Optimal Immunization Policy Using Dynamic Programming

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Abstract—Decisions in public health are almost always made in the context of uncertainty. Policy makers responsible for making important decisions are faced with the daunting task of choosing from amongst many possible options. This task is called planning under uncertainty, and is particularly acute when addressing complex systems, such as issues of global health and development. Decision making under uncertainty is a challenging task, and all too often this uncertainty is averaged away to simplify results for policy makers. A popular way to approach this task is to formulate the problem at hand as a (partially observable) Markov decision process, (PO)MDP. This work aims to apply these AI efforts to challenging problems in health and development. In this paper, we developed a framework for optimal health policy design in a dynamic setting. We apply a stochastic dynamic programing approach to identify both the optimal time to change the health intervention policy and the optimal time to collect decision relevant information.

Keywords: health policy, dynamic programming, optimal control, reinforcement learning

I. INTRODUCTION

The field of health economics is focused on maximizing the impact and cost effectiveness of health interventions. These economic evaluations rely on mathematical or computational models to estimate the impact of candidate intervention packages, and associated incremental costs. Because funds are limited and data is expensive to collect, good information is of significant value to health policy making programs, and we must weigh the potential value of information (VOI) against other opportunity costs such as treatment, prevention, and system strengthening. Thus, balancing investments in information and investments on interventions will be important. The big question, here, is where more valuable to invest on gathering data rather than investing on intervention. When a lot of money, people, and process change would be involved, then careful methodical value of information modeling is likely to be necessary. One example is the design of the measles control program: what is the right fraction of lab-confirmed case to aim for? Should sentinel sites be organized to complement their case-based surveillance or is it enough as is? How should the system change as better disease control is achieved and eradication is approached? What is the necessary level of surveillance to achieve, to verify, and to maintain eradication? Should we invest more in case based surveillance to achieve better vaccination coverage, or is it the other way around that better operational excellence is needed before better data can have an impact? In this work, we will find a balance between the value of such information with the value of more disease control efforts: what fraction of the funds of a disease control program should be allocated to conducting disease surveillance or collecting accurate data versus directed to increasing efforts to control the disease? Furthermore, if we decide to vaccinate a region, when is the optimal time to do so.

Cost effectiveness analysis in health is an economic method that compares the lifetime costs and benefits associated with different health interventions. The optimal allocation of resources across health interventions is determined by solving a constrained optimization problem with the objective of maximizing health benefits subject to a budget constraint [1]. The health intervention we considered in this paper is vaccination, which is one of the primary intervention strategies used by public health agents to control infectious diseases. Most of the current deterministic approaches focus on pre-determining vaccination strategies to reduce the expected number of infected population for a given budget. We extend the literature on optimal vaccination policies by developing an approach to identify the optimal information acquisition policy and optimal vaccination policy together when we deal with decision making under uncertainty. Finally, as an example, we apply our framework to the timely vaccination problem for the classic stochastic Susceptible-Infected-Recovered disease model.

Value of information and planning under uncertainty have a long history in fields such as control theory, computer science (AI), and health economics. A common theme that emerges is that uncertainty leads to cautious or incorrect decisions that cost time, money, and human life. It is with this understanding that we pursue greater clarity on, and methods to address, optimal policy making in health. Our goal in this work is to implement dynamic programming which provides basis

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for compiling planning results into reactive strategies. We present here a description of an AI-based method and illustrate how this method can improve our ability to find an optimal vaccination strategy.

II. METHODS

A classic planning problem in AI is specified as follows: Given a description of the current state of some system, a set of actions that can be performed on the system and a description of cost (or reward) of states and actions for the system, find a sequence of actions that can be performed to minimize the cumulative cost (or maximize the cumulative reward). The optimal planner must balance the potential of some plan achieving our goal to maximize the benefits against the cost of performing the plan.

Markov decision processes (MDPs) is an approach to planning under uncertainty, that combines the following elements:

- Set of states \((\mathcal{X})\),
- Set of actions \((\mathcal{U})\),
- Immediate cost for choosing action \(u \in \mathcal{U}\) in state \(x \in \mathcal{X}\), \(l(x,u)\),

and the following distribution:

- Transition model \(T(x',x,u) = \Pr(x' | x,u)\). The transition matrix determines the outcome probabilities for each action in each possible state. In this framework, transitions are Markovian, which means that transition to state \(x'\) only depends on the current state \(x\), and not the history of earlier states.

