1. Introduction

The ambient metric, introduced in [FG1], has proven to be an important object in conformal geometry. To a manifold $M$ of dimension $n$ with a conformal class of metrics $[g]$ of signature $(p, q)$ it associates a formal expansion of a metric $\widetilde{g}$ of signature $(p + 1, q + 1)$ on a space $\tilde{G}$ of dimension $n + 2$. This generalizes the realization of the conformal sphere $S^n$ as the space of null lines for a quadratic form of signature $(n + 1, 1)$, with associated Minkowski metric $\tilde{g}$ on $\mathbb{R}^{n+2}$. The ambient space $\tilde{G}$ carries a family of dilations with respect to which $\tilde{g}$ is homogeneous of degree $2$. The other conditions determining $\tilde{g}$ are that it be Ricci-flat and satisfy an initial condition specified by the conformal class $[g]$.

The ambient metric behaves differently in even and odd dimensions, reflecting an underlying distinction in the structure of the space of jets of metrics modulo conformal rescaling. When $n$ is odd, the equations determining $\tilde{g}$ have a smooth infinite-order formal power series solution, uniquely determined modulo diffeomorphism. But when $n$ is even and $\geq 4$, there is an obstruction at order $n/2$ to the existence of a smooth formal solution, which is realized as a conformally invariant natural tensor called the ambient obstruction tensor. It is possible to continue the expansion of $\tilde{g}$ to higher orders by including log terms ([K]), but this destroys the differentiability of the solution so is problematic for applications requiring higher order differentiation.

In this article, we describe a modification to the form of the ambient metric in even dimensions which enables us to obtain invariantly defined, smooth infinite order “ambient metrics”. We introduce what we call inhomogeneous ambient metrics, which are formally Ricci-flat and have an asymptotic expansion involving the logarithm of a defining function for the initial surface which is homogeneous of degree $2$. Such metrics are themselves no longer homogeneous, and of course are not smooth. However, we are able to define the smooth part of such a metric in an invariant way, and the smooth part is homogeneous and of course smooth and can be used in applications just as the ambient metric itself is used in odd dimensions. A significant difference, though, is that an inhomogeneous ambient metric is no longer

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uniquely determined up to diffeomorphism by the conformal class \([g]\) on \(M\): there is a family of inhomogeneous ambient metrics, and therefore of their smooth parts, parametrized by the choice of an arbitrary trace-free symmetric 2-tensor field on \(M\) which we call the ambiguity tensor.

There are two main motivations for the introduction of terms involving the logarithm of a defining function homogeneous of degree 2 in the expansion of an ambient metric. The simplest is the observation that in flat space, the null cone of the Minkowski metric has an invariant defining function homogeneous of degree 2, namely the quadratic form \(Q\) itself, but no invariant defining function homogeneous of degree 0. In this flat space setting, the theory of invariants of conformal densities initiated in [EG] can be extended to all orders by the introduction of \(\log |Q|\) terms in the expansion of the ambient harmonic extension of a density. We will report on this work on invariants of densities elsewhere.

Another motivation is the construction of the potential of the inhomogeneous CR ambient metric in [H], which also involves the log of a defining function homogeneous of degree 2 and an invariantly defined smooth part. The existence of this construction in the CR case was compelling evidence that there should be a conformal analogue. Nonetheless, despite having the CR construction as a guide, we did not find the conformal construction to be straightforward. The relation between the constructions is still not completely clear; further discussion of this issue is contained in [3].

We also indicate how inhomogeneous ambient metrics can be used to complete the description of scalar invariants of conformal structures in even dimensions. One can form scalar invariants as Weyl invariants, defined to be linear combinations of complete contractions of covariant derivatives of the curvature tensor of the smooth part of an inhomogeneous ambient metric, and also its volume form and a modified volume form in the case of odd invariants. Such invariants generally depend on the choice of ambiguity. Nonetheless, in dimensions \(n \equiv 2 \mod 4\), all scalar conformal invariants arise as Weyl invariants which are independent of the ambiguity. If \(n \equiv 0 \mod 4\), there are exceptional odd invariants which are not of this form. In this case, following [BG], we provide a construction of a finite set of basic exceptional invariants, which, together with ambiguity-independent Weyl invariants, span all scalar conformal invariants. The study of scalar invariants was initiated in [F2] in the CR case. Fundamental algebraic results were derived in [BEG1], resulting in the full description of scalar invariants in odd dimensional conformal geometry and below the order of the obstruction in CR and even dimensional conformal geometry. The completion of the description in the CR case was given in [H]. A different approach to the study of conformal invariants is developed in [Go] using tractor calculus. In [A1], [A2], Alexakis has recently derived a theory handling invariants of a density coupled with a conformal structure, below the order of the obstruction.
of the ambient metric in even dimensions, and below the order of the obstruction of harmonic extension of the density.

In §2 we review the construction of smooth homogeneous ambient metrics. Inhomogeneous ambient metrics are introduced in §3 and the main theorem asserting the existence and uniqueness of inhomogeneous ambient metrics with prescribed ambiguity tensor is formulated. We describe in some detail the transformation law for the ambiguity tensor under conformal change. We state a necessary and sufficient condition for the asymptotic expansion of an inhomogeneous ambient metric to have a simpler form, and show by explicit identification of the obstruction tensor that Fefferman metrics of nondegenerate integrable CR manifolds always satisfy this condition. We also briefly discuss the Poincaré metrics associated to inhomogeneous ambient metrics. In §4 we describe the results concerning scalar invariants and give some examples of invariants.

2. Smooth homogeneous ambient metrics

In this section we review the usual ambient metric construction in odd dimensions and up to the obstruction in even dimensions. Let $[g]$ be a conformal class of metrics of signature $(p, q)$ on a (paracompact) manifold $M$ of dimension $n \geq 3$. Here $g$ is a smooth metric of signature $(p, q)$ on $M$ and $[g]$ consists of all metrics of the form $\Omega^2 g$ with $0 < \Omega \in C^\infty(M)$. The metric bundle $G \subset \mathcal{O}^2 T^*M$ of $[g]$ is the set of all pairs $(x, g)$, where $x \in M$ and $g \in \mathcal{O}^2 T^*_x M$ is of the form $g = t^2 g(x)$ for some $t > 0$. The fiber variable $t$ on $G$ so defined is associated to the choice of metric $g$ and provides an identification $G \cong \mathbb{R}^+ \times M$. Sections of $G$ are precisely metrics in the conformal class. There is a tautological symmetric 2-tensor $g_0$ on $G$ defined by $g_0(X, Y) = g(\pi_* X, \pi_* Y)$, where $\pi : G \to M$ is the projection and $X, Y$ are tangent vectors to $G$ at $(x, g) \in G$. The family of dilations $\delta_s : G \to G$ defined by $\delta_s(x, g) = (x, s^2 g)$ defines an $\mathbb{R}^+$ action on $G$, and one has $\delta^* g_0 = s^2 g_0$. We denote by $T = \frac{d}{ds}\delta_s|_{s=1}$ the vector field on $G$ which is the infinitesimal generator of the dilations $\delta_s$.

