Dynamical and spectral Dirac systems: response function and inverse problems

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Abstract

We establish simple connections between response functions of the dynamical Dirac systems and $A$-amplitudes and Weyl functions of the spectral Dirac systems. Using these connections we propose a new and rigorous procedure to recover a general-type dynamical Dirac system from its response function as well as a procedure to construct explicit solutions of this problem.

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1 Introduction

Dynamical systems and corresponding control problems are of great interest and three books on dynamical systems [33, 34, 37] appeared already in 2015 (see also interesting references therein). In particular, hyperbolic dynamical systems (to which class the dynamical Dirac system belongs) are important. For dynamical systems and related inverse problems (including inverse problems for dynamical Schrödinger and Dirac systems) see, for instance, [6, 8, 9, 12, 30, 41, 46] and numerous references therein. At the same time inverse problems for the classical or spectral (frequency domain) self-adjoint Dirac system, also called AKNS, ZS or Dirac type system, had been actively studied since 1950s (see [38, 42]). Various interesting results were published last years (see, e.g., [1, 2, 16, 20, 22, 35, 45, 47, 54]).
The interconnections between dynamical and spectral Dirac systems, which we establish here, provide a new and rigorous way to recover a general-type dynamical Dirac system from the response function and to construct explicit solutions of this problem as well. These interconnections open also further possibilities for the study of both dynamical and spectral Dirac systems.

Spectral Dirac system has the form

\[ y'(x, z) = i(zj + jV(x))y(x, z), \quad x \geq 0 \]

\[ y'(x, z) := \frac{dy}{dx}(x, z) \], \hspace{1cm} (1.1) \]

where

\[ j = \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{bmatrix}, \quad V = \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}, \quad m_1 + m_2 =: m. \] \hspace{1cm} (1.2) \]

Here \( I_{m_i} \) is the \( m_i \times m_i \) identity matrix and \( v(x) \) is an \( m_1 \times m_2 \) matrix function.

As we already mentioned, dynamical Schrödinger and Dirac systems as well as their connections with response functions and boundary control are of growing interest [5, 6, 8, 9]. Dynamical Dirac system (Dirac system in the time-domain setup) was studied in the important recent paper [9]. The dynamical Dirac system considered in [9] is an evolution system of hyperbolic type and has the following form:

\[ iu_t + Ju_x + Vu = 0 \quad (x \geq 0, \quad t \geq 0); \]

\[ u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} p & q \\ q & -p \end{bmatrix}, \quad u_t := \frac{\partial u}{\partial t}, \] \hspace{1cm} (1.3) \]

where \( p = p(x) \) and \( q = q(x) \) are real-valued functions of \( x \), and initial-boundary conditions are given by the equalities

\[ u(x, 0) = 0, \quad x \geq 0; \quad u_1(0, t) = f(t), \quad t \geq 0. \] \hspace{1cm} (1.5) \]

Here \( f \) is a complex-valued function (so called boundary control) and the input-output map (response operator) \( R : u_1(0, \cdot) \to u_2(0, \cdot) \) is of the convolution form \( Rf = if + r * f \). The inverse problem consists in recovery
of the potential $\mathcal{V}$ from the response function $r$. This inverse problem was considered in [9] using boundary control methods.

We note that recent results on dynamical Schrödinger and Dirac equations are based on several earlier works. In his paper [10] from 1971, A.S. Blagoveščenskii considered dynamical system

$$u_{tt} - u_{xx} + \mathcal{Q}(x)u_x = 0 \quad (1.6)$$

with boundary control $u(0, t) = f(t)$, and solved inverse problem to recover $\mathcal{Q}$ from $f$. A.S. Blagoveščenskii established important connections between his problem and spectral theory of string equations. This work was developed further in [7] (see also references therein), where response operator appears in inverse problem. Finally, the inverse problem to recover the matrix potential $\mathcal{Q}(x)$ of the dynamical Schrödinger equation

$$u_{tt} - u_{xx} + \mathcal{Q}(x)u = 0 \quad (1.7)$$

from the response function was considered in [4] by S. Avdonin, M. Belishev, and S. Ivanov.

Similar to the case of the spectral Dirac and Schrödinger equations, the dynamical Dirac equation is a more general object than the dynamical Schrödinger equation. More precisely, setting in (1.3)

$$p(x) = 0, \quad q(x) = g_x(x)/g(x), \quad \text{where} \quad g_{xx}(x) = \mathcal{Q}(x)g(x), \quad (1.8)$$

and rewriting (1.3), (1.4) in the form

$$(u_1)_t = i((u_2)_x + qu_2), \quad (u_2)_t = i(-(u_1)_x + qu_1), \quad (1.9)$$

we obtain a dynamical Schrödinger equation

$$(u_1)_{tt} = (u_1)_{xx} - (q_x + q^2)u_1 = (u_1)_{xx} - (g_{xx}/g)u_1 = (u_1)_{xx} - \mathcal{Q}u_1. \quad (1.10)$$

For interesting applications of the interconnections between spectral Dirac and Schrödinger equations see, for instance, the papers [11,20,21,26,27] and references therein.
In our paper, we come from the dynamical to the spectral Dirac system taking Fourier transformation of both parts of (1.3). For that we should be able to estimate the behavior of solutions $u$ of (1.3)–(1.5). Our next section is dedicated to the necessary estimates on $u$. A simple connection between response function of the dynamical Dirac system and Weyl function of the corresponding spectral Dirac system is derived in Section 3. Recovery of the dynamical Dirac system from the response function is described in Corollary 4.5 in Section 4. The connection between response function and so-called $A$-amplitude is described in Section 4 as well. We note that interrelations between dynamical and spectral Dirac systems provide a procedure for solving the general-type inverse problem for dynamical Dirac system in much greater generality than in [9].

