$L^p$-$L^q$ ESTIMATES FOR THE DAMPED WAVE EQUATION AND THE CRITICAL EXponent FOR THE NONLINEar PROBLEM WITH SLOWLY DECAYING DATA

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Abstract. We study the Cauchy problem of the damped wave equation
\[ \partial_t^2 u - \Delta u + \partial_t u = 0 \]
and give sharp $L^p$-$L^q$ estimates of the solution for $1 \leq q \leq p < \infty$ ($p \neq 1$) with derivative loss. This is an improvement of the so-called Matsumura estimates. Moreover, as its application, we consider the nonlinear problem with initial data in $(H^s \cap H^\beta_r) \times (H^{s-1} \cap L^r)$ with $r \in (1, 2]$, $s \geq 0$, and $\beta = (n-1)|\frac{1}{2} - \frac{1}{r}|$, and prove the local and global existence of solutions. In particular, we prove the existence of the global solution with small initial data for the critical nonlinearity with the power $1 + \frac{2}{n}$, while it is known that the critical power $1 + \frac{2}{n}$ belongs to the blow-up region when $r = 1$. We also discuss the asymptotic behavior of the global solution in supercritical cases. Moreover, we present blow-up results in subcritical cases. We give estimates of lifespan by an ODE argument.

1. Introduction. The damped wave equation
\[ \partial_t^2 u - \Delta u + \partial_t u = 0 \]
is known as a model describing the wave propagation with friction, and studied for long years. In particular, for the Cauchy problem
\[
\begin{cases}
\partial_t^2 u - \Delta u + \partial_t u = 0, \\
u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x), \\
(t, x) \in (0, \infty) \times \mathbb{R}^n,
\end{cases}
\]
(1)

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where \((u_0, u_1)\) is a given function, the asymptotic behavior of the solution has been investigated by many mathematicians after the pioneering work by Matsumura [24]. Matsumura [24] applied the Fourier transform to \((1)\) and obtained the formula

\[
 u(t, x) = D(t)(u_0 + u_1) + \partial_t D(t)u_0,
\]

where \(D(t)\) is defined by

\[
 D(t) := e^{-\frac{t}{2} F^{-1} L(t, \xi) F}
\]

with

\[
 L(t, \xi) := \begin{cases} 
 \sinh(t \sqrt{1/4 - |\xi|^2}) & (|\xi| < 1/2), \\
 \frac{1}{2} \sin(t \sqrt{|\xi|^2 - 1/4}) & (|\xi| > 1/2). 
\end{cases}
\]  

(2)

Using the above formula, he proved the so-called Matsumura estimates \((L^p-L^q\) estimates)\)

\[
 \| u(t) \|_{L^p} \lesssim \langle t \rangle^{-\frac{n}{2} \left(1 - \frac{1}{q}\right)} (\| u_0 \|_{L^\infty} + \| u_1 \|_{L^\infty}) + e^{-t/4} \left( \| u_0 \|_{H_{\beta}^{1/2} + 1} + \| u_1 \|_{H_{\beta}^{1/2}} \right),
\]

(3)

where \(1 \leq q \leq 2 \leq p \leq \infty\), \(\langle t \rangle := (1 + |t|^2)^{1/2}\), \([n/2]\) denotes the integer part of \(n/2\), and the notation \(f \lesssim g\) stands for \(f \leq Cg\) with some constant \(C > 0\). The first and second terms on the right-hand side are corresponding to the low and the high frequency parts of the solution, respectively. The estimates \((3)\) indicate that the low frequency part of the solution behaves like that of the heat equation

\[
 \partial_t v - \Delta v = 0.
\]

(4)

Here, we recall the well-known \(L^p-L^q\) estimates for the heat equation

\[
 \| G(t)g \|_{L^p} \lesssim \langle t \rangle^{-\frac{n}{2} \left(1 - \frac{1}{q}\right)} \| g \|_{L^\infty},
\]

(5)

where \(1 \leq q \leq p \leq \infty\), \(g \in L^2(\mathbb{R}^n)\), and

\[
 G(t) := F^{-1} e^{-t|\xi|^2} F.
\]

Namely, \(G(t)g\) is the solution of \((4)\) with \(v(0) = g\) (see [7]). Also, we see from \((3)\) that the high frequency part causes derivative losses like the wave equation

\[
 \partial_t^2 w - \Delta w = 0.
\]

(6)

We also recall the estimates for the wave equation:

\[
 \| W(t)g \|_{L^p} \lesssim \langle t \rangle^{\delta_p'} \| g \|_{H_{\beta}^{\beta'-1}}
\]

(7)

with some constant \(\delta_p' \geq 0\), where \(1 < p < \infty\), \(g \in H_{\beta}^{\beta-1}(\mathbb{R}^n)\), \(\beta = (n-1)(\frac{1}{2} - \frac{1}{p})\), and

\[
 W(t) := F^{-1} \frac{\sin(t|\xi|)}{|\xi|} F,
\]

namely, \(W(t)g\) is the solution of \((6)\) with \((w, \partial_t w)(0) = (0, g)\) (see [33, 25]). Here, we set \(H_{\beta}^{\beta}(\mathbb{R}^n) := \{ f \in S'(\mathbb{R}^n) : \| f \|_{H_{\beta}^{\beta}} = \| (\nabla)^s f \|_{L^p} < \infty \}\). However, in contrast to the estimates \((5)\) and \((7)\), the Matsumura-type estimate \((3)\) requires the restriction \(q \leq 2 \leq p\), and the derivative losses of the high frequency part seems not sharp. Therefore, we expect that the estimate \((3)\) can be improved.

Indeed, in the following we give an improvement of the Matsumura-type estimate \((3)\). Let \(\chi_{<c}(\nabla)\) and \(\chi_{>c}(\nabla)\) be the cut-off Fourier multipliers defined by \((18)\) for low and high frequency, respectively. Our first result reads as follows.
**Theorem 1.1** \((L^p-L^q\) estimates). Let \(1 \leq q \leq p < \infty\) with \(p \neq 1\), \(s_1 \geq s_2\), and \(\beta = (n-1)|\frac{1}{2} - \frac{1}{p}|\). Then, there exists \(\delta_p > 0\) such that
\[
\|\nabla^{s_1}(D(t)g)\|_{L^p} \lesssim \langle t \rangle^{-\frac{n}{2} - \frac{1}{2} - \frac{\beta}{2}} \|\nabla^{s_2} \chi \leq 1(\nabla)g\|_{L^q} + e^{-\frac{\delta_p}{2}} \|\nabla^{s_1} \chi > 1(\nabla)g\|_{H^{-\beta}_{p}},
\]
for \(t > 0\), provided that the right-hand side is finite.

We will prove this theorem in the next section. The main ideas of the proof are the following. To remove the restriction of the exponents \(1 \leq q \leq 2 \leq p \leq \infty\), we derive a pointwise estimate for the convolution kernel for the low frequency part. Also, to make the derivative losses sharp, we apply the estimate (7) to the high frequency part.

**Remark 1.1.** Chen, Fan and Zhang \([2, 3]\) stated similar \(L^p-L^q\) estimates for the damped fractional wave equation. However, unfortunately, the proof seems incomplete. For the damped wave equation, we give a complete proof. Moreover, our argument remains valid for the damped fractional wave equation, through minor modifications.

In the proof of Theorem 1.1, we explicitly write a leading part of the convolution kernel of \(D(t)g\) in the low frequency part, which has the same coefficient as that of the heat kernel (see Lemmas 2.3 and 2.8 below). Accordingly, the difference in the low frequency part satisfies a better time-decay estimate.

**Theorem 1.2.** Let \(1 \leq q \leq p < \infty\) with \(p \neq 1\), \(s_1 \geq s_2\), and \(\beta = (n-1)|\frac{1}{2} - \frac{1}{p}|\). Then, there exists \(\delta_p > 0\) such that
\[
\|\nabla^{s_1}(D(t) - G(t))g\|_{L^p} \lesssim \langle t \rangle^{-\frac{n}{2} + \frac{1}{2} - \frac{\beta}{2} - \frac{1}{2}} \|\nabla^{s_2} \chi \leq 1(\nabla)g\|_{L^q} + e^{-\frac{\delta_p}{2}} \|\nabla^{s_1} \chi > 1(\nabla)g\|_{H^{-\beta}_{p}},
\]
for \(t \geq 1\), provided that the right-hand side is finite.

**Remark 1.2.** Nishihara \([30]\) gives an improvement of the estimate of (3) in the 3-dimensional case of the form
\[
\|D(t)f - G(t)f - e^{-t/2}W(t)f\|_{L^p} \lesssim t^{-\frac{n}{2} - \frac{1}{2} - \frac{\beta}{2} - 1} \|f\|_{L^p},
\]
where \(1 \leq q \leq p \leq \infty\) (for other space dimensions, see \([23, 13, 27, 32]\)). In other words, \(D(t)f\) is asymptotically expressed as
\[
D(t)f \sim G(t)f + e^{-t/2}W(t)f \quad (t \to \infty),
\]
and it implies the high frequency part causing the derivative loss is explicitly given by \(W(t)f\) when \(n = 3\). Therefore, combining (11) and (7), we can obtain similar estimates as (8), (10). However, our approach is direct and has broad utility.

Our next purpose is the application of Theorems 1.1 and 1.2 to the Cauchy problem of the nonlinear damped wave equation
\[
\begin{align*}
\partial_t^2 u - \Delta u + \partial_t u &= N(u), \\
u(0, x) = \varepsilon u_0(x), \quad \partial_t u(0, x) &= \varepsilon u_1(x), \quad x \in \mathbb{R}^n,
\end{align*}
\]
where $N(u)$ denotes the nonlinearity, $(u_0, u_1)$ is a given function, which denotes the shape of the initial data, and $\varepsilon$ is a positive parameter, which denotes the size of the initial data.

Our concern is to prove the local and global existence of the solution, asymptotic behavior, and blow-up of solutions when the initial data do not belong to $L^1(\mathbb{R}^n)$ in general. More precisely, we show the existence of the global solution with small data even for the critical nonlinearity.

Based on the linear estimates (3), many mathematicians studied the global existence and the blow-up of solutions (see [9, 10, 11, 12, 13, 15, 16, 18, 19, 20, 21, 22, 23, 24, 27, 30, 34, 36, 37] and the references therein). In particular, from these studies, the critical exponent was determined as $p_c(n) = 1 + \frac{2}{n}$, provided that the initial data decay sufficiently fast at the spatial infinity. Here, the critical exponent means the threshold of the global existence and the blow-up of solutions for small initial data. More precisely, if $p$ is larger than the critical exponent $p_c$, then for any shape $(u_0, u_1)$, there exists $\varepsilon_0 > 0$ such that the solution exists globally in time for any $\varepsilon \in (0, \varepsilon_0)$, and if $p$ is smaller than the critical exponent $p_c$, then there exists a shape $(u_0, u_1)$ and $\varepsilon_0 > 0$ such that the solution blows up in finite time for any $\varepsilon \in (0, \varepsilon_0)$.

However, there are only few results when the initial data slowly decay, namely, do not belong to $L^1(\mathbb{R}^n)$, at the spatial infinity. Nakao and Ono [26] studied the case $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and they proved the global well-posedness with small data when $p \geq 1 + \frac{2}{n}$, We also refer the reader to [28] for the global existence of solutions with slowly decaying initial data in modulation spaces, but the nonlinearity should be a polynomial of $u$. Ikehata and Ohta [17] proved the global existence of solutions for small data $(u_0, u_1) \in (H^1(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)) \times (L^2(\mathbb{R}^n) \cap L^r(\mathbb{R}^n))$ when $p > 1 + \frac{2}{n}$. Here

$$r \in [1, 2] \quad (n = 1, 2),$$
$$r \in \left[ \sqrt{n^2 + 16n - n}, \min \left\{ 2, \frac{n}{n - 2} \right\} \right] \quad (3 \leq n \leq 6).$$

They also proved for any $n \geq 1$ and for the nonlinearity $N(u) = |u|^{p-1}u$ with $1 < p < 1 + \frac{2}{n}$, there exists $(u_0, u_1) \in (H^1(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)) \times (L^2(\mathbb{R}^n) \cap L^r(\mathbb{R}^n))$ such that there is no global solution even if the size of the initial data $\varepsilon$ is arbitrary small. Here we shall give a remark on their results. In the supercritical case $p > 1 + \frac{2}{n}$, their solution belongs only to $C([0, \infty); H^1(\mathbb{R}^n)) \cap C^1([0, \infty); L^2(\mathbb{R}^n))$ and we do not know whether $u(t) \in L^r(\mathbb{R}^n)$. It is a natural question whether the solution $u$ has the same integrability near the spatial infinity as the initial data. Narazaki and Nishihara [29] further considered the asymptotic profile of the solution under the assumption $(u_0, u_1) \sim \langle x \rangle^{-kn}$ with $k \in (0, 1]$. They proved that when $n \leq 3$ and $p > 1 + \frac{2}{n}$ (which corresponds to the condition $p > 1 + \frac{2}{n}$ in terms of the Lebesgue space $L^r(\mathbb{R}^n)$), the small data global existence holds and the solution is approximated by $\varepsilon G(t)(u_0 + u_1)$. Moreover, in [14], we extended the above results to higher dimensional cases in terms of the weighted Sobolev spaces

$$H^{s, \alpha}(\mathbb{R}^n) = \langle x \rangle^{-\alpha} H^s(\mathbb{R}^n) := \{ f \in S'(\mathbb{R}^n); \| (x)^\alpha \langle \nabla \rangle^s f \|_{L^2} < \infty \},$$

where the symbol $\langle \nabla \rangle^s$ stands for the Fourier multiplier $\mathcal{F}^{-1} \left[ \xi^s \hat{f}(\xi) \right] (x)$. We showed that if $p > 1 + \frac{2}{n}$ and if the initial data satisfy $(u_0, u_1) \in (H^{s,0} \cap H^{0,0})(\mathbb{R}^n) \times (H^{s-1,0} \cap H^{0,0})(\mathbb{R}^n)$ with $\alpha > n(\frac{1}{2} - \frac{1}{p})$ and sufficiently small, then the global
solution uniquely exists. However, in this setting, we cannot treat the critical case $p = 1 + \frac{2n}{n-1}$.

In the present paper, based on the improved $L^p$-$L^q$ estimates given in Theorem 1.1, we further generalize the results of [14] when the initial data belong to $L^r(\mathbb{R}^n)$ with $r \in (1, 2)$. In particular, we prove the small data global existence in the critical case $p = 1 + \frac{2}{n}$. This result is completely new when $r \in (1, 2)$. We recall that when $r = 1$, $p = 1 + \frac{2}{n}$, and $\mathcal{N}(u) = |u|^p$, the local solution blows up in a finite time even if the size of the initial data $\varepsilon$ is arbitrary small, provided that the shape of the initial data $(u_0, u_1) \in (H^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)) \times (L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n))$ has a positive integral average (see [37]). Namely, when $r = 1$, the critical power $p = 1 + \frac{2}{n}$ belongs to the blow-up case. On the other hand, when $r = 2$, as we explained before, Nakao and Ono [26] showed that the critical power $p = 1 + \frac{n}{n-1}$ belongs to the global-existence case. Our main result for the nonlinear problem (Theorem 1.4) shows that for $r \in (1, 2)$, the critical exponent $p = 1 + \frac{n}{n-1}$ belongs to the global-existence case, although some restriction on the range of $r$ is imposed. Also, we refer the reader to [35] in which the global existence of solutions to the critical semilinear heat equation $v_t - \Delta v = v^{1+\frac{2}{n}}$ was proved, when the initial datum belongs to $L^r(\mathbb{R}^n)$ and is sufficiently small.

To state our results, we first define a solution. We say that a function $u \in L^\infty(0, T; L^2(\mathbb{R}^n))$ is a mild solution of (12) if $u$ satisfies the integral equation

$$u(t) = \mathcal{D}(t)(u_0 + u_1) + \varepsilon \partial_t \mathcal{D}(t) u_0 + \int_0^t \mathcal{D}(t-\tau) \mathcal{N}(u(\tau)) d\tau$$

in $L^\infty(0, T; L^2(\mathbb{R}^n))$. Moreover, we shall call $u$ an $H^s$-mild solution (resp. an $H^s \cap L^r$-mild solution) if $u$ is a mild solution of (12) satisfying $u \in C([0, T); H^s(\mathbb{R}^n))$ (resp. $u \in C([0, T); H^s(\mathbb{R}^n) \cap L^r(\mathbb{R}^n))$).

We assume that there exists $p > 1$ such that $\mathcal{N} \in C^\infty(\mathbb{R})$ with some integer $p_0 \in [0, p)$ and

$$\begin{cases} 
\mathcal{N}^{(l)}(0) = 0, \\
|\mathcal{N}^{(l)}(u) - \mathcal{N}^{(l)}(v)| \leq C|u - v|(|u| + |v|)^{p-l-1} & (l = 0, 1, \ldots, p_0).
\end{cases} \quad (13)$$

Before going to the global existence results, we first prepare the local existence of a unique $H^s \cap L^r$-mild solution based on the linear estimates in Theorem 1.1. We note that an $H^s \cap L^r$-mild solution is also an $H^s$-mild solution for any $r \in (1, 2]$. After introducing the existence of the $H^s \cap L^r$-mild solution, we also discuss the blow-up alternative for $H^s$-mild solutions.

**Theorem 1.3** (Local existence). Let $n \geq 1$, and assume (13). Let $s \geq 0$ and $r \in (1, 2]$ satisfy $|s| \leq p_0$, $r \geq \frac{2(n-1)}{n+1}$ and

$$1 < p < \infty, \quad 1 < p \leq 1 + \frac{\min\{n, 2\}}{n-2s} \quad \text{if} \quad 2s \geq n, \quad \text{if} \quad 2s < n.$$ 

Let $\beta = (n-1)\left(\frac{1}{r} - \frac{1}{2}\right)$ and let the initial data satisfy

$$(u_0, u_1) \in (H^s(\mathbb{R}^n) \cap H^2(\mathbb{R}^n)) \times (H^{s-1}(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)).$$

Then, for any $\varepsilon > 0$, there exists $T > 0$ such that the problem (12) admits a unique $H^s \cap L^r$-mild solution

$$u \in C([0, T); H^s(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)), \quad \partial_t u \in C([0, T); H^{s-1}(\mathbb{R}^n)).$$
Moreover, for the lifespan of the $H^s$-mild solutions defined by
\[
T_2(\varepsilon) := \sup \{ T \in (0, \infty) ; \text{ there exists a unique } H^s \text{-mild solution of (12)} \]
with $u \in C([0, T); H^s(\mathbb{R}^n))$, $\partial_t u \in C([0, T); H^{s-1}(\mathbb{R}^n))$,
\[
\text{we have the blow-up alternative: if } T_2(\varepsilon) < \infty, \text{ then the solution satisfies}
\]
\[
\liminf_{t \to T_2(\varepsilon)} \| (u, \partial_t u)(t) \|_{H^{s} \times H^{s-1}} = \infty.
\]

Remark 1.3. (i) We remark that the assumption $r \geq \frac{2(n-1)}{n+1}$ implies $\beta = (n - 1) \left(\frac{1}{2} - \frac{1}{p}\right) \leq 1$, namely, the derivative loss in the linear estimate does not exceed 1.

(ii) In the previous result [14], the local existence requires $p \geq \max\{1 + \frac{2}{n}, 1 + \frac{2}{\sigma}\}$, which comes from estimates involving weighted Sobolev norms. Theorem 1.3 removes this condition and we do not need any restriction from below on $p$.

(iii) In Theorem 1.3, we show the blow-up criterion only for the $H^s$-mild solution. It is difficult to obtain the blow-up criterion for the $H^s \cap L^r$-mild solution for $r \in [1, 2)$ because the derivative loss prevents us from extending the local solution. Namely, the solution does not have the persistence property, which means that the solution $u(t)$ belongs to the same space as the initial data with continuous dependence on the time variable.

We prove Theorem 1.3 in Section 3. Our proof is based on the $L^p$-$L^q$ estimates given in Theorem 1.1 and the contraction mapping principle. To control the nonlinear term, we introduce an appropriate norm for the nonlinearity (see (38)), which is inspired by Hayashi, Kaikina and Naumkin [9]. Then, to estimate the derivative of the nonlinearity, we apply the fractional chain rule.

Moreover, in the critical or supercritical case $p \geq 1 + \frac{2r}{n}$, we have the global existence of the $H^s \cap L^r$-mild solution for the small initial data.

**Theorem 1.4 (Global existence of $H^s \cap L^r$-mild solution for small data).** In addition to the assumption of Theorem 1.3, we suppose
\[
1 + \frac{2r}{n} \leq p.
\]
Then, there exists $\varepsilon_0 = \varepsilon_0(n, p, r, s, \| u_0 \|_{H^s \cap L^r}, \| u_1 \|_{H^{s-1} \cap L^r}) > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, the problem (12) admits a unique global $H^s \cap L^r$-mild solution satisfying
\[
u \in C([0, \infty); H^s(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)), \quad \partial_t u \in C([0, \infty); H^{s-1}(\mathbb{R}^n)).
\]

The reason why the global solution exists even in the critical case $p = 1 + \frac{2r}{n}$ is that the nonlinearity $\mathcal{N}(u)$ decays faster than the linear part at the spatial infinity. More precisely, we see that $\mathcal{N}(u) \in L^{\sigma_1}(\mathbb{R}^n)$ with $\sigma_1 = \max\{1, \frac{2}{n}\} < r$ (see Section 3), while the linear part of the solution satisfies $\varepsilon \partial_t(u_0 + u_1) + \varepsilon \partial_t \mathcal{D}(t) u_0 \in L^r(\mathbb{R}^n)$. This enables us to control the nonlinearity even in the critical case $p = 1 + \frac{2r}{n}$.

On the other hand, if we consider the global existence of the $H^s$-mild solution and do not require that $u \in C([0, \infty); L^r(\mathbb{R}^n))$, then we do not need to impose $r \geq \frac{2(n-1)}{n+1}$. 


