ON A QUESTION OF DRINFELD ON THE WEIL REPRESENTATION: THE FINITE FIELD CASE

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Abstract. Let $F$ be a finite field of odd cardinality, and let $G = \text{GL}_2(F)$. The group $G \times G \times G$ acts on $F^2 \otimes F^2 \otimes F^3$ via symplectic similitudes, and has a natural Weil representation. Answering a question raised by V. Drinfeld, we decompose that representation into irreducibles. We also decompose the analogous representation of $\text{GL}_2(A)$, where $A$ is a cubic algebra over $F$.

Introduction

Let $F$ be a finite field of odd cardinality $q$, and let $W$, $(,)$ be a symplectic vector space over $F$ of dimension $2n$. The Heisenberg group $H(W)$, attached to $W$ and $F$, is a set $W \oplus F$ with the group law: $(w, t)(w', t') = (w + w', t + t' + \frac{(w', t)}{2})$. Let $\text{Sp}(W)$ be the isometry group of $(W, (,))$. Define a semi-direct product group $H(W) \rtimes \text{Sp}(W)$ by $[(w, t), g][w', t', g'] = [(w, t) + (g \cdot w', t'), gg']$. Fix a non-trivial character $\psi$ of $F$. According to the Stone-Von Neumann theorem, there is only one equivalence class of irreducible complex representation $\omega_\psi$ of $H(W)$ with central character $\psi$. By Weil’s celebrated paper [14], in fact $\omega_\psi$ is a representation of $H(W) \rtimes \text{Sp}(W)$ in the finite field case. The restriction of $\omega_\psi$ to $\text{Sp}(W)$, now is well-known as the Weil representation; in [6], Gérardin investigated fully this representation. Following Shimoda [12], we extend it to the symplectic similitude group $\text{GSp}(W)$ by setting $\rho = \text{Ind}_{\text{Sp}(W)}^{\text{GSp}(W)} \omega_\psi$, which does not depend on the choice of $\psi$ ([12], p. 270, Theorem).

The initial question raised by V. Drinfeld, in the finite field case, is understood roughly in the following way. Let $F^2$, $(,)$ be a symplectic space over $F$ of dimension $2$. Consider now $W = F^2 \otimes F^2 \otimes F^2$, a symplectic vector space over $F$ of dimension $8$ endowed with the symplectic form $(,)_F \otimes (,)_F \otimes (,)_F$. So there is a map from $\text{GL}_2(F) \times \text{GL}_2(F) \times \text{GL}_2(F)$ to $\text{GSp}(W)$. In this way we can define a Weil representation $\pi$ for $\text{GL}_2(F) \times \text{GL}_2(F) \times \text{GL}_2(F)$ via the restriction of $\rho$, where $\rho$ is the Weil representation of $\text{GSp}(W)$. The question is asked about the set of the quotients of $\pi$. Does it contain the representations of the form $\sigma \otimes \sigma \otimes \sigma$ for any irreducible representation $\sigma$ of $\text{GL}_2(F)$? In this paper, we answer this question and also consider its variant version. To be precise, suppose now that $E/F$ (resp. $K/F$) is a field extension of degree $2$ (resp. $3$). Take $A$ to be an étale algebra over $F$ of degree $3$, so $A$ is isomorphic to one of the algebras $F \times F \times F, F \times E, K$. We shall construct a homomorphism from $\text{GL}_2(A)$ to $\text{GSp}_2(F)$. If $A = F \times F \times F$, then $\text{GL}_2(A) \cong \text{GL}_2(F) \times \text{GL}_2(F) \times \text{GL}_2(F)$. This goes back to Drinfeld’s question. If $A = F \times E$, then $\text{GL}_2(A) \cong \text{GL}_2(F) \times \text{GL}_2(E)$, which we use the Galois descent, we construct a quadratic vector space $Q = \{ \left( \begin{array}{cc} x & \alpha \\ \alpha & y \end{array} \right), x, y \in F, \alpha \in E \}$ over $F$ of dimension $4$, with the quadratic form $Q$ defined by the determinant of the matrix. Clearly there is a map from $\text{GL}_2(E)$ to $\text{GO}(Q)$, which is defined by $h \cdot m = hmn^t$, where $h \in \text{GL}_2(E), m \in M$ and $n^t$ is the conjugate transpose of $h$. So $F^2 \otimes M$ is a symplectic vector space over $F$ of dimension $8$, and there is a map from $\text{GL}_2(F) \times \text{GL}_2(E)$ to $\text{GSp}_2(F)$. If $A = K$, in this situation, we also need to use Weil’s Galois descent to construct a map from $\text{GL}_2(K)$ to $\text{GSp}_2(F)$. The map from $\text{GL}_2(A)$ to $\text{GSp}_2(F)$ leads us to define a representation $\pi_A$ of $\text{GL}_2(A)$ via the restriction of $\rho$. The main purpose of this paper is to obtain the complete decomposition of $\pi_A$ in each case.
For the group \( G = \text{GL}_2(F) \), we write \( 1_G \) for the trivial representation of \( G \), and \( \text{St}_G \) for the Steinberg representation of \( G \). Let \( T = \{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in G \}, B = \{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G \}. \) Let \( \chi_1 \otimes \chi_2 \) be the character of \( T \) defined by
\[
\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mapsto \chi_1(a) \chi_2(d)
\]
for two characters \( \chi_1, \chi_2 \) of \( F^* \). We will denote the principal series representation \( \text{Ind}_{\mathbb{A}^1}^G(\chi_1 \otimes \chi_2) \) of \( G \) by \( \pi_{\chi_1, \chi_2} \). If \( (\sigma, V) \) is a representation of \( G \) and \( \psi \) a character of \( F^* \), we write \( \psi \cdot \sigma \) for the representation \( \psi \cdot \sigma(g) = \psi(\det g) \sigma(g) \). Let \( \text{Irr}(G) \) denote the class of all irreducible complex representations of the group \( G \). Let \( L \) be a field extension of \( F \). By Shintani’s work \([13]\), one knows that there exists the base-change map \( \text{Bc}_{L/F} : \text{Irr}(\text{GL}_2(F)) \rightarrow \text{Irr}(\text{GL}_2(L)) \), which is determined by character equalities. Our main results may be formulated as follows:

**Theorem (1).** If \( A = F \times F \times F \), \( \text{GL}_2(A) \cong \text{GL}_2(F) \times \text{GL}_2(F) \times \text{GL}_2(F) \), then
\[
\pi_A \cong \bigoplus_{\sigma \in \text{Irr}(\text{GL}_2(F))} \sigma \otimes \sigma \otimes \sigma \otimes \bigoplus_{\varphi \in \text{Irr}(F^* \times F^* \times F^*)} \left( \psi \text{St}_{\text{GL}_2(F)} \otimes \psi \text{St}_{\text{GL}_2(F)} \otimes \psi \text{St}_{\text{GL}_2(F)} \right).
\]

**Theorem (2).** If \( A = F \times E \), \( \text{GL}_2(A) \cong \text{GL}_2(F) \times \text{GL}_2(E) \), then
\[
\pi_A \cong \bigoplus_{\sigma \in \text{Irr}(\text{GL}_2(E))} \sigma \otimes \text{Bc}_{E/F}(\sigma) \otimes \bigoplus_{\varphi \in \text{Irr}(F^* \times F^* \times F^*)} \left( \psi \text{St}_{\text{GL}_2(F)} \otimes \psi \text{St}_{\text{GL}_2(F)} \right).
\]

**Theorem (3).** If \( A = K \), \( \text{GL}_2(A) \cong \text{GL}_2(K) \), then
\[
\pi_A \cong \bigoplus_{\sigma \in \text{Irr}(\text{GL}_2(K))} \text{Bc}_{K/F}(\sigma).
\]

Let us briefly review the whole story. Theorems (1) is obtained mainly by using the method in \([1]\) to decompose reducible representations. In \([1]\), Andrade considered higher rank groups. We first formulate the representation \( \pi_A \) of \( \text{GL}_2(F) \times \text{GL}_2(F) \times \text{GL}_2(F) \) concerned in this case. This can be done by following works of Gérardin and of Shinoda on the Weil representations(cf. \([6], [12]\)). Then we take two irreducible representations \( \pi_1, \pi_2 \) of \( \text{GL}_2(F) \), and determine the dimension of \( \text{Hom}_{\text{Irr}(\text{GL}_2(F) \times \text{GL}_2(F))} \left( \pi_1, \pi_1 \otimes \pi_2 \right) \). One key ingredient is that \( \pi_A \) is in fact a representation of the group \( \left( \text{GL}_2(F) \times \text{GL}_2(F) \times \text{GL}_2(F) \right) \rtimes S_3 \), where \( S_3 \) is the permutation group of \( 3 \) variables. So if we put \( \mathcal{R}(\pi_A) = \{ \pi_1 \otimes \pi_2 \otimes \pi_3 | \text{Hom}_{\text{Irr}(\text{GL}_2(F) \times \text{GL}_2(F))} \left( \pi_A, \pi_1 \otimes \pi_2 \otimes \pi_3 \right) \neq 0 \} \), by Clifford theory, \( \mathcal{R}(\pi_A) \) is \( S_3 \)-invariant. This together with the above calculations of dimension derives Theorem (1). For Theorem (2), following the method in \([1]\), we first write down the Weil representation \( \pi_A \) of \( \text{GL}_2(F) \times \text{GL}_2(E) \) in this case, and then decompose the canonical representation \( \left( \text{GL}_2(F), \text{Hom}_{\text{Irr}(\text{GL}_2(F))} \left( \pi_1, \Pi \right) \right) \) into irreducibles for each \( \Pi \in \text{Irr}(\text{GL}_2(E)) \). We did this by checking the irreducible representations of \( \text{GL}_2(E) \) one by one. The main difficulty is when \( \Pi \) is cuspidal. For that case, we use the explicit models given by \([1]\). The étale algebra \( A = K \) is a new case. We use Weil’s Galois descent to construct a map \( i \) from \( \text{GL}_2(K) \) to \( \text{GSp}_4(F) \). Through this map, we shall define a new Weil representation \( \pi_A \) for the group \( \text{GL}_2(K) \). However the explicit realisation of this representation is somehow complex, this causes the difficulty to study its irreducible components. One point is that the map \( i \) sends the standard Borel subgroup of \( \text{GL}_2(K) \) to that of \( \text{GSp}_4(F) \). By virtue of Frobenius reciprocity, we obtain the results for the principal series representations. For the cuspidal representations, we use a technique so-called “base change” to reduce to deal with some principal series representations. We should mention that this technique has been used in Gan’s paper \([5]\) to obtain Howe correspondences for exceptional groups.

Another approach to the results of this paper maybe use character theory in representations and it sometimes involves to solve certain equations. In practice, giving such equations in some sense is also complex.

The structure of this paper is as follows. The first section is devoted to giving some notations and recalling some known results. In the second section, we consider the étale algebra \( A = F \times F \times F \). In the third section, we deal with the case \( A = F \times E \). In the fourth section, we consider the case \( A = K \); there we put some calculations in two appendices.

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1. Notation and Preliminaries

1.1. The following notations will be standard through the whole paragraph, and used repeatedly without recalling their meanings:

- $F$ = a finite field with odd cardinality $q$;
- $E$ = a fixed field extension of $F$ of degree 2;
- $K$ = a fixed field extension of $F$ of degree 3;
- $\phi$ = a fixed non-trivial character of the additive group $F$;
- $\psi$ = the character of $F$, defined by $\psi(b) := \phi(ab)$ for $b \in F$, $a \in F^\times$;
- $X_A$ = the set of all non-trivial irreducible complex representations of an abelian group $A$;
- $\text{Rep}(G)$ = the category of complex representations of a finite group $G$;
- $\text{Irr}(G)$ = the class of all irreducible complex representations of a finite group $G$, up to isomorphism;
- $\hat{\sigma}$ = the contragredient representation of $\sigma$, for $\sigma \in \text{Rep}(G)$;
- If $(\sigma, V)$ is a representation of $G$, then we will denote its $G$-invariant set by $V^G$.

1.2. For later use, we regroup some results of the Weil representation of $\text{GSp}_{2n}(F)$ (cf. [9], [10], [12]).

Let $V$ be a 2n-dimensional $F$-vector space, endowed with a non-degenerate symplectic form $(\cdot, \cdot)$. To each non-trivial character $\psi$ of the additive group $F$, one can associate the Weil representation $(\omega_\psi, W_\psi)$ of the metaplectic group $\text{Mp}_{2n}(F)$ (cf. [10] Chapter 2). The exact sequence

$$1 \rightarrow \mathbb{C}^\times \rightarrow \text{Mp}_{2n}(F) \xrightarrow{p} \text{Sp}_{2n}(F) \rightarrow 1$$

is splitting. Except $n = 1$, $F = \mathbb{F}_3$, the group $\text{Sp}_{2n}(F)$ is perfect, so there exists a unique section of morphism $i$ from $\text{Sp}_{2n}(F)$ to $\text{Mp}_{2n}(F)$, such that $p \circ i = 1_{\text{Sp}_{2n}(F)}$. In the case $n = 1$, $F = \mathbb{F}_3$, we choose a certain section $i$ in the sense of Gérardin (cf. [6] p. 63, Theorem 2.4 (a’)). Via the map $i$, one obtains a representation $(\omega_\psi, W_\psi)$ of $\text{Sp}_{2n}(F)$ with respect to $\psi$, called the Weil representation. One can extend it as a representation of $\text{GSp}_{2n}(F)$ by setting $\rho_\psi = \text{Ind}_{\text{Sp}_{2n}(F)}^{\text{GSp}_{2n}(F)} \omega_\psi$. It is observed that $\rho_\psi$ is independent of $\psi$ (see [12] p. 270, Theorem). Hence we could omit $\psi$, and only write $\rho$ briefly.

The study of the Weil representation often involves an explicit realized model. We recall one so-called “the Schrödinger model”: Let $V = V_+ \oplus V_-$ be a complete polarization. Let $\{v_1, \ldots, v_n\}$ be a $F$-basis of $V_+$, and $\{v'_1, \ldots, v'_n\}$ its dual basis with respect to $(\cdot, \cdot)$. Every element $g \in \text{GSp}(V)$ can be written in the following form:

$$g = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$$

where $\alpha \in \text{End}_F(V_+), \beta \in \text{Hom}_F(V_-, V_+), \gamma \in \text{Hom}_F(V_+, V_-), \delta \in \text{End}_F(V_-)$. The group $\text{GSp}(V)$ is generated by the set $(h(a), u(b), h'(t), \omega)$ (see [1], p. 163), where $h(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, a^v$ is the contragredient of $a; u(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ for a symmetric morphism $b \in \text{Hom}_F(V_+, V_+); h'(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, t \in F^\times; \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ with $\omega(v) = -v'_i, \omega(v'_i) = v_i$. The Weil representation $\rho$ of $\text{GSp}(V)$ can be realized in the space $W_- = \mathbb{C}[V_+ \times X_F]$ of complex functions on $V_+ \times X_F$. More precisely the action of $\text{GSp}(V)$ on $W_-$ is determined by the following formulas (cf. [12] p. 270):

$$\rho(h(a))F(y, \psi) = \chi_a^v((\text{det}v, a))F(a^{\psi^{-1}}y, \psi), \quad (1.1)$$

$$\rho(u(b))F(y, \psi) = \psi(\frac{1}{2}(by, y))F(y, \psi), \quad (1.2)$$

$$\rho(\omega)F(y, \psi) = \gamma(\psi^{-\frac{1}{2}})^{-n} \sum_{z \in V_-} F(z, \psi)\psi((z, \omega^{-1}(y))), \quad (1.3)$$

$$\rho(h'(t))F(y, \psi) = F(y, \psi^{t^{-1}}), \quad (1.4)$$

where $y \in V_-, \psi \in X_F, \gamma(\psi) = \sum_{\omega \in \text{End}_F} \psi(x^2), x^\omega_q = \text{Legendre symbol } (\frac{x}{q}).$
1.3. We summarize some facts about the irreducible representations of \( GL_2(F) \) and its Borel subgroup (cf. [2, Chapter 2] and [11]).

We write \( G = GL_2(F), B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G \right\}, N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in G \right\}, T = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in G \right\}, M = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in G \right\}, Z = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in G \right\} \). Recall that \( \iota_G \) is the trivial representation of \( G \), and \( St_G \) is the Steinberg representation of \( G \). Let \( \theta \) be a regular character of \( E^\times \); the irreducible cuspidal representation of \( G \) corresponding to \( \theta \) will be denoted by \( \pi_\theta \). If \( (\sigma, V) \) is a representation of \( G \) and \( \psi \) a character of \( F^\times \), we define the representation \( (\psi \cdot \sigma, V) \) of \( G \) by \( \psi \cdot \sigma(g) = \psi(\det g)\sigma(g) \).

**Theorem 1.1** ([2, Chapter 2]). The following is a complete list of the isomorphism classes of the irreducible representations of \( G \):

1. \( \pi_{\chi_1, \chi_2} \), where \( \chi_1 \neq \chi_2 \) are characters of \( F^\times \);
2. \( \psi \cdot \iota_G \), where \( \psi \) ranges over the characters of \( F^\times \);
3. \( \psi \cdot St_G \), where \( \psi \) ranges over the characters of \( F^\times \);
4. \( \pi_\theta \), where \( \theta \) ranges over the regular characters of \( E^\times \).

The classes in the list are all distinct except that in (1) \( \pi_{\chi_1, \chi_2} \approx \pi_{\chi_2, \chi_1} \), and in (4) \( \pi_\theta \approx \pi_{\theta^{-1}} \).

**Lemma 1.2** ([2, Chapter 2]). Notations being in above Theorem, we then have \( (\pi_{\chi_1, \chi_2})^\vee \approx (\pi_{\chi_2, \chi_1})^\vee, (\psi \cdot \iota_G)^\vee \approx \psi^{-1} \cdot \iota_G, (\psi \cdot St_G)^\vee \approx \psi^{-1} \cdot St_G \) and \( (\pi_\theta)^\vee \approx \pi_{\theta^{-1}} \).

Now we investigate the representations of the group \( B \). Let \( \sigma_{\chi_1, \chi_2} \) be the character of \( B \), defined by \( \sigma_{\chi_1, \chi_2}(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}) = \chi_1(a)\chi_2(d) \). Let \( \sigma \) be the unique irreducible representation of \( M \) of the highest dimension and \( \psi \) a character of \( F^\times \). Attached to \( \sigma \) and \( \psi \), there is an irreducible representation \( \psi \otimes \sigma \) of \( B \), defined by \( (\psi \otimes \sigma)(z m) = \psi(z)\sigma(m) \) for \( z \in Z, m \in M \).

**Theorem 1.3** ([11, Theorem 7.1]). The following is a complete list of the isomorphism classes of the irreducible representations of \( B \):

1. \( \sigma_{\chi_1, \chi_2} \) for any pair \( (\chi_1, \chi_2) \) of characters of \( F^\times \);
2. \( \psi \otimes \sigma \) for any character \( \psi \) of \( Z \).

For convenience use, we describe the decomposition of the restriction to \( B \) of any irreducible representation of \( G \).

**Proposition 1.4.** (1) \( \text{Res}_B^G(\psi \cdot \iota_G) = \sigma_{\psi, \psi} \).
2. \( \text{Res}_B^G(\psi \cdot St_G) = (\sigma_{\psi, \psi}) \boxplus (\chi^2 \otimes \sigma) \).
3. \( \text{Res}_B^G \pi_{\chi_1, \chi_2} = (\sigma_{\chi_1, \chi_2}) \boxplus (\sigma_{\chi_2, \chi_1}) \boxplus (\chi_1 \chi_2 \otimes \sigma) \).
4. \( \text{Res}_B^G \pi_\theta = (\theta m^\times) \otimes \sigma \).

