THE DOORWAYS PROBLEM AND STURMIAN WORDS

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Abstract. The doorways problem considers adjacent parallel hallways of unit width each with a single doorway (aligned with integer lattice points) of unit width. It then asks, what are the properties of lines that pass through each doorway? Configurations of doorways closely correspond to Sturmian words, and so properties of these configurations may be lifted to properties of Sturmian words. This paper classifies the slopes of lines of sight, lines that pass through each doorway, for both the case of a finite number of parallel hallways and an infinite number and their consequences for Sturmian words. We then produce a metric on configurations with an infinite number of hallways that preserves the property of admitting a line of sight under limits. Pulling back this metric to \( \mathbb{R} \), we produce the Baire metric under which the irrational numbers form a complete metric space. Pulling back this metric to the set of all Sturmian sequences, we show that the set of all Sturmian sequences is complete with this metric (unlike with the standard metric).

1. The Doorways Problem

Imagine a series of \( n + 1 \) infinitely long parallel walls spaced one unit apart, creating \( n \) hallways. Further imagine that each wall has infinitely many doors of unit width, but that only one door per wall is open.

Standing to one side of the hallways, you could imagine certain arrangements of open doors you could see through and certain arrangements you could not. This is precisely stated in the following definitions.

Definition 1 (Hallway). An \( n \)-hallway is the set \( H_n \subset \mathbb{R}^2 \) defined by

\[
H_n = \bigcup_{i \in \{0, \ldots, n\}} \{i\} \times (\mathbb{R} \setminus D_i)
\]

where \( D_i = (d_i, d_i + 1) \) is an open interval of width one and left point \( d_i \in \mathbb{Z} \). The set \( D_i \) is called the \( i \)th doorway.

Definition 2 (Line of Sight). Given an \( n \)-hallway \( H_n \), we can see through \( H_n \) if there exists some line \( \ell_{\alpha\beta} = \{(x, \alpha x + \beta) : x \in \mathbb{R}\} \) with slope \( \alpha \) and \( y \)-intercept \( \beta \) so that \( \ell_{\alpha\beta} \cap H_n = \emptyset \). If \( \ell_{\alpha\beta} \cap H_n = \emptyset \), we call \( \ell_{\alpha\beta} \) a line of sight and we say \( H_n \) admits the line of sight \( \ell_{\alpha\beta} \). If \( \ell_{\alpha\beta} \) is a line of sight and \( \alpha \in \mathbb{Q} \), we call \( \ell_{\alpha\beta} \) a rational line of sight.

Note that Definition 2 captures the idea that “no light is blocked” by a hallway. This could be equivalently phrased as an \( n \)-hallway \( H_n \) with doorways \( D_i \) admits the line of sight \( \ell_{\alpha\beta} \) if \( \ell_{\alpha\beta} \cap \{(i) \times D_i\} \neq \emptyset \) for all \( i \), which would capture the idea that “light passed through every doorway.” However, upon the introduction of infinite hallways, the “no light is blocked” definition will be more useful.

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The doorways problem in general asks what types of \( n \)-hallways can be seen through, and what are the properties of lines of sight. This question is closely related to rotation sequences, balanced sequences, and Sturmian sequences \([2,4]\), and it is from this context that the following motivating question arises.

**Question 3.** For an \( n \)-hallway \( H_n \) that can be seen through, is there always a line of sight \( \ell_{\alpha,\beta} \) with slope \( \alpha = \frac{p}{q} \) where \( q \leq n \)?

1.1. **Connection to Sturmians.** Sturmian sequences and Sturmian words have many equivalent definitions in terms of rotation sequences, billiard sequences, balanced words, complexity, and invariant measures \([2,3,4]\). For the sake of brevity, we provide only two equivalent definitions.

**Definition 4 (Complexity).** For a sequence \( x \in \{0,1\}^\mathbb{N} \), the complexity function is

\[
L_n(x) = \#\{ \text{distinct subwords of } x \text{ of length } n \}.
\]

**Definition 5 (Periodic and Eventually Periodic).** For a sequence \( x \in \{0,1\}^\mathbb{N} \), let \( (x)_i \) be the \( i \)th coordinate of \( x \). The sequence \( x \in \{0,1\}^\mathbb{N} \) is called periodic if there exists \( m > 0 \) so that \( (x)_i = (x)_{i+m} \) for all \( i \) and is called aperiodic otherwise. The sequence is called eventually periodic if there exists \( m > 0 \) and some \( I \) so that \( (x)_i = (x)_{i+m} \) for all \( i > I \).

**Definition 6 (Sturmian Sequence).** Let \( x \in \{0,1\}^\mathbb{N} \). The sequence \( x \) is a Sturmian sequence if it is periodic and satisfies \( L_n(x) \leq n + 1 \) for all \( n \) or if it satisfies \( L_n(x) = n + 1 \) for all \( n \). A Sturmian word is a subword of a Sturmian sequence.

A sequence \( x \in \{0,1\}^\mathbb{N} \) satisfying \( L_n(x) = n + 1 \) is always aperiodic and never eventually periodic. Thus, an eventually periodic Sturmian sequence must be periodic. Hedlund and Morse \([\text{5}]\) proved that for any \( x \in \{0,1\}^\mathbb{N} \), \( x \) is eventually periodic if and only if there exists an \( n \) such that \( L_n(x) < n + 1 \). Viewed this way, aperiodic Sturmian sequences are the aperiodic sequences of the lowest possible complexity.

**Definition 7 (Rotation Sequence).** For a pair \( (\alpha, \beta) \in [0,1]\times\mathbb{R} \), the rotation sequences \( s = R_{[c]}(\alpha, \beta) \in \{0,1\}^\mathbb{N} \) and \( s' = R_{[-]}(\alpha, \beta) \in \{0,1\}^\mathbb{N} \) are the sequences whose \( i \)th coordinates are given by

\[
(s)_i = \lfloor (i+1)\alpha + \beta \rfloor - \lfloor i\alpha + \beta \rfloor
\]

and

\[
(s')_i = \lceil (i+1)\alpha + \beta \rceil - \lceil i\alpha + \beta \rceil,
\]

where \( \lfloor \cdot \rfloor \) and \( \lceil \cdot \rceil \) are the floor and ceiling functions, respectively. A sequence \( x \in \{0,1\}^\mathbb{N} \) is called a rotation sequence if \( x = R_{[c]}(\alpha, \beta) \) or \( x = R_{[-]}(\alpha, \beta) \) for some \( (\alpha, \beta) \in [0,1]\times\mathbb{R} \).

As shown in \([2,4]\), a sequence \( x \in \{0,1\}^\mathbb{N} \) is Sturmian if and only if it is a rotation sequence. Further, every Sturmian word appears as the starting word of a rotation sequence (equivalently Sturmian sequence).

Given an \( n \)-hallway \( H_n \) with doorways \( D = (d_i, d_i + 1) \), there is a natural correspondence between \( H_n \) and elements in \( \mathbb{Z}^n \). Namely, associate \( H_n \) with the \( n \)-word \( (d_1 - d_0, d_2 - d_1, \ldots, d_n - d_{n-1}) \) given by the differences between positions of consecutive doorways. Let \( \Phi : \{\text{hallways}\} \to \{\text{words}\} \) denote this correspondence.
The question of whether a hallway admits a line of sight only depends on the relative placement of each doorway and is therefore translation invariant. Thus $H_n$ admits a line of sight if and only if every hallway in $\{H'_n : \Phi(H'_n) = \Phi(H_n)\}$ admits a line of sight.

Fix an $n$-hallway $H_n$ with initial doorway $D_0 = (0, 1)$ and suppose $\Phi(H_n)$ is a Sturmian word. Further suppose $\Phi(H_n)$ appears as the initial word for the rotation sequence $s = R_{\lfloor \cdot \rfloor}(\alpha, \beta)$ and that $(i\alpha + \beta) \not\in \mathbb{Z}$ for $0 \leq i \leq n$. We can now conclude that

$$D_i = ([i\alpha + \beta], [i\alpha + \beta] + 1)$$

and $H_n$ admits the line of sight $\ell_{\alpha\beta}$. The converse of this statement also holds, and with the technical assumptions minimized, we get Theorem 8.

**Theorem 8.** Let $\Psi_a : \{a, a+1\}^n \to \{0, 1\}^n$ be the map that sends $a \mapsto 0$ and $(a+1) \mapsto 1$. The $n$-hallway $H_n$ admits a line of sight if and only if $\Phi(H_n) \in \{a, a+1\}^n$ for some $a$ and $\Psi_a \circ \Phi(H_n)$ is a Sturmian word.

We will not prove Theorem 8 in the context of Sturmian sequences, however, studying the hallway problem directly we will arrive at equivalent results. Theorem 8 also gives context as to why Question 3 might be interesting.

Consider the following: given a finite Sturmian word $w$, is $w$ always contained in a periodic Sturmian word? If so, what is the minimum period of such a word? Translating from hallways to rotation sequences to Sturmian sequences, Question 3 asks, “Is a finite Sturmian word $w$ always contained in a periodic Sturmian sequence with period bounded by the length of $w$?”

