The asymptotic number of lattice zonotopes

Olivier Bodini, Théophile Buffière

Abstract

We provide a sharp estimate for the asymptotic number of lattice zonotopes inscribed in \([0, n]^d\) when \(n\) tends to infinity. Our estimate refines the logarithmic equivalent established by Barany, Bureaux, and Lund when the sum of the generators of the zonotope is prescribed. Since a 2-dimensional zonotope is a centrally-symmetric polygon, this can also be considered a generalization to higher dimension of similar estimates that have been established for 2-dimensional convex chains. As we shall see, the exponential part of our estimate is composed of a polynomial of degree \(d\) in \(n^{1/(d+1)}\), and involves Riemann’s zeta function and depends on the Riemann hypothesis. Our analysis is based on a mapping between sums of coprime numbers and Eulerian polynomials. We also analyse some combinatorial properties of lattice zonotopes. In particular, we provide their average asymptotic diameter when \(n\) goes to infinity.

A zonotope is a convex geometric object defined as the Minkowski sum of \(k\) segments, called its generators. More particularly, a lattice zonotope \(Z\) (or integral zonotope) is a polytope for whom there exists \(k \in \mathbb{N}\) and \(v_1, ..., v_k \in \mathbb{Z}^d\) such that, up to translation, \(Z\) is:

\[
Z = \left\{ \sum_{i=1}^k \alpha_i v_i \mid \alpha_1, ..., \alpha_k \in [0, 1] \right\}.
\]

The idea of enumerating geometric objects of this kind is a long-standing question: in the late ‘70s, Arnold bounded the number of convex lattice polytopes with area \(A\) up to the affine transformations of \(\mathbb{Z}^2\) [1], showing that it behaves roughly like \(A^{1/3}\). In 1992, Bárány and Pach narrowed those bounds [4], before Bárány and Vershik extended the result to higher dimensions [5]. Vershik later simply asked for the number of lattice polytopes in a box [26]: e.g. in 2 dimensions, the number of convex lattice polygons inscribed in the square \([-n, n]^2\).

A natural intermediate stage is to enumerate the number \(p(n)\) of convex lattice chains from \((0, 0)\) to \((n, n)\). In 1994, an asymptotic equivalent of its logarithm is found independently by Bárány [2], Vershik [26], and Sinai [24]. They showed that:

\[
p(n) = \exp \left(3\kappa^{1/3}n^{2/3}(1 + o(1))\right), \quad \text{where } \kappa = \frac{\zeta(3)}{\zeta(2)}.
\]

More recently, Bodini et al. [7] and Bureaux and Enriquez [10] computed the asymptotic equivalent of the number of convex lattice chains. The former used analytic combinatorics tools that we are using in this paper, and starts from the study of the convex polyominoes. The latter started from Sinai’s model, and use the local limit theorem from Bogatchev and Zarbaliev [9]. They both got:

\[
p(n) \sim \frac{e^{-2\zeta'(-1)}}{(2\pi)^{7/6}\sqrt{3\kappa^{1/18}n^{17/18}}} \exp \left(3\kappa^{1/3}n^{2/3} + I \left(\left(\frac{\kappa}{n}\right)^{1/3}\right)\right),
\]
with \( I \) is a sum over the non-trivial zeros of \( \zeta \) and under Riemann hypothesis, \( I \left( \left( \frac{\kappa}{\pi} \right)^{1/3} \right) \) is of order \( O(n^{1/6}) \).

Eventually, computing the number of lattice polytopes still remains an open question as stepping up from convex lattice chains to convex polygons requires a thorough work over the convergence of the connections of the chains. More broadly zonotopes, as Minkowski sums of segments, are more of a combinatorial object than a geometric one. They remain important objects. They can be used when solving systems of polynomial equations [19] and for computing simplifications in spatial representation [21]. From a theoretical perspective, they appear in the theory of the simplex method [11], in the study of polytope diameters [14], and in relation with primitive Erhrart theory [13, 22]. Evaluating their number is one more step in our comprehension of those objects and their relation to convex polytopes.

Finally, we wish to stress that the question of enumerating polytopes in a convex body \( K \) is closely related to the limit shape of the \( \frac{1}{n} \mathbb{Z}^d \)-lattice polytopes in \( K \). Bárány, Bureaux, and Lund [3, Section 6] proved the existence of such a limit shape for zonotopes in \([0,1]^d\), while extending to higher dimensions the logarithmic asymptotic estimator of zonotopes in a cone, ending at a given point. They computed the logarithmic number of lattice zonotopes inside a cone \( C \) that ends at a prescribed point \( nk \) which gives, in the case of the zonotopes inscribed in a cube and ending at diagonally opposed points:

\[
\log \left( z_d'((\mathbb{R}^+), n1) \right) \sim (d + 1) n^{d+1} (d + 1)(d + \kappa_d) n^{1/d+1} .
\]

It is to be pointed up that our result takes into account every zonotope only considering their size, and considers the size of the hypercube they’re contained in rather than a cone or a set of prescribed vertices, as we place ourselves on the line of Vershik’s question. We hereby establish an asymptotic estimate in any given dimension, obtaining in the process interesting results such as a general mapping between sums of coprime numbers and Eulerian polynomials, estimation of the average and the variance of the number of generators and other similar results, and the asymptotic equivalent itself, which involves a certain polynomial in the exponential. We use generating functions, broadly spread in analytic combinatorics (see for instance Flajolet and Sedgewick [16]). This method can be seen as a combinatorial equivalent to Sinai’s model using Boltzmann-like probabilities and then applying local limit theorems (see [10, 24] and Bogatchev and Zarbaliev [9]). It is used for instance to study graphs (and in particular trees or graphs embedded in surfaces [6]) or in \( \lambda \)-calculus [8]. Here the result of [3] is refined, and the results of [10] generalized as follows:

**Theorem 1.** Let \( d \in \mathbb{N}, d \geq 2 \). Denote by \( z_d(n) \) the number of convex lattice zonotopes inscribed in \([0,n]^d\). As \( n \) grows large:

\[
z_d(n) \sim \alpha_d n^{\beta_d} \exp \left( Q_d(n^{1/d}) + J_{\text{crit}}(d,n) \right),
\]

where \( Q_d \) is a polynomial of degree \( d \). All the parameters, \( \alpha_d, \beta_d \in \mathbb{R}, Q_d \) and \( J_{\text{crit}}(d,n) \), are defined in (15) in Section 4.

**Remark 1.** Denote the generalization of the \( \kappa \) defined in (7) \( \kappa_d = 2^{d-1} \frac{\zeta(d+1)}{\zeta(d)} \). The leading term of the polynomial \( Q_d(n^{1/d}) \) is \((d + 1)(d + \kappa_d n^{1/d})\), which echoes the different results presented before.

This result naturally suggests two major remarks. Firstly, unexpectedly, the generalization of the exponential part is not a monomial as is in 2 and 3 dimensions (see below) but a polynomial of the \( d + 1 \)-th root of the size of the box.
Moreover, the use of the $\zeta$ function in the denominator forces us to keep the residues of the non-trivial zeros of $\zeta$, as was already observed in 2 dimensions. In the final section, we discuss the value of this term, and extend the results of [7, 10] that show that $J_{\text{crit}}$ is negligible in 2 dimensions for "computably" large $n$ (typically $< 10^{50}$) due to the decrease of $\Gamma$ function when it gets away from the real line.

To illustrate this theorem, we give the 2, 3, and 4 dimensional asymptotic number of zonotopes in the hypercube of side $n$:

$$z_2(n) = \frac{2^{1/9} 3^{13/18} \zeta(3)^{2/9} e^{-4\zeta'(1)}}{6 \pi^{16/9}} n^{-1/9} \exp \left( \frac{2^{2/3} 3^{4/3} \zeta(3)^{1/3}}{\pi^{2/3}} n^{2/3} + J_{\text{crit}}(2, n) \right)$$

$$z_3(n) = \frac{2^{5/8} 3^{3/4} 5^{8/9} e^{-4\zeta'(2)}}{120\zeta(3)^{1/8} \pi^{13/8}} n^{-13/8} \exp \left( \frac{2^{9/4} \pi}{3^{1/2} 5^{1/3} \zeta(3)^{1/4}} n^{3/4} + J_{\text{crit}}(3, n) \right)$$

$$z_4(n) = \left( \frac{2^{189/2} 3^{142} \zeta(5)^{7/4}}{5^{83} \pi^{379}} \right) \frac{1}{2^{27}} e^{-8\zeta'(3)} \zeta(3)^{-1} n^{-521/229} \times \exp \left( \frac{(5720)^{1/5} \zeta(5)^{1/5}}{\pi^{4/5}} n^{4/5} + \frac{(16720)^{2/5} \zeta(5)^{2/5} \zeta(3)^{1/5}}{\pi^{18/5}} n^{2/5} + J_{\text{crit}}(4, n) \right)$$

For which $J_{\text{crit}}(2, n)$ (resp. $J_{\text{crit}}(3, n)$ and $J_{\text{crit}}(4, n)$) are numerically approximated in the final Section (see [11]).

