S.x \) and contains \( x \).

Let \( y \in \Omega \). Then, \( S \) is a maximal torus of \( H^S.y \). Let \( S_y \) be a maximal torus of \( H_y \) containing \( S \). Then, \( S_y \) is contained in \( H^S \). Therefore \( S = S_y \). Then, Lemma 3.1 shows that \( \rho_{H,y} = \text{rk}(H) - \text{rk}(S) \). Therefore, since \( (H,y)^S \) is not empty, the type of \( H.y \) is \( \tau \). By Corollary 4.1, this proves Assertion 4.

By Proposition 4.2, each orbit of type \( \tau \) intersects \( G^S.x \) in a unique orbit of \( H^S \). Hence, Assertion 3 shows that the set of \( H^S \)-orbit in \( \Omega \) is finite. So, \( H^S \) has a dense orbit in \( \Omega \) and in \( G^S.x \). The first assertion follows. The second one is now a consequence of Assertion 3. \( \square \)
5 Knop’s action of $W$ on $H(B)$ and orbit type

5.1 — Keep the notation as above. In particular, $\tau$ is a $W_H$-conjugacy class of subtori of $T^H$ such that $H(B)_{\tau}$ is not empty and $S$ belongs to $\tau$. Set $w_{N_H(S)G^S} \setminus W = \{w_{N_H(S)G^S}w : w \in W\}$. By Proposition 4.3, we can define a map

$$\Theta : H(B)_{\tau} \rightarrow w_{N_H(S)G^S} \setminus W$$

$$V \mapsto C(V, S).$$

We consider on $w_{N_H(S)G^S} \setminus W$ the action of the Weyl group $W$ by right multiplication.

In this section we show the following

**Theorem 1** The subset $H(B)_{\tau}$ of $H(B)$ is stable by Knop’s action of $W$. Moreover, the map $\Theta$ is $W$-equivariant.

5.2 — Start with

**Lemma 5.1** Let $V \in H(B)_{\tau}$, $x \in V^S$ and $\alpha \in \Delta$. Consider $\phi_\alpha : B \rightarrow P_\alpha$. Let $w \in W$ be such that $G^S.x = G^S.wB/B$. Then one of the two following cases occurs:

**Case 1:** $\phi_\alpha^{-1}(\phi_\alpha(x))$ is pointwise fixed by $S$.

Then, we have $G^S.wS_aB/B = G^S.wB/B$.

**Case 2:** There exists $y \neq x$ such that $\phi_\alpha^{-1}(\phi_\alpha(x))^S = \{x, y\}$.

Then, $G^S.x \neq G^S.y$ and $G^S.y = G^S.wS_aB/B$.

**Proof:** Set $F = \phi_\alpha^{-1}(\phi_\alpha(x))$. The variety $F$ is isomorphic to the projective line $\mathbb{P}^1$. Moreover, $F$ is stable by the action of the torus $S$. Then, the image of $S$ in $\text{Aut}(F) \simeq \text{PSL}(2)$ is either trivial or a maximal torus of $\text{Aut}(F)$. In particular, one of the following cases occurs.

Case 1: $F^S = F$.

Case 2: There exists $y \neq x$ such that $F^S = \{x, y\}$.

In either case, consider the $G^S$-orbit $G^S.\phi_\alpha(x)$ and the flag variety $B_{G^S}$ of the group $G^S$. Since $G^S.\phi_\alpha(x)$ is the image by $\phi_\alpha$ of $G^S.x \simeq B_{G^S}$, it is a complete $G^S$-homogeneous space. Moreover, since $\phi_\alpha$ is a $\mathbb{P}^1$-fibration, we have: $\dim(B_{G^S}) = \dim(G^S.\phi_\alpha(x)) \geq \dim(B_{G^S}) - 1$. Then, two cases occur.

Case a: $G^S_{\phi_\alpha(x)}$ is a non solvable minimal parabolic subgroup of $G^S$ and $G^S.x$ contains $F$.

Case b: $G^S_{\phi_\alpha(x)}$ is a Borel subgroup of $G^S$ and $F \cap G^S.x = \{x\}$.

In Case 1, $F$ is contained in the irreducible component of $B^S$ which contains $x$; that is in $G^S.x$. So, Case 1 implies Case a. In Case 2, we cannot have that $F$ contains $G^S.x$. So, Case 2 implies Case b. In particular, $G^S.x \neq G^S.y$.

