Symmetry analysis of the high-order equations for the description of the Fermi – Pasta – Ulam problem

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Abstract. Recently some new nonlinear equations for the description of the Fermi – Pasta – Ulam problem have been derived. The main aim of this work is to use the symmetry test to investigate these equations. We consider equations for the description of the $\alpha$ and $\alpha + \beta$ Fermi – Pasta – Ulam model. We find the infinitesimal operators and Lie groups, admitted by the equations. Using the groups we find the self-similar variables as well as the reductions to the ordinary differential equations. Some exact solutions are also constructed.

1. Introduction
The Fermi – Pasta – Ulam (FPU) model was introduced about 60 years ago in paper [5]. It describes perturbations in a one-dimensional lattice with a nonlinear potential of interaction. The authors supposed to observe a thermalization but the energy distribution was seen to be recurrent. This fact was called the Fermi – Pasta – Ulam paradox and have been studied until now.

Some results were achieved using the continuous limit approximation. In paper [1] it was shown that the $\alpha$ FPU model can be approximated by the Korteweg-de Vries equation. Then the recurrent dynamic can be explained. In [4] Kudryashov found out that the recurrence disappears if one uses more accurate approximation. Thus, it is important to study equations, for more accurate description of the Fermi – Pasta – Ulam problem. The fifth- and seventh-order equations for the description of the $\alpha$ FPU problem were derived in [3, 4]. The sixth-order equation was obtained to study two waves propagating in different directions in the $\alpha$ FPU model in [6]. And the fifth-order equation for the description of the $\alpha + \beta$ FPU problem was derived in [2]. It was shown that such equations does not pass the Painlevé test. Some exact solutions were found. The symmetry analysis of these equations has not been done so far. This is the main aim of the current work.

The rest of the work is organized as follows. In section 2 we apply the symmetry analysis for the fifth-order equation for the description of the $\alpha$ Fermi – Pasta – Ulam problem from the paper [4]. The seventh-order equation, derived in [3] is analyzed in section 3. Symmetries for the extended Boussinesq equation, derived in [6] are investigated in section 4. In section 5 we apply the symmetry analysis to investigate the equation for the description of the $\alpha + \beta$ FPU problem from the work [2]. In section 6 we briefly discuss our results.
2. Symmetry test of the fifth-order differential equation for the description of the 
α FPU model

If we consider the quadratic potential of interaction in the Fermi – Pasta – Ulam mass chain, 
then we can describe it using the fifth-order differential equation [4]. It has the form:

\[
u_t + uu_x + \delta^2 u_{xxx} + 2\delta^2 u_x u_{xx} + \delta^2 u_{xxxx} + \delta^4 u_{xxxxx} = 0. \tag{1}\]

Let us search for the infinitesimal generator in the form:

\[X = \xi_x \frac{\partial}{\partial x} + \xi_t \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u}. \tag{2}\]

Its prolongation is:

\[X_5 = X + \xi_x \frac{\partial}{\partial u_x} + \xi_{2x} \frac{\partial}{\partial u_{2x}} + \xi_{3x} \frac{\partial}{\partial u_{3x}} + \xi_{4x} \frac{\partial}{\partial u_{4x}} + \xi_{5x} \frac{\partial}{\partial u_{5x}} + \xi_t \frac{\partial}{\partial u_t}. \tag{3}\]

We use generator (3) for equation (1) and the prolongation formulas for the components 
\(\xi_x, \xi_{2x}, \xi_{3x}, \xi_{4x}, \xi_{5x}, \xi_t,\) then we obtain the overdetermined linear system of partial differential 
equations:

\[\frac{\partial \xi_t}{\partial t} = \frac{\partial \xi_t}{\partial x} = \frac{\partial \xi_t}{\partial u} = \frac{\partial \xi_x}{\partial t} = \frac{\partial \xi_x}{\partial x} = \frac{\partial \xi_x}{\partial u} = \eta_u = 0. \tag{4}\]

As we solve (4) we derive components of the tangent vector field:

\[\xi_t = C_1, \quad \xi_x = C_2, \quad \eta_u = 0. \tag{5}\]

Thus the operator, admitted by (1) has the form:

\[X = C_1 \frac{\partial}{\partial t} + C_2 \frac{\partial}{\partial x}. \tag{6}\]

We can find from (6) that equation (1) admits shift transformation \(x' = x + a\) or \(t' = t + a.\) 
Thus we can use the traveling wave variables \(u(x, t) = v(z), z = x - C_0 t\) to find the solution of 
the partial differential equation (1).

