Enhanced Random Walk with Choice: An Empirical Study

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Abstract. The random walk with choice is a well known variation to the random walk that first selects a subset of \( d \) neighbours nodes and then decides to move to the node which maximizes the value of a certain metric; this metric captures the number of (past) visits of the walk to the node. In this paper we propose an enhancement to the random walk with choice by considering a new metric that captures not only the actual visits to a given node, but also the intensity of the visits to the neighbourhood of the node. We compare the random walk with choice with its enhanced counterpart. Simulation results show a significant improvement in cover time, maximum node load and load balancing, mainly in random geometric graphs.

Keywords: Random Walk, Power of Choice, Cover Time, Load Balancing, Wireless networks.

1 Introduction

There is a growing interest in random walk-based algorithms, especially for a variety of networking tasks (such as searching, routing, self stabilization and query processing in wireless networks, peer-to-peer networks and other distributed systems [5, 1]), due to its locality, simplicity, low overhead and inherit robustness to structural changes. Many wireless and mobile networks are subjected to dramatic structural changes caused by sleep modes, channel fluctuations, mobility, device failures and other factors. Topology driven algorithms are inappropriate for such networks, as they incur high overhead to maintain up-to-date topology and routing information and also have to provide recovery mechanisms for critical points of failure. By contrast, algorithms that require no knowledge of network topology, such as random walks, are advantageous.

A random walk on a graph is a process of visiting the nodes of a graph in some sequential random order. The walk starts at some fixed node and at each

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step it moves to a randomly chosen neighbour of the current node. The cover time $C_G$ of a graph $G$ is the expected time taken by a simple random walk to visit all nodes in $G$ and the partial cover time $C_G(c)$ is the expected time to visit a fixed fraction $c$ of the nodes $n$; these metrics are widely used to evaluate the effectiveness of a random walk. An optimal cover time of a general graph is $\Theta(n \log n)$, i.e. has the same order as the cover time of the complete graph. Another performance metric is load balancing, which is the act of distributing objects among a set of locations as evenly as possible.

One class of graphs that has received particular attention in this context is the class of random geometric graphs widely adopted for modelling wireless ad hoc and sensor networks. A random geometric graph is a graph $G(n, r)$ resulting from placing $n$ points uniformly at random on the unit square and connecting two points if and only if their Euclidean distance is at most $r$. Recently, it has been proven that, when $r = \Theta(r_{con})$ then w.h.p. $G(n, r)$ has optimal cover time of $O(n \log n)$ and optimal partial cover time of $O(n)$ [6], where $r_{con}$ grows as $O\left(\frac{\log n}{n}\right)$ and is the critical radius to guarantee connectivity w.h.p. [12].

The basic idea behind the power of choice is to make some decision process more efficient by selecting the best among a small number of randomly generated alternatives. The most basic results [16] about the power of choice are as follows: Suppose that $n$ balls are placed into $n$ bins, with each ball being placed into a bin chosen independently and uniformly at random. Let the load of a bin be the number of balls in that bin after all balls have been thrown. What is the maximum load over all bins once the process is terminated? It is well known that, with high probability, the maximum load upon completion will be approximately $\log n / \log \log n$. We now state a surprising result proved in a seminal paper by Azar, Broder, Karlin, and Upfal [7]. Suppose that the balls are placed sequentially so that for each ball we choose 2 bins independently and uniformly at random and place the ball into the less full bin (breaking ties arbitrarily). In this case, the maximum load drops to $\log \log n + O(1)$ with high probability. If each ball has $d \geq 2$ choices instead, then the maximum load will be $\log \log n / \log d + O(1)$ with high probability. Having two choices hence yields a qualitatively different type of behavior from the single choice case, leading to an exponential improvement in the maximum load; having more than two choices further improves the maximum load by only a constant factor.

Random Walk with Choice, RWC($d$): Chen Avin and Bhaskar Krishnamachari combine the power of choice with Random Walks in [4] by introducing the Random walk with Choice, the ($RWC(d)$) in which, instead of selecting one neighbour at each step, the random walk selects $d$ neighbours uniformly at random and then chooses to move to the node that minimizes a certain metric (to be described). For the complete graph the analytical results show that the cover time of $RWC(d)$ is reduced by a factor of $d$. For general graphs the lack of the Markov property suggests that the analytical results may be harder to obtain. The simulation based study shows a consistent improvement in the cover time, cover time distribution and the load balancing at cover time for different graphs and different sizes.
Enhanced Random Walk with Choice, ERWC(d): In this work an enhanced random walk with choice is introduced. We refer to this random walk as the Enhanced Random Walk with Choice. At each node a metric is defined that captures not only the actual visits to a given node, but also the intensity of the visits to the neighbourhood of the node. The random walk updates at each step this metric as follows. At the current node of the walk, the metric is increased by a value \( h (h > 1) \) (which will be defined later), whereas the metric associated with the neighbouring nodes increases by 1. The walk selects \( d \) neighbours uniformly at random and then chooses to move to the node with the least value of the metric, divided by the degree of the node. The comparison of RWC(d), with ERWC(d), using simulation results, shows a significant improvement in cover time, maximum node load and load balancing.

