MINIMAL ASYMPTOTIC TRANSLATION LENGTHS
ON CURVE COMPLEXES
AND HOMOLOGY OF MAPPING TORI

HYUNGRYUL BAIK, DONGRYUL M. KIM, AND CHENXI WU

Abstract. Let \( S_g \) be a closed orientable surface of genus \( g \geq 1 \). We consider the minimal asymptotic translation length \( L_T(k, g) \) on the Teichmüller space of \( S_g \), among pseudo-Anosov mapping classes of \( S_g \) acting trivially on a \( k \)-dimensional subspace of \( H_1(S_g) \), \( 0 \leq k \leq 2g \). The asymptotes of \( L_T(k, g) \) for extreme cases \( k = 0, 2g \) have been known by several authors. Jordan Ellenberg asked whether there is a lower bound for \( L_T(k, g) \) interpolating the known results on \( L_T(0, g) \) and \( L_T(2g, g) \), which was affirmatively answered by Agol, Leininger, and Margalit.

In this paper, we study an analogue of Ellenberg’s question, replacing Teichmüller spaces with curve complexes. We provide lower and upper bound on the minimal asymptotic translation length \( L_C(k, g) \) on the curve complex, whose lower bound interpolates the known results on \( L_C(0, g) \) and \( L_C(2g, g) \).

Finally, we construct non-Torelli pseudo-Anosovs that are unable to normally generate the whole mapping class groups. As Lanier and Margalit proved that pseudo-Anosovs with small translation lengths on the Teichmüller spaces normally generate mapping class groups, our observation provides a restriction on how small the asymptotic translation lengths on curve complexes should be if the similar phenomenon holds for curve complexes.

1. Introduction

Let \( S_g \) be a closed connected orientable surface of genus \( g \geq 1 \), \( \text{Mod}(S_g) \) be its mapping class group, and \( \mathcal{C}(S_g) \) be its curve complex. Then \( \text{Mod}(S_g) \) isometrically acts on \( \mathcal{C}(S_g) \), hence the asymptotic translation length \( \ell_C(f) \) of \( f \in \text{Mod}(S_g) \) on \( \mathcal{C}(S_g) \) is defined:

\[
\ell_C(f) := \liminf_{n \to \infty} \frac{d_C(x, f^n(x))}{n}
\]

for any \( x \in \mathcal{C}(S_g) \) where \( d_C \) is the standard metric on \( \mathcal{C}(S_g) \). The asymptotic translation length is also called stable translation length.

Note that \( \text{Mod}(S_g) \) also acts on \( H_1(S_g) \), the first homology group of \( S_g \) with real coefficients. For \( f \in \text{Mod}(S_g) \), we denote the dimension of a maximal subspace of \( H_1(S_g) \) on which \( f \) is trivial by \( m(f) \). In particular, \( m(f) = 2g \) if and only if \( f \) is in the Torelli group \( \mathcal{T}_g \leq \text{Mod}(S_g) \), the subgroup

\( Date: \) November 23, 2022.
consisting of elements which act trivially on $H_1(S_g)$. As an application of Mayer-Vietoris sequence, one can observe that $m(f) + 1$ is same as the first betti number of the mapping torus of $f$, which is hyperbolic if and only if $f$ is pseudo-Anosov by Thurston [Thu98].

In this paper, we mainly study the minimal asymptotic translation lengths among pseudo-Anosov mapping classes acting trivially on some subspaces of homology groups. Namely, for $0 \leq k \leq 2g$, we define

$$L_C(k, g) := \inf \{ \ell_C(f) : f \in \text{Mod}(S_g), f \text{ is pseudo-Anosov}, m(f) \geq k \}.$$

Then we investigate asymptotes of $L_C(k, g)$ with varying $k$ and $g$.

By replacing the curve complex $C(S_g)$ with Teichmüller space $T(S_g)$, one can also define $L_T(k, g)$ analogously. Note that $\ell_T(f)$ for a pseudo-Anosov element $f$ is same as the logarithm of the stretch factor [B+78], hence coincides with the topological entropy of $f$ [FLP12, Exposé Ten].

In each setting, there are two extreme cases; the case $k = 0$ where it is about the minimal asymptotic translation length over the entire mapping class group and the case $k = 2g$ where it is about the minimal asymptotic translation length for the elements of the Torelli subgroup. These four cases have been resolved by various authors as below. Throughout the paper, we write $A(x) \asymp B(x)$ if there exists a uniform constant $C > 0$ such that $A(x) \leq CB(x)$ for all $x$.

(1) (Penner [Pen91])

$$L_T(0, g) \asymp 1/g$$

(2) (Farb-Leininger-Margalit [FLM08])

$$L_T(2g, g) \asymp 1$$

(3) (Gadre-Tsai [GT11])

$$L_C(0, g) \asymp 1/g^2$$

(4) (Baik-Shin [BS20])

$$L_C(2g, g) \asymp 1/g$$

Ellenberg [Ell] asked if $L_T(k, g)$ interpolates $L_T(0, g)$ and $L_T(2g, g)$ in the sense that there exists $C > 0$ such that

$$L_T(k, g) \geq C(k + 1)/g$$

for all $g > 1$ and $0 \leq k \leq 2g$. This was answered affirmatively by Agol-Leininger-Margalit in [ALM16]. Indeed, they actually showed $L_T(k, g) \asymp (k + 1)/g$.

We ask an analogous question whether $L_C(k, g)$ interpolates $L_C(0, g)$ and $L_C(2g, g)$ in a similar sense as Ellenberg’s question (1.1). We show that this is indeed the case and more concretely we obtain the following:
Theorem 1.1. There exists $C, C' > 0$ such that
\[
\frac{C}{g(2g - k + 1)} \leq L_C(k, g) \leq C' \frac{k + 1}{g \log g}
\]
for all $g > 1$ and $0 \leq k \leq 2g$.

From the statement, if $k$ grows at least $2g - C'$ for some constant $C' > 0$, then $L_C(k, g) \gtrsim 1/g$ while $L_C(0, g) \asymp 1/g^2$. Observing this, we ask about minimal $k$ with $L_C(k, g) \asymp 1/g$. For this discussion, see Section 4.

