Thermally assisted quantum cavitation in solutions of $^3$He in $^4$He

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Abstract

We have investigated the quantum-to-thermal crossover temperature $T^*$ for cavitation in liquid helium mixtures up to 5% $^3$He concentrations. With respect to the pure $^4$He case, $T^*$ is sizeably reduced, to a value below 50 mK for $^3$He concentrations above 2%. As in pure $^4$He, the homogeneous cavitation pressure is systematically found close to the spinodal pressure.

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Quantum cavitation in superfluid liquid $^4$He has been experimentally observed by Balibar and coworkers [1]. They have used an ultrasound technique by means of which, a large pressure oscillation is created in a small bulk region of the experimental sample. This prevents cavitation on the walls of the experimental cell. Moreover, as superfluid $^4$He can be made perfectly pure, heterogeneous cavitation can be avoided. Due to the nature of these experiments, it is very difficult to determine the pressure (P) and temperature (T) at the cavitation site. Consequently, a quantitative analysis of the experimental results relies on theoretical estimates of the equation of state and cavitation barriers in the spinodal region [2,3]. Depending on these theoretical inputs and on experimental conditions, such as volume and observation time, it is inferred that quantum cavitation likely takes over thermal cavitation at a temperature in the 200-240 mK range. The agreement between experiment [1] and theory [2,3] can be considered as satisfactory given the inherent limitations of both.

In this work we address the problem of thermally assisted quantum cavitation in low $^3$He-concentration, liquid helium mixtures (up to 5%). The relevance of this problem is twofold. First, the planned experiments on quantum cavitation in these mixtures [4]. Second, the complexity introduced in the theoretical description of the still superfluid liquid $^4$He, when a small but sizeable $^3$He amount is present. To our knowledge, no investigation of this kind, either experimental or theoretical, exists in the literature. Some effort has been concentrated in the study of supersaturated $^3$He-$^4$He mixtures at positive pressures and low temperatures (see for example Refs. [5-8]), or near the tricritical point (see Ref. [9] and refs. therein).

The method we use here is conceptually simple and technically workable. It is based, on the one hand, in using a density functional [10] that reproduces the thermodynamical properties of the $^3$He-$^4$He liquid mixture at zero temperature. This functional has been slightly modified to reproduce the surface properties of the $^3$He-$^4$He interface, and those of the mixture free-surface [11,12]. All these properties are relevant for a quantitative description of the cavitation process. On the other hand, we make use of a functional-integral approach especially well suited to find the crossover temperature $T^*$ with the only further approximation of imposing irrotational motion for the growing of the critical bubbles.
Let us first generalize the functional-integral approach as used in [3], to the mixture case (see also [13] for another example of how it applies). We recall that above a crossover temperature $T^*$, the cavitation rate, i.e., the number of bubbles formed per unit time and volume, is given by

$$J_T = J_{0T} e^{-\Delta \Omega_{\text{max}}/T},$$

where $\Delta \Omega_{\text{max}}$ is the barrier height for thermal activation. Below $T^*$, assuming $\Delta \Omega_{\text{max}} \gg T$, one can write

$$J_Q = J_{0Q} e^{-S_{\text{min}}},$$

being $S_{\text{min}}$ the minimum of the imaginary-time action

$$S(T) = \int d\tau \int d\vec{r} \mathcal{L}.$$  

$\mathcal{L}$ is the imaginary-time classical Lagrangian density of the system. For a given $T$, the time-integration is extended over a "period" $\tau$ in the inverted barrier potential well with $\tau = \hbar/T$. The crossover temperature $T^*$ is determined from the angular frequency $\omega_p$ of the small amplitude oscillations around the minimum of the potential well (see [14] and Refs. therein). The prefactors $J_{0T}$ and $J_{0Q}$ depend on the dynamics of the cavitation process. In actual calculations, they are often estimated as the number of cavitation sites per unit volume times an attempting frequency.

