Abstract

In this paper we discuss some special generalizations of equationally Noetherian property which naturally arise in the universal algebraic geometry. We introduce weakly equationally Noetherian, $q_\omega$-compact, $u_\omega$-compact, and weakly $u_\omega$-compact algebras and then examine properties of such algebras. Also we consider the connections between five classes: the class of equationally Noetherian algebras, the class of weakly equationally Noetherian algebras, the class of $u_\omega$-compact algebras, the class of weakly $u_\omega$-compact algebras, and the class of $q_\omega$-compact algebras.

Keywords: Compactness Theorem, universal closure, quasivariety, algebraic structure, algebraic set, coordinate algebra, (weakly) $u_\omega$-compact algebra, $q_\omega$-compact algebra, (weakly) equationally Noetherian algebra, logically irreducible set.

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1 Introduction

This paper deals with the universal algebraic geometry. The universal algebraic geometry is a young branch of mathematics. The subject of universal algebraic geometry lies in the solutions of systems of equations over an arbitrary algebraic structure.

Investigations in universal algebraic geometry were started in works by B.I. Plotkin [22, 23, 24] and papers on algebraic geometry over groups by G. Baumslag, O.G. Kharlampovich, A.G. Myasnikov, and V.N. Remeslennikov [14, 15, 16, 17]. After that there were a lot of papers on algebraic geometry over concrete groups, algebras, monoids and so on. Among them there are the famous works by O.G. Kharlampovich,
A. G. Myasnikov [14, 15, 16, 17] and Z. Sela [25, 26, 27] on algebraic geometry over free groups.

In recent years we have achieved more general and systematic point of view on the universal algebraic geometry as on a formalized theory. In this respect we have started a series of works on universal algebraic geometry. This paper is the third one of that series along with [3, 4].

According to [6, 19, 20], in [3] we give a framework of universal algebra and model theory as much as we need it in universal algebraic geometry. At the same time we discuss how notions and ideas from model theory work in universal algebraic geometry. In [4] we introduce the foundation of universal algebraic geometry, basic definitions and constructions of the algebraic geometry over an arbitrary algebraic structure \( B \).

This paper is supposed to be read after the previous ones [3, 4], however for the sake of convenience we present in here some of the most essential notations and definitions (see Section 2).

We consider only first-order functional languages (signatures). Recall that algebraic structures in a functional language are called *algebras*. Typically we denote algebraic structures by capital calligraphic letters (\( \mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots \)) and their universes by the corresponding capital Latin letters (\( A, B, C, \ldots \)).

The main results of papers [3, 4] are so-called the Unification Theorems (Theorem A and Theorem C) which give a description of coordinate algebras by means of several languages.

**Theorem A.** Let \( \mathcal{B} \) be an equationally Noetherian algebra in a functional language \( L \). Then for a finitely generated algebra \( \mathcal{C} \) of \( L \) the following conditions are equivalent:

1) \( \text{Th}_v(\mathcal{B}) \subseteq \text{Th}_v(\mathcal{C}), \) i.e., \( \mathcal{C} \in \text{Ucl}(\mathcal{B}) \);

2) \( \text{Th}_3(\mathcal{B}) \supseteq \text{Th}_3(\mathcal{C}) \);

3) \( \mathcal{C} \) embeds into an ultrapower of \( \mathcal{B} \);

4) \( \mathcal{C} \) is discriminated by \( \mathcal{B} \);
5) $C$ is a limit algebra over $B$;

6) $C$ is an algebra defined by a complete atomic type in the theory $\text{Th}_q(B)$ in $L$;

7) $C$ is the coordinate algebra of an irreducible algebraic set over $B$ defined by a system of equations in the language $L$.

**Theorem C.** Let $B$ be an equationally Noetherian algebra in a functional language $L$. Then for a finitely generated algebra $C$ of $L$ the following conditions are equivalent:

1) $C \in \text{Qvar}(B)$, i.e., $\text{Th}_q(B) \subseteq \text{Th}_q(C)$;

2) $C \in \text{Pvar}(B)$;

3) $C$ embeds into a direct power of $B$;

4) $C$ is separated by $B$;

5) $C$ is a subdirect product of a finitely many limit algebras over $B$;

6) $C$ is an algebra defined by a complete atomic type in the theory $\text{Th}_q(B)$ in $L$;

7) $C$ is the coordinate algebra of an algebraic set over $B$ defined by a system of equations in the language $L$.

Note that items 5) in both Theorem A and Theorem C give a description of coordinate algebras by means of limit algebras. The limit algebraic structures (groups, as the rule) become the object of intense interest in modern algebra \[2, 5, 7, 8, 9, 10, 11\].

Theorems A and C are formulated for so-called equationally Noetherian algebras (the definition see in Section 2). Equationally Noetherian algebras possess the best opportunity to study the algebraic geometry over them. If a given algebra $B$ is equationally Noetherian then we have an advantage when investigating the algebraic geometry over $B$. In this case we may use:

(i) Unification Theorems;
(ii) the decomposition of any algebraic set over $\mathcal{B}$ into a finite union of irreducible algebraic sets (Theorem 5.11 below);

(iii) the possibility to study only finite system of equations;

(iv) and some more results [3, 4].

In the case when a given algebra $\mathcal{B}$ is not equationally Noetherian we lose some results for equationally Noetherian algebras, while some of them may remain in force. In this paper we introduce four generalizations of the equationally Noetherian property which naturally arise in universal algebraic geometry. These are

(N'): weak equationally Noetherian property that retains (iii);

(Q): $q_\omega$-compactness that retains Unification Theorem C;

(U): $u_\omega$-compactness that retains Unification Theorems A and C;

(U'): and weak $u_\omega$-compactness that retains (iv), namely, some weak form of Unification Theorem A.

We denote by $\mathcal{N}$ the class of all equationally Noetherian algebras in a given functional language $L$. By $\mathcal{N}'$, $\mathcal{Q}$, $\mathcal{U}$, $\mathcal{U}'$, correspondingly, we denote the classes of algebras with properties above. The picture of connections between classes $\mathcal{Q}$, $\mathcal{U}$, $\mathcal{U}'$, $\mathcal{N}'$ and $\mathcal{N}$ is presented in Section 6.

There exist several equivalent approaches to $q_\omega$- and $u_\omega$-compact algebras. We introduce them in Section 4. One of these approaches rises from some ideas of model theory. It relates to the Compactness Theorem and the notion of compact algebra.

Recall that a set of formulas $T$ in a language $L$ is called satisfyable in a class $\mathcal{K}$ of algebraic structures in $L$ (or $T$ is realized in $\mathcal{K}$) if one can assign some elements from a particular algebraic structure from $\mathcal{K}$ as values to the variables which occur in $T$ in such a way that all formulas from $T$ become true. The set $T$ is called finitely satisfyable in $\mathcal{K}$ if every finite subset of $T$ is realized in $\mathcal{K}$.
Compactness Theorem (K. F. Gödel, A. I. Malcev [6]). If a set of first-order formulas $T$ in a language $L$ is finitely satisfiable in a class $K$ of algebraic structures in $L$, then $T$ is satisfiable in an ultraproduct of structures from $K$.

Class $K$ is called compact if every finitely satisfiable in $K$ set of formulas $T$ is satisfiable in $K$. This definition occurs in the book by V. A. Gorbunov [6]. It is natural to name an algebraic structure $B$ compact if the class $\{B\}$ is compact. However, according to W. Hodges [13], algebraic $L$-structure is called compact if its universe is a Hausdorff topological space, in such a way that each function from $L$ is interpreted by a continuous function. The same algebraic structures appear in [6] under the name of topologically compact structures.

Trying to avoid an ambiguity we call an $L$-algebra $B$ logically compact if every finitely satisfiable in $B$ set of formulas $T$ in the language $L$ is satisfiable in $B$. When we modify this definition and consider only special types of sets of formulas $T$ we get definitions of special compactness, such as $q_\omega$- and $u_\omega$-compactness. Short review of the history of “$q_\omega$-compact” notion is represented in Subsection 4.1.

First and foremost in this article we generalize the Unification Theorems to $u_\omega$- and $q_\omega$-compact algebras. In Section 3 we give geometric definitions of $u_\omega$- and $q_\omega$-compactness. In Subsection 3.1 we prove that Theorem A is true for any $u_\omega$-compact algebra $B$ and every algebra $B$ which satisfies Theorem A is $u_\omega$-compact. The similar result that connects $q_\omega$-compact algebras and Theorem C is presented in Subsection 3.2. In Subsection 3.3 for weakly $u_\omega$-compact algebras we formulate and prove a weak analog of Theorem A.

Section 4 is devoted to $q_\omega$- and $u_\omega$-compact algebras. In Subsection 4.1 we put definitions of $q_\omega$- and $u_\omega$-compact algebras in different equivalent forms and prove the equivalence of them in Subsection 4.4. For $u_\omega$-compact algebra $B$ Unification Theorems give a global view to all (irreducible) coordinate algebras over $B$. However, it may happen that one has no $u_\omega$-compact property but some “local $u_\omega$-compact property” which gives result of Theorem A for a certain algebra $C$ (not for all $C$). This idea is developed in Subsection 4.3.

In Section 5 we discuss weak properties: weak equationally Noetherian property (Sub-
section 5.1) and weak $u_\omega$-compactness (Subsection 5.3). In Subsection 5.2 we introduce logically irreducible algebraic sets. Those sets naturally arise as generalization of irreducible ones. In particularly, we show that the notions of irreducible algebraic set and logically irreducible algebraic set over an algebra $B$ coincide if and only if $B$ is weakly $u_\omega$-compact.

In the last Section 7 we continue discussion about connections between $u_\omega$- and $q_\omega$-compact algebras with the Compactness Theorem and corresponding technique from the model theory. By the way, we construct $u_\omega$-compact elementary extension for an arbitrary algebra $B$.

2 Preliminaries

In this section we remind basic notions and facts from universal algebraic geometry according to [3, 4].

Let $L$ be a first-order functional language, $X = \{x_1, x_2, \ldots, x_n\}$ a finite set of variables, $T_L(X)$ the set of all terms of $L$ with variables in $X$, $T_L(X)$ the absolutely free $L$-algebra with basis $X$ and $A_L(X)$ the set of all atomic formulas of $L$ with variables in $X$.

In universal algebraic geometry atomic formulas from $A_L(X)$ are named equations in $L$ and subsets $S \subseteq A_L(X)$ are named systems of equations in the language $L$.

For a system of equations $S \subseteq A_L(X)$ and an algebra $B$ in the language $L$ we denote by $V_B(S)$ the set of all solutions of $S$ in $B$:

$$V_B(S) = \{(b_1, \ldots, b_n) \in B^n \mid B \models (t(b_1, \ldots, b_n) = s(b_1, \ldots, b_n)) \quad \forall (t = s) \in S\}.$$  

It is called the algebraic set over $B$ defined by the system $S$. If $S$ contains of only one equation $(t = s)$ we write $V_B(t = s)$ instead of $V_B(\{(t = s)\})$.

Algebraic set is irreducible if it is not a finite union of proper algebraic subsets; otherwise it is reducible. The empty set is not considered to be irreducible. Hence, according to R. Hartshorne [12], all irreducible algebraic sets are non-empty in our paper.

Two systems $S_1, S_2 \subseteq A_L(X)$ are equivalent over $B$ if $V_B(S_1) = V_B(S_2)$. The radical $Rad_B(S)$ of a system of equations $S \subseteq A_L(X)$ is the maximal system which is equivalent
to $S$ over $\mathcal{B}$. It is also called the radical of algebraic set $Y = V_{\mathcal{B}}(S)$ and denoted by $\text{Rad}(Y)$. By $[S]$ we denote the congruent closure of $S$, i.e., the least congruent subset of $\text{At}_L(X)$ that contains $S$.

