NON-MANIFOLD MONODROMY SPACES OF BRANCHED COVERINGS BETWEEN MANIFOLDS

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Abstract. By a construction of Berenstein and Edmonds every proper branched cover \( f \) between manifolds is a factor of a branched covering orbit map from a locally connected and locally compact Hausdorff space called the monodromy space of \( f \) to the target manifold. For proper branched covers between 2-manifolds the monodromy space is known to be a manifold. We show that this does not generalize to dimension 3 by constructing a self-map of the 3-sphere for which the monodromy space is not a locally contractible space.

1. Introduction

A map \( f: X \to Y \) between topological spaces is a branched covering, if \( f \) is open, continuous and discrete map. The branch set \( B_f \subset X \) of \( f \) is the set of points in \( X \) for which \( f \) fails to be a local homeomorphism. The map \( f \) is proper, if the pre-image in \( f \) of every compact set is compact.

Let \( f: X \to Y \) be a proper branched covering between manifolds. Then the codimension of \( B_f \subset X \) is at least two by Väisälä [14] and the restriction map

\[
f' := f|X \setminus f^{-1}(f(B_f)): X \setminus f^{-1}(f(B_f)) \to Y \setminus f(B_f)
\]

is a covering map between open connected manifolds, see Church and Hemmingsen [6]. Thus there exists, by classical theory of covering maps, an open manifold \( X_f' \) and a commutative diagram of proper branched covering maps

\[
\begin{array}{ccc}
X_f' & \xrightarrow{\varphi'} & X_f' \\
\downarrow{\varphi} & & \downarrow{\varphi'} \\
X & \xrightarrow{f} & Y
\end{array}
\]

where \( \varphi': X_f' \to X \setminus f^{-1}(f(B_f)) \) and \( \varphi: X_f' \to Y \setminus f(B_f) \) are normal covering maps and the deck-transformation group of the covering map \( \varphi': X_f' \to Y \setminus f(B_f) \) is isomorphic to the monodromy group of \( f' \).

Further, by Berstein and Edmonds [3], there exists a locally compact and locally connected second countable Hausdorff space \( X_f \supset X_f' \) so that \( X_f \setminus X_f' \) does not locally separate \( X_f \) and the maps \( \varphi' \) and \( \varphi \) extend to proper covers.

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normal branched covering maps \( p: X_f \to X \) and \( \bar{f} := q: X_f \to Y \) so that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{p} & & \downarrow{q} \\
X_f & \xrightarrow{\bar{f}} & Y
\end{array}
\]

commutes, and \( p \) and \( \bar{f} \) are the Fox-completions of \( p': X'_f \to X \) and \( q': X'_f \to Y \), see also [4], [8] and [11]. In this paper the triple \((X_f, p, \bar{f})\) is called the monodromy representation, \( \bar{f}: X_f \to Y \) the normalization and the space \( X_f \) the monodromy space of \( f \).

The monodromy space \( X_f \) is a locally connected and locally compact Hausdorff space and, by construction, all points in the open and dense subset \( X_f \setminus B_{\bar{f}} \subset X_f \) are manifold points. The natural question to ask regarding the monodromy space \( X_f \) is thus the following: What does the monodromy space \( X_f \) look like around the branch points of \( \bar{f} \)?

When \( X \) and \( Y \) are 2-manifolds, Stoïlow's Theorem implies, that the points in \( B_{\bar{f}} \) are manifold points and the monodromy space \( X_f \) is a manifold. We further know by Fox [8] that the monodromy space \( X_f \) is a locally finite simplicial complex, when \( f: X \to Y \) is a simplicial branched covering between piecewise linear manifolds. It is, however, stated as a question in [8] under which assumptions the locally finite simplicial complex obtained in Fox' completion process is a manifold. We construct here an example in which the locally finite simplicial complex obtained in this way is not a manifold.

**Theorem 1.1.** There exists a simplicial branched cover \( f: S^3 \to S^3 \) for which the monodromy space \( X_f \) is not a manifold.

Theorem 1.1 implies that the monodromy space is not in general a manifold even for proper simplicial branched covers between piecewise linear manifolds. Our second theorem states further, that in the non-piecewise linear case the monodromy space is not in general even a locally contractible space. We construct a branched covering, which is a piecewise linear branched covering in the complement of a point, but for which the monodromy space is not a locally contractible space.

**Theorem 1.2.** There exists a branched cover \( f: S^3 \to S^3 \) for which the monodromy space \( X_f \) is not a locally contractible space.

We end this introduction with our results on the cohomological properties of the monodromy space. The monodromy space of a proper branched covering between manifolds is always a locally orientable space of finite cohomological dimension. However, in general the monodromy space is not a cohomology manifold in the sense of Borel [5]; there exist a piecewise linear branched covering \( S^3 \to S^3 \) for which the monodromy space is not a cohomology manifold. This shows, in particular, that the theory of normalization maps of proper branched covers between manifolds is not covered by Smith-theory in [5] and completes [2] for this part.

This paper is organized as follows. In Section 4, we give an example \( f: S^2 \to S^2 \) of an open and discrete map for which the monodromy space is not a two sphere. In Section 5 we show that the suspension \( \Sigma f: S^3 \to S^3 \) of
$f$ prove Theorem 1.1, and that the monodromy space of $f$ is not a cohomology manifold. In Section 6 we construct an open and discrete map $g: S^3 \to S^3$. In Section 7 we show that $g: S^3 \to S^3$ proves Theorem 1.2.

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2. Preliminaries

In this paper all topological spaces are locally connected and locally compact Hausdorff spaces if not stated otherwise. Further, all proper branched coverings $f: X \to Y$ between topological spaces are also branched coverings in the sense of Fox [8] and completed coverings in the sense of [1]; $Y' := Y \setminus f(B_f)$ and $X' := X \setminus f^{-1}(B_f)$ are open dense subsets so that $X \setminus X'$ does not locally separate $X$ and $Y' \setminus Y$ does not locally separate $Y$. We say that the proper branched covering $f: X \to Y$ is normal, if $f' := f|X': X' \to Y'$ is a normal covering. By Edmonds [7] every proper normal branched covering $f: X \to Y$ is an orbit map for the action of the deck-transformation group $\text{Deck}(f)$ i.e. $X/\text{Deck}(f) \approx Y$.

