Numerical analysis of elliptic inverse problems with interior data

Mark S. Gockenbach
Department of Mathematical Sciences
Michigan Technological University
Houghton, MI 49931
USA
E-mail: msgocken@mtu.edu

Abstract. A number of algorithms have been proposed and analyzed for estimating a coefficient in an elliptic boundary value problem when interior data is available. Most of the analysis has been done for the simple scalar BVP

\[-\nabla \cdot (a \nabla u) = f \text{ in } \Omega,\]
\[a \frac{\partial u}{\partial n} = g \text{ on } \partial \Omega.\]

However, some methods and the associated analysis have been extended to the problem of estimating the Lamé moduli in the system of linear, isotropic elasticity. Under certain idealized conditions, convergence of estimates to the exact Lamé moduli has been proved for two techniques, the output least-squares method and a variational method similar to the equation error approach.

1. Introduction
There are a number of applications which call for estimating a coefficient in an elliptic partial differential equation (PDE) from measurements of the solution to an associated elliptic boundary value problem (BVP). For instance, the BVP

\[-\nabla \cdot (a \nabla u) = f \text{ in } \Omega,\]  \[(1)\]
\[a \frac{\partial u}{\partial n} = h \text{ on } \partial \Omega,\]  \[(2)\]

is a simple model of steady-state groundwater flow, in which \(a = a(x, y)\) is the transmissivity at \((x, y) \in \Omega\), \(f = f(x, y)\) is the recharge, and \(u = u(x, y)\) is the piezometric head. The direct problem is to compute \(u\) given \(a\) and \(f\); a related inverse problem is to estimate \(a\) from measurements of \(u\) and \(f\).

Another elliptic BVP, and the one that is of most interest in this paper, is the system describing a linear, isotropic elastic membrane:\(^1\)

\[-\nabla \cdot \sigma = 0 \text{ in } \Omega,\]  \[(3)\]
\[\sigma n = g \text{ on } \partial \Omega,\]  \[(4)\]

\(^1\) For applications, it is of more interest to consider a three-dimensional body rather than a two-dimensional membrane. However, all of the analysis to date has been carried out for the two-dimensional case.
where

\[
\sigma = 2\mu \varepsilon + \lambda \text{tr}(\varepsilon)I, \quad (5)
\]

\[
\varepsilon = \frac{1}{2} (\nabla u + \nabla u^T). \quad (6)
\]

Here \( u \) is the displacement of an elastic membrane (which occupies the planar region \( \Omega \) when at rest) due to the boundary traction \( g \), \( \epsilon \) is the corresponding (linearized) strain (a measure of the local relative change in position), and \( \sigma \) is the associated stress tensor. The stress-strain law (5) expresses the assumption that the material is isotropic and that the strain is sufficiently small that a linear relationship holds to good approximation. The Lamé moduli \( \mu \) and \( \lambda \) characterize the elastic properties of the material: \( \mu \) is the shear modulus, which determines how resistant the material is to shearing, and \( \mu + \lambda \) is the bulk modulus, which determines how resistant the membrane is to expansion.

The direct problem defined by (3–6) is to compute the displacement \( u \) from a knowledge of the traction \( g \) and the Lamé moduli \( \mu \) and \( \lambda \). The associated inverse problem seeks estimates of \( \mu \) and \( \lambda \) from measurements of \( u \) and \( g \).

This paper describes several related optimization-based algorithms for solving the inverse problems described above. We focus our attention on algorithms for which a convergence analysis has appeared in the literature. The following section describes two variants of the output least-squares (OLS) method, Section 3 presents the method of equation error, and Section 4 discusses a variational approach that can be viewed as a variant of the equation error method. Finally, Section 5 analyzes the simple case in which a constant coefficient is to be estimated; in this case, it is possible to compare the various techniques directly.

### 2. The output-least squares approach

The OLS approach is perhaps the most natural optimization-based approach to inverse problem, and it is broadly applicable. It assumes that the observable data are related to the desired parameters by a mathematical model (usually a differential equation), and that the data can be simulated for any meaningful estimate of the parameters. The parameters are then estimated by choosing them so that the simulated data are as close as possible to the observed data in some norm.

In the case of either (1–2) or (3–6), the solution BVP can be simulated by the finite element method. In the first case, we will write \( u_h(a_h) \) for the finite element solution defined by the coefficient \( a = a_h \); here \( h \) represents the mesh size, and it is assumed that both \( u_h \) and \( a_h \) are represented as piecewise polynomials functions. The unknown coefficient \( a^* \) is then estimated by a minimizer \( a_h^* \) of

\[
J_{OLS}(a_h) = \frac{1}{2} ||u_h(a_h) - z||^2,
\]

where \( z \) is the observed data. The objective function \( J_{OLS} \) is minimized subject to pointwise constraints \( 0 < a \leq a_h \leq \beta \) on the coefficient that guarantee that the simulation of \( u_h(a_h) \) is well-defined.

