Configurations representing a skew perspective

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Abstract

A combinatorial object representing schemas of, possibly skew, perspectives, called a configuration of skew perspective is defined. Some classifications of skew perspectives are presented.

key words: Veblen (Pasch) configuration, (generalized) Desargues configuration, binomial configuration, complete (free sub)subgraph, perspective.

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Introduction

The term perspective, the title subject of this paper, is used, primarily, in architecture drawings and, after that, in descriptive and projective geometry. It refers, in fact, to a (central) projection i.e. to a correspondence between objects of one space (points, lines, planes, spheres, ...) and objects of another space, while both two are subspaces of a third one (“the real world”). Such a projection is central if the lines which join corresponding points meet in a “center” (an ‘eye’). In investigations of projective geometry central projections between lines, planes etc. play a crucial role; e.g. they are used to characterize so called projective collineations, projective correspondence, and similar notions of projective geometry (see standard textbooks like [13], [14], textbooks on general geometry like [20], or more advanced investigations in [16], [12]). Projections are also used to characterize Pasch property (invariance of an order) in projective and chain geometries (see e.g. [15]). And in many other places. Roughly speaking, a projection is a local linear collineation.

One can talk about perspective in a more general settings of (finite) systems of points: of configurations, or even: of graphs. General requirements that should be met by such a perspective we formulate in (P) in Section 1. And the most ‘instructive’ and ‘vivid’ example that one should have in mind is the classical Desargues configuration considered as a perspective between two triangles (see e.g. [20] Ch. III, §19). This configuration was generalized in many directions, e.g. to take into account a perspective between m-simplices that may be realized in a projective space (see [3], [17], [2] Generalized Desargues Configuration]). As we said, a perspective $\pi$ is a local collineation: while defined primarily on the points it extends uniquely to a map $\pi$ defined on more complex objects such as lines (planes, chains and so on).

It appears even in the smallest reasonable case of $10_3$-configurations that the Desargues configuration has two cousins, both two realizable in a projective plane
over a field, such that the associated perspectives $\pi$ are skew. Namely, the points in which intersect sides of a triangle and their images under $\pi$ do not colline. However, another correspondence $\xi$ can be found, $\xi \neq \pi$ such that a side $e$ of a triangle and $\xi(e)$ intersect on a fixed line (an ‘axis’).

Is it possible to generalize this class of perspectives to ‘bigger’ simplices? Formally, the answer is trivial: it suffices to introduce suitable (‘constructive’) definition (see Construction 1.1). After that, the natural question aries, how to classify the obtained structures, how to characterize them and their geometry. Some partial answers to these questions are given in this paper. First, we note that, from a general perspective, our perspectives are exactly the binomial partial Steiner triple systems which freely contain at least two maximal complete graphs (see [5]). From this point of view, a complete characterization of all the skew perspectives seems far to reach. Applying the requirement that $\xi$ extends to the line graphs of the respective simplices (i.e. $\xi$ maps concurrent edges onto concurrent edges) we arrive much closer to a complete classification of skew perspectives. In particular, we obtain such a classification for perspectives whose axes are generalized Desargues configurations (Prop. 3.4). In Section 4 we compare the obtained perspectives with some other known $(15_2^{20_3})$-configurations. Examples show that a variety of quirks may appear, a configuration in question may be represented in a ‘regular’ way and - with other centre chosen - as a perspective with quite irregular skew.

The question which of our perspectives are simultaneously multiveblen configurations (another class of partial Steiner triple systems generalizing a projective perspective, introduced in [4]) is discussed in Proposition 4.8. The natural question which of so generalized perspectives can be realized in a Desarguesian projective space is discussed in Section 5 and is completely solved in case of $(15_2^{20_3})$-skew-perspectives. Some remarks on configurational axioms associated with our configurations are made in Section 6.

1 Underlying ideas and basic definitions

Let us begin with introducing some, standard, notation. Let $X$ be an arbitrary set. The symbol $S_X$ stands for the family of permutations of $X$. Let $k$ be a positive integer; we write $\mathcal{P}_k(X)$ for the family of $k$-element subsets of $X$. Then $K_X = \langle X, \mathcal{P}_2(X) \rangle$ is the complete graph on $X$; $K_n$ is $K_X$ for any $X$ with $|X| = n$. Analogously, $S_n = S_X$.

A $(\nu, b, \kappa)$-configuration is a configuration (a partial linear space i.e. an incidence structure with blocks (lines) pairwise intersecting in at most a point) with $\nu$ points, each of rank $\tau$, and $b$ lines, each of rank (size) $\kappa$. A partial Steiner triple system (in short: a PSTS) is a partial linear space with all the lines of size 3. A $\left(\binom{n}{2}, \binom{n}{3}_2, \binom{n}{3}_3\right)$-configuration is a partial Steiner triple system, it is called a binomial partial Steiner triple system.

We say that a graph $G$ is freely contained in a configuration $\mathcal{B}$ iff the vertices of $G$ are points of $\mathcal{B}$, each edge $e$ of $G$ is contained in a line $\tau$ of $\mathcal{B}$, the above map $e \mapsto \tau$ is an injection, and lines of $\mathcal{B}$ which contain disjoint edges of $G$ do not intersect in $\mathcal{B}$. If $\mathcal{B}$ is a $\left(\binom{n}{2}, \binom{n}{3}_2, \binom{n}{3}_3\right)$-configuration and $G = K_X$ then $|X| + 1 \leq n$. Consequently,
$K_{n-1}$ is a maximal complete graph freely contained in a binomial $\binom{n}{2} - \binom{n}{3}$ configuration. Further details of this theory are presented in [5], relevant results will be quoted in the text, when needed.

In the paper we aim to develop a theory of configurations which characterize abstract properties of a perspective between two graphs. Let us start with the following general (evidently: unprecise yet) requirements.

When we talk about a perspective between two graphs $G_1$ and $G_2$, where $X_i$ is the set of vertices and $E_i$ is the set of edges of $G_i$ then we have

- a perspective center: a point $p$ such that perspective rays, lines through $p$ establish a one-to-one correspondence $\pi$ (point perspective) between $X_1$ and $X_2$, and

- an axis: a configuration (disjoint with $X_1 \cup X_2$) such that a one-to-one correspondence $\xi$ (line perspective) between $E_1$ and $E_2$ is characterized by the condition:

  an edge in $E_1$ and its counterpart in $E_2$, suitably extended, intersect on the axis.

(comp. [5, Prop. 2.6], [8, Repr. 2.4, Repr. 2.5] or standard textbooks on projective geometry, e.g. [13], [14]).

The associated configuration consists of the points in $X_1 \cup X_2$ completed by the center and the intersections of extended edges, and the minimal amount of the lines which join these intersection points.

This approach is, however, too general. We want our perspective to yield a regular configuration i.e. a one with all the points of the same rank. It is seen that the size of the lines must be 3. The rank of the perspective center is $n = |X_1| = |X_2|$, therefore the rank of $a \in X_1$ in $G_1$ must be $n - 1$ and therefore $G_1$ and $G_2$ both are complete $K_n$-graphs. So, unhappily, only perspectives between complete graphs can be characterized in accordance with our requirements (P). On the other hand, this restriction leads us to a quite nice part of the theory of configurations.

So, let us pass to a more exact formulation of requirements (P).

**Construction 1.1.** Let $I$ be a nonempty finite set, $n := |I|$. In most parts, without loss of generality, we assume that $I = I_n = \{1, \ldots, n\}$. Let $A = \{a_i: i \in I\}$ and $B = \{b_i: i \in I\}$ be two disjoint $n$-element sets, let $p \not\in A \cup B$.

Then we take a $(\binom{n}{2})$-element set $C = \{c_u: u \in \wp_2(I)\}$ disjoint with $A \cup B \cup \{p\}$. Set

$\mathcal{P} = A \cup B \cup \{p\} \cup C$.

Let us fix a permutation $\sigma$ of $\wp_2(I)$ and write

- $\mathcal{L}_p := \{\{p, a_i, b_i\}: i \in I\}$,
- $\mathcal{L}_A := \{\{a_i, a_j, c_{\{i,j\}}\}: \{i, j\} \in \wp_2(I)\}$,
- $\mathcal{L}_B := \{\{b_i, b_j, c_{\sigma^{-1}(\{i,j\})}\}: \{i, j\} \in \wp_2(I)\}$.
Finally, let \( \mathcal{L}_C \) be a family of 3-subsets of \( C \) such that \( \mathfrak{N} = (\mathcal{L}, \mathcal{L}_C) \) is a \( \left( \binom{n}{2} \mathcal{L}_{n-2} \right) \text{-configuration} \). Set

\[
\mathcal{L} = \mathcal{L}_p \cup \mathcal{L}_A \cup \mathcal{L}_B \cup \mathcal{L}_C \text{ and } \Pi(n, \sigma, \mathfrak{N}) := (\mathcal{P}, \mathcal{L}).
\]

The structure \( \Pi(n, \sigma, \mathfrak{N}) \) will be referred to as a skew perspective with the skew \( \sigma \).