The ambient space is $\tilde{G} = G \times \mathbb{R}$; the coordinate in the $\mathbb{R}$ factor is typically written $\rho$. The dilations $\delta_s$ extend to $\tilde{G}$ acting on the $G$ factor and we denote also by $T$ the infinitesimal generator of the $\delta_s$ on $\tilde{G}$. We embed $G$ into $\tilde{G}$ by $\iota : z \to (z, 0)$ for $z \in G$, and we identify $G$ with its image under $\iota$. As described above, a choice of representative metric $g$ induces an identification $G \cong \mathbb{R}^+ \times M$, so also an identification $\tilde{G} \cong \mathbb{R}^+ \times M \times \mathbb{R}$. We use capital Latin indices to label objects on $\tilde{G}$, '0' indices to label the $\mathbb{R}^+$ factor, lower case Latin indices to label the $M$ factor, and '$\infty$' indices for the $\mathbb{R}$ factor.

In the even-dimensional case, different components of the ambient metric are determined to different orders. If $S_{IJ}$ is a smooth symmetric 2-tensor field on an open neighborhood of $G$ in $\tilde{G}$, then for $m \geq 0$ we write $S_{IJ} = O_{IJ}(\rho^m)$ if:
(i) \( S_{IJ} = O(\rho^m) \); and
(ii) For each point \( z \in G \), the symmetric 2-tensor \((\iota^* (\rho^{-m} S))(z)\) is of the form \( \pi^* s \) for some symmetric 2-tensor \( s \) at \( x = \pi(z) \in M \) satisfying \( \text{tr}_{\pi_*} s = 0 \).

The symmetric 2-tensor \( s \) is allowed to depend on \( z \), not just on \( x \).

In terms of components relative to a choice of representative metric \( g \), \( S_{IJ} = \Omega + \Omega_{IJ}(\rho^m) \) if and only if all components satisfy \( S_{IJ} = O(\rho^m) \) and if in addition one has that \( S_{00}, S_{0i} \) and \( g^{ij} S_{ij} \) are \( O(\rho^{m+1}) \).

The condition \( S_{IJ} = O(\rho^m) \) is easily seen to be preserved by diffeomorphisms on a neighborhood of \( G \) which restrict to the identity on \( G \).

We consider metrics \( \tilde{g} \) of signature \((p+1, q+1)\) defined on a neighborhood of \( G \) in \( \tilde{G} \). We will assume that the neighborhood is homogeneous, i.e., it is invariant under the dilations \( \delta_s \) for \( s > 0 \). A smooth diffeomorphism defined on such a neighborhood will be said to be homogeneous if it commutes with the \( \delta_s \). The metric \( \tilde{g} \) (or more generally a symmetric 2-tensor field) will be said to be homogeneous (of degree 2) if it satisfies \( \delta_s^* \tilde{g} = s^2 \tilde{g} \).

The main existence and uniqueness result for smooth ambient metrics is the following.

**Theorem 2.1.** Let \([g]\) be a conformal class on a manifold \( M \). If \( n \) is odd, then there is a smooth metric \( \tilde{g} \) on a homogeneous neighborhood of \( G \) in \( \tilde{G} \), uniquely determined up to a homogeneous diffeomorphism of a neighborhood of \( G \) in \( \tilde{G} \) which restricts to the identity on \( G \), and up to a homogeneous term vanishing to infinite order along \( G \), by the requirements:

\begin{enumerate}
  \item \( \delta_s^* \tilde{g} = s^2 \tilde{g} \)
  \item \( \iota^* \tilde{g} = g_0 \)
  \item \( \text{Ric}(\tilde{g}) = 0 \) to infinite order along \( G \).
\end{enumerate}

If \( n \geq 4 \) is even, the same statement holds except that (3) is replaced by \( \text{Ric}(\tilde{g}) = O(\rho^{n/2-1}) \), and \( \tilde{g} \) is uniquely determined up to a homogeneous diffeomorphism and up to a smooth homogeneous term which is \( O(\rho^{n/2}) \).

A metric \( \tilde{g} \) satisfying the conditions of Theorem 2.1 is called a smooth ambient metric for \((M, [g])\). The diffeomorphism indeterminacy in \( \tilde{g} \) can be fixed by the choice of a metric \( g \) in the conformal class. As described above, the choice of such a metric \( g \) determines an identification \( \tilde{G} \cong \mathbb{R}_+ \times M \times \mathbb{R} \).

**Definition 2.2.** A smooth metric \( \tilde{g} \) on \( \tilde{G} \) satisfying (1) and (2) above is said to be in normal form relative to \( g \) if in the identification \( \tilde{G} = \mathbb{R}_+ \times M \times \mathbb{R} \) induced by \( g \),

\begin{enumerate}
  \item \( \tilde{g} = 2t \, dt \, d\rho + g_0 \) at \( \rho = 0 \), and
  \item The lines \( \rho \rightarrow (t, x, \rho) \) are geodesics for \( \tilde{g} \) for each choice of \( (t, x) \in \mathbb{R}_+ \times M \).
\end{enumerate}

As in the construction of Gaussian coordinates relative to a hypersurface, it follows by straightening out geodesics that if \( \tilde{g} \) is any smooth metric satisfying (1) and
(2) and \( g \) is a representative metric in the conformal class, then there is a unique homogeneous diffeomorphism \( \Phi \) defined in a homogeneous neighborhood of \( \mathcal{G} \) and which restricts to the identity on \( \mathcal{G} \), such that \( \Phi^*\tilde{g} \) is in normal form relative to \( g \). It follows that in Theorem 2.1, the metric \( \tilde{g} \) can be chosen to be in normal form relative to \( g \), and it is then uniquely determined up to \( O(\rho^\infty) \) for \( n \) odd, and up to \( O_{IJ}(\rho^{n/2}) \) for \( n \) even.

Theorem 2.1 is proved by a formal power series analysis of the equation \( \text{Ric}(\tilde{g}) = 0 \) for metrics \( \tilde{g} \) in normal form relative to some metric \( g \) in the conformal class. In carrying out this analysis, one finds that the solution has an additional property.

**Definition 2.3.** A metric \( \tilde{g} \) on a homogeneous neighborhood \( \mathcal{U} \) of \( G \) in \( \tilde{G} \) is said to be straight if for each \( p \in \mathcal{U} \), the dilation orbit \( s \to \delta_s p \) is a geodesic for \( \tilde{g} \).

**Proposition 2.4.** In Theorem 2.1, the metric \( \tilde{g} \) can be chosen to be straight.

Since the condition that \( \tilde{g} \) be straight is invariant under smooth homogeneous diffeomorphisms, it follows that any smooth ambient metric \( \tilde{g} \) agrees with a straight metric to infinite order when \( n \) is odd and \( \text{mod} \ O_{IJ}(\rho^{n/2}) \) when \( n \) is even. That is, any smooth ambient metric is straight to the order that it is determined.

