ON THE EXISTENCE OF MAXIMIZING CURVES FOR THE CHARGED-PARTICLE ACTION

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AbSTRACT. The classical Avez-Seifert theorem is generalized to the case of the Lorentz force equation for charged test particles with fixed charge-to-mass ratio. Given two events $x_0$ and $x_1$, with $x_1$ in the chronological future of $x_0$, and a ratio $q/m$, it is proved that a timelike connecting solution of the Lorentz force equation exists provided there is no null connecting geodesic and the spacetime is globally hyperbolic. As a result, the theorem answers affirmatively to the existence of timelike connecting solutions for the particular case of Minkowski spacetime. Moreover, it is proved that there is at least one $C^1$ connecting curve that maximizes the functional $I[\gamma] = \int_0^1 ds + q/(mc^2)\omega$ over the set of $C^1$ future-directed non-spacelike connecting curves.

1. INTRODUCTION

Let $\Lambda$ be a Lorentzian manifold endowed with the metric $g$ having signature $(+ - - -)$. A point particle of rest mass $m$ and electric charge $q$, moving in the electromagnetic field $F$, has a timelike worldline that satisfies the Lorentz force equation (cf. [8])

$$D_s \left( \frac{dx}{ds} \right) = \frac{q}{mc^2} \hat{F}(x) \left[ \frac{dx}{ds} \right].$$

Here $x = x(s)$ is the world line of the particle parameterized with proper time, $\frac{dx}{ds}$ is the four-velocity, $D_s \left( \frac{dx}{ds} \right)$ is the covariant derivative of $\frac{dx}{ds}$ along $x(s)$ associated to the Levi-Civita connection of $g$, and $\hat{F}(x)[:]$ is the linear map on $T_x\Lambda$ defined by

$$g(x)[v, \hat{F}(x)[w]] = F(x)[v, w],$$

for any $v, w \in T_x\Lambda$.

Let $x_0$ and $x_1$ be two chronologically related events, $x_1 \in I^+(x_0)$. If the manifold $\Lambda$ is globally hyperbolic, the Avez-Seifert theorem [1] assures the existence of at least a timelike connecting solution of the Lorentz force equation in the $q = 0$ case. We are looking for a suitable generalization to $q/m \neq 0$ cases.

Works in this direction [5, 4] have shown that in a globally hyperbolic manifold $\Lambda$, and for an exact electromagnetic field $F = d\omega$ (i.e. in absence of monopoles), connecting solutions exist for any ratio $q/m$ in a suitable neighborhood $[-R, R]$. $R$ is a gauge invariant quantity that depends on the extremals $x_0$ and $x_1$ and on the potential one-form. That result was satisfying from the physical point of view since for sufficiently weak field, compatible with the absence of quantum pair creation effects, the electron’s charge-to-mass ratio is less than $R$.

From a mathematical point of view, however, the problem in the strong field case was still open. Here we prove that under the same conditions as above $R = +\infty$ provided there is no null connecting geodesic.
Like in previous papers on the subject \[4, 5\], the strategy is to introduce a Kaluza-
Klein spacetime \[9, 7\] and to regard the solutions of the Lorentz force equation as
projections of null geodesics of a higher dimensional manifold. In this way one can
take advantage of causal techniques. Here the reference text for most notations and
results on causal techniques is \[6\].

So assume that \( F \) is an exact two-form and let \( \omega \) be a potential one-form for
\( F \).

Let us consider a trivial bundle \( P = \Lambda \times \mathbb{R}, \pi : P \to \Lambda \), with the structure group
\( T_1 : b \in T_1, \ p = (x, y), \ p' = pb = (x, y + b) \), and \( \tilde{\omega} \) the connection one-form on \( P \):
\[
\tilde{\omega} = i(dy + \frac{e}{\hbar c}\omega).
\]

