Abstract Robert de Montessus de Ballore proved in 1902 his famous theorem on the convergence of Padé approximants of meromorphic functions. In this paper, we will first describe the genesis of the theorem, then investigate its circulation. A number of letters addressed to Robert de Montessus by different mathematicians will be quoted to help determining the scientific context and the steps that led to the result. In particular, excerpts of the correspondence with Henri Padé in the years 1901-1902 played a leading role. The large number of authors who mentioned the theorem soon after its derivation, for instance Nörlund and Perron among others, indicates a fast circulation due to factors that will be thoroughly explained.

Key words Robert de Montessus, circulation of a theorem, algebraic continued fractions, Padé’s approximants.

MSC : 01A55 ; 01A60

1 Introduction

This paper aims to study the genesis and circulation of the theorem on convergence of algebraic continued fractions proven by the French mathematician Robert de Montessus de Ballore (1870-1937) in 1902. The main issue is the following : which factors played a role in the elaboration then the use of this new result ? Inspired by the study of Sturm’s theorem by Hourya Sinaceur [52], the scientific context of Robert de Montessus’ research will be described. Additionally, the correlation with the other topics on which he worked will be highlighted, as well as the major points that led him to the result. In [20], Catherine Goldstein raised the following issues for any research in history of mathematics : “What is the historical description of a theorem in mathematics ? ”; “How was the theorem read and understood ? How was it used ?”and : “understanding a paper, historical as well as mathematical, means determining on which knowledge it relies ; on which options, for what public; which context is required to interpret the text ; which points are needed to build an answer.” In the case of Robert de Montessus, the challenge is to find the mathematical approaches of the theorem and the reasons for other mathematicians to refer to his work.

Robert de Montessus [30], aged 32, began his thesis in 1902 at the Faculté des Sciences of Paris, under the supervision of Paul Appell. His defense[1] on the 8th of Mai 1905[2] led to a publication in the

[1]His research on algebraic continued fractions was awarded by the Grand Prix de l’Académie des Sciences Mathématiques in 1906. The price was shared with Robert de Montessus, Henri Padé and André Auric.

[2]Paul Appell (chairman), Henri Poincaré and Edouard Goursat were present in the jury.
Does a sequence of consecutive convergents from a table constituted by normal fractions define a function similar to the function defined by the series that has led to the table? And, if yes, does the continuous fraction corresponding to the progression extend the series outside its circle of convergence?

Robert de Montessus gave a positive answer to these questions in the particular case of the following function: a meromorphic function, analytic around zero, and for which one consider as the sequence of rational fractions, the elements in a row of the Padé table associated to the function.

During the last forty years, Robert de Montessus de Ballore has been quoted a large number of times in the mathematic field in connection with the approximation using rational fractions or with continued fractions. His name appears in the statement of a new result or as a reference to the theorem of 1902. As an example, E.B. Saff published in 1972 An extension of Montessus de Ballore’s theorem on the convergence of interpolating rational functions. This article states the theorem using Padé approximants and the bibliography includes a reference to Montessus’ article published in 1902 in the Bulletin de la Société Mathématique de France. Moreover, Annie Cuyt and Doron Lubinsky published in 1997 A de Montessus theorem for multivariate homogeneous Padé approximants. Here, the name of Montessus de Ballore appears several times through a less specific book: it is the work of Perron on continued fractions that is mentioned as a reference, even if Montessus’ theorem is cited. More recently in 2011, on can find the theorem stated with a direct reference to the article of 1902 in From QD to LR and QR, or, how were the QD and LR algorithms discovered? by M. H. Gutknecht M.H. and B.N. Parlett.

In which context was the theorem of 1902 elaborated? Who else was doing research on the convergence of algebraic continued fractions and what was the goal at that time? These are the challenging issues we will try to answer in the first part. To make it easier for the reader, we will explain at the beginning of this section some definitions and results on continued fractions and Padé approximants. In the second part, the circulation of the theorem will be studied: which authors cited Robert de Montessus? When? What did Robert de Montessus to further improve the dissemination of his work?

Methodologically, we proceeded as following: we used three types of sources. The books of C. Brezinski were the starting point. Then we also find a lot of valuable informations in electronic sources and in letters received by Robert de Montessus from other mathematicians working on topics close to algebraic continued fractions. These letters represent a crucial and a new source of information on the process of elaboration of the new theorem and its circulation. To illustrate this, excerpts will be given. Moreover, for the study of the circulation of the theorem, we linked different sources.
electronic databases, such as Jahrbuch \[2\], Numdam \[1\], Jstor \[3\], Gallica \[4\], Internet Archive \[6\] and Göttinger Digitalisierungszentrum \[7\]. The goal is to gather, with the most exhaustivity possible, all the mathematical articles that mention in any way the theorem of Robert de Montessus. The book written by C. Brezinski \[15\] on Henri Padé’s work, will enable us to position Robert de Montessus’ work as regard to Henri Padé’s.

2 Robert de Montessus and continued fractions

2.1 Mathematical objects: continued fractions, algebraic continued fractions, Padé approximants

The mathematical objects used by Robert de Montessus will be described the same way they were at the beginning of the 20th century. First, a continued fraction is given by two sequences of natural integers, real or complex numbers \(q_0, q_1, q_2, \ldots, p_1, p_2,\ldots\), and by a calculation procedure:

\[
q_0 + \frac{p_1}{q_1 + \frac{p_2}{q_2 + \frac{p_3}{q_3 + \ldots}}}
\]

For example, starting with a fraction and with the use of Euclide’s algorithm, one gets:

\[
\frac{105}{24} = 4 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2}}}.\]

A similar procedure but using the floor function gives for the number \(\sqrt{3}\):

\[
\sqrt{3} = 1 + (\sqrt{3} - 1) = 1 + \frac{1}{\sqrt{3} - 1} = 1 + \frac{1}{\frac{\sqrt{3} + 1}{2}} = 1 + \frac{1}{\frac{1}{2 + \frac{1}{1 + \frac{1}{2 + \ldots}}}}.
\]

The convergent of order \(k\) of the continued fraction is the usual fraction:

\[
\frac{A_k}{B_k} = q_0 + \frac{p_1}{q_1 + \frac{p_2}{q_2 + \ldots + \frac{p_k}{q_k}}}.
\]

The convergents give in certain circumstances a good approximations of the value of the continued fraction. As an illustration, in the 17th century, the Dutch astronomer Christian Huygens used continued fractions and their convergents to build planetary automaton\[13\] in *Descriptio automati planetarii* \[26\].