We frequently shorten \( c_{i,j} \) to \( c_{i,j} \). In many cases, the parameter \( \mathfrak{N} \) will not be essential and then it will be omitted, we shall write simply \( \Pi(n, \sigma) \). In essence, the names "\( a_i \)", "\( c_{i,j} \)" are – from the point of view of mathematics – arbitrary, and could be replaced by any other labelling (cf. analogous problem of labelling in [4, Constr. 3, Repr. 3] or in [10, Rem 2,11, Rem 2,13], [4, Exmpl. 2]). Formally, one can define \( J = I \cup \{a, b\} \), \( x_i = \{x, i\} \) for \( x \in \{a, b\} =: \mathcal{P} \) and \( i \in I \), and \( c_u = u \) for \( u \in \mathcal{P}(J) \). After this identification \( \Pi(n, \sigma) \) becomes a structure defined on \( \mathcal{P}(J) \).

Then, it is easily seen that

\[
\Pi(n, \sigma, \mathfrak{N}) \text{ is a } \left( \binom{n+2}{2} \mathcal{L}_{n+3} \right) \text{-configuration.} \tag{1}
\]

In particular, it is a partial Steiner triple system (a partial linear space), so we can use standard notation: \( \overline{x, y} \) stands for the line which joins two collinear points \( x, y \in \mathcal{P} \), and then we define the partial operation \( \oplus \) with the following requirements:

\[
x \oplus x = x, \{x, y, x \oplus y\} \in \mathcal{L} \text{ whenever } x, y \text{ exists.}
\]

Observe then that (cf. [3, Eq. (1), the definition of combinatorial Grassmannian \( \mathbf{G}_2(n) \)]

\[
\mathbf{G}_2(n+2) = \mathbf{G}_2(J) = (\mathcal{P}(J), \mathcal{P}_3(J), \subset) \cong \Pi(n, \text{id}_I, \mathbf{G}_2(I_n)). \tag{2}
\]

It is clear that \( A^* = A \cup \{p\} \) and \( B^* = B \cup \{p\} \) are two \( K_{n+1} \)-graphs freely contained in \( \Pi(n, \sigma) \). Applying the results [5, Prop. 2.6 and Thm. 2.12] we immediately obtain the following.

**Fact 1.2.** Let \( N = n+2 \). The following conditions are equivalent.

(i) \( \mathfrak{M} \) is a binomial \( \left( \binom{N}{2} \mathcal{L}_{N-2} \right) \text{-configuration} \) which freely contains two \( K_{N-1} \)-graphs.

(ii) \( \mathfrak{M} \cong \Pi(n, \sigma, \mathfrak{N}) \) for a \( \sigma \in \mathcal{P}(J) \) and a \( \left( \binom{n}{2} \mathcal{L}_{n-2} \right) \text{-configuration} \) \( \mathfrak{N} \) defined on \( \mathcal{P}(J) \).

Consequently, the configurations defined by \( \mathfrak{M} \) are essentially known, but no general classification of them is known, though.

The map

\[
\pi = (a_i \mapsto b_i, i \in I)
\]

is a point-perspective of \( K_A \) onto \( K_B \) with center \( p \). Moreover, the map

\[
\xi = (a_i, a_j \mapsto b_i', b_j', \sigma(\{i, j\}) = \{i', j'\} \in \mathcal{P}(J))
\]

is a line perspective, where \( \mathfrak{N} \) is the axial configuration of our perspective. Consequently, \( \Pi(n, \sigma, \mathfrak{N}) \) satisfies the requirement \( (P) \) i.e. it is a schema of a perspective of some type. Contrary to the approach of [5], following the approach of this paper we can better analyze some particular properties of the perspective \( (\pi, \xi) \).
Lemma 1.3. The map $\xi$ maps intersecting edges of $K_A$ onto intersecting edges of $K_B$ iff either

(i) there is a permutation $\sigma_0 \in S_I$ such that
$$\sigma({i,j}) = \sigma_0({i,j}) = \{\sigma_0(i), \sigma_0(j)\}$$
for every $\{i,j\} \in \varphi_2(I)$, or

(ii) $n = 4$ and $\sigma(u) = I \setminus \sigma_0(u)$ for every $u \in \varphi_2(I)$, where $\sigma_0$ is defined by (3) for some $\sigma_0 \in S_I$.

In case (i), $\xi$ preserves the (ternary) concurrency of edges, and in case (ii), the concurrency is not preserved.

Proof. One can identify an edge $\{a_i, a_j\}$ of $K_A$ with $\{i,j\} \in \varphi_2(I)$; analogously we identify $\varphi_2(B) \ni \{b_i, b_j\} \mapsto \{i,j\} \in \varphi_2(I)$. After this identification $\xi \in S\varphi_2(I)$, and $\xi$ preserves the edge-intersection iff it preserves set-intersection. The claim is just a reformulation of the folklore (cf. [7], [3, Prop. 1.5], [4, Prop. 15]).

A more detailed analysis of the case (1.3(ii)) is addressed to another paper.

Note 1. If $\sigma_0 \in S_I$ we frequently identify $\sigma_0$, $\sigma_0$, and the corresponding map $\xi$. Consequently, if $\sigma \in S_I$ we write $\Pi(n, \sigma, \mathfrak{N})$ in place of $\Pi(n, \sigma, \mathfrak{N})$.

Proposition 1.4. Let $f \in S\mathfrak{M}$, $f(p) = p$, $\sigma_1, \sigma_2 \in S\varphi_2(I)$, and $\mathfrak{M}_1, \mathfrak{M}_2$ be two $\left(\begin{array}{c} n \\alpha \\beta \\ 2 \\alpha \\beta \end{array}\right)$-configurations defined on $\varphi_2(I)$. The following conditions are equivalent.

(i) $f$ is an isomorphism of $\Pi(n, \sigma_1, \mathfrak{M}_1)$ onto $\Pi(n, \sigma_2, \mathfrak{M}_2)$.

(ii) There is $\varphi \in S_I$ such that one of the following holds

$$f(x_i) = x_{\varphi(i)}, \quad x = a, b,$$

$$f(c_{i,j}) = c_{(\varphi(i), \varphi(j))}, \quad i, j \in I, i \neq j,$$

$$\varphi \circ \sigma_1 = \sigma_2 \circ \varphi,$$

or

$$f(a_i) = b_{\varphi(i)}, \quad f(b_i) = a_{\varphi(i)},$$

$$f(c_{i,j}) = c_{(\varphi(i), \varphi(j))}, \quad i, j \in I, i \neq j,$$

$$\varphi \circ \sigma_1 = \sigma_2^{-1} \circ \varphi.$$

Proof. Write $\mathfrak{M}_l = \Pi(n, \sigma_l, \mathfrak{M}_l)$ for $l = 1, 2$.

Assume (i). Since exactly two free $K_{n+1}$ subgraphs of $\mathfrak{M}_l$ ($l = 1, 2$) pass through $p$ (cf. [5, Prop.s 2.6, 2.7]), one of the following holds

(a) $f(A) = A$ and $f(B) = B$, or
(b) $f(A) = B$ and $f(B) = A$. 

Assume, first, (4). Consequently, there is a permutation \( \varphi \in S_I \) such that \( f(a_i) = a_{\varphi(i)} \) for each \( i \in I \). This yields \( f(b_i) = f(p) \oplus f(a_i) = b_{\varphi(i)} \), and, finally \( f(c_{\{i,j\}}) = f(a_i \oplus a_j) = \ldots = c_{\varphi(i) \oplus \varphi(j)} \). This justifies (5). Since \( f \) preserves the lines of \( \mathfrak{M} \), from (5) we infer (4). Finally, the equation \( c_{\varphi^{-1}(\{i,j\})} = f(c_{\varphi^{-1}(\{i,j\})}) = f(b_i \oplus b_j) = f(b_i) \oplus f(b_j) = b_{\varphi(i)} \oplus b_{\varphi(j)} = c_{\varphi^{-1}(\{\varphi(i), \varphi(j)\})} \) justifies (6).

In case (b) the reasoning goes analogously. We only need to note that \( f(c_{\{i,j\}}) = f(b_{\varphi(i)} \oplus b_{\varphi(j)}) = c_{\varphi^{-1}(\{\varphi(i), \varphi(j)\})} \), which justifies the last condition in (8) and yields (7).

Conversely, if (ii) is assumed we directly verify that \( f(x \oplus y) = f(x) \oplus f(y) \) holds for all \( x, y \in (A \cup B) \), which proves (i).

**Lemma 1.5.** Assume that \( \Pi(n, \sigma, \mathfrak{M}) \) freely contains a complete \( K_{n+1} \)-graph \( G \neq K_{A^*}, K_{B^*} \), \( \sigma \in S_{\mathcal{V}_2(I)} \). Then there is \( i_0 \in I \) such that \( S(i_0) = \{c_a : i_0 \in u \in \mathcal{V}_2(I)\} \) is a collinearity clique in \( \mathfrak{M} \) freely contained in it. Moreover,

\[
G = G_{(i_0)} := \{a_{i_0}, b_{i_0}\} \cup S(i_0). \tag{10}
\]

**Proof.** Let \( G \neq K_{A^*}, K_{B^*} \) be a complete \( K_{n+1} \)-graph freely contained in \( \Pi(n, \sigma, \mathfrak{M}) =: \mathfrak{M} \). Then \( p, G \cap A, \) and \( G \cap B \) form a triple of collinear points (cf. [3] Prop. 2.7]). So, there is \( i_0 \in I \) such that \( a_{i_0}, b_{i_0} \in G \). And \( G \setminus \{a_{i_0}, b_{i_0}\} \subset C \). The set of points in \( C \) which are collinear with \( a_{i_0} \) is exactly \( S(i_0) \); it contains \( G \) and its cardinality is \( n - 1 \), and therefore \( G = G_{i_0} \). Since \( G \) is a clique, we conclude with: \( S(i_0) \) is a clique in \( \mathfrak{M} \). Clearly, it is freely contained in \( \mathfrak{M} \).

