Entropy of eternal black holes

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Abstract

The entropy of a quantum-statistical system which is classically approximated by a general stationary eternal black hole is studied by means of a microcanonical functional integral. This approach opens the possibility of including explicitly the internal degrees of freedom of a physical black hole in path integral descriptions of its thermodynamical properties. If the functional integral is interpreted as the density of states of the system, the corresponding entropy equals \( \mathcal{S} = A_H/4 - A_H/4 = 0 \) in the semiclassical approximation, where \( A_H \) is the area of the black hole horizon. The functional integral reflects the properties of a pure state.

1 Introduction

The relationship between the entropy of a physical black hole and its internal degrees of freedom remains a subject of active research. A natural question to ask in this regard is: can these degrees of freedom be effectively included in a functional integral description of black hole entropy? In an attempt to give an affirmative answer to this question, we investigate in this paper a microcanonical functional integral when applied to a quantum self-gravitating statistical system that includes spacetimes whose topology and boundary conditions coincide with the ones of (either distorted or Kerr-Newman) eternal black holes.

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A proposal for the density of states of a gravitational system defined in terms of a microcanonical functional integral has been suggested recently in Ref. [1]. This integral is defined as a formal sum over Lorentzian geometries. The black hole density of states is obtained from this functional integral when the latter is approximated semiclassically by using a complex metric whose boundary data at its single boundary surface coincide with the boundary data of a Lorentzian, stationary black hole. The density of states defined accordingly equals the exponential of one fourth of the area of the black hole horizon. This proposal opens the possibility of determining the thermodynamical properties of black hole systems starting from a sum over real Lorentzian geometries. However, several problems remain in this approach. First, a spacelike hypersurface Σ that describes the initial data of a Lorentzian black hole has to cross necessarily the event horizon and eventually intersect the interior singularity. Second, a Lorentzian, stationary black hole is not a extremum of the microcanonical action for a spacetime region with a single timelike boundary surface. This implies that the black hole density of states whose boundary data correspond to the ones of a Lorentzian, stationary black hole cannot be approximated semiclassically by using the same Lorentzian metric that motivates its boundary conditions. It is therefore necessary to use complex metrics to evaluate the Lorentzian functional integral in a steepest descents approximation. This procedure yields the correct result for the entropy but conceals its origin: the interior of the Lorentzian black hole literally disappears by virtue of this procedure, leaving effectively only a periodically identified Euclidean version of the “right” wedge region of a Kruskal diagram. The properties of the black hole interior become encoded in a set of conditions at the so-called “bolt” of the complex geometry. In this approach the statistical origin of entropy and its relationship to the internal degrees of freedom of a black hole remain obscure.

The problems mentioned above and the role of internal degrees of freedom in functional integral descriptions of black hole thermodynamics can be addressed by explicitly considering the eternal version of a black hole [2]. The excitations of the physical black hole can be associated with the deformations of an initial global Cauchy surface Σ of the eternal black hole. In general, the spatial slices Σ that foliate an eternal black hole are (deformed) Einstein-Rosen bridges with wormhole topology $R^1 \times S^2$. The spacetime is composed of two wedges $M_+$ and $M_-$ located in the right and left sectors of a Kruskal diagram. Internal and external degrees of freedom of the black hole can be easily identified in this approach since the hypersurfaces Σ are
naturally divided in two parts $\Sigma_+$ and $\Sigma_-$ by a bifurcation two-surface $S_0$ where the lapse function $N$ vanishes. While the “external” degrees of freedom of the original black hole are naturally given by the initial data at $\Sigma_+$, its “internal” degrees of freedom can be identified with initial data defined at $\Sigma_-$. 

## 2 Microcanonical action and functional integral

Consider a spacetime region whose three-dimensional timelike boundary surface $B$ consists of two disconnected parts $B_+$ and $B_-$. The microcanonical action $S_m$ is the action appropriate to a variational principle in which the fixed boundary conditions at the timelike boundaries $B_\pm$ are not the spacetime three-geometry but the surface energy density $\varepsilon$, surface momentum density $j_\alpha$, and boundary two-metric $\sigma_{ab}$. The covariant form of the microcanonical action for a general spacetime whose timelike surfaces $B_+$ and $B_-$ are located in the regions $M_+$ and $M_-$ respectively has been presented in Ref. [2]. Its Hamiltonian form is easily obtained under a $3 + 1$ spacetime split by recognizing that there exists a direction of time at the boundaries $B_\pm$. By introducing the momentum $P_{ij}$ conjugate to the three-metric $h_{ij}$ of $\Sigma$ for the so-called “tilted” foliation [3] and integrating the kinetic part of the volume integral, the action becomes