The following recurrence relations link the numerators and the denominators of the convergents:

\[
A_k = q_kA_{k-1} + p_kA_{k-2} ; \quad B_k = qkB_{k-1} + pkB_{k-2}.
\]

with first terms:

\[
A_{-1} = 1, \quad A_0 = q_0, \quad A_1 = q_1q_0 + p_1
\]

\[12\]In *Additions au mémoire sur la résolution des équations numériques*, published in 1770, Lagrange proved that every quadratic number, like \(\sqrt{3}\), can be rewritten as a continuous and periodic fraction.

\[13\]To fix the number of gear teeth of the notched wheels of his machines, Huygens faced fractions with huge numbers, namely astronomical distances. He could approximate in a satisfactory way these quantities by quotients with small integer, using convergents of the developments in continued fractions.
and

\[ B_{-1}, \ B_0 = 1, \ B_1 = q_1. \]

These two equations show in the first place a relation between the continued fractions and the equations with finite linear differences. If in [16] there are different references on the relation between continued fractions and equations with differences, we will mention here the thesis of Niels Nörlund entitled \textit{Fractions continues et différences réciproques} and published in 1911 in Acta Mathematica. Indeed, as we will see later on, Nörlund contacted Robert de Montessus in 1910 about his work on algebraic continued functions. In the second place and with certain hypotheses, it is possible to write the \( p_k \) and \( q_k \) with the quantities \( A_i \) and \( B_j \). It is thus possible to build continued fractions whose convergents are a sequence of given fractions.

Furthermore, if the sequences \( p_i \) and \( q_i \) are sequences of polynomials, an \textit{algebraic continued fraction} has a similar definition. For example, Heinrich Lambert obtained in 1768 a development of \( \tan(x) \)\footnote{The result can be found in \textit{Mémoire sur quelques propriétés remarquables des quantités transcendantes, circulaires et logarithmiques}, Académie des Sciences of Berlin, 1768. Besides, H. Lambert used the development of \( \tan(x) \) to prove \( \pi \) was irrational.}, without proving the convergence\footnote{There has been a debate concerning this. It will be mentioned later when we will talk about Alfred Pringsheim.}:

\[
\tan(x) = \frac{x}{1 - \frac{x^2}{3 - \frac{x^2}{5 - \frac{x^2}{7 - \cdots}}}}
\]

This development in algebraic continued fraction can be obtained with a software performing algebraic computation such as Maple. Thus, using \textit{cfrac} in Maple, one gets:

\[
\exp(z) = 1 + \frac{z}{1 + \frac{z}{2 + \frac{z}{3 + \frac{z}{4 + \cdots}}}}
\]

This equation is an example of a development in continued fraction that is part of the theory of Padé approximants\footnote{Henri Padé used in his thesis [43] the terms of \textit{fraction continue simple}, pages 42-93}. These convergents namely correspond to a certain displacement in Padé’s table associated with the exponential function. But what is the a Padé approximant of a function?

Consider \( f(z) = \sum_{i=0}^{+\infty} a_i z^i \) a function\footnote{We use \( z \) for the variable. Instead, Henri Padé and Robert de Montessus used the letter \( x \).} that can be developed in power series at the origin. The Padé’s approximant of \( f \), \( [L/M](z) \), is a rational fraction with a numerator having a degree inferior to \( L \) and the denominator a degree inferior to \( M \), with

\[
f(z) - [L/M](z) = O\left(z^{L+M+1}\right)
\]

which means that the development of \( f \) and of \( [L/M] \) are the same until the order \( z^{L+M} \) is reached. For example, the approximant \([3/4]\) of \( \exp(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \) is:

\[
\begin{align*}
&1 + \frac{3}{4}z + \frac{1}{12}z^2 + \frac{1}{210}z^3 \\
&1 - \frac{1}{7}z + \frac{1}{17z^2} - \frac{2}{105}z^3 + \frac{1}{840}z^4 = \\
&1 + z + \frac{1}{6}z^2 + \frac{1}{24}z^3 + \frac{1}{120}z^4 + \frac{1}{720}z^5 + \frac{1}{5040}z^6 + \frac{1}{40320}z^7 + \cdots \\
&+ \frac{17}{705600}z^8 + \cdots
\end{align*}
\]
These rational fractions are placed in an array called Padé table. A good choice in this array of a sequence of approximants, generated a continued fraction whose convergents are the chosen fractions. This result can be found in the systematic study by Padé on these approximants in his thesis defended in 1892. Thus, the convergents of the continued fraction of the function \(e\) that were given few lines before are its Padé approximants, \([n/n]\) or \([(n+1)/n]\).

In the middle of the 18\textsuperscript{th} century, Euler used an algebraic continued fraction to "sum" the series

\[1! - 2! + 3! - \cdots\]

(see [13, 16, 56]). Then during the 19\textsuperscript{th} century, studies using algebraic continued fractions will be produced in a large quantity, as it is highlighted in [16] pages 190-259. For instance, the work of Lagrange on arithmetic and algebraic continued fractions are numerous. Besides, C. Brezinski commented in [16] page 139 the dissertation of Lagrange called *Sur l’usage des fractions continues dans le calcul intégral* and talked about birth certificate Padé approximants. Indeed, Lagrange developed the function \((1 + x)^m\) as an algebraic continued fraction. And he moreover noticed that consecutive convergents have developments in power series that are the same until a certain order to those of \((1 + x)^m\). It is Henri Padé in his thesis [41], who achieved a systematic study of Padé approximants. He defined then studied the array formed by these fractions. He pointed out in particular the structure in blocks of the array. Then he linked the algebraic continued fractions with the Padé approximants. The major question was: starting with a sequence of rational fractions, how is it possible to build an algebraic continued fraction whose convergents are the initial fractions? Finally Padé devoted his thesis to the particular case of the exponential function. He dedicated it to Charles Hermite. We remind the reader that Charles Hermite used Padé approximants in his derivation of the transcendence of \(e\) in 1873.

2.2 The context

Robert de Montessus was awarded his degree in science in 1901. He started soon after his thesis and it is thanks to Paul Appell that he began to work on the issue of the convergence of algebraic continued fractions. This has been found in letters that we will give some excerpts. This leads Robert de Montessus to get insight into the work of Henri Padé, but also of Edmond Laguerre. These three mathematicians Laguerre, Appell and Padé have also in common to be close to Charles Hermite. Indeed, Laguerre has graduated from the Ecole polytechnique and was a close friend of Joseph Bertrand, the brother-in-law of Hermite. Appell got Hermite as his Professor and he married one of his nieces. And Hermite supervised Padé’s doctoral dissertation.