**Example 1.6.** Let us define \( \zeta : \mathcal{V}_2(I_4) \rightarrow \mathcal{V}_2(I_4) \) by the following formula:

\[
\zeta(\{u\}) = \begin{cases} 
\{u\} & \text{when } u \neq \{1, 2\}, \{3, 4\}, \\
I_4 \setminus u & \text{when } u \in \{\{1, 2\}, \{3, 4\}\}.
\end{cases} \tag{11}
\]

Note that \( \zeta^{-1} = \zeta \).

Clearly, \( \zeta \) does not preserve edge-intersection. It is easy to verify that \( \mathfrak{M} = \Pi(4, \zeta, G_2(I_4)) \) has no free \( K_5 \)-subgraph distinct from \( A^* \) and \( B^* \). Any isomorphism of \( \mathfrak{M} \) onto \( \mathfrak{M}_0 = \Pi(4, \mathfrak{G}_0, \mathfrak{M}_0) \) \( \zeta_0 = \mathfrak{G}_0 \) or \( \zeta_0 = \zeta \mathfrak{G}_0 \), \( \sigma_0 \in S_{I_4}, \mathfrak{G}(u) = I_4 \setminus u \), notation of [1.3] maps \( p \) onto \( p \), so it determines (use [1.4]) a permutation \( \varphi \in S_{I_4} \) such that \( \zeta = \zeta_0^\varphi \). Since no such \( \zeta_0, \varphi \) exist, *there is no skew perspective that preserves edge intersection and is isomorphic to \( \mathfrak{M} \).*

## 2 Perspectivities associated with permutations of indices: general properties

**Note 2.** Let \( \mathfrak{M} = \Pi(n, \sigma, \mathfrak{M}) \) be a skew perspective with \( \sigma \in S_{I_4} \). If \( n = 1 \) then \( \mathfrak{M} \) is a single line. If \( n = 2 \) then \( \mathfrak{M} \) is a single point and \( \sigma = id_{\mathcal{V}_2(I_2)} \), and then \( \mathfrak{M} \) is the Veblen configuration \( G_2(I_4) \) (the configuration in question is also frequently called the Pasch configuration, cf. e.g. [11]). If \( n = 3 \) then \( \mathfrak{M} \) is a single 3-line. The configurations \( \Pi(3, \sigma) \) were determined and characterized in [3]; these are exactly:

- the Desargues configuration \( \Pi(3, id_{I_4}) \),
- the fez configuration \( \Pi(3, (1, 2, 3)) \),
the Kantor configuration \( \Pi(3, (1)(2, 3)) \);

cf. [6] Repr. 2.6

In this section we consider structures \( \Pi(n, \sigma) \) where \( \sigma \in S_n \) and \( n > 3 \). Two very useful formulas will be frequently used without explicit quotation:
\[
    a_i \oplus a_j = c_{i, j}, \quad \text{and} \quad b_i \oplus b_j = c_{\sigma^{-1}(i), \sigma^{-1}(j)} \quad \text{for each} \quad \{i, j\} \in \mathcal{V}_2(I)
\]
so, \( a_i, a_j \) crosses \( b_{\sigma(i)}, b_{\sigma(j)} \) in \( c_{i, j} \).

**Lemma 2.1.** The following conditions are equivalent.

(i) \( \Pi(n, \sigma, \mathcal{N}) \) freely contains a complete \( K_{n+1} \)-graph \( G \neq K_{A^*}, K_{B^*} \).

(ii) There is \( i_0 \in \text{Fix}(\sigma) \) such that \( S(i_0) = \{c_u: i_0 \in u \in \mathcal{V}_2(I)\} \) is a collinearity clique in \( \mathcal{N} \) freely contained in it.

In case (i),
\[
    G_{(i_0)} := \{a_{i_0}, b_{i_0}\} \cup S(i_0)
\]
is a complete graph freely contained in \( \Pi(n, \sigma, \mathcal{N}) \).

**Proof.** Assume (i). From (15), \( G \) has form (10), so \( b_{i_0} \) must be collinear with each point in \( S(i_0) \). In other words, for each \( j \in I \setminus \{i_0\} \) there is \( j' \) such that \( a_{i_0} \oplus a_j = c_{i_0, j} = b_{\sigma(i_0)} \oplus b_{\sigma(j)} = b_{i_0} \oplus b_{j'} \). From this we infer \( \{\sigma(i_0), \sigma(j)\} = \{i_0, j'\} \), and thus \( \sigma(i_0) = i_0 \). So, from (i) we have arrived to (i).

It is a trivial task to prove that under assumptions (ii) the set defined by (12) is a required \( K_{n+1} \)-graph, which proves (ii). \( \square \)

Let us note, as a particular case of (11) the following characterization.

**Proposition 2.2.** Let \( f \in S_p \), \( f(p) = p \), \( \sigma_1, \sigma_2 \in S_f \), and \( \mathcal{N}_1, \mathcal{N}_2 \) be two \( \left( \binom{n}{2}, \binom{n}{3} \right) \)-configurations defined on \( \mathcal{V}_2(I) \). The following conditions are equivalent.

(i) \( f \) is an isomorphism of \( \Pi(n, \sigma_1, \mathcal{N}_1) \) onto \( \Pi(n, \sigma_2, \mathcal{N}_2) \).

(ii) There is \( \varphi \in S_I \) such that
\[
    \varphi \quad (\text{comp.} \; (3)) \quad \text{is an isomorphism of} \; \mathcal{N}_1 \; \text{onto} \; \mathcal{N}_2,
\]
and one of the following holds
\[
    f(x_i) = x_{\varphi(i)}, \quad x = a, b, \quad f(c_{i, j}) = c_{\varphi(i), \varphi(j)}, \quad i, j \in I, i \neq j, \quad \varphi \circ \sigma_1 = \sigma_2 \circ \varphi \quad (14),
\]
or
\[
    f(a_i) = b_{\varphi(i)}, \quad f(b_i) = a_{\varphi(i)}, \quad f(c_{i, j}) = c_{\varphi(i), \varphi(j)}, \quad i, j \in I, i \neq j, \varphi \circ \sigma_1 = \sigma_2^{-1} \circ \varphi. \quad (15)
\]

As we know (cf. [6] Prop. 2.6), in case 2.1 there is a permutation of the edges of \( K_{A^* \setminus \{a_{i_0}\}} \) such that \( \mathcal{M} \cong \Pi(n, \sigma', \mathcal{N}') \) for an adequate configuration \( \mathcal{N}' \): \( \mathcal{M} \) is a skew perspective of \( K_{A^* \setminus \{a_{i_0}\}} \) onto \( G_{(i_0)} \). In the case we frequently say “\( \mathcal{M} \cong \Pi(n, \sigma', \mathcal{N}') \) and \( a_{i_0} \) is the perspective center in \( \Pi(n, \sigma', \mathcal{N}') \)”. However, \( \sigma' \) need not to be determined by a permutation of the vertices (cf. 1.3) neither \( \sigma \) and \( \sigma' \) are necessarily conjugate (cf. 2.2).
Proposition 2.3. Let $S(i_0)$ be a clique in $\mathcal{R}$ for some $i_0 \in \text{Fix}(\sigma)$, $\sigma \in S_I$, $|I| = n + 1 \geq 4$ (cf. 2.1). The following conditions are equivalent.

(i) $\Pi(n + 1, \sigma, \mathcal{R}) \cong \Pi(n + 1, \sigma', \mathcal{R}')$, for a $\sigma' = \sigma_0$, $\sigma_0 \in S_{n+1}$ and a suitable configuration $\mathcal{R}'$, where $a_{i_0}$ is the perspective center in $\Pi(n + 1, \sigma', \mathcal{R}')$ of the graphs $G_{(i_0)}$ and $K_{A-\{a_{i_0}\}}$.

(ii) There is $\tau \in S_I \setminus \{i_0\}$ such that

$$c_{\{i_0, \tau(i)\}} \oplus c_{\{i_0, \tau(j)\}} = c_{\{i, j\}}$$

(18)

for all $i, j \in I$, $i, j \neq i_0$.