By $\Phi_{qf,L}(X)$ we denote the set of all quantifier-free formulas in $L$ with variables in $X$. We say that a formula $\phi \in \Phi_{qf,L}(X)$ is a consequence of a system of equations $S \subseteq \text{At}_L(X)$ over an $L$-algebra $\mathcal{B}$, if $\mathcal{B} \models \phi(b_1, \ldots, b_n)$ for all $(b_1, \ldots, b_n) \in V_{\mathcal{B}}(S)$. For example, an atomic formula $(t = s)$, $t, s \in T_L(X)$, is a consequence of $S$ over $\mathcal{B}$ if and only if $(t = s) \in \text{Rad}_B(S)$.

For an arbitrary algebraic set $Y \subseteq B^n$ over $\mathcal{B}$ the radical $\text{Rad}(Y)$ defines the congruence $\theta_{\text{Rad}(Y)}$ on $T_L(X)$:

$$t_1 \sim_{\theta_{\text{Rad}(Y)}} t_2 \iff (t_1 = t_2) \in \text{Rad}(Y), \quad t_1, t_2 \in T_L(X).$$

The factor-algebra $\Gamma(Y) = T_L(X)/\theta_{\text{Rad}(Y)}$ is called the coordinate algebra of the algebraic set $Y$.

Let $Y \subseteq B^n$ and $Z \subseteq B^m$ be algebraic sets over $\mathcal{B}$. One has $\Gamma(Y) \cong \Gamma(Z)$ if and only if algebraic sets $Y$ and $Z$ are isomorphic (we omit here the definition of isomorphism between algebraic sets). Isomorphic algebraic sets are irreducible and reducible simultaneously.

We say that an $L$-algebra $\mathcal{C}$ is a coordinate algebra over $\mathcal{B}$ if $\mathcal{C} \cong \Gamma(Y)$ for some algebraic set $Y$ over $\mathcal{B}$, and $\mathcal{C}$ is an irreducible coordinate algebra over $\mathcal{B}$ if $\mathcal{C} \cong \Gamma(Y)$ for some irreducible algebraic set $Y$ over $\mathcal{B}$.

One of the principal goals of algebraic geometry over a given algebraic structure $\mathcal{B}$ is the problem of classification of algebraic sets over $\mathcal{B}$ up to isomorphism. This problem is equivalent to the problem of classification of coordinate algebras of algebraic sets over $\mathcal{B}$. Also it is important to classify coordinate algebras of irreducible algebraic sets over $\mathcal{B}$. Formulated in Introduction Unification Theorems A and C are very useful for solution of those problems.

In Theorems A and C we claim an algebra $\mathcal{B}$ is equationally Noetherian. Thus, let us remind that an $L$-algebra $\mathcal{B}$ is called equationally Noetherian, if for every finite set $X$ and every system of equations $S \subseteq \text{At}_L(X)$ there exists a finite subsystem $S_0 \subseteq S$ such that $V_{\mathcal{B}}(S_0) = V_{\mathcal{B}}(S)$. Properties of equationally Noetherian algebras are discussed in [3, 4].
An $L$-algebra $C$ is *separated* by $L$-algebra $B$ if for any pair of non-equal elements $c_1, c_2 \in C$ there is a homomorphism $h: C \to B$ such that $h(c_1) \neq h(c_2)$. An algebra $C$ is *discriminated* by $B$ if for any finite set $W$ of elements from $C$ there is a homomorphism $h: C \to B$ whose restriction onto $W$ is injective. We are interested in a familiar form of results, so it is useful to put by definition that the trivial algebra $E$ is separated by an algebra $B$ anyway, and $E$ is discriminated by $B$ if and only if $B$ has a trivial subalgebra.

The definitions of limit algebras and algebras defined by complete atomic types need a large introduction, so we omit them (see [3]).

In this paper we use some operators which image a class $K$ of $L$-algebras into another one. For the sake on convenience we collect here the list of all these operators:

- $S(K)$ — the class of subalgebras of algebras from $K$;
- $P(K)$ — the class of direct products of algebras from $K$;
- $P_\omega(K)$ — the class of finite direct products of algebras from $K$;
- $P_s(K)$ — the class of subdirect products of algebras from $K$;
- $P_f(K)$ — the class of filterproducts of algebras from $K$;
- $P_u(K)$ — the class of ultraproducts of algebras from $K$;
- $L_{\rightarrow}(K)$ — the class of direct limits of algebras from $K$;
- $L_{\rightarrow} s(K)$ — the class of epimorphic direct limits of algebras from $K$;
- $L_{fg}(K)$ — the class of algebras in which all finitely generated subalgebras belong to $K$;
- $P\text{var}(K)$ — the least prevariety including $K$;
- $Q\text{var}(K)$ — the least quasi-variety including $K$, i.e., $Q\text{var}(K) = \text{Mod}(\text{Th}_{qi}(K))$;
- $U\text{cl}(K)$ — the universal class of algebras generated by $K$, i.e., $U\text{cl}(K) = \text{Mod}(\text{Th}_\forall(K))$;
- $\text{Res}(K)$ — the class of algebras which are separated by $K$;
- $\text{Dis}(K)$ — the class of algebras which are discriminated by $K$;
- $K_e$ — the addition of the trivial algebra $E$ to $K$, i.e., $K_e = K \cup \{E\}$;
- $K_\omega$ — the class of finitely generated algebras from $K$.

Here we denote by $\text{Th}_{qi}(K)$ (correspondingly, $\text{Th}_\forall(K)$, $\text{Th}_\exists(K)$) the set of all quasi-identities (correspondingly, universal sentences, existential sentences) which are true in all structures from $K$. 

9
For an arbitrary class $K$ of $L$-algebras one has:

$$\text{Ucl}(K) = \text{SP}_u(K), \quad \text{Dis}(K) \subseteq \text{Ucl}(K),$$
$$\text{Res}(K) = \text{Pvar}(K) = \text{SP}(K), \quad \text{Pvar}(K) \subseteq \text{Qvar}(K).$$

According to Gorbunov [6] and in contrast to [3], we assume that the direct product for the empty set of indexes coincides with the trivial $L$-algebra $E$. In particularly, when we say that an algebra $C$ is a finite direct product of algebras from $K$ (or a subdirect product of a finitely many algebras from $K$) then $C$ may be just the trivial algebra. However, while defining an filterproduct we assume that the set of indexes is non-empty.

### 3 Generalizations of the Unification Theorems

Unification Theorems A and C are formulated in Introduction above for an equationally Noetherian algebra $B$. Those theorems have been proven in [3, 4].

**Question:** Suppose that the algebra $B$ is not equationally Noetherian. When Unification Theorems remain true for $B$?

To answer this question we need to analyze the proofs of Theorems A and C. As it was mentioned in [4], for the reasoning of some implications in Theorems A and C the equationally Noetherian property is not required, namely, one has the following remark.

**Remark 3.1.** Let $B$ be an algebra in a functional language $L$ and $C$ a finitely generated $L$-algebra. Then

- $C$ is the coordinate algebra of an irreducible algebraic set over $B$ defined by a system of equations in the language $L$ IF AND ONLY IF $C$ is discriminated by $B$ (Theorem A: $7 \iff 4$);

- IF $C$ is discriminated by $B$ THEN $C$ is a limit algebra over $B$ (Theorem A: $4 \implies 5$);

- $C$ is the coordinate algebra of an algebraic set over $B$ defined by a system of equations in the language $L$ IF AND ONLY IF $C \in \text{Pvar}(B)$ (Theorem C: $7 \iff 2$);
• IF \( C \) is a subdirect product of a finitely many limit algebras over \( B \) THEN \( C \in Qvar(B) \) (Theorem C: 5 \( \Rightarrow \) 1); and so on.

The complete set of implications in Theorems A and C which always remain true is represented as follows:

**Theorem A:** \( \{4 \Leftrightarrow 7\} \implies \{1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 5 \Leftrightarrow 6\} \);

**Theorem C:** \( \{5\} \implies \{1 \Leftrightarrow 6\} \iff \{2 \Leftrightarrow 3 \Leftrightarrow 4 \Leftrightarrow 7\} \).

Further, when proving 1) \( \implies \) 4) in both Theorems A and C, we use not equationally Noetherian property itself, but some weaker properties. What properties exactly? These are \( u_\omega \)-compactness and \( q_\omega \)-compactness.

**Definition 3.2.** We say \( L \)-algebra \( B \) is \( q_\omega \)-compact if for any finite set \( X \), any system of equations \( S \subseteq At_L(X) \), and any equation \((t_0 = s_0) \in At_L(X)\) such that
\[
V_B(S) \subseteq V_B(t_0 = s_0)
\]
there exists a finite subsystem \( S_0 \subseteq S \) such that
\[
V_B(S) \subseteq V_B(S_0) \subseteq V_B(t_0 = s_0).
\]
Here the finite subsystem \( S_0 \) may alter depending on equation \((t_0 = s_0)\).

**Definition 3.3.** An \( L \)-algebra \( B \) is termed \( u_\omega \)-compact if for any finite set \( X \), any system of equations \( S \subseteq At_L(X) \), and any equations \((t_1 = s_1), \ldots, (t_m = s_m) \in At_L(X)\) such that
\[
V_B(S) \subseteq V_B(t_1 = s_1) \cup \ldots \cup V_B(t_m = s_m)
\]
there exists a finite subsystem \( S_0 \subseteq S \) such that
\[
V_B(S) \subseteq V_B(S_0) \subseteq V_B(t_1 = s_1) \cup \ldots \cup V_B(t_m = s_m).
\]
Here the finite subsystem \( S_0 \) may alter depending on equations \((t_1 = s_1), \ldots, (t_m = s_m)\).

It is clear that any equationally Noetherian algebra \( B \) is \( u_\omega \)-compact, and any \( u_\omega \)-compact algebra is \( q_\omega \)-compact.

The definitions of \( u_\omega \)-compactness and \( q_\omega \)-compactness above are given in geometric form. We know some other approaches to these notions that will be discussed in Section 4.

In that section will be also represented the etymology of the notion of \( u_\omega(q_\omega) \)-compactness.
3.1 The generalization of Unification Theorem A

The significance of $u_\omega$-compact algebras in universal algebraic geometry is shown in the following theorem.

**Theorem 3.4** (analog of Theorem A). Let $\mathcal{B}$ be $u_\omega$-compact algebra in a functional language $L$. Then for a finitely generated algebra $\mathcal{C}$ of $L$ the following conditions are equivalent:

1) $\text{Th}_\forall(\mathcal{B}) \subseteq \text{Th}_\forall(\mathcal{C})$, i.e., $\mathcal{C} \in \text{Ucl}(\mathcal{B})$;

2) $\text{Th}_\exists(\mathcal{B}) \supseteq \text{Th}_\exists(\mathcal{C})$;

3) $\mathcal{C}$ embeds into an ultrapower of $\mathcal{B}$;

4) $\mathcal{C}$ is discriminated by $\mathcal{B}$;

5) $\mathcal{C}$ is a limit algebra over $\mathcal{B}$;

6) $\mathcal{C}$ is an algebra defined by a complete atomic type in the theory $\text{Th}_\forall(\mathcal{B})$ in $L$;

7) $\mathcal{C}$ is the coordinate algebra of an irreducible algebraic set over $\mathcal{B}$ defined by a system of equations in the language $L$.

Moreover, if for an $L$-algebra $\mathcal{B}$ and for all finitely generated $L$-algebras $\mathcal{C}$ the conditions above are equivalent then $\mathcal{B}$ is $u_\omega$-compact.

**Proof.** It follows from Remark 3.1 that conditions 1)–7) are equivalent if and only if one has equivalence 1) $\iff$ 4). The latter means that a finitely generated algebra $\mathcal{C}$ is discriminated by $\mathcal{B}$ if and only if $\mathcal{C} \in \text{Ucl}(\mathcal{B})$, i.e., $\text{Ucl}(\mathcal{B})_\omega = \text{Dis}(\mathcal{B})_\omega$. By Theorem 4.2 below, one has the equality $\text{Ucl}(\mathcal{B})_\omega = \text{Dis}(\mathcal{B})_\omega$ if and only if an algebra $\mathcal{B}$ is $u_\omega$-compact.

