Perturbation Theory for Parent Hamiltonians of Matrix Product States

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This article investigates the stability of the ground state subspace of a canonical parent Hamiltonian of a Matrix product state against local perturbations. We prove that the spectral gap of such a Hamiltonian remains stable under weak local perturbations even in the thermodynamic limit, where the entire perturbation might not be bounded. Our discussion is based on preceding work by D.A. Yarotsky that develops a perturbation theory for relatively bounded quantum perturbation of classical Hamiltonians. We exploit a renormalization procedure, which on large scale transforms the parent Hamiltonian of a Matrix product state into a classical Hamiltonian plus some perturbation. We can thus extend D.A. Yarotsky’s results to provide a perturbation theory for parent Hamiltonians of Matrix product states and recover some of the findings of the independent contributions \cite{4,9}.

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I. INTRODUCTION

The purpose of this article is to investigate the low energy sector of certain models of many-body quantum systems with local interaction. We are interested in the stability of quantum phases when small perturbations act on the system. In particular, we aim at understanding the conditions under which certain physical properties of the ground state change smoothly when an interaction is added to the model Hamiltonian. While for general models this question is intractably hard, in
In this article we focus our eyesight on a restricted class, namely on parent Hamiltonians of Matrix product states.

Matrix product states (MPS) have been an extremely useful tool in the study of the ground state physics of many-body quantum systems. With their local structure MPS provide an efficient description of states arising from local interactions and constitute a natural framework for the analysis of local gapped Hamiltonians in 1D. In fact, the matrix product state representation lies at the heart of the very successful density matrix renormalization group method \[13, 20\]. To any MPS a local frustration-free and gapped Hamiltonian having this MPS as a unique ground state can be associated. A canonical choice of such Hamiltonians was introduced in \[5\] and is referred to as *parent Hamiltonian* of the MPS. On the one hand the local structure of the MPS endows the canonical parent Hamiltonian with the structure necessary for a rigorous analysis. On the other hand canonical parent Hamiltonians constitute a wide class of local Hamiltonians and include many important special cases such as the AKLT-Hamiltonian \[1\].

We are interested in how the parent Hamiltonian model behaves under small perturbations, as this allows one to use the idealization to predict the behaviour of actual physical systems. It seems generally expected that if a ground state of a quantum many-body system is in a non-critical regime characterized by the presence of a local spectral gap and exponential decay of correlations, then the system remains in this phase under sufficiently weak perturbations. We prove that for translationally invariant parent Hamiltonians of generic MPS this is indeed the case i.e. we show that the spectral gap of such a Hamiltonian is stable under arbitrary local perturbations even in the thermodynamic limit. This result itself is not new. It was shown in \[9\] that local Hamiltonians that satisfy the *Local Topological Quantum Order* (LTQO) condition and that are *locally gapped* are stable under local perturbations. It was also claimed in \[9\] and shown in \[4\] that parent Hamiltonians of MPS have LTQO. (However, in spin systems of higher dimension the presence of LTQO is hard to verify.) The fact that parent Hamiltonians are locally gapped was already known from \[10\]. Hence, the stability of the spectral gap against sufficiently weak perturbations follows.

The contribution at hand contains a new proof of this result. Our derivation is based on the observation that with increasing system scale a matrix product state “looks more and more classical” \[16\]. We exploit a renormalization group flow on parent Hamiltonians to prove that on sufficiently large scale a (generic) parent Hamiltonian can be seen as a perturbation of a classical system. Hence, any sufficiently small quantum perturbation of a parent Hamiltonian is equivalent to a relatively bounded perturbation of a classical model. We then draw on the theory for ground states in quantum perturbations of classical lattice systems by D.A. Yarotsky \[23\] to conclude our proof.

The results presented in this article were achieved independently of the contributions \[4, 9\] as a part of the doctoral thesis of the first author in the summer of 2011, before the publication of \[4, 9\].

### II. PRELIMINARIES

As mentioned in the introduction, this article investigates how the ground state subspace of an MPS parent Hamiltonian behaves under small perturbations. This section reviews the required definitions and basic results.

#### A. Notation

We model quantum spin chains as connected subsets \(\Lambda \subset \mathbb{Z}\), where each site \(x \in \Lambda\) is equipped with a \(d\)-dimensional, complex Hilbert space \(\mathcal{H}_x\). The total Hilbert space associated with a finite
subset $\Lambda \subset \mathbb{Z}$ will be denoted by $H_\Lambda = \bigotimes_{x \in \Lambda} H_x$. The interactions on the spin chain are given by a translationally invariant (TI) Hamiltonian with some fixed interaction range $\Lambda_0$. Such Hamiltonians can formally be written as

$$H_\Lambda = \sum_{x \in \Lambda} h_x,$$

where $h_x$ is a positive semi-definite operator acting (non-trivially) on $H_{\Lambda_0+x}$ and $\Lambda_0 + x$ is a translate of $\Lambda_0$ by $x$. We will assume that $H_\Lambda$ has a non-degenerate ground state $|\Omega\rangle_\Lambda$ and that $H_\Lambda$ has a spectral gap $\gamma > 0$ above the ground state energy

$$H_\Lambda | \Omega\rangle_\Lambda \geq \gamma \mathbb{1}.$$ 

Moreover, the Hamiltonians considered in this article will be frustration free, that is each interaction term $h_x$ minimizes the global ground state energy: for all $x$ we have $h_x |\Omega\rangle_\Lambda = 0$. We analyse how the spectral gap behaves if the Hamiltonian is perturbed with local interactions. Formally, we add a perturbation

$$\Phi_\Lambda = \sum_{x \in \Lambda} \phi_x,$$

where each of the terms $\phi_x$ acts locally on a finite subset of $\Lambda$. Often, we will find it convenient to identify the first and last site of $\Lambda$ to impose periodic boundary conditions (PBC) on the system.

To distinguish particular Hilbert subspaces of $H_\Lambda$ we will add Latin subscripts, for example $H_A$ and $H_B$. For any operator $X$ acting on a finite subset of the chain we denote by $\|X\|_p$ the Schatten $p$-norm of $X$. If $X$ acts on an infinite subsets we will only employ the $\| \cdot \|_\infty$-norm, which coincides with the usual operator norm.

As mentioned before we will consider a renormalization group flow that transforms the MPS parent Hamiltonian into a classical Hamiltonian. This flow will be modeled using a consecutive application of a linear map $T$ acting on matrices $X$. More precisely, we define the map $T$ by

$$T(X) := \sum_i A_i X A_i^\dagger,$$

where the summation goes over a set of so-called Kraus operators $\{A_i\}$. Maps with this structure are completely positive (CP). For each such map the dual map $T^*$ is defined by $T^*(X) := \sum_i A_i^\dagger X A_i$. $T^*$ is simply the adjoint of $T$ with respect to the Hilbert-Schmidt inner product $\langle X|Y \rangle = \text{tr}(X^\dagger Y)$. $T$ is called unital (CPU) iff it preserves the identity operator $T(\mathbb{1}) = \mathbb{1}$ and $T$ is called trace-preserving (CPTP) iff $T^*(\mathbb{1}) = \mathbb{1}$.

### B. Matrix Product States

We consider a finite subset $\Lambda \subset \mathbb{Z}$ consisting of $N$ sites, whose Hilbert spaces are each of dimension $d$. Every pure state of the spin system of $\Lambda$ can be written as

$$|\Psi\rangle = \sum_{i_1, \ldots, i_N}^d \text{tr}(A_{i_1}^{[1]} \cdot A_{i_2}^{[2]} \cdot \ldots \cdot A_{i_N}^{[N]}) |i_1 \ldots i_N\rangle,$$

with site dependent $D_k \times D_{k+1}$ matrices $A_{i_k}^{[k]}$. States of this structure are called Matrix product states. In the case of periodic boundary conditions and translational invariance of the MPS it is possible to show [12] that the matrices can be chosen in a site-independent way, i.e.

$$|\Psi\rangle = \sum_{i_1, \ldots, i_N}^d \text{tr}(A_{i_1} \cdot A_{i_2} \cdot \ldots \cdot A_{i_N}) |i_1 \ldots i_N\rangle.$$
with $D \times D$ matrices $\{A_i\}_{i=1,...,d}$. In our consecutive discussion a special class of MPS will be of particular importance. This class is characterized by the following generic condition.

**Condition (G1):**
There is a finite number $L_0$ such that for all $L \geq L_0$ the list of matrices
$$\{A_{i_1} \cdot ... \cdot A_{i_L}\}_{i_j \in \{1...d\}}$$
spans the entire algebra of $D \times D$ matrices.

