Optimizing Synchronization Stability of the Kuramoto Model in Complex Networks and Power Grids

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Maintaining the stability of synchronization state is crucial for the functioning of many natural and artificial systems. In this study, we develop methods to optimize the synchronization stability of the Kuramoto model by minimizing the dominant Lyapunov exponent. With the help of the recently proposed cut-set space approximation of the steady states, we greatly simplify the objective function, and further derive its gradient and Hessian with respect to natural frequencies, which leads to an efficient algorithm with the quasi-Newton’s method. The optimized systems are demonstrated to achieve better synchronization stability for the Kuramoto model with or without inertia in certain regimes. Hence our method is applicable in improving the stability of power grids. It is also viable to adjust the coupling strength of each link to improve the stability of the system. Various operational constraints can also be easily integrated into our scope by employing the interior point method in convex optimization. The properties of the optimized networks are also discussed.

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I. INTRODUCTION

Synchronization occurs widely in many natural and artificial systems, such as firefly flashes, pacemaker cells of heart, Josephson junctions and power grids [1–4]. In general, the synchronous states are subject to different kinds of perturbations, and maintaining the stability of the systems against these perturbations is crucial for the functioning of the systems under consideration. For instance, the power grids are subject to various disturbances and real time active controls are needed to maintain a stable synchronization state [5]. The future power grids will sustain larger and larger fluctuations with the introduction of more and more renewable energies such as wind and solar power, which raise needs to enhance the robustness and stability of existing power networks [5].

To describe these synchronization phenomena, statistical physicists have proposed many simple but explanatory models, e.g., chaotic oscillator systems, the Kuramoto model and their various generalizations [3, 6–8]. A remarkable relation between spectral aspects of network structure and synchronizability in a broad range of coupled oscillator models has been developed in the master stability function (MSF) framework [9, 10]. In particular, the second smallest eigenvalues of the graph Laplacian matrix \( \lambda_2 \), namely the graph algebraic connectivity, is crucial in the synchronizability of models with unbounded MSF [9]. The graph algebraic connectivity is an interesting measure of network connectivity [10, 11], whose role in dynamical stability can be simplified in consensus dynamics or diffusion on networks \( \dot{x}_i = - \sum_j L_{ij} x_j \), where \( \lambda_2 \) determines the rate of convergence of the slowest mode [9].

In this study, we focus on the stability of the Kuramoto model on general networks. The stability of the frequency synchronization state of this nonlinear dynamical model is no longer determined by the graph algebraic connectivity or network structure itself, but has an intricate dependence on the system steady state [12]. The optimization of synchronization stability should take into account both the graph connectivity and the dynamical parameters.

Enhancing the synchronization stability in these settings has been stressed in a few recent studies [13–14], where the effects of network structures or power grid parameters, e.g., the damping coefficients and power injections, on the system stability were explored. However, only gradient descent methods have been applied to the optimization of stability with power scheduling or line impedance modification and the nonlinear power flow equation needs to be solved in each update step in both cases, both of which are less efficient. In this paper, we introduce the cut-set space approximation [15] to greatly simplify the calculation of power flow and the evaluation of the objective function, which is further facilitated by the much more efficient Newton’s method and interior point algorithm for optimization.

II. THE MODEL

A. First Order Kuramoto Model

We focus on the non-uniform first order Kuramoto model on a connected network in the form of

\[
\dot{\theta}_i = \omega_i + \sum_j K_{ij} \sin(\theta_j - \theta_i),
\]

where \( \theta_i \) denotes the phase angle of node \( i \), \( \omega_i \) the natural frequency and \( K_{ij} \) the coupling strength between node \( i \) and node \( j \). Without loss of generality, we assume \( \sum_i \omega_i = 0 \). The steady state is given by

\[
0 = \omega_i + \sum_j K_{ij} \sin(\theta_j^* - \theta_i^*).
\]
In the leading order, the small deviation from the steady state $\delta \theta_i = \theta_i - \theta^*_i$ follows \cite{12}

$$
\delta \dot{\theta}_i \approx \sum_j K_{ij} \cos(\theta^*_j - \theta^*_i)(\delta \theta_j - \delta \theta_i) \\
= - \sum_j L(\theta^*)_ij \delta \theta_j,
$$

