Nonsingular universe in massive gravity’s rainbow

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One of the fundamental open questions in cosmology is whether we can regard the universe evolution without singularity like a Big Bang or a Big Rip. This challenging subject stimulates one to regard a nonsingular universe in the far past with an arbitrarily large vacuum energy. Considering the high energy regime in the cosmic history, it is believed that Einstein gravity should be corrected to an effective energy dependent theory which could be acquired by gravity’s rainbow. On the other hand, employing massive gravity provided us with solutions to some of the long standing fundamental problems of cosmology such as cosmological constant problem and self acceleration of the universe. Considering these aspects of gravity’s rainbow and massive gravity, in this paper, we initiate studying FRW cosmology in the massive gravity’s rainbow formalism. At first, we show that although massive gravity modifies the FRW cosmology, it does not itself remove the big bang singularity. Then, we generalize the massive gravity to the case of energy dependent spacetime and find that massive gravity’s rainbow can remove the early universe singularity. We bring together all the essential conditions for having a nonsingular universe and the effects of both gravity’s rainbow and massive gravity generalizations on such criteria are determined.

I. INTRODUCTION

The possible existence of big bang singularity at the early universe and its challenging physical properties (at still earlier epochs near the Planck energy, $E_P$) urge one to look for a quantum theory of gravity [1–7]. It is believed that the quantum gravity admits a semi-classical regime [8–10] in which the quantum corrections could be viewed as dependency of the spacetime metric on the energy of particles probing it. Thus, we should regard essentially an upper bound on the energy scale of such particles which is Planck energy. The possible connection between the energy of probing particles and the energy-dependent spacetime is proposed in the context of gravity’s rainbow. In the gravity’s rainbow framework, it is considered that particles with different energies are affected differently by the structure of spacetime, depending on their wavelengths [11].

One of the basic elements for building gravity’s rainbow theory is modified energy-momentum dispersion relation. This modification can be arisen from applying nonlinear Lorentz transformations with the following explicit result

$$E^2 f(\varepsilon)^2 - p^2 g(\varepsilon)^2 = m_0^2,$$

where the energy ratio is $\varepsilon = E/E_P$ in which $E$ is the energy of test particle. Besides, both $f(\varepsilon)$ and $g(\varepsilon)$ are rainbow functions, and $m_0$ is the mass of the test particle. It is also notable that in the limit of $\lim_{\varepsilon \to 0} f(\varepsilon) = 1$ and $\lim_{\varepsilon \to 0} g(\varepsilon) = 1$, the standard energy-momentum dispersion relation is recovered. Such modification in energy-momentum dispersion relation has been supported in the context of several topics, such as discrete spacetime [12], spacetime foam [13], spin-network in loop quantum gravity (LQG) [14], ghost condensation [15] and non-commutative geometry [16, 17]. Moreover, the observational data of the Pierre Auger Collaboration and the High Resolution Fly’s Eye experiment [18] have confirmed the requirement for such modification and also the necessity of an upper limit on the energy of cosmic rays.

Another cornerstone of considering the gravity’s rainbow is based on the generalization of doubly special relativity [19] in curved spacetime [11]. The doubly special relativity is a generalization of the special relativity which considers an upper limit for energies that a particle can acquire. Particularly, its generalization to non-flat spacetimes is the so-called doubly general relativity. Besides, using the gravity dictionary, one may rename the doubly general relativity to gravity’s rainbow. Among the interesting achievements of the gravity’s rainbow, one can regard the UV completion of general relativity [11], existence of remnants for black holes after evaporation [20, 21], admitting usual uncertainty principle [10, 22] and providing solutions for information paradox [23, 24]. Moreover, the gravity’s rainbow has been employed to investigate the thermodynamical properties of black holes [25, 26] and the structure of neutron stars.

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In the context of cosmology, it was shown that employing this formalism will provide the possibility of removing the big bang singularity and the big bounce of a cyclic universe. The mentioned subjects motivate us to consider the gravity’s rainbow in the context of modified Einstein theory. It is worth mentioning that through several works, it was shown that by using the nonlinear electrodynamics, one can remove the possibility of existence of singularity in early universe. Here, our idea is to employ the approach of gravity’s rainbow alongside of massive gravity instead of the nonlinear electromagnetic field for removing the big bang singularity.

Einstein theory of gravity incorporates massless spin−2 gravitons as intermediate couriers of gravitational interactions. But studies that are conducted in the context of brane-world gravity (especially regarding hierarchy problem) emphasize on the existence of massive gravitons. Therefore, it is logical to generalize Einstein gravity to a new theory with massive gravitons. This generalization can improve our insight concerning the gravitational concepts. In this regard, the first attempts were done by Fierz and Pauli. This theory of massive gravity suffers the existence of vDVZ (van Dam-Veltman-Zakharov) discontinuity which indicates that for the limit of vanishing mass, the propagators of massive and massless theories are not consistent. One of the solutions of this problem is Vainshtein mechanism which requires a system to be considered in a nonlinear framework. But the generalization to nonlinearity leads to the presence of extra degree of freedom which provides a ghost instability. To overcome this problem, several theories of the massive gravity have been proposed, in which among them one can name new massive gravity, bi-gravity and dRGT (de Rham, Gabadadze and Tolley) theory. The dRGT massive gravity is a ghost free theory which employs a reference metric to build massive terms. The modification in the reference metric’s ansatz leads to different theories of massive gravity which are dRGT-like. One of these modifications with specific interest in gauge/gravity duality was done by Vegh where the graviton plays the role of lattice. This theory of massive gravity has been employed to conduct studies regarding super massive objects such as black holes. It was shown that the presence of massive gravitons has important contributions into geometrical, thermodynamical and structure of these super massive objects. Also, the existence of massive gravitons can generate a new range of phase transitions for topological black holes. From cosmological perspective, massive gravity has also a long history. Its effects on the existence of flat and open FRW universes were studied before. In addition, the ghostlike (in)stabilities of massive cosmology have been investigated in literature. Also, it was pointed out that the massive theory can be used to explain the cosmological constant problem and provides an interesting basis for self-acceleration of the universe without introducing the cosmological constant. In other words, it was shown that the massive graviton term in cosmological solutions can be equivalent to a cosmological constant. Furthermore, the existence of massive gravitons provides extra polarization for gravitational waves, and affects the propagation’s speed of gravitational waves and production of gravitational waves during inflation. Therefore, it is reasonable to take the massive gravity into account for the cosmological systems. It is expected that investigation of the early universe in the context of massive gravity opens a window to discuss cosmological implications at high energy regime.

