Asymptotic Integration of Certain Differential Equations in Banach Space

P. N. Nesterov*
Demidov State University, Yaroslavl, 150003 Russia
*e-mail: nesterov.pn@gmail.com
Received March 26, 2017

Abstract—In this work, we investigate the problem of constructing asymptotic representations for weak solutions of a certain class of linear differential equations in the Banach space as an independent variable tends to infinity. We consider the class of equations that represent a perturbation of a linear autonomous equation, in general, with an unbounded operator. The perturbation takes the form of a family of bounded operators that, in a sense, oscillatorily decreases at infinity. It is assumed that the unperturbed equation satisfies the standard requirements of the center manifold theory. The essence of the proposed asymptotic integration method is to prove the existence of a center-like manifold (a critical manifold) for the initial equation. This manifold is positively invariant with respect to the initial equation and attracts all trajectories of the weak solutions. The dynamics of the initial equation on the critical manifold is described by the finite-dimensional system of ordinary differential equations. The asymptotics of the fundamental matrix of this system can be constructed by using the method developed by P.N. Nesterov for asymptotic integration of systems with oscillatory decreasing coefficients. We illustrate the proposed technique by constructing the asymptotic representations for solutions of the perturbed heat equation.

Keywords: asymptotics, differential equation, Banach space, oscillatory decreasing coefficients, center manifolds method, perturbed heat equation
DOI: 10.3103/S0146411619070150

FORMULATION OF THE PROBLEM

We study the equation
\[ \dot{u} = (A + G(t))u, \; t \geq t_0, \]  
where \( u \) is an element of a complex Banach space \( \mathcal{B} \). Here, \( A \) is a closed linear operator with the dense domain in \( \mathcal{B} \); this operator is a generator of a strongly continuous semigroup of bounded linear operators \( T(t): \mathcal{B} \to \mathcal{B} \ (t \geq 0) \). In this case, \( G(t) \ (t \geq t_0) \) is a family of linear bounded operators acting from \( \mathcal{B} \) to \( \mathcal{B} \); here,
\[ G(t) = B(t) + R(t). \]  

In (2), the family of linear bounded operators \( B(t) \) is such that the operator function \( B(t) \) is strongly measurable on any interval \( [t_0, T] \), \( T \geq t_0 \), while \( \|B(t)u\| \) oscillatory tends to zero when \( t \to \infty \) for \( u \in \mathcal{B} \). More accurately, the structure of this family of operators will be defined later. The family of linear bounded operators \( R(t) \) is also strongly measurable on any interval \( [t_0, T] \), \( T \geq t_0 \); in addition, there exists a function \( \gamma(t) \in L_1[t_0, \infty) \) such that
\[ \|R(t)u\|_{\mathcal{B}} \leq \gamma(t)\|u\|_{\mathcal{B}} \]  
for any \( u \in \mathcal{B} \). The feasibility of studying equations of form (1), where an operator function \( G(t) \) is understood as a certain parametric perturbation with a continuous spectrum, is noted, among other things, in the monograph [6, p. 230].

Everywhere in this work, the solution of Eq. (1) with the initial condition \( u(t_0) = u_0 \) is understood in the weak sense (see [9]), more precisely, as a solution of the integral equation
From the results of [9, 10] (see also [8]) it follows that for any \( u_0 \in \mathcal{B} \) there exists an unambiguous continuous (on the interval \([t_0, T]\) \((T \geq t_0)\)) weak solution of Eq. (1) with the initial condition \( u(t_0) = u_0 \), and this solution is described by formula (4). The question we are interested in is the asymptotic behavior of solutions of Eq. (4) as \( t \to \infty \) if the operator \( A \) is subjected to the following additional conditions. Suppose

\[
\mathcal{B} = \mathcal{X} \oplus \mathcal{Y},
\]

where the finite-dimensional linear subspace \( \mathcal{X} \) is a linear span of generalized eigenvectors of the operator \( A \) corresponding to the eigenvalues \( \lambda_1, \ldots, \lambda_N \) with the zero real part (with account of their multiplicities);

\( \text{ii) the closed linear subspace} \ \mathcal{Y} \ \text{is invariant under the semigroup} \ \mathcal{T}(t) \ \text{and, in addition, for any} \ \gamma \in \mathcal{Y}, \ \text{the following inequality holds:} \)

\[
\|T(t)\gamma\| \leq Ke^{-\alpha t}\|\gamma\|, \ t \geq 0,
\]

where \( K, \alpha > 0 \).

Formulated conditions (i) and (ii) represent standard requirements of the centre manifold theory (see, e.g., [3, 11, 20]). We use the basic ideas of this theory, along with the averaging-method version proposed in [4] for constructing the asymptotics of weak solutions of Eq. (1) as \( t \to \infty \).

**CRITICAL MANIFOLD AND ITS PROPERTIES**

We begin this section by clarifying the type of the operator function \( B(t) \) in (2). In accordance with [4, 6], we assume that

\[
B(t) = \sum_{j=1}^{n} v_j(t)B_j(t) + \sum_{1 \leq i_1 \leq 2 \leq n} v_{i_1}(t)v_{i_2}(t)B_{i_1i_2}(t) + \ldots + \sum_{1 \leq i_1 \leq 2 \leq \ldots \leq n} v_{i_1}(t)\ldots v_{i_n}(t)B_{i_1\ldots i_n}(t).
\]

Here, \( B_{i_1\ldots i_n}(t) \) are operator functions for which it is assumed that

\[
B_{i_1\ldots i_n}(t) = \sum_{j=1}^{M} e^{\lambda_{i_1j}t}B_{j_{i_1\ldots i_n}},
\]

where \( B_{j_{i_1\ldots i_n}} \) are linear bounded operators that do not depend on \( t \) and act from \( B \) to \( B \). Finally, \( v_1(t), \ldots, v_n(t) \) are scalar functions absolutely continuous on \([t_0, \infty)\) such that

\(1^0\) \quad \( v_1(t) \to 0, v_2(t) \to 0, \ldots, v_n(t) \to 0 \) when \( t \to \infty \);

\(2^0\) \quad \( \psi_1(t), \psi_2(t), \ldots, \psi_n(t) \in L_1[t_0, \infty) \);

\(3^0\) \quad the product \( v_{i_1}(t)v_{i_2}(t)\ldots v_{i_n}(t) \in L_1[t_0, \infty) \) for any set \( 1 \leq i_1 \leq i_2 \leq \ldots \leq i_{k+1} \leq n \).