Studying $n$-hallways will provide a geometric way to answer this question. Further, the extension of $n$-hallways to infinite hallways will allow us to arrive at several results without the subtleties of working with Sturmian sequences or rotation sequences directly. In particular, the distinction between *aperiodic* and *not eventually periodic* and the need to include both $[\cdot]$ and $\lceil \cdot \rceil$ (as in the definition of rotation sequences) is avoided.

## 2. Answering the Question

As discussed earlier, the question of whether an $n$-hallway admits a line of sight is translation invariant. Thus, we will assume that all $n$-hallways satisfy $D_0 = (0, 1)$. Now, we will tackle the question of whether or not there exists lines of sight.

**Definition 9.** Let $\text{proj}_\gamma : \mathbb{R}^2 \to \mathbb{R}$ be parallel projection onto the $y$-axis along a line of slope $\gamma$. That is,

$$\text{proj}_\gamma(x, y) = y - \gamma x.$$

**Proposition 10.** If $H_n$ is an $n$-hallway that admits a line of sight, then there is an interval of slopes corresponding to lines of sight for $H_n$.

**Proof.** Fix $H_n$, an $n$-hallway, and let $\ell_{\alpha\beta}$ be a line of sight. We now have $\text{proj}_\alpha(\ell_{\alpha\beta}) = \{\beta\}$. Let $D_i = (d_i, d_i + 1)$ be the $i$th doorway of $H_n$, and let

$$D = \bigcap_{0 \leq i \leq n} \text{proj}_\alpha(\{i\} \times D_i).$$
Since \( \ell_{\alpha \beta} \) is a line of sight, \( \beta \in D \neq \emptyset \). Since \( D \) is a finite intersection of open intervals, \( D = (d_l, d_r) \) is an open interval. It directly follows that the “tube” \( T = \text{proj}_{-1}^{-1}(D) \cap ([0, n] \times \mathbb{R}) \) safely passes through every doorway in \( H_n \), and as a consequence any line contained in \( T \) will be a line of sight. See Figure 1 for an example.

![Figure 1. A hallway with four doorways and the slope-\( \alpha \) tube \( T = \text{proj}_{-1}^{-1}(D) \cap ([0, n] \times \mathbb{R}) \) along with a line of sight \( \ell \) having slope greater than \( \alpha \).](image)

Since the width of \( T \) is \( n \) and the height of \( T \) is \( d_r - d_l > 0 \), we know there must be lines of sight for every slope in the interval \( (\alpha - \frac{d_r - d_l}{n}, \alpha + \frac{d_r - d_l}{n}) \).

**Corollary 11.** If \( H_n \) is an \( n \)-hallway that admits a line of sight, then \( H_n \) admits a rational line of sight.

Since every interval of real numbers contains a rational number, Corollary 11 follows immediately.

The proof of Proposition 10 gives some insight into what types of hallways admit lines of sight. In particular, a hallway admits a line of sight if and only if \( D_{\alpha} = \bigcap \text{proj}_{\alpha}(\{i\} \times D_i) \) is non-empty for some \( \alpha \).

Suppose \( \alpha \in [0, 1) \), and consider 1-hallways. Recall that we always assume \( D_0 = (0, 1) \) is the first doorway. Now, if \( \ell_{\alpha \beta} \) is a line of sight for a 1-hallway \( H_1 \), because \( 0 \leq \alpha < 1 \), there are only two possibilities for \( D_1 \). Namely, \( D_1 = (0, 1) \) or \( D_1 = (1, 2) \). Since we are assuming \( \ell_{\alpha \beta} \) is a line of sight for \( H_1 \), we can completely determine what \( D_1 \) is by the following procedure: Let \( s_1 = \text{proj}_{\alpha}(1, 1) \). If \( \beta \), the \( y \)-intercept of \( \ell_{\alpha \beta} \), satisfies \( \beta < s_1 \), then \( D_1 = (0, 1) \). If \( \beta > s_1 \) then \( D_1 = (1, 2) \). See Figure 2 for an illustration.

Inductively, we may determine all possible hallways corresponding to lines of sight with a particular slope. For illustration, consider 2-hallways and assume we have a line of sight \( \ell_{\alpha \beta} \). If \( D_1 = (0, 1) \), we know \( D_2 = (0, 1) \) or \( (1, 2) \). To find out which, let \( s_2 = \text{proj}_{\alpha}(2, 1) \). If \( \beta < s_2 \), \( D_2 = (0, 1) \), and if \( \beta > s_2 \), \( D_2 = (1, 2) \). Similarly, if \( D_1 = (1, 2) \), the options for \( D_2 \) are \( (1, 2) \) or \( (2, 3) \). Now letting \( s'_2 = \text{proj}_{\alpha}(2, 2), \beta < s'_2 \) implies \( D_2 = (1, 2) \) and \( \beta > s'_2 \) implies \( D_2 = (2, 3) \).
If we continue this process, we will notice that we always consider \( s = \text{proj}_\alpha(x_0, y_0) \) as a bifurcation point. That is, \( s \) allows us to decide which doorway must be open if we presuppose a certain line of sight.

**Definition 12.** Let \( Y_{\alpha,n} = (0, 1) \setminus \bigcup_{i \leq n} \text{proj}_\alpha(i \times \mathbb{Z}) \) and let \( Y_{\alpha,n} \) be the partition of \( Y_{\alpha,n} \) consisting of its connected components.

As the next proposition shows, \( Y_{\alpha,n} \) exactly classifies which sequence of doors must be open for a given line of sight. In this respect, \( Y_{\alpha,n} \) can be seen as generating an equivalence relation on intercepts of lines of sight with slope \( \alpha \).

**Proposition 13.** Fix \( \alpha \in [0, 1) \) and suppose \( \ell_{\alpha\beta} \) is a line of sight for the \( n \)-hallway \( H_n \) and \( \ell_{\alpha\beta'} \) is a line of sight for the \( n \)-hallway \( H'_n \). Then, \( \beta, \beta' \in Y \) for some \( Y \in Y_{\alpha,n} \) if and only if \( H_n = H'_n \).

**Proof.** Notice that \( H_n = H'_n \) if and only if the sequences of doorways for \( H_n \) and \( H'_n \) are the same.

Suppose \( H_n = H'_n \) and that \( \ell_{\alpha\beta} \) and \( \ell_{\alpha\beta'} \) are lines of sight for \( H_n \) and \( H'_n \), respectively. By construction \( Y = (\text{proj}_\alpha H_n)^c = (\text{proj}_\alpha H'_n)^c \in Y_{\alpha,n} \), and we must have \( \beta, \beta' \in Y \).

Now suppose that \( \beta, \beta' \in Y \) for some \( Y \in Y_{\alpha,n} \). We will proceed by induction.

The base case is clear. \( H_1 = H'_1 \), since \( Y_{\alpha,1} \) precisely partitions the \( y \)-intercepts for lines of sight through \((D_0, D_1) = \{(0, 1), (0, 1)\}\) and \((D_0, D_1) = \{(0, 1), (1, 2)\}\).

Assume the proposition holds for \( n - 1 \). This means that \( D_i = D'_i \) for \( i < n \). Fix \( a \) so that \( D_{n-1} = D'_{n-1} = (a, a+1) \). Since \( \alpha \in [0, 1) \), there can only be two possibilities for \( D_n \) or \( D'_n \). Namely, \( (a, a+1) \) or \( (a+1, a+2) \). Let \( s = \text{proj}_\alpha(n, a+1) \) and suppose \( \ell_{\alpha\gamma} \) is a line of sight for \( H_n \). If \( \gamma < s \), \( D_n = (a, a+1) \) and if \( \gamma > s \), \( D_n = (a+1, a+2) \).
We complete the proof by noticing that \( s \) lies on the boundary of a partition element of \( \mathcal{Y}_{\alpha,n} \) (or completely outside the interval \((0, 1)\)). Thus, if \( \beta, \beta' \in Y \), we have \( \beta, \beta' > s \) or \( \beta, \beta' < s \), and so \( D_n = D'_n \). □

Propositions like Proposition 13 can be extended to handle lines of sights with slopes in \( \mathbb{R} \) without too much trouble, so we will mainly focus on lines of sights with slopes in \([0, 1)\) to make our arguments simpler.

**Corollary 14.** For a fixed line \( \ell_{\alpha \beta} \), there is at most one \( n \)-hallway such that \( \ell_{\alpha \beta} \) is a line of sight.

**Proof.** This follows directly from an application of Proposition 13 with \( \beta = \beta' \). □

**Proposition 15.** For a fixed \( \alpha \), the number of elements in the partition \( \mathcal{Y}_{\alpha,n} \) is at most \( n + 1 \).

**Proof.** First, notice that since \( \text{proj}_\alpha(x, y) - \text{proj}_\alpha(x, y') = y - y' \), for a fixed \( i \), \((0, 1) \cap \text{proj}_\alpha(\{i\} \times \mathbb{Z})\) contains at most one point. This follows from the fact that integers are one unit apart and \((0, 1)\) is an open interval of width one.

