Two boundary Hecke Algebras and combinatorics of type $C$

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Abstract

This paper gives a Schur-Weyl duality approach to the representation theory of the affine Hecke algebras of type $C$ with unequal parameters. The first step is to realize the affine braid group of type $C_k$ as the group of braids on $k$ strands with two poles. Generalizing familiar methods from the one pole (type A) case, this provides commuting actions of the quantum group $U_qg$ and the affine braid group of type $C_k$ on a tensor space $M \otimes N \otimes V^\otimes k$. Special cases provide Schur-Weyl pairings between the affine Hecke algebra of type $C_k$ and the quantum group of type $gl_n$, resulting in natural labelings of many representations of the affine Hecke algebras of type $C$ by partitions. Following an analysis of the structure of weights of affine Hecke algebra representations (extending the one parameter case to the three parameter case necessary for affine Hecke algebras of type $C$), we provide an explicit identification of the affine Hecke algebra representations that appear in tensor space (essentially by identifying their Langlands parameters).

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1 Introduction

This paper explores a Schur-Weyl duality approach to the representations of the affine Hecke algebras of type C with unequal parameters. Following Kazhdan-Lusztig [KL], the irreducible representations of the affine Hecke algebra are usually constructed via the K-theory of generalized Springer fibers. This method works well when an algebraic group is available, which is only for special cases of the three parameters $t, t_0, t_k$ of the affine Hecke algebras of type C.

G. Lusztig gave a general approach to the unequal parameter case using Kazhdan-Lusztig bases and cells. In [Lu2], the challenges for pushing this method through in type C are outlined in a set of conjectures, many of which have now been settled in work of Geck, Bonnafé, and others (see [Ge, Bo, Gu] and references there). Another analytic approach, closer to the original classification and construction of Kazhdan-Lusztig, is given by Opdam and Solleveld (see [OS] and [So] and the references there). In the type C case, Kato [Kt] explained that the “exotic nilpotent cone” can be used to replace the Kazhdan-Lusztig geometry and obtain a complete geometric classification of the irreducible representations of affine Hecke algebras (with mild restrictions on parameters).

In the type A case, there is a powerful alternative to the geometric method via Schur-Weyl duality (see for example [AS, OR, VV]). In this paper we provide an analogue of this Schur-Weyl duality approach for the type C case, with unequal parameters. This is a generalization of the degenerate case studied by Daugherty [Da].

The method is the following: Let $U_q\mathfrak{g}_n$ be the Drinfeld-Jimbo quantum group corresponding to the general linear Lie algebra, and let $V = \mathbb{C}^n$ be the standard representation of $U_q\mathfrak{g}_n$. Write $L(\lambda)$ for the irreducible polynomial representation of $U_q\mathfrak{g}_n$ indexed by the partition $\lambda$, let $M = L((a^c))$ and $N = L((b^d))$ be irreducible representations of $U_q\mathfrak{g}_n$ indexed by $a \times c$ and $b \times d$ rectangles. There is an action of an extension of the affine Hecke algebra of type $C_k$, denoted $H^\text{ext}_k$, with parameters

$$t_0^\frac{1}{2} = q, \quad t_0^\frac{1}{2} = -iq^{b+d}, \quad \text{and} \quad t_k^\frac{1}{2} = -iq^{a+c} \quad (\text{where } i = \sqrt{-1}),$$

such that

$$M \otimes N \otimes V^\otimes k$$

is a $(U_q\mathfrak{g}_n, H^\text{ext}_k)$-bimodule.

We show that the commuting actions of $U_q\mathfrak{g}_n$ and $H^\text{ext}_k$ provide a Schur-Weyl duality, which can be used to derive the representation theory of $H^\text{ext}_k$ from the quantum group $U_q\mathfrak{g}_n$. We work out the combinatorics of this correspondence, relating the natural indexing of $H^\text{ext}_k$-modules coming from the Schur-Weyl duality to the other indexings, by describing the weights for the action of the polynomial part (generated by Bernstein generators) on each irreducible module.

A significant portion of the work in identifying the centralizer of the $U_q\mathfrak{g}_n$ action on $M \otimes N \otimes V^\otimes k$ as an extended affine Hecke algebra of type C is in relating Coxeter and Bernstein presentations, and putting the parameter conversions into focus. The relationships between these presentations are given in Theorem 2.1 for the affine braid group of type C, and in Theorem 2.2 for the affine...
Hecke algebra of type C. Sections 3, 4 and 5 could, perhaps have stood as papers on their own. In Section 3 we give the combinatorics of local regions and standard tableaux for the case of type C with unequal parameters (following the equal parameter case done in [Ra2]). The main result of Section 3, Theorem 3.3, provides a classification and a construction of all irreducible calibrated $H_k^{\text{ext}}$-modules. As in [Ra2], this classification is via \textit{skew local regions}, whose precise definition of skew local regions depends on the careful analysis of the structure of the irreducible representations of rank two affine Hecke algebras. This analysis was done in the single parameter case in [Ra1]. Since the corresponding analysis for \textit{three distinct parameters in the type $C_2$ case} is, to our knowledge, not available in the literature, we have provided it in Section 4. This will ensure that our classification of calibrated irreducible representations for $H_k^{\text{ext}}$ with distinct parameters, as given in Theorem 3.3, is on firm footing. The construction of the action of $H_k^{\text{ext}}$ on tensor space is completed in Theorems 5.1 and 5.4. Finally, armed with these tools we prove the main result, Theorem 5.5, which determines exactly which representations of $H_k^{\text{ext}}$ appear in tensor space, comparing the natural indexing from the highest weight theory for $\mathfrak{gl}_n$ to the combinatorics of the weights of the action of the polynomial part of $H_k^{\text{ext}}$.

Following the schematic from [OR], one would like to generalize the analysis in this paper by replacing finite-dimensional $M$ and $N$ with, for example, other modules from category $\mathcal{O}$. In the finite-dimensional case, the key is that $R$-matrices for $M \otimes V$ and $N \otimes V$ have only two eigenvalues. This strongly restricts the choices for $M$ and $N$. Non-finite-dimensional choices of modules $M$ and $N$ that satisfy these conditions exist in category $\mathcal{O}$, but additional work toward understanding the combinatorics of $M \otimes N \otimes V \otimes k$ in these cases is needed. This understanding would yield an interesting generalization of the work in this paper.

The seeds of the idea for this paper were sown during conversations of A. Ram with P. Pyatov and V. Rittenberg in Bonn in 2005. They suggested that one should analyze two boundary spin chains by $R$-matrices, thus implying the possibility for Schur-Weyl duality approach to representations of affine braid groups of type C. This idea was completed in the degenerate case in [Da], and significant information was obtained in the Temperley-Lieb case in [GN] (see also references there). In [DR] we shall complete the connection to the statistical mechanics by using the results of this paper to identify the representations of the two boundary Temperley-Lieb algebra given, in a diagrammatic form, by de Gier and Nichols in [GN].

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\section{The two boundary Hecke algebra}

In this section we define the two boundary braid group and Hecke algebras and establish multiple presentations of each. The conversion between presentations is important for matching the algebraic approach to the representation theory with the Schur-Weyl duality approach that we give in Section 5.
For generators $g_i, g_j$, encode relations graphically by
\[ g_i \circ g_j \quad \text{means} \quad g_ig_j = g_jg_i, \] \begin{equation}
(2.1)
\end{equation}
\[ g_i \circ g_j \quad \text{means} \quad g_ig_jg_jg_i, \] \[ g_i \circ g_j \quad \text{means} \quad g_ig_jg_jg_i = g_jg_i. \] 

For example, the group of signed permutations,
\[ \mathcal{W}_0 = \left\{ \text{bijections } w: \{-k, \ldots, -1, 1, \ldots, k\} \to \{-k, \ldots, -1, 1, \ldots, k\} \right\}, \] \begin{equation}
(2.2)
\end{equation}
has a presentation by generators $s_0, s_1, \ldots, s_{k-1}$, with relations
\[ s_1 s_k^{-1} s_2 s_k^{-2} \ldots s_k = 1 \quad \text{for} \quad i = 0, 1, 2, \ldots, k - 1. \] 
\begin{equation}
(2.3)
\end{equation}

### 2.1 The two boundary braid group

The two boundary braid group is the group $B_k$ generated by $\bar{T}_0, \bar{T}_1, \ldots, \bar{T}_k$, with relations
\[ \bar{T}_0 \bar{T}_1 \bar{T}_2 \ldots \bar{T}_{k-2} \bar{T}_{k-1} \bar{T}_k. \] 
\begin{equation}
(2.4)
\end{equation}
Pictorially, the generators of $B_k$ are identified with the braid diagrams
\[ \bar{T}_k = \quad \bar{T}_0 = \quad \text{and} \]
\[ \bar{T}_i = \quad \text{for } i = 1, \ldots, k - 1, \] 
\begin{equation}
(2.5)
\end{equation}
and the multiplication of braid diagrams is given by placing one diagram on top of another.

To make explicit the Schur-Weyl duality approach to representations of $B_k$ appearing in Section 5 it is useful to move the rightmost pole to the left by conjugating by the diagram
\[ \sigma = \quad \text{and} \] 
\begin{equation}
(2.6)
\end{equation}
Define
\[ T_i = \sigma \bar{T}_i \sigma^{-1} \quad \text{and} \quad Y_1 = \sigma \bar{T}_0 \sigma^{-1}. \] 
\begin{equation}
(2.7)
\end{equation}
and

\[ X_1 = T_1^{-1}T_2^{-1} \cdots T_{k-1}^{-1} \sigma \tilde{T}_k \sigma^{-1} T_{k-1} \cdots T_1 = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \end{bmatrix}. \]  

(2.8)

Define

\[ Z_1 = X_1 Y_1 \quad \text{and} \quad Z_i = T_{i-1} T_{i-2} \cdots T_1 X_1 Y_1 T_1 \cdots T_{i-1} = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \end{bmatrix}, \]  

(2.9)

for \( i = 2, \ldots, k \).

**Theorem 2.1.** The two boundary braid group \( B_k \) is presented in the following three ways, using the notation defined in (2.1).

(a) \( B_k \) is presented by generators \( X_1, Y_1, Z_1, T_1, \ldots, T_{k-1} \) and relations

\[ X_1 = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \end{bmatrix}. \]  

(b) \( B_k \) is presented by generators \( X_1, Y_1, T_1, \ldots, T_{k-1} \) and relations

\[ \begin{align*}
T_i T_j = T_j T_i & \quad \text{for } i, j = 1, \ldots, k, \\
Y_i Z_j = Z_j Y_i & \quad \text{for } i = 2, \ldots, k, \quad \text{and}
\end{align*} \]  

and

\[ Z_1 = X_1 Y_1. \]  

(c) \( B_k \) is presented by generators \( Z_1, \ldots, Z_k, Y_1, T_1, \ldots, T_{k-1} \) and relations

\[ Z_i Z_j = Z_j Z_i \quad \text{for } i, j = 1, \ldots, k, \]  

(2.1)

\[ Y_1 Z_i = Z_i Y_1 \quad \text{for } \]  

(2.2)

\[ T_i Z_j = Z_j T_i \quad \text{for } j \neq i, i + 1, \]  

(2.3)

and

\[ Z_{i+1} = T_i Z_i T_i. \]  

(2.4)

**Proof.** With \( \sigma \) as in (2.6) let \( T_k = \sigma \tilde{T}_k \sigma^{-1} \), so that the original generators are the \( \sigma \)-conjugates of

\[ T_0, T_1, \ldots, T_k. \]  

(o)
Conjugate the relations in (2.4) by $\sigma$ to rewrite them in the form

$$
\begin{array}{cccc}
Y_1 & T_1 & T_2 & T_{k-2} & T_{k-1} \\
\circ & \circ & \circ & \circ & \circ
\end{array}
,$$  

\(T_k T_{k-1} T_k T_{k-1} = T_{k-1} T_k T_{k-1} T_k, \quad (o1)\)

and

$$
T_k Y_1 = Y_1 T_k, \quad \text{and} \quad T_k T_i = T_i T_k, \text{ for } i = 1, \ldots, k - 2.
$$  

(02)

The conversions between the generators in presentations (a), (b), and (c) are given in (2.7), (2.8), and (2.9). For generators (a) and (b) in terms of generators (c), the key relations are

$$
Y_1 = T_0, \quad X_1 = T_1^{-1} \cdots T_{k-1}^{-1} T_k T_{k-1} \cdots T_1 \quad \text{and} \quad T_k = T_{k-1} \cdots T_1 X_1 T_1^{-1} \cdots T_{k-1}^{-1}.
$$

Relations (a) from relations (b): Relation (a4) is the conversion from generators (b) to generators (a). The relations in (a3) then follow from

$$
T_i Z_1 = T_i X_1 Y_1 = X_1 T_i Y_1 = X_1 Y_1 T_i = Z_1 T_i, \quad \text{for } i = 2, \ldots, k - 1,
$$

and

$$
T_i Z_1 T_i Z_1 = T_i X_1 Y_1 T_i X_1 Y_1 = T_i X_1 (Y_1 T_i X_1 T_1^{-1}) Y_1 T_i
= X_1 T_i X_1 T_1^{-1} Y_1 T_i Y_1 = X_1 T_i X_1 T_1^{-1} Y_1 T_i Y_1 T_i
= X_1 Y_1 T_i X_1 T_1^{-1} T_1 Y_1 T_i = Z_1 T_i Z_1 T_i.
$$

Relations (b) from relations (a): Multiplying

$$
T_i X_1 (T_1 X_1 T_1^{-1} Y_1) T_i Y_1 = X_1 T_i X_1 T_1^{-1} Y_1 T_i Y_1 = X_1 T_i X_1 T_1^{-1} Y_1 T_i Y_1
= X_1 T_i X_1 T_1^{-1} Y_1 T_i Y_1
= T_i Z_1 Y_1 T_i X_1 T_1 Y_1 = T_i X_1 (Y_1 T_i X_1 T_1^{-1}) Y_1 T_i
$$

on the left by \((T_i X_1)^{-1}\) and on the right by \((T_i Y_1)^{-1}\) gives \(T_i X_1 T_1^{-1} Y_1 = Y_1 T_i X_1 T_1^{-1}\), establishing \(b3\).

Relations (b) from relations (o): The pictorial computations

$$
\begin{array}{l}
\begin{array}{l}
\text{and}
\end{array}
\end{array}
$$

show that \(X_1 T_i = T_i X_1\) for \(i = 1, 2, \ldots, k - 1\), \(Y_1 T_i X_1 T_1^{-1} = T_i X_1 T_1^{-1} Y_1\), and \(X_1 T_i X_1 T_i = T_1 X_1 T_1 X_1\). Hence the relations \(a1\) and \(a2\) follow from the relations in \(o1\) and \(o2\).
Relations (o) from relations (b): The first set of relations in (o1) are the same as the relations in (a2). Let $A = T_{k-1} \cdots T_1$ and $B = T_{k-1} \cdots T_2$. Since $X_1$ commutes with $T_i$ for $i = 2, \ldots, k-1$, then $BX_1B^{-1} = X_1$ so that

$$ABX_1B^{-1}A^{-1} = \begin{pmatrix} & & & \vdots \end{pmatrix} = T_k,$$

and

$$ABT_1B^{-1}A^{-1} = \begin{pmatrix} & & & \vdots \end{pmatrix} = T_{k-1}.$$

Thus, by conjugation by $AB$, the relation $X_1T_1X_1T_1 = T_1X_1T_1X_1$ becomes $T_kT_{k-1}T_kT_{k-1} = T_{k-1}T_kT_{k-1}T_k$, establishing the second relation in (o1). For $i = 1, \ldots, k-2$,

$$T_iT_k = T_iT_{k-1} \cdots T_1X_1T_i^{-1} \cdots T_k^{-1} = T_{k-1} \cdots T_{i+2}T_{i+1}T_{i+1} \cdots T_iX_1T_i^{-1} \cdots T_k^{-1}
= T_{k-1} \cdots T_1X_1T_1^{-1} \cdots T_1^{-1}T_{i+1}T_i^{-1} \cdots T_k^{-1}
= T_{k-1} \cdots T_1X_1T_1^{-1} \cdots T_{k-1}^{-1}T_{i+2}T_{i+1} \cdots T_k^{-1}
= T_{k-1} \cdots T_1X_1T_1^{-1} \cdots T_{k-1}^{-1}T_{i+2}T_{i+1} \cdots T_k^{-1} = T_kT_i.

Similarly, (b3) gives

$$Y_iT_k = Y_iT_{k-1} \cdots T_2T_1X_iT_1^{-1}T_2^{-1} \cdots T_{k-1}^{-1} = T_{k-1} \cdots T_2(Y_iT_1X_1T_1^{-1})T_2^{-1} \cdots T_{k-1}^{-1}
= T_{k-1} \cdots T_2T_iX_1T_1^{-1}Y_1T_2^{-1} \cdots T_{k-1}^{-1} = T_{k-1} \cdots T_2T_iX_1T_1^{-1}T_2^{-1} \cdots T_{k-1}^{-1}Y_1 = T_kY_i,$$
giving the relations in (o2).

Relations (c) from relations (o): The first set of relations in (o1) are the same as the relations in (a2). Relations (c4) are exactly the definitions in the second part of (2.9). The pictorial computation

$$Z_jZ_i = \begin{pmatrix} & & & \vdots \end{pmatrix} = Z_iZ_j$$

give relations (c1). Similarly, pictorial computations readily show that $Y_1Z_i = Z_iY_1$ for $i > 1$ and $T_iZ_j = Z_jT_i$ for $i \neq j, j+1$, proving relations (c2) and (c3).
Generators \((o)\) from generators \((c)\): The key formula for the generator \(T_k\) is

\[
T_k = T_{k-1} \cdots T_1 (T_{k-1}^{-1} \cdots T_{k-1}^{-1} T_k T_{k-1} \cdots T_1) Y_1 (T_1 \cdots T_{k-1}) (T_{k-1}^{-1} \cdots T_{k-1}^{-1}) Y_1^{-1} (T_1^{-1} \cdots T_{k-1}^{-1}) = (T_{k-1} \cdots T_1) X_1 Y_1 (T_1 \cdots T_{k-1}) T_{s \varphi} = Z_k T_{s \varphi},
\]

where

\[
T_{s \varphi} = T_{k-1} T_{k-2} \cdots T_1 Y_1 T_1 \cdots T_{k-2} T_{k-1} = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 \\
\end{bmatrix}.
\]

Relations \((o)\) from relations \((c)\): The first set of relations in \((o1)\) are the same as the relations in \((a2)\). The relations

\[
T_{s \varphi} Y_1 = Y_1 T_{s \varphi} \quad \text{and} \quad T_{s \varphi} T_i = T_i T_{s \varphi}, \quad \text{for } i = 1, \ldots, k - 2,
\]

are verified pictorially by

\[
\begin{bmatrix}
\begin{array}{c}
1 \\
1 \\
1 \\
1 \\
\end{array}
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 \\
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
\begin{array}{c}
1 \\
1 \\
1 \\
1 \\
\end{array}
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 \\
\end{bmatrix}
\]

or by direct computation using the relations in \((a2)\).

By \((2.8)\) and \((2.9)\), \(Z_k = T_k T_{s \varphi}\) and, by \((e3)\) and \((e2)\) respectively,

\[
T_k T_i = Z_k T_{s \varphi}^{-1} T_i = Z_k T_i T_{s \varphi}^{-1} = T_i T_{s \varphi}^{-1} = T_i T_k, \quad \text{for } i = 1, \ldots, k - 2, \quad \text{and}
\]

\[
T_k Y_1 = Z_k T_{s \varphi}^{-1} Y_1 = Z_k Y_1 T_{s \varphi}^{-1} = Y_1 Z_k T_{s \varphi}^{-1} = Y_1 T_k,
\]

which proves the relations in \((o2)\).

By the relations in \((2.11)\) and the second set of relations in \((2.10)\),

\[
(T_{k-1}^{-1} T_{s \varphi} T_{k-1}^{-1}) T_k = T_k (T_{k-1}^{-1} T_{s \varphi} T_{k-1}^{-1}) \quad \text{and} \quad (T_{k-1}^{-1} T_{s \varphi} T_{k-1}^{-1}) T_{s \varphi} = T_{s \varphi} (T_{k-1}^{-1} T_{s \varphi} T_{k-1}^{-1}),
\]

so that \((T_{k-1}^{-1} T_{s \varphi} T_{k-1}^{-1}) (T_k T_{s \varphi}) = (T_k T_{s \varphi}) (T_{k-1}^{-1} T_{s \varphi} T_{k-1}^{-1}).\) Using these and the equality

\[
T_{k-1} Z_k Z_{k-1} = T_{k-1} Z_{k-1} Z_k = Z_k T_{k-1}^{-1} Z_k = Z_k Z_{k-1} T_{k-1},
\]

we have

\[
T_{k-1} Z_k Z_{k-1} = T_{k-1} Z_k (T_{k-1}^{-1} Z_k T_{k-1}^{-1}) = T_{k-1} (T_k T_{s \varphi}) T_{k-1}^{-1} (T_k T_{s \varphi}) T_{k-1}^{-1} = T_{k-1} T_k T_{k-1} (T_{k-1}^{-1} T_{s \varphi} T_{k-1}^{-1}) T_k T_{s \varphi} T_{k-1}^{-1} = (T_{k-1} T_k T_{k-1} T_k) (T_{k-1}^{-1} T_{s \varphi} T_{k-1}^{-1} T_k T_{s \varphi} T_{k-1}^{-1})
\]

\[
= Z_k Z_{k-1} T_{k-1} = Z_k (T_{k-1}^{-1} Z_k T_{k-1}^{-1}) = T_k T_{s \varphi} (T_{k-1}^{-1} T_{s \varphi} T_{k-1}^{-1}) (T_k T_{s \varphi}) = T_k T_{k-1} (T_{k-1}^{-1} T_{s \varphi} T_{k-1}^{-1}) (T_k T_{s \varphi}) = T_k T_{s \varphi} (T_{k-1}^{-1} T_{s \varphi} T_{k-1}^{-1}) (T_k T_{s \varphi})
\]

Multiplying on the right by \((T_{k-1}^{-1} T_{s \varphi} T_{k-1}^{-1} T_{s \varphi} T_{k-1}^{-1})^{-1}\) gives \(T_k T_{k-1} T_k T_{k-1} = T_{k-1} T_k T_{k-1} T_k\), establishing the last relation in \((o1)\). \(\square\)
If
\[ P^{1/2} = \begin{array}{cccccc}
\circ & & & & & \\
\circ & & & & & \\
\circ & & & & & \\
\circ & & & & & \\
\circ & & & & & \\
\circ & & & & & \\
\end{array} \tag{2.12} \]
then
\[ P^{1/2}Y_1P^{-1/2} = \begin{array}{cccccc}
\circ & & & & & \\
\circ & & & & & \\
\circ & & & & & \\
\circ & & & & & \\
\circ & & & & & \\
\circ & & & & & \\
\end{array} = Y_1^{-1}X_1Y_1 \tag{2.13} \]
and
\[ P^{1/2}X_1P^{-1/2} = \begin{array}{cccccc}
\circ & & & & & \\
\circ & & & & & \\
\circ & & & & & \\
\circ & & & & & \\
\circ & & & & & \\
\circ & & & & & \\
\end{array} = Y_1 \tag{2.14} \]
Following these pictorial computations, the extended affine braid group is the group \( B_{\text{ext}}^k \) generated by \( B_k \) and \( P \) with the additional relations
\[ PX_1P^{-1} = Z_1^{-1}X_1Z_1, \quad PY_1P^{-1} = Z_1^{-1}Y_1Z_1, \quad PZ_1P^{-1} = Z_1, \quad \text{and} \quad PT_iP^{-1} = T_i \text{ for } i = 1, \ldots, k - 1. \tag{2.15} \]
Note that the element \( Z_0 = PZ_1 \cdots Z_k \) is central in \( B_{\text{ext}}^k \) since the group \( B_{\text{ext}}^k \) is a subgroup of the braid group on \( k + 2 \) strands, and \( Z_0 \) is the generator of the center of the braid group on \( k + 2 \) strands (see [GM, Theorem 4.2]). So
\[ \text{if } D = \{ Z_0^j \mid j \in \mathbb{Z} \} \text{ then } B_{\text{ext}}^k = D \times B_k, \text{ with } D \cong \mathbb{Z}. \tag{c0} \]

\section*{2.2 The two boundary Hecke algebra \( H_{\text{ext}}^k \)}
In this subsection we define the two boundary Hecke algebras and relate it to the presentation of the affine Hecke algebra of type C that is found, for example, in [Lu1, Proposition 3.6] and [Mac2, (4.2.4)].

