Stability and Sensitivity Analysis of Non-Newtonian Flow through an Axisymmetric Expansion

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Abstract. This paper deals with a linear stability analysis of the flow in a circular pipe with a sudden expansion. We consider both Newtonian and non-Newtonian fluid models and a thorough comparison is presented. The stability analysis is completed by an adjoint-based investigation on the sensitivity characteristics of perturbations. The results are discussed and compared, when it is possible, to those already published in the pertinent literature.

1. Introduction

The main goal of this paper is the stability analysis of the flow of Newtonian and non-Newtonian fluids through a sudden expansion of an axisymmetric tube. The problem is interesting either for the physics of the fluid mechanics and for the applications. Very recent papers (Rojas et al. 2010, Cantwell et al. 2010, Mullin et al. 2009) put the attention on the behavior of the first instabilities of the flow which are observed in the domain. In particular experimental measurements carried out by Mullin et al. showed the existence of an instability at Reynolds number around 1140 (where Re is here based on the diameter of the upstream tube and the mean velocity before the expansion). From a numerical point of view the linear stability analysis (Sanmiguel-Rojas et al. 2010) showed an exponential unstable mode at $Re = 3273$, while a transient growth analysis (Cantwell et al. 2010) studied the amplification of the perturbation for symmetric and non-symmetric modes.

In addition to these basic issues, this simple geometry arises frequently also in many industrial applications and as a prototype for more complex configurations existing in biological systems such as, for example, the flow of blood in arteries with stenosis (Jung et al. 2004).

The behavior of the fluid in such geometry is characterized by the existence of a recirculating region whose size scales with the Reynolds number and with the value of the expansion ratio.

In many industrial processes, as well as in some biological flows, the assumption of Newtonian behavior is often not acceptable. This is an important issue since the characteristics of the flow can be strongly affected by the peculiar properties of the fluid used in the application. In particular, for the flow in a circular pipe with a sudden expansion, the recirculation bubble can show pronounced changes when the fluid is non-Newtonian. This behavior is well known in both the rheological (Nag & Datta 2007, Ternik 2010) and biomechanics (Forrest & Young 1970) communities.
For this reason, we consider a non-Newtonian fluid described by the Carreau-Yasuda model (Bird 1987) which has sufficient flexibility to fit a wide variety of experimental viscosity/rate of strain curves.

In particular, in this paper, we study the stability of a stationary base flow without swirl. The presence of the spatial non homogeneity due to the sudden expansion does not allow the use of a quasi parallel approach to the analysis of the stability which, for this reason, can be classified as “global” (Theofillis 2011).

In this paper we report a comparison between the stability characteristics of flow for different parameters of the viscosity model up to a Reynolds number of 3000, where this time the non dimensional number is based on the diameter and the maximum velocity at the inlet. For a Newtonian fluid, thus, the reference Reynolds number used in this paper is twice that used in Cantwell et al. (2010) and in Sanmiguel-Rojas et al. (2010). In particular here we inspect the spatial characteristics of some global modes and we report the results obtained through an adjoint-based analysis (Giannetti & Luchini 2007) which shows the sensitivity characteristics of the flow. In particular the sensitivity of the eigenvalue to a structural perturbation of the differential operators and the mode sensitivity to initial condition and/or to forcing term are presented. During the oral presentation a more complete set of results, including the determination of the critical Reynolds number for both the Newtonian and non-Newtonian cases and a comparison with the experimental data, will be discussed in details.

2. Mathematical formulation and numerical method

With reference to Fig. 1 the geometry is provided by an inlet tube of diameter $d$ and length $L_1$ which expands into a tube of diameter $D$ and length $L_2$. Here only the case $d/D = 0.5$ will be discussed. A cylindrical system of coordinates $(r, \theta, x)$ has its origin $O$ in correspondence of the sudden expansion. The fluid flows from left to right. An inlet pipe length of $10d$ was used for the simulation, as in Sanmiguel-Rojas et al. (2010).