Explicit methods in spectral theory and in construction of solutions of equations is an interesting domain, which is actively developed (to a great extent independently from the general theory). In particular, the methods include different versions of Bäcklund-Darboux transformations, Crum-Krein transformations and commutation methods (see, e.g., [14,15,17,19,25,29,36,40,44,51,56] and various references therein). In Section 5, we recover explicitly dynamical Dirac system from response function using GBDT (generalized Bäcklund-Darboux transformation) technique from [32,49,51,54] (see also references therein).

In order to avoid the usage of the same letters for different items, we often use the calligraphic font (e.g., $V$) for functions and operators corresponding to dynamical system (1.3). As usual, $\mathbb{R}$ stands for the real axis, $\mathbb{C}$ stands for the complex plane, $\mathbb{C}_+$ is the open upper half-plane $\{z : \Im(z) > 0\}$ and $\mathbb{C}_M$ is the half-plane $\{z : \Im(z) > M > 0\}$. The notation $\overline{\theta}$ stands for the complex conjugate of $\theta$ when $\theta$ is a complex number, and for the vector with the entries which are complex conjugates of the entries of $\theta$ when $\theta$ is a vector. By l.i.m. we denote the entrywise limit of matrix functions in the $L^2$ norm on finite intervals, semi-axes or $\mathbb{R}$ depending on the context. We say that the matrix function is boundedly (continuously) differentiable when its derivative is bounded in the matrix norm (continuous). By $B(H)$ we denote the class of bounded linear operators acting in the Hilbert space $H$, $I$ is the identity operator, $I_m$ is the $m \times m$ identity matrix and $A^*$ stands for the
2 Preliminaries and estimates on the solutions \( u \) of the dynamical Dirac systems

According to [9, Theorem 1], in the case where \( p, q, f \) are continuously differentiable (i.e., \( p, q, f \in C^1 \)) and \( f(0) = f'(0) = 0 \), there is a unique classical solution \( u \) of (1.3), (1.5) and this solution admits representation

\[
\begin{align*}
  u &= u_f^T + w_f; \quad u_f^T(x, t) = f(t - x) \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad f(t) = 0 \ (t < 0); \\
  w_f^T(x, t) &= w(x, t) = \int_x^t f(t - s)\kappa(x, s)ds \ (t \geq x \geq 0), \\
  w(x, t) &= 0 \ (x > t \geq 0),
\end{align*}
\]

where \( \kappa(x, s) \ (x \leq s) \) is continuously differentiable. In particular, formulas (2.1) and (2.3) yield:

\[
  u(x, t) = 0 \text{ for } 0 \leq t < x \quad \text{(finiteness of the domain of influence).} \tag{2.4}
\]

Representation (2.1)–(2.3) is proved in [9] using Duhamel formula. Moreover, the second term \( w_f^T = w \) in the representation (2.1) of \( u \) admits (see [9, p. 6]) an expansion

\[
\begin{align*}
  w(x, t) &= \sum_{k \geq 0} A^{k+1} u_f^T(x, t) \text{ for } t \geq x \geq 0, \\
  (A g)(x, t) &= -S(V(x)g(x, t)),
\end{align*}
\]
and the operator $S$ is given by the right-hand side integrals in [9, formulas (1.8)–(1.11)]. Namely, for $t \geq x$ we have

\[ (Sh)_1(x,t) = -\frac{1}{2\sqrt{2}} \left( \int_{\lambda_1}^{\lambda_2} (ih_1 + h_2)d\ell - \int_{\lambda_2}^{\lambda_3} (ih_1 + h_2)d\ell \right. \]
\[ + \int_{\lambda_1}^{\lambda_4} (ih_1 - h_2)d\ell \left. \right) , \quad h(x,t) = \begin{bmatrix} h_1(x,t) \\ h_2(x,t) \end{bmatrix} ; \quad (2.7) \]

\[ (Sh)_2(x,t) = \frac{1}{2\sqrt{2}} \left( \int_{\lambda_1}^{\lambda_2} (h_1 - ih_2)d\ell - \int_{\lambda_2}^{\lambda_3} (h_1 - ih_2)d\ell \right. \]
\[ - \int_{\lambda_1}^{\lambda_4} (h_1 + ih_2)d\ell \left. \right) , \quad (2.8) \]

where $\lambda_i \in \mathbb{R}^2$ for $i = 1, 2, 3, 4$ and $\lambda_1 = (x,t)$, $\lambda_2 = (0, t-x)$, $\lambda_3 = (t-x, 0)$, $\lambda_4 = (x+t, 0)$. Here $\int_{\lambda_i}^{\lambda_k} d\ell$ stands for the integral along the interval $[\lambda_i, \lambda_k]$ and $\ell$ is the length.