When \( n \) is even, in general there is an obstruction to the existence of a smooth ambient metric solving \( \text{Ric}(\tilde{g}) = O(\rho^{n/2}) \). Let \( \tilde{g} \) be a smooth ambient metric. We denote by \( r\# \) the function \( r\# = \|T\|^2 = \tilde{g}(T,T) \). Since \( g_0(T,T) = 0 \), it follows that \( r\# = 0 \) on \( \tilde{G} \), and it turns out (as a consequence of the Einstein condition or the straightness condition) that \( dr\# \neq 0 \) on \( \tilde{G} \). Thus \( r\# \) is a defining function for \( G \subset \tilde{G} \) invariantly associated to \( \tilde{g} \) which is homogeneous of degree 2: \( \delta_s r\# = s^2 r\# \). Now since \( \text{Ric}(\tilde{g}) = O_{IJ}(\rho^{n/2-1}) \), the quantity \( (r\#^{-n/2} \text{Ric} \tilde{g})|_{\mathcal{G}} \) defines a tensor field on \( \mathcal{G} \), homogeneous of degree \( 2 - n \), which annihilates \( T \). It therefore defines a symmetric 2-tensor-density on \( M \) of weight \( 2 - n \), which is trace-free. If \( g \) is a metric in the conformal class, evaluating this tensor-density at the image of \( g \) viewed as a section of \( \mathcal{G} \) defines a 2-tensor on \( M \) which we denote by \( (r\#^{-n/2} \text{Ric} \tilde{g})|_g \). The obstruction tensor of the metric \( g \) is defined to be

\[
(2.1) \quad \mathcal{O} = c_n (r\#^{-n/2} \text{Ric} \tilde{g})|_g, \quad c_n = (-1)^{n/2-1} \frac{2^{n-2}(n/2 - 1)!}{n - 2}.
\]

We then have:

**Proposition 2.5.** Let \( n \geq 4 \) be even. The obstruction tensor \( \mathcal{O}_{ij} \) of \( g \) is independent of the choice of smooth ambient metric \( \tilde{g} \) and has the following properties:

1. \( \mathcal{O} \) is a natural tensor invariant of the metric \( g \); i.e. in local coordinates the components of \( \mathcal{O} \) are given by universal polynomials in the components of \( g \), \( g^{-1} \) and the curvature tensor of \( g \) and its covariant derivatives. The
expression for $O_{ij}$ takes the form

$$O_{ij} = \Delta^{n/2-2} \left( P_{ij,k}^k - P_k^k,_{ij} \right) + \text{lots}$$

$$= (3 - n)^{-1} \Delta^{n/2-2} W_{kijl, kl} + \text{lots},$$

where $\Delta = \nabla^i \nabla_i$,

$$P_{ij} = \frac{1}{n - 2} \left( R_{ij} - \frac{R}{2(n - 1)} g_{ij} \right),$$

$W_{ijkl}$ is the Weyl tensor, and lots denotes quadratic and higher terms in curvature involving fewer derivatives.

(2) One has

$$O_i^i = 0 \quad \text{and} \quad O_{ij, j} = 0.$$

(3) $O_{ij}$ is conformally invariant of weight $2 - n$; i.e. if $0 < \Omega \in C^\infty(M)$ and $\hat{g}_{ij} = \Omega^2 g_{ij}$, then $\hat{O}_{ij} = \Omega^{2-n} O_{ij}$.

(4) If $g_{ij}$ is conformal to an Einstein metric, then $O_{ij} = 0$.

(5) If $n = 4$, then $O_{ij} = B_{ij}$ is the classical Bach tensor.

Here the Bach tensor is defined in any dimension $n \geq 3$ by

$$B_{ij} = C_{ijk}^k - P^{kl} W_{kijl},$$

where $C_{ijk}$ is the Cotton tensor

$$C_{ijk} = P_{ij,k}^k - P_{ik,j}^j.$$

Clearly, if the obstruction tensor is nonzero, then it is impossible to find a smooth ambient metric $\tilde{g}$ solving $\text{Ric}(\tilde{g}) = O(\rho^{n/2}).$

3. INHOMOGENEOUS AMBIENT METRICS

Restrict now to the case $n$ even. In order to find ambient metrics beyond order $n/2$, we broaden the class of allowable metrics $\tilde{g}$. The $\tilde{g}$ that we consider are neither homogeneous nor smooth. However, $\tilde{g}$ will always be required to satisfy the initial condition

$$\iota^* \tilde{g} = g_0.$$

Let $r$ be a smooth defining function for $\mathcal{G} \subset \tilde{\mathcal{G}}$ homogeneous of degree 0 (one could for example choose $r = \rho$) and let $r_\#$ be a smooth defining function homogeneous of degree 2 (for example, $r_\# = 2 \rho t^2$). Denote by $\mathcal{M}$ the space of formal asymptotic expansions along $\mathcal{G}$ of metrics on $\tilde{\mathcal{G}}$ of signature $(p + 1, q + 1)$ of the form

$$\tilde{g} \sim \tilde{g}^{(0)} + \sum_{N \geq 1} \tilde{g}^{(N)} r(r^{n/2-1} \log |r_\#|)^N$$

where each $\tilde{g}^{(N)}$, $N \geq 0$, is a smooth symmetric 2-tensor field on $\tilde{\mathcal{G}}$ satisfying $\delta_s^* \tilde{g}^{(N)} = s^2 \tilde{g}^{(N)}$, such that the initial condition (3.1) holds. Observe that $\tilde{g} \in \mathcal{M}$
is the expansion of a smooth metric on $\tilde{\mathcal{G}}$ if and only if $\tilde{g}^{(N)} = 0$ to infinite order for $N \geq 1$. We will say in this case that $\tilde{g}$ is smooth. Note that if $\tilde{g}$ is smooth, then $\tilde{g}$ agrees to infinite order with a homogeneous metric; we will also say that $\tilde{g}$ is homogeneous. Denote by $\mathcal{A}$ the space of formal asymptotic expansions of scalar functions $f$ on $\tilde{\mathcal{G}}$ of the form

$$f \sim f^{(0)} + \sum_{N \geq 1} f^{(N)} r^{(n/2-1) \log |r\#|} N,$$

where each $f^{(N)}$ is a smooth function on $\tilde{\mathcal{G}}$ homogeneous of degree 0. Observe that these spaces of asymptotic expansions are independent of the choice of defining functions $r$, $r\#$. If $\Phi$ is a formal smooth, homogeneous diffeomorphism of $\tilde{\mathcal{G}}$ satisfying $\Phi|_{\mathcal{G}} = Id$, then pullback by $\Phi$ preserves $\mathcal{M}$ and $\mathcal{A}$. As already initiated above, in the following we will refer to such asymptotic expansions for metrics, functions, and formal smooth diffeomorphisms as if they were actually defined in a neighborhood of $\mathcal{G}$, and when we say that such an object satisfies a certain condition, we will mean that the condition holds formally to infinite order along $\mathcal{G}$. For example, we will say that $\tilde{g}$ is straight if the ordinary differential equations which state that the orbits $s \to \delta_s p$ are geodesics hold formally to infinite order along $\mathcal{G}$.

For inhomogeneous ambient metrics, we impose the straightness condition at the outset.

**Definition 3.1.** An inhomogeneous ambient metric for $(\mathcal{M}, [g])$ is a straight metric $\tilde{g} \in \mathcal{M}$ satisfying $\text{Ric}(\tilde{g}) = 0$.

The straightness condition is crucial in the inhomogeneous case because of the following proposition.

**Proposition 3.2.** Let $\tilde{g} \in \mathcal{M}$ be straight. Then $T \mathcal{J} \tilde{g}$ is smooth and $\tilde{g}(T, T)$ is a smooth defining function for $\mathcal{G}$ homogeneous of degree 2.

Observe that for $\tilde{g} \in \mathcal{M}$, $T \mathcal{J} \tilde{g}$ and $\tilde{g}(T, T)$ will in general have asymptotic expansions involving $\log |r\#|$, so will be neither smooth nor homogeneous. Proposition 3.2 asserts that the requirement that $\tilde{g}$ be straight has as a consequence that no log terms occur in the expansions of these quantities. The proof of Proposition 3.2 is a straightforward analysis of the geodesic equations for the dilation orbits.

