Construction of a Channel Code from an Arbitrary Source Code with Decoder Side Information

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Abstract

The construction of a channel code by using a source code with decoder side information is introduced. For the construction, any pair of encoder and decoder is available for a source code with decoder side information. A constrained-random-number generator, which generates random numbers satisfying a condition specified by a function and its value, is used to construct a stochastic channel encoder. The result suggests that we can divide the channel coding problem into the problems of channel encoding and source decoding with side information.

Index Terms

Shannon theory, channel coding, source code with decoder side information, constrained-random-number generator

I. INTRODUCTION

The source coding with decoder side information (Fig. 1) is a special case of the distributed coding of correlated sources introduced by Slepian and Wolf [38]. Let \((X^n, Y^n)\) be a pair of correlated sources. We consider a source code where an encoder transmits a codeword obtained from a source output \(X^n\) and a decoder reproduces \(X^n\) from the codeword and the side information \(Y^n\), where it is expected that the decoding error probability is close to zero. From the Slepian-Wolf theorem [38], the asymptotically optimum encoding rate for stationary memoryless sources is given by the conditional entropy \(H(X|Y)\). The result is extended to general correlated sources \((X, Y)\) in [25][40], where conditions such as stationarity and ergodicity are not assumed and the fundamental limit is given by the conditional spectral sup-entropy rate \(\overline{H}(X|Y)\). This paper considers a pair of general correlated sources \((X, Y)\), and the results can be applied to the stationary memoryless case.

The main result of this paper is that we can construct a channel code (Fig. 2) from a given source code of \(X\) with decoder side information \(Y\), where the channel input and output are given by \(X\) and \(Y\), respectively. We can construct a code that achieves the capacity by letting \(X\) be an optimum channel input random variable and using a source code achieving the limit \(\overline{H}(X|Y)\).

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Historically, the Slepian-Wolf codes are constructed by using channel codes. In [38], the code is given by using a set of randomly-generated channel codewords that covers the conditionally typical set of $X$ for a given $Y$. Wyner [42] introduced the Slepian-Wolf code by using parity check matrices, where it is shown by Elias [13] that the capacity of a binary symmetric channel is achievable by using a linear or a convolutional code. In accordance with this idea, the Slepian-Wolf codes are constructed in [3], [17], [21], [35] from turbo codes [5], polar codes [2], and low density parity check (LDPC) codes [16] with practical decoding algorithms. It should be noted that the correlation of two sources are assumed to be binary-symmetric. Codes for an asymmetric channel can be constructed by using the channel-input alphabet extension [4], [15, Sec. 6.2] or polar codes [20].

On the other hand, Cover [9] introduced the random binning method for constructing the Slepian-Wolf code, where conditions such as symmetric correlations are not assumed for two sources. Following this idea, Csiszár [11] proved that the fundamental limit is achievable by using a linear code. In [34], it is proved that the fundamental limit is achievable by using an LDPC code. These results are unified by using the 2-universal class of hash functions [7] and its extensions [28], [29]. It should be noted that the use of a typical-set decoder or a maximum-likelihood decoder is assumed in these results.

Based on the concept of hash functions, in this paper we adopt an approach where we construct a channel code from a source code with decoder side information, which is a kind of Slepian-Wolf code. A similar approach is investigated in the context of the linear codebook-level duality of channel codes and the Slepian-Wolf codes [8], where the symmetric correlation of two sources (channel input and output) is assumed. This paper does not assume such correlations. It should be noted that this approach is investigated in [24], [26], [28], [43], where these papers prove that there is a pair consisting of a source code with decoder side information and a encoding map to construct a channel code. However, a maximal-likelihood decoder is assumed, and it is unknown whether for an arbitrary given source code with decoder side information there is a good encoding map with which to construct a channel code. In [34], it is proved by assuming a stationary memoryless condition that for a given arbitrary linear source code with decoder side information there is a good encoding map with which to construct a channel code, where the encoding map is intractable. In contrast, this paper introduces a
tractable encoding map by using a constrained-random-number generator [26]. We can use any source code with decoder side information, where it is confirmed theoretically or empirically that the decoding error probability is small. Neither a typical-set decoder nor a maximal-likelihood decoder is assumed for the source code with decoder side information. Our result suggests that we can divide channel coding problem into the problems of channel encoding and source decoding with side information. It should be noted that the similar results have been appeared in [36, Remark 2], [43] when the output distribution of the encoder with side information is close to a uniform distribution. In contrast, this paper clarifies that such an assumption is unnecessary.

This paper is organized as follows. In Section II, we review the formula of the channel capacity. In Section III, we introduce the construction of a channel code by using an arbitrary source code with decoder side information. Based on the results in Section III, Section IV revisits the channel code using the constrained-random-number generator introduced in [26].

II. CHANNEL CAPACITY

This section reviews the definition of the capacity of a general channel. All the results in this paper are presented by using the information spectrum method introduced in [18], [19], [41], where the consistency and stationarity are not assumed.

Let $\mathcal{X}$ and $\mathcal{Y}$ be the alphabets of a channel input and output, respectively. Then product sets $\mathcal{X}^n$ and $\mathcal{Y}^n$ are the alphabets of a channel input vector $X^n$ and a channel output vector $Y^n$, respectively. It should be noted that $\mathcal{X}$ and $\mathcal{Y}$ are allowed to be infinite/uncountable/continuous sets on condition that probability distributions/measures $\mu_{X^n}$ and $\mu_{Y^n|X^n}(\cdot|x), x \in \mathcal{X}^n$ are well-defined.

We consider a general source and a general channel. A general source $X$ is defined by a sequence $X \equiv \{\mu_{X^n}\}_{n=1}^{\infty}$ of probability distributions and a general channel is defined by a sequence $W \equiv \{\mu_{Y^n|X^n}\}_{n=1}^{\infty}$ of conditional probability distributions.

Here, we define the operational channel capacity with a channel input constraint specified by a set $\mathcal{P}_n$ of probability distributions on $\mathcal{X}^n$. A typical example of a channel input constraint is the cost constraint, where any distribution $\mu \in \mathcal{P}_n$ satisfies $\int c_n(x)\mu(x)dx < C$ for a given cost function $c_n : \mathcal{X}^n \rightarrow [0, \infty)$ and $C \in [0, \infty)$.

**Definition 1:** Let $\mathcal{P} \equiv \{\mathcal{P}_n\}_{n=0}^{\infty}$ be a sequence of the set of probability distributions on $\mathcal{X}^n$. For a general channel $W$, we call a rate $R$ achievable if for all $\delta > 0$ and all sufficiently large $n$ there is a pair consisting of an encoder $\varphi_n : \mathcal{M}_n \rightarrow S^n$ and a decoder $\psi_n : \mathcal{Y}^n \rightarrow \mathcal{M}_n$ such that

$$\frac{1}{n} \log |\mathcal{M}_n| \geq R$$

$$\mu_{X^n} \in \mathcal{P}_n$$

$$P(\psi_n(Y^n) \neq M_n) \leq \delta,$$

where we call a subset $S$ of $\mathcal{X}$ a signaling alphabet\(^1\), $\mathcal{M}_n$ is a set of messages, $\lfloor 1/n \rfloor \log |\mathcal{M}_n|$ represents the rate of the code, $M_n$ is a random variable of the message corresponding to the uniform distribution on $\mathcal{M}_n$.

\(^1\)This terminology comes from [6].
$Y^n$ is the random variable of a channel output with an input $X^n \equiv \varphi_n(M_n)$, and the joint distribution $\mu_{M_nY^n}$ is given as

$$\mu_{M_nY^n}(m, y) \equiv \frac{\mu_{Y^n|X^n}(y|\varphi_n(m))}{|M_n|}.$$ 

The channel capacity $C_S(W)$ is defined by the supremum of the achievable rate, where the signaling alphabet $S$ is specified.