Edmond Laguerre is in particular well known for the orthogonal polynomials to which he gave his name. Despite a short life—he died at 52- and health problems, he published not less than 150 articles [32] and [33]. Henri Poincaré wrote in the preface of *Oeuvres complètes de Laguerre* [32]:

\[\text{18This is equivalent to choosing a displacement in the table.}\]
\[\text{19Besides, the software Maple gives this information.}\]
\[\text{20We advise to consult the bibliography given in [16] and the website Mathdoc, http://portail.mathdoc.fr/OEUVRES/, where the a link to reach the work of Lagrange that has been digitized and can be read on Gallica.}\]
\[\text{21This dissertation was read the July the 18\textsuperscript{th} in 1776 by the Académie royale des Sciences et Belles Lettres of Berlin.}\]
\[\text{22For a given convergent, this order is the sum of the degrees of the numerator and denominator.}\]
\[\text{23The analysis of the derivation of Hermite is very well explained by Michel Waldschmidt on Bibnum, http://bibnum.education.fr/}\]
\[\text{24There were no real thesis' supervisor at that time, so that here supervising means more following.}\]
Circulation of a theorem

The study of algebraic functions will without any doubt help us one day to write the
functions as more convergent developments than power series; however, few geometers
have tried to investigate in this field that will certainly give us plenty of surprises; Laguerre
went into this field thanks to his research on polynomials solving linear differential equations.
He obtained many results but I want to cite only one that is the most amazing and the most
suggestive. From a divergent series, a convergent continued fractions can be deduced: this
is the new legitimate use of divergent series and it has a great future ahead.

In the same publication can be found two articles entitled Sur la réduction en fractions continues d’une
fonction qui satisfait à une équation linéaire du premier ordre à coefficients rationnels that describe
work that Robert de Montessus will further investigate on [36]. If Henri Poincaré underlines at the
end of the 19th century the extreme importance of studying algebraic continued fractions, it is because
he is working on it. First of all he published articles on continued fractions such as [47, 48] and then
used these fractions to solve differential equations [49]. Moreover, he also investigated how to sum
a divergent series, like in the second volume of méthodes nouvelles de la mécanique céleste published
in 1892. Emile borel, in [13], dedicated a chapter to explain the theory of asymptotic series that he
attributed the creation to both Henri Poincaré and Thomas Stieltjes.

Paul Appell was also interested by continued fractions. One of his first article is Sur les fractions
continues périodiques [10]. Nevertheless his influence on Robert de Montessus was certainly more due
to his researches in mathematical analysis particularly his work on developments in series of functions
of one or several variables

Henri Padé on his side, underlined the importance of the issue of the convergence of algebraic
continued fractions in Recherches sur la convergence des développements en fractions continues d’une
certaine catégorie de fonctions [44]. He wrote:

The question of the convergence is not mentioned in the researches on the transformation
of continued fractions in power series of Lagrange and Laplace; there are not either
mentioned in the study of Gauss on the well known continued fraction resulting from the
quotient of two hyper geometrical series. Nothing can be found on this topic in the huge
work of Cauchy; and, near the middle of the 19th century, if we can cite the large number
of studies -not all very fruitful- of Stern and Seidel and those of Heine, we need to reach
the posthumous work of Riemann (1863): Sullo svolgimento del quoziente di due serie
ipergeometric in frazione continua infinita to find the first really relevant work. However,
this work was still not complete, despite the restoration of Mr. Schwarz, and the nice and
highlighting Mémoires by Thomé on the same topic and published in the volumes 66 and 67
(1866, 1867) of the Journal by Crel (...)

Since, the studies of Laguerre, Halphen, Mr. Pincherle and Markoff followed one after
the other: and, for the last ten years, those of Stieltjes (1894), MM. von Koch, van Vleck
and Pringsheim which are the main studies; finally and recently, two notes to the Comptes
rendus and two short dissertations published in the Bulletin de la Société Mathématique
of France (1902) and in the Annales de la Société Scientifique of Bruxelles (1903), by Mr. R.
de Montessus de Ballore.

For Henri Padé, the first relevant results on the convergence of an algebraic continued function appeared
at the beginning of 1860. It could be that for him, the term of Oeuvres meant a group of several works.

25 Appell analyzed first his work in [11]. It is very likely that this note has been written for his application to the
Academy of Science. Appell pointed out the importance of these development in series for the cases of the potential
theory, the differential linear equations and the study of hyper geometrical functions.
Else his history is not complete. Why did he not mention Adrien Legendre and his proof of the convergence of the development of the function tangent in algebraic continued fraction, as it is shown above by Lambert? This derivation can be found for example in the *Eléments de Géométrie* by Legendre [29] published in 1837. The works of Bernhard Riemann and of Ludwig Wilhelm Thomé are related to the convergence of continued fractions that are build on quotients of hypergeometrical fractions [53]. The American mathematician Edward Burr Van Vleck defended his thesis entitled *Zur Kettenbruchentwicklung Hyperelliptischer und Ähnlicher Integrale* in Göttingen under the supervision of Félix Klein [54], and he developed in 1903 the questions raised above as well as the studies of mathematicians mentioned by Padé [26] [55]. Henri Padé quoted from Andrei Andreyevich Markov but it is worth pointing out both Pafnuty Lvovich Chebyshev (Tchebychev) and Konstantin Aleksandrovich Possé who was as Markov a student of Chebyshev. Georges Henri Halphen studied particularly the convergence of continued fractions associated with $\sqrt{X}$, where $X$ is a polynomial of degree 3 with real coefficients and roots [16] pp 200-201. Henri Padé mentioned also the Italian mathematician Salvatore Pincherle because he linked the results obtained for linear equations with finite differences with algebraic continued fractions (see notice sur travaux [46]). Niels Fabian Helge von Koch, student of Mittag-Leffler, generalized a result of convergence found by Thomas Stieltjes [28]. Emile Borel devoted a chapter [27] to continued fractions and to the theory of Stieltjes in his *Leçons sur les divergentes* [14], at the page 63, Emile Borel wrote:

**The starting point of the research of Stieltjes is the continued fraction**

$$F = \frac{1}{a_1 z + \frac{1}{a_2 + \frac{1}{a_3 z + \frac{1}{a_4 z + \cdots}}}}$$

where $a_n$ are real positive numbers and $z$ is a complex variable.

then

**The only case that will matter here is the one where the series $\sum a_n$ is divergent. The continued fraction is then convergent and defines an holomorphic function in the plane of the complex variable, points with a negative real part excepted (...)**

The continued fraction can be developed in series with power of $\frac{1}{z}$ (...) This development can be then obtained as following:

$$F = \frac{c_0}{z} - \frac{c_1}{z^2} + \frac{c_2}{z^3} \cdots$$

The numbers $c_0, c_1, c_2, \cdots$ are positive ; Stieltjes gives a way to get their expressions in function of $a_n$ but these expressions are complicated. On the contrary, the $a_n$ can be written using the $c_n$ in very elegant form (...)

