Ultra-hard fluid and scalar field in the Kerr-Newman metric

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An analytic solution for the accretion of ultra-hard perfect fluid onto a moving Kerr-Newman black hole is found. This solution is a generalization of the previously known solution by Petrich, Shapiro and Teukolsky for a Kerr black hole. We show that the found solution is applicable for the case of a non-extreme black hole, however it cannot describe the accretion onto an extreme black hole due to violation of the test fluid approximation. We also present a stationary solution for a massless scalar field in the metric of a Kerr-Newman naked singularity.

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I. INTRODUCTION

The only known three-dimensional exact solution for the accretion flow onto the Kerr black hole is the analytical solution by Petrich, Shapiro and Teukolsky [1]. This solution describes the stationary accretion of a perfect fluid with the ultra-hard equation of state, \( p = \rho \), with \( \rho \) being the pressure and \( \rho \) being the energy density, onto a moving Kerr black hole (see also [2, 3, 4, 5]). Here we generalize this solution to the case of a moving Kerr-Newman black hole. For the Kerr-Newman metric with naked singularity, we present the stationary solution for a massless scalar field.

The problem of a steady-state accretion onto a (moving) black hole can be formulated as follows. Consider a black hole moving through a fluid with a given equation of state, \( p = p(\rho) \) [27]. Usually it is assumed that the back-reaction of fluid to the metric is negligible, in which case the problem is being solved in the test fluid approximation. It is also assumed that the black hole mass changes in time slowly enough, such that the steady-state accretion is established (so called quasi-stationary process). The goal is to find a stationary solution to the equations of motion for the flowing fluid in the gravitational field of the black hole.

The Fig. 1 illustrates an example of this solution for the case of fluid passing the Schwarzschild black hole. It is unlikely that the Kerr-Newman black hole (or a naked singularity) is found in astrophysical context, as well as the accreting fluid with the ultra-hard equation of state. However, the theoretical study of these questions can be useful for better understanding both the principal General Relativity problems, e.g. the approaching to the extreme black hole state, and the real matter accretion onto the astrophysical compact objects.

The velocity of relativistic perfect fluid, \( u^{\mu} \), in the absence of vorticity can be expressed as the gradient of the scalar potential \( \psi \), see, e.g. [3],

\[
h u_{\mu} = \psi_{,\mu},
\]

where \( h = d\rho/d\nu = (p + \rho)/\nu \) is the fluid enthalpy properly normalized, \( h = (\psi^{,\alpha}\psi,_{\alpha})^{1/2} \), \( \nu \) is the particle number density. When the equation of state for the perfect fluid has a simple form, \( p = \rho(p) \), then the number density can be expressed in terms of the energy density and the pressure:

\[
\frac{n(\rho)}{n_{\infty}} = \exp \left[ \int_{\rho_{\infty}}^{\rho} \frac{d\rho'}{\rho' + p(\rho')} \right],
\]

where \( n_{\infty} \) and \( \rho_{\infty} \) are correspondingly the number density and the energy density at the infinity. The above equation can be seen as a formal definition of the “number density” for a perfect fluid with an arbitrary equation of state \( p = p(\rho) \).

A specific feature of a fluid with the ultra-hard equation of state is that the resulting equation for the scalar potential is linear [7]. For \( p = \rho \), from Eq. (2) one easily obtains, \( \rho \propto n^{2} \). As a result, the number density conservation, \( (nu^{p})_{,p} = 0 \), gives the Klein-Gordon equation for a massless scalar field, \( \Box \psi = 0 \). This reveals a way to obtain an exact solution of the accretion problem in the Kerr metric by separation of variables and decomposition in spherical harmonic series [1]. Our study of the perfect fluids and scalar fields in the Kerr-Newman metric closely follows the analysis of Petrich et al. [1].

The introduction of the potential \( \psi \), Eq. (1), provides a way to solve a problem of hydrodynamics dealing with an equation for a scalar function. This is not surprising, since any scalar field \( \psi \) is fully equivalent to the hydrodynamical description, provided that the vector \( \psi^{,\mu} \) is timelike. For example, the canonical massless scalar field \( \psi \) with the energy-momentum tensor,

\[
T_{\mu\nu} = \psi^{,\mu}\psi_{,\nu} - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \psi_{,\rho}\psi_{,\sigma}.
\]
is equivalent to a perfect fluid with the ultra-hard equation of state, provided the following identifications are taken into account:

\[ u_\mu = \frac{\psi_\mu}{\sqrt{\psi_\mu \psi^\mu}}, \quad p = \rho = \frac{1}{2} \psi_\mu \psi^\mu. \tag{4} \]

Therefore, in what follows we do not make a distinction between massless scalar field and ultra-hard perfect fluid, as far as vector \( \psi^\mu \) is timelike. For example, the problem of accretion of a perfect fluid can be viewed as accretion of a scalar field, with a specific form of the boundary condition at the infinity, \( \psi \to \psi_\infty t \). Here \( \psi_\infty \) is expressed in terms of the fluid density at the infinity, \( \rho_\infty \), as follows, \( \psi_\infty = (2\rho_\infty)^{1/2} \). However, if a solution for the scalar field contains spacelike \( \psi^\mu \) somewhere then such a solution can be identified with no perfect fluid analogue.

