Gödel-type Universes and the Landau problem

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Abstract: We point out a close relation between a family of Gödel-type solutions of 3+1 General Relativity and the Landau problem in $S^2$, $\mathbb{R}^2$ and $\mathbb{H}^2$; in particular, the classical geodesics correspond to Larmor orbits in the Landau problem. We discuss the extent of this relation, by analyzing the solutions of the Klein-Gordon equation in these backgrounds. For the $\mathbb{R}^2$ case, this relation was independently noticed in hep-th/0306148. Guided by the analogy with the Landau problem, we speculate on the possible holographic description of a single chronologically safe region.

Keywords: Gödel universe, Landau Problem, Holographic Principle.
1. Introduction.

Renewed interest in spacetimes with closed timelike curves has been sparked in part by the work of Gauntlett et al. [1], who describe all SUSY solutions of $N = 1$ 5d supergravity with a timelike Killing vector, and note that the generic solution has closed timelike curves (CTCs). One of the surprises of this analysis was the discovery of a maximally supersymmetric 5d relative of the original 4d Gödel solution [2]. Similar techniques have been applied to the classification of SUSY solutions of 11d supergravity [3] with a timelike Killing vector, and again, solutions with closed timelike curves are commonplace. In particular, SUSY Gödel type metrics were also found.

The existence of Gödel type solutions of 10d and 11d supergravity raises the issue of whether these are valid backgrounds in string/M theory\footnote{To the best of our knowledge, the first appearance of a Gödel type solution in the string theory literature was in [4], as the KK reduction of a 5d chiral model, although the interpretation as a Gödel-type metric was not explicitly mentioned there.}, a question addressed in a number of recent works [4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. The validity of other supergravity solutions with closed timelike curves has been discussed in [14, 15, 16, 17, 18].

In particular, in our previous work [8], we argued that a class of Gödel type solutions of type II supergravity are not, despite $g_5 = 0$ appearances, valid backgrounds in string theory. We argued for this by showing that a specific D-brane probe, a supertube [19], develops negative mass modes in this background, following a similar computation performed in [17]. Since the worldvolume dynamics of D-branes captures the spacetime where they live, we interpreted this sickness as a problem of the spacetime itself, not just of the probe.
Nevertheless, there are many questions unanswered concerning Gödel-type universes. In this respect, the authors of [6] made an intriguing observation for these spacetimes: If one applies Bousso’s prescription [20] for constructing holographic screens (see [21] for a review) to these solutions, one finds that starting at any point in spacetime, the screen is inside the corresponding chronologically safe region. The precise meaning of these holographic screens when they are not at asymptotia has yet to be fully clarified, but the take of [6] seems to be that in a fully quantum gravity theory on these backgrounds, any individual observer has a unitary description of the physics, encoded in the holographic screen.

For the reasons mentioned above, the Gödel type metric is not a valid solution of string theory; however, the question of holography for this background can be rephrased as follows: in [8], we presented a new solution that locally is a Gödel type space, but at the radius where closed timelike curves would start to appear, we placed a domain wall made out of supertubes, such that the overall solution is free of CTCs. This domain wall clearly breaks the translation invariance of the original solution. Bousso’s prescription still yields the same holographic screen as in [6] for an observer at the origin of this new solution, and the question of which are the degrees of freedom at the holographic screen, still remains.

The possibility of having holographic screens that are not at asymptotia raises many questions that were not present in the more familiar case of AdS/CFT holography. A very first question is what are the bulk (string/gravity) degrees of freedom that one should associate to a particular screen. It is worth noting that even when the metric can be embedded in a string theory solution, and the string spectrum computed at $g_s = 0$ [22, 23, 7], the free string spectrum shows no immediate hint of a holographic description, or the possibility of a compact holographic screen. For relevant spacetimes as de Sitter and FRW cosmologies, Bousso’s prescription yields observer dependent holographic screens (although in the case of de Sitter it is also possible to place the screen at asymptotia [20]), so it would be clearly desirable to obtain concrete examples of observer dependent holography.

In the present work we will be interested in a family of solutions of 3+1 General Relativity, that extend Gödel’s original solution [4]. That solution corresponds to a homogeneous spacetime, with rigid rotation, negative cosmological constant and a perfect fluid as source. Gödel’s original solution was generalized, among others, by Rebouças and Tiomno [24], who found all the spacetime homogeneous solutions of General Relativity with rigid rotation. Their solutions typically have non-zero cosmological constant and non-vanishing stress energy tensor, and the metric in polar coordinates can be written as

$$ds^2 = -\left(dt + \frac{\Omega}{l^2} \sinh^2 lr d\phi\right)^2 + \frac{\sinh^2 2lr}{4l^2} d\phi^2 + dr^2 + dx^2.$$  \hspace{1cm} (1.1)

The main observation of the present paper is that the physics in these metrics have a close relationship with the problem of a charged particle in $\mathbb{H}_2$, $\mathbb{R}^2$ or $S^2$—depending on the sign of $l^2$—coupled to a magnetic field of strength given by $\Omega$ and an effective charge given by the energy (the conserved quantity associated to the time isometry). For the $l^2 \to 0$ case, this similarity was independently noticed and discussed in [7]. As we will see, the presence of closed timelike curves in...
these metrics translates into the absence of non-periodic orbits in the analogous particle problem. G. Gibbons has suggested \cite{25} that this might be a more general phenomenon.

At the classical level, the analogy between the Gödel-type solutions and the Landau problem has a first manifestation in the fact, shown in section 2, that all the geodesics project to circles, in the surface with constant \(t\) and constant \(x_3\). In particular, the projection of some spacelike geodesics are closed timelike curves. These projections of course are reminiscent of the Larmor orbits for an electron moving in a magnetic field. At the quantum level, we analyze in section 3 the solutions to the Klein-Gordon equation in these backgrounds (first studied in \cite{26}), and note that the wavefunctions closely resemble those of the Quantum Hall Effect (QHE) on \(S^2, \mathbb{R}^2\) and \(\mathbb{H}_2\) respectively. The main difference is that there is a rescaling in the wavefunctions, that causes a particular subset of them to have most of their support within a region free of closed timelike curves. This leads to the suggestion, discussed in section 4, that when we restrict to a single chronologically safe region (e.g. by considering our solution \cite{5} with a domain wall) the relevant modes should be similar to the wavefunctions supported mostly inside it. We further argue that taking into account gravity imposes a high energy cut-off in the bulk spectrum. We then assume the existence of a holographic description of this single chronologically safe region, and deduce that the number of boundary degrees of freedom scales linearly with the cut-off in the energy level. We conclude presenting a simple model—based on considering a fuzzy version of the holographic screen—that reproduces this scaling.

