Why Newton’s gravity is practically reliable in the large-scale cosmological simulations

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(November 11, 2018)

Until now, it has been common to use Newton’s gravity to study the non-linear clustering properties of the large-scale structures. Without confirmation from Einstein’s theory, however, it has been unclear whether we can rely on the analysis, for example, near the horizon scale. In this work we will provide a confirmation of using Newton’s gravity in cosmology based on relativistic analysis of weakly non-linear situations to the third order in perturbations. We will show that, except for the gravitational wave contribution, the relativistic zero-pressure fluid equations perturbed to the second order in a flat Friedmann background coincide exactly with the Newtonian results. We will also present the pure relativistic correction terms appearing in the third order. The third-order correction terms show that these are the linear-order curvature perturbation strength higher than the second-order relativistic/Newtonian terms. Thus, the pure general relativistic corrections in the third order are independent of the horizon scale and are small in the large-scale due to the low-level temperature anisotropy of the cosmic microwave background radiation. Since we include the cosmological constant, our results are relevant to currently favoured cosmology. As we prove that the Newtonian hydrodynamic equations are valid in all cosmological scales to the second order, and that the third-order correction terms are small, our result has a practically important implication that one can now use the large-scale Newtonian numerical simulation more reliably as the simulation scale approaches and even goes beyond the horizon.

I. INTRODUCTION

In order to interpret results from Einstein’s gravity theory properly we often need corresponding results in Newton’s theory. On the other hand, in order to use results from Newton’s gravity theory reliably we need confirmation from Einstein’s theory. The observed large-scale structures show non-linear processes are working; see the 2dF galaxy redshift survey final data release (Colless et al. 2003) and the SDSS data release 3 (Abazajian et al. 2004). Currently, studies of such structures are mainly based on Newtonian physics in both analytical and numerical approaches (for reviews, see Sahni & Coles 1995; Bertschinger 1998; Bernardeau et al. 2002; Cooray & Sheth 2002). One may admit its incompleteness as the simulation scale becomes bigger because, first, Newton’s gravity is an action-at-a-distance, i.e., the gravitational influence propagates instantaneously thus violating causality. And, second, Newton’s theory is ignorant of the presence of horizon where the relativistic effects are supposed to dominate: near horizon we have $GM/\lambda c^2 \sim \lambda^2/H^2 \sim 1$ where $\lambda H \sim c/H$ is the dynamic horizon scale with $H$ Hubble’s constant. One other reason we may add is that Einstein’s gravity apparently has quite different structure from Newton’s one. The causality of gravitational interactions and consequent presence of the horizon in cosmology are naturally taken into account in the relativistic gravity theories where Einstein’s gravity is the prime example. In this work we will present the similarity and difference between the two gravity theories in the weakly non-linear regimes in cosmological situation.

In the literature, however, independently of such possible shortcomings of Newton’s gravity in the cosmological situation, the sizes of Newtonian simulations, in fact, have already reached Hubble horizon scale (Colberg et al. 2000; Jenkins et al. 2002; Evrard et al. 2002; Bode & Ostriker 2003; Dubinski et al. 2003; Park et al. 2005, in preparation). Common excuses often made by people working in this active field of large-scale numerical simulation are, first in the small scale one may rely on Newton’s theory and, second, as the scale becomes large the large-scale distribution of galaxies looks linear in which case Einstein’s gravity gives the same result as the Newtonian one. In such a situation, in order to have proper confirmation, we still need to investigate Einstein’s case in the non-linear or weakly non-linear situations. While the general relativistic cosmological simulation is currently not available, in this work, we will shed light on the situation by a perturbative study of the non-linear regimes in Einstein’s gravity. We will show that even to the second order in perturbations, except for coupling to gravitational waves, Einstein’s gravity gives the same equations known in Newton’s theory and the pure relativistic corrections appearing in the third-order perturbations are independent of the horizon and are small. Thus, now our relativistic analysis assures that Newton’s gravity is practically reliable even in the weakly non-linear regimes in cosmology. Such a comforting conclusion comes from
a thorough relativistic analysis of the weakly non-linear situations to the third order in perturbations. Despite our simply expressed final conclusion the results still look surprising and important. We set \( c = 1 \).