Here \( y \) is a dimensionless coordinate on the fibre, \(-e (e > 0)\) is the electron charge
and \( \hbar = h/2\pi \), with \( h \) the Planck constant. Henceforth we will denote by \( \bar{\omega} \)
and \( \bar{F} \) respectively the one-form \( e\hbar c\omega \) and the two-form \( e\hbar c F \). Let us endow \( P \) with the
Kaluza-Klein metric
\[
g_{kk} = g + a^2 \tilde{\omega}^2 \quad (2)
\]
or equivalently, using the notation \( z \) for the points in \( P \) and the identification
\( z = (x, y) \in \Lambda \times \mathbb{R} \),
\[
g_{kk}(z)[w, w] = g_{kk}(x, y)[(v, u), (v, u)] = g(x)[v, v] - a^2(u + \tilde{\omega}(x)[v])^2,
\]
for every \( w = (v, u) \in T_x\Lambda \times \mathbb{R} \). The positive constant \( a \) has the dimension of a
length and has been introduced for dimensional consistency of definition \( \[2\] \).

Let \( x_1 \) be an event in the chronological future of \( x_0 \). The set \( \mathcal{N}_{x_0, x_1} \), includes
the \( C^1 \) future-pointing non-spacelike connecting curves. With connecting curve we
mean a map \( x \) from an interval \([a, b] \subset \mathbb{R} \) to \( \Lambda \) such that \( x(a) = x_0 \) and \( x(b) = x_1 \)
and any other map \( w \) such that \( w = x \circ \lambda \) with \( \lambda \) a \( C^1 \) function from an interval
\([c, d] \) to the interval \([a, b] \), having positive derivative.

The functional \( I[\gamma] \) defined on the space \( \mathcal{N}_{x_0, x_1} \) is
\[
I[\gamma](x_0, x_1) = \int_\gamma (ds + \frac{q}{mc^2}\omega).
\]
The timelike solutions of the Lorentz force equation \( \[1\] \), if they exist, are critical
points of this functional as it follows from a computation of the Euler-Lagrange
equation.

Let us now consider the geodesics over \( P \). They are \( C^1 \) curves \( z(\lambda) = (x(\lambda), y(\lambda)) \)
that are critical points of the functional
\[
S = S(z) = \int_0^{11} \frac{1}{2}g_{kk}(z(\lambda))[\dot{z}(\lambda), \dot{z}(\lambda)] d \lambda.
\]
Taking into account that \( g_{kk} \) is independent of \( y \) we find that the following quantity
is conserved
\[
p_z = -a^2(\dot{y} + \tilde{\omega}(x)[\dot{x}]).
\]
Moreover taking variations with respect to the variable \( x \) we obtain
\[
D_\lambda \dot{x} = p_z \frac{\partial}{\partial x} F(x)[\dot{x}]. \quad (3)
\]
If \( x \) is non-spacelike we define
\[
g(x)[\dot{x}, \dot{x}] = C^2.
\]
Moreover, since $z$ is a geodesic, $g^{kk}(z)[\dot{z}, \dot{z}]$ is conserved too and
\begin{equation}
  g^{kk}(z)[\dot{z}, \dot{z}] = C^2 - \frac{p_z^2}{a^2}.
\end{equation}
From this formula it follows that if $z$ is timelike (non-spacelike) then $x$ is timelike (non-spacelike). If $z$ is a null geodesic then $C^2 = p_z^2/a^2$ and $x$ is timelike iff $p_z \neq 0$.

In case $x$ is timelike its proper time is given by
\begin{equation}
ds = C d\lambda,
\end{equation}
and parameterizing with respect to proper time Eq. (3) becomes
\begin{equation}
D_s \left( \frac{d x}{d s} \right) = \frac{p_z}{C} \hat{F}(x) \left[ \frac{d x}{d s} \right] = p_z \frac{e}{hc} \hat{F}(x) \left[ \frac{d x}{d s} \right].
\end{equation}
This is exactly the Lorentz force equation for a charge-to-mass ratio
\begin{equation}
\frac{q}{m} = \frac{p_z}{C} \frac{e}{h}.
\end{equation}
Notice that, a solution of Eq. (3) must be timelike ($p_z \neq 0$) in order to represent a charged particle. Only in this case it can be parameterized with respect to proper time.