In 2016, for the first time, gravitational waves produced by a binary black hole merger were detected by the LIGO collaboration by using laser interferometer method. This confirmed the validation of the general relativity with two of its predictions, known as the existence of black holes and gravitational waves. In addition, as it was pointed out in Ref., realization of a gravitational wave astronomy provides us with the possibility of discriminating among general relativity and other gravity theories. In other words, by improving the results of LIGO and Virgo scientific collaborations, we are able to determine the validation of Einstein gravity extensions. Here, two of such extensions are considered alongside each other; gravity’s rainbow and massive gravity. In this paper, we take into account the modified FRW universe in the massive gravity’s rainbow. The main motivations for such consideration is given as follows; first of all, the physical properties of early universe dictate the necessity of regarding semi-quantum corrections which could be achieved by considering geometry of the spacetime as an energy dependent one. To do so, we employ gravity’s rainbow. This consideration has specific contributions such as absence of the big bang in the history of universe. Since the evolution of universe is governed by gravity, it is crucial to examine and understand the effects of massive gravitons on different stages of the universe’s evolution such as big bang, inflation and etc. Especially, it is important to understand the effects of massive gravitons on gravitational waves which were produced during inflation. Therefore, in this paper, we intend to provide preliminaries for such studies and investigate massive gravity, too. In essence, when we are dealing with a physical system, we should impose maximum number of generalizations, in order to have more general results and reliable predications. This is another motivation why we are considering such set up for our gravitational system.

The outline of the paper is as follows. In the next section, we study the Einstein-massive gravity and show that although Friedmann equations are modified in the presence of massive gravity, the singular big bang remains unsolved. After that, we modify FRW universe in Einstein-massive gravity’s rainbow, and then we show that there is no big bang singularity. We investigate nonsingular rainbow universe by considering three cases of rainbow functions. We finish our paper with some concluding remarks.
II. MODIFIED FRW COSMOLOGY IN THE MASSIVE GRAVITY

Here, we are interested in the structure of FRW universe in the context of massive gravity. We consider the Lagrangian of Einstein-massive gravity with a matter field as

$$\mathcal{L}_E = \mathcal{R} + m^2 \sum_{i=1}^{4} c_i \mathcal{U}_i(g, f) + \mathcal{L}_m,$$

where $\mathcal{R}$ and $\mathcal{L}_m$ are the Lagrangians of Einstein gravity and matter field, respectively, and also $m$ is the massive parameter. The second term in the Lagrangian of system produces massive terms in which $f$ is a fixed symmetric tensor, $c_i$'s are some constants, and $\mathcal{U}_i$'s are symmetric polynomials of the eigenvalues of matrix $K_{\mu\nu} = \sqrt{g} f^{\alpha\nu} K_{\mu\alpha}$.

$$\begin{align*}
\mathcal{U}_1 &= [K], \\
\mathcal{U}_2 &= [K]^2 - [K^2], \\
\mathcal{U}_3 &= [K]^3 - 3[K][K^2] + 2[K^3], \\
\mathcal{U}_4 &= [K]^4 - 6[K^2][K^2] + 8[K^3][K] + 3[K^2]^2 - 6[K^4].
\end{align*}$$

Using the variational principle, one can find following field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + m^2 \chi_{\mu\nu} = 8\pi G T_{\mu\nu},$$

in which $\chi_{\mu\nu}$ is the massive term with the following form

$$\begin{align*}
\chi_{\mu\nu} &= -\frac{c_1}{2} (\mathcal{U}_1 g_{\mu\nu} - K_{\mu\nu}) - \frac{c_2}{2} (2\mathcal{U}_1 K_{\mu\nu} + 2K_{\mu\nu}^2) \\
&- \frac{c_3}{2} (3\mathcal{U}_2 K_{\mu\nu} + 6\mathcal{U}_1 K_{\mu\nu}^2 - 6K_{\mu\nu}^3) \\
&- \frac{c_4}{2} (4\mathcal{U}_2 K_{\mu\nu} - 12\mathcal{U}_1 K_{\mu\nu}^2 - 24\mathcal{U}_1 K_{\mu\nu}^3 + 24K_{\mu\nu}^4).
\end{align*}$$