A policy, \(\pi(x) : x \rightarrow u\), for an MDP is a mapping from state to action that selects an action for each state. Now given a policy, we can define its value function, \(v(x)\), which is equal to the immediate cost for the action selected by the policy at the current state plus the value function at the next state. A solution to an MDP is a policy that minimizes a term called the Hamiltonian,

\[ H_\alpha(x,u,v^*(\cdot)) \triangleq l(x,u) + \alpha \mathbb{E}_{x' \sim \Pr(\cdot|x,u)}[v(x')] , \]

where \(\alpha\) is discount factor to ensure that the cost-to-go remains finite. All problems of perfect information have an optimal value function \(v^*(x)\), which determines the cost from every state \(x\) under perfect implementation of the optimal policy.

Two popular methods for solving this equation and finding an optimal policy for an MDP are value iteration and policy iteration. In policy iteration, the current policy is repeatedly improved by finding some action in each state that has a higher value than the action chosen by the current policy for that state. The policy is initially chosen at random, and the process terminates when no improvement can be found [2]. In value iteration, optimal policies are produced for successively longer finite horizons, until they converge, terminating when the maximum change in values between the current and previous value functions is below some threshold [2].

Both of these methods, mentioned above, solve the MDP problem by recursively computing the optimal value function in a search tree containing approximately \(|\mathcal{U}|^d\) possible sequences of moves, where \(|\mathcal{U}|\) is the number of legal actions per state, and \(d\) is the planning horizon. For large population, exhaustive search is hardly feasible. David Silver et al. introduced new search algorithm that successfully combines neural network evaluations with Monte Carlo rollouts [3], [4], [5].

In a Markov decision process the environment’s dynamics are fully determined by its current state, \(x\). For any state and for any action, the transition probability determines the next state distribution, and the reward function determines the expected reward. In a partially observed Markov decision process (POMDP), the state cannot be directly observed by the agent. Instead, the agent receives an observation, determined by the observation probabilities. In POMDP, the policy maps a history to a probability distribution over actions, where, history is a sequence of actions and observations.

In addition of the an MDP elements mentioned previously, a POMDP has the following elements:

- Set of observations \((\mathcal{Y})\),
- Belief state \(b\), posterior distribution over states. The belief at each time \(t\) can be computed using the initial belief and the history of observations \(y(0 : t-1)\) and actions \(u(0 : t-1)\),
- Observation model \(O(y,x,u) = \Pr(y | x,u)\). This model determines the probability of perceiving the observation \(y \in \mathcal{Y}\) in state \(x\) after choosing action \(u\).

In POMDP, a policy, \(\pi(b) : b \rightarrow u\), is a mapping from belief to action that selects an action at each time. In POMDP we apply the very same idea as in MDP, but since the full states, \(x\), are not observable, then the agent needs to choose the optimal policy only considering the belief state, \(b\).
The optimal policy at time $t$ defined as:

$$
\pi_t^*(b) \triangleq \arg\min_{u \in U} \left[ l(b, u) + \int v_{t-1}(b') \Pr(b' | u, b) db' \right],
$$

$$
v_t(b) = \alpha \cdot \min_{u \in U} \left[ l(b, u) + \int v_{t-1}(b') \Pr(b' | u, b) db' \right],
$$

$$
v_1(b) = \alpha \cdot \min_{u \in U} [\mathbb{E}_x [b(x, u)]] .
$$

The POMDP algorithms compute these value functions recursively. Finding optimal solution for POMDP is challenging. Some methods, like Monte Carlo POMDP [6], can approximate the value function of POMDP, and the optimal action can be read from the value function for any belief state. But the time complexity of solving POMDP value iteration is exponential in number of possible actions and observations, and the dimensionality of the belief space grows with number of states.

Policy making in an uncertain environment often involves a trade off between exploratory actions, whose goal is to gather data, and regular actions which exploit the information gathered so far and pursue the objectives. Now consider the stochastic control problem for a POMDP in which there are a number of observation options available to us, with varying associated costs. Therefore, the observation cost is added to the running cost and the optimal policy which is a combination of the optimal control and optimal sensor query is obtained from minimizing the total cost. The performance of the system depends on the level of uncertainty presented in the states estimation (belief). Thus, the controller needs to balance between performance and the penalty of requesting information.

There exists works in literature studying optimal controlled observation for stochastic linear systems with quadratic running penalty [7]. In [8], a computationally tractable algorithm has proposed to solve the stochastic sensor scheduling problem for the finite horizon linear quadratic Gaussian problem. The problem of finding the optimal locations for observation has also been considered in Environment Monitoring and informative path planning [9], which uses Bayesian optimization to select sensing locations. The most popular approach to choose sampling locations in the literature is based on information theory principles, where the goal is to place observations at higher entropy locations or in locations with higher information gain [10]. Some works consider optimal sensor query by minimizing the uncertainty of state estimates from belief-state MDP [11].

