Moments of the first descending epoch for a random walk with negative drift

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June 7, 2022

Abstract

We consider the first descending ladder epoch \( \tau = \min\{n \geq 1 : S_n \leq 0 \} \) of a random walk \( S_n = \sum_{i=1}^{n} \xi_i, n \geq 1 \) with i.d.d. summands having a negative drift \( \mathbb{E}\xi = -a < 0 \). Let \( \xi^+ = \max(0, \xi_1) \). It is well-known that, for any \( \alpha > 1 \), the finiteness of \( \mathbb{E}(\xi^+)^\alpha \) implies the finiteness of \( \mathbb{E}\tau^\alpha \) and, for any \( \lambda > 0 \), the finiteness of \( \mathbb{E}\exp(\lambda\xi^+) \) implies that of \( \mathbb{E}\exp(c\tau) \) where \( c > 0 \) is, in general, another constant that depends on the distribution of \( \xi_1 \). We consider the intermediate case, assuming that \( \mathbb{E}\exp(g(\xi^+)) < \infty \) for a positive increasing function \( g \) such that \( \lim_{x \to \infty} g(x)/\log x = \infty \) and \( \lim_{x \to \infty} g(x)/x = 0 \), and that \( \mathbb{E}\exp(\lambda\xi^+) = \infty \), for all \( \lambda > 0 \). Assuming a few further technical assumptions, we show that then \( \mathbb{E}\exp((1 - \varepsilon)(1 - \delta)a\tau)) < \infty \), for any \( \varepsilon, \delta \in (0, 1) \).

Keywords: random walk, negative drift, descending ladder epoch, existence of moments, heavy tail.

AMS classification: 60G50, 60G40, 60K25.

1 Introduction and the main result

Let \( \xi, \xi_1, \xi_2, \ldots, \xi_n, \ldots \) be independent and identically distributed (i.i.d.) random variables (r.v.’s) with a common distribution function \( F \) having a finite negative mean \( \mathbb{E}\xi = -a < 0 \). Let \( S_0 = 0, S_n = \sum_{k=1}^{n} \xi_k, n \geq 1 \) be a random walk, and \( \tau = \min\{n \geq 1 : S_n \leq 0 \} < \infty \) a.s. its first descending ladder epoch.

The descending ladder epoch \( \tau \) plays an important role in theoretical and applied probability. In particular, \( \tau \) represents the length of a busy cycle in a \( GI/GI/1 \) queueing system. Namely, consider a FIFO single-server queue with i.i.d. interarrival times \( \{t_n\} \) with a finite mean \( \mathbb{E}t_1 = a \) and independent of them i.i.d. service times \( \{\sigma_n\} \) with a finite mean \( \mathbb{E}\sigma_1 = b < a \). Let \( W_n \) be the waiting time of customer \( n \). Assume \( W_1 = 0 \), i.e. customer 1 arrives at an empty queue. The sequence \( \{W_n\} \) satisfies the Lindley recursion

\[
W_{n+1} = \max(0, W_n + \sigma_n - t_n) \quad n \geq 1.
\]

We may let \( \xi_n = \sigma_n - t_n \) and conclude that \( \tau \) is the number of customers served in the queue during the first busy cycle, i.e. customer \( \tau + 1 \) is the next customer after customer 1 that finds the queue empty.
We are interested in the existence (finiteness) of moments of \( \tau \) in terms of moments of the common distribution \( F \) of the summands. In particular, the existence of a power (or an exponential) moment of \( \tau \) implies corresponding convergence rates in stability and continuity theorems for various single- and multi-server queueing systems, see e.g. Theorems 2 and 11 in Chapter 4 of [1].

The following results are known (see, e.g., Theorems III.3.1 and 3.2 in [5], and also [6]). Let \( \alpha > 1 \) and \( \lambda > 0 \).

