Chern-Simons Superconductivity at finite magnetic field

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Abstract

We study Chern-Simons (CS) superconductivity in the presence of uniform external magnetic field of arbitrary strength for a system of fermions in two spatial dimensions, which are minimally coupled both to the CS and Maxwell gauge fields. We have carried out the computation within the mean field ansatz. Analysing only the mean field (i.e., ignoring the fluctuations of the gauge fields), we find that chemical potential, susceptibility and magnetization show discontinuities for integer number of filled Landau levels. Taking into account the fluctuations of the gauge fields, we find that the masses of the excitations increase with the magnetic field, and that the presence of nonlinear magnetic susceptibilities show the absence of any critical or pseudo critical magnetic field. Finally, an interesting result is that, unlike ordinary superconductors, the system is magnetically asymmetric.

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I. INTRODUCTION

The relevance of Chern-Simons (CS) gauge theory of planar phenomena like quantum Hall effect is by now well established. It has also been recognized that the CS interaction can lead to a novel kind of superconductivity – characterized by Parity and Time reversal ($P$, $T$) violation, no Cooper pair formation, two penetration depths in Meissner effect, and finally, an antisymmetric (super) conductivity tensor. Proposed originally as a model for high-$T_c$ superconductors, it attracts continued theoretical interest, partly due to the novelty of the mechanism and partly due to the distinct possibility of the existence of such real systems.

This paper is devoted to a study of CS superconductivity (CSS) in uniform finite (external) magnetic field. Recall that the conventional superconducting phase gets destroyed beyond both a critical temperature and a critical field. While there have been extensive studies of CSS at finite temperatures ($T$), there is not much work at finite magnetic field ($B$). We intend to fill this gap here.

CSS was first established at $T = 0$ in the pioneering works of Laughlin and Chen, Wilczek, Witten and Halperin, who considered spinless fermions. This was followed by an extension to spin 1/2 by Hosotani and Chakravarty and Chakraborty, Ramaswamy and Ravishankar. The latter found a unique possibility for the existence of CSS with a ferromagnetic ground state. There exists also an extensive literature on the $T \neq 0$ properties of CSS. See Ref. 6 for details and for references to earlier works. To put it concisely, it was found that CSS gives a normal insulating nonmagnetic state beyond a certain temperature. The transition to the normal state is over a rather narrow range of temperatures, but does not appear sharp enough to qualify unambiguously to be a phase transition. Thus it is not clear whether we have at hand a critical or a pseudo critical temperature.

Based on a mean field (MF) analysis, Hetric, Hosotani and Lee conclude that there is also a pseudo critical magnetic field beyond which CSS would not survive. We propose to study this in detail by computing explicitly the one loop effective action for the system in a
background magnetic field. Further there is the interesting question of the system’s response when the sign of $B$ is flipped. We anticipate the system to be magnetically asymmetric around $B = 0$. The interesting region around $B = -b$, ($b$ being the mean CS magnetic field), where the particles exhibit net zero mean field will also be examined here.

Finally, we remark here that the study of CSS here and elsewhere relies on the MF ansatz plus perturbative one loop correction. The MF ansatz requires justification, which has been attempted in Ref. 6. Alternatively, one might appeal to the success of the MF picture in fractional quantum Hall effect (FQHE)\textsuperscript{9} which is in agreement with experiment.\textsuperscript{9} We return to a discussion of its validity here as well, although very briefly and only contextually.

The paper is organized as follows. Sec. IIA displays briefly the formalism of the MF ansatz. Mean field results are presented in Sec. IIB. In Sec. IIC, the form factors are evaluated. The effective Lagrangian for the magnetic field is obtained by the integration over the fluctuating part of the gauge fields in sec. IIA. The behaviour of low-lying excitations are then discussed in Sec. IIB. In sec. IV, we compute nonlinear magnetic susceptibilities and conclude the paper in Sec. V.