Fix \( a_1, a_2, b_1, b_2, t^{1/2} \in \mathbb{C}^\times \). The extended two boundary Hecke algebra \( H_{\text{ext}}^k \) is the quotient of \( B_{\text{ext}}^k \) by the relations
\[ (X_1 - a_1)(X_1 - a_2) = 0, \quad (Y_1 - b_1)(Y_1 - b_2) = 0, \quad \text{and} \quad (T_i - t^{1/2})(T_i + t^{-1/2}) = 0, \tag{h} \]
for \( i = 1, \ldots, k - 1 \). Let
\[ t_k^{\frac{1}{2}} = a_1^{\frac{1}{2}}(-a_2)^{-\frac{1}{2}} \quad \text{and} \quad t_0^{\frac{1}{2}} = b_1^{\frac{1}{2}}(-b_2)^{-\frac{1}{2}}. \tag{2.17} \]
With \( Z_i \in H_{\text{ext}}^k \) as in \( \text{(2.9)} \), define
\[ T_0 = b_1^{-\frac{1}{2}}(-b_2)^{-\frac{1}{2}}Y_1, \quad W_i = -(a_1a_2b_1b_2)^{-\frac{1}{2}}Z_i \text{ for } i = 1, \ldots, k, \text{ and } \]
\[ W_0 = PW_1 \cdots W_k = (-1)^k(a_1a_2b_1b_2)^{-\frac{k}{2}}PZ_1 \cdots Z_k = (-1)^k(a_1a_2b_1b_2)^{-\frac{k}{2}}Z_0. \tag{2.18} \]
Then
\[ X_1 = Z_1Y_1^{-1} = a_1^{\frac{1}{2}}(-a_2)^{\frac{1}{2}}W_1T_0^{-1}. \tag{2.20} \]
Theorem 2.2. Fix $t_0, t_k, t \in \mathbb{C}^\times$ and use notations for relations as defined in (2.1). The extended affine Hecke algebra $H_k^{\text{ext}}$ defined in (1) is presented by generators, $T_0, T_1, \ldots, T_{k-1}, W_0, W_1, \ldots, W_k$ and relations

\[ W_0 \in Z(H_k^{\text{ext}}), \quad T_0 \quad T_1 \quad T_2 \quad T_{k-2} \quad T_{k-1}; \quad (B1) \]

\[ W_iW_j = W_jW_i, \quad \text{for } i, j = 0, 1, \ldots, k; \quad (B2) \]

\[ T_0W_j = W_jT_0, \quad \text{for } j \neq 1; \quad (B3) \]

\[ T_iW_j = W_jT_i \quad \text{for } i = 1, \ldots, k-1 \quad \text{and } j = 1, \ldots, k \quad \text{with } j \neq i, i+1; \quad (B4) \]

\[ (T_0 - t_0^\frac{1}{2})(T_0 + t_0^{-\frac{1}{2}}) = 0, \quad \text{and} \quad (T_i - t_i^\frac{1}{2})(T_i + t_i^{-\frac{1}{2}}) = 0 \quad \text{for } i = 1, \ldots, k - 1. \quad (H) \]

For $i = 1, \ldots, k - 1,$

\[ T_iW_i = W_{i+1}T_i + (t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}}) \frac{W_i - W_{i+1}}{1 - W_iW_{i+1}}, \quad T_iW_{i+1} = W_iT_i + (t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}}) \frac{W_{i+1} - W_i}{1 - W_iW_{i+1}}, \quad (C1) \]

and

\[ T_0W_1 = W_1^{-1}T_0 + \left( (t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}) + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}}) \right) \frac{W_1 - W_1^{-1}}{1 - W_1^{-2}}. \quad (C2) \]

Proof. The conversion between the different sets of generators of $H_k^{\text{ext}}$ is provided by (2.18).

Equivalence between (c0–c4) and the second and third relations of (H) with the relations (B1–B4) and (H). Since $T_0$ and $Y_1$ differ by a constant, and $W_i$ and $Z_i$ differ by a constant, the relations in (c0–c4) are equivalent to the relations in (B1–B4), respectively. Since

\[
0 = (Y_1 - b_1)(Y_1 - b_2) = b_1^\frac{1}{2}(-b_2)^\frac{1}{2} (T_0 - b_1^\frac{1}{2}(-b_2)^{-\frac{1}{2}})b_1^\frac{1}{2}(-b_2)^\frac{1}{2} (T_0 + b_1^\frac{1}{2}(-b_2)^{\frac{1}{2}})
\]

\[= -b_1b_2(T_0 - t_0^{-\frac{1}{2}})(T_0 + t_0^{-\frac{1}{2}}), \]

the relations (H) are equivalent to the second and third relations in (H).

Relations (C1–C2) from relations (c0–c4) and (H): From (2.9) and (2.18), $W_{i+1} = T_iW_iT_i$, and by the last relation in (H), $T_i^{-1} = T_i - (t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}})$. So

\[ T_iW_i = W_{i+1}T_i^{-1} = W_{i+1}(T_i - (t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}})) = W_{i+1}T_i + (t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}}) \frac{W_i - W_{i+1}}{1 - W_iW_{i+1}} \]

and

\[ T_iW_{i+1} = T_i^2W_iT_i = (t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}})W_{i+1} + W_iT_i = W_iT_i + (t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}}) \frac{W_{i+1} - W_i}{1 - W_iW_{i+1}}, \]

which establishes the relations in (C1).
By the first relation in (h), \( X_1^{-1} = -a_1^{-1}a_2^{-1}X_1 + (a_1^{-1} + a_2^{-1}) \). Since \( W_1 = a_1^{-\frac{1}{2}}(-a_2)^{-\frac{1}{2}}X_1T_0 \) and \( T_0 - T_0^{-1} = t^{-\frac{1}{2}}_0 - t^{-\frac{1}{2}}_0 \),

\[
T_0W_1 - W_1^{-1}T_0 = a_1^{-\frac{1}{2}}(-a_2)^{-\frac{1}{2}}(T_0X_1T_0 - a_1(-a_2)T_0^{-1}X_1^{-1}T_0) = a_1^{-\frac{1}{2}}(-a_2)^{-\frac{1}{2}}(T_0X_1T_0 + a_1a_2^{-1}T_0^{-1}(-a_1^{-1}a_2^{-1}X_1 + (a_1^{-1} + a_2^{-1}))T_0) = a_1^{-\frac{1}{2}}(-a_2)^{-\frac{1}{2}}((T_0 - T_0^{-1})X_1T_0 + (a_1 - (-a_2))) = (t_0^{-\frac{1}{2}} - t_0^{-\frac{1}{2}})W_1 + (t_0^\frac{1}{2} - t_0^\frac{1}{2}),
\]

which establishes (C2).

The first relation in (h) from the relations (B1–B4), (H) and (C1–C2). By (C2),

\[
a_1^{-\frac{1}{2}}(-a_2)^{-\frac{1}{2}}(T_0X_1T_0 - a_1(-a_2)T_0^{-1}X_1^{-1}T_0) = T_0W_1 - W_1^{-1}T_0 = (t_0^\frac{1}{2} - t_0^{-\frac{1}{2}})W_1 + (t_0^\frac{1}{2} - t_0^{-\frac{1}{2}})
\]

\[
= a_1^{-\frac{1}{2}}(-a_2)^{-\frac{1}{2}}((T_0 - T_0^{-1})X_1T_0 + (a_1 - (-a_2))) = a_1^{-\frac{1}{2}}(-a_2)^{-\frac{1}{2}}((T_0X_1T_0 + a_1a_2^{-1}T_0^{-1}(-a_1^{-1}a_2^{-1}X_1 + (a_1^{-1} + a_2^{-1}))T_0),
\]

giving \( X_1^{-1} = -a_1^{-1}a_2^{-1}X_1 + (a_1^{-1} + a_2^{-1}) \), which establishes the first relation in (h). \( \square \)

As vector spaces,

\[
H_k^{\text{ext}} = \mathbb{C}[W_0^{\pm 1}, W_1^{\pm 1}, \ldots, W_k^{\pm 1}] \otimes H_k^{\text{fin}}, \tag{2.21}
\]

where \( H_k^{\text{fin}} \) is the subalgebra of \( H_k^{\text{ext}} \) generated by \( T_0, T_1, \ldots, T_{k-1} \). The algebra \( H_k^{\text{fin}} \) is the Iwahori-Hecke algebra of finite type \( C_k \). If \( s_0, s_1, \ldots, s_{k-1} \) are the generators of \( W_0 \) as given in (2.3), write \( T_w = T_{s_{i_{k-1}}} \cdots T_{s_{i_1}} \) for a reduced expression \( w = s_{i_1} \cdots s_{i_k} \), so that

\[
\{ T_w \mid w \in W_0 \} \quad \text{is a } \mathbb{C}-\text{basis of } H_k^{\text{fin}}.
\]

Thus (2.21) means that any element \( h \in H_k^{\text{ext}} \) can be written uniquely as

\[
h = \sum_{w \in W_0} h_w T_w, \quad \text{with } \ h_w \in \mathbb{C}[W_0^{\pm 1}, W_1^{\pm 1}, \ldots, W_k^{\pm 1}].
\]

Let

\[
W^\lambda = W_0^{\lambda_0}W_1^{\lambda_1}W_2^{\lambda_2} \cdots W_k^{\lambda_k} \quad \text{for } \lambda = (\lambda_0, \lambda_1, \ldots, \lambda_k) \in \mathbb{Z}^{k+1}. \tag{2.22}
\]

Relations (C1) and (C2) produce an action of \( W_0 \) on

\[
\mathbb{C}[W_0^{\pm 1}, W_1^{\pm 1}, \ldots, W_k^{\pm 1}] = \text{span}_\mathbb{C}\{ W^\lambda \mid \lambda = (\lambda_0, \lambda_1, \ldots, \lambda_k) \in \mathbb{Z}^{k+1} \}.
\]

Namely, for \( w \in W_0 \) and \( \lambda \in \mathbb{Z}^{k+1} \),

\[
w W^\lambda = W^{s_0\lambda}, \quad \text{where } \ s_0\lambda = s_0(\lambda_0, \lambda_1, \ldots, \lambda_k) = (\lambda_0, -\lambda_1, \ldots, \lambda_k), \quad \text{and}
\]

\[
s_i\lambda = s_i(\lambda_0, \lambda_1, \ldots, \lambda_k) = (\lambda_0, \lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \lambda_i, \lambda_{i+2}, \ldots, \lambda_k), \tag{2.23}
\]
for $i = 1, 2, \ldots, k-1$ (see \cite[1.12]{Ra2}). With this notation, for $\lambda \in \mathbb{Z}^{k+1}$, the relations (C1) and (C2) give
\[ T_i W^\lambda = W^{s_i \lambda} T_i + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \frac{W^\lambda - W^{s_i \lambda}}{1 - W_i W_{i+1}^{-1}} \quad \text{and} \]
\[ T_0 W^\lambda = W^{s_0 \lambda} T_0 + \left( (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) W_1^{-1} \right) \frac{W^\lambda - W^{s_0 \lambda}}{1 - W_1^{-2}}, \]
and, replacing $s_i \lambda$ by $\mu$,
\[ W^\mu T_i = T_i W^{s_i \mu} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \frac{W^\mu - W^{s_i \mu}}{1 - W_i W_{i+1}^{-1}} \quad \text{and} \]
\[ W^\mu T_0 = T_0 W^{s_0 \mu} + \left( (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) W_1^{-1} \right) \frac{W^\mu - W^{s_0 \mu}}{1 - W_1^{-2}}, \quad \text{for } \mu \in \mathbb{Z}^{k+1}. \]

The subalgebra $H_k \subseteq H_k^{\text{ext}}$ generated by $W_1, \ldots, W_k$ and $T_0, \ldots, T_{k-1}$ is the affine Hecke algebra of type C considered, for example, in \cite{Lu1}. The following theorem determines the center of $H_k^{\text{ext}}$ and shows that, as algebras, $H_k^{\text{ext}}$ is a tensor product of $H_k$ by the algebra of Laurent polynomials in one variable. It follows that the irreducible representations of $H_k^{\text{ext}}$ are indexed by $\mathbb{C}^\times \times \hat{H}_k$, where $\hat{H}_k$ is an indexing set for the irreducible representations of $H_k$.

**Theorem 2.3.** Let $H_k$ be the subalgebra of $H_k^{\text{ext}}$ generated by $W_1, \ldots, W_k$ and $T_0, \ldots, T_{k-1}$. As algebras,
\[ H_k^{\text{ext}} \cong \mathbb{C}[W_0^{\pm 1}] \otimes H_k, \]

The center of $H_k^{\text{ext}}$ is
\[ Z(H_k^{\text{ext}}) = \mathbb{C}[W_0^{\pm 1}] \otimes \mathbb{C}[W_1^{\pm 1}, \ldots, W_k^{\pm 1}] W_0, \]
and $H_k^{\text{ext}}$ is a free module of rank $\text{Card}(W_0) = 2^{2k}(k!)^2$ over $Z(H_k^{\text{ext}})$.

**Proof.** As observed in \cite{0}, $Z_0$ is central in $H_k^{\text{ext}}$, and therefore $W_0 = (-1)^k (a_1 a_2 b_2)^{k/2} Z_0$ is central in $H_k^{\text{ext}}$. Thus
\[ H_k^{\text{ext}} = \mathbb{C}[W_0^{\pm 1}] \otimes H_k. \]

By the formulas in (2.23), the Laurent polynomial ring $\mathbb{C}[W_1^{\pm 1}, \ldots, W_k^{\pm 1}]$ is a $W_0$-submodule of $\mathbb{C}[W_0^{\pm 1}, W_1^{\pm 1}, \ldots, W_k^{\pm 1}]$, and
\[ \mathbb{C}[W_0^{\pm 1}, W_1^{\pm 1}, \ldots, W_k^{\pm 1}] W_0 = \mathbb{C}[W_1^{\pm 1}, \ldots, W_k^{\pm 1}] W_0 \otimes \mathbb{C}[W_0^{\pm 1}]. \]

The proof that $Z(H_k^{\text{ext}}) = \mathbb{C}[W_0^{\pm 1}, W_1^{\pm 1}, \ldots, W_k^{\pm 1}] W_0$ is exactly as in \cite[Thm. 4.12]{RR}. The fact that $H_k^{\text{ext}}$ is a free module of rank $\text{Card}(W_0)^2$ over $\mathbb{C}[W_0^{\pm 1}, W_1^{\pm 1}, \ldots, W_k^{\pm 1}] W_0$ follows from (2.21) and \cite[Theorem 1.17]{Ra2}. 

### 2.3 Weights of representations and intertwiners

Let $t^{\frac{1}{2}} \in \mathbb{C}^\times$ be such that $(t^{\frac{1}{2}})^\ell \neq 1$ for $\ell \in \mathbb{Z}$. All irreducible complex representations $\gamma$ of the algebra $\mathbb{C}[W_0^{\pm 1}, W_1^{\pm 1}, \ldots, W_k^{\pm 1}]$ are one-dimensional. Identify the sets
\[ \mathcal{C} = \{ \text{irreducible representations } \gamma \text{ of } \mathbb{C}[W_0^{\pm 1}, W_1^{\pm 1}, \ldots, W_k^{\pm 1}] \} \]
\[ \leftrightarrow \{ \text{sequences } (z, \gamma_1, \ldots, \gamma_k) \in (\mathbb{C}^\times)^{k+1} \} \]
\[ \leftrightarrow \{ \text{sequences } (\zeta, c_1, \ldots, c_k) \in \mathbb{C}^{k+1} \} \]
\[
\gamma(W_0) = z = (-1)^k t^\zeta \quad \text{and} \quad \gamma(W_i) = -t c^i \quad \text{for} \quad i = 1, \ldots, k \quad (2.32)
\]

The choice of sign in the last equation is an artifact of equations (5.32) and (5.33) and an effort to make the combinatorics of contents of boxes Section 5 optimally helpful. The action of \( \mathcal{W}_0 \) from (2.23) induces an action of \( \mathcal{W}_0 \) on \( C \) by

\[
(w\gamma)(W^\lambda) = \gamma(W^{w^{-1}\lambda}), \quad \text{for} \quad w \in \mathcal{W}_0 \quad \text{and} \quad \lambda \in \mathbb{Z}^{k+1}. \quad (2.33)
\]

Equivalently, on sequences \((\zeta, c_1, \ldots, c_k)\), this action is given by

\[
w(\zeta, c_1, \ldots, c_k) = (\zeta, c_{w^{-1}(1)}, \ldots, c_{w^{-1}(k)}), \quad \text{for} \quad w \in \mathcal{W}_0. \quad (2.34)
\]

Let \( \tilde{H}^\text{ext}_k \) be the extensions of \( H^\text{ext}_k \) by the rational functions in \( W_1, \ldots, W_k \):

\[
\tilde{H}^\text{ext}_k = \mathbb{C}[W_0^{\pm1}] \otimes \mathbb{C}(W_1, \ldots, W_k) \otimes H^\text{fin}_k,
\]

where \( H^\text{fin}_k \) is the subalgebra of \( H^\text{ext}_k \) generated by \( T_0, T_1, \ldots, T_{k-1} \). The intertwining operators for \( \tilde{H}^\text{ext}_k \) are

\[
\tau_0 = T_0 - \left( \frac{t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}}{1 - W_1^{-2}} \right) + \left( \frac{t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}}}{1 - W_1^{-2}} \right) W_1^{-1} \quad \text{and} \quad \tau_i = T_i - \frac{t_0^{\frac{1}{2}} - t_i^{\frac{1}{2}}}{1 - W_i W_{i+1}^{-1}} \quad (2.35)
\]

for \( i = 1, 2, \ldots, k - 1 \). Proposition 2.4 shows that these operators satisfy \( \tau_0 W^\lambda = W^s_{\gamma_0} \tau_0 \) and \( \tau_i W^\lambda = W^s_{\gamma_i} \tau_i \) so that, for \( w \in \mathcal{W}_0 \) and \( \lambda = (\lambda_0, \ldots, \lambda_k) \in \mathbb{Z}^{k+1} \),

\[
\tau_w W^\lambda = W^w_{\gamma_w} \tau_w, \quad \text{where} \quad \tau_w = \tau_{i_1} \ldots \tau_{i_t} \quad (2.36)
\]

for a reduced expression \( w = s_{i_1} \ldots s_{i_t} \).

Each \( \tilde{H}^\text{ext}_k \)-module \( M \) can be written as \( M = \bigoplus_{\gamma \in \mathcal{C}} M^\text{gen}_\gamma \), where for each \( \gamma = (z, \gamma_1, \ldots, \gamma_k) \in \mathcal{C} \),

\[
M^\text{gen}_\gamma = \left\{ m \in M \bigg| \begin{array}{l}
\text{there exists} \ N \in \mathbb{Z}_{>0} \ \text{such that} \\
(W_0 - z)^N m = 0 \quad \text{and} \quad (W_i - \gamma_i)^N m = 0 \quad \text{for} \quad i = 1, \ldots, k
\end{array} \right\} \quad (2.37)
\]

is the generalized weight space associated to \( \gamma \). The intertwiners (2.35) define vector space homomorphisms

\[
\tau_0: M^\text{gen}_\gamma \longrightarrow M^\text{gen}_{s_0 \gamma} \quad \text{and} \quad \tau_i: M^\text{gen}_\gamma \longrightarrow M^\text{gen}_{s_i \gamma} \quad \text{for} \quad i = 1, \ldots, k - 1, \quad (2.38)
\]

where

\[
\tau_0 \text{ is defined only when } \gamma_1 \neq 1, \quad \text{so that } (1 - W_1^{-1})^{-1} \text{ is well-defined on } M^\text{gen}_\gamma \quad \text{and} \quad \\
\tau_i \text{ is defined only when } \gamma_i \neq \gamma_{i+1}, \quad \text{so that } (1 - W_i W_{i+1}^{-1})^{-1} \text{ is well-defined on } M^\text{gen}_\gamma
\]

for \( i = 1, \ldots, k - 1 \).
Proposition 2.4. (Intertwiner presentation) The algebra $\tilde{H}_k^{\text{ext}}$ is generated by $\tau_0, \ldots, \tau_k$, $W_0$, and $\mathbb{C}(W_1, \ldots, W_k)$ with relations

$$\tau_0 \tau_1 \tau_2 \ldots \tau_{k-2} \tau_{k-1}$$

(2.39)

in the notation of (2.1):

$$\tau_0 W_1 = W_1^{-1} \tau_0 \quad \text{and} \quad \tau_0 W_j = W_j \tau_0 \text{ for } j \neq 1;$$

(2.40)

for $i = 1, \ldots, k - 1,$

$$\tau_i W_i = W_{i+1} \tau_i \quad \text{and} \quad \tau_i W_i + 1 = W_i \tau_i \quad \text{for } i > 0, \quad \text{and} \quad \tau_i W_j = W_j \tau_i \text{ for } j \neq i, i + 1;$$

(2.41)

Using (C1),

$$\tau_0^2 = \left(1 - t_0^2 W_1^{-1}\right) \left(1 + t_0^2 W_1^{-1}\right) \left(1 + t_0^2 W_1^{-1}\right) \left(1 + t_0^2 W_1^{-1}\right);$$

(2.42)

and

$$\tau_i^2 = \left(t_i^2 - t_i^{-1} W_i W_{i+1}\right) \left(t_i^2 - t_i^{-1} W_{i+1} W_i\right) \text{ for } i = 1, \ldots, k - 1.$$ 

(2.43)

Proof. The proof of the relations in (2.39) is accomplished exactly as in the proof of [Ra2 Proposition 2.14(e)]; relation (2.43) is [Ra2 Proposition 2.14(c)]. Let us check the relations in (2.41) and (2.42).

