Static and dynamic analysis of isotropic shell structures by the spectral finite element method

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Abstract. The paper deals with certain problems related to static and modal analysis of isotropic shell structures by the use of the approach known in the literature as the time-domain spectral finite element method. Although recently this spectral approach has been widely reported as a very powerful numerical tool used to solve various wave propagation problems, its properties make it very well suited to solve static and modal problems. The robustness and effectiveness of the spectral approach has been successfully demonstrated by the authors in the case of a thin-walled spherical shell structure representing a pressure vessel. Static and modal responses of the structure have been investigated by the use of transversally deformable shell-type spectral finite elements and the results of this investigation have been compared to known analytical solutions as well as those obtained by the use of commercially available software for the finite element method.

1. Introduction

The spectral finite element method, also known as the time-domain spectral finite element method, as a numerical technique has been known for a few decades. Its origins come back to the first publications by Patera [1] who by its use investigated a two-dimensional problem of a laminar flow in a channel expansion. Since that time the spectral finite element has been successfully adopted by various physical and applied sciences and nowadays its application fields cover not only fluid dynamics [2] but also heat transfer [3], acoustics [4], seismology [5], and more recently mechanical engineering [6, 7, 8, 9, 10]. The method originates from the application of spectral series for solution of partial differential equations, while at the same time its base ideas are analogous to the classical finite element approach. Its main assumption is the application of orthogonal Lobatto polynomials as approximation functions defined at appropriate Gauss-Lobatto-Legendre integration points. As a consequence of that the inertia matrix obtained in this spectral approach is diagonal making the total cost of numerical calculations much less demanding. Additionally, thanks to the orthogonality of the approximation polynomials the spectral finite element method is characterised by exponential convergence.

In the field of mechanical engineering the time-domain spectral finite element method has established a firm position as a modelling tool used to investigate wave propagation related problems especially for SHM purposes [11, 12]. On the other hand the applicability of this spectral approach to solve static and modal problems has not been widely demonstrated and reported in the literature.
The aim of this paper is to show the robustness and effectiveness of the time-domain spectral finite element method in the case of thin-walled isotropic elastic spherical shell structures representing a pressure vessel and hemispherical dome. Their static and/or modal responses have been investigated by the authors by the use of transversally deformable shell-type spectral finite elements [9] and the results of this investigation have been compared to the known analytical solutions [13, 14, 15] as well as to the solutions obtained by the use of commercially available software for the finite element method.

2. Transversally deformable shell-type spectral finite element

A key factor of the time-based spectral finite element method is an appropriate selection of element nodes. Their coordinates are a key factor that strongly influences the element performance and is directly linked with the type of assumed approximation within the element [10]. The coordinates must be also appropriately selected to avoid Runge’s phenomenon associated with the use of uniform grids of element nodes together with higher orders of approximation polynomials.

In the normalised (curvilinear) coordinate system of the element \(\xi\eta\zeta\) the coordinates of element nodes can be defined as the roots of the following well-known polynomial expression:

\[
(1 - t^2)P_n^m(t) = 0, \quad t \in [-1, +1]
\]  

where \(P_n^m(t)\) is the \(n^{th}\) order Legendre polynomial and the symbol ‘\(^t\)’ denotes its first derivative. For the 5th order approximation polynomials used by the authors the nodal coordinates of the elements in the normalised (curvilinear) coordinate system \(\xi\eta\zeta\) can be found as:

\[
\xi_m, \eta_n \in \{\pm 1, \pm \sqrt{\frac{1}{3} \pm \frac{2}{3\sqrt{3}}}\}, \quad m, n = 1, \ldots, 6
\]  

Next a set of elemental shape functions can be built in the local coordinate system spanned on the specified nodes. A Lagrange interpolation function \(f(\xi, \eta)\) supported on the nodes can be defined in the following manner:

\[
f(\xi, \eta) = \sum_{m=1}^{6} \sum_{n=1}^{6} N_m(\xi)N_n(\eta)f_{mn}, \quad m, n = 1, \ldots, 6
\]  

where \(N_m(\xi)\) and \(N_n(\eta)\) are one-dimensional shape functions of the element, while \(f_{mn}\) are the nodal values of the function \(f(\xi, \eta)\). The approximation shape functions \(N_m(\xi)\) and \(N_n(\eta)\) are orthogonal in a discrete sense:

\[
\int_{-1}^{1} N_m(\zeta)N_n(\zeta)d\zeta = \sum_{k=1}^{6} w_kN_m(\zeta_k)N_n(\zeta_k) = w_{mn}\delta_{mn}, \quad k, m, n = 1, \ldots, 6
\]  

where \(w_{mn}\) is the Gauss-Lobatto weight and \(\delta_{mn}\) is the Kronecker delta.