3.2 The generalization of Unification Theorem C

To prove an analog of Theorem C for $q_\omega$-compact algebras we need the following results.
Lemma 3.5 ([3]). Let $\mathcal{C}$ be a limit algebra over an $L$-algebra $\mathcal{B}$. Then there exists an ultrapower $\mathcal{B}^*$ of $\mathcal{B}$ such that $\mathcal{C}$ embeds into $\mathcal{B}^*$.

Lemma 3.6 ([4]). A finitely generated $L$-algebra $\mathcal{C}$ is the coordinate algebra of an algebraic set over $L$-algebra $\mathcal{B}$ if and only if $\mathcal{C}$ is a subdirect product of the coordinate algebras of irreducible algebraic sets over $\mathcal{B}$.

Theorem 3.7 (analog of Theorem C). Let $\mathcal{B}$ be $q_\omega$-compact algebra in a functional language $L$. Then for a finitely generated algebra $\mathcal{C}$ of $L$ the following conditions are equivalent:

1) $\mathcal{C} \in \text{Qvar}(\mathcal{B})$, i.e., $\text{Th}_{qi}(\mathcal{B}) \subseteq \text{Th}_{qi}(\mathcal{C})$;

2) $\mathcal{C} \in \text{Pvar}(\mathcal{B})$;

3) $\mathcal{C}$ embeds into a direct power of $\mathcal{B}$;

4) $\mathcal{C}$ is separated by $\mathcal{B}$;

5') $\mathcal{C}$ is a subdirect product of limit algebras over $\mathcal{B}$;

6) $\mathcal{C}$ is an algebra defined by a complete atomic type in the theory $\text{Th}_{qi}(\mathcal{B})$ in $L$;

7) $\mathcal{C}$ is the coordinate algebra of an algebraic set over $\mathcal{B}$ defined by a system of equations in the language $L$.

Moreover, if for an $L$-algebra $\mathcal{B}$ and for all finitely generated $L$-algebras $\mathcal{C}$ the conditions above are equivalent then $\mathcal{B}$ is $q_\omega$-compact.

Proof. By Remark 3.1 it is sufficient to prove implications 1) $\implies$ 2), 5') $\implies$ 1), and 7) $\implies$ 5') for $q_\omega$-compact algebra $\mathcal{B}$. By Theorem 4.1 below, we have the identity $\text{Qvar}(\mathcal{B})_\omega = \text{Pvar}(\mathcal{B})_\omega$ that gives proof of 1) $\implies$ 2). For implication 5') $\implies$ 1) we refer to Lemma 3.5 and the fact that every quasi-variety is closed under ultraproducts, direct products and subalgebras.

For proving 7) $\implies$ 5') suppose that $\mathcal{C}$ is the coordinate algebra of an algebraic set over $\mathcal{B}$. By Lemma 3.6, $\mathcal{C}$ is a subdirect product of coordinate algebras of irreducible algebraic
sets over \( B \). By Remark 3.1 (Theorem A: 7 \( \implies \) 5), coordinate algebras of irreducible algebraic sets over \( B \) are limit algebras over \( B \).

Suppose now that for some \( L \)-algebra \( B \) we have equivalence 1) \( \iff \) 2) for all finitely generated \( L \)-algebras \( C \). It means that \( \text{Qvar}(B)_\omega = \text{Pvar}(B)_\omega \) and, by Theorem 4.1 below, the algebra \( B \) is \( q_\omega \)-compact.

**Remark 3.8.** Unfortunately, we are not in a position to formulate Theorem C for \( q_\omega \)-compact algebras in all its fullness, because item 5) essentially needs equationally Noetherian property. We have to weaken 5), namely we should erase words “finitely many”.

To establish Remark 3.8 we formulate the following problem.

**Embedding Problem.** Let \( B \) be \( q_\omega \)-compact algebra in a functional language \( L \). The question: whether or not every coordinate algebra over \( B \) subdirectly embeds into a finite direct product of algebras from \( \text{Ucl}(B) \)? If the answer is “not”, then we ask whether or not the same holds for at least \( u_\omega \)-compact algebras.

A. N. Shevlyakov in [28] gives the negative answer to the Embedding Problem both for \( q_\omega \)-compact and \( u_\omega \)-compact algebras.

Let us put an addition to Remark 3.1.

**Remark 3.9.** The following implications and equivalencies from Theorem 3.7 hold for an arbitrary algebra \( B \):

\[
\begin{align*}
&\{1 \iff 6\} \quad \{2 \iff 3 \iff 4 \iff 7\} \\
&\{5'\}
\end{align*}
\]

Theorem 3.7 gives a classification of coordinate algebras in terms of quasivarieties. Thereby, any characterizations of quasivariety \( \text{Qvar}(K) \) of a class \( K \) of \( L \)-algebras are helpful in universal algebraic geometry. In [6] [19] one can find the identities:

\[
\begin{align*}
\text{Qvar}(K) &= \text{SP}_t(K)_\omega = \text{SPP}_u(K) = \text{SP}_u \text{P}(K) = \text{SP}_u \text{P}_\omega(K) = \\
&= \text{SL}_\omega \text{P}(K) = L_\omega \text{SP}(K) = L_\omega \text{P}_\omega(K) = L_\omega \text{SP}(K).
\end{align*}
\]
3.3 Weak generalization of Unification Theorem A

Let $B$ be an algebra in a functional language $L$. Let us consider the class $\text{Ucl}(B)_\omega$.

By Remark 3.1 for any irreducible algebraic set $Y$ over $B$ the coordinate algebra $\Gamma(Y)$ belongs to $\text{Ucl}(B)_\omega$. If $B$ is $u_\omega$-compact algebra then, by Theorem 3.4 every algebra $\mathcal{C}$ from $\text{Ucl}(B)_\omega$ is the coordinate algebra of some irreducible algebraic set $Y$ over $B$.

Let us apply a weak mode to $u_\omega$-compactness and require that every coordinate algebra $\mathcal{C}$ from $\text{Ucl}(B)_\omega$ is irreducible. Suppose that some algebras from $\text{Ucl}(B)_\omega$ are not coordinate algebras for algebraic sets over $B$ at all, however, if $\Gamma(Y) \in \text{Ucl}(B)$ then $Y$ is irreducible. Let us introduce a specific name for algebra $B$ with this type of property.

**Definition 3.10.** We name an $L$-algebra $B$ *weakly $u_\omega$-compact* if each non-empty algebraic set $Y$ over $B$ which coordinate algebra $\Gamma(Y)$ belongs to $\text{Ucl}(B)$ is irreducible.

By Theorem 3.4 every $u_\omega$-compact algebra is weakly $u_\omega$-compact. We will discuss weakly $u_\omega$-compact algebras, their properties and equivalent approaches to them in Subsection 5.3.

For weakly $u_\omega$-compact algebras we have just the following weak analog of Theorem A. It allows to describe irreducible coordinate algebras inside the class of all coordinate algebras.

**Theorem 3.11** (weak analog of Theorem A). Let $B$ be a weakly $u_\omega$-compact algebra in a functional language $L$ and $Y$ a non-empty algebraic set over $B$. Then the following conditions are equivalent:

1) $\text{Th}_\forall(B) \subseteq \text{Th}_\forall(\Gamma(Y))$, i.e., $\Gamma(Y) \in \text{Ucl}(B)$;

2) $\text{Th}_\exists(B) \supseteq \text{Th}_\exists(\Gamma(Y))$;

3) $\Gamma(Y)$ embeds into an ultrapower of $B$;

4) $\Gamma(Y)$ is discriminated by $B$;

5) $\Gamma(Y)$ is a limit algebra over $B$;
6) $\Gamma(Y)$ is an algebra defined by a complete atomic type in the theory $\text{Th}_\nu(B)$ in $L$;

7) $Y$ is irreducible.

Moreover, if for an $L$-algebra $B$ and for every non-empty algebraic set $Y$ the conditions above are equivalent then $B$ is weakly $u_\omega$-compact.

Proof. It follows from Remark 3.1 that conditions 1)–7) are equivalent if and only if one has implication 1) $\implies$ 7). By definition, implication 1) $\implies$ 7) take place if and only if $B$ is weakly $u_\omega$-compact.

4 $q_\omega$-compact and $u_\omega$-compact algebras

In Section 3 we gave the definitions of $q_\omega$- and $u_\omega$-compact algebras in geometric language. In Subsection 4.1 we gather the numerous another approaches to these notions into two theorems. We will prove these theorems in Subsection 4.4.

In Subsection 4.3 we introduce “local $q_\omega(u_\omega)$-compact property” and show its use in universal algebraic geometry. Subsection 4.2 contains some accessory materials.

4.1 Criteria of $q_\omega$- and $u_\omega$-compactness

At first we formulate the theorems and then give the necessary explanations.

**Theorem 4.1.** For an algebra $B$ in a functional language $L$ the following conditions are equivalent:

1) $B$ is $q_\omega$-compact;

2) for any finite set $X$, any system of equations $S \subseteq \text{At}_L(X)$, and any consequence $c = (t_0 = s_0) \in \text{Rad}_B(S)$ there exists a finite subsystem $S_c \subseteq S$ such that $c \in \text{Rad}_B(S_c)$;

3) for any finite set $X$, any subset $S \subseteq \text{At}_L(X)$, and any atomic formula $(t_0 = s_0) \in \text{At}_L(X)$ if an (infinite) formula

$$\forall x_1 \ldots \forall x_n \left( \bigwedge_{(t=s) \in S} t(\bar{x}) = s(\bar{x}) \implies t_0(\bar{x}) = s_0(\bar{x}) \right)$$
holds in $B$ then for some finite subsystem $S_c \subseteq S$ the quasi-identity
\[
\forall x_1 \ldots \forall x_n \left( \bigwedge_{(t=s) \in S_c} t(\bar{x}) = s(\bar{x}) \rightarrow t_0(\bar{x}) = s_0(\bar{x}) \right)
\]
also holds in $B$;

4) for any finite set $X$, any subset $S \subseteq \text{At}_L(X)$, and any atomic formula $(t_0 = s_0) \in \text{At}_L(X)$ if the set of formulas
\[
T = S \cup \{ -(t_0 = s_0) \}
\]
is finitely satisfiable in $B$ then it is satisfiable in $B$;

5) every finitely generated algebra from $\text{Qvar}(B)$ is the coordinate algebra of an algebraic set over $B$;

6) $\text{Qvar}(B)_{\omega} = \text{Pvar}(B)_{\omega}$;

7) $\text{Qvar}(B) = L_{fg} \text{Res}(B)$;

8) $L_{fg}^{SP}(B) = L_{fg} \text{SP}(B)$;

9) $L_{fg} \text{SP}(B) = L_{fg} \text{SP}(B)$;

10) for any finite set $X$ and any system of equations $S \subseteq \text{At}_L(X)$ one has:
\[
\text{Rad}_B(S) = \bigcup_{S_0 \subseteq S} \text{Rad}_B(S_0),
\]
where $S_0$ runs all finite subsystems of $S$;

11) for any finite set $X$ and any directed system $\{ S_i, i \in I \}$ of radical ideals over $B$ from $\text{At}_L(X)$ the union $S = \bigcup_{i \in I} S_i$ is a radical ideal over $B$;

12) for any finite set $X$ and any epimorphic direct system $\Lambda = (I, C_i, h_{ij})$ of coordinate algebras over $B$ with generating set $X$, and $h_{ij}(x) = x, x \in X$, the epimorphic direct limit $\varinjlim C_i$ is a coordinate algebra over $B$. 

