THE SECOND CONVEX HULL OF EVERY OPTIMAL RECTILINEAR DRAWING OF $K_n$ IS A TRIANGLE

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Abstract. A rectilinear drawing of a graph $G$ is optimal if it has the smallest number of crossings among all rectilinear drawings of $G$. In this paper it is shown that for $n \geq 8$, the second convex hull of every optimal rectilinear drawing of the complete graph $K_n$ is a triangle.

1. Introduction

The rectilinear crossing number $\text{rc}(G)$ of a graph $G$, is the minimum number of edge crossings in a rectilinear drawing of $G$ in the plane, that is, a drawing of $G$ in the plane where the vertices are points in general position and the edges are straight segments. Determining $\text{rc}(K_n)$, where $K_n$ is the complete graph with $n$ vertices, is a well-known open problem in combinatorial geometry initiated by R. K. Guy [10] which has been attracting a great amount of attention during the last ten years, see for instance [1, 2, 3, 4, 5, 7, 9, 10, 12, 13, 14, 16]. A more recent line of research related to this problem is the study of the structural properties of the optimal rectilinear drawings of the complete graphs. The first work [8] in this sense, due to O. Aichholzer, D. Orden and P. Ramos, established that the convex hull of every optimal rectilinear drawing of $K_n$ is a triangle. A year later, in 2007, J. Balogh, J. Leaños, S. Pan, R. B. Richter, and G. Salazar verified that such a result remains valid for the case of the optimal pseudolinear drawings of $K_n$ [11]. Another structural property which has been conjectured for the optimal rectilinear drawings of $K_n$ is the, so-called 3–decomposability [2]. This conjecture states that every optimal rectilinear drawing of $K_n$ is 3-decomposable, that is, there is a triangle $T$ enclosing the drawing, and a balanced partition $A, B, C$ of the underlying set of points $P$, such that the orthogonal projections of $P$ onto sides of $T$ show $A$ between $B$ and $C$ on one side, $B$ between $A$ and $C$ on another side, and $C$ between $A$ and $B$ on the third side. The 3-decomposability of rectilinear drawings of $K_n$ has been studied in [2, 6, 13].

Let $D$ be a drawing of $K_n$ and let $P$ be its underlying set of points. The convex hull of $D$, denoted by $C(D)$, is defined as the frontier of the intersection of all convex sets in $\mathbb{R}^2$ containing $P$. In particular, $C(D)$ is a Jordan curve formed by some vertices and some edges of $D$.

As usual, we shall denote by $\text{CH}(P)$ the set of vertices of $D$ which are in $C(D)$. In this context we define the 2nd–convex hull of $D$ as the convex hull of $P \setminus \text{CH}(P)$, the 3rd–convex hull of $D$ as the convex hull of $P \setminus (\text{CH}(P) \cup \text{CH}(P \setminus \text{CH}(P)))$, and

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so on. By convention, the 1st-convex hull of $D$ will be the convex hull of $D$. We use $C_k(D)$ to denote the $k$th-convex hull of $D$. Analogously, we use $CH_k(D)$ to denote the vertex set of $D$ in $C_k(D)$. See Figure 1.

Our aim in this paper is to study the 2nd-convex hull of the set of points corresponding to optimal rectilinear drawings of $K_n$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{This set of points induces an optimal rectilinear drawing $D$ of $K_{10}$. In this case $C(D)$ and $C_2(D)$ are triangles, and $C_3(D)$ is a quadrilateral.}
\end{figure}

Our main result is the following.

**Theorem 1.** Let $n \geq 8$ be an integer. If $D$ is an optimal rectilinear drawing of $K_n$, then $C_2(D)$ is a triangle.

In Figure 2 we show an optimal rectilinear drawing of $K_7$ which has a quadrilateral as its 2nd-convex hull. The existence of such a drawing justifies our hypothesis that $n \geq 8$ in Theorem 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure2.png}
\caption{Optimal rectilinear drawings of $K_7$ and $K_8$ with a quadrilateral and a triangle respectively as its 2nd-convex hull.}
\end{figure}

We conjecture the following generalization:

**Conjecture 2.** There exists a positive constant $c$ such that, for every integer $n \geq 8$ the following holds. If $k$ is an integer such that $1 \leq k \leq cn$ and $D$ is an optimal rectilinear drawing of $K_n$, then $C_k(D)$ is a triangle.
The motivation behind Conjecture 2 is the evidence of all the best crossing-wise known rectilinear drawings of $K_n$. In particular, from the optimal rectilinear drawing of $K_{15}$ reported in [5] it follows that if such a constant $c$ exists, then $c \leq 1/5$ (we have deduced this upper bound from all known optimal rectilinear drawing of $K_n$ [1] [2] [3]).

