NORMS OF SCHUR MULTIPLIERS

KENNETH R. DAVIDSON AND ALLAN P. DONSIG

Abstract. A subset $\mathcal{P}$ of $\mathbb{N}^2$ is called Schur bounded if every infinite matrix with bounded entries which is zero off of $\mathcal{P}$ yields a bounded Schur multiplier on $B(\mathcal{H})$. Such sets are characterized as being the union of a subset with at most $k$ entries in each row with another that has at most $k$ entries in each column, for some finite $k$. If $k$ is optimal, there is a Schur multiplier supported on the pattern with norm $O(\sqrt{k})$, which is sharp up to a constant.

The same techniques give a new, more elementary proof of results of Varopoulos and Pisier on Schur multipliers with given matrix entries of random sign.

We consider the Schur multipliers for certain matrices which have a large symmetry group. In these examples, we are able to compute the Schur multiplier norm exactly. This is carried out in detail for a few examples including the Kneser graphs.

Schur multiplication is just the entrywise multiplication of matrices or operators in a fixed basis. These maps arise naturally as the (weak-∗ continuous) bimodule maps for the algebra of diagonal matrices (operators). They are well-behaved completely bounded maps that play a useful role in the theory of operator algebras.

As in the case of operators themselves, the actual calculation of the norm of any specific Schur multiplier is a delicate task; and is often impossible. This has made it difficult to attack certain natural, even seemingly elementary, questions.

This study arose out of an effort to understand norms of Schur multipliers supported on certain patterns of matrix entries. The question of which patterns have the property that every possible choice of bounded entries supported on the pattern yield bounded Schur multipliers was raised by Nikolskaya and Farforovskaya in [13]. We solve this problem completely. The answer is surprisingly elegant. The pattern must decompose into two sets, one with a bound on the number of entries in each row, and the other with a bound on the number of entries in each column.

There is a close relationship of these results with work of Varopoulos [20] and Pisier [16]. We had overlooked this work and only discovered

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it late in our study. Perhaps this is just as well, as we may well have stopped had we realized how close their results were to the ones we were seeking. The upshot is that we also obtain a much more elementary proof of the bulk of their results, though without the probabilistic component. Indeed our main tool in the decomposition is an elementary, albeit powerful, combinatorial result known as the min-cut-max-flow theorem.

In Section 3 we recover results of [13] on patterns of Hankel and Toeplitz forms. Actually the Toeplitz case is classical, and we compare the bounds from our theorem with the tighter bounds available from a deeper use of function theory.

Sections 4 and 5 deal with exact computation of the Schur norm of certain matrices that have lots of symmetry. More precisely, let $G$ be a finite group acting transitively on a set $X$. We obtain an explicit formula for the Schur multiplier norm of matrices in the commutant of the action, i.e., matrices constant on each orbit of $G$. This uses a result of Mathias [12]. We carry this out for one nontrivial case—the adjacency matrix of the Kneser graph $K(2n+1, n)$, which has $\binom{2n+1}{n}$ vertices indexed by $n$-element subsets of $2n+1$, with edges between disjoint sets.

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1. Background

If $A = [a_{ij}]_{i,j \in S}$ is a finite or infinite matrix, the Schur (a.k.a. Hadamard) multiplier is the operator $S_A$ on $B(l^2(S))$ that acts on an operator $T = [t_{ij}]$ by pointwise multiplication: $S_A(T) = [a_{ij}t_{ij}]$. To distinguish from the norm on bounded operators, we will write $\|A\|_m$ for the norm of a Schur multiplier. In general it is very difficult to compute the norm of a Schur multiplier. Nevertheless, much is known in a theoretical sense about the norm. In this section, we will quickly review some of the most important results.
The following classical result owes most credit to Grothendieck. For a proof, see the books by Pisier [17, Theorem 5.1] and Paulsen [14, Theorem 8.7].

**Theorem 1.1.** For $X$ an arbitrary set, let $S = [s_{ij}]$ be an $|X| \times |X|$ matrix with bounded entries considered as a Schur multiplier on $B(l^2(X))$. Then the following are equivalent:

1. $\|S\|_m \leq 1$.
2. $\|S\|_{cb} \leq 1$.
3. There are contractions $V$ and $W$ from $l^2(X)$ to $l^2(X) \otimes l^2(Y)$ such that $S(A) = W^*(A \otimes I)V$.
4. There are unit vectors $x_i$ and $y_j$ in $l^2(Y)$ so that $s_{ij} = x_i^*y_j$.
5. $\gamma_2(S) \leq 1$ where $\gamma_2(S) = \inf_{S=AB} \|A\|_{2,\infty}\|B\|_{1,2}$.

Recall that the complete bound norm of $S$ is the norm of the inflation of $S$ acting on operators with operator entries. The most elegant proof of (1) implies (2) is due to Smith [13]. The converse is trivial. The equivalence of (2) and (2') is Wittstock's Theorem for representing completely bounded maps.

The equivalence of (1), (3) and (4) is due to Grothendieck. (3) follows from (2') by taking $y_j = (E_{ij} \otimes I)V e_j$ and $x_i = (E_{i1} \otimes I)W e_i$. Conversely, (3) implies (2') by taking $V e_j = e_j \otimes y_j$ and $W e_i = e_i \otimes x_i$. This condition was rediscovered by Haagerup, and became well-known as his observation. So we shall refer to these as the Grothendieck–Haagerup vectors for the Schur multiplier.

The $\gamma_2$ norm is the optimal factorization through Hilbert space of $S$ considered as a map from $l^1$ to $l^\infty$. The norm $\|A\|_{2,\infty}$ is the maximum of the 2-norm of the rows; while $\|B\|_{1,2}$ is the maximum of the 2-norm of the columns. Thus (3) implies (4) follows from $A = \sum_i e_i x_i^*$ and $B = \sum_j y_j e_j^*$. And this implication is reversible.

The equivalence of (5) is due to Paulsen, Power and Smith [15]. This follows from (3) by taking $a_{ij} = x_i^* x_j$ and $b_{ij} = y_i^* y_j$. Conversely, assume first that $X$ is finite. Then the positive matrix $P$ decomposes as a sum of positive rank one matrices, and thus have the form $[\overline{z}_i z_j]$ which can be seen to be a scalar version of (3). Indeed it is completely positive. Hence the sum $S_P$ is also a completely positive Schur multiplier. Consequently $\|S_P\|_{cb} = \|S_P(I)\| = \max\{a_{ii}, b_{ii}\} = 1$. So (2) holds. The case of general $X$ is a routine limit argument.

The $\gamma_2$ norm is equivalent to the norm in the Haagerup tensor product $\ell^\infty(X) \otimes_h \ell^\infty(X)$, where we identify an elementary tensor $a \otimes b$ with
the matrix \([a_i b_j]\). The Haagerup norm of a tensor \(\tau\) is given by taking the infimum over all representations \(\tau = \sum_k a_k \otimes b_k\) of

\[
\left\| \sum_k a_k a_k^* \right\|^{1/2} \left\| \sum_k b_k^* b_k \right\|^{1/2}.
\]

See [14, Chapter 17]. Of course, since \(\ell^\infty\) is abelian, the order of the adjoints is irrelevant. One can see the equivalence by taking a factorization \(S = AB\) from (4). Consider \(A\) as a matrix with columns \(a_k \in \ell^\infty(X)\) and \(B\) as a matrix with rows \(b_k \in \ell^\infty(X)\). Identify the product with the tensor \(\sum_k a_k \otimes b_k\). The norm \(\left\| \sum_k a_k a_k^* \right\|^{1/2}\) can be seen to be \(\|A\|_{2,\infty}\) and the norm of \(\left\| \sum_k b_k^* b_k \right\|^{1/2}\) to be \(\|B\|_{1,2}\).

Generally, it is difficult to compute the norm of a Schur multiplier. The exception occurs when the matrix \(S\) is positive definite. Then it is a classical fact that \(S\) is a completely positive map. Consequently, \(\|S\|_{cb} = \|S(I)\| = \sup_{|s_{ii}| < \infty} s_{ii}\).