It should be noted that the standard definition of channel capacity can be denoted by $C_X(W)$, where the signaling alphabet $S$ is equal to $X$. It should also be noted that we can let $P^n$ be the set of all probability distributions on $X^n$ when it is assumed that there is no channel input constraint.

Next, let us define the capacity $C^q_X(W)$ of a channel with a finite signaling alphabet as

$$C^q_X(W) \equiv \sup_{S \subset \mathcal{X} : |S| \leq q} C_S(W).$$

Since $\{S : |S| \leq q\} \subset \{S : |S| \leq q + 1\}$, we have the fact that $C^q_X(W)$ is a non-decreasing function of $q$. We have the following lemma.

**Lemma 1 ([32, Theorem 2]):** When $C_X(W) < \infty$, we have

$$C_X(W) = \lim_{q \to \infty} C^q_X(W).$$

The proof is given in Section V-A for the completeness of this paper.

For a general channel $W$, the channel capacity $C_X(W)$ is derived in [41], [19, Theorem 3.6.1] as

$$C_X(W) = \sup_{X \in \mathcal{P}} J(X; Y),$$

where the supremum is taken over all general sources $X = \{\mu_{X^n}\}_{n=1}^\infty$ such that $\mu_{X^n} \in P_n$ for every $n$, and the joint distribution $\mu_{X^nY^n}$ is given as

$$\mu_{X^nY^n}(x, y) \equiv \mu_{Y^n|X^n}(y|x)\mu_{X^n}(x).$$

Furthermore, similarly to the proof in [26], we can show the formula

$$C_X(W) = \sup_{X \in \mathcal{P}} [H(X) - \mathcal{P}(X|Y)],$$

when $\mathcal{X}$ is finite, where the supremum is taken over all general sources $X$ and the joint distribution of $(X, Y)$ is given by (5). From Lemma 1, we have the following lemma.

**Lemma 2:** When $C_X(W) < \infty$, we have

$$C_X(W) = \lim_{q \to \infty} \sup_{S \subset \mathcal{X} : |S| \leq q} \sup_{X \in \mathcal{P} : X^n \in S^n \text{ for all } n} J(X; Y)$$

$$= \lim_{q \to \infty} \sup_{S \subset \mathcal{X} : |S| \leq q} \sup_{X \in \mathcal{P} : X^n \in S^n \text{ for all } n} [H(X) - \mathcal{P}(X|Y)],$$

where the condition $X^n \in S^n$ implies that the support of the probability distribution of a channel input is a subset of $S^n$.  

\footnote{In [19, Theorem 3.6.1], it is assumed that $\mathcal{P}$ is a cost constraint. However, we can easily extend the result to an arbitrary channel input constraint.}
Remark 1: For many channels it is known that an optimal input distribution in a channel coding has a discrete support, where a support is defined as the set of all elements with positive measure. For example, for an additive white Gaussian noise (AWGN) channel, it is shown in [39] that the optimal input distribution has a discrete and finite support under the maximum power constraint. It should be noted that the above lemma implies that we can approach the capacity with a sufficiently large signaling alphabet for any channel with an uncountable/continuous channel input alphabet (e.g. AWGN channel under the average power constraint).

In this paper, we show the fact that for a given finite set $X$ there is a code such that the rate of the code is close to the right hand side of (6). Then, from the above lemma, we have the fact that the capacity of a channel with an uncountable channel input alphabet is achievable with the code by optimizing the finite signaling alphabet $S$ and letting $|S| \to \infty$.

Remark 2: In [6, Section 7.8], the optimal signaling alphabet $S \in X$ is derived for an additive white Gaussian noise channel, where it is assumed that all the symbols in $S$ are used equally often, that is, the input distribution is uniform on $S$. This assumption is natural when we use conventional linear codes. On the other hand, it is unnecessary to assume that the input distribution is uniform on $S$ in the code construction introduced in [26], where the encoding rate may increase.

III. Channel Code by Using Source Code with Decoder Side Information

In this section, we construct a channel code by using a source code with decoder side information. To this end, we review a balanced-coloring property [27], which is a variant of the hash property introduced in [26], [28], [30], [31].

A. $(\alpha, \beta)$-Balanced-Coloring Property

Throughout this paper, we use the following definitions and notations. The complement of $U$ is denoted by $U^c$ and the set difference is defined as $U \setminus V \equiv U \cap V^c$. Let $Bx$ denote a value taken by a function $B$ at $x \in X^n$, where $B$ may be nonlinear. When $B$ is a linear function expressed by an $l \times n$ matrix, we assume that $X \equiv \text{GF}(q)$ is a finite field and the range of functions is $X^l$. For a function $B$ and a set $B$ of functions, let $\text{Im}B$ and $\text{Im}B$ be defined as

$\text{Im}B \equiv \{Bx : x \in X^n\}$

$\text{Im}B \equiv \bigcup_{B \in B} \text{Im}B.$

We define a set $C_B(m)$ as

$C_B(m) \equiv \{x : Bx = m\}.$

The random variables of a function $B$ and a vector $c$ are denoted by the sans serif letters $B$ and $c$, respectively. It should be noted that the random variable of an $n$-dimensional vector $x \in X^n$ is denoted by the Roman letter $X^n$ that does not represent a function, which is the way it has been used so far. The symbol $E$ denotes the expectation. For example, $E_{B_0}[^{\cdot}]$ denotes the expectation with respect to random variables $B$ and $c$.

Here, we introduce the balanced-coloring property for an ensemble of functions. It requires weaker conditions than the hash property introduced in [26], [30], [31].
Definition 2 ([27]): Let $B_n$ be a set of functions on $\mathcal{X}^n$ and $p_{B,n}$ be a probability distribution on $B_n$. We call a pair $(B_n, p_{B,n})$ an ensemble. Then a sequence $(B, p_B) \equiv \{(B_n, p_{B,n})\}_{n=1}^{\infty}$ has an $(\alpha_B, \beta_B)$-balanced-coloring property if there are two sequences $\alpha_B \equiv \{\alpha_B(n)\}_{n=1}^{\infty}$ and $\beta_B \equiv \{\beta_B(n)\}_{n=1}^{\infty}$, depending on $\{p_{B,n}\}_{n=1}^{\infty}$, such that

$$
\limsup_{n \to \infty} \alpha_B(n) = 1 \quad \text{(BC1)}
$$

$$
\limsup_{n \to \infty} \frac{1}{n} \log(\beta_B(n) + 1) = 0 \quad \text{(BC2)}
$$

and

$$
\sum_{x' \in \mathcal{X}^n \setminus \{x\} : p_{B,n}((B : Bx = Bx')) > \frac{\alpha_B(n)}{|\{x' \in \mathcal{T}\}|}} p_{B,n}\left(\left\{B : Bx = Bx'\right\}\right) \leq \beta_B(n) \quad \text{(BC3)}
$$

for all sufficiently large $n$ and all $x \in \mathcal{X}^n$. Throughout this paper, we omit the dependence of $B$, $p_B$, $\alpha_B$ and $\beta_B$ on $n$.

Here, let us introduce examples satisfying the balanced-coloring property. When $B$ is a two-universal class of hash functions [7] and $p_B$ is the uniform distribution on $B$, then $(B, p_B)$ has a $(1, 0)$-balanced-coloring property, where $1$ and $0$ denote the constant sequences of 1 and 0, respectively. Random binning [9] and a set of all linear functions [11] are examples of the two-universal class of hash functions. It is proved in [27] that an ensemble of systematic sparse matrices has a balanced-coloring property, where a matrix has an identity sub-matrix with the same number of rows.

