Asymptotically Exact Constants in Natural Convergence Rate Estimates in the Lindeberg Theorem

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Abstract: Following (Shevtsova, 2013) we introduce detailed classification of the asymptotically exact constants in natural estimates of the rate of convergence in the Lindeberg central limit theorem, namely in Esseen’s, Rozovskii’s, and Wang–Ahmad’s inequalities and their structural improvements obtained in our previous works. The above inequalities involve algebraic truncated third-order moments and the classical Lindeberg fraction and assume finiteness only the second-order moments of random summands. We present lower bounds for the introduced asymptotically exact constants as well as for the universal and for the most optimistic constants which turn to be not far from the upper ones.

Keywords: central limit theorem; Lindeberg’s theorem; normal approximation; asymptotically exact constant; asymptotically best constant; uniform distance; Lindeberg fraction; truncated moment; absolute constant

1. Introduction

In various applications of probability theory, one has to approximate an unknown distribution of a sum of independent random variables with some known law. Such problems arise, for example, in insurance, financial mathematics, reliability theory, queueing theory, and many other areas. The most common approximation is the normal one which is based on the central limit theorem. The adequacy of the normal approximation can be estimated with the help of convergence rate estimates in the central limit theorem such as the celebrated Berry–Esseen [1,2] inequality (in terms of full moments and under the additional moment-type assumptions), or Osipov–Petrov’s [3,4], Esseen’s [5], Rozovskii’s [6], Wang–Ahmad’s [7] inequalities and their generalizations [8–11] (in terms of truncated moments without any additional assumptions). However, the most natural estimates, such as Esseen’s, Rozovskii’s and Wang–Ahmad’s inequalities contained unknown constants, and their application in practice was made possible only by the results of [8,11], where in particular, the unknown constants in the above inequalities were evaluated. A detailed overview of the cited inequalities can be found in [11] and for brevity, we do not duplicate it here. Since the crucial role in estimation of the adequacy of the normal approximation is played by the values (upper bounds, in fact) of appearing absolute constants, it is very important to understand how accurate the existing upper bounds for the constants are, how much they might be lowered and if it is worth trying to improve the method of their evaluation. The problem becomes much deeper as soon as we observe that estimates of the accuracy of the normal approximation are usually used with large sample sizes or when the majorizing expressions are assumed to be small, so that, in fact, not only the absolute values of the appearing constants are of interest, but also their presumably more optimistic
(smaller) values which may be used under the corresponding asymptotic assumptions. Every set of asymptotic assumptions generates the corresponding asymptotic constant. Hence, we can introduce a whole classification of asymptotic constants. The present study is devoted to investigation of the various asymptotic constants and the main purpose is to construct their lower bounds.

Let \( X_1, X_2, \ldots, X_n \) be independent random variables (r.v.’s) with distribution functions (d.f.’s) \( F_k(x) = P(X_k < x), x \in \mathbb{R} \), expectations \( E X_k = 0 \), variances \( \sigma_k^2 = \text{Var} X_k, k = 1, \ldots, n \), and such that

\[
B_n^2 := \sum_{k=1}^n \sigma_k^2 > 0.
\]

For \( n = 1, 2, \ldots \) denote

\[
S_n = X_1 + X_2 + \cdots + X_n, \quad \tilde{S}_n = S_n - ES_n = \sum_{k=1}^n \frac{X_k}{B_n},
\]

\[
\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt, \quad x \in \mathbb{R}, \quad \Delta_n = \Delta_n(F_1, \ldots, F_n) = \sup_{x \in \mathbb{R}} \left| P(\tilde{S}_n < x) - \Phi(x) \right|,
\]

\[
\sigma_k^2(z) = EX_k^21(|X_k| \geq z), \quad \mu_k(z) = EX_k^21(|X_k| < z), \quad k = 1, \ldots, n
\]

\[
M_n(z) := \frac{1}{B_n} \sum_{k=1}^n \mu_k(zB_n) = \frac{1}{B_n^2} \sum_{k=1}^n EX_k^21(|X_k| < zB_n),
\]

\[
\Lambda_n(z) := \frac{1}{B_n^2} \sum_{k=1}^n EX_k^21(|X_k| < zB_n),
\]

\[
L_n(z) = \frac{1}{B_n^2} \sum_{k=1}^n \sigma_k^2(zB_n) = \frac{1}{B_n^2} \sum_{k=1}^n EX_k^21(|X_k| \geq zB_n), \quad z > 0.
\]

The function \( L_n(z) \) is called the Lindeberg fraction. It is easy to see that \( |M_n(z)| \leq \Lambda_n(z) \), \( z > 0 \). In case of independent identically distributed (i.i.d.) r.v.’s \( X_1, \ldots, X_n \) we denote their common d.f. by \( F \) and write \( \Delta_n(F) := \Delta_n(F, \ldots, F) \).

In [8] it was proved that

\[
\Delta_n \leq A_\varepsilon(\varepsilon, \gamma) \sup_{0 < \varepsilon < \varepsilon} \{ \gamma |M_n(z)| + zL_n(z) \}, \quad (1)
\]

\[
\Delta_n \leq A_\varepsilon(\varepsilon, \gamma) \left( \gamma |M_n(z)| + \sup_{0 < \varepsilon < \varepsilon} zL_n(z) \right), \quad \varepsilon, \gamma > 0, \quad n \in \mathbb{N}, \quad (2)
\]

where the functions \( A_\varepsilon(\varepsilon, \gamma), A_\varepsilon(\varepsilon, \gamma) \) depend only on \( \varepsilon \) and \( \gamma \) (that is, they turn into absolute constants as soon as \( \varepsilon \) and \( \gamma \) are fixed), both are monotonically non-increasing with respect to \( \gamma \geq 0 \), and \( A_\varepsilon(\varepsilon, \gamma) \) is also non-increasing with respect to \( \varepsilon > 0 \). The question on the boundedness of \( A_\varepsilon(\varepsilon, \gamma) \) as \( \varepsilon \to \infty \) is still open, while \( A_\varepsilon(0+, \gamma) = A_\varepsilon(0+, \gamma) = \infty \) for every \( \gamma > 0 \) and \( A_\varepsilon(0+, 0) = A_\varepsilon(0+, 0) = \infty \) for every \( \varepsilon > 0 \). To avoid ambiguity, in what follows by constants appearing here in various inequalities we mean their exact values; in particular, in majorizing expressions — their least possible values. Upper bounds for the constants \( A_\varepsilon(\varepsilon, \gamma) \) and \( A_\varepsilon(\varepsilon, \gamma) \) for some \( \varepsilon \) and \( \gamma \) computed in [8] are presented in Tables 1 and 2, respectively. Here the symbol \( \gamma_\ast \) stands for the point of minimum of the upper bound for \( A_\varepsilon(\varepsilon, \gamma) \), obtained within the framework of the method used in [8], i.e., the upper bound for \( A_\varepsilon(\varepsilon, \gamma) \) found in [8] remains constant as \( \gamma \geq \gamma_\ast \) grows for every fixed \( \varepsilon > 0 \). More precisely, the quantity \( \gamma_\ast \) is defined as follows:

\[
\gamma_\ast = 1/\sqrt{6\pi} = (1 - t^2 + t^4)^{1/4} / (\sqrt{3}) = 0.5599 \ldots
\]
where \( t \in (\pi/2, \pi) \) is the unique root of the equation \( \tan t = t/(1 - t^2) \),

\[ x = x_0 = 5.487414 \ldots \text{ is the unique root of the equation} \]

\[ 8(\cos x - 1) + 8x \sin x - 4x^2 \cos x - x^3 \sin x = 0, \quad x \in (\pi, 2\pi). \]

Table 1. Two-sided bounds for the constants \( A_\varepsilon(\varepsilon, \gamma) \) from inequality (1), \( \gamma_* = 0.5599 \ldots \) Upper bounds were obtained in [8], and the lower ones in Theorem 3 below.

| \( \varepsilon \) | \( \gamma \) | \( A_\varepsilon(\varepsilon, \gamma) \leq \) | \( A_\varepsilon(\varepsilon, \gamma) \geq \) |
|---------|----|------------------|------------------|
| 1.21    | 0.2 | 2.8904           | 0.5006           |
| 1.24    | 0.2 | 2.8900           | 0.4889           |
| \infty  | 0.2 | 2.8846           | 0.4876           |
| 1.76    | 0.4 | 2.7360           | 0.4606           |
| 5.94    | 0.4 | 2.7300           | 0.4606           |
| \infty  | 0.4 | 2.7299           | 0.4606           |
| 1       | \gamma_* | 2.7367           | 0.5795           |
| 1.87    | \gamma_* | 2.6999           | 0.4359           |
| \infty  | 0.72 | 2.7298           | 0.5746           |
| 1       | \infty | 2.7286           | 0               |
| 4.35    | 1   | 2.6600           | 0.3703           |

Table 2. Two-sided bounds for the constants \( A_\varepsilon(\varepsilon, \gamma) \) from inequality (2), \( \gamma_* = 0.5599 \ldots \) Upper bounds were obtained in [8], and the lower ones in Theorem 3 below.

| \( \varepsilon \) | \( \gamma \) | \( A_\varepsilon(\varepsilon, \gamma) \leq \) | \( A_\varepsilon(\varepsilon, \gamma) \geq \) |
|---------|----|------------------|------------------|
| 1.21    | 0.2 | 2.8700           | 0.5048           |
| 5.39    | 0.2 | 2.8635           | 0.5000           |
| 1.76    | 0.4 | 2.6999           | 0.4485           |
| 2.63    | 0.4 | 2.6933           | 0.4675           |
| 0.5     | \gamma_* | 3.0396           | 1.1329           |
| 1       | \gamma_* | 2.7286           | 0.5795           |

In the same paper [8] there were found sharpened upper bounds for the constants \( A_\varepsilon(\varepsilon, \gamma) \) and \( A_\varepsilon(\varepsilon, \gamma) \) provided that the corresponding fractions

\[ L_{\varepsilon,n}(\varepsilon, \gamma) := \sup_{0 < z < \varepsilon} \{ \gamma | M_n(z) | + z L_n(z) \}, \quad L_{\gamma,n}(\varepsilon, \gamma) := \gamma | M_n(\varepsilon) | + \sup_{0 < z < \varepsilon} z L_n(z). \]

take small values. In particular, there were introduced asymptotically exact constants

\[
A_\varepsilon^*(\varepsilon, \gamma) := \limsup_{\varepsilon \to 0} \mathop{\sup}_{n \in F_1, \ldots, F_n} \left\{ \frac{\Delta_n(F_1, \ldots, F_n)}{\ell} : L_{\varepsilon,n}(\varepsilon, \gamma) = \ell \right\}, \quad \varepsilon, \gamma > 0, \tag{3}
\]

\[
A_\gamma^*(\varepsilon, \gamma) := \limsup_{\varepsilon \to 0} \mathop{\sup}_{n \in F_1, \ldots, F_n} \left\{ \frac{\Delta_n(F_1, \ldots, F_n)}{\ell} : L_{\varepsilon,n}(\varepsilon, \gamma) = \ell \right\}, \quad \varepsilon, \gamma > 0, \tag{4}
\]
and the following upper bounds were obtained for them:

\[
A^\Phi_n(\epsilon, \gamma) \leq \frac{4}{\sqrt{2\pi}} + \frac{1}{\pi} \left[ \frac{\sqrt{\epsilon}}{\epsilon} \sqrt{1 + \frac{1}{2\pi}} + \frac{\epsilon}{12} \sqrt{1 + \frac{1}{2\pi}} + \sqrt{\frac{e^2(6\epsilon^2 + 1)}{6\pi}} \right].
\]

(5)

\[
A^\Phi_k(\epsilon, \gamma) \leq \frac{4}{\sqrt{2\pi}} + \frac{1}{\pi} \left[ \frac{\sqrt{\epsilon}}{\epsilon} \sqrt{1 + \frac{1}{2\pi}} + \frac{\epsilon}{12} \sqrt{1 + \frac{1}{2\pi}} + \sqrt{\frac{e^2(6\epsilon^2 + 1)}{6\pi}} \right] + \sqrt{\frac{\epsilon}{\gamma}} \left[ \frac{\sqrt{\epsilon}}{\epsilon} \sqrt{1 + \frac{1}{2\pi}} - \sqrt{\frac{1}{2} \gamma} \right],
\]

(6)

where \(\Gamma(r, x) := \int_0^\infty t^{r-1}e^{-x t} dt\), \(Y(r, x) := \int_0^\infty t^{r-1}e^{-x t} dt = \Gamma(r) - \Gamma(r, x), r, x > 0\), are the upper and the lower gamma-functions, respectively,

\[t_\gamma := \frac{2}{\gamma} \left( (\gamma/\gamma) + 1 \right), \quad t_\gamma := 2 \max \left\{ \gamma^{-1}, \gamma^{-1} \right\}, \quad t_{\gamma} := t_{\gamma} (1 - \sqrt{(1 - (\gamma/\gamma)^2)}),\]

\(\gamma = 0.5315 \ldots, \gamma_+ = 0.5599 \ldots\) were defined above. The values of the upper bounds for the asymptotically exact constants \(A^\Phi_n(\epsilon, \gamma)\) and \(A^\Phi_n(\epsilon, \gamma)\) in (5) and (6) for some \(\epsilon > 0\) and \(\gamma > 0\) are given in the third and the seventh columns of Table 3, respectively.

Moreover, in [8] it was shown that the asymptotically exact constants \(A^\Phi_n \in \{A^\Phi_n, A^\Phi_n\}\) are unbounded as \(\gamma \to 0\):

\[A^\Phi_n(\epsilon, \gamma) \geq \sup_{F} \sup_{n \to \infty} \frac{\Delta_n(F)}{L_n(\epsilon, \gamma)} \to \infty, \quad \gamma \to 0, \quad \forall \epsilon > 0.\]

Let us note that estimate (2) with \(\epsilon = \gamma = 1\) coincides with the Rozovskii inequality [6] ([Corollary 1]) and establishes an upper bound for the appearing absolute constant \(A_k(1, 1) \leq A_k(1, \gamma_+) \leq 2.73\). Estimate (1) with \(\epsilon = \gamma = 1\) and \(\epsilon \to \infty, \gamma = 1\) improves both Esseen’s inequalities from [5], where the absolute value argument and the least upper bound with respect to \(z \in (0, \epsilon)\) stand inside the sum in comparison with (1). In particular, estimate (1) yields upper bounds for the absolute constants in Esseen’s inequalities \(A_k(1, 1) \leq A_k(1, 0.72) \leq 2.73\) and \(A_k(1, \gamma) \leq A_k(\gamma, 0.97) \leq 2.66\) which were remaining unknown for a long time. Esseen’s inequality where the least upper bound is taken over a bounded range (see (1) with \(\epsilon = \gamma = 1\)) yields, in its turn, the Osipov inequality [3]:

\[
\Delta_n \leq A_k(1, 1) \sup_{0 < z < 1} \left\{ |M_n(z) + zL_n(z)| \right\} \leq A_k(1, 1) \sup_{0 < z < 1} \left\{ \Delta_n(z) + zL_n(z) \right\} = A_k(1, 1)(\Delta_n(1) + L_n(1)) = A_k(1, 1) \inf_{\epsilon > 0} (\Delta_n(\epsilon) + L_n(\epsilon))
\]

(7)

(for details see [8]). The latest bound obtained by Osipov [3] with some constant \(A_k(1, 1)\) (whose best known value 1.87 is published in [12]) yields, in its turn, the Lindeberg theorem: indeed, under the Lindeberg condition \(\sup_{\epsilon > 0} n \to \infty\) \(L_n(\epsilon) = 0\) and with the account of \(\Delta_n(\epsilon) \leq \epsilon\), from (7) we have

\[\Delta_n \leq A_k(1, 1) \inf_{\epsilon > 0} (\epsilon + L_n(\epsilon)) \to 0, \quad n \to \infty.\]

Hence, by Feller’s theorem, in case of uniformly infinitesimal random summands (in particular, in the i.i.d. case) the right- and the left-hand sides of (7) are either both infinitesimal or both do not tend to zero. According to the terminology, introduced by Zolotarev [13], such convergence rate estimates are called natural. Together with (1), inequality (2) is a natural convergence rate estimate in the Lindeberg–Feller theorem certainly under the additional assumption of existence of such an \(\epsilon_0 > 0\) that \(M_n(\epsilon_0) = 0\) for all sufficiently large \(n\) (see, e.g., [11]), in particular, if the r.v.’s \(X_1, \ldots, X_n\) have symmetric distributions.
Table 3. Values of the two-sided bounds of the asymptotically exact constants $A_{AE}^\varepsilon(\varepsilon, \gamma)$ and $A_{AE}^\varepsilon(\varepsilon, \gamma)$ in (5) and (6) for some $\varepsilon > 0$ and $\gamma > 0$. Recall that $\gamma^* = 0.5599 \ldots$. Upper bounds were obtained in [8], and the lower ones in Corollary 1 below.

| $\varepsilon$ | $\gamma$ | $A_{AE}^\varepsilon(\varepsilon, \gamma) \leq$ | $\varepsilon$ | $\gamma$ | $A_{AE}^\varepsilon(\varepsilon, \gamma) \geq$ |
|--------------|---------|----------------------------------|--------------|---------|----------------------------------|
| 0.6          | 0.3     | 1.9225                           | 1.21         | 0.2     | 1.9348                           |
| 1.21         | 0.2     | 1.9546                           | 1.89         | 0.2     | 1.9300                           |
| 2.06         | 0.2     | 1.9500                           | 2.77         | 0.2     | 1.9289                           |
| $\infty$     | 0.2     | 1.9488                           | 5.39         | 0.2     | 1.9584                           |
| 1.48         | 0.4     | 1.8100                           | 1.41         | 0.4     | 1.7798                           |
| $\infty$     | 0.4     | 1.8001                           | 1.76         | 0.4     | 1.7725                           |
| 1.89         | $\gamma^*$ | 1.7714                           | 1.99         | 0.4     | 1.7713                           |
| 2.03         | $\gamma^*$ | 1.7700                           | 2.63         | 0.4     | 1.7785                           |
| $\infty$     | $\gamma^*$ | 1.7638                           | 0.5          | $\gamma^*$ | 1.9475                           |
| 1            | $\gamma^*$ | 1.8060                           | 0.4685       | 1.52    | $\gamma^*$ | 1.7500                           |
| 1            | $\infty$   | 1.7915                           | 0.1994       | 1.89    | $\gamma^*$ | 1.7439                           |
| 2.24         | 1        | 1.7400                           | 0.4097       | 1.99    | $\gamma^*$ | 1.7442                           |
| $\infty$     | 1        | 1.7319                           | 0.4097       | 2.12    | $\gamma^*$ | 1.7455                           |
| 3.07         | $\infty$ | 1.7200                           | 0.1994       | 3       | $\gamma^*$ | 1.7710                           |
| 3.2          | 5        | 1.7200                           | 0.1994       | 5       | $\gamma^*$ | 1.8650                           |
| 3.28         | 4        | 1.7200                           | 0.2240       |         |                                   |
| 4            | 2.4      | 1.7200                           | 0.3059       |         |                                   |
| 5            | 2.06     | 1.7200                           | 0.3284       |         |                                   |
| 5.37         | 2        | 1.7200                           | 0.3324       |         |                                   |
| $\infty$     | $\infty$ | 1.7146                           | 0.1994       |         |                                   |

Let us also note that Esseen-type inequality (1) not only links the criteria of convergence with the rate of convergence, as Osipov’s inequality does, but also provides a numerical demonstration of the Ibragimov’s criteria [14] of the rate of convergence in the CLT to be of order $O(n^{-1/2})$. According to [14], in the i.i.d. case we have $\Delta_n = O(n^{-1/2})$ as $n \to \infty$ if and only if $\max\{|\mu_1(z)|, z \sigma_1^2(z)\} = O(1)$ as $z \to \infty$. Inequality (1) trivially yields the sufficiency of the Ibragimov condition to $\Delta_n = O(n^{-1/2})$ as $n \to \infty$.

