THE DISTRIBUTION OF VALUES IN THE QUADRATIC ASSIGNMENT PROBLEM

ALEXANDER BARVINOK AND TAMON STEPHEN

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Abstract. We obtain a number of results regarding the distribution of values of a quadratic function \( f \) on the set of \( n \times n \) permutation matrices (identified with the symmetric group \( S_n \)) around its optimum (minimum or maximum). In particular, we estimate the fraction of permutations \( \sigma \) such that \( f(\sigma) \) lies within a given neighborhood of the optimal value of \( f \). We identify some “extreme” functions \( f \) (there are 4 of those for \( n \) even and 5 for \( n \) odd) such that the distribution of every quadratic function around its optimum is a certain “mixture” of the distributions of the extremes and describe a natural class of functions (which includes, for example, the objective function in the Traveling Salesman Problem) with a relative abundance of near-optimal permutations. In particular, we identify a large class of functions \( f \) with the property that permutations in the vicinity of the optimal permutation (in the Hamming metric of \( S_n \)) tend to produce near optimal values of \( f \) (such is, for example, the objective function in the symmetric Traveling Salesman Problem) and show that for general \( f \), just the opposite behavior may take place: an average permutation in the vicinity of the optimal permutation may be much worse than an average permutation in the whole group \( S_n \).

1. Introduction

The Quadratic Assignment Problem (QAP for short) is an optimization problem on the symmetric group \( S_n \) of \( n! \) permutations of an \( n \)-element set. The QAP is one of the hardest problems of combinatorial optimization, whose special cases include the Traveling Salesman Problem (TSP) among other interesting problems.

Recently the QAP has been of interest to many people. An excellent survey of recent results is found in [5]. Despite this work, it is still extremely difficult to solve QAP’s of size \( n = 20 \) to optimality, and the solution to a QAP of size \( n = 30 \) is considered noteworthy, see, for example, [1] and [4]. Moreover, it appears that essentially no positive approximability results for the general QAP are known.

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although some “bad news” (non-approximability) and approximability for special classes have been established, see [3] and [2].

The goal of this paper is to study the distribution of values of the objective function of the QAP. We hope that our results would allow one on one hand to understand the behavior of the local search heuristic, and, on the other hand, to get guaranteed approximations to the optimum using some simple algorithms based on random or partial enumeration with guaranteed complexity bounds. In particular, we estimate how well the sample optimum from a random sample of a given size approximates the global optimum.

(1.1) The Quadratic Assignment Problem. Let Mat$_n$ be the vector space of all real $n \times n$ matrices $A = (a_{ij})$, $1 \leq i, j \leq n$ and let $S_n$ be the set of all permutations $\sigma$ of the set $\{1, \ldots, n\}$. There is an action of $S_n$ on the space Mat$_n$ by simultaneous permutations of rows and columns: we let $\sigma(A) = B$, where $A = (a_{ij})$ and $B = (b_{ij})$, provided $b_{\sigma(i)\sigma(j)} = a_{ij}$ for all $i, j = 1, \ldots, n$. One can check that $(\sigma \tau)A = \sigma(\tau A)$ for any two permutations $\sigma$ and $\tau$. There is a standard scalar product on Mat$_n$:

$$\langle A, B \rangle = \sum_{i,j=1}^{n} a_{ij} b_{ij} \quad \text{where} \quad A = (a_{ij}) \quad \text{and} \quad B = (b_{ij}).$$

Let us fix two matrices $A = (a_{ij})$ and $B = (b_{ij})$ and let us consider a real-valued function $f : S_n \rightarrow \mathbb{R}$ defined by

$$(1.1.1) \quad f(\sigma) = \langle B, \sigma(A) \rangle = \sum_{i,j=1}^{n} b_{\sigma(i)\sigma(j)} a_{ij} = \sum_{i,j=1}^{n} b_{ij} a_{\sigma^{-1}(i)\sigma^{-1}(j)}$$

The problem of finding a permutation $\sigma$ where the maximum or minimum value of $f$ is attained is known as the Quadratic Assignment Problem. It is one of the hardest problems of Combinatorial Optimization. From now on we assume that $n \geq 4$.

In this paper, we study the distribution of values of $f$ from the optimization perspective:

• How “steep” or how “flat” can the optimum of $f$ be?
• How many values of $f$ lie within a given distance to the optimum?
• When can we hope to improve the value of $f(\sigma)$ by modifying $\sigma$ slightly?

To formulate the questions rigorously (and to answer them), we introduce the standard Hamming metric on the symmetric group $S_n$. 
(1.2) **Definitions.** For two permutations $\tau, \sigma \in S_n$, let the distance $\text{dist}(\sigma, \tau)$ be the number of indices $1 \leq i \leq n$ where $\sigma$ and $\tau$ disagree:

$$\text{dist}(\tau, \sigma) = |i : \sigma(i) \neq \tau(i)|.$$ 

One can observe that the distance is invariant under the left and right actions of $S_n$:

$$\text{dist}(\sigma_1 \sigma, \sigma_2 \sigma) = \text{dist}(\sigma_1, \sigma_2) = \text{dist}(\sigma_1 \sigma, \sigma_2 \sigma)$$ 

for all $\sigma_1, \sigma_2, \sigma \in S_n$.

For a permutation $\tau$ and an integer $k > 1$, we consider the “$k$-th ring” around $\tau$:

$$U(\tau, k) = \{\sigma \in S_n : \text{dist}(\sigma, \tau) = k\}.$$ 

In particular, we are interested in the distribution of values of $f$ in the set $U(\tau, k)$, where $\tau$ is an optimal permutation.

(1.3) **The generalized problem.** Our approach produces essentially identical results for a more general problem, where we are given a 4-dimensional array $C = \{c_{ij}^{kl} : 1 \leq i, j, k, l \leq n\}$ of $n^4$ real numbers and the function $f$ is defined by

$$(1.3.1) \quad f(\sigma) = \sum_{i,j=1}^{n} c_{\sigma(i)\sigma(j)}^{ij}.$$ 

If $c_{ij}^{kl} = a_{ij} b_{kl}$ for some matrices $A = (a_{ij})$ and $B = (b_{kl})$, in which case we write $C = A \otimes B$, we get the special case (1.1.1) we started with.

The main idea of our approach is as follows. Let

$$\bar{f} = \frac{1}{n!} \sum_{\sigma \in S_n} f(\sigma)$$

be the average value of $f$ on the symmetric group and let $f_0 = f - \bar{f}$. Hence the average value of $f_0$ is 0 and we study the distribution of values of $f_0$ around its maximum (the problem with minimum instead of maximum is completely similar).

Now, as long as the distribution of values of $f_0$ is concerned, without loss of generality we may assume that $f_0$ attains its maximum on the identity permutation $e$, so that $f_0(e) \geq f_0(\sigma)$ for all $\sigma \in S_n$. Let us define a function $g : S_n \rightarrow \mathbb{R}$, which we call the **central projection** (with the term coming from the representation theory) of $f$ by

$$(1.4) \quad g(\sigma) = \frac{1}{n!} \sum_{\omega \in S_n} f_0(\omega^{-1}\sigma\omega).$$
It turns out that $g$ attains its maximum on the identity permutation, that the average value of $g$ on $S_n$ is 0 and, moreover, the average values of $f_0$ and $g$ on the $k$-th ring $U(e, k)$ coincide for all $k$. In short, $g$ captures some important information about the distribution of $f$. The set of all functions $g$ obtained by central projection (1.4) from all functions $f_0$ having maximum at the identity forms a 3-dimensional convex polyhedral cone. We describe this cone, identifying its extreme rays (there are 4 for even $n$ and 5 for odd $n$), which provide us with some “extreme” types of distribution. Hence we study the distribution of values of $g$, which is a much easier problem. Once the distribution of values of $g$ is understood, using (1.4), we infer various facts about the distribution of values of $f$.

We remark that it is easy to compute the average value $\bar{f}$ of $f$ given by (1.1.1) or by (1.3.1).

(1.5) Lemma. Let $f : S_n \rightarrow \mathbb{R}$ be a function defined by

$$f(\sigma) = \langle B, \sigma(A) \rangle$$

for some matrices $A = (a_{ij})$ and $B = (b_{ij})$. Let

$$\bar{f} = \frac{1}{n!} \sum_{\sigma \in S_n} f(\sigma)$$

be the average value of $f$ on the symmetric group $S_n$. Let us define

$$\alpha_1 = \sum_{1 \leq i \neq j \leq n} a_{ij}, \quad \alpha_2 = \sum_{i=1}^{n} a_{ii} \quad \text{and}$$

$$\beta_1 = \sum_{1 \leq i \neq j \leq n} b_{ij}, \quad \beta_2 = \sum_{i=1}^{n} b_{ii}.$$  

Then

$$\bar{f} = \frac{\alpha_1 \beta_1}{n(n-1)} + \frac{\alpha_2 \beta_2}{n}.$$ 

Similarly, if $f$ is a function (1.3.1) of the generalized problem, then

$$\bar{f} = \frac{1}{n(n-1)} \sum_{1 \leq i \neq j \leq n} \sum_{1 \leq k \neq l \leq n} c_{ij}^{kl} + \frac{1}{n} \sum_{1 \leq i \leq n} \sum_{1 \leq l \leq n} c_{ii}^{ll}.$$ 

We prove Lemma 1.5 in Section 6.

(1.6) Notation. We often denote by $c$ some positive constant whose precise value is not of particular importance to us. If $F$ and $G$ are non-negative functions of a positive integer $n$, we write $F = O(G)$ if $F(n) \leq cG(n)$ for some $c > 0$ and all sufficiently large $n$. Similarly, we write $F(n) = \Omega(G)$ if $F(n) \geq cG(n)$ for some
constant $c > 0$ and all sufficiently large $n$. We denote by $e$ the identity permutation in $S_n$. We denote by $|X|$ the cardinality of a finite set $X$ and by $\text{conv } A$ the convex hull of the set $A$ in Euclidean space. Given a function $f : S_n \rightarrow \mathbb{R}$, we denote by $\overline{f}$ its average value on $S_n$:

$$\overline{f} = \frac{1}{n!} \sum_{\sigma \in S_n} f(\sigma)$$

and by

$$f_0 = f - \overline{f}$$

the “shifted” function with 0 average. Our results concern the function $f_0$.

The paper is organized as follows. In Sections 2-5, we state our results about the number of near-optimal permutations. In Sections 6-11, we prove those results and describe certain “extreme” distributions. We give an informal preview of our results below. In what follows, $\tau$ is an optimal permutation such that $f_0(\tau) \geq f_0(\sigma)$ for all $\sigma \in S_n$. Since we consider the shifted function, the minimization and maximization problems are completely similar.

In Section 2, we consider a special case of the problem where matrix $A$ is symmetric, has constant row and column sums and a constant diagonal (of course, $A$ and $B$ are interchangeable). For example, the symmetric TSP belongs to this class. The interesting feature of this special case is what we call the “bullseye” distribution of values of $f_0$ around its maximum. It turns out that the average value of $f_0$ over the $k$-th ring $U(\tau, k)$ (see Definitions 1.2) around an optimal permutation $\tau$ steadily improves as the ring contracts to $\tau$. The proof is given in Section 8. This is also the simplest case to analyze. It turns out that the set of all possible central projections $g$ (see (1.4)) is one-dimensional.