While the lower bound in Theorem 1.1 interpolates $L_C(0, g) \asymp 1/g^2$ and $L_C(2g, g) \asymp 1/g$, one can see that the upper bound does not. Indeed, we construct some examples showing that $\frac{k + 1}{g \log g}$ is larger than the actual asymptote.

We also show that $k/g^2$ can partially serve as an upper bound for $L_C(k, g)$, which interpolates $L_C(0, g) \asymp 1/g^2$ and $L_C(2g, g) \asymp 1/g$.

Theorem 1.2. There is a uniform constant $C > 0$ satisfying the following:
For any integers $g, k \geq 0$, there exists a pseudo-Anosov $f : S_{g'} \to S_{g'}$ such that $g' > g$, $m(f) = k' > k$, and
\[
\ell_C(f) \leq C \frac{k'}{g'^2}.
\]

Applying Theorem 1.2 inductively, it follows that there is a diverging sequence $(k_j, g_j) \to \infty$ so that $L_C(k_j, g_j) \lesssim k_j/g_j^2$. See Corollary 3.1. Based on (3) and (4), we conjecture that the upper bound in Theorem 1.2 is actually the asymptote for $L_C(k, g)$.

Conjecture 1.3.
\[
L_C(k, g) \asymp \frac{k}{g^2}
\]
for $g > 1$ and $0 \leq k \leq 2g$.

We focus on specific dimensions of maximal invariant subspaces. In [BS20], Torelli pseudo-Anosovs are constructed in a concrete way based on Penner’s or Thurston’s construction. In a similar line of thought, we utilize finite cyclic covers of $S_2$ so that we get pseudo-Anosovs $f \in \text{Mod}(S_g)$ with $m(f) = 2g - 1$ and satisfying the upper bound in Theorem 1.2. As a consequence, it figures out the asymptote of $L_C(2g - 1, g)$; only two extreme cases $\text{Mod}(S_g)$ and $I_g$ were previously known.

Theorem 1.4. There exists a uniform constant $C > 0$ and pseudo-Anosovs $f_g \in \text{Mod}(S_g)$ such that
\[
m(f_g) = 2g - 1 \quad \text{and} \quad \ell_C(f_g) \leq C \frac{g}{g}
\]
for all $g > 1$. Moreover, the following asymptote holds:
\[
L_C(2g - 1, g) \asymp \frac{1}{g}.
\]
The construction involved in Theorem 1.4 can be modified to deal with the Torelli case. Such a modification gives an asymptote for $L_C(2g,g)$ which was already shown by [BS20] in a different way. See Remark 4.1. Further, only the last assertion can also be deduced from Theorem 1.1 and [BS20]. See Section 4 for details.

In [LM18], Lanier and Margalit showed that a pseudo-Anosov with small asymptotic translation length on the Teichmüller space has an entire mapping class group as its normal closure. The first and the third authors, Kin, and Shin, made an analogous question for asymptotic translation lengths on curve complexes in [BKSW19] (see [BKSW19, Question 1.2]). We later show that pseudo-Anosovs $f_g$ constructed in Theorem 1.4 never normally generate the mapping class groups. As $\ell_C(f_g)$ is concretely estimated in Section 4, it provides how small the asymptotic translation length should be in order to observe the similar phenomenon as in [LM18]. In other words, we prove the following.

**Theorem 1.5.** Suppose that there exists a universal constant $C$ so that if a non-Torelli pseudo-Anosov $f \in \text{Mod}(S_g)$ has $\ell_C(f) < C/g$ then $f$ normally generates $\text{Mod}(S_g)$ for large $g$. Then $C \leq 1152$.

**Organization.** In Section 2, we prove Theorem 1.1. Theorem 1.2 is proved in Section 3. In Section 4, explicit construction of pseudo-Anosovs realizing the asymptote for $L_C(2g-1,g)$ is provided, implying Theorem 1.4. The discussion on small asymptotic translation lengths on curve complexes and normal generation of mapping class groups is provided in Section 5.

**Acknowledgements.** The authors greatly appreciate Changsub Kim and Yair N. Minsky for helpful discussions. We truly appreciate Hee Oh for reading through the previous version of this paper and for her helpful comments. The first author was partially supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT) (No. 2020R1C1C1A01006912).

2. Proof of the main theorem

In this section, we prove Theorem 1.1.

**Lower bound.** The main idea of showing the lower bound is similar to the one used in the proof in [BS20] of that $L_C(2g,g) \geq C/g$ for some constant. First note that for any $f : S_g \to S_g$, the Lefschetz number $L(f)$ is $2 - \text{Tr}(f_*)$ where $\text{Tr}(f_*)$ is the trace of the induced map $f_* : H_1(S_g) \to H_1(S_g)$.

Let us fix a pseudo-Anosov $f : S_g \to S_g$ whose restriction onto a $k$-dimensional subspace of $H_1(S_g)$ is the identity.

Pick a suitable basis for $H_1(S_g)$, the matrix for $f_*$ can be written as

$$
\begin{pmatrix}
I_k & * \\
0 & M
\end{pmatrix}
$$
Suppose first that $k > 0$. When $k$ is odd, let $m = 2g - k$ and when $k$ is even, let $m = 2g - (k - 1)$. By taking the upper left block to be $I_{k-1}$ in case $k$ is even, one may assume $M$ is a $m \times m$ square matrix with determinant 1 and $m$ is odd (determinant 1 comes from the fact that $f_*$ is actually a symplectic matrix).

Recall that there is a relation between trace and determinant as follows:

**Lemma 2.1** ([KK92, Appendix B]). For any $m \times m$ matrix $A$,

$$(-1)^m \det A = \sum_{c_1, \ldots, c_m \geq 0, c_1 + 2c_2 + \cdots + mc_m = m} \prod_{i=1}^{m} \frac{1}{c_i!} \left( - \frac{\text{Tr}(A^i)}{i} \right)^{c_i}$$

Observe that at least one of the matrices $M, M^2, \ldots, M^m$ must have positive trace. Otherwise the right-hand side of the equality in Lemma 2.1 is always non-negative when we plug in $M$ in the place of $A$ in the lemma. On the other hand, since $\det(M) = 1$ and $m$ is always odd by our choice, the left-hand side is $-1$, a contradiction.

