We investigate the scattering behavior between a unit cell with PT-symmetry and its integrated system composed of the N-cells. In particular, we discuss the formations of propagating transmitted wave, bi-directional reflectionless, and coherent perfect absorber-laser (CPAL) occurred at a finite periodic optical waveguide network. Through the use of parametric space obtained from considerations of PT-symmetric transfer matrix and Lorentz reciprocity theorem, our outcome is regardless of operating frequency, system configurations, and materials embedded, also valid for any two-port wave systems. Hence, we can interpret the relationship among the Bloch phase and the consequent propagating transmitted wave. We observe that when its building unit cell is operated at PT broken symmetric phase or an exceptional point, the integrated system would always have a propagating transmitted wave, independent of the number and transmission phase of the unit cell. On the other hand, when the unit cell is operated at PT-symmetric phase, the formation of propagating transmitted waves would depend on the transmission phase of that. We find that even though the unit cell is not operated at the exceptional point, with a proper number of unit cells and operation of specific PT-phase, it can eventually achieve the reflectionless with bi-directionality. To implement CPAL in an integrated system, there are two approaches. One is by unit cells having CPAL, while its construction number has to be odd. Another one is through operation of specific broken symmetric phases and a proper number of unit cells, while the transmission phase is required to be null. We believe this work could offer an alternative means to observe extraordinary wave phenomena of PT-symmetric photonics by a replacement of a finite periodic structure.

PACS numbers:

I. INTRODUCTION

Inspired by an extension of quantum mechanics [1], systems with parity-time (PT) symmetry had been studied in a variety of wave physics, such as photonics [2], acoustics [3–5], electric circuits [6–9], elastic plate [10, 11], coupled mechanical oscillators [12, 13], and so on. One dimensional photonics with parity-time (PT) symmetry would demand that the real part of refractive index in spatial placement has an even symmetry, while the imaginary part has an odd symmetry, corresponding to gain (amplification) and loss media (attenuation) embedded [14]. Such system belongs to non-Hermitian, while is invariant under a combined PT operation. The principal studying schemes in one-dimensional PT-symmetric photonics could be classified into longitudinal and transverse. In the longitudinal scheme, the propagating wave governed by paraxial equation of diffraction (Schrödinger-like equation) as well as coupled-mode formalism, would simultaneously experience gain and loss media. It had been experimentally and theoretically explored in coupled optical waveguides based on Fe-doped LiNbO$_3$ and AlGaAs [15–17]. In the transverse one, the propagating wave described by transfer matrix or scattering matrix, would go through an ordering spatial placement for gain and loss slab media [18–21]. Although it seems that foregoing systems are intrinsically distinct, the scattering behavior can be in general categorized into PT symmetry and broken symmetry phases. In between, there has an exceptional point with degenerate eigenvalues, corresponding to the onset of symmetry-breaking transition.

More specifically, in the transverse, once the PT-symmetric system is operated at PT symmetry phase (broken symmetry phase), the corresponding magnitudes of scattering eigenvalues form unimodulus and distinguishable (reciprocal pairs), and transmittance is lower (higher) than unity. Systems with eigenstates of the symmetry phase would exhibit either no net amplification or no net attenuation. On the contrary, at the symmetry phase, it would correspond to either amplification or attenuation for these eigenstates. At an exceptional point, the system could display unity transmittance as well as uni-directional reflectionless [21] or bi-directional reflectionless [22]. Recently, we observe the possibility of symmetry reflectances upon two opposite incidences, when the system is operated at specific PT symmetry phase or exceptional point. We note that PT-symmetric systems violates parity alone [23]. In an exceptional point, the delay time is divergence due to the absence of reflected waves [24]. The emergence of exceptional points is not a signature of PT-symmetric systems, while it is associated with non-Hermiticity systems involving gain, loss or unbalance gain-loss [25–26].