In Section 3, we consider a more general case of a not necessarily symmetric matrix $A$ with constant row and column sums and a constant diagonal. For example, the asymmetric TSP belongs to this class. We call this case “pure” since the objective function $f$ lacks the component that can be attributed to the Linear Assignment Problem. Although we don’t have the bullseye distribution of Section 2, we can provide some guarantees for the number of reasonably good permutations $\sigma$. Thus, for any $\alpha > 1$ the probability that a random permutation $\sigma \in S_n$ satisfies $f_0(\sigma) \geq \frac{\alpha}{n^2} f_0(\tau)$ is at least $\Omega(n^{-2})$. Furthermore, for any $\epsilon > 0$ the probability that a random permutation $\sigma$ satisfies $f_0(\sigma) \geq n^{-\epsilon} f_0(\tau)$ is “mildly exponential”, that is at least of the order of $\exp\{-n^c\}$ for some constant $c = c(\epsilon) < 1$. The proof is given in Section 9. It turns out that the set of all central projections $g$, maximized at the identity, forms a 2-dimensional cone. The extreme rays provide us with the extreme types of distributions, which, although not as good as the “bullseye” distribution of Section 2, still quite reasonable, especially compared with types of distributions we encounter in general symmetric QAP.

In Section 4, we consider the symmetric Quadratic Assignment Problem, where matrix $A$ (or, equivalently $B$) is symmetric. This case turns out to be very different
in many respects from the special cases of Sections 2 and 3. It turns out that the “bullseye” distribution is no longer the law. We present a simple example of function $f_0$ where the average value of $f_0$ over the $k$-th ring $U(\tau, k)$ of an optimal permutation $\tau$ is much worse than the average over the whole group $S_n$ even for small $k$. We call such a distribution a “spike”. We argue that at least for the generalized problem (1.3), the number of near-optimal permutations is much smaller than in the pure case of Section 3. The proofs are given in Section 10. It turns out that the set of all central projections (1.4) forms a 2-dimensional cone whose extreme rays provide us with the extreme types of distributions. One of those rays turns out to have an extreme “spike” distribution.

In Section 5, we consider the general Quadratic Assignment Problem. As in Section 3, we prove that for any $\alpha > 1$ the probability that a random permutation $\sigma \in S_n$ satisfies $f_0(\sigma) \geq \frac{\alpha}{n^2} f_0(\tau)$ is at least $\Omega(n^{-2})$, although with a worse constant than in Section 3. We prove that for any $\epsilon > 0$ there is a constant $c(\epsilon) < 1$ such that the probability that a random permutation $\sigma$ satisfies $f_0(\sigma) \geq n^{-1-\epsilon} f_0(\tau)$ is at least of the order $\exp\{-n^{c}\}$ (mildly exponential). The proofs are given in Section 11. It turns out that the set of central projections (1.4) forms a 3-dimensional polyhedral cone with 4 extreme rays when $n$ is even and 5 extreme rays when $n$ is odd. In a sense, those extreme rays describe all “extreme” distributions that one may encounter in the general Quadratic Assignment Problem.

In Section 6, we prove some preliminary technical results. In Section 7, we review the necessary facts from the representation theory of the symmetric group, which we use essentially in our approach.

2. The Bullseye Case

Our analysis of the Quadratic Assignment Problem is the simplest in the following special case (it also exhibits some features absent in the general case). Suppose that the matrix $A = (a_{ij})$ is symmetric and has constant row and column sums and
a constant diagonal:

\[ a_{ij} = a_{ji} \quad \text{for all} \quad 1 \leq i, j \leq n; \]

for some \( a \)

\[ \sum_{i=1}^{n} a_{ij} = a \quad \text{for all} \quad j = 1, \ldots, n \quad \text{and} \]
\[ \sum_{j=1}^{n} a_{ij} = a \quad \text{for all} \quad i = 1, \ldots, n; \]

\[ a_{ii} = b \quad \text{for some} \quad b \quad \text{and all} \quad i = 1, \ldots, n. \]

For example,

\[
A = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 & 1 \\
1 & 0 & 1 & \ldots & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 1 \\
1 & 0 & 0 & \ldots & 1 & 0
\end{pmatrix}, \quad a_{ij} = \begin{cases} 1 & \text{if } |i-j| = 1 \mod n \\ 0 & \text{otherwise} \end{cases}
\]

satisfies these properties and the corresponding optimization problem is the Symmetric Traveling Salesman Problem. It turns out that the optimum has a characteristic “bullseye” feature in the Hamming metric on \( S_n \) (see Definition 1.2).

**Theorem 2.1.** Suppose that the matrix \( A \) is symmetric and has constant row and column sums and a constant diagonal. Let \( f : S_n \rightarrow \mathbb{R} \) be the function defined by (1.1.1) for \( A \) and some matrix \( B \). Let \( \bar{f} \) be the average value of \( f \) on \( S_n \), let \( f_0 = f - \bar{f} \) and let \( \tau \in S_n \) be an optimal permutation: \( f_0(\tau) = \max_{\sigma \in S_n} f_0(\sigma) \). For \( k \geq 0 \) let

\[ U(\tau; k) = \{ \sigma : \text{dist}(\sigma, \tau) = k \} \]

be the \( k \)-th “ring” around \( \tau \) and let

\[ \alpha(n, k) = \frac{(n-k)^2 - 3(n-k)}{n^2 - 3n}. \]

Then

\[ \frac{1}{|U(\tau; k)|} \sum_{\sigma \in U(\tau; k)} f_0(\sigma) \geq \alpha(n, k) f_0(\tau). \]

We prove Theorem 2.1 in Section 8.
(2.2) The “bullseye” distribution. Connections with the local search. It follows from our proof that we have almost equality in the formula of Theorem 2.1. We observe that as the ring $U(\tau; k)$ contracts to the optimal permutation $\tau$, the average value of $f$ on the ring steadily improves.

![Distribution of values of the objective function with respect to the Hamming distance from the maximum point](image)

It is easy to construct examples where some values of $f$ in a very small neighborhood of the optimum are particularly bad, but as follows from Theorem 2.1, such values are relatively rare. In our opinion, this provides some justification for the local search heuristic, where one starts from a permutation and tries to improve the value of the objective function by searching a small neighborhood of the current solution. Indeed, if we had the value of $f_0(\sigma)$ for each $\sigma \in U(\tau, k)$ equal to $\alpha(n, k)f_0(\tau)$, then the local search would have converged to the optimum in $O(n)$ steps, since each step would have brought us to a smaller neighborhood of the optimal solution. Instead, we have that the average value over $U(\tau, k)$ is (almost) equal to $\alpha(n, k)f_0(\tau)$. We can no longer guarantee that the local search converges fast (or even converges) to the optimal solution (after all, our problem includes the Traveling Salesman Problem as a special case and hence is NP-hard), but it plausible that the local search behaves reasonably well for an “average” optimization problem. This agrees with the empirical evidence that the local search works well for the Traveling Salesman Problem.

Incidentally, one can prove that the same type of the “bullseye” behavior is observed for the Linear Assignment Problem and some other polynomially solvable problems, such as the weighted Matching Problem.

Estimating the size of the ring $U(\tau, k)$, we get the following result.

(2.3) Theorem. Suppose that the matrix $A$ is symmetric and has constant row and column sums and a constant diagonal. Let $f : S_n \to \mathbb{R}$ be the function defined
by (1.1.1) for $A$ and some matrix $B$, let $\overline{f}$ be the average value of $f$ on $S_n$ and let $f_0 = f - \overline{f}$. Let $\tau$ be an optimal permutation: $f_0(\tau) = \max_{\sigma \in S_n} f_0(\sigma)$. Let us choose an integer $3 \leq k \leq n - 5$ and a number $0 < \gamma < 1$ and let

$$\beta(n, k) = \frac{k^2 - 3k}{n^2 - 3n}.$$  

The probability that a random permutation $\sigma \in S_n$ satisfies the inequality

$$f_0(\sigma) \geq \gamma \beta(n, k) f_0(\tau)$$

is at least

$$\frac{(1 - \gamma) \beta(n, k)}{3k!}.$$  

We prove Theorem 2.3 in Section 8.

Our results can be generalized in a quite straightforward way to functions $f$ defined by (1.3.1), if we assume that for any $k$ and $l$ the matrix $A = (a_{ij})$, where $a_{ij} = c_{kl}^{ij}$, is symmetric with constant row and column sums and has a constant diagonal.

3. The Pure Case

In this Section, we consider a more general case of a not necessarily symmetric matrix $A$ having constant row and column sums and a constant diagonal:

for some $a$

$$\sum_{i=1}^{n} a_{ij} = a \quad \text{for all} \quad j = 1, \ldots, n$$

and

$$\sum_{j=1}^{n} a_{ij} = a \quad \text{for all} \quad i = 1, \ldots, n;$$

$$a_{ii} = b \quad \text{for some} \quad b \quad \text{and all} \quad i = 1, \ldots, n.$$

For example, matrix

$$A = \begin{pmatrix}
0 & 1 & 0 & \ldots & \ldots & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & \ldots & 0 & 1 \\
1 & 0 & \ldots & \ldots & 0 & 0
\end{pmatrix}, \quad a_{ij} = \begin{cases}
1 & \text{if } j = i + 1 \mod n \\
0 & \text{otherwise}
\end{cases}$$
satisfies these properties and the corresponding optimization problem is the Asymmetric Traveling Salesman Problem.

We call this case pure, because as we remark in Sections 7 and 9, the objective function $f$ lacks the component attributed to the Linear Assignment Problem. More generally, an arbitrary objective function $f$ in the Quadratic Assignment Problem can be represented as a sum $f = f_1 + f_2$, where $f_1$ is the objective function in a Linear Assignment Problem and $f_2$ is the objective function in some pure case.

In this case we can no longer claim the bullseye distribution of Section 2 (the reasons are explained in Section 9), the distribution in this case is not as bad as, for example, in the general symmetric QAP (see Section 4) and the estimates of the number of relatively good values we are able to prove are almost as good as those of Section 2.

(3.1) Theorem. Suppose that the matrix $A$ has constant row and column sums and a constant diagonal. Let $f : S_n \rightarrow \mathbb{R}$ be the function defined by (1.1.1) for $A$ and some matrix $B$, let $\overline{f}$ be the average value of $f$ on $S_n$ and let $f_0 = f - \overline{f}$. Let $\tau$ be an optimal permutation, so $f_0(\tau) = \max_{\sigma \in S_n} f_0(\sigma)$. Let us choose an integer $3 \leq k \leq n - 5$ and a number $0 < \gamma < 1$ and let

$$\beta(n, k) = \frac{k^2 - 3k + 1}{n^2 - 3n + 1}.$$

The probability that a random permutation $\sigma \in S_n$ satisfies the inequality

$$f_0(\sigma) \geq \gamma \beta(n, k) f_0(\tau)$$

is at least

$$\frac{(1 - \gamma) \beta(n, k)}{10k!}.$$

In particular, by choosing an appropriate $k$, we obtain the following corollary.

(3.2) Corollary.

(1) Let us fix any $\alpha > 1$. Then there exists a $\delta = \delta(\alpha) > 0$ such that for all sufficiently large $n \geq N(\alpha)$ the probability that a random permutation $\sigma$ in $S_n$ satisfies the inequality

$$f_0(\sigma) \geq \frac{\alpha}{n^2} f_0(\tau)$$

is at least $\delta n^{-2}$. In particular, one can choose $\delta = \exp\{-c\sqrt{\alpha \ln \alpha}\}$ for some absolute constant $c > 0$.