We recall some elementary properties of proper branched coverings needed in the forthcoming sections. Let $f: X \to Y$ be a proper normal branched covering and $V \subset Y$ an open and connected set. Then each component of $f^{-1}(V)$ maps onto $V$, see [1]. Further, if the pre-image $D := f^{-1}(V)$ is connected, then $f|D: D \to V$ is a normal branched covering and the map

$$(1) \quad \text{Deck}(f) \to \text{Deck}(f|D), \tau \mapsto \tau|D,$$

is an isomorphism, see [1].

Lemma 2.1. Let $f: X \to Y$ be a branched covering between manifolds. Suppose $p: W \to X$ and $q: W \to Y$ are normal branched coverings so that $q = p \circ f$. Then $\text{Deck}(p) \subset \text{Deck}(q)$ is a normal subgroup if and only if $f$ is a normal branched covering.

Proof. Let $X' := Y \setminus f(B_f)$, $X' := X \setminus f^{-1}(f(B_f))$ and $W' = W \setminus q^{-1}(f(B_f))$. Let $f' := f|X': X' \to Y'$, $p' := p: W' \to X'$ and $q' := q|W': W' \to Y'$ and let $w_0 \in W'$, $x_0 = p'(w_0)$ and $y_0 = q'(w_0)$. Then $\text{Deck}(p) \subset \text{Deck}(q)$ is a normal subgroup if and only if $\text{Deck}(p') \subset \text{Deck}(q')$ is a normal subgroup and the branched covering $f$ is a normal branched covering if and only if the covering $f'$ is normal. We have also a commutative diagram

\[
\begin{array}{ccc}
W' & \xrightarrow{q'} & Y' \\
\downarrow{p'} & & \downarrow{f'} \\
X' & \xrightarrow{f} & Y'
\end{array}
\]

of covering maps, where

$q'_* (\pi_1(W', w_0)) \subset f'_* (\pi_1(X', x_0)) \subset \pi_1(Y', y_0)$.

The deck-homomorphism

$\pi_{(q', w_0)}: \pi_1(Y', y_0) \to \text{Deck}(q')$
now factors as

\[ \pi_1(Y', y_0) \xrightarrow{\pi(q', w_0)} \pi_1(Y', y_0)/q'_*(\pi_1(W', w_0)) \xrightarrow{\pi(q', w_0)} \text{Deck}(q') \]

for an isomorphism \( \pi(q', w_0) \) and

\[ \pi(q', w_0)(f'_*(\pi_1(X', x_0))) = \pi(q', w_0)(f'_*(\pi_1(X', x_0))/q'_*(\pi_1(W', w_0))) = \text{Deck}(p'). \]

In particular, \( \text{Deck}(p') \subset \text{Deck}(q') \) is a normal subgroup if and only if

\[ f'_*(\pi_1(X', x_0))/q'_*(\pi_1(W', w_0)) \subset \pi_1(Y', y_0)/q'_*(\pi_1(W', w_0)) \]

is a normal subgroup. Since \( q'_*(\pi_1(W', w_0)) \subset f'_*(\pi_1(X', x_0)) \), this implies that \( \text{Deck}(p') \subset \text{Deck}(q') \) is a normal subgroup if and only if

\[ f'_*(\pi_1(X', x_0)) \subset \pi_1(Y', y_0) \]

is a normal subgroup. We conclude that \( f \) is a normal branched covering if and only if \( \text{Deck}(p) \subset \text{Deck}(q) \) is a normal subgroup.

Let \( f : X \to Y \) be a proper branched covering. We say that \( D \subset X \) is a normal neighbourhood of \( x \) if \( f^{-1}\{f(x)\} \cap D = \{x\} \) and \( f|D : D \to V \) is a proper branched covering. We note that for every \( x \in X \) there exists a neighbourhood \( U \) of \( f(x) \) so that for every open connected neighbourhood \( V \subset U \) of \( f(x) \) we have the x-component \( D \) of \( f^{-1}(V) \) a normal neighbourhood of \( x \) and the pre-image \( f^{-1}(E) \subset X \) is connected for every open connected subset \( E \subset Y \) satisfying \( Y \setminus E \subset U \). This follows from the following lemma.

**Lemma 2.2.** Let \( f : X \to Y \) be a proper branched covering. Then for every \( y \in Y \) there exists such a neighbourhood \( U \) of \( y \) that the pre-image \( f^{-1}(V) \subset X \) is connected for every open connected set \( V \subset Y \) satisfying \( Y \setminus V \subset U \).

**Proof.** Let \( y_0 \in Y \setminus \{y\} \). Since \( f \) is proper the subsets \( f^{-1}\{y\}, f^{-1}\{y_0\} \subset X \) are finite. Since \( W \setminus f^{-1}\{y\} \) is connected, there exists a path \( \gamma : [0, 1] \to W \setminus f^{-1}\{y\} \) so that \( f^{-1}\{y_0\} \subset f(\gamma[0, 1]) \). Let \( U \subset Y \) be a neighbourhood of \( y \) satisfying \( U \setminus f(\gamma[0, 1]) = \emptyset \).

Suppose that \( V \subset Y \) is an open connected subset satisfying \( Y \setminus V \subset U \). Then \( f^{-1}(y_0) \subset f(\gamma[0, 1]) \) is contained in a component of \( f^{-1}(V) \), since \( f(\gamma[0, 1]) \subset V \). Since \( V \subset Y \) is connected, every component of \( f^{-1}(V) \) maps onto \( V \). Thus \( f^{-1}(V) = D \).

We end this section with introduction the terminology and elementary results for the part of singular homology. Let \( X \) be a locally compact and locally connected second countable Hausdorff space. In this paper \( H_i(X; Z) \) is the \( i \)th singular homology group of \( X \) and \( \overline{H}_i(X; Z) \) the \( i \)th reduced singular homology group of \( X \) with coefficients in \( Z \), see [10]. We recall that \( H_i(X; Z) = \overline{H}_i(X; Z) \) for all \( i \neq 0 \) and \( \overline{H}_0(X; Z) = \mathbb{Z}^{k-1} \), where \( k \) is the number of components in \( X \). We recall that for open subsets \( U, V \subset X \) with \( X = U \cup V \) and \( X = U \cap V \) connected the reduced Mayer-Vietoris sequence is a long exact sequence of homomorphisms that terminates as follows:

\[ \to H_1(X; \mathbb{Z}) \to \overline{H}_0(U \cap V; \mathbb{Z}) \to \overline{H}_0(U; \mathbb{Z}) \oplus \overline{H}_0(V; \mathbb{Z}) \to \overline{H}_0(X; \mathbb{Z}). \]
3. Local orientability and cohomological dimension

In this section we show that the monodromy space of a proper branched covering between manifolds is a locally orientable space of finite cohomological dimension. We also introduce Alexander-Spanier cohomology following the terminology of Borel [5] and Massey [10] and define a cohomology manifold in the sense of Borel [5].