The work of Stieltjes on théorie analytique des fractions continues [28] is not separable of the moment problem, i.e. given a sequence of real numbers $(m_n)_{n=0}^\infty$, does a positive measure $\mu$ over $\mathbb{R}$ exists, so that for every $n$,

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26 The bibliography given by Van Vleck is particularly interesting.
27 Moreover in this book, Emile Borel talked about the work of Henri Padé and thus about the link between Padé approximants and algebraic continued fractions.
28 Several authors consider Stieltjes as the father of the analytic theory of continued fractions [16] [27].
Emile Borel further developed this point:

Therefore, the main goal of the Mémoire de Stieltjes is to introduce an analytical element of a defined integral that is more easy to deal with and that leads to the divergent series by its development in powers of $\frac{1}{z}$. On the other hand, for a given series, is it possible to build an integral using the continued fraction as an intermediate to connect the series and the integral (...) 

Stieltjes considered the integral

$$ J = \int_{0}^{\infty} \frac{f(u)du}{z+u} $$

where $f(u)$ is a function supposed to be nonnegative (...)

It is easy to find its formal development. We have

$$ J = \int_{0}^{\infty} \left( \frac{1}{z} - \frac{u}{z^2} + \frac{u^2}{z^3} - \frac{u^3}{z^4} + \cdots \right) f(u)du $$

So if we consider

$$ c_n = \int_{0}^{\infty} f(u)u^n du \quad (n = 0, 1, 2, \ldots) $$

we obtain

$$ F = \frac{c_0}{z} - \frac{c_1}{z^2} + \frac{c_2}{z^3} \cdots $$

After some hypothesis and according to Emile Borel, we have:

Indeed in that case, the continued fraction deducted from the series is convergent and defines an analytical function $F(z)$ that is regular in all the plane except on the negative part of the real axis; the equality

$$ F(z) = \int_{0}^{\infty} \frac{f(u)du}{z+u} $$

enables us to determine the function $f(u)$.

Thus Stieltjes solved the moment problem in a special case.

The German mathematician Alfred Pringsheim demonstrated several results of convergence of continued fractions [16], but also studied the proofs of Lambert and Legendre on the irrationality of $\pi$ in an article published in 1901. One critic can be found in the second part of the volume 25, year 1901 of the Bulletin des Sciences Mathématiques (pp 86-88).

Every study on the convergence of algebraic continued fractions is linked to the question of the analytic continuation. This is one of the fundamental topic in Analysis at the end of the 19th century and the beginning of the 20th. In this paper we will not give a complete history on it, but we still

29 Alfred Pringsheim explained that Lambert demonstrated the convergence of the algebraic continued fraction associated with $\tan(x)$, see the article by R Walisser, On Lambert’s proof of the irrationality of $\pi$, in Algebraic number theory and Diophantine analysis, Graz, 1998 (de Gruyter, Berlin, 2000,) pp 521-530.
need to mention that Walter William Rouse Ball tackles this issue in the paragraph *Analyse*. Robert de Montessus contributed to the writing. Two other mathematician have corresponded with Robert de Montessus, namely Jacques Hadamard and Eugène Fabry. It his well-known that Jacques Hadamard obtained important results in this field, between 1888 and 1902. His dissertation defended in 1892 is indeed entitled *Essai sur l'étude des fonctions données par leur développement de Taylor*. We do not know how much Hadamard and Robert de Montessus have written each other. However, Robert de Montessus used the work of Jacques Hadamard for the proof of his theorem in 1902. As for the French mathematician Eugène Fabry, Professor at the university of Montpellier, one letter addressed to Robert de Montessus in 1901 explained the results he got in this field. We have to cite the article of Fabry published in 1869 in the Annales scientifiques de l’Ecole Normale Supérieure, *Sur les points singuliers d’une fonction donnée par son développement en série et l’impossibilité du prolongement analytique dans des cas très généraux*.

### 2.3 Genesis

How did Robert de Montessus start his work on algebraic continued fractions? According to the classifications in [34] and [19], the result found by Robert de Montessus is part of the field named Analysis. However, Juliette Leloup wrote page 68 of [34]:

*As for the dissertation in arithmetics, these form by themselves a special corpus since the reports are always full of praise as soon as it has strong links with the theory of functions: the dissertations of Cotty, Châtelet, Got and Chapelon are such examples, the applications of the new tools of the theory of functions to the theory of continued fractions in the thesis of Montessus de Ballore is another example.*

The author might consider that continued fractions are at the border of arithmetics and analysis.

Robert de Montessus was awarded his degree in mathematical science October the 24th, 1901. He started afterwards a doctoral thesis under the supervision of Paul Appell. Paul Appell has been a member of the Academy of Science since 1892 and became the dean of the Faculty of Science of Paris in 1903. He supported Robert de Montessus during all his career. A letter from Charles-Ange Laisant written on the 30/10/1900 shows that Robert de Montessus wanted to be closer to Paul Appell:

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30 This paragraph has an asterisk because it has been added by Robert de Montessus. He quoted in particular Hadamard and the Finnish mathematician Ernst Lindelöf.

31 We have only found one letter, without a date on it, where Hadamard indicated to Robert de Montessus how to find his article *Sur certaines surfaces minima*.

32 See the following for an excerpt of this letter.

33 According to what is written in the statement of his dissertation.

34 We are able to claim this fact according to the bunch of letters from Paul Appel found in the collection Robert de Montessus. It is letters of recommendation, or for example, letters in which he ask for the authorization to give a lecture at the Sorbonne.

35 Charles-Ange Laisant, 1841-1920, graduated from the Ecole Polytechnique in Paris, was a mathematician, French politician and founder of the *Intermédiaire des mathématiciens* and with Henri Fehr of the *Enseignement mathématique*.