In Section 2 we formalize the relevant notions at play and establish some elementary facts. The proof of Theorem 1 is given in Section 3.

2. PRELIMINARIES

The aim of this section is to introduce the notions that will be used in our proofs and establish some basic facts.

Throughout this section, $S$ denotes a set of $m \geq 3$ points in the plane $\mathbb{R}^2$ in general position, that is, no three points lie on a common line. If $\ell$ is a fixed directed line in $\mathbb{R}^2$, then we denote by $\ell^+(S)$ (respectively, $\ell^-(S)$) the subset of points of $S$ lying on the right (respectively, left) open halfplane defined by $\ell$. Thus $S$ is the disjoint union of $\ell^+(S)$, $\ell^-(S)$ and the set of points of $S$ which are in $\ell$.

We will denote by $\mathcal{L}(S)$ to the set of $\binom{m}{2}$ lines spanned by the points of $S$. If $p$ and $q$ are distinct points of $S$, we use $\ell_{pq}$ (respectively, $\overline{pq}$) to denote the line (respectively, closed segment) spanned by $p$ and $q$.

The connected regions of $\mathbb{R}^2 \setminus \mathcal{L}(S)$, i.e., the 2-dimensional faces, into which $\mathcal{L}(S)$ divide the plane are the cells of $S$. For $p \in S$ we define the cell of $p$ in $S$ as the cell of $\mathcal{L}(S \setminus \{p\})$ which contains $p$. We shall use $\text{cell}_S(p)$ to denote the cell of $p$ in $S$.

Although the following two propositions can be deduced easily from some well-known facts in the context of the order types (see e.g. [3] [17]), here we give elementary arguments.

**Proposition 3.** Let $p$ be a point of $S$ and let $D$ be the rectilinear drawing of $K_m$ induced by $S$. Let $p'$ be any fixed point contained in $\text{cell}_S(p)$. If $D'$ is the rectilinear drawing of $K_m$ induced by $(S \setminus \{p\}) \cup \{p'\}$, then two edges of $D$ cross each other if and only if the corresponding two edges in $D'$ cross each other.

**Proof.** If $p = p'$ there is nothing to prove. Similarly, if two edges in $D$ are not both incident to $p$, then their corresponding edges in $D'$ are themselves and we are done. Then it is enough to prove that the edges $\overline{pq}$ and $\overline{v_1v_2}$ cross each other in $D$ iff $\overline{p'q'}$ and $\overline{v_1'v_2'}$ cross each other in $D'$, where $p \neq p'$.

Suppose that $\overline{pq}$ and $\overline{v_1v_2}$ cross each other in $D$. Let $R := \{v, v_1, v_2, p\}$ and $R' := \{v, v_1, v_2, p'\}$. Since $\text{cell}_S(p) \subseteq \text{cell}_R(p)$ and by hypothesis $p, p' \in \text{cell}_S(p)$, then $p, p' \in \text{cell}_R(p)$, and hence $\text{cell}_R(p) = \text{cell}_{R'}(p')$. Now as $\overline{pq}$ crosses $\overline{v_1v_2}$ in $D$ and $p' \in \text{cell}_{R'}(p)$, then clearly, $\overline{p'q'}$ also crosses $\overline{v_1'v_2'}$ in $D'$. Similarly, if $\overline{p'q'}$ crosses $\overline{v_1'v_2'}$ in $D'$ then $p \in \text{cell}_{R'}(p')$ (as $p' \in \text{cell}_R(p)$) and clearly $\overline{pq}$ also crosses $\overline{v_1v_2}$ in $D$. \hfill \square

