Statistical Entropy of Three-dimensional Kerr-De Sitter Space

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ABSTRACT
I consider the (2+1)-dimensional Kerr-De Sitter space and it’s statistical entropy computation. It is shown that this space has only one (cosmological) event horizon and there is a phase transition between the stable horizon and the evaporating horizon at a point $M^2 = \frac{1}{3}J^2/l^2$ together with a lower bound of the horizon temperature. Then, I compute the statistical entropy of the space by using a recently developed formulation of Chern-Simons theory with boundaries, and extended Cardy’s formula. This is in agreement with the thermodynamics formula.

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I. Introduction

Recently, there has been tremendous interests in the statistical origin of the entropy for the (2+1)-dimensional space with a negative cosmological constant which is asymptotically the Anti-de Sitter space \( AdS_3 \) [1]. Independently on the string paradigm, two typical approaches have been known: One is Carlip’s approach [2] and the other is Strominger’s one [3]. In the Calip’s approach, one uses several assumptions. A first is the existence of a well defined conformal field theory with the Kac-Moody current algebra and it’s related Virasoro algebra through the Sugawara construction. A second is the appropriate boundary conditions which yield the desired value for the entropy. Third, one assumes that all statistical degrees of freedom of black hole live on the black-hole event horizon.

The Strominger’s approach is a drastically different one which concerns the Brown-Henneaux’s asymptotic isometry group \( SO(2, 2) \) [4] which preserves the asymptotic metric \( g^{\mu \nu} \) of \( AdS_3 \). In this approach, the fact that there is a central charge as \( c = 12l \) even at the “classical” level is a basic ingredient. [Here, Newton’s constant is set \( G \equiv 1/8 \) and cosmological constant is \( \Lambda = -1/l^2 \).]

However, it was not clear how these two extremal approaches are connected. Involved with this problem, recently Bañados, Brotz and Ortiz (BBO) have considered the Chern-Simons gravity theory with boundaries (finite or infinite) [5, 6]. (See also Ref. [7] for recent compact review and comparison with other various formulations.) In the Chern-Simons theory with boundaries, there are Kac-Moody and it’s related Virasoro algebras with the central terms at the “classical” level, which was first argued \(^2\) by Bañados [5] and recently proved [9] in the symplectic method [10] and this algebra has a crucial role in their formulation. Their formulation produces, “independently on the radius of the outer boundary, which envelopes all the space”, the Bekenstein-Hawking’s thermodynamics entropy for the BTZ black hole [1]

\[
S = 2\pi \sqrt{l(M + J)} + 2\pi \sqrt{l(M - J)}
\]

with black hole mass \( M \) and angular momentum \( J \) [1, 2, 3]. In this derivation, it is a basic ingredient that the central charge of the Virasoro algebra is completely determined by matching the isometries asymptotically; The central charge is found to be the same as that of asymptotic isometries.

Now, with this powerful formulation, the previous two extremal approaches can be understood as some limiting cases. Moreover, it provides a simple answer about the reason for the same result of the two previous approaches: They treated an identical object which lives only

\(^2\) For the Kac-Moody algebra, it was known in a different context of Yang-Mills theory with Chern-Simons term in Ref. [8].
on the boundary! However, contrast to the $\Lambda < 0$ case, the analysis of statistical entropy for the (2+1)-dimensional space with $\Lambda > 0$ which is asymptotically De Sitter space ($DS_3$) [11, 12], has not been well studied. Actually, the $\Lambda > 0$ case is quite different to the $\Lambda < 0$ case. In the $DS_3$ space which is the simplest case of $\Lambda > 0$
\[
\begin{align*}
  ds^2 &= -\left(1 - \frac{r^2}{l^2}\right)dt^2 + \left(1 - \frac{r^2}{l^2}\right)^{-1}dr^2 + r^2 d\varphi^2
\end{align*}
\] (2)
there is no black-hole event horizon for an observer moving on a timelike world line, but there is a cosmological event horizon $r_+ = l$ separating the outside region which the observer can never see from the inside region that he can see if he waits long enough. This is an opposite situation to the usual black hole spacetimes. However, as have been shown by Gibbons and Hawking, the cosmological event horizon has many formal similarities with the black-hole event horizon. Furthermore the ideas of thermodynamics for the black-hole event horizons whose areas can be interpreted as the entropies to the cosmological event horizons, but by abandoning the concept of particle as being observer-independent [11]. For the special case $DS_3$, the statistical analysis was recently done by Maldacena and Strominger [13] by applying the Carlip’s approach instead of the Strominger’s one. In this analysis they were able to show a good agreement with the Gibbons-Hawkings formula. However, there remains some gap to the complete understanding of the statistical entropy for far-horizon region and more general cases with $M$ and $J$.