Grothendieck proved another remarkable result about Schur multipliers. Recall that the projective tensor product \(\ell^\infty(X) \hat{\otimes} \ell^\infty(X)\) norms a tensor \(\tau\) as the infimum over representations \(\tau = \sum_k a_k \otimes b_k\) of the quantity \(\sum_k \|a_k\| \|b_k\|\). It is a surprising fact that this norm and the \(\gamma_2\) or Haagerup norm are equivalent. We will need this connection to understand the relevance of work of Varopoulos. For the moment, we state this result in a way that makes a stronger connection to Schur multipliers. An elementary tensor \(a \otimes b\) yields a rank one matrix \([a_i b_j]\).

Thus Grothendieck’s result is equivalent to:

**Theorem 1.2** (Grothendieck). The convex hull of the rank one Schur multipliers of norm one contains the ball of all Schur multipliers of norm at most \(K_G^{-1}\), where \(K_G\) is a universal constant.

In terms of the projective tensor product norm for a tensor \(\tau\) and the corresponding Schur multiplier \(S_\tau\), this result says that

\[
K_G^{-1} \|\tau\|_{\ell^\infty(S) \hat{\otimes} \ell^\infty(S)} \leq \|S_\tau\|_m \leq \|\tau\|_{\ell^\infty(S) \hat{\otimes} \ell^\infty(S)}
\]

The constant \(K_G\) is not known exactly. In the complex case Haagerup [9] showed that 1.338 < \(K_G\) < 1.405; and in the real case Krivine [10] obtained the range [1.676, 1.783] and conjectured the correct answer to be \(\frac{\pi}{2 \log(1+\sqrt{2})}\).

We turn to the results of Varopoulos [20] and Pisier [16] which relate to our work. The paper of Varopoulos is famous for showing that three commuting contractions need not satisfy the von Neumann inequality. Proofs of this, including the one in the appendix of Varopoulos’s paper, are generally constructive. But the argument in the main part of
his paper instead establishes a result about $\ell^\infty(X) \hat{\otimes} \ell^\infty(X)$. He does not establish precise information about constants. This result was extended and sharpened by Pisier, who casts it in the language of Schur multipliers, to deal with multipliers and lacunary sets on nonamenable groups.

Consider $\{\pm 1\}^{X \times X}$ to be the space of functions from $X \times X$ to $\{1, -1\}$ with the product measure $\mu$ obtained from $p(1) = p(-1) = .5$.

**Theorem 1.3** (Varopoulos–Pisier). Let $S = [s_{ij}]$. The following are equivalent.

1. For all $\varepsilon \in \{\pm 1\}^{X \times X}$, $\|\varepsilon_{ij}s_{ij}\|_m < \infty$.
2. For almost all $\varepsilon \in \{\pm 1\}^{X \times X}$, $\|\varepsilon_{ij}s_{ij}\|_m < \infty$.
3. $S = A + B$ and there is a constant $M$ so that
   \[ \sup_i \sum_j |a_{ij}|^2 \leq M^2 \quad \text{and} \quad \sup_j \sum_i |b_{ij}|^2 \leq M^2. \]
4. There is a constant $M$ so that for every pair of finite subsets $R, C \subset X$, $\sum_{i \in R, j \in C} |s_{ij}|^2 \leq M^2 \max\{|R|, |C|\}$.

Pisier shows that if the average Schur multiplier norm
\[ \int \|\varepsilon_{ij}s_{ij}\|_m \, d\mu(\varepsilon) \leq 1, \]
then one can take $M = 1$ in (3). Our results are not quite so sharp, as we require a constant (Lemma 2.9) of approximately $1/4$. The constant $M$ in the two conditions (3) and (4) are not the same. The correct relationship replaces $\max\{|R|, |C|\}$ by $|R| + |C|$ (see Lemma 2.7); but they are related within a constant. If $M$ is the bound in (3), it is not difficult to obtain a bound of $2M$ for (1) (see Corollary 2.6). Thus one obtains that the average Schur norm is within a factor of 2 of the maximum.

## 2. Schur Bounded Patterns

A pattern $P$ is a subset of $\mathbb{N} \times \mathbb{N}$. An infinite matrix $A = [a_{ij}]$ is supported on $P$ if $\{(i, j) : a_{ij} \neq 0\}$ is contained in $P$. We let $S(P)$ denote the set of Schur multipliers supported on $P$ with matrix entries $|s_{ij}| \leq 1$.

More generally, we will also consider Schur multipliers dominated by a given infinite matrix $A = [a_{ij}]$ with nonnegative entries. Let $S(A)$ denote the set of all Schur multipliers with matrix entries $|s_{ij}| \leq a_{ij}$.

**Definition 2.1.** Say that a pattern $P \subset \mathbb{N} \times \mathbb{N}$ is Schur bounded if every $X \in S(P)$ yields bounded Schur multiplier $S_X$. The Schur bound
of $\mathcal{P}$ is defined as $s(\mathcal{P}) := \sup_{X \in \mathcal{S}(\mathcal{P})} \|X\|_m$. Similarly, for a matrix $A$ with nonnegative entries, define $s(\mathcal{S}(A)) = \sup_{X \in \mathcal{S}(A)} \|X\|_m$; and say that $\mathcal{S}(A)$ is Schur bounded if this value is finite.

It is easy to see that if $\mathcal{S}(\mathcal{P})$ is Schur bounded, then $s(\mathcal{P})$ is finite. Note that if $A_P$ is the matrix with 1s on the entries of $\mathcal{P}$ and 0s elsewhere, then $\mathcal{S}(A_P) = \mathcal{S}(\mathcal{P})$. We will maintain a distinction because we will require integral decompositions when working with a pattern $\mathcal{P}$.

Certain patterns are easily seen to be Schur bounded and this is the key to our result. The following two definitions of row bounded for patterns and matrices are not parallel, as the row bound of $A_P$ is actually the square root of the row bound of $\mathcal{P}$. Each definition seems natural for its context, so we content ourselves with this warning.

**Definition 2.2.** A pattern is row bounded by $k$ if there are at most $k$ entries in each row; and row finite if it is row bounded by $k$ for some $k \in \mathbb{N}$. Similarly we define column bounded by $k$ and column finite.

A nonnegative matrix $A = [a_{ij}]$ is row bounded by $L$ if the rows of $A$ are bounded by $L$ in the $l^2$-norm: $\sup_{i \geq 1} \sum_{j \geq 1} |a_{ij}|^2 \leq L^2 < \infty$. Similarly we define column bounded by $L$.

The main result of this section is:

**Theorem 2.3.** For a pattern $\mathcal{P}$, the following are equivalent:

1. $\mathcal{P}$ is Schur bounded.
2. $\mathcal{P}$ is the union of a row finite set and a column finite set.
3. $\sup_{R,C \text{ finite}} \frac{|\mathcal{P} \cap (R \times C)|}{|R| + |C|} < \infty$.

Moreover, the optimal bound $m$ on the size of the row and column finite sets in (2) coincides with the least integer dominating the supremum in (3); and the Schur bound satisfies

$$\sqrt{m}/4 \leq s(\mathcal{P}) \leq 2\sqrt{m}.$$

This theorem has a direct parallel for nonnegative matrices.

**Theorem 2.4.** For a nonnegative infinite matrix $A = [a_{ij}]$, the following are equivalent:

1. $\mathcal{S}(A)$ is Schur bounded.
2. $A = B + C$ where $B$ is row bounded and $C$ is column bounded.
3. $\sup_{R,C \text{ finite}} \frac{\sum_{i \in R, j \in C} a_{ij}^2}{|R| + |C|} < \infty$.