**Proof.** See the table in [11, p. 87].

1.4. Let \( L \) be the Galois field extension of \( F \) of degree \( n \). One knows that there exists the base-change map \( \text{Bc}_{L/F} : \text{Irr}(GL_2(F)) \rightarrow \text{Irr}(GL_2(L)) \) (cf. [13]). Now we describe the explicit behaviour of this map in terms of the classification of the irreducible representations of the group \( GL_2 \) in the cases \( n = 2, 3 \).

**Theorem 1.5.** (1) If \( [L : F] = 2 \), then

i. \( \text{Bc}_{L/F}(\xi_1) = \Pi_{L_{2, \xi_1}} \) where \( L_{2, \xi_1} \) as characters of \( L^\times \) for \( i = 1, 2 \);
ii. \( \text{Bc}_{L/F}(\psi \cdot 1_{GL_2(F)}) = \Psi \cdot 1_{GL_2(L)} \) where \( \Psi = \psi \circ N_{L/F} \) as characters of \( L^\times \);
iii. \( \text{Bc}_{L/F}(\theta \cdot St_{GL_2(F)}) = \Psi \cdot St_{GL_2(L)} \) where \( \Psi = \psi \circ N_{L/F} \) as characters of \( L^\times \);
iv. \( \text{Bc}_{L/F}(\pi_\theta) = \Pi_{L^\times} \).

(2) If \( [L : F] = 3 \), then

i. \( \text{Bc}_{L/F}(\xi_1) = \Pi_{L_{2, \xi_1}} \) where \( L_{2, \xi_1} \) as characters of \( L^\times \) for \( i = 1, 2 \);
ii. \( \text{Bc}_{L/F}(\psi \cdot 1_{GL_2(F)}) = \Psi \cdot 1_{GL_2(L)} \) where \( \Psi = \psi \circ N_{L/F} \) as characters of \( L^\times \);
iii. \( \text{Bc}_{L/F}(\theta \cdot St_{GL_2(F)}) = \Psi \cdot St_{GL_2(L)} \) where \( \Psi = \psi \circ N_{L/F} \) as characters of \( L^\times \);
ON A QUESTION OF DRINFELD ON THE WEIL REPRESENTATION: THE FINITE FIELD CASE

5

(iv) $B_{G/F}(\pi_0) = \Pi_{\theta}$ where $[F_1 : F] = 2, [L_1 : L] = 2, L_1 \supseteq F_1, \theta \in \text{Irr}(F_1^\times) - \text{Irr}(F^\times), \Theta \in \text{Irr}(L_1^\times) - \text{Irr}(L^\times)$, and $\Theta = \theta \circ N_{L_1/F}$, as characters of $L_1^\times$.

Proof. See [13] Section 4, p. 410—414]. □

1.5. As is known that one can generalize the above base-change map defined for other groups, called Shintani lifting or Shintani descent (e.g. [3]). In the article [7], Gyoda studied systematically Shintani lifting for connected linear algebraic groups. We will recall his certain results below. In addition, we also present one main result in [8] about the behaviour of the Weil representations with respect to Shintani lift.

Let $F$ be a fixed algebraic closure of $F$ with Frobenius map $\sigma$. Let $G$ be a connected linear algebraic group over $F$. Denote by $F_i$ the $\sigma^i$-fixed points of $F$. Let $Y$ be a set on which there exists a $\sigma$-action; we denote the set of $\sigma$-fixed points by $Y_\sigma$. Denote by $G(F_i)$ the $F_i$-geometric points of $G$ and $C(G(F_i))$ the set of complex valued class functions of $G(F_i)$. Fix a positive integer $m$. Via the map $\text{Gal}(F/F) \to \text{Gal}(F/F_0)$, we view the Frobenius element $\sigma$ as one generator for the group $\text{Gal}(F/F_0)$. For $0 \leq i \leq m - 1$, let us denote by $G(F_m) \rtimes \sigma^i$, the subset of the semi-direct product $G(F_m) \rtimes \text{Gal}(F/F_0)$ consisting of $(g, \sigma^i)$'s for $g \in G(F_m)$. In the article [7], following Kawanaka [9], Gyoda defined the norm maps $N_i$ as follows:

$$N_i : G(F_m) \rtimes \sigma^i \to G(F_i);$$

$[x, \sigma^i] \mapsto \alpha(x)\sigma^i(x) \cdots \sigma^i(\frac{m}{i}-1)(x)\alpha(x)^{-1},$

where $\alpha(x)$ is an element in $G(F_i)$ such that $\alpha(x)^{-1}\sigma^d(\alpha(x)) = \sigma^d(x) \cdots \sigma^d(\frac{m}{i}-1)(x)$ and $d$ are the integers given by $d = (m, i)$ and $ti \equiv d \pmod{m}$. Here $(m, i)$ denotes the greatest common divisor of $m$ and $i$.

Each norm map $N_i$ induces a bijection from the set of $G(F_m)$-conjugacy classes of $G(F_m) \rtimes \sigma^i$ onto the set of conjugacy classes of $G(F_m, \sigma^i) = G(F_m)$. Through $N_i$, one defines the $i$-restriction map from $C(G(F_m) \rtimes \text{Gal}(F/F_0))$ to $C(G(F_m, \sigma^i) \rtimes \text{Gal}(F/F_0))$ such that $(i-\text{res}(f)) \circ N_i = f|_{G(F_m) \rtimes \sigma^i}$ for any $f \in C(G(F_m) \rtimes \text{Gal}(F/F_0))$.

Lemma 1.6. (i) For any $f, g \in C(G(F_m, \sigma^i))$, we have $(f \circ g)|_{G(F_m, \sigma^i)} = (f \circ N_i \circ g \circ N_i)|_{G(F_m, \sigma^i)}$.

(ii) The $i$-restrictions define an isomorphism: $C(G(F_m) \rtimes \text{Gal}(F/F_0)) \simeq \bigoplus_{\sigma \in \text{Gal}(F/F_0)} C(G(F_m, \sigma^i))$.

Proof. See [7] p.11, Corollary 3.3 and p.1, Introduction]. □

Lemma 1.7. Let $H$ be a connected closed subgroup of $G$ defined over $F$. Then the following diagram is commutative

$$
\begin{array}{ccc}
C(G(F) \rtimes \text{Gal}(F/F_0)) & \xrightarrow{\text{Res}} & C(H(F) \rtimes \text{Gal}(F/F_0)) \\
\downarrow{\text{i-res}} & & \downarrow{\text{i-res}} \\
C(G(F_{(m, 0)})_{\sigma} & \xrightarrow{\text{Res}} & C(H(F_{(m, 0)})_{\sigma})
\end{array}
$$

Proof. See [7] p. 12, Lemma 3.6]. □

Now let $V, (\cdot, \cdot)$ be a symplectic space over $F$ and let $G = \text{GSp}_m$ be the algebraic group of symplectic similitudes of $(V, (\cdot, \cdot))$. For $0 \leq i \leq m - 1$, write $\Xi_{F_{(m, 0)}}$ for the Weil representation of $G(F_{(m, 0)})$.

Proposition 1.8. There exists a unique representation $\Xi_{F_i}$ of $G(F_i) \rtimes \text{Gal}(F/F_0)$ such that $i-\text{res}(\Xi_{F_i}) = \Xi_{F_{(m, 0)}}$ for $0 \leq i \leq m - 1$.

Proof. See [8] Theorem 4.2]. □

2. The decomposition of the Weil representation of $\text{GL}_2(F) \times \text{GL}_2(F) \times \text{GL}_2(F)$

2.1. We give some notations and formulate the mainly studied representation in this section.

In this section, we use the following notations: $G = \text{GL}_2(F), H = G \times G, B = \{(a, b) \in G\}, N = \{(1, b) \in G\}, T = \{(a, 0) \in G\}, Z = \{(a, 0) \in G\}; h(r) = \left(\begin{array}{cc}r & 0 \\ 0 & r^{-1}\end{array}\right), \ u(b) = \left(\begin{array}{cc}1 & b \\ 0 & 1\end{array}\right), h'(t) = \left(\begin{array}{cc}1 & 0 \\ 0 & 1\end{array}\right), \omega' = \{(0, 1, 0) \in G\}$.
\(\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\) in \(G, S_3 = \) the permutation group of 3 variables.

Let \(V\) be a vector space over \(F\) of dimension 2, endowed with a non-degenerate symplectic form \(\langle \cdot, \cdot \rangle\). Let \([e_1, e_2]\) be a symplectic basis of \(V\) i.e. \(\langle e_1, e_2 \rangle = 1, \langle e_2, e_1 \rangle = -1\). We attach the vector space \(V^{\otimes 3} = V \otimes_F V \otimes_F V\) with the natural symplectic form \(\langle \cdot, \cdot \rangle \otimes \langle \cdot, \cdot \rangle \otimes \langle \cdot, \cdot \rangle\), so there exists a homomorphism \(p\) from \(\left(\text{GS}(V) \times \text{GS}(V) \times \text{GS}(V)\right) \rightarrow S_3\) to \(\text{GS}(V^{\otimes 3})\). The above group \(S_3\) acts on \(V \otimes V \otimes V\) by permutations. By the fixed basis \([e_1, e_2]\) of \(V\) and \([e_1 \otimes e_j \otimes e_k|1 \leq i, j, k \leq 2]\) of \(V^{\otimes 3}\), we could identify the group \(G\) with \(\text{GL}(V)\), and the group \(\text{GS}_p(F)\) with \(\text{GS}(V^{\otimes 3})\).

Let \(\rho\) be the Weil representation of \(\text{GS}(V^{\otimes 3})\). Through the above morphism \(p\), we get a representation \(\rho'\) of the group \(\left(\text{GS}(V) \times \text{GS}(V) \times \text{GS}(V)\right) \rightarrow S_3\). Let \(\pi\) denote the restriction of \(\rho'\) to \(\text{GS}(V) \times \text{GS}(V) \times \text{GS}(V)\).

Write \(+ V^{\otimes 3} = \{x \in V^{\otimes 3}|x \in F e_1 \otimes V \otimes V\} \) and \(- V^{\otimes 3} = \{y \in V^{\otimes 3}|y \in F e_2 \otimes V \otimes V\}.\ Every element \(y \in - V^{\otimes 3}\) has the form \(y = \sum_{j=1}^2 a_{j} e_1 \otimes e_j \otimes e_k\), which corresponds to a matrix \(m = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\). So we could identify \(- V^{\otimes 3}\) with the matrix ring \(M_2(F)\) as vector space over \(F\). The representation \(\pi\) of \(G \times G \times G\) can be realized in the vector space \(W = \mathbb{C}[M_2(F) \times X_F]\) of complex functions on \(M_2(F) \times X_F\).

**Proposition 2.1.** The representation \((\pi, G \times G \times G, W)\) is given by the following formulas:

\[
\langle \pi[\rho(a), 1, 1]\rangle f(m, \psi) = f(\alpha m, \psi),
\]

\[
\langle \pi[\rho(b), 1, 1]\rangle f(m, \psi) = \phi(b \det(m)) f(m, \psi),
\]

\[
\langle \pi[\rho'(t), 1, 1]\rangle f(m, \psi) = f(m, \psi^{t^{-1}}),
\]

\[
\langle \pi[\omega, 1, 1]\rangle f(m, \psi) = \varphi^{-2}(B(m, n)) f(n, \psi),
\]

\[
\langle \pi[1, g_2, g_3]\rangle f(m, \psi) = f(\det(g_2 g_3)^{-1} m(g_3^{-1}, \psi^{\det(g_2 g_3)^{-1}}),
\]

where \(g_2, g_3 \in G, m \in M_2(F), g_3^{-1} = \) the transpose of \(g_3\), \(B(m, n) = m_{11} m_{22} + m_{21} m_{12} - m_{12} m_{21} - m_{21} m_{12}\) for \(m = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, n = \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} \in M_2(F)\).

**Proof.** (2.1) — (2.4) come directly from the formulas (1.1) — (1.4) in Section I. Consider now the formula (2.5).

Recall, for \(g \in G, \cdot e_1 := (e_1, e_2) g \begin{pmatrix} 1 \\ 0 \end{pmatrix}\) and \(g \cdot e_2 := (e_1, e_2) g \begin{pmatrix} 0 \\ 1 \end{pmatrix}\). By the fixed basis \([e_2 \otimes e_j \otimes e_k|1 \leq i, j, k \leq 2]\), we obtain \(g_2 \otimes g_3 \cdot m := g_2 g_3 m_{ij}\) for \(g_2, g_3 \in G, m \in M_2(F)\). Then, by (1.1), (1.4) in Section I we have

\[
\langle \pi[1, g_2, g_3]\rangle f(m, \psi) = \varphi\left(\begin{pmatrix} 1 & 0 \\ 0 & \det(g_2 g_3) \end{pmatrix} \begin{pmatrix} g_2 \otimes g_3 \\ 0 \end{pmatrix} \right) f(m, \psi)
\]

\[
= \varphi\left(\begin{pmatrix} g_2 \otimes g_3 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & \det(g_2 g_3)^{-1} g_2 \otimes g_3 \end{pmatrix} \right) f(m, \psi^{\det(g_2 g_3)^{-1}})
\]

\[
= \chi_{\psi}(\det(g_2 \otimes g_3, g_2^{-1} g_3^{-1} \det(g_2 g_3), m, \psi^{\det(g_2 g_3)^{-1}}) = f(\det(g_2 g_3)^{-1} m(g_3^{-1}, \psi^{\det(g_2 g_3)^{-1}})).
\]

\[\square\]

### 2.2. To decompose the representation \(\pi\), it involves to describe the \(1 \times G \times G\)-invariant part of the vector space \(W\).

We consider the set \(S = \{\pi_1, \pi_2\}\) for \(i = 1, 2, (\pi_i, V_i) \in \text{Rep}(G)\) such that \(\langle \pi_1 \otimes \pi_2\rangle_{\mathbb{C}} = \mathbb{I}_{V_i \otimes V_i}\). For each pair \((\pi_1, \pi_2) \in S\), it determines a representation \(\pi_1 \otimes \pi_2, V_1 \otimes V_2\) of the group \(H\). We write \(\text{Irr}_H(H)\) for the set of the isomorphism classes of all irreducible representations \((\pi_1 \otimes \pi_2, V_1 \otimes V_2\) of \(H\) for which \((\pi_1, \pi_2) \in S\).

Now we concentrate on the decomposition of the representation \((\pi, G \times G \times G, W)\). Following the method in Section I, we first associate a representation \((\pi'_{\alpha}, G, W[\pi_1 \otimes \pi_2])\) to any representation \(\pi_1 \otimes \pi_2 \in \text{Irr}_H(H)\), where the vector space \(W[\pi_1 \otimes \pi_2]\) consists of all functions \(f : M_2(F) \times X_F \rightarrow V_1 \otimes V_2\) such that

\[
f(\det(g_1^{-1} g_2^{-1}) g_1 m g_2, \psi^{\det(g_1^{-1} g_2^{-1})}) = (\pi_1(g_1) \otimes \pi_2(g_2)) f(m, \psi),
\]

(2.6)
for \((g_1, g_2) \in H, m \in M_2(F), \psi \in X_F\), and the action of \(G\) on \(W[\pi_1 \otimes \pi_2]\) is given by the formulas (2.1)—(2.4) in Proposition 2.1.

**Proposition 2.2.** For the representation \((\pi, G \times G \times G, W)\), we have

\[
W = \bigoplus_{\pi_1 \otimes \pi_2 \in \text{Irr}(H)} W[\pi_1 \otimes \pi_2] \otimes \tilde{V}_1 \otimes \tilde{V}_2.
\]

**Proof.** Since the representation \((\pi, W)\) of \(G \times G \times G\) is semi-simple and arises from the restriction of \(\rho\), we have

\[
W = \oplus_{\pi_1 \otimes \pi_2 \in \text{Irr}(H)} W_{\pi_1 \otimes \pi_2} \otimes \tilde{V}_1 \otimes \tilde{V}_2.
\]

Here \(W_{\pi_1 \otimes \pi_2}\) is the greatest \(\pi_1 \otimes \pi_2\)-isotypic quotient of \(W\) (cf. [10] p. 46, III.4)). Note that

\[
W_{\pi_1 \otimes \pi_2} \cong [W \otimes (V_1 \otimes V_2)]^{G \times G} \cong W[V_1 \otimes V_2]
\]

as \(G\)-module. In (2.7), we treat \(W\) as \(G \times G\)-module via the embedding \(G \times G = 1 \times G \times G \hookrightarrow G \times G \times G\).

Recall the Cartan involution: \(\theta : G \rightarrow G; g \mapsto (g')^{-1}\). It is well-known that \(\sigma = (\sigma \circ \theta)\) for \(\sigma \in \text{Irr}(G)\). Let \((H[\pi_1 \otimes \pi_2], \text{Hom}_C(V_2, V_1))\) be a representation of \(G \times G\), defined by

\[
[p_1, p_2 \circ \theta](g_1, g_2)(\varphi) = p_1(g_1) \circ \varphi \circ p_2(g_2^{-1}), \quad g_1, g_2 \in G, \varphi \in \text{Hom}_C(V_2, V_1).
\]

Define an isomorphism of vector spaces:

\[
\lambda : V_1 \otimes V_2^* \rightarrow \text{Hom}_C(V_2, V_1);
\]

\[
v_1 \otimes v_2^* \mapsto (\varphi_{v_1, v_2^*} : v_2 \mapsto (v_2^*, v_2)v_1).
\]

It can be checked that \(\lambda\) defines an intertwining operator between \((\pi_1 \otimes (\pi_2 \circ \theta))^\vee, V_1 \otimes V_2^*\) and \(\{(p_1, p_2 \circ \theta) : (H[\pi_1 \otimes \pi_2], \text{Hom}_C(V_2, V_1))\}\). To simplify calculation, we replace \((\pi_1 \otimes \pi_2, V_1 \otimes V_2^*)\) with \((\pi_1, \pi_2 \circ \theta, \text{Hom}_C(V_2, V_1))\) in (2.7), and get an isomorphic representation of \((\pi_0, G, W[\pi_1 \otimes \pi_2])\), say \((\pi_0, G, W[\pi_1, \pi_2])\), where \(W[\pi_1, \pi_2]\) is a vector space over \(\mathbb{C}\) consisting of all functions \(f : M_2(F) \times X_F \rightarrow \text{Hom}_C(V_2, V_1)\) such that

\[
f(g_1^{-1} g_2^{-1} g_{1} m g_2^t, \phi^{\text{det}(g_1^{-1}g_2^{-1})}) = \pi_1(g_1) \circ f(m, \phi) \circ \pi_2(g_2^{-1}), \quad g_1, g_2 \in G, \quad (2.8)
\]

and the action of \(G\) on \(W[\pi_1, \pi_2]\) arises naturally from the above formulas (2.1)—(2.4) in Proposition 2.1.

### 2.3. Continuing the above discussion, determine the dimension of the vector space \(W[\pi_1, \pi_2]\).

For \(\pi_1 \otimes \pi_2 \in \text{Irr}(H)\), we write \(W[\pi_1, \pi_2](\xi) = \{f(\xi) | f \in W[\pi_1, \pi_2], \xi \in M_2(F) \times X_F\}\). Now we define an \(H\)-action on the set \(M_2(F) \times X_F\) as follows:

\[
(g_1, g_2)(m, \psi) = (g_1^{-1} g_2^{-1} g_1 m g_2^t, \phi^{\text{det}(g_1^{-1}g_2^{-1})}),
\]

where \((g_1, g_2) \in H, \psi \in X_F, m \in M_2(F)\). It is observed that \(W[\pi_1, \pi_2](\xi) = \text{Fix}_{\text{Hom}_C(V_2, V_1)}(\text{Stab}_H(\xi))\) for \(\xi \in M_2(F) \times X_F\), more precisely

\[
W[\pi_1, \pi_2](\xi) = \{\phi : V_2 \rightarrow V_1 : \pi_1(g_1) \circ \phi = \pi_2(g_2^{-1} \phi)\}, \quad (g_1, g_2) \in \text{Stab}_H(\xi). (2.10)
\]

Let us determine the \(H\)-orbits in \(M_2(F) \times X_F\). They are of the following three kinds:

- **i)** Orbit \([\xi_a]\), where \(\xi_a = (\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \phi^a)\) for any \(a \in F^\times\);

- **ii)** Orbit \([\eta]\), where \(\eta = (\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \phi)\);

- **iii)** Orbit \([\delta]\), where \(\delta = (\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \phi)\).