The paper is organized as follows. Accretion of ultra-hard perfect fluid (massless scalar field) onto the moving Kerr-Newman black hole is considered in Sec. III. The basic equation of motion for the scalar potential is derived in Sec. IIIA in Sec. IIIB we solve this equation for the case of the moving non-extreme black hole; in Sec. IIIC we analyze in detail the case of the extreme black hole. In Sec. III we find a stationary solution for the scalar field in the Kerr-Newman singularity. In Sec. IV we briefly describe the results of the work.

II. ACCRETION OF ULTRA-HARD FLUID

A. Metric and equation of motion

The Kerr-Newman metric can be written as \[ ds^2 = \frac{\Sigma \Delta}{A} dt^2 - \frac{A \sin^2 \theta}{\Sigma} (d\phi - \omega dt)^2 - \frac{\Sigma}{\Delta} dr^2 - \Sigma d\theta^2, \tag{5} \]

where

\[ \Delta = r^2 - 2Mr + a^2 + Q^2; \tag{6} \]
\[ \Sigma = r^2 + a^2 \cos^2 \theta; \tag{7} \]
\[ A = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta; \tag{8} \]
\[ \omega = \frac{2Mr - Q^2}{A} a. \tag{9} \]

Here \( M \) is the mass of a black hole or a naked singularity, \( a \) is the specific angular momentum, \( Q \) is the electric charge and \( \omega \) is the angular dragging velocity. The event horizon of the Kerr-Newman black hole, \( r = r_+ \), is the larger root of the equation \( \Delta = 0 \), i.e., \( r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2} \). The event horizon exists only if \( M^2 \geq a^2 + Q^2 \). When \( M^2 < a^2 + Q^2 \) the metric \( ds^2 \) describes a naked singularity.

The Klein-Gordon equation for a massless scalar field in the background gravitational field \( g_{\alpha\beta} \) is given by

\[ \psi^{;\alpha}_{;\alpha} = \frac{1}{\sqrt{-g} \partial x^\alpha} \left( \sqrt{-g} \partial_\alpha (\partial^\alpha \psi) \right) = 0. \tag{10} \]

For the Kerr-Newman metric \( ds^2 \) the above equation can be rewritten as

\[ \left\{ \frac{1}{\Delta} \left[ (r^2 + a^2) \partial_r + a \partial_\phi \right]^2 - \frac{1}{\sin^2 \theta} \left( \partial_\theta + a \sin^2 \theta \partial_\phi \right)^2 \right\} \psi = 0. \tag{11} \]

Eq. (11) is the main equation we will use to find analytic solutions.

B. Accretion onto moving black hole

First we consider a stationary accretion of an ultra-hard perfect fluid onto a non-extreme black hole, i.e., we assume \( M^2 > a^2 + Q^2 \). The first boundary condition for
Eq. (11) is a relation at the space infinity \( [1] \), \( r \to \infty \):

\[
\psi = -u^0_\infty t + u_\infty r [\cos \theta \cos \theta_0 + \sin \theta \sin \theta_0 \cos(\phi - \phi_0)],
\]

where \( u^0_\infty \) is the 0-component of the black hole 4-velocity at infinity

\[
u^0 \equiv (u^0, u_\infty) = (1 - v^2)^{-1/2}(1, v_\infty),
\]

and \( u_\infty \equiv |u_\infty| \); while \( v_\infty \) is the 3-velocity vector of the black hole at the infinity with the space orientation specified by the two arbitrary angles, \( \theta_0 \) and \( \phi_0 \).

There are several approaches to formulate the second boundary condition for Eq. (11) in the case of the stationary accretion onto black hole.

The classical approach, originated from the Bondi problem \( [10, 11] \), fixes the value of the energy flux onto a black hole at the critical sound surface \( [12, 13, 14, 15] \). The critical point is fixed by the requirement, that the physical solution is double valued neither in the velocity of the fluid nor in the radial coordinate. This requirement picks up one physical solution from the infinite number of formal solutions, parameterized by the flux of the energy onto a black hole. In Fig. 2 several solutions with different fluxes are shown, along with the “correct” (critical) one.

An alternative way to fix the flux is to ask that the energy density is finite at the black hole event horizon \( r = r_+ \) (see e.g. \([1]\)). In most cases both approaches give the same result. In particular, this is true for a non-extreme Kerr-Newman black hole. As we see later, however, in the specific case of the extreme Kerr-Newman black hole the requirement for the flow to have a trans-sonic point gives a solution for which the energy density is infinite at the horizon (see details in the next Section II C).

