A Note on Dressed S-Matrices
in Models with Long-range Interactions

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ABSTRACT

The dressed Scattering matrix describing scattering of quasiparticles in various models with long-range interactions is evaluated by means of Korepin’s method[1]. For models with $\frac{1}{\sin^2(r)}$-interactions the S-matrix is found to be a momentum-independent phase, which clearly demonstrates the ideal gas character of the quasiparticles in such models. We then determine S-matrices for some models with $\frac{1}{\sinh^2(r)}$-interaction and find them to be in general nontrivial. For the $\frac{1}{r^2}$-limit of the $\frac{1}{\sinh^2(r)}$-interaction we recover trivial S-matrices, thus exhibiting a crossover from interacting to noninteracting quasiparticles. The relation of the S-matrix to fractional statistics is discussed.

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1. Introduction

Haldane\cite{2} recently has put forward an interpretation of the Haldane-Shastry model as a generalized ideal gas with fractional statistics (“ideal gas of spinons”). The Calogero-Sutherland (CS) model is another example for a system of free particles with fractional statistics\cite{3-7}. All methods employed so far in exhibiting the ideal gas character and the nature of the statistics are based on the knowledge of the exact wave-functions\cite{8} for the HS and CS models. For the multitude of other models with long-range interactions, in particular models solvable by Asymptotic Bethe Ansatz (ABA)\cite{9,10}, exact wave-functions are not known. It would be useful to have a method based merely on the ABA to decide whether or not those systems fall into Haldane’s category of ideal gases with fractional statistics. The most direct way to determine whether a system of quasiparticles is an ideal gas is to evaluate the dressed scattering matrix describing scattering of the elementary excitations in the model. If it is a momentum-independent phase, then we are indeed dealing with an ideal gas. Furthermore, if the phase it not $\pm 1$, the quasiparticles have fractional statistics in the sense that the phase of the wave-function under interchange of two particles is neither bosonic ($+1$) nor fermionic ($-1$). This follows from the observation that for noninteracting particles \textit{i.e.} for momentum-independent S-matrices the scattering phase is precisely equal to the phase picked up by interchanging the two particles.

The plan of this note is as follows: in sections 2 and 3 we determine the S-matrix for scattering of quasiparticles in $SU(2)$ Haldane-Shastry chains. It is found to be a momentum-independent phase, which shows both the ideal-gas character and the fractional statistics of the quasiparticles. In section 4 we repeat this analysis for the case of the Calogero-Sutherland model. In section 5 we discuss the generalization of our results to other models with $\frac{1}{\sin^2(\tau)}$-interactions. In section 6 we consider the $\frac{1}{\sinh^2(\tau)}$-CS model and its exchange generalizations and show that quasiparticles in these models are \textit{interacting}, and become free in the $\frac{1}{r^2}$ limit only. In section 7 we summarize and discuss our results.

2. $SU(2)$ Haldane-Shastry Chain

The hamiltonian of the $SU(2)$ HS chain is given by\cite{11,12}

$$H = 2 \sum_{i<j} \frac{1}{(\frac{\pi}{N} \sin(\frac{\pi}{N}(i-j)))^2} (P_{ij} - 1) , \quad (2.1)$$

where $P_{ij}$ is a permutation operator exchanging the spins at sites $i$ and $j$. Ha and Haldane\cite{13} proposed to characterize eigenstates of (2.1) by means of sets of spectral variables $k^n_{\alpha}$ obeying the following set of “Bethe-like equations”

$$N k^n_{\alpha} = 2 N i^n_{\alpha} + \pi \sum_{m\beta} t_{nm} \text{sgn}(k^n_{\alpha} - k^m_{\beta}) , \quad (2.2)$$
where \( k^n_\alpha \) is the position of the center of a “string of length \( n \)” (\( \alpha \) labels different strings of the same length), \( t_{nm} = 2\min(n, m) - \delta_{mn} \), and \( I^n_\alpha \) are integer or half-odd integer quantum numbers with range
\[
|I^n_\alpha| \leq \frac{1}{2}(N - \sum_{m=1}^{\infty} t_{nm}M_m - 1), \tag{2.3}
\]
where \( M_m \) is the number of strings of given length \( m \). Thus \( \sum_{n=1}^{\infty} nM_n = M \), where \( M \) is the total number of down spins. Energy and momentum are given as
\[
E = \sum_{n=1}^{\infty} \frac{1}{2} \left[ (k^n_\alpha)^2 - \pi^2 \right]
\]
and
\[
P = \sum_{n=1}^{\infty} \left( k^n_\alpha + \pi \right) .
\]
The above equations are very similar to the Bethe equations for the spin-\( \frac{1}{2} \) Heisenberg XXX (nearest neighbour) antiferromagnet\[14–17\]. Ha and Haldane proceed to show that ground state and excitations as well as the thermodynamics of the HS chain are correctly described by the above equations if one considers them as “true Bethe equations” for an integrable system. The ground state is a filled “Fermi-sea”, where all vacancies for the integers \( I^1_\alpha \) (allowed by (2.3)) are taken. More precisely we have \( M = M_1 = \frac{N}{2} \), and there are \( \frac{N}{2} \) vacancies for the integers \( I^1_\alpha \), all of which are filled. This corresponds to filling all vacancies for the momenta \( k^1_\alpha \) between \(-\pi\) and \( \pi \). The Ha-Haldane equations take the form
\[
Nk^1_\alpha = 2\pi I^1_\alpha + \frac{1}{2} \sum_{\beta=1, \beta\neq\alpha}^{M} \text{sgn}(k^1_\alpha - k^1_\beta), \quad \alpha = 1 \ldots M, \tag{2.4}
\]
Subtracting (2.4) for \( \alpha \) and \( \alpha + 1 \) one obtains an equation for the density of \( k \)'s \( \rho_1(k^1_\alpha) = \frac{1}{k^1_{\alpha+1} - k^1_\alpha} \), which in the thermodynamic limit \( N \to \infty \) turns into the following integral equation
\[
\frac{1}{2\pi} = \rho_1(k) + \int_{-\pi}^{\pi} dk' \delta(k - k')\rho_1(k') , \tag{2.5}
\]
which can be solved trivially with the result \( \rho_1(k) = \frac{1}{4\pi} \). This shows that the ground state is of much simpler nature than for the Heisenberg antiferromagnet.

The elementary excitations or “quasiparticles” are identified as two spin-\( \frac{1}{2} \) objects, called spinons, with dispersion (using the conventions of [13]) \( \epsilon(p) = p(\pi - p), \ p \in [0, \pi] \). The situation is thus very similar to the nearest neighbour Heisenberg model, where there are also two elementary excitations\[15,16\] carrying spin \( \frac{1}{2} \), but with dispersion* \( \epsilon_{XXX}(p) = \pi \sin(p), \ p \in [0, \pi] \). The similarity is not surprising due to the fact that (2.3) are the same for both models, and the ground states of both models are given by filling a Fermi sea of real spectral parameters. The \( SU(2) \) structure of excitations is the same for the HS chain\[18\] and the nearest neighbour model: all excited states over the true ground state are scattering states of an even number of quasiparticles. A scattering state is characterized by having energy and momentum equal to the sum of the energies/momenta of its constituent quasiparticles. The simplest excited states are in the two-particle sector, and their \( SU(2) \)-representation theory

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\*In order to compare the results of [15] with the Haldane-Shastry case we should set \( J = 2 \) in the hamiltonian of [15].
is simply given by tensoring two fundamental representations (the quasiparticles transform in the fundamental representation $\frac{1}{2}$): $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$. In other words there are four excited states in the two-particle sector, three of which form a $SU(2)$ triplet and one a $SU(2)$ singlet. Their energies/momenta are degenerate and given by the sum of the quasiparticle energies/momenta $E = \epsilon(p_1) + \epsilon(p_2)$, $P = p_1 + p_2$. In the four-particle sector we obtain the $SU(2)$ representation content $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = 2 \oplus 1 \oplus 1 \oplus 1 \oplus 0 \oplus 0$ and so on.