By assuming the flow incompressible, its motion is governed by the Navier-Stokes equations

$$\nabla \cdot \mathbf{u} = 0,$$  

(1)
\[\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \nabla \mathbf{u} \right) = -\nabla p + \nabla \mathbf{T}. \]  

(2)

In Eqs. (1,2) \(\mathbf{u}\) is the fluid velocity vector with components \((u_r, u_\theta, u_x)\), \(\rho\) the density and \(p\) the pressure. The stress tensor \(\mathbf{T}\) depends linearly on the rate of deformation tensor \(\mathbf{D} = \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^T)\) through the relation \(\mathbf{T} = 2\mu(S)\mathbf{D}\), where \(S = 2tr(D)^2\).

While for a Newtonian fluid the dynamic viscosity \(\mu\) does not depend on the shear rate \(S\), for a non-Newtonian fluid this dependence cannot be ignored and the functional form of \(\mu = \mu(S)\) changes with the particular fluid considered (Bird 1987). For the Carreau-Yasuda model, the viscosity law has the following form

\[
\frac{\eta - \eta_\infty}{\eta_0 - \eta_\infty} = \left[1 + \left(\frac{\lambda a}{S a/2}\right)^{n-1}/a\right]^{1/(n-1)} \]  

(3)

where \(\eta_\infty\) and \(\eta_0\) are the viscosities at infinite and zero shear rates, respectively. \(\lambda\) is the relaxation time and the two parameters \(a, n\) give the possibility to have the power-law region behavior.

Leuprecht et al. (1999) reported a satisfactory comparison of the Yasuda-Carreau model with experimental data concerning the blood properties.

The total flow field \((\mathbf{u}, p)\) is decomposed into an axisymmetric stationary base flow \((U_r, U_x, P)\) and a small amplitude time-dependent three-dimensional perturbation \((u'_r, u'_\theta, u'_x, p')\). The normal mode hypothesis gives then the following expression for the fluctuating field

\[u'_r, u'_\theta, u'_x, p'(r, \theta, x, t) = [\hat{u}_r, \hat{u}_\theta, \hat{u}_x, \hat{p}](r, x) \exp(\sigma t + m\theta) \]  

(4)

where the complex \(\sigma = \sigma_r + i\sigma_i\) is the eigenvalue of the problem which gives the growth rate (real part) and the possible oscillations of the mode. The real \(m\) fixes the azimuthal mode (symmetric or non symmetric).

The governing equations, both for the base flow and the perturbation, written in cylindrical coordinates, are discretized by means of a second order finite difference conservative scheme on a non-uniform smoothly-varying staggered mesh. A domain of \([0:1]\times[-10:400]\) with a resolution of \((120\times3300)\) grid points were used for the simulations. The nodes are uniformly spaced in the radial direction, while in the axial direction the mesh spacing is smoothly increased in the downstream direction.

The base flow is governed by the axisymmetric and stationary version of the Navier-Stokes equations, while the small perturbation satisfies, upon a linearization and the Fourier expansion in the azimuthal angle \(\theta\), a linearized version of the full three-dimensional governing equations. The discrete problem describing the axisymmetric base flow is solved numerically by a Newton-Raphson algorithm and a sparse LU decomposition (UMFPACK package). At the inlet of the tube we give the velocity profile which corresponds to the fully developed flow in an infinite tube, in the Newtonian case this profile is parabolic. on the external wall the adherence condition is given on the velocity, while the classical outflow condition is given in terms of vanishing \(x\)–derivative of the velocity field. On the axis the condition is

\[U_r = \frac{\partial U_x}{\partial r} = 0.\]  

(5)

We observe that no assumptions on the streamwise behavior of the perturbation are used to simplify the stability problem. As a consequence the global stability analysis, which is here performed, leads to a large generalized eigenvalue problem (Theofillis 2011) which has been solved by the use of the ARPACK package (Lehoucq et al. 1998).
The boundary conditions for the stability problem are the same of base flow at the inlet, the outlet and the walls, but homogeneous. On the axis the boundary condition depends on the azimuthal number \( m \). For \( m = 1 \) the proper condition are

\[
\frac{\partial u'_r}{\partial r} = \frac{\partial u'_\theta}{\partial r} = u'_x = 0.
\]

In order to highlight the importance and the effects of a proper fluid model on the general characteristics of the flow, results obtained using a Newtonian and Non-Newtonian model will be compared at different values of the Reynolds numbers.
Figure 2. Streamlines distribution for Newtonian (top) and non-Newtonian fluids at $Re = 200$. (Middle) $\lambda = 5, a = 2, n = 0.8$. (Bottom) $\lambda = 5, a = 2, n = 0.5$.

