Multivalued backward doubly stochastic differential equations with time delayed coefficients

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Abstract

In this paper, we deal with a class of multivalued backward doubly stochastic differential equations with time delayed coefficients. Based on a slight extension of the existence and uniqueness of solutions for backward doubly stochastic differential equations with time delayed coefficients, we establish the existence and uniqueness of solutions for these equations by means of Yosida approximation.

Keywords: backward doubly stochastic differential equation; time delayed coefficients; subdifferential operator; Yosida approximation.

MSC 60H10, 60G40, 60H30

1 Introduction

Backward Stochastic Differential Equations (BSDEs in short) have been first introduced in Pardoux and Peng \([11]\) in order to give a probabilistic interpretation (Feynman-Kac formula) for the solutions of semilinear parabolic PDEs. In addition, in order to give a probabilistic representation for a class of quasilinear stochastic partial differential equations, Pardoux and Peng \([12]\) introduced a new class of BSDEs, called as backward doubly stochastic differential equations (BDSDEs in short). This equation involves two different directions of stochastic integrals and has also appeared as a powerful tool to give probabilistic formulas for solutions of stochastic PDEs/PDIEs. One can see Bally and Matoussi \([1]\); Matoussi \([10]\); Zhang and Zhao \([16]\); Ren et al. \([15]\) and the references therein.

On the other hand, BSDEs involving a subdifferential operator has been treated by Pardoux and Raşcanu \([13]\), that they used to give a probabilistic representation for a class of parabolic (and elliptic)

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variational inequalities. Furthermore, Maticiuc and Raşcanu [9] gave a probability interpretation of the viscosity solution of the parabolic variational inequality (PVI in short) with a mixed nonlinear multivalued Neumann–Dirichlet boundary condition. Moreover, Boufoussi and Mrhardy [2] established the existence result to stochastic viscosity solution for some multivalued parabolic stochastic partial differential equation via BDSDEs whose coefficient contains the subdifferential of a convex function.

Recently, Delong and Imkeller [5] introduced a class of BSDEs with time delayed generators of the form

\[ Y(t) = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW(s), \quad (1.1) \]

where the generator \( f \) at time \( s \) depends arbitrary on the past values of a solution \( (Y_s, Z_s) = (Y(s + u), Z(s + u)), -T \leq u \leq 0 \). They proved in [5] the existence and uniqueness of a solution for (1.1).

Moreover, in Delong and Imkeller [6], they established the existence and uniqueness as well as the Malliavin’s differentiability of the solution for BSDEs with time delayed generators driven by Brownian motions and Poisson random measures. Following this, Reis et al. [14] extended the results of [5] and [6] in \( L^p \)-spaces. For the applications of BSDEs with time delayed coefficients in insurance and finance, one can see Delong [4]. Very recently, Diomande and Maticiuc [7] established the existence and uniqueness result for multivalued BSDEs with time delayed generators.

Besides, Lu and Ren [8] proved the existence and uniqueness of the solutions for a class of BDSDEs with time delayed coefficients under Lipschitz condition. Based on an extension of the existence result of Lu and Ren [8], the present paper is to establish the existence and uniqueness of the solutions for multivalued BDSDEs with time delayed coefficients.

The paper is organized as follows. In Section 2, we give some preliminaries. Section 3 is concerned with BDSDEs with time delayed coefficients. In Section 4, we prove the existence and uniqueness of the solution for multivalued BDSDEs with time delayed coefficients.

2 Notations, preliminaries and basic assumptions

In this section, we provide some spaces and notations used in the sequel. More precisely, consider two mutually independent \( d \)-dimensional Brownian motions \( \{ W_t, 0 \leq t \leq T \} \) and \( \{ B_t, 0 \leq t \leq T \} \) defined on the probability spaces \( (\Omega_1, \mathcal{F}_1, \mathbb{P}_1) \) and \( (\Omega_2, \mathcal{F}_2, \mathbb{P}_2) \), respectively, where \( T < \infty \) is a finite time horizon. We denote

\[ \mathcal{F}_r^B := \sigma \{ B_r - B_s, s \leq r \leq t \}, \quad \mathcal{F}_r^W := \sigma \{ W_r, 0 \leq r \leq t \}. \]

Moreover, we define \( \Omega = \Omega_1 \times \Omega_2, \mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2 \) and \( \mathbb{P} = \mathbb{P}_1 \otimes \mathbb{P}_2 \). We put

\[ \mathcal{F}_t := \mathcal{F}_t^W \otimes \mathcal{F}_{s,t}^B \otimes \mathcal{N}, \]

where \( \mathcal{N} \) is the collection of \( \mathbb{P} \)-null sets of \( \mathcal{F} \). We use the usual Euclidian norm \( \| \cdot \| \) in \( \mathbb{R}^k \) and \( \mathbb{R}^{k \times d} \).

In what follows, we need the following spaces.

- \( L^2_{-T} (\mathbb{R}^{k \times d}) \): the space of measurable functions \( z : [-T, 0] \to \mathbb{R}^{k \times d} \) such that \( \int_{-T}^0 |z(t)|^2 dt < \infty \).
- \( L^\infty_{-T} (\mathbb{R}^k) \): the space of measurable functions \( y : [-T, 0] \to \mathbb{R}^k \) such that \( \sup_{-T \leq t \leq 0} |y(t)|^2 < \infty \).
- \( H^2_T (\mathbb{R}^m) \): the space of \( \mathcal{F} \)-predictable processes \( \psi : \Omega \times [0, T] \to \mathbb{R}^m \) such that \( E \int_0^T |\psi(t)|^2 dt < \infty \).
Now, we give the following assumptions.

