An Application of a Log Version of the Kodaira Vanishing Theorem to Embedded Projective Varieties

by

Aaron Bertram

0. Introduction. Let $Y \subset \mathbb{CP}^n$ be a smooth complex projective subvariety of codimension $r$, and let $\mathcal{I}_Y$ be the ideal sheaf of the embedding, with $\mathcal{I}_Y^k \subset \mathcal{O}_{\mathbb{P}^n}$ denoting its $k$th power. In this paper, we will be interested in the following two integer invariants of the embedding:

- $d_Y$, the minimum of the degrees $d$ such that $Y$ is a scheme-theoretic intersection of hypersurfaces of degree at most $d$, and given $d_Y$,

- $e_Y$, the minimum of the integers $e$ such that:

$$H^i(\mathbb{P}^n, \mathcal{I}_Y^k(p)) = 0 \text{ for all } i > 0, k \geq 0 \text{ and } p \geq e + (k - 1)d_Y.$$  

An upper bound for $e_Y$ (which is sharp if $Y$ is a complete intersection) was computed by the author in collaboration with Ein and Lazarsfeld:

Theorem 1: ([BEL], Proposition 1) Suppose $Y$ is scheme-theoretically cut out by equations of degrees

$$d_Y = d_1 \geq d_2 \geq ... \geq d_m.$$  

Then $e_Y \leq d_1 + ... + d_r - n$. (Recall that $r$ is the codimension of $Y$.)

The idea in this paper is to show how a generalized “log” version of the Kodaira vanishing theorem can be employed to improve the results of Theorem 1 (which was also proved by Kodaira vanishing) when we have more knowledge about the equations for $Y$. The idea is to find a hypersurface $F \subset \mathbb{P}^n$ which has high multiplicity along $Y$, is “log canonical” near $Y$, and has relatively small degree, then to invoke Kodaira vanishing on the blow-up of $\mathbb{P}^n$ along $Y$. In the context of Theorem 1, the hypersurface $F$ is approximately a divisor with normal crossings (see its proof in §2). However one of the main points of this paper is the observation that even in the most familiar of projective embeddings, log canonical divisors quite different from normal crossings divisors seem to play an important role.

1Partially supported by a Sloan research fellowship.
All the new cases we consider are determinantal, in the sense that $d_Y = 2$ and a collection of quadrics which scheme-theoretically cut out $Y$ arise either as $2 \times 2$ minors or $4 \times 4$ Pfaffians of a matrix of linear forms on $\mathbf{P}^n$. In each of these cases, the hypersurface $F$ is constructed out of minors or Pfaffians of all sizes. The results are most satisfactory for the “universal” determinantal varieties where the key is to observe that the theories of complete linear maps and quadrics (and a version involving Pfaffians in the skew case, which seems not to have been previously worked out) give us the information we need to check whether hypersurfaces built out of minors have mild enough singularities. We apply the same idea to curves embedded by a line bundle of large degree, obtaining similarly satisfactory results in genus 0 and 1. However in higher genus, some complications arise, and the results obtained here are probably not the best.

The contents of the paper are as follows. In §1, we review some of the relevant definitions and results of the log minimal model program leading up to the log version of the Kodaira vanishing theorem, due to Nadel. In addition, we state a useful Bertini property, due to Kollár. In §2 we explain how a hypersurface $F \subset \mathbf{P}^n$ yields an upper bound on $e_Y$, and use it in subsequent sections to give:

(i) A reproof of Theorem 1 taking $F$ to be a sum of hypersurfaces of degree $d_i$ (for $1 \leq i \leq r$) which are general among those vanishing on $Y$.

(ii) An upper bound for $e_Y$ which is independent of the dimensions of vector spaces $V$ and $W$ for each of the three universal determinantal varieties:

(a) (Generic) $Y = \mathbf{P}(V) \times \mathbf{P}(W)$, Segre embedding. $e_Y \leq -1$.

(b) (Symmetric) $Y = \mathbf{P}(V)$, quadratic Veronese embedding. $e_Y \leq 0$.

(c) (Skew) $Y = G(2,V)$, Plücker embedding. $e_Y \leq -3$.

(iii) A proof that $e_Y \leq 1$ when $Y = C$ is a Riemann surface of genus $g$ embedded by a complete linear series in the following cases:

(a) $g = 0$ or 1.

(b) the degree of the embedding is sufficiently large ($> \frac{8g+2}{3}$ will do).

(c) the degree of the embedding is at least $2g + 3$, but with “gaps”.

(Notice that in (ii) and (iii), Theorem 1 would give only $e_Y \leq 2r - n$.)
Remarks: In [W], Wahl proves the vanishing $H^1(P^n, \mathcal{I}_X(p)) = 0$ for all $p \geq 3$ and $X \subset P^n$ embedded by a complete linear series in the following cases:

1. $X$ is projective space,
2. $X$ is arbitrary, but the linear series is sufficiently ample, and
3. $X$ is a general canonical curve of genus $\geq 3$.

Since in all these cases the embedding is projectively normal and $X$ is scheme-theoretically cut out by quadrics, Wahl’s results may be a special case of the more general property $e_Y \leq 1$. This we know to be the case for (1) and (2) when $X$ is a curve by the results of §4. It would be interesting to know whether or not this property does indeed hold in this generality (as well as the case of an embedding of a curve of degree $2g + 3$ or more).

The next remark is more of a confession, really. The invariant $e_Y$ defined here probably ought to be modified to conform with Castelnuovo-Mumford regularity. Recall that if $\mathcal{F}$ is a sheaf on $P^n$, then $\mathcal{F}$ is defined to be $m$-regular if $H^i(P^n, \mathcal{F}(m - i)) = 0$ for all $i > 0$. The main feature of regularity is that $m$-regular implies $m + 1$-regular. But of course this pattern of vanishing in case $\mathcal{F}$ is a power of the ideal sheaf of $X$ does not lend itself to proof by vanishing theorems as outlined in this paper.

On the other hand, in [T] §6, Thaddeus obtains some similar vanishing results using ordinary Kodaira vanishing in case $X$ is a curve embedded by a line bundle of large degree. In his case, the vanishing takes place on spaces obtained from $Y$ by a sequence of flips. These flips are shown to be log flips in the sense of the log minimal model program in [B2] using log canonical divisors of precisely the sort we use here to prove vanishing. Probably the sharpest results would be obtained by applying the generalized Kodaira vanishing theorem on these flipped spaces and transferring the vanishing results back to $Y$. It seems entirely possible that such a procedure will yield a pattern of vanishing which does conform with Castelnuovo-Mumford regularity.
§1. Log Kodaira Vanishing: Let $X$ be a smooth complex projective variety of dimension $n$. The following definitions are standard to the experts, but are perhaps not widely known:

**Definitions:**

(a) A finite $\mathbb{Q}$-linear combination $F = \sum \alpha_i F_i$ of distinct prime divisors of $X$ is called a $\mathbb{Q}$-divisor. It is **effective** if each $\alpha_i \geq 0$. Intersection with divisors extends by linearity to give well-defined rational numbers $F^n$ and $F.B$, given a $\mathbb{Q}$-divisor $F$ and a curve $B \subset X$. In particular, numerical equivalence extends to an equivalence relation on $\mathbb{Q}$-divisors.

(b) A(n equivalence class of) $\mathbb{Q}$-divisor(s) $A$ is **nef and big** if:

(i) $A.B \geq 0$ for all curves $B \subset X$ and

(ii) $A^n > 0$.