Moreover, the optimal bound $M$ on the row and column bounds in (2) coincides with the square root of the supremum $M^2$ in (3); and the
Schur bound satisfies
\[ M/4 \leq s(P) \leq 2M. \]

**Lemma 2.5.** If \( P \) is row (or column) bounded by \( n \), then \( s(P) \leq \sqrt{n} \).
Likewise if \( A \) is row (or column) bounded by \( L \), then \( s(S(A)) \leq L \).

**Proof.** The pattern case follows from the row bounded case for the nonnegative matrix \( A = AP \) with \( L = \sqrt{n} \). Suppose that \( S(A) \) is row bounded by \( L \). Consider any \( S \in S(A) \). Then \( \sup_{i\geq1} \sum_{j\geq1} |s_{ij}|^2 \leq L^2 \).
Define vectors \( x_i = \sum_{j\geq1} s_{ij} e_j \) for \( i \geq 1 \). Then \( \sup_{i\geq1} \|x_i\| \leq L \); and \( \langle x_i, e_j \rangle = s_{ij} \). So by the Grothendieck–Haagerup condition,
\[ \|S\|_m \leq \sup_{i,j} \|x_i\| \|e_j\| \leq L. \]
Thus \( s(S(A)) \leq L \).

**Corollary 2.6.** If \( P \) is the union of a set row bounded by \( n \) and a set column bounded by \( m \), then \( P \) is Schur bounded with bound \( \sqrt{n} + \sqrt{m} \).
Likewise, if \( A = B + C \) such that \( B \) is row bounded by \( L \) and \( C \) is column bounded by \( M \), then \( s(S(A)) \leq L + M \).

We require a combinatorial characterization of sets which are the union of an \( n \)-row bounded set and an \( m \)-column bounded set. This will be a consequence of the min-cut-max-flow theorem (see [5], for example). This is an elementary result in combinatorial optimization that has many surprising consequences. For example, it has been used by Richard Haydon to give a short proof of the reflexivity of commutative subspace lattices [7]. It should be more widely known.

**Lemma 2.7.** A pattern \( P \) is the union of a set \( P_r \) row bounded by \( m \) and a set \( P_c \) column bounded by \( n \) if and only if for every pair of finite subsets \( R, C \subset \mathbb{N} \),
\[ |P \cap R \times C| \leq m|R| + n|C|. \]
Similarly, a matrix \( A = [a_{ij}] \) with nonnegative entries decomposes as a sum \( A = A_r + A_c \) where \( A_r \) is row bounded by \( M^{1/2} \) and \( A_c \) is column bounded by \( N^{1/2} \) if and only if for every pair of finite subsets \( R, C \subset \mathbb{N} \),
\[ \sum_{i\in R} \sum_{j\in C} a_{ij}^2 \leq M|R| + N|C|. \]

**Proof.** The two proofs are essentially identical. However the decomposition of \( P \) must be into two disjoint subsets. This means that the decomposition \( A_P = A_{P_1} + A_{P_2} \) is a split into 0,1 matrices. We will
work with $A$, but will explain the differences in the pattern version when it arises.

The condition is clearly necessary.

For the converse, we first show that it suffices to solve the finite version of the problem. For $p \in \mathbb{N}$, let $A_p$ be the restriction of $A$ to the first $p$ rows and columns. Suppose that we can decompose $A_p = A_{r,p} + A_{c,p}$ where $A_{r,p}$ is row bounded by $M^{1/2}$ and $A_{c,p}$ column bounded by $N^{1/2}$ for each $p \in \mathbb{N}$. Fix $k$ so that $A_k \neq 0$. For each $p \geq k$, the set of such decompositions for $A_p$ is a compact subset of $\mathbb{M}_p \times \mathbb{M}_p$. In the pattern case, we consider only 0, 1 decompositions.

The restriction to the $k \times k$ corner is also a compact set, say $X_{k,p}$. Observe that this is a decreasing sequence of nonempty compact sets. Thus $\bigcap_{p \geq k} X_{k,p} = X_k$ is nonempty. Therefore there is a consistent choice of a decomposition $A = A_r + A_c$ so that the restriction to each $k \times k$ corner lies in $X_k$ for each $k \geq 1$. In the pattern case, the entries are all zeros and ones.

So now we may assume that $A = [a_{ij}]$ is a matrix supported on $R_0 \times C_0$, where $R_0$ and $C_0$ are finite. We may also suppose that the $l^2$-norm of each row is greater than $M^{1/2}$ and the $l^2$-norm of each column is greater than $N^{1/2}$. For otherwise, we assign all of those entries in the row to $A_r$ (or all entries in the column to $A_c$) and delete the row (column). Solving the reduced problem will suffice. If after repeated use of this procedure, the matrix is empty, we are done. Otherwise, we reach a reduced situation in which the $l^2$-norm of each row is greater than $M^{1/2}$ and the $l^2$-norm of each column is greater than $N^{1/2}$.

Define a graph $G$ with vertices $\alpha$, $r_i$ for $i \in R_0$, $c_j$ for $j \in C_0$, and $\omega$. Put edges from each $r_i \in R_0$ to each $c_j \in C_0$, from $\alpha$ to $r_i$, $i \in R_0$, and from $c_j$ to $\omega$, $j \in C_0$. Consider a network flow on the graph in which the edge from $r_i$ to $c_j$ may carry $a_{ij}$ units; edges leading out of $\alpha$ can carry up to $M$ units; and the edge from $c_j$ to $\omega$ can carry $v_j - N$ units, where $v_j = \sum_{i \in R_0} a^2_{ij}$. In the pattern case, these constraints are integers.

The min-cut-max-flow theorem states that the maximal possible flow from $\alpha$ to $\omega$ across this network equals the minimum flow across any cut that separates $\alpha$ from $\omega$. Moreover, when the data is integral, the maximal flow comes from an integral solution. A cut $\mathcal{X}$ is just a partition of the vertices into two disjoint sets $\{\alpha\} \cup R_1 \cup C_1$ and $\{\omega\} \cup R_2 \cup C_2$. The flow across the cut is the total of allowable flows on each edge between the two sets.
The flow across the cut $\mathcal{X}$ is
\[
f(\mathcal{X}) = \sum_{i \in R_1} \sum_{j \in C_2} a_{ij}^2 + M|R_2| + \sum_{j \in C_1} (v_j - N)
\]
\[
= \sum_{i \in R_1} \sum_{j \in C_2} a_{ij}^2 + M|R_2| - N|C_1| + \sum_{i \in R_0} \sum_{j \in C_1} a_{ij}^2
\]
\[
= \sum_{i \in R_1} \sum_{j \in C_2} a_{ij}^2 - \sum_{i \in R_2} \sum_{j \in C_1} a_{ij}^2 + M|R_2| + N|C_2| - N|C_0|
\]
\[
\geq \sum_{i \in R_0} \sum_{j \in C_0} a_{ij}^2 - N|C_0|
\]
The last inequality uses the hypothesis on $A$ with $R = R_2$ and $C = C_2$. On the other hand, the cut separating $\omega$ from the rest has flow exactly
\[
\sum_{j \in C_0} (v_j - N) = \sum_{i \in R_0} \sum_{j \in C_0} a_{ij}^2 - N|C_0|.
\]
Therefore there is a network flow that achieves this maximum. In the pattern case, the solution is integral. Necessarily this will involve a flow of exactly $v_j - N$ from each $j \in C_0$ to $\omega$. Let $b_{ij}$ be the optimal flow from $r_i$ to $c_j$. So $0 \leq b_{ij} \leq a_{ij}$. The flow out of each $r_i$ equals the flow into $r_i$ from $\alpha$, whence \[ \sum_{j \in C_0} b_{ij} \leq M. \]
Define the matrix $A_r = [\begin{array}{c} b_{ij} \end{array}]$ and $A_c = [\begin{array}{c} a_{ij} - b_{ij} \end{array}]$. In the pattern case, these entries are 0 or 1. Then the rows of $A_r$ are bounded by $M^{1/2}$. The $j$th column of $A_c$ has norm squared equal to
\[
\sum_{i \in R_0} a_{ij} - b_{ij} = v_j - (v_j - N) = N.
\]
This is the desired decomposition and it is integral for patterns.