\textbf{Note added:} As this manuscript was near completion, we learned from G. Gibbons that some of the results presented in sections 2 and 3 were independently obtained by himself and C. Herdeiro in unpublished work, as mentioned in \cite{14}.

\textbf{Note added:} After the first version of this work appeared, we learned from S.J. Rey that the analogy between Gödel type metrics with \(l^2 = 0\) with the Landau problem in the plane, was already noticed and discussed in a recent paper by Y. Hikida and S.J. Rey \cite{9}. Furthermore, the authors of \cite{9} consider supergravity solutions with \(l^2 = 0\) Gödel type metrics and background field strengths turned on, which allows them to discuss also the geodesics and wavefunctions of particles charged under those fields. This introduces a qualitatively new phenomenon, as compared to the discussion in the present paper, since some of these charged geodesics can go beyond the radius of the chronologically safe region. We briefly consider the possible implications of this observation in the last section.

\section{A family of Gödel-type solutions}

We will study a family of 3+1 metrics discussed by Rebouças and Tiomno \cite{24} that generalizes the original solution due to Gödel. These are spacetime homogeneous metrics, with rigid rotation, and are characterized by two parameters \((l, \Omega)\), both with the dimensions of inverse length. As in the original Gödel solution, these spacetimes are of the form \(M \times \mathbb{R}\), where \(M\) has signature \((2, 1)\) and \(\mathbb{R}\) is a spatial direction that mostly plays no role in the discussion. These solutions in general...
have non-zero cosmological constant and stress-energy tensor; if we use as sources a combination of perfect fluid, \(U(1)\) gauge fields and/or a massless scalar, one can cover the range \([-\infty, \Omega^2]\). The solutions with \(l^2 > \Omega^2\) can be realized \([27]\), if we allow for solutions with torsion; we will touch upon these metrics only briefly.

It is useful to write the solution in different coordinate systems. The cylindrical symmetry of this family of solutions is manifest in polar coordinates

\[
ds^2 = -\left(dt + \frac{\Omega}{l^2} \sinh^2 \frac{l r}{l^2} d\phi\right)^2 + \frac{\sinh^2 \frac{2lr}{l^2} d\phi^2 + dr^2 + dx_3^2,}
\]
which will also be convenient when discussing the causal structure of these solutions, and the location of the holographic screens. We can also rewrite the metric in Cartesian coordinates \([24]\)

\[
ds^2 = -\left(dt + \frac{\Omega}{\sqrt{2l}} e^{2lx} dy\right)^2 + \frac{1}{2} e^{4lx} dy^2 + dx^2 + dx_3^2.
\]
Finally, for \(l^2 > 0\), if we further make the change of coordinates

\[2lx = -\ln 2lY, \quad y = \sqrt{2}X,\]
we obtain the metric in hyperbolic coordinates

\[
ds^2 = -\left(dt + \frac{\Omega}{2lY} dX\right)^2 + \frac{1}{(2lY)^2} (dX^2 + dY^2) + dx_3^2.
\]
For some particular values of the parameters, this solution reduces to well-known ones. For \(l^2 = \Omega^2/2\) we recover the Gödel solution. For \(l^2 = \Omega^2\) we get \(AdS_3 \times \mathbb{R}\) \([28]\).

A relevant feature of these metrics is that they have closed timelike curves. This is easiest to establish in polar coordinates, by considering curves of constant \(t, r, x_3\). The metric component \(g_{\phi\phi}\) is

\[g_{\phi\phi} = \frac{\sinh^2 2lr}{4l^2} - \frac{\Omega^2}{l^4} \sinh^4 lr,\]
and changes sign at the radius

\[\tanh lr_c = \frac{l}{\Omega}.\]
Outside this radius \(r_c\), there are closed timelike curves around the origin. Therefore there are closed timelike curves for the range \(l^2 < \Omega^2\). We call the region of \(r \leq r_c\) the chronologically safe cavity. Although for this discussion we fixed the origin, by the homogeneity of the metric it is clear that exactly the same picture must hold at any point in space-time: there is a chronologically safe cylinder of radius \(r_c\) and closed timelike curves outside it.

As we will see below, certain features of the solutions depend markedly on the sign of \(l^2\), so we briefly outline the different possibilities.
**Sphere** ($l^2 < 0$): It is convenient to introduce the new coordinates $R = i/2l$ and $\theta = r/R$. The metric (1.1) becomes

$$ds^2 = - \left( dt + 4\Omega R^2 \sin^2 \frac{\theta}{2} d\phi \right)^2 + R^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) + dx_3^2. \tag{2.7}$$

Therefore, the 2+1 manifold $\mathcal{M}$ is a real line bundle over $S^2$.

**Flat plane** ($l^2 = 0$): If we take the limit $l \to 0$ in the metric (1.1), we get a geometry originally obtained by Som and Raychaudhuri [29], in which $\mathcal{M}$ is now a real bundle over $\mathbb{R}^2$

$$ds^2 = - (dt + \Omega^2 r^2 d\phi)^2 + r^2 d\phi^2 + dr^2 + dx_3^2. \tag{2.8}$$

This metric has appeared a number of times in solutions of string theory [4, 22, 23] and more recently in [6, 7], where it was reinterpreted as a Gödel-type solution in string theory.

**Hyperbolic plane** ($l^2 > 0$): In this case, the manifold $\mathcal{M}$ is a real line bundle over $\mathbb{H}_2$. The range $0 < l^2 \leq \Omega^2 / 2$ can be realized with the stress energy tensor of a perfect fluid and/or a $U(1)$ gauge field, whereas the range $\Omega^2 / 2 < l^2 \leq \Omega^2$ can be realized by adding a scalar field to the sources [24].

After these preliminary remarks, we come to an important observation: all those metrics are of the generic form

$$ds^2 = -(dt + A_i(x) dx^i)^2 + h_{ij}(x) dx^i dx^j, \tag{2.9}$$

where $x^i$ denote all the coordinates that are not $t$. For these $D$ dimensional metrics (2.9) the geodesic equation, in affine parametrization, is identical to the equations of motion of a charged particle moving in a $D - 1$ surface with metric $h_{ij}$ coupled to a magnetic field with gauge potential one-form given by $A(x) = A_i(x) dx^i$. Furthermore, the charge of the particle (or coupling) in this analogous system, is given by $\Pi_t$, the conjugate momentum to the timelike coordinate $t$. The metrics (1.1) considered above correspond to the particular cases when, apart from the trivial extra direction $x_3$, the aforementioned surfaces have constant curvature ($S^2$, $\mathbb{R}^2$ and $\mathbb{H}_2$) and the magnetic field is constant, i.e. proportional to the volume form. In light of this remark, we anticipate that there will be many similarities between physics in these spacetimes and the Landau problem in surfaces of constant curvature, which is reviewed briefly in the appendix.