This implies that for some $j$ satisfying $1 \leq j \leq m \leq 2g - k + 1$, $\text{Tr}(M^j)$ is positive, i.e., at least 1 since it is an integral matrix. $\text{Tr}(f_j^*)$ is the sum of $\text{Tr}(M^j)$ and the trace of the upper left block which is $2g - m \geq 1$. Therefore, $\text{Tr}(f_j^*)$ is at least 2 in general. But in fact $2g - m \geq 3$ as long as $k \geq 3$.

Temporarily let assume $k \geq 3$. Now we have that $L(f_j^*) = 2 - \text{Tr}(f_j^*) < 0$ and we can apply a result of Tsai [Tsa09]. Then $\ell_C(f_j^*) \geq C/g$ for some constant $C > 0$ and consequently,

$$\ell_C \geq \frac{C}{gj} \geq \frac{C}{g(2g - k + 1)}.$$

For any fixed $k$, $\ell_C \geq L_C(0, g) \times 1/g^2$. Hence $\ell_C \geq \frac{C_k}{g(2g - k + 1)}$ for some constant $C_k$. Since above argument works for any $k \geq 3$, replacing $C$ by $\min\{C, C_0, C_1, C_2\}$, we obtain the lower bound in Theorem 1.1.

**Upper bound.** We now prove the upper bound provided in Theorem 1.1.

Recall that the Teichmüller space $T(S_g)$ is a space of marked hyperbolic structures on $S_g$ and vertices of the curve complex $\mathcal{C}(S_g)$ are essential simple closed curves on $S_g$. Hence, we can associate each point $x \in T(S_g)$ with systoles on $S_g$, the shortest simple closed geodesics, in the hyperbolic structure $x$. As systoles are within a uniformly bounded distance in the curve complex, it gives a coarsely well-defined map $\pi_g : T(S_g) \to \mathcal{C}(S_g)$.

Masur-Minsky studied $\pi_g : T(S_g) \to \mathcal{C}(S_g)$ in [MM99] and showed that $\pi_g$ is coarsely Lipschitz.

**Proposition 2.2** ($(K_g, D_g)$-coarsely Lipschitz, [MM99]). There exist constants $K_g, D_g > 0$ such that for any $x, y \in T(S_g)$ we have

$$d_C(\pi_g(x), \pi_g(y)) \leq K_g d_T(x, y) + D_g$$

where $d_T$ is the Teichmüller metric.
Furthermore, $\pi_g$ is coarsely $\text{Mod}(S_g)$-equivariant in the sense that there exists a constant $A_g$ such that $d_C((\pi_g \circ f)(x), (f \circ \pi_g)(x)) \leq A_g$ for any $x \in T(S_g)$ and $f \in \text{Mod}(S_g)$. Then for $f \in \text{Mod}(S_g)$, $n > 0$, and $x \in T(S_g)$, we have
\[
d_C(\pi_g(x), f^n(\pi_g(x))) \leq d_C(\pi_g(x), \pi_g(f^n(x))) + A_g \leq K_g d_T(x, f^n(x)) + D_g + A_g.
\]
Hence, we now have the comparison between asymptotic translation lengths of $f \in \text{Mod}(S_g)$ measured on $C(S_g)$ and $T(S_g)$:
\[
\ell_C(f) \leq K_g \ell_T(f)
\]
In particular, we have
\[
(2.1) \quad L_C(k, g) \leq K_g L_T(k, g).
\]
Due to the work [ALM16] of Agol-Leininger-Margalit, we already know the asymptote of $L_T(k, g)$. Hence, it remains to figure out the asymptote of $K_g$.

In [GHKL13], Gadre-Hironaka-Kent-Leininger considered the minimal possible Lipschitz constant $K_g$ which is defined as
\[
\kappa_g := \inf \{ K_g \geq 0 : \pi_g \text{ is } (K_g, D_g)-\text{coarsely Lipschitz for some } D_g > 0 \}.
\]
Then they showed that
\[
\kappa_g \asymp \frac{1}{\log g}.
\]
Combining this with [ALM16] and Inequality (2.1), we deduce the upper bound in Theorem 1.1.

3. Upper bound interpolates $L_C(0, g)$ and $L_C(2g, g)$

The upper bound provided in Theorem 1.1 does not interpolate $L_C(0, g)$ and $L_C(2g, g)$, and it is not sharp enough as one can see in Section 4. As stated in Theorem 1.2, we improve the upper bound in a weaker setting. This section is devoted to prove Theorem 1.2.

Proof of Theorem 1.2 Let $f_0$ be a pseudo-Anosov map in the Torelli group of genus $g_0 > 1$. Let $M$ be its mapping torus, $\alpha \in H^1(M)$ be the first cohomology class of $M$ corresponding to $f_0$, $\beta$ be an element in $H^1(M)$ which is restricted to a cohomology class dual to a simple closed curve $\gamma$ on $S_{g_0}$. For large enough $n > g + k$, let $f_n$ be the pseudo-Anosov monodromy corresponding to $2^n \alpha + \beta$. Then $f_n$ has the fiber of genus $O(2^n)$, and $\ell_C(f_n)$ is $O(2^{-2n})$. (cf. [BSW18])

We know that a way to construct the surface $S_n$ and map $f_n$ corresponding to $2^n \alpha + \beta$ is as follows: Let $\tilde{S}$ be the $\mathbb{Z}$-fold cover corresponding to $\beta$ restricted to $S_{g_0}$, $\tilde{f}$ a lift of $f_0$, $h$ the deck transformation, then with a suitable choice of $\tilde{f}$ we have $S_n = \tilde{S}/(h^{2^n} \tilde{f})$ and $f_n$ is lifted to $h$. Now consider a simple closed curve on a fundamental domain of $\tilde{S}$ which is not homologous to the boundary, such that the homology class $c$ represented by
this curve $\gamma$ is preserved by $\hat{f}$. The existence of such a homology class is due to the construction in Baik-Shin [BS20]. Then $\sum_{i=0}^{2^n-1} f_n^i c$ is invariant under $f_n$, and for $k < n$, let $c_k = \sum_{i=0}^{2^n-k-1} f_n^i c$. Now $\text{Span}\{c, f_n c, \ldots, f_n^{2^n-1} c\}$ is a $2^k$ dimensional invariant subspace of $f_n^{2^k}$. This proves Theorem 1.2.

