On the correspondence between Light-Front Hamiltonian approach and Lorentz-covariant formulation for Quantum Gauge Theory

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The problem of the restoring of the equivalence between Light-Front (LF) Hamiltonian and conventional Lorentz-covariant formulations of gauge theory is solved for QED(1+1) and (perturbatively to all orders) for QCD(3+1). For QED(1+1) the LF Hamiltonian is constructed which reproduces the results of Lorentz-covariant theory. This is achieved by bosonization of the model and by analysing the resulting bosonic theory to all orders in the fermion mass. For QCD(3+1) we describe nonstandard regularization that allows to restore mentioned equivalence with finite number of counterterms in LF Hamiltonian.

1. Introduction

Light-Front (LF) Hamiltonian approach to Quantum Field Theory, proposed by P. Dirac\textsuperscript{a}, uses, instead of usual Lorentz coordinates $x^0, x^1, x^2, x^3$, the coordinate $x^+ = (x^0 + x^3)/\sqrt{2}$ as a "time", and $x^- = (x^0 - x^3)/\sqrt{2}$, $x^\perp = (x^1, x^2)$ as "spatial" coordinates. The quantization surface is $x^+ = \text{const}$. Corresponding Hamiltonian $P_+ = (P_0 + P_3)/\sqrt{2}$ acts in LF Fock space. The "mathematical" vacuum in this space coincides with physical vacuum state, defined by LF momentum operator $P_- \geq 0$. This simplification of the vacuum description essentially facilitates nonperturbative eigenvalue problem for the $P_+$. However there is general difficulty related with the singularity at $p_- \to 0$, specific for LF formulation. So called naive regularization of this singularity is achieved by the cutoff $|p_-| \geq \varepsilon > 0$. It cuts out zero modes ($p_- = 0$) of fields and breaks Lorentz and gauge symmetries. The other known regularization, Discretized Light Cone Quantization (DLCQ), uses the cutoff in the $x^-$ ($|x^-| \leq L$) plus periodic boundary conditions for fields in $x^-$. Then the momentum $p_-$ becomes discrete ($p_- = p_n = \pi n/L$ with integer $n$) and zero modes ($p_- = 0$) are expressed through nonzero modes via solving canonical constraints. Gauge invariance can be kept in this regularization scheme. However the solution of the constraints for zero modes is very complicated due to the nonlinearity of these constraints\textsuperscript{1}. Furthermore there is ordering problem of quantum operators in these constraints.

Both mentioned regularizations break Lorentz invariance. This can lead to a violation of the equivalence between LF and Lorentz-covariant formulations (after removing of the regularization). The question about this equivalence can be answered in perturbation theory. It was supposed in\textsuperscript{1} and proved to all orders in\textsuperscript{2} that for nongauge theories, like Yukawa model, it is sufficient to modify only the parameters, already present in naive LF Hamiltonian, to restore this equivalence. It was also found for these theories that the discussed equivalence can be maintained without any modification of LF Hamiltonian if one uses Pauli-Villars "ghost" fields for ultraviolet (UV) regularization. A possibility to use LF nonperturbative Hamiltonian approach with Pauli-Villars fields was also investigated recently in\textsuperscript{3}.

For gauge theories naively formulated on the LF (in the Light Cone gauge, $A_- = 0$) it is much more difficult to restore the equivalence with Lorentz-covariant formalism. The comparison of corresponding Feynman diagrams of LF and Lorentz-covariant perturbation theories (for the simplest choices of UV regularization) shows a difference between these perturbation theories. This difference can not be compensated by adding
finite number of counterterms to the naive LF Hamiltonian \( H \). Only the complication of the regularization scheme by including ghost fields (similar to Pauli-Villars ones) allows to restore the equivalence with finite (but large enough) number of such counterterms in LF Hamiltonian \( H \). We describe this scheme in sect. 3.

The question about mentioned equivalence beyond the perturbation theory in usual coupling constant can be answered for simpler gauge field model like two dimensional QED (QED(1+1)), however, only to all orders in different perturbation theory (in fermionic mass). In this model we can bosonize the theory and analyze the equivalent scalar field model within corresponding perturbation theory (which is nonperturbative with respect to original coupling constant).

Let us consider this more simple example of gauge theory at first, and then return to more realistic gauge theories like QCD.