\( S^2_F(\mathbb{R}^k) \): the space of \( F \)-adapted, product measurable processes \( Y : \Omega \times [0, T] \to \mathbb{R}^k \) such that 
\[ E \left[ \sup_{0 \leq t \leq T} |Y(t)|^2 \right] < \infty. \]

The spaces \( H^2_T(\mathbb{R}^{k \times d}) \) and \( S^2_T(\mathbb{R}^k) \) are endowed with the norm
\[
\|Z\|_{H^2_T}^2 = E \int_0^T e^{2\beta |Z(t)|^2} dt \quad \text{and} \quad \|Y\|_{S^2_T}^2 = E \left[ \sup_{0 \leq t \leq T} e^{\beta |Y(t)|^2} \right]
\]
respectively with some \( \beta > 0 \).

The purpose of the present paper is to consider the following multivalued BDSDE with time delayed coefficients:
\[
\begin{aligned}
-dY(t) + \partial \phi(Y(t))dt &\ni f(t, Y(t), Z(t), Y_t, Z_t)dt + g(t, Y_t, Z_t)dB(t) - Z(t)dW(t), \quad 0 \leq t \leq T, \\
Y_T &= \xi,
\end{aligned}
\]
(2.1)
here the coefficients \( f \) and \( g \) at time set can depend on the past values of the solution denoted by \( Y_s := (Y(s + \theta))_{-T \leq \theta \leq 0} \) and \( Z_s := (Z(s + \theta))_{-T \leq \theta \leq 0} \).

**Remark 2.1** Throughout this paper, we always assume that \( Y(t) = 0 \) and \( Z(t) = 0 \) for \( t < 0 \).

We mention that \( \partial \phi \) in Eq. (2.1) is the subdifferential operator of the function \( \phi : \mathbb{R}^k \to (-\infty, +\infty] \) which satisfies:
(i) \( \phi \) is proper (\( \phi \not\equiv \infty \)), convex and lower semicontinuous (l.s.c. for short),
(ii) without loss generality, \( \phi(y) \geq \phi(0) = 0, \forall y \in \mathbb{R}^k \).

For \( \phi \), let’s define:
- \( \text{Dom} \phi := \{ u \in \mathbb{R}^k : \phi(u) < \infty \} \),
- \( \partial \phi(u) := \{ u^* \in \mathbb{R}^k : \langle u^*, v - u \rangle + \phi(u) \leq \phi(v), \forall v \in \mathbb{R}^k \} \),
- \( \text{Dom}(\partial \phi) := \{ u \in \mathbb{R}^k : \partial \phi(u) \neq \emptyset \} \),
- \( (u, u^*) \in \partial \phi \iff u \in \text{Dom}(\partial \phi), u^* \in \partial \phi(u) \).

**Remark 2.2** It is well known that the subdifferential operator \( \partial \phi \) is a maximal monotone operator, i.e.,
is maximal in the class of operators which satisfy the condition
\[
\langle u^* - v^*, u - v \rangle \geq 0, \forall (u, u^*), (v, v^*) \in \partial \phi.
\]

Now, we give the following assumptions.

(H1) \( E \left[ |\xi|^2 + \phi(\xi) \right] < \infty. \)

(H2) The coefficients \( f : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \times L^\infty_{-T}(\mathbb{R}^k) \times L^2_{-T}(\mathbb{R}^{k \times d}) \to \mathbb{R}^k \) and \( g : \Omega \times [0, T] \times L^\infty_{-T}(\mathbb{R}^k) \times L^2_{-T}(\mathbb{R}^{k \times d}) \to \mathbb{R}^{k \times d} \) are product measurable, \( F \)-adapted and Lipschitz continuous in the sense that there exist positive constant \( K, L \) and \( R \) such that, for a non-random, finitely valued measure \( \alpha \) supported on \([ -T, 0 ] \) and for any \( t \in [0, T], (y^1, z^1), (y^2, z^2) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}, (y^1_t, z^1_t), (y^2_t, z^2_t) \in L^\infty_{-T}(\mathbb{R}^k) \times L^2_{-T}(\mathbb{R}^{k \times d}) \), \( \mathbb{P} \text{-a.s.} \)
Definition 2.1

\( k \) is \(\alpha\) if

\[
Y(t) = \xi + \int_t^T f(s, Y(s), Z(s), Y_\theta, Z_\theta) ds + \int_t^T g(s, Y(s), Z(s), Y_\theta, Z_\theta) dB(s) - \int_t^T Z(s) dW(s), \quad 0 \leq t \leq T.
\]

Remark 2.3 We remark that, taking the measure \( \alpha \) as Dirac measure \( \delta_\nu \), with \( \nu \in (0, T] \) or as Lebesgue measure, the coefficients could be of the form \( k(t, y, z) = L\theta(t - r) \) or \( k(t, y, z) = L \int_0^t \theta(s) ds \) with \( k = f, g \) and \( \theta = y, z \).

We end this section by introduce the definition of the solution for multivalued BDSDE (2.1).