Since the constant $C$ in Theorem 1.2 does not depend on the choice of given $g$ and $k$, we can apply the theorem inductively: At each $j$-th step with $g_j$ and $k_j$, Theorem 1.2 applied to $g_{j+1}$ and $k_{j+1}$ gives $g_{j+1} > g_j$, $k_{j+1} > k_j$, and a pseudo-Anosov $f_{j+1} : S_{g_j} \to S_{g_{j+1}}$ with $\ell_C(f_{j+1}) \leq C k_j'/g_j'^2$. Then we set $g_{j+1} := g_j'$ and $k_{j+1} := k_j'$. As a consequence, we obtain the following corollary that interpolates $L_C(0,g)$ [GT11] and $L_C(2g,g)$ [BS20] in a partial way.

**Corollary 3.1.** There is a constant $C$ and a diverging sequence $(k_j, g_j) \to \infty$ as $j \to \infty$ such that

$$L_C(k_j, g_j) \leq C \frac{k_j}{g_j^2}.$$  

For the construction of pseudo-Anosov maps satisfying Theorem 1.2 and Corollary 3.1, one can refer to Section 4.

4. Pseudo-Anosovs with specified invariant homology dimension

Up to the authors’ knowledge, asymptotes for $L_C(k,g)$ are known only when $k = 0$ (whole mapping class groups) and $k = 2g$ (Torelli groups). In this section, we construct pseudo-Anosovs $f_g \in \text{Mod}(S_g)$ with $m(f_g) = 2g-1$ and realizing the asymptote for $L_C(2g-1,g)$.

From the definition of $L_C(k,g)$, $L_C(k,g) \leq L_C(k',g)$ if $k \leq k'$. Since $L_C(2g,g) \asymp 1/g$ from [BS20], the lower bound in Theorem 1.1 implies that $L_C(k,g) \asymp 1/g$ if $k$ behaves like $2g$: for instance, $k \geq 2g - C$ for some constant $C > 0$. However, $L_C(0,g) \asymp 1/g^2$ by [GT11]. In this regard, we ask whether there is a sort of threshold for $k$ that $L_C(k,g)$ becomes strictly smaller than $1/g$, such as $1/g^2$.

As a potential approach for this question, we think of constructing pseudo-Anosovs of specified maximal invariant homology dimensions on surfaces of large genera with small asymptotic translation lengths. In order to get pseudo-Anosov maps on surfaces of large genera, [ALM16], [BSW18], and [BKSW19] employ a fixed hyperbolic mapping torus and consider its monodromy obtained from a fibered cone. Since the first betti number of a mapping torus of $f$ is same as $m(f)+1$, such monodromies in a fibered cone share the same $m$-value.

In contrast, we come up with a finite cyclic cover of a genus 2 surface to get the desired pseudo-Anosov maps on large genera surfaces as lifts of a fixed map. From the concrete estimation on how covering maps distort the distances on curve complexes [APT18], asymptotic translation lengths of such lifts would grow at least reciprocal of degrees. Hence we morally think
if we can construct such lifts with specified maximal invariant homology dimensions, then it would help to figure out the minimal $k = k(g)$ with $L_C(k, g) \sim 1/g$.

We start with a non-separating curve $\alpha$ on the genus 2 surface $S_2$, and take $g$ copies of $S_2 \setminus \alpha$ for $g > 1$. Gluing two different copies of $S_2 \setminus \alpha$ along one boundary component in a cyclic way, we obtain the finite cyclic cover $p_{g+1}$ of degree $g$ as in Figure 1. Let us denote the resulting cover by $S_{g+1}$ since it is of genus $g + 1$.

```
Figure 1. $g$-fold finite cyclic covering
```

This cover $p_{g+1}$ corresponds to the kernel of the composed map

$$
\pi_1(S_2) \xrightarrow{\hat{i}(\cdot, \alpha)} \mathbb{Z} \xrightarrow{\text{mod } g} \mathbb{Z}/g\mathbb{Z}
$$

where $\hat{i}(\cdot, \cdot)$ stands for the algebraic intersection number. To see this, one can observe that an element of $\pi_1(S_2)$ can be lifted to $\pi_1(S_{g+1})$ via $p_{g+1}$ if and only if its lift departs one copy of $S_2 \setminus \alpha$ and then returns to the same copy. If the lift departs and returns through the same boundary component of $S_2 \setminus \alpha$, then the element of $\pi_1(S_2)$ has the algebraic intersection number 0 with $\alpha$. Otherwise, if the lift departs and returns through the different boundary components, then the algebraic intersection number is an integer multiple of $g$.

In [BS20], the first author and Shin directly constructed pseudo-Anosovs on $S_g$ that are Torelli and of small asymptotic translation lengths on curve complexes. In the following, we construct pseudo-Anosovs with specific maximal invariant homology dimensions and satisfying the upper bound provided in Theorem 1.2 and Corollary 3.1. As a result, we obtain Theorem 1.4. Our strategy is fixing a suitable pseudo-Anosov on $S_2$ and then lift it via $p_{g+1}$. Due to the symmetry of the covering, we can find a number of invariant homology classes proportional to the degree of the cover.

Proof of Theorem 1.4. The last assertion is a direct consequence of the first assertion and Theorem 1.1. By Theorem 1.1 there exists $C' > 0$ so that $L_C(2g - 1, g) \geq \frac{C'}{g}$ for all $g > 1$. Hence, it remains to show the existence of the desired pseudo-Anosovs.