II. FULLY NON-LINEAR EQUATIONS AND PERTURBATIONS

We start from the completely non-linear and covariant \((1 + 3)\) equations (Ehlers 1961; Ellis 1971, 1973). We need the energy conservation equation and the Raychaudhury equation. For a zero-pressure medium in the energy frame, we have (Noh \& Hwang 2004)

\[
\ddot{\mu} + \mu \ddot{\theta} = 0, \tag{1}
\]

\[
\ddot{\theta} + \frac{1}{3} \dot{\theta}^2 + \ddot{\sigma}_{ab} - \ddot{\omega}_{ab} + 4\pi G \mu - \Lambda = 0, \tag{2}
\]

where \( \Lambda \) is the cosmological constant; \( \dot{\theta} \equiv \dot{u}_a^a \) is the expansion scalar, and \( \dot{\sigma}_{ab} \) and \( \dot{\omega}_{ab} \) are the shear and the rotation tensors. Tildes indicate the covariant quantities based on the spacetime metric \( \tilde{g}_{ab} \). We have \( \ddot{\mu} \equiv \ddot{\mu}_a \tilde{u}^a \) and \( \ddot{\theta} \equiv \ddot{\theta}_a \tilde{u}^a \) which are the covariant derivatives along \( \tilde{u}^a \). From these we have

\[
\left( \frac{\ddot{\mu}}{\ddot{\theta}} \right)^2 - \frac{1}{3} \left( \frac{\ddot{\theta}}{\ddot{\mu}} \right)^2 - \dot{\sigma}_{ab} - \dot{\omega}_{ab} + 4\pi G \ddot{\mu} + \Lambda = 0. \tag{3}
\]

Equations (1)-(3) are valid to all orders, i.e., these equations are fully non-linear and still covariant. Equations (1)-(3) are not complete: to the second and higher order perturbations we will need other equations in Einstein’s theory.

We consider the scalar- and tensor-type perturbations in the Friedmann background without pressure; we ignore the vector-type perturbation (rotation) because it always decays in the expanding phase even to the second order (Noh \& Hwang 2004). As the metric we take

\[
ds^2 = -(1 + 2\alpha) \, dt^2 - 2a_{\beta \alpha} \, dt \, dx^\alpha + a^2 \left[ \theta_{(3)}^{(1)}(1 + 2\varphi) + 2\gamma_{\alpha \beta} + 2C_{(t)}^{(1)} \right] \, dx^\alpha \, dx^\beta, \tag{4}
\]

where \( a(t) \) is the scale factor; \( \alpha, \beta, \gamma \) and \( \varphi \) are spacetime dependent scalar-type perturbed-order variables; \( C_{(t)}^{(1)} \) is the transverse and tracefree tensor-type metric perturbation (gravitational waves). We take the metric convention in Bardeen (1988) extended to the third order (Noh \& Hwang 2004). The Greek and Latin indices indicate the space and spacetime indices, respectively; the spatial indices of perturbed order variables are raised and lowered by \( \delta_{\alpha \beta} \) which becomes \( \delta_{\alpha \beta} \) if we take Cartesian coordinates in the flat Friedmann background. A vertical bar indicates a covariant derivative based on \( \delta_{\alpha \beta} \). We will take \( \varphi \equiv 0 \) as the spatial gauge condition which makes all the remaining variables spatially gauge-invariant to the all orders in perturbations (Noh \& Hwang 2004).

The fluid quantities are ordinarily defined based on the fluid four-vector \( \tilde{u}_a \) in the energy-frame. Our comoving gauge condition takes vanishing flux \( \tilde{q}_a \equiv 0 \) (the energy-frame), and \( \tilde{u}_a \equiv 0 \) for the fluid four-vector; here, we ignored the vector-type perturbation. Thus, the fluid four-vector coincides with the normal-frame four-vector \( \tilde{n}_a \) which has \( \tilde{n}_a \equiv 0 \). The condition \( \tilde{u}_a = 0 \) implies vanishing rotation tensor \( \tilde{\omega}_{ab} = 0 \). We lose no generality by imposing the gauge condition. Since the comoving gauge condition fixes the temporal gauge mode completely the remaining variables under this gauge condition are equivariant gauge-invariant; this is the case to all orders in perturbations (Noh \& Hwang 2004). In our gauge condition the energy-momentum tensor becomes

\[
\tilde{T}^0_0 = -\tilde{\mu}, \quad \tilde{T}^0_\alpha = 0 = \tilde{T}_\beta^\alpha, \tag{5}
\]

where \( \tilde{\mu} \) is the energy density.