It remains to determine $G^S.wS_aB/B$ in each case. The fiber $\phi_\alpha^{-1}(\phi_\alpha(B/B))$ of $\phi_\alpha$ is the closure $\overline{B_{S_a}B/B}$ of $B_{S_a}B/B$ in $B$. Let $g \in G^S$ be such that $x = gwB/B$. Then, $F = gw\overline{B_{S_a}B/B}$. 

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In Case 1, $F$ is contained in $G^S wB/B$. In particular, $gws_\alpha$ belongs to $G^S wB/B$. Therefore, $G^S w_\alpha B/B = G^S wB/B$.

In Case 2, we can notice that $gws_\alpha B/B$ is fixed by $S$ and belongs to $F$; Therefore, $y = gws_\alpha B/B$. Then, $G^S . y = G^S w_\alpha B/B$. □

**5.3 — Proof of Theorem 1.** Let $V \in H(B)_\tau$ and $\alpha \in \Delta$. We will prove that $C(V, S)s_\alpha = C(s_\alpha V, S)$. Let $w \in C(V, S)$. By Proposition 4.3, it is sufficient to show that $w_\alpha$ belongs to $C(s_\alpha V, S)$.

We fix $x$ in $V^S \cap G^S wB/B$ and we set $F = \phi_\alpha^{-1}(\phi_\alpha(x))$. Then, one of the following 4 cases occurs.

**Case 1:** $\alpha$ raises $V$ on $s_\alpha V$ (type $U$).

Since $V \cap F = \{x\}$, $(s_\alpha V)^S$ is not empty. Since $V$ and $s_\alpha V$ have the same rank, Corollary 4.1 implies that $S$ belongs to the type of $s_\alpha V$.

Let us assume that there exists $y \neq x$ such that $F^S = \{x, y\}$. Necessarily, $y$ belongs to $s_\alpha V$. But Lemma 5.1 shows that $G^S . y = G^S w_\alpha B/B$. So, $w_\alpha$ belongs to $C(s_\alpha V, S)$.

If $F^S = F$ then Lemma 5.1 shows that $F$ is contained in $G^S wB/B = G^S w_\alpha B/B$. Then, since $F$ intersects $s_\alpha V$, $w_\alpha$ belongs to $C(s_\alpha V, S)$.

**Case 2:** $\alpha$ raises $V$ and $s_\alpha V$ on a third $H$-orbit $V_1$ (type $T$).

Since $\text{rk}(V_1) = \text{rk}(V) + 1$, Corollary 4.1 shows that $V_1^S$ is empty. Then, by Lemma 5.1 there exists $y \neq x$ such that $F^S = \{x, y\}$. On the other hand, $F \cap V_1$ is equal to $F$ with two points removed (type $T$). Since, $V_1^S$ is empty it follows that $F \cap V_1 = F - \{x, y\}$, $F \cap V = \{x\}$ and $F \cap s_\alpha V = \{y\}$. But, Lemma 5.1 shows that $G^S . y = G^S w_\alpha B/B$. Therefore, $w_\alpha$ belongs to $C(s_\alpha V, S)$.

**Case 3:** $\alpha$ raises $V$ and $s_\alpha V = V$ (type $N$).

The same proof as in Case 2 shows that $F^S = \{x, y\} = F \cap V$ and $G^S . y = G^S w_\alpha B/B$. It follows that $w_\alpha$ belongs to $C(V, S) = C(s_\alpha V, S)$.

**Case 4:** $F \cap V$ is open in $F$.

If $F^S = F$ then $V$ is the only $H$-orbit in $\phi_\alpha^{-1}(V)$ of maximal rank (type $T$ or $N$). Therefore, $s_\alpha V = V$. Moreover, by Lemma 5.1, we have $G^S . wB/B = G^S w_\alpha B/B$; therefore, $w_\alpha \in C(V, S) = C(s_\alpha V, S)$.

We may assume that there exists $y \neq x$ such that $F^S = \{x, y\}$. Then, since $F \cap V$ is open in $F$, stable by $S$ and contains $x$, $F \cap V$ is either $F$ or $F - \{y\}$. If $F \cap V = F - \{y\}$ then $\alpha$ raises $s_\alpha V$ to $V$ by an edge of type $U$. By exchanging $V$ and $s_\alpha V$ we come back to Case 1. Assume that $V$ contains $F$. Since $G^S w_\alpha B/B$ intersects $F$, it intersects $V$. Then, $V = s_\alpha V$ and $w_\alpha \in C(V, S) = C(s_\alpha V, S)$.