### 2.1 Isometries

The spacetimes considered in this paper are homogeneous and have five independent Killing vectors [24]. In Cartesian coordinates, these are given by

$$K_0 = \frac{\partial}{\partial t}, \quad K_1 = \frac{\partial}{\partial x_3}, \quad K_2 = \frac{\partial}{\partial y},$$

$$K_3 = -2ly \frac{\partial}{\partial y} + \frac{\partial}{\partial x},$$

$$K_4 = \frac{\Omega}{\sqrt{2l^2}} e^{-2l_2} \frac{\partial}{\partial t} - y \frac{\partial}{\partial x} + \left( ly^2 - \frac{1}{2l} e^{-4l_2} \right) \frac{\partial}{\partial y}. \tag{2.10}$$
They satisfy the commutation relations

\[ [K_0, K_r] = [K_1, K_r] = 0, \quad r = 0, \ldots, 4, \quad (2.11) \]
\[ [K_2, K_3] = -2lK_2, \quad [K_2, K_4] = -K_3, \quad [K_3, K_4] = -2lK_4. \quad (2.12) \]

Notice that the subset \{K_2, K_3, K_4\} form an \textit{su}(2), h_2, \textit{sl}(2, \mathbb{R}) subalgebra depending on the sign of \(l^2\). It was checked in [28] that when (1.1) describes AdS_3 \times \mathbb{R}, that is when \(l^2 = \Omega^2\), the isometry group contains a second \textit{SL}(2, \mathbb{R}) factor, thus matching SO(2, 2).

2.2 Geodesics

The connection of the physics in these spaces with the Landau problem has a first manifestation when we consider the classical geodesics. As we will show, the projection of any geodesic on the \((r, \phi)\) plane is a circle, which is of course reminiscent of the Larmor orbits of an electron in a constant magnetic field. This fact was noted for the original Gödel solution in [30], and for the Som-Raychaudhuri solution in [31]. The geodesics of higher dimensional versions of the \(l^2 = 0\) case were discussed recently in [9] where the analogy with Larmor orbits of the Landau problem is also pointed out. For the full family of solutions we are considering, the geodesics were qualitatively studied in [32], using polar coordinates. However, in that choice of coordinates, it is not manifest that the geodesics project to circles with arbitrary centers in the \((r, \phi)\) plane. We shall prove this statement in hyperbolic coordinates.

**Flat plane** \((l^2 = 0)\): The analysis of geodesics is easiest if we rewrite the metric in Cartesian coordinates

\[ ds^2 = - (dt + \Omega (y dx - x dy))^2 + dx^2 + dy^2 + dx_3^2. \quad (2.13) \]

Let’s denote by \(\Pi_t\) and \(\Pi_3\) the conserved quantities associated to the \(\partial_t\) and \(\partial_3\) isometries. The geodesics satisfy the differential equations

\[ \ddot{x} = -2\Omega \Pi_t \dot{y}, \quad \ddot{y} = 2\Omega \Pi_t \dot{x}, \quad (2.14) \]

with general solutions

\[ x(\lambda) = A \cos (2\Omega \Pi_t \lambda) - B \sin (2\Omega \Pi_t \lambda) + x_0, \quad (2.15) \]
\[ y(\lambda) = A \sin (2\Omega \Pi_t \lambda) + B \cos (2\Omega \Pi_t \lambda) + y_0. \quad (2.16) \]

Therefore, the projection of all geodesics on the \((x, y)\) plane are circles with arbitrary origin \((x_0, y_0)\), and radius

\[ r^2 = A^2 + B^2 = \frac{\Pi_t^2 - m^2 - \Pi_3^2}{4\Omega^2 \Pi_t^2}. \quad (2.17) \]

All null \((m^2 = 0)\) or timelike \((m^2 > 0)\) geodesics project on circles in the \((x, y)\) plane with radii smaller than 1/2\(\Omega\). For \(\Pi_3 = 0\), all null geodesics have \(r_g = 1/2\Omega\), independently of \(\Pi_t\), and
all spacelike \((m^2 < 0)\) geodesics project to circles with \(r > r_g\). Furthermore, when \(r > r_c\) the projections of these spacelike geodesics are closed timelike curves.

After presenting the geodesics in Cartesian coordinates, it is easy to make contact with the discussion in polar coordinates of [32, 31]. As a consequence of homogeneity, it is obvious that all geodesics with the same values of \(\Pi_t, m^2, \Pi_3\) are just copies of a single one, only shifted in the \((r, \phi)\) plane. For instance, the geodesics centered at the origin have angular momentum \(L = \frac{\Pi_t^2 - m^2 - \Pi_3^2}{4\Omega\Pi_t}\), and as we shift their center away from the origin, their angular momentum increases, becoming \(L = 0\) when they go through the origin. Furthermore, one can prove that as we shift the geodesics from being centered at the origin, they touch the chronologically safe region for \(L = \Pi_t \left(1 - \sqrt{1 - m^2/\Pi_t^2}\right) / \Omega\). This is the range of classical geodesics contained inside the chronologically safe region; note in particular that for null geodesics \((m^2 = 0)\), the geodesic with \(L = 0\), goes both through the origin and the boundary of the chronologically safe region (see figure 1).

It is worth noting that the \(l^2 = 0\) metrics appear in supergravity solutions where other background fields are turned on [1, 4]. This opens up the possibility of having particles charged under those fields, whose motion is determined by the combined effects of the metric and their coupling to the rest of the background. In this broader context, our discussion applies to neutral particles only, since we concentrate in solutions of General Relativity, with no further fields turned on. On the other hand, the geodesics of charged particles in supergravity backgrounds have been discussed in [3], and quite remarkably, it is possible to find causal charged geodesics with \(L = 0\) that leave the chronologically safe region.

**Hyperbolic plane \((l^2 > 0)\):** The analysis of the geodesics is easiest in hyperbolic coordinates. Taking advantage of the isometries of the metric, we derive that the geodesics satisfy

\[
\begin{align*}
\dot{t} &= \frac{\Omega^2 - l^2}{l^2} \Pi_t - 2\Omega Y \Pi_x, \\
\dot{X} &= -2\Omega Y \Pi_t + 4l^2 Y^2 \Pi_x, \\
\dot{Y} &= 4l^2 Y \Pi_x (X - X_0), \\
\dot{X}_3 &= \Pi_3,
\end{align*}
\]

where \(\Pi_t, \Pi_x, \Pi_3\) are conserved quantities associated to the respective isometries and \(X_0\) is an arbitrary constant. From this and the “on-shell” condition it follows that the projections of the geodesics on the plane are circles

\[
(X - X_0)^2 + \left(Y - \frac{\Omega \Pi_t}{2l^2 \Pi_x}\right)^2 = \frac{\Pi_t^2 - m^2 - \Pi_3^2}{4l^2 \Pi_x^2},
\]

Let us concentrate on \(m = \Pi_3 = 0\). Compared to the \(l^2 = 0\) case, there is a new feature. Since in hyperbolic coordinates \(Y \geq 0\), the geodesic projects into a full circle only if the radius is smaller than \(Y_0 = \frac{\Omega \Pi_t}{2l^2 \Pi_x}\). For null geodesics this condition implies \(l < \Omega\), which precisely coincides with the range where there are closed timelike curves. So, if \(l \leq \Omega\), all null or timelike geodesic project to
full circles, while only some of the spacelike geodesics do; among these, the ones with \( r > r_c \) project to the closed timelike curves. On the other hand, if we consider \( l > \Omega \), some timelike geodesics will still project to circles, but no null or spacelike geodesics will, and there are no closed timelike curves.