The Lagrangian density can be easily obtained from the zero-temperature density functional of [11,12], whose use is justified in view of the low temperatures involved (below 200 mK). The critical cavity density profiles $\rho_3^{\theta}(r)$, $\rho_4^{\theta}(r)$ are obtained solving the coupled Euler-Lagrange equations

$$\frac{\delta \omega(\rho_3, \rho_4)}{\delta \rho_q} = 0 \quad , q = 3, 4 ,$$

where $\omega(\rho_3, \rho_4)$ is the grand potential density and $\rho_q$ are the particle densities of each helium isotope. $\Delta \Omega_{\text{max}}$ is given by
\[ \Delta \Omega_{\text{max}} = \int \dd{\vec{r}} \left[ \omega(\rho_3^0, \rho_4^0) - \omega(\rho_m^3, \rho_m^4) \right], \]  

(5)

where \( \rho_{maq} \) are the corresponding densities of the metastable homogeneous liquid (see [11,12] for details). Assuming that only spherical bubbles are developing, the collective velocities \( \vec{u}_q(\vec{r}, t) \) of both helium fluids are irrotational and one can then define for each isotope a velocity potential field \( s_q(\vec{r}, t) \) such that \( \vec{u}_q(\vec{r}, t) \equiv \nabla s_q(\vec{r}, t) \). It follows that

\[ \mathcal{L} = \sum_q m_q \dot{\rho}_q s_q - \mathcal{H}(\rho_3, \rho_4, s_3, s_4), \]  

(6)

where \( \mathcal{H}(\rho_q, s_q) \) is the imaginary-time Hamiltonian density

\[ \mathcal{H}(\rho_3, \rho_4, s_3, s_4) = \frac{1}{2} \sum_q m_q \rho_q \vec{u}_q^2 - \left[ \omega(\rho_3, \rho_4) - \omega(\rho_m^3, \rho_m^4) \right]. \]  

(7)

Hamilton’s equations yield the following four equations:

\[ m_q \dot{\rho}_q = \frac{\delta \mathcal{H}}{\delta s_q} = -m_q \nabla (\rho_q \vec{u}_q) \]  

(8)

\[ m_q \dot{s}_q = -\frac{\delta \mathcal{H}}{\delta \rho_q}. \]  

(9)

Eqs. (8) are the continuity equations. Taking the gradient of Eqs. (8) one gets

\[ m_q \frac{d\vec{u}_q}{dt} = -\nabla \left\{ \frac{1}{2} m_q \vec{u}_q^2 - \frac{\delta \omega}{\delta \rho_q} \right\}. \]  

(10)

To determine \( T^* \) one has to find the small amplitude, periodic solutions of Eqs. (8) and (10) linearized around \( \rho_3^0 \) and \( \rho_4^0 \) [3,13]. Defining the transition densities \( \rho_q^1(\vec{r}) \) as

\[ \rho_q(\vec{r}, t) \equiv \rho_q^0(\vec{r}) + \rho_q^1(\vec{r}) \cos(\omega_p t), \]  

(11)

where \( \rho_q^1(\vec{r}) \) is much smaller than \( \rho_q^0(\vec{r}) \), and keeping only first order terms in \( \vec{u}_q(\vec{r}, t) \) and in \( \rho_q^1(\vec{r}) \), one gets:

\[ \omega_p^2 \rho_q^1(\vec{r}) = \frac{1}{m_q} \nabla \left[ \rho_q^0(\vec{r}) \nabla \left( \sum_{q' = 3, 4} \frac{\delta^2 \omega}{\delta \rho_q \delta \rho_{q'}} \cdot \rho_q^1(\vec{r}) \right) \right], \quad q = 3, 4. \]  

(12)

In Eq. (12), \( \frac{\delta^2 \omega}{\delta \rho_q \delta \rho_{q'}} \cdot \rho_q^1(\vec{r}) \) means that \( \delta \omega/\delta \rho_q \) has to be linearized, keeping only terms in \( \rho_3^1 \) and \( \rho_4^1 \), and their derivatives.
Eq. (12) is a fourth-order linear differential, eigenvalue equation for the "vector"
$(\rho_3^l(r), \rho_4^l(r))$, whose right-hand side term is straightforward but very cumbersome to obtain.
We have done it using the Mathematica software package [15]. As in [3], a careful analysis
shows that the physical solutions of Eqs. (12) have to fulfill $(\rho_4^l)'(0) = (\rho_4^l)'''(0) = 0$, and
have to fall exponentially to zero at large distances. From the linearized continuity equation
$\rho_4^l(r) \propto -\nabla(\rho_0^0 \vec{u}_q)$, it is obvious that the integral of $\rho_4^l(r)$ over the whole space is zero.

We have solved the eigenvalue Eq. (12) as in [3]. For a given pressure and $^3$He-
concentration $x \equiv \rho_{m3}/(\rho_{m3} + \rho_{m4})$, only a positive eigenvalue $\omega_p^2$ is found, out of which
we get $T^* = \hbar \omega_p/2\pi$.

