THE DE RHAM COHOMOLOGY OF THE SUZUKI CURVES

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ABSTRACT. For a natural number \( m \), let \( S_m/\mathbb{F}_2 \) be the \( m \)th Suzuki curve. We study the 2-torsion group scheme and the Dieudonné module of \( S_m \). This is accomplished by studying the de Rham cohomology group \( H^{1}_{\text{dR}}(S_m) \). In particular, \( \text{Jac}(S_m) \) is isogenous over \( \overline{\mathbb{F}}_2 \) to a product of supersingular elliptic curves. In particular, \( \text{Jac}(S_m) \) has 2-rank 0; it has no points of order 2 over \( \mathbb{F}_2 \).

1. Introduction

The structure of the de Rham cohomology of the Hermitian curves as a representation of PGU(3, \( q \)) was studied in [2, 3, 10]. The structures of the Dieudonné module and the Ekedahl-Oort type of the Hermitian curves were determined in [18]. In this paper, we study the analogous structures for the Suzuki curves.

For \( m \in \mathbb{N} \), let \( q_0 = 2^m \), and let \( q = 2^{2m+1} \). The Suzuki curve \( S_m \) is the smooth projective connected curve over \( \mathbb{F}_2 \) given by the affine equation:

\[
z^q + z = y^{q_0}(y^q + y).
\]

It has genus \( g_m = q_0(q - 1) \).

The number of points of \( S_m \) over \( \mathbb{F}_q \) is \( \#S_m(\mathbb{F}_q) = q^2 + 1 \); which is optimal in that it reaches Serre’s improvement to the Hasse-Weil bound [12] Proposition 2.1. In fact, \( S_m \) is the unique \( \mathbb{F}_q \)-optimal curve of genus \( g_m \). Because of the large number of rational points relative to their genus, the Suzuki curves provide good examples of Goppa codes [7, Section 4.3], [8], [12].

The automorphism group of \( S_m \) is the Suzuki group \( Sz(q) \), whose order \( q^2(q - 1)(q^2 + 1) \) is very large compared with \( g_m \). The Suzuki curve \( S_m \) is the Deligne-Lusztig curve associated with the group \( Sz(q) = 2B_2(q) \) [11] Proposition 4.3.

The \( L \)-polynomial of \( S_m/\mathbb{F}_q \) is \((1 + \sqrt{2qt + qt^2})^{g_m} \) and \( S_m \) is supersingular for each \( m \in \mathbb{N} \) [11] Proposition 4.3. This implies that the Jacobian \( \text{Jac}(S_m) \) is isogenous over \( \mathbb{F}_2 \) to a product of supersingular elliptic curves. In particular, \( \text{Jac}(S_m) \) has 2-rank 0; it has no points of order 2 over \( \mathbb{F}_2 \).

The 2-torsion group scheme \( \text{Jac}(S_m)[2] \) is a BT-1-group scheme of rank \( 2^{2g_m} \). In [5], the authors show that the \( a \)-number of \( \text{Jac}(S_m)[2] \) is \( q_0(q_0 + 1)(2q_0 + 1)/6 \). However, the Ekedahl-Oort type of \( \text{Jac}(S_m)[2] \) is not known. Understanding the Ekedahl-Oort type is equivalent to understanding the structure of the de Rham cohomology or the (mod 2 reduction of) the Dieudonné module as a module under the actions of the Frobenius and Verschiebung operators.

In this paper, we study the de Rham cohomology group \( H^{1}_{\text{dR}}(S_m) \) of the Suzuki curves. Using results in [14], [9], [13], [11], we determine the multiplicity of each irreducible 2-modular representation of \( Sz(q) \) in \( H^{1}_{\text{dR}}(S_m) \) in Corollary 2.2.

Let \( D_m \) denote the (mod 2 reduction of) the Dieudonné module of \( \text{Jac}(S_m)[2] \). We identify a submodule \( D_{m,0} \) of \( D_m \) corresponding to the trivial eigenspace under the action of the cyclic group of order \( q - 1 \) on \( S_m \). It has dimension \( 2q_0 \). Using [4] Section 5, we determine the structure of \( D_{m,0} \) and its Ekedahl-Oort type in Section 3.1. This leads to the following result, where \( \mathcal{E} \) is defined in Section 3.1.

**Corollary 1.1.** (Corollary 3.7) If \( 2^m \equiv 2^e \mod 2^{e+1} + 1 \), then the \( \mathcal{E} \)-module \( \mathcal{E}/\mathcal{E}(V^{e+1} + F^{e+1}) \) occurs as an \( \mathcal{E} \)-submodule of the Dieudonné module \( D_m \) of \( S_m \). In particular,

1. \( \mathcal{E}/\mathcal{E}(V^{m+1} + F^{m+1}) \) occurs as an \( \mathcal{E} \)-submodule of \( D_m \) for all \( m \);
2. \( \mathcal{E}/\mathcal{E}(V + F) \) occurs as an \( \mathcal{E} \)-submodule of \( D_m \) if \( m \) is even; and

2010 Mathematics Subject Classification. Primary: 11G10, 11G20, 14F40, 14H40, 20C20. Secondary: 14L15, 20C33.

Key words and phrases. Suzuki curve, Suzuki group, Ekedahl-Oort type, de Rham cohomology, Dieudonné module, modular representation.
(3) $E/E(V^2 + F^2)$ occurs as an $E$-submodule of $D_m$ if $m \equiv 1 \mod 4$.

In Section 11, we explicitly compute a basis for $H^1_{dR}(S_m)$ for all $m \in \mathbb{N}$. As an application, we determine the complete structure of the Dieudonné module $D_m$ for $m = 1$ and $m = 2$ in Section 13.

Malmskog was partially supported by NSA grant H98230-16-1-0300. Pries was partially supported by NSF grant DMS-15-02227.

1.1. Notation. Due to strict page limitations, we refer to [18, Section 2] verbatim for the definitions of the Frobenius and Verschiebung operators, the $p$-torsion group scheme, the Dieudonné module, the $p$-rank and $a$-number, the Ekedahl-Oort type, and the de Rham cohomology. We use the abbreviation Dieudonné module for the $p$-torsion reduction of the Dieudonné module.