Let $\alpha$ be a non-separating curve on $S_2$ and let $\beta$ be a separating curve on $S_2$ disjoint from $\alpha$. Then each lift of $\beta$ through $p_{g+1}$ is also separating. For instance, see Figure 2.
Figure 2. A non-separating curve $\alpha$ and a separating curve $\beta$ on $S_2$ with $\alpha \cap \beta = \emptyset$

Now let $\varphi \in \text{Mod}(S_2)$ be a Torelli pseudo-Anosov mapping class on $S_2$. Note that $\lim_{n \to \infty} d_C(\beta, \varphi^n \beta)/n = \ell_C(\varphi) > 0$ by [MM99]. Then taking powers if necessary, we may assume that $d_C(\beta, \varphi \beta) \geq 3$. It means that $\beta$ and $\varphi \beta$ are separating simple closed curves on $S_2$ that fill the surface. Such $\varphi$ can be obtained in a constructive way. Indeed, we can take any pseudo-Anosov $\psi \in \text{Mod}(S_2)$ with $d_C(\beta, \psi \beta) \geq 3$. Then $\varphi$ can be chosen, for instance, as a sufficient power of $T_\beta T_\psi^{-1} \varphi$ which is pseudo-Anosov by Thurston [Thu88] or Penner [Pen88].

Now let $f = T_\beta T_\psi^{-1} T_\varphi^{-1}$. Since $\beta$, $\varphi \beta$, and $\varphi \alpha$ fill the surface and $\varphi \beta \cap \varphi \alpha = \emptyset$, $f$ is pseudo-Anosov again by Thurston [Thu88] and Penner [Pen88]. Since $\beta$ and $\varphi \beta$ are separating, $T_\beta$ and $T_\psi^{-1}$ are Torelli, in particular, they preserve the homology class $[\alpha]$ of $\alpha$. Furthermore, since $\varphi$ is Torelli, $[\varphi \alpha] = [\alpha]$, which implies that $T_\varphi^{-1}$ also preserves $[\alpha]$. Hence, $f$ preserves $[\alpha]$, and thus $\hat{i}(f(\cdot),\alpha) = \hat{i}(\cdot,\alpha)$. In particular, $f$ preserves the kernel of $\pi_1(S_2) \xrightarrow{\hat{i}(\cdot,\cdot)} \mathbb{Z} \xrightarrow{\text{mod } g} \mathbb{Z}/g\mathbb{Z}$. Consequently, $f$ can be lifted through $p_{g+1}$.

Let $\tilde{f} = T_{p_{g+1}\beta} T_{p_{g+1}\varphi}^{-1} T_{p_{g+1}\psi}^{-1}$ be a lift of $f$ via $p_{g+1}$. We now estimate $\ell_C(\tilde{f})$. Recall the construction of $p_{g+1}$: Take $g$ copies $X_1, \ldots, X_g$ of $S_2 \setminus \alpha$ and glue $X_i$ and $X_{i+1}$ along one of their boundary components. Throughout, we write each index $i$ modulo $g$. Let $\tilde{\alpha} = \partial X_0 \cap \partial X_1$. That is, $\tilde{\alpha}$ is a boundary component of $X_0$ and $X_1$ where they are glued. Due to the construction, $\tilde{\alpha}$ is a lift of $\alpha$.

Noting that $\hat{i}(\varphi \alpha,\alpha) = 0$ since $\varphi$ is Torelli, we get

$$T_{p_{g+1}\varphi}^{-1} \tilde{\alpha} \subseteq \bigcup_{j=-i(\varphi \alpha,\alpha)/2}^{i(\varphi \alpha,\alpha)/2} X_j$$

where $i(\cdot,\cdot)$ is the geometric intersection number. (cf. Figure 3) Similarly, $\hat{i}(\varphi \beta,\alpha) = 0$ and

$$T_{p_{g+1}\varphi}^{-1} T_{p_{g+1}\psi}^{-1} \tilde{\alpha} \subseteq \bigcup_{j=-i(\varphi \beta,\alpha)+i(\varphi \alpha,\alpha)}^{i(\varphi \beta,\alpha)+i(\varphi \alpha,\alpha)} X_j.$$
Since $T_{p_{g+1}(\beta)}$ fixes each $X_j$, we have
\[ \tilde{f}\tilde{\alpha} \subseteq \bigcup_{j=-(\varphi\beta,\alpha)+i(\varphi\alpha,\alpha)}^{(\varphi\beta,\alpha)+i(\varphi\alpha,\alpha)} X_j. \]

Conducting this procedure inductively, we finally get
\[ \tilde{f}^n\tilde{\alpha} \subseteq \bigcup_{j=-(\varphi\beta,\alpha)+i(\varphi\alpha,\alpha)}^{n(\varphi\beta,\alpha)+i(\varphi\alpha,\alpha)} X_j. \]

Hence, there exists $\tilde{j}$ such that $\tilde{f}^{\frac{g-2}{2}}(\varphi\beta,\alpha) + i(\varphi\alpha,\alpha) \tilde{\alpha} \cap X_{\tilde{j}} = \emptyset$. Since there exists an essential simple closed curve in $X_{\tilde{j}}$ which is a 2-holed torus, we have $d_{C}(\tilde{\alpha}, \tilde{f}^{\frac{g-2}{2}}(\varphi\beta,\alpha) + i(\varphi\alpha,\alpha) \tilde{\alpha}) \leq 2$ so $\ell_{C}(\tilde{f}^{\frac{g-2}{2}}(\varphi\beta,\alpha) + i(\varphi\alpha,\alpha) \tilde{\alpha}) \leq 2$. It concludes the following estimation. Note that $\varphi$, $i(\varphi\beta,\alpha)$, and $i(\varphi\alpha,\alpha)$ are universal quantities independent on $p_{g+1}$ and $g$.