Theorem 1.5 (Global existence of $H^s$-mild solution for small data). Let $n \geq 1$, and assume \((13)\). Let $s \geq 0$ and $r \in (1, 2]$ satisfy $|s| \leq p_0$, $r \left(\frac{\sqrt{n^2 + 16n - n}}{4} - \frac{n}{2} \right)$, and

\[
1 + \frac{2r}{n} \leq p < \infty \quad \text{if} \quad 2s \geq n, \quad \text{and} \quad 1 + \frac{2r}{n} \leq p \leq 1 + \frac{\min\{n, 2\}}{n - 2s} \quad \text{if} \quad 2s < n.
\]

Let the initial data satisfy

\[
(u_0, u_1) \in (H^s(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)) \times (H^{s-1}(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)).
\]

Then, there exists $\varepsilon_0 = \varepsilon_0(n, p, r, s, \|u_0\|_{H^\sigma L^r}, \|u_1\|_{H^{\sigma-1} L^r}) > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, the problem \((12)\) admits a unique global $H^s$-mild solution satisfying

\[
u C([0, \infty); H^s(\mathbb{R}^n)), \quad \partial_t u \in C([0, \infty); H^{s-1}(\mathbb{R}^n)).
\]

Remark 1.4. Theorem 1.5 states that we can relax the condition $r \in \left[\frac{2(n-1)}{n+1}, 2\right]$ in the case $n \geq 5$ if we do not require that $u \in C([0, \infty); L^r(\mathbb{R}^n))$. Indeed, concerning the assumption on the range of $r$, we see that

\[
(1, 2] \cap \left[\frac{2(n-1)}{n+1}, 2\right] \subset (1, 2] \cap \left(\frac{\sqrt{n^2 + 16n - n}}{4}, 2\right) \quad \text{if and only if} \quad n \geq 5.
\]

Therefore, for $n \geq 5$, the assumption of $r$ in Theorem 1.5 is weaker than that of Theorem 1.4.

Furthermore, in the supercritical case $p > 1 + \frac{2r}{n}$, we prove that the solution is approximated by that of the linear heat equation \((4)\) with the initial data $\varepsilon(u_0 + u_1)$. This extends the results by [29] to all space dimensions.

Theorem 1.6 (Asymptotic behavior of global solutions). Let $u$ be the global $H^s \cap L^r$-mild solution constructed in Theorem 1.4, and let $\sigma_1 = \max\{1, \frac{p}{r}\}$. We further assume $p > 1 + \frac{2r}{n}$. Then, for any $\delta > 0$ and $t \geq 1$,

\[
\|\nabla^\sigma (u(t) - \varepsilon G(t)(u_0 + u_1))\|_{L^2} \lesssim \langle t \rangle^{-\frac{\sigma}{2} - \frac{\delta}{2}} \min\{1, \frac{p}{2} \}^{-1} \langle t \rangle^{-\frac{\sigma}{2} - \frac{\delta}{2}} \frac{\langle t \rangle^{\frac{p}{2} - \frac{\delta}{2}}}{\langle t \rangle^{\frac{p}{2} - \frac{\delta}{2}}}
\]

\times \varepsilon \left(\|u_0\|_{H^{\sigma-1} L^r} + \|u_1\|_{H^{\sigma-1} L^r} \right),

\[
\|u(t) - \varepsilon G(t)(u_0 + u_1)\|_{L^r} \lesssim \langle t \rangle^{-\frac{\sigma}{2} - \frac{\delta}{2}} \min\{1, \frac{p}{2} \}^{-1} \langle t \rangle^{-\frac{\sigma}{2} - \frac{\delta}{2}} \frac{\langle t \rangle^{\frac{p}{2} - \frac{\delta}{2}}}{\langle t \rangle^{\frac{p}{2} - \frac{\delta}{2}}}
\]

\times \varepsilon \left(\|u_0\|_{H^{\sigma-1} L^r} + \|u_1\|_{H^{\sigma-1} L^r} \right),

\[
\|u(t) - \varepsilon G(t)(u_0 + u_1)\|_{L^r} \lesssim \langle t \rangle^{-\frac{\sigma}{2} - \frac{\delta}{2}} \min\{1, \frac{p}{2} \}^{-1} \langle t \rangle^{-\frac{\sigma}{2} - \frac{\delta}{2}} \frac{\langle t \rangle^{\frac{p}{2} - \frac{\delta}{2}}}{\langle t \rangle^{\frac{p}{2} - \frac{\delta}{2}}}
\]

\times \varepsilon \left(\|u_0\|_{H^{\sigma-1} L^r} + \|u_1\|_{H^{\sigma-1} L^r} \right),
\]

where $q = r$ if $2s \geq n$ and $q = \min\{r, \frac{2n}{p(n-2s)}\}$ if $2s < n$, and the implicit constant depends on $\delta$.

We next handle the subcritical case $p < 1 + \frac{2r}{n}$. We have the sharp lower bound and an almost sharp upper bound of the lifespan. To state the result, we define the lifespan of $H^s \cap L^r$-mild solutions by

\[
T_\varepsilon := \sup \{T \in (0, \infty) : \text{there exists a unique } H^s \cap L^r \text{-mild solution of } (12) \}.
\]
Theorem 1.7 (Lower bound of the lifespan for $H^s \cap L^r$-mild solutions). In addition to the assumption of Theorem 1.3, we suppose that
\[ p < 1 + \frac{2r}{n}. \]
Then, there exists $\varepsilon_1 = \varepsilon_1(n, p, r, s, \|u_0\|_{H^s \cap H^r}, \|u_1\|_{H^{s-1} \cap L^r}) > 0$ such that for any $\varepsilon \in (0, \varepsilon_1]$, the lifespan of $H^s \cap L^r$-mild solutions $T_r(\varepsilon)$ is estimated as
\[ T_r(\varepsilon) \gtrsim \varepsilon^{-1/\omega}, \quad (17) \]
where $\omega = \frac{1}{p-1} - \frac{n}{2r}$ and the implicit constant is independent of $\varepsilon$.

We prove Theorem 1.7 in Section 6. The proof is a slight modification of Theorem 1.3.

The rate $-1/\omega$ of $\varepsilon$ in the estimate (17) is optimal in the sense that we cannot obtain the estimate $T_r(\varepsilon) \gtrsim \varepsilon^{-1/\omega-\delta}$ for any $\delta > 0$ in general. More precisely, we give the following upper estimate of $T_2(\varepsilon)$ (see also Remark 1.5).

Theorem 1.8 (Upper bound of the lifespan for $H^s$-mild solutions). In addition to the assumption of Theorem 1.3, we assume that $N(u) = \pm |u|^p$ with $p < 1 + \frac{2r}{n}$. Then, for any $\delta > 0$, there exist initial data $(u_0, u_1) \in (H^s(\mathbb{R}^n) \cap H^r(\mathbb{R}^n)) \times (H^{s-1}(\mathbb{R}^n) \cap L^r(\mathbb{R}^n))$ and a constant $\varepsilon_2 = \varepsilon_2(n, p, r, s, \delta) > 0$ such that for any $\varepsilon \in (0, \varepsilon_2]$, the lifespan of $H^s$-mild solutions defined by (14) is estimated as
\[ T_2(\varepsilon) \lesssim \varepsilon^{-1/\omega-\delta}, \]
where the implicit constant is dependent on $\delta$ but independent of $\varepsilon$.

We will prove more general blow-up results in Section 7. The proof is based on the argument of [5], in which the blow-up of solutions to the semilinear wave equation with time-dependent damping was studied via an analysis of an ordinary differential inequality.

Remark 1.5. By the definitions of lifespans of $H^s$-mild solutions (14) and $H^s \cap L^r$-mild solutions (16), we immediately have $T_r(\varepsilon) \leq T_2(\varepsilon)$. From this and Theorems 1.7 and 1.8, we see that
\[ \varepsilon^{-1/\omega} \lesssim T_r(\varepsilon) \leq T_2(\varepsilon) \lesssim \varepsilon^{-1/\omega-\delta}, \]
which gives an almost optimal estimate for both $T_r(\varepsilon)$ and $T_2(\varepsilon)$.

The rest of the paper is organized as follows. In Section 2, we prove our $L^p$-$L^q$ estimates. Theorems 1.3 and 1.4, namely, the local and global existence of an $H^s \cap L^r$-mild solution are proved in Section 3. Then, Section 4 is devoted to the proof of Theorem 1.5, that is, the global existence of $H^s$-mild solution. In Section 5, we give a proof of Theorem 1.6. Finally, in Sections 6 and 7, we give proofs of Theorems 1.7 and 1.8, respectively.

We introduce the notation used throughout this paper. For the variable $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we use the notation of derivatives $\partial_j = \frac{\partial}{\partial x_j} \ (j = 1, \ldots, n)$. Let $1_I$ be the characteristic function of $I \subset \mathbb{R}$, the notation $X \sim Y$ stands for $X \lesssim Y$ and $Y \lesssim X$.

Let $\chi \in C_0^{\infty}(\mathbb{R})$ be a cut-off function satisfying $\chi(r) = 1$ for $|r| \leq 1$ and $\chi(r) = 0$ for $|r| \geq 2$. We write
\[ \chi_{<\alpha}(r) := \chi \left( \frac{r}{\alpha} \right), \quad \chi_{\geq\alpha}(r) := 1 - \chi_{<\alpha}(r), \quad \chi_{\alpha \leq \cdot \leq \beta}(r) := \chi_{< \beta}(r) - \chi_{< \alpha}(r) \quad (18) \]
for $0 < a < b$.

For a function $f : \mathbb{R}^n \to \mathbb{C}$, we define the Fourier transform and the inverse Fourier transform by

$$
\mathcal{F}[f](\xi) = \hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} f(x) \, dx,
$$

$$
\mathcal{F}^{-1}[f](x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi} f(\xi) \, d\xi.
$$

Moreover, for a measurable function $m = m(\xi)$, we denote the Fourier multiplier $m(\nabla)$ by

$$
m(\nabla)f(x) = \mathcal{F}^{-1} \left[ m(\xi) \hat{f}(\xi) \right](x).
$$

For $s \in \mathbb{R}$ and $p \in (1, \infty)$, we denote the usual Sobolev space by $H^s_p(\mathbb{R}^n)$ and its homogeneous version by $\dot{H}^s_p(\mathbb{R}^n)$.

2. Proof of the $L^p$-$L^q$ estimates. We divide $\mathcal{D}(t)$ into the low and high frequency parts $\mathcal{D}(t) = \mathcal{D}_1(t) + \mathcal{D}_2(t)$, where

$$
\mathcal{D}_1(t) = \mathcal{F}^{-1} \left[ \chi_{<1}(|\xi|) e^{-\frac{1}{2}L(t,\xi)} \right], \quad \mathcal{D}_2(t) = \mathcal{F}^{-1} \left[ \chi_{\geq 1}(|\xi|) e^{-\frac{1}{2}L(t,\xi)} \right].
$$

Let $\vartheta$ be the multiplier of $\mathcal{D}_1$, namely

$$
\vartheta(t,x) := \mathcal{F}^{-1} \left[ \chi_{<1}(|\xi|) e^{-\frac{1}{2}L(t,\xi)} \right](x).
$$

2.1. $L^p$-$L^q$ estimates for linear damped wave equation. First, we focus on the low frequency part. The $L^p$-$L^q$ estimates of the low frequency part is similar to that of the heat propagator. The first step is to get the pointwise estimate for the kernel $\vartheta$, which gives the value of the $L'$-norm of the kernel $\vartheta$. The second step is to get the $L^p$-$L^q$ estimates whose proof is based on Young’s inequality and the value of the $L'$-norm of the kernel $\vartheta$.

We have the following pointwise estimate of the kernel $\vartheta$.

Proposition 2.1. For $s \geq 0$, we have

$$
||\nabla|^s \vartheta(t,x)|| \lesssim \min \left( |x|^{-1}, |t|^{-\frac{1}{2}} \right)^{s+n}. \tag{19}
$$

Moreover, for any $j \in \mathbb{N}$,

$$
|\vartheta(t,x)| \lesssim (t)^{-\frac{n}{2}} \min \left( |t|^\frac{1}{2} |x|^{-1}, 1 \right)^j. \tag{20}
$$

For the proof of Proposition 2.1, we observe the following lemmas.

Lemma 2.2. For $t \geq 0$, $a \in \mathbb{R}$, and $\sigma \in (0, \frac{1}{2})$, we have

$$
\int_{\sigma \leq |\xi| \leq \frac{1}{2}} |\xi|^a e^{-t|\xi|^2} \, d\xi \lesssim \begin{cases} 
|t|^{-\frac{a+n}{2}}, & \text{if } a > -n, \\
|\log(|t|^\frac{1}{2} \sigma)|, & \text{if } a = -n, \\
\sigma^{a+n}, & \text{if } a < -n.
\end{cases}
$$

Moreover, $\sigma = 0$ is allowed if $a > -n$.

Proof. By changing variable $\eta = (t)^{\frac{1}{2}} \xi$, we have

$$
\int_{\sigma \leq |\xi| \leq \frac{1}{2}} |\xi|^a e^{-t|\xi|^2} \, d\xi = (t)^{-\frac{a+n}{2}} \int_{(t)^{\frac{1}{2}} \sigma \leq |\eta| \leq (t)^{\frac{1}{2}}} |\eta|^a e^{-\frac{|\eta|^2}{2t}} \, d\eta.
$$
When \((t)^{\frac{1}{2}} \sigma \geq \frac{1}{2}\), the integral on the right-hand side is bounded by

\[
\int_{\frac{1}{2} \leq |\eta| \leq \frac{1}{2} (t)^{\frac{1}{2}}} |\eta|^a e^{-\frac{1}{2} |\eta|^2} d\eta \leq \begin{cases} 
\int_{\frac{1}{2} \leq |\eta| \leq 2} |\eta|^a d\eta \lesssim 1, & \text{if } 0 \leq t \leq 1, \\
\int_{|\eta| \geq \frac{1}{2}} |\eta|^a e^{-\frac{1}{2} |\eta|^2} d\eta \lesssim 1, & \text{if } t \geq 1.
\end{cases}
\]

For \((t)^{\frac{1}{2}} \sigma \leq \frac{1}{2}\), we have

\[
\int_{(t)^{\frac{1}{2}} \sigma \leq |\eta| \leq \frac{1}{2} (t)^{\frac{1}{2}}} |\eta|^a e^{-\frac{1}{2} |\eta|^2} d\eta \leq \int_{(t)^{\frac{1}{2}} \sigma \leq |\eta| \leq \frac{1}{2}} |\eta|^a d\eta + \int_{\frac{1}{2} \leq |\eta| \leq (t)^{\frac{1}{2}}} |\eta|^a e^{-\frac{1}{2} |\eta|^2} d\eta 
\lesssim \begin{cases} 
1, & \text{if } a > -n, \\
(\log((t)^{\frac{1}{2}} \sigma)), & \text{if } a = -n, \\
(t)^{\frac{n+2}{2}} \sigma^{n}, & \text{if } a < -n,
\end{cases}
\]

which concludes the proof.

Lemma 2.3. For \(k \in \mathbb{Z}_{\geq 0}\), there exist some constants \(C_{l,m}^{(k)} (k - \lfloor \frac{k}{2} \rfloor) \leq l \leq k, 0 \leq m \leq l)\) satisfying

\[
\partial_1^l \left( \frac{e^{t \sqrt{\frac{1}{4} - |\xi|^2}}}{\sqrt{\frac{1}{4} - |\xi|^2}} \right) = e^{t \sqrt{\frac{1}{4} - |\xi|^2}} \sum_{l=0}^{l} \sum_{m=0}^{m} C_{l,m}^{(k)} \xi^{2l-k} (\frac{1}{4} - |\xi|^2)^{-l+\frac{m+1}{2}}
\]

for \(t \in \mathbb{R}\), where \(\partial_1 = \partial / \partial \xi_1\).

Proof. For \(k = 0\), we have \(C_{0,0}^{(0)} = 1\). We assume that (21) holds for some \(k \in \mathbb{Z}_{\geq 0}\).

For simplicity, we define \(C_{l,m}^{(k)} = 0\) for \((l, m) \notin \{(l, m) \in \mathbb{Z}^2_{\geq 0}: k - \lfloor \frac{k}{2} \rfloor \leq l \leq k, 0 \leq m \leq l\} \). Then, a direct calculation yields

\[
e^{-t \sqrt{\frac{1}{4} - |\xi|^2}} \partial_1^{k+1} \left( \frac{e^{t \sqrt{\frac{1}{4} - |\xi|^2}}}{\sqrt{\frac{1}{4} - |\xi|^2}} \right) = e^{-t \sqrt{\frac{1}{4} - |\xi|^2}} \partial_1 \left\{ e^{t \sqrt{\frac{1}{4} - |\xi|^2}} \sum_{l=0}^{l} \sum_{m=0}^{m} C_{l,m}^{(k)} \xi^{2l-k} (\frac{1}{4} - |\xi|^2)^{-l+\frac{m+1}{2}} \right\}
\]

\[
= \sum_{l=0}^{l} \sum_{m=0}^{m} C_{l,m}^{(k)} \left\{ -t^{m+1} \xi^{2l-k} (\frac{1}{4} - |\xi|^2)^{-l+\frac{m+2}{2}} \\
+ (2l - k) t^{m} \xi^{2l-k-1} (\frac{1}{4} - |\xi|^2)^{-l+\frac{m+1}{2}} \\
+ (2l - m + 1) t^{m} \xi^{2l-k+1} (\frac{1}{4} - |\xi|^2)^{-l+\frac{m+1}{2}} \right\}
\]

\[
= \sum_{l=0}^{l} \sum_{m=0}^{m} \left\{ -t^{m+1} \xi^{2l-k} (\frac{1}{4} - |\xi|^2)^{-l+\frac{m+2}{2}} \\
+ (2l - k) t^{m} \xi^{2l-k-1} (\frac{1}{4} - |\xi|^2)^{-l+\frac{m+1}{2}} \\
+ (2l - m + 1) t^{m} \xi^{2l-k+1} (\frac{1}{4} - |\xi|^2)^{-l+\frac{m+1}{2}} \right\}
\]

\[
\quad + \sum_{l=0}^{l} \sum_{m=0}^{m} \left\{ -t^{m+1} \xi^{2l-k} (\frac{1}{4} - |\xi|^2)^{-l+\frac{m+2}{2}} \\
+ (2l - k) t^{m} \xi^{2l-k-1} (\frac{1}{4} - |\xi|^2)^{-l+\frac{m+1}{2}} \\
+ (2l - m + 1) t^{m} \xi^{2l-k+1} (\frac{1}{4} - |\xi|^2)^{-l+\frac{m+1}{2}} \right\}
\]
\[ + \sum_{l=k+1-\lceil \frac{k+1}{2} \rceil}^{k+1} \sum_{m=0}^{l-1} (2l - m - 1)C_{l-1,m}^{(k)} \xi_1^{2l-(k+1)} (\frac{1}{4} - |\xi|^2)^{-l + \frac{m-1}{2}} \]

\[ = \sum_{l=k+1-\lceil \frac{k+1}{2} \rceil}^{k+1} \sum_{m=0}^{l} \left\{ -C_{l-1,m-1}^{(k)} + (2l - k)C_{l,m}^{(k)} + (2l - m - 1)C_{l-1,m}^{(k)} \right\} \times t^m \xi_1^{2l-(k+1)} (\frac{1}{4} - |\xi|^2)^{-l + \frac{m-1}{2}}. \]

Hence, the constants \( C_{l,m}^{(k+1)} \) are defined by

\[ C_{l,m}^{(k+1)} := -C_{l-1,m-1}^{(k)} + (2l - k)C_{l,m}^{(k)} + (2l - m - 1)C_{l-1,m}^{(k)}, \quad (22) \]

which shows (21).

**Proof of Proposition 2.1.** We prove the inequality with respect to the right side in the minimum in (19) and (20), i.e.

\[ |||\nabla||| \mathfrak{d}(t,x)| \lesssim \langle t \rangle^{-\frac{1}{2} + \frac{1}{s}} \]

for any \( s \geq 0 \). Since

\[ -\frac{1}{2} + \sqrt{\frac{1}{4} - |\xi|^2} = -\frac{|\xi|^2}{\frac{1}{2} + \sqrt{\frac{1}{4} - |\xi|^2}} \leq -|\xi|^2 \quad (23) \]

for \( |\xi| \leq \frac{1}{2} \), Lemma 2.2 gives

\[ |||\nabla||| \mathfrak{d}(t,x)| = (2\pi)^{-\frac{1}{2}} \left| \int_{\mathbb{R}^n} e^{ix\xi} \chi_{<1}(|\xi|) |\xi|^s e^{-\frac{1}{2}} L(t,\xi) d\xi \right| \]

\[ \lesssim \int_{|\xi| < \frac{1}{2} - \delta} |\xi|^s e^{-l|\xi|^2} d\xi + \int_{\frac{1}{2} - \delta \leq |\xi| \leq 2} t e^{-\frac{1}{2}} d\xi \]

\[ \lesssim \langle t \rangle^{-\frac{1}{2} + \frac{1}{s}}, \quad (24) \]

where \( \delta > 0 \) is a sufficiently small number.

It remains to prove the inequality with respect to the left side in the minimum in (19) and (20). To obtain the decay with respect to \( |x| \), we divide \( \mathfrak{d} \) into two parts \( \mathfrak{d} = \mathfrak{d}_1 + \mathfrak{d}_2 \):

\[ \mathfrak{d}_1(t,x) := \mathcal{F}^{-1} \left[ \chi_{< \frac{1}{2}}(|\xi|) e^{-\frac{1}{2}} L(t,\xi) \right](x), \]

\[ \mathfrak{d}_2(t,x) := \mathcal{F}^{-1} \left[ \chi_{\frac{1}{2} \leq |\xi| < 1}(|\xi|) e^{-\frac{1}{2}} L(t,\xi) \right](x). \]

Without loss of generality, we may assume that \( |x| \sim |x_1| \). First, we prove the estimate for \( \mathfrak{d}_1 \) with respect to the left side in the minimum in (20). Namely, we show

\[ |\mathfrak{d}_1(t,x)| \lesssim \langle t \rangle^{-\frac{1}{2} + \frac{1}{2}} |x|^{-j}, \]

for any \( j \in \mathbb{N} \).