Using (C1),

$$\tau_i W_i = \left(1 - \frac{t_i^2 - t_i^{-2}}{1 - W_i W_{i+1}}\right) W_i = W_{i+1} \tau_i + \left(t_i^2 - t_i^{-2}\right) - \left(t_i^2 - t_i^{-2}\right) W_i W_{i+1}^{-1}$$

$$= W_{i+1} \left(1 - \frac{t_i^2 - t_i^{-2}}{1 - W_i W_{i+1}}\right) \tau_i.$$

Similarly, using (C2),

$$\tau_0 W_1 = \left(1 - \frac{t_0^2 - t_0^{-2}}{1 - W_1^{-2}}\right) W_1$$

$$= W_1^{-1} T_0 + \left(t_0^2 - t_0^{-2}\right) W_1 + \left(t_0^2 - t_0^{-2}\right) - \left(t_0^2 - t_0^{-2}\right) W_1 W_2^{-1}$$

$$= W_1^{-1} T_0 + \left(t_0^2 - t_0^{-2}\right) W_1 + \left(t_0^2 - t_0^{-2}\right) W_2^{-1} W_1^{-1}$$

$$= W_1^{-1} T_0 + \left(t_0^2 - t_0^{-2}\right) W_1 + \left(t_0^2 - t_0^{-2}\right) W_2^{-1} W_1^{-1}.$$

For $i = 0, \ldots, k - 1$ and $j \neq i, i + 1$, $\tau_i$ and $W_j$ commute by the second set of relations in (C1). These computations establish the relations in (2.40) and (2.41).
By the first relation in (2.42), $T_0 = T_0^{-1} + (t_0^\frac{1}{2} - t_0^{-\frac{1}{2}})$, so that

$$\tau_0 = T_0 - \frac{(t_0^\frac{1}{2} - t_0^{-\frac{1}{2}}) + (t_k^\frac{1}{2} - t_k^{-\frac{1}{2}})W_1^{-1}}{1 - W_1^{-2}} = T_0^{-1} + (t_0^\frac{1}{2} - t_0^{-\frac{1}{2}}) + \frac{(t_0^\frac{1}{2} - t_0^{-\frac{1}{2}})W_1^2 + (t_k^\frac{1}{2} - t_k^{-\frac{1}{2}})W_1}{1 - W_1^2}.$$

$$= T_0^{-1} + \frac{(t_0^\frac{1}{2} - t_0^{-\frac{1}{2}}) + (t_k^\frac{1}{2} - t_k^{-\frac{1}{2}})W_1}{1 - W_1^2}.$$

Then

$$\tau_0^2 = \tau_0 \left( T_0 - \frac{(t_0^\frac{1}{2} - t_0^{-\frac{1}{2}}) + (t_k^\frac{1}{2} - t_k^{-\frac{1}{2}})W_1^{-1}}{1 - W_1^{-2}} \right) = \tau_0 T_0 - \left( \frac{(t_0^\frac{1}{2} - t_0^{-\frac{1}{2}}) + (t_k^\frac{1}{2} - t_k^{-\frac{1}{2}})W_1}{1 - W_1^2} \right) \tau_0$$

$$= \left( T_0^{-1} + \frac{(t_0^\frac{1}{2} - t_0^{-\frac{1}{2}}) + (t_k^\frac{1}{2} - t_k^{-\frac{1}{2}})W_1}{1 - W_1^2} \right) T_0 - \left( \frac{(t_0^\frac{1}{2} - t_0^{-\frac{1}{2}}) + (t_k^\frac{1}{2} - t_k^{-\frac{1}{2}})W_1}{1 - W_1^2} \right) T_0$$

$$= 1 + \left( \frac{(t_0^\frac{1}{2} - t_0^{-\frac{1}{2}}) + (t_k^\frac{1}{2} - t_k^{-\frac{1}{2}})W_1}{1 - W_1^2} \right) \left( \frac{(t_0^\frac{1}{2} - t_0^{-\frac{1}{2}}) + (t_k^\frac{1}{2} - t_k^{-\frac{1}{2}})W_1}{1 - W_1^2} \right)$$

$$= 1 - \left( \frac{(t_0^\frac{1}{2} - t_0^{-\frac{1}{2}})W_1^{-2} + (t_k^\frac{1}{2} - t_k^{-\frac{1}{2}})W_1^{-1}}{1 - W_1^{-2}} \right) \left( \frac{(t_0^\frac{1}{2} - t_0^{-\frac{1}{2}}) + (t_k^\frac{1}{2} - t_k^{-\frac{1}{2}})W_1^{-1}}{1 - W_1^{-2}} \right)$$

$$= (1 - 2W_1^{-2} + W_1^{-4} - ((t_0^\frac{1}{2} - t_0^{-\frac{1}{2}})^2 + (t_k^\frac{1}{2} - t_k^{-\frac{1}{2}})^2)W_1^{-2}$$

$$- (t_0^\frac{1}{2} - t_0^{-\frac{1}{2}})(t_k^\frac{1}{2} - t_k^{-\frac{1}{2}})W_1^{-1} - (t_0^\frac{1}{2} - t_0^{-\frac{1}{2}})(t_k^\frac{1}{2} - t_k^{-\frac{1}{2}})W_1^{-3}$$

$$= \frac{(1 - t_0^\frac{1}{2} t_k^\frac{1}{2} W_1^{-1}) (1 + t_0^\frac{1}{2} t_k^\frac{1}{2} W_1^{-1}) (1 + t_0^\frac{1}{2} t_k^\frac{1}{2} W_1^{-1}) (1 - t_0^\frac{1}{2} t_k^\frac{1}{2} W_1^{-1})}{1 + W_1^{-1}}.$$

establishing (2.42).

\[\square\]

3 Calibrated representations of $H_k^{\text{ext}}$

A calibrated $H_k^{\text{ext}}$-module is an $H_k^{\text{ext}}$-module $M$ such that $W_0, W_1, \ldots, W_k$ are simultaneously diagonalizable as operators on $M$. In the context of (2.37), $M$ is calibrated if

$$M = \bigoplus_{\gamma \in \mathcal{C}} M_\gamma,$$

where $M_\gamma = \{m \in M \mid W_0 m = zm \text{ and } W_i m = \gamma_i m \text{ for } i = 1, \ldots, k\}$ \hspace{0.1cm} (3.1)

for $\gamma = (z, \gamma_1, \ldots, \gamma_k) \in \mathcal{C}$. Another formulation is that $M$ is calibrated if $M$ has a basis of simultaneous eigenvectors for $W_0, \ldots, W_k$. This section follows the framework of [Ra2] in developing...
combinatorial tools for describing the structure and the classification of irreducible calibrated $H_k^{ext}$-modules. In Section 5 we will use this combinatorics to analyze and classify the $H_k^{ext}$-modules arising in the Schur-Weyl duality settings.

With notations as in the definition of $\mathcal{W}_0$ in (2.2), the reflection representation of $\mathcal{W}_0$ is the action of $\mathcal{W}_0$ on $\mathfrak{h}_R = \mathbb{R}^k$ given by

$$w(c_1, \ldots, c_k) = (c_{w^{-1}(1)}, \ldots, c_{w^{-1}(k)}), \quad \text{where } c_{-i} = -c_i \text{ for } i = 1, 2, \ldots, k.$$ 

The dual space $\mathfrak{h}_R^* \cong \mathbb{R}^k$ has basis $\varepsilon_1, \ldots, \varepsilon_k$, where $\varepsilon_i : \mathfrak{h}_R \to \mathbb{R}$ is the $\mathbb{R}$-linear map given by $\varepsilon_i(\gamma_1, \ldots, \gamma_k) = \gamma_i$. With $\varepsilon_{-i} = -\varepsilon_i$, the action of $\mathcal{W}_0$ on $\mathbb{R}^k$ produces an action on $\mathfrak{h}_R^*$ given by $w\varepsilon_i = \varepsilon_{w^{-1}(i)}$.

Let

$$R^+ = \{\varepsilon_1, \ldots, \varepsilon_k\} \cup \{\varepsilon_j - \varepsilon_i, \varepsilon_j + \varepsilon_i \mid 1 \leq i < j \leq k\}$$

$$= \{\varepsilon_1, \ldots, \varepsilon_k\} \cup \{\varepsilon_j - \varepsilon_i \mid 1 \leq j < k\} \cup \{\varepsilon_j - \varepsilon_i \mid 1 \leq i < j \leq k\}$$

$$= \{\varepsilon_1, \ldots, \varepsilon_k\} \cup \{\varepsilon_j - \varepsilon_i \mid i, j \in \{-k, -1, 1, \ldots, k\}, i < j, i \neq -j\}.$$

If $w \in \mathcal{W}_0$, the inversion set of $w$ is

$$R(w) = \{\alpha \in R^+ \mid w\alpha \notin R^+\} \quad \text{(3.2)}$$

$$= \{\varepsilon_i \mid \text{if } i > 0 \text{ and } w(i) < 0\} \cup \{\varepsilon_j - \varepsilon_i \mid \text{if } 0 < i < j \text{ and } w(i) > w(j)\} \quad \text{(3.3)}$$

$$\cup \{\varepsilon_j + \varepsilon_i \mid \text{if } 0 < i < j \text{ and } -w(i) > w(j)\}.$$ 

The chambers are the connected components of $\mathfrak{h}_R \setminus \bigcup_{\alpha \in R^+} \mathfrak{h}^\alpha$, where $\mathfrak{h}^\alpha = \{\gamma \in \mathfrak{h}_R \mid \alpha(\gamma) = 0\}$. The fundamental chamber in $\mathfrak{h}_R$ is

$$C = \{c \in \mathfrak{h}_R \mid \alpha(\gamma) \in \mathfrak{h}_{>0} \text{ for } \alpha \in R^+\} = \{(c_1, \ldots, c_k) \in \mathbb{R}^k \mid 0 < c_1 < c_2 < \cdots < c_k\},$$

and the group $\mathcal{W}_0$ can be identified with the set of chambers via the bijection

$$\begin{array}{ccc}
\mathcal{W}_0 & \leftrightarrow & \{\text{chambers}\} \\
\longleftrightarrow & w^{-1}C & .
\end{array}$$

Since

$$w^{-1}C = \left\{c \in \mathfrak{h}_R \mid \begin{array}{l}
\alpha(c) \in \mathbb{R}_{<0} \text{ if } \alpha \in R(w) \text{ and } \\
\alpha(c) \in \mathbb{R}_{>0} \text{ if } \alpha \in R^+ \setminus R(w)
\end{array} \right\},$$

the set $R(w)$ determines $w$.

### 3.1 Local regions

For $\gamma = (\gamma_1, \ldots, \gamma_k) \in (\mathbb{C}^x)^k$, define

$$Z(\gamma) = \{\varepsilon_i \mid \gamma_i = \pm 1\} \cup \{\varepsilon_j - \varepsilon_i \mid 0 < i < j, \gamma_i\gamma_j^{-1} = 1\} \cup \{\varepsilon_j + \varepsilon_i \mid 0 < i < j, \gamma_i\gamma_j = 1\},$$

$$P(\gamma) = \{\varepsilon_i \mid \gamma_i \in \{(t_0^{1 \frac{1}{2}}t_k^{\pm 1}), (-t_0^{\frac{-1}{2}}t_k^{\pm 1})\} \cup \{\varepsilon_j - \varepsilon_i \mid 0 < i < j, \gamma_i\gamma_j^{-1} = t^{\pm 1}\}$$

$$\cup \{\varepsilon_j + \varepsilon_i \mid 0 < i < j, \gamma_i\gamma_j = t^{\pm 1}\}. \quad (3.4)$$

Using the conversion from $\gamma_i$ to $c_i$ as in (2.32), let

$$\gamma_i = -t^{c_i}, \quad \text{and set } -t^{r_1} = -t_k^{\frac{1}{2}}t_0^{\frac{1}{2}} \quad \text{and } \quad -t^{r_2} = t_k^{\frac{1}{2}}t_0^{-\frac{1}{2}}, \quad (3.5)$$

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so that \(-t^{\pm r_1}\) and \(-t^{\pm r_2}\) are the eigenvalues of \(W_1\) that cause \(\tau_0^2\) to have a nonzero kernel (see (2.42)). Then, for \(c = (c_1, \ldots, c_k) \in \mathbb{C}^k\) let \(c_{-i} = -c_i\) and define

\[
\begin{align*}
Z(c) &= \{\varepsilon_i \mid c_i = 0\} \cup \{\varepsilon_j - \varepsilon_i \mid 0 < i < j \text{ and } c_j - c_i = 0\} \\
P(c) &= \{\varepsilon_i \mid c_i \in \{\pm r_1, \pm r_2\}\} \cup \{\varepsilon_j - \varepsilon_i \mid 0 < i < j \text{ and } c_j - c_i = \pm 1\} \cup \{\varepsilon_j + \varepsilon_i \mid 0 < i < j \text{ and } c_j + c_i = \pm 1\}.
\end{align*}
\]

(3.6)

(3.7)

A local region is a pair \((c, J)\) with \(c \in \mathbb{C}^k\) and \(J \subseteq P(c)\). The set of standard tableaux of shape \((c, J)\) is

\[\mathcal{F}(c, J) = \{w \in W_0 \mid R(w) \cap Z(c) = \emptyset, \ R(w) \cap P(c) = J\}.\]

(3.8)

As in [Ra2, §5 and §8] the local regions \((c, J)\) and standard tableaux \(w \in \mathcal{F}(c, J)\) can be converted to configurations of boxes \(\kappa\) and standard tableaux \(S\) of shape \(\kappa\) similar to those that are familiar in the literature on irreducible representations of Weyl groups of classical types. As explained in [Ra2, §5.11], the definitions of \(Z(c)\) and \(P(c)\) make it possible to view the general case \(c \in \mathbb{C}^k\) as pieced together from the cases \(c \in (\mathbb{Z} + \beta)^k\) where \(\beta\) runs over a set of representatives of the \(\mathbb{Z}\)-cosets in \(\mathbb{C}\). Below we make the conversion between local regions and configurations of boxes explicit for the cases when \(c \in \mathbb{Z}^k\) and \(c \in (\mathbb{Z} + \frac{1}{2})^k\). These are the cases that appear in the Schur-Weyl duality approach to the representations of \(H_k^{\text{ext}}\) explored in Section 5. As in [Ra2, §8], it is also true that these cases are sufficient to determine the general \(c \in (\mathbb{Z} + \beta)^k\) setting.

Let \((c, J)\) be a local region with \(c = (c_1, \ldots, c_k)\),

\[c \in \mathbb{Z}^k \quad \text{or} \quad c \in (\mathbb{Z} + \frac{1}{2})^k, \quad \text{and} \quad 0 \leq c_1 \leq \cdots \leq c_k.\]

(3.9)

Start with an infinite arrangement of NW to SE diagonals, numbered consecutively from \(\mathbb{Z}\) or \(\mathbb{Z} + \frac{1}{2}\), increasing southwest to northeast (see Example 1). The configuration \(\kappa\) of boxes corresponding to the local region \((c, J)\) has \(2k\) boxes (labeled \(\text{box}_{-k}, \ldots, \text{box}_{-1}, \text{box}_1, \ldots, \text{box}_k\)) with the following conditions.

\((\kappa 1)\) Location: box\(_i\) is on diagonal \(c_i\), where \(c_{-i} = -c_i\) for \(i \in \{-k, \ldots, -1\}\).

\((\kappa 2)\) Same diagonals: box\(_i\) is NW of box\(_j\) if \(i < j\) and box\(_i\) and box\(_j\) are on the same diagonal.

\((\kappa 3)\) Adjoining diagonals:

If \(\varepsilon_j - \varepsilon_i \in J\), then box\(_j\) is NW (strictly north and weakly west) of box\(_i\):

\[
\begin{array}{c}
j \\
i
\end{array}
\]

If \(\varepsilon_j - \varepsilon_i \in P(c) - J\), then box\(_j\) is SE (weakly south and strictly east) of box\(_i\):

\[
\begin{array}{c}
i \\
j
\end{array}
\]

\((\kappa 4)\) Markings: There is a marking on each of the diagonals \(r_1, -r_1, r_2\) and \(-r_2\).

If \(\varepsilon_i \in J\), box\(_i\) is NW of the marking on diagonal \(c_i\):

\[
\begin{array}{c}
\downarrow
\end{array}
\]

If \(\varepsilon_i \in P(c) - J\), then box\(_i\) is SE of the marking in diagonal \(c_i\) : 

\[
\begin{array}{c}
\uparrow
\end{array}
\]

Condition \((\kappa 1)\) enables the values \((c_{-k}, \ldots, c_{-1}, c_1, \ldots, c_k)\) to be read off of configuration \(\kappa\). The
sets $Z(c)$, $P(c)$, and $J$ can also be determined from the configuration $\kappa$ since

\[
Z(c) = \{\varepsilon_i \mid 0 < i \text{ and } \text{box}_i \text{ is in diagonal } 0\}
\]

\[
\sqcup \{\varepsilon_j - \varepsilon_i \mid 0 < i < j \text{ and } \text{box}_i \text{ and } \text{box}_j \text{ are in the same diagonal}\}
\]

\[
\sqcup \{\varepsilon_j + \varepsilon_i \mid 0 < i < j \text{ and box}_i \text{ and box}_j \text{ are both in diagonal } 0\},
\]

\[
P(c) = \{\varepsilon_i \mid 0 < i \text{ and box}_i \text{ is in diagonal } r_1 \text{ or } r_2\},
\]

\[
\sqcup \{\varepsilon_j - \varepsilon_i \mid 0 < i < j \text{ and box}_i \text{ and box}_j \text{ are in adjacent diagonals}\}
\]

\[
\sqcup \{\varepsilon_j + \varepsilon_i \mid 0 < i < j \text{ and box}_{i-1} \text{ and box}_j \text{ are in adjacent diagonals}\}, \quad \text{and}
\]

\[
J = \{\varepsilon_i \in P(c) \mid \text{box}_i \text{ is NW of the marking}\}
\]

\[
\sqcup \{\varepsilon_j - \varepsilon_i \in P(c) \mid \text{box}_j \text{ is northwest of box}_i\}
\]

\[
\sqcup \{\varepsilon_j + \varepsilon_i \in P(c) \mid \text{box}_j \text{ is northwest of box}_{i-1}\}.
\]

A **standard filling** of the boxes of $\kappa$ is a bijective function $S: \kappa \rightarrow \{-k, \ldots, -1, 1, \ldots k\}$ such that

(S1) **Symmetry:** $S(\text{box}_{-i}) = -S(\text{box}_i)$.

(S2) **Same diagonals:**
If $0 < i < j$ and box$_i$ and box$_j$ are on the same diagonal then $S(\text{box}_i) < S(\text{box}_j)$.

(S3) **Adjacent diagonals:**
If $0 < i < j$, box$_i$ and box$_j$ are on adjacent diagonals, and box$_j$ is NW of box$_i$, then $S(\text{box}_j) < S(\text{box}_i)$.
If $0 < i < j$, box$_i$ and box$_j$ are on adjacent diagonals, and box$_j$ is SE of box$_i$, then $S(\text{box}_j) > S(\text{box}_i)$.

(S4) **Markings:**
If box$_i$ is on a marked diagonal and is SE of the marking, then $S(\text{box}_i) > 0$.
If box$_i$ is on a marked diagonal and is NW of the marking, then $S(\text{box}_i) < 0$.

The **identity filling** of a configuration $\kappa$ is the filling $F$ of the boxes of $\kappa$ given by $F(\text{box}_i) = i$, for $i = -k, \ldots, -1, 1, \ldots, k$. The identity filling of $\kappa$ is usually not a standard filling of $\kappa$ (see Example 1).

**Example 1.** Let $k = 4$, $r_1 = 1$, and $r_2 = 3$. Consider $c = (-3, -2, -2, 2, 3)$. Then

\[
Z(c) = \{\varepsilon_2 - \varepsilon_1\} \quad \text{and} \quad P(c) = \{\varepsilon_3, \varepsilon_3 - \varepsilon_1, \varepsilon_3 - \varepsilon_2\}.
\]

The box configurations corresponding to $J = \{\varepsilon_3 - \varepsilon_2\}$ and $J = \{\varepsilon_3, \varepsilon_3 - \varepsilon_1, \varepsilon_3 - \varepsilon_2\}$ (filled with their identity fillings) are
For both configurations, the identity filling is not a standard filling. Examples of standard fillings of the configuration corresponding to $J = \{\varepsilon_3 - \varepsilon_2\}$ include

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad \text{but not} \quad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

The proof of the following proposition is a straightforward, though slightly tedious, check that the conditions $R(w) \cap Z(c) = \emptyset$ and $R(w) \cap P(c) = J$ from (3.8) convert to the conditions (S2), (S3), (S4) on standard fillings of shape $\kappa$. The proof is similar to the proof of [Ra2, Thm. 5.9].

**Proposition 3.1.** Let $\kappa$ be a configuration of boxes corresponding to a local region $(c, J)$ with $c \in \mathbb{Z}^k$ or $c \in (\mathbb{Z} + \frac{1}{2})^k$. For $w \in \mathcal{W}_0$ let $S_w$ be the filling of the boxes of $\kappa$ given by $S_w(\text{box}_i) = w(i)$, for $i = -k, \ldots, -1, 1, \ldots, k$.

The map

$$\mathcal{F}^{(c, J)} : w \mapsto \{\text{standard fillings } S \text{ of the boxes of } \kappa\}$$

is a bijection.

**Example 2.** Let $k = 12$, $r_1 = \frac{3}{2}$, $r_2 = \frac{15}{2}$, $c = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}, \frac{15}{2}, \frac{17}{2})$ and

$$J = \left\{ \varepsilon_3, \varepsilon_{10}, \varepsilon_3 - \varepsilon_2, \varepsilon_4 - \varepsilon_2, \varepsilon_5 - \varepsilon_4, \varepsilon_8 - \varepsilon_7, \varepsilon_{10} - \varepsilon_8, \varepsilon_{10} - \varepsilon_9, \varepsilon_{11} - \varepsilon_9, \varepsilon_{12} - \varepsilon_{10}, \varepsilon_{12} - \varepsilon_{11} \right\}$$

Let

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ -9 & 10 & -8 & 7 & 6 & 3 & 4 & 1 & 5 & -11 & 2 & -12 \end{pmatrix} \in \mathcal{F}^{(c, J)}.$$

Then, for the corresponding configuration of boxes $\kappa$, the identity filling $F$, and the standard filling $S_w$ corresponding to $w$ are

$$F = \begin{pmatrix} \frac{17}{2} & \frac{15}{2} & 12 & 10 & 8 & 11 \\ -\frac{17}{2} & -\frac{15}{2} & -2 & 1 & 3 & 5 \\ -4 & -5 & -6 & -7 & -8 & -9 \\ -10 & -11 & -12 \end{pmatrix} \quad \text{and} \quad S_w = \begin{pmatrix} -10 & -9 & -8 & 6 \\ -7 & -6 & -5 & -4 & -3 \\ -2 & -1 & 11 & 12 \\ 7 & 8 & 9 & 10 \end{pmatrix}.$$
Remark 3.2. Borrowing a physical intuition, configurations are invariant under sliding boxes along
diagonals like beads on an abacus, so long as boxes that run into each other are not allowed to
exchange places, i.e. for most \( c \in \mathbb{Z} \),
\[
\begin{array}{c c c}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\neq
\begin{array}{c c c}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\]
Then by arranging configurations so that the boxes are packed together, standard fillings of con-
figurations are exactly analogous to standard tableaux for partitions.