If \( \mathbb{E}(\xi^+)^\alpha < \infty \), then \( \mathbb{E}\tau^\alpha < \infty \). (2)

If \( \mathbb{E}\exp(\lambda \xi) < \infty \), then there exists \( c > 0 \) (that depends on \( F \)) such that \( \mathbb{E}\exp(c \tau) < \infty \). (3)

One can view (2) and (3) as two particular cases of the following implication:

If \( \mathbb{E}G(\xi^+) < \infty \), then \( \mathbb{E}G(C \tau) < \infty \), for a certain \( C > 0 \). (4)

Indeed, (2) is a particular case of (4) with \( G(x) = x^\alpha \), and (3) a particular case of (4) with \( G(x) = \exp(\lambda x) \) (clearly, for \( \lambda > 0 \), exponential moments \( \mathbb{E}\exp(\lambda \xi) \) and \( \mathbb{E}\exp(\lambda \xi^+) \) are either finite or infinite simultaneously).

In this article, we consider the intermediate case where \( G \) is a monotone function that increases faster than any power function and slower than any exponential function. It is convenient to us to use representation \( G(x) = e^{g(x)} \) and work with function \( g \) instead. Here is our main result.

**Theorem 1.** Assume that \( \mathbb{E}\exp(c \xi) = \infty \), for any \( c > 0 \). If a function \( g \) satisfies conditions \((C1)−(C3)\), introduced below, and if

\[ \mathbb{E}\exp(g(\xi)) < \infty, \]  

then

\[ \mathbb{E}\exp((1 - \varepsilon)g((a - \delta)\tau)) < \infty, \text{ for any } \varepsilon \in (0, 1) \text{ and } \delta \in (0, a). \] (6)

The conditions \((C1)−(C3)\) are as follows:

- \((C1)\) function \( g \) is positive, increasing and differentiable;
- \((C2)\) \( \lim_{x \to \infty} g'(x) = 0 \);
- \((C3)\) there exist a constant \( \gamma \in (0, 1) \) such that

\[ \int_1^\infty \exp(-(1 - \gamma)g(x))dx < \infty \]  

and positive constants \( x_0 \) and \( A \) such that, for any \( x_0 < y \leq x/2 \),

\[ g(x) - g(x - y) \leq \gamma g(y) + A. \] (8)

It follows from condition \((C2)\) that \( \sup_{x \geq x_0} g'(x) \downarrow 0 \) as \( x_0 \to \infty \). Therefore, we may choose \( x_0 \) in condition \((C3)\) and constant \( B > 0 \) such that

\[ g'(x) < B, \text{ for } x > x_0. \] (9)
Remark 1. Conditions \((C1) - (C3)\) are given in the form that are convenient to us, they may be weakened. For example, it is not necessary to assume differentiability, and condition \((C2)\) can be adjusted to ‘dying’ growth rate that also gives us inequality \((9)\). However, inequalities \((7)\) and \((8)\) are more substantial since they target heavy-tailed “Weibull-type” and “lognormal-type” distributions.

Example 1. Here are examples of functions \(g\) that satisfy conditions \((C1) - (C3)\):
\[
g_1(x) = (\log \max(x,1))^\alpha, \ g_2(x) = (x^+)^\beta \text{ and } g_3(x) = (x^+)^\beta \log(\max(x,1)), \text{ where } \alpha > 1 \text{ and } \beta \in (0, 1).
\]
More generally, the functions \(g_1\) and \(g_3\) continue to satisfy condition \((C1) - (C3)\) if the logarithmic function therein is replaced by a “sufficiently smooth” increasing and slowly varying function.

Remark 2. Note that one can represent \((6)\) in an equivalent form as:
\[
\mathbb{E} \exp((1 - \varepsilon)g((1 - \varepsilon)a\tau)) < \infty, \text{ for any } \varepsilon \in (0, 1).
\]
On the other hand, given condition \((6)\), the inequality in \((6)\) also holds for function \(g_1\) from Example 1 with \(\delta = 0\) and any \(\varepsilon \in (0, 1)\), and for functions \(g_2\) and \(g_3\) with \(\varepsilon = 0\) and any \(\delta \in (0, a)\). Let us show this for \(g_3\). Indeed, for any \(\delta_1 \in (0, a)\) there exist \(\varepsilon_2 \in (0, 1)\) and \(\delta_2 \in (0, a)\) such that
\[
\lim_{x \to \infty} \frac{g_3((a - \delta_1)x)}{(1 - \varepsilon_2)g_3((a - \delta_2)x)} = \frac{(a - \delta_1)^\beta}{(1 - \varepsilon_2)(a - \delta_2)^\beta} < 1.
\]
Then there exists a constant \(c > 0\) such that
\[
\mathbb{E} \exp(g_3((a - \delta_1)\tau)) \leq c\mathbb{E} \exp((1 - \varepsilon_2)g_3((a - \delta_2)\tau)) < \infty.
\]

Our proof of the theorem includes two steps. First, we show the existence of a r.v. \(\tilde{\xi} \geq_{st} \xi\) that has a strong subexponential distribution, negative mean and certain finite moments. Second, we prove that the stopping time for the random walk with new increments \(\{\xi_n\}\) satisfies the conditions of the theorem.