By (23) and Lemma 2.3, for \( k \in \mathbb{Z}_{\geq 0} \) and \( |\xi| \leq \frac{1}{4} \), we have

\[ \left| e^{-\frac{1}{2}} \partial_x^k L(t,\xi) \right| \leq \frac{1}{2} \sum_{l=k-\lceil \frac{k+1}{2} \rceil}^{k+1} \sum_{m=0}^{l-1} (2l - m - 1)C_{l-1,m}^{(k)} \xi_1^{2l-(k+1)} (\frac{1}{4} - |\xi|^2)^{-l + \frac{m-1}{2}} \]

\[ + \sum_{l=k+1-\lceil \frac{k+1}{2} \rceil}^{k+1} \sum_{m=0}^{l} \left\{ -C_{l-1,m-1}^{(k)} + (2l - k)C_{l,m}^{(k)} + (2l - m - 1)C_{l-1,m}^{(k)} \right\} \times t^m \xi_1^{2l-(k+1)} (\frac{1}{4} - |\xi|^2)^{-l + \frac{m-1}{2}}. \]
\[
\frac{1}{2} e^{i \left( \frac{x^2}{4} - \frac{y^2}{4} - |\xi|^2 \right)} \sum_{l=k}^{k} \sum_{m=0}^{l} C_{l,m}^{(k)} |t|^m |\xi|^{2l-k \left( \frac{1}{2} - |\xi|^2 \right) - t + \frac{m-1}{2}} \lesssim \sum_{l=k}^{k} \langle t \rangle^l |\xi|^{2l-k-e^{-t}|\xi|^2}.
\]  

(25)

Since \( |\partial^\ell \left( \chi_{< \frac{1}{2}} (|\xi|) \right) | \lesssim 1 \) for \( k \in \mathbb{Z}_{\geq 0} \), integration by parts, (25), and Lemma 2.2 yield
\[
|\partial_1 (t, x)| = (2\pi)^{-\frac{n}{2}} \left| \int_{\mathbb{R}^n} e^{ix\xi} \chi_{< \frac{1}{2}} (|\xi|) e^{-\frac{1}{2} L(t, \xi)} d\xi \right|
\]
\[
= (2\pi)^{-\frac{n}{2}} \frac{1}{|x|^j} \sum_{k=0}^{j} \left( \frac{j}{k} \right) \int_{\mathbb{R}^n} e^{ix\xi} \partial_1^{1-k} \left( \chi_{< \frac{1}{2}} (|\xi|) \right) e^{-\frac{1}{2} L(t, \xi)} d\xi \lesssim |x|^{-j} \sum_{k=0}^{j} \sum_{l=k-\left( \frac{n}{2} \right)}^{k} \langle t \rangle^l \int_{|\xi| \leq \frac{1}{2}} |\xi|^{2l-k-e^{-t}|\xi|^2} d\xi
\]
\[
\lesssim |x|^{-j} \langle t \rangle^{\frac{1}{2} - \frac{2}{n}}
\]  

(26)

for any \( j \in \mathbb{N} \).

Secondly, we show the inequality for \( \partial_1 \) with respect to the left side in the minimum in (19) \( \text{i.e.} \ |\nabla^s \partial_1 (t, x)| \lesssim |x|^{-s(n+1)} \). For \( s \geq 0 \), we assume \( |x| \geq \langle t \rangle^\frac{1}{2} \), otherwise the desired bound follows from (24). Then, we further divide the multiplier \( \partial_1 \) into two parts \( \partial_1 = \partial_1' + \partial_1'' \): \( \partial_1' (t, x) := \mathcal{F}^{-1} \left[ \chi_{\frac{1}{2} \leq |\xi|} (|\xi|) e^{-\frac{1}{2} L(t, \xi)} \right] (x) \), \( \partial_1'' (t, x) := \mathcal{F}^{-1} \left[ \chi_{|\xi| > \frac{1}{2}} (|\xi|) e^{-\frac{1}{2} L(t, \xi)} \right] (x) \).

By (25), we have
\[
|\nabla^s \partial_1' (t, x)| = (2\pi)^{-\frac{n}{2}} \left| \int_{\mathbb{R}^n} e^{ix\xi} \chi_{\frac{1}{2} \leq |\xi|} (|\xi|) |\xi|^s e^{-\frac{1}{2} L(t, \xi)} d\xi \right|
\]
\[
\lesssim \int_{|\xi| \leq \frac{1}{2}} |\xi|^s d\xi \lesssim |x|^{-s(n+1)}.
\]  

(27)

We set \( j := [s] + n + 1 \) for simplicity. Integration by parts \( j \)-times yields
\[
|\nabla^s \partial_1'' (t, x)| = (2\pi)^{-\frac{n}{2}} \left| \int_{\mathbb{R}^n} e^{ix\xi} \chi_{|\xi| > \frac{1}{2}} (|\xi|) |\xi|^s e^{-\frac{1}{2} L(t, \xi)} d\xi \right|
\]
\[
= (2\pi)^{-\frac{n}{2}} \frac{1}{|x|^j} \sum_{k=0}^{j} \left( \frac{j}{k} \right) \int_{\mathbb{R}^n} e^{ix\xi} \partial_1^{1-k} \left( \chi_{|\xi| > \frac{1}{2}} (|\xi|) |\xi|^s \right) e^{-\frac{1}{2} L(t, \xi)} d\xi.
\]

Since \( |\partial_1^\ell (\chi_{|\xi| > \frac{1}{2}} (|\xi|) |\xi|^s) | \lesssim |\xi|^{-k} \) for \( k \in \mathbb{Z}_{\geq 0} \), from (25) we further obtain
\[
|\nabla^s \partial_1'' (t, x)| \lesssim |x|^{-j} \sum_{k=0}^{j} \sum_{l=k-\left( \frac{n}{2} \right)}^{k} \langle t \rangle^l \int_{|\xi| \leq \frac{1}{2}} |\xi|^{s-j+2l-e^{-t}|\xi|^2} d\xi.
\]
Finally, Lemma 2.2 concludes
\[ ||\nabla||^a \partial_2^j(t, x) || \]
\[ \lesssim |x|^{-j} \sum_{l=0}^{j}(t)^{j-1} \left\{ \langle t \rangle^{-1} \log(\langle t \rangle^{1/2} |x|^{-1}) |1_{(t-\varepsilon_{-n})} (l) + |x|^{-2j+n} 1_{[0, \varepsilon_{-n})} (l) \right\} \]
\[ \lesssim |x|^{-j} (t)^{j-1} \left( |x|^{-s-n} + |x|^{-j} (t)^{j-1} \log(\langle t \rangle^{1/2} |x|^{-1}) \right) \]
\[ \lesssim |x|^{-s-n}, \quad (28) \]
provided that \(|x| \geq \langle t \rangle^{1/2} \). Here we note that \(|x|^{-j} \langle t \rangle^{-s-n} |\log(\langle t \rangle^{1/2} |x|^{-1})| \lesssim 1 \)
holds, since \(|x|^{-1} \langle t \rangle^{1/2} \in [0, 1] \).

At last, we go on to the estimate for \( \partial_2 \) with respect to the left side in the minimum in (19) and (20). More precisely, we prove the better estimate \(||\nabla||^a \partial_2(t, x) || \lesssim |x|^{-j} \) for any \( j \in \mathbb{N} \). Since \( L(t, \xi) \) is smooth with respect to \( \xi \),
\[ ||\partial_k^j L(t, \xi)|| \lesssim \langle t \rangle^{k+1} \left\{ e^{\sqrt{\frac{1}{4} - |\xi|^2}} 1_{[0, \frac{1}{4})} (|\xi|) + 1_{[\frac{1}{4}, 1]} (|\xi|) \right\} \]
for \( |\xi| \leq 2 \) and \( k \in \mathbb{Z}_{\geq 0} \). Since
\[ \frac{1}{4} + \sqrt{\frac{1}{4} - |\xi|^2} \leq \frac{4 + \sqrt{17}}{8} < 0 \]
for \( \frac{1}{8} \leq |\xi| \leq \frac{1}{2} \), we have
\[ e^{-\frac{1}{2}} \int_{|\xi| \leq \frac{1}{8}} ||\partial_k^j L(t, \xi)|| d\xi \lesssim \langle t \rangle^{k+1} e^{-\frac{4 + \sqrt{17}}{8} t}. \]

Hence, integration by parts yields
\[ ||\nabla||^a \partial_2(t, x) || \]
\[ = (2\pi)^{-\frac{d}{2}} \left| \int_{\mathbb{R}^n} e^{ix\xi} \chi_{\frac{1}{8} \leq \cdot < 1} (|\xi|) |\xi|^k e^{-\frac{1}{2}} L(t, \xi) d\xi \right| \]
\[ = (2\pi)^{-\frac{d}{2}} \frac{1}{|x|^j} \sum_{k=0}^{j} \frac{1}{j!} \int_{\mathbb{R}^n} e^{ix\xi} \partial_1^{j-k} \left( \chi_{\frac{1}{8} \leq \cdot < 1} (|\xi|) |\xi|^k \right) e^{-\frac{1}{2}} \partial^j L(t, \xi) d\xi \]
\[ \lesssim |x|^{-j} \langle t \rangle^{j+1} e^{-\frac{4 + \sqrt{17}}{8} t} \quad (29) \]
for any \( j \in \mathbb{N} \). Combining (24) and (26)–(29), we obtain the desired bound. \( \Box \)

**Remark 2.1.** We divide \( \partial_1 \) into two parts \( \partial_1 = \partial_1^{(1)} - \partial_1^{(2)} \):
\[ \partial_1^{(1)}(t, x) := F^{-1} \left[ \chi_{< 1}(|\xi|) e^{\left( -\frac{4 + \sqrt{17}}{2} |\xi|^2 \right)} \frac{1}{2 \sqrt{1 - |\xi|^2}} \right](x), \]
\[ \partial_1^{(2)}(t, x) := F^{-1} \left[ \chi_{< 1}(|\xi|) e^{\left( -\frac{4 - \sqrt{17}}{2} |\xi|^2 \right)} \frac{1}{2 \sqrt{1 - |\xi|^2}} \right](x). \]
In (25), we neglect the decay factor \( e^{-\frac{s}{n}} \) of \( \mathcal{D}_1^{(2)} \). By keeping this factor, the proof of Proposition 2.1 yields that for any \( s \geq 0 \) and \( j \in \mathbb{Z}_{\geq 0} \), we have

\[
|\nabla|^s \mathcal{D}_1^{(2)}(t, x) | \lesssim \min \left( |x|^{-1}, \langle t \rangle^{-\frac{1}{2}} \right)^{s+n} e^{-\frac{s}{n}},
\]

\[
|\mathcal{D}_1^{(2)}(t, x) | \lesssim \langle t \rangle^{-\frac{n}{2}} \min \left( \langle t \rangle^{\frac{1}{2}} |x|^{-1}, 1 \right)^j e^{-\frac{s}{2}}.
\]

Proposition 2.1 and Young’s inequality lead to the following linear estimate for the low frequency.

**Proposition 2.4.** Let \( 1 \leq q \leq p \leq \infty \) and \( s_1 \geq s_2 \geq 0 \). Then,

\[
|||\nabla||^s \mathcal{D}_1(t) g||_{L^p} \lesssim \langle t \rangle^{-\frac{q}{2}(\frac{1}{p} - \frac{1}{q}) - \frac{s_1 - s_2}{p}} |||\nabla||^{s_2} g||_{L^q}
\]

for \( |||\nabla||^s g \in L^q(\mathbb{R}^n) \).

**Proof.** It reduces to show the bound

\[
|||\nabla||^s \mathcal{D}(t)||_{L^p} \lesssim \langle t \rangle^{-\frac{q}{2}(1 - \frac{1}{p}) - \frac{s}{2}}
\]

for \( s \geq 0 \) and \( 1 \leq p \leq \infty \). Indeed, (30) and Young’s inequality show

\[
|||\nabla||^s \mathcal{D}_1(t) g||_{L^p} = |||\nabla||^{s_1 - s_2} \mathcal{D}(t) * |||\nabla||^{s_2} g||_{L^p} \leq |||\nabla||^{s_1 - s_2} \mathcal{D}(t)||_{L^p} |||\nabla||^{s_2} g||_{L^q}
\]

\[
\lesssim \langle t \rangle^{-\frac{q}{2}(1 - \frac{1}{p}) - \frac{s_1 - s_2}{p}} |||\nabla||^{s_2} g||_{L^q} \sim \langle t \rangle^{-\frac{q}{2}(\frac{1}{p} - \frac{1}{q}) - \frac{s_1 - s_2}{p}} |||\nabla||^{s_2} g||_{L^q}
\]

with \( \frac{1}{q} = \frac{1}{p} - \frac{1}{q} + 1 \).

It remains to show (30). The case \( p = \infty \) is a direct consequence of Proposition 2.1. For \( s > 0 \) and \( 1 \leq p < \infty \), Proposition 2.1 implies

\[
|||\nabla||^s \mathcal{D}(t)||_{L^p} \lesssim \langle t \rangle^{-\frac{s+n}{2}} \left( \int_{|x| < \langle t \rangle^{\frac{1}{2}}} dx \right)^{\frac{1}{p}} + \left( \int_{|x| \geq \langle t \rangle^{\frac{1}{2}}} |x|^{-p(s+n)} dx \right)^{\frac{1}{p}}
\]

\[
\lesssim \langle t \rangle^{-\frac{s}{2}(1 - \frac{1}{p}) - \frac{s}{2}}.
\]

On the other hand, for \( s = 0 \), Proposition 2.1 with \( j = n + 1 \) implies

\[
||\mathcal{D}(t)||_{L^p} \lesssim \langle t \rangle^{-\frac{n}{2}} \left( \int_{|x| < \langle t \rangle^{\frac{1}{2}}} dx \right)^{\frac{1}{p}} + \langle t \rangle^{\frac{n}{2}} \left( \int_{|x| \geq \langle t \rangle^{\frac{1}{2}}} |x|^{-p(n+1)} dx \right)^{\frac{1}{p}}
\]

\[
\lesssim \langle t \rangle^{-\frac{n}{2}(1 - \frac{1}{p})}.
\]

This finishes the proof.

Next, we consider the high frequency part of \( \mathcal{D}(t) \). To estimate the high frequency part, we reduce the high frequency part to the wave propagator by using Mikhlin’s multiplier theorem and apply the \( L^p \)-estimate of the wave propagator.

**Proposition 2.5.** Let \( 1 < p < \infty \) and \( \beta = (n-1)(\frac{1}{2} - \frac{1}{p}) \). Then, there exists \( \delta_p > 0 \) such that

\[
||\mathcal{D}_2(t) g||_{L^p} \lesssim e^{-\frac{s}{2}} \langle t \rangle^{\delta_p} ||g||_{H^{\beta}_{p-1}}
\]

for \( g \in H^{\beta-1}_{p-1}(\mathbb{R}^n) \) and \( t \geq 0 \).

This proposition follows from the \( L^p \) bound for linear wave solutions, which was proved by Sjöstrand [33] and improved by Miyachi [25] and Peral [31].
for \( g \in H^2_p(\mathbb{R}^n) \) and \( t \in \mathbb{R} \).

**Proof of Proposition 2.5.** Since we focus on the high frequency and we see that
\[
sin(t \sqrt{|\xi|^2 - 1/4}) = \frac{1}{2!} (e^{it \sqrt{|\xi|^2 - 1/4}} - e^{-it \sqrt{|\xi|^2 - 1/4}})
\]
holds, it suffices to show that
\[
\|e^{it \sqrt{-\Delta - 1/4} - |\xi|^2/4}}\chi_{\geq 1} F g\|_{L^p} \lesssim (t \delta_p) \|g\|_{H^2_p}
\]
for \( g \in H^2_p(\mathbb{R}^n) \). We note that
\[
e^{it \sqrt{-\Delta - 1/4}} = e^{it |\xi|} e^{it \sqrt{-\Delta - 1/4} - |\xi|^2/4}}\chi_{\geq 1} F g\|_{L^p} \lesssim (t \delta_p) \|g\|_{H^2_p}
\]
Because
\[
\left| \sqrt{|\xi|^2 - 1/4} - |\xi| \right| = \frac{1}{4(\sqrt{|\xi|^2 - 1/4} + |\xi|)} \sim |\xi|^{-1}
\]
holds for \( |\xi| \geq 1 \), a simple calculation shows
\[
\|D^\alpha (\chi_{\geq 1} F g)\|_{L^p} \lesssim \|g\|_{H^2_p}
\]
for \( |\xi| \neq 0 \) and \( \alpha \in \mathbb{Z}^n_{\geq 0} \). Hence, Mikhlin’s multiplier theorem (see [8, Theorem 6.2.7]) shows that
\[
\|F^{-1} e^{it \sqrt{|\xi|^2 - 1/4} - |\xi|} \chi_{\geq 1} F g\|_{L^p} \lesssim \|g\|_{H^2_p}
\]
Owing to Theorem 2.6, this estimate yields (31).

The estimate (8) in Theorem 1.1 follows from Propositions 2.4 and 2.5.

2.2. \( L^p \)-\( L^q \) estimates for the derivative of the solution. Next, we prove the second statement (9) of Theorem 1.1. We use the same notation as in Remark 2.1.

We define the Fourier multipliers \( D_1^{(j)} \) by \( D_1^{(j)} g = \delta_1^{(j)} * g \) for \( j = 1, 2 \). From (23), Mikhlin’s multiplier theorem shows that
\[
\|\partial_1 D_1^{(1)} g\|_{L^p} = \left\| \left( \frac{1}{2} + \sqrt{\frac{1}{4} + \Delta} \right)^{-1} \Delta D_1^{(1)} g \right\|_{L^p} \lesssim \|\Delta D_1^{(1)} g\|_{L^p},
\]
\[
\|\partial_1 D_1^{(2)} g\|_{L^p} = \left\| \left( \frac{1}{2} + \sqrt{\frac{1}{4} + \Delta} \right) D_1^{(2)} g \right\|_{L^p} \lesssim \|D_1^{(2)} g\|_{L^p}.
\]
Therefore, Remark 2.1 and Proposition 2.4 imply the desired estimate for the low frequency part \( \partial_1 D_1(t)g \). Moreover, the same argument as in the proof of Proposition 2.5 shows the estimate for the high frequency part \( \partial_1 D_2(t)g \).

2.3. \( L^p \)-\( L^q \) estimates for the difference. Now, we prove Theorem 1.2. Set
\[
m(t, x) := \delta(t, x) - F^{-1} [\chi_{\leq 1} (|\xi|) e^{-it |\xi|^2}] (x).
\]
We recall that \( \delta \) is the multiplier of the low frequency part of \( D \).

We show the pointwise decay estimates for \( m \).

**Proposition 2.7.** For \( s \geq 0 \), we have
\[
\|\nabla^s m(t, x)\| \lesssim \min \left( |x|^{-1}, \langle t \rangle^{-\frac{3}{2}} \right)^{s+n+2}.
\]
For the proof of Proposition 2.7, we observe the following two lemmas.

**Lemma 2.8.** For $k \in \mathbb{N}$, there exist some constants $D_{l,m}^{(k)}$ $(k - \lfloor \frac{k}{2} \rfloor \leq l \leq k$, $1 \leq m \leq l)$ satisfying

$$\partial_k^l \left( e^{-t|\xi|^2} \right) = e^{-t|\xi|^2} \sum_{l=k-\lfloor \frac{k}{2} \rfloor}^{k} \sum_{m=1}^{l} D_{l,m}^{(k)} t^m \xi_1^{2l-k}. \quad (32)$$

In particular, $D_{l,m}^{(k)} = 2^l C_{l,t}^{(k)}$ for $k - \lfloor \frac{k}{2} \rfloor \leq l \leq k$, where $C_{k,l}^{(k)}$ are the constants in Lemma 2.3.