Note, that at the critical surface the infall velocity of the accreting fluid reaches the sound speed \( c_s \), which for the ultra-hard fluid is equal to the speed of light, \( c_s = 1 \). For this reason in the considered case a sound surface coincides with the event horizon \( r = r_+ \).

Following Petrich et al. \([1]\), we search a solution of Eq. (11) separating variables and decomposing \( \psi \) in the spherical harmonic series:

\[
\psi = -u^0_\infty t + \sum_{l,m} A_{l,m} R_l(r) Y_{lm}(\theta, \phi), \quad (14)
\]

where the radial part \( R_l(r) \) satisfies the equation

\[
\frac{d}{dr} \left( \Delta \frac{dR_l(r)}{dr} \right) + \left[ -l(l+1) + \frac{m^2 a^2}{\Delta} \right] R_l(r) = 0. \quad (15)
\]

It is convenient to define a new variable \( \xi \) by the relation,

\[
r = M + \xi \sqrt{M^2 - a^2 - Q^2}. \quad (16)
\]

Note that the definition of \( \xi \), Eq. (16), does not work in the case of the extreme black hole. Using (16) we can rewrite Eq. (15) in the form

\[
(1 - \xi^2)R''_{\xi} - 2\xi R'_{\xi} + \left[ l(l+1) - \frac{m^2(\alpha^2)}{1 - \xi^2} \right] R = 0, \quad (17)
\]

where \( \alpha = a/\sqrt{M^2 - a^2 - Q^2} \). This is the Legendre equation with an imaginary second index \([16]\). The general solution of Eq. (14) is

\[
\psi = -u^0_\infty t + \sum_i [A_1 P_i(\xi) + B_i Q_i(\xi)] Y_{10}(\theta, \phi) + \sum_{lm} [A^l_{lm} P^{lna}(\xi) + A^l_{lm} P^{l-ima}(\xi)] Y_{lm}(\theta, \phi). \quad (18)
\]

In the above equation the prime (‘) denotes that we do not include the term with \( m = 0 \); and \( P_l \) and \( Q_l \) are the Legendre functions of the first and second kind correspondingly; and \( P^{ima}_l \) is the associated Legendre function which can be expressed in terms of the hypergeometric function \([16]\):

\[
P^{ima}_l(\xi) \propto e^{ima} F(-l, l+1; 1 - ima; (1 - \xi)/2) \quad (19)
\]

where

\[
\chi = \frac{\alpha}{2} \ln \frac{\xi + 1}{\xi - 1} = \frac{\alpha}{2\sqrt{M^2 - a^2 - Q^2}} \ln \frac{r - r_-}{r - r_+}. \quad (20)
\]

Then the components of the 4-velocity are
\[ \begin{align*}
nu_t &= -u^0_\infty \\
nu_r &= \left\{ \sum_l [A_l(P_l)_\xi + B_l(Q_l)_\xi] Y_0 + \sum_{lm} [A^+_l(P_{l+m})_\xi + A^-_l(P_{l-m})_\xi] Y_{lm} \right\} / \sqrt{M^2 - a^2 - Q^2} \\
nu_\theta &= \left\{ \sum_l [A_l P_l + B_l Q_l] \partial \xi Y_0 + \sum_{lm} \left[ A^+_l P_{l+m} + A^-_l P_{l-m} \right] \partial Y_{lm} \right\} / \partial \theta \\
nu_\phi &= \left\{ \sum_{lm} \left[ A^+_l P_{l+m} + A^-_l P_{l-m} \right] \partial Y_{lm} \right\} / \partial \phi. \tag{21} \end{align*} \]

where subindex \( \xi \) denotes a derivative on the variable \( \xi \). Using the normalization condition \( u_\alpha u^\alpha = 1 \), we obtain

\[ n^2 = (\Sigma \Delta)^{-1} [(r^2 + a^2)u^0_\infty - anu_\phi]^2 - (\Sigma \sin^2 \theta)^{-1} [nu_\phi - a \sin^2 \theta u^0_\infty]^2 - \frac{\Delta}{\Sigma} (nu_r)^2 - \Sigma^{-1} (nu_\theta)^2. \tag{22} \]

In the limit \( r \to r_+ \) we find,

\[ (n^2 \Delta \Sigma)|_{r \to r_+} = \left[ (r^2 + a^2)u^0_\infty - \sum_{l,m} \im A^+_l e^{im\phi} + A^-_l e^{-im\phi} Y_{lm}(\theta, \phi) \right]^2 \]

\[ - \left[ a \sum_{l,m} \im (-A^+_l e^{im\phi} + A^-_l e^{-im\phi}) Y_{lm}(\theta, \phi) - \sqrt{M^2 - a^2 - Q^2} \sum_l B_l Y_{lm}(\theta, \phi) \right]^2. \tag{23} \]