**Definition 1.** The linear \( N \)-dimensional subspace \( \mathcal{W}(t) \subset \mathcal{B} \) we shall call a critical manifold for Eq. (1) when \( t \geq t_0 \geq t_0 \) if the following conditions are met:

1. There exists a row vector \( H(t) = (h_1(t), \ldots, h_N(t)) \in \mathcal{Y}^N \) composed of functions continuous for \( t \geq t_0 \) with the values from the subspace \( \mathcal{Y} \) such that \( \|H(t)\|_{\mathcal{Y}^N} \to 0 \) when \( t \to \infty \), where

\[
\|H(t)\|_{\mathcal{Y}^N} = \|h_1(t)\|_\mathcal{Y}, \ldots, \|h_N(t)\|_\mathcal{Y} \]

and \( \|\cdot\|_\mathcal{Y} \) is a certain norm in the space of row-vectors of length \( N \).

2. The set \( \mathcal{W}(t) \) for \( t \geq t_0 \) is described by the formula

\[
\mathcal{W}(t) = \{u \in \mathcal{B} | u = \Phi w + H(t)w, w \in \mathbb{C}^N \}.
\]
Here, the row vector $\Phi = (\varphi_1, \ldots, \varphi_N) \in \mathbb{R}^N$ is composed of generalized eigenvectors of the operator $A$ that correspond to the eigenvalues $\lambda_1, \ldots, \lambda_N$ from condition (i), which form the basis of a subspace $\mathcal{X}$. In this case, by $\mathbb{C}^N$ we designate a space of complex-valued column vectors of length $N$; the product of the row vector $U = (u_1, \ldots, u_N) \in \mathbb{R}^N$ and the column vector $w = (w_1, \ldots, w_N)^T$ is understood in the standard sense: $Uw = u_1w_1 + \ldots + u_Nw_N$.

3. The set $\mathcal{W}(t)$ when $t \geq t_0$ is positively invariant relative to the solutions of Eq. (1); i.e., if $u(T) \in \mathcal{W}(T)$, $T \geq t_0$, then $u(t) \in \mathcal{W}(t)$ for all $t \geq T$.

Before we derive the system of equations describing the dynamics of solutions of Eq. (1) on the critical manifold $\mathcal{W}(t)$, recall the following facts from the theory of conjugate operators in Banach spaces. Suppose $\mathcal{B}^*$ is a space conjugate to $\mathcal{B}$ (a space of linear bounded functionals assigned on $\mathcal{B}$) and $\{\cdot, \cdot\}$ are brackets of duality between the spaces $\mathcal{B}$ and $\mathcal{B}^*$ such that

$$\langle \varphi, \psi \rangle = \psi(\varphi) \in \mathbb{C}$$

for any $\varphi \in \mathcal{B}$ and $\psi \in \mathcal{B}^*$. Here, if $U = (u_1, \ldots, u_l) \in \mathcal{B}'$ and $V = (v_1, \ldots, v_m)^T \in (\mathcal{B}^*)^m$, then $\langle U, V \rangle$ is an $m$-by-$l$ matrix such that

$$\langle U, V \rangle = \{v_i(u_j)\}_{1 \leq i \leq m, \ 1 \leq j \leq l}.$$ 

Suppose $A$ is a conjugate to $A$ linear closed operator acting from the space $\mathcal{B}^*$ to $\mathcal{B}^*$. As is well known (see, e.g., [1, 14]) from assumption (i) on the operator $A$, the operator $A'$ also has eigenvalues $\lambda_1, \ldots, \lambda_N$; here, the dimensions of matching generalized eigenspaces of these operators that correspond to the same eigenvalues, coincide. Denote by $\Psi = (\psi_1, \ldots, \psi_N)^T \in (\mathcal{B}^*)^N$ a column vector of length $N$ consisting of linearly independent generalized eigenvectors of the operator $A'$ that correspond to the eigenvalues $\lambda_1, \ldots, \lambda_N$. Suppose the column vector $\Psi$ is chosen in such a way that

$$\langle \Phi, \Psi \rangle = I,$$ 

where $I$ is an $N$-by-$N$ unit matrix (see [7, 13]). Note in particular that one of the possible choices of the complementary subspace for $\mathcal{X}$ in condition (i) is the choice of the set $\mathcal{Y} = \{u \in \mathcal{B} | \langle u, \Psi \rangle = 0\}$ for this subspace.

We construct a system that describes the dynamics of solutions of Eq. (1) on a critical manifold $\mathcal{W}(t)$ assuming the existence of this manifold for sufficiently large $t$. Suppose $P$ is a projector on the subspace $\mathcal{X}$ along $\mathcal{Y}$. Note that $P$ is a bounded linear operator defined in the entire space $\mathcal{B}$ (see, e.g., [14]). On account of condition (i) we conclude that if $u \in \mathcal{B}$, then

$$u(t) = u_x(t) + u_y(t), \ t \geq t_0,$$ 

where

$$u_x(t) = Pu(t) \in \mathcal{X}, \ u_y(t) = (1 - P)u(t) \in \mathcal{Y}$$ 

and by $I$ we denote an identity operator. Note that since the subspaces $\mathcal{X}$ and $\mathcal{Y}$ are invariant under the operator $A$ and the semigroup $T(t)$, for all $t \geq 0$ and $u \in \mathcal{B}$ we have the equalities

$$PT(t)u = T(t)Pu, \ (1 - P)T(t)u = T(t)(1 - P)u.$$ 

Substituting (13) in Eq. (4) and taking into account (14) and (15), we obtain

$$u_x(t) = T(t - t_0)u_x(t_0) + \int_{t_0}^{t} T(t - s)PG(s)u(s)ds,$$ 

$$u_y(t) = T(t - t_0)u_y(t_0) + \int_{t_0}^{t} T(t - s)(1 - P)G(s)u(s)ds.$$ 

Since

$$u_x(t) = Pu(t) = \Phi w(t)$$ 

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for certain \( w(t) \in \mathbb{C}^N \), we have

\[
\Phi w(t) = T(t - t_0)\Phi w(t_0) + \int_{t_0}^{t} T(t - s)PG(s)u(s)ds.
\] (19)

In addition, from (i) it follows that \( A\mathcal{X} \subseteq \alpha\mathcal{X} \); hence, there exists an \( N \)-by-\( N \) matrix \( D \) whose spectrum is the set \( \{\lambda_1, \ldots, \lambda_N\} \) such that

\[
A\Phi = \Phi D.
\] (20)

Note that if an element \( u \in \mathcal{B} \) belongs to the domain of the operator \( A \), then from [18, p. 4, Theorem 2.4] we have

\[
\frac{d}{dt}T^T u = TAu = ATu.
\] (21)

It is clear that all elements of the subspace \( \mathcal{X} \) belong to the domain of the operator \( A \). Differentiating expression (19) and considering (21), we obtain

\[
\Phi \dot{w}(t) = AT(t - t_0)\Phi w(t_0) + PG(t)u(t) + A\int_{t_0}^{t} T(t - s)PG(s)u(s)ds = A\Phi w(t) + PG(t)u(t).
\]

With allowance for (20), we have

\[
\Phi \dot{w} = \Phi Dw(t) + PG(t)u(t)
\] (22)
or taking into account (12),

\[
\dot{w} = Dw(t) + \langle PG(t)u(t), \Psi \rangle.
\] (23)

Suppose \( u(t_0) \in \mathcal{W}(t_0) \); then from the invariance of a critical manifold with account of (10), we conclude that if \( t \geq t_0 \) then

\[
u(t) = \Phi w(t) + H(t)w(t).
\] (24)

Substituting (24) in (23), we have

\[
\dot{w} = [D + \langle PG(t)(\Phi + H(t)), \Psi \rangle]w(t), \quad t \geq t_0, \quad w \in \mathbb{C}^N.
\] (25)

We shall call system (25) a projection of Eq. (1) on a critical manifold \( \mathcal{W}(t) \), or simply a system on a critical manifold.