An other significant unanswered question is the largest possible diameter of the graph of a polytope of a given size, a question that lead, for instance, to the Hirsch conjecture. Along with its own scientific interest, its connection to the complexity of the simplex algorithm makes upper bounds of this quantity actively looked for (see [11] for more references). Zonotopes are widely studied in this perspective as it has been conjectured that the largest possible diameter of a polytope contained in $[0, n]^d$ is achieved by a zonotope [12] Conjecture 3.3. The largest possible diameter of a lattice zonotope contained $[0, n]^d$ is given in [13] for all $n$ and $d$, and its exact asymptotic behavior when $d$ is fixed and $n$ goes to infinity in [14]. The method used in this paper and the result established in Theorem 1 leads to estimate statistical parameters of lattice zonotopes such as the average number of generators in a zonotope, which is the diameter of their graph [12]. We hereby computed the asymptotic estimates of the mean and the variance of the distribution of the diameter of zonotopes in $[0, n]^d$.

**Theorem 2.** Let $\mu_{\text{diam}}^n$ the mean and $(\sigma^2)_{\text{diam}}^n$ the variance of the distribution of the diameter of a lattice zonotope inscribed in $[0, n]^d$. They are asymptotically equivalent to:

$$\mu_{\text{diam}}^n \underset{n \to +\infty}{\sim} \frac{d^{d/2} \sqrt{\zeta(d)}}{(d + 1)\pi^{d/2 + 1}} n^{d/2 + 1}, \quad (\sigma^2)_{\text{diam}}^n \underset{n \to +\infty}{\sim} \left( 1 - \frac{1}{2^d} \right) \mu_{\text{diam}}^n.$$

We also establish the similar result for the number of lattice points crossed by a prescribed generator (that we will see as the number of occurrences of a primitive generator, meaning non-zero coordinates are coprime).

This article is organized as follows. We begin with a section devoted to establishing the generating function associated to the combinatorial class of zonotopes using the symbolic method. The resulting infinite product is asymptotically estimated in the third section but previously, we spend some time on matching sums of coprime numbers with Eulerian polynomials which makes it possible to carry out the calculations in any dimension. Theorem 1 is established in fourth section using multi-dimensional saddle-point methods (using the extension to higher dimensions by Gittenberg and Mandlburer’s work [18]) over...
the dominant singularity of the generating function in order to obtain the asymptotic equivalent of its series coefficients. Finally the last two sections are respectively dedicated to discussing the obtained asymptotic equivalent and to computing the estimated average and variance of the number of non-collinear generators and the number of occurrences of a given generator.

1 Class of zonotopes and generating function

Let \( d \in \mathbb{N} \) be the dimension. As said before, a lattice zonotope is, up to translation, a Minkowski sum of segments, one end of whose is in the lattice \( \mathbb{Z}^d \), and the other is 0. Looking for inscribed zonotopes brings us to consider lattice zonotopes up to translation thereafter. Given a lattice zonotope \( Z \) and a set \( v_1, \ldots, v_k \in \mathbb{Z}^d \setminus \{0\} \) such that

\[
Z = \left\{ \sum_{i=1}^k \alpha_i v_i \mid \alpha_1, \ldots, \alpha_k \in [0, 1] \right\}.
\]

A set of generators of \( Z \) defines it uniquely but the inverse is not true. We denote \( \text{Zon} \) the set of lattice zonotopes up to translation and define the size of a lattice zonotope:

**Definition 1.** The function \( s \) defined over \( \text{Zon} \) as

\[
s(Z) = \sum_{i=0}^k ||v_i||_1
\]

is a size function, in particular, it is independent of the chosen set \( (v_1, \ldots, v_k) \) of generators.

The combinatorial specification that is done in this section is known as the symbolic method (see for instance [16, Chapter 1]). With the size function previously defined, \( \text{Zon} \) becomes a combinatorial class (with a finite number of object for a given the size). This class arises naturally from the set of generators given the definition of lattice zonotopes. Yet, without restrictions on this set, the generator set of one zonotope is not unique. This leads us to be a little more specific about that set in order to have one-to-one correspondence between a lattice zonotope and its set of generators.

Firstly, we restrain generators to the set of primitive vectors denoted \( \mathbb{P}^d \) (and respectively denote the set of positive primitive vectors \( \mathbb{P}^d_+ \)). A segment is said primitive if its intersection with the lattice \( \mathbb{Z}^d \) is exactly its extremities. In the case of vectors, it implies that the non-zero coordinates are coprime (in case of one nonzero coordinate, it is 1).

Secondly, we consider zonotopes up to translation. In order to do that, following the convention from [12], we restrict ourselves to the set of primitive vectors whose first non-zero coordinate is positive (denoted \( \mathbb{P}^*(\mathbb{Z}^d) \)). By this convention, two generators in this set are never collinear.

Now given a lattice zonotope \( Z \), the set \( \{v_1, \ldots, v_k\} \subset \mathbb{P}^*(\mathbb{Z}^d) \) that generates \( Z \) is unique. This leads to the following:

**Proposition 1** (Analytic combinatorics specification of the class of lattice zonotopes). Let \( \mathbb{P}^*(\mathbb{Z}^d) \subset \mathbb{Z}^d \) be the set of primitive vectors defined previously, the class of lattice convex zonotopes \( \text{Zon} \) is specified as follow:

\[
\text{Zon} = \text{MSet}(\mathbb{P}^*(\mathbb{Z}^d)).
\]

We now introduce the generating function of lattice zonotopes, the construction of which is very naturally based on the specification, matching the \( n^{\text{th}} \) coefficient of the series with the number of \( n \)-sized objects of the class. We use in the following proposition the variable \( \omega \) to encode the total size of lattice zonotopes, and the variable \( x_i \) the size along the \( i^{\text{th}} \) coordinate.
Proposition 2. The generating function of the class ZON is

\[ Zon(x, \omega) = \prod_{i=1}^{d} (1 - x_i \omega)^{-1} \prod_{\delta=2}^{d} \left( \prod_{1 \leq i_1 < ... < i_\delta \leq d} \left( \prod_{n_{i_1}, ..., n_{i_\delta} > 0} \left( 1 - x_{i_1}^{n_{i_1}} ... x_{i_\delta}^{n_{i_\delta}} \omega \sum_{n_{i_k}} \right)^{-1} \right) \right)^{2^{d-1}}. \]

Proof. A multiset over \( P^*(\mathbb{Z}^d) \) can be seen as the product of the sequences of each terms over \( P^*(\mathbb{Z}^d) \) (where each of these terms are defined in \([10]\)), e.g. the sequence of an object \( \alpha \) is the set \( \{ \emptyset, \alpha, (\alpha, \alpha), (\alpha, \alpha, \alpha), ... \} \).

Then \( Zon(x, \omega) \) can be written \( Zon(x, \omega) = \prod_{v \in P^*(\mathbb{Z}^d)} \left( 1 - \prod_{1 \leq i \leq d} x_i^{||v||_1} \omega^{||v||_1} \right)^{-1} \).

One can see that for one \( v \) with \( \delta \) non-zero coordinate, there are exactly \( 2^{d-1} \) vectors \( w \) in \( P^*(\mathbb{Z}^d) \) that verify \( ||v||_1 = ||w||_1 \) and for each \( i, |v_i| = |w_i| \). They can therefore be grouped together as they have the same contribution in the generating function.

In summary, \([x_1^{n_1}][x_2^{n_2}][x_d^{n_d}]Zon(x, 1)\) is equal to the number of lattice zonotopes inscribed in a box of size \( n_1 \times n_2 \times ... n_d \).

In the sequel, to lighten the notation and facilitate calculations, we will use the change of variable \( x_i = e^{-\theta_i} \). This choice can be seen as a standard calculus tool for the following derivations but in our case, it also shows the deep connection between the generating function and Boltzmann-like probabilities used in Sinai’s model \([10]\). Besides, as the variable \( \omega \) is redundant with \( (x_i)_{1 \leq i \leq d} \), we’ll fix \( \omega = 1 \) for the rest of the paper. Therefore, in the absence of further information in the beginning of each section, the generating function becomes

\[ Zon(e^{-\theta}, 1) = Zon(e^{-\theta}) = \prod_{v \in P^d_+} \left( 1 - e^{-<\theta, v>} \right)^{-2 \sum_{i \neq 0}^d 1_{v_i \neq 0} - 1}, \tag{3} \]

where \( \theta = (\theta_1, ..., \theta_d), e^{-\theta} = (e^{-\theta_1}, ..., e^{-\theta_d}), \) and "\(<, >" denotes the canonical inner product of \( \mathbb{R}^d \).

In order to obtain the asymptotic equivalent of the coefficients of this function for any \( d \) (obtained through a multidimensional saddle point method), we set in the following section general results for sums of type \( \sum_{p_1+p_2+...+p_r=n} p_1 \nu_1^1 p_2^2 ... p_r^r \). It will start with a brief reminder about Eulerian polynomials.