Now, we may proceed by induction. Clearly \( \mathcal{Y}_{\alpha,0} = \{(0, 1)\} \) consists of one interval. Suppose \( \mathcal{Y}_{\alpha,n-1} \) consists of no more than \( n \) intervals. \( \mathcal{Y}_{\alpha,n} \) can be obtained from \( \mathcal{Y}_{\alpha,n-1} \) by slicing the partition elements of \( \mathcal{Y}_{\alpha,n-1} \) by the points in \((0, 1) \cap \text{proj}_\alpha(\{n\} \times \mathbb{Z})\). But, there is at most one point in \((0, 1) \cap \text{proj}_\alpha(\{n\} \times \mathbb{Z})\) and so at most one interval in \( \mathcal{Y}_{\alpha,n-1} \) could be sliced into two intervals. Thus the number of elements in \( \mathcal{Y}_{\alpha,n} \) cannot exceed \( n + 1 \). □

**Corollary 16.** For a fixed \( \alpha \), the number of distinct \( n \)-hallways having \( D_0 = (0, 1) \) and admitting a line of sight of slope \( \alpha \) is at most \( n + 1 \).

**Proof.** From Proposition 13, \( \mathcal{Y}_{\alpha,n} \) is in one-to-one correspondence with \( n \)-hallways having lines of sight of slope \( \alpha \). Applying Proposition 15 shows \( |\mathcal{Y}_{\alpha,n}| \leq n + 1 \), which completes the proof. □

Recalling the correspondence between \( n \)-hallways and finite words, Corollary 16 can be applied to show that rotation sequences satisfy the complexity conditions required of Sturmian sequences. We might also ask the total number of \( n \)-hallways admitting lines of slope of any \( \alpha \in [0, 1) \).

**Theorem 17** (Mignosi [6]). Let \( C(n) \) be the number of distinct \( n \)-hallways with \( D_0 = (0, 1) \) and admitting a line of sight with slope in \([0, 1)\). Then,

\[
C(n) = 1 + \sum_{i=1}^{n} (n + 1 - i)\phi(i)
\]

where \( \phi(i) \) is Euler’s totient function, which counts the number of integers in \( \{1, \ldots, i\} \) that are relatively prime to \( i \).

In [6], Mignosi uses combinatoric properties to count subwords of Sturmian sequences, In [1], Berstel and Pocchiola use geometric arguments to arrive at the same conclusion.

Let’s get a slightly better idea of what the set of all \( n \)-hallways looks like.
Definition 18. Let \( S_n \subset [0, 1] \times (0, 1) \) be the set of pairs \((\alpha, \beta)\) such that \(\ell_{\alpha\beta}\) is a line of sight for some \(n\)-hallway. Let \(P_n\) be the partition of \(S_n\) where \((\alpha, \beta)\) and \((\alpha', \beta')\) are in the same partition element if \(\ell_{\alpha\beta}\) and \(\ell_{\alpha'\beta'}\) are lines of sight for the same \(n\)-hallway.

Corollary 14 ensures that \(P_n\) is well defined. Drawing \(P_n\) as a subset of \(\mathbb{R}^2\), we see that the vertical fiber of \(P_n\) with \(x\)-coordinate \(\alpha\) is precisely \(Y_{\alpha,n}\).

Looking at \(Y_{\alpha,n}\) as a function of \(\alpha\), we see that \(Y_{\alpha,n}\) must be split at the point

\[
 f_i(\alpha) = \text{proj}_\alpha(\{i\} \times \mathbb{Z}) = \text{proj}_\alpha(i, 0) \mod 1 = -\alpha i \mod 1
\]

for every \(i \in \{0, \ldots, n\}\). In other words, \(P_n\) looks like \([0, 1] \times (0, 1)\) cut by the lines \((\mod 1)\) of slope \(-i\) for \(i \in \{0, \ldots, n\}\). See Figure 3.

![Figure 3. The partition \(P_5\).](image)

Using this description of \(P_n\), we can answer our question about rational lines of sight.

**Theorem 19.** Given an \(n\)-hallway \(H_n\) admitting a line of sight, there is a rational line of sight \(\ell_{\alpha\beta}\) with \(\alpha = \frac{p}{q}\) and \(q \leq n\).

**Proof.** Fix an \(n\)-hallway \(H_n\) admitting a line of sight at let \(P \in P_n\) be the corresponding partition element. That is, for every \((\alpha, \beta) \in P\), \(\ell_{\alpha\beta}\) is a line of sight for \(H_n\).

Our proof would be complete if we could show that \(P\) contained a point \((\alpha, \beta)\) where \(\alpha = \frac{p}{q}\) with \(q \leq n\). To this end, consider the corners of \(P\), when \(P\) is interpreted as a polygon. Since \(P_n\) is formed by cutting \([0, 1] \times (0, 1)\) by lines of slope \(-1, \ldots, -n\), the edges of \(P\) are segments of lines of the same slope and so the corners are intersections of such lines.

We will now compute the intersection of two lines of the form \(y = -ax \mod 1\). Notice that any connected segment of the graph of such a line is identical to the graph of the line \(y = -ax + b\) restricted to \([0, 1] \times (0, 1)\) for some \(1 \leq b \leq a\).
Let \( L_{-a,b}(x) = -ax + b \) be a line with slope \(-a\) and \(y\)-intercept \(b\). Then, the intersection of the graphs of \( L_{-a,b} \) and \( L_{-a',b'} \) occurs at

\[
x = \frac{b' - b}{a' - a} \quad \text{and} \quad L_{-a,b}(x) = \frac{a'b - ab'}{a' - a}.
\]

Now, if \( a, a' \in \{0, \ldots, n\}, \ |a' - a| \leq n \). This shows that the \(x\)-coordinate of every corner of \( P \) is of the form \( \frac{p}{q} \) with \( q \leq n \).

To complete the proof, notice that either \( P \) is one of the two extreme cases—the triangles with corners \((0, 0), (1, 1), (1/n, 0)\) or \((1, 0), (1, 1), (1/n, 1)\)—or \( P \) has a corner directly above or below its interior (see Figure 3). If \( P \) is the left extremal triangle, then there is a line of sight \( \ell_{0,\beta} \) for some \( \beta \) and if \( P \) is the right extremal triangle, there is a line of sight \( \ell_{1,\beta} \) for some \( \beta \).

Finally, if \( P \) has a corner with coordinates \((\alpha, b)\) above or below its interior, then there must be some point \((\alpha, \beta)\) \(\in P\). Thus \( \ell_{\alpha,\beta} \) is a line of sight and as shown, \( \alpha = \frac{p}{q} \) with \( q \leq n \).

\[\blacksquare\]

### 3. Infinite Hallways

Diagrams like Figure 3 show that if an \(n\)-hallway admits a line of sight \( \ell_{\alpha,\beta} \), then it admits lines of sign \( \ell_{\gamma,\delta} \) for a host of real numbers \( \gamma \) and \( \delta \). However, things become a bit more restricted when we pass to infinite hallways.

An infinite hallway is defined analogously to a finite hallway and can be thought of as the union of finite hallways that get longer and longer. It would now seem natural to say that an infinite hallway \( H^\infty \) has a line of sight if and only if \( \ell_{\alpha,\beta} \cap H^\infty = \emptyset \) for some \( \alpha \) and \( \beta \). However, this definition rules out a very desirable property.

**Desirable Property:**

If \( H^\infty \) is an infinite hallway admitting a line of sight, then there exists a doorway \( D_{-1} = (d_{-1}, d_{-1} + 1) \) such that the infinite hallway \( \{-1\} \times (\mathbb{R} \setminus D_{-1}) \cup H^\infty \) admits a line of sight.

That is, if a hallway admits a line of sight, we should be able to add a doorway to it and have it still admit a line of sight. In the finite hallway case, this is always true and there are always infinitely many lines of sight. However, for infinite hallways, there may be a unique line of sight and the naïve formulation of infinite hallways does not always allow a door to be added while preserving visibility. For example, consider the infinite hallway \( H^\infty \) whose \(i\)th doorway is \( D_i = (\lfloor n\pi \rfloor, \lfloor n\pi \rfloor + 1) \) for \( i \geq 1 \) and where \( D_0 \) is undefined. As will be shown in Theorem 27, \( \ell_{x_0} \) is the only line such that \( \ell_{x_0} \cap H^\infty = \emptyset \). However, \( \ell_{x_0} \) contains the point \((0,0)\) on the integer lattice and so there is no acceptable choice of \( D_0 \) if we would like \( H^\infty \) to admit a line of sight (since every doorway excludes every lattice point).

We will solve this issue by introducing infinitesimals.

**Definition 20 (Infinitesimals).** Let \( \epsilon \) represent a positive infinitesimal. Formally, let \( \mathbb{R}^\epsilon = \{a + b\epsilon : a, b \in \mathbb{R}\} \) be the two-dimensional vector space with basis \( \{1, \epsilon\} \) over the field \( \mathbb{R} \).
endowed with the following total order:
\[ a + b\epsilon < c + d\epsilon \quad \text{if} \quad a < c \quad \text{or} \quad a = c \quad \text{and} \quad b < d \]
and
\[ a + b\epsilon = c + d\epsilon \quad \text{if} \quad a = c \quad \text{and} \quad b = d. \]

A number \( r = a + b\epsilon \in \mathbb{R}^{\epsilon} \) is called real if \( b = 0 \).