The only exception to this physical intuition is for boxes on the diagonals \( \pm \frac{1}{2} \). Note that if
\( c_i = \frac{1}{2} \), then box \( i \) and box \(-i\) are on adjacent diagonals. However, since \( 2\varepsilon_i = \varepsilon_i - \varepsilon_{-i} \notin R^+ \) and
therefore never in \( P(c) \), the relative positions of box \( i \) and box \(-i\) will never be recorded in the set \( J \). For example, in Figure 2 the point where \( (c_1, c_2) = (\frac{1}{2}, \frac{1}{2}) \) has two configurations, each with
two boxes overlapping in indication that box \( i \) and box \(-i\) may “slide past each other”. The drawing
\[
\begin{array}{c c c}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\]
(with boxes filled in the identity filling) where box \( 1 \) and box \(-1\) can move freely past each other, and
\[
\begin{array}{c c c}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\]
where box \( 2 \) and box \(-2\) can move freely past each other. In these two examples \( \varepsilon_1 - \varepsilon_{-2} \in P(c) \) and
\( \varepsilon_2 - \varepsilon_{-1} \in P(c) \) and so the relative orientation of box \( 2 \) and box \(-1\) and the relative orientation of
box \( 1 \) and box \(-2\) are recorded in \( J \). Each configuration has exactly two standard fillings.

3.2 Classifying and constructing calibrated representations

Theorem 3.3 below provides an indexing of the calibrated irreducible \( H_k^{\text{ext}} \)-modules by skew local
regions. A skew local region is a local region \((c, J)\), \( c = (c_1, \ldots, c_k) \), such that
if \( w \in \mathcal{F}(c, J) \) then \( wc = ((wc)_1, \ldots, (wc)_n) \) satisfies
\[
(wc)_1 \neq 0, \quad (wc)_2 \neq 0, \quad (wc)_1 \neq -(wc)_2,
\]
\[
(wc)_i \neq (wc)_{i+1} \quad \text{for} \quad i = 1, \ldots, k - 1, \quad \text{and} \quad (wc)_i \neq (wc)_{i+2} \quad \text{for} \quad i = 1, \ldots, k - 2.
\]
Theorem 3.3 is completely analogous to the same theorem for the case \( t^2 = t_0^2 = t_k^2 \) in [Ra2, Theo-
rem 3.5]. As explained in the discussion and remarks before [Ra2, Lemma 3.1] in [Ra2, §3], getting
exactly the right definition of skew local region for the purpose of Theorem 3.3 is accomplished by
detailed computation of the irreducible representations in rank two cases. More specifically, for
\( I \subseteq \{0, \ldots, k\} \), let \( H_I \) be the subalgebra of \( H_k^{\text{ext}} \) generated by \( \{T_i\}_{i \in I} \) and \( \mathbb{C}[W_{i}^{1}, \ldots, W_{i}^{\pm 1}] \). Then the conditions in (3.10) guarantee that for \( w \in \mathcal{F}(c;J) \) and \( i, j \in \{0, 1, \ldots, k-1\} \),

there exists a calibrated \( H_{\{i,j\}} \)-module \( M \) with \( M_{w}^{\text{gen}} \neq 0 \).

The cases where \( H_{\{i,j\}} \) is of type \( A \times A \) or of type \( A_2 \) are checked in [Ra1]. However, when \( H_{\{i,j\}} \) is of type \( C_2 \) and there are three distinct parameters, we do not know a reference for this. So in the effort to provide a more complete presentation, we have done the appropriate analysis in Section 4 for all generic choices of the three parameters \( t_1^{\frac{1}{2}}, t_0^{\frac{1}{2}}, \) and \( t_k^{\frac{1}{2}} \), as given in the following theorem (see also (4.1)).

**Theorem 3.3.** Assume \( t_1^{\frac{1}{2}}, t_0^{\frac{1}{2}}, \) and \( t_k^{\frac{1}{2}} \) are invertible, \( t_2^{\frac{1}{2}} \) is not a root of unity, and

\[
\begin{align*}
t_0^{\frac{1}{2}}t_k^{\frac{1}{2}}, -t_0^{-\frac{1}{2}}t_k^{\frac{1}{2}} \notin \{1, -1, t^{\pm \frac{1}{2}}, -t^{\pm \frac{1}{2}}, t^{\pm 1}, -t^{\pm 1}\} \quad \text{and} \quad t_0^{\frac{1}{2}}t_k^{\frac{1}{2}} \neq (-t_0^{-\frac{1}{2}}t_k^{\frac{1}{2}})^{\pm 1}.
\end{align*}
\]

(a) Let \((c, J)\) be a skew local region and let \( z \in \mathbb{C}^\times \). Define

\[
H_k^{(z,c,J)} = \text{span}_\mathbb{C} \{v_w \mid w \in \mathcal{F}(c;J)\},
\]

so that the symbols \( v_w \) are a labeled basis of the vector space \( H_k^{(z,c,J)} \). Let

\[
\gamma_i = -t^{e_i} \quad \text{for } i = 1, 2, \ldots, k, \quad \text{and} \quad \gamma_0 = z\gamma_{w^{-1}(1)}^{-1} \cdots \gamma_{w^{-1}(k)}^{-1}.
\]

Then the following formulas make \( H_k^{(z,c,J)} \) into an irreducible \( H_k^{\text{ext}} \)-module:

\[
\begin{align*}
P W_1 \cdots W_k v_w &= z v_w, \quad P v_w = \gamma_0 v_w, \quad W_i v_w = \gamma_{w^{-1}(i)} v_w, \quad (3.12) \\
T_i v_w &= [T_{i}]_{ww} v_w + \sqrt{-([T_{i}]_{ww} - t_i^2)([T_{i}]_{ww} + t_i^{-2})} v_{s_iw}, \quad \text{for } i = 1, \ldots, k-1, \quad (3.13) \\
T_0 v_w &= [T_0]_{ww} v_w + \sqrt{-(T_0)_{ww} - t_0^{-2}}(T_0)_{ww} + t_0^{-2}) v_{s_0w}, \quad (3.14)
\end{align*}
\]

where \( v_{s_iw} = 0 \) if \( s_iw \notin \mathcal{F}(c;J) \), and

\[
[T_{i}]_{ww} = \frac{t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}}}{1 - \gamma_{w^{-1}(i)} \gamma_{w^{-1}(i+1)}^{-1}} \quad \text{and} \quad [T_0]_{ww} = \frac{(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}) + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}}) \gamma_{w^{-1}(1)}}{1 - \gamma_{w^{-1}(1)}^{-2}}. \quad (3.15)
\]

(b) The map

\[
\mathbb{C}^\times \times \{\text{skew local regions } (c, J)\} \leftrightarrow \{\text{irreducible calibrated } H_k^{\text{ext}} \text{-modules}\}
\]

\[
(z, c, J) \quad \mapsto \quad H_k^{(z,c,J)}
\]

is a bijection.

**Proof.** This result follows from [Ra2] Theorems 3.2 and 3.5. It is only necessary to establish that the formulas in (3.12), (3.13), and (3.14) are correct. These are derived in a similar manner to [Ra2].
Proposition 3.3] as follows. As in [Ra2, Theorem 3.2], if $M$ is an irreducible calibrated $H_k^{\text{ext}}$-module
then
$$M = \bigoplus_{w \in W_0} M^\text{gen}_{w\gamma}, \quad \text{with } \dim(M^\text{gen}_{w\gamma}) = 1 \text{ if } M^\text{gen}_{w\gamma} \neq 0.$$ 

For $w \in W_0$, if $M^\text{gen}_{w\gamma} \neq 0$, let $v_w$ be a nonzero vector in $M^\text{gen}_{w\gamma}$, otherwise if $M^\text{gen}_{w\gamma} = 0$, let $v_w = 0$. By (2.35), $\tau_i v_w = [T_i]_{w,w} v_{s_i w}$ for some constant $[T_i]_{w,w}$ and the definition of $\tau_i$ in (2.35) gives that
$$T_i v_w = \frac{t_i^2 - t_i^{-1}}{1 - \gamma_{w-1}^{(i)} \gamma_{w-1}^{-(i+1)}} v_w + [T_i]_{s_i w,w} v_{s_i w} \quad \text{for } i = 1, \ldots, k,$$
and
$$T_0 v_w = \left(\frac{\frac{3}{2} \gamma_{w} - \frac{1}{2}}{1 - \gamma_{w}^{-2} (1)}\right) v_w + [T_0]_{s_0 w,w} v_{s_0 w}.$$ 

Thus $T_0$ is an operator on the subspace $\text{span}_C\{v_w, v_{s_0 w}\}$ satisfying $(T_0 - \frac{3}{2} \gamma_{w} - \frac{1}{2})(T_0 + \frac{3}{2} \gamma_{w} - \frac{1}{2}) = 0$ by [H]. Restricting to the action on $\text{span}_C\{v_w, v_{s_0 w}\}$, the formulas in (3.14) now follow from the following argument about general $2 \times 2$ matrices.

If a $2 \times 2$ matrix $[T_0]$ has eigenvalues $\alpha_1$ and $\alpha_2$,
$$[T_0] = \begin{pmatrix} [T_0]_{w,w} & [T_0]_{w,s_0 w} \\ [T_0]_{s_0 w,w} & [T_0]_{s_0 w,s_0 w} \end{pmatrix},$$
then $([T_0] - \alpha_1)([T_0] - \alpha_2) = 0$ is the characteristic polynomial for $[T_0]$, and it follows that
$$\text{Tr}([T_0]) = [T_0]_{w,w} + [T_0]_{s_0 w,s_0 w} = \alpha_1 + \alpha_2,$$
and
$$\det([T_0]) = [T_0]_{w,w}[T_0]_{s_0 w,s_0 w} - [T_0]_{w,s_0 w}[T_0]_{s_0 w,w} = \alpha_1 \alpha_2.$$ 

Thus
$$-[T_0]_{w,s_0 w}[T_0]_{s_0 w,w} = \alpha_1 \alpha_2 - [T_0]_{w,w} [T_0]_{s_0 w,s_0 w} = \alpha_1 \alpha_2 - [T_0]_{w,w}((\alpha_1 + \alpha_2) - [T_0]_{w,w})$$
$$= \alpha_1 \alpha_2 - (\alpha_1 + \alpha_2)[T_0]_{w,w} + ([T_0]_{w,w})^2 = ([T_0]_{w,w} - \alpha_1)([T_0]_{w,w} - \alpha_2).$$

Choosing a normalization of $v_{s_0 w}$ so that the matrix of $[T_0]$ is symmetric, we have $[T_0]_{w,s_0 w} = [T_0]_{s_0 w,w}$ and
$$[T_0]_{s_0 w,w} = \sqrt{([T_0]_{s_0 w,w})^2} = \sqrt{[T_0]_{w,s_0 w}[T_0]_{s_0 w,w}} = \sqrt{-(T_0)_{w,w} - \alpha_1)}([T_0]_{w,w} - \alpha_2).$$

\[ \square \]

4 Classification of irreducible representations of $H_2$

In this section we do a complete classification of the irreducible representations of the algebra $H_2^{\text{ext}}$. An important reason for doing this classification of $H_2^{\text{ext}}$ representations is to provide a sound basis for the definition of a skew local region (see the remarks immediately after the definition of skew local region in (3.10)). The classification and construction of calibrated representations of
In terms of skew local regions in Theorem 3.3, $H_k^{\text{ext}}$ is important for determining the irreducible representations of $H_k^{\text{ext}}$ that arise in the Schur-Weyl duality framework (see Theorem 4.5). We will do the classification of irreducible $H_2^{\text{ext}}$ representations under genericity assumptions on the parameters: $t^\frac{1}{2}$ is not a root of unity and

$$t^\frac{1}{2} t^\frac{1}{2} - t^\frac{1}{2} t^\frac{1}{2} \notin \{ 1, -1, t^\pm \frac{1}{2}, -t^\pm \frac{1}{2}, t^\pm 1, -t^\pm 1 \} \quad \text{and} \quad t^\frac{1}{2} t^\frac{1}{2} \neq (-t^\frac{1}{2} t^\frac{1}{2})^\pm 1. \quad (4.1)$$

More specifically, this condition is used for the (rank 2) computation in equation (4.4). Similar methods apply to the nongeneric cases but the final classification needs to be stated differently and we will not treat the nongeneric cases here. The nongeneric case $t^\frac{1}{2} = t^\frac{1}{2} = t^\frac{1}{2}$ is done in [Ra1, Ra2] and [Re]; the case where $t^\frac{1}{2} = t^\frac{1}{2} \neq t^\frac{1}{2}$ appears in [En] (see also [KR]).

The algebra $H_2$ is generated by $W_1^{\pm 1}, W_2^{\pm 1}, T_0, \text{ and } T_1,$ and the Weyl group $W_0$ is generated by $s_0$ and $s_1$ with relations $s_0^2 = 1$ and $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$. By (2.29),

$$H_2^{\text{ext}} = C[W_0^{\pm 1}] \otimes H_2 \quad \text{as algebras},$$

and therefore it is sufficient to do the classification of irreducible representations of $H_2$. This is because all irreducible representations of $C[W_0^{\pm 1}]$ are one dimensional and determined by the image of $W_0$; and all irreducible representations of $H_2^{\text{ext}}$ are the tensor product of an irreducible representation of $C[W_0^{\pm 1}]$ and an irreducible representation of $H_2$.

The group $W_0$ acts on $(C^\times)^2$ by

$$s_0(\gamma_1, \gamma_2) = (\gamma_1^{-1}, \gamma_2) \quad \text{and} \quad s_1(\gamma_1, \gamma_2) = (\gamma_2, \gamma_1). \quad (4.2)$$

By (2.35), the intertwiners are

$$\tau_0 = T_0 - \frac{(t^\frac{1}{2} - t^\frac{1}{2}) + (t^\frac{1}{2} - t^\frac{1}{2}) W_1^{-1}}{1 - W_1^{-2}} \quad \text{and} \quad \tau_1 = T_1 - \frac{t^\frac{1}{2} - t^\frac{1}{2}}{1 - W_1 W_2^{-1}}.$$

### 4.1 Classification of central characters

Following [Ra1], the classification of irreducible $H_k^{\text{ext}}$-modules begins with a classification of possible pairs $(Z(c), P(c)) = (Z(\gamma), P(\gamma))$ (where $\gamma$ and $c$ are related as in (2.32)). It is straightforward (though slightly tedious) to enumerate all the possibilities by taking note of the following:

1. Since $(Z(w \gamma), P(w \gamma)) = (w Z(\gamma), w P(\gamma))$, it is sufficient to do the analysis for a single representative $\gamma$ of each $W_0$-orbit on $(C^\times)$. 

2. The $W_0$-orbits of roots are $\{ \pm \varepsilon_1, \pm \varepsilon_2 \}$ and $\{ \pm (\varepsilon_2 \pm \varepsilon_1) \}$, and our preferred representative of the $W_0$-orbit will have $\varepsilon_1$ or $\varepsilon_2 - \varepsilon_1$ in $Z(\gamma)$ if $Z(\gamma) \neq \emptyset$.

3. If $Z(\gamma) = \emptyset$ and $P(\gamma) \neq \emptyset$ then our preferred representative of the $W_0$-orbit will have $\varepsilon_1$ or $\varepsilon_2 - \varepsilon_1$ in $Z(\gamma)$.

With these preferences, the classification of $(Z(\gamma), P(\gamma))$ is accomplished by noting that

(a) if $\gamma \in \{(1, 1), (-1, -1)\}$ then $(Z(\gamma), P(\gamma)) = (\{ \varepsilon_1, \varepsilon_2, \varepsilon_2 \pm \varepsilon_1 \}, \emptyset)$;
We shall freely use the conversion between chambers that are on the negative side of the hyperplanes in \( J \) are labeled by the equation that defines them. If \( \varepsilon_1 \in Z(\gamma) \) if and only if \( \gamma = (\gamma_1, \gamma_2) \) or \( \gamma = (\gamma_1, 1) \) or \( \gamma = (1, \gamma_2) \); \( \varepsilon_2 \in Z(\gamma) \) if and only if \( \gamma = (\gamma_1, 1) \) or \( \gamma = (1, \gamma_2) \); \( \varepsilon_1 \in P(\gamma) \) if and only if \( \gamma = (\gamma_1, \gamma_2) \) with \( \gamma_1 \in \{ \frac{1}{t_0^2} \frac{1}{t_0^2}, \frac{1}{t_0^2} \frac{1}{t_0^2}, \frac{1}{t_0^2} \frac{1}{t_0^2}, \frac{1}{t_0^2} \frac{1}{t_0^2} \} \); \( \varepsilon_2 - \varepsilon_1 \in P(\gamma) \) if and only if \( \gamma = (\gamma_1, \gamma_2) \) with \( \gamma_2 = \gamma_1 t^{\pm 1} \); \( \varepsilon_2 + \varepsilon_1 \in P(\gamma) \) if and only if \( \gamma = (\gamma_1, \gamma_2) \) with \( \gamma_1 \gamma_2 = t^{\pm 1} \).

We shall freely use the conversion between \( \gamma = (\gamma_1, \gamma_2) \) and \( c = (c_1, c_2) \) given by (2.32),

\[ \gamma_1 = -t^{\alpha_1}, \quad \gamma_2 = -t^{\alpha_2}, \quad \text{and write } (Z(c), P(c)) = (Z(\gamma), P(\gamma)). \]

Representatives of the 12 possible \((Z(c), P(c))\) with \( Z(c) = \emptyset \) are displayed in Figure 1. Representatives of the 9 possible \((Z(c), P(c))\) with \( Z(c) \neq \emptyset \) are displayed in Figure 2. It works out that, in each case, the pair \((Z(c), P(c))\) is attained by an element \( c \) that has real coordinates (the one complex character in the equal parameter case that behaves differently from the real characters, namely the point \( t_0 \) in \([Ra1\), Figure 5.1], does not appear in the generic unequal parameter case assumed in (4.1)).

With notation as at the beginning of Section 3, in Figures 1 and 2, the fundamental region \( C \) is the shaded area, the solid lines are the hyperplanes \( h^\alpha \) for \( \alpha \in R^+ \), and the dotted hyperplanes are labeled by the equation that defines them. If \( c = (c_1, c_2) \in C \), so that \( 0 \leq c_1 \leq c_2 \), then

\[ Z(c) = \{ \text{solid hyperplanes through } c \} \quad \text{and} \quad P(c) = \{ \text{dotted hyperplanes through } c \}. \]

The bijection

\[ W_0 \leftrightarrow \{ \text{chambers} \} \quad w \mapsto w^{-1}C \]

identifies each \( F(c, \gamma) \) with a set of chambers, \( (4.3) \) a local region in \( h^\alpha_\mathbb{R} \). As illustrated by the example at the bottom right of Figures 1 and 2, \( F(c, \gamma) \) is identified with the set of chambers that are on the negative side of the hyperplanes in \( J \) and on the positive side of the hyperplanes in \( P(c) - J \). For each \( (c, \gamma) \) the corresponding configuration of boxes \( \kappa \) is displayed in the local region of chambers corresponding to the elements of \( F(c, \gamma) \) by \( (4.3) \). In Figure 1 only the boxes on positive diagonals are shown, since they determine the entire doubled configuration when \( Z(c) = \emptyset \). The diagram at the bottom right of each figure gives an example of the correspondence between chambers corresponding to \( F(c, \gamma) \), the elements of \( F(c, \gamma) \), and the standard fillings of the corresponding configuration of boxes \( \kappa \): the point \( c = (r_1 - 1, r_1) \) in the bottom right of Figure 1 and the point \( c = (0, 1) \) in the bottom right of Figure 2.

In Figure 2 the small graphs nearby each marked \( c = (c_1, c_2) \) indicate the structure (generalized weight spaces and intertwiner maps) of the irreducible modules \( M \) of central character \( c \). This structure is determined below in Section 4.2. There is a vertex in the chamber \( w^{-1}C \) for each element of a basis of \( M_{wc}^\text{gen} \) and there is an edge if the matrix of \( \tau_i \) (or \( T_i \) if \( \tau_i \) is not defined on \( M_{wc}^\text{gen} \)) is nonzero in the entry corresponding to the two vertices that are connected.
Figure 1: Regular central characters in rank 2. See the description in Section 4.1.
Figure 2: Non-regular points

\[ c_1 = 0 \]
\[ c_2 = r_2 \]
\[ c_1 = r_1 \]
\[ c_1 = c_2 \]
\[ c_2 = r_1 \]
\[ c_2 = c_1 + 1 \]
\[ c_2 = -c_1 + 1 \]
\[ J = \emptyset \]
\[ J = \{ \varepsilon_2 - \varepsilon_1 \} \]
\[ J = \{ \varepsilon_2 \pm \varepsilon_1 \} \]
4.2 Construction of the irreducible $H_2$-modules

The group $W_0$ acts on $(\mathbb{C}^*)^2$ as in (4.2) and the central characters are the $W_0$-orbits on $(\mathbb{C}^*)^2$. The regular central characters are the $W_0$-orbits of $\gamma = (\gamma_1, \gamma_2) \in (\mathbb{C}^*)^2$ that have $Z(\gamma) = \emptyset$, i.e. where the intertwining operators in (2.38) are defined. Let $C[W] = C[W_1^{\pm 1}, W_2^{\pm 1}] \subseteq H_2$. By Kato’s criterion (see [Ra2, Proposition 2.11b]), for central characters $\gamma = (\gamma_1, \gamma_2)$ with $P(\gamma) = \emptyset$ there is a single irreducible module of dimension eight given by

$$L_{(\gamma_1, \gamma_2)} = \text{Ind}^{H_2}_{C[W]}(\mathbb{C}_{\gamma_1, \gamma_2}),$$

where $\mathbb{C}_{\gamma_1, \gamma_2} = \mathbb{C}v$ with $W_1v = \gamma_1 v$ and $W_2v = \gamma_2v$.

All irreducible modules with $Z(\gamma) = \emptyset$ are calibrated and can be constructed as in Theorem 3.3.