2 Study of sums of coprime numbers

Eulerian polynomials were introduced by Leonhard Euler \([15]\), and they have been studied and applied since then. They come from considering the number \( A(n, m) \) of permutations of \( 1, 2, ..., n \) such that the image of exactly \( m \) integer are greater than the one preceding them (for example for the identity \( m = n - 1 \)). The Eulerian polynomials are defined recursively as:

\[
\left\{
\begin{array}{l}
A_0(y) = y, \\
A_n(y) = \sum_{m=0}^{n} A(n, m) y^{n-m}.
\end{array}
\right.
\]

Hence the first Eulerian polynomials are \( A_1(y) = y \), \( A_2(y) = y^2 + y \), \( A_3(y) = y^3 + 4y^2 + y \).
Euler found these polynomials while studying the series form of \( \frac{ye^y}{1-y e^y} \) [17]. In particular, the formula that interests us is the following:

\[
\sum_{k=1}^{\infty} y^k k^n = \frac{A_n(y)}{(1 - y)^{n+1}}
\]

(4)

### 2.1 Eulerian summation of coprimes

Let be \( r \in \mathbb{N}, i \in \mathbb{N}^r \) and \( n \in \mathbb{N} \). We introduce the sum \( \sum_{p_1+p_2+...+p_r=n} p_1^{i_1} p_2^{i_2} ... p_r^{i_r} \) with the aim to asymptotically estimate it. We denote it \( S_n(i) \), and \( j = 1 \cdot i \). Note that \( S_n(i) \) is the sum of a power function over the primitive points contained in the surface of the dilates of the \((r-1)\)-dimensional standard simplex. It is noteworthy that the number of such points has been computed in [13] and that more generally a coprime Ehrhart theory is introduced in [22].

**Property 1.**

\[
\sum_{k \geq 0} \sum_{n \geq 1} k^j S_n(i) y^{nk} = \frac{\prod_{k=1}^{r} A_{i_k}(y)}{(1 - y)^{j+r}}.
\]

(5)

**Proof.** This relation relies on the partition \( \mathbb{Z}^r_+ = \bigcup_{k \geq 0} \{ kp, p \in \mathbb{P}_r \} \). We write

\[
\sum_{k \geq 0} k^j \sum_{n \geq 1} S_n(i) (y^n)^k = \sum_{k \geq 0} \sum_{n \geq 1} \left( \sum_{p_1+p_2+...+p_r=n \atop p_1 \wedge ... \wedge p_r=1} \prod_{r} (kp_1)^{i_1} y^{kp_1}...(kp_r)^{i_r} y^{kp_r} \right) \]

\[
= \sum_{l \geq 1} \left( \sum_{l_1+l_2+...+l_r=l} \prod_{p \geq 1} (kp_1)^{i_{p1}} y^{l_1} (l_2)^{i_2} y^{l_2} ...(l_r)^{i_r} y^{l_r} \right) = \prod_{p=1}^{r} \left( \sum_{l_p \geq 1} i_{p1} l_p \right) = \frac{\prod_{p=1}^{r} A_{i_p}(y)}{(1 - y)^{j+r}}.
\]

The \( n^{th} \) Eulerian polynomial is of degree \( n \) except for \( A_0(y) = y \) and has no constant term which causes the degree of \( \prod_{\lambda=1}^{r} A_{i_\lambda}(y) \) to be no more than \( j + r \). We write it as follows:

\[
\prod_{\lambda=1}^{r} A_{i_\lambda}(y) = B_t(y) = b_{i,j+r} y^{j+r} + ... b_{i,1} y.
\]

Despite this heavy notation, the sums under study in the zonotopal case are usually \( S_n(0) \), \( S_n(1,0,...,0) \) or \( S_n(2,0,...,0) \), making \( B_t(y) \) equal to \( y^d \), \( y^{d+1} \), or \( y^{d+2} + y^{d+1} \).

### 2.2 Mellin transform

Mellin transforms appear naturally in the asymptotic analysis of infinite products (see [7, 16, 18]). In the following, the analysis of zonotopes will require to apply a Mellin transform on sums of coprime numbers. The following proposition is of primary importance in the sequel:
Property 2. For \( t > 0 \), the Mellin transform of \( \sum_{n \geq 1} S_n(i) e^{-nt} \) is

\[
\frac{1}{\zeta(s-j)} \left( \sum_{n=0}^{j+r} \sum_{k \geq 1} b_{k,n} \left( \frac{k+j+r-n-1}{j+r-1} \right) \Gamma(s) \right).
\]

Proof. As we do throughout the paper, we use the change of variables \( y = e^{-t} \) and will always try to write products as sums, to have an expression compatible with the Mellin transform. This leads to:

\[
\sum_{k \geq 0} \sum_{n \geq 1} S_n(i) (e^{-nt})^k = \prod_{\lambda=1}^{r} A_{\lambda}(e^{-t}) \left( \sum_{k \geq 0} \left( \frac{k+j+r-1}{j+r-1} \right) e^{-kt} \right) = \sum_{n=0}^{j+r} b_{k,n} \sum_{k \geq 1} \left( \frac{k+j+r-n-1}{j+r-1} \right) e^{-kt}.
\]

We can now make use of the Mellin transform which is the transform that matches the exponential with the Gamma function:

\[
\forall s, \Re(s) > 0, \quad \Gamma(s) = \int_{0}^{+\infty} e^{-t} t^{s-1} dt.
\]

Under conditions that will be detailed just below, the Mellin transform of a function \( f \) the expression is defined as

\[
\mathcal{M} \left[ f \left( e^{-t} \right) \right] (s) = \int_{0}^{+\infty} f \left( e^{-t} \right) t^{s-1} dt.
\]

For \( \Re(s) > j + r - 1 \), the term \( \left| \sum_{n \geq 0} \left( \frac{k+j+r-n-1}{j+r-1} \right) \right| \) is bounded, which makes the use of Fubini’s theorem possible, so the Mellin function of \( \sum_{n \geq 1} S_n(i) e^{-nt} \) can be written, on the one hand:

\[
\int_{0}^{+\infty} \sum_{n=0}^{j+r} b_{k,n} \sum_{k \geq 1} \left( \frac{k+j+r-n-1}{j+r-1} \right) e^{-kt} t^{s-1} dt = \sum_{n=0}^{j+r} b_{k,n} \sum_{k \geq 1} \left( \frac{k+j+r-n-1}{j+r-1} \right) \Gamma(s),
\]

and on the other hand:

\[
\int_{0}^{+\infty} \sum_{k \geq 0} \sum_{n \geq 1} S_n(i) e^{-nt} t^{s-1} dt = \zeta(s-j) \int_{0}^{+\infty} \sum_{n \geq 1} S_n(i) e^{-nt} t^{s-1} dt.
\]

These holomorphic functions being defined and equal on \( \Re(s) > j + r - 1 \), we can extend it to its meromorphic continuation on \( \mathbb{C} \).

The Mellin transform is widely used to estimate asymptotic behaviors using its correspondence with the poles of the transformed function, as we will do in next subsection (and as it is done in \([7]\) and \([10]\)). Property 2 make it possible to compute the asymptotic equivalent of any of the sums \( \sum_{n \geq 1} S_n(i) y^n \) near \( y = 1 \).

2.3 Riemann’s non trivial zeros challenge, with the example \( \sum_{p<n \text{ prime}} p^2 \)

We use this subsection to detail more specifically the way the Mellin inversion formula is handled, as we will encounter it multiple times. Riemann’s \( \zeta \) function intervenes in it, and its non-trivial zeros too. We will use the integral of \([10]\) to manage all the zeros in the critical strip with an integral, but we could also write a sum over all non-trivial zeros. This term is numerically discussed in the Section 5.
Proposition 3. We denote the contour \( \gamma = \gamma_{\text{right}} \cup \gamma_{\text{left}} \), with \( \gamma_{\text{right}} = 3 - \frac{A}{\log(2+|t|)} \) and \( \gamma_{\text{left}} = 2 + \frac{A}{\log(2+|t|)} \) it as \( t \) runs from \(-\infty\) to \(+\infty\), with \( A > 0 \) such that \( \gamma \) surrounds all the zeros in the critical strip of \( \zeta(s-2) \).

Hereafter we will denote

\[
I_{\text{crit}} = \frac{1}{2i\pi} \int_{\gamma} \mathcal{M}\left[ \sum_{n \geq 1} S_n(2,0)e^{-nt} \right] t^{-s} ds
\]

and

\[
I_{\text{err}} = \frac{1}{2i\pi} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \mathcal{M}\left[ \sum_{n \geq 1} S_n(2,0)e^{-nt} \right] t^{-s} ds
\]

We have for all \( t > 0 \):

\[
\sum_{n \geq 1} S_n(2,0)e^{-nt} = \frac{2}{\zeta(2)t^4} + I_{\text{crit}}(t) + \frac{1}{3t^2} - \frac{1}{90\zeta'(-1)} \log(t) + I_{\text{err}}(t). \tag{8}
\]

When \( t \) goes to 0, \( I_{\text{err}}(t) = O(t^{1/2}) \), \( I_{\text{err}}'(t) = o(t^{-1}) \) and the \( k \)-th derivative of \( I_{\text{crit}} \) is \( o(t^{-k-1}) \).

Proof. We take the general set of sums \( \left( \sum_{p<n \atop p \wedge n = 1} p^i \right)_{i \geq 0} \).

