Quantization and Renormalization and the Casimir Energy of a Scalar Field Interacting with a Rotating Ring

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Effects due to vacuum fluctuations in a semi-classical model of a massless scalar field interacting with a rotating ring are investigated by introducing a collective coordinate for the motion of the background field (potential) \( V(\varphi) \). The model is solved for a repulsive periodic \( \delta \)-distribution background of arbitrary strength. The Casimir energy of this system is calculated in the co-rotating and, by Legendre transformation, in the stationary laboratory frame. The zero-point contribution to the angular momentum in this model is bounded below by \( |\ell_{ZP}| \leq \hbar/24 \) and the ground state of the entire system thus generally is non-rotating with a positive moment of inertia that decreases only slightly with increasing angular rotation frequency. There is no transfer between the zero-point and classical contributions to the total angular momentum and energy of this system at zero temperature.

Recently Chernodub observed \[1, 2\] that zero-point fluctuations of a scalar field contribute negatively to the classical contributions to the total angular momentum and energy of this system at zero temperature. All Casimir systems could (and perhaps should) be interpreted in this manner \[8\]. The presence of a “classical” contribution \( I\theta^2/2 \) to the rotational energy in this case is due to the motion of the soliton on the circle (or that of a ring) and is not at all surprising. The original Chernodub model of a scalar on a circle subject to a rotating Dirichlet boundary condition \[1\] is the limit of this extended model for very large \( I \gg \hbar R/c \) and a “thin-wall” soliton \( V(\varphi) \) proportional to a periodic \( \delta \)-distribution at strong coupling \( \lambda \to \infty \). The main qualitative modification to Chernodub’s original model thus is the presence of a dynamical collective coordinate giving the location and dynamics of the wall.

Let us for example consider the relatively simple and instructive example model in which a scalar field on a circle of radius \( R \) interacts with an everywhere positive background field (potential) \( V(\varphi) \geq 0 \) whose position on the circle is referenced by the collective coordinate \( \theta(t) \).

The Lagrangian for this model is,

\[
L(\varphi, \dot{\varphi}, \theta, \dot{\theta}) = \frac{I}{2} \dot{\theta}^2 + \int_{S_1} R d\sigma \frac{1}{2} \left( \dot{\varphi}^2 - R^{-2} \dot{\varphi}'^2 - V(\varphi - \theta(t)) \right) \phi^2.
\]

(1)

where \( I \) is the moment of inertia for the collective coordinate \( \theta(t) \). Eq.\[1\] may be interpreted as describing quantum fluctuations to quadratic order of a scalar field \( \phi(\varphi, t) \) on a circle in the background of a classical soliton solution \( V(\varphi) \) located at \( \theta(t) \). This only omits to specify the originally highly nonlinear model \( V(\varphi) \) is the soliton of. All Casimir systems could (and perhaps should) be interpreted in this manner \[8\]. The presence of a “classical” contribution \( I\theta^2/2 \) to the rotational energy in this case is due to the motion of the soliton on the circle (or that of a ring) and is not at all surprising. The original Chernodub model of a scalar on a circle subject to a rotating Dirichlet boundary condition \[1\] is the limit of this extended model for very large \( I \gg \hbar R/c \) and a “thin-wall” soliton \( V(\varphi) \) proportional to a periodic \( \delta \)-distribution at strong coupling \( \lambda \to \infty \). The main qualitative modification to Chernodub’s original model thus is the presence of a dynamical collective coordinate giving the location and dynamics of the wall.

The extended Lagrangian of eq.\[1\] does not explicitly depend on time and conserves total angular momentum. We are interested in the lowest (vacuum) energy of this model for a given total angular momentum \( \ell \). For a quantum time crystal, the energy is minimal at \( \ell \neq 0 \).

