GEOMETRY OF SMOOTH EXTREMAL SURFACES

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ABSTRACT. We study the geometry of the smooth projective surfaces that are defined by Frobenius forms, a class of homogeneous polynomials in prime characteristic recently shown to have minimal possible F-pure threshold among forms of the same degree. We call these surfaces extremal surfaces, and show that their geometry is reminiscent of the geometry of smooth cubic surfaces, especially non-Frobenius split cubic surfaces of characteristic two, which are examples of extremal surfaces. For example, we show that an extremal surface $X$ contains $d^2(d^2 - 3d + 3)$ lines where $d$ is the degree, which is notable since the number of lines on a complex surface is bounded above by a quadratic function in $d$. Whenever two of those lines meet, they determine a $d$-tangent plane to $X$ which consists of a union of $d$ lines meeting in one point; we count the precise number of such "star points" on $X$, showing that it is quintic in the degree, which recovers the fact that there are exactly 45 Eckardt points on an extremal cubic surface. Finally, we generalize the classical notion of a double six for cubic surfaces to a double $2d$ on an extremal surface of degree $d$. We show that, asymptotically in $d$, smooth extremal surfaces have at least $\frac{1}{16}d^{14}$ double $2d$'s. A key element of the proofs is using the large automorphism group of extremal surfaces which we show acts transitively on many sets, such as the set of (triples of skew) lines on the extremal surface. Extremal surfaces are closely related to finite Hermitian geometries, which we recover as the $\mathbb{F}_q^2$-rational points of special extremal surfaces defined by Hermitian forms over $\mathbb{F}_{q^2}$.

1. Introduction

Let $k$ be an algebraically closed field of positive characteristic $p$. Our goal is to study the geometry of smooth extremal surfaces over $k$.

An extremal surface is a surface in $\mathbb{P}^3$ defined by a homogeneous polynomial of smallest possible F-pure-threshold among reduced forms of the same degree. While not obvious such forms exist, a sharper lower bound on the F-pure threshold in terms of degree was proved in [KKP+21a, 1.1], where the forms achieving it were classified and dubbed Frobenius forms. The F-pure threshold is a measurement of singularities\footnote{The F-pure threshold was first defined as a "characteristic $p$ analog" of the log canonical threshold, a well-known invariant of complex singularities, by Takagi and Watanabe [TW04], who were building on the work of Hara and Yoshida [HY03]. See also [MTW05], [BMS08] or [BFS13].} with smaller thresholds representing "worse singularities," so forms with minimal F-pure threshold cut out "maximally singular" cones in affine space. Thus it is natural to expect the corresponding projective hypersurfaces to exhibit some extremal geometric properties as well.

The simplest case of an extremal surface is a non-Frobenius split cubic surface of characteristic two, which were studied in depth in [KKP+21b]. Geometrically, extremal cubic surfaces can be characterized among all cubic surfaces as those that admit no triangles. To understand this statement, recall that each smooth cubic surface admits exactly forty-five plane sections consisting of a union of three lines, typically forming a "triangle". Some special cubic surfaces

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admit one or more such tri-tangent sections in which the three lines meet at some point (called an Eckardt point). An extremal cubic surface has the highly unusual property that each and every one of the forty-five tri-tangent plane sections consists of three concurrent lines. Such "triangle-free" cubic surfaces do not exist over \( \mathbb{C} \) nor indeed over any field of odd characteristic. Extremal cubic surfaces exist only in characteristic two, precisely when the cubic form cutting out the surface is a Frobenius form. These results are all worked out in [KKP+21b]; see also [KKP+21a, DD19, Har98 5.5], [Hom97 1.1] and [Hir85 20.2] for related work.

This paper explores ways in which smooth projective surfaces defined by Frobenius forms have "extremal" geometric features analogous to the abundance of concurrent configurations of lines on a smooth extremal cubic surface. Like extremal cubics, higher degree extremal surfaces contain no triangles: if two lines on a smooth extremal surface \( X \) meet at some point \( p \), then the tangent plane section \( T_pX \cap X \) at \( p \) consists of \( q + 1 \) distinct lines meeting at \( p \) (Corollary 2.3.2). We call such a configuration of lines on an extremal surface a star and the point of concurrency a star point; note that a star point is precisely an Eckardt point in the case of cubic surfaces.

We prove that smooth extremal surfaces have a large number of stars—indeed the number of stars grows like \( d^5 \) where \( d \) is the degree of the surface—which means that extremal surfaces have a large number of lines—exactly \( d^2(d^2 - 3d + 3) \) to be precise (see Corollary 3.2.3). This contrasts sharply with the characteristic zero case, where the number of lines on a smooth surface in \( \mathbb{P}^3 \) is bounded above by a quadratic function in the degree; see [Seg43, RS15b] or [BS07]. Bauer and Rams recently showed that a quadratic bound holds even in characteristic \( p \), provided \( p > d \) [BR20]. On the other hand, the quadratic bound has been known to be false in non-zero characteristic (see e.g. [RS15a]). Corollary 3.2.3 confirms that it is wildly false in every characteristic, even for \( d = p + 1 \).

The main theme of this paper is that extremal surfaces exhibit fascinating geometry reminiscent of the geometry of lines on cubic surfaces. Most substantially, in Section 6, we generalize the classical notion of a "Double Six" on a cubic surface, showing that an extremal surface of degree \( d \) admits many configurations of "Double 2d" (Theorem 6.0.2). To construct and establish the abundance of such Double 2d’s, we use pairs of quadric configurations, a concept investigated in Section 5. A quadric configuration is a collection of \( 2d \) lines on a surface of degree \( d \) all lying on the same quadric. While most surfaces do not contain any quadric configuration, we show that, like cubic surfaces, a degree \( d \) extremal surface contains many quadric configurations—roughly \( d^6 \) for large \( d \) (Corollary 5.1.3). In Conjecture 6.0.3, we speculate that, as is classically known for cubic surfaces, every double 2d on an extremal surface is a union of two quadric configurations. In Theorem 6.0.4 we prove this conjecture for \( d > 10 \) and \( d < 5 \).

A key tool used throughout is that the symmetry group of an extremal surface is quite large. Indeed, we prove it acts transitively on several large sets, including the set of all stars (Proposition 3.1.1) and the set of all ordered triples of skew lines on \( X \) (Theorem 5.1.2). We find an explicit description of the automorphism group of an extremal surface (Theorem 3.3.1) that recovers a theorem of Shioda on the automorphism group of certain Fermat hypersurfaces, and of Duncan and Dolgachev for cubics "with no canonical point" (see Remark 3.3.3).

Extremal varieties are closely connected to finite Hermitian geometry, although our approach is completely independent (see [BC66, Hir85]). Indeed, a Hermitian form is a (very) special type of Frobenius form defined over \( \mathbb{F}_{q^2} \); see § 2.3. For this reason, several basic facts established in Sections 3 and 4 will sound familiar to experts in finite geometry where Hermitian forms over \( \mathbb{F}_{q^2} \) play a starring role. For example, our counts of star points and lines on an extremal
surface (Corollary 3.2.3) produce well-known numbers of points and lines in a Hermitian sub-
geometry of the finite projective geometry \( \text{PG}(3,q^2) \). Proposition 3.4.1 explains why: we show
that if the extremal variety happens to be defined by a Hermitian form over \( \mathbb{F}_{q^2} \), then its star
points are simply its \( \mathbb{F}_{q^2} \)-points. Our paper is independent of the vast theory of finite geometries
(e.g., [Hir85]), and uses only standard algebraic geometry over an algebraically closed field as
one might find in a text such as Shafarevich [Sha13]; indeed, we discovered the connection
with finite geometry only after our work was complete. None-the-less, we include self-contained
proofs of some results, such as the structure of the automorphism group of an extremal variety
(Theorem 3.3.1), which could, a postieri, be deduced from well-known results in finite geometry.

Our work connects extremal varieties to a diverse array of active research groups throughout
pure and applied mathematics including in coding and design theory [Gop83, ES16, TVZ82],
rational points on curves and varieties [HK16, HKT08], graph theory [FK83], cryptography [KST11],
group theory [Gro02, Tit76], and the combinatorics of hyperspace arrangements and generalized
quadrangles [P109]. Nearly all this research is written from a dramatically different perspective
from our paper. We hope to inspire algebraic geometers to investigate some of the many open
problems, for example, in [HT15], and conversely, help researchers in diverse fields gain access to
new techniques. A small sample of related literature includes [Seg67], [HT16], [TVN07], [Tit79],
[vM89, Koll15 § 35], and the references therein.

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2. Basics of Frobenius forms

This section consolidates needed known facts and terminology about Frobenius forms.

Fix a field \( k \) of positive characteristic \( p \), and let \( q \) denote \( p^e \) for some fixed positive integer
\( e \). A Frobenius form (in \( n \) variables, say) is a homogeneous polynomial of degree \( p^e + 1 \) in the
"Frobenius power" \( \langle x_1^{q^e}, x_2^{q^e}, \ldots, x_n^{q^e} \rangle \) of the unique homogenous maximal ideal of the polynomial
ring. Put differently, a Frobenius form is a polynomial \( h \) that can be written \( \sum_{i=1}^n x_i^{q^e} L_i \), where
\( L_i \) are linear forms. In particular, every Frobenius form admits a matrix factorization

\[
 h = \begin{bmatrix} x_1^q & x_2^q & \ldots & x_n^q \end{bmatrix} A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = (\bar{x}^{(q)})^\top A \bar{x},
\]

where \( A \) is the unique \( n \times n \) matrix whose \( i \)-th row is made up of the coefficients of the linear form
\( L_i \). Here, for a matrix \( B \) of any size, the notation \( B^{[q]} \) denotes the matrix obtained by raising all
entries to the \( p^e \)-th power, and \( B^\top \) denotes the transpose of \( B \). The notation \( \bar{x} \) denotes a column
vector of size \( n \).

2.1. Changes of Coordinates. The set of Frobenius form is preserved by arbitrary linear
changes of coordinates, since both degree and the ideal \( \langle x_1^{q^e}, x_2^{q^e}, \ldots, x_n^{q^e} \rangle \) are preserved. If \( g \) is an
invertible \( n \times n \) matrix representing some linear change of coordinates, and the Frobenius form \( F \)
is represented by the matrix $A$, then the Frobenius form $g^*F$ obtained after changing coordinates is represented by the matrix
\[
[g^{ij'}]^T A g.
\]

See [KKP+21a § 5] for details.

A Frobenius form is said to be non-degenerate if it cannot be written as a polynomial in fewer variables after any linear change of coordinates.

The rank of a Frobenius form is the rank of the representing matrix. The rank is the same as the codimension of the singular locus of the corresponding hypersurface [KKP+21a 5.3].

**Theorem 2.1.1.** [KKP+21a 6.1] [Bea90]. All maximal rank Frobenius forms of fixed degree and number of variables over a fixed algebraically closed field $k$ are the same up to linear change of variables.

Theorem 2.1.1 says there is a unique smooth extremal hypersurface of each dimension and allowable degree. In particular, a smooth extremal surface in $\mathbb{P}^3$ can be assumed, after suitable choice of projective coordinates, to be defined by $x^{q+1} + y^{q+1} + z^{q+1} + w^{q+1}$ or, equivalently, by $x^q w + w^q x + y^q z + z^q y$ or any full rank Frobenius form in $x, y, z, w$.

More generally, Frobenius forms are classified up to linear changes of coordinates over an algebraically closed field. For each $n$, the number of distinct projective equivalence classes of Frobenius forms of a fixed degree $q+1$ is equal to the number of partitions of $n$. See [KKP+21a 7.1] for the precise statement.

**Example 2.1.2.** There are three equivalence classes of non-degenerate Frobenius forms in three variables and of degree $q+1$, corresponding, respectively, to the three matrices
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix}.
\]

These determine, respectively, the forms $x^{q+1} + y^{q+1} + z^{q+1}$, $x^{q+1} + y^q z$ and $x^q y + y^q z$.

### 2.2. Stars.

Smooth extremal surfaces have distinguished points with special geometry analogous to the Eckardt points on a cubic surface:

**Definition 2.2.1.** A star on a smooth surface $X$ of degree $d$ is a configuration of $d$ lines on $X$, all meeting at one point $p$ called the center of the star, or a star point.

If $L$ is a line on a smooth surface $X$ and $p$ is a point on $L$, then $L \subset T_pX$, the tangent plane to $X$ at $p$. Thus the $d$ lines forming a star on $X$ are coplanar—all lie in $T_pX$ where $p$ is the center of the star. In this case, the plane section $T_pX \cap X$ is the reduced union of the $d$ lines of the star. A plane containing a star is called a star plane. Star planes are uniquely determined by their centers and vice versa, since each star plane is the tangent plane to $X$ at the center of its star. Stars are defined and studied for higher dimensional hypersurfaces in [CC10].

**Example 2.2.2.** Consider the extremal surface $X$ defined by $x^q w + w^q x + y^{q+1} + z^{q+1}$. Intersecting with the plane $H$ defined by $w$, we see a star $X \cap H$ consisting of $q + 1$ distinct lines
\[
\{ \nabla(w, y - \nu z) \mid \nu^{q+1} = -1 \},
\]
all intersecting in the star point $p = [1 : 0 : 0 : 0]$. Thus $H$ is a star plane with center $p$. 


Remark 2.2.3. The lines in Example 2.2.2 are indistinguishable up to projective transformation. Indeed, the projective linear changes of coordinates \([x : y : z : w] \mapsto [x : y : \mu z : w]\) (for each \(\mu \in \mu_{q+1}\)) stabilizes the surface \(X\) and its star plane \(H\) while transitively permuting around the lines in \(H\).

2.3. Plane Sections of Extremal Surfaces.

Proposition 2.3.1. [KKP+21b] A plane section of a smooth extremal surface is one of the following types of divisors, all defined by Frobenius forms:

1. A smooth extremal curve.
2. A singular extremal curve with an isolated cuspidal singularity.
3. The reduced sum of a line and an irreducible curve tangent at one point.
4. A star of lines on the surface.

In particular, the plane section \(H \cap X\) is a star if and only if the Frobenius form \(F\) defining \(X \cap H\) in \(H\) is degenerate—that is, if and only if \(F\) can be written as a Frobenius form in two (of three) homogeneous coordinates for the projective plane \(H\).