Definition 2.1 The triple \((Y, Z, U)\) is a solution of multivalued BDSDE (2.1) with subdifferential operator if

(i) \( (Y, Z, U) \in \mathcal{L}^2(\mathbb{R}^k) \times \mathcal{H}^2_1(\mathbb{R}^{k \times d}) \times \mathcal{H}^2_1(\mathbb{R}^k) \),

(ii) \( E \int_0^T e^{\lambda t} \varphi(Y(t)) dt < \infty \),

(iii) \((Y(t), U(t)) \in \partial \varphi, d\mathbb{P} \times dt\text{-a.e. on } \Omega \times [0, T] \),

(iv) \( Y(t) + \int_t^T U(s) ds = \xi + \int_t^T f(s, Y(s), Z(s), Y_\theta, Z_\theta) ds + \int_t^T g(s, Y(s), Z(s), Y_\theta, Z_\theta) dB(s) - \int_t^T Z(s) dW(s) \), \( t \in [0, T] \).

3 BDSDEs with time delay coefficients

In this part, we consider a class of BDSDEs with time delayed coefficients as the form:

\[
Y(t) = \xi + \int_t^T f(s, Y(s), Z(s), Y_\theta, Z_\theta) ds + \int_t^T g(s, Y(s), Z(s), Y_\theta, Z_\theta) dB(s) - \int_t^T Z(s) dW(s), 0 \leq t \leq T.
\]

We mention that the above equation is an extension of that of Lu and Ren [8], since the coefficients \( f \) and \( g \) can depend on both the present and the past values of a solution \((Y, Z)\).

Now, we propose the definition of the solution for BDSDE (3.1).
Definition 3.1 A solution to the BDSDE (3.1) is a pair of \((Y, Z) = (Y(t), Z(t))_{0 \leq t \leq T}\) satisfying that the BDSDE (3.1) such that \((Y, Z) \in S^2_T(\mathbb{R}^k) \times H^2_T(\mathbb{R}^{k \times d}).\)

Next, we list some results on BDSDE (3.1). Since their proofs are similar to that of Lemma 3.1 and Theorem 3.2 of Lu and Ren [8] with only a few slight changes, so we prefer to omit them.

Lemma 3.1 Assume \(E|\xi|^2 < \infty\) and the assumptions (H2)–(H4) hold, and \((Y, Z) \in S^2_T(\mathbb{R}^k) \times H^2_T(\mathbb{R}^{k \times d})\) be a solution of the BDSDE (3.1). If the Lipschitz constant \(L\) of the coefficients \(f\) and \(g\) is small enough, then there exist two positive constants \(\beta\) and \(C\) such that

\[
E \left[ \sup_{0 \leq s \leq T} e^{\beta s} |Y(s)|^2 + \int_0^T e^{\beta s} |Z(s)|^2 ds \right] \leq C E \left[ e^{\beta T} |\xi|^2 + \int_0^T e^{\beta s} |f(s, 0, 0, 0, 0)|^2 ds + \int_0^T e^{\beta s} |g(s, 0, 0, 0, 0)|^2 ds \right]. \tag{3.2}
\]

Here and in the sequel, \(C > 0\) denotes a constant whose value may change from one line to another.

Theorem 3.1 Assume \(E|\xi|^2 < \infty\) and the assumptions (H2)–(H4) hold. If the Lipschitz constant \(L\) of the coefficients \(f\) and \(g\) is small enough, then the BDSDE (3.1) has a unique solution.

4 Existence and uniqueness of the solution

This section is devoted to the study of the existence and uniqueness result of multivalued BDSDE (2.1). The main result of this section is the following

Theorem 4.1 Assume that the assumptions (H1)–(H4) hold. If the Lipschitz constant \(L\) of the coefficients \(f\) and \(g\) is small enough, then there exists a unique solution of BDSDE (2.1).

We mention that our proof is based on the Yosida approximations. First of all, let’s introduce an approximation of the function \(\varphi\) by a convex \(C^1\)–function \(\varphi_\varepsilon, \varepsilon > 0\), defined by

\[
\varphi_\varepsilon(u) = \inf \left\{ \frac{1}{2\varepsilon} |u - v|^2 + \varphi(v) : v \in \mathbb{R}^k \right\} = \frac{1}{2\varepsilon} |u - J_\varepsilon(u)|^2 + \varphi(J_\varepsilon u), \tag{4.1}
\]

where \(J_\varepsilon(u) = (I + \varepsilon \partial \varphi)^{-1}(u)\). For convenience, we illustrate some properties of this approximation, for more details, one can see Brézis [3].

Proposition 4.1 For all \(\varepsilon, \delta > 0, u, v \in \mathbb{R}^k\), it holds that

(i) \(\varphi_\varepsilon\) is a convex function with the gradient being a Lipschitz function;

(ii) \(\varphi_\varepsilon(u) \leq \varphi(u)\);

(iii) \(\nabla \varphi_\varepsilon(u) = \partial \varphi_\varepsilon(u) = \frac{u - J_\varepsilon(u)}{\varepsilon} \in \partial \varphi(J_\varepsilon u)\);

(iv) \(|J_\varepsilon(u) - J_\varepsilon(v)| \leq |u - v|\);

(v) \(0 \leq \varphi_\varepsilon(u) \leq \langle \nabla \varphi_\varepsilon(u), u \rangle\);
Let us consider the approximating equation as the form:

\[
Y_t^\varepsilon + \int_t^T \nabla \varphi_e(Y^\varepsilon(s)) ds = \xi + \int_t^T f(s, Y^\varepsilon(s), Z^\varepsilon(s), Y_s^\varepsilon, Z_s^\varepsilon) ds + \int_t^T g(s, Y^\varepsilon(s), Z^\varepsilon(s), Y_s^\varepsilon, Z_s^\varepsilon) dB(s) - \int_t^T Z^\varepsilon(s) dW(s), t \in [0, T].
\]  

where \( \xi, f \) and \( g \) satisfy assumptions (H1)–(H4). Since \( \nabla \varphi_e \) is Lipschitz continuous, we know that from Theorem [3.1] for sufficiently small \( L \) and \( R \), BDSDE (4.1) has a unique solution \( (Y^\varepsilon, Z^\varepsilon) \in S^2_t(\mathbb{R}^k) \times H^2_t(\mathbb{R}^k 	imes d) \).