### III. Optimal Vaccination Policy for SIR Model

A first test problem of this AI-based approach to planning under uncertainty is finding the best vaccination policy for a stochastic SIR. The classic stochastic SIR model tracks the people in a community of population $N$ in one of three categories: Susceptible ($S$), Infectious ($I$), and Recovered ($R$).

The SIR diagram below shows how individuals move through each compartment in the model.

**Fig. 1: Susceptible - Infectious - Recovered model**

At the first step, we model the SIR dynamics as an MDP. The state defined as $x = (S, I, R)$ and action (decision) is the vaccination fraction. The stochastic discrete time formulation for SIR epidemic model is given by:

$$
S(t + \delta) = S(t) - \Delta N_u - \Delta N_i ,
$$

$$
I(t + \delta) = I(t) + \Delta N_i - \Delta N_r ,
$$

$$
R(t + \delta) = R(t) + \Delta N_r + \Delta N_u ,
$$

where,

$$
\Delta N_u \sim Binomial(N, 1 - e^{-\delta u}) ,
$$

$$
\Delta N_i \sim Binomial(S - \Delta N_u, 1 - e^{-\delta I/N}) ,
$$

$$
\Delta N_r \sim Binomial(I, 1 - e^{-\gamma \delta}) .
$$

where $\beta$ is the infectious rate, which is the probability of transmitting disease between a susceptible and an infectious individual, and $\delta = \frac{1}{\gamma} > 0$ is the duration of infection.

In this formulation, $\Delta N_i$ and $\Delta N_r$ are the number of new cases ($S \rightarrow I$) and number of recovered people ($I \rightarrow R$). The infection and recovery process are represented as probabilistic binomial draws in this model. The control $u(t)$ is the fraction of population being vaccinated at time $t$. The vaccinated people, $\Delta N_u$ are selected on a binomial basis and all individuals have the same probability of being included.

We modeled the SIR dynamics as an MDP. Our goal, here, is to find the optimal vaccination fraction, $u^*(t)$. A simple form of the optimization problem can be written
as:
\[
\begin{align*}
\text{minimize} & \quad \sum_t \alpha^t [c_1 I(t) + c_u u(t) N] \\
\text{subject to} & \quad 0 \leq u(t) \leq \text{maximum coverage,}
\end{align*}
\]
where \(c_1\) is the social cost of one person being infected, and \(c_2\) is the cost of one vaccine. This optimization can be solved for:

- **Infinite Horizon** in this case the discount factor should be smaller than one \((0 < \alpha < 1)\).
- **Finite Horizon** in this case the discount factor can be equal to one \((0 < \alpha \leq 1)\).
- **First Exit** the policy terminate as soon as we reach to a specific goal. In this work, we define the goal state as any state with prevalence less than some threshold \((\frac{I}{N} < T)\).

**IV. Optimal Vaccination for Imperfect Observation**

When a policy maker is attempting to solve a planning problem, the survey data are used to obtain noisy information. By modeling this problem at a high level as a POMDP, the policy maker is able to account for the inherent uncertainty in the measurements. When faced with decisions in the presence of uncertainties, we should select the option with highest expected utility. Here we answer the question of if we need surveillance versus increasing efforts on controlling disease (intervention vs. surveillance).

In the previous section we presented the SIR model as an MDP. In order to demonstrate the value of information, we model the problem as a POMDP. In the last section we assumed that the number of susceptible, infected, and recovered population are perfectly observed in all time steps. But the information we receive is not perfect in practice. Depending on surveys, we only have access to part of states, e.g. we can observe a noisy measurement of the prevalence.

The observation comes from a test with binary test characteristics \(q = (q_1, q_2)\), where \(q_1\) is the sensitivity, \(q_2\) is the specificity, and \(q_1 + q_2 > 1\). To model the problem as a POMDP, we can use the same formulation we had for MDP. We only need to define the conditional observation. Assume that for a given survey coverage, \(c_t\), we test \(n_t = c_t \cdot N\) people for a disease at time \(t\). The test could be imperfect. For a given sensitivity and test specificity, the observation can be modeled as:

\[
y(t) \sim Binomial \left( n_t, \frac{q_1 I(t) + (1 - q_2)(S(t) + R(t))}{N} \right)
\]

Using this model, we can compute \(\Pr(y \mid x)\).

Now we have stochastic SIR model and stochastic survey model to compute the transition matrix and the conditional observation matrices. The optimal vaccination policy is obtained from solving the proposed POMDP.

**V. Optimal Vaccination and Surveillance Policies**

In the previous section, we assumed that the survey coverage is given. In this section, we define augmented control which includes control on observation. Thus, the control is \(u = (u_x, u_y)\), where \(u_x\) is the vaccination fraction, similar to what we had in regular POMDP, and \(u_y\) determines the survey coverage.