III. BACKGROUND AND LINEAR PERTURBATIONS

To the background order, we have \( \tilde{\mu} = \mu \) and \( \tilde{\theta} = \frac{3H}{\Omega} \) where an overdot indicates a time derivative based on \( t \). Equations (1) and (2) give
\[
\dot{\mu} + 3\frac{\dot{a}}{a}\mu = 0,
\]
\[\tag{6}
3\frac{\ddot{a}}{a} + 4\pi G\mu - \Lambda = 0.
\]

Combining these equations we have the Friedmann equation
\[
\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\mu - \text{const.} + \frac{\Lambda}{3},
\]
\[\tag{8}
\]
with \(\mu \propto a^{-3}\). This equation was first derived based on Einstein’s gravity by Friedmann (1922, 1924) and Robertson (1929), and Newtonian study followed later by Milne (1934) and McCrea & Milne (1934). In the Newtonian context the relativistic energy density \(\mu\) can be identified with the mass density \(\rho\). The “const.” part is interpreted as the spatial curvature \((K)\) in Einstein’s gravity (Friedmann 1922, 1924) and the total energy in Newton’s gravity (McCrea & Milne 1934).

To the linear-order perturbations in the metric and energy-momentum variables, we introduce
\[
\dot{\mu} \equiv \mu + \delta\mu, \quad \dot{\theta} \equiv 3\frac{\dot{a}}{a} + \delta\theta,
\]
\[\tag{9}
\]
where \(\mu\) and \(\delta\mu\) are the background and perturbed energy density, respectively, and \(\delta\theta\) is the perturbed part of expansion scalar. We emphasize that our spatial \(\gamma = 0\) gauge and temporal comoving gauge conditions defined above eq. (5) fix the spatial and temporal gauge degrees of freedom completely. Thus, all variables in these gauge conditions are equivalently gauge invariant to the linear order: i.e., each of the variables has a unique corresponding gauge-invariant combination (Bardeen 1988; Hwang 1991). It is important to point out that the above two statements are valid even in the second and all higher order perturbations, see §VI in Noh & Hwang (2004). To the background order we already identified \(\mu \equiv \rho\). Now, to the linear order we identify
\[
\delta\mu \equiv \delta\rho, \quad \delta\theta \equiv \frac{1}{a}\nabla \cdot \mathbf{u}.
\]
\[\tag{10}
\]
To the linear order the perturbed parts of eqs. (1) and (2) give
\[
\dot{\delta} + \frac{1}{a}\nabla \cdot \mathbf{u} = 0,
\]
\[\tag{11}
\]
\[
\frac{1}{a}\nabla \cdot \left( \mathbf{u} + \frac{\dot{a}}{a}\mathbf{u} \right) + 4\pi G\mu\delta = 0.
\]
\[\tag{12}
\]
Combining these equations we have
\[
\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = 0,
\]
\[\tag{13}
\]
which is the well known density perturbation equation in both relativistic and Newtonian contexts; we set \(\delta \equiv \delta\mu/\mu\). This equation was first derived based on Einstein’s gravity by Lifshitz (1946), and Newtonian study followed later by Bonnor (1957). Notice that the relativistic result is identical to the Newtonian result. The gravitational wave perturbation present in the relativistic theory simply decouples from the density perturbation and follows the wave equation (Lifshitz 1946)
\[
\ddot{C}_{\alpha\beta}^{(t)} + 3\frac{\dot{a}}{a}C_{\alpha\beta}^{(t)} - \frac{\Delta - 2K}{a^2}C_{\alpha\beta}^{(t)} = 0,
\]
\[\tag{14}
\]
where \(K\) is the sign of the background spatial curvature.