(4.1) \[ \ell_{C}(\tilde{f}) \leq \frac{2}{\frac{g-2}{2} + i(\varphi\beta,\alpha) + i(\varphi\alpha,\alpha)} \]

We now show that $m(\tilde{f}) = 2g + 1$. Recall that $\beta \subseteq S_2$ is a separating curve and $\alpha \subseteq S_2$ is a non-separating curve disjoint from $\beta$. Temporarily let us denote $Y$ be a component of $S_2 \setminus \beta$ that does not contain $\alpha$. Let $\gamma$ and $\delta$ be non-separating simple closed curves whose homology classes form a basis for $H_1(Y) \cong \mathbb{R}^2$. Let us also denote $\eta$ be a non-separating curve on $S_2 \setminus \beta$ with $i(\eta,\alpha) = 1$. Then $p_{g+1}^{-1}(\gamma)$, $p_{g+1}^{-1}(\delta)$, $p_{g+1}^{-1}(\eta)$, and one component of $p_{g+1}^{-1}(\alpha)$ form a basis for $H_1(S_{g+1}) \cong \mathbb{R}^{2g+2}$. See Figure 3 for instance.

Let $\tilde{\gamma}_j = p_{g+1}^{-1}(\gamma) \cap X_j$, $\tilde{\delta}_j = p_{g+1}^{-1}(\delta) \cap X_j$, and $\tilde{\eta} = p_{g+1}^{-1}(\eta)$. Further, let $\tilde{\alpha} = \partial X_0 \cap \partial X_1$ which is a component of $p_{g+1}^{-1}(\alpha)$. Since $\varphi$ is Torelli, it has a lift $\tilde{\varphi}$ through $p_{g+1}$. Hence, homology classes $\{[\tilde{\varphi}\tilde{\gamma}_j], [\tilde{\varphi}\tilde{\delta}_j], [\tilde{\varphi}\tilde{\eta}], [\tilde{\varphi}\tilde{\alpha}]\}$ also form a basis for $H_1(S_{g+1})$.\[ \]
Recall that $\tilde{f} = T^{-1}_{p_g+1}(β) \cdot T^{-1}_{p_g+1}(φβ) \cdot T^{-1}_{p_g+1}(φα)$.

Since $γ \cap (α \cup β) = \emptyset$, we have $\tilde{φ}_β \cap (p_g^{-1}(φα)) \cap p_g^{-1}(φβ) = \emptyset$. Here, note that $\tilde{φ}_β$ is a lift of $φγ$ which is a component of $p_g^{-1}(φγ)$. Hence it follows that $\tilde{f}\tilde{φ}_β = T^{-1}_{p_g+1}(β) \cdot \tilde{φ}_β$.

Since each component of $p_g^{-1}(β)$, which is a lift of $β$, is separating, $T^{-1}_{p_g+1}(β)$ is Torelli. As a result, $\tilde{f} \tilde{φ}_β = [\tilde{φ}_β](i)$. Similarly, we have $\tilde{f} \tilde{φ}_β = [\tilde{φ}_β](i)$.

Now we consider $\tilde{f} \tilde{φ}_β$. As $\tilde{φ}_β$ is a lift of $φγ$, $T^{-1}_{p_g+1}(φα) \tilde{φ}_β = \tilde{φ}_α$. Furthermore, since $α \cap β = \emptyset$, $\tilde{φ}_α$, a lift of $φα$, does not intersect $p_g^{-1}(φβ)$. It implies that $T^{-1}_{p_g+1}(φα) \tilde{φ}_β = \tilde{φ}_α$. Finally, since $T^{-1}_{p_g+1}(β)$ is Torelli again, we conclude $\tilde{f} \tilde{φ}_β = [\tilde{φ}_β]$.

So far, we have proved $m(\tilde{f}) ≥ 2g + 1$. Suppose to the contrary that $m(\tilde{f}) = 2g + 2$, which means that $\tilde{f}$ is Torelli. Then $\tilde{f} \tilde{φ}_β$ should be homologous to $\tilde{φ}_α$. It implies $T^{-1}_{p_g+1}(φα) \tilde{φ}_β = [\tilde{φ}_β]$ since $T^{-1}_{p_g+1}(β)$ is Torelli.

As any two components of $p_g^{-1}(φα)$ bound a subsurface, they are homologous. In particular, as $\tilde{φ}_α$ is a component of $p_g^{-1}(φα)$, each of its component is homologous to $\tilde{φ}_α$. Hence, $T^{-1}_{p_g+1}(φβ) \tilde{φ}_β = [\tilde{φ}_β]$. Noting that $T^{-1}_{p_g+1}(φβ)$ can be isotoped into arbitrary neighborhood of $\tilde{φ}_γ \cup \tilde{φ}_β$, $T^{-1}_{p_g+1}(φβ)$ can also be isotoped into arbitrary neighborhood of $\tilde{φ}_γ \cup \tilde{φ}_β$. Since $\tilde{φ}_γ \cup \tilde{φ}_β$ and $p_g^{-1}(φβ)$ are disjoint compact sets, we have $T^{-1}_{p_g+1}(φβ) \tilde{φ}_β = T^{-1}_{p_g+1}(φβ) \tilde{φ}_β = T^{-1}_{p_g+1}(φβ) \tilde{φ}_β$.

Summing up the above argument, we obtain

$$[\tilde{φ}_β] = [\tilde{φ}_β] = [\tilde{φ}_β] = [\tilde{φ}_β] = [\tilde{φ}_β]$$

where the first equality is the assumption. However,

$$[T^{-1}_{p_g+1}(φβ) \tilde{φ}_β] = [\tilde{φ}_β]$$
which implies that $i(\hat{\alpha}, \hat{\alpha}) = 0$. It contradicts our choice of $\eta$ that $i(\hat{\eta}, \hat{\alpha}) = 1$. Therefore, $m(\hat{f}) = 2g + 1$. Setting $f_{g+1} = \hat{f}$ completes the proof of Theorem 1.4.

The lower bound on $\ell_C(f_g)$ for $f_g$ constructed in the proof can also be calculated in a concrete way by Aougab-Patel-Taylor [APT18] as follows.

$$\frac{\ell_C(f)}{(g-1) \cdot 13 \cdot 80 \cdot 2^{13} e^{34} \pi} \leq \ell_C(f_g)$$

**Remark 4.1.** In the proof, all figures describe one specific example. Any choice of $\alpha$, $\beta$, $\gamma$, $\delta$, and $\eta$ works if it satisfies the condition we provide. Furthermore, if we modify the map on $S_2$ to be $f = T_\beta T_\gamma^{-1}$, then its lift via $p_{g+1}$ is Torelli, which gives another proof of $L_C(2g, g) \approx 1/g$.