Let $X$ be a locally compact and locally connected second countable Hausdorff space. In this paper $H^i_c(X; \mathbb{Z})$ is the $i$:th Alexander-Spanier cohomology group of $X$ with coefficients in $\mathbb{Z}$ and compact supports. Let $A \subset X$ be a closed subset and $U = X \setminus A$. The standard push-forward homomorphism $H^i_c(U; \mathbb{Z}) \rightarrow H^i_c(X; \mathbb{Z})$ is denoted $\tau^i_{XU}$, the standard restriction homomorphism $H^i_c(X; \mathbb{Z}) \rightarrow H^i_c(U; \mathbb{Z})$ is denoted $\iota^i_{UX}$ and the standard boundary homomorphism $H^i_c(A; \mathbb{Z}) \rightarrow H^{i+1}_c(X \setminus A; \mathbb{Z})$ is denoted $\partial^i_{(X/A)A}$ for all $i \in \mathbb{N}$.

We recall that the exact sequence of the pair $(X, A)$ is a long exact sequence

$$\rightarrow H^i_c(X \setminus A; \mathbb{Z}) \rightarrow H^i_c(X; \mathbb{Z}) \rightarrow H^i_c(A; \mathbb{Z}) \rightarrow H^{i+1}_c(X \setminus A; \mathbb{Z}) \rightarrow$$

where all the homomorphisms are the standard ones. We also recall that $\tau^i_{UX} = \tau^i_{XU} \circ \tau^i_{UV}$ for all open subsets $V \subset U$ and $i \in \mathbb{N}$.

The cohomological dimension of a locally compact and locally connected Hausdorff space $X$ is $\leq n$, if $H^{n+1}_c(U; \mathbb{Z}) = 0$ for all open subsets $U \subset X$.

**Theorem 3.1.** Let $f: X \rightarrow Y$ be a proper branched covering between $n$-manifolds. Then the monodromy space $X_f$ of $f$ has dimension $\leq n$.

**Proof.** Let $B_f \subset X_f$ be the branch set of the normalization map $\overline{f}: X_f \rightarrow Y$ of $f$. Let $U \subset X_f$ be a connected open subset and $B_f \mid U$ the branch set of $\bar{f} \mid U$. The cohomological dimension of $B_f \mid U$ is at most $n - 2$ by [5], since $B_f \mid U$ does not locally separate $U$. Thus $H^i_c(B_f \mid U; \mathbb{Z}) = 0$ for $i > n - 2$ and the part

$$\rightarrow H^{i-1}_c(B_f \mid U; \mathbb{Z}) \rightarrow H^i_c(U \setminus B_f \mid U; \mathbb{Z}) \rightarrow H^i_c(U; \mathbb{Z}) \rightarrow H^i_c(B_f \mid U; \mathbb{Z}) \rightarrow$$

of the long exact sequence of the pair $(U, B_f \mid U)$ gives us an isomorphism $H^i_c(U \setminus B_f \mid U; \mathbb{Z}) \rightarrow H^i_c(U; \mathbb{Z})$ for $i \geq n$. Since $U \setminus B_f \mid U$ is a connected $n$-manifold, $H^{n+1}_c(U \setminus B_f \mid U; \mathbb{Z}) = 0$. Thus $H^{n+1}_c(U; \mathbb{Z}) \cong H^{n+1}_c(U \setminus B_f \mid U; \mathbb{Z}) = 0$. We conclude that $X_f$ has dimension $\leq n$. \hfill $\square$

The $i$:th local Betti-number $\rho^i(x)$ around $x$ is $k$, if given a neighbourhood $U$ of $x$, there exists open neighbourhoods $W \subset V \subset U$ with $W \subset V$ and $\overline{V} \subset U$ so that $\text{Im}(\tau^i_{XW}) = \text{Im}(\tau^i_{XV})$ and has rank $k$. The space $X$ is called a Wilder manifold, if $X$ is finite dimensional and for all $x \in X$ the local Betti-numbers satisfy $\rho^i(x) = 0$ for all $i < n$ and $\rho^n(x) = 1$.

A locally compact and locally connected Hausdorff space $X$ with cohomological dimension $\leq n$ is orientable, if there exists for every $x \in X$ a neighbourhood basis $\mathcal{U}$ of $x$ so that $\text{Im}(\tau^n_{XU}) = \mathbb{Z}$ for all $U \in \mathcal{U}$, and locally orientable if every point in $X$ has an orientable neighbourhood.

**Theorem 3.2.** Let $\bar{f}: X_f \rightarrow Y$ be a normalization map of a branched covering $f: X \rightarrow Y$ so that $Y$ is orientable. Then $X_f$ is orientable.
the push-forward homomorphism connected open subset and cohomology manifold proper branched covering between manifolds is always locally orientable. A see [15], p.255, the space branched cover and [8]. We first present the proof of this fact in the case we use it for completion to a branched covering between surfaces is always a surface as mentioned in an isomorphism and \( \tau \) cohomological dimension of \( x \) isomorphism for every \( x \in X_f \) and normal neighbourhood \( W \) of the pairs \( (X_f, B_f) \) and \( (W, B_f \cap W) \), the push-forward homomorphisms \( \tau^n_{X_f(X_f \setminus B_f)} \) and \( \tau^n_W(W \setminus B_f) \) are isomorphisms. Since \( W \setminus B_f \subset X_f \setminus B_f \) is a connected open subset and \( X_f \setminus B_f \) is a connected orientable manifold, the push-forward homomorphism \( \tau_{(X_f \setminus B_f)((W \setminus B_f))} \) is an isomorphism. Since \( \tau^n_{X_fW} \circ \tau^n_{W(W \setminus B_f)} = \tau^n_{X_f(X_f \setminus B_f)} \circ \tau^n_{(X_f \setminus B_f)(W \setminus B_f)}, \) we conclude that \( \tau^n_{X_fW} \) is an isomorphism and \( \text{Im}(\tau^n_{X_fW}) = \text{Im}(\tau^n_{X_fX'_f}) \cong \mathbb{Z}. \)

We note that a similar argument shows that a monodromy space of a proper branched covering between manifolds is always locally orientable. A cohomology manifold in the sense of Borel [5] is a locally orientable Wilder manifold.