The 4-dimensional FRW metric with flat horizon ($k = 0$) is given by

$$ds^2 = -dt^2 + R(t)^2 dx_i^2, \quad i = 1, 2, 3,$$

where $R(t)$ is the scale factor. The energy-momentum tensor of our cosmological system is constructed based on perfect fluid as

$$T_{\mu\nu} = \rho(t) u_\mu u_\nu + P(t) (g_{\mu\nu} + u_\mu u_\nu),$$

where $\rho(t)$ is the energy density and $P(t)$ is the pressure of perfect fluids in which we will use $\rho$ and $P$ for simplicity. $u_\mu$ is four vector velocity which is defined as

$$u_\mu = (1, 0, 0, 0),$$

where $u_\mu$ satisfies the following restriction

$$g^{\mu\nu} u_\mu u_\nu = -1.$$  

The massive terms are constructed by considering a reference metric. Here, we employ the following ansatz for the reference metric [84]

$$f_{\mu\nu} = \text{diag}(0, 0, c^2 h_{ij}),$$

in which $c$ is a positive constant. It is worthwhile to mention that this reference metric preserves general covariance in temporal and radial coordinates but not in the transverse spatial coordinates [62], and therefore, the massive terms will have a Lorentz breaking property. In other words, we relax the Lorentz invariance property of massive gravity [85], which may be a necessary requirement for quantum description of gravity [12, 13, 86-88, 91]. In addition, we are not concerned to encounter singular properties of the reference metric (which is not as a physical metric, but
as a tool), since we use the line element $g_{\mu\nu}$ for raising and lowering indices. Using this metric ansatz [11], $U_i$’s are constructed in the following forms [84]

$$U_1 = \frac{2c}{r}, \quad U_2 = \frac{2c^2}{r^2}, \quad U_3 = U_4 = 0,$$

in which $r$ is the co-moving coordinate. Considering the metric [5] and field equation [3], one can find the modified FRW equations in the massive gravity as

$$3H^2 + \frac{cm^2}{r^2} (c_1 r + cc_2) = 8\pi G \rho,$$

$$\dot{H} = -4\pi G (\rho + P),$$

where $H = \dot{R}/R$ is the Hubble parameter and we used the notation $\dot{A} = \frac{dA}{dt}$. It is notable that, in the absence of massive parameter ($m = 0$), the Friedmann equations in Einstein-massive gravity [11 and 12] reduce to the ones in Einstein gravity. Now, we consider the conservation equation of energy-momentum tensor as

$$\nabla_\mu T^\mu_\nu = \partial_\mu T^\mu_\nu - \Gamma^\lambda_\mu_\nu T^\mu_\lambda + \Gamma^\mu_\mu_\lambda T^\lambda_\nu = 0,$$

where after some calculations, one finds

$$\dot{\rho} + 3H(\rho + P) = 0.$$

Here, we are interested in a large range of ultra relativistic particles which are in thermal equilibrium with a typical or an average energy $\epsilon \sim T$. Using the concept of continuity equation, one can find following the first law of thermodynamics in standard cosmology

$$d(\rho V) = -P dV,$$

where the volume, $V$, is given by $V = [R(t)]^3$. It is worthwhile to mention that regarding the integrability condition ($\frac{\partial^2 S}{\partial \rho \partial V} = \frac{\partial^2 S}{\partial V \partial P}$) [89] with the first law of thermodynamics [13] will lead to a constant entropy in the following form

$$S = \frac{V(\rho + P)}{T} = \text{const.}$$

In order to obtain the properties of FRW cosmology, it is necessary to consider an equation of state (EoS). Here, EoS is given by

$$P = (\gamma - 1)\rho,$$

which leads to a singular FRW spacetime in standard cosmology at $t = 0$ ($\gamma$ known as EoS parameter which is $4/3$ for the radiation dominated era). The average energy, $\epsilon$, is obtained as

$$\epsilon \sim T = c' \gamma \rho V,$$

where $c'$ is a constant which is equal to $1/S$. Substituting the EoS [17] in the conservation equation [14], one can find

$$\frac{d\rho}{d\ln(R)} = -3\gamma \rho,$$

which leads to energy density as $\rho = R^{-3\gamma}$. It is a matter of calculation to find average energy as

$$\epsilon = c' \gamma \rho \frac{\dot{R}}{R},$$

where the relation between the average energy and density is governed by EoS parameter, $\gamma$.

Now, we are in a position to investigate the condition regarding singular and nonsingular universe. To do so, we are going to follow the same analysis that has been used in Refs. [31, 34]. Using the obtained modified Friedmann equation of massive gravity (Eq. [11]) with conservation equation (Eq. [14]) and EoS (Eq. [17]), we obtain

$$\dot{\rho} = \pm\gamma\rho \sqrt{24\pi G \rho - \frac{3cm^2}{r^2}}.$$
where $\mathcal{C} = c(c_1r + c_2)$. The time for reaching a potential singularity could be obtained by integrating Eq. (21) (starting from an initial finite density $\rho^*$ to an infinite one). Therefore, the integration is given by

$$t = \pm \frac{1}{\gamma} \int_{\rho^*}^{\infty} \frac{1}{\rho} \left[ 24\pi G \rho - \frac{3Cm^2}{r^2} \right]^{-\frac{1}{2}} d\rho,$$

and it is a matter of calculation to find

$$t = \pm \frac{2r}{\gamma m \sqrt{3\mathcal{C}}} \arctan \left( \frac{8\pi G \rho r^2}{Cm^2 - 1} \right)_{\rho^*}^\infty,$$  \hspace{1cm} (23)