In the Landau problem on \( \mathbb{H}_2 \), this can be phrased as saying that the magnetic field has to be strong enough, in order to force the electrons to follow closed orbits.

Going back to polar coordinates, one can show that the radius of the projection of a null geodesic satisfies \( \tanh 2r_g = l/\Omega \), and again we have the relation \( r_g = r_c/2 \). This relation is easy to understand geometrically. The radius \( r_g \) is that of null geodesics centered at a given point. On the other hand, the chronologically safe cavity is constructed by considering the family of null geodesics going through that same point. By homogeneity, all these null geodesics, project to circles or radii \( r_g \), so as illustrated in figure 1, they form a cavity or radius \( r_c = 2r_g \).

3. The Klein-Gordon equation

After analyzing the classical geodesics in these backgrounds, and comparing them with the classical orbits for the Landau problem, we turn in this section to the quantum version of the relation between Gödel type metrics and the Landau problem, by studying the solutions to the Klein Gordon equation in these backgrounds, first considered in [26]. For the \( l^2 = 0 \) case, the solutions to the massless Klein-Gordon equation were also discussed in [9]. For the different signs of \( l^2 \), we will see many similarities, but also important differences, with the respective Landau problems. The reason for the difference is that there is a coupling between energy and angular momentum for probes in these backgrounds, so in the Landau analogy, the energy plays the role of effective charge (this was already observed in [22]).

**Flat plane (\( l^2 = 0 \)):** The Laplacian on this space is given in polar coordinates by

\[
\Delta = \frac{1}{r} \partial_r (r \partial_r) + \left( \frac{1}{r} \partial_\phi - \Omega r \partial_t \right)^2 - \partial_t^2 + \partial_3^2. \tag{3.1}
\]

Taking the ansatz

\[
\Psi = \Phi(r) e^{i\omega t + im\phi + ik_3 x_3}, \quad m \in \mathbb{Z} \quad w > 0, \tag{3.2}
\]

results in the following radial equation for a field of mass \( M \)

\[
\frac{1}{r} \partial_r (r \partial_r) \Phi - \left( \frac{m^2}{r^2} + \Omega^2 \omega^2 r^2 \right) \Phi + \left( \omega^2 + 2m\Omega \omega - M^2 - k_3^2 \right) \Phi = 0. \tag{3.3}
\]

The \( r \) dependent terms are (twice) the Schrödinger operator of a two dimensional harmonic oscillator with angular momentum \( m \) and frequency \( \hat{\omega} = \Omega \omega \). One can thus use the information
regarding the solution to the Schrödinger equation for a two dimensional harmonic oscillator in polar coordinates. The wave functions can be written as

$$\Phi_{n_r,m}(r) = C_{n_r,m} r^{|m|} e^{-\Omega \omega r^2/2} F_1\left(-n_r, |m| + 1; \Omega \omega r^2\right). \quad (3.4)$$

If we require regularity of the wave functions at the origin and normalizability at infinity, it forces the first two arguments of the confluent series to be integers; in this case the confluent series truncates to a polynomial, which turns out to be proportional to the associated Laguerre polynomial. The boundary conditions would be modified if we take the domain wall [8] into account, but we still expect this to be a good approximation. All in all, our wave functions can be written as follows:

$$\Psi_{n_r,m}(t, \phi, r) = C_{n_r,m} r^{|m|} e^{-\Omega \omega r^2/2} L^{|m|}_n(\Omega \omega r^2) e^{im\phi} e^{i\omega t} e^{ik_3 x_3}. \quad (3.5)$$

The energy is quantized, and it depends on the angular momentum \(m\) and the radial quantum number \(n_r\)

$$E = \hat{\omega} (|m| + 2n_r + 1), \quad n_r = 0, 1, \ldots \quad (3.6)$$

The degeneracy at level \(N \equiv |m| + 2n_r\) is \(N + 1\).

The wavefunctions of (3.5) are the same as those of the single particle states of the QHE on the plane. To make contact with the Landau problem, we define \(n = (N - m)/2\), which equals \(n_r\) for \(m > 0\) and \(N - n_r\) for \(m < 0\). This quantum number \(n\) is the ordinary Landau level, and at each level \(n\) we have infinite degeneracy, with \(m = -n, \ldots, \infty\). Note however that the previous wavefunctions have a level dependent rescaling compared to the ordinary wavefunctions of the QHE, since they depend on \(\Omega \omega r^2\), rather than just \(r^2\). We comment below on the consequences of this rescaling.

The frequency \(\omega\) of a given level \(n\) is determined by equating the constant term in the Laplacian with twice the energy eigenvalues of the 2d harmonic oscillator

$$\omega^2 + 2m\Omega \omega - M^2 - k_3^2 = 2\Omega \omega (N + 1), \quad (3.7)$$

which determines the set of allowed frequencies to be

$$\omega = (2n + 1) \Omega \pm \sqrt{(2n + 1)^2 \Omega^2 + M^2 + k_3^2}, \quad n = 0, 1, \ldots \quad (3.8)$$

In what follows, we restrict to solutions with the positive sign of the square root, and focus on the sector \(M^2 = k_3^2 = 0\). The allowed frequencies are

$$\omega = 2\Omega (2n + 1), \quad n = 0, 1, \ldots \quad (3.9)$$

and the levels are equally spaced.