We use the following notation and conventions. For a distribution function \(F\) on the real line, \(\overline{F}(x) = 1 - F(x)\) is its tail distribution function. For two strictly positive functions \(h_1\) and \(h_2\), equivalence \(h_1(x) \sim h_2(x)\) means that \(\lim_{x \to \infty} h_1(x)/h_2(x) = 1\). For two r.v.s \(\eta_1\) and \(\eta_2\), stochastic inequality \(\eta_1 \leq_{st} \eta_2\) means that \(\mathbb{P}(\eta_1 > x) \leq \mathbb{P}(\eta_2 > x)\), for all \(x\). For an increasing function \(g\), its (generalised) inverse function \(g^{-1}\) is defined as \(g^{-1}(t) = \inf\{x : g(x) > t\}\). Then the sets \(\{g(x) > t\}\) and \(\{x > g^{-1}(t)\}\) do coincide. A function \(f\) is slowly varying if \(f(\lambda x)/x \to 1\), as \(x \to \infty\), for \(\lambda > 0\), and regularly varying with exponent \(\alpha\) if \(f(\lambda x)/x \to \lambda^\alpha\).

## 2 Proof of the theorem

Recall the following definitions. Let \(F\) be a distribution on the real line with right-unbounded support. We say that \(F\) is long-tailed if \(\lim_{x \to -\infty} \overline{F}(x - 1)/\overline{F}(x) = 1\). Since the tail function \(\overline{F}\) is monotone non-increasing, its long-tailedness implies that \(\lim_{x \to \infty} \overline{F}(x - y)/\overline{F}(x) = 1\), for any \(y > 0\).

Further, let a distribution \(F\) have right-unbounded support and finite mean \(m = \int_0^\infty \overline{F}(y)dy\) on the positive half line. We say that \(F\) is strong subexponential and write \(F \in S^*\) if \(\int \overline{F}(x - y)\overline{F}(y)dy \sim 2m\overline{F}(x)\), as \(x \to \infty\). The strong subexponentiality is a tail property: if a distribution function \(F\) is strong subexponential and if \(G\) is another distribution function such that \(\overline{F}(x) \sim \overline{G}(x)\), then \(G\) is also strong subexponential (see, e.g., [3], Theorem 3.11).
Lemma 1. Under the assumptions (C1) – (C3), the r.v. $\hat{\xi} = g^{-1}(\ln(\hat{\xi}))$ has a strong subexponential distribution $\hat{F}$.

Theorem 3.30 from [3]. Let $F$ be a long-tailed distribution on the real line. Let $R(x) = -\ln F(x)$. Suppose that there exist $\gamma < 1$ and $A' < \infty$ such that

$$R(x) - R(x - y) \leq \gamma R(y) + A',$$

for all $x > 0$ and $y \in [0, x/2]$. If, in addition,

$$\exp(-(1 - \gamma) R(x))$$

is integrable over $[0, \infty)$,

then $F \in S^*$. 

To apply Proposition 1 we need to verify the long-tailedness of $\hat{F}$ and conditions (11) and (12). First, we show the long-tailedness of $\hat{F}$. For a fixed $y > 0$ and large $x$, we have $\mathbb{P}\{\hat{\xi} > x + y\} = K \exp(-g(x + y))$.

From the first-order Taylor expansion $g(x + y) = g(x) + yg'(z)$, for some $z \in (x, x + y)$, and from condition (C2) we get

$$1 \geq \frac{\mathbb{P}\{\hat{\xi} > x + y\}}{\mathbb{P}\{\hat{\xi} > x\}} \geq \frac{\exp(-g(x) - yg'(z))}{\exp(-g(x))} = \exp(-yg'(z)) = \exp(o(1)) = (1 + o(1)),
$$
as $x \to \infty$. Thus, the distribution of $\hat{\xi}$ is long-tailed.