Let us denote by $\mathcal{G}$ a set of all non-decreasing functions $g: [0, \infty) \to [0, \infty)$ such that $g(z) > 0$ for $z > 0$ and $z / g(z)$ is also non-decreasing for $z > 0$. The set $\mathcal{G}$ was initially introduced by Katz [15] and used later in the works [4,9–12,16]. In [9] it was proved that

(i) For every function $g \in \mathcal{G}$ and $a > 0$

$$g_0(z,a) := \min\left(\frac{z}{a}, 1\right) \leq \frac{g(z)}{g(a)} \leq \max\left(\frac{z}{a}, 1\right) := g_1(z,a), \quad z > 0,$$

with $g_0(\cdot,a)$, $g_1(\cdot,a) \in \mathcal{G}$.

(ii) Every function from $\mathcal{G}$ is continuous on $(0, \infty)$.
Moreover, in [11] (Theorem 2) it was proved that for all $g$ in $\mathcal{G}$, the class $\mathcal{G}$ also includes the following functions:

$$g_c(z) \equiv 1, \quad g_*(z) = z, \quad c \cdot z^\delta, \quad c \cdot g(z), \quad z > 0,$$

for all $c > 0$, $\delta \in [0, 1]$ and $g \in \mathcal{G}$.

For $g \in \mathcal{G}$ we set

$$L_{n,n}(g, \varepsilon, \gamma) = \frac{1}{B_{n}^2 g(B_{n})} \sup_{0 < z < \varepsilon B_{n}} \frac{g(z)}{z} \left\{ \gamma \sum_{k=1}^{n} \mu_k(z) + z \sum_{k=1}^{n} \sigma_k^2(z) \right\}$$

$$= \sup_{0 < z < \varepsilon} \frac{g(zB_{n})}{zg(B_{n})} \left( \gamma |M_n(z)| + zL_n(z) \right), \quad (9)$$

$$L_{n,n}(g, \varepsilon, \gamma) = \frac{1}{B_{n}^2 g(B_{n})} \left( \gamma \frac{g(\varepsilon B_{n})}{\varepsilon B_{n}} \sum_{k=1}^{n} \mu_k(\varepsilon B_{n}) + \sup_{0 < z < \varepsilon B_{n}} g(z) \sum_{k=1}^{n} \sigma_k^2(z) \right)$$

$$= \gamma \frac{g(\varepsilon B_{n})}{\varepsilon g(B_{n})} |M_n(\varepsilon)| + \sup_{0 < z < \varepsilon} \frac{g(zB_{n})}{g(B_{n})} L_n(z). \quad (10)$$

Please note that the introduced fractions with $g = g_*$ coincide with the fractions in the Esseen- and Rozovskii-type inequalities (1), (2) considered above:

$$L_{n,n}(g_*, \varepsilon, \gamma) = L_{n,n}(\varepsilon, \gamma), \quad L_{n,n}(g_*, \varepsilon, \gamma) = L_{n,n}(\varepsilon, \gamma), \quad \varepsilon, \gamma > 0.$$ 

Inequalities (1) and (2) were generalized in [11] in the following way:

$$\Delta_n \leq C_*(\varepsilon, \gamma) \cdot L_{n,n}(g, \varepsilon, \gamma), \quad (11)$$

$$\Delta_n \leq C_*(\varepsilon, \gamma) \cdot L_{n,n}(g, \varepsilon, \gamma), \quad \varepsilon, \gamma > 0, \quad g \in \mathcal{G}, \quad (12)$$

where

$$C_*(\varepsilon, \cdot) = A_*(\varepsilon, \cdot), \quad \varepsilon \in (0, 1], \quad \text{in particular, } C_*(0, \cdot) = \infty,$$

$$C_*(\varepsilon, \cdot) \leq A_*(1, \cdot), \quad \varepsilon > 1;$$

$$C_*(\varepsilon, \cdot) = A_*(\varepsilon, \cdot), \quad \varepsilon \in (0, 1], \quad \text{in particular, } C_*(0, \cdot) = \infty,$$

$$C_*(\varepsilon, \cdot) \leq \varepsilon A_*(\varepsilon, \cdot), \quad \varepsilon > 1; \quad C_*(\infty, \cdot) = \infty$$

(recall that according to the above convention, all the equalities between the constants including $C_*(0, \cdot) = C_*(0, \cdot) = C_*(\infty, \cdot) = \infty$ are exact with formal definitions of $C_*(\varepsilon, \gamma)$ and $C_*(\varepsilon, \gamma)$ being given in (19) below).

It is easy to see that the both fractions $L_{n,n} \in \{ L_{n,n}, L_{n,n} \}$ are invariant with respect to scale transformations of $g \in \mathcal{G}$:

$$L_{n,n}(c g, \varepsilon, \gamma) = L_{n,n}(g, \varepsilon, \gamma), \quad c > 0.$$ 

Moreover, in [11] (Theorem 2) it was proved that for all $\varepsilon, \gamma > 0$

$$1 \leq L_{n,n}(g_1, \varepsilon, \gamma) \leq \max \{ \varepsilon, 1 \} \cdot \max \{ \gamma, 1 \}, \quad (13)$$

$$1 \leq L_{n,n}(g_1, \varepsilon, \gamma) \leq \max \{ \varepsilon, 1 \} \cdot (\gamma + 1). \quad (14)$$

Extreme properties of the functions

$$g_0(z) := B_n g_0(z, B_n) = \min \{ z, B_n \}, \quad g_1(z) := B_n g_1(z, B_n) = \max \{ z, B_n \}, \quad z > 0,$$
We consider various statements of the problem of construction of the lower bounds, namely with the extreme functions $g_{11}, (12)$ with inequalities of Katz [15], Petrov [4], Osipov [3], Esseen [5], Rozovskii [6], second order moments. Denote constants in inequalities (11), (12). Let $C\in\mathcal{G}$ and Wang–Ahmad [7] can be found in papers [8,11].

Inequality (11) also generalizes and improves up to the values of the appearing absolute constant the classical Katz–Petrov inequality [4,15] (which is equivalent to the Osipov inequality [3]) because of involving the algebraic truncated third order moments instead of the absolutes ones and also a recent result of Wang and Ahmad [7] at the expense of moving the modulus and the least upper bound signs outside of the sum sign. In particular, inequality (11) established an upper bound of the constant in the Wang–Ahmad inequality $C_0(\infty, 1) \leq A_k(1, 1) \leq 2.73$.

A detailed survey and the analysis of the relationships between inequalities (1), (2), (11), (12) with inequalities of Katz [15], Petrov [4], Osipov [3], Esseen [5], Rozovskii [6], and Wang–Ahmad [7] can be found in papers [8,11].

The main goal of the present work is construction of the lower bounds of the absolute constants $C_0(\varepsilon, \gamma)$, $C_k(\varepsilon, \gamma)$ in inequalities (11), (12), and also of the constants $A_0(\varepsilon, \gamma)$, $A_k(\varepsilon, \gamma)$ in inequalities (1), (2), in particular, we show that even in the i.i.d. case

$$C_0(1, 1) = A_0(1, 1) > 0.5685, \quad C_k(1, 1) = A_k(1, 1) > 0.5685,$$

$$C_0(\infty, 1) > 0.5685, \quad A_k(\infty, 1) > 0.3703.$$  

We consider various statements of the problem of construction of the lower bounds, namely we introduce a detailed classification of the asymptotically exact constants and construct their lower bounds. As a corollary, we obtain two-sided bounds for the asymptotically exact constants $A_0(\varepsilon, \gamma)$ and $A_k(\varepsilon, \gamma)$ defined in (3) and (4), in particular, we show that

$$0.4097 \ldots = \frac{\sqrt{10} + 3}{6\sqrt{2\pi}} \leq A_0(1, 1) \leq 1.80.$$

$$0.3989 \ldots = \frac{1}{\sqrt{2\pi}} \leq A_k(1, 1) \leq 1.80.$$  

The paper is organized as follows. In Section 2 we introduce exact, asymptotically exact and asymptotically best constants defining the corresponding statements for the construction of the lower bounds. Sections 4–6 are devoted namely to the construction of the lower bounds for the introduced constants. Section 3 contains some auxiliary results which might represent an independent interest, in particular, the values of the fractions $L_{n, n}(g, \varepsilon, \gamma)$ and $L_{n, n}(g, \varepsilon, \gamma)$ are found for all $n, \varepsilon, \gamma > 0$ and some $g \in \mathcal{G}$ in the case where $X_1, \ldots, X_n$ have identical two-point distribution.

2. Exact, Asymptotically Exact and Asymptotically Best Constants

Following [13,17–25], let us define exact, asymptotically exact and asymptotically best constants in inequalities (11), (12). Let $F$ be a set of all d.f.’s with zero means and finite second order moments. Denote

$$C_0(g, \varepsilon, \gamma) = \sup_{F_1, \ldots, F_n \in F, n \in \mathbb{N}} \frac{\Delta_0(F_1, \ldots, F_n)}{L_{n, n}(g, \varepsilon, \gamma)}, \quad C_k(g, \varepsilon, \gamma) = \sup_{F_1, \ldots, F_n \in F, n \in \mathbb{N}} \frac{\Delta_k(F_1, \ldots, F_n)}{L_{n, n}(g, \varepsilon, \gamma)}.$$  

(18)
Please note that the fractions $L_{\varepsilon,n}(g,\varepsilon,\gamma)$, $L_{\varepsilon,n}(g,\varepsilon,\gamma)$ also depend on d.f.'s $F_1, \ldots, F_n$, but we omit these arguments for the sake of brevity.

The constants $C_\varepsilon(g,\varepsilon,\gamma)$ and $C_\varepsilon(g,\varepsilon,\gamma)$ are the minimal possible (exact) values of the constants $C_\varepsilon(\varepsilon,\gamma)$ and $C_\varepsilon(\varepsilon,\gamma)$ in inequalities (11), (12) for the fixed function $g \in \mathcal{G}$, while their universal values $\sup_{g \in \mathcal{G}} C_\varepsilon(g,\varepsilon,\gamma)$, $\sup_{g \in \mathcal{G}} C_\varepsilon(g,\varepsilon,\gamma)$, that provide the validity of the inequalities under consideration for all $g \in \mathcal{G}$ are called exact constants and namely they are the minimal possible (exact) values of the constants $C_\varepsilon(\varepsilon,\gamma)$ and $C_\varepsilon(\varepsilon,\gamma)$ in (11) and (12), respectively. In order not to introduce excess indexes, we use identical notation for the asymptotic below to the non-i.i.d. case is not difficult (see, for example, definitions (3) and (4) of the already introduced in (3), (4). Since we are interested in the lower bounds, we assume not only the absolute the concrete estimates of its accuracy are needed. That is why it is interesting to study the constants $\varepsilon, \gamma > 0$, hence, every lower bound for the constants $A_\varepsilon(\varepsilon,\gamma)$, $A_\varepsilon(\varepsilon,\gamma)$ serves as a lower bound for the constants $C_\varepsilon(\varepsilon,\gamma)$, $C_\varepsilon(\varepsilon,\gamma)$ as well.

The least upper bounds in (18) are taken without any restrictions on the values of the fractions $L_{\varepsilon,n}(g,\varepsilon,\gamma)$, $L_{\varepsilon,n}(g,\varepsilon,\gamma)$, while inequalities (11), (12) represent the most interest with small values of these fractions, when the normal approximation is adequate and only the only the asymptotically exact constants $\Delta_\varepsilon(\varepsilon,\gamma)$, $\Delta_\varepsilon(\varepsilon,\gamma)$ appearing in inequalities(11), (23) do not have identical distributions. Generalization of the definitions introduced below to the non-i.i.d. case is not difficult (see, for example, definitions (3) and (4) of the asymptotically exact constants for the general case).

For each of the fractions $L_\varepsilon = L_\varepsilon(F, g, \varepsilon, \gamma) \in \{ L_{\varepsilon,n}, L_{\varepsilon,n} \}$ appearing in inequalities (11), (12) we define the asymptotically best constants

$$ C_{ab}(\varepsilon, \gamma) = \sup_{g \in \mathcal{G}} \limsup_{n \to \infty} \Delta_\varepsilon(F) / L_\varepsilon(F, g, \varepsilon, \gamma), $$

the upper asymptotically exact constants

$$ \overline{C}_{ab}(g, \varepsilon, \gamma) = \limsup_{n \to \infty} \sup_{F \in \mathcal{F}} \Delta_\varepsilon(F) / L_\varepsilon(F, g, \varepsilon, \gamma), $$

the asymptotically exact constants

$$ C_{ab}(g, \varepsilon, \gamma) = \limsup_{F \in \mathcal{F}} \sup_{\ell \to 0} \sup_{n \in \mathbb{N}, F \in \mathcal{F} : L_\varepsilon(F, g, \varepsilon, \gamma) = \ell} \Delta_\varepsilon(F) / \ell, $$

the lower asymptotically exact constants

$$ \underline{C}_{ab}(g, \varepsilon, \gamma) = \limsup_{F \in \mathcal{F}} \sup_{\ell \to 0} \sup_{n \to \infty} \Delta_\varepsilon(F) / \ell, $$

the conditional upper asymptotically exact constants

$$ C_{ab}^*(g, \varepsilon, \gamma) = \sup_{F \in \mathcal{F}} \sup_{\ell \to 0} \limsup_{n \to \infty} \Delta_\varepsilon(F) / \ell. $$

In order not to introduce excess indexes, we use identical notation for the asymptotic constants in (11), (12), in what follows every time specifying the inequality in question. All the introduced constants aim to improve the structure of the simplest upper bounds of
the form \( \Delta_n \leq C(g, \varepsilon, \gamma) L_n(g, \varepsilon, \gamma) \) with the help of introducing an additional term \( o(L_n) \) which is allowed to be infinitesimal of a higher order than \( L_n \) as \( L_n \to 0 \):

\[
\Delta_n \leq C_\varepsilon(g, \varepsilon, \gamma)L_n(F, g, \varepsilon, \gamma) + o(L_n(F, g, \varepsilon, \gamma)), \quad n \to \infty,
\]

where the assumptions on \( o(L_n) \) define the value of the corresponding minimal constant \( C_\varepsilon \). For example, in (20) it is supposed that the distribution of the random summands does not depend on the number of summands \( n \) and, hence, \( L_n \to 0 \) if and only if \( n \to \infty \), so that \( o(L_n(F, g, \varepsilon, \gamma)) \) in (25) is not obliged to tend to zero uniformly for all d.f.'s \( F \in \mathcal{F} \) as \( n \to \infty \). All the rest introduced constants allow a double array scheme. The values of \( L_n \) are obliged to be infinitesimal in (20), (22), and (23). The upper asymptotically exact (24) constants are linked by the following relation (21) and the conditional upper asymptotically exact (24) constants are linked by the following relation (20), (22), and (23). The upper asymptotically exact (24) constants are linked by the following relation (22) and (23) is in the upper bound and the limit with respect to \( n \), so that \( C_{AB}(g, \varepsilon, \gamma) \geq C_{\alpha \epsilon}(g, \varepsilon, \gamma) \), where the strict inequality may also take place, as it happens indeed, for example, with the similar constants in the classical Berry–Esseen inequality [19–24]. In terms of inequalities, this means that inequality (23) with \( C_\varepsilon = C_{\alpha \epsilon} \) assumes that \( o(L_n(F, g, \varepsilon, \gamma)) \) tends to zero uniformly for all d.f.'s \( F \in \mathcal{F} \) with fixed value of \( L_n(F, g, \varepsilon, \gamma) = \ell \) as \( \ell \to 0 \), while taking \( C_\varepsilon = C_{\alpha \epsilon} \) in (25) leads to separating of \( o(L_n(F, g, \varepsilon, \gamma)) \) in (25) into two terms:

\[
\sup_{F \in \mathcal{F} : L_n(F, g, \varepsilon, \gamma) = \ell} \Delta_n = C_{\alpha \epsilon}(g, \varepsilon, \gamma) \ell + r_n(\ell, g, \varepsilon, \gamma),
\]

where \( \lim \sup \lim \sup |r_n(\ell, g, \varepsilon, \gamma)| = 0 \). The constants in the classical Berry–Esseen inequality similar to those defined in (20)–(24) were first considered in [18] for (20), [17] for (22), [13,26] for (21), [23] for (23), and [25] for (24). The upper asymptotically exact (21) and the conditional upper asymptotically exact (24) constants are linked by the following relation \( C_{\alpha \epsilon} \leq C_{AB} \) by definition, and we shall construct lower bounds namely for \( C_{\alpha \epsilon} \). As for the constants \( \overline{C}_{AB} \), we introduce them here to pay tribute to the classical works [13,26]. The function \( g \) in (20), (21), (24) may be arbitrary from the class \( \mathcal{G} \), while the constants \( C_{AB}(g, \varepsilon, \gamma) \) and \( C_{\alpha \epsilon}(g, \varepsilon, \gamma) \) (see (22) and (23)) are defined not for all \( g \in \mathcal{G} \). For example, \( C_{AB}(g_1, \varepsilon, \gamma) \) and \( C_{\alpha \epsilon}(g_1, \varepsilon, \gamma) \) are not defined in any of the inequalities (11), (12), since the corresponding fractions \( L_{\alpha \epsilon}(g_1, \varepsilon, \gamma) \), \( L_{\alpha \epsilon}(g_1, \varepsilon, \gamma) \) are bounded from below by one uniformly with respect to \( \varepsilon \) and \( \gamma \) (see (13) and (14)) and, hence, cannot be infinitesimal.