Proof. Assume (i). Without loss of generality we can assume that $I = \{0, 1, \ldots, n\}$ and $i_0 = 0$. So, we relabel the points of $\Pi(n + 1, \sigma, \mathcal{R}) =: \mathcal{R}$ so as $q = a_0$ becomes a perspective center and $a_i : i = 1, \ldots, n + 1$ and $d_i : i = 1, \ldots, n + 1$ be the complete subgraphs that are in the respective perspective. Finally, we take $e_{i,j} = a_i \oplus a_j$ for $\{i, j\} \in \mathcal{P}_2(T)$, $T = \{1, \ldots, n + 1\}$. So, we obtain

$$a_{n+1} = p, \quad d_i = q \oplus a_i = c_{0,i} \quad \text{for } i \in T, \quad i \neq 0, \quad d_{n+1} = q \oplus a_{n+1} = b_0,$$

$$e_{i,j} = c_{0,i} \oplus c_{0,j} \quad \text{(computed in } \mathcal{R}) \quad \text{for } i, j \in T, \quad i, j \neq 0,$$

$$e_{i,n+1} = b_i \quad \text{for } i \in T, \quad i \neq 0. \quad (19)$$

Let $\tau \in S_I$ be the corresponding skew i.e. assume that

$$a_i \oplus a_j = e_{i,j} = d_{\tau(i)} \oplus d_{\tau(j)} \quad (20)$$

for all $\{i, j\} \in \mathcal{P}_2(T)$. In particular, this yields for $i \in T$, $i \neq n + 1$ the following:

$$a_i \oplus a_{n+1} =$$

$$b_i = d_{\tau(i)} \oplus d_{\tau(n+1)} = \begin{cases} c_{0,\tau(i)} \oplus c_{0,\tau(n+1)} & \text{or} \\ c_{0,\tau(i)} \oplus b_0 & \tau(n + 1) = n + 1, \tau(i) \neq n \\ b_0 \oplus c_{0,\tau(n+1)} & \tau(i) = n + 1, \tau(n + 1) = 0 \end{cases} \quad (21)$$

Since $\mathcal{R}$ does not contain any line with exactly one point in $B$ and two points in $C$, the first possibility is inconsistent. So, we end up with $\tau(n + 1) = n + 1$ and therefore, $\tau \in S_n$. If so, we obtain $e_{i,j} = a_i \oplus a_j = e_{i,j} = d_{\tau(i)} \oplus d_{\tau(j)} = c_{0,\tau(i)} \oplus c_{0,\tau(j)}$ for distinct $1 \leq i, j \leq n$. This justifies (18).

The converse reasoning consists in a simple computation: the reasoning above defines, in fact, a required isomorphism. It also defines the configuration $\mathcal{R}'$: the formulas $e_{i,n+1} \oplus e_{j,n+1} = b_i \oplus b_j = c_{\sigma^{-1}(i),\sigma^{-1}(j)} = e_{\sigma^{-1}(i),\sigma^{-1}(j)}$ for $1 \leq i, j \leq n$ and $e_u \oplus e_v = e_y$ iff $e_u \oplus e_v = e_y$ for $u, v, y \in \mathcal{P}_2(T \setminus \{n + 1\})$ determine the lines of $\mathcal{R}'$.

\[\Box\]

3 Particular case: $\mathcal{R}$ is a generalized Desargues configuration

In the class of skew perspectives one type of them seems “most similar to the classical geometrical perspective”: when the perspective axis is a generalized Desargues configuration i.e. when $\mathcal{R} = G_2(n)$ (cf. [1], [2]). So, in this subsection we set

$$\mathcal{R} = \Pi(n, \sigma, G_2(n)), \quad \sigma \in S_I, \quad n \geq 4.$$
Proposition 3.1. Either \( \mathcal{M} = G_2(n + 2) = \Pi(n, \text{id}) \) and then each point of \( \mathcal{M} \) can be chosen as a center of a skew perspective, or \( \mathcal{M} \) does not contain any point \( q \neq p \) such that \( \mathcal{M} \cong \Pi(n, \sigma, \mathcal{B}) =: \mathcal{M}' \) for a suitable configuration \( \mathcal{B} \), such that \( q \) is the perspective center in \( \mathcal{M}' \).

Proof. Assume that \( \sigma \neq \text{id}_I \). Suppose that such a point \( q \) exists, then – comp. 2.3 and 2.1 – there is \( i_0 \in I \) such that \( \sigma(i_0) = i_0 \). Moreover, in view of \([18]\), there is a permutation \( \tau \) such that \( c_{i_0,\tau(i)} + c_{i_0,\tau(j)} = c_{i,j} \) for all \( i,j \in I \), \( i,j \neq i_0 \). On the other hand, in \( G_2(I) \) we have \( c_{i_0,\tau(i)} + c_{i_0,\tau(j)} = c_{\tau(i),\tau(j)} \) for all \( i,j \) as above. This, finally, gives \( \{i,j\} = \{\tau(i),\tau(j)\} \), from which we deduce \( \tau = \text{id} \) and then \( \mathcal{M}' \cong G_2(n + 2) \).

Corollary 3.2. Let \( S_I \ni \sigma_1 \neq \text{id}_I \). If \( f \) is an isomorphism between \( \Pi(n, \sigma_1, G_2(n)) \) and \( \Pi(n, \sigma_2, G_2(n)) \) then \( f(p) = p \) and \( \sigma_2 \neq \text{id}_I \). Moreover, \( f \) is determined by a permutation \( \varphi \in S_I \) (comp. \([14], [16]\)) so as either \( f \) fixes \( A \) and \( B \) and then \( \sigma_2 = \varphi \circ \sigma_1 \circ \varphi^{-1} = \sigma_1^\varphi \), or \( f \) interchanges \( A \) and \( B \) and \( \sigma_2^{-1} = \sigma_1^\varphi \) (see Prop. 2.2).

Let us recall a few facts from the folklore of group theory. Let \( \sigma \in S_I \), then \( \sigma \) has a unique (up to an order) decomposition \( \sigma = \sigma_1 \circ \ldots \circ \sigma_k \) where \( \sigma_1, \ldots, \sigma_k \) are pairwise disjoint cycles. Let \( x_i \) be the length of \( \sigma_i \), then \( n = \sum_{i=1}^k x_i \). Without loss of generality we can assume that \( x_1 \leq \ldots \leq x_k \) and we can set \( C(\sigma) := \{x_1, \ldots, x_k\} \). So, \( C(\sigma) \) is an unordered partition of the integer \( n \) into \( k \) components (see e.g. [18, Ch. 4], [19]). The following is known:

Fact 3.3. \( \sigma_1 \) and \( \sigma_2 \) are conjugate in \( S_I \) (i.e. \( \sigma_2 = \varphi \circ \sigma_1 \circ \varphi^{-1} = \sigma_1^\varphi \) for a \( \varphi \in S_I \)), iff \( C(\sigma_1) = C(\sigma_2) \).

In particular, \( \sigma \) and \( \sigma^{-1} \) are conjugate for every \( \sigma \in S_I \).

Permutations \( \sigma \) and \( \text{id}_I \) are conjugate iff \( \sigma = \text{id}_I \).

As an immediate consequence of 3.3 and 3.2 we obtain

Proposition 3.4. Let \( \sigma_1, \sigma_2 \in S_I \). \( \Pi(n, \sigma_1, G_2(n)) \cong \Pi(n, \sigma_2, G_2(n)) \) iff \( \sigma_1 \) and \( \sigma_2 \) are conjugate.

Consequently, there are \( P(n) = \sum_{k=1}^n P(n,k) \) types of the skew perspectives whose axial configurations are the generalized Desargues configuration, where \( P(n,k) \) is the number of unordered partitions of \( n \) into \( k \) components.

4. A few examples and counterexamples: some \( (15, 20, 3) \)-configurations

In this Section we discuss some \((15, 20, 3)\)-configurations which appear to be skew perspectives. Some of them were (up to an isomorphism) defined elsewhere, they fall into some other classes of configurations. Then we use the notation of the papers where ‘origins’ can be found without definite explanation. But original definitions are useless in this place (sometimes we briefly quote the idea of a respective definition): we merely want to show what ‘name’ has the structure in that other papers. No general important result follows from investigations of this Section: the reader will stay more familiar with technical apparatus used in our paper and with some fundamental examples of (really ‘skew’) perspectives.
Example 4.1. Let \( n = 2k, I = I_{2k} \), and \( \sigma = (1,2)(3,4) \ldots (2k - 1, 2k) \), or \( n = 2k + 1, I = I_{2k} \cup \{0\}, \) and \( \sigma = (0)(1,2)(3,4) \ldots (2k - 1, 2k) \), for an integer \( k \geq 2 \). So, \( \sigma \) is, in fact, a family of disjoint transpositions. The following is a direct consequence of [8, Repr. 2.4]

\[
\Pi(n, \sigma, G_2(n)) \text{ is the combinatorial quasi Grassmannian } \mathcal{R}_n \text{ of } [5].
\]

In accordance with our theory developed in Subsection 3, \( \mathcal{R}_{2k} \) has exactly two \( K_{2k+1} \) subgraphs and \( \mathcal{R}_{2k+1} \) has three \( K_{2k+2} \)-subgraphs (see also [5, Cor. 4.4]).

In particular, \( \mathcal{R}_4 \) is a \((15_4 20_3)\)-configuration.

All the \((15_4 20_3)\)-configurations with at least three free \( K_5 \)-subgraphs inside were listed in [10, Classif. 2.8]. In particular, each of them is a binomial configuration which contains two maximal complete subgraphs so, it is a perspective of two \( K_5 \) with an additional free \( K_5 \). Let us analyse some, concrete, examples, which appear in accordance with [2,1]

Let \( \mathcal{M} = \Pi(4, \sigma, G_2(I_4)) \); suppose that \( \text{Fix}(\sigma) \neq \emptyset, I_4 \) for a \( \sigma \in S_{I_4} \).