Representatives of the $W_0$-orbits of $\gamma = (\gamma_1, \gamma_2) \in (\mathbb{C}^*)^2$ that have $Z(\gamma) \neq \emptyset$ and $P(\gamma) \neq \emptyset$ are as follows:

| $\gamma = (\gamma_1, \gamma_2)$ | $Z(\gamma)$ | $P(\gamma)$ |
|-----------------------------|-------------|-------------|
| $(t^2, t^2), (-t^2, -t^2)$ | $\{\varepsilon_2 - \varepsilon_1\}$ | $\{\varepsilon_2 + \varepsilon_1\}$ |
| $(t^2, t^2), (-t^2, -t^2), (-t_0^{-1}t_k^2, -t_0^2t_k^2)$ | $\{\varepsilon_2 - \varepsilon_1\}$ | $\{\varepsilon_1, \varepsilon_2\}$ |
| $(1, t), (-1, -t)$ | $\{\varepsilon_1\}$ | $\{\varepsilon_2 - \varepsilon_1, \varepsilon_2 + \varepsilon_1\}$ |
| $(\pm 1, t_0^2t_k^2), (\pm 1, t_0^{-2}t_k^2)$ | $\{\varepsilon_1\}$ | $\{\varepsilon_2\}$ |

This classification is valid under the genericity assumption on the parameters (4.1), which guarantees that none of these representatives are in the $W_0$-orbit of another.

The following analysis of modules of central character $\gamma = (\gamma_1, \gamma_2)$ in (4.4) shows that no irreducible calibrated $H_2$-modules appear at these central characters. As in (3.5), the values $r_1$ and $r_2$ are defined by

$$-t^{r_1} = -t_k^2t_0^{-\frac{1}{2}}$$

and

$$-t^{r_2} = t_k^{\frac{1}{2}}t_0^{rac{1}{2}}.$$

**Case** $(\gamma_1, \gamma_2) = (-1, t^{r_i})$ for $i = 1$ or 2: Let $H_{(0)}$ be the subalgebra of $H_2$ generated by $T_0, W_0^{\pm 1}, W_2^{\pm 1}$. For each $i = 1$ and $i = 2$, there are two irreducible modules of central character $c = (0, r_i)$:

$$L^+_{(0, r_i)} = \text{Ind}^{H_2}_{H_{(0)}}(\mathbb{C}_{(r_i, 0)}),$$

where $\mathbb{C}_{(r_i, 0)} = \mathbb{C}v$ with $W_1v = -t^{r_i}v$, $W_2v = -v$, $T_0v = t_0^{\frac{1}{2}}v$,

and

$$L^-_{(0, r_i)} = \text{Ind}^{H_2}_{H_{(0)}}(\mathbb{C}_{(-r_i, 0)}),$$

where $\mathbb{C}_{(-r_i, 0)} = \mathbb{C}v$ with $W_1v = -t^{-r_i}v$, $W_2v = -v$, $T_0v = -t_0^{-\frac{1}{2}}v$.

With $M = L^+_{(0, r_i)}$, the generalized weight space decomposition is

$$M = M_{(r_i, 0)}^{\text{gen}} \oplus M_{(0, r_i)}^{\text{gen}}$$

with $\dim(M_{(r_i, 0)}^{\text{gen}}) = \dim(M_{(0, r_i)}^{\text{gen}}) = 2$.

The element $W_1W_2^{-1}$ acts on $M_{(r_i, 0)}^{\text{gen}}$ with eigenvalues $t^{r_i}$. Since the parameters are generic (see (4.1)), $t^{r_i} \neq t^{\pm 1}$ and thus, by (2.43), $t_1^2$ has no kernel. Thus the intertwiner $\tau_1: M_{(r_i, 0)}^{\text{gen}} \rightarrow M_{(0, r_i)}^{\text{gen}}$ is
invertible and \( M = L^+_{(0,r_1)} \) is irreducible. Replacing \( r_i \) with \(-r_i\) in (4.5) yields the decomposition of \( M = L^-_{(0,r_i)} \) analogously.

**Case** \((\gamma_1, \gamma_2) = (-t^{\frac{1}{2}}, -t^{\frac{1}{2}})\): Let \( H_{\{1\}} \) be the subalgebra of \( H_2 \) generated by \( T_1, W_{1}^{1}, W_{2}^{1} \). There are two irreducible modules of central character \( c = (\frac{1}{2}, \frac{1}{2}) \):

\[
L^+_{(\frac{1}{2}, \frac{1}{2})} = \text{Ind}_{H_{\{1\}}}^{H_2} (\mathbb{C}_{-\frac{1}{2}, \frac{1}{2}}), \quad \text{where} \quad \mathbb{C}_{-\frac{1}{2}, \frac{1}{2}} = Cv \quad \text{with} \quad \begin{align*}
W_1 v &= -t^{-\frac{1}{2}} v, \\
W_2 v &= -t^{\frac{1}{2}} v, \\
T_1 v &= t^{\frac{1}{2}} v,
\end{align*}
\]

and

\[
L^-_{(\frac{1}{2}, \frac{1}{2})} = \text{Ind}_{H_{\{1\}}}^{H_2} (\mathbb{C}_{\frac{1}{2}, -\frac{1}{2}}), \quad \text{where} \quad \mathbb{C}_{\frac{1}{2}, -\frac{1}{2}} = Cv \quad \text{with} \quad \begin{align*}
W_1 v &= -t^{\frac{1}{2}} v, \\
W_2 v &= -t^{-\frac{1}{2}} v, \\
T_1 v &= -t^{-\frac{1}{2}} v.
\end{align*}
\]

With \( M = L^+_{(\frac{1}{2}, \frac{1}{2})} \), the generalized weight space decomposition is

\[
M = M^{\text{gen}}_{(\frac{1}{2}, \frac{1}{2})} \oplus M^{\text{gen}}_{(-\frac{1}{2}, -\frac{1}{2})}, \quad \text{with} \quad \dim(M^{\text{gen}}_{(\frac{1}{2}, \frac{1}{2})}) = \dim(M^{\text{gen}}_{(-\frac{1}{2}, -\frac{1}{2})}) = 2.
\] (4.6)

The element \( W_1^{-1} \) acts on \( M^{\text{gen}}_{(\frac{1}{2}, \frac{1}{2})} \) with eigenvalues \(-t^{\frac{1}{2}}\). Since the parameters are generic (see (4.1)), \(-t^{\frac{1}{2}} \notin \{-t^{r_1}, -t^{r_2}\}\) and thus, by (2.42), \( \tau_0 \) has no kernel. Thus the intertwiner \( \tau_0 : M^{\text{gen}}_{(-\frac{1}{2}, -\frac{1}{2})} \to M^{\text{gen}}_{(\frac{1}{2}, \frac{1}{2})} \) is invertible and \( M = L^-_{(\frac{1}{2}, \frac{1}{2})} \) is irreducible. Similarly, the structure of \( M = L^+_{(\frac{1}{2}, \frac{1}{2})} \) is given by swapping \( \frac{1}{2} \) and \(-\frac{1}{2}\) in (4.6).

**Case** \((\gamma_1, \gamma_2) = (-t^{r_i}, -t^{r_i}) \text{ for } i = 1 \text{ or } 2\): Let \( H_{\{0\}} \) be the subalgebra of \( H_2 \) generated by \( T_0, W_{1}^{1}, W_{2}^{1} \). For each of \( i = 1 \) and \( i = 2 \), there are two irreducible modules of central character \( c = (r_i, r_i) \):

\[
L^+_{(r_i, r_i)} = \text{Ind}_{H_{\{0\}}}^{H_2} (\mathbb{C}_{r_i, -r_i}), \quad \text{where} \quad \mathbb{C}_{r_i, -r_i} = Cv \quad \text{with} \quad \begin{align*}
W_1 v &= -t^{r_i} v, \\
W_2 v &= -t^{-r_i} v, \\
T_0 v &= t^{\frac{1}{2}} v,
\end{align*}
\]

and

\[
L^-_{(r_i, r_i)} = \text{Ind}_{H_{\{0\}}}^{H_2} (\mathbb{C}_{-r_i, r_i}), \quad \text{where} \quad \mathbb{C}_{-r_i, r_i} = Cv \quad \text{with} \quad \begin{align*}
W_1 v &= -t^{-r_i} v, \\
W_2 v &= -t^{r_i} v, \\
T_0 v &= -t^{-\frac{1}{2}} v.
\end{align*}
\]

The irreducibility of \( L^+_{(r_i, r_i)} \) and \( L^-_{(r_i, r_i)} \) is not immediate. We will show that \( M = L^+_{(r_i, r_i)} \) is irreducible; the irreducibility of \( L^-_{(r_i, r_i)} \) is proved analogously.

The generalized weight space decomposition of \( M = L^+_{(r_i, r_i)} \) is

\[
M = M^{\text{gen}}_{(r_i, -r_i)} \oplus M^{\text{gen}}_{(-r_i, r_i)} \oplus M^{\text{gen}}_{(r_i, r_i)} \quad \text{with} \quad \begin{align*}
\dim(M^{\text{gen}}_{(r_i, -r_i)}) &= \dim(M^{\text{gen}}_{(-r_i, r_i)}) = 1, \\
\dim(M^{\text{gen}}_{(r_i, r_i)}) &= 2.
\end{align*}
\]

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Thus (1.41), \( t^{2r_i} \neq t^{\pm 1} \) and thus, by (2.43), \( \tau_i^2 \) has no kernel. Thus the intertwiner \( \tau_1 : M_{r_i}^{\text{gen}} \to M_{-r_i}^{\text{gen}} \) is invertible. As a \( H(0) \)-module, \( M_{r_i}^{\text{gen}} \) is irreducible (2-dimensional). So either \( N = M_{r_i}^{\text{gen}} \) is an \( H_2 \)-submodule or \( M \) is irreducible.

For the purpose of deriving a contradiction, assume that \( N = M_{r_i}^{\text{gen}} \) is an \( H_2 \)-submodule of \( M \). The space \( N \) has a basis \( \{n_\gamma, T_1n_\gamma\} \) with \( W_1n_\gamma = -t^{r_i}n_\gamma \), and \( W_2n_\gamma = -t^{r_i}n_\gamma \).

By (2.24), \( W_1^{-1}T_1n_\gamma = T_1W_2^{-1}n_\gamma + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})W_1^{-1}n_\gamma = T_1(-t^{r_i})n_\gamma + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})(-t^{r_i})n_\gamma \) and the action of \( W_1^{-1} \) and \( W_2^{-1} \) on the basis \( \{n_\gamma, T_1n_\gamma\} \) are given by the matrices

\[
\rho(W_1^{-1}) = (-t^{r_i}) \begin{pmatrix} t^{\frac{1}{2}} - t^{-\frac{1}{2}} & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \rho(W_2^{-1}) = \rho(W_1^{-1})^2 = -t^{2r_i} \begin{pmatrix} 2(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) & 1 \\ 0 & 1 \end{pmatrix}.
\]

Thus

\[
\rho(1 - W_1^{-2}) = (1 - t^{-2r_i}) \begin{pmatrix} 1 & -2(t^{\frac{1}{2}} - t^{-\frac{1}{2}})t^{-2r_i} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho(1 - W_2^{-2}) = \frac{1}{(1 - t^{-2r_i})} \begin{pmatrix} 2(t^{\frac{1}{2}} - t^{-\frac{1}{2}})t^{-2r_i} & 1 \\ 0 & 1 \end{pmatrix}.
\]

Since \( N \) is a submodule of \( M \), we have \( 0 = \tau_0 = T_0 - \frac{(t_0^{1/2} - t_0^{-1/2}) + (t_k^{1/2} - t_k^{-1/2})W_1^{-1}}{1 - W_1^{-2}} \) (see (2.35)) for the formula for \( \tau_0 \), and so

\[
\rho(T_0) = \left( (t_0^{1/2} - t_0^{-1/2}) + (t_k^{1/2} - t_k^{-1/2})W_1^{-1} \right)(1 - W_1^{-2})^{-1}
\]

\[
= \left( t_0^{1/2} - t_0^{-1/2} \right) + \frac{(t_k^{1/2} - t_k^{-1/2})(-t^{-r_i})}{1 - t^{-2r_i}} \begin{pmatrix} (t_0^{1/2} - t_0^{-1/2})(t^{\frac{1}{2}} - t^{-\frac{1}{2}})(-r_i) & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2(t^{\frac{1}{2}} - t^{-\frac{1}{2}})t^{-2r_i} \\ 0 & 1 \end{pmatrix}
\]

\[
= \left( t_0^{1/2} - t_0^{-1/2} \right) + \frac{(t_k^{1/2} - t_k^{-1/2})(-t^{-r_i})}{1 - t^{-2r_i}} \begin{pmatrix} 1 & 2(-t^{-r_i})t^{-2r_i} \\ 0 & 1 \end{pmatrix} + \frac{(t_k^{1/2} - t_k^{-1/2})}{1 - t^{-2r_i}} \begin{pmatrix} 1 & 2(t^{\frac{1}{2}} - t^{-\frac{1}{2}})t^{-2r_i} \\ 0 & 1 \end{pmatrix}.
\]

Recall, from (3.5), that \( -t^{r_i} = \pm t_k^{\pm 2/3} t_0^{1/2} \), so that

\[
\frac{(t_0^{1/2} - t_0^{-1/2}) + (t_k^{1/2} - t_k^{-1/2})(-t^{-r_i})}{1 - t^{-2r_i}} = \frac{(t_0^{1/2} - t_0^{-1/2}) + (t_k^{1/2} - t_k^{-1/2})(\pm \frac{2}{3} t_0^{1/2} - t_0^{-1/2})}{1 - t_k^{1/2} t_0^{-1/2}} = t_0^{-1/2}.
\]

The eigenvalues of \( \rho(T_0) \) are \( t_0^{-1/2} \) and, since \( (T_0 - t_0^{-1/2})(T_0 + t_0^{-1/2}) = 0 \), the Jordan blocks of \( \rho(T_0) \) are of size 1, forcing

\[
0 = \frac{2(-t^{-r_i})}{1 - t^{-2r_i}} + \frac{(t_k^{1/2} - t_k^{-1/2})}{(t_0^{1/2} - t_0^{-1/2}) + (t_k^{1/2} - t_k^{-1/2})(-r_i)} = \frac{2(-t^{-r_i})}{1 - t^{-2r_i}} + \frac{(t_k^{1/2} - t_k^{-1/2})}{(1 - t^{2r_i})t_0^{-1/2}}
\]

\[
= \frac{2(-t^{-r_i})t_0^{1/2} + (t_k^{1/2} - t_k^{-1/2})}{(1 - t^{-2r_i})t_0^{1/2}} = \frac{2(\pm t_k^{2/3} t_0^{1/2} - t_0^{-1/2})t_0^{1/2} + (t_k^{1/2} - t_k^{-1/2})}{(1 - t^{-2r_i})t_0^{1/2}} = \pm (t_k^{2/3} + t_k^{-1/2})/(1 - t^{-2r_i})t_0^{1/2}.
\]
This is a contradiction since, by the generic condition on parameters in (4.1), \( 1 \neq (-t')^2 = (-t_0^2 t^{-1})^2 = -(t_0^2)^2 \). Thus \( N \) is not a submodule of \( M \), and so \( M \) is irreducible.

**Case** \((\gamma_1, \gamma_2) = (-1, -t)\): Let \( H_{(1)} \) be the subalgebra of \( H_2 \) generated by \( T_1, W_1 \pm 1, W_2 \pm 1 \). There are two irreducible modules of central character \( c = (0, 1) \):

\[
L^+_{(0,1)} = \Ind_{H_{(1)}}^{H_2} (\mathbb{C}_{(-1,0)}), \quad \text{where } \mathbb{C}_{(-1,0)} = \mathbb{C} \text{v with } W_1 \text{v} = -t^{-1} \text{v}, \quad W_2 \text{v} = -\text{v}, \quad T_1 \text{v} = t^{1/2} \text{v},
\]

and

\[
L^-_{(0,1)} = \Ind_{H_{(1)}}^{H_2} (\mathbb{C}_{(1,0)}), \quad \text{where } \mathbb{C}_{(1,0)} = \mathbb{C} \text{v with } W_1 \text{v} = -t \text{v}, \quad W_2 \text{v} = -\text{v}, \quad T_1 \text{v} = -t^{-1/2} \text{v}.
\]

The irreducibility of \( L^+_{(0,1)} \) and \( L^-_{(0,1)} \) is not immediate. We will show that \( M = L^+_{(0,1)} \) is irreducible; the irreducibility of \( L^-_{(0,1)} \) is proved analogously.

The generalized weight space decomposition of \( M = L^+_{(0,1)} \) is

\[
M = M^{\text{gen}}_{(-1,0)} \oplus M^{\text{gen}}_{(1,0)} \oplus M^{\text{gen}}_{(0,1)} \quad \text{with } \dim(M^{\text{gen}}_{(-1,0)}) = \dim(M^{\text{gen}}_{(1,0)}) = 1,
\]

\[
\dim(M^{\text{gen}}_{(0,1)}) = 2.
\]

The element \( W_1^{-1} \) acts on \( M^{\text{gen}}_{(-1,0)} \) with eigenvalue \(-t\). Since the parameters are generic (see (4.1)), \(-t \notin \{-t^{\pm r_1}, -t^{\pm r_2}\} \) and thus, by (2.42), \( \tau_0 \) has no kernel. Thus the intertwiner \( \tau_0 : M^{\text{gen}}_{(-1,0)} \to M^{\text{gen}}_{(1,0)} \) is invertible. Since \( M^{\text{gen}}_{(0,1)} \) is irreducible as a \( H_{(0)} \)-module, we have either \( N = M^{\text{gen}}_{(0,1)} \) is an \( H_2 \)-submodule or \( M \) is irreducible.

For the purpose of deriving a contradiction, assume that \( N = M^{\text{gen}}_{(0,1)} \) is an \( H_2 \)-submodule of \( M \). The space \( N \) has a basis

\[
\{n_\gamma, T_0 n_\gamma\} \quad \text{with } W_1 n_\gamma = -n_\gamma, \quad \text{and } W_2 n_\gamma = -t n_\gamma.
\]

By (C2) and (B3),

\[
W_1 W_2^{-1} T_0 n_\gamma = T_0 W_1^{-1} W_2^{-1} n_\gamma + ((t_{1/2}^{0} - t_{0}^{-1/2}) + (t_{k}^{1/2} - t_{k}^{-1/2}) W_1^{-1}) W_1 - W_2^{-1} W_{2}^{-1} n_\gamma
\]

\[
= T_0 t^{-1} n_\gamma + ((t_{1/2}^{0} - t_{0}^{-1/2}) + (t_{k}^{1/2} - t_{k}^{-1/2})(-1)) t^{-1} n_\gamma,
\]

and the action of \( W_1 W_2^{-1} \) on the basis \( \{n_\gamma, T_0 n_\gamma\} \) is given by the matrix

\[
\rho(W_1 W_2^{-1}) = \begin{pmatrix} t^{-1} & ((t_{1/2}^{0} - t_{0}^{-1/2}) + (t_{k}^{1/2} - t_{k}^{-1/2})(-1)) t^{-1} \\ 0 & t^{-1} \end{pmatrix}.
\]

Thus

\[
\rho(1 - W_1 W_2^{-1}) = \begin{pmatrix} 1 - t^{-1} & -((t_{1/2}^{0} - t_{0}^{-1/2}) + (t_{k}^{1/2} - t_{k}^{-1/2})(-1)) t^{-1} \\ 0 & 1 - t^{-1} \end{pmatrix}
\]

\[
= (1 - t^{-1}) \begin{pmatrix} 1 & -((t_{1/2}^{0} - t_{0}^{-1/2}) + (t_{k}^{1/2} - t_{k}^{-1/2})(-1)) t^{-1} \\ 0 & 1 - t^{-1} \end{pmatrix}
\]

\( = (1 - t^{-1}) \begin{pmatrix} 1 & 0 \\ 0 & 1 - t^{-1} \end{pmatrix}
\]

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and \[
\rho(1 - W_1W_2^{-1})^{-1} = \frac{1}{(1-t^{-1})} \begin{pmatrix}
1 & \left((t_0^{1/2} - t_0^{-1/2}) + (t_k^{1/2} - t_k^{-1/2})(-1)t^{-1}\right)
0 & 1 - t^{-1}
\end{pmatrix}.
\]

If \( N \) is a submodule of \( M \) then \( 0 = \tau_1 = T_1 - \frac{t_0^{1/2} + t_k^{-1/2}}{1-W_1W_2} \) (see (2.33) for the formula for \( \tau_1 \)). Thus
\[
\rho(T_1) = t_1^\frac{1}{2} \begin{pmatrix}
1 & \left((t_0^{1/2} - t_0^{-1/2}) + (t_k^{1/2} - t_k^{-1/2})(-1)t^{-1}\right)
0 & 1 - t^{-1}
\end{pmatrix}.
\]

Since \((T_1 - t_1^\frac{1}{2})(T_1 + t_1^{-\frac{1}{2}}) = 0\) the Jordan blocks of \( \rho(T_1) \) are of size one, forcing
\[
0 = (t_0^{1/2} - t_0^{-1/2}) - (t_k^{1/2} - t_k^{-1/2}) = t_0^{-\frac{1}{2}}(t_0^{1/2} + t_k^{-1/2})(t_0^{1/2} - t_k^{-1/2}).
\]

This is a quadratic equation in \( t_0^{\frac{1}{2}} \) with two solutions, \( t_0^\frac{1}{2} = t_k^\frac{1}{2} \) and \( t_0^{-\frac{1}{2}} = -t_k^{-\frac{1}{2}} \). This is a contradiction since, by the generic condition on parameters in [4,1], \(-t^{-r_1} = -t_0^\frac{1}{2}t_k^{-\frac{1}{2}} \neq -1\) and \(-t^{r_2} = t_0^\frac{1}{2}t_k^\frac{1}{2} \neq -1\). Thus \( N \) is not a submodule of \( M \), and so \( M \) is irreducible.

5 Representations of \( B_k^{\text{ext}} \) in tensor space

In this section we give a Schur-Weyl duality approach to the representations of the two boundary Hecke algebras \( H_k^{\text{ext}} \). More generally, in Theorem 5.1 we show that, for a quantum group or quasitriangular Hopf algebra \( U_q\mathfrak{g} \) and three \( U_q\mathfrak{g} \)-modules \( M, N \) and \( V \), there is an action of the two boundary braid group \( B_k^{\text{ext}} \) on tensor space \( M \otimes N \otimes V^{\otimes k} \) that commutes with the \( U_q\mathfrak{g} \)-action. This means that there is a weak Schur-Weyl duality pairing between \( U_q\mathfrak{g} \)-modules and \( B_k^{\text{ext}} \)-modules, so that if \( M \otimes N \otimes V^{\otimes k} \) is completely reducible as a \( U_q\mathfrak{g} \)-module then
\[
(M \otimes N \otimes V^{\otimes k}) \cong \bigoplus_{\lambda} L(\lambda) \otimes B_k^{\lambda} \quad \text{as } (U_q\mathfrak{g}, B_k^{\text{ext}})\text{-modules},
\]
where \( L(\lambda) \) are irreducible \( U_q\mathfrak{g} \)-modules and \( B_k^{\lambda} \) are \( B_k^{\text{ext}} \)-modules. In Section 5.4 we will explain that when \( \mathfrak{g} = \mathfrak{g}\mathfrak{l}_n \) and \( M \) and \( N \) and \( V \) are appropriately chosen, the \( B_k^{\text{ext}} \)-action provides an action of the two boundary Hecke algebra \( H_k^{\text{ext}} \) (where the parameters depend on the choice of \( M \) and \( N \)). Our main theorem, Theorem 5.5, proves that the \( H_k^{\text{ext}} \)-modules \( B_k^{\lambda} \) that appear in tensor space \( M \otimes N \otimes V^{\otimes k} \) are irreducible, and identifies them in terms of the classification of irreducible calibrated \( H_k^{\text{ext}} \)-modules which is given in Theorem 3.3.