Proof. Any plane section of an extremal surface \(X\) is extremal [KKP+21a, 8.1], so to understand a plane section \(X \cap H\), we look at the classification of Frobenius forms in three variables given in [KKP+21a, 7.1]. The non-degenerate ones are in one-one correspondence with the three partitions of 3: these are described in Example 2.1.2 (up to projective change of coordinates) and produce the first three types of divisors listed above. It is also possible that \(X \cap H\) is defined by a degenerate Frobenius form. These are classified by partitions of 2 and of 1:

1. A star, projectively equivalent to \(x^qy + y^qx\)
2. The non-reduced scheme projectively equivalent to \(x^qy\)
3. The non-reduced scheme projectively equivalent to \(x^{q+1}\).

However, because \(X\) is smooth, the plane sections \(X \cap H\) are reduced (by e.g. [Zak93, 1.15]) and so only the first of these possibilities occurs.

Proposition 2.3.1 has the following consequences; we prove only the second as the first is immediate;

Corollary 2.3.2. [KKP+21a, 8.11] Any collection of coplanar lines on an extremal surface is concurrent. In particular, an extremal surface contains no triangles.

Corollary 2.3.3. Every line on a smooth extremal surface is in some star.

Proof of Corollary 2.3.3. Fix a line \(L\) on a smooth extremal surface \(X\). There is some plane \(H\) such that \(X \cap H\) is a star, as there is no loss of generality in assuming \(X\) as in Example 2.2.2 (Theorem 2.1.1). Now, if \(L\) lies in \(H\), then the Corollary is proved, as \(L\) is a line in the star \(X \cap H\). But if \(L\) does not lie in \(H\), then \(L\) meets \(H\) at some point \(p'\). Because point \(p' \in X \cap H\), we know \(p'\) lies on some line \(L'\) in the star \(X \cap H\). Since the two lines \(L\) and \(L'\) intersect at \(p'\), the plane \(H'\) they span is a star plane centered at \(p'\) (Proposition 2.3.1). The star \(X \cap H'\) thus contains our line \(L\).

2.4. Extremal Collections of Points. The automorphism group of any zero dimensional smooth extremal hypersurface \(Y\) (that is, of a reduced extremal configuration of points in \(\mathbb{P}^1\)) acts transitively on the points in \(Y\); this follows immediately from Remark 2.2.3 by projectivizing the the star plane. However, a stronger symmetry holds that we record for future reference:
Proposition 2.4.1. Let $Y \subset \mathbb{P}^1$ be a reduced extremal configuration of points—that is, assume $Y$ is defined by a rank two Frobenius form in two variables. Then the projective linear automorphism group of $Y$ acts three-transitively on the points of $Y$. Furthermore, $\text{Aut}(Y)$ is isomorphic to $\text{PGL}(2, \mathbb{F}_q)$.

Proof. We may assume that $Y$ is defined by the form $yz^q - zy^q$ (Theorem 2.1.1), so $Y$ consists of the points $[\mu : 1]$ where $\mu^q = \mu$, together with the "point at infinity" $[1 : 0]$. So the points of $Y$ are precisely the $\mathbb{F}_q$-points of $\mathbb{P}^1$.

Now, given an ordered triple of three distinct points in $Y$, there is a unique automorphism $g$ of $\mathbb{P}^1$ sending them to any other ordered triple in $Y$. Because all six points are defined over $\mathbb{F}_q$, the automorphism $g \in \text{PGL}(2, \mathbb{k})$ is represented by a $2 \times 2$ matrix with entries in $\mathbb{F}_q$, so that $g \in \text{PGL}(2, \mathbb{F}_q)$. In particular, it must send every $\mathbb{F}_q$-point of $\mathbb{P}^1$ to another $\mathbb{F}_q$-point of $\mathbb{P}^1$. That is, $g$ is an automorphism of $Y$. This establishes both claims of Proposition 2.4.1. \[\blacksquare\]

2.5. Hermitian Forms over Finite Fields. A Hermitian form is a special kind of Frobenius form in which the representing matrix $A$ satisfies $(A^q)^\top = A$. In this case, all entries of $A$ satisfy $a_{ij}^q = a_{ij}$, which means they are in the finite field $\mathbb{F}_q^2$. Thus a Hermitian form is defined over the finite field $\mathbb{F}_q^2$. In this case, the Frobenius map $(x \mapsto x^q)$ is an involution on the set of $\mathbb{F}_q^2$ points, so can play a role analogous to complex conjugation. See [Hir85, § 19.1].

The classification of Hermitian forms is well-known and simple: there is only one invariant, rank [BC66, 4.1], where as the classification of Frobenius forms is more subtle [KKP+21a, 7.1]. On the other hand, every smooth projective hypersurface defined by a Frobenius form is projectively equivalent (over the algebraically closed field $\mathbb{k}$) to one defined by a Hermitian form (Theorem 2.1.1), although of course, the needed change of coordinates is not usually defined over $\mathbb{F}_q^2$.

3. Configurations of Lines and Stars on Extremal Surfaces

3.1. Symmetry of Extremal surfaces. Smooth extremal surfaces are highly symmetric:

Proposition 3.1.1. The automorphism group of a smooth extremal surface acts transitively on its set of stars.

By automorphisms here, we mean projective linear transformations of the surface in $\mathbb{P}^3$. Thus $\text{Aut}(X)$ is a subgroup of PGL(4, $\mathbb{k}$); in Section 3.3, this group is discussed in detail.

Before proving the proposition, we deduce a corollary:

Corollary 3.1.2. The automorphism group of a smooth extremal surface $X$ acts transitively on the set of all pairs $(H, L)$ consisting of a star plane $H$ and a line $L$ in the star $H \cap X$. In particular, $\text{Aut}(X)$ acts transitively on the set of all lines on $X$.

Proof of Corollary 3.1.2. Without loss of generality, assume $X$ is defined by $x^q w + w^q x + y^{q+1} + z^{q+1}$ (Theorem 2.1.1). Given an arbitrary pair $(H, L)$, Proposition 3.1.1 says there is an automorphism of $X$ taking $H$ to the star plane $H'$ defined by $w = 0$ (see also Example 2.2.2). But then we can compose with an automorphism of $X$ preserving $H'$ while taking the image of $L$ to any line in the star $H' \cap X$ (Remark 2.2.3).

To prove Proposition 3.1.1 we make use of the following lemma.
Lemma 3.1.3. Given an arbitrary star \( X \cap H \) with center \( p \) on a smooth extremal surface \( X \), we may choose coordinates for \( \mathbb{P}^3 \) so that

\[
p = [0 : 0 : 0 : 1], \quad H = \mathbb{V}(x), \quad \text{and} \quad X = \mathbb{V}(x^q\ell + xw^q + y^qz + z^qy),
\]

for some linear form \( \ell = ax + by + cz + w \).

Proof. Choose coordinates so that the star plane \( H \) is defined by \( x = 0 \). In this case, the form \( F \) defining \( X \) is

\[
F = xG + G'(y, z, w)
\]

where \( G \) is some form of degree \( q \) and \( G' \) is a Frobenius form in the variables \( y, z, w \). The form \( G' \) defines the star \( X \cap H \) in the hyperplane \( H \cong \mathbb{P}^2 \). In particular, \( G' \) is degenerate (Proposition 2.3.1). So there is a linear change of coordinates involving only \( y, z, w \) such that \( G' \) transforms into a rank two Frobenius form in two variables. Since all reduced Frobenius forms in 2 variables are projectively equivalent, we can assume without loss of generality, that

\[
F = xG + yz^q + zy^q.
\]

Observe that \( xG \in \langle x^q, y^q, z^q, w^q \rangle \), which implies that \( G \in \langle x^{q-1}, y^q, z^q, w^q \rangle \). Because \( \deg G = q \), we can write

\[
G = x^{q-1}\ell + (\alpha_1y + \alpha_2z + \alpha_3w)^q
\]

for some scalars \( \alpha_i \) and linear form \( \ell \). That is,

\[
F = x^q\ell + x(\alpha_1y + \alpha_2z + \alpha_3w)^q + zy^q + yz^q.
\]

The scalar \( \alpha_3 \) can not be zero. Indeed, if \( \alpha_3 = 0 \), then \( F \in \langle x^q, y^q, z^q \rangle \), so the rank of \( F \) would be at most three and \( F \) could not define a smooth surface [KKP⁺21a, 5.3]. So we may replace the form \( \alpha_1y + \alpha_2z + \alpha_3w \) by \( w \) (which changes \( \ell \) but nothing else) to assume without loss of generality that

\[
(4) \quad F = x^q\ell + xw^q + zy^q + yz^q.
\]

The linear form \( \ell = ax + by + cz + dw \) must satisfy \( d \neq 0 \), for otherwise \( F \in \langle x, y, z \rangle \) and again \( F \) would have rank at most 3. Finally, the change of coordinates

\[
[x : y : z : w] \mapsto [\lambda x : y : z : \lambda^{-1/q}w]
\]

where \( \lambda^{q^2-1} = \frac{1}{d^q} \) transforms \( F \) (formula (4)) into

\[
(\lambda x)^q(a\lambda x + by + cz + d\lambda^{-1/q}w) + xw^q + zy^q + yz^q\]

which has the desired form since the coefficient of \( x^qw \) is \( d\lambda^{q-\frac{1}{q}} = 1 \). Lemma 3.1.3 is proved. □

Proof of Proposition 3.1.1. It suffices to show that given an arbitrary star point \( p \) on an arbitrary smooth extremal surface \( X \) in \( \mathbb{P}^3 \) of degree \( q + 1 \), there is a choice of coordinates for \( \mathbb{P}^3 \) so that \( p = [0 : 0 : 0 : 1] \) and the defining form of \( X \) is \( w^q x + x^q w + y^q z + z^q y \).

Let \( H \) be the star plane centered at \( p \). Use Lemma 3.1.3 to assume that \( p = [0 : 0 : 0 : 1] \), \( H \) is defined by \( x = 0 \) and that the Frobenius form defining \( X \) looks like

\[
F = x^q(ax + by + cz + w) + w^qx + y^qz + z^qy.
\]

We will perform a sequence of changes of coordinates that all fix \( p \) and its tangent plane \( H \), but eventually bring \( F \) into the desired anti-diagonal form.
First, we show we can change coordinates so as to assume \( b = 0 \). Consider an indeterminate scalar \( \lambda \). Perform the change of coordinates
\[
[x : y : z : w] \xrightarrow{\phi} [x : y : z + \lambda^q x : w - \lambda y].
\]
The map \( \phi \) fixes \( p \) and \( H \) but \( \phi^* \) transforms the form \( F \) into
\[
\begin{align*}
x^q(ax + by + cz + \lambda^q cx + w - \lambda y) + (w - \lambda y)^q x + y^q(z + \lambda^q x) + (z^q + \lambda^q x^q)y \\
= x^q((a + c\lambda^q)x + (b + \lambda^q - \lambda)y + cz + w) + w^q x + y^q z + z^q y.
\end{align*}
\]
So any choice of \( \lambda \) such that \( \lambda^q - \lambda + b = 0 \) will transform \( F \) into
\[
F_1 = x^q(a' x + cz + w) + w^q x + y^q z + z^q y,
\]
where \( a' \in k \), without moving \( p \) or \( H \). Similarly, interchanging the roles of \( y \) and \( z \), we can transform \( F_1 \) into
\[
F_2 = x^q(a' x + w) + w^q x + y^q z + z^q y
\]
without moving star point \( p \) or its star plane \( H \).

Finally, again let \( \lambda \) be an indeterminate scalar and consider the change of coordinates
\[
[x : y : z : w] \xrightarrow{\phi} [x : y : z + \lambda x].
\]
This fixes \( p \) and \( H \) but transforms \( F_2 \) to
\[
(a' + \lambda + \lambda^q)x^{q+1} + x^q w + w^q x + y^q z + z^q y.
\]
So choosing \( \lambda \) to be any root of the polynomial \( t^q + t + a' \), the form \( F_2 \) is transformed into the standard form \( x^q w + w^q x + y^q z + z^q x \) without changing the star point \( p = [0 : 0 : 0 : 1] \). This completes the proof of Proposition 3.1.1.

3.2. Star Points and Star Planes. We now count configurations of star points on star planes and on lines on the extremal surface:

**Theorem 3.2.1.** Let \( L \) be an arbitrary line on a smooth extremal surface \( X \). Then
(a) There are exactly \( q^2 + 1 \) star points on \( L \). Equivalently, there are exactly \( q^2 + 1 \) stars on the surface \( X \) containing \( L \).
(b) There are exactly \( q(q^2 + 1) \) lines on \( X \) that intersect \( L \), not counting \( L \) itself.
(c) There are exactly \( q^4 \) lines on \( X \) skew to \( L \).

Before proving Theorem 3.2.1, we deduce a few corollaries.

**Corollary 3.2.2.** Each star plane of an extremal surface contains exactly \( q^3 + q^2 + 1 \) star points—that is, each star contains \( q^3 + q^2 \) star points other than its center.

**Proof of Corollary 3.2.2.** Let \( p \) be the center of the star \( H \cap X \). Each of the \( q + 1 \) lines in this contains exactly \( q^2 \) star points other than \( p \) by Theorem 3.2.1. So \( H \) contains exactly \( q^2(q + 1) + 1 \) star points.

**Corollary 3.2.3.** Let \( X \) be a smooth extremal surface of degree \( q + 1 \).
(a) There are a total of \( q^4 + q^3 + q + 1 = (q^3 + 1)(q + 1) \) distinct lines on \( X \), each containing exactly \( q^2 + 1 \) star points.
(b) There are a total of \( q^5 + q^3 + q^2 + 1 = (q^3 + 1)(q^2 + 1) \) distinct stars on \( X \), each containing exactly \( q + 1 \) lines.
Proof. (a). Fix one line $L$ on $X$. There are exactly $q(q^2 + 1)$ lines on $X$ which intersect $L$ by Theorem 3.2.1(b). On the other hand, there are $q^4$ lines on $X$ disjoint from $L$ by Theorem 3.2.1(c). So the total number of lines, counting $L$, is $q^4 + q^3 + q + 1$.