Example 4.2. \( \sigma = (1)(2,3,4) \). Then \( \mathcal{M} \) coincides with the configuration defined in [10, Classif. 2.8(ii)]. To see this it suffices to represent it in the form of a system of triangle perspectives in accordance with Figure 1

Example 4.3. \( \sigma = (1)(2)(3,4) \). Then \( \mathcal{M} \) coincides with the configuration defined in [10, Rem. 2.10(iii)] – cf. Figure 2. Consequently, \( \mathcal{M} \) is isomorphic to the so called multi veblen configuration \( \mathcal{W}_{I_4 \mathcal{L}_4}^P G_2(I_4) \), where \( L_4 \) is a linear graph on \( I_4 \).

Example 4.4. Let \( \mathcal{M} = \mathcal{W}_{I_4 \mathcal{L}_4}^P G_2(I_4) \). It is known that \( \mathcal{W}_{I_4 \mathcal{L}_4}^P G_2(I_4) \cong \mathcal{W}_{I_4 \mathcal{K}_4 \{\{2,3\}\}}^P G_2(I_4) \) (cf. [3, Thm. 4]). Without coming into details let us quote (after [4] Constr. 4), compare with Construction [1,1] that in an arbitrary multiveblen configuration \( \mathcal{W}_{I_4 \mathcal{L}_4}^P \mathcal{R} \), we have a centre \( p \), the lines through \( p \) with the points \( a_i, b_i, i \in I \) as in \( L_p \), and a graph \( \mathcal{P} \) defined on \( I \) which determines whether \( c_i, j = a_i \oplus a_j = b_i \oplus b_j \) \((i, j) \in \mathcal{P})\).
Fig. 2: The diagram of the line \( \{c_{2,3}, c_{3,4}, c_{2,4}\} \) in \( \Pi(4, (1)(2)(3,4), G_2(I_4)) \).

\[
\begin{align*}
\Delta_1 & : c_{1,2} \\
\Delta_2 & : a_2 \\
\Delta_3 & : b_2
\end{align*}
\]

\[
\begin{align*}
\Delta_1 & : c_{1,3} \\
\Delta_2 & : a_3 \\
\Delta_3 & : b_3 \\
\Delta_4 & : a_4
\end{align*}
\]

\[
\begin{align*}
\Delta_1 & : c_{1,4} \\
\Delta_2 & : a_4 \\
\Delta_3 & : b_4
\end{align*}
\]

or \( c_{i,j} = a_i \oplus b_j = b_i \oplus a_j \) \((i,j) \notin \mathcal{P})\). Then the axis \( \mathfrak{A} \) is used as in the definition of \( \Pi(n, \sigma, \mathfrak{N}) \) to get \( \mathcal{L}_C \).

Let us quote after [5, Cor. 2.13] the following characterization, which will be needed in the sequel.

A \( \left( \binom{n}{2}, \binom{n}{3}\right) \)-configuration is a multiveblen configuration with the axis \( G_2(n-2) \) iff it contains at least \( n-2 \) free \( K_{n-1} \)-subgraphs. \([22]\)

\( \mathfrak{A} \) can be represented as a perspective of two graphs \( G_1 = \{a_1, c_{1,2}, c_{1,3}, b_1\} \) and \( G_2 = \{a_4, c_{2,4}, c_{3,4}, b_4\} \) with centre \( q = c_{1,4} \).

**Fact.** \( \mathfrak{A} \cong \Pi(4, \text{id}, \mathfrak{N}) \), where \( \mathfrak{N} \cong \mathcal{B}(2) \) is the Veblen configuration with the lines \( \{\{e_{1,4}, e_{1,2}, e_{2,4}\}, \{e_{1,4}, e_{1,3}, e_{3,4}\}, \{e_{1,2}, e_{2,3}, e_{3,4}\}, \{e_{1,3}, e_{2,3}, e_{2,4}\}\} \).

\( e_{i,j} = x_i \oplus x_j \), and \( x_i \) are the vertices of \( G_1 \).

Gathering together [4,3] and [4,4] we see that \( \Pi(4, \text{id}, \mathcal{B}(2)) \cong \Pi(4, (1)(2)(3,4), G_2(I_4)) \) so, a skew perspective does not determine, geometrically, its centre and a labelling of the points in axial configuration.

**Example 4.5.** Let \( G_2^*(I_4) \) be the Veblen configuration whose lines are the \( \mathcal{N} \)-images (see [6,6]) of the lines of \( G_2(I_4) \). Then, for every graph \( \mathcal{P} \) defined on \( I_4 \) the structure \( \mathfrak{M} = \mathcal{W}_4^{\mathcal{P}} \triangleright\triangleright_N G_2(I_4) \) contains four \( K_5 \)-graphs: \( G_i = \{a_i, b_i\} \cup \{c_{i,j} : j \in I_4 \setminus \{\} \} \) with \( i \in I_4 \). However, no one of the \( G_i \) is freely contained in \( \mathfrak{M} \) and one can directly verify that \( \mathfrak{M} \) cannot be presented as a \((15_4, 20_3)\)-perspective.

**Example 4.6.** Let \( \mathfrak{M} = \mathcal{W}_4^{\mathcal{P}} \triangleright\triangleright N_4 G_2(I_4) \) (cf. [3], Rem. 2.10(ii), [4, Constr. 2]), where \( N_4 \) is the empty graph on 4 vertices. The structure \( \mathfrak{M} \) freely contains four \( K_5 \)-subgraphs and it is homogeneous: any two points in \( C \) can be interchanged by an automorphism of \( \mathfrak{M} \). Let us represent \( \mathfrak{M} \) in the form \( \Pi(4, \sigma, \mathfrak{N}) \) with the centre \( q = c_{1,2} \) chosen as an example. Then the perspective graphs are \( G_1 = \{a_1, b_1, c_{1,3}, c_{1,4}\} \) and \( G_2 = \{b_2, a_2, c_{2,3}, c_{2,4}\} \). We find then the following representation.
**Fact.** $\mathfrak{M} \cong \Pi(4,(1,2)(3)(4),\mathfrak{N})$, where $\mathfrak{N} \cong \mathbb{P}(2)$ is the Veblen configuration with the lines 

$$\{\{e_{1,2},e_{1,3},e_{2,3}\},\{e_{1,2},e_{1,4},e_{2,4}\},\{e_{1,3},e_{2,4},e_{3,4}\},\{e_{1,4},e_{2,3},e_{3,4}\}\},$$
e

$e_{i,j}$ are defined as in (4.4).

\[ \circ \]

**Example 4.7.** Examples 4.4 and 4.6 both can be generalized with the following computation. Let $\mathfrak{M} = \mathbb{N}^*_X_\mathcal{P} G_2(X)$ where $\mathcal{P}$ is a graph defined on $X$, $|X| = n$. Consider two complete free subgraphs $G_1, G_2$ of $\mathfrak{M}$ intersecting in a point $q = c_{i,j}$. Without loss of generality we can assume that $i = 1$, $j = 2$ and $X = \{1, \ldots, n\}$. Set $I_0 = \{3, \ldots, n\}$.

Then we have 

$$G_1 = \{x_1 = a_1, x_2 = b_1, x_j = c_{1,j}, j \in I_0\} \text{ and } G_2 = \{y_1 = a_2, y_2 = b_2, y_j = c_{2,j}, j \in I_0\}. \quad (23)$$

Define 

$$\therefore e_{i,j} = x_i \oplus x_j \text{ for } \{i, j\} \in \mathcal{P}_2(X). \text{ Then we have }$$

$$e_{1,2} = p = y_1 \oplus y_2, \quad e_{i,j} := c_{i,j} = y_i \oplus y_j \text{ for all } \{i, j\} \in \mathcal{P}_2(I_0). \quad (24)$$

Let us consider the two following cases:

(A) $\{1, 2\} \in \mathcal{P}$

(B) $\{1, 2\} \notin \mathcal{P}$.

Assume (A). One can easily compute that $q = x_i \oplus y_i$ for $i \in I$. Moreover, we compute for $j \geq 3$ as follows:

$$e_{1,j} = \begin{cases} a_j & \text{when } \{1, j\} \in \mathcal{P} \\ b_j & \text{when } \{1, j\} \notin \mathcal{P} \end{cases} \quad \text{and } e_{2,j} = \begin{cases} b_j & \text{when } \{1, j\} \in \mathcal{P} \\ a_j & \text{when } \{1, j\} \notin \mathcal{P} \end{cases}.$$ 

Analogously, we compute 

$$y_1 \oplus y_j = \begin{cases} a_j & \text{when } \{2, j\} \in \mathcal{P} \\ b_j & \text{when } \{2, j\} \notin \mathcal{P} \end{cases} \quad \text{and } y_2 \oplus y_j = \begin{cases} b_j & \text{when } \{2, j\} \in \mathcal{P} \\ a_j & \text{when } \{2, j\} \notin \mathcal{P} \end{cases}.$$ 

The formulas above and the formula (24) determine the skew:

$$\sigma(\{i, j\}) = \{i, j\} \text{ for } \{i, j\} \in \mathcal{P}_2(I_0) \cup \{\{1, 2\}\}, \text{ let } j \geq 3 :$$

$$\sigma: \{1, j\} \mapsto \{2, j\} \mapsto \{1, j\}$$

when $\{1, j\} \in \mathcal{P}, \{2, j\} \in \mathcal{P}$ or $\{1, j\} \notin \mathcal{P}, \{2, j\} \notin \mathcal{P},$$

$$\sigma: \{1, j\} \mapsto \{1, j\}, \sigma: \{2, j\} \mapsto \{2, j\}$$

when $\{1, j\}, \{2, j\} \in \mathcal{P}$ or $\{1, j\}, \{2, j\} \notin \mathcal{P}. \quad (25)$$

Finally, let $\mathcal{P}_0$ be the restriction of $\mathcal{P}$ to $\mathcal{P}_2(I_0)$. We conclude with

**Fact.** 4.7.1. In case (A), $\mathfrak{M} \cong \Pi(n,\sigma,\mathfrak{N})$, where $\mathfrak{N} = \mathbb{N}^*_I_0 \mathcal{P}_0 G_2(I_0)$ and $\sigma$ is defined by (25).