Our strategy is to search a future-directed null geodesic in $P$ that projects on a connecting timelike curve on $\Lambda$. To this end we have to choose the following value for $a$
\begin{equation}
a = \frac{p_z}{|C|} = \frac{h}{ec} \frac{q}{m}.
\end{equation}

2. The theorem

We state the theorem.

**Theorem 2.1.** Let $(\Lambda, g)$ be a time-oriented Lorentz manifold. Let $\omega$ be a one-form ($C^2$) on $\Lambda$ and $F = d\omega$. Assume that $(\Lambda, g)$ is a globally hyperbolic manifold. Let $x_1$ be an event in the chronological future of $x_0$ and $q/m$ any charge-to-mass ratio. There exists at least one future-directed non-spacelike $C^1$ curve $x(\lambda)$ connecting $x_0$ and $x_1$ that maximizes the functional $I[\gamma](x_0, x_1)$ on the space $N_{x_0, x_1}$. If $x$ is timelike, once parameterized with respect to proper time, it becomes a solution of the Lorentz force equation (1); if it is null, it is a null geodesic.

We need some lemmas.

**Lemma 2.2.** The manifold $P = \Lambda \times \mathbb{R}$ endowed with the metric (2) is a time-oriented globally hyperbolic Lorentzian manifold.

**Proof.** See [5] or [4].

**Remark 2.3.** Let $E^+(p_0) = J^+(p_0) - J^+(p_0)$, $p_0 \in P$. It is well known (see [6, p. 112, 184]) that if $q \in E^+(p_0)$ there exists a null geodesic connecting $p_0$ and $q$.

**Lemma 2.4.** Any globally hyperbolic Lorentzian manifold $\Lambda$ is causally simple, i.e. for every compact subset $K$ of $\Lambda$, $\dot{J}^+(K) = E^+(K)$, where $\dot{J}^+(K)$ denotes the boundary of $J^+(K)$.