As \( p \wedge n = 1 \) being equivalent to \( p \wedge (n-p) = 1 \), those sums can be rewritten as \( \sum_{p+q=n \atop p,q=1} p^i = S_n(i,0) \).

We fix \( i = 2 \). Applying properties \( \mathbb{I} \) and \( \mathbb{II} \) we have:

\[
\sum_{k \geq 0} k^2 \sum_{n \geq 1} S_n(2,0)e^{-kt} = \frac{A_0(e^{-t})A_2(e^{-t})}{(1-e^{-t})^4}
= (e^{-3t} + e^{-2t}) \sum_{k \geq 0} \binom{k+3}{3} e^{-kt}.
\]

For \( t > 0, s > 1 \), we have the following Mellin transform:

\[
\mathcal{M}\left[ \sum_{n \geq 1} S_n(2,0)e^{-nt} \right](s) = \frac{1}{\zeta(s-2)} \left( \sum_{k \geq 1} \frac{(k+1)_3}{k^s} \Gamma(s) \right)
= \frac{2\zeta(s-3) - 3\zeta(s-2) + \zeta(s-1)}{6\zeta(s-2)} \Gamma(s).
\]

Therefore we have, using the Mellin inversion formula, for all \( c > 4 \) (to ensure \( \sup_{s \in c+i\mathbb{R}} \zeta(s-3) < \infty \)):

\[
\sum_{n \geq 1} S_n(2,0)e^{-nt} = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{2\zeta(s-3) - 3\zeta(s-2) + \zeta(s-1)}{6\zeta(s-2)} \Gamma(s)t^{-s} ds.
\]

We use the residue theorem to shift the line of integration to the left as \( s = 4 \) is a location of a pole of the Mellin transform. It follows that, for all \( c_1 \in [3,4] \):

8
\[
\sum_{n \geq 1} S_n(2, 0) e^{-nt} = \frac{2}{\zeta(2)t^4} + \frac{1}{2it\pi} \int_{c_1-i\infty}^{c_1+i\infty} \frac{2\zeta(s-3) - 3\zeta(s-2) + \zeta(s-1)}{6\zeta(s-2)} \Gamma(s)t^{-s} ds.
\]

The remaining poles of the Mellin transform are the double pole at \( s = 0 \), and simple ones at \( s = 0 \) and at the non-trivial zeros of \( \zeta(s-2) \), which are in the critic strip \( 2 < \Re(z) < 3 \), and in the left side of the complex domain.

Now consider the contour \( \gamma \) as described in the proposition. The existence of \( A \), such that \( \gamma \) surrounds all the zeros of \( \zeta(s-2) \) in the critical strip is a consequence of Theorem 3.8 in [25].

\[
I_{\text{crit}}(t) = \frac{1}{2it\pi} \int_{\gamma} \frac{2\zeta(s-3) - 3\zeta(s-2) + \zeta(s-1)}{6\zeta(s-2)} \Gamma(s)t^{-s} ds.
\]

For any \( c_2 \in ]0, 2[ \), we have:

\[
\sum_{n \geq 1} S_n(2, 0) e^{-nt} = \frac{2}{\zeta(2)t^4} + I_{\text{crit}}(t) + \frac{1}{3t^2} + \frac{1}{2it\pi} \int_{c_2-i\infty}^{c_2+i\infty} \frac{2\zeta(s-3) - 3\zeta(s-2) + \zeta(s-1)}{6\zeta(s-2)} \Gamma(s)t^{-s} ds.
\]

Finally denoting \( I_{\text{err}}(t) = \frac{1}{2it\pi} \int_{\frac{1}{2}+i\infty}^{\frac{1}{2}+i\infty} \frac{2\zeta(s-3) - 3\zeta(s-2) + \zeta(s-1)}{6\zeta(s-2)} \Gamma(s)t^{-s} ds \), we have for \( t > 0 \)

\[
\sum_{n \geq 1} S_n(2, 0) e^{-nt} = \frac{2}{\zeta(2)t^4} + I_{\text{crit}}(t) + \frac{1}{3t^2} - \frac{1}{90\zeta'(-1)} \log(t) + I_{\text{err}}(t).
\]

The bounds over the order of \( I_{\text{crit}} \) and \( I_{\text{err}} \) are coming from Lemma 2.2 from [10].

3 Analysis of the generating function

In this section, we will compute the asymptotic equivalent of the generating function when all variables \( x_i \) equally tend to 1 which correspond to the main singularity of the generating function. This will lead us in the next section to complete the saddle point method using the equivalent found here. To simplify the notation, we denote, throughout the section:

\[
Zon(x_1 = x, ..., x_d = x) = Zon(x) = \prod_{\delta=1}^{d} \left( \prod_{v \in \mathbb{P}_+^d} (1 - x ||v||_1)^{-\binom{d-1}{\delta}} \right).
\]

Again, in order to establish a general result regarding the dimension \( d \), we need to introduce a sequence of polynomials that will be directly related to the polynomial of Theorem 1. Consequently, we present them along with the notation used in the subsequent paragraph.

3.1 Intermediate sequence of polynomials

As we make use of the Mellin inversion formula over the logarithm of the generating function, for each \( \delta \) between 1 and \( d \), the coefficient \( \binom{k-1}{\delta-1} \) from Property 2 will appear with \( S_n(0, ..., 0) \). In the rest of the paper (as well as in Theorem 1), we denote:
\[
\sum_{\delta=1}^{d} \binom{d}{\delta} 2^{\delta-1} \binom{k-1}{\delta-1} = P_d(k) = p_{d,d-1}k^{d-1} + \ldots + p_{d,1}k + p_{d,0}.
\]

The first terms of \((P_d(X))_{d \geq 2}\) are:

\[
P_1(X) = 1 \\
P_2(X) = 2X \\
P_3(X) = 2X^2 + 1
\]

These polynomials verify the relation \(P_{d+2}(X) = \frac{2X}{d+1}P_{d+1}(X) + P_d(X)\) for \(d \geq 1\).

Two properties naturally follow:

- The odd polynomials only contain even powers of \(i\) and have a constant term equal to 1. Similarly, the even polynomials only contain odd powers of \(i\).

- The leading term of \(P_d\) is of degree \(d - 1\) and its coefficient is \(\frac{2^{d-1}}{(d-1)!}\).

As each exponent \(\delta\) of \(k\) is reflected in the Riemann \(\zeta\) function as \(\zeta(s - \delta)\) when using Mellin transform, we introduce an operator over complex functions:

**Definition 2.** Let \(M(\mathbb{C})\) be the field of meromorphic functions in \(\mathbb{C}\). We define the operator \(\Pi_d\) as

\[
\Pi_d : M(\mathbb{C}) \rightarrow M(\mathbb{C}) \quad \phi \mapsto p_{d,d-1}\phi(\cdot - (d-1)) + \ldots + p_{d,1}\phi(\cdot - 1) + p_{d,0}\phi.
\]

It is important to note that we can write, for \(s \neq 1\), \(\Pi_d(\zeta)(s) = \sum_{k \geq 1} \frac{P_d(k)}{k^s}\).

### 3.2 Asymptotic analysis of the generating function

**Lemma 1.** Let \(A\) be a positive number such that \(\gamma = \gamma_{\text{right}} \cup \gamma_{\text{left}}\), with \(\gamma_{\text{right}} = 1 - \frac{4}{\log(2+|\theta|)} + it\) and \(\gamma_{\text{left}} = \frac{A}{\log(2+|\theta|)} + it\) as \(\theta\) goes from \(-\infty\) to \(+\infty\), surrounds all the zeros of \(\zeta\) in the critical strip.

For \(\theta > 0\), as \(\theta \rightarrow 0\), we have:

\[
Z_{\text{on}}(e^{-\theta}) = \theta^{2\Pi_d(\zeta)(0)} e^{2\Pi_d(\log(2\pi)\zeta-\zeta')(0)} \exp \left( \sum_{\delta=1}^{d-1} \left( p_{d,\delta} \frac{\zeta(\delta + 2)\Gamma(\delta + 1)}{\zeta(\delta + 1)\theta^{\delta+1}} \right) + I_{\text{crit}}(d, \theta) + I_{\text{err}}(d, \theta) \right),
\]

with

\[
I_{\text{crit}}(d, \theta) = \frac{1}{2i\pi} \int_{\gamma} \frac{\Pi_d(\zeta)(0)}{\zeta(s)} \zeta(s + 1)\Gamma(s)\theta^{-s}ds
\]

and

\[
I_{\text{err}}(d, \theta) = \frac{1}{2i\pi} \int_{-\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \frac{\Pi_d(\zeta)(0)}{\zeta(s)} \zeta(s + 1)\Gamma(s)\theta^{-s}ds = O(\theta^{1/2})
\]
Proof. We apply the logarithm to the generating function with the change of variables $x = e^{-\theta}$, which gives:

$$
\log \left( Z_{\text{on}} (e^{-\theta}) \right) = -\sum_{\delta=1}^{d} \left( \frac{d}{\delta} \right) 2^{d-1} \sum_{\nu \in \mathbb{Z}_{+}^d} \log \left( 1 - e^{-\nu \theta} \right)
$$

$$
= -d \log(1 - e^{-\theta}) - \sum_{\delta=2}^{d} \left( \frac{d}{\delta} \right) 2^{d-1} \left( \sum_{n=2}^{+\infty} \left( \sum_{n_1 + \ldots + n_\delta = n \atop n_1, \ldots, n_\delta \in \mathbb{Z}^+} 1 \right) \log \left( 1 - e^{-n \theta} \right) \right)
$$