5.1 Quantum groups and \( R \)-matrices

Let \( \mathfrak{g} \) be a finite-dimensional complex Lie algebra with a symmetric nondegenerate ad-invariant bilinear form, and let \( U_q\mathfrak{g} \) be the Drinfel’d-Jimbo quantum group corresponding to \( \mathfrak{g} \). The quantum group \( U_q\mathfrak{g} \) is a ribbon Hopf algebra with invertible \( R \)-matrix
\[
R = \sum_{R} R_1 \otimes R_2 \quad \text{in } U_q\mathfrak{g} \otimes U_q\mathfrak{g}, \quad \text{and ribbon element } v = q^{-2\rho u},
\]
where \( u = \sum_R S(R_2)R_1 \) and \( \rho \) is the staircase weight (see [LR, Corollary (2.15)]). For \( U_q\mathfrak{g} \)-modules \( M \) and \( N \), the map

\[
\tilde{R}_{MN}: \quad N \otimes M \rightarrow N \otimes M
\]

\[
\quad n \otimes m \rightarrow \sum_R R_2 m \otimes R_1 n
\]

is a \( U_q\mathfrak{g} \)-module isomorphism. The quasitriangularity of a ribbon Hopf algebra provides the relations (see, for example, [OR, (2.9), (2.10), and (2.12)],

\[
(\varphi \otimes \text{id}_N)\tilde{R}_{MN} = \tilde{R}_{MN}(\text{id}_N \otimes \varphi),
\]

\[
(\tilde{R}_{MN} \otimes \text{id}_V)(\text{id}_N \otimes \tilde{R}_{MV})(\tilde{R}_{NV} \otimes \text{id}_M) = (\text{id}_M \otimes \tilde{R}_{NV})(\tilde{R}_{MV} \otimes \text{id}_N)(\text{id}_V \otimes \tilde{R}_{MN}),
\]

\[
(\tilde{R}_{M \otimes N,V}) = (\text{id}_M \otimes \tilde{R}_{NV})(\tilde{R}_{MV} \otimes \text{id}_N)
\]

\[
(\tilde{R}_{M \otimes N,V}) = (\text{id}_M \otimes \tilde{R}_{NV})(\tilde{R}_{MV} \otimes \text{id}_N). \quad (5.2)
\]

For a \( U_q\mathfrak{g} \)-module \( M \) define

\[
C_M: \quad M \rightarrow M \quad m \mapsto v_m
\]

so that

\[
C_M \otimes N = (\tilde{R}_{MN} \tilde{R}_{NM})^{-1}(C_M \otimes C_N) \quad (5.3)
\]

(see [Dr, Prop. 3.2]). Let \( L(\lambda) \) denote the simple \( U_q\mathfrak{g} \)-module generated by a highest weight vector \( v_\lambda^+ \) of weight \( \lambda \). Then

\[
C_{L(\lambda)} = q^{-(\lambda,\lambda+2\rho)}\text{id}_{L(\lambda)} \quad (5.4)
\]
Proof. Using the notation \( \Phi(X, Y, Z) \), it follows that if \( M = L(\mu) \) and \( N = L(\nu) \) are finite-dimensional irreducible \( U_q \mathfrak{g} \)-modules of highest weights \( \mu \) and \( \nu \) respectively, then \( \tilde{R}_{MN} \tilde{R}_{NM} \) acts on the \( L(\lambda) \)-isotypic component \( L(\lambda)^{\oplus c_{\lambda}} \) of the decomposition

\[
L(\mu) \otimes L(\nu) = \bigoplus_{\lambda} L(\lambda)^{\oplus c_{\lambda}}
\]

by the scalar

\[
q^{(\lambda, \lambda + 2\rho) - (\mu, \mu + 2\rho) - (\nu, \nu + 2\rho)}.
\]

Proposition 5.1. Let \( \mathfrak{g} \) be a finite-dimensional complex Lie algebra with a symmetric nondegenerate ad-invariant bilinear form, let \( U_q \mathfrak{g} \) be the corresponding Drinfeld-Jimbo quantum group, and let \( Z = Z(U_q) \) be the center of \( U \). Let \( M, N, \) and \( V \) be \( U_q \mathfrak{g} \)-modules. Then \( M \otimes N \otimes V^\otimes k \) is a \( \mathcal{ZB}^{\text{ext}}_k \)-module with action given by

\[
\Phi: \mathcal{ZB}^{\text{ext}}_k \longrightarrow \text{End}_{U_q \mathfrak{g}}(M \otimes N \otimes V^\otimes k)
\]

\[
\begin{align*}
T_i & \mapsto \tilde{R}_i, & \text{for } i = 1, \ldots, k - 1, \\
X_1 & \mapsto \tilde{R}_M^2, \\
Y_1 & \mapsto \tilde{R}_N^2, \\
Z_1 & \mapsto \tilde{R}_0, \\
P & \mapsto (\tilde{R}_{MN} \tilde{R}_{NM}) \otimes \text{id}_V^{\otimes (k-1)}.
\end{align*}
\]

(5.6)

where

\[
\tilde{R}_0^2 = (\tilde{R}_{(M \otimes N)V} \tilde{R}_{V(M \otimes N)}) \otimes \text{id}_V^{\otimes (k-1)}, \quad \tilde{R}_i = \text{id}_M \otimes \text{id}_V^{\otimes (i-1)} \otimes \tilde{R}_{VV} \otimes \text{id}_V^{\otimes (k-i-1)}
\]

for \( i = 1, \ldots, k - 1 \),

\[
\tilde{R}_M^2 = ((\text{id}_M \otimes \tilde{R}_{NV})(\text{id}_M \otimes \tilde{R}_{NM}) \otimes \text{id}_N)(\text{id}_M \otimes \tilde{R}_{VV}^{-1})) \otimes \text{id}_V^{\otimes k-1}, \quad \text{and}
\]

\[
\tilde{R}_N^2 = \text{id}_M \otimes (\tilde{R}_{NV} \tilde{R}_{VN}) \otimes \text{id}_V^{\otimes (k-1)},
\]

with \( \tilde{R}_{MV} \) as in (5.1). Moreover, this \( \mathcal{ZB}^{\text{ext}}_k \) action commutes with the \( U_q \mathfrak{g} \)-action on \( M \otimes N \otimes V^\otimes k \).

Proof. This proof follows the proof of [LR Prop. 3.1], checking that the images of the generators \( T_i, X_1, Y_1, \) and \( Z_1 \) under the map \( \Phi \) satisfy the relations of presentation (a) of the two boundary braid group in Theorem 2.1 as well as relations (2.15) and (2.16) for the extended two boundary braid group. For \( i \in \{1, \ldots, k-2\} \),

\[
\Phi(T_i) \Phi(T_{i+1}) \Phi(T_i) = \tilde{R}_i \tilde{R}_{i+1} \tilde{R}_i = \quad \quad \quad \quad = \quad \quad = \quad = \quad \Phi(T_{i+1}) \Phi(T_i) \Phi(T_{i+1}).
\]

Using the notation \( \tilde{R}_{M \otimes N} \) for the endomorphism \( \tilde{R}_0 \), we have that, for \( L = M, N, \) or \( M \otimes N \),

\[
\tilde{R}_L^2 \tilde{R}_1 \tilde{R}_L^2 \tilde{R}_1 = \quad \quad \quad \quad = \quad \quad = \quad = \quad = \quad \tilde{R}_1 \tilde{R}_L^2 \tilde{R}_1 \tilde{R}_L^2,
\]

\[
33
\]
which establishes
\[ \Phi(A)\Phi(T_1)\Phi(A)\Phi(T_1) = \Phi(T_1)\Phi(A)\Phi(T_1)\Phi(A) \quad \text{for } A = X_1, Y_1, \text{ and } Z_1, \text{ respectively.} \]

The formula
\[ \Phi(Z_1) = R^2_0 = R^2_M R^2_N = \Phi(X_1)\Phi(Y_1) \]
is a consequence of the third set of relations (cabling relations) in (5.2). Finally, the relations
\[ \Phi(P)\Phi(Y_1)\Phi(P) = \Phi(Z^{-1}_1)\Phi(Y_1)\Phi(Z_1) \quad \text{and} \quad \Phi(P)\Phi(X_1)\Phi(P) = \Phi(Z^{-1}_1)\Phi(X_1)\Phi(Z_1) \]
follow from the first and second sets of relations for \( R \)-matrices in (5.2) by the same braid computation by which the identities (2.13) were derived. The remainder of the relations (commuting generators) follow directly from the definitions of \( \Phi(T_i) \), \( \Phi(X_1) \), \( \Phi(Y_1) \), \( \Phi(Z_1) \), and \( \Phi(P) \).

\[ \square \]

### 5.2 The \( B^\text{ext}_k \)-modules \( B^\lambda_k \)

Assume that \( M, N, \text{ and } V \) are finite-dimensional \( U_q\mathfrak{g} \)-modules and that \( \omega \) is the highest weight of \( V \) so that
\[ V = L(\omega) \quad \text{is irreducible of highest weight } \omega. \]

Let \( \mathcal{P}^{(j)} \) be an index set for the irreducible \( U_q\mathfrak{g} \)-modules that appear in \( M \otimes N \otimes V^{\otimes j} \) and let \( \mathcal{P}^{(-1)} \) be an index set for the irreducible \( U_q\mathfrak{g} \)-modules in \( M \). The Bratteli diagram for the sequence of \( U_q\mathfrak{g} \)-modules
\[ M, \quad M \otimes N, \quad M \otimes N \otimes V, \quad M \otimes N \otimes V \otimes V, \quad \ldots \quad (5.7) \]
is the graph with

- vertices on level \( j \) labeled by \( \mu \in \mathcal{P}^{(j)} \), for \( j \in \mathbb{Z}_{\geq -1} \),
- \( m_{\mu\lambda} \) edges \( \mu \to \lambda \) for \( \mu \in \mathcal{P}^{(j)} \) and \( \lambda \in \mathcal{P}^{(j+1)} \), and where \( L(\mu) \otimes V \cong \bigoplus_{\lambda \in \mathcal{P}^{(j+1)}} L(\lambda)^{\oplus m_{\mu\lambda}} \),
- each edge \( \mu \to \lambda \) labeled with \( \frac{1}{2}(\langle \lambda, \lambda + 2\rho \rangle - \langle \omega, \omega + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle) \).

A specific example in the case where \( \mathfrak{g} = \mathfrak{gl}_n \) is given in Figure 3.

If \( M \) and \( N \) are finite-dimensional then \( M \otimes N \otimes V^{\otimes k} \) is completely decomposable as a \( U_q\mathfrak{g} \)-module. If \( B^\lambda_k \) is the space of highest weight vectors of weight \( \lambda \) in \( M \otimes N \otimes V^{\otimes k} \), then
\[ M \otimes N \otimes V^{\otimes k} \cong \bigoplus_{\lambda \in \mathcal{P}^{(k)}} L(\lambda) \otimes B^\lambda_k, \quad \text{as } (U_q\mathfrak{g}, B^\text{ext}_k)-\text{bimodules}. \quad (5.8) \]

The \( B^\text{ext}_k \)-modules \( B^\lambda_k \) are not necessarily irreducible and not necessarily nonisomorphic, though they will be in the (mostly rare but very important) settings where \( \Phi(\mathbb{C}B^\text{ext}_k) = \text{End}_{U_q\mathfrak{g}}(M \otimes N \otimes V^{\otimes k}) \).

Recall from (2.9) that
\[ Z_i = T_{i-1} \cdots T_1 Z_1 T_1 \cdots T_{i-1} \quad \text{for } i = 1, \ldots, k. \]

The following proposition shows that, as operators on \( B^\lambda_k \), the \( Z_i \) are simultaneously diagonalizable and have eigenvalues determined by the edges on the Bratteli diagram. The proof follows the same schematic that is used, for example, in the proof of [OR, Proposition 3.2].
Proposition 5.2. Assume $M$, $N$, and $V$ are finite-dimensional $U_q\mathfrak{g}$ modules with $V$ irreducible. For $\lambda \in \mathcal{P}(k)$, let $B_k^\lambda$ be the $B_k^\text{ext}$-module in (5.8) and let

$$\mathcal{T}_k^\lambda = \{ \text{paths } S = (S^{(-1)} \xrightarrow{e_0} S^{(0)} \xrightarrow{e_1} \cdots \xrightarrow{e_k} S^{(k)} = \lambda) \text{ in the Bratteli diagram} \}.$$Then

$$B_k^\lambda \text{ has a basis } \{ v_S \mid S \in \mathcal{T}_k \}$$of simultaneous eigenvectors for the action of $P, Z_1, \ldots, Z_k$, with

$$Pv_S = q^{2e_0}v_S \quad \text{and} \quad Z_i v_S = q^{2e_i}v_S, \quad \text{for } i = 1, \ldots, k,$$so that the eigenvalues of $P$ and $Z_1, \ldots, Z_k$ on $v_S$ are determined by the labels on the edges of the path $S$.

Proof. The basis $\{ v_S \mid S \in \mathcal{T}_k^\lambda \}$ is constructed inductively. For the initial case, choose any basis $\hat{B}_{k-1}$ of the highest weight vectors in $M$, and let $\hat{B}_{k-1}^\nu$ be the set of basis elements in $\hat{B}_{k-1}$ of weight $\nu$. For the inductive step, assume that $\hat{B}_{k-1}^\nu = \{ v_T \mid T \in \mathcal{T}_{k-1}^\nu \}$ has been constructed so that

$$M \otimes N \otimes V^{\otimes(k-1)} = \bigoplus_{\mu \in \mathcal{P}(k-1)} L(\mu) \otimes \hat{B}_{k-1}^\mu = \bigoplus_{\mu \in \mathcal{P}(k-1)} L(\mu) \otimes \left( \sum_{T \in \mathcal{T}_{k-1}^\mu} \mathbb{C}v_T \right),$$The set $\hat{B}_{k-1}^\mu = \{ v_T \mid T \in \mathcal{T}_{k-1}^\mu \}$ is a basis of the vector space of highest weight vectors of weight $\mu$ in $M \otimes N \otimes V^{\otimes(k-1)}$ that is indexed by the paths $T = (T^{(-1)} \rightarrow \cdots \rightarrow T^{(k-1)} = \mu)$ of length $k$ in the Bratteli diagram that end at $\mu$. In this form $L(\mu) \otimes \mathbb{C}v_T$ denotes the irreducible $U_q\mathfrak{g}$-submodule of $M \otimes N \otimes V^{\otimes(k-1)}$ with highest weight vector $v_T$ of weight $\mu$.

Then, for each $T = (T^{(-1)} \rightarrow \cdots \rightarrow T^{(k-1)} = \mu)$ in $\mathcal{T}_{k-1}^\mu$, choose a basis

$$\hat{B}_k^{T \rightarrow \lambda} = \{ v_S \mid S = (T^{(-1)} \rightarrow \cdots \rightarrow T^{(k-1)} = \mu \rightarrow \lambda) \}$$of highest weight vectors in the submodule of $M \otimes N \otimes V^{\otimes k}$ given by

$$(L(\mu) \otimes \mathbb{C}v_T) \otimes V = L(\mu) \otimes V \otimes \mathbb{C}v_T = \sum_{\mu \rightarrow \lambda} L(\lambda) \otimes \mathbb{C}v_S.$$The basis $\hat{B}_k^{T \rightarrow \lambda}$ is indexed by the edges in the Bratteli diagram from $\mu$ to a partition $\lambda$ on level $k$. Then

$$\hat{B}_k^{\lambda} = \bigcup_{\mu} \bigcup_{T \in \mathcal{T}_{k-1}^\mu} \mathcal{T}_k^{T \rightarrow \lambda} \quad \text{is a basis of } B_k^\lambda.$$The central element $q^{-2p}u$ in $U_q\mathfrak{g}$ acts on the submodule $L(\mu) \otimes \mathbb{C}v_T$ of $M \otimes N \otimes V^{\otimes(k-1)}$ by the constant $q^{-\langle \mu, \mu+2p \rangle}$. From (5.2), (5.3), and (5.4) it follows that $Z_i$ acts on $M \otimes N \otimes V^{\otimes k}$ by

$$\Phi(Z_i) = \hat{R}_{i-1} \cdots \hat{R}_1 \hat{R}_0 \hat{R}_1 \cdots \hat{R}_{i-1} = \hat{R}_{M \otimes N \otimes V^{\otimes(i-1)}, V} \hat{R}_{V, M \otimes N \otimes V^{\otimes(i-1)}} \otimes \text{id}_V^{\otimes(k-i)}$$

$$= (C_{M \otimes N \otimes V^{\otimes(i-1)}} \otimes C_V) \hat{C}^{-1}_{M \otimes N \otimes V^{\otimes i}} \otimes \text{id}_V^{\otimes(k-i)}$$

$$= \sum_{\lambda, \mu, \nu} q^{\langle \lambda, \lambda + 2p \rangle - \langle \mu, \mu + 2p \rangle - \langle \nu, \omega + 2p \rangle} t_{\mu \nu}^\lambda \otimes \text{id}_V^{\otimes(k-i)}, \quad (5.9)$$

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where \( \pi_{\mu\nu} : M \otimes N \otimes \text{id}_V^\otimes \to M \otimes N \otimes \text{id}_V^\otimes \) is the projection onto the \( L(\lambda) \) isotypic component of \( (L(\mu) \otimes B_{\mu-1}^\lambda) \otimes V \). Thus \( Z_i \) acts diagonally on the basis \( \tilde{B}_k^\lambda \) and, by the definition of the labels of edges in the Bratteli diagram in (5.7), the eigenvalues of \( Z_i v_S = q^{2e_i v_S} \) where \( e_i \) is the label on the edge \( S(i) \to S(i+1) \) in the Bratteli diagram.

5.3 Some tensor products for \( g = \mathfrak{gl}_n \)

The finite-dimensional irreducible polynomial representations \( L(\lambda) \) of \( U_q \mathfrak{gl}_n \) are indexed by elements of \( P^+_{\text{poly}} = \{ \lambda = \lambda_1 \varepsilon_1 + \cdots + \lambda_n \varepsilon_n, \ | \lambda_i \in \mathbb{Z}, \lambda_1 \geq \cdots \geq \lambda_n \geq 0 \} \).

Use

\[ \rho = (n-1)\varepsilon_1 + (n-2)\varepsilon_2 + \cdots + \varepsilon_{n-1} = \sum_{i=1}^{n} (n-i)\varepsilon_i, \quad (5.10) \]

as in [Mac1 I (1.13)]. Identify each element \( \lambda = \lambda_1 \varepsilon_1 + \cdots + \lambda_n \varepsilon_n \) in \( P^+_{\text{poly}} \) with the corresponding partition having \( \lambda_i \) boxes in row \( i \) so that, for example,

\[ \lambda = 3\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3 = \begin{array}{|c|c|c|} \hline & & \\
\hline & & \\
\hline \end{array} \]

The content of the box in row \( i \) and column \( j \) of a partition \( \lambda \) is

\[ c(\text{box}) = j - i = (\text{diagonal number of box}), \quad (5.11) \]

where the diagonals are numbered by the elements of \( \mathbb{Z} \) from southwest to northeast, with the northwest corner box of a partition being in diagonal 0.

The representation \( L(\varepsilon_1) = L(\square) \) is the standard \( n \)-dimensional representation of \( U_q \mathfrak{gl}_n \). When \( \nu = \varepsilon_1 \), the decomposition in (5.5) is given by

\[ L(\mu) \otimes L(\square) \cong \bigoplus_{\lambda \in \mu^+} L(\lambda), \quad (5.12) \]

where \( \mu^+ \) is the set of partitions obtained by adding a box to \( \mu \). If \( \lambda \in \mu^+ \) and \( \lambda/\mu \) is the box added to \( \mu \) to obtain \( \lambda \), then the action in (5.5) is given by

\[ \langle \lambda, \lambda + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle - \langle \varepsilon_1, \varepsilon_1 + 2\rho \rangle = (\mu_1 + 2\rho_1 - 1) - 2\rho_1 = 2\mu_1 + 2(n-i) - 2(n-1) = 2\mu_1 - 2i + 2 = 2c(\lambda/\mu) \]

(see [Mac1 I (5.16) and (8.4)]). Since \( \langle \varepsilon_1, \varepsilon_1 + 2\rho \rangle = 2(n-1) + 1 = 2n - 1 \), it follows by induction on the number of boxes in a partition \( \lambda \) that

\[ \langle \lambda, \lambda + 2\rho \rangle = (2n-1)|\lambda| + \sum_{\text{box} \in \lambda} 2c(\text{box}). \quad (5.14) \]

For \( \mu, \nu \in P^+_{\text{poly}} \), the decomposition of the tensor product \( L(\mu) \otimes L(\nu) \) can be calculated using the Littlewood-Richardson rule (see [Mac1 Ch. I (9.2)]). When \( \mu \) and \( \nu \) are rectangles the
decomposition is multiplicity free by the following theorem. In equation (5.15), \(A\) consists of the boxes that are in the union of the rectangles \((a^c)\) and \((b^d)\) (placed with northwest corner at \((1, 1)\)), and the dashed rectangular regions are the \(\min(a, b) \times d\) rectangle \(B\) with northwest corner box at \((\max(a, b) + 1, 1)\), and the \(d \times \min(a, b)\) rectangle \(B'\) with northwest corner at \((1, c + 1)\).

**Proposition 5.3.** (See [St, Lem. 3.3], [Ok, Thm 2.4]) Let \(a, b, c, d \in \mathbb{Z}_{\geq 0}\) such that \(c \geq d\). For \(\mu \subseteq (\min(a, b)^d)\) let

\[
if a \geq b:
\]

\[
\tilde{\mu} = \begin{array}{c}
\mu^c \\
A \\
\mu \\
B \\
B'
\end{array}
\]

\[
if a \leq b:
\]

\[
\tilde{\mu} = \begin{array}{c}
\mu^c \\
A \\
\mu \\
B \\
B'
\end{array}
\]

(5.15)

so that \(\mu^c\) is the 180° rotation of the complement of \(\mu\) in \(a \min(a, b) \times d\) rectangle. Denote the rectangular partition with \(c\) rows of length \(a\) by \((a^c)\). Then

\[
L((a^c)) \otimes L((b^d)) \cong \bigoplus_{\mu \subseteq (\min(a, b)^d)} L(\tilde{\mu}) \cong \bigoplus_{\nu \in \mathcal{P}(0)} L(\nu),
\]

(5.16)

where \(\mathcal{P}(0) = \{\tilde{\mu} \mid \mu \subseteq ((\min(a, b)^d)\}).