If $\tilde{g} \in \mathcal{M}$ is straight, then $\tilde{g}(T, T)$ is a canonically determined smooth defining function for $\mathcal{G}$ homogeneous of degree 2. We may therefore take $r\# = \tilde{g}(T, T)$ in (3.2). Having fixed $r\#$, the smooth symmetric 2-tensor fields $\tilde{g}^{(0)}$ and $r^{(n/2-1)N+1} \tilde{g}^{(N)}$ for $N \geq 1$ are then uniquely determined by $\tilde{g}$ independently of any choices. In particular, $\tilde{g}^{(0)}$ is an invariantly determined smooth part of $\tilde{g}$, which is itself a smooth metric in $\mathcal{M}$. If $\Phi$ is a smooth homogeneous diffeomorphism satisfying $\Phi|_{\mathcal{G}} = Id$ and $\tilde{g} \in \mathcal{M}$ is straight, then $(\Phi^* \tilde{g})^{(0)} = \Phi^*(\tilde{g}^{(0)})$.

Next we formulate the notion of normal form for straight metrics in $\mathcal{M}$. 
Definition 3.3. Let $g$ be a metric in the conformal class and let $\tilde{g} \in \mathcal{M}$ be straight. Then $\tilde{g}$ is said to be in normal form relative to $g$ if its smooth part is in normal form relative to $g$.

It is clear from the existence and uniqueness of a diffeomorphism putting a smooth metric into normal form that if $\tilde{g} \in \mathcal{M}$ is straight, then there is a unique smooth homogeneous diffeomorphism $\Phi$ such that $\Phi|_G = \text{Id}$ and such that $\Phi^*\tilde{g}$ is in normal form relative to $g$.

The following proposition makes explicit the normal form condition. Its proof is an analysis of the geodesic equations for the straightness and normal form conditions using the initial condition (a) of Definition 2.2.

Proposition 3.4. Let $g$ be a representative for the conformal structure. A straight metric $\tilde{g} \in \mathcal{M}$ is in normal form relative to $g$ if and only if in the identification $\tilde{G} = \mathbb{R}_+ \times M \times \mathbb{R}$ induced by $g$, $\tilde{g}$ takes the form:

\begin{equation}
\tilde{g}_{IJ} = \begin{pmatrix}
2\rho & 0 & t \\
0 & t^2g_{ij} & t^2g_{i\infty} \\
t & t^2g_{j\infty} & t^2g_{\infty\infty}
\end{pmatrix}
\end{equation}

where $g_{ij}|_{\rho=0}$ is the given metric $g$ on $M$, and where the expansions for the components $g_{j\infty}$ and $g_{\infty\infty}$ have zero smooth part when expanded using $r^2 = 2\rho t^2$. That is, for $I = i, \infty$ we have

\begin{equation}
g_{I\infty} \sim \sum_{N \geq 1} g^{(N)}_{I\infty} \rho^{n/2-1} \log(2\rho t^2)^N
\end{equation}

where the $g^{(N)}_{I\infty}$ are smooth and homogeneous of degree 0.

It is a consequence of Proposition 3.4 that if $\tilde{g} \in \mathcal{M}$ is straight, then its smooth part is also straight.

The next theorem is our main theorem concerning the existence and uniqueness of inhomogeneous ambient metrics. As described in §2 in odd dimensions, given a representative metric $g$ in the conformal class, there is to infinite order a unique smooth ambient metric in normal form relative to $g$. For $n$ even, inhomogeneous ambient metrics in normal form are no longer uniquely determined by a representative for the conformal class. One has exactly the freedom of a smooth trace-free symmetric 2-tensor field on $M$, which we call the ambiguity tensor.

Theorem 3.5. Let $(M^n, [g])$ be a manifold with a conformal structure, $n \geq 4$ even. Up to pull-back by a smooth homogeneous diffeomorphism which restricts to the identity on $\mathcal{G}$, the inhomogeneous ambient metrics for $(M, [g])$ are parametrized by the choice of an arbitrary trace-free symmetric 2-tensor field $A_{ij}$ on $M$. Precisely, for each representative metric $g$ and choice of ambiguity tensor $A_{ij} \in \Gamma(\bigodot^2 T^*M)$,
there is a unique inhomogeneous ambient metric \( \tilde{g} \) in normal form relative to \( g \) such that

\[
(3.5) \quad \text{tf } \left( (\partial_\rho)^{n/2} g_{ij}^{(0)} \right) \mid_{\rho=0} = A_{ij}.
\]

Here we have written \( \tilde{g} \) in the form (3.3), \( g_{ij}^{(0)} \) denotes the smooth part of \( g_{ij} \), and tf the trace-free part.

There is a natural pseudo-Riemannian invariant 1-form \( D_i \) so that the metric \( \tilde{g} \) determined by initial metric \( g \) and ambiguity \( A_{ij} \) is smooth if and only if \( O_{ij} = 0 \) and \( A_{ij,j} = D_i \), where \( O_{ij} \) denotes the obstruction tensor of \( g \).

The 1-form \( D_i \) is defined as follows. A straight smooth ambient metric in normal form takes the form

\[
(3.6) \quad \tilde{g}_{ij} = \begin{pmatrix} 2\rho & 0 & t \\ 0 & t^2 g_{ij} & 0 \\ t & 0 & 0 \end{pmatrix}
\]

where \( g_{ij} = g_{ij}(x, \rho) \) is a smooth 1-parameter family of metrics on \( M \). The derivatives \( \partial_\rho^m g_{ij} \) for \( m \leq n/2 - 1 \) and \( g^{ij} \partial_\rho^{n/2} g_{ij} \) are determined at \( \rho = 0 \) and are natural invariants of the initial metric \( g_{ij}(x, 0) \). Fix the indeterminacy of \( g_{ij}(x, \rho) \) to higher orders by fixing \( g_{ij} \) to be the Taylor polynomial of degree \( n/2 \) whose \((n/2)\)th derivative is pure trace:

\[
(3.7) \quad g_{ij}(x, \rho) = \sum_{m=0}^{n/2-1} \frac{1}{m!} \partial_\rho^m g_{ij} \rho^m + \frac{1}{n(n/2)!} (g^{kl} \partial_\rho^{n/2} g_{kl}) g_{ij} \rho^{n/2},
\]

where on the right hand side, \( g_{ij} \) and its derivatives are evaluated at \((x, 0)\). The Ricci curvature component \( \tilde{R}_{i\infty} \) of \( \tilde{g} \) vanishes to order \( n/2 - 1 \), and \( D_i \) is given by:

\[
D_i = -2(\partial_\rho^{n/2-1} \tilde{R}_{i\infty}) \mid_{\rho=0, t=1}.
\]

When \( n = 4 \), this reduces to:

\[
D_i = 4P^{jk} P_{ij,k} - 3P^{jk} P_{jk,i} + 2P^j P_{k,j}.
\]

If the metric \( \tilde{g} \) in Theorem 3.5 is written in the form (3.3), (3.4) and \( g_{ij} \) is also expanded as in (3.4) (with expansion including a smooth term \( g_{ij}^{(0)} \)), then \( g_{ij} \mid_{\rho=0} = c_1 O_{ij} \) and \( g_{ij}^{(1)} \mid_{\rho=0} = c_2 (A_{ij,j} - D_i) \) for nonzero constants \( c_1, c_2 \).