(b). There are a total of $q^4 + q^3 + q + 1$ lines, and each line is contained in exactly $q^2 + 1$ stars. So the number of pairs $(L, H)$ consisting of a line $L$ on a star $H \cap X$ must be $(q^4 + q^3 + q + 1)(q^2 + 1)$. On the other hand, each star contains exactly $q + 1$ lines, so the total number of stars is
\[
\frac{(q^4 + q^3 + q + 1)(q^2 + 1)}{q + 1} = \frac{(q^3 + 1)(q + 1)(q^2 + 1)}{q + 1} = (q^3 + 1)(q^2 + 1) = q^5 + q^3 + q^2 + 1.
\]
\[\square\]

Proof of Theorem 3.2.1 (a). The line $L$ belongs to some star $H \cap X$ by Corollary 2.3.3. By Corollary 3.1.2, we can choose coordinates so that $X$ is defined by
\[
F = x^q w + xw^q + y^q z + z^q y,
\]
and $L \subset H$ are cut out by $x, y$ and $x$ respectively.

Consider the pencil of planes containing the line $L$. Each plane $H_\lambda$ in the pencil is defined by the vanishing of some linear form $\lambda x - y$. The plane $H$ itself is defined by $x = 0$ (the case where $\lambda = \infty$), which we already assumed is a star.

Restricting the Frobenius form $F$ to the plane $H_\lambda$, we can set $y = \lambda x$ and view the plane section $X \cap H_\lambda$ as defined by the Frobenius form
\[
\overline{F} = x^q w + xw^q + \lambda^q x^q z + \lambda x z^q,
\]
in the variables $x, z, w$. The plane section $X \cap H_\lambda$ is a star if and only if the form $\overline{F}$ is degenerate (Cf. Proposition 2.3.1).

We claim that $\overline{F}$ is degenerate precisely when $\lambda$ is a root of the separable polynomial $t^{q^2} - t$. This will imply that there are precisely $q^2$ planes (besides $H$) which contain $L$ as a component of a star, so the proof of (a) will be complete once we have proved the claim.

To this end, consider the change of coordinates
\[
[x : z : w] \mapsto [x : z : w - \lambda^q z].
\]
This transformation sends $\overline{F}$ to
\[
\overline{F}_1 = x^q w + xw^q + (\lambda - \lambda^q^2)x z^q,
\]
which is clearly degenerate if $\lambda^q^2 - \lambda = 0$. On the other hand, if $\lambda^q^2 - \lambda \neq 0$, then $\overline{F}$ is not degenerate. Indeed, in this case
\[
\overline{F}_1 = x((\gamma z + w)^q + x^{q-1} w)
\]
for some non-zero $\gamma$, which is projectively equivalent to $x(z^q + x^{q-1} w)$, so defines a union of the line $L$ and an irreducible curve of degree $q$, not a star. This completes the proof of (a).

(b). A line $M$ on $X$ intersects $L$ if and only if $L$ and $M$ appear together in a star. There are $q^2 + 1$ stars containing $L$ and each of them contains $q$ distinct lines (other than $L$). Of course, a pair $L$ and $M$ can not appear together in more than one star, since the plane producing a star is uniquely determined by any two lines in it. So there must be $q(q^2 + 1)$ distinct lines $M$ which intersect $L$ on our extremal surface.

(c). Fix a star $H \cap X$ containing $L$ (this is possible by Corollary 2.3.3). There are $q$ other lines in this star. Pick one, $M$. Now $M$ appears in exactly $q^2$ other stars besides $H$ by Theorem 3.2.1(a).
For each of these stars, each of the other $q$ lines in the star is a line $L'$ which does not meet $L$. Indeed, if $L'$ meets $L$, then the lines $L, L', M$ form a triangle, contradicting Corollary 2.3.2. In this way, we produce $q^3$ distinct lines $L'$ on $X$ which meet $M$ but not $L$. Now, varying over each of the $q$ lines $M$ in the star $H \cap X$ (other than $L$), we produce $q^3$ new lines for each of the $q$ choices of line $M$. In total, we found $q^4$ lines skew to $L$.

Finally, we need to show that our above count includes every line $L'$ on $X$ skew to $L$. Say $L'$ is skew to $L$. Pick any star point $p$ on $L$. The star plane $H$ at $p$ contains $L$ but not $L'$ (otherwise $L'$ would meet $L$). So $L'$ must meet $H$ at some point $p'$, necessarily in the star $H \cap X$. So $p'$ is a star point on some line $M$ in the star $X \cap H$, and the star centered at $p'$ contains $L'$. This means $L'$ is a line of the type we already counted in the previous paragraph. So there are exactly $q^4$ lines on the extremal surface skew to any fixed line on the surface. □

**Corollary 3.2.4.** Fix any pair of skew lines on an extremal surface. Then there are exactly $q^2 + 1$ lines on the surface that meet both.

**Proof of Corollary 3.2.4.** Fix arbitrary skew lines $L$ and $L'$ on the extremal surface $X$. We claim that for each star point $p$ on $L$, there is exactly one line through $p$ meeting $L'$. Because there are exactly $q^2 + 1$ star points on $L$ (Theorem 3.2.1(a)) and any intersection point of lines on $X$ is a star point, the claim proves the corollary.

To prove the claim, observe that $L'$ is not in the star plane $H$ centered at $p$, since that would imply $L'$ meets $L$. Thus $L'$ meets $H$ at a unique point $p'$, which means $p'$ is in the star $X \cap H$, and hence in (exactly) one of the lines $M$ in the star $X \cap H$. The line $M$ meets both $L$ and $L'$. There is no other line through $p$ meeting both $L$ and $L'$, for if $M'$ is another, then $M, M', L'$ form a triangle, contrary to Corollary 2.3.2. □

### 3.3. The Automorphism Group of an Extremal Surface

We now use the geometry of extremal surfaces to describe their automorphism groups. The results in this section are (essentially) known, albeit in somewhat different contexts with slightly stronger hypotheses; we include straightforward new proofs for completeness. The first result is due to Shioda when $d > 3$ [Shi88, p97] and Duncan and Dolgachev when $d = 3$ [DDT9, § 5.1].

**Theorem 3.3.1.** Let $X$ be a smooth extremal hypersurface of degree $q+1$ and dimension $n-2 \geq 0$ over algebraically closed field $k$. The group $\text{Aut}(X)$ of projective linear automorphisms of $X$ is isomorphic to the finite group $\text{PU}(n, \overline{F}_q)$, where $\text{PU}(n, \overline{F}_q)$ is the quotient of the finite unitary group

\[ U(n, \overline{F}_q^2) = \{ g \in \text{GL}(n, \overline{F}_q^2) \mid (g[q])^\top g = I_n \} \]

by its center,

\[ \{ \lambda I_n \mid \lambda^{q+1} = 1 \}, \]

the cyclic group of scalar matrices of order $q+1$.

**Remark 3.3.2.** We have already computed that when $X$ is zero dimensional, $\text{Aut}(X)$ is isomorphic to $\text{PGL}(2, \overline{F}_q)$ (Proposition 2.4.1), so Theorem 3.3.1 confirms that $\text{PU}(2, \overline{F}_q^2) \cong \text{PGL}(2, \overline{F}_q)$.

**Proof of Theorem 3.3.1.** Choose coordinates so that the extremal hypersurface $X$ is defined by $F = x_1^{q+1} + x_2^{q+1} + \cdots + x_n^{q+1}$ (Theorem 2.1.1). The group $\text{Aut}(X)$ is the subgroup of $\text{PGL}(n, k)$ represented by matrices $g \in \text{GL}_n(k)$ such that $g^*F = \lambda F$ for some non-zero scalar $\lambda$. Because $k$ is algebraically closed, the class of $g$ in $\text{PGL}(n, k)$ can be represented by the scalar multiple $\mu g$ where $\mu^{q+1} = \frac{1}{\lambda}$, so without loss of generality we assume that $g^*F = F$. Such $g$ satisfy

\[ (g[q])^\top I_n g = I_n, \]
by formula (2) in Section 2. Raising \(6\) to the \(q\)-th power and transposing, we have \((g^{[q]})^\top g^{[q]} = I_n\). In particular, both \(g\) and \(g^{[q]}\) are inverses of the matrix \((g^{[q]})^\top\), so that \(g = g^{[q]}\). Thus each entry of \(g\) is fixed by the Frobenius map \(x \mapsto x^q\), and hence in \(\mathbb{F}_{q^2}\)—that is, we can assume \(g \in \text{PGL}(4, \mathbb{F}_{q^2})\). This means that the naturally induced group map

\[
U(n, \mathbb{F}_{q^2}) \longrightarrow \text{Aut}(X)
\]

is surjective, and it remains only to compute its kernel.

An element \(g \in U(n, \mathbb{F}_{q^2})\) induces the identity map on \(X\) if and only if \(g = \lambda I_n\) for some \(\lambda \in k^*\). But a scalar matrix \(\lambda I_n\) is in \(U(n)\) if and only if \((\lambda I_n)^{[q]}(\lambda I_n) = \lambda^{q+1}g^{[q]}g^\top = I_n\)—that is, if and only if \(\lambda^{q+1} = 1\). On the other hand, all \((q + 1)\)-st roots \(\lambda\) of unity in \(k\) are in \(\mathbb{F}_{q^2}\): if \(\lambda^{q+1} = 1\), then \(\lambda^q = \lambda^{q-1}\lambda = (\lambda^{q+1})^{q-1}\lambda = \lambda\). So the kernel is the cyclic group of order \(q + 1\) consisting of the scalar matrices of \(q + 1\)-st roots of unity, as claimed. The theorem is proved. 

\[\square\]

**Remark 3.3.3.** Shioda’s theorem is actually concerned with Fermat hypersurfaces—projective hypersurfaces defined by \(x_0^d + x_1^d + \cdots + x_n^d\) over an arbitrary algebraically closed field (although he omits the case where \(d = 3\), which our result includes). Shioda shows that, with the exception of the case where \(d = p^e + 1\), the automorphism group of the Fermat hypersurface is generated by the "obvious" automorphisms: the \(S_p\) permuting the coordinates and the group \(\mu_{q^d}/\mu_d\) scaling the coordinates by \(d\)-th roots of unity. When \(d = p^e + 1 > 3\), he proves Theorem 3.3.1 for the Fermat hypersurface using a different method than our argument. Likewise, Duncan and Dolgachev are concerned with the automorphism group of cubic surfaces in general; in the special case of the Fermat cubic surface of characteristic two (the extremal case), their proof uses a result of Beauville to show there is no "canonical point", and then appeals to known facts about the automorphism group of a finite Hermitian geometry [Hir85, 19.1.7, 19.1.9].

**Remark 3.3.4.** In the special case that extremal surface defined by a Hermitian form (meaning that its matrix \(A\) satisfies \(A^{[q]} = A^\top\)), the proof of Theorem 3.3.1 shows that

\[
\text{Aut}(X) = \{g \in \text{GL}(n, \mathbb{F}_{q^2}) \mid (g^{[q]})^\top Ag = A\}/\{\lambda I_n \mid \lambda^{q+1} = 1\}.
\]

For an arbitrary smooth extremal surface \(X\), the automorphism group \(\text{Aut}(X)\) is conjugate to \(\text{PU}(n, \mathbb{F}_{q^2})\) via the automorphism in \(\text{PGL}(n, k)\) taking it to the Fermat hypersurface.

### 3.4. Star Points and \(\mathbb{F}_{q^2}\)-rational points.

We next give a straightforward proof of a familiar fact that connects extremal surfaces to finite geometry:

**Proposition 3.4.1.** The star points on a extremal surface of degree \(q + 1\) defined by a Hermitian form are precisely its \(\mathbb{F}_{q^2}\) points.

**Remark 3.4.2.** Proposition 3.4.1 (together with Corollary 3.2.3) recovers the known fact that there are precisely \(q^5 + q^3 + q^2 + 1\) points in a Hermitian sub-geometry of the finite projective 3-space over \(\mathbb{F}_{q^2}\) [Seg67, Hir85, 19.1.5].

**Proof.** We first verify that the star points of extremal surface \(X\) defined by \(x^qw + w^qy + y^qz + z^qy\) are precisely its \(\mathbb{F}_{q^2}\) points. By symmetry, it suffices show this for the open set \(U\) where \(x \neq 0\).

Consider a point \(p = [1 : a : b : c] \in U\) whose coordinates are in \(\mathbb{F}_{q^2}\). The tangent plane \(T_pX\) at \(p\) is \(c^q x + b^qy + a^qz + w\), so the plane section \(T_pX \cap X\) is defined by the Frobenius form in \(x, y, z\)

\[
-x^q(c^q x + b^qy + a^qz) - x(c^q x + b^qy + a^qz)^q + y^qz + z^qy.
\]
Thus $p$ is a star point if and only if (7) is a degenerate Frobenius form (Proposition 2.3.1). Because the projective transformation

$$\phi : [x : y : z] \mapsto [x : y + ax : z + bx],$$

transforms the form (7) into the degenerate form $y^2z + z^2y$ (remember that $c + c^a + a^q b + b^q a = 0$), we conclude that all $\mathbb{F}_q^2$-points of $U$ are star points.

To check that the $\mathbb{F}_q^2$-points comprise all star points of $U$, we count them. Note that for each choice of the pair $(a, b) \in \mathbb{F}_q^2$, the polynomial $t^q + t + (a^q b + b^q a)$ has $q$ distinct solutions in $\mathbb{F}_q^2$. So there are exactly $q^5$ $\mathbb{F}_q^2$-rational points $[1 : a : b : c]$ in $U$, all of which are star points. On the other hand, there are exactly $q^5$ star points in $U$ as well—of the total $q^5 + q^3 + q^2 + 1$ star points on $X$ (Theorem 3.2.3(b)), there are precisely $q^3 + q^2 + 1$ in the complement of $U$, because $X \setminus U = \mathcal{V}(w)$ is a star plane (Corollary 3.2.2). By the pigeon hole principle, the star points of $U$ (and hence of $X$) are precisely its $\mathbb{F}_q^2$-points.

Now consider an arbitrary smooth variety $X'$, given by some Frobenius form that is Hermitian—that is, whose representing matrix $A$ satisfies $(A^q)^\top = A$. There is a change of coordinates $g \in \text{PGL}(4, k)$ transforming $X'$ to the extremal surface $X = \mathcal{V}(x^q w + w^q x + y^q z + z^q y)$, so that using formula (2) in § 2.1, we have

$$g^{[q]}\top J g = A$$

where $J = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ is the matrix of the Frobenius form defining $X$. Raising (8) to the $q$-th power and transposing, we have

$$(g^{[q]}\top (J^{[q]}\top g^{[q^2]} = (A^{[q]}\top = A).$$

Setting (8) and (9) equal and remembering that $(J^{[q]}\top = J$, we have

$$(g^{[q]}\top J g^{[q^2]} = (g^{[q]}\top J g,$$

from whence it follows that $g^{[q^2]} = g$. This tells us that $g \in \text{PGL}(4, \mathbb{F}_q^2)$.