In the case that \( J_{OLS} \) is defined by the \( L^2(\Omega) \) norm, this method was analyzed by Falk [5]. He showed that there exists a constant \( C \) such that, if \( a_h^* \) is any minimizer of \( J_{OLS} \), then

\[
||a_h^* - a^*||_{L^2(\Omega)} \leq C \left( h^r + \frac{||z - u^*||_{L^2(\Omega)}}{h^2} \right)
\]

for all \( h \) sufficiently small. This result assume that the coefficient \( a \) is represented by continuous piecewise polynomials of degree \( r \) and the solution \( u \) by continuous piecewise polynomials of
degree \( r + 1 \). Moreover, a critical assumption is that the true coefficient \( a^* \) is smooth, namely, that \( a^* \in H^{r+1}(\Omega) \). It is also assumed that there exists a constant unit vector \( \nu \) such that
\[
\nabla u^* \cdot \nu \geq \gamma \text{ in } \Omega. \tag{8}
\]
This may be viewed as a nondegeneracy condition for the experiment under which the data are measured. In the application of groundwater flow, this means that there is always some flow in the \( \nu \) direction, and it allows the PDE (1) to be viewed as a hyperbolic PDE for \( a \).

The bound (7) suggests that the inverse problem is ill-posed, as the bound for the error in the estimated parameter blows up as the mesh is refined. Indeed, Alessandrini [1] has demonstrated by explicit example the discontinuous dependence of \( a^* \) on \( u^* \). However, (7) does provide a convergence result: If \( z \) is the continuous piecewise polynomial interpolant of degree \( r + 1 \) of \( u^* \) (exact pointwise data), then
\[
\| z - u^* \|_{L^2(\Omega)} = O(h^{r+2})
\]
by standard approximation results, and we obtain
\[
\| a_h^* - a^* \|_{L^2(\Omega)} = O(h^r).
\]

There are two major steps to Falk’s proof of (7). The first is to show that \( J_{OLS} \) can be made small when the data is exact; the result is that for \( z = u^* \), we have
\[
\inf J_{OLS}(b_h) \leq C h^{r+2} \tag{9}
\]
(where the infimum is taken over all \( b_h \) satisfying appropriate pointwise bounds). The proof uses the fact that
\[
J_{OLS}(a_h^*) = \inf J_{OLS}(b_h) \leq J_{OLS}(\tilde{a}_h),
\]
where \( \tilde{a}_h \) is the \( L^2 \)-projection of \( a^* \) onto the finite element space, along with standard approximation results. This part of the proof can be extended to more general problems, such as the system of isotropic elasticity, in a straightforward manner.

The second part of the proof hinges on a clever choice of a test function, namely, \( v = p e^{-2k_{\nu} \cdot (x,y)}(\tilde{a}_h - a_h^*) \), where \( k \) is a constant and \( \rho \) is the solution of a hyperbolic PDE (defined by the principal part of \( -\nabla \cdot (a \nabla u) = f \), viewed as a PDE in \( a \)). This test function allows Falk to prove that
\[
C \| \tilde{a}_h - a_h^* \|_{L^2(\Omega)}^2 \leq \int_\Omega (a_h^* - \tilde{a}_h) \nabla u^* \cdot \nabla v. \tag{10}
\]
The right-hand side of (10) can be bounded using (9) and standard approximation results; then, since \( \tilde{a}_h \) is close to \( a^* \), the desired result follows.

Turning our attention to the system of isotropic elasticity, we begin by pointing out that estimating \( \mu \) and \( \lambda \) is equivalent to estimating the bulk modulus \( \rho = \mu + \lambda \) along with the shear modulus \( \mu \). We will henceforth write \( m = (\mu, \rho) \). As described above, Falk’s result is based on the fact that (1) defines a hyperbolic PDE for \( a \) under certain conditions on the \( a \). In Cox and Gockenbach [4], we showed that (3) can be viewed as a hyperbolic PDE for \( (\mu, \lambda) \) (or, equivalently, for \( (\mu, \rho) \)) if the displacement \( u \) satisfies the nondegeneracy condition
\[
\min \{|\epsilon_{12}^*|, |\text{tr}(\epsilon^*)|\} \geq c > 0 \tag{11}
\]
(where \( \epsilon^* \) is the linearized strain associated with \( u^* \)). Condition (11) restricts the conditions under which \( u^* \) is observed, and implies that, at each point in \( \Omega \), the displacement is neither a pure expansion nor a pure shear. Therefore, it seems reasonable that it might be possible to estimate both the bulk modulus and shear modulus at each point in \( \Omega \).

