ON THE APPROXIMATION PROPERTIES OF CESÀRO MEANS OF NEGATIVE ORDER OF VILENKIN-FOURIER SERIES

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Abstract. In this paper we establish approximation properties of Cesàro \((C, -\alpha)\) means with \(\alpha \in (0, 1)\) of Vilenkin-Fourier series. This result allows one to obtain a condition which is sufficient for the convergence of the means \(\sigma_n^{-\alpha}(f, x)\) to \(f(x)\) in the \(L^p\) metric.

Let \(N_+\) denote the set of positive integers, \(N := N_+ \cup \{0\}\). Let \(m := (m_0, m_1, \ldots)\) denote a sequence of positive integers not less than 2. Denote by \(Z_{m_k} := \{0, 1, \ldots, m_k - 1\}\) the additive group of integers modulo \(m_k\). Define the group \(G_m\) as the complete direct product of the groups \(Z_{m_j}\) with the product of the discrete topologies of \(Z_{m_j}\)’s.

The direct product of the measures

\[
\mu_k(\{j\}) := \frac{1}{m_k}, \quad (j \in Z_{m_k}).
\]

Is the Haar measure on \(G_m\) with \(\mu(G_m) = 1\). If the sequence \(m\) is bounded, then \(G_m\) is called a bounded Vilenkin group. In this paper we will consider only bounded Vilenkin group. The elements of \(G_m\) can be represented by sequences \(x := (x_0, x_1, \ldots, x_j, \ldots)\), \((x_j \in Z_{m_j})\). The group operation \(+\) in \(G_m\) is given by

\[
x + y = ((x_0 + y_0 \mod m_0, \ldots, x_k + y_k \mod m_k, \ldots),
\]

where \(x := (x_0, \ldots, x_k, \ldots)\) and \(y := (y_0, \ldots, y_k, \ldots) \in G_m\).

The inverse of \(+\) will be denoted by \(-\). It is easy to give a base for the neighborhoods of \(G_m\):

\[
I_0(x) := G_m
\]

\[
I_n(x) := \{y \in G_m | y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\}.
\]

for \(x \in G_m\), \(n \in N\) define \(I_n = I_n(0)\). Set \(e_n := (0, \ldots, 0, 1, 0, \ldots) \in G_m\) the \(n\) th coordinate of which is 1 and the rest are zeros \((n \in N)\).

If we define the so-called generalized number system based on \(m\) in the following way: \(M_0 := 1\), \(M_{k+1} := m_k M_k\) \((k \in N)\), then every \(n \in N\) can
be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j} \ (j \in N_+)$ and only a finite number of $n_j$’s differ from zero. We use the following notation.

Let $|n| := \max\{k \in N : n_k \neq 0\}$ (that is, $M_{|n|} \leq n < M_{|n|+1}$). Next, we introduce of $G_m$ an orthonormal system which is called Vilenkin system. At first define the complex valued functions $r_k(x) : G_m \to C$. the generalized Rademacher functions in this way

$$r_k(x) := \exp \left( \frac{2\pi i x_k}{m_k} \right), \quad (i^2 = -1, \ x \in G_m, \ k \in N).$$

Now define the Vilenkin system $\psi := (\psi_n : n \in N)$ on $G_m$ as follows.

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x), \quad (n \in N).$$

In particular, we call the system the Walsh-Paley if $m = 2$. The Vilenkin system is orthonormal and complete in $L^1(G_m)$. Now, introduce analogues of the usual definitions of the Fourier analysis. If $f \in L^1(G_m)$ we can establish the following definitions in the usual way:

Fourier coefficients:

$$\hat{f}(k) := \int_{G_m} f \overline{\psi_k} d\mu, \quad (k \in N),$$

partial sums:

$$S_n f := \sum_{k=0}^{n-1} \hat{f}(k) \psi_k, \quad (n \in N_+, \ S_0 f := 0),$$