The dressed S-matrix can also be obtained from the equations (2.2) and (2.3), if we treat the Ha-Haldane equations the same way we treat Bethe equations for Bethe-Ansatz solvable models. This is not as straightforward as it may seem, as Bethe Ansatz equations have a direct connection to the exact eigenfunctions of the Hamiltonian, which is something missing for the case of the HS chain and the Ha-Haldane equations. However, this connection is not vital for determining the scattering phase-shifts\cite{1,19}. Adopting (2.2) as “Bethe equations” Korepin’s method\cite{1} can then be applied in a way completely analogous to the nearest neighbour XXX chain\cite{15,16}, for a more detailed explanation of the method we use we refer to [1,20].

In order to determine the two-quasiparticle scattering matrix we need to determine the two-quasiparticle eigenstates of the Hamiltonian (the triplet and the singlet), which by construction are also eigenstates of the scattering operator we seek. We proceed again in a way analogous to the XXX case: the spin-triplet $SU(2)$-highest weight state is obtained by choosing $M_1 = \frac{N}{2} - 1$. The allowed range of integers is $|I_\alpha| \leq \frac{N}{4}$, which means that there are $\frac{N}{2} + 1$ vacancies and thus 2 holes. We take the holes to have momenta $k_1^h$ and $k_2^h$. The Ha-Haldane equations for this excitation are

$$N \tilde{k}_\alpha^1 = 2\pi \tilde{I}_\alpha + \pi \sum_{\beta=1, \beta \neq \alpha}^{\frac{N}{2} - 1} \text{sgn}(\tilde{k}_\alpha^1 - \tilde{k}_\beta^1) - \pi \sum_{j=1}^{2} \text{sgn}(\tilde{k}_\alpha^1 - k_j^h) \, .$$  \hspace{1cm} (2.6)

Here we have used the notation $\tilde{k}$ to indicate that the momenta of the particles in the Fermi sea are slightly different for the excited state as compared to the ground state\cite{20}. Subtracting (2.6) from the equation (2.4) for the ground state and taking the thermodynamic limit we obtain an integral equation for the shift function $F_T(k)\cite{21}$ of the spin-triplet state (which is the limit of the finite-lattice quantity $F_T(k_\alpha^1) = \frac{k_\alpha^1 - k_{\alpha + 1}^1}{k_\alpha^1 - k_{\alpha}}$)

$$F_T(k) = 1 - \int_{-\pi}^{\pi} dk' \delta(k - k') F_T(k') - \frac{1}{2} \sum_{j=1}^{2} \text{sgn}(k - k_j^h) \, .$$  \hspace{1cm} (2.7)

The solution to this equation is

$$F_T(k) = \frac{1}{2} - \frac{1}{4} \sum_{j=1}^{2} \text{sgn}(k - k_j^h) \, .$$  \hspace{1cm} (2.8)
The phase shift for the spin-triplet state is (for an explanation see e.g. [20])

\[ \delta_T(k^h_1, k^h_2) = 2\pi F_T(k^h_1) \bigg|_{k^h_1 > k^h_2} = \pi - \frac{\pi}{2} \text{sgn}(k^h_1 - k^h_2) = \frac{\pi}{2}. \quad (2.9) \]

Here we have used that \( k^h_1 \) must be larger than \( k^h_2 \) (or vice versa, in which case \( \delta_T(k^h_1, k^h_2) = 2\pi F_T(k^h_1) \bigg|_{k^h_2 > k^h_1} = \pi - \frac{\pi}{2} \text{sgn}(k^h_1 - k^h_2) = \frac{\pi}{2} \)) for scattering to occur. The spin-singlet excitation is constructed by taking \( M_1 = N_2 - 2, M_2 = 1 \). Now there are \( \frac{N}{2} \) vacancies for the 1-strings and thus again 2 holes, whereas there is only 1 vacancy for the 2-string. Denoting the positions of the 2 holes again by \( k^h_j \) and the position of the 2-string by \( \kappa \), we find that the Bethe-equation for the 2-string leads to the condition \( k^h_1 > \kappa > k^h_2 \) or \( k^h_2 > \kappa > k^h_1 \), whereas the equations for the 1-strings lead to an integral equation for the singlet shift function \( F_S(k) \), which has the solution

\[ F_S(k) = -\frac{\pi}{2} \sum_{j=1}^{2} \text{sgn}(k - k^h_j) + \pi \text{sgn}(k - \kappa). \quad (2.10) \]

The leads to the following result for the singlet phase-shift

\[ \delta_S(k^h_1, k^h_2) = 2\pi F_S(k^h_1) \bigg|_{k^h_1 > k^h_2} = \frac{\pi}{2}. \]

Our result for the complete dressed S-matrix is thus

\[ S_{HS}(k^h_1, k^h_2) = i \text{ id}. \quad (2.11) \]

This result ought to be compared with the exact S-matrix for the nearest neighbour Heisenberg model\textsuperscript{[15,16]}

\[ S_{XXX}(\mu) = -\frac{\Gamma(1+i\mu)\Gamma(1-i\mu)}{\Gamma(1-i\mu)\Gamma(1+i\mu)} \left( \frac{\mu}{\mu + i} \text{id} + \frac{i}{\mu + i} P \right), \quad (2.12) \]

where \( P \) is the 4 \times 4 permutation matrix and \( \mu = \lambda_1 - \lambda_2 \) is the difference of the spectral parameters of the two quasiparticles. We see that (2.11) is the \( \mu \to \infty \) limit of the XXX S-matrix. Note that this does not imply (and it is not true either) that the low-energy spinon-spinon scattering is the same in HS and XXX models: the low-energy region of the XXX chain is defined by taking \( \lambda_j \to \pm \infty, j = 1, 2 \), which still leaves the difference \( \mu \) as a free parameter, the HS chain corresponds to the \( \mu \to \infty \) limit of the XXX low-energy physics. This fact does not contradict the identification of the conformal limits of both XXX and HS chains with the \( SU(2)_1 \) WZWN CFT. Whereas the conformal limit of the XXX S-matrix precisely coincides with the WZWN result of [22] (with trivial LR scattering), the HS result (2.11) corresponds to “soft” scattering in the WZWN model only (the conformal momenta are taken to be very small).
Like in the case of the XXX antiferromagnet the result (2.11) is a priori exact up to a possible overall constant factor, which stems from the fact that there are no one-spinon states and we thus cannot determine the one-particle phase shift directly.

The form (2.11) of the dressed S-matrix indicates that the spinons behave like an ideal gas as the S-matrix is both momentum independent and proportional to the identity. Furthermore, if we believe that there is no additional constant phase factor for the S-matrix, the phase $i$ can be interpreted as exhibiting the *semionic* character of the spinons as it is in between the phase shifts for an ideal Bose and Fermi gas.