This completes the proof of Theorem 1. □

**5.4 —** Let $\sigma$ be in $W$ and $\overline{\sigma}$ be its class in $w_{S_H(s)\cap G^S} \setminus W$. We are now interested in the fiber $\Theta^{-1}(\overline{\sigma})$ of $\Theta$. By definition of $C(V, S)$, $\Theta^{-1}(\overline{\sigma})$ is the set of the orbits $V$ in $H(B)_\tau$ which intersects $G^S aB/B$. Let $H^S(B_{G^S})$ denote the set of the $H^S$-orbits in the flag manifold $B_{G^S}$ of
\(G^S\), and let \(H^S(B_{G^S})_{\max}\) denote the set of the \(H^S\)-orbits of maximal rank. By Proposition 4.4, the map

\[
\eta_\sigma : \Theta^{-1}(\sigma) \rightarrow H^S(B_{G^S})_{\max}
\]

\[
V \rightarrow V \cap G^S \sigma B / B
\]

is a bijection.

The subgroup \(\sigma^{-1}W_{N_H(S)G^S}\) stabilizes \(\Theta^{-1}(\sigma)\). Moreover, \(W_{N_H(S)G^S}\) contains \(W_{G^S}\). Therefore, the group \(W_{G^S}\) acts on \(\Theta^{-1}(\sigma)\) through the morphism \(W_{G^S} \rightarrow W, w \mapsto \sigma^{-1}w\). On the other hand, \(W_{G^S}\) acts on \(H^S(B_{G^S})_{\max}\) by Knop’s action. Is the bijection \(\eta_\sigma\) \(W_{G^S}\)-equivariant? The answer is NO in general, but YES for at least one \(\sigma\).

**Proposition 5.1** There exists \(\sigma\) such that \(\eta_\sigma\) is \(W_{G^S}\)-equivariant.

**Proof:** Actually, the map \(\Theta\) depends on the choice of the Borel subgroup \(B\) made in Paragraph 1. To prove the proposition, it is sufficient to prove that for a good choice of \(B\), \(\eta_1\) is \(W_{G^S}\)-equivariant. Let us make such a choice.

Let \(P\) be a parabolic subgroup of \(G\) with Levi subgroup \(G^S\). Let \(B\) be a Borel subgroup of \(G\) such that \(T \subset B \subset P\).

Notice that \(B^S = B \cap G^S\) is a Borel subgroup of \(G^S\). Denote by \(\Delta^S\) the set of conjugacy classes of minimal non solvable parabolic subgroups of \(G^S\). Let \(\alpha \in \Delta^S\) and \(P^S_\alpha\) denote the \(G^S\)-homogeneous space with isotropy \(\alpha\). If \(P^S_\alpha\) is a minimal parabolic subgroup of \(G^S\) containing \(B^S\) corresponding to \(\alpha\), then \(P^S_\alpha B\) is a minimal parabolic subgroup of \(G\). Moreover, \(P^S_\alpha = (P^S_\alpha B) \cap G^S\). Therefore, we obtain an immersion (from now on implicit) of \(\Delta^S\) in \(\Delta\). In particular \(P_\alpha = P^S_\alpha B\). Consider the following commutative diagram \(\mathcal{D}\):

\[
\begin{array}{ccc}
B_{G^S} \simeq G^S B / B & \xrightarrow{\text{inclusion}} & B \\
\downarrow & & \downarrow \phi_{\alpha_i} \\
P^S_\alpha & \xrightarrow{\phi_{\alpha_i}} & P_{\alpha_i}
\end{array}
\]

The restriction of \(\phi_{\alpha}\) to \(G^S B / B\) is obviously the unique \(G^S\)-equivariant map \(\phi_{\alpha,S}\) from \(B_{G^S}\) onto \(P_{\alpha,S}\).

Let \(x \in G^S B / B\) such that \(H^S x\) belongs to \(H^S(B_{G^S})_{\max}\). It remains to prove the following

**Claim:** \(G^S B / B \cap (s_\alpha H x) = s_\alpha (H^S x)\).

Since Diagram \(\mathcal{D}\) is commutative, we have

\[
\phi_{\alpha}^{-1}(\phi_{\alpha}(x)) = \phi_{\alpha,S}^{-1}(\phi_{\alpha,S}(x));
\]

we denote by \(F\) this subvarity of \(\mathcal{B}\). Moreover, since the rank of \(H^S x\) is maximal in \(H^S(B_{G^S})\), Proposition 4.4 shows
\[ G^SB/B \cap Hx = H^Sx. \]  

(2)

Four cases can occur:

Case 1: \( \alpha \) raises \( H^Sx \) in \( \Gamma(G^S/H^S) \).