Condition (G1) is generic in the sense that $d$ matrices chosen randomly according to some reasonable measure comply with this condition with probability one. It is not hard to see that (G1) holds iff the map
$$\Gamma_L : X \mapsto \sum_{i_1,...,i_L} \text{tr}(XA_{i_1}A_{i_2}...A_{i_L})|i_1...i_L)$$
is injective for $L \geq L_0$. The correspondence between sets $\{A_i\}_{i=1,...,d}$ and MPS is not bijective; for example the set $\{XA_iX^{-1}\}_{i=1,...,d}$ with invertible $X$ belongs to the same state. It is shown in [12], Chapter 3 that the matrices of any MPS satisfying (G1) can be chosen to constitute a CPU map $T$. More precisely, we can choose $\{A_i\}_{i=1,...,d}$ such that the map $T(X) = \sum_i A_i X A_i^\dagger$ satisfies $T(\mathbb{1}) = \mathbb{1}$ and $T^\gamma(\Xi) = \Xi$ for some diagonal and strictly positive matrix $\Xi$. In addition, $\mathbb{1}$ is the only fixed point of $T$. For a more detailed discussion of MPS we refer to [12].

### C. Canonical Parent Hamiltonians

We consider a TI state $|\Psi\rangle = \sum_{i_{1}...i_{N}} \text{tr}(A_{i_{1}} \cdot ... \cdot A_{i_{N}})|i_{1}...i_{N}\rangle$ of a spin system with PBC on a chain $\Lambda$. For fixed $L \in \mathbb{N}$ we define $\mathcal{G}_L \subset (\mathbb{C}^d)^{\otimes L}$ to be the subspace spanned by the vectors $|\Psi(X)\rangle = \sum_{i_{1}...i_{L}} \text{tr}(XA_{i_{1}} \cdot ... \cdot A_{i_{L}})|i_{1}...i_{L}\rangle$, where $X$ are complex $D \times D$ matrices. Note that if condition (G1) holds for the matrices $A_i$ then for $L \geq L_0$ the space spanned by $|\Psi(X)\rangle$ has dimension $D^2$. We write $h_{\mathcal{G}_L}$ for the projector onto the orthogonal complement of $\mathcal{G}_L$ in $(\mathbb{C}^d)^{\otimes L}$. The canonical parent Hamiltonian for $|\Psi\rangle$ (and fixed $L$) is defined as the formal expression

$$H_{\Lambda} = \sum_{i}^{N} \tau^i(h_{\mathcal{G}_L})$$

where $\tau$ denotes the translation operation by one site [3, 12]. For a parent Hamiltonian with nearest neighbour interaction ($L=2$) we will write $H_{\Lambda} = \sum_{k} h_{k,k+1}$ to emphasize this fact. It is clear from the definition that $H_{\Lambda}|\Psi\rangle = 0$ and that $H_{\Lambda}$ is frustration free. Moreover, as a result of condition (G1) $|\Psi\rangle$ is the unique ground state of $H_{\Lambda}$ if $L > L_0$ and $N \geq 2L_0$, [12, Theorem 10]. More generally, under (G1) $H_{\Lambda}$ can be shown to have a spectral gap $\gamma > 0$ above the ground state energy $\frac{3}{2} [12]$ even in the limit of an infinite chain. Let $\Lambda_1 \subset \Lambda$ and let $G_{\Lambda_1}$ denote the projector onto the kernel of $H_{\Lambda_1} = \sum_{\{i+(i+1,...,i+L)\} \subset \Lambda_1} \tau^i(h_{\mathcal{G}_L})$. The local gap is defined to be the largest number $\gamma_{\Lambda_1}$ such that

$$H_{\Lambda_1} \geq \gamma_{\Lambda_1}(1 - G_{\Lambda_1})$$.

The local gap does not depend on $\Lambda$ but only on the number of sites in $\Lambda_1$. The "Local-Gap condition" of [9] refers to the property of a general frustration-free Hamiltonian that the local gap decays at most polynomially in the number of lattice sites. It is one core assumption for the stability proof for frustration-free Hamiltonians (the other one being LTQO). In [10, 13] a constant lower bound on the local gap of one-dimensional, frustration-free Hamiltonians is derived. In particular, this implies that parent Hamiltonians satisfy the Local-Gap condition and we will naturally make use of this in our derivation. A more detailed discussions of parent Hamiltonians for MPS can be found in [12].
D. Stability of the spectral gap under quantum Perturbations of classical Hamiltonians

In this section we recall a fundamental result by D.A. Yarotsky [23] that asserts the stability of the spectral gap of a classical Hamiltonian under certain local perturbations. The effect of small quantum perturbations to classical Hamiltonians was discussed for example in [2, 7, 21, 22]. In [6, 23] this was extended to perturbations that need not necessarily be small but are required to consist of a small bounded part and a term that is bounded relatively to the unperturbed Hamiltonian. In the following we describe rigorously this perturbation theory.

We start with a chain \( \Lambda \subset \mathbb{Z} \) with PBC and we consider a TI frustration-free Hamiltonian \( H_\Lambda = \sum_{x \in \Lambda} h_x \). We will call \( H_\Lambda \) classical if in each space \( H_x \) there is a preferred vector \( |\Omega_x\rangle \) and an orthogonal basis containing that vector such that the product basis in \( H_\Lambda \) diagonalizes \( h_x \). Furthermore we assume that \( H_\Lambda \) has non-degenerate ground state \( |\Omega\rangle_\Lambda = \bigotimes_{x \in \Lambda} |\Omega_x\rangle \) and strictly positive spectral gap above \( |\Omega\rangle_\Lambda \). We consider perturbations \( \Phi_\Lambda = \sum_{x \in \Lambda} \phi_x \) whose local terms act on finite subchains and that can be split into a purely bounded part and a relatively bounded part as

\[
\phi_x = \phi_x^{(r)} + \phi_x^{(b)}.
\]

The bounded part is characterized by

\[
\| \phi_x^{(b)} \|_\infty \leq \beta.
\]

For the relatively bounded part we suppose that for any \( |\psi\rangle \) and any \( I \subset \Lambda \)

\[
\left| \sum_{x \in I} \langle \psi | \phi_x^{(r)} | \psi \rangle \right| \leq \alpha \langle \psi | H_A | \psi \rangle.
\]

**Theorem 1** [23, Theorem 2]). Let \( H_\Lambda = \sum_{x} h_x \) be a classical Hamiltonian on a chain \( \Lambda \) with PBC and non-degenerate gapped ground state \( |\Omega\rangle_\Lambda \). Consider the perturbed Hamiltonian \( \tilde{H}_\Lambda = H_\Lambda + \Phi \), where \( \Phi = \sum_x \phi_x \) is a perturbation that satisfies (11)–(12). For any \( \kappa > 1 \) there is \( \delta(\kappa) > 0 \) such that for any \( \alpha \in (0, 1) \) and \( \beta = \delta(1 - \alpha)^{2\kappa} \) the following conclusions hold:

1. \( \tilde{H}_\Lambda \) has a non-degenerate gapped ground state \( |\tilde{\Omega}\rangle_\Lambda \):

\[
\tilde{H}_\Lambda |\tilde{\Omega}\rangle_\Lambda = \tilde{E}_\Lambda |\tilde{\Omega}\rangle_\Lambda
\]

and for some \( \gamma > 0 \) that does not depend on \( \Lambda \)

\[
\tilde{H}_\Lambda |_{H_\Lambda \otimes |\tilde{\Omega}\rangle_\Lambda} \geq (\tilde{E}_\Lambda + \gamma) 1.
\]

2. There exists a thermodynamic weak*-limit of the ground states \( |\tilde{\Omega}\rangle_\Lambda \): For \( \Lambda \to \mathbb{Z} \) one has that

\[
\langle A |\tilde{\Omega}\rangle_\Lambda \to \omega(A), \quad A \in \bigcup_{|A| < \infty} \mathcal{B}(\mathcal{H}_\Lambda),
\]

where \( \mathcal{B}(\mathcal{H}_\Lambda) \) denotes the bounded operators on \( \mathcal{H}_\Lambda \).

3. There is an exponential decay of correlations in the infinite volume ground state \( \omega \): for some positive \( c \) and \( \epsilon < 1 \)

\[
|\omega(A_1 A_2) - \omega(A_1) \omega(A_2)| \leq c^{\|A_1\| + \|A_2\|} \epsilon^{\text{dist}(A_1, A_2)} \|A_1\|_{\infty} \|A_2\|_{\infty}, \quad A_i \in \mathcal{B}(\mathcal{H}_\Lambda).
\]
4. If within the allowed range of perturbations the term $\phi_x$ depends analytically on some parameters, then the ground state $\omega$ is also weakly $^\ast$-analytic in these parameters.

Theorem 1 establishes that the spectral gap of a classical Hamiltonian is stable under perturbations that comply with the above assumptions. We will use this result to prove that parent Hamiltonians of MPS have a spectral gap that is stable under sufficiently weak bounded perturbations. To achieve this we will view the MPS parent Hamiltonian as a perturbation of a classical Hamiltonian, which is within a parameter range where Theorem 1 applies. The bounded part of this perturbation will decay faster under scaling of the system size than $\delta (1-\alpha)^2$. For sufficiently large systems this implies that under a small bounded perturbation $\phi_x'$ the parent Hamiltonian remains a perturbation of a classical Hamiltonian such that Theorem 1 applies. This provides us with the desired perturbation result.