The sum $\sum_{n_1 + \ldots + n_\delta = n} 1$ is of the form of the sums studied in section 2 as being $S_n(0, \ldots, 0)$. In a geometrical point of view, it is exactly the number of primitive points. Therefore we compute the Mellin transform accordingly (using logarithmic series), for all $s > d$

$$
\mathcal{M} \left[ \log \left( Z_{\text{on}} (e^{-\theta}) \right) \right] (s) = \sum_{\delta=1}^{d} \left( \frac{d}{\delta} \right) 2^{d-1} \left( \sum_{k=1}^{\infty} \left( \frac{k-1}{\delta-1} \right) \zeta(s+1) \Gamma(s) \right)
$$

With the notation introduced in definition 2, the transform simplifies to the following form:

**Proposition 4.** For every $d > 1$, we have:

$$
\mathcal{M} \left[ \log \left( Z_{\text{on}} (e^{-\theta}) \right) \right] (s) = \frac{\Pi_d \left( \zeta(s) \right)}{\zeta(s)} \zeta(s+1) \Gamma(s).
$$

(11)

We can extend this function into a meromorphic one on $\mathbb{C}$, and use successive residue theorems to scan from the right to the left each pole of the Mellin transform, as we did in the previous section. We start with $c > d$ and

$$
\log(Z_{\text{on}}(e^{-\theta})) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} p_{d,d-1} \zeta(s-d+1) + \ldots + p_{d,1} \zeta(s-1) + p_{d,0} \zeta(s) \frac{\zeta(s+1) \Gamma(s) \theta^{-s} ds}{\zeta(s)}
$$

which leads to

$$
\log(Z_{\text{on}}(e^{-\theta})) = \sum_{\delta=1}^{d-1} \left( p_{d,\delta} \frac{\zeta(\delta + 2) \Gamma(\delta + 1)}{\zeta(\delta + 1) \theta^{\delta+1}} \right) + \frac{1}{2i\pi} \int_{\gamma} \Pi_d \left( \zeta(s) \right) \frac{\zeta(s+1) \Gamma(s) \theta^{-s} ds}{\zeta(s)} + 2\Pi_d \left( \log(2\pi) \zeta - \zeta' \right) (0)
$$

$$
+ 2\Pi_d \left( \zeta \right) (0) \log(\theta) + \frac{1}{2i\pi} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Pi_d \left( \zeta(s) \right) \frac{\zeta(s+1) \Gamma(s) \theta^{-s} ds}{\zeta(s)}
$$

We denote $I_{\text{crit}}(d, \theta)$ and $I_{\text{err}}(d, \theta)$ respectively the first and the second integral above.
Corollary 1. The asymptotic equivalent of $Zon(e^{-\theta})$ is:

$$Zon\left(e^{-\theta}\right) \sim \theta^{2\Pi_d(\zeta)(0)} e^{2\Pi_d(\log(2\pi)\zeta-\zeta')(0)} \exp\left(\sum_{\delta=1}^{d-1} \left(\frac{p_d \zeta(\delta + 2)\Gamma(\delta + 1)}{\zeta(\delta + 1)\theta^{\delta+1}} + I_{crit}(d, \theta)\right)\right), \text{ as } \theta \to 0.$$ 

4 Proof of the Theorem

The proof of our result relies on the saddle point method in $d$ dimensions. Saddle point analysis consists in fetching the coefficient of the series by using Cauchy's integral formula:

$$\text{given } \mathcal{C} \subset \mathbb{C} \text{ a circle centered at } 0, \forall i \in \mathbb{N}, 1 \leq i \leq d, \quad [x^n_i]Zon(x) = \frac{1}{2i\pi} \int_{\mathcal{C}} Zon(x)x^{-n-1}dx.$$ 

At this point, three steps remain. We first resolve the saddle-point equation, which determines the coordinate $\tilde{\theta}_n$ of the saddle point (the point where the derivative of the integrand is null), and then prove the H-admissibility of the generating function, i.e. that the Cauchy integral can be approximated by a Gaussian integral. Eventually, we simply compute the Gaussian approximation, using the asymptotic equivalent of the generating function previously calculated.

4.1 The saddle-point equation

Lemma 2. Denoting $\tilde{\theta}_n = \left(\frac{2d-1\zeta(d+1)}{\zeta(d)}\right)^{-1}$, we have:

$$\frac{\partial \log \left( Zon\left(e^{-\theta}\right) \right)}{\partial \theta_i} \bigg|_{\theta=\tilde{\theta}_n, i=1} = -n + o(n)$$

$\tilde{\theta}_n$ is called the asymptotic solution of the saddle-point equation.

Proof. We fetch the coefficient of the multivariate series of the generating function by using Cauchy’s integral formula and therefore we look for parameters $\bar{\theta}_n = (\bar{\theta}_{n,1}, \ldots, \bar{\theta}_{n,d})$ such that for all $1 \leq i \leq d$,

$$\frac{\partial}{\partial \theta_i} \left( \log \left( Zon\left(e^{-\theta}\right) \right) + (n + 1)\theta_i \right) \bigg|_{\theta=\bar{\theta}_n} = 0. \quad \text{(12)}$$

The symmetry of the variables in $Zon(\theta)$ implies $\theta_{n,i} = \theta_n$ for $1 \leq i \leq d$. The system (12) is then restricted to $\theta_1$ only:

$$\frac{\partial}{\partial \theta_1} \log \left( Zon\left(e^{-\theta}\right) \right) \bigg|_{\theta=\bar{\theta}_n} = -\sum_{\delta=1}^{d} 2^{\delta-1} \binom{d-1}{\delta-1} \sum_{v \in \mathbb{P}_+^\delta} v_1 \frac{e^{-<\bar{\theta}_n,v>}}{1-e^{-<\bar{\theta}_n,v>}}$$

$$= -\sum_{\delta=1}^{d} 2^{\delta-1} \binom{d-1}{\delta-1} \sum_{m=\delta}^{\infty} \sum_{v \in \mathbb{P}_+^\delta} v_1 \frac{e^{-m\theta_n}}{1-e^{-m\theta_n}} \quad \text{under } S_m(1,0_{d-1})$$

12
Using the analysis of the sums $S_m$ in Section 2, we have the following Mellin transform, for $\Re(s) > d + 1$:

$$
\mathcal{M} \left[ \frac{\partial}{\partial \theta_1} \log \left( \text{Zon} \left( e^{-\theta_1} \right) \right) \right]_{\theta = \theta_n} (s) = - \sum_{\delta = 1}^{d} 2^{\delta - 1} \left( d - 1 \right) \left( \sum_{k \geq d} \frac{(k)}{k^s} \right) \frac{\zeta(s) \Gamma(s)}{\zeta(s - 1)}.
$$

This function is a meromorphic function on $\mathbb{C}$, so we are taking $c \in \mathbb{C}$ with $\Re(c) > d + 1$ to use the Mellin inversion formula

$$
\frac{\partial}{\partial \theta_1} \log \left( \text{Zon} \left( e^{-\theta_1} \right) \right) \bigg|_{\theta = \theta_n} = - \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} 2^{\delta - 1} \left( d - 1 \right) \left( \sum_{k \geq d} \frac{(k)}{k^s} \right) \frac{\zeta(s) \Gamma(s)}{\zeta(s - 1)} \theta_n^{-s} ds.
$$

With the residue theorem, we can shift the line of the integral to the left of $\Re(s) = d + 1$. The resulting integral is $o \left( \frac{1}{\theta_n^d} \right)$, which leads to:

$$
\frac{\partial}{\partial \theta_1} \log \left( \text{Zon} \left( e^{-\theta_1} \right) \right) \bigg|_{\theta = \theta_n} = - \frac{2^{d - 1} \zeta(d + 1)}{\zeta(d) \theta_n^{d + 1}} + o \left( \frac{1}{\theta_n^{d + 1}} \right).
$$

To generalize the notation of [10], we denote $\kappa_d = \frac{2^{d - 1} \zeta(d + 1)}{\zeta(d)}$. Finally get the parameter which is solution of the saddle equation:

$$
\theta_n = \left( \frac{\kappa_d}{n} \right)^{\frac{1}{d + 1}} \quad (13)
$$

### 4.2 H-admissibility

The saddle point method is suitable when the integrand is subject to a abrupt variation, with a major negligible part, and a small one that can be approximated by a Gaussian integral. It can be typically the case if the path along which a function is integrated passes close to a pole. In the 50’s, Hayman defined a class of analytic complex functions $\sum_n a_n x^n$, called H-admissible functions, for which one can use this method to estimate the coefficients $a_n$ [20]. Since then, many extensions of this concept have been made. As we have here a $d$ dimensional space, we will use the work of Gittenberger and Mandlburger [18] that extends the H-admissible class for functions on $d$-dimensional spaces.