To construct large norm Schur multipliers on certain patterns, we will make use of the following remarkable result by Françoise Lust-Piquard [11, Theorem 2]. While the method of proof is unexpected, it is both short and elementary.

**Theorem 2.8 (Lust-Piquard).** Given any (finite or infinite) nonnegative matrix $X = [x_{ij}]$ satisfying
\[
\max_i \sum_j x_{ij}^2 \leq 1 \quad \text{and} \quad \max_j \sum_i x_{ij}^2 \leq 1 \quad \text{for all} \quad i, j,
\]
there is an operator $Y = [y_{ij}]$ so that
\[
\|Y\| \leq \sqrt{6} \quad \text{and} \quad |y_{ij}| \geq x_{ij} \quad \text{for all} \quad i, j.
\]
The constant of $\sqrt{6}$ is optimal, as shown in an addendum to [11].
Lemma 2.9. Let $A = [a_{ij}]$ be a nonnegative $m \times m$ matrix such that 
$\sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij}^2 = m\alpha$. Then there is a Schur multiplier $S \in S(A)$ such that $\|S\|_m \geq \frac{1}{2} \sqrt{\frac{2}{3}}$.

Proof. We may assume that there are no nonzero rows or columns. Let

$$r_i = \sum_{j=1}^{m} a_{ij}^2 \quad \text{and} \quad c_j = \sum_{i=1}^{m} a_{ij}^2.$$ 

Define

$$x_{ij} = \frac{a_{ij}}{\sqrt{r_i + c_j}}.$$ 

Let $X = [x_{ij}]$. The row norms of $X$ satisfy

$$\sum_{j=1}^{m} x_{ij}^2 \leq \sum_{j=1}^{m} a_{ij}^2 r_i = 1;$$

and similarly the column norms are bounded by 1.

By Theorem 2.8, there is a matrix $Y$ such that

$$\|Y\| \leq \sqrt{6} \quad \text{and} \quad |y_{ij}| \geq x_{ij} \quad \text{for all} \quad i, j.$$ 

Define $s_{ij} = a_{ij}x_{ij}/y_{ij}$ (where $0/0 := 0$). Then $S = [s_{ij}]$ belongs to $S(A)$. Observe that

$$S(Y) = Z := [a_{ij}x_{ij}] = \left[ \frac{a_{ij}^2}{\sqrt{r_i + c_j}} \right].$$

Hence $\|S\|_m \geq \|Z\|/K$.

Define vectors $u = (u_i)$ and $v = (v_j)$ by

$$u_i = \left( \frac{r_i}{m\alpha} \right)^{1/2} \quad \text{and} \quad v_j = \left( \frac{c_j}{m\alpha} \right)^{1/2}.$$ 

Then $\|u\|_2^2 = \frac{1}{m\alpha} \sum_{i=1}^{m} r_i = 1$ and similarly $\|v\|_2 = 1$. Compute

$$\|Z\| \geq u^*Zv = \frac{1}{m\alpha} \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij}^2 \sqrt{\frac{r_i c_j}{r_i + c_j}}.$$
Observe that $\sqrt{\frac{r_i c_j}{r_i + c_j}} = \left(\frac{1}{r_i} + \frac{1}{c_j}\right)^{-1/2}$. Also

$$
\sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij}^2 \left(\frac{1}{r_i} + \frac{1}{c_j}\right) = \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{a_{ij}^2}{r_i} \sum_{j=1}^{m} \frac{a_{ij}^2}{c_j} = \sum_{i=1}^{m} 1 + \sum_{j=1}^{m} 1 = 2m.
$$

A routine Lagrange multiplier argument shows that if $\alpha_k \geq 0$ are constants, $t_k > 0$ are variables, and $\sum_{k=1}^{m^2} \alpha_k t_k = 2m$, then $\sum_{k=1}^{m^2} \alpha_k t_k^{-1/2}$ is minimized when all $t_k$ are equal. Hence if $\sum_{k=1}^{m^2} \alpha_k = m\alpha$, then

$$
\sum_{k=1}^{m^2} \alpha_k t_k^{-1/2} \geq m\alpha \left(\frac{2m}{m\alpha}\right)^{-1/2} = m\alpha \sqrt{\frac{\alpha}{2}}.
$$

Applying this to the numbers $\frac{1}{r_i} + \frac{1}{c_j}$ yields

$$
\|Z\| \geq \frac{1}{m\alpha} \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij}^2 \left(\frac{1}{r_i} + \frac{1}{c_j}\right)^{-1/2} \geq \sqrt{\frac{\alpha}{2}}.
$$

Thus $\|S\|_m \geq \sqrt{\frac{\alpha}{\sqrt{6\sqrt{2}}}} = \frac{1}{2} \sqrt{\frac{m}{3}}$.

**Proof of Theorem 2.3 and Theorem 2.4.**

Statements (2) and (3) are equivalent by Lemma 2.7, taking $m = n$ and $M = N$.

Assuming (2), Corollary 2.6 shows that $P$ or $S(A)$ is Schur bounded by $2\sqrt{m}$ or $2M$. Assuming (3) in the pattern case, the supremum exceeds $m - 1$; so Lemma 2.9 shows that

$$
\mathfrak{s}(P) \geq \frac{\sqrt{m-1}}{2\sqrt{3}} \geq \frac{\sqrt{m}}{4}
$$

for $m \geq 4$. For $m \leq 16$, $\sqrt{m}/4 \leq 1$; and 1 is also a lower bound for any pattern. For the matrix case, we use the exact supremum in Lemma 2.9, so we obtain a lower bound of $M/4$.

Conversely, if the supremum in (3) is infinite, the same argument shows that the Schur bound is infinite. In fact it is easy to see that this implies that $S(P)$ or $S(A)$ contains unbounded Schur multipliers. It is not difficult to produce disjoint finite rectangles $R_n \times C_n$ on which the ratio in (3) exceeds $n^2$. So by Lemma 2.9, we construct a Schur multiplier $S_n$ in $S(P)$ or $S(A)$ supported on $R_n \times C_n$ with Schur norm...
at least \( n/4 \). Take \( S \) to be defined on each rectangle as \( S_n \) and zero elsewhere. Then \( S \) is an unbounded Schur multiplier in this class.

Remark 2.10. One might suspect, from the \( \sqrt{n} \) arising in Lemma 2.5, that if two matrices are supported on pairwise disjoint patterns, there might be an \( L^2 \) estimate on the Schur norm of the sum. This is not the case, as the following example shows.

Let \( 1 = (1, 1, 1, 1)^t \in \mathbb{C}^4 \) and \( A = 11^* - I \). If \( U = \text{diag}(1, i, -1, -i) \), then the diagonal expectation is

\[
\Delta(X) = S_I(X) = \frac{1}{4} \sum_{k=0}^{3} U^k X U^{*k}.
\]

We use a device due to Bhatia–Choi–Davis [4]. Observe that

\[
S_{A+tI}(X) = X + (t - 1)\Delta(X)
\]

\[
= (1 + \frac{t-1}{4})X + \frac{t-1}{4} \sum_{k=1}^{3} U^k X U^{*k}.
\]

Therefore

\[
\|S_{A+tI}\|_m \leq \left| 1 + \frac{t-1}{4} \right| + \frac{3|t-1|}{4}
\]

\[
= \begin{cases} 
|t| & \text{if } t \geq 1 \text{ or } t \leq -3 \\
\frac{3}{2}|3-t| & \text{if } -3 \leq t \leq 1
\end{cases}
\]

On the other hand, \( S_{A+tI}(I) = tI \); so \( \|S_{A+tI}\|_m \geq |t| \). Observe that \( \frac{1}{2}11^* \) is a projection. Hence \( A + tI = 11^* + (t-1)I \) has spectrum \( \{t-1, t+3\} \); and thus

\[
\|A + tI\| = \max\{|t-1|, |t+3|\}.
\]

So \( \|A - I\| = 2 \). If \( -3 \leq t \leq 1 \), then \( S_{A+tI}(A - I) = A - tI \) has norm \( |3-t| \) and so \( \|S_{A+tI}\|_m \geq |3-t|/2 \).