Figure 3. Downstream reattachment coordinate $x_r$ vs the Reynolds number for Newtonian and non-Newtonian fluids. (1) $\lambda = 5, a = 2, n = 0.8$. (2) $\lambda = 5, a = 2, n = 0.5$. (3) $\lambda = 20, a = 2, n = 0.5$
3. Results

3.1. Baseflow
As an example, we report here some results concerning the base flow. In particular Fig.2 shows the streamlines at \( Re = 200 \), for Newtonian and non-Newtonian fluid models. The dashed lines represents streamlines outside of the recirculation bubbles. Note that the computational domain length \( L_2 \) after the expansion extends up to 400 diameters \( d \) and, for sake of clarity, only a close up view of the recirculation region is shown in the figures. The size of the recirculating bubble increases when the fluid is less newtonian.

The figures show the streamlines distributions for a Yasuda-Carreau fluid (Fig.1) and a Newtonian fluid (Fig.2). The color scale is related to the value of the axial velocity component. Both computations have been performed on a domain whose length extends to \( 400 \times d \).

The larger extent of the recirculation region in the non-Newtonian case is just an example of the importance, in some applications, of selecting a proper fluid model. Figure 3 shows the behavior of the reattachment point for different Reynolds numbers and for different parameters of the viscosity model. As previously observed in other studies, the reattachment point \( x_r \) scales linearly with the Reynolds number both for the Newtonian and non-Newtonian case.

In the Newtonian case the scaling law is given by \( x_r = 0.0221 Re \), which compares quite well with the value \( 0.0219 Re \) of Cantwell et al. (2010) and with 0.022 obtained from the experiments of Hammad et al. (1999).

3.2. Stability
Fig. 4 shows a close up of the spectra in the region close to the real axis at \( Re = 2800 \) for Newtonian and non Newtonian fluids. The least damped eigenvalues are stationary and can
Figure 5. Spatial distribution of the least damped zero-frequency mode. $Re = 2800$, $m = 1$.

be compared with the numerical analyses of Cantwell (2010) and Rojas (2010): the results show an excellent agreement between the data obtained by our finite-difference code and their spectral-element discretization.

By inspecting the figure, we observe that the least stable stationary eigenvalues for the non Newtonian fluid are lightly shifted to left, that’s to say their behavior is more stable. For those eigenvalues which oscillate the effect is opposite. To a less newtonian fluid corresponds a less stable set of eigenvalues. Preliminary results show that for the Reynolds number considered here the flow is globally stable in case 1 and 2, while is unstable in case 3, due to the existence of unstable modes with frequencies larger than those shown in the figure. A more complete analysis of the spectra will be discussed during the conference presentation.

Figure 5 shows the spatial distribution of the first non axially-symmetric mode ($m = 1$) where the absolute value of the perturbation velocity is plotted. The dotted line refers to the separation bubble. It can be observed that in the Newtonian case (top of Fig. 5) the disturbances are localized near to the recirculating region. This characteristics is common to other geometries
where large regions of separated flow are present (see for example Marino & Luchini 2009). For the two non Newtonian fluids (middle and bottom of Fig. 5) the mode extension becomes larger and its maximum progressively shifts towards the exit of the tubes, partially due to a major extension of the recirculation bubble. Note in particular that for $\lambda = 5$, $a = 2$, $n = 0.5$, the domain used for the computations ([0:1]x[-10:400]) was not large enough to capture the point where the direct mode reaches its maximum. Nevertheless, computations performed with smaller domains, always gave the same eigenvalues, confirming in this way the validity of the analysis presented here.

The eigenvalues spectra and the corresponding spatial structure of the modes can explain only partially the characteristics of the instabilities of the flow. As well established in many papers (Giannetti & Luchini 2007, Chomaz 2005, among the others) the adjoint analysis can give important additional information regarding the sensitivity of the flow. Such information can be used also to explain experimental results which are not in perfect agreement with the numerical
simulations. In particular, the adjoint of the mode gives an insight about the sensitivity to inlet or initial disturbances, while the direct-adjoint product is a measure of the eigenvalue sensitivity to perturbations in the structure of the differential operator, which physically can be thought as small variations in the boundary conditions or in the fluid model.