III. STABILITY OF THE SPECTRAL GAP OF A CANONICAL PARENT HAMILTONIAN

In this section we state our main theorem. We consider a MPS that satisfies the generic condition (G1) and prove that the spectral gap of the corresponding parent Hamiltonian is stable under sufficiently weak perturbations. In the following corollary we extend this result and show that our discussion includes D.A. Yarotsky’s perturbation theory for the AKLT model [23] as an important special case.

**Theorem 2.** Let $|\Psi\rangle$ be a TI MPS on a finite ring $\Lambda$ with PBC and suppose that for the matrices of $|\Psi\rangle$ condition (G1) holds. Choose $L \geq L_0$ and let $H_\Lambda = \sum_i \tau^i(h_{G_L})$ be the canonical parent Hamiltonian for $|\Psi\rangle$. Furthermore let $\Phi_\Lambda = \sum_k \phi_k$ be any finite range interaction with $||\phi_k||_\infty \leq \beta$ for a sufficiently small $\beta$ depending on the range of $\Phi$. Then all conclusions of Theorem 1 hold for the perturbed parent Hamiltonian $\tilde{H}_\Lambda = H_\Lambda + \Phi_\Lambda$.

Note that the above does not apply to important special cases as the AKLT model. There one considers a Hamiltonian with local nearest neighbour interaction but the matrices at each site do not span the whole algebra. The following simple corollary is to remedy this issue.

**Corollary 3.** Let $H_\Lambda = \sum_i \tau^i(h_{G_L})$ be a canonical parent Hamiltonian such that Theorem 2 applies. Consider a Hamiltonian $\hat{H}_\Lambda = \sum_i h_{i,i+1}$ and suppose that there are positive constants $c_1$ and $c_2$ such that

$$c_1 h_{G_L} \leq \sum_{j=1}^{L-1} h_{j,j+1} \leq c_2 h_{G_L}.$$

Then all conclusions of Theorem 1 also hold for $\hat{H}$.

The ground states of the AKLT model are MPS with $\{A_i\} = \{\sigma^z, \sqrt{2}\sigma^+, -\sqrt{2}\sigma^-\}$, where the $\sigma$'s are the Pauli matrices. If we choose $\hat{H}$ to be he AKLT Hamiltonian Corollary 3 applies with $L = 3$ and implies the stability of the spectral gap of the AKLT model.

IV. PROOF OF STABILITY

We start this section with an outline of the proof of Theorem 2. In Section IV B we prove some lemmas from the theory of quantum channels and MPS. The following Subsection IV C contains a proof of Theorem 2 under the stronger assumption that the matrices $\{A_i\}_{i=1,\ldots,d}$ at each site of the chain span the whole algebra of $D \times D$ matrices. However, this assumption is not principal and in Section IV D we extend the previous discussion to prove stability under (G1).
A. Outline of the proof

For the readers convenience, before we proceed with the derivation of Theorem 2, we start with an exposition of core observations that will provide us with the proof.

1. We are given a MPS parent Hamiltonian \( H_\Lambda \). We divide \( \Lambda \) into subchains \( \Lambda_k \) and we consider local sub-Hamiltonians \( H_{\Lambda_k \cup \Lambda_{k+1}} \) of \( H_\Lambda \) acting on \( \Lambda_k \cup \Lambda_{k+1} \). We analyze the behavior of the ground state subspace of \( H_{\Lambda_k \cup \Lambda_{k+1}} \) under scaling of \( \Lambda_k \). To this end we introduce density matrices \( \rho_{\Lambda_k \cup \Lambda_{k+1}} \) whose image subspace is exactly the kernel of \( H_{\Lambda_k \cup \Lambda_{k+1}} \).

2. Using a renormalization group flow we construct local unitaries \( W_{\Lambda_k} \) such that on sufficiently large scale the image of \( W_{\Lambda_k} \otimes W_{\Lambda_{k+1}} \rho_{\Lambda_k \cup \Lambda_{k+1}} W_{\Lambda_k}^\dagger \otimes W_{\Lambda_{k+1}}^\dagger \) has particularly simple structure. It turns out that in the asymptotic limit of large system size this image corresponds to the ground state subspace of a classical Hamiltonian.

3. We use convergence estimates from the theory of quantum Markov chains to show that the projectors \( G_{\Lambda_k \cup \Lambda_{k+1}} \) onto the kernel of \( H_{\Lambda_k \cup \Lambda_{k+1}} \) and \( G_{\Lambda_k \cup \Lambda_{k+1}} \) onto the kernel of the asymptotic classical Hamiltonian can be made exponentially close. We prove that \( \| W_{\Lambda_k} \otimes W_{\Lambda_{k+1}} G_{\Lambda_k \cup \Lambda_{k+1}} W_{\Lambda_k}^\dagger \otimes W_{\Lambda_{k+1}}^\dagger - G_{\Lambda_k \cup \Lambda_{k+1}} \|_\infty \leq O(\| \lambda \|^L/2) \).

4. We provide an explicit perturbation consisting of a bounded part \( \sum_k \phi_k^{(b)} \) and a relatively bounded part \( \sum_k \phi_k^{(r)} \) that transform the classical Hamiltonian into \( \bigotimes_k W_{\Lambda_k} H_\Lambda \bigotimes_k W_{\Lambda_k}^\dagger \). Using the estimate from 3. we show that these perturbations are in accordance with the conditions of Theorem 2. When adding a sufficiently small bounded perturbation to \( \sum_k \phi_k^{(b)} \) the total perturbation remains in the range where Theorem 2 applies. Hence, the ground state subspace of \( H_\Lambda \) is stable.

B. Some Lemmas

We already mentioned (Section IIB) that to any TI MPS we can associate a certain CPU map \( \mathcal{T} \). To better keep track of the kernel of the canonical parent Hamiltonian it will be useful to introduce the operator \( \rho_{EE'} = \frac{1}{T} \sum_{i_1\cdot i_2\cdot j_1\cdot j_2} \text{tr}(A_{i_1} A_{i_2} A_{j_2}^\dagger A_{j_1}^\dagger) |i_1\rangle \langle j_1|_E \otimes |i_2\rangle \langle j_2|_{E'} \), which is defined via the Kraus operators of \( \mathcal{T} \). The subscripts \( E \) and \( E' \) have no physical significance but are introduced to more conveniently distinguish the systems involved. The following lemma shows that if two CPU maps \( \mathcal{T} \) and \( \mathcal{T}' \) are close, then the corresponding operators \( \rho_{EE'} \) and \( \tilde{\rho}_{EE'} \) can be made close using a local unitary transformation.

Lemma 4. Let \( \mathcal{T}(X) = \sum_i A_i X A_i^\dagger \) and \( \mathcal{T}'(X) = \sum_i \tilde{A}_i X \tilde{A}_i^\dagger \) be CPU maps. Consider the operators \( \rho_{EE'} = \frac{1}{T} \sum_{i_1\cdot i_2\cdot j_1\cdot j_2} \text{tr}(A_{i_1} A_{i_2} A_{j_2}^\dagger A_{j_1}^\dagger) |i_1\rangle \langle j_1|_E \otimes |i_2\rangle \langle j_2|_{E'} \) and \( \tilde{\rho}_{EE'} := \frac{1}{T} \sum_{i_1\cdot i_2\cdot j_1\cdot j_2} \text{tr}(\tilde{A}_{i_1} \tilde{A}_{i_2} \tilde{A}_{j_2}^\dagger \tilde{A}_{j_1}^\dagger) |i_1\rangle \langle j_1|_E \otimes |i_2\rangle \langle j_2|_{E'} \). The following conclusions hold:

1. The operators \( \rho_{EE'} \) and \( \tilde{\rho}_{EE'} \) are positive semidefinite and \( \text{tr}(\rho_{EE'}) = \text{tr}(\tilde{\rho}_{EE'}) = 1 \) (i.e. they are density operators).