In particular, \( \|S_A\|_m = \frac{3}{2} \) and \( \|S_I\|_m = 1 \), but

\[
\|S_{A-I}\|_m = 2 > (\|S_A\|_m^2 + \|S_I\|_m^2)^{1/2}.
\]

Remark 2.11. In [3], Bennett, Goodman and Newman show that if \( A \) is an \( n \times n \) matrix with entries taking the values \( \pm 1 \) with probability \(.5\), then on average the norm of \( A \) is bounded by \( K\sqrt{n} \), where \( K \) is a universal constant. This is best possible as each row and column has norm \( \sqrt{n} \). This minimum can be achieved in certain cases, for example by tensoring copies of \( \begin{bmatrix} 0 & 1 \end{bmatrix} \) together. The maximum norm occurs for
the matrix $11^*$ for which all entries are 1, in which case the norm is $n$. So we see that the average norm is within a constant of the minimum.

This can be used to show that, on average, the Schur norm $\|A\|_m$ is near the maximum $\sqrt{n}$. Indeed, $S_A(A) = 11^*$. So

$$\|A\|_m \geq \frac{n}{\|A\|} \geq K^{-1} \sqrt{n}$$
on average.

3. Hankel and Toeplitz Patterns

A Hankel pattern is a set of the form

$$\mathcal{H}(S) = \{(i, j) : i, j \in \mathbb{N}, \ i + j \in S\} \quad \text{for} \quad S \subset \mathbb{N}.$$ 

A Toeplitz pattern is a set of the form

$$\mathcal{T}(S) = \{(i, j) : i, j \in \mathbb{N}_0, \ i - j \in S\} \quad \text{for} \quad S \subset \mathbb{Z}.$$ 

Recall that a set $S = \{s_1 < s_2 < \ldots\}$ is lacunary if there is a constant $q > 1$ so that $s_{i+1}/s_i > q$ for all $i \geq 1$.

Nikolskaya and Farforovskaya show that a Hankel pattern is Schur bounded if and only if it is a finite union of lacunary sets [13, Theorem 3.8], by considering Fejér kernels and Toeplitz extensions. We give an elementary proof based on Theorem 2.3.

**Proposition 3.1.** Consider a Hankel pattern $\mathcal{H}(S)$ of a set $S \subset \mathbb{N}$. Then the following are equivalent:

1. $\mathcal{H}(S)$ is Schur bounded.
2. $\mathcal{H}(S)$ is the union of a row finite and a column finite set.
3. $\sup_{k \geq 0} |S \cap (2^{k-1}, 2^k]| < \infty$.
4. $S$ is the union of finitely many lacunary sets.

**Proof.** By Theorem 2.3, (1) and (2) are equivalent.

Let $a_k = |S \cap (2^{k-1}, 2^k]|$ for $k \geq 0$. If (3) holds, $\max_{k \geq 0} a_k = L < \infty$. So $S$ splits into $2L$ subsets with at most one element in every second interval $(2^{k-1}, 2^k)$; which are therefore lacunary with ratio at least 2. Conversely, suppose that $S$ is the union of finitely many lacunary sets. A lacunary set with ratio $q$ may be split into $d$ lacunary sets of ratio 2 provided that $q^d \geq 2$. So suppose that there are $L$ lacunary sets of ratio 2. Then each of these sets intersects $(2^{k-1}, 2^k]$ in at most one element. Hence $\max_{k \geq 0} a_k \leq L < \infty$. Thus (3) and (4) are equivalent.

Suppose that $S$ is the union of $L$ sets $S_i$ which are each lacunary with constant 2. Split each $\mathcal{H}(S_i)$ into the subsets $R_i$ on or below the diagonal and $C_i$ above the diagonal. Observe that $R_i$ is row bounded by 1, and $C_i$ is column bounded by 1. Hence (4) implies (2).
Consider the subset of $\mathcal{H}(S)$ in the first $2^k$ rows and columns $R_k \times C_k$. This square will contain at least $2^{k-1}a_k$ entries corresponding to the backward diagonals for $S \cap (2^{k-1}, 2^k]$, which all have more than $2^{k-1}$ entries. Thus
\[
\sup_{k \geq 0} \frac{|\mathcal{H}(S) \cap (R_k \times C_k)|}{|R_k| + |C_k|} \geq \sup_{k \geq 0} \frac{2^{k-1}a_k}{2^k + 2^k} = \sup_{k \geq 0} \frac{a_k}{4}.
\]
Hence if (3) fails, this supremum if infinite. Thus $\mathcal{H}(S)$ is not the union of a row finite and a column finite set. So (2) fails.

The situation for Toeplitz patterns is quite different. It follows from classical results, as we explain below, and Nikolskaya and Farforovskaya outline a related proof [13, Remark 3.9]. But first we show how it follows from our theorem.

**Proposition 3.2.** The Toeplitz pattern $\mathcal{T}(S)$ of any infinite set $S$ is not Schur bounded. Further,
\[
\frac{1}{4}|S|^{1/2} \leq s(\mathcal{T}(S)) \leq |S|^{1/2}.
\]

**Proof.** Since $\mathcal{T}(S)$ is clearly row bounded by $|S|$, the upper bound follows from Lemma 2.5. Suppose that $S = \{s_1 < s_2 < \cdots < s_n\}$. Consider the $m \times m$ square matrix with upper left hand corner equal to $(s_1, 0)$ if $s_1 \geq 0$ or $(0, -s_1)$ if $s_1 < 0$. Then beginning with row $m - (s_n - s_1)$, there will be $n$ entries of $\mathcal{T}(S)$ in each row. Thus the total number of entries is at least $n(m - (s_n - s_1))$. For $m$ sufficiently large, this exceeds $(n - 1)m$. Hence by Lemma 2.3,
\[
s(\mathcal{T}(S)) \geq \frac{\sqrt{n - 1}}{2\sqrt{3}} \geq \frac{\sqrt{n}}{4}
\]
provided $n \geq 4$. The trivial lower bound of 1 yields the lower bound for $n < 4$.

To see how this is done classically, we recall the following [2, Theorem 8.1]. Here, $\mathcal{T}$ denotes the space of Toeplitz operators.

**Theorem 3.3 (Bennett).** A Toeplitz matrix $A = [a_{i-j}]$ determines a bounded Schur multiplier if and only if there is a finite complex Borel measure $\mu$ on the unit circle $\mathbb{T}$ so that $\hat{\mu}(n) = a_n, n \in \mathbb{Z}$. Moreover
\[
\|A\|_m = \|SA|\mathcal{T}\| = \|\mu\|.
\]

We combine this with estimates obtained from the Khintchine inequalities.
Theorem 3.4. Let \((a_k)_{k \in \mathbb{Z}}\) be an \(l^2\) sequence and let \(A = [a_{i-j}]\). Then
\[
\frac{1}{\sqrt{2}} \| (a_k) \|_2 \leq s(A) \leq \| (a_k) \|_2.
\]

Proof. Suppose \(S \in \mathcal{S}(A)\), that is, \(S = [s_{ij}]\) with \(|s_{ij}| \leq a_{i-j}\). Then each row of \(S\) has norm bounded by \(\| (a_k) \|_2\). Hence by Lemma 2.5\(\| S \|_m \leq \| (a_k) \|_2\). So \(s(A) \leq \| (a_k) \|_2\).