Theorem 3.5 is proved by an inductive perturbation analysis of the equation of vanishing Ricci curvature for metrics of the form given by Proposition 3.4. In the proof, one sees that the smooth part \( \tilde{g}^{(0)} \) of any inhomogeneous ambient metric is a smooth ambient metric for \((M, [g])\) in the sense of §2

The ambiguity tensor has a well-defined transformation law under conformal change. Let \( \tilde{g} \) be the inhomogeneous ambient metric in normal form determined by initial metric \( g \) and ambiguity tensor \( A_{ij} \). Suppose we choose another metric
\[ \hat{g} = e^{2\Upsilon} g \] in the conformal class. There is a uniquely determined smooth homogeneous diffeomorphism \( \Phi \) with \( \Phi|_g = Id \) so that \( \Phi^*\hat{g} \) is in normal form relative to \( \hat{g} \).

Now \( \Phi^*\hat{g} \) determines an ambiguity tensor which we denote by \( \hat{A}_{ij} \), defined by the version of the equation \( (3.5) \) for the Taylor coefficient of the smooth part of \( \Phi^*\hat{g} \) in the identification \( \hat{G} = \mathbb{R}_+ \times M \times \mathbb{R} \) induced by \( \hat{g} \). We describe next the expression for \( \hat{A}_{ij} \) in terms of \( A_{ij} \) and \( \Upsilon \). This transformation law is best understood in terms of another trace-free symmetric 2-tensor on \( M \) which is a modification of the ambiguity tensor.

According to Proposition 3.4, the smooth part \( \tilde{g}_{ij}^{(0)} \) of an inhomogeneous ambient metric \( \tilde{g}_{ij} \) in normal form relative to \( g \) takes the form

\[
\tilde{g}_{ij}^{(0)} = \begin{pmatrix}
2\rho & 0 & t \\
0 & t^2g_{ij}^{(0)} & 0 \\
t & 0 & 0
\end{pmatrix},
\]

where \( g_{ij}^{(0)}(x, \rho) \) is a smooth 1-parameter family of metrics on \( M \). Since \( \tilde{g}^{(0)} \) is itself a smooth ambient metric, the derivatives \( \partial^m_\rho g_{ij}^{(0)}|_{\rho=0} \) for \( m < n/2 \) and the trace \( g^{kl}\partial^m_\rho g_{kl}^{(0)}|_{\rho=0} \) are the same natural tensors \( \partial^m_\rho g_{ij}|_{\rho=0} \) and \( g^{kl}\partial^m_\rho g_{kl}|_{\rho=0} \) which occur in the expansion \( (3.7) \). Consider the value at \( \rho = 0, t = 1 \) of the component \( \tilde{R}^{(0)}_{\infty ij\infty \ldots \infty \infty} \) of the iterated covariant derivative of the curvature tensor of \( \tilde{g}_{ij}^{(0)} \). This component depends on derivatives of orders \( \leq n/2 \) of the components of \( \tilde{g}_{ij}^{(0)} \). An inspection of the formula for this covariant derivative component yields the following proposition.

**Proposition 3.6.**

\[
\tilde{R}^{(0)}_{\infty ij\infty \ldots \infty} \big|_{\rho=0, t=1} = \frac{1}{2}(A_{ij} + K_{ij}),
\]

where \( K_{ij} \) is a natural trace-free symmetric tensor which can be expressed algebraically in terms of the tensors \( \partial^m_\rho g_{ij}|_{\rho=0}, m < n/2 \).

The tensors \( \partial^m_\rho g_{ij}|_{\rho=0}, m < n/2 \), are determined by the inductive solution of the equation \( \text{Ric}(\tilde{g}) = 0 \) for a smooth ambient metric in normal form, and \( K_{ij} \) is expressed in terms of these via the formula for covariant differentiation. To the extent that \( K_{ij} \), and therefore also its conformal transformation law, may be regarded as known, the conformal transformation law of \( A_{ij} \) is determined by that of \( \tilde{R}^{(0)}_{\infty ij\infty \ldots \infty} \big|_{\rho=0, t=1} \). Henceforth we shall write \( A_{ij} = A_{ij} + K_{ij} \) for this modified ambiguity tensor. For \( n = 4 \), one has \( K_{ij} = -2 t f(P^i_k P_{jk}) \) so that

\[
(3.8) \quad A_{ij} = A_{ij} - 2 t f(P^i_k P_{jk}).
\]
The conformal transformation law of $A_{ij}$ can be expressed succinctly. Define the strength of lists of indices in $\mathbb{R}^{n+2}$ as follows. Set $\|0\| = 0, \|i\| = 1$ for $1 \leq i \leq n$, and $\|\infty\| = 2$. For a list, write $\|I \ldots J\| = \|I\| + \cdots + \|J\|$. An inductive analysis of the formula for covariant differentiation shows that if $r \geq 0$ and $\|ABCDF_1 \cdots F_r\| \leq n+1$, then the curvature component $\tilde{R}_{ABCDF_1 \cdots F_r|_{\rho=0}, t=1}^{(0)}$ defines a natural tensor on $M$ as the indices between 1 and $n$ vary and those which are 0 or $\infty$ remain fixed. Set

$$p^A_I = \begin{pmatrix} 1 & \Upsilon_i & -\frac{1}{2} \Upsilon_k \Upsilon^k \\ 0 & \alpha_i & -\Upsilon^a \\ 0 & 0 & 1 \end{pmatrix}.$$ 

**Proposition 3.7.** Set $r = n/2 - 2$. Under the conformal change $\hat{g} = e^{2\Upsilon} g$, the modified ambiguity tensor transforms by:

$$e^{(n-2)\Upsilon} \hat{A}_{ij} = A_{ij} + 2 \sum' \tilde{R}_{ABCDF_1 \cdots F_r|_{\rho=0}, t=1}^{(0)} p^A_{ij} p^B_{ij} p^C_{ij} p^D_{ij} p^F_{ij} \cdots p^F_{ij},$$

where $\sum'$ indicates the sum over all indices except $ABCDF_1 \cdots F_r = \text{inid} \cdot \text{inid} \cdots \text{inid}$.

The upper-triangular form of $p^A_I$ implies that any component $\tilde{R}_{ABCDF_1 \cdots F_r|_{\rho=0}, t=1}^{(0)}$ which occurs with nonzero coefficient in $\sum'$ necessarily satisfies $\|ABCDF_1 \cdots F_r\| \leq n+1$. So all these components are natural tensors, which can be calculated algorithmically using the expansion of $g_{ij}(x, \rho)$ through order $< n/2$. Thus Proposition 3.7 expresses the conformal transformation law of $A_{ij}$ in terms of “known” natural tensors and $\Upsilon$ and its first derivatives. For example, for $n = 4$ this becomes

$$e^{2\Upsilon} \hat{A}_{ij} = A_{ij} - 2\Upsilon^l (C_{ijl} + C_{jil}) + 2\Upsilon^l \Upsilon^l W_{kijl}.$$ 

Fix an even integer $n \geq 4$. In dimension $d > n$, there is a trace-free symmetric natural tensor satisfying the transformation law of Proposition 3.7, namely $2\tilde{R}_{\infty ij \infty, \infty \cdots \infty|_{n/2-2}^{(0)}} \rho=0, t=1$. For instance,

$$\tilde{R}_{\infty ij \infty|_{n/2-2}^{(0)}} \rho=0, t=1 = -\frac{1}{d-4} B_{ij}$$

if $d \neq 4$. Considered formally in the dimension $d$, the component $\tilde{R}_{\infty ij \infty, \infty \cdots \infty|_{n/2-2}^{(0)}} \rho=0, t=1$ has a simple pole at $d = n$ whose residue is a multiple of $O_{ij}$. The formal continuation to $d = n$ of the transformation law for this component is the statement of conformal invariance of the obstruction tensor. The modified ambiguity tensor $A_{ij}$ provides a substitute for the natural tensor $\tilde{R}_{\infty ij \infty, \infty \cdots \infty|_{n/2-2}^{(0)}} \rho=0, t=1$, which does not occur in dimension $n$ because of the conformal invariance of the obstruction tensor.