where $L(\theta^*)_{ij} = \delta_{ij} \sum_l K_{il} \cos(\theta^*_l - \theta^*_i) - K_{ij} \cos(\theta^*_i - \theta^*_j)$ is a state dependent Laplacian matrix with edge weight $W(\theta^*)_{ij} = K_{ij} \cos(\theta^*_i - \theta^*_j)$. Thus the Jacobian matrix is $J = -L(\theta^*)$, which has a null-space of dimension one, corresponding to the rotational symmetry of the model. If $|\theta^*_i - \theta^*_j| < \pi/2$ holds for every edge $(i,j)$, then all the edge weights $W_{ij}$ are positive and all the other eigenvalues of $L(\theta^*)$ are positive, making the dynamical system locally exponentially stable. In this case, the slowest mode corresponds to the second smallest eigenvalue of $L(\theta^*)$, that is, the negative of the largest Lyapunov exponent excluding the null exponent of $J$. We denote it as $\lambda_2(L(\theta^*))$ and call it the state algebraic connectivity to distinguish it from the usual graph algebraic connectivity $\lambda_2(L[K])$. To improve the stability, it is reasonable to maximize $\lambda_2(L(\theta^*))$ as in \cite{13}.

### B. Second Order Kuramoto Model

The second order Kuramoto model is gaining attention due to its resemblance to the swing equation of power grids neglecting the transmission losses \cite{12}

$$
M_i \ddot{\theta}_i + D_i \dot{\theta}_i = P_i + \sum_j \frac{|V_i||V_j|}{X_{ij}} \sin(\theta_j - \theta_i), \quad (3)
$$

where $M_i$ and $D_i$ are the inertia and damping coefficient of node $i$ respectively. $P_i$ and $|V_i|$ are the mechanical power and voltage magnitude of node $i$, and $X_{ij}$ is the line reactance of edge $(i,j)$. The connection to the Kuramoto model is obvious if $P_i$ is identified as the natural frequency $\omega_i$ and $|V_i||V_j|/X_{ij}$ is identified as coupling $K_{ij}$. For simplicity, we consider uniform inertia and damping coefficient $M_i = M$ and $D_i = D$ and focus on the following model

$$
M \ddot{\theta}_i + D \dot{\theta}_i = \omega_i + \sum_j K_{ij} \sin(\theta_j - \theta_i). \quad (4)
$$

The steady state $(\dot{\theta}^* = 0, \theta^*)$ is again given by Eq. \cite{14}, with Jacobian matrix evaluated at this point as \cite{13, 14}

$$
J(\dot{\theta}^* = 0, \theta^*) = \begin{bmatrix} -\frac{D}{M} I & -\frac{1}{M} L(\theta^*) \\ \frac{1}{M} & 0 \end{bmatrix}.
$$

As derived in \cite{14}, $J(\dot{\theta}^*, \theta^*)$ can be diagonalized by the eigenvectors of $L(\theta^*)$, with corresponding eigenvalues

$$
\mu_{j\pm}(\lambda_j, D, M) = -\frac{D}{2M} \pm \frac{1}{2} \sqrt{\left(\frac{D}{M}\right)^2 - \frac{4}{M} \lambda_j(L(\theta^*)}. \quad (5)
$$

The maximal nontrivial eigenvalue will be $\mu_{2+} = -\frac{D}{2M} + \frac{1}{2} \sqrt{\left(\frac{D}{M}\right)^2 - \frac{4}{M} \lambda_2(L(\theta^*)}$. When $\lambda_2(L(\theta^*)) < D^2/4M$, increasing $\lambda_2(L(\theta^*))$ will always make $\mu_{2+}$ larger. In this regime, optimizing $\lambda_2(L(\theta^*))$ is also applicable to stabilize the uniform second order Kuramoto model, therefore it may have potential applications in the study of power grid stability. This regime can correspond to large damping, small inertia or close to bifurcation.

### III. METHOD

#### A. Variation of State Algebraic Connectivity

Viewing $\omega_i$ and $K_{ij}$ as control variables, we aim at maximizing $\lambda_2(L(\theta^*))$ in order to improve the stability of both Eq. \cite{1} and Eq. \cite{2}. We first derive the variation of state algebraic connectivity due to change of natural frequency. We assume that the state algebraic connectivity is non-degenerate throughout optimization, which usually holds when the corresponding graph algebraic connectivity $\lambda_2(L[K])$ is non-degenerate.