**Theorem 1.** For sufficiently large \( t \), system (1) has a critical manifold \( \mathcal{W}(t) \) determined by formula (10).

*Proof.* From representation (24) we have

\[
(1 - P)u(t) = u_0(t) = H(t)w(t).
\]

We use this equality along with formula (24) in (17), suppose \( t_0 = t_0 \), and obtain the equation

\[
H(t)w(t) = T(t - t_0)H(t_0)w(t_0) + \int_{t_0}^{t} T(t - s)(1 - P)G(s)[\Phi + H(s)]w(s)ds.
\] (26)

Suppose \( W_H(t, s)(t, s \geq t_0) \) is the Cauchy matrix of system (25) \( (W_H(s, s) = I) \). Then, taking into account that \( w(t) = W_H(t, t_0)w(t_0) \) and using the properties of the matrix \( W_H(t, s) \), we write Eq. (26) in the following operator form:

\[
H(t) = \Gamma H(t),
\] (27)

\[
\Gamma H(t) = T(t - t_0)H(t_0)W_H(t_0, t) + \int_{t_0}^{t} T(t - s)(1 - P)G(s)[\Phi + H(s)]W_H(s, t)ds.
\] (28)

We assume that the domain of the operator $\Gamma$ is the Banach space $\mathcal{C}$ of continuous in $t \geq t_0$ row vectors $H(t)$ of length $N$ with the values from the space $\mathcal{Y}^N$ and the fixed initial condition $H(t_0)$, such that $\|H(t)\|_{\mathcal{B}^N} \to 0$ when $t \to \infty$. The norm in this space we define as follows:

$$\|H\|_{\mathcal{B}} = \sup_{t \geq t_0} \|H(t)\|_{\mathcal{B}^N}. \quad (29)$$

From (6), for any $U \in \mathcal{B}^N$ we have the estimate

$$\|G(t)(1 - P)U\|_{\mathcal{B}^N} \leq Ke^{-\alpha t} \|U\|_{\mathcal{B}^N}, \quad t \geq 0. \quad (30)$$

Here and further the various constants, whose exact values are not important to us, we will denote by the same symbols. In addition, from (2), (3), (7), and (8) it follows that

$$\|G(t)U\|_{\mathcal{B}^N} \leq p(t) \|U\|_{\mathcal{B}^N}, \quad p(t) = f(t) + \gamma(t). \quad (31)$$

Here, $f(t) \to 0$ as $t \to \infty$ and $\gamma(t)$ is a certain function from the class $L_1[t_0, \infty)$.

Further steps in proving rest on the principle of contracting mappings. In an analogous way as it is done in proving Theorem 1 in [5] (see also [17, Theorem 4.3]), it is easy to show that the operator $\Gamma$ maps a certain closed ball $\mathcal{B}(\mathcal{B}^N, \mathcal{B}^N)$ to itself and represents a contracting operator in this ball if $t_0$ is sufficiently large and the chosen initial condition $H(t_0)$ is sufficiently small.

For approximately finding the row vector $H(t)$, which describes the critical manifold $\mathcal{W}(t)$ by formula (10), we act as follows. Substitute representation (24) in Eq. (1) and take into account (20), (22), and (25). We have

$$\Phi Dw(t) + PG(t)(\Phi + H(t))w(t) + \dot{H}w(t) + H[D + \langle PG(t)(\Phi + H(t)), \Psi \rangle]w(t) = A\Phi w(t) + AH(t)w(t) + G(t)[\Phi + H(t)]w(t). \quad (32)$$

With allowance for (20) we obtain

$$\dot{H} = AH - HD + (1 - P)G(t)(\Phi + H(t)) - H \langle PG(t)(\Phi + H(t)), \Psi \rangle. \quad (32)$$

We shall try to satisfy Eq. (32) with the accuracy of terms $\hat{R}(t)$ such that $\|\hat{R}(t)\|_{\mathcal{B}^N} \in L_1[t_0, \infty)$. We put

$$\hat{H}(t) = \sum_{j=1}^{n} v_j(t)H_j(t) + \sum_{1 \leq j_1 < \ldots < j_k \leq n} v_{j_1}(t)v_{j_2}(t)H_{j_2}(t) + \ldots + \sum_{1 \leq j_1 < \ldots < j_k \leq n} v_{j_1}(t) \cdots v_{j_k}(t)H_{j_{k-1}}(t). \quad (33)$$

Here, the elements of the row vectors $H_{j_{k-1}}(t)$ of length $N$, to be determined, belong to the space $\mathcal{Y}$ for all $t \in \mathbb{R}$ and represent trigonometric polynomials in the sense

$$H_{j_{k-1}}(t) = \sum_j \beta_j^{(k-1)} e^{in\gamma t}, \quad (34)$$

where $\beta_j^{(k-1)} \in \mathcal{Y}^N$. In addition, a value of the integer $k \geq 0$ is determined by property $3^0$ of the functions $v_1(t), \ldots, v_n(t)$.

Substitute expression (33) into Eq. (32) instead of $H(t)$ and gather the terms with the same multipliers $v_j(t) \cdots v_{j_k}(t)$ ($l \leq k$). We obtain the following one-type equations for determining the row vectors $H_{j_{k-1}}(t)$:

$$\hat{H}_{j_{k-1}} = AH_{j_{k-1}} - H_{j_{k-1}}D + F_{j_{k-1}}(t). \quad (35)$$

Here we take into account formulas (2), (7), and (8), from which it follows, specifically, that in order to solve Eq. (35), it is necessary to determine, first, all row vectors $H_{j_{k-1}}(t)$ with $s < l$. In this case, $F_{j_{k-1}}(t)$ is a certain known row vector, which from (8) and (34) has the form analogous to (34):

$$F_{j_{k-1}}(t) = \sum_j \beta_j^{(l-k)} e^{in\gamma t}. \quad (36)$$
where \( f_{j}^{(l-\delta)} \in \mathcal{Y}^N \). In addition, the row vectors \( f_{j}^{(l-\delta)} \) belong to the space \( \mathcal{Y}^N \). In fact, this follows from the fact that the row vectors \( H_{j-\delta} (t) \) take values from the space \( \mathcal{Y}^N \) for all \( s < l \) and, in addition, in Eq. (32) the row vector \((1 - PG(t)(\Phi + H(t)))\) takes values from the space \( \mathcal{Y}^N \).