From the finiteness of the fluid density at the event horizon it follows that all \( B_l \) equal to zero except \( B_0 \), and \( B_0 > 0 \). All coefficients \( A^+_l \) are also zero, since the terms containing \( A^-_l \) are irregular at the horizon. The corresponding solution, which is finite at the horizon, reduces to

\[ \psi = -u^0_\infty t + \frac{(r^2 + a^2)u^0_\infty}{2 \sqrt{M^2 - a^2 - Q^2}} \ln \frac{r - r_+}{r - r_+} + \sum_{l,m} A_{lm} F(-l, l + 1; 1 + im; (1 - \xi)/2) Y_{lm}(\theta, \phi - \chi). \tag{24} \]

where the second term comes from the Legendre function \( Q_0(x) \):

\[ Q_0(x) = \frac{1}{2} \ln \frac{1 + x}{1 - x}. \tag{25} \]

Using the boundary condition at the infinity, Eq. (12), we finally obtain,

\[ \psi = -u^0_\infty t + \frac{(r^2 + a^2)u^0_\infty}{2 \sqrt{M^2 - a^2 - Q^2}} \ln \frac{r - r_+}{r - r_+} + u_\infty (r - M) \cos \theta \cos \theta_0 + u_\infty \Re [(r - M + ia) \sin \theta \sin \theta_0 e^{i(\phi - \phi_0 - \chi)}]. \tag{26} \]

In the case of the Kerr black hole the solution (26) reduces to the known solution by Petrich et al. From (2) one can find the components of the 4-velocity:

\[ \begin{align*}
nu_t &= -u^0_\infty, \\
nu_r &= -u^0_\infty \frac{r^2 + a^2}{\Delta} + u_\infty \cos \theta \cos \theta_0 + u_\infty \Re \left\{ 1 + \frac{ia}{\Delta} (r - M + ia) \right\} \sin \theta \sin \theta_0 e^{i(\phi - \phi_0 - \chi)}; \\
nu_\theta &= -u_\infty (r - M) \sin \theta \cos \theta_0 + u_\infty \Re [(r - M + ia) \cos \theta \sin \theta_0 e^{i(\phi - \phi_0 - \chi)}]; \\
nu_\phi &= -u_\infty \Im [(r - M + ia) \sin \theta \sin \theta_0 e^{i(\phi - \phi_0 - \chi)}]. \tag{27} \end{align*} \]

The accretion rate of the fluid onto the black hole is given by

\[ \dot{N} = -\int_S \nu^t \sqrt{-g} dS = 4\pi (r^2 + a^2) n_\infty u^0_\infty. \tag{28} \]

While, the corresponding rate of the energy flux onto the black hole is

\[ \dot{M} = 4\pi (r^2 + a^2) (\rho_\infty + p_\infty) u^0_\infty. \tag{29} \]

Note that as in the Kerr case, the flux (28) is independent
on the direction of the black hole motion.

From Eq. (20) one can see, that in the limit of the extreme black hole, $M^2 - a^2 - Q^2 \to 0$, the solution for the accretion is problematic, since $\psi$ contains a divergent term proportional to $(M^2 - a^2 - Q^2)^{-1/2}$. Moreover, in this limit $n \to \infty$ while $u^r \to 0$ at the event horizon, which clearly signals on the violation of the test fluid approximation. However, it might happen, that going to the limit $M^2 - a^2 - Q^2 \to 0$ we missed a “correct” regular solution, valid only in the extreme case. To check this, in the next Section II C we will redo all the calculation for the extreme case, $M^2 = a^2 + Q^2$.

Let us study in more details some particular cases. First, we set $a = 0$, i.e. we consider a moving non-rotating charged black hole. As in the case of a Schwarzschild black hole (see [1]), there is a stagnation point: the point where the fluid velocity is zero relative to the black hole. It is not difficult to find the location of the stagnation point:

$$r_{stag} = M \left\{ 1 + \left( 1 + \frac{2r_+ M - Q^2 (1 + v_\infty)}{M^2 v_\infty} \right)^{1/2} \right\}$$

As an example, in the Fig. 3 we show the three-velocity of the fluid in the case of Reissner-Nordström black hole with $Q = 0.99M$ and $v_\infty = 0.5$.

Let us turn to another particular case. Substituting $v_\infty = 0$ into Eq. (22) we find the radial density distribution of the accreting ultra-hard fluid onto the Kerr-Newman black hole at rest:

$$n^2 = \frac{A - (r_+^2 + a^2)^2}{\Sigma \Delta}.$$ (30)

This radial distribution is shown in Fig. 4. At the event horizon $r = r_+$ the ratio of densities at the equator ($\theta = \pi/2$) and the pole ($\theta = 0$) is

$$\frac{\rho(r_+ \pi/2)}{\rho(r_+, 0)} = \frac{4r_+ (r_+^2 + a^2) - a^2 (r_+ - r_-)}{4r_+}.$$ (31)

In Appendix A we provide an alternative approach to solve a problem for the stationary accretion, in the case when a black hole is at rest.

