BEREZIN REGULARITY OF DOMAINS IN $\mathbb{C}^n$ AND THE ESSENTIAL NORMS OF TOEPLITZ OPERATORS

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Abstract. For the open unit disc $D$ in the complex plane, it is well known that if $\varphi \in C(D)$ then its Berezin transform $\tilde{\varphi}$ also belongs to $C(D)$. We say that $D$ is BC-regular. In this paper we study BC-regularity of more general domains in $\mathbb{C}^n$ and show that the boundary geometry plays an important role. We also establish a relationship between the essential norm of a Toeplitz operator acting on a pseudoconvex domain and the norm of its Berezin transform.

The Berezin transform plays an important role in operator theory, especially in regard to compactness of certain classes of operators. A prime example is a well-known Axler-Zheng theorem [AZ98] that characterizes compactness of Toeplitz operators on the Bergman space of the unit disc. The Axler-Zheng theorem, together with its extension done by Suarez [Suá07], says that an operator $T$ in the Toeplitz algebra is compact if and only if the Berezin transform of $T$, denoted by $\tilde{T}(z)$ (to be defined in the next section) goes to 0 as $z$ approaches the boundary of the unit disc. There were many extensions and versions of this theorem including the recent contribution [ČSZ18] for the case the domain is pseudoconvex in $\mathbb{C}^n$.

Another example of the importance of the Berezin transform is the result by Békkolé-Berger-Coburn-Zhu [BBCZ90] that characterizes compactness of Hankel operators $H_f$ and $H_{\overline{f}}$ in terms of the mean oscillation $(|f|^2 - |\overline{f}|^2)^{1/2}$ near the boundary of the unit ball in $\mathbb{C}^n$, where $\overline{f}$ and $|f|^2$ are the Berezin transforms of $T_f$ and $T_{|f|^2}$, respectively.

In this paper we are interested in Toeplitz operators acting on pseudoconvex domains that are not compact, but we want to find a relationship between their essential norms and the sup norms of their Berezin transforms on the boundary of the domain. This led us to the question of the regularity of the Berezin transform. It is well known that on the unit disc $D$, if the function $\phi \in C(D)$, then $\tilde{\phi} \in C(D)$ and moreover $\tilde{\phi} = \phi$ on $\partial D$. It was shown in [AE01] that the second part of this result is true on more general domains. In particular, they showed that when $\Omega$ is a bounded pseudoconvex domain, then $\lim_{z \to p} \tilde{\phi}(z) = \phi(p)$, when $p$ is strongly pseudoconvex point in $\partial \Omega$ (see also [ČSZ18, Lemma 15]). However the important question now is this: Is $\tilde{\phi} \in C(\Omega)$ whenever $\phi \in C(\Omega)$? The following two related results are worth mentioning. The first is the work by Engliš [Eng07] where he proved that the Berezin transform can have a singularity inside the domain in case the domain is unbounded. The second result is due to Coburn [Cob05] regarding the Lipschitz regularity of the Berezin transform of any bounded linear operator on the Bergman space of a bounded domain in $\mathbb{C}^n$ the Bergman metric.

In this paper, we answer the question of regularity of the Berezin transform on several classes of bounded domains in $\mathbb{C}^n$ and on a subalgebra of the Toeplitz algebra. We show that the answer is
negative for bounded convex domains with discs in the boundary as well as dense strongly pseudo-convex points. However, the answer is positive for products of strongly pseudoconvex domains and for bounded convex domains with no discs in the boundary. Finally, we establish the relationship between the essential norms of operators in certain Toeplitz subalgebra and the sup norm of the Berezin transform on the set of strongly pseudoconvex points.

This paper is organized as follows. In the next section we provide the basic set up and the main results. In section 2 we prove Theorems 4 and 5. In section 3 we construct a smooth bounded convex domain \( \Omega \) in \( \mathbb{C}^2 \) such that \( \| \tilde{T}_\phi \|_{L^\infty(\Omega)} < \| T_\phi \|_e \) for some \( \phi \in C^\infty(\overline{\Omega}) \). Finally in Section 4 we provide the proofs for Theorems 1, 2, and 3.

1. Preliminaries and Results

Let \( \Omega \) be a bounded domain in \( \mathbb{C}^n \). The space of square integrable holomorphic functions on \( \Omega \), denoted by \( A^2(\Omega) \), is called the Bergman space of \( \Omega \). Since \( A^2(\Omega) \) is a closed subspace of \( L^2(\Omega) \) there exists a bounded orthogonal projection \( P : L^2(\Omega) \to A^2(\Omega) \), called the Bergman projection. We denote the set of bounded linear operators on \( A^2(\Omega) \) by \( B(A^2(\Omega)) \). The Toeplitz operator \( T_{\phi} \in B(A^2(\Omega)) \) with symbol \( \phi \in L^\infty(\Omega) \) is defined as
\[
T_{\phi}f = P(\phi f)
\]
for \( f \in A^2(\Omega) \). We will work on the norm closed subalgebra of \( B(A^2(\Omega)) \) generated by \( \{ T_{\phi} : \phi \in C(\overline{\Omega}) \} \) and we will denote it as \( \mathcal{T}(\Omega) \).

Next we define the Berezin transform. Let \( K(\xi, z) \) denote the Bergman kernel of \( \Omega \). We define the normalized kernel as
\[
k_z(\xi) = \frac{K(\xi, z)}{K(z, z)}
\]
for \( \xi, z \in \Omega \). The Berezin transform of a bounded linear operator \( T \) is defined as
\[
\tilde{T}(z) = \langle Tk_z, k_z \rangle
\]
for any \( z \in \Omega \). For \( \phi \in L^\infty(\Omega) \) we define \( \tilde{\phi} = \tilde{T}_\phi \).

Let \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) be the unit disc. It is well known that if \( \varphi \in C(\overline{\mathbb{D}}) \) then its Berezin transform \( \tilde{\varphi} \) also belongs to \( C(\overline{\mathbb{D}}) \). In this paper we want to study conditions that guarantee continuity of the Berezin transform of operators on domains in \( \mathbb{C}^n \). To be more specific, we want to find necessary and/or sufficient conditions on the domain \( \Omega \subset \mathbb{C}^n \) such that \( \tilde{T} \in C(\overline{\Omega}) \) whenever \( T \in \mathcal{T}(\Omega) \).

**Definition 1.** Let \( \Omega \) be a bounded domain in \( \mathbb{C}^n \). We say that \( \Omega \) is BC-regular if \( \tilde{T} \) has a continuous extension onto \( \overline{\Omega} \) (that is, \( \tilde{T} \in C(\overline{\Omega}) \)) whenever \( T \in \mathcal{T}(\overline{\Omega}) \).

In this context we should mention the work of Arazy and Engliš about a weaker version of BC-regularity. More specifically, in [AE01, Theorem 2.3] they proved that if \( \Omega \) is either a bounded domain in \( \mathbb{C} \) with \( C^1 \)-smooth boundary or a strongly pseudoconvex domain in \( \mathbb{C}^n \) with \( C^3 \)-smooth boundary, then \( \tilde{T}_\phi \in C(\overline{\Omega}) \) whenever \( \phi \in C(\overline{\Omega}) \).

The following are the main theorems in our paper. These results show that the geometry of the domain plays an important role in understanding the BC-regularity of the domain.

Theorem 1 gives a positive answer to the question above in case the domain is convex and its boundary does not contain any analytic structure. We note that such domains satisfy property (P) of Catlin or equivalently B-regularity of Sibony (see, [Cat84, Sib87, FS98, Str10]).
We say a set $X \subset \mathbb{C}$ contains a non-trivial analytic disc if there exists a non-constant holomorphic mapping $f : D \to X$.

**Theorem 1.** Let $\Omega$ be a bounded convex domain in $\mathbb{C}^n$. Assume that the boundary of $\Omega$ does not contain any non-trivial analytic disc. Then $\Omega$ is BC-regular.

The next theorem gives a negative answer to the BC-regularity question on bounded convex domains that contain discs in the boundary yet have dense strongly pseudoconvex points.