Case 2: \( \alpha \) raises an orbit \( H^Sy \) of \( H^S(B_{Gs})_{\text{max}} \) on \( H^Sx \).

Case 3: \( \alpha \) raises an orbit of \( H^S(B_{Gs}) \) on \( H^Sx \) by an edge of type \( T \) or \( N \).

Case 4: \( H^Sx = \phi^{-1}_{\alpha,S}(\phi_{\alpha,S}(H^Sx)) \).

In Case 1, \( F \cap H^Sx = \{x\} \) and \( H^Sx \), and hence \( Hx \), acts transitively on \( F - \{x\} \). Moreover, by Equality 2, \( F \cap Hx = \{x\} \). Therefore, \( \alpha \) raises \( Hx \) by an edge of type \( U \) in \( \Gamma(G/H) \). The claim follows.

In Case 2, \( H^Sy \) is in Case 1. The claim follows.

In Case 3, \( F \cap H^Sx = F \cap Hx \) is equal to \( F \) with two points removed. Therefore, \( \alpha \) raises an orbit of \( H(B) \) on \( Hx \) by an edge of type \( T \) or \( N \), and \( s_\alpha.Hx = Hx \).

In Case 4, \( F \) is contained in \( H^Sx \) and hence in \( Hx \). As a consequence, \( Hx = \phi_{\alpha}^{-1}(\phi_{\alpha}(Hx)) \) and \( s_\alpha.Hx = Hx \). This completes the proof of the proposition.

Here, comes our main result.

**Theorem 2** Two elements of \( H(B) \) are in the same \( W \)-orbit for Knop’s action if and only if they have the same type.

**Proof:** By Theorem 1, it is sufficient to prove that one (or any) fiber of \( \Theta \) is an orbit of \( W_{S_H(S)G^S} \). Then, by Proposition 5.1, it is sufficient to prove the theorem for the orbits of maximal rank. Let \( V_0 \) be such an orbit. There exist a sequence \( \alpha_1, \ldots, \alpha_k \) in \( \Delta \) and a sequence \( V_0, V_1, \ldots, V_k \) of \( H \)-orbits such that \( \alpha_i \) raises \( V_{i-1} \) on \( V_i \) for all \( i = 1, \ldots, k \), and \( V_k \) is the open \( H \)-orbit in \( B \). Since the rank of \( V_0 \) is maximal, all the orbits \( V_i \) have the same rank and the edges joining these orbits are of type \( U \). Therefore, we have \( (s_{\alpha_k} \cdots s_{\alpha_1})V_0 = V_k \). The theorem is proved.

6 Some consequences

6.1 — Theorem 2 has a nice corollary about the character groups of the elements of \( B(H) \):

**Corollary 6.1** Let \( V \) and \( V' \) in \( H(B) \). Then, \( \mathcal{X}(V) = \mathcal{X}(V') \) if and only if \( \mathcal{X}(V) \otimes \mathbb{Q} = \mathcal{X}(V') \otimes \mathbb{Q} \).

**Proof:** Let us fix \( T \subset B \). By identifying \( \mathcal{X}(B) \) with \( \mathcal{X}(T) \), we obtain an action of \( W \) on \( \mathcal{X}(B) \). Assume that \( \mathcal{X}(V) \otimes \mathbb{Q} = \mathcal{X}(V') \otimes \mathbb{Q} \). By Proposition 4.1, the orbits \( V \) and \( V' \) have the same type. Then, by Theorem 2, there exists \( w \) in \( W \) such that \( V = wV' \).
Then, by [Kno95, Theorem 4.3], $\mathcal{X}(V) = w\mathcal{X}(V')$. Now, $\mathcal{X}(V) \otimes \mathbb{Q} = \mathcal{X}(V') \otimes \mathbb{Q}$ implies $\mathcal{X}(V) = \mathcal{X}(V')$. □

6.2 — We can also apply Theorem 2 to the description of the isotropy subgroups of the action of $H$ in $\mathcal{B}$.

**Corollary 6.2** Let $x$ and $y$ be in $\mathcal{B}$ such that $Hx$ and $Hy$ have the same type. Then, $(H_x/H_x^o)$ and $(H_y/H_y^o)$ are isomorphic.