These distinctive PT phases can serve as exotic functionalities, for example, uni-directional invisibility [21], subdiffraction imaging [27, 28], coherent perfect absorber-laser [19, 20], PT-symmetric whispering gallery mode resonators [32, 33], uni-directional optical pulling force [34, 35], superior sensing capability [7, 8, 37, 38], single-mode microrings lasing [39, 40], and Bloch oscilla-
In particular, many works were established in a simple gain-loss architecture. Although several works involving a finite periodic PT-symmetric structures in one dimension had been explored in Refs. [21, 28, 32–44], the relationship of PT phases, Bloch phase, and propagating transmittance between the building unit cell and its integrated system still remains open. In this work, we investigate the scattering behavior occurred at a finite periodic optical waveguide network in which its unit cell has PT symmetry. By a means of parametric space obtained from PT-symmetric transfer matrix and Lorentz reciprocity theorem, we can not only interpret the scattering behavior, but also discuss the formations of propagating transmission, bi-directional reflectionless, and coherent perfect absorber-laser (CPAL). We could observe that when the building unit cell is operated at broken symmetric phase or an exceptional point, the resultant integrated system would enable a propagating transmitted wave, independent of total number and transmission phase of the unit cell. However, it is not the case for the unit cell operated at symmetry phase. Moreover, we find that even the unit cell is not operated at an exceptional point, by embedding a proper number of cells and operation of specific PT-phase, it can eventually achieve reflectionless with bi-directionality. In addition, to implement CPAL, there are two approaches. One is by the unit cell having CPAL, while its construction number has to be odd. Another one is through a proper number of unit cells as well as operation of specific broken symmetric phases with null transmission phase of the unit cell. This work could provide an alternative method to realize extraordinary waves by using a finite periodic structure.

II. THEORY

We consider a PT-symmetric periodic structure by an optical waveguide network, as shown in Fig. 1(a). Its construction is formed by finite periodic unit cells in which there are two separated waveguide segments. These segmental waveguide are in absence of wave coupling. Each segment is made of two sub-segmental waveguides connected where one is gain and another is loss. Here the corresponding refractive indexes for gain and loss are \( n_g \) and \( n_l = n_g^* \), respectively, and the lengths for each sub-segmental waveguides are \( \frac{1}{2} \), in order to meet PT-symmetry condition. Although our studying system is an optical waveguide network, the description of wave propagation is equivalent to that in the transverse scheme. We note that the study of optical waveguide networks has a related extensive in neural deep learning [45] and artificial photonic crystals [46, 47]. We let the transfer matrix of the unit cell be \( M_1 \), and due to PT-symmetry embedded, it obeys \( M_1^* = M_1^{-1} \). In addition, since the left and right leads of the optical waveguide network are connected to identical lossless waveguides, according to Lorentz reciprocity theorem, we have 

\[
Det[M_1^N] = 1.
\]

Here \( N \) denotes total number of unit cells. The refractive index for the lossless waveguide is \( n_1 \). Moreover, since \( Det[M_1^N] = 1 \) is valid for any \( n \), we can argue that it would lead to \( Det[M_1] = 1 \). Combined with \( M_1^* = M_1^{-1} \) and \( Det[M_1] = 1 \) for the unit cell, we can also find \( (M_1^N)^* = (M_1^N)^{-1} \). The detailed analysis is placed in appendix A. This result means that the whole optical waveguide network has PT-symmetry invariant. The transfer matrix for the unit cell can be parametrized by

\[
M_1 = \left[ \begin{array}{c c}
\sqrt{1 - \alpha_1 \beta_1 e^{i\phi_1}} & i \alpha_1 \\
i \beta_1 & \sqrt{1 - \alpha_1 \beta_1 e^{-i\phi_1}}
\end{array} \right]
\]

(1)

here \( \alpha_1 \) and \( \beta_1 \) are reals and \( \phi_1 \) is transmission phase bound by \([0, 2\pi]\), Refs [23, 36]. The exact description of \( \alpha_1, \beta_1, \) and \( \phi_1 \) would depend on system configurations. The use of parameterization is valid not only for any PT-symmetric systems with two-ports input and output, but also for a variety of wave physics. In appendix B, we derive the corresponding transfer matrix for our optical waveguide networks.

Due to Lorentz reciprocity theorem established in a unit cell, i.e., \( Det[M_1] = 1 \), there has a constraint for \( \alpha_1 \) and \( \beta_1 \), i.e., \( \alpha_1 \beta_1 \leq 1 \). For an optical waveguide network composed of identical N-unit cells, with the help of Chebyshev identity, the corresponding transfer matrix reads

\[
M_1^N = \left[ \begin{array}{c c}
\sqrt{1 - \alpha_1 \beta_1 e^{i\phi_1}} & i \alpha_1 \\
i \beta_1 & \sqrt{1 - \alpha_1 \beta_1 e^{-i\phi_1}}
\end{array} \right]^N
\]

\[
M_1^N = \left[ \begin{array}{c c}
\sqrt{1 - \alpha_1 \beta_1 e^{i\phi_1} U_{N-1} - U_{N-2}} & i \alpha_1 U_{N-1} \\
i \beta_1 U_{N-1} & \sqrt{1 - \alpha_1 \beta_1 e^{-i\phi_1} U_{N-1} - U_{N-2}}
\end{array} \right] = \left[ \begin{array}{c c}
\frac{1}{L_N} & r_{R,N} \\
\frac{1}{L_N} & \frac{1}{L_N}
\end{array} \right]
\]