2. QED(1+1) on the LF

The QED(1+1), defined originally by the Lagrangian

\[
L = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi} (i \gamma^m D_\mu - M) \Psi,
\]

(1)

can be transformed to its bosonized form \( \Phi \), described by scalar field Lagrangian

\[
L = \frac{1}{2} \left( \partial_\mu \Phi \partial^\mu \Phi - m^2 \Phi^2 \right) + \frac{M m e C}{2\pi} \cos(\theta + \sqrt{4\pi} \Phi),
\]

(2)

where \( m = e/\sqrt{\pi} \) is Schwinger boson mass (the \( e \) is original coupling), \( C = 0.577 \ldots \) is Euler constant, and the \( \theta \) is "\( \theta \)-vacuum parameter. Here fermion mass \( M \) plays the role of the coupling in bosonized theory so that perturbation theory in this coupling corresponds to chiral perturbation theory in QED(1+1). The nonpolynomial form of scalar field interaction leads in perturbation theory to infinite sums of diagrams in each finite order. It can be proved \( \Phi \) that some partial sums of these infinite sums are UV divergent in the 2nd order, whereas for full (Lorentz-covariant) Green functions these divergencies cancel (remaining only for vacuum diagrams). Therefore physical quantities are UV finite in this theory. Only at intermediate steps of our analysis we need some UV regularization.

We compare LF and Lorentz-covariant perturbation theories for such bosonized model using an effective resummation of perturbation series in coordinate representation for Feynman diagrams \( \Phi \) and also using the methods of the paper \( \Phi \). The results of this comparison can be formulated as follows.

The difference between considered perturbation theories can be eliminated in the limit of removing regularizations if we use instead of the naive LF Hamiltonian

\[
H = \int dx^2 \left( \frac{1}{8\pi} m^2 \varphi^2 : e^{i\varphi} ; e^{-i\varphi} ; -\frac{\gamma}{2} e^{i\theta} ; e^{-i\varphi} \right),
\]

\[
\gamma = \frac{M m e C}{2\pi}, \quad \varphi = \sqrt{4\pi} \Phi, \quad |p_-| \geq \varepsilon > 0, \quad (3)
\]

the "corrected" LF Hamiltonian:

\[
H = \int dx^2 \left( \frac{1}{8\pi} m^2 \varphi^2 : -B : e^{i\varphi} ; -B^* : e^{-i\varphi} : \right) -
\]

\[
-2\pi e^{-2\varepsilon} \frac{|B|^2}{m^2} \int dx^2 dy^- \left( : e^{i\varphi(x^-)} e^{-i\varphi(y^-)} : -1 \right) \times
\]

\[
\times \theta(|x^- - y^- | - \varepsilon) \frac{v(x^- - y^-)}{|x^- - y^- |}. \quad (4)
\]

Here the terms, linear in \( B \) and \( B^* \) (new coupling constants), are of the same form as in naive Hamiltonian, only the term, containing the \( |B|^2 \), is of new form (nonlocal in \( x^- \)). The \( \alpha \) is the UV regularization parameter, and the \( v(x^-) \) is some arbitrary continuous rapidly decreasing at the infinity function, going to unity as \( z \to 0 \). The coupling \( B \) can be perturbatively written as a series in \( \gamma \):

\[
B = \frac{\gamma}{2} e^{i\theta} + \sum_{n=2}^{\infty} \gamma^n B_n. \quad (5)
\]