Written in terms of the relative angle \( \sigma = \varphi - \theta(t) \), the Lagrangian of eq.\[1\] reads,

\[
L(\varphi, \dot{\varphi}, \theta, \dot{\theta}) = \frac{I}{2} \dot{\theta}^2 + \int_{S_1} R d\sigma \frac{1}{2} \left( (\dot{\varphi} - \dot{\theta}\phi')^2 - R^{-2} \phi'^2 - V(\varphi) \phi^2 \right),
\]

(2)

and \( \theta \) is a cyclical coordinate. The canonical momenta are,

\[
(3a)
\]

\[
(3b)
\]

where \( \ell \) defined by eq.\[3b\] is the conserved total angular momentum conjugate to \( \theta \). The Hamiltonian \( H_s \) in the stationary frame in these coordinates is obtained from...
eq. (2) as,
\[ H_s = -L + \ell \dot{\theta} + \oint_{S_1} R \sigma \pi(\sigma, t) \dot{\phi}(\sigma, t) \]
\[ = \frac{1}{2} \left( \ell + \oint_{S_1} \sigma R \pi \dot{\phi} \right)^2 + \oint_{S_1} R \sigma \pi \frac{1}{2} (\pi^2 + R^{-2} \phi''^2 + V(\sigma) \phi^2) . \]

The quartic term of \( H_s \) is linearized by a Legendre transformation to the energy \( H_c \) in the co-rotating frame. With the angular rotation frequency \( \Omega \neq \theta \) given as,
\[ \Omega = \frac{\partial H_s}{\partial \ell} = \frac{1}{\ell} \left( \ell + \oint_{S_1} \sigma R \pi \dot{\phi} \right) \]
we have that (see footnote on p.74 of [9]),
\[ H_c(\Omega) = H_s(\ell(\Omega)) - \ell(\Omega)\Omega \]
\[ = -\frac{1}{2} \Omega^2 \ell + \oint_{S_1} R \sigma \pi \frac{1}{2} (\pi^2 + \Omega \pi' + \phi' \pi) + R^{-2} \phi''^2 + V(\sigma) \phi^2) , \]
is quadratic in fields and momenta. Canonical quantization of this model proceeds by promoting fields and momenta to operators with equal time commutator,
\[ [\phi(\sigma, t), \pi(\sigma', t)] = \frac{i}{\hbar} \delta_{\text{per}}(\sigma - \sigma') , \]
where \( \delta_{\text{per}}(\sigma) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{i n \sigma} \) denotes the periodic \( \delta \)-distribution. With the ordering given in eq. (6), \( H_c \) is hermitian and Hamilton’s equations,
\[ \dot{\phi} = \frac{\delta H_c}{\delta \pi} = \pi + \Omega \phi' \]
\[ \dot{\pi} = -\frac{\delta H_c}{\delta \phi} = \Omega \pi' + R^{-2} \phi'' - V(\sigma) \phi , \]
coincide with Heisenberg’s equations of motion for the operators. Solving for \( \pi \) in eq. (8a) and inserting in eq. (8b) gives the separable equation of motion in the co-rotating frame,
\[ \ddot{\phi} - 2\Omega \dot{\phi} + \Omega^2 (\pi^2 - R^{-2}) \phi'' + V(\sigma) \phi = 0 , \]
whose general solution is,
\[ \phi(\sigma, t) = \sum_{\omega_m \geq 0} a_m e^{-i \omega_m t} u_m(\sigma) + a_m^\dagger e^{i \omega_m t} \bar{u}_m(\sigma) . \]

Upon quantization, the coefficients \( a_m \) and \( a_m^\dagger \) are interpreted as annihilation and creation operators of quanta in mode \( m \). Inserting the general solution of eq. (9) in eq. (8a), the momentum is expressed in terms of operators \( a \) and \( a^\dagger \) as,
\[ \pi(\sigma, t) = \dot{\phi} - \Omega \phi' = \sum_{\omega_m \geq 0} a_m e^{-i \omega_m t} (-i \omega_m u_m - \Omega u_m^\dagger) + a_m^\dagger e^{i \omega_m t} (i \omega_m \bar{u}_m - \Omega \bar{u}_m^\dagger) . \]

eq. (9) implies that the mode functions \( u_m(\sigma) \) satisfy the homogeneous ODE,
\[ (\Omega^2 - R^{-2}) u_m'' + 2i \omega_m \Omega u_m' + (V(\sigma) - \omega_m^2) u_m = 0 . \]

of Bloch waves [10]. The frequencies \( \omega_m \) at which eq. (12) has non-trivial periodic solutions are discrete. For a finite periodic potential \( \infty > V(\sigma) = V(\sigma + 2\pi) \geq 0 \), the frequencies \( \omega_m \) and corresponding solutions \( u_m(\sigma) \) of eq. (12) are determined by requiring that,
\[ \left( u_m(\sigma) \right) = \lim_{\epsilon \to 0^+} \left( u_m(\sigma + 2\pi - \epsilon) \right) \]