By straightforward calculation, the corresponding stabilizer of the given representative element in each orbit has the following form:

- **i)** \(\text{Stab}_{H}(\xi_a) = \{(g, (g^{-1})') | g \in G\};

- **ii)** \(\text{Stab}_{H}(\eta) = \{(m_1, s^{-1} n_2) | s \in T, n_1, n_2 \in N\};

- **iii)** \(\text{Stab}_{H}(\delta) = \{(g_1, g_2) | g_1, g_2 \in G \text{ and } \det(g_1g_2) = 1\}.

To obtain the dimension of the vector space \(W[\pi_1, \pi_2]\), we state the lemma:
Lemma 2.3. (1) \( W[\pi_1, \pi_2](\xi_0) \neq 0 \) if and only if \( \pi_1 \simeq \pi_2 \), in which case \( \dim_C W[\pi_1, \pi_2](\xi_0) = 1 \);
(2) \( W[\pi_1, \pi_2](\eta) = 0 \) except the following cases:
   a) \( \dim_C W[\tau_1, \tau_2, \pi_1, \pi_2](\eta) = 2 \),
   b) \( \dim_C W[\psi \cdot 1_G, \psi \cdot 1_G](\eta) = 1 \),
   c) \( \dim_C W[\psi \cdot St_G, \psi \cdot St_G](\eta) = 1 \),
   d) \( \dim_C W[\psi \cdot 1_G, \psi \cdot St_G](\eta) = 1 \),
   e) \( \dim_C W[\psi \cdot St_G, \psi \cdot 1_G](\eta) = 1 \),
for the characters \( \chi_1 \neq \psi, \chi_2, \psi \) of \( F^\infty \).
(3) \( W[\pi_1, \pi_2](\delta) = 0 \) except \( \pi_1 = \pi_2 = \psi \cdot 1_G \), in that case \( \dim_C W[\psi \cdot 1_G, \psi \cdot 1_G](\delta) = 1 \) for any character \( \psi \) of \( F^\infty \).

Proof. 1) By the formula (2.10), the vector space \( W[\pi_1, \pi_2](\xi_0) \) consists of the functions \( \varphi : V_2 \rightarrow V_1 \) such that \( \pi_1(g_1) \circ \varphi = \varphi \circ \pi_2(g_2^{-1}) \) for \( (g_1, g_2) \in \text{Stab}(\xi_0) \). Hence \( W[\pi_1, \pi_2](\xi_0) \) is isomorphic to \( \text{Hom}_G(V_2, V_1) \).
2) Note that \( W[\pi_1, \pi_2](\eta) = 0 \) unless \( \pi_1, \pi_2 \) both are induced representations. Consider the induced representation \( (\pi_1, \pi_2, \psi \cdot V) = \text{Ind}^{G}_{1, \phi} (\chi_1 \otimes \chi_2) \) for \( \chi_1, \chi_2 \in \text{Irr}(F^\infty) \). The vector space \( V^N \) is generated by the following two functions \( f_{\chi_1, \chi_2} \) and \( g_{\chi_1, \chi_2} \), where
   a) the support of \( f_{\chi_1, \chi_2} \) belongs to \( B_1 \), and \( f_{\chi_1, \chi_2}(t) = \chi_1 \otimes \chi_2(t) \) for \( t \in T, n \in N \);
   b) the support of \( g_{\chi_1, \chi_2} \) belongs to \( B_2 \), and \( g_{\chi_1, \chi_2}(n \cdot t \cdot n^{-1}) = \chi_1 \otimes \chi_2(t) \) for \( t \in T, n_1, n_2 \in N \).
The action of \( T \) on \( V^N = \{ f_{\chi_1, \chi_2}, g_{\chi_1, \chi_2} \} \) is simply given by the formulas:
\[
\begin{align*}
t \cdot f_{\chi_1, \chi_2} &= \chi_1 \otimes \chi_2(t) f_{\chi_1, \chi_2} \quad \text{and} \quad t \cdot g_{\chi_1, \chi_2} &= \chi_2(t) g_{\chi_1, \chi_2}.
\end{align*}
\]
Thus, \( \dim_C \text{Hom}_G(\pi_1^N, \pi_2^N) = 2 \) for \( \chi_1 \neq \chi_2 \) in \( \text{Irr}(F^\infty) \), and (a) follows. On the other hand \( \dim_C \text{Hom}_G(\pi_1^N, \pi_2^N) = 4 \) for \( \psi \in \text{Irr}(F^\infty) \). Clearly \( (St_G)^N = (\eta_1 + \eta_2) \) and \( (\text{Id})^N = \eta_1 + \eta_2 \).
3) By the formula (2.10), \( W[\pi_1, \pi_2](\delta) \) consists of the functions \( \varphi : V_2 \rightarrow V_1 \) such that \( \pi_1(g_1) \circ \varphi = \varphi \circ \pi_2((g_2^{-1})') \) for \( (g_1, g_2) \in \text{Stab}(\delta) \). Since \( \text{Stab}(\delta) = \{(g_1, g_2) \mid g_1, g_2 \in G \text{ and } \det(g_1 g_2) = 1 \} \), we know \( \varphi = 0 \) except that \( \pi_1 = \psi_1 \circ \text{det}, \pi_2 = \psi_2 \circ \text{det} \), in which case \( W[\pi_1, \pi_2](\delta) \simeq \text{Hom}_G(\psi_1, \psi_1) \), and we get the result.

Corollary 2.4. For any irreducible representation \( \pi_1 \otimes \pi_2 \in \text{Irr}_0(H) \), the dimension of the representation \( (\pi_0, G, W[\pi_1, \pi_2]) \) has the following form:
\[
\begin{align*}
(i) &\quad \dim_C W[\tau_1, \tau_2, \pi_1, \pi_2] = q + 1, \\
(ii) &\quad \dim_C W[\psi \cdot St_G, \psi \cdot St_G] = q, \\
(iii) &\quad \dim_C W[\psi \cdot 1_G, \psi \cdot 1_G] = q + 1, \\
(iv) &\quad \dim_C W[\pi_0, \pi_0] = q - 1, \\
(v) &\quad \dim_C W[\psi \cdot St_G, \psi \cdot 1_G] = 1, \\
(vi) &\quad \dim_C W[\psi \cdot 1_G, \psi \cdot St_G] = 1,
\end{align*}
\]
for the characters \( \chi_1 \neq \chi_2, \psi \) of \( F^\infty \), the regular character \( \theta \) of \( E^\infty \). And the above lists are all the representations \( \pi_1 \otimes \pi_2 \in \text{Irr}_0(H) \), such that \( W[\pi_1, \pi_2](\delta) = 0 \).

Proof. Note that \( \dim_C W[\pi_1, \pi_2] = \sum_{\sigma \in F^\infty} \dim_C W[\pi_1, \pi_2](\xi_{\sigma}) + \dim_C W[\pi_1, \pi_2](\eta) + \dim_C W[\pi_1, \pi_2](\delta) \), so the corollary results immediately from above Lemma 2.3. □

2.4. We have already calculated the dimension of the vector space \( W[\pi_1, \pi_2] \), and it suffices to prove the main theorem in this section.

Theorem 2.5. For \( (\pi, G \times G \times G, W) \), we have:
\[
\pi \simeq \bigoplus_{\sigma \in \text{Irr}(G)} \left( \sigma \otimes \sigma \otimes \sigma \right) \oplus \bigoplus_{\phi \in \text{Irr}(F^3)} \left( \psi \cdot St_G \otimes \psi \cdot 1_G \otimes \psi \cdot 1_G \right) \oplus \left( \psi \cdot 1_G \otimes \psi \cdot 1_G \otimes \psi \cdot 1_G \otimes \psi \cdot St_G \right).
\]

Proof. Since \( \pi = \text{Res}_{G \times G \times G}^{G \times G \times G} \pi' \), by Clifford theory, the representation \( \pi \) is the direct sum of the following three kinds of representations:
\[
\begin{align*}
(1) &\quad \tau_0 \otimes \tau_0 \otimes \tau_0 \text{ for } \tau_0 \in \text{Irr}(G); \\
(2) &\quad \tau_1 \otimes \tau_1 \otimes \tau_2 + \tau_1 \otimes \tau_2 \otimes \tau_1 + \tau_2 \otimes \tau_1 \otimes \tau_1 \text{ for } \tau_1 \neq \tau_2 \in \text{Irr}(G); \\
(3) &\quad \tau_1 \otimes \tau_1 \otimes \tau_2 + \tau_1 \otimes \tau_2 \otimes \tau_1 + \tau_2 \otimes \tau_1 \otimes \tau_1 \text{ for } \tau_1 = \tau_2 \in \text{Irr}(G).
\end{align*}
\]
Now let \( \tau'_0 \otimes \tau'_1 \otimes \tau'_2 + \tau'_0 \otimes \tau'_1 \otimes \tau'_2 + \tau'_1 \otimes \tau'_2 \otimes \tau'_0 + \tau'_2 \otimes \tau'_0 \otimes \tau'_1 + \tau'_2 \otimes \tau'_1 \otimes \tau'_0 \) for three different representations \( \tau'_0, \tau'_1, \tau'_2 \) in \( \text{Irr}(G) \).
Comparing with the results in Corollary 2.4 gives the theorem. \( \square \)

3. The decomposition of the Weil representation of \( \text{GL}_2(F) \times \text{GL}_2(E) \)

3.1. We first give some notations and formulate the representation concerned in this section.

In this section, we use the following notations: \( G = \text{GL}_2(F), B = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G \), \( N = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in G \), \( T = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in G \), \( Z = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in G \); \( H = \text{GL}_2(E), B' = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in H \), \( N' = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in H \), \( T' = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in H \), \( \mathcal{Z}' = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in H \); \( h(r) = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}, u(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, h'(r) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) in \( G \) or \( H \); \( \text{Gal}(E/F) = \langle \sigma \rangle \).

If \( h \in H \) or \( M_2(E) \), we will denote its conjugate by \( h' \) or \( \overline{h} \), its transpose by \( h^t \), and let \( h^* := \overline{h} \).

Let \( V \) be a vector space over \( F \) of dimension 2, endowed with a symplectic form \( \langle , \rangle \). Let \( \{e_1, e_2\} \) be a symplectic basis of \( V \). Consider the \( E \)-vector space \( V_E = E \otimes_F V \), endowed with the symplectic form \( \langle , \rangle_{V_E} \) induced from \( V \). Define a \( \text{Gal}(E/F) \)-action on \( V_E \) by

\[
\text{Gal}(E/F) \times E \otimes_F V \rightarrow E \otimes_F V; (\sigma, \sum_i t_i \otimes e_i) \mapsto \sum_i t_i^\sigma \otimes e_i.
\]

Now let \( \mathcal{W} = V_E \otimes_E V_E \) endowed with the symmetric form \( \langle , \rangle_{\mathcal{W}} = \langle , \rangle_{V_E} \otimes \langle , \rangle_{V_E} \). On \( \mathcal{W} \), we consider the twisted Galois action defined by

\[
\text{Gal}(E/F) \times \mathcal{W} \rightarrow \mathcal{W}; (\sigma, w = \sum_i u_i \otimes v_i) \mapsto \sigma w = \sum_i v_i^\sigma \otimes u_i^\sigma.
\]

We will let \( \mathcal{W}_0 \) denote the set \( \{w \in \mathcal{W} | w = w\} \). It can be checked that the restriction of \( \langle , \rangle_{\mathcal{W}} \) to \( \mathcal{W}_0 \) defines an \( F \)-symmetric form, denoted by \( \langle , \rangle_{\mathcal{W}_0} \). Let \( q \) be its associative quadratic form given by

\[
(w_0, w'_0)_{\mathcal{W}_0} = q(w_0 + w'_0) - q(w_0) - q(w'_0), \quad w_0, w'_0 \in \mathcal{W}_0.
\]

By calculation, each \( w_0 \in \mathcal{W}_0 \) may be expressed in the form

\[
w_0 = xe_1 \otimes e_1 + ae_1 \otimes e_2 + \overline{a}e_2 \otimes e_1 + ye_2 \otimes e_2 \text{ for } x, y \in F, a \in E.
\]

Every element \( w_0 \) corresponds to a matrix \( \begin{pmatrix} x & a \\ \overline{a} & y \end{pmatrix} \). So we can identify \( \mathcal{W}_0 \) with \( M = \begin{pmatrix} x & a \\ \overline{a} & y \end{pmatrix} | x, y \in F, a \in E \).

The symmetric form \( q \) is transferred as \( q(m) = (m) \) for \( m \in M \).

Let \( \text{GO}(\mathcal{W}) \) denote the group of symmetric similitudes of \( (\mathcal{W}, \langle , \rangle_{\mathcal{W}}) \). By the definition of \( \mathcal{W} \), there exists a morphism of groups: \( \text{GL}(V_E) \times \text{GL}(V_E) \rightarrow \text{GO}(\mathcal{W}) \). We define a twisted Galois action of \( \text{Gal}(E/F) \) on \( \text{GL}(V_E) \times \text{GL}(V_E) \) by

\[
\text{Gal}(E/F) \times \left( \text{GL}(V_E) \times \text{GL}(V_E) \right) \rightarrow \text{GL}(V_E) \times \text{GL}(V_E); h = (g_1, g_2) \mapsto h^\sigma := (g_2^\sigma, g_1^\sigma).
\]

Write \( \text{GL}(V_E) \equiv \{ h \in \text{GL}(V_E) \times \text{GL}(V_E) | h^\sigma = h \} \). Then there exists an isomorphism of groups \( \text{GL}(V_E) \rightarrow \text{GL}(V_E); g \mapsto (g, g^\sigma) \). If given \( h \in \text{GL}(V_E) \times \text{GL}(V_E), w \in \mathcal{W} = V_E \otimes_E V_E \), one can verify that \( (h \cdot w)^\sigma = (\sigma \cdot (h \cdot w)) \).

So it induces a morphism from \( \text{GL}(V_E) \cong \text{GL}(V_E) \) to \( \text{GO}(M) \). By the fixed basis \( \{e_1, e_2\} \), we obtain a morphism \( i : H = \text{GL}_2(E) \rightarrow \text{GO}(M) \).

**Lemma 3.1.** The morphism \( i : H = \text{GL}_2(E) \rightarrow \text{GO}(M) \) is defined by \( H \times M \rightarrow M; (h, m) \mapsto h \overline{m^n} \), where \( \overline{n} \) is the transpose conjugate of \( h \).

**Proof.** Let \( h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H \). By definition, \( h \cdot (e_1, e_2) := (e_1, e_2) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ae_1 + ce_2; be_1 + de_2). \) So \( (h, 1) \cdot (ae_1 \oplus e_1 + \beta e_1 \oplus e_2 + \gamma e_2 \oplus e_1 + \delta e_2 \oplus e_2) = (a(a + b)y) + (e_1 \oplus e_1 + (b \beta + b\delta)e_1 \oplus e_2 + (c + d)y)e_2 \oplus e_1 + (\beta \beta + d\delta)e_2 \oplus e_2. \) Forgetting
the basis we obtain \((h, 1) \cdot \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\). Similarly, \((1, \overline{h}) \cdot (\alpha e_1 \otimes e_1 + \beta e_2 \otimes e_2 + \gamma e_3 \otimes e_1 + \delta e_2 \otimes e_2) = (\alpha \overline{a} + \beta \overline{b})e_1 \otimes e_1 + (\beta \overline{a} + \alpha \overline{c})e_1 \otimes e_2 + (\gamma \overline{a} + \delta \overline{b})e_2 \otimes e_1 + (\delta \overline{a} + \gamma \overline{c})e_2 \otimes e_2\), i.e. \((1, \overline{h}) \cdot \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha \overline{a} + \beta \overline{b} & \beta \overline{a} + \alpha \overline{c} \\ \gamma \overline{a} + \delta \overline{b} & \delta \overline{a} + \gamma \overline{c} \end{pmatrix}\).

Now we consider the symplectic vector space \(V \otimes M\) over \(F\) of dimension 8. By the above discussion, there is a map from \(G \times H\) to \(\text{GSp}(V \otimes F, M)\). Similarly as in Section 3 we consider the restriction of the Weil representation \((\pi, \text{GSp}(F, W))\) to the group \(G \times H\), denoted by \((\pi, G \times H, W)\).

**Proposition 3.2.** The Weil representation \((\pi, G \times H, W)\) can be realized in the space \(W = \mathbb{C}[M \times X_F]\), and the action of \(G \times H\) on \(W\) is given by the following formulas:

\[
(\pi([h(a), 1])F(m, \psi) = F(\rho(m, \psi), (3.1)
\]

\[
(\pi([u(b), 1])F(m, \psi) = \psi(b \det(m))F(m, \psi), (3.2)
\]

\[
(\pi([\omega, 1])F(m, \psi) = -q^{-2} \sum_{m \in M} (F(m, \psi)\psi(B(m, n)), (3.3)
\]

\[
(\pi([h, 1])F(m, \psi) = F(m, \psi)^{-1}, (3.4)
\]

\[
(\pi([h', t])F(m, \psi) = F(h^{-1}mh^{*1}N_{E/F}(\det(h))), \psi_N^{e,r(\det(h)^{-1})}), (3.5)
\]

where \(h(a), u(b), h'(t) \in G; h \in H, m \in M, \psi \in X_F; B(m, n) := q(m + n) - q(m) - q(n)\) for \(m, n \in M\).

**Proof.** (3.1), (3.2) and (3.4) follow directly from (1.1)—(1.4) in Section II

For (3.3):

\[
(\pi([1, h])F(m, \psi) = F(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} F(m, \psi) = -q^{-2} \sum_{n \in M} (\rho(\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix})F(e_2 \otimes n, \psi)((e_2 \otimes n, \overline{w}^{-1}(e_2 \otimes m)))
\]

\[
= q^{-2} \cdot \chi^2(2^2 \xi^2) \sum_{n \in M} F((A^{-1} \begin{pmatrix} 0 & A \end{pmatrix})e_2 \otimes n, \psi((e_2 \otimes n, \overline{w}^{-1}(e_2 \otimes m)))\)
\]

\[
= q^{-2} \cdot \chi^2(2^2 \xi^2) \sum_{n \in M} F(e_2 \otimes n, \psi((e_2 \otimes n, \overline{w}^{-1}(e_2 \otimes m)))\)
\]

\[
= -q^{-2} \sum_{n \in M} F(e_2 \otimes n, \psi((e_2 \otimes n, \overline{w}^{-1}(e_2 \otimes m)))\)
\]

Forgetting the basis, we get the formula (3.3). □

**Remark 3.3.** The above representation \(\pi\) of \(G \times H\) is compatible with \(\rho_Q\) constructed by Andrade in [1] p.35, Theorem 5) (There the \(\rho_Q\) is defined more directly).
3.2. To decompose the representation \( \pi \) of \( G \times H \), let us calculate the dimension of the vector space \( W[\pi_1] \) (see below for its definition).