17
Theorem 4.2. For an algebra \( B \) in a functional language \( L \) the following conditions are equivalent:

1) \( B \) is \( u_\omega \)-compact;

2) for any finite set \( X \), any system of equations \( S \subseteq \text{At}_L(X) \), and any consequence \( c \) of \( S \) over \( B \) of the form \( c = (t_1 = s_1) \lor \ldots \lor (t_m = s_m) \), \( t_i, s_i \in T_L(X) \), there exists a finite subsystem \( S_c \subseteq S \) such that \( c \) is a consequence of \( S_c \) over \( B \);

3) for any finite set \( X \), any subset \( S \subseteq \text{At}_L(X) \), and any atomic formulas \((t_1 = s_1), \ldots, (t_m = s_m) \in \text{At}_L(X)\) if an (infinite) formula

\[
\forall x_1 \ldots \forall x_n \left( \bigwedge_{(t=s) \in S} t(x) = s(x) \implies \bigvee_{i=1}^m t_i(x) = s_i(x) \right)
\]

holds in \( B \) then for some finite subsystem \( S_c \subseteq S \) the universal sentence

\[
\forall x_1 \ldots \forall x_n \left( \bigwedge_{(t=s) \in S_c} t(x) = s(x) \implies \bigvee_{i=1}^m t_i(x) = s_i(x) \right)
\]

also holds in \( B \);

4) for any finite set \( X \), any subset \( S \subseteq \text{At}_L(X) \), and any atomic formulas \((t_1 = s_1), \ldots, (t_m = s_m) \in \text{At}_L(X)\) if the set of formulas

\[
T = S \cup \{ \neg (t_1 = s_1), \ldots, \neg (t_m = s_m) \}
\]

is finitely satisfiable in \( B \) then it is satisfiable in \( B \);

5) every finitely generated algebra from \( \text{Ucl}(B) \) is the coordinate algebra of an irreducible algebraic set over \( B \);

6) \( \text{Ucl}(B)_\omega = \text{Dis}(B)_\omega \);

7) \( \text{Ucl}(B) = \text{LfgDis}(B) \).
Item 2) in Theorem 4.1 (correspondingly, in Theorem 4.2) gives the definition of $q_\omega$-compact (correspondingly, $u_\omega$-compact) algebra in terms of radicals; item 3) — in terms of infinite formulas; item 5) — in terms of coordinate algebras.

Item 4) shows that the definition of $q_\omega(u_\omega)$-compactness is a compact property relating to special types of sets of formulas $T$, as it is discussed in Introduction. The background of this notion is detailed in [21] for groups. Here we will tell just a few words about it.

The answer for the following question has been attained by V. A. Gorbunov [6].

**Malcev Problem.** When the prevariety $\text{Pvar}(K)$ generated by class $K$ is a quasivariety?

V. A. Gorbunov has introduced the notion of quasi-compact (q-compact) class $K$ and proved that $\text{Pvar}(K) = \text{Qvar}(K)$ if and only if $K$ is q-compact. Let us compare that result with item 6) in Theorem 4.1.

The definition of q-compact algebra $B$ is much the same as the definition of $q_\omega$-compact algebra given in item 4) of Theorem 4.1. We just bound the set of variables $X$ for defining $q_\omega$-compact algebras: $X$ must be finite. For q-compact algebras $X$ runs sets of all possible cardinalities.

While items 1)–7) in Theorems 4.1 and 4.2 are symmetric, items 10)–12) in Theorem 4.1 are specific for $q_\omega$-compact algebras; 8) and 9) in Theorem 4.1 are just corollaries of 7).

Items 10) and 11) in Theorems 4.1 are close. The family $\{\text{Rad}(S_0)\}$, where $S_0$ runs all finite subsystems of a system $S$, gives an example of a directed system. Let us remind concerned definitions.

A partial ordering $(I, \leq)$ is directed if any two elements from $I$ have an upper bound. A family $\{\theta_i, i \in I\}$ of congruencies on an $L$-algebra $M$ with $i \leq j \Leftrightarrow \theta_i \subseteq \theta_j$ is called directed system of congruencies.

A system $S \subseteq \text{At}_L(X)$ is radical ideal over $B$ if $S = \text{Rad}_B(S)$.

**Definition 4.3.** We say that a family $\{S_i, i \in I\}$ of radical ideals from $\text{At}_L(X)$ is a directed system if the family $\{\theta_{S_i}, i \in I\}$ is a directed system of congruencies on $\mathcal{T}_L(X)$.

Let us prove just a little part of Theorems 4.1 and 4.2.
Lemma 4.4. Let $\mathcal{B}$ be an $L$-algebra, $X$ a finite set, $|X| = n$, $S \subseteq \text{At}_L(X)$ a system of equations, and $(t_1 = s_1), \ldots, (t_m = s_m) \in \text{At}_L(X)$ atomic formulas. Then the following conditions are equivalent:

1) $V_{\mathcal{B}}(S) \subseteq V_{\mathcal{B}}(t_1 = s_1) \cup \ldots \cup V_{\mathcal{B}}(t_m = s_m)$;

2) $(t_1 = s_1) \vee \ldots \vee (t_m = s_m)$ is a consequence of $S$ over $\mathcal{B}$;

3) the (infinite) formula

$$\forall x_1 \ldots \forall x_n \left( \bigwedge_{(t = s) \in S} t(\vec{x}) = s(\vec{x}) \rightarrow \bigvee_{i=1}^m t_i(\vec{x}) = s_i(\vec{x}) \right)$$

holds in $\mathcal{B}$;

4) the set of formulas

$$T = S \cup \{ \neg(t_1 = s_1), \ldots, \neg(t_m = s_m) \}$$

is not satisfiable in $\mathcal{B}$;

5) there is no homomorphism $h: \langle \{c_1, \ldots, c_n\} \mid S \rangle \rightarrow \mathcal{B}$ such that

$$h(t_i(c_1, \ldots, c_n)) \neq h(s_i(c_1, \ldots, c_n)) \text{ for all } i \in \{1, \ldots, m\}.$$ 

Proof. Straightforward.

Corollary 4.5. One has equivalencies 1) $\iff$ 2), 1) $\iff$ 3), 3) $\iff$ 4) in both Theorems 4.1 and 4.2.

Proof. Equivalencies 1) $\iff$ 2), 1) $\iff$ 3) are easy. Note that the statement in item 3) has a form “$A$ implies $B$”. The equivalent statement is “$\neg B$ implies $\neg A$” which gives 4). So we have 3) $\iff$ 4).

From now on, we will use not only geometric definition of $q_\omega$-compact (correspondingly, $u_\omega$-compact) algebra, but also the definitions that items 2), 3), 4) in Theorem 4.1 (correspondingly, in Theorem 4.2) give us.
4.2 $\mathcal{E}$-compact algebras

This subsection is a special excursion. We consider here the following problem.

**Problem.** When the conditions “$\mathcal{E} \in \text{Ucl}(\mathcal{B})$” and “$\mathcal{B}$ has a trivial subalgebra” are equivalent?

It is important to note that for a large class of algebras the conditions “$\mathcal{E} \in \text{Ucl}(\mathcal{B})$” and “$\mathcal{B}$ has a trivial subalgebra” are equivalent, but not for all algebras.

**Definition 4.6.** We say an $\mathcal{L}$-algebra $\mathcal{B}$ is $\mathcal{E}$-compact if finite satisfiability in $\mathcal{B}$ of the set of all atomic formulas $\text{At}_L(\{x\})$ in one variable $x$ implies its satisfiability in $\mathcal{B}$.

**Lemma 4.7.** An $\mathcal{L}$-algebra $\mathcal{B}$ is $\mathcal{E}$-compact if and only if the conditions “$\mathcal{E} \in \text{Ucl}(\mathcal{B})$” and “$\mathcal{B}$ has a trivial subalgebra” are equivalent.

**Proof.** It is sufficient to show that $\text{At}_L(\{x\})$ is satisfiable in $\mathcal{B}$ if and only if $\mathcal{B}$ has a trivial subalgebra, and $\text{At}_L(\{x\})$ is finitely satisfiable if and only if $\mathcal{E} \in \text{Ucl}(\mathcal{B})$.

Suppose that $\text{At}_L(\{x\})$ is satisfiable in $\mathcal{B}$. Then there exists an element $b \in \mathcal{B}$ with $\mathcal{B} \models (t(b) = s(b))$ for all $t, s \in T_L(\{x\})$. Therefore, subalgebra of $\mathcal{B}$ generated by the element $b$ is trivial. Conversely, if $\mathcal{B}$ has a trivial subalgebra $\mathcal{E} = \{e\}$ then the set of all atomic formulas $\text{At}_L(\{x\})$ is realized in $\mathcal{B}$ on the element $e$.

Assume now that $\text{At}_L(\{x\})$ is not finitely satisfiable in $\mathcal{B}$. Then there exists a finite set $S_0$ of atomic formulas such that the universal sentence

$$\forall x \left( \bigvee_{(t=s)\in S_0} \neg (t(x) = s(x)) \right)$$

holds in $\mathcal{B}$. However $\Pi$ is false in $\mathcal{E}$, so $\mathcal{E} \not\in \text{Ucl}(\mathcal{B})$. Conversely, if the set of all atomic formulas $\text{At}_L(\{x\})$ is finitely satisfiable in $\mathcal{B}$ then by Compactness Theorem it is realized in some ultrapower $\mathcal{B}^\ast$ of $\mathcal{B}$. Hence, $\mathcal{E} \in \text{Ucl}(\mathcal{B})$. \hfill $\square$

**Corollary 4.8.** The condition “algebra $\mathcal{B}$ is $\mathcal{E}$-compact” means that $\mathcal{B}$ has a trivial subalgebra or $\mathcal{E} \not\in \text{Ucl}(\mathcal{B})$.

Let us note that in “good” signatures all algebras are $\mathcal{E}$-compact.

21
Lemma 4.9. Suppose a functional language $L$ has at least one constant symbol. Then every algebra in $L$ is $E$-compact.

Proof. Let $B$ be an $L$-algebra. We need to show that condition $E \in \text{Ucl}(B)$ implies that $B$ has a trivial subalgebra. Consider the set of formulas

$$T = \{c = c'\} \cup \{F(c, \ldots, c) = c\},$$

where $c, c'$ run all constant symbols from $L$ and $F$ runs all functional symbols from $L$. If $E \in \text{Ucl}(B)$, then $B \models T$. Therefore, there exists an element $b \in B$ such that $c^B = b$ for all constant symbol $c$ from $L$, and $F(b, \ldots, b) = b$ for all functional symbol $F$ from $L$. Thereby, the element $b$ generates the trivial subalgebra in $B$. □

Lemma 4.10. Suppose $L$ is a finite functional language. Then every algebra in $L$ is $E$-compact.

Proof. After Lemma 4.9 we may assume that $L$ has no constant symbols. Let $B$ an $L$-algebra. If $E \in \text{Ucl}(B)$ then the existential sentence

$$\exists x \left( \bigwedge_{F \in L} F(x, \ldots, x) = x \right)$$

holds in $B$. Thereby, $B$ has a trivial subalgebra. □

If $L$ is an infinite functional language with no constant symbols, then it is easy to construct an $L$-algebra $B$ that is not $E$-compact (see Example 5.18 below).

It follows from the definition that all equationally Noetherian algebras are $E$-compact. Now we state that all $q_L$- and $u_L$-compact algebras are $E$-compact. We need the following facts and definitions.

According to V. A. Gorbunov [6], an $L$-algebra $B$ is weakly atomic compact, if for any set $X$ and any subset $S \subseteq \text{At}_L(X)$ finite satisfiability of $S$ in $B$ implies realizability of $S$ in $B$. We say that an $L$-algebra $B$ is weakly atomic $\omega$-compact, if for any finite set $X$ and any subset $S \subseteq \text{At}_L(X)$ finite satisfiability of $S$ in $B$ implies realizability of $S$ in $B$. It is obvious that weak atomic $\omega$-compactness implies $E$-compactness.
The following result has been proven by M. Kotov [18].