The frequency spectrum \( \{ \omega_m \} \) is real and the complex conjugate mode function \( \bar{u}_m(\sigma) \) is a solution of eq. (12) to frequency \(-\omega_m \). It thus suffices in eq. (10) to sum over non-negative frequencies only. Eq. (12) is consistent with the normalization conditions,
\[ R \oint_{S_1} d\sigma [ (\omega_m + \omega_n) u_m u_n - 2i \Omega u_m u_n^\dagger] = \delta_{mn} \]
\[ R \oint_{S_1} d\sigma [ (\omega_m - \omega_n) u_m u_n - 2i \Omega u_m u_n^\dagger] = 0 , \]
and their complex conjugates. Mode functions to different frequencies are orthogonal in this sense and eq. (14) can be satisfied when some frequencies happen to be degenerate. Using eq. (14) in eq. (10) and eq. (11) the annihilation operators are related to field operators as,
\[ a_n = R \oint_{S_1} d\sigma (\phi(\sigma, 0) \omega_n + i \pi(\sigma, 0)) \bar{u}_n \]
\[ a_n^\dagger = R \oint_{S_1} d\sigma (\phi(\sigma, 0) \omega_n - i \pi(\sigma, 0)) \bar{u}_n \].

Eq. (15), eq. (7) and eq. (14) imply the usual commutation relations,
\[ [a_m, a_n] = [a_m^\dagger, a_n^\dagger] = 0 ; \quad [a_m, a_n^\dagger] = \delta_{mn} , \]
\[ [a_m, a_n] = [a_m^\dagger, a_n^\dagger] = 0 \]
of creation and annihilation operators. Inserting eq. (10) and eq. (11) in eq. (9) and using eq. (14), the Hamiltonian \( H_c(\Omega) \) of the co-rotating system is seen to be diagonal,
\[ H_c(\Omega) = \frac{1}{2} \sum_{\omega_n \geq 0} \omega_n(\Omega, R)(a_n a_n + a_n^\dagger a_n) \]
and the construction of the Fock-space is analogous to the non-rotating case: at any given angular frequency \( \Omega \), the lowest energy state of the co-rotating system is annihilated by all \( a_n \) and has the zero-point energy given by the formal sum,
\[ E_{\text{ZP}}(R, \Omega) = -\frac{1}{2} \sum_{\omega_n \geq 0} \omega_n(R, \Omega) \].
finite difference,
\[ \mathcal{E}_c(R, \Omega) = E_{ZP}(R, \Omega) - E_{ZP}(\infty, 0) , \]
which one may refer to as the Casimir energy of the co-rotating system. Various methods have been developed to extract the parameter-dependent part of the infinite zero-point energy. Here this is straightforward only if the intermediate regularization of eq. [18] respects the symmetries of the co-rotating system. Many regularizations, ranging from the insertion of an exponential cutoff in the zero-point sum of eq. [18] to generalized zeta-function regularization, to point-splitting, meet this criterion. However, in the latter regularization method, the point-splitting should be invariant under the time-transformation symmetry of the co-rotating frame. This is not the same as time-splitting in the lab-frame. The point-splitting regularization otherwise explicitly breaks rotational invariance, which would have to be explicitly restored for the total angular momentum to be conserved.