**Proof:** Set $V = Hx$ and $V' = Hy$. Let $\alpha \in \Delta$. Since $W$ is generated by the simple reflections, by Theorem 2 it is sufficient to prove the corollary for $V' = s_{\alpha}V \neq V$. Two cases occur:

- Type $T$: $V$ and $V'$ are raised on a third orbit $V''$.
- Type $U$: $\alpha$ raises $V$ on $V'$ (up to re-indexing).

In the first case, the restrictions of $\phi_\alpha$ to $V$ and $V'$ are isomorphisms onto $\phi_\alpha(V'')$. The corollary follows.

Assume that $\alpha$ raises $V$ on $V' = s_{\alpha}V$. By replacing $y$ by another point of $Hy$, we may assume that $\phi_\alpha(x) = \phi_\alpha(y)$. Since the restriction of $\phi_\alpha$ to $V$ is an isomorphism onto $\phi_\alpha(V)$ and $\phi_\alpha(V') = \phi_\alpha(V)$, $Hy$ is contained in $H_x$. This inclusion induces a morphism $\psi : H_y/H_y^o \rightarrow H_x/H_x^o$. But, $H_x/H_y$ is isomorphic to $\mathbb{A}^1$ and hence irreducible. We deduce that $\psi$ is surjective.

It remains to show that $H_y \cap H_x^o = H_x^o$ to prove that $\psi$ is injective. Obviously, $H_x^o \subset (H_y \cap H_x^o)$; and we can define a morphism $H_x^o/H_x^o \rightarrow H_x^o/(H_y \cap H_x^o)$. Since $H_x^o/(H_y \cap H_x^o)$ is isomorphic to $\mathbb{A}^1$, it is simply connected and $H_y \cap H_x^o = H_y^o$. □

6.3 — We are going to apply Theorem 2 to the $H$-orbits in $\mathcal{B}$ of minimal rank. We keep notation as above. In particular, $H(\mathcal{B})_{\{T^H\}}$ is the set of the orbits of $H$ in $\mathcal{B}$ of minimal rank.

**Proposition 6.1** We assume that $H$ is connected. Then, we have:

(i) The group $H^{T_H}/T^H$ is a maximal unipotent subgroup of $G^{T_H}/T^H$.

(ii) The stabilizers in $W$ (for Knop’s action) of the elements of $H(\mathcal{B})_{\{T^H\}}$ are isomorphic to the Weyl group $W_H$ of $H$.

(iii) Let $V$ be in $H(\mathcal{B})_{\{T^H\}}$. The stabilizers in $H$ of the points of $V$ are connected.

**Proof:** Since $T^H$ is maximal in $H$, $H^{T_H}/T^H$ is unipotent. But it is a spherical subgroup of $G^{T_H}/T^H$. Assertion 1 follows.

We claim that the cardinality of the set $H(\mathcal{B})_{\{T^H\}}$ is $\frac{|W|}{|W_H|}$. By Proposition 4.3, we have to prove that the set of irreducible components of the $V^{T^H}$ for $V \in H(\mathcal{B})_{\{T^H\}}$ has the same
cardinality as $W$. But, by Proposition 4.4, this set is in natural bijection with the set of orbits of $H^T H$ in $B^T H$. Moreover, by Assertion 1, $H^T H$ has $|W_{G^T H}|$ orbits in each one of the $|W_{G^T H}|$ irreducible components of $B^T H$. The claim follows.

By Proposition 5.1, we may assume that $\eta_1$ is $W_{G^T H}$-equivariant to prove Assertion 2. Let $V$ be in $H(B)_{(T^H)}$ such that $\Theta(V) = T$. We have to prove that the stabilizer $W_V$ of $V$ in $W$ is isomorphic to $W_H$. Since $\Theta$ is $W$-equivariant, $W_V$ is contained in $W_{N_H(T^H)G^T H}$ and by Lemma 4.2 maps on $W_H$. Moreover, the claim shows that $|W_V| = |W_H|$. So, by Lemma 4.2 it is sufficient to prove that $W_V \cap W_G$ is trivial. By Proposition 5.1, this is a consequence of Assertion 1.

By Corollary 6.2, it is sufficient to prove the last assertion for a closed orbit $V$ of $H$ in $B$. Let $x$ be in $V$. Since $V$ is closed in $B$, it is projective. So, $H_x$ is a parabolic subgroup of $H$. In particular, it is connected. □

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