Now because $g^{-1}$ is an isomorphism from $X$ to $X'$, it defines a bijection between their respective star points. Because $g$ has entries in $\mathbb{F}_q^2$, it also preserves the $\mathbb{F}_q^2$-rationality of points. So the star points of $X'$ are precisely its $\mathbb{F}_q^2$-points. $\square$

4. STAR CHORDS

Star chords are auxiliary lines not on the extremal surface but none-the-less intimately related:

**Definition 4.0.1.** A star chord for a smooth extremal surface $X$ is a line in $\mathbb{P}^3$ not on $X$ which passes through (at least) two star points of $X$.

**Remark 4.0.2.** Despite the name, a star chord $\ell$ through star point $p$ is never in the star plane $T_pX$ centered at $p$. Otherwise, assume $\ell \subset T_pX$. Because there is another star point $p' \in \ell$, the point $p'$ would then be on some line $L$ in the star $T_pX \cap X$. But then both $L$ and $\ell$ contain both $p$ and $p'$, which means $\ell = L$, contrary to the fact that $\ell \not\subset X$.

---

2Proof: The map $\mathbb{F}_q^2 \to \mathbb{F}_q^2$ sending $t \mapsto t^q + t$ is linear over the subfield $\mathbb{F}_q$, and its kernel consists of $q - 1$ distinct $(q - 1)$-st roots of $-1$ together with 0. So there are exactly $q$ solutions to $\gamma(t) = \gamma(-a^q b) = -a^q b - ab^q$ in $\mathbb{F}_q^2$ as well.
Remark 4.0.3. In the special case where the extremal surface is defined by a Hermitian form over \( \mathbb{F}_{q^2} \), star chords are *Baer sublines* or *hyperbolic lines* in the terminology of finite geometry (see e.g. [BD12, p4] or [Mas10, p102]). In this context, lines on the surface are called its generators.

The basic facts about star chords are the following:

**Theorem 4.0.4.** Let \( \ell \) be an arbitrary star chord for a smooth extremal surface. Then

(i) The stars centered at points on \( \ell \) share no lines.

(ii) The star planes of all stars centered along \( \ell \) intersect in a common line \( \ell' \) which is skew to \( \ell \) and also a star chord for \( X \).

(iii) The star planes of all stars centered along \( \ell' \) intersect in the original star chord \( \ell \).

(iv) The star chords \( \ell \) and \( \ell' \) each intersect \( X \) in \( q+1 \) distinct star points.

Before proving Theorem 4.0.4, we observe that it ensures that the next definition makes sense.

**Definition 4.0.5.** The dual of a star chord \( \ell \) for an extremal surface is the unique star chord \( \ell' \) contained in all star planes centered along \( \ell \), or equivalently, the intersection of all star planes centered along \( \ell \).

Duality between star chords is a symmetric relationship: Theorem 4.0.4(iii) implies that \( \ell' \) is the dual star chord of \( \ell \) if and only if \( \ell \) is the dual star chord of \( \ell' \).

**Example 4.0.6.** The lines \( \ell = \mathbb{V}(x,y) \) and \( \ell' = \mathbb{V}(z,w) \) are a pair of dual star chords on the Fermat extremal surface \( X = \mathbb{V}(x^{q+1} + y^{q+1} + z^{q+1} + w^{q+1}) \). Indeed, \( \ell \) is not on \( X \) but contains the \( q+1 \) star points \( p_a = [0 : 0 : a : 1] \), where \( a^{q+1} = -1 \). To check that \( p_a \) is a star point, observe that the tangent plane to \( p_a \) is \( T_{p_a}X = \mathbb{V}(a^qz + w) = \mathbb{V}(z - aw) \), which intersects \( X \) in a star. These star planes \( \mathbb{V}(z - aw) \) all obviously contain \( \ell' \), so \( \ell' \) is their common intersection, as promised by Theorem 4.0.4. Note that, dually, the star points on \( \ell' \) are the points \( p_b' = [b : 1 : 0 : 0] \) where \( b^{q+1} = -1 \), and the corresponding star planes \( \mathbb{V}(x - by) \) intersect in \( \ell \).

**Proof of Theorem 4.0.4** Fix any two star points \( p_1 \) and \( p_2 \) on \( \ell \). Since \( p_1 \) is on every line in the star centered at \( p_1 \), and likewise for \( p_2 \), any shared line shared line between these stars would contain both \( p_1 \) and \( p_2 \) and hence be \( \ell \) itself. But by definition, the star chord \( \ell \) is not on \( X \). So stars centered on \( \ell \) can not share any lines, proving (i).

Now, let \( \ell' = T_{p_1}X \cap T_{p_2}X \). Note that \( \ell' \not\subset X \): otherwise, \( \ell' \subset T_{p_1}X \cap X \) and \( \ell' \subset T_{p_2}X \cap X \), making \( \ell' \) a shared line between these stars, which would contradict (i).

We claim that \( \ell' \) is skew to \( \ell \). First note that \( \ell \neq \ell' \), for otherwise the star chord \( \ell \) lies in the star plane \( T_{p_1}X \), contradicting Remark 4.0.2. So at least one of \( p_1 \) or \( p_2 \)—say \( p_1 \)—is not on \( \ell' \). Now, if \( \ell \) and \( \ell' \) are not skew, the unique plane they span is necessarily the plane \( T_{p_1}X \), since both planes contain \( \ell' \) and \( p_1 \not\subset \ell' \). But now the star chord \( \ell' \) is in the star plane \( T_{p_1}X \), again contradicting Remark 4.0.2.

We now claim \( \ell' \) is a star chord intersecting \( X \) in \( q+1 \) distinct star points. Observe that because \( \ell' \subset T_{p_1}X \), it meets each line in the star \( T_{p_1}X \cap X \). But since the center \( p_1 \) is not on \( \ell' \), we know \( \ell' \) must meet each of the \( q+1 \) lines in the star \( T_{p_1}X \cap X \) in a distinct point. These \( q+1 \) points make up the full intersection \( \ell' \cap X \), since \( X \) has degree \( q+1 \). Similarly, since also \( p_2 \not\subset \ell' \), the points of \( \ell' \cap X \) are the \( q+1 \) distinct intersection points of \( \ell' \) with the lines in the star \( T_{p_2}X \cap X \). Thus each \( p' \) in \( \ell' \cap X \) lies on at least two lines of \( X \). So \( \ell' \) is a star chord and meets \( X \) in \( q+1 \) distinct star points.
Next, we show that $\ell \subset T_{p'}X$ for all $p' \in \ell' \cap X$, which will establish (iii). As we saw in the preceding paragraph, the star $X \cap T_{p'}X$ contains a line in each of the two stars $T_{p_1}X \cap X$ and $T_{p_2}X \cap X$. In particular, both $p_1$ and $p_2$ are in $T_{p'}X$, so also $\ell = \overline{p_1p_2} \subset T_{p'}X$.

We now claim $\ell$ meets $X$ in $q + 1$ distinct star points. To see this, take an arbitrary $p' \in \ell' \cap X$. Since $\ell \subset T_{p'}X$ (using (iii)) but $p' \notin \ell$ (by skewness of $\ell$ and $\ell'$), each line in the star $T_{p'}X \cap X$ meets $\ell$ in a distinct point. These are the $q + 1$ points of $X \cap \ell$. They are star points because each lies on a line in every other star $T_{p''}X \cap X$ with $p'' \in \ell'$.

The proof will be complete once we have shown that $\ell'$ is independent of the choice of the star points $p_1$ and $p_2$ on $\ell$. For this, it suffices to show that $\ell' \subset T_pX$ for each star point $p$ on $\ell$, so that $\ell'$ is the intersection of all $q + 1$ star planes centered along $\ell$ (or any two of them). But taking any $p' \in \ell' \cap X$, we have seen that $p' \in \ell'$ lies on a line in the star $T_pX \cap X$. So the $q + 1$ star points of $\ell'$, and hence $\ell'$ itself, are in $T_pX$.

4.1. Symmetry of star chords.

**Theorem 4.1.1.** The automorphism group of a smooth extremal surface induces a natural transitive action on the set of all its star chords.

**Proof of Theorem 4.1.1.** A star chord is determined by two star points not spanning a line on $X$, so any projective linear automorphism of the surface induces a permutation of the star chords for the surface. To prove this action is transitive, it suffices to show that $\text{Aut}(X)$ acts transitively on the set of ordered pairs $(p_1, p_2)$ of star points spanning star chords.

Fix an arbitrary ordered pair $(p_1, p_2)$ of star points spanning a star chord. Since $\text{Aut}(X)$ acts transitively on star points (Proposition 3.1.1), there is no loss of generality in assuming

$$X = \mathbb{V}(x^qw + w^qx + y^qz + z^qy)$$

and $p_1 = [0 : 0 : 0 : 1]$.

The theorem will be proved if we show that, in addition, we can choose coordinates so that $p_2$ is the star point $[1 : 0 : 0 : 0]$.

First note that we can assume that $p_2 = [1 : a : b : c]$. Indeed, otherwise $p_2 \in \mathbb{V}(x) = T_{p_1}X$, so that $p_2$ would be in the star centered at $p_1$. In this case, the line $\overline{p_1p_2}$ is in that star and hence on $X$, contrary to the assumption that $p_1$ and $p_2$ span a star chord. Note also that $a, b, c \in \mathbb{F}_{q^2}$ (Proposition 3.4.1).

Consider the change of coordinates $\phi$

$$[x : y : z : w] \mapsto [x : y - ax : z - bx : w + c^q x + b^q y + a^q z].$$

Clearly $\phi$ fixes $[0 : 0 : 0 : 1]$ and takes $p = [1 : a : b : c]$ to $[1 : 0 : 0 : 0]$. Furthermore, $\phi^*$ fixes the polynomial $x^qw + w^qx + y^qz + z^qy$: Remembering that $a^{q^2} = a$, $b^{q^2} = b$, $c^{q^2} = c$, and $c^q + c = -(ab + ba)$, we easily verify that $\phi^*(x^qw + w^qx + y^qz + z^qy) = x^qw + w^qx + y^qz + z^qy$. This completes the proof of Theorem 4.1.1.

Theorem 4.1.1 has the following consequence:

**Corollary 4.1.2.** The automorphism group of a smooth extremal surface $X$ acts transitively on the set of pairs of skew lines on $X$.

**Proof of Corollary 4.1.2.** Fix an arbitrary pair of skew lines $L$ and $L'$ on an extremal surface $X$. It suffices to show that we can choose projective coordinates so that $X$ is defined by the Frobenius form $F = x^qw + w^qx + y^{q+1} + z^{q+1}$ and the two lines are $L_a = \mathbb{V}(x, y - az)$ and $L_b = \mathbb{V}(w, y - bz)$ where $a$ and $b$ are distinct fixed $q + 1$ roots of $-1$. 

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We can choose star points \( p \in L \) and \( p' \in L' \) that span a star chord \( \ell \). Indeed, there are \((q^2 + 1)^2\) lines in \( \mathbb{P}^3 \) connecting star points on \( L \) to star points on \( L' \) (Theorem 3.2.1(a)) but only \( q^2 + 1 \) of them lie on \( X \) (Corollary 3.2.4).

Now by (the proof of) Theorem 4.1.1, the automorphism group of an extremal surface acts transitively on ordered pairs of star points spanning star chords. So we can choose coordinates so that the extremal surface is \( X = \mathbb{V}(x^q w + w^q x + y^{q+1} + z^{q+1}) \), \( p = [0 : 0 : 0 : 1] \), and \( p' = [1 : 0 : 0 : 0] \) (note that \( \overline{pp'} = \mathbb{V}(y, z) \), which is not on \( X \)). In this case, the star planes at \( p \) and \( p' \), respectively, are defined by \( x = y \) and \( w = y \). The line \( L \) is therefore in the star \( T_p X \cap X = \mathbb{V}(x, y^{q+1} + z^{q+1}) \) and \( L' \) is the star \( T_{p'} X \cap X = \mathbb{V}(w, y^{q+1} + z^{q+1}) \). In particular, \( L = \mathbb{V}(x, y - \nu_1 z) \) and \( L' = \mathbb{V}(w, y - \nu_2 z) \) where \( \nu_1^{q+1} = \nu_2^{q+1} = -1 \). The assumption that \( L \) and \( L' \) are skew means that \( \nu_1 \neq \nu_2 \).

Finally, we need a change of coordinates that fixes \( x \) and \( w \), while taking the forms \( \{y - \nu_1 z, y - \nu_2 z\} \) to our chosen ones \( \{y - az, y - bz\} \). Equivalently, we need the automorphism group of the extremal configuration \( Y = \mathbb{V}(y^{q+1} + z^{q+1}) \) in \( \mathbb{P}^1 \) to act two-transitively on \( Y \). This is immediate from Proposition 2.4.1. \( \square \)

4.2. Application: The order of the automorphism group. The group \( PU(n, \mathbb{F}_{q^2}) \) is well-studied in representation theory, and its order is classically known [Dic58 pp131-144], [Hir85, 19.1.6]. Still, we can give a cute computation of the order of \( PU(4, \mathbb{F}_{q^2}) \) as a corollary of Theorem 4.1.1.

**Corollary 4.2.1.** The order of the automorphism group of a smooth extremal surface, and hence the order of \( PU(4, \mathbb{F}_{q^2}) \), is

\[
q^6(q^2-1)(q^3+1)(q^4-1).
\]

**Proof of Corollary 4.2.1.** Fix a smooth extremal surface \( X \). We compute the order of \( Aut(X) \) using the orbit-stabilizer theorem for its action on the set \( S = \{(p, p') \mid p, p' \text{ star points, } \overline{pp'} \not\subset X\} \) of ordered pairs of star points spanning a star chord. This action is transitive by (the proof of) Theorem 4.1.1.

The cardinality of the orbit \( S \) is \( q^5(q^3+1)(q^2+1) \). Indeed, there are \((q^3+1)(q^2+1)\) star points on \( X \) (Corollary 3.2.3), so it suffices to show that for each choice of star point \( p \), there are \( q^5 \) star points \( p' \) such that \( \overline{pp'} \) is a star chord. For this, note that of the total \( q^5 \) star points other than \( p \), there are \( q^3 + q^2 \) in the star plane centered at \( p \) (Corollary 3.2.2). This leaves \( q^5 \) star points \( p' \) such that \( \overline{pp'} \) is a star chord, establishing that the cardinality of \( S \) is as claimed.