We further restrict our considerations to the example of a periodic \( \delta \)-distribution background \( V(\sigma) = \lambda \delta_{\text{per}}(\sigma) \). The previous considerations for finite potentials are readily adapted to this singular case. Eq. [13] and the boundary conditions of eq. [16] for a periodic \( \delta \)-distribution potential become,
\[ 0 = (1 - \beta^2)u''_m - 2i\alpha_m \beta u'_m + \alpha^2 m u_m \]  
\[ \text{with } u_m(0) = u_m(2\pi) \]  
\[ \text{and } u'_m(0) - u'_m(2\pi) = \frac{\lambda R^2}{1 - \beta^2} u_m(0) , \]
where \( \alpha_m = \omega_m R/c \) and \( \beta = \Omega R/c \) are the dimensionless frequency and rotation speed. Eq. [20c] ensures that the discontinuity in the derivative of \( u_m \) compensates for the singular potential in eq. [12]. The mode function satisfying eq. [20] to the dimensionless frequency \( \alpha_m \) is of the form,
\[ u_m(\sigma) \propto (1 - e^{-\frac{2\pi i m \sigma}{\lambda R}}) e^{i \sigma \frac{\pi \alpha}{1 + \beta}} - (1 - e^{\frac{2\pi i m \sigma}{\lambda R}}) e^{-i \sigma \frac{\pi \alpha}{1 - \beta^2}} , \]
with \( \alpha = \alpha_m \) a solution to the secular equation,
\[ \sin \frac{\pi \alpha}{1 - \beta} \sin \frac{\pi \alpha}{1 + \beta} = \frac{\lambda R^2}{4 \alpha} \sin \frac{2\pi \alpha}{1 - \beta^2} . \]
Note that for \( \lambda \sim \infty \) the mode function in eq. [21], satisfies the Dirichlet condition \( u_m(0) = 0 \). Using the generalized argument principle [12], eq. [22] gives the Casimir energy in the co-rotating frame of a scalar field interacting with a rotating ring by a periodic \( \delta \)-distribution potential of strength \( \lambda \) for any radius \( R \) and angular rotation frequency \( \Omega \) as the finite integral,
\[ \mathcal{E}_c(R, \beta = \Omega R/c, \lambda) = \]
\[ \frac{\hbar c}{2\pi R} \int_{0}^{\infty} d\zeta \ln \left( 1 - \frac{4\zeta}{\zeta^2 - 1} \right) + \frac{(\lambda R^2 - 2\zeta) e^{-\frac{2\pi \zeta}{1 - \beta^2}}}{(2\zeta + \lambda R^2) e^{\frac{2\pi \zeta}{1 - \beta^2}}} . \]
One can perform the integral analytically in the limits of vanishing and very strong coupling: \( \mathcal{E}_c(R, \beta, \lambda = 0) = -\frac{\hbar c}{2\pi R} \), and \( \mathcal{E}_c(R, \beta, \lambda \sim \infty) = -\frac{\hbar c}{4\pi R} (1 - \beta^2) \). For vanishing interaction strength, the frequencies solving eq. [22] of left and right-moving modes are Doppler-shifted by factors \( 1 \pm \beta \). Their sum and thus the Casimir energy of the co-rotating frame do not depend on \( \Omega \). In the limit of very strong interaction strength on the other hand, the spectrum of frequencies solving eq. [22] is \( \{\omega_m = (1 - \Omega^2 R^2) m \pi / R; m \in \mathbb{N} \} \) and the dependence of the Casimir energy on \( \Omega \) is quadratic in the co-rotating frame,
\[ \mathcal{E}_c = -\frac{\hbar c}{48 R} (1 - \beta^2) - \frac{\hbar c}{2} \Omega^2 = -\frac{\hbar c}{48 R} + \frac{\hbar R \Omega^2}{48 c} - \frac{\hbar}{2} \Omega^2 . \]
The inverse Legendre transform of \( \mathcal{E}_C \) in eq. [24] gives the dependence of the ground state energy on the total angular momentum,
\[ \ell(\lambda \sim \infty) = -\frac{\partial \mathcal{E}_c}{\partial \Omega} \bigg|_{\lambda \to \infty} = \left( I - \frac{\hbar c}{24 R} \right) \Omega , \]
as,
\[ \mathcal{E}_s = \mathcal{E}_c(\Omega) + \ell \Omega \]
\[ \lambda \to \infty \rightarrow -\frac{\hbar c}{48 R} + \frac{\ell}{2} (1 - \frac{\hbar c}{\pi R}) R = -\frac{\hbar c}{48 R} - \frac{\hbar R \Omega^2}{48 c} + \frac{\hbar}{2} \Omega^2 . \]
Apart from the classical contribution proportional to \( I \), eq. [26] reproduces the zero-point energy of a scalar field with rotating Dirichlet boundary conditions obtained in [1]. Although the computation of [1] in the stationary frame for general potentials does not conserve energy and angular momentum, our results do agree for a rotating Dirichlet boundary conditions. As Chernodub pointed out [1], eq. [26] implies that zero-point fluctuations of a scalar field reduce the moment of inertia of the device. However, the classical contribution in general is not negligible and the semi-classical treatment of this system becomes questionable when the zero-point contribution to the total moment of inertia is larger in magnitude than the classical one. We argue below that this in fact never occurs.