where \( t_N \) is transmission coefficient, \( r_{R,N} \) is right reflection coefficient, \( r_{L,N} \) is left reflection coefficient, \( U_N = \frac{\sin((N+1)\Phi)}{\sin \Phi} \), and \( \Phi \) is the Bloch phase [48]. The Bloch phase is related to the eigenvalues \( (\lambda_1 \) and \( \lambda_2 \) of \( M_1 \),

\[
\cos \Phi = \frac{1}{2}(\lambda_1 + \lambda_2) = \frac{1}{2} Tr[M_1] = \sqrt{1 - \alpha_1 \beta_1 \cos[\phi_1]}.
\]
The occurrence of $\text{Im} [\Phi] = 0$ enables incident waves propagating through periodic systems.

We now turn to discuss general scattering behaviors from any two-ports system with PT-symmetry, whose the transfer matrix is defined as $M$. The eigenvalues of the scattering matrix $S$, which is linear relation between outgoing and incoming waves, can be expressed as $S_{\pm} = \frac{e^{i \text{Arg} [M_{11}]} \left(1 \pm \sqrt{1 - \text{Im} [M_{12}] \text{Im} [M_{21}]}\right)}{\sqrt{1 - \text{Im} [M_{12}] \text{Im} [M_{21}]} \left(1 \pm \sqrt{1 - \text{Im} [M_{12}] \text{Im} [M_{21}]}\right)}$. The transmission and reflection coefficients from right or left incidences are $t = \frac{\text{Im} [M_{12}] e^{i \text{Arg} [M_{11}]} \left(1 \pm \sqrt{1 - \text{Im} [M_{12}] \text{Im} [M_{21}]}\right)}{\sqrt{1 - \text{Im} [M_{12}] \text{Im} [M_{21}]} \left(1 \pm \sqrt{1 - \text{Im} [M_{12}] \text{Im} [M_{21}]}\right)}$, and $r_L = \frac{\text{Im} [M_{21}] e^{i \text{Arg} [M_{11}]} - \frac{\pi}{2}}{-\sqrt{1 - \text{Im} [M_{12}] \text{Im} [M_{21}]}}$, respectively. In PT-symmetric phase, the eigenvalues of scattering matrix are unimodular in magnitudes and distinguishable, while the transmittance has $T < 1$. Here $T = |t|^2$. The symmetry phase requires $\text{Im} [M_{12}] \text{Im} [M_{21}] < 0$. In broken PT-symmetric phase, the scattering eigenvalues become a reciprocal pair in magnitudes, while $T > 1$. It requires $0 < \text{Im} [M_{12}] \text{Im} [M_{21}] \leq 1$. System with these eigenvectors would exhibit either amplification or attenuation. However, eigenvalues can also exist for extreme zero and infinity occurred at this broken symmetric phase, supporting the hybrid functionality of CPAL. It obeys $\text{Im} [M_{12}] \text{Im} [M_{21}] = 1$. We note that the extrinsic scattering behavior cannot perform CPAL simultaneously, but it can be switchable by proper incoming waves. PT-symmetric systems with CPAL can display high sensitivity to detect extremely small admittance perturbations over standard electromagnetic sensors [37].

Between symmetry and broken symmetry phases, there has an exceptional point with degenerate eigenvalues $S_+ = S_- = e^{i \text{Arg} [M_{11}]}$. The system can exhibit unity transmittance as well as unidirectional reflectionless ($R_L = |R_L|^2 = 0$ or $R_L = |R_L|^2 = 0$) or bidirectional reflectionless, ($R_L = R_L = 0$). It can be $\text{Im} [M_{12}] = 0$, $\text{Im} [M_{21}] = 0$, or $\text{Im} [M_{12}] = \text{Im} [M_{21}] = 0$, respectively. As our demonstration in Refs. [23], only $\text{Im} [M_{12}]$ and $\text{Im} [M_{21}]$ are responsible for symmetry phase, broken symmetry phase, and an exceptional point, irrespective of $\text{Arg} [M_{11}]$. As a result, we provide a parametric space valid for any PT-symmetric unit cells related to $\text{Im} [M_{12}]$ and $\text{Im} [M_{21}]$ with indication of symmetry phase, broken symmetry phase, exceptional point, and CPAL marked by brown region, yellow region, blue dashed line, green dashed line, and red solid line, respectively, as shown in Fig. 1 (b). The parametric space reflects all scattering behaviors for any PT-symmetric systems, irrespective of system configuration, materials, and operating frequency. We note that the white region represents inaccessible parametrization.