5. SMALL TRANSLATION LENGTH AND NORMAL GENERATION

In this section, we discuss pseudo-Anosov mapping classes with small asymptotic translation lengths and normal generation of mapping class groups. For a general group $G$ and $g \in G$, the normal closure $\langle\langle g \rangle\rangle$ of $g$ is the smallest normal subgroup of $G$ containing $g$. The normal closure can be described in various way:

$$\langle\langle g \rangle\rangle = \bigcap_{g \in N \trianglelefteq G} N = \langle hgh^{-1} : h \in G \rangle$$

In a particular case that $\langle\langle g \rangle\rangle = G$, we say $g$ normally generates $G$, and $g$ is said to be a normal generator of $G$.

Normal generators of mapping class groups of surfaces have been studied by various authors. In [Lon86], Long asked whether there is a pseudo-Anosov that normally generates a mapping class group. This question recently answered affirmatively by Lanier and Margalit in [LM18]. Their strategy is to show that there is a universal constant so that pseudo-Anosovs with stretch factors less than the constant should be normal generators. Then the asymptote $L_T(0, g) \sim 1/g$ by Penner [Pen91] deduces the answer. Precisely, Lanier and Margalit proved the following.

**Theorem 5.1** (Lanier-Margalit [LM18]). If a pseudo-Anosov $\varphi \in \text{Mod}(S_g)$ has the stretch factor less than $\sqrt{2}$, then $\varphi$ normally generates $\text{Mod}(S_g)$.

As the logarithm of stretch factor of a pseudo-Anosov equals to the translation length of the pseudo-Anosov on the Teichmüller space, Lanier and Margalit’s result also means that the small translation length on the Teichmüller space implies the normal generation of the mapping class group. One natural question in this philosophy is whether the same holds in the circumstance of curve complexes. There are several ways to formalize this question:

1. Is there a universal constant $C > 0$ so that if a pseudo-Anosov $\varphi \in \text{Mod}(S_g)$ has $\ell_C(\varphi) < C/g$ then $\langle\langle \varphi \rangle\rangle = \text{Mod}(S_g)$?
Hence, once we fix $\alpha$ configuration as in Figure 5. In the above questions, the factor $1/g$ is necessary since $L_C(2g, g) \propto 1/g$ and due to Proposition 1.3. Furthermore, we separately state above two questions in order to forbid the trivial (Torelli) case in (2) and deal with the same problem.

Remark 5.2. In the above questions, the factor $1/g$ is necessary since $L_C(2g, g) \propto 1/g$ and due to Proposition 1.3. Furthermore, we separately state above two questions in order to forbid the trivial (Torelli) case in (2) and deal with the same problem.

Proof of Theorem 1.5. The family of pseudo-Anosovs constructed in Theorem 1.4 actually consists of non-normal generators, that is, $\langle\langle f_g \rangle\rangle \neq \text{Mod}(S_g)$. To see this, recall that $f_g = T_{p_g^{-1}(\beta)} T_{p_g^{-1}(\varphi\beta)} T_{p_g^{-1}(\varphi\alpha)}$. It can be rewritten as

$$f_g = T_{p_g^{-1}(\beta)} \left( \tilde{\varphi} T_{p_g^{-1}(\beta)}^{-1} \tilde{T}^{-1} \right) \left( \varphi T_{p_g^{-1}(\alpha)}^{-1} \tilde{T}^{-1} \right).$$

Hence, it follows that $\langle\langle f_g \rangle\rangle \leq \langle\langle T_{p_g^{-1}(\beta)}, T_{p_g^{-1}(\alpha)} \rangle\rangle$ where the right-hand-side means the smallest normal subgroup containing $T_{p_g^{-1}(\beta)}$ and $T_{p_g^{-1}(\alpha)}$.

Since each component of $p_g^{-1}(\beta)$ is separating, $T_{p_g^{-1}(\beta)}$ is Torelli, namely, contained in the kernel of the symplectic representation $\text{Mod}(S_g) \to \text{Sp}(2g, \mathbb{Z})$. Moreover, any two components of $p_g^{-1}(\alpha)$ bound an essential subsurface so they are homologous, which means that $T_{p_g^{-1}(\alpha)}$ acts similarly as $T_{p_g^{-1}(\beta)}$ on $H_1(S_g; \mathbb{Z})$. As such, $T_{p_g^{-1}(\alpha)}$ acts trivially on the mod $(g - 1)$ homology $H_1(S_g, \mathbb{Z} / (g - 1) \mathbb{Z})$. Hence, we have that the symplectic representation of $T_{p_g^{-1}(\alpha)}$ is contained in the kernel of $\text{Sp}(2g, \mathbb{Z}) \to \text{Sp}(2g, \mathbb{Z} / (g - 1) \mathbb{Z})$. Consequently, the normal closure $\langle\langle T_{p_g^{-1}(\beta)}, T_{p_g^{-1}(\alpha)} \rangle\rangle$ is contained in the kernel of the composition

$$\text{Mod}(S_g) \to \text{Sp}(2g, \mathbb{Z}) \to \text{Sp}(2g, \mathbb{Z} / (g - 1) \mathbb{Z})$$

which is surjective. It concludes that

$$\langle\langle f_g \rangle\rangle \leq \langle\langle T_{p_g^{-1}(\beta)}, T_{p_g^{-1}(\alpha)} \rangle\rangle \neq \text{Mod}(S_g)$$

so $f_g$ is not a normal generator as desired.

Note that we have a concrete upper bound for $\ell_C(f_g)$ in (4.1):

$$\ell_C(f_g) \leq \frac{2}{g - \frac{g - 3}{i(\varphi\beta, \alpha) + i(\varphi\alpha, \alpha)}} \leq \frac{2(i(\varphi\beta, \alpha) + i(\varphi\alpha, \alpha))}{g - 3 - (i(\varphi\beta, \alpha) + i(\varphi\alpha, \alpha))}$$

Hence, once we fix $\alpha$, $\beta$, and $\varphi$, we get a quantitative restriction on the constant $C$ in the above questions. For instance, we can consider the configuration as in Figure 5.