Conversely, let \(X := \{1,-1\}^\mathbb{Z}\). Put the measure \(\mu\) on \(X\) which is the product of measures on \([-1,1]\) assigning measure 1/2 to both \(\pm 1\). For \(\varepsilon = (\varepsilon_k)_{k \in \mathbb{Z}}\) in \(X\), define \(f_\varepsilon(\theta) = \sum_{k \in \mathbb{Z}} \varepsilon_k a_k e^{ik\theta}\). Then \(f_\varepsilon \in L^2(\mathbb{T}) \subset L^1(\mathbb{T})\). Hence \(S_\varepsilon := S_{T_\varepsilon}\) defines a bounded Schur multiplier with
\[
\| S_\varepsilon \|_m = \| f_\varepsilon \|_1 \leq \| f_\varepsilon \|_2 = \| (a_k) \|_2.
\]

Then we make use of the Khintchine inequality [19] [8]:
\[
\frac{1}{\sqrt{2}} \| (a_k) \|_2 \leq \int_X \| f_\varepsilon \|_1 d\mu(\varepsilon) \leq \| (a_k) \|_2.
\]
It follows that on average, most \(f_\varepsilon\) have \(L^1\)-norm comparable to the \(L^2\)-norm. In particular, there is some choice of \(\varepsilon\) with \(\| f_\varepsilon \|_1 \geq \frac{1}{\sqrt{2}} \| (a_k) \|_2\).

Thus \(s(A) \geq \| S_\varepsilon \|_m \geq \frac{1}{\sqrt{2}} \| (a_k) \|_2\). \(\Box\)

Remark 3.5. In the case of a finite Toeplitz pattern \(T(S)\), say \(S = \{s_1 < s_2 < \cdots < s_n\}\), \(f_\varepsilon = \sum_{k=1}^n \varepsilon_k e^{is_k \theta}\). We can use the Khintchine inequality for \(L^\infty\):
\[
\| (a_k) \|_2 \leq \int_X \| f_\varepsilon \|_\infty d\mu(\varepsilon) \leq \sqrt{2} \| (a_k) \|_2.
\]
Thus there will be choices of \(\varepsilon\) so that \(\| f_\varepsilon \|_\infty \leq \sqrt{2n}\). Then note that \(S_{T_\varepsilon}(T_{f_\varepsilon}) = T_{f_\varepsilon}\), where \(f_1 = \sum_{k=1}^n e^{is_k \theta}\). Clearly \(\| f_1 \|_\infty = f_1(0) = n\). Thus \(\| S_{T_\varepsilon} |_{T(S)} \| \geq \sqrt{n}/2\).

4. Patterns with a Symmetry Group

Consider a finite group \(G\) acting transitively on a finite set \(X\). Think of this as a matrix representation on the Hilbert space \(\mathcal{H}_X\) with orthonormal basis \(\{e_x : x \in X\}\). Let \(\pi\) denote the representation of \(G\) on \(\mathcal{H}_X\) and \(T\) the commutant of \(\pi(G)\). The purpose of this section is to compute the norm of \(S_T\) for \(T \in T\).

Decompose \(X^2\) into \(G\)-orbits \(X_i\) for \(0 \leq i \leq n\), beginning with the diagonal \(X_0 = \{(x,x) : x \in X\}\). Let \(T_i \in \mathcal{B}(\mathcal{H}_X)\) denote the matrix with 1s on the entries of \(X_i\) and 0 elsewhere. Then it is easy and well-known that \(T\) is span\{\(T_i : 0 \leq i \leq n\}\}. In particular, \(T\) is a C*-algebra. Also observe that every element of \(T\) is constant on the main diagonal.
Since $G$ acts transitively on $X$, $r_i := \{|y \in X : (x,y) \in X_i\}$ is independent of the choice of $x \in X$. Thus the vector $1$ of all ones is a common eigenvector for each $T_i$, and hence for all elements of $\mathcal{T}$, corresponding to a one-dimensional reducing subspace on which $G$ acts via the trivial representation.

First we establish an easy, general upper bound for $\|T\|_m$ where $T \in \mathcal{T}$. As usual, $\Delta$ is the expectation onto the diagonal.

**Proposition 4.1.** For a matrix $T$,
\[
\|T\|_m \leq \|\Delta(|T^*|)\|^{1/2} \|\Delta(|T|)\|^{1/2} = \| |T^*| \|_m^{1/2} \| |T| \|_m^{1/2}.
\]

**Proof.** Use polar decomposition to factor $T = U|T|$. Define vectors $x_i = |T|^{1/2} e_i$ and $y_j = |T|^{1/2} U^* e_j$. Then
\[
\langle x_i, y_j \rangle = \langle |T|^{1/2} e_i, |T|^{1/2} U^* e_j \rangle = \langle Te_i, e_j \rangle.
\]
This yields a Grothendieck–Haagerup form for $S_T$. Now
\[
\|x_i\|^2 = \langle |T|^{1/2} e_i, |T|^{1/2} e_i \rangle = \langle |T| e_i, e_i \rangle.
\]
Hence $\max_i \|x_i\| = \|\Delta(|T|)\|^{1/2}$. Similarly, since $|T|^{1/2} U^* = U^* |T^*|^{1/2}$
\[
\|y_j\|^2 = \langle U^* |T^*|^{1/2} e_j, U^* |T^*|^{1/2} e_j \rangle = \langle |T^*| e_j, e_j \rangle.
\]
So $\max_j \|y_j\| = \|\Delta(|T^*|)\|^{1/2}$. Therefore
\[
\|T\|_m \leq \max_{i,j} \|x_i\| \|y_j\| = \|\Delta(|T^*|)\|^{1/2} \|\Delta(|T|)\|^{1/2}.
\]
Since $|T|$ and $|T^*|$ are positive, the Schur norm is just the sup of the diagonal entries.

**Corollary 4.2.** If $T = T^*$, then $\|T\|_m \leq \|\Delta(|T|)\|$.

**Remark 4.3.** In general this is a strict inequality. If $T = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}$, then $|T| = \begin{bmatrix} 2\sqrt{5} & \sqrt{5} \\ \sqrt{5} & \sqrt{5} \end{bmatrix}$. But $\|S_T\|_m = 4 < 2\sqrt{5}$. Indeed, take $x_1 = y_1 = 2e_1$ and $x_2 = \frac{3}{2} e_1 + \frac{\sqrt{5}}{2} e_2$ and $y_2 = \frac{3}{2} e_1 - \frac{\sqrt{5}}{2} e_2$.

The main result of this section is:

**Theorem 4.4.** Let $X$ be a finite set with a transitive action by a finite group $G$. If $T$ belongs to $\mathcal{T}$, the commutant of the action of $G$, then for any $x_0 \in X$,
\[
\|T\|_m = \|S_T|_{\mathcal{T}}\| = |X|^{-1} \text{Tr}(|T|) = \langle |T| e_{x_0}, e_{x_0} \rangle.
\]
This result is a special case of a nice result of Mathias [12]. As far as we know, the application of Mathias’ result to the case of matrices invariant under group actions has not been exploited. As Mathias’s argument is short and elegant, we include it.

**Theorem 4.5** (Mathias). If \( T \) is an \( n \times n \) matrix with \( \Delta(|T^*|) \) and \( \Delta(|T|) \) scalar, then

\[
\|T\|_m = \frac{1}{n} \text{Tr}(|T|).
\]

**Proof.** For an upper bound, Proposition 4.1 shows that

\[
\|T\|_m \leq \|\Delta(|T^*|)\|^{1/2} \|\Delta(|T|)\|^{1/2} = \left(\frac{1}{n} \text{Tr}(|T^*|)\right)^{1/2} \left(\frac{1}{n} \text{Tr}(|T|)\right)^{1/2} = \frac{1}{n} \text{Tr}(|T|),
\]

because \( |T| \) and \( |T^*| \) are constant on the main diagonal, and \( |T^*| \) is unitarily equivalent to \( |T| \), and so has the same trace.