36 This letter can be found in the collection of letters received by Robert de Montessus de Ballore from C. A. Laisant during the period 1897-1937, lettres de C. A. Lais
Robert de Montessus was appointed lecturer in 1902 at the Catholic University of Lille, and he was then nominated assistant professor in 1903 and gave lectures in Special Mathematics and Rational Mechanics.\footnote{37}

The first signs of the theorem that we found appear in a letter from Henri Padé sent on November the 26th, 1901\footnote{38}:  

\footnote{37}Thus Robert d’Adhémar, at that time assistant professor there, mentioned in the minutes of the council of the Faculté des Sciences de l’Université de Lille on November the 7th, 1902, that a substitute teacher was needed for the class of Special Mathematics. He brought the name of Robert de Montessus. The minutes of these meetings between 1886 and 1924 are preserved in the archives of the Faculté des Sciences de l’Université Catholique de Lille at the number S7E.\footnote{38}Collection of letters received by Robert de Montessus de Ballore during the period 1897-1937, from Padé.
J’ai d’ailleurs vu M. Appell au commencement ce mois, il m’a parlé de vous et m’avait fait parvenir votre lettre.

La théorie des fractions continue offre un champ très vaste de recherches, mais où il n’est pas toujours facile de se rendre compte à l’avance des difficultés que l’on rencontrera. C’est donc avec toutes sortes de réserves que je vous indiquerai, comme devant présenter un grand intérêt, une étude approfondie de la généralisation des fractions continues. Je n’ai fait qu’effleurer le sujet dans un mémoire qui a paru, il y a quelques années dans le journal de M. Jordan, que je me fais un plaisir de vous envoyer en même temps que cette lettre. Vous y trouverez l’indication d’un mémoire de M. Hermite, sur le même sujet. Dans ce mémoire, M. Hermite arrive à la méthode des polynômes associés par des considérations [...] de calcul integral : il serait, sans doute, aussi bien intéressant d’approfondir davantage le rapport entre le calcul integral et les lois de récurrence de la théorie des fractions continues.

Le résultat que vous m’annoncez avoir obtenu me paraît des plus remarquables, mais doit être soumis à des exceptions assez nombreuses. Je lirai avec plaisir votre démonstration quand vous l’aurez publiée.

Moreover, I have seen Mr Appell at the beginning of the months. We talked about you and he forwarded your letter to me.

The theory of continued fractions opens door to a large field of research but where it is not always that easy to figure out in advance the difficulties that will be encountered. It is thus with a pinch of salt that I will recommend you to read a detailed study of the generalization of continued fractions that should interest you. I only touched upon this topic in a paper that was published few years ago in the journal of Mr. Jordan, which I am happy to enclose to this letter. You will find inside a reference to a paper from Mr. Hermite where he deals with the same topic. In this last paper, Mr. Hermite explains the method of associated polynomials using [...] integrals: it would be for sure that interesting to go further into the relationship between the calculations using integrals and the recursive rules of the theory of continued fractions.

The result that you said you found seems to be really outstanding but there might be numerous exceptions to it. I would be very happy to read your demonstration when it will be published.
Henri Padé was then professor at the University of Poitiers.

Furthermore, notes from Robert de Montessus in a file named *Continued fractions* enable us to know which work related to continued fraction he consulted at the beginning of his research:

*Note du 29 Mars 1901*

Mr. Hermite gave the expression of the general convergents of the function exp [...] 

little bit further in this note:

*Sur la fraction de Stieltjes 19 Avril [...]*

In the margin of the note are also written the names of Hurwitz and Minkowski. Still in the same file is the article of Salvator Pincherle *Sur la généralisation de systèmes récurrents au moyen d’une équation différentielle* published in 1892 in Acta Mathematica. Alfred Pringsheim gave to Robert de Montessus several references for articles in a letter on February the fourth, 1902.

At the same time, Robert de Montessus was in touch with Eugène Fabry who worked on Taylor series. Fabry sent two letters to Robert de Montessus on October the 12th, 1901 and January the 7th, 1902. Émile Borel described and explained the importance of the work of Fabry in his obituary notice read at the Academy of Science on October the 23rd, 1944. Indeed this notice shows a thorough overview of Fabry’s research. In particular, Émile Borel wrote:

_Eugène Fabry dedicated then his research to an important and difficult problem that has been already tackled by a lot of mathematicians, such as our distinguished fellow member Jacques Hadamard. It relates to looking for and staying the singularities located on the convergence circle of a Taylor development that defines an analytical function inside the circle. The brilliant idea of Fabry was to substitute the study of the complete sequence of the coefficients of the Taylor series by the study of the partial sequences of coefficients extracted from this complete sequence._

The two letters from Fabry to Robert de Montessus are indeed related to Taylor series and their singularities:
Le théorème que vous m'indiquez ne peut pas être exact sous la forme la plus générale. Il est possible que le second énoncé soit exact mais le premier ne l’est pas. Si la série n’a qu’un point singulier sur la circonférence de convergence, on ne peut affirmer que $\frac{s_n}{s_{n+1}}$ ait une limite. Cela résulte des théorèmes que j’ai indiqué (…) dans les Acta Math (Tome 22, page 86). J’ai en effet montré qu’il existe des séries incomplètes n’ayant qu’un seul point singulier sur la circonférence de convergence. $\frac{s_n}{s_{n+1}}$ a alors des valeurs nulles et infinies, et ne peut avoir aucune limite. L’exemple que j’ai donné est le suivant :

$$\sum x^n e^{n[-1+\cos(Ln)^θ]} \quad 0 < θ < 1$$

[…]

En résumé, je ne peux pas vous donner de réponse absolument précise sur l’exactitude du théorème que vous énoncez ; mais je ne serais pas étonné qu’il soit exact en prenant le second énoncé.

Cette question me paraît très intéressante et doit conduire à des résultats importants.

Si vous voulez avoir une idée des travaux publiés sur la série de Taylor, vous pourrez trouver des renseignements très complets dans un petit traité publié par M. Hadamard au mois de mai dernier sur “la série de Taylor et son prolongement analytique” dans la collection Scienta (…).
This second letter by Fabry enlightens the first one. Fabry gave to Robert de Montessus the book of Hadamard as a new reference. Indeed Hadamard took interest in [23] in the singularities of a function that can be developed in a power series around zero, with a development

\[ a_0 + a_1 x + \cdots + a_m x^m + \cdots \]

having a radius \( R \) and a circle of convergence \( C \). As did Lecornu\(^{42}\), Hadamard considered the ratio \( \frac{a_m}{a_{m+1}} \) and precised on the page 19 of [23] :

\[ \text{Indeed, on the one hand, as we already mentioned, } \frac{a_m}{a_{m+1}} \text{ has in general no limit. On the other hand, there could be several singularities on } C. \text{ Then one can see three different meaning in the statement:} \]

1. If \( x_0 \) is the only singularity on \( C \), the ratio \( \frac{a_m}{a_{m+1}} \) has \( x_0 \) for limit.
2. If \( \frac{a_m}{a_{m+1}} \) has \( x_0 \) for limit, then the point with affix \( x_0 \) is the only singularity of the function on \( C \).
3. If \( \frac{a_m}{a_{m+1}} \) has \( x_0 \) for limit, then the point with affix \( x_0 \) is a singularity of the function.