The solution of Eq. (35) we find in the form of (34). Substituting this expression in (35), with allowance for (36), we obtain the following operator equation for determining an element \( \beta_{j}^{(l-\delta)} \in \mathcal{Y}^N \):

\[
A\beta_{j}^{(l-\delta)} - \beta_{j}^{(l-\delta)} D - i\omega \beta_{j}^{(l-\delta)} = -f_{j}^{(l-\delta)}. 
\]  

Lemma 1. Equation (37) has a unique solution in the space \( \mathcal{Y}^N \) for any right side \( f_{j}^{(l-\delta)} \in \mathcal{Y}^N \).

Proof. Determine the operator \( L : \mathcal{Y}^N \rightarrow \mathcal{Y}^N \) by the formula

\[
L\beta = A\beta - \beta D, 
\]

where \( \beta \in \mathcal{Y}^N \). In effect of conditions imposed on the operator \( A \), the operator \( L \) is a closed linear operator with the dense in \( \mathcal{Y}^N \) domain of definition. In addition, it is easy to check that the operator \( L \) is a generator of a strongly continuous semigroup of bounded linear operators \( U(t) : \mathcal{Y}^N \rightarrow \mathcal{Y}^N \) \((t \geq 0)\), which is determined by the rule

\[
U(t)\beta = T(t)\beta e^{-\Delta t}. 
\]

Since all eigenvalues of the matrix \( D \) are located on the imaginary axis, the following inequality holds from (6) for all \( \beta \in \mathcal{Y}^N \):

\[
\|U(t)\beta\| \leq Ke^{-\nu t} \|\beta\|, \quad t \geq 0,
\]

where \( K, \nu > 0 \). From (40) we obtain

\[
(\sigma(U(t))) \leq Ke^{-\nu t}, \quad t \geq 0,
\]

where \( \sigma(U(t)) \) is the spectrum of the operator \( U(t) \). Inequality (41) is understood in the sense that it holds for any \( \mu \in \sigma(U(t)) \). According to [18, p. 45, Theorem 2.3], we have

\[
e^{-\nu t} \subset \sigma(U(t)), \quad t \geq 0.
\]

From (41) and (42) we conclude that a number of the form \( i\omega \), where \( \omega \in \mathbb{R} \), cannot belong to the set \( \sigma(L) \), i.e., the spectrum of the operator \( L \). Thus, the entire imaginary axis belongs to the resolvent set of the operator \( L \). Hence (see, e.g., [11]), for any \( \omega \in \mathbb{R} \) there exists a linear continuous operator \((L - i\omega I)^{-1}\) defined in the entire space \( \mathcal{Y}^N \). Thus, Eq. (37) has a unique solution

\[
\beta_{j}^{(l-\delta)} = -(L - i\omega I)^{-1} f_{j}^{(l-\delta)}.
\]

Corollary 1. The unique solution of Eq. (37) is determined by the formula

\[
\beta_{j}^{(l-\delta)} = \int_{0}^{\infty} e^{-i\omega s} U(s) f_{j}^{(l-\delta)} ds,
\]

where the semigroup of operators \( U(t) \) has the form of (39).

Proof. The convergence of the integral in (43) follows from the inequality (40). In an analogous manner to that which is done in [18, p. 8, Theorem 3.1], it is possible to show that integral (43) belongs to the domain of the operator \( L \) described by formula (38). In addition, the following equality holds:

\[
L\beta_{j}^{(l-\delta)} = i\omega \beta_{j}^{(l-\delta)} - f_{j}^{(l-\delta)},
\]

which is equivalent to (37). Hence, the constructed row vector \( \hat{H}(t) \) defined by formula (33) satisfies the following equation:

\[
\frac{d\hat{H}}{dt} = A\hat{H} - \hat{H} D + (1 - PG(t)(\Phi + \hat{H}(t))) - \hat{H} \left< PG(t)(\Phi + \hat{H}(t)), \Psi \right> + \hat{R}(t),
\]
where \( \hat{R}(t) \) is a certain row vector that takes values from the space \( \mathcal{Y}^N \) such that \( \| \hat{R}(t) \|_{\mathcal{Y}^N} \in L_t[t_0, \infty) \). The following approximation theorem holds.

**Theorem 2.** Suppose \( \mathcal{W}(t) \) is a critical manifold (for Eq. (1)) existing according to Theorem 1 for sufficiently large \( t \). Then there exists sufficiently large \( t_0 \) such that for \( t \geq t_0 \) the row vector \( \hat{H}(t) \) from (10) allows the representation in the form

\[
H(t) = \hat{H}(t) + Z(t), \quad t \geq t_0, \tag{45}
\]

where the row vector \( \hat{H}(t) \) is described by formula (33) and satisfies Eq. (44), while the row vector \( Z(t) \) taking values from \( \mathcal{Y}^N \) is such that \( \| Z(t) \|_{\mathcal{Y}^N} \to 0 \) when \( t \to \infty \) and \( \| Z(t) \|_{\mathcal{Y}^N} \in L_t[t_0, \infty) \).

**Proof.** The row vector \( H(t) \), which is the solution of operator Eq. (27) and (28), we present as sum (45). Then Eq. (27) can be presented in the unknown row vector:

\[
Z(t) = \Pi Z(t), \tag{46}
\]

\[
\Pi Z(t) = \Gamma (\hat{H}(t) + Z(t)) - \hat{H}(t, \theta), \tag{47}
\]

where the operator \( \Gamma \) is defined by formula (28). We consider the operator \( \Pi \) acting in the space \( \mathcal{C}_E \mathcal{L} \) of continuous for \( t \geq t_0 \) row vectors \( Z(t) \) taking values in the space \( \mathcal{Y}^N \) with a fixed initial condition \( \hat{H}(t_0) \) and \( \Pi(\hat{H}(t_0)) = 0 \). Here the function \( p(t) \) is defined by (31). In this case, \( \mathcal{C}_E \mathcal{L} \) is the Banach space if we introduce a norm in it by the rule

\[
\| Z \|_{\mathcal{C}_E \mathcal{L}} = \| Z \|_E + \| Z \|_F, \quad \| Z \|_E = \int_{t_0}^{\infty} p(t) \| Z(t) \|_{\mathcal{Y}^N} \, dt, \tag{48}
\]

where the norm \( \| \cdot \|_E \) is introduced according to (29).