If $h(x,t) = 0$ for $t < x$, we have $(Sh)(x,t) = 0$ for $t < x$, and therefore, in view of (2.1), we shall need here only the expression for $Sh$ when $t \geq x$. Moreover, since we assume that $f(t) = 0$ for $t < 0$, it follows from (2.3) and (2.6) that (2.5) holds for $t < x$ as well.

Let us also assume that $V$, $f$ and $f'$ are bounded:

\[ \sup_{x>0} \|V(x)\| < M_1, \quad \sup_{t>0} \left\| f(t) \left[ \begin{array}{c} 1 \\ i \end{array} \right] \right\| < c_0, \quad \sup_{t>0} \left\| f'(t) \left[ \begin{array}{c} 1 \\ i \end{array} \right] \right\| < \tilde{c}_0 . \quad (2.9) \]

**Proposition 2.1** Let $p$, $q$, $f$ be continuously differentiable and let equalities $f(t) = f'(t) = 0$ hold for $t \leq 0$. Assume that (2.9) is valid. Then the solution $u$ of the dynamical Dirac system (1.3), such that (1.5) and (2.4) are valid, satisfies the following inequalities

\[ \|u(x,t)\| \leq c_0 e^{Mt}, \quad \|u_t(x,t)\| \leq \tilde{c}_0 e^{Mt}, \quad \|u_x(x,t)\| \leq M_2 e^{Mt}; \quad (2.10) \]

where $x \geq 0$ and $t \geq 0$, $M_2 > 0$ is some constant, and $M = 2\sqrt{2}M_1$.

Moreover, the functions $u(x,t)$, $u_t(x,t)$ and $u_x(x,t)$ are continuous in the quarterplane $x \geq 0$, $t \geq 0$.

**Proof.** Using (2.6)–(2.8) and the first inequality in (2.9), it is easy to show that

\[ \|(Ag)(x,t)\| \leq 2\sqrt{2}c_k M_1 t^{k+1}/(k+1) \]

where $\lambda_i \in \mathbb{R}^2$ for $i = 1, 2, 3, 4$ and $\lambda_1 = (x,t)$, $\lambda_2 = (0, t-x)$, $\lambda_3 = (t-x, 0)$, $\lambda_4 = (x+t, 0)$. Here $\int_{\lambda_i}^{\lambda_k} d\ell$ stands for the integral along the interval $[\lambda_i, \lambda_k]$ and $\ell$ is the length.
for the case that \( \|g(x,t)\| \leq c_k t^k, \ t \geq x \) (recall that we are interested in the case \( g(x,t) = 0 \) for \( t < x \)). Hence, relation (2.1) and the second inequality in (2.9) imply that

\[
\|(A^{k+1}u^*_f)(x,t)\| \leq c_0 (2\sqrt{2}M_1)^{k+1} t^{k+1}/(k+1)!. \tag{2.11}
\]

It follows from (2.5) and (2.11) that

\[
\|w(x,t)\| \leq c_0 (e^{Mt} - 1) \quad (M = 2\sqrt{2}M_1). \tag{2.12}
\]

Formulas (2.1), (2.9) and (2.12) yield the inequality \( \|u(x,t)\| \leq c_0 e^{Mt} \).

Recall that \( u(x,t) \) is continuous at \( x = t \). According to (2.1) and (2.2), the functions \( u_x(x,t) \) and \( u_t(x,t) \) are continuous at \( x = t \). It is evident that \( u, u_x \) and \( u_t \) are continuous outside \( x = t \) as well. Thus, \( u, u_x \) and \( u_t \) are continuous in the quarterplane. □

Proposition 2.1 yields the following corollary.

**Corollary 2.2** We can apply to \( u \) the transformation:

\[
\hat{u}(x,z) = \int_0^\infty e^{izt} u(x,t)dt, \quad z \in \mathbb{C}_M = \{z : \Im(z) > M\}, \tag{2.14}
\]

where \( \hat{u} \) stands for the Fourier transformation of \( u \) (and Fourier transformation is taken, for the sake of convenience, for the fixed values \( x, z \)). Moreover, the same transformation can be applied to \( u_t \) and we have

\[
i \int_0^\infty e^{izt} u_t(x,t)dt = z\hat{u}(x,z), \quad z \in \mathbb{C}_M. \tag{2.15}
\]

7
Using the mean value theorem and the fact that \( u_x \) is continuous and satisfies the third inequality in (2.10), we obtain our next corollary.

**Corollary 2.3** Fourier transformation can be applied to \( u_x \) and
\[
\int_0^\infty e^{izt}u_x(x,t)dt = \frac{d}{dx} \hat{u}(x,z) = \hat{u}'(x,z), \quad z \in \mathbb{C}_M. \tag{2.16}
\]

Now, applying the Fourier transformation to the dynamical Dirac system (1.3), we derive
\[
z\hat{u}(x,z) + J\hat{u}'(x,z) + \mathcal{V}(x)\hat{u}(x,z) = 0. \tag{2.17}
\]

We note that various generalized Fourier transformations are successfully used (see, e.g., [3, 18, 54]) for solving inverse problems.