The following lemma is an extension of the leftover hash lemma [22], the balanced-coloring lemma [1, Lemma 3.1], [12, Lemma 17.3], and the output statistics of random binning [43]. This lemma implies that there is a function $B$ such that $\mathcal{T}$ is almost equally partitioned by $B$ with respect to a measure $Q$.

Lemma 3 ([26, Lemma 5],[31, Lemma 4]): If $(B, p_B)$ satisfies (BC3), then

$$
E_B \left[ \frac{\sum_{y} Q(T \cap C_B(m))}{Q(T)} - \frac{1}{|\text{Im}B|} \right] \leq \sqrt{\alpha_B - 1 + \frac{[\beta_B + 1] |\text{Im}B| \max_{u \in \mathcal{T}} Q(u)}{Q(T)}}
$$

for any function $Q : \mathcal{X}^n \to [0, \infty)$ and $\mathcal{T} \subset \mathcal{X}^n$, where

$$
Q(T) \equiv \sum_{x \in \mathcal{T}} Q(x).
$$

B. Construction of Channel Code

This section introduces a channel code. The idea for the construction is drawn from [28], [31], [34]. We assume that the channel input alphabet $\mathcal{X}^n$ is a finite set but allow the channel output alphabet $\mathcal{Y}^n$ to be an arbitrary (infinite, continuous) set.

We assume that the channel distribution $\mu_{Y^n | X^n}$ and the input distribution $\mu_{X^n}$ are given. Let $\mu_{X^n | Y^n}$ be defined as

$$
\mu_{X^n | Y^n}(x | y) \equiv \frac{\mu_{Y^n | X^n}(y | x) \mu_{X^n}(x)}{\sum_x \mu_{Y^n | X^n}(y | x) \mu_{X^n}(x)}
$$

3In [27], an ensemble is required to satisfy (BC3) for all $n$ and all $x \in \mathcal{X}^n$. However, it is sufficient to assume that an ensemble satisfies (BC3) for sufficiently large $n$ and all $x \in \mathcal{X}^n$ because we finally let $n \to \infty$. 

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Here, we assume that an arbitrary fixed-length source code \((A, x_A)\) with decoder side information (Fig. 1) is given, where \(A : X^n \rightarrow \text{Im} A\) is an encoding function and \(x_A : \text{Im} A \times Y^n \rightarrow X^n\) is a decoding function. Then the coding rate \(r\) of the code is given as
\[
r = \frac{1}{n} \log |\text{Im} A|.
\]
(7)
The decoding error probability \(\text{Error}(A)\) of this code is given as
\[
\text{Error}(A) \equiv \mu_{X^n Y^n} \{ (x, y) : x_A(Ax|y) \neq x \}.
\]
(8)

It should be noted that the condition \(\lim_{n \rightarrow \infty} \text{Error}(A) = 0\) is not assumed for this code, and this code may be sub-optimal in the sense that the coding rate \(r\) is not close to the fundamental limit described as the conditional spectral sup-entropy rate \(\overline{H}(X|Y)\).

For a given \(R > 0\), let \((B, p_B)\) be an ensemble of functions on the set \(X^n\) satisfying
\[
R = \frac{1}{n} \log |\text{Im} B|,
\]
(9)
where we define \(M_n \equiv \text{Im} B\) and \(R\) represents the rate of the code. We obtain a function \(B \in \mathcal{B}\) and a vector \(c \in \text{Im} A\) generated at random subject to the distribution \(p_B\) and \(\{\mu_{X^n}(C_A(c))\}_{c \in \text{Im} A}\), respectively. It should be noted that we can obtain \(c \equiv Ax\) generated at random subject to the distribution \(\{\mu_{X^n}(C_A(c))\}_{c \in \text{Im} A}\) by generating \(x\) at random subject to the distribution \(\mu_{X^n}\) and operating \(A\) on \(x\).

We fix \(B\) and \(c\) so that they are shared by the channel encoder and the channel decoder. To summarize, the channel encoder has functions \(A, B\) and a vector \(c\), and the channel decoder has functions \(x_A, B,\) and a vector \(c\).

We use a constrained-random-number generator to construct a stochastic encoder. Let \(\tilde{X}_n \equiv \tilde{X}_n^{AB}(c, m)\) be a random variable corresponding to the distribution
\[
\nu_{\tilde{X}_n|M_n}(x|m) = \begin{cases} \frac{\mu_{X^n}(x)}{\mu_{X^n}(C_{AB}(c, m))}, & \text{if } x \in C_{AB}(c, m), \\ 0, & \text{if } x \notin C_{AB}(c, m), \end{cases}
\]
where \(C_{AB}(c, m) \equiv C_A(c) \cap C_B(m)\). The encoder generates \(x\) that satisfies \(Ax = c\) and \(Bx = m\) with probability \(\nu_{\tilde{X}_n|M_n}(x|m)\). It should be noted that we can use the sum-product algorithm or the Markov-Chain-Monte-Carlo method to implement the constrained-random-number generator [26], [27]. We define the stochastic channel encoder \(\Phi_n : \text{Im} B \rightarrow X^n\) as
\[
\Phi_n(m) \equiv \begin{cases} \tilde{X}_n^{AB}(c, m), & \text{if } \mu_{X^n}(C_{AB}(c, m)) > 0, \\ \text{"error"}, & \text{if } \mu_{X^n}(C_{AB}(c, m)) = 0. \end{cases}
\]
Let \(y \in Y^n\) be a channel output. We define the channel decoder \(\psi_n : Y^n \rightarrow \text{Im} B\) as
\[
\psi_n(y) \equiv Bx_A(c|y),
\]
where the decoder reproduces \(x\) that satisfies \(Ax = c\) by using \(x_A\) and obtains a reproduced message \(m = Bx\). The flow of vectors is illustrated in Fig. 3.

The error probability \(\text{Error}(A, B, c)\) is given by
\[
\text{Error}(A, B, c) \equiv \sum_{m : \mu_{X^n}(C_{AB}(c, m)) = 0} \frac{1}{|M_n|} + \sum_{m, x : y} \frac{\mu_{Y^n|(X^n(x))}(y|x)\mu_{X^n}(x)}{|M_n|\mu_{X^n}(C_{AB}(c, m))}.
\]
We have the following theorem, where the proof is given in Section V-B.

**Theorem 1:** Let \((A, x_A)\) be a source code with decoder side information, where the encoding rate and the decoding error probability are given by (7) and (8), respectively. Assume that \((B, p_B)\) has an \((\alpha_B, \beta_B)\)-balanced-coloring property for a given \(R\) satisfying (9) and

\[
 r + R < H(X),
\]

where \(H(X)\) is the spectral inf-entropy rate. Then for any \(\delta > 0\) and all sufficiently large \(n\) there is a function \(B \in \mathcal{B}\), and a vector \(c \in \text{Im}A\) such that

\[
 \text{Error}(A, B, c) \leq \text{Error}(A) + \delta.
\]

**Remark 3:** In [36, Remark 2], [43], inequality (11) is shown by assuming that the output distribution of an encoding function \(A\) is close to a uniform distribution. In contrast, such an assumption is not assumed in the above theorem.

From the above theorem and (6), the channel capacity is achievable with the proposed code by letting \(X\) be a source that attains the supremum on the right hand side of (6), \(R \to H(X) - I(X; Y), \delta \to 0\), and assuming that \((A, x_A)\) is optimum in the sense that \(r \to I(X; Y)\) and \(\text{Error}(A) \to 0\).

**IV. CHANNEL CODE USING CONSTRAINED-RANDOM-NUMBER GENERATOR REVISITED**

In this section, we revisit the channel code using the constrained-random-number generator introduced in [26]. To this end, we introduce a variant of the hash property [26], [28] and a construction of source code with decoder side information based on this property.