Assuming small strains the displacement field in the mid-plane of the element in the local coordinate system \(x y z\) can be expressed in a manner typical for shell-type elements as:

\[
\begin{align*}
    u(x, y, z) &= u_0(x, y) + z \cdot \phi(x, y) \\
    v(x, y, z) &= v_0(x, y) + z \cdot \psi(x, y) \\
    w(x, y, z) &= w_0(x, y) + z \cdot \theta(x, y)
\end{align*}
\]  

It can be noticed that the element defined in such a manner is transversely deformable \((\epsilon_{zz} \neq 0)\), has six degrees of freedom per node including three membrane displacement components \(u_0(x, y)\), \(v_0(x, y)\) and \(\theta(x, y)\) and three flexural displacement components \(\phi(x, y)\), \(\psi(x, y)\) and \(w_0(x, y)\).

Based on the assumed displacement field given by (5) the strains within the element can be easily expressed according to the first order shear deformation theory [16, 17] leading to the well-known definitions of the characteristic inertia (diagonal) and stiffness matrices of the element in the global coordinate system \(X Y Z\), as shown in [10].
3. Numerical computations
Numerical simulations were divided into two parts and followed modal analysis and static responses of thin-walled elastic and isotropic spherical shell structure representing a pressure vessel and hemispherical dome. Initially it was assumed that the spherical shell structure is made of aluminium alloy (Young’s modulus $E = 72.7$ GPa, Poisson ratio $\nu = 0.33$, density $\rho = 2700$ kg/m$^3$). The thickness $t$ and the diameter $D$ of the structure were assumed as $t = 5$ mm and $D = 2$ m, respectively.

![Figure 1. Natural frequencies of a thin-walled elastic, isotropic spherical shell.]

It is interesting to note that in the case of a perfectly symmetrical and isotropic elastic sphere there are two main branches of multiple (degenerated) modes of vibrations. The first branch represents purely torsional behaviour ($u_r = 0$), while the second branch corresponds to coupled behaviour due to shearing and stretching [18]. These branches are supplemented by a single purely extensional natural vibration mode ($u_\theta = u_\phi = 0$). The multiplicity (degeneration degree) of particular natural vibration modes (either torsional or coupled) increases linearly with the natural frequency number as $k = 2m + 1$, where $m = 2, 3, \ldots$. For this reason for a given value of the natural frequency (either torsional or coupled) there exists a set of substantially different modes of vibrations (either torsional or coupled), as presented in figure 1.

| Mode number | Mode multiplicity | Mesh density |
|-------------|------------------|--------------|
| $m$ | $k$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | $d_6$ |
| 1 | 5 | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% |
| 2 | 7 | -1.41% | 0.04% | 0.04% | 0.04% | 0.04% | 0.04% |
| 3 | 9 | -1.69% | 0.10% | 0.10% | 0.10% | 0.10% | 0.10% |
| 4 | 11 | -1.25% | 0.21% | 0.20% | 0.20% | 0.20% | 0.20% |
| 5 | 13 | -1.41% | 0.04% | 0.04% | 0.04% | 0.04% | 0.04% |

First a convergence analysis was performed. During this analysis the first 45 natural frequencies of the structure under investigation were calculated assuming free type boundary
conditions. The results obtained by the use of the time-domain spectral finite element method and the transversally deformable shell-type spectral finite elements [9] were compared to the known analytical solutions [13, 14, 15] as well as to the solutions obtained by the use of commercially available software (Patran/Nastran) for the finite element method. In that case 6-node triangle CTRIA6 shell finite elements were used.

Figure 2. Multiple (degenerated) modes of natural vibrations of a thin-walled elastic, isotropic spherical shell for $k = 6$, calculated based on the time-domain spectral finite element method.