The rest of the paper is organized as follows: Section 2 gives background and formal definitions. In Section 3 the enhanced random walk with choice is introduced. In Section 4 we describe the metrics of interest and the graphs under investigation. Section 5 presents the experimental results for random geometric and two dimensional torus graphs. We present our conclusions in Section 6.

2 Definitions - Background

Let \( G = (V, E) \) be an undirected graph with \( V \) the set of nodes and \( E \) the set of edges. Let \( |V| = n \) and \( |E| = m \). For \( v \in V \) let \( N(v) = \{v \in V | (vu) \in E\} \) the set of neighbours of \( v \) and \( \delta(v) = |N(v)| \) the degree of \( v \). A \( \delta \)-regular graph is a graph in which the degree of all nodes is \( \delta \).

The Random Walk (SRW) is a walk where the next node is chosen uniformly at random from a set of neighbours. That is, when the walk is at node \( v \) the probability to move in the next step to \( u \) is \( P(v, u) = \frac{1}{\delta(v)} \) for \((v, u) \in E \) and 0 otherwise. If \( \{v_t : t = 0, 1, 2, \ldots \} \) denotes the node visited by the SRW at step \( t \) then is a Markov chain.

The Random Walk with \( d \) Choice, RWC(d) in [4], is a walk whose next node to move to is determined as follows:

Let \( v \) denote the node reached by the walk at time \( t \); let \( c^t(v) \) be the number of visits to \( v \) to that node until time \( t \).

Algorithm 1: Upon visiting node \( v \) at time \( t \), the RWC(d):

1. Selects \( d \) nodes from \( N(v) \) independently and uniformly at random.
2. Steps to node \( u \) that minimizes \( \frac{c^t(u)+1}{\delta(u)} \) (break ties in an arbitrary way)

Cover Steps (CS): The Cover Steps \( C_v \) of a graph is the expected number of steps for the simple random walk starting at \( v \) to visit all the nodes in \( G \). In the last decade much work has been devoted to find the cover steps for different graphs and to give general upper and lower bounds. It was shown by Feige in [10, 11] that \((1 + o(1))n \log n < C_v < (1 + o(1)) \frac{1}{\delta} n^3\).

Cover Time (CT): The Cover Time \( C_g \) of a graph \( G \) is the maximum (over all starting nodes \( v \)) expected time taken by a random walk on \( G \) to visit all
nodes in $G$. Formally, for $v \in V$ let $C_v$ be the expected number of steps for the simple random walk starting at $v$ to visit all the nodes in $G$, and the cover time of $G$ is $C_g = \max_v C_v$. In other words, the cover time is the maximum of all cover steps. The cover time of graphs have been widely investigated [15, 2, 9, 8, 3, 17, 6, 13]. Results for the cover time of specific graphs vary from optimal cover time of $\Theta(n \log n)$ associated with the complete graph, to the worse case of $\Theta(n^3)$ associated with the lollipop graph. The best known cases correspond to dense, highly connected graphs; on the other hand, when connectivity decreases and bottlenecks exist in the graph, the cover time increases.

Load balancing (LB): Load balancing is the act of distributing objects, among a set of locations as evenly as possible. In random walks on graphs, load balancing refers to ..... a the number of visits per node as evenly distributed as possible. The problem is to traverse the graph in such a way that, at cover steps, all nodes will have about the same number of visits. Load balancing is of crucial interest in energy-limited wireless networks, where such protocols may be implemented.

3 Enhanced Random Walk with Choice, ERWC($d$)

The Enhanced Random Walk with $d$ Choice, ERWC($d$), is a walk whose next node to move to is determined as follows: Let $v$ be the node visited by the walk at time $t$; let $h^t(v)$ denote the value of a metric (to be specified below) associated with node $v$ at time $t$; let $i(v)$ denote an indicator function ..... whether node $v$ has been visited or not in the past; let $h > 1$.