In this paper, I consider a Kerr-De Sitter space with the general $M, J$ with $\Lambda > 0$ and a computation of it’s statistical entropy. Following the Gibbons-Hawking’s approach, it is found that this space has only one (cosmological) event horizon and there is a phase transition between a stable horizon and an (unstable) evaporating horizon at a point $M^2 = \frac{1}{3}J^2/l^2$ together with a lower bound of horizon temperature. Then, I compute the statistical entropy of the space by a direct adaptation of the approach of BBO [5 - 7, 9] and extending the Cardy’s formula to the complex valued central charge and eigenvalues of $L_0$, $\bar{L}_0$. My result agrees with the thermodynamics formula exactly.

II. Kerr-$DS_3$ solution

In order to proceed parallel to $\Lambda < 0$ case [1], I start by considering the Kerr metric in $\Lambda > 0$ case [11, 12]. The (2+1)-dimensional gravity for a cosmological constant $\Lambda = -1/l^2$ is described by the action
\[
I = \frac{1}{2\pi} \int d^3x \sqrt{-g}(R + 2l^{-2}) + I_m,
\] (3)
where $I_m$ is a presumed matter action [The details are not important in this paper] and I have omitted the surface terms in the pure gravity part as usual. This theory has a constant
curvature $R = -2l^{-2}$ outside the matters. Regardless of the sign of $l^2$, the vacuum line element for the rotationally symmetric and stationary metric can be written as

$$ds^2 = -(N_\perp)^2(r)dt^2 + f^{-2}(r)dr^2 + r^2(N^\varphi(r)dt + d\varphi)^2,$$  \hspace{1cm} (4)

where $\varphi$ has period $2\pi$. Then, the Hamiltonian is expressed by $H = \int dr (N\mathcal{H} + N^\varphi\mathcal{H}_\varphi)$ with the constraints

$$\mathcal{H} \equiv -2l^2 \frac{p^2}{r^3} + (f^2)' + \frac{2r}{l^2} \approx 0,$$

$$\mathcal{H}_\varphi \equiv -2ilp' \approx 0,$$

$$N(r) = f^{-1}N_\perp,$$  \hspace{1cm} (5)

where $N, N^\varphi$ are the Lagrange multipliers and prime ($'$) denotes the derivative with respect to the radial coordinate $r$. The solutions of (5) depend on the sign of $l^2$. In my interesting case of $l^2 < 0$, the solutions of $p, f^2, N_\perp, N^\varphi$ are given as follow:

$$p = -\frac{J}{2il},$$

$$f^2 = N_\perp^2 = M - \left(\frac{r}{l}\right)^2 + \frac{J^2}{4r^2},$$

$$N^\varphi = -\frac{J}{2r^2},$$  \hspace{1cm} (6)

where I have renamed $l$ by $il$ such that $l$ is a positive real number and I have set $N|_{\partial D_2} = 1, N^\varphi|_{\partial D_2} = 0$ in order to get the $DS_3$ space (2) asymptotically [1]. Here, two constants of integration $J$ and $M$, which characterize a $Kerr - DS_3$ space, are identified as the total angular momentum and mass because they appear as the conjugates to the boundary (rescaled) lapse and shift displacements $N|_{\partial D_2}$ and $N^\varphi|_{\partial D_2}$, respectively, in the variation of the action (3) with an appropriate boundary action [1, 14].\footnote{This definition of $M$ and $J$ is valid even for finite space. Moreover, $M$ and $J$ converge into the quasi-local or ADM definitions of mass and angular momentum asymptotically [14] though the gravitational energy vanishes. There are other several methods of identifying the mass and angular momentum. See Ref. [15] for these other methods. I thank Prof. S. Carlip for suggesting me to consider this problem.} Note that there is an additional sign change in front of $M$ as well as the $l^2$ terms. In this paper I will focus mainly on the statistical entropy of the solution (4), (6). The geometric structure is not main issue for this purpose and will not be provided in this paper.