**Theorem 2.** Let $\Omega$ be a bounded convex domain in $\mathbb{C}^n$ for $n \geq 2$. Assume that the boundary of $\Omega$ contains a non-trivial analytic disc and the set of strongly pseudoconvex points is dense in the boundary of $\Omega$. Then the Berezin transform does not map $C(\overline{\Omega})$ to itself (and hence $\Omega$ is not BC-regular).

BC-regularity makes sense on a large class of domains in $\mathbb{C}^n$ called pseudoconvex domains. These domains include convex domains and are natural homes for holomorphic functions in the sense that for every boundary point $p$ of a pseudoconvex domain $\Omega$ there exists a holomorphic function $f_p$ on $\Omega$ such that $f_p$ has no local holomorphic extension beyond $p$. For $C^2$-smooth domains pseudoconvexity is defined as the Levi form of the domain being positive semi-definite on the boundary. Similarly, strongly pseudoconvex domains are domains whose Levi form is positive definite on the boundary. There is a large literature about these domains and we refer the reader to [CS01, Kra01, Ran86] for information about pseudoconvexity and general questions in several complex variables. Due to an example of Kohn and Nirenberg [KN73] we know that the class of pseudoconvex domains is much larger than the locally convexifiable domains. In the following theorem we show that if the domain is a product of strongly pseudoconvex domains such as the polydisc, then the domain is BC-regular. We note that strongly pseudoconvex domains are not necessarily convex but locally convexifiable (see [Kra01, Lemma 3.2.2]) and are well understood in terms of the $\overline{\partial}$-Neumann problem and the Bergman kernel (see [CS01, Str10]).

**Theorem 3.** Let $\Omega$ be a finite product of $C^2$-smooth bounded strongly pseudoconvex domains. Then $\Omega$ is BC-regular.

The following is an immediate corollary of Theorem 3.

**Corollary 1.** Let $\Omega = \Omega_1 \times \cdots \times \Omega_m$ such that each $\Omega_j$ is either a ball or the unit disc. Then $\Omega$ is BC-regular.

Next we will introduce the $\overline{\partial}$-Neumann problem and its relationship to Hankel operators.

Let $\Box = \overline{\partial}\partial + \partial\overline{\partial} : L^2_{(0,1)}(\Omega) \to L^2_{(0,1)}(\Omega)$ be the complex Laplacian, where $\overline{\partial}$ is the Hilbert space adjoint of $\overline{\partial} : L^2_{(0,1)}(\Omega) \to L^2_{(0,2)}(\Omega)$. When $\Omega$ is a bounded pseudoconvex domain in $\mathbb{C}^n$, Hörmander [Hör65] showed that $\Box$ has a bounded inverse $N$ called the $\overline{\partial}$-Neumann operator. The $\overline{\partial}$-Neumann operator is an important tool in several complex variables that is closely connected to the boundary geometry of the domains. Kohn in [Koh63] showed that the Bergman projection is connected to the $\overline{\partial}$-Neumann operator by the formula $P = I - \overline{\partial}N\overline{\partial}$. We refer the reader to the books [CS01, Str10] for more information about the $\overline{\partial}$-Neumann operator.

Kohn’s formula implies that the Hankel operator $H_\psi$ with symbols $\psi \in C^1(\overline{\Omega})$ satisfies the formula $H_\psi f = \overline{\partial}N(f\overline{\partial}\psi)$ for any $f \in A^2(\Omega)$. Furthermore, one can show that compactness of the $\overline{\partial}$-Neumann operator implies that $H_\psi$ is compact for all $\psi \in C(\overline{\Omega})$ (see, [Str10, Proposition 4.1]). We have used the formula $H_\psi f = \overline{\partial}N(f\overline{\partial}\psi)$ to study compactness of Hankel operators on some domains in $\mathbb{C}^n$ on which $N$ is not necessarily compact (see, for instance, [CS09, ČS17, ČS12, Sah12]).
In this paper we are also interested in the essential norm estimates of operators in $\mathcal{S}(\Omega)$. Recall that if $T$ is a bounded linear map on $A^2(\Omega)$ then its essential norm is defined by

$$\|T\|_e = \inf\{\|T - K\| : K \text{ is compact on } A^2(\Omega)\}.$$  

In this context we mention the work in [ACM82] where the authors were the first to study compactness and the essential norms of Toeplitz operators, with symbols continuous up to the boundary, acting on the Bergman space of a bounded domain in $\mathbb{C}$.

The following theorem establishes the relation between the essential norm of $T$ and the sup-norm of $\tilde{T}$ on the set of strongly pseudoconvex points under the assumption that the $\overline{\partial}$-Neumann operator is compact.

**Theorem 4.** Let $\Omega$ be a $C^\infty$-smooth bounded pseudoconvex domain in $\mathbb{C}^n$ and $\Gamma$ denote the set of strongly pseudoconvex points in $b\Omega$. Assume that the $\overline{\partial}$-Neumann operator is compact on $L^2_{(0,1)}(\Omega)$ and $T \in \mathcal{S}(\overline{\Omega})$. Then $\tilde{T}$ has a continuous extension onto $\Omega \cup \Gamma$ and $\|T\|_e = \|\tilde{T}\|_{L^\infty(\Gamma)}$.

In the absence of compactness of the $\overline{\partial}$-Neumann operator, we use the new essential norm, denoted by $\|.\|_{e_*}$, that measures the distance of the operator to a new class of operators containing Hankel operators with symbols continuous on the closure. This essential norm was introduced in [ČSZ18]. Next we make the relevant definitions.

**Definition 2.** Let $\Omega$ be a $C^2$-smooth bounded pseudoconvex domain in $\mathbb{C}^n$. We say

i. a sequence $\{f_j\} \subset A^2(\Omega)$ converges to $f \in A^2(\Omega)$ weakly about strongly pseudoconvex points if $f_j \to f$ weakly and $\|f_j - f\|_{L^2(\Gamma \cap \Omega)} \to 0$ for some neighborhood $U$ of the set of the weakly pseudoconvex points in $b\Omega$,

ii. a bounded linear operator $T$ on $A^2(\Omega)$ is compact about strongly pseudoconvex points if $Tf_j \to Tf$ in $A^2(\Omega)$ whenever $f_j \to f$ weakly about strongly pseudoconvex points.

Let $K^*$ denote the operators on $A^2(\Omega)$ that are compact about strongly pseudoconvex points and $T : A^2(\Omega) \to A^2(\Omega)$. Then we denote

$$\|T\|_{e_*} = \inf\{\|T - S\| : S \in K^*\}.$$  

**Theorem 5.** Let $\Omega$ be a $C^2$-smooth bounded pseudoconvex domain in $\mathbb{C}^n$ and $\Gamma$ denote the set of strongly pseudoconvex points in $b\Omega$. Assume that $T \in \mathcal{S}(\overline{\Omega})$. Then $\tilde{T}$ has a continuous extension onto $\Omega \cup \Gamma$ and $\|T\|_{e_*} = \|\tilde{T}\|_{L^\infty(\Gamma)}$.

**2. Proofs of Theorems 4 and 5**

We start this section with the proof of Theorem 4.

**Proof of Theorem 4.** Let $\Gamma$ denote the set of strongly pseudoconvex points in $b\Omega$. First we will show that $\tilde{T}$ has a continuous extension onto $\Gamma$. That is, $\tilde{T} \in C(\Omega \cup \Gamma)$. We just need to prove continuity at any point in $\Gamma$ as Berezin transform is real analytic on $\Omega$. Since the $\overline{\partial}$-Neumann operator is compact, Hankel operators with symbols continuous on the closure are compact (see [Str10, Proposition 4.1]). If $T$ is a finite sum of finite products of Toeplitz operators with symbols continuous on $\Omega$, then $T = T_\phi + K$ where $\phi \in C(\overline{\Omega})$ and $K$ is a compact operator (see proof of Theorem 1 in [ČS13]). Hence for $T \in \mathcal{S}(\overline{\Omega})$ there exist sequences of functions $\{\phi_j\} \subset C(\overline{\Omega})$ and compact operators $\{K_j\}$ such that $\|T - T_{\phi_j} + K_j\| \to 0$ as $j \to \infty$. Then

$$|\tilde{T}(z) - \tilde{T}_{\phi_j}(z) + \tilde{K}_j(z)| \leq \|T - T_{\phi_j} + K_j\|$$
for any \( z \in \Omega \). Then \( \{ \widetilde{T}_{\phi_j} + \widetilde{K}_j \} \) is a Cauchy sequence in \( L^\infty(\Omega) \). Compactness of \( K_j \) implies that \( \widetilde{K}_j \) has a continuous extension up to the boundary of \( \Omega \) and \( \widetilde{K}_j = 0 \) on \( b\Omega \) because \( k_z \to 0 \) weakly as \( z \to b\Omega \) (see [CS18, Lemma 4.9]). Furthermore, \( \widetilde{T}_{\phi_j} \) has continuous extensions onto \( \Omega \cup \Gamma \) (see [CSZ18, Lemma 15] and [Ran86, Theorem 1.13 in Ch VI]) and \( \widetilde{T}_{\phi_j} = \phi_j \) on \( \Gamma \).