\[ 1 \text{ A finite single-particle Casimir energy can be defined only if a certain coefficient of the asymptotic heat kernel expansion of the differential operator vanishes \textendash\textemdash\text{for scalar fields on } S_1 \text{ this is the case and the Casimir energy is finite for any positive potential as well as for a Dirichlet condition } [11].} \]

\[ 2 \text{ The generalized } \zeta \text{-function techniques of } [11] \text{ allow one to numerically obtain this energy for any well-behaved potential } V(\sigma).} \]
Since the smallest observable change in the total angular momentum in this quantum system is \(\pm \hbar\), the upper bound of eq. (29) together with eq. (27) imply,

\[
I \frac{\Delta \Omega}{\Delta \ell} = 1 - \frac{\Delta I_{ZP}}{\Delta \ell} \geq (1 - 1/24) > 0.
\] (30)

We therefore have that either \(\frac{\Delta \Omega}{\Delta \ell} = I_{\text{tot}} > 0\) or that the moment of inertia \(I\) of the collective coordinate is itself negative. Since the latter case would contradict the model assumptions, we arrive at the conclusion that the total effective moment of inertia of the simplest Chernobub-device is always positive and its ground state is non-rotating once the classical contribution to the total angular momentum of the device is included. This classical contribution is necessary to conserve the total angular momentum of this semi-classical model.

Noting that only the total moment of inertia of the entire device can be measured and not the contribution from quantum fluctuations alone, one can always renormalize and decompose the total moment of inertia of the device as,

\[
I_{\text{tot}}(\beta^2) = I + I_{ZP}(\beta^2) = I_{\text{tot}}(\beta_0^2) + (I_{ZP}(\beta^2) - I_{ZP}(\beta_0^2))
\] (31)

where \(\beta_0\) is a reference rotation speed. If we choose \(\beta_0 = 1\), the second term in eq. (31) is positive for all \(\beta\) due to the lower bound of eq. (28). Whether or not the total moment of inertia of the device is negative therefore depends exclusively on phenomenological input and can only be determined by a measurement. The renormalized form of eq. (31) pays tribute to the fact that the quantum fluctuations are not the whole story and also makes sense when \(I_{ZP}(\beta_0^2) \rightarrow \infty\) but differences remain finite. Note that the negative contribution from quantum fluctuations in this model is irrelevant in eq. (31).

The conclusion could be different only if quantum corrections to the total moment of inertia were unbounded below – in this case \(I_{\text{tot}}(\beta^2)\) invariably turns negative for some value of \(|\beta| < 1\) and measuring \(I_{\text{tot}}(\beta_0^2)\) determines only at which rotation speed this occurs.

This simple and transparent model thus demonstrates that a negative zero-point contribution to the moment of inertia does not imply that the ground state of the complete quantum system could be self-rotating – much as negative contributions to the mass from quantum fluctuations do not imply the existence of tachyons.

Note further that due to the relation in eq. (5), a self-rotating ground state would imply that \(\Omega(\ell) = 0\) at some finite value \(\ell \neq 0\). Assuming that \(\Omega(\ell = 0) = 0\) as well, \(E_c(\Omega \sim 0)\) would have to be multi-valued at \(\Omega = 0\). This is not the case for a quadratic Hamiltonian such as \(H_c\) with a unique ground state.

The collective coordinate allows one to relate the Casimir energy in the stationary system to the one in the co-rotating frame by a Legendre transformation. Since \(H_c\) of eq. (6) and the total angular momentum \(\ell\) are commuting hermitian operators, one therefore can conclude
that the Casimir energies of the co-rotating and station-
ary frame are both real and that total angular momen-
tum is conserved. No vacuum friction slows the rotation
of this device. There is no transfer of angular momentum
between the zero-point and classical contributions to the
total angular momentum of this device.

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