For the lower bound, use the polar decomposition \( T = W|T| \). Let \( \overline{W} \) have matrix entries which are the complex conjugates of the matrix entries of \( W \). Write \( T = [t_{ij}] \) and \( W = [w_{ij}] \) as \( n \times n \) matrices in the given basis. Set \( \mathbf{1} \) to be the vector with \( n \) 1’s. Then

\[
\|T\|_m \geq \|S_T(|T|)\| \geq \frac{1}{n} \langle S_T(|T|)\mathbf{1}, \mathbf{1} \rangle
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{w}_{ij}t_{ij} = \frac{1}{n} \sum_{j=1}^{n} \langle W^*Te_j, e_j \rangle = \frac{1}{n} \text{Tr}(|T|)
\]

Thus \( \|T\|_m = \frac{1}{n} \text{Tr}(|T|) \).

**Proof of Theorem 4.4.** We have already observed that elements of \( T \) are constant on the diagonal. Thus \( \|T\|_m = \frac{1}{n} \text{Tr}(|T|) = \langle |T|e_{x_0}, e_{x_0} \rangle \).

For the rest, observe that \( W \) belongs to \( C^*(T) \). Hence so does \( \overline{W} \) because the basis \( T_i \) of \( T \) has real entries.

We will provide an interesting example in the next section. For now we provide a couple of more accessible ones.

**Example 4.6.** Consider the action of the symmetric group \( S_n \) acting on a set \( X \) with \( n \) elements in the canonical way. Then the orbits in \( X^2 \) are just the diagonal \( X_0 \) and its complement \( X_1 \). So \( S_{X_1} \) is the projection onto the off-diagonal part of the matrix.

Observe that \( X_1 = \mathbf{1} \mathbf{1}^* - I \), where \( \mathbf{1} \) is the vector of \( n \) ones. Since \( \mathbf{1} \mathbf{1}^* = nP \), where \( P \) is the projection onto \( \mathbb{C} \mathbf{1} \), \( X_1 = (n-1)P - P^\perp \).
Therefore we obtain a formula due to Bhatia, Choi and Davis \[4\]

\[
\|X_1\|_m = \frac{1}{n} \text{Tr}(|X_1|) = \frac{1}{n} \text{Tr} ((n-1)P + P^\perp) \\
= \frac{1}{n} (n-1 + n-1) = 2 - \frac{2}{n}.
\]

**Example 4.7.** Consider the cyclic group \(C_n\) acting on an \(n\)-element set, \(n \geq 3\). Let \(U\) be the unitary operator given by \(U e_k = e_{k+1}\) for \(1 \leq k \leq n\), working modulo \(n\). The powers of \(U\) yields a basis for the commutant of the group action.

Consider \(T = U + I\). The spectrum of \(U\) is just \(\{\omega^k : 0 \leq k \leq n-1\}\) where \(\omega = e^{2\pi i/n}\). Thus the spectrum of \(|T|\) consists of the points

\[
|1 + \omega^k| = 2|\cos \left(\frac{k\pi}{n}\right)| \quad \text{for} \quad 0 \leq k \leq n-1.
\]

Hence

\[
\|T\|_m = \frac{1}{n} \text{Tr}(|T|) = \frac{2}{n} \sum_{k=0}^{n-1} \left|\cos \left(\frac{k\pi}{n}\right)\right| = \begin{cases} \\
\frac{2 \cos \left(\frac{\pi}{n}\right)}{n \sin \left(\frac{\pi}{2n}\right)} & \text{if } n \text{ even} \\
\frac{2 \cos \left(\frac{\pi}{n}\right)}{n \sin \left(\frac{\pi}{2n}\right)} & \text{if } n \text{ odd}
\end{cases}
\]

Thus the limit as \(n\) tends to infinity is \(\frac{4}{\pi}\). The multiplier norms for the odd cycles decrease to \(\frac{4}{\pi}\), while the even cycles increase to the same limit.

**Example 4.8.** Mathias \[12\] considers polynomials in the circulant matrices \(C_z\) given by \(C_z e_k = e_{k+1}\) for \(1 \leq k < n\) and \(C_z e_n = z e_1\), where \(|z| = 1\). This falls into our rubric because there is a diagonal unitary \(D\) so that \(DC_z D^* = wU\) where \(U\) is the cycle in the previous example and \(w\) is any \(n\)th root of \(z\). It is easy to see that conjugation by a diagonal unitary has no effect on the Schur norm. Thus any polynomial in \(C_z\) is unitarily equivalent to an element of \(C^*(U)\) via the diagonal \(D\). Hence the Schur norm equals the normalized trace of the absolute value.

The most interesting example of this was obtained with \(z = -1\) and \(S_n = \sum_{k=0}^{n-1} C_{-1}^k\) which is the matrix with entries \(\text{sgn}(i-j)\). So the Schur multiplier defined by \(S_n\) is a finite Hilbert transform. Mathias shows that

\[
\|S_n\|_m = \frac{2}{n} \sum_{j=1}^{\lfloor n/2 \rfloor} \cot \frac{(2j-1)\pi}{2n}.
\]

From this, he obtains sharper estimates on the norm of triangular truncation than are obtained in \[11\].
5. Kneser and Johnson Graph Patterns

In this section, we consider an interesting family of symmetric patterns which arise commonly in graph theory and combinatorial codes. The Johnson graphs $J(v, n, i)$ have $\binom{v}{n}$ vertices indexed by $n$ element subsets of a $v$ element set, and edges between $A$ and $B$ if $|A \cap B| = i$. Thus $0 \leq i \leq n$. We consider only $1 \leq n \leq v/2$ since, if $n > v/2$, one obtains the same graphs by considering the complementary sets of cardinality $v - n$. We will explicitly carry out the calculation for the Kneser graphs $K(v, n) = J(v, n, 0)$, and in particular, for $K(2n + 1, n)$. For more on Johnson and Kneser graphs, see [6].

We obtained certain Kneser graphs from Toeplitz patterns. Take a finite subset $S = \{s_1 < s_2 < \cdots < s_{2n+1}\}$ and consider the Toeplitz pattern $P$ with diagonals in $S$, namely $P = \{(i, j) : j - i \in S\}$. Consider $R$ to be the set of all sums of $n$ elements from $S$ and $C$ to be the set of all sums of $n+1$ elements from $S$. Index $R$ by the corresponding subset $A$ of $\{1, 2, \ldots, 2n+1\}$ of cardinality $n$; and likewise index each element of $C$ by a subset $B$ of cardinality $n+1$. Then for each entry $A$ in $R$, there are exactly $n+1$ elements of $C$ which contain it. The difference of the sums is an element of $S$. It is convenient to re-index $C$ by sets of cardinality $n$, replacing $B$ by its complement $\{1, 2, \ldots, 2n+1\} \setminus B$. Then the pattern can be seen to be the Kneser graph $K(2n + 1, n)$ with $\binom{2n+1}{n}$ vertices indexed by $n$ element subsets of a $2n + 1$ element set, with an edge between vertices $A$ and $B$ if $A \cap B = \emptyset$. In general, unfortunately, $P \cap (R \times C)$ will contain more than just these entries, because two subsets of $S$ of size $n+1$ can have the same sum.

The adjacency matrix of a graph $G$ is a $v \times v$ matrix with a 1 in each entry $(i, j)$ corresponding to an edge from vertex $i$ to vertex $j$, and 0’s elsewhere. This is a symmetric matrix and its spectral theory is available in the graph theory literature; see, for example, [6]. We prove the simple facts we need.

Fix $(v, n)$ with $n \leq v$ and let $X$ denote the set of $n$ element subsets of $\{1, \ldots, v\}$. Define a Hilbert space $\mathcal{H} = \mathcal{H}_X$ as in the previous section but write the basis as $\{e_A : A \in X\}$. Observe that there is a natural action $\pi$ of the symmetric group $\mathfrak{S}_v$ on $X$. The orbits in $X^2$ are

$$X_i = \{(A, B) : A, B \in X, |A \cap B| = i\} \quad \text{for} \quad 0 \leq i \leq n.$$

The matrix $T_i$ is just the adjacency matrix of the Johnson graph $J(v, n, i)$ and, in particular, $T_n = I$.