We present the operator \( \Pi \) in a slightly different form. Assume that the vector function \( w(t) \in \mathcal{C}^N \) is the solution of system (25), in which \( H(t) \) represents sum (45). From the form of \( \hat{H}(t) \) (see formulas (33) and (34)) with allowance for Corollary 1, we conclude that elements of the row vector \( \hat{H}(s) \) belong to the domain of definition of the operator \( A \). Then, having in mind the absolute continuity of the functions \( v_i(t), v_s(t) \) and \( [18, \text{p. 4, Theorem 2.4}] \), we come to the conclusion that

\[
\hat{H}(t)w(t) = T(t - t_0)\hat{H}(t_0)w(t_0) + \int_{t_0}^{t}(T(t - s)\hat{H}(s)w(s)) ds, \tag{49}
\]

where

\[
\frac{d}{ds} (T(t - s)\hat{H}(s)w(s)) = -(T(t - s)A\hat{H}(s)w(s) + T(t - s)\frac{d}{ds} \hat{H}(s)w(s). \tag{50}
\]

Taking into account (25), (44), and (45), we have

\[
\frac{d}{ds} (T(t - s)\hat{H}(s)w(s)) = T(t - s)((1 - P)G(s)(\Phi + \hat{H}(s)) + H \{ PG(s)Z(s), \Psi \} + \hat{R}(s))w(s). \tag{51}
\]

We present the vector function \( w(t) \) as \( w(t) = W_{H+Z}(t, t_0)w(t_0) \), where \( W_{H+Z}(t, s) (t, s \geq t_0) \) is the Cauchy matrix of system (25) \( W_{H+Z}(t, s) = I \) \), in which \( H(t) \) represents sum (45). We substitute (51) in (49), return to (47), and obtain (with allowance for (28)) the following representation for the operator \( \Pi \):

\[
\Pi Z(t) = T(t - t_0)Z(t_0)U_{H+Z}(t_0, t) + \int_{t_0}^{t} T(t - s)((1 - P)G(s)Z(s) - \hat{H}(s) \{ PG(s)Z(s), \Psi \} - \hat{R}(s))W_{H+Z}(s, t) ds. \tag{51}
\]

In the same way as is done in proving Theorem 2 in [5] (see also [17, Theorem 4.4]), it can be established that the operator \( \Pi \) maps a certain closed ball \( \| Z \|_{\mathcal{C}_E \mathcal{L}} \leq r_0 \) of the space \( \mathcal{C}_E \mathcal{L} \) to itself and represents
contracting operator in this ball if \( t_0 \) is sufficiently large and \( \|Z(t_0)\|_{\beta^\alpha} \) is sufficiently small. Here, it turns out that not only the function \( p(t)\|Z(t)\|_{\beta^\alpha} \), but also the function \( \|Z(t)\|_{\beta^\alpha} \) belongs to the class \( L_1([t_0, \infty)) \). Note that estimates (6) and (30) play a significant role in proving.

**Corollary 2.** Assume that we have the estimate

\[
\sum_{1 \leq i \leq n} |V_i(t)\| + \sum_{i=1}^n |\gamma(t)| + \beta(t) \leq \phi(t), \quad t \geq t_0,
\]

where \( \phi(t) \) is a certain function positive for \( t \geq t_0 \), while the function \( \gamma(t) \) is defined by (3). Suppose there exists \( \beta(t_0) \leq \phi(t_0) \), \( t_0 \leq t_1 \leq t_2 \).

Then for the row vector \( Z(t) \) from representation (45) when \( t \geq t_0 \) we have the inequality

\[
\|Z(t)\|_{\beta^\alpha} \leq K\phi(t)
\]

with a certain constant \( K \).

The proof of this proposition is absolutely identical to the establishment of the analogous result from [5]. The critical manifold \( \{W(t) \} \) has the property of global attraction in the following sense.

**Theorem 3.** Suppose \( u(t) \) is a weak solution of Eq. (1), i.e., a solution Eq. (4), determined for \( t \geq T \geq t_0 \).

Then there exists a sufficiently large \( t_0 \geq T \) such that for \( t \geq t_0 \) the following asymptotic representation holds:

\[
u(t) = \Phi_w(t)w(t) + H(t)w(t) + O(e^{-(\alpha + \epsilon)\gamma}), \quad t \to \infty.
\]

Here, \( \alpha > 0 \) is chosen on account of inequality (6), \( \epsilon \in (0, \alpha) \) is an arbitrary value, and \( w(t) \) is a certain solution of the system on critical manifold (25).

**Proof.** On account of (13) and (18), the solution \( u(t) \) can be presented as

\[
u(t) = \Phi w(t) + u(t_0(t)), \quad t \geq t_0,
\]

where the function \( u(t_0) \), which takes values from the subspace \( \mathcal{Y} \), satisfies Eq. (17) with \( t_0 = t_0 \), while the function \( w(t) \) is the solution of system (23) with the initial condition \( w(t_0) = \{pu(t_0), \Psi\} \). Suppose \( \{W(t) \} \) is a critical manifold (for Eq. (1)) existing, due to Theorem 1, for \( t \geq t_0 \), where \( t_0 \) is sufficiently large. We note that this manifold is determined by formula (10). Suppose \( w_H(t) \) is a solution of the system on critical manifold (25) with the initial condition \( w_H(t_0) = w(t_0) \); then, the function

\[
\tilde{u}(t) = \Phi w_H(t) + H(t)w_H(t)
\]

represents a certain weak solution of Eq. (1), which lies for \( t \geq t_0 \) on the manifold \( \{W(t) \} \). We show that

\[
u(t) = \tilde{u}(t) + O(e^{-(\alpha + \epsilon)\gamma}).
\]

Assuming that

\[
\rho(t) = u_{\rho}(t) - H(t)w_H(t), \quad r(t) = w(t) - w_H(t)
\]

and subtracting (54) from (53), we obtain

\[
u(t) - \tilde{u}(t) = \Phi r(t) + z(t), \quad t \geq t_0.
\]

Note that the function \( H(t)w_H(t) \) satisfies l Eq. (26) with \( w(t) = w_H(t) \); subtracting (26) from (17) (with \( t_0 = t_0 \)) and taking into account (53), we obtain the following equation for finding \( z(t) \):

\[
z(t) = T(t - t_0)z(t_0) + \int_{t_0}^t T(t - s)(1 - P)G(s)(\Phi r(s) + z(s))ds, \quad t \geq t_0.
\]