in which by substituting the bounds, one can find

$$t = \pm \frac{2r}{\gamma m \sqrt{-3\mathcal{C}}} \sinh^{-1} \left( \frac{m}{2r} \sqrt{-\frac{\mathcal{C}}{2\pi G \rho^*}} \right),$$  \hspace{1cm} (24)

where $\mathcal{C}$ should be negative ($\mathcal{C} < 0$). Considering negative $\mathcal{C}$, it is possible to obtain a limit on massive coefficients. Regarding the fact that the co-moving coordinate is zero at the big bang, one can conclude that whether $c$ and $c_2$, both must be negative or only $c_2$ itself must be negative while $c$ is positive. As we drift away from big bang, the effects of $c_1r$ term start to grow, so we get another set of condition indicating that $c$ and $c_2$ should be negative and $c_1$ must be positive, so Eq. (24) has positive term for its square root function. Another possible case for having real function in Eq. (24) is by setting $c$ as positive constant while $c_1$ and $c_2$ are negative. These two cases enable us to put limit on the signs that massive coefficients could acquire. Regarding Eq. (24), one can conclude that the time is finite as density of universe goes from an initial density $\rho^*$ (the present time) to infinity (the big bang time). This means that the time to reach the potential singularity is finite, so we have the big bang singularity.

III. MODIFIED FRW RAINBOW COSMOLOGY IN EINSTEIN-MASSIVE GRAVITY’S RAINBOW

In this section, we are going to modify usual FRW universe by considering Einstein-massive gravity’s rainbow and study its effect as a semi-classical approach in the early universe. In other words, since dRGT massive gravity is a low energy effective theory, one may use its modifications to address issues such as the big bang singularity. In addition, one can keep the gravitational structure of field equations and instead consider an energy-dependent ansatz for the metric as a stand-in for any high energy or quantum effects. In this prescription, we focus on the energy dependent nature of spacetime, which is affected by the energy of moving particles. As a starting point, we consider the following 4-dimensional metric

$$ds^2 = -\frac{dt^2}{k(\varepsilon)^2} + \frac{R(t)^2}{g(\varepsilon)^2} dx_i^2, \hspace{1cm} i = 1, 2, 3,$$

and here, we define $u_\mu$ as

$$u_\mu = (k(\varepsilon)^{-1}, 0, 0, 0),$$

in which

$$g^{\mu\nu} u_\mu u_\nu = -1.$$  \hspace{1cm} (27)

We obtain the conservation equation in the following form

$$\dot{\rho} + 3 \left( H - \frac{\dot{g}(\varepsilon)}{g(\varepsilon)} \right) (\rho + P) = 0.$$  \hspace{1cm} (28)

One can use the same procedure, like previous section, to show that the average energy has the following form

$$\epsilon = c' \gamma \rho^{\frac{2}{\gamma}}.$$  \hspace{1cm} (29)

It is notable that, in Eq. (29) we used $V = (R(t)/g(\varepsilon))^3$, and interestingly, Eq. (29) (such as Eq. (20)) does not depend on the rainbow functions.
Considering Eq. (26) and using the metric (25) with field equation (3), one can show that the modified FRW equations in Einstein-massive gravity’s rainbow are

\[
\left( H - \frac{\dot{g}(\varepsilon)}{g(\varepsilon)} \right)^2 + \frac{C m^2}{3 r^2 k(\varepsilon)^2} = \frac{8 \pi G \rho}{3 k(\varepsilon)^2},
\]

\[
\dot{H} + \frac{\dot{g}(\varepsilon)^2}{g(\varepsilon)^2} - \frac{\ddot{g}(\varepsilon)}{g(\varepsilon)} + \frac{k(\varepsilon)}{k(\varepsilon)} \left( H - \frac{\dot{g}(\varepsilon)}{g(\varepsilon)} \right) = -\frac{4 \pi G (\rho + P)}{k(\varepsilon)^2},
\]

in which Eqs. (30) and (31) turn into Eqs. (11) and (12) for $g(\varepsilon) = k(\varepsilon) = 1$, respectively. On the other hand, they reduce to the FRW equations in Einstein gravity’s rainbow [31] in the absence of massive parameter ($m = 0$).

**IV. WHEN A NONSINGULAR RAINBOW UNIVERSE IS POSSIBLE?**

In this section, we are going to show that how a nonsingular rainbow universe is possible. Here, we want to discuss this possibility in different cases.

**A. Case 1: the constants are independent from energy**

In this case, we consider all the constants as energy independent ones. Substituting the modified Friedmann equation of massive gravity’s rainbow (30) in the conservation equation (28) and using the EoS, we obtain

\[
\dot{\rho} = \pm \frac{\gamma \rho}{k(\varepsilon)} \sqrt{24 \pi G \rho - \frac{3 C m^2}{r^2}}. \tag{32}
\]

Now, we are in a position to study the resolution of finite-time singularities. This is done by showing that the existence of an upper bound for the density $\rho$ is reached at an infinite time. In other words, there exists a divergence point for density which is acquired in an infinite time. Therefore, it is not a physical singularity.