Fejér means:

$$\sigma_n f := \sum_{k=0}^{n} \left( 1 - \frac{k}{n+1} \right) \hat{f}(k) \psi_k, \quad (n \in N_+).$$

Dirichlet kernels:

$$D_n := \sum_{k=0}^{n-1} \psi_k, \quad (n \in N_+).$$

Fejér kernels:

$$K_n(x) := \frac{1}{n} \sum_{k=1}^{n} D_k(x).$$

Recall that (see [3] or [13])

$$(1) \quad D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n. \end{cases}$$
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It is well known that

\[ \sigma_n f(x) = \int_{G_m} f(t) K_n(x - t) \, d\mu(t). \]

The \((C, -\alpha)\) means of the Vilenkin-Fourier series are defined as

\[ \sigma_{-\alpha} (f, x) = \frac{1}{A_{n-\alpha}} \sum_{k=0}^{n} A_{n-\alpha-k} \hat{f}(k) \psi_k(x), \]

where

\[ A_0^\alpha = 1, \quad A_n^\alpha = \frac{(\alpha + 1) \cdots (\alpha + n)}{n!}. \]

It is well known that [16]

(2) \[ A_n^\alpha = \sum_{k=0}^{n} A_k^{\alpha-1}. \]

(3) \[ A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1}. \]

(4) \[ A_n^\alpha \sim n^\alpha. \]

The norm of the space \(L^p(G_m)\) is defined by

\[ \|f\|_p := \left( \int_{G_m} |f(x)|^p \, d\mu(x) \right)^{1/p}, \quad (1 \leq p < \infty). \]

Denote by \(C(G_m)\) the class of continuous functions on the group \(G_m\), endowed with the supremum norm.

For the sake of brevity in notation, we agree to write \(L^\infty(G_m)\) instead of \(C(G_m)\).

Let \(f \in L^p(G_m), 1 \leq p \leq \infty\). The expression

\[ \omega \left( \frac{1}{M_n}, f \right)_p = \sup_{h \in I_n} \| f(\cdot + h) - f(\cdot) \|_p \]

is called the modulus of continuity.

The problems of summability of partial sums and Cesàro means for Walsh-Fourier series were studied in [2], [5]-[12], [14]. In his monography [15] Zhizhinashvili investigated the behavior of Cesàro method of negative order for trigonometric Fourier series in detail. Goginava [5] studied analogous question in case of the Walsh system. In particular, the following theorem is proved.
Theorem G. Let $f$ belong to $L^p(G_2)$ for some $p \in [1, \infty]$ and $\alpha \in (0, 1)$. Then for any $2^k \leq n < 2^{k+1}$ ($k, n \in \mathbb{N}$) the inequality

$$\left\| \sigma_{\alpha n}^{-} (f) - f \right\|_p \leq c(p, \alpha) \left\{ 2^{k\alpha} \omega \left( 1/2^{k-1}, f \right)_p + \sum_{r=0}^{k-2} 2^{r-k} \omega (1/2^r, f)_p \right\}$$

holds true.

In this paper we establish approximation properties of Cesàro $(C, -\alpha)$ means with $\alpha \in (0, 1)$ of Vilenkin-Fourier series.

Theorem 1. Let $f$ belong to $L^p(G_m)$ for some $p \in [1, \infty]$ and $\alpha \in (0, 1)$. Then for any $M_k \leq n < M_{k+1}$ ($k, n \in \mathbb{N}$) the inequality

$$\left\| \sigma_{\alpha n}^{-} (f) - f \right\|_p \leq c(p, \alpha) \left\{ M_k^{\alpha} \omega (1/M_{k-1}, f)_p + \sum_{r=0}^{k-2} M_r \omega (1/M_r, f)_p \right\}$$

holds true.

This result allows one to obtain the condition which is sufficient for the convergence of the means $\sigma_{\alpha n}^{-} (f, x)$ to $f(x)$ in the $L^p$-metric.