Figure 6 reports the sensitivity of the mode to momentum forcing and to initial conditions. It is interesting to observe that in all the cases considered here, the sensitivity is sharply localized near to the step of the expansion, with values which get larger and larger as the fluid becomes more non-Newtonian.

Finally, in Figure 7 the spatial distribution of the structural sensitivity (the product between the direct and the adjoint mode) is shown. Note that larger values of the sensitivity are well confined inside the initial region of the recirculating bubble. Such results explain why we obtained correct eigenvalues even in cases for which the computational domain was too small to contain the maximum of the direct mode. According to the theory, in fact, the eigenvalues changes considerably only when the border of the computational domain crosses the regions where the structural sensitivity attains larger values.

Figure 7. Spatial distribution of the adjoint-direct product. $Re = 2800$, $m = 1$. 
References

[1] BIRD, R.B., ARMSTRONG, R.C. AND HASSAGER, O. 1987 Dynamics of Polymeric Liquids. vol. I. Fluid Mechanics. Wiley, New York.

[2] CANTWELL, C. D., BARKLEY, D., BLACKBURN, H.M. 2010 Transient growth analysis of flow through a sudden expansion in a circular pipe, Phys. Fluids 22.

[3] CHOMAZ, J.M. 2005 Global instabilities in spatially developing flows: Non-normality and nonlinearity”. Annual Review of Fluid Mechanics, 37, 357–392.

[4] FORREST, J.H., YOUNG, D.F. 1970 Flow through a converging-diverging tube and its implications in occlusive vascular disease. J. Biomechanics 3,307-316.

[5] GIANNETTI, F. LUCHINI P. Structural sensitivity of the first instability of the cylinder wake. J. Fluid Mech. 581, 167–197.

[6] HAMMAD, K.J., OTUGEN, M. V., ARIK, E. B. 1999 A PIV study of the laminar axisymmetric sudden expansion flow, Exp. Fluids 26, 266.

[7] LEHOUQ, R. B., SORENSEN, D. C., YANG, C. 1998 ARPACK Users Guide: Solution of Large-Scale Eigenvalue Problems with Implicitly Restarted Arnoldi Methods. Philadelphia, SIAM.

[8] LEUPRECHT A. AND K. PERKTOLE, 1999 Numerical Studies of Viscoelastic Blood Flow Behaviour in Large Arteries, Proceedings of 4th International Symposium on computer Methods in Biomechanics & Biomedical Engineering, Lisbon, Portugal, 737 – 742.

[9] MARINO, L., LUCHINI, P. 2009 Adjoint analysis of the flow over a forward-facing step, Theoretical and Computational Fluid Dynamics, 23.

[10] MULLIN, T., SEDDON, J. R. T., MANTLE, M.D., SEDERMAN, A. J. 2009 Bifurcation phenomena in the flow through a sudden expansion in a circular pipe, Phys. Fluids 21.

[11] NAG, D., DATTA, A. 2007 Variation of the recirculating length of Newtonian and non-Newtonian Power-Law through a suddenly expanded axisymmetric geometry. J. Fluid Eng. 129,245-250.

[12] SANMIGUEL-ROJAS, E., DEL PINO, GUTIERREZ-MONTES, C. 2010 Global mode analysis of a pipe flow through a 1:2 axisymmetric sudden expansion, Phys. Fluids 22.

[13] TERNIK, P. 2010 New contributions on laminar flow of inelastic non-Newtonian fluid in the two dimensional symmetry expansion: Creeping and slowly moving conditions. J. Non-Newtonian Fluid Mech. 165, 1400-1411.

[14] THEOFILLIS, V. 2011 Global Linear Instability. Annu. Rev. Fluid Mech. 43, 319-352.

[15] JUNG, H., CHOI, J.W., PARK, C.G. 2004 Axisymmetric flows of non-Newtonian fluids in symmetric stenosed artery, Korea-Australia Rheology Journal 16,101-108.