**C. Accretion onto extreme black hole**

In this section we will concentrate specifically on the accretion onto the extreme Kerr-Newman black hole, trying to find a good solution. We search a stationary solution of the wave equation (11) in the form (14), like in the non-extreme case. The radial part $R_l(r)$ now satisfies Eq. (15) with $\Delta = (r - M)^2$. Using a new variable,

$$\zeta = r/M - 1,$$

we transform (15) to a simpler equation

$$\zeta^2 R''_{\zeta \zeta} + 2 \zeta R'_{\zeta} + \left[ \frac{m^2 a^2}{M^2 \zeta^2} - l(l + 1) \right] R = 0.$$ (32)
For $m = 0$ Eq. \((32)\) reduces to the Euler’s equation with a solution
\[
R = C_1 \left( \frac{r}{M} - 1 \right)^l + C_2 \left( \frac{r}{M} - 1 \right)^{-l-1}. \tag{33}
\]

The corresponding potential is
\[
\psi = -u_\infty^0 t + \sum_{l,m} \left[ C_{lm}^1 \left( \frac{r}{M} - 1 \right)^l + C_{lm}^2 \left( \frac{r}{M} - 1 \right)^{-l-1} \right] Y_{lm}. \tag{34}
\]

Using the boundary condition at infinity we find:
\[
\psi = -u_\infty^0 t + \frac{u_\infty^0 M^2}{r - M} + u_\infty (r - M) \cos \theta. \tag{35}
\]

For $m \neq 0$ the solution of Eq. \((15)\) is
\[
R = \sqrt{\frac{ma}{M} \xi} \left[ C_j \sqrt{\frac{ma}{M} \xi} + C_\nu \sqrt{\frac{ma}{M} \xi} \right], \tag{36}
\]

where $J$ and $\Upsilon$ is Bessel function of the first and second kind correspondingly. The general solution of Eq. \((32)\) is
\[
\psi = -u_\infty^0 t + \sum_{l,m} \sqrt{\frac{ma}{M} \xi} \left[ C_{lm}^1 J_{l+1/2} \left( \frac{ma}{M} \xi \right) + C_{lm}^2 \Upsilon_{l+1/2} \left( \frac{ma}{M} \xi \right) \right] Y_{lm}, \tag{37}
\]

where it used the Bessel functions \([17]\):
\[
J_\nu(x) = \sum_{k=0}^\infty \frac{(-1)^k (x/2)^{\nu+2k}}{k! \Gamma(\nu + k + 1)}, \tag{38}
\]
\[
\Upsilon_\nu(x) = \frac{J_\nu(x) \cos \pi \nu - J_{-\nu}(x)}{\sin \pi \nu}, \tag{39}
\]
\[
J_{-\nu}(x) = \sqrt{\frac{2}{\pi x}} \left( \frac{\cos x}{x} - \sin x \right). \tag{40}
\]

From the first boundary condition at space infinity we find $C_{10}^1 = C_{-1m}^2 = 0$, $C_{10}^1 = M u_\infty \cos \theta_0$, $C_{11}^{-2} = -\sqrt{\pi/2} u_\infty a \sin \theta_0$. While, the second boundary condition fixes the value of the influx at the sound surface: $C_{0m}^2 = C_{10}^2 = 0$, $C_{00}^1 = u_\infty^0(M^2 + a^2)/M$. With these boundary conditions we obtain
\[
\psi = -u_\infty^0 t + u_\infty (r - M) \cos \theta \cos \theta_0 + \frac{(M^2 + a^2)u_\infty^0}{r - M} \cos \theta \sin \theta_0 \sin \theta_0 \cos(\phi - \phi_0) \left( \frac{r - M}{a} \cos \frac{a}{r - M} + \sin \frac{a}{r - M} \right). \tag{41}
\]

The components of the 4-velocity are given by
\[
\begin{align*}
n u_t &= -u_\infty^0, \\
n u_r &= -\frac{(M^2 + a^2)u_\infty^0}{(r - M)^2} + u_\infty \cos \theta \cos \theta_0 \\
&{} + u_\infty \sin \theta_0 \sin \theta \cos(\phi - \phi_0) \cos \left( \frac{a}{r - M} \right) \left[ 1 - \frac{a^2}{(r - M)^2} + \frac{a}{r - M} \tan \left( \frac{a}{r - M} \right) \right], \\
n u_\theta &= -u_\infty (r - M) \sin \theta \cos \theta_0 + u_\infty a \cos \theta \sin \theta_0 \cos(\phi - \phi_0) \left[ \frac{r - M}{a} \cos \frac{a}{r - M} + \sin \frac{a}{r - M} \right], \\
n u_\phi &= -u_\infty a \sin \theta_0 \sin \theta \sin(\phi - \phi_0) \left[ \frac{r - M}{a} \cos \frac{a}{r - M} + \sin \frac{a}{r - M} \right]. \tag{42}
\end{align*}
\]