For an example of the decomposition in (5.16), see Figure 3, where the decomposition of \(L(a^c) \otimes L(2^2)\) for \(a, c \geq 2\) is indicated in level 0 of the Bratteli diagram (see the description following (5.23) for explanation of the Bratteli diagram).

The value in (5.5) for the product in (5.16) is given by using (5.14) to compute

\[
\langle \tilde{\mu}, \tilde{\mu} + 2\rho \rangle - \langle (a^c), (a^c) + 2\rho \rangle - \langle (b^d), (b^d) + 2\rho \rangle
\]

\[
= (2n - 1)(|\tilde{\mu}| - |(a^c)| - |(b^d)|) + \left(\sum_{\text{box} \in \tilde{\mu}} 2c(\text{box})\right) - \sum_{\text{box} \in (a^c)} 2c(\text{box}) - \sum_{\text{box} \in (b^d)} 2c(\text{box})
\]

\[
= 0 + \sum_{\text{box} \in \tilde{\mu}} 2c(\text{box}) - ac(a - c) - bd(b - d).
\]

(5.17)

### 5.4 Irreducible \(H_k^{\text{ext}}\)-modules in \(M \otimes N \otimes V^{\otimes k}\)

In this subsection we provide, for \(g = \mathfrak{gl}_n\), specific highest weight modules \(M, N,\) and \(V\) such that the \(B_k^{\text{ext}}\)-action factors through the extended two boundary Hecke algebra \(H_k^{\text{ext}}\). In these cases the \(B_k^{\text{ext}}\)-modules \(B_k^{\text{ext}}\) in (5.5) are calibrated \(H_k^{\text{ext}}\)-modules. Theorem 5.5 identifies the \(B_k^{\text{ext}}\) for these cases explicitly in terms of the indexings of calibrated \(H_k^{\text{ext}}\)-modules given in Theorem 3.3 and Proposition 3.1.
Recall that, as defined in Section 2.2, the extended two boundary Hecke algebra $H_k^{\text{ext}}$ is the quotient of the group algebra of the extended two boundary braid group $\mathbb{C}B_k^{\text{ext}}$ by the relations

$$(X_i - a_i)(X_i - a_2) = 0, \quad (Y_i - b_1)(Y_i - b_2) = 0, \quad \text{and} \quad (T_i - t_i^2)(T_i + t_i^2) = 0, \quad (5.18)$$
i = 1, \ldots, k - 1, \text{for fixed } a_1, a_2, b_1, b_2, t_i^\frac{1}{2} \in \mathbb{C}^\times.

**Theorem 5.4.** If $\mathfrak{g} = \mathfrak{gl}_n, M = L((a^c)), N = L((b^d)), \text{and } V = L(\square),$

$$a_1 = q^{2a}, \quad a_2 = q^{-2c}, \quad b_1 = q^{2b}, \quad b_2 = q^{-2d}, \quad \text{and} \quad t_i^\frac{1}{2} = q, \quad (5.19)$$

then the map $\Phi$ from Proposition 5.1 gives an action of $H_k^{\text{ext}}$ on $M \otimes N \otimes V^{\otimes k}$ commuting with that of $U_q\mathfrak{gl}_n$.

**Proof.** The module $M \otimes V$ decomposes as

$$M \otimes V = L \left( \begin{array}{c} a \\ c \\
\hline b \\
\end{array} \right) \oplus L \left( \begin{array}{c} a \\ c \\
\hline d \\
\end{array} \right). \quad (5.20)$$

By (5.5) and (5.13), $\check{R}_{MV} R_{VM}$ acts on the first summand by the constant $q^{2a}$ and on the second summand by the constant $q^{-2c}$. So

$$(\Phi(X_1) - q^{2a})(\Phi(X_1) - q^{-2c}) = 0; \quad \text{similarly} \quad (\Phi(Y_1) - q^{2b})(\Phi(Y_1) - q^{-2d}) = 0$$

by replacing $(a^c)$ with $(b^d)$. The relation

$$(\Phi(T_i) - q)(\Phi(T_i) + q^{-1}) = 0$$

follows similarly by considering the tensor product $V \otimes V = L(\square) \otimes L(\square)$. \hfill \square

From (2.17), (5.19), and (3.5),

$$a_1 = q^{2a}, \quad a_2 = q^{-2c}, \quad b_1 = q^{2b}, \quad b_2 = q^{-2d}, \quad t_i^\frac{1}{2} = q, \quad (5.21)$$

Using these conversions, the genericity conditions in (4.1) become requirements that $q$ is not a root of unity and

$$-q^{(a+c)-(b+d)}, -q^{a+c+b+d} \notin \{1, -1, q^{\pm 1}, -q^{\pm 1}, q^{\pm 2}, -q^{\pm 2}\} \quad \text{and} \quad -q^{(a+c)-(b+d)} \neq -q^{(a+c+b+d)}.$$

In the context of Theorem 5.4 these genericity conditions are

$$q \text{ is not a root of unity, } a, b, c, d \in \mathbb{Z}_{>0} \text{ and } (a + c) - (b + d) \notin \{0, \pm 1, \pm 2\}. \quad (5.22)$$

In the setting of Theorem 5.4 equation (5.8) provides $H_k^{\text{ext}}$-modules $B_\lambda^k$ with

$$M \otimes N \otimes V^{\otimes k} \cong \bigoplus_{\lambda \in \mathfrak{p}(k)} L(\lambda) \otimes B_\lambda^k, \quad \text{as } (U_q\mathfrak{g}, H_k^{\text{ext}})-\text{bimodules.} \quad (5.23)$$
Theorem 5.5 below will accomplish our primary goal for this paper by identifying the module $B_k$ explicitly as a calibrated $H_k^{(c,d,j)}$ as constructed in Theorem 3.3. The results of (5.12), (5.13), and Proposition 5.3 show that the Bratteli diagram of (5.7) has $P^{(-)} = \{(a^c)\}$, $P^{(0)} = \{\mu \mid \mu \subseteq (\min(a,b))^d\}$ as in Proposition 5.3 and, for $j \in \mathbb{Z}_{\geq 0}$,

$$P^{(j)} = \{\text{partitions obtained by adding } j \text{ boxes to a partition in } P^{(0)}\}.$$ 

By (5.17), if $\mu \in P^{(0)}$ then there is an edge

$$\begin{align*}
(a^c) e_0(\mu) \rightarrow \mu &\text{ with label } e_0(\mu) = -\frac{ac}{2} (a - c) - \frac{bd}{2} (b - d) + \sum_{\text{box } \in \mu} c(\text{box}). \quad (5.24)
\end{align*}$$

For $j \geq 0$, the edges $\mu \rightarrow \lambda$ from level $j$ to level $j + 1$ correspond to adding a single box to $\mu$ to get $\lambda$, and are labeled by $c(\lambda/\mu)$, the content of the box $\lambda/\mu$:

$$\mu \xrightarrow{c(\lambda/\mu)} \lambda \quad \text{for edges from level } j \text{ to level } j + 1. \quad (5.25)$$

The case when $M = L(a^c)$ and $N = L(2^2)$ with $a,c > 2$ is illustrated in Figure 3. Let $\lambda \in \mathcal{P}(\lambda)$. Define

$$c_0 = -\frac{1}{2} (k(a - c + b - d) + ac(a - c) + bd(b - d)) + \sum_{\text{box } \in \lambda} c(\text{box}) \quad \text{and} \quad z = (-1)^k q^{2 \epsilon_0}. \quad (5.26)$$

Using notation as in (5.15), let

$$\mu^c = \lambda \cap B' \quad \text{and let } S^{(0)}_{\text{max}} \text{ be the corresponding } \hat{\mu}. \quad (5.27)$$

Define the shifted content of a box by

$$\bar{c}(\text{box}) = c(\text{box}) - \frac{1}{2} (a - c + b - d), \quad \text{and let } c = (c_1, \ldots, c_k) \text{ with } 0 \leq c_1 \leq c_2 \leq \cdots \leq c_k \quad (5.28)$$

be the sequence of absolute values of the shifted contents of the boxes in $\lambda/S^{(0)}_{\text{max}}$ arranged in increasing order. Index the boxes of $\lambda/S^{(0)}_{\text{max}}$ with $1, 2, \ldots, k$ so that

(a) if $i < j$ then $|\bar{c}(\text{box}_i)| \leq |\bar{c}(\text{box}_j)|$,
(b) if $i < j$ and $\bar{c}(\text{box}_i) = \bar{c}(\text{box}_j) < 0$ then box$_i$ is SE of box$_j$,
(c) if $i < j$ and $\bar{c}(\text{box}_i) = \bar{c}(\text{box}_j) \geq 0$ then box$_i$ is NW of box$_j$,
(d) if $i < j$ and $\bar{c}(\text{box}_i) = -\bar{c}(\text{box}_j)$, then $\bar{c}(\text{box}_i) \leq 0 \leq \bar{c}(\text{box}_j)$,

and define

$$J = \{\varepsilon_i \mid \bar{c}(\text{box}_i) \in \{-r_1, -r_2\}\} \quad \text{ where } P(c) \text{ is as defined in (3.7). See Examples 3 and 4 following the proof of Theorem 5.5.}$$
Figure 3: Levels $-1$, $0$, and $1$ of a Bratteli diagram encoding isotypic components of $M \otimes N \otimes V$ where $a, c > 2$ and $b = d = 2$. The edges from level $-1$ to level $0$ are labeled by $e_0(T^{(0)})$ as in (5.17); the edges from level $0$ to $1$ are labeled by the content of the box added.
**Theorem 5.5.** Let $\mathfrak{g} = \mathfrak{gl}_n$ and let $M = L(a^c)$, $N = L(b^d)$ and $V = L(\square)$ so that $H_k^{\text{ext}}$ acts on $M \otimes N \otimes V^{\otimes k}$ as in Theorem 5.4. Assume that the genericity conditions of (5.22) hold so that $q$ is not a root of unity, $a, b, c, d \in \mathbb{Z}_{>0}$ and $(a + c) - (b + d) \not\in \{0, \pm1, \pm2\}$. For $\lambda \in \mathcal{P}(k)$, let $B_k^\lambda$ be the $H_k^{\text{ext}}$-module of (5.23) and define $z, c$ and $J$ as in (5.26), (5.28), and (5.29). Then

$$B_k^\lambda \cong H_k^{(z,c,J)}$$

as $H_k^{\text{ext}}$-modules. (5.30)

**Proof.** By Proposition 5.2, $B_k^\lambda$ is a calibrated $H_k^{\text{ext}}$ module. Therefore $B_k^\lambda$ has a composition series with factors that are irreducible calibrated $H_k^{\text{ext}}$-modules. By Theorem 3.3, each factor is isomorphic to some $H_k^{(z,c,J)}$ where $(c, J)$ is a skew local region, and $(z, c, J)$ is determined by the eigenvalues of the action of $W_0, W_1, \ldots, W_k$. By Proposition 5.2, the simultaneous eigenbasis $\{v_S \mid S \in T_k^\lambda\}$ $B_k^\lambda$ is indexed by

$$T_k^\lambda = \{\text{paths } S = (a^c) \to S^{(0)} \to S^{(1)} \to \cdots \to S^{(k)} = \lambda\}$$

in the Bratteli diagram. (5.31)

To determine which $H_k^{(z,c,J)}$ appear as composition factors of $B_k^\lambda$, it is necessary to compute the eigenvalues of the action of the $W_i$'s on the basis vectors $v_S$ as follows.

By (5.24), (5.25), and the formulas in Proposition 5.2

$$\Phi(P)v_S = q^{2v_0(S^{(0)})}v_S \quad \text{and} \quad \Phi(Z_i)v_S = q^{2v(S^{(i)}/S^{(i-1)})}v_S \quad \text{for } i = 1, \ldots, k.$$ 

Using (2.18) and (5.19), $W_i = -(a_1a_2b_1b_2)^{-\frac{1}{2}}Z_i$ with $a_1 = q^{2a}$, $a_2 = q^{-2c}$, $b_1 = q^{2b}$, and $b_2 = q^{-2d}$, and thus

$$\Phi(W_i)v_S = -(a_1a_2b_1b_2)^{-\frac{1}{2}}\Phi(Z_i)v_S = -q^{-(a-c+b-d)}q^{2v(S^{(i)}/S^{(i-1)})}v_S = -q^{2v(S^{(i)}/S^{(i-1)})}v_S.$$ (5.32)

Then $\Phi(PW_1 \cdots W_k)v_S = (-1)^k q^{2v_0(S^{(0)}) + c(S^{(1)}/S^{(0)}) + \cdots + c(S^{(k)}/S^{(k-1)}) - k(a-c+b-d)}v_S$ so that, with $c_0$ and $z$ as in (5.26),

$$\Phi(W_0) = \Phi(PW_1 \cdots W_k)v_S = (-1)^k q^{2v_0v_S} = zv_S.$$ (5.33)

Let $S = (a^c) \to S^{(0)} \to S^{(1)} \to \cdots \to S^{(k)} = \lambda$ be a path to $\lambda$ in the Bratteli diagram. In the context of the diagrams in (5.15), the partitions $S^{(0)}$ and $S^{(0)}_{\text{max}}$ differ by moving some boxes from $\mu$ to $\mu^c$ (from the NW border of $\lambda/S^{(0)}_{\text{max}}$ in $B$ to the NW border of $\lambda/S^{(0)}$ in $B'$). Thus the sequence $c = (c_1, \ldots, c_k)$, where

$$c_1, \ldots, c_k$$

are the values $|\tilde{c}(S^{(1)}/S^{(0)})|, \ldots, |\tilde{c}(S^{(k)}/S^{(k-1)})|$ arranged in increasing order, coincides with $c$ as defined in (5.28). Let $w_S \in W_0$ be the minimal length element such that

$$w_Sc = w_S(c_1, \ldots, c_k) = (c_{w_S^{-1}(1)}, \ldots, c_{w_S^{-1}(k)}) = (\tilde{c}(S^{(1)}/S^{(0)}), \ldots, \tilde{c}(S^{(k)}/S^{(k-1)})),$$ (5.34)

where $c_{-i} = -c_i$ for $i \in \{1, \ldots, k\}$. The signed permutation $w_S$ is the unique signed permutation such that

$$w_Sc = (\tilde{c}(S^{(1)}/S^{(0)}), \ldots, \tilde{c}(S^{(k)}/S^{(k-1)}))$$

and $R(w_S) \cap Z(c) = \emptyset$. 


where $Z(c)$ is as in (3.6). If the boxes of $\lambda/S(0)$ are indexed according to the same conditions as just before (5.29), then $w_S$ is the signed permutation given by

$$w_S(i) = \text{sgn}(\tilde{c}(\text{box}_i))(\text{entry in box}_i \text{ of } S),$$

where the path $S$ is identified with the standard tableau of shape $\lambda/S(0)$ that has $S^{(j)}/S^{(j-1)}$ filled with $j$.

The basis vector $v_S$ appears in a composition factor isomorphic to $H_k^{(\varepsilon,c,J)}$ where

$$J = R(w_S) \cap P(c), \quad \text{where } R(w_S) = R_1 \cup R_2 \cup R_3 \quad \text{and } P(c) = P_1 \cup P_2 \cup P_3,$$

as defined in (3.2) and (3.7), are given by

- $R_1 = \{\varepsilon_i \mid i > 0 \text{ and } w_S(i) < 0\},$
- $P_1 = \{\varepsilon_i \mid c_i \in \{r_1, r_2\}\},$
- $R_2 = \{\varepsilon_j - \varepsilon_i \mid i < j \text{ and } w_S(i) > w_S(j)\},$
- $P_2 = \{\varepsilon_j - \varepsilon_i \mid 0 < i < j, c_j = c_i + 1\},$
- $R_3 = \{\varepsilon_j + \varepsilon_i \mid i < j \text{ and } -w_S(i) > w_S(j)\},$
- $P_3 = \{\varepsilon_j + \varepsilon_i \mid 0 < i < j, c_j = -c_i + 1\}.$

To describe $J = (R_1 \cap P_1) \cup (R_2 \cap P_2) \cup (R_3 \cap P_3)$ in terms of the boxes in $\lambda$, first record that

$$R_1 \cap P_1 = \{\varepsilon_i \mid i > 0 \text{ and } w_S(i) < 0\} \cap \{\varepsilon_i \mid c_i \in \{r_1, r_2\}\} = \{\varepsilon_i \mid \tilde{c}(\text{box}_i) = \{-r_1, -r_2\}\}.$$

Next analyze

$$R_2 \cap P_2 = \{\varepsilon_j - \varepsilon_i \mid i < j \text{ and } w(i) > w(j)\} \cap \{\varepsilon_j - \varepsilon_i \mid 0 < i < j, c_j = c_i + 1\}.$$

Since $0 \leq c_i$ and $c_j = c_i + 1$, we have $c_j \geq 1$.

Case 1: $\tilde{c}(\text{box}_i) \geq 0$, so that $\tilde{c}(\text{box}_j) = \pm(\tilde{c}(\text{box}_i) + 1)$.

Case 1a: $\tilde{c}(\text{box}_j) = \tilde{c}(\text{box}_i) + 1$.

If box$_j$ is NW of box$_i$ then $w(j) < w(i)$ and $\varepsilon_j - \varepsilon_i \in J$.

If box$_j$ is SE of box$_i$ then $w(j) > w(i)$ and $\varepsilon_j - \varepsilon_i \notin J$.

Case 1b: $\tilde{c}(\text{box}_j) = -\tilde{c}(\text{box}_i) + 1$.

Then $w(j) < 0 < w(i)$ so that $w(j) < w(i)$ and $\varepsilon_j - \varepsilon_i \in J$.

Case 2: $\tilde{c}(\text{box}_j) < 0$, so that $\tilde{c}(\text{box}_j) = \pm(-\tilde{c}(\text{box}_i) + 1)$.

Case 2a: $\tilde{c}(\text{box}_j) = \tilde{c}(\text{box}_i) - 1 < \tilde{c}(\text{box}_i) < 0$.

If box$_j$ is NW of box$_i$ then $-w(j) < -w(i)$ so that $w(i) < w(j)$ and $\varepsilon_j - \varepsilon_i \notin J$.

If box$_j$ is SE of box$_i$ then $-w(j) > -w(i)$ so that $w(i) > w(j)$ and $\varepsilon_j - \varepsilon_i \in J$.

Case 2b: $\tilde{c}(\text{box}_j) = -\tilde{c}(\text{box}_i) + 1 > 0 > \tilde{c}(\text{box}_i)$.

Then $w(i) < 0$ and $0 < w(j)$ so that $\varepsilon_j - \varepsilon_i \notin J$.

Finally, analyze

$$R_3 \cap P_3 = \{\varepsilon_j + \varepsilon_i \mid i < j \text{ and } -w(i) > w(j)\} \cap \{\varepsilon_j + \varepsilon_i \mid 0 < i < j, c_j = -c_i + 1\}.$$

Since $0 \leq c_i$ and $c_j = -c_i + 1 \geq c_i$, we have $0 \leq c_i \leq 1/2$. Since the entries of $c$ are in $\mathbb{Z}$ or in $\frac{1}{2} + \mathbb{Z}$, the possibilities for $(c_i, c_j)$ are $(0,1)$ and $(\frac{1}{2}, \frac{1}{2})$, and the possibilities for $(\tilde{c}(\text{box}_i), \tilde{c}(\text{box}_j))$...
are $(0, 1)$, $(0, -1)$, $(\frac{1}{2}, \pm \frac{1}{2})$, or $(-\frac{1}{2}, \pm \frac{1}{2})$.

Case 1: $\tilde{c}(\text{box}_j) = 1$ and $\tilde{c}(\text{box}_i) = 0$.
If box$_j$ is NW of box$_i$ then $0 < w(j) < w(i)$ so that $-w(i) < 0 < w(j)$ and $\varepsilon_j + \varepsilon_i \notin J$.
If box$_j$ is SE of box$_i$ then $0 < w(i) < w(j)$ so that $-w(i) < 0 < w(j)$ and $\varepsilon_j + \varepsilon_i \notin J$.

Case 2: $\tilde{c}(\text{box}_j) = -1$ and $\tilde{c}(\text{box}_i) = 0$.
If box$_j$ is NW of box$_i$ then $-w(j) < w(i)$ so that $-w(i) < w(j)$ and $\varepsilon_j + \varepsilon_i \notin J$.
If box$_j$ is SE of box$_i$ then $-w(i) > w(j)$ so that $-w(i) > w(j)$ and $\varepsilon_j + \varepsilon_i \in J$.

Case 3: $\tilde{c}(\text{box}_j) = \frac{1}{2}$ and $\tilde{c}(\text{box}_i) = \frac{1}{2}$.
Then $0 < w(i) < w(j)$ so that $-w(i) < 0 < w(j)$ and $\varepsilon_j + \varepsilon_i \notin J$.

Case 4: $\tilde{c}(\text{box}_j) = -\frac{1}{2}$ and $\tilde{c}(\text{box}_i) = \frac{1}{2}$.
This case cannot occur since, when indexing the boxes of $\lambda/S^{(0)}$,
the boxes of shifted content $-\frac{1}{2}$ are numbered before the boxes of shifted content $\frac{1}{2}$.

Case 5: $\tilde{c}(\text{box}_j) = \frac{1}{2}$ and $\tilde{c}(\text{box}_i) = -\frac{1}{2}$.
If box$_j$ is NW of box$_i$ then $w(i) < 0$ and $w(j) < -w(i)$ so that $\varepsilon_j + \varepsilon_i \in J$.
If box$_j$ is SE of box$_i$ then $w(i) < w(j)$ so that $\varepsilon_j + \varepsilon_i \notin J$.

Case 6: $\tilde{c}(\text{box}_j) = -\frac{1}{2}$ and $\tilde{c}(\text{box}_i) = -\frac{1}{2}$.
Then $0 < -w(j) < -w(i)$ and $w(j) < 0 < -w(i)$ so that $\varepsilon_j + \varepsilon_i \in J$.

This analysis shows that $J = R(w_S) \cap P(c) = (R_1 \cap P_1) \cup (R_2 \cap P_2) \cup (R_3 \cap P_3)$ is as given in (5.29).
A consequence of the description of $J$ in (5.29) is that $J = R(w_S) \cap P(c)$ is independent of the choice of $S \in T^\lambda_k$. It follows that all composition factors of $B^\lambda_k$ are isomorphic to $H_k^{(z,c,J)}$.

Let $S, T \in T^\lambda_k$ such that $v_S$ and $v_T$ have the same eigenvalues for $W_0, \ldots, W_k$. By definition of $T^\lambda_k$, $S^{(k)} = T^{(k)} = \lambda$. Since $W_kv_S = -q^{c(S^{(k)})/S^{(k-1)}}v_S = -q^{c(\lambda/S^{(k-1)})}v_S$ and $W_kv_T = -q^{c(T^{(k)})/T^{(k-1)}}v_T = -q^{c(\lambda/T^{(k-1)})}v_T$, we have $\tilde{c}(\lambda/T^{(k-1)}) = \tilde{c}(\lambda/S^{(k-1)})$ which implies that $T^{(k-1)} = S^{(k-1)}$. Using this and the fact that the eigenvalues of $W_{k-1}$ on $v_S$ and $v_T$ are the same, implies similarly that $T^{(k-2)} = S^{(k-2)}$. Induction gives that

$S^{(0)} = T^{(0)}, \ldots, S^{(k)} = T^{(k)}$ so that $S = T$.