From the perturbation theory in quantum mechanics, in case $\lambda_2(L(\theta^*))$ is non-degenerate, the variation of $\lambda_2(L(\theta^*))$ is given by \cite{16}

$$
\delta \lambda_2(L(\theta^*)) = \langle v_2(\theta^*)|\delta L(\theta^*)|v_2(\theta^*) \rangle \\
= v_2(\theta^*)^T \delta L(\theta^*) v_2(\theta^*), \quad (6)
$$

where $v_2(\theta^*)$ is the eigenvector of $L(\theta^*)$ corresponding to $\lambda_2(L(\theta^*))$. Since $L(\theta^*)$ is a Laplacian matrix with edge weight $W(\theta^*)_{ij} = K_{ij} \cos(\theta^*_i - \theta^*_j)$, one has $\delta L(\theta^*)_{ij} = \delta_{ij} \sum_l \delta W(\theta^*)_{il} - \delta W(\theta^*)_{ij}$ and

$$
\delta \lambda_2(L(\theta^*)) = \sum_{(i,j)} \delta W(\theta^*)_{ij} \langle v_2(\theta^*)_i - v_2(\theta^*)_j \rangle^2. \quad (7)
$$

So the gradient of state algebraic connectivity with respect to $\omega$ is

$$
\langle \nabla_\omega \lambda_2(L(\theta^*)) \rangle_k = \sum_{(i,j)} \frac{\delta W(\theta^*)_{ij}}{\delta \omega_k} \langle v_2(\theta^*)_i - v_2(\theta^*)_j \rangle^2. \quad (8)
$$

The computational difficulty comes from the implicit dependence between shift of steady state $\delta \theta^*$ and change of natural frequency $\delta \omega$. In \cite{13}, $\delta \theta^*/\delta \omega$ is proved to be related to the pseudo-inverse of $L(\theta^*)$. These expressions lead to a gradient ascent method to maximize $\lambda_2(L(\theta^*))$ by scheduling $\omega$. However, this method requires solving the steady state equation Eq. \cite{2} and computing the pseudo-inverse of $L(\theta^*)$ in every iteration, both of which
B. Cut-set Space Approximation of Network Flows

\(\omega_i\) can be viewed as supply/demand of node \(i\) in a supply network as \(P_i\) in the power grid, and \(K_{ij}\sin(\theta_j - \theta_i)\) is the resource/power transported from node \(j\) to node \(i\). The steady state equation Eq. (2) implies the flow conservation on each node.

Solving the nonlinear steady state equation can be computational costly. Recently it has been shown that the cut-set space approximation of power flows can be rather accurate in many regimes \[13, 17\]. The idea is to view \(\beta_{ij} = \sin(\theta_j^* - \theta_i^*)\) as a cochain, which can be decomposed as the sum of a potential difference part \(\beta_{ij}^{\text{cut}} = \phi_j - \phi_i\), and a circular flow part \(\beta_{ij}^{\text{cycle}}\) satisfying \(\sum_{j \in \partial i} K_{ij} \beta_{ij}^{\text{cycle}} = 0 \forall i\) \[13\]. In the language of graph theory, \(\beta_{ij}^{\text{cut}}\) and \(\beta_{ij}^{\text{cycle}}\) are said to live in the cut-set space and cycle space respectively. It turns out that \(\phi\) coincides with the DC approximation of AC power flow in power engineering \[\theta_{DC}\] \[13\]. Let \(L[K]\) denote the graph Laplacian matrix of the underlying weighted network with edge weight \(K_{ij}\), such that \(L[K]_{ij} = \delta_{ij} \sum_{k} K_{ik} - K_{ij}\), then \(\phi\) satisfies \(L[K] \phi = \omega\). The cut-set space component of \(\beta_{ij}\) can be expressed as

\[
\beta_{ij}^{\text{cut}} = \phi_j - \phi_i = \sum_I (L[K]_{ij}^I - L[K]_{ij}^I) \omega_I,
\]

where \(L[K]^I\) is the pseudo-inverse of \(L[K]\). To simplify the calculation, it is proposed to approximate \(\beta\) by its cut-set space component \(\beta_{ij}^{\text{cut}}\), i.e., \(\sin(\theta_j^* - \theta_i^*) \approx \phi_j - \phi_i = \sum_I (L[K]_{ij}^I - L[K]_{ij}^I) \omega_I\). Such an approximation is exact in some specific systems, such as acyclic graphs and systems with cut-set inducing frequencies, while it has also been tested numerically in many generic networks that the approximation is surprisingly accurate \[13, 17\]. We demonstrate two examples in Fig. (1).