**Theorem 3.** The function $\text{Zon} : \mathbb{C}^d \rightarrow \mathbb{C}$ is H-admissible on $\mathcal{R} = \{ x \in \mathbb{R}^d, \forall i \ |x_i| < 1 \}$ in the sense of Gittenberger et Mandlburger [18].

This theorem relies on five conditions that we will prove in what follows. In order to prove them, we use the following notation:

Recall that $\overline{\theta_n} = \left( \frac{2^{d - 1} \zeta(d + 1)}{\zeta(d)n} \right)^{\frac{1}{d + 1}}$ is the solution of the saddle equation and denote $x_n = \exp(-\overline{\theta_n})$ ($x_n \rightarrow 1$ as $n \rightarrow \infty$). To be H-admissible, a function needs to be approximately its Gaussian distribution (i.e. its logarithm is approximately its own Taylor expand) on a small path.

This path is defined as $\Delta(\overline{\theta_n}) := \left\{ \theta = (\theta_i)_{1 \leq i \leq d}, \forall i \ |\theta_i| < \overline{\theta_n}^\alpha \right\}$ for $\alpha \in [1 + \frac{d}{3}, 1 + \frac{d}{2}]$.

Finally, $a(x) = (a_i(x))_{1 \leq i \leq d}$, $B(x) = (B_{i,j}(x))_{1 \leq i,j \leq d}$, and $C(x) = (C_{i,j,k}(x))_{1 \leq i,j,k \leq d}$ respectively stand for the sets of the first, second, and third order partial derivative of the logarithm of $\text{Zon}(x)$. 

13
Lemma 3. $\text{Zon}(x_n e^{i\theta}) \sim \text{Zon}(x_n) \exp \left( i \theta \alpha'(x_n) - \frac{\theta B(x_n) \theta'}{2} \right)$ uniformly as $\theta, \theta' \in \Delta(\tilde{\theta}_n)$.

Proof. This is a consequence of the Lagrange form of Taylor’s expansion theorem. We compute the asymptotic equivalent of the second and third partial derivatives of the generating function’s logarithm (using Mellin transform analysis as in the previous sections):

\begin{align}
B_{i,j}(x_n) &\sim (1 + \delta_{i,j}) \frac{2^{d-1} \zeta(d+1)}{\zeta(d)} \frac{1}{\theta_n^{d+2}}, \\
C_{i,j,k}(x_n) &\sim (1 + \delta_{i,j} + \delta_{j,k} + \delta_{k,i} + 2\delta_{i,j,k}) \frac{2^{d-1} \zeta(d+1)}{\zeta(d)} \frac{1}{\theta_n^{d+3}},
\end{align}

where $\delta_{ij}$ is Kronecker’s function (and $\delta_{i,j,k} = 1$ if $i = j = k$ and $0$ otherwise).

To conclude, we need $\alpha$ to ensure

$$B_{i,j}(x_n)(\tilde{\theta}_n^2) \to \infty \quad \text{and} \quad C_{i,j,k}(x_n)(\tilde{\theta}_n^3) \to 0$$

which results in $\frac{d+3}{3} < \alpha < \frac{d+2}{2}$. The third order Taylor polynomial with the Lagrange form of the remainder at $\theta = \tilde{\theta}_n$ ends the proof. \hfill $\square$

Lemma 4. $|\text{Zon}(x_n e^{i\theta})| = o \left( \frac{\text{Zon}(x_n)}{\sqrt{|B(x_n)|}} \right)$ as $n \to \infty$, holds uniformly for $\theta \notin \Delta(\tilde{\theta}_n)$.

Proof. As we have $\tilde{\theta}_n < \tilde{\theta}_n^\alpha$, proving the result for $\tilde{\theta}_n \leq \theta_i \leq \pi$, for all $i$ will ensure it for $\tilde{\theta}_n^\alpha \leq \theta_i \leq \pi$.

$$\log \left( \frac{|\text{Zon}(x_n e^{i\theta})|}{\text{Zon}(x_n)} \right) = \log \left( \prod_{\delta=1}^{d} \prod_{1 \leq i_1 < \cdots < i_d \leq d} \left( \prod_{n=(n_1, \ldots, n_d) \atop n_i \geq 0} \frac{1-x^{n_1}}{|1-x_n^{n_1} e^{i n \theta}|} \right)^{2^{d-1}} \right).$$

Using $\frac{1-x}{1-x e^{i\theta}} = \sqrt{\frac{(1-x)^2}{(1-x)^2 + 4x \sin^2(\theta/2)}}$ for $x \in [0, 1[$, we have

$$\log \left( \frac{|\text{Zon}(x_n e^{i\theta})|}{\text{Zon}(x_n)} \right) = -\frac{1}{2} \log \left( \prod_{\delta=1}^{d} \prod_{1 \leq i_1 < \cdots < i_d \leq d} \left( \prod_{n=(n_1, \ldots, n_d) \atop n_i \geq 0} \frac{1+4x_n^{n_1} \sin^2 \left( \frac{n \theta}{2} \right)}{(1-x_n^{n_1})^2} \right)^{2^{d-1}} \right) \leq -\frac{1}{2} \left( \sum_{\delta=1}^{d} 2^{\delta-1} \sum_{1 \leq i_1 < \cdots < i_d \leq d} \log \left( 1+4x_n^{n_1} \sin^2 \left( \frac{n \cdot \theta}{2} \right) \right) \right) \leq -\frac{\log(5)}{2} \left( \sum_{\delta=1}^{d} 2^{\delta-1} \sum_{1 \leq i_1 < \cdots < i_d \leq d} \sum_{n=(n_1, \ldots, n_d) \atop n_i \geq 0} x_n^{n_1} \sin^2 \left( \frac{n \cdot \theta}{2} \right) \right) = -\frac{\log(5)}{2} U_n,$$
In the last inequality, we use the fact that \( \log(5) x \leq \log(1 + x) \) as \( 0 \leq x \leq 4 \). We simply need to find a lower bound for \( U_N \). A polynomial bound is enough. Rewriting \( u_i = \frac{\theta_i}{\pi} \), we have \( u_i \in ]\frac{\pi}{2}, \frac{\pi}{2} [ \). We can consider the family of primitive vectors \( (v_i = (1, i, 0))_i \geq 1 \) :

\[
\sin^2(\pi v_i \cdot u) = \sin^2((u_1 + iu_2)\pi)
\]

We therefore look at the solutions of \( \sin^2((u_1 + xu_2)\pi) \geq \frac{1}{4} \). Let \( T_u \) be \( \frac{1}{u_2} \). The solutions are \( T_u \)-periodic sets of length \( \frac{2}{3} T_u \). As \( \frac{1}{u_2} > 2 \), the \( k \)-th \( v_i \) satisfying the lower bound under the sinus satisfies \( i \leq 2k \). Finally, as \( x_n = 1 - \tilde{\theta}_n + o(\tilde{\theta}_n) \),

\[
U_n \geq 2 \sum_{i=1}^{+\infty} x_n^{1+i} \sin^2((u_1 + iu_2)\pi)
\]

\[
\geq \frac{1}{2} \sum_{i=1}^{+\infty} x_n^{1+2i}
\]

\[
\geq \frac{1}{4\theta_n} (1 + o(1)) \sim \frac{1}{4} \left( \frac{\zeta(d)n}{2^{d-1}d\zeta(d+1)} \right)^{\frac{1}{d+1}},
\]

which gives

\[
|\text{Zon}(x_ne^{i\theta})| = O \left( \text{Zon}(x_n)e^{-c_1/d+1} \right).
\]

We denote \( |\mathbf{B}(x_n)| \) the determinant of \( \mathbf{B}(x_n) \). Using expression (14), we get

\[
|\mathbf{B}(x_n)| \sim (d + 1) \left( \frac{2^{d-1}\zeta(d+1)}{\zeta(d)} \right) \frac{1}{\theta_n^{d+2}} d,
\]

which finally gives the following result, ending the proof:

\[
\frac{1}{\sqrt{|\mathbf{B}(x_n)|}} = O \left( n^{-\frac{d(d+2)}{2(d+1)}} \right).
\]

\( \square \)

**Lemma 5.** The following properties hold:

- Eigenvalues of \( \mathbf{B}(x_n) \) all tend to \( +\infty \) as \( n \) grows large.
- For sufficiently large \( x_n \), and \( \theta \in [-\pi, \pi]^d \setminus 0 \), one has \( |\text{Zon}(x_ne^{i\theta_1},...x_ne^{i\theta_d})| < \text{Zon}(x_n) \).
- \( B_{ii}(x_n) = o(a_i(x_n)^2) \) as \( n \) grows large.