Finally, note that definitions (20)–(24) immediately yield the relations

\[
\max\{C_{AB}, C_{\alpha \epsilon}\} \leq \min\{C_{AB}, C_{\alpha \epsilon}\}, \quad \max\{C_{AB}, C_{\alpha \epsilon}\} \leq \overline{C}_{AB}
\]

where we omitted the arguments \( g, \varepsilon, \gamma \) for clarity.

### 3. Two-Point Distributions

Most of the lower bounds will be obtained by the choice of a two-point distribution

\[
P\left(X_k = \sqrt{\frac{q}{p}}\right) = 1 - P\left(X_k = -\sqrt{\frac{p}{q}}\right) = p, \quad q = 1 - p \in (0,1), \quad k = 1, \ldots, n,
\]

for the random summands. The present section contains the corresponding required results. In particular, we will find values of the fractions \( L_{\alpha \epsilon}(g, \varepsilon, \gamma) \), \( L_{\alpha \epsilon}(g, \varepsilon, \gamma) \) with \( g = g_{\varepsilon}, g_{\gamma}, g_{0} \), \( g_{1} \) for all \( \varepsilon, \gamma > 0 \) (Theorem 1). We will also investigate the uniform distance between the d.f. of (27) and the standard normal d.f. \( \Phi \) and find the corresponding extreme values of the argument of d.f.’s (see Theorem 2):

\[
\Delta_1(p) := \sup_{x \in \mathbb{R}}|P(X_1 < x) - \Phi(x)| = \begin{cases} \Phi(\sqrt{\frac{p}{q}}) - p, & 0 < p < \frac{1}{2}, \\ \Phi(\sqrt{\frac{q}{p}}) - q, & \frac{1}{2} \leq p < 1. \end{cases}
\]
3.1. Computation of the Fractions

For distribution (27) we have $EX_k = 0$, $EX_k^2 = 1$, $k = 1, \ldots, n$, $B^2 = n$,

$g_0(z) = \min\{z, \sqrt{n}\}$, $g_1(z) = \max\{z, \sqrt{n}\}$, $g_*(z) = z$, $g_c(z) = 1$, $z \geq 0$.

Recall that for $\varepsilon \leq 1$

$L_{\varepsilon,n}(g_0, \varepsilon, \cdot) = L_{\varepsilon,n}(g_*, \varepsilon, \cdot) = L_{\varepsilon,n}(\varepsilon, \cdot), \quad L_{\varepsilon,n}(g_1, \varepsilon, \cdot) = L_{\varepsilon,n}(g_c, \varepsilon, \cdot), \quad L_{\varepsilon,n}(g_0, \varepsilon, \cdot) = L_{\varepsilon,n}(g_*, \varepsilon, \cdot) = L_{\varepsilon,n}(\varepsilon, \cdot), \quad L_{\varepsilon,n}(g_1, \varepsilon, \cdot) = L_{\varepsilon,n}(g_c, \varepsilon, \cdot).

That is why in the formulation of the next Theorem 1, we do not indicate values of $L_{\varepsilon,n}(g_0, \varepsilon, \cdot)$, $L_{\varepsilon,n}(g_1, \varepsilon, \cdot)$, $L_{\varepsilon,n}(g_0, \varepsilon, \cdot)$, $L_{\varepsilon,n}(g_1, \varepsilon, \cdot)$ for $\varepsilon \leq 1$ separately.

Theorem 1. (i) For all $n \in \mathbb{N}$, $p \in [1/2, 1]$ and $\varepsilon, \gamma > 0$ we have

$L_{\varepsilon,n}(g_*, \varepsilon, \gamma) = \begin{cases} \varepsilon, & ne^2 \leq \frac{q}{p}, \\ \max \left\{ \frac{\gamma}{\sqrt{np}}, \gamma \sqrt{\frac{q}{np}} + \varepsilon p \right\}, & \frac{q}{p} < ne^2 \leq \frac{p}{q}, \\ \max \left\{ q, \left( \gamma q^2 + p^2 \right) \mathbf{1}(p > 1/2), \gamma(p - q) \right\} / \sqrt{npq}, & ne^2 > \frac{p}{q}, \end{cases}$ (29)

$L_{\varepsilon,n}(g_c, \varepsilon, \gamma) = \begin{cases} 1, & ne^2 \leq \frac{q}{p}, \\ \max \{ 1, \gamma q + p \}, & \frac{q}{p} < ne^2 \leq \frac{p}{q}, \\ \max \left\{ 1, \left( \gamma q + p \right) \mathbf{1}(p > 1/2), \gamma(p - q) \right\}, & ne^2 > \frac{p}{q}. \end{cases}$ (30)

$L_{\varepsilon,n}(g_*, \varepsilon, \gamma) = \begin{cases} \gamma \sqrt{\frac{q}{np}} + \max \left\{ \frac{\gamma}{\sqrt{np}}, \varepsilon p \right\}, & \frac{q}{p} < ne^2 \leq \frac{p}{q}, \\ \gamma(p - q) \varepsilon / \sqrt{npq}, & ne^2 > \frac{p}{q}. \end{cases}$ (31)

$L_{\varepsilon,n}(g_c, \varepsilon, \gamma) = \begin{cases} 2 \sqrt{\frac{q}{np}} + 1, & \frac{q}{p} < ne^2 \leq \frac{p}{q}, \\ \gamma(p - q) \varepsilon / \sqrt{npq} + 1, & ne^2 > \frac{p}{q}. \end{cases}$ (32)

(ii) For all $\varepsilon > 1$, $\gamma > 0$, $n \in \mathbb{N}$

$L_{\varepsilon,n}(g_1, \varepsilon, \gamma) = 1$, if $p = 1/2$,

$L_{\varepsilon,n}(g_1, \varepsilon, \gamma) = \begin{cases} \max \left\{ 1, \gamma q + p, 2 \sqrt{\frac{q}{np}}, 2 \varepsilon p \right\}, & ne^2 \leq \frac{p}{q}, \\ \max \left\{ 1, \gamma q + p, 2 q \sqrt{\frac{q}{np}}, \gamma(p - q) \sqrt{npq} \right\}, & n \leq \frac{p}{q} < ne^2, \quad \text{if } p > 1/2, \\ \max \left\{ 1, \gamma q + p, 2 \varepsilon(p - q) \right\}, & n > \frac{p}{q}. \end{cases}$

(iii) For all $\varepsilon > 1$, $\gamma > 0$, $n \in \mathbb{N}$, and $p \in [1/2, 1)$

$L_{\varepsilon,n}(g_0, \varepsilon, \gamma) = \begin{cases} 2 \sqrt{\frac{q}{np}} + 1, & n \leq \frac{q}{p} < ne^2 \leq \frac{p}{q}, \\ \frac{\gamma(p - q)}{\varepsilon \sqrt{npq}} + 1, & n \leq \frac{q}{p} \leq \frac{p}{q} < ne^2, \end{cases}$
Proof. Let us find $\mu_{\frac{1}{2}}$ and all $n$.

In particular,

$$L_{x,n}(g_1,\varepsilon,\gamma) = \begin{cases} \sqrt{\frac{p^2}{np}} + \max\left\{ \sqrt{\frac{p}{np}}, \varepsilon p \right\}, & n \leq \frac{q}{p} < n \varepsilon^2 \leq \frac{p}{q}, \\ \frac{p}{q} + \max\left\{ \sqrt{\frac{p}{np}}, \sqrt{\frac{q}{np}} \right\}, & \frac{p}{q} < n \varepsilon^2 \leq \frac{p}{q}, \end{cases}$$

for $p \geq 1/2$ and all $\varepsilon, \gamma > 0$, $n \in \mathbb{N}$:

$$L_{x,n}(g_s,\varepsilon,\gamma) = L_{x,n}(g_0,\varepsilon,\gamma) = L_{x,n}(g_1,\varepsilon,\gamma) = L_{x,n}(g_0,\varepsilon,\gamma) = \min\left\{ \varepsilon, \frac{1}{\sqrt{n}} \right\},$$

for $p > 1/2$ and all $n \in \mathbb{N}$:

$$L_{x,n}(g_s,1,1) = L_{x,n}(g_0,1,1) = L_{x,n}(g_0,\infty,1) = \begin{cases} \frac{q^2}{npq} + p, & n \leq \frac{p}{q}, \\ \frac{q^2}{npq} + p^2, & n > \frac{p}{q}, \end{cases}$$

$$L_{x,n}(g_s,\infty,1) = \frac{q^2}{\sqrt{npq}},$$

$$L_{x,n}(g_0,\infty,1) = \max\left\{ \sqrt{\frac{q}{np}}, \sqrt{\frac{q}{np}} + p, \frac{p^2}{q} \right\}, \quad n \leq \frac{p}{q},$$

$$L_{x,n}(g_1,\varepsilon,\gamma) = 1, \quad \varepsilon > 0, \gamma \leq 1.$$

Let us find $\mu_1(\cdot), \sigma_1^2(\cdot)$. For all $z \geq 0$ we have

$$|\mu_1(z)| = |\mathbb{E}X_1^2(\mathbb{1}_{|X_1| < z})| = \begin{cases} 0, & z \leq \sqrt{\frac{q}{p}}, \\ \sqrt{\frac{q}{p}}, & \sqrt{\frac{q}{p}} < z \leq \sqrt{\frac{p}{q}}, \\ \frac{p^2}{\sqrt{npq}}, & \sqrt{\frac{p}{q}} > z \leq \sqrt{\frac{p}{q}}, \end{cases}$$

$$\sigma_1^2(z) = \mathbb{E}X_1^2(\mathbb{1}_{|X_1| \geq z}) = \begin{cases} 0, & z \leq \sqrt{\frac{q}{p}}, \\ \sqrt{\frac{q}{p}}, & \sqrt{\frac{q}{p}} < z \leq \sqrt{\frac{p}{q}}, \\ 1, & \sqrt{\frac{p}{q}} > z \leq \sqrt{\frac{p}{q}}, \end{cases}$$
\[ \gamma |\mu_1(z)| + z\sigma_1^2(z) = \begin{cases} z, & z \leq \sqrt{\frac{p}{p'}} \\ \gamma \sqrt{\frac{q}{p'}} + zp, & \sqrt{\frac{p}{p'}} < z \leq \sqrt{\frac{q}{q'}} \\ \gamma (\frac{p-q}{\sqrt{q'}}), & z > \sqrt{\frac{q}{q'}}. \end{cases} \]

(1) Compute \( L_{s,n}(g_s, \varepsilon, \gamma) \) and \( L_{s,n}(g_0, \varepsilon, \gamma) \). For all \( n \in \mathbb{N}, \varepsilon, \gamma > 0 \) we have
\[
L_{s,n}(g_s, \varepsilon, \gamma) = \frac{1}{\sqrt{n}} \sup_{0 < z < \varepsilon \sqrt{n}} \left\{ \gamma |\mu_1(z)| + z\sigma_1^2(z) \right\} =
\]
\[
= \begin{cases} \varepsilon, & ne^2 \leq \frac{q}{p}, \\
\max \left\{ \sqrt{\frac{q}{p'}} \gamma \sqrt{\frac{q}{p'}} + pe \right\}, & \frac{q}{p} < ne^2 \leq \frac{q}{q'} \\
\max \left\{ \sqrt{\frac{q}{p'}} \left( \gamma \sqrt{\frac{q}{p'}} + \sqrt{\frac{p'}{q'}} \right) 1(p > \frac{1}{2}), \frac{\gamma(p-q)}{\sqrt{p'q'}} \right\}, & ne^2 > \frac{p}{q}. \end{cases}
\]