Corollary 4.2.1 then follows immediately from the following computation:

**Lemma 4.2.2.** The stabilizer of an ordered pair of star points \( (p_1, p_2) \) spanning a star chord on a smooth extremal surface is isomorphic to

\[
\mathbb{F}_{q^2}^* \times U(2, \mathbb{F}_{q^2}) \big/ \{ (\lambda, \lambda I_2) \mid \lambda^{q+1} = 1 \},
\]

and has order \( q(q^2-1)^2 \).

**Proof of Lemma.** We may assume that \( X = \mathbb{V}(x^q w + y^q z + z^q y + w^q x), p_1 = [1 : 0 : 0 : 0] \), and \( p_2 = [0 : 0 : 0 : 1] \). Then any automorphism in the stabilizer of \( (p_1, p_2) \) stabilizes also the tangent
planes at $p_1$ and $p_2$ so is represented by a matrix of the form

$$
\begin{bmatrix}
a_{11} & 0 & 0 & 0 \\
0 & a_{22} & a_{23} & 0 \\
0 & a_{32} & a_{33} & 0 \\
0 & 0 & 0 & a_{44}
\end{bmatrix}
$$

fixing $x^q w + y^q z + z^q y + w^q x$. In particular, the submatrix $h = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$ fixes $y^q z + z^q y$, so is in $U(2, \mathbb{F}_{q^2})$. Furthermore, because $g$ fixes $x^q w + w^q x$, we get $a_{11} = a_{11}$ and $a_{44} = a_{44} = \frac{1}{a_{11}}$. Thus there is a natural surjective map

$$
\mathbb{F}_{q^2}^* \times U(2, \mathbb{F}_{q^2}) \to \text{Stab}\{p, p'\} \quad (a_{11}, h) \mapsto [g] = \begin{bmatrix}
a_{11} & 0 & 0 & 0 \\
0 & a_{22} & a_{23} & 0 \\
0 & a_{32} & a_{33} & 0 \\
0 & 0 & 0 & a_{11}^{-q}
\end{bmatrix},
$$

and we easily compute that the kernel is $(\lambda, \lambda I_2 \mid \lambda^{q+1} = 1)$. So (invoking Remark 3.3.2), the order of the stabilizer of $(p_1, p_2)$ is

$$
\frac{|\mathbb{F}_{q^2}^*| \cdot |U(2, \mathbb{F}_{q^2})|}{|\mu_{q+1}|} = |\mathbb{F}_{q^2}^*| \cdot |\text{PU}(2, \mathbb{F}_{q^2})| = (q^2 - 1) \cdot |\text{PGL}(2, \mathbb{F}_q)| = q(q^2 - 1)^2.
$$

\[\square\]

## 5. Quadric Configurations

Extremal surfaces contain interesting line configurations we call **quadric configurations**:

**Definition 5.0.1.** A quadric configuration on a surface of degree $d \geq 3$ in projective three space is a collection of $2d$ lines on the surface consisting of two sets of $d$ skew lines with the property that each line in either set meets every line of the other set.

The next proposition justifies the name:

**Proposition 5.0.2.** A quadric configuration on an irreducible surface $X$ is equal to $X \cap Q$ for some unique smooth quadric surface $Q$.

**Proof.** Let $\mathcal{L} \cup \mathcal{M}$ be a configuration of lines, where $\mathcal{L}$ (respectively $\mathcal{M}$) consists of $d$ skew lines intersecting every line in $\mathcal{M}$ (respectively $\mathcal{L}$). Choose any three skew lines $L_1, L_2, L_3 \in \mathcal{L}$, and let $Q$ be the unique smooth quadric they determine [Har95, 2.12]. The lines of $\mathcal{M}$ intersect all lines in $\mathcal{L}$, including $L_1, L_2,$ and $L_3$, which lie on $Q$. So each line $M \in \mathcal{M}$ intersects the quadric $Q$ in at least three points, which means $M \subset Q$. But now each line $L \in \mathcal{L}$ intersects all lines in $\mathcal{M}$, so $L$ intersects $Q$ in at least three points. Again, we conclude $L \subset Q$. So $\mathcal{L} \cup \mathcal{M} \subset Q$.

Now if $\mathcal{L} \cup \mathcal{M} \subset X$, then $\mathcal{L} \cup \mathcal{M} \subset X \cap Q$. So since $X \cap Q$ and $\mathcal{L} \cup \mathcal{M}$ both have degree $2d$, and $X \cap Q$ is a complete intersection, we conclude that $X \cap Q$ is precisely the reduced union of the $2d$ lines in $\mathcal{L} \cup \mathcal{M}$. \[\square\]
Example 5.0.3. Let $Q_\mu$ be the quadric surface $Q_\mu = \mathbb{V}(\mu wx - yz)$, where $\mu \in k$ is a fixed $(q+1)$-st root of unity. The quadric $Q_\mu$ defines a quadric configuration on the Fermat extremal surface. Indeed, the lines in the sets

$$\mathcal{L}_\mu = \{ \mathbb{V}(x - \alpha y, z - \mu \alpha w) \mid \alpha^{q+1} = -1 \}$$

$$\mathcal{M}_\mu = \{ \mathbb{V}(x - \beta z, y - \mu \beta w) \mid \beta^{q+1} = -1 \}$$

all lie on the quadric $Q_\mu$ (with the lines in $\mathcal{L}_\mu$ and $\mathcal{M}_\mu$ in opposite rulings), as well as on the extremal surface $X = \mathbb{V}(x^{q+1} + y^{q+1} + z^{q+1} + w^{q+1})$. Thus $X \cap Q_\mu$ is the quadric configuration $\mathcal{L}_\mu \cup \mathcal{M}_\mu$.

Quadric configurations are rare on an arbitrary surface—for example, a generic surface of degree greater than three admits no lines at all [Har95, 12.8]. Remarkably, extremal surfaces contain many quadric configurations:

**Theorem 5.0.4.** Any triple of skew lines on a smooth extremal surface determines a unique quadric configuration.

**Proof.** Fix three skew lines, $L$, $L'$, and $L''$ on the extremal surface $X$ of degree $d = q + 1$. Without loss of generality, assume $X$ is defined by the form $x^q w + w^q x + y^q z + z^q y$, $L$ by $x = y = 0$, and $L'$ by $z = w = 0$ (Corollary 4.1.2). In this case, $L''$ can be defined by linear equations of the form

$$x = az + bw \quad y = cz + dw,$$

where the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is full rank, and is parametrized as $\{(as+bt : cs+dt : s : t) \mid [s : t] \in \mathbb{P}^1\}$. Furthermore, the condition that $L''$ lies on $X$ means that

$$(as + bt)^q t + t^q (as + bt) + (cs + dt)^q s + s^q (cs + dt) = 0$$

for all $s, t$. This imposes the constraints

$$c^q + c = b^q + b = a^q + d = a + d^q = 0.$$  
(11)

The quadric $Q$ defined by

$$cxz + dxw - ayz - byw$$

contains $L$, $L'$, and $L''$. Note that $Q$ is the image of the Segre map

$$\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3: ([s_1 : s_2], [t_1, t_2]) \mapsto [(as_1 + bs_2) t_1 : (cs_1 + ds_2) t_1 : s_1 t_2 : s_2 t_2].$$

Now, consider an arbitrary line in one of the rulings on $Q$, say

$$\ell = \{(a \lambda_1 + b \lambda_2) t_1 : (c \lambda_1 + d \lambda_2) t_1 : \lambda_1 t_2 : \lambda_2 t_2) \mid [t_1 : t_2] \in \mathbb{P}^1\}.$$

The line $\ell$ is on $X$ if and only if, plugging into the Frobenius form defining $X$, the form

$$\lambda_2 (a \lambda_1 + b \lambda_2)^q t_1^q t_2 + \lambda_1^2 (a \lambda_1 + b \lambda_2) t_1 t_2^q + \lambda_1 (c \lambda_1 + d \lambda_2)^q t_1^q t_2 + \lambda_1^2 (c \lambda_1 + d \lambda_2) t_1^q t_2,$$

is uniformly zero for all values of $t_1, t_2$. Equivalently, $\ell$ is on $X$ precisely when the coefficients of $t_1^q t_2$ and of $t_1^q t_2^q$ in expression (13) satisfy

$$\lambda_2 (a \lambda_1 + b \lambda_2)^q + \lambda_1 (c \lambda_1 + d \lambda_2)^q = 0$$
$$\lambda_2^2 (a \lambda_1 + b \lambda_2) + \lambda_1^2 (c \lambda_1 + d \lambda_2) = 0.$$

In light of the constraints (11), these equations simplify to

$$c \lambda_1^{q+1} + d \lambda_1^q \lambda_2 + a \lambda_1 \lambda_2^q + b \lambda_2^{q+1} = 0$$

(14)
Because the form in (14) is a Frobenius form in $\lambda_1, \lambda_2$ with the full rank matrix $\begin{bmatrix} c & d \\ a & b \end{bmatrix}$, there are precisely $q + 1$ distinct solutions to (14) in $\mathbb{P}^1$. We conclude that there are precisely $q + 1$ lines $\ell$ of the form $\sigma((\lambda_1 : \lambda_2) \times \mathbb{P}^1)$ lying on both $X$ and $Q$. These are $q + 1$ different skew lines on the extremal surface.

Now consider a line in the other ruling, say

$$m = \{(as_1 + bs_2)\lambda_1 : (cs_1 + ds_2)\lambda_1 : s_1\lambda_2 : s_2\lambda_2 \mid [s_1 : s_2] \in \mathbb{P}^1\}.$$  

The line $m$ lies on $X$ if and only if

$$\lambda^q_2\lambda_2(as_1 + bs_2)^q s_2 + \lambda^q_1\lambda^q_2(as_1 + bs_2)s_2^q + \lambda^q_1\lambda_2(cs_1 + ds_2)^q s_1 + \lambda^q_1\lambda_2(cs_1 + ds_2)s_1^q = 0$$

for all values of $s_1, s_2$. That is, $m \subset X$ if and only if

$$c^q\lambda^q_1\lambda_2 + c\lambda_1\lambda^q_2 = a^q\lambda^q_1\lambda_2 + d\lambda_1\lambda^q_2 = a\lambda_1\lambda^q_2 + d\lambda_1\lambda^q_2 = b^q\lambda^q_1\lambda_2 + b\lambda_1\lambda^q_2 = 0.$$  

Again making use of the relations (11), these four equations all boil down to one,

$$\lambda^q_2\lambda_2 - \lambda_1\lambda^q_2.$$  

Because there are exactly $q + 1$ points $[\lambda_1 : \lambda_2] \in \mathbb{P}^1$ satisfying (15), there are precisely $q + 1$ lines $m$ in this ruling of $Q$ which lie $X$. These form a set of $q + 1$ skew lines, each of which meets every line in the other set of $q + 1$ skew lines on $X$.  

Remark 5.0.5. In the finite geometry setting, Hirschfeld proves an analog of Theorem 5.0.4 for Hermitian geometries using different techniques and language [Hir85, 19.3.1].

5.1. Symmetry of Quadric Configurations.

Theorem 5.1.1. The automorphism group of a smooth extremal surface acts transitively on its set of quadric configurations.

In light of Theorem 5.0.4, Theorem 5.1.1 is an immediate consequence of the following:

Theorem 5.1.2. The automorphism group of a smooth extremal surface $X$ acts transitively on the set of triples of skew lines on $X$.

Proof of Theorem 5.1.2. It suffices to show that Aut($X$) acts transitively on the set $\mathcal{S}$ of all ordered sextuples $(L_1, L_2, L_3, M_1, M_2, M_3)$ of lines on $X$, consisting of two triples of skew lines $\{L_1, L_2, L_3\}$ and $\{M_1, M_2, M_3\}$ with $L_i \cap M_j \neq \emptyset$ for all $i, j$.

Fix an ordered sextuple $(L_1, L_2, L_3, M_1, M_2, M_3) \in \mathcal{S}$. First note that its stabilizer, even in PGL$(4, k)$, is trivial. Indeed, the intersection points $p_{ij} = L_i \cap M_j$ must be fixed by any element in the stabilizer of $(L_1, L_2, L_3, M_1, M_2, M_3)$. These nine points contain five points in general linear position (no three on a line, no four on a plane). But an automorphism of $\mathbb{P}^3$ fixing five points in general linear position is trivial.

Next, we compute the cardinality of $\mathcal{S}$. There are $(q^3 + 1)(q + 1)$ choices for $L_1$ by Corollary 3.2.3(a), and fixing $L_1$, there are $q^4$ choices for a skew line $L_2$ on $X$ by Theorem 3.2.1(c). The number of choices for $L_3$ is the total number of lines on $X$ minus the number of lines meeting $L_1$ or $L_2$. Accounting for the double-counting of lines meeting both $L_1$ and $L_2$, the number of choices for $L_3$ is

$$[(q^3 + 1)(q + 1)] - 2[q^3 + q + 1] + [q^2 + 1] = q(q^2 + 1)(q - 1),$$
using Corollary\textsuperscript{3.2.3} (a), Theorem\textsuperscript{3.2.1} (b), and Corollary\textsuperscript{3.2.4} The choice of the triple $L_1, L_2, L_3$ determines the quadric, and hence $q + 1$ lines in $Q \cap X$ that all intersect $L_1, L_2, L_3$ by Theorem\textsuperscript{5.0.4}. There are thus $(q + 1)q(q - 1)$ ways to choose the triple $M_1, M_2, M_3$. In total, the number of ordered sextuples is thus

$$\left[(q^3 + 1)(q + 1)\right] \cdot [q^4] \cdot [q(q^2 + 1)(q - 1)] \cdot [(q + 1)q(q - 1)] = q^6(q^4 - 1)(q^3 + 1)(q^2 - 1).$$

This is precisely the order of the automorphism group Aut($X$) by Corollary\textsuperscript{4.2.1}. So Aut($X$) must act transitively on the set $S$, and hence on the set of all triples of skew lines on $X$. \hfill \Box

For future reference, we record the following corollary of the proof of Theorem\textsuperscript{5.1.2}.