Given an effective $\mathbb{Q}$-divisor $F = \sum \alpha_i F_i$ and a birational morphism $f : \tilde{X} \to X$, let $E$ be the $f$-exceptional divisor, and let $\{E_j\}$ be the components of $E$. Also let $f^*(F_i)$ and $f^{-1}_*(F_i)$ be the total and strict transforms, respectively, of $F_i$ on $\tilde{X}$, extending the usual notions by linearity.

(c) If $f : \tilde{X} \to X$ has the property that $\tilde{X}$ is smooth and $\sum E_j + \sum f^{-1}_*(F_i)$ is a normal crossings divisor with smooth components, then $f$ is called a **log resolution** of the pair $(X, F)$. Given such an $f$, one attaches a rational number to each $E_j$ and $f^{-1}_*(F_i)$, called the **discrepancy** of $f$, as follows:

(i) The discrepancy at $f^{-1}_*(F_i)$ is $-\alpha_i$.

(ii) The discrepancy at $E_j$ is its coefficient in the difference:

$$ (K_{\tilde{X}} + f^{-1}_*(F)) - f^*(K_X + F) $$

(d) Given an effective $\mathbb{Q}$-divisor $F$ and a log resolution $f : \tilde{X} \to X$,

$\text{discrep}(X, F, f)$ is the minimum of the discrepancies of type (ii), and $\text{totaldiscrep}(X, F, f)$ is the minimum of all the discrepancies.

**Remark:** We have limited ourselves here to smooth $X$, since that is all we will need to consider. See [Keta] or [K] for the general definitions when $X$ is not assumed to be smooth, as well as a proof of the following:
**Basic Observation:** The following definitions are intrinsic to a pair \((X, F)\) (i.e. they do not depend upon the log resolution \(f : \widetilde{X} \to X\)):

\((X, F)\) is **log canonical** (or lc) if \(\text{totaldiscrep}(X, F, f) \geq -1\).

\((X, F)\) is **Kawamata log terminal** (or klt) if \(\text{totaldiscrep}(X, F, f) > -1\).

\((X, F)\) is **purely log terminal** (or plt) if it is log canonical, and if, in addition, \(\text{discrep}(X, F, f) > -1\).

**Examples:**

1. \((X, F)\) is log canonical when \(F\) is a Cartier divisor with smooth components and normal crossings. (the identity is a log resolution of \((X, F)\), and all the discrepancies are \(-1\) or 0.)

2. Suppose \(Z_1 \subset Z_2 \subset \ldots \subset Z_k \subset X\) are closed subvarieties, and \(F\) is an effective (Cartier) divisor on \(X\). Suppose blowing up the strict transforms of each \(Z_j\) in order is a sequence of blow-ups along smooth centers so that the composition of blow-downs \(f : \widetilde{X} \to X\) is a log resolution of \((X, F)\). Let \(m_j\) be the multiplicity of \(F\) at the generic point of \(Z_j\). Then:

   \((X, F)\) is lc if \(m_j \leq \text{codim}_X(Z_j)\) for all \(j\) and
   \((X, F)\) is plt if \(m_j < \text{codim}_X(Z_j)\) (it is only klt if \(F = \emptyset\)).

   (This is an immediate consequence of Riemann-Hurwitz.)

3. Given a log canonical pair \((X, F)\) and a rational number \(0 < \epsilon < 1\), then the pair \((X, (1 - \epsilon)F)\) is klt. (Immediate from the definitions.)

**More Definitions:**

Let \(F\) be an effective \(\mathbb{Q}\)-divisor on \(X\). For each \(x \in X\), one says \((X, F)\) is **not lc** (resp. **not klt**) **at** \(x\) if there is a subvariety \(x \in Z \subset X\), a log resolution \(f : \widetilde{X} \to X\) and an exceptional (or strict-transform) divisor \(E_Z \subset \widetilde{X}\) such that \(f(E_Z) = Z\) and the discrepancy at \(E_Z\) is \(< -1\) (resp. \(\leq -1\)). The following subsets of \(X\) are known to be closed:

\[ \text{Nklt}(X, F) := \{ x \in X \mid (X, F) \text{ is not klt at } x \}, \]

\[ \text{Nlc}(X, F) := \{ x \in X \mid (X, F) \text{ is not lc at } x \} \]

(closed by (3) above).

We will use the following very simple case of a Bertini property due to Kollár which tells us that the Nlc and Nklt loci for general members of linear series can be detected “pointwise” (again, see [K] for a much more general version).
Suppose $F$ is an effective $\mathbb{Q}$-divisor and $|B_1|, \ldots, |B_k|$ are linear series on $X$. Let $B^g_i$ denote a general member of $|B_i|$, let $B^g := B^g_1 + \ldots + B^g_k$, and let $b_1, \ldots, b_k$ be rational numbers between 0 and 1. Then using (4.8.1-2) of [K], we obtain:

**Bertini Property:** If $x \not\in \text{Nlc}(X, F + B^g)$ for each $x$ in some subset $W \subset X$, then $\text{Nlc}(X, F + B^g) \cap W = \emptyset$. The same is true with lc replaced by klt provided that the $b_i$ are strictly less than 1.

(The point is that a priori the choice of $B^g$ could depend upon $x$.)

**Example:** If $(X, F)$ is log canonical and the $|B_i|$ are all base-point-free, then the Bertini property shows that $(X, F + \sum_{i=1}^k B^g_i)$ is log canonical.

The following theorem is due to Alan Nadel (see Kollár’s notes, Theorem 2.16 for a more general version when $X$ is allowed some singularities).

**Theorem (Log Kodaira Vanishing):** Suppose that $F$ is an effective $\mathbb{Q}$-divisor on $X$, $A$ is another $\mathbb{Q}$-divisor which is nef and big, and that $L$ is a line bundle on $X$ satisfying:

$$L \equiv K_X + F + A.$$ 

Then there is an ideal sheaf $\mathcal{J}$ on $X$ (called Nadel’s multiplier ideal sheaf) with the following properties:

(i) $\mathcal{O}_X/\mathcal{J}$ is supported on $\text{Nklt}(X, F)$ (which is therefore closed!), and
(ii) $H^i(X, \mathcal{J} \otimes L) = 0$ for all $i > 0$.

And the obvious corollary:

**Corollary:** If $(X, F)$ is klt in the theorem, then:

$$H^i(X, L) = 0 \text{ for all } i > 0.$$
§2. The Strategy (and Reproof of Theorem 1): We return now to the set-up from the introduction. \( Y \subset \mathbf{P}^n \) is a smooth projective subvariety of codimension \( r > 0 \). Let:

\[ X := \text{bl} (\mathbf{P}^n, Y), \]  

the blow-up of \( \mathbf{P}^n \) along \( Y \),

and let \( H \) and \( E \) be hyperplane and exceptional divisors on \( X \)

Here are a few standard observations about \( X \):

1. \( K_X \equiv -(n+1)H + (r-1)E \) (Riemann-Hurwitz).
2. \( H - \epsilon E \) is ample for \( 0 < \epsilon << 1 \) (Kleiman’s Criterion).

Our strategy for seeking upper bounds for \( e_Y \) rests on the following proposition, which is the essential observation of the paper.

**Proposition 2.1:** If there is an effective \( \mathbb{Q} \)-divisor \( F \) on \( X \) such that:

(i) \( F \equiv (e+n)H - rE \) and

(ii) \( \text{Nlc}(X,F) \cap E = \emptyset \),

then \( e_Y \leq e \).