2. There is a local unitary \( U_E \) such that

\[
\| U_E \otimes U_{E'} \rho_{EE'} U_E^\dagger \otimes U_{E'}^\dagger - \tilde{\rho}_{EE'} \|_1 \leq 4d^2 \| \mathcal{T} - \mathcal{T}' \|_{CB}^{1/2},
\]

where by \( \| \cdot \|_{CB} \) we denote the norm of complete boundedness \([14, 18]\).
Proof. The first assertion of the lemma follows by straightforward computations. For the second assertion we extend the CPU maps $\mathcal{T}$ and $\tilde{\mathcal{T}}$ using Stinespring representations $V := \sum_i A_i^\dagger \otimes |i\rangle_E$ and $\tilde{V} := \sum_i \tilde{A}_i^\dagger \otimes |\tilde{i}\rangle_E$, respectively. Since

$$\mathcal{T}(\rho) = V^\dagger (\rho \otimes 1_E) V \quad \forall \rho,$$

$V$ is indeed a Stinespring extension of $\mathcal{T}$ with dilation space $\mathcal{H}_E$. By assumption $\mathcal{T}$ is unital and thus $V$ is an isometry, i.e. $V^\dagger V = 1$. Moreover, it is not hard to verify that the operator $\rho_{EE'}$ can be rewritten as

$$\rho_{EE'} = \left( \frac{1}{D} \text{tr}_{CD} \left( (V \otimes 1_E)VV^\dagger (V^\dagger \otimes 1_E) \right) \right)^T,$$

where $(\cdot)^T$ denotes transposition with respect to the computational basis. The corresponding statements hold for the operators $\tilde{V}$ and $\tilde{\rho}_{EE'}$. To shorten the notation we introduce the isometry $W := (1 \otimes (U_E)^\dagger) V$, where $U_E$ denotes a unitary acting on the $E$ subsystem. Using the above expression for $\rho_{EE'}$ and the monotonicity of the Schatten 1-norm under the partial trace, we conclude that

$$\begin{align*}
\|U_e^\dagger \otimes U_{E'}^\dagger (\rho_{EE'})U_E \otimes U_{E'} - \tilde{\rho}_{EE'}\|_1 & = \| (U_E \otimes U_{E'})^T (\rho_{EE'})^T (U_e^\dagger \otimes U_{E'})^T - (\tilde{\rho}_{EE'})^T \|_1 \\
& \leq \frac{1}{D} \left\| (W \otimes 1_E) W W^\dagger (W^\dagger \otimes 1_E) - (\tilde{V} \otimes 1_E) \tilde{V} V^\dagger (\tilde{V}^\dagger \otimes 1_E) \right\|_1 \\
& \leq d^2 \left\| (W \otimes 1_E) W W^\dagger (W^\dagger \otimes 1_E) - (\tilde{V} \otimes 1_E) \tilde{V} V^\dagger (\tilde{V}^\dagger \otimes 1_E) \right\|_\infty \\
& \leq 4d^2 \| W - \tilde{V} \|_\infty.
\end{align*}$$

It follows from the continuity of the Stinespring extension (see [8, Theorem 1]) that the unitary $(U_E)^\dagger$ acting on the dilation space can be chosen such that

$$\| W - \tilde{V} \|_\infty^2 = \| (1 \otimes (U_E)^\dagger) V - \tilde{V} \|_\infty^2 \leq \| \mathcal{T} - \tilde{\mathcal{T}} \|_{CB}.$$

\hfill \Box

As mentioned before the operators $\rho_{EE'}$ will help us to keep track of the behaviour of the kernels of local parent Hamiltonians under scaling. The images of $\rho_{EE'}$ will correspond to the kernels of the Hamiltonians. We write $P_{EE'}$ and $\tilde{P}_{EE'}$ for the projectors onto the images of $\rho_{EE'}$ and $\tilde{\rho}_{EE'}$. In the following we shall obtain conditions under which the distance of these projectors is small, i.e. the kernels of the parent Hamiltonians are almost the same.

**Lemma 5.** Let $\rho$ and $\tilde{\rho}$ be two Hermitian operators and let $\rho^{-1}$ and $\tilde{\rho}^{-1}$ be their (Moore-Penrose-) pseudo inverses. Let $P = \rho \rho^{-1}$ and $\tilde{P} = \tilde{\rho} \tilde{\rho}^{-1}$ denote the projectors onto the images of $\rho$ and $\tilde{\rho}$. Then for any Schatten $p$-norm $\| \cdot \|_p$ we have that

$$\| P - \tilde{P} \|_p \leq \| \rho - \tilde{\rho} \|_p (\| \rho^{-1} \|_\infty + \| \rho^{-2} \|_\infty + \| \rho^{-2} \|_\infty + \| \rho^{-1} \|_\infty \| \tilde{\rho}^{-1} \|_\infty).$$

**Proof.** We rewrite the projectors $P$ and $\tilde{P}$ using $\rho^{-1}$ and $\tilde{\rho}^{-1}$ to conclude that

$$\begin{align*}
\| P - \tilde{P} \|_p & = \| \rho \rho^{-1} - \tilde{\rho} \tilde{\rho}^{-1} - \rho \rho^{-1} + \tilde{\rho} \tilde{\rho}^{-1} \|_p \\
& \leq \| \rho^{-1} \|_\infty \| \rho - \tilde{\rho} \|_p + \| \tilde{\rho} \|_\infty \| \rho^{-1} - \tilde{\rho}^{-1} \|_p.
\end{align*}$$
The distance $\|\rho^{-1} - \tilde{\rho}^{-1}\|_p$ can be bounded using the fact that

$$\rho^{-1} - \tilde{\rho}^{-1} = \rho^{-2}(\rho - \tilde{\rho})(\mathbb{1} - \tilde{\Pi}) + (\mathbb{1} - \Pi)(\rho - \tilde{\rho})\tilde{\rho}^{-2} - \rho^{-1}(\rho - \tilde{\rho})\tilde{\rho}^{-1}. $$

Applying the triangle inequality and the Hölder Inequality yields

$$\|\rho^{-1} - \tilde{\rho}^{-1}\|_p \leq \|\rho - \tilde{\rho}\|_p (\|\rho^{-2}\|_\infty + \|\tilde{\rho}^{-2}\|_\infty + \|\rho^{-1}\|_\infty \|\tilde{\rho}^{-1}\|_\infty)$$

which implies that

$$\|P - \tilde{P}\|_p \leq \|\rho - \tilde{\rho}\|_p (\|\rho^{-1}\|_\infty + \|\rho^{-2}\|_\infty + \|\tilde{\rho}^{-2}\|_\infty + \|\rho^{-1}\|_\infty \|\tilde{\rho}^{-1}\|_\infty).$$

Proof.

In our main derivation we will encounter the situation, where $\tilde{\rho}$ is fixed whereas $\rho$ depends on an integer, $\rho = \rho(L)$, and approaches $\tilde{\rho}$ as $L$ goes to infinity. All operators $\rho(L)$ as well as the asymptotic operator $\tilde{\rho}$ will be density operators of the same rank. We write $\mu = \mu(L)$ for the smallest non-zero eigenvalue of $\rho(L)$ and accordingly $\tilde{\mu}$ for smallest non-zero eigenvalue of $\tilde{\rho}$. By Lemma 5 the convergence behaviour of the projectors $P = P(L)$ towards $\tilde{P}$ is governed by the distance $\|\rho - \tilde{\rho}\|_p$, and the largest eigenvalues $1/\mu$ and $1/\tilde{\mu}$ of $\rho^{-1}$ and $\tilde{\rho}^{-1}$. The upper bound for the distance between the projectors $P$ and $\tilde{P}$ obtained from Lemma 5 depends explicitly on $1/\mu$. However, when $\|\rho - \tilde{\rho}\|_\infty$ is small enough it follows from the continuity of eigenvalues that one can replace the dependence on $1/\mu$ by $1/\tilde{\mu}$.

Lemma 6. Let $\rho$ and $\tilde{\rho}$ be two density matrices of the same rank and let $\tilde{\mu}$ be the smallest positive eigenvalue of $\tilde{\rho}$. If $\|\rho - \tilde{\rho}\|_\infty < \tilde{\mu}$ then

$$\|P - \tilde{P}\|_\infty \leq \frac{4\|\rho - \tilde{\rho}\|_\infty}{(\tilde{\mu} - \|\rho - \tilde{\rho}\|_\infty)^2}. $$

Proof. An application of Weyl’s Perturbation Theorem 3 under exploitation of the fact that $\rho$ and $\tilde{\rho}$ have the same rank shows that $|\mu - \tilde{\mu}| \leq \|\rho - \tilde{\rho}\|_\infty$. This yields an upper bound on the operator norm of $\rho^{-1}$:

$$\|\rho^{-1}\|_{\infty} = \frac{1}{\mu} \leq \frac{1}{\tilde{\mu} - \|\rho - \tilde{\rho}\|_\infty}. $$

We use Lemma 5 to conclude that

$$\|P - \tilde{P}\|_\infty \leq \|\rho - \tilde{\rho}\|_\infty \left( \frac{1}{\tilde{\mu} - \|\rho - \tilde{\rho}\|_\infty} + \frac{1}{\tilde{\mu} - \|\rho - \tilde{\rho}\|_\infty} \right) \leq \frac{4\|\rho - \tilde{\rho}\|_\infty}{(\tilde{\mu} - \|\rho - \tilde{\rho}\|_\infty)^2}. $$

The proof of Theorem 2 relies on a renormalization group technique as introduced in 16. We define local Hamiltonians acting on subchains of $\Lambda$. We then group the sites upon which these Hamiltonians act to blocks. The core observation is that the number of matrices required for the representation of the MPS will not increase from a certain point on. On the other hand with each grouping the blocked Hamiltonians “look more and more classical”. The following lemma is taken from 16 and describes this blocking procedure more precisely. The consecutive application of this result to larger and larger subchains of $\Lambda$ will be referred to as the renormalization group flow.
Lemma 7. Let \( \{A_i\}_{i=1,...,d} \) be a set of \( D \times D \) matrices and consider the set \( \{A_{i_1} \cdot ... \cdot A_{i_L}\}_{i_j=1,...,d} \) of all matrix products formed by matrices from \( \{A_i\}_{i=1,...,d} \). There is a \( d^L \times d^L \) unitary matrix \( U \) and matrices \( A_m^{(L)} \) with
\[
\sum_{i_1,...,i_L} U_{m(i_1,...,i_L)} A_{i_1} \cdot ... \cdot A_{i_L} = A_m^{(L)}
\]
such that \( A_m^{(L)} = 0 \) for all \( m > \min\{D^2,d^L\} \). Moreover, it holds that \( T^L = T^{(L)} \), where \( T^{(L)} \) denotes the CP map with Kraus operators \( A_m^{(L)} \).