Let \( U = \{ x \in E^\times \mid N_{E/F}(x) = 1 \} \). We regard \( U \) as a subgroup of \( H \). Let \( \text{Irr}_0(H) \) be the set of the isomorphism classes of the irreducible representations \( \pi_1 \) of \( H \), such that \( \pi_1 \) is trivial over \( U \). For each representation \( (\pi_1, V_1) \in \text{Irr}_0(H) \), we associate a representation \( (\pi_0, W[\pi_1]) \) of \( G \), where the vector space \( W[\pi_1] \) consists of functions: \( f : M \times X_F \to V_1 \) such that

\[
\mathcal{f}(h \eta^* N_{E/F}(\det(h))^{-1}, \eta^*) = \pi_1(h) \circ f(m, \eta)
\]

(3.6) for \( h \in H, (m, \eta) \in M \times X_F \) and the action of \( G \) on \( W[\pi_1] \) is given by the formulas (3.3) – (3.4).

**Proposition 3.4.** For the representation \( (\pi, G \times H, W) \), we have the following decomposition:

\[
\pi \simeq \bigoplus_{(\pi_1, V_1) \in \text{Irr}_0(H)} W[\pi_1] \otimes V_1.
\]

**Proof.** The representation \( W \) has the decomposition \( \pi = \oplus_{(\pi_1, V_1) \in \text{Irr}_0(H)} W_{\pi_1} \otimes V_1 \) and then \( W_{\pi_1} \simeq (W \otimes V_1)^H \simeq W[\pi_1] \). The action of \( G \) on \( W[\pi_1] \) arises from the definition of \( \pi \) and above isomorphisms. \( \square \)

For \( (\pi_1, V_1) \in \text{Irr}_0(H) \), we define \( W[\pi_1](\xi) = \{ f(\xi) \mid f \in W[\pi_1] \} \) and an \( H \)-action on the set \( M \times X_F \) as follows:

\[
h \cdot (m, \psi) := (h \eta^* N_{E/F}(\det(h))^{-1}, \psi^* N_{E/F}(\det(h)) \eta)
\]

\( h \in H, \psi \in X_F, m \in M \). (3.7)

It is observed that \( W[\pi_1](\xi) = \bigoplus_{(\pi_1, V_1) \in \text{Irr}_0(H)} \bigoplus_{(\pi_1, V_1) \in \text{Irr}_0(H)} W[\pi_1] \otimes V_1 \) for any \( \xi \in M \times X_F \).

**Proposition 3.5.** Consider the action of \( H \) on \( M \times X_F \).

1) The distinct orbit of this action can be described as follows:

(i) Orbit(\( \xi_a \)), where \( \xi_a = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \phi \) for any \( a \in F^\times \);

(ii) Orbit(\( \eta \)), where \( \eta = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \phi \);

(iii) Orbit(\( \delta \)), where \( \delta = \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right), \phi \).

2) The corresponding stabilizer of the canonical element in each orbit is presented as follows:

(i) \( \text{Stab}_H(\xi_a) = U_2(E) \), where \( U_2(E) = \{ h \in H | h \eta^* = 1 \} \);

(ii) \( \text{Stab}_H(\eta) = H_1 \), where \( H_1 = \{ h = \left( \begin{array}{cc} u & b \\ 0 & v \end{array} \right) | u, v \in U, b \in E \} \);

(iii) \( \text{Stab}_H(\delta) = H_2 \), where \( H_2 = \{ h = \left( \begin{array}{cc} u & b \\ 0 & v \end{array} \right) | h \in H, \det(h) \in U \} \).

**Proof.** We transfer the \( H \)-action \( \cdot \) to another \( H \)-action \( \circ \), where \( \circ \) is defined by \( h \circ (m, \psi) := (h \eta^* N_{E/F}(\det(h))^{-1}, \psi^* N_{E/F}(\det(h)) \eta) \).

Since \( \alpha : H \to H; h \mapsto h \eta^* \) is a group isomorphism and \( \alpha(h) \circ (m, \psi) = h \cdot (m, \psi) \), it reduces to consider the action \( \circ \).

1) Every element \( m \in M \) corresponds to a hermitian form on a 2-dimensional \( E \)-vector space \( V \). By the property of hermitian form over finite fields and the surjection of the morphism \( N_{E/F} : E^\times \to F^\times \), one can find \( h \in H \) such that \( h \eta^* N_{E/F}(\det(h))^{-1} \) (\( a, b \in U, b \in E, N_{E/F}(a) + N_{E/F}(b) = 1 \)).

2) It is straightforward. \( \square \)

**Lemma 3.6.**

(i) \( U_2(E) = \left( \begin{array}{cc} u & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} a & b \\ -b^* & a^* \end{array} \right) | u, a, b \in E, N_{E/F}(a) + N_{E/F}(b) = 1 \};

(ii) \( |U_2(E)| = (q - 1)q(q + 1)^2 \);

(iii) If we choose an element \( e_{-1} \in E^\times \), such that \( e_{-1}^q = -1 \), then the element \( \left( \begin{array}{cc} 0 & 1 \\ -1 & e_{-1}^q \end{array} \right) \notin B'U_2(E) \);
Lemma 3.7. Let \((\pi_1, V_1)\) be an irreducible representation of \(H\) in \(\text{Irr}_0(H)\). Then:

(1) In the case \(\pi_1 = \Psi \cdot 1_H\) with \(\Psi \in \text{Irr}(E^\times)\), we have

(a) \(W[\pi_1] = \Psi \cdot 1_H\), \(\psi \neq \Psi^q\).
(b) \(\dim C W[\pi_1]|_{\xi_0} = \dim C W[\pi_1]|_{\eta} = \dim C W[\pi_1]|_{\delta} = 1\), if \(\Psi = \Psi^q\).

(2) In the case \(\pi_1 = \Psi \cdot \text{St}_H\) with \(\Psi \in \text{Irr}(E^\times)\), we have

(a) \(W[\pi_1] = 0\), \(\psi \neq \Psi^q\).
(b) \(\dim C W[\pi_1]|_{\xi_0} = \dim C W[\pi_1]|_{\eta} = 1\) and \(\dim C W[\Psi \cdot \text{St}_H]|_{\delta} = 0\), if \(\Psi = \Psi^q\).

(3) In the case \(\pi_1 = \Pi_{\Lambda, \Sigma}^1 \Lambda \neq \Sigma \in \text{Irr}(E^\times)\), we have

(a) \(\dim C W[\Pi_{\Lambda, \Sigma}^1]|_{\xi_0} = 1\), \(\dim C W[\Pi_{\Lambda, \Sigma}^1]|_{\eta} = 2\), \(\dim C W[\Pi_{\Lambda, \Sigma}^1]|_{\delta} = 0\), if \(\Lambda = \Lambda^q\) and \(\Sigma = \Sigma^q\).
(b) \(\dim C W[\Pi_{\Lambda, \Sigma}^1]|_{\xi_0} = 1\), \(\dim C W[\Pi_{\Lambda, \Sigma}^1]|_{\eta} = \dim C W[\Pi_{\Lambda, \Sigma}^1]|_{\delta} = 0\), if \(\Lambda = \Sigma^q\) and \(\Sigma = \Lambda^q\).

Proof. (i)(ii) follow from [1] pp. 242-243.

(iii) \[
B'U_2(E) = B' \cdot SU_2(E) = \left\{\begin{array}{cc} x & y \\ z & \end{array} \right\}, \quad x, z \in E^\times; a, b, y \in E; N_{E/F}(a) + N_{E/F}(b) = 1
\]

\[
=(\begin{array}{cc} xa - yb^q & xb + ya^q \\ -zb^q & za^q \end{array}) | x, z \in E^\times; a, b, y \in E; N_{E/F}(a) + N_{E/F}(b) = 1.
\]

So, \(N_{E/F}(-b^q) + N_{E/F}(a^q) = N_{E/F}(c) \neq 0\). On the other hand, \(N_{E/F}(-1) + N_{E/F}(-c^q) = 0\); this implies the result.

(iv) It is enough to check that they have the same cardinality, i.e.

\[
|H| = |B'U_2(E)| + |B' \cdot (q - 1)q.
\]

By calculation,

\[
B' \cap U_2(E) = \left\{x \begin{array}{c} \ u \\ \ 0 \end{array} \right\}, \quad [a, a \in U].
\]

So

\[
|B'U_2(E)| = |B'||U_2(E)\cap B'| = |B'| \cdot (q - 1)q.
\]

Now let

\[
g = \left(\begin{array}{cc} aa & ub \\ -b^q & a^q \end{array} \right) \in U_2(E), \quad g_0 = \left(\begin{array}{c} 0 \\ 1 \end{array} \right)
\]

Then

\[
g_0 g_0 g_0^{-1} = \left(g_0\begin{array}{cc} (a - be_{-1})^q & b^q \\ -a \cdot e_{-1} b^q & a^q \end{array}\right)\g_0^{-1} = \left(\begin{array}{cc} (a - be_{-1})^q & b^q \\ -a \cdot e_{-1} & a^q \end{array}\right)\]|N_{E/F}(a) + N_{E/F}(b) = 1.
\]

So

\[
|B' \cap g_0 U_2(E)g_0^{-1}| = |SU_2(E)| = (q - 1)q(q + 1).
\]

From this, we obtain

\[
|g_0 U_2(E)g_0^{-1}/B' \cap g_0 U_2(E)g_0^{-1}| = q + 1,
\]

and

\[
|B'g_0 U_2(E)| = |B'||g_0 U_2(E)g_0^{-1}/B' \cap g_0 U_2(E)g_0^{-1}| = |B'| \cdot (q + 1).
\]

Finally

\[
|B'U_2(E)| + |B'g_0 U_2(E)| = |B'| \cdot (q - 1)q + |B'| \cdot (q + 1) = |B'| \cdot (q^2 + 1) = |H|.
\]

\[\blacksquare\]

Let us determine the dimension of the vector space \(W[\pi_1]\) for each \(\pi_1 \in \text{Irr}_0(H)\).
For the other kind of \( \Lambda \neq \Sigma \in \text{Irr}(E^\times) \), \( W[\Pi_{\Lambda, \Sigma}] = 0 \).

(4) In the case \( \pi_1 = \Pi_0 \), where \( \Theta \in \text{Irr}(E^\times) \) for some quadratic extension \( E_1 \) of \( E \), we have

\[ W[\Pi_{\Theta}] = 0. \]

Proof. 1) By Proposition 3.3(2), we know that the image of the map \( \text{det} : \text{Stab}_H(\xi) \rightarrow E^\times \) is \( U \) for \( \xi = \xi_a, \eta, \delta \).

So \( V^\lambda_{\text{Stab}_H(\xi)} \) is trivial over \( U \), otherwise.

2) Let \( (\pi_2, V_2) = \text{Ind}_H^U(\Psi \cdot 1_B) \). By definition, \( W[\pi_2](\xi) = W[\Psi \cdot 1_H](\xi) \oplus W[\Psi \cdot \text{St}_H](\xi) \).

The dimension of \( W[\Psi \cdot 1_H](\xi) \) is known, so it remains to calculate the dimension \( W[\pi_2](\xi) \) for \( \xi = \xi_a, \eta, \delta \).

(a) \( \xi = \xi_a \). In this case, \( \text{Stab}_H(\xi) = U_2(E) \) and \( H = B'U_2(E) \cup B' \left( \begin{array}{cc} 0 & 1 \\ 1 & -1 \end{array} \right) U_2(E) \) for \( e \in E^\times \) satisfying

\[ \text{N}_{E/F}(e_{-1}) = -1 \]. So \( V^\lambda_{U_2(E)} \) is generated by the following functions \( \alpha, \beta \):

1. The support of \( \alpha \) is \( B'U_2(E), \alpha(bu) = \Psi \cdot 1_b(b) \) for \( b \in B', u \in U_2(E) \), and \( \Psi \cdot 1_b \) is trivial over \( B' \cap U_2(E) \).

2. The support of \( \beta \) is \( B' \left( \begin{array}{cc} 0 & 1 \\ 1 & -1 \end{array} \right) U_2(E) \), \( \beta(b \left( \begin{array}{cc} 0 & 1 \\ 1 & -1 \end{array} \right) u) = \Psi \cdot 1_b(b) \) and \( \Psi \cdot 1_b \) is trivial over \( B' \cap \left( \begin{array}{cc} 0 & 1 \\ 1 & -1 \end{array} \right) U_2(E) \).

By Lemma 3.6 \( B' \cap U_2(E) = \left\{ \begin{array}{cc} u & 0 \\ 0 & 1 \end{array} \right\} \alpha, a, u \in U \), then \( \alpha \neq 0 \) otherwise. On the other hand, by Lemma 3.6 \( \left( \begin{array}{cc} 0 & 1 \\ 1 & -1 \end{array} \right) \bar{u} \left( \begin{array}{cc} e^q_{-1} & -1 \\ 1 & 0 \end{array} \right) = \left( \begin{array}{cc} 0 & 1 \\ 1 & -1 \end{array} \right) \bar{u} \left( \begin{array}{cc} e^q_{-1} & -1 \\ 1 & 0 \end{array} \right) \in B' \), which implies that

\[ u(ae^q_{-1} + b) = (ae^q_{-1} - (be^q_{-1})^2) \text{N}_{E/F}(a) + \text{N}_{E/F}(b) = 1 \] and \( u \in U \). In particular, in case \( a = 0, u = (e_{-1})^{q-1} \) and \( \text{N}_{E/F}(e_{-1}) = -1 \); in case \( b = 0, u = a^{q-1} \) and \( \text{N}_{E/F}(a) = 1 \). By calculation, we see \( U = \{ au \mid a = a^q \} \) with \( a \in E^\times \) and \( \text{N}_{E/F}(a) = \pm 1 \). Hence \( \text{det}(g) = u \left( \begin{array}{cc} 0 & 1 \\ 1 & -1 \end{array} \right) \bar{u} \left( \begin{array}{cc} q^{q-1} & -1 \\ 1 & 0 \end{array} \right) \in B', g = \left( \begin{array}{cc} u & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} a & b \\ -b & a^q \end{array} \right) \in U_2(E) \).

Finally, we see \( \beta \neq 0 \) if \( \Psi = \Psi^q \).

(b) \( \xi = \eta \). In this case, \( \text{Stab}_H(\eta) = H_1 = N' \rtimes T'' \) and \( V^\lambda_{U_2} = (V^\lambda_{T''})^{T''} \). By Lemma 2.3, as shown above, the vector space \( V^\lambda_{U_2} \) is generated by the functions \( f_{\eta, \Psi}, \bar{g}_{\eta, \Psi} \). We know that \( t \cdot f_{\eta, \Psi} = \Psi \otimes \Psi(t) f_{\eta, \Psi} \), and \( t \cdot \bar{g}_{\eta, \Psi} = \Psi \otimes \Psi(t) \bar{g}_{\eta, \Psi} \) for \( t \in T' \). So \( V^\lambda_{U_2} = \left\{ \begin{array}{cc} f_{\eta, \Psi}, \bar{g}_{\eta, \Psi} \end{array} \right\} \text{if } \Psi = \Psi^q, \) otherwise.

(c) \( \xi = \delta \). In this case, we have \( \text{Stab}_H(\delta) = H_2, H = B'H_2 \) and \( B' \cap H_2 = \left\{ \begin{array}{cc} a & b \\ 0 & d \end{array} \right\} \mid ad \in U \). So \( \text{dim}_C V^\lambda_{U_2} = \)

\[ \left\{ \begin{array}{cc} 1 \text{ if } \Psi = \Psi^q, \\ 0 \text{ otherwise.} \end{array} \right. \]

3) (a) In case \( \xi = \xi_a \), \( V^\lambda_{U_2(E)} \) is generated by the following two functions \( \alpha, \beta \) in \( V_1 \):

1. The support of \( \alpha \) belongs to \( B'U_2(E), \alpha(bu) = \Lambda \Sigma(b) \) for \( b \in B', u \in U_2(E) \) and \( \Lambda \otimes \Sigma \) is trivial over \( B' \cap U_2(E) \);

2. The support of \( \beta \) belongs to \( B' \left( \begin{array}{cc} 0 & 1 \\ 1 & -1 \end{array} \right) U_2(E), \beta(b \left( \begin{array}{cc} 0 & 1 \\ 1 & -1 \end{array} \right) u) = \Lambda \otimes \Sigma(b) \) and \( \Lambda \otimes \Sigma \) is trivial over \( B' \cap \left( \begin{array}{cc} 0 & 1 \\ 1 & -1 \end{array} \right) U_2(E) \).

By Lemma 3.6 \( B' \cap U_2(E) = \left\{ \begin{array}{cc} u & 0 \\ 0 & 1 \end{array} \right\} \alpha, \alpha, u \in U \). Therefore \( \left\{ \begin{array}{cc} \alpha \neq 0 \text{ if } \Lambda = \Lambda^q, \Sigma = \Sigma^q, \end{array} \right. \) On the other hand, by Lemma 3.6 \( B' \cap \left( \begin{array}{cc} 0 & 1 \\ 1 & -1 \end{array} \right) U_2(E) \) is trivial over \( B' \cap \left( \begin{array}{cc} 0 & 1 \\ 1 & -1 \end{array} \right) U_2(E) \), \( \left( \begin{array}{cc} a & b \\ -b & a^q \end{array} \right) = \text{N}_{E/F}(a) + \text{N}_{E/F}(b) = 1 \). Let \( t = a - 2b + 1 \), then \( \text{N}_{E/F}(a) + \text{N}_{E/F}(b) = 1 \) is equivalent to \( t^q + t^q = 2 \). And \( \beta \neq 0 \) if and only if \( \left( \begin{array}{cc} \beta & b^q \\ 0 & t^{-1} \end{array} \right) = \text{N}_{E/F}(t) = 1 \) for \( t \in E^\times \) satisfying \( t^q + t^q = 2 \). Considering \( t = s^q \), we know
\[ \begin{cases} \beta \neq 0 & \text{if } \Sigma = \Lambda^g, \Sigma \neq \Lambda, \\ \beta = 0 & \text{otherwise.} \end{cases} \]

(b) In case \( \xi = \eta \), \( \text{Stab}_H(\xi) = N' \rtimes T'' \) and \( V_{1'}^N \) is generated by the functions \( f_{\Lambda, \Sigma}, g_{\Lambda, \Sigma} \) defined in Lemma 3.3.

Considering the \( T'' \)-action on \( V_{1'}^N \), we know \( V_{1'}^H = \begin{cases} \{ f_{\Lambda, \Sigma}, g_{\Lambda, \Sigma} \} & \text{if } \Lambda = \Lambda^g, \Sigma = \Sigma^g, \\ \{0\} & \text{otherwise.} \end{cases} \)

(c) In case \( \xi = \delta \), \( \text{Stab}_H(\xi) = H_2 \), and \( H = B'H_2 \), \( B' \cap H_2 = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in U \). \( V_{1'}^{H_2} \) is generated by the function \( f \), where \( f(bh) = (\Lambda \otimes \Sigma)(b) \) for \( b \in B', h \in H_2 \) such that \( \Lambda \otimes \Sigma \) is trivial over \( B' \cap H_2 \); this implies \( V_{1'}^{H_2} = 0 \).

4) In case \( \xi = \xi_a, U_2(E) \supseteq SU_2(E) = SL_2(E) / U_2(E) \), and there exists \( h \in H \) such that \( hSU_2(E)h^{-1} = SL_2(F) \) (see [1], p. 242, Proposition 4). Hence \( V_{1'}^{H_2(E)} \subseteq V_{1'}^{SL_2(F)} \), which vanishes by [1], p. 82, Proposition 1).

If \( \xi = \eta, \delta \), then \( H_i \supseteq N' \) and \( V_{1'}^{N'} = 0 \). So \( V_{1'}^{H_i} = 0 \) for \( i = 1, 2 \).

\[ \square \]

**Corollary 3.8.** Let \( \pi_i \) be an irreducible representation of \( H \) in \( \text{Irr}(H) \). Then:

(i) \( \dim_{\mathbb{C}} W[\pi_i] = q + 1 \), if \( \pi_i = \Psi \rtimes 1_H \Psi \in \text{Irr}(E^\times) \) with \( \Psi = \Psi^q \).