**Lemma 4.1** Assume the assumptions (H1)–(H4) hold. If the Lipschitz coefficients \( L \) and \( R \) are small enough, then it holds that

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} e^{\beta t} |Y^\varepsilon(t)|^2 + \int_0^T e^{\beta s} |Z^\varepsilon(s)|^2 ds \right] \leq CM_1,
\]

where \( M_1 := \mathbb{E} \left[ e^{\beta T} |\xi|^2 + \int_0^T e^{\beta s} |f(s, 0, 0, 0)|^2 ds + \int_0^T e^{\beta s} |g(s, 0, 0, 0)|^2 ds \right] \).

**Proof.** For any \( \beta > 0 \), applying Itô’s formula to \( e^{\beta t} |Y^\varepsilon(t)|^2 \) yields that

\[
e^{\beta T} |\xi|^2 + 2 \int_t^T e^{\beta s} \langle Y^\varepsilon(s), f(s, Y^\varepsilon(s), Z^\varepsilon(s), Y_s^\varepsilon, Z_s^\varepsilon) \rangle ds + \int_t^T e^{\beta s} |g(s, Y^\varepsilon(s), Z^\varepsilon(s), Y_s^\varepsilon, Z_s^\varepsilon)|^2 ds
\]

\[
+ 2 \int_t^T e^{\beta s} \langle Y^\varepsilon(s), g(s, Y^\varepsilon(s), Z^\varepsilon(s), Y_s^\varepsilon, Z_s^\varepsilon) dB(s) \rangle - 2 \int_t^T e^{\beta s} \langle Y^\varepsilon(s), Z^\varepsilon(s) dW(s) \rangle.
\]

By Young’s inequality and (H2), we have

\[
2 \int_t^T e^{\beta s} \langle Y^\varepsilon(s), f(s, Y^\varepsilon(s), Z^\varepsilon(s), Y_s^\varepsilon, Z_s^\varepsilon) \rangle ds \\
\leq \gamma \int_t^T e^{\beta s} |Y^\varepsilon(s)|^2 ds + \frac{1}{\gamma} \int_t^T e^{\beta s} |f(s, Y^\varepsilon(s), Z^\varepsilon(s), Y_s^\varepsilon, Z_s^\varepsilon)|^2 ds \\
\leq \gamma \int_t^T e^{\beta s} |Y^\varepsilon(s)|^2 ds + \frac{3}{\gamma} \int_t^T e^{\beta s} |f(s, 0, 0, 0, 0)|^2 ds + \frac{6K^2}{\gamma} \int_t^T e^{\beta s} (|Y^\varepsilon(s)|^2 + |Z^\varepsilon(s)|^2) ds
\]

\[
+ \frac{3L}{\gamma} \int_t^T e^{\beta s} \int_0^s (|Y^\varepsilon(s + \theta)|^2 + |Z^\varepsilon(s + \theta)|^2) \alpha(d\theta) ds
\]

and

\[
\int_t^T e^{\beta s} |g(s, Y^\varepsilon(s), Z^\varepsilon(s), Y_s^\varepsilon, Z_s^\varepsilon)|^2 ds \leq 6R^2 \int_t^T e^{\beta s} (|Y^\varepsilon(s)|^2 + |Z^\varepsilon(s)|^2) ds \\
+ 3 \int_t^T e^{\beta s} |g(s, 0, 0, 0, 0)|^2 ds + 3L \int_t^T e^{\beta s} \int_0^s (|Y^\varepsilon(s + \theta)|^2 + |Z^\varepsilon(s + \theta)|^2) \alpha(d\theta) ds.
\]
By a change of integration order argument and Remark 2.1, we obtain
\[
\int_{0}^{T} e^{\beta s} \int_{0}^{T} |Y(s + \theta)|^2 d\alpha(d\theta) ds = \int_{-T}^{0} \int_{0}^{T} e^{\beta t} |Y(s + \theta)|^2 d\alpha(d\theta) = \int_{-T}^{0} e^{-\beta \theta} \int_{\theta}^{T+\theta} e^{\beta t} |Y(t)|^2 d\alpha(d\theta) \leq \min \left\{ T \bar{\alpha} \sup_{0 \leq t \leq T} e^{\beta t} |Y(t)|^2, \bar{\alpha} \int_{0}^{T} e^{\beta t} |Y(t)|^2 dt \right\}. \tag{4.7}
\]
and
\[
\int_{0}^{T} e^{\beta s} \int_{-T}^{0} |Z(s + \theta)|^2 d\alpha(d\theta) ds = \int_{-T}^{0} e^{-\beta \theta} \int_{0}^{T+\theta} e^{\beta t} |Z(t)|^2 d\alpha(d\theta) \leq \bar{\alpha} \int_{0}^{T} e^{\beta t} |Z(t)|^2 dt, \tag{4.8}
\]
where \(\bar{\alpha} = \int_{-T}^{0} e^{-\beta \theta} \alpha(d\theta)\).