**Proof:** Given such an \( F \), then for each \( \epsilon \in \mathbb{Q} \) satisfying \( 0 < \epsilon < 1 \), we would have \( \text{Nklt}(X,(1-\epsilon)F) \cap E = \emptyset \), and using (1),

\[ pH - E \equiv K_X + (1-\epsilon)F + A \]

where \( A \equiv (p+1-e+(e+n)\epsilon)H - r\epsilon E \). If additionally, \( \epsilon << 1 \), then by (2), \( A \) is ample (hence nef and big) provided that \( p \geq e \).

Moreover, \( d_Y H - E \) (and all positive multiples) is base-point-free on \( X \), by definition of \( d_Y \), so that for each positive integer \( k \),

\[ pH - kE \equiv K_X + (1-\epsilon)F + A \]

where \( A \equiv (p+1-e-(k-1)d_Y + (e+n)\epsilon)H - r\epsilon E + (k-1)(d_Y H - E) \) is ample provided that \( p \geq e + (k-1)d_Y \).

Thus, the log Kodaira vanishing theorem (using (ii)) tells us that

\[ H^i(X, \mathcal{J} \otimes \mathcal{O}_X(pH - kE)) = 0 \text{ for all } i > 0, p \geq e + (k-1)d_Y \]
where \( J \) is an ideal sheaf on \( X \) with the property that the support of \( \mathcal{O}_X/J \) is disjoint from \( E \).

We can therefore identify \( J \) with its direct image in \( \mathbb{P}^n \), and it is a consequence of the theorem of formal functions ([H],III.11) that:

\[
H^i(\mathbb{P}^n, J \otimes \mathcal{I}_Y^k \otimes \mathcal{O}_{\mathbb{P}^n}(p)) = 0 \quad \text{for all } i > 0, p \geq e + (k - 1)dy.
\]

To conclude the vanishing without \( J \), we use the fact that the ideal sheaves \( J \) and \( \mathcal{I}_Y^k \) have disjoint cosupport to conclude that \( \mathcal{O}_{\mathbb{P}^n}/J \) is a direct summand of \( \mathcal{O}_{\mathbb{P}^n}/\mathcal{I}_Y^k \). We also use the disjoint cosupport in the first of the following two exact sequences:

\[
0 \to J \otimes \mathcal{I}_Y^k \otimes \mathcal{O}_{\mathbb{P}^n}(p) \to \mathcal{I}_Y^k \otimes \mathcal{O}_{\mathbb{P}^n}(p) \to \mathcal{O}_{\mathbb{P}^n}(p)/J \to 0,
\]

\[
0 \to J \otimes \mathcal{I}_Y^k \otimes \mathcal{O}_{\mathbb{P}^n}(p) \to \mathcal{O}_{\mathbb{P}^n}(p) \to \mathcal{O}_{\mathbb{P}^n}(p)/(J \mathcal{I}_Y^k) \to 0.
\]

If it were the case that \( H^i(\mathbb{P}^n, \mathcal{I}_Y^k \otimes \mathcal{O}_{\mathbb{P}^n}(p)) \neq 0 \), then from the long exact sequence on cohomology associated to these two short exact sequences and the vanishing above, we would have \( H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(p)/J) \neq 0 \), hence \( H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(p)/(J \mathcal{I}_Y^k)) \neq 0 \) and \( H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(p)) \neq 0 \), a contradiction.

With this proposition, we now have a very fast

**Proof of Theorem 1:** Let \( I \subset \mathbb{C}[x_0, \ldots, x_n] \) be the ideal generated by the given homogeneous polynomials of degree \( dy = d_1, \ldots, d_m \) which scheme-theoretically cut out \( Y \). For each \( i = 1, \ldots, r \), let \( I_{d_i} \) be the homogeneous part of degree \( d_i \), and let \( |B_i| \) be the corresponding sub-linear series of \( |d_i H - E| \) on \( X \). It follows that for each \( x \in E \), the sum of general elements \( B^g := B^g_1 + \ldots + B^g_r \) is a normal-crossings divisor, hence log canonical at \( x \). Thus the Bertini property tells us that \( \text{Nlc}(X, B^g) \cap E = \emptyset \), so \( F := B^g \) satisfies the conditions of Proposition 2.1 with \( e = d_1 + \ldots + d_r - n \).

**Remark:** This proof is essentially the same as the proof in [BEL]. However, by making the dependence upon a suitable hypersurface \( F \) explicit in Proposition 2.1, a general strategy has emerged which was not apparent in [BEL]. Namely, given an embedding \( Y \subset \mathbb{P}^n \), one wants to find hypersurfaces which are highly singular along \( Y \) relative to their degree but whose strict transform on \( X \) is log canonical near \( E \). We will see in the next sections that certain determinantal varieties fit nicely into this strategy.
§3. Universal Determinantal Varieties: Let $V$ and $W$ be vector spaces of dimension $k$ and $m$ respectively, suppose that $k \leq m$, and let $Y$ be one of the following:

(a) $P(V) \times P(W)$ embedded in $P^n := P(V \otimes W)$ ($n = km - 1$) by the Segre embedding,

(b) $P(V)$ embedded in $P^n := P(\text{Sym}^2(V))$ ($n = \frac{1}{2}(k^2 + k) - 1$) by the quadratic Veronese embedding, or

(c) $G(2, V^*)$ embedded in $P^n := P(\wedge^2(V))$ ($n = \frac{1}{2}(k^2 - k) - 1$) by the Plücker embedding.

Then $Y$ is the rank one locus of a universal map $\phi$ of vector bundles. $\phi : V \otimes O_{P^n} \to W^* \otimes O_{P^n}(1)$ in case (a), and $\phi : V \otimes O_{P^n} \to V^* \otimes O_{P^n}(1)$ in cases (b) and (c), where $\phi$ is, respectively, symmetric and skew symmetric. Alternatively, one can, of course, choose bases for $W$ and $V$ and think of $\phi$ as a matrix of linear forms. In each case, $Y$ is the last of a nested sequence of degeneracy loci in $P^n$ determined by the map $\phi$. In (a) and (b), let $\Delta_i \subset P^n$ be the zero locus of $\wedge^i \phi$, while in (c), let $\Delta_i$ be the zero locus of $\wedge^{2i} \phi$, (to get this right scheme-theoretically, one needs to take the “square root” of this map...see below). Then it is a standard fact that:

(a) $Y = \Delta_2 \subset \Delta_3 \subset \ldots \subset \Delta_k$

and each $\Delta_i$ is irreducible of codimension $(k - i + 1)(m - i + 1)$ in $P^n$,

(b) $Y = \Delta_2 \subset \Delta_3 \subset \ldots \subset \Delta_k$

and each $\Delta_i$ is irreducible of codimension $\binom{k-i+2}{2}$ in $P^n$,

(c) $Y = \Delta_2 \subset \Delta_3 \subset \ldots \subset \Delta_{\lfloor \frac{k}{2} \rfloor}$

and each $\Delta_i$ is irreducible of codimension $\binom{k-2i+2}{2}$ in $P^n$.

If one takes $F_i$ to be an $i \times i$ minor of $\phi$ (in cases (a) and (b)...we’ll do case (c) later), then $F_i$ has degree $i$ and multiplicity $i - 1$ along $Y$. The strategy we take here for constructing the $F$ to use in Proposition 2.1 is therefore to sum general linear combinations of minors of the largest size until we hit an obstruction (determined by the corresponding degeneracy locus), then to decrease the size of the minor and continue. Amazingly (at least, to the author), we will finish with a log canonical divisor $F$ of multiplicity $r$ along $Y$ and degree $e + n$ where $e$ is independent of $k$ and $m$. 

9
We need to invoke two aspects of the theories of complete linear maps and quadrics (see, for example, [L] for an exposition and specific references).