Let us switch to complex coordinates \(z = re^{i\phi}\). The metric and Laplacian read

$$ds^2 = -(dt + \frac{\Omega}{2}(z \bar{z} - \bar{z} z))^2 + dz d\bar{z}, \quad (3.10)$$

$$\Delta = (-1 + \Omega^2 z \bar{z}) \partial_t^2 + 2i \Omega (z \bar{\partial} - z \partial) \partial_t + 4\partial \bar{\partial}. \quad (3.11)$$
The wavefunctions of the lowest level \((n = 0)\) are given by

\[
\Phi_{n=0,m} = z^m e^{-\Omega^2 z \bar{z}}, \tag{3.12}
\]

and except for the first ones, they are peaked outside the chronologically safe region. We will also be interested in the states with the minimal value of \(m\) at each level \(n\), i.e. the lowest weight states with \(m = -n\). Their eigenfunctions are

\[
\Phi_{n,m=-n} = z^n e^{-\Omega^2 (2n+1)z \bar{z}}. \tag{3.13}
\]

These are strongly peaked at

\[
r = \sqrt{\frac{2n - 1}{2n + 1}} \frac{1}{\Omega}, \tag{3.14}
\]

so that these radii approach, for large \(n\), the classical radius of null geodesics, \(r_g = 1/2\Omega\). This is to be contrasted with the ordinary Landau problem in the plane, where the lowest weight wavefunctions are strongly peaked at a radius that is unbounded as the level grows. The difference is caused by the fact that the energy of the state plays the role of an effective charge. As a result of the rescaling, all the lowest weight states fit inside the chronologically safe region.

The wave function of the harmonic oscillator is a polynomial of degree \(N\) times the exponential \(\exp -\Omega \omega r^2/2\). For a massless field this has a maximum at

\[
r = \sqrt{\frac{N - 1}{2n + 1}} \frac{1}{\Omega}. \tag{3.15}
\]

Beyond that radius, the wave function decays exponentially. Therefore the wavefunction is localized within the chronologically safe region as long as \(N \lesssim 2n + 1\), or \(m \lesssim 0\). We saw the same phenomenon at the classical level, where all the null geodesics with angular momentum \(L \leq 0\) were fully within the radius \(r_c\).

We conclude that wavefunctions with \(m \leq 0\), are exponentially suppressed outside the chronologically safe region. On the other hand, wavefunctions with \(m > 0\) may have a large support outside the radius of the chronologically safe region, \(r_c\).

The \(l^2 = 0\) solutions can be embedded in string theory, and the spectrum computed at weak coupling. This metric appears in [4] as a KK reduction of a 5d metric with a compact direction; this model is exact to all orders in \(\alpha'\). The spectrum was derived in the bosonic case in [22] and in the type II and heterotic cases in [23]. An important feature is that in the world-sheet, the vorticity \(\Omega\) only couples to, say, the right-movers, so only the right-moving zero modes appear in the Hamiltonian. The left-moving zero modes don’t appear neither in the Hamiltonian nor the level matching conditions, so at every stringy level, there is infinite discrete degeneracy, just as in the ordinary Landau problem in the plane.
**Sphere** \((l^2 < 0)\): With the standard change of coordinates \(z = 2R \tan \frac{\theta}{2} e^{i\phi}\), and defining \(K = (1 + z\bar{z}/4R^2)^{-1}\), the metric and Laplacian read now

\[
ds^2 = - \left( dt + i \frac{\Omega}{2} K(zd\bar{z} - \bar{z}dz) \right)^2 + K^2 d\bar{z}d\bar{z}, \tag{3.16}
\]

\[
\Delta = (\Omega^2 z\bar{z} - 1) \partial_t^2 + \frac{2i\Omega}{K} (\bar{z}\bar{\partial} - z\partial_t) + \frac{4}{K^2} \partial^2. \tag{3.17}
\]

The allowed frequencies are

\[
\omega = (2n + 1)\Omega + \sqrt{(2n + 1)^2\Omega^2 + \frac{n(n + 1)}{R^2} + M^2 + k_3^2}, \quad n = 0, 1, \ldots \tag{3.18}
\]

and for each frequency (Landau level), we have finite degeneracy \(-n \leq m \leq 4\Omega w(n)R^2\). At each level, the wavefunctions with the minimal value of \(m\), i.e. the lowest weight states, are particularly easy to find, and from them one can build in principle all the wavefunctions (see [26] for expressions). They are

\[
\Phi_{n,m=-n} = \frac{\bar{z}^n}{(1 + \frac{z\bar{z}}{4R^2})^{2\omega(n)\Omega R^2 + n}}. \tag{3.19}
\]

For instance, if \(M^2 = k_3^2 = 0\), the lowest Landau level has \(\omega = 2\Omega\) and wavefunctions

\[
\Phi_{n=0,m} = \frac{z^m}{(1 + \frac{z\bar{z}}{4R^2})^{4\Omega^2 R^2}}, \quad m = 0, \ldots, 8\Omega^2 R^2. \tag{3.20}
\]

The wavefunctions for lowest weight states are strongly peaked at

\[
\tan^2 \frac{\theta}{2} = \frac{n/4R^2}{\omega(n)\Omega + n/4R^2}, \tag{3.21}
\]

which again in the large \(n\) limit reproduce the classical value, \(\cot \theta = 2\Omega R\).

This spectrum shares some common features with the Landau problem on the sphere (reviewed in the appendix): It has an infinite number of levels (labeled by \(n\)), and finite degeneracy for each of them. As in the \(l^2 = 0\) case, a first difference is that at higher levels, there is a rescaling of the wavefunctions, such that all the lowest weight states peak over the same classical radius. On top of that, there is another important difference with the spectrum of the Landau problem on the sphere. In that case, the curvature correction is quadratic in the level (compared to the ordinary linear term), while here, for large \(n\), the frequencies are linear in \(n\).

**Hyperbolic plane** \((l^2 > 0)\): Finally, in this case we define \(z = \frac{\tanh \frac{l}{2} e^{i\phi}}{1 - l^2 z\bar{z}}\), and the metric is identical to the \(l^2 < 0\) case, but now with

\[
K = \frac{1}{1 - l^2 z\bar{z}}. \tag{3.22}
\]

The frequencies are

\[
\omega = (2n + 1)\Omega + \sqrt{(2n + 1)^2\Omega^2 - 4l^2 n(n + 1) + M^2 + k_3^2}. \tag{3.23}
\]
Again, in the regime $l^2 \leq \Omega^2$, the frequencies are linear in $n$, for large $n$. The wavefunctions for the lowest weight states and the lowest level are very similar to those of the $l^2 < 0$ case. For $M^2 = k_3^2 = 0$ they are

\begin{align}
\Phi_{n,m=-n} &= \frac{\bar{z}^n}{(1-l^2\bar{z}z)^\frac{\Omega^2}{2\pi^2}n}, \\
\Phi_{n=0,m} &= \frac{z^m}{(1-l^2\bar{z}z)^\frac{\Omega^2}{l^2}},
\end{align}

As it happened in the discussion of the classical geodesics, here we have to distinguish again between the $l < \Omega$ and $l \geq \Omega$ regimes. For $l^2 < \Omega^2$—the range where the metrics have CTCs—there is no upper bound on $n$, and we have again an infinite number of levels. On the other hand, if we allow for $l^2 > \Omega^2$, there is an upper bound on the level $n$, given by

\begin{equation}
2n + 1 \leq \frac{\Omega}{l} \frac{\sqrt{M^2 + k_3^2}}{\sqrt{l^2 - \Omega^2}},
\end{equation}

and furthermore, above that level, states with a continuous energy spectrum appear. This matches the behavior of classical geodesics discussed in section 2: for $l \leq \Omega$, all timelike or null geodesics project to closed orbits, and that is reflected at the quantum level by a discrete energy spectrum. On the other hand, for $l > \Omega$ some timelike geodesics and all null geodesics project to unbounded orbits in the hyperbolic plane. At the quantum level, these correspond to the states in the continuum of energy. This is completely analogous to the Landau problem on the hyperbolic plane [33] (see also the appendix), where there is a finite number of discrete levels, and above it, a continuum of energy states.