in particular, \( L_{s,n}(g_s, \varepsilon, \gamma) = \min \{ \varepsilon, \frac{1}{\sqrt{n}} \} \) for \( p = \frac{1}{2} \). Now let us find \( L_{s,n}(g_0, \varepsilon, \gamma) \) for \( \varepsilon > 1 \). We have
\[
L_{s,n}(g_0, \varepsilon, \gamma) = \frac{1}{\sqrt{n}} \sup_{0 < z < \varepsilon \sqrt{n}} \left\{ \gamma |\mu_1(z)| + z\sigma_1^2(z) \right\} =
\]
\[
= \max \left\{ \frac{1}{\sqrt{n}} \sup_{0 < z < \varepsilon \sqrt{n}} \left\{ \gamma |\mu_1(z)| + z\sigma_1^2(z) \right\}, \sup_{\sqrt{n} \leq z < \varepsilon \sqrt{n}} \frac{1}{z^2} \left\{ \gamma |\mu_1(z)| + z\sigma_1^2(z) \right\} \right\} =
\]
\[
= \max \left\{ L_{s,n}(g_s, 1, \gamma), \sup_{\sqrt{n} \leq z < \varepsilon \sqrt{n}} \left\{ \frac{\gamma}{z} |\mu_1(z)| + \sigma_1^2(z) \right\} \right\}, \quad \gamma > 0.
\]
If \( p = q = 1/2 \), then \( L_{s,n}(g_s, \varepsilon, \gamma) = 1/\sqrt{n} \),
\[
\sup_{\sqrt{n} \leq z < \varepsilon \sqrt{n}} \left\{ \frac{\gamma}{z} |\mu_1(z)| + \sigma_1^2(z) \right\} = \sup_{\sqrt{n} \leq z < \varepsilon \sqrt{n}} \left\{ 1(z \leq 1) = 1(n = 1) \right\}
\]
and hence,
\[
L_{s,n}(g_0, \varepsilon, \gamma) = \max \left\{ \frac{1}{\sqrt{n}}, 1(n = 1) \right\} = \frac{1}{\sqrt{n}}, \quad \varepsilon > 1,
\]
so that we may write for all \( \varepsilon > 0 \)
\[
L_{s,n}(g_0, \varepsilon, \gamma) = \min \{ \varepsilon, \frac{1}{\sqrt{n}} \} = L_{s,n}(g_s, \varepsilon, \gamma) = L_{s,n}(g_s, \varepsilon \wedge 1, \gamma).
\]
If \( p > 1/2 \), then
\[
\sup_{\sqrt{n} \leq z < \varepsilon \sqrt{n}} \left\{ \frac{\gamma}{z} |\mu_1(z)| + \sigma_1^2(z) \right\} = \sup_{\sqrt{n} \leq z < \varepsilon \sqrt{n}} \left\{ \frac{\gamma}{z} \sqrt{\frac{p'}{p'}} + p, \quad z \leq \sqrt{\frac{p}{p'}}, \quad \frac{\gamma(p-q)}{\sqrt{p'q'}} \right\} =
\]
\[
= \begin{cases} \gamma \sqrt{\frac{q}{p'}} + p, & ne^2 \leq \frac{p}{q}, \\
\max \left\{ \gamma \sqrt{\frac{q}{p'}} + p, \frac{\gamma(p-q)}{\sqrt{q'}} \right\}, & \frac{p}{q} < ne^2, \\
\max \left\{ \gamma \sqrt{\frac{q}{p'}} + p, \frac{\gamma(p-q)}{\sqrt{q'}} \right\}, & n > \frac{p}{q}. \end{cases}
\]
and with the account of \( p \leq \sqrt{\frac{\sigma}{\mu}} \), \( \frac{\gamma(p-q)}{p} \leq \frac{\gamma(p-q)}{\sqrt{\mu p q}} \) we finally obtain
\[
L_{n,n}(g_0, \epsilon, \gamma) = L_{n,n}(g_*, 1, \gamma) = \begin{cases} 
\max \left\{ \frac{\epsilon}{\sqrt{n p}}, \frac{\sqrt{q}}{\sqrt{n p}} + p \right\}, & \text{for } n^2 \leq \frac{p}{\sqrt{q}}, \\
\max \left\{ \frac{\epsilon}{\sqrt{n p}}, \frac{\sqrt{q}}{\sqrt{n p}} + \sqrt{\frac{p}{\mu}} \frac{\gamma(p-q)}{\sqrt{\mu p q}} \right\}, & \text{for } n^2 > \frac{p}{\sqrt{q}}, 
\end{cases}
\]
for all \( \epsilon > 1 \) and \( \gamma > 0 \), i.e., for all \( \epsilon > 0 \) the identity
\[
L_{n,n}(g_0, \epsilon, \cdot) = L_{n,n}(g_*, \epsilon \wedge 1, \cdot)
\]
holds also for \( p > \frac{1}{2} \). In particular, with \( \gamma = 1 \) and \( p > \frac{1}{2} \) for all \( \epsilon > 0 \) we have
\[
L_{n,n}(g_*, 1, 1) = L_{n,n}(g_0, 1, 1) = L_{n,n}(g_0, \infty, 1) = \begin{cases} 
\frac{q^2 + p^2}{\sqrt{\mu p q}}, & \text{for } n \leq \frac{p}{\sqrt{q}}, \\
\frac{q^2}{\sqrt{\mu p q}}, & \text{for } n > \frac{p}{\sqrt{q}}, 
\end{cases}
\]
(2) Let us find \( L_{n,n}(g_C, \epsilon, \gamma) \) and \( L_{n,n}(g_1, \epsilon, \gamma) \). For all \( n \in \mathbb{N}, \epsilon, \gamma > 0 \) we have
\[
L_{n,n}(g_C, \epsilon, \gamma) = \sup_{0 < z < \epsilon \sqrt{n}} \left\{ \frac{\gamma}{2} |\mu_1(z)| + \sigma_1^2(z) \right\} = \sup_{0 < z < \epsilon \sqrt{n}} \left\{ \frac{\gamma}{2} \sqrt{\frac{\mu}{p}}, \sqrt{\frac{p}{\mu}} z \leq \frac{\sqrt{\mu}}{p} \right\} = \begin{cases} 
1, & \frac{\epsilon}{\sqrt{n p}} > \frac{p}{\sqrt{q}}, \\
\max \left\{ 1, \gamma q + p \right\}, & \frac{p}{\sqrt{q}} < \frac{\epsilon}{\sqrt{n p}} \leq \frac{p}{\sqrt{q}}, \\
\max \left\{ 1, (\gamma q + p) \gamma \right\}, & \frac{p}{\sqrt{q}} > \frac{\epsilon}{\sqrt{n p}}, 
\end{cases}
\]
in particular, \( L_{n,n}(g_C, \epsilon, \gamma) \equiv 1 \) for \( p = \frac{1}{2} \) and all \( \epsilon, \gamma > 0 \), as well as for \( p > \frac{1}{2}, \epsilon > 0, \gamma \leq 1 \). For \( \epsilon > 1 \) we have
\[
\sqrt{n} L_{n,n}(g_1, \epsilon, \gamma) = \sup_{0 < z < \epsilon \sqrt{n}} \left\{ \frac{\gamma}{2} \left| \mu_1(z) \right| + z \sigma_1^2(z) \right\} = 
= \max \left\{ \sup_{0 < z < \epsilon \sqrt{n}} \left\{ \frac{\gamma}{2} \left| \mu_1(z) \right| + z \sigma_1^2(z) \right\}, \sup_{\sqrt{n} \in \mathbb{Z} \cap \epsilon \sqrt{n}} \left\{ \frac{\gamma}{2} \left| \mu_1(z) \right| + z \sigma_1^2(z) \right\} \right\} = 
= \max \left\{ \sqrt{n} L_{n,n}(g_C, 1, \gamma), \sup_{\sqrt{n} \in \mathbb{Z} \cap \epsilon \sqrt{n}} \left\{ \frac{\gamma}{2} \left| \mu_1(z) \right| + z \sigma_1^2(z) \right\} \right\}, \gamma > 0.
\]
If \( p = q = 1/2 \), then
\[
\sup_{\sqrt{n} \in \mathbb{Z} \cap \epsilon \sqrt{n}} \left\{ \frac{\gamma}{2} \left| \mu_1(z) \right| + z \sigma_1^2(z) \right\} = \sup_{\sqrt{n} \in \mathbb{Z} \cap \epsilon \sqrt{n}} z \mathbf{1}(z \leq 1) = \mathbf{1}(n = 1),
\]
and hence,
\[
L_{n,n}(g_1, \epsilon, \gamma) = \max \left\{ 1, \frac{\epsilon}{\sqrt{n}}(n = 1) \right\} = 1, \quad \epsilon > 1, \gamma > 0, n \in \mathbb{N};
\]
otherwise
\[
\operatorname{sup}_{\sqrt{\pi} \leq z < \sqrt{\pi}} \left\{ \gamma |\mu_1(z)| + z\sigma_1^2(z) \right\} =
\]
\[
= \operatorname{sup}_{\sqrt{\pi} \leq z < \sqrt{\pi}} \left\{ \gamma \sqrt{\frac{p}{\pi}} + zp, \quad z \leq \sqrt{\frac{p}{\pi}}, \right\} \]
\[
= \operatorname{sup}_{\sqrt{\pi} \leq z < \sqrt{\pi}} \left\{ \gamma \frac{q^{3/2}}{\sqrt{\pi}} + p, \quad z > \sqrt{\frac{p}{\pi}}, \right\} = \left\{ \begin{array}{ll}
\gamma q^{3/2} \sqrt{\pi} + p, & \text{if } n^2 \leq \frac{p}{q}, \\
\max\{\gamma q^2 + p^2, \gamma(p - q)\} \sqrt{\frac{p}{\pi}}, & \text{if } \frac{p}{q} \leq n^2, \\
\gamma(p - q) \sqrt{\frac{p}{\pi}}, & \text{if } \frac{p}{q} > n^2, \\
\end{array} \right.
\]
and hence,
\[
L_{\varepsilon,n}(g_1, \varepsilon, \gamma) = \left\{ \begin{array}{ll}
\max\{1, \gamma q + p, \frac{q^{3/2}}{\sqrt{\pi}} + p\varepsilon\}, & n^2 \leq \frac{p}{q}, \\
\max\{1, \gamma q + p, \max\{\gamma q^2 + p^2, \gamma(p - q)\} \sqrt{\frac{p}{\pi}}, & \frac{p}{q} \leq n^2, \\
\max\{1, \gamma q + p, \frac{2(q - p)}{p}, \gamma(p - q) \sqrt{\frac{p}{\pi}}\} = \max\{1, \gamma q + p, \frac{2(q - p)}{p}\}, & n > \frac{p}{q}, \\
\end{array} \right.
\]
for all \( \varepsilon > 1 \) and \( \gamma > 0 \). In particular,
\[
L_{\varepsilon,n}(g_1, \varepsilon, 1) = \left\{ \begin{array}{ll}
\max\{1, \frac{q^{3/2}}{\sqrt{\pi}} + p\varepsilon\}, & n^2 \leq \frac{p}{q}, \\
\max\{1, \frac{p}{\sqrt{\pi}}\varepsilon, & \frac{p}{q} < n^2, \quad p > \frac{1}{2}, \quad \varepsilon > 1.
\end{array} \right.
\]

(3) Let us compute \( L_{\varepsilon,n}(g_s, \varepsilon, \gamma) \) and \( L_{\varepsilon,n}(g_0, \varepsilon, \gamma) \). With the account of
\[
\sup_{0 < z \leq \sqrt{\pi}} \max\{z, z \leq \sqrt{\frac{p}{\pi}}\} \varepsilon(z) = \sup_{0 < z \leq \sqrt{\pi}} \left\{ \begin{array}{ll}
z, & z \leq \sqrt{\frac{p}{\pi}}, \\
0, & z > \sqrt{\frac{p}{\pi}}, \\
\end{array} \right\} = \left\{ \begin{array}{ll}
\epsilon \sqrt{\pi}, & n^2 \leq \frac{p}{q}, \\
\max\left\{ \sqrt{\frac{p}{\pi}}p \varepsilon, \frac{p}{q} \right\} \varepsilon, & \frac{p}{q} < n^2 \leq \frac{p}{q}, \\
\max\left\{ \sqrt{\frac{p}{\pi}} \varepsilon, \sqrt{\pi} \varepsilon \right\}, & n > \frac{p}{q}, \\
\end{array} \right.
\]
and \( \max\left\{ \sqrt{\frac{p}{\pi}} \varepsilon, \sqrt{\pi} \varepsilon(p > \frac{1}{2}) \right\} = \max\left\{ \sqrt{\frac{p}{\pi}} \varepsilon, \sqrt{\pi} \varepsilon \right\}, p \in (\frac{1}{2}, 1), \) for all \( n \in \mathbb{N}, \varepsilon, \gamma > 0 \) we obtain
\[
L_{\varepsilon,n}(g_s, \varepsilon, \gamma) = \frac{1}{\sqrt{n}} \left\{ \gamma |\mu_1(\varepsilon \sqrt{n})| + \sup_{0 < z \leq \varepsilon \sqrt{n}} z\sigma_1^2(z) \right\} =
\]
\[
= \left\{ \begin{array}{ll}
\left\{ \frac{1}{\sqrt{n}} \varepsilon |\mu_1(\varepsilon \sqrt{n})|, & n^2 \leq \frac{p}{q}, \\
\max\left\{ \frac{q^{3/2}}{\sqrt{\pi}} \varepsilon, \frac{p}{q} \right\} \varepsilon, & \frac{p}{q} < n^2 \leq \frac{p}{q}, \\
\max\left\{ \frac{2}{\sqrt{\pi}} \varepsilon, \sqrt{\pi} \varepsilon \right\}, & n > \frac{p}{q}, \\
\end{array} \right.
\]
in particular, \( L_{\varepsilon,n}(g_s, \varepsilon, \gamma) = \min\{\epsilon, \frac{1}{\sqrt{n}}\} \) for \( p = \frac{1}{2} \). Now let \( \varepsilon > 1 \). Taking into account that \( \sigma_1^2(z) \) does not increase with respect to \( z \geq 0 \), and using the just computed \( \sup_{0 < z \leq \varepsilon \sqrt{n}} z\sigma_1^2(z) \), we obtain
\[
L_{\varepsilon,n}(g_0, \varepsilon, \gamma) = \frac{1}{\sqrt{n}} \left\{ \frac{1}{\varepsilon} |\mu_1(\varepsilon \sqrt{n})| + \sup_{0 < z \leq \varepsilon \sqrt{n}} \min\{z, \sqrt{n}\} \sigma_1^2(z) \right\} =
\]
\[
= \frac{1}{\sqrt{n}} \left\{ \frac{1}{\varepsilon} |\mu_1(\varepsilon \sqrt{n})| + \sup_{0 < z \leq \varepsilon \sqrt{n}} z\sigma_1^2(z) \right\} =
\]
3.2. Computation of the Uniform Distance

In particular, for $p = \frac{1}{2}$ we have $L_{n,n}(g_0, \varepsilon, \cdot) = \frac{1}{\sqrt{n}} = \min\left\{ \varepsilon, \frac{1}{\sqrt{n}} \right\} = L_{n,n}(g_0, \varepsilon, \cdot)$ for all $\varepsilon > 0$, and hence, for all $\varepsilon > 0$.

(4) Let us compute $L_{n,n}(g_C, \varepsilon, \gamma)$ and $L_{n,n}(g_1, \varepsilon, \gamma)$. Taking into account that $\sigma_1^2(z)$ does not increase with respect to $z \geq 0$ and $\sigma_1^2(0) = \sigma_1^2 = 1$, for all $n \in \mathbb{N}, \varepsilon, \gamma > 0$ and $p \in [\frac{1}{2}, 1)$ we obtain

$$L_{n,n}(g_C, \varepsilon, \gamma) = \frac{\gamma}{\varepsilon \sqrt{n}} |\mu_1(\varepsilon \sqrt{n})| + \sup_{0 < z < \sqrt{n}} \sigma_1^2(z) = \frac{\gamma}{\varepsilon \sqrt{n}} |\mu_1(\varepsilon \sqrt{n})| + 1 = \begin{cases} 1, & n \varepsilon^2 \leq \frac{p}{q}, \\ \frac{\gamma}{\varepsilon \sqrt{n}} \sqrt{n \varepsilon^2 \cap \frac{p}{q}}, & \frac{p}{q} < n \varepsilon^2 \leq \frac{p}{q}, \\ \frac{\gamma}{\varepsilon \sqrt{n}} \sqrt{n} + 1, & n \varepsilon^2 > \frac{p}{q}, \end{cases}$$

in particular, $L_{n,n}(g_C, \cdot, \cdot) \equiv 1$ for $p = \frac{1}{2}$ and $L_{n,n}(g_C, \infty, \cdot) \equiv 1$ for all $p \in [\frac{1}{2}, 1)$. Now let $\varepsilon > 1$. With the account of

$$\sup_{\sqrt{n} \varepsilon^2 < z < \sqrt{n}} z \sigma_1^2(z) = \begin{cases} z, & z \leq \sqrt{\frac{p}{q}}, \\ \sqrt{\frac{p}{q}}, & \sqrt{\frac{p}{q}} \leq z \leq \sqrt{\frac{p}{q}}, \\ 0, & z > \sqrt{\frac{p}{q}}, \end{cases}$$

we have

$$\sqrt{n}L_{n,n}(g_1, \varepsilon, \gamma) = \gamma |\mu_1(\varepsilon \sqrt{n})| + \sup_{0 < z < \sqrt{n}} (z \sqrt{n}) \sigma_1^2(z) = \gamma |\mu_1(\varepsilon \sqrt{n})| + \max_{\sqrt{n} \varepsilon^2 < z < \sqrt{n}} z \sigma_1^2(z) = \begin{cases} \gamma \sqrt{n} + \max\left\{ \frac{\gamma}{\varepsilon \sqrt{n}}, \sqrt{n} \right\}, & n \varepsilon^2 \leq \frac{p}{q}, \\ \frac{\gamma}{\varepsilon \sqrt{n}} \sqrt{n} + \max\left\{ \frac{\gamma}{\varepsilon \sqrt{n}}, \sqrt{n} \right\}, & \frac{p}{q} < n \varepsilon^2 \leq \frac{p}{q}, \\ \gamma \sqrt{n} + \max\left\{ \sqrt{n} \right\}, & n \varepsilon^2 > \frac{p}{q}, \end{cases}$$

in particular, $L_{n,n}(g_1, \varepsilon, \gamma) \equiv 1$ for $p = \frac{1}{2}$.

3.2. Computation of the Uniform Distance

In the present section, we compute the uniform distance $\Delta_1(p)$ between the d.f. of (27) and the standard normal d.f. Let us denote

$$\Psi(x) = \frac{1}{1 + x^2} - \Phi(-|x|) = \Phi(|x|) - \frac{x^2}{1 + x^2}, \quad x \in \mathbb{R}, \quad (33)$$
\( \Psi(p) = \Psi\left(\sqrt{\frac{1-p}{p}}\right) = \Phi\left(\sqrt{\frac{1-p}{p}}\right) - (1-p), \quad 0 < p < 1. \) \hspace{1cm} (34)

Please note that \( \Psi(x) \) is an even function, therefore, it suffices to investigate it only for \( x \geq 0 \).

**Lemma 1** (see [27]). The function \( \Psi(x) \) is positive for \( x \geq 0 \), increases for \( 0 < x < x_\Phi \), decreases for \( x > x_\Phi \) and attains its maximal value \( C_\Phi = 0.54093 \ldots \) in the point \( x_\Phi = 0.213105 \ldots \), where \( x_\Phi \) is the unique root of the equation

\[ xe^{x^2/2}(1+x^2)^{-2} = (8\pi)^{-1/2}. \]

**Remark 1.** The statement about the interval of monotonicity is absent in the formulation of the corresponding lemma in [27], but these intervals were investigated in the proof.

The definition of \( \tilde{\Psi} \) and lemma 1 immediately imply that the function

\[ \tilde{\Psi}(p) := \Phi\left(\sqrt{\frac{1-p}{p}}\right) - (1-p) \]

is positive for \( p \in (0,1) \), increases for \( 0 < p < p_\Phi \), decreases for \( p_\Phi < p < 1 \) and

\[ \max_{p \in (0,1)} \tilde{\Psi}(p) = \tilde{\Psi}(p_\Phi) = C_\Phi, \quad \text{where} \quad p_\Phi = \frac{1}{x_\Phi + 1} = 0.9565 \ldots. \]

**Lemma 2.** On the interval \( p \in (0,1) \) the function

\[ \phi(p) := \tilde{\Psi}(1-p) + p = \Phi\left(\sqrt{\frac{p}{1-p}}\right) \]

monotonically increases, while its derivative

\[ \phi'(p) = -\frac{e^{-\frac{p}{2(1-p)}}}{2\sqrt{2\pi}\sqrt{p(1-p)^3}} \]

monotonically decreases.