**A. \((\alpha, \beta)\)-Collision-Resistance Property**

In this section, we revisit [28, Remark 1], which mentions that some ensembles of sparse matrices satisfy the weaker condition \(\lim_{n \to \infty} [1/n] \log \alpha_A(n) = 0\). We introduce the collision-resistance property as follows.
Definition 3: Let $A_n$ be a set of functions on $X^n$ and $p_{A,n}$ be a probability distribution on $A_n$. Then a sequence $(A, p_A) \equiv \{(A_n, p_{A,n})\}_{n=1}^\infty$ has an $(\alpha_A, \beta_A)$-collision-resistance property if there are two sequences $\alpha_A \equiv \{\alpha_A(n)\}_{n=1}^\infty$ and $\beta_A \equiv \{\beta_A(n)\}_{n=1}^\infty$, depending on $p_{A,n}$, such that

$$\limsup_{n \to \infty} \frac{1}{n} \log \alpha_A(n) = 0 \quad \text{(CR1)}$$

$$\limsup_{n \to \infty} \beta_A(n) = 0 \quad \text{(CR2)}$$

and

$$\sum_{x' \in X^n \setminus \{x\}: p_{A,n}(\{A : Ax = Ax'\}) > \frac{\alpha_A(n)}{\|\text{Im}A\|}} \leq \beta_A(n) \quad \text{(CR3)}$$

for all sufficiently large $n$ and all $x \in X^n$, where (CR3) is the same as (BC3).

It should be noted that when $A$ is a two-universal class of hash functions [7] and $p_A$ is the uniform distribution on $A$, then $(A, p_A)$ also has a $(1,0)$-collision-resistance property. Random binning [9] and a set of all linear functions [11] are examples of the two-universal class of hash functions. The motivation for introducing the collision-resistance property is that we can include the expurgated ensemble of linear functions introduced in the next section.

The following lemma is related to the collision-resistance property, that is, if the number of bins is greater than the number of items then there is an assignment such that every bin contains at most one item.

Lemma 4 ([26, Lemma 4],[28, Lemma 1]): If $(A, p_A)$ satisfies (CR3), then

$$p_A(\{A : [G \setminus \{u\}] \cap C_A(Au) \neq \emptyset\}) \leq \frac{|G|\alpha_A}{|\text{Im}A|} + \beta_A$$

for all $G \subset U^n$ and $u \in U^n$.

B. Expurgated Ensemble of Linear Functions

In this section, we consider the expurgated ensemble of linear functions. The idea comes from [4], [14], [16], [23].

Let $X$ be a finite field and $A$ be a set of linear functions $A : X^n \to X^l$, where $A$ can be expressed by an $l \times n$ matrix. Let $t(x)$ be the type of $x \in U^n$, which is characterized by the empirical probability distribution of the sequence $x$. For a type $t$, let $C_t$ be defined as

$$C_t \equiv \{x \in X^n : t(x) = t\}.$$ 

Let $H$ be the set of all types of length $n$ except $t(0)$, where $0$ is the zero vector. For a probability distribution $p_A$ on a set of $l \times n$ matrices and a type $t$, let $S(p_A, t)$ be defined as

$$S(p_A, t) \equiv \sum_{A \in A} p_A(A) |\{x \in X^n : Ax = 0, t(x) = t\}|$$

$$= \sum_{A \in A} p_A(A) |C_A(0) \cap C_t|,$$

which is the expected number of codewords that have type $t$. For a given $H \subset H$, we define $\alpha_A(n)$ and $\beta_A(n)$ as

$$\alpha_A(n) = \frac{|\text{Im}A|}{|X|^n} \cdot \max_{t \in H} \frac{S(p_A, t)}{S(p_A, t)} \quad \text{(12)}$$
and $\beta$ (\(l\) and $C$ minimum Hamming distance between two different vectors in requirement that the weight (the number of non-zero symbols) is greater than $\hat{H}$ where (15) implicitly assumes that $\hat{H}$ is usually defined as the set of $t$ satisfying the requirement that the weight (the number of non-zero symbols) is greater than $\gamma n$, where $\gamma$ depends only on $|X|$ and $l$. Then the condition $t(x) \in \hat{H} \cup \{t(0)\}$ for all $x \in C_A(0)$ implies that the minimum distance of $A$ (the minimum Hamming distance between two different vectors in $C_A(0)$) is greater than $\gamma n$. That is, the ensemble $(A, \tilde{p}_A)$ is obtained from $(A, p_A)$ by expurgating functions with the minimum distance smaller than $\gamma n$.

We have the following lemma, where the proof is given in Section V-C.

**Lemma 5:** We first assume that for an ensemble $(A, p_A)$ of linear functions there is $\hat{H}$ such that a pair $\alpha_A$ and $\beta_A$ defined by (12) and (13), respectively, satisfy

\[
\limsup_{n \to \infty} \frac{1}{n} \log \alpha_A(n) = 0
\]

(14)

\[
\limsup_{n \to \infty} \frac{1}{n} \log \frac{1}{1 - \beta_A(n)} = 0,
\]

(15)

where (15) implicitly assumes that $\beta_A < 1$ for all sufficiently large $n$. Then the sequence $(\tilde{A}, \tilde{p}_A)$ of expurgated ensembles satisfies the $(\tilde{\alpha}_A, 0)$-collision-resistance property, where $\tilde{\alpha}_A \equiv \{\tilde{\alpha}_A(n)\}_{n=1}^\infty$ is given as

\[
\tilde{\alpha}_A(n) \equiv \frac{\alpha_A(n)}{1 - \beta_A(n)}.
\]

Furthermore, we have

\[
\tilde{p}_A(\{A: G \setminus \{x\} \cap C_A(Ax) \neq 0\}) \leq \frac{|G|\alpha_A}{|\text{Im}A| |1 - \beta_A|}
\]

(16)

for all sufficiently large $n$, all $G \subset X^n$, and all $x \in X^n$.

**C. Source Code with Decoder Side Information**

Here, we consider the source coding with decoder side information illustrated in Fig. 1. The fundamental limit for this problem is given as the conditional spectral sup-entropy rate $\overline{H}(X|Y)$.

The achievability of this problem is proved via the Slepian-Wolf theorem using random binning [9], [25], [40] or the ensemble of all $q$-ary linear matrices [11]. The construction of an encoder using sparse matrix is studied in [35], [37] and the achievability is proved in [29], [34] by using a maximum-likelihood or minimum-divergence decoding. We obtain the coding theorem based on the collision-resistance property as a corollary of [28, Theorem 7] for stationary memoryless sources.

We fix a function $A: X^n \to \text{Im}A$ which is used as a encoding function. The encoding rate $r$ is given as

\[
r = \frac{\log |\text{Im}A|}{n}
\]

(17)
We define the decoder $x_A : \text{Im}A \times Y^n \rightarrow X^n$ as
\[
x_A(c|y) \equiv \arg \max_{x \in C_A(c)} \mu_{X^n Y^n}(x, y)
\] (18)
\[
= \arg \max_{x \in C_A(c)} \mu_{X^n|Y^n}(x|y).
\] (19)
The decoding error probability $\text{Error}(A)$ is given as
\[
\text{Error}(A) \equiv \mu_{X^n Y^n}(\{(x, y) : x_A(Ax|y) \neq x\}).
\]
It should be noted that the construction is analogous to the syndrome encoding/decoding when $A$ is a linear function.

We have the following theorem. It should be noted that $Y$ is allowed to be an infinite/continuous set and the correlation of the two sources is allowed to be asymmetric.