Combining (4.5)-(4.8) together with (v) of Proposition 4.1, (4.2) becomes
\[
e^{\beta T} |Y^E(t)|^2 + \beta \int_{0}^{T} e^{\beta s} |Y^E(s)|^2 ds + \int_{0}^{T} e^{\beta s} |Z^E(s)|^2 ds \leq e^{\beta T} |\xi|^2 + \gamma \int_{0}^{T} e^{\beta s} |Y^E(s)|^2 ds + \frac{6K^2}{\gamma} \int_{0}^{T} e^{\alpha s} (|Y^E(s)|^2 + |Z^E(s)|^2) ds + \frac{3}{\gamma} \int_{0}^{T} e^{\beta s} |f(s, 0, 0, 0)|^2 ds + 3 \int_{0}^{T} e^{\beta s} |g(s, 0, 0, 0)|^2 ds + 6R^2 \int_{0}^{T} e^{\beta s} (|Y^E(s)|^2 + |Z^E(s)|^2) ds + \left( \frac{3L\bar{\alpha}}{\gamma} + 3L\bar{\alpha} \right) \int_{0}^{T} e^{\beta s} (|Y^E(s)|^2 + |Z^E(s)|^2) ds + 2 \int_{0}^{T} e^{\beta s} (Y^E(s), g(s, Y^E(s), Z^E) dB(s)) - 2 \int_{0}^{T} e^{\beta s} (Y^E(s), Z^E(s) dW(s)). \tag{4.9}
\]
For \(t = 0\), taking expectation on both sides of above gives
\[
E |Y^E(0)|^2 + K_1 E \int_{0}^{T} e^{\beta s} |Y^E(s)|^2 ds + K_2 E \int_{0}^{T} e^{\beta s} |Z^E(s)|^2 ds \leq E [e^{\beta T} |\xi|^2] + \frac{3}{\gamma} E \int_{0}^{T} e^{\beta s} |f(s, 0, 0, 0)|^2 ds + 3E \int_{0}^{T} e^{\beta s} |g(s, 0, 0, 0)|^2 ds. \tag{4.10}
\]
where \(K_1 := \beta - \gamma - \frac{6K^2}{\gamma} - \frac{3L\bar{\alpha}}{\gamma} - 3L\bar{\alpha} - 6R^2, K_2 := 1 - \frac{6K^2}{\gamma} - \frac{3L\bar{\alpha}}{\gamma} - 3L\bar{\alpha} - 6R^2\).

For sufficiently small \(L\) and \(R\), choosing \(\beta > 0, \gamma > 0\) such that \(K_1 > 0\) and \(K_2 > 0\), by (4.10), we obtain that there exists a constant \(C > 0\) depending on \(\beta, \gamma, K, L, R\) and \(\bar{\alpha}\) such that
\[
E \int_{0}^{T} e^{\beta s} |Y^E(s)|^2 ds + E \int_{0}^{T} e^{\beta s} |Z^E(s)|^2 ds \leq C \left\{ E [e^{\beta T} |\xi|^2] + E \int_{0}^{T} e^{\beta s} |f(s, 0, 0, 0)|^2 ds + E \int_{0}^{T} e^{\beta s} |g(s, 0, 0, 0)|^2 ds \right\}. \tag{4.11}
\]
On the other hand, for \(\beta\) and \(\gamma\) choosing above, from (4.9), we have
\[
\sup_{0 \leq t \leq T} e^{\beta t} |Y^E(t)|^2 \leq e^{\beta T} |\xi|^2 + \frac{3}{\gamma} \int_{0}^{T} e^{\beta s} |f(s, 0, 0, 0)|^2 ds + 3 \int_{0}^{T} e^{\beta s} |g(s, 0, 0, 0)|^2 ds + 2 \sup_{0 \leq t \leq T} \left| \int_{0}^{T} e^{\beta s} (Y^E(s), g(s, Y^E(s), Z^E(s), Y^E(s), Z^E(s)) dB(s)) \right| + 2 \sup_{0 \leq t \leq T} \left| \int_{0}^{T} e^{\beta s} (Y^E(s), Z^E(s) dW(s)) \right|. \tag{4.12}
\]
By the Burkholder–Davis–Gundy inequality and Young’s inequality, together with (4.5)–(4.6) and (H2), there exists a constant $d_1 > 0$ such that

\[
2\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t e^{\beta s} \langle Y^e(s), g(s, Y^e(s), Z^e(s), Y^e_s, Z^e_s) \rangle dB(s) \right| \right] \\
\leq d_1 \left[ \rho_1 \mathbb{E} \left( \sup_{0 \leq t \leq T} e^{\beta t} |Y^e(t)|^2 \right) + \frac{6R^2 + 3L\tilde{\alpha}}{\rho_1} \mathbb{E} \int_0^T e^{\beta s} |Z^e(s)|^2 ds \\
+ \frac{3}{\rho_1} \mathbb{E} \int_0^T e^{\beta s} |g(s, 0, 0, 0, 0)|^2 ds \right],
\]

(4.13)

Similarly, there exists a constant $d_2 > 0$ such that

\[
2\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t e^{\beta s} \langle Y^e(s), Z^e(s) \rangle dW(s) \right| \right] \\
\leq d_2 \left[ \rho_2 \mathbb{E} \left( \sup_{0 \leq t \leq T} e^{\beta t} |Y^e(t)|^2 \right) + \frac{1}{\rho_2} \mathbb{E} \int_0^T e^{\beta s} |Z^e(s)|^2 ds \right],
\]

(4.14)

where $\rho_1$ and $\rho_2$ are two positive constants.