Corollary 1. Let $f$ belong to $L^p(G_m)$ for some $p \in [1, +\infty]$ and let

$$M_k^{\alpha} \omega (1/M_{k-1}, f)_p \to 0, \quad \text{as} \quad k \to \infty, \quad (0 < \alpha < 1).$$

Then

$$\left\| \sigma_{\alpha n}^{-} (f) - f \right\|_p \to 0 \quad \text{as} \quad n \to \infty.$$
where $M_k \leq n \leq M_{k+1}$.

**Proof of Lemma 2.** Applying Abel’s transformation, from (5) we get

\begin{equation}
\frac{1}{A_n^\alpha} \left\| \int_{G_m} \sum_{v=0}^{M_{k+1}-1} A_n^{-\alpha} \psi_v (u) \left[ f (\cdot + u) - f (\cdot) \right] d\mu (u) \right\|_p
\end{equation}

\begin{equation}
= \frac{1}{A_n^\alpha} \left\| \int_{G_m} \sum_{v=1}^{M_k} A_n^{-\alpha} \psi_{v-1} (u) \left[ f (\cdot + u) - f (\cdot) \right] d\mu (u) \right\|_p
\end{equation}

\begin{equation}
\leq \frac{1}{A_n^\alpha} \left\| \int_{G_m} \sum_{v=1}^{M_k-1} A_n^{-\alpha} \psi_{v-1} D_v (u) \left[ f (\cdot + u) - f (\cdot) \right] d\mu (u) \right\|_p
\end{equation}

\begin{equation}
+ \frac{1}{A_n^\alpha} \left\| \int_{G_m} A_n^{-\alpha} D_{M_k-1} (u) \left[ f (\cdot + u) - f (\cdot) \right] d\mu (u) \right\|_p
\end{equation}

\[= I_1 + I_2.\]

From the generalized Minkowski’s inequality, and by (1) and (4) we obtain

\begin{equation}
I_2 \leq c(\alpha) M_{k-1} \int_{I_{k-1}} \left\| [f (\cdot + u) - f (\cdot)] \right\|_p d\mu (u)
\end{equation}

\[= O (\omega (1/M_{k-1}, f))_p,\]

\begin{equation}
I_1 \leq \frac{1}{A_n^\alpha} \sum_{r=0}^{k-2} \left\| \int_{G_m} \sum_{v=M_r}^{M_{r+1}-1} A_n^{-\alpha} \psi_{v-1} D_v (u) \left[ f (\cdot + u) - f (\cdot) \right] d\mu (u) \right\|_p
\end{equation}

\begin{equation}
\leq \frac{1}{A_n^\alpha} \sum_{r=0}^{k-2} \left\| \int_{G_m} \sum_{v=M_r}^{M_{r+1}-1} A_n^{-\alpha} \psi_{v-1} D_v (u) \left[ f (\cdot + u) - S_{M_r} (\cdot + u, f) \right] d\mu (u) \right\|_p
\end{equation}

\begin{equation}
+ \frac{1}{A_n^\alpha} \sum_{r=0}^{k-2} \left\| \int_{G_m} A_n^{-\alpha} \psi_{v-1} D_v (u) \left[ S_{M_r} (\cdot + u, f) - S_{M_r} (\cdot , f) \right] d\mu (u) \right\|_p
\end{equation}

\begin{equation}
+ \frac{1}{A_n^\alpha} \sum_{r=0}^{k-2} \left\| \int_{G_m} \sum_{v=M_r}^{M_{r+1}-1} A_n^{-\alpha} \psi_{v-1} D_v (u) \left[ S_{M_r} (\cdot , f) - f (\cdot) \right] d\mu (u) \right\|_p
\end{equation}
Since
\[ \| f - S_{M_r} (f) \|_p \leq \omega (1/M_r, f)_p, \]
using Lemma 1 for \( I_{11} \) we can write