Lemma ([18]). Every \( q_\omega \)-compact algebra in a functional language \( L \) is weakly atomic \( \omega \)-compact.

Corollary 4.11. Let \( B \) be \( q_\omega \)-compact \( L \)-algebra (in particularly, \( B \) may be \( u_\omega \)-compact). Then the universal closure \( Ucl(B) \) contains the trivial algebra \( E \) if and only if \( B \) has a trivial subalgebra.

Let us note that M. Kotov has proven more general result in his work. We formulate it on geometric language.

Lemma ([18]). Let \( B \) be an \( L \)-algebra and \( S \) a system of equations in \( L \). If \( B \) is \( q_\omega \)-compact and \( V_B(S) \) is a singleton set or the empty set, then there exists a finite subsystem \( S_0 \subseteq S \) which is equivalent to \( S \) over \( B \). If \( B \) is \( u_\omega \)-compact and \( V_B(S) \) is a finite set or the empty set, then there exists a finite subsystem \( S_0 \subseteq S \) which is equivalent to \( S \) over \( B \).

4.3 Local compact properties

Let \( X \) be a finite set. Fix a subset \( S \subseteq At_L(X) \). We will give the definitions of local compact properties with respect to fixed \( S \).

Definition 4.12. An \( L \)-algebra \( B \) is called \( q_S \)-compact if for each atomic formula \( (t_0 = s_0) \in At_L(X) \) if the set of formulas

\[
T = S \cup \{ \neg(t_0 = s_0) \}
\]

is finitely satisfiable in \( B \) then it is satisfiable in \( B \).

Definition 4.13. An \( L \)-algebra \( B \) is called \( u_S \)-compact if for any atomic formulas \( (t_1 = s_1), \ldots, (t_m = s_m) \in At_L(X) \) if the set of formulas

\[
T = S \cup \{ \neg(t_1 = s_1), \ldots, \neg(t_m = s_m) \}
\]

is finitely satisfiable in \( B \) then it is satisfiable in \( B \).
It is clear that algebra $B$ is $q_{\omega}(u_{\omega})$-compact if and only if it is $q_{S}(u_{S})$-compact for every finite set $X$ and every $S \subseteq \text{At}_{L}(X)$.

The main results on local compact properties are the following.

**Proposition 4.14.** Let $B$ be an algebra in a functional language $L$, $X$ a finite set, $S \subseteq \text{At}_{L}(X)$, and $C = \langle X \mid S \rangle$. Then the following conditions are equivalent:

1) $C$ is the coordinate algebra of an algebraic set over $B$ defined by a system of equations in the language $L$;

2) $C$ is separated by $B$;

3) $C \in \text{Qvar}(B)$ and $B$ is $q_{S}$-compact.

**Proposition 4.15.** Let $B$ be an algebra in a functional language $L$, $X$ a finite set, $S \subseteq \text{At}_{L}(X)$, such that $\left[S\right] \neq \text{At}_{L}(X)$, and $C = \langle X \mid S \rangle$. Then the following conditions are equivalent:

1) $C$ is the coordinate algebra of an irreducible algebraic set over $B$ defined by a system of equations in the language $L$;

2) $C$ is discriminated by $B$;

3) $C \in \text{Ucl}(B)$ and $B$ is $u_{S}$-compact.

Before giving a proof of these propositions we need some remarks. Firstly, equivalence 1) $\iff$ 2) in both Propositions 4.14 and 4.15 have been proven in [4]. Secondly, let us answer the question: when the set of formulas (2) is not finitely satisfiable in $B$? It happens if and only if there exists a finite subset $S_0 \subseteq S$ such that the universal sentence

\[
\forall y_1 \ldots \forall y_n \left( \bigwedge_{(t=s) \in S_0} t(\bar{y}) = s(\bar{y}) \longrightarrow \bigvee_{i=1}^{m} t_i(\bar{x}) = s_i(\bar{y}) \right), \quad \text{where} \ |X| = n,
\]

(3) holds in $B$. For example, if $(t_i = s_i) \in [S]$ for some $i \in \{1, \ldots, m\}$, then there exists a finite subset $S_0 \subseteq S$ such that $S_0 \vdash (t_i = s_i)$, in particularly, universal formula (3) holds in $B$. 

24
Thirdly, note that in Propositions 4.15 we claim $[S] \neq \text{At}_L(X)$, but in Propositions 4.14 such restriction is omitted. If $[S] = \text{At}_L(X)$ then $\mathcal{C} = \langle X | S \rangle$ is the trivial algebra $\mathcal{E}$. Moreover, in this case every algebra $\mathcal{B}$ is $q_S$- and $u_S$-compact. Since the trivial algebra $\mathcal{E}$ is the coordinate algebra of an algebraic set over $\mathcal{B}$ anyway and $\mathcal{E}$ belongs to each quasi-variety $[4]$, we have no difficulties with $\mathcal{E}$ in Propositions 4.14.

**Remark 4.16.** One can omit restriction $[S] \neq \text{At}_L(X)$ in Proposition 4.15 if and only if $\mathcal{B}$ is $\mathcal{E}$-compact algebra. Indeed, the trivial algebra $\mathcal{E}$ is the coordinate algebra of an irreducible algebraic set over $\mathcal{B}$ if and only if $\mathcal{B}$ has a trivial subalgebra $[4, Lemma 3.22]$. By Lemma 4.7, the conditions “$\mathcal{E} \in \text{Ucl}(\mathcal{B})$” and “$\mathcal{B}$ has a trivial subalgebra” are equivalent if and only if $\mathcal{B}$ is $\mathcal{E}$-compact.

Now we are going to prove Propositions 4.14 and 4.15. Arguments for them are the similar, so we will prove only Propositions 4.15.

**Proof of Propositions 4.15.** Let $\mathcal{C} \simeq \mathcal{T}_L(X)/\theta_S$, $X = \{c_1, \ldots, c_n\}$, and $[S] \neq \text{At}_L(X)$. By definition $\mathcal{C}$ is discriminated by $\mathcal{B}$ if for any finite set of atomic formulas $(t_1 = s_1), \ldots, (t_m = s_m) \in \text{At}_L(X) \setminus [S]$ there exists a homomorphism $h: \mathcal{C} \to \mathcal{B}$, such that $h(t_i(c_1, \ldots, c_n)) \neq h(s_i(c_1, \ldots, c_n))$ for all $i \in \{1, \ldots, m\}$. The existence of such homomorphism $h: \mathcal{C} \to \mathcal{B}$ means that the set $T$ in (2) is realized in $\mathcal{B}$. Note that if we take $(t_i = s_i) \in [S]$ for some $i \in \{1, \ldots, m\}$, then $T$ is not finitely satisfiable in $\mathcal{B}$. Anyway, we shown that if $\mathcal{C}$ is discriminated by $\mathcal{B}$ then $\mathcal{B}$ is $u_S$-compact. The occurrence $\mathcal{C} \in \text{Ucl}(\mathcal{B})$ follows from the inclusion $\text{Dis}(\mathcal{B}) \subseteq \text{Ucl}(\mathcal{B})$.

Suppose now that $\mathcal{C} = \langle X | S \rangle$ is not discriminated by $\mathcal{B}$ and show that $\mathcal{C} \not\in \text{Ucl}(\mathcal{B})$ or $\mathcal{B}$ is not $u_S$-compact. In this case for some atomic formulas $(t_1 = s_1), \ldots, (t_m = s_m) \in \text{At}_L(X) \setminus [S]$ the set $T$ from (2) is not realized in $\mathcal{B}$. If at the same time $T$ is finitely satisfiable in $\mathcal{B}$ then $\mathcal{B}$ is not $u_S$-compact. Assume that $T$ is not finitely satisfiable in $\mathcal{B}$. Therefore, there exists a finite subset $S_0 \subseteq S$ such that the universal formula (3) holds in $\mathcal{B}$. On the other hand, the formula

$$\bigwedge_{(t=s) \in S_0} t(\bar{y}) = s(\bar{y}) \longrightarrow \bigvee_{i=1}^{m} t_i(\bar{x}) = s_i(\bar{y})$$
is false in $C$ under the interpretation $y_i \mapsto c_i$, $i = 1, \ldots, n$, hence $C \not\in \text{Ucl}(B)$. □

4.4 Proof of the criteria

In this subsection we prove Theorems 4.1 and 4.2 that have been formulated in Subsection 4.1. Remain that equivalencies 1) $\iff$ 2), 1) $\iff$ 3), 3) $\iff$ 4) in both theorems have been proven in Subsection 4.1.

At first we prove the following easy lemma that will be useful below.

Lemma 4.17. Let $B, C$ be $L$-algebras, $C \in \text{Ucl}(B)$, and $T$ a set of quantifier-free formulas in $L$. If $T$ is finitely satisfiable in $C$ then it is finitely satisfiable in $B$.

Proof. Suppose $T$ is finitely satisfiable in $C$. Then for every finite subset $\{\phi_1, \ldots, \phi_m\} \subseteq T$ the existential sentence
\[
\exists x_1 \ldots \exists x_n \left( \phi_1(x_1, \ldots, x_n) \land \ldots \land \phi_m(x_1, \ldots, x_n) \right)
\]
holds in $C$. Since $C \in \text{Ucl}(B)$ then [1] holds in $B$ too. Thereby, $T$ is finitely satisfiable in $B$. □

We start with Theorem 4.2. Consider item 6). It states that $\text{Ucl}(B)_\omega = \text{Dis}(B)_\omega$. As inclusion $\text{Ucl}(B)_\omega \supseteq \text{Dis}(B)_\omega$ holds for an arbitrary algebra $B$, then item 6) is equivalent to inclusion $\text{Ucl}(B)_\omega \subseteq \text{Dis}(B)_\omega$. On the other hand, $\text{Dis}(B)_\omega$ is the class of all irreducible coordinate algebras over $B$ [4, Corollary 3.39]. Hence, we have equivalence 5) $\iff$ 6).

Now let us show equivalence 4) $\iff$ 6). Suppose $B$ is $u_\omega$-compact and $M$ is a finitely generated algebra from $\text{Ucl}(B)$. If $M$ is a trivial algebra then, by Corollary 4.11, $B$ has a trivial subalgebra, therefore, $M$ is discriminated by $B$.

For non-trivial algebra $M$ let us find a presentation $\langle X \mid S \rangle$, where $X$ is a finite set and $S \subseteq \text{At}_L(X)$, $[S] \neq \text{At}_L(X)$. As $B$ is $u_S$-compact we have $M \in \text{Dis}(B)$, by Proposition 4.15. Thus we proved inclusion $\text{Ucl}(B)_\omega \subseteq \text{Dis}(B)_\omega$ and implication 4) $\implies$ 6).

We prove the converse implication 6) $\implies$ 4) by contradiction. Suppose that $B$ is not $u_\omega$-compact. Then there exists a finite set $X$, a subset $S \subseteq \text{At}_L(X)$, and atomic formulas
\((t_1 = s_1), \ldots, (t_m = s_m) \in \text{At}_L(X)\), such that the set of formulas
\[ T = S \cup \{\neg(t_1 = s_1), \ldots, \neg(t_m = s_m)\}, \]
is not realized in \(B\), but every its finite subset is realized in \(B\).

By Compactness Theorem \(T\) is realized in some ultrapower \(B^I/D\) of \(B\). Let \(c_1, \ldots, c_n\) be elements from \(B^I/D\), such that \(B^I/D \models T(c_1, \ldots, c_n)\), and \(C\) subalgebra of \(B^I/D\) generated by the set \(\{c_1, \ldots, c_n\}\). Clearly, \(C\) is finitely generated algebra from \(\text{Ucl}(B)\).