Then, choosing $\rho_1 = \frac{1}{3d_1}$ and $\rho_2 = \frac{1}{3d_2}$, for sufficiently small $L > 0$ and $R > 0$, there exists a constant $C > 0$ depending on $\beta, \gamma, L, K, \tilde{\alpha}, d_1$ and $d_2$ such that

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} e^{\beta t} |Y^e(t)|^2 + \int_0^T e^{\beta s} |Z^e(s)|^2 ds \right] \\
\leq C \mathbb{E} \left[ e^{\beta T} |\xi|^2 + \int_0^T e^{\beta s} |f(s, 0, 0, 0, 0)|^2 ds + \int_0^T e^{\beta s} |g(s, 0, 0, 0, 0)|^2 ds \right].
\]

The lemma is proved.

**Lemma 4.2** Assume that the conditions of Lemma 4.1 hold. Then, for all $0 \leq t \leq T$, it holds that

(i) $\mathbb{E} \left( \int_0^T e^{\beta s} |\nabla \Phi_e(Y^e(s))|^2 ds \right) \leq CM_2$,

(ii) $\mathbb{E} \left[ e^{\beta T} \Phi_e(J_e(Y^e(t))) + E \int_0^T e^{\beta s} \Phi_e(J_e(Y^e(s))) ds \right] \leq CM_2$,

(iii) $\mathbb{E} [e^{\beta T} |Y^e(t) - J_e(Y^e(t))|^2 ds] \leq C\epsilon M_2$,

where $M_2 := M_1 + \mathbb{E}[e^{\beta T} \Phi(\xi)]$.

**Proof.** The stochastic subdifferential inequality in Pardoux and Răşcanu [13] gives that

\[
e^{\beta T} \Phi_e(\xi) \geq e^{\beta T} \Phi_e(Y^e(t)) + \int_0^T e^{\beta s} \langle \nabla \Phi_e(Y^e(s)), dY^e(s) \rangle + \int_0^T \Phi_e(Y^e(s)) d(e^{\beta s}).
\]
Moreover, since

\[
\begin{align*}
    e^{\beta t}\varphi_e(Y^e(t)) + \beta \int_t^T e^{\beta s}\varphi_e(Y^e(s))ds + \int_t^T e^{\beta s}\left|\nabla\varphi_e(Y^e(s))\right|^2ds \\
    \leq \ e^{\beta T}\varphi_e(\xi) + \int_t^T e^{\beta s}\langle \nabla\varphi_e(Y^e(s)), f(s, Y^e(s), Z^e(s), Y^e_s, Z^e_s) \rangle ds \\
    + \int_t^T e^{\beta s}\langle \nabla\varphi_e(Y^e(s)), g(s, Y^e(s), Z^e(s), Y^e_s, Z^e_s) dB(s) \rangle \\
    - \int_t^T e^{\beta s}\langle \nabla\varphi_e(Y^e(s)), Z^e(s)dW(s) \rangle. 
\end{align*}
\]

Therefore,

\[
e^{\beta t}\varphi_e(Y^e(t)) + \beta \int_t^T e^{\beta s}\varphi_e(Y^e(s))ds + \int_t^T e^{\beta s}\left|\nabla\varphi_e(Y^e(s))\right|^2ds \\
    \leq \ e^{\beta T}\varphi_e(\xi) + \int_t^T e^{\beta s}\langle \nabla\varphi_e(Y^e(s)), f(s, Y^e(s), Z^e(s), Y^e_s, Z^e_s) \rangle ds \\
    + \int_t^T e^{\beta s}\langle \nabla\varphi_e(Y^e(s)), g(s, Y^e(s), Z^e(s), Y^e_s, Z^e_s) dB(s) \rangle \\
    - \int_t^T e^{\beta s}\langle \nabla\varphi_e(Y^e(s)), Z^e(s)dW(s) \rangle. 
\]

Since

\[
\begin{align*}
    \int_t^T e^{\beta s}\langle \nabla\varphi_e(Y^e(s)), f(s, Y^e(s), Z^e(s), Y^e_s, Z^e_s) \rangle ds \\
    \leq \ \frac{1}{2} \int_t^T e^{\beta s}\left|\nabla\varphi_e(Y^e(s))\right|^2ds + \frac{3}{2} \int_t^T e^{\beta s}\left|f(s, 0, 0, 0, 0)\right|^2ds \\
    + 3K^2 \int_t^T (|Y^e(s)|^2 + |Y^e(s)|^2)ds + \frac{3L\alpha}{2} \int_0^T e^{\beta s}(|Y^e(s)|^2 + |Z^e(s)|^2)ds.
\end{align*}
\]

Then, by Lemma 4.1 and the nonnegative property of $\varphi_e(y)$, (i) hold for sufficiently small $L$ and $R$. From (4.15), for sufficiently small $L$ and $R$, we get

\[
\mathbb{E}[e^{\beta T}\varphi_e(Y^e(t))] + \mathbb{E} \int_t^T e^{\beta s}\varphi_e(Y^e(s))ds \leq CM_2.
\]

since $\varphi(J_e(y)) \leq \varphi_e(y)$ (see 4.11), it follows that

\[
\mathbb{E} \left[ e^{\beta T}\varphi(J_e(Y^e(t))) \right] + \mathbb{E} \int_t^T e^{\beta s}\varphi(J_e(Y^e(s)))ds \leq CM_2.
\]

Moreover, since

\[
\frac{1}{2\varepsilon} e^{\beta T}|Y^e(t) - J_e(Y^e(t))|^2 \leq e^{\beta T}\varphi_e(Y^e(t)),
\]

we then have

\[
\mathbb{E} \left[ e^{\beta T}|Y^e(t) - J_e(Y^e(t))|^2 \right] \leq C\varepsilon M_2.
\]

The proof is complete.