If we neglect term at \(\delta^4,\) then equation (1) becomes the third-order partial differential 
equation:

\[u_t + uu_x + \delta^2 u_{xxx} + 2\delta^2 u_x u_{xx} + \delta^2 u_{xxxx} = 0. \tag{7}\]

In this case, the prolongation form for the infinitesimal generator has the form:

\[X_3 = X + \xi_x \frac{\partial}{\partial u_x} + \xi_{2x} \frac{\partial}{\partial u_{2x}} + \xi_{3x} \frac{\partial}{\partial u_{3x}} + \xi_t \frac{\partial}{\partial u_t}. \tag{8}\]

System of PDEs for the components of the tangent vector field is:

\[\frac{\partial^2 \xi_t}{\partial t^2} = \frac{\partial \xi_t}{\partial x} = \frac{\partial \xi_t}{\partial u} = \frac{\partial \xi_x}{\partial x} = \frac{\partial \xi_x}{\partial u} = 0, \quad \frac{\partial \xi_x}{\partial t} = -\frac{\partial \xi_t}{\partial t}, \quad \eta = -\frac{\partial \xi_t}{\partial t}(u + 1). \tag{9}\]

From (9) we get:

\[\xi_t = C_1 t + C_2, \quad \xi_x = -C_1 t + C_3, \quad \eta_u = -C_1 u - C_1, \tag{10}\]

and the following infinitesimal generator:

\[X = (C_1 t + C_2) \frac{\partial}{\partial t} + (-C_1 t + C_3) \frac{\partial}{\partial x} - C_1 (u + 1) \frac{\partial}{\partial u}. \tag{11}\]
Without loss of generality, we can take $C_1 = 1$, $C_2 = C_3 = 0$ in (11) and find the nontrivial transformations for (7):

$$x' = x + t(1 - e^a), \quad t' = te^a, \quad u' = (u + 1)e^{-a} - 1.$$

(12)

The following invariants correspond to this group:

$$I_1 = x + t, \quad I_2 = tu - x.$$

(13)

Thus we can search for the self-similar solution of equation (7) in the form:

$$u = \frac{f(\theta) + x}{t}, \quad \theta = t + x.$$

(14)

Substituting (14) into (7), we get ordinary differential equation for $f(\theta)$:

$$\theta f' + f f' + \delta^2 \theta f''' + 2\delta^2 f' (1 + f') + \delta^2 f''' = 0 \ (15)$$

Equation (15) is not an autonomous. Its infinitesimal generator has the form:

$$X = C_1 \frac{\partial}{\partial \theta} - C_1 \frac{\partial}{\partial f}.$$

(16)

Its corresponds to the shift transformation group:

$$\theta' = \theta + a, \quad f' = f - a.$$

(17)

However, we can not find any new solutions, using transformations (17).

Obvious solution for equation (15) is $f = c$, where $c$ is a constant. We can use it to generate solution for equation (7) through transformation (14) in the form:

$$u = \frac{c + x}{t}.$$

(18)

It is a self-similar solution for equation (7).