The lapse function $N_\perp$ vanishes for “one” value of $r$ given by

$$r_+ = \frac{l}{\sqrt{2}} \sqrt{M + \sqrt{M^2 + \frac{J^2}{l^2}}},$$  \hspace{1cm} (7)
This is the cosmological event horizon in $Kerr - DS_3$ and there is no black-hole event horizon. Here, there is no additional condition for $M$ in order that the horizon exists unless $J$ vanishes: Even the negative values of $M$ and $J$ are allowed. So, in the $J \neq 0$ case the whole mass spectrums (ranging form $-\infty$ to $\infty$) are continuous and there is no mass gap; This is contrast to $Kerr - AdS_3$ called BTZ solution [1]. For $J = 0$ case, there is no horizon when $M < 0$; One is left just with the outside region which is filled with negative masses. Moreover, I define $r_- \equiv \frac{l}{\sqrt{2}} \sqrt{M - \sqrt{M^2 + l^2}} \equiv i r_{(-)}$ which is a pure imaginary number. With these two parameters, the $Kerr$ metric (4), for a positive cosmological constant $1/l^2$, can be conveniently written in the proper radial coordinates as

$$ds^2 = \sinh^2 \rho \left( \frac{r_+ dt}{l} - r_{(-)} d\varphi \right)^2 - l^2 d\rho^2 + \cosh^2 \rho \left( \frac{r_{(-)} dt}{l} + r_+ d\varphi \right)^2$$

(8)

with

$$M = \frac{r_+^2 - r_{(-)}^2}{l^2}, \quad J = \frac{2 r_+ r_{(-)}}{l},$$

$$r^2 = r_+^2 \cosh^2 \rho + r_{(-)}^2 \sinh^2 \rho.$$  \hspace{1cm} (9)

In these coordinates, the cosmological event horizon is at $\rho = 0$ and hence this metric represents the exterior of the horizon for real value $\rho$ and represents the interior for imaginary value $\rho$. [This is completely opposite situation to Schwarzshild black hole.] The interior and exterior regions are casually disconnected and so the cosmological event horizon acts like as a black-hole horizon \footnote{Because of this fact, I posit that the sources of $M$ and $J$ are isotropically distributed matters within cosmological horizon and outer boundary in accordance with black hole analogy where the sources hide also inside the (black-hole) event horizon. The centrifugal terms will be a result of Mach effect for the observer surrounded by the rotating mass shell \cite{16}. Of course, my calculation of statistical entropy is independent on the precise physical setting for the metric solution (4), (6). However, if one accepts this interpretation, the BTZ black hole can be also interpreted as the rotation of the space filled with isotropically distributed negative mass matters.}. Here, I note that the sign of $M$ is controlled by the relative magnitudes of $r_+^2$ and $r_{(-)}^2$. By considering $J = 0$ case, the metric (4) can be identified with the $DS_3$ space (2) \cite{11, 12} with $M = 1$.

Finally, let me applies the Gibbons-Hawking’s thermodynamics theory to the interior region of $Kerr - DS_3$ where the signature of metric is the same as ours. Then, the observers in the interior will calculate the Bekenstein-Hawking entropy \cite{11}

$$S = 2 \cdot Area \ of \ event \ horizon$$

$$= 4 \pi r_+$$ \hspace{1cm} (10)
and detect an isotropic background of thermal radiation with a temperature
\[ T = \left( \frac{\partial S}{\partial M} \right)^{-1}_J = \frac{r_+^2 + r_-^2}{2\pi l^2 r_+}. \]  

Moreover, according to a semiclassical argument of Gibbons-Hawking, a stability of the cosmological event horizon can be analyzed by considering the change of temperature \( T \) upon varying \( M \) or in a more compact way by the heat capacity \( C_J \equiv (\partial M/\partial T)_J \) (with \( J \) fixed) [14]:

From (11), one obtains
\[ C_J = 2\sqrt{2\pi l} \sqrt{M} \frac{\sqrt{1 + x} \left(1 + \sqrt{1 + x}\right)^{3/2}}{1 - x + \sqrt{1 + x}}, \]  

where \( x = J^2/(M^2l^2) \) is a dimensionless parameter. This shows an infinite discontinuity at the point \( M^2 = \frac{1}{3} J^2/l^2 \) (i.e., \( x = 3 \)) and shows a critical phenomena; A physical interpretation is as follow: For \( J = 0, M > 0 \) case, if one absorb the thermal radiation at the expense of the mass of the horizon, the area of the horizon \( (2\pi r_+) \) will go down, \( T = (r_+/(2\pi l^2)) \) goes down \( (C_J = 4\pi l\sqrt{M} > 0) \), and hence the (cosmological) horizon is stable. On the other hand, for \( J \neq 0, M > 0 \) case, as one absorbing the radiation from horizon, \( T \) goes down \( (C_J > 0) \) for \( M^2 > \frac{1}{3} J^2/l^2 \) (i.e., \( x < 3 \)) but \( T \) goes up \( (C_J < 0) \) for \( M^2 < \frac{1}{3} J^2/l^2 \) (i.e., \( x > 3 \)). From the fact that \( C_J > 0 \) and/or \( C_J < 0 \) imply the stability and/or instability of the horizon, one finds that there is a phase transition between the stable horizon and evaporating (unstable) event horizon at the critical point. \(^5\) This is contrast to the BTZ black-hole event horizon which is always stable [14] and the Schwarzschild black-hole horizon which always evaporate upon thermal radiation in the vacuum. Moreover, in this case there is also a lower bound of temperature as
\[ T \geq T_c, \]
\[ T_c = \sqrt{\frac{2}{3\pi l}} \sqrt{M}, \]  

where \( T_c \) is the horizon temperature at the critical point, which is lower than the temperature for the extremal point \( M^2 = J^2/l^2 \) (\( x = 1 \)).

\(^5\) This implies that the concept of particle is observer dependent [11].

\(^6\) This looks like a second-order phase transition in the usual (equilibrium) thermodynamics because of an (infinite) discontinuity in the second derivatives of the Gibbs free energy \( G = M - TS - \Omega J \) (\( \Omega \equiv -T(\partial S/\partial J)_M \)) even though \( G \) and it’s first derivatives are continuous. Similar phenomena have observed also in the (3+1)-dimensional Kerr-Newmann black holes [20]. However, according to a recently developed non-equilibrium thermodynamics there is the second-order phase transition at \( M = 0, J = 0 \) point, but not at \( M^2 = \frac{1}{3} J^2/l^2 \) and corresponding critical exponents satisfy the scaling laws [21]. The details will be appeared in a separate paper [22].
III. Chern-Simons gravity with boundaries

The (2+1)-dimensional pure gravity with the positive cosmological constant $\Lambda = 2l^{-2}$ can be written as a $SL(2, \mathbb{C})$ Chern-Simons gauge theory [17, 18]. The action for this theory is, up to the surface terms [1],

$$I_g[A] = \frac{is}{4\pi} \int_{D_2 \times R} d^3x \epsilon^{\mu
u\rho} \left( A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right) - \frac{is}{4\pi} \int_{D_2 \times R} d^3x \epsilon^{\mu
u\rho} \left( \bar{A}_\mu \partial_\nu \bar{A}_\rho + \frac{2}{3} \bar{A}_\mu \bar{A}_\nu \bar{A}_\rho \right),$$

(14)

on the manifold $\Sigma = D_2 \times R$. [$D_2$ is a 2-dimensional disc of space and $R$ is a 1-dimensional infinite real manifold of time. $\bar{A}_\mu$ is complex conjugate of $SL(2, \mathbb{C})$ gauge field $A_\mu$ and $\langle \cdots \rangle$ denotes the trace.] Here, the topological mass parameter ‘$s$’ needs not be quantized in the non-compact group $SL(2, \mathbb{C})$ irrespective of the existence of the boundaries. Action (14) can be identified to (3) by the gauge connections [1]

$$A_\mu^a = \omega_\mu^a + \frac{e_\mu^a}{il}, \quad \bar{A}_\mu^a = \omega_\mu^a - \frac{e_\mu^a}{il} \quad (a = 0, 1, 2),$$