II. MEAN FIELD THEORY

A. Formalism

Consider a system of non-relativistic spinless fermions in (2+1) dimensions whose dynamics is governed by the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\nu}{2} e^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \psi^\dagger i D_0 \psi - \frac{1}{2m} |D_k \psi|^2 + \psi^\dagger \mu \psi - e A_0 \rho , \quad (2.1)$$

where $A(a)$ denotes the Maxwell(CS) gauge field and the covariant derivative $D_\mu = \partial_\mu - ie (A_\mu + a_\mu)$. The condition of a fixed density of the fermions is implemented by introducing the chemical potential $\mu$ (note that we use $\mu$ as a space time index as well; this should cause no confusion). The last term represents the background neutralizing ‘classical’ charge density.
We employ the path integral formalism to evaluate the partition function
\[ Z = \int [dA][da][d\psi][d\psi^\dagger] e^{i \int d^3 x L}. \] (2.2)

We proceed with the evaluation of \( Z \) by performing the integration over fermionic field first. This gives the effective action for the gauge fields incorporating the accumulated effect of fermions on the system. The standard method of evaluation of \( Z \) is the MF ansatz in which one smears out the CS magnetic field to obtain a uniform background (in which the particles move). At \( T = 0 \), this approach can be justified for large \( N \) (\( N \) being related to the CS coefficient \( \nu = Ne^2/2\pi \)) and for the parameters used in our analysis. In this case the external magnetic field will have to also be included along with the mean CS magnetic field. The effect of the external magnetic field would depend on its direction relative to the CS field. We expand the gauge fields around the configuration
\[ A_0 = A_2 = 0 ; a_0 = a_2 = 0 ; A_1 = A_1^{\text{ex}} = -Bx_2 ; a_1 = -bx_2. \] (2.3)

Keeping terms up to second order in fluctuations, we find
\[ Z = Z_{\text{MF}} \int [dA][da] e^{i S}, \] (2.4)

where the mean field action
\[ S_{\text{MF}} = -i \ln Z_{\text{MF}} = -i \text{Tr} \ln (i\partial_0 - H + \mu) - \frac{1}{2} \int d^3 x B^2, \] (2.5)

with \( H = -D_k^2/2m \). Also, the one-loop effective action is given by
\[ S = \int d^3 x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\nu}{2} \epsilon^{\mu\nu\lambda} a_{\mu} \partial_{\nu} a_{\lambda} \right) \\
- \frac{1}{2} \int d^3 x \int d^3 y (A_{\mu}(x) + a_{\mu}(x)) \Pi^{\mu\nu}(x,y) (A_{\nu}(y) + a_{\nu}(y)) , \] (2.6)

where we have represented the fluctuating fields by \( a \) and \( A \) again. The current correlators are given by
\[ \Pi^{\mu\nu}(x,y) = -\frac{\delta \langle j^\mu(x) \rangle}{\delta A_{\nu}(y)} \bigg|_{MF} ; A_{\mu} = A_{\mu} + a_{\mu} , \] (2.7)
The fermionic currents are given by

\begin{align}
  j_0(x) &= e\psi^\dagger\psi, \\
  j_k(x) &= -ie\frac{c}{2m} \left( \psi^\dagger D_k\psi - D^*_k\psi^\dagger\psi \right). \quad (2.8a, b)
\end{align}

The single particle Greens function \( G(x, y) = -i\langle T \psi(x)\psi^\dagger(y) \rangle \) can be obtained by solving the differential equation

\[(i\partial_0 - H + \mu) G(x, y) = \delta^{(3)}(x - y), \quad (2.9)\]

subject to appropriate boundary conditions. The boundary conditions which we use for evaluating \( G(x, y) \) will be discussed in the next subsection. \( T \) represents the time ordering of two fermionic fields. Thus using a suitable limiting procedure one can express fermionic current and current correlator respectively in terms of \( G(x, y) \) as follows:

\begin{align}
  \langle j_0(x) \rangle &= iG(x, x') \bigg|_{x' = X, t' = t + 0^+}, \\
  \langle j_k(x) \rangle &= \frac{e}{2m} (D_k - D^*_k) G(x, x') \bigg|_{x' = X, t' = t + 0^+}, \quad (2.10a, b)
\end{align}

\begin{align}
  \Pi_{00}(x, y) &= ie^2 G(x, y) G(y, x), \\
  \Pi_{0k}(x, y) &= \frac{e^2}{2m} \left[ G(x, y) D^*_k G(y, x) - (D^*_k G(x, y))^2 \right], \\
  \Pi_{kl}(x, y) &= -ie^2 \frac{e^2}{4m^2} \left[ D^*_k G(x, y) D^*_l G(y, x) - (D^*_k D^*_l G(x, y))^2 \right] \\
  &\quad + D^*_k G(x, y) D^*_l G(y, x) - G(x, y) D^*_k D^*_l G(y, x) \right] \\
  &\quad - ie^2 \frac{e^2}{2m} \delta_{kl} (\delta(x - y) + \delta(x' - y)) G(x, x') \bigg|_{x' = X, t' = t + 0^+}. \quad (2.11a, b, c)
\end{align}

In the MF approximation the current correlators will be obtained in terms of the Greens functions satisfying (2.9) with the MF configuration. Thus, we pause to discuss the MF ground state first before discussing the fluctuations of the gauge fields.
B. Mean Field Results

The mean field configuration in this case is rather involved since the external magnetic field changes the degeneracy as well as the cyclotron frequency. Since the levels are otherwise completely filled, the highest Landau level (LL) is now only partially filled which makes the ground state degenerate.