(ii) \( \dim_{\mathbb{C}} W[\pi_i] = q \), if \( \pi_i = \Psi \rtimes \text{Stab}_H \Psi \in \text{Irr}(E^\times) \) with \( \Psi = \Psi^q \).

(iii) \( \dim_{\mathbb{C}} W[\pi_i] = q + 1 \), if \( \pi_i = \Pi_{\Lambda, \Sigma} \) for \( \Lambda \not\subseteq \Sigma \in \text{Irr}(E^\times), \Lambda = \Lambda^q, \Sigma = \Sigma^q \).

(iv) \( \dim_{\mathbb{C}} W[\pi_i] = q - 1 \), if \( \pi_i = \Pi_{\Lambda, \Sigma} \) for \( \Lambda \not\subseteq \Sigma \in \text{Irr}(E^\times), \Lambda = \Sigma, \Sigma = \Lambda^q \).

And the above lists are all the representations \( \pi_i \in \text{Irr}(H) \), such that \( W[\pi_i] \neq 0 \).

**Proof.** As is known that \( \dim_{\mathbb{C}} W[\pi_i] = \sum_{a \in E^\times} \dim_{\mathbb{C}} W[\pi_i](\xi_a) + \dim_{\mathbb{C}} W[\pi_i](\eta) + \dim_{\mathbb{C}} W[\pi_i](\delta) \), so the results follow from above Lemma 3.7. \( \square \)

3.3. **The representation** \( (\pi_0, W[\pi_1]) \). In this subsection, let \( \pi_1 = \Psi \rtimes 1_H \Psi \), where \( \Psi = \Psi^q \in \text{Irr}(E^\times) \) and \( \Psi = \psi \circ \text{N}_{E/F} \) for some \( \psi \in \text{Irr}(E^\times) \). The vector space \( W[\pi_1] \) is generated by the functions \( F_a, R, S : M \times X_F \rightarrow V_1 \) for any \( a \in E^\times \). Namely they all satisfy the equality (3.6) and

1. \( \text{supp} F_a = \text{Orbit}(\xi_a), \text{Orbit}(\overline{\xi}_a) = V_{1'}^{\Psi(\xi_a)} \) for any \( a \in E^\times \),

2. \( \text{supp} R = \text{Orbit}(\eta), \text{Orbit}(\overline{\eta}) = V_{1'}^{\Psi(\eta)} \),

3. \( \text{supp} S = \text{Orbit}(\delta), \text{Orbit}(\overline{\delta}) = V_{1'}^{\Psi(\delta)} \).

**Lemma 3.9.** For \( t_i, t_j, r, t_k \neq t_k \in E^\times \), we have

\[
\begin{align*}
\pi_0(h(r))F_{ar^2} &= \Psi(r^{-2})F_a & (I) \\
\pi_0(h(t))F_{at^{-1}} &= F_a & (II) \\
\pi_0(a(b))F_{a^t} &= \Psi(b)F_a & (III) \\
\pi_0(h(r))R &= R & (IV) \\
\end{align*}
\]

and

\[
\begin{align*}
\pi_0(h'(t))R &= \Psi(r^{-1})R & (V) \\
\pi_0(u(b))R &= R & (VI) \\
\pi_0(h(r))S &= S & (VII) \\
\pi_0(h'(t))S &= \Psi(r^{-1})S & (VIII) \\
\pi_0(a(b))S &= S & (IX) \\
\end{align*}
\]

**Proof.** Firstly we know that \( \text{supp}(\pi_0(h(r))F_{ar^2}) \subseteq \{ (g, \psi) | g \in M, \det(g) \neq 0, \psi \in X_F \} \). Fix \( i \in E^\times \) such that \( N_{E/F}(i) = r \). Then \( \pi_0(h(r))F_{ar^2}(\xi_b) = F_{ar^2}(\xi_b) \).

\[
\begin{align*}
\pi_0(h(r))F_{ar^2}(\xi_b) &= F_{ar^2}(\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, \phi^b) = F_{ar^2}(N_{E/F}(r)\begin{pmatrix} r^{-1} & 0 \\ 0 & i^{-1} \end{pmatrix}, \phi^b) = \pi_1(\begin{pmatrix} r^{-1} & 0 \\ 0 & i^{-1} \end{pmatrix}v_0) = \pi_1(\begin{pmatrix} r^{-1} & 0 \\ 0 & i^{-1} \end{pmatrix}v_0 = \Psi(i^{-2})v_0 = \Psi(r^{-2})F_a(\xi_b) \quad \text{if } b = a, \\
&= \pi_1(\begin{pmatrix} i^{-1} & 0 \\ 0 & i^{-1} \end{pmatrix}) \circ F_{ar^2}(\xi_b) = \pi_1(\begin{pmatrix} 0 & r^{-1} \\ 0 & 0 \end{pmatrix}v_0) = \Psi(i^{-2})v_0 = \Psi(r^{-2})F_a(\xi_b) \quad \text{if } b = a, \\
&= \pi_1(\begin{pmatrix} i^{-1} & 0 \\ 0 & i^{-1} \end{pmatrix}v_0 = \Psi(i^{-2})v_0 = \Psi(r^{-2})F_a(\xi_b) \quad \text{otherwise.} \\
\end{align*}
\]

Hence (I) follows. Similarly, \( \pi_0(h'(t))F_{at^{-1}}(\xi_b) = F_{at^{-1}}(\xi_b) = \Psi(v_1) = \Psi(\overline{\xi}_a) \). So we obtain (II), and (III) is clear.

For (IV), by (3.1)—(3.4), we have \( \pi_0(h(r))R(\eta) = R(\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}, \phi) = R(\begin{pmatrix} i & 0 \\ 0 & i^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) = R(\begin{pmatrix} i & 0 \\ 0 & i^{-1} \end{pmatrix}) = R(\begin{pmatrix} i & 0 \\ 0 & i^{-1} \end{pmatrix}v_1 = v_1 = R(\eta), \) which implies our (IV). Now we let \( x \in E^\times \) satisfying \( N_{E/F}(x) = t \). Then

\[
\begin{align*}
\pi_0(h'(t))R(\eta) &= R(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}x^{-1} 0 \\ 0 & 1 \end{pmatrix}R(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}x^{-1} 0 \\ 0 & 1 \end{pmatrix}) = \Psi(x^{-1})R(\eta) = \Psi(r^{-1})R(\eta); \\
\end{align*}
\]
thus we obtain (V). The following (VI), (VII) and (IX) are easy to verify. However in this way we verify (VIII).

**Corollary 3.10.** For \( t_1, r_1, r_2 \neq r_2 \in F^x \), we have

1. \( \text{Tr} \left( \pi_0 \left( \begin{array}{cc} r & 0 \\ 0 & 0 \end{array} \right) \right) = (q + 1) \psi(r^{-2}). \)
2. \( \text{Tr} \left( \pi_0 \left( \begin{array}{cc} r_1 & 0 \\ 0 & r_2 \end{array} \right) \right) = 2 \psi(r_1^{-1} r_2^{-1}). \)
3. \( \text{Tr} \left( \pi_0 \left( \begin{array}{cc} r & 0 \\ 0 & 1 \end{array} \right) \right) = \psi(r^{-2}). \)

**Proposition 3.11.** \( \pi_0 \cong \psi \cdot \text{Ind}_H^G 1_G. \)

**Proof.** By Corollary [3.10] we know \( \text{Res}_H^G \pi_0 \cong (2 \sigma_{\psi^{-1}, \psi^{-1}}) \oplus (\psi^{-2} \otimes \sigma) \cong \text{Res}_H^G (\psi^{-1} \cdot \text{Ind}_H^G 1_H) \) for irreducible representations \( \sigma_{\psi^{-1}, \psi^{-1}}, \psi^{-1} \otimes \sigma \) of \( B \) defined in Theorem [3.3] and the isotypic components \( 2 \sigma_{\psi^{-1}, \psi^{-1}} \) are spanned by the functions \( R, S \). By Proposition [1.4] we have (i) \( \pi_0 \cong \psi^{-1} \cdot \text{Ind}_H^G 1_G \), or (ii) \( \pi_0 \cong (2 \psi^{-1} \cdot 1_G) \oplus \pi_0 \) for certain regular character \( \theta \) of \( E^x \). But \( \pi_0(\omega)S | (\delta) = q^2 \sum_{n \in M} S(n, \phi, \phi(B(0, n)) = q^{2} S(\delta) \) so that \( \pi_0(\omega)S \neq S \); this means that \( \pi_0 \) has at most one isotypic component \( \psi^{-1} \cdot 1_G \). Therefore the above case (ii) is impossible. \( \square \)

### 3.4. The representation \((\pi_0, W[\pi_1])\) II

In this subsection, let \( \pi_1 = \Psi \cdot \text{St}_H \), where \( \Psi = \Psi' \in \text{Irr}(E^x) \). The vector space \( W[\pi_1] \) is generated by the functions: \( F_a, R : M \times X_F \rightarrow V_1 \) for any \( a \in F^x \). They all satisfy the equality (5.6), and

1. \( \text{supp}(F_a) = \text{Orbit}(\xi_a), F_a(\xi_a) = v_0 \in V^1_{(F^x)} \) for any \( a \in F^x \).
2. \( \text{supp}(R) = \text{Orbit}(\eta), R(\eta) = v_1 = q^2 f_{\Psi, \Psi} - g_{\Psi, \Psi} \in V^1_{\Psi} \) by Lemma [2.3]

Similarly as in Section [3.3] we obtain:

**Lemma 3.12.**

\[
\begin{align*}
\pi_0(h(r))F_{a^{\infty}} & \cong (\psi^{-1})F_a \\
\pi_0(h'(r))F_{a^{\infty}} & \cong F_a \\
\pi_0(\text{u}(b))F_a & \cong (\psi^{q^2})F_a
\end{align*}
\]

**Lemma 3.13.** Let \( M^{(1)} = \{ n \in M | \text{rank } n = 1 \} \). Then:

(a) \( M^{(1)} = \left\{ \begin{array}{ll}
s & 0 \\
0 & 0
\end{array} \right\} \cup \left\{ \begin{array}{ll}
\frac{a}{b} & 0 \\
0 & 1
\end{array} \right\} \mid s \in F^x, b \in E \}.

(b) \( s \left( \begin{array}{ll}
\frac{a}{b} & 0 \\
0 & 1
\end{array} \right) u(b) \left( \begin{array}{ll}
0 & 0 \\
0 & s
\end{array} \right) u(b)^* \) for \( s \in F^x \).

**Proof.** See [2.3] p.246—247. \( \square \)

Now, let us consider

\[
\begin{align*}
(\pi_0(\omega)R)(\eta) & = -q^{-2} \sum_{n \in M} R(n, \phi)B \left( \begin{array}{ll}
1 & 0 \\
0 & 0
\end{array} \right) \text{supp} R \in M^{(1)} \\
& = -q^{-2} \sum_{n \in M} R(n, \phi)B \left( \begin{array}{ll}
1 & 0 \\
0 & 0
\end{array} \right) \text{supp} R \in M^{(1)}
\end{align*}
\]

\[
\begin{align*}
\pi_0(\omega(\text{u}(b))) & \cong \phi(b) \pi_0(\text{u}(b)) \\
\pi_0(\text{u}(b)) & \cong (\psi^{q^2})\pi_0(\text{u}(b)) \quad (\text{XVI})
\end{align*}
\]

\[
\begin{align*}
\pi_0(\omega)R & \cong R \quad (\text{XVII})
\end{align*}
\]
Proposition 3.14. $\pi_0 \simeq \psi \cdot S\eta_G$.

Proof. By the formulas (XII)—(XVI), we obtain $\text{Res}_G^G \pi_0 = \text{Res}_G^G (\psi^{-1} \cdot S\eta_G)$. Consequently by the formula (XVII), $\pi_0$ has no $\psi^{-1} \cdot 1_G$ isotypic component. Comparing this with Proposition 3.4 gives the result. □

3.5. The representation $(\pi_0, W[\pi_1])$ III. In this subsection, let $\pi_1 = \Pi_{A \Sigma}$ for $A \neq \Sigma$ and $A = \lambda \circ N_{E/F}, \Sigma = \sigma \circ N_{E/F} \in \text{Irr}(E^\Sigma)$. The vector space $V_1^{Ht}(\equiv V_1^\Sigma)$ is generated by the two functions $f_{A \Sigma}, g_{A \Sigma}$ defined in Lemma 2.3. Let $\Delta: M \times X_F \rightarrow V_1 \triangleright \text{Irr}(F)$ satisfying (3.6) and $\supp(\Delta) = \text{Orbit}[\eta], \Delta(\eta) = f_{A \Sigma}$. Then

(i) \[
\left(\pi_0(h(r)\Delta)(\eta) = \Delta\left(\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}\right), \phi = \Delta\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}\right)\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)\left(\begin{pmatrix} x^q & 0 \\ 0 & x \end{pmatrix}\right)\phi = \Pi_{A \Sigma}\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}\right)\Delta(\eta)
\right)
\]

(ii) \[
\left(\pi_0(h(r)\Delta)(\eta) = \Delta\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right), \phi = \Delta\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\left(\begin{pmatrix} x^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right)\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)\left(\begin{pmatrix} x^{-q} & 0 \\ 0 & 1 \end{pmatrix}\right)\phi^N_{E/F}(r(x^{-1}))
\right)
\]

By the above (i) and (ii), we obtain that $\pi_0(h(r)\Delta) = \lambda(r)\sigma(r^{-1})\Delta, \pi_0(h(t)\Delta) = \lambda(t^{-1})\Delta$. In particular, $\pi_0\left(\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}\right)\Delta = \lambda(t_2^{-1})\sigma(t_1^{-1})\Delta$. It follows that $\text{Hom}_G(\pi_0, \text{Ind}_H^G(\lambda^{-1} \otimes \sigma^{-1})) = \text{Hom}_G(\pi_0, \text{Ind}_H^G(\sigma^{-1} \otimes \lambda^{-1})) \neq 0$. Since $\text{dim}_C \pi_0 = q + 1$, surely $\pi_0 = \pi_{A \Sigma, \sigma^{-1}}.$

3.6. The representation $(\pi_0, W[\pi_1])$ IV. In this subsection, let $\pi_1 = \Pi_{A \Sigma}$ where $A \neq \Sigma \in \text{Irr}(E^\Sigma)$ and $A = \Sigma^\lambda, \Sigma = \lambda^\sigma$. We start with recalling some explicit models for certain representations(cf. III).

3.6.1. I. Model for $\Pi_{A \Sigma}$. By II p.21, Definition 2], $\Pi_{A \Sigma}$ can be realized in the vector space $V_1$ spanned by all the functions $\nu: E^2 \times E^\Sigma \rightarrow \mathbb{C}$ such that

\[
\nu(a(e_1, e_2); a^{-1}b^{-1}e_3) = \Lambda(\mu(b))v(e_1, e_2; e_3) \quad \cdots (\ast),
\]

\[
(\Pi_{A \Sigma}(h)\nu)(e_1, e_2; e_3) = \nu(e_1, e_2)h; e_3 \text{det}(h)^{-1} \quad \cdots (\ast\ast),
\]

for $e_1, e_2 \in E; a, b, e_3 \in E^\Sigma; h \in H$.

3.6.2. II. Model for $\pi_\lambda$. Let $\pi_\lambda$ be a cuspidal representation of $G$ corresponding to a regular character $\lambda$ of $E^\Sigma$. Invoking II p.53, Proposition 4], we know that $\pi_\lambda$ can be realized in the vector space $\mathbb{C}[X_F]$ as follows:

\[
\pi_\lambda\left(\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}\right)f = \lambda(r)f
\]

(1)

\[
(\pi_\lambda\left(\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}\right)f)(\psi) = f(\psi^{-t})
\]

(2)

\[
(\pi_\lambda\left(\begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}\right)f)(\psi) = \phi(s)f(\phi)
\]

(3)

\[
(\pi_\lambda(\psi)f)(\psi) = -q^{-1} \sum_{y \in E^\Sigma} \phi(\text{Tr}_{E/F}(y))\Lambda(y)f(\psi^{N_{E/F}(y)})
\]

(4)

where $\psi \in X_F, t, r \in F^\times, s \in F$. 
3.6.3. The representation \( (\pi_0, W[\pi_1]) \). The vector space \( W[\pi_1] \) is generated by those functions \( F_a : M \times X_F \to V_1 \) satisfying (3.6.4) and \( \text{supp}(F_a) = \text{Orbit}(\xi_a), F_a(\xi_a) = v_1 \in V_1^{U_2(E)} \) for any \( a \in F^\times \). Using the above model, we choose an element \( v_1 \) as follows: \( v_1 : E^2 \times E^2 \to \mathbb{C} \) satisfies above (\( \ast \)) and

1. \( \text{supp}(v_1) = \bigcup_{u \in U} \text{Orbit}([(1, u_{e_1}; 1)], \)

2. \( v_1(1, u_{e_1}; 1) = \Lambda(u), \)

where \( \text{Orbit}([(1, u_{e_1}; 1)] = \{(a, u_{e_1}; a^{-1}b^{-1}) \in E^2 \times E^2 \mid a, b \in E^\times \}, \) and \( e_{-1} \) is a fixed element in \( E^\times \) such that \( \mathbb{N}_{E/F}(e_{-1}) = -1. \)

**Lemma 3.15.** The above constructed \( v_1 \) belongs to \( V_1^{U_2(E)}. \)

**Proof.** 1) For \( g = \left( \begin{array}{cc} a & b \\ 0 & a^t \end{array} \right) \in U_2(E), \) we have \( \text{supp}(g \cdot v_1) = \text{supp}(v_1), \) and \( g \cdot v_1(1, u_{e_1}; 1) = v_1(a, u_{0}e_1; u^{-1}) = \Lambda(a)v_1(1, u_{0}e_1^{-1}u_{e_1}; 1) = \Lambda(a)\Lambda(u_0a^{-1}) = \Lambda(u_0) = v_1(1, u_{0}e_1; 1); \) thus \( g \cdot v_1 = v_1. \)

2) For \( g = \left( \begin{array}{cc} a & b \\ -b^t & a^t \end{array} \right) \in U_2(E), \) we have \( \text{supp}(g \cdot v_1) = \text{supp}(v_1), \) and \( g \cdot v_1(1, u_{e_1}; 1) = v_1(a + b^tue_1, b + u_{e_1}a^t; 1) = v_1(a - b^tue_1, a - b^tue_1)^{-1} \Lambda(a - b^tue_1)\Lambda(a - b^tue_1)^{e_{-1}}(a - b^tue_1) = \Lambda(a) = v_1(1, u_{e_1}; 1); \) therefore \( g \cdot v_1 = v_1 \) in this case.