**Proof.** For $k = 1$, we have $D_{1,1}^{(1)} = -2$, because $\partial_1 \left( e^{-t|\xi|^2} \right) = -2t\xi_1 e^{-t|\xi|^2}$. We assume that (32) holds for some $k \in \mathbb{N}$. For simplicity, we define $D_{l,m}^{(k)} = 0$ for $(l, m) \notin \{(l, m) \in \mathbb{N}^2 : k - \lfloor \frac{k}{2} \rfloor \leq l \leq k, 1 \leq m \leq l \}$. Then, a direct calculation yields

$$e^{t|\xi|^2} \partial_k^{k+1} \left( e^{-t|\xi|^2} \right) = e^{t|\xi|^2} \partial_k \left\{ e^{-t|\xi|^2} \sum_{l=k-\lfloor \frac{k}{2} \rfloor}^{k} \sum_{m=1}^{l} D_{l,m}^{(k)} t^m \xi_1^{2l-k} \right\}$$

$$= \sum_{l=k-\lfloor \frac{k}{2} \rfloor}^{k} \sum_{m=1}^{l} D_{l,m}^{(k)} \left\{ -2t^{m+1} \xi_1^{2l+1} + (2l-k) t^m \xi_1^{2l-k-1} \right\}$$

$$= \sum_{l=k+1-\lfloor \frac{k}{2} \rfloor}^{k+1} \sum_{m=2}^{l} (-2) D_{l-1,m-1}^{(k)} t^m \xi_1^{2l-(k+1)}$$

$$+ \sum_{l=k+1-\lfloor \frac{k}{2} \rfloor}^{k} \sum_{m=1}^{l} (2l-k) t^m \xi_1^{2l-(k+1)}$$

$$= \sum_{l=k+1-\lfloor \frac{k+1}{2} \rfloor}^{k+1} \sum_{m=1}^{l} \left\{ -2D_{l-1,m-1}^{(k)} + (2l-k) D_{l,m}^{(k)} \right\} t^m \xi_1^{2l-(k+1)}.$$}

Hence, the constants $D_{l,m}^{(k+1)}$ are defined by

$$D_{l,m}^{(k+1)} := -2D_{l-1,m-1}^{(k)} + (2l-k) D_{l,m}^{(k)},$$

which shows (32). In particular, $2^{-t} D_{l,t}^{(k)}$ satisfies (22) with $m = l$ because $C_{l-1,l}^{(k)} = 0$. Since $C_{l,l}^{(1)} = -1$ and $D_{l,l}^{(1)} = -2$, we obtain $D_{l,l}^{(k)} = 2^l C_{l,t}^{(k)}$ for $k - \lfloor \frac{k}{2} \rfloor \leq l \leq k$. \hfill \Box

**Lemma 2.9.** For $|\xi| \leq \frac{1}{4}$, we have

$$\left| \frac{e^{t \left( \frac{1}{2} + \sqrt{\frac{1}{4} - |\xi|^2} \right)}}{2 \sqrt{\frac{1}{4} - |\xi|^2}} - e^{-t|\xi|^2} \right| \lesssim \left( |\xi|^2 + (t)|\xi| \right) e^{-t|\xi|^2} + e^{-\frac{1}{4} t} \mathbf{1}_{(t^{-\frac{1}{4}}, \infty)}(|\xi|).$$

Furthermore, for $k \in \mathbb{N}$ and $|\xi| \leq \frac{1}{4}$, we have

$$\left| \partial_k^l \left( \frac{e^{t \left( \frac{1}{2} + \sqrt{\frac{1}{4} - |\xi|^2} \right)}}{2 \sqrt{\frac{1}{4} - |\xi|^2}} - e^{-t|\xi|^2} \right) \right|$$
In the same way as before, the right-hand side is bounded by

\[
\sum_{l=k-\lfloor \frac{1}{2}\rfloor}^{k+1} \langle t \rangle^{l} |\xi|^{2(l+1)-k} e^{-t|\xi|^2} + \langle t \rangle^{k} e^{-t^2 \frac{1}{4}} 1_{(t-\frac{1}{4}, \infty)} (|\xi|). 
\]

**Proof.** We note that

\[
\left| \frac{1}{\sqrt{1 - 4|\xi|^2}} - 1 \right| = \frac{4|\xi|^2}{\sqrt{1 - 4|\xi|^2} (1 + \sqrt{1 - 4|\xi|^2})} \lesssim |\xi|^2,
\]

\[
e^{-t|\xi|^2} \left| e^{t \left( -\frac{1}{4} + \sqrt{\frac{1}{4} - |\xi|^2} \right)} - e^{-t|\xi|^2} \right| = e^{-t|\xi|^2} \left| \exp \left( - \frac{4t|\xi|^4}{(1 + \sqrt{1 - 4|\xi|^2})^2} \right) - 1 \right|
\]

\[
\lesssim t|\xi|^4 e^{-t|\xi|^2} 1_{[0,(t-\frac{1}{4})]}(|\xi|) + e^{-t(t-\frac{1}{4})} 1_{((t-\frac{1}{4}), \infty)}(|\xi|)
\]

\[
\lesssim (t)|\xi|^4 e^{-t|\xi|^2} + e^{-t^2 \frac{1}{4}} 1_{((t-\frac{1}{4}), \infty)}(|\xi|)
\]

for $|\xi| \leq \frac{1}{2}$, where in the second inequality we used

\[
-\frac{1}{2} + \sqrt{\frac{1}{4} - |\xi|^2} + |\xi|^2 = -\frac{1}{4} \left( 1 - \sqrt{1 - 4|\xi|^2} \right)^2 = -\frac{4|\xi|^4}{(1 + \sqrt{1 - 4|\xi|^2})^2}.
\]

The triangle inequality with (23) implies

\[
\left| e^{t \left( -\frac{1}{4} + \sqrt{\frac{1}{4} - |\xi|^2} \right)} - 2 \frac{1}{\sqrt{1 - |\xi|^2}} \right| \lesssim e^{-t|\xi|^2} \left( |\xi|^2 + \langle t \rangle |\xi|^4 \right) + e^{-t^2 \frac{1}{4}} 1_{((t-\frac{1}{4}), \infty)}(|\xi|).
\]

Similarly, Lemmas 2.3 and 2.8 yield that, for $k \in \mathbb{N}$,

\[
\left| \partial_t^k \left( e^{t \left( -\frac{1}{4} + \sqrt{\frac{1}{4} - |\xi|^2} \right)} - e^{-t|\xi|^2} \right) \right|
\]

\[
\leq e^{-t|\xi|^2} \sum_{l=k-\lfloor \frac{1}{2}\rfloor}^{k} |C^{(k)}_{l,l} t^{|\pi|} |\xi|^{2l-k} \left\{ \left| \frac{1}{2} \left( \frac{1}{4} - |\xi|^2 \right) - \frac{1}{4} \right| - 2 \right\}
\]

\[
+ \frac{e^{-t|\xi|^2}}{2} \sum_{l=k-\lfloor \frac{1}{2}\rfloor}^{k} \sum_{m=0}^{l-1} |C^{(k)}_{l,m} t^m |\xi|^{2l-k} (\frac{1}{4} - |\xi|^2)^{-l+\frac{m+1}{2}}
\]

\[
+ e^{-t|\xi|^2} \sum_{l=k-\lfloor \frac{1}{2}\rfloor}^{k} \sum_{m=1}^{l-1} |D^{(k)}_{l,m} t^m |\xi|^{2l-k}.
\]

In the same way as before, the right-hand side is bounded by

\[
e^{-t|\xi|^2} \sum_{l=k-\lfloor \frac{1}{2}\rfloor}^{k} \langle t \rangle^{l} |\xi|^{2l-k} (|\xi|^2 + t|\xi|^4) + \langle t \rangle^{k} e^{-t^2 \frac{1}{4}} 1_{((t-\frac{1}{4}), \infty)}(|\xi|)
Proof of Proposition 2.7. Lemmas 2.2 and 2.9 give
\[
\|\nabla^s m(t, x)\| = (2\pi)^{-\frac{n}{2}} \left| \int_{\mathbb{R}^n} e^{ix\xi} \chi_{\tilde{\Omega}}(|\xi|) |\xi|^s \left\{ \frac{e^{i\frac{t}{2} + \sqrt{\frac{t}{2} - |\xi|^2}}}{2\sqrt{\frac{t}{2} - |\xi|^2}} - e^{-t|\xi|^2} \right\} d\xi \right|
\]
\[
\lesssim \int_{|\xi| \leq \frac{1}{4}} \left\{ (|\xi|^{s+2} + \langle t \rangle |\xi|^{s+4}) e^{-t|\xi|^2} + e^{-\frac{t}{4}} \right\} d\xi
\]
\[
+ \int_{\frac{1}{4} \leq |\xi| \leq 2} \left( \langle t \rangle e^{-\frac{t}{2}} + e^{-t|\xi|^2} \right) d\xi
\]
\[
\lesssim (t)^{-\frac{s+n}{2}+1}. \tag{33}
\]

To obtain the decay with respect to $|x|$, we assume $|x| \geq (t)^{\frac{1}{2}}$, otherwise the desired bound follows from (33). We divide $m$ into four parts

\[ m = m_1 + m_2 + m_3 + m_4, \]

where

\[ m_1(t, x) := \mathcal{F}^{-1} \left[ \chi_{\frac{1}{2\pi t} \leq \frac{1}{4}}(|\xi|) \left( \frac{e^{i\left(\frac{t}{2} + \sqrt{\frac{t}{2} - |\xi|^2}\right)}}{2\sqrt{\frac{t}{2} - |\xi|^2}} - e^{-t|\xi|^2} \right) \right](x), \]

\[ m_2(t, x) := \mathcal{F}^{-1} \left[ \chi_{\frac{1}{2\pi t} \leq \frac{1}{4}}(|\xi|) \left( \frac{e^{i\left(\frac{t}{2} + \sqrt{\frac{t}{2} - |\xi|^2}\right)}}{2\sqrt{\frac{t}{2} - |\xi|^2}} - e^{-t|\xi|^2} \right) \right](x), \]

\[ m_3(t, x) := -\mathcal{F}^{-1} \left[ \chi_{\frac{1}{8} \leq |\xi| \leq \frac{1}{4}}(|\xi|) \frac{e^{-i\left(\frac{t}{2} + \sqrt{\frac{t}{2} - |\xi|^2}\right)}}{2\sqrt{\frac{t}{2} - |\xi|^2}} \right](x), \]

\[ m_4(t, x) := -\mathcal{F}^{-1} \left[ \chi_{\frac{1}{8} \leq |\xi| \leq \frac{1}{4}}(|\xi|) \left( e^{-\frac{t}{2} L(t, \xi)} - e^{-t|\xi|^2} \right) \right](x). \]

Without loss of generality, we may assume that $|x| \sim |x_1|$. Since $\frac{1}{4\pi t} \leq t^{-\frac{1}{2}} \leq (t)^{-\frac{1}{4}}$, Lemma 2.9 yields

\[
\|\nabla^s m_1(t, x)\| = (2\pi)^{-\frac{n}{2}} \left| \int_{\mathbb{R}^n} e^{ix\xi} \chi_{\frac{1}{2\pi t} \leq \frac{1}{4}}(|\xi|) |\xi|^s \left( \frac{e^{i\left(\frac{t}{2} + \sqrt{\frac{t}{2} - |\xi|^2}\right)}}{2\sqrt{\frac{t}{2} - |\xi|^2}} - e^{-t|\xi|^2} \right) d\xi \right|
\]
\[
\lesssim \int_{|\xi| \leq \frac{1}{4\pi t}} (|\xi|^{s+2} + \langle t \rangle |\xi|^{s+4}) d\xi
\]
\[
\lesssim |x|^{-s-n-2} + t|x|^{-s-n-4} \sim |x|^{-s-n-2}. \tag{34}
\]

We set $j := \lceil a \rceil + n + 3$ for simplicity. Integration by parts $j$-times yields

\[
\|\nabla^s m_2(t, x)\| = (2\pi)^{-\frac{n}{2}} \left| \int_{\mathbb{R}^n} e^{ix\xi} \chi_{\frac{1}{8} \leq |\xi| \leq \frac{1}{4}}(|\xi|) |\xi|^s \left( \frac{e^{i\left(\frac{t}{2} + \sqrt{\frac{t}{2} - |\xi|^2}\right)}}{2\sqrt{\frac{t}{2} - |\xi|^2}} - e^{-t|\xi|^2} \right) d\xi \right|
\]
Proof. Let \( \|m_2(t,x)\| \lesssim |t|^{s-n-2} + \sum_{j=0}^{l} \langle |t|^{-j} \rangle \left[ \langle |t|^{-l+1} \rangle \leq 1 \right] \). Furthermore, Lemma 2.2 gives \( \|m_2(t,x)\| \lesssim |t|^{s-n-2} \). Then, we proceed the calculation with Lemma 2.9 to obtain

\[
\|\nabla^s m_2(t,x)\| \lesssim |t|^{-j} \left( \langle |t|^{-j} \rangle \leq 1 \right)
\]

\[
\lesssim |t|^{-j} (|t|^{-j} \langle |t|^{-l+1} \rangle \leq 1) \left[ \langle |t|^{-l+1} \rangle \leq 1 \right] \left[ \langle |t|^{-j} \rangle \leq 1 \right] + |t|^{-j} e^{-t^{\frac{1}{2}}}
\]

\[
\lesssim |t|^{-s-n-2} + |t|^{-j} (|t|^{-j} \langle |t|^{-l+1} \rangle \leq 1) \left[ \langle |t|^{-l+1} \rangle \leq 1 \right] + |t|^{-j} e^{-t^{\frac{1}{2}}}
\]

\[
\lesssim |t|^{-s-n-2},
\]

provided that \( |t| \geq (t)^{\frac{1}{2}} \). Here we note that \( |t|^{-j} \langle |t|^{-l+1} \rangle \leq 1 \) holds, since \( |t|^{-j} (t)^{\frac{1}{2}} \in [0,1] \).

The calculation used in (29) yields

\[
\|\nabla^s m_3(t,x)\| \lesssim |t|^{-j} (t)^{\frac{1}{2}} e^{-t^{\frac{1}{2}}}, \quad \|\nabla^s m_4(t,x)\| \lesssim |t|^{-j} (t)^{j+1} e^{-t^{\frac{1}{2}}}
\]

\[
\text{(35)}
\]

for any \( j \in \mathbb{N} \). Combining (33)–(35) we obtain the desired bound. \( \square \)

Proposition 2.7 leads to the \( L^p-L^q \) estimate for the difference.

**Proposition 2.10.** Let \( 1 \leq q \leq p \leq \infty \) and \( s_1 \geq s_2 \geq 0 \). Then,

\[
\|\nabla^s (D_1(t)-G(t)) g\|_{L^p} \lesssim \langle t \rangle^{-\frac{n}{2} (\frac{1}{q} - \frac{1}{p}) - \frac{n-2}{2} - 1} \|\nabla^s g\|_{L^q}
\]

for \( \|\nabla^s g\| \in L^q(\mathbb{R}^n) \) and \( t \geq 1 \).

**Proof.** We divide \( G(t) \) into low and high frequency parts \( G(t) = G_1(t) + G_2(t) \), where

\[
G_1(t) = F^{-1} [ \chi_{(1)}(\xi) e^{-t|\xi|^2} ] , \quad G_2(t) = F^{-1} [ \chi_{(2)}(\xi) e^{-t|\xi|^2} ] .
\]

Since Proposition 2.7 gives

\[
\|\nabla^s m(t)\|_{L^p} \lesssim \langle t \rangle^{-\frac{n}{2} (1 - \frac{1}{q}) - \frac{n-2}{2} - 1}
\]
for \( s \geq 0 \) and \( 1 \leq p \leq \infty \), the same argument as in the proof of Proposition 2.4 yields
\[
\| |\nabla|^{s_1} (D_1(t) - G_1(t)) |\|_{L^p} \lesssim \langle t \rangle^{-\frac{s}{2} \left( \frac{1}{2} - \frac{1}{p} \right)} \langle t \rangle^{-\frac{s}{2} \left( \frac{1}{2} - \frac{1}{p} \right)} \| |\nabla|^{s_2} g |\|_{L^s}.
\]

Set \( r := -(s_1 - s_2) + 2 \left( [s_1 - s_2] + n + 1 \right) \). Since \( r > n \), integration by parts gives
\[
|F^{-1}[| \cdot |^{-r} \chi_{\geq 1}]| = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i|x|} |\xi|^{-r} \chi_{\geq 1}(|\xi|) d\xi \lesssim \min(1, |x|^{-n-1}),
\]
which yields that \( F^{-1}[| \cdot |^{-r} \chi_{\geq 1}] \in L^1(\mathbb{R}^n) \). Hence, Young’s inequality and the well-known \( L^p-L^q \) estimates for the heat equation imply
\[
\| |\nabla|^{s_1} G_2(t) g |\|_{L^p} = \| F^{-1}[| \cdot |^{-2N-(s_1-s_2)} \chi_{\geq 1}] * (|\nabla|^{2N} g(t)) |\|_{L^p} \lesssim \| \Delta^N g(t) |\|_{L^p} \lesssim \langle t \rangle^{-\frac{s}{2} \left( \frac{1}{2} - \frac{1}{p} \right)} \| |\nabla|^{s_2} g |\|_{L^s}
\]
for \( t \geq 1 \) and for any large \( N \in \mathbb{N} \).

Theorem 1.2 follows from Propositions 2.5 and 2.10.

3. Local and global existence. Based on the linear estimates, we define the following function spaces. For \( T \in (0, \infty) \), \( s \geq 0 \), and \( r \in (1, 2) \), we define
\[
X(T) := \{ \phi \in L^\infty(0, T; H^s(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)); \| \phi \|_{X(T)} < \infty \}
\]
with the norm
\[
\| \phi \|_{X(T)} := \sup_{t \in (0, T)} \left\{ \langle t \rangle^s \| |\nabla|^{s} \phi(t) |\|_{L^2} + \langle t \rangle^\frac{s}{2} \| \phi(t) |\|_{L^2} + \| \phi(t) |\|_{L^r} \right\}.
\]

(36)

It is obvious that \( X(T) \) is a Banach space. Let \( M > 0 \). We consider the closed ball
\[
X(T, M) := \{ u \in X(T); \| u \|_{X(T)} \leq M \}.
\]
We also consider a wider function space
\[
Z(T) := L^\infty(0, T; L^2(\mathbb{R}^n))
\]
with the norm
\[
\| u \|_{Z(T)} := \| u \|_{L^\infty(0, T; L^2(\mathbb{R}^n))}.
\]

Then, we can see that \( X(T, M) \) is a closed subset of \( Z(T) \) for \( T \in (0, \infty) \) (see Lemma A.1). We shall find a local solution by constructing an approximate sequence in the ball \( X(T, M) \) and prove its convergence with respect to the metric
\[
d(u, v) := \| u - v \|_{Z(T)}.
\]

(37)

To this end, for the estimate of the nonlinear term satisfying (13) and \( |\nabla(u)| \lesssim |u|^p \), we define an auxiliary space. For \( T \in (0, \infty) \), \( s \geq 0 \), \( r \in (1, 2) \), and \( 1 < p \leq 2n/(n-2s) \) if \( n > 2s \) and \( 1 < p < \infty \) if \( n \leq 2s \), we define the space \( Y(T) \) as follows. When \( s > 1 \), we define
\[
Y(T) := \{ \psi \in L^\infty(0, T; \dot{H}^{s-1}(\mathbb{R}^n) \cap L^{\frac{2n}{n-2s}}(\mathbb{R}^n) \cap L^s(\mathbb{R}^n)); \| \psi \|_{Y(T)} < \infty \},
\]
where
\[
\| \psi \|_{Y(T)} := \sup_{t \in (0, T)} \left\{ \langle t \rangle^s \| |\nabla|^{s-1} \psi(t) |\|_{L^2} + \sup_{\gamma \in [\sigma_1, \sigma_2]} \langle t \rangle^\frac{s}{2} \| \psi(t) |\|_{L^r} \right\},
\]

(38)
and when $0 \leq s \leq 1$, we define
\[
Y(T) := \{ \psi \in L^\infty(0, T; L^{s_1}(\mathbb{R}^n) \cap L^{s_2}(\mathbb{R}^n)); \| \psi \|_{Y(T)} < \infty \},
\]
where
\[
\| \psi \|_{Y(T)} := \sup_{t \in (0, T)} \left\{ \langle t \rangle^n \| \nabla^{s_1-1} \chi_1(\nabla) \psi(t) \|_{L^2} + \sup_{\gamma \in [\sigma_1, \sigma_2]} \langle t \rangle^{\frac{n}{2}(\frac{\gamma}{s} - \frac{1}{2})} \| \psi(t) \|_{L^{\gamma}} \right\}
\]
and
\[
\eta = -\frac{1}{2} + \frac{s}{2} + \frac{n}{2} \left( \frac{r}{p} - \frac{1}{2} \right),
\]
\[
\sigma_1 = \max \left\{ 1, \frac{r}{p} \right\},
\]
\[
\sigma_2 = \begin{cases} 
2 & \text{if } 2s \geq n, \\
\min \left\{ 2, \frac{2n}{p(n-2s)} \right\} & \text{if } 2s < n.
\end{cases}
\]
We remark that the condition $p \leq 1 + \frac{\min(n, 2)}{n-2s}$ if $2s < n$ implies $\sigma_1 \leq \sigma_2$. We also note that the choices of the parameters $\eta, \sigma_1, \sigma_2$ are quite natural. Indeed, in the proof of Theorems 1.3 and 1.4, we use the norm of $Y(T)$ with $\psi = \mathcal{N}(u)$, that is, the nonlinear term of the equation (12). Roughly speaking, if $u$ belongs to $X(\infty)$, then $\| |u(t)|^p \|_{L^2} = \| u(t) \|_{L^{2p}}$ decays like $\langle t \rangle^{-\frac{n}{2} + (\frac{p}{2} - \frac{1}{2})}$, and hence, we expect $\| \nabla |^{s-1} |u(t)|^p \|_{L^2} \lesssim \langle t \rangle^{-n}$. Also, roughly speaking, if $u \in X(\infty)$, then the Sobolev embedding implies $u(t) \in L^r(\mathbb{R}^n) \cap L^{2n/(n-2s)}(\mathbb{R}^n)$, and hence, we expect $|u(t)|^p \in L^{r/p}(\mathbb{R}^n) \cap L^{2n/(p(n-2s))}(\mathbb{R}^n)$ and $\| |u(t)|^p \|_{L^{\gamma}}$ decays like $\langle t \rangle^{-\frac{n}{2} + (\frac{p}{2} - \frac{1}{2})}$ for $\gamma \in [\sigma_1, \sigma_2]$.

3.1. Local and global existence. Hereafter, we assume the condition in Theorem 1.3.

**Lemma 3.1.** Under the assumption of Theorem 1.3, we have
\[
\left\| \int_0^t \mathcal{D}(t - \tau) \psi(\tau) \, d\tau \right\|_{X(T)} \lesssim \int_0^T \| \psi \|_{Y(\tau)} \, d\tau
\]
for any $0 < T \leq 1$ and $\psi \in Y(T)$, and
\[
\left\| \int_0^T \mathcal{D}(t - \tau) \psi(\tau) \, d\tau \right\|_{X(T)} \lesssim \| \psi \|_{Y(T)} \begin{cases} 
1 & (p \geq 1 + \frac{2n}{n}) \\
(T)^{1-\frac{n}{2}(p-1)} & (p < 1 + \frac{2n}{n})
\end{cases}
\]
for any $T > 0$ and $\psi \in Y(T)$.