Now, let us pass to the case (B). In this case we only slightly renumber the elements of $G_1$ and $G_2$ (cf. (23)):

$$G_1 = \{x_1 = a_1, x_2 = b_1, x_j = c_{1,j}, j \in I_0\} \text{ and } G_2 = \{y_1 = b_2, y_2 = a_2, y_j = c_{2,j}, j \in I_0\}. \quad (26)$$
Clearly, $e_{i,j}$ take values as in \(A\). Differences appear when we compute for $j \geq 3$
\[
y_i \oplus y_j = \begin{cases} a_j & \text{when } \{2, j\} \notin \mathcal{P} \\
 b_j & \text{when } \{2, j\} \in \mathcal{P}
\end{cases}
\text{ and } y_2 \oplus y_j = \begin{cases} b_j & \text{when } \{2, j\} \notin \mathcal{P} \\
 a_j & \text{when } \{2, j\} \in \mathcal{P}.
\end{cases}
\]
Now, the skew is determined by the following conditions:
\[
\sigma((i, j)) = (i, j) \text{ for } \{i, j\} \in \varphi_2(I_0) \cup \{\{1, 2\}\}, \text{ let } j \geq 3:
\]
\[
\sigma: \{1, j\} \mapsto \{2, j\} \mapsto \{1, j\}
\]
when $\{1, j\}, \{2, j\} \in \mathcal{P}$ or $\{1, j\}, \{2, j\} \notin \mathcal{P}$,\]
\[
\sigma: \{1, j\} \mapsto \{1, j\}, \sigma: \{2, j\} \mapsto \{2, j\}
\]
when $\{1, j\} \in \mathcal{P}, \{2, j\} \in \mathcal{P}$ or $\{1, j\} \notin \mathcal{P}, \{2, j\} \notin \mathcal{P}$. \(27\)

We conclude with

**Fact 4.7.2.** In case \(1\), \(\mathcal{M} \cong \Pi(n, \sigma, \mathfrak{N})\), where \(\mathfrak{N} = \mathbb{W}_{I_0}^{p_{I_0}} G_2(I_0)\) and \(\sigma\) is defined by \(27\).

In particular, we obtain the following generalizations of \(4.6\) and a folklore.

**Fact 4.7.3.**
\begin{enumerate}
  \item \(\mathbb{W}_{X, N_X}^{p_{X}} G_2(X) \cong \Pi(n, \sigma, \mathfrak{N})\), where \(\mathfrak{N} = \mathbb{W}_{I_0}^{p_{I_0}} G_2(I_0)\) and \(\sigma = (1, 2)(3) \ldots (n)\).
  \item \(\mathbb{W}_{X, K_X}^{p_{K_X}} G_2(X) \cong \Pi(n, \sigma, \mathfrak{N})\), where \(\mathfrak{N} = \mathbb{W}_{I_0}^{p_{I_0}} G_2(I_0)\) and \(\sigma = \text{id}_X\). \(\square\)
\end{enumerate}

Finally, combining \(2.1\), \(2.2\), and \(4.7\) we obtain the following.

**Proposition 4.8.** Let \(I = I_0\). Assume that \(\mathfrak{M}\) is not a generalized Desargues configuration. If a multiveblen configuration \(\mathfrak{M} = \mathbb{W}_{I_0}^{p_{I_0}} G_2(I)\) is isomorphic to \(\Pi(n, \sigma, \mathfrak{N})\) where \(\sigma = \varphi_0\), \(\varphi_0 \in S_I\) and \(\mathfrak{N}\) is a binomial \(\Pi\) defined on \(\varphi_2(I)\) then, up to an isomorphism, \(\varphi_0 = (1, 2)(3) \ldots (n)\) and either \(\{1, 2\} \in \mathcal{P}\), \(\{1, i\} \in \mathcal{P}\) iff \(\{2, i\} \in \mathcal{P}\) for all \(j = 3, \ldots, n\), or \(\{1, 2\} \notin \mathcal{P}\), \(\{1, i\} \in \mathcal{P}\) iff \(\{2, i\} \notin \mathcal{P}\) for all \(j = 3, \ldots, n\), and \(\mathfrak{N}\) is a multiveblen configuration determined by the graph obtained by deleting from \(\mathcal{P}\) the vertices \(1\) and \(2\).

**Remark 1.** The two cases of \(\{1, 2\} \in \mathcal{P}\) or \(\notin \mathcal{P}\) above are, in fact, superfluous. From \(4.1\) Prop. 9 we know that, up to an isomorphism we can always assume that \(\{1, 2\} \in \mathcal{P}\).

Consequently, \(4.7\) characterizes all the binomial configurations which are simultaneously multiveblen configurations and skew perspectives preserving edge-concurrency.

Another example which is worth to consider is a combinatorial Veronesian \(V_k(3)\) of \(9\). This example shows, primarily, that not every "sensibly roughly presented" perspective \(\Pi(n, \sigma, \mathfrak{N})\) between complete graphs necessarily has a Desarguesian axis' nor its skew preserves the adjacency of edges of the graphs in question.

**Example 4.9.** Let us adopt the notation of \(9\). Let \(|X| = 3\), \(X = \{a, b, c\}\). Then the combinatorial Veronesian \(V_k(X) =: \mathfrak{M}\) is a \(\left(\binom{k+2}{2}, \binom{k+2}{3}\right)\)-configuration; its point set is the set \(\eta_k(X)\) of the \(k\)-element multisets with elements in \(X\) and the lines have form \(eX\), \(e \in \eta_{k-\alpha}(X)\). \(V_1(X)\) is a single line, \(V_2(X)\) is the Veblen configuration, and \(V_3(X)\) is the known Kantor configuration (comp. \(9\) Prop’s.
2.2, 2.3], [Repr. 2.7]). Consequently, we assume \( k > 3 \). The following was noted in [\text{Fct. 4.1}]:

The \( K_{k+1} \) graphs freely contained in \( V_k(X) \) are the sets \( X_{a,b} := \eta_k(\{a, b\}) \), \( X_{b,c} := \eta_k(\{b, c\}) \), and \( X_{c,a} := \eta_k(\{c, a\}) \).

In particular, \( \mathfrak{M} \) contains two complete subgraphs \( X_{a,b}, X_{c,a}, \) which cross each other in \( p = a^k \). Let us present \( \mathfrak{M} \) as a perspective between these two graphs. Let us re-label the points of \( V_k(X) \):

\[
e_i = b_i a^{k-i}, \quad b_i = c_i a^{k-i}, \quad i \in \{1, \ldots, k\} =: I, \quad e_{i,j} = c_i \oplus c_j, \quad \{i, j\} \in \mathcal{P}_2(I).
\]

Clearly, \( p \oplus e_i = b_i \) so, the map \( (c_i \mapsto b_i, \ i \in I) \) is a point-perspective. Let us define the permutation \( \sigma \) of \( \mathcal{P}_2(I) \) by the formula

\[
\sigma(\{i, j\}) = \{j - 1, j\} \text{ when } 1 \leq i < j \leq k.
\]

It is seen that \( \sigma = \sigma^{-1} \). After routine computation we obtain \( b_i \oplus b_j = c_{\sigma(\{i,j\})} \) whenever \( i < j \); moreover, in this representation the axial configuration consists of the points in \( bct \mathcal{P}_{k-2}(X) \) so, it is isomorphic to \( V_{k-2}(X) \). Consequently, \( V_k(X) \cong \Pi(k, \sigma, V_{k-2}(X)) \). It is seen that there is no permutation \( \varphi \in S_I \) such that \( \{\varphi(i), \varphi(j)\} = \{j - i, j\} \) for all \( i < j \), unless \( |I| = 2 \leq 4 \). This can be summarized in the following

**FACT.** The binomial configuration \( V_k(3) \) with \( k > 3 \) cannot be presented as a skew perspective, with the skew determined by a permutation or by the complementing in the set of indices. Though it represents a perspective of two simplices. 

\[\Box\]

5 Few remarks on projective realizability of skew perspectives

Our construction \([11]\) a generalization of a projective perspective, originates in studying arrangements of points and lines of a (real) projective space. So, the question whether (an which) skew perspectives can be realized in a Desarguesian projective space is quite natural. For 103-configurations of the type \( \Pi(3, \sigma, \mathbb{G}_2(I_3)) \) the answer is affirmative (all three are realizable!) and is known for ages. For structures \( \Pi(4, \sigma, \mathbb{G}_2(I_4)) \), which are primarily investigated in this Section, situation is more complex. Let us begin with results easily derivable from known facts.