To handle this, we shall introduce a fictitious spin-like internal degree of freedom, represented by the operator $\hat{U}$ in the Hamiltonian, to the particle which couples to the magnetic field and splits the degeneracy. (Alternatively, one could also introduce a background harmonic oscillator potential to split the degeneracy. However, we find the former choice more convenient). Indeed, if the degeneracy per unit area is $\rho_l = 1/2\pi l^2$ ($l = |e|b + B|^{-1/2}$ being the magnetic length of the system), then the ‘pseudospin’ operator $\hat{U}$ belongs to that representation which has exactly as many eigenvalues as $\rho_l A$, where $A$ is the area of the system.

Thus, the Hamiltonian

$$H \rightarrow H' = H + \lambda \omega_c \hat{U},$$

where $\omega_c = 2\pi \rho_l / m$ is the cyclotron frequency and $\lambda$ is the dimensionless strength. As mentioned, $\hat{U}$ has eigenvalues given by

$$- \frac{(\rho_l A - 1)}{2} \leq u_i \leq \frac{(\rho_l A - 1)}{2}; \quad u_{i+1} = u_i + 1.$$  \hspace{1cm} (2.13)

The modified spectrum for $H$ is shown schematically in Fig. 1. Since $\lambda$ is to be a small parameter (which will be switched off at the end of our calculation), it is necessary that

$$\lambda |1 + x| \ll \frac{N}{\rho A}; \quad x = \frac{B}{b}.$$  \hspace{1cm} (2.14)

The spectrum of $H'$ is given by

$$\epsilon_{ni} = \left(n + \frac{1}{2} + \lambda u_i\right) \omega_c; \quad n = 0, 1, 2, \cdots, \quad i = 1, 2, \cdots \rho_l A.$$  \hspace{1cm} (2.15)

The MF Lagrangian (2.5) becomes
\[ \mathcal{L}_{MF} = \frac{1}{A} \sum_{n=0}^{\infty} \sum_i \int \frac{dk_0}{2\pi i} \left[ \ln (k_0 - \epsilon_{ni} + \mu) \right] - \frac{B^2}{2}. \] (2.16)

In evaluating the density \( \rho = \partial \mathcal{L}_{MF}/\partial \mu \), we encounter the integral of the form
\[ \lim_{\delta \to 0} \int \frac{dk_0}{2\pi i} e^{ik_0\delta} = \theta(x), \] (2.17)
where an additional convergence term \( \exp (ik_0\delta) \) has been inserted. The integral is thus equal to the heaviside function \( \theta(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases} \). Therefore, expression for the density of fermions follows:
\[ \rho \equiv \frac{\partial \mathcal{L}_{MF}}{\partial \mu} = \frac{1}{A} \sum_{n=0}^{\infty} \sum_i \theta(\mu - \epsilon_{ni}). \] (2.18)

Using the same technique, \( k_0 \) integral in (2.16) can be determined to obtain
\[ \mathcal{L}_{MF} = \frac{1}{A} \sum_{n=0}^{\infty} \sum_i (\mu - \epsilon_{ni}) \theta(\mu - \epsilon_{ni}) - \frac{B^2}{2}. \] (2.19)

Note that the chemical potential (which at \( T = 0 \) equals the Fermi energy) can be now determined. In the limit \( \lambda \to 0 \),
\[ \mu = \begin{cases} ([K + 1] - \frac{1}{2}) \omega_c \\ K \omega_c \end{cases}, \] (2.20)
where \( K = N/|1 + x| \). The upper case corresponds to fractional filling factor \( K \) and the lower case to the integer values of \( K \). \([x]\) denotes the largest integer less than \( x \). Note that \( \mu \) is discontinuous at those values of \( B \), where it corresponds to integral number of fully filled LL. In between the two integer filling fractions \( \mu \) varies linearly with \( B \) as \( \omega_c \) varies in the same way. In fact, \( \omega_c(x) = \omega_c(0)|1 + x| \). Fig. 2 shows the variation of the chemical potential with the application of external magnetic field parallel to the mean CS magnetic field.