Jacques Hadamard showed indeed that the first two statements were wrong, contrary to the third one which has been proven by Fabry.

Robert de Montessus moved closer to Padé. Indeed, two other letters by Henri Padé\(^{43}\) show that Robert de Montessus shared his results with him. Henri Padé\(^{44}\) was at that time professor at the University of Poitiers \(^{15}\) but we think that Robert de Montessus and him could have met during the international congress of mathematicians that was held in Paris in August 1900. Robert de Montessus got a signed copy of the announcement made by Henri Padé during this congress, entitled *Aperçu sur les développements récents de la théorie des fractions continues*. Here we should highlight a point related to the vocabulary used: Padé talked of *fraction approchée* in his articles to refer to the approximants that have his name now. Van Vleck used the English word *approximant* in 1903 in one of his articles, cited below: it might be this change to the English word that made the French word to be forgotten.

The following excerpts of these two letters underline the mathematical communication between the two men:

\(^{42}\) Lecornu stated in 1887 in a note to the Comptes Rendus de l’Académie des Sciences that the existence of a limit for the ratio \( \frac{a_m}{a_{m+1}} \) would imply that this limit is the unique singularity of the function on \( C \).

\(^{43}\) Collection of letters received by Robert de Montessus de Ballore during the period 1897-1937, from Padé.

\(^{44}\) We should keep in mind that Paul Appell was among the jury of Henri Padé’s doctoral defense. He was also one of his professors in Paris.
Henri Padé suggested to Robert de Montessus to use new notations but the last one will not follow his advice. Robert de Montessus wrote the convergents \((n, p)\) by \(\frac{U_n}{V_p}\) with \(U_n\) a polynomial of degree \(n\) and \(V_p\) a polynomial of degree \(p\) (in the case of a normal table). This convergents corresponds to the Padé approximant \([n, p]\) of the function \(f(x)\).

Henri Padé referred to his article published in 1902, *Recherches nouvelles sur la distribution des fractions approchées d’une fonction*, Annales Scientifiques de l’ENS, volume 19, 1902, pages 153-189. The last sentence of the excerpt should be put in parallel with the controversy of the priority between...
Robert de Montessus and Henri Padé that will be further developed later.

2.4 The theorem

What is the content of the article published in 1902? On which mathematical elements did Robert de Montessus build his demonstration?

Robert de Montessus started to publish in 1902 articles where he tackled the issue of the convergence of algebraic continued fractions. Thus, before the publication of the theorem in the Bulletin de la Société Mathématique de France, a note from Robert de Montessus was presented at the Academy of Science by Paul Appell on June the 23rd, 1902. Robert de Montessus got inspiration from a study by Laguerre [31] on the development in continued fractions of the function \( \frac{x + 1}{x - 1} \). He proved the convergence of the continued fraction obtained by Laguerre outside the interval \([-1, 1]\).

The theorem of Robert de Montessus was thus published in the Bulletin de la Société Mathématique de France in 1902. The title of the article was not precise: *Sur les fractions continues algébriques*. The article was short, with 9 pages. The writing was quite dynamic since the author went directly to the main points. The way how Robert de Montessus has written his article might be one of the reasons why his results has had and still had a special echo. Robert de Montessus derived his theorem as well in his doctoral dissertation: there the writing is even denser. This can be explained by the paragraphs before the result where Robert de Montessus developed with lots of details the links between continued fractions and Padé approximants.

How did Robert de Montessus build his article? He started from the power series

\[ y = s_0 + s_1 x + \cdots + s_h x^h + \cdots \quad (s_0 \neq 0), \]

then reminded the reader several notions from Padé: Padé approximants table; normality of the table and the necessary condition to have the normality; link between Padé approximants, seen as convergents, and continued fraction. Robert de Montessus explained that the study of a sequence of fractions well chosen in Padé table, is similar to the study of a continued fraction. The study of the convergence of the continued fraction, which is the convergence of its convergents, is actually the study of the convergence of the series associated to this sequence.

Robert de Montessus imposed some conditions on the degrees of the numerators and denominators of the fractions from Padé table to simplify the problem: if every fraction is *in advance of the previous one*, that is to say that \( \frac{U_p}{V_q} \) is in advance of \( \frac{U_p}{V_q} \) if \( p + n > m + q \), then the convergents of the sequence have to be all consecutive. This means that

\[ \frac{U_i}{V_i} = \frac{U^m}{V^m}, \quad \frac{U_{i+1}}{V_{i+1}} = \frac{U^q}{V^q} \]

then \( p + n + 1 = q + m \) or \( p + n + 2 = q + m \).

Robert de Montessus considered only the particular case of the rows in the table, namely the case of a sequence:

\[ \frac{U^0_{p_0}}{V^0_{p_0}}, \quad \frac{U^1_{p_1}}{V^1_{p_1}}, \quad \ldots, \quad \frac{U^n_{p_n}}{V^n_{p_n}}, \quad \ldots \]

Then he tackled the two major issues raised in our introduction which are the rebuilding of the function given by the series and its possible analytical continuation. Robert de Montessus considered
a meromorphic function, analytical at the origin with a development given by $y$. His proof used the results found by Hadamard [22] on the asymptotical behavior of polynomials called polynomials of Hadamard [23] and of determinants of Hankel $H^m_p$ associated to the series $y$. The general formulation of the determinants of Hankel is:

$$H^m_p = \begin{vmatrix} s_m & s_{m+1} & \cdots & s_{m+p-1} \\ s_{m+1} & s_{m+2} & \cdots & s_{m+p} \\ \vdots & \vdots & \ddots & \vdots \\ s_{m+p-1} & s_{m+p} & \cdots & s_{m+2p-2} \end{vmatrix}.$$