We define an intertwining operator between \( \pi_{\lambda_1} \) and \( \pi_0 \) by

\[
j : \pi_{\lambda_1} \to W[\pi_1]; f \mapsto j(f) = \sum_{d \in F^\times} f(\phi^d)F_a, \text{ i.e. } j(f)(\xi_a) = f(\phi^d)v_1.
\]

Claim: \( j(\pi_{\lambda_1}(g) f) = \pi_0(g) f \) for \( g \in G. \)

**Proof:** (1) Let \( g = \left( \begin{array}{cc} x & y \\ 0 & z \end{array} \right) \in B. \)

\[
\pi_0(\left( \begin{array}{cc} x & y \\ 0 & z \end{array} \right))j(f)(\xi_a) = \sum_{d \in F^\times} f(\phi^d)\pi_0(\left( \begin{array}{cc} x & y \\ 0 & z \end{array} \right))F_a(\xi_a)
\]

\[
= \sum_{d \in F^\times} f(\phi^d)\pi_0(\left( \begin{array}{cc} x & 0 \\ 0 & x^{-1} \end{array} \right)\left( \begin{array}{cc} 1 & 0 \\ 0 & x \end{array} \right)\left( \begin{array}{cc} 1 & x^{-1} \\ 0 & 1 \end{array} \right))F_a(\xi_a)
\]

\[
= \sum_{d \in F^\times} f(\phi^d)F_i(\left( \begin{array}{cc} x & 0 \\ 0 & x \end{array} \right), \phi^{ax^{-1}z^{-1}})\phi^d(\phi^z^{-1})
\]

\[
= \sum_{r \in F^\times} f(\phi^d)F_i(\left( \begin{array}{cc} r^{-1} & 0 \\ 0 & r^{-1} \end{array} \right))\left( \begin{array}{cc} 1 & 0 \\ 0 & r^q \end{array} \right)\left( \begin{array}{cc} 0 & 1 \\ 0 & r^{-q} \end{array} \right)N_{E/F}(r^2), \phi^{N_{E/F}(r^{-3})\phi^d(\phi^z^{-1})}
\]

\[
= \sum_{d \in F^\times} f(\phi^d)\pi_1(\left( \begin{array}{cc} r^{-1} & 0 \\ 0 & r^{-1} \end{array} \right))F_i(\left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \phi^{ax^{-1}})\phi^d(\phi^z^{-1})
\]

\[
= f(\phi^{ax^{-1}})\pi_0(\left( \begin{array}{cc} r^{-1} & 0 \\ 0 & r^{-1} \end{array} \right))v_1|\phi^d(\phi^z^{-1})
\]

\[
= f(\phi^{ax^{-1}})\Lambda(r^{-1})\Sigma(r^{-1})v_1\phi^d(\phi^z^{-1}) = f(\phi^{ax^{-1}})\Lambda(\lambda(x^{-1}))\phi^d(\phi^z^{-1})v_1.
\]

So

\[
\pi_0(\left( \begin{array}{cc} x & y \\ 0 & z \end{array} \right))j(f) = \sum_{d \in F^\times} f(\phi^{ax^{-1}})\Lambda(\lambda(x^{-1}))\phi^d(\phi^z^{-1})F_a.
\]

On the other hand,

\[
j(\pi_{\lambda_1}(\left( \begin{array}{cc} x & y \\ 0 & z \end{array} \right))) = \sum_{d \in F^\times} \pi_{\lambda_1}(\left( \begin{array}{cc} x & y \\ 0 & z \end{array} \right))f(\phi^d)F_a
\]

\[
= \sum_{d \in F^\times} \pi_{\lambda_1}(\left( \begin{array}{cc} x & 0 \\ 0 & x^{-1}z \end{array} \right)\left( \begin{array}{cc} 1 & x^{-1} \\ 0 & 1 \end{array} \right))f(\phi^d)F_a
\]

\[
= \sum_{d \in F^\times} \Lambda(\lambda(x^{-1}))f(\phi^{ax^{-1}})\phi^{ax^{-1}}(x^{-1}y)F_a
\]
(2) Let $g = \omega$.

\[ (\pi(\omega)(j(f))(\xi_\omega) = -q^{-2} \sum_{n_{\omega}} \phi^\omega(B(1d_{\omega}, n))(f)(n, \phi^\omega) \]

consider $\supp (j, f)$.

\[ = -q^{-2} |\mathcal{U}(2(E))|^{-1} \sum_{h \in H} \phi^\omega \left( B(1d_{\omega}, N_{E/F}(\det(h^{-1}))h^*), \phi^\omega \right) \]

replace $h$ by $h^{-1}/\det(h)$.

\[ = -q^{-2} |\mathcal{U}(2(E))|^{-1} \sum_{h \in H} \phi^\omega \left( B(1d_{\omega}, h^{-1}(h^{-1})) \right) \]

Let $\kappa_u = -q^{-2} |\mathcal{U}(2(E))|^{-1} \sum_{h \in H} \phi^\omega \left( B(1d_{\omega}, h^{-1}(h^{-1})) \right) (\phi^\omega N_{E/F}(\det(h))) \]

and $\kappa_u^2 = -q^{-2} |\mathcal{U}(2(E))|^{-1} \sum_{h \in H} \phi^\omega \left( B(1d_{\omega}, h^{-1}(h^{-1})) \right) (\phi^\omega N_{E/F}(\det(h))) (\phi^\omega N_{E/F}(\det(h))) \]

Then $\kappa_u = \sum_{\alpha \in F^*} \kappa_u^\alpha$, and

\[ \kappa_u^2 = -q^{-2} |\mathcal{U}(2(E))|^{-1} \sum_{h \in H} \phi^\omega \left( N_{E/F}(\alpha) + N_{E/F}(\beta) + N_{E/F}(\gamma) + N_{E/F}(\delta) \right) (\phi^\omega) \]

where $\mathcal{M} = \{ h = (a, b) \in H| N_{E/F}(\alpha^2 - \beta \gamma) = 1; -\beta + \alpha e_{-1} = u_1 e_{-1} (\delta - \gamma e_{-1}), \delta - \gamma e_{-1} \neq 0 \}$. Moreover, by the equations in \{(a - b \gamma = u_2; -\beta + \alpha e_{-1} = u_1 e_{-1} (\delta - \gamma e_{-1}) \} and $\delta - \gamma e_{-1} = z$ for $u_1, u_2 \in U(z \in \mathbb{E})$, we change the variables $\alpha, \beta, \gamma, \delta$ by $u_1, u_2, z, \gamma$. Note that this is reasonable.

By $-\beta + \alpha e_{-1} = u_1 e_{-1} (\delta - \gamma e_{-1})$, we get $-\beta e_{-1}^\alpha + u_1 \gamma e_{-1} = u_1 \delta - \alpha$. Then $-\beta e_{-1}^\alpha + u_1 \gamma e_{-1}(\gamma^\alpha + u_1 e_{-1} - \beta \gamma e_{-1}) = (u_1 \delta - \alpha)(u_1 \delta - \alpha)^\alpha$. By calculation, we have $N_{E/F}(\alpha) + N_{E/F}(\beta) + N_{E/F}(\gamma) + N_{E/F}(\delta) = Tr_{E/F}(u_1 (\alpha^2 - \gamma^2 e_{-1}^\alpha))$.

Set $A = \alpha^2 \delta - \gamma^2 e_{-1}^\alpha$. Now we consider

\[ u_1^2 e_{-1}^\alpha z = u_1^2 e_{-1}^\alpha = u_1(\alpha^2 \delta^2 - \gamma^2 e_{-1}^\alpha) \]

and also

\[ u_1 A e_{-1}^\alpha = u_1(\alpha^2 \delta^2 - \gamma^2 e_{-1}^\alpha) \]

\[ = u_1^2(\alpha^2 \delta^2 - \gamma^2 e_{-1}^\alpha) \]

\[ = u_1^2(\alpha^2 \delta^2 - \gamma^2 e_{-1}^\alpha) \]

\[ = u_1^2(\alpha^2 \delta^2 - \gamma^2 e_{-1}^\alpha) \]

So

\[ u_1 A = u_1^2 z^\alpha + (\delta^2 e_{-1}^\alpha) \]

In this way, we obtain

\[ Tr_{E/F}(u_1 A) = Tr_{E/F}(u_1^2 z^\alpha) \]

Hence

\[ \begin{align*}
\end{align*} \]
4.1. In this section, we use the following notations: \( G = \text{GL}_2(K) \), \( B = \left( \begin{array}{cc} a & b \\ 0 & d \end{array} \right) \in G \), \( N = \left( \begin{array}{cccc} 1 & 0 \\ 0 & 1 \end{array} \right) \in G \), \( T = \left( \begin{array}{cccc} 1 & 0 \\ 0 & 1 \end{array} \right) \in G \), \( Z = \left( \begin{array}{cccc} 1 & 0 \\ 0 & 1 \end{array} \right) \in G \); \( \text{Gal}(K/F) = \langle \sigma \rangle \); \( u(b) = \left( \begin{array}{cccc} 1 & 0 \\ 0 & 1 \end{array} \right) \) for \( b \in K \), \( h(a, d) = \left( \begin{array}{cccc} a & 0 \\ 0 & d \end{array} \right) \) for \( a, d \in K^\times \), \( \omega = \left( \begin{array}{cccc} 0 & 1 \\ -1 & 0 \end{array} \right) \in G \).

4.2. We recall the technique of Weil’s Galois descent to construct a morphism from \( G \) to \( \text{GSp}_4(F) \).

Let \( V_0 \) be a vector space over \( F \) of dimension 2, endowed with a symplectic form \( \langle \cdot, \cdot \rangle_{V_0} \). Let \( \{e_1, e_2\} \) be a symplectic base of \( V_0 \). Namely \( V = V_0 \otimes_F K \) is a symplectic \( K \)-vector space, endowed with the symplectic form \( \langle \cdot, \cdot \rangle_V \) induced from \( V_0 \) by scalar extension. Let us define a \( \text{Gal}(K/F) \)-action on \( V \) by

\[
\text{Gal}(K/F) \times K \otimes_F V_0 \longrightarrow K \otimes F V_0; (\sigma, \sum_i k_i \otimes e_i) \longmapsto \sum_i k_i^\sigma \otimes e_i.
\]

Let \( W = V \otimes_K V \otimes_K V \), and we assign \( W \) a symplectic form \( \langle \cdot, \cdot \rangle_W = \langle \cdot, \cdot \rangle_V \otimes \langle \cdot, \cdot \rangle_V \cup \langle \cdot, \cdot \rangle_V \). On \( W \), we will consider the twisted Galois action defined by

\[
\text{Gal}(K/F) \times W \longrightarrow W; (\sigma, w = \sum_{i=1}^n u_i \otimes v_i \otimes w_i) \longmapsto \sigma w = \sum_{i=1}^n w_i^\sigma \otimes u_i^\sigma \otimes v_i^\sigma.
\]

We will let \( W_0 \) denote the set \( \{w \in W \mid \sigma w = w \} \). By calculation, each \( w_0 \in W_0 \) may be expressed in the form

\[
w_0 = x e_1 \otimes e_1 \otimes e_1 + a e_1 \otimes e_1 \otimes e_2 + a^\sigma e_2 \otimes e_1 \otimes e_1 + a e_1 \otimes e_2 \otimes e_1 + a^\sigma e_1 \otimes e_2 \otimes e_1 + a^\sigma e_1 \otimes e_2 \otimes e_1 + \beta e_1 \otimes e_2 \otimes e_1 + \beta e_1 \otimes e_2 \otimes e_1 + \beta e_2 \otimes e_1 \otimes e_2 + \beta e_2 \otimes e_1 \otimes e_2 + x e_2 \otimes e_2 \otimes e_2 \text{ for } x, y \in F, a, \beta \in K.
\]
Every element \( w_0 \) of this form is well-defined by its corresponding coefficients. For simplicity, we write \( w_0 = \begin{pmatrix} x & \alpha \\ \beta & y \end{pmatrix} \) instead of the whole term. One can check that the restriction of \( \langle , \rangle_W \) to \( W_0 \) defines an \( F \)-symplectic form, denoted by \( \langle , \rangle_{W_0} \). More precisely,

\[
\langle w_0, w_0' \rangle_{W_0} = xy' - x'y = \text{Tr}_{K/F}(\alpha \beta') + \text{Tr}_{K/F}(\alpha' \beta)
\]

for \( w_0 = \begin{pmatrix} x & \alpha \\ \beta & y \end{pmatrix}, w_0' = \begin{pmatrix} x' & \alpha' \\ \beta' & y' \end{pmatrix}. \)

Let \( \text{GSp}(W) \) denote the group of symplectic similitudes of \((W, \langle , \rangle_W)\). By definition, there actually exists a morphism of groups

\[
\left( \text{GL}(V) \times \text{GL}(V) \times \text{GL}(V) \right) \rightarrow \text{GSp}(W).
\]

Here the group \( S_3 \) acts on \( W \) by permuting its three variables. Now we define a twisted Galois action of \( \text{Gal}(K/F) \) on \( \text{GL}(V) \times \text{GL}(V) \times \text{GL}(V) \) by

\[
\text{Gal}(K/F) \times \left( \text{GL}(V) \times \text{GL}(V) \times \text{GL}(V) \right) \rightarrow \text{GL}(V) \times \text{GL}(V) \times \text{GL}(V); h = (g_1, g_2, g_3) \mapsto ^h = (g_1^\sigma, g_2^\sigma, g_3^\sigma).
\]

Write \( \text{GL}(V) = \{ h \in \text{GL}(V) \times \text{GL}(V) \times \text{GL}(V) | ^h = h \} \). Then there exists an isomorphism of groups \( \text{GL}(V) \rightarrow \text{GSp}(W) \); \( g \mapsto (g, g^\sigma, g^\sigma) \). If given \( h \in \text{GL}(V) \times \text{GL}(V) \times \text{GL}(V) \), \( w \in W = V \otimes_K V \otimes_K V \), one can verify that \( ^w = ^h \cdot ^w = ^h \cdot w \). So it induces a morphism from \( \text{GL}(V) \rightarrow \text{GSp}(W_0) \). By the fixed basis \( \{e_1, e_2\} \), we obtain a morphism: \( G \rightarrow \text{GSp}(W_0) \).

4.3. We interpret the above construction of the morphism \( G \rightarrow \text{GSp}(W_0) \) in terms of the language of algebraic groups.

Let \( V \) be the \( K \)-algebraic vector space associated to \( V \). That is to say:

\[
V : \text{Alg}_K \rightarrow \text{Vect}_K; R \mapsto V \otimes_K R,
\]

a functor from the category of unital commutative associative \( K \)-algebras to the category of \( K \)-vector spaces. Namely \( V \otimes_K R \) inherits the \( R \)-symplectic structure from \( V \). We define a \( \text{Gal}(K/F) \)-action on \( V \) in the following way:

\[
\text{Gal}(K/F) \times V \otimes_K R \rightarrow V \otimes_K R; (\sigma, \sum_{i=1}^n v_i \otimes r_i) \mapsto \sum_{i=1}^n v_i^\sigma \otimes r_i^\sigma.
\]

Now let \( W \) be the \( K \)-algebraic vector space associated to \( W \), and \( W_0 \) the \( F \)-algebraic vector space associated to \( W_0 \). We define a twisted \( \text{Gal}(K/F) \)-action on \( W \) in the following way:

\[
\text{Gal}(K/F) \times V \otimes_K V \otimes_K V \otimes_K V \otimes_K V \otimes_K R; \quad (\sigma, \sum_{i=1}^n u_i \otimes v_i \otimes w_i \otimes r_i) \mapsto \sum_{i=1}^n w_i^\sigma \otimes u_i^\sigma \otimes v_i^\sigma \otimes r_i^\sigma.
\]

So \( W_0 \) is the \( \text{Gal}(K/F) \)-invariant algebraic scheme of \( W \) in the following sense:

1. \( W_0 \cong W_0 \times_F K \).
2. \( W(R)^{\text{Gal}(K/F)} \cong W_0(R^{\text{Gal}(K/F)}) \) for any \( R \in \text{Alg}_K \).

On the other hand, we also define a twisted Galois action of \( \text{Gal}(K/F) \) on \( \text{GL}_2/K \times \text{GL}_2/K \times \text{GL}_2/K \) as

\[
\text{Gal}(K/F) \times \left( \text{GL}_2(R) \times \text{GL}_2(R) \times \text{GL}_2(R) \right) \rightarrow \text{GL}_2(R) \times \text{GL}_2(R) \times \text{GL}_2(R);
\]

\[
(\sigma, (g_1, g_2, g_3)) \mapsto (g_1^\sigma, g_2^\sigma, g_3^\sigma).
\]

We denote by \( \text{H}^{\text{Gal}(K/F)} \), the \( \text{Gal}(K/F) \)-invariant algebraic scheme of \( H = \text{GL}_2/K \times \text{GL}_2/K \times \text{GL}_2/K \). Indeed, by definition,

\[
\text{H}^{\text{Gal}(K/F)} \cong \text{Res}_{K/F}(\text{GL}_2/K).
\]

There exists an action of \( \text{H}^{\text{Gal}(K/F)} \) on \( W_0 \), and it preserves the symplectic form up to the similitude factors. Thus we obtain a morphism of algebraic group schemes:

\[
i : \text{Res}_{K/F}(\text{GL}_2/K) \rightarrow \text{GSp}_{W_0}.
\]
4.4. Let $X_0 = \{w_0 = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} | w_0 \in W_0\}$, $Y_0 = \{w_0 = \begin{pmatrix} 0 & 0 \\ \beta & y \end{pmatrix} | w_0 \in W_0\}$. Then $X_0$, $Y_0$ are two vector spaces over $F$ and $W_0 = X_0 \oplus Y_0$ is a complete polarization of $W_0$. Via the morphism $i : G \to GSp(W_0)$, it gives rise to a $G$-action on $W_0$ by the following formulas:

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, $w_0 = \begin{pmatrix} x & 0 \\ \beta & y \end{pmatrix}$, write $g \cdot w_0 = \begin{pmatrix} x' & 0 \\ \beta' & y' \end{pmatrix}$. Then

$x' = N_{K/F}(a)x + Tr_{K/F}(ab^2 \beta^2 x) + Tr_{K/F}(bb^2 \beta^2 y) + N_{K/F}(b)y$;

$\alpha' = d d^r b \beta y + (dd^r a \beta \beta + cd^r b \beta \beta + dc^r b \beta \beta) + (cc^r b \beta \beta + dc^r a \beta \beta + cd^r \beta \beta \beta) + cc^r a \beta x$;

$\gamma' = N_{K/F}(d)y + Tr_{K/F}(dd^r c \beta \beta) + Tr_{K/F}(cc^r d \beta \beta) + N_{K/F}(c)x$.

We write each element $h \in GSp(W_0)$ in the form of $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a \in End_F(X_0), b \in Hom_F(Y_0, X_0), c \in Hom_F(X_0, Y_0), d \in End_F(Y_0)$.

**Corollary 4.1.** Through the map $i : G \to GSp(W_0)$, the actions of $(u(b), h(a, d), \omega)$ on $W_0$ are described as follows:

1. $i(u(b)) = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}$, where $m = \begin{pmatrix} x & \alpha \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x + Tr_{K/F}(b^2 \alpha) & \alpha \\ 0 & 0 \end{pmatrix}$;

2. $i(h(a, d)) = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}$, where $m = \begin{pmatrix} x & \alpha \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} N_{K/F}(a)x + (a^d d^r a \alpha^r) & a \alpha \alpha \\ 0 & 0 \end{pmatrix}$;

3. $i(\omega) = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}$, where $u = \begin{pmatrix} y & -\beta \\ 0 & 0 \end{pmatrix}$, $v = \begin{pmatrix} x & \alpha \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \alpha & -x \end{pmatrix}$.

Let $(\rho, V)$ be the Weil representation of the symplectic similitude group $GSp(W_0)$. Via the map $i$, it gives rise to a representation $(\pi, V)$ of $G$ which can be realized in the vector space $V = \mathbb{C}[Y_0 \times X_F]$ of complex functions on $Y_0 \times X_F$.

**Proposition 4.2.** For the representation $(\pi, G, \mathbb{C}[Y_0 \times X_F])$, the action is determined by the following formulas:

1. $[\pi(u(b))F] \begin{pmatrix} 0 & 0 \\ \beta & y \end{pmatrix} \psi = \psi \left( Tr_{K/F}(bb^2 \beta y) - N_{K/F}(b)y^2 - Tr_{K/F}(b \beta \beta) \right) \begin{pmatrix} 0 & 0 \\ \beta & y \end{pmatrix} \psi$;

2. $[\pi(h(a, d))F] \begin{pmatrix} 0 & 0 \\ \beta & y \end{pmatrix} \psi = \chi_{\beta}(a) \psi \left( N_{K/F}(a) \begin{pmatrix} 0 & \beta \\ 0 & y \end{pmatrix} \psi \right)$;

3. $[\pi(\omega)] \psi = q^{-2} \sum_{\beta \in \mathbb{R}_+ \times F} \psi(y \psi + Tr_{K/F}(\beta \beta') \psi)$.