\textbf{Corollary 5.1.3.} A smooth extremal surface $X$ contains exactly $\frac{1}{2}(q^3 + 1)(q^2 + 1)q^4$ quadric configurations, where the degree of $X$ is $q + 1$.

\textbf{Proof of Corollary.} By Theorem\textsuperscript{5.0.4} each quadric configuration on a smooth extremal surface $X$ is uniquely determined by an ordered triple of skew lines $(L_1, L_2, L_3)$ on $X$. The number of such ordered triples is

$$(q^3 + 1)(q + 1) \cdot q^4 \cdot q(q^2 + 1)(q - 1),$$

as we computed in the proof of Theorem\textsuperscript{5.1.2}. To determine the number of quadric configurations, then, we must determine the number of ordered triples determining the same quadric. To this end, first note that there are $2(q + 1)$ choices of a line $L_1$ in $Q$. Once $L_1$ is fixed, the lines $L_2$ and $L_3$ are among the $q$ lines in same ruling of $Q$, so there are $q(q - 1)$ choices for $(L_2, L_3)$. We conclude that there are

$$\frac{(q^3 + 1)(q + 1)q^5(q - 1)(q^2 + 1)}{2(q + 1)q(q - 1)} = \frac{1}{2}(q^3 + 1)(q^2 + 1)q^4$$

quadric configurations on a smooth extremal surface. \hfill \Box

### 5.2. Star Chords in Quadric Configurations

We record some observations about star chords and quadric configurations that will be useful in Section\textsuperscript{6}.

\textbf{Lemma 5.2.1.} Let $Q$ be a quadric defining a quadric configuration on a smooth extremal surface $X$. Let $\ell$ be a line on $Q$ but not on $X$. Then $\ell$ intersects $X$ in $q + 1$ distinct points, and if any one of these intersection points is a star point of $X$, then they all are.

\textbf{Proof.} Because the automorphism group of $X$ acts transitively on quadric configurations (Theorem\textsuperscript{5.1.1}), we may assume that $X$ is given by the Fermat Frobenius form and $Q$ by $xw = yz$. The lines on $Q$ have the following parametrizations

$$\{[\lambda s : s : \lambda t : t] \mid [s : t] \in \mathbb{P}^1\} \quad \text{and} \quad \{[\lambda s : \lambda t : s : t] \mid [s : t] \in \mathbb{P}^1\}.$$ 

Without loss of generality, let $\ell = \{[\lambda s : s : \lambda t : t] \mid [s : t] \in \mathbb{P}^1\}$ for some fixed $\lambda$. The condition that a point $[\lambda s_0 : s_0 : \lambda t_0 : t_0]$ of $\ell$ lies on $X$ is that

$$(\lambda s_0)^q + 1 + s_0^{q+1} + (\lambda t_0)^q + 1 + t_0^{q+1} = (\lambda^{q+1} + 1)(s_0^{q+1} + t_0^{q+1}) = 0.$$

There are two ways this can happen. Either $\lambda^{q+1} = -1$, which means \textsuperscript{(16)} holds for all values of $[s_0 : t_0]$, so the line $\ell$ lies on $X$. Or $\lambda^{q+1} \neq -1$, and there are exactly $q + 1$ points $[s_0 : t_0]$ satisfying $s_0^{q+1} + t_0^{q+1} = 0$. In this case, there are exactly $q + 1$ distinct points of $\ell \cap X$, all of the form $[\lambda \mu : \mu : \lambda : 1]$ where $\mu$ ranges through the $q + 1$ distinct roots of $-1$. In particular, $\mu \in \mathbb{F}_{q^2}$. Now if one of these points $[\lambda \mu : \mu : \lambda : 1]$ is a star point, then it is defined over $\mathbb{F}_{q^2}$ (Proposition\textsuperscript{3.4.1}), so $\lambda \in \mathbb{F}_{q^2}$ as well. Thus all $q + 1$ points of $X \cap \ell$ are defined over $\mathbb{F}_{q^2}$ and hence all are star points. \hfill \Box
Proposition 5.2.2. Let $Q$ be a smooth quadric defining a quadric configuration on a smooth extremal surface $X$. Then there are exactly $q^2 - q$ star chords in each ruling of $Q$, and those in opposite rulings meet off $X$.

Proof. Consider a star chord $\ell$ on $Q$. Write $Q \cap X = L \cup M$ where $L$ and $M$ are the two skew sets of lines on $X$ in opposite rulings of $Q$.

Because $\ell$ must lie in one of the rulings of $Q$, it intersects each of the $q + 1$ lines in, say, $M$. For each $M \in M$, the intersection point $\ell \cap M$ is a star point (Theorem 4.0.4(iv)). Conversely, through each star point on $M$, the unique line in the opposite ruling of $Q$ is either a line in $L$, or a star chord, depending on whether or not it is on $X$ (Lemma 5.2.1). Since there are $q^2 + 1$ total star points on $M$ (Theorem 3.2.1(a)), this leaves $q^2 - q$ possible points of intersection of the star chord $\ell$ with $M$. Thus there are exactly $q^2 - q$ possibilities for the star chord $\ell$ in this ruling of $Q$. By symmetry, the same holds in the other ruling.

Now suppose $\ell$ and $m$ are star chords in opposite rulings on $Q$. If $p = \ell \cap m$ lies on $X$, then it must be one of the $q + 1$ points on $\ell \cap X$, and hence $p$ is some star point on some line $M \subset Q \cap X$ in the ruling opposite $\ell$. In this case, $M$ is the unique line through $p$ on $Q$ in the ruling opposite $\ell$, forcing $m = M$. This contradicts our assumption that $m$ is not on $X$. \hfill \square

Remark 5.2.3. Proposition 5.2.2 and Lemma 5.2.1 together say the complete set of lines on $Q$ passing through star points of $X$ consists of two sets of $q^2 + 1$ skew lines (one on each ruling); in each of these skew sets, there are $q + 1$ lines on $X$ and $q^2 - q$ star chords.

Theorem 5.2.4. The automorphism group of a smooth extremal surface acts transitively on the set of triples $(Q, \ell, m)$ consisting of a quadric $Q$ defining a quadric configuration, together with a choice star chords $\ell$ and $m$, one in each ruling of $Q$.

Proof. We may assume that the extremal surface $X$ is defined by $x^{q+1} + y^{q+1} + z^{q+1} + w^{q+1}$ and $Q$ by $xw - yz$ (Theorem 5.1.1). Let $\ell$ and $m$ be an arbitrary pair of star chords on $Q$, lying in opposite rulings. It suffices to show that there is an automorphism of $X$ which stabilizes $Q$ and sends $\ell$ and $m$ to the star chords $V(x, z)$ and to $V(z, w)$, respectively.

The lines in the two rulings of $Q$ have the form

$$V(\lambda x - \mu y, \lambda z - \mu w) \quad \text{and} \quad V(\alpha x - \beta z, \alpha y - \beta w);$$

the star chords among them are precisely those where $[\lambda : \mu]$ (respectively $[\alpha : \beta]$) is an $\mathbb{F}_q^2$ point of $\mathbb{P}^1$ not on $V(s^{q+1} + t^{q+1})$. Indeed, all such lines are on $Q$, but not on $X$, and since there are $q^2 - q$ in each ruling, we have found the complete list of star chords on $Q$ (Proposition 5.2.2).

Suppose that $\ell = V(\lambda x - \mu y, \lambda z - \mu w)$. The change of coordinates $g$ where $g^{-1} = \begin{bmatrix} \lambda^q & \mu & 0 & 0 \\ -\mu^q & \lambda & 0 & 0 \\ 0 & 0 & \lambda^q & \mu \\ 0 & 0 & -\mu^q & \lambda \end{bmatrix}$ is in $\text{Aut}(X) \cap \text{Aut}(Q)$, since it simply scales the defining equation of both $X$ and $Q$ by a non-zero scalar (remember $\lambda^{q+1} + \mu^{q+1} \neq 0$). In addition, $g$ sends $\ell$ to $V(x, z)$, as

$$g(\ell) = V(\lambda(\lambda^q x + \mu y) - \mu(-\mu^q x + \lambda y), \lambda(\lambda^q z + \mu w) - \mu(-\mu^q z + \lambda w)) = V((\lambda^{q+1} + \mu^{q+1}) x, (\lambda^{q+1} + \mu^{q+1}) z) = V(x, z).$$

Of course, $g$ sends $m$ to some star chord on $Q$ in the opposite ruling from $g(\ell)$. So $g(m) = V(\alpha x - \beta z, \alpha y - \beta w)$ for some $\mathbb{F}_q^2$ point $[\alpha : \beta] \in \mathbb{P}^1$ not on $V(s^{q+1} + t^{q+1})$. Now observe that the
change of coordinates $h$ where $h^{-1} = \begin{bmatrix} \beta & 0 & \alpha q & 0 \\ 0 & \beta & 0 & \alpha q \\ \alpha & 0 & -\beta q & 0 \\ 0 & \alpha & 0 & -\beta q \end{bmatrix}$ preserves the Fermat extremal surface and the quadric $Q$ defined by $xw = yz$. In addition, $h$ preserves the line $V(x, z)$, since $h(x)$ and $h(z)$ are forms in only $x$ and $z$. Finally, the line $g(m) = V(\alpha x - \beta z, \alpha y - \beta w)$ is sent to

$$h(g(m)) = V(\alpha(\beta x + \alpha q z) - \beta(\alpha x - \beta q z), \alpha(\beta y + \alpha q w) - \beta(\alpha y - \beta q w)) = V((\alpha q^1 + \beta q^1)z, (\alpha q^1 + \beta q^1)w) = V(z, w)$$

We conclude that the composition $h \circ g$ is an automorphism of $X$ which preserves $Q$, and takes $\ell$ and $m$ to $V(x, z)$ and $V(z, w)$, respectively. This completes the proof. \qed

**Corollary 5.2.5.** If a star chord $\ell$ is in a quadric $Q$ defining a quadric configuration on a smooth extremal surface, then its dual star chord $\ell'$ (Definition 4.0.5) is also on $Q$, necessarily in the same ruling as $\ell$.

**Proof.** Assume that the extremal surface is the Fermat surface, and $\ell$ is the star chord $V(x, y)$ on the quadric $Q = V(xw - yz)$ (Theorem 5.2.4). The dual chord of $\ell$ is $\ell' = V(z, w)$ (Example 4.0.6), which clearly lies on $Q$ as well. The lines $\ell$ and $\ell'$ lie in the same ruling of $Q$ because they are skew (Theorem 4.0.4(ii)). \qed

**Corollary 5.2.6.** Let $\ell$ and $m$ be star chords for a smooth extremal surface $X$. If $\ell$ and $m$ lie on a quadric that defines a quadric configuration on $X$, then $\ell$ and $m$ lie on exactly $q + 1$ quadrics that define quadric configurations on $X$.

**Proof.** By Theorem 5.2.4, we can assume that $X$ is the Fermat extremal surface, $\ell = V(x, y)$ and $m = V(x, z)$. The quadrics defining quadric configurations that contain $\ell$ and $m$ must also contain their dual star chords, $\ell' = V(z, w)$ and $m' = V(y, w)$, respectively, by Corollary 5.2.5.

The quadrics containing $\{\ell, \ell', m, m'\}$ are defined by degree two polynomials in the ideal

$$\langle x, z \rangle \cap \langle y, w \rangle \cap \langle z, w \rangle \cap \langle x, y \rangle = \langle xw, yz \rangle.$$

But a quadratic form $\mu xw - yz$ (where $\mu$ is a non-zero scalar) defines quadric containing lines of $X$ if and only if $\mu q^1 = 1$. Indeed, the lines in one of the rulings are parametrized by $[a : b] \in \mathbb{P}^1$:

$$L_{ab} = \{[as : \mu bs : at : bt] | [s : t] \in \mathbb{P}^1\},$$

which lies on the Fermat surface only if $\mu q^1 = 1$ and $a q^1 + b t^1 = 0$. Thus there are $q + 1$ quadrics that contain the four star chords $\{\ell, \ell', m, m'\}$. \qed

### 6. Double 2d Configurations

One fascinating classical feature of the geometry of a cubic surface is the existence of thirty six “double sixes” [Sch58]. A double six consists of two collections of six skew lines on the cubic, with the property that each line in one collection intersects exactly five lines in the other. A choice of double six is equivalent to a labeling of the twenty-seven lines on the cubic so that one of the collections of six skew lines is the set of six exceptional divisors, thinking of the cubic surface as the blow up of the plane at six points, and the other collection is the set of strict transforms of the six conics through five of the points. In this section, we present a generalization of a “double six” which exists on all extremal surfaces.
**Definition 6.0.1.** For any $d \geq 2$, a **double $2d$** is a collection of two sets, $A$ and $B$, each consisting of $2d$ lines in projective three space, such that

1. Each line in $A$ (resp. $B$) is skew to every other line in $A$ (resp. $B$); and
2. Each line in $A$ (resp. $B$) intersects exactly $d+2$ lines in $B$ (resp. $A$).

Typically, we do not expect a surface of degree $d$ to contain any double $2d$—for example, a general surface in $\mathbb{P}^3$ of degree greater than three contains no line [Har95, 12.8]. The next result guarantees, however, that like cubic surfaces, extremal surfaces always contain double $2d$'s.

**Theorem 6.0.2.** Every smooth extremal surface of degree $d$ contains double $2d$ configurations of lines.

In fact, there are a great many double $2d$'s on an extremal surface: Corollary 6.3.2 will eventually show that their number grows asymptotically to $\frac{1}{16}d^{14}$ as $d$ grows large.

We will prove Theorem 6.0.2 by constructing explicit pairs of quadric configurations whose union is a double $2d$. First, we speculate that every double $2d$ arises from pairs of quadrics:

**Conjecture 6.0.3.** Every double $2d$ on an extremal surface $X$ of degree $d$ consists of $4d$ lines that are the union of two quadric configurations on $X$.

Towards Conjecture 6.0.3, we have proven

**Theorem 6.0.4.** Every double $2d$ on a degree $d$ extremal surface is a union of two quadric configurations when $d > 10$ or $d < 5$. Moreover, for $d \geq 5$, if two quadrics determine some double $2d$, then no other pair of quadrics determines the same double $2d$.

We have also verified the conjecture by computer when $d = 5$.

Before proving Theorems 6.0.2 and 6.0.4, we review the motivating example of cubic surfaces.