**Proof.** See [6, p. 188, 207].
Proof of Theorem 2.1. Let $P$ be the Kaluza-Klein principal bundle constructed in the introduction having $a$ given by Eq. (4). Given a parameterized curve $\sigma(\lambda) : [0,1] \rightarrow \Lambda$ belonging to $\mathcal{N}_{x_{0},x_{1}}$, define its lifts $\tilde{\sigma}^{\pm}(\lambda)$ and $\hat{\sigma}^{\pm}(\lambda)$ of starting point $p_{0} = (x_{0},y_{0})$ by requiring $p_{\bar{y}^{\pm}} = \pm a \int_{\sigma} ds$ and $\hat{\sigma}^{\pm}(0) = p_{0}$. In other words $\hat{\sigma}^{\pm}(\lambda) = (\sigma(\lambda), y^{\pm}(\lambda))$ satisfies the condition
\[
\dot{y}^{\pm} + \tilde{\omega}[\sigma] = -\frac{p_{\bar{y}^{\pm}}}{a},
\] (6)
$\hat{\sigma}^{\pm}(\lambda)$ is a null curve that depends on both $\sigma$ and its parameterization. Let $y_{1}^{\pm}(\sigma) = \hat{\sigma}^{\pm}(1)$ and $\Delta y^{\pm}(\sigma) = y_{1}^{\pm}(\sigma) - y_{0}$. Integrating Eq. (6) over $\sigma$
\[
p_{\bar{y}^{\pm}} = -a^{2}(\Delta y^{\pm}(\sigma) + \int_{\sigma} \omega),
\] or
\[
\Delta y^{\pm}(\sigma) = y_{1}^{\pm}(\sigma) - y_{0} = \mp \frac{1}{a} \left( \int_{\sigma} ds + \frac{\pm |q/m|}{c^2} \int_{\sigma} \omega \right).
\] (7)
Notice that the final point $p_{1} = (x_{1},y_{1}^{\pm})$ does not depend on the specific parameterization of $\sigma$. A maximization on $\mathcal{N}_{x_{0},x_{1}}$ of the functional $I$ relative to the ratio $+|q/m|$, corresponds to a minimization of $y_{1}^{+}(\sigma)$. Analogously, a maximization on $\mathcal{N}_{x_{0},x_{1}}$ of the functional $I$ relative to the ratio $-|q/m|$ corresponds to a maximization of $y_{1}^{-}$. Let
\[
\hat{s} = \sup_{\sigma \in \mathcal{N}_{x_{0},x_{1}}} y_{1}^{-}(\sigma),
\]
\[
\tilde{s} = \inf_{\sigma \in \mathcal{N}_{x_{0},x_{1}}} y_{1}^{+}(\sigma),
\]
we show that $\hat{s} > \tilde{s}$. Indeed, for a given $\sigma$ we have
\[
y_{1}^{-}(\sigma) - y_{1}^{+}(\sigma) = \frac{2}{a} \int_{\sigma} ds \geq 0
\] (8)
thus
\[
\hat{s} - \tilde{s} \geq \frac{2}{a} \sup_{\sigma \in \mathcal{N}_{x_{0},x_{1}}} \int_{\sigma} ds = \frac{2l(x_{0},x_{1})}{a} \geq 0,
\]
with $l(x_{0},x_{1})$ the Lorentzian distance function. Moreover, both $\int_{\sigma} ds$ and $\int_{\sigma} |\tilde{\omega}|$ are bounded on $\mathcal{N}_{x_{0},x_{1}}$ [5], therefore $\hat{s}$ and $\tilde{s}$ are finite.
Let $\eta : [0,1] \rightarrow P$ be a non-spacelike future-directed $C^{1}$ curve that starts in $p_{0}$ and ends in $p_{1} : \pi(p_{1}) = x_{1}$. Let $p_{1} = (x_{1},y_{1})$, and consider the projection $x(\lambda)$ of $\eta(\lambda)$. Since $\eta$ in a non-spacelike curve
\[
g(\dot{x},\dot{x}) - a^{2} (\dot{y} + \omega(\dot{x}))^{2} \geq 0.
\]
Taking the square-root and integrating over $x(\lambda)$
\[
|y_{1} - y_{0}| \leq \frac{1}{a} \int_{x} ds + \int_{x} |\tilde{\omega}| < M < +\infty,
\]
where $M$ is a suitable positive constant. Hence $y_{1}$ is finite. Now we consider the set $W = J^{+}(p_{0}) \cap \pi^{-1}(x_{0})$ and define
\[
\hat{s}' = \sup_{p \in W} y_{1}(p),
\]
\[
\tilde{s}' = \inf_{p \in W} y_{1}(p).
\]
where \( y_1(p) \) is defined through \( p = (x_1, y_1) \). Since for any non-spacelike curve \( y_1 \) is bounded, \( \hat{s}' \) and \( \tilde{s}' \) are bounded too. Since \( P \) is globally hyperbolic \( J^+(p_0) \) is closed and the set \( W \), being limited and closed, is compact. The points \( \hat{p}_1 = (x_1, \hat{s}') \) and \( \tilde{p}_1 = (x_1, \tilde{s}') \), being accumulation points, belong to \( W \). Moreover they can’t be points of the open set \( I^+(p_0) \). Thus, they belong to \( E^+(p_0) \) and therefore (remark 2.5) there are two null geodesics \( \hat{\eta}(\lambda) = (\hat{x}(\lambda), \hat{y}(\lambda)) \), \( \tilde{\eta}(\lambda) = (\tilde{x}(\lambda), \tilde{y}(\lambda)) \), that join \( p_0 \) with \( \hat{p}_1 \) and \( \tilde{p}_1 \) respectively. Let \( \lambda \) be that affine parameter that has values 0 and 1 at the endpoints. For a null geodesic \( \eta = (x, y) \) as those under consideration, \( p_\eta \) is conserved, hence for a suitable choice of sign \( \eta = \hat{x}^\pm \). Let \( p_1 = (x_1, y_1) \) be its final point. For the definition of \( \hat{s} \) and \( \tilde{s} \)
\[
\hat{s} \leq y_1 \leq \tilde{s}.
\]
But in the case \( \eta = \tilde{\eta} \) it is \( \hat{s}' \geq \tilde{s} \), otherwise there would be a null curve \( \beta \) having final point \( \beta(1) \) strictly above \( \hat{p}_1 \) and \( \tilde{p}_1 \) respectively. Let \( \lambda \) be that affine parameter that has values 0 and 1 at the endpoints. For a null geodesic \( \eta = (x, y) \) as those under consideration, \( p_\eta \) is conserved, hence for a suitable choice of sign \( \eta = \hat{x}^\pm \). Let \( p_1 = (x_1, y_1) \) be its final point. For the definition of \( \hat{s} \) and \( \tilde{s} \)
\[
\hat{s} \leq y_1 \leq \tilde{s}.
\]