It is curious to notice that in both the expanding world model and its linear structures the first studies were made in the context of Einstein’s gravity (Friedmann 1922; Lifshitz 1946), and the much simpler and, in hindsight, more intuitive Newtonian studies followed later (Milne 1934; Bonnor 1957). Perhaps these historical developments reflect that people did not have confidence in using Newton’s gravity in cosmology before the result was already known in, and the method was ushered by, Einstein’s gravity. This is also reflected in the historical development of modern cosmology which began only after the advent of Einstein’s gravity theory (Einstein 1917). Furthermore, it is known in the literature that the results in Newtonian cosmology are, in fact, guided ones by previously known relativistic results; i.e., without the guidance of the relativistic analyses Newtonian theory could have lead to other results (Layzer
1954; Lemons 1988). It may be also true that only after having a Newtonian counterpart we could understand better what the often arcane relativistic analysis shows. For the second-order perturbations, however, the history is different from the two previous cases. Currently we only have the Newtonian result known in the literature. Thus, the result only known in Newton’s gravity still awaits confirmation from Einstein’s theory. Here, we are going to fill the gap by presenting the much needed relativistic confirmation to the second order and the pure relativistic corrections start appearing from the third order.

Although eq. (13) is also valid with general spatial curvature, in the following we consider the flat background only. As we include the cosmological constant $\Lambda$, however, our zero-pressure background and perturbations describe remarkably well the current expanding stage of our universe and its large-scale structures (Spergel et al. 2003; Tegmark, et al. 2004), which are believed to be in the near linear stage. In the small scale, however, the structures are apparently in non-linear stage, and even in the large scale weakly non-linear study is needed. Until now, such a weakly non-linear stage has been studied in Newton’s gravity only. In the following we plan to investigate whether such a usage of Newtonian gravity in handling the large-scale structure can be justified in the relativistic standpoint by studying the relativistic behaviours of higher-order perturbations.

IV. SECOND-ORDER PERTURBATIONS AND NEWTONIAN CORRESPONDENCE

Now, we consider equations perturbed to the second order in the metric and the energy-momentum tensor. Even to the second order we introduce perturbations as in eq. (9) which are always allowed. We will also take the same identifications made in eq. (10); this point will be justified by our results soon. To the second order the perturbed parts of eqs. (1) and (2) give (Hwang & Noh 2005a)

$$\dot{\delta} + \frac{1}{a} \nabla \cdot \mathbf{u} = -\frac{1}{a} \nabla \cdot (\delta \mathbf{u}),$$  (15)

$$\frac{1}{a} \nabla \cdot (\dot{\mathbf{u}} + \frac{\dot{\alpha}}{a} \mathbf{u}) + 4\pi G\mu \delta = -\frac{1}{a^2} \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) - \dot{C}_{\alpha\beta}^{(t)} \left( \frac{2}{a} \nabla_{\alpha} u_{\beta} + \dot{C}_{\alpha\beta}^{(t)} \right),$$  (16)

where the gravitational wave part comes from the shear term in eq. (2) (Noh & Hwang 2004, 2005; Hwang & Noh 2005a) and it follows eq. (14); in order to derive these equations we also used the $G^0_\alpha$ component (momentum constraint) of Einstein’s field equations. By combining these equations we have

$$\ddot{\delta} + 2\frac{\dot{\alpha}}{a} \delta - 4\pi G\mu \delta = -\frac{1}{a^2} \frac{\partial}{\partial t} [a \nabla \cdot (\delta \mathbf{u})] + \frac{1}{a^2} \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) + \dot{C}_{\alpha\beta}^{(t)} \left( \frac{2}{a} \nabla_{\alpha} u_{\beta} + \dot{C}_{\alpha\beta}^{(t)} \right),$$  (17)

which also follows from eq. (3). Equations (15)-(17) are our extension of eqs. (11)-(13) to the second-order perturbations in Einstein’s theory. We will show that, except for gravitational waves, exactly the same equations also follow from Newton’s theory. The presence of the gravitational waves, however, can be regarded as one of the truly relativistic effects of gravitation.