Figure 1: (Color online) \(\phi_j - \phi_i\) vs \(\sin(\theta_j^* - \theta_i^*)\). (a) Erdős-Rényi graph of 50 nodes (ER50), where \(\omega\) is drawn from a Gaussian distribution and \(K_{ij} = 1\). Inset: root mean square error (RMSE) of estimation \(\phi_j - \phi_i\) for \(\sin(\theta_j^* - \theta_i^*)\). (b) IEEE Reliability Test System 96 (RTS96), where \(\omega\) is modified from the power injection data in the test system and \(K_{ij}\) is defined to be the inverse of line reactance of edge \((i, j)\). Inset: RMSE of estimation \(\phi_j - \phi_i\) for \(\sin(\theta_j^* - \theta_i^*)\). \(\phi_j - \phi_i\) approximates \(\sin(\theta_j^* - \theta_i^*)\) quite well even in the stress cases with large \(|\omega||\omega|_2\).

where \(A^{(ij)}\) is defined to be a matrix with entry \(A_{kl}^{(ij)} = (L[K]_{ij}^i - L[K]_{ik}^i)(L[K]_{kj}^j - L[K]_{jk}^j)\) and we have made use of the fact that \(\phi_{ij} = L[K]^i \omega\). Provided that \(L[K]^i\) is calculated and recorded, we only need to solve for \(\phi\) by simple matrix multiplication instead of solving the nonlinear steady state equation Eq. (2). Now we work on the state algebraic connectivity \(\lambda_2(L(\phi))\) which corresponds to the state dependent Laplacian matrix with edge weight \(W(\phi)_{ij} = K_{ij} \sqrt{1 - \sum_k A_{kl}^{(ij)} \omega_k}\). We assume in the following discussion that \(|\theta_j^* - \theta_i^*|\) is closed to \(\pi/2\) along some edges, in which case a preprocess to destroy the system before optimization is needed.

The gradient in Eq. (8) can be estimated by

\[
[\nabla_{\omega} \lambda_2(L(\phi))]_k = \sum_{(i,j)} K_{ij} \frac{-\sum_{l} A_{kl}^{(ij)} \omega_l}{\sqrt{1 - \omega^T A^{(ij)} \omega}} [v_2(\phi)_i - v_2(\phi)_j]^2,
\]

where \(v_2(\phi)\) is the eigenvector corresponding to \(\lambda_2(L(\phi))\).

Similarly, the Hessian of the state algebraic connectivity is estimated by

\[
H_{kl} = \frac{\partial^2 \lambda_2(L(\phi))}{\partial \omega_k \partial \omega_l} = \sum_{(i,j)} \frac{\partial^2 W(\phi)_{ij} \omega_l}{\partial \omega_k \partial \omega_l} [v_2(\phi)_i - v_2(\phi)_j]^2 + \sum_{(i,j)} 2 \frac{\partial W(\phi)_{ij}}{\partial \omega_k} [v_2(\phi)_i - v_2(\phi)_j] \left[ \frac{\partial v_2(\phi)_i}{\partial \omega_l} - \frac{\partial v_2(\phi)_j}{\partial \omega_l} \right],
\]

where \(\partial v_2(\phi)/\partial \omega\) can also be obtained from the non-degenerate perturbation theory, which is computational costly. We found in all our numerical experiments that truncating the second term of the Hessian can still lead to
to be zero-sum by solving the steady state equation Eq. (2) with the step size $s$ determined by back tracking line search \[19\], after which $\omega$ is enforced to be zero-sum by $\omega_i \rightarrow \omega_i - 1/N \sum_j \omega_j$ so that it admits a steady state.