III. NUMERICAL RESULTS AND DISCUSSION- FORMATIONS OF PROPAGATING WAVES, BIDIRECTIONAL REFLECTIONLESS AND CPAL

Propagating wave: The absence of the imaginary part of the Bloch phase, i.e., $\text{Im} [\Phi] = 0$, denotes that incident waves enable to propagate through the finite periodic systems. We should note that Eq. (3) reveals that the Bloch phase depends on the unit cell configurations. Combined with Eq. (3) and parametrization, we provide a 3D plot to display solutions supporting propagating waves, i.e., $\text{Im} [\Phi] = 0$, marked by orange color in Fig. 1 (c). Interestingly, not every PT phase can support propagating waves.

More specifically, we mark the non-imaginary value of the Bloch phases by using $\phi_1 = 0, 1, 4, 6$ in Fig. 1 (d). We can observe that some regions of the symmetry phase would have complex numbers of the Bloch phase. In order to understand the behind mechanism, with the help of parametrization, we provide analysis in the following. In the broken symmetry phase and exceptional point, it requires $0 < \alpha_1 \beta_1 \leq 1$ and $\alpha_1 \beta_1 = 0$, respectively. These conditions would have non-imaginary values of the Bloch phase, because $0 \leq \sqrt{1 - \alpha_1 \beta_1} < 1$. Thus, any PT-symmetric systems operated at the broken symmetry phase or exceptional point would always have propagating waves. However, in the symmetry phase, it corresponds to $\alpha_1 \beta_1 < 0$, so the factor of Eq. (3) would be subject to $\sqrt{1 - \alpha_1 \beta_1} > 1$, leading to the occurrence of complex numbers of the Bloch phase, as clearly shown in the white color of Fig. 1 (d). This means that incident waves through such PT-symmetric systems would become evanescent waves.

To demonstrate our findings, we use $n_1 = 1.5$, $n_1 = 3.1 + 0.7i$ and $k_0 = 1$ in our finite periodic waveguide network as in Figs. 2 (a) and (b). In the Fig. 2 (a), we use $l = 2.2$. This set of parameters has an absence of the imaginary part of the Bloch phase. Moreover, the unit cell is operated at broken symmetry phase. Now, we use the unit cell as the construction unit and calculate the resultant transmittance $T_N$ with $N$, in the right panel of Fig. 2 (a). This case supports that incident waves can propagate through the periodic system with any N-cells. In Fig. 2 (b), we use $l = 2.1$. From the parametric space, we can see that this set of parameters has a complex number of the Bloch phase and the unit cell is operated at symmetry phase. In the right panel of Fig. 2 (b), we calculate $T_N$ with $N$, supporting non-propagating wave results.

Transmittance relation: To figure out the transmittance relation between a unit cell and its integrated system, [42] had proposed

$$1 - \frac{1}{T_N} = \left(1 - \frac{1}{T_1}\right) \frac{\sin^2 |N \Phi|}{\sin^2 \Phi}$$

obtained from an extension of generalized conservation relation.
FIG. 1: (a) Schematic of an optical waveguide network with a finite periodic PT-symmetric structure. For each building element, i.e., unit cell, it is composed of two segments of waveguides. Each segment is made of gain and loss waveguides connected, as marked by red and blue lines, respectively. The total arc-lengths of gain and loss waveguides are \( l_2 \) and the corresponding refractive indexes for gain and loss are \( n_g \) and \( n_l = n_g^* \), respectively. In the left and right leads, it is constructed by lossless waveguide with refractive index \( n_1 \). Here \( A \) and \( B \) (\( C \) and \( D \)) are complex amplitudes of scalar plane waves in the left lead (right lead) toward right and left propagating, respectively. Description of wave propagation in our waveguide network can be equivalent to two-ports systems as shown in the right panel. In (b), we plot a parametric space valid for any PT-symmetric unit cell. We mark symmetry phase, broken symmetry phase, exceptional point, and CPAL by brown region, yellow region, blue dashing line, green dashing line, and red solid line, respectively. The white region denotes forbidden parametrization. For the Bloch phase, Eq.(3), in terms of parametrization \( \alpha_1, \beta_1, \) and \( \phi_1 \), we can construct a 3D plot with orange color in (c) to show accessible parametrization supporting propagating waves. By taking transmission phases \( \phi_1 = 0, 1, 4, 6 \), we mark the non-imaginary complex values of the Bloch phase \( \Phi \) in the parametric space in (d). Obviously, some accessible parametrization now have complex numbers of the Bloch phase illustrated in the white color.