(8)

\[ I_{11} \]

\[ \leq \frac{1}{A_{\alpha}^n} \sum_{r=0}^{k-2} \left( \sum_{v=M_r}^{M_r+1-1} A_{n-v-1}^{-\alpha-1} D_v (u) \right) \| f (\cdot + u) - S_{M_r} (\cdot + u, f) \|_p \, d\mu (u) \]

\[ \leq \frac{1}{A_{\alpha}^n} \sum_{r=0}^{k-2} \omega (1/M_r, f)_p \int_{G_m} \left( \sum_{v=M_r}^{M_r+1-1} A_{n-v-1}^{-\alpha-1} D_v (u) \right) d\mu (u) \]

\[ \leq c(\alpha) n^\alpha \sum_{r=0}^{k-2} \omega (1/M_r, f)_p \sqrt{M_{r+1}} \left( \sum_{v=M_r}^{M_r+1-1} (n-v-1)^{-2\alpha-2} \right)^{1/2} \]

\[ \leq c(\alpha) n^\alpha \sum_{r=0}^{k-2} \omega (1/M_r, f)_p \sqrt{M_{r+1}} (n-M_{r+1})^{-\alpha-1} \sqrt{M_{r+1}} \]

\[ \leq c(\alpha) \sum_{r=0}^{k-2} \frac{M_r}{M_k} \omega (1/M_r, f)_p. \]

Analogously, we can prove that

(9)

\[ I_{13} \leq c(\alpha) \sum_{r=0}^{k-2} \frac{M_r}{M_k} \omega (1/M_r, f)_p. \]

It is evident that

(10)

\[ \int_{G_m} \sum_{v=M_r}^{M_{r+1}-1} A_{n-v-1}^{-\alpha-1} D_v (u) \left[ S_{M_r} (x + u, f) - S_{M_r} (x, f) \right] d\mu (u) \]

\[ = \sum_{v=M_r}^{M_{r+1}-1} A_{n-v-1}^{-\alpha-1} \int_{G_m} S_{M_r} (x + u, f) D_v (u) d\mu (u) - \sum_{v=M_r}^{M_{r+1}-1} A_{n-v-1}^{-\alpha-1} S_{M_r} (x, f) \]

\[ = \sum_{v=M_r}^{M_{r+1}-1} A_{n-v-1}^{-\alpha-1} S_v (x, S_{M_r} (f)) - \sum_{v=M_r}^{M_{r+1}-1} A_{n-v-1}^{-\alpha-1} S_{M_r} (x, f). \]
\[ I_{12} = 0. \]

Combining (5)–(11) we receive the proof of Lemma 2.

**Lemma 3.** Let \( f \in L^p(G_m) \) for some \( p \in [1, \infty) \). Then for every \( \alpha \in (0, 1) \) the following estimations hold

\[
\frac{1}{A_n^{-\alpha}} \left\| \int_{G_m} \sum_{v=M_{k-1}}^{M_k-1} A_{n-v}^{-\alpha} \psi_v (u) [f(\cdot + u) - f(\cdot)] d\mu(u) \right\|_p \\
\leq c(p, \alpha) \omega(1/M_{k-1}, f)_p M_k^\alpha
\]

and

\[
\frac{1}{A_n^{-\alpha}} \left\| \int_{G_m} \sum_{v=M_k}^{n} A_{n-v}^{-\alpha} \psi_v (u) [f(\cdot + u) - f(\cdot)] d\mu(u) \right\|_p \\
\leq c(p, \alpha) \omega(1/M_k, f)_p M_k^\alpha
\]

where \( M_k \leq n < M_{k+1} \).

**Proof of Lemma 3.** We can write

(12) \[ II = \frac{1}{A_n^{-\alpha}} \left\| \int_{G_m} \sum_{v=M_{k-1}}^{M_k-1} A_{n-v}^{-\alpha} \psi_v (u) [f(\cdot + u) - f(\cdot)] d\mu(u) \right\|_p \]

\[ = \frac{1}{A_n^{-\alpha}} \left\| \int_{G_m} \sum_{v=M_{k-1}}^{M_k-1} A_{n-v}^{-\alpha} \psi_v (u) f(\cdot + u) d\mu(u) \right\|_p \]

