RIGIDITY OF THE SHARP BEZOUT ESTIMATE ON NONNEGATIVELY CURVED RIEMANN SURFACES

CHENGJIE YU\textsuperscript{1} AND CHUANGYUAN ZHANG

Abstract. In this short note, by using a general three circle theorem, we show the rigidity of the sharp Bezout estimate first found by Gang Liu on nonnegatively curved Riemann surfaces.

1. Introduction

On a complete noncompact Kähler manifold \((M, g)\), the space of holomorphic functions on \(M\) is denoted as \(\mathcal{O}(M)\). A function \(f\) on \(M\) is said to be of polynomial growth if there are some positive constants \(C\) and \(d\) such that
\[
|f(x)| \leq C(1 + r^d(x)) \quad \forall x \in M,
\]
where \(r(x)\) the distance function to some fixed point. The space of holomorphic functions of polynomial growth is denoted as \(\mathcal{P}(M)\), and the space of holomorphic functions on \(M\) satisfying (1.1) for some \(C > 0\) is denoted as \(\mathcal{O}_d(M)\). For any nonzero \(f \in \mathcal{P}(M)\), the degree of \(f\) is defined as
\[
\deg(f) = \inf\{d > 0 \mid f \in \mathcal{O}_d(M)\}.
\]
For \(x \in M\) and \(f \in \mathcal{O}(M)\), \(\text{ord}_x(f)\) means the vanishing order of \(f\) at \(x\).

The purpose of this short note is to give a proof of the following result.

Theorem 1.1. Let \((M, g)\) be a noncompact Riemann surface equipped with a complete conformal metric \(g\) such that the Guassian curvature of \(g\) is nonnegative. Then, for any nonzero holomorphic function \(f\) of polynomial growth on \(M\),
\[
\sum_{x \in M} \text{ord}_x(f) \leq \deg(f).
\]
Moreover, the equality of (1.3) holds for some nonconstant holomorphic function \(f\) of polynomial growth if and only if \((M, g)\) is biholomorphically isometric to \(\mathbb{C}\) equipped with the standard metric.

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The estimate (1.3) was first obtained in the unpublished preprint [3] by Liu. It is clear that the estimate (1.3) is sharp because the equality of the estimate holds for any nonzero polynomial on \( \mathbb{C} \). In this short note, we give an alternative proof of (1.3) using the following general three circle theorem on Riemann surfaces, in a similar spirit of [4] and characterize the rigidity of the estimate (1.3).

**Theorem 1.2.** Let \( M \) be a Riemann surface and \( \Omega \) be an open subset of \( M \). Let \( u \) and \( v \) be two continuous functions on \( \Omega \) that are subharmonic and supharmonic respectively. Suppose that \( v : \Omega \to (\inf \Omega v, \sup \Omega v) \) is proper and \( M_v(u, t) = \max_{x \in S_v(t)} u(x) \) is an increasing function for \( t \in (\inf \Omega v, \sup \Omega v) \) where \( S_v(t) = \{x \in \Omega \mid v(x) = t\} \). Then, \( M_v(u, t) \) is a convex function of \( t \in (\inf \Omega v, \sup \Omega v) \).

It seems that the original proof in [3] can not characterize the rigidity of the estimate (1.3). In higher dimension, there is an estimate in similar spirit which is not sharp in [1, Proposition 7.2].

In algebraic geometry, the Bezout theorem relates the number of roots and the degrees of polynomials. So, an estimate such as (1.3) is called a Bezout estimate (See [5, P. 244]). The Bezout estimate (1.3) is stronger than the sharp vanishing order estimate:

\[(1.4) \quad \text{ord}_x(f) \leq \deg(f)\]

for any nonzero \( f \in \mathcal{P}(M) \) and any \( x \in M \). Vanishing order estimate that is not sharp was first obtained by Mok [5] on complete noncompact Kähler manifolds with certain geometric conditions. In [4], Ni obtained the sharp estimate (1.4) under the assumptions of nonnegative holomorphic bisectional curvature and maximal volume growth. The second assumption of Ni’s result was later removed by Chen-Fu-Yin-Zhu [2] and finally the first assumption of Ni’ result was relaxed to nonnegative holomorphic sectional curvature by Liu [4]. According to all these works, it seems that there must be a higher dimensional analogue of the sharp Bezout estimate (1.3).

2. **Proof of the theorem**

Although the proof of the general three circle theorem (Theorem 1.2) is almost the same as that of the classical Hadamard’s three circle theorem on \( \mathbb{C} \), we give a proof to it for completeness.