(2) Let us fix any $\epsilon > 0$. Then there exists a $\delta = \delta(\epsilon) < 1$ such that for all sufficiently large $n \geq N(\alpha)$ the probability that a random permutation $\sigma$ in $S_n$ satisfies the inequality

$$f_0(\sigma) \geq n^{-\epsilon} f_0(\tau)$$

is at least $\exp\{-n^\delta\}$. In particular, one can choose any $\delta > 1 - \epsilon/2$. 


We prove Theorem 3.1 in Section 9.

From Corollary 3.2, it follows that to get a permutation \( \sigma \) which satisfies (1) for any fixed \( \alpha \), we can use the following straightforward randomized algorithm: sample \( O(n^2) \) random permutations \( \sigma \in S_n \), compute the value of \( f \) and choose the best permutation. With the probability which tends to 1 as \( n \to +\infty \), we will hit the right permutation. The complexity of the algorithm is quadratic in \( n \) for any \( \alpha \), but the coefficient of \( n^2 \) grows as \( \alpha \) grows. If we are willing to settle for an algorithm of a mildly exponential complexity of the type \( \exp\{n^\beta\} \) for some \( \beta < 1 \) we can achieve a better approximation (2) by searching through the set of randomly selected \( \exp\{n^\beta\} \) permutations. We remark that no algorithm solving the Quadratic Assignment Problem (even in the special case considered in this section) with an exponential in \( n \) complexity \( \exp\{O(n)\} \) is known, although there is a dynamic programming algorithm solving the Traveling Salesman Problem in \( \exp\{O(n)\} \) time.

Again, our results can be generalized in a quite straightforward way to functions \( f \) defined by (1.3.1), if we assume that for any \( k \) and \( l \) the matrix \( A = (a_{ij}) \), where \( a_{ij} = c_{kl}^{ij} \) has constant row and column sums and has a constant diagonal.

### 4. The Symmetric Case

In this section, we assume that the matrix \( A = (a_{ij}) \) is symmetric, that is

\[
a_{ij} = a_{ji} \quad \text{for all} \quad 1 \leq i, j \leq n.
\]

Overall, the distribution of values of \( f \) turns out to be much more complicated when in the special cases described in Sections 2 and 3. First, we observe that generally one can not hope for the “bullseye” feature described in Section 2.2.

*(4.1) The “spike” distribution.* Let us choose an \( n \times n \) matrix \( A = (a_{ij}) \), where

\[
a_{ij} = \begin{cases} 
1 & \text{if } (ij) = (12) \text{ or } (ij) = (21) \\
0 & \text{otherwise}
\end{cases}
\]

so

\[
A = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0
\end{pmatrix}.
\]

Let

\[
\gamma = \frac{-n^2 + 5n - 8}{8(n-2)}
\]
and let $B = (b_{ij})$, where

$$b_{ij} = \begin{cases} 
0 & \text{if } i = j \\
\gamma & \text{if } i \leq 2 \text{ and } j \geq 3 \text{ or } j \leq 2 \text{ and } i \geq 3 \\
1/2 & \text{otherwise},
\end{cases}$$

so

$$B = \begin{pmatrix}
0 & 1/2 & \gamma & \gamma & \cdots & \gamma \\
1/2 & 0 & \gamma & \gamma & \cdots & \gamma \\
\gamma & \gamma & 0 & 1/2 & \cdots & 1/2 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\gamma & \gamma & 1/2 & 0 & \cdots & 1/2 \\
\gamma & \gamma & 1/2 & \cdots & 1/2 & 0
\end{pmatrix}.$$ 

Let $f : S_n \rightarrow \mathbb{R}$ be the function defined by (1.1.1). In Section 10, we prove the following properties of $f$.

- We have $\bar{f} = 0$ for the average value of $f$ on $S_n$;

- The maximum value of $f$ on $S_n$ is 1 and is attained, in particular, on the identity permutation $e$;

- For the $k$-th ring $U(e, k)$ centered at the identity permutation $e$, we have

$$\frac{1}{|U(e, k)|} \sum_{\sigma \in U(e, k)} f(\sigma) \leq \frac{-nk + k^2 + 3n - k - 4}{2n - 4}.$$ 

We observe that already for $k = 4$ (a more careful analysis yields $k = 3$) the average value of $f$ over $U(e, k)$ is negative for all sufficiently large $n$. Thus an average permutation in $U(e, 4)$ presents us with a choice worse than an average permutation in $S_n$. The distribution of values of $f$ turns out to be of the opposite
nature to the bullseye distribution of Figure 1. We call it the “spike” distribution.

Figure 2

Distribution of values of the objective function with respect to the Hamming distance from the maximum point

\[ f(\sigma) = \frac{-n(p(\sigma) - 1) + p(\sigma)p(\sigma) + 2t(\sigma) - 4}{2n - 4}, \]

where \( p(\sigma) = |\{i : \sigma(i) = i\}| \) is the number of fixed points of the permutation and \( t(\sigma) = |\{i < j : \sigma(i) = j \text{ and } \sigma(j) = i\}| \) is the number of 2-cycles in the permutation. We show that \( \bar{f} = 0 \) and that the maximum value 1 of \( f \) is attained at the identity permutation \( e \) (where \( p(\sigma) = n \) and \( t(\sigma) = 0 \)) and, for even \( n \), on the permutations that consist of \( n/2 \) transpositions (where \( p(\sigma) = 0 \) and \( t(\sigma) = n/2 \)). On the other hand, for any fixed \( n - 3 \geq k \geq 3 \) and all \( n \geq 5 \), the value of \( f(\sigma) \) with \( \sigma \in U(e, k) \) is negative.

(4.2) Scarcity of relatively good values. Unfortunately, we are unable to present an example of the symmetric QAP which beats the bound of Theorem 3.1 but we can construct such an example for the generalized problem (1.3). In Section 10, we prove that for any \( 3 \leq m \leq n \), there exists a tensor \( c_{kl}^{ij} \) such that \( c_{kl}^{ij} = c_{kl}^{ji} \) for all \( k \) and \( l \) and such that for the corresponding function \( f \) defined by (1.3.1), we have

\[ f(\sigma) = \frac{p^2(\sigma) - mp(\sigma) + 2t(\sigma) + m - 3}{n^2 - nm + m - 3}, \]
where \( p(\sigma) \) is the number of fixed points in \( \sigma \) and \( t(\sigma) \) is the number of 2-cycles in \( \sigma \). We show that \( \overline{f} = 0 \) and that \( f(e) = 1 \) is the maximum value of \( f \).

Let us fix any \( 0 < \delta < 1 \) and let us choose some \( m \) such that \( n^{1-\epsilon} > m > n^\delta \) for some \( \epsilon > 0 \). Then, for all sufficiently large \( n \), the value \( f(\sigma) > 2/n \) can be achieved only on permutations \( \sigma \) with \( p(\sigma) > m \). The number of such permutations \( \sigma \) does not exceed \( \frac{n!}{m!} \), that is, the probability that a random permutation \( \sigma \) satisfies \( f(\sigma) > 2/n \) does not exceed \( \exp\{-n^\delta\} \) for large \( n \).

5. The General Case

It appears that the difference between the general case and the symmetric case of Section 4 is not as substantial as the difference between the symmetric case and the special cases of Sections 2 and 3. Our main result is:

\textbf{(5.1) Theorem.} Let \( f : S_n \rightarrow \mathbb{R} \) be the function defined by (1.1.1) or (1.3.1), let \( \overline{f} \) be the average value of \( f \) on \( S_n \) and let \( f_0 = f - \overline{f} \). Let \( \tau \) be an optimal permutation: \( f_0(\tau) = \max_{\sigma \in S_n} f_0(\sigma) \). Let us choose an integer \( 3 \leq k \leq n-5 \) and a number \( 0 < \gamma < 1 \). Let

\[ \beta(n, k) = \frac{k-2}{n^2 - nk + k-2}. \]

The probability that a random permutation \( \sigma \in S_n \) satisfies

\[ f_0(\sigma) \geq \gamma \beta(k, n)f_0(\tau) \]

is at least

\[ \frac{(1-\gamma)\beta(k, n)}{5k!}. \]

In particular, by choosing an appropriate \( k \), we obtain the following corollary.

\textbf{(5.2) Corollary.}

\begin{enumerate}
  \item Let us fix any \( \alpha > 1 \). Then there exists a \( \delta = \delta(\alpha) > 0 \) such that for all sufficiently large \( n \geq N(\alpha) \) the probability that a random permutation \( \sigma \) in \( S_n \) satisfies the inequality

  \[ f_0(\sigma) \geq \frac{\alpha}{n^2}f_0(\tau) \]

  is at least \( \delta n^{-2} \). In particular, one can choose \( \delta = \exp\{-c\alpha \ln \alpha\} \) for some absolute constant \( c > 0 \).
  \item Let us fix any \( \epsilon > 0 \). Then there exists a \( \delta = \delta(\epsilon) < 1 \) such that for all sufficiently large \( n \geq N(\epsilon) \) the probability that a random permutation \( \sigma \) in \( S_n \) satisfies the inequality

  \[ f_0(\sigma) \geq n^{-1-\epsilon}f_0(\tau) \]

  is at least \( \exp\{-n^\delta\} \). In particular, one can choose any \( \delta > 1-\epsilon \).
\end{enumerate}
We prove Theorem 5.1 in Section 11. As in Section 2, we conclude that for any fixed $\alpha > 1$ there is a randomized $O(n^2)$ algorithm which produces a permutation $\sigma$ satisfying (1). If we are willing to settle for an algorithm of mildly exponential complexity, we can achieve the bound of type (2), which is weaker than the corresponding bound of Corollary 3.2.

In Section 11, we construct an example of a function of type (1.3.1) with an even sharper spike distribution than in example 4.1.

6. Preliminaries

First, we prove Lemma 1.5.

Proof of Lemma 1.5. Let us choose a pair of indices $1 \leq i \neq j \leq n$. Then, as $\sigma$ ranges over the symmetric group $S_n$, the ordered pair $(\sigma(i), \sigma(j))$ ranges over all ordered pairs $(k, l)$ with $1 \leq k \neq l \leq n$ and each such a pair $(k, l)$ appears $(n - 2)!$ times. Similarly, for each index $1 \leq i \leq n$, the index $\sigma(i)$ ranges over the set $\{1, \ldots, n\}$ and each $j \in \{1, \ldots, n\}$ appears $(n - 1)!$ times. Therefore,

$$
\bar{f} = \frac{1}{n!} \sum_{\sigma \in S_n} \sum_{i,j=1}^{n} b_{\sigma(i)\sigma(j)}a_{ij} = \sum_{i,j=1}^{n} \left( a_{ij} \frac{1}{n!} \sum_{\sigma \in S_n} b_{\sigma(i)\sigma(j)} \right) 
= \frac{1}{n(n-1)} \sum_{i \neq j} a_{ij} + \frac{1}{n} \sum_{i=1}^{n} a_{ii} \beta_{2} = \frac{\alpha_1 \beta_1}{n(n-1)} + \frac{\alpha_2 \beta_2}{n}
$$

and the proof follows. \qed

Suppose that $f(\sigma) = \langle B, \sigma(A) \rangle$ for some matrices $A$ and $B$ and all $\sigma \in S_n$ and suppose that the maximum value of $f$ is attained at a permutation $\tau$. Let $A_1 = \tau(A)$ and let $f_1(\sigma) = \langle B, \sigma(A_1) \rangle$. Then $f_1(\sigma) = f(\sigma \tau)$, hence the maximum value of $f_1$ is attained at the identity permutation $e$ and the distribution of values of $f$ and $f_1$ is the same. We observe that if $A$ is symmetric then $A_1$ is also symmetric, and if $A$ has constant row and column sums and a constant diagonal then so does $A_1$ (see also Section 7). Hence, as long as the distribution of values of $f$ is concerned, without loss of generality we may assume that the maximum of $f$ is attained at the identity permutation $e$.