The MF Lagrangian for \( K \) fractionally filled levels becomes
\[ \mathcal{L}_{MF} = \frac{1}{A} \left( \sum_{n=0}^{[K-1]} \sum_i (\mu - \epsilon_{ni}) + \sum_{i \leq i_0} (\mu - \epsilon_{[K]\i}) \right) - \frac{B^2}{2}. \] (2.21)
Here \( i_0 \) denotes the quantum number of the highest occupied level corresponding to the eigenvalue \( u_{i_0} = u_0 \) of \( \hat{U} \). The summation over \( n \) and \( i \) can be easily carried out. In the limit \( \lambda \to 0 \), (2.21) gets a simple form
\[ \mathcal{L}_{MF} = \begin{cases} \frac{e^2}{4\pi m} \left| b + B \right|^2 ([K]^2 + [K]) - \frac{B^2}{2} \\ \frac{e^2}{4\pi m} \left| b + B \right|^2 K^2 - \frac{B^2}{2} \end{cases} . \] (2.22)

The last term in Eq. (2.22) is due to the kinetic term of the Maxwell field and does not contribute in the calculation of electro-magnetic response. The MF magnetic susceptibility can be readily evaluated (by the omission of Maxwell kinetic term) as

\[ \chi_{MF} \equiv \frac{\partial^2 \mathcal{L}_{MF}}{\partial B^2} = \begin{cases} \frac{e^2}{2\pi m} ([K]^2 + [K]) \\ \frac{e^2}{2\pi m} K^2 \end{cases} . \] (2.23)

This is also discontinuous when \([K]\) passes from one value to the other. This is the well known de Haas-van Alphen effect for diamagnetism. Recall that susceptibility diverges quadratically for \(B \to 0\) in the conventional de Haas-van Alphen effect where there is no internal field. Similarly, here, it diverges for \(B \to -b\). The variation of \(\chi_{MF}\) with the magnetic field is shown in Fig. 3. We note that a similar behaviour was also obtained by Hetrich et al. in their calculation of MF magnetization, which we obtain as

\[ \mathcal{M}_{MF} = -\frac{e\rho}{2m} \left( 1 + 2[K] - \frac{2[K]}{N} ([K] + 1) \left| 1 + \frac{B}{b} \right| \right) \text{sign}(b + B) . \] (2.24)

However, they do not consider the fluctuations of the gauge fields which should play an important role in the response of the system to the applied magnetic field. We shall pursue this issue in the next section.

C. Current Correlation Functions

The evaluation of the form factors requires the Green function (2.9). We first define the frequency transformed Greens function \(G_\omega(\vec{X}, \vec{X}')\) to be

\[ G(x, x') = \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} G_\omega(\vec{X}, \vec{X}') , \] (2.25)

which clearly satisfies the differential equation

\[ [\omega - H' + \mu] G_\omega(\vec{X}, \vec{X}') = \delta^{(2)}(\vec{X} - \vec{X}') . \] (2.26)
This can be solved, as usual, in terms of the complete set of eigen functions \( \psi_{ni}(\vec{X}) \) of \( H' \) giving

\[
G(x, x') = \int_C \frac{d\omega}{2\pi} \sum_{ni} \frac{\psi_{ni}(\vec{X})\psi_{ni}^*(\vec{X}')}{\omega - \epsilon_{ni} + \mu} e^{-i\omega(t-t')} .
\] (2.27)

The contour \( C \) for the frequency integration has to be chosen so that \( G(x, x') \) satisfies the boundary conditions

\[
G(x, x') \sim \begin{cases} 
\sum_{n=0}^{[K-1]} \sum_{i} e^{-i(\epsilon_{ni}-\mu)(t-t')} + \sum_{i \leq i_0} e^{-i(\epsilon_{[K]}-\mu)(t-t')}, & t < t' , \\
\sum_{n>[K]} \sum_{i} e^{-i(\epsilon_{ni}-\mu)(t-t')} + \sum_{i > i_0} e^{-i(\epsilon_{[K]}-\mu)(t-t')}, & t > t' .
\end{cases}
\] (2.28)

More explicitly, the contour \( C \) must pass below the poles at \( \omega = \epsilon_{ni} - \mu \) for \( n \leq [K-1] \), for all \( i \) and \( n = [K] \), \( i \leq i_0 \) and above otherwise.