Algorithm 2: Upon visiting node $v$ at time $t$, the ERWC($d$):

1. Selects $d$ nodes from $N(v)$ independently and uniformly at random. Let $M(v, d)$ denote the selected set of neighbours of $v$, $M(v, d) \subseteq N(v)$.
2. Modify $M(v, d)$ as follows
   
   $$M(v, d) = \begin{cases} 
   u, & \forall u \in M(v, d) \text{ and } i(u) = 0 \\
   M(v, d) \text{ remains the same.}
   \end{cases}$$

3. Steps to node $u \in M(v, d)$ that minimizes $\frac{h^t(u)}{i(u)}$ (break ties in an arbitrary way)

4. $i(u) = 1$, $h^t(u) = h^t(u) + h$, and $h^t(k) = h^t(k) + 1$, $\forall k \in N(v) - \{u\}$.

If $d > 1$ the Markov property does not hold any more since the current step depends on all the past steps. The lack of this property suggests that the analytical results may be harder to obtain. Therefore in the current work, we turn to a simulation based study of the behavior of the RWC($d$) and the ERWC($d$).
4 Metrics - Graphs under investigation

The main metrics of interest to be investigated are the CS and CT. In addition, the following metrics will be explored.

**Maximum Node Load at Cover Steps (MNLCS):** Is the number of visits of the most visited node at cover steps. The mean value of MNLCS expresses the load balancing. When MNLCS decreases, then the load balancing improves.

**Maximum Node Load at Cover Time (MNLCT):** Is the expected number of visits to the most visited node at cover time.

In this paper the simulation study considers random geometric graphs $G(n,r)$ (presented earlier) with $r = r_{\text{con}}$ [12] and two dimensional torus $T(m,n)$ described below.

**Two dimensional torus $T(m,n)$:** A 2-dimensional mesh $M(m,n)$ is a graph in which the vertices are arranged in a rectangular 2 dimensional array with dimensions $m$ and $n$. A 2-dimensional torus $T(m,n)$, is formed by connecting the vertices on opposite sides of the boundaries of the mesh. Every vertex in a 2-dimensional torus is connected to 4 other vertices. This graph provides a deterministic graph, which also has geometric locality and is also used to model wireless sensor networks. It is known that it has non-optimal cover time of $\Theta(n \log^2 n)$ [9].

5 Experimental Results

5.1 Simulation on Random geometric graphs

In the enhanced random walk with choice, we have to define the value of $h$. We look for a value of $h$ that, when is used in ERWC, minimizes both the mean value of cover steps and the mean value of maximum node load at cover steps, in comparison with other values of $h$.

We define $h$ for the random geometric graphs $G(900, 2r_{\text{con}})$. We run this simulation using ERWC with $d = 2, 3, 4$ for all $h$ from 2 to 133. For each $h$ we run the ERWC for 100 different graphs $G(900, 2r_{\text{con}})$. On each graph the walk starts at 2 different nodes, 2 times for each node. Finally, for each $h$, we calculate the mean value of ERWC cover steps and the mean value of ERWC maximum node load at cover steps.

From the Table of Figures 5.1.1 and the similar simulation data for $n = 100, 400, 1600$, we see that $h$ (when is used in ERWC) does not minimize the mean maximum node load at cover steps, in the same interval with the mean cover steps. We also have noticed that in the interval where $h$ minimizes the mean cover steps, the mean maximum node load at cover steps is bigger as compared to the mean maximum node load at cover steps in the RWC. So, we make the decision to choose the interval where $h$ minimizes the mean maximum node load at cover steps. We conclude that $h$ minimizes the ERWC maximum node load at cover steps on $G(900, 2r_{\text{con}})$ near 9.

Generally, we believe that the value of $h$, which minimizes the mean MNLCS for ERWC, is related to the degree of the nodes. Examining random geometric
graphs with $n = 100, 400, 900, 1600$, we highly speculate that if $d_n$ is the mean degree of the nodes of a $G(n, 2r_{con})$, then $h$ is near the area of $\frac{d_n}{n}$ and $\frac{d_n}{2}$. This is left to be examined in the future. For the random geometric graph $G(n, r)$ we use $n = 900$, $r = 2r_{con}$ and $h = 9$ for the simulations.