Now, due to non-negative property of \( \frac{\sin^2[N\Phi]}{\sin^2\Phi} \geq 0 \), we can further obtain \( T_N \leq 1 \), when the unit cell operated at the symmetry phase with \( T_1 < 1 \). This result indicates that the scattering behaviors from any periodic systems made of unit cells with the symmetry phase would have the resultant symmetry phase or exceptional point, for any \( N \). Now, if the unit cell is operated at the exceptional point, it obeys \( T_1 = 1 \). Again, with non-negative property of \( \frac{\sin^2[N\Phi]}{\sin^2\Phi} \geq 0 \), we can obtain \( T_N = 1 \). Thus, any
periodic systems composed of unit cells with exceptional points would always be at the exceptional point. For the unit cell with the broken symmetry phase, we can obtain $T_N \geq 1$ due to that $T_1 > 1$. As a result, any periodic systems made of the broken symmetry phase cells would eventually perform the broken symmetry or exceptional point, as indicated in Fig. 2 (a).

**Exceptional point.** Any PT-symmetric systems operated at the exceptional point would accompany by unidirectional or bidirectional reflectionless, as well as unity transmittance [22, 30]. The corresponding transmission phase can be arbitrary, so any system operated at exceptional points can not be regarded as invisibility [22]. Any unit cells with the exceptional point can have ($M_1$)$_{12} = 0$ or ($M_1$)$_{21} = 0$ or ($M_1$)$_{12} = (M_1)_{21} = 0$, corresponding to $\alpha_1 = 0$ or $\beta_1 = 0$ or $\alpha_1 = \beta_1 = 0$. By employing these conditions to $M_1^N$, it can lead to the resultant periodic PT-symmetric systems having exceptional point. Another possibility to have the exceptional point in $M_1^N$, in the absence of the exceptional points for unit cells, is

$$\sin[N\Phi] = 0,$$  \hspace{1cm} (5)

obtained from Eq. (2), already proposed by Refs. [43, 44]. We note that this condition would support the bidirectional reflectionless, because it results in ($M_1^N$)$_{12} = 0$ and ($M_1^N$)$_{21} = 0$ ($r_{R,N} = 0$ and $r_{L,N} = 0$).

Alternatively, it is desirable to see the Eq. (5) depicted in the parametric space combined with consideration of the non-imaginary Bloch phase, which can understand PT phase relations between the unit cells and its integrated systems. In Fig. 3 (a), we use $n_1 = 1.5$, $n_1 = 3.1 + 0.7i$, $k_0 = 1$, and $l = 1.8$. The corresponding PT phase of the unit cell is depicted by a yellow dot, belonging to the symmetry phase and real Bloch phase. We show the solution of $\sin[N\Phi] = 0$ with $N = 5$ marked by the black lines, that occur at the symmetry phase and broken symmetry phase. We can see that our considering system can meet $\sin[5\Phi] = 0$. The scattering of the periodic system can perform exceptional points, even the PT-symmetric unit cell is not at the exceptional point. In Fig. 3 (b), we calculate the corresponding transmittance $T$, left reflectance $R_{L,N} = |r_{L,N}|^2$, and right reflectance $R_{R,N} = |r_{R,N}|^2$ with $N$. We note that two reflectances for the unit cell are asymmetry, but when $N = 5$, we find $T = 1$ and $R_{L,N} = R_{R,N} = 0$, supporting the bidirectional reflectionless.

**Coherent perfect absorber-laser (CPAL)-** PT-symmetric systems with CPAL would have zero and infinity eigenvalues of the scattering matrix, corresponding to coherent perfect absorber and laser, respectively [19, 20]. To have CPAL occurred at periodic PT-symmetric systems, it requires ($M_1^N$)$_{11} = (M_1^N)_{22} = 0$ and $\text{Im}[(M_1^N)_{12}] \text{Im}[(M_1^N)_{21}] = 1$. There are two solutions: (a) the unit cell performs CPAL and the corresponding construction number is odd,

$$\alpha_1\beta_1 = 1 \hspace{1cm} N = 1, 3, 5, ..$$

(b) The unit cell is not at CPAL, but there have specific PT phases for the unit cell and construction number,

$$\alpha_1\beta_1 \neq 1 \hspace{1cm} \cos[N\Phi] = 0.$$ \hspace{1cm} (7)

The derivation is placed at appendix C.