Second, we verify condition (11). It is equivalent to

$$\frac{\overline{\Pi}(x - y)}{\overline{\Pi}(x)} \leq \frac{\exp(A')}{\overline{\Pi}'(y)},$$

where $\overline{\Pi}(x) = \mathbb{P}\{\hat{\xi} > \exp(g(x))\}$. We take $\gamma$ from condition (C3). Next we show the existence of an appropriate constant $A'$.

Let $x_1 = \inf\{x : \overline{\Pi}(x) < 1\}$. Since we have chosen $K > \exp(g(x_0))$, we get $x_1 \geq x_0$. We consider four cases depending on whether $\overline{\Pi}(x) = 1$ or $\overline{\Pi}(x) = K \exp(-g(x))$.

Assume $x \leq x_1$. Then inequality (13) holds if we take $A' \geq 0$.

Assume $x - y \leq x_1 < x$. Then (13) is equivalent to $K^{-1} \exp(g(x)) \leq \exp(A')$. Since $x/2 \leq x - y \leq x_1$, inequality $A' \geq g(2x_1) - \ln K$ is a sufficient condition on $A'$ to satisfy (13).
Assume $y \leq x_1 < x - y$. Then (13) is equivalent to $\exp(g(x) - g(x - y)) \leq \exp(A')$. Since $g(x - y) = g(x) - yg'(z)$, for $z \in (x - y, x)$, we have $g(x) - g(x - y) = yg'(z) < Bx_1$. Therefore, it is sufficient to assume $A' \geq Bx_1$.

Next, assume $y > x_1$. Then (13) is equivalent to $\exp(g(x) - g(x - y)) \leq K^{-\gamma} \exp(\gamma g(y) + A')$. From condition (C3) it is sufficient to assume $A' \geq A + \gamma \ln K$ for the Proposition 2 to hold.

Finally, condition (12) follows directly from (C3).

By construction, $E \exp((1 - \varepsilon)g(\xi)) = E \xi^{1-\varepsilon} < \infty$. However, we need our upper-bound to have sufficiently close mean to the original. Thus, we need the following lemma.

**Lemma 2.** Assume that conditions (C1) – (C3) hold. For any $\delta \in (0, a)$, we can introduce a r.v. $\tilde{\xi}$ such that $\tilde{\xi} = g^{-1}(\ln(\xi))$ has a strong subexponential distribution, $\tilde{\xi} \geq_{st} \xi$ and, in addition, $E\tilde{\xi} < E\xi + \delta = -a + \delta < 0$.

**Proof.** Since the distributions of $\xi$ and $\tilde{\xi}$ have right-unbounded support, for all $V > 0$ we can find $V' > V$ such that there exists r.v. $\xi$ with right tail

$$P\{\tilde{\xi} > t\} = \begin{cases} P\{\xi > t\}, & t < V, \\ P\{\xi > V\}, & V \leq t < V', \\ P\{\tilde{\xi} > t\}, & t \geq V'. \end{cases}$$

Clearly, $\xi \leq_{st} \tilde{\xi} \leq_{st} \tilde{\xi}$. Since $\tilde{\xi}$ and $\tilde{\xi}$ have the same right tail, $\tilde{\xi}$ has a strong subexponential distribution. By choosing sufficiently large $V$ we can make $E\tilde{\xi} = E(\tilde{\xi}; \tilde{\xi} \leq V) + \int_{V}^{\infty} P\{\tilde{\xi} > t\} dt = E(\xi; \xi \leq V) + \int_{V}^{\infty} P\{\xi > t\} dt < -a + \delta$. ■

### 2.2 Step two: existence of moments of the first descending epoch for strong subexponential distributions

We have introduced a r.v. $\tilde{\xi}$ with negative drift $E\tilde{\xi} = -\bar{a} = -a + \delta < 0$ and a finite moment $E \exp((1 - \varepsilon)g(\tilde{\xi})) < \infty$, such that $\xi \leq_{st} \tilde{\xi}$. Now we want to show that the stopping time $\tau$ satisfies $E \exp((1 - \varepsilon)g((a - \delta)\tau)) < \infty$.