Indeed, the following lemma implies that \( m = (\mu, \rho) \) are uniquely determined by \( u^* \).
Lemma 2.1 Suppose \( u^* \in H^3(\Omega)^2 \) satisfies (11), and \( m \in H^1(\Omega)^2 \). Then there exists a \( a > 0 \) such that \( v = \sigma(m, u^*) q \), \( q(x) = (e^{ax_1}, e^{ax_2}) \), satisfies

\[
\|\sigma(m, u^*)\|_{L^2(\Omega)}^2 \leq C \left\{ \|\sigma(m, u^*) n\|_{L^2(\partial\Omega)} \|\sigma(m, u^*)\|_{L^2(\partial\Omega)} + \left| \int_{\Omega} \sigma(m, u^*) \cdot \epsilon(v) \right| \right\}.
\]

The constant \( C \) depends on \( u^* \) and a but is independent of \( m \).

Lemma 2.1 is a generalization of Lemma 2.1 of Chen and Gockenbach [3] and was proved in our recent paper [10]. The notation \( \sigma(m, u) \) represents the stress tensor \( \sigma = 2\mu \varepsilon + \lambda \text{tr}(\varepsilon)I \) determined by \( m = (\mu, \rho) \) and the strain \( \epsilon = \epsilon(u) \).

Using Lemma 2.1, we can prove a result similar to Falk’s error bound for the scalar problem. Analogous to his analysis, we assume that the displacement \( u \) is simulated by the Galerkin finite element method, and that \( u \) is approximated by a continuous piecewise polynomial \( u_h \) of degree \( r + 1 \) \((r \geq 1)\), while \( m \) is approximated by a continuous piecewise polynomial \( m_h \) of degree \( r \).

To be more precise, let \( \{T_h\} \) be a regular, quasi-uniform family of triangulations of the domain \( \Omega \). Here \( h \) denotes the maximum diameter of any triangle in \( T_h \), and there exists \( \nu > 0 \) such that

\[
\nu h \leq \rho_T \leq h_T \leq h \text{ for all } T \in T_h, h > 0,
\]

where \( h_T \) is the diameter of \( T \) and \( \rho_T \) is the diameter of the largest ball contained in \( T \). Define

\[
L_h = \left\{ u \in C(\Omega) : |u|_{T} \in \mathcal{P}_r \text{ for all } T \in T_h \right\}^2,
\]

\[
K_h = \{(\mu, \rho) \in L_h : c_0 < \mu, \rho < c_1 \},
\]

\[
U_h = \left\{ u \in C(\Omega) : |u|_{T} \in \mathcal{P}_{r+1} \text{ for all } T \in T_h \right\}^2,
\]

where \( \mathcal{P}_k \) is the space of polynomials (in two variables) of degree at most \( k \) and \( c_0, c_1 \) are given positive constants, \( 0 < c_0 < c_1 \).

We write \( u_h(m_h) \) for the finite element simulation of the solution to (3–6), where \( m = m_h \). We use (12) with \( m = m_h^* - \bar{m}_h \), where \( m_h^* \) is a minimizer of the OLS functional and \( \bar{m}_h \) is the \( L^2 \) projection of the exact coefficient \( m^* \) onto the finite element space. We can easily bound \( \|m_h^* - m^*\|_{L^2(\Omega)} \) above by \( \|\sigma(m, u^*)\|_{L^2(\Omega)} \) (plus a small error): the goal is then to bound the left-hand side of (12) to obtain an error estimate for \( \|m_h^* - m^*\|_{L^2(\Omega)} \). Because (12) involves the boundary term \( \|\sigma(m, u^*) n\|_{L^2(\partial\Omega)} \), it turns out that we need more control over

\[
\|\sigma(m, u_h(m_h)) n - g\|_{L^2(\partial\Omega)}
\]

than is provided by the fact that \( m_h \) and \( u_h(m_h) \) together satisfy the weak form of the BVP (of course, \( \sigma(m^*, u^*) n - g = 0 \) on \( \partial\Omega \)). We therefore define the OLS functional by

\[
J_h(m) = \|u_h(m) - z\|_{L^2(\Omega)}^2 + h^3 \|\sigma(m, u_h(m)) n - g\|_{L^2(\partial\Omega)}^2,
\]

where \( z \) is the observed data (a measurement of \( u^* \)). The OLS problem is to minimize \( J_h(m) \) subject to pointwise bounds on the coefficients \( \mu \) and \( \rho \): \( m \in K_h \).

In [10], we prove the following results under the assumption that the exact \( u^* \) and \( m^* \) are sufficiently smooth, namely, \( u^* \in W^{r+3}_\infty(\Omega)^2 \) and \( m^* = (\mu^*, \rho^*) \in W^{r+1}_\infty(\Omega)^2 \). We also assume that the boundary traction \( g \) is chosen so that \( u^* \) satisfies the nondegeneracy condition (11).

**Theorem 2.2** There exists a constant \( C \) such that, with \( z = u^* \),

\[
\inf_{m \in K_h} J_h(m) \leq Ch^{2r+4}.
\]

The constant \( C \) depends on \( c_0, c_1, \|m^*\|_{W^{r+1}_\infty(\Omega)}, \|u^*\|_{H^{r+2}(\Omega)} \), and \( \nu \), but is independent of \( h \).
The previous result, which holds for \( z = u^* \), is easily turned into an estimate for inexact data.