We can consider that \( \|z(t_0)\|_{\beta^\alpha} \) is so small as we need. In fact, on account of the linearity of Eq. (4), we can always pass from the solution \( u(t) \) to considering the solution \( \delta u(t)/\|u(t_0)\|_{\beta^\alpha} \) for any preassigned \( \delta > 0 \).
Subtracting (25) (where \( w(t) = w_p(t) \)) from (23) and taking into account (53), we conclude that the vector function \( r(t) \) is the solution of the following Cauchy problem:

\[
\dot{r} = Dr(t) + \langle PG(t) (\Phi r(t) + z(t)), \Psi \rangle, \quad r(t_0) = 0. \tag{57}
\]

Without limiting the generality, we can assume that the matrix \( D \) has the Jordan normal form. Hence, it can be presented as

\[
D = D_1 + D_2,
\]

where \( D_1 = \text{diag} \; D \) and \( D_2 \) is a nilpotent matrix. Note that we can always suppose \( |D_2| < \delta \) for any preassigned \( \delta > 0 \). In fact, it is possible to perform in Eq. (57) a replacement with the constant matrix \( r = C_\delta \hat{r} \), where the matrix \( C_\delta \) brings the matrix \( \delta^{-1}D \) to the Jordan form. Then this replacement leaves the matrix \( D_1 \) without changes and different from zero elements of the matrix \( D_2 \) can represent only elements \( d_{i,i+1} = \delta \). Passing from (57) to the corresponding integral equation, we obtain

\[
r(t) = \int_{t_0}^{t} e^{D_1(t-s)} [D_2 r(s) + \langle PG(s) (\Phi r(s) + z(s)), \Psi \rangle] ds. \tag{58}
\]

Consider the space \( \mathcal{D} \) whose elements are pairs \((z(t), r(t))\). The functions \( z(t) \) and \( r(t) \) are continuous when \( t \geq t_0 \). Consider that the initial element \( z(t_0) \) is fixed and belongs to the space \( \mathcal{D} \). In addition, the following inequalities hold:

\[
\|z(t)\|_\mathcal{D} \leq Ke^{(\alpha+\gamma)(t-t_0)}, \quad \|r(t)\| \leq Ke^{(\alpha+\gamma)(t-t_0)}, \quad t \geq t_0, \tag{59}
\]

where a certain constant \( K > 0 \) and arbitrarily taken \( \varepsilon \in (0, \alpha) \) are used. The space \( \mathcal{D} \) becomes the Banach space if we introduce in it the norm by the rule

\[
\|z(t), r(t)\|_\mathcal{D} = \sup_{t \geq t_0} e^{(\alpha-\varepsilon)(t-t_0)} (\|z(t)\| + \|r(t)\|).
\]

Note that if system (56) and (58) has a solution \((z(t), r(t)) \in \mathcal{D}\), then Eq. (58) can be written in the following equivalent form. In this equation, direct the variable \( t \) to infinity; take into account the right of inequalities in (59), along with the fact that all eigenvalues of the matrix \( D_1 \) have zero real parts. We obtain

\[
\int_{t_0}^{\infty} e^{-D_1 t} [D_2 r(s) + \langle PG(s) (\Phi r(s) + z(s)), \Psi \rangle] ds = 0.
\]

Using this ratio in (58), rewrite the last expression:

\[
r(t) = -\int_{t_0}^{\infty} e^{-D_1 t} [D_2 r(s) + \langle PG(s) (\Phi r(s) + z(s)), \Psi \rangle] ds, \quad t \geq t_0. \tag{60}
\]

We present system of Eqs. (56) and (60) in the form of an operator equation in the space \( \mathcal{D} \):

\[
(z(t), r(t)) = \Sigma(z(t), r(t)) = (\Sigma_1(z(t), r(t)), \Sigma_2(z(t), r(t))), \tag{61}
\]

where the operators \( \Sigma_1 : \mathcal{D} \to \mathcal{Y} \) and \( \Sigma_2 : \mathcal{D} \to \mathbb{C}^N \) are defined by the right sides of Eqs. (56) and (60), respectively. It can be shown that the operator \( \Sigma \) is contractive in the space \( \mathcal{D} \) if \( \|z(t_0)\|_\mathcal{D} \) is sufficiently small, \( t_0 \) is sufficiently large, and the constant \( K \) in (59) is suitably chosen. This is done in exactly the same way as in the proof of Theorem 3 in [5] (see also [17, Theorem 4.7]).

Suppose \( w^{(1)}(t), \ldots, w^{(N)}(t) \) are fundamental solutions of the system on critical manifold (25), while \( u(t) \) is an arbitrary weak solution of Eq. (1) (this solution is determined for \( t \geq T \)). Then from Theorem 3, the following asymptotic representation holds when \( t \to \infty \):

\[
u(t) = (\Phi + H(t)) \sum_{i=1}^{N} c_i w^{(i)}(t) + O(e^{-\alpha t}), \tag{62}
\]
where $c_1, \ldots, c_N$ are arbitrary complex constants and $\tilde{\alpha} > 0$ is a certain real number. Thus, the question of constructing asymptotics for weak solutions of Eq. (1) is reduced to the problem of asymptotic integration of the $N$-dimensional system of ordinary differential Eqs. (25).

With allowance for Theorem 3 and formulas (33) and (34), which define the row vector $\hat{H}(t)$, the system on the critical manifold has the following form:

$$\dot{w} = [D + \sum_{i=1}^{n} v_i(t)D_i(t) + \sum_{1 \leq i_1 \leq \ldots \leq i_N} v_{i_1}(t)v_{i_2}(t)D_{i_1i_2}(t) + \ldots + \sum_{1 \leq i_1, \ldots, i_N \leq n} v_{i_1}(t) \cdots v_{i_N}(t)D_{i_1 \cdots i_N}(t) + L(t)]w, \ w \in \mathbb{C}^N.$$  \hspace{1cm} (63)

In this system, $N$-by-$N$ matrices $D_{i_1 \cdots i_N}(t)$ represent matrices whose elements are trigonometric polynomials, i.e., matrices of type (8), where $b_{i_1 \cdots i_N}^{(l_1 \cdots l_N)}$ are certain constant matrices. In addition, $L(t)$ is a certain matrix from the class $L_1[t_0, \infty)$. System (63) belongs to the class of systems with oscillatory decreasing coefficients. The asymptotic method for integrating systems of such a type is proposed in [4]. The essence of this method is to conduct some special substitutions, which diagonalize the main part of system (63). More precisely, with the help of such substitutions, system (63) is brought to the $L$-diagonal form. The asymptotics of fundamental solutions of $L$-diagonal systems can be built using the theorem of N. Levinson (see [2, 12, 16]). The detailed description of the method for asymptotic integration of systems having form (63) can be found in [4, 5, 17].