We now evaluate current correlation functions in the momentum space following the elegant procedure given by Randjbar-Daemi et al.\(^{11}\) Gauge and rotational invariance requires that the current correlator has the form

\[
\Pi^{\mu\nu}(\omega, \vec{q}^2) = \Pi_0(\omega, \vec{q}^2) (q^2g^{\mu\nu} - q^\mu q^\nu) + (\Pi_2 - \Pi_0)(\omega, \vec{q}^2) \delta^{\mu\nu} \delta^{ij} (q^2 \delta^{ij} - q^i q^j) + i\Pi_1(\omega, \vec{q}^2) \epsilon^{\mu\nu\lambda\chi} q_\chi ,
\] (2.29)

where the last term is the parity and time reversal violating contribution. Here \( q^2 = \omega^2 - \vec{q}^2 \).

For the purpose of evaluating low energy effective Lagrangian, it is sufficient to compute the form factors \( \Pi_0, \Pi_1 \) and \( \Pi_2 \) in the limit \( \omega \rightarrow 0, \vec{q}^2 \rightarrow 0 \). The limits commute in the evaluation of form factors. For simplicity, we take the limit \( \vec{q}^2 \rightarrow 0 \) first. In the limit \( \vec{q}^2 \rightarrow 0 \), the form factors are given by

\[
\Pi_0(\omega, 0) = \frac{e^2}{2\pi \rho_0 A} \int_0^{\omega'} \frac{d\omega'}{2\pi i} \sum_{m} \sum_{j} \frac{(n\delta_{n,m+1} + (n+1)\delta_{n,m-1}) \delta_{ij}}{(\omega' - \epsilon_{mj} + \mu)(\omega' - \epsilon_{ni} + \mu - \omega)} ,
\] (2.30a)

\[
\Pi_1(\omega, 0) = \Pi_0(\omega, 0) ,
\] (2.30b)

\[
\Pi_2(\omega, 0) = \frac{e^2\omega_c}{2\pi m_0 \rho_0 A} \int_0^{\omega'} \frac{d\omega'}{2\pi i} \sum_{m} \sum_{j} \frac{1}{(\omega' - \epsilon_{mj} + \mu)(\omega' - \epsilon_{ni} + \mu - \omega)}
\times \left[ n(n-1)\delta_{n,m+2} + 3n^2\delta_{n,m+1} + (2n+1)^2\delta_{nm}
+ 2(n+1)^2\delta_{n,m-1} + (n+1)(n+2)\delta_{n,m-2} \right] \delta_{ij}.
\] (2.30c)
The integral which we encounter in the evaluation of (2.30) is

\[
\int \frac{d\omega'}{2\pi i} \frac{1}{(\omega' - \epsilon_{mj} + \mu)(\omega' - \epsilon_{ni} + \mu - \omega)}
\]

\[= (\epsilon_{ni} - \epsilon_{mj} + \omega)^{-1}, \quad \text{for } \begin{cases} m \geq [K + 1], \text{ for all } j; & m = [K], \ j > i_0; \\ n < [K], \text{ for all } i; & n = [K], \ i \leq i_0 \end{cases} \] (2.31)

\[= -(\epsilon_{ni} - \epsilon_{mj} + \omega)^{-1}; \quad \text{for } \begin{cases} n \geq [K + 1], \text{ for all } i; & n = [K], \ i > i_0; \\ m < [K], \text{ for all } j; & m = [K], \ j \leq i_0 \end{cases} \]

\[= 0 \quad \text{otherwise.} \]

Thus, the form factors (in the limit \(\omega \to 0\), \(\vec{q}^2 \to 0\)) are obtained as

\[\Pi_0 = \frac{e^2K}{2\pi\omega_c}, \ \Pi_1 = \frac{e^2K}{2\pi} \cdot \Pi_2 = \frac{e^2K^2}{2\pi m}. \] (2.32)

Note that they are dependent on the applied magnetic field. In terms of their values at \(B = 0\), they are expressed as follows:

\[\Pi_0(x) = \frac{e^2mN^2}{4\pi^2\rho|1+x|^2} = \frac{\Pi_0(0)}{|1+x|^2}, \] (2.33a)

\[\Pi_1(x) = \frac{e^2N}{2\pi|1+x|} = \frac{\Pi_1(0)}{|1+x|}, \] (2.33b)

\[\Pi_2(x) = \frac{e^2N^2}{2\pi m|1+x|^2} = \frac{\Pi_2(0)}{|1+x|^2}. \] (2.33c)

The behaviour of the form factors for negative \(x\), specially for \(x = -1\), will be discussed later in section (IV).