Using the above solution, one can calculate the accretion rate
\[
\dot N = 4\pi (M^2 + a^2)u_\infty^0 n_\infty, \tag{43}
\]

and radial density distribution (for $u_\infty = 0$)
\[
n^2 = \frac{(r^2 + a^2)^2 - (M^2 + a^2)^2}{\Sigma (r - M)^2} - \frac{a^2 \sin^2 \theta}{\Sigma}. \tag{44}
\]
From (12) and (14) one can see that at the event horizon of the extreme black hole \( r_+ = M \) both the radial 4-velocity \( u^r \) and the energy density \( \rho \) behave as: \( u^r \to 0 \) and \( \rho \propto n^2 \propto (r - M)^{-1} \to \infty \). And the full mass of fluid near the black hole tends to infinity too. This behavior is an indication of violation of the test fluid approximation. For this reason the solution (11), (12) and (14) is not self-consistent, and back reaction of the accreting fluid must be taken into account to obtain a physically relevant solution.

III. SCALAR FIELD AROUND NAKED SINGULARITY

It is worthwhile to note that the spacetime of the Kerr-Newman naked singularity is pathological when both \( Q \) and \( a \) are nonzero [18]. In general, this spacetime is not globally hyperbolic, which means, in particular, that the Cauchy problem for the scalar field is not well-posed. The origin of the pathology is inside the region \( \mathcal{A} < 0 \), where function \( \mathcal{A} \) is given by Eq. (8). This region contains closed causal curves, and it is possible to start from a point at the asymptotically flat region far from the singularities, to go into the “pathological” region and to travel back in time there and then come back to the starting point. Thus a time machine can be constructed. If, however, \( Q = 0 \) or \( a = 0 \) the space-time is not pathological anywhere, except for the point of the physical singularity at \( r = 0 \).

Below we describe an analytic solution for the stationary distribution of the scalar field in the Kerr-Newman geometry. For simplicity we consider the case when the naked singularity is at rest, with respect to the scalar field at the infinity. Thus we impose the boundary condition at the infinity in the following form,

\[
\psi \to \dot{\psi}_\infty t + \psi_\infty \quad \text{at} \quad r \to \infty, \tag{45}
\]

where \( \dot{\psi}_\infty \) and \( \psi_\infty \) are constants.

The corresponding solution of the wave equation (11) is given by Eq. (18), provided that the following replacements are made,

\[
\xi \to -i\xi, \quad \alpha \to i\alpha, \quad -u^0 \to \dot{\psi}_\infty.
\]

The boundary condition at the infinity gives \( A_{lm}^+ = A_{lm}^- = 0 \) for all \( l \) and \( m \), since the solution should not depend on \( \phi \) as \( r \to \infty \). To satisfy Eq. (15) it is also necessary to put \( A_l = 0 \) and \( B_l = 0 \) for all \( l \neq 0 \), since the terms corresponding \( l \neq 0 \) contain solutions which diverge at \( r \to \infty \). The term containing \( A_0 \) which can be identified with \( \dot{\psi}_\infty \). The full solution which satisfies the boundary condition (45) can be written as

\[
\psi = \dot{\psi}_\infty t + \psi_\infty + B \left[ \arctan \left( \frac{r - M}{\sqrt{a^2 + Q^2 - M^2}} \right) - \frac{\pi}{2} \right], \tag{46}
\]

where the last term comes from \( Q_0 \), and \( B \) is an arbitrary constant. The components of the energy-momentum tensor are:

\[
\sqrt{-g} T^r_t = -\dot{\psi}_\infty B \sin \theta \sqrt{a^2 + Q^2 - M^2}, \tag{47}
\]

\[
T^t_t = -T^r_r = \frac{B^2 (a^2 + Q^2 - M^2) + A \dot{\psi}^2_\infty}{2 \Sigma \Delta}, \tag{48}
\]

\[
T^\theta_\theta = T^\phi_\phi = \frac{B^2 (a^2 + Q^2 - M^2) - A \dot{\psi}^2_\infty}{2 \Sigma \Delta}. \tag{49}
\]

The energy flux onto the singularity is nonzero when \( B \dot{\psi}_\infty \neq 0 \). The corresponding radial 4-velocity of the inflowing fluid is

\[
u^r = -\frac{B \sqrt{a^2 + Q^2 - M^2} \Delta \psi^{2}_{\infty} A - B^2 (a^2 + Q^2 - M^2)}{\sqrt{\psi^{2}_{\infty} A - B^2 (a^2 + Q^2 - M^2)}}. \tag{50}
\]

Note that the influx (11) is not fixed when both constants \( B \) and \( \dot{\psi}_\infty \) are nonzero, because in the case of the naked singularity one physical boundary condition is missing, which for the black hole was the boundary condition at the event horizon or at the critical point. The energy flux onto the central singularity is zero in particular cases, \( B = 0 \) or \( \dot{\psi}_\infty = 0 \).