An estimate for \( \kappa(x,t) \) is also necessary for our further considerations. According to [9, p. 7], the formula
\[
\kappa(x,t) = \sum_{k \geq 0} A^k (Au^\delta_x); \quad (Au^\delta_x)(x,t) = -\lim_{\varepsilon \to +0} S \left( (p + iq)\delta_\varepsilon \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right); \tag{2.18}
\]

\[
\delta_\varepsilon(t) := \frac{1}{\varepsilon} \text{ for } 0 \leq t \leq \varepsilon, \quad \delta_\varepsilon(t) := 0 \text{ for } t > \varepsilon
\]
is valid. Using (2.7)–(2.9) and the second equality in (2.18), it is easy to show that
\[
\|(Au^\delta_x)(x,t)\| \leq 2\sqrt{2}M_1. \tag{2.19}
\]

From the first equality in (2.18) and from (2.19), similar to the proof of Proposition 2.1, we obtain the following proposition.

**Proposition 2.4** Let the conditions of Proposition 2.1 hold. Then
\[
\|\kappa(x,t)\| \leq Me^{Mt}. \tag{2.20}
\]

### 3 Response and Weyl functions

Recall that the \( m_2 \times m_1 \) Weyl matrix function (Weyl function) \( \varphi(z) \) of the spectral Dirac system (1.1) with the locally summable potential \( V \) is uniquely
determined by the inequality
\[
\int_0^\infty [I_{m_1} \varphi(z)]^* Y(x,z)^* Y(x,z) \left[ I_{m_1} \varphi(z) \right] dx < \infty, \quad z \in \mathbb{C}_+,
\] (3.1)

where \(Y(x,z)\) is the fundamental \(m \times m\) solution of (1.1) normalized by the condition \(Y(0,z) = I_m\) (see [22], [54, Section 2.2] and some references therein). Here \(\mathbb{C}_+\) is the open upper half-plane. Weyl functions \(\varphi\) introduced by (3.1) are analytic and contractive in \(\mathbb{C}_+\).

In view of (2.1) and (2.14) we see that
\[
\hat{u}(x,z) = \hat{f}(z)e^{izx} \begin{bmatrix} 1 \\ i \end{bmatrix} + \hat{w}(x,z),
\] (3.2)

where (according to (2.12))
\[
\|\hat{w}(x,z)\| = \left\| \int_x^\infty e^{itz} w(x,t) dt \right\| \leq \frac{c_0}{3z-M} e^{-(3z-M)x}.
\] (3.3)

Hence, for \(z \in \mathbb{C}_M\), the relation \(\hat{w}(x,z) \in L_2^2(0,\infty)\) holds and implies \(\hat{u}(x,z) \in L_2^2(0,\infty)\). In other words \(\hat{u}\) is the Weyl solution of the Dirac equation (2.17). The transformation of the spectral (i.e., frequency-domain) Dirac system (2.17) (and its solutions) into the equivalent form (1.1) is given by the formula
\[
y = \mathcal{K} \hat{u}, \quad v = iq - p, \quad \mathcal{K} = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix},
\] (3.4)

which means that \(\mathcal{K} \hat{u}\) is the Weyl solution of (1.1), (1.2), where \(m_1 = m_2 = 1\).

Finally, in view of (1.5), (2.1) and (2.2) we obtain
\[
u_1(0,t) = f(t), \quad u_2(0,t) = if(t) + \int_0^t r(t-s)f(s)ds,
\] (3.5)

where \(r(t) = \kappa_2(0,t)\) is the response function [9]. Taking into account (3.5) and estimates (2.9) and (2.20), we have
\[
\hat{u}_1(0,z) = \hat{f}(z), \quad \hat{u}_2(0,z) = \hat{f}(z)(\hat{r}(z) + i).
\] (3.6)
Now, using (3.6), we express $\hat{K}\hat{u}$ in terms of the normalized fundamental solution $Y$:

$$y(x, z) = \hat{K}\hat{u}(x, z) = \hat{f}(z)Y(x, z)K\left[\frac{1}{\hat{r}(z) + i}\right].$$

(3.7)

Recall that $y = \hat{K}\hat{u}$ is the Weyl solution, that is, $y \in L^2_2(0, \infty)$ and that the Weyl function $\varphi(z)$ is uniquely determined (see definition (3.1)) by the condition

$$Y(x, z)\left[\frac{1}{\varphi(z)}\right] \in L^2_2(0, \infty).$$

Therefore, formula (3.7) implies that

$$\varphi(z) = \hat{r}(z)/(\hat{r}(z) + 2i).$$

(3.8)

**Proposition 3.1** The response function $r(t)$ of the dynamical Dirac system (1.3) is connected with the Weyl function $\varphi(z)$ of the corresponding spectral Dirac system (1.1) (where $m_1 = m_2 = 1$) via equality (3.8).

We note that, for the classical case $m_1 = m_2 = k$, it is often more convenient to introduce Weyl functions $\varphi_H(z)$ (see, e.g., Definition 1.51 [54]) which belong to Herglotz class instead of being contractive. (Recall that Herglotz matrix functions in $\mathbb{C}_+$ are such matrix functions $\psi(z)$ that $i(\psi(z)^* - \psi(z)) \geq 0$ for all $z \in \mathbb{C}_+$.) Namely, Weyl matrix function $\varphi_H(z)$ is uniquely determined by the inequality:

$$\int_0^\infty \left[\begin{array}{cc} I_k & i\varphi_H(z)^* \end{array}\right] KY(x, z)^*Y(x, z)K^* \left[\begin{array}{c} I_k \\ -i\varphi_H(z) \end{array}\right] dx < \infty, \quad (3.9)$$

$$K := \frac{1}{\sqrt{2}} \left[\begin{array}{cc} I_k & -I_k \\ I_k & I_k \end{array}\right], \quad z \in \mathbb{C}_+. \quad (3.10)$$

A simple connection between $\varphi$ and $\varphi_H$, which easily follows from (3.1) and (3.9), implies (for $k = 1$) that (3.8) may be rewritten in the form

$$\varphi_H(z) = \hat{r}(z) + i. \quad (3.11)$$

Thus, our procedure [50, 52, 54] to recover potentials $V$ of Dirac systems (1.1) can be used to recover potentials $\mathcal{V}$ of dynamical Dirac systems from
response functions. Moreover, even for the case of continuously differentiable potentials, equality (3.11) generates important new results, especially on explicit recovery of potentials.