(6.1) Definition. Let $f : S_n \rightarrow \mathbb{R}$ be a function. Let us define function $g : S_n \rightarrow \mathbb{R}$ by

$$
g(\sigma) = \frac{1}{n!} \sum_{\omega \in S_n} f(\omega^{-1}\sigma\omega).
$$

We call $g$ the central projection of $f$.

The following simple observation is quite important for our approach.
Lemma. Let $f : S_n \rightarrow \mathbb{R}$ be a function such that $f(e) \geq f(\sigma)$ for all $\sigma \in S_n$ and let $g$ be the central projection of $f$. Then $g(e) = f(e) \geq g(\sigma)$ for all $\sigma \in S_n$ and the average values of $f$ and $g$ are equal: $\bar{f} = \bar{g}$.

Proof. We observe that $\omega^{-1}e = e$ for all $\omega \in S_n$ and hence $g(e) = f(e)$. Moreover, for any $\sigma \in S_n$

$$g(\sigma) = \frac{1}{n!} \sum_{\omega \in S_n} f(\omega^{-1}\sigma) \leq \frac{1}{n!} \sum_{\omega \in S_n} f(e) = g(e).$$

Finally,

$$\bar{g} = \frac{1}{n!} \sum_{\sigma \in S_n} g(\sigma) = \frac{1}{n!} \sum_{\sigma \in S_n} \frac{1}{n!} \sum_{\omega \in S_n} f(\omega^{-1}\sigma) = \frac{1}{n!} \sum_{\omega \in S_n} \left( \frac{1}{n!} \sum_{\sigma \in S_n} f(\omega^{-1}\sigma) \right)$$

$$= \frac{1}{n!} \sum_{\sigma \in S_n} \bar{f} = \bar{f}$$

and the proof follows. $\square$

Moreover, one can observe that the averages of $f$ and $g$ on the $k$-th ring $U(e,k)$ coincide for all $k = 0, \ldots, n$, see Definition 1.2.

We will rely on a Markov type estimate, which asserts, roughly, that a function with a sufficiently large average takes sufficiently large values sufficiently often.

(6.3) Lemma. Let $X$ be a finite set and let $f : X \rightarrow \mathbb{R}$ be a function. Suppose that $f(x) \leq 1$ for all $x \in X$ and that

$$\frac{1}{|X|} \sum_{x \in X} f(x) \geq \beta$$

for some $\beta > 0$.

Then for any $0 < \gamma < 1$ we have

$$|\{x \in X : f(x) \geq \beta \gamma\}| \geq \beta(1 - \gamma)|X|.$$ 

Proof. We have

$$\beta \leq \frac{1}{|X|} \sum_{x \in X} f(x) = \frac{1}{|X|} \sum_{x : f(x) < \beta \gamma} f(x) + \frac{1}{|X|} \sum_{x : f(x) \geq \beta \gamma} f(x)$$

$$\leq \beta \gamma + \frac{|\{x : f(x) \geq \beta \gamma\}|}{|X|}.$$ 

Hence

$$|\{x : f(x) \geq \beta \gamma\}| \geq \beta(1 - \gamma)|X|.$$ 

$\square$

Finally, we need some facts about the structure of the symmetric group $S_n$ (see, for example, [6]).
(6.4) The conjugacy classes of $S_n$

Let us fix a permutation $\rho \in S_n$. As $\omega$ ranges over the symmetric group $S_n$, the permutation $\omega^{-1}\rho\omega$ ranges over the conjugacy class of $X(\rho)$ of $\rho$, that is the set of permutations that have the same cycle structure as $\rho$.

We will be using the following facts.

(6.4.1) Central projections and conjugacy classes. If $f : S_n \rightarrow \mathbb{R}$ is a function and $g : S_n \rightarrow \mathbb{R}$ its central projection, then

$$ g(\rho) = \frac{1}{|X(\rho)|} \sum_{\sigma \in X(\rho)} f(\sigma). $$

If $X \subset S_n$ is a set which splits into a union of conjugacy classes $X(\rho_i) : i \in I$, and for each such a class we have

$$ \frac{1}{|X(\rho_i)|} \sum_{\sigma \in X(\rho_i)} f(\sigma) \geq \alpha $$

for some number $\alpha$, then

$$ \frac{1}{|X|} \sum_{\sigma \in X} f(\sigma) \geq \alpha. $$

(6.4.2) Permutations with no fixed points and 2-cycles. Let us fix some positive integers $c_i : i = 1, \ldots, m$ and let $a_n$ be the number of permutations in $S_n$ that have no cycles of length $c_i$ for $1 \leq i \leq m$. The exponential generating function for $a_n$ is given by

$$ \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n = \frac{1}{1 - x} \exp \left\{ - \sum_{i=1}^{m} \frac{x^{c_i}}{c_i} \right\}, $$

where we agree that $a_0 = 1$, see, for example, pp. 170–173 of [7]. It follows that the number of permutations $\sigma \in S_n$ without fixed points is asymptotically $e^{-1}n!$ and without fixed points and 2-cycles is $e^{-3/2}n!$. We will use that the first number exceeds $n!/3$ and the second number exceeds $n!/5$ for $n \geq 5$.

(6.4.3) Permutations with many fixed points and 2-cycles. The number of permutations $\sigma \in S_n$ with at least $k$ fixed points is at most $n!/k!$, since to choose such a permutation, we can first choose $k$ fixed points in $\binom{n}{k}$ ways and then choose an arbitrary permutation of the remaining $(n-k)$ elements in $(n-k)!$ ways (some permutations will be counted several times). Similarly, the number of permutations $\sigma \in S_n$ with at least $k$ transpositions (2-cycles) is at most $\frac{n!}{k!2^k}$, since to choose such a permutation, we first choose some $k$ pairs in $\frac{n!}{(n-2k)!k!2^k}$ ways and then an arbitrary permutation of the remaining $n-2k$ elements in $(n-2k)!$ ways (again, some permutations will be counted several times).
7. Action of the Symmetric Group in the Space of Matrices

The crucial observation for our approach is that the vector space of all central projections $g$ of functions $f$ defined by (1.1.1) or (1.3.1) is 4-, 3-, or 2-dimensional depending on whether we consider the general case, the cases of Sections 3 and 4 or the special case of Section 2. If we require, additionally, that $\overline{f} = 0$ then the dimensions drop by 1 to 3, 2 and 1, respectively. This fact is explained by the representation theory of the symmetric group (see, for example, [6]). In this section, we review some facts that we need. Our notation is inspired by the generally accepted notation of the representation theory.

We describe some important invariant subspaces of the action of $S_n$ in the space of $n \times n$ matrices $\text{Mat}_n$ by simultaneous permutations of rows and columns. We recall that $n \geq 4$.

(7.1) Subspace $L_n^1$. Let $L_n^1$ be the space of constant matrices $A$:

$$a_{ij} = \alpha \quad \text{for some } \alpha \text{ and all } 1 \leq i, j \leq n.$$ 

Let $L_n^2$ be the subspace of scalar matrices $A$:

$$a_{ij} = \begin{cases} \alpha & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \text{for some } \alpha.$$ 

Finally, let $L_n = L_n^1 + L_n^2$. One can observe that $\dim L_n = 2$ and that $L_n$ is the subspace of all matrices that remain fixed under the action of $S_n$.

(7.2) Subspace $L_{n-1,1}$. Let $L_{n-1,1}^1$ be the subspace of matrices with identical rows and such that the sum of entries in each row is 0:

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1 & \alpha_2 & \ldots & \alpha_n \end{pmatrix}, \quad \text{where } \alpha_1 + \ldots + \alpha_n = 0.$$ 

Similarly, let $L_{n-1,1}^2$ be the subspace of matrices with identical columns and such that the sum of entries in each column is 0:

$$A = \begin{pmatrix} \alpha_1 & \alpha_1 & \ldots & \alpha_1 \\ \alpha_2 & \alpha_2 & \ldots & \alpha_2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n & \alpha_n & \ldots & \alpha_n \end{pmatrix}, \quad \text{where } \alpha_1 + \ldots + \alpha_n = 0.$$ 

Finally, let $L_{n-1,1}^3$ be the subspace of diagonal matrices with the zero sum on the diagonal:

$$A = \begin{pmatrix} \alpha_1 & 0 & \ldots & 0 & 0 \\ 0 & \alpha_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & \alpha_n \end{pmatrix}, \quad \text{where } \alpha_1 + \ldots + \alpha_n = 0.$$
Let $L_{n-1,1} = L_{n-1,1}^1 + L_{n-1,1}^2 + L_{n-1,1}^3$. One can check that the dimension of each of $L_{n-1,1}^1$, $L_{n-1,1}^2$ and $L_{n-1,1}^3$ is $n - 1$ and that $\dim L_{n-1,1} = 3n - 3$. Moreover, the subspaces $L_{n-1,1}^1$, $L_{n-1,1}^2$ and $L_{n-1,1}^3$ do not contain non-trivial invariant subspaces. The action of $S_n$ in $L_{n-1,1}$, although non-trivial, is not very complicated. One can show that if $A \in L_{n-1,1} + L_n$, then the problem of optimizing $f(\sigma)$ defined by (1.1) reduces to the Linear Assignment Problem.

**Subspace $L_{n-2,2}$.** Let us define $L_{n-2,2}$ as the subspace of all symmetric matrices $A$ with row and column sums equal to 0 and zero diagonal

$$a_{ij} = a_{ji} \quad \text{for all} \quad 1 \leq i, j \leq n;$$

$$\sum_{i=1}^{n} a_{ij} = 0 \quad \text{for all} \quad j = 1, \ldots, n;$$

$$\sum_{j=1}^{n} a_{ij} = 0 \quad \text{for all} \quad i = 1, \ldots, n \quad \text{and}$$

$$a_{ii} = 0 \quad \text{for all} \quad i = 1, \ldots, n.$$

One can check that $L_{n-2,2}$ is an invariant subspace and that $\dim L_{n-2,2} = (n^2 - 3n)/2$. Besides, $L_{n-2,2}$ contains no non-trivial invariant subspaces.

**Subspace $L_{n-2,1,1}$.** Let us define $L_{n-2,1,1}$ as the subset of all skew symmetric matrices $A$ with row and column sums equal to 0:

$$a_{ij} = -a_{ji} \quad \text{for all} \quad 1 \leq i, j \leq n;$$

$$\sum_{i=1}^{n} a_{ij} = 0 \quad \text{for all} \quad j = 1, \ldots, n \quad \text{and}$$

$$\sum_{j=1}^{n} a_{ij} = 0 \quad \text{for all} \quad i = 1, \ldots, n.$$

One can check that $L_{n-2,1,1}$ is an invariant subspace and that $\dim L_{n-2,1,1} = (n^2 - 3n)/2 + 1$. Similarly, $L_{n-2,1,1}$ contains no non-trivial invariant subspaces.