Considering Eq. (29), it is possible to write $k$ as a function of $\rho$ instead of $\varepsilon$ in Eq. (32). Now, it is a matter of calculation to show that all finite-time singularities (including big bang singularity) are removed if $k$ grows asymptotically as $\rho^{1/2}$, or faster such as $k \sim \rho^s$ where $s \geq 1/2$. In this case, one can calculate the time for reaching a potential singularity by integrating Eq. (32) (starting from a finite density $\rho^*$ to an infinite one) which leads to

\[
t = \pm \frac{1}{\gamma} \int_{\rho^*}^{\infty} \rho^{s-1} \left[ 24 \pi G \rho - \frac{3 C m^2}{r^2} \right]^{-\frac{1}{2}} d\rho. \tag{33}
\]

After some calculations, we obtain

\[
t = \pm \frac{\left( \frac{C m^2}{2 r^2} \right)^{s-1} \sqrt{24 \pi G \rho - \frac{3 C m^2}{r^2}} \mathcal{W}_1}{3(2)^{3s-1} \pi^s G} \bigg|_{\rho^*}^{\infty},
\]

\[
= \infty, \quad s \geq \frac{1}{2}, \tag{34}
\]

in which

\[
\mathcal{W}_1 =_2 F_1 \left( \left[ \frac{1}{2}, 1 - s \right], \left[ \frac{3}{2} \right], 1 - \frac{8 \pi G \rho^2}{C m^2} \right), \tag{35}
\]

where $_2 F_1$ is a hypergeometric function and $C < 0$. Considering (34), we conclude that the time to reach the potential singularity is infinite for $s \geq 1/2$, so it is not a finite-time singularity, i.e., not physical. Evidently, the energy function, $k(\varepsilon)$, plays a crucial role to remove the big bang singularity.
B. Case 2: energy dependent constants

In this case, we consider the two constants $G$ and $m$ as functions of energy ($G(\varepsilon)$ and $m(\varepsilon)$). With this assumption, Eq. (32) becomes

$$
\dot{\rho} = \pm \frac{\gamma \rho}{k(\varepsilon)} \sqrt{\frac{24\pi G(\varepsilon)\rho - \frac{3Cm(\varepsilon)^2}{r^2}}}. \tag{36}
$$

According to Eq. (29), we can write $G$ and $m$ as functions of $\rho$ instead of $\varepsilon$. Following the steps of previous section, the time for reaching a potential singularity by integrating Eq. (36) is obtained as

$$
t = \pm \frac{1}{\gamma} \int_{\rho_*}^{\infty} \rho^{s-1} \left[ 24\pi \rho G(\rho) - \frac{3Cm(\rho)^2}{r^2} \right]^{-\frac{1}{2}} d\rho
= \pm \frac{1}{\gamma} \int_{\rho_*}^{\infty} \rho^{s-1} \left[ 24\pi \rho^{a+1} - \frac{3C\rho^{2b}}{r^2} \right]^{-\frac{1}{2}} d\rho, \tag{37}
$$

where in the above equation we considered $G(\rho) = \rho^a$ and $m(\rho) = \rho^b$. One may note that it is not possible to compute this integration analytically. Indeed, we can solve this integration without any bounds, but it is not possible to find the asymptotic behavior of solution because we need to know whether $a+1$ is larger (smaller) than $2b$, or they are equal. So, we have to restrict ourselves to the special case, $G(\rho) = m(\rho)^2 = \rho^2$. After some calculations, we obtain

$$
t = \pm \frac{1}{\gamma} \int_{\rho_*}^{\infty} \rho^{s-b-1} \left[ 24\pi \rho - \frac{3C}{r^2} \right]^{-\frac{1}{2}} d\rho
= \pm \frac{2\pi^2}{3\gamma C} \left( \frac{C}{8\pi r^2} \right)^{s-b} \sqrt{24\pi \rho - \frac{3C}{r^2}}
\times \, _2F_1\left( \left[ \frac{1}{2}, 1+b-s \right], \left[ \frac{3}{2}, 1 - \frac{8\pi \rho r^2}{C} \right], \rho_* \right) \bigg|_{\rho_*}^{\infty}
= \infty, \quad s - b \geq \frac{1}{2}, \tag{38}
$$

where $C < 0$. Considering (38), one can reach to the conclusion that the potential singularity is achieved at infinity for $s - b \geq 1/2$, so it is not a finite-time singularity, i.e., not physical.

On the other hand, we can consider $\rho^{a+1}$ or $\rho^{2b}$ as a dominant term in the integration (37) to show that one can have a nonsingular universe with special constraint $for \ a+1 > 2b$

$$
t = \pm \frac{1}{\gamma \sqrt{24\pi}} \int_{\rho_*}^{\infty} \rho^{s-1-\frac{a+1}{2}} d\rho
= \pm \frac{\rho^s}{\gamma \sqrt{24\pi} \left( s - \frac{a+1}{2} \right) \rho_*} \bigg|_{\rho_*}^{\infty} = \infty, \quad s - \frac{a}{2} \geq \frac{1}{2}, \tag{39}
$$

$for \ a+1 < 2b$

$$
t = \pm \frac{r}{\gamma \sqrt{-3C}} \int_{\rho_*}^{\infty} \rho^{s-1-b} d\rho
= \pm \frac{r \rho^{s-b}}{\gamma \sqrt{-3C} \left( s - b \right) \rho_*} \bigg|_{\rho_*}^{\infty} = \infty, \quad s - b \geq 0, \tag{40}
$$

where $C < 0$. This means that when $a+1 > 2b$ ($a+1 < 2b$), the time to reach the potential singularity is infinite for $s - \frac{a}{2} \geq \frac{1}{2}$ ($s - b \geq 0$).
V. NONSINGULAR RAINBOW UNIVERSE