Let $k = \mathbb{F}_p$. The only non-standard notation is that $E = k[F, V]$ denotes the non-commutative ring generated by semi-linear operators $F$ and $V$ with the relations $FV = VF = 0$ and $F\lambda = \lambda F$ and $V\lambda = V\lambda$ for all $\lambda \in k$ and $E(A_1, \ldots)$ denotes the left ideal of $E$ generated by $A_1, \ldots$.

There is an equivalence of categories between the $p$-torsion group schemes of principally polarized abelian varieties of dimension $g$ and symmetric $E$-modules of dimension $2g$ over $k$. Furthermore, these can be described combinatorially by the Ekedahl-Oort type, which is a $g$-tuple $[\nu_1, \ldots, \nu_g]$.

Let $I_{i,1}$ denote the $p$-torsion group scheme of rank $p^{2t}$ having $p$-rank $0$ and $a$-number $1$. Then $I_{i,1}$ has Dieudonné module $E/E(F^2 + V^t)$ and Ekedahl-Oort type $[0, 1, \ldots, t-1]$ [17, Lemma 3.1].

In the rest of the paper, $p = 2$. The key reference is [16, Section 5], which states that there is an isomorphism of $E$-modules between the Dieudonné module of the 2-torsion group scheme $\text{Jac}(S_m)[2]$ and the de Rham cohomology group $H^1_{dR}(S_m)$.

2. The de Rham cohomology as a module for the Suzuki group

Suzuki determined the irreducible ordinary characters and representations of $Sz(q)$ [20]. The absolutely irreducible 2-modular representations of $Sz(q)$ are well-understood [11,15,19].

2.1. Modular representations of the Suzuki group. Let $q = 2^{2m+1}$. We recall some results about the 2-modular representations of the Suzuki group $Sz(q)$ from [15]. Let $V$ be the natural 4-dimensional module for $G = Sz(q)$. Let $\tau \in Sz(q)$ be an element of order $q - 1$. Let $\zeta = e^{\pi i/(q-1)}$. Let $\theta \in \text{Aut}(\mathbb{F}_q)$ be such that $\theta^2(\alpha) = \alpha^2$ for all $\alpha \in \mathbb{F}_q$, i.e., $\theta$ is the square root of Frobenius. Then $Sz(q)$ has an irreducible 4-dimensional 2-modular representation $V_0$ in which $\tau \mapsto M$, where $M \in \text{GL}_4(\mathbb{F}_q)$ is the matrix

\[
M = \begin{pmatrix}
\zeta^{\theta + 1} & 0 & 0 & 0 \\
0 & \zeta & 0 & 0 \\
0 & 0 & \zeta^{-1} & 0 \\
0 & 0 & 0 & \zeta^{-(\theta + 1)}
\end{pmatrix}.
\]

For $0 \leq i \leq 2m$, consider the automorphism $\alpha_i$ of $G$ induced by the automorphism $x \mapsto x^{2^i}$ of $\mathbb{F}_q$. Let $V_i$ be the 4-dimensional $\mathbb{F}_q Sz(q)$-module where $g \in Sz(q)$ acts as $g^{\alpha_i}$ on $V$.

Let $I$ be a subset of $N = \mathbb{Z}/(2m+1)\mathbb{Z}$. Define $V_I = \bigoplus_{i \in I} V_i$, with $V_0$ being the trivial module. Then $V_I$ is an absolutely irreducible 2-modular representation of $Sz(q)$. By [15, Lemma 1], if $I \neq J$ then $V_I$ and $V_J$ are geometrically non-isomorphic and $\{V_I \mid I \subset N\}$ is the complete set of simple $\mathbb{F}_2 Sz(q)$-modules. Note that $V_I$ has dimension $4^{|I|}$ and that $V_N$ is the Steinberg module.

Note that $\bigoplus_{I \subset N} V_I$ is an $\mathbb{F}_2 Sz(q)$-module if and only if $I$ is invariant under the Frobenius map $x \mapsto x^2$ or, equivalently, if and only if $\{I \mid I \subset \mathbb{Z}\}$ is invariant under the translation $i \mapsto i + 1$.

For $i \in N$, let $\phi_i$ denote the Brauer character associated to the 4-dimensional module $V_i$. For $I \subseteq N$, let $\phi_I = \prod_{i \in I} \phi_i$, so $\phi_I$ is the character associated to the module $V_I$. Then $\{\phi_{I} : I \subseteq N\}$ is a complete set of Brauer characters for $Sz(q)$.

By [13, Theorem 1.7], adapted from [1, Theorem 3.4], $\phi_I^2 = 4 + 2\phi_{i+m+1} + \phi_{i+1}$. Using this relation, Liu constructs a graph with vertex set $N$ and edge set $\{(i, i+1) \mid i \in N\}$. Edges of the form $(i, i+1)$ are called short edges and edges of the form $(i, i+1+m)$ are called long edges. Two vertices $i, j$ are called adjacent if they are connected by a long edge, i.e., if $i - j \equiv \pm m \mod 2m + 1$. A set $I' \subseteq N$ is called circular if no vertices of $I' = N \setminus I'$ are adjacent. A set $I \subseteq S$ is called good if $I' = N \setminus I$ is circular.
2.2. Modular representation of the de Rham cohomology. The de Rham cohomology $H^1_{\text{dR}}(S_m)$ is an $\mathbb{F}_q[S_3(q)]$-module with dimension $2g_m = 2q_0(q-1)$. It is the reduction modulo 2 of the crystalline cohomology. We consider the decomposition of $H^1_{\text{dR}}(S_m)$ into irreducible 2-modular representations of the Suzuki group $Sz(q)$.

Consider the following four unipotent representations of $Sz(q)$. Let $W_N$ denote the Steinberg representation of dimension $q^2$. Let $W_0$ be the trivial representation of dimension 1. Let $W_+$ and $W_-$ be the irreducible representations of $Sz(q)$ associated to the two ordinary characters of $Sz(q)$ of degree $q_0(q-1)$.

Then $W_+$ and $W_-$ each have dimension $q_0(q-1)$.