**Complete Linear Maps and Quadrics...Classical Construction:** In both cases (a) and (b), let $U = \mathbb{P}^n - \Delta_k$. The space $P$ of complete objects is the smooth, projective variety obtained as the closure of the graph of $U$ under the morphism:

$$\wedge := (\wedge^2, ..., \wedge^k)$$

**Explanation:** A point $\alpha \in \mathbb{P}^n$ is a linear map (modulo scalars). In case (a), it is represented by a map $\alpha : V \to W^*$, while in case (b), it is represented by a symmetric map $\alpha : V \to V^*$. Thus $\wedge^i \alpha$ is a map from $\wedge^i V$ to $\wedge^i W^* \cong (\wedge^i W)^*$ in (a), and a symmetric map from $\wedge^i V$ to $\wedge^i V^* \cong (\wedge^i V)^*$ in (b). The locus $U \subset \mathbb{P}^n$ is the set of $\alpha$ such that $\wedge^i \alpha \neq 0$ for all $i \leq k$, thus it is where the map:

$$\wedge : P(V \otimes W) - \to P(\wedge^2 V \otimes \wedge^2 W) \times ... \times P(\wedge^k V \otimes \wedge^k W) \text{ in (a), or}$$

$$\wedge : P(S^2(V)) - \to P(S^2(\wedge^2 V)) \times ... \times P(S^2(\wedge^k V)) \text{ in (b)}$$

is defined, and $P$ is embedded in a product of $k$ projective spaces.

**Remarks:** $P$ comes equipped with projection morphisms:

$$\rho_i : P \to P(\wedge^i V \otimes \wedge^i W) \text{ in (a), and } \rho_i : P \to P(S^2(\wedge^i V)) \text{ in case (b).}$$

Thus, provided $i < k$ in (b) or $i \leq k$ and $i < m$ in (a), the projection $\rho_i$ maps to a positive-dimensional projective space, giving rise to a base-point-free linear series on $P$. By definition, the restriction of this linear series to $U$ is spanned by the $i \times i$ minors of $\phi$ (principal minors in case (b)). To pin down the linear series on $P$ associated to the map $\rho_i$, we use a second construction of complete linear maps and quadrics, due to Vainsencher:

**Complete Linear Maps and Quadrics...Blow-Up Construction:** Recall that we set $X := \text{bl}(P^n, Y)$. The fact that $i$ matrices of rank one sum to a matrix of rank at most $i$ implies that in cases (a) and (b),

$$\Delta_i = \Sigma_{i-1}(Y),$$

where $\Sigma_i(Y)$ is the secant variety defined as the closure of the union of projective planes spanned by $i$ distinct points of $Y$.

One blows up the degeneracy loci as follows:
\[ f_2 : X_2 \to X_1 := X \text{ blows up the strict transform of } \Sigma_2(Y) = \Delta_3, \]
\[ f_3 : X_3 \to X_2 \text{ blows up the strict transform of } \Sigma_3(Y) = \Delta_4 \]
\[ \vdots \]
\[ f_{k-1} : \tilde{X} := X_{k-1} \to X_{k-2} \text{ blows up the strict transform of } \Sigma_{k-1}(Y). \]

For consistency, let \( f_1 : X \to \mathbb{P}^n \) also be the blow-down. Then:

**Theorem:** ([N], Theorem 6.3) (a) Each strict transform of \( \Delta_{i+2} \) in \( X_i \) is smooth, so in particular \( \tilde{X} \) is smooth, because:

\[ f = f_2 \circ f_3 \circ \ldots \circ f_{k-1} : \tilde{X} \to X \]

is a sequence of blow-ups along smooth centers. In addition, if we let \( E_i \) denote the strict transform in \( \tilde{X} \) of the (smooth, irreducible) \( f_i \)-exceptional divisor (for all \( 1 \leq i \leq k-1 \)), then the divisor:

\[ E_1 + \ldots + E_{k-1} \]

is a normal crossings divisor on \( \tilde{X} \).

(b) The inclusion \( i : U \hookrightarrow P \) extends to an isomorphism \( \tau : \tilde{X} \sim P. \)

(There is also a precise recursive description of \( \tau \) which we will not need here.)

Suppose now that \( A_i \subset \mathbb{P}^n \) is the hypersurface cut out by some \( i \times i \) minor of \( \phi \) (or principal minor in case (b)). Then \( A_i \) has degree \( i \) and its multiplicity along \( \Delta_j \) is (at least) \( i - j + 1 \) for all \( j \leq i \). Thus, \( A_i \) determines a divisor:

\[ B_i \equiv iH - (i - 1)E_1 - \ldots - E_{i-1} \]

on \( \mathbb{P} \) by subtracting \( i - j + 1 \) copies of \( E_j \) from \( f^*(A_i) \).

I claim that the projection \( \rho_i \) determines a base-point-free sub-linear series of \( |B_i| \). To see this, it suffices to show that the generic multiplicity of \( A_i \) along \( \Delta_j \) is precisely \( i - j + 1 \) (so that no \( E_j \) is in the base locus of \( |B_i| \)). But given \( A_i \), choose \( \alpha \in \Delta_j - \Delta_{j-1} \) so that some \( j \times j \) minor (or principal minor) of \( \alpha \) contained in the \( i \times i \) minor defining \( A_i \) has nonzero determinant. Then it is immediate that \( A_i \) has multiplicity exactly \( i - j + 1 \) at \( \alpha \).

Here, then, is our main proposition to cover cases (a) and (b):
Proposition 3.1: Given nonnegative integers \( n_2, \ldots, n_k \), let \( A^g_{i,1}, \ldots, A^g_{i,n_i} \) be the zero loci of general linear combinations of the determinants of \( i \times i \) minors of \( \phi \) (principal in case (b)). Let \( F^g_{i,j} \) be the strict transform of \( A^g_{i,j} \) in \( X \), and let \( F^g = \sum_{i=2}^k (F^g_{i,1} + \ldots + F^g_{i,n_i}) \). Then:

(a) In case (a), suppose either \( k < m \) or \( k = m \) and \( n_k \leq 1 \). Then \( f : \widetilde{X} \to X \) is a log resolution of \( (X, F^g) \), and the discrepancy at \( E_j \) is:

\[
(k - j)(m - j) - 1 - \sum_{i>j}(i - j)n_i
\]

for each \( 2 \leq j \leq k - 1 \).

(b) In case (b), suppose \( n_k \leq 1 \). Then \( f : \widetilde{X} \to X \) is a log resolution of \( (X, F^g) \), and the discrepancy at \( E_j \) is

\[
\left( \frac{k - j + 1}{2} \right) - 1 - \sum_{i>j}(i - j)n_i
\]

for each \( 2 \leq j \leq k - 1 \).

Proof: Vainsencher’s theorem tells us the exceptional divisors have normal crossings. In case (b), and if \( k = m \) in case (a), the last exceptional divisor \( E_{k-1} \) is itself the strict transform of \( A_k \). Otherwise, the strict transforms of the \( F^g_{i,j} \) in \( \widetilde{X} \) are smooth members of \( |B_i| \), intersecting each other and the exceptional divisors transversely by (ordinary) Bertini. Thus \( f \) is a log resolution of the pair \( (X, F^g) \).

The discrepancies are computed using Riemann-Hurwitz (and the count for the codimensions at the beginning of this section) as well as the linear series computation for \( |B_i| \), which yields \( \sum_{i>j}(i - j)E_j = f^*(F^g_{i,j}) - f_*^{-1}(F^g_{i,j}) \).

The following corollary picks out the optimal choices for the \( n_i \) in order to produce an \( e_Y \) which is as small as possible.