Proof. We write \( (A_{i_1} \cdot ... \cdot A_{i_L})_{\alpha,\beta} \) with \( \alpha,\beta \in \{1,...,D\} \) for the entry of the matrix \( A_{i_1} \cdot ... \cdot A_{i_L} \) in row \( \alpha \) and column \( \beta \). Let \( \tilde{A} \) be the \( d^L \times D^2 \) matrix which has the entry \( (A_{i_1} \cdot ... \cdot A_{i_L})_{\alpha,\beta} \) in its \((i_1...i_L)\)-th row and \((\alpha,\beta)\)-th column. We perform a singular value decomposition of \( \tilde{A} \) writing
\[
\tilde{A}_{(i_1...i_L),\alpha\beta} = \sum_{l=1}^{\min\{D^2,d^L\}} (U^\dagger)_{(i_1...i_L),l} \rho_l V_l(\alpha\beta).
\]

For the \( m \)-th row of \( U \tilde{A} \), \((U \tilde{A})^{(m)} \), then
\[
(U \tilde{A})^{(m)} = \begin{cases} 
\rho_m V^{(m)} & \text{; } m \leq \min\{d^L,D\} \\
0 & \text{; } m > \min\{d^L,D\}
\end{cases}
\]

holds. The rows of the matrix \( U \tilde{A} \) now correspond to the matrices \( A_i^{(L)} \) and thus the first assertion of the lemma follows.

For the second assertion simply observe that for any \( X \) the quantity
\[
T^L(X) = \sum_{i_1,...,i_L} A_{i_1} \cdot ... \cdot A_{i_L} X A_{i_L}^\dagger \cdot ... \cdot A_{i_1}^\dagger
\]
is invariant under unitary summations i.e.
\[
T^L(X) = \sum_m A_m^{(L)} X (A_m^{(L)})^\dagger = T^{(L)}(X).
\]

\( \square \)

In the following lemma we analyse the asymptotic behaviour of the renormalization group flow and show that at large scale a generic TI MPS “looks classical”. To achieve this, we consider large powers of the CPU map associated to the MPS and prove that the corresponding Kraus operators have a certain structure. It is well known that condition (G1) implies that the peripheral spectrum of \( T \) is trivial i.e. 1 is the only eigenvalue of \( T \) whose magnitude is one \([3,12,13]\).

Lemma 8. Let \( T(X) = \sum_i A_i X A_i^\dagger \) be a CPU map such that 1 is the unique eigenvalue of magnitude one and suppose that \( \Xi = \text{diag}(\xi_1,...,\xi_n) \) with \( \xi_i > 0 \) is the corresponding fixed point of \( T^* \). Then the following conclusions hold:

1. The limit \( T^{\infty} := \lim_{n \to \infty} T^n \) exists and we can write \( T^{\infty}(X) = \sum_{i=1}^{D^2} A_i^{(\infty)} X (A_i^{(\infty)})^\dagger \) with matrices \( A_{(pq)}^{(\infty)} = \sqrt{\xi_p} |p\rangle \langle q| \) and \( p,q \in \{1,...,D\} \).
2. The projector $P^{(\infty)}_{EE'}$ onto the image of

$$\rho^{(\infty)}_{EE'} := \frac{1}{D} \sum_{i_1 i_2 j_1 j_2} D^2 \text{tr} \left( A^{(\infty)}_{i_1} A^{(\infty)}_{j_2} \left( A^{(\infty)}_{j_1} \right)^\dagger \right) |i_1\rangle \langle j_1|_E \otimes |i_2\rangle \langle j_2|_{E'}$$

can be written as

$$P^{(\infty)}_{EE'} = I_A \otimes |\varphi\rangle \langle \varphi|_{BC} \otimes I_D,$$

where $|\varphi\rangle = \sum_i \sqrt{\xi_i}|ii\rangle$, each of the subsystems $A, B, C, D$ is isomorphic to $\mathbb{C}^D$, and $E = AB, E' = CD$.

Proof. All eigenvalues of a CPU map are contained in the closed unit disc in the complex plane. By assumption $T$ has only one eigenvalue on the boundary and this eigenvalue is 1. Those eigenvalues of $T^n$, which are contained in the open unit disc decay with increasing $n$, while 1 is an eigenvalue of $T^n$ for any $n$. Hence, $\lim_{n \to \infty} T^n$ simply converges to the projector onto the eigenvector $1$ corresponding to the eigenvalue 1 of $T$. The fact that $A^{(\infty)}_{pq} = \sqrt{\xi_q}|p\rangle\langle q|$ is then straightforward since the dual map $(T^*)^\infty$ acts as $(T^*)^\infty(X) = tr(X)\Xi$.

It follows from the first assertion of the lemma and the fact that $\{ A^{(\infty)}_i \}_{i=1, \ldots, D^2}$ span the entire matrix algebra that the vectors $|\mu^{(\infty)}(X)\rangle = \sum_{i_1 i_2} D^2 \text{tr} (X A^{(\infty)}_{i_1} A^{(\infty)}_{i_2}) |i_1 i_2\rangle$ span the image of $\rho^{(\infty)}_{EE'}$. Furthermore they can be written as

$$|\mu^{(\infty)}(X)\rangle = (I \otimes \sqrt{\Xi}X)_{AD} |\omega\rangle_{AD} |\varphi\rangle_{BC},$$

where $|\omega\rangle_{AD} = \sum_i |ii\rangle_{AD}$. Observe that $P^{(\infty)}_{EE'}$ as defined in the lemma has rank $D^2$ and $P^{(\infty)}_{EE'} |\mu^{(\infty)}(X)\rangle = |\mu^{(\infty)}(X)\rangle$. Therefore $P^{(\infty)}_{EE'}$ projects onto the image of $\rho^{(\infty)}_{EE'}$. \hfill \Box

C. The core argument

In this subsection we consider the stability of the spectral gap of a parent Hamiltonian with nearest neighbour interaction $H_\Lambda = \sum_k h_{k,k+1}$. We prove that the spectral gap is stable under the assumption that at each site $\{ A_i \}_{i=1, \ldots, d}$ span the entire algebra of $D \times D$ matrices. In the following subseetions we extend this argument to show that stability holds more generally for generic MPS in the sense of (G1).

Proof of stability (Theorem 2) under strong assumptions. We show that at large scale the parent Hamiltonian $H_\Lambda$ is a perturbation of a classical model and apply Theorem 1 to obtain the perturbation result. For this we divide $\Lambda$ into blocks $\Lambda_k$ of length $L$ and block the terms of $H_\Lambda$ into Hamiltonians $H_{\Lambda_k} := \sum_{j:j+1} h_{j,j+1}$. Acting locally on $H_{\Lambda_k} := \sum_{j:j+1} h_{j,j+1}$ such that

$$H_\Lambda = \frac{1}{2} \sum_k \left( H_{\Lambda_k} + h_{kL,kL+1} \right).$$

For notational convenience we shall abbreviate $H_{k,k+1} := \frac{1}{2} \left( H_{\Lambda_k} + h_{kL,kL+1} \right)$. Clearly it holds that

$$\text{Kern} H_{k,k+1} = \text{Kern} H_{\Lambda_k}$$
An application of Lemma 7 shows that there is a unitary with the property that

The local Hamiltonians $H$ denote the projector onto $\text{Kern}L$ space that is isomorphic to $\text{Kern}L_{C2}\text{space is non-zero. In the sequel we shall assume that}$

We introduce the density matrix $G$ of the $2$-dimensional space. For any given

$$\rho_{k+1} := \frac{1}{D} \sum_{(i_1 \ldots i_{2L})} \text{tr}(A_{i_1} \cdot \cdots \cdot A_{i_{2L}} A_{j_{2L}} \cdot \cdots \cdot A_{j_1}) |i_1 \ldots i_{2L}, j_1 \ldots j_{2L}|.$$  