Let \( p \in \Gamma \) and \( r > 0 \) such that \( \overline{B(p,r)} \cap b\Omega \subset \Gamma \). Hence \( \{ \widetilde{T}_{\phi_j} + \widetilde{K}_j \} \) is a Cauchy sequence in \( C(\Omega \cap \overline{B(p,r)}) \) and it converges to \( \widetilde{T} \) uniformly on \( \overline{\Omega \cap B(p,r)} \). Hence \( \widetilde{T} \in C(\Omega \cap B(p,r)) \) and since \( p \in \Gamma \) is arbitrary we conclude that \( \widetilde{T} \) has a continuous extension onto \( \overline{\Omega \cap B(p,r)} \).

We note that compactness of the \( \partial \)-Neumann operator implies that \( \Gamma \) is dense in the boundary (see, for example, [SS06, Corollary 1] or [Str10, Corollary 4.24]). Now we assume that \( T \) is a finite sum of finite products of Toeplitz operators with symbols continuous on \( \Omega \). Then \( T = T_\phi + K \) where \( \phi \in C(\overline{\Omega}) \) and \( K \) is a compact operator. Hence \( \widetilde{K} = 0 \) on \( b\Omega \) and

\[
(1) \quad \|T\|_e = \|\phi\|_{L^\infty(\Omega)} = \|\tilde{\phi}\|_{L^\infty(\Gamma)} = \|\tilde{\phi}\|_{L^\infty(\Gamma)} = \|\tilde{T}\|_{L^\infty(\Gamma)}.
\]

The first equality above is due to [CS13, Corollary 3]. The second equality comes from the fact that \( \Gamma \) is dense in \( b\Omega \) and the third equality is due to the fact that \( \phi(p) = \tilde{\phi}(p) \) for any strongly pseudoconvex point.

Finally, we assume that \( T \in \mathcal{S}(\overline{\Omega}) \). For \( \varepsilon > 0 \) there exists \( \phi \in C(\overline{\Omega}) \) and a compact operator \( K \) such that \( \|T - T_\phi + K\| < \varepsilon \). Then for any compact operator \( S \) we have

\[
\|T\|_e \leq \|T + K - S\| \leq \|T - T_\phi + K\| + \|T_\phi - S\| \leq \varepsilon + \|T_\phi - S\|.
\]

If we take infimum over \( S \), by \( (1) \) we get

\[
(2) \quad \|T\|_e \leq \|T_\phi\|_e + \varepsilon = \|\tilde{T}_\phi\|_{L^\infty(\Gamma)} + \varepsilon.
\]

For any \( z \in \Omega \) we have

\[
|\tilde{T}(z) - \tilde{T}_\phi(z) + \tilde{K}(z)| = |\langle (T - T_\phi + K)k_z, k_z \rangle| \leq \|T - T_\phi + K\| \leq \varepsilon.
\]

Then

\[
\|\tilde{T} - \tilde{T}_\phi\|_{L^\infty(\Gamma)} \leq \varepsilon.
\]

Combining, the inequality above with \( (2) \) we get

\[
\|T\|_e \leq \|\tilde{T}_\phi\|_{L^\infty(\Gamma)} + \varepsilon \leq \|\tilde{T}\|_{L^\infty(\Gamma)} + \|\tilde{T} - \tilde{T}_\phi\|_{L^\infty(\Gamma)} + \varepsilon \leq \|\tilde{T}\|_{L^\infty(\Gamma)} + 2\varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary we have

\[
(3) \quad \|T\|_e \leq \|\tilde{T}\|_{L^\infty(\Gamma)}.
\]

To prove the converse, for \( \varepsilon > 0 \), we choose a compact operator \( K \) so that

\[
\|T - K\| \leq \|T\|_e + \varepsilon.
\]

Then for any \( z \in \Omega \) we have

\[
|\tilde{T}(z) - K(z)| = |\langle (T - K)k_z, k_z \rangle| \leq \|T - K\| \leq \|T\|_e + \varepsilon.
\]

Then \( \|\tilde{T}\|_{L^\infty(\Gamma)} \leq \|T\|_e + \varepsilon \) (as \( \tilde{K} = 0 \) on \( b\Omega \)) and since \( \varepsilon \) is arbitrary we get \( \|\tilde{T}\|_{L^\infty(\Gamma)} \leq \|T\|_e \).

Combining the last inequality with \( (3) \) we get \( \|T\|_e = \|\tilde{T}\|_{L^\infty(\Gamma)} \). \qed
Let $\Omega$ be a $C^\infty$-smooth bounded pseudoconvex domain in $\mathbb{C}^n$ that is finite type in the sense of D’Angelo (see, [D’A82, D’A02]). Then the $\partial$-Neumann operator is compact (see, [Cat84]), and one can use [Bel86, Boa87] and the proof of [ČŠ13, Lemma 1] to prove that $T_\phi(z) \to \phi(p)$ as $z \to p$ for any $p \in b\Omega$ and $\phi \in C(\overline{\Omega})$. Then these facts together with the proof of Theorem 4 imply the following corollary.

**Corollary 2.** Let $\Omega$ be a $C^\infty$-smooth bounded pseudoconvex domain in $\mathbb{C}^n$ that is finite type in the sense of D’Angelo. Assume that $T \in \mathcal{F}(\overline{\Omega})$. Then $\tilde{T}$ has a continuous extension onto $\overline{\Omega}$ and

$$\|T\|_\ast = \|\tilde{T}\|_{L^\infty(\partial \Omega)}.$$ 

Next we present the proof of Theorem 5.

**Proof of Theorem 5.** Let $T$ be a finite sum of finite products of Toeplitz operators with symbols continuous on $\overline{\Omega}$. Then $T = T_\phi + K$ where $K$ is an operator that is compact about strongly pseudoconvex points (see proof of Theorem 4 in [ČŠZ18]) and $\phi \in C(\overline{\Omega})$.

Let $\alpha > \|\tilde{T}\|_{L^\infty(\Gamma)}$. Since $\phi \in C(\overline{\Omega})$ and

$$\|\tilde{T}\|_{L^\infty(\Gamma)} = \|\tilde{T}_\phi\|_{L^\infty(\Gamma)} = \|\phi\|_{L^\infty(\Gamma)}$$

there exists an open neighborhood $U$ of $\Gamma$ such that $|\phi| < \alpha$ on $U \cap \overline{\Omega}$. We choose a function $\chi \in C_0^\infty(U)$ such that $0 \leq \chi \leq 1$ and $\chi = 1$ on a neighborhood of $\Gamma$. One can check that $S = T_{(1-\chi)\phi}$ is an operator compact about strongly pseudoconvex points (see [ČŠZ18, Lemma 12]). Then

$$\|T\|_{\ast \ast} \leq \|T_\phi - S\| \leq \|\chi \phi\|_{L^\infty(U)} \leq \|\phi\|_{L^\infty(U)} \leq \alpha.$$ 

Since $\alpha$ is arbitrary we conclude that

$$\|T\|_{\ast \ast} \leq \|\tilde{T}\|_{L^\infty(\Gamma)}.$$

To prove the converse, let $p \in b\Omega$ be a strongly pseudoconvex point. Then for every $\varepsilon > 0$ there exists an operator $S_\varepsilon$ compact about strongly pseudoconvex points such that $\|T - S_\varepsilon\| < \|T\|_{\ast \ast} + \varepsilon$. Then

$$\lim_{z \to p} |\tilde{T}(z)| = \lim_{z \to p} |\tilde{T}(z) - S_\varepsilon(z)| = \lim_{z \to p} |\langle (T - S_\varepsilon)k_z, k_z \rangle| \leq \|T - S_\varepsilon\| < \|T\|_{\ast \ast} + \varepsilon.$$ 