Corollary 3.2: (a) In case (a), let \( n_k = m - k + 1 \) and \( n_i = 2 \) for \( 2 \leq i < k \). Then \( (X, F^g) \) is lc and \( F^g \equiv (n - 1)H - rE \).

(b) In case (b), let \( n_i = 1 \) for all \( i \). Then \( (X, F^g) \) is lc and \( F^g \equiv nH - rE \).

So using Proposition 2.1, we get \( e_Y \leq -1 \) in case (a) and \( e_Y \leq 0 \) in case (b).
**Proof:** A direct application of the Proposition tells us that the discrepancies are all $-1$ for these choices of the $n_i$, so $(X,F^9)$ is lc. These and the other computations (the coefficients of $H$ and $E$) are straightforward, and left to the reader.

For case (c), we need versions of the classical construction and blow-up construction for complete skew forms. Specifically, we’ll prove the theorem below in an appendix to this paper:

**Complete Skew Forms ... A “Classical” Construction:** Given $V$, let $l = \lfloor \frac{1}{2} \dim(V) \rfloor$, let $U = \mathbb{P}(\wedge^2 V) - \Delta_l$, and consider the rational map

$$\wedge : \mathbb{P}(\wedge^2 V) \dashrightarrow \mathbb{P}(\wedge^4 V) \times \ldots \times \mathbb{P}(\wedge^{2l}(V));$$

$$\alpha \mapsto (\alpha \wedge \alpha, \alpha \wedge \alpha \wedge \alpha, \ldots).$$

It is straightforward to check that for each $i = 2, \ldots, l$, the degeneracy locus $\Delta_i$ is the locus of indeterminacy of the map $\mathbb{P}(\wedge^2 V) \dashrightarrow \mathbb{P}(\wedge^{2i}(V))$ obtained by composing $\wedge$ with the projection, so in particular, $\wedge$ is regular on $U$, and we define:

**Definition:** The closure $P := \overline{\Gamma} \subset \mathbb{P}(\wedge^2 V) \times \ldots \times \mathbb{P}(\wedge^{2l}(V))$ of the graph of $\wedge$ restricted to $U$ is the space of complete skew forms on $V$.

**Remark:** This $\wedge$ map is not the same as the map we obtain by regarding a skew form as a linear map and restricting the wedge map. Firstly, it does not involve the odd wedge powers of $V$, and secondly it is a “square root” of the even part of the wedge map in the following sense. Notice that $\wedge^{2i} \wedge^2 V \subset \text{Sym}^2 \wedge^{2i} V$ as representations of $\text{GL}(V)$, so we can think of $\wedge^{2i} \alpha$ as being a quadratic form on $\wedge^{2i} V$. The value of this quadratic form on a decomposable wedge (i.e. a point of the Grassmannian $G(2i, V^*)$) is always a square, as it is the determinant of a principal (skew) minor of the skew form $\alpha : V \to V^*$ with the Pfaffian as a square root. (This is NOT to say, however, that each $\wedge^{2i} \alpha$ is of rank one.) One checks that these Pfaffians of $2i \times 2i$ principal minors give the linear series associated to the rational maps:

$$\mathbb{P}(\wedge^2 V) \dashrightarrow \mathbb{P}(\wedge^{2i}(V))$$

defined above. The common zero scheme of these Pfaffians is reduced (unlike the principal determinants), equal to the degeneracy locus $\Delta_i$. These linear series will be used as before to construct a log canonical divisor.
Complete Skew Forms ... The Construction by Blowing Up: Let $X = \text{bl}(\mathbb{P}(\wedge^2 V), G(2, V^*))$ and blow up the other degeneracy loci in order as before, letting $X_1 := X$, inductively letting:

$$ f_i : X_i \to X_{i-1} $$

be the blow up of the strict transform of $\Delta_{i+1}$, and letting $f = f_2 \circ ... \circ f_{i-1} : \tilde{X} \to X$. Finally, let $E_i \subset \tilde{X}$ be the strict transform of the $f_i$-exceptional divisor for all $1 \leq i \leq l - 1$. Then:

**Theorem 3.3:** (a) Each strict transform of $\Delta_{i+2}$ in $X_i$ is smooth, so $\tilde{X}$ is smooth, and moreover, $E_1 + ... + E_{l-1}$ is a normal crossings divisor on $\tilde{X}$.

(b) The inclusion $\iota : U \hookrightarrow P$ extends to an isomorphism $\iota : \tilde{X} \cong P$.

**Proof:** See the appendix.

Then as before, we conclude that for each $i < \frac{1}{2}\dim(V)$, the zero loci $A_i$ of the Pfaffians of principal $2i \times 2i$ minors of $\phi$ give elements of the linear series:

$$ |B_i| := |iH - (i - 1)E_1 - ... - E_{i-1}| $$

on $\tilde{X}$ which contains the base-point-free linear series associated to the projection $\rho_i : P \to \mathbb{P}(\wedge^{2i}(V))$. Finally, we obtain analogues of 3.1 and 3.2 in case (c):

**Proposition 3.4:** Let $n_2, ..., n_l$ be nonnegative integers, let $A_{i,1}^g, ..., A_{i,n_i}^g$ be zero loci of general linear combinations of Pfaffians of $2i \times 2i$ principal minors of $\phi$, let $F_{i,j}^g$ be the strict transform of $A_{i,j}^g$ in $X$, and let $F^g = \sum_{i=1}^{l} (F_{i,1}^g + ... + F_{i,n_i}^g)$.

If $2l < k = \dim(V)$ or $2l = k$ and $n_l = 1$, then $f : \tilde{X} \to X$ is a log resolution of $(X, F^g)$ and the discrepancy at $E_j$ is:

$$ \left( \frac{k - 2j}{2} \right) - 1 - \sum_{i>j} (i - j)n_i $$

for each $2 \leq j \leq l$.

**Proof:** Just as in Proposition 3.1.

**Corollary 3.5:** Let $n_l = 1$ if $k$ is even and $n_l = 3$ if $k$ is odd. Otherwise, let $n_i = 4$ for $2 \leq i < l$. Then $(X, F^g)$ is lc and $F^g \equiv (n - 3)H - rE$.

Thus using Proposition 2.1, we get $e_Y \leq -3$ in case (c).
4. Curves. Let \( C \) be a smooth, irreducible projective curve over the complex numbers of genus \( g \), let \( K_C \) be a canonical divisor, and \( D \) be a divisor of degree \( d \geq 3 \). This restriction on \( d \) assures us that the linear series map:

\[
\phi_{|K_C + D|} : C \to |K_C + D| \cong \mathbb{P}^{d+g-2}
\]

is an embedding (and we will set \( Y = C \) and \( n = d + g - 2 \) in this section). Notice that the degree of the embedding is \( d + 2g - 2 \), not \( d \).

We set \( X \) to be the blow-up of \( \mathbb{P}^n \) along \( Y \) as before, and recall the standard result (see, for example [ACGH]) that if \( d \geq 4 \), then the embedded curve \( C \) is a scheme-theoretic intersection of quadric hypersurfaces, from which it follows that:

(a) \( d_Y = 2 \), and

(b) the ample cone of \( X \) is spanned (in the \( H, E \)-plane) by \( H \) and \( 2H - E \).

The following result is not standard. It is proved in [B2] using Thaddeus’ stable pairs ([B]) and the author’s blow-up of secant varieties ([B1]). Indeed, these techniques closely resemble the two constructions of the spaces of complete objects in the previous section!