Here, we are going to investigate nonsingular rainbow universe by considering some special cases of rainbow functions. The energy functions of gravity’s rainbow are motivated from different branches of the physics. The first model comes from the hard spectra of gamma rays motivation with the following form \[91\]

\[ k(\varepsilon) = \exp\left(\varepsilon - \frac{1}{\varepsilon}\right), \quad g(\varepsilon) = 1. \]

(41)

Taking the constancy of the velocity of light into account, one can find following relations for the rainbow functions as second model

\[ k(\varepsilon) = g(\varepsilon) = \frac{1}{1 - \varepsilon}. \]

(42)

The third model is motivated from loop quantum gravity and non-commutative geometry in which rainbow functions are \[92\]

\[ k(\varepsilon) = 1, \quad g(\varepsilon) = \sqrt{1 - \varepsilon^n}, \]

(43)

Considering Eq. (28), one can convert the above equations into

| Model       | \( k(\mathcal{G}) \)                                      | \( g(\mathcal{G}) \)                                      |
|-------------|-----------------------------------------------------------|-----------------------------------------------------------|
| first model | \( \exp\left(\gamma \mathcal{G}^{\frac{1}{\gamma}}\right) - 1 \) | 1                                                         |
| second model| \( \frac{1}{1 - \gamma \mathcal{G}^{\frac{1}{\gamma}}} \) | \( \frac{1}{1 - \gamma \mathcal{G}^{\frac{1}{\gamma}}} \) |
| third model | 1                                                        | \( \sqrt{1 - \gamma \mathcal{G}^{\frac{n(\gamma-1)}{\gamma}}} \) |

(44)

where \( \mathcal{G} = \rho / \rho_p \), and \( E_p = c' \rho_p^{(\gamma-1)/\gamma} \) is the Planck energy versus density \( \rho_p \). Using the modified Friedmann equation \[30\], on can show that

\[ \left( H - \frac{\dot{g}(\varepsilon)}{g(\varepsilon)} \right) = \pm \frac{1}{k(\mathcal{G})} \sqrt{\frac{8\pi G \rho_p \mathcal{G}}{3} - \frac{Cm^2}{3r^2}}. \]

(45)

Considering the modified Friedmann equation \[45\] with Eq. \[28\], following relation is obtained

\[ \dot{\mathcal{G}} = \pm \frac{\gamma \mathcal{G}}{k(\mathcal{G})} \sqrt{24\pi G \rho_p \mathcal{G} - \frac{3Cm^2}{r^2}}, \]

(46)

in which \( \dot{\mathcal{G}} = \dot{\rho} / \rho_p \). Now, considering the previous discussion, we are in a position to discuss the possibility of nonsingular universe for three cases of rainbow functions \[44\].

A. First model

Substituting the first model of rainbow functions \[44\] in \(46\), on can get

\[ \dot{\mathcal{G}} = \pm \frac{\gamma^2 \mathcal{G}^{2\gamma - 1}}{\exp\left(\gamma \mathcal{G}^{\frac{1}{\gamma}}\right) - 1} \sqrt{24\pi G \rho_p \mathcal{G} - \frac{3Cm^2}{r^2}}. \]

(47)

Now, we intend to elaborate the infinity of time for going from an initial finite density \( \mathcal{G}^* \) to an infinite one in special case \( \gamma = 4/3 \), (i.e., radiation). To do so, we integrate Eq. \(47\)

\[ t = \pm \frac{9}{16} \int_{\mathcal{G}^*}^{\infty} \frac{\exp\left(\frac{4\mathcal{G}^{\frac{1}{3}}}{3} - 1\right)}{\mathcal{G}^{\frac{2}{3}}\sqrt{24\pi G \rho_p \mathcal{G} - \frac{3Cm^2}{r^2}}} d\mathcal{G}. \]

(48)
which is not possible to obtain analytical solution, but numerical evaluation shows that it does not converge on \( \mathcal{G}^*, \infty \), which leads to infinity of time to reach infinite density. It is worthwhile to mention that this result is valid for \( \gamma > 1 \). In order to make more clarification, we consider \( G^- \) term as dominant one in denominator which yields

\[
t = \pm \frac{9}{32 \sqrt{6\pi \rho_p G^*}} \int_{G^*}^{\infty} \frac{\exp \left( \frac{4}{3} G^\frac{1}{4} \right) - 1}{G^\frac{7}{4}} dG
\]

\[= \pm \frac{K_1 - 32 G^\frac{7}{4} \mathcal{E} \left( \frac{4}{3} G^\frac{1}{4} \right) |_{G^*}^{\infty}}{72 \sqrt{6\pi \rho_p G^*}} = \infty,
\]

in which

\[
K_1 = 3 \left( 6 G^\frac{7}{4} + 8 G^\frac{11}{4} + 9 \right) \exp \left( \frac{4}{3} G^\frac{1}{4} \right) - 9,
\]

where \( \mathcal{E} \) is the exponential integration. Obtained result indicates that the time to reach this infinite density is infinite, so there is no finite-time singularity which confirms the consequence of Eq. (48).