**Proof of Theorem 1.2.** For any \( \inf \Omega v < t_1 < t_2 < t_3 < \sup \Omega v \), let

\[(2.1) \quad \tilde{v} = \frac{M_v(u, t_3) - M_v(u, t_1)}{t_3 - t_1} v + \frac{t_3 M_v(u, t_1) - t_1 M_v(u, t_3)}{t_3 - t_1} t_1.

Because \( M_v(u, t_3) - M_v(u, t_1) \geq 0 \), \( \tilde{v} \) is supharmonic. Moreover, it is clear that

\[\tilde{v} \geq u\]
on $S_v(t_1) \cup S_v(t_3)$. Hence, by maximum principle,
\[ \tilde{v} \geq u \]
in $A_v(t_1, t_3) := \{ x \in \Omega \mid t_1 < v(x) < t_3 \}$. Thus, for any $x \in S_v(t_2)$,
\[ u(x) \leq \tilde{v}(x) \leq \frac{M_v(u, t_3) - M_v(u, t_1)}{t_3 - t_1} t_2 + \frac{t_3 M_v(u, t_1) - t_1 M_v(u, t_3)}{t_3 - t_1} t_2 + \frac{t_2 - t_1}{t_3 - t_1} M_v(u, t_3) + \frac{t_3 - t_2}{t_3 - t_1} M_v(u, t_1). \]

Hence
\[ M_v(u, t_2) \leq \frac{t_2 - t_1}{t_3 - t_1} M_v(u, t_3) + \frac{t_3 - t_2}{t_3 - t_1} M_v(u, t_1). \]

This completes the proof of the theorem. \qed

Before the proof of Theorem 1.1 we need the following two lemmas.

**Lemma 2.1.** Let $(M, g)$ be a noncompact Riemann surface equipped a conformal metric $g$ and $f$ be a nonzero holomorphic function on $M$. Let $p_1, p_2, \ldots, p_n$ be $n$ distinct roots of $f$, $m_i = \text{ord}_{p_i}(f)$ and $\rho = \prod_{i=1}^{n} r_{p_i}^{m_i}$. Then,
\[ \lim_{r \to 0^+} \frac{\log M_{\rho}(|f|, r)}{\log r} = 1. \]

*Proof.* Let $r_0 = \frac{1}{4} \min\{r(p_i, p_j) \mid 1 \leq i < j \leq n\}$ and $\delta_0 \in (0, r_0]$ be such that for any $i = 1, 2, \ldots, n$ and $x \in B_{p_i}(\delta_0)$,
\[ c_1 r_{p_i}^{m_i}(x) \leq |f(x)| \leq C_2 r_{p_i}^{m_i}(x) \]
for some positive constants $c_1$ and $C_2$. Let $\rho_0 = \delta_0^N$ where $N = \sum_{i=1}^{n} m_i$. Then, for any $x \in M$ such that $\rho(x) = r < \rho_0$, $x \in B_{p_i}(\delta_0)$ for some $i = 1, 2, \ldots, n$. Then, for any $j \neq i$, by the triangle inequality,
\[ \delta_0 < r(p_i, p_j) - r_{p_i}(x) \leq r_{p_j}(x) \leq r(p_i, p_j) + r_{p_i}(x) \leq R_0 + \delta_0 \]
where $R_0 = \max\{r(p_i, p_j) \mid 1 \leq i < j \leq n\}$. Thus,
\[ \frac{r}{(R_0 + \delta_0)^{N-m_i}} \leq r_{p_i}^{m_i}(x) = \frac{\rho(x)}{r_{p_i}^{m_i}(x) \cdots r_{p_i}^{m_i}(x) \cdots r_{p_i}^{m_i}(x)} \leq \frac{\rho}{\delta_0^{N-m_i}}. \]

Combining this with (2.5), we know that
\[ \frac{c_1 r}{(R_0 + \delta_0)^{N-m_i}} \leq |f(x)| \leq \frac{C_2 r}{\delta_0^{N-m_i}}. \]

So, for any $x \in M$ with $\rho(x) = r < \rho_0$, we have
\[ c_3 r \leq |f(x)| \leq C_4 r \]
where
\[ c_3 = \min \left\{ \frac{c_i}{(R_0 + \delta_0)^{N - m_i}} \mid i = 1, 2, \ldots, n \right\} \]
and
\[ C_4 = \max \left\{ \frac{C_2}{\delta_0^{N - m_i}} \mid i = 1, 2, \ldots, n \right\}. \]

Hence
\[ \frac{C_4}{\log r} + 1 \leq \frac{\log M_\rho(|f|, r)}{\log r} \leq 1 + \frac{c_3}{\log r} \]
for any \( r > 0 \) small enough. Then, by the squeezing rule, we complete the proof of the lemma. \( \square \)

**Lemma 2.2.** Let \((M, g)\) be a noncompact Riemann surface equipped with a complete conformal metric \(g\). Let \(p_1, p_2, \ldots, p_n\) be \(n\) distinct points on \(M\), \(m_1, m_2, \ldots, m_n \in \mathbb{N}\) and \(\rho = r_1^{m_1}r_2^{m_2} \cdots r_n^{m_n}\). Let \(o\) be a fixed point on \(M\), \(R_0 = \max\{r(o, p_i) \mid 1 \leq i \leq n\}\).