**Corollary 2.3** Let \( m_h^* \) be a minimizer of \( J_h \). Then

\[
\| u_h(m_h^*) - z \|_{L^2(\Omega)} \leq C \left( h^{r+2} + \| z - u^* \|_{L^2(\Omega)} \right)
\]

and

\[
\| \sigma(m_h^*, u_h(m_h^*)) - g \|_{L^2(\partial \Omega)} \leq C \left( h^{r+1/2} + \frac{\| z - u^* \|_{L^2(\Omega)}}{h^{3/2}} \right),
\]

where \( C \) depends on \( c_0, c_1, \| m^* \|_{W^{r+1}(\Omega)}, \| u^* \|_{H^{r+2}(\Omega)}, \) and \( \nu \), but is independent of \( h \).

Based on the preceding results, we can prove the following bound on \( \| m_h^* - m^* \|_{L^2(\Omega)} \).

**Theorem 2.4** There exists a constant \( C \) such that if \( m_h^* \) is a minimizer of \( J_h \), then

\[
\| m_h^* - m^* \|_{L^2(\Omega)} \leq C \left( h^r + \frac{\| z - u^* \|_{L^2(\Omega)}}{h^2} \right).
\]

The constant \( C \) depends on \( c_0, c_1, \nu, c, \| m^* \|_{W^{r+1}(\Omega)} \), and \( \| u^* \|_{W^{r+3}(\Omega)} \), but is independent of \( h \).

This result is entirely analogous to Falk’s and, as in the case of the scalar inverse problem, shows that the algorithm converges as \( h \to 0 \) if \( z \) represents the piecewise polynomial interpolant of degree \( r + 1 \) of \( u^* \). For less accurate data, the error bound blows up as \( h \to 0 \), reflecting the underlying instability of the inverse problem.

### 2.1. A modified OLS functional

Zou [14] and Knowles [12] independently proposed applying the OLS method with a coefficient-dependent energy norm. Their work was in the context of the scalar problem (1–2) and the modified OLS (MOLS) functional takes the form

\[
J_{	ext{MOLS}}(a) = \int_{\Omega} a \nabla (u(a) - z) \cdot \nabla (u(a) - z).
\]

Although \( J_{	ext{MOLS}} \) might appear more complicated than \( J_{	ext{OLS}} \), in fact it is simpler: \( J_{	ext{MOLS}} \) is convex, as shown by Zou.

In Gockenbach and Khan [8] (see also [9]), we showed how to extend the MOLS functional to any inverse problem defined by a linear elliptic BVP. The variational form of a linear elliptic BVP can be written as

\[
T(a, u, v) = \ell(v) \quad \text{for all } v \in V,
\]

where \( T \) is trilinear and satisfies the following boundedness and coercivity conditions defined by positive constants \( \alpha \) and \( \beta \):

\[
T(a, u, v) \leq \beta |a|_B \| u \|_V \| v \|_V \quad \text{for all } u, v \in V, a \in B,
\]

\[
T(a, u, u) \geq \alpha \| u \|_V^2 \quad \text{for all } u \in V, a \in A.
\]

In this abstract setting, the coefficient \( a \) is chosen from a convex subset \( A \) of a Banach space \( B \) (for the examples discussed in this paper, \( A \) would incorporate the pointwise bounds on the coefficient(s) that guarantee the conditions (19–20)). The abstract form of the MOLS functional is

\[
J_{	ext{MOLS}}(a) = T(a, u(a) - z, u(a) - z),
\]
where \( u = u(a) \) satisfies (18).

Working in the abstract framework outlined above, which encompasses both the scalar BVP and the system of linear elasticity, we showed in [8] that a regularized version of \( J_{MOLS} \) has a minimizer, even if subjected to BV-regularization.\(^2\) Moreover, we defined (abstractly) a sequence of finite-dimensional approximate problems whose minimizers converge to a minimizer of \( J_{MOLS} \). These finite-dimensional problems are defined in a manner compatible with finite element discretization. It should be noted that all these optimization problems are based on convex functionals, so there is no question of local, nonglobal minimizers.

We expect to be able to show that the convergence analysis described above for the \( L^2 \) OLS functional extends directly to the MOLS functional; however, this analysis has not been completed yet.

We remark that although the analysis of MOLS in [8] allows for nonsmooth or even discontinuous coefficients, this analysis extends only to the existence of minimizers of \( J_{MOLS} \) (in the infinite-dimensional setting) and the convergence of certain discretizations. The author is not aware of any error bounds that apply to the problem of estimating nonsmooth or discontinuous coefficients in the context of elliptic inverse problems.