In general, $\lambda_2(\tilde{L}(\phi))$ is an increasing function with $\tilde{W}(\phi)_{ij}$, which favors small phase angle difference across each edge.  Under imposing any constraint, the optimal solution should take place at $\omega = 0$, in which case the optimum $\lambda_2(L(\theta^* = 0))$ coincides with the graph algebraic connectivity. In Fig. (2) we show the the optimization process for the RTS96 power network with gradient ascent update and quasi-Newton update. It is observed in this case that (i) $\lambda_2(\tilde{L}(\phi))$ is close to the exact state algebraic connectivity $\lambda_2(L(\theta^*))$ at the same $\omega$ (obtained by solving the steady state equation Eq. (2) with $\omega$ given at that iteration); (ii) the Newton’s method is much more efficient than the gradient ascent, approaching the optimum within only a few steps, despite the extra efforts for computing the Hessian $H$ and solving the linear equation $H \Delta \omega_{\text{Newton}} = \nabla_{\omega} \lambda_2(\tilde{L}(\phi))$ to obtain $\Delta \omega_{\text{Newton}}$.

D. Optimization by Tuning for Coupling Strengths

Instead of optimizing the natural frequencies, one can also tune the coupling strengths of edges to improve the stability. In power grids, this corresponds to the change of line reactance of each edge, which may be implemented by tuning the transmission lines or using FACTS devices \[20\]. Similarly, we can also derive the gradient and Hessian of $\lambda(\tilde{L}(\phi))$ with respect to the coupling strength

\[
[\nabla_K \lambda_2(\tilde{L}(\phi))]_{(k,l)} = \sum_{(i,j)} \frac{\partial \tilde{W}(\phi)_{ij}}{\partial K_{kl}} [v_2(\phi)_i - v_2(\phi)_j]^2
\]

\[
= \sum_{(i,j)} \left\{ \delta_{(i,j),(k,l)} \sqrt{1 - \omega_i^2 A(ij)\omega} + \frac{\partial A(ij)}{\partial K_{kl}} \frac{\omega_i}{\sqrt{1 - \omega_j^2 A(ij)\omega}} \right\} [v_2(\phi)_i - v_2(\phi)_j]^2,
\]

where the evaluation of $\partial A(ij)/\partial K_{kl}$ relies on the computation of $\partial L[K]/\partial K_{kl}$ which is attainable as long as the rank of $L[K]$ remains unchanged \[21\]. The gradient ascent update is simply given by $K \leftarrow K + s \nabla_K \lambda_2(\tilde{L}(\phi))$. The Hessian matrix and update of Newton’s method can also be obtained straightforwardly, although the expression is extremely tedious. The update of coupling strength renders the modification of $L[K]$ and recalculation of $L[K]^T$, making it much more time consuming than the update of natural frequencies.

IV. RESULTS

A. Behavior at Optimal Natural Frequencies

To obtain a non-trivial solution with optimal stability, we introduce an additional Euclidean norm constraint

\[
\|\omega\|^2 = \sum_i \omega_i^2 \geq c,
\]

which treats all nodes in equal footing and doesn’t emphasize the role of import nodes, say, hubs. The constraint optimization is solved by the barrier method, a particular interior point algorithm, which in fact finds a solution satisfying the KKT conditions \[19\]. Although the constraint Eq. (12) is nonconvex and global optimum may not be attainable, we find in our numerical experiments that the barrier method can efficiently achieve a satisfactory local minimum.

In Fig. (3) we plot the optimization process of the RTS96 power network with constraint parameter $c = 0.99\|\omega_0\|_2^2$, where $\omega_0$ is the same as the initial natural frequency in Fig. (2). The corresponding unoptimized and optimized system is shown in Fig. (3) (b). It is observed that for most edges, $\cos(\theta^* - \theta^*)$ in the optimized system is larger than the one in the unoptimized system, especially for two inter-module connections edge (318, 223) and edge (325, 121). To illustrate the improved stability of the optimized system related to an unoptimized one, we impose a small disturbance $\delta_i$ to the steady state at
much more common than increase. of phase angle differences that there is a correlation between the increments of higher algebraic connectivity. Thus it is not surprising state dependent network will be better connected with a weight $W$ model. In Fig. 3(c) and 3(d) we monitor the discrepancy RTS96 power network under Euclidean norm constraint. (b) steady state more rapidly than the unoptimized system. It is observed that the optimized system converges to the smaller phase angle differences $|\theta_i^* - \theta_j^*|$. In fact, more cohesive phase angles in general imply smaller phase angle differences $|\theta_i^* - \theta_j^*|$ and larger edge weight $W(\theta^*)_{ij} = K_{ij} \cos(\theta_i^* - \theta_j^*)$, in which case the state dependent network will be better connected with a higher algebraic connectivity. Thus it is not surprising that there is a correlation between the increments of $r$ and $\lambda_2(L(\theta^*))$. We show in Fig. 4(b) that the decrease of phase angle differences $|\theta_i^* - \theta_j^*|$ after optimization is much more common than increase.