Thus $\dim((B^\lambda_k)_r) \leq 1$ (in the notation of (3.1)) and $B^\lambda_k \cong H_k^{(z,c,J)}$ as $H_k^{\text{ext}}$-modules. \hfill \square

In the course of the proof of Theorem 5.5, we have also established the following result, which deserves mention.

**Corollary 5.6.** Keeping the notations of Theorem 5.5, let $\lambda \in P^{(k)}$ and $S \in T^\lambda_k$, and let $w_S$ be the signed permutation defined in (5.34). Then

$$T^\lambda_k \xrightarrow{S} F^{(c,J)} \xrightarrow{w_S} \text{is a bijection.}$$

**Example 3.** Let $M = L(a^c) = L(6)$ and $N = L(b^d) = L(3)$ so that

$$a = 6, \quad c = 1, \quad b = 3, \quad d = 1, \quad r_1 = \frac{3}{2}, \quad \text{and} \quad r_2 = \frac{11}{2}.$$ 

The partition $\lambda = (10, 8)$ is in $P^{(k)}$ with $k = 9$. Then we draw $\lambda$ as the (marked) partition

$$\lambda = (10, 8) = \begin{array}{c|c|c|c|c|c|c|c|c|c}
\hline
& & & & & & & & & \\
\hline
& & & & & & & & & \\
\hline
& & & & & & & & & \\
\hline
\end{array}. \quad \text{Here,} \quad S^{(0)}_{\text{max}} = (6, 3)$$
is indicated by the shaded boxes. The boxes of $\lambda/S_{\text{max}}^{(0)}$ have

and shifted contents

Example 4. Let $M = L(a^c) = L(5^4)$ and $N = L(b^d) = L(3^3)$ so that

$$a = 5, \quad c = 4, \quad b = 3, \quad d = 3, \quad r_1 = \frac{3}{2}, \quad \text{and} \quad r_2 = \frac{15}{2}.$$

The partition $\lambda = (9,9,6,6,6,2,1,1,1)$ is in $\mathcal{P}^{(k)}$ with $k = 12$. For this partition $S_{\text{max}}^{(0)} = (7,6,5,5,3,2,1)$; and one tableau $S \in \mathcal{T}_k^\lambda$ with $S^{(0)} = S_{\text{max}}^{(0)}$ (where the shaded portion of $\lambda$ corresponds to $S^{(0)}$) is

indicates the indexing of the boxes in $\lambda/S_{\text{max}}^{(0)}$. The contents of the boxes $S^{(i)}/S^{(i-1)}$ for $i = 1, \ldots, k$ are $7,8,5,6,7,3,2,-1,0,1,-7,-8$; and since $-\frac{1}{2}(a-c+b-d) = -\frac{1}{2}$, the shifted contents $\tilde{c}(S^{(i)}/S^{(i-1)})$ for $i = 1, \ldots, k$ are

$$\frac{13}{2}, \frac{15}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}, \frac{5}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{15}{2}, \frac{17}{2},$$

respectively. The sum of the contents of the boxes in $S_{\text{max}}^{(0)}$ is 1, the sum of the contents of the boxes in $\lambda$ is 23, $c_0 = -\frac{1}{2}(12(5-4+3-3)+5\cdot4(5-4)+3\cdot3(3-3)+24) = 8$,

$$z = q^{16}, \quad \text{and} \quad c = \left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{5}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}, \frac{13}{2}, \frac{15}{2}, \frac{15}{2}, \frac{17}{2}\right)$$
is the sequence of absolute values of the shifted contents, arranged in increasing order. Using \((5.34)\),

$$w_S = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ -9 & 10 & -8 & 7 & 6 & 3 & 4 & 1 & 5 & -11 & 2 & -12 \end{pmatrix},$$

$$P(c) = \begin{cases} \varepsilon_3, \varepsilon_4, \varepsilon_{10}, \varepsilon_{11}, \varepsilon_2 - \varepsilon_1, \\ \varepsilon_3 - \varepsilon_1, \varepsilon_4 - \varepsilon_1, \varepsilon_5 - \varepsilon_2, \varepsilon_4 - \varepsilon_2, \varepsilon_5 - \varepsilon_3, \varepsilon_5 - \varepsilon_4, \\ \varepsilon_7 - \varepsilon_6, \varepsilon_8 - \varepsilon_7, \varepsilon_9 - \varepsilon_7, \\ \varepsilon_{10} - \varepsilon_8, \varepsilon_{11} - \varepsilon_8, \varepsilon_{10} - \varepsilon_9, \varepsilon_{11} - \varepsilon_9, \varepsilon_{12} - \varepsilon_{10}, \varepsilon_{12} - \varepsilon_{11} \end{cases},$$

$$R(w_S) = \begin{cases} \varepsilon_1, \varepsilon_3, \varepsilon_{10}, \varepsilon_{12}, \\ \varepsilon_{10} - \varepsilon_1, \varepsilon_{12} - \varepsilon_1, \varepsilon_3 - \varepsilon_2, \varepsilon_4 - \varepsilon_2, \varepsilon_5 - \varepsilon_2, \varepsilon_6 - \varepsilon_2, \varepsilon_7 - \varepsilon_2, \varepsilon_8 - \varepsilon_2, \varepsilon_9 - \varepsilon_2, \varepsilon_{10} - \varepsilon_2, \\ \varepsilon_{11} - \varepsilon_2, \varepsilon_{12} - \varepsilon_2, \varepsilon_{10} - \varepsilon_3, \varepsilon_{12} - \varepsilon_3, \varepsilon_5 - \varepsilon_4, \varepsilon_6 - \varepsilon_4, \varepsilon_7 - \varepsilon_4, \varepsilon_8 - \varepsilon_4, \varepsilon_9 - \varepsilon_4, \varepsilon_{10} - \varepsilon_4, \\ \varepsilon_{11} - \varepsilon_4, \varepsilon_{12} - \varepsilon_4, \varepsilon_6 - \varepsilon_5, \varepsilon_7 - \varepsilon_5, \varepsilon_8 - \varepsilon_5, \varepsilon_9 - \varepsilon_5, \varepsilon_{10} - \varepsilon_5, \varepsilon_{11} - \varepsilon_5, \varepsilon_{12} - \varepsilon_5, \varepsilon_8 - \varepsilon_6, \\ \varepsilon_{10} - \varepsilon_6, \varepsilon_{11} - \varepsilon_6, \varepsilon_{12} - \varepsilon_6, \varepsilon_8 - \varepsilon_7, \varepsilon_{10} - \varepsilon_7, \varepsilon_{11} - \varepsilon_7, \varepsilon_{12} - \varepsilon_7, \varepsilon_{10} - \varepsilon_8, \varepsilon_{12} - \varepsilon_8, \\ \varepsilon_{10} - \varepsilon_9, \varepsilon_{11} - \varepsilon_9, \varepsilon_{12} - \varepsilon_9, \varepsilon_{12} - \varepsilon_{10}, \varepsilon_{12} - \varepsilon_{11}, \varepsilon_3 + \varepsilon_1, \varepsilon_4 + \varepsilon_1, \varepsilon_5 + \varepsilon_1, \varepsilon_6 + \varepsilon_1, \varepsilon_7 + \varepsilon_1, \varepsilon_8 + \varepsilon_1, \varepsilon_9 + \varepsilon_1, \varepsilon_{10} + \varepsilon_1, \varepsilon_{11} + \varepsilon_1, \varepsilon_{12} + \varepsilon_1, \\ \varepsilon_{10} + \varepsilon_2, \varepsilon_{12} + \varepsilon_2, \varepsilon_4 + \varepsilon_3, \varepsilon_5 + \varepsilon_3, \varepsilon_6 + \varepsilon_3, \varepsilon_7 + \varepsilon_3, \varepsilon_8 + \varepsilon_3, \varepsilon_9 + \varepsilon_3, \varepsilon_{10} + \varepsilon_3, \varepsilon_{11} + \varepsilon_3, \varepsilon_{12} + \varepsilon_3, \varepsilon_{10} + \varepsilon_5, \varepsilon_{12} + \varepsilon_5, \varepsilon_{10} + \varepsilon_6, \varepsilon_{12} + \varepsilon_6, \varepsilon_{10} + \varepsilon_7, \varepsilon_{12} + \varepsilon_7, \\ \varepsilon_{10} + \varepsilon_8, \varepsilon_{12} + \varepsilon_8, \varepsilon_{10} + \varepsilon_9, \varepsilon_{12} + \varepsilon_9, \varepsilon_{11} + \varepsilon_{10}, \varepsilon_{12} + \varepsilon_{10}, \varepsilon_{12} + \varepsilon_{11} \end{cases},$$

and \(J = R(w_S) \cap P(c)\) consists of the outlined elements of \(P(c)\) (which are the same as the outlined elements of \(R(w_S)\)). Another \(T \in T^\lambda_k\) is (again, with \(T^{(0)}\) indicated by the shaded boxes)

![Diagram](image)

Keeping the setting of Theorem 5.5, Proposition 3.1 associates a configuration of \(2k\) boxes to \((c, J)\). This configuration can be described in terms of the data of \(\lambda \in \mathcal{P}^{(k)}\) as follows. With \(S^{(0)}_{\max}\) as defined just before \((5.28)\), let \(\text{rot}(\lambda/S_{\max}^{(0)})\) be the \(180^\circ\) rotation of the skew shape \(\lambda/S_{\max}^{(0)}\). Then

the configuration of boxes \(\kappa\) corresponding to \((c, J)\) is \(\kappa = \text{rot}(\lambda/S_{\max}^{(0)}) \cup \lambda/S_{\max}^{(0)}\), \((5.35)\)

so that it is the (disjoint) union of two skew shapes \(\lambda/S_{\max}^{(0)}\) and \(\text{rot}(\lambda/S_{\max}^{(0)})\), placed with

\(\text{rot}(\lambda/S^{(0)})\) northwest of \(\lambda/S^{(0)}\),

\(\lambda/S^{(0)}\) positioned so that the contents of its boxes are \((\tilde{c}(S^{(1)}/S^{(0)}), \ldots, \tilde{c}(S^{(k)}/S^{(k-1)})))\),

\(\text{rot}(\lambda/S^{(0)})\) positioned so that the contents of its boxes are \((-\tilde{c}(S^{(k)}/S^{(k-1)}), \ldots, -\tilde{c}(S^{(1)}/S^{(0)})))\),
and with markings placed at the NE corners of the rectangles $B$ and $B'$ corresponding to $\lambda/S(0)$ (in the notation of (5.15)). The resulting doubled skew shape is symmetric under the 180° rotation which sends a box on diagonal $c_i$ to a box on diagonal $-c_i$. In the case of Example 4 the corresponding configuration of boxes is

This configuration of boxes also appeared in Example 2.

For generically large $a, b, c, d$, there will be examples of $\lambda, \mu \in \mathcal{P}(k)$ with $\lambda \neq \mu$ and $B_{\lambda}^k \cong B_{\mu}^k$ as $H_k^{\text{ext}}$-modules; see Example 5. This is because the eigenvalues of $P$ on $M \otimes N$ are not sufficient to distinguish the components of $M \otimes N$ as a $\mathfrak{gl}_n$-module. It could be helpful to further extend $H_k^{\text{ext}}$ and consider an algebra $Z(U_q \mathfrak{gl}_n) \otimes H_k$ acting on $M \otimes N \otimes V^\otimes k$.

**Example 5.** Let $a = c = 6$ and $b = d = 4$,

$$\lambda(k) = (11 + k, 10, 8, 8, 6, 6, 5, 3, 3, 1) \quad \text{and} \quad \mu(k) = (11 + k, 9, 9, 8, 7, 6, 4, 3, 2, 2), \quad \text{i.e.}$$

Then $\lambda(k) \neq \mu(k)$ but, as $H_k^{\text{ext}}$-modules,

$$B_{\lambda}^k \cong B_{\mu}^k \cong H_k^{\text{ext}}(z, c, \emptyset), \quad \text{where} \ c = (11, 12, \ldots, 11 + k - 1) \ \text{and} \ z = q^{28 + k(k+21)}.$$

Recall from (5.23) that

$$M \otimes N \otimes V^\otimes k \cong \bigoplus_{\lambda \in \mathcal{P}(k)} L(\lambda) \otimes B_{\lambda}^k$$

as $(U_q \mathfrak{g}, H_k^{\text{ext}})$-bimodules.

A consequence of Theorem 3.3(b) is the following construction of the irreducible $H_k^{\text{ext}}$-modules $B_{\lambda}^k$.

Keeping the setting and notation of (5.31), for $\lambda \in \mathcal{P}(k)$ and $S \in \mathcal{T}_k^\lambda$, let

$$s_j S \text{ be the path from } (a^c) \text{ to } \lambda \text{ that differs from } S \text{ only at } S^{(j)}. \quad (5.36)$$
The path $s_j S$ is unique if it exists: if $S = ((a^c) \to S^{(0)} \to S^{(1)} \to \cdots \to S^{(k)})$ then $S^{(j+1)}$ is obtained by adding a box to $S^{(j)}$, and $(s_j S)^{(j)}$ is obtained by moving a box of $S^{(j)}$ to the position of the added box in $S^{(j+1)}$. In the case that $j = 0$, the paths $s_0 S$ and $S$ satisfy $(s_0 S)^{(1)} = S^{(1)}$ and the partitions $(s_0 S)^{(0)}$ and $S^{(0)}$ in $\mathcal{P}^{(0)}$ differ by the placement of one box, with
\[
\hat{c}((s_0 S)^{(1)}/(s_0 S)^{(0)}) = -\hat{c}(S^{(1)}/S^{(0)}),
\]
where the shifted content of a box $\hat{c}$ (box) is as defined in (5.28).

**Corollary 5.7.** Keep the conditions of Theorems 5.4 and 5.5. In particular, assume that the genericity conditions of (5.22) hold so that $q$ is not a root of unity, $a, b, c, d \in \mathbb{Z}_{>0}$ and $(a + c) - (b + d) \not\in \{0, \pm 1, \pm 2\}$. Let $\lambda \in \mathcal{P}^{(k)}$. Then $B_k^\lambda$ has a basis $\{v_S \mid S \in \mathcal{T}_k^\lambda\}$ such that the $H_{ext}$-action is given by
\[
P v_S = q^{2c_0(T)} v_S, \quad Z i v_S = q^{2c(S^{(i)}/S^{(i-1)})} v_S,
\]
\[
T_i v_S = [T_i]_{S,S} v_S + \sqrt{-([T_i]_{S,S} - q) ([T_i]_{S,S} + q^{-1})} v_{s_i S}, \quad \text{for } i = 1, \ldots, k - 1,
\]
\[
Y_1 v_S = [Y_1]_{S,S} v_S + \sqrt{-([Y_1]_{S,S} - q^{-2d}) ([Y_1]_{S,S} - q^{2d})} v_{s_0 S},
\]
\[
X_1 v_S = [X_1]_{S,S} v_S + q^{-2c(S^{(1)}/S^{(0)})} q^{(a-c+b-d)} \sqrt{-([X_1]_{S,S} - q^{2a}) ([X_1]_{S,S} - q^{-2c})} v_{s_0 S},
\]
where $v_{s_j S} = 0$ if $s_j S$ does not exist, and
\[
[T_i]_{S,S} = \frac{q - q^{-1}}{1 - q^{2c(S^{(i)}/S^{(i-1)}) - c(S^{(i+1)}/S^{(i)})}},
\]
\[
[Y_1]_{S,S} = \frac{(q^{2b} + q^{-2d}) - (q^{2a} + q^{-2c}) q^{2(b-d)} q^{-2c(S^{(1)}/S^{(0)})}}{1 - q^{2(a-c+b-d) q^{-4c(S^{(1)}/S^{(0)})}}},
\]
\[
[X_1]_{S,S} = \frac{(q^{2a} + q^{-2c}) - (q^{2b} + q^{-2d}) q^{2(a-c) q^{-2c(S^{(1)}/S^{(0)})}}}{1 - q^{2(a-c+b-d) q^{-4c(S^{(1)}/S^{(0)})}}}.\]

**Proof.** The appropriate basis of $B_k^\lambda$ is the one given in Proposition 5.2 and used also in the proof of Theorem 5.5. It is only necessary to convert from the notation $v_w$ in Theorem 3.3 to the notation $v_S$ using the bijection in Corollary 5.6. Recall from (5.21) that
\[
a_1 = q^{2a}, \quad a_2 = q^{-2c}, \quad b_1 = q^{2b}, \quad b_2 = q^{-2d},
\]
\[
t^{\frac{1}{2}} = q, \quad t^{\frac{1}{2}} = a_1^{\dagger} (-a_2)^{-\dagger} = -iq^{t+c}, \quad \text{and} \quad t^{\dagger} = b_1^{\dagger} (-b_2)^{-\dagger} = -i q^{b+d}.
\]
From (3.12) and (5.32),
\[
\gamma^{w-1(i)} v_w = \Phi(W_i v_S) = -q^{-(a-c+b-d)} q^{2c(S^{(i)}/S^{(i-1)})} v_S.
\]
From (2.18), (2.9), and (5), $Y_1 = b_1^{\dagger} (-b_2)^{-\dagger} T_0 = iq^{b-d} T_0$ and $X_1 = (a_1 + a_2) - a_1 a_2 Y_1 Z_1^{-1} = q^{2a} + q^{-2c} - q^{2(a-c)} Y_1 Z_1^{-1}$. With these conversions, the formulas from (3.13) and (3.14) become
\[
T_i v_S = T_i v_w = [T_i]_{S,S} v_S + [T_i]_{S,S} v_{s_i S}, \quad \text{for } i = 1, \ldots, k - 1,
\]
\[
Y_1 v_S = iq^{b-d} T_0 v_w = [Y_1]_{S,S} v_S + [Y_1]_{s_0 S,S} v_{s_0 S} \quad \text{and}
\]
\[
X_1 v_S = \left(q^{2a} + q^{-2c} - q^{2(a-c)} Y_1 Z_1^{-1}\right) v_S = \left(q^{2a} + q^{-2c} - q^{2(a-c) q^{-2c(S^{(1)}/S^{(0)})}} Y_1\right) v_S
\]
\[
= [X_1]_{S,S} v_S - [X_1]_{s_0 S,S} v_{s_0 S},
\]
47
with

\[ [T_1]_{S,S} = [T_1]_{w,w} = \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - \gamma_{w-1(1)} t^{-\frac{1}{2} - \frac{1}{2}}} = \frac{q - q^{-1}}{1 - q^{2(c(S^{(1)}/S^{(e)}) - c(S^{(d)}/S^{(e)}))}}, \]

\[ [Y_1]_{S,S} = i q^{b-d}[T_0]_{w,w} = i q^{b-d} \frac{(t^{\frac{1}{2}} - t^{-\frac{1}{2})} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})}{1 - \gamma_{w-1(1)}} \]

\[ = i q^{b-d} \left( \frac{q^{(b-d)} + q^{(b+d)}}{1 - q^{2(a-c+b-d)}q^{-4c(S^{(1)}/S^{(0)})}} \right), \]

and

\[ [X_1]_{S,S} = q^{2a} + q^{2c} - q^{2c}q^{-2c(S^{(1)}/S^{(0)})}[Y_1]_{S,S} \]

\[ = q^{2a} + q^{2c} - q^{2c}q^{-2c(S^{(1)}/S^{(0)})} \left( \frac{q^{2b} + q^{2d}}{1 - q^{2(a-c+b-d)}q^{-4c(S^{(1)}/S^{(0)})}} \right), \]

\[ = \frac{(q^{2b} + q^{2d}) - (q^{2a} + q^{2c})q^{2(b-d)}q^{-2c(S^{(1)}/S^{(0)})}}{1 - q^{2(a-c+b-d)}q^{-4c(S^{(1)}/S^{(0)})}}. \]

On the two-dimensional subspace spanned by \( \{v_S, v_{S_0S}\} \) the action of \( T_0 \) in the basis \( \{v_S, v_{S_0S}\} \) is a symmetric matrix \( [T_0] \), and so the matrix of \( Y_1 \) in this basis is \( [Y_1] = i q^{b-d}[T_0] \) is also symmetric. The action of \( Z_1 \) is by a diagonal matrix \( [Z_1] \), so \( [Z_1] = [Z_1] \). Therefore, using \( X_1 = Z_1Y_1^{-1} \) from (2.9) and \( X_1 = (a_1 + a_2) - a_1 a_2 X_1^{-1} \) from (5.3), we have \( ([X_1]^{-1})^{t} = ([Y_1][Z_1]^{-1})^{t} = ([Z_1]^{-1})^{t} [Y_1]^{t} = [Z_1]^{-1}[Y_1] \) and so

\[ [Z_1][X_1][Z_1]^{-1} = [Z_1]((a_1 + a_2) - a_1 a_2[Z_1]^{-1}[Y_1])[Z_1]^{-1} = [X_1]. \]

Thus

\[ [Z_1]_{S,S}[X_1]_{S_0S,S}[Z_1]^{-1}_{S_0L,S_0S} = [X_1]_{S_0S,S} \]

and

\[ -[X_1]_{S_0S,S}[X_1]_{S_0S,S} = ([X_1]_{S_0S,S} - a_1) ([X_1]_{S_0S,S} - a_2), \]

since \( [X_1] \) is a 2x2 matrix with eigenvalues \( a_1 \) and \( a_2 \) (as in the proof of Theorem 3.3). Thus

\[ [X_1]_{S_0S,S} = \sqrt{([X_1]_{S_0S,S})^2} = \sqrt{[X_1]_{S_0S,S}[Z_1]^{-1}_{S_0S,S}[X_1]_{S_0S,S}[Z_1]_{S_0S,S}} \]

\[ = \sqrt{[Z_1]_{S_0S,S}([X_1]_{S_0S,S} - a_1) ([X_1]_{S_0S,S} - a_2)}. \]

By (5.37), \( c((s_0S)^{(1)}/(s_0S)^{(0)}) = -c(S^{(1)}/S^{(0)}) + (a - c + b + d) \), so that

\[ \sqrt{[Z_1]_{S_0S,S}[X_1]_{S_0S,S}} = q^{-c(S^{(1)}/S^{(0)})} q^{c((s_0S)^{(1)}/(s_0S)^{(0)})} = q^{-2c(S^{(1)}/S^{(0)}) + (a - c + b + d)}. \]

Thus

\[ [X_1]_{S_0S,S} = q^{-2c(S^{(1)}/S^{(0)})} q^{(a-c+b+d)} \sqrt{([X_1]_{S_0S,S} - a_1) ([X_1]_{S_0S,S} - a_2)}. \]
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