(15)

with $s = -l$. Here, $e_\mu^a = e_\mu^a dx^\mu$, $\omega_\mu^a = \frac{1}{2} e^{abc} \omega_{\mu bc} dx^\mu$ are the triads and the $SL(2, \mathbb{R})$ spin connections, respectively. From now on, I will only consider the $A_\mu$-part, unless otherwise stated, because the $\bar{A}_\mu$-part can be obtained by complex conjugation of $A_\mu$-part. It is easily checked that the 1-form gauge connections are given, in the proper coordinates, by

$$A^0 = -\frac{r_+ + i r_-}{l} \left( \frac{dt}{l} + i d\varphi \right) \sinh \rho,$$

$$A^1 = d\rho,$$

$$A^2 = -\frac{r_+ + r_-}{l} \left( \frac{dt}{l} + i d\varphi \right) \cosh \rho.$$

(16)

[ The superscript indices denote the group indices $a = 0, 1, 2$.] For the $DS_3$ space (2), these reduce to

$$A^0 = \mp \sqrt{1 - \frac{r^2}{l^2}} \left( \frac{dt}{l} - d\varphi \right),$$

$$A^1 = \mp \frac{i}{\sqrt{l^2 - r^2}} dr,$$

$$A^2 = \frac{r}{l^2} dt + \frac{i r}{l} d\varphi,$$

(17)

7Recently, Bañados and Mendez proved that the surface terms in the covariant form of Chern-Simons gravity action like as (14) are exactly the same as the required surface terms in the pure gravity action [19].

8I take $t_0 = \frac{i}{2} \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right)$, $t_1 = \frac{i}{2} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)$, $t_2 = \frac{i}{2} \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$ so that $[t_a, t_b] = \epsilon_{abc} t_c$ and $\langle t_a t_b \rangle = \frac{1}{2} \eta_{ab}$, where $\epsilon_{012} = 1$ and $\eta_{ab} = \text{diag}(-1, 1, 1)$. These are the same conventions as Ref. [6].
using the coordinates \((t, r, \varphi)\) and on-shell mass \(M = 1\) [13]. In general cases, (16) becomes, in matrix form,

\[
A = \frac{1}{2} \begin{pmatrix}
    d\rho & -z e^\rho \left( \frac{dt}{l} + i d\varphi \right) \\
    -z e^\rho \left( \frac{dt}{l} + i d\varphi \right) & -d\rho
\end{pmatrix},
\]  

(18)

where \(z \equiv (r_+ + i r_-)/l\). From this, the polar components \(A_\rho, A_\varphi, A_t\) in the proper coordinates can be obtained as

\[
A_\rho = t_1, \quad A_\varphi = -iz \left( U^{-1} t_2 U \right), \quad A_t = i A_\varphi,
\]

(19)

where

\[
U = \begin{pmatrix}
    e^{\rho/2} & 0 \\
    0 & e^{-\rho/2}
\end{pmatrix}.
\]

A. Symmetry algebra and classical central terms

The Chern-Simons action has the gauge and diffeomorphism \((\text{Diff})\) symmetries. If there are boundaries, the central terms appear in the symmetry algebras even at the classical level. This was first argued by Bañados [5] and proved recently [9] in the symplectic method [10]. Especially, for the time-independent and spatial \(\text{Diff}\)

\[
\delta_f \alpha^\mu = -\delta^\mu_k f^k, \\
\delta_f A_i^a = f^k \partial_k A_i^a + (\partial_i f^k) A_k^a, \\
\delta_f A_0^a = f^k \partial_k A_0^a,
\]

(21)

the conserved Noether charge becomes

\[
Q(f) = -\frac{i s}{4 \pi} \oint_{\partial D_2} d\varphi \eta_{ab} (2 f^\rho A_\rho^a A_\varphi^b + f^\varphi A_\varphi^a A_\rho^b + f^\rho A_\rho^a A_\varphi^b),
\]

(22)

where a boundary condition “\(A_\rho^a |_{\partial D_2} = A_\rho^a |_{\partial D_2} = \text{constant} \)” is imposed, \(A_\varphi\) is a pure gauge form \(A_\varphi = g^{-1} \partial_\varphi g\), \(f^k\) is a real function of spatial coordinates and the boundary \(\partial D_2\) is a circle. [The last constant term in (22) was included to obtain the standard Virasoro central term, which procedure can be always done according to the definition of Noether charge.] Here, the boundary condition about \(A_\rho\) is crucial for the existence of central term in the Virasoro algebra but, in a general text, one can not discard other boundary conditions which do not produce the central term [9]. However, in our analysis of spacetime with event horizons, it is quite natural

\(^9\)Here, \(A_\rho = \hat{\rho}^i A_i, A_\varphi = \hat{\varphi}^i A_i\), for the orthogonal unit vectors \(\hat{\rho}, \hat{\varphi}\) on \(\partial D_2\).