**Proof.** The first and the third points directly follow the asymptotic equivalent we have in (14). As for the second point, it is an outcome of Lemma 4. \( \square \)

### 4.3 Asymptotic number of lattice zonotopes

As \( \text{Zon} \) is \( H \)-admissible, and given the solution of the saddle-point equation \( \tilde{\theta}_n \), we can write:

\[
z_d(n) := \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} \text{Zon} \left( e^{-\tilde{\theta}_n+i\theta} \right) e^{(n+1)(d\tilde{\theta}_n+i<1,\theta>)} d\theta \sim \frac{\text{Zon} \left( e^{-\tilde{\theta}_n} \right)}{\sqrt{(2\pi)^d |\mathbf{B}(e^{-\tilde{\theta}_n})|}},
\]

\]
with the notation from the H-admissibility demonstration (B is the matrix of the second partial derivatives of the generating function’s logarithm). We recall that:

\[ |B(x_n)| \sim (d + 1) \left( \frac{\kappa_d}{\theta_{n+2}} \right)^d. \]

Using \( \kappa_d = \frac{2^{d-1} \zeta(d+1)}{\zeta(d)} \), the contour \( \gamma = \gamma_{\text{right}} \cup \gamma_{\text{left}} \) with \( \gamma_{\text{right}} = 1 - \frac{A}{\log(2 + |t|)} + it \) and \( \gamma_{\text{left}} = \frac{A}{\log(2 + |t|)} + it \), \( A > 0 \) such that \( \gamma \) surrounds the critical strip of the \( \zeta \) function, and the operator \( \Pi_d \) defined in Definition 2, we finally have:

\[ z_d(n) \sim \alpha_d n^{\beta_d} \exp \left( Q_d(n^{\frac{1}{d+1}}) + I_{\text{crit}}(d, t) \right), \quad (15) \]

with \( \alpha_d = \frac{\kappa_d^{d+1}}{(2\pi)^{d/2} \sqrt{d+1}} \),

\[ \beta_d = \frac{-1}{d+1} \left( d(d+2)/2 + 2\Pi_d(\zeta)(0) \right), \]

\[ Q_d(X) = (d+1)\kappa_d^{\frac{1}{d+1}} X^d + \sum_{\delta=2}^{d-1} p_{d,\delta-1} \frac{\zeta(\delta+1)(\delta-1)!}{\zeta(\delta)} \kappa_d^{\frac{-\delta}{d+1}} X^\delta, \]

\[ J_{\text{crit}}(d, n) = \frac{1}{2i\pi} \int_{\gamma} \frac{\Pi_d(\zeta)(s)}{\zeta(s)} \zeta(s+1)\Gamma(s) t^{-s} ds \bigg|_{t=(\frac{n}{\kappa_d})^{\frac{1}{d+1}}}. \]

Note that \( J_{\text{crit}}(d, n) \) is just the evaluation of \( I_{\text{crit}}(d, \theta) \) at the saddle point value \( \theta = (\frac{n}{\kappa_d})^{\frac{1}{d+1}} \).

5 Numerical elements about the speed of convergence to the asymptotic regime

In this section, we make two numerical remarks about our result, in order to ease the understanding of the behaviour of our result when \( d \) and \( n \) grow large.

5.1 Order of size of \( J_{\text{crit}} \)

Throughout this paper, we left \( J_{\text{crit}} \) under its integral form but we could also write it as an infinite sum of residues as we did in the introduction, with \( r \) ranging over the non-trivial zeros of \( \zeta \):

\[ \sum_r \text{Res} \left( \frac{1}{\zeta(r)} \right) \frac{\Pi_d(\zeta)(r)}{\zeta(r+1)\Gamma(r)} \left( \frac{\kappa_d}{n} \right)^{\frac{r}{d+1}}. \]

This sum is actually invisible for "computably large" numbers \( n \). The density of poles \( r \) is known to be strongly bounded since Selberg in the beginning of the 1940’s [23]. Due to this and to the exponential decrease of \( \Gamma \) function, the value of \( J_{\text{crit}} \) is with a good approximation the first term of the sum. It is composed of the term at the first non-trivial zero, approximately at \( r = \frac{1}{2} + 14.13472i \) and of the term at its conjugate, which gives for respectively \( J_{\text{crit}}(2, n) \), \( J_{\text{crit}}(3, n) \), and \( J_{\text{crit}}(4, n) \) the following approximations:
\[ J_{\text{crit}}(2, n) \approx -1.3579 \times 10^{-10} n^{1/6} \cos (4.7116 \ln(0.6842n)) - 1.4236 \times 10^{-9} n^{1/6} \sin (4.7116 \ln(0.6842n)), \]
\[ J_{\text{crit}}(3, n) \approx -1.2325 \times 10^{-10} n^{1/8} \cos (3.5337 \ln(0.2777n)) - 1.2921 \times 10^{-9} n^{1/8} \sin (3.5337 \ln(0.2777n)), \quad (16) \]
\[ J_{\text{crit}}(4, n) \approx -3.1764 \times 10^{-9} n^{1/10} \cos (2.8269 \ln(0.1305n)) - 8.0628 \times 10^{-9} n^{1/10} \sin (2.8269 \ln(0.1305n)). \]

We can see that the size of these numbers mainly relies on \( \Gamma(r) \). The second non-trivial zero of \( \zeta \) is approximately in \( r = \frac{1}{2} + 21,02203i \), which is smaller by \( 10^{-4} \) for the first dimensions.

5.2 About moving up in dimension

For a more down-to-earth analysis of the logarithmic equivalent, i.e. the leading term of the exponential term, we can use the expansion of \( \zeta \).

For a more down-to-earth analysis of the logarithmic equivalent, i.e. the leading term of the exponential term, we can use the expansion of \( \zeta(d) = 1 + 2^d + o(2^d) \) as \( d \) grows large.

One can see that:

\[
\left( \frac{2^{d-1} \zeta(d+1)}{\zeta(d)} \right)^{\frac{1}{d+1}} = 2 + O \left( \frac{1}{d} \right), \quad \text{as } d \to +\infty.
\]

Therefore, when we get that, in higher dimension, the logarithm of \( z(n) \) is nearly equivalent to \( 2(d+1)n^d \pi^{d+1} \).

6 The average diameter of Zonotopes

The method we have just applied to the asymptotic number of zonotopes is very generic and can be easily adapted to obtain information on certain parameters (called additives in the symbolic method).

In this section, we establish the asymptotic behavior of some of these parameters that can be computed with our generating function approach. To do that, we add a variable \( u \) that acts as a counting variable for the parameter. Then the partial derivative along this variable gives us the average value of the quantity under study.

We begin by giving a result similar to Lemma 4.3 of [3] about the number of generators. It gave the average number of generators of a zonotope contained in a given cone and ending at a given point. In the case when the zonotope is contained in a hypercube, this average can also be computed, along with the variance of the number of generators. As said before, the number of generators of a zonotope is the diameter of its graph, which gives Theorem 2.

**Proof of Theorem 2** In the generating function, each term \( \frac{1}{1-e^{-\theta, v}} \) represents as a series the possibility of having \( k \) times the generator \( v \) (this is the generating function of the sequence set in the symbolic method [16]). Therefore the variable \( u \) that encodes the number of generator in a zonotope turns each term into:

\[
\sum_{k=0}^{+\infty} e^{-k<\theta, v>} \rightarrow 1 + \sum_{k=1}^{+\infty} ue^{-k<\theta, v>}. \]

We call \( Zon_{\text{gen}} \left( e^{-\theta}, u \right) \) the generating function defined as such, with the notation \( \theta = (\theta_1, ..., \theta_d) \) and \( x = e^{-\theta} = (e^{-\theta_1}, ..., e^{-\theta_d}) \). The main idea of the rest of the proof is to grasp that the average number of generator for lattice zonotope inscribed in \([0, n]^d\) is given by (see for instance [16] Chapter 3):

\[
\mu^n_{\text{gen}} = \left. \frac{[x^n_1]...[x^n_d]}{[x^n_1]...[x^n_d]} \partial_u Zon_{\text{gen}}(x, u) \right|_{u=1}. \]
We therefore compute the partial derivative along $u$:

$$\left. \frac{\partial Z_{on\, gen}(e^{-\theta}, u)}{\partial u} \right|_{u=1} = \left( \sum_{v \in P(\mathbb{Z}_+^d)} \left( 2\sum_i 1_{v_i \neq 0} - 1 \right) e^{-\langle \theta, v \rangle} \right) Z_{on}(e^{-\theta}).$$

To get the series coefficient, we use the saddle-point method as was done in previous sections. Then we get the asymptotic equivalent of the uni-variate series through the Mellin transform (using $S_n(0, ..., 0)$ and then $P_d$ from previous sections):

$$\sum_{v \in P(\mathbb{Z}_+^d)} 2\sum_i 1_{v_i \neq 0} - 1 e^{-\langle t, v \rangle} \sim \frac{2^{d-1}}{\zeta(d)t^d}.$$  

As $\left. \frac{\partial Z_{on\, gen}(e^{-\theta}, u)}{\partial u} \right|_{u=1}$ is exponentially equivalent to $Z_{on}$, it is trivially H-admissible, and the solution of the saddle equation is the same $\theta_n$. In short, the saddle point method gives for the numerator the same result as the denominator with multiplying term which is the equivalent freshly calculated. We have:

$$\frac{[x_1^n]...[x_d^n]}{[x_1^n]...[x_d^n]} \left. \frac{\partial Z_{on\, gen}(x, u)}{\partial u} \right|_{u=1} \sim \frac{2^{d-1}}{\zeta(d)\theta_n^{d+1}} \frac{n}{n^{d+1}}.$$  