**Proof.** The first assertion is easily derived by modifying the following argument, and we omit its proof.

For the second assertion, first, we estimate
\[
\langle t \rangle^{\frac{n}{2}(\frac{p}{2} - \frac{1}{2})} \left\| \nabla \int_0^t \mathcal{D}(t - \tau) \psi(\tau) \, d\tau \right\|_{L^2} + \langle t \rangle^{\frac{n}{2}(\frac{p}{2} - \frac{1}{2})} \left\| \int_0^t \mathcal{D}(t - \tau) \psi(\tau) \, d\tau \right\|_{L^2}
\]
\[
\leq I + II + III,
\]
where
where

\[
I := \langle t \rangle^{\frac{1}{2}} (t^{-\frac{1}{2}} + \frac{1}{2}) \int_0^{t/2} \| \nabla |s| \mathcal{D}(t-\tau) \psi(\tau) \|_{L^2} d\tau \\
+ \langle t \rangle^{\frac{1}{2}} (t^{-\frac{1}{2}} + \frac{1}{2}) \int_0^{t/2} \| \mathcal{D}(t-\tau) \psi(\tau) \|_{L^2} d\tau + \int_0^{t/2} \| \mathcal{D}(t-\tau) \psi(\tau) \|_{L^r} d\tau,
\]

\[
II := \langle t \rangle^{\frac{1}{2}} (t^{-\frac{1}{2}} + \frac{1}{2}) \int_{t/2}^t \| \nabla |s| \mathcal{D}(t-\tau) \psi(\tau) \|_{L^2} d\tau \\
+ \langle t \rangle^{\frac{1}{2}} (t^{-\frac{1}{2}} + \frac{1}{2}) \int_{t/2}^t \| \mathcal{D}(t-\tau) \psi(\tau) \|_{L^2} d\tau,
\]

and

\[
III := \int_{t/2}^t \| \mathcal{D}(t-\tau) \psi(\tau) \|_{L^r} d\tau.
\]

For the term \( I \), applying Theorem 1.1 with \( p = 2 \) and \( q = \sigma_1 \), we have

\[
I \lesssim \langle t \rangle^{\frac{1}{2}} (t^{-\frac{1}{2}} + \frac{1}{2}) \int_0^{t/2} \left( (t-\tau)^{-\frac{1}{2}} \left( \frac{1}{2} - \frac{1}{2} \right) \right) \| \psi(\tau) \|_{L^{\sigma_1}} + e^{-\frac{\tau}{t^{\frac{1}{2}}}} (t-\tau)^{\frac{1}{2}} \| \nabla |s-1| \chi_{>1}(\nabla) \psi(\tau) \|_{L^2} d\tau \\
+ \langle t \rangle^{\frac{1}{2}} (t^{-\frac{1}{2}} + \frac{1}{2}) \int_0^{t/2} \left( (t-\tau)^{-\frac{1}{2}} \left( \frac{1}{2} - \frac{1}{2} \right) \right) \| \psi(\tau) \|_{L^{\sigma_1}} + e^{-\frac{\tau}{t^{\frac{1}{2}}}} (t-\tau)^{\frac{1}{2}} \| \nabla |s-1| \chi_{>1}(\nabla) \psi(\tau) \|_{L^2} d\tau \\
+ \int_0^{t/2} (t-\tau)^{-\frac{1}{2}} \left( \frac{1}{2} - \frac{1}{2} \right) \| \psi(\tau) \|_{L^{\sigma_1}} + e^{-\frac{\tau}{t^{\frac{1}{2}}}} (t-\tau)^{\frac{1}{2}} \| \nabla |s-1| \chi_{>1}(\nabla) \psi(\tau) \|_{H^{\beta-1}} d\tau.
\]

We note that \( \sigma_1 < r \),

\[
\| \psi(\tau) \|_{L^{\sigma_1}} \leq (\tau)^{-\frac{1}{2}} \left( \frac{\tau}{t} \right) \| \psi \|_{Y(T)}
\]

and

\[
\| \nabla |s-1| \chi_{>1}(\nabla) \psi(\tau) \|_{L^2} \leq \| \nabla |s-1| \chi_{>1}(\nabla) \psi(\tau) \|_{L^2} \leq (\tau)^{-\frac{1}{2}} \| \psi \|_{Y(T)},
\]

(42)

\[
\| \chi_{>1}(\nabla) \psi(\tau) \|_{H^{\beta-1}} \lesssim \| \psi \|_{L^p} \lesssim (\tau)^{-\frac{1}{2}} (\frac{\tau}{t}) \| \psi \|_{Y(T)}
\]

(43)

with \( \mu = \max\{\sigma_1, (\frac{1}{t} + \frac{1-\beta}{n})^{-1}\} \in [\sigma_1, \sigma_2] \), because \( p \leq 1 + \frac{2}{n-2s} \). Here, we note that \( \beta \leq 1 \) holds (see Remark 1.3). Therefore, a straightforward calculation gives

\[
I \lesssim \| \psi \|_{Y(T)} \begin{cases} 1 & (p \geq 1 + \frac{2}{n}), \\
(T)^{1-\frac{1}{p}(p-1)} & (p < 1 + \frac{2}{n}).
\end{cases}
\]

Next, we estimate \( II \). We divide the estimate of \( II \) into two cases. First, when \( s > 1 \), from Theorem 1.1 with \( (q, s_1, s_2) = (2, s, s-1) \) and \( (q, s_1, s_2) = (\sigma_2, 0, 0) \), we obtain

\[
II \lesssim \langle t \rangle^{\frac{1}{2}} (t^{-\frac{1}{2}} + \frac{1}{2}) \int_{t/2}^t \left( (t-\tau)^{-\frac{1}{2}} \| \nabla |s-1| \psi(\tau) \|_{L^2} + e^{-\frac{\tau}{t^{\frac{1}{2}}}} (t-\tau)^{\frac{1}{2}} \| \nabla |s-1| \chi_{>1}(\nabla) \psi(\tau) \|_{L^2} \right) d\tau
\]
determined later implies which completes the proof.

Under the assumption of Theorem 1.3, we have

We compute

\[ \| \nabla \|^{-1} \psi \|_{L^2} \leq \langle \tau \rangle^{-\eta} \|\psi\|_{Y(T)}, \]

\[ \| \psi \|_{L^2} \leq \langle \tau \rangle^{-\frac{s}{2}} \|\psi\|_{Y(T)} \]

and we have (42). Then, by a simple calculation, we deduce

\[ II \lesssim \|\psi\|_{Y(T)} \begin{cases} 1, & (p \geq 1 + \frac{2r}{n}), \\ \langle T \rangle^{1 - \frac{2r}{p-1}}, & (p < 1 + \frac{2r}{n}). \end{cases} \]

On the other hand, when \( 0 \leq s \leq 1 \), applying Theorem 1.1 with \((q,s_1,s_2) = (\sigma_2,s,0)\) and \((q,s_1,s_2) = (\sigma_2,0,0)\), we compute

\[ II \leq (t)^{\frac{s}{2}}(\frac{1}{2} - \frac{1}{2}) \int_{t/2}^t (t - \tau)^{-\frac{3}{2}} \|\psi\|_{L^2} \]

\[ + e^{-\frac{t-\tau}{r^2}} \langle \tau \rangle^s \|\psi\|_{Y(T)} \] 

\[ + (t)^{\frac{s}{2}}(\frac{1}{2} - \frac{1}{2}) \int_{t/2}^t (t - \tau)^{-\frac{3}{2}} \|\psi\|_{L^2} \]

\[ + e^{-\frac{t-\tau}{r^2}} \langle \tau \rangle^s \|\psi\|_{Y(T)} \] 

Noting that \(-\frac{s}{2} \left( \frac{1}{2} - \frac{1}{2} \right) - \frac{3}{2} > -1\) since \( 0 \leq s \leq 1 \), and using (44) and (42), we conclude

\[ II \lesssim \|\psi\|_{Y(T)} \begin{cases} 1, & (p \geq 1 + \frac{2r}{n}), \\ \langle T \rangle^{1 - \frac{2r}{p-1}}, & (p < 1 + \frac{2r}{n}). \end{cases} \]

Thus, for both cases we obtain the desired estimate.

Finally, we estimate III. Theorem 1.1 with \( p = r, s_1 = s_2 = 0 \), and \( q \in [\sigma_1, \sigma_2] \) determined later implies

\[ III \lesssim \int_{t/2}^t (t - \tau)^{-\frac{3}{2}} \|\psi\|_{L^2} + e^{-\frac{t-\tau}{r^2}} \langle \tau \rangle^s \|\psi\|_{H^{1} \chi_{>1}(\nabla)} \] 

First, we have (43). Next, for the first term, we choose \( q \) as

\[ q = \begin{cases} \sigma_1 + \epsilon \left( -\frac{1}{2} \left( \frac{1}{\sigma_1} - \frac{1}{2} \right) = -1 \right), \\ \sigma_1 \quad \text{(otherwise)} \end{cases} \]

with sufficiently small \( \epsilon > 0 \). Consequently, we have

\[ III \lesssim \|\psi\|_{Y(T)} \begin{cases} 1, & (p \geq 1 + \frac{2r}{n}), \\ \langle T \rangle^{1 - \frac{2r}{p-1}}, & (p < 1 + \frac{2r}{n}). \end{cases} \]

which completes the proof. \( \square \)

**Lemma 3.2.** Under the assumption of Theorem 1.3, we have

\[ \| \mathcal{N}(u) \|_{Y(T)} \lesssim \|u\|_{X(T)} \]

for any \( T > 0 \) and \( u \in X(T) \).
Proof. Let $\gamma \in [\sigma_1, \sigma_2]$. Then, by the assumption (13), we have $|N(u)| \lesssim |u|^p$ and we calculate
\[
(t)^{\frac{s}{2} - \frac{1}{2}} \|N(u)\|_{L^2} \lesssim \left( (t)^{\frac{s}{2} - \frac{1}{2}} \|u\|_{L^p} \right)^p.
\]
By the definition of $\sigma_1$ and $\sigma_2$, we see that
\[
r \leq p\sigma_1 \quad \text{and} \quad p\sigma_2 \begin{cases} < \infty & (2s \geq n), \\ \leq \frac{2n}{n-2s} & (2s < n).
\end{cases}
\]

This and the interpolation between $L^r$ and $L^2$ in the case that $p\gamma \in [r, 2]$ and between $H^s$ and $L^2$ in the other case imply $(t)^{\frac{s}{2} - \frac{1}{2}} \|u\|_{L^p} \leq \|u\|_{X(T)}$ (see also [9, Lemma 2.3] or [14, Lemma 2.3]), which leads to
\[
\langle t \rangle^{\frac{s}{2} - \frac{1}{2}} \|N(u)\|_{L^2} \lesssim \|u\|_{X(T)}^p.
\]

Next, we prove the following inequalities.
\[
\begin{align*}
\langle t \rangle^s \|\nabla^{s-1}N(u)\|_{L^2} & \lesssim \|u\|_{X(T)}^p & (s > 1), \\
\langle t \rangle^s \|\nabla^{s-1}\chi_1(\nabla)N(u)\|_{L^2} & \lesssim \|u\|_{X(T)}^p & (0 \leq s \leq 1).
\end{align*}
\]

First, we consider the case of $0 \leq s \leq 1$. Now, we have
\[
\|\nabla^{s-1}\chi_1(\nabla)N(u(t))\|_{L^2} \lesssim \|N(u(t))\|_{L^p} \lesssim \|u(t)\|_{L^p}^p,
\]
provided that $\rho$ satisfies
\[
\max \left\{ 1, \frac{2n}{n-2s+2} \right\} \leq \rho \leq 2, \tag{45}
\]
since $\|\nabla^{s-1}\chi_1(\nabla)f\|_{L^2} \sim \|\nabla^{s-1}f\|_{L^2}$ and $\|f\|_{L^2} \lesssim \|\nabla^{1-s}f\|_{L^p}$ hold for $\rho$ satisfying (45) by the Sobolev embedding. Since $1 < p \leq 1 + \frac{\min(n,2)}{n-2s} \leq \min\{\frac{2n}{n-2s}, 1 + \frac{2}{n-2s}\}$, there exists $\rho$ satisfying (45) such that
\[
r \leq \rho p \leq \frac{2n}{n-2s}.
\]

In fact, the family of intervals $\{[r/\rho, 2n/\rho(n-2s)]\}_{\rho \leq 2}$ covers the interval $(1, \min\{2n/(n-2s), 1 + 2/(n-2s)\})$. Namely,
\[
\left( 1, \min \left\{ \frac{2n}{n-2s}, 1 + \frac{2}{n-2s} \right\} \right] \subset \bigcup_{\rho \leq 2} \left[ \frac{r}{\rho}, \frac{2n}{\rho(n-2s)} \right].
\]

Then, we obtain
\[
\|u(t)\|_{L^p}^p \lesssim \langle t \rangle^{-\frac{s}{2} - \frac{1}{2}} \|u\|_{X(T)}^p.
\]

Next, we consider the case of $s > 1$. Let $\tilde{s}$ be the fractional part of $s$, namely, $\tilde{s} := s - [s]$. Using the Faa di Bruno formula (see the proof of Lemma 2.5 in [14]), the fractional Leibniz rule, and the fractional chain rule (see for example [4, Proposition 3.1, 3.3]), we see that
\[
\|\nabla^{s-1}N(u)\|_{L^2} = \|\nabla^{\tilde{s}}\nabla^{[s]-1}N(u)\|_{L^2} \\
\lesssim \|u\|_{L^p}^{[s]} \sum_{k=0}^{[s]} \|\nabla^{k+\tilde{s}}u\|_{L^{q_1(k)}} \prod_{j=2}^{[s]} \|\nabla^{k_j}u\|_{L^{r_j(k)}}
\]
Proof of Theorem 1.3. Let $C$ with some constant $s$ note that when $[s] = 1$, the above inequality is interpreted as

$$
\|u\|_{L^{p-[s]}(0,1)} \leq \sum_{k=(k_1,\ldots,k_{[s]}) \in \mathbb{Z}^{|s|-1}_{\geq 0}} \|\nabla|^{k_1+s+n\left(\frac{1}{2} - \frac{1}{q_1(k)}\right)}u\|_{L^2} \prod_{j=2}^{[s]} \|\nabla|^{k_j+n\left(\frac{1}{2} - \frac{1}{q_j(k)}\right)}u\|_{L^2},
$$

where $q_0$ and $q_j(k) (j = 1, 2, \ldots, [s])$ are given so that

$$
\begin{align*}
\frac{1}{2} = q_0 + \frac{1}{q_1(k)} + \cdots + \frac{1}{q_{[s]}(k)}, \\
2 < q_j(k) < \infty \text{ for } j = 1, 2, \ldots, [s], \\
\frac{r}{p-[s]} \leq q_0 \begin{cases} < \infty & \text{if } 2s \geq n, \\
\leq \frac{2n}{(p-[s])(n-2s)} & \text{if } 2s < n,
\end{cases} \\
k_1 + s + n \left(\frac{1}{2} - \frac{1}{q_1(k)}\right) \leq s, \\
k_j + n \left(\frac{1}{2} - \frac{1}{q_j(k)}\right) \leq s \text{ for } j = 2, 3, \ldots, [s].
\end{align*}
$$

We postpone the proof of the existence of such exponents to Appendix B. We also note that when $[s] = 1$, the above inequality is interpreted as

$$
\|\nabla|^{s-1}N(u)\|_{L^2} \lesssim \|u\|_{L^{p-[s]}(0,1)} \|\nabla|^{s+n\left(\frac{1}{2} - \frac{1}{q_1(k)}\right)}u\|_{L^2}.
$$

Finally, by the interpolation, we obtain

$$
\langle t\rangle^{\gamma} \|\nabla|^{s-1}N(u)\|_{L^2} \lesssim \|u\|^p_{X(T)}.
$$

The proof is complete. \hfill \Box

Now we prove the local existence of the solution. 

**Proof of Theorem 1.3.** Let $T \in [0,1]$. First, we note that Theorem 1.1 implies

$$
\|\dot{u} + D(t)\|_{X(T)} \leq C_0\varepsilon(\|u_0\|_{H^{s} \cap H^{s}} + \|u_1\|_{H^{s-1} \cap L^r})
$$

with some constant $C_0 = C_0(n, s, r) > 0$, which is independent of $T$. We put

$$
M(\varepsilon) := 2C_0\varepsilon(\|u_0\|_{H^{s} \cap H^{s}} + \|u_1\|_{H^{s-1} \cap L^r})
$$

and consider $X(T, M(\varepsilon))$. For $u \in X(T, M(\varepsilon))$, we define a mapping

$$
\Psi(u)(t) := (\dot{u} + D(t))u_0 + D(t)u_1 + \int_0^t D(t-\tau)N(u(\tau))d\tau.
$$

For $u, v \in X(T, M(\varepsilon))$, by Lemmas 3.1 and 3.2, we have

$$
\|\Psi(u)\|_{X(T)} \leq \frac{M(\varepsilon)}{2} + C_1TM(\varepsilon)^p
$$

with some constant $C_1 = C_1(n, s, r, p) > 0$. Moreover, we have

$$
\|\Psi(u) - \Psi(v)\|_{Z(T)} \leq C_2T(2M(\varepsilon))^{p-1}\|u - v\|_{Z(T)}
$$

with some constant $C_2 = C_2(n, s, r, p) > 0$. Indeed, we put

$$
\gamma = \frac{1}{q} = \frac{1}{q} + \frac{1}{2},
$$

and

$$
\frac{1}{q} = \frac{1}{q} + \frac{1}{2}.
$$
with sufficiently small $\delta > 0$. We note that the condition $p \leq 1 + \frac{n}{2s}$ if $2s < n$ implies $\gamma \geq 1$. Then, we calculate

$$\left\| \int_0^t \mathcal{D}(t - \tau) (\mathcal{N}(u(\tau)) - \mathcal{N}(v(\tau))) \, d\tau \right\|_{L^2} \leq \int_0^t \left\langle \left( t - \tau \right)^{-\frac{3}{2} - \frac{1}{2}} \left\| \mathcal{N}(u(\tau)) - \mathcal{N}(v(\tau)) \right\|_{L^\gamma} \right. $$

$$\left. + e^{-\frac{1}{\delta t}} \left( t - \tau \right)^{\delta_2} \left\| \nabla^{-1}(\mathcal{N}(u(\tau)) - \mathcal{N}(v(\tau))) \right\|_{L^2} \right) \, d\tau$$

$$\leq \int_0^t \left\| \mathcal{N}(u(\tau)) - \mathcal{N}(v(\tau)) \right\|_{L^\gamma} \, d\tau$$

$$\leq \int_0^t \left\| u(\tau) - v(\tau) \right\|_{L^2} \left( \left\| u(\tau) \right\|_{L^p(\mathbb{R}^n)} + \left\| v(\tau) \right\|_{L^p(\mathbb{R}^n)} \right) \, d\tau$$

$$\leq T \left\| u - v \right\|_{L^2(T)} \left( \left\| u \right\|_{L^p(T)} + \left\| v \right\|_{L^p(T)} \right).$$

(54)

Here, we have used the embedding

$$\left\| \phi \right\|_{L^2} \lesssim \left\| \nabla \phi \right\|_{L^\gamma}$$

with noting that $\frac{1}{2} \geq \frac{1}{\gamma} - \frac{1}{n}$, for the second inequality. Moreover, we have also used the Hölder inequality with the relation $\frac{1}{\gamma} = \frac{1}{2} + \frac{1}{\gamma}$.

Therefore, from (50) and (51), taking $T > 0$ sufficiently small depending on $C_1, C_2$, and $M(\varepsilon)$, we see that $\Psi$ is a contraction mapping in $X(T, M(\varepsilon))$ with the metric of $Z(T)$, and it has a unique fixed point $u$, which is a $H^s \cap L^r$-mild solution of (12).

We show that $u \in C([0, T); H^s(\mathbb{R}^n) \cap L^r(\mathbb{R}^n))$. The solution $u$ satisfies the integral equation

$$u(t) = (\partial_t + 1)\mathcal{D}(t)\varepsilon u_0 + \mathcal{D}(t)\varepsilon u_1 + \int_0^t \mathcal{D}(t - \tau)\mathcal{N}(u(\tau)) \, d\tau.$$

Since the linear part of the solution obviously satisfies this property, it suffices to show that

$$\int_0^t \mathcal{D}(t - \tau)\mathcal{N}(u(\tau)) \, d\tau \in C([0, T); H^s(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)).$$

By Theorem 1.1, we have

$$\left\| \nabla^\alpha \mathcal{D}(t - \tau)\mathcal{N}(u(\tau)) \right\|_{L^2} \lesssim \left\| \chi_{\leq 1}(\nabla)\mathcal{N}(u(\tau)) \right\|_{L^2} + \left\| \nabla^{\alpha - 1} \chi_{\geq 1}(\nabla)\mathcal{N}(u(\tau)) \right\|_{L^2},$$

$$\left\| \mathcal{D}(t - \tau)\mathcal{N}(u(\tau)) \right\|_{L^2} \lesssim \left\| \chi_{\leq 1}(\nabla)\mathcal{N}(u(\tau)) \right\|_{L^2} + \left\| \nabla^{\alpha - 1} \chi_{\geq 1}(\nabla)\mathcal{N}(u(\tau)) \right\|_{L^2},$$

$$\left\| \mathcal{D}(t - \tau)\mathcal{N}(u(\tau)) \right\|_{L^r} \lesssim \left\| \chi_{\leq 1}(\nabla)\mathcal{N}(u(\tau)) \right\|_{L^r} + \left\| \chi_{\geq 1}(\nabla)\mathcal{N}(u(\tau)) \right\|_{H^s_r}$$

for $t \in [0, T)$ and $\tau \in [0, t]$ and the right-hand sides are bounded independently of $t$. Therefore, we can apply the dominated convergence theorem in the Bochner integral and thus the continuity holds.

We also prove that $\partial_t u \in C([0, T); H^{s-1}(\mathbb{R}^n))$. Since the linear part of the solution obviously satisfies this property, it suffices to show that

$$\partial_t \int_0^t \mathcal{D}(t - \tau)\mathcal{N}(u(\tau)) \, d\tau \in C([0, T); H^{s-1}(\mathbb{R}^n)).$$

(55)

We note that $\left\| \mathcal{N}(u) \right\|_{Y(T)}$ is bounded. This and Theorem 1.1 implies

$$\left\| \nabla^\alpha \mathcal{D}(t - \tau)\mathcal{N}(u(\tau)) \right\|_{L^2} \lesssim \left\| \chi_{\leq 1}(\nabla)\mathcal{N}(u(\tau)) \right\|_{L^2} + \left\| \nabla^{\alpha - 1} \chi_{\geq 1}(\nabla)\mathcal{N}(u(\tau)) \right\|_{L^2},$$
\[ \| \mathcal{D}(t - \tau)\mathcal{N}(u(\tau)) \|_{L^2} \lesssim \| \chi_{\leq 1}(\nabla)\mathcal{N}(u(\tau)) \|_{L^{\sigma_2}} + \| \nabla^{-1}\chi_{> 1}(\nabla)\mathcal{N}(u(\tau)) \|_{L^2} \]

for \( t \in [0, T) \) and \( \tau \in [0, t] \), and the right-hand sides are bounded. Therefore, for any fixed \( t \in [0, T) \), \( \mathcal{D}(t - \cdot)\mathcal{N}(u(\cdot)) \in L^\infty(0, t; H^s(\mathbb{R}^n)) \) holds. Moreover, this also leads to \( \partial_t \mathcal{D}(t - \cdot)\mathcal{N}(u(\cdot)) \in L^\infty(0, t; H^{s-1}(\mathbb{R}^n)) \). Indeed, by (9) in Theorem 1.1, we have

\[ \| \nabla^{s-1}\partial_t \mathcal{D}(t - \tau)\mathcal{N}(u(\tau)) \|_{L^2} \lesssim \| \chi_{\leq 1}(\nabla)\mathcal{N}(u(\tau)) \|_{L^{\sigma_2}} + \| \nabla^{-1}\chi_{> 1}(\nabla)\mathcal{N}(u(\tau)) \|_{L^2}, \]

\[ \| \partial_t \mathcal{D}(t - \tau)\mathcal{N}(u(\tau)) \|_{L^2} \lesssim \| \chi_{\leq 1}(\nabla)\mathcal{N}(u(\tau)) \|_{L^{\sigma_2}} + \| \nabla^{-1}\chi_{> 1}(\nabla)\mathcal{N}(u(\tau)) \|_{L^2} \]

for \( t \in [0, T) \) and \( \tau \in [0, t] \), and the right-hand sides are bounded independently of \( t \). Therefore, by the Lebesgue convergence theorem in the Bochner integral and \( \mathcal{D}(0) = 0 \), we see that

\[ \partial_t \int_0^t \mathcal{D}(t - \tau)\mathcal{N}(u(\tau)) \, d\tau = \int_0^t \partial_t \mathcal{D}(t - \tau)\mathcal{N}(u(\tau)) \, d\tau, \]

which implies (55).