In the first equality above we used the fact that $k_z \to 0$ weakly about strongly pseudoconvex points as $z \to p$ (see the proof of [ČŠZ18, Theorem 4]). Hence $\|\tilde{T}\|_{L^\infty(\Gamma)} \leq \|T\|_{\ast \ast}$. Combining this inequality with (4) we conclude that

$$\|T\|_{\ast \ast} = \|\tilde{T}\|_{L^\infty(\Gamma)}.$$

Next if $T \in \mathcal{F}(\overline{\Omega})$ then there exist sequences $\{\phi_j\} \subset C(\overline{\Omega})$ and operators compact about strongly pseudoconvex points $\{K_j\}$ such that

$$\|T - T_{\phi_j} + K_j\| \to 0 \text{ as } j \to \infty$$

(again see the proof of Theorem 4 in [ČŠZ18]). Then as in the proof of Theorem 4 we conclude that the sequence $\{\tilde{T}_{\phi_j} + K_j\}$ is Cauchy in $C(\overline{\Omega} \cap \overline{B(p,r)})$ for any $p \in \Gamma$ and $r > 0$ such that $\overline{B(p,r)} \cap b\Omega \subset \Gamma$ and it converges to $\tilde{T}$ uniformly on $\Omega$ as $j \to \infty$. Hence $\tilde{T}$ has a continuous extension onto $\Omega \cup \Gamma$. That is, $\tilde{T} \in C(\overline{\Omega} \cup \Gamma)$.

Then we complete the proof just as in the second half of the proof of Theorem 4 by replacing compact operators with operators compact about strongly pseudoconvex points. Therefore,

$$\|T\|_{\ast \ast} = \|\tilde{T}\|_{L^\infty(\Gamma)}$$.
3. Example

In general, we have the following inequality
\[ \|\tilde{T}\|_{L^\infty(\Omega)} \leq \|T\| \]
for any \( T \in \mathcal{B}(A^2(\Omega)) \). However, there is no relation between \( \|\tilde{T}\|_{L^\infty(\Omega)} \) and \( \|T\|_e \). This can be seen as follows. In Example 1 below we construct a domain \( \Omega \) such that \( \|\tilde{T}_\phi\|_{L^\infty(\Omega)} < \|T_\phi\|_e \) for some \( \phi \in C^\infty(\overline{\Omega}) \). However, for any \( \phi \in C^\infty_0(\Omega) \) and \( \phi \neq 0 \) we have \( \|T_\phi\|_e = \|\tilde{T}_\phi\|_{L^\infty(\Omega)} \).

Example 1. In this example, we construct a smooth bounded pseudoconvex complete Reinhardt domain \( \Omega \) in \( \mathbb{C}^2 \) and a symbol \( \phi \in C^\infty(\Omega) \) such that
\[ \|\tilde{T}_\phi\|_{L^\infty(\Gamma)} \leq \|\tilde{T}_\phi\|_{L^\infty(\Omega)} < \|T_\phi\|_e = \|\phi\|_{L^\infty(\Omega)} \]
even though \( \tilde{T}_\phi \in C(\Omega \cup \Gamma) \). Let us choose an even function \( \chi \in C^\infty_0(0,1) \) such that \( \chi \geq 0 \) and
\[ \int_0^1 \chi(r)rdr < 2 \int_0^1 \chi(r)r^3dr. \]
One can show the existence of \( \chi \) as follows. We first note that \( \int_0^1 rdr = (1 - \alpha_0^2)/2 \) and \( 2 \int_0^1 r^3dr = (1 - \alpha_0^4)/2 \). Then for any \( 0 < \alpha_0 < 1 \) we have
\[ \int_0^{\alpha_0} rdr = \frac{1 - \alpha_0^2}{2} < \frac{1 - \alpha_0^4}{2} = 2 \int_0^{\alpha_0} r^3dr. \]
Then we choose \( 0 < \alpha_0 < \alpha < 1 \) and \( \chi \in C^\infty_0(0,1) \) by approximating the characteristic function of \( (\alpha_0,1) \) so that \( \int_0^1 \chi(r)rdr < 2 \int_0^1 \chi(r)r^3dr \) and \( \text{supp}(\chi) \subset (\alpha_0,\alpha) \). Let \( J = [0,\alpha] \). Then
\[ (5) \quad \int_J \chi(r)rdr < 2 \int_J \chi(r)r^3dr. \]
Now we define the domain \( \Omega \) as follows: First let \( H \subset [0,1) \times [0,1) \) be a smooth domain in \( \mathbb{R}^2 \) defined by
\[ H = \{(x,y) \in \mathbb{R}^2 : 0 \leq y < h(x), 0 \leq x < 1\}. \]
\[
\begin{array}{c}
\text{y} \\
\includegraphics{fig.png}
\end{array}
\]
where \( h \) is a smooth function such that \( 0 < h \leq 1 \) and \( J = \{h = 1\} \subset [0,1) \). Then
\[ \Omega = \{(z,w) \in \mathbb{C}^2 : (|z|,|w|) \in H\}. \]
We define \( \phi(z,w) = \chi(|z|) \) and
\[ \lambda_{n,m} = \frac{\int_0^1 \chi(r)r^{2n+1}(h(r))^{2m+2}dr}{\int_0^1 r^{2n+1}(h(r))^{2m+2}dr} \geq 0 \]
for all \( n, m = 0, 1, 2, \ldots \). Then one can check that
\[
(6) \quad \langle \phi(z, w)z^n w^m - \lambda_{nm} z^n w^m, z^j w^k \rangle = 0
\]
for all \( n, m, j, k = 0, 1, 2, \ldots \). Then
\[
T_\phi z^n w^m = \lambda_{nm} z^n w^m
\]
for all \( n \in \mathbb{Z}, m = 0, 1, 2, \ldots \). We note that the Bergman kernel of \( \Omega \) is
\[
K((\xi, \eta), (z, w)) = \sum_{n,m=0}^{\infty} \frac{\xi^n \eta^m z^n w^m}{\delta_{nm}}
\]
where \( \delta_{nm} = \|z^n w^m\| \). Then, using (6) in the second equality below, we get
\[
\langle \phi K(., (z, w)), K(., (z, w)) \rangle = \sum_{n,m=0}^{\infty} \frac{|z|^{2n}|w|^{2m}}{\delta_{nm}^2} \langle \phi(\xi) \xi^n \eta^m, \xi^n \eta^m \rangle
\]
\[
= \sum_{n,m=0}^{\infty} \frac{|z|^{2n}|w|^{2m}}{\delta_{nm}^2} \lambda_{nm} \langle \xi^n \eta^m, \xi^n \eta^m \rangle
\]
\[
= \sum_{n,m=0}^{\infty} \frac{|z|^{2n}|w|^{2m}}{\delta_{nm}^2} \lambda_{nm} \delta_{nm}^2
\]
\[
= \sum_{n,m=0}^{\infty} \lambda_{nm} \frac{|z|^{2n}|w|^{2m}}{\delta_{nm}^2}.
\]
Hence we have
\[
\tilde{T}_\phi(z, w) = \frac{\sum_{n,m=0}^{\infty} \lambda_{nm} \frac{|z|^{2n}|w|^{2m}}{\delta_{nm}^2}}{\sum_{n,m=0}^{\infty} \frac{|z|^{2n}|w|^{2m}}{\delta_{nm}^2}}
\]
Next we will compute the norm and essential norm of \( T_\phi \). Since \( \chi \) has a compact support in the interior of \( J = \{ h = 1 \} \) we have
\[
\lambda_{n,m} = \frac{\int_J \chi(r)r^{2n+1}dr}{\int_0^1 r^{2n+1} (h(r))^{2m+2}dr}
\]
for \( n, m = 0, 1, 2, \ldots \) and
\[
\lim_{m \to \infty} \lambda_{n,m} = \lambda_{n,\infty} = \frac{\int_J \chi(r)r^{2n+1}dr}{\int_J r^{2n+1}dr} < \infty.
\]
Furthermore, since \( 0 \leq h < 1 \) on \( [0, 1] \) \( \setminus \) \( J \) we have
\[
(7) \quad \lambda_{n,m} < \lambda_{n,m+1} < \lambda_{n,\infty} < \|\phi\|_{L^\infty(\Omega)}
\]
for all \( n, m \). The last inequality above is due to the fact that the probability measure \( r^{2n+1}(\int_J r^{2n+1}dr)^{-1} \) is absolutely continuous with respect to the Lebesgue measure on \( J \) and \( \chi \) is a non-negative compactly supported function on \( J \). Also, one can show that the sequence \( \{ r^{2n+1}(\int_J r^{2n+1}dr)^{-1} \} \) converges to point mass measure \( \delta_n \) weakly as \( n \to \infty \). Hence the fact that the support of \( \chi \) is in the interior of \( J \) implies that
\[
\lim_{n \to \infty} \lambda_{n,\infty} = \lambda_{\infty,\infty} = 0.
\]
Using the fact that the monomials are eigenfunctions for $T_\phi$ and the set of monomials is a basis for $A^2(\Omega)$ one can show that
\[ \|(T_\phi - \lambda I)f\| \geq \inf\{|\lambda_{n,m} - \lambda| : m, n = 0, 1, 2, \ldots \}|f| \]
for all $f \in A^2(\Omega)$. Hence the spectrum of $T_\phi$ is the closure of $\{\lambda_{n,m} : m, n = 0, 1, 2, \ldots \}$. We note that $\lambda_{0,0} = \tilde{T}_\phi(0,0)$ and (5) implies that $\lambda_{0,0} < \lambda_{1,\infty}$. Therefore, there exists $n_0 \geq 1$ such that
\[ \|T_\phi\| = \lambda_{n_0,\infty} = \max\{\lambda_{n,\infty} : n = 0, 1, 2, \ldots \} > 0. \]
Hence
\[ (8) \quad \|\tilde{T}_\phi\|_{L^\infty(\Omega)} = \sup \left\{ \frac{\sum_{n,m=0}^\infty \lambda_{nm} |z|^{2n} |w|^{2m}}{\sum_{n,m=0}^\infty |z|^{2n} |w|^{2m}} : (z, w) \in \Omega \right\} < \lambda_{n_0,\infty} = \|T_\phi\|. \]
Since $\tilde{T}_\phi$ has continuous extension to $\Gamma$ we have
\[ \|\tilde{T}_\phi\|_{L^\infty(\Gamma)} \leq \|\tilde{T}_\phi\|_{L^\infty(\Omega,\Gamma)} = \|\tilde{T}_\phi\|_{L^\infty(\Omega)}. \]
We note that each $\lambda_{n,\infty}$ is in the essential spectrum because they are limits of eigenvalues and $T_\phi$ is self-adjoint (see [EE87, Theorem 1.6 in ch IX]). So $\|T_\phi\|e = \lambda_{n_0,\infty}$ and, therefore, using (7) and (8) we get
\[ \|\tilde{T}_\phi\|_{L^\infty(\Gamma)} \leq \|\tilde{T}_\phi\|_{L^\infty(\Omega)} < \|T_\phi\|e = \|T_\phi\| = \lambda_{n_0,\infty} < \|\phi\|_{L^\infty(b\Omega)} = \|\chi\|_{L^\infty(J)}. \]