From the above discussion the following relation between $SU(2)$ HS chain and XXX model emerges: the quasiparticles in both models are spin $\frac{1}{2}$ spinons, and the $SU(2)$ representation content is identical in both models. The difference is that the spinons in the HS model are noninteracting, whereas the spinons in the XXX chain are interacting. This follows directly from the form of the S-matrices and agrees with the picture previously put forward by Haldane.

### 3. $SU(N)$ Haldane-Shastry Chain

The $SU(N)$ case can be dealt with in an analogous manner: the “Bethe equations” given by Haldane and Ha$^{13}$ are again very similar to the ones of the nearest-neighbour $SU(N)$ Sutherland model$^{23}$. The quasiparticle interpretation$^{24}$ and exact S-matrix$^{25,26}$ for the $SU(N)$ Sutherland model have recently been derived and can be used to analyze the $SU(N)$ HS chain (the important equations determining the ranges of integers are identical). For simplicity we we only discuss the $SU(3)$ case, the general case can be treated analogously. The Ha-Haldane equations for the $SU(3)$ case read

\[
N k_{n}^{(1)} = 2\pi I_{n}^{(1)} + \pi \sum_{m \beta} t_{nm} \text{sgn}(k_{a}^{(1)n} - k_{\beta}^{(1)m}) - \pi \sum_{m \beta} \text{min}(n, m) \text{sgn}(k_{a}^{(1)n} - k_{\beta}^{(2)m}) \\
\pi \sum_{m \beta} \text{min}(n, m) \text{sgn}(k_{a}^{(2)n} - k_{\beta}^{(1)m}) = 2\pi I_{n}^{(2)} + \pi \sum_{m \beta} t_{nm} \text{sgn}(k_{a}^{(2)n} - k_{\beta}^{(2)m}) .
\]

(3.1)

The ground state is obtained by choosing $M_{1}^{(1)} = \frac{2N}{3}$ and $M_{1}^{(2)} = \frac{N}{3}$ and filling all vacancies for the integers $I_{a}^{(1)}$ and $I_{a}^{(2)}$, which corresponds to filling two “Fermi seas” of spectral parameters $k_{a}^{(1)}$ and $k_{a}^{(2)}$ between $-\pi$ and $\pi$. In the thermodynamic limit we can describe the ground state by densities of spectral parameters $\rho_{1}^{(1)}(k)$ and $\rho_{1}^{(2)}(k)$ (like we did for the $SU(2)$ case above) subject to a set of two coupled integral equations. The solution of these integral equations is straightforward as the integral kernels are again $\delta$-functions. We find that both densities are constant over the interval $[-\pi, \pi]$

\[
\rho_{1}^{(1)}(k) = \frac{1}{3\pi}, \quad \rho_{1}^{(2)}(k) = \frac{1}{6\pi} .
\]

(3.2)
This extends straightforwardly to the general $SU(N)$ case. Excitations over this ground state can be constructed in an analogous way to the $SU(2)$-case above. One finds that the only “dynamical” objects are holes in the two Fermi seas of spectral parameters $k^{(1)}_1$ and $k^{(2)}_1$, i.e. only these holes carry energy and momentum, whereas longer strings (described by spectral parameters $k^{(j)n}_1$, $n > 1, j = 1, 2$) contribute to neither energy nor momentum and are only counting degeneracies. All excited states (for the $SU(N)$ case) can be interpreted as scattering states of $N - 1$ types quasiparticles subject to superselection rules. The quasiparticles are associated with a hole in one of the $N - 1$ Fermi seas respectively, and transform in the $N - 1$ fundamental representations of $SU(N)$. The $SU(N)$-structure of the excited states as well as the superselection rules are the same as in the $SU(N)$ Sutherland model, we refer to [26] for a detailed discussion with proofs of our assertions.

In the $SU(3)$-case there are a total of six quasiparticles, three of which form the fundamental representations $3$ and $\overline{3}$ respectively. The quasiparticles in $3$ have energy and momentum

$$\epsilon_3(k) = \frac{1}{3}(\pi^2 - k^2), \quad p_3(k) = \frac{2}{3}(\pi - k) ,$$

whereas the quasiparticles in $\overline{3}$ have energy and momentum

$$\epsilon_{\overline{3}}(k) = \frac{1}{6}(\pi^2 - k^2), \quad p_{\overline{3}}(k) = \frac{1}{3}(\pi - k) .$$

The superselection rules are that the number of quasiparticles of type $3$ plus twice the number of quasiparticles of type $\overline{3}$ must be a multiple of (the integer number) 3. That means that the only two-particle states are given by $3 \otimes \overline{3} = 8 \oplus 1$. In the 3-particle sector only the states $3 \otimes 3 \otimes 3$ and $\overline{3} \otimes \overline{3} \otimes 3$ are allowed. It can be shown that all excited states are scattering states of quasiparticles subject to the superselection rules. Let us now turn to the evaluation of the phase shifts for the octet and singlet states in the two-quasiparticle sector.

The octet is characterized by choosing $M^{(1)}_1 = \frac{2N}{3} - 1$ and $M^{(2)}_1 = \frac{N}{3} - 1$, which leads to one hole in the sea of $k^{(1)}_1$’s and $k^{(2)}_1$ with spectral parameters $k^{(1)}_h$ and $k^{(2)}_h$ respectively. The Ha-Haldane equations read

$$N \tilde{k}^{(1)}_\alpha = 2\pi \tilde{I}^{(1)}_\alpha + \pi \sum_{\beta=1}^{2N} \text{sgn}(\tilde{k}^{(1)}_\alpha - \tilde{k}^{(1)}_\beta) - \pi \sum_{\beta=1}^{2N} \text{sgn}(\tilde{k}^{(1)}_\alpha - \tilde{k}^{(2)}_\beta) - \pi \sum_{\beta=1}^N \text{sgn}(\tilde{k}^{(1)}_\alpha - k^{(1)}_h) + \pi \sum_{\beta=1}^N \text{sgn}(\tilde{k}^{(1)}_\alpha - k^{(2)}_h) - \pi \sum_{\beta=1}^N \text{sgn}(\tilde{k}^{(2)}_\alpha - k^{(1)}_h) - \pi \sum_{\beta=1}^N \text{sgn}(\tilde{k}^{(2)}_\alpha - k^{(2)}_h) ,$$

$$\pi \sum_{\beta=1}^N \text{sgn}(\tilde{k}^{(2)}_\alpha - k^{(1)}_h) = 2\pi \tilde{I}^{(2)}_\alpha + \pi \sum_{\beta=1}^N \text{sgn}(\tilde{k}^{(2)}_\alpha - \tilde{k}^{(1)}_\beta) + \pi \sum_{\beta=1}^N \text{sgn}(\tilde{k}^{(2)}_\alpha - \tilde{k}^{(2)}_\beta) + \pi \sum_{\beta=1}^N \text{sgn}(\tilde{k}^{(1)}_\alpha - k^{(1)}_h) - \pi \sum_{\beta=1}^N \text{sgn}(\tilde{k}^{(2)}_\alpha - k^{(2)}_h) .$$