They naturally appear when one transforms the coefficients of the denominator of the approximate (or of the convergents) in a linear system, with a condition like:

$$f(z) - [L/M](z) = O\left(z^{L+M+1}\right)$$

As for the polynomials of Hadamard, they have been invented by Carl Jacobi in 1846. Their general formulation is:

$$\begin{vmatrix} s_m & s_{m+1} & \cdots & s_{m+p-1} \\ s_{m+1} & s_{m+2} & \cdots & s_{m+p} \\ \vdots & \vdots & \ddots & \vdots \\ s_{m+p-1} & s_{m+p} & \cdots & s_{m+2p-2} \end{vmatrix}.$$ 

Indeed, the denominators of the convergents, $V^p_n$, are nothing else but the Hadamard polynomials of the same degree $p$. Thus, Robert de Montessus raised the hypothesis on the poles

$$|\alpha_1| \leq |\alpha_2| \leq \cdots \leq |\alpha_p| < |\alpha_{p+1}| \leq |\alpha_{p+2}| \leq \cdots$$

and deduced that the $V^p_n$ approach a polynomial of degree $p$ whose roots are the poles. Here stands the key point of the result: Robert de Montessus connected the denominators of the particular Padé approximants that he considered to the result found by Jacques Hadamard page 41 of [23]. Robert de Montessus continued his reasoning on the series related to the continued fraction:

$$\frac{U^0_p}{V^0_p} + \left(\frac{U^1_p}{V^1_p} - \frac{U^0_p}{V^0_p}\right) + \left(\frac{U^2_p}{V^2_p} - \frac{U^1_p}{V^1_p}\right) + \cdots$$

It is a series of rational fractions with a known asymptotical behavior of their denominators. This enables us to study the convergence of a series expressed without these denominators. Once it is proved convergent, one should prove that the limit is a meromorphic function. Robert de Montessus thus derived the following theorem [47]:

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45 P. Henrici in [24], pages 622-633, uses these terms.
46 Jacobi, C. G. J., Über die Darstellung einer Reihe gegebner Werthe durch eine gebrochne rationale Function, J. Reine Angew. Math. 30, 127-156.
47 The statement of the theorem is given in the conclusion of this article.
\[ ... \] given a Taylor series representing a function \( f(x) \) whose \( p \) poles the closest to the origin are inside the circle \( (C) \), this circle being itself inside the next poles, each multiple pole counted as many as simple poles as it exists unities in his degree, the continued fraction deduced from the row of rank \( p \) of Padé table, this table constituted of normal convergents represents a function \( f(x) \) in a circle of radius \( |\alpha_{p+1}| \), where \( \alpha_{p+1} \) is the affix of the pole the closest to the origin among those outside the circle \( (C) \).

Seventy years later, E.B. Saff \[51\] stated the theorem of Robert de Montessus as:

Let \( f(z) \) be analytic at \( z = 0 \) and meromorphic with precisely \( \nu \) poles (multiplicity counted) in the disk \( |z| < \tau \). Let \( D \) the domain obtained from \( |z| < \tau \) by deleting the \( \nu \) poles of \( f(z) \). Then, for all \( n \) sufficiently large, there exits a unique rational function \( R_{\mu,\nu} \), of type \( (\mu, \nu) \), which interpolates to \( f(z) \) in the point \( z = 0 \) considered of multiplicity \( \eta + \nu + 1 \). Each \( R_{\eta,\nu} \) has precisely \( \nu \) finite poles and, as \( n \to \infty \), these pole approach, respectively, the \( \nu \) poles of \( f(z) \) in \( |z| < \tau \). The sequence \( R_{\mu,\nu} \) converges throughout \( D \) to \( f(z) \), uniformly on any compact subset of \( D \).

He did not mention continued fractions in the article: interpolation and approximation notions were highlighted instead. Saff referred to the article of J.L. Walsh\[48\] in a previous article\[49\]. In the introduction of this article, J.L. Walsh wrote:

The \( W_{\mu} \) form a table of double entry (...) known as the Walsh array which is similar in form and properties to the table of Padé. Indeed, J.L. Walsh has for the rows of this array established (...) the following analogue of the important result (...) of Montessus de Ballore concerning the convergence of the rows of the Padé table.

2.5 The work of Robert de Montessus and of Henri Padé

As we mentioned it before, Henri Padé and Robert de Montessus were in touch since the beginning of the 20\textsuperscript{th} century. This did not lead to any collaboration excepted the request of off-print article\[50\]. Let us give some quantitative elements on the work of these two authors in the field of continued fractions. Henri Padé published 28 articles between 1890 and 1907\[51\] whereas Robert de Montessus published 9 articles\[52\] on algebraic continued fractions between 1902 and 1909. Robert de Montessus has preferentially worked on the convergence whereas Padé more on the creation of his approximants and on the methods to obtain them. Since 1908, Padé occupied several jobs of state superintendent of education; he thus stopped his research. However we do not know why Robert de Montessus did not published any other work on the algebraic continued fractions after the year 1909.

Robert de Montessus referred several times as expected to the work of Padé. However, how did Henri Padé received the results of Robert de Montessus? What could the reasons for Robert de Montessus to...
Circulation of a theorem

3.1 Before 1914

Three factors promoted the rapid circulation of the theorem before 1914. The first one is the references to the results of Robert de Montessus in the work of Padé, Van Vleck and Nörlund. It is worth reminding that Robert de Montessus shared his result with Henri Padé at the beginning of the year 1902. Van Vleck mentioned the result in a seminar in Boston in 1903. Indeed, he wrote in Padé’s Table of Approximants and its Applications. \[55\]:
In investigating the convergence of the horizontal lines the first case to be considered is naturally that of a function having a number of poles and no other singularities within a prescribed distance of the origin. It is just this case that Montessus [33, a] has examined very recently. Some of you may recall that four years ago in the Cambridge colloquium Professor Osgood took Hadamard’s thesis as the basis of one of his lectures. This notable thesis is devoted chiefly to series defining functions with polar singularities. Montessus builds upon this thesis and applies it to a table possessing a normal character. Although his proof is subject to this limitation, his conclusion is nevertheless valid when the table is not normal, as I shall show in some subsequent paper.

Van Vleck clarified what we have previously pointed out, that is to say that Robert de Montessus used the results of Hadamard on the polar singularities. On top of that, Van Vleck mentioned a new aspect, namely that the theorem is still accurate even if Padé table is not normal. Moreover, N.E. Nörlund sent the following[^4] to Robert de Montessus in 1910:

Les “Rendiconti di Palermo” ne se trouvent à aucune bibliothèque publique de Copenhague, mais j’ai obtenu aujourd’hui vos thèses.