**Theorem 2:** Let $(X, Y)$ be a pair of correlated general sources. Assume that an ensemble $(A, p_A)$ has an $(\alpha_A, \beta_A)$-collision-resistance property for a given $r$ satisfying (17) and
\[
r > T(X|Y).
\] (20)
Then for any $\delta > 0$ and all sufficiently large $n$ there is a function (sparse matrix) $A \in A$ such that
\[
\text{Error}(A) \leq \delta.
\]
It should be noted that the theorem is implicitly proved in [26, Eq. (58)] from Lemma 4. The proof is given in Section V-D for the completeness of the paper.

**Remark 4:** We can use a stochastic decoder instead of the decoder defined by (18) or (19) in the above coding scheme. Let $\chi(\cdot)$ be defined as
\[
\chi(S) \equiv \begin{cases} 1, & \text{if the statement } S \text{ is true} \\ 0, & \text{if the statement } S \text{ is false} \end{cases}.
\] (21)
Then the joint distribution $p_{X^n Y^n C_n}$ is given as
\[
p_{X^n Y^n C_n}(x, y, c) = \mu_{X^n Y^n}(x, y)\chi(Ax = c),
\]
and the reproduction is determined at random subject to the distribution
\[
p_{X^n|Y^n C_n}(x|y, c) \equiv \frac{\mu_{X^n Y^n}(x, y)\chi(Ax = c)}{\sum_x \mu_{X^n Y^n}(x, y)\chi(Ax = c)}
\]
\[
= \frac{\mu_{X^n|Y^n}(x|y)\chi(Ax = c)}{\mu_{X^n|Y^n}(C_A(x)|y)}.
\] (22)
Then, from Lemma 7 in Appendix, the decoding error probability is at most twice the error probability of the decoder defined by (18) or (19). Thus, from Theorem 2, we have the fact that the fundamental limit $\overline{H}(X|Y)$ is also achievable with the code by using this stochastic decoder. It should be noted that we can consider (22) as the output distribution of a constrained-random-number generator [26], which is implementable by using the sum-product algorithm or the Markov-Chain-Monte-Carlo method when $(X^n, Y^n)$ is memoryless (see [26], [27], [33]).
D. Channel Code Using Constrained-Random-Number Generator

From Theorems 1 and 2, we have the following corollary, which is an improvement of [26, Theorem 1]. It should be noted that the conditions for \((\mathcal{A}, p_A)\) and \((\mathcal{B}, p_B)\) are weaker than those in [26, Theorem 1].

**Corollary 3:** Assume that \((\mathcal{A}, p_A)\) and \((\mathcal{B}, p_B)\) have an \((\alpha_A, \beta_A)\)-collision-resistance property and an \((\alpha_B, \beta_B)\)-balanced-coloring property, respectively, for given \(r\) and \(R\) satisfying
\[
\begin{align*}
    r &= \frac{1}{n} \log |\text{Im}\, A| \\
    R &= \frac{1}{n} \log |\text{Im}\, B| \\
    r + R &< H(X|Y)
\end{align*}
\]
Then for any \(\delta > 0\) and all sufficiently large \(n\) there are functions \(A \in \mathcal{A}, B \in \mathcal{B}\), and a vector \(c \in \text{Im}\, A\) such that the decoding error probability is less than \(\delta\). The channel capacity is achievable with the proposed code by letting \(X\) be a source that attains the supremum on the right hand side of (6).

It should be noted that we can use the stochastic decoder subject to the distribution defined by (22) instead of the decoder defined by (18) or (19). This implies that both encoding and decoding functions can be constructed by using constrained-random-number generators\(^4\).

V. PROOFS

In the following proofs, we omit the dependence on \(n\) of \(X\) and \(Y\) when they appear in the subscripts of \(\mu\), \(\overline{T}\), and \(\underline{T}\). The integral over the alphabet \(\mathcal{Y}^n\) is denoted by \(\sum\).

A. Proof of Lemma 1

The proof is completed by showing that
\[
C_X(W) \leq \lim_{q \to \infty} C_X^q(W)
\]
because we can show
\[
C_X(W) \geq \lim_{q \to \infty} C_X^q(W)
\]
from the trivial fact \(C_X(W) \geq C_X^q(W)\).

From the definition of \(C_X(W)\) and the assumption \(C_X(W) < \infty\), we have the fact that for any \(\varepsilon > 0\) there are a number \(R\) and a pair consisting of an encoder \(\varphi_n : M_n \to \mathcal{X}^n\) and a decoder \(\psi_n : \mathcal{Y}^n \to M_n\) satisfying
\[
C_X(W) - \varepsilon \leq \frac{1}{n} \log |M_n| \leq R < \infty
\]
and the conditions (2) and (3). Let \(S\) be defined as
\[
S_n \equiv \bigcup_{(x_1, \ldots, x_n) \in \varphi_n(M_n)} \bigcup_{i=1}^n \{x_i\}.
\]
\(^4\)We would like to call this type of codes CoCoNuTS (Codes based on Constrained Numbers Theoretically-achieving the Shannon limit).
From the fact that $[1/n] \log |\mathcal{M}_n| \leq R < \infty$, we have $|S_n| \leq n2^n R$, that is, $S_n$ is a finite set. Furthermore, the image of $\varphi_n$ is a subset of $S_n^a$. Let $q_n \equiv n2^n R$ and fix the set $S_n$ defined as above. Then, we have the fact that

$$C_X(W) - \varepsilon \leq C_X^n(W)$$

$$\leq \lim_{q \to \infty} C_X^q(W)$$

(23)

for all $\varepsilon > 0$ and all sufficiently large $n$, where the first inequality comes from the fact that $S_n \subset X$ and $|S_n| \leq q_n$ and the last inequality comes from the fact that $C_X^q(W)$ is a non-decreasing function of $q$. Hence we have

$$C_X(W) \leq \lim_{q \to \infty} C_X^q(W) + \varepsilon$$

for all $\varepsilon > 0$ and $C_X(W) \leq \lim_{q \to \infty} C_X^q(W)$ by letting $n \to \infty$ and $\varepsilon \to 0$.

**B. Proof of Theorem 1**

From (10), we have the fact that there is $\varepsilon > 0$ satisfying

$$r + R < H(X) - \varepsilon.$$  

(24)

Let $\mathcal{I}_X \subset \mathcal{X}^n$ be defined as

$$\mathcal{I}_X = \left\{ x : \frac{1}{n} \frac{1}{\mu_X(x)} \geq H(X) - \varepsilon \right\}.$$

For all $A$, we have

$$E_{\mathcal{F}_B} \left[ \sum_{m} \left| \frac{\mu_X(\mathcal{C}_{AB}(c, m))}{\mu_X(\mathcal{C}_A(c))} - \frac{1}{\text{Im}B} \right| \right]$$

$$= E_B \left[ \sum_{m,c} \left| \mu_X(\mathcal{C}_{AB}(c, m)) - \frac{\mu_X(\mathcal{C}_A(c))}{\text{Im}B} \right| \right]$$

$$\leq E_B \left[ \sum_{m,c} \left| \mu_X(\mathcal{C}_{AB}(c, m)) \cap \mathcal{I}_X \right| - \frac{\mu_X(\mathcal{C}_A(c)) \cap \mathcal{I}_X}{\text{Im}B} \right]$$

$$+ E_B \left[ \sum_{m,c} \left| \mu_X(\mathcal{C}_{AB}(c, m)) \cap [\mathcal{I}_X]^c \right| \right] + E_B \left[ \sum_{m,c} \left| \mu_X(\mathcal{C}_A(c)) \cap [\mathcal{I}_X]^c \right| \right]$$

$$= E_B \left[ \sum_{m,c} \left| \mu_X(\mathcal{C}_{AB}(c, m)) \cap \mathcal{I}_X \right| - \frac{\mu_X(\mathcal{C}_A(c)) \cap \mathcal{I}_X}{\text{Im}B} \right]$$

$$+ E_B \left[ \sum_{m,c} \mu_X(\mathcal{C}_{AB}(c, m)) \cap [\mathcal{I}_X]^c \right] + E_B \left[ \sum_{m,c} \mu_X(\mathcal{C}_A(c)) \cap [\mathcal{I}_X]^c \right]$$