### 6.1. Double Sixes on Cubics

Every double six on a cubic surface—whether extremal or not—is a union of two quadric configurations. For an arbitrary double six $A \cup B$ on a cubic surface $X$, there is a choice of coordinates making $X$ the blowup of six points on $\mathbb{P}^2$ (no three on a line, not all on a conic), and so that $A$ consists of the six lines of exceptional divisors $\{E_1, \ldots, E_6\}$ and $B$ consists of the proper transforms $\{\tilde{C}_1, \ldots, \tilde{C}_6\}$ of the six conics in $\mathbb{P}^2$ through five of the six points [Nag00, Thm 8, pg. 366]. Here, $\tilde{C}_i$ denotes the proper transform of the conic that misses the point blown up to get $E_i$.

Now, given any three lines in $A$, say $\{E_1, E_2, E_3\}$, there are three lines, $\{\tilde{C}_4, \tilde{C}_5, \tilde{C}_6\}$, in $B$ that meet all of them. This says that the unique quadric surface $Q$ containing $\{E_1, E_2, E_3\}$ must also contain $\{\tilde{C}_4, \tilde{C}_5, \tilde{C}_6\}$. Likewise, the unique quadric $Q'$ containing $\{E_4, E_5, E_6\}$ must contain $\{\tilde{C}_1, \tilde{C}_2, \tilde{C}_3\}$. So the quadrics $Q$ and $Q'$ both produce quadric configurations on $X$:

$$Q \cap X = \{E_1, E_2, E_3, \tilde{C}_4, \tilde{C}_5, \tilde{C}_6\} \quad \text{and} \quad Q' \cap X = \{E_4, E_5, E_6, \tilde{C}_1, \tilde{C}_2, \tilde{C}_3\},$$

which together produce the double six

$$A = \{E_1, E_2, E_3, E_4, E_5, E_6\} \quad \text{and} \quad B = \{\tilde{C}_1, \tilde{C}_2, \tilde{C}_3, \tilde{C}_4, \tilde{C}_5, \tilde{C}_6\}.$$

So every double six on a cubic surface is the union of two quadric configurations.

**Remark 6.1.1.** The quadric configurations $Q$ and $Q'$ determining the double six $A \cup B$ on a cubic surface are not unique: there is a quadric containing any three of the six skew lines in $A$ and another containing the remaining three, and the lines of $B$ lie three in each of these two quadrics.
Thus there are \( \frac{1}{2} \binom{6}{3} = 10 \) different pairs of quadrics determining the double six \( \mathcal{A} \cup \mathcal{B} \). This confirms that some restriction on \( d \) is necessary in the uniqueness statement in Theorem 6.0.4 above.

**Remark 6.1.2.** The previous discussion applies to an arbitrary smooth cubic surface: each of its thirty-six double sixes is a union of two quadric configurations. However, for an extremal cubic surface, the double sixes come from two quadrics of a particular form. Specifically, if \( Q \) and \( Q' \) are quadrics on an extremal cubic surface which together give a double six, then \( Q \cap Q' \) is the union of four lines.

To see this, observe that \( Q \cap Q' \cap X \) consists of twelve distinct points—otherwise, one line of \( Q \cap X \) would intersect a line from both rulings of \( Q' \cap X \) (or vice versa), violating the skewness condition for a double six. These twelve points are star (Eckardt) points as they lie at the intersection of a line in \( Q \cap X \) with a line in \( Q' \cap X \). Now, each of these twelve star points lies on only one line in \( X \cap Q \), again by skewness, so these twelve star points lie on star chords of \( Q \cap X \). Since there are only two star chords in each ruling (Remark 5.2.3), each containing exactly \( q + 1 = 3 \) star points, these twelve points lie three each on the four star chords on \( Q \). Likewise, the same argument replacing \( Q \) by \( Q' \) shows that the twelve star points lie three each on the four star chords on \( Q' \). We conclude that \( Q \cap Q' \) consists of the four shared star chords for \( X \).

### 6.2. The existence of double \( 2d \)'s on extremal surfaces.

**Proof of Theorem 6.0.2.** Choose coordinates so that the extremal surface \( X \) is defined by \( x^{q+1} + y^{q+1} + z^{q+1} + w^{q+1} \).

Fix \( \mu \), a \((q+1)\)-st root of unity. As we saw in Example 5.0.3, the lines

\[
\mathcal{L}_\mu := \{ V(x - \alpha y, z - \mu \alpha w) \mid \alpha^{q+1} = -1 \}
\]

and

\[
\mathcal{M}_\mu = \{ V(x - \beta z, y - \mu \beta w) \mid \beta^{q+1} = -1 \}
\]

form a quadric configuration cut out by the quadric \( Q_\mu = V(\mu x w - y z) \).

We claim that if \( \mu_1 \) and \( \mu_2 \) are distinct \((q+1)\) roots of unity, then the sets

\[
\mathcal{A} := \mathcal{L}_{\mu_1} \cup \mathcal{M}_{\mu_2} \quad \text{and} \quad \mathcal{B} := \mathcal{L}_{\mu_2} \cup \mathcal{M}_{\mu_1}
\]

together form a double \( 2(q+1) \).

To see that \( \mathcal{A} \) consists of skew lines, first observe that the lines of \( \mathcal{L}_{\mu_1} \) are mutually skew, as they lie in the same ruling of a quadric. To see that each \( L \in \mathcal{L}_{\mu_1} \) is skew to every \( M \in \mathcal{M}_{\mu_2} \), we check that the ideal of their intersection, \( (x - \alpha y, z - \mu_1 \alpha w, x - \beta z, y - \mu_2 \beta w) \), is generated by four linearly independent linear forms. For this, it suffices to show that the matrix

\[
\begin{bmatrix}
1 & -\alpha & 0 & 0 \\
0 & 0 & 1 & -\mu_1 \alpha \\
1 & 0 & -\beta & 0 \\
0 & 1 & 0 & -\mu_2 \beta
\end{bmatrix}
\]

whose rows are the coefficients of the linear forms, has full rank. But this is clear, since its determinant is \( \alpha \beta (\mu_2 - \mu_1) \). A symmetric argument shows that also \( \mathcal{B} \) consists of skew lines.

Now that we know \( \mathcal{A} \) and \( \mathcal{B} \) are skew sets, the proof of Theorem 6.0.2 will be complete once we have proved the following general lemma.
Lemma 6.2.1. Let $X$ be a smooth extremal surface of degree $d$. Let $Q_1$ and $Q_2$ be two quadric configurations on $X$ that do not share a line (on $X$). Write $Q_1 = L_1 \cup M_1$ and $Q_2 = L_2 \cup M_2$ for the decomposition of each quadric configuration into the lines of the two rulings. Then $A = L_1 \cup M_2$ and $B = L_2 \cup M_1$ form a double $2d$ on $X$ if (and only if) both $A$ and $B$ are skew sets.

Proof of Lemma 6.2.1. Since $Q_1$ and $Q_2$ have no common line, there are $4d$ lines in $Q_1 \cup Q_2$, and $2d$ lines in each of $A$ and $B$. Because we are given that $A$ and $B$ are each skew sets, we need only check condition (2) of Definition 6.0.1 to verify that $A \cup B$ is a double $2d$.

To this end, take any $N \in A$. Without loss of generality, assume $N \in L_1$. We need to show that $N$ intersects exactly $d+2$ lines in $B$. Since $N$ lies in one ruling of the quadric $Q_1$ determining $Q_1$, the line $N$ intersects the $d$ lines of the opposite ruling $M_1 \subset B$. Thus we need to show that $N$ intersects exactly two lines of $L_2$.

Since $N$ does not lie on the quadric $Q_2$ determining $Q_2$ (remember $Q_1 \cap Q_2 = \emptyset$), its intersection multiplicity with $Q_2$ is two. If $N$ meets $Q_2$ in two distinct points, we are done: $N$ must meet exactly two of the lines in the ruling $L_2$ since it does not meet any line of the ruling $M_2$ by our assumption that $A$ is a skew set.

It remains to show that $N$ can not be tangent to $Q_2$. If, on the contrary, $N$ is tangent to $Q_2$ at some point $p$, then $N \subset T_p Q_2$. Because $p \in Q_2 \cap X$, and $Q_2 \cap X$ is a union of lines, the point $p$ lies on some line $M$ in $Q_2 \cap X$. In particular, $p$ is a star point since it is the intersection of the two lines $M$ and $N$ on $X$. Furthermore, since both $N$ and $M$ are in the tangent plane $T_p Q_2$, as well as in the star plane $T_p X$, we have $T_p X = T_p Q$. But now consider the unique line $\ell$ through the star point $p$ on $Q_2$ in the opposite ruling from $M$. We know $\ell$ is not on $X$, for otherwise, $p \in \ell \subset Q_2 \cap X$, which means $p$ lies on lines in both rulings of $Q_2$, violating skewness. By Lemma 5.2.1 we conclude that $\ell$ is a star chord through $p$, and being on $Q_2$, also $\ell \subset T_p Q_2 = T_p X$. But no star chord through a star point $p$ can lie in the star plane $T_p X$ (Remark 4.0.2). This contradiction ensures that $N$ is not tangent to $Q_2$, and the proof is complete.

6.3. Pairs of Quadrics containing a common line. The double $2d$ constructed in the proof of Theorem 6.0.2 is obtained from two quadric configurations whose quadric surfaces intersect in four lines. These are an abundant type of double $2d$’s—encompassing all the double sixes in the case of extremal cubics.

Theorem 6.3.1. Let $X$ be a smooth extremal surface of degree $d$, and let $Q$ and $Q'$ be distinct quadrics defining quadric configurations on $X$.

Assume that $Q$ and $Q'$ share a common line, but share no line on $X$. Then

(i) The $4d$ lines of $(Q \cap X) \cup (Q' \cap X)$ can be split into two sets of $2d$ lines forming a double $2d$;

(ii) The intersection $Q \cap Q'$ consists of four star chords $\{\ell, m, \ell', m'\}$, where $\{\ell, \ell'\}$ and $\{m, m'\}$ are dual chord pairs in opposite rulings.

Importantly, not all double $2d$’s are of the type guaranteed by Theorem 6.3.1; see Example 6.3.4.

Before proving Theorem 6.3.1 we deduce the following corollary bounding below the total number of double $2d$s on an extremal surface.
Corollary 6.3.2. An extremal surface of degree \( d = q + 1 \geq 5 \) contains at least
\[
\frac{1}{16}(q^3 + 1)(q^2 + 1)(q - 1)^2 q^7
\]
collections of double \( 2d \)'s.

Proof of Corollary. By Theorem 6.0.4, if two quadrics determine a double \( 2d \) on an extremal surface of degree \( d \geq 5 \), then they are unique. So we can prove Corollary 6.3.2 by counting the pairs of quadrics \( \{Q, Q'\} \) determining quadric configurations whose intersection consists of four star chords (Theorem 6.3.1).

Fix one quadric \( Q \) giving a quadric configuration on \( X \). There are \( (q^2 - q)^2 \) choices of pairs of star chords \( \{\ell, m\} \) on \( Q \), one in each ruling, by Proposition 5.2.2. Since the dual of each star chord on \( Q \) is also on \( Q \), there are \( \frac{(q^2 - q)^2}{4} \) choices for sets of star chords \( \{\ell, \ell', m, m'\} \) on \( Q \), where \( \ell \) and \( m \) are in opposite rulings and \( \ell', m' \) are their duals.

There are exactly \( q \) additional quadrics, besides \( Q \), that contain \( \{\ell, \ell', m, m'\} \) and define a quadric configuration (Corollary 5.2.6). So there are exactly \( q^3(q - 1)^2 \) quadrics \( Q' \) defining quadric configurations such that \( Q \cap Q' \) is the union of two star chords and their duals.

Finally, multiplying by the total number of choices for \( Q \) (provided by Corollary 5.1.3), we get
\[
\frac{1}{2}(q^3 + 1)(q^2 + 1)q^4 \cdot \frac{1}{4}q(q - 1)^2 = \frac{1}{8}(q^3 + 1)(q^2 + 1)(q^3 + 1)(q - 1)^2 q^7
\]
ordered pairs of quadric configurations whose intersection is four star chords. This counts each pair twice so the result follows.

Proof of Theorem 6.3.1. Suppose \( \ell \subset Q \cap Q' \) but \( \ell \nsubseteq X \). Because \( \ell \) is in some ruling on each of \( Q \) and \( Q' \), \( \ell \) must intersect \( d \) lines on \( X \cap Q \) and \( d \) lines on \( X \cap Q' \). By hypothesis, these lines are distinct, so \( \ell \) intersects \( 2d \) lines on \( X \). Now because \( \ell \cap X \) can be at most \( d \) points, \( \ell \) simultaneously intersects \( X \) at a line on \( X \cap Q \) and a line on \( X \cap Q' \), so \( \ell \) intersects \( X \) at a star point. So \( \ell \) passes through \( d \) star points and is a star chord.

Let \( \ell' \) be the dual star chord to \( \ell \). We know \( \ell' \subset Q \cap Q' \), by Corollary 5.2.5. Since \( \ell \) and \( \ell' \) are skew, they are in the same ruling on \( Q \) and also in the same ruling on \( Q' \), which means that \( \ell \cup \ell' \) is a curve of bi-degree \((2,0)\) on each quadric. Since \( Q \cap Q' \) is a curve of bi-degree \((2,2)\) on each quadric, the residual intersection curve has bi-degree \((0,2)\) in each quadric. Since homogeneous polynomials in two variables over an algebraically closed field factor into linear terms, this residual curve is either two distinct lines, or a double line. In particular, it contains some line \( m \), which, by the argument above, must be a star chord. Now again by Corollary 5.2.5 the residual intersection must be two dual star chords \( m \) and \( m' \). This proves (ii).

To prove (i), we use Theorem 5.2.4 to chose coordinates so that \( X \) is the Fermat extremal surface, \( Q \) is the quadric defined by \( xw = yz \), and \( \ell \) and \( m \) are the lines \( V(x,z) \) and \( V(w,z) \), respectively. In this case, we have already computed (in the proof of Corollary 5.2.6) that the quadrics containing \( \{\ell, \ell', m, m'\} \) and defining quadric configurations are all of the form \( V(\mu xw - yz) \) where \( \mu^d = 1 \), and that any two such quadrics define a double \( 2d \) (in the proof of Theorem 6.0.2).