Now, like for the case of the Hubbard model[27,28], we have to deal with two shift functions

$$F_j(\tilde{k}^{(j)}_{\alpha}) = \frac{\tilde{k}^{(j)}_{\alpha} - k^{(j)}_{\alpha}}{k^{(j)}_{\alpha+1} - k^{(j)}_{\alpha}}, \quad j = 1, 2 ,$$

which in the thermodynamic limit are found to obey
a system of two coupled integral equations. The solution of these integral equations is elementary due to the occurrence of \( \delta \)-function integral kernels:

\[
F_1(k) = \frac{1}{6} \left( \text{sgn}(k - k_h^{(2)}) - \text{sgn}(k - k_h^{(1)}) \right) = -F_2(k) .
\]

The dispersion for the octet states is found to be \( E = \epsilon_3(k_h^{(1)}) + \epsilon_\bar{3}(k_h^{(2)}) \), \( P = p_3(k_h^{(1)}) + p_\bar{3}(k_h^{(2)}) \), in accordance with the quasiparticle interpretation. The octet phase-shift is

\[
\delta_8 = -2\pi F_1(k_h^{(1)}) + 2\pi F_2(k_h^{(1)}) + \pi = \frac{\pi}{3} ,
\]

where we have use that \( k_h^{(1)} > k_h^{(2)} \) for scattering to occur. The \( SU(3) \)-singlet in \( 3 \otimes \bar{3} \) is obtained by choosing \( M_1^{(1)} = \frac{2N}{3} - 2, \ M_2^{(1)} = 1, \ M_1^{(2)} = \frac{N}{3} - 2 \) and \( M_2^{(2)} = 1 \). Energy and momentum of the singlet are the same as for the octet. The shift-functions can be determined analogously to the octet case, although the computation is slightly more difficult due to the presence of the 2-strings. The result is

\[
\delta_1 = \frac{\pi}{3} = \delta_8 ,
\]

i.e. the singlet phase shift is the same (constant) as the octet phase shift. The phase-shifts for scattering of quasiparticles of type 3 (3) on quasiparticles of type 3 (3) can be extracted from the three-particle states \( 3 \otimes 3 \otimes 3 \) (\( 3 \otimes \bar{3} \otimes \bar{3} \)), with the result that the phase-shifts for scattering of 3 on 3 and 3 on 3 are also equal to \( \frac{\pi}{3} \). This implies that the quasiparticles are an ideal gas with fractional statistics \( \frac{\pi}{3} \). For the \( SU(N) \) case we conjecture the phase to be \( \frac{\pi}{N} \). The interesting new phenomenon is the “decoloration” of physical excitations: the superselection rules force the quasiparticles to combine to either “mesons” \( (3 \otimes \bar{3}) \) or “baryons” \( (3 \otimes 3 \otimes 3) \text{ and } 3 \otimes 3 \otimes 3 \). In this way the \( SU(3) \) HS chain (as well as the \( SU(3) \) Sutherland model\cite{26}) reminds an ideal (1-d) gas of (confined) quarks. Like for the \( SU(2) \) case, the constant S-matrix found for the HS model is precisely the limit \( \mu \to \infty \) of the corresponding S-matrix of the nearest neighbour Sutherland model\cite{26}, where \( \mu \) is the difference of the spectral parameters of the two scattering quasiparticles.

4. **Calogero-Sutherland Model**

The Calogero-Sutherland model\cite{9,10,29} is given by the following hamiltonian

\[
H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{j<k} \frac{2\lambda(\lambda - 1)}{(x_k - x_j)^2} . \tag{4.1}
\]

Ground state, excitations, and thermodynamics for the CS model are all constructed from the following set of “asymptotic” Bethe equations\cite{10}

\[
\exp(-ik_j L) = \prod_{l \neq j} S(k_j, k_l) , \quad \tag{4.2}
\]
where \( S(k) = -\exp(-i\pi(\lambda - 1)\text{sgn}(k)) \) is the bare S-matrix describing scattering of two bare particles over the bare (trivial) vacuum. Following the logic of Bethe Ansatz, Sutherland used (4.2) to construct the true ground state and dressed excitations over it, by “filling the Fermi sea”. By this we mean the following: taking the logarithm of (4.2) we arrive at the set of equations

\[
Lk_j = 2\pi I_j + (\lambda - 1)\pi \sum_{l \neq j} \text{sgn}(k_j - k_l), \quad j + 1 \ldots N, \tag{4.3}
\]

where \( I_j \) are all integer of half-odd integer numbers, which can be chosen as a set of quantum numbers that completely determines an eigenstate of the hamiltonian. The ground state is characterized by filling all vacancies for the \( k_j \)'s symmetrically around zero in the interval \([-k_F, k_F]\). The “Fermi-momentum” is \( k_F = \sqrt{\mu} \), where \( \mu \) is the chemical potential. In analogy to the delta-function Bose gas\(^{30,31}\) dressed (particle-hole) excitations can be constructed by removing one particle with rapidity \( k_h \) from the Fermi-sea and placing it on a vacancy \( k_p \) outside the Fermi sea. Energy and momentum of a particle-hole excitation are given by

\[
E_{ph} = (k_p^2 - \mu) + (\mu - k_h^2), \quad P_{ph} = (k_p - k_F) + (k_F - k_h). \quad \text{Equations (4.3) can also be used to determine the phase shifts for scattering of dressed excitations over the ground state. The computations are completely analogous to the ones for the \( \delta \)-function Bose gas\(^{20,21}\), so that we will only give the results here. It is again straightforward to show that all phase shifts are momentum-independent constants, which proves the ideal-gas nature of the quasiparticles. In determining the constants we follow\(^{20}\) (where the \( \delta \)-function Bose gas was treated, see p.23) and change the boundary conditions for one-particle and one-hole excitations (the situation here is quite analogous to the Bose gas case). We then find that particles do not get any dressing through the ground state, \emph{i.e.} still behave like bare particles, which scatter off each other with the bare phase-shift \( \delta_{pp} = -\pi \lambda \). This is in agreement with the results previously obtained in\(^{[3]}\) by means of different methods. Particles scatter off holes with phase-shift \( \delta_{ph} = \pi \lambda \), which means that to the scattering particle a hole is nothing by the absence of a bare particle. Last but not least the hole-hole phase-shift is \( \delta_{hh} = \frac{\pi}{\lambda} \).

We note that we recover the correct scattering phases for free fermions \( e^{\delta_{pp}} = e^{\delta_{hh}} = -1 \) for \( \lambda = 1 \) and free bosons \( e^{\delta_{pp}} = 0 \) for \( \lambda = 0 \) (in this limit there are no holes but only particles).

5. Other \( \sin^2(r) \)-Type Models

Other candidates for applying Korepin’s dressed S-matrix method would be for example the \( gl(n,1) \) supersymmetric “\( t-J \)” models with long-range exchange interactions\(^{[32-34]}\), or Kawakami’s hierarchy of \( SU(N) \) electron models\(^{[35]}\). The asymptotic Bethe equations (ABE)\(^{[33-35]}\) have to be complemented by a “squeezed-string” prescription\(^{[13]}\) in order to give the correct degeneracies of the spectrum. If we take into account only the states given by the ABE we find that all phase-shifts will be constants, and the states described by
the ABE will thus describe mixtures of ideal gases with fractional statistics. This can be seen as follows: from the computations above it is clear that the ideal-gas character of the quasiparticles is “caused” by the \( \delta \)-function kernel in the integral equations, or alternatively the \( \text{sgn}(x) \)-kernels in the Asymptotic Bethe equations. The occurrence of \( \text{sgn}(x) \)-kernels in the ABA equations is generic feature of models with \( \frac{1}{\sin^2(\tau)} \)-type interactions, so that by analyzing only ABA states we would conclude that all these models describe mixtures of noninteracting quasiparticles.