**Proposition 4.** Let $p_0$ and $q_0$ be two distinct points of $S$ and let $\ell := \ell_{p_0q_0} \in \mathcal{L}(S)$ be directed from $p_0$ to $q_0$. Let $R_1, R_2$ be two cells of $\mathcal{L}(S)$ sharing a boundary segment $s \subseteq (\ell \setminus \overline{p_0q_0})$ such that $R_1$ lies on the right of $\ell$ and $R_2$ on its left. Let $r$ be a fixed point of $R_1$, $l := |\ell^- (S)|$ and $k := |\ell^+(S)|$. If $D_1$ denotes the rectilinear drawing of $K_{m+1}$ induced by $S \cup \{r\}$ and $D_2$ denotes the rectilinear drawing of $K_{m+1}$ which results by moving $r$ from $R_1$ to $R_2$, then $\text{cc}(D_2) = \text{cc}(D_1) + (l - k)$.

**Proof.** Let $s_0$ be the middle point of $s$. By relabelling $p_0$ and $q_0$ if necessary, we may assume that $s_0$ is closer to $p_0$ than $q_0$. See Figure 3 (left). Let $\epsilon > 0$ be
small enough that the open ball $O$ of radius $\epsilon$ centered at $s_0$ satisfies the following conditions: (1) $O$ is totally contained in $cell_{S'}(s_0)$, where $S' := \{s_0\} \cup (S \setminus \{p_0\})$, and (2) if $x$ is any point in $R_1 \cap O$ (respectively, $R_2 \cap O$) and $p \in \ell^+(S)$ (respectively, $p \in \ell^-(S)$) then $\overline{p p_0}$ crosses $\overline{p q_0}$.

For clarity, we call $r_1$ (respectively, $r_2$) to $r$ before (respectively, after) it is moved. Moreover, by Proposition 3 we may assume without loss of generality that $r_i \in R_i \cap O$, for $i = 1, 2$.

Since the difference between $D_1$ and $D_2$ are the edges incident to $r_1$ and $r_2$, we need only compare the number of crossings of $D_1$ involving edges incident with $r_1$ with the number of crossings of $D_2$ involving edges incident with $r_2$.

If $e$ is an edge of $D_1 \cap D_2$ which is not incident with $p_0$, then by condition (1) and Proposition 3 we have that for any $v \in S \setminus \{p_0\}$, the edge $\overline{v w}$ crosses $e$ in $D_1$ if and only if the edge $\overline{v' w'}$ crosses $e$ in $D_2$. Thus, we need only compare the number of crossings between the edges of $p_0$ with $\overline{r_1 q_0}$ and with $\overline{r_2 q_0}$. But condition (2) implies that there are exactly $k$ crossings of the first kind and exactly $l$ crossings of the second kind. See Figure 3 (right). Hence $\overline{r_1 q_0} - k = \overline{r_2 q_0} - l$, and so $\overline{r_1 q_0} = \overline{r_2 q_0} + (l - k)$. □

![Figure 3](image)

The proof of our next statement is a routine exercise.

**Proposition 5.** Let $D'$ be a rectilinear drawing of $K_m$ with vertex set $P'$. Then by perturbing the points of $P'$ we can get another set of points $P$ such that the rectilinear drawing $D$ of $K_m$ induced by $P$ satisfies the following properties:

(C1) Two edges of $D'$ cross each other if and only if the corresponding two edges in $D$ cross each other. In particular, $\overline{\text{CH}(D')} = \overline{\text{CH}(D)}$.

(C2) $|\text{CH}^k(D')| = |\text{CH}^k(D)|$ for each $k = 1, 2, \ldots$

(C3) $L(P)$ has no parallel lines.

(C4) No point of $\mathbb{R}^2 \setminus P$ belongs to three lines of $L(P)$.

3. **Proof of main theorem**

For the rest of the paper, we assume that $D$ is an optimal rectilinear drawing of $K_n$ with $n \geq 8$ an integer, and that $P$ is its underlying set of points. Moreover, by Proposition 4 we also assume that $L(P)$ has no parallel lines and that no point of $\mathbb{R}^2 \setminus P$ belongs to three lines of $L(P)$. If $p \in P$ and $x \in \mathbb{R}^2$, then we will say that $p$ sees $x$ if the straight segment $\overline{px}$ does not cross any line of $L(P)$.

We start by showing that every vertex in the 2nd-convex hull of $D$ can see at least one point of $C(D)$ (not necessarily a vertex of $\text{CH}(D)$).