We run the simulation of random walk algorithms 2000 times for one instance of the graph $G(900, 2r_{con})$ with $h = 9$ and $d = 2$. Firstly, we examine the mean cover steps, Mean CS and the mean maximum node load at cover steps, (Mean MNLCS). The mean cover steps are normalized by $n$. We work with one graph, so we have cover time, CT and maximum node load at cover time, MNLCT. The cover time is also normalized by $n$. Table 5.1.1 shows the improvement in mean CS, mean MNLCS, CT, MNLCT comparing ERWC(2) with the RWC(2) running on one instance of the graph.

Table 5.1.1: Simulation results of random walks and comparison of the ERWC(2) with the RWC(2) on $G(900, 2r_{con})$, $d = 2$, $h = 9$

|            | Mean CS | Mean MNLCS | CT  | MNLCT |
|------------|---------|------------|-----|-------|
| SRW        | 15.67   | 40.29      | 71.71 | 141   |
| RWC(2)     | 6.24    | 11.30      | 19.24 | 30    |
| ERWC(2)    | 4.64    | 8.61       | 10.86 | 17    |
| Improvement| 25.64%  | 23.8%      | 43.55%| 43.33%|

Table of Figures 5.1.1: Calculation of $h$ on $G(900, 2r_{con})$
Cover time is the worse case among all cover steps. Cover time and maximum node load at cover time are defined for one graph, but as we want results not depending to one instance of a graph but to a family of graphs $G(900, 2r_{con})$, $d = 2, 3, 4, h = 9$, we run the simulation of random walks algorithms on 1000 different graphs. On each graph the walk starts at 2 different nodes, 2 times for each one. The mean cover steps are normalized by $n$. Table 5.1.2 shows the improvement in mean CS, mean MNLCS, comparing ERWC(2,3) with the RWC(2,3) running on many instances of the graph.

|                | RWC(2) | ERWC(2) | Improvement |
|----------------|--------|---------|-------------|
| Mean Cover Steps | 6.44   | 4.87    | 24.37%      |
| Mean MNLCS     | 12.12  | 8.93    | 26.32%      |

In Figure 5.1.1 we can see graphically the improvement in mean cover steps and mean partial cover steps comparing SRW, RWC(2) and ERWC(2) on $G(900, 2r_{con})$, $d = 2, h = 9$.

Figure 5.1.2 is very important as it shows the improvement in distribution of mean number of visits per node, which is very closely related to load balancing, on $G(900, 2r_{con})$, $d = 2, h = 9$. We clearly see that the enhanced random walk with choice, pushes to left the distribution of the mean number of visits per node, causing the load of visits to be distributed more evenly in comparison with the random walk with choice.

Figure 5.1.3 shows the sorted cover steps distribution of the 4000 runs of the algorithms SRW, RWC(2) and ERWC(2) on random geometric graphs $G(900, 2r_{con})$, $d = 2, h = 9$. We can see graphically the improvement in cover steps, by comparing the ERWC(2) with the RWC(2). Figure 5.1.4 shows the sorted maximum node load at cover steps distribution, of the 4000 runs of the algorithms SRW, RWC(2) and ERWC(2), on random geometric graphs $G(900, 2r_{con})$, $d = 2, h = 9$. We can see graphically the improvement in maximum node load at cover steps, by comparing the ERWC(2) with the RWC(2). This distribution is very closely related to load balancing at cover steps and shows the improvement in load balancing at cover steps.

5.2 Simulation on 2-dimensional torus

As previous on random geometric graphs, we will define the value of $h$ which minimizes the ERWC mean maximum node load at cover steps, for the 2-dimensional
Random Walks on Random Geometric Graphs with \( n = 900 \)

**Mean cover steps normalized**

| Fraction of cover | \( \text{SRW} \) | \( \text{RWC(2)} \) | \( \text{ERWC(2)} \) |
|-------------------|----------------|----------------|----------------|
| 0.00              | 2.00           | 2.00           | 2.00           |
| 0.05              | 1.95           | 1.95           | 1.95           |
| 0.10              | 1.90           | 1.90           | 1.90           |
| 0.15              | 1.85           | 1.85           | 1.85           |
| 0.20              | 1.80           | 1.80           | 1.80           |

**Mean Number of visits per node**

| Probability | \( \text{SRW} \) | \( \text{RWC(2)} \) | \( \text{ERWC(2)} \) |
|-------------|----------------|----------------|----------------|
| 0.00        | 2.00           | 2.00           | 2.00           |
| 0.05        | 1.95           | 1.95           | 1.95           |
| 0.10        | 1.90           | 1.90           | 1.90           |
| 0.15        | 1.85           | 1.85           | 1.85           |
| 0.20        | 1.80           | 1.80           | 1.80           |

**Figure 5.1.1:** Mean cover steps and mean partial cover steps on \( G(900, 2r_{\text{con}}) \), \( d = 2 \), \( h = 9 \)

**Figure 5.1.2:** Distribution of number of visits per node on \( G(900, 2r_{\text{con}}) \), \( d = 2 \), \( h = 9 \)

We run this simulation using ERWC with \( d = 2, 3, 4 \) for all \( h \) from 2 to 133. For each \( h \) we run the ERWC for the graph \( T(30, 30) \) 100 times. Each time the walk starts at 2 different nodes, 2 times for each node.