**Proposition 5.1.** Let \( \sigma \in S_{I_n} \) and \( C(\sigma) \) be one of the following: \( (1, \ldots, 1), (1, 2, \ldots, 2) \). Then \( \Pi(n, \sigma, \mathbb{G}_2(I_n)) \) can be realized in a real projective space.

**Proof.** Write \( \mathfrak{M} = \Pi(n, \sigma, \mathbb{G}_2(I_n)) \) Note that in the first case \( \sigma = \text{id}_{I_n} \), and \( \mathfrak{M} \) is the generalized Desargues configuration, see \([2]\). In the second and the third case \( \sigma \) can be written in the form \( (n)(1, 2)(3, 4) \ldots (n - 2, n - 1) \) and \( (1, 2) \ldots (n - 1, n) \) resp. and then \( \mathfrak{M} \) is a combinatorial quasi Grassmannian, see Example \([4,1]\) In all these cases the claim follows from the results of \([17]\) Thm. 2.17 and \([8]\) Prop. 1.6 and Prop’s 3.6-3.8. \[\Box\]

We have also an evident lemma:
Figure 3: The structure $\mathbf{\Pi}(4, (1, 2)(3, 4), G_2(I_4)) = \mathcal{R}_4$, the smallest not commonly known example of the structures defined in 5.1.
Lemma 5.2. Let $\sigma \in S_1$, $J \subset I$, and $\sigma(J) = J$; set $\sigma_0 := \sigma \upharpoonright J$. Then $\Pi(|J|, \sigma_0, G_2(J))$ is a subconfiguration of $\Pi(|I|, \sigma, G_2(I))$.

Proposition 5.3. Let $\sigma \in S_{I_n}$. Assume that $C(\sigma)$ contains the sequence $(1, 1, 2)$ as its subsequence. Then $\Pi(n, \sigma, G_2(I_n))$ cannot be realized in any Desarguesian projective space.

Proof. Write $\mathfrak{M} = \Pi(n, \sigma, G_2(I_n))$, $\sigma_0 = (1)(2)(3, 4)$, and $\mathfrak{M}_0 = \Pi(n, \sigma_0, G_2(I_4))$. Clearly, $\mathfrak{M}_0$ is a subconfiguration of $\mathfrak{M}$. From Example 4.3 and [17, Prop. 2.3] we know that $\mathfrak{M}_0$ cannot be realized in any Desarguesian projective space, which closes our proof.

We say that a configuration $\mathfrak{M}$ is planar if for any realization of $\mathfrak{M}$ in a projective space $\mathfrak{P}$ this realization lies on a plane of $\mathfrak{P}$. Note that, anyway, even if $\mathfrak{M}$ cannot be realized in any Desarguesian projective space then it can be extended to a projective plane. So, in fact, in the definition above we can restrict ourselves to Desarguesian $\mathfrak{P}$. And a configuration nonrealizable in a Desarguesian projective space is, by definition, planar.

Lemma 5.4. Let $\sigma \in S_n$ be a cycle of length $n$, $n \geq 3$. The configuration $\Pi(n, \sigma, G_2(I_n))$ is planar.

Proof. Consider a realization of $\Pi(n, \sigma, G_2(I_n))$ in a projective space $\mathfrak{P}$. As it is commonly accepted, we do not distinguish a point of a configuration and its image under a realization in question.

Let $A$ be the plane of $\mathfrak{P}$ which contains $p, a_1, a_2$. Then $b_2 = p \oplus a_2$ and $e_{1,2} = a_1 \oplus a_2$ are on $A$. So, $b_3 = e_{1,2} \oplus b_2 \in A$ and then $a_3 = p \oplus b_3 \in A$. Inductively, we come to $a_i, b_i \in A$ for all $i \in I_n$, which closes our proof.

Lemma 5.5. Let $\sigma \in S_{I_n}$, $\sigma(i_0) = i_0$, and $\sigma(i_1) = i_2 \neq i_1$ for some $i_0, i_1, i_2 \in I_n$. Set $J := I \setminus \{i_0\}$. If $\mathfrak{M} = \Pi(n, \sigma, G_2(I_n))$ is embedded via $\gamma$ into a Desarguesian projective space $\mathfrak{P}$ and the image under $\gamma$ of the subconfiguration $\mathfrak{M} = \Pi(n - 1, \sigma \upharpoonright J, G_2(J))$ of $\mathfrak{M}$ lies on a plane $A$ of $\mathfrak{P}$ then the image of $\mathfrak{M}$ under $\gamma$ lies on $A$.

In particular, if $\mathfrak{M}$ is planar then $\mathfrak{M}$ is planar as well.

Proof. Suppose that $a_{i_0} \notin A$. Then the plane $B$ spanned in $\mathfrak{P}$ by the points $p, a_{i_0}, a_{i_1}$ is distinct from $A$ and it contains $b_{i_0}$. However, the lines $a_{i_0}b_{i_0}, a_{i_1}$ and $b_{i_0}b_{i_2}$ intersect in $c_{i_0,i_1} \in B$, so $b_{i_2} \in B$. Consequently, $p, a_{i_1}, b_{i_1}, b_{i_2} \in A, B$ so, they are collinear and $\mathfrak{M}$ degenerate.

As an direct consequence of 5.5 and 5.4 we obtain.

Lemma 5.6. Let $C(\sigma) = (\underbrace{1, \ldots, 1}_{(n-k)-\text{times}}, k)$, $k \geq 3$. Then $\Pi(n, \sigma, G_2(I_n))$ is planar.

Finally, with the help of the computer program Maple we can decide which of the remaining perspectives $\Pi(4, \sigma, G_2(I_4))$ can be projectively realized. These cases are $C(\sigma) = (1, 3)$ and $C(\sigma) = (4)$.
**Lemma 5.7.** Let a system of points \( p, a_i, b_i, 1 \leq i \leq 4 \) of the real projective plane \( \mathcal{P} \) be characterized by the following parametric equations.

\[
p = [1, 0, 0], a_1 = [0, 0, 1], b_1 = [1, 0, 1], a_2 = [1, \alpha_1, \alpha_2], b_2 = [1, \alpha_1 x, \alpha_2 x],
\]

\[
a_3 = [0, 1, 0], b_3 = [1, 1, 0], a_4 = [1, \beta_1, \beta_2], b_4 = [1, \beta_1 y, \beta_2 y].
\]

(i) If \( \sigma = (1, 2, 3, 4) \) then the above system of points yields in \( \mathcal{P} \) a configuration isomorphic to \( \Pi(4, \sigma, G_2(I_4)) \) iff

\[
\beta_1 = -\frac{1 + \beta_2 y}{y}, \alpha_1 = -\frac{1 - 2\beta_2 y + \beta_2^2 y^2}{xy(\beta_2 - 1)},
\]

\[
\alpha_2 = \frac{\beta_2 y(\beta_2^2 y^2 - 2\beta_2 y + 1 - xy + xy\beta_2)}{(\beta_2^2 y^2 - \beta_2 y - y + 1)}
\]

(29)

and a (terribly long) equation which assures that \( c_1, c_2, c_3, c_4 \) are not collinear holds.

**Fact 5.7.1.** As an example we can quote that substituting \( \beta_2 := 2; \alpha_2 := -1, x := 2, y := 2 \) and taking into account (29) we do obtain a realization of \( \Pi(4, \sigma, G_2(I_4)) \). Consequently,

\[
\Pi(4, (1, 2, 3, 4), G_2(I_4)) \text{ can be realized in the real projective plane.}
\]

(ii) If \( \sigma = (1, 2, 3)(4) \) then the above system of points yields in \( \mathcal{P} \) a configuration isomorphic to \( \Pi(4, \sigma, G_2(I_4)) \) iff

\[
\alpha_1 = -\frac{1 + \alpha_2 x}{x},
\]

(30)

which guarantees that the points \( p, a_i, b_i, i \leq 3 \) yield the fez configuration \( \Pi(3, (1, 2, 3), G_2(I_3)) \), and

\[
\alpha_2 = -\frac{\beta_2(-1 + x)}{x\beta_1}, x = \frac{\beta_2^2 - \beta_2 \beta_1 + \beta_1^2}{\beta_2^2}, \beta_1 \neq -\frac{\beta_2 y - 1}{y}.
\]

(31)

The last relation in (31) assures that \( c_1, c_2, c_3, c_4 \) are not collinear.

**Fact 5.7.2.** Substituting, concretely, \( \beta_1 := 5, \beta_2 := 2, y := 2 \) and using (30), (31) we arrive to an example of concrete realization of \( \Pi(4, \sigma, G_2(I_4)) \). Consequently

\[
\Pi(4, (1, 2, 3)(4), G_2(I_4)) \text{ can be realized in the real projective plane.}
\]

**Note 3.** Due to the homogeneity of desarguesian planes the system (28) characterizes, in fact, an arbitrary system of points \( a_i, b_i, i \leq 4 \) that are point-perspective with the centre \( p \).