**EXAMPLE**

As a simple example that illustrates the use of the technique described above, consider the perturbed heat equation

$$\frac{\partial u}{\partial t} = \Delta u + g(x)\frac{\sin \omega t}{t^\rho}u, \ x \in \Omega, \ t \geq t_0 > 0$$  \hspace{1cm} (64)

with the initial condition

$$u(t_0, x) = \varphi(x)$$  \hspace{1cm} (65)

and the Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0, \ x \in \partial \Omega.$$  \hspace{1cm} (66)

Here the function $u(t, x)$ is considered in a bounded area $\Omega$ of space $\mathbb{R}^m$ with a smooth boundary $\partial \Omega$. We denote by $\partial u/\partial \nu$ the derivative along the exterior normal to $\partial \Omega$. The real-valued functions $\varphi(x)$ and $g(x)$ are considered as belonging to the space $L_2(\Omega)$, the parameters $\omega$ and $\rho$ are positive, and $\Delta$ is the Laplace operator in components of the vector $x$. The question of constructing the asymptotics of solutions of Eq. (64) with conditions (65) and (66) is discussed in [15]. Here we show another approach to this problem.

We consider initial boundary value problem (64)—(66) in the Hilbert space $\mathcal{B} = L_2(\Omega)$. As the domain of the operator $\Delta$ we consider the set $C_0^2(\Omega)$ of twice continuously differentiable in $\Omega$ functions that meet boundary condition (66) in the boundary $\partial \Omega$. It is known (see, e.g., [19, 20]) that the operator $\Delta$ defined in such a way, allows in the space $L_2(\Omega)$ the closure in the form of the operator $A$. In turn, the operator $A$ is a generator of a strongly continuous (even analytic) compact semigroup of the operators $T(t)$. Note that the point spectrum of the operator $(-A)$ has the form

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots.$$  

For this reason, present the space $L_1(\Omega)$ as direct sum (5), taking for the space $\mathcal{X}$ the eigenspace (of the operator $A$) that corresponds to the eigenvalue $\lambda_0$. Then $\mathcal{X} = \{f(x) \equiv \text{const}, x \in \Omega\}$. Because $\mathcal{B}$ is a Hilbert space, $\mathcal{B}^* = \mathcal{B}$ and duality bracket (11) is a scalar product in the space $\mathcal{B}$

$$(\varphi, \psi) = \int_{\Omega} \varphi(x)\overline{\psi(x)}dx.$$
Note that $A$ is a self-conjugate operator; therefore, $A^* = A$. The elements $Φ$ and $Ψ$ of the space $𝔅$ we choose in such a way that normalization condition (12) holds:

$$Φ(x) ≡ 1, \ Ψ(x) ≡ \frac{1}{|Ω|},$$

(67)

where $|Ω|$ is the Lebesgue measure of the set $Ω$. Finally, we put

$$ℭ = B^{-1} = \{y(x) ∈ B|ι(y(x), Ψ) = 0\},$$

(68)

Then the validity of estimate (6) follows from Lemmas 7.4.1 and 7.4.2 in [7] and the compactness of the semigroup $T(t)$.

Since from (68) for any $u ∈ B$ we have the equality

$$(u, Ψ) = (Pu, Ψ),$$

equation on the critical manifold (25) in the considered case is presented as follows:

$$w = \frac{sin \omega t}{\rho} (g(x)(Φ + H(t, x)), Ψ)w(t), \ \ t ≥ t_0, \ \ w ∈ ℝ.$$

(69)

Here the function $H(t, ·)$ for all $t ≥ t_0$ belongs to the space $L_1(Ω)$; in addition, $\|H(t, ·)\|_{L_1(Ω)} → 0$ for $t → ∞$. By Theorem 2, for this function we have the following representation:

$$H(t, x) = H(t, x) + Z(t, x), \ \ t ≥ t_0 ≥ t_0, \ \ -h ≤ θ ≤ 0.$$  

(70)

Here the function $Z(t, ·)$, which belongs to the subspace $ℭ$, is such that $\|Z(t, ·)\|_{L_1(Ω)} → 0$ when $t → ∞$ and $\|Z(t, ·)\|_{L_1(Ω)} ∈ L_1[t_0, ∞)$. The function $\hat{H}(t, x)$, which belongs in the variable $x$ to the subspace $ℭ$, represents an approximate solution of the equation

$$\frac{∂H}{∂t} = AH + g(x)\frac{sin \omega t}{\rho} (Φ + H(t, x)) - \frac{sin \omega t}{\rho} (Φ + H(t, x))(g(x)(Φ + H(t, x)), Ψ)$$

(71)

with an accuracy of terms $\hat{R}(t, x)$ such that $\|\hat{R}(t, ·)\|_{L_1(Ω)} ∈ L_1[t_0, ∞)$. Here we also take into account the fact that for any $u ∈ B$ the following equality holds:

$$u - Pu = u - Φ(u, Ψ).$$

(72)

We seek the function $\hat{H}(t, x)$ as

$$\hat{H}(t, x) = t^\varphi H_1(t, x) + t^{\varphi} H_2(t, x) + ..., $$

(73)

where the functions $H_1(t, x), H_2(t, x), ...$ in the variable $x$ belong to the subspace $ℭ$ and represent trigonometric polynomials of the variable $t$. We substitute expression (73) in Eq. (71) and gather the terms with $t^\varphi$; in doing so, we omit terms $\hat{R}(t, x)$ such that $\|\hat{R}(t, ·)\|_{L_1(Ω)} ∈ L_1[t_0, ∞)$. By using (67), we obtain the following equation for finding the function $H_1(t, x)$:

$$\frac{∂H_1}{∂t} = AH_1 + g(x)sin \omega t - sin \omega t(g(x), Ψ).$$

(74)

The solution of Eq. (74) we seek in the form

$$H_1(t, x) = h_1(x)e^{iωt} + \bar{h}(x)e^{-iωt},$$

(75)

where the function $h_1(x)$ belongs to the subspace $ℭ$ of the space $L_2(Ω)$. Substituting (75) in (74) and gathering the terms with $e^{iωt}$, we obtain the equation for the function $h_1(x)$:

$$Ah_1 - iωh_1 = \frac{ig(x)}{2} - \frac{1}{2}(g(x), Ψ).$$

(76)

By Lemma 1, this equation has a unique solution in the subspace $ℭ$. Using (70), (73), and (75) in Eq. (69), with allowance for Corollary 2, we obtain the following representation for the equation on a critical manifold:
\[
\dot{w} = [t^{-p}a_1(t) + t^{-2p}a_2(t) + O(t^{-3p}) + O(t^{-2p+1})]w(t).
\]  
(77)

Here,

\[
a_1(t) = \sin \omega t (g(x) \Phi, \Psi) = \frac{\sin \omega t}{|\Omega|} \int_{\Omega} g(x)dx,
\]  
(78)

\[
a_2(t) = \sin \omega t (g(x)H_1(t, x), \Psi) = \frac{\sin \omega t}{|\Omega|} \left[ e^{i\omega t} \int_{\Omega} g(x)h_1(x)dx + e^{-i\omega t} \int_{\Omega} g(x)h_2(x)dx \right].
\]  
(79)