\(^{10}\)In this derivation, the additional gauge fixing conditions are not needed. See Refs. [5. 6] for comparison.
choice according to the solution (19) [5 - 7]. From the symplectic structure of the action (14) for the pure gauge $A_i = g^{-1} \partial_i g$, one finds the Poisson bracket algebra for $A^a_\varphi$ who lives on $\partial D_2$:

$$\{ A^a_\varphi(\varphi), A^b_\varphi(\varphi') \} = \frac{2\pi}{is} \epsilon^{ab}_c A^c_\varphi(\varphi) \delta(\varphi - \varphi') + \frac{2\pi}{is} \eta^{ab} \partial_\varphi \delta(\varphi - \varphi')$$

$$= \frac{2\pi}{is} (D_\varphi \delta(\varphi - \varphi'))^{ab}, \quad (23)$$

which is the $SL(2, \mathbb{C})$ Kac-Moody algebra in the density form [23, 24]. This is an explicit realization of the assumed $SL(2, \mathbb{C})$ current algebra in Ref. [13]. ($D_\varphi$ is the $\varphi$-th component of the covariant derivative $D_i^{ab} = \eta^{ab} \partial_i + \epsilon^{abc} A^c_i$.) [See Ref. [9] for further details.] Using this Poisson bracket, one finds

$$\{ Q(f), Q(g) \} = Q([f, g]) - \frac{is}{\pi} \langle A_\rho A_\rho \rangle \int_{\partial D_2} d\varphi (f^\rho \partial_\varphi g^\rho - f^\varphi \partial_\varphi g^\rho), \quad (24)$$

where $[f, g]^k = f^\varphi \partial_\varphi g^k - g^\varphi \partial_\varphi f^k$ is Lie bracket on the boundary circle ($\partial D_2$). In general, this algebra does not satisfy the Jacobi identity and so the Noether charge $Q(f)$ as a symmetry generator can not be accepted. Therefore, the only way to avoid this undesirable situation is to consider the subset of transformation with particular $f^\rho|_{\partial D_2} \propto \partial_\varphi f^\varphi|_{\partial D_2}$ and $g^\rho|_{\partial D_2} \propto \partial_\varphi g^\varphi|_{\partial D_2}$ [5, 6, 9] such that only the third and first order derivatives appear in the central term and hence (24) satisfies the Jacobi identity [25]: Here, this particular form corresponds to the Diff which deforms across the boundary with proportionality to the steepness ($\partial_\varphi f^\varphi$) of Diff along the circle ($\partial D_2$); The boundary $\partial D_2$ responds as an elastic medium to the deformations. Then, (24) will become the Virasoro algebra with central term even at the classical level, but with an undetermined proportionality constant. This will be determined by matching the asymptotic isometries [5 - 7] in the next section. Before ending this sub-section, I note that the fact of the existness of the central term in (24) is a purely Abelian effect which is contained in any non-Abelian gauge theories with the non-degenerate ($\langle t_a t_b \rangle \neq 0$) Lie groups.

**B. Asymptotic isometries and central charge**

The gauge field (19) has the information about the metric on $\partial D_2$ through the relation (15). So, the isometries which preserve the metric on $\partial D_2$ can be described by Diff generated by the symmetry generator $Q$. Using (23), it is found that the transformations of $A_\varphi$ who lives on $\partial D_2$, generated by $Q(f)$ [9, 26] are

$$\delta_f A_\varphi = \{ Q(f), A_\varphi(\varphi) \}$$

$$= D_\varphi (f^k A_k),$$

$$\delta_f A_\rho = 0, \quad (25)$$
and, using the specific $SL(2, \mathbb{C})$ gauge field (19), the $\varphi$ part of (25) becomes

$$\delta_f A_\varphi = \frac{1}{2} \left( \begin{array}{cc} \partial_\varphi f^\rho & iz e^{-\rho} (f^\rho - \partial_\varphi f^\varphi) \\ -iz e^\rho (f^\rho + \partial_\varphi f^\varphi) & -\partial_\varphi f^\rho \end{array} \right).$$

(26)