3. The method of equation error

The equation error method takes a different approach altogether, choosing a value of \( a \) that, together with \( u = z \), satisfies the governing BVP as closely as possible. Expressed in terms of the strong form (1–2) of the BVP, this amounts to choosing \( a \) so that some combination of

\[
\| \nabla \cdot (a \nabla z) + f \| \text{ and } \left\| a \frac{\partial z}{\partial n} - g \right\|
\]

(where appropriate norms are chosen) is as small as possible. Combining the residuals in the PDE and the boundary condition is straightforward in the weak form of the BVP, which is

\[
u \in \hat{V}, \int_{\Omega} a \nabla u \cdot \nabla v = \int_{\Omega} f v + \int_{\partial \Omega} g v \text{ for all } v \in \hat{V}.
\]

Here \( \hat{V} \) is the space of mean-zero functions:

\[
\hat{V} = \left\{ v \in H^1(\Omega) : \int_{\Omega} v = 0 \right\}.
\]

Assuming \( f \) and \( g \) satisfy the compatibility condition

\[
\int_{\Omega} f + \int_{\partial \Omega} g = 0,
\]

\(^2\) If \( a \) is sufficiently regular, the total variation of \( a \) is

\[
\int_{\Omega} |\nabla a|,
\]

where \( |\nabla a| \) is the Euclidean norm of \( \nabla a \). Using the total variation as a regularization term is usually referred to BV-regularization, since the space of \( L^1 \) functions with finite total variation is called the space of functions of bounded variation. BV-regularization should be contrasted with the more common \( H^1 \) seminorm regularization, in which the regularization term is

\[
\int_{\Omega} |\nabla a|^2.
\]

\( H^1 \) seminorm regularization penalizes large gradients and is therefore suitable for problems in which the coefficient to be estimated is assumed to be smooth. On the other hand, BV-regularization is effective for estimated rapidly varying or even discontinuous coefficients.
the BVP (1–2) has infinitely many solutions, any two differing by a constant. The variational problem (21) defines a unique solution, namely, the one with mean zero.

We want the linear functionals defined by the left and right sides of (21), namely,

\[ \ell_a(v) = \int_{\Omega} a \nabla z \cdot \nabla v \]

and

\[ \ell(v) = \int_{\Omega} fv + \int_{\partial \Omega} gv, \tag{22} \]

to be as close as possible in the dual norm (notice that \(u\) has been replaced by the data \(z\)). This is actually implemented in the discrete spaces, so we define

\[ J_{EE}(a_h) = \frac{1}{2} \| \ell_a - \ell \|_{V_h}^2. \]

This functional is even simpler than \(J_{MOLS}: J_{EE}\) is convex and quadratic, so minimizing it requires only the solution of a symmetric positive (semi)definite system. The method of equation error is, like OLS, a general approach to inverse problems. The analysis presented below is due to Kärkkäinen [11].

Although regularization is essential for practical implementations of all four methods discussed in this paper, it plays a part in the analysis only of the method of equation error. In order to obtain an error estimate, Kärkkäinen minimizes the regularized functional

\[ J_{EE}(a_h) + \frac{\beta}{2} \| a_h \|_{H^1(\Omega)}. \tag{23} \]

Kärkkäinen does not assume any nondegeneracy condition comparable to (8). Consequently, the error \(\| (a_h^* - a^*) \nabla z \|\) (where \(a_h^*\) now represents a minimizer of the regularized equation error functional) is bounded instead of simply \(\| a_h^* - a^* \|\). Including \(\nabla z\) in the expression for the error means that the error estimate provides no information about the error in \(a_h^*\) in any region of \(\Omega\) in which \(\nabla z\) is zero. To express Kärkkäinen’s result, we must define the quotient space \((L^2(\Omega))^2/\text{rot},\) consisting of the equivalence classes of functions under the equivalence relation \(\text{rot}(v) = \text{rot}(w)\) (where, for \(u \in (L^2(\Omega))^2\), \(\text{rot}(v) = \partial u_2/\partial x - \partial u_1/\partial y\)). The quotient norm is defined by

\[ \| w \|_{(L^2(\Omega))^2/\text{rot}} = \inf_{v \in (L^2(\Omega))^2} \| v \|_{(L^2(\Omega))^2} \quad \text{rot}(v - w) = 0. \]

Then, if \(a_h^*\) minimizes the regularized equation error functional (23), we have

\[ \| (a_h^* - a^*) \nabla z \|_{(L^2(\Omega))^2/\text{rot}} \leq C \left( h^{r+1} + \| z - u^* \|_{H^1(\Omega)} + \sqrt{\beta} + \frac{\beta}{\sqrt{h}} \right). \tag{24} \]

In this quotient norm, this error estimate is actually better than those given previously. If \(z\) is the piecewise polynomial interpolant of \(u^*\) of degree \(r + 1\) and the regularization parameter is chosen appropriately (\(\beta = O(h^{2r+2})\)), then we obtain

\[ \| (a_h^* - a^*) \nabla z \|_{(L^2(\Omega))^2/\text{rot}} = O \left( h^{r+1} \right), \]

one order of \(h\) better than the previous results. (However, the results are not directly comparable because of the different norms.)