**Proof.** See Appendix 1. □

4.5. The whole goal of this section is to determine the different isotypic components of $\pi$. We first consider the principal series representations.

Let $\alpha, \beta \in \text{Irr}(K^+)$. To determine the principal series components of $\pi$, it involves to calculate the dimension of the vector space $\text{Hom}_G(V, \text{Ind}_G^H(\alpha \otimes \beta))$. Applying Frobenius reciprocity, we see

$\text{Hom}_G(V, \text{Ind}_G^H(\alpha \otimes \beta)) \cong \text{Hom}_T(V_N, \alpha \otimes \beta) \cong \text{Hom}_T(V^N, \alpha \otimes \beta)$.

Therefore we shall first describe the vector space $V^N$, and then consider the $T$-action on it. Once we regard the action of $N$ on the vector space $V$, as described in Proposition 4.2.1, we should consider the following action:

$N \times (Y_0 \times X_F) \to Y_0 \times X_F: \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \cdot \left( \begin{pmatrix} 0 & 0 \\ \beta & y \end{pmatrix}, \psi \right) \to \left( \begin{pmatrix} 0 & 0 \\ \beta & y \end{pmatrix}, \psi \right)$.

The orbits of this action are following:

(i) $\text{Orbit}(\xi_{(0,0,\beta)}), \quad \text{where} \quad \xi_{(0,0,\beta)}(\psi) = \begin{pmatrix} 0 & 0 \\ \beta & y \end{pmatrix} \psi$ for any $\beta \in K, \psi \in X_F$;

(ii) $\text{Orbit}(\eta_{(0,y,\psi)}), \quad \text{where} \quad \eta_{(0,y,\psi)}(\psi) = \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix} \psi$ for any $y \in F^N, \psi \in X_F$. 

The stabilizer of the chosen element in each orbit is described as follows:

The function $F$ belongs to $V^N$ if and only if it satisfies the equality:

$$\psi(\text{Tr}_{K/F}(bb^\sigma y) - N_{K/F}(b)y^2 - \text{Tr}_{K/F}(b\beta\beta^\sigma))F\left(\begin{pmatrix} 0 & 0 \\ \beta - b\beta^\sigma y & y \end{pmatrix}, \psi\right) = F\left(\begin{pmatrix} 0 & 0 \\ \beta & y \end{pmatrix}, \psi\right)$$

(4.1)

for any $b \in K$.

**Proposition 4.3.** (1) The vector space $V^N$ is generated by the following functions:

(i) $F_{(0,0,0)}$, where $\text{supp}(F_{(0,0,0)}) = \text{Orbit}\{\xi_{(0,0,0)}\}$, $F_{(0,0,0)}(\xi_{(0,0,0)}) = 1$ and it satisfies the equation (4.1) for any $\psi \in X_F$;

(ii) $G_{(0,0,0)}$, where $\text{supp}(G_{(0,0,0)}) = \text{Orbit}\{\eta_{(0,0,0)}\}$, $G_{(0,0,0)}(\eta_{(0,0,0)}) = 1$ and it satisfies the equation (4.1) for any $y \in \mathbb{F}_q$, any $\psi \in X_F$.

(2) Let $t = h(a, d) \in T$. Then the action of $t$ on the vector space $V^N$ is given as follows:

(i) $\pi(h(a, d))F_{(0,0,0)} = \chi_q(N_{K/F}(ad))F_{(0,0,0)}(\xi_{(0,0,0)});

(ii) $\pi(h(a, d))G_{(0,0,0)} = \chi_q(N_{K/F}(ad))G_{(0,0,0)}(\eta_{(0,0,0)}).

**Proof.**

1) Every element $F$ in $V^N$, that satisfies the equation (4.1), is completely determined by its values at the points in $\{\xi_{(0,0,0)}, \eta_{(0,0,0)}\}$. Let $\delta$ be one point among them. Then $F(\delta)$ can be nonzero if and only if the coefficient on the left-hand side of the equation (4.1) is trivial over the stabilizer of $\delta$. After checking each such point, we obtain the result.

2) It is straightforward. □

Let $\Phi$ be an element in $\text{Hom}_T(V^N, \alpha \otimes \beta)$. Then it is determined by the following two equations:

1. $\chi_q(N_{K/F}(ad))\Phi(F_{(0,0,0)}(\xi_{(0,0,0)})) = \alpha(a)\beta(b)\Phi(F_{(0,0,0)})$, $a, d \in K^\times$.

2. $\chi_q(N_{K/F}(ad))\Phi(G_{(0,0,0)}(\eta_{(0,0,0)})) = \alpha(a)\beta(b)\Phi(G_{(0,0,0)}))$, $a, d \in K^\times$.

Now let us define a $T$-action on the vector space $V^N$:

$$t \cdot F_{(0,0,0)} := F_{(0,0,0)}(\xi_{(0,0,0)}), \quad t \cdot G_{(0,0,0)} = G_{(0,0,0)}(\eta_{(0,0,0)}), \quad t = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.

$$

For such action, there are two kinds of orbits:

(i) Orbit$[F_{(0,0,0)}]$ and (ii) Orbit$[G_{(0,1,0)}]$, for the fixed $\phi \in X_F$.

The stabilizer of the representative element in each orbit has the following form:

(i) $\text{Stab}_T(F_{(0,0,0)}) = \{h(a, d) \in T | N_{K/F}(ad) = 1\};$

(ii) $\text{Stab}_T(G_{(0,1,0)}) = \{h(a, d) \in T | N_{K/F}(a) = N_{K/F}(d) = 1\}$.

Now we present one statement about the principal series components of the representation $\pi$:

**Proposition 4.4.** Let $\alpha, \beta \in \text{Irr}(K^\times)$. Then

1. If $\alpha = \chi_1 \circ N_{K/F}, \beta = \chi_2 \circ N_{K/F}$ for some characters $\chi_1 \neq \chi_2 \in \text{Irr}(F^\times)$, then $\dim_{\mathbb{C}} \text{Hom}_G(V, \text{Ind}_G^H(\alpha \otimes \beta)) = 1$.

2. If $\alpha = \beta = \chi \circ N_{K/F}$ for a character $\chi \in \text{Irr}(F^\times)$, then $\dim_{\mathbb{C}} \text{Hom}_G(V, \text{Ind}_G^H(\alpha \cdot 1_H)) = 2$.

For the other kind of $\alpha, \beta \in \text{Irr}(K^\times)$, $\text{Hom}_G(V, \text{Ind}_G^H(\alpha \otimes \beta)) = 0$.

**Proof.** By Frobenius reciprocity, we see $\text{Hom}_G(V, \text{Ind}_G^H(\alpha \otimes \beta)) = \text{Hom}_T(V^N, \alpha \otimes \beta)$. Let $\Phi \in \text{Hom}_T(V^N, \alpha \otimes \beta)$. The function $\Phi$ is completely determined by its values at the points $F_{(0,0,0)}$ and $G_{(0,1,0)}$. The value $\Phi(F_{(0,0,0)})$ can be any complex number if and only if $\alpha \otimes \beta(1) = 1$ for $h(a, d) \in \text{Stab}_T(F_{(0,0,0)})$, which is equivalent to $\alpha = \beta = \chi \circ N_{K/F}$ for some character $\chi \in \text{Irr}(F^\times)$. Similarly the value $\Phi(G_{(0,1,0)})$ can be any complex number if and only if $\alpha = \chi_1 \circ N_{K/F}, \beta = \chi_2 \circ N_{K/F}$ for two characters $\chi_1, \chi_2 \in \text{Irr}(F^\times)$; thus we obtain the results. □
4.6. Now it reduces to check whether the representation \( \chi \circ N_{E/F} \cdot 1_G \) of \( G \) is a sub-representation of \( \pi \).

Let \((\alpha \cdot \pi, V_\pi)\) be the representation of \( \pi \) twisted by the character \( \alpha = \chi \circ N_{K/F} \in \text{Irr}(G) \). Since \( \text{Hom}_G(\pi, \alpha^{-1}1_G) \cong (V_\pi)^{\alpha} \), it suffices to determine the dimension of \((V_\pi)^{\alpha} \) for the representation \((\alpha \cdot \pi, G, V_\pi)\). Notice that \((V_\pi)^{\alpha} \cong V^N \) which is generated by two functions \( F_{(0,0,\phi)} \), \( G_{(0,0,\phi)} \); the action of \( T \) on \((V_\pi)^{\alpha} \) is given by the following formulas:

1. \( \alpha \cdot (h(a,d))F_{(0,0,\phi)} = \chi \cdot \chi_{q}^\alpha(N_{K/F}(ad))F_{(0,0,\phi)} \).
2. \( \alpha \cdot (h(a,d))G_{(0,0,\phi)} = \chi \cdot \chi_{q}^\alpha(N_{K/F}(ad))G_{(0,0,\phi)} \).

Proposition 4.5. The vector space \((V_\pi)^{\alpha} \) is generated by two non-zero functions \( A = \sum_{\alpha \in T} \alpha \cdot \pi(t)F_{(0,0,\phi)} \) and \( B = \sum_{\alpha \in T} \alpha \cdot \pi(t)G_{(0,0,\phi)} \) for the fixed \( \phi \in X_F \).

Proof. It is straightforward.

Our final task for this subsection is to consider the action of \( \omega \) on the vector space \((V_\pi)^{\alpha} \). Observe that \( [\alpha \cdot \pi](\omega)A, [\alpha \cdot \pi](\omega)B \) both belong to \((V_\pi)^{\alpha} \). Consider the \( T \)-action on the set \( Y_0 \times X_F \): (we treat the vector space \((V_\pi)^{\alpha} \) similarly as \((V_\pi)^{\beta} \).)

\[
T \times (Y_0 \times X_F) \longrightarrow Y_0 \times X_F: \left( \begin{pmatrix} 0 & 0 \\ d & y \end{pmatrix}, \psi \right) \longmapsto \left( \begin{pmatrix} 0 & N_{K/F}(ad) \\ \phi \end{pmatrix} \right).
\]

The orbits of this action are following:

1. \text{Orbit (}x_{00}, x_{00} = \left( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \phi \right) \text{ and Orbit (}x_{10}, x_{10} = \left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \phi \right).}

2. \text{Orbit (}x_{11}, x_{11} = \left( \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \phi \right) \text{ and Orbit (}y_k, y_k = \left( \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \phi \right).}

for the fixed character \( \phi \in X_F \) and any \( k \in \mathbb{F}_k \). By the calculations in Appendix 2, we obtain the following table for the values of the functions \( A, B, [\alpha \cdot \pi](\omega)A, [\alpha \cdot \pi](\omega)B \) at the points \( (1) \ x_{00}; (2) \ x_{10}; (3) \ x_{11}; (4) \ y_k \).

| \( x_{00} \) | \( x_{01} \) | \( x_{10} \) | \( y_k \) |
|----------|----------|----------|----------|
| \( A \) | \( (q-1)(q^2 + q + 1)^2 \) | \( (q^2 + q + 1)^2 \) | \( 0 \) | \( \chi \chi_{q}^\alpha(k)\phi(-k)(q^2 + q + 1)^2 \) |
| \( B \) | \( 0 \) | \( (q^2 + q + 1)^2 \) | \( 0 \) | \( \chi \chi_{q}^\alpha(kq^{-2}(q-1)(q^2 + q + 1)^2 \) |
| \( \alpha \cdot \pi(\omega)A \) | \( q^{-2}(q-1)(q^2 + q + 1)^2 \) | \( q^{-2}(q-1)(q^2 + q + 1)^2 \) | \( q^{-2}(q-1)(q^2 + q + 1)^2 \) | \( \chi \chi_{q}^\alpha(kq^{-2}(q-1)(q^2 + q + 1)^2 \) |
| \( \alpha \cdot \pi(\omega)B \) | \( q^{-2}(q-1)(q^2 + q + 1)^2 \) | \( q^{-2}(q-1)(q^2 + q + 1)^2 \) | \( q^{-2}(q^2 + q + 1)^2 \) | \( \chi \chi_{q}^\alpha(kq^{-2}(q-1)(q^2 + q + 1)^2 \) |

Corollary 4.6. The element \( qA - (q-1)B \in V_\pi^{\alpha} \) is \([\alpha \cdot \pi](\omega)\)-invariant.

Proof. Let us consider \( C = \sum_{\pi \in G}[\alpha \cdot \pi]_1 g \in F_{(0,0,\phi)} \). Then

\[
C(x_{00}) = \sum_{n \in N, b \in B}[\alpha \cdot \pi](nab)F_{(0,0,\phi)}(x_{00}) + \sum_{b \in B}[\alpha \cdot \pi](b)F_{(0,0,\phi)}(x_{00})
\]

As \([\alpha \cdot \pi](\omega)A \neq A \), this means that \( \text{dim}(V_\pi) = 1 \). So there exists two constants \( a, b \in \mathbb{C} \) such that \( aA + bB \) is \([\alpha \cdot \pi](\omega)\)-invariant. By the above diagram, we can let \( a = q, b = -(q-1) \).

Corollary 4.7. For any character \( \chi \in \text{Irr}(\mathbb{F}_k) \) and \( \alpha^{-1} = \chi^{-1} \circ N_{K/F} \), we have:

1. \( \text{dim}_C \text{Hom}_C(V, \alpha^{-1} \cdot 1_G) = 1 \);
2. \( \text{dim}_C \text{Hom}_C(V, \alpha^{-1} \cdot \text{St}_G) = 1 \).

Proof. 1) \( \text{Hom}_C(V, \alpha^{-1} \cdot 1_G) \subseteq (V_\pi)^{\alpha^{-1}} \), which is of dimension smaller than 2. As \( \alpha \cdot \pi(\omega)A \neq A \) and \( \alpha \cdot \pi(\omega)(qA - (q-1)B) = qA - (q-1)B, \) we know that \( \text{dim}_C(V_\pi)^{\alpha^{-1}} = 1 \).

2) It follows from the above (1) and Proposition 4.4.
Proposition 4.8. The non-cuspidal part of the Weil representation $\pi$ is presented as follows:

$$\pi_{\text{non-cusp}} = \bigoplus_{\sigma \in \text{Irr}(\text{GL}_2(F))} \text{Bc}_{K/F}(\sigma),$$

where $\pi_{\text{non-cusp}}$ is the non-cuspidal part of the representation $\pi$ and $\text{Bc}_{K/F}$ is the map of base change from $\text{Irr}(\text{GL}_2(F))$ to $\text{Irr}(\text{GL}_2(K))$.

Proof. It follows from Theorem 1.5(2), Proposition 4.4 and Corollary 4.7. □

Corollary 4.9. The total dimension of the cuspidal part of $\pi$ equals

$$\frac{(q^2 - 1)(q^2 - 1)}{2}.$$ 

Proof. By Proposition 4.8 the dimension of the non-cuspidal part of $\pi$ equals

$$(q^2 + 1) \cdot \frac{(q - 1)(q - 2)}{2} + 1 \cdot (q - 1) + q^3 \cdot (q - 1) = \frac{(q - 1)(q^4 - 1)}{2};$$

the dimension of $\pi$ is $(q - 1)q^4$, and $(q - 1)q^4 - \frac{(q - 1)(q^4 - 1)}{2} = \frac{(q - 1)(q^4 - 1)}{2}$. □

4.7. We continue the above discussion and determine the cuspidal part of $\pi$.

Now let $K_1$ (resp. $F_1$) be a quadratic field extension of $K$ (resp. $F$). Assume $K_1 \supset F_1$. Let $\rho(\text{resp. } \rho_1)$ denote the Weil representation of $\text{GSp}_{2n}(F)$ (resp. $\text{GSp}_{2n}(F_1)$). Denote by $\pi = \rho_{\text{GL}_2(K)}$ and $\pi_1 = \rho_{1_{\text{GL}_2(K_1)}}$. By Proposition 1.8 in Section 1.5 there exists a unique representation $\tilde{\rho}_1$ of the group $\text{GSp}_{2n}(F_1) \rtimes \text{Gal}(F/F_1)$ such that $0 - \text{res}(\tilde{\rho}_1) = \rho_1$, and $1 - \text{res}(\tilde{\rho}_1) = \rho$. By the result in Section 4.2 there exists a morphism from $\text{Res}_{K/F}(\text{GL}_2)$ to $\text{GSp}_{2n}$, which induces a map $\tilde{\rho}_1 : \text{GL}_2(K_1) \rtimes \text{Gal}(K_1/K) \cong \text{Res}_{K/F}(\text{GL}_2)(F_1) \rtimes \text{Gal}(F_1/F) \longrightarrow \text{GSp}_{2n}(F_1) \rtimes \text{Gal}(F_1/F)$.

Via the map $\tilde{\rho}_1$, we let $\tilde{\pi}_1 = \tilde{\rho}_1|_{\text{GL}_2(K_1) \rtimes \text{Gal}(K_1/K)}$. By Lemma 1.7 one sees $0 - \text{res}(\tilde{\pi}_1) = \pi_1$, and $1 - \text{res}(\tilde{\pi}_1) = \pi$. For a cuspidal representation $\Pi_\lambda$ of $\text{GL}_2(K)$, by Theorem 1.5 we know $\text{Bc}_{K_1/K}(\Pi_\lambda) = \Pi_{\lambda \circ \lambda'}$. Let $\Pi_{\lambda \circ \lambda'}$ denote the unique representation of the group $\text{GL}_2(K_1) \rtimes \text{Gal}(K_1/K)$ such that $0 - \text{res}(\Pi_{\lambda \circ \lambda'}) = \Pi_{\lambda \circ \lambda'}$ and $1 - \text{res}(\Pi_{\lambda \circ \lambda'}) = \Pi_\lambda$. By Proposition 4.4 $(\Pi_{\lambda \circ \lambda'}, \Pi_{\lambda \circ \lambda'})|_{\text{GL}_2(K_1)} = 1$ for $\lambda = \lambda \circ \text{N}_{K_1/F_1}$, where $\lambda$ is a regular character of $F_1^*$. By Lemma 1.6(i), we have

$$\langle \text{Tr} \tilde{\pi}_1, \text{Tr} \Pi_{\lambda \circ \lambda'} \rangle_{\text{GL}_2(K_1) \rtimes \text{Gal}(K_1/K)}$$

$$= \frac{1}{|\text{GL}_2(K_1) \rtimes \text{Gal}(K_1/K)|} \sum_{g \in \text{GL}_2(K_1)} \text{Tr} \tilde{\pi}_1((1, g)) \text{Tr} \Pi_{\lambda \circ \lambda'}((1, g)) + \sum_{g \in \text{GL}_2(K_1)} \text{Tr} \tilde{\pi}_1((\sigma, g)) \text{Tr} \Pi_{\lambda \circ \lambda'}((\sigma, g))$$

$$= \frac{|\text{GL}_2(K_1)|}{|\text{GL}_2(K_1) \rtimes \text{Gal}(K_1/K)|} \langle \text{Tr} \pi_1, \text{Tr} \Pi_{\lambda \circ \lambda'} \rangle_{\text{GL}_2(K_1)} + \frac{|\text{GL}_2(K_1)|}{|\text{GL}_2(K_1) \rtimes \text{Gal}(K_1/K)|} \langle \text{Tr} \pi, \text{Tr} \Pi_{\lambda} \rangle_{\text{GL}_2(K_1)}$$

$$= \frac{1}{2} \langle \Sigma \text{Tr} \pi_1, \Sigma \text{Tr} \Pi_{\lambda \circ \lambda'} \rangle_{\text{GL}_2(K_1)} + \langle \Sigma \text{Tr} \pi, \Sigma \text{Tr} \Pi_{\lambda} \rangle_{\text{GL}_2(K_1)}$$

$$= 1 + 1$$

for $\lambda = \lambda \circ \text{N}_{K_1/F_1}$. It follows that for such $\lambda$, $\langle \pi, \Pi_{\lambda} \rangle_{\text{GL}_2(K_1)} \geq 1$. By Corollary 4.9 we see $\langle \pi, \Pi_{\lambda} \rangle_{\text{GL}_2(K_1)} = 1$ and it will also turn out that there are no other kind of cuspidal sub-representations of $\pi$. Finally we achieve the main theorem in this section.