Next, we show that under the assumptions of Theorem 1.3, for any fixed \( T_0 > 0 \), the \( H^s \cap L^r \)-mild solution on the interval \([0, T_0]\) is unique. This also implies the uniqueness of \( H^s \cap L^r \)-mild solution, because an \( H^s \cap L^r \)-mild solution is also an \( H^s \)-mild solution. Let \( T_0 > 0 \) and fix it, and let \( u, v \) be \( H^s \)-mild solutions of (12) with same initial data \( \varepsilon(u_0, u_1) \in H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n) \). Let \( T_1 \in (0, T_0) \) be an arbitrary number. We define

\[ \| \phi \|_{X_2(T_1)} := \sup_{0 < t < T_1} \left\{ \{ t \} \frac{1}{2} \| \nabla^s \phi(t) \|_{L^2} + \| \phi(t) \|_{L^2} \right\}. \]

Then, there exists a constant \( M > 0 \) such that \( \| u \|_{X_2(T_1)} + \| v \|_{X_2(T_1)} \leq M \). From this and the same argument as deriving (51) with \( r = 2 \), we can see that

\[ \| u - v \|_{Z(T)} \leq C_3 M^{p-1} \int_0^T \| u - v \|_{Z(\tau)} \, d\tau \]

for any \( T \in [0, T_1] \) with some constant \( C_3 = C_3(n, s, p) > 0 \). Indeed, for \( t \in [0, T] \), we can obtain

\[ \int_0^t \mathcal{D}(t - \tau) (\mathcal{N}(u(\tau)) - \mathcal{N}(v(\tau))) \, d\tau \lesssim \int_0^t \| u(\tau) - v(\tau) \|_{L^2} (\| u(\tau) \|_{L^p(n-1)} + \| v(\tau) \|_{L^p(n-1)})^{p-1} \, d\tau \]

\[ \lesssim (\| u \|_{X_2(T)} + \| v \|_{X_2(T)})^{p-1} \int_0^t \| u - v \|_{Z(\tau)} \, d\tau \]

\[ \lesssim M^{p-1} \int_0^t \| u - v \|_{Z(\tau)} \, d\tau \]

in the same manner as (54), where \( q \) and \( \gamma \) are defined in (52) and (53). Thus, the Gronwall inequality implies \( u \equiv v \) on \([0, T_1]\). Since \( T_1 \in (0, T_0) \) is arbitrary, we have \( u \equiv v \) on \([0, T_0]\).

We next prove the locally Lipschitz continuity of the solution map

\[ (H^s(\mathbb{R}^n) \cap H^r(\mathbb{R}^n)) \times (H^{s-1}(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)) \to C([0, T); L^2(\mathbb{R}^n)), \]

\[ \varepsilon(u_0, u_1) \mapsto u. \]
Let $M > 0$ and we consider the ball

$$B(M) := \{ \varepsilon(u_0, u_1) \in (H^s(\mathbb{R}^n) \cap H^0_1(\mathbb{R}^n)) \times (H^{s-1}(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)) ; \\
\|\varepsilon(u_0, u_1)\|_{(H^s \cap H^0_1) \times (H^{s-1} \cap L^r)} \leq M\},$$

for the initial data. Then, by the proof of the existence part above, we find $T > 0$ depending only on $M$ such that for each $\varepsilon(u_0, u_1) \in B(M)$, there exists a unique solution $u \in X(T, 2M)$. Let $\varepsilon(u_0, u_1), \varepsilon(v_0, v_1) \in B(M)$ and let $u, v$ be the associated solutions, respectively. From this and the same argument before, we see that

$$\|u - v\|_{Z(T)} \lesssim \|\varepsilon(u_0 - v_0)\|_{H^s \cap H^0_1} + \|\varepsilon(u_1 - v_1)\|_{H^{s-1} \cap L^r},$$

and

$$\|u - v\|_{Z(T)} + (2M)^{p-1}\int_0^T \|u - v\|_{Z(T)} \, d\tau.$$ 

Therefore, the Gronwall inequality implies

$$\|u - v\|_{Z(T)} \lesssim \|\varepsilon(u_0 - v_0)\|_{H^s \cap H^0_1} + \|\varepsilon(u_1 - v_1)\|_{H^{s-1} \cap L^r},$$

which shows the locally Lipschitz continuity of the solution map.

Finally, we prove the blow-up alternative for $H^s$-mild solution, namely, $T_2(\varepsilon) < \infty$ implies (15). Let us suppose $T_2(\varepsilon) < \infty$ and

$$\liminf_{t \to T_2(\varepsilon)} \|(u, \partial_t u)(t)\|_{H^r \times H^{r-1}} < \infty.$$ 

Then, there exist a constant $M > 0$ and a sequence $\{t_m\}_{m=1}^\infty \subset [0, T_2(\varepsilon))$ such that $t_m \to T_2(\varepsilon)$ ($m \to \infty$) and

$$\|(u, \partial_t u)(t_m)\|_{H^r \times H^{r-1}} \leq M \quad (m \geq 1).$$

We note that, from the above proof of the local existence of the $H^s$-mild solution in the case $r = 2$, we deduce that there exists $T_1 > 0$ independent of $\{t_m\}_{m=1}^\infty$ such that we can construct the solution

$$u \in C([t_m, t_m + T_1); H^r(\mathbb{R}^n)), \quad \partial_t u \in C([t_m, t_m + T_1); H^{s-1}(\mathbb{R}^n)).$$

However, letting $m \to \infty$, this contradicts the definition of the lifespan $T_2(\varepsilon)$. This completes the proof. 

**Proof of Theorem 1.4.** Let $p \geq 1 + \frac{2s}{n}$. Let $T > 0$ be an arbitrary finite number. We define $M(\varepsilon)$ by (48) and consider the mapping (49) on $X(T, M(\varepsilon))$. Then, applying Lemmas 3.1 and 3.2, we have for $u, v \in X(T, M(\varepsilon))$,

$$\|\Psi(u)\|_{X(T)} \leq \frac{M(\varepsilon)}{2} + C_1 M(\varepsilon)^p,$$

$$\|\Psi(u) - \Psi(v)\|_{Z(T)} \leq C_2 (2M(\varepsilon))^{p-1}\|u - v\|_{Z(T)}$$

with some constants $C_1, C_2 > 0$ independent of $T$, instead of (50) and (51), respectively. Indeed, the first estimate is a direct consequence of Lemmas 3.1 and 3.2. The second estimate is obtained by a similar way to the proof of (51). More precisely, we have

$$\frac{1}{2} \|D(t - \tau) (N(u(\tau)) - N(v(\tau))) \, d\tau\|_{L^2} \leq \int_0^t (t - \tau)^{-\frac{p}{2}} (\varepsilon - \frac{1}{2}) \|N(u(\tau)) - N(v(\tau))\|_{L^2} \, d\tau$$

$$+ e^{-\frac{t}{2} \varepsilon} (t - \tau)^{\delta_2} \|\nabla\|^{-1} (N(u(\tau)) - N(v(\tau))) \|_{L^2} \, d\tau$$

$$+ C_1 M(\varepsilon)^p \int_0^t \|u - v\|_{Z(T)} \, d\tau$$

$$\|\varepsilon(u_0, u_1)\|_{L^2} = \varepsilon(t_0, u_0, u_1)$$

$$= \int_0^T \|N(u(\tau)) - N(v(\tau))\|_{L^2} \, d\tau \leq C_2 (2M(\varepsilon))^{p-1}\|u - v\|_{Z(T)}$$

$$\leq C_2 (2M(\varepsilon))^{p-1}\|u - v\|_{Z(T)}.$$
The term \( \approx \) is estimated as

\[
\langle \tau \rangle^{\frac{1}{\gamma} - \frac{1}{q(p-1)}} \| u(\tau) \|_{L^{q(p-1)}} + \| v(\tau) \|_{L^{q(p-1)}}^{p-1} \, d\tau
\]

where \( \gamma \) and \( q \) are defined by (53) and (52) and in the last inequality we have used the Sobolev inequality with \( s' = n \left( \frac{1}{2} - \frac{1}{q(p-1)} \right) \) and the interpolation inequality to obtain

\[
\begin{align*}
\langle \tau \rangle^{\frac{1}{\gamma} - \frac{1}{q(p-1)}} \| u(\tau) \|_{L^{q(p-1)}} & \\
\leq & \langle \tau \rangle^{\frac{1}{\gamma} - \frac{1}{q(p-1)}} \| \nabla^s u(\tau) \|_{L^2} \\
= & \langle \tau \rangle^{\frac{1}{\gamma} - \frac{1}{q(p-1)}} \| \nabla^s u(\tau) \|_{L^2} \\
\leq & \left( \langle \tau \rangle^{\frac{1}{\gamma} - \frac{1}{q(p-1)}} \| u(\tau) \|_{L^2} \right)^{1 - \frac{s'}{s}} \left( \langle \tau \rangle^{\frac{1}{\gamma} - \frac{1}{q(p-1)}} \| \nabla^s u(\tau) \|_{L^2} \right)^{\frac{s'}{s}} \\
\leq & \| u(\tau) \|_{X(T)}.
\end{align*}
\]

Here we also note that the definition of \( q \) implies \( 0 \leq s' \leq s \). Therefore, it suffices to show

\[
\int_0^t \langle t - \tau \rangle^{-\frac{\gamma}{2} \left( \frac{1}{\gamma} - \frac{1}{q(p-1)} \right)} \langle \tau \rangle^{-\frac{\gamma}{2} \left( \frac{1}{\gamma} - \frac{1}{q(p-1)} \right)} (p-1) \, d\tau \lesssim 1
\]  

(57)

holds under the condition \( p \geq 1 + \frac{2\gamma}{n} \). To prove (57), we divide the integral into

\[
\begin{align*}
\int_0^t & \langle t - \tau \rangle^{-\frac{\gamma}{2} \left( \frac{1}{\gamma} - \frac{1}{q(p-1)} \right)} \langle \tau \rangle^{-\frac{\gamma}{2} \left( \frac{1}{\gamma} - \frac{1}{q(p-1)} \right)} (p-1) \, d\tau \\
= & \int_0^{t/2} \langle t - \tau \rangle^{-\frac{\gamma}{2} \left( \frac{1}{\gamma} - \frac{1}{q(p-1)} \right)} \langle \tau \rangle^{-\frac{\gamma}{2} \left( \frac{1}{\gamma} - \frac{1}{q(p-1)} \right)} (p-1) \, d\tau \\
& + \int_{t/2}^t \langle t - \tau \rangle^{-\frac{\gamma}{2} \left( \frac{1}{\gamma} - \frac{1}{q(p-1)} \right)} \langle \tau \rangle^{-\frac{\gamma}{2} \left( \frac{1}{\gamma} - \frac{1}{q(p-1)} \right)} (p-1) \, d\tau \\
=: & A + B.
\end{align*}
\]

The term \( A \) is estimated as

\[
A \lesssim \langle t \rangle^{-\frac{\gamma}{2} \left( \frac{1}{\gamma} - \frac{1}{q(p-1)} \right)} \int_0^{t/2} \langle \tau \rangle^{-\frac{\gamma}{2} \left( \frac{1}{\gamma} - \frac{1}{q(p-1)} \right)} (p-1) \, d\tau.
\]

By noting that \( -\frac{\gamma}{2} \left( \frac{1}{\gamma} - \frac{1}{q(p-1)} \right) < 0 \), if \( -\frac{\gamma}{2} \left( \frac{1}{\gamma} - \frac{1}{q(p-1)} \right) (p - 1) \leq -1 \), then we immediately have \( A \lesssim 1 \). Otherwise, we also easily compute

\[
A \lesssim \langle t \rangle^{-\frac{\gamma}{2} \left( \frac{1}{\gamma} - \frac{1}{q(p-1)} \right)} \langle t \rangle^{-\frac{\gamma}{2} \left( \frac{1}{\gamma} - \frac{1}{q(p-1)} \right)} (p-1) + 1 = \langle t \rangle^{-\frac{\gamma}{2} (p-1) + 1} \lesssim 1,
\]

since \( \frac{1}{\gamma} = \frac{1}{p} + \frac{1}{2} \) and \( p \geq 1 + \frac{2\gamma}{n} \). Next, the term \( B \) is estimated as

\[
B \lesssim \langle t \rangle^{-\frac{\gamma}{2} \left( \frac{1}{\gamma} - \frac{1}{q(p-1)} \right)} \int_{t/2}^t \langle t - \tau \rangle^{-\frac{\gamma}{2} \left( \frac{1}{\gamma} - \frac{1}{q(p-1)} \right)} (p-1) \, d\tau.
\]
When $2s \geq n$, $\frac{1}{2} = \frac{1}{2} + \delta$ with sufficiently small $\delta$, and hence, $-\frac{n}{2} \left(\frac{1}{2} - \frac{1}{2}\right) > -1$ and we have

$$B \lesssim \langle t \rangle^{-\frac{2}{3}(\frac{1}{16} - \frac{1}{8})} \langle p \rangle^{-\frac{2}{3}(\frac{1}{16} - \frac{1}{8})+1} = \langle t \rangle^{-\frac{2}{3}(\frac{1}{16} - \frac{1}{8})+1} \lesssim 1,$$

since $\frac{1}{7} = \frac{1}{q} + \frac{1}{2}$ and $p \geq 1 + \frac{2r}{n}$. When $2s < n$, the definition of $\gamma$ is $\frac{1}{\gamma} = \left(\frac{2n}{(p-1)(n-2n)}\right)^{-1} + \frac{1}{2}$, which leads to $-\frac{2}{3} \left(\frac{1}{\gamma} - \frac{1}{2}\right) = -\frac{n-2s}{1} \geq -\frac{1}{2}$, since $p \leq 1 + \frac{2r}{n-2n}$. Hence, we have

$$B \lesssim \langle t \rangle^{-\frac{2}{3}(\frac{1}{16} - \frac{1}{8})} \langle p \rangle^{-\frac{2}{3}(\frac{1}{16} - \frac{1}{8})+1} = \langle t \rangle^{-\frac{2}{3}(\frac{1}{16} - \frac{1}{8})+1} \lesssim 1,$$

since $\frac{1}{\gamma} = \frac{1}{q} + \frac{1}{2}$ and $p \geq 1 + \frac{2r}{n}$. Thus, we have (57). Therefore, by taking $\varepsilon_0 > 0$ so that

$$C_1 M(\varepsilon)^p \leq \frac{M(\varepsilon)}{2}, \quad C_2 (2M(\varepsilon))^{n-1} \leq \frac{1}{2},$$

(58)

holds, the mapping $\Psi$ becomes a contraction mapping on $X(T, M(\varepsilon))$ with respect to the metric of $Z(T)$, and thus we have the solution $u \in X(T, M(\varepsilon))$. Moreover, the uniqueness has been already proved in the proof of Theorem 1.3. Since $T$ is arbitrary, the solution is global. Moreover, $u \in X(\infty, M(\varepsilon))$. Indeed, since we have $\|u\|_{X(T)} \leq M(\varepsilon)$ for arbitrary $T \in (0, \infty)$ from (56) and (58), we get $\|u\|_{X(\infty)} \leq M(\varepsilon)$. We also have

$$\|\mathcal{N}(u)\|_{Y(\infty)} \lesssim \|u\|_{\tilde{X}(\infty)} \lesssim M(\varepsilon) = C\varepsilon(\|u_0\|_{H^{s-1} \cap H^2} + \|u_1\|_{H^{s-1} \cap L^2})$$

(59)

since Lemma 3.2 holds for any $T \in (0, \infty)$. \hfill \Box

4. Global existence of an $H^s$-mild solution for small data. In this section, we give a sketch of the proof of Theorem 1.5. The argument is the same as that of Theorem 1.4 and we give only the difference. For $T \in (0, \infty)$, $s \geq 0$, and $r \in (1, 2]$, we define

$$\tilde{X}(T) := \left\{ \phi \in L^\infty(0, T; H^s(\mathbb{R}^n)); \|\phi\|_{\tilde{X}(T)} < \infty \right\}$$

with the norm

$$\|\phi\|_{\tilde{X}(T)} := \sup_{t \in [0, T]} \left\{ \langle t \rangle^\frac{3}{2} \langle \gamma - \frac{3}{2} \rangle^+ \|\nabla|^{s-1} \phi(t)\|_{L^2} + \langle t \rangle^{-\frac{3}{2}} \langle \gamma - \frac{3}{2} \rangle \|\phi(t)\|_{L^2} \right\}.$$

For $T \in (0, \infty)$, $s \geq 0$, and $r \in (1, 2]$, and $1 < p \leq 2n/(n - 2s)$ if $n > 2s$ and $1 < p < \infty$ if $n \leq 2s$, we also define the function space

$$\tilde{Y}(T) := \left\{ \psi \in L^\infty(0, T; \dot{H}^{s-1}(\mathbb{R}^n) \cap L^{r_1}(\mathbb{R}^n) \cap L^{r_2}(\mathbb{R}^n)); \|\psi\|_{\tilde{Y}(T)} < \infty \right\}$$

with the norms

$$\|\psi\|_{\tilde{Y}(T)} := \sup_{t \in [0, T]} \left\{ \langle t \rangle^\eta \|\nabla|^{s-1} \psi(t)\|_{L^2} + \sup_{\gamma \in [\sigma_1, \sigma_2]} \langle t \rangle^{-\frac{3}{2}} \langle \gamma - \frac{3}{2} \rangle \|\psi(t)\|_{L^2} \right\}$$

for $s > 1$ and

$$\|\psi\|_{\tilde{Y}(T)} := \sup_{t \in [0, T]} \left\{ \langle t \rangle^\eta \|\nabla|^{s-1} \chi_{>1} \nabla \psi(t)\|_{L^2} + \sup_{\gamma \in [\sigma_1, \sigma_2]} \langle t \rangle^{-\frac{3}{2}} \langle \gamma - \frac{3}{2} \rangle \|\psi(t)\|_{L^2} \right\}.$$
for $0 \leq s \leq 1$, where $\eta = -\frac{1}{2} + \frac{p}{2} + \frac{\sigma_1}{2} - \frac{1}{2}$, $\sigma_1 = \max\{1, \frac{2}{p}\}$, and $\sigma_2$ is the same as (41). We remark that the assumption $r > \frac{\sqrt{n^2 + 16n}}{4}$ implies that

$$1 + \frac{2r}{n} > \frac{2}{r}.$$  

From this and the assumption $p \geq 1 + \frac{2r}{n}$, it follows that $r > \frac{2}{p}$, which also implies $\sigma_1 < r$. Therefore, we can apply the same argument as Lemmas 3.1 and 3.2. We note that the norm of $\tilde{X}(T)$ does not involve $L^r$-norm, and hence, we do not need to estimate the $L^r$-norm of the solution. From this and repeating the same procedure as the proof of Theorem 1.4, we can prove the existence of a global solution.