Assume that \( \hat{s} = \hat{x}^- \) then, from the definition of \( \hat{s}' \) (\( = \hat{s} \))
\[
\hat{s}' = \hat{x}^+(1) \geq \hat{x}^-(1).
\]
Equation (3) implies that \( \hat{x}^- (1) \geq \hat{x}^+ (1) \) where the equality holds if and only if \( \hat{x} \) is a null curve. From the hypothesis we find that \( \hat{x} \) is a null curve and, since in this case both lifts coincide with the horizontal lift, \( \hat{x}^- (\lambda) = \hat{x}^+ (\lambda) \). Thus \( - \) is always the right sign whereas \( + \) is right if and only if \( \hat{x} \) is a null curve, in which case \( \hat{x}^- = \hat{x}^+ \).

With an analogous reasoning for \( \tilde{\eta} \) we find
\[
\tilde{\eta}(\lambda) = \tilde{x}^- (\lambda),
\tilde{\eta}(\lambda) = \tilde{x}^+ (\lambda).
\]
We conclude that the functional \( I[\gamma](x_0, x_1) \) is maximized in \( N_{x_0, x_1} \) by \( \hat{x} \) if \( q/m < 0 \) or by \( \tilde{x} \) if \( q/m > 0 \). The curve \( \hat{x} \), being the projection of a null geodesic, is a connecting solution of Eq. (3), moreover if timelike it is a connecting solution of the Lorentz force equation with charge-to-mass ratio \( -|q/m| \). If it is null, from \( |p_\eta| = a \int_{\hat{x}} ds = 0 \) and Eq. (3) we conclude that it is a null geodesic. An analogous conclusion holds for \( \tilde{x} \).

In many cases the spacetime \( \Lambda \) has the property that no two chronologically related events are joined by a null geodesic. Minkowski spacetime is the most important example.

**Corollary 2.5.** Let \((M, \eta)\) be the Minkowski spacetime. Let \( F \) be an electromagnetic tensor field (closed two-form). Let \( x_1 \) be an event in the chronological future of \( x_0 \) and \( q/m \) a charge-to-mass ratio, then there exist at least one future-directed timelike solution to (1) connecting \( x_0 \) and \( x_1 \).
Proof. Since \( M \) is contractible \( F \) is exact. Moreover, in Minkowski spacetime, if \( x_1 \in I^+(x_0) \) there is no null geodesic connecting \( x_0 \) with \( x_1 \). \( \square \)

3. Conclusions

We have shown that in Minkowski spacetime the existence of at least a timelike connecting solution to the Lorentz force equation is assured by corollary 2.5. Notice that theorem 2.1 holds more generally for any chronologically related pair \( x_0, x_1 \), belonging to a globally hyperbolic set \( N \subset M \). Thus in a generic spacetime the existence of a timelike connecting solution to the Lorentz force equation is assured whenever \( x_0 \) and \( x_1 \) belong to a globally hyperbolic set and there is no connecting null geodesic. Finally, we have proved the existence of at least one \( C^1 \) connecting curve that maximizes the functional \( I[\gamma] = \int_\gamma ds + q/(mc^2)\omega \) over the set of \( C^1 \) future-directed non-spacelike connecting curves.

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