3. Symmetry test of the seventh-order differential equation for the description of the $\alpha$ FPU model

If we take into account more terms in the Taylor series, then we can derive the seventh-order partial differential equation for the description of the Fermi – Pasta – Ulam model [3]. It has the form:

$$u_t + uu_x + \delta^2 u_{3x} + 2\delta^2 u_x u_{2x} + \delta^2 uu_{3x} + \frac{2}{5}\delta^4 u_{5x} + 2\delta^4 u_{2x}u_{3x} + \frac{6}{5}\delta^4 u_xu_{4x} + \frac{3}{35}\delta^6 u_{7x} = 0. \ (19)$$

As equation (1), equation (19) admits only trivial groups of transformations. If we take into account only members up to $\delta^3$, then equation (19) takes the form:

$$u_t + uu_x + \delta^2 u_{3x} + 2\delta^2 u_x u_{2x} + \delta^2 uu_{3x} + \frac{2}{5}\delta^4 u_{5x} + 2\delta^4 u_{2x}u_{3x} + \frac{6}{5}\delta^4 u_xu_{4x} = 0. \ (20)$$

Equation (20) has nontrivial set of infinitesimals:

$$\xi_x = -t, \quad \xi_t = t, \quad \eta = -u - 1.$$

(21)
which corresponds to the generator:

\[ X = t \frac{\partial}{\partial t} - t \frac{\partial}{\partial x} - (u + 1) \frac{\partial}{\partial u}, \]  

(22)

and the Lie group:

\[ x' = x + t(1 - e^a), \quad t' = te^a, \quad u' = (u + 1) e^{-a} - 1. \]  

(23)

Invariant transformation for equation (20) has the form:

\[ u = f(\theta) + \frac{x}{t}, \quad \theta = x + t. \]  

(24)

Transformation (24) is similar to (14). This fact can be explained as equation (19) is an extension of equation (1). Using (24) we can transform equation (20) to the ordinary differential equation:

\[ f'(\theta + f) + 2\delta^2 (1 + f') f'' + 6\delta^4 (1 + f) f'^V + 2\delta^4 (\theta + f) f^V = 0. \]  

(25)

Equation (25) admits only trivial transformations. We can find a solution for equation (25) in the form

\[ u = c, \]  

where \( c \) is an arbitrary constant. Using (24) we get:

\[ u = \frac{c + x}{t}. \]  

(26)

We see that an self-similar solution for equation (20) is similar to the solution for equation (7).

4. Symmetry test of the sixth-order differential equation for the description of the \( \alpha \) FPU model

If we study two waves, propagating to the different sides in the \( \alpha \) or \( \alpha + \beta \) Fermi – Pasta – Ulam model to the different sides, we can get the sixth-order differential equation [2, 6]:

\[ \frac{d^2 y}{dt^2} = y_{xx} + y_{xx} y_{xx} - \mu y_x^2 y_{xx} + \delta y_{xxxx} + 2\delta y_{xx} y_{xxx} + \delta y_{x} y_{xxxx} - 4\mu \delta y_{x} y_{xx} y_{xxx} - \mu \delta y_{xx}^3 - \mu \delta y_{x}^2 y_{xxx} + 2\delta y_{xxxx} + \frac{2}{5} \delta^2 y_{xxxxxx}. \]  

(27)

We differentiate both parts of (27) with respect to \( x \) and substitute \( y_x = u \):

\[ \frac{d^2 u}{dt^2} = u_{xx} + (u u_x)_x - \mu (u^2 u_x)_x - \mu u_{xxxx} + 2\delta (u_x u_{xx})_x + \delta (u u_{xxx})_x - 4\mu \delta (u u_x u_{xx})_x - \mu \delta (u_x^3)_x - \mu \delta (u^2 u_{xxx})_x + \frac{2}{5} \delta^2 u_{xxxxxx}. \]  

(28)

Equation (28) at any values of \( \mu \) admits only trivial groups of transformations. We take \( \mu = 0 \) and take into account only members up to \( \delta^2 \). As the result, equation (28) takes the form:

\[ \frac{d^2 u}{dt^2} = u_{xx} + (u u_x)_x + \delta u_{xxxx} + 2\delta (u_x u_{xx})_x + \delta (u u_{xxx})_x. \]  

(29)