4. THE MONODROMY SPACE OF BRANCHED COVERS BETWEEN SURFACES

A surface is a closed orientable 2-manifold. The monodromy space related to a branched covering between surfaces is always a surface as mentioned in [8]. We first present the proof of this fact in the case we use it for completion of presentation and then we show that there exists a branched cover \( f: S^2 \to S^2 \) so that \( X_f \neq S^2 \) towards proving Theorems 1.1 and 5.1.

**Lemma 4.1.** Let \( F \) be an orientable surface and \( f: F \to S^2 \) be a proper branched cover and \( \tilde{f}: X_f \to S^2 \) the normalization of \( f \). Then \( X_f \) is an orientable surface.

**Proof.** Since \( S^2 \) is orientable, the space \( X_f \) is orientable by Theorem 3.2. Since the domain \( F \) of \( f \) is compact, the normalization \( \tilde{f} \) has finite multiplicity and the space \( X_f \) is compact. Let \( x \in X_f \). By Stoilow’s theorem, see [15], \( f(B_f) = \tilde{f}(B_f) \) is a discrete set of points. Thus there exists a normal neighbourhood \( V \subset X_f \) of \( x \) so that \( \tilde{f}(V) \cap f(B_f) \subset \{ f(x) \} \) and \( \tilde{f}(V) \approx \mathbb{R}^2 \). Now \( \tilde{f}(V \setminus \{ x \} : V \setminus \{ x \} \to \tilde{f}(V \setminus \{ f(x) \}) \) is a cyclic covering of finite multiplicity, since \( \tilde{f}(V \setminus \{ f(x) \}) \) is homeomorphic to the complement of a point in \( \mathbb{R}^2 \). We conclude from this that \( x \) is a manifold point of \( X_f \). Thus \( X_f \) is a 2-manifold and a surface.

We record as a theorem the following result in the spirit of Fox [8, p.255].

**Theorem 4.2.** Let \( F \) be an orientable surface and \( f: F \to S^2 \) a proper branched cover and \( \tilde{f}: X_f \to S^2 \) the normalization of \( f \). Assume \( |fB_f| > 3 \). Then \( X_f \neq S^2 \).

**Proof.** The space \( X_f \) is \( S^2 \) if and only if the Euler characteristic \( \chi(X_f) \) is 2. By Riemann Hurwitz formula

\[
\chi(X_f) = (\deg \tilde{f})\chi(S^2) - \sum_{x \in X_f} (i(x, \tilde{f}) - 1),
\]
where \( i(x, f) \) is the local index of \( f \) at \( x \). Since \( f \) is a normal branched cover, \( i(x', f) = i(x, f) \) for \( x, x' \in X_f \) with \( f(x) = f(x') \). We define for all \( y \in S^2 \),

\[
n(y) := i(x, f), x \in \overline{f}^{-1}\{y\}.
\]

Then for all \( y \in S^2 \)

\[
\deg f = \sum_{x \in \overline{f}^{-1}\{y\}} i(x, f) = n(y)|\overline{f}^{-1}\{y\}|
\]

and thus for all \( y \in S^2 \)

\[
|\overline{f}^{-1}\{y\}| = \frac{\deg f}{n(y)}
\]

Hence,

\[
\chi(X_f) = (\deg f) \left( \chi(S^2) - \sum_{y \in fB_f} \frac{n(y) - 1}{n(y)} \right),
\]

where \( \chi(S^2) = 2 \) and \( \deg f := N \in \mathbb{N} \). Since \( n(y) \geq 2 \) for all \( y \in fB_f = fB_f \) and \( \frac{k - 1}{k} \to 1 \) as \( k \to \infty \), we get the estimate

\[
\chi(X_f) \leq N \left( 2 - \frac{|fB_f|}{2} \right).
\]

Thus \( \chi(X_f) \leq 0 < 2 \), since \( |fB_f| \geq 4 \) by assumption. Thus \( X_f \neq S^2 \). □

We end this section with two independent easy corollaries.

**Corollary 4.3.** Let \( F \) be an orientable surface and \( f: F \to S^2 \) be a proper branched cover so that \( |fB_f| > 3 \). Then \( f \) is not a normal covering.

**Corollary 4.4.** Let \( F \) be an orientable surface and \( f: F \to S^2 \) be a proper branched cover so that \( |fB_f| < 3 \). Then \( f \) is a normal covering.

**Proof.** Since the first fundamental group of \( S^2 \setminus fB_f \) is cyclic, the monodromy group of \( f|F \setminus f^{-1}(fB_f): F \setminus f^{-1}(fB_f) \to S^2 \setminus fB_f \) is cyclic. Thus every subgroup of the deck-transformation group of the normalization map \( \tilde{f}: X_f \to X \) is a normal subgroup. Thus \( \bar{f} = f \) and in particular, \( f \) is a normal branched covering. □

5. **The suspension of a branched cover between orientable surfaces**

In this section we prove Theorem 1.1 in the introduction and the following theorem.

**Theorem 5.1.** There exists a simplicial branched cover \( f: S^3 \to S^3 \) for which the monodromy space \( X_f \) is not a cohomology manifold.

More precisely, we show that there exists a branched cover \( S^2 \to S^2 \) for which the monodromy space is not a manifold or a cohomology manifold for the suspension map \( \Sigma S^2 \to \Sigma S^2 \).

Let \( F \) be an orientable surface. Let \( \sim \) be the equivalence relation in \( F \times [-1, 1] \) defined by the relation \( (x, t) \sim (x', t) \) for \( x, x' \in F \) and \( t \in \{-1, 1\} \). Then the quotient space \( \Sigma F := F \times [-1, 1]/ \sim \) is the suspension space of \( F \).
and the subset $CF := \{(x,t) : x \in F, t \in [0,1]\} \subset \Sigma F$ is the cone of $F$. We note that $\Sigma S^2 \simeq S^3$. Let $f : F_1 \to F_2$ be a piecewise linear branched cover between surfaces. Then $\Sigma f : \Sigma F_1 \to \Sigma F_2, (x,t) \mapsto (f(x), t)$, is a piecewise linear branched cover and called the suspension map of $f$. We note that the suspension space $\Sigma F$ is a polyhedron and locally contractible for all surfaces $F$.

We begin this section with a lemma showing that the normalization of a suspension map of a branched cover between surfaces is completely determined by the normalization of the original map.