Fig. 1 shows $T^*$ (mK) as a function of P(bar) for $x = 0.1, 1, 2, 3, 4$ and $5\%$. Compared
to the pure $^4$He case [3], $T^*(P)$ has now a more complex structure. It is worth to note that
the maximum of the $T^*(P)$ curve has decreased from $\sim 240$ mK for pure $^4$He [3] down to $\sim
140$ mK for $^3$He-concentrations as small as $1\%$.

Fig. 2 shows two different bubble configurations for $x = 1\%$. Configuration (a) corre-
sponds to $P = -8.23$ bar and $T^* = 62.7$ mK, and configuration (b) to $P = -5.17$ bar and
$T^* = 115.7$ mK. The solid lines represent the $^3$He and $^4$He critical bubble particle densities
in $\text{Å}^{-3}$, and the dashed (dash-dotted) lines represent $\rho_3^l(r)$ $(\rho_4^l(r))$ in arbitrary units. Near
the spinodal region, the "bubble" configuration is filled with $^3$He: the surface tension that
matters for bubble formation is that of the $^3$He-$^4$He interface. Away from the spinodal region
(configuration (b)), the critical bubble is a true bubble covered with $^3$He (Andreev states):
the surface tension that matters now is that of the $^3$He-$^4$He liquid free-surface, which is
about ten times larger than the previous one. The different surface tensions involved in
these processes, together with the existence of a $^3$He-$^4$He saturation curve at negative pres-
sures down to $x \sim 2.4\%$ [11] are the cause of the structures displayed in Fig. 1. We will not
give here any further detail since, as in the pure $^4$He case, only the part of the $T^*(P)$ curve
near the spinodal region is relevant for the cavitation problem.

It is interesting to see that the transition densities $\rho_4^l$ evolve from those corresponding to
"volume oscillations" (Fig. 2a) to "surface oscillations" for $^4$He, and a mixed surface-volume
type for \(^{3}\)He (Fig. 2b), to eventually become pure ”surface oscillations” for both isotopes when moving from the spinodal towards the saturation line. This fact has been already found in pure \(^{4}\)He [3].

To establish which \(T^\ast(P)\) in Fig. 1 corresponds to that of the experimental conditions, we have to determine the homogeneous cavitation pressure \(P_h\), which is the one the system can sustain before bubbles nucleate at an appreciable rate. \(P_h\) can be obtained solving the equation

\[ 1 = (Vt)_e J, \]  

(13)

taking \(J = J_T\) and \(J_0T = (k_B T)/(hV_0)\) if \(T>T^\ast\), where \(k_B T/h\) is a thermal attempting frequency, and \(V_0 = 4\pi R_c^3 / 3\) is the inverse of the number of possible nucleation centers per unit volume, with \(R_c=10\ \text{Å}\) as a typical radius of the critical bubble. For \(T \leq T^\ast\), one takes \(J = J_Q\) and, lacking of a better choice, \(J_0Q = J_0T(T=T^\ast)\). It yields \(J_0Q\) of the order of \(10^5\ \text{Å}^{-3}\) s\(^{-1}\). Another possible guess for the quantum attempting frequency is to equal it to \(v_s/R_c\), where \(v_s\) is the sound velocity, which is of the order of \(10^{12}\ \text{Å} \text{ s}^{-1}\) in the region of interest. This choice differs from the previous one in two orders of magnitude. However, this difference in the prefactor value does not alter in practice the results. Indeed, we have chosen two extreme experimental values for the factor \((Vt)_e\) (experimental volume \(\times\) time) which differ ten orders of magnitude without much influencing the obtained \(T^\ast\)-value.

\(P_h(T^\ast)\) is shown as circles (squares) on the curves in Fig 1. The circles correspond to \((Vt)_e = 10^4\ \text{Å}^3\) s, and the squares to \(10^4\ \text{Å}^3\) s. Compared to the pure \(^{4}\)He case [3], and depending on the \((Vt)_e\)-value, for \(x = 1\%\) \(T^\ast\) has been reduced a factor of 4 or 5, respectively.