Let \(\langle \{c_1, \ldots, c_n\} \mid R \rangle\) be a presentation of \(C\), i.e., \(C \simeq T_L(X)/\theta_R\), \(R \subseteq \text{At}_L(X)\). Since \(C \models T(c_1, \ldots, c_n)\), one has \(S \subseteq R\) and \((t_i = s_i) \notin [R]\), \(i = 1, \ldots, m\). Put
\[ T' = R \cup \{\neg(t_1 = s_1), \ldots, \neg(t_m = s_m)\}. \]
Since \(T'\) is realized in \(C\) and \(C \in \text{Ucl}(B)\), then, by Lemma 4.17, \(T'\) is finitely satisfiable in \(B\). However, \(T'\) is not satisfiable in \(B\). Thus \(B\) is not \(u_R\)-compact. Hence, by Proposition 4.15, \(C\) is not discriminated by \(B\). We proved 6) \(\Rightarrow\) 4).

Equivalence 6) \(\iff\) 7) is true in more general case. Let \(K\) and \(K'\) be two classes of \(L\)-algebras (let us have in mind \(K = \text{Ucl}(B)\) and \(K' = \text{Dis}(B)\)), \(K\) is universal axiomatizable and \(K'\) is closed under taking \(L\)-subalgebras. Then \(K = \text{Lfg}K'\) is equivalent to \(K_\omega = K'_\omega\).

Indeed, \(K = \text{Lfg}K'\) easy implies \(K_\omega = K'_\omega\). Inversely, if \(K_\omega = K'_\omega\) then \(K = \text{Lfg}K_\omega = \text{Lfg}K'_\omega = \text{Lfg}K'\).

Now we begin to prove Theorem 4.1

Equivalences 5) \(\iff\) 6), 4) \(\iff\) 6), 6) \(\iff\) 7) may be proven by means of the similar reasoning as in Theorem 4.2 (remind that \(\text{Pvar}(B) = \text{Res}(B)\)).

Let us show equivalence 7) \(\iff\) 8) \(\iff\) 9). For an arbitrary algebra \(B\) we have \(\text{Qvar}(B) = \text{LfgSP}(B) = \text{LfgSP}(B)\) [6, Corollary 2.3.4] and \(\text{Res}(B) = \text{SP}(B)\). So the identity \(\text{Qvar}(B) = \text{LfgRes}(B)\) is equivalent to \(\text{LfgSP}(B) = \text{LfgSP}(B)\) or \(\text{LfgSP}(B) = \text{LfgSP}(B)\).

Equivalence 2) \(\iff\) 10) is easy. Equivalence 11) \(\iff\) 12) is due to V. A. Gorbunov [6, Proposition 1.4.9]. So, it remains to prove implications 2) \(\implies\) 11) and 11) \(\implies\) 10).
Let $B$ be $\omega$-compact algebra, $\{S_i, i \in I\}$ a directed system of radical ideals from $At_L(X)$ and $S = \bigcup_{i \in I} S_i$. We show that $S = \text{Rad}(S)$, i.e., $\text{Rad}(S) \subseteq \bigcup_{i \in I} S_i$. Indeed, if $c$ is a consequence of $S$ then there exists a finite subsystem $S_0 \subseteq S$ with $c \in \text{Rad}(S_0)$. Since $I$ is directed there exists an index $i \in I$ such that $S_0 \subseteq S_i$, therefore $c \in S_i$. Thus we have implication $2) \implies 11$).

To prove implication $11) \implies 10)$ consider an arbitrary system $S \subseteq At_L(X)$. The family $\{\text{Rad}(S_0)\}$, where $S_0$ runs all finite subsystems of a system $S$, forms a directed system of radical ideals from $At_L(X)$. Hence $\bigcup_{S_0 \subseteq S} \text{Rad}(S_0)$ is a radical ideal over $B$. Also we have

$S \subseteq \bigcup_{S_0 \subseteq S} \text{Rad}_B(S_0) \subseteq \text{Rad}_B(S),$

therefore $\bigcup_{S_0 \subseteq S} \text{Rad}(S_0) = \text{Rad}_B(S)$. So, implication $11) \implies 10)$ has been proven.

5 Weakly equationally Noetherian and weakly $u_\omega$-compact algebras

A weak form of the equationally Noetherian property naturally arises in practice. We discuss algebras with this property in Subsection 5.1.

In Subsection 3.3 we have introduced weakly $u_\omega$-compact algebras. Now in Subsection 5.3 we present some equivalent approaches to weakly $u_\omega$-compact algebras.

In Subsection 5.2 we study logically irreducible algebraic sets. It is important to note that logically irreducible algebraic sets inspired the notion of weakly $u_\omega$-compact algebras.

5.1 Weak equationally Noetherian property

**Definition 5.1.** An $L$-algebra $B$ is said to be *weakly equationally Noetherian*, if for any finite set $X$ every system $S \subseteq At_L(X)$ is equivalent over $B$ to some finite system $S_0 \subseteq At_L(X)$. Here we do not assume that $S_0$ is a subsystem of $S$.

To make comparison equationally Noetherian and weakly equationally Noetherian properties it is required to reformulate corresponding definitions in the following form.
An L-algebra $\mathcal{B}$ is termed \textit{weakly equationally Noetherian}, if for any finite set $X$ and any system $S \subseteq \text{At}_L(X)$ there exists finite system $S_0 \subseteq \text{Rad}_B(S)$ such that $V_B(S) = V_B(S_0)$.

An L-algebra $\mathcal{B}$ is termed \textit{equationally Noetherian}, if for any finite set $X$ and any system $S \subseteq \text{At}_L(X)$ there exists finite system $S_0 \subseteq [S]$ such that $V_B(S) = V_B(S_0)$.

Indeed, for every atomic formula $c = (t = s) \in [S]$ there exists a finite subsystem $S_c \subseteq S$ such that $S_c \vdash (t = s)$. Therefore, if $V_B(S) = V_B(S_0)$ for a finite system $S_0 \subseteq [S]$ then one has

$$V_B(S) = V_B(\bigcup_{c \in S_0} S_c). \quad (5)$$

\textbf{Lemma 5.2.} If an L-algebra $\mathcal{B}$ is weakly equationally Noetherian and $q_{\omega}$-compact then it is equationally Noetherian.

\textit{Proof.} As $\mathcal{B}$ is weakly equationally Noetherian, for each system of equations $S$ there exists a finite system $S_0 \subseteq \text{Rad}_B(S)$ with $V_B(S) = V_B(S_0)$. As $\mathcal{B}$ is $q_{\omega}$-compact, for each equation $c = (t_0 = s_0) \in S_0$ there exists a finite subsystem $S_c \subseteq S$ with $V_B(S_c) \subseteq V_B(t_0 = s_0)$. Thereby, one has (5). It means that $\mathcal{B}$ is equationally Noetherian algebra. \hfill \qed

\textbf{Lemma 5.3.} If an L-algebra $\mathcal{B}$ is weakly equationally Noetherian and $\mathcal{C}$ a subalgebra of some direct power of $\mathcal{C}$ then $\mathcal{C}$ is weakly equationally Noetherian too.

\textit{Proof.} It follows from \cite[Lemma 3.7]{4}.

It is clear that every weakly equationally Noetherian algebra is $E$-compact.

\textbf{Lemma 5.4.} If an L-algebra $\mathcal{B}$ is weakly equationally Noetherian then

$$\text{Ucl}(\mathcal{B}) \cap \text{Res}(\mathcal{B})_\omega = \text{Dis}(\mathcal{B})_\omega.$$ 

\textit{Proof.} Since $\text{Dis}(\mathcal{B}) \subseteq \text{Res}(\mathcal{B})$, $\text{Dis}(\mathcal{B}) \subseteq \text{Ucl}(\mathcal{B})$, and $\text{Res}(\mathcal{B}) = \text{Pvar}(\mathcal{B})$ for any algebra $\mathcal{B}$ \cite{3}, we should check that $\text{Ucl}(\mathcal{B}) \cap \text{Pvar}(\mathcal{B})_\omega \subseteq \text{Dis}(\mathcal{B})_\omega$. Let us assume that $\mathcal{C}$ is a finitely generated algebra such that $\mathcal{C} \in \text{Pvar}(\mathcal{B}) \setminus \text{Dis}(\mathcal{B})$ and prove $\mathcal{C} \notin \text{Ucl}(\mathcal{B})$. If $\mathcal{C}$ is the trivial algebra $E$ then, by definition, condition $\mathcal{C} \notin \text{Dis}(\mathcal{B})$ implies that $\mathcal{B}$ has not a trivial subalgebra. Since $\mathcal{B}$ is weakly equationally Noetherian, then $\mathcal{B}$ is $E$-compact, and, by Lemma \cite[4.7]{7} $E \notin \text{Ucl}(\mathcal{B})$. Thereby, we may assume that $\mathcal{C}$ is non-trivial.

29
Let $\langle \{c_1, \ldots, c_n \mid S \rangle$ be a presentation of $C$, i.e., $C \simeq T_L(X)/\theta_{S}$, $S \subseteq At_L(X)$, $X = \{x_1, \ldots, x_n\}$. Since $C \notin \text{Dis}(B)$, there exits atomic formulas $(t_1 = s_1), \ldots, (t_m = s_m) \in At_L(X) \setminus [S]$ such that the (infinite) formula

$$
\forall x_1 \ldots \forall x_n \left( \bigwedge_{(t=s) \in S} t(\bar{x}) = s(\bar{x}) \rightarrow \bigvee_{i=1}^{m} t_i(\bar{x}) = s_i(\bar{x}) \right)
$$

holds in $B$. As one can find a finite system $S_0 \subseteq At_L(X)$ with $V_B(S_0) = V_B(S)$ then the universal sentence

$$
\forall x_1 \ldots \forall x_n \left( \bigwedge_{(t=s) \in S_0} t(\bar{x}) = s(\bar{x}) \rightarrow \bigvee_{i=1}^{m} t_i(\bar{x}) = s_i(\bar{x}) \right)
$$

holds in $B$.

Since $V_B(S_0) = V_B(S)$ we have $V_C(S_0) = V_C(S)$ [Lemma 3.7]. Hence, $(c_1, \ldots, c_n) \in V_C(S_0)$ but $t_i(c_1, \ldots, c_n) \neq s_i(c_1, \ldots, c_n)$ for all $i = 1, \ldots, m$. Therefore, universal formula (6) is not true in $C$, and $C \notin \text{Ucl}(B)$.

5.2 Logically irreducible algebraic sets

One of the approaches to $u_\omega$-compact algebras deals with so-called logically irreducible algebraic sets.

**Definition 5.5.** We say that an algebraic set $Y$ over $B$ is **logically irreducible** if its coordinate algebra $\Gamma(Y)$ belongs to $\text{Ucl}(B)$.

In Section 3 we have discussed that every irreducible algebraic set over an arbitrary algebra $B$ is logically irreducible. In Subsection 5.3 we will show that the notions of irreducible and logically irreducible algebraic sets coincide if and only if $B$ is weakly $u_\omega$-compact algebra.

**Lemma 5.6.** Let $B$ be an $L$-algebra. For a finitely generated $L$-algebra $C$ the following conditions are equivalent:

- $C$ is the coordinate algebra of a logically irreducible algebraic set over $B$;
• \( C \) belongs to \( \text{Ucl}(B) \cap \text{Pvar}(B) \).

**Proof.** Indeed, \( C \) is the coordinate algebra of an algebraic set over \( B \) if and only if \( C \in \text{Pvar}(B) \) [4, Proposition 3.22]. \( \square \)

**Corollary 5.7.** The class of all coordinate algebras of logically irreducible algebraic sets over \( B \) coincides with \( \text{Ucl}(B) \cap \text{Pvar}(B)_\omega \).

For irreducible algebraic sets we have the following result.