Note that at the point where the continuum appears, $l^2 = \Omega^2$, the metric (1.1) becomes $AdS_3 \times \mathbb{R}$. For string theory on $AdS_3$ the continuous representations of $sl(2,\mathbb{R})$ correspond to long strings [35,36], as explained in [37]. However, the long strings can be understood as coming from the spectral flow of spacelike geodesics [37], while here the continuous representations correspond to massive particles (and therefore timelike geodesics). It would be nice to understand better the relation, if any, between these two appearances of $SL(2,\mathbb{R})$ continuous representations.

## 4. Holographic screens.

As mentioned in the introduction, we don’t expect the full Gödel type solutions to be a valid background in string theory. On the other hand, in previous work [8], we presented a 10d solution that patches a single chronologically safe region to an outside metric, separated by a domain wall made of supertubes. This solution is free of closed timelike curves, and for an observer at the origin, Bousso’s prescription associates a holographic screen with the same radius as in the original solution. This solution with a domain wall allows us to address the existence of a holographic description for a region of the universe that is locally Gödel type, freeing us from all the conceptual troubles associated with closed timelike curves. This holographic screen is presumed to encode the physics localized inside the domain wall, although the precise meaning of “localized” in this
context is not completely clear, since the region on the inside of the domain wall is not causally disconnected from the rest.

In the Landau problem analogy, restricting to a single chronologically safe region, bears some resemblance to considering a Hall droplet of finite size \[38\], instead of the Quantum Hall effect in the full plane. For instance, in the matrix quantum mechanics description of the QHE for a finite size droplet, due to Polychronakos \[39\], one introduces a harmonic potential to confine the electrons into a finite domain, breaking the translation invariance of the system. In the Gödel type solutions, the domain wall has a similar effect.

Bousso’s prescription for constructing holographic screens is completely classical. As such, there is no guarantee that given a classical solution of general relativity, there is a quantum gravity version of that background and much less a holographic description. In what follows, we assume that such a holographic description exists, for a single chronologically safe region of the family of 4d metrics we have been studying, and present some heuristic ideas on what this holographic theory might be. Eventually, it is plausible that these ideas will only make sense when these metrics (or rather, metrics with domain walls surrounding a single chronologically safe region) are embedded in solutions of string/M theory.

First we address the question of which bulk states we expect inside the domain wall. Properly, we should impose boundary conditions consistent with the presence of a domain wall, but we believe that for states inside the cavity, the analysis of the previous section is a good approximation. A natural prescription is to restrict to states whose wavefunctions start decaying exponentially inside the chronologically safe region, or just at the boundary. For instance, in the previous section we saw that for massless states, those wavefunctions have arbitrary \(n\) and \(m \leq 0\) (they correspond to null geodesics with \(L \leq 0\), which are fully inside the chronologically safe region).

Although the presence of the domain wall naturally suggests a way to restrict the bulk states, a pure QFT analysis misses the effects of gravity. Once we consider the backreaction of these states to the metric, we expect a cut-off in the allowed frequencies: the Klein Gordon wavefunctions are extended in \(x_3\), and localized in the plane of constant \(t, x_3\), so once the backreaction is taken into account, the new solution will be similar to a cosmic string. In particular, it will introduce a deficit angle, and we should require that this deficit angle is smaller than \(2\pi\), i.e. we require that \(w\) is smaller than the mass of a particle that would close the universe in 2+1 dimensions, \(w \leq 1/\ell_P\). This translates into \(n_{\text{max}} \sim r_c/\ell_P\).

Having discussed the spectrum of modes we expect inside the domain wall, we address next the boundary degrees of freedom. First, let’s recall where the holographic screen is, according to Bousso’s prescription. If we restrict to the non-trivial 3d part of the metrics, the position of the holographic screen was computed in \[3\], and it is best described in polar coordinates. By the symmetries of the problem, it was argued in \[4\] that the radius of the screen is determined by \(dg_{\phi\phi}(r)/dr = 0\), or

\[
\sinh lr_h = \frac{1}{\sqrt{2} \sqrt{\frac{\ell^2}{\ell^2} - 1}}.
\]

(4.1)
This is always smaller than the radius of the cavity $r_c$ and bigger than the radius of the null geodesics, $r_g < r_h < r_c$. In fact,

$$\sinh lr_h = \frac{1}{\sqrt{2}} \sinh lr_c.$$  \hspace{1cm} (4.2)

For the full Gödel type solutions, the holographic screen breaks the isometry of the full solution, as happens in other solutions with observer dependent holographic screens (e.g., the static observer approach to holography for de Sitter space \cite{40}). In this case, we have $\mathbb{R} \times U(1)$ associated to time translations times rotations on the plane (plus an additional $\mathbb{R}$ if we consider translations in the $x_3$ direction). On the other hand, if we start with a solution with a domain wall, the holographic screen breaks no further symmetries.

Next, we are going to relate the number of boundary degrees of freedom with the Landau level structure of the bulk spectrum. In general, the number of degrees of freedom in the holographic screen scales with the area $A$ of the screen, $N \sim A/\ell_P^2$, and since the screen extends indefinitely in the $x_3$ direction, the area—and the number of degrees of freedom—is infinite. We instead consider a screen with a cut-off length $h$ in its height, and concern ourselves on how the different quantities scale with $r_c$, for fixed $h$. The area goes like $A \sim r_ch$ and $N \sim r_ch/\ell_P^2$, so the expected number of degrees of freedom (for fixed $h$) grows linearly with $r_c$.

Now we use this relation $N \sim r_c$ and $n_{\text{max}} \sim r_c/\ell_P$ to conclude that $N_{\text{dof}} \sim n_{\text{max}}$. Namely, we roughly get a boundary degree of freedom per level per field$^2$. This can be realized at least in a couple of ways: The independent degrees of freedom map to the states with $n = n_{\text{max}}$, or alternatively, for each level there is an independent boundary degree of freedom.