To verify the case (a), we prepare another set of parameters for our waveguide network by $n_1 = 2$, $n_1 = 3.1 + 0.5i$, $k_0 = 1$, and $l = 6.7$ in Fig. 3 (c). The corresponding transmission phase for the unit cell is $\phi_1 = 5.07$. We note that the scattering behavior of the unit cell is not perfect, but close to, CPAL. We analyze the PT phase and the Bloch phase for the unit cell in Fig. 3 (c). The unit cell is operated at CPAL (almost). In Fig. 3 (d),

![Image](image_url)
we provide the corresponding transmittance and two reflectances with respect to $N$. As expectation, when $N$ is odd, the values of $T$, $R_{R,N}$, and $R_{L,N}$ become huge.

To verify the case (b), the parameters $\eta_1 = 0.85$, $\eta_2 = 1.61 + 1.49i$, $k_0 = 1$, and $l = 0.75$ are used in Fig. 3 (c). The corresponding transmission phase of the unit cell is null, $\phi_1 = 0$. We analyze the PT phase of the unit cell and the Bloch phase in Fig. 3 (c), marked by a Magenta dot and light blue region, respectively. This unit cell is not operated at CPAL. Now, we calculate the $T$, $R_{R,N}$, and $R_{L,n}$ with respect to $N$. When $N = 7$, the transmittance and two reflectances become sudden huge. This result supports the CPAL occurred at this system. We should stress that as discussed in the theory section, the solutions of $\cos[N\Phi] = 0$ can not be occurred at the symmetry phase.

IV. CONCLUSION

With Lorentz reciprocity theorem and PT-symmetry condition, we employ generalized parametric space to analyze the scattering behavior between unit cells and its integrated systems. We mark the non-imaginary complex values of the Bloch phase in the framework of the parametric space, to indicate propagating waves. When the unit cell is operated at the broken symmetry phase and exceptional point, it can guarantee to have incident waves propagating through periodic structures of any $N$. However, in the symmetry phase, the occurrence of the non-imaginary value of the Bloch phase would depend on the transmission phase of the unit cell. We also investigate the formations of bi-directional reflectionless and CPAL in finite periodic PT-symmetric systems, with the help of the parametric space with consideration of the non-imaginary complex values of the Bloch phase. Even the unit cell is at the absence of the exceptional point, with specific PT phase and N-cells, the N-cells system can perform the exceptional point with bi-directional reflectionless. To implement CPAL, there are two solutions. One is by unit cells having CPAL, while its cell number needs to be odd. Another is by unit cells having specific broken symmetry phase and N-cells. Interestingly, its transmission phase has to be null. We provide several examples to support our finding. We believe this work can provide an alternative means to achieve extraordinary wave phenomena by using finite periodic PT-symmetric systems.

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FIG. 3: (a) Realization of the exceptional points at finite periodic PT-symmetric systems. (b) The corresponding transmittance and two reflectances with N. Here the parameters we use are $n_1 = 1.5, n_l = 3.1 + 0.7i, k_0 = 1$, and $l = 1.8$, whose the PT phase is depicted by a yellow dot. Obviously, this unit cell is operated at the symmetry phase and the corresponding transmission phase is $\phi_1 = 5.34$. The black lines are the solutions of $\sin(5\Phi) = 0$ occurred at the symmetry and broken symmetry phases. We can see that two reflectances are asymmetry at $N = 1$, but at $N = 5$ they become null and $T_N = 1$, supporting the exceptional point with bi-directional reflectionless. In (c), we consider unit cells having CPAL. The transmission phase is $\phi_1 = 5.07$. When we employ this unit cell as the construction, with odd N number, the $T_N, R_{L,N}$, and $R_{R,N}$ become huge in (d). This result supports one of solutions to realize CPAL. Here the parameters are $n_1 = 2, n_l = 3.1 + 0.5i, k_0 = 1$, and $l = 6.7$. In (e), we consider another solution to CPAL. The unit cell is not operated at CPAL, but with specific PT phase and construction number $N = 7$ to meet $\cos(7\Phi) = 0$, the resultant system made of this unit cell can perform CPAL. In (f), we calculate the $T_N, R_{L,N}$, and $R_{R,N}$ with N. We note that this solution is required to have the null transmission phase, i.e., $\phi_1 = 0$, in the unit cell. Here the parameters we use are $n_1 = 0.85, n_l = 1.61 + 1.49i, k_0 = 1$, and $l = 0.75$. 