**Lemma 4.3** Assume that the conditions of Lemma 4.1 hold. Then, it holds that

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} e^{\beta T}|Y^e(t) - Y^\delta(t)|^2 + \int_0^T e^{\beta s}|Z^e(s) - Z^\delta(s)|^2ds \right] \leq C(\varepsilon + \delta)M_2.
\]
Proof. Applying Itô’s formula to $e^{\beta t}|Y^\varepsilon(t)|^2$ yields

$$
e^{\beta t}|Y^\varepsilon(t)|^2 + \beta \int_t^T e^{\beta s}|Y^\varepsilon(s)|^2 ds + \int_t^T e^{\beta s}|Z^\varepsilon(s) - Z^\delta(s)|^2 ds$$

$$+ 2 \int_t^T e^{\beta s} (Y^\varepsilon(s) - Y^\delta(s), \nabla \varphi_\varepsilon(Y^\varepsilon(s)) - \nabla \varphi_\delta(Y^\delta(s))) ds$$

$$= 2 \int_t^T e^{\beta s} (Y^\varepsilon(s) - Y^\delta(s), f(s, Y^\varepsilon(s), Z^\varepsilon(s), Y^\varepsilon, Z^\varepsilon) - f(s, Y^\delta(s), Z^\delta(s), Y^\delta, Z^\delta)) ds$$

$$+ 2 \int_t^T e^{\beta s} (g(s, Y^\varepsilon(s), Z^\varepsilon(s), Y^\varepsilon, Z^\varepsilon) - g(s, Y^\delta(s), Z^\delta(s), Y^\delta, Z^\delta)) dB(s)$$

$$- 2 \int_t^T e^{\beta s} (Y^\varepsilon(s) - Y^\delta(s), (Z^\varepsilon(s) - Z^\delta(s)) dW(s)). \tag{4.16}$$

Since

$$\langle Y^\varepsilon(s) - Y^\delta(s), \nabla \varphi_\varepsilon(Y^\varepsilon(s)) - \nabla \varphi_\delta(Y^\delta(s)) \rangle \geq -(\varepsilon + \delta) |\nabla \varphi_\varepsilon(Y^\varepsilon(s))||\nabla \varphi_\delta(Y^\delta(s))|,$$

by Young’s inequality and the assumptions of $f$ and $g$, we get

$$2 \int_t^T e^{\beta s} (Y^\varepsilon(s) - Y^\delta(s), f(s, Y^\varepsilon(s), Z^\varepsilon(s), Y^\varepsilon, Z^\varepsilon) - f(s, Y^\delta(s), Z^\delta(s), Y^\delta, Z^\delta)) ds$$

$$\leq \gamma \int_t^T e^{\beta s} |Y^\varepsilon(s) - Y^\delta(s)|^2 ds + \frac{4K^2}{\gamma} \int_t^T e^{\beta s} (|Y^\varepsilon(s) - Y^\delta(s)|^2 + |Z^\varepsilon(s) - Z^\delta(s)|^2) ds$$

$$+ \frac{2M}{\gamma} \int_t^T e^{\beta s} \left[ \int_0^t (Y^\varepsilon(s) + \theta) - Y^\delta(s + \theta) \right]^2 + |Z^\varepsilon(s) - \alpha d\theta \right] ds$$

$$\leq \gamma \int_t^T e^{\beta s} |Y^\varepsilon(s) - Y^\delta(s)|^2 ds + \frac{4K^2}{\gamma} \int_t^T e^{\beta s} (|Y^\varepsilon(s) - Y^\delta(s)|^2 + |Z^\varepsilon(s) - Z^\delta(s)|^2) ds$$

$$+ \frac{2M}{\gamma} \int_0^T e^{\beta s} (|Y^\varepsilon(s) - Y^\delta(s)|^2 + |Z^\varepsilon(s) - Z^\delta(s)|^2) ds \tag{4.17}$$

and

$$\int_t^T e^{\beta s} |g(s, Y^\varepsilon(s), Z^\varepsilon(s), Y^\varepsilon, Z^\varepsilon) - g(s, Y^\delta(s), Z^\delta(s), Y^\delta, Z^\delta)|^2 ds$$

$$\leq 4R^2 \int_t^T e^{\beta s} (|Y^\varepsilon(s) - Y^\delta(s)|^2 + |Z^\varepsilon(s) - Z^\delta(s)|^2) ds$$

$$+ 2M \int_0^T e^{\beta s} (|Y^\varepsilon(s) - Y^\delta(s)|^2 + |Z^\varepsilon(s) - Z^\delta(s)|^2) ds. \tag{4.18}$$