Fig. 3 shows \(P_h\) as a function of \(T\) for the above mentioned \(x\)-values and \((Vt)_e = 10^4\ \text{Å}^3\) s. Thermal and quantum regimes are displayed. The dashed line is the extrapolation of the thermal regime to temperatures close to \(T=0\). Finally, Fig. 4 displays the \(T=0, P_{sp}(x)\) spinodal line together with the \(P_h(x,T^\ast)\) curves for the indicated \((Vt)_e\) values. Notice that the smallest \(x\)-value displayed in this figure is 0.1%. For \(^{3}\)He concentrations closer to zero, it is very unlike that \(^{3}\)He has time enough to diffuse and develop the critical configurations.
that constitute the starting point of the present calculations. Also shown in that figure (dots on the y-axis) are the pure $^4$He values [3].

In conclusion, we have thoroughly investigated thermally-assisted quantum cavitation in $^3$He-$^4$He liquid mixtures at small $^3$He concentrations. Our quantitative predictions rely on a robust density functional that reproduces the relevant characteristics of the mixture and its interfaces, and on a sound formalism, the functional integral theory. Besides the approximations inherent to the method (use of a density functional, irrotational flows), no further approximation has been introduced to obtain the quantum-to-thermal crossover temperature. The present results might thus be a valuable guide to the planned experiments [4].

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REFERENCES

[1] S. Balibar, C. Guthmann, H. Lambare, P. Roche, E. Rolley and H.J. Maris, J. Low Temp. Phys. 101, 271 (1995).

[2] H.J. Maris, J. Low Temp. Phys. 98, 403 (1995).

[3] M. Guilleumas, M. Barranco, D. Jezek, R. Lombard and M. Pi, to be published in Phys. Rev. B (1996).

[4] S. Balibar, private communication.

[5] I.M. Lifshitz, V.M. Polesskii and V.A. Khokhlov, Sov. Phys. JETP 47, 137 (1978).

[6] T. Satoh, M. Morishita, M. Ogata and S. Katoh, Phys. Rev. Lett. 69, 335 (1992).

[7] S.N. Burmistrov, L.B. Dubovskii and V.L. Tsymbalenko, J. Low Temp. Phys. 90, 363 (1993).

[8] D.M. Jezek, M. Guilleumas, M. Pi and M. Barranco, Phys. Rev. B51, 11981 (1995).

[9] J.K. Hoffer and D.N. Sinha, Phys. Rev. A33, 1918 (1986).

[10] F. Dalfovo and S. Stringari, Phys. Lett. 112A, 171 (1985).

[11] M. Guilleumas, D. Jezek, M. Pi, M. Barranco and J. Navarro, Phys. Rev. B51, 1140 (1995).

[12] M. Guilleumas, M. Pi, M. Barranco, D. Jezek and J. Navarro, Phys. Rev. B52, 1210 (1995).

[13] E. M. Chudnovsky, A. Ferrera and A. Vilenkin, Phys. Rev. B51, 1181 (1995).

[14] E. M. Chudnovsky, Phys. Rev. A46, 8011 (1992).

[15] S. Wolfram, Mathematica, A System for Doing Mathematics by Computer , (Addison-Westley Publishing Company, Inc., California, 1991).
FIGURES

FIG. 1. $T^*$ as a function of $P$. The homogeneous cavitation pressure $P_h(T^*)$ is shown as circles (squares) for $(Vt)_c = 10^4 \text{Å}^3 \text{s}$ ($10^{14} \text{Å}^3 \text{s}$), respectively.

FIG. 2. Particle densities $\rho_0^0(r)$ and $\rho_0^0(r)$ of the critical bubbles (solid lines), and the $\rho_1^1(r)$ (dash-dotted lines) and $\rho_3^1(r)$ (dashed lines) transition densities for $x= 1\%$, corresponding to: (a) $P= -8.23$ bar and $T^* = 62.7$ mK. (b) $P= -5.17$ bar and $T^* = 115.7$ mK. $\rho_q^0(r)$ are drawn in arbitrary units, $\rho_q^0(r)$ in $\text{Å}^{-3}$.

FIG. 3. Homogeneous cavitation pressure $P_h$ as a function of $T$ for the same $x$-values as in Fig. 1.

FIG. 4. The $T=0$ spinodal line $P_{sp}(x)$ (solid line), and the homogeneous cavitation pressure $P_h(x,T^*)$ (dashed and dash-dotted curves) for the indicated $(Vt)_c$ values. The dots on the $y$-axis are the corresponding pure $^4\text{He}$ values.
\((V_t)_e = 10^{14} \text{Å}^3 \text{s}\)