It is found in previous studies that natural frequencies which optimize $r$ subject to constraint of the form $\|\omega\|^2 = \text{constant}$ have negative correlations between neighboring frequencies, and align with eigenvectors corresponding to large eigenvalues of graph Laplacian $L$. We show in Fig. 4(c) and (d) that such properties are also observed in natural frequencies which optimize $\lambda_2(L(\theta^*))$.

B. Properties of Optimized Systems

In the following, we explore some general properties of the optimal systems under the Euclidean norm constraint. The networks are ER random graphs with 50 nodes and every pair of nodes are connected with probability $p = 0.1$. As found in Fig. 4(a), not only does the optimization result in improving the objective function $\lambda_2(L(\theta^*))$, but also the Kuramoto order parameter $r$. In fact, more cohesive phase angles in general imply smaller phase angle differences $|\theta_i^* - \theta_j^*|$ and larger edge weight $W(\theta^*)_{ij} = K_{ij} \cos(\theta_i^* - \theta_j^*)$, in which case the state dependent network will be better connected with a higher algebraic connectivity. Thus it is not surprising that there is a correlation between the increments of $r$ and $\lambda_2(L(\theta^*))$. We show in Fig. 4(b) that the decrease of phase angle differences $|\theta_i^* - \theta_j^*|$ after optimization is much more common than increase.

C. Difference Between $\lambda_2(L(\theta^*))$ and $r$

Observing the similarity of the results of optimizing $\lambda_2(L(\theta^*))$ with the Euclidean norm constraint and those of optimizing $r$ with the same constraint, it is tempting to conclude that the more synchronized a system the more stable and one can improve the system stability by just increasing the order parameter $r$, which can be much simpler. However, we argue that while such a judgement is valid in many cases like the above homogeneous ER graphs, it is not necessarily a universal rule. In fact, there is a conceptual difference between the two quantities. The Kuramoto order parameter $r$ is a measure of coherence of phase angles of all oscillators in a global and average sense, which cannot identify the role of critical edges in maintaining stability, e.g., the interconnections between modules. To be more concrete, we consider a
the state algebraic connectivity. This simple example highlights the essence of using $\lambda_2(L(\theta^*))$ as a cost function for measuring stability in general networks.

#### D. Inclusion of Practical Power Grid Constraints

The Euclidean norm constrained optimization problem above treats all nodes on equal footing where a supplier can become a consumer and vice versa. This will not be realistic if we consider the power grid applications. In this section, we consider two problems regarding practical constraints of power grid operations.

In Problem 1, both the supply and the demand are allowed to vary within a certain range. Furthermore, regulating both the generation and consumption may be necessary in future grids with the introduction of renewable energy. Hence specifically we consider the constraint $\omega_i - \alpha|\omega_0| \leq \omega_i \leq \omega_0 + \alpha|\omega_0|$ for $i$ to be either a supply node or demand node, where $\omega_0$ is the natural frequency of the original system and the parameter $\alpha$ satisfies $0 < \alpha \leq 1$. For the relay node with $\omega_0 = 0$, the natural frequency will remain unchanged throughout optimization $\omega_i = \omega_0 = 0$.

In Problem 2, only the supply nodes with $\omega_0 > 0$ are allowed to schedule their productions with fraction $\alpha$, while the demands must be satisfied and the relay nodes should also be fixed, i.e., $\omega_i = \omega_0$ for $\omega_0 \leq 0$. To deal with both the inequality and equality constraints, the primal-dual interior point method in convex optimization is applied in these problems. Although we always make the supply and demand balanced in every iteration, we discovered that imposing the additional constraint $\sum \omega_i = 0$ into the definition of the problem can significantly facilitate the convergence of the algorithm.