Equation (29) admits nontrivial set of infinitesimals:

\[ \xi_x = 0, \quad \xi_t = t, \quad \eta_u = -2u - 2. \]  

(30)
It corresponds to the infinitesimal generator:

\[ X = t \frac{\partial}{\partial t} - 2(u + 1) \frac{\partial}{\partial u}, \]  

\[ (31) \]

and transformation group:

\[ x' = x, \quad t' = te^a, \quad u' = (u + 1)e^{-2a} - 1. \]  

\[ (32) \]

We have the following invariants:

\[ I_1 = x, \quad I_2 = t^2(u + 1). \]  

\[ (33) \]

Thus we can search for the self-similar solution of equation (29) in the form

\[ u = \frac{v(x)}{t^2} - 1. \]  

\[ (34) \]

Using (34) we transform equation (29) to the ordinary differential equation for \( v(x) \):

\[ -6v + (vv')' + 2\delta(v'v'')' + \delta(vv''')' = 0. \]  

\[ (35) \]

We search for the particular solution of equation (35) in the form:

\[ v = A + Bx + Cx^2. \]  

\[ (36) \]

where \( A, B, C \) are some constants. We substitute (36) into (35) and determine coefficients:

\[ \frac{1}{4}B^2 + 2\delta, \quad C = 1. \]  

\[ (37) \]

Then we use (34) to get the solution for equation (29):

\[ u = \frac{4x^2 + Bx + B^2 + 2\delta}{4t^2} - 1. \]  

\[ (38) \]

5. Symmetry test of the fifth-order differential equation for the description of the \( \alpha + \beta \) FPU model

If we consider \( \alpha + \beta \) Fermi – Pasta – Ulam problem, we can derive the following fifth-order partial differential equation [2]:

\[ u_t + uu_x - \mu u^2 u_x + \delta^2 u_{xxx} + 2\delta^2 u_x u_{xx} + \delta^2 u_{xxxx} - \\
4 \delta^2 \mu u_x u_{xx} - \delta^2 \mu u_x^2 - \delta^2 \mu u^2 u_{xx} + \frac{2}{5} \delta^4 u_{xxxxx} = 0. \]  

\[ (39) \]

The prolongation of the infinitesimal generator for equation (39) has the form:

\[ X_5 = X + \xi_1 \frac{\partial}{\partial u_x} + \xi_{11} \frac{\partial}{\partial u_{xx}} + \xi_{111} \frac{\partial}{\partial u_{xxx}} + \xi_{1111} \frac{\partial}{\partial u_{xxxx}} + \xi_{11111} \frac{\partial}{\partial u_{xxxxx}} + \xi_2 \frac{\partial}{\partial u_t}. \]  

\[ (40) \]

We get the system of partial differential equations:

\[ \frac{\partial \xi_t}{\partial t} = \frac{\partial \xi_t}{\partial x} = \frac{\partial \xi_t}{\partial u} = \frac{\partial \xi_x}{\partial t} = \frac{\partial \xi_x}{\partial x} = \frac{\partial \xi_x}{\partial u} = \eta_u = 0. \]  

\[ (41) \]

Then we solve (41) and get components of the tangent vector field:

\[ \xi_t = C_1, \quad \xi_x = C_2, \quad \eta_u = 0. \]  

\[ (42) \]

Infinitesimal generator, admitted by equation (39) is:

\[ X = C_1 \frac{\partial}{\partial t} + C_2 \frac{\partial}{\partial x}. \]  

\[ (43) \]

We see from (43) that equation (39) admits only shift transformations and has invariant solution in the form of the traveling wave \( u(x, t) = v(z), z = x - C_0 t \). These solutions have already been investigated in [2].
6. Conclusion
Nonlinear partial differential equations for the description of the Fermi – Pasta – Ulam model have been investigated using the symmetry analysis. We have found components of the tangent vector field, and groups, admitted by the equations. We have found the self-similar variables and the reductions to the ordinary differential equations. Using exact solutions for the ordinary differential equations, we have constructed new self-similar solutions for the investigated equations.

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