**Proposition 4A:** There exist log canonical divisors on \( X \) that are:

(a) numerically equivalent to \( (d - 1)H - (d - 3)E \) (if \( g \geq 0 \))

(b) numerically equivalent to \( dH - (d - 2)E \) (if \( g > 0 \))

(c) numerically equivalent to \( \left( \frac{d+g-5}{d-1} \right)(dH - (d - 2)E) \) (if \( g > 0 \) and \( d > 4 \)).

The divisors in (a) and (b) can be taken to be the strict transforms of hypersurfaces in \( \mathbb{P}^n \), but in case (c), one obviously needs to stick with \( \mathbb{Q} \)-coefficients.

We can apply our strategy directly now in genus 0 and 1:

**Proposition 4.1:** If \( g = 0 \) or \( g = 1 \), then \( e_Y \leq 1 \).

**Proof:** Recall that by Proposition 2.1, we are searching for log canonical divisors \( F \equiv (n + 1)H - (n - 1)E \) on \( X \) (and here \( n = d + g - 2 \)). But this is just what Proposition 4A (a) and (b) produce for us in genus 0 and 1.
To handle higher genus, we need to improve Proposition 2.1 a bit.

Namely, thanks to (b) above, we know that as soon as $\epsilon < \frac{1}{2}$, then $H - \epsilon E$ is in the ample cone of $X$. Recall that the key point of Proposition 2.1 was the observation that the desired vanishing occurs when $p$ and $k$ satisfy:

$$pH - kE \equiv K_X + (1 - \epsilon)F + A$$

where $0 < \epsilon < 1$, $F$ is log canonical (at least along $E$) and $A$ is big and nef. Since Proposition 4A (c) gives us a “very efficient” log canonical $\mathbb{Q}$-divisor on $X$, we’ll use this divisor and our better knowledge of the ample cone to get better vanishing results.

Assume throughout that $d > 4$ and that a log canonical divisor $F$ is given satisfying $F \equiv \left(\frac{d + g - 5}{d - 4}\right)(dH - (d - 2)E)$. Then

$$K_X + F \equiv \left(\frac{2g - 2}{d - 4}\right)(2H - E) - E,$$

which means that if we rewrite $pH - kE$ as above and let $\epsilon' = \frac{d + g - 5}{d - 4}\epsilon$, then

$$A \equiv (p + d\epsilon')H - (k - 1 + (d - 2)\epsilon')E - \left(\frac{2g - 2}{d - 4}\right)(2H - E).$$

Thus we see that $A$ is big and nef if the following two inequalities are satisfied, and at least one of them is strict:

(i) $p + d\epsilon' \geq 2(k - 1 + (d - 2)\epsilon')$ and

(ii) $k - 1 + (d - 2)\epsilon' \geq \frac{2g - 2}{d - 4}$.

We now get the following proposition by choosing $\epsilon'$ carefully:

**Proposition 4.2:** If $d > \frac{2g + 8}{3}$, then $e_Y \leq 1$.

**Proof:** For $\epsilon' \leq \frac{1}{d - 4}$, condition (i) is satisfied whenever $p \geq 2k - 1$. If there were no additional conditions on $k$, then this would imply $e_Y \leq 1$. (Recall that $d_Y = 2$). If we choose $\epsilon'$ to be very close to (and less than) $\frac{1}{d - 4}$, then condition (ii) becomes: $k > \frac{2g - d}{d - 4} + 1$.

By a theorem of Castelnuovo, $C \subset |K_C + D|$ is projectively normal, which is to say that vanishing holds when $k = 1$ and $p \geq 1$, so that we only need to prove vanishing for $k \geq 2$. But moreover, by the following:
Proposition (Rathmann): If \( d \geq 5 \), then

\[
H^i(P^n, I_C^2(p)) = 0 \text{ for all } i > 0 \text{ and } p \geq 3.
\]

we only need to prove vanishing when \( k \geq 3 \), which we get since both conditions are satisfied if \( k \geq 3 \) and \( d > \frac{2g+8}{3} \), hence \( A \) is ample, and log Kodaira vanishing applies.

Observation: If vanishing is proven for \( p \geq 2k - 1 \) and \( k \leq k_0 \), then there will be a corresponding improvement in the lower bound for \( d \) in Proposition 4.2. However, these will all be linear in \( g \), while I suspect the correct lower bound is actually \( d \geq 5 \), for which I submit the following “gap” as evidence.

Proposition 4.3: If \( d \geq 5 \) and \( g \) are fixed, then with at most finitely many exceptions for the values of \( p \) and \( k \),

\[
H^i(P^{d+g-2}, I_C^2(p)) = 0 \text{ for all } i > 0, k \geq 0 \text{ and } p \geq 2k - 1.
\]

Proof: By the proof of Proposition 4.2 and the two Propositions cited therein, any exceptions must lie in the region in the \( k, p \)-plane bounded on the left by \( k = 3 \) and on the right by \( k = \frac{2g-d}{d-4} + 1 \). This is of course infinite because there is no upper bound on \( p \). But notice that if \( p \geq 2k - 1 + a \), then we may boost \( \epsilon' \) to \( \frac{a}{d-4} \) (up to a maximum of \( \frac{d+g-5}{d-4} \)), preserving the inequality in condition (i). This gives a corresponding lowering of the upper bound for \( k \) to \( k < \frac{2g-d-a(d-2)}{d-4} + 1 \) beneath which the exceptions may occur. Thus any exceptions are constrained to lie in a roughly triangular region of the plane.
Appendix. Two constructions of complete skew forms. Let me begin by arguing why complete skew forms (definition in §3) are entirely analogous to complete quadrics.

Suppose \((R, m)\) is a discrete valuation ring over a field \(k\) with quotient field \(K\), residue field \(k\) and uniformizing parameter \(t\), and let \(\alpha\) be an \(R\)-valued skew 2-form on a vector space \(V\) over \(k\) of dimension \(2l\) or \(2l + 1\), with the following properties:

- the induced “generic” 2-form on \(V \otimes_k K\) is of rank \(2l\), and
- the induced “special” 2-form on \(V\) is nonzero.

In other words, suppose \(\alpha\) is the lift of a morphism \(f : \text{Spec}(R) \to \mathbf{P}(\wedge^2 V)\) with the property that \(\wedge \circ f : \text{Spec}(R) \to \mathbf{P}(\wedge^2 V) \times \ldots \times \mathbf{P}(\wedge^2 V)\) is defined at the generic point of \(\text{Spec}(R)\). I want to investigate the extension of \(\wedge \circ f\) across the special point.

Choosing a basis \(x_1, \ldots, x_n\) for \(V^*\) gives a straightforward description of the extension, since:

\[
\alpha = \sum_{i < j} a_{i,j} x_i \wedge x_j
\]

with \(a_{i,j} \in R\), and an \(r\)-fold wedge \(\alpha \wedge \ldots \wedge \alpha\) is of the form \(\sum_I f_I x_{i_1} \wedge \ldots \wedge x_{2r}\) with \(f_I \in R\). There will be a maximal \(d_r\) such that each \(f_I\) is divisible by \(t^{d_r}\), and the nonzero images of \(t^{-d_r} \alpha \wedge \ldots \wedge \alpha\) in \(\wedge^{2r} V^* \otimes R/m \cong \wedge^{2r} V^*\) for each \(r\) will give the image of the special point.