### III. GAUGE FIELD FLUCTUATIONS

#### A. Effective Lagrangian

Now we choose the Coulomb gauge, \(\partial_iA_i = 0; \ \partial_i a_i = 0\) in order to evaluate the fluctuating part of the partition function (2.4) by an integration over the gauge fields. Basically one computes the contribution to the partition function from the effective gauge field modes (collective excitations). We obtain
\[ \ln Z_{\text{eff}} = \frac{i}{2} \text{Tr} \ln \left[ (\omega^2 - \omega_+^2) (\omega^2 - \omega_-^2) \right] , \]  

(3.1)

The collective modes have the dispersion relation \( \omega = \omega_\pm(q^2) \), with

\[ \omega_\pm^2 = \frac{1}{2C_1} \left( C_2 \pm \sqrt{C_2^2 - 4C_1C_2} \right) , \]  

(3.2)

and

\[ \begin{align*}
C_1 &= \Pi_0^2 , \\
C_2 &= \Pi_0(\Pi_0 + \Pi_2)q^2 + \nu^2 \left( \Pi_0^2 + 2\Pi_0 \right) + (\nu - \Pi_1)^2 , \\
C_3 &= \Pi_0\Pi_2q^4 + \left[ \nu^2 (\Pi_0 + \Pi_2 + \Pi_0\Pi_2) + (\nu - \Pi_1)^2 \right] q^2 + \nu^2\Pi_1^2 .
\end{align*} \]  

(3.3a, 3.3b, 3.3c)

Therefore, one finds the effective Lagrangian

\[ \mathcal{L}_{\text{eff}} = \frac{i}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d^2q}{(2\pi)^2} \ln \left[ (\omega^2 - \omega_+^2)(\omega^2 - \omega_-^2) \right] . \]  

(3.4)

The divergent frequency integral can be regularized as in Eq. (2.18) to obtain

\[ \mathcal{L}_{\text{eff}} = -\frac{1}{2} \int \frac{d^2q}{(2\pi)^2} (\omega_+ + \omega_-) . \]  

(3.5)

This is the effective Lagrangian for the system coming from the fluctuations of the gauge fields, where the dependence on the external magnetic field arises through the form factors in the dispersion relations. Below we discuss the behaviour of \( \omega_\pm \) as functions of the magnetic field.

### B. Low-lying Excitations

We first consider, for simplicity, the neutral system (i.e., excluding the internal Maxwell gauge field). The dispersion relation that we obtain, for the neutral system, from the equation of motion of the CS gauge field is given by

\[ \Pi_0 \left( \Pi_0\omega^2 - \Pi_2q^2 \right) - (\nu - \Pi_1)^2 = 0 . \]  

(3.6)
Recall that in the absence of an external magnetic field, the tree level CS term exactly canceled with the dynamically generated CS term (i.e., $\nu = \Pi_1$). Thus, the system possessed a super-fluid mode with the low-lying massless phononic excitation,

$$\omega^2 = \frac{\Pi_2(0)}{\Pi_0(0)} q^2. \quad (3.7)$$

(This in fact is the collective pseudo-Goldstone mode which is absorbed by the Maxwell gauge field upon coupling the particles to the EM field thereby producing a gap and leading to the Meissner effect).

When we introduce external magnetic field, this mode acquires a mass ($M$),

$$M^2 = \omega^2(0) x^2 (1 + x)^2, \quad (3.8)$$

as the cancellation of the CS term no longer holds and hence the super-fluidity diminishes.

The dispersions are shown in Fig. 4 for three different values of $B$. We have chosen $N = 10, e^2 = 10^5 \text{cm}^{-1}$ and the value of two dimensionless parameters $e^2/m$ and $\rho/me^2$ to be $10^{-5}$ and $10^{-1}$ respectively. Observe that the mass gap increases with the value of $B$ although the dispersions are parallel. Thus, the velocity of the mode decreases with the increase of $B$.

We now come back for a discussion of excitations of the charged system. The low-lying excitations of the system are two massive photonic modes. These two modes are nothing but $\omega_{\pm}$ given in equations (3.2 and 3.3). Fig. 5 and Fig. 6 show how the frequencies disperse with momentum for three chosen values of $B$ for the modes. The masses $M^2_{\pm}$ increase with $B$. Note however, that the damping lengths $\lambda^2_{\pm} = M^2_{\pm}$ have to be distinguished from the penetration depth which in fact decreases with increasing $B$, signifying the absence of the Meissner effect. This is seen from the calculation of the magnetic susceptibility below.