\[ \leq \frac{1}{A_n^{-\alpha}} \left\| \int_{G_m} \sum_{v=M_{k-1}}^{M_k-1} A_{n-v}^{-\alpha} \psi_v (u) [f(\cdot + u) - S_{M_{k-1}} (\cdot + u, f)] d\mu(u) \right\|_p \]

\[ + \frac{1}{A_n^{-\alpha}} \left\| \int_{G_m} \sum_{v=M_{k-1}}^{M_k-1} A_{n-v}^{-\alpha} \psi_v (u) S_{M_{k-1}} (\cdot + u, f) d\mu(u) \right\|_p = II_1 + II_2. \]
Since

\[
\int_{G_m} \sum_{v=M_{k-1}}^{M_k-1} A_{n-v}^{-\alpha} \psi_v (u) S_{M_{k-1}} (x + u, f) \, d\mu (u)
\]

\[
= \sum_{j=0}^{M_{k-1}-1} \hat{f} (j) \psi_j (x) \sum_{v=M_{k-1}}^{M_k-1} A_{n-v}^{-\alpha} \int_{G_m} \psi_v (u) \psi_j (u) \, d\mu (u) = 0,
\]

for \(II_2\) we obtain

\[
(14) \quad II_2 = 0.
\]

Using generalized Minkowski’s inequality we have

\[
(15) \quad II_1 \leq \frac{1}{A_n^\alpha} \int_{G_m} \left| \sum_{v=M_{k-1}}^{M_k-1} A_{n-v}^{-\alpha} \psi_v (u) \right| \times \left\| \left[ f (\cdot \oplus u) - S_{M_{k-1}} (\cdot \oplus u, f) \right] \right\|_p \, d\mu (u)
\]

\[
\leq c (\alpha) n^\alpha \omega (1/M_{k-1}, f)_p \int_{G_m} \left| \sum_{v=M_{k-1}}^{M_k-1} A_{n-v}^{-\alpha} \psi_v (u) \right| \, d\mu (u).
\]

Let \(t \in I_{A-1} \setminus I_A, \ A = 1, 2, ..., k-1\) and \(M_k = pM_A + q, \) where \(0 \leq q < M_A.\) Then we have

\[
(16) \quad \sum_{v=M_{k-1}}^{M_k-1} A_{n-v}^{-\alpha} \psi_v (t)
\]

\[
= \sum_{v=M_{k-1}}^{pM_A-1} A_{n-v}^{-\alpha} \psi_v (t) + \sum_{v=pM_A}^{M_k-1} A_{n-v}^{-\alpha} \psi_v (t)
\]

\[
= \sum_{r=M_{k-1}/M_A}^{p-1} \left( r+1 \right) A_{n-v}^{-\alpha} \psi_v (t)
\]

\[
+ \sum_{v=0}^{q-1} A_{n-v-pM_A}^{-\alpha} \psi_v (t + pM_A)
\]

\[
= \sum_{r=M_{k-1}/M_A}^{p-1} \sum_{v=0}^{M_A-1} A_{n-v-rM_A}^{-\alpha} \psi_v (t + rM_A)
\]

\[
+ \sum_{v=0}^{q-1} A_{n-v-pM_A}^{-\alpha} \psi_v (t + pM_A)
\]
\[\begin{equation}
\sum_{r=M_{k-1}/M}^{p-2} \psi_{rM} (t) \sum_{v=0}^{M_{k-1}/M} (n-v-rM) \psi_v (t)
\end{equation}\]
\[+ \psi_{(p-1)M} (t) \sum_{v=0}^{M_{k-1}/M} (n-(p-1)rM) \psi_v (t)
\]
\[+ \psi_{rM} (t) \sum_{v=0}^{q-1} (n-q-v-rM) \psi_v (t)
\]
\[= A_1 + A_2 + A_3.
\]
Since \(D_{M} (t) = 0, t \in I_{A-1} \setminus I_A\) and \(|D_{k} (t)| \leq k\), from Abel’s transformation, we have