In [14] Theorem 6.1, Lusztig studied the compactly supported ℓ-adic cohomology of the affine Deligne-Lusztig curve. He proved that $W_N$, $W_+$, $W_-$, $W_0$ are the eigenspaces under Frobenius and that each appears with multiplicity 1. In [9] pages 2535-2536, Gross uses this to prove that

$$H^1(S_m) = W_+ \oplus W_-.$$ (2.1)

The decompositions of $W_+$ and $W_-$ into irreducible 2-modular representations are known.

Theorem 2.1. [13] Theorem 3.4] The irreducible 2-modular representation $V_i$ appears in $W_{\pm}$ if and only if $i$ is good, i.e., if and only if there do not exist $i, j \in I$ such that $j-i \equiv \pm m \mod 2m+1$. In this case, the multiplicity of $V_i$ in $W_{\pm}$ is $2^{m-|I|}$.

Theorem 2.1 and (2.1) thus yield the following result.

Corollary 2.2. The irreducible 2-modular representation $V_i$ appears in $H^1_{\text{dR}}(S_m)$ if and only if there do not exist $i, j \in I$ such that $j-i \equiv \pm m \mod 2m+1$. If $V_i$ appears in $H^1_{\text{dR}}(S_m)$ then its multiplicity is $2^{m+1-|I|}$.

Thus

$$H^1_{\text{dR}}(S_m) = \bigoplus_{i \text{ good}} V_i^{2^{m+1-|I|}}.$$ (2.2)

Example 2.3. When $m = 1$, then $H^1_{\text{dR}}(S_m) = (V_0 \oplus V_1 \oplus V_2)^2$.

Example 2.4. When $m = 2$, then

$$H^1_{\text{dR}}(S_m) = (V_0 \oplus V_1 \oplus V_2 \oplus V_3 \oplus V_4)^2 \oplus (V_{\{0,1\}} \oplus V_{\{1,2\}} \oplus V_{\{2,3\}} \oplus V_{\{3,4\}} \oplus V_{\{4,0\}})^2.$$ (2.2)

Remark 2.5. For small $m$, Corollary 2.2 can be verified using the multiplicity of the eigenvalues for Frobenius on $H^1_{\text{dR}}(S_m)$ as an $\mathbb{F}_q[S_3(q)]$-module.

3. A submodule of the de Rham cohomology

The chosen element $\tau \in Sz(q)$ of order $q-1$ acts on $H^1_{\text{dR}}(S_m)$. The eigenvalues for the action of $\tau$ are the $q-1$ distinct powers of $\zeta = e^{2\pi i/(q-1)}$. Let $L_i$ be the eigenspace with eigenvalue $\zeta^i$.

The Frobenius $F$ commutes with the action of $\tau$. So $F$ acts on $\{L_i \mid 1 \leq i \leq q-1\}$, taking $L_i$ to $L_{2i}$. In particular, the trivial eigenspace $L_0$ is invariant under $F$.

The non-trivial eigenspaces are each in an orbit of length $2m+1$ under $F$. This is because the orbit of $L_i$ contains $L_j$ if and only if $j \equiv 2^m i \mod q-1$. If $i \neq 0$, the orbits have length $2m+1$ because $i, 2i, 4i, \ldots, 2^m i$ are distinct modulo $q-1$ if $i \neq 0$. One can show that $L_i$ has dimension $2q_0$.

3.1. The trivial eigenspace. The eigenspace $L_0$ is the subspace of $H^1_{\text{dR}}(S_m)$ of elements fixed by $\tau$. Since $\tau$ acts fixed point freely on the 4-dimensional module $V_i$ for each $i$ [13] proof of Lemma 3], the generators of $H^1_{\text{dR}}(S_m)$ which are fixed by $\tau$ are exactly those in $V_i$ for $I = \emptyset$. In other words, the representation for $L_0$ consists of the $2^{m+1} = 2^{q_0}$ copies of the trivial representation in (2.2).

Let $C_{m,0}$ be the quotient curve of $S_m$ by the subgroup $\langle \tau \rangle$. This is a hyperelliptic curve of genus $q_0$ [8 Theorem 6.9]. The de Rham cohomology $H^1_{\text{dR}}(C_{m,0})$ of $C_{m,0}$ is isomorphic as an $\mathbb{E}$-module to $L_0$.

Let $D_{m,0}$ be the Dieudonné module of $C_{m,0}$ and let $G_{m,0}$ be the 2-torsion group scheme $\text{Jac}(C_{m,0})[2]$.

Proposition 3.1. The $a$-number of $G_{m,0}$ is $2^{m-1}$ and the Ekedahl-Oort type is $[0,1,1,2,2,\ldots,q_0-1,q_0]$.

Proof. Since $S_m$ has 2-rank 0, so does $C_{m,0}$. The result follows from [4 Corollary 5.3].

The structure of the Dieudonné module for the Ekedahl-Oort type $[0,1,1,2,2,\ldots,q_0-1,q_0]$ is determined in [4] Section 5].
Proposition 3.2. [4] Proposition 5.8] The Dieudonné module $D_{m,0}$ is the $E$-module generated as a $k$-vector space by $\{X_1, \ldots, X_{q_0}, Y_1, \ldots, Y_{q_0}\}$ with the actions of $F$ and $V$ given by:

1. $F(Y_j) = 0$.
2. $V(Y_j) =$ \begin{cases} 
    Y_{2j} & \text{if } j \leq q_0/2, \\
    0 & \text{if } j > q_0/2.
  \end{cases}
3. $F(X_j) =$ \begin{cases} 
    X_{j/2} & \text{if } j \text{ is even}, \\
    Y_{q_0-(j-1)/2} & \text{if } j \text{ is odd}.
  \end{cases}
4. $V(X_j) =$ \begin{cases} 
    0 & \text{if } j \leq (q_0 - 1)/2, \\
    -Y_{2q_0-2j+1} & \text{if } j > (q_0 - 1)/2.
  \end{cases}

By [4] Proposition 5.10, $D_{m,0}$ has $2^{m-1}$ generators subject to some complicated relations.