This all amounts to saying that \( \epsilon \) satisfies \( 0 < \epsilon < a \) for all positive real numbers \( a \) and that addition of infinitesimals and multiplication of infinitesimals by real numbers makes sense. We define open and closed intervals in the usual way: the open interval \((a, b) \subseteq \mathbb{R}^{\epsilon}\) is defined as \( (a, b) = \{ r \in \mathbb{R}^{\epsilon} : a < r < b \} \) and the closed interval \([a, b] \subseteq \mathbb{R}^{\epsilon}\) is defined as \( [a, b] = \{ r \in \mathbb{R}^{\epsilon} : a \leq r \leq b \} \) and in general we endow \( \mathbb{R}^{\epsilon} \) with the order topology.

Now we will precisely define what it means to be an infinite hallway, systematically replacing \( \mathbb{R} \) (in the finite case) with \( \mathbb{R}^{\epsilon} \) (in the infinite case).

**Definition 21 (Infinite Hallway).** Let \( D_i = (d_i, d_i + 1) \subset \mathbb{R}^{\epsilon} \) for \( d_i \in \mathbb{Z} \) be a sequence of open unit intervals and define the corresponding infinite hallway \( H^{\infty} \) as
\[
H^{\infty} = \bigcup_{i \in \mathbb{Z}} \{i\} \times (\mathbb{R}^{\epsilon} \setminus D_i).
\]

We will notate the restriction of \( H^{\infty} \) to the first \( n \) hallways it contains by
\[
H^{\infty}_n = H^{\infty} \cap ([0, n] \times \mathbb{R}^{\epsilon}).
\]

**Definition 22 (Infinite Lines).** For \( \alpha, \beta \in \mathbb{R} \), define the infinite line \( \ell^{\epsilon}_{\alpha\beta} \subset \mathbb{R} \times \mathbb{R}^{\epsilon} \) by
\[
\ell^{\epsilon}_{\alpha\beta} = \{(x, \alpha(x + t\epsilon) + \beta) : x, t \in \mathbb{R}\}.
\]

Given a subset \( X \subset \mathbb{R} \times \mathbb{R}^{\epsilon} \), for any \( t \in \mathbb{R}^{\epsilon} \), we define \( X + t = \{(a, b + t) : (a, b) \in X\} \). In this notation, we can alternatively define \( \ell^{\epsilon}_{\alpha\beta} = \bigcup_{t \in \mathbb{R}} \ell^{\epsilon}_{\alpha\beta} + t\epsilon \).

Infinite lines are “fattened up” lines. As such, we need to slightly modify how we define a line of sight. Whereas, \( \ell_{\alpha\beta} \cap H = \emptyset \) captures the idea “no light is blocked,” we would like to capture the idea “not all light is blocked.”

**Notation 23.** Given a line or infinite line \( \ell^*_{\alpha\beta} \) and a hallway or infinite hallway \( H^* \), the visibility operator \( \lor \) is defined as
\[
\ell^*_{\alpha\beta} \lor H^* = \text{real part of } (\text{proj}_{\alpha^*} \ell^*_{\alpha\beta}) \cap (\text{proj}_{\alpha^*} H^*)^c.
\]

**Definition 24 (Infinite Line of Sight).** The infinite line \( \ell^{\epsilon}_{\alpha\beta} \) is a line of sight for the infinite hallway \( H^{\infty} \) if \( \ell^{\epsilon}_{\alpha\beta} \lor H^{\infty} \neq \emptyset \).

Notice that infinite lines of sight are still defined by real parameters, they are just infinitesimally “fattened up.” To see this, consider the following example. Let \( H^{\infty} \) be the infinite hallway with doorways \( D_i = (i, i + 1) \). \( H^{\infty} \) has infinite lines of sight \( \ell^{\epsilon}_{1\beta} \) for every \( \beta \in [0, 1] \). In particular, \( \ell^{\epsilon}_{10} \) and \( \ell^{\epsilon}_{11} \) are infinite lines of sight, but we would not consider the real lines \( \ell_{10} \) and \( \ell_{11} \) to be lines of sight.

From now on we will refer to infinite lines of sight simply as lines of sight and use the term real line of sight if we need to draw a careful distinction between infinite and non-infinite lines of sight.
Proposition 25. If $H^\infty$ is an infinite hallway with line of sight $\ell_{\alpha\beta}$, then either $(\ell_{\alpha\beta} + \epsilon) \cap H^\infty \neq \emptyset$ or $(\ell_{\alpha\beta} - \epsilon) \cap H^\infty \neq \emptyset$.

Proof. Observe that if $\ell_{\alpha\beta} \cup H^\infty \neq \emptyset$, then for some $r \in \mathbb{R}$ we have $(\ell_{\alpha\beta} + r\epsilon) \cap H^\infty = \emptyset$. If $r \geq 0$, then we must have $(\ell_{\alpha\beta} + \epsilon) \cap H^\infty \neq \emptyset$ and if $r \leq 0$ we must have $(\ell_{\alpha\beta} - \epsilon) \cap H^\infty \neq \emptyset$.

Next we will show that the using infinite lines of sight gives us our desired property. In fact, it gives us something slightly stronger.

Proposition 26. Suppose $\hat{H}^\infty$ is an infinite hallway missing its first door and admitting a line of sight $\ell_{\alpha\beta}$. If $D_i$ for $i \geq 1$ are the doorways for $\hat{H}^\infty$, then there exists a doorway $D_0$ such that the infinite hallway $H^\infty$ with doorways $D_i$ for $i \geq 0$ admits the line of sight $\ell_{\alpha\beta}$.

Proof. Suppose $D_i$ for $i \geq 1$ and $\hat{H}^\infty$ are as in the statement of the proposition. Given a definition for $D_0$, let $H^\infty$ be the infinite hallway with doorways $D_i$ for $i \geq 0$. Let $z = [\beta]$. From Proposition 25, we know either $(\ell_{\alpha\beta} + \epsilon) \cap H^\infty = \emptyset$ or $(\ell_{\alpha\beta} - \epsilon) \cap H^\infty = \emptyset$.

Suppose $(\ell_{\alpha\beta} + \epsilon) \cap \hat{H}^\infty = \emptyset$. In this case, define $D_0 = (z, z + 1)$. We now have $z + 1 > \text{proj}_a(\ell_{\alpha\beta} + \epsilon) = \beta + \epsilon > z$, and so $(\ell_{\alpha\beta} + \epsilon) \cap H^\infty = \emptyset$. It immediately follows that $\ell_{\alpha\beta}$ is a line of sight for $H^\infty$.

If $(\ell_{\alpha\beta} - \epsilon) \cap \hat{H}^\infty = \emptyset$, a similar argument shows that if $D_0 = (z - 1, z)$, then $H^\infty$ admits the line of sight $\ell_{\alpha\beta}$.

Theorem 27. Let $H^\infty$ be an infinite hallway. If $\ell_{\alpha\beta}$ and $\ell_{\gamma\delta}$ are both lines of sight for $H^\infty$, then $\alpha = \gamma$. In other words, all lines of sight for $H^\infty$ have the same slope.

Proof. Let $H^\infty$ be an infinite hallway with doorways $D_i = (d_i, d_i + 1)$ and suppose $\ell_{\alpha\beta}$ is a line of sight for $H^\infty_n$. Let $D_i = [d_i, d_i + 1]$ and notice that $\ell_{\alpha\beta}$ must pass through $D_0$ and $D_n$. From this, we conclude

$$\frac{d_0 - (d_n + 1)}{n} = \frac{d_0 - d_n}{n} - \frac{1}{n} \leq \alpha \leq \frac{d_0 - d_n}{n} + \frac{1}{n} = \frac{d_0 + 1 - d_n}{n},$$

since a line with any slope outside of that range would not pass through $D_0$ and $D_n$.

Now, if $\ell_{\alpha\beta}$ is a line of sight for $H^\infty$, it is a line of sight for every $H^\infty_n$. Thus for any $r > 0$ and $n > 1/r$,

$$\alpha - r \leq \frac{d_0 - d_n}{n} \leq \alpha + r$$

and so $\lim_{n \to \infty}(d_0 - d_n)/n = \alpha$ exists. Since $\lim_{n \to \infty}(d_0 - d_n)/n$ exists and is unique, there can only be one slope for lines of sight for $H^\infty$.

Definition 28 (Periodic Hallways). An infinite hallway $H^\infty$ with doors $D_i = (d_i, d_i + 1)$ is called periodic with period $m > 0$ if there exists some $k$ such that

$$D_{i+m} = D_i + k = (d_i + k, d_i + k + 1).$$

If there exists an $m$ such that $H^\infty$ is periodic with period $m$, $H^\infty$ is called periodic, otherwise $H^\infty$ is called aperiodic.

If $H^\infty$ is periodic, the minimum period is the smallest $m$ such that $H^\infty$ is periodic with period $m$. 
Theorem 29. An infinite hallway $H^\infty$ which admits a line of sight is periodic if and only if it admits a rational line of sight.