**Lemma 5.8** ([3]). *Let \( B \) be an \( L \)-algebra. Every non-empty algebraic set \( Y \) over \( B \) is a union of maximal with respect to inclusion irreducible algebraic sets over \( B \).*

Now we try to find a similar decomposition for algebraic sets into a union of maximal logically irreducible algebraic sets. It is clear that Lemma 5.8 gives a decomposition. However, maximal with respect to inclusion irreducible algebraic set may be a proper subset of some logically irreducible algebraic set.

**Lemma 5.9.** Let \( Y_1 \subset Y_2 \subset \ldots \) be an ascending chain of logically irreducible algebraic sets in \( B^n \) and \( Y \) the least algebraic set containing all these sets. Then \( Y \) is logically irreducible algebraic set.

**Proof.** Note that \( Y = V_B(\text{Rad}(\bigcup_i Y_i)) \) and \( \text{Rad}(Y) = \bigcap_i \text{Rad}(Y_i) \). Hence, there exists embedding \( h: \Gamma(Y) \to \prod_i \Gamma(Y_i) \) [3, Lemma 3.1]. Index \( i \) runs the linearly ordered set \( I \). For each \( i \in I \) denote by \( J_i \) the set \( \{ j \in I, j \geq i \} \). The family of subsets \( \{ J_i, i \in I \} \) is centered, hence there exists an ultrafilter \( D \) on \( I \) containing \( J_i \) for all \( i \in I \). Let \( f: \prod_i \Gamma(Y_i) \to \prod_i \Gamma(Y_i)/D \) be a canonical homomorphism. Let us show that composition \( f \circ h: \Gamma(Y) \to \prod_i \Gamma(Y_i)/D \) is embedding.

Indeed, we have \( \Gamma(Y) = T_L(X)/\theta_{\text{Rad}(Y)} \), where \( X = \{ x_1, \ldots, x_n \} \). If \( t_1/\theta_{\text{Rad}(Y)}, t_2/\theta_{\text{Rad}(Y)} \) are distinct elements from \( \Gamma(Y) \) then \( (t_1 = t_2) \in \text{At}_L(X) \setminus \text{Rad}(Y) \). Since \( \text{Rad}(Y_1) \supset \text{Rad}(Y_2) \supset \ldots \), then there exists an index \( i_0 \in I \) such that \( (t_1 = t_2) \not\in \text{Rad}(Y_i) \) for all \( i \in J_{i_0} \). It implies that \( f \circ h(t_1/\theta_{\text{Rad}(Y)}) \neq f \circ h(t_2/\theta_{\text{Rad}(Y)}) \). Thus \( f \circ h \) is injective.
Since $\Gamma(Y_i) \in \text{Ucl}(\mathcal{B})$ for each $i \in I$ and $\Gamma(Y) \in \text{SP}_u(\{\Gamma(Y_i), i \in I\})$, then $\Gamma(Y) \in \text{Ucl}(\mathcal{B})$, i.e., $Y$ is logically irreducible algebraic set.

Lemma 5.10. Let $\mathcal{B}$ be an $L$-algebra. Every non-empty algebraic set $Y$ over $\mathcal{B}$ is a union of maximal with respect to inclusion logically irreducible algebraic sets over $\mathcal{B}$.

Proof. We will show that for each point $p \in Y$ there exists logically irreducible algebraic set $Z$ such that $p \in Z \subseteq Y$ and $Z$ is maximal with these properties. Denote by $\Omega$ the family of logically irreducible algebraic sets $Z$ with $p \in Z \subseteq Y$ and show that $\Omega$ is not empty and has maximal elements.

Denote by $Z_p$ the closure in the Zariski topology of the set $\{p\}$. One has $p \in Z_p \subseteq Y$. Furthermore, $Z_p$ is irreducible algebraic set [4, Lemma 3.34]. Hence, $Z_p \in \Omega$.

By Zorn Lemma it is sufficiently to show now that family $\Omega$ contains upper boundary for each ascending chain $Y_1 \subset Y_2 \subset \ldots$ of element from $\Omega$. Let $Y_p$ be the least algebraic set that contains union $\bigcup_i Y_i$. By Lemma 5.9, $Y_p$ is logically irreducible. As $Y_p \subseteq Y$ one has $Y_p \in \Omega$.

Thereby, the union $\bigcup_{p \in Y} Y_p$ is desired.

Let us remind that for equationally Noetherian algebras we have the next result.

Theorem 5.11 ([3]). Let $\mathcal{B}$ be an equationally Noetherian algebra. Then any non-empty algebraic set $Y$ over $\mathcal{B}$ is a finite union of irreducible algebraic sets (irreducible components): $Y = Y_1 \cup \ldots \cup Y_m$. Moreover, if $Y_i \nsubseteq Y_j$ for $i \neq j$ then this decomposition is unique up to a permutation of components.

It is natural to ask the following question.

Decomposition Problem. Let $\mathcal{B}$ be a “good” algebra ($u_\omega$-, $q_\omega$-compact, weakly equationally Noetherian, for instance). Is it true that every non-empty algebraic set over $\mathcal{B}$ is a finite union of logically irreducible algebraic sets?

In spite of the fact that $u_\omega$-compact and weakly equationally Noetherian algebras are the closest algebras to equationally Noetherian ones we give for them the negative answer to the question above.

32
Indeed, a decomposition $Y = Y_1 \cup \ldots \cup Y_m$ of algebraic set $Y$ into a union of algebraic sets $Y_1, \ldots, Y_m$ implies the existence of a subdirect embedding $h : \Gamma(Y) \to \Gamma(Y_1) \times \ldots \times \Gamma(Y_m)$ \cite{4}. Suppose that the Decomposition Problem has the positive answer for $u_\omega$-compact algebras. It involves that the Embedding Problem for $u_\omega$-compact algebras has the positive answer too. However, A. N. Shevlyakov has proven the inverse result (see Subsection 3.2). Moreover, he has proven also that the Decomposition Problem for weakly equationally Noetherian algebras has the negative answer \cite{28}.

5.3 Weak $u_\omega$-compactness

In the proposition below we gather the different approaches to weakly $u_\omega$-compact algebras.

**Proposition 5.12.** For an algebra $B$ in a functional language $L$ the following conditions are equivalent:

1) $B$ is weakly $u_\omega$-compact;

2) every non-empty logically irreducible algebraic set over $B$ is irreducible;

3) every non-trivial coordinate algebra over $B$ that belongs to $\text{Ucl}(B)$ is irreducible;

4) $\text{Ucl}(B) \cap \text{Res}(B) = (\text{Dis}(B) e)_\omega$.

**Proof.** Equivalence 1) $\iff$ 2) is evident by definition. Remind that the trivial algebra $\mathcal{E}$ is a coordinate algebra over $B$ anyway, moreover, if $Y$ is an algebraic set over $B$ such that $\mathcal{E} = \Gamma(Y)$ then $Y$ is irreducible or $Y = \emptyset$ \cite{4} Lemma 3.22. It implies that we have equivalence 2) $\iff$ 3).

Since $\text{Dis}(B) \subseteq \text{Res}(B)$, $\text{Dis}(B) \subseteq \text{Ucl}(B)$, $\text{Res}(B) = \text{Pvar}(B)$, and $\mathcal{E} \in \text{Res}(B)$ for any algebra $B$, then item 4) means that every non-trivial algebra $C$ from $\text{Ucl}(B) \cap \text{Pvar}(B)_\omega$ belongs to $\text{Dis}(A)_\omega$.

As the class of all coordinate algebras of irreducible algebraic sets over $B$ coincides with $\text{Dis}(B)_\omega$ \cite{1} Corollary 3.37, and, by Corollary 5.7, the class of all coordinate algebras
of logically irreducible algebraic sets over $\mathcal{B}$ coincides with $\text{Ucl}(\mathcal{B}) \cap \text{Pvar}(\mathcal{B})_\omega$, we have equivalence 3) $\iff$ 4).

\[
\square
\]

**Remark 5.13.** Every $u_\omega$-compact (as well as $q_\omega$-compact, weakly equationally Noetherian) algebra is $\mathcal{E}$-compact. However, there exist weakly $u_\omega$-compact algebras that are not $\mathcal{E}$-compact (see Example 5.18 below). Suppose an algebra $\mathcal{B}$ is $\mathcal{E}$-compact. In this case one can omit “non-empty” in item 2), omit “non-trivial” in item 3), and write \( \text{Ucl}(\mathcal{B}) \cap \text{Res}(\mathcal{B})_\omega = \text{Dis}(\mathcal{B})_\omega \) instead of \( \text{Ucl}(\mathcal{B}) \cap \text{Res}(\mathcal{B})_\omega = (\text{Dis}(\mathcal{B})_e)_\omega \) in the formulation of Proposition 5.12. In this case the empty set is not algebraic over $\mathcal{B}$, or if it is algebraic then its coordinate algebra $\mathcal{E}$ does not belong to $\text{Ucl}(\mathcal{B})$.

**Lemma 5.14.** If an $\mathcal{L}$-algebra $\mathcal{B}$ is weakly $u_\omega$-compact and $q_\omega$-compact then it is $u_\omega$-compact.

**Proof.** We need to show that $\text{Ucl}(\mathcal{B})_\omega \subseteq \text{Dis}(\mathcal{B})_\omega$. Assume that $\mathcal{C}$ is a finitely generated algebra and $\mathcal{C} \not\subseteq \text{Dis}(\mathcal{B})$. Since $\text{Dis}(\mathcal{B})_\omega = \text{Ucl}(\mathcal{B}) \cap \text{Pvar}(\mathcal{B})_\omega$, then $\mathcal{C} \not\subseteq \text{Ucl}(\mathcal{B})$, and we have required, or $\mathcal{C} \not\subseteq \text{Pvar}(\mathcal{B})$. By Theorem 5.11, $\mathcal{C} \not\subseteq \text{Pvar}(\mathcal{B})$ implies that $\mathcal{C} \not\subseteq \text{Qvar}(\mathcal{B})$, hence $\mathcal{C} \not\subseteq \text{Ucl}(\mathcal{B})$.

The next question is naturally arises. Is there a geometric definition of weak $u_\omega$-compactness?

**Definition 5.15.** We name an $\mathcal{L}$-algebra $\mathcal{B}$ **geometrically weakly** $u_\omega$-**compact** if for any finite set $X$, any system of equations $S \subseteq \text{At}_\mathcal{L}(X)$, and any equations $(t_1 = s_1), \ldots, (t_m = s_m) \in \text{At}_\mathcal{L}(X)$ such that

\[
\text{V}_\mathcal{B}(S) \subseteq \text{V}_\mathcal{B}(t_1 = s_1) \cup \ldots \cup \text{V}_\mathcal{B}(t_m = s_m)
\]

and for each $i \in \{1, \ldots, m\}$

\[
\text{V}_\mathcal{B}(S) \not\subseteq \text{V}_\mathcal{B}(t_i = s_i)
\]

there exists a finite subsystem $S_0 \subseteq \text{Rad}_\mathcal{B}(S)$ such that

\[
\text{V}_\mathcal{B}(S_0) \subseteq \text{V}_\mathcal{B}(t_1 = s_1) \cup \ldots \cup \text{V}_\mathcal{B}(t_m = s_m).
\]
The definition above is evident generalization of both weak equationally Noetherian property and \( u_\omega \)-compactness. It also has analogs in terms of radical, in terms of infinite formulas, and in terms of compactness.