From the Table of Figures 5.2.1 we conclude that \( h \) minimizes the ERWC mean maximum node load at cover steps on \( T(30, 30) \) near 3.

More generally we believe that the value of \( h \), which minimizes the mean MNLCS for ERWC on 2-dimensional torus, is related with the mean degree of the nodes. Examining 2-dimensional torus with nodes = 100, 400, 900, 1600, we highly speculate that if \( d_n \) is the degree of every node, then \( h \) is near the area of \( \frac{d_n}{3} \) and \( \frac{d_n}{2} \). This is left to be examined in future. We also high speculate that this value of \( h \) is a lower threshold. Above this value the algorithm ERWC has about the same results (mean value of cover steps and mean value of maximum node load). Again this is left to be examined in future.

We run the simulation of the random walk algorithms 1000 times for the 2-dimensional torus \( T(30, 30) \) with \( h = 3 \). Each time the walk starts at 2 different nodes, 2 times for each node. The mean cover steps are normalized by \( n \).

Table 5.2.1 shows the improvement in mean CS and mean MNLCS comparing the ERWC(2,3,4) with the RWC(2,3,4).
Table 5.2.1: Comparison of RWC(2,3,4) and ERWC(2,3,4) on $T(30, 30)$, $h = 3$

|                | RWC(2) | ERWC(2) | Improvement  |
|----------------|--------|---------|-------------|
| Mean Cover Steps | 4.20   | 3.66    | 12.85%      |
| Mean MNLCS      | 7.32   | 6.44    | 12.01%      |
| Mean Cover Steps | 2.75   | 2.36    | 14.18%      |
| Mean MNLCS      | 4.80   | 4.10    | 14.58%      |

The cover time, CT and the best case of cover steps (bcCS), are normalized. The bcMNLCS means mean maximum node load at best case of cover steps.

Table 5.2.2: Comparison of ERWC(2) with RWC(2) on $T(30, 30)$, $d = 2$ $h = 3$

|                | CT    | MNLCT | bcCS | bcMNLCS |
|----------------|-------|-------|------|---------|
| SRW            | 36.33 | 74    | 8.98 | 32      |
| RWC(2)         | 7.77  | 11    | 2.77 | 6       |
| ERWC(2)        | 7.46  | 10    | 2.49 | 5       |
| Improvement    | 3.98% | 9.09% | 10.1%| 16.66%  |

Table 5.2.2 and 5.2.3 show the improvement in CT, MNLCT, bcCS and bcMNLCS comparing the ERWC(2,3) with the RWC(2,3). The improvement comparing ERWC with RWC, for the 2-dimensional torus, is not analogous to the improvement of the same algorithms on random geometric graphs. The reason is that, in 2-dimensional torus, the neighbourhood of a node has 4 nodes. So if in random walk with choice we sample 2 nodes, actually we sample the half neighborhood.
|                | CT   | MNLCT | bcCS | bcMNLCS |
|----------------|------|-------|------|--------|
| SRW            | 35.53| 78    | 7.97 | 27     |
| RWC(3)         | 4.48 | 7     | 1.92 | 4      |
| ERWC(3)        | 3.74 | 6     | 1.79 | 3      |
| Improvement    | 16.51%| 14.28%| 6.77%| 25%    |

of a node and the enhanced random walk with choice makes no big difference in selecting a better next step.

6 Conclusions

In this work we introduce the Enhanced Random Walk with Choice, ERWC(d), to further improve the Random Walk with Choice, RWC(d). The comparison of the ERWC(d) with the RWC(d), using the simulation results, shows a significant improvement in cover time, maximum node load and load balancing at cover time, mainly on random geometric graphs and on 2-dimensional torus. For example the improvement of our algorithm ERWC with respect to the cover time is about 44%, 10% and with respect to maximum node load at cover time is about 44%, 12% in comparison with RWC, on random geometric graphs and 2-dimensional torus, respectively.

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