However, there are a number of open problems concerning the supersymmetric models: the squeezed-string prescription seems to be not available, but more importantly, the ground state will in general not be a \( gl(n,1) \) singlet and thus no \( Y(gl(n,1)) \) Yangian singlet. This is easily seen for the case of the long-range supersymmetric \( (gl(2,1)) \ t-J \) model: in the \( 3^L \)-dimensional Hilbert space without doubly occupied sites there exists no \( gl(2|1) \) singlet! Thus, very much like for the case of the nearest neighbour model\(^{[36]} \), the ground state will belong to a larger \( gl(2,1) \) multiplet. This raises the question how to interpret the other states in the multiplet containing the ground state in terms of a quasiparticle picture, which is necessary for identifying the model as a gas.

6. INTERACTING QUASIPARTICLES: \( \frac{1}{\sinh^2(\tau)} \)-MODELS

Let us now demonstrate that not all models with long-range interactions describe noninteracting quasiparticles. To this end let us consider the \( \frac{1}{\sinh^2(\tau)} \)-CS model\(^{[37]} \), defined in terms of the Hamiltonian \((\lambda \geq 1)\)

\[
H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{j<k} \frac{2\lambda(\lambda - 1)}{\sinh^2(\frac{x_k-x_j}{a})}.
\]  

(6.1)

In the limit \( a \to \infty \) (6.1) reduces to the CS model with coupling \( 2\lambda(\lambda - 1)a^2 \). The ABA as well as ground state and excitations were constructed in [37]. The ABA equations are

\[
Lk_j = 2\pi I_j + \sum_{l\neq j} \theta(k_j - k_l), \quad j + 1 \ldots N,
\]  

(6.2)

where \( a \) has been set to 1 and where

\[
\theta(k) = i \ln \left( \frac{\Gamma(1 + \frac{ik}{2}) \Gamma(\lambda - \frac{ik}{2})}{\Gamma(1 - \frac{ik}{2}) \Gamma(\lambda + \frac{ik}{2})} \right).
\]  

(6.3)

In terms of the variables \( k_j \) the effect of \( a \) is recovered by a rescaling \( k_j \to ak_j \). Excitations over the ground state are (like in the CS case above) of particle-hole type. Their energy and momentum are \( E_{ph} = \epsilon(k_p) - \epsilon(k_h) \) and \( P_{ph} = p_{k_p} - p_{k_h} \), where \( k_p \) and \( k_h \) are the rapidities of
the particle and hole respectively, and where $\epsilon(k)$ and $p(k)$ are given in terms of the integral equations

$$\epsilon(k) = k^2 - \mu - \frac{1}{2\pi} \int_{-B}^{B} dk' \epsilon(k') \theta'(k-k'),$$

$$p(k) = k - \frac{1}{2\pi} \int_{-B}^{B} dk' p(k') \theta'(k-k'),$$

where $\theta'(x) = \frac{d}{dx} \theta(x)$. Here $\mu$ is the chemical potential, and the integral boundary $B$ is determined as a function of $\mu$ through the requirement that $\epsilon(\pm B) = 0$. The computations of the S-matrices for scattering of particles on holes, particles on particles, and holes on holes are again completely analogous to the ones for the Bose gas (see p.23 of [20]). The result is that all phase-shifts can be expressed in terms of a function $\delta(\lambda, \mu)$ subject to the integral equation

$$\delta(k_1, k_2) + \frac{1}{2\pi} \int_{-B}^{B} dk \theta'(k_1 - k) \delta(k, k_2) = \pi + \theta(k_1 - k_2).$$

(6.4)

The S-matrix for particle-hole scattering ($k_p > B, -B \leq k_h \leq B$, the constraint $k_p > B$ is only a matter of convenience) is given as

$$S_{ph}(k_p, k_h) = e^{i\delta(k_p, k_h)}.$$

Similarly particle-particle and hole-hole S-matrices are found to be

$$S_{pp}(k_{p,2}, k_{p,1}) = e^{-i\delta(k_{p,2}, k_{p,1})}, \quad k_{p,2} > k_{p,1} > B$$

$$S_{hh}(k_{h,2}, k_{h,1}) = e^{-i\delta(k_{h,2}, k_{h,1})}, \quad B \geq k_{h,2} > k_{h,1} \geq -B.$$

As noted above the CS limit is obtained by rescaling $k_j \to ak_j$ and then taking $a \to \infty$ keeping $2\lambda(\lambda - 1)a^2 =: 2g(g - 1)$ fixed. In this limit one obtains $\theta(k_1 - k_2) \to \pi(g - 1)\text{sgn}(k_1 - k_2)^{[31]}$, and the expression for the S-matrices reduce to the ones found for the CS model in section 4 (if $g$ is identified with $\lambda$ of section 4) as can be seen directly from (6.4). In general the integral equation (6.4) can only be solved numerically, the result being a nontrivial function of $k_1$ and $k_2$.

Physically our results for the S-matrices imply that the quasiparticles in the $\frac{1}{\sinh^2(\tau)}$-CS model are interacting as the S-matrices are momentum dependent and nontrivial. In the $\frac{1}{\tau^2}$-limit they become noninteracting.

A very interesting extension of the $\frac{1}{\sinh^2(\tau)}$-CS model is the $\frac{1}{\sinh^2(\tau)}$-CS model with exchange$^{[38-40]}$. The Hamiltonian of the model is$^{[41]}

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{j<k}^{N} \kappa^2 \frac{(\lambda^2 - \lambda P_{jk})}{\sinh^2((x_k - x_j)\kappa)}.$$

(6.5)

where $P_{jk}$ is a permutation operator exchanging the spins of the particles at positions $x_j$ and $x_k$. We will consider only the simplest case of $SU(2)$ spins. $N$ is the number of particles in
a box of length $L$ and we are interested in the limit $L \to \infty$ keeping the density $\frac{N}{L}$ fixed. In the inverse square limit $\kappa \to 0$ the interaction becomes $\sum_{j<k} \frac{\lambda(\lambda - \rho_{jk})}{(x_j - x_k)^2}$ and the model reduces to the CS model with inverse square exchange$^{[38,41,8,42]}$. The ABA equations for (6.5) are $^{[39]}
abla i k_j = \prod_{l \neq j}^{N_j} \frac{\Gamma(1 - \frac{i(k_j - k_l)}{2})}{\Gamma(1 + \frac{i(k_j - k_l)}{2})} \frac{\Gamma(\lambda + \frac{i(k_j - k_l)}{2})}{\Gamma(\lambda - \frac{i(k_j - k_l)}{2})} \prod_{s=1}^{N_j} \frac{k_j - \alpha_s + i\lambda}{k_j - \alpha_s - i\lambda},
abla \prod_{j=1}^{N_j} \frac{\alpha_s - k_j + i\lambda}{\alpha_s - k_j - i\lambda} = -\prod_{j=1}^{N_j} \frac{\alpha_s - \alpha_t + 2i\lambda}{\alpha_s - \alpha_t - 2i\lambda}. \quad (6.6)