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For a unit cell with PT-symmetry, we have $M^{-1} = M^*$. On the other hand, once an integrated system has same opposite leads, we can have $\text{Det}[M^N] = 1$ from Lorentz reciprocity theorem.

In the following analysis, we would infer $\text{Det}[M] = 1$ and $(M^N)^{-1} = (M^*)^N$, based on two given conditions $M^{-1} = M^*$ and $\text{Det}[M^N] = 1$.

Because $\text{Det}[M^N] = 1$ is valid for any N, we argue

$$\text{Det}[M^N] = \text{Det}[M] \cdot \text{Det}[M] \cdot \ldots \cdot \text{Det}[M] = 1.$$

Consider $AAA^{-1}A^{-1} = I = (AA)(A^{-1}A^{-1}) = A^2B$ where A is invertible matrix and I is unit matrix. From linear algebra, $B = (A^{-1})^2 = (A^2)^{-1}$. Thus, $M^{-1}M^{-1} \ldots M^{-1} = (M^{-1})^N = (M^N)^{-1}$. Using PT symmetry, i.e., $M^{-1} = M^*$, we have

$$(M^{-1})^N = (M^N)^{-1} \tag{9}$$

$$\vdots \quad (M^*)^N = (M^N)^* = (M^N)^{-1}.$$

Thus we obtain $(M^N)^* = (M^N)^{-1}$.

VI. APPENDIX B

As shown in Fig. 4, we consider an optical waveguide network where is constructed by a finite number of unit cells. Each unit has two segments of waveguides in which there has no wave coupling. Each segment of the waveguide has two subsegments of gain and loss waveguides connected, marked by blue and red lines. We discuss scalar wave equation and time evolution of propagating waves is $e^{-i\omega t}$. At each connecting interface, the wave function and its derivative are continuous. The latter condition is related to energy flux conservation, [43] [44].
The corresponding wave function for the gain waveguide is \( \psi_g(s) = ce^{ik_o n_g s} + de^{-ik_o n_g s}, 0 \leq s \leq \frac{l}{2} \). Here \( s \) is arc-length, \( n_g \) is gain index of refraction, \( k_o \) is vacuum wave number, \( c \) and \( d \) denote complex amplitudes for right and left propagating waves. The total arc-lengths for the gain and loss waveguides in the unit cell are \( \frac{l}{2} \).

The corresponding wave function for loss waveguide is \( \psi_l(s) = ae^{ik_o n_l s} + be^{-ik_o n_l s}, -\frac{l}{2} \leq s \leq 0 \) here \( n_l \) is loss index of refraction.

At the interface, the corresponding boundary conditions are the continuity of the wave functions and its derivative, i.e., \( \psi_g(0^+) = \psi_l(0^-) \) and \( \frac{d\psi_g}{ds}(0^+) = \frac{d\psi_l}{ds}(0^-) \),
\[
c + d = a + b \\
\alpha k_o n_g (c - d) = \lambda k_o n_l (a - b).
\]
(10)

As a result, we can construct the transfer matrix for the subsegmental waveguides at the interface,
\[
\begin{pmatrix}
    c \\
    d
\end{pmatrix} = \begin{pmatrix}
    \frac{1}{2} (1 + \frac{n_l}{n_g}) & \frac{1}{2} (1 - \frac{n_l}{n_g}) \\
    \frac{1}{2} (1 - \frac{n_l}{n_g}) & \frac{1}{2} (1 + \frac{n_l}{n_g})
\end{pmatrix} \begin{pmatrix}
    a \\
    b
\end{pmatrix}.
\]
(11)

\[
M_1 = \begin{pmatrix}
    \left(1 + \frac{n_l}{n_g}\right) & \left(1 - \frac{n_l}{n_g}\right) \\
    \left(1 - \frac{n_l}{n_g}\right) & \left(1 + \frac{n_l}{n_g}\right)
\end{pmatrix} e^{ik_o n_g \frac{l}{2}} \begin{pmatrix}
    0 \\
    0
\end{pmatrix} = \begin{pmatrix}
    \frac{1}{2} (1 + \frac{n_l}{n_g}) & \frac{1}{2} (1 - \frac{n_l}{n_g}) \\
    \frac{1}{2} (1 - \frac{n_l}{n_g}) & \frac{1}{2} (1 + \frac{n_l}{n_g})
\end{pmatrix} e^{ik_o n_l \frac{l}{2}} \begin{pmatrix}
    0 \\
    0
\end{pmatrix}.
\]
(13)

\[
\begin{pmatrix}
    a \\
    b
\end{pmatrix} = \begin{pmatrix}
    \frac{1}{4} (1 + \frac{n_l}{n_g}) & \frac{1}{4} (1 - \frac{n_l}{n_g}) \\
    \frac{1}{4} (1 - \frac{n_l}{n_g}) & \frac{1}{4} (1 + \frac{n_l}{n_g})
\end{pmatrix} \begin{pmatrix}
    A \\
    B
\end{pmatrix}.
\]
(12)