**Lemma 6.** If $p \in \text{CH}^2(D)$ then $p$ sees at least one point of $C(D)$. 
Proof. We know that $T := C(D)$ is a triangle $[8]$. Let $a, b,$ and $c$ be the vertices of $P$ which form $T$.

We proceed by contradiction. Suppose that for $q \in CH^2(D)$ the following is true: if $x$ is any fixed point of $T$, the straight segment $xq$ crosses at least one line delimiting $\Gamma := cell_P(q)$. For brevity, we use $F$ to denote the subset of lines of $\mathcal{L}(P)$ delimiting $\Gamma$.

Since $q \in CH^2(D)$, then there is a straight line $\ell$ passing through $q$ and avoiding $P \setminus \{q\}$ which leaves all the points of $P \setminus \{a, b, c, q\}$ on the same open halfplane. By perturbing $\ell$ around $q$, if necessary, we may assume that $\ell$ is not parallel to any line of $\mathcal{L}(P)$. Now we rotate $P$ and $\ell$ around $q$ in such a way that $\ell$ becomes vertical and direct it upwards. Let $L$ and $R$ be the left and right open halfplane of $\ell$. By reflecting $P$ through $\ell$, if necessary, we also assume that all the points of $P \setminus \{a, b, c, q\}$ are in $R$.

Let $F_{<q}$ (respectively, $F_{>q}$) be the subset of lines of $F$ which intersect $\ell$ below (respectively, above) $q$. Thus $F$ is the disjoint union of $F_{<q}$ and $F_{>q}$.

By hypothesis, for any fixed point $x \in T$, the straight segment $xq$ crosses at least one line of $F$. This implies that: (1) neither $F_{<q}$ or $F_{>q}$ is empty, and (2) there is a line $\ell_1$ of $F_{<q}$ and a line $\ell_2$ of $F_{>q}$ such that the intersection point $\times$ between $\ell_1$ and $\ell_2$ is in $L$. See Figure 4.

![Figure 4](image)

For $i = 1, 2$ let $x_i, y_i$ be the points of $P$ defining $\ell_i$ and let $s_i$ be the segment of $\ell_i$ which is frontier of $\Gamma$. We also assume that $x_i$ is closer to $\times$ than $y_i$, and that $\ell_i$ is directed from $\times$ to $y_i$.

From the definition of $\ell$ it follows that $L$ contains at least one and at most two points of $P$. Moreover, such points must be elements of $\{a, b, c\}$. Without loss of generality we may assume that $c \in R$.

For brevity, for $i = 1, 2$, we will omit the reference to $P$ in $\ell_i^+(P)$ and $\ell_i^-(P)$, and simply write $\ell_i^+$ and $\ell_i^-$, respectively.

Claim 7. If $\Delta := |\ell_2^+| - |\ell_1^+|$, then $\Delta \geq 2$.

Let $P_1 := \ell_2^+ \cap \ell_2^-$, $P_2 := (\ell_1^+ \cap \ell_2^-) \cup \{x_2, y_2\}$, $P_3 := (\ell_1^- \cap \ell_2^+) \cup \{x_1, y_1\}$, and $P_4 := \ell_1^+ \cap \ell_2^-$. Thus $P$ is the disjoint union of $P_1, P_2, P_3$ and $P_4$. As $q, x_1, y_1 \in P_3$, $|P_3| \geq 3$.

Each of the following statements is easy to see:
(A1) The contribution of a point in \( P_1 \) to \( \Delta \) is \(-1\).
(A2) The contribution of a point in \( P_2 \) to \( \Delta \) is \(0\).
(A3) The contribution of a point in \( P_3 \) to \( \Delta \) is \(+1\).
(A4) The contribution of a point in \( P_4 \) to \( \Delta \) is \(0\).

Then it is enough to show that \( |P_3| \geq |P_1| + 2 \). If \( |P_1| \leq 1 \) we are done. On the other hand, remember that \( P_1 \subset L \), \( L \) has at most two points of \( \{a, b, c\} \), and \( c \in R \). Thus the only remaining case is when \( P_1 = \{a, b\} \). Since \( q \) is in the interior of \( T \) (the triangle defined by \( a, b \) and \( c \)), then \( c \) cannot be in any of \( \ell_1^+ \cup \{x_1, y_1\} \) or \( \ell_2^- \cup \{x_2, y_2\} \). This implies that \( c \) must be in \( P_3 \setminus \{x_1, y_1\} \), \( |P_3| \geq 4 \). This proves the claim.