**IV. NONLINEAR RESPONSE**

Here we study the nonlinear response of the system to an external field by computing the higher order susceptibilities defined by
\( \chi^{(r)} \equiv -\frac{1}{(r+1)!} \frac{\partial^{r+2} F}{\partial B^{r+2}} \bigg|_{B=0} ; \quad r = 0, 1, 2, \cdots \), \hspace{1cm} (4.1)

where \( F \) is the free energy density of the system which can be computed from the Lagrangian (3.5) as \( F \equiv -L_{\text{eff}} \) and \( \chi^{(0)} \) is recognized to be the linear response susceptibility. Note that the complete response of the system is obtained by adding the corresponding MF values which may be inferred from Eq. (2.23).

We expand the sum \( \omega_{+} + \omega_{-} \) up to order \( q^2 \), since it is a low energy-momentum theory, and obtain

\[ \omega_{+} + \omega_{-} = F_1 + F_2 q^2 \]  

(4.2)

with

\[
F_1 = \nu \left[ 1 + 2 \frac{\Pi_2}{\Pi_0} + \frac{1}{\Pi_0} \left( 1 - \frac{\Pi_1}{\nu} \right)^2 + \frac{2\Pi_1}{\Pi_0\nu} \right]^{\frac{1}{2}},
\]

(4.3a)

\[
F_2 = \frac{1}{2F_1} \left[ \left( 1 + \frac{\Pi_2}{\Pi_0} \right) + \frac{\nu}{\Pi_1} \left( \Pi_0 + \Pi_2 + \Pi_0\Pi_2 + \left( 1 - \frac{\Pi_1}{\nu} \right)^2 \right) \right].
\]

(4.3b)

We introduce a momentum cut-off \( \Lambda \) to obtain

\[
F = \frac{1}{8\pi} \left[ F_1 \Lambda^2 + \frac{1}{2} F_2 \Lambda^4 \right].
\]

(4.4)

The susceptibilities \( \chi^{(r)} \), which are apparently a function of \( \Lambda \), are calculated to be

\[
\chi^{(r)}(B, \Lambda) = -\frac{\Lambda^2}{8\pi(r+1)!} \left[ \frac{\partial^{r+2} F_1}{\partial B^{r+2}} + \frac{\Lambda^2 \partial^{r+2} F_2}{2 \partial B^{r+2}} \right]_{B=0}.
\]

(4.5)

To determine \( \Lambda \), we demand that the linear magnetic susceptibility

\[
\chi^{(0)} \equiv -\frac{\partial^2 F}{\partial B^2} \bigg|_{B=0} = -1,
\]

(4.6)

which we know already and independently from linear response theory\(^{12} \). This fixes \( \Lambda^2 \) to be

\[
\Lambda^2 \approx \frac{8\pi^2 \rho}{N} \left[ \left( \frac{2\rho}{e^2 m N} \right)^{\frac{3}{2}} - \frac{\rho}{m^2 N} \right].
\]

(4.7)

This value of \( \Lambda \) is appropriate as a cut-off momentum for high magnetic field also since \( \Lambda^2 l^2 \approx 1.8/|1 + x| \), using our chosen values of parameters. Therefore, unless \( x \) is very high,
Λ is quite reasonable as a cut-off momentum. The cut-off independent non-linear magnetic susceptibilities can now be readily evaluated from (4.5) by the substitution of the value of Λ^2 (4.7). They can be analytically determined, to a high degree of approximation to be

\[ \chi^{(r)} = (-1)^{r+1} \frac{r + 2}{2b^r} . \] (4.8)

Notice that higher order susceptibilities are nonvanishing and do not even have numerically small values, which clearly shows that there is neither a critical nor a pseudo critical field which would characterize the phase of the system. However, as \( B \to \infty \), the free energy has only a linear dependence on \( B \), which means that all the susceptibilities vanish. The system returns to its normal state asymptotically. We also notice that since both even and odd order susceptibilities survive, the system is magnetically asymmetric around \( B = 0 \), as a consequence of the \( \mathcal{P}, \mathcal{T} \) violation inherent in the theory.

It is instructive to study the behaviour of free energy as a function of applied magnetic field. It has the (approximate) form

\[ F(B) \approx \frac{b^2}{2(1 + B/b)} - \frac{b^2}{2}(1 - B/b) . \] (4.9)

Clearly, the leading order contribution comes from \( B^2 \) since, as we know from the linear response analysis, the system does not possess any spontaneous magnetization. The curvature of \( F(B) \) at \( B = 0 \) gives the linear susceptibility. The behaviour of \( F(B) \) is shown in Fig. 7. It may be seen (see also Eq. (4.9)) that the free energy diverges at \( B = -b \). This requires some discussion, as it corresponds to a zero mean field.