Following the methodology of the work [4], we perform the averaging change of variable in Eq. (77)

\[
w = [1 + t^{-p}y_1(t) + t^{-2p}y_2(t) + \ldots + t^{-kp}y_k(t)]w_1(t).
\]  
(80)

Here, the nonnegative integer parameter \( k \) is chosen in such a way that \( kp \leq 1 < (k + 1)p \) and the functions \( y_1(t), \ldots, y_k(t) \) are trigonometric polynomials with the zero mean value. As a result of this change, Eq. (77) is converted to the averaged form

\[
\dot{w}_1 = [t^{-p}a_1 + t^{-2p}a_2 + O(t^{-3p}) + O(t^{-2p+1})]w_1(t).
\]  
(81)

In this case,

\[
a_1 = M[a_1(t)], \quad a_2 = M[a_2(t) + a_1(t)y_1(t)],
\]

where by \( M[\cdot] \) we denote the mean value of the almost periodic function \( p(t) \); i.e.,

\[
M[p(t)] = \lim_{T \to \infty} \frac{1}{T} \int_0^T p(t)dt.
\]

The trigonometric polynomial \( y_1(t) \) is defined as the solution of the equation

\[\dot{y}_1 = a_1(t) - a_1\]

with the zero mean value. By the simple calculations using formulas (78) and (79) we obtain the following expressions for \( a_1 \) and \( a_2 \):

\[
a_1 = 0, \quad a_2 = M[a_2(t)] = \frac{1}{|\Omega|} Re \left\{ i \int_{\Omega} g(x)h_1(x)dx \right\}.
\]  
(82)

We show that \( a_2 \) is a nonnegative quantity. Note that because the function \( h_1(x) \) belongs to the subspace \( \mathcal{Y} \), we have by (68)

\[
(h_1, \Psi) = \frac{1}{|\Omega|} \int_{\Omega} h_1(x)dx = 0.
\]  
(83)

We calculate the complex conjugation of both sides of Eq. (76) and multiply each of them by the function \( h_1(x) \). Multiplying scalarly each side by the element \( \Psi \) and equating the real parts of the obtained equality, with allowance for (83) and the self-conjugacy of the operator \( A \), we obtain the expression

\[
a_2 = -\frac{2}{|\Omega|} Re \left\{ \int_{\Omega} Ah_1(x)\overline{h_1(x)}dx \right\}.
\]  
(84)

First assume that the function \( h_1(x) \) belongs to the domain of definition of the operator \( \Delta \). Therefore, in (84), the equality \( Ah_1 = \Delta h_1 \) holds. Using the formula of integration by parts for multidimensional cases, we have

\[
\int_{\Omega} \Delta h_1(x)\overline{h_1(x)}dx = \int_{\Omega} \frac{\partial h_1}{\partial \nu} \overline{h_1(x)}d\sigma - \int_{\Omega} \nabla h_1(x)\nabla \overline{h_1(x)}dx,
\]  
(85)

where \( d\sigma \) is a measure on \( \partial \Omega \). Since \( h_1(x) \) belongs to the domain of definition of the operator \( \Delta \), this function satisfies boundary condition (66). Thus,

\[
\int_{\Omega} \Delta h_1(x)\overline{h_1(x)}dx = -\int_{\Omega} |\nabla h_1(x)|^2dx.
\]  
(86)
By the definition of the operator \(A\), passing to the limit in (86), it is easy to see that for any function \(h\) from the definition domain of this operator, the following equality holds:

\[
\int_{\Omega} Ah(x)h(x)dx = -\int_{\Omega} \nabla h(x)^2 dx.
\]

(87)

Finally, we obtain

\[
a_2 = \frac{2}{|\Omega|} \int_{\Omega} \nabla h(x)^2 dx \geq 0.
\]

(88)

Now integrate Eq. (81) with account of formulas (82) and (88). We obtain the following asymptotic representations for its solutions when \(t \to \infty\):

\[
w(t) = \begin{cases} 
  c(1 + O(t^\rho)), & \rho \geq 1, \\
  c(1 + O(t^{-2\rho+1})), & \frac{1}{2} < \rho < 1, \\
  c(1 + O(t^{-1/2})) t^{d_2}, & \rho = \frac{1}{2}, \\
  c(1 + O(t^{-2\rho+1})) \exp\left\{ \frac{a_2}{1-2\rho} t^{-2\rho+1} \right\}, & \frac{1}{3} < \rho < \frac{1}{2}, \\
  c(1 + O(t^{-k+1/\rho+1})) \exp\left\{ \frac{a_2}{1-2\rho} t^{-2\rho+1} (1 + o(1)) \right\}, & \frac{1}{k+1} < \rho \leq \frac{1}{k}, \quad k \geq 3,
\end{cases}
\]

(89)

where \(c\) is an arbitrary real constant. By Theorem 3, with allowance for (67), for weak solutions of original problem (64)–(66), we have the following asymptotic representation when \(t \to \infty\):

\[
u(t, x) = (1 + H(t, x)) w(t) + r(t, x).
\]

(90)

Here, the function \(w(t)\) is determined by formulas (89), the function \(H(t, x)\) has the property \(\|H(t, x)\|_{L_2(\Omega)} \to 0\) when \(t \to \infty\), and the function \(r(t, x)\) allows the estimation \(\|r(t, x)\|_{L_2(\Omega)} = O(e^{-\alpha t})\) (where \(\alpha > 0\) is a certain real number).

Analyzing formulas (89) and (90), we come to the conclusion that for \(\rho > 1/2\) all solutions of problem (64)–(66) are bounded as \(t \to \infty\) (in the norm of the space \(L_2(\Omega)\)), while in the case \(\rho \leq 1/2\), the solutions, generally speaking, are unbounded. Especially note that the existence of unbounded solutions is a consequence of the spatial heterogeneity of the perturbation factor in Eq. (64). In fact, suppose that in this equation \(g(x) \equiv g = \text{const}\). Then integral Eq. (26) with allowance for formulas (67) and (72), has the solution \(H(t, x) \equiv 0\). Therefore, equation on the critical manifold (69) takes the form

\[
\dot{w} = \frac{g \sin \omega t}{r^p} w(t).
\]

It is easy to show that all solutions of this equation have the following asymptotic representation as \(t \to \infty\):

\[
w(t) = c(1 + O(t^\rho)),
\]

where \(c\) is an arbitrary real constant.

### FUNDING

The reported study was funded by RFBR according to the research project 18-29-10055.

### CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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*Translated by L. Kartvelishvili*