5. Asymptotic behavior of the global solution. In this section, we give a proof of Theorem 1.6.

Proof of Theorem 1.6. Let $u$ be the global solution constructed in Theorem 1.4 and let $p > 1 + 2r/n$. We see that

$$u(t) - \varepsilon G(t)(u_0 + u_1) = w_L(t) + w_{NL}(t),$$

where

$$w_L(t) = \varepsilon (D(t) - G(t))(u_0 + u_1) + \varepsilon \partial_t D(t) u_0,$$

$$w_{NL}(t) = \int_0^t D(t - \tau) N(u(\tau)) \, d\tau.$$

From Theorems 1.1 and 1.2, we obtain

$$\|\nabla^s w_L(t)\|_{L^2} \lesssim \varepsilon \langle t \rangle^{-\frac{p}{2} + \frac{s}{2} - \frac{1}{2}} \left( \|u_0\|_{H^{s-1} \cap L^r} + \|u_1\|_{H^{s-1} \cap L^r} \right),$$

$$\|w_L(t)\|_{L^2} \lesssim \varepsilon \langle t \rangle^{-\frac{p}{2} - \frac{s}{2} - 1} \left( \|u_0\|_{L^2 \cap L^r} + \|u_1\|_{H^{s-1} \cap L^r} \right),$$

$$\|w_L(t)\|_{L^r} \lesssim \varepsilon \langle t \rangle^{-1} \left( \|u_0\|_{H^2} + \|u_1\|_{L^r} \right).$$

Therefore, it suffices to estimate $w_{NL}(t)$. We note that Lemma 3.2 gives

$$\|N(u)\|_{Y(\infty)} \leq C \varepsilon \left( \|u_0\|_{H^{s-1} \cap L^r} + \|u_1\|_{H^{s-1} \cap L^r} \right)$$

since $u$ belongs to $X(\infty, M(\varepsilon))$ (see Theorem 1.4 and (59)). In the same manner as in the proof of Lemma 3.1, we see that

$$\langle t \rangle^{\frac{s}{2} + \frac{1}{2} - \frac{1}{2}} \|\nabla^s w_{NL}(t)\|_{L^2}$$

$$\lesssim \|N(u)\|_{Y(\infty)} \langle t \rangle^{\frac{s}{2} + \frac{1}{2} - \frac{1}{2}} \left[ \int_0^t \langle t - \tau \rangle^{-\frac{s}{2} + \frac{1}{2}} \langle \tau \rangle^{\frac{s}{2} - \frac{1}{2}} \, d\tau \right]$$

$$+ \int_0^t \langle t - \tau \rangle^{-\frac{s}{2} + \frac{1}{2}} \langle \tau \rangle^{-\frac{s}{2} - \frac{1}{2}} \, d\tau + \int_0^t e^{-\frac{t}{\varepsilon}} \langle t - \tau \rangle^{-\delta} \langle \tau \rangle^{-\eta} \, d\tau$$

$$\lesssim \varepsilon \left( \|u_0\|_{H^{s-1} \cap L^r} + \|u_1\|_{H^{s-1} \cap L^r} \right) \langle t \rangle^{-\min\left(\frac{s}{2}, (p-1)\right)} \langle \varepsilon \rangle^{-\frac{s}{2} - \frac{1}{2}}.$$  

Hereafter, $\delta$ denotes an arbitrary small positive number, and the implicit constants are dependent on $\delta$. We also have

$$\langle t \rangle^{\frac{s}{2} + \frac{1}{2} - \frac{1}{2}} \|w_{NL}(t)\|_{L^2}$$

$$\lesssim \|N(u)\|_{Y(\infty)} \langle t \rangle^{\frac{s}{2} + \frac{1}{2} - \frac{1}{2}} \left[ \int_0^t \langle t - \tau \rangle^{-\frac{s}{2} + \frac{1}{2}} \langle \tau \rangle^{\frac{s}{2} - \frac{1}{2}} \langle \varepsilon \rangle^{-\frac{s}{2} - \frac{1}{2}} \, d\tau \right]$$
By Lemma 3.1, when \(1 < p < 1 + 2r/n\), we see that there exists a constant \(C_1 > 0\) such that

\[
\left\| \int_0^t D(t - \tau)\psi(\tau) d\tau \right\|_{X(T)} \leq C_1 \|\psi\|_{Y(T)} \langle T \rangle^{1 - \frac{2}{p}(p-1)}
\]

for any \(T > 0\) and \(\psi \in Y(T)\). Also, it follows from Lemma 3.2 that \(\|N(u)\|_{Y(T)} \leq C_1 \|u\|^p_{X(T)}\) for any \(T > 0\) and \(u \in X(T)\). Based on these estimates, we repeat the argument in the proof of Theorem 1.3. First, we have (47) for any \(T > 0\). We define a constant \(M(\varepsilon)\) by (48). Let \(\Psi\) be a mapping on \(X(T, M(\varepsilon))\) given by (49). Then, instead of (50) and (51), in this case we obtain

\[
\|\Psi(u)\|_{X(T)} \leq \frac{M(\varepsilon)}{2} + C_1 \langle T \rangle^{1 - \frac{2}{p}(p-1)} M(\varepsilon)^p,
\]

\[
\|\Psi(u) - \Psi(v)\|_{Z(T)} \leq C_2 \langle T \rangle^{1 - \frac{2}{p}(p-1)} M(\varepsilon)^p \|u - v\|_{Z(T)}
\]

with some constant \(C_2 > 0\) for any \(u, v \in X(T, M(\varepsilon))\) and \(T > 0\). Therefore, as long as

\[
\max\{C_1, C_2\} \langle T \rangle^{1 - \frac{2}{p}(p-1)} (2C_0(\|u_0\|_{H^r \cap H^\beta} + \|u_1\|_{H^{r-1} \cap L^r}))^{p-1} \varepsilon^{p-1} \leq \frac{1}{2}
\]

we have

\[
\|w_{NL}(t)\|_{L^r} \lesssim \|N(u)\|_{Y(\infty)} \left[ \int_0^{t/2} (t - \tau)^{-\frac{2}{r}(1 + \frac{r}{p})} - \frac{2}{q} \left( \frac{\tau}{r} - \frac{1}{2} \right) d\tau \right] \lesssim \varepsilon(\|u_0\|_{H^r \cap H^\beta} + \|u_1\|_{H^{r-1} \cap L^r})(t)^{-\min\left\{ \frac{2}{p}(p-1) - 1, \frac{2}{q} \left( \frac{1}{2} - \frac{1}{r} \right) \right\} + \delta},
\]

where \(q = \min\{r, \sigma_2\} = \min\{r, \frac{2n}{p(n-2r)}\} \in [\sigma_1, \sigma_2]\) and \(\sigma_1, \sigma_2\) are defined in (40), (41). Here we note that \(q > r/p\) holds under the assumption of Theorem 1.6. Consequently, we have

\[
\|\nabla^\alpha (u(t) - \varepsilon \Phi(u_0 + u_1))\|_{L^2} \lesssim (t)^{-\frac{2}{q} \left( \frac{1}{2} - \frac{1}{r} \right) - \frac{2}{q} \min\left\{ \frac{2}{p}(p-1) - 1, \frac{2}{q} \left( \frac{1}{2} - \frac{1}{r} \right) \right\} + \delta}
\]

\[
\times \varepsilon(\|u_0\|_{H^r \cap H^\beta} + \|u_1\|_{H^{r-1} \cap L^r}),
\]

\[
\|u(t) - \varepsilon \Phi(u_0 + u_1)\|_{L^r} \lesssim (t)^{-\frac{2}{q} \left( \frac{1}{2} - \frac{1}{r} \right) - \frac{2}{q} \min\left\{ \frac{2}{p}(p-1) - 1, \frac{2}{q} \left( \frac{1}{2} - \frac{1}{r} \right) \right\} + \delta}
\]

\[
\times \varepsilon(\|u_0\|_{H^r \cap H^\beta} + \|u_1\|_{H^{r-1} \cap L^r}),
\]

\[
\|u(t) - \varepsilon \Phi(u_0 + u_1)\|_{L^\infty} \lesssim (t)^{-\min\left\{ \frac{2}{p}(p-1) - 1, \frac{2}{q} \left( \frac{1}{2} - \frac{1}{r} \right) \right\} + \delta}
\]

\[
\times \varepsilon(\|u_0\|_{H^r \cap H^\beta} + \|u_1\|_{H^{r-1} \cap L^r}).
\]

This completes the proof. 

\[\square\]

6. Lower bound of the lifespan. In this section, we prove Theorem 1.7.

Proof of Theorem 1.7. By Lemma 3.1, when \(1 < p < 1 + 2r/n\), we see that there exists a constant \(C_1 > 0\) such that

\[
\left\| \int_0^t D(t - \tau)\psi(\tau) d\tau \right\|_{X(T)} \leq C_1 \|\psi\|_{Y(T)} \langle T \rangle^{1 - \frac{2}{p}(p-1)}
\]
holds, the mapping $\Psi$ is contractive on $X(T, M(\varepsilon))$ with respect to the metric of $Z'(T)$, and we can construct a unique local solution. We take $\varepsilon_1 > 0$ sufficiently small so that
\[
\max\{C_1, C_2\} (2C_0(\|u_0\|_{H^r} + \|u_1\|_{H^{r-1}(\mathbb{R}^n)}))^{\frac{1}{2}} < 1,
\]
where $C_3 = C_3(n, s, r, p, \|u_0\|_{H^r}, \|u_1\|_{H^{r-1}(\mathbb{R}^n)}) > 0$ is a constant independent of $\varepsilon$. Since we can construct a solution until the time $\hat{T}(\varepsilon)$, we have $\hat{T}(\varepsilon) \leq T_r(\varepsilon)$. Finally, retaking $\varepsilon_1 > 0$ smaller if needed so that $C_3\varepsilon_1^{-1/\omega} \geq \sqrt{4/3}$, we have for any $\varepsilon \in (0, \varepsilon_1)$,
\[
T_r(\varepsilon) \geq \hat{T}(\varepsilon) = \sqrt{(C_3\varepsilon^{-1/\omega})^2 - 1} \geq \frac{C_3}{2} \varepsilon^{-1/\omega},
\]
which gives the desired estimate (17).

\[
\square
\]

7. Upper bound of the lifespan. In this section, we prove Theorem 1.8. More precisely, we will give more general blow-up result (see Proposition 7.2), and Theorem 1.8 will follow as a corollary of it. A similar result in $L^1(\mathbb{R}^n)$-data setting and in the Fujita-subcritical case, i.e. $p < 1 + \frac{2}{n}$ was obtained in [5].

We define a smooth compactly supported function $\tilde{\psi} \in C^\infty([0, \infty); [0, 1])$ as
\[
\tilde{\psi}(y) = \begin{cases} 
1 & \text{if } y \leq 1, \\
\frac{1}{\sqrt{2}} & \text{if } 1 < y < 2, \\
0 & \text{if } y \geq 2.
\end{cases}
\]
Set $\psi(x) := \tilde{\psi}(|x|)$ for $x \in \mathbb{R}^n$, and for $R > 0$ let $\psi_R(x) = \psi(x/R)$. For $p > 1$, and $A > 0$, we define $\mu = \mu(p, A)$ by
\[
\mu = \mu(p, A) := \min\left\{1, \frac{p-1}{2} A \right\}.
\]

For $n \in \mathbb{N}$, $p > 1$, $l \in \mathbb{N}$ satisfying $l > 2p'$, where $p' := p/(p - 1)$, and for $\phi \in C_0^\infty(\mathbb{R}^n)$ with $\phi \geq 0$, we also define $A(n, p, l, \phi)$ as
\[
A = A(n, p, l, \phi) := 2^{p'-1}p' \frac{1 - p}{p} \frac{1}{p'} \left[\frac{\|\phi\|_{L^1(\mathbb{R}^n)}}{\|\phi\|_{L^p(\mathbb{R}^n)}}\right] \left[\|\phi\|_{L^1(\mathbb{R}^n)}\right] \left[\|\phi\|_{L^p(\mathbb{R}^n)}\right],
\]
where
\[
\Phi := \phi^{-1}\Delta(\phi) = l(l - 1)\nabla\phi \cdot \nabla\phi + l\phi\Delta\phi.
\]
We derive an ordinary differential inequality for the weighted average of the solution, i.e.
\[
I_\phi(t) = I_\phi[u](t) := \int_{\mathbb{R}^n} u(t, x)\phi(x)dx,
\]
up to the constant $A(n, p, l, \phi)$ via the method in [5].

At first, we note that for the local $H^s$-mild solution $u(t) \in H^s(\mathbb{R}^n)$ constructed in Theorem 1.3, we see that the linear part $\varepsilon D(t)(u_0 + u_1) + \varepsilon \partial_t D(t)u_0$ satisfies the equation $u_{tt} - \Delta u + u_t = 0$ in $H^{s-2}(\mathbb{R}^n)$. Also, from the proof of Theorem 1.3, we know that $\mathcal{N}(u) \in C([0, T_2(\varepsilon)]; H^{s-1}(\mathbb{R}^n) \cap L^{\sigma_2})$ and hence, $\mathcal{N}(u) \in C([0, T_2(\varepsilon)]; H^{-\frac{n(1/\sigma_2-1/2)}{2}}(\mathbb{R}^n))$. Thus, the nonlinear part $\int_0^t D(t - \tau)\mathcal{N}(u(\tau))d\tau$
satisfies the equation \( u_{tt} - \Delta u + u_t = \mathcal{N}(u) \) in \( H^{-n(1/\sigma_2 - 1/2)}(\mathbb{R}^n) \). Consequently, the local \( H^s \)-mild solution \( u \) satisfies the equation (12) in \( H^{\min(2-n, -n(1/\sigma_2 - 1/2))}(\mathbb{R}^n) \) for each \( t \in (0, T_2(\varepsilon)) \). This enables us to consider the coupling of the equation (12) with a test function, and we can derive an ordinary differential inequality by the argument in [5, Section 3].

We recall a blow-up result obtained in [5], which gives an upper estimate of the lifespan of solutions to (12) in a general setting. We note that in the following theorem, we do not need any condition on \( p \) such as \( p < 1 + \frac{2r}{n} \), but we impose certain condition on the test function \( \phi \).

**Proposition 7.1** (Proposition 1.3 in [5]). Let \( s \geq 0 \), \( p \in (1, \infty) \) and \((u_0, u_1) \in H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n) \), and let \( u \) be the associated \( H^s \)-mild solution to (12) constructed in Theorem 1.3 (the solution obtained by applying Theorem 1.3 with \( r = 2 \)). Assume that there exists \( \phi \in S(\mathbb{R}^n; [0, \infty)) \) such that the inequalities

\[
0 < I_{\phi}(0) - A(n, p, l, \phi) < 2\frac{1}{l} \|\phi\|_{L^1(\mathbb{R}^n)}, \quad I'_{\phi}(0) > 0 \tag{63}
\]

holds, where \( l \in \mathbb{N} \) with \( l > 2r' \). Let

\[
J_{\phi}(t) := I_{\phi}(t) - A(n, p, l, \phi), \quad \text{for } t \in [0, T_2(\varepsilon)),
\]

\[
J_{\phi}(0) := 2^{-\frac{1}{l}} \|\phi\|_{L^1(\mathbb{R}^n)}^{-1} I_{\phi}(0),
\]

\[
A_1 := \frac{J'_{\phi}(0)}{I'_{\phi}(0)} = \frac{I'_{\phi}(0)}{I_{\phi}(0) - A(n, p, l, \phi)}.
\]

Then, the estimate

\[
J_{\phi}(t) \geq J_{\phi}(0) \left( 1 - \mu(p, A_1) J_{\phi}(0)^{p-1} t \right)^{-\frac{1}{p-1}} \tag{64}
\]

is valid for \( t \in [0, T_2(\varepsilon)) \). Moreover, the lifespan \( T_2(\varepsilon) \) of the \( H^s \)-mild solution \( u \) is estimated as

\[
T_2(\varepsilon) \leq \mu(p, A_1)^{-1} J_{\phi}(0)^{1-p}.
\]

**Remark 7.1.** From the estimate (64), we can expect that the blow-up rate of the solution \( u \) is similar to that of the second order ordinary differential equation \( y''(t) = y(t)^p \), which indicates the wave-like behavior of the solution near the blow-up time. However, we remark that the estimate (64) does not directly imply the blow-up rate of the solution, because there is a possibility that the blow-up time of \( \| (u, \partial_t u)(t) \|_{H^1 \times L^2} \) is earlier than that of \( J_{\phi}(t) \).

Proposition 7.1 means that the condition (63) is a sufficient condition for the blow-up of a solution. Indeed, we prove that if \( p \in (1, 1 + \frac{2r}{n}) \), we take the test function \( \phi = \psi_{R(\varepsilon)} \) with an appropriate scaling parameter \( R(\varepsilon) \), which will be defined later, we ensure the condition (63) for any \( \varepsilon > 0 \) and show an upper estimate of the lifespan of solutions to (12) for sufficiently small \( \varepsilon > 0 \).

To state our main result, we introduce several notation. We denote by \( |S_{n-1}| \) the surface area of the unit sphere \( S_{n-1} \) in \( \mathbb{R}^n \). For \( \varepsilon > 0 \), \( r \in (1, 2) \), \( p \in (1, 1 + \frac{2r}{n}) \), \( k \in (\frac{2}{r}, \min\{n, \frac{2}{p-1}\}) \), and \( c_0, C_0 > 0 \), set

\[
R(\varepsilon) := \max \left( 2\frac{1}{l}, \left\{ \frac{C_0 |S_{n-1}| 2^{n-k} \varepsilon}{(n-k)2^{\frac{1}{l}} \|\psi\|_{L^1}} \right\}^\frac{1}{2}, \left\{ \frac{4(n-k)A(n, p, l, \psi)}{c_0 |S_{n-1}| \varepsilon} \right\}^\frac{1}{p-1} \right).
\]
where $A = A(n, p, l, \psi)$ is defined by (62) with $\phi = \psi$.

**Proposition 7.2.** Let $n \in \mathbb{N}$, $r \in (1, 2]$, $p \in (1, 1 + \frac{2r}{n})$, $l \in \mathbb{N}$ with $l > 2p'$, $\frac{2}{r} < k < \min\{n, \frac{2}{p-1}\}$ and $\varepsilon > 0$. We assume that $(u_0, u_1) \in H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)$ satisfies

$$C_0(1+|x|)^{-k} \geq u_0(x) \geq \begin{cases} c_0|x|^{-k}, & \text{if } |x| \geq 1, \\ 0, & \text{if } |x| \leq 1, \end{cases}$$

and

$$u_1(x) \geq \begin{cases} c_1|x|^{-k}, & \text{if } |x| \geq 1, \\ 0, & \text{if } |x| \leq 1, \end{cases}$$

with some positive constants $c_0$, $c_1$ and $C_0$. Then there exists $\varepsilon_2 > 0$ depending only on $n, k, p, l, c_0, C_0$ such that for any $\varepsilon \in (0, \varepsilon_2]$ the associated $H^s$-mild solution $u$ of (12) satisfies

$$\int_{\mathbb{R}^n} u(t, x) \psi_{R(\varepsilon)}^l(x) \, dx \geq \frac{c_0|S_{n-1}|}{2(n-k)} R(\varepsilon)^{-k} \varepsilon \log R(\varepsilon) \left(1 - \frac{\mu_0\varepsilon}{2^{n-1}k} t\right)^{-\frac{2}{n-1}},$$

for $t \in [0, T_2(\varepsilon))$, with some constant $\mu_0 = \mu_0(n, p, k, c_1, C_0) > 0$ independent of $\varepsilon$. Moreover, the lifespan $T_2(\varepsilon)$ is estimated as

$$T_2(\varepsilon) \leq \mu_0^{-1} \varepsilon^{-\frac{1}{n-1} - \frac{2}{n}}.$$  

**Remark 7.2.** (i) The condition $\frac{2}{r} < k < n$ implies $(x)^{-k} \in L^r(\mathbb{R}^n) \setminus L^1(\mathbb{R}^n)$.

(ii) We can also consider the case $k = n$ if $p < 1 + \frac{2}{n}$ holds. In this case, with

$$R(\varepsilon) := \max \left\{ 2, \frac{c_0|S_{n-1}|}{2^{n-1} \|\psi\|_{L^1}} \right\},$$

we have the estimates

$$\int_{\mathbb{R}^n} u(t, x) \psi_{R(\varepsilon)}^l(x) \, dx \geq \frac{c_0|S_{n-1}|}{2(n-k)} \varepsilon \log R(\varepsilon) \left(1 - \frac{\mu_0\varepsilon}{2^{n-1}k} t\right)^{-\frac{2}{n-1}},$$

and

$$T_2(\varepsilon) \leq \mu_0^{-1} \varepsilon^{-\frac{1}{n-1} - \frac{2}{n}}.$$  

(iii) The case $k > n$, i.e. $u_0, u_1 \in L^1(\mathbb{R}^n)$, has already been studied in [5], and it follows that

$$\int_{\mathbb{R}^n} u(t, x) \psi_{R(\varepsilon)}^l(x) \, dx \geq \frac{c_0|S_{n-1}|}{4(k-n)^2} \left(1 - \frac{\mu_0\varepsilon}{2^{n-1}k} t\right)^{-\frac{2}{n-1}},$$

and (69).

Our proof of Proposition 7.2 is based on the combination of the argument of the proof of Corollary 1.4 in [5] and the proof of Corollary 1.8 in [6]. In Corollary 1.4 in [5], blow-up mechanism (upper estimate of lifespan and blow-up rate) to (12) in the Fujita-subcritical case, i.e. $1 < p < 1 + \frac{2}{n}$ and $L^1(\mathbb{R}^n)$-setting is studied. In Corollary 1.8 in [6], a similar result to the nonlinear Schrödinger equation is proved.

**Proof of Proposition 7.2.** We apply Proposition 7.1 with $\phi = \psi_{R(\varepsilon)}$ for sufficiently small $\varepsilon > 0$. To do so, we first prove that the condition (63) with $\phi = \psi_{R(\varepsilon)}$ is valid for any $\varepsilon > 0$. For any $\varepsilon > 0$, let $R(\varepsilon)$ be given by (65). By the definition (62) of $A$ and using changing variable, the identity

$$A(n, p, l, \psi_{R(\varepsilon)}) = A(n, p, l, \psi) R(\varepsilon)^{-2\frac{p}{p'}}$$

(70)
holds for any \( \varepsilon > 0 \). Since \( u \) satisfies the initial condition of (12), by this identity, the equality

\[
I_{\psi(R(\varepsilon))}(0) - A(n, p, l, \psi_R(\varepsilon)) = \varepsilon \int_{\mathbb{R}^n} u_0(x) \psi_R'(x) dx - A(n, p, l, \psi) R(\varepsilon)^{n-2} \frac{\varepsilon^{p}}{p} \tag{71}
\]

holds for any \( \varepsilon > 0 \). By the definition of \( R(\varepsilon) \), the estimates

\[
R(\varepsilon) \geq 2^{\frac{n-k}{2-k}} > 1 \tag{72}
\]

hold for any \( \varepsilon > 0 \). Since \( u_0 \) is non-negative and satisfies \( u_0(x) \geq c_0|x|^{-k} \) for \( |x| \geq 1 \), by the properties of the function \( \psi \), the estimates

\[
\int_{\mathbb{R}^n} u_0(x) \psi_R'(x) dx = \int_{0 \leq |x| \leq 2R(\varepsilon)} u_0(x) \psi_R'(x) dx \geq c_0 \int_{1 \leq |x| < R(\varepsilon)} |x|^{-k} dx
\]

\[
= \frac{c_0|S_{n-1}|}{n-k} (R(\varepsilon)^{-k} - 1) \geq \frac{c_0|S_{n-1}|}{2(n-k)} R(\varepsilon)^{n-k} \tag{73}
\]

hold for any \( \varepsilon > 0 \), where we have used the estimate (72) to obtain the last inequality. By the definition of \( R(\varepsilon) \), the estimate

\[
R(\varepsilon) \geq \left\{ \frac{4(n-k)A(n, p, l, \psi)}{c_0|S_{n-1}|} \right\}^{\frac{1}{n-k}} \tag{74}
\]

is true for any \( \varepsilon > 0 \). By the assumption \( k < \frac{2}{p-1} \) and the estimates (71) and (73), the inequalities

\[
I_{\psi(R(\varepsilon))}(0) - A(n, p, l, \psi_R(\varepsilon)) \geq \frac{c_0|S_{n-1}|}{2(n-k)} R(\varepsilon)^{-k} - A(n, p, l, \psi) R(\varepsilon)^{n-2} \frac{\varepsilon^{p}}{p}
\]

\[
= R(\varepsilon)^{-k} \left\{ \frac{c_0|S_{n-1}|}{2(n-k)} - A(n, p, l, \psi) R(\varepsilon)^{-(\frac{2}{p-1} - k)} \right\}
\]

\[
\geq \frac{c_0|S_{n-1}|}{4(n-k)} R(\varepsilon)^{n-k} \varepsilon > 0
\]

hold for any \( \varepsilon > 0 \).