### B. Second model

Now, we are going to use the second model to obtain nonsingular universe. Using the second form of rainbow functions (44) with (46), on can show that

\[
\dot{\mathcal{G}} = \pm \gamma \left( \mathcal{G} - \gamma \mathcal{G}^{2/3 + 1} \right) \sqrt{24\pi \rho_p G - \frac{3Cm^2}{\rho^2}},
\]

where for a special case \( \gamma = 4/3 \), (i.e., radiation) it reduces to

\[
t = \pm \frac{3}{4} \int_{G^*}^{\infty} \frac{dG}{\left( \mathcal{G} - \frac{4}{3} \mathcal{G}^{\frac{1}{3}} \right) \sqrt{24\pi \rho_p G - \frac{3Cm^2}{\rho^2}}}
\]

which is not possible to obtain analytical solution. When we choose the term with \( G \) as dominant term, the equation (52) will be

\[
t = \pm \frac{3}{8 \sqrt{6\pi \rho_p G^*}} \int_{G^*}^{\infty} \frac{dG}{\left( \mathcal{G}^\frac{7}{4} - \frac{4}{3} \mathcal{G}^{\frac{1}{4}} \right)}
\]

\[= \pm \frac{1}{12 \sqrt{6\pi \rho_p G^*}} \left[ 3 \left( \mathcal{G}^* \right)^{-\frac{1}{4}} \left( 3 + 8 \left( \mathcal{G}^* \right)^{\frac{1}{4}} \right) \right.
\]

\[+ 32 \ln \left( 1 - \frac{3}{4} \left( \mathcal{G}^* \right)^{-\frac{1}{4}} \right) \] \( \left. \right|_{G^*}^{\infty} \) \( + 32 \ln \left( 1 - \frac{3}{4} \left( \mathcal{G}^* \right)^{-\frac{1}{4}} \right) \] \( \left. \right|_{G^*}^{\infty} \)

where shows that this integration has finite value for \( \mathcal{G}^* > \left( \frac{3}{4} \right)^4 \) and it does not converge on \( (0, (\frac{3}{4})^4] \). So, this divergency in integration is because of initial density \( \mathcal{G}^* \) and one concludes that we have big bang singularity.

In order to obtain nonsingular universe, one may follow the previous discussion and consider the two constants \( G \) and \( m \) as functions of energy. So, one can rewrite the integration (52) as

\[
t = \pm \frac{3}{8 \sqrt{6\pi \rho_p G^*}} \int_{G^*}^{\infty} \frac{dG}{\left( \mathcal{G} - \frac{4}{3} \mathcal{G}^{\frac{1}{3}} \right) \sqrt{24\pi \rho_p G - \frac{3Cm^2}{\rho^2}}}
\]

\[= \pm \frac{3}{4 \sqrt{6\pi \rho_p G^*}} \left[ 3 \left( \mathcal{G}^* \right)^{-\frac{1}{4}} \left( 3 + 8 \left( \mathcal{G}^* \right)^{\frac{1}{4}} \right) \right.
\]

\[+ 32 \ln \left( 1 - \frac{3}{4} \left( \mathcal{G}^* \right)^{-\frac{1}{4}} \right) \] \( \left. \right|_{G^*}^{\infty} \) \( + 32 \ln \left( 1 - \frac{3}{4} \left( \mathcal{G}^* \right)^{-\frac{1}{4}} \right) \] \( \left. \right|_{G^*}^{\infty} \)

where \( G(G) = \mathcal{G}^\alpha \) and \( m(G) = \mathcal{G}^{b'} \). As it was mentioned before, we may consider one dominant term in denominator of Eq. (54) which yields

\[
for \ a' + 1 > 2b':
\]
\[
\begin{align*}
\dot{t} &= \pm \frac{3}{8\sqrt{6\pi G_\rho G}} \int_{G^*}^{\infty} dG \left( G^\frac{\dot{G}}{2} - \frac{4}{3} G^\frac{\dot{G}}{2} \right) \\
&= \pm \frac{3(2a' + 1) + 8G^\frac{\dot{G}}{2}(a' + 1)}{4(a' + 1)(2a' + 1)} \frac{dG}{\sqrt{6\pi G_\rho G}} \bigg|_{G^*}^{\infty} \\
&= \infty, \quad a' \leq -\frac{3}{2} \quad \& \quad G^* > \left( \frac{3}{4} \right)^4 
\end{align*}
\] (55)

for \( a' + 1 < 2b' \):

\[
\begin{align*}
t &= \pm \frac{3r}{4\sqrt{-3C}} \int_{G^*}^{\infty} \frac{dG}{G^{\nu+1}} \left( 1 - \frac{4}{3} G^\frac{\dot{G}}{2} \right) \\
&= \pm \frac{3(1 - 4b')}{4b'(4b' - 1)} \frac{dG}{\sqrt{-3C G^{\nu}}} \bigg|_{G^*}^{\infty} \\
&= \infty, \quad b' \leq -\frac{1}{4} \quad \& \quad G^* > \left( \frac{3}{4} \right)^4,
\end{align*}
\] (56)
in which

\[
\begin{align*}
W_2 &= 2F_1 \left( [1, -2a' - 1], [-2a'], \frac{4}{3} G^\frac{\dot{G}}{2} \right), \\
W_3 &= 2F_1 \left( [1, 1 - 4b'], [2(1 - 2b')], \frac{4}{3} G^\frac{\dot{G}}{2} \right),
\end{align*}
\] (57)

where \( C < 0 \) and the initial density \( G^* \) should be larger than \( (3/4)^4 \). The above equations mean that when \( a' + 1 > 2b' \) \( (a' + 1 < 2b') \), the time to reach the potential singularity is infinite for \( a' \leq -\frac{3}{2} \) \( (b' \leq -\frac{1}{4}) \).