The transformation law of Proposition 3.7 can be reinterpreted in terms of tractors. Recall (see, for example, [BEGa]) that a cotractor field of weight $w$ on a
conformal manifold \((M, [g])\) can be expressed upon choosing a representative metric \(g\) as \(vl = (v_0, v_1, v_\infty)\), where \(v_0, v_\infty\) are functions on \(M\) and \(v_i\) is a 1-form, such that under the conformal change \(\tilde{g} = e^{2T}g\), \(vl\) transforms by
\[
(3.9) \quad \tilde{v}_l = e^{(w+1-2Tc)}v_l A^I P^A_l.
\]
For \(r < n/2 - 2\), the components of the tensor \(\widetilde{\nabla}^r \tilde{R}^{(0)}|_{\rho=0, t=1}\) define natural tensors on \(M\) and satisfy the correct transformation laws under conformal change to define a (natural) cotractor field of rank \(4 + r\) and weight \(-2 - r\). (A detailed discussion of the relation between ambient curvature and tractors can be found in \([CG]\).) For \(r = n/2 - 2\), the same is true of all components except for \(\tilde{R}^{(0)}|_{\rho=0, t=1}\) (and components obtained from one via the symmetries of the curvature tensor). The transformation law of Proposition 3.7 is precisely that required by (3.9) so that the field of rank \(n/2 + 2\) and weight \(-n/2\). Thus, choosing an ambiguity tensor \(A_{ij}\) is entirely equivalent to completing \(\widetilde{\nabla}^n/2 - 2 \tilde{R}^{(0)}|_{\rho=0, t=1}\) to a cotractor field.

When \(n\) is even, one can construct homogeneous ambient metrics with asymptotic expansions involving \(\log |r|\) ([K]). Such nonsmooth homogeneous ambient metrics all have asymptotic expansions of the form \(\sum_{N \geq 0} \tilde{g}^{(N)}(r^{n/2}\log |r|)^N\), where the \(\tilde{g}^{(N)}\) are smooth and homogeneous of degree 2 ([FG2]). This suggests that the inhomogeneous ambient metrics considered here might actually have expansions of the form
\[
(3.10) \quad \sum_{N \geq 0} \tilde{g}^{(N)}(r^{n/2}\log |r_\#|)^N,
\]
again with smooth homogeneous coefficients \(\tilde{g}^{(N)}\). But this is not the case: for inhomogeneous ambient metrics, the coefficient \(\tilde{g}^{(2)}\) does not in general vanish at \(\rho = 0\). In fact, we have

**Proposition 3.8.** Let \(\tilde{g}_{IJ}\) be an inhomogeneous ambient metric in normal form. When \(\tilde{g}_{IJ}\) is written in the form (3.3) and its components are expanded as in (3.4) (with a corresponding expansion for \(g_{ij}\) including a smooth term \(g_{ij}^{(0)}\)), then \(g_{ij}^{(2)}|_{\rho=0} = 0\), \(g_{ij}^{(2)}|_{\rho=0} = cO_{ij}O_{ij}\), for a nonzero constant \(c\). Thus if \(\tilde{g}\) has the form (3.10), then \(O^{ij}O_{ij} = 0\). Conversely, if \(O^{ij}O_{ij} = 0\), then \(\tilde{g}\) has the form (3.10) for any choice of ambiguity.

In Proposition 3.8 one would expect that there is a condition analogous to \(O^{ij}O_{ij} = 0\) arising from the coefficient of each of \((\log |2\rho^2|)^N\) for \(N \geq 2\). The fact that all of these are satisfied once the condition holds for \(N = 2\) is surprising.

Proposition 3.8 raises the question of whether there are interesting classes of conformal manifolds for which \(O_{ij}\) is nonvanishing but \(O^{ij}O_{ij} = 0\). Of course, this cannot happen in definite signature. But the Fefferman conformal structure of
any nondegenerate integrable CR manifold satisfies $O^{ij}O_{ij} = 0$, and typically they satisfy also that $O_{ij}$ is nonvanishing. These observations follow from an identification of the obstruction tensor of a Fefferman metric.

The analogue of the obstruction tensor in CR geometry is the scalar CR invariant, denoted here by $L$, which obstructs the existence of smooth solutions of Fefferman’s complex Monge-Ampère equation. It is defined as follows. Let $M \subset \mathbb{C}^n$ be a hypersurface with smooth defining function $u$ whose Levi form $-u_{ij} \vert_{T_z \circ M}$ has signature $(p, q)$, $p + q = n - 1$. Fefferman showed in [F1] that there is such a $u$ uniquely determined mod $O(u^{n+2})$ such that $J(u) = 1 + O(u^{n+1})$, where

$$J(u) = (-1)^{p+1} \det \left( \begin{array}{ll} u & u_j \\ u_i & u_{ij} \end{array} \right)_{1 \leq i, j \leq n}.$$ 

The invariant $L$ is defined to be a constant multiple of $(J(u) - 1)/u^{n+1} \vert_M$, and is independent of the choice of $u$ satisfying $J(u) = 1 + O(u^{n+1})$. The fundamental properties of $L$ are derived in [L], [Gr1], [Gr2].

**Proposition 3.9.** Let $\theta$ be a pseudohermitian form for an integrable nondegenerate CR manifold $M$ and let $g$ be the associated representative of the Fefferman conformal structure on the circle bundle $\mathcal{C}$. Then the obstruction tensor $O$ of $g$ is a nonzero constant multiple of the pullback to $\mathcal{C}$ of $L\theta^2$.

**Proof.** It suffices to assume that $M \subset \mathbb{C}^n$ is embedded. The circle bundle $\mathcal{C} = S^1 \times M$ has dimension $2n$. Let $u$ be a smooth defining function for $M$ satisfying $J(u) = 1 + O(u^{n+1})$ as above. According to Fefferman’s original definition, the representative $g$ associated to $\theta = i^2(\partial u - \bar{\partial} u)$ is the pullback to

$$S^1 \times M = \{(z^0, z) : \vert z^0 \vert = 1, z \in M\} \subset \mathbb{C}^* \times \mathbb{C}^n$$

of the Kähler metric $\tilde{g}$ on a neighborhood of $\mathbb{C}^* \times M$ in $\mathbb{C}^* \times \mathbb{C}^n$ given by

$$\tilde{g} = \partial^2_{\bar{z}z}(-\vert z^0 \vert^2 u) dz^I d\bar{z}^I.$$ 

The Ricci curvature of $\tilde{g}$ is

$$\partial^2_{\bar{z}z} \left( \log J(u) \right) dz^I d\bar{z}^I = cL u^{n-1} \partial u \bar{\partial} u + O(u^{n}).$$

Since the pullback to $\mathcal{G} = \mathbb{C}^* \times M$ of $\partial u \bar{\partial} u$ is a multiple of $\theta^2$, which is trace-free with respect to $g$, it is evident that this Ricci curvature is $O^{ij}_L(p^{n-1})$. Now $\tilde{g}$ is clearly homogeneous, and it satisfies the initial condition (3.1) by the definition of the Fefferman metric $g$. Therefore $\tilde{g}$ is a smooth ambient metric for the Fefferman conformal structure in the sense of our earlier definition. Now $T = 2 \Re z^0 \partial_{z^0}$, and an easy calculation gives $\tilde{g}(T, T) = -|z^0|^2 u$. The definition (2.1) of the obstruction tensor thus reduces to a constant multiple of $L\theta^2$. \qed