One can check that $\text{Mat}_n = L_n + L_{n-1,1} + L_{n-2,2} + L_{n-2,1,1}$. The importance of the subspaces (7.1)–(7.4) is explained by the fact that they are the isotypical components of the irreducible representations of the symmetric group in the space of matrices. The following proposition follows from the representation theory of the symmetric group [6].

**Proposition.** For an $n \times n$ matrices $A$ and $B$, where $n \geq 4$, let $f : S_n \rightarrow \mathbb{R}$ be the function defined by (1.1) and let $g : S_n \rightarrow \mathbb{R}$,

$$g(\sigma) = \frac{1}{n!} \sum_{\omega \in S_n} f(\omega^{-1} \sigma \omega)$$
be the central projection of \( f \). Given a permutation \( \sigma \in S_n \), let

\[
p(\sigma) = |\{ i : \sigma(i) = i \}| \quad \text{and} \quad t(\sigma) = |\{ i < j : \sigma(i) = j \ \text{and} \ \sigma(j) = i \}|
\]

be the number of fixed points of the permutation and the number of 2-cycles in the permutation correspondingly.

1. If \( A \in L_n \) then \( g \) is a scalar multiple of the constant function

\[
\chi_n(\sigma) = 1 \quad \text{for all} \quad \sigma \in S_n;
\]

2. If \( A \in L_{n-1,1} \) then \( g \) is a scalar multiple of the function

\[
\chi_{n-1,1}(\sigma) = p(\sigma) - 1 \quad \text{for all} \quad \sigma \in S_n;
\]

3. If \( A \in L_{n-2,2} \) then \( g \) is a scalar multiple of the function

\[
\chi_{n-2,2}(\sigma) = t(\sigma) + \frac{1}{2}p^2(\sigma) - \frac{3}{2}p(\sigma) \quad \text{for all} \quad \sigma \in S_n;
\]

4. If \( A \in L_{n-2,1,1} \) then \( g \) is a scalar multiple of the function

\[
\chi_{n-1,1,1}(\sigma) = \frac{1}{2}p^2(\sigma) - \frac{3}{2}p(\sigma) - t(\sigma) + 1 \quad \text{for all} \quad \sigma \in S_n.
\]

The functions \( \chi_n, \chi_{n-1,1}, \chi_{n-2,2} \) and \( \chi_{n-1,1,1} \) are the characters of corresponding irreducible representations of \( S_n \) for \( n \geq 4 \). They are linearly independent, and, moreover orthogonal: \( \sum_{\sigma \in S_n} \chi_i(\sigma)\chi_j(\sigma) = 0 \) for two characters of different irreducible representation of \( S_n \). In particular,

\[
\sum_{\sigma \in S_n} \chi_{n-1,1}(\sigma) = \sum_{\sigma \in S_n} \chi_{n-2,2}(\sigma) = \sum_{\sigma \in S_n} \chi_{n-2,1,1}(\sigma) = 0,
\]

hence the average value of all but the trivial character \( \chi_n \) is 0.

(7.6) Remark. It follows [6] that each of the functions \( \chi_n, \chi_{n-1,1}, \chi_{n-2,2} \) and \( \chi_{n-2,1,1} \) is the objective function (1.3.1) in some generalized problem with a tensor \( c_{kl}^{ij} \) (see Section 1.3) with the property that for all \( k \) and \( l \) the matrix \( A = (a_{ij}) \) for \( a_{ij} = c_{kl}^{ij} \) belongs to the corresponding subspace. Since the set of all functions (1.3.1) is closed under linear combinations, it follows that every function \( f \in \text{span}\{\chi_n, \chi_{n-1,1}, \chi_{n-2,2}, \chi_{n-2,1,1}\} \) is an objective function in the generalized problem.
8. The Bullseye Case. Proofs

In this section, we prove Theorem 2.1 and Theorem 2.3. An important observation is that \( A \) satisfies the conditions of Section 2 if and only if \( A \in L_n + L_{n-2,2} \) (see Section 7).

**Proof of Theorem 2.1.** Without loss of generality, we may assume that the maximum of \( f_0(\sigma) \) is attained at the identity permutation \( e \) (see Section 6). Excluding the non-interesting case of \( f_0 \equiv 0 \), by scaling, if necessary, we can assume that \( f_0(e) = 1 \). Let \( g \) be the central projection of \( f_0 \). Then by Lemma 6.2, \( g = 0 \) and \( 1 = g(e) \geq g(\sigma) \) for all \( \sigma \in S_n \). Moreover, since \( A \in L_n + L_{n-2,2} \), by Parts 1 and 3 of Proposition 7.5, \( g \) must be a linear combination of the constant function \( \chi_n \) and \( \chi_{n-2,2} \). Since \( g = 0 \), \( g \) should be proportional to \( \chi_{n-2,2} \) and since \( g(e) = 1 \), we have

\[
g = \frac{2}{n^2 - 3n} \chi_{n-2,2} = \frac{2t + p^2 - 3p}{n^2 - 3n}.
\]

Now \( \sigma \in U(e, k) \) if and only if \( p(\sigma) = n-k \). Hence \( g(\sigma) \geq \alpha(n, k) \) for all \( \sigma \in U(e, k) \). The set \( U(e, k) \) splits into disjoint union of conjugacy classes \( X(\rho) \) and, using (6.4.1), we conclude that for each such \( X(\rho) \)

\[
g(\rho) = \frac{1}{|X(\rho)|} \sum_{\sigma \in X(\rho)} f_0(\sigma) \geq \alpha(n, k)
\]

and, therefore,

\[
\frac{1}{|U(n, k)|} \sum_{\sigma \in U(n, k)} f_0(\sigma) \geq \alpha(n, k),
\]

hence the proof follows. \( \square \)

Using estimates of (6.4.2), one can show that the input of the number of 2-cycles \( t(\sigma) \) into the average of \( f_0 \) over \( U(e, k) \) is asymptotically negligible, so there is an “almost equality” in the formula of Theorem 2.1.

By estimating the cardinality of the \( k \)-th ring \( U(\tau, k) \), we deduce Theorem 2.3.

**Proof of Theorem 2.3.** As in the proof of Theorem 2.1, we assume that the maximum value of \( f_0 \) is equal to 1.

Let us estimate the cardinality \( |U(\tau, n-k)| = |U(e, n-k)| \). Since \( \sigma \in U(e, n-k) \) if and only if \( \sigma \) has \( k \) fixed points, to choose a \( \sigma \in U(e, n-k) \) one has to choose \( k \) points in \( \binom{n}{k} \) ways and then choose a permutation of the remaining \( n-k \) points without fixed points. Using (6.4.2), we get

\[
|U(\tau, n-k)| \geq \binom{n}{k}(n-k)!/3 = \frac{n!}{3k!}.
\]
Applying Lemma 6.3 with \( \beta = \beta(n, k) \) and \( X = U(\tau, n - k) \), from Theorem 2.1, we conclude that
\[
\mathbb{P}\{\sigma \in S_n : f_0(\sigma) \geq \gamma \beta(n, k)\} \geq \frac{(1 - \gamma)\beta(n, k)|U(\tau, n - k)|}{n!} \\
\geq \frac{(1 - \gamma)\beta(n, k)}{3k!}.
\]

\[\square\]

9. The Pure Case. Proofs

In this case, \( A \in L_n + L_{n-2,1,1} + L_{n-2,2} \) (see Section 7). As in Section 8, the \( L_n \) component contributes just a constant to \( f \). Since the \( L_{n-1,1} \) component attributed to the Linear Assignment Problem (see Section 7.2) is absent, we call this case “pure”.

We choose a more convenient basis \( g_1 \) and \( g_2 \) in the vector space spanned by \( \chi_{n-2,2} \) and \( \chi_{n-2,1,1} \), namely:

\[g_1 = \chi_{n-2,2} + \chi_{n-2,1,1} = p^2 - 3p + 1 \quad \text{and} \quad g_2 = \chi_{n-2,1,1} - \chi_{n-2,2} = 1 - 2t.\]

(9.1) Definition. Let \( K_p \) (where \( p \) stands for “pure”) be the set of all functions \( g : S_n \to \mathbb{R} \) such that \( g \in \text{span}\{g_1, g_2\} \), where \( g_1 = p^2 - 3p + 1 \) and \( g_2 = 1 - 2t \) and \( g(e) \geq g(\sigma) \) for all \( \sigma \in S_n \), where \( e \) is the identity permutation. We call \( K_p \) the central cone.

Identifying \( \text{span}\{g_1, g_2\} \) with two-dimensional vector space \( \mathbb{R}^2 \) (plane), we see that the conditions \( g(e) \geq g(\sigma) \) define the central cone \( K \) as a convex cone in \( \mathbb{R}^2 \). Our goal is to find the extreme rays \( r_1 \) and \( r_2 \) of \( K \), so that every function \( g \in K \) can be written as a non-negative linear combination of \( r_1 \) and \( r_2 \).

First, we prove a useful technical result.

(9.2) Lemma. For a permutation \( \sigma \in S_n \), \( \sigma \neq e \), let \( a_\sigma \in \mathbb{R}^2 \) be the point
\[
a_\sigma = \left(p(\sigma), \frac{2t(\sigma)}{n - p(\sigma)}\right).
\]

Let \( P = \text{conv}\{a_\sigma : \sigma \neq e\} \) be the convex hull of all such points \( a_\sigma \).

If \( n \) is even, the extreme points of \( P \) are
\[(0,0), \ (n-3,0), \ (n-2,1) \quad \text{and} \quad (0,1).
\]

If \( n \) is odd, the extreme points of \( P \) are
\[(0,0), \ (n-3,0), \ (n-2,1), \ (0, (n-3)/n) \quad \text{and} \quad (1,1).
\]
Proof. The set of all possible values \((p(\sigma), t(\sigma))\), where \(\sigma \neq \epsilon\), consists of all pairs of non-negative integers \((p, t)\) such that \(p \leq n - 2, 2t \leq n\) and, additionally, \(p + 2t \leq n - 3\) or \(p + 2t = n\). To find the extreme points of the set of feasible points \((p, 2t/(n - p))\), we choose a generic vector \((\gamma_1, \gamma_2)\) and investigate for which values of \(p\) and \(t\) the maximum of 
\[
\gamma_1 p + \gamma_2 \frac{2t}{n - p}
\]
is attained.

Clearly, we can assume that \(\gamma_2 \neq 0\). If \(\gamma_2 < 0\) then we should choose the smallest possible \(t\) which would be \(t = 0\) unless \(p = n - 2\) when we have to choose \(t = 1\). Depending on the sign of \(\gamma_1\), this produces the following pairs
\[
(p, t) = \{(0, 0), \ (n - 3, 0), \ (n - 2, 1)\}.
\]
If \(\gamma_2 > 0\) then the largest possible value of \(2t/(n - p)\) is 1. If \(\gamma_1 > 0\) this produces the (already included) point
\[
(p, t) = (n - 2, 1).
\]
If \(\gamma_1 < 0\) we get
\[
(p, t) = (0, n/2) \quad \text{for even} \ n
\]
and
\[
(p, t) = \{(0, (n - 3)/2), (1, (n - 1)/2)\} \quad \text{for odd} \ n.
\]
Summarizing, the extreme points of \(P\) are
\[
(0, 0), \ (n - 3, 0), \ (n - 2, 1), \ (0, 1) \quad \text{for even} \ n
\]
and
\[
(0, 0), \ (n - 3, 0), \ (n - 2, 1), \ (0, (n - 3)/n), \ (1, 1) \quad \text{for odd} \ n
\]
as claimed. 