4. Proofs of Theorems 1, 2, and 3

First we will prove a sufficient condition for BC-regularity.

**Lemma 1.** Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ on which the $\overline{\partial}$-Neumann operator is compact. Assume that $k_z \to 0$ weakly in $A^2(\Omega)$ as $z \to b\Omega$ and $|k_z|^2 \to \delta_p$ weakly as $z \to p$ for any $p \in b\Omega$. Then $\Omega$ is BC-regular.

**Proof.** Assume that $T \in \mathcal{T}(\overline{\Omega})$. Then, by [ČS13, Proof of Theorem 1], there exists $\{T_j\}$, a sequence of finite sum of finite products of Toeplitz operators with symbols continuous on $\overline{\Omega}$, such that $\|T - T_j\| \to 0$. Since the $\overline{\partial}$-Neumann operator is compact, we have $T_j = T_{\phi_j} + K_j$ for some $\phi_j \in C'\Omega$ and compact operator $K_j$ for every $j$. The assumption $|k_z|^2 \to \delta_p$ weakly as $z \to p$ for any $p \in b\Omega$ implies that $T_{\phi_j} = \phi_j$ on $b\Omega$ for every $j$. Furthermore, since $k_z \to 0$ weakly as $z \to b\Omega$ and $K_j$ is compact we have $K_j = 0$ on $b\Omega$. Then $\tilde{T}_j \in C'\Omega$ for every $j$, $\tilde{T}_j \to \tilde{T}$ in $L^\infty(\Omega)$ as $j \to \infty$, and $\{\tilde{T}_j\}$ form a Cauchy sequence in $C'\Omega$. Hence $\tilde{T}$ has a continuous extension onto $\overline{\Omega}$. That is, $\Omega$ is BC-regular. □

**Remark 1.** We note that the condition $k_z \to 0$ weakly in $A^2(\Omega)$ as $z \to b\Omega$ in Lemma 1 is automatic if the boundary is $C^\infty$-smooth (see, [ČS18, Lemma 4.9]). However, it is not automatic if we do not assume any regularity of the boundary even if $N$ is compact. For instance, if $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1, z_1 \neq 0\}$ then $k_z \not\to 0$ weakly as $z \to 0$ because the Bergman kernel of $\Omega$ is the same as the Bergman kernel of the ball. However, the $\overline{\partial}$-Neumann operator on $\Omega$ is compact (see, [FS01, section 4] or [Str10, pg 99, Example]).

When $\Omega$ is bounded convex the condition $k_z \to 0$ weakly as $z \to b\Omega$ is automatic (see, [ČS18, Lemma 4.9]). Then we have the following corollary.
Corollary 3. Let $\Omega$ be a bounded convex domain in $\mathbb{C}^n$ on which the $\overline{\partial}$-Neumann operator is compact. Assume that $|k_z|^2 \to \delta_p$ weakly as $z \to p$ for any $p \in b\Omega$. Then $\Omega$ is BC-regular.

Furthermore, if we assume a smooth boundary then we get a necessary and sufficient condition for BC-regularity under the assumption that the $\overline{\partial}$-Neumann operator is compact.

Proposition 1. Let $\Omega$ be a $C^\infty$-smooth bounded pseudoconvex domain in $\mathbb{C}^n$ on which the $\overline{\partial}$-Neumann operator is compact. Then $\Omega$ is BC-regular if and only if $|k_z|^2 \to \delta_p$ weakly as $z \to p$ for any $p \in b\Omega$.

Proof. First let us assume that $\Omega$ is BC-regular. Let $p \in b\Omega$ and $\phi \in C(\overline{\Omega})$ such that $\phi(p) = 1$ and $0 \leq \phi < 1$ on $\overline{\Omega} \setminus \{p\}$. We note that compactness of $N$ implies that the set of strongly pseudoconvex points $\Gamma$ is dense in $b\Omega$ (see [SS06, Corollary 1] and [Str10, Corollary 4.24]) and $\lim_{z \to p} T_{\phi} = \phi(p) = 1$ for any strongly pseudoconvex point $p \in b\Omega$. Since $T_{\phi}$ is continuous on $\overline{\Omega}$ and it is equal to $\phi$ on $\Gamma$ a dense set in $b\Omega$ we conclude that $\tilde{T}_{\phi} = \phi$ on $b\Omega$. Furthermore, since $\phi$ peaks at $p$ one can show that $|k_z|^2 \to \delta_p$ weakly as $z \to p$. This can be seen as follows: If $|k_z|^2$ does not converge to $\delta_p$ weakly then, by Alaoglu Theorem, there exist a probability measure $\mu \neq \delta_p$ and a sequence $\{z_j\} \subset \Omega$ such that $z_j \to p$ and $|k_{z_j}|^2 \to \mu$ weakly as $j \to \infty$. Then $\tilde{T}_{\phi}(z_j) \to \int \phi d\mu \neq 1$ as $j \to \infty$ which is a contradiction with $\lim_{z \to p} T_{\phi} = \phi(p) = 1$.