Next we prove the inequality \( I'_{\psi(R(\varepsilon))}(0) > 0 \) for any \( \varepsilon > 0 \). In the same way as (73), we have for any \( \varepsilon > 0 \),

\[
I'_{\psi(R(\varepsilon))}(0) = \varepsilon \int_{\mathbb{R}^n} u_1(x) \psi_R'(x) dx > \frac{\varepsilon c_1|S_{n-1}|}{2(n-k)} R(\varepsilon)^{n-k} > 0. \tag{75}
\]

Finally, we prove that for any \( \varepsilon > 0 \), the inequality

\[
I_{\psi(R(\varepsilon))}(0) - A(n, p, l, \psi_R(\varepsilon)) < 2^{\frac{1}{p-1}} \| \psi_R' \|_{L^1(\mathbb{R}^n)}
\]

holds. Since \( A \) is positive, and \( u_0 \) satisfies

\[
u_0(x) \leq C_0(1 + |x|)^{-k}, \quad \text{for} \ x \in \mathbb{R}^n,
\]

due to the assumption of \( u_0 \), by the properties of the function \( \psi \), the estimates

\[
I_{\psi(R(\varepsilon))}(0) - A(n, p, l, \psi_R(\varepsilon)) < \varepsilon \int_{\mathbb{R}^n} u_0(x) \psi_R'(x) dx
\]

\[
\leq C_0 \varepsilon \int_{|x| < 2R(\varepsilon)} (1 + |x|)^{-k} dx \leq \frac{C_0|S_{n-1}|2^{n-k} \varepsilon}{n-k} R(\varepsilon)^{n-k}
\]

\[
\leq 2^{\frac{1}{p-1}} \| \psi' \|_{L^1(\mathbb{R}^n)} R(\varepsilon)^n = 2^{\frac{1}{p-1}} \| \psi_R' \|_{L^1(\mathbb{R}^n)} \tag{76}
\]
hold for any \( \varepsilon > 0 \), where we have used the conditions \( k < n \) and

\[
R(\varepsilon) \geq \left\{ \frac{C_0|S_{n-1}|2^{n-k}\varepsilon}{(n-k)2^{\frac{n-k}{2}}\|\psi\|_{L^1}} \right\}^{\frac{2}{p-1}}.
\]

Therefore, we find that the function \( \psi_{R(\varepsilon)} \) satisfies the condition (63) with \( \phi = \psi_{R(\varepsilon)} \) for any \( \varepsilon > 0 \). Thus we can apply Proposition 7.1 with \( \phi = \psi_{R(\varepsilon)} \), to obtain the estimate

\[
J_{\psi_{R(\varepsilon)}}(t) \geq \left( 1 - \mu(p, A_1(\varepsilon)) \right) J_{\psi_{R(\varepsilon)}}(0)^{p-1} t^{-\frac{2}{p-1}},
\]

for any \( t \in (0, T_2(\varepsilon)) \), and the lifespan \( T_2(\varepsilon) \) is estimated as

\[
T_2(\varepsilon) \leq \mu(p, A_1(\varepsilon))^{-1} J_{\psi_{R(\varepsilon)}}(0)^{1-p},
\]

where \( A_1(\varepsilon) \) satisfies

\[
A_1(\varepsilon) := \frac{I'_{\psi_{R(\varepsilon)}}(0)}{I_{\psi_{R(\varepsilon)}}(0) - A(n, p, l, \psi_{R(\varepsilon)})} \geq \frac{c_1}{C_02^{n-k+1}} =: \tilde{A}_1,
\]

for any \( \varepsilon > 0 \), where we have used the estimates (75) and (76). We note that \( \tilde{A}_1 \) is independent of \( \varepsilon > 0 \). Moreover, by changing variable and the estimate (74), the inequalities

\[
J_{\psi_{R(\varepsilon)}}(0)^{p-1} = 2^{-1}\|\psi\|_{L^1(\mathbb{R}^n)}^{-(p-1)} J_{\psi_{R(\varepsilon)}}(0)^{p-1}
\]

\[
\geq 2^{-1}\|\psi\|_{L^1(\mathbb{R}^n)}^{-(p-1)} \left\{ \frac{c_0|S_{n-1}|}{4(n-k)R(\varepsilon)^{n-k}\varepsilon} \right\}^{p-1}
\]

\[
= \left\{ \frac{c_0|S_{n-1}|}{2^{\frac{n-k}{p-1}+2(n-k)}\|\psi\|_{L^1}} \right\}^{p-1} R(\varepsilon)^{-k(p-1)\varepsilon} \quad (79)
\]

hold for any \( \varepsilon > 0 \). Here we take sufficiently small \( \varepsilon_2 = \varepsilon_2(n, k, l, \psi, c_0, C_0) > 0 \) so that

\[
R(\varepsilon) = \left\{ \frac{4(n-k)A(n, p, l, \psi)}{c_0|S_{n-1}|} \right\}^{-\frac{1}{p-1}}
\]

holds for \( \varepsilon \in (0, \varepsilon_2] \). Thus by combining (79) and (80), the estimate

\[
J_{\psi_{R(\varepsilon)}}(0)^{p-1} \geq 2^{-1+\frac{k-1}{p-1}} \left( \frac{n-k}{|S_{n-1}|} \right)^{\frac{2(k-1)(k-\frac{1}{p-1})}{p-1}} \times \|\psi\|_{L^1(\mathbb{R}^n)}^{k(p-1)} A \left( \psi, \varepsilon^{\frac{1}{p-1} - \frac{1}{2}} \right)
\]

holds for any \( \varepsilon \in (0, \varepsilon_2] \). Therefore, by combining the estimates (79) and (81), we obtain (67) with \( \mu_0 = \mu(p, A_1) \), which completes the proof.

**Appendix A. Closedness of \( X(T, M) \) in \( Z(T) \).** In this appendix, we prove that \( X(T, M) \) is a closed subset of \( Z(T) \) if \( T > 0 \) is finite. Let \( s \geq 0, r \in (1, 2], \) \( T \in (0, \infty) \). Since \( T \) is finite, we note that the topology of \( X(T) \) with respect to the norm (36) is the same as the usual topology of \( L^\infty(0, T; H^s(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)) \).

**Lemma A.1.** \( X(T, M) \) is a closed subset of \( Z(T) \).
Proof. First, it is obvious that $X(T, M) \subset Z(T)$. Therefore, it suffices to show that for any sequence in $X(T, M)$ converging in $Z(T)$, its limit belongs to $X(T, M)$. Let $\{u_j\}_{j=1}^\infty \subset X(T, M)$ converge in $Z(T)$ and let $u$ be its limit. We note that

$$L^\infty(0, T; H^s(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)) = \left( L^1(0, T; H^{-s}(\mathbb{R}^n) + L^r'(\mathbb{R}^n)) \right)^*,$$

where $r' = r/(r-1)$. This and the separability of $L^1(0, T; H^{-s}(\mathbb{R}^n) + L^r'(\mathbb{R}^n))$ (in general, the sum of two separable normed spaces is separable) enable us to apply the sequential Banach–Alaoglu theorem [1, Theorem 3.16]. From this theorem and

$$\|u_j\|_{X(T)} \leq M,$$

we can take a subsequence $\{u_{j(l)}\}_{l=1}^\infty$ and $v \in L^\infty(0, T; H^s(\mathbb{R}^n) \cap L^r(\mathbb{R}^n))$ such that

$$u_{j(l)} \overset{\ast}{\rightharpoonup} v \quad \text{in} \quad L^\infty(0, T; H^s(\mathbb{R}^n) \cap L^r(\mathbb{R}^n))$$

as $l \to \infty$. Moreover, we have

$$\|v\|_{X(T)} \leq \liminf_{l \to \infty} \|u_{j(l)}\|_{X(T)} \leq M$$

and hence, we see $v \in X(T, M)$. On the other hand, both $\{u_j\}_{j=1}^\infty$ and $\{u_{j(l)}\}_{l=1}^\infty$ converge in the space of the distribution $D'((0, T) \times \mathbb{R}^n)$ and hence, we obtain

$$u_{j(l)} \to v \quad \text{in} \quad D'((0, T) \times \mathbb{R}^n) \quad (l \to \infty),$$

$$u_j \to u \quad \text{in} \quad D'((0, T) \times \mathbb{R}^n) \quad (j \to \infty).$$

Thus, by the uniqueness of the limit of distribution implies $u = v$, which shows $u \in X(T, M)$.

Appendix B. Verification of (46). In the appendix, we prove the existence of the exponents $q_0$ and $q_j(k)$, $j \in \{1, \ldots, [s]\}$ satisfying (46).

Lemma B.1. Let $n$ be a positive integer. Let $s_1, \ldots, s_n > 0$ and $A > 0$ satisfy $A < \sum_{j=1}^n s_j$ (resp. $A \leq \sum_{j=1}^n s_j$). Then, there exist $a_1, \ldots, a_n > 0$ such that

$$a_j < s_j \quad \text{for any} \quad j \in \{1, \ldots, n\},$$

$$\sum_{j=1}^n a_j = A.$$

Proof. Let $\theta \in [0, 1]$ satisfy $\theta \sum_{j=1}^n s_j = A$, and we define $a_j := \theta s_j$ for $j = 1, \ldots, n$. Then, we easily see that $\{a_j\}_{j=1}^n$ has the desired property.\hfill $\square$

We apply this lemma to show the existence of $q_0$ and $q_j(k)$ $(j = 1, 2, \ldots, [s])$ satisfying (46).

First, we consider the case of $n > 2s$. Then, take $q_0$ satisfying

$$\frac{r}{p - |s|} \leq q_0 \leq \frac{2n}{(p - |s|)(n - 2s)},$$

$$\sum_{j=1}^{[s]} k_j + \delta + n \left( \frac{[s]}{2} - 1 + \frac{1}{q_0} \right) \leq s[\delta].$$
Namely, in this case, it suffices to take $q_0 = \frac{2n}{(p-|s|)(n-2s)}$, where we note that $p \leq 1 + 2/(n-2s)$. Then, noting $\sum_{j=1}^{|s|} k_j = |s| - 1$, we have

$$\frac{|s|}{2} - \frac{1}{2} + \frac{1}{q_0} \leq \frac{s - \bar{s} - k_1}{n} + \sum_{j=2}^{|s|} \frac{s - k_j}{n}.$$  

Applying Lemma B.1 with $A = \frac{|s|}{2} - \frac{1}{2} + \frac{1}{q_0}$, $s_1 = (s - \bar{s} - k_1)/n$, and $s_j = (s - k_j)/n$, we find $a_1, a_2, \ldots, a_{|s|} > 0$ such that

$$a_1 \leq \frac{s - \bar{s} - k_1}{n},$$  

$$a_j \leq \frac{s - k_j}{n} \text{ for any } j \in \{2, 3, \ldots, |s|\},$$  

$$\sum_{j=1}^{|s|} a_j = \frac{|s|}{2} - \frac{1}{2} + \frac{1}{q_0}.$$  

Since $n > 2s$, we have

$$\frac{s - \bar{s} - k_1}{n} < \frac{1}{2},$$  

$$\frac{s - k_j}{n} < \frac{1}{2} \text{ for any } j \in \{1, 2, \ldots, |s|\}.$$  

Therefore, $a_j < 1/2$ for all $j$. We define $q_j(k)$ such that

$$\frac{1}{q_j(k)} = \frac{1}{2} - a_j.$$  

Then, $2 < q_j(k) < \infty$. Moreover, we have

$$\sum_{j=1}^{|s|} \frac{1}{q_j(k)} = \frac{1}{2} - \frac{1}{q_0},$$  

$$k_1 + \bar{s} + n \left(\frac{1}{2} - \frac{1}{q_1(k)}\right) \leq s,$$

$$k_j + n \left(\frac{1}{2} - \frac{1}{q_j(k)}\right) \leq s \text{ for } j = 2, 3, \ldots, |s|,$$

where these come from (82), (83), and (84).

Next, we consider the case of $n \leq 2s$. Take $q_0$ satisfying

$$\frac{r}{p - |s|} \leq q_0 < \infty,$$

$$\frac{|s|}{2} - \frac{1}{2} + \frac{1}{q_0} < \min \left\{ \frac{1}{2}, s - \bar{s} - k_1 \right\} + \sum_{j=2}^{|s|} \min \left\{ \frac{1}{2}, s - k_j \right\}.$$  

We show that there exists $q_0$ satisfying the above inequality. The second inequality is equivalent to

$$\frac{1}{q_0} < \frac{1}{2} + \min \left\{ 0, \frac{2(s - \bar{s} - k_1)}{2n} \right\} + \sum_{j=2}^{|s|} \min \left\{ 0, \frac{2(s - k_j)}{2n} \right\}.$$  

Define the set $J$ by

$$J := \{ j \in \{2, 3, \ldots, |s|\} ; 2(s - k_j) - n < 0 \}.$$
We denote the number of the elements of \( J \) by \( \# J \) and \( \{2, 3, \ldots, [s]\} \setminus J \) by \( J^c \).

**Case 1.** We consider the case of \( 2(s - \tilde{s} - k_1) - n < 0 \). Then, we have

\[
\text{(R.H.S of (85))} = \frac{1}{2} + \frac{2(s - \tilde{s} - k_1) - n}{2n} + \sum_{j \in J} \frac{2(s - k_j) - n}{2n}
\]

\[
= \frac{1}{2} + \frac{2[s] - 2k_1 - n}{2n} + \frac{\# J(2s - n) - 2 \sum_{j \in J} k_j}{2n}
\]

\[
\geq \frac{2[s] + \# J(2s - n) - 2([s] - 1)}{2n}
\]

\[
= \frac{\# J(2s - n) + 2}{2n} > 0.
\]

**Case 2-I.** We consider the case of \( 2(s - \tilde{s} - k_1) - n \geq 0 \) and \( \# J \geq 1 \). Then, we have

\[
\text{(R.H.S of (85))} \geq \frac{1}{2} + \sum_{j \in J} \frac{2(s - k_j) - n}{2n}
\]

\[
= \frac{n + \# J(2s - n) - 2 \sum_{j \in J} k_j}{2n}
\]

\[
\geq \frac{n + \# J(2s - n) - 2([s] - 1)}{2n}
\]

\[
\geq \frac{\# J(2s - n) + n - 2s + 2}{2n}
\]

\[
= \frac{(\# J - 1)(2s - n) + 2}{2n} > 0.
\]

**Case 2-II.** We consider the case of \( 2(s - \tilde{s} - k_1) - n \geq 0 \) and \( \# J = 0 \). Then,

\[
\text{(R.H.S of (85))} = \frac{1}{2}.
\]

Therefore, \( \text{(R.H.S of (85))} \) is positive and thus it is enough to take sufficiently large \( q_0 \). Applying Lemma B.1 with \( A = [s]/2 - 1/2 + 1/q_0 \), \( s_1 = \min\{1/2, (s - \tilde{s} - k_1)/n\} \), and \( s_j = \min\{1/2, (s - k_j)/n\} \), we find \( a_1, a_2, \ldots, a_{[s]} > 0 \) such that

\[
a_1 < \min \left\{ \frac{1}{2}, \frac{s - \tilde{s} - k_1}{n} \right\}, \tag{86}
\]

\[
a_j < \min \left\{ \frac{1}{2}, \frac{s - k_j}{n} \right\} \text{ for any } j \in \{2, 3, \ldots, [s]\}, \tag{87}
\]

\[
\sum_{j=1}^{[s]} a_j = \frac{[s]}{2} - \frac{1}{2} + \frac{1}{q_0}. \tag{88}
\]

We define \( q_j(k) \) such that

\[
\frac{1}{q_j(k)} = \frac{1}{2} - a_j.
\]

Then, \( 2 < q_j(k) < \infty \). Moreover, we have the desired properties

\[
\sum_{j=1}^{[s]} \frac{1}{q_j(k)} = \frac{1}{2} - \frac{1}{q_0},
\]

\[
\sum_{j=1}^{[s]} \frac{1}{q_j(k)} > \frac{1}{2} - \frac{1}{q_0},
\]

\[
\sum_{j=1}^{[s]} \frac{1}{q_j(k)} < \frac{1}{2} - \frac{1}{q_0}.
\]
\[ k_1 + \hat{s} + n \left( \frac{1}{2} - \frac{1}{q_1(k)} \right) \leq s, \]
\[ k_j + n \left( \frac{1}{2} - \frac{1}{q_j(k)} \right) \leq s \text{ for } j=2,3,\ldots,[s], \]

where these come from (86), (87), and (88).

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REFERENCES

[1] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext. Springer, New York, 2011.
[2] J. Chen, D. Fan and C. Zhang, Space-time estimates on damped fractional wave equation, *Abstr. Appl. Anal.*, (2014).
[3] J. Chen, D. Fan and C. Zhang, Estimates for damped fractional wave equations and applications, *Electronic Journal of Differential Equations*, 2015 (2015), 1–14.
[4] F. M. Christ and M. I. Weinstein, Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation, *J. Funct. Anal.*, 100 (1991), 87–109.
[5] K. Fujiwara, M. Ikeda and Y. Wakasugi, Estimates of lifespan and blow-up rates for the wave equation with a time-dependent damping and a power-type nonlinearity, *Funkt. Ekvac.*, to appear, arXiv:1609.01035v2.
[6] K. Fujiwara and T. Ozawa, Finite time blowup of solutions to the nonlinear Schrödinger equation without gauge invariance, *J. Math. Phys.*, 57 (2016), 1–8.
[7] M.-H. Giga, Y. Giga and J. Saal, *Nonlinear Partial Differential Equations*, Progress in Nonlinear Differential Equations and their Applications, 79, Birkhäuser, Boston, MA, 2010.
[8] L. Grafakos, *Classical Fourier Analysis*, Third edition, Graduate Texts in Mathematics, 249, Springer, New York, 2014. xviii+638 pp.
[9] N. Hayashi, E. I. Kaikina and P. I. Naumkin, Damped wave equation with super critical nonlinearities, *Differential Integral Equations*, 17 (2004), 637–652.
[10] N. Hayashi, E. I. Kaikina and P. I. Naumkin, Damped wave equation with a critical nonlinearity, *Trans. Amer. Math. Soc.*, 358 (2006), 1165–1185.
[11] N. Hayashi, E. I. Kaikina and P. I. Naumkin, On the critical nonlinear damped wave equation with large initial data, *J. Math. Anal. Appl.*, 334 (2007), 1400–1425.
[12] N. Hayashi and P. I. Naumkin, Damped wave equation with a critical nonlinearity in higher space dimensions, *J. Math. Appl. Anal.*, 446 (2017), 801–822.
[13] T. Hosono and T. Ogawa, Large time behavior and \( L^p-L^q \) estimate of solutions of 2-dimensional nonlinear damped wave equations, *J. Differential Equations*, 203 (2004), 82–118.
[14] M. Ikeda, T. Inui and Y. Wakasugi, The Cauchy problem for the nonlinear damped wave equation with slowly decaying data, *NoDEA Nonlinear Differential Equations Appl.*, 24 (2017), no. 2, Art. 10, 53 pp.
[15] R. Ikehata, Y. Miyaoaka and T. Nakatake, Decay estimates of solutions for dissipative wave equations in \( \mathbb{R}^N \) with lower power nonlinearities, *J. Math. Soc. Japan*, 56 (2004), 365–373.
[16] R. Ikehata, K. Nishihara and H. Zhao, Global asymptotics of solutions to the Cauchy problem for the damped wave equation with absorption, *J. Differential Equations*, 226 (2006), 1–29.
[17] R. Ikehata and M. Ohta, Critical exponents for semilinear dissipative wave equations in \( \mathbb{R}^N \), *J. Math. Anal. Appl.*, 269 (2002), 87–97.
[18] R. Ikehata and K. Tanizawa, Global existence of solutions for semilinear damped wave equations in \( \mathbb{R}^N \) with noncompactly supported initial data, *Nonlinear Anal.*, 61 (2005), 1189–1208.
[19] G. Karch, Selfsimilar profiles in large time asymptotics of solutions to damped wave equations, *Studia Math.*, 143 (2000), 175–197.
[20] S. Kawashima, M. Nakao and K. Ono, On the decay property of solutions to the Cauchy problem of the semilinear wave equation with a dissipative term, *J. Math. Soc. Japan*, 47 (1995), 617–653.
[21] M. Kirane and M. Qafsaoui, Fujita’s exponent for a semilinear wave equation with linear damping, Adv. Nonlinear Stud., 2 (2002), 41–49.

[22] T.-T. Li and Y. Zhou, Breakdown of solutions to $\square u + u_t = |u|^{1+\alpha}$, Discrete Contin. Dynam. Syst., 1 (1995), 503–520.

[23] P. Marcati and K. Nishihara, The $L^p-L^q$ estimates of solutions to one-dimensional damped wave equations and their application to the compressible flow through porous media, J. Differential Equations, 191 (2003), 445–469.

[24] A. Matsumura, On the asymptotic behavior of solutions of semi-linear wave equations, Publ. Res. Inst. Math. Sci., 12 (1976), 169–189.

[25] A. Miyachi, On some estimates for the wave equation in $L^p$ and $H^p$, J. Fac. Sci. Univ. Tokyo Sect. IA Math., 27 (1980), 331–354.

[26] M. Nakao and K. Ono, Existence of global solutions to the Cauchy problem for the semilinear dissipative wave equations, Math. Z., 214 (1993), 325–342.

[27] T. Narazaki, $L^p-L^q$ estimates for damped wave equations and their applications to semi-linear problem, J. Math. Soc. Japan, 56 (2004), 585–626.

[28] T. Narazaki, Global solutions to the Cauchy problem for a system of damped wave equations, Differential and Integral Equations, 24 (2011), 569–600.

[29] T. Narazaki and K. Nishihara, Asymptotic behavior of solutions for the damped wave equation with slowly decaying data, J. Math. Anal. Appl., 338 (2008), 803–819.

[30] K. Nishihara, $L^p-L^q$ estimates of solutions to the damped wave equation in 3-dimensional space and their application, Math. Z., 244 (2003), 631–649.

[31] J. C. Peral, $L^p$ estimates for the wave equation, J. Funct. Anal., 36 (1980), 114–145.

[32] S. Sakata and Y. Wakasugi, Movement of time-delayed hot spots in Euclidean space, Math. Z., 285 (2017), 1007–1040.

[33] S. Sjöstrand, On the Riesz means of the solutions of the Schrödinger equation, Ann. Scuola Norm. Sup. Pisa, 24 (1970), 331–348.

[34] G. Todorova and B. Yordanov, Critical exponent for a nonlinear wave equation with damping, J. Differential Equations, 174 (2001), 464–489.

[35] F. B. Weissler, Existence and non-existence of global solutions for a semilinear heat equation, Israel J. Math., 38 (1981), 29–40.

[36] H. Yang and A. Milani, On the diffusion phenomenon of quasilinear hyperbolic waves, Bull. Sci. Math., 124 (2000), 415–433.

[37] Qi S. Zhang, A blow-up result for a nonlinear wave equation with damping: the critical case, C. R. Acad. Sci. Paris Sér. I Math., 333 (2001), 109–114.

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