Without loss of generality, we may assume that the distribution $\bar{F}$ of the r.v.’s $\bar{\xi}_k$ is bounded below, i.e. $\bar{\xi}_k \geq -L$ a.s., for some $L \in (0, \infty)$. Indeed, let us choose an arbitrary $L > 0$ and take $\xi_i = \max(\xi_i, -L)$, $i \geq 1$. Then the random walk $S'_0 = 0$, $S'_n = \sum_{k=1}^{n} \xi_k$ satisfies $S'_n \geq S_n$ a.s., for all $n$ and, therefore, $\tau' = \inf\{n \geq 1 : S'_n \leq 0\} \geq \tau$ a.s.

By taking $L$ large enough, we can make $E\xi' = E\tilde{\xi} - E(\tilde{\xi} + L; \tilde{\xi} \leq -L)$ as close to $E\tilde{\xi}$ as one wishes and, in particular, smaller than zero. Since $\sup_{x \leq 0} g(x) < \infty$, condition (5) implies the finiteness of $E \exp(g(\xi'))$ too. If we prove the statement of Theorem[1] for the random walk with increments $\xi_n$, then we prove it for the initial random walk, too.

We write $h(\cdot) = (1 - \varepsilon)g(\cdot)$ for short. We prove now that $E \exp(h((a - \delta)\tau)) < \infty$. Let $\chi = S_\bar{\tau}$, $\chi \in [-L, 0]$. We have

$$(a - \delta)\tau = (a - \delta)\bar{\tau} + \chi - \chi \leq ((a - \delta)\bar{\tau} + \chi) + L = \sum_{i=1}^{\bar{\tau}} (\bar{\xi}_i + a - \delta) + L.$$

Let $\psi_i = \bar{\xi}_i + a - \delta$. Thus, $E\psi_1 < 0$ and, since $P\{\psi_1 > x\} \sim P\{\xi_1 > x\}$, r.v. $\psi_1$ has a strong subexponential distribution. From inequality (9) and the first-order Taylor expansion for $h$ we get $h(x +
Further comments

In our theorem, the coefficients $(1 - \varepsilon)$ and $(1 - \delta)$ appear because the first moment of the upper-bound distribution in (11) is infinite. The following nice result may help to eliminate the coefficients under certain assumptions discussed below.

Proposition 3. (Corollary 1 in [2]) Let $\zeta$ be a nonnegative r.v. and $\mathbb{E}\zeta^\alpha < \infty$ for some $\alpha > 0$. Then there exists a r.v. $\hat{\zeta}$ such that $\mathbb{E}\hat{\zeta}^\alpha < \infty$, $\mathbb{P}\{\hat{\zeta} > t\}$ is a function of regular variation with exponent $-\alpha$, and $\zeta \leq \hat{\zeta}$.

We can apply Proposition 3 with $\alpha = 1$, $\zeta = \exp(g(\xi))$, and then the upper bound $\hat{\zeta}$ has the tail distribution $\mathbb{P}\{\hat{\zeta} \geq x\} \sim l(x)/x$, which is integrable. Here $l(x)$ is a slowly varying function. If in addition $l(x)$ is sufficiently smooth (to be justified), there is a chance to show that $\hat{\xi} = g^{-1}(\ln \hat{\zeta})$ has a strong subexponential distribution and $\mathbb{E}\exp(g(\hat{\xi})) < \infty$. Then the statement of the theorem holds with $\varepsilon = \delta = 0$.

Another way to apply Proposition 3 is to provide an alternative proof of Theorem III.3.1 in [5]. Indeed, in this case the distribution of $(\xi^+)^\alpha$ possesses an integrable majorant having a regularly varying distribution. Since any power of a regularly varying function is also a regularly varying function, the distribution of $\xi^+$ possesses a majorant having a regularly varying distribution with finite moment of order $\alpha$. And it is known that any regularly varying distribution with finite mean is strong subexponential.
Acknowledgment. The work is supported by Mathematical Center in Akademgorodok under agreement No. 075-15-2022-282 with the Ministry of Science and Higher Education of the Russian Federation.

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