4.8.

Theorem 4.10. The representation $(\pi, V)$ has the following decomposition:

$$\pi = \bigoplus_{\sigma \in \text{Irr}(\text{GL}_2(F))} \text{Bc}_{K/F}(\sigma),$$

where $\text{Irr}(\text{GL}_2(F))$ is the set of the classes of the irreducible representations of $\text{GL}_2(F)$, and $\text{Bc}_{K/F}$ is the base change from $\text{Irr}(\text{GL}_2(F))$ to $\text{Irr}(\text{GL}_2(K))$.

Proof. It follows from Proposition 4.8 for non-cuspidal representations and the above discussion for cuspidal representations. □
4.9. Appendix 1. In the following, we explain how to get the formulas in Proposition 4.2:

(1):
\[
[\pi(u(b)F)](\begin{pmatrix} 0 \\ \beta \\ y \end{pmatrix}, \psi) = [\rho(i(u(b))F)](\begin{pmatrix} 0 \\ \beta \\ y \end{pmatrix}, \psi) = [\rho(\begin{pmatrix} m \\ 0 \\ m \end{pmatrix})]F(\begin{pmatrix} 0 \\ \beta \\ y \end{pmatrix}, \psi) = \chi^*_\omega(det_X(m))\rho(\begin{pmatrix} m^{-1}n \\ 0 \\ 1 \end{pmatrix})F(\begin{pmatrix} m^{-1}n \\ \beta \\ y \end{pmatrix}, \psi) = [\rho(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix})]F(\begin{pmatrix} 0 \\ \beta \\ -y \end{pmatrix}, \psi) = \psi(\frac{1}{2}(m^{-1}n)\begin{pmatrix} \beta \\ 0 \\ -y \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ y \end{pmatrix})F(\begin{pmatrix} 0 \\ \beta \\ y \end{pmatrix}, \psi) = \psi(\frac{1}{2}(\text{Tr}_{K/F}(bb\sigma\beta) - 2 \text{N}_{K/F}(b)y) - y, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix})F(\begin{pmatrix} 0 \\ \beta \\ y \end{pmatrix}, \psi).
\]

(2):
\[
[\pi(h(a,d))F](\begin{pmatrix} 0 \\ \beta \\ y \end{pmatrix}, \psi) = [\rho(i(h(a,d)))F](\begin{pmatrix} 0 \\ \beta \\ y \end{pmatrix}, \psi) = [\rho(\begin{pmatrix} m \\ 0 \\ n \end{pmatrix})]F(\begin{pmatrix} 0 \\ \beta \\ y \end{pmatrix}, \psi) = [\rho(\begin{pmatrix} 0 \\ \beta \end{pmatrix})]F(\begin{pmatrix} 0 \\ \beta \end{pmatrix}, \psi) = \chi^*_\omega(det_X(m))\rho(\begin{pmatrix} 0 \\ \beta \end{pmatrix})F(\begin{pmatrix} 0 \\ \beta \end{pmatrix}, \psi).
\]

(3): Assume $K = F(\xi)$. For a matrix $X$, we denote its transpose by $X^T$. Choose a basis $\mathcal{A} = \{m_0 = e_1 \otimes e_1, m_1 = \xi e_1 \otimes e_1, m_2 = \xi^2 e_1 \otimes e_1, \ldots\}$. If we denote by $\rho_{N_{K/F}(ad)}(-1)$ the determinant of $X$, then by Corollary 4.1(3), we know $\rho(\omega)(m_i) = n_i$ and $\rho(\omega)(n_j) = -m_j$ for $0 \leq i \leq 3$. By calculation, we obtain
\[
\begin{pmatrix}
\langle m_0, m_0 \rangle & \ldots & \langle m_0, m_3 \rangle & \langle m_0, n_0 \rangle & \ldots & \langle m_0, n_3 \rangle \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\langle m_3, m_0 \rangle & \ldots & \langle m_3, m_3 \rangle & \langle m_3, n_0 \rangle & \ldots & \langle m_3, n_3 \rangle \\
\langle n_0, m_0 \rangle & \ldots & \langle n_0, m_3 \rangle & \langle n_0, n_0 \rangle & \ldots & \langle n_0, n_3 \rangle \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\langle n_3, m_0 \rangle & \ldots & \langle n_3, m_3 \rangle & \langle n_3, n_0 \rangle & \ldots & \langle n_3, n_3 \rangle
datac{4.9.1}{1}
\end{pmatrix} = \begin{pmatrix} 0 & A \\
-A & 0 \end{pmatrix}.
\]

where $A = \begin{pmatrix}
-1 & 0 & 0 \\
0 & -\text{Tr}_{K/F}(\xi^2) & -\text{Tr}_{K/F}(\xi^2) \\
0 & -\text{Tr}_{K/F}(\xi^2) & -\text{Tr}_{K/F}(\xi^2) \\
0 & 0 & -\text{Tr}_{K/F}(\xi^2)
\end{pmatrix}$.

Suppose $A = \{P_1 \mid P_1\}$ and $(g_0, \ldots, g_3; h_0, \ldots, h_3) = (m_0, \ldots, m_3; n_0, \ldots, n_3) \varphi$ for some $P = \begin{pmatrix}
P_1^{-1} & 0 \\
0 & P_1^{-1}
\end{pmatrix}$. Then:

\[
\begin{pmatrix}
(g_0, g_0) & \ldots & (g_0, g_3) & (g_0, h_0) & \ldots & (g_0, h_3) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
(g_3, g_0) & \ldots & (g_3, g_3) & (g_3, h_0) & \ldots & (g_3, h_3) \\
(h_0, g_0) & \ldots & (h_0, g_3) & (h_0, h_0) & \ldots & (h_0, h_3) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
(h_3, g_0) & \ldots & (h_3, g_3) & (h_3, h_0) & \ldots & (h_3, h_3)
\end{pmatrix} = \begin{pmatrix}
(m_0, m_0) & \ldots & (m_0, m_3) & (m_0, n_0) & \ldots & (m_0, n_3) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
(m_3, m_0) & \ldots & (m_3, m_3) & (m_3, n_0) & \ldots & (m_3, n_3) \\
(n_0, m_0) & \ldots & (n_0, m_3) & (n_0, n_0) & \ldots & (n_0, n_3) \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
(n_3, m_0) & \ldots & (n_3, m_3) & (n_3, n_0) & \ldots & (n_3, n_3)
\end{pmatrix} \begin{pmatrix}
P_1^{-1} & 0 \\
0 & P_1^{-1}
\end{pmatrix}
\]

i.e. the set $(g_0, \ldots, g_3; h_0, \ldots, h_3)$ is a symplectic basis of $W_0$. Moreover $\iota(\iota)(g_0, \ldots, g_3; h_0, \ldots, h_3) = (g_0, \ldots, g_3; h_0, \ldots, h_3) P^{-1} \begin{pmatrix}0 & -I \\
I & 0\end{pmatrix}$. And $P^{-1} \begin{pmatrix}0 & -I \\
I & 0\end{pmatrix} P = \begin{pmatrix}0 & -I \\
I & 0\end{pmatrix} = \omega_{GSp(W_0)}^{-1} \in \text{GSp}(W_0)$ with respect to the symplectic basis $(g_0, \ldots, g_3; h_0, \ldots, h_3)$. Now let $\alpha = a_1 \xi + a_2 \xi^2 + a_3 \xi^3, \beta = b_1 \xi + b_2 \xi^2 + b_3 \xi^3 \in K$. Put $b = (b_0, \ldots, b_3)$ and $a = (a_0, \ldots, a_3)$. Then

\[
[q(\omega)F](\begin{pmatrix}0 & 0 \\
\beta & -b_0\end{pmatrix} \psi) = \rho(\iota(\omega)F)(\begin{pmatrix}0 & 0 \\
\beta & -b_0\end{pmatrix} \psi)
\]

\[
= q^{-2} \sum_{(n_0, \ldots, n_3) \in Y_0} F((n_0, \ldots, n_3) a, \psi) \varphi((n_0, \ldots, n_3) a, \omega_{GSp(W_0)}((n_0, \ldots, n_3) b))
\]

\[
= q^{-2} \sum_{(n_0, \ldots, n_3) \in Y_0} F((n_0, \ldots, n_3) a, \psi) \varphi((n_0, \ldots, n_3) a, \iota(\iota)^{-1}((n_0, \ldots, n_3) b))
\]

\[
= q^{-2} \sum_{(n_0, \ldots, n_3) \in Y_0} F((-a_0 e_2 \otimes e_2 + a_1 e_2 \otimes e_1 + a_2 e_2 \otimes e_1 + a_3 e_2 \otimes e_1 + a_1 e_2 \otimes e_1 + a_2 e_2 \otimes e_1 + a_3 e_2 \otimes e_1 + a_1 e_2 \otimes e_1 + a_2 e_2 \otimes e_1 + a_3 e_2 \otimes e_1) \psi)
\]

\[
= q^{-2} \sum_{a \in F, \psi \in K} F((-a_0 e_2 \otimes e_2 + a_1 e_2 \otimes e_1 + a_2 e_2 \otimes e_1 + a_3 e_2 \otimes e_1 + a_1 e_2 \otimes e_1 + a_2 e_2 \otimes e_1 + a_3 e_2 \otimes e_1 + a_1 e_2 \otimes e_1 + a_2 e_2 \otimes e_1 + a_3 e_2 \otimes e_1) \psi)(a b_0 + \text{Tr}_{K/F}(a \beta))
\]

Finally, we obtain

\[
[q(\omega)F](\begin{pmatrix}0 & 0 \\
\beta & y\end{pmatrix} \psi) = q^{-2} \sum_{\beta \in K, \gamma \in F} F(\begin{pmatrix}0 & 0 \\
\beta & \gamma\end{pmatrix} \psi) \varphi((\beta \gamma) + \text{Tr}_{K/F}(\beta \gamma))
\]

4.10 Appendix 2. We put the calculations for the table in Section 4.6 in this appendix. From the definition, we see:

\[
A(\begin{pmatrix}0 & 0 \\
\beta & \gamma\end{pmatrix} \psi) = \sum_{t \in T} \alpha \cdot \pi(t) F(\begin{pmatrix}0 & 0 \\
\beta & \gamma\end{pmatrix} \psi)
\]

\[
= x X(\begin{pmatrix}0 & 0 \\
\beta & \gamma\end{pmatrix} \psi) \sum_{a \in K} F(\begin{pmatrix}0 & 0 \\
\beta & \gamma\end{pmatrix} \psi) F(\begin{pmatrix}0 & 0 \\
\beta & \gamma\end{pmatrix} \psi)
\]

\[
= \sum_{a \in K} x X(\begin{pmatrix}0 & 0 \\
\beta & \gamma\end{pmatrix} \psi) F(\begin{pmatrix}0 & 0 \\
\beta & \gamma\end{pmatrix} \psi)
\]
\[
B\left(\begin{pmatrix} 0 \\ \beta \\ 0 \end{pmatrix}, \phi^k\right) = \sum_{t \in T} [\alpha \cdot \pi](t)G_{(0,1,\phi)}\left(\begin{pmatrix} 0 \\ \beta \\ 0 \end{pmatrix}, \phi^k\right)
\]
\[
= \sum_{a, d \in K^*} \chi_{q^+}(N_{K/F}(ad))G_{(0,1,\phi)}\left(\begin{pmatrix} 0 \\ \beta \\ 0 \end{pmatrix}, \phi^kN_{K/F}(ad)^{-1}\right)
\]
\[
= \sum_{a, d \in K^*, N_{K/F}(ad) = k} \chi_{q^+}(k)G_{(0,1,\phi)}\left(\begin{pmatrix} 0 \\ \beta \\ 0 \end{pmatrix}, \phi^kN_{K/F}(ad)^{-1}\right);
\]
\[
(\alpha \cdot \pi(\omega)B)\left(\begin{pmatrix} 0 \\ \beta \\ 0 \end{pmatrix}, \phi^k\right) = \sum_{t \in T} [\alpha \cdot \pi(t)[\alpha \cdot \pi(\omega)]G_{(0,1,\phi)}\left(\begin{pmatrix} 0 \\ \beta \\ 0 \end{pmatrix}, \phi^k\right)
\]
\[
= \sum_{a, d \in K^*} \chi_{q^+}(N_{K/F}(ad))\left(\begin{pmatrix} 0 \\ \beta \\ 0 \end{pmatrix}, \phi^kN_{K/F}(ad)^{-1}\right)
\]
\[
= q^{-2} \sum_{a, d \in K^*, N_{K/F}(ad) = k} \chi_{q^+}(k) = q^{-2} \chi_{q^+}(k)(q^3 - 1)(q^2 + q + 1);
\]
Notice: \(G_{(0,1,\phi)}\left(\begin{pmatrix} 0 \\ \beta \\ 0 \end{pmatrix}, \phi\right) = \phi(-N_{K/F}(\beta))\) by the formula (4.1).

\[
A(x_0) = \sum_{a, d \in K^*, N_{K/F}(ad) = 1} F_{(0,0,\phi)}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \phi\right)
\]
\[
= \sum_{a, d \in K^*, N_{K/F}(ad) = 1} 1 = (q^3 - 1)(q^2 + q + 1).
\]
\[
A(x_{10}) = A(x_0) = A(y_0) = 0.
\]
(2)
\[
B(x_0) = \sum_{a, d \in K^*, N_{K/F}(ad) = 1} G_{(0,1,\phi)}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \phi\right) = 0 = B(x_{10}).
\]
\[
B(x_0) = \sum_{a, d \in K^*, N_{K/F}(ad) = 1} G_{(0,1,\phi)}\left(\begin{pmatrix} 0 \\ 0 \\ N_{K/F}(a) \end{pmatrix}, \phi\right)
\]
\[ B(y_k) = \sum_{\beta \in \mathbb{K}^+, \mathbb{N}_{K/F}(\beta) = N_{K/F}(a)} \sum_{\alpha \in \mathbb{K}^+, \mathbb{N}_{K/F}(\alpha) = 1} 1 = (q^2 + q + 1)^2. \]

\[ = \sum_{\alpha, \beta \in \mathbb{K}^+, \mathbb{N}_{K/F}(\alpha) = 1, \mathbb{N}_{K/F}(\beta) = k} \chi_{\hat{\mathbb{K}}}^*(\beta)G(0,1,\phi)(y_k) \]

\[ = \sum_{\alpha, \beta \in \mathbb{K}^+, \mathbb{N}_{K/F}(\alpha) = 1, \mathbb{N}_{K/F}(\beta) = k} \chi_{\hat{\mathbb{K}}}^*(\beta)G(0,1,\phi) \left( \begin{array}{cc} 0 & \mathbb{N}_{K/F}(\beta) \\ \mathbb{d} & 1 \end{array} \right) \]

\[ = \sum_{\alpha, \beta \in \mathbb{K}^+, \mathbb{N}_{K/F}(\alpha) = 1, \mathbb{N}_{K/F}(\beta) = k} \chi_{\hat{\mathbb{K}}}^*(\beta) \phi \left( -\mathbb{N}_{K/F}(\alpha) \mathbb{d} \mathbb{d}^* \right) \]

\[ = \phi(-\mathbb{N}_{K/F}(\beta))(q^2 + q + 1)^2. \]

(3) \[ [\alpha \cdot \pi](\omega)A(x_{10}) = [\alpha \cdot \pi](\omega)A(x_{10}) = [\alpha \cdot \pi](\omega)A(x_{10}) = q^2 - (q^3 - 1)(q^2 + q + 1). \]

(4) \[ \alpha \cdot \pi(\omega)B(x_{10}) = q^{-2} \sum_{\alpha \in \mathbb{K}^+, \mathbb{N}_{K/F}(\alpha) = 1} \phi(-\mathbb{N}_{K/F}(\beta)) = \sum_{\beta \in \mathbb{K}^+, \mathbb{N}_{K/F}(\beta) = 1} (q^2 - q). \]

(Since \( \sum_{\beta \in \mathbb{K}^+, \mathbb{N}_{K/F}(\beta) = 1} \phi(-\mathbb{N}_{K/F}(\beta)) = \sum_{\beta \in \mathbb{K}^+, \mathbb{N}_{K/F}(\beta) = 1} (q^2 - q). \)

\[ = \sum_{\alpha \in \mathbb{K}^+, \mathbb{N}_{K/F}(\alpha) = 1} \phi(-\mathbb{N}_{K/F}(\beta)) \phi(\mathbb{d} \mathbb{d}^* \beta) \]

\[ = \sum_{\alpha \in \mathbb{K}^+, \mathbb{N}_{K/F}(\alpha) = 1} \phi(-\mathbb{N}_{K/F}(\beta)) \phi(\mathbb{d} \mathbb{d}^* \beta) \]

\[ = q^{-2} \sum_{\alpha \in \mathbb{K}^+, \mathbb{N}_{K/F}(\beta) = 1} \phi(-\mathbb{N}_{K/F}(\beta)) \phi(\mathbb{d} \mathbb{d}^* \beta) \]

(i) If \( \beta = 0, \sum_{\alpha \in \mathbb{K}^+, \mathbb{N}_{K/F}(\alpha) = 1} \phi(-\mathbb{N}_{K/F}(\beta)) = 1 = \sum_{\alpha \in \mathbb{K}^+, \mathbb{N}_{K/F}(\alpha) = 1} \phi(-\mathbb{N}_{K/F}(\beta)) \).

(ii) If \( \beta \neq 0, \sum_{\alpha \in \mathbb{K}^+, \mathbb{N}_{K/F}(\alpha) = 1} \phi(-\mathbb{N}_{K/F}(\beta)) = \sum_{\alpha \in \mathbb{K}^+, \mathbb{N}_{K/F}(\alpha) = 1} \phi(-\mathbb{N}_{K/F}(\beta)) \)

\[ = (q^2 + q + 1)(-q^2 - q - 1) = (q^2 + q + 1)^2. \]

\[ \sum_{\beta \in \mathbb{K}^+, \mathbb{N}_{K/F}(\beta) = 1} \phi(-\mathbb{N}_{K/F}(\beta)) = (q^2 + q + 1)(q^2 - 1) + (q^2 + q + 1)^2 = q^{-2}(q^2 + q + 1)^2 = q^{-1}(q^2 + q + 1)^2. \]

\[ \alpha \cdot \pi(\omega)B(x_{10}) = q^{-2} \sum_{\alpha \in \mathbb{K}^+, \mathbb{N}_{K/F}(\alpha) = 1} \phi(-\mathbb{N}_{K/F}(\beta)) = q^{-2} \sum_{\alpha \in \mathbb{K}^+, \mathbb{N}_{K/F}(\alpha) = 1} \phi(-\mathbb{N}_{K/F}(\beta)) \]

\[ = q^{-2} \sum_{\alpha \in \mathbb{K}^+, \mathbb{N}_{K/F}(\alpha) = 1} \phi(-\mathbb{N}_{K/F}(\beta))(-q^2 - q) \]

\[ = q^{-2}(q^2 + q + 1)(-q^2 - q - 1)(-q^2 - q) = q^{-1}(q + 1)(q^2 + q + 1)^2. \]
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