We would like to conclude by presenting a simple model of a holographic description that captures this scaling, although clearly leaves many things out. Simply put, we want to substitute the classical holographic screen by a fuzzy version of it, a non-commutative cylinder in the case at hand, with the scale of non-commutativity given by the Planck area, $\ell_P^2$. Our chief motivation is that in general, when the classical screens given by Bousso’s prescription have finite proper area, the covariant version of the holographic principle assigns them a finite number of degrees of freedom at the quantum level, and a natural way to assign a finite number of degrees of freedom to a surface is by considering a matrix regularization. Similar ideas have been proposed in \cite{41,42}, in the context of the static observer approach to holography in de Sitter space \cite{40}, by replacing the classical spherical horizon with a fuzzy sphere.

In the context of the matrix model for M-theory, non-commutative cylinders were constructed in \cite{43} providing a matrix realization of supertubes. Recall that in the 10d solution of \cite{8}, the domain wall separating the Gödel type region from the exterior metric is made out of supertubes; however, the matrix theory of \cite{43} doesn’t quite describe this domain wall, since the latter was constructed with smeared supertubes. It would be interesting to construct the matrix model of the IIA solution of \cite{8}, with a $l^2 = 0$ Gödel type region inside the supertube domain wall; alternatively,

$^2$Although in the previous section we only discussed the Klein Gordon equation for a scalar field, in string theory realization of the $l^2 = 0$ case, all fields (graviton, NS form) also display a Landau level type spectrum \cite{22}.
one could consider the matrix model formulation for the full $l^2 = 0$ solution, and try to extract from it the physics of a single chronologically safe region.

The non-commutative cylinder is a solution of the matrix regularized version of membrane theory, with three time independent non-zero matrices satisfying

\[
[X_1, X_2] = 0, \quad [X_1, X_3] = -iaX_2, \quad [X_2, X_3] = aX_1,
\]

where $a$ is a length scale. Moreover, $X_1^2 + X_2^2$ is a Casimir of this algebra, and we consider representations with a fixed value $X_1^2 + X_2^2 = R^2$. These matrices correspond to a non-commutative cylinder of radius $R$ and non-commutativity scale $a$ in the $x_3$ direction. The quantum of area of this non-commutative cylinder is $2\pi a R$, and we set $a R \sim \ell_P^2$. Note that the two sides of this quantum of area are very different, if $R \gg \ell_P$.

As already noticed in [43], the scalar fluctuations around this matrix configuration are also similar to the Landau problem on the plane. This is intuitively clear, since a supertube is supported against collapse by an electric field along the tube, and a magnetic field transverse to it. The equation that describes fluctuations of a massless scalar is [43]

\[
\left( \partial_t - \frac{a}{\ell_P^2} \partial_\theta \right)^2 \Phi - \frac{a^2}{\ell_P^4} \partial_\theta^2 \Phi - \frac{a^2 R^2}{\ell_P^4} \partial_3^2 \Phi = 0. \tag{4.4}
\]

The ansatz $\Phi = e^{iwt} e^{im\theta} e^{ik_3 x_3}$ yields the frequencies

\[
w = m a/l_P^2 + \sqrt{(m a/l_P^2)^2 + a^2 R^2 k_3^2}, \tag{4.5}
\]

which reproduce the spectrum of frequencies for $l^2 = 0$, [38], if we identify

\[
a/l_P^2 \leftrightarrow \Omega. \tag{4.6}
\]

The difference is that now the spectrum has no degeneracy for a given frequency, which agrees with the scaling we deduced previously, roughly one boundary degree of freedom per Landau level. Note however, that at this level of discussion, we don’t see the existence of a cut-off, $n_{max}$.

Using $a R \sim \ell_P^2$, the previous identification leads to $R \sim 1/\Omega$. These preliminary considerations are clearly not enough to reproduce Bousso’s prescription for the position of the holographic screen in the $l^2 = 0$ case, $r_h = 1/\sqrt{2}\Omega$.

To sum up, in the full homogeneous Gödel type solutions, the Klein Gordon equation gives a degeneracy of solutions at each frequency, given by the quantization of momentum, characteristic of the Landau problem. The states of a given frequency form full representations of $SU(2)$, $H_2$ or $SL(2, \mathbb{R})$. Once we restrict to a single chronologically safe region, it is natural to consider a truncation of the spectrum at each level, keeping states that are localized inside the region. These states clearly no longer form full representations of the corresponding groups. So far, everything we said refers to quantum field theory in these backgrounds.

Finally, on a more speculative note, we tried to relate the boundary degrees of freedom of a possible dual description with the states that are localized inside a single chronologically safe region.
We seem to be a long way from understanding holography when the holographic screens are not at asymptotia, assuming such a thing makes sense. Making more precise the speculations presented in this last section will hopefully provide some of the much needed insight.

5. Discussion

We have pointed out the close relation between a family of 3+1 Gödel type solutions and the Landau problem of a charged particle moving on a surface, in the presence of a constant magnetic field, a relation discussed in a particular case in [9]. This analogy has allowed us to suggest a heuristic picture of a potential holographic description for a single chronologically safe region.

Our discussion generalizes immediately to other solutions. For instance, the 5d Gödel solution found in [1] would be related to a charged particle moving in $\mathbb{R}^4$ with magnetic fields transverse to two planes. This system was considered recently in [15]. Going in the opposite direction, a system that has received a great amount of interest is a 4+1d generalization of the QHE [4]. In light of our discussion, we can ask whether there is any (higher dimensional) GR solution related to this model. Since the 4+1d QHE is based on the second Hopf map (in the same way the ordinary QHE is based on the first Hopf map), natural candidates are metrics of $SU(2)$ bundles over $S^4$, but to obtain a Lorentzian signature, it seems more natural to consider the reformulation of this higher dimensional QHE as a $U(1)$ fibration over $\mathbb{CP}^3$ [17].

Our suggestions about a holographic description of these backgrounds are far from conclusive. Another approach to holography for general backgrounds, has been suggested by Banks [48], who put forward the idea that there is an approximation to quantum gravity, dubbed asymptotic darkness, where the possible black holes in that background are stable, and the microstates of all possible black holes in that background provide a basis of the Hilbert space of the quantum theory. In the case of the supersymmetric 5d Gödel solution [1], there has been some work on the possible black holes [19]. On the other hand, one might hope to identify the string states that become black holes, once the interactions are taken into account. This would require making precise the bound on frequencies mentioned in section 4. The 5d Gödel solution (or more properly, a single chronologically safe cavity of it) might be a candidate to make more concrete the idea of asymptotic darkness.