Then, for any holomorphic function \(f\) on \(M\) and any \(r > (2R_0)^N\),
\[ M_o(|f|, r + f - R_0) \leq M_\rho(|f|, r) \leq M_o(|f|, r + R_0) \]
where \(N = \sum_{i=1}^{n} m_i\) and \(M_o(|f|, r) = \max_{x \in S_o(r)} |f(x)|\) with \(S_o(r) = \partial B_o(r)\).

**Proof.** By the triangle inequality
\[ r_o(x) - R_0 \leq r_{p_i}(x) \leq r_o(x) + R_0, \quad \forall x \in M, \]
for \(i = 1, 2, \ldots, n\). Then, for any \(x\) with \(\rho(x) = r > (2R_0)^N\), we have
\[ (r_o(x) + R_0)^N \geq \rho(x) = r > (2R_0)^N \]
which implies that
\[ r_o(x) > R_0. \]

Then, by the left hand side of (2.12),
\[ (r_o(x) - R_0)^N \leq \rho(x) = r. \]

Thus
\[ r_o(x) \leq r_o + R_0 \]
which means that \(S_\rho(r) \subset B_o(r_o + R_0)\) for any \(r > (2R_0)^N\). Therefore, by the principle of maximal modulus for holomorphic functions,
\[ M_\rho(|f|, r) \leq M_o(|f|, r_o + R_0). \]

On the other hand, for any \(x \in S_o(r)\), by the right hand side of (2.12),
\[ \rho(x) \leq (r + R_0)^N. \]
So, \( S_o(r) \subset B_o((r + R_0)^N) := \{ x \in M \mid \rho(x) \leq (r + R_0)^N \} \) and hence
\[
M_o(|f|, r) \leq M_o(|f|, (r + R_0)^N) \tag{2.19}
\]
by the principle of maximal modulus for holomorphic functions again. Thus
\[
M_o(|f|, r) \leq M_o(|f|, r^{\frac{1}{N}} - R_0) \tag{2.20}
\]
for any \( r > R_0^N \). This completes the proof of the theorem. \( \square \)

We are now ready to prove Theorem 1.4.

**Proof of Theorem 1.4.** Let \( p_1, p_2, \ldots, p_n \) be \( n \) distinct roots of \( f \) with \( m_i = \text{ord}_{p_i}(f) \) for \( i = 1, 2, \ldots, n \). Let \( \rho = r_{p_1}^{m_1} r_{p_2}^{m_2} \cdots r_{p_n}^{m_n} \). Then, by the Laplacian comparison, we know that
\[
\Delta r_{p_i} \leq \frac{1}{r_{p_i}} \tag{2.21}
\]
for \( i = 1, 2, \ldots, n \) in the sense of distribution. So,
\[
\Delta \log r_{p_i} \leq 0 \tag{2.22}
\]
on \( M \setminus \{ p_i \} \) in the sense of distribution which means that \( \log r_{p_i} \) is a supharmonic function on \( M \setminus \{ p_i \} \) for \( i = 1, 2, \ldots, n \). So \( \log \rho = \sum_{i=1}^{n} m_i \log r_{p_i} \) is a supharmonic function on \( \Omega = M \setminus \{ p_1, p_2, \ldots, p_n \} \). Moreover, it is clear that \( \log(|f|^2 + \epsilon) \) is a subharmonic function on \( M \). Then, by Theorem 1.2 \( \log M_o(|f|^2 + \epsilon, r) \) is a convex function of \( \log r \). Letting \( \epsilon \to 0^+ \), we know that \( \log M_o(|f|, r) \) is a convex function of \( \log r \). Then, \( \log M_o(|f|, r) - \log r \) is also a convex function of \( \log r \). Therefore, for any \( 0 < r_1 < r_2 < r_3 \),
\[
\log M_o(|f|, r_2) - \log r_2 \leq \log M_o(|f|, r_3) - \log r_3 \tag{2.23}
\]
Letting \( r_1 \to 0^+ \) in the last inequality, and using Lemma 2.1, we get
\[
\log M_o(|f|, r_2) - \log r_2 \leq \log M_o(|f|, r_3) - \log r_3 \tag{2.24}
\]
for any \( 0 < r_2 < r_3 \). Thus, letting \( r_2 = 1 \) in (2.24), we have
\[
M_o(|f|, r) \geq M_o(|f|, 1)r \tag{2.25}
\]
for any \( r > 1 \). Let \( a \in M \) be a fixed point and \( R_0 = \max\{ r(a, p_i) \mid 1 \leq i \leq n \} \). Then, by Lemma 2.2 and (2.25), for any \( r \) large enough,
\[
M_o(|f|, r^{1/N} + R_0) \geq M_o(|f|, r) \geq M_o(|f|, 1)r. \tag{2.26}
\]
Hence, for any \( r \) large enough,
\[
M_o(|f|, r) \geq M_o(|f|, 1)(r - R_0)^N. \tag{2.27}
\]
So