Notation 3.3. [4] Notation 5.9] Fix $c = q_0 \in \mathbb{N}$. Consider the set $I = \{j \in \mathbb{N} \mid \lceil (c + 1)/2 \rceil \leq j \leq c\}$, which has cardinality $\lceil (c + 1)/2 \rceil$. For $j \in I$, let $\ell(j)$ be the odd part of $j$ and let $e(j) \in \mathbb{Z}_{\geq 0}$ be such that $j = 2^e(j)\ell(j)$. Let $s(j) = c - (\ell(j) - 1)/2$. Then $s(j) | j \in I = I$. Also, let $m(j) = 2c - 2j + 1$ and let $e(j) \in \mathbb{Z}_{\geq 0}$ be such that $t(j) := 2^e(j)m(j) \in I$. Then $\{t(j) \mid j \in I\} = I$. Thus, there is a unique bijection $\iota: I \to I$ such that $\iota(j) = s(j)$ for each $j \in I$.

Proposition 3.4. [4] Proposition 5.10] The set $\{X_j \mid j \in I\}$ generates the Dieudonné module $D_{m,0}$ as an $E$-module subject to the relations: $F^{e(j)+1}(X_j) + V^{e(j)+1}(X_{\iota(j)})$ for $j \in I$.

Example 3.5. (1) When $m = 1$ and the Ekedahl-Oort type is $[0, 1]$, then $L_0 = I_{2,1}$ which has Dieudonné module $D_{1,0} = E/E(F^2 + V^2)$.

(2) When $m = 2$ and the Ekedahl-Oort type is $[0, 1, 2]$, then $L_0 = I_{1,1} \oplus I_{3,1}$ which has Dieudonné module $D_{2,0} = E/E(F + V) \oplus E/E(F^3 + V^3)$.

The next result determines some of the $E$-submodules of $D_{m,0}$ for general $m$.

Proposition 3.6. The $E$-module $E/(V^{ce+1} + F^{ce+1})$ occurs as an $E$-submodule of the Dieudonné module $D_{m,0}$ if and only if $2^m \equiv 2^e \pmod{2^{e+1} + 1}$. In particular:

1. $E/(V^{m+1} + F^{m+1})$ occurs for all $m$;
2. $E/(V + F)$ occurs if and only if $m$ is even; and
3. $E/(V^2 + F^2)$ occurs if and only if $m \equiv 1 \pmod{4}$.

Proof. Let $e \in \mathbb{Z}_{\geq 0}$. By Proposition 5.4 the relation $(V^{ce+1} + F^{ce+1})X_j = 0$ is only possible if $j = 2^e\ell$ where $\ell$ is odd. Write $s(j) = c - (\ell - 1)/2$. Then $F^{e+1}(X_j) = F(X_{\ell}) = Y_{s(j)}$. Now $V(X_j) = -Y_{m(j)}$ where $m(j) = 2c - 2j + 1$. Also $V^{e+1}(X_j) = 2^em(j)$. Thus we need $s(j) = 2^em(j)$. This is equivalent to $2^e\ell - (j - 2^e) = 2^{e+1}(2c - 2j + 1)$, which is equivalent to

$$j = \frac{c2^{e+1}(2^{e+1} - 1) + 2^e(2^{2e+1} - 1)}{2^{2e+2} - 1} = \frac{c2^{e+1} + 2^e}{2^{e+1} + 1}. $$

This value of $j$ is integral if and only if $c \equiv 2^e \pmod{2^{e+1} + 1}$. Thus, the relation $(V^{ce+1} + F^{ce+1})X_j = 0$ occurs if and only if $2^m \equiv 2^e \pmod{2^{e+1} + 1}$ and $j = (2^{e+1}q_0 + 2^e)/(2^{e+1} + 1)$. In particular, one checks that:

1. $(V^{m+1} + F^{m+1})X_{2m} = 0$;
2. the relation $(V + F)X_j = 0$ occurs if and only if $m$ is even and $j = (2 \cdot 2^m + 1)/3$;
3. the relation $(V^2 + F^2)X_j = 0$ occurs if and only if $m \equiv 1 \pmod{4}$ and $j = (4 \cdot 2^m + 2)/5$.

Here is the main result of the paper.

Corollary 3.7. If $2^m \equiv 2^e \pmod{2^{e+1} + 1}$, then the $E$-module $E/E/(V^{ce+1} + F^{ce+1})$ occurs as an $E$-submodule of the Dieudonné module $D_m$ of $S_m$. In particular,

1. $E/E/(V^{m+1} + F^{m+1})$ occurs as an $E$-submodule of $D_m$ for all $m$;
2. $E/(V + F)$ occurs as an $E$-submodule of $D_m$ if $m$ is even; and
3. $E/(V^2 + F^2)$ occurs as an $E$-submodule of $D_m$ if $m \equiv 1 \pmod{4}$.

Proof. By Proposition 5.6 $E/E/(V^{ce+1} + F^{ce+1})$ occurs as an $E$-submodule of the Dieudonné module $D_{m,0}$. The result follows since $L_0$ is an $F$-invariant subspace of $H^1(S_m)$.
3.2. A conjecture. We include a conjecture about the $E$-module structure of the Dieudonné module of $S_m$, which is motivated by Corollary 2.2.

Consider the following 2-modular representation of $S_4(q)$. Let $I_m = \{0, \ldots, m - 1\}$. Let $W_m = \oplus_{i=0}^{2m} F_i(V_{I_m})$. Then $\dim(W_m) = (2m + 1)4^m$. For example, $W_1 = V_0 \oplus V_1 \oplus V_2$ and

$$W_2 = (V_0 \oplus V_1) \oplus (V_1 \oplus V_2) \oplus (V_2 \oplus V_3) \oplus (V_3 \oplus V_4) \oplus (V_4 \oplus V_0).$$

By Corollary 2.2, the representation $W_m$ appears with multiplicity 2 in $H^1_{dR}(S_m)$. We conjecture that $W_m$ corresponds to an indecomposable factor of $D_m$ with structure $E/E(F^{2m+1} + V^{2m+1})$.

Conjecture 3.8. The multiplicity of $E/E(F^{2m+1} + V^{2m+1})$ in the Dieudonné module $D_m$ of $\text{Jac}(S_m)[2]$ is $4^m$.