As \( B \to -b \), the inter Landau level spacings squeeze and hence the energy spectrum approaches the continuum; more and more number of LL will be filled up even as the particle density remains the same. Therefore, the values of the form factors increase and they diverge leading to the divergence in \( F \). The correctness of the above observations hinges crucially on the validity of the MF ansatz in this region. Some indirect support for the validity can be obtained from the related phenomena of FQHE, where the composite fermion model predicts a zero mean field at the filling fraction \( \nu = 1/2 \), thus leading to a
free fermion like system. The analysis of Halperin, Lee and Read\cite{13} and a recent experiment 
by Du et al\cite{9} do support this prediction. More specifically for the system at hand, we recall 
the argument given in Ref. 6. One simple criterion is that MF would be plausible if the
interparticle seperation is less than the magnetic length $l$ which is a measure of the scale 
over which the single particle wave function extends, i.e., 
\[\sqrt{\rho^{-1}} < l \Rightarrow \frac{N}{|1 + x|} > 2\pi. \] 
(4.10)
Therefore, the MF theory will be invalid for very high $|x|$. However for moderate $|x|$, the MF 
ansatz works very well. Note that (4.10) suggests that the MF theory is valid at 
$x = -1$.

Before we end this section, we remark that the MF value for $F$ approaches a constant 
value, with an ever increasing number of discontinuities as $x \to -1$. In contrast, the contribution 
from the fluctuations diverges linearly and thus dominates over the MF contribution. 
To illustrate this, if the generalized susceptibility defined as $\chi(x) \equiv \partial M(x)/\partial B$ is con-
sidered, it is easy to see that the fluctuation part diverges as $-1/(1 + x)^3$ unlike the MF 
contribution which diverges only quadratically in the same limit. Finally, note that even 
around $x = 0$, the susceptibilities get a dominant contribution from the fluctuations.

V. CONCLUSION

To conclude, the mean field properties such as chemical potential, magnetization and
magnetic susceptibility oscillate as functions of the external magnetic field. Considering the 
fluctuations of the gauge fields we find that the masses of the mode of excitations increase 
with magnetic field. The nonlinear susceptibilities arising from the fluctuations of the gauge 
fields are computed which show the absence of any critical or pseudo critical field. Further, 
there is a unique asymmetry in the system at $B = 0$ as far as the magnetic properties are 
concerned, and would be an interesting property to look for in case candidates for CSS are 
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FIG. 1. The energy levels for MF ground state. $N$ Landau Levels are filled at $B = 0$. At $B \neq 0$, there are $K$ fractionally filled (topmost level is not fully filled) levels. Split levels arise due to the switching on of the fictitious interaction. Each split level can accommodate only one particle. The number of split levels for each LL are equal to the degeneracy of each LL. The horizontal dotted lines represent the Fermi levels.

FIG. 2. Chemical potential $\mu$ (solid line) is plotted against the applied magnetic field $B$, $N = 6$, in the units of $2\pi \rho/m = \mu(B = 0)$. $B$ is in the units of CS magnetic field $b$. Note that $\mu$ shows discontinuities at those values of $B$ where the number of filled levels are exactly integral in number. Also, note that the values of $\mu$ at integer filling (denoted by dark dots) are the same and equal that at $B = 0$.

FIG. 3. MF magnetic susceptibility $\chi_{\text{MF}}$ (solid lines) is shown, for $N = 6$, against $B/b$. It also shows the discontinuities in those values of $B$ for which filled levels are integers. Its values for those values of $B$ are shown by dark dots.

FIG. 4. The dispersions of the phononic modes for different values of $B$ are shown. Here $l_0 = |eb|^{-1/2}$ is the magnetic length of the system in absence of $B$. The numbers associated to each curve are the applied magnetic field $B$ in units of $b$. Note that the mass gap of the mode increases with $B$.

FIG. 5. The dispersion relation $\omega^2_\text{+}$ as a function of $q^2$ for different values of $B$. The numbers associated to each curve are the applied magnetic field $B$ in units of $b$.

FIG. 6. The dispersion relation $\omega^2_\text{-}$ for different values of $B$. The numbers associated to each curve are the applied magnetic field $B$ in units of $b$.

FIG. 7. Free energy is shown as a function of magnetic field.