(25) and (26) are Diff of gauge fields on $\partial D_2$ regardless of the radius of $\partial D_2$ and $\rho$ is the proper radius of $\partial D_2$. The radius may be finite or infinite; This can be even $\rho = 0$ which corresponds to the event horizon. Let us consider these transformation at infinite boundary $\partial D_2$ i.e., $\rho \to \infty$. Then, in the leading order (26) becomes

$$\delta_f A_\varphi = \frac{1}{2} \left( \begin{array}{cc} \mathcal{O}(1) & \mathcal{O}(e^{-\rho}) \\ -iz e^\rho (f^\rho + \partial_\varphi f^\varphi) & -\mathcal{O}(1) \end{array} \right),$$

(27)

where $\mathcal{O}(1)$ represents the order of $\partial_\varphi f^\rho$. Therefore, this transformation (27) gives the isometries $\delta_f A_i = 0$ on $\partial D_2$ when

$$f^\rho|_{\partial D_2} = -\partial_\varphi f^\varphi|_{\partial D_2}$$

(28)

is satisfied. Contrary to the fact of the existness of the central term itself, this result is a purely non-Abelian effect which comes from the off-diagonal parts. Now, by substituting (28) with the insertion of $\langle A_\rho A_\rho \rangle = 1/2$ for the black hole solution (19), the algebra (24) becomes the standard Virasoro algebra with imaginary number central charge

$$c = -i24 \ s \langle A_\rho A_\rho \rangle = -12 \ is.$$  

(29)

By defining $Q(f) \equiv \frac{1}{2\pi} \int_{\partial D_2} d\varphi f^\varphi (\sum_n L_n e^{-in\varphi})$, $L_n$‘s satisfy the momentum space Virasoro algebra

$$\{L_m, L_n \} = i(m - n)L_{m+n} + \frac{ic}{12} m(m^2 - 1)\delta_{m+n,0}$$

(30)

with the imaginary value central charge $c$ of (29). In the application of this algebra to Kerr – $DS_3$, it is peculiar that the central charge can be completely determined only by considering the exterior region which can not be seen by an observer moving on a timelike worldline in the interior region. This is a physically different situation to Kerr – $AdS_3$ (BTZ) [5 - 7]. Moreover, in a general context the central charge might depend on $\partial D_2$. But this is impossible because all kinds of $\partial D_2$ can be smoothly deformed by Diff which allows the radial as well as the angular deformations, and hence all kinds of $\partial D_2$ are equivalent; If the central charge might depend on the radius of circle ($\partial D_2$), an absolute length scale must exist in the model but this is contrast to the Diff invariance, which includes the scale invariance of course, of the boundary Chern-Simons theory (14)
IV. Statistical entropy

In the computation of the statistical entropy, the zero-mode generators $L_0, \bar{L}_0$ have a crucial role. From the definition, they become

$$L_0 = \frac{-is}{2\pi} \oint_{\partial D_2} (A_\varphi A_\varphi + A_\rho A_\rho) = (iM - J + il),$$
$$\bar{L}_0 = (-iM - J - il).$$

(31)

Now, by adjusting the additive constants in $L_0, \bar{L}_0$ so that they vanish for the $M = J = 0$ case (vacuum solution) [3], one obtains

$$M = \frac{1}{il}(L_0 - \bar{L}_0),$$
$$J = -(L_0 + \bar{L}_0).$$

(32)

Here, there is no condition about the Hermicity for $L_0, \bar{L}_0$ in general to insure the Hermicity of $M$ and $J$. So, in the general context, one can assume that $L_0, \bar{L}_0$ have the complex (eigen)values $\mathcal{N}, \bar{\mathcal{N}}$, then $M$ and $J$ become

$$M = \frac{1}{il}(\mathcal{N} - \bar{\mathcal{N}}) = \frac{2}{l} \text{Im}(\mathcal{N}),$$
$$J = -(\mathcal{N} + \bar{\mathcal{N}}) = -2 \text{Re}(\mathcal{N}).$$

(33)

$M$ and $J$ are controlled by the imaginary and real part of $\mathcal{N}$, respectively. The eigenvalues $\mathcal{N}, \bar{\mathcal{N}}$ are expressed as

$$\mathcal{N} = \frac{i}{2}(lM + iJ), \quad \bar{\mathcal{N}} = -\frac{i}{2}(lM - iJ).$$