We verify this conjecture for $m = 1$ and $m = 2$ in Section 4.3.

4. AN EXPLICIT BASIS FOR THE DE RHAM COHOMOLOGY

In this section, we compute an explicit basis for $H^1_{dR}(S_m)$ with the goal of describing the action of $F$ and $V$ on the basis elements. As an application, we determine the Dieudonné module of $S_m$ when $m = 1$ and $m = 2$ in Section 4.3.

4.1. Preliminaries. Let $P_\infty$ be the point at infinity on $S_m$. Let $P_{(y,z)}$ denote the point $(y, z)$ on $S_m$. Define the functions $h_1, h_2 \in F_2(S_m)$ by:

$$h_1 := z^{2q_0} + y^{2q_0+1}, \quad h_2 := z^{2q_0}y + h_1^{2q_0}.$$

Lemma 4.1. (1) The function $y$ has divisor

$$\text{div}(y) = \sum_{z \in F_q} P_{(0,z)} - qP_\infty.$$

(2) The function $z$ has divisor

$$\text{div}(z) = \sum_{y \in F_q^*} P_{(y,0)} + (q_0 + 1)P_{(0,0)} - (q + q_0)P_\infty.$$

(3) Let $S = \{(y, z) \in F_q^2 : y^{2q_0+1} = z^{2q_0}, (y, z) \neq (0, 0)\}$. The function $h_1$ has divisor

$$\text{div}(h_1) = \sum_{(y,z) \in S} P_{(y,z)} + (2q_0 + 1)P_{(0,0)} - (q + 2q_0)P_\infty.$$

(4) The function $h_2$ has divisor

$$\text{div}(h_2) = (q + 2q_0 + 1)(P_{(0,0)} - P_\infty).$$

Proof. The pole orders of these functions are determined in [12, Proposition 1.3]. The orders of the zeros can be determined using the equation for the curve and the definitions of $h_1$ and $h_2$. □

Let $E_m$ be the set of $(a, b, c, d) \subset \mathbb{Z}^4$ satisfying

$$0 \leq b \leq 1, \quad 0 \leq c \leq q_0 - 1, \quad 0 \leq d \leq q_0 - 1,$$

$$aq + b(q + q_0) + c(q + 2q_0) + d(q + 2q_0 + 1) \leq 2g - 2.$$

Lemma 4.2. The following set is a basis of $H^0(S_m, \Omega^1)$:

$$B_m := \{g_{a,b,c,d} := y^az^bh_1^ch_2^d dy \mid (a, b, c, d) \in E_m\}.$$

Proof. See [5, Proposition 3.7]. □

A basis for $H^1(S_m, \mathcal{O})$ can be built similarly.

Lemma 4.3. The following set is a basis of $H^1(S_m, \mathcal{O})$:

$$A_m := \left\{f_{a,b,c,d} := \frac{1}{y^az^bh_1^ch_2^d} \left| h_1^{q_0-1}h_2^{q_0-1} \right| y \mid (a, b, c, d) \in E_m\right\}.$$
In other words, these functions have a pole at \( d\kappa \) where

\[
0 \to Q \to \alpha \to \gamma \to 0.
\]

Constructing the de Rham cohomology.

Let \( U_\infty = S_m \setminus \pi^{-1}(\infty) \) and \( U_0 = S_m \setminus \pi^{-1}(0) \). The elements of \( H^1(S_m, \mathcal{O}) \) can be represented by classes of functions that are regular on \( U_\infty \cap U_0 \), but are not regular on \( U_\infty \) or regular on \( U_0 \). In other words, these functions have a pole at \( P_\infty \) and at some point in \( \pi^{-1}(0) \).

Let \( f = f_{a,b,c,d} \) for some \( (a, b, c, d, d) \in \mathcal{E}_m \). Then \( f \) has poles only in \( \{ P_\infty, \pi^{-1}(0) \} \) by Lemma \[\text{[4.3]}\]. Let \( Q = (0, \alpha) \) for some \( \alpha \in \mathbb{F}_q^\times \). Then \( v_Q(f) = -(a + 1) \leq -1 \).

So \( f \) is regular on \( U_\infty \cap U_0 \) but not on \( U_\infty \) or \( U_0 \). By a calculation similar to \[\text{[5, Proposition 3.7]}\], the elements of \( \mathcal{A}_m \) are independent because each element has a different pole order at \( P_\infty \). The cardinality of \( \mathcal{A}_m \) is \( g_m = \dim(H^1(S_m, \mathcal{O})) \). Thus \( \mathcal{A} \) is a basis for \( H^1(S_m, \mathcal{O}) \).

\[\square\]

4.1.1. Constructing the de Rham cohomology. Let \( \mathcal{U} \) be the open cover of \( S_m \) given by \( U_\infty \) and \( U_0 \) in the previous proof. For a sheaf \( \mathcal{F} \) on \( S_m \), let

\[
C^0(\mathcal{U}, \mathcal{F}) := \{ g = (g_\infty, g_0) \mid g_i \in \Gamma(U_i, \mathcal{F}) \},
\]

\[
C^1(\mathcal{U}, \mathcal{F}) := \{ \phi \in \Gamma(U_\infty \cap U_0, \mathcal{F}) \}.
\]

Define the coboundary operator \( \delta : C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F}) \) by \( \delta g = g_\infty - g_0 \).

Then the closed de Rham cocycles are the set

\[
Z^1_{\text{dR}}(\mathcal{U}) := \{ (f, g) \in C^1(\mathcal{U}, \mathcal{O}) \times C^0(\mathcal{U}, \Omega^1) \mid df = \delta g \},
\]

that is, where \( df = g_0 - g_\infty \). The de Rham coboundaries are the set

\[
B^1_{\text{dR}}(S_m) := \{ (\delta \kappa, d\kappa) \in Z^1_{\text{dR}}(\mathcal{U}) : \kappa \in C^0(\mathcal{U}, \mathcal{O}) \},
\]

where \( d\kappa = (d(\kappa_0), d(\kappa_\infty)) \). In other words, coboundaries are cocycles that arise from functions regular on \( U_0 \cap U_\infty \) which are the difference of functions which are each regular on one of the open covering sets \( U_i \).