As a somewhat tricky generalization of (3) let us note the following

**Fact 5.8.** Let \( C(\sigma) \) contain the sequence \( (1, 1, 3) \) as its subsequence for a permutation \( \sigma \in S_n \). Then \( \mathfrak{M} = \Pi(n, \sigma, G_2(I_n)) \) cannot be realized in any Desarguesian projective space.
Figure 4: The structures $\Pi(4, (1, 2, 3, 4), G_2(I_4))$ (left) and $\Pi(4, (1, 2, 3)(4), G_2(I_4))$ (right), see [5.7] and [5.7]. Schemas!: lines are drawn here as curved segments.

**Proof.** Let $\sigma = (1, 2, 3)(4)(5)$. Suppose that $\mathcal{R} = \Pi(5, \sigma, G_2(I_5))$ can be realized in a Desarguesian projective space; from [5.6], $\mathcal{R}$ is realizable on a Desarguesian plane $A$. Without loss of generality we can assume that the points $p, a_i, b_i$ are defined by the system [23], and $a_5 = [1, \delta_1, \delta_2], b_5 = [1, \delta_1 z, \delta_2 z]$. Since both systems of points: $p, a_1, a_2, a_3, a_4$ and $p, a_1, a_2, a_3, a_5$ yield on $A$ (together with the respective $b_i$) subconfigurations of $\mathcal{R}$ isomorphic to $\Pi(4, (1, 2, 3)(4), G_2(I_4))$, from [5.7] we infer $\frac{\beta_2}{\alpha_1} = -\frac{\alpha_2}{\alpha_1} = \frac{\beta_2}{\alpha_1}$, which yields that $p, a_4, a_5$ are collinear, and this is impossible. Now the claim is evident, as $\mathcal{R}$ contains $\mathcal{N}$.

6 A few configurational properties: an analogue of the desargues Axiom

Other group of problems which are commonly related to configurations similar to the Desargues configuration are so called configurational axioms. Let us briefly quote a formulation of the Desargues Axiom in the form which is suitable for our purposes here:

*Let $\mathcal{A}$ be a family of 10 points in a (Desarguesian) projective space $\mathcal{P}$ such that after an identification $\gamma$ of the points in $\mathcal{A}$ and the points of $G_2(I_5)$ $\gamma$ maps 9 of the collinear triples of $G_2(I_5)$ onto triples collinear in $\mathcal{P}$ and no noncollinear triple is mapped onto a collinear one. Then the last, remaining, collinear triple in $G_2(I_5)$ is mapped by $\gamma$ onto a collinear one.*

We say that the Desargues configuration *closes* in Desarguesian projective spaces. Clearly, such an elegant formulation of the Desargues axiom is possible because of
the symmetries of the Desargues configuration. Analogous statement (with ‘10’ and ‘9’ replaced by suitable values ‘(n)’ and ‘(n) − 1’ is valid for generalized Desargues configuration $G_2(n)$ (comp. Prop. 1.9]). Nevertheless, one can prove that, in a sense, every (not too small) skew perspective associated with a permutation of indices closes in every Desarguesian space.

Let us begin with an evident observation.

**Lemma 6.1.** Let $\sigma \in S_{I_n}$. Then (in the notation of 1.1) each of two sets $A \cup C$ and $B \cup C$ yields in $\Pi(n, \sigma, G_2(I_n))$ a subconfiguration that is a generalized Desargues configuration of the type $G_2(n + 1)$.

From this we easily obtain the following form of “configurational closeness” of skew perspectives.

**Theorem 6.2.** Let $\mathcal{D}$ be a set of $\binom{n+2}{2}$ points of a Desarguesian projective space $\mathfrak{P}$ and let $\gamma$ be a bijection of $\mathcal{D}$ and the points of $\mathfrak{M} = \Pi(n, \sigma, G_2(I_n)), \sigma \in S_{I_n}, n \geq 4$. Assume that

(i) $\gamma$ maps collinear triples of the form $p, a_i, b_i$ onto triples collinear in $\mathfrak{P}$,

(ii) $\gamma$ maps $\binom{n+2}{3} - n - 1$ of the remaining collinear triples of $\mathfrak{M}$ onto triples collinear in $\mathfrak{P}$, and

(iii) no noncollinear triple of points of $\mathfrak{M}$ is mapped onto a collinear one.

Then the last triple of collinear points of $\mathfrak{M}$ (recall: $\mathfrak{M}$ has $\binom{n+2}{3}$ triples of collinear points) is mapped by $\gamma$ onto a collinear triple.

**Proof.** From assumptions, this ‘last’ triple $L$ of collinear points has one of the following forms:

$$L = \{a', a'', c\}, \quad L = \{b', b'', c\} \quad \text{or} \quad L = \{c, c', c''\}$$

for $a', a'' \in A$, $b', b'' \in B$, $c, c', c'' \in C$. In any case, by 6.1 $L$ is contained in a generalized Desargues configuration $\mathcal{G} \cong G_2(m)$ with $m \geq 5$, contained in $\mathfrak{M}$. From assumptions, $\gamma$ maps all the collinear triples of $\mathcal{G}$ except possibly $L$ onto collinear triples. So, $\gamma(L)$ is collinear as well.

**Remark 2.**

(i) One cannot formulate 6.2 as a full analogue of (Des). Namely, the conditions 6.2 must be placed in the assumptions. Indeed, there is an embedding of $\mathfrak{M} = \Pi(4, \{1, 2, 3\}(4), G_2(I_4))$ into a real projective plane so as all the collinear triples of $\mathfrak{M}$ except the triple $L = \{p, a_4, b_4\}$ are mapped into collinear triples, but $\gamma(L)$ is not collinear. Even a more impressive is the fact (a folklore, in fact), that there is an embedding of the fez configuration which preserves all the collinearities except $\{p, a_3, b_3\}$.

(ii) It is a folklore, again, that 6.2 is not valid for $n = 3$; consider equation (30) in 5.7 which is not a tautology on the real plane.

**References**

[1] A. Doliwa, The affine Weil group symmetry of Desargues maps and the noncommutative Hirota-Miwa system, Phys. Lett. A 375 (2011), 1219–1224.
A. Doliwa, *Desargues maps and the Hirota-Miwa equation*, Proc. R. Soc. A **466** (2010), 1177–1200.

M. Prażmowska, *Multiple perspectives and generalizations of the Desargues configuration*, Demonstratio Math. **39** (2006), no. 4, 887–906.

M. Prażmowska, K. Prażmowski, *Some generalization of Desargues and Veronese configurations*, Serdica Math. J. **32** (2006), no 2–3, 185–208.

M. Prażmowska, K. Prażmowski, *Binomial partial Steiner triple systems containing complete graphs*, Graphs Combin. **32**(2016), no. 5, 2079–2092.

K. Petelczyc, M. Prażmowska, *103-configurations and projective realizability of multiplied configurations*, Des. Codes Cryptogr. **51**, no. 1 (2009), 45–54.

M. Ch. Klin, R. Pöschel, K. Rosenbaum, *Angewandte Algebra für Mathematiker und Informatiker*, VEB Deutcher Verlag der Wissenschaften, Berlin 1988.

M. Prażmowska, *On some regular multi-Veblen configurations, the geometry of combinatorial quasi Grassmannians*, Demonstratio Math. **42**(2009), no.1 2, 387–402.

M. Prażmowska, K. Prażmowski, *Combinatorial Veronese structures, their geometry, and problems of embeddability*, Results Math. **51** (2007), 275–308.

K. Petelczyc, M. Prażmowska, *A complete classification of the (154203)-configurations with at least three K5-graphs*, Discrete Math. **338** (2016), no 7, 1243–1251.

A. C. H. Ling, C. J. Colbourn, M. J. Granell, T. S. Griggs, *Construction techniques for anti-Pasch Steiner triple systems*, Jour. London Math. Soc. **61** (2000), no. 3, 641–657.

H. S. M. Coxeter, *Desargues configurations and their collineation groups*, Math. Proc. Camb. Phil. Soc. **78**(1975), 227–246.

H. S. M. Coxeter, *Introduction to Geometry*, John Wiley, 1989.

R. Hartshorne, *Foundations of projective geometry*, Lecture Notes, Harvard University, 1967.

H. Karzel, H.-J. Kroll, *Perspectivities in Circle Geometries*, in *Geometry – von Staudt’s point of view*, P. Plaumann, K. Strambach (Eds), D. Reidel Publ. Co., 1981, pp. 51–100.

G. Pickert, *Projectivities in Projective Planes*, in *Geometry – von Staudt’s point of view*, P. Plaumann, K. Strambach (Eds), D. Reidel Publ. Co., 1981, pp. 1–50.

M. M. Prażmowska, *On the existence of projective embeddings of multiveblen configurations*, Bull. Belg. Math. Soc. Simon-Stevin, **17**, (2010), no 2, 1–15.

M. Hall Jr., *Combinatorial Theory*, J. Wiley 1986.

G. E. Andrews, *The theory of Partitions*, Cambride Univ. Press, 1998.

D. Hilbert, S. COHN-VOSSEN, *Geometry and the Imagination*, AMS Chelsea Publishing, 1999.

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