To prove the converse we use Lemma 1 together with that fact that $k_z \to 0$ weakly as $z \to b\Omega$ if $\Omega$ is a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ (see, [CS18, Lemma 4.9]).

Lemma 2. Let $\Omega$ be a bounded $C^\infty$-smooth pseudoconvex domain in $\mathbb{C}^n$ on which the $\overline{\partial}$-Neumann operator is compact. Assume that $W$ is the set of weakly pseudoconvex points and $p \in W$. Furthermore, assume that there exists $\{p_j\} \subset \Omega$ such that $|k_{p_j}|^2 \to \mu$ weakly as $j \to \infty$ for some measure $\mu$. Then $\text{supp}(\mu) \subset W$.

Proof. We note that Bell’s version of Kerzman’s result [Bel86] implies that if $\Omega$ has condition R (implied by compactness of $N$) and $q$ is a strongly pseudoconvex point. Then the Bergman kernel $K \in C^\infty((U \cap \Omega) \times (V \cap \overline{\Omega})$ for small enough disjoint neighborhoods $U$ of $p$ and $V$ of $q$. Then $|k_z|^2 \to 0$ on $V \cap \overline{\Omega}$ as $z \to p$. Then $\text{supp}(\mu) \cap V = \emptyset$. Since $q$ was arbitrary strongly pseudoconvex point the support $\mu$ lies on the set of weakly pseudoconvex points.

We note that the domains in the following corollary are not necessarily (products of) finite type domains as in Corollaries 2 and 5 or convex as in Theorem 1 (see [KN73]). They are not even necessarily locally convexifiable.

Corollary 4. Let $\Omega$ be a bounded $C^\infty$-smooth pseudoconvex domain in $\mathbb{C}^n$. Assume that $\Omega$ has a single weakly pseudoconvex boundary point. Then $\Omega$ is BC-regular.

Proof. We will use Sibony’s B-regularity (see [Sib87]). One can show that any single point and any compact set of strongly pseudoconvex points are B-regular. Then $b\Omega$ is B-regular as a compact set that is a countable union of B-regular sets is B-regular (see [Sib87, Proposition 1.9]). One can also use that fact that a single point has two-dimensional Hausdorff measure zero to prove that $b\Omega$ is B-regular (see [Boa88]). Then the $\overline{\partial}$-Neumann operator is compact on $\Omega$.

Let $p \in b\Omega$ be the weakly pseudoconvex point and $\{|k_{z_j}|^2\}$ be a weakly convergent sequence where $z_j \to p$ as $j \to \infty$. Then Lemma 2 implies that $\{|k_{z_j}|^2\}$ converges weakly to $\delta_p$. Now let $z_j \to p$ as $j \to \infty$. Then, by Alaoglu theorem and Lemma 2, there exists a subsequence $|k_{z_{j_k}}|^2 \to \delta_p$ weakly as $j_k \to \infty$. Since every sequence $\{|k_{z_j}|^2\}$ has a sequence weakly convergent to $\delta_p$ we conclude that $|k_z|^2 \to \delta_p$ as $z \to p$. \(\square\)
Proof of Theorem 3. First we will prove the theorem for a finite product of Toeplitz operators with symbols continuous on $\Omega$. We will use induction on the number of product of domains. If $\Omega = \Omega_1$ is a $C^2$-smooth bounded strongly pseudoconvex domain and $T$ is a finite products of Toeplitz operators with symbols continuous on $\Omega$ then the proof of Theorem 4 implies that $\tilde{T} \in C(\Omega)$. We note that the proof of Theorem 4 simplifies for $C^2$-smooth bounded strongly pseudoconvex domains as $G = b\Omega$. As the induction step, we assume that $\tilde{T} \in C(\Omega)$ whenever $T$ is a finite product of Toeplitz operators with symbols continuous on $\Omega$ and $\Omega$ is the product of $m - 1$ domains that are $C^2$-smooth bounded strongly pseudoconvex.

Let $\Omega = \Omega_1 \times \cdots \times \Omega_m$ where $\Omega_j$ is a $C^2$-smooth bounded strongly pseudoconvex domain in $C^0$ and $\phi_1, \ldots, \phi_m$ be continuous on $\Omega$. Let $q = (q', q_m) \in b\Omega$ and $z = (z', z_m)$ where $z' \in \Omega'$ and $\Omega_1 \times \cdots \times \Omega_{m-1}, z_m \in \Omega_m$, and, without loss of generality, $q_m \in b\Omega_m$ as there exists $1 \leq l \leq m$ such that $q_l \in b\Omega_l$. Let us define $\phi_j(z', z_m) = \phi_j(z', q_m)$ and $\phi_{j,2} = \phi_j - \phi_{j,1}$ for $j = 1, \ldots, m$. Then $\phi_{j,2} = 0$ on $\Omega' \times \{q_m\}$. We note that below we use the fact that $\langle f, \phi g \rangle = \langle f, \phi g \rangle$ for any $\phi \in L^\infty(\Omega)$ and $f, g \in A^2(\Omega)$. Furthermore, we denote the inner product and the $L^2$-norm on a set $V$ by $\langle \cdot, \cdot \rangle_V$ and $\|\cdot\|_V$, respectively. Let $T = T_{\phi_m} \cdots T_{\phi_1}$ be a product of Toeplitz operators. Then

\[
\langle T_{\phi_m} \cdots T_{\phi_1} k(z', z_m), k(z', z_m) \rangle_\Omega = \langle T_{\phi_m-1} \cdots T_{\phi_1} k(z', z_m), T_{\phi_m,1} k(z', z_m) \rangle_\Omega \\
+ \langle T_{\phi_m-1} \cdots T_{\phi_1} k(z', z_m), T_{\phi_m,2} k(z', z_m) \rangle_\Omega \\
= \langle T_{\phi_m-1} \cdots T_{\phi_1} k(z', z_m), T_{\phi_m,1} k(z', z_m) \rangle_\Omega \\
+ \langle T_{\phi_m-1} \cdots T_{\phi_1} k(z', z_m), T_{\phi_m,2} k(z', z_m) \rangle_\Omega.
\]

(9)

The second inner product on the right hand side of the second equality in (9) goes to zero as $z \to q$. This can be seen using Cauchy-Schwarz inequality together with the following argument. Let $U_m, \varepsilon = \{\xi \in \Omega : |\phi_{m,2}(\xi)| < \varepsilon\}$. Then for $\varepsilon > 0$ given we have $\Omega' \times \{q_m\} \subset U_m, \varepsilon$ and

\[
\|k(z', \phi_{m,2} k(z', z_m) \|_\Omega^2 \leq \|k(z', \phi_{m,2} k(z', z_m) \|_{\Omega \setminus U_m, \varepsilon}^2 + \|k(z', \phi_{m,2} k(z', z_m) \|_{U_m, \varepsilon}^2 \\
\leq \sup_{\Omega \setminus U_m, \varepsilon} \{\phi_{m,2}(\xi) k(z', \phi_{m,2}(\xi)) \}^2 : \xi \in \Omega \setminus U_m, \varepsilon \|k(z', \phi_{m,2}(\xi)) \|_{\Omega}^2 \\
+ \varepsilon^2 \|k(z', \phi_{m,2}(\xi)) \|_{U_m, \varepsilon}^2 \\
\leq \sup_{\Omega \setminus U_m, \varepsilon} \{\phi_{m,2}(\xi) k(z', \phi_{m,2}(\xi)) \}^2 : \xi \in \Omega \setminus U_m, \varepsilon \} V(\Omega_m) + \varepsilon^2
\]

where $V(\Omega_m)$ is the volume of $\Omega_m$. Then $\lim \sup_{z \to q} \|k(z', \phi_{m,2} k(z', z_m) \|_{\Omega}^2 \leq \varepsilon^2$ because $k(z', z_m) \to 0$ uniformly on $\Omega \setminus U_m, \varepsilon$ as $z \to q$ (see the proof of Lemma 1 in [ČS13]). Since $\varepsilon$ was arbitrary we conclude that $\|k(z', \phi_{m,2} k(z', z_m) \|_{\Omega} \to 0$ as $z \to q$.