\[(2.28) \quad \deg(f) \geq N = \sum_{i=1}^{n} \text{ord}_{p_i}(f). \]

This completes the proof of the Bezout estimate \((1.3)\).

We next come to prove the rigidity of \((1.3)\). Let \(f\) be a nonconstant holomorphic function of polynomial growth such that equality of \((1.3)\) holds. Let \(d = \deg(f)\). It is then clear from Cheng’s Liouville theorem for harmonic functions that \(d \geq 1\). Let \(p_1, p_2, \cdots, p_n\) be all the distinct roots of \(f\), and \(m_i = \text{ord}_{p_i}(f)\) for \(i = 1, 2, \cdots, n\). Then \(d = \sum_{i=1}^{n} m_i\). By Lemma 2.2, we know that

\[(2.29) \quad \frac{\log M_\rho(|f|, r_{\frac{d}{2}} - R_0)}{\log r} \leq \frac{\log M_\rho(|f|, r)}{\log r} \leq \frac{\log M_\rho(|f|, r_{\frac{d}{2}} + R_0)}{\log r}\]

when \(r\) is large enough. Moreover, note that

\[(2.30) \quad \lim_{r \to +\infty} \frac{\log M_\rho(|f|, r_{\frac{d}{2}} + R_0)}{\log r} = \lim_{t \to +\infty} \frac{\log M_\rho(|f|, t)}{d \log(t + R_0)} = 1\]

and

\[(2.31) \quad \lim_{r \to +\infty} \frac{\log M_\rho(|f|, r_{\frac{d}{2}} - R_0)}{\log r} = \lim_{t \to +\infty} \frac{\log M_\rho(|f|, t)}{d \log(t - R_0)} = 1\]

where we have used the following conclusion proved in [4]:

\[(2.32) \quad \deg(f) = \lim_{r \to +\infty} \frac{\log M_\rho(|f|, r)}{\log r}\]

for any nonzero \(f \in \mathcal{P}(M)\). Thus, by the squeezing rule,

\[(2.33) \quad \lim_{r \to +\infty} \frac{\log M_\rho(|f|, r)}{\log r} = 1.\]

Then, by letting \(r_3 \to +\infty\) in \((2.23)\), we get

\[(2.34) \quad \log M_\rho(|f|, r_2) - \log r_2 \leq \log M_\rho(|f|, r_1) - \log r_1\]

for any \(0 < r_1 < r_2\). Combining this with \((2.24)\), we know that

\[\log M_\rho(|f|, r) - \log r = C\]

for some constant \(C\). For each \(r > 0\), let \(z_x \in S_\rho(r)\) achieve the maximal modulus of \(f\) on \(S_\rho(r)\) and

\[(2.35) \quad F(x) = \log |f(x)| - \log \rho(x) - C.\]
Then $F \leq 0$ and $F(z_r) = 0$. So, $F$ as a subharmonic function on $\Omega$ achieves its maximum value on $z_r$. By strong maximum principle for subharmonic functions, we know that $F \equiv 0$ and hence

\begin{equation}
\log \rho(x) = \log |f(x)| - C
\end{equation}

is harmonic $\Omega$ since $\Delta \log |f| = 0$ on $\Omega$. So

\begin{equation}
\Delta \log r_{p_1} = \Delta \log r_{p_2} = \cdots = \Delta \log r_{p_n} = 0
\end{equation}

and the equality of the Laplacian comparison \eqref{2.21} holds. Thus $(M, g)$ is flat and hence is biholomorphically isometric to $\mathbb{C}$ or a cylinder. Note that a cylinder admits no nonconstant holomorphic function of polynomial growth. So, $(M, g)$ is biholomorphically isometric to $\mathbb{C}$. \hfill \square

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\textbf{Department of Mathematics, Shantou University, Shantou, Guangdong, 515063, China}

\textit{Email address:} cjyu@stu.edu.cn

\textbf{Department of Mathematics, Shantou University, Shantou, Guangdong, 515063, China}

\textit{Email address:} 12cyzhang@stu.edu.cn