An important point is that in the present work we considered only solutions to General Relativity. In Bousso’s prescription, the position of the holographic screen is determined by the metric alone, or in other words, by the geodesics of neutral particles. Once we consider solutions in extended theories, like supergravity, we face the possibility of having solutions with the same metric but differing in the rest of the fields turned on, and Bousso’s prescription is insensitive to that difference. Since the holographic principle refers only to the number of degrees of freedom, but not their nature, we see not immediate contradiction: the putative holographic descriptions of these backgrounds could be two different theories, with the same number of degrees of freedom. On the other hand, as the results of [1] nicely illustrate, the geodesics of charged objects can have

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3We would like to thank D. Karabali and V.P. Nair for pointing out this possibility.
dramatically new behavior compared to the neutral ones, and it seems important to understand their relevance for holography in spacetimes with closed timelike curves, and the holographic principle in general.

In the present work, we suggested that if there is a holographic description of a single chronologically safe region for Gödel type universes, it will involve the quantum mechanics of the matrix regularization of the classical holographic screen, i.e. a non-commutative cylinder. The obvious generalization of this idea is that whenever Bousso’s prescription yields a classical holographic screen with finite proper area, the quantum holographic description involves a matrix regularization of the classical screen.

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A. The Landau problem in surfaces of constant curvature.

In this appendix we review the Landau problem of a charged particle moving under the presence of a constant magnetic field, perpendicular to a surface of constant curvature, following [33, 50]. We consider the sphere, the plane, and the hyperbolic plane. The spectrum is related to representations of $SU(2)$, $H_2$ and $SL(2, \mathbb{R})$ respectively.

In each case, we will have a set of discrete energy levels. For the sphere, the degeneracy of each level is finite, while for the plane and the hyperbolic plane, each level has infinite degeneracy. On the other hand, the number of discrete levels is infinite for the sphere and the plane, but finite for the hyperbolic plane. Finally, in the hyperbolic plane, there is a continuum of states, above the finite range of discrete levels.

We have a surface with metric

$$ds^2 = g_{zz}dzd\bar{z}. \quad (A.1)$$

The volume form is

$$dv = \frac{i g_{zz}}{2} dz \wedge d\bar{z}. \quad (A.2)$$
The natural definition of constant magnetic field is $F = Bdv$, and the Hamiltonian for a charged particle moving in this surface is

$$H = \frac{1}{\sqrt{g}} (P - A) g^{zz} \sqrt{g} (P - A). \quad (A.3)$$

We start considering the 2d surface to be $S^2$. In this case, $N \equiv 2BR^2$ has to be an integer, $g_{zz} = 1/(1 + z\bar{z}/4R^2)^2$, and with the gauge choice

$$A_z = -\frac{iB}{4} \frac{z}{1 + z\bar{z}/4R^2}, \quad (A.4)$$

the Hamiltonian reads

$$H = \frac{1}{2R^2} \left( -1 + \frac{z\bar{z}}{4R^2} \partial\bar{\partial} + \frac{B}{4} \left( 1 + \frac{z\bar{z}}{4R^2} \right) (z\bar{\partial} - z\partial) + \frac{B^2}{16} z\bar{z} \right). \quad (A.5)$$

We introduce

$$L_+ = -\frac{1}{2R} z^2 \partial - 2R \bar{\partial} + \frac{BR}{2} z, \quad L_- = \frac{1}{2R} \bar{z}^2 \bar{\partial} + 2R \partial + \frac{BR}{2} \bar{z}, \quad L_3 = z\partial - \bar{z}\bar{\partial} - BR^2, \quad (A.6)$$

which form an $SU(2)$ algebra. In terms of these operators, the Hamiltonian can be rewritten as

$$H = \frac{1}{2R^2} \left( \frac{1}{2} (L_+ L_- + L_- L_+) + L_3^2 - B^2 R^4 \right) = \frac{C_2 - B^2 R^4}{2R^2}, \quad (A.7)$$

where $C_2$ is the second Casimir of $SU(2)$. This allows for a purely algebraic deduction of the spectrum. The energy spectrum is

$$E_n = B \left( n + \frac{1}{2} \right) + \frac{n(n+1)}{2R^2}. \quad (A.8)$$

There is the familiar linear term, plus a quadratic correction due to the curvature. The unnormalized lowest Landau level wavefunctions are

$$\Phi_{n=0,m} = \frac{z^m}{\left( 1 + \frac{z\bar{z}}{4R^2} \right)^{N/2}}, \quad (A.9)$$

i.e. the wavefunctions are the wavefunction of the vacuum times a holomorphic polynomial. Each Landau level is finite dimensional, the degeneracy of the $n$-th level being $N + 2n + 1$, the states are in $SU(2)$ representations. The generic wavefunctions can be obtained by applying $L_+$ on the lowest weight states of each level, defined by $L_- \Phi = 0$. The lowest weight wavefunctions are

$$\Phi_{n,m=-n} = \frac{z^n}{\left( 1 + \frac{z\bar{z}}{4R^2} \right)^{N+n}}. \quad (A.10)$$
Having discussed the Landau problem on the sphere, one can recover the Landau problem on the plane as a limit $N, R \to \infty$, keeping $B = N/2R^2$ fixed. By an appropriate rescaling of the operators, the $SU(2)$ algebra contracts to a Heisenberg algebra. The curvature correction term drops from the energy spectrum, and we are left with the familiar

$$E_n = B \left( n + \frac{1}{2} \right).$$

(A.11)

Now each Landau level has infinite degeneracy (since we sent $N \to \infty$), and the lowest Landau level and lowest weight wavefunctions are just the limit of the ones on the sphere

$$\Phi_{n=0,m} = z^m e^{-\frac{Bz}{4}},$$

(A.12)

$$\Phi_{n,m=-n} = \bar{z}^m e^{-\frac{B\bar{z}}{4}}.$$  

(A.13)

Finally, we can consider the Landau problem in the hyperbolic plane. Now $g_{z\bar{z}} = 1/(1 - z\bar{z}/4R^2)^2$, and most of the previous discussion goes through. There is in principle no quantization on $B$, the $SU(2)$ is replaced by an $SL(2, \mathbb{R})$ algebra and the discrete spectrum is

$$E_n = B \left( n + \frac{1}{2} \right) - \frac{n(n+1)}{2R^2}, \quad 0 \leq n < \left\lfloor BR^2 \right\rfloor - 1.$$  

(A.14)

There is only a finite number of discrete Landau levels, each with infinite degeneracy, with states forming discrete $SL(2, \mathbb{R})$ representations. Beyond that range of energies, there is a continuous part to the spectrum \cite{33}, with states in the continuous principal series $\mathcal{C}_j$

$$j = -\frac{1}{2} + i\nu, \quad 0 \leq \nu \leq \infty,$$

(A.15)

$$E(\nu) = \frac{1}{2R^2} \left( \frac{1}{4} + BR^2 + \nu^2 \right).$$

(A.16)

Note that $E_n < (1/4 + B^2R^4)/2R^2$, while $E(\nu) \geq (1/4 + B^2R^4)/2R^2$, so the discrete and continuous spectra don’t overlap.

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