**Lemma 5.2.** Let $F$ be an orientable surface and $f : F \to S^2$ a branched cover and $\tilde{f} : X_f \to S^2$ the normalization of $f$. Then $\Sigma \tilde{f} : \Sigma X_f \to \Sigma S^2$ is the normalization of $\Sigma f : \Sigma F \to \Sigma S^2$.

**Proof.** Let $p : X_f \to F$ be a normal branched covering so that $\tilde{f} = f \circ p$. Then $\Sigma \tilde{f}$ is a normal branched cover so that $\Sigma \tilde{f} = \Sigma p \circ \Sigma f$ and $\varphi : \text{Deck}(\Sigma \tilde{f}) \to \text{Deck}(f) ; \tau \mapsto \tau |_{X_f}$ is an isomorphism. We need to show that, if $G \subset \text{Deck}(\Sigma \tilde{f})$ is a subgroup so that $f \circ (\Sigma p/G)$ is normal. Then $G$ is trivial.

Suppose $G \subset \text{Deck}(\Sigma \tilde{f})$ is a group so that $f \circ \Sigma/G$ is normal. Then $f \circ (p/G')$ is normal for the quotient map $p/G' : X_f/G' \to S^2$, where $G' = \varphi(G)$. Since $\tilde{f}$ is the normalization of $f$, the group $G'$ is trivial. Thus $G = \varphi^{-1}(G')$ is trivial, since $\varphi^{-1}$ is an isomorphism. \qed

We then characterize the surfaces for which the suspension space is a manifold or a cohomology manifold in the sense of Borel.

**Lemma 5.3.** Let $F$ be an orientable surface. Then $\Sigma F$ is a manifold if and only if $F = S^2$.

**Proof.** Suppose $F = S^2$. Then $\Sigma F \simeq S^3$. Suppose then that $F \neq S^2$. Then there exists a (cone) point $x \in \Sigma F$ and a contractible neighbourhood $V \subset \Sigma F$ of $x$ so that $F \subset V$ and $V \setminus \{x\}$ contracts to $F$. Now $\pi_1(V \setminus \{x\}, x_0) \cong \pi_1(F, x_0) \neq 0$ for $x_0 \in F$. Suppose that $\Sigma F$ is a 3-manifold. Then $\pi_1(V \setminus \{x\}, x_0) = \pi_1(F, x_0) = 0$, which is a contradiction. Thus $\Sigma F$ is not a manifold. \qed

**Lemma 5.4.** Let $F$ be an orientable surface. Then $\Sigma F$ is not a Wilder manifold if $F \neq S^2$.

**Proof.** We show that then the second local Betti-number is non-trivial around a point in $\Sigma F$. Let $CF \subset \Sigma F$ be the cone of $F$. Let $\pi : F \times [0,1] \to \Sigma F, (x,t) \mapsto (x, t)$, be the quotient map to the suspension space and $\bar{x} = \pi(F \times \{1\})$. We first note that $H^1_c(CF, \mathbb{Z}) = H^2_c(CF, \mathbb{Z}) = 0$, since $CF$ contracts properly to a point. Further, by Poincare duality $H^4_c(F, \mathbb{Z}) = \mathbb{Z}^{2g}$, where $g$ is the genus of $F$. In the exact sequence of the pair $(CF, F)$ we have the short exact sequence

$$\to 0 \to H^2_c(CF \setminus F; \mathbb{Z}) \to H^1_c(F; \mathbb{Z}) \to 0.$$ 

Thus $H^2_c(CF \setminus F; \mathbb{Z}) \cong H^1_c(F; \mathbb{Z})$ and $H^2_c(CF \setminus F; \mathbb{Z}) = 0$ if and only if $g = 0$ for the genus $g$ of $F$. Thus $H^2_c(CF \setminus F; \mathbb{Z}) = 0$ if and only if $F = S^2$.

We then show that the rank of $H^2_c(CF \setminus F; \mathbb{Z})$ is the local Betti-number $\rho^2(\bar{x})$ around $\bar{x}$. For this it is sufficient to show that given any neighbourhood
$U \subset CF$ of $\tilde{x}$, there exists open neighbourhoods $W \subset V \subset U$ of $\tilde{x}$ with $V, \tilde{V} \subset U$, so that for any open neighbourhood $W' \subset W$ of $\tilde{x}$, $\text{Im}(\tau_{W'}) = \text{Im}(\tau_{W}) \equiv H_z^2(CF \setminus F; \mathbb{Z})$.

Denote $\Omega_t = \varphi(F \times [0, t])$ for all $t \in (0, 1)$. We note that then $\tau_{\Omega_t}$ is an isomorphism for all $t, s \in \mathbb{R}, t < s$, since $\iota: \Omega_t \to \Omega_s$ is properly homotopic to the identity. Let $U \subset CF$ be any neighbourhood of $\tilde{x}$. We set $V = \Omega_t$ for such $t \in (0, 1)$ that $\Omega_t \subset U$ and $W = \Omega_t/2$. Then for any neighbourhood $W' \subset \Omega_t$ of $\tilde{x}$ there exists $t' \in (0, t/2)$ so that $\Omega_{t'} \subset W'$. Now $\tau_{\Omega_{2t'}}$ is surjective, since $\tau_{\Omega_{t'}} = \tau_{\Omega_{W'}} \circ \tau_{W'\Omega_{t'}}$ is an isomorphisms. Thus

$$\text{Im}(\tau_{\Omega_{t'}}) = \text{Im}(\tau_{\Omega_{t'}}) = H_z^2(\Omega_t; \mathbb{Z}) \equiv H_z^2(CF \setminus F; \mathbb{Z}),$$

since $\tau_{\Omega_{t'}}$ and $\tau_{\Omega_{t'}}(CF/F)$ are isomorphisms. Thus $\rho^2(\tilde{x}) \neq 0$, since $F \neq S^2$. 

**Corollary 5.5.** Let $f: S^2 \to S^2$ be a branched cover with $|f| > 3$, $\Sigma f: S^3 \to S^3$ the suspension map of $f$ and $\Sigma f: X_{\Sigma f} \to S^3$ the normalization of $\Sigma f$. Then $X_{\Sigma f}$ is not a manifold and not a Wilder manifold.