Remark 6.3.3. The bound in Corollary 6.3.2 is not valid when \( d \) is less than \( 5 \) because in this case, there can be multiple pairs of quadric configurations that determine the same double \( 2d \). For example, every double six on a cubic surface can be split into the union of two quadric configuration in ten different ways (Remark 6.1.1). Note that dividing the bound provided by


Corollary 6.3.2 by ten, we get a lower bound of 36 double sixes on a cubic surface, recovering the fact that all double sixes on an extremal cubic comes from quadrics sharing star chords (Remark 6.1.2).

Similarly, when \( d = 4 \), there are double eights that split into the union of two quadric configurations in multiple ways. For example, the double eight on the Fermat quartic defined by the two quadrics \( Q_1 = V(xw - yz) \) and \( Q_2 = V(xw + yz) \) can also be given by two different quadrics \( Q_3 \) and \( Q_4 \), as one can check by examining the intersection matrix for the sixteen lines of \((Q_1 \cap X) \cup (Q_2 \cap X)\) to find a different grouping into lines in two quadrics.

Example 6.3.4. We now construct an example of a double eight on a quartic extremal surface that can not be given two quadrics sharing a line. This shows that not every double 2d on an extremal surface is of the special type in Theorem 6.3.1.

We work on the Fermat quartic, \( X = V(x^4 + y^4 + z^4 + w^4) \) in characteristic three. The quadrics \( Q_1 = V(xw - yz) \) and \( Q_2 = V(x^2 + xy + xz - xw - y^2 + yz + yw + z^2 - zw - w^2) \) both give quadric configurations on \( X \). The quadric configuration \( X \cap Q_1 \) is the union \( L \cup M \) where

\[
L = \{ V(x - ay, z - aw) \mid \alpha^4 = -1 \} \quad \text{and} \quad M = \{ V(x - az, y - aw) \mid \alpha^4 = -1 \},
\]

as we computed in Example 5.0.3. The quadric configuration \( X \cap Q_2 \) is the union \( N \cup P \) where \( N = \{ V(x - aw, y - az), V(x - \alpha w, y - \alpha z), V(-x - y + w, x - y - z), V(-x - y - w, x - y + z) \} \) and \( P = \{ V(x + ay, z - \alpha w), V(x + \alpha y, z - aw), V(-x + y + w, x - y + z), V(-x + y - w, x - y - z) \} \),

where \( a \) and \( \alpha \) are the roots in \( k \) of the polynomial \( T^2 - T - 1 \) over \( \mathbb{F}_3 \). We leave it to the reader to directly verify these eight lines all lie on both \( Q_2 \) and \( X \).

The set \( A \cup B \) is a double eight, where \( A = L \cup N \) and \( B = M \cup P \). To check this, it suffices to check that the lines in \( L \) and skew to those in \( N \), and similarly that the lines in \( M \) are skew to those in \( P \) (Lemma 6.2.1), which can be directly verified.

It remains to check that \( Q_1 \) and \( Q_2 \) do not share any line. If they did, then there are two shared lines in each ruling (Theorem 6.3.1). So it suffices to show an arbitrary line \( \ell = \{ \lambda s : s : \ell t \mid [s : t] \in \mathbb{P}^1 \} \) in one of the rulings of \( Q_1 \) can not lie on \( Q_2 \). If \( \ell \subset Q_2 \), then the points \([0 : 0 : \lambda : 1], [\lambda : 1 : 0 : 0] \) and \([\lambda : 1 : -\lambda : -1] \) in \( \ell \) must all lie on \( Q_2 \). Plugging into the equation for \( Q_2 \) produces the constraints

\[
\lambda^2 - \lambda - 1 = 0, \quad \lambda^2 + \lambda - 1 = 0, \quad \text{and} \quad \lambda^2 = 0.
\]

Because these three equations are inconsistent, we conclude that \( \ell \) does not lie on \( Q_2 \).

Finally, we must show that the double eight \( A \cup B \) can not be given by any other pair of quadrics that do share a line. To this end, assume on the contrary that \( A \cup B \) is given by quadrics \( Q_3 \) and \( Q_4 \), and that \( \ell \subset Q_3 \cap Q_4 \) for some line \( \ell \). Furthermore, since \( Q_1 \) and \( Q_2 \) share no line, we may assume that \( \ell \not\subset Q_1 \); in particular, \( \ell \) intersects two lines in each ruling of \( Q_1 \). Because \( \ell \) lies in one ruling of each of \( Q_3 \) and of \( Q_4 \), \( \ell \) meets each in a set of eight skew lines in the double eight \((Q_3 \cap X) \cup (Q_4 \cap X) = A \cup B \). Since at most two of these eight intersection points are on \( Q_1 \), we know \( \ell \) intersects at least six of the lines in \( Q_2 \), and so \( \ell \subset Q_2 \). But this is impossible: \( \ell \) lies in one of the rulings of \( Q_2 \) (and is not on \( X \)), so it intersects exactly four of the lines on \( Q_2 \cap X \).

6.4. Progress towards Conjecture 6.0.3. We now prove Theorem 6.0.4. The proof will mainly use the combinatorics of the intersection matrix between the two skew sets of size \( 2d \) and the properties of quadrics.
Lemma 6.4.1. Let $\mathcal{A} \cup \mathcal{B}$ be a double 2$d$ on a smooth surface $X$ of degree $d \geq 5$. If $\mathcal{A}$ contains three lines $A_1, A_2, A_3$ and $\mathcal{B}$ contains five lines that all meet each $A_i$ for $i = 1, 2$ and 3, then the double 2$d$ is the union of two unique quadric configurations.

Proof. Let $Q$ be the unique smooth quadric containing the three skew lines $A_1, A_2, A_3$. Let $B_1, B_2, B_3, B_4,$ and $B_5 \in \mathcal{B}$ be the five lines meeting each of $A_1, A_2, A_3$. Since each $B_i$ meets $Q$ in three points—namely $B_i \cap A_1, B_i \cap A_2,$ and $B_i \cap A_3$—$B_i$ lies on $Q$ for $i = 1, 2, \ldots, 5$.

Label the lines in $\mathcal{A}$ so that $A_i$ lies on $Q$ if and only if $i \leq t$. We first show that $t \geq 5$. For any $A \in \mathcal{A}$, note that $A$ meets all five $\{B_1, \ldots, B_5\}$ if $A$ lies on $Q$ and at most two of $\{B_1, \ldots, B_5\}$ if $A$ is not on $Q$. So

$$
\sum_{i=1}^{2d} A_i \cdot \left( \sum_{j=1}^{5} B_j \right) \leq 5t + 2(2d - t) = 3t + 4d.
$$

On the other hand, each $B_j$ must intersect exactly $d + 2$ lines in $\mathcal{A}$, so

$$
\sum_{i=1}^{2d} A_i \cdot \left( \sum_{j=1}^{5} B_j \right) = 5(d + 2) = 5d + 10.
$$

Thus, $3t + 4d \geq 5d + 10$ so $t \geq 3 + \frac{d+1}{3}$. Since $d \geq 5$, we get $t \geq 5$. Thus, the double 2$d$ must contain at least five lines of each ruling of $Q$.

Next we show that in fact each set $\mathcal{A}$ and $\mathcal{B}$ contains $d$ lines on $Q$. If not, let $k$ be maximal such that $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ lie on $Q$, and assume without loss of generality that $A_{k+1} \not\in Q$. Since each $B_j$ intersects exactly $d + 2$ lines in $\mathcal{A}$, of which are $A_1, \ldots, A_k$, we have that

$$
(17) \quad \sum_{j=1}^{k} B_j \cdot \left( \sum_{i=k+1}^{2d} A_i \right) = k(d + 2) - k^2 = k(d + 2 - k).
$$

Since $A_i$ lies on $Q$ if and only if $i \leq k$, each line $A_j$ for $j > k$ can meet at most two of $\{B_1, \ldots, B_k\}$. So

$$
(18) \quad \sum_{j=1}^{k} B_j \cdot \left( \sum_{i=k+1}^{2d} A_i \right) \leq 2(2d - k) = 4d - 2k.
$$

Comparing (17) and (18), we have $(4d - 2k) - k(d + 2 - k) = (k - d)(k - 4) \geq 0$, which (since we’ve shown $k \geq 5$) is a contradiction unless $k \geq d$. We conclude that $\mathcal{L} = \{A_1, \ldots, A_d\}$ and $\mathcal{M} = \{B_1, \ldots, B_d\}$ lie on the same quadric $Q$. That is, half the lines of $\mathcal{A}$, together with half the lines of $\mathcal{B}$, form a quadric configuration $\mathcal{L} \cup \mathcal{M}$ on $X$.

It suffices to show that the remaining halves of $\mathcal{A}$ and $\mathcal{B}$ also form a quadric configuration. Now fix $B \in \mathcal{B} \setminus \mathcal{M}$. Because $B$ intersects at most two of the lines of $\mathcal{L}$, we know $B$ intersects every line in $\mathcal{A} \setminus \mathcal{L}$. So each line of the skew set $\mathcal{B} \setminus \mathcal{M}$ meets every line of the skew set $\mathcal{A} \setminus \mathcal{L}$—that is $\mathcal{A} \setminus \mathcal{L}$ and $\mathcal{B} \setminus \mathcal{M}$ form a quadric configuration, as desired.

□

Remark 6.4.2. Even if $d$ is three or four, the final paragraph of the proof of Lemma 6.4.1 shows that if half the lines of a double 2$d$ lie on some quadric $Q$, then the complementary half lies on some other quadric $Q'$. The quadrics $\{Q, Q'\}$ may not be the only quadrics determining the double 2$d$ in this case, however. See Remark 6.1.1 and Remark 6.3.4

We are now ready to prove Theorem 6.0.4.
**Proof of Theorem 6.0.4.** Let \( X \) be a smooth surface of degree \( d \). The case where \( d = 3 \) is dealt with in § 6.1. We next handle the case \( d \geq 11 \).

Let \( \mathcal{A} := \{A_1, \ldots, A_{2d}\} \) and \( \mathcal{B} := \{B_1, \ldots, B_{2d}\} \) denote the two skew sets of the double \( 2d \). By Lemma 6.4.1, it suffices to show that there are three skew lines in \( \mathcal{A} \) all intersecting each of five skew lines in \( \mathcal{B} \). Let \( M \) denote the intersection matrix \( M_{ij} = A_i \cdot B_j \). By definition of a double \( 2d \), \( M \) has exactly \( d + 2 \) ones and exactly \( d - 2 \) zeros in every row and column.

For any subset \( S \subseteq \mathcal{A} \), let

\[
\text{IntersectionSet}(S, \mathcal{B}) := \{ B_i \in \mathcal{B} \mid B_i \cdot A_j = 1 \text{ for all } A_j \in S \}.
\]

We want to show that there exists some \( A_i, A_j, A_k \) such that \( |\text{IntersectionSet}(\{A_i, A_j, A_k\}, \mathcal{B})| \geq 5 \). After a possible relabeling of the \( B_i \), we may assume

\[
\text{IntersectionSet}(A_1, \mathcal{B}) = \{B_1, \ldots, B_{d+2}\}.
\]

Let

\[
k := \max \{ |\text{IntersectionSet}(\{A_1, A_i\}, \mathcal{B})| \}.
\]

Then by assumption, the number of ones in rows \( 2, \ldots, 2d \) and columns \( 1, \ldots, d + 2 \) of \( M \) is at most \( k(2d - 1) \) since there are at most \( k \) ones in each of these rows. However, by looking at columns, we see that there are exactly \( (d + 1)(d + 2) \) ones in this submatrix of \( M \). Thus, we see

\[
\frac{(d + 1)(d + 2)}{2d - 1} \leq k
\]

and since we may assume \( k \) is an integer, we have

\[
\left\lfloor \frac{(d + 1)(d + 2)}{2d - 1} \right\rfloor \leq k.
\]

By relabelling \( A_2, \ldots, A_{2d} \), we may assume \( |\text{IntersectionSet}(\{A_1, A_2\}, \mathcal{B})| = k \). By relabelling \( B_1, \ldots, B_{d+2} \), we may assume \( \text{IntersectionSet}(\{A_1, A_2\}, \mathcal{B}) = \{B_1, \ldots, B_k\} \). Now note that \( M \) has exactly \( kd \) ones in columns \( 1, \ldots, k \) and rows \( 3, \ldots, 2d \).

Let

\[
\ell := \max \{ |\text{IntersectionSet}(\{A_1, A_2, A_i\}, \mathcal{B})| \}.
\]

Then the number of ones in columns \( 1, \ldots, k \) and rows \( 3, \ldots, 2d \) is at most \( \ell(2d - 2) \), so we have

\[
\ell \geq \frac{kd}{2d - 2} \geq \left\lfloor \frac{(d + 1)(d + 2)}{2d - 1} \right\rfloor \frac{d}{2d - 2}
\]

and since \( \ell \) is an integer, we have

\[
(19) \quad \ell \geq \left\lfloor \frac{(d + 1)(d + 2)}{2d - 1} \right\rfloor \frac{d}{2d - 2}
\]

From (19), it follows that when \( d \geq 11 \), we have \( \ell \geq 5 \), as desired.

Finally, when \( d = 4 \), formula (19) implies that \( \ell \geq 4 \), so that there exists a set of three skew lines \( A_1, A_2, \) and \( A_3 \in \mathcal{A} \) that all intersect four skew lines \( B_1, B_2, B_3, \) and \( B_4 \in \mathcal{B} \). In particular, each of the four \( B_i \) must lie on the unique quadric \( Q \) determined by \( A_1, A_2, \) and \( A_3 \), since they each intersect this quadric in 3 points. We claim one more line in \( \mathcal{A} \) lies on \( Q \), in which case it follows that every double eight is the union of two quadric configurations (Remark 6.4.2). To
verify the claim, observe that if no $A_i$ lies on $Q$ for $i > 3$, then these $A_i$ intersect at most two of $B_1, B_2, B_3$, and $B_4 \in B$. Thus
\[
\sum_{i=1}^{4} B_i \cdot \sum_{A_i \in A} A_i = \sum_{i=1}^{4} B_i \cdot \sum_{i=1}^{3} A_i + \sum_{i=1}^{4} B_i \cdot \sum_{i=4}^{8} A_i \leq 12 + 10 = 22,
\]
contrary to the fact that $\sum_{i=1}^{4} B_i \cdot \sum_{A_i \in A} A_i = 24$, since each line in $B$ intersects exactly six lines in $A$.

\[\square\]

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