**Open Problem.** Show that equality (3.11) holds for much wider classes of potentials \( V \) then continuously differentiable potentials and is also valid for the non-scalar case \( m_1 = m_2 > 1 \).

### 4 Response function, \( A \)-amplitude and general-type inverse problem

#### 4.1 Inverse problem for the spectral Dirac system

Since \( V \) is bounded, the procedure to recover locally bounded potentials from Weyl functions, which is given in [50, Theorem 5.4], suffices for our purposes. A quick summary of this procedure (with some functions and operators multiplied, for convenience, by corresponding constant scalar factors) is presented below (and is close to the summary in [24, Section 2]).

First, introduce a family of convolution operators:

\[
S_l = \frac{d}{dx} \int_0^l s(x-t) \cdot dt, \quad s(x) = -s(-x)^*,
\]

where the index "\( l \)" in the notation of the operator \( S_l \) indicates the space \( L^2_k(0,l) \), in which \( S_l \) is acting, and the matrix function \( s(x) \) is associated with Dirac system. More precisely, \( s \) is introduced (for \( x > 0 \)) via the Weyl function \( \phi_H \):

\[
\begin{align*}
\phi_H & = \frac{i}{4\pi} \lim_{\eta \to \infty} \int_{-a}^a e^{-i\xi x} (\xi + i\eta)^{-2} \phi_H(\xi + i\eta) d\xi,
\end{align*}
\]

Here \( \eta > 0 \) and \( \phi_H \) is the Weyl function of a spectral Dirac system (1.1), where \( m_1 = m_2 = k \) and \( V \) is locally bounded on \([0, \infty)\). Recall that \( \phi_H \) is uniquely determined via the inequality (3.9). The notation i.m. denotes in (4.2) the entrywise limit in the norm of \( L^2(0, \infty) \). It was proved in [50] that

\[
(\xi + i\eta)^{-2} \phi(\xi + i\eta) \in L^2_{k \times k}(-\infty, \infty)
\]
for every fixed $\eta > 0$, that l.i.m. on the right-hand side of (4.2) is differentiable (and does not depend on $\eta > 0$), and so $s(x)$ is well-defined. Moreover, $s(x)$ is boundedly differentiable on the intervals $(0, l), s(+0) = \frac{1}{2}I_k$, and the operators $S_l$ are bounded and positive (i.e., $S_l > 0$) for all $l > 0$. Thus, we have

$$S_l = I + \int_0^l \omega(x - t) \cdot dt > 0, \quad \omega(x) := s'(x), \quad (4.3)$$

$$\omega(x) = \omega(-x)^*, \quad s' := \frac{d}{dx} s, \quad (4.4)$$

and $S_l, S_l^{-1} \in B\left(L^2_k(0,l)\right)$.

Next, introduce the matrix-functions

$$\theta_1(x) = [I_k \ 0]Y(x,0)K^*, \quad \theta_2(x) = [0 \ I_k]Y(x,0)K^*. \quad (4.5)$$

According to [50, formula (4.16)] we have

$$\theta_2(x) = \frac{1}{\sqrt{2}} \left( [-I_k \ I_k] - \int_0^{2x} \omega(t)^* S_k^{-1} [2s(t) \ I_k] dt \right), \quad (4.6)$$

where $S_l^{-1} (l = 2x)$ is applied to $[2s(t) \ I_k]$ columnwise. Recall that $Y$ satisfies (1.1) and is normalized by $Y(0,z) = I_m$, and that $K$ is given by (3.10). Hence,

$$KY(x,0)^*jY(x,0)K^* \equiv J, \quad Y(x,0)K^*JKY(x,0)^* \equiv j, \quad J := \begin{bmatrix} 0 & I_k \\ I_k & 0 \end{bmatrix}. \quad (4.7)$$

Using (1.1), (4.5) and the second relation in (4.7), we obtain

$$v(x) = i\theta_1'(x)J\theta_2(x)^*. \quad (4.8)$$

Finally, from (1.1), (3.10), (4.5) and the second relation in (4.7) we derive the equalities

$$\theta_1(0) = \frac{1}{\sqrt{2}} [I_k \ I_k], \quad \theta_1(x)J\theta_2(x)^* \equiv 0, \quad \theta_1'(x)J\theta_1(x)^* \equiv 0, \quad (4.9)$$

which uniquely determine $\theta_1$, assuming that $\theta_2$ is already given. We can now formulate Theorem 5.4 from [50].
Theorem 4.1 Let $\varphi_H$ be the Weyl function of a spectral Dirac system (1.1), where $m_1 = m_2 = k$ and $V$ is locally bounded, that is,

$$\sup_{0<x<l} \|V(x)\| < \infty \quad \text{for any} \quad l > 0. \quad (4.10)$$

Then $V$ can be uniquely recovered from $\varphi_H$ via the formulas (1.2), (4.8) and (4.6), (4.9). Here $\omega$ on the right-hand side of (4.6) is obtained from (4.2) (after using $\omega = s'$) and $S_{2x}$ on the right-hand side of (4.6) is given (using $\omega$) in (4.3).