To compute the variance, we need a second order derivative, as one can see that it can be written as

$$(\sigma^2)^n_{gen} = \frac{[x_1^n]...[x_d^n]}{[x_1^n]...[x_d^n]} \left. \frac{\partial_u^2 Z_{on\, gen}(x, u)}{\partial_u^2} \right|_{u=1} - \mu_{gen}^2.$$  

Evaluating the derivative brings

$$\left. \frac{\partial_u^2 Z_{on\, gen}(e^{-\theta}, u)}{\partial_u^2} \right|_{u=1} = \left( \sum_{v \in P(\mathbb{Z}_+^d)} 2\sum_i 1_{v_i \neq 0} - 1 \left( e^{-\langle \theta, v \rangle} - e^{-2\langle \theta, v \rangle} \right) \right) Z_{on}(e^{-\theta}).$$

We see that the second sum is exactly the quantity $\mu_{gen}^2$ which means that the equivalent of the variance will be given by the first sum

$$\sum_{v \in P(\mathbb{Z}_+^d)} 2\sum_i 1_{v_i \neq 0} - 1 \left( e^{-\langle t, v \rangle} - e^{-2\langle t, v \rangle} \right) \sim \frac{2^d - 1}{2\zeta(d)t^d}.$$  

For the same reasons as before the saddle-point method is giving the following:

$$\frac{[x_1^n]...[x_d^n]}{[x_1^n]...[x_d^n]} \left. \frac{\partial_u^2 Z_{on\, gen}(x, u)}{\partial_u^2} \right|_{u=1} - \mu_{gen}^2 \sim \frac{2}{2^{d+1}} \frac{\zeta(d)}{\theta_n^{d+1}} \frac{n}{n^{d+1}}.$$  

Finally, the diameter of the graph of a zonotope is equal to its number of generators, by construction, which is to say

$$\mu_{gen}^n = \mu^n_{diam}, \quad (\sigma^2)^n_{gen} = (\sigma^2)^n_{diam}.$$  

$\square$
Theorem 4. The number of occurrences of a primitive generator \( v_0 \) in a lattice zonotope inscribed in \([0, n]^d\) is distributed with mean \( \mu_{\text{occ}}^n \) and variance \( (\sigma^2)_{\text{occ}}^n \) such as:

\[
\mu_{\text{occ}}^n \xrightarrow{n \to +\infty} \frac{n^{-\frac{1}{d+1}}}{K_d^{\frac{1}{d+1}} \|v_0\|_1}, \quad \text{and} \quad (\sigma^2)_{\text{occ}}^n \xrightarrow{n \to +\infty} \left( \frac{n^{-\frac{1}{d+1}}}{K_d^{\frac{1}{d+1}} \|v_0\|_1} \right)^2.
\]

Proof. Without loss of generality, we can choose \( v_0 \in \mathbb{Z}_d^d \). As for the previous parameter, we stress that the term \( \frac{1}{1-e^{-\theta, v_0}} \) holds in its series form the possibility of having \( k \) times \( v_0 \). To insert a variable \( u \) counting for the number of occurrences of \( v_0 \), it is substituted for:

\[
\sum_{k=0}^{+\infty} e^{-k<\theta, v_0>} \rightarrow \sum_{k=0}^{+\infty} u^k e^{-k<\theta, v_0>}.
\]

Therefore, the generating function \( Zon_{\text{occ}}(e^{-\theta}, u) \) is defined as \( Zon(e^{-\theta}) \frac{1-e^{-\theta, v_0}}{1-ue^{-\theta, v_0}} \).

The mean and variance are established as in the previous proof, computing

\[
\mu_{\text{occ}}^n = \frac{[x_1^n]\ldots[x_d^n]}{[x_1^n]\ldots[x_d^n]Zon(x)} \partial_u Zon_{\text{occ}}(x, u) \bigg|_{u=1} \quad \text{and} \quad (\sigma^2)_{\text{occ}}^n = \frac{[x_1^n]\ldots[x_d^n]}{[x_1^n]\ldots[x_d^n]Zon(x)} \partial_u \partial_u Zon_{\text{occ}}(x, u) \bigg|_{u=1} - \mu_{\text{occ}}^2.
\]

\[\square\]

7 Acknowledgments

The authors wish to thank Philippe Marchal and Lionel Pournin for their support, their valuable comments, remarks and corrections on the first version of this paper.

References

[1] Vladimir I. Arnold, Statistics of integral convex polygons, Functional Analysis and its Applications 14 (1980), no. 2, 1–3.

[2] Imre Bárány, The limit shape of convex lattice polygons, Discrete and Computational Geometry 13 (1995), no. 3-4, 279–295.

[3] Imre Bárány, Julien Bureaux, and Ben Lund, Convex cones, integral zonotopes, limit shape, Advances in Mathematics 331 (2018), 143–169.

[4] Imre Bárány and Janos Pach, On the number of convex lattice polygons, Combinatorics, Probability and Computing 1 (1992), no. 4, 295–302.

[5] Imre Bárány and Anatoli Moiseevitch Vershik, On the number of convex lattice polytopes, Geometric and Functional Analysis 2 (1992), no. 4, 381–393.

[6] Olivier Bodini, Julien Courtiel, Sergey Dovgal, and Hsien-Kuei Hwang, Asymptotic distribution of parameters in random maps, 29th International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms, AofA 2018, June 25-29, 2018, Uppsala, Sweden (James Allen Fill and Mark Daniel Ward, eds.), LIPIcs, vol. 110, Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2018, pp. 13:1–13:12.
[7] Olivier Bodini, Philippe Duchon, Alice Jacquot, and Ljuben R. Mutafchiev, *Asymptotic analysis and random sampling of digitally convex polyominoes*, Discrete Geometry for Computer Imagery, Gonzalez-Diaz R., Jimenez MJ., Medrano B. (eds), Lecture Notes in Computer Science 7749 (2013).

[8] Olivier Bodini, Danièle Gardy, Bernhard Gittenberger, and Alice Jacquot, *Enumeration of generalized BCI lambda-terms*, Electronic Journal of Combinatorics 20 (2013), no. 4, P30.

[9] Leonid V. Bogachev and Sakhavat M. Zarbaliev, *Universality of the limit shape of convex lattice polygonal lines*, Annals of Probability 39 (2011), no. 6, 2271–2317.

[10] Julien Bureaux and Nathanaël Enriquez, *On the number of lattice convex chains*, Discrete Analysis 19 (2016).

[11] Alberto Del Pia and Carla Michini, *Short simplex paths in lattice polytopes*, Discrete and Computational Geometry (to appear).

[12] Antoine Deza, George Manoussakis, and Shmuel Onn, *Primitive zonotopes*, Discrete and Computational Geometry 60 (2018), 27–39.

[13] Antoine Deza and Lionel Pournin, *Primitive point packing*, arXiv e-prints (2020), arXiv:2006.14228.

[14] Antoine Deza, Lionel Pournin, and Noriyoshi Sukegawa, *The diameter of lattice zonotopes*, Proceedings of the American Mathematical Society 148 (2020), no. 8, 3507–3516.

[15] Leonhard Euler, *Institutiones calculi differentialis cum eius usu in analysi finitorum ac Doctrina serierum*, Opera Omnia, 1755.

[16] Philippe Flajolet and Robert Sedgewick, *Analytic Combinatorics*, Cambridge University Press, 2009.

[17] Dominique Foata, *Les polynômes Eulériens, d'Euler à Carlitz*, Leonhard Euler, mathématicien, physicien et théoricien de la musique, Hascher X., Papadopoulos A., (eds) (2007).

[18] Bernhard Gittenberger and Johannes Mandlburger, *Hayman admissible functions in several variables*, The electronic journal of combinatorics 13 (2006).

[19] Leonidas J. Guibas, An Nguyen, and Li Zhang, *Zonotopes as bounding volumes*, SODA 3 (2003), 803–812.

[20] Walter K. Hayman, *A generalisation of Stirling’s formula*, Journal für die reine und angewandte Mathematik 196 (1956), 67–95.

[21] Birkett Huber and Bernt Sturmfels, *A polyhedral method for solving sparse polynomial systems*, Math. of Computation 64 (1995), 1541–1555.

[22] Sebastian Manecke and Raman Sanyal, *Coprime erhart theory and counting free segments*, International Mathematics Research Notices - IMRN (to appear).

[23] Atle Selberg, *On the zeros of riemann’s zeta-function*, Skr. Norske Vid.-Akad. Oslo I 10 (1942), 59 pp., MR10712 10.0X.

[24] Iakov Grigorievitch Sinaï, *A probabilistic approach to the analysis of the statistics of convex polygonal lines*, Functional Analysis and its Applications 28 (1994), no. 2, 41–48.

[25] Edward C. Titchmarsh, *The theory of the Riemann zeta-function, 2nd ed.*, The Clarendon Press, Oxford University Press, New York., MR882550, (88c:11049), 1986.

[26] Anatoli M. Vershik, *The limit form of convex integral polygons and related problems*, Functional Analysis and its Applications 28 (1994), 13–20.