Using the same idea again on the first term on the right hand side of the second equality of (9) we get the following

\[
\langle T_{\phi_m-1} \cdots T_{\phi_1} k(z', z_m), T_{\phi_m,1} k(z', z_m) \rangle_\Omega \\
= \langle T_{\phi_m-2} \cdots T_{\phi_1} k(z', z_m), T_{\phi_m-1,1} k(z', z_m) \rangle_\Omega \\
+ \langle T_{\phi_m-2} \cdots T_{\phi_1} k(z', z_m), T_{\phi_m-1,2} k(z', z_m) \rangle_\Omega.
\]
Similarly the second term on the right hand side converges to zero as $z \to q$ because
\[
\| (T_{\phi_{m,1}} k_{z'}) \overline{\phi}_{m-1,2} k_{z_m} \|^2 \\
\leq \sup \{ \overline{\phi}_{m-1,2} k_{z_m}(\xi)^2 : \xi \in \Omega \setminus U_{m-1, \epsilon} \} \| \phi_{m,1} \|^2_{L^\infty(\Omega)} V(\Omega_m) \\
+ \varepsilon^2 \| (T_{\phi_{m,1}} k_{z'}) k_{z_m} \|_{U_{m-1, \epsilon}}^2.
\]

Using similar arguments as before one can show that
\[
\| (T_{\phi_{m,1}} k_{z'}) \overline{\phi}_{m-1,2} k_{z_m} \| \to 0 \text{ as } z \to q.
\]
Inductively, we get
\[
\langle T_{\phi_{m,1}} \cdots T_{\phi_{1}} k_{(z', z_m)}, k_{(z', z_m)} \rangle_{\Omega} = \langle k_{z'} k_{z_m}, (T_{\phi_{m,1}} \cdots T_{\phi_{1}} k_{z'}) k_{z_m} \rangle_{\Omega}
\\
+ R_{z_m}
\\
= \langle (T_{\phi_{m,1}} \cdots T_{\phi_{1}} k_{z'}) k_{z_m}, k_{z_m} \rangle_{\Omega} + R_{z_m}
\\
= \langle T_{\phi_{m,1}} \cdots T_{\phi_{1}} k_{z'}, k_{z'} \rangle_{\Omega} + R_{z_m}
\\
= T'(z') + R_{z_m}
\]
where $T' = T_{\phi_{m,1}} \cdots T_{\phi_{1}}$ on $A^2(\Omega')$ and $R_{z_m} \to 0$ as $z \to q$. By the induction assumption, we have $T' \in C(\Omega')$. Therefore, $T \in C(\Omega)$.

If $T$ is a finite sum of finite products of Toeplitz operators with symbols continuous on $\overline{\Omega}$ then $\widetilde{T} \in C(\overline{\Omega})$ because we can apply the previous step to every finite products in the sum.

Finally, if $T \in \mathcal{F}(\overline{\Omega})$, then there exists $\{T_j\}$, a sequence of finite sum of finite products of Toeplitz operators with symbols continuous on $\overline{\Omega}$, such that $\| T - T_j \| \to 0$ as $j \to \infty$. Then $\widetilde{T}_j \in C(\overline{\Omega})$ for every $j$, $\{\widetilde{T}_j\}$ form a Cauchy sequence in $C(\overline{\Omega})$, and $\widetilde{T}_j \to \widetilde{T}$ in $L^\infty(\Omega)$ as $j \to \infty$. Hence $\widetilde{T}$ has a continuous extension onto $\overline{\Omega}$.

\[\Box\]

Example 2. We note that $\phi$ and $\overline{\phi}$ might not match on $b\Omega$ even if $\overline{\phi}$ is continuous up to the boundary. One can produce an example on the bidisc as follows. Let $\phi(z_1, z_2) = |z_1|^2 \in C^\infty(\mathbb{D}^2)$. Then $\phi(0, 0) = 0$ and by Theorem 3 we have $\overline{\phi} \in C(\overline{\mathbb{D}^2})$. However,
\[
\overline{\phi}(0, 0) = \frac{1}{\pi} \int_{\mathbb{D}} |z_1|^2 dV(z_1) = \frac{1}{2}.
\]

Hence $\phi \neq \overline{\phi}$ on $b\Omega$.

The proof of Theorem 3 can be applied, with minor modifications and Corollary 2, to products of finite type domains. Hence we have the following corollary.

Corollary 5. Let $\Omega = \Omega_1 \times \cdots \times \Omega_m$ such that each $\Omega_j$ is a $C^\infty$-smooth bounded pseudoconvex domain that is finite type in the sense of D’Angelo. Then $\Omega$ is BC-regular. That is, $\widetilde{T} \in C(\overline{\Omega})$ whenever $T \in \mathcal{F}(\overline{\Omega})$.

We continue with the proof of Theorem 1.

Proof of Theorem 1. Assume that $0 \in b\Omega$, the negative $x_n$-axis is outward normal direction at 0, and

\[
\Omega \subset \{(z', x_n + iy_n) \in \mathbb{C}^n : z' \in \mathbb{C}^{n-1}, x_n > 0, y_n \in \mathbb{R}\}.
\]

Let us denote $L = b\Omega \cap \{(z', 0) \in \mathbb{C}^n : z' \in \mathbb{C}^{n-1}\}$ and $f(z', z_n) = \frac{1-z_n}{1+z_n}$. Then $f$ is a holomorphic function on $\Omega$ that is continuous on $\overline{\Omega}$ such that $f = 1$ on $L$ and $|f| < 1$ on $\overline{\Omega} \setminus L$. Now assume
that there exists a sequence \( \{p_j\} \subset \Omega \) such that \( p_j \to 0 \in L \) and \( \{|k_p|^2\} \) converges to a measure \( \mu \) weakly. Then \( \mu \) is a representing measure for \( A^2(\Omega) \cap C(\overline{\Omega}) \) such that \( g(0) = \int_{\Omega} g(z) d\mu(z) \) for all \( g \in A^2(\Omega) \cap C(\overline{\Omega}) \). Then \( \int_{\Omega} f(z) d\mu(z) = f(0) = 1 \). We conclude that \( \text{supp}(\mu) \subset L \) because if \( \text{supp}(\mu) \cap \overline{\Omega} \setminus L \neq \emptyset \) we would have \( \text{Re}(\int_{\Omega} f(z) d\mu(z)) < 1 \).

If the domain \( \Omega \) is strongly pseudoconvex then, by Theorem 3, it is BC regular. So assume that \( 0 \in b\Omega \) is a weakly pseudoconvex point. Since \( \Omega \) is convex and there is no (affine) analytic variety in the boundary of \( \Omega \), the intersection is either a single point or a totally real affine space. This can be seen as follows: Assume that \( L \) is not a singleton and nor a totally real surface. Then there exists \( p \in L \) and \( X \subset \partial L \), the tangent space of \( b\Omega \) at \( p \), such that \( JX \subset T_p L \) where \( J \) is the complex structure. Since \( L \) is an affine compact convex set, there exists \( \varepsilon > 0 \) such that

\[
S = \{ \alpha X + \beta JX + p : -\varepsilon < \alpha, \beta < \varepsilon \} \subset L.
\]

Then \( X \) and \( JX \) span the tangent space of \( S \) and using [BER99, Proposition 1.3.19] we conclude that \( S \) is a complex manifold. Hence \( L \) and \( b\Omega \) contain a one dimensional affine analytic variety. We reach a contradiction with the assumption that \( b\Omega \) does not contain an analytic variety.

If \( L = \{0\} \) then \( |k_z|^2 \to \delta_0 \) as \( z \to 0 \). This can be seen as follows. Assume that \( |k_p|^2 \to \mu_0 \). Then, by the argument at the beginning of this proof, \( \text{supp}(\mu_0) \subset \delta_0 \). So any convergent subsequence of \( \{|k_z|^2\} \) converges to \( \delta_0 \). Therefore, \( |k_z|^2 \to \delta_0 \) as \( z \to 0 \) and hence \( \tilde{T}_p(0) = \phi(0) \) for any \( \phi \in C(\overline{\Omega}) \).