**Proof.** The function \( \phi(p) \) monotonically increases as a superposition of monotonically increasing functions. Please note that the derivative \( \phi'(p) \) takes only positive values, hence, we may define its logarithm

\[ \ln \phi'(p) = -\frac{p}{2(1-p)} - \ln(2\sqrt{2\pi}) - \frac{1}{2} \ln p - \frac{3}{2} \ln(1-p) =: u(p). \]

The derivative

\[ u'(p) = -\frac{1}{2(1-p)^2} - \frac{1}{2p} + \frac{3}{2(1-p)} = 0 \]

changes its sign in the points \( p \in (0,1) \) that are the roots of the equation

\[ -p - (1-p)^2 + 3p(1-p) = 0, \]

which has a unique solution \( p = 1/2 \) on \((0,1)\). Since

\[ \frac{d}{dp} u(p) \big|_{p=\frac{1}{2}} = -\frac{8}{9}, \quad \frac{d}{dp} u(p) \big|_{p=\frac{1}{2}} = -\frac{8}{3}. \]
then \( u(p) \) is strictly decreasing on \((0, 1)\), and hence, \( \phi'(p) \) is also strictly decreasing on \((0, 1)\). \( \square \)

**Theorem 2.** Let \( X_1 \) have distribution (27), then

\[
\Delta_1(p) := \sup_{x \in \mathbb{R}} |P(X_1 < x) - \Phi(x)| = \Psi(p) = \begin{cases} \Phi(\sqrt{p/q}) - p, & 0 < p < \frac{1}{2}, \\ \Phi(\sqrt{q/p}) - q, & \frac{1}{2} \leq p < 1. \end{cases}
\]

**Proof.** Please note that

\[
\Delta_1(p) = \max \left\{ \Phi(-\sqrt{p/q}), |\Phi(-\sqrt{p/q}) - q|, |\Phi(\sqrt{q/p}) - q|, 1 - \Phi(\sqrt{q/p}) \right\} = \max \{1 - \phi(p), \Psi(q), \Psi(p), 1 - \phi(q)\} = \max \{f_1(p), f_2(p), f_3(p), f_4(p)\},
\]

where

\[
\begin{align*}
&f_1(p) = 1 - \phi(p), \\
&f_2(p) = \Psi(1 - p) = \phi(p) - p, \\
&f_3(p) = \Psi(p) = \phi(1 - p) - (1 - p), \\
&f_4(p) = 1 - \phi(1 - p).
\end{align*}
\]

Let \( p \geq q \). Then \( f_1(p) \leq f_4(p) \), and it suffices to show that

\[
\max \{f_2(p), f_4(p)\} \leq f_3(p).
\]

Let us prove that \( f_5(p) := f_3(p) - f_4(p) \geq 0 \). Using lemma 2, we conclude that the derivative

\[
f'_5(p) := \frac{d}{dp} (2\phi(1 - p) + p - 2) = 2\frac{d\phi(1 - p)}{dp} + 1 = 1 - \frac{e^{-\frac{(1-p)}{2p}}}{\sqrt{2\pi}\sqrt{(1-p)p}}
\]

decreases on \((0, 1)\) with

\[
f'_5(\frac{1}{2}) = 1 - \frac{4}{\sqrt{2\pi}} = 0.0321 \ldots, \quad \lim_{p \to 1^-} f'_5(p) = -\infty,
\]

hence, \( f_5(p) \) has a unique stationary point on \((0, 1)\), which is a point of maximum, so that

\[
\inf_{\frac{1}{2} \leq p < 1} f_5(p) = \min \left\{ f_5(\frac{1}{2}), \lim_{p \to 1^-} f_5(p) \right\} = \min \{2\Phi(1) - \frac{3}{2}, 0\} = \min \{0.1826 \ldots, 0\} = 0.
\]

The inequality \( f_2(p) \leq f_3(p) \) follows from the properties of the function \( \Psi \) with the account of

\[
\Psi(\frac{1}{2}) \wedge \Psi(1 - 0) = \Psi(\frac{1}{2}).
\]

Thus, the theorem has been proved in the case \( p \geq q \). The validity of the statement of the theorem for \( p < q \) follows from that \( f_1(p) = f_4(q) \) and \( f_2(p) = f_3(q) \). \( \square \)

**Lemma 3** (see [27,28]). For an arbitrary d.f. \( F \) with zero mean and unit variance we have

\[
\sup_{x \in \mathbb{R}} |F(x) - \Phi(x)| \leq \sup_{x \in \mathbb{R}} \Psi(x) = \Psi(x_{\Phi}) = \Psi(p_{\Phi}) = \Delta_1(p_{\Phi}) =: C_{\Phi} = 0.54093 \ldots
\]

**Remark 2.** In [28] [(2.32), (2.33)] it is proved that

\[
|F(x) - \Phi(x)| \leq \Psi(x), \quad x \in \mathbb{R}.
\]
In [27] it is proved that in the inequality
\[
\sup_{x \in \mathbb{R}} |F(x) - \Phi(x)| \leq \sup_{x \in \mathbb{R}} \Psi(x) = C_\Phi
\]
the equality is attained at a two-point distribution.

4. Lower Bounds for the Exact Constants

In the present section, we construct lower bounds for the quantities

\[
\inf_{g \in \mathcal{G}} C_\varepsilon(g, \varepsilon, \gamma), \quad \inf_{g \in \mathcal{G}} C_\gamma(g, \varepsilon, \gamma),
\]

\[
C_\varepsilon(g, \varepsilon, \gamma) = \sup_{g \in \mathcal{G}} C_\varepsilon(g, \varepsilon, \gamma), \quad C_\gamma(g, \varepsilon, \gamma) = \sup_{g \in \mathcal{G}} C_\gamma(g, \varepsilon, \gamma)
\]

and

\[
A_\varepsilon(\varepsilon, \gamma) = C_\varepsilon(\varepsilon, \varepsilon, \gamma), \quad A_\varepsilon(\varepsilon, \gamma) = C_\gamma(\varepsilon, \varepsilon, \gamma)
\]

for all \(\varepsilon, \gamma > 0\). Recall that

\[
g_0(z) = \min\{z, B_n\}, \quad g_1(z) = \max\{z, B_n\}, \quad g_\varepsilon(z) = z, \quad g_\gamma(z) = 1, \quad z \geq 0.
\]

By virtue of invariance of the fractions \(L_{\varepsilon, n}(g, \varepsilon, \gamma), L_{\gamma, n}(g, \varepsilon, \gamma)\) with respect to scale transform of \(g\) and extreme property of \(g_1\) (see (15)) we have

\[
\inf_{g \in \mathcal{G}} C_\varepsilon(g, \varepsilon, \gamma) = \inf_{g \in \mathcal{G}} \sup_{F_1, \ldots, F_n \in \mathcal{F}, n \in \mathbb{N}, B_n > 0} \frac{\Delta_n(F_1, \ldots, F_n)}{L_{\varepsilon, n}(g, \varepsilon, \gamma)} = \inf_{g \in \mathcal{G}} \sup_{F_1, \ldots, F_n \in \mathcal{F}, n \in \mathbb{N}, B_n = 1} \frac{\Delta_n(F_1, \ldots, F_n)}{L_{\varepsilon, n}(g_1, \varepsilon, \gamma)}
\]

on one hand, and \(\inf_{g \in \mathcal{G}} C_\varepsilon(g, \varepsilon, \gamma) \leq C_\varepsilon(g_1, \varepsilon, \gamma)\) by definition of the lower bound, on the other hand. Therefore

\[
\inf_{g \in \mathcal{G}} C_\varepsilon(g, \varepsilon, \gamma) = C_\varepsilon(g_1, \varepsilon, \gamma), \quad \inf_{g \in \mathcal{G}} C_\gamma(g, \varepsilon, \gamma) = C_\gamma(g_1, \varepsilon, \gamma),
\]

where the second equality is proved similar to the first one.

Let us show that

\[
C_\varepsilon(\varepsilon, \gamma) = C_\varepsilon(g_0, \varepsilon, \gamma), \quad C_\gamma(\varepsilon, \gamma) = C_\gamma(g_0, \varepsilon, \gamma).
\]

Indeed, for arbitrary \(n \in \mathbb{N}, F_1, \ldots, F_n \in \mathcal{F}\) and \(g \in \mathcal{G}\) due to the extremality of \(g_0\) (see (15)) we have

\[
\Delta_n \leq C_\varepsilon(g_0, \varepsilon, \gamma) L_{\varepsilon, n}(g_0, \varepsilon, \gamma) \leq C_\varepsilon(g_0, \varepsilon, \gamma) L_{\varepsilon, n}(g, \varepsilon, \gamma),
\]

hence, \(C_\varepsilon(\varepsilon, \gamma) \leq C_\varepsilon(g_0, \varepsilon, \gamma)\), on one hand. On the other hand, this inequality may hold true only with the equality sign, since \(C_\varepsilon(\varepsilon, \gamma) = \sup_{g \in \mathcal{G}} C_\varepsilon(g, \varepsilon, \gamma)\) by definition. The same reasoning holds also true for \(C_\gamma(\varepsilon, \gamma)\).

Thus, \(C_\varepsilon(g_1, \varepsilon, \gamma), C_\gamma(g_1, \varepsilon, \gamma)\) are the most optimistic constants, while \(C_\varepsilon(g_0, \varepsilon, \gamma), C_\gamma(g_0, \varepsilon, \gamma)\) are the most pessimistic, but universal (exact) ones. The next theorem establishes lower bounds for the exact constants \(C_\varepsilon(\varepsilon, \gamma) = C_\varepsilon(g_0, \varepsilon, \gamma), C_\gamma(\varepsilon, \gamma) = C_\gamma(g_0, \varepsilon, \gamma)\), and also for the constants \(A_\varepsilon(\varepsilon, \gamma)\) and \(A_\varepsilon(\varepsilon, \gamma)\) appearing in inequalities (1) and (2).
Theorem 3. (i) For all $\varepsilon, \gamma > 0$ we have
\[
\min\{C_\varepsilon(\varepsilon, \gamma), C_\varepsilon(\varepsilon, \gamma), A_\varepsilon(\varepsilon, \gamma), A_\varepsilon(\varepsilon, \gamma)\} \geq \frac{\Phi(1) - 0.5}{\min\{1, \varepsilon\}} > 0.3413 \frac{1}{\min\{1, \varepsilon\}}.
\]
(ii) For $\frac{1}{2} < p < 1$, $q = 1 - p$ set
\[
\Delta_1(p) := \Phi(\sqrt{q/p}) - q.
\]
Take any $\gamma > 0$. Let us denote for $\varepsilon \leq 1$ :
\[
K_{\varepsilon,0}(p, \varepsilon, \gamma) = \begin{cases}
\Delta_1(p)/\varepsilon, & \frac{1}{2} < p \leq \frac{1}{\varepsilon + \gamma}, \\
\Delta_1(p)/\max\{\sqrt{q/p}, \gamma \sqrt{q^2/p + \varepsilon p}\}, & \frac{1}{\varepsilon + \gamma} < p < 1,
\end{cases}
\]
and for $\varepsilon > 1$ :
\[
K_{\varepsilon,0}(p, \varepsilon, \gamma) = \begin{cases}
\Delta_1(p)/\varepsilon, & \frac{1}{\varepsilon + \gamma} \leq p < 1,
\end{cases}
\]
\[
K_{\varepsilon,1}(p, \varepsilon, \gamma) = \begin{cases}
\Delta_1(p)/\max\{\sqrt{q/p}, \gamma \sqrt{q^2/p + \varepsilon p}\}, & \frac{1}{\varepsilon + \gamma} \leq p < 1,
\end{cases}
\]
\[
K_{\varepsilon,0}(p, \varepsilon, \gamma) = \begin{cases}
\Delta_1(p)/\max\{\sqrt{q/p}, \gamma \sqrt{q^2/p + \varepsilon p}\}, & \frac{1}{\varepsilon + \gamma} \leq p < 1,
\end{cases}
\]
Then for all $\varepsilon, \gamma > 0$ we have
\[
C_\varepsilon(\varepsilon, \gamma) \geq \sup_{\frac{1}{2} < p < 1} K_{\varepsilon,0}(p, \varepsilon, \gamma), \quad C_\varepsilon(\varepsilon, \gamma) \geq \sup_{\frac{1}{2} < p < 1} K_{\varepsilon,0}(p, \varepsilon, \gamma),
\]
and for $\varepsilon > 1$ also
\[
A_\varepsilon(\varepsilon, \gamma) \geq \sup_{\frac{1}{2} < p < 1} K_{\varepsilon,1}(p, \varepsilon, \gamma), \quad A_\varepsilon(\varepsilon, \gamma) \geq \sup_{\frac{1}{2} < p < 1} K_{\varepsilon,0}(p, \varepsilon, \gamma).
\]
In particular,
\[
A_\varepsilon(\infty, 1) \geq \sup_{\frac{1}{2} < p < 1} \frac{(\Phi(\sqrt{q/p} - q))\sqrt{q/p}}{q^2 + p^2} > 0.3703,
\]
and for the both constants $C_\varepsilon \in \{C_\varepsilon, C_\varepsilon\}$ with $\varepsilon = 1, \gamma = 1$ the following lower bound holds:
\[
\min\{C_\varepsilon(1, 1), C_\varepsilon(\infty, 1)\} \geq \sup_{\frac{1}{2} < p < 1} \frac{\Phi(\sqrt{q/p} - q)}{\sqrt{q^2/p + p}} > 0.5685 \quad (p = 0.9058 \ldots).
\]
Values of the greatest lower bounds of $K_{\varepsilon,1}(p, \varepsilon, \gamma)$ and $K_{\varepsilon,0}(p, \varepsilon, \gamma)$ for some $\varepsilon \geq 1$ and $\gamma > 0$ are given in Tables 1 and 2, respectively. Values of the greatest lower bounds of $K_{\varepsilon,0}(p, \varepsilon, \gamma)$ and $K_{\varepsilon,0}(p, \varepsilon, \gamma)$ with the corresponding extremal values of $p$ are given in Table 4 for some $\varepsilon > 0$ and $\gamma > 0$. 
Table 4. Values of the lower bounds for the constants $C_\varepsilon(\varepsilon, \gamma)$ and $C_\eta(\varepsilon, \gamma)$ obtained in Theorem 3 for some $\varepsilon > 0$ and $\gamma > 0$.

| $\varepsilon$ | $\gamma$ | $C_\varepsilon(\varepsilon, \gamma)$ | $p$ | $\varepsilon$ | $\gamma$ | $C_\varepsilon(\varepsilon, \gamma)$ | $p$ |
|--------------|----------|-------------------------------|-----|--------------|----------|-------------------------------|-----|
| 0.2          | 0.2      | 2.78472                       | 0.96368 | 0.2          | 0.2      | 2.78535                       | 0.96415 |
| 0.5          | 0.2      | 1.17226                       | 0.84124 | 0.5          | 0.2      | 1.17196                       | 0.84770 |
| $\geq 1$     | 0.2      | 0.60108                       | 0.74349 | 1             | 0.2      | 0.60108                       | 0.74349 |
| 0.2          | 0.4      | 2.76543                       | 0.96318 | 1.21          | 0.2      | 0.60572                       | 0.70768 |
| 0.5          | 0.4      | 1.14539                       | 0.89145 | 2             | 0.2      | 0.60674                       | 0.80000 |
| $\geq 1$     | 0.4      | 0.58619                       | 0.83356 | 3             | 0.2      | 0.59194                       | 0.68233 |
| 0.2          | $\gamma_*$ | 2.74959                      | 0.96276 | 4             | 0.2      | 0.60265                       | 0.68233 |
| 0.5          | $\gamma_*$ | 1.13293                      | 0.91254 | 5.39          | 0.2      | 0.61121                       | 0.68233 |
| $\geq 1$     | $\gamma_*$ | 0.57952                      | 0.86446 | 8             | 0.2      | 0.61947                       | 0.68232 |
| 0.2          | 0.72     | 2.73337                       | 0.96233 | 0.2           | 0.4      | 2.76388                       | 0.96415 |
| 0.5          | 0.72     | 1.12389                       | 0.92595 | 0.5           | 0.4      | 1.14539                       | 0.89145 |
| $\geq 1$     | 0.72     | 0.57469                       | 0.88402 | 1             | 0.4      | 0.58619                       | 0.83356 |
| 0.2          | 0.97     | 2.71102                       | 0.96906 | 1.76          | 0.4      | 0.59815                       | 0.76397 |
| 0.5          | 0.97     | 1.11352                       | 0.93958 | 2             | 0.4      | 0.59934                       | 0.80000 |
| $\geq 1$     | 0.97     | 0.56910                       | 0.90399 | 2.63          | 0.4      | 0.59228                       | 0.87369 |
| $\geq 1$     | 1        | 0.56854                       | 0.90586 | 4             | 0.4      | 0.57272                       | 0.94118 |
| $\geq 1$     | 1.43     | 0.56207                       | 0.92560 | 0.2           | $\gamma_*$ | 2.74835                       | 0.96415 |
| $\geq 1$     | 1.5      | 0.56122                       | 0.92795 | 0.5           | $\gamma_*$ | 1.13293                       | 0.91254 |
| $\geq 1$     | 1.62     | 0.55987                       | 0.93161 | 1             | $\gamma_*$ | 0.57952                       | 0.86446 |
| $\geq 1$     | 2        | 0.55624                       | 0.94085 | 1.5           | $\gamma_*$ | 0.58759                       | 0.82642 |
| $\geq 1$     | 3        | 0.54955                       | 0.95569 | 1.99          | $\gamma_*$ | 0.59346                       | 0.79839 |
| $\geq 1$     | 4        | 0.54507                       | 0.96419 | 2.12          | $\gamma_*$ | 0.59407                       | 0.81800 |
| $\geq 1$     | 5        | 0.54176                       | 0.96978 | 3             | $\gamma_*$ | 0.58546                       | 0.90000 |
| $\geq 1$     | $\infty$ | 0                             | 5      | $\gamma_*$ | 0.56465                | 0.68232 |

Proof. Let $n = 1$ and the r.v. $X_1$ have the two-point distribution (27) with $p \in [\frac{1}{2}, 1)$. Then, by Theorem 2, we have $\Delta_1(F_1) = \Delta_1(p)$, and all the constants $C_\bullet \in \{C_\varepsilon, C_\eta\}, A_\bullet \in \{A_\varepsilon, A_\eta\}$ can be bounded from below as

$$C_\bullet(\varepsilon, \gamma) \geq \sup_{\frac{1}{2} \leq p < 1} \frac{\Delta_1(p)}{L_{\bullet, 1}(g_0, \varepsilon, \gamma)}, \quad A_\bullet(\varepsilon, \gamma) \geq \sup_{\frac{1}{2} \leq p < 1} \frac{\Delta_1(p)}{L_{\bullet, 1}(g^*, \varepsilon, \gamma)}.$$

In particular, for $p = \frac{1}{2}$ we have $\Delta_1(\frac{1}{2}) = \Phi(1) - 0.5 = 0.3413\ldots$, and by Theorem 1

$$L_{\varepsilon, n}(g_0, \varepsilon, \gamma) = L_{\eta, n}(g_0, \varepsilon, \gamma) = L_{\varepsilon, n}(g^*, \varepsilon, \gamma) = L_{\eta, n}(g^*, \varepsilon, \gamma) = \min\{1, \varepsilon\}, \quad \varepsilon, \gamma > 0,$$

whence the statement of point (i) follows immediately.