Combining (4.16)–(4.18), we obtain

$$|Y^\varepsilon(0) - Y^\delta(0)|^2 + K_3 \int_0^T e^{\beta s} |Y^\varepsilon(s) - Y^\delta(s)|^2 ds + K_4 \int_0^T e^{\beta s} |Z^\varepsilon(s) - Z^\delta(s)|^2 ds$$

$$\leq 2(\varepsilon + \delta) \int_t^T e^{\beta s} |\nabla \varphi_\varepsilon(Y^\varepsilon(s))||\nabla \varphi_\delta(Y^\delta(s))| ds. \tag{4.19}$$
where $K_3 := \beta - \gamma - \frac{4K^2}{T} - \frac{2L\bar{\alpha}}{T} - 2L\bar{\alpha} - 4R^2$, $K_4 := 1 - \frac{4K^2}{T} - \frac{2L\bar{\alpha}}{T} - 2L\bar{\alpha} - 4R^2$. For sufficiently small $L$ and $R$, choosing $\beta, \gamma > 0$ such that $K_3 > 0$ and $K_4 > 0$, by (i) of Lemma 4.2 we have

$$
\mathbb{E} \int_0^T e^{\beta s} |Y^\varepsilon(s) - Y^\delta(s)|^2 ds + \mathbb{E} \int_0^T e^{\beta s} |Z^\varepsilon(s) - Z^\delta(s)|^2 ds \leq C(\varepsilon + \delta)M_2. \tag{4.20}
$$

Therefore, as the same procedure as (4.13)–(4.14), we can get the desired result from the Burkholder–Davis–Gundy inequality and (4.20).

Now, let’s give a proof of Theorem 4.1.

**Proof. Existence.** Lemma 4.3 implies that there exist $Y \in S^2_T(\mathbb{R}^2)$ and $Z \in H^2_T(\mathbb{R}^{k \times d})$ such that

$$
\lim_{\varepsilon \to 0} (Y^\varepsilon, Z^\varepsilon) = (Y, Z).
$$

Then Lemma 4.2 shows that

$$
\lim_{\varepsilon \to 0} J_\varepsilon(Y^\varepsilon) = y \text{ in } H^2_T(\mathbb{R}^k)
$$

and

$$
\lim_{\varepsilon \to 0} E[e^{\beta t} |J_\varepsilon(Y^\varepsilon(t)) - y(t)|^2] = 0, \quad 0 \leq t \leq T.
$$

Moreover, Fatou’s lemma, Lemma 4.2, Proposition 4.1 and the lower semicontinuity of $\varphi$ shows that (ii) of Definition 2.1 is satisfied.

In addition, (i) of Lemma 4.2 shows that $U^\varepsilon(t) := \nabla \varphi_\varepsilon(Y^\varepsilon(t))$ are bounded in the space $H^2_T(\mathbb{R}^k)$, so there exists a subsequence $\varepsilon_n \to 0$ such that

$$
U^{\varepsilon_n} \to U, \text{ weakly in } H^2_T(\mathbb{R}^k).
$$

Furthermore, we have

$$
\mathbb{E} \int_0^T |U(s)|^2 ds \leq \liminf_{n \to \infty} \mathbb{E} \int_0^T |U^{\varepsilon_n}(s)|^2 ds \leq CM_2.
$$

In virtue of (H2), by passing limit in BDSDE (4.2), we deduce that the triple $(Y, Z, U)$ satisfies (iv) of Definition 2.1.

Finally, let us show (iii) of Definition 2.1 is satisfied. Since $U^\varepsilon(t) \in \partial \varphi(J_\varepsilon(Y^\varepsilon(t)))$, $t \in [0, T]$, it follows that, for all $V \in H^2_T(\mathbb{R}^k)$,

$$
e^{\beta t} \langle U^\varepsilon(t), V(t) - J_\varepsilon(Y^\varepsilon(t)) \rangle + e^{\beta t} \varphi(J_\varepsilon(Y^\varepsilon(t))) \leq e^{\beta t} \varphi(V(t)), d\mathbb{P} \times dt - a.e.
$$

Taking the liminf in the probability in the above inequality, (iii) of Definition 2.1 holds.

**Uniqueness.** Let $(Y^i(t), Z^i(t), U^i(t))$, $i = 1, 2$ be two solutions of multivalued BDSDE (2.1). Denote

$$(\Delta Y(t), \Delta Z(t), \Delta U(t)) := (Y^1(t) - Y^2(t), Z^1(t) - Z^2(t), U^1(t) - U^2(t)).$$
By Itô’s formula, we have
\[
e^{\beta t} |\Delta Y(t)|^2 + \beta \int_t^T e^{\beta s} |\Delta Y(s)|^2 ds + \int_t^T e^{\beta s} |\Delta Z(t)|^2 ds + 2 \int_t^T e^{\beta s} \langle \Delta Y(s), \Delta U(s) \rangle ds
\]
\[
= 2 \int_t^T e^{\beta s} \langle \Delta Y(s), (f(s, Y^1(s), Z^1(s), Y^1_s, Z^1_s) - f(s, Y^2(s), Z^2(s), Y^2_s, Z^2_s)) \rangle ds
\]
\[
+ \int_t^T e^{\beta s} |g(s, Y^1(s), Z^1(s), Y^1_s, Z^1_s) - g(s, Y^2(s), Z^2(s), Y^2_s, Z^2_s)|^2 ds
\]
\[
+ \int_t^T e^{\beta s} \langle \Delta Y(s), (g(s, Y^1(s), Z^1(s), Y^1_s, Z^1_s) - g(s, Y^2(s), Z^2(s), Y^2_s, Z^2_s)) dB(s) \rangle
\]
\[
- 2 \int_t^T e^{\beta s} \langle \Delta Y(s), \Delta Z(s) dW(s) \rangle.
\]

Since
\[
\langle \Delta Y(s), \Delta U(s) \rangle \geq 0, \quad d\mathbb{P} \times dt - a.e.
\]

Thus, as the same procedure as Lemma 4.3, we can derive the uniqueness of the solution. The proof is complete.

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