This action has additional structure that does not hold for arbitrary transitive actions.
Lemma 5.1. The commutant $\mathcal{T} = \text{span}\{T_i : 0 \leq i \leq n\}$ of $\pi(\mathfrak{S}_v)$ is abelian. Thus $\pi$ decomposes into a direct sum of $n + 1$ distinct irreducible representations.

Proof. Equality with the span was observed in the last section. To see that the algebra $\mathcal{T}$ is abelian, observe that $T_i T_j = \sum_{k=0}^{n} a_{ijk} T_k$ where we can find the coefficients $a_{ijk}$ by fixing any two sets $A, B \subset V$ of size $n$ with $|A \cap B| = k$ and computing

$$a_{ijk} = |\{C \subset V : |C| = n, |A \cap C| = i, |C \cap B| = j\}|.$$ 

This is clearly independent of the order of $i$ and $j$. As $\mathcal{T}$ is abelian and $n + 1$ dimensional, the representation $\pi$ decomposes into a direct sum of $n + 1$ distinct irreducible representations.

Corollary 5.2. $\|T_i\| = \binom{n}{i} \binom{v-n}{n-i}$ and this is an eigenvalue of multiplicity one. The spectrum of $T_i$ contains at most $n + 1$ points.

Proof. Observe that if $|A| = n$, then the number of subsets $B \in X$ with $|A \cap B| = i$ is $\binom{n}{i} \binom{v-n}{n-i}$. Thus $T_i$ has this many 1’s in each row. Hence

$$T_i 1 = \binom{n}{i} \binom{v-n}{n-i} 1.$$ 

Clearly $T_i$ has nonnegative entries and is indecomposable (except for $i = n$, the identity matrix). So by the Perron–Frobenius Theorem, $\binom{n}{i} \binom{v-n}{n-i}$ is the spectral radius and 1 is the unique eigenvector; and there are no other eigenvalues on the circle of this radius. Since $T = T^*$, the norm equals spectral radius. As $\mathcal{T}$ is $n + 1$ dimensional, the spectrum can have at most $n + 1$ points.

We need to identify the invariant subspaces of $\mathfrak{S}_v$ as they are the eigenspaces of $T_i$. The space $V_0 = \mathbb{C}1$ yields the trivial representation. Define vectors associated to sets $C \subseteq \{1, \ldots, v\}$ of cardinality at most $n$, including the empty set, by

$$v_C := \sum_{|A|=n, A \cap C = \emptyset} e_A.$$ 

Then define subspaces $V_i = \text{span}\{v_C : |C| = i\}$ for $0 \leq i \leq n$. It is obvious that each $V_i$ is invariant for $\mathfrak{S}_v$. Given $C$ with $|C| = i$, we have

$$\sum_{C \subset D, |D| = i+1} v_D = (v - n - i)v_C,$$
as the coefficient of $e_A$ counts the number of choices for the $(i + 1)$st element of $D$ disjoint from an $A$ already disjoint from $C$. Therefore
\[ \mathbb{C}1 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n. \]

So the $n + 1$ subspaces $W_i = V_i \cap V_{i−1}$ are invariant for $\mathcal{S}_v$.

Let $E_i$ denote the idempotent in $\mathcal{T}$ projecting onto $W_i$. Observe that $\mathcal{T} = \text{span}\{E_i : 0 \leq i \leq n\}$. We need to know the dimension of these subspaces.

**Lemma 5.3.** The vectors $\{v_C : |C| = i\}$ are linearly independent. Hence $\dim W_i = \binom{n}{i} - \binom{n}{i−1}$.

**Proof.** Suppose that $v_{C_0} + \sum_{|C| = i, C \neq C_0} \gamma Cv_C = 0$. By averaging over the subgroup of $\mathcal{S}_v$ which fixes $C_0$, namely $\mathcal{S}_i \times \mathcal{S}_{v−i}$, we may assume that the coefficients are invariant under this action. Hence $\gamma_C = \alpha_j$ where $j = |C \cap C_0|$. So with $w_j := \sum_{|C| = i, |C \cap C_0| = j} v_C$, we have $\sum_{j=0}^i \alpha_j w_j = 0$ where $\alpha_i = 1$. We also define vectors $x_k = \sum_{|A \cap C_0| = k} e_A$, which are clearly linearly independent for $0 \leq k \leq i$. Compute for $0 \leq j \leq i$ (here $A$ implicitly has $|A| = n$)
\[
    w_j = \sum_{|C| = i} \sum_{|C \cap C_0| = j} e_A = \sum_{k=0}^{i−j} b_{jk} x_k
\]
where the coefficients are obtained by counting, for a fixed set $A$ with $|C_0 \cap A| = k$ and $k \leq i − j$:
\[
    b_{jk} = |\{C : |C| = i, |C \cap C_0| = j, A \cap C = \emptyset\}| = \binom{i−k}{j} \binom{v+k−n−i}{i−j}.
\]

It is evident by induction that
\[
    \text{span}\{w_j : i−k \leq j \leq i\} = \text{span}\{x_j : 0 \leq j \leq k\}.
\]

So $\{v_C : |C| = i\}$ are linearly independent. $\blacksquare$

We write $T_i = \sum_{j=0}^n \lambda_{ij} E_j$ be the spectral decomposition of each $T_i$. The discussion above shows that if $|C| = j$, then $v_C$ is contained in $V_j$ but not $V_{j−1}$. Thus $\lambda_{ij}$ is the unique scalar so that $(T_i − \lambda_{ij} I)v_C \in V_{j−1}$. This idea can be used to compute the eigenvalues, but the computations are nontrivial. We refer to [6, Theorem 9.4.3] for the Kneser graph $K(2n+1, n)$ which is the only one we work out in detail.

**Lemma 5.4.** The adjacency matrix for the Kneser graph $K(2n+1, n)$ has eigenvalues are $(-1)^i(n + 1 − i)$ with eigenspaces $W_i$ for $0 \leq i \leq n$. 

Theorem 5.5. If $T$ is the adjacency matrix of $K(2n + 1, n)$, then
\[
\|T\|_m = \|S_T\| = \frac{2^{2n}}{2^{2n+1}} = \frac{(4)(6)\ldots(2n+2)}{(3)(5)\ldots(2n+1)} > \frac{1}{2} \log(2n + 3).
\]

Proof. By Theorem 4.4 and Lemma 5.4,
\[
\|T\|_m = \|\Delta(|T|)\| = \left(\frac{2n+1}{n}\right)^{-1} \sum_{i=0}^{n} (n+1-i) \text{Tr}(E_i)
\]
\[
= \left(\frac{2n+1}{n}\right)^{-1} \sum_{i=0}^{n} (n+1-i) \left(\frac{2n+1}{i} - \frac{2n+1}{i-1}\right)
\]
\[
= \left(\frac{2n+1}{n}\right)^{-1} \sum_{i=0}^{n} \left(\frac{2n+1}{i}\right)
\]
\[
= \left(\frac{2n+1}{n}\right)^{-1} \frac{1}{2} \sum_{i=0}^{2n+1} \left(\frac{2n+1}{i}\right)
\]
\[
= \left(\frac{2n+1}{n}\right)^{-1} 2^{2n} = \frac{2^{2n}n!(n+1)!}{(2n+1)!}
\]
\[
= \frac{2 \cdot 4 \cdot \ldots (2n)}{1 \cdot 3 \cdot \ldots (2n-1)(2n+1)} \frac{2 \cdot 4 \cdot \ldots (2n)}{2 \cdot 4 \cdot \ldots (2n)} \frac{2n+2}{2n+1}
\]
\[
= \prod_{i=0}^{n} \left(1 + \frac{1}{2i+1}\right) > \frac{1}{2} \log(2n + 3).
\]

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Pure Math. Dept., U. Waterloo, Waterloo, ON N2L–3G1, CANADA
E-mail address: krdavidson@math.uwaterloo.ca

Math. Dept., University of Nebraska, Lincoln, NE 68588, USA
E-mail address: adonsig@math.unl.edu