Then the de Rham cohomology \( H^1_{\text{dR}}(S_m) \) is given by

\[
H^1_{\text{dR}}(S_m) \cong H^1_{\text{dR}}(S_m)(\mathcal{U}) := Z^1_{\text{dR}}(\mathcal{U}) / B^1_{\text{dR}}(\mathcal{U}).
\]

There is an injective homomorphism \( \lambda : H^0(S_m, \Omega^1) \to H^1_{\text{dR}}(S_m) \) denoted informally by \( g \mapsto (0, g) \), where the second coordinate is a tuple \( g = (g_\infty, g_0) \) defined by \( g_i = g|_{U_i} \). Define another homomorphism \( \gamma : H^1_{\text{dR}}(S_m) \to H^1(S_m, \mathcal{O}) \) with \( (f, g) \mapsto f \). These create a short exact sequence

\[
0 \to H^0(S_m, \Omega^1) \xrightarrow{\lambda} H^1_{\text{dR}}(S_m) \xrightarrow{\gamma} H^1(S_m, \mathcal{O}) \to 0.
\]

Let \( A \) be a basis for \( H^1(S_m, \mathcal{O}) \) and \( B \) a basis for \( H^0(S_m, \Omega^1) \). A basis for \( H^1_{\text{dR}}(S_m) \) is then given by \( \psi(A) \cup \lambda(B) \), where \( \psi \) is defined as follows. Given \( f \in H^1(S_m, \mathcal{O}) \), one can write \( df = df_\infty + df_0 \), where \( df_i \in \Gamma(U_i, \Omega^1) \) for \( i \in \{0, \infty\} \). For convenience, define \( d_f = (df_\infty, df_0) \). Define a section \( \psi : H^1(S_m, \mathcal{O}) \to H^1_{\text{dR}}(S_m) \) of \( \lambda \) by \( \psi(f) = (f, df) \). The image of \( \psi \) is a complement in \( H^1_{\text{dR}}(S_m) \) to \( \lambda(H^0(S_m, \Omega^1)) \) in \( H^1(S_m, \mathcal{O}) \).
4.1.2. The Frobenius and Verschiebung operators. The Frobenius $F$ and Verschiebung $V$ act on $H^1_{dR}(S_m)$ by

$$F(f, g) := (f^p, (0, 0)) \text{ and } V(f, g) := (0, \mathcal{C}(g))$$

where $\mathcal{C}$ is the Cartier operator, which acts componentwise on $g$. The Cartier operator is defined by the properties that it annihilates exact differentials, preserves logarithmic differentials, and is $p^{-1}$-linear. It follows from the definitions that

$$\ker(F) = \lambda(H^0(S_m, \Omega^1)) = \text{im}(V).$$

4.2. The case $m = 1$. When $m = 1$, then $g_0 = 2$, $q = 8$, $g = 14$, and $2g - 2 = 26$. The Suzuki curve $S_1$ has affine equation

$$z^8 + z = y^2(y^8 + y).$$

The set $\mathcal{E}_1$ consists of the 14 tuples

$$\mathcal{E}_1 = \{(0, 0, 0, 0), (0, 0, 0, 1), (0, 0, 1, 0), (0, 0, 1, 1), (0, 1, 0, 0), (0, 1, 0, 1), (0, 1, 1, 0), (1, 0, 0, 0), (1, 0, 0, 1),$$

$$
(1, 0, 1, 0), (1, 1, 0, 0), (2, 0, 0, 0), (2, 1, 0, 0), (3, 0, 0, 0)\}.$$

By Lemmas 4.3 and 4.2 $B_1$ is a basis for $H^0(S_1, \Omega^1)$ and $A_1$ is a basis for $H^1(S_1, \mathcal{O})$. Based on the action of Frobenius and Verschiebung, the following sets make more convenient bases:

**Lemma 4.4.**

1. The set

$$A = \{f(0,0,0,0), f(2,0,0,0), f(0,1,0,0) + f(3,0,0,0), f(2,1,0,0) + f(0,0,0,1), f(0,0,0,1) + f(1,0,1,0),$$

$$f(1,0,0,0), f(2,1,0,0), f(1,0,1,1), f(0,0,1,1), f(1,0,0),$$

$$f(3,0,0,0), f(1,0,1,0), f(1,1,0,0), f(0,1,0,1)\}$$

is a basis for $H^1(S_1, \mathcal{O})$.

2. The set

$$B = \{g(0,0,0,0), g(2,0,0,0), g(0,1,0,0) + g(3,0,0,0), g(2,1,0,0) + g(0,0,0,1), g(0,0,0,1) + g(1,0,1,0),$$

$$g(1,0,0,0), g(2,1,0,0), g(1,0,1,1), g(0,0,1,1), g(1,0,1,0),$$

$$g(3,0,0,0), g(1,1,0,0), g(0,1,0,1)\}$$

is a basis for $H^0(S_1, \Omega^1)$.

**Proof.** Using Lemmas 4.3 and 4.2 these functions have distinct pole orders at $P_\infty$ and are therefore linearly independent. They each span a subspace of dimension 14 and thus each form a basis. \qed

It is now possible to calculate the action of $F$ and $V$ on $\psi(A) \cup \lambda(B)$, a basis for $H^1_{dR}(S_m)$.

4.2.1. The action of Frobenius when $m = 1$. The action of $F$ is summarized in the right column of Table 4.2. Note that $F(g) = 0$ for $g \in B$ since $\ker(F) = \text{im}(V) \cong H^0(S_1, \Omega^1)$. For the action of $F$ on $\psi(f)$ for $f \in A$, note that $F(\psi(f)) = (f^2, (0, 0))$. Then

$$f^2 = (f(a,b,c,d))^2 = (y^{-1-a}z^{-b}h_1^{-c}h_2^{-d})^2 = (y^{-2})^{1-a}(yb_1 + h_2)^{1-b}(z + y^3)^{1-c}(h_1 + zy^2)^{1-d}.$$ 

To do these calculations, we simplify $f^2$ and write it as a sum of quotients of monomials in $\{y, z, h_1, h_2\}$. These monomials can then be classified as belonging to $\Gamma(U_0)$ or $\Gamma(U_\infty)$, or can otherwise be rewritten in terms of the basis for $H^1(S_1, \mathcal{O})$. It is then possible to use coboundaries to write $(f^2, (0, 0))$ in terms of the given basis for $H^1_{dR}(S_1)$.