At the beginning of each period \(t\), the algorithm determines whether to invest in an intervention at time step \(t\) and whether to conduct a survey of size \(u_y\) over the period to obtain a better estimate of the belief \(b\). Information, if sought is used together with the known stochastic dynamics of the disease to update the belief. Let \(d_t = u_x(t) \cdot N \in \{0, 1, 2, \ldots, N\}\) denote the intervention decision at time \(t\), where \(d_t = 0\) indicates No Intervention and \(d_t = 1\) \(\neq 0\) indicates Intervention with \(i\)th level of vaccination coverage. The cost associated to this decision is monotonically increasing with the level of vaccination coverage, and is linear with respect to the number of vaccines required for the intervention. Here, we assumed that the vaccination cost is \(c_u \cdot d_t\).

The amount of information collected is measured in terms of the survey coverage and the sensitivity and specificity of the medical test. The survey coverage determines the number of people being tested \(n_t = u_y(t) \cdot N \in \{0, 1, 2, \ldots, N\}\); it is obtained at the cost \(c_y \cdot n_t\) where \(c_y\) is the cost of testing a person for the disease. Thus, at each time \(t\) the policy maker implements the control \(u = (u_x, u_y)\). The immediate cost for the current combination of state and control \(l(x, u)\) is

\[
l(x, u) = c_i(I_t) + c_v d_t + c_y n_t,
\]

where, \(I_t = I(t) - I(t - 1)\) is the number of new cases from the previous time step \(t - 1\) to the current time \(t\).

Given a discount factor \(\alpha \in (0, 1]\), the policy maker’s objective is to minimize the net cost of the policy over a given horizon (in case of finite horizon), and the given initial belief \(b_0(x)\), and admissible possible decisions \(u_t = (d_t, n_t)\).

Solving POMDP with the augmented control action, we have the optimal vaccination policy and optimal survey coverage at each time step. To simplify the problem, we can solve the POMDP for \(u_x\) and \(u_y\) are
binary, which determines if we need to gather data (with a fixed survey coverage) and whether we need to have a vaccination campaign with a given fixed coverage.

VI. RESULTS

Consider a small community of 10 people with stochastic SIR model with infectious probability $\beta = 0.5$ and the recovery rate $\gamma = 0.15$. We also have a description of the goal states, which is any states with less than 10% disease prevalence. We are going to find the optimal sequence of actions that can be performed to move into one of the goal states. The cost is defined as the sum of the number of infectious people plus the cost of vaccine times the number of vaccinated people. Here the set of actions is vaccination coverage of 0%, 25%, 50%, 75%, and 100%. The solution of MDP gives us the optimal action for each state as a lookup table. In figures 2 and 3 we have the expected optimal cost starting from any state to the goal states. In this case, we assumed that the social cost of one person being infected, $c_i$, is equal to the cost of one dose of vaccine, $c_v$.

![Fig. 2: Optimal policy starting from each initial state and follow the optimal policy afterwards.](image)

We can summarize the optimal policy given in figure 2 as: If the prevalence is zero or the susceptible population is less than 25%, or the immune population is more than 35%, then no intervention is needed, otherwise, full power intervention would be required.

A. Benefit of smart vaccination

Here we compare two policies: optimal policy and zero vaccination policy. Assume the population is 50 and the current disease prevalence is 12%, we know that 24% of the population are already immune. The cost of one infection per unit time is $1$ and the cost of one vaccine is $1$ as well. The result of deploying these two policies are given in figure 4. Each blue dot in this figure represents a simulation corresponding to the optimal policy, and each black point represents the cost of zero-vaccination policy. If we do not vaccinate anyone, the average cost is $103$ while the average cost of the optimal policy reduces the cost to $56$. If we implement the optimal policy, then the prevalence will increase to 14% initially and then eradicate after around 25 time steps. If we implement zero-vaccination policy, then the prevalence will increase to 26% and then the disease will be eradicated after about 50 time steps. So we can conclude that spending $33$ for vaccination will survive 6 people and decrease the time to eradicate by 50%, which means significant cost saving.

VII. CONCLUSION

This paper proposed a new technique for informative policy making in health. The developed method in this work addresses practical policy and program problems encountered by funders, governments, health planners, and program implementers to help them allocating limited resources more efficiently. The technique presented in this paper can be applied in settings in which the decision maker wishes to identify the optimal time to stop the current intervention and initiate a new intervention.

The main challenge in dynamic programming is solving for the value function via either value iteration or policy iteration. Active research over the past decade
has revealed practical approximate solutions to dynamic programming problems. Bertsekas and Tsitsiklis pioneered the use of artificial neural networks to learn these functions, which mitigates the curse of large state space dimensionality. Deep convolutional neural network approaches to dynamic programming were the solution to Googles amazing Alpha-Go artificial intelligence.

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