The method of equation error has been extended to problem of estimating the Lamé moduli (3–6); for instance, Gockenbach, Jadamba, and Khan [6, 7] prove the existence of minimizers in the infinite-dimensional setting and prove the convergence of minimizers to a sequence of finite-dimensional approximations. However, no error bounds have been derived for the resulting equation error estimate of the Lamé moduli.
4. A variational method

We finally describe a variational method due to Kohn and Lowe \[13\] that can be viewed as a modification of the method of equation error. It is based on separating the constitutive and balance laws that result in the PDE (1): $-\nabla \cdot (a \nabla u) = f$ is equivalent to $\sigma = a \nabla u$ and $-\nabla \cdot \sigma = f$.

A new variable $\sigma$ is therefore introduced, and (in discretized form) the following functional is minimized:

$$ J_{\text{var}}(\sigma_h, a_h) = \frac{1}{2} \|\sigma_h - a_h \nabla z\|_{L^2(\Omega)}^2 + \frac{h^2}{2} \|\nabla \cdot \sigma_h + f\|_{L^2(\Omega)}^2 + \frac{h}{2} \|\sigma_h \cdot n - g\|_{L^2(\partial \Omega)}^2. $$

The continuous form of this functional, which we use below, is

$$ J_{\text{var}}(\sigma, a) = \frac{1}{2} \|\sigma - a \nabla z\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\ell_\sigma - \ell\|_{V^*}^2, $$

where $\ell$ is defined by (22) and $\ell_\sigma$ is the functional define by

$$ \ell_\sigma(v) = \int_\Omega \sigma \cdot \nabla v. $$

Like $J_{\text{EE}}$, $J_{\text{var}}$ is convex quadratic and is therefore relatively simple to minimize; however, $J_{\text{var}}$ depends on three times as many unknowns as does $J_{\text{EE}}$ and so the method of equation error is less costly.

Kohn and Lowe \[13\] prove an error estimate for the minimizer $a_h^*$ of $J_{\text{var}}$. In place of (8), they assume the less restrictive condition

$$ \inf \max \{ |\nabla u|, \Delta u \} > 0, \quad (25) $$

and they prove their result for $r = 1$, meaning that $a_h$ is represented by piecewise linear functions and $u_h$ by piecewise quadratic. Under certain smoothness assumptions on $a^*$ and $u^*$, Kohn and Lowe prove

$$ \|a_h^* - a^*\|_{L^2(\Omega)} \leq C \left( h + \frac{\|z - u^*\|_{H^1(\Omega)}}{h} \right), \quad (26) $$

(To obtain (26), Kohn and Lowe’s result has been specialized to the case in which $f$ and $g$ are known exactly.)

Chen and Gockenbach \[3\] have extended the variational method of Kohn and Lowe to the problem of estimating Lamé moduli. The key is a version of Lemma 2.1 and result is the bound

$$ \|m_h^* - m^*\|_{L^2(\Omega)} \leq C \left( h + \frac{\|z - u^*\|_{H^1(\Omega)}}{h} \right). $$

The error bounds for OLS and the variational method can be compared in the case $r = 1$, that is, when the solution of the BVP is approximated by continuous piecewise quadratic functions and the unknown coefficient(s) by continuous linear functions. If $z$ is the piecewise quadratic interpolant of the exact data $u^*$, then

$$ \|z - u^*\|_{L^2(\Omega)} = O\left(h^3\right), \quad \|z - u^*\|_{H^1(\Omega)} = O\left(h^2\right). $$

Then an estimate of $a^*$ produced by either the OLS method or the variational method satisfies

$$ \|a_h - a^*\|_{L^2(\Omega)} = O(h). $$
In general, if z any piecewise quadratic estimate of the exact (smooth) solution $u^*$, then an inverse estimate yields

$$\|z - u^*\|_{H_1(\Omega)} \leq C \frac{\|z - u^*\|_{L^2(\Omega)}}{h},$$

and the bounds for the errors in both the OLS and variational methods reduce to

$$\|a_h^* - a^*\|_{L^2(\Omega)} \leq C \left( h + h^2 \frac{\|z - u^*\|_{L^2(\Omega)}}{h^2} \right).$$  (27)

This suggests that the two methods might perform similarly. On the other hand, obtaining these error bounds requires many estimates involving unknown constants and there is no reason to assume that the magnitude of the final constant $C$ appearing in (27) is the same for the two approaches. Therefore, the analysis alone does not allow us to conclude which method will produce more accurate estimates of the coefficient $a$ in (1–2). Similar considerations apply to the OLS and variational methods when used to estimate the Lamé moduli in the system of elasticity.