**Example 4.5.** To compute that $\lambda(\psi(1,0,1,0)) = \lambda(g(0,0,0,0))$, note first that

$$(f(0,1,0,1))^2 = y^{-2}(z + y^3) = \frac{z}{y^2} + y.$$ 

Also,

$$d \left( \frac{z}{y^2} \right) = \frac{1}{y^2}dz - 2\frac{z}{y^2}dy = dy \text{ and } d(y) = dy.$$
Since \( y \in \Gamma(U_\infty, \mathcal{O}) \) and \( \frac{z}{y} \in \Gamma(U_0, \mathcal{O}) \), the pair \( \left( \frac{z}{y}, y \right) \) is in \( C^0(\mathcal{U}, \mathcal{O}) \) and \( (\frac{z}{y} + y, (dy, dy)) \) is a coboundary. Thus

\[
F \left( \psi \left( f_{(0,1,0,1)} \right) \right) = \left( \frac{z}{y} + y, (0,0) \right) + \left( \frac{z}{y^2} + y, (dy, dy) \right) = (0, (dy, dy)) = \lambda(dy) = (0, g_{(0,0,0,0)}).
\]

**Example 4.6.** We compute that \( F \left( \psi(f_{(0,0,1,1)}) \right) = \psi \left( f_{(0,1,0,1)} \right) \). This is true because

\[
(f_{(0,0,1,1)})^2 = y^{-2}(y h_1 + h_2) = \frac{h_1}{y} + \frac{h_2}{y^2}.
\]

Note that \( \frac{z}{y} \in \Gamma(U_0, \mathcal{O}) \), so \( (\frac{z}{y}, 0) \in C^0(\mathcal{U}, \mathcal{O}) \), and \( d \left( \frac{z}{y} \right) = \frac{z}{y^2} dy \). So \( \left( \frac{h_1}{y}, (\frac{z}{y} dy, 0) \right) \) is a coboundary. Also, \( d \left( \frac{h_1}{y} \right) = \frac{z^4}{y^4} dy \). Thus

\[
F \left( \psi \left( f_{(0,0,1,1)} \right) \right) = \left( \frac{h_1}{y} + \frac{h_2}{y^2}, (0,0) \right) + \left( \frac{h_2}{y^2}, \left( \frac{z^4}{y^4} dy, 0 \right) \right) = \left( \frac{h_1}{y}, \left( \frac{z^4}{y^4} dy, 0 \right) \right) = \psi \left( f_{(0,1,0,1)} \right).
\]

4.2.2. The action of Verschiebung when \( m = 1 \). The action of \( V \) is summarized in the middle column of Table 2. In \[5\], the authors calculate the action of the Cartier operator \( \mathcal{C} \) (see Table 1). This determines the action of \( V \) on \( \lambda(g) \) for \( g \in B \). It also helps determine the action of \( V \) on \( \psi(f) \) for \( f \in A \).

**Example 4.7.** We compute that \( V \left( \psi(f_{(0,1,0,1)}) \right) = (0, 0) \). Writing \( f = f_{(0,1,0,1)} = \frac{h_1}{y} = \frac{z^4}{y} + y^4 \), then

\[
df = \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = \left( -\frac{z^4}{y^2} + 4y^3 \right) dy + 4z^3 dy = \frac{z^4}{y^2} dy.
\]

Considering the pole orders of \( y, z, \) and \( dy \), define \( df = df_0 \in \Omega_0 \) and \( df_\infty = 0 \), so \( df = (0, df) \). Thus \( \mathcal{C}(df) = \frac{z^4}{y^2} \mathcal{C}(dy) = 0 \). Thus \( \mathcal{C}(df) = (0, 0) = 0 \) and \( V(\psi(f_{(0,1,0,1)})) = (0, 0) \).

**Example 4.8.** We compute that \( V \left( \psi(f_{(2,1,0,0)}) \right) = (0, g_{(0,1,0,0)}) \). This is because

\[
f_{(2,1,0,0)} = \frac{h_1 h_2}{y^3},
\]

so

\[
df = y^{-3} d(h_1 h_2) + y^{-4} h_1 h_2 dy = y^{-3} h_1 d(h_2) + y^{-3} h_2 d(h_1) + y^{-4} h_1 h_2 dy.
\]

Then

\[
d(h_1) = d(z^4 + y^3) = y^4 dy \quad \text{and} \quad d(h_2) = d(z^4 + h_1^4) = z^4 dy,
\]

so

\[
df = y^{-3} z^4 h_1 dy + y h_2 dy + y^{-4} h_1 h_2
= y^{-3} h_1 (h_1 + y^5) dy + y h_2 dy + y^{-4} h_1 h_2
= \frac{h_1^2}{y^3} dy + \frac{h_1 h_2}{y^4} dy + y^2 h_1 dy + y h_2 dy,
\]

using the fact that \( z^4 = h_1 + y^5 \). Considering the poles orders, define \( df_0 = \frac{h_1^2}{y^3} dy + \frac{h_1 h_2}{y^4} dy \in \Omega_0 \) and \( df_\infty = y^2 h_1 dy + y h_2 dy \in \Omega_\infty \). Using Table 1 and the fact that \( h_1^2 = z + y^3 \), then

\[
\mathcal{C}(df_\infty) = y \mathcal{C}(h_1 dy) + \mathcal{C}(yh_2 dy)
= y^3 dy + h_1^2 dy = (y^3 + z + y^3) dy = zdy.
\]

Thus \( V \left( \psi(f_{(2,1,0,0)}) \right) = (0, g_{(0,1,0,0)}) \).