Je suis heureux maintenant de pouvoir citer votre mémoire en reconnaissant votre priorité. Le mémoire dont j’ai eu l’honneur de vous envoyer un tirage à part, ne paraîtra que dans le tome 34 des Acta Mathematica en 1911 […]

I cannot find the “Rendiconti di Palermo” in any public library of Copenhagen, but I managed to get your dissertations today.

I am glad now to be able to cite your thesis and to acknowledge your priority. The thesis I have the pleasure to send to you an off-print will be published in the volume 34 of Acta Mathematica in 1911 […]

This dissertation of more than a hundred pages is entitled Fractions continues et différences réciproques.

The second factor of the circulation is given by Robert de Montessus himself. Indeed, after his defense in 1905, he got in touch with Mittag-Leffler[^5]. His results on continued fractions were published in Acta Mathematica, the journal owned by Mittag-Leffler in 1909. His dissertation was also published in a foreign magazine in 1905, Rendiconti di Palermo.

Finally, the academic network contributed in the usability of the result. The award of a part of the Grand Prix de l’Académie des Sciences in 1906[^6] has also certainly boosted the larger circulation of his result.

The table underneath gives the articles that contain a reference to the theorem of Robert de Montessus. One can find the article of Nörlund that we already mentioned. As for the work of Oskar Perron, it is the first version of the paper that have been published several times after the first world war.

[^4]: Collection of letters received by Robert de Montessus de Ballore, period 1897-1937, letter from N.E. Nörlund sent on March the 29th, 1910.
[^5]: Collection of letters received by Robert de Montessus de Ballore, period 1897-1937, letter sent by Mittag-Leffler on December the 30th, 1905. Robert de Montessus wrote on the letter: dissertation not sent.
[^6]: The Grand Prix of 1906 corresponded in fact in a throw in of a Grand Prix proposed in 1904.
Table 1: publications by Robert de Montessus in 1905 and 1909

| year | author             | title                                | publisher                                       |
|------|--------------------|--------------------------------------|-------------------------------------------------|
| 1905 | Robert de Montessus| Sur les fractions continues algébriques | volume XIX, pages 185-257, Rendiconti del Circolo Matematico di Palermo. |
| 1909 | Montessus de Ballore R. | Les fractions continues algébriques | Acta mathematica, volume 32, pages 257-282 |

3.2 In between the two world wars

The result was not forgotten after the first world war. Several English or American authors took interest in the theorem. As an example, the English mathematician R. Wilson contacted Robert de Montessus in 1923:

(letter of October the 19th, 1923) *I am interested in the development of M. Padés work on the representation of a function by means of a continued fractions, and find that you have contributed much recent to this subject [...]*

and

(letter of October the 30th, 1923) *Your important work on the continued fractions, and the singularities of this function, is special interest to me [...]*

The American mathematician J.L. Walsh referred to the theorem in a review of the A.M.S. published in 1935. It appears that he was a contributor to the circulation of the result. Indeed, he was in Paris in 1920-1921, where he worked with Paul Montel. He defended his thesis in 1927 under the supervision of Maxime Böcher. We shall mention here that Maxime Böcher stayed in Paris in the years 1913-1914 at the occasion of an exchange between the Universities of Harvard and Paris. Maxime Böcher had F. Klein for his thesis supervision. Yet, E.B. Van Vleck who we mentioned several times already, defended as well his thesis under the supervision of F. Klein. Then he supervised himself H.S. Wall in 1927, who mentioned then the theorem of Robert de Montessus de Ballore. In the sixties J.L. Walsh referred to the theorem of Robert de Montessus in a series of articles. Thus, J.L. Walsh induced the circulation of the theorem after the second world war.

The following table gives the articles containing a reference to the theorem of Robert de Montessus and published during the period 1924-1940. Nörlund quoted the theorem again in 1924; Perron as well in the second edition of his book on continued fractions.

57Collection of letters received by Robert de Montessus de Ballore, period 1897-1937, letters from Wilson sent on October the 19th and 30th, 1923.
58In 1893.
59To be more precise, in 1964, 1965 and 1967.
60Maybe he is not the only one?
4 Conclusion

To conclude this study, it appeared that the result of Robert de Montessus is part of a group of research on the issue of the convergence of algebraic continued fractions led by several mathematicians since the middle of the 19th century. We have also linked this issue to the problem of the analytical continuation and to other questions in the analysis domain, such as the summation of divergent series or the moment problem. Robert de Montessus summarized in his article of 1902 the work of Padé on the link between taylor series, continued fractions and fractions approchées, and the work of Hadamard on polar singularities. These are the strength of his work and the reason it has been quickly circulated.

The circulation took place in different ways: by publishing in foreign magazines, thanks to the will of Robert de Montessus to communicate with the mathematical community of his time, and at last by choosing a relevant topic.

After the second world war, the result of Robert de Montessus tackled already more domains than the continued fractions only, namely interpolation and approximation. The American mathematician J. L. Walsh, who also wrote in French and in German, played an important role. No communications with Robert de Montessus are known.

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62 Hélène Gispert points out that French mathematician did not published in foreign magazines, except Henri Poincaré. [19].
Circulation of a theorem

| year | author           | title                                                      | publisher                                      |
|------|------------------|------------------------------------------------------------|------------------------------------------------|
| 1924 | Nörlund N. E.    | Vorlesungen über Differenzenrechnung                      | Berlin Verlag von Julius Springer              |
| 1927 | Wilson R.        | Divergent continued fractions and polar singularities     | Proceedings L. M. S. (2) 26, 159-168           |
| 1928 | Wilson R.        | Divergent continued fractions and polar singularities     | Proceedings L. M. S. (2) 27, 497-512; (2) 28, 128-144 |
| 1929 | Perron O.        | Die Lehre von den Kettenbruchen                            | Second edition, revised. Leipzig and Berlin, Teubner |
| 1935 | Walsh J.L.       | Interpolation and approximation by rational functions in the complex domain | New York, American Mathematical Society (Amer. Math. Soc. Colloquium Publ. Vol. XX) |
| 1939 | Scott W.T., Wall H.S. | Continued Fractions                                      | National Mathematics Magazine, Vol. 13, No. 7, pp. 305-322 |
| 1940 | Mall J.          | Beitrag zur Theorie der mehrdimensionalen Padéschen Tafel | [J] Math. Z. 46, 337-349                        |

Table 3: Author referring to the theorem in the years 1924-1940

The position of Robert de Montessus at the editorial board of the Journal de Mathématiques Pures et Appliquées might have played a part in the circulation of his result as well.

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