Here $\kappa$ has been set to 1 and the effect of $\kappa$ corresponds to a rescaling $k_j \to \frac{k_j}{\kappa}$ and $\alpha_s \to \frac{\alpha_s}{\kappa}$. All $k_j$’s are real (complex $k$’s do not lead to bound states in the bare scattering amplitudes on the r.h.s. of the first equation in (6.6)), whereas the $\alpha_s$’s can form bound states of the form $\alpha_s^\alpha = \alpha_s + i(n + 1 - 2j)\lambda$ with $\alpha_s^\alpha \in \mathbb{R}$. This not surprising as the second set of equations in (6.6) is nothing but the set of Bethe equations for an inhomogeneous Heisenberg model. Inserting this “string hypothesis” into (6.6) and then taking the logarithm we obtain

$$Lk_j = 2\pi I_j + \sum_{l \neq j} \theta(k_j - k_l) - \sum_s \vartheta\left(\frac{k_j - \alpha_s^n}{n\lambda}\right),$$

$$0 = 2\pi J_s^n - \sum_l \vartheta\left(\frac{\alpha_s - k_l}{n\lambda}\right) + \sum_{(m,t) \neq (n,s)} \vartheta_{nm}\left(\frac{\alpha_s^m - \alpha_t^m}{\lambda}\right), \quad (6.7)$$

where $I_j$ and $J_s^n$ are integer or half-odd integer numbers, $\vartheta(x)$ is given by (6.3), $\vartheta(x) = 2\arctan(x)$ and where

$$\vartheta_{nm}(x) = \begin{cases} \vartheta\left(\frac{x}{|n-m|}\right) + 2 \vartheta\left(\frac{x}{|n-m+2|}\right) + \ldots + 2 \vartheta\left(\frac{x}{n+m-2}\right) + \vartheta\left(\frac{x}{n+m}\right) & \text{if } n \neq m \\ 2 \vartheta\left(\frac{x}{2}\right) + 2 \vartheta\left(\frac{x}{4}\right) + \ldots + 2 \vartheta\left(\frac{x}{2n-2}\right) + \vartheta\left(\frac{x}{2n}\right) & \text{if } n = m. \end{cases}$$

The range of $J_s^n$ follows from (6.7) to be

$$|J_s^n| \leq \frac{1}{2}(N - \sum_{m=1}^{\infty} t_{nm} M_m - 1), \quad (6.8)$$

where $M_m$ is the number of $\alpha$-strings of length $m$.

The construction of ground state and excitations is rather similar to the less than half-filled Hubbard model$^{[43]}$. The ground state is obtained by filling two Fermi seas of spectral parameters $k_j$ and $\alpha_s^1$. In the thermodynamic limit it is described in terms of two densities (of spectral parameters) $\rho(k)$ and $\sigma(\alpha)$ subject to the coupled integral equations

$$\rho(k) = \frac{1}{2\pi} - \frac{1}{2\pi} \int_{-A}^{A} dk' \theta'(k - k') \rho(k') + \frac{1}{2\pi} \int_{-\infty}^{\infty} d\alpha \frac{2\lambda}{\lambda^2 + (k - \alpha)^2} \sigma(\alpha),$$

$$\sigma(\alpha) = \frac{1}{2\pi} \int_{-A}^{A} dk \frac{2\lambda}{\lambda^2 + (k - \alpha)^2} \rho(k) - \frac{1}{2\pi} \int_{-\infty}^{\infty} d\alpha' \frac{4\lambda}{4\lambda^2 + (\alpha - \alpha')^2} \sigma(\alpha'),$$

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where \( \int_A^A dk \rho(k) = \frac{N}{2} \) and where the integration boundary \( A \) is a function of the chemical potential \( \mu \). The ground state energy density is given by \( E_{GS} = \int_A^A dk \rho(k)k^2 \) (\( \mu = \frac{dE_{GS}}{dN} \), which fixes \( A \) as a function of \( \mu \)). We note that for \( \kappa > 0, \sigma(\alpha) > 0 \) on the whole real axis ("the sea of \( \alpha \)'s is completely filled"), whereas in the inverse square limit \( \kappa \to 0, \sigma(\alpha) = 0 \ \forall |\alpha| > A \). This is in agreement with [35]. There are two classes of low-lying excitations over the ground state: particle-hole excitations in the Fermi sea of \( \alpha \)'s, which are very similar to the excitations in the \( \frac{1}{\sinh(r)} \) CS model (see above; the only difference is that now there will be a dressing through the second Fermi sea of \( \lambda \)'s) and spin-excitations in the second Fermi sea. We will constrain ourselves to a discussion of the spin-excitations here. Inspection of (6.8) shows that the situation for \( \kappa > 0 \) is very similar to the one for the HS chain treated in section 2: the simplest low-lying excitations are a spin-triplet \( \left( M_1 = \frac{N}{2} - 1 \right) \) and a spin-singlet \( \left( M_1 = \frac{N}{2} - 2, M_2 = 1 \right) \) two-hole excitation. In the inverse square limit the Fermi sea of \( \alpha \)'s is not completely filled, so that the simplest spin excitations are of particle-hole type. We consider only the case \( \kappa > 0 \) as it is the far more interesting one. Like in sections 2 and 3 we describe the excitations in terms of shift-functions \( F_1(k) \) and \( F_2(\alpha) \) (the construction is very similar to the one of spin excitations in the Hubbard model, which was treated in detail in [27]). After some manipulations we find for the triplet

\[
F^T_1(k) = \frac{1}{2\pi} \sum_{p=1}^2 \left( 2 \arctan \left( e^{\frac{k - \alpha_{h,p}}{2\lambda}} \right) - \frac{\pi}{2} \right) - \frac{1}{2\pi} \int_A^A dk' F^T_1(k') \Theta(k - k') ,
\]

\[
F^T_2(\alpha) = \frac{1}{2} - \frac{i}{2\pi} \sum_{p=1}^2 \ln \left( \frac{\Gamma \left( \frac{1+i(\alpha - \alpha_{h,p})}{2\lambda} \right)}{\Gamma \left( \frac{1-i(\alpha - \alpha_{h,p})}{2\lambda} \right)} \right) + \frac{1}{4\lambda} \int_A^A dk \frac{F^T_1(k)}{\cosh \left( \frac{\pi}{2\lambda}(\alpha - k) \right)} ,
\]

(6.9)

where \( \alpha_{h,p} \) are the rapidities of the two holes and where

\[
\Theta(x) = \Re \left\{ \frac{1}{2\lambda} \Psi \left( \frac{1}{2} + \frac{i}{4\lambda} \right) - \frac{1}{2\lambda} \Psi \left( 1 + \frac{i}{4\lambda} \right) + \Psi \left( \lambda + \frac{i}{4\lambda} \right) - \Psi \left( 1 + \frac{i}{2} \right) \right\} .
\]

Here \( \Psi(x) \) is the Digamma function. Energy and momentum of the spin-triplet are given by

\[
E_{ST}(\alpha_{h,1}, \alpha_{h,2}) = \int_A^A dk \ 2k \ F^T_1(k) ,
\]

\[
P_{ST}(\alpha_{h,1}, \alpha_{h,2}) = \int_A^A dk \ F^T_1(k) .
\]