Remark 4.2 For the case $m_1 = m_2 = k = 1$, equalities (4.9) are equivalent to a much simpler relation

$$\theta_1 \equiv -\frac{\theta_2}{j}, \quad (4.11)$$

and we recover $\theta_1$ from (4.11) instead of using (4.9) (if $\theta_2$ is given).

In the seminal paper [28] (see also [55] as well as some references in [28, 55]), the high energy asymptotics of the Weyl functions of Schrödinger operators is expressed in terms of the so called $A$-amplitudes. High energy asymptotics of the Weyl functions of self-adjoint (spectral) Dirac systems was studied in [16, 50] (see also [48, formula (32)]). According to [50, formula (3.25)], the asymptotic equalities

$$\varphi_H(z) = iI_k + 2i \int_0^l e^{izx} (s'(x))^* dx + o(ze^{izl}), \quad |z| \to \infty \quad (l > 0) \quad (4.12)$$

hold in all the angles $c\Im(z) \geq |\Re(z)|$ in $\mathbb{C}_+$. Formula (4.12) is an analog of the asymptotic expression for the Weyl function of a Schrödinger operator in terms of the $A$-amplitude.

Remark 4.3 Since $2i(s'(x))^*$ is an analogue of the $A$-amplitude for the Schrödinger operator case, we call this expression the $A$-amplitude of the spectral Dirac system. We note that, in M.G. Krein’s terminology, $\omega = s'$ is called the accelerant [39] (see further explanations in [2]).
4.2 Response function and $A$-amplitude

Recall that dynamical Dirac system (1.3) generates corresponding spectral Dirac system via formulas

$$V = \begin{bmatrix} p & q \\ q & -p \end{bmatrix}, \quad \bar{V} = \begin{bmatrix} 0 & v \\ \bar{v} & 0 \end{bmatrix}, \quad v = iq - p.$$  \hfill (4.13)

**Theorem 4.4** Let $r(t)$ be the response function of a dynamical Dirac system (1.3) and let the conditions of Proposition 2.1 hold. Then $s$ given by formula (4.2) is well-defined and differentiable, and we have the equality

$$r(t) = 2is'(t),$$  \hfill (4.14)

that is, the response function $r$ of the dynamical Dirac system coincides with the $A$-amplitude of the corresponding spectral Dirac system.

**Proof.** We rewrite (3.11) in the form

$$\varphi_H(z) - i = \int_0^\infty e^{itz} r(t) dt, \quad z \in \mathbb{C}_M.$$  \hfill (4.15)

In view of the equality $r(t) = \kappa_2(0, t)$ and inequality (2.20), formula (4.15) yields

$$\varphi_H(\xi + i\eta) - i = \text{l.i.m.}_{a \to \infty} \int_0^a e^{i(\xi+i\eta)t} r(t) dt, \quad \text{where } \eta > M \text{ is fixed} \quad (4.16)$$

and $\xi \in \mathbb{R}$, that is, l.i.m. is taken in the norm $L^2(-\infty, \infty)$. Hence, putting

$$r(t) = 0 \quad \text{for } t < 0,$$  \hfill (4.17)

and taking inverse Fourier transformation, we derive

$$e^{-\eta t} r(t) = \text{l.i.m.}_{a \to \infty} \frac{1}{2\pi} \int_{-a}^a e^{-i\xi t} (\varphi_H(\xi + i\eta) - i) d\xi$$  \hfill (4.18)

for each fixed $\eta > M$. Thus, it is immediate that on each semiaxis $(-\infty, l)$ we have

$$r(t) = \text{l.i.m.}_{a \to \infty} \frac{1}{2\pi} \int_{-a}^a e^{-i\xi t} (\varphi_H(\xi + i\eta) - i) d\xi.$$  \hfill (4.19)
By standard calculations, using (4.19), we obtain

\[
\begin{align*}
\int_0^x \int_0^y r(t)dt \, dy &= -\frac{1}{2\pi} \lim_{a \to \infty} \int_{-a}^a e^{-i(\xi + i\eta)x} \frac{\varphi H(\xi + i\eta) - i}{(\xi + i\eta)^2} \, d\xi + C_1x + C_2 \\
&= -\frac{1}{2\pi} e^{\eta x} \text{l.i.m.}_{a \to \infty} \int_{-a}^a e^{-i\xi x} \frac{\varphi H(\xi + i\eta) - i}{(\xi + i\eta)^2} \, d\xi + C_1x + C_2,
\end{align*}
\]

where l.i.m. is taken on the intervals \((0, l)\) with respect to \(x\).