Given the comments of the last paragraphs and the fact that the error estimate for the method of equation error involves a different norm, one cannot predict which method will work better in practice. Indeed, this point has been addressed in the literature. For example, Kohn and Lowe state:

Although the variational method might appear more unstable than the [output] least-squares method, as it requires differentiation of the measurement [z], the estimates give no such indication. ([13], pages 129–130)

Kärkkäinen states:

We are able to prove the stability estimate including the rate of convergence for the equation error method only in a quotient space. However, if we compare our results to other works including such estimates . . . we notice that in this space we obtain an estimate one order better with respect to the observation error than before. ([11], page 1042)

In the next section, we compare the four methods by quite a different criterion: Which method estimates a constant coefficient most robustly? By this simple analysis, we see that the four methods must perform quite differently in practice, at least when the observation z contains random measurement errors.

5. Estimating a constant coefficient

We present a simple analysis of the inverse problem for the scalar equation (1–2) in the special case that $a$ is known to be constant. We continue to write $u = u(a)$ for the solution of (1–2), where now $a$ represents a positive constant. It is easy to see that $u(a) = a^{-1}u(1) = a^{-1}u_1$, where $u_1$ is the solution of (1–2) for $a = 1$.

Given the condition $u(a) = a^{-1}u_1$, there is no loss in generality in assuming that the true value of $a$ is $a^* = 1$, and hence that the exact data is $u_1$. The conclusions given below are essentially unchanged if the true coefficient is an arbitrary $a^* > 0$. Throughout this section, $z$ will denote the data, a measurement of $u_1$.

The OLS approach estimates $a$ by minimizing

$$J_1(a) = \frac{1}{2} \|u(a) - z\|_{L^2(\Omega)}^2.$$
Because of the simple form of \( u(a) \), we can easily show that
\[
a_{z,1} = \frac{\|u_1\|_{L^2(\Omega)}^2}{(u_1, z)_{L^2(\Omega)}}
\]
is the unique global minimizer of \( J_1 \) on \((0, \infty)\). In the case that the noise in \( z \) is random (and hence high-frequency), it is reasonable to assume that \( z = u_1 + \eta \), where
\[
(u_1, \eta)_{L^2(\Omega)} \ll (u_1, u_1)_{L^2(\Omega)}.
\]
It follows that
\[
a_{z,1} = \frac{(u_1, u_1)_{L^2(\Omega)}}{(u_1, u_1)_{L^2(\Omega)} + (u_1, \eta)_{L^2(\Omega)}} = \frac{(u_1, u_1)_{L^2(\Omega)}}{(u_1, u_1)_{L^2(\Omega)}} = 1,
\]
and this method gives a robust estimate of the true value of \( a \).

The MOLS approach estimates \( a \) by minimizing
\[
J_2(a) = \frac{1}{2} \int_{\Omega} a (\nabla u(a) - \nabla z) \cdot (\nabla u(a) - \nabla z).
\]
Again using the formula \( u(a) = a^{-1} u_1 \), we obtain the unique minimizer
\[
a_{z,2} = \frac{\|\nabla u_1\|_{L^2(\Omega)}}{\|\nabla z\|_{L^2(\Omega)}}.
\]
Since \( a_{z,2} \to 0 \) as \( \|\nabla z\|_{L^2(\Omega)} \to \infty \), it appears that MOLS approach will not give an accurate estimate when the data is noisy. The reader should notice that \( \|\nabla z\|_{L^2(\Omega)} \gg \|z\|_{L^2(\Omega)} \) is expected when the noise in \( z \) is high-frequency.

The method of equation error minimizes
\[
J_3(a) = \frac{1}{2} \|\ell_a - \ell\|^2_{\hat{V}^*}.
\]
The functional \( J_3 \) is convex quadratic, with the unique minimizer
\[
a_{z,3} = \frac{(\nabla z, \nabla u_1)_{L^2(\Omega)}}{\|\nabla z\|_{L^2(\Omega)}}.
\]
When \( z = u_1 + \eta \), where \( \eta \) represents high-frequency noise, it is reasonable to assume that \( (\nabla \eta, \nabla u_1)_{L^2(\Omega)} \ll (\nabla u_1, \nabla u_1)_{L^2(\Omega)} \), and hence that
\[
a_{z,3} = \frac{(\nabla u_1, \nabla u_1)_{L^2(\Omega)}}{\|\nabla z\|^2_{L^2(\Omega)}} = a_{z,2}^2.
\]
This shows that \( a_{z,3} \to 0 \) much faster than does \( a_{z,2} \) as the noise in \( z \) increases, and hence that the method of equation error is less robust than MOLS.