Proof. Suppose $H^\infty$ is an aperiodic infinite hallway with doorways $D_i = (d_i, d_i + 1)$ and line of sight $\ell_{\alpha\beta}^\epsilon$.

If $H^\infty$ is periodic, then there is some $m$ such that $d_{km} = d_0 + k(d_m - d_0)$. Now,
$$\alpha = \lim_{n \to \infty} \frac{d_0 - d_n}{n} = \lim_{k \to \infty} \frac{d_0 - d_{km}}{km} = \lim_{k \to \infty} \frac{d_0 - (d_0 + k(d_m - d_0))}{km} = \frac{r}{m},$$
where $r = d_m - d_0$ and so $\alpha \in \mathbb{Q}$.

Conversely, suppose $\alpha = \frac{p}{q} \in \mathbb{Q}$. Now, $\beta + n\frac{p}{q} + \epsilon \in (d_n, d_n + 1)$ or $\beta + n\frac{p}{q} - \epsilon \in (d_n, d_n + 1)$, and so in particular, $d_n$ is the unique integer such that
$$\beta + n\frac{p}{q} - 1 < d_n < \beta + n\frac{p}{q}.$$

Letting $k = n + 1$, we additionally have
$$\beta + n\frac{p}{q} + 1 < d_{n+1} < \beta + n\frac{p}{q},$$
and so $d_{n+1}$ is the unique integer such that $\beta + n\frac{p}{q} - 1 < d_{n+1} < \beta + n\frac{p}{q}$, which shows $H^\infty$ is periodic with period $q$. ■

The proof of Theorem 29 actually gives us an additional result.

Theorem 30. An infinite hallway $H^\infty$ admitting a line of sight has period $m$ if and only if it admits a rational line of sight with slope $p/m$ for some $p \in \mathbb{Z}$.

Corollary 31. Suppose $H^\infty$ is an infinite periodic hallway with minimal period $m$. If $\ell_{\alpha\beta}^\epsilon$ is a line of sight for $H^\infty$, then $p/m$ is in lowest terms.

Proof. We will prove the contrapositive. Suppose $H^\infty$ is an infinite hallway admitting a line of sight $\ell_{\alpha\beta}^\epsilon$. If $p/m$ is not in lowest terms, then $p/m = p'/m'$ where $0 < m' < m$.

Since $m'$ is a period for $H^\infty$, we know $m$ is not a minimal period. ■

We now know quite a bit about the slopes of lines of sight for infinite hallways. Let us introduce a lemma dealing with the intercepts.

Lemma 32. Suppose $H^\infty$ is an infinite hallway with doorways $D_i$ and admitting a line of sight $\ell_{\alpha\beta}^\epsilon$. Let $P_n = (p_i^n)_{i \in \mathbb{Z}}$ be an enumeration of the points in $\text{proj}_\alpha(\{0, 1, \ldots, n\} \times \mathbb{Z})$ satisfying $p_i^n < p_{i+1}^n$. Let $B_n = \{[p_i^n, p_{i+1}^n] : p_i^n \in P^n\}$ and let $D = \{\gamma : \ell_{\alpha\gamma}^\epsilon$ is a line of sight for $H^\infty\}$. Then $D$ is a (possibly degenerate) interval such that for all $n$, $D \subseteq B$ for some $B \in B_n$.

Proof. Suppose $\ell_{\alpha\beta}^\epsilon$ is a line of sight for an infinite hallway $H^\infty$ with doorways $D_i = (d_i, d_i + 1)$. Let $\bar{D}_i = [d_i, d_i + 1]$. If $\ell_{\alpha\gamma}^\epsilon$ is a line of sight for $H^\infty$, then $\ell_{\alpha\gamma}^\epsilon$ must intersect $\{i\} \times D_i$ for every $i$. By Proposition 23, $\ell_{\alpha\gamma}^\epsilon + \epsilon$ or $\ell_{\alpha\gamma}^\epsilon - \epsilon$ must intersect $\{i\} \times D_i$ for each $i$ and so $\ell_{\alpha\gamma}^\epsilon$ intersects $\{i\} \times \bar{D}_i$ for each $i$. Thus, we have the equality
$$D = \bigcap_{i \geq 0} \text{proj}_\alpha(\{i\} \times \bar{D}_i).$$

Since $D$ can be written as an intersection of intervals, it is an interval. Lastly, since $D_i \cap \mathbb{Z} = \emptyset$ for all $i$, we know that for every $n$, $D \subseteq B$ for some $B \in B_n$. ■
From Lemma 32 we can get a bound on the size of $D$.

**Theorem 33.** Let $H^\infty$ be an infinite hallway. If $H^\infty$ is periodic with minimal period $m$ and $\ell_{\alpha,\beta}^\epsilon$ and $\ell_{\alpha,\delta}^\epsilon$ are lines of sight for $H^\infty$, then $|\beta - \delta| \leq 1/m$.

**Proof.** This follows quickly from Lemma 32. If $\ell_{\alpha,\beta}^\epsilon$ is a line of sight for an infinite periodic hallway with minimal period $m$, then $\alpha = p/m$ for some $p$. This means, for all $i, j \geq m$,

$$\text{proj}_\alpha(\{0, 1, \ldots, i\} \times \mathbb{Z}) = \text{proj}_\alpha(\{0, 1, \ldots, j\} \times \mathbb{Z}),$$

and so $B_i = B_j$ where $B_n$ is defined as in the statement of Lemma 32. Since $\alpha = p/m$ must be in lowest terms, a quick calculation shows every interval in $B_i$ for $i \geq m$ has width no greater than $1/m$, which completes the proof. ■

**Theorem 34.** Let $H^\infty$ be an infinite hallway. If $H^\infty$ is aperiodic, then there is at most one line of sight for $H^\infty$.

**Proof.** Suppose $H^\infty$ is an aperiodic infinite hallway with doorways $D_i = (d_i, d_i + 1)$ and line of sight $\ell_{\alpha,\beta}^\epsilon$. Necessarily we have $\alpha \notin \mathbb{Q}$, and by Theorem 27 $\alpha$ is unique.

Let $D = \{ \gamma : \ell_{\alpha,\gamma}^\epsilon \text{ is a line of sight for } H^\infty \}$. Since $\beta \in D$, $D$ is a non-empty (but possibly degenerate) interval. Further, for every $n$ we have that $D \subseteq B$ for some $B \in B_n$ where $B_n$ is as in the statement of Lemma 32. But, since $\alpha \notin \mathbb{Q}$, $\text{proj}_\alpha(\mathbb{N} \times \mathbb{Z})$ is dense the diameter if every interval in $B_n$ tends towards zero as $n \to \infty$. We conclude that $D = \{ \beta \}$ must be a singleton and so there is only one line of sight for $H^\infty$. ■

The converse to Theorem 34 is also true. If there is a unique line of sight for an infinite hallway, it must be aperiodic.

3.1. **Metrics on Hallways.** The theorems in the preceding section, taken together, show a correspondence between infinite hallways and a symbolic shift space. Consider the map $\Phi : \{\text{infinite hallways}\} \to \mathbb{Z}^\mathbb{N}$ where the $i$th coordinate of $\Phi(H)$ is $d_i - d_{i+1}$. $\Phi$ is an extension of the identically-named map between $n$-hallways and $n$-words from earlier. Under the assumption that the initial doorway of every infinite hallway is $D_0 = (0, 1)$, $\Phi$ is a bijection between hallways and sequences.

Let $T : \mathbb{Z}^\mathbb{N} \to \mathbb{Z}^\mathbb{N}$ be the shift map. That is $T(a_0, a_1, a_2, \ldots) = (a_1, a_2, \ldots)$ deletes the first coordinate of a sequence. Let $\Omega$ be the image under $\Phi$ of all infinite hallways admitting a line of sight. Now, $T(\Omega) \subseteq \Omega$ is immediate, and Proposition 20 (the proposition that gives us our desirable property) shows that $T(\Omega) = \Omega$. Thus, $\Omega$ is $T$-invariant. The word closed is almost always used in conjunction with the word invariant, so we might ask if $\Omega$ is also closed.

The shift space $(\mathbb{Z}^\mathbb{N}, T)$ is typically endowed with the product topology on $\mathbb{Z}^\mathbb{N}$ where $\mathbb{Z}$ has the discrete topology. This is the same topology arising from the standard metric on sequences, $d$. Namely, if $x, y \in \mathbb{Z}^\mathbb{N}$, $d(x, y) = 1/n$ where $n$ is the index of the first coordinate where $x$ and $y$ differ.

Using $\Phi$, the standard metric on sequences induces a metric on infinite hallways.
Definition 35 (Standard Metric on Infinite Hallways). Let \( H, H' \) be infinite hallways. The standard metric on infinite hallways, notated \( d_S \), is defined as
\[
d(\Phi(H), \Phi(H')) = d_S(H, H') = \frac{1}{n} \quad \text{where} \quad n = \inf\{k \in \mathbb{N} : H_k \neq H'_k\},
\]
with the convention \( 1/\infty = 0 \) and \( 1/0 = \infty \).