On the other hand, Residue Theorem implies that, uniformly for \(x \geq 0\), the equality

\[
\lim_{a \to \infty} \int_{-a}^a e^{-i\xi x} (\xi + i\eta)^{-2} \, d\xi = -2\pi x e^{-\eta x}
\]

(4.21)

holds. Hence, formula (4.2) can be rewritten in the form

\[
s(x)^* = \frac{1}{2} + \frac{d}{dx} \left( \frac{i}{4\pi} e^{\eta x} \text{l.i.m.}_{a \to \infty} \int_{-a}^a e^{-i\xi x} \frac{\varphi H(\xi + i\eta) - i}{(\xi + i\eta)^2} \, d\xi \right).
\]

(4.22)

Finally, (4.14) follows from (4.20) and (4.22). \(\blacksquare\)

**Corollary 4.5** Let \(r(t)\) be the response function of a dynamical Dirac system (1.3) and let the conditions of Proposition 2.1 hold. Then the function \(v(x)\) is uniquely recovered from \(r(t)\) using the formula

\[
s(x) = \frac{1}{2} \left( 1 + i \int_0^x \overline{r(t)} \, dt \right), \quad x > 0,
\]

(4.23)

and the procedure given in Theorem 4.1 (more precisely, via the formulas (4.8) and (4.3), (4.6), (4.11)). Next, the potential \(V(x)\) is uniquely recovered from \(v(x)\) using (4.13) or, equivalently, using the first equality in (4.13) and the formulas

\[
p = -\Re(v), \quad q = \Im(v).
\]

(4.24)
4.3 Extensions and conditions on the $A$-amplitude and on response function

From (4.3) and (4.6) it is clear that in order to recover the values of the potential $V$ (of the spectral Dirac system) on $[0, l]$ we need to know $\omega(t)$ or, equivalently, $s(t)$ on the interval $[0, 2l]$. In the paper [9], the dynamical Dirac system is considered on the square $\{(x, t) : 0 \leq x \leq T, 0 \leq t \leq T\}$. In a similar way, this yields the necessity to introduce an "extended" problem (and to extend boundary condition $f$ and response function $r$ on the interval $[0, 2T]$) in order to recover $V(x)$ on $[0, T]$ from $r$. In other words, from $r$ given on the interval $[0, T]$, the potential $V$ is recovered on $[0, T/2]$ only.

It is also interesting to compare the conditions on the Weyl function and $A$-amplitude for the spectral Dirac system with the conditions on the response function for the dynamical Dirac system. For the case that $v$ is a square matrix function (or a scalar function), sufficient conditions for a Herglotz function $\varphi_H$ to be a Weyl function can be given in terms of the spectral function [42, 43], and the spectral function is connected with $\varphi_H$ via Herglotz representation. On the other hand, positive operators $S_l$ are also recovered from the spectral function (see [54, Theorem 2.11]), and in this way the conditions from [42, 43] are related to the condition of positivity of $S_l$. The invertibility of the convolution operators (which up to constant factors coincide with our operators $S_l$) is required in [2, 38] and provides the positivity of $S_l$ as well. Finally, for the case of the $m_1 \times m_2$ matrix functions $v$, sufficient conditions for $\varphi(z)$ to be a Weyl function are local boundedness of the $A$-amplitude and invertibility (positivity) of the corresponding operators $S_l$, see [54, Theorem 2.54]. We see that the positivity of the operators $S_l$ acting in $L^2(0, l)$ is required from $\varphi_H$ to be a Weyl function of a spectral Dirac system. The condition for $r$ to be a response function, which is given in [9], is again the positivity of a structured operator $S$ acting in the vector space $L^2_2(0, 2T)$, although certain discrepancies in the definition of $S$ exist (see [9, pp. 18 and 23]).

We note that necessary and sufficient conditions for the solvability of the inverse problem, which was studied in [10] for system (1.6), are again formulated in terms of positivity of certain convolution operators (Krein integral
operators).

5 A special type of inverse problem: explicit recovery of the potential

Explicit construction of the Weyl function (direct problem) and explicit recovery of the potential $V$ of the spectral Dirac system (1.1) from the Weyl function (inverse problem) were dealt with in [23,31,32,51]. We shall use the formulation of these results (for the case $m_1 = m_2 = k$) from [51, Subsection 5.1.1].

The class of potentials that we recover is called pseudo-exponential [31]. Each potential from this class is determined by a fixed number $n \in \mathbb{N}$, by an $n \times n$ matrix $A$ and by two $n \times k$ matrices $\vartheta_1$ and $\vartheta_2$ such that

$$A - A^* = i\Lambda(0)j\Lambda(0)^*, \quad \Lambda(0) := \begin{bmatrix} \vartheta_1 & \vartheta_2 \end{bmatrix}. \quad (5.1)$$

Then $V$ corresponding to the triple of parameter matrices $\{A, \vartheta_1, \vartheta_2\}$ is determined by the second equality in (1.2) and by the formula

$$v(x) = -2i\vartheta_1^*e^{ixA^*}S(x)^{-1}e^{ixA}\vartheta_2, \quad (5.2)$$

where $S(x)$ is given by the equations

$$S'(x) = \Lambda(x)\Lambda(x)^*, \quad S(0) = I_n; \quad \Lambda'(x) = -iA\Lambda(x)j, \quad \Lambda(0) = \begin{bmatrix} \vartheta_1 & \vartheta_2 \end{bmatrix}. \quad (5.3)$$

The formula defining $S$ and $\Lambda$ above is easily rewritten in the form

$$S(x) = I_n + \int_0^x \Lambda(t)\Lambda(t)^*dt > 0, \quad (5.3)$$

$$\Lambda(x) = \begin{bmatrix} \Lambda_1(x) & \Lambda_2(x) \end{bmatrix} = \begin{bmatrix} e^{-ixA}\vartheta_1 & e^{ixA}\vartheta_2 \end{bmatrix}. \quad (5.4)$$

Weyl functions $\varphi_H$ of Dirac systems with pseudo-exponential potentials are rational functions belonging to Herglotz class (i.e., $i(\varphi_H(z)^* - \varphi_H(z)) \geq 0$ for $z \in \mathbb{C}_+$). More precisely, the next statement is valid.