**Proof.** By Lemmas 4.1 and 5.2 we know that $X_f$ is a surface and $X_{\Sigma f} = \Sigma X_f$.

Further, by Lemma 4.2 we know that $X_f \not\subset S^2$, since $|f| > 3$. Thus, by Lemma 5.3 $X_f$ is not a manifold. Further, by Lemma 5.4 $X_{\Sigma f}$ is not a Wilder manifold. Thus $X_f$ is not a cohomology manifold in the sense of Borel. \hfill \lceil \\

**Proof of Theorems 1.1 and 5.1.** By Corollary 5.5 it is sufficient to show that there exists a branched cover $f: S^2 \to S^2$ so that $|f(B_f)| = 4$. Such a branched cover we may easily construct as $f = f_1 \circ f_2$, where $f_1: S^2 \to S^2$ is a winding map with branch points $x_i^1$ and $x_i^2$ for $i \in \{1, 2\}$ satisfying $x_1^1, x_2^1 \not\in \{f_1(x_1^1), f_1(x_2^1)\}$ and $f_2(f_1(x_1^1)) \neq f_2(f_1(x_1^2))$. \hfill \lceil \\

6. AN EXAMPLE OF A NON-LOCALLY CONTRACTIBLE MONODROMY SPACE

In this section we introduce an example of a branched cover $S^3 \to S^3$ for which the related monodromy space is not a locally contractible space. The construction of the example is inspired by Heinones and Rickmans construction in [9] of a branched covering $S^3 \to S^3$ containing a wild Cantor set in the branch set. We need the following result originally due to Berstein and Edmonds [4] in the extent we use it.

**Theorem 6.1** ([12], Theorem 3.1). Let $W$ be a connected, compact, oriented piecewise linear 3-manifold whose boundary consists of $p \geq 2$ components $M_0, \ldots, M_{p-1}$ with the induced orientation. Let $W' = N \setminus \text{int}B_j$ be an oriented piecewise linear 3-sphere $N$ in $\mathbb{R}^4$ with $p$ disjoint, closed, polyhedral 3-balls removed, and have the induced orientation on the boundary. Suppose that $n \geq 3$ and $\varphi_j: M_j \to \partial B_j$ is a sense preserving piecewise linear branched cover of degree $n$, for each $j = 0, 1, \ldots, p - 1$. Then there exists a sense preserving piecewise linear branched cover $\varphi: W \to W'$ of degree $n$ that extends $\varphi_j$.

Let $x \in S^3$ be a point in the domain and $y \in S^3$ a point in the target. Let $X \subset S^3$ be a closed piecewise linear ball with center $x$ and let $Y \subset S^3$
be a closed piecewise linear ball with center \( y \). Let \( T_0 \subset \text{int} X \) be a solid piecewise linear torus so that \( x \in \text{int} T_0 \). Now let \( T = (T_n)_{n \in \mathbb{N}} \) be a sequence of solid piecewise linear tori in \( T_0 \) so that \( T_{k+1} \subset \text{int} T_k \) for all \( k \in \mathbb{N} \) and \( \bigcap_{n=1}^{\infty} T_n = \{ x \} \). Let further \( B_0 \subset \text{int} Y \) be a closed piecewise linear ball with center \( y \) and let \( B = (B_n)_{n \in \mathbb{N}} \) be a sequence of closed piecewise linear balls with center \( y \) so that \( B_{k+1} \subset \text{int} B_k \) for all \( k \in \mathbb{N} \) and \( \bigcap_{n=1}^{\infty} B_n = \{ y \} \). See illustration in Figure 1.

We denote \( \partial X = \partial T_{-1} \) and \( \partial Y = \partial B_{-1} \) and choose an orientation to all boundary surfaces from an outward normal. Let \( f_n : \partial T_n \to \partial B_n, n \in \{-1\} \cup \mathbb{N} \), be a collection of sense preserving piecewise linear branched coverings so that

(i) the degree of all the maps in the collection are the same and greater than 2,
(ii) \( f_{-1} \) has an extension to a branched covering \( g : S^3 \setminus \text{int} X \to S^3 \setminus \text{int} Y \),
(iii) the maps \( f_n \) are for all even \( n \in \mathbb{N} \) normal branched covers with no points of local degree three and
(iv) the branched covers \( f_n \) have for all uneven \( n \in \mathbb{N} \) a point of local degree three.

We note that for an example of such a collection of maps of degree 18, we may let \( f_{-1} \) be a 18-to-1 winding map, \( f_n \) be for even \( n \in \mathbb{N} \) as illustrated in Figure 2 and \( f_n \) be for all uneven \( n \in \mathbb{N} \) as illustrated in Figure 3.

Let then \( n \in \mathbb{N} \) and let \( F_n \subset X \) be the compact piecewise linear manifold with boundary \( \partial T_{n-1} \cup \partial T_n \) that is the closure of a component of \( X \setminus (\bigcup_{k=-1}^{\infty} \partial T_k) \). Let further, \( G_n \subset Y \) be the compact piecewise linear manifold with boundary \( \partial B_{n-1} \cup \partial B_n \) that is the closure of a component of \( Y \setminus (\bigcup_{k=-1}^{\infty} \partial B_k) \). Then \( F_n \subset X \) is a compact piecewise linear manifold with two boundary components and \( G_n \subset Y \) is the complement of the interior of two distinct piecewise linear balls in \( S^3 \). Further, \( f_{n-1} : \partial T_{n-1} \to \partial B_{n-1} \). 

\[ X \]

\[ Y \]

**Figure 1.**
and \(f_n: \partial T_n \to \partial B_n\) are sense preserving piecewise linear branched covers between the boundary components of \(F_n\) and \(G_n\). Since the degree of \(f_n\) is the same as the degree of \(f_n - 1\) and the degree is greater than 2, there exists by 6.1 a piecewise linear branched cover \(g_n: F_n \to G_n\) so that \(g_n|_{\partial T_n} = f_n - 1\) and \(g_n|_{\partial T_n} = f_n\).

Now \(X = \bigcup_{k=0}^{\infty} F_n\) and \(Y = \bigcup_{k=0}^{\infty} G_n\) and \(g: S^3\setminus \text{int} X \to S^3\setminus \text{int} Y\) satisfies \(g|_{\partial X} = g_0|_{\partial X}\). Hence we may define a branched covering \(f: S^3 \to S^3\) by setting \(f(x) = g_n(x)\) for \(x \in G_n, n \in \mathbb{N}\), and \(f(x) = g(x)\) otherwise.