\[\begin{equation}
|A_1| = \left| \sum_{r=M_{k-1}/M}^{p-2} \psi_{rM} (t) \sum_{v=0}^{M_{k-1}/M} (n-v-rM) D_v (t) \right|
\end{equation}\]
\[\leq c(\alpha) M_A \sum_{r=M_{k-1}/M}^{p-2} \sum_{v=0}^{M_{k-1}/M} (n-rM-A)^{-\alpha-1}
\]
\[\leq c(\alpha) M_A (n-(p-1)M_A)^{-\alpha}
\]
\[\leq c(\alpha) M_A^{1-\alpha}
\]
For \(A_2\) we have

\[\begin{equation}
|A_2| \leq c(\alpha) \sum_{v=0}^{M_{k-1}/M} (n-(p-1)M_A-M)^{-\alpha}
\end{equation}\]
\[\leq c(\alpha) \sum_{v=0}^{M_{k-1}/M} (n-q-v)^{-\alpha} \leq c(\alpha) M_A^{1-\alpha}.
\]
We can write

\[\begin{equation}
|A_3| \leq c(\alpha) \sum_{v=0}^{q-1} (n-v)^{-\alpha}
\end{equation}\]
\[\leq c(\alpha) q^{1-\alpha} \leq c(\alpha) M_A^{1-\alpha}.
\]
Combining (16)-(19) we obtain

\[\begin{equation}
\sum_{v=M_{k-1}}^{M_{k-1}+M} A_{n-v}(u) \psi_v (u) \leq c(\alpha) M_{A-1}^{1-\alpha}, t \in I_A \setminus I_A, A = 1, ..., k - 1.
\end{equation}\]
Hence, we can write
Combining (20)-(21) we have

\[ II_1 \leq c(\alpha) \omega \left( \frac{1}{M_k^{-1}}, f \right)_p M_k^\alpha. \]

Combining (15), (12) and (22) we conclude that

\[ \frac{1}{A_n^{-\alpha}} \left\| \int_{G_m} \sum_{v=M_{k-1}}^{M_k-1} A_{n-v}^{-\alpha} \psi_v(u) \left[ f(\cdot + u) - f(\cdot) \right] d\mu(u) \right\|_p \]

\[ \leq c(\alpha) \omega \left( \frac{1}{M_k^{-1}}, f \right)_p M_k^\alpha. \]

Analogously, we can prove that

\[ \frac{1}{A_n^{-\alpha}} \left\| \int_{G_m} \sum_{v=M_k}^{M_n} A_{n-v}^{-\alpha} \psi_v(u) \left[ f(\cdot + u) - f(\cdot) \right] d\mu(u) \right\|_p \]

\[ \leq c(\alpha) \omega \left( \frac{1}{M_k}, f \right)_p M_k^\alpha. \]

Lemma proved.

\[ \square \]

Proof of Theorem. We can write
\[ \sigma_n^{-\alpha}(f, x) - f(x) \]
\[ = \frac{1}{A_n^{-\alpha}} \int \sum_{v=0}^{G_m-1} A_{n^{-\alpha} \psi_v}(x) [f(\cdot + u) - f(\cdot)] d\mu(u) \]
\[ + \frac{1}{A_n^{-\alpha}} \int \sum_{v=M_k+1}^{M_{k-1}} A_{n^{-\alpha} \psi_v}(x) [f(\cdot + u) - f(\cdot)] d\mu(u) \]
\[ + \frac{1}{A_n^{-\alpha}} \int \sum_{v=M_k}^{n} A_{n^{-\alpha} \psi_v}(x) [f(\cdot + u) - f(\cdot)] d\mu(u) \]
\[ = I + II + III \]

Since
\[ \| \sigma_n^{-\alpha}(f, \cdot) - f(\cdot) \|_p \leq \| I \|_p + \| II \|_p + \| III \|_p \]
From Lemmas 2 and 3 the proof of theorem is complete. \( \square \)

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