By assumption the matrices $\{A_i\}_{i=1}^d$ span the entire matrix algebra. Hence, for any $L$ the image of $\rho_{k+1}$ is spanned by the $D^2$-dimensional manifold of vectors

$$|\mu(X)\rangle = \sum_{i_1 \ldots i_{2L}} \text{tr}(XA_{i_1} \cdot \cdots \cdot A_{i_{2L}}) |i_1 \ldots i_{2L},$$

where $X$ is a $D \times D$ matrix with complex entries (see Section IIIB). On the other hand these vectors exactly span the kernel of $H_{\Lambda_k \cup \Lambda_{k+1}}$ (see Section IIIC and [12]) and we obtain

$$\text{Im} \rho_{k+1} = \text{Kern}H_{\Lambda_k \cup \Lambda_{k+1}}.$$  

The local Hamiltonians $H_{\Lambda_k \cup \Lambda_{k+1}}$ have a positive spectral gap (see Section IIIC). Let $G_{\Lambda_k \cup \Lambda_{k+1}}$ denote the projector onto $\text{Kern}H_{\Lambda_k \cup \Lambda_{k+1}}$ then there is a $\gamma > 0$ that does not depend on $L$ such that

$$H_{\Lambda_k \cup \Lambda_{k+1}} \geq \gamma (1 - G_{\Lambda_k \cup \Lambda_{k+1}}).$$  

(5)

An application of Lemma 7 shows that there is a unitary $U_{\Lambda_k}$ acting non-trivially on $\mathcal{H}_{\Lambda_k}$ only, with the property that

$$U_{\Lambda_k} \otimes U_{\Lambda_k} \rho_{\Lambda_k \cup \Lambda_{k+1}} U_{\Lambda_k}^\dagger \otimes U_{\Lambda_{k+1}}^\dagger = \begin{pmatrix} \rho_{EE'}^{(L)} & 0 \\ 0 & 0 \end{pmatrix},$$  

(6)

where

$$\rho_{EE'}^{(L)} := \frac{1}{D} \sum_{(i_1 j_1) (i_2 j_2)} \text{tr}(A_{i_1}^{(L)} A_{i_2}^{(L)} (A_{j_2}^{(L)})^\dagger (A_{j_1}^{(L)})^\dagger) |i_1 j_1 E \otimes i_2 j_2 E'|.$$  

and the matrices $A_{ij}^{(L)}$ are as in Lemma 7. The matrix $U_{\Lambda_k} \otimes U_{\Lambda_k} \rho_{\Lambda_k \cup \Lambda_{k+1}} U_{\Lambda_k}^\dagger \otimes U_{\Lambda_{k+1}}^\dagger$ acts on a space that is isomorphic to $(C^d)^{\otimes L} \otimes (C^d)^{\otimes L}$ but only the action on a $(\min\{D^2, dL\})^2$ dimensional subspace is non-zero. In the sequel we shall assume that $L$ is chosen large such that $\rho_{EE'}^{(L)}$ acts on a $(D^2)^2$ dimensional space. For any given $L$ we fix this space and define the matrix $\rho_{\Lambda_k \cup \Lambda_{k+1}}^{(\infty)}$ by replacing $\rho_{EE'}^{(L)}$ in that space by $\rho_{EE'}^{(\infty)}$ i.e.

$$\rho_{\Lambda_k \cup \Lambda_{k+1}}^{(\infty)} = \begin{pmatrix} \rho_{EE'}^{(\infty)} & 0 \\ 0 & 0 \end{pmatrix}.$$  

We denote by $G_{\Lambda_k \cup \Lambda_{k+1}}^{(\infty)}$ the projector onto the image of $\rho_{\Lambda_k \cup \Lambda_{k+1}}^{(\infty)}$. Note that since the orientation of the $(D^2)^2$ dimensional subspace in $(C^d)^{\otimes L} \otimes (C^d)^{\otimes L}$ can depend on $L$ it follows that $\rho_{\Lambda_k \cup \Lambda_{k+1}}^{(\infty)}$
and \( G_{A_k \cup A_{k+1}}^{(\infty)} \) can depend on \( L \).

We will now discuss the asymptotic properties of the matrices \( \rho_{A_k \cup A_{k+1}}^{(L)} \). We will prove that with a suitable unitary transformation acting locally on the spaces \( \mathcal{H}_{A_k} \) and with \( L \) chosen large the operators \( \rho_{A_k \cup A_{k+1}}^{(L)} \) and \( \rho_{A_k \cup A_{k+1}}^{(\infty)} \) can be made arbitrarily close. This will provide us with an explicit unitary acting locally on (sufficiently large) spaces \( \mathcal{H}_{A_k} \) that transforms the kernel of \( H_{A_k \cup A_{k+1}} \) into a shape determined by \( \rho_{A_k \cup A_{k+1}}^{(\infty)} \).

Let us consider the CPU map \( \mathcal{T} \) associated with the MPS \( |\Psi\rangle \) and let \( \lambda_2 \) denote its largest in magnitude subdominant eigenvalue. We note that \( \sup_{k \geq 0} \| \mathcal{T}^k \|_{CB} = 1 \), i.e. \( \mathcal{T} \) is power-bounded with respect to the \( CB \)-norm and constant 1. Hence, the discussion in [15] applies and yields that there is \( C \) that does not depend on \( L \) such that

\[
\| \mathcal{T}^L - \mathcal{T}^\infty \|_{CB} \leq C|\lambda_2|^L.
\]

By Lemma 7 this is equivalent to

\[
\| \mathcal{T}^{(L)} - \mathcal{T}^{(\infty)} \|_{CB} \leq C|\lambda_2|^L,
\]

where the maps \( \mathcal{T}^{(L)} \) are defined as in the lemma. We apply Lemma 8 to conclude that there is a unitary \( V_E \) such that

\[
\| V_E \otimes V_{E'} \rho_{EE'}^{(L)} V_E^\dagger \otimes V_{E'}^\dagger - \rho_{EE'}^{(\infty)} \|_\infty \leq 4D^4 \sqrt{C}|\lambda_2|L/2.
\]

By Lemma 9 it holds for \( L \) chosen sufficiently large that

\[
\| V_E \otimes V_{E'} \rho_{E'E}^{(L)} V_E^\dagger \otimes V_{E'}^\dagger - \rho_{E'E}^{(\infty)} \|_\infty \leq \frac{16D^4 \sqrt{C}|\lambda_2|L/2}{(\mu - 4D^4 \sqrt{C}|\lambda_2|L/2)^2},
\]

where \( \mu \) is the smallest non-zero eigenvalue of \( \rho_{EE'}^{(\infty)} \). A straightforward computation shows that in fact \( \mu \) equals the smallest eigenvalue of the fixed point matrix \( \Lambda \).

Taken together, the inequalities (7) and (8) imply that the projectors onto the images of \( \rho_{A_k \cup A_{k+1}}^{(\infty)} \) and \( \rho_{A_k \cup A_{k+1}}^{(\infty)} \) can be made exponentially close with a local unitary operation: There is a unitary \( W_{A_k} \) such that

\[
\| W_{A_k} \otimes W_{A_{k+1}} G_{A_k \cup A_{k+1}} W_{A_k}^\dagger \otimes W_{A_{k+1}}^\dagger - G_{A_k \cup A_{k+1}}^{(\infty)} \|_\infty \leq \frac{16D^4 \sqrt{C}|\lambda_2|L/2}{(\mu - 4D^4 \sqrt{C}|\lambda_2|L/2)^2}
\]

In terms of the Hamiltonians \( H_{A_k \cup A_{k+1}} \) this means that we have achieved to construct a unitary acting locally on spaces \( \mathcal{H}_{A_k} \) that on sufficiently large scale transforms the ground state space of \( H_{A_k \cup A_{k+1}} \) into a certain subspace determined by \( G_{A_k \cup A_{k+1}}^{(\infty)} \). In the next step we construct a classical Hamiltonian with this ground state subspace. For each \( L \) the structure of the operators \( G_{A_k \cup A_{k+1}}^{(\infty)} \) is known from Lemma 9. We have that

\[
G_{A_k \cup A_{k+1}}^{(\infty)} = \begin{pmatrix} 1_A \otimes |\varphi\rangle \langle \varphi|_{BC} \otimes 1_D & 0 \\ 0 & 0 \end{pmatrix}
\]

with \( |\varphi\rangle = \sum_i \sqrt{c_i} |ii\rangle \). Thus \( G_{A_k \cup A_{k+1}}^{(\infty)} \) induces a natural decomposition of \( \mathcal{H}_{A_k \cup A_{k+1}} \) into a subspace \( \mathcal{H}_X \) on which \( G_{A_k \cup A_{k+1}}^{(\infty)} \) acts as the zero operator and a subspace which is isomorphic to \( \mathbb{C}^{D^2} \otimes \mathbb{C}^{D^2} \).
The latter can further be decomposed according to the structure of $G_{\Lambda_k \cup \Lambda_{k+1}}^{(\infty)}$ into $\mathbb{C}^{D^2} \otimes \mathbb{C}^{D^2} \cong \mathbb{C}_A^D \otimes \mathbb{C}_B^D \otimes \mathbb{C}_C^D \otimes \mathbb{C}_D^D$. By an additional decomposition of $\mathcal{H}_X$ and choosing $L$ even we achieve the decomposition

$$\mathcal{H}_{\Lambda_k \cup \Lambda_{k+1}} \cong (\mathbb{C}_A^D \oplus \mathbb{C}_X^A) \otimes (\mathbb{C}_B^D \oplus \mathbb{H}_{X_B}) \otimes (\mathbb{C}_C^D \oplus \mathbb{H}_{X_C}) \otimes (\mathbb{C}_D^D \oplus \mathbb{H}_{X_D}).$$