However, we want the map \(f: S^3 \to S^3\) to satisfy one more technical condition, namely the existence of collections of properly disjoint open sets \((M_k)_{k \in \mathbb{N}}\) of \(X\) and \((N_k)_{k \in \mathbb{N}}\) of \(Y\) so that \(M_k \subset X\) is a piecewise linear regular neighbourhood of \(\partial T_k\) and \(N_k \subset Y\) is a piecewise linear regular neighbourhood of \(\partial B_k\) and \(M_k = f^{-1}N_k\), and \(f|M_k: M_k \to N_k\) has a product structure of \(f_k\) and the identity map for all \(k \in \mathbb{N}\). We may require this to hold for the \(f: S^3 \to S^3\) defined, since in other case we may by cutting \(S^3\) along the boundary surfaces of \(\partial T_k\) and \(\partial B_k\) and adding regular neighbourhoods \(M_k\) of \(\partial T_k\) and \(N_k\) of \(\partial B_k\) in between for all \(k \in \mathbb{N}\) arrange this to hold without loss of conditions (i)–(iv), see [13].

In the last section of this paper we prove the following theorem.

**Theorem 6.2.** Let \(f: S^3 \to S^3\) and \(y \in S^3\) be as above and \(\overline{f}: X_f \to Y\) the normalization of \(f\). Then \(H_1(W; \mathbb{Z}) \neq 0\) for all open sets \(W \subset X_f\) satisfying \(\overline{f}^{-1}\{y\} \cap W \neq \emptyset\).

Theorem 1.1 in the introduction then follows from Theorem 6.2 by the following easy corollary.
Corollary 6.3. Let \( f : S^3 \to S^3 \) and \( y \in S^3 \) be as above. Then the monodromy space \( X_f \) of \( f \) is not locally contractible.

Proof. Let \( x \in \tilde{f}^{-1}(y) \) and \( W \) a neighbourhood of \( x \). Then \( H_1(W;\mathbb{Z}) \neq 0 \) and \( W \) has non-trivial fundamental group by Hurewicz Theorem, see [10]. Thus \( W \) is not contractible. Thus the monodromy space \( X_f \) of \( f \) is not a locally contractible space. \( \square \)

7. Destructive points

In this section we define destructive points and prove Theorem 6.2.

Let \( X \) be a locally connected Hausdorff space. We call an open and connected subset \( V \subset X \) a domain. Let \( V \subset X \) be a domain. A pair \( \{A, B\} \) is called a domain covering of \( V \), if \( A, B \subset X \) are domains and \( V = A \cup B \). We say that a domain covering \( \{A, B\} \) of \( V \) is strong, if \( A \cap B \) is connected. Let \( x \in V \) and let \( U \subset V \) be a neighbourhood of \( x \). Then we say that \( \{A, B\} \) is \( U \)-small at \( x \), if \( x \in A \subset U \) or \( x \in B \subset U \).

Let then \( f : X \to Y \) be a branched covering between manifolds, \( y \in Y \) and \( V_0 \subset Y \) a domain containing \( y \). Then \( V_0 \) is a destructive neighbourhood of \( y \) with respect to \( f \), if \( f|f^{-1}(V_0) \) is not a normal covering to its image, but there exists for every neighbourhood \( U \subset V_0 \) of \( y \) a \( U \)-small strong domain covering \( \{A, B\} \) of \( V_0 \) at \( y \) so that \( \{f^{-1}(A), f^{-1}(B)\} \) is a strong domain cover of \( f^{-1}(V_0) \) and \( f|(f^{-1}(A) \cap f^{-1}(B)) \) is a normal covering to its image.

We say that \( y \) is a destructive point of \( f \), if \( y \) has a neighbourhood basis consisting of neighbourhoods that are destructive with respect to \( f \).

Theorem 7.1. The map \( S^3 \to S^3 \) of the example in section 6 has a destructive point.

Proof. We show that \( y \in \bigcap_{n=1}^{\infty} B_n \) is a destructive point of \( f \). We first show that \( V_0 = \text{int} B_0 \) is a destructive neighbourhood of \( y \).

We begin this by showing that \( g := f|f^{-1}(V_0) : f^{-1}(V_0) \to V_0 \) is not a normal branched cover. Towards contradiction suppose that \( g \) is a normal branched cover. Then Deck(\( g \)) \( \cong \text{Deck}(g|M_1) \) and Deck(\( g \)) \( \cong \text{Deck}(g|M_2) \), since \( M_1 = f^{-1}(N_1) \) and \( M_2 = f^{-1}(N_2) \) are connected. On the other hand \( (iii) \) and \( (iv) \) imply that Deck(\( g|M_1 \)) \( \cong \) Deck(\( g|M_2 \)) and we have a contradiction.

Let then \( V_1 \subset V_0 \) be any open connected neighbourhood of \( y \). Then there exists such \( k \in \mathbb{N} \), that \( B_{2k} \cup N_{2k} \subset V_1 \). Let \( B := B_{2k} \cup N_{2k} \) and \( A = (V_0 \setminus B_{2k}) \cup N_{2k} \). Then \( \{A, B\} \) is a strong \( V_1 \)-small domain cover of \( V_0 \) at \( y \) and \( A \cap B = N_{2k} \). In particular, \( \{f^{-1}(A), f^{-1}(B)\} \) is a strong domain cover of \( f^{-1}(V_0) \). Further,

\[
 f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(N_{2k}) = M_{2k}
\]

and \( f|(f^{-1}(A) \cap f^{-1}(B)) = f|M_{2k} \to M_{2k} \) is a normal branched covering by \( (iii) \). Thus \( V_0 \) is a destructive neighbourhood of \( y \). The same argument shows that \( V_k := \text{int} B_k \) is a destructive neighbourhood of \( y \) for all \( k \in \mathbb{N} \). Thus \( y \) has a neighbourhood basis consisting of neighbourhoods that are destructive with respect to \( f \). \( \square \)

Theorem 7.1 implies that Theorems 6.2 and 1.1 follow from the following result.
Theorem 7.2. Let $f: X \to Y$ be a proper branched covering between manifolds and let 
\[
\begin{array}{c}
X_f \\
\downarrow^{p} \quad \downarrow^{f} \\
X \\
\downarrow^{f} \\
Y
\end{array}
\]
be a commutative diagram of branched coverings so that $X_f$ is a connected, locally connected Hausdorff space and $p: X_f \to X$ and $q: X_f \to Y$ are proper normal branched coverings. Suppose there exists a destructive point $y \in Y$ of $f$. Then $H_1(W; \mathbb{Z}) \neq 0$ for all open sets $W \subset X_f$ satisfying
\[
\overline{f}^{-1}\{y\} \cap W \neq \emptyset.
\]

We begin the proof of Theorem 7.2 with two lemmas. The following observation is well known for experts.