A particular solution (19) with \( B = 0 \) and \( \dot{\psi}_\infty \neq 0 \) describes the stationary distribution of the massless scalar field (or, equivalently, the ultra-hard fluid with \( p = \rho \)) without the influx around the naked singularity. This solution has a very specific feature: the energy density of the scalar field \( \rho(r, \theta) = T^r_t \) from Eq. (18) becomes
This solution does not have a perfect fluid analogue, and the total mass of the scalar field for this solution is finite. Since the space-time with both $a$ and $Q$ non-zero is pathological, in what follows we consider only the Kerr and Reissner-Nordström cases, correspondingly $Q = 0$ and $a = 0$, where noncausal region $\mathcal{A} < 0$ is absent. For example, in Fig. 4 the distribution of the energy density of ultra-hard fluid with zero flux around the Kerr naked singularity is shown.

An another particular solution ([19] with $\psi_{\infty} = 0$ and $B \neq 0$, describes a stationary distribution of the massless scalar field with zero energy density at infinity. This solution does not have a perfect fluid analogue. In Fig. 6 the energy density distribution of the solution (46) in the quasi-static frame $(t, r, \theta, \phi)$:

$$\mathcal{T}^t_t = -\frac{1}{2} \hat{g}^{rr}(\psi, r)^2 = \frac{1}{2} B^2 \frac{r^2}{\Sigma \Delta} M^2,$$

where we define a super-extreme parameter

$$\epsilon = \sqrt{\frac{a^2 + Q^2 - M^2}{M^2}}, \quad a^2 + Q^2 - m^2 > 0.$$  

Substituting $\mathcal{T}^t_t$ from (53) into (52), we obtain

$$M_f = 2\pi B^2 \epsilon^2 M^2 \int_0^\infty \frac{dr}{\Delta} = \pi B^2 (\pi + 2 \arccot \epsilon) \epsilon M. \quad \text{(55)}$$

This equation relates the constant $B$ in (46) with a total mass of the scalar field.

**IV. CONCLUSION**

We found an analytic solution for the stationary accretion of ultra-hard perfect fluid onto the moving Kerr-Newman black hole. The presented solution is a generalization of Petrich, Teukolsky and Shapiro solution for the accretion onto the moving Kerr black hole. Our solution describes the velocity, and the density distribution of the perfect fluid with the equation of state $p(\rho) = \rho$ in space, when the Kerr-Newman black hole moves through the fluid. The solution is uniquely determined in terms of the scalar potential, the exact solution for which is given by Eq. (20). One can think of the found solution as of one to the problem for a canonical massless scalar field in the metric of a moving Kerr-Newman black hole, since these two description are equivalent for the considered problem. In this case the solution for the scalar field coincides with the scalar potential for the fluid, Eq. (20).

Our results are only valid in the test fluid approximation, put differently, the back reaction of the accreting fluid is ignored. In the case of a non-extreme Kerr-Newman black hole one can always find such a range of parameters that the back reaction can indeed be safely ignored. While this is not the case for the extreme black hole. We showed that for the extreme Kerr-Newman black hole the test fluid approximation is always violated, since the energy density of ultra-hard fluid diverges at the event horizon. A similar behavior of the energy density for the perfect fluids with sound velocity $c_s \geq 1$ occurs for the extreme Reissner-Nordström black hole (24). Apparently, if the black hole is almost extreme, $M^2 = Q^2 - a^2 \to +0$, the value of the energy density at the infinity must be tuned considerably, to avoid the violation of the test fluid approximation. Thus to solve correctly the problem of accretion in the case of an (almost) extreme black hole the back reaction of the fluid must be.
taken into account. This is in accord with [24, 25], where it was suggested that the back reaction is important for a description of a scalar particle absorption with a large angular momentum by a near extreme black hole. We will investigate this problem in a separate work [26].

We also presented an analytic solution for the massless scalar field in the metric of the Kerr-Newman naked singularity, i.e. when \(M^2 < Q^2 + a^2\). This solution describes the scalar field distribution around the naked singularity, in general with non-zero influx. The found solution contains some pathologies which are the consequences of the causality violation in the vicinity of the Kerr-Newman space-time with a naked singularity. For the Kerr or the Reissner-Nordström naked singularity the presented solution is well behaved in the whole spacetime, except for the point of the physical singularity.

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APPENDIX A: ALTERNATIVE FORMALISM FOR ACCRETION

Here we present an alternative approach to solve the problem of accretion in a specific case when a black hole is at rest, \(u_0^0 = 1, u_\infty = 0\). The way we solve the problem here is closer to the original “hydrodynamic” approach, described in [10, 11, 12], rather than the “scalar field approach” by Pertich et al. [1].