Standard arguments now show that the set of all infinite hallways is complete with respect to \( d_S \).

Let \( V : \{\text{infinite hallways}\} \to \{0, 1\} \) be the visibility function. That is, \( V(H) = 1 \) if \( H \) admits a line of sight and 0 otherwise. Now, suppose \( H \) is an infinite hallway that admits a line of sight and let \( H' \) be \( H \) with a single doorway changed. Since \( V(H) = 1 \) and \( V(H') = 0 \), we cannot hope that \( V \) is continuous. However, we might hope that \( V \) would be upper-semicontinuous. That is, we might hope that if \( H^{(n)} \to H \) is a convergent sequence of infinite hallways and \( V(H^{(n)}) = 1 \), then \( V(H) = 1 \). Alas, this is not so with the standard metric.

Proposition 36. The visibility function \( V \) is not upper-semicontinuous with respect to the standard metric on infinite hallways, \( d_S \).

Proof. Let \( H^{(n)} \) be the periodic infinite hallway admitting a line of sight with slope \( 1/n \) and with doorways \( D_0^{(n)} = (0, 1), D_1^{(n)} = (1, 2), \) and \( D_i^{(n)} = (\lfloor \frac{i-1}{n} \rfloor + 1, \lfloor \frac{i-1}{n} \rfloor + 2) \). Geometrically, \( H^{(n)} \) has a jump of size 1 between \( D_0 \) and \( D_1 \), then has \( n \) identical doorways in a row before another jump of size 1.

Now, for all \( m, n \), we have
\[
(0, 1) = D_0^{(m)} = D_0^{(n)} \quad \text{and} \quad (1, 2) = D_1^{(m)} = D_1^{(n)}
\]
and for every \( n, m > k > 1 \) and \( 1 \leq i \leq k \),
\[
(1, 2) = D_i^{(n)} = D_i^{(m)}.
\]
From this description, we see \( H^{(n)} \to H^\infty \) which has doorways \( D_0^\infty = (0, 1) \) and \( D_i^\infty = (1, 2) \) for all \( i \geq 1 \). But \( V(H^{(n)}) = 1 \) and \( V(H^\infty) = 0 \), so \( V \) is not semi-continuous. ■

Since \( \Omega = \Phi^{-1} \circ V^{-1}(1) \), Proposition 36 shows that \( \Omega \) is not closed with respect to the standard metric. Similarly, the set of all Sturmian sequences is not closed under the standard metric (because, as in Proposition 36, limits of periodic points may be aperiodic but eventually periodic) and the property of being a rotation sequence is not closed under limits.

All hope is not lost, though. There may be a different metric that \( V \) is upper-semicontinuous with respect to. The counterexample used in Proposition 36 relied on a sequence of periodic infinite hallways. In particular, the lines of sight had slope converging to a rational number, so we might seek to prevent hallways from doing this.

Definition 37 (Common Initial Segment). Given two infinite hallways \( H \) and \( H' \), their common initial segment is
\[
\text{comm}(H, H') = H_{1/d_S(H, H')} = H'_{1/d_S(H, H')}.
\]
It is not immediately obvious that Proposition 42.

Definition 38 (Unframed Hallway). Given a hallway $H$ with $i$th doorway $(d_i, d_i + 1)$, the corresponding unframed hallway is the hallway $\bar{H}$ whose $i$th doorway is the closed interval $[d_i, d_i + 1]$.

Definition 39 (Rational Metric on Infinite Hallways). Let $H, H'$ be infinite hallways. The rational metric on infinite hallways, notated $d_R$, is defined as follows.

If $H = H'$, then $d_R(H, H') = 0$; if $\text{comm}(H, H')$ admits a line of sight,

$$ d_R(H, H') = \max \left\{ \frac{1}{q} : \ell_{\frac{p}{q}} \text{ is a line of sight for } \text{comm}(\bar{H}, \bar{H}') \text{ for some } \beta \in \mathbb{R}, p \in \mathbb{Z} \right\}; $$

and if $\text{comm}(H, H')$ admits no line of sight and $H \neq H'$, then $d_R(H, H') = \infty$.

Proposition 40. The rational metric, $d_R$, is a metric on infinite hallways.

Proof. By definition $d_R(H, H') = 0$ if $H = H'$. Suppose $H \neq H'$. If $\text{comm}(H, H')$ admits no line of sight, $d_R(H, H') = \infty > 0$. If $\text{comm}(H, H')$ admits a line of sight, because it is a finite hallway, it admits a rational line of sight and so $d_R(H, H') > 0$. Further, since $\text{comm}(H, H') = \text{comm}(H', H)$, $d_R(H, H') = d_R(H', H)$ and so conditions (i) and (ii) are satisfied.

Now we consider condition (iii). Let $H, H', H''$ be infinite hallways and notice that either

$$ \text{comm}(H, H'') \subseteq \text{comm}(H, H') \quad \text{or} \quad \text{comm}(H', H'') \subseteq \text{comm}(H, H'). $$

To see this, let $n_{XY} = 1/d_S(X, Y)$ be the number of doorways that hallways $X$ and $Y$ agree for, and consider the three choices: (a) $n_{HH'} = n_{HH''}$, (b) $n_{HH'} > n_{HH''}$, or (c) $n_{HH'} < n_{HH''}$.

In case (a), $\text{comm}(H, H'') = \text{comm}(H, H')$ and so $\text{comm}(H, H'') \subseteq \text{comm}(H, H')$; in case (b), we must have $n_{HH'} = n_{HH''} < n_{HH}$, and so $\text{comm}(H', H'') \subseteq \text{comm}(H, H')$; and, in case (c), we must have $n_{HH'} = n_{HH''}$ which means $\text{comm}(H', H'') = \text{comm}(H, H')$ and so $\text{comm}(H', H'') \subseteq \text{comm}(H, H')$.

Now, for finite hallways $X, Y$ where $X \subset Y$, the set of lines of sight for $X$ is a superset of the set of lines of sight for $Y$. Thus, the above set inclusions give us either

$$ d_R(H, H'') \geq d_R(H, H') \quad \text{or} \quad d_R(H', H'') \geq d_R(H, H'), $$

and so certainly $d_R(H, H'') + d_R(H', H'') \geq d_R(H, H')$. ■

Notation 41. Let $\mathcal{H}$ be the set of all infinite hallways; let $\mathcal{Q}$ be the set of all infinite periodic hallways that admit lines of sight; let $\mathcal{Q} \subseteq \mathcal{H}$ be the closure of $\mathcal{Q}$ under the metric $d_R$; and let $\mathcal{Q}^c = \mathcal{Q} \setminus \mathcal{Q}$.

It is not immediately obvious that $\mathcal{Q}^c$ contains anything at all, but we will show that $\mathcal{Q}^c$ is precisely the set of all aperiodic hallways admitting lines of sight. Not only that, but we will show that $\mathcal{Q}^c$ is a closed set with respect to $d_R$, from which it will follow that $V$, the visibility function, is upper-semicontinuous.

Proposition 42. The set $\mathcal{Q}^c$ is closed with respect to $d_R$. 

**Proof.** By definition $\tilde{Q}$ is closed with respect to $d_R$. Thus, if we can show $Q$ is open with respect to $d_R$, the proof will be complete.

To that end, suppose $H \in Q$ admits a line of sight $\ell_{\alpha,\beta}^e$. We necessarily have that $\ell_{\alpha,\beta}^e$ is a line of sight for $H_k$ for all $k$. Thus, $d_R(H, H') \geq 1/q$ for all $H' \neq H$, and so the singleton $\{H\}$ is open with respect to $d_R$ (it is the only element in the $d_R$-ball of radius $1/(2q)$ with center $H$). We conclude that $\tilde{Q}$ is the union of open sets and is therefore open. ■

**Lemma 43.** If $H$ is an infinite hallway that admits a unique line of sight $\ell_{\alpha,\beta}^e$, then the unframed infinite hallway $\tilde{H}$ admits $\ell_{\alpha,\beta}^e$ as its unique line of sight.

**Proof.** Let $H$ be an infinite hallway with unique line of sight $\ell_{\alpha,\beta}$ and $\tilde{H}$ the corresponding unframed infinite hallway. Notice that the proof of Theorem 27 applies equally well to unframed infinite hallways, and so the slope of any line of sight for $\tilde{H}$ must be $\alpha$.

Let $(d_i, d_i + 1)$ be the $i$th doorway for $H$. Similarly to the proof of Lemma 32,

$$\{\beta\} = B = \bigcap_{n \in \mathbb{N}} (\text{proj}_{\alpha}(\{n\} \times [d_i, d_i + 1]),$$

is the complete set of intercepts for the infinite hallway $H$ and is also the complete set of intercepts for the unframed infinite hallway $\tilde{H}$. Thus $\ell_{\alpha,\beta}^e$ is unique. ■

**Proposition 44.** Suppose $H$ is an aperiodic infinite hallway admitting a line of sight. Then $H \in Q^c$.

**Proof.** Since $H$ is an aperiodic infinite hallway, $H \notin Q$. To show $H \in Q^c$, we must show that there is a sequence $H^{(i)}$ of periodic infinite hallways such that $H^{(i)} \to H$ with respect to $d_R$.