Let $\lambda = T_\xi \tilde{T}$. As $\beta$ and $\xi$ fill the surface $S_2$, $\beta$ and $\lambda = T_\xi \tilde{T}$ also fill the surface. Since $\beta$ is separating, $\lambda = T_\xi \tilde{T}$ is also separating. Hence, due to Penner or Thurston, $\varphi = T_\lambda T_\beta^{-1}$ is a Torelli pseudo-Anosov.
Furthermore, it follows that $\beta$ and $\varphi\beta$ also fill the surface. Therefore, we can construct $f_g$ as in Theorem 1.4 starting with $\alpha$, $\beta$, and $\varphi$ depicted above.

Since $i(\xi, \beta) = 6$, $i(\lambda, \beta) = i(T\xi\beta, \beta) = i(\xi, \beta)^2 = 36$ by [FM11, Proposition 3.2]. Now from $\varphi\alpha = T\lambda\alpha$ and $\varphi\beta = T\lambda\beta$, we have:

\[
i(\varphi\alpha, \alpha) = i(T\lambda\alpha, \alpha) = i(\lambda, \alpha)^2 = 144
\]

\[
i(\varphi\beta, \alpha) = i(T\lambda\beta, \alpha) = i(\lambda, \beta)i(\lambda, \alpha) = 432
\]

Hence, for the resulting $f_g$,

\[
\ell_C(f_g) \leq \frac{1152}{g - 579}
\]

for $g > 579$. Consequently, we conclude Theorem 1.5.

References

[ALM16] Ian Agol, Christopher J Leininger, and Dan Margalit. Pseudo-anosov stretch factors and homology of mapping tori. Journal of the London Mathematical Society, 93(3):664–682, 2016.

[APT18] Tarik Aougab, Priyam Patel, and Samuel J Taylor. Covers of surfaces, kleinian groups, and the curve complex. arXiv preprint arXiv:1810.12953, 2018.

[B+78] Lipman Bers et al. An extremal problem for quasiconformal mappings and a theorem by thurston. Acta Mathematica, 141:73–98, 1978.

[BKSW19] Hyungryul Baik, Eiko Kin, Hyunshik Shin, and Chenxi Wu. Asymptotic translation lengths and normal generations of pseudo-anosov monodromies for fibered 3-manifolds. arXiv preprint arXiv:1909.00974, 2019.

[BS20] Hyungryul Baik and Hyunshik Shin. Minimal asymptotic translation lengths of torelli groups and pure braid groups on the curve graph. International Mathematics Research Notices, 2020(24):9974–9987, 2020.

[BSW18] Hyungryul Baik, Hyunshik Shin, and Chenxi Wu. An upper bound on the asymptotic translation lengths on the curve graph and fibered faces. to appear in Indiana University Mathematics Journal, arXiv preprint arXiv:1801.06638, 2018.

[Ell] Jordan Ellenberg. Pseudo-anosov puzzle 2: homology rank and dilatation. Quomodocumque, http://quomodocumque.wordpress.com/2010/04/26, April 26, 2010.

[FLM08] Benson Farb, Christopher J Leininger, and Dan Margalit. The lower central series and pseudo-anosov dilatations. American journal of mathematics, 130(3):799–827, 2008.

[FLP12] Albert Fathi, François Laudenbach, and Valentin Poéna. Thurston’s Work on Surfaces (MN-48), volume 48. Princeton University Press, 2012.

[FM11] Benson Farb and Dan Margalit. A primer on mapping class groups (pms-49). Princeton university press, 2011.
[GHKL13] V. Gadre, E. Hironaka, R. P. Kent, IV, and C. J. Leininger. Lipschitz constants to curve complexes. *Math. Res. Lett.*, 20(4):647–656, 2013.

[GT11] Vaibhav Gadre and Chia-Yen Tsai. Minimal pseudo-Anosov translation lengths on the complex of curves. *Geom. Topol.*, 15(3):1297–1312, 2011.

[KK92] LA Kondratyuk and MI Krivoruchenko. Superconducting quark matter in su(2) colour group. *Zeitschrift für Physik A Hadrons and Nuclei*, 344(1):99–115, 1992.

[LM18] Justin Lanier and Dan Margalit. Normal generators for mapping class groups are abundant. *arXiv preprint arXiv:1805.03666*, 2018.

[Lon86] DD Long. A note on the normal subgroups of mapping class groups. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 99, pages 79–87. Cambridge University Press, 1986.

[MM99] Howard A. Masur and Yair N. Minsky. Geometry of the complex of curves I: Hyperbolicity. *Invent. Math.*, 138(1):103–149, 1999.

[Pen88] Robert C Penner. A construction of pseudo-Anosov homeomorphisms. *Transactions of the American Mathematical Society*, 310(1):179–197, 1988.

[Pen91] Robert C Penner. Bounds on least dilatations. *Proceedings of the American Mathematical Society*, 113(2):443–450, 1991.

[Thu88] William P Thurston. On the geometry and dynamics of diffeomorphisms of surfaces. *Bulletin (new series) of the American Mathematical Society*, 19(2):417–431, 1988.

[Thu98] William P Thurston. Hyperbolic structures on 3-manifolds, ii: Surface groups and 3-manifolds which fiber over the circle. *arXiv preprint math/9801045*, 1998.

[Tsa09] Chia-Yen Tsai. The asymptotic behavior of least pseudo-Anosov dilatations. *Geom. Topol.*, 13(4):2253–2278, 2009.

Department of Mathematical Sciences, KAIST, 291 Daehak-ro Yuseong-gu, Daejeon, 34141, South Korea

Email address: hrbaik@kaist.ac.kr

Department of Mathematics, Yale University, 10 Hillhouse Avenue, New Haven, CT 06511, USA

Email address: dongryul.kim@yale.edu

Department of Mathematics, University of Wisconsin–Madison, 480 Lincoln Drive, Madison, WI 53706, USA

Email address: cwu367@math.wisc.edu