(25)

where the first equality comes from the fact that the distribution of the random variable $c$ is $\{\mu_X(\mathcal{C}_A(c))\}_{c \in \text{Im}A}$, the first inequality comes from the triangular inequality and the fact that

$$\mu_X(\mathcal{C}_{AB}(c, m)) = \mu_X(\mathcal{C}_{AB}(c, m) \cap \mathcal{I}_X) + \mu_X(\mathcal{C}_{AB}(c, m) \cap [\mathcal{I}_X]^c)$$

$$\mu_X(\mathcal{C}_A(c)) = \mu_X(\mathcal{C}_A(c) \cap \mathcal{I}_X) + \mu_X(\mathcal{C}_A(c) \cap [\mathcal{I}_X]^c),$$

and the second equality comes from the fact that

$$\mu_X(\mathcal{C}_{AB}(c, m) \cap [\mathcal{I}_X]^c) \geq 0.$$
\[ \mu_X(\mathcal{C}_A(e) \cap [T_X]') \geq 0. \]

Since \( \{C_{AB}(c, m)\}_{c,m} \) and \( \{C_A(e)\}_c \) form a partition, the second and the third terms on the right hand side of (25) are evaluated as

\[ E_B \left[ \sum_{m,c} \mu_X(C_{AB}(c, m) \cap [T_X]^c) \right] = \mu_X([T_X]^c) \quad (26) \]

\[ E_B \left[ \sum_{m,c} \frac{\mu_X(C_A(e) \cap [T_X]^c)}{|\text{Im}B|} \right] = \mu_X([T_X]^c). \quad (27) \]

On the other hand, the first term on the right hand side of (25) is evaluated as

\[ E_B \left[ \sum_{m,c} \mu_X(C_{AB}(c, m) \cap [T_X]) - \frac{\mu_X(C_A(e) \cap [T_X])}{|\text{Im}B|} \right] \]

\[ = \sum_e \mu_X(C_A(e) \cap [T_X]) E_B \left[ \sum_m \mu_X(C_{AB}(m) \cap C_A(e) \cap [T_X]) - \frac{1}{|\text{Im}B|} \right] \]

\[ \leq \sum_e \mu_X(C_A(e) \cap [T_X]) \sqrt{\alpha_B - 1 + \frac{[\beta_B + 1]|\text{Im}B|2^{-n[H(X)]-\epsilon}}{\mu_X(C_A(e) \cap [T_X])}} \]

\[ \leq \mu_X([T_X]) \sqrt{\sum_e \frac{\mu_X(C_A(e) \cap [T_X])}{\mu_X([T_X])} \left[ \alpha_B - 1 + \frac{[\beta_B + 1]|\text{Im}A||\text{Im}B|2^{-n[H(X)]-\epsilon}}{\mu_X([T_X])} \right]} \]

\[ \leq \sqrt{\alpha_B - 1 + [\beta_B + 1]2^{-n[H(X)]-r-R-\epsilon}} \quad (28) \]

where the first inequality comes from Lemma 3 and the fact that \( \mu_X(x) \leq 2^{-n[H(X)]-\epsilon} \) for all \( x \in [T_X] \), the second inequality comes from the Jensen inequality, and the last inequality comes from (7), (9), and the fact that \( \mu_X([T_X]) \leq 1 \). Then we have

\[ E_{Bc}[\text{Error}(A, B, c)] \]

\[ = E_{Bc} \left[ \sum_{m,c} \frac{1}{|\text{Im}B|} + \sum_{m,x,y} \mu_X(C_{AB}(c, m) \cap C_A(c) \cap [T_X]) \right] \]

\[ = E_{Bc} \left[ \sum_{m,c} \frac{1}{|\text{Im}B|} + \sum_{m,x,y} \mu_X(C_{AB}(c, m) \cap C_A(c) \cap [T_X]) \right] \]

\[ \leq \sum_{x,y} \mu_X(C_{AB}(c, m)) \left[ \frac{1}{\mu_X(C_A(c))} + \frac{1}{|\text{Im}B|} \right] \]

\[ \leq \text{Error}(A) + \sqrt{\alpha_B - 1 + [\beta_B + 1]2^{-n[H(X)]-r-R-\epsilon}} + 2\mu_X([T_X]^c) \quad (29) \]
where the first inequality comes from the fact that

\[ E_{\text{BC}} \left[ \sum_{m : x, y : x \in C_A(c, m)} \frac{\mu_{XY}(x, y)}{\mu_X(C_A(c))} \right] = E_{\text{BC}} \left[ \sum_{c, m : x, y : x \in C_A(c, m), x_A(Ax \neq y) \neq x} \frac{\mu_{XY}(x, y)}{\mu_X(C_A(c))} \right] = \sum_{x, y} \mu_{XY}(x, y) \chi(x_A(Ax) \neq x) \quad (30) \]

and

\[
\sum_{m : x, y : \mu_X(C_A(c, m)) > 0} \mu_X(C_A(c, m)) \frac{1}{|\text{Im}B|} \frac{1}{\mu_X(C_A(c, m))} - \frac{1}{\mu_X(C_A(c))}
\]

\[
= \sum_{m} \left| \frac{\mu_X(C_A(c, m))}{\mu_X(C_A(c))} - \frac{1}{|\text{Im}B|} \right| - \sum_{m : \mu_X(C_A(c, m)) = 0} \frac{1}{|\text{Im}B|}, \quad (31)
\]

and the second inequality comes from (8) and (25)–(28). From (BC1), (BC2), (24), (29), and the fact that \( \mu_X([\ell\chi]') \to 0 \) as \( n \to \infty \), we have the fact that there is a pair consisting of a function \( B \in \mathcal{B} \) and a vector \( c \in \text{Im}A \) that satisfy (11).

\[ \square \]

C. Proof of Lemma 5

Since \( p_A \left( \{ A : t(x) \in \hat{\mathcal{H}} \cup \{ t(0) \} \text{ for all } x \in \mathcal{C}_A(0) \} \right) \) depends only on \( \hat{\mathcal{H}} \) and \( p_A(\{ A : Ax = 0 \}) \) depends only through the type \( t(x) \), we have the fact that \( \tilde{p}_A(\{ A : Ax = 0 \}) \) also depends on \( x \) only through the type \( t(x) \). We use the following lemma.

Lemma 6 ([26, Lemma 6],[29, Theorem 1]): Let \((A, p_A)\) be an ensemble of matrices. We assume that \( p_A(\{ A : Ax = 0 \}) \) depends on \( x \) only through the type \( t(x) \). Let \((\alpha_A, \beta_A)\) be defined by (12) and (13). Then \((A, p_A)\) satisfies (CR3).