Let $\ell_{\alpha,\beta}^e$ be the unique line of sight for $H$ and let $(d_i, d_i + 1)$ be the $i$th doorway of $H$. Without loss of generality, assume $\alpha \in [0,1]$. By Theorem 29 $\alpha \notin Q$. By Lemma 43, the unframed infinite hallway $\tilde{H}$ admits the unique line of sight $\ell_{\alpha,\beta}^e$.

Recall that $H_n$ is the restriction of $H$ to the first $n$ doorways. Fix $q \in \mathbb{N}$. Now, since there are only a finite number of rationals of the form $\frac{p}{q} \in [0,1]$, it must be the case that for large enough $n$, $H_n$ admits no line of sight of the form $\ell_{\alpha,\beta}^e$, lest $\alpha = \frac{p}{q}$. Similarly, for large enough $n$, $\tilde{H}_n$ admits no line of sight of the form $\ell_{\alpha,\beta}^e$.

To complete the proof, notice that since $H_n$ is a finite hallway that admits a line of sight, it admits a rational line of sight $\ell_{\gamma,\delta}$. Thus, there exists a periodic infinite hallway $H^{(n)}$ admitting the line of sight $\ell_{\gamma,\delta}$ and satisfying

$$H_n \subseteq \text{comm}(H^{(n)}, H).$$

Further, by construction $d_R(H^{(n)}, H) \to 0$. ■

An immediate corollary of Proposition 44 is that $Q^c$ is non-empty. Next, we will show that every hallway in $Q^c$ admits a line of sight.

**Lemma 45.** Let $H$ be an infinite hallway and $H_n$ be the restriction of $H$ to the first $n$ doorways. If $H_n$ admits a line of sight for every $n$, then the unframed hallway $\tilde{H}$ admits a line of sight.
Proof. For a hallway $K$, let

$$A(K) = \{ \alpha : \ell_{\alpha \beta}^k \text{ is a line of sight for } K \text{ for some } \beta \}. $$

By assumption, $A(H_n) \neq \emptyset$, and $A(H_m) \subseteq A(H_n)$ for all $m > n$. Further notice that $A(H) = \bigcap_{n \in \mathbb{N}} A(H_n)$. Let $H_n$ be the unframed hallway corresponding to $H_n$. Since $A(H_n)$ is closed and bounded and $\{A(H_n)\}$ satisfies the finite intersection property, $A(H) = \bigcap_{n \in \mathbb{N}} A(H_n) \neq \emptyset$, and so $H$ admits a line of sight. \qed

**Proposition 46.** If $H \in \mathcal{Q}^c$, then $H$ admits a line of sight.

**Proof.** Let $H \in \mathcal{Q}^c$ and suppose the sequence $H^{(i)} \in \mathcal{Q}$ satisfies $H^{(i)} \to H$ with respect to $d_R$. Let $D_i = (d_i, d_i + 1)$ be the $i$th doorway of $H$.

Let $|\text{comm}(H^{(n)}, H)|$ denote the number of doorways in $\text{comm}(H^{(n)}, H)$. Since $\text{comm}(H^{(n)}, H)$ admits a line of sight and is necessarily a finite hallway, $\text{comm}(H^{(n)}, H)$ must admit a rational line of sight with slope $p/q$ where $q \leq |\text{comm}(H^{(n)}, H)|$. Since $d_R(H^{(i)}, H) \to 0$, we must have that $|\text{comm}(H^{(n)}, H)| \to \infty$.

Now, by Lemma 45 the unframed hallway $\bar{H}$ must admit a line of sight $\ell_{\alpha \beta}^k$ since $H_k \subseteq \text{comm}(H^{(n)}, H)$ for large enough $n$. Further, $\alpha \notin \mathbb{Q}$ and so $\ell_{\alpha \beta}^k$ must be unique.

If we can show that $\ell_{\alpha \beta}^k$ is a line of sight for $H_k$, regardless of $k$, then $\ell_{\alpha \beta}^k$ will be a line of sight for $H$. Suppose this is not the case, and let $k$ be the smallest number such that $\ell_{\alpha \beta}^k$ is not a line of sight for $H_k$. Trivially, $k \geq 1$, and since $H^{(n)} \to H$ with respect to $d_R$ and $H_k \subseteq \text{comm}(H^{(n)}, H)$ for large enough $n$, we must have that $\ell_{\alpha \beta}^k$ is a line of sight for $H_k$.

Since $\ell_{\alpha \beta}^k$ is a line of sight for $H_{k-1}$ but not for $H_k$, we must have either $\alpha k + \beta = d_k$ or $\alpha k + \beta = d_k + 1$.

Assume $\alpha k + \beta = d_k$. Since $\ell_{\alpha \beta}^k$ is not a line of sight for $H_k$, we have that $\ell_{\alpha \beta} + \epsilon$ is not a line of sight for $H_k$. Since $\alpha k + \beta + \epsilon \in D_k$, this means $\ell_{\alpha \beta} + \epsilon$ is not a line of sight for $H_{k-1}$.

So, by Proposition 25 $\ell_{\alpha \beta} - \epsilon$ must be a line of sight for $H_{k-1}$. We conclude that for some $0 \leq i < k$, we must have $\alpha i + \beta = d_i + 1$. Thus, $\alpha = \frac{d_k - d_{k-1}}{k-i} \in \mathbb{Q}$, which is a contradiction.

Assuming $\alpha k + \beta = d_k + 1$, the proof follows similarly. \qed

**Corollary 47.** If $H \in \mathcal{Q}$ then $H$ admits a line of sight.

**Proof.** By definition $\bar{\mathcal{Q}} = \mathcal{Q} \cup \mathcal{Q}^c$. If $H \in \mathcal{Q}$, then by definition it admits a line of sight and by Proposition 46 if $H \in \mathcal{Q}^c$, $H$ admits a line of sight. \qed

We are almost ready to prove the semi-continuity of $V$. But first, let us completely characterize the set of hallways that admit lines of sight.

**Theorem 48.** Let $\mathcal{V} = \{ H \in \mathcal{H} : V(H) = 1 \}$ be the set of infinite hallways that admit lines of sight. Then, $\mathcal{Q} = \mathcal{V}$.

**Proof.** By Corollary 47 $\mathcal{Q} \subseteq \mathcal{V}$. Now, suppose $H \in \mathcal{V}$. If $H$ is a periodic hallway, then $H \in \mathcal{Q}$.

Suppose $H$ is an aperiodic hallway. Since $H$ admits a line of sight, so does the finite hallway $H_k$. Thus $H_k$ admits a rational line of sight $\ell_{\alpha_k \beta_k}$ where $\alpha_k = \frac{d_k}{\beta_k}$ by Theorem 20.
Let $H^{(k)}$ be the infinite periodic hallway admitting the line of sight $\ell_{\alpha_k, \delta_k}$. We will now show $H^{(k)} \to H$ with respect to $d_R$.

Since $H^{(k)}_k = H_k$, we have $H^{(k)} \to H$ with respect to $d_S$. Let $\ell_{\bar{\eta}_k \delta_n}$ be a line of sight for $H_k$ such that $q_k$ is as small as possible. If $q_k \to \infty$, we are done. Suppose $\{q_k\}$ is bounded and let $\bar{D}_0 = [d_0, d_0 + 1]$ and $\bar{D}_1 = [d_1, d_1 + 1]$ be the first two doorways of $H$. Since $d_1 - d_0 - 1 \leq \bar{\eta}_k \leq d_1 + 1 - d_0$ and there are only finitely many $q_k$, there exists $p/q \in \{pk/q_k : k \in \mathbb{N}\}$ such that $p/q = pk/q_k$ for infinitely many $k$. It follows that $\text{proj}_{p/q} H_k$ is a non-empty closed interval for all $k$ and so $\text{proj}_{p/q} H$ is non-empty. Thus, $H$ admits a rational line of sight with slope $p/q$, which is a contradiction. ■

**Theorem 49.** $V : \mathcal{H} \to \{0, 1\}$ is upper-semicontinuous with respect to $d_R$.

**Proof.** Let $H^{(n)} \to H$ with respect to $d_R$ and suppose $V(H^{(n)}) = 1$. By Theorem 48, $H^{(n)} \in \mathcal{Q}$, and so by definition $H \in \mathcal{Q}$. Now, by Corollary 47, $V(H) = 1$. ■

### 4. Applications of $d_R$

Theorem 27 states that if an infinite hallway admits a line of sight, its slope is unique. Thus, we may define a function $s : \mathcal{Q} \to \mathbb{R}$ by

$$s(H) = \alpha \text{ where } \ell_{\alpha, \beta} \text{ is a line of sight for } H.$$ 

Let $s^{-1}$ be the set-valued right-inverse to $s$. That is

$$s^{-1}(\alpha) = \{H : s(H) = \alpha\},$$

and we have the equality $s \circ s^{-1} = \text{id}$. Now, any metric $d$ on infinite hallways induces a metric $\tilde{d}$ on $\mathbb{R}$ via

$$\tilde{d}(\alpha, \gamma) = d(s^{-1}(\alpha), s^{-1}(\gamma)).$$

Two metrics $d_X$ and $d_Y$ are said to be equivalent if their convergent sequences are the same. That is, $d_X(x_i, x) \to 0$ if and only if $d_Y(x_i, x) \to 0$ for all sequences $(x_i)$.