$$S_m = \int_M d^4x \left[ P_{ij} \dot{h}_{ij} - N\mathcal{H} - V^i \mathcal{H}_i \right],$$

where the dot denotes differentiation with respect to time, $V^i$ denotes the shift vector, and the gravitational contributions to the Hamiltonian and momentum constraints are given by the usual expressions. Observe that the action vanishes identically for stationary solutions of the vacuum Einstein equations describing stationary eternal black holes. In this case $\dot{h}_{ij} = 0$, the constraint equations are satisfied, and no boundary terms remain in the action.

Consider now the microcanonical functional integral for a gravitational system whose timelike boundary surfaces $B_\pm$ are located in $M_\pm$. The path integral

$$\bar{\nu}[\varepsilon_+, j_+, \sigma_+; \varepsilon_-, j_-, \sigma_-] = \sum_M \mathcal{D}H \exp(iS_m)$$

is a functional of the energy density $\varepsilon_\pm$, momentum density $j_\pm$, and two-metric $\sigma_\pm$ at the boundaries $B_\pm$ and $B_-$. The sum over $M$ refers to a sum over manifolds of different topologies whose boundaries have topologies $B_+ \equiv S_+ \times S^1 = S^2 \times S^1$ and $B_- \equiv S_- \times S^1 = S^2 \times S^1$. The element $S^1$ is due to the periodic identification in the global time direction at the boundaries.
when the initial and final hypersurfaces are identified. The integral is a sum over periodic Lorentzian metrics that satisfy the boundary conditions at $B_+$ and $B_-$. The eternal black hole functional integral $\bar{\nu}^*$ is obtained when the boundary data $(\varepsilon_+, j_+, \sigma_+)$ and $(\varepsilon_-, j_-, \sigma_-)$ of the geometries summed over correspond to the boundary data of a general Lorentzian, stationary eternal black hole. The boundary data of this solution can be determined at $S_+$ and $S_-$ for each slice $\Sigma$. Observe that by virtue of the gravitational constraint equations, these data determine uniquely the size of the black hole horizon and are such that the two-metric is continuous at this horizon.

We evaluate now the functional integral in the semiclassical approximation. This requires finding a four-metric that extremizes the action $S_m$ and satisfies the boundary conditions $(\varepsilon_+, j_+, \sigma_+)$ at $S_+$ and $(\varepsilon_-, j_-, \sigma_-)$ at $S_-$. Since the classical Lorentzian eternal black hole metric can be periodically identified and placed on a manifold whose two spatial boundaries have the desired topologies, the resulting metric can be used to approximate the path integral. The periodic identification alters neither the constraint equations nor the boundary data. In the semiclassical approximation the functional integral $\bar{\nu}^*$ becomes

$$\bar{\nu}^*[\varepsilon_+, j_+, \sigma_+; \varepsilon_-, j_-, \sigma_-] \approx \exp (i S_m [\bar{\mathcal{N}}, \bar{\mathcal{V}}, \tilde{h}]) \approx \exp (0) ,$$

since the microcanonical action $S_m[\bar{\mathcal{N}}, \bar{\mathcal{V}}, \tilde{h}]$ evaluated at the periodically identified geometry vanishes identically if the stationarity condition and the constraints are satisfied.

It is illustrative to consider now a complex four-metric which also extremizes the microcanonical action for eternal black hole boundary conditions and which can be used to reevaluate the path integral in a steepest descent approximation. This alternative approximation of the quantity $\bar{\nu}^*$ is useful in understanding the relationship of the result with the density of states for an ordinary (that is, non-eternal) black hole. The complex metric can be obtained from the Lorentzian eternal black hole metric by a complexification map $\Psi$ defined by $\Psi(N) = -iN$, $\Psi(V^i) = -iV^i$. This map preserves the reflection symmetry and the canonical variables $h_{ij}$ and $P^{ij}$ of the Lorentzian solution. The complex geometry consists of two complex sectors $\bar{M}_+$ and $\bar{M}_-$ which join at the locus of points at which the lapse vanishes. This geometry is also a solution of Einstein equations if one requires that conical singularities do not exist at that locus for every $\Sigma$. To do this, it is necessary to puncture each complex sector and to close smoothly.
the geometry at the inner boundaries \( \Sigma^{\pm} \) of \( M^{\pm} \), where \( 2H^{\pm} \) denotes the intersection of the slices \( \Sigma^{\pm} \) with the black hole horizon for the Lorentzian metric \( [2] \). After imposing these regularity conditions, the topology of each sector \( M^{\pm} \) becomes \( R^{2} \times S^{2} \). However, each element \( 3H^{\pm} \) does contribute a term to the microcanonical action for the complex geometry.