C. Third model

Substituting the third model of rainbow functions (44) in (46), it is easy to show that

\[
\dot{G} = \pm \gamma G \sqrt{24\pi G_\rho G - \frac{3C m^2}{r^2}},
\] (59)

which leads to the following integration

\[
\begin{align*}
t &= \pm \frac{1}{\gamma} \int_{G^*}^{\infty} \frac{dG}{G \sqrt{24\pi G_\rho G - \frac{3C m^2}{r^2}}} \\
&= \pm \frac{2r}{\gamma m \sqrt{-3C}} \sinh^{-1} \left( \frac{m}{2r} \sqrt{\frac{C}{2\pi G G^*}} \right),
\end{align*}
\] (60)

where \( C < 0 \) and shows that we have singular universe. It is worthwhile to mention that the integration (60) is like Eq. (22) and one may note that the rainbow function, \( g(\varepsilon) \), does not affect the singularity of universe. But here, there is another story because of rainbow functions. Indeed, we can follow the previous discussion to consider the two constants \( G \) and \( m \) as functions of energy in order to obtain nonsingular universe. So, the integration (60) will convert to

\[
\begin{align*}
t &= \pm \frac{1}{\gamma} \int_{G^*}^{\infty} \frac{dG}{G \sqrt{24\pi G_\rho G^{\alpha'+1} - \frac{3C m^2}{r^2}}},
\end{align*}
\] (61)
where $G(G) = G''$ and $m(G) = G''$. As it has mentioned before, one may consider a dominant term to solve the integration

$$t = \pm \frac{1}{2\gamma \sqrt{6\pi r_g}} \int_{G_*}^{\infty} \frac{dG}{G^{a''+1}} = \pm \frac{1}{\gamma (a''+1) \sqrt{6\pi r_g} G^{a''+1}} \bigg|_{G_*}^{\infty} = \infty, \quad a'' \leq -1,$$

for $a'' + 1 > 2b''$ :

$$t = \pm \frac{1}{2\gamma \sqrt{6\pi r_g}} \int_{G_*}^{\infty} \frac{dG}{G^{b''+1}} = \pm \frac{r}{\gamma b'' \sqrt{-3C G^{b''+1}}} \bigg|_{G_*}^{\infty} = \infty, \quad b'' \leq 0,$$

where $C < 0$. This means that when $a'' + 1 > 2b''$ ($a'' + 1 < 2b''$), the time to reach the potential singularity is infinite for $a'' \leq -1$ ($b'' \leq 0$) which is true for all values of $\gamma$ including $\gamma = 4/3$.

### D. Density of states

Our final study is devoted to the possible divergency of density of states at the Planck scale [32, 35]. Employing the modified dispersion relation (1), the density of states could be obtained as

$$a(E) dE \approx p^2 dp = \frac{k(\varepsilon)^3}{g(\varepsilon)^3} \left[ 1 + E \left( \frac{k(\varepsilon)'}{k(\varepsilon)} - \frac{g(\varepsilon)'}{g(\varepsilon)} \right) \right] E^2 dE,$$

in which by remembering the fact that energy cannot be larger than Planck energy, the density of states yield to a finite value for all rainbow function models (Eqs. (41)–(43))

$$a(E) \approx \begin{cases} \frac{E^2}{1 - \varepsilon^2} \exp(\varepsilon) \left[ \exp(\varepsilon) - 1 \right]^2, & \text{first model}, \\ E^2, & \text{second model}, \\ \frac{E^2 (\varepsilon)^n + 2}{2(1 - \varepsilon^n)^{1/2}}, & \text{third model}, \end{cases}$$

which show that the density of states have regular behavior without any divergency (note: $\varepsilon < 1$).

### VI. CLOSING REMARKS

Motivated by the high energy regime at the early universe, developing Einstein gravity has been applied in the context of cosmology. FRW cosmology in the presence of massive gravity and massive gravity’s rainbow have been separately investigated. First, the massive gravity modification was investigated and it was shown that generalization to massive gravity does not remove the big bang singularity. Then, the generalization to gravity’s rainbow was imposed for two different cases; in one case, the constants were considered independent of energy while in the other case the energy dependency of constants was taken into account. It was pointed out that in order to remove the big bang singularity in an energy dependent spacetime, certain conditions are required to be satisfied.

Using the method which was inscribed in [91] (choosing the suitable rainbow functions), it was possible to study the effects of rainbow functions on FRW-like cosmology. It was shown that Friedmann equations were modified in the
presence of massive gravity’s rainbow which lead to the absence of big bang singularity. Such property was derived for large varying range of equation of state parameter, $\gamma > 4/3$. The absence of singularity was shown by using the analysis in \[31, 34\]. It was found that the universe takes infinite time to reach $\rho \rightarrow \infty$ from an initial finite value of $\rho$. Then, we have investigated two other models of rainbow functions (see Eqs. (12) and (13)) and we found that the universe will be singular in these two cases. In order to obtain nonsingular universe for these two models, we had to consider two constants $G$ and $m$ as functions of energy. Finally, the possibility of divergency of density of state at the Planck scale was investigated for three models of rainbow functions. It was pointed out that considering the energy conditions of gravity’s rainbow, the density of state does not diverge and a possible resolution regarding the big bang singularity is obtained.

Here, we have provided a preliminary to study the effects of massive gravitons on different stages of the universe’s evolution, especially gravitational waves which were produced in these stages. It is interesting to study the effects of gravity’s rainbow and massive gravity on the inflation mechanism and the age of different eras in the standard cosmology. In addition, it is worthwhile to see how these two generalizations could address the old cosmological constant problem and accelerating expansion of the universe.

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