**Proposition 50.** The metric $\tilde{d}_S$ on $\mathbb{R}$ induced by the metric $d_S$ is equivalent to the standard metric $d$ on $\mathbb{R}$ given by $d(\alpha, \gamma) = |\alpha - \gamma|$.

**Proof.** We will first show that convergence in $d$ implies convergence in $\tilde{d}_S$. Let $(x_i)$ be a sequence and suppose $d(x_i, x) = |x_i - x| \to 0$. Fix $k > 0$ and let $H_k$ be a finite hallway with doorways $D_i$ admitting a line of sight $\ell_{x, \beta}$ for some $\beta$. Now, for any $y \in \mathbb{R}$, define

$$D^y = \bigcap_{i \leq k} \text{proj}_y(\{i\} \times D_i)$$

and note that $D^y = (l^y, r^y)$ is always an open interval or the empty set. Further, $D^{x_i} \to D^x$ in the sense that $l^{x_i} \to l^x$ and $r^{x_i} \to r^x$. Necessarily we have $\beta \in D^x$, but we also see that since $D^{x_i} \to D^x$, for all large enough $i$, we have $\beta \in D^{x_i}$. Thus, for large enough $i$, $\ell_{x_i, \beta}$ and $\ell_{x, \beta}$ are lines of sight for $H_k$ and so $\tilde{d}_S(x_i, x) \leq 1/k$. But $k$ was arbitrary, so $\tilde{d}_S(x_i, x) \to 0$. ■
Now, suppose $\tilde{d}_S(x_i, x) \to 0$. Fix $k > 0$. Now for all sufficiently large $i$, $\tilde{d}_S(x_i, x) \leq 1/k$. Supposing $i$ is sufficiently large, we necessarily have that for some $\beta, \beta_i$, there exists a $k$-hallway, $H_k$, for which $\ell_{x, \beta}$ and $\ell_{x, \beta_i}$ are both lines of sight. In particular, $\ell_{x, \beta}$ and $\ell_{x, \beta_i}$ both pass through the $k$th doorway of $H_k$ and so

$$|(kx + \beta) - (kx_i + \beta_i)| \leq 1.$$ 

By the reverse triangle inequality we have

$$k|x - x_i| - |\beta - \beta_i| \leq |(kx + \beta) - (kx_i + \beta_i)| \leq 1.$$ 

Since $\ell_{x, \beta}$ and $\ell_{x, \beta_i}$ both pass through the initial doorway of $H_k$, we know $|\beta - \beta_i| \leq 1$ and so $k|x - x_i| \leq 2$. Thus, $|x - x_i| \leq 2/k$ and so, since $k$ was arbitrary, $|x - x_i| \to 0$. ■

The set $\tilde{d}_S$ is equivalent to what we are used to in a metric on $\mathbb{R}$, but the metric $\tilde{d}_R$ is much stranger.

**Proposition 51.** The set $\mathbb{R}\setminus \mathbb{Q}$ of irrational numbers is closed with respect to $\tilde{d}_R$.

**Proof.** Suppose $(x_i)$ is a sequence of irrational real numbers and $x \in \mathbb{Q}$. Further, suppose $\tilde{d}_R(x_i, x) \to 0$. This implies the existence of hallways $H^{(i)} \in s^{-1}(x_i)$ so that $H^{(i)} \to H \in s^{-1}(x)$ with respect to $d_R$. However, since $x \in \mathbb{Q}$, $H$ is a periodic infinite hallway, and so by Proposition 12 $H^{(i)} \not\to H$, a contradiction. ■

The set $\mathbb{R}\setminus \mathbb{Q}$ is clearly not closed under $\tilde{d}_S$. We now have an unusual situation. The set $\mathbb{Q}$ is dense in $\mathbb{R}$ (its closure is $\mathbb{R}$) under both $\tilde{d}_S$ and $\tilde{d}_R$, however $\mathbb{R}\setminus \mathbb{Q}$ is dense in $\mathbb{R}$ under $\tilde{d}_S$, but not $\tilde{d}_R$. Stranger still, according to the following proposition, $\tilde{d}_R$ does not change very much.

**Proposition 52.** The metrics $\tilde{d}_R$ and $\tilde{d}_S$ are equivalent when restricted to the set $\mathbb{R}\setminus \mathbb{Q}$ of irrational numbers.

**Proof.** First note that convergence in $d_R$ implies convergence in $d_S$ since $d_R(H^{(i)}, H) \to 0$ implies $|\text{comm}(H^{(i)}, H)| \to \infty$. Thus convergence in $\tilde{d}_R$ implies convergence in $\tilde{d}_S$.

Now, fix $\alpha \in \mathbb{R}\setminus \mathbb{Q}$ and $q \in \mathbb{N}$ and choose $\kappa > 0$ so that the interval

$$B_\kappa(\alpha) = (\alpha - \kappa, \alpha + \kappa) \subset \mathbb{R}$$
contains no rational points with denominator less than $q$. Since $\tilde{d}_S$ is equivalent to the standard metric on $\mathbb{R}$, there exists a $k$ so that $\tilde{d}_S(\alpha, \gamma) < 1/k$ implies $\gamma \in B_\kappa(\alpha)$.

Now, if $H$ is an infinite hallway admitting a line of sight of slope $\alpha$, then $d_S(H, H') < 1/k$ implies $d_R(H, H') \leq 1/q$. Since $q$ was arbitrary, if sequence of hallways converges to $H$ with respect to $d_S$, the same sequence converges to $H$ with respect to $d_R$. Thus, a sequence converging to $\alpha$ with respect to $d_S$ converges to $\alpha$ with respect to $d_R$. This holds on all of $\mathbb{R}$ so long as $\alpha \in \mathbb{R}\setminus \mathbb{Q}$, and therefore it holds on all of $\mathbb{R}\setminus \mathbb{Q}$. ■

We can also use $d_R$ to induce a metric on the set of sequences, $\mathbb{Z}^\mathbb{N}$, and in particular, the set of Sturmian sequences.

**Theorem 53.** For an infinite hallway $H$, $\Phi(H)$ is a Sturmian sequence if and only if $H$ admits an infinite line of sight $\ell'_{\alpha, \beta}$ with $\alpha \in [0, 1]$. 

Proof. Suppose $H$ is an infinite hallway with doorways $D_i$. Further, suppose $\Phi(H)$ is a Sturmian sequence. Then $\Phi(H) = R_{\lfloor \cdot \rfloor}((\alpha, \beta))$ or $\Phi(H) = R_{\lceil \cdot \rceil}((\alpha, \beta))$ for some $\alpha, \beta \in [0, 1] \times \mathbb{R}$. If $\Phi(H) = R_{\lfloor \cdot \rfloor}((\alpha, \beta))$ then, because we assume the initial doorway of $H$ is $D_0 = (0, 1)$, we have

$$D_i = ([i \alpha + \beta], [i \alpha + \beta] + 1).$$

Similarly, if $\Phi(H) = R_{\lceil \cdot \rceil}((\alpha, \beta))$,

$$D_i = ([i \alpha + \beta], [i \alpha + \beta] + 1).$$

In either case, $H$ admits the infinite line of sight $\ell_{\alpha \gamma}$ where $\gamma = \beta \mod 1$.

Now suppose that $H$ admits the infinite line of sight $\ell_{\alpha \beta}$. By Proposition 25, $(\ell_{\alpha \beta} + \epsilon) \cap H = \emptyset$ or $(\ell_{\alpha \beta} - \epsilon) \cap H = \emptyset$. Suppose $(\ell_{\alpha \beta} + \epsilon) \cap H = \emptyset$. Then

$$D_i = ([i \alpha + \beta], [i \alpha + \beta] + 1)$$

and so $\Phi(H) = R_{\lfloor \cdot \rfloor}((\alpha, \beta))$. Alternatively, suppose $(\ell_{\alpha \beta} - \epsilon) \cap H = \emptyset$. Then

$$D_i = ([i \alpha + \beta], [i \alpha + \beta] + 1)$$

and so $\Phi(H) = R_{\lceil \cdot \rceil}((\alpha, \beta))$. In either case, $\Phi(H)$ is a rotation sequence and therefore a Sturmian sequence.

Recall that $\Omega = \Phi(\{H : V(H) = 1\})$. In light of Theorem 53, $S = \Omega \cap \{0, 1\}^\mathbb{N}$ is the set of all Sturmian sequences. $S$ is $T$-invariant ($T(S) = S$), but it is not closed with respect to the standard metric. Let $\hat{d}_R$ be the metric on sequences induced by $d_R$. Again, $\hat{d}_R$ induces the same topology on $S$ as $d$, the standard metric on sequences, however under $\hat{d}_R$, $S$ is closed, and the set of periodic Sturmian sequences is dense.

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