The scattering phase shift is given as \( \delta_T(\alpha_{h,1}, \alpha_{h,2}) = 2\pi F^T_2(\alpha_{h,1}) \) with \( \alpha_{h,1} - \alpha_{h,2} > 0 \). For the spin-singlet we find

\[
F^S_1(k) = F^T_1(k) ,
\]

\[
F^S_2(\alpha) = F^T_2(\alpha) + \frac{1}{\pi} \arctan \left( \frac{\alpha - \alpha_{h,1} + \alpha_{h,2}}{\lambda} \right) - \frac{1}{2} ,
\]

(6.10)
where \( F^T_{1,2} \) are given by (6.9). From the first equality in (6.10) it follows immediately that energy and momentum of triplet and singlet are identical (as it must be). The singlet scattering phase-shift is found to be \( \delta_S(\alpha_{h,1}, \alpha_{h,2}) = 2\pi F^S_2(\alpha_{h,1}) \). The resulting two particle S-matrix describing scattering of spinons in the \( 1 \text{sinh}^2 \) CS model with exchange is

\[
S(\alpha_{h,1}, \alpha_{h,2}) = e^{2\pi i F^T_2(\alpha_{h,1})} \left( \frac{\nu}{\nu + i} \text{id} + \frac{i}{\nu + i} P \right), \nu = \frac{\alpha_{h,1} - \alpha_{h,2}}{2\lambda} > 0, \tag{6.11}
\]

where \( P \) is the \( 4 \times 4 \) permutation matrix and where \( F^T_2(\alpha) \) is given by (6.9). The result we get is extremely similar to the spinon-spinon S-matrix of the Hubbard model\[27,28,44]: the rapidities get renormalized by a factor of \( 2\lambda \) (2\( U \) in the Hubbard model) as compared to the pure XXX scattering matrix (2.12), and the common overall phase gets an additional contribution from the dynamical degrees of freedom (i.e. the Fermi sea of \( k \)'s). We can rewrite (6.11) in terms of the XXX S-matrix \( S_{XXX} \) given by (2.12) as

\[
S(\alpha_{h,1}, \alpha_{h,2}) = S_{XXX}(\nu) \exp \left( \frac{2\pi i}{4\lambda} \int^A_{-A} \frac{dk}{2\lambda} \frac{F^T_1(k)}{\cosh \left( \frac{\pi}{2\lambda}(\alpha_{h,1} - k) \right)} \right),
\]

where \( \nu \) is as in (6.11). In the limit \( \nu \to \infty \) this reduces to the Haldane-Shastry result (2.11). We see that in (6.11) there are two distinct contributions: one from pure spin-spin scattering (given by \( S_{XXX}(\nu) \)), and one from coupling of spin and dynamical degrees of freedom (given by the second factor).

As was noted by Sutherland, Römer and Shastry, it is possible to freeze out the dynamical degrees of freedom in (6.5) by taking the limit \( \lambda \to \infty \)[39]. In this limit the particles freeze into an equidistant lattice \( x_j = \frac{j}{d} \) (recall that \( d = \frac{N}{L} \) is the fixed density of particles), and the hamiltonian (6.5) separates into \( H_{\text{dyn}} + 2\lambda H_{\text{latt}} \), where \( H_{\text{dyn}} \) is of the form (6.1) with coupling \( \lambda(\lambda - 1) \) and where

\[
H_{\text{latt}} = -\frac{1}{2} \sum_{j>k} \frac{1 + P_{jk}}{\sinh^2 \left( \frac{x_j - x_k}{d} \right)} \tag{6.12}
\]

Ground state and excitations of the lattice model (6.12) can be obtained by rescaling and expanding the spectral parameters in (6.7) according to \( k_j = 2\lambda \xi_j^{(0)} + \xi_j^{(1)} + \frac{1}{2\lambda} \xi_j^{(2)} + \ldots \), \( \alpha_n^a = 2\lambda \beta_n^a + \ldots \), and then expanding the ABE in inverse powers of \( \lambda \). This procedure yields

\[
L \xi_j^{(0)} = \sum_l \theta_0(\xi_j^{(0)} - \xi_l^{(0)})
\]

\[
0 = 2\pi I_j - \sum_l \theta_0(\xi_j^{(0)} - \xi_l^{(0)})\xi_l^{(1)} - \sum_{(n,s)} \vartheta \left( \frac{2}{n} (\xi_j^{(0)} - \beta_n^a) \right) + \sum_l \theta_1(\xi_j^{(0)} - \xi_l^{(0)}) \tag{6.13}
\]

\[
0 = 2\pi J_n^a - \sum_l \vartheta \left( \frac{2}{n} (\beta_n^a - \xi_l^{(0)}) \right) + \sum_{(m,t) \neq (n,s)} \vartheta_{nm}(2(\beta_n^a - \beta_t^m))
\]

\[14\]
where \(\theta_0(x) = \frac{i}{2} \ln \left(1 + \frac{1}{x^2}\right) + \frac{i}{2} \ln \left(\frac{1 - i x}{1 + i x}\right)\) and \(\theta_1(x) = -\frac{x}{2} - \frac{i}{2} \ln \left(\frac{1 - i x}{1 + i x}\right)\). The first set of equations (6.13) is of order \(\lambda\), and leads in the thermodynamic limit to an integral equation for the ground state density of the dynamical part [39] \(\rho(x)\) (defined to be the limit \(N \to \infty\) of \(\rho(\xi_j^{(0)}) = \frac{1}{N(\xi_j^{(0)} - \xi_j^{(0)})}\)). The second and third sets of equations (6.13) are of order 1 and can be used to construct ground state and excitations of the spin model (6.12). This has already been done (for the general \(SU(N)\) case in [39]). Our goal here is to determine the exact S-matrix for the \(SU(2)\) case, for which we need to construct all two particle excitations in the framework of the \(F\)-function formalism. This is easily done as the integers \(J_k^n\) are actually the same as in the dynamical model treated above. Before we get to this let us review some results of [39] that we will need later on. The ground state of (6.5) in the limit \(\lambda \to \infty\) is obtained by taking \(M_1 = \frac{N}{2}\), \(M_k = 0\) \(\forall k > 1\). In the thermodynamic limit \(N \to \infty\) \((d = \frac{N}{L}\) fixed) the ABA equations turn into a set of three coupled integral equations [39]

\[
\frac{1}{d} = \int_{-a}^{a} d\xi' \rho(\xi')\theta_0(\xi - \xi') ,
\]

\[
0 = 2\pi \rho(\xi) - \int_{-a}^{a} d\xi' \gamma(\xi')\theta_0'(\xi - \xi') + \int_{-a}^{a} d\xi' \rho(\xi')\theta_1'(\xi - \xi') - \int_{-\infty}^{\infty} d\beta \frac{2\sigma(\beta)}{1 + 4(\xi - \beta)^2} \quad (6.14)
\]

\[
0 = 2\pi \sigma(\beta) - \int_{-a}^{a} d\xi \rho(\xi) \frac{2}{1 + 4(\xi - \beta)^2} - \int_{-\infty}^{\infty} d\beta' \sigma(\beta') \frac{2}{1 + (\beta - \beta')^2} ,
\]

where \(a\) is a function of the fixed density \(d\) and where \(\sigma(\beta)\) and \(\gamma(\xi)\) are the infinite volume limits of the densities \(\frac{1}{N(\beta_{s+1} - \beta_l)}\) and \(\frac{\xi^{(0)}_j}{N(\xi_j^{(0)} - \xi_j^{(0)})}\). In order to describe only the ground state of (6.12) it is necessary to decouple the dynamical degrees of freedom by hand [39]. We note that for the excitations no such decoupling has to be carried out because the structure of (6.14) is such that the dynamical degrees of freedom decouple automatically. Eqns (6.14) and (6.13) are all we need to determine the S-matrix. Let us start with the spin-triplet phase shift. The spin-triplet excitation is obtained by taking \(M_1 = \frac{N}{2} - 1\) and all other \(M_k = 0\).