To prove point (ii) it suffices to make sure that for all $p \in (\frac{1}{2}, 1)$ and $\varepsilon, \gamma > 0$

$$K_{\bullet, 0}(p, \varepsilon, \gamma) = \frac{\Delta_1(p)}{L_{\bullet, 1}(g_0, \varepsilon, \gamma)}, \quad K_{\bullet, \gamma}(p, \varepsilon, \gamma) = \frac{\Delta_1(p)}{L_{\bullet, 1}(g^*, \varepsilon, \gamma)}, \quad \bullet \in \{s, \varepsilon\}.$$
Theorem 1 (i) with \( n = 1 \) implies that for \( p > \frac{1}{2}, \varepsilon, \gamma > 0 \)

\[
L_{n,1}(g^*, \varepsilon, \gamma) = \begin{cases}
\varepsilon, & e^2 \leq \frac{q}{p}, \\
\max\left\{\sqrt[\frac{p}{2}]\gamma \sqrt[\frac{p}{2}] + e p\right\}, & \frac{q}{p} < e^2 \leq \frac{p}{q}, \\
\max\left\{q, \gamma q^2 + p^2, \gamma(p - q)\right\} / \sqrt{pq}, & e^2 > \frac{p}{q}.
\end{cases}
\]

\[
L_{n,1}(g^*, \varepsilon, \gamma) = \begin{cases}
\varepsilon, & e^2 \leq \frac{q}{p}, \\
\gamma \sqrt[\frac{p}{2}] + \max\left\{\sqrt[\frac{p}{2}]p, e p\right\}, & \frac{q}{p} < e^2 \leq \frac{p}{q}, \\
\gamma(p-q) \sqrt[\frac{p}{2}] + \max\left\{\sqrt[\frac{p}{2}]p, \sqrt[\frac{p}{2}]q\right\}, & e^2 > \frac{p}{q}.
\end{cases}
\]

\[
L_{n,1}(g_0, \varepsilon, \gamma) = L_{n,1}(g^*, \varepsilon \land 1, \gamma), \text{ while for } \varepsilon > 1, \text{ by the same Theorem 1 (iii), we have}
\]

\[
L_{n,1}(g_0, \varepsilon, \gamma) = \begin{cases}
\varepsilon, & e^2 \leq \frac{q}{p}, \\
\gamma \sqrt[\frac{p}{2}] + \max\left\{\sqrt[\frac{p}{2}]p, e p\right\}, & 1 < e^2 \leq \frac{p}{q}, \\
\gamma(p-q) \sqrt[\frac{p}{2}] + \max\left\{\sqrt[\frac{p}{2}]p, \sqrt[\frac{p}{2}]q\right\}, & 1 < \frac{p}{q} < e^2.
\end{cases}
\]

(recall that \( L_{n,1}(g_0, \varepsilon, \gamma) = L_{n,1}(g^*, \varepsilon, \gamma) \) for \( \varepsilon \leq 1 \)). Please note that for all \( \varepsilon > 0 \) and \( p \in (\frac{1}{2}, 1) \)

\[
\frac{q}{p} < e^2 \leq \frac{p}{q} \implies \begin{cases}
p > \frac{1}{\varepsilon^2 + 1}, & 1 < e^2 \leq \frac{p}{q}, \\
p > \frac{1}{\varepsilon^2 + 1}, & 1 < \frac{p}{q} < e^2.
\end{cases}
\]

so that from the three conditions in (35) under the additional condition \( p \in (\frac{1}{2}, 1) \) there remain only two:

\[
\begin{align*}
\frac{1}{2} < p & \leq \frac{1}{\varepsilon^2 + 1} \quad \text{for } \varepsilon \leq 1, & \frac{\varepsilon^2}{\varepsilon^2 + 1} \leq p & \leq 1, \\
\frac{1}{\varepsilon^2 + 1} < p & < 1 \quad \text{for } \varepsilon > 1.
\end{align*}
\]

In particular, for \( \varepsilon \leq 1 \)

\[
L_{n,1}(g_0, \varepsilon, \gamma) = L_{n,1}(g^*, \varepsilon, \gamma) = \begin{cases}
\varepsilon, & 1 < p \leq \frac{1}{\varepsilon^2 + 1}, \\
\max\left\{\sqrt[\frac{p}{2}]\gamma \sqrt[\frac{p}{2}] + e p\right\}, & \frac{1}{\varepsilon^2 + 1} < p < 1,
\end{cases}
\]

\[
L_{n,1}(g^*, \varepsilon, \gamma) = \begin{cases}
\varepsilon, & 1 < p \leq \frac{1}{\varepsilon^2 + 1}, \\
\gamma \sqrt[\frac{p}{2}] + \max\left\{\sqrt[\frac{p}{2}]p, e p\right\}, & \frac{1}{\varepsilon^2 + 1} < p < 1,
\end{cases}
\]

and for \( \varepsilon > 1 \)

\[
L_{n,1}(g^*, \varepsilon, \gamma) = \begin{cases}
\max\left\{\sqrt[\frac{p}{2}]\gamma \sqrt[\frac{p}{2}] + e p\right\}, & \frac{\varepsilon^2}{\varepsilon^2 + 1} \leq p < 1, \\
\max\left\{q, \gamma q^2 + p^2, \gamma(p - q)\right\} / \sqrt{pq}, & \frac{1}{2} < p \leq \frac{\varepsilon^2}{\varepsilon^2 + 1}.
\end{cases}
\]

\[
L_{n,1}(g_0, \varepsilon, \gamma) = L_{n,1}(g^*, 1, \gamma) = \max\left\{\sqrt[\frac{p}{2}]\gamma \sqrt[\frac{p}{2}] + p\right\}, & \frac{1}{2} < p < 1,
\]

\[
L_{n,1}(g^*, \varepsilon, \gamma) = \begin{cases}
\gamma \sqrt[\frac{p}{2}] + \max\left\{\sqrt[\frac{p}{2}]p, e p\right\}, & \frac{\varepsilon^2}{\varepsilon^2 + 1} \leq p < 1, \\
\gamma(p-q) \sqrt[\frac{p}{2}] + \max\left\{\sqrt[\frac{p}{2}]p, \sqrt[\frac{p}{2}]q\right\}, & \frac{1}{2} < p \leq \frac{\varepsilon^2}{\varepsilon^2 + 1}.
\end{cases}
\]
\[ L_{t,1}(g_0, \epsilon, \gamma) = \begin{cases} \frac{2}{\epsilon} \sqrt{\frac{q}{p}} + \max \left\{ \sqrt{\frac{q}{p}}, p \right\}, & \frac{2}{\epsilon+1} \leq p < 1, \\ \gamma^{(p-q)} + \max \left\{ \sqrt{\frac{q}{p}}, p \right\}, & \frac{1}{\epsilon+1} < p < \frac{2}{\epsilon+1}, \end{cases} \]
whence we obtain the above expressions for \( K_{s,1}, K_{s,1} \).

Now let us consider the particular cases. The coinciding lower bounds for \( C_t(\infty, 1), C_t(1, 1) \) and \( C_t(1, 1) \) follow from that \( L_{t,1}(g_0, \infty, 1) = \sqrt{q^3/p} + p \) for all \( p \in (\frac{1}{2}, 1) \), and also from, as it can easily be made sure,

\[ L_{t,1}(g_0, 1, 1) = L_{t,1}(g_0, \infty, 1) = \sqrt{q^3/p} + p = L_{t,1}(g_0, 1, 1) \]
for \( p \geq p_0 \), where \( p_0 = 0.6823 \ldots \) is the unique root of the equation \( p^3 + p - 1 = 0 \) on the segment \([0.5, 1] \). The lower bound for \( A_t(\infty, 1) \) follows from that \( L_{t,1}(g_*, \infty, 1) = (q^2 + p^2)/\sqrt{pq} \). \( \Box \)

Now let us find lower bounds for the most optimistic constants \( C_t(g_1, \epsilon, \gamma) \) and \( C_t(g_1, \epsilon, \gamma) \).

**Theorem 4.** (i) For all \( \epsilon, \gamma > 0 \) we have

\[ \min\{C_t(g_1, \epsilon, \gamma), C_t(\epsilon, \gamma)\} \geq \Phi(1) - 0.5 > 0.3413. \]

(ii) For \( \frac{1}{2} < p < 1, q = 1 - p \) set

\[ \Delta_1(p) := \Phi(\sqrt{q/p}) - q. \]

Take any \( \gamma > 0 \). Let us denote for \( \epsilon \leq 1 \):

\[ K_{t,1}(p, \epsilon, \gamma) = \begin{cases} \Delta_1(p), & \frac{1}{\epsilon} < p \leq \frac{1}{\epsilon+1}, \\ \Delta_1(p) / \max\{1, \gamma p + q\}, & \frac{1}{\epsilon+1} < p < 1, \end{cases} \]

\[ K_{t,1}(p, \epsilon, \gamma) = \begin{cases} \Delta_1(p), & \frac{1}{\epsilon} < p \leq \frac{1}{\epsilon+1}, \\ \Delta_1(p) / \left(1 + \frac{2}{\epsilon} \sqrt{\frac{p}{q}}\right), & \frac{1}{\epsilon+1} < p < 1, \end{cases} \]
and for \( \epsilon > 1 \):

\[ K_{t,1}(p, \epsilon, \gamma) = \begin{cases} \Delta_1(p) / \max\left\{1, \gamma p + q, \frac{\gamma^2 + p^2}{\sqrt{pq}}, \frac{\gamma^{(p-q)}}{\sqrt{pq}}\right\}, & \frac{1}{\epsilon} < p < \frac{2}{\epsilon+1}, \\ \Delta_1(p) / \max\{1, \gamma p + q, \frac{\sqrt{\frac{p}{q}}}{\gamma^{(p-q)}}\}, & \frac{2}{\epsilon+1} \leq p < 1, \end{cases} \]

\[ K_{t,1}(p, \epsilon, \gamma) = \begin{cases} \Delta_1(p) / \left(\frac{2(p-q)}{\sqrt{pq}} + \max\left\{\sqrt{\frac{2}{q}}, 1\right\}\right), & \frac{1}{\epsilon} < p < \frac{2}{\epsilon+1}, \\ \Delta_1(p) / \left(\gamma \sqrt{\frac{q}{p}} + \max\{ep, 1\}\right), & \frac{2}{\epsilon+1} \leq p < 1. \end{cases} \]

Then for all \( \epsilon, \gamma > 0 \) we have

\[ C_{t}(g_1, \epsilon, \gamma) \geq \sup_{\frac{1}{2} < p < 1} K_{t,1}(p, \epsilon, \gamma), \quad C_{t}(g_1, \epsilon, \gamma) \geq \sup_{\frac{1}{2} < p < 1} K_{t,1}(p, \epsilon, \gamma). \]

In particular,

\[ C_{t}(g_1, \epsilon, \gamma) \geq \sup_{\frac{1}{2} < p < 1} \Delta_1(p) = \Delta_1(p_\Phi) =: C_\Phi = 0.5409 \ldots, \]
if \( \gamma \leq 1, \epsilon \leq 1 \) or \( \epsilon \leq x_\Phi, \) where \( x_\Phi = 0.213105 \ldots \) is defined in Lemma 1, \( p_\Phi = (x_\Phi^2 + 1)^{-1} = 0.9565 \ldots, \)

\[ C_{t}(g_1, \infty, 1) \geq \sup_{\frac{1}{2} < p < 1} \frac{(\Phi(\sqrt{q/p}) - q) \sqrt{pq}}{q^2 + p^2} > 0.3703, \]
\[ C_\varepsilon(g_1, \varepsilon, \gamma) \geq C_\Phi \quad \text{for} \quad \varepsilon \leq x_\Phi, \]

\[ C_\varepsilon(g_1, 1, 1) \geq \sup_{1 < p < 1} K_{s,1}(p, 1, 1) = \left. \frac{\Delta_1(p)}{1 + \sqrt{(1 - p)^3 / p}} \right|_{p = 0.9678...} > 0.5370. \]

Values of the greatest lower bounds of \( K_{s,1}(p, \varepsilon, \gamma) \) and \( K_{s,1}(p, \varepsilon, \gamma) \) with the corresponding extremal values of \( p \) are given in Table 5 for some \( \varepsilon > 0 \) and \( \gamma > 0 \).

**Table 5.** Values of the lower bounds for the most optimistic constants \( C_\varepsilon(g_1, \varepsilon, \gamma) \) and \( C_\varepsilon(g_1, \varepsilon, \gamma) \) obtained in Theorem 3 for some \( \varepsilon > 0 \) and \( \gamma > 0 \).

| \( \varepsilon \) | \( \gamma \) | \( C_\varepsilon(g_1, \varepsilon, \gamma) \) | \( p \) | \( \varepsilon \) | \( \gamma \) | \( C_\varepsilon(g_1, \varepsilon, \gamma) \) | \( p \) |
|---|---|---|---|---|---|---|---|
| 0.2 | 0.2 | 0.54093 | 0.95655 | 0.2 | 0.2 | 2.78355 | 0.96415 |
| 0.5 | 0.2 | 0.54093 | 0.95655 | 0.5 | 0.2 | 1.17196 | 0.84770 |
| \( \geq 1 \) | 0.2 | 0.54093 | 0.95655 | 1 | 0.2 | 0.60108 | 0.74349 |
| 0.2 | 0.4 | 0.54093 | 0.95655 | 1.21 | 0.2 | 0.60572 | 0.70768 |
| \( \geq 1 \) | 0.4 | 0.54093 | 0.95655 | 2 | 0.2 | 0.60674 | 0.80000 |
| 0.2 | \( \gamma_* \) | 0.54093 | 0.95655 | 4 | 0.2 | 0.60265 | 0.68233 |
| 0.5 | \( \gamma_* \) | 0.54093 | 0.95655 | 5.39 | 0.2 | 0.61121 | 0.68233 |
| \( \geq 1 \) | \( \gamma_* \) | 0.54093 | 0.95655 | 8 | 0.2 | 0.61947 | 0.68232 |
| 0.2 | 0.72 | 0.54093 | 0.95655 | 0.2 | 0.4 | 2.76388 | 0.96415 |
| 0.5 | 0.72 | 0.54093 | 0.95655 | 0.5 | 0.4 | 1.14539 | 0.89145 |
| \( \geq 1 \) | 0.72 | 0.54093 | 0.95655 | 1 | 0.4 | 0.58619 | 0.83356 |
| 0.2 | 0.97 | 0.54093 | 0.95655 | 1.76 | 0.4 | 0.59815 | 0.76397 |
| 0.5 | 0.97 | 0.54093 | 0.95655 | 2 | 0.4 | 0.59934 | 0.80000 |
| \( \geq 1 \) | 0.97 | 0.54093 | 0.95655 | 2.63 | 0.4 | 0.59228 | 0.87369 |
| 1 | 1 | 0.54093 | 0.95655 | 4 | 0.4 | 0.57272 | 0.82642 |
| \( \geq 1 \) | 1.43 | 0.53297 | 0.97214 | 0.2 | \( \gamma_* \) | 2.74835 | 0.96415 |
| \( \geq 1 \) | 1.5 | 0.53197 | 0.97383 | 0.5 | \( \gamma_* \) | 1.13293 | 0.91254 |
| \( \geq 1 \) | 1.62 | 0.53041 | 0.97637 | 1 | \( \gamma_* \) | 0.57952 | 0.86446 |
| \( \geq 1 \) | 2 | 0.52637 | 0.98230 | 1.5 | \( \gamma_* \) | 0.58759 | 0.82642 |
| \( \geq 1 \) | 3 | 0.51963 | 0.99024 | 1.99 | \( \gamma_* \) | 0.59346 | 0.79839 |
| \( \geq 1 \) | 4 | 0.51568 | 0.99379 | 2.12 | \( \gamma_* \) | 0.59407 | 0.81800 |
| \( \geq 1 \) | 5 | 0.51307 | 0.99569 | 3 | \( \gamma_* \) | 0.58546 | 0.90000 |
| \( \geq 1 \) | \( \infty \) | 0 | 5 | \( \gamma_* \) | 0.56465 | 0.68232 |

**Remark 3.** Theorem 4 yields the following lower bound for the exact constant \( A_\varepsilon(\infty, 1) \) in the Esseen-type inequality (1):

\[ A_\varepsilon(\infty, 1) \geq C_\varepsilon(g_1, \infty, 1) \geq \sup_{1 < p < 1} \frac{\Phi(\sqrt{q/p} - q)}{q^2 + p^2} > 0.3703, \]

which coincides with the one obtained directly in Theorem 3.
Proof. Let \( n = 1 \) and the r.v. \( X_1 \) have the two-point distribution (27) with \( p \in \left[ \frac{1}{2}, 1 \right) \). Then, by Theorem 2 we have \( \Delta_1(F_1) = \Delta_1(p) \), and all the constants \( C_* \in \{ C_!, C_\} \) are bounded from below as

\[
C_*(g_1, \epsilon, \gamma) \geq \sup_{\frac{1}{2} \leq p < 1} \frac{\Delta_1(p)}{L_{*,1}(g_1, \epsilon, \gamma)}.
\]

In particular, for \( p = \frac{1}{2} \) we have \( \Delta_1\left(\frac{1}{2}\right) = \Phi(1) - 0.5 = 0.3413 \ldots \), while by Theorem 1

\[
L_{e,n}(g_1, \epsilon, \gamma) = L_{e,n}(g_1, \epsilon, \gamma) \equiv 1,
\]

whence the statement of point (i) follows immediately.