The variational method of Kohn-Lowe, in the continuous form suggested above, seeks to minimize
\[
J_4(\sigma, a) = \frac{1}{2} \|\sigma - a \nabla z\|^2_{L^2(\Omega)} + \frac{1}{2} \|\ell_{\sigma} - \ell\|^2_{\hat{V}^*},
\]
where the functionals \( \ell_{\sigma} \) and \( \ell \) are defined in Section 4. Minimizing \( J_4 \) is more involved than the first three cases above, because \( \sigma \) is a function even when \( a \) is constant. However, it can be shown (see Andreev [2]) that the unique minimizer of \( J_4 \) corresponds to
\[
a_{z,4} = a_{z,3} = \frac{(\nabla z, \nabla u_1)_{L^2(\Omega)}}{\|\nabla z\|^2_{L^2(\Omega)}},
\]
and hence that the variational method shares the same shortcomings as the method of equation error.
An example  To illustrate the above analysis, we take $\Omega$ to be the unit square and choose $f$ and $g$ so that the exact solution to (21) (with $a = 1$) is

$$u_1(x, y) = 5x + 7y - \cos(\pi(2x + y)) - 6.$$  

We solved the finite element equations representing (21), using piecewise linear finite elements on a uniform mesh consisting of 3200 triangular elements, to obtain $U_1$ (the vector of nodal values). We then added uniformly distributed random noise, varying from 0% to 50% in the $L^\infty$ norm, to $U_1$ to create noisy data sets. Figure 1 shows the resulting values of $a_{z,1}$, $a_{z,2}$, and $a_{z,3}$. For comparison, $a_{z,2}^2$ is also plotted.

![Figure 1.](image)

Figure 1. Estimates of $a$ by various methods and for various levels of (random) noise. The horizontal axis is the relative error in the data $z$, measured in the $L^\infty$ norm. The solid curve gives the estimates produced by the OLS method ($a_{z,1}$), the dashed curved by MOLS ($a_{z,2}$), and the dot-dashed curve by equation error ($a_{z,3}$). Also shown, as a dashed curve, is $a_{z,2}^2$, but this is indistinguishable from $a_{z,3}$, as predicted by the analysis. The true value of $a$ is 1.

As predicted by the analysis, the OLS result, $a_{z,1}$, robustly estimates the true value of $a$, even when the noise level is large. The MOLS value, $a_{z,2}$, is much less accurate when the noise level is large, while the equation error results, $a_{z,3}$, is still worse and satisfies $a_{z,3} = a_{z,2}^2$ quite accurately.

These results cannot be predicted by the error estimates presented in the previous sections, which suggest that all four methods might be expected to behave similarly. On the other hand, we regard this analysis as merely suggestive; to conclude that the OLS method is preferable in practice would require careful numerical testing on problems with nonconstant coefficients and realistic noise.

References

[1] G. Alessandrini. On the identification of the leading coefficient of an elliptic equation. *Bollettino U.M.I.*, Analisi Funzionale e Applicazioni, Serie VI, IV-C:87–111, 1985.
[2] Anatoli Andreev. *Comparison of Numerical Methods for Elliptic Inverse Problems*. PhD thesis, Michigan Technological University, Houghton, MI, U.S.A., 2006.
[3] J. Chen and M. S. Gockenbach. A variational method for recovering planar Lamé moduli. *Mathematics and Mechanics of Solids*, 7:191–202, 2002.
[4] S. J. Cox and M. S. Gockenbach. Recovering planar Lamé moduli from a single traction experiment. *Mathematics and Mechanics of Solids*, 2(3):297–306, 1997.
[5] R. Falk. Error estimates for the numerical identification of a variable coefficient. *Mathematics of Computation*, 40:537–546, 1983.
[6] M. S. Gockenbach, B. Jadamba, and A. A. Khan. Numerical estimation of discontinuous coefficients by the method of equation error. *International Journal of Mathematics and Computer Science, 1*(3):343–359, 2006.

[7] M. S. Gockenbach, B. Jadamba, and A. A. Khan. Equation error approach for elliptic inverse problems with an application to the identification of lamé parameters. *Inverse Problems in Science and Engineering*, to appear.

[8] M. S. Gockenbach and A. A. Khan. An abstract framework for elliptic inverse problems: Part 1. an output least-squares approach. *Mathematics and Mechanics of Solids, 12*:259–276, 2007.

[9] M. S. Gockenbach and A. A. Khan. An abstract framework for elliptic inverse problems: Part 2: An augmented Lagrangian approach. *Mathematics and Mechanics of Solids*, to appear.

[10] Mark S. Gockenbach. The output least-squares approach to estimating lamé moduli. *Inverse Problems, 23*:2437–2455, 2007.

[11] T. Kärkkäinen. An equation error method to recover diffusion from distributed observations. *Inverse Problems, 13*:1033–1051, 1997.

[12] Ian Knowles. Parameter identification for elliptic problems. *J. Comp. Appl. Math.*, 131:175–194, 2001.

[13] R. V. Kohn and B. Lowe. A variational method for parameter identification. *Math. Modeling & Num. Anal.*, 22(1):119–158, 1988.

[14] Jun Zou. Numerical methods for elliptic inverse problems. *International Journal of Computer Mathematics, 20*:211–232, 1998.