We then derive the transfer matrix for the unit cell, i.e., between two neighboring nodes,

\[
\begin{pmatrix}
    a \\
    b
\end{pmatrix} = \begin{pmatrix}
    \alpha \beta & \gamma \\
    \gamma & -\alpha \beta
\end{pmatrix} \begin{pmatrix}
    A \\
    B
\end{pmatrix}.
\]
(13)

**VII. APPENDIX C**

To perform CPAL in periodic PT-symmetric systems, as our above discussion in the theory section, the necessary conditions are \( (M_1^N)_{11} = (M_1^N)_{22} = \frac{1}{2} \) and \( Im[(M_1^N)_{12}]Im[(M_1^N)_{21}] = 1 \). There are two solutions to meet these conditions.

(a) As the unit cell is operated at CPAL, i.e., \( \alpha = 1 \), the remaining conditions to have CPAL in periodic PT-symmetric systems are \( U_{N-2} = 0 \) and \( (U_{N-1})^2 = 1 \), therefore,
\[
\sin[(N - 1)\Phi] = 0
\]
(14)
and
\[
\sin[N\Phi] = \pm 1
\]
(15)

We note that due to symmetrical geometry for upper and lower segmental waveguides and as assumption of non-coupling wave, the wave function for two segments in the unit cell are identical.

Now, we turn to derive the corresponding transfer matrix in the nexus, i.e., node, marked by black dots. For node number 1, it is formed by a lossless waveguide and two lossy waveguides. The index of refraction for lossless waveguide is \( n_1 \). The boundary conditions at the node are the continuity of wave function and its derivative, so we have

\[
\begin{pmatrix}
    a \\
    b
\end{pmatrix} = \begin{pmatrix}
    \frac{1}{4} (1 + \frac{n_l}{n_l}) & \frac{1}{4} (1 - \frac{n_l}{n_l}) \\
    \frac{1}{4} (1 - \frac{n_l}{n_l}) & \frac{1}{4} (1 + \frac{n_l}{n_l})
\end{pmatrix} \begin{pmatrix}
    A \\
    B
\end{pmatrix}.
\]
(12)

, respectively. To simultaneously satisfy above equations, we need \( \Phi = \frac{\pi}{2}, \frac{3\pi}{2} \) and \( N \) is odd, \( N = 1, 3, 5, 7, \ldots \)

(b) If the unit cell is not operated at CPAL, i.e., \( \alpha \beta \neq 1 \), we need to require
\[
Im[(M_1^N)_{12}]Im[(M_1^N)_{21}] = 1
\]
\[
\rightarrow \alpha \beta \sin^2[N\Phi] = \sin^2 \Phi
\]
(16)

Now we turn to consider \( (M_1^N)_{11} = 0 \) and \( (M_1^N)_{22} = 0 \), so that it further requires
\[
\sqrt{1 - \alpha \beta e^{i\phi} U_{N-1}} = U_{N-2}
\]
\[
\sqrt{1 - \alpha \beta e^{-i\phi} U_{N-1}} = U_{N-2}.
\]
(17)

Thus to satisfy above equations simultaneously, we need \( \phi = 0 \). Thus, the transmission phase of the unit cell is
null. Based on $\phi = 0$, the corresponding Bloch phase would become

$$\cos \Phi = \sqrt{1 - \alpha_1 \beta_1}. \quad (18)$$

Then, the last remaining condition for $(M_1^N)_{11} = 0$ and $(M_1^N)_{22} = 0$ is

$$\sqrt{1 - \alpha_1 \beta_1} U_{N-1} - U_{N-2} = \frac{\cos \Phi \sin[N\Phi] - \sin[N\Phi] \cos \Phi + \cos[N\Phi] \sin \Phi}{\sin \Phi} \quad (19)$$

$$= \cos[N\Phi] \Rightarrow 0$$

The last condition is $\cos[N\Phi] = 0$. We note that $\cos[N\Phi] = 0$ is consistent with Eqs. (16) and (18).

In short, if the unit cell is not at CPAL, to achieve CPAL for periodic PT-symmetric system, the corresponding conditions are $\phi = 0$ and $\cos[N\Phi] = 0$. 