Now we describe the central cone \(K_p\).

(9.3) Lemma. For \(n \geq 4\) let us define the functions \(r_1, r_{2e}\) and \(r_{2o} : S_n \rightarrow \mathbb{R}\) by
\[
\begin{align*}
r_1 &= 1 - 2t, \\
r_{2e} &= \frac{p^2 - 3p - n - 6t + 2tn + 4}{n^2 - 4n + 4} \quad \text{and} \\
r_{2o} &= \frac{p^2 - 3p - n - 4t + 2tn + 3}{n^2 - 4n + 3}.
\end{align*}
\]
Then

(1) If $n$ is even then $K_p$ is a 2-dimensional convex cone with the extreme rays spanned by $r_1$ and $r_{2e}$;
(2) If $n$ is odd then $K_p$ is a 2-dimensional convex cone with the extreme rays spanned by $r_1$ and $r_{2o}$. Cone $K_p$ contains $r_{2e}$;
(3) If $e \in S_n$ is the identity, then

$$r_1(e) = r_{2e}(e) = r_{2o}(e) = 1.$$ 

Proof. A function $g \in K_p$ can be written as a linear combination $g = \alpha_1 g_1 + \alpha_2 g_2$. Since $p(e) = n$ and $t(e) = 0$, we have $g(e) = \alpha_1 (n^2 - 3n + 1) + \alpha_2$. Therefore, the inequalities $g(e) \geq g(\sigma)$ can be written as

$$\alpha_1 (n^2 - 3n + 1) + \alpha_2 \geq \alpha_1 (p(\sigma)^2 - 3p(\sigma) + 1) + \alpha_2 (1 - 2t(\sigma)),$$

which, for $g \neq e$ is equivalent to

$$\alpha_1 (n + p(\sigma) - 3) + \alpha_2 \frac{2t(\sigma)}{n - p(\sigma)} \geq 0.$$ 

Using Lemma 9.2, we conclude that for even $n$, the system is equivalent to

$$(9.3.1) \quad \alpha_1 \geq 0$$

$$(n - 3)\alpha_1 + \alpha_2 \geq 0$$

and for odd $n$, the system is equivalent to

$$(9.3.2) \quad \alpha_1 \geq 0$$

$$(n - 2)\alpha_1 + \alpha_2 \geq 0.$$ 

Consequently, every solution $(\alpha_1, \alpha_2)$ of (9.3.1) is a non-negative linear combination of $(0, 1)$ and $(1, 3 - n)$ and every solution of (9.3.2) is a non-negative linear combination of $(0, 1)$ and $(1, 2 - n)$.

The functions $r_1, r_{2e}$ and $r_{2o}$ are obtained from $g_2, g_1 + (3-n)g_2$ and $g_1 + (2-n)g_2$ respectively by scaling so that the value at the identity becomes equal to 1.

Since every solution of (9.3.1) is a solution of (9.3.2), we conclude that $r_{2e} \in K_p$ for odd $n$ as well. □

(9.4) Remark. If $n$ is even, then $r_{2o} \notin K_p$. Indeed, if $\sigma$ is a product of $n/2$ commuting transpositions, so that $p(\sigma) = 0$ and $t(\sigma) = n/2$, then $r_{2o}(\sigma) = (n^2 -
\[ \frac{3n + 3}{n^2 - 4n + 3} > 1 = r_{2o}(e). \]

The central (pure) cone

The functions \( r_{2o} \) and \( r_{2e} \) have the bullseye distribution of Section 2. The distribution type of \( r_1 \) may be characterized as that of a “damped oscillator” with the averages over the \( k \)-ring \( U(e, k) \) changing sign and going fast to 0 as \( k \) grows. Hence a typical function from the central cone has a “weak” bullseye type distribution, which becomes weaker as the function becomes closer to \( r_1 \).

(9.5) Lemma. Let \( g \in K_p \) be a function such that \( g(e) = 1 \). For any \( 1 \leq k \leq n - 2 \), let \( \sigma_k \) be a permutation such that \( p(\sigma_k) = k \) and \( t(\sigma_k) = 0 \) and let \( \theta_k \) be a permutation such that \( p(\theta_k) = k \) and \( t(\theta_k) = 1 \). Then

\[ \max \{ g(\sigma_k), g(\theta_k) \} \geq \frac{k^2 - 3k + 1}{n^2 - 3n + 1}. \]

Proof. Applying Lemma 9.3, we may assume that \( g \) is a convex combination of \( r_1 \) and \( r_{2o} \), hence \( g = \alpha_1 r_1 + \alpha_2 r_{2o} \), for some \( \alpha_1, \alpha_2 \geq 0 \) and \( \alpha_1 + \alpha_2 = 1 \). Then

\[ g(\sigma_k) = \alpha_1 + \alpha_2 \frac{k^2 - 3k - n + 3}{n^2 - 4n + 3} \]

and

\[ g(\theta_k) = -\alpha_1 + \alpha_2 \frac{k^2 - 3k + n - 1}{n^2 - 4n + 3}. \]

We observe that if \( \alpha_1 = 1 \) and \( \alpha_2 = 0 \) then \( g(\sigma_k) > g(\theta_k) \) and if \( \alpha_1 = 0 \) and \( \alpha_2 = 1 \) then \( g(\sigma_k) < g(\theta_k) \). Moreover, as \( (\alpha_1, \alpha_2) \) change from \( (1, 0) \) to \( (0, 1) \) function \( g(\sigma_k) \)
decreases and function \( g(\theta_k) \) increases. Hence the minimum of \( \max\{g(\sigma_k), g(\theta_k)\} \) is attained when \( g(\sigma_k) = g(\theta_k) \). This produces the system of linear equations

\[
\alpha_1 + \alpha_2 \frac{k^2 - 3k - n + 3}{n^2 - 4n + 3} = -\alpha_1 + \alpha_2 \frac{k^2 - 3k + n - 1}{n^2 - 4n + 3}
\]

and

\[
\alpha_1 + \alpha_2 = 1
\]

with the solution

\[
\alpha_1 = \frac{n - 2}{n^2 - 3n + 1} \quad \text{and} \quad \alpha_2 = \frac{n^2 - 4n + 3}{n^2 - 3n + 1}.
\]

The corresponding value of \( g(\theta_k) = g(\sigma_k) \) is

\[
\frac{k^2 - 3k + 1}{n^2 - 3n + 1},
\]

which completes the proof. \( \square \)

Now we are ready to prove Theorem 3.1.

**Proof of Theorem 3.1.** Without loss of generality, we may assume that the maximum value of \( f_0 \) is attained at the identity permutation \( e \) (see Section 6). Excluding an obvious case of \( f_0 \equiv 0 \), by scaling \( f \), if necessary, we may assume that \( f_0(e) = 1 \). Let \( g \) be the central projection of \( f_0 \). By Lemma 6.2, \( g(e) = f_0(e) = 1 \geq g(\sigma) \) for all \( \sigma \in S_n \) and \( \overline{g} = 0 \). Moreover, since \( A \in L_n + L_{n-2,2} + L_{n-2,1,1} \), by Proposition 7.5, \( g \) must be a linear combination of the constant function \( \chi_n \) and functions \( \chi_{n-2,2} \) and \( \chi_{n-2,1,1} \). Since \( \overline{g} = 0 \), \( g \) is a linear combination of \( \chi_{n-2,2} \) and \( \chi_{n-2,1,1} \) alone. Therefore, \( g \) lies in the central cone: \( g \in K_p \), see Definition 9.1.

Let us choose a \( 3 \leq k \leq n - 5 \) and let \( X_k \) be the set of permutations \( \sigma \) such that \( p(\sigma) = k \) and \( t(\sigma) = 0 \) and let \( Y_k \) be the set of permutations \( \theta \) such that \( p(\theta) = k \) and \( t(\theta) = 1 \). To choose a permutation \( \sigma \in X_k \), one has to choose \( k \) fixed points in \( \binom{n}{k} \) ways and then a permutation without fixed points or 2-cycles on the remaining \( (n - k) \) points. Then, by (6.4.2)

\[
|X_k| \geq \frac{1}{5} \binom{n}{k} (n-k)! = \frac{1}{5} \frac{n!}{k!}.
\]

Similarly, to choose a permutation \( \theta \in Y_k \), one has to choose a 2-cycle in \( \binom{n}{2} \) ways, \( k \) fixed points in \( \binom{n-2}{k} \) ways and a permutation without fixed points or 2-cycles on the remaining \( (n - k - 2) \) points. Then, by (6.4.2)

\[
|Y_k| \geq \frac{1}{5} \binom{n}{2} \binom{n-2}{k} (n-k-2)! = \frac{n!}{10k!}.
\]
Let us choose a permutation \( \sigma \in X_k \) and a permutation \( \theta \in Y_k \) and let \( Z = X_k \) if \( g(\sigma) \geq g(\theta) \) and \( Z = Y_k \) otherwise. Then

\[
|Z| \geq \frac{n!}{10k!}
\]

and by Lemma 9.5,

\[
g(\sigma) \geq \frac{k^2 - 3k + 1}{n^2 - 3n + 1}
\]

for all \( \sigma \in Z \).

The set \( Z \) is a disjoint union of some conjugacy classes \( X(\rho) \) and for each \( X(\rho) \) by (6.4.1), we have

\[
g(\rho) = \frac{1}{|X(\rho)|} \sum_{\sigma \in X(\rho)} f_0(\sigma) \geq \frac{k^2 - 3k + 1}{n^2 - 3n + 1}
\]

and hence

\[
\frac{1}{|Z|} \sum_{\sigma \in X(\rho)} f_0(\sigma) \geq \frac{k^2 - 3k + 1}{n^2 - 3n + 1}.
\]

Applying Lemma 6.3 with \( X = Z \) and \( \beta = \beta(n, k) \), we get that

\[
P\{\sigma \in S_n : f_0(\sigma) \geq \gamma \beta(n, k)\} \geq \frac{(1 - \gamma)\beta(n, k)}{10k!}.
\]

10. The Symmetric Case. Proofs

In this case, \( A \in L_n + L_{n-1,1} + L_{n-2,2} \) (see Section 7). As in Sections 8 and 9, the \( L_n \) component contributes a just a constant to \( f \). We choose a more convenient basis \( g_1 \) and \( g_2 \) in the vector space spanned by \( \chi_{n-1,1} \) and \( \chi_{n-2,2} \), namely

\[
g_1 = \chi_{n-1,1} = p - 1 \quad \text{and} \quad g_2 = 2\chi_{n-2,2} + 3\chi_{n-1,1} = p^2 + 2t - 3,
\]

where \( p(\sigma) \) is the number of fixed points of \( \sigma \) and \( t(\sigma) \) is the number of 2-cycles in \( \sigma \).

(10.1) Definition. Let \( K_s \) (where \( s \) stands for “symmetric”) be the set of all functions \( g : S_n \to \mathbb{R} \) such that \( g \in \mathop{\mathrm{span}}\{g_1, g_2\} \), where \( g_1 = p - 1 \) and \( g_2 = p^2 + 2t - 3 \) and \( g(e) \geq g(\sigma) \) for all \( \sigma \in S_n \), where \( e \) is the identity permutation. We call \( K \) the central cone.