Here the spaces $\mathbb{H}_{X_B}, \ldots, \mathbb{H}_{X_D}$ are chosen to have dimension $d^{L/2} - D$. In the decomposition of $\mathcal{H}_{\Lambda_k \cup \Lambda_{k+1}}$ we identify the “half-shifted” spaces $\mathcal{H}_{\Lambda_k \cup \Lambda_{k+1}}^{HS} := (\mathbb{C}_B^D \oplus \mathbb{H}_{X_B}) \otimes (\mathbb{C}_C^D \oplus \mathbb{H}_{X_C})$. Note that $\mathcal{H}_{\Lambda_k \cup \Lambda_{k+1}}^{HS} \cong \mathcal{H}_{\Lambda_k}$ and that the following inclusions hold:

$$\mathcal{H}_{\Lambda_k \cup \Lambda_{k+1}}^{HS} \subset \mathcal{H}_{\Lambda_k \cup \Lambda_{k+1}} \subset \mathcal{H}_{\Lambda_k-1 \cup \Lambda_k}^{HS} \otimes \mathcal{H}_{\Lambda_k \cup \Lambda_{k+1}} \otimes \mathcal{H}_{\Lambda_k+1 \cup \Lambda_{k+2}}^{HS}.$$

Let $P_{\Lambda_k \cup \Lambda_{k+1}}^{HS}$ denote the projector in $\mathcal{H}_{\Lambda_k \cup \Lambda_{k+1}}^{HS}$ onto the orthogonal complement of $|\varphi\rangle$. The above inclusions translate into the estimates

$$H_{\Lambda_k \cup \Lambda_{k+1}}^{HS} \leq 1 - C_{\Lambda_k \cup \Lambda_{k+1}}^{(\infty)} \leq H_{\Lambda_k-1 \cup \Lambda_k}^{HS} + H_{\Lambda_k \cup \Lambda_{k+1}}^{HS} + H_{\Lambda_k+1 \cup \Lambda_{k+2}}^{HS}. \quad (9)$$

Consider the operator

$$H_{\Lambda_k}^{CL} := 3L \sum_k H_{\Lambda_k \cup \Lambda_{k+1}}^{HS}.$$

This operator is classical in the sense of Theorem [1] with respect to the half-shifted spaces $H_{\Lambda_k \cup \Lambda_{k+1}}^{HS}$. We claim that for $L$ chosen large enough $(\bigotimes_k W_{\Lambda_k})H_{\Lambda}(\bigotimes_k W_{\Lambda_k})^\dagger$ is a perturbation of $H_{\Lambda_k}^{CL}$ satisfying the assumptions of Theorem [1]. We construct this perturbation explicitly. It consists of a bounded part

$$\phi^{(b)}_{k,k+1} := W_{\Lambda_k} \otimes W_{\Lambda_{k+1}}(1 - G_{\Lambda_k \cup \Lambda_{k+1}})H_{k,k+1}(1 - G_{\Lambda_k \cup \Lambda_{k+1}})W_{\Lambda_k}^\dagger \otimes W_{\Lambda_{k+1}}^\dagger$$

and a relatively bounded part

$$\phi^{(r)}_{k,k+1} := (1 - G_{\Lambda_k \cup \Lambda_{k+1}})(W_{\Lambda_k} \otimes W_{\Lambda_{k+1}})H_{k,k+1}W_{\Lambda_k}^\dagger \otimes W_{\Lambda_{k+1}}^\dagger (1 - G_{\Lambda_k \cup \Lambda_{k+1}})^{\infty}$$

and a relatively bounded part

$$\phi^{(r)}_{k,k+1} := (1 - G_{\Lambda_k \cup \Lambda_{k+1}})(W_{\Lambda_k} \otimes W_{\Lambda_{k+1}})H_{k,k+1}W_{\Lambda_k}^\dagger \otimes W_{\Lambda_{k+1}}^\dagger (1 - G_{\Lambda_k \cup \Lambda_{k+1}})^{\infty}$$

and a relatively bounded part

$$\phi^{(r)}_{k,k+1} := (1 - G_{\Lambda_k \cup \Lambda_{k+1}})(W_{\Lambda_k} \otimes W_{\Lambda_{k+1}})H_{k,k+1}W_{\Lambda_k}^\dagger \otimes W_{\Lambda_{k+1}}^\dagger (1 - G_{\Lambda_k \cup \Lambda_{k+1}})^{\infty}$$

Taking both together yields

$$(\bigotimes_k W_{\Lambda_k})H_{\Lambda}(\bigotimes_k W_{\Lambda_k})^\dagger = H_{\Lambda_k}^{CL} + \sum_k \phi^{(b)}_{k,k+1} + \sum_k \phi^{(r)}_{k,k+1}.$$

First we estimate

$$\|\phi^{(b)}_{k,k+1}\| = \left\| W_{\Lambda_k} \otimes W_{\Lambda_{k+1}}(1 - G_{\Lambda_k \cup \Lambda_{k+1}})H_{k,k+1}(1 - G_{\Lambda_k \cup \Lambda_{k+1}})W_{\Lambda_k}^\dagger \otimes W_{\Lambda_{k+1}}^\dagger \right\|$$

$$\leq \left\| H_{k,k+1}(1 - G_{\Lambda_k \cup \Lambda_{k+1}})W_{\Lambda_k}^\dagger \otimes W_{\Lambda_{k+1}}^\dagger - H_{k,k+1}W_{\Lambda_k}^\dagger \otimes W_{\Lambda_{k+1}}^\dagger \right\| + \left\| W_{\Lambda_k} \otimes W_{\Lambda_{k+1}}(1 - G_{\Lambda_k \cup \Lambda_{k+1}})H_{k,k+1}W_{\Lambda_k} \otimes W_{\Lambda_{k+1}}H_{k,k+1} \right\|$$

$$\leq 2 \left\| W \otimes W_{\Lambda_k \cup \Lambda_{k+1}}(1 - G_{\Lambda_k \cup \Lambda_{k+1}})W_{\Lambda_k} \otimes W_{\Lambda_{k+1}}H_{k,k+1} \right\|$$

where

$$\mu = 32L^2 \sqrt{C} |\lambda_2|^{L/2}.$$
The last inequality makes use of (8) and the fact that \(|H_{k,k+1}\|_\infty \leq L\). Thus we have shown that the norm of \(\phi_{b,k,k+1}\) decays exponentially fast with increasing size of the blocks \(\Lambda_k\).

To verify that \(\phi_{b,k,k+1}\) is in accordance with the conditions of Theorem 1, we need to estimate \(|\sum_{x \in I} \phi_{b,k}^I|\) for any \(I \subset \{1, \ldots, N/L\}\). The maximum is attained when \(I = \Lambda\) since

\[
\phi_{b,k,k+1}^I \leq L \left(1 - G_{\Lambda_k \cup \Lambda_{k+1}}^{(\infty)}\right) - L \left(H_{\Lambda_k \cup \Lambda_{k+1}}^{\text{SUB}} + H_{\Lambda_k \cup \Lambda_{k+1}}^{\text{SUB}} + H_{\Lambda_k \cup \Lambda_{k+1}}^{\text{SUB}}\right) \leq 0,
\]

where the second inequality makes use of (9). A lower bound on \(\phi_{b,k,k+1}\) follows from the gappedness of \(H_{k,k+1}\) (5):

\[
(1 - G_{\Lambda_k \cup \Lambda_{k+1}}^{(\infty)}) W_{\Lambda_k} \otimes W_{\Lambda_{k+1}} H_{k,k+1} W_{\Lambda_k}^\dagger \otimes W_{\Lambda_{k+1}}^\dagger (1 - G_{\Lambda_k \cup \Lambda_{k+1}}^{(\infty)}) \geq
\]

\[
\frac{\gamma}{2} (1 - G_{\Lambda_k \cup \Lambda_{k+1}}^{(\infty)}) \left(1 - G_{\Lambda_k \cup \Lambda_{k+1}}^{(\infty)} - \frac{16D^4 \sqrt{C} |\lambda_2| L/2}{(\mu - 4D^4 \sqrt{C} |\lambda_2| L/2)^2} \right) \left(1 - G_{\Lambda_k \cup \Lambda_{k+1}}^{(\infty)}\right) \geq
\]

\[
\frac{\gamma}{2} (1 - \frac{16D^4 \sqrt{C} |\lambda_2| L/2}{(\mu - 4D^4 \sqrt{C} |\lambda_2| L/2)^2}) (1 - G_{\Lambda_k \cup \Lambda_{k+1}}^{(\infty)}) \geq \frac{\gamma}{2} \left(1 - \frac{16D^4 \sqrt{C} |\lambda_2| L/2}{(\mu - 4D^4 \sqrt{C} |\lambda_2| L/2)^2}\right) H_{\Lambda_k \cup \Lambda_{k+1}}^{\text{HS}}.
\]