The actions of \( F \) and \( V \) are summarized in Table 2.
### Table 1. Cartier Operator on $H^0(S_1, \Omega^1)$

| $f$ | $\mathcal{C}(f \, dy)$ |
|-----|---------------------|
| 1   | 0                   |
| $y$ | $dy$                |
| $z$ | $y^{\psi_1/2} \, dy$|
| $h_1$ | $y^{\psi_0} \, dy$ |
| $h_2$ | $(y h_1)^{\psi_0/2} + h_2) \, dy$ |
| $y h_1$ | $(y h_1)^{\psi_0/2} + h_2) \, dy$ |
| $z h_1$ | $(y h_2)^{\psi_0/2} \, dy$ |
| $z h_2$ | $(h_1 h_2)^{\psi_0/2} \, dy$ |
| $h_1 h_2$ | $(h_1 + z y^{\psi_0}) \, dy$ |
| $y z h_1$ | $(y^{\psi_0/2} z + (h_1 h_2)^{\psi_0/2}) \, dy$ |
| $y z h_2$ | $(z h_1^{\psi_0/2} + y^{\psi_0} + h_2^{\psi_0/2}) \, dy$ |
| $z h_1 h_2$ | $(z y^{\psi_0/2} h_2^{\psi_0/2} + h_1^{\psi_0/2} + 1) \, dy$ |
| $y h_1 h_2$ | $(y h_1)^{\psi_0/2} z + h_2^{\psi_0/2} \, dy$ |
| $y z h_1 h_2$ | $(y h_1)^{\psi_0/2} h_2 + z h_1^{\psi_0/2} h_2^{\psi_0/2} \, dy$ |

#### 4.3. Applications

As an application, we determine the Dieudonné modules of $S_1$ and $S_2$.

**Proposition 4.9.** When $m = 1$, then the Dieudonné module of $S_1$ is

$$D_1 = \mathbb{E}/\mathbb{E}(F^2 + V^2) \oplus (\mathbb{E}/\mathbb{E}(F^3 + V^3))^4.$$

**Proof.** The Dieudonné module is isomorphic as an $\mathbb{E}$-module to $H^1_{dR}(S_1)$. Examining Table 1, we find that $H^1_{dR}(S_1)$ has a summand of rank 4 generated by $X_1 = \psi(f(1,0,1,0))$ with relation $(F^2 + V^2) X_1 = 0$. There are 4 summands of rank 6 generated by $X_2 = \psi(f(2,1,0,0)), X_3 = \psi(f(2,0,0,0)), X_4 = \psi(f(3,0,0,0))$, and $X_5 = \psi(f(0,0,0,0))$ with the relations $(F^3 + V^3) X_1 = 0$. This yields the $\mathbb{E}$-module structure $\mathbb{E}/\mathbb{E}(F^2 + V^2) \oplus (\mathbb{E}/\mathbb{E}(F^3 + V^3))^4$. \qed

Note that the trivial eigenspace $D(L_0)$ appears as the summand $\mathbb{E}/(F^2 + V^2)$. It is spanned by $\{\psi(f(1,0,1,0)), \psi(f(0,0,0,1) + f(1,0,1,0)), \psi(f(1,0,1,0)), \psi(f(0,0,0,1) + f(1,0,1,0))\}$.

We determined the structure of the Dieudonné module of $S_2$ by implementing the explicit computation of $F$ and $V$ on a basis for $H^1_{dR}(S_2)$ using the computer package MAGMA. Write $\overline{F} = F^{-1}$ and consider the rank 20 group scheme $Z$ given by the word $\overline{F}^4 V^3 \overline{F}^3 V^4 \overline{F}^4 V^3$. Let $\mathbb{E}(Z)$ denote the Dieudonné module corresponding to $Z$.

**Proposition 4.10.** When $m = 2$, then the Dieudonné module of $S_2$ is

$$D_2 = (\mathbb{E}/\mathbb{E}(F^5 + V^5))^4 \oplus (\mathbb{E}/\mathbb{E}(F^5 + V^5))^4 \oplus (\mathbb{E}/\mathbb{E}(F^5 + V^5))^4 \oplus (\mathbb{E}/\mathbb{E}(F^5 + V^5))^4.$$
Table 2. Action of Vershiebung and Frobenius on $H^1_{\text{dR}}(S_1)$

| $(f, g)$ | $V(f, g)$ | $F(f, g)$ |
|---------|-----------|-----------|
| $(0, g(0,0,0))$ | $(0, 0)$ | $(0, 0)$ |
| $(0, g(2,0,0))$ | $(0, 0)$ | $(0, 0)$ |
| $(0, g(1,0,1) + g(3,0,0))$ | $(0, 0)$ | $(0, 0)$ |
| $(0, g(2,1,0) + g(0,0,1))$ | $(0, 0)$ | $(0, 0)$ |
| $(0, g(0,0,1) + g(1,0,1))$ | $(0, 0)$ | $(0, 0)$ |
| $(0, g(1,0,0))$ | $(0, g(0,0,0))$ | $(0, 0)$ |
| $(0, g(1,1,0))$ | $(0, g(2,0,0))$ | $(0, g(0,1,0) + g(3,0,0))$ |
| $(0, g(0,1,1))$ | $(0, g(2,1,0) + g(0,0,1))$ | $(0, 0)$ |
| $(0, g(1,0,1))$ | $(0, g(0,0,1) + g(1,0,1))$ | $(0, 0)$ |
| $(0, f(0,1,1))$ | $(0, 0)$ | $(0, g(0,0,0))$ |
| $(0, f(1,1,0))$ | $(0, 0)$ | $(0, g(2,0,0))$ |
| $(0, f(1,0,1) + f(3,0,0))$ | $(0, 0)$ | $(0, g(0,1,0) + g(3,0,0))$ |
| $(0, f(0,0,1) + f(1,0,1))$ | $(0, 0)$ | $(0, g(2,1,0) + g(0,0,1))$ |
| $(0, f(0,1,1))$ | $(0, f(1,0,1))$ | $(0, f(0,1,1))$ |
| $(0, f(1,1,0))$ | $(0, f(1,0,1))$ | $(0, f(1,1,0))$ |
| $(0, f(2,1,0) + f(0,0,1))$ | $(0, f(1,0,1))$ | $(0, f(2,1,0) + f(0,0,1))$ |
| $(0, f(1,0,1))$ | $(0, f(1,0,1))$ | $(0, f(1,0,1))$ |

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