On the other side, it is related with the sum of all connected "generalized tadpole" diagrams (i.e. diagrams with external lines attached to only one vertex), which is described by the "condensate" parameter \( A = \frac{1}{2} \langle \Omega | : e^{i(\varphi + \theta)} : |\Omega \rangle \) of the
Lorentz-covariant formulation (the |Ω⟩ is physical vacuum state in this formulation):
\[ B + |B|^2w = A, \]  
\[ w = \frac{2\pi e^{-2C}}{m^2} \int dx \frac{\theta(|x| - \varepsilon \alpha)}{|x|} \nu(x). \]  
The eq. (6) can be solved with respect to the B:
\[ B = -\frac{1}{2w} + \sqrt{\frac{1}{4w^2} + \frac{A'}{w} - A'' + iA''}, \]  
where \( A = A' + iA'' \), and the sign before the root respects the perturbation theory. Within the perturbation theory in \( \gamma \) one cannot remove UV regularization (\( \alpha \to 0 \) and therefore \( w \to \infty \)) in this expression due to UV divergencies of the coefficients \( B_n \). However, taking into account the validity of the eq. (6) to all orders in \( \gamma \), we can consider it beyond the perturbation theory. Then we use the estimation for the \( A \) at \( \alpha \to 0 \):  
\[ A = \frac{\gamma^2}{4}w + \text{const} \]  
and get for the \( B \) in \( \alpha \to 0 \) limit UV finite result:
\[ B = \frac{\gamma}{2}e^{i\hat{\theta}(\theta, M/e)} \]  
so that all information about the condensate is contained in the phase factor \( e^{i\hat{\theta}} \):
\[ \sin \hat{\theta} = \frac{2\text{Im}A}{\gamma} = \langle \Omega | \sin(\varphi + \theta) : |\Omega \rangle. \]  
Then we can make a transformation, inverse to the bosonization, but on the LF. Actually we need the expression only for one independent component \( \psi_+ \) of the bispinor field \( \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \) due to the LF constraint, permitting to write the \( \psi_- \) in terms of \( \psi_+ \). One can use the exact expression for the \( \psi_+ \) in terms of the \( \varphi \) obtained in the theory on the interval \( |x^-| \leq L \) with periodic boundary conditions [11]. We need only to modify our corrected bosonized theory by using discretized LF momentum \( p_- \) instead of continuous one and hence replacing the cutoff parameter \( \varepsilon \) by \( \pi/L \). The necessary formulae for the \( \psi_+ \) has the following form
\[ \psi_+(x) = \frac{1}{\sqrt{2L}}e^{-i\omega x - i\pi/4}Qe^{i\pi/4}e^{-i\varphi(x)} :. \]  
The operator \( \omega \) and the charge operator \( Q \) are canonically conjugated so that the \( \psi_+ \) defined by the eq. (12) has proper commutation relation with the charge. On the other side the operator \( e^{i\omega} \) shifts Fourier modes \( \psi_+ \):  
\[ e^{i\omega}\psi_n e^{-i\omega} = \psi_{n+1}. \]  
If we separate the modes related with creation and annihilation operators on the LF:
\[ \psi_+(x) = \frac{1}{\sqrt{2L}} \left( \sum_{n \geq 1} b_n e^{-i\frac{i\pi}{4}(n+\frac{1}{2})x^-} + \right. \]  
\[ + \left. \sum_{n \geq 0} d_n^+ e^{i\frac{i\pi}{4}(n+\frac{1}{2})x^-} \right), \quad b_n|0\rangle = d_n^+|0\rangle = 0, \]  
we can define the operator \( e^{i\omega} \) uniquely by specifying its action on the LF vacuum \( |0\rangle \) as follows:
\[ e^{i\omega}|0\rangle = b_0^+|0\rangle, \quad e^{-i\omega}|0\rangle = d_0^+|0\rangle. \]  
In such sense this operator is similar to a fermion.
We can now rewrite our corrected boson LF Hamiltonian in terms of \( \psi_+ \) and \( e^{i\omega} \). The result is remarkably simple:
\[ H = \int_{-L}^{L} dx^- \left( \frac{e^2}{2} \left( \partial_-^{-1}|\psi_+^+\psi_+\rangle \right)^2 - \frac{iM^2}{2} \right. \]  
\[ \times \left. \psi_+^+\partial_-^{-1}\psi_+ - \left( \frac{Me^C}{4\pi^{3/2}}e^{-i\hat{\theta}} e^{i\omega}d_0^+ + h.c. \right) \right). \]  
This fermionic LF Hamiltonian differs from canonical one (in corresponding DQLCQ scheme) only by last term, depending on zero modes and vacuum condensate parameter \( \hat{\theta} \) which can be related with chiral condensate by transforming the variables in the eq. (11):
\[ \sin \hat{\theta} = -\frac{2\pi^{3/2}}{e^C}e^{i\omega}(\Omega : \bar{\Psi}\gamma^5\Psi : |\Omega\rangle. \]  
Our result can be formally reproduced if we modify (by proper additional zero mode contribution)
the constraint equation, connecting the ψ− with the ψμ on the LF. An analogous modification of this constraint was got in the paper [4] where the method of exact operator solution of massless Schwinger model was applied.

We have some preliminary results of nonperturbative calculations of the spectrum of lowest bound states with our LF Hamiltonian. These results agree with lattice calculations in usual coordinates [1] for all values of coupling $M/e (\theta = 0)$. This confirms the hope that our LF Hamiltonian is valid nonperturbatively.

3. QCD(3+1) on the LF

Now let us discuss our results for (3+1)-dimensional QCD reported in [4]. They concern the question about the equivalence of LF and Lorentz (and gauge) covariant perturbation theories for QCD. The main difficulty in the analysis of this problem for Green functions results from additional pole singularity at $p_{-} \rightarrow 0$ in the gluon propagator in Light Cone gauge $A_{-} = 0$ [1] (we use the Mandelstam-Leibbrandt form):

$$\frac{-i\delta^{ab}}{p^2 + i0} \left( g_{\mu\nu} - \frac{p_\mu n_\nu + p_\nu n_\mu}{2p_{-} + i0} 2p_{+} \right).$$