The regular complex metric is not included in the sum over Lorentzian geometries \( \nu^{*} \) in \( (1) \) but can be used to approximate it by distorting the contours of integration for both lapse and shift into the complex plane \( [1] \). In this approximation the path integral becomes \( \nu^{*} \approx \exp(iS_{m}[-i\tilde{N},-i\tilde{V},\tilde{h}]) \), where \( S_{m}[-i\tilde{N},-i\tilde{V},\tilde{h}] \) is the action of the complex metric when the smoothness of the geometries at \( 3H^{+} \) and \( 3H^{-} \) is enforced. This action turns out to be \( S_{m}[-i\tilde{N},-i\tilde{V},\tilde{h}] = -iA^{+/4} + iA^{-/4} \), where \( A^{+} \) and \( A^{-} \) denote the surface area of the elements \( 2H^{+} \) and \( 2H^{-} \). Since the periodic identification and the complexification \( \Psi \) do not alter the boundary data nor the gravitational constraint equations, the area \( A^{+} \) of \( 2H^{+} \) coincides with the area \( A^{-} \) of \( 2H^{-} \). This implies that, in agreement with \( (2) \), the eternal black hole functional integral is

\[
\bar{\nu}^{*}[\varepsilon^{+},\sigma^{+};\varepsilon^{-},\sigma^{-}] \approx \exp(A_{H}/4 - A_{H}/4) = \exp(0)
\]

in the “zero-loop” approximation.

If the microcanonical functional integral \( (1) \) is interpreted as the density of states of the statistical system, it is possible to express \( \bar{\nu}^{*} \) approximately as

\[
\bar{\nu}^{*}[\varepsilon^{+},\sigma^{+};\varepsilon^{-},\sigma^{-}] \approx \exp(S[\varepsilon^{+},\sigma^{+};\varepsilon^{-},\sigma^{-}]),
\]

where \( S \) represents the total entropy of the system. The above result implies that the entropy for the system in the semiclassical approximation is

\[
S \approx \frac{1}{4}A_{H} - \frac{1}{4}A_{H} = 0,
\]

where \( A_{H} \) is the area of the horizon of the physical eternal black hole solution that classically approximates the system \( [2] \). The total entropy is given formally by the subtraction \( S = S_{+}[\varepsilon^{+},\sigma^{+}] - S_{-}[\varepsilon^{-},\sigma^{-}] \), where \( S_{+} \) and \( S_{-} \) can be interpreted as the semiclassical entropies associated with the external \( (M^{+}) \) and internal \( (M^{-}) \) regions respectively of the eternal black hole system.

3 Conclusions

The functional integral \( (2) \) refers to a quantum-statistical system which is classically approximated by a general stationary, eternal black hole solution.
of Einstein equations within a region bounded by two timelike surfaces $B_+$ and $B_-$. Its semiclassical value is a consequence of the choice of boundary data, the gravitational constraint equations, and the vanishing of the microcanonical action for the four-geometries that satisfy the boundary conditions and approximate the path integral. The calculation presented above applies to any distorted black hole in the strong gravity regime. It indicates that a pure state (of zero entropy) can be defined not only for matter fields perturbations propagating in the spacetime of an eternal black hole but also for the gravitational field itself. This is physically appealing: the initial data for the eternal black hole specified at the spacelike hypersurface $\Sigma$ contain all the information required for the evolution of both the exterior and interior parts of a physical black hole. The entropy associated with $\Sigma$ must therefore equal zero. Since in a microcanonical description it seems natural to relate the external and internal degrees of freedom of a black hole with the boundary data at the surfaces $B_+$ and $B_-$ respectively \[3\], we believe that the microcanonical functional integral for eternal black hole systems opens the possibility of extending path integral formulation of black hole thermodynamics to situations when internal degrees of freedom are present and allows the study of gravitational statistical properties in terms of a single pure state \[2\]. These conclusions are in complete agreement with thermofield dynamics descriptions of quantum processes and, in particular, with the application of this approach to black hole thermodynamics developed originally by Israel \[6\] for small perturbations. They strongly suggest that the thermofield dynamics description of quantum field processes in a curved background can be extended beyond perturbations to the gravitational field itself of distorted eternal black holes.

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