In Fig. 5(a), we plot the optimization process of the RTS96 power network with constraints of Problem 1. The primal-dual interior point algorithm can bring the system to optimum effectively. During optimization, the system is also destressed as indicated by the decreasing $||\omega||_1$. In Fig. 5(b), we plot $\lambda_2(L(\theta^*))$ and $||\omega||_1$ as a function of $\alpha$ with constraints of both Problem 1 and Problem 2. It is observed that $\lambda_2(L(\theta^*))$ increases with $\alpha$ for both cases with variable demands and fixed demands. This is not surprising since the feasible region of the problem with larger $\alpha$ is a superset of the one with smaller $\alpha$, and a larger feasible region gives the system more flexibility to search for more stable state. The system can achieve higher stability with variable demands in Problem 1 than the fixed demand in Problem 2, which is also due to more degrees of freedom to vary in Problem 1. Our method can solve both problems satisfactorily.

#### E. Behavior at Optimal Coupling Strengths

Lastly, we consider behavior at the optimal state algebraic connectivity by updating the coupling strengths. To avoid indefinite solutions, we impose a simple constraint

$$\sum_{(i,j)} K_{ij} = K_{\text{total}},$$  \hspace{1cm} (13)

Figure 5: (Color online) Phase angles $\theta^*$ and state dependent edge weights $W(\theta^*)$ in a two-module network. In both cases, the $L^2$-norm of natural frequency is $||\omega||_2 = 4.26$. (a) Phases of the system depicted on the unit circle in Case 1. (b) The state dependent edge weight $W(\theta^*)_{ij} = K_{ij} \cos(\theta_i^* - \theta_j^*)$ in Case 1. (c) Phases of the system depicted on the unit circle in Case 2. (d) The state dependent edge weight $W(\theta^*)_{ij} = K_{ij} \cos(\theta_i^* - \theta_j^*)$ in Case 2.
where $K_{\text{total}}$ represents the availability of the total capacity, and $K_{ij}$ is constrained to be non-negative. Due to the high complexity of computing the Hessian, we only consider the gradient ascent update. To preserve the resource constraint, the approximate gradient $\nabla_K \lambda_2(L(\phi))$ as calculated by Eq. (11) is projected onto the feasible region, after which the coupling strengths are updated. In Fig. 7(a), we plot the optimization process of the projected gradient update on the two-module network discussed in Section IV C and the initial condition is the same as Case 1 in Section IV C. It is shown that redistributing the coupling strengths can significantly improve both the graph algebraic connectivity and state algebraic connectivity, reaching a more stable state. In Fig. 7(b), we sketch the state dependent edge weight in the optimal state. Contrary to the un-optimized system in Fig. 5(b), the optimized system exhibits large edge weight $W(\theta^*)_{ij}$ in edge (1,16) and edge (0,15), the interconnections between the two modules, which favors higher state algebraic connectivity. For each module, the nodes are well connected and the need for transporting resource is modest. Thus the coupling strengths inside each module are sacrificed so that the system can invest more on the critical edges.

V. DISCUSSION

In this paper, we studied the optimization of synchronization stability of Kuramoto model by updating the natural frequencies or coupling strengths. The proposed cut-set space approximation can accurately estimate the network flows of steady states and thus simplify the objective function, i.e., the state algebraic connectivity whose increment can increase the stability of the phase locked steady states of both the first and second order Kuramoto model. Such an approximation leads to compact expressions of gradient and Hessian of the cost function. Together with the interior point algorithm or projected gradient ascent, our method can cope with various constraints, which is shown to be effective and efficient. There is a general correlation between the optimization of the Kuramoto order parameter and the state algebraic connectivity, especially in the homogeneous networks. However, the Kuramoto order parameter cannot represent the role of critical links, e.g., inter-module connections, which is crucial to the synchronization stability. In light of this consideration, the state algebraic connectivity is a more appropriate cost function for the measure of stability. Our framework has potential applications in improving the stability of power grids which are usually simplified to a second order Kuramoto model.

Nevertheless, there are many other aspects to consider concerning the application of power grids, such as extending our formalism to non-uniform inertia or damping, lossy transmissions, effect of changes of network topology due to breakdown of grid elements, etc. In addition, our method is based on the assumption of non-degenerate state algebraic connectivity, which may not hold in highly symmetric networks, and how to achieve an optimum under general constraints in these networks remains to be explored. Lastly, our study considers only linear stability and the critical edges. More on the the critical edges.

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