Identifying \( \mathop{\mathrm{span}}\{g_1, g_2\} \) with two-dimensional vector space \( \mathbb{R}^2 \) (plane), we see that the conditions \( g(e) \geq g(\sigma) \) define the central cone \( K_s \) as a convex cone in \( \mathbb{R}^2 \). Our immediate goal is to find the extreme rays \( r_1 \) and \( r_2 \) of \( K_s \), so that every function \( g \in K_s \) can be written as a non-negative linear combination of \( r_1 \) and \( r_2 \).
**Lemma.** For \( n \geq 4 \) let us define the functions \( r_1, r_{2e} \) and \( r_{2o} : S_n \rightarrow \mathbb{R} \) by

\[
\begin{align*}
    r_1 &= \frac{2np - 2n - p^2 - 3p - 2t + 6}{n^2 - 5n + 6}, \\
    r_{2e} &= \frac{-np + n + p^2 + p + 2t - 4}{2n - 4} \quad \text{and} \\
    r_{2o} &= \frac{-n^2p + np^2 + n^2 + np + 2nt - 4n - 3p + 3}{2n^2 - 7n + 3}.
\end{align*}
\]

Then

1. If \( n \) is even then \( K_s \) is a 2-dimensional convex cone with the extreme rays spanned by \( r_1 \) and \( r_{2e} \);
2. If \( n \) is odd then \( K_s \) is a 2-dimensional convex cone with the extreme rays spanned by \( r_1 \) and \( r_{2o} \). Cone \( K_s \) contains \( r_{2e} \);
3. If \( e \in S_n \) is the identity, then \( r_1(e) = r_{2e}(e) = r_{2o}(e) = 1 \).

**Proof.** A function \( g \in K_s \) can be written as a linear combination \( g = \alpha_1 g_1 + \alpha_2 g_2 \). Since \( p(e) = n \) and \( t(e) = 0 \), we have \( g(e) = \alpha_1(n-1) + \alpha_2(n^2 - 3) \). Therefore, the inequalities \( g(e) \geq g(\sigma) \) can be written as

\[
\alpha_1(n - 1) + \alpha_2(n^2 - 3) \geq \alpha_1(p(\sigma) - 1) + \alpha_2(p^2(\sigma) + 2t(\sigma) - 3),
\]

which, for \( \sigma \neq e \), is equivalent to

\[
(10.2.1) \quad \alpha_1 + \alpha_2 \left( n + p(\sigma) - \frac{2t(\sigma)}{n - p(\sigma)} \right) \geq 0.
\]

Applying Lemma 9.2, we observe that (10.2.1) is equivalent to the system of two inequalities:

\[
\alpha_1 + (2n - 3)\alpha_2 \geq 0
\]

and

\[
\alpha_1 + (n - 1)\alpha_2 \geq 0 \quad \text{if} \ n \text{ is even},
\]

\[
na_1 + (n^2 - n + 3)\alpha_2 \geq 0 \quad \text{if} \ n \text{ is odd}.
\]

Thus every pair \( (\alpha_1, \alpha_2) \) satisfying (10.2.1) can be written as a non-negative linear combination of \((2n - 3, -1)\) and \((1 - n, 1)\) when \( n \) is even and \((2n - 3, -1)\) and \((-n^2 + n - 3, n)\) when \( n \) is odd.

The generators \( r_1, r_{2e} \) and \( r_{2o} \) are obtained from \((2n - 3)g_1 - g_2, (1 - n)g_1 + g_2\) and \((-n^2 + n - 3)g_1 + ng_2\) respectively by scaling so that the value at the identity becomes equal to 1.
It remains to check that \( r_{2e} \in K \) for \( n \) odd as well. Indeed, using that \( 2t + p \leq n \) we have
\[
(2n - 4)(r_{2e} - 1) = -n(p - 1) + p(p + 1) + 2t - 4 - 2n + 4
\]
\[
= -n(p + 1) + p(p + 1) + 2t(p + 1)(-n + p) + 2t
\]
\[
\leq (p + 1)(-n + p) + n - p = p(-n + p) \leq 0.
\]

\[\square\]

(10.3) Remark. The average value of \( r_1, r_{2e} \) and \( r_{2o} \) on \( S_n \) is 0.

The function \( r_1 : S_n \rightarrow \mathbb{R} \) provides an example of the “bullseye” distribution (see Section 2.2). The maximum value of 1 is attained at the identity and at any transposition. The positive values of \( r_1 \) occur on permutations with at least two fixed points and \( r_1(\sigma) = \Omega(p(\sigma)/n) \) if \( p(\sigma) \geq 3 \).

In contrast, \( r_{2e} \) and \( r_{2o} \) exhibit a spike type distribution of Section 4.1. The maximum value of 1 is attained at the identity and, for \( r_{2e} \), on the product of \( n/2 \) transpositions, or, for \( r_{2o} \), on the product of \( (n - 3)/2 \) transpositions. On the other hand, no permutation other than \( e \) with at least 2 fixed points yields a positive value.

One can observe that if \( n \) is even then \( r_{2o} \notin K \). Indeed, if \( \sigma \) is a product of \( n/2 \) transpositions then \( r_{2o}(\sigma) = (2n^2 - 4n + 3)/(2n^2 - 7n + 3) > 1 \).

The picture of \( K_s \) is very similar to that of \( K_p \), see Section 9.4.

(10.4) Remark. The spike distribution. Let us consider Example 4.1. It is seen that \( f(\sigma) = 2b_{\sigma(1)\sigma(2)} \) and hence the maximum value of \( f \) is indeed 1 and obtained, in
particular, on the identity permutation $e$. Applying Lemma 1.5, we get

$$\mathcal{f} = \frac{1 + 4\gamma(n-2) + 0.5(n-2)(n-3)}{n(n-1)} = 0.$$ 

Let us prove that the central projection of $f$ is the function $r_{2e}$ of Lemma 10.2.

Suppose that $g$ is the central projection of $f$. It follows that $g$ can be written as a linear combination $g = \alpha_1 r_1 + \alpha_2 r_{2e}$. Since $g(e) = r_1(e) = r_{2e}(e) = 1$, we must have $\alpha_1 + \alpha_2 = 1$. Let $\theta = (12)$ be a transposition, hence $p(\theta) = n-2$ and $t(\theta) = 1$. Then $r_1(\theta) = 1$ and $r_{2e}(\theta) = 0$, hence $\alpha_1 = g(\theta)$. Denoting by $X$ the set of all transpositions in $S_n$, by (6.4.1) we get

$$g(\theta) = \frac{1}{|X|} \sum_{\sigma \in X} f(\sigma) = 2 \binom{n}{2}^{-1} \sum_{\sigma \in X} b_{\sigma(1)\sigma(2)}$$

$$= \binom{n}{2}^{-1} \left( 1 + \binom{n-2}{2} + 4(n-2)\gamma \right) = 0.$$

Therefore, $g = r_{2e}$. Let $X(\rho)$ be a conjugacy class with $p(\rho) = n-k$. Then

$$\frac{1}{|X(\rho)|} \sum_{\sigma \in X(\rho)} f(\sigma) = r_{2e}(\rho) = \frac{-n(n-k) + n + (n-k)^2 + (n-k) + 2t(\rho) - 4}{2n-4}$$

$$\leq \frac{-nk + k^2 - k + 3n - 4}{2n-4}.$$

Since the $k$-th ring $U(e, k)$ splits into a disjoint union of conjugacy classes $X(\rho)$ with $p(\rho) = n-k$, we conclude by (6.4.1) that

$$\frac{1}{|U(e,k)|} \sum_{\sigma \in U(e,k)} f(\sigma) \leq \frac{-nk + k^2 - k + 3n - 4}{2n-4}$$

as claimed.

More generally, one can prove that for any function $g \in K_s$ there is a function $f$ of type (1.1.1) with symmetric $A$, such that $\mathcal{f} = 0$, $f$ attains its maximum at the identity and the central projection of $f$ is $g$.

(10.5) Remark. Scarcity of relatively good values. Let us consider the function $f$ of Example 4.2. We observe that

$$f = \alpha_1 r_1 + \alpha_2 r_{2e}$$

for

$$\alpha_1 = \frac{n^2 - nm - 4n + 3m + 3}{n^2 - nm + m - 3} \quad \text{and} \quad \alpha_2 = \frac{4n - 2m - 6}{n^2 - nm + m - 3}.$$

Thus $f$ is a convex combination of $r_1$ and $r_{2e}$, hence $1 = f(e) \geq f(\sigma)$ for all $\sigma \in S_n$ and $\mathcal{f} = 0$. Remark 7.6 implies that $f$ is a generalized function (1.3.1) of the required type.
11. The General Case. Proofs

In this Section, we prove Theorem 5.1 and describe the “extreme” distributions. Let us choose a convenient basis in span\{\chi_{n-1,1}, \chi_{n-2,2}, \chi_{n-2,1,1}\}:

\[ g_1 = \chi_{n-1,1} - 1, \quad g_2 = \chi_{n-2,2} + \chi_{n-2,1,1} + 3\chi_{n-1,1} - p^2 - 2 \quad \text{and} \]
\[ g_3 = \chi_{n-2,1,1} - \chi_{n-2,2} = 1 - 2t. \]

(11.1) Definition. Let \( K \) be the set of all functions \( g \in \text{span}\{g_1, g_2, g_3\} \) such that \( g(e) \geq g(\sigma) \) for all \( \sigma \in S_n \). We call \( K \) the central cone.

Identifying \( \text{span}\{g_1, g_2, g_3\} \) with a 3-dimensional vector space \( \mathbb{R}^3 \), we see that conditions \( g(e) \geq g(\sigma) \) define the central cone \( K \) as a convex polyhedral cone in \( \mathbb{R}^3 \). The condition \( g(e) = 1 \) defines a plane \( H \) in \( \mathbb{R}^3 \) and the intersection \( B = H \cap K \) is a base of \( K \), that is, a polygon such that every \( g \in K \) can be uniquely represented in the form \( g = \lambda h \) for some \( h \in B \).

Our goal is to determine the structure of \( K \). This is somewhat more complicated than in the 2-dimensional situations of Sections 9-10.