The distortion of this pole due to LF cutoff $|p_{-}| \geq \varepsilon > 0$ does not disappear in the limit $\varepsilon \rightarrow 0$, and infinite number of new counterterms are required to compensate this distortion [1]. The simplest way to avoid this difficulty is to add small mass-like parameter $\mu^2$ in the denominator:

$$\frac{1}{2p_{+} + i0} \rightarrow \frac{1}{2p_{+} + p_{-} - \mu^2 + i0},$$

and take the limit $\varepsilon \rightarrow 0$ before $\mu \rightarrow 0$). To describe this modification with local Lagrangian we need to introduce ghost fields $A'_{\mu}$ in addition to conventional $A_{\mu}$. We write the free part of pure gluon Lagrangian as follows (using higher derivatives and the parameter $\Lambda$ for UV regularization):

$$L_0 = -\frac{1}{4} \left( f^{a,\mu\nu} \left( 1 + \frac{\partial^2}{\Lambda^2} \right) f_{\mu\nu}^a - f^{a,\mu\nu} \left( 1 + \frac{\partial^2}{\Lambda^2} + \frac{2\partial_+ \partial_-}{\mu^2} \right) f_{\mu\nu}^a \right),$$

where $f^{a,\mu\nu} = \partial_\mu A_{\nu}^a - \partial_\nu A_{\mu}^a$, $f_{\mu\nu}^a = \partial_\mu A_{\nu}^a - \partial_\nu A_{\mu}^a$ and $A_{-} = A_{-}^a = 0$. Interaction terms depend only on summary field $A_{\mu} = A_{\mu}^a + A_{\mu}^a$.

At fixed $\mu$ and $\Lambda$ we get a theory with broken gauge invariance but with preserved global $SU(3)$-invariance. We put into the Lagrangian all necessary interaction terms (but with unknown coefficients) including those that are needed for UV renormalization (these terms are local and can be taken in Lorentz covariant form due to the restoring of this symmetry in the $\varepsilon \rightarrow 0, \mu \rightarrow 0$ limit):

$$L = L_0 + c_0 \partial_{\mu} A_{\nu}^a \partial^a A_{\mu}^a + c_1 \partial_{\mu} A_{\nu}^a \partial^a A_{\mu}^a + c_2 A_{\mu}^a \partial^a \partial^a A_{\mu}^a + c_3 f^{abc} A_{\mu}^a \partial_\mu A_{\nu}^b \partial_\nu A_{\mu}^c + A_{\mu}^a A_{\rho}^b A_{\alpha}^c (c_4 f^{abc} f^{def} \partial_\mu g^{\rho_\nu} g^{\alpha_\delta} + \delta^{\alpha_\beta} \delta^{\rho_\nu} g^{\sigma_\delta} + \delta^{\rho_\nu} \delta^{\sigma_\delta} g^{\alpha_\beta}) \right).$$

In such a theory one can apply the methods of the paper [4] to compare (at $\varepsilon \rightarrow 0$) the LF perturbation theory and that taken in Lorentz coordinates within the same regularization scheme. We find that the difference between mentioned perturbation theories can be compensated by changing of the value of coefficient $c_2$ before the term of gluon mass form $A_{\mu}^a A_{\mu}^a$ in naive LF Hamiltonian of this theory. After that we can analyse further our regularized theory in Lorentz coordinates and even make Euclidean continuation.

It is possible to prove by induction to all orders [5] that in the limit $\mu \rightarrow 0$, $\Lambda \rightarrow \infty$ our theory can be made finite and coinciding with the usual renormalized (dimensionally regularized) theory in Light Cone gauge [13] (for all Green functions). To get this result one needs to choose the unknown coefficients $c_i$ before all counterterms so that the Green functions in each order coincided (after removing the regularization) with those obtained in conventional dimensionally regularized formulation and therefore satisfied Ward identities. Besides, we need to correlate the limits $\mu \rightarrow 0$ and $\Lambda \rightarrow \infty$ to avoid infrared divergencies at $\mu \rightarrow 0$. It is sufficient to take $\mu = \mu (\Lambda)$ and to require that $\mu \Lambda \rightarrow 0$ and $(\log \mu) / \Lambda \rightarrow 0$.

Our resulting LF Hamiltonian for pure $SU(3)$ gluon fields contains 7 unknown coefficients, including coefficient before gluon mass term that takes also into account the difference between LF
and Lorentz coordinate formulations of our regularized theory. The generalization of our scheme for full QCD with fermions is described in \[6\]. In this case there are 10 unknown coefficients in the LF Hamiltonian. We hope that it is possible to find an analog of Ward identities relating the coefficients \(c_i\) at fixed \(\Lambda\). This problem seems very important for our approach.

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