If \( |\ell_1^+| < (n - 3)/2 \) then by Proposition 4 the drawing obtained by crossing \( q \) through \( s_1 \) has fewer crossings than \( D \), which is a contradiction. Thus \( |\ell_1^+| \geq (n - 3)/2 \).

By Claim 7 \( |\ell_2^-| \geq 2 + |\ell_1^+| \geq (n + 1)/2 \). But \( |\ell_2^-| = n - (2 + |\ell_2^-|) \) and then \( |\ell_2^-| < (n - 3)/2 \). As before, by Proposition 4 the drawing obtained by crossing \( q \) through \( s_2 \) has fewer crossings than \( D \), which is a contradiction. \( \square \)

We are ready to prove our main result.

**Proof of Theorem 7**  Again we proceed by contradiction. Suppose that \( U := \text{CH}^2(D) \) has at least 4 vertices. By the Pigeonhole Principle and Lemma 6 there are at least two distinct vertices \( p, q \) of \( U \) such that for some side \( t \) of the triangle \( T := C(D) \) the following is true: both \( p \) and \( q \) see some point (not necessarily the same) of \( t \). Let \( a, b \) be the vertices of \( P \) defining \( t \) and let \( c \) be the third vertex of \( T \). Without loss of generality we may assume that \( T, t, a, b \) and \( c \) look like Figure 5.

![Figure 5](image)

**Figure 5.** Without loss of generality, we may assume that \( T, t, a \) and \( b \) look like this.

**Case 1.** The convex hull of \( \{p, q, a, b\} \) is a quadrilateral. By relabelling \( p \) and \( q \), if necessary, we may assume that \( \overline{aq} \) and \( \overline{bp} \) are the diagonals of such a quadrilateral. See Figure 6 (left). Thus the line \( \ell_{aq} \) prohibits \( p \) to see any point of \( t \setminus \{a\} \) and so \( a \) is the only point of \( t \) which is seen by \( p \). Similarly, \( \ell_{bp} \) prohibits \( q \) to see any point of \( t \setminus \{b\} \) and so \( b \) is the only point of \( t \) which is seen by \( q \). Note that the interior of triangle \( apc \) (respectively, \( bqc \)) cannot contain any point \( r \) of \( P \); otherwise, the line \( \ell_{cr} \) prohibits \( p \) (respectively, \( q \)) to see \( a \) (respectively, \( b \)). These facts have the following immediate consequences:
Figure 6. If $|U| \geq 4$, then $U$ contains points in the interior of triangle $Q$ which contradicts Lemma \[6\]

(B1) $\ell_{pq}$ separates $\{a, b\}$ from $P \setminus \{a, b, p, q\}$.
(B2) $\ell_{cp}$ separates $\{a\}$ from $P \setminus \{a, c, p\}$.
(B3) $\ell_{cq}$ separates $\{b\}$ from $P \setminus \{b, c, q\}$.

Let $v$ be the vertex of $U \setminus \{p, q\}$ such that the segment $\overline{av}$ forms the smallest angle $\alpha$ with $\overline{ac}$. Since $P$ is in general position and $|U| \geq 4$ such a $v$ exists. Thus the elements of $U \setminus \{p, q, v\}$ are in the interior of the triangle $Q$ formed by $\ell_{av}, \ell_{cq}$ and $\ell_{pq}$, see Figure 6 (right), and no vertex of $U \setminus \{p, q, v\}$ can see any point of $T$, which contradicts Lemma \[6\].

Case 2. The convex hull of $\{p, q, a, b\}$ is a triangle $T'$. By relabelling $p$ and $q$ if necessary we may assume that $a, b$ and $p$ are the vertices of $T'$. Then $q$ is in the interior of $T'$ and the lines $\ell_{aq}$ and $\ell_{bq}$ prohibits $p$ to see any point of $t$, which is a contradiction. See Figure 7.

Figure 7. The lines $\ell_{aq}$ and $\ell_{pq}$ prohibits $p$ to see any point of $t$.

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