Since $\theta$ is null with respect to $g$, Proposition 3.9 implies that the obstruction tensor of a Fefferman metric satisfies $O^{ij}O_{ij} = 0$. In particular $O^{ij}O_{ij} = 0$, so Fefferman metrics satisfy the condition in Proposition 3.8.
Upon applying Theorem 3.5 to the Fefferman conformal structure of a CR manifold, one obtains a family of inhomogeneous ambient metrics with ambiguity an arbitrary trace-free symmetric tensor on the circle bundle. In [H], a family of “inhomogeneous CR ambient metrics” associated to a nondegenerate hypersurface in $\mathbb{C}^n$ was constructed with ambiguity a scalar function on the hypersurface. The relation between these constructions is not clear. The metrics constructed in [H] are not straight in general. It seems reasonable to guess that there are inhomogeneous, nonsmooth diffeomorphisms with expansions involving $\log |r|_\#$ relating the constructions (for an appropriately restricted ambiguity in the conformal construction). We intend to investigate this issue in the future.

A homogeneous ambient metric for a conformal manifold $(M, [g])$ gives rise to an asymptotically hyperbolic “Poincaré” metric by a procedure described in [FG1]. Namely, if $\tilde{g}$ is a homogeneous ambient metric, then the pullback $g_+$ of $\tilde{g}$ to the hypersurface $\{\tilde{g}(T,T) = -1\}$ is a metric of signature of $(p + 1, q)$ which in suitable coordinates is asymptotically hyperbolic with conformal infinity $(M, [g])$. The homogeneity of $\tilde{g}$ implies that the condition $\text{Ric}(\tilde{g}) = 0$ is equivalent to $\text{Ric}(g_+) = -ng_+$. The same construction can also be carried out for an inhomogeneous ambient metric. Proposition 3.2 shows that $\tilde{g}(T,T)$ is still a smooth homogeneous defining function. When we take $r_\# = \tilde{g}(T,T)$, all log terms in the asymptotic expansion (3.2) vanish on the hypersurface $\tilde{g}(T,T) = -1$, so the Poincaré metric $g_+$ which one obtains agrees with that defined by the smooth part $\tilde{g}^{(0)}$. In particular, $g_+$ has a smooth conformal compactification with no log terms. The $g_+$ which arise as $\tilde{g}$ varies over the family of inhomogeneous ambient metrics associated to $(M, [g])$ form a family of smoothly compactifiable Poincaré metrics invariantly associated to $(M, [g])$ up to choice of ambiguity and up to a smooth diffeomorphism restricting to the identity on $M$. However, $\tilde{g}^{(0)}$ is not usually Ricci flat, so $g_+$ will not in general be Einstein. We do not know a direct characterization of either the smooth parts $\tilde{g}^{(0)}$ or the resulting family of Poincaré metrics $g_+$, other than to say that they arise from inhomogeneous ambient metrics by taking smooth parts.

4. Scalar invariants

An original motivation for the ambient metric construction in [FG1] was to construct scalar conformal invariants. In even dimensions, the construction of invariants in [FG1] terminates at finite order owing to the failure of the existence of infinite order smooth ambient metrics. Inhomogeneous ambient metrics can be used to extend the construction of invariants to all orders. The new invariants generally depend both on the initial metric and the ambiguity.

By a scalar invariant $I(g)$ of metrics we mean a polynomial in the variables $(\partial^a g_{ij})_{|a| \geq 0}$ and $|\det g_{ij}|^{-1/2}$, which is coordinate-free in the sense that its value is independent of orientation-preserving changes of the coordinates used to express and differentiate $g$. Such a scalar invariant is said to be even if it is also unchanged
under orientation-reversing changes of coordinates, and \textit{odd} if it changes sign under orientation-reversing coordinate changes. It is said to be conformally invariant of weight \( w \) if \( I(\Omega^2 g) = \Omega^w I(g) \) for smooth positive functions \( \Omega \). By a scalar conformal invariant we will mean a scalar invariant of metrics which is conformally invariant of weight \( w \) for some \( w \).

First recall the characterization of scalar conformal invariants in odd dimensions. Denote by \( \tilde{\nabla}^r \tilde{\mathcal{R}} \) the \( r \)-th iterated covariant derivative of the curvature tensor of a smooth ambient metric, by \( \tilde{\mu} \in \Lambda^{n+2} T^* \tilde{\mathcal{G}} \) the volume form of \( \tilde{g} \), and set \( \tilde{\mu}_0 = T \downarrow \tilde{\mu} \). Consider complete contractions of these tensors:

\begin{equation}
\begin{aligned}
\text{contr}(\tilde{\nabla}^{r_1} \tilde{\mathcal{R}} \otimes \cdots \otimes \tilde{\nabla}^{r_L} \tilde{\mathcal{R}}) \\
\text{contr}(\tilde{\mu} \otimes \tilde{\nabla}^{r_1} \tilde{\mathcal{R}} \otimes \cdots \otimes \tilde{\nabla}^{r_L} \tilde{\mathcal{R}}) \\
\text{contr}(\tilde{\mu}_0 \otimes \tilde{\nabla}^{r_1} \tilde{\mathcal{R}} \otimes \cdots \otimes \tilde{\nabla}^{r_L} \tilde{\mathcal{R}})
\end{aligned}
\end{equation}

where the contractions are taken with respect to \( \tilde{g} \). Each such complete contraction defines a homogeneous function on \( \tilde{\mathcal{G}} \) whose restriction to \( \mathcal{G} \) is independent of the diffeomorphism indeterminacy in the smooth ambient metric. If \( g \) is a metric in the conformal class, evaluating this homogeneous function at the image of \( g \) viewed as a section of \( \mathcal{G} \) defines a function on \( M \) which depends polynomially on the derivatives of \( g \). Every such function is a scalar conformal invariant. Contractions of the first type in (4.1) define even invariants and contractions of the second and third types define odd invariants. A linear combination of such complete contractions, all having the same weight, is called a Weyl invariant. The characterization of conformal invariants for \( n \) odd states that every scalar conformal invariant is a Weyl invariant. This follows from the invariant theory of [BEGr] together with the jet isomorphism theorem of [FG2], as outlined in [BEGr]. Full details will appear in [FG2].

When \( n \) is even, we carry out the same construction, but now replacing the \( \tilde{\nabla}^r \tilde{\mathcal{R}} \) in the complete contractions (4.1) by the corresponding covariant derivatives of curvature of the smooth part of an inhomogeneous ambient metric. Of course, the resulting functions on \( M \) generally depend on the choice of ambiguity tensor as well as the metric \( g \); they can be regarded as conformal invariants of the pair \((g, A)\). We still refer to linear combinations of such complete contractions as Weyl invariants. Some such Weyl invariants may actually be independent of the ambiguity, in which case they define scalar conformal invariants. We call such Weyl invariants ambiguity-independent Weyl invariants. We do not have a systematic general way of constructing ambiguity-independent Weyl invariants, but we do have examples and conditions for Weyl invariants to be ambiguity-independent in interesting cases. Also, we can extend the jet isomorphism theorem of [FG2] and the invariant theory of [BEGr] to prove that all scalar conformal invariants are of this form if \( n \equiv 2 \mod 4 \):
Theorem 4.1. If \( n \equiv 2 \mod 4 \), then every scalar conformal invariant is an ambiguity-independent Weyl invariant.