In the Newtonian context, the mass and the momentum conservations, and Poisson’s equation give (Peebles 1980)

$$\dot{\delta} + \frac{1}{a} \nabla \cdot \mathbf{u} = -\frac{1}{a} \nabla \cdot (\delta \mathbf{u}),$$  (18)

$$\dot{\mathbf{u}} + \frac{\dot{\alpha}}{a} \mathbf{u} + \frac{1}{a} \nabla \delta \Phi = -\frac{1}{a} \mathbf{u} \cdot \nabla \mathbf{u},$$  (19)

$$\frac{1}{a^2} \nabla^2 \delta \Phi = 4\pi G\rho \delta,$$  (20)

where $\delta \Phi$ is the perturbed gravitational potential, $\mathbf{u}$ is the perturbed velocity, and $\delta \equiv \delta \rho / \rho$. Equation (18) is the same as eq. (18), and eq. (16) ignoring gravitational waves follows from eqs. (19) and (20). Thus, eq. (17) also naturally follows in Newton’s theory (Peebles 1980). This shows the exact relativistic-Newtonian correspondence to the second order, except for the gravitational wave contribution which is a pure relativistic effect. This also justifies our identifications made in eq. (10) to the second order. Although we identified the relativistic density and velocity perturbation variables we cannot identify a relativistic variable which corresponds to $\delta \Phi$ to the second order (Hwang & Noh 2005a). We believe this can be understood naturally because Poisson’s equation indeed reveals the action-at-a-distance nature and the static nature of Newton’s gravity theory compared with Einstein’s gravity (Fock 1964; Rindler 1977). Poisson’s equation was formulated in 1812 which was 125 years after the publication of Newton’s Principia in 1687. Notice that eqs. (18)-(20) are valid to fully non-linear order. In our relativistic case, however, eqs. (15)-(17) are valid only to the second order in perturbations.
V. THIRD-ORDER PERTURBATIONS AND PURE RELATIVISTIC CORRECTIONS

Since the zero-pressure Newtonian system is exact to the second order in non-linearity, all non-vanishing third and higher order perturbation terms in the relativistic analysis can be regarded as the pure relativistic corrections. Thus, we have a clear reason to go to the third order which was not previously attempted. For simplicity we ignore gravitational wave contribution; for a complete presentation, see Hwang & Noh (2005b). Based on our success in the second-order perturbations, we continue identifying eq. (10) is valid even to the third order and will take the consequent additional third-order terms as the pure relativistic corrections. To the third order the perturbed parts of eqs. (1) and (2) give (Hwang & Noh 2005b):

\[
\dot{\delta} + \frac{1}{a} \nabla \cdot \mathbf{u} = -\frac{1}{a} \nabla \cdot (\delta \mathbf{u}) + \frac{1}{a^2} \left[2 \varphi \mathbf{u} - \nabla (\Delta^{-1}X)\right] \cdot \nabla \delta,
\]

(21)

\[
\frac{1}{a} \nabla \cdot \left(\dot{\mathbf{u}} + \frac{\dot{a}}{a} \mathbf{u}\right) + 4\pi G\mu \delta = -\frac{1}{a^2} \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) - \frac{2}{3a^2} \varphi \mathbf{u} \cdot \nabla (\nabla \cdot \mathbf{u}) + \frac{4}{a^2} \nabla \cdot \left[\varphi \left(\mathbf{u} \cdot \nabla \mathbf{u} - \frac{1}{3} \mathbf{u} \nabla \cdot \mathbf{u}\right)\right] - \frac{\Delta}{a^2} \left[\mathbf{u} \cdot \nabla (\Delta^{-1}X)\right] + \frac{1}{a^2} \mathbf{u} \cdot \nabla X + \frac{2}{3a^2} X \nabla \cdot \mathbf{u},
\]

(22)

where

\[
X \equiv 2\varphi \nabla \cdot \mathbf{u} - \mathbf{u} \cdot \nabla \varphi + \frac{3}{2} \Delta^{-1} \nabla \cdot [\mathbf{u} \cdot \nabla (\nabla \varphi) + \mathbf{u} \Delta \varphi].
\]

(23)