The basis-free approach sets up the analogy with complete quadrics. Given \(\alpha\), let \(\alpha_0 \in \wedge^2 V^* \otimes R/m \cong \wedge^2 V^*\) be its residue modulo \(m\), and suppose \(\alpha_0\) has rank \(2r_1\). Then \(\alpha_0\) is induced from a nondegenerate skew form on a quotient \(V \to T_{r_1}\), and we let \(W_{r_1}\) be the kernel of this map. The image of \(\alpha\) in \(\wedge^2 W_{r_1}^* \otimes R\) lies in \(\wedge^2 W_{r_1}^* \otimes m\), and we let \(\alpha_1 \in \wedge^2 W_{r_1}^* \otimes m/m^2 \cong \wedge^2 W_{r_1}^*\) be the residue modulo \(m^2\). Continuing in this manner, we produce from \(\alpha\) the following data:

(D1) A flag of (strict) subspaces:

\[
W_{r_m} \subset W_{r_{m-1}} \subset \ldots \subset W_{r_1} \subset W_{r_0} = V
\]

such that \(\dim(W_{r_{i-1}}/W_{r_i}) = 2r_i\) and \(r_1 + \ldots + r_m = l\).

(D2) Skew 2-forms \(\alpha_i\) on \(W_{r_i}\), induced from nondegenerate skew forms on the quotients \(W_{r_i}/W_{r_{i+1}}\).
The data (D1) and (D2) determine elements of each $P(\wedge^{2r}V)$ as follows. For each $\alpha_i$, let $\tilde{\alpha}_i \in \wedge^2 V^*$ be an arbitrary lift, and take the $r$-fold wedge product:

$$\omega_r := \alpha_0 \wedge ... \wedge \alpha_0 \wedge \tilde{\alpha}_1 \wedge ... \wedge \tilde{\alpha}_1 \wedge \tilde{\alpha}_2 \wedge ...$$

taking up to $r_1$ copies of $\alpha_0$ followed by up to $r_2$ copies of $\tilde{\alpha}_1$, etc. until $r$ terms in all have been taken. It is now an easy exercise to see that:

(i) $\omega_r \in \wedge^{2r} V^*$ is nonzero if $r \leq l$, and

(ii) $\omega_r$ does not depend upon the choice of lifts, and

(iii) if the data (D1) and (D2) come from $\alpha \in \wedge^2 V^* \otimes R$ as described above, then the $\tilde{\omega}_r \in P(\wedge^{2r}V)$ extend $\wedge \circ f$ across the special point of Spec($R$).

In fact, we have the following Lemma, which should look familiar to anyone who has thought about complete quadrics:

**Lemma A1:** The map to $P(\wedge^2 V) \times ... \times P(\wedge^{2l}V)$ is a bijection from the set of data (D1) and (D2) (modulo scalars) to the set of complete skew forms.

**Proof:** Given a subspace $W \subset V$ of codimension $2r$ and quotient $T$, the canonical inclusion $\wedge^{2r}T^* \otimes \wedge^2 W^* \subset \wedge^{2r+2}V^*$ is the key to recovering $\alpha_0, ..., \alpha_m$ (modulo scalars) from its image in $P(\wedge^2 V) \times ... \times P(\wedge^{2l}V)$. Precisely, if $(\beta_1, ..., \beta_l)$ is the image, then $\beta_1 = \alpha_0$ (up to scalar multiple) and therefore determines $W_{r_1}$. Then $\beta_{r_1+1}$ determines $\alpha_1$ by the inclusion above, determining $W_{r_2}$, and $\beta_{r_1+r_2+1}$ determines $\alpha_2$, etc. proving injectivity.

One can always “smooth” $\alpha_0, ..., \alpha_m$, taking $\alpha = \alpha_0 + t\tilde{\alpha}_1 + ... + t^m\tilde{\alpha}_m$ (where $\tilde{\alpha}_i$ denotes a lift to $\wedge^2 V^*$) exhibiting the image of the sequence of $\alpha_i$ as a specialization of $\wedge \circ f$ and proving that the set of data (D1) and (D2) maps to the complete skew forms. Surjectivity follows from the valuative criterion for properness.

The basic idea behind Theorem 3.3 is the same as in the case of complete quadrics. The normal bundle to the smooth subvariety $\Delta_r - \Delta_{r-1} \subset P(\wedge^2 V)$ is naturally a (twisted) bundle of skew forms on the distinguished subspaces $W_r \subset V$. Thus, the information consisting of a rank $2r$ form $\alpha_0$ and a point in the projectivized normal bundle to $\Delta_r - \Delta_{r-1}$ at $\tilde{\alpha}_0$ is part of the data (D1) and (D2). Blowing up the degeneracy loci in order turns out to give a natural variety structure to the data (D1) and (D2) (modulo scalars) which one then proves is isomorphic to the variety of complete skew forms.
We will thus need to consider complete skew forms in a relative setting. It will suffice for our purposes to generalize the above discussion to the case where \( V \) is a vector bundle over a smooth base scheme \( X \) over an algebraically closed field, though as in the case of complete bilinear forms (see [KT]) much of what is proved here can presumably be further generalized.

Given the vector bundle \( V \), we introduce the following cast of characters:

**Definitions:**

1. \( \pi_{r,V} : \mathbf{P}(\wedge^2 V) \to X \) (whenever \( 2r \leq \text{rk}(V) \)).
2. \( \rho_{r,V} : \mathbf{G}(V, 2r) \to X \), the bundle of \( 2r \)-dimensional quotients of the fibers of \( V \) (with relative Plücker embedding \( \mathbf{G}(V, 2r) \subset \mathbf{P}(\wedge^2 V) \)).
3. \( 0 \to S_{r,V} \to \rho_{r,V}^* V \to Q_{r,V} \to 0 \), the universal sequence on \( \mathbf{G}(V, 2r) \).
4. \( f_{r,V} : \mathbf{P}(\wedge^2 Q_{r,V}) \to \mathbf{P}(\wedge^2 V) \) inducing a 2-form on \( V \) from one on the quotient. This is the Plücker embedding when \( r = 1 \). Note that \( \mathbf{P}(\wedge^2 Q_{r,V}) \) is a projective bundle over \( \mathbf{G}(V, 2r) \), with projection map \( \pi_{1,Q_{r,V}} \).
5. \( \Delta_{r,V} \subset \mathbf{P}(\wedge^2 V) \) is the “bundle of degeneracy loci” of the fibers, well-defined since rank is independent of basis.

All the important identifications are made in the following lemma.

**Lemma A2:**

(a) \( \Delta_{r,V} = f_{r,V}(\mathbf{P}(\wedge^2 Q_{r,V})) \).

(b) \( f_{r,V} \) is an embedding when restricted to the complement of \( \Delta_{r-1,Q_{r,V}} \). The normal bundle of the embedding is \( \pi^* \wedge^2 S_{r,V}^* \otimes \mathcal{O}(1) \).

(c) For each \( r \leq s \), the fiber product of \( f_{r,V} \) and \( f_{s,V} \) satisfies:

\[
\mathbf{P}(\wedge^2 Q_{r,V}) \times_{\mathbf{P}(\wedge^2 V)} \mathbf{P}(\wedge^2 Q_{s,V}) \cong \mathbf{P}(\wedge^2 Q_{r,Q_{s,V}})
\]

which is a projective bundle over the flag variety \( \text{Fl}(V, 2s, 2r) \). The map to \( \mathbf{P}(\wedge^2 Q_{s,V}) \) is \( f_{r,Q_{s,V}} \) (given by (iv) above) and if \( \sigma : \text{Fl}(V, 2s, 2r) \to G(V, 2r) \) is the forgetful map, then the map to \( \mathbf{P}(\wedge^2 Q_{r,V}) \) is the projection from \( \mathbf{P}(\wedge^2 Q_{r,Q_{s,V}}) \cong \sigma^* \mathbf{P}(\wedge^2 Q_{r,V}) \).