We sum the terms \(\phi_{b,k,k+1}^I\) to conclude that

\[
\sum_{k,k+1} \phi_{b,k,k+1}^I \geq
\]

\[
\sum_{k,k+1} \left(\frac{\gamma}{2} \left(1 - \frac{16D^4 \sqrt{C} |\lambda_2| L/2}{(\mu - 4D^4 \sqrt{C} |\lambda_2| L/2)^2}\right) H_{\Lambda_k \cup \Lambda_{k+1}}^{\text{HS}} - L \left(H_{\Lambda_k \cup \Lambda_{k+1}}^{\text{HS}} + H_{\Lambda_k \cup \Lambda_{k+1}}^{\text{HS}} + H_{\Lambda_k \cup \Lambda_{k+1}}^{\text{HS}}\right)\right)
\]

\[
= \left(-1 + \frac{\gamma}{6L} - \frac{8\gamma D^4 \sqrt{C} |\lambda_2| L/2}{3L(\mu - 4D^4 \sqrt{C} |\lambda_2| L/2)^2}\right) H_{\Lambda}^{\text{CL}}.
\]

Thus for Theorem 1 we have that

\[
\left|\sum_{k,k+1} \langle \psi | \phi_{b,k,k+1}^I | \psi \rangle\right| \leq \alpha \langle \psi | H_{\Lambda}^{\text{CL}} | \psi \rangle
\]

with \(\alpha = (1 - \frac{\gamma}{6L} + O(|\lambda_2| L/2))\) and \(\beta = \delta (\frac{\gamma}{6L} - O(|\lambda_2| L/2))^{2\kappa}\), where the constants \(\delta \) and \(\kappa\) still have to be chosen appropriately. As long as \(\gamma\) decays sub-exponentially fast with \(L\), for \(L\) sufficiently large \(|\phi_{b,k,k+1}^I|_\infty \leq \beta\) holds. For parent Hamiltonians, which have a constant local gap this is certainly the case.

Applying Theorem 1 we could recover the well-known fact that \(H_{\Lambda}\) has a gapped ground state. However, the conditions of Theorem 1 are “open” in the sense that adding sufficiently small bounded perturbation to \(\phi_{b,k,k+1}^I\) still results in a total perturbation, which is within the range where Theorem 1 can be applied. This provides us with a perturbation result for Hamiltonians in the neighbourhood of \(H_{\Lambda}\). More precisely, let \(\Phi' := \sum_{k,k+1} \phi_{k,k+1}'\) be a finite range interaction with \(|\phi_{b,k,k+1}^I|_\infty \leq \beta'\) and \(\beta' > 0\) small enough. We analyse the spectral gap of \(H_{\Lambda}' = H_{\Lambda} + \Phi'\). Suppose for the moment that \(\phi_{k,k+1}'\) acts exactly on \(H_{\Lambda_k \cup \Lambda_{k+1}}^{\text{SUB}}\) and let

\[
\phi_{k,k+1}'' := W_{\Lambda_k} \otimes W_{\Lambda_{k+1}}' \phi_{k,k+1}' W_{\Lambda_k}^\dagger \otimes W_{\Lambda_{k+1}}^\dagger.
\]
Consider the Hamiltonian
\[ (\bigotimes_k W_{\Lambda_k}) H_{\Lambda} (\bigotimes_k W_{\Lambda_k})^\dagger + \sum_k \phi''_{k,k+1} = (\bigotimes_k W_{\Lambda_k})(H_{\Lambda} + \Phi')(\bigotimes_k W_{\Lambda_k}). \]

If \( \beta' > 0 \) is chosen sufficiently small Theorem 1 applies and proves the stability of the spectral gap of \( H_{\Lambda} + \Phi' \). In general, though, we want to allow an arbitrary (finite) interaction range for \( \phi'_{k,k+1} \). If \( \phi'_{k,k+1} \) acts nontrivially on a subchain of \( \Lambda_k \cup \Lambda_{k+1} \) only it is possible to group the \( \phi_{k,k+1} \) terms in such a way that in total one gets a finite range interaction on \( \Lambda_k \cup \Lambda_{k+1} \). Choosing \( \beta' \) we make sure that the grouped perturbation is sufficiently small for an application of Theorem 1. On the other hand if the perturbation has interaction range exceeding the subchain \( \Lambda_k \cup \Lambda_{k+1} \) one simply chooses \( L \) larger and the previous discussion applies to the larger subchains.

D. Proof of Theorem 2 and Corollary 3

Proof of Theorem 2. The proof is a simple upgrade of the restricted discussion of the previous subsection. By condition (G1) there is finite \( P_0 \) such that the matrices \( \{A_{i_1} \cdots A_{i_{P_0}}\} \) span the whole algebra of \( D \times D \) matrices. Hence, \( H_{\Lambda} = \sum_i \tau^i(h_{G_P}) \) has a unique ground state for any \( P > P_0 \), see Section II C. We proceed as in the proof of the theorem and divide \( \Lambda \) into chains \( \Lambda_k \) of length \( L \). In addition we assume that the chains are sufficiently large to support \( h_{G_P} \) i.e. \( L \geq P \).

We define the operators
\[ H_{\Lambda_k \cup \Lambda_{k+1}} := \sum_{i:i+1 \in \Lambda_k \cup \Lambda_{k+1}} \tau^i(h_{G_P}), \]
which are sums of all the translates of \( h_{G_P} \) that act locally on \( \Lambda_k \cup \Lambda_{k+1} \). There are \( P - 1 \) terms in the above Hamiltonian that partially act on block \( \Lambda_k \) and partially on \( \Lambda_{k+1} \). We define the operators \( H_{k,k+1} \) by adding these terms to \( H_{\Lambda_k \cup \Lambda_{k+1}} \). Formally
\[ H_{k,k+1} = \frac{1}{2} H_{\Lambda_k \cup \Lambda_{k+1}} + \frac{1}{2} \sum_{i:i+1 \in \Lambda_k \cup \Lambda_{k+1}} \tau^i(h_{G_P}). \]

As before, we have the properties
\[ H_{k,k+1} \geq H_{\Lambda_k \cup \Lambda_{k+1}}, \]
\[ \text{Kern}(H_{k,k+1}) = \text{Kern}(H_{\Lambda_k \cup \Lambda_{k+1}}) \]
and
\[ H_{\Lambda} = \sum_k H_{k,k+1}. \]

The kernel of \( H_{\Lambda_k \cup \Lambda_{k+1}} \) is given by the image (see also [12, Section 4.1.1]) of
\[ \rho_{\Lambda_k \cup \Lambda_{k+1}} = \sum^{d}_{i_1 \cdots i_{2L}} \text{tr}(A_{i_1} \cdots A_{i_{2L}} A^\dagger_{j_1} \cdots A^\dagger_{j_{2L}} |i_1 \cdots i_{2L} \rangle \langle j_1 \cdots j_{2L}|). \]

As before, the spectral gap of \( H_{\Lambda_k \cup \Lambda_{k+1}} \) can be lower bounded by some constant. With \( G_{\Lambda_k \cup \Lambda_{k+1}} \) and \( G_{\Lambda_k \cup \Lambda_{k+1}}(\infty) \) defined as in Subsection IV C the derivation follows the same lines as before. Hence, stability follows under condition (G1), which completes the proof of Theorem 2. \( \square \)
Proof of Corollary 3. As before we choose $L \geq P$ and divide $\Lambda$ into subchains of length $L$. The restrictions of $\hat{H}_\Lambda = \sum_j h_{j,j+1}$ and $H_\Lambda = \sum_i \tau^i(h_{G_P})$ to $\Lambda_k \cup \Lambda_{k+1}$ are given by $\hat{H}_{\Lambda_k \cup \Lambda_{k+1}} = \sum_{j,j+1} h_{j,j+1}$ and $H_{\Lambda_k \cup \Lambda_{k+1}} = \sum_{i:j+1,\ldots,i+P} \tau^i(h_{G_P})$. The condition $c_1 h_{G_P} \leq \sum_{j,j+1} h_{j,j+1} \leq c_2 h_{G_P}$ implies that

$$c_1 H_{\Lambda_k \cup \Lambda_{k+1}} \leq \sum_{i:j+1,\ldots,i+P} \tau^i \left( \sum_{j=1}^{P-1} h_{j,j+1} \right) \leq c_2 H_{\Lambda_k \cup \Lambda_{k+1}}.$$ 

It follows that $\sum_{i:j+1,\ldots,i+P} \tau^i \left( \sum_{j=1}^{P-1} h_{j,j+1} \right)$ has the same kernel as $H_{\Lambda_k \cup \Lambda_{k+1}}$. Thus the kernels of $H_{\Lambda_k \cup \Lambda_{k+1}}$ and $\hat{H}_{\Lambda_k \cup \Lambda_{k+1}}$ are identical and Corollary 3 follows from the derivation of Theorem 2.

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