There are two holes with corresponding spectral parameters \(\beta_{h,j}, j = 1, 2\) in the distribution of \(\beta\)'s. The ABA equations (6.13) read (our convention is \(\bar{J}_s - J_s = \frac{1}{2}\))

\[
0 = 2\pi I_j - \sum_l \theta_0(\xi_j^{(0)} - \xi_l^{(0)})\xi_l^{(1)} - \sum_s \partial \left(2(\xi_j^{(0)} - \bar{\beta}_s)\right) + \sum_l \theta_1(\xi_j^{(0)} - \xi_l^{(0)})
\]

\[
+ \sum_{j=1}^2 \partial(2(\xi_j^{(0)} - \beta_{h,j})) - \pi \quad \text{(6.15)}
\]

\[
0 = 2\pi \bar{J}_s - \sum_l \partial(2(\bar{\beta}_s - \xi_l^{(0)})) + \sum_t \partial(\bar{\beta}_s - \bar{\beta}_t) + \pi - \sum_{j=1}^2 \partial(\bar{\beta}_s - \beta_{h,j}) .
\]

Subtracting the corresponding ground state equations from (6.15) we obtain coupled equations for the shift functions \(F_2(\beta_j) = \frac{\bar{\beta}_j - \beta_j}{\beta_{j+1} - \beta_j}\) and \(F_1(\xi_j) = \frac{\xi_l^{(1)} - \xi_l^{(0)}}{\xi_j^{(0)} - \xi_j^{(0)}}\), which in the thermody-
namic limit turn into coupled integral equations

\[ F_2(\beta) = 1 - \frac{1}{2\pi} \int_{-\infty}^{\infty} d\beta' \frac{2}{1 + (\beta - \beta')^2} F_2(\beta') - \frac{1}{2\pi} \sum_{j=1}^{2} \vartheta(\beta - \beta_{h,j}) \]

\[ 0 = -\int_{-a}^{a} d\xi' F_1(\xi') \theta_0(\xi - \xi') + \int_{-\infty}^{\infty} d\beta \frac{2}{1 + 4(\beta - \xi)^2} F_2(\beta) - \pi + \sum_{j=1}^{2} \vartheta(2(\xi - \beta_{h,j})). \]

(6.16)

Note that in order to obtain (6.16) we used the ground state equations (6.14). The equation for \( F_2 \) is readily solved by Fourier techniques

\[ F_2(\beta) = 1 - \frac{i}{2\pi} \sum_{p=1}^{2} \ln \left( \frac{\Gamma \left( \frac{1 + i\beta - \beta_{h,p}}{2} \right) \Gamma \left( 1 - i\frac{\beta - \beta_{h,p}}{2} \right)}{\Gamma \left( \frac{1 - i\beta - \beta_{h,p}}{2} \right) \Gamma \left( 1 - i\frac{\beta - \beta_{h,p}}{2} \right)} \right). \]

The triplet phase shift is \( \delta_T = 2\pi F_2(\beta_{h,1}) \) with \( \beta_{h,1} - \beta_{h,2} > 0 \) and is identical to the triplet phase shift in the nearest neighbour XXX model! The excitation energy of the triplet states is

\[ E_T = \int_{-a}^{a} d\xi \ 2\xi F_1(\xi) = -\sum_{j=1}^{2} \int_{-a}^{a} d\xi \frac{e(\xi)}{2 \cosh \left( \pi (\xi - \beta_{h,j}) \right)} \]

\[ \xi^2 - \mu = \int_{-a}^{a} d\xi' e(\xi') \theta_0(\xi - \xi'), \]

where \( e(\xi) \) is the "classical ground state energy density" of Sutherland, Römer and Shastri\cite{39}. Repeating the above steps for the spin singlet \( (M_1 = \frac{N}{2} - 2, M_2 = 1) \) we find that the excitation energy is the same as for the triplet, and the phase-shift is \( \delta_S = \delta_T + 2 \arctan \left( \beta_{h,1} - \beta_{h,2} \right) - \pi \), which results in an S-matrix identical to the nearest neighbour Heisenberg XXX S-matrix (2.12) with \( \mu = \beta_{h,1} - \beta_{h,2} \). This shows that the spinons in the nearest neighbour Heisenberg model and its \( \frac{1}{\sin^2(r)} \) analog are very similar: in both models they are interacting with the same S-matrix, the only difference is the dispersion. Our result for the S-matrix furthermore leads to the conclusion that the conformal limit of the \( \frac{1}{\sin^2(r)} \) model (6.12) is given by the \( SU(2)_1 \) WZNW conformal field theory.

7. Discussion

In this note we have determined the dressed scattering matrices for several models with long-range interactions by applying a method invented by Korepin for models solvable by (normal) Bethe Ansatz. We would like to stress that this method can be applied to any model, for which the Asymptotic Bethe Ansatz can be formulated. Our results show very directly that models with \( \frac{1}{\sin^2(r)} \) interaction are ideal gases with fractional statistics. Long-range models with \( \frac{1}{\sin^2(r)} \) interactions describe interacting elementary excitations and are
close in nature to integrable nearest-neighbour models. Our analysis in section 7 can readily be generalized from $SU(2)$ to $SU(N)$. The structure of the ABE relevant for the spin degrees of freedom is that of an inhomogeneous $SU(N)$ Sutherland model\cite{23}. On the basis of our results for $SU(2)$ we conjecture that the resulting dressed S-matrix for the $SU(N)$ spin chain with \( \frac{1}{\sinh^2(r)} \) hopping is identical to the one for the nearest neighbour model. The fact that elementary excitations in \( \frac{1}{\sinh^2(r)} \) models are interacting in basically the same way as in their nearest neighbour analogs indicates that the evaluation of correlation functions may be rather more difficult than for the \( \frac{1}{\sin^2(r)} \) case, in which elementary excitations are free.

Finally we would like to point out a close relation between fractional statistics and the fractional “charge” previously observed in many solvable models. As was first observed by Korepin for the case of the Massive Thirring model (MTM)\cite{1}, elementary excitations over the true ground state will in general carry a fractional charge. Here charge is the eigenvalue of the fermion number operator defined in terms of the (fermionic) quantum fields entering the hamiltonian. The relation to fractional statistics is most easily seen for the simple example of $SU(2)$ XXX model: the analog of charge is the third component of the spin. A one-particle excitation over the bare vacuum corresponds to flipping one spin, and thus carries “charge”, i.e. spin 1. By construction this excitation has bosonic statistics. From the discussion above we see that flipping one spin over the true (antiferromagnetic) ground state leads to a two-spinon excitation, and that one spinon thus carries “charge” \( \frac{1}{2} \), and carries fractional statistics. Analogously we can deduce that the quasiparticles with fractional charge in the MTM ought to be thought of as objects of fractional statistics as well.

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