In case, \( L \) is a totally real affine surface, \( p \in L \), and \( |k_p|^2 \to \mu_p \) weakly as \( j \to \infty \) for some \( \{p_j\} \subset \Omega \) then \( \text{supp}(\mu_p) \subset L \). We note that \( \mu_p \) is a representing measure for \( C(\overline{\Omega}) \cap A^2(\Omega) \) because it is the limit of \( |k_z|^2 \). Then \( f(p) = \int_{L} f(z) d\mu_p(z) \) for any \( f \in C(\overline{\Omega}) \cap A^2(\Omega) \). Let \( \psi \in C(L) \) such that \( \psi(p) = 1 \) and \( \psi < 0 \) on \( L \setminus \{p\} \). Next we will use [HW68, Theorem 1.1] and choose a sequence of holomorphic polynomials \( \{P_k\} \) that converges to \( \psi \) uniformly on \( L \). Then

\[
P_k(p) = \int_{L} P_k(z) d\mu_p(z) \to \int_{L} \psi(z) d\mu_p(z) \text{ as } k \to \infty
\]

and \( \lim_{k \to \infty} P_k(p) = 1 = \psi(p) \). Then we have \( \psi(p) = \int_{L} \psi(z) d\mu_p(z) \) for any continuous peak function at \( p \). Then \( \text{supp}(\mu_p) = \{p\} \). That is \( \mu_p = \delta_p \). So any convergent subsequence of \( \{|k_z|^2\} \) converges to \( \delta_p \). Hence \( \tilde{T}_\phi(0) = \phi(p) \) for any \( \phi \in C(\overline{\Omega}) \). Since the \( \overline{\partial} \)-Neumann operator is compact on convex domains with no analytic discs in the boundary (see [FS98, Theorem 1.1]) we use Corollary 3 to conclude that \( \Omega \) is BC-regular.

We finish the paper with the proof of Theorem 2.

**Proof of Theorem 2.** We first use [ČS09, Lemma 2] (see also [FS98, Section 2]) to conclude that there exists an affine analytic variety \( \Delta \) in \( b\Omega \). Hence, without loss of generality, we assume that

\[
\{\xi \in \mathbb{C}^n : ||\xi|| < 2\varepsilon \} \times \{0\} \subset \Delta = \{z \in \mathbb{C}^n : z_{m+1} = \cdots = z_n = 0\} \cap b\Omega
\]

for some \( \varepsilon > 0 \) and \( 1 \leq m \leq n - 1 \). Furthermore, we assume that

\[
\Omega \subset \{(z', x_n + iy_n) \in \mathbb{C}^n : z' \in \mathbb{C}^{n-1}, x_n > 0, y_n \in \mathbb{R}\},
\]

the negative \( x_n \)-axis is outward normal direction of \( b\Omega \) on \( \Delta \), and the set of strongly pseudoconvex points are dense in \( b\Omega \).

Let \( \phi \in C(\overline{\Omega}) \) such that \( 0 \leq \phi \) on \( \overline{\Omega} \), \( \phi(0) = 1 \), and \( \phi(z) < 1 \) for \( z \in \overline{\Omega} \setminus \{0\} \). Then, by the assumption, there exists a sequence of strongly pseudoconvex points \( \{p_j\} \subset b\Omega \setminus \Delta \) such that
\[ \lim_{j \to \infty} p_j = 0 \text{ and } \tilde{\phi}(p_j) = \phi(p_j) \text{ for all } j. \] This is a result of the fact that at any strongly pseudoconvex boundary point, the Berezin transform of \( \phi \) equals to \( \phi \). Hence we have

\[ \lim_{j \to \infty} \tilde{\phi}(p_j) = \lim_{j \to \infty} \phi(p_j) = \phi(0) = 1. \]

Let

\[ \Delta \Omega = \{ \xi \in \mathbb{C}^m : (\xi, z_{m+1}, \ldots, z_n) \in \Omega \text{ for some } (z_{m+1}, \ldots, z_n) \in \mathbb{C}^{n-m} \}, \]

\[ \Omega_{\xi} = \{(z_{m+1}, \ldots, z_n) \in \mathbb{C}^{n-m} : (\xi, z_{m+1}, \ldots, z_n) \in \Omega \} \]

where \( \xi \in \Delta \Omega \), and

\[ \mu_q(\xi) = \int_{\Omega_{\xi}} |k_q(\xi, z_{m+1}, \ldots, z_n)|^2 dV(z_{m+1}, \ldots, z_n). \]

Then for every \( q \in \Omega \) the measure \( \mu_q dV \) is a finite measure on \( \Delta \Omega \) as

\[ \int_{\Delta \Omega} \mu_q(\xi) dV(\xi) \leq \int_{\Omega} |k_q(\xi, z_{m+1}, \ldots, z_n)|^2 dV(\xi, z_{m+1}, \ldots, z_n) = 1. \]

Furthermore, \( \mu_q \) is a continuous plurisubharmonic function on \( \Delta \Omega \). We choose \( q_j = (0, \ldots, 0, 1/j) \). Alaoglu Theorem implies that there exists a finite measure \( \mu \) such that \( \mu_{q_j} \to \mu \) weakly as \( k \to \infty \) for some subsequence \( \{j_k\} \). We note that

\[ M_q(r) = \int_{S^{2m-1}} \mu_q(rz)r^{2m-1} d\sigma(z) \]

is increasing in \( r \). This can be seen as follows: Let \( 0 < r_1 < r_2 \) and \( u_{r_2,h} \) be the harmonic extension of \( \mu_q|_{r_2S^{2m-1}} \) onto the ball

\[ B_m(0, r_2) = \{ \xi \in \mathbb{C}^m : \|\xi\| < r_2 \}. \]

Then \( \mu_q = u_{r_2,h} \) on \( r_2S^{2m-1} \) and \( \mu_q \leq u_{r_2,h} \) on \( B_m(0, r_2) \). Then

\[ \int_{S^{2m-1}} \mu_q(r_1z)r_1^{2m-1} d\sigma(z) \leq \int_{S^{2m-1}} u_{r_2,h}(r_2z)r_2^{2m-1} d\sigma(z) \]

\[ = \int_{S^{2m-1}} \mu_q(r_2z)r_2^{2m-1} d\sigma(z). \]

Next we will show that \( \mu \neq \delta_0 \) where \( \delta_0 \) is the Dirac delta measure. Let \( g(r) = 1 - \varepsilon^{-1}r \). Then using the change of variables \( r \to 2\varepsilon - r \) on the second integral of the right hand side of the first
equality below, we get
\[
\int_{B_m(0,2\varepsilon)} g(\|\xi\|) \mu_q(\xi) dV(\xi) = \int_0^{2\varepsilon} g(r) M_q(r) dr \\
= \int_0^{\varepsilon} g(r) M_q(r) dr + \int_{\varepsilon}^{2\varepsilon} g(r) M_q(r) dr \\
\leq \int_0^{\varepsilon} (g(r) M_q(r) + g(2\varepsilon - r) M_q(2\varepsilon - r)) dr \\
= \int_0^{\varepsilon} g(r) (M_q(r) - M_q(2\varepsilon - r)) dr \\
\leq 0.
\]
In the last step above we used the fact that \( M_q(2\varepsilon - r) \geq M_q(r) \) for \( 0 \leq r \leq \varepsilon \). Hence, if we let \( k \to \infty \) we get
\[
\int_\Delta g(\|\xi\|) d\mu(\xi) \leq 0 < 1 = g(0) = \int_\Delta g(\|\xi\|) d\delta_0.
\]
That is \( \mu \neq \delta_0 \).

Since \( \mu \neq \delta_0 \) and \( \int_\Omega d\mu(\xi) \leq 1 \) we have \( \int_\Omega \phi(z) d\mu(z) < 1 \). That is,
\[
\lim_{k \to \infty} \tilde{\phi}(q_{jk}) = \int_\Omega \phi(z) d\mu(z) \neq 1.
\]
Hence,
\[
\lim_{k \to \infty} \tilde{\phi}(q_{jk}) \neq \lim_{j \to \infty} \tilde{\phi}(p_j).
\]
Therefore, \( \tilde{\phi} \not\in C(\overline{\Omega}) \). \( \square \)

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