(11.2) Proposition. Let us define functions

\[ r_1 = \frac{-np + n + p^2 - 2}{n - 2}, \]
\[ r_2 = 1 - 2t, \]
\[ r_3 = \frac{2np - 3p - 2n - p^2 - 2t + 6}{n^2 - 5n + 6}, \]
\[ r_4 = \frac{p + 2t - 2}{n - 2} \quad \text{and} \]
\[ r_5 = \frac{-2np + 3p^2 - 3p + 2tn - n - 3}{n^2 - 2n - 3}. \]

Then

1. If \( e \in S_n \) is the identity, then
\[ r_1(e) = r_2(e) = r_3(e) = r_4(e) = r_5(e) = 1; \]

2. If \( n \) is even then \( r_1, r_2, r_3 \) and \( r_4 \) are the vertices (in consecutive order) of the planar quadrilateral \( B = \text{conv}\{r_1, r_2, r_3, r_4\} \) which is a base of the central cone \( K \);

3. If \( n \) is odd then \( r_1, r_2, r_3, r_4 \) and \( r_5 \) are the vertices (in consecutive order) of the planar pentagon \( B = \text{conv}\{r_1, r_2, r_3, r_4, r_5\} \) which is a base of the central cone \( K \).
Proof. A function $g \in \text{span}\{g_1, g_2, g_3\}$ can be written as a linear combination $g = \alpha_1 g_1 + \alpha_2 g_2 + \alpha_3 g_3$. Then $g(e) = \alpha_1(n - 1) + \alpha_2(n^2 - 2) - \alpha_3$ and the conditions $g(e) \geq g(\sigma)$ are written as

$$
\alpha_1(n - 1) + \alpha_2(n^2 - 2) + \alpha_3 \geq \alpha_1(p(\sigma) - 1) + \alpha_2(p^2(\sigma) - 2) + \alpha_3(1 - 2t(\sigma)),
$$

which, for $\sigma \neq e$ are equivalent to

$$
\alpha_1 + \alpha_2(n + p(\sigma)) + \alpha_3 \frac{2t(\sigma)}{n - p(\sigma)} \geq 0.
$$

Applying Lemma 9.2, we see that for even $n$, the system is equivalent to

$$
\begin{align*}
\alpha_1 + n\alpha_2 & \geq 0 \\
\alpha_1 + (2n - 3)\alpha_2 \geq 0 \\
\alpha_1 + (2n - 2)\alpha_2 + \alpha_3 \geq 0 \\
\alpha_1 + n\alpha_2 + \alpha_3 \geq 0
\end{align*}
$$

(11.2.1)

whereas for odd $n$, the system is equivalent to

$$
\begin{align*}
\alpha_1 + n\alpha_2 & \geq 0 \\
\alpha_1 + (2n - 3)\alpha_2 \geq 0 \\
\alpha_1 + (2n - 2)\alpha_2 + \alpha_3 \geq 0 \\
\alpha_1 + (n + 1)\alpha_2 + \alpha_3 \geq 0 \\
n\alpha_1 + n^2\alpha_2 + (n - 3)\alpha_3 \geq 0.
\end{align*}
$$

(11.2.2)

The set of all feasible 3-tuples $(\alpha_1, \alpha_2, \alpha_3)$ is a polyhedral cone, which, for even $n$, has at most 4 extreme rays and for odd $n$ has at most 5 extreme rays. We call an inequality of (11.2.1)–(11.2.2) active on a particular tuple if it holds with equality.

It is readily verified that for even $n$ the following tuples span the extreme rays of the set of solutions to (11.2.1):

$$
\begin{align*}
(-n, & ~1, ~0) & \text{4th and 1st inequalities are active} \\
(0, & ~0, ~1) & \text{1st and 2nd inequalities are active} \\
(2n - 3, & ~1, ~1) & \text{2nd and 3d inequalities are active} \\
(1, & ~0, ~-1) & \text{3d and 4th inequalities are active}
\end{align*}
$$

and that for odd $n$ the following tuples span the extreme rays of the set of solutions to (11.2.1):

$$
\begin{align*}
(-n, & ~1, ~0) & \text{5th and 1st inequalities are active} \\
(0, & ~0, ~1) & \text{1st and 2nd inequalities are active} \\
(2n - 3, & ~1, ~1) & \text{2nd and 3d inequalities are active} \\
(1, & ~0, ~-1) & \text{3d and 4th inequalities are active} \\
(-2n - 3, & ~3, ~-n) & \text{4th and 5th inequalities are active}
\end{align*}
$$
We obtain \( r_1, r_2, r_3, r_4 \) and \( r_{5_0} \) by scaling the corresponding linear combinations \( \alpha_1 g_1 + \alpha_2 g_2 + \alpha_3 g_3 \) so that the value at the identity is equal to 1 and hence \( r_1, r_2, r_3, r_4 \) and \( r_{5_0} \) lie on the same plane in \( \text{span}\{g_1, g_2, g_3\} \).

\((11.3)\) Remark. One can observe that if \( n \) is even then \( r_{5_0} \notin K \), for if \( \sigma \) is a product of \( n/2 \) commuting transpositions, so that \( p(\sigma) = 0 \) and \( t(\sigma) = n/2 \), then \( r_{5_0}(\sigma) = (n^2 + n - 3)/(n^2 - 2n - 3) > 1 = r_{5_0}(e) \).

The base of the central cone

\[ \begin{array}{c}
\text{n is even} \\
\text{n is odd}
\end{array} \]

We observe that function \( r_3 \) coincides with function \( r_1 \) of Lemma 10.2 (the symmetric QAP) and that function \( r_2 \) coincides with function \( r_1 \) of Lemma 9.3 (the pure QAP). Function \( r_4 \) has a bullseye type distribution (see Section 2.2) whereas \( r_1 \) is a sharp spike (see Section 4.1). We have \( r_1(\sigma) = 1 \) if and only if \( \sigma = e \) or
\[ \text{dist}(e, \sigma) = n \text{ and } r_1(\sigma) < 0 \text{ for all other } \sigma. \]

Distribution of values of the objective function with respect to the Hamming distance from the maximum point

Function \( r_{5o} \) resembles a spike, but diluted.

Now we are getting ready to prove Theorem 5.1.

**Lemma.** Let \( g \in \text{span}\{g_1, g_2, g_3\} \) be a function such that \( g(e) = 1 \). For a \( 2 \leq k \leq n - 2 \), let \( \sigma_k \) be a permutation such that \( p(\sigma_k) = k \) and \( t(\sigma_k) = 0 \), let \( \eta \) be a permutation such that \( p(\eta) = 0 \) and \( t(\eta) = 1 \) and let \( \theta \) be permutation such that \( p(\theta) = t(\theta) = 0 \). Then

\[
\max\{g(\sigma_k), g(\eta), g(\theta)\} \geq \frac{k - 2}{n^2 - kn + k - 2}.
\]

**Proof.** We can write

\[
g = \alpha_1 \frac{p - 1}{n - 1} + \alpha_2 \frac{p^2 - 2}{n^2 - 2} + \alpha_3 (1 - 2t)
\]

for some \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) such that \( \alpha_1 + \alpha_2 + \alpha_3 = 1 \). Then

\[
g(\sigma_k) = \alpha_1 \frac{k - 1}{n - 1} + \alpha_2 \frac{k^2 - 2}{n^2 - 2} + \alpha_3
\]

\[
g(\eta) = -\alpha_1 \frac{1}{n - 1} - \alpha_2 \frac{2}{n^2 - 2} - \alpha_3
\]

\[
g(\theta) = -\alpha_1 \frac{1}{n - 1} - \alpha_2 \frac{2}{n^2 - 2} + \alpha_3.
\]
We observe that \( g(\sigma_k), g(\eta) \) and \( g(\theta) \) are linear functions of \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) and hence
\[
\ell(\alpha_1, \alpha_2, \alpha_3) = \max\{g(\sigma_k), g(\eta), g(\theta)\}
\]
is a convex function on the plane \( \alpha_1 + \alpha_2 + \alpha_3 = 1 \).

Moreover, for
\[
(11.4.1) \quad \alpha_1 = \frac{k(1-n)}{n^2-nk+k-2}, \quad \alpha_2 = \frac{n^2-2}{n^2-nk+k-2} \quad \text{and} \quad \alpha_3 = 0
\]
we have
\[
(11.4.2) \quad g(\sigma_k) = g(\eta) = g(\theta) = \frac{k-2}{n^2-nk+k-2}.
\]

Let us prove that the minimum of \( \ell(\alpha_1, \alpha_2, \alpha_3) \) on the plane \( \alpha_1 + \alpha_2 + \alpha_3 = 1 \) is attained at (11.4.1). Let
\[
\lambda_1 = \frac{n^2-2n}{k^2-2k}, \quad \lambda_2 = \frac{n^2-nk}{2k-4} \quad \text{and} \quad \lambda_3 = \frac{n^2-kn-2n+2k}{2k}.
\]
Then
\[
\lambda_1 g(\sigma_k) + \lambda_2 g(\eta) + \lambda_3 g(\theta) = \alpha_1 + \alpha_2 + \alpha_3 = 1 \quad \text{and} \quad \lambda_1, \lambda_2, \lambda_3 > 0.
\]
Comparing this with (11.4.2), we conclude that there is no point \((\alpha_1, \alpha_2, \alpha_3)\) with \( \alpha_1 + \alpha_2 + \alpha_3 = 1 \) such that
\[
g(\sigma_k), g(\eta), g(\theta) < \frac{k-2}{n^2-nk+k-2}.
\]

□

Now we are ready to prove Theorem 5.1.

Proof of Theorem 5.1. Without loss of generality, we may assume that the maximum value of \( f_0 \) is attained at the identity permutation \( e \). Excluding an obvious case of \( f_0 \equiv 0 \), by scaling \( f \), if necessary, we may assume that \( f_0(e) = 1 \). Let \( g \) be the central projection of \( f_0 \). By Lemma 6.2, \( g(e) = f_0(e) = 1 \geq g(\sigma) \) for all \( \sigma \in S_n \) and \( \overline{g} = 0 \). By Proposition 7.5, \( g \) must be a linear combination of the functions \( \chi_n, \chi_{n-1,1}, \chi_{n-2,2} \) and \( \chi_{n-2,1,1} \). Since \( \overline{g} = 0 \), \( g \) is a linear combination of non-trivial characters \( \chi_{n-1,1}, \chi_{n-2,2} \) and \( \chi_{n-2,1,1} \) alone. Therefore, \( g \) lies in the central cone: \( g \in K \), see Definition 11.1.

Let \( X_k \) be the set of all permutations \( \sigma \) such that \( p(\sigma) = k \) and \( t(\sigma) = 0 \). As in the proof of Theorem 4.1, we conclude that
\[
|X_k| \geq \frac{1}{5} \frac{n!}{k!}.
\]
Let $Y$ be the set of all permutations $\sigma$ such that $p(\sigma) = 0$ and $t(\sigma) = 1$. To choose a permutation $\sigma \in Y$, one has to choose a transpositions in $\binom{n}{2}$ ways and then an arbitrary permutation of the remaining $(n-2)$ symbols without fixed points and 2-cycles. Using (6.4.2), we estimate

$$|Y| \geq \frac{n!}{5 \cdot 2(n-2)!} (n-2)! = \frac{1}{10} n!.$$ 

Let us choose a permutation $\sigma_k \in X_k$, a permutation $\eta \in Y$ and a permutation $\theta \in X_0$. Let us choose $Z$ to be one of $X_k$, $X_0$ and $Y$, depending where the maximum value of $g(\sigma_k)$, $g(\eta)$ or $g(\theta)$ is attained. Hence

$$|Z| \geq \frac{n!}{5k!}.$$ 

The set $Z$ is a disjoint union of some conjugacy classes $X(\rho)$ and for each $X(\rho)$ by (6.4.1) and Lemma 11.4, we have

$$g(\rho) = \frac{1}{|X(\rho)|} \sum_{\sigma \in X(\rho)} f_0(\sigma) \geq \frac{k - 2}{n^2 - kn + k - 2}$$

and hence

$$\frac{1}{|Z|} \sum_{\sigma \in Z} f_0(\sigma) \geq \frac{k - 2}{n^2 - kn + k - 2}.$$ 

Applying Lemma 6.3 with $X = Z$ and $\beta = \beta(n,k)$, we conclude that

$$P\{\sigma \in S_n : f_0(\sigma) \geq \gamma \beta(n,k)\} \geq \frac{(1 - \gamma) \beta(n,k)}{5k!}.$$ 

for all $n \geq 5$. \hfill \Box

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Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1109

E-mail address: barvinok@umich.edu, tamon@umich.edu