If \( n \) is a multiple of 4, there are exceptional odd invariants which do not arise as ambiguity-independent Weyl invariants. The existence of exceptional invariants for a linearized problem was observed in [BEG] and in this context they were studied systematically in [BG]. We follow the approach of [BG] to complete the description of scalar conformal invariants when \( n \equiv 0 \mod 4 \) as follows.

Choose a representative metric \( g \) and corresponding identification \( \tilde{G} = \mathbb{R}_+ \times M \times \mathbb{R} \). Define an \( n \)-form \( \eta \) on \( \tilde{G} \) by \( \eta = t^{-2} \partial_\rho \mathbf{1} \mu_0 \). Consider a partial contraction

\[
L = \text{pcontr}(\eta \otimes \tilde{R} \otimes \cdots \otimes \tilde{R})^{n/2},
\]

where all the indices of \( \eta \) are contracted. While \( \eta \) depends on the choice of \( g \), it can be shown that \( L|_G \) is independent of this choice and also of the choice of ambiguity tensor (for \( n > 4 \) this is obvious since \( \tilde{R}|_G \) is already independent of the ambiguity).

A scalar invariant can be obtained from \( L \) by applying an ambient realization of the tractor \( D \) operator (see again [CG]). This is a differential operator \( D \) defined initially as a map

\[
D : \Gamma(\otimes^p T^* \tilde{G})(w) \to \Gamma(\otimes^{p+1} T^* \tilde{G})(w),
\]

where \( \Gamma(\otimes^p T^* \tilde{G})(w) \) denotes the space of covariant \( p \)-tensor fields homogeneous of degree \( w \) in the sense of the previous sections \( (\delta_s f = s^w f) \), by:

\[
Df = \tilde{\nabla} f - \frac{1}{2(n+2(w-p-1))} dr_\# \otimes \tilde{\Delta} f.
\]

Here \( \tilde{\nabla} \) and \( \tilde{\Delta} = \tilde{\nabla}^I \tilde{\nabla}_I \) are defined with respect to \( \tilde{g}^{(0)} \) and it is assumed that \( w \neq p+1-n/2 \). One checks that \( D \) acts tangentially to \( G \) in the sense that \( Df \) vanishes on \( G \) if \( f \) does, so \( D \) induces a map

\[
D : \Gamma(\otimes^p T^* \tilde{G}|_G)(w) \to \Gamma(\otimes^{p+1} T^* \tilde{G}|_G)(w).
\]

The homogeneities are such that the expression

\[
D^I D^J \cdots D^K L_{(IJ \cdots K)}
\]

is defined and the above discussion implies that it restricts to \( G \) to give a homogeneous function independent of the choice of ambiguity tensor. (Here \( (IJ \cdots K) \) denotes symmetrization over the enclosed indices.) As in the case of Weyl invariants, when evaluated at the image of \( g \) as a section of \( G \), one obtains a scalar conformal invariant.

Such conformal invariants are odd and are called basic exceptional invariants. It is easy to see that there are only finitely many basic exceptional invariants in any dimension \( n \equiv 0 \mod 4 \) (the construction works just as well when \( n \equiv 2 \mod 4 \), but all basic exceptional invariants vanish in this case). For \( n = 4 \) there are only two nonzero basic exceptional invariants:

\[
\eta^{IJKL} \tilde{R}_{I,J}^{AB} \tilde{R}_{K,LAB}^{L}
\]
of weight $-4$, and
\[ D^A D^B (\eta^{IJKLM} \tilde{R}_{IJAC} \tilde{R}_{KLB}^C ) \]
of weight $-6$. The first is a multiple of $|W^+|^2 - |W^-|^2$, where $W^\pm$ denote the ± self-dual parts of the Weyl tensor, and the second is due to Bailey, Eastwood, and Gover (see p. 1207 of [BEGo]).

By extending the jet isomorphism theorem and invariant theory of [FG2], [BEG], and [BG], we can prove:

**Theorem 4.2.** If $n \equiv 0 \pmod{4}$, then every even scalar conformal invariant is an ambiguity-independent Weyl invariant, and every odd scalar conformal invariant is a linear combination of an ambiguity-independent Weyl invariant and basic exceptional invariants.

The above constructions use only the smooth part $\tilde{g}^{(0)}$ of an inhomogeneous ambient metric. It is also possible to construct invariants using the tensors $r^{(n/2-1)N+1} \tilde{g}^{(N)}$ for $N \geq 1$ in (3.2) (with $r_\# = \tilde{g}(T, T)$). However, the above theorems show that this is not necessary: the invariants already can be constructed using just the smooth part.

We conclude with some examples. The smooth part $\tilde{g}^{(0)}$ of an inhomogeneous ambient metric is not Ricci flat; the leading part of its Ricci tensor can be identified with the obstruction tensor. So Weyl invariants involving the Ricci tensor of $\tilde{g}^{(0)}$ give rise to conformal invariants involving the obstruction tensor. Define an ambient version of the obstruction tensor by
\[ \tilde{\mathcal{O}}_{IJ} = \frac{1}{n-2} \tilde{\Delta}^{n/2-1} \tilde{R}_{IJ}, \]
where $\tilde{\Delta}$ and $\tilde{R}_{IJ}$ are with respect to $\tilde{g}^{(0)}$. One checks that $T^j \tilde{\mathcal{O}} = 0$ on $G$ and that $\tilde{\mathcal{O}} |_{T^G}$ is the tensor on $G$ homogeneous of degree $2 - n$ defined by $\mathcal{O}$. Otherwise put, in the identification $\tilde{G} = \mathbb{R}^+ \times M \times \mathbb{R}$ induced by a representative $g$, one has $\tilde{\mathcal{O}}_{ij} = 0$ and $\tilde{\mathcal{O}}_{ij} = \tilde{t}^{2-n} \mathcal{O}_{ij}$ at $\rho = 0$. From this it is easy to see that the contraction $\tilde{\mathcal{O}}_{IJ} \tilde{\mathcal{O}}^{IJ}$ determines the conformal invariant $\mathcal{O}_{ij} \mathcal{O}^{ij}$, and similarly that $\tilde{R}^{IJKL} \tilde{\mathcal{O}}_{IK, JL}$ determines the conformal invariant $W^{ijkl} W_{ijkl} \mathcal{O}_{ab}$. A more interesting example is to consider $\tilde{R}^{IJKL} \tilde{\mathcal{O}}_{IK, JL}$. One finds that for even $n \geq 6$, the restriction of this quantity to $G$ is ambiguity-independent and determines the conformal invariant
\[ W^{ijkl} \mathcal{O}_{ik, jl} - (n - 1) W^{ijkl} P_{ik} \mathcal{O}_{jl} + 2 n C^{ijkl} \mathcal{O}_{jk, l} + \frac{n(n-1)}{n-4} B^{jk} \mathcal{O}_{jk}. \]
For $n = 4$, the Weyl invariant determined by $\tilde{R}^{IJKL} \tilde{\mathcal{O}}_{IK, JL}$ depends on the ambiguity tensor. It is given by the same formula, except that in the last term $\frac{n-4}{n-4} B^{jk}$ is replaced by $-\frac{1}{2} A^{jk}$ with $A$ given by (3.8).
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