The equations for stationary distribution of ultra-hard fluid with the equation of state \(p = \rho\) in the Kerr-Newman metric is integrated directly, similar to the analogous problem in the Schwarzschild [12, 21, 22] and Reissner-Nordström metrics [23]. From (12) it follows that the specific azimuthal and longitudinal angular momentum are both zero:

\[
L_\phi = u_\phi = 0, \quad L_\theta = u_\theta = u_\theta = 0. \quad (A1)
\]

This means, in particular, that \(u_\theta = 0\), and so \(\theta = \text{const}\) along the lines of flow. Using this property of the stationary ultra-hard fluid, we find the first integrals of the energy momentum conservation

\[
T^\alpha_{\beta, \alpha} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} (T^\alpha_{\beta} \sqrt{-g}) = \frac{1}{2} g_{\alpha \gamma, \beta} T^{\alpha \gamma} = 0. \quad (A2)
\]

Integration of this equation for \(\beta = 0\) gives the first integral of motion (the relativistic Bernoulli energy conservation equation):

\[
(p + \rho) u_0 u^i \sqrt{-g} = C_1(\theta) M^2, \quad (A3)
\]

where \(u = u^i\) is a radial 4-velocity component and \(C_1(\theta)\) is a function of \(\theta\). To find the second integral of motion we write a projection equation

\[
u_\mu T^\mu_{\nu} = \rho, u^i (p + \rho) \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} (\sqrt{-g} u^\alpha) = 0, \quad (A4)
\]

which can be expressed as

\[
\rho \frac{u^\alpha}{p + \rho} + \frac{\partial}{\partial r} (\sqrt{-g} u^r) = 0. \quad (A5)
\]

Integration of (A5) gives the second integral of motion (the energy flux conservation):

\[
\sqrt{-g} u \exp \left[ \int_{r_\infty}^{r} \frac{d\rho'}{\rho' + p(\rho')} \right] = -A(\theta) M^2, \quad (A6)
\]

where \(A(\theta)\) is a function of \(\theta\). Using normalization condition \(u^\alpha u_\alpha = 1\), from integrals of motion (A3) and (A6) we find

\[
(p + \rho) \exp \left[ \int_{r_\infty}^{r} \frac{d\rho'}{\rho' + p(\rho')} \right] \sqrt{\Sigma(\Delta + \Sigma u^2)} \frac{A}{A} = -C_1(\theta) \quad (A7)
\]

where function \(A\) is given by (8). From Eqs. (A6) and (A7) for \(a = 0\) and \(Q = 0\), we can fix the functions \(C_1(\theta)\) and \(A(\theta)\).

\[
C_1(\theta) = -A(\theta)(p_\infty + \rho_\infty), \quad A(\theta) = A_0 \sin \theta, \quad (A8)
\]

where \(A_0 = \text{const}\) is a dimensionless constant. Following Michel [12], we find relations at the critical sound point \(r = r_*\):

\[
\Delta(r_*) = 0, \quad u_*^2 = \frac{(r_* + a^2)^2 (r_* - M)}{\Sigma(2r_* (r_*^2 + a^2) - a^2 \sin^2 \theta (r_* - M))}. \quad (A9)
\]

where \(u_* = u(r_*)\). From the first equation in (A8) it follows that for \(M^2 \geq a^2 + Q^2\), there are two critical points \(r_1 = r_-\) and \(r_2 = r_+\). The critical point at smaller radius, \(r_1 = r_-\), is inside the event horizon. The two points coincide only for the extreme black hole, \(M^2 = a^2 + Q^2\). This means that for the extreme black hole the boundary conditions must be different from ones in the non-extreme case. Using the general relation \(b = 2 + i - s\), derived in [13, 14, 15], between the number of boundary conditions \(b\), the number of invariants \(i\) and the number of critical surfaces \(s\), we can see that in the extreme case we must put only 3 boundary conditions, very similar to the case of a stationary atmosphere. Thus, a stationary atmosphere seems to be a more adequate description in the case of the extreme black hole, rather than a stationary accretion.
Using the relations (A8) for critical point \( r = r_+ \), we calculate the constant \( A_0 \), which determines the value of the accretion flow:

\[
A_0 = \frac{r_+^2 + a^2}{M^2}.
\]

Now, from (A6) and (A7) one can find the space distribution of the energy density, \( \rho = \rho(r, \theta) \), and the radial component of the 4-velocity, \( u = \left( \frac{A_0 M^2}{\Sigma} \right) \left( \frac{\rho}{\rho_\infty} \right) - \frac{1}{2} \Sigma (r - r_-) \sin^2 \theta \),

\[
\rho = \frac{(r + r_+)(r^2 + r_+^2 + 2a^2) - a^2(r - r_-)\sin^2 \theta}{\Sigma (r - r_-)},
\]

(A10)

The solution (A10) coincides with the one found in Sec. II B in the case when \( u_\infty = 0 \).

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