**Proposition 5.1** Let $v$ be a pseudo-exponential potential (i.e., let $v$ admit representation (5.2)-(5.4), where (5.1) holds). Then the Weyl function $\varphi_H$
of the spectral Dirac system (1.1), (1.2) with this \( m_1 = m_2 = k \) is given by the equality

\[
\varphi_H(z) = i I_k + 2 \vartheta_2^* (z I_n - \alpha)^{-1} \vartheta_1, \quad \alpha := A - i \vartheta_1 (\vartheta_1 + \vartheta_2)^*.
\]  

(5.5)

Vice versa, each proper rational matrix function \( \phi(z) \) such that

\[
\lim_{z \to \infty} \phi(z) = I_k; \quad i (\phi(z)^* - \phi(z)) \geq 0 \quad \text{for} \quad z \in \mathbb{C}_+
\]

(5.6)
is the Weyl function of a unique spectral Dirac system with a pseudo-exponential potential.

In order to deduce which class of response functions corresponds to Weyl functions of the form (5.5) \((k = 1)\), we take into account the equality

\[
\frac{1}{2\pi i} \int_{\Gamma} e^{-izx} (z I_n - \mathcal{L})^{-1} dz = \exp(-ix \mathcal{L}),
\]

(5.7)

which holds [53, p. 557] for anti-clockwise oriented contours \( \Gamma \) and matrices \( \mathcal{L} \) such that the spectrum of \( \mathcal{L} \) is situated inside \( \Gamma \). In view of (5.5), we rewrite (4.19) in the form

\[
\hat{r}(t) = (-2i) \lim_{a \to \infty} \frac{\vartheta_2^*}{2\pi i} \int_{\Gamma_a} e^{-izt} (z I_n - \alpha)^{-1} dz \vartheta_1,
\]

(5.8)

where \( \Gamma_a \) are anti-clockwise oriented contours consisting of the points

\[
\{ z : |z - \text{i} \eta| = a, \Im(z) < \eta \} \cup \{ z : -a \leq z - \text{i} \eta \leq a \},
\]

\( \eta > 0 \) is sufficiently large and fixed, \( \vartheta_1 \) and \( \vartheta_2 \) are column vectors \((\vartheta_1, \vartheta_2 \in \mathbb{C}^n)\), and l.i.m. may be considered on any finite interval \((0, l)\). It is immediate from (5.7) and (5.8) that

\[
r(t) = -2i \vartheta_2^* \exp(-\text{i} \alpha \cdot t) \vartheta_1.
\]

(5.9)

We easily check also directly that for \( r \) of the form (5.9) we, indeed, have

\[
\hat{r}(z) = 2 \vartheta_2^* (z I_n - \alpha)^{-1} \vartheta_1 \quad (\Im(z) > \| \alpha \|),
\]

(5.10)

that is,

\[
\varphi_H(z) = \hat{r}(z) + i = i + 2 \vartheta_2^* (z I_n - \alpha)^{-1} \vartheta_1 \quad (\Im(z) > \| \alpha \|).
\]

(5.11)

Thus, Proposition 5.1, relations (3.11) and (5.10) and uniqueness in the recovery of \( v \) from \( \varphi_H \) (see Theorem 4.1) imply the following theorem.
**Theorem 5.2** Let \( r(t) \) be the response function of a dynamical Dirac system satisfying conditions of Proposition 2.1 and assume that \( r(t) \) admits representation (5.9), where the matrix \( A = \alpha + i \vartheta_1 (\vartheta_1 + \vartheta_2)^* \) and column vectors \( \vartheta_i \in \mathbb{C}^n \) \((i = 1, 2)\) satisfy (5.1). Then the potential \( V \) of this dynamical Dirac system is given by the first equality in (4.13) and formulas (4.24) and (5.2)-(5.4).

We note that identity (5.1) may be rewritten in an equivalent form

\[
\alpha - \alpha^* = -i (\vartheta_1 + \vartheta_2) (\vartheta_1 + \vartheta_2)^*.
\]

**Remark 5.3** The requirement (in Theorem 5.2) that \( r \) should admit representation (5.9) could be substituted by some requirements on the Fourier transform \( \hat{r} \) of \( r \) (more precisely, by the conditions (5.6) on \( \phi(z) := \hat{r}(z) + i \)). In that case \( \phi \) is the Weyl function \( \varphi_H \) of some spectral Dirac system with a pseudo-exponential potential, and so matrices \( \alpha \) and \( \vartheta_i \) are recovered from \( \varphi_H \) following the procedure from [51, Theorem 5.4] (see also references therein).

It would be of interest also to construct explicit solutions \( u \) of dynamical Dirac systems similar to the construction of explicit fundamental solutions of the spectral Dirac systems in [23, 31, 32, 51].

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