(d) The map from the pull-back of the conormal bundle of \( f_{r,V} \) to the conormal bundle of \( f_{r,Q_{s,V}} \), is, after the identifications from (b), the natural map:

\[
\sigma^* \left( \wedge^2 S_{r,V} \right) (-1) \to \wedge^2 S_{r,Q_{s,V}} (-1)
\]

on the complements of \( \Delta_{r-1,s} \) in the fiber product from (c).
**Proof:** A skew form of rank \(\leq r\) on a fiber of \(V\) is always induced from a skew form on an \(r\)-dimensional quotient of the fiber. This gives (a).

In (b), injectivity is clear. Via the Euler sequences for the tangent bundles to \(P(\wedge^2 V)\) and \(P(\wedge^2 Q_{r,V})\), one obtains a sheaf map:

\[
0 \rightarrow S^*_{r,V} \otimes Q^*_{r,V} \xrightarrow{\phi} \wedge^2 V^*(1)/ \wedge^2 Q^*_{r,V}(1)
\]

over \(P(\wedge^2 Q_{r,V})\) with the property that \(f_{r,V}\) is an immersion with normal bundle isomorphic to the cokernel of \(\phi\) wherever \(\phi\) is fiberwise injective. But \(\phi\) factors through the natural map:

\[
S^*_{r,V} \otimes Q^*_{r,V} \rightarrow S^*_{r,V} \otimes Q^*_{r,V}(1)
\]

in the obvious way, exhibiting \(f_{r,V}\) as an immersion with desired normal bundle precisely on the complement of \(\Delta_{r-1,Q_{r,V}}\).

(c) is straightforward. The proposed fiber product embeds naturally in the product. And (d) follows from a diagram chase.

Now that the identifications (a)-(d) in Lemma A2 have been established, it is a formal consequence of the recursive nature of the conormal bundles and maps among them that:

1. The blow up, in order, of the strict transforms of the degeneracy loci \(\Delta_{r,V}\) produces a smooth variety \(\tilde{X}\) with normal crossings exceptional divisors, and

2. The set of data \((D1)\) and \((D2)\) (modulo scalars) corresponds to the points of \(\tilde{X}\), with each new subspace \(W_i\) and skew form \(\alpha_i\) corresponding to a normal direction of the strict transform of \(\Delta_{r,V}\) (modulo scalars).

This is proved, for instance, in Proposition 2.2 of [B1] with Lemma 1.3 of that paper playing the role of Lemma A2 here. The steps of that proof are readily adapted to handle this situation (or, for that matter, the case of complete linear maps and quadrics). This takes care of the first part of Theorem 3.3, leaving us to prove that the rational map:

\[
t: P(\wedge^2 V) \dashrightarrow P(\wedge^2 V) \times ... \times P(\wedge^2 V)
\]

extends to an embedding of \(\tilde{X}\) which agrees with the map on the set of data \((D1)\) and \((D2)\) (modulo scalars) considered in Lemma A1.
To prove that \( \iota \) extends to a map from \( \tilde{X} \), we use:

**Lemma A3:** Suppose \( X \) is a normal variety and \( Y \) is a projective variety over an algebraically closed field \( k \). If \( f : X \to Y \) is a rational map and \( \overline{f} : X \to \overline{Y} \) extends \( f \) as a map of sets, then \( \overline{f} \) is a morphism if and only if the following “valuative” criterion is satisfied:

\[ (*) \text{ For all discrete valuation rings } R \text{ over } k \text{ with residue field } k, \text{ and all morphisms } \alpha : \text{Spec}(R) \to X \text{ sending the generic point } \xi \in \text{Spec}(R) \text{ to the domain of } f, \text{ the image of the special point under } \overline{f} \circ \alpha \text{ agrees with the specialization of } f(\alpha(\xi)). \]

**Proof:** This is an immediate consequence of Zariski’s Main Theorem applied to the graph of \( f \) (see [H], V.5.2).

We can apply this to the extension of \( \iota \) via the identification of \( \tilde{X} \) with the set of data (D1) and (D2) (modulo scalars). The discussion preceding Lemma A1 tells us that the conditions of Lemma A3 are satisfied, implying that the bijection of Lemma A1 is a morphism \( \tau : \tilde{X} \to P \) to the space of complete skew forms. Thus it only remains to prove that \( \tau \) is an immersion. But we can prove this by induction on the rank of \( V \) (again considering the relative setting). Namely, we know that \( \tau \) is an immersion on the complement of exceptional divisors, since that locus is included in \( P(\wedge^2 V) \). On the other hand, by induction, the exceptional divisor over \( \Delta_{r,V} \) embeds in the complete skew forms on \( P(\wedge^2 Q_r) \), hence in the complete skew forms on \( V \) via the embedding:

\[
P(\wedge^2 Q_r) \times_{G(V,2r)} \ldots \times P(\wedge^{2r-2} Q_r) \hookrightarrow P(\wedge^2 V) \times_X \ldots \times P(\wedge^{2r-2} V) \times G(V, 2r).
\]

This only leaves normal vectors to the exceptional divisors to worry about. But such a normal vector is either tangent to some other exceptional divisor, and we have already dealt with it, or else it maps to a nonzero normal vector to the smooth part of a \( \Delta_{r,V} \subset P(\wedge^2 V) \) under the blow-down. Thus in all cases, nonzero tangent vectors to \( \tilde{X} \) remain nonzero under \( \tau_* \), and \( \tau \) is indeed an embedding.
References

[ACGH] E. Arbarello, M. Cornalba, P.A. Griffiths and J. Harris, Geometry of Algebraic Curves Volume I, Springer-Verlag (1985).

[B1] A. Bertram, *Moduli of rank-2 vector bundles, theta divisors, and the geometry of curves in projective space*, J. Diff. Geo. **35**, No.2 (1992) 429-470.

[B2] A. Bertram, *Stable pairs and log flips*, to appear in Algebraic Geometry (Santa Cruz 1995), Proc. Symp. Pure Math, AMS.

[BEL] A. Bertram, L. Ein and R. Lazarsfeld, *Vanishing theorems, a theorem of Severi, and the equations defining projective varieties*, Journal of the AMS **4**, No.3 (1991) 587-602.

[CKM] H. Clemens, J. Kollár and S. Mori, Higher Dimensional Complex Geometry, Astérisque **166** (1988).

[H] R. Hartshorne, Algebraic Geometry, Springer-Verlag (1977).

[K] J. Kollár, *Singularities of pairs*, to appear in Algebraic Geometry (Santa Cruz 1995), Proc. Symp. Pure Math, AMS.

[Ketal] J. Kollár (with 14 coauthors), Flips and Abundance for Algebraic Threefolds, Astérisque **211** (1993).

[KT] S. Kleiman and A. Thorup, *Complete bilinear forms*, Algebraic Geometry (Sundance 1986), Lecture Notes in Math **1311**, Springer (1988) 253-320.

[L] D. Laksov, *Completed quadrics and linear maps*, in Algebraic Geometry (Bowdoin 1985), Proc. Symp. Pure Math **46**, Part 2, AMS (1987) 371-387.

[T] M. Thaddeus, *Stable pairs, linear systems and the Verlinde formula*, Invent. Math. **117** (1994), 317-353.

[R] J. Rathmann, *An infinitesimal approach to a conjecture of Eisenbud and Harris*, preprint.
I. Vainsencher, *Schubert calculus for complete quadrics*, in Enumerative Geometry and Classical Algebraic Geometry, P. Le Barz, Y. Hervier eds., Birkhauser (1982), 199-236.

J. Wahl, *On Cohomology of the square of an ideal sheaf*, preprint, [alg-geom/9601027].

University of Utah, Salt Lake City, UT 84112

*email address*: bertram@math.utah.edu