(34)

There is an argument, by Witten, of the unitarity for the theory with the pure imaginary central charges [18]. But it is not well established whether the usual Cardy’s formula [7, 27] for the entropy of a conformal field theory

$$S = 2\pi \sqrt{\frac{cN}{6}} + 2\pi \sqrt{\frac{c\bar{N}}{6}}$$

(35)

is valid for the complex valued $c(\bar{c})$ and $\mathcal{N}(\bar{\mathcal{N}})$ in general. But I will assume this formula and see what happens in my case. Using the two main results (29) and (34), one finds the statistical entropy of $Kerr - DS_3$ as

$$S = 2\pi \sqrt{l(lM + iJ)} + 2\pi \sqrt{l(lM - iJ)}.$$ 

(36)
For $J = 0$ state or a semiclassical regime of large $M$, $l$ with small $J$, the entropy becomes

$$S = 4\pi l \sqrt{M}. \quad (37)$$

This will be the statistical entropy for non-rotating $Kerr - DS_3$ and this agrees with the Bekenstein-Hawking’s entropy (10) [11, 13]. In this case, the state of negative $M$ has no real value entropy which is connected with non-existence of the event horizon. From the fact that the result (37) is highly sensitive to the central charge $c$ and the desired $c$ is exactly obtained by matching isometries at $\rho \to \infty$, one finds that the central charge should be independent on the radius of $\partial D_2$ in order to get a consistency with the thermodynamics formula; This fact is consistent with $Diff$ invariance of our theory as I have noted at the end of the Sec. III.

On the other hand, when $J \neq 0$, a negative entropy is also a possible solution. However, if I assume the smoothness of the entropy change when there is smooth change from $J = 0$, where $S > 0$, to $J \neq 0$, it seems to natural to consider only the positive entropy solution which agrees exactly with (10). In this case, the state of negative $M$ has a real value entropy which is connected with the existence of the event horizon. Moreover, for the positive entropy solution, (36) is exactly what can be obtained from BTZ black hole entropy (1) by simple replacements: 

(i) $l \to il$, (ii) $M \to -M$. The part ‘(i)’ corresponds to an analytic continuation and part ‘(ii)’ is connected to the sign change in $M$ which has been noted below (6).

From this coincidence, the assumed formula (35) must have some meaning.

V. Summary and discussions

I have considered the $Kerr - DS_3$ space and a statistical evaluation of the entropy for the space in the $SL(2, \mathbb{C})$ Chern-Simons gravity formulation. It is shown that the space has only one (cosmological) event horizon and there is a phase transition between the stable horizon and (unstable) evaporating horizon at the point $M^2 = \frac{1}{3} J^2 / l^2$. It is shown also that there is a lower bound on the temperature as (13). Then, it is shown that the Chern-Simons gauge theory with boundaries produces the $SL(2, \mathbb{C})$ Virasoro algebra with imaginary value central term at the classical level; In this derivation, it is a basic ingredient that the boundary $\partial D_2$ behaves as an elastic medium to the deformations, i.e., $f^\rho |_{\partial D_2} \propto \partial_\phi f^\phi |_{\partial D_2}$. Using this Virasoro algebra, and following the recent approach of BBO, I have shown that the statistical entropy for the $Kerr - DS_3$ space can be calculated by assuming the Cardy’s formula (35) even in the imaginary value central charge $c$ and complex eigenvalues of generators $L_0$, $\bar{L}_0$. This entropy agrees with the Bekenstein-Hawking’s formula. My result is independent on the radius of boundary because of a $Diff$ invariance of the theory. It would be interesting to study the Cardy’s formula with complex value $c$ and $\mathcal{N}$ in a general context and understand why it works in my case. It would be also interesting to extend to (a) a complex value $c$ which might

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be related to the inclusion of other gauge fields or matter sectors [18], (b) supersymmetric and higher dimensional Chern-Simons gravity theories [28], and to understand the analysis of $Kerr - DS_3$ solution within the context of string theory [29]. These remain outstanding challenges.

Note added: After completing this work, I received a paper [30] which computes a statistical entropy of the $DS_3$ space (2) using $SU(2) \times SU(2)$ Chern-Simons formulation in the Euclidean signature and it’s result agrees with my result (37) with $M = 1$. I thank M. Ortiz for kindly sending the paper to me before submitting.

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