To prove point (ii), it suffices to make sure that for all \( p \in \left( \frac{1}{2}, 1 \right), \epsilon, \gamma > 0 \) we have

\[
K_{*,1}(p, \epsilon, \gamma) = \frac{\Delta_1(p)}{L_{*,1}(g_1, \epsilon, \gamma)}, \quad \bullet \in \left\{ \nu, x \right\}.
\]

Theorem 1 (i) (see (30) and (32)) with \( n = 1 \) implies that for \( p > \frac{1}{2}, \gamma > 0 \) and \( \epsilon \leq 1 \) we have

\[
L_{n,1}(g_1, \epsilon, \gamma) = L_{n,1}(g_C, \epsilon, \gamma) = \begin{cases} 1, & \epsilon^2 \leq \frac{q}{p} \iff p = \frac{1}{\epsilon^2}, \quad \epsilon \leq 1, \quad \frac{\Delta_1(p)}{K_{n,1}(p, \epsilon, \gamma)}\, \Delta_1(p) \\
\max \left\{ 1, \gamma q + p, \sqrt[3]{\frac{1}{p}} + \sqrt{\frac{1}{p}} \right\}, & \epsilon^2 > \frac{q}{p} \iff p > \frac{1}{\epsilon^2}, \quad \epsilon \leq 1, \quad \frac{\Delta_1(p)}{K_{n,1}(p, \epsilon, \gamma)}\, \Delta_1(p) \end{cases}
\]

while for \( \epsilon > 1 \), by Theorem 1 (ii) and (iii), respectively, we have

\[
L_{n,1}(g_1, \epsilon, \gamma) = \left\{ \begin{array}{ll}
\max \left\{ 1, \gamma q + p, \sqrt[3]{\frac{1}{p}} + \sqrt{\frac{1}{p}} \right\}, & \epsilon^2 \leq \frac{q}{p} \iff p \geq \frac{\epsilon^2}{\epsilon^2 + 1}, \\
\max \left\{ 1, \gamma q + p, \sqrt[3]{\frac{1}{p}} + \sqrt{\frac{1}{p}} \right\}, & \epsilon^2 > \frac{q}{p} \iff p < \frac{\epsilon^2}{\epsilon^2 + 1}, \quad \frac{\Delta_1(p)}{K_{n,1}(p, \epsilon, \gamma)}\, \Delta_1(p) \end{array} \right.
\]

which coincides with the statement of point (ii).

Now let us consider the particular cases. The lower bound for \( C_*(g_1, \epsilon, \gamma) \) with \( \gamma \leq 1 \) follows from that, for the specified \( \gamma \) and \( \epsilon \leq 1 \), we have \( K_{n,1}(p, \epsilon, \gamma) = \Delta_1(p), \frac{1}{2} < p \leq 1 \). For \( \epsilon \leq x_\Phi \) we have \( \frac{1}{\epsilon^{2+1}} \geq \frac{1}{x_\Phi^{2+1}} = p_\Phi \) and, hence,

\[
C_*(g_1, \epsilon, \gamma) \geq \sup_{\frac{1}{2} < p \leq \frac{1}{\epsilon^{2+1}}} K_{n,1}(p, \epsilon, \gamma) = \sup_{\frac{1}{2} < p \leq \frac{1}{\epsilon^{2+1}}} \Delta_1(p) = \Delta_1(p_\Phi) = C_\Phi.
\]

The lower bound for \( C_*(g_1, \epsilon, \gamma) \geq C_\Phi \) with \( \epsilon < x_\Phi \) is obtained similarly. The given values of the constants \( C_*(g_1, \epsilon, \gamma) \geq C_\Phi \) with \( \epsilon < x_\Phi \) are computed trivially. \( \square \)

5. Lower Bounds for the Asymptotically Best Constants

Let us investigate the constants \( C_{ab}(g, \epsilon, \gamma) \) in (11) and (12) and construct their lower bounds. Due to the extremity of the functions \( g_0(z) = \min \{ z, B_n \} \) and \( g_1(z) = \max \{ z, B_n \} \) (see (15)) we have

\[
\sup_{g \in \tilde{U}} C_{ab}(g, \epsilon, \gamma) = C_{ab}(g_0, \epsilon, \gamma), \quad \inf_{g \in \tilde{U}} C_{ab}(g, \epsilon, \gamma) = C_{ab}(g_1, \epsilon, \gamma).
\]

in both inequalities (11) and (12).
Theorem 5. For all $\varepsilon$, $\gamma > 0$:

(i) in inequality (11)

$$\inf_{\varepsilon > 0} C_{\text{AB}}(g_0, \varepsilon, \gamma) \geq \frac{1}{3\sqrt{2\pi}} \sup_{\frac{1}{2} < p < 1} \max \{1 - p, \gamma(1-p)^2 + p^2, \gamma(2p-1)\} =: C_{\text{AB}}(\gamma), \quad (36)$$

in particular, $\inf_{\varepsilon > 0} C_{\text{AB}}(g_0, \varepsilon, 1) \geq \frac{\sqrt{3} + 3}{6\sqrt{2\pi}} = 0.4097 \ldots$;

(ii) in inequality (12)

$$C_{\text{AB}}(g_0, \varepsilon, \gamma) \geq \begin{cases} \frac{1 + \sqrt{5}}{\sqrt{2\pi}} \frac{3\sqrt{2\pi}(2\gamma(\varepsilon^{-1} \wedge 1)(\sqrt{5} - 2) + 3 - \sqrt{5})}{\varepsilon \sqrt{1}} & \gamma < \frac{2}{3}, \\
\frac{1}{\sqrt{2\pi}} = 0.3989 \ldots , & \gamma = \frac{2}{3}, \\
\frac{1}{\varepsilon \sqrt{1}} & \gamma \geq \frac{2}{3}. \end{cases} \quad (37)$$

(iii) in both inequalities (11) and (12)

$$C_{\text{AB}}(g_1, \varepsilon, \gamma) = 0.$$

Values of $C_{\text{AB}}(\gamma)$ for some $\gamma$ and the corresponding extreme values of $p$ are given in Table 6.

**Table 6.** Values of the lower bound $C_{\text{AB}}(\gamma)$ (see (36)), rounded down, for the asymptotically best constant $C_{\text{AB}}(g_0, \varepsilon, \gamma)$ from inequality (11) for some $\gamma$. The second line contains rounded extreme values of $p$ in (36).

| $\gamma$ | 0.1 | 0.2 | 0.4 | 0.56 | 1 | 1.5 | 2 | 3 | 4 | 5 |
|---------|-----|-----|-----|------|---|-----|---|---|---|---|
| $p$     | 0.6112 | 0.6039 | 0.5871 | 0.5710 | 0.5812 | 0.6733 | 0.6666 | 0.6340 | 0.6202 | 0.6126 |
| $C_{\text{AB}}(\gamma)$ | 0.5511 | 0.5384 | 0.5111 | 0.4868 | 0.4097 | 0.3627 | 0.3324 | 0.2703 | 0.2240 | 0.1904 |

**Proof.** Let us show that $C_{\text{AB}}(g_1, \varepsilon, \gamma) = 0$. According to (13) and (14), $L_{\varepsilon,h}(g_1, \varepsilon, \gamma) \geq 1$, $L_{\varepsilon,h}(g_1, \varepsilon, \gamma) \leq 1$ for all $\varepsilon, \gamma > 0$, so that for the both inequalities (11) and (12) we have $C_{\text{AB}}(g_1, \varepsilon, \gamma) \leq \sup \lim \sup_{\varepsilon,h} \Delta_n(F) = 0$, whence, with the account of non-negativity of $C_{\text{AB}}(g_1, \varepsilon, \gamma)$, the statement of point (iii) follows.

Now let us estimate from below the constants $C_{\text{AB}}(g_0, \varepsilon, \gamma)$. Take i.i.d. r.v.’s $X_1, \ldots, X_n$ with distribution (27), where $p > 1/2$. Then, by virtue of Theorem 1, for $n(\varepsilon^2 \wedge 1) > \frac{L}{q}$ we have

$$L_{\varepsilon,h}(g_0, \varepsilon, \gamma) = \frac{\max \{q, \gamma q^2 + p^2, \gamma(p-q)\}}{\sqrt{n pq} \gamma^2}.$$

The r.v. $X_1$ is lattice with a span $h = 1/\sqrt{pq}$. The Esseen asymptotic expansion [18] for lattice distributions with the span $h$ implies that

$$\lim_{n \to \infty} \sup \Delta_n \sqrt{n} = \frac{|EX_1^3| + 3h\sigma_1^2}{6\sqrt{2\pi} \sigma_1^3}.$$  

For the chosen distribution of $X_1$ we have $\sigma_1^2 = 1$, $EX_1^3 = (p-q)/\sqrt{pq}$, therefore

$$\lim_{n \to \infty} \Delta_n \sqrt{n} = \frac{p - q + 3}{6\sqrt{2\pi pq}} = \frac{p + 1}{3\sqrt{2\pi pq}},$$

and, hence, in inequality (11) we have

$$C_{\text{AB}}(g_0, \varepsilon, \gamma) \geq \lim_{n \to \infty} L_{\varepsilon,h}(g_0, \varepsilon, \gamma) = \lim_{n \to \infty} \max \{q, \gamma q^2 + p^2, \gamma(p-q)\} = \Delta_n \sqrt{n pq}.$$
for all $p \in (1/2, 1)$, $\varepsilon, \gamma > 0$, whence, with the account of arbitrariness of the choice of $p$, the statement of point (ii) follows immediately. In particular, for $\gamma = 1$ we obtain

$$C_{\infty}(g_0, \varepsilon, 1) \geq \sup_{1/2 < p < 1} \frac{p + 1}{3 \sqrt{2\pi \cdot \max \{q, \gamma q^2 + p^2, \gamma(p - q)\}}}$$

for $p = \sqrt{5}/2 - 1 = 0.5811 \ldots$ (extremality of the specified $p$ was proved in [18]).

Now let us consider inequality (12). Taking into account that for $n(\varepsilon^2 \wedge 1) > \frac{p}{q}$ we have

$$L_{\infty}(g_0, \varepsilon, \gamma) = \frac{\gamma(p - q)(\varepsilon^{-1} \wedge 1) + \max \{q, p^2\}}{\sqrt{n pq}}, \quad \max \{q, p^2\} = \begin{cases} q, & p \in \left(\frac{5}{2}, \frac{\sqrt{5} - 1}{2}\right), \\ p^2, & p \in \left(\frac{\sqrt{5} - 1}{2}, 1\right), \end{cases}$$

and denoting

$$a = \gamma(\varepsilon^{-1} \wedge 1) = \frac{\gamma}{\varepsilon \vee 1},$$

we obtain

$$f(p) := 3 \sqrt{2\pi} \cdot \lim_{n \to \infty} \frac{\Delta_n}{L_{\infty}(g_0, \varepsilon, \gamma)} = \begin{cases} \frac{p + 1}{a(2p - 1) + 1 - p}, & p \in \left(\frac{1}{2}, \frac{\sqrt{5} - 1}{2}\right), \\ \frac{p + 1}{a(2p - 1) + p^2}, & p \in \left(\frac{\sqrt{5} - 1}{2}, 1\right). \end{cases}$$

Let us investigate the behaviour of $f(p)$ in dependence of $a > 0$. For $p \in \left(\frac{1}{2}, \frac{\sqrt{5} - 1}{2}\right]$ the numerator of $f'(p)$ takes the form

$$a(2p - 1) + 1 - p - (p + 1)(2a - 1) = 2 - 3a,$$

so that $f(p)$ is monotonically decreasing, if $a > \frac{2}{3}$, and monotonically increasing, if $a < \frac{2}{3}$, while for $p \in (\frac{\sqrt{5} - 1}{2}, 1)$ the numerator of $f'(p)$ has the form

$$a(2p - 1) + p^2 - 2(p + 1)(a + p) = -p^2 - 2p - 3a < 0, \quad 1/2 < p < 1,$$

and hence, $f(p)$ decreases strictly monotonically. Therefore,

$$3 \sqrt{2\pi} \cdot C_{\infty}(g_0, \varepsilon, \gamma) \geq \sup_{1/2 < p < 1} f(p) = \begin{cases} f\left(\frac{\sqrt{5} - 1}{2}\right) = \frac{1 + \sqrt{5}}{2a(\sqrt{5} - 2) + 3 - \sqrt{5}}, & a < \frac{2}{3}, \\ f\left(\frac{1}{2}\right) = 3, & a \geq \frac{2}{3}, \end{cases}$$

whence the statement of point (ii) follows immediately. □

6. Lower Bounds for the Asymptotically Exact Constants

First of all, please note that asymptotically exact constants $C_{\infty}(g_1, \varepsilon, \gamma)$ and the lower asymptotically exact constants $C_{\infty}(g_1, \varepsilon, \gamma)$ are defined for none of the inequalities (11), (12), since the corresponding fractions $L_{\infty}(g_1, \varepsilon, \gamma)$, $L_{\infty}(g_1, \varepsilon, \gamma)$ are bounded from below by one uniformly with respect to $\varepsilon$ and $\gamma$ (see (13) and (14)) and, hence, cannot be infinitesimal.

**Theorem 6.** In the both inequalities (11), (12) for all $\varepsilon, \gamma > 0$ we have

$$C_{\infty}(g_0, \varepsilon, \gamma) \geq \frac{1}{2 \sqrt{2\pi}}, \quad (38)$$
Therefore, it remains to prove the lower bounds for \( C_{\alpha} \) and \( C_{\beta} \) and thus staying always greater than \((2\sqrt{2\pi})^{-1}\) to that for \( \gamma \) varying within the range from \((\sqrt{2\pi})^{-1}\) as \( \gamma / (\varepsilon \lor 1) \to \frac{5}{2} \) to

\[
\frac{1 + \sqrt{5}}{3\sqrt{2\pi}(3 - \sqrt{5})} = 0.5633 \ldots \quad \text{as} \quad \frac{\gamma}{\varepsilon \lor 1} \to 0
\]

and thus always greater than \((2\sqrt{2\pi})^{-1}\). The minorant to the asymptotically best constant in (36) monotonically decreases with respect to \( \gamma \) and does not depend on \( \varepsilon \). Hence, there exists a unique value \( \gamma_0 > 0 \) such that \( C_{\alpha}(\gamma_0) = (2\sqrt{2\pi})^{-1} \) (and even \( \gamma_0 > 1 \) due to that for \( \gamma = 1 \) we have \( C_{\alpha}(1) = (\sqrt{10} + 3)/(6\sqrt{2\pi}) = 0.4097 \ldots > (2\sqrt{2\pi})^{-1} \)), so that for all \( \gamma < \gamma_0 \) we have \( C_{\alpha}(\gamma) > (2\sqrt{2\pi})^{-1} \). It is easy to make sure that \( \gamma_0 = 4.7010\ldots \) Therefore, as a lower bound for the constant \( C_{\alpha}(\gamma_0) \) in Esseen-type inequality (11) it is reasonable to choose the lower bound in (38) for \( \gamma \leq \gamma_0 \) and the lower bound in (36) for \( \gamma > \gamma_0 \). Let us formulate this as a corollary.