Lemma 7.3. Let $X$ be a locally connected Hausdorff space and $W \subset X$ an open and connected subset. Suppose there exists open and connected subsets $U, V \subset W$ so that $W = U \cup V$ and $U \cap V$ is not connected. Then the first homology group $H_1(W; \mathbb{Z})$ is not trivial.

Proof. Towards contradiction we suppose that $H_1(W; \mathbb{Z}) = 0$. Then the reduced Mayer-Vietoris sequence
\[
\to H_1(W; \mathbb{Z}) \to \tilde{H}_0(U \cap V; \mathbb{Z}) \to \tilde{H}_0(U; \mathbb{Z}) \oplus \tilde{H}_0(V; \mathbb{Z}) \to \tilde{H}_0(W; \mathbb{Z})
\]
takes the form
\[
0 \to \tilde{H}_0(U \cap V; \mathbb{Z}) \to 0.
\]
Thus, $\tilde{H}_0(U \cap V; \mathbb{Z}) = 0$. Thus $U \cap V$ is connected, which is a contradiction. Thus, $H_1(W; \mathbb{Z})$ is not trivial. $\square$

The following lemma is the key observation in the proof of Theorem 7.2.

Lemma 7.4. Suppose $f: X \to Y$ is a branched covering between manifolds. Suppose $W$ is a connected locally connected Hausdorff space and $p: W \to X$ and $q: W \to Y$ are normal branched coverings so that the diagram
\[
\begin{array}{c}
W \\
\downarrow^{p} \quad \downarrow^{q} \\
X \\
\downarrow^{f} \\
Y
\end{array}
\]
commutes. Suppose there exists an open and connected subset $C_1 \subset Y$ so that $D_1 = f^{-1}(C_1)$ is connected and $f|D_1: D_1 \to C_1$ is a normal branched covering. Then $f$ is a normal branched covering, if $E_1 = q^{-1}(C_1)$ is connected.

Proof. Since $E_1 := q^{-1}(C_1)$ is connected, we have
\[
\text{Deck}(q) = \{ \tau \in \text{Deck}(q) : \tau(E_1) = E_1 \} \cong \text{Deck}(q|E_1: E_1 \to C_1)
\]
and
\[
\text{Deck}(p) = \{ \tau \in \text{Deck}(p) : \tau(E_1) = E_1 \} \cong \text{Deck}(p|E_1: E_1 \to D_1),
\]
where the isomorphisms are canonical in the sense that they map every deck-homomorphism $\tau: W \to W$ to the restriction $\tau|E_1: E_1 \to E_1$. 

Since \( f|D_1 : D_1 \to C_1 \) is a normal branched covering,
\[
\text{Deck}(p|E_1 : E_1 \to D_1) \subset \text{Deck}(q|E_1 : E_1 \to C_1)
\]
is a normal subgroup. Hence, \( \text{Deck}(p) \subset \text{Deck}(q) \) is a normal subgroup. Hence, the branched covering \( f : X \to Y \) is normal.

**Proof of Theorem 7.2.** Let \( W \subset X_f \) be a open set and \( y \in f(W) \) a destructive point and \( x \in f^{-1}(y) \). By Lemma 7.3, to show that there exists a domain cover of \( W \) that is not strong.

Let \( V_0 \) be a destructive neighbourhood of \( y \) so that the \( x \)-component \( W_0 \) of is a normal neighbourhood of \( x \) in \( W \). Let \( \{A, B\} \) be a strong domain cover of \( V_0 \) so that \( y \in B \subset V_0 \) and \( \{W_A^0, W_B^0\} \) is a domain cover of \( W_0 \) for \( W_A^0 := (f|W_0)^{-1}(A) \) and \( W_B^0 := (f|W_0)^{-1}(B) \), (see Lemma 2.2).

We first show that \( \{W_A^0, W_B^0\} \) is not strong. Suppose towards contradiction that \( \{W_A^0, W_B^0\} \) is strong. Then \( A \cap B, f^{-1}(A) \cap f^{-1}(B) \) and \( W_A^0 \cap W_B^0 \) are connected and

\[
\begin{align*}
\text{Deck}(\overline{A} \cap \overline{B}) &\quad \text{factors} & \text{Deck}(\overline{A} \cap \overline{B}) \quad \text{is a commutative diagram of branched covers. In particular, since } f|f^{-1}(A) \cap f^{-1}(B) \text{ is a normal branched cover } \text{Deck}(p|W_A^0 \cap W_B^0) \subset \text{Deck}(f|W_A^0 \cap W_B^0) \text{ is a normal subgroup. On the other hand, since }\end{align*}
\]

\[
W_A^0 \cap W_B^0 = (f|W_0)^{-1}(A \cap B) = (p|W_0)^{-1}(f^{-1}(A) \cap f^{-1}(B)) \subset W_0
\]
is connected, \( \text{Deck}(p|W_A^0 \cap W_B^0) \cong \text{Deck}(p|W_0), \text{Deck}(f|W_A^0 \cap W_B^0) \cong \text{Deck}(f|W_0) \) and in particular, \( \text{Deck}(p|W_0) \subset \text{Deck}(f|W_0) \) is a normal subgroup. Thus the factor \( f|f^{-1}(V_0) : f^{-1}(f(V_0)) \to V_0 \) of \( f|W_0 : W_0 \to V_0 \) is a normal branched covering. This is a contradiction, since \( V_0 \) is a destructive neighbourhood of \( y \) and we conclude that \( W_A^0 \cap W_B^0 \subset W_0 \) is not connected.

Since \( B \subset V_0 \) there exists a connected neighbourhood \( W' \subset W \) of \( W \setminus W_B^0 \) so that \( W' \cap W_B^0 = \emptyset \). Now \( W_A^0 \cup W' \) is connected, since \( W_A^0 \) is connected and every component of \( W' \) has a non-empty intersection with \( W_A^0 \). Further, \( W = W_B^0 \cup (W_A^0 \cup W') \) and \( W_B^0 \cap (W_A^0 \cup W') = W_A^0 \cap W_B^0 \). Thus \( \{W_B^0, W_A^0 \cup W'\} \) is a domain cover of \( W \) that is not strong and by Lemma 7.3 we conclude \( H_1(W; \mathbb{Z}) \neq 0 \).

This concludes the proof of Theorem 6.2 and further by Corollary 6.3 the proof of Theorem 1.2 in the introduction.

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