Now, we prove the lemma. Let \( \chi(\cdot) \) be defined by (21). From the fact that \( \mathcal{H} \setminus \hat{\mathcal{H}} \) and \( \hat{\mathcal{H}} \cup \{ t(0) \} \) are the disjoint union of the set of all types of length \( n \), we have

\[
p_A \left( \{ A : t(x) \in \hat{\mathcal{H}} \cup \{ t(0) \} \text{ for all } x \in \mathcal{C}_A(0) \} \right) = 1 - p_A \left( \{ A : \exists x \in \mathcal{C}_A(0) \text{ s.t. } t(x) \in \mathcal{H} \setminus \hat{\mathcal{H}} \} \right)
\]

\[
\geq 1 - \sum_{t \in \mathcal{H} \setminus \hat{\mathcal{H}}} \sum_{x} p_A(\{ A : Ax = 0, t(x) = t \})
\]

\[
= 1 - \sum_{t \in \mathcal{H} \setminus \hat{\mathcal{H}}} \sum_{A} p_A(A) \chi(Ax = 0, t(x) = t)
\]

\[
= 1 - \sum_{t \in \mathcal{H} \setminus \hat{\mathcal{H}}} \sum_{A} p_A(A) \chi(Ax = 0, t(x) = t)
\]

\[
= 1 - \sum_{t \in \mathcal{H} \setminus \hat{\mathcal{H}}} \sum_{A} p_A(A) \chi(Ax = 0, t(x) = t)
\]

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\[
= 1 - \sum_{t \in \mathcal{H} \setminus \tilde{\mathcal{H}}} S(p_A, t)
\]
\[
= 1 - \beta_A
\] (32)

and
\[
\frac{1}{p_A \left( \left\{ A : t(x) \in \tilde{\mathcal{H}} \cup \{t(0)\} \text{ for all } x \in \mathcal{C}_A(0) \right\} \right)} \leq \frac{1}{1 - \beta_A}
\]

for all sufficiently large \(n\) satisfying \(\beta_A < 1\). We have
\[
\tilde{\rho}_A(A) \leq \frac{p_A \left( \left\{ A : t(x) \in \tilde{\mathcal{H}} \cup \{t(0)\} \text{ for all } x \in \mathcal{C}_A(0) \right\} \right)}{\beta_A}
\]
\[
\leq \frac{p_A(A)}{1 - \beta_A}
\] (33)

for all \(A\) and all sufficiently large \(n\). Let \(\tilde{\alpha}_A^*\) be defined as
\[
\tilde{\alpha}_A^* = \frac{|\text{Im}A|}{|X|^l} \max_{t \in \tilde{\mathcal{H}}} S(\tilde{\rho}_A, t).
\]

Then we have the relation
\[
\tilde{\alpha}_A^* = \frac{|\text{Im}A|}{|X|^l} \max_{t \in \tilde{\mathcal{H}}} \frac{\tilde{\rho}_A(A)[\{x : Ax = 0, t(x) = t\}]}{S(\tilde{\rho}_A, t)}
\]
\[
= \frac{|\text{Im}A|}{|X|^l} \max_{t \in \tilde{\mathcal{H}}} \frac{\rho_A(A)[\{x : Ax = 0, t(x) = t\}]}{[1 - \beta_A]} \cdot \max_{t \in \tilde{\mathcal{H}}} S(\tilde{\rho}_A, t)
\]
\[
= \frac{\alpha_A}{1 - \beta_A}
\]
\[
= \tilde{\alpha}_A^*
\] (34)

for all sufficiently large \(n\), where the inequality comes from (33).

When \(t \in \mathcal{H} \setminus \tilde{\mathcal{H}}\) and \(|\{x : Ax = 0, t(x) = t\}| > 0\), there is \(x \in \mathcal{C}_A(0)\) such that \(t(x) \in \mathcal{H} \setminus \tilde{\mathcal{H}}\). Then, from the definition of \(\tilde{\rho}_A\), we have \(\tilde{\rho}_A(A) = 0\) when \(t \in \mathcal{H} \setminus \tilde{\mathcal{H}}\) and \(|\{x : Ax = 0, t(x) = t\}| > 0\). This implies that
\[
\sum_{t \in \mathcal{H} \setminus \tilde{\mathcal{H}}} S(\tilde{\rho}_A, t) = \sum_{t \in \mathcal{H} \setminus \tilde{\mathcal{H}}} \sum_A \tilde{\rho}_A(A)[\{x : Ax = 0, t(x) = t\}]
\]
\[
= 0.
\] (35)

Then we have
\[
\sum_{x' \in \mathcal{X} \setminus \{x\} : \tilde{\rho}_A((A, Ax = Ax')) > \tilde{\alpha}_A^*} \tilde{\rho}_A((A, Ax = Ax')) \leq \sum_{x' \in \mathcal{X} \setminus \{x\} : \tilde{\rho}_A((A, Ax = Ax')) > \tilde{\alpha}_A^*} \tilde{\rho}_A(A, Ax = Ax')
\]
\[
\leq \sum_{t \in \mathcal{H} \setminus \tilde{\mathcal{H}}} S(\tilde{\rho}_A, t)
\]
\[
= 0
\] (36)

for all sufficiently large \(n\) satisfying \(\beta_A < 1\), where the first inequality comes from (34), the second inequality is obtained by applying Lemma 6 to the ensemble \((\mathcal{A}, \tilde{\rho}_A)\) and \(\tilde{\alpha}_A^*\), and the equality comes from (35).
We have

$$\limsup_{n \to \infty} \frac{1}{n} \log \tilde{\alpha}_A(n) = \limsup_{n \to \infty} \frac{1}{n} \log \frac{\alpha_A(n)}{1 - \beta_A(n)} \leq \limsup_{n \to \infty} \frac{1}{n} \log \alpha_A(n) + \limsup_{n \to \infty} \frac{1}{n} \log \frac{1}{1 - \beta_A(n)} = 0,$$

(37)

where the last equality comes from (14) and (15). Then, from (36) and (37), we have the fact that \((\mathcal{A}, \tilde{p}_A)\) satisfies the \((\tilde{\alpha}_A, 0)\)-collision-resistance property. Inequality (16) is shown directly by applying Lemma 4 to \((\mathcal{A}, \tilde{p}_A)\).  

D. Proof of Theorem 2

From (20), we have the fact that there is \(\varepsilon > 0\) satisfying

$$r > H(X|Y) + \varepsilon.$$

(38)

Let \(T_{X|Y} \subset X^n \times Y^n\) be defined as

$$T_{X|Y} := \left\{ (x, y) : \frac{1}{n} \log \frac{\mu_{X|Y}(x|y)}{\mu_{X|Y}(x')} \leq H(X|Y) + \varepsilon \right\}.$$

Assume that \((x, y) \in T_{X|Y}\) and \(x_A(Ax|y) \neq x\). Then we have the fact that there is \(x' \in C_A(Ax)\) such that \(x' \neq x\) and

$$\mu_{X|Y}(x'|y) \geq \mu_{X|Y}(x|y) \geq 2^{-n[H(X|Y) + \varepsilon]}.$$

This implies that \([T_{X|Y} \setminus \{x\}] \cap C_A(Ax) \neq \emptyset\), where \(T_{X|Y} := \{x : (x, y) \in T_{X|Y}\}\). We have

$$E_A[\chi(x_A(Ax|y) \neq x)] \leq p_A\left(\{A : [T_{X|Y} \setminus \{x\}] \cap C_A(Ax) \neq \emptyset\}\right)$$

$$\leq \frac{T_{X|Y}(y)}{\lim_{n \to \infty} A} + \beta_A$$

$$\leq 2^{-n[H(X|Y) + \varepsilon]} + \beta_A$$

(39)

for all \((x, y) \in T_{X|Y}\), where \(\chi()\) is defined by (21), the second inequality comes from Lemma 4, and the third inequality comes from (20) and the fact that \([T_{X|Y}(y)] \leq 2^n[H(X|Y) + \varepsilon]\).