In order to derive these equations we also used the $\delta^{\alpha}_\alpha$-component of Einstein’s field equations. Equations (21) and (22) extend eqs. (15) and (16) to the third order. By combining eqs. (21) and (22) we can derive

\[
\dot{\delta} + 2\frac{\dot{a}}{a} \delta - 4\pi G\mu \delta = -\frac{1}{a^2} \frac{\partial}{\partial t} \left[a \nabla \cdot (\delta \mathbf{u})\right] + \frac{1}{a^2} \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) + \frac{1}{a^2} \frac{\partial}{\partial t} \left\{a \left[2 \varphi \mathbf{u} - \nabla (\Delta^{-1}X)\right] \cdot \nabla \delta\right\} + \frac{2}{3a^2} \varphi \mathbf{u} \cdot \nabla (\nabla \cdot \mathbf{u}) - \frac{4}{a^2} \nabla \cdot \left[\varphi \left(\mathbf{u} \cdot \nabla \mathbf{u} - \frac{1}{3} \mathbf{u} \nabla \cdot \mathbf{u}\right)\right] + \frac{\Delta}{a^2} \left[\mathbf{u} \cdot \nabla (\Delta^{-1}X)\right] - \frac{1}{a^2} \mathbf{u} \cdot \nabla X - \frac{2}{3a^2} X \nabla \cdot \mathbf{u},
\]

(24)

which extends eq. (17) to the third order. The last two lines of eq. (24) are pure third-order terms. The variable $\varphi$ is a perturbed-order metric variable in eq. (4) in our comoving gauge condition.

The third-order correction terms in eqs. (21)-(24) reveal that all of them are simply of $\varphi$-order higher than the second-order terms. Thus, the pure general relativistic effects are at least $\varphi$-order higher than the relativistic/Newtonian ones in the second order. Our $\varphi$ is related to the perturbed three-space curvature (in our comoving gauge) and dimensionless (Bardeen 1980). As we mentioned earlier, $\varphi$ in the comoving gauge is the same as a unique gauge-invariant combination. To the linear order such a combination was first introduced by Field & Shepley (1968). For an explicit form of the combination to the second order, see eq. (281) in Noh & Hwang (2004). Notice that we only need the behavior of $\varphi$ to the linear order. To the linear order, in terms of known Newtonian variables we have (Hwang & Noh 2005b)

\[
\varphi = -\delta \Phi + \dot{a} \Delta^{-1} \nabla \cdot \mathbf{u},
\]

(25)

and it satisfies (Hwang & Noh 1999a)

\[
\dot{\varphi} = 0,
\]

(26)

thus $\varphi = C(x)$ with no decaying mode; this is true considering the presence of the cosmological constant. For $\Lambda = 0$, the temperature anisotropy of cosmic microwave background radiation gives (Sachs & Wolfe 1967; Hwang & Noh 1999b)

\[
\frac{\delta T}{T} \sim \frac{1}{3} \frac{\delta \Phi}{\Phi} \sim \frac{1}{5} \varphi.
\]

(27)
The observations of cosmic microwave background radiation give $\delta T/T \sim 10^{-5}$ (Smoot et al. 1992; Spergel, et al. 2003), thus

$$\varphi \sim 5 \times 10^{-5},$$

in the large-scale limit near the horizon scale where $GM/(\lambda c^2) \sim \lambda^2/\lambda_H^2$ approaches unity. Therefore, to the third order, the pure relativistic corrections are independent of the horizon scale and depend on the linear-order curvature $\varphi$ ($\sim$ gravitational potential $\delta \Phi$) perturbation strength only, and are small. That is, compared with the second-order terms, the third-order correction terms in eqs. (21)-(23) only involve $\varphi$, and do not contain terms like $(aH)^{-1}\nabla \varphi$, etc.

**VI. DISCUSSION**

We have shown that to the second order, except for the gravitational wave contribution, the zero-pressure relativistic cosmological perturbation equations can be exactly identified with the known equations in Newton’s theory. As a consequence, to the second order, we identified correct relativistic variables which can be interpreted as density $\delta \mu$ and velocity $\delta \theta$ perturbations in eq. (10), and we showed that to the second order the Newtonian hydrodynamic equations remain valid in all cosmological scales including the super-horizon scale. It might as well happen that our relativistic results give relativistic correction terms appearing to the second order which become important as we approach and go beyond the horizon scale which are strongly relativistic regimes. Our results show that there are no such correction terms appearing to the second order, and except for gravitational waves, the correspondence is exact to that order. Ignoring gravitational waves, the pure relativistic correction terms, however, start appearing from the third order. Our study shows that to the third order the correction terms only involve $\varphi$ which is again independent of the the horizon scale and is small in the large scale.