In the case of numerical calculations similar mesh densities were used in order to keep the number of nodes at the same level, but in favour of the finite element method. The obtained results are presented in table 1 and table 2. They express the relative errors between the average values of selected natural frequencies, calculated numerically, and the corresponding values calculated analytically. It should be mentioned that the stabilised and small constant values of the relative errors are a direct consequence of the multiplicity of particular natural vibration modes as well as applied numerical procedures used to calculate them. The multiplicity of natural vibration modes presents a serious numerical challenge to all natural frequency extraction procedures. This results in the fact that the values of the calculated natural frequencies, and corresponding to higher multiplicity numbers, are spread around certain average value, as illustrated in figure 2.
Table 2. Natural frequency relative error as a function of the mesh density calculated based on the finite element method (CTRIA6 triangle shell elements) ($d_1 - 258/128$, $d_2 - 768/392$, $d_3 - 1602/800$, $d_4 - 2706/1352$, $d_5 - 4098/2048$, $d_6 - 61506/30752$ nodes/elements).

| Mode number | Mode multiplicity | Mesh density |
|-------------|-------------------|--------------|
| $m$         | $k$               | $d_1$        | $d_2$        | $d_3$        | $d_4$        | $d_5$        | $d_6$        |
| 1           | 5                 | -10.6%       | -5.87%       | -2.08%       | -0.79%       | -0.31%       | 0.01%        |
| 2           | 7                 | -15.5%       | -7.54%       | -2.40%       | -0.87%       | -0.34%       | 0.02%        |
| 3           | 9                 | -18.5%       | -8.39%       | -2.51%       | -0.88%       | -0.32%       | 0.05%        |
| 4           | 11                | -20.1%       | -8.98%       | -2.51%       | -0.83%       | -0.26%       | 0.10%        |
| 5           | 13                | -20.8%       | -9.64%       | -2.43%       | -0.72%       | -0.15%       | 0.20%        |

Static analysis concerned a hemispherical thin-walled elastic and isotropic shell structure representing a dome. In this case it was assumed that all material properties as well as the geometry of the structure follow the classical benchmark test data for the shell type of finite elements [20] (Young’s modulus $E = 68.25$ MPa, Poisson ratio $\nu = 0.3$). The thickness $t$ and the diameter $D$ of the structure were assumed as $t = 40$ mm and $D = 20$ m, respectively.

Figure 3. Initial and deformed configurations of a hemispherical thin-walled elastic, isotropic spherical dome calculated based on the time-domain spectral finite element method.

In this static test the structure was loaded by a set of inward and outward symmetrically acting concentrated forces $P$ of 2 N placed on the structure free edge at points $A$ and $B$, as shown in figure 3. Boundary conditions were provided by fixing the pole point of the structure.

Numerical calculations were carried out for various mesh densities and compared against the known analytical solution, as presented in table 3. It can be clearly seen also in this case that the results obtained thanks to the application of the time-domain spectral finite element method and the transversally deformable shell-type spectral finite elements are characterised by very high accuracy even for relatively low mesh densities.

Table 3. Deflection at point $A$ as a function of the mesh density calculated based on the time-domain spectral finite element method ($d_1 - 321/12$, $d_2 - 1241/48$, $d_3 - 2761/108$, $d_4 - 4881/192$, $d_5 - 7610/300$, $d_6 - 30201/1200$ nodes/elements).

| Mesh density | Deflection | Relative error | Mesh density | Deflection | Relative error |
|--------------|------------|----------------|--------------|------------|----------------|
| $d_1$        | 90.13 mm   | -2.46%         | $d_4$        | 92.02 mm   | -0.41%         |
| $d_2$        | 91.68 mm   | -0.78%         | $d_5$        | 92.04 mm   | -0.39%         |
| $d_3$        | 91.98 mm   | -0.45%         | $d_6$        | 92.09 mm   | -0.34%         |
4. Conclusions

Based on the results obtained by the authors and presented in this work it can be stated that the time-domain spectral finite element method is a very effective and robust numerical tool. It can be successfully used not only to solve various wave propagation problems, but also can be applied for static and modal problems. It has been shown that the application of the transversally deformable shell-type spectral finite element, developed by the authors, helps to reduce significantly errors of numerical investigation providing excellent accuracy, especially in comparison to the finite element method and typical finite elements commonly used for that purpose.

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