We have the fact that

$$E_A[\text{Error}(A)]$$

$$= E_A \left[ \sum_{x, y} \mu_{X|Y}(x, y) \chi(x_A(Ax|y) \neq x) \right]$$

$$= \sum_{(x, y) \in T_{X|Y}} \mu_{X|Y}(x, y) E_A[\chi(x_A(Ax|y) \neq x)] + \sum_{(x, y) \notin T_{X|Y}} \mu_{X|Y}(x, y) E_A[\chi(x_A(Ax|y) \neq x)]$$

$$\leq 2^{-n[H(X|Y) - \varepsilon]} + \beta_A + \mu_{X|Y}([T_{X|Y}]^c),$$

(40)

where the last inequality comes from (39). From (CR1), (CR2), (38) and the fact that \(\mu_{X|Y}([T_{X|Y}]^c) \to 0\) as \(n \to \infty\), we have the fact that there is a function \(A \in A\) satisfying \(\text{Error}(A) \leq \delta\) for all \(\delta > 0\) and all sufficiently large \(n\).
APPENDIX

Error Probability of Stochastic Decision

This appendix reviews the result of [33], which investigates the error probability of the stochastic decision. It should be noted that stochastic decoding is a stochastic decision in the context of a coding scheme.

Let \( \mathcal{U} \) and \( \mathcal{V} \) be the alphabets of random variable \( U \) and \( V \), respectively. We assume that the joint distribution \( p_{UV} \) of \((U, V)\) is known.

Let us assume the situation where a decoder make a stochastic decision of the invisible state \( U \) after the observation \( V \). We use a random number generator \( \hat{U} \in \mathcal{U} \) after observing \( V \) and let \( \hat{U} \) be a decision (guess) about the state \( U \). Formally, we generate \( \hat{U} \) subject to the conditional distribution \( q_{\hat{U}|V}(\cdot|V) \) on \( \mathcal{U} \) depending on an observation \( V \) and let an output be a decision of \( U \), where \( U \) and \( \hat{U} \) are conditionally independent for a given \( V \), that is, \( U \leftrightarrow V \leftrightarrow \hat{U} \) forms a Markov chain. The joint distribution \( p_{UV\hat{U}} \) of \((U, V, \hat{U})\) is given as

\[
p_{UV\hat{U}}(u, v, \hat{u}) = q_{\hat{U}|V}(\hat{u}|v)p_{U|V}(u|v)p_{V}(v).
\]

Let us call \( q_{\hat{U}|V} \) a stochastic decision rule. As a special case, when \( q_{\hat{U}|V} \) is given by using a function \( f : \mathcal{U} \to \mathcal{V} \) and is defined as

\[
q_{\hat{U}|V}(\hat{u}|v) = \begin{cases} 
1 & \text{if } \hat{u} = f(v) \\
0 & \text{if } \hat{u} \neq f(v),
\end{cases} \tag{41}
\]

we call \( q_{\hat{U}|V} \) or \( f \) a deterministic decision rule. It should be noted that the maximum a posteriori decision rule is deterministic.

Let \( \chi \) be a support function defined by (21). Then the error probability \( \text{Error}(q_{\hat{U}|V}) \) of a (stochastic) decision rule \( q_{\hat{U}|V} \) is given as

\[
\text{Error}(q_{\hat{U}|V}) = \sum_{v} p_{V}(v) \sum_{u} p_{U|V}(u|v) \sum_{\hat{u}} q_{\hat{U}|V}(\hat{u}|v) \chi(\hat{u} \neq u).
\]  

In the last equality, \( 1 - q_{\hat{U}|V}(u|v) \) corresponds to the error probability of the decision rule \( q_{\hat{U}|V} \) after the observation \( v \in \mathcal{V} \), and \( \text{Error}(q_{\hat{U}|V}) \) corresponds to the average of this error probability. When \( q_{\hat{U}|V} \) is defined by using \( f : \mathcal{V} \to \mathcal{U} \) and (41), the decision error probability \( \text{Error}(f) \) of a deterministic decision rule \( f \) is given as

\[
\text{Error}(f) \equiv \sum_{v} p_{V}(v) \sum_{u} p_{U|V}(u|v) \chi(f(v) \neq u)
\]

\[
= \sum_{v} p_{V}(v) [1 - p_{U|V}(f(v)|v)]. \tag{43}
\]

It should be noted that the right hand side of the first equality can be derived directly from (42) and the fact that

\[
q_{\hat{U}|V}(u|v) = \chi(f(v) = u) = 1 - \chi(f(v) \neq u).
\]  

That is, we have \( \text{Error}(f) = \text{Error}(q_{\hat{U}|V}) \) when \( f \) and \( q_{\hat{U}|V} \) satisfy (41).
It is well-known fact (see [33, Lemma 2]) that an optimal strategy for guessing the state $U$ is finding $\hat{u}$ which maximize the conditional probability $p_{U|V}(\hat{u}|v)$ depending on a given observation $v$. Formally, by taking $\hat{u}$ that maximizes $p_{U|V}(\hat{u}|v)$ for each $v \in \mathcal{V}$, we can define the function $f_{\text{MAP}}: \mathcal{V} \rightarrow \hat{u}$ as
\[
f_{\text{MAP}}(v) \equiv \arg \max_{\hat{u}} p_{U|V}(\hat{u}|v) \quad \text{(45)}
\]
\[
= \arg \max_{\hat{u}} p_{UV}(\hat{u}, v). \quad \text{(46)}
\]
We call (45) and (46) a maximum a posteriori decoder and a maximum likelihood decoder, respectively. It should be noted that the discussion does not depend on the choice of states with the same maximum probability.

Here, let us consider the case $q_{U|V}(\hat{u}|v) = p_{U|V}(\hat{u}|v)$ for all $(\hat{u}, v)$, that is, we make a stochastic decision with the conditional distribution $p_{U|V}$ of a state $U$ for a given observation $V$. It should be noted that the joint distribution $p_{UV\hat{U}}$ of $(U, V, \hat{U})$ is given as
\[
p_{UV\hat{U}}(u, v, \hat{u}) = p_{U|V}(\hat{u}|v)p_{U|V}(u|v)p_{V}(v).
\]
We call this type of decision rule a stochastic decision with the a posteriori distribution.

We have the following lemma.

**Lemma 7 ([10, Eq. (29)] [33, Lemma 3]):** Let $(U, V)$ be a pair consisting of a state $U$ and an observation $V$ and $p_{UV}$ be the joint distribution of $(U, V)$. When we make a stochastic decision with $p_{U|V}$, the decision error probability of this rule is at most twice the decision error probability of the maximum a posteriori decision rule $f_{\text{MAP}}$. That is, we have
\[
\text{Error}(p_{U|V}) \leq 2\text{Error}(f_{\text{MAP}}).
\]

**Proof:** In this proof, we assume that $\mathcal{U}$ and $\mathcal{V}$ are finite sets. It should be noted that the result does not change when $\mathcal{V}$ is an infinite/continuous set, where the summation should be replaced with the integral. We have
\[
\text{Error}(p_{U|V}) = \sum_{v} p_{V}(v) \sum_{u} p_{U|V}(u|v)[1 - p_{U|V}(u|v)]
\]
\[
\leq \sum_{v} p_{V}(v) \left[1 - \sum_{u} p_{U|V}(u|v)^2\right]
\]
\[
\leq \sum_{v} p_{V}(v) \left[1 - p_{U|V}(f_{\text{MAP}}(v)|v)^2\right]
\]
\[
= \sum_{v} p_{V}(v)[1 - p_{U|V}(f_{\text{MAP}}(v)|v)][1 + p_{U|V}(f_{\text{MAP}}(v)|v)]
\]
\[
\leq 2 \sum_{v} p_{V}(v)[1 - p_{U|V}(f_{\text{MAP}}(v)|v)]
\]
\[
= 2\text{Error}(f_{\text{MAP}}), \quad \text{(47)}
\]
where the second inequality comes from the fact that $p_{U|V}(f(v)|v) \leq 1$ and the fourth equality comes from (43).

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