In the non-linear clustered regions we may have $\varphi \sim \delta \Phi \sim GM/(Rc^2)$ where $M$ and $R$ are characteristic mass and length scales involved. In such clustered regimes the post-Newtonian approximation would complement our non-linear perturbation approach presented here. The post-Newtonian approximation takes $v/c \ll 1$, and gravity is weak $GM/(Rc^2) \ll 1$. A complementary result, showing the relativistic-Newtonian correspondence in the Newtonian limit of the post-Newtonian approach, can be found in Kofman & Pogosyan (1995), (see also Bertschinger & Hamilton 1994; Ellis & Dunsby 1997). In fact, the Newtonian hydrodynamic equations naturally appear in the zeroth-order post-Newtonian approximation (Chandrasekhar 1965). Recently, we presented the fully nonlinear cosmological hydrodynamic equations with first-order post-Newtonian correction terms (Hwang, Noh & Puetzfeld 2005); we showed that these correction terms have typically $GM/(Rc^2) \sim v^2/c^2 \sim 10^{-5}$ order smaller than the Newtonian terms in the non-linearly clustered regions.

Therefore, our general relativistic results allow us to draw the following important practical conclusion which is stated in our title. As we prove that the Newtonian hydrodynamic equations are valid on all cosmological scales to the second order, and that the third-order pure relativistic correction terms are small and independent of the horizon, one can now use the large-scale Newtonian numerical simulation more reliably as the simulation scale approaches and even goes beyond the horizon. The fluctuations near the horizon scale are supposed to be linear or weakly nonlinear; otherwise, it is difficult to introduce the spatially homogeneous and isotropic background world model which is the basic assumption of the modern cosmology. In the small-scale but fully non-linear stage, the post-Newtonian approximation also shows that the relativistic correction terms are small, thus the Newtonian simulations can be trusted again. The sub-horizon scale Newtonian non-linear inhomogeneities are not supposed to affect the homogeneous and isotropic background world model (Siegel & Fry 2005). The other side of this conclusion is that it might be difficult to find testable signatures of Einstein’s gravity theory based on such large-scale weakly non-linear structures (with relativistic corrections) or small-scale fully non-linear structures (with post-Newtonian corrections). However, it would be interesting to find cosmological situations where the pure relativistic correction terms in eqs. (21)-(24) or the first-order post-Newtonian corrections terms derived in Hwang, Noh & Puetzfeld (2005) could have observationally distinguishable consequences. Since our equations include the cosmological constant our equations and conclusions are relevant to the currently favoured world models.

In our relativistic-Newtonian correspondence to the second order, the relativistic equations are identified with the continuity equation and the divergence of the Euler equation replacing the Newtonian gravitational potential using Poisson’s equation. It is important to be remember that we showed the relativistic-Newtonian correspondence for the density and velocity perturbations, but not for the gravitational potential. Therefore, although our result assures that one can trust cold dark matter simulations at all scales for the density and velocity fields, it does not imply that one can trust the Newtonian simulations for effects involving the gravitational potential, like the weak gravitational lensing effects. In order to handle the lensing effects properly we often require an extra factor of two which, indeed, comes from the post-Newtonian effects.
Our relativistic-Newtonian correspondence to the second order perturbation assuming a single component zero-pressure irrotational fluid in the flat cosmological background. Dropping any of these conditions could potentially lead to relativistic corrections. The genuine relativistic correction terms appear as we consider the gravitational waves to the second order. We showed that pure relativistic correction terms in the scalar-type perturbation appear in the third order; we showed that these correction terms do not involve the horizon scale and are small in our observable patch of the universe. Extensions to include the pressure, the rotation, the non-flat background, and the multi-component situation will be investigated in future occasions.

Acknowledgments

HN was supported by grants No. R04-2003-10004-0 from the Basic Research Program of the Korea Science and Engineering Foundation. JH was supported by the Korea Research Foundation Grant No. 2003-015-C00253.

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