**Corollary 1.** For the asymptotically exact constant in inequality (11) the lower bound

\[
\inf_{\varepsilon > 0} C_{\alpha}(\varepsilon, \gamma) \geq C_{\alpha}(\gamma \land \gamma_0),
\]

holds, where \( C_{\alpha}(\gamma) \) is defined in (36), and \( \gamma_0 = 4.7010\ldots \) is the unique root of the equation \( C_{\alpha}(\gamma) = (2\sqrt{2\pi})^{-1} \) for \( \gamma > 0 \). In particular, with the account of Table 3 the following two-sided bounds hold:

\[
0.4097 \ldots = \frac{\sqrt{10} + 3}{6\sqrt{2\pi}} \leq C_{\alpha}(\varepsilon_0, 1, 1) \leq 1.80.
\]

For the asymptotically exact constant in (12) we have

\[
C_{\alpha}(\varepsilon_0, \gamma) \geq \begin{cases} 
\frac{1 + \sqrt{5}}{3\sqrt{2\pi}(2\gamma(\varepsilon^{-1} \land 1)(\sqrt{5} - 2) + 3 - \sqrt{5})}, & \gamma < \frac{2}{3}, \\
\frac{1}{\sqrt{2\pi}}, & \gamma \lor 1 \geq \frac{2}{3}.
\end{cases}
\]

In particular, with the account of Table 3 the following two-sided bounds hold:

\[
0.3989 \ldots = \frac{1}{\sqrt{2\pi}} \leq C_{\alpha}(\varepsilon_0, 1, 1) \leq 1.80.
\]

**Proof of Theorem 6.** The relations between the constants follow from their definitions (see also (26)). Therefore, it remains to prove the lower bounds for \( C_{\alpha}(\varepsilon_0, \gamma) \), \( C_{\alpha}(\varepsilon_0, \gamma) \), and \( C_{\alpha}(\varepsilon_1, \gamma) \).

Following [29] ([§2.3.2]), consider i.i.d. r.v.’s \( X_1, \ldots, X_n \) with a symmetric tree-point distribution

\[
P(|X_1| = 1) = p = 1 - P(X_1 = 0) \in (0, 1),
\]
whose d.f. will be denoted by $F_p(x) = P(X_1 < x), x \in \mathbb{R}$. Then

$$\begin{align*}
EX_1 &= 0, \quad EX_1^2 = p, \quad c_1^2(z) = EX_1^2 \mathbf{1}(|X_1| \geq z) = \begin{cases} p, & z \leq 1, \\ 0, & z > 1, \end{cases}, \quad \mu_1(\cdot) \equiv 0,
\end{align*}$$

$$B_n^2 = np, \quad g_0(z) = \min\{z, \sqrt{np}\}, \quad g_1(z) = \max\{z, \sqrt{np}\},$$
and the fractions $L_{n_2}, L_{n_3}$ coincide, do not depend on $\gamma$ and take the form

$$L_n(g, \varepsilon) := L_{n_2}(g, \varepsilon, \cdot) = L_{n_3}(g, \varepsilon, \cdot) = \sup_{0 < z < \sqrt{np}} g(z) \frac{g(\varepsilon \sqrt{np} \cdot 1 - 1)}{g(\varepsilon \sqrt{np})} = \sup_{0 < z < \sqrt{np}} \frac{g(z) c_1^2(z)}{g(\sqrt{np})}.$$  

Due to the monotonicity of $g \in G$, we have

$$\sup_{0 < z < \sqrt{np}} g(z) c_1^2(z) = pg(\varepsilon \sqrt{np} \cdot 1 - 1) = p \min\{g(1), g(\varepsilon \sqrt{np})\},$$

and hence,

$$L_n(g, \varepsilon) = \frac{\min\{g(1), g(\varepsilon \sqrt{np})\}}{g(\sqrt{np})}, \quad \varepsilon > 0, \quad n \in \mathbb{N}, \quad p \in (0, 1),$$
in particular,

$$L_n(g_0, \varepsilon) = \min\left\{\frac{1}{\sqrt{np}}, \varepsilon, 1\right\}, \quad L_n(g_1, \varepsilon) = \min\left\{\frac{1}{\sqrt{np}} \vee 1, \varepsilon \vee 1\right\}, \quad \varepsilon > 0,$$

and $L_n(g_1, \varepsilon) = 1$ for $\varepsilon \leq 1$. Moreover, for all $n > \ell^{-2} \vee \varepsilon^{-2} \vee 1$ we have

$$\{p \in (0, 1) : L_n(g_0, \varepsilon) = \ell\} = \{p \in (0, 1) : \min\left\{\frac{1}{\sqrt{np}}, \varepsilon, 1\right\} = \ell\} = \begin{cases} \{\ell^{-2}/n =: p(\ell)\}, \quad \ell < \varepsilon \wedge 1, \\ (0, (\ell^{-2} \vee 1)/n), \quad \ell = \varepsilon \wedge 1, \quad \text{for all } \varepsilon > 0, \\ \emptyset, \quad \ell > \varepsilon \wedge 1, \end{cases}$$

$$\{p \in (0, 1) : L_n(g_1, \varepsilon) = \ell\} = \begin{cases} \{p : \ell = 1\} = \begin{cases} (0, 1), \quad \ell = 1, \\ \emptyset, \quad \text{otherwise,} \end{cases} \quad \varepsilon \leq 1, \\ \{p : \min\left\{\frac{1}{\sqrt{np}} \vee 1\right\} = \ell\} = \begin{cases} p(\ell), \quad \ell \in [1, \varepsilon], \\ \emptyset, \quad \text{otherwise,} \quad \varepsilon > 1. \end{cases} \end{cases}$$

As a result, the fractions do not depend on $\gamma$, we obtain the lower bounds

$$\inf_{\gamma > 0} C_{n_2}(g_0, \varepsilon, \gamma) \geq \limsup_{\substack{\gamma \to 0 \\ \varepsilon \to 0}} \limsup_{\ell \to 0} \sup_{\varepsilon \in (0, 1)} \sup_{p(\ell) = \ell} \Delta_\varepsilon(F_p)/\ell,$$

in particular, for all $\gamma > 0$ we have

$$C_{n_2}(g_0, \varepsilon, \gamma) \geq \max\left\{\sup_{0 < \ell < \varepsilon \vee 1} \limsup_{\ell \to 0} \sup_{\varepsilon \in (0, 1)} \sup_{p(\ell) = \ell} \Delta_\varepsilon(F_p)/\ell, \quad \limsup_{\ell \to 0} \sup_{0 < \varepsilon \in (0, 1)} \sup_{0 < p < 1} \Delta_\varepsilon(F_p) / (\ell \wedge 1)\right\},$$

$$C_{n_2}(g_1, \varepsilon, \gamma) \geq \begin{cases} \limsup_{\varepsilon \to 0} \sup_{0 < p \leq 1} \Delta_\varepsilon(F_p), \quad \varepsilon \leq 1, \\ \sup_{1 < \varepsilon \to 0} \limsup_{\ell \to 0} \sup_{0 < p \leq 1} \Delta_\varepsilon(F_p)/\ell, \quad \varepsilon > 1. \end{cases}$$
Let us find the lower bound for the uniform distance $\Delta_n(F_p)$. Due to the symmetry, we have $2P(S_n < 0) + P(S_n = 0) = 1$, i.e., $P(S_n < 0) = (1 - P(S_n = 0))/2$, whence for even $n$ we obtain

$$\Delta_n(F_p) \geq \Phi(0) - P(S_n = 0) = \frac{P(S_n = 0)}{2} = \frac{(1 - p)^n n!/2}{2} \sum_{k=0}^{n/2} \frac{n!}{(n - 2k)!(k)!^2} \left(\frac{p/2}{1 - p}\right)^{2k}.$$  

Please note that $\limsup_{p \to 0} \Delta_n(F_p) \geq \frac{1}{2}$, while with $p = \alpha/n, \alpha \in (0, n)$ we have

$$\Delta_n(F_{\alpha/n}) \geq \frac{1}{2} \left(1 - \frac{\alpha}{n}\right) \sum_{k=0}^{n/2} \frac{n!}{(n - 2k)!(k)!^2} \left(\frac{1/2}{n/\alpha - 1}\right)^{2k} = e^{-\alpha} + \delta_\alpha(n) \sum_{k=0}^{n/2} u_{k,n}(\alpha),$$

where

$$u_{k,n}(\alpha) = \frac{n!}{(n - 2k)!(k)!^2} \left(\frac{1/2}{n/\alpha - 1}\right)^{2k}, \quad \lim_{n \to \infty} \delta_\alpha(n) = 0, \quad \alpha > 0.$$  

In [29] ([pp. 268–269]) it was shown that for every $\alpha > 0$

$$\limsup_{n \to \infty} \sum_{k=0}^{n/2} u_{k,n}(\alpha) \geq \sum_{k=0}^{\infty} \left(\frac{\alpha/2}{k!}\right)^{2k} = I_0(\alpha).$$

Therefore,

$$\limsup_{n \to \infty} \Delta_n(F_{\alpha/n}) \geq \frac{1}{2} e^{-\alpha} I_0(\alpha), \quad \ell > 0. \quad (39)$$  

Let us bound from below expressions such as $\sup_{p} \Delta_n(F_p)$ as

$$\sup_{p} \Delta_n(F_p) \geq \limsup_{p \to 0} \Delta_n(F_p) \geq \frac{1}{2}. \quad (40)$$  

From (39) with $\alpha = \ell^2$ we obtain

$$C_{\infty}(g_0, \varepsilon, \gamma) \geq \limsup_{n \to \infty} \limsup_{n \to \infty} \frac{\Delta_n(F_p(\ell))/\ell - \frac{1}{2} \lim_{n \to \infty} \sqrt{\pi} e^{-\alpha} I_0(\alpha)}{\varepsilon, \gamma > 0}.$$  

Inequalities (39) and (40) imply that

$$C_{\infty}(g_0, \varepsilon, \gamma) \geq \max \left\{ \sup_{0 < \ell < 1} \limsup_{n \to \infty} \frac{\Delta_n(F_p(\ell))/\ell}{\ell}, \limsup_{n \to \infty} \sup_{0 < p < (\ell^2 + 1)/n} \Delta_n(F_p)/\ell \right\} \geq \max \left\{ \sup_{\alpha > e^{-2}/\ell} \sqrt{\pi} e^{-\alpha} I_0(\alpha), \frac{1}{2}(\ell^2 + 1) \right\},$$

$$C^*_{\infty}(g_1, \varepsilon, \gamma) \geq \left\{ \begin{array}{ll}
\limsup_{n \to \infty} \sup_{0 < p < 1} \Delta_n(F_p) \geq \frac{1}{2}, & \ell \leq 1, \\
\limsup_{n \to \infty} \sup_{1 < \ell < \frac{\varepsilon}{2}, \varepsilon < \alpha} \Delta_n(F_p(\ell))/\ell \geq \sup_{\varepsilon < \alpha < 1} \frac{1}{2} \sqrt{\pi} e^{-\alpha} I_0(\alpha), & \ell > 1.
\end{array} \right.$$  

The plot of the function $f(\alpha) = \sqrt{\pi} e^{-\alpha} I_0(\alpha)$ looks monotonically increasing for $\alpha \leq 0.78$ and monotonically decreasing for $\alpha > 0.79 =: \alpha_*$ with $f(\alpha_*) > 0.4688$, therefore it is reasonable to estimate upper bounds $\sup_{p} f(\alpha)$ from below as

$$\sup_{\alpha > e^{-2}/\ell} f(\alpha) \geq f(1) = e^{-1} I_0(1) = 0.4657 \ldots,$$

$$\sup_{e^{-2} < \alpha < 1} f(\alpha) \geq \left\{ \begin{array}{ll}
f(\alpha_*) > 0.4688, & e^{-2} < \alpha_*, \quad \equiv \quad \varepsilon > 1/\sqrt{\alpha_*} = 1.1250 \ldots, \\
f(\varepsilon^{-2}) = \exp(-\varepsilon^{-2}) I_0(\varepsilon^{-2}), & \varepsilon^{-2} \geq \alpha_*, \quad \equiv \quad \varepsilon \leq 1/\sqrt{\alpha_*}.
\end{array} \right.$$
As we can see, the obtained lower bounds are not far from the upper ones, so that, from the point of practical use, there is no high motivation to improve the method of construction of the values of the constants under the “best” choice of the function \( g \).

Recall that \( C_N \) bounds for \( \gamma \) asymptotically exact constructed in terms of the upper asymptotically exact \( C_N \) values. As regards the historical values \( \gamma = 1 \) in both inequalities (11) and (12), \( \gamma = \infty \), \( \gamma = 1 \) in (11), we obtained the following results for the absolute constants:

\[
\min\{C_\infty(1, 1), C_\infty(1, 1), C_\infty(\infty, 1)\} \geq \Phi\left(\frac{\sqrt{(1-p)p}}{p} - 1 + p\right) \left|_{p=0.9058...}^{p=0.9678...}\right| > 0.5685.
\]

Recall that

\[
C_\infty(\infty, 1) \leq C_\infty(1, 1) = A_\infty(1, 1) \leq 2.73, \quad C_\infty(1, 1) = A_k(1, 1) \leq 2.73
\]

(see Tables 1 and 2), and, as it follows from the presented results, these upper bounds cannot be lowered more than by 5 times. As regards the most optimistic constants, i.e., the values of the constants under the “best” choice of the function \( g \in G \), we have shown that they cannot nevertheless be less than

\[
C_k(g_1, 1, 1) \geq \frac{\Phi\left(\frac{\sqrt{(1-p)p}}{p} - 1 + p\right)\left|_{p=0.9678...}^{p=0.6090...}\right|}{1 + \sqrt{(1-p)p}} > 0.5370.
\]

\[
C_k(g_1, \infty, 1) \geq C_\Phi = 0.5409..., \quad C_\infty(g_1, 1, 1) \geq 0.5370.
\]

Since \( A_k(\infty, 1) \geq C_k(g_1, \infty, 1) \), the latest inequality immediately yields the lower bound \( A_k(\infty, 1) > 0.3703 \) that was also obtained directly in Theorem 3. Recall that \( A_k(\infty, 1) \leq 2.66\) (see Table 1).

\textit{Asymptotic} lower bounds for \( C_k \) and \( C_\infty \) for infinitely large sample sizes \( n \) may be constructed in terms of the upper asymptotically exact \( C_\infty \) and the conditional upper asymptotically exact \( C_k \) constants which are linked as \( C_k \leq C_\Phi \). We constructed the lower bounds for \( C_k(g, \varepsilon, \gamma) \) with \( g = g_0, g_1 \) which turned out to coincide for \( \varepsilon = \gamma = 1 \):

\[
C_k(g_0, 1, 1) \geq C_k(g_1, 1, 1) \geq 0.5
\]
in both inequalities (11) and (12) (the same lower bound 0.5 was obtained directly for \( C_{\text{AB}}(g_0, 1, 1) \) in Theorem 6).

The next series of the lower bounds for \( C_e, C_i \) was obtained under the additional assumptions on the smallness of the fractions \( L_{e,n}, L_{i,n} \), which allow constructing more optimistic upper bounds, for example, in terms of the asymptotically exact constants, i.e., to consider estimates such as (25) with \( C_e = C_{\text{AB}} \). Recall that in this situation the following upper bounds are known from [11]: \( A_{\text{AB}}^{23}(1, 1) \leq 1.80, A_{\text{AB}}^{24}(1, 1) \leq 1.80 \). Observation that the constants \( A_{\text{AB}}^{23}(1, 1), A_{\text{AB}}^{24}(1, 1) \) coincide with \( C_{\text{AB}}(g_0, 1, 1) \) in (22) with \( L_{i} = L_{e,n}, L_{i,n} \), respectively, allows taking as a lower bound to \( A_{\text{AB}}^{23}(1, 1), A_{\text{AB}}^{24}(1, 1) \) (or, more generally, to \( C_{\text{AB}}(g_0, 1, 1) \)) any of \( C_{\text{AB}}(g_0, 1, 1) \) and \( C_{\text{AB}}(g_0, 1, 1): C_{\text{AB}}(g_0, 1, 1) \geq \max\{C_{\text{AB}}(g_0, 1, 1), C_{\text{AB}}(g_0, 1, 1)\} \). However, the lower bound \( C_{\text{AB}}(g_0, 1, 1) \geq (2\sqrt{2\pi})^{-1} = 0.1994 \ldots \) in the both inequalities (11) and (12) is less than each of the lower bounds \( C_{\text{AB}}(g_0, 1, 1) \geq \frac{\sqrt{11} + 1}{6\sqrt{2\pi}} = 0.4097 \ldots \) in (11) and \( C_{\text{AB}}(g_0, 1, 1) \geq 1/\sqrt{2\pi} = 0.3989 \ldots \) in (12), so that the best choice of the lower bound for \( C_{\text{AB}}(g_0, 1, 1) \) is \( C_{\text{AB}}(g_0, 1, 1) \) in both inequalities (11) and (12). Hence, the following two-sided bounds hold true:

\[
0.4097 < A_{\text{AB}}^{23}(1, 1) \leq 1.80, \quad 0.3989 < A_{\text{AB}}^{24}(1, 1) \leq 1.80.
\]

Therefore, the asymptotic values \( A_{\text{AB}}^{23}(1, 1), A_{\text{AB}}^{24}(1, 1) \) of the constants \( C_e(1, 1) \) and \( C_i(1, 1) \) in (11), (12), valid for small \( L_{e,n}, L_{i,n} \), can neither be improved more than by 4.5 times.

Further research might include construction of the estimates of the rate of convergence in non-classical Lindeberg’s theorem whose conditions were obtained in a recent paper [30] or some extensions to the case of dependent random summands (see [31]).

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**Abbreviations**

The following abbreviations are used in this manuscript:

- r.v. random variable
- i.i.d. independent identically distributed
- d.f. distribution function
- w.r.t. with respect to

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