**Lemma 5.16.** For an algebra \( B \) in a functional language \( L \) the following conditions are equivalent:

1) \( B \) is geometrically weakly \( u_\omega \)-compact;

2) for any finite set \( X \), any radical ideal \( S \subseteq \text{At}_L(X) \) over \( B \), and any atomic formulas \((t_1 = s_1), \ldots, (t_m = s_m) \in \text{At}_L(X) \setminus \text{Rad}_B(S)\) if \( c = (t_1 = s_1) \lor \ldots \lor (t_m = s_m) \) is a consequence of \( S \) over \( B \) then there exists a finite subsystem \( S_c \subseteq S \) such that \( c \) is a consequence of \( S_c \) over \( B \);

3) for any finite set \( X \), any radical ideal \( S \subseteq \text{At}_L(X) \) over \( B \), and any atomic formulas \((t_1 = s_1), \ldots, (t_m = s_m) \in \text{At}_L(X) \) if an (infinite) formula

\[
\forall x_1 \ldots \forall x_n \left( \bigwedge_{(t = s) \in S} t(\bar{x}) = s(\bar{x}) \longrightarrow \bigvee_{i=1}^{m} t_i(\bar{x}) = s_i(\bar{x}) \right)
\]

holds in \( B \), and for each \( i \in \{1, \ldots, m\} \) an (infinite) formula

\[
\forall x_1 \ldots \forall x_n \left( \bigwedge_{(t = s) \in S} t(\bar{x}) = s(\bar{x}) \longrightarrow t_i(\bar{x}) = s_i(\bar{x}) \right)
\]

does not hold in \( B \), then for some finite subsystem \( S_c \subseteq S \) the universal sentence

\[
\forall x_1 \ldots \forall x_n \left( \bigwedge_{(t = s) \in S_c} t(\bar{x}) = s(\bar{x}) \longrightarrow \bigvee_{i=1}^{m} t_i(\bar{x}) = s_i(\bar{x}) \right)
\]

holds in \( B \);

4) for any finite set \( X \), any radical ideal \( S \subseteq \text{At}_L(X) \) over \( B \), and any atomic formulas \((t_1 = s_1), \ldots, (t_m = s_m) \in \text{At}_L(X) \) if the set of formulas

\[
T = S \cup \{ \neg(t_1 = s_1), \ldots, \neg(t_m = s_m) \}
\]
is finitely satisfiable in \( B \) and for each \( i \in \{1, \ldots, m\} \) the set of formulas

\[
T_i = S \cup \{\neg (t_i = s_i)\}
\]

is realized in \( B \) then \( T \) is satisfiable in \( B \).

Proof. Equivalences 1) \( \iff \) 2), 1) \( \iff \) 3), 3) \( \iff \) 4) follows from Lemma 4.4. Note that the statement in item 3) has a form “\( A \& \neg C \) implies \( B \)”. The equivalent statement is “\( \neg B \& \neg C \) implies \( \neg A \)” which gives 4). So we have 3) \( \iff \) 4).

Unfortunately, for weak \( u_\omega \)-compactness we have no an analog of Theorem 4.2 that holds for \( u_\omega \)-compact algebras.

**Lemma 5.17.** If an \( L \)-algebra \( B \) is geometrically weakly \( u_\omega \)-compact that it is weakly \( u_\omega \)-compact. The converse statement does not hold.

Proof. Suppose that \( B \) is geometrically weakly \( u_\omega \)-compact and \( Y \) a non-empty algebraic set over \( B \) such that \( \Gamma(Y) \in Ucl(B) \). We need to show that \( \Gamma(Y) \in \text{Dis}(B) \). Let \( S = \text{Rad}(Y) \), then \( \Gamma(Y) \) has the presentation \( \langle X \mid S \rangle \). If \( \Gamma(Y) \) is the trivial algebra, i.e., \( S = \text{At}_L(X) \), then \( Y \) is irreducible [4, Lemma 3.22] and \( \Gamma(Y) \in \text{Dis}(B) \).

Assume now that \( \Gamma(Y) \) is non-trivial, i.e., \( S \neq \text{At}_L(X) \). As the coordinate algebra \( \Gamma(Y) \) is separated by \( B \), hence for each atomic formula \( (t = s) \in \text{At}_L(X) \setminus S \) the set of formulas \( S \cup \{\neg (t = s)\} \) is realized in \( B \). Take atomic formulas \( (t_1 = s_1), \ldots, (t_m = s_m) \in \text{At}_L(X) \setminus S \). As the set of formulas \( T = S \cup \{\neg (t_1 = s_1), \ldots, \neg (t_m = s_m)\} \) is satisfiable in \( \langle X \mid S \rangle \), and \( \langle X \mid S \rangle \in Ucl(B) \), then, by Lemma 4.17, \( T \) is finitely satisfiable in \( B \). It follows from item 4) of Lemma 5.16 that \( T \) is satisfiable in \( B \). Thereby, algebra \( \langle X \mid S \rangle \) is discriminated by \( B \).

Example 5.18 below shows that the converse statement does not hold.

The following example is similar to the example by M. V. Kotov [18].
Example 5.18. Let $L = \{g_n, n \in \mathbb{N}\}$ be the infinite signature with unary functional symbols and $A$ the $L$-algebra with the universe $\mathbb{N}$ and

$$g_n(x) = \begin{cases} 
2n, & x = 2n + 1, \\
2n + 1, & x = 2n, \\
x, & \text{otherwise}.
\end{cases}$$

It is clear that $A$ has no trivial subalgebra. At the same time, the set of formulas $\{g_n(x) = x, n \in \mathbb{N}\}$ is finitely satisfiable in $A$, therefore, by Compactness Theorem, it is satisfiable in some ultrapower $A^* \in \text{Ucl}(A)$, then $E \in \text{Ucl}(A)$. Thereby, $A$ is not $E$-compact.

We state that $A$ is weakly $\omega$-compact. Indeed, take a non-trivial algebra $C$ from $\text{Ucl}(A) \cap \text{Pvar}(A)$. Since $\text{Pvar}(A) = \text{SP}(A)$ then $C$ is a subalgebra of a direct power of $A$. For any $n, m \in \mathbb{N}$, $n \neq m$, the universal formula

$$\forall x \ (g_n(x) = x \lor g_m(x) = x)$$

holds in $A$. Therefore, $C$ has a finite universe $\{c_1, c'_1, \ldots, c_d, c'_d\}$ with $c_i = g_n(c'_i)$ for all $i = 1, d$. The map $h: C \to A$, $h(c_i) = 2n_i$, $h(c'_i) = 2n_i + 1$, $i = 1, d$, is a monomorphism. Thus, $C \in \text{Dis}(A)$, and $A$ is weakly $\omega$-compact.

Let us check that $A$ is not geometrically weakly $\omega$-compact. Consider the systems of equations $S'(x) = \{g_n(x) = x, n \in \mathbb{N} \setminus \{0\}\}$ and $S(x, y) = S'(x) \cup S'(y)$. We have $V_A(S) = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. Therefore,

$$V_A(S) \subseteq V_A(x = y) \cup V_A(x = g_0(y)),$$

$$V_A(S) \not\subseteq V_A(x = y), \quad V_A(S) \not\subseteq V_A(x = g_0(y)).$$

Furthermore, it is not hard to see that

$$\text{Rad}_A(S) = \left\{ \begin{array}{l}
x = g_{n_1}(g_{n_2}(\ldots g_{n_m}(x)\ldots)), \\
y = g_{n_1}(g_{n_2}(\ldots g_{n_m}(y)\ldots)), \ n_i \neq 0, \\
x = g_{n_1}(g_{n_2}(\ldots g_{n_m}(y)\ldots)).
\end{array} \right\}$$

It is obvious that for any finite subsystem $S_0 \subseteq \text{Rad}_A(S)$ we have

$$V_A(S_0) \not\subseteq V_A(x = y) \cup V_A(x = g_0(y)).$$
6 Connections between the classes of algebras $Q$, $U$, $U'$, $N'$, and $N$

Let $L$ be a functional language. We use the following denotations:

- $N$ — the class of all equationally Noetherian $L$-algebras;
- $N'$ — the class of all weakly equationally Noetherian $L$-algebras;
- $Q$ — the class of all $q_\omega$-compact $L$-algebras;
- $U$ — the class of all $u_\omega$-compact $L$-algebras;
- $U'$ — the class of all weakly $u_\omega$-compact $L$-algebras.

It is clear that

$$Q \supseteq U \supseteq N \subseteq N'.$$

Moreover, by Lemma 5.2,

$$N = N' \cap Q = N' \cap U.$$

So, we have exactly the following picture for co-location of classes $N$, $N'$, $Q$, $U$:

![Venn diagram showing the relationships between $Q$, $U$, $N$, and $N'$]

Let us find the place of the class $U'$ in the picture above. By Theorem 3.3, Lemma 5.4 and Proposition 5.12, we have

$$U \subseteq U' \quad \text{and} \quad N' \subseteq U'.$$

It follows from Lemma 5.14 that

$$Q \cap U' = U.$$

Hence, co-location of classes $N$, $N'$, $Q$, $U$, and $U'$ are exactly the following:
In paper [21] A. G. Myasnikov and V. N. Remeslennikov asked the questions for the class of groups:

**Question 1:** \( N = Q \) or \( N \neq Q \) ?

**Question 2:** \( Q = U \) or \( Q \neq U \) ?

Now we add new questions:

**Question 3:** \( N = N' \) or \( N \neq N' \) ?

**Question 4:** \( N = U \) or \( N \neq U \) ?

**Question 5:** \( U' = U \cup N' \) or \( U' \neq U \cup N' \) ?

The answer to the first question has been given by B. I. Plotkin in [24]. He has constructed \( q_\omega \)-compact group that is not equationally Noetherian. We will discuss that construction in this section below. Note that B. I. Plotkin uses notation *logically Noetherian* for \( q_\omega \)-compact algebras and *geometrically Noetherian* for equationally Noetherian algebras.

The second and third questions have been solved by M. V. Kotov [18]. He has constructed examples that show \( Q \neq U \) and \( N \neq N' \). His examples are original algebraic structures in the language \( L = \{ g_n, n \in \mathbb{N} \} \) with countable set of unary functional symbols and with universe-sets \( \mathbb{R} \) and \( \mathbb{N} \).

At these results the fourth question remains open as well as the problem of differentiation of classes \( Q, U, N, N' \) for classical varieties: groups, rings, monoids, semigroups. In [28] A. N. Shevlyakov finds the neat examples in the variety of commutative idempotent semigroups in the language with countable set of constants. His examples distinguish classes \( N, N', Q, U \).

The algebra \( A \) from Example 5.18 gives an answer to the fifth question. It has been shown that \( A \in U' \), but \( A \) is not \( \mathcal{E} \)-compact. Since all \( u_\omega \)-compact and weakly equa-
tionally Noetherian algebras are $\mathcal{E}$-compact, then $\mathcal{A} \notin U \cup N'$. Another example for $U' \neq U \cup N'$ has been constructed by A.N. Shevlyakov \cite{28} in the class of commutative idempotent semigroups in the language $L$ with countable set of constants. It is important to note that all algebras in the language $L$ are $\mathcal{E}$-compact, by Lemma \ref{lem:4.9}.

Let us return to the construction given by B.I. Plotkin. He denotes by $H$ the discrete direct product of all finitely generated groups (in the language of groups $L = \{\cdot, -1, e\}$). Since every finitely generated group $G$ imbeds into $H$, then $G$ is a coordinate group over $H$. By \ref{thm:4.1} below, $H$ is $q_\omega$-compact. As there exists a finitely generated group $G$ that is not finitely presented, hence $H$ is not equationally Noetherian.

It is evident that this construction of $H$ may be repeated in other varieties of algebras, where exist finitely generated, not finitely presented algebras. Clearly, the algebraic geometry over objects like $H$ is quite elementary.

7 $q_\omega$-compact and $u_\omega$-compact extensions

In Introduction it is given the formulation of the Compactness Theorem and the notion of logical compactness. The Compactness Theorem has a great importance in model theory \cite{13}.

For an arbitrary algebra $B$ it is possible with a use of the Compactness Theorem to construct an elementary extension $B^\ast$ of $B$ such that $B^\ast$ is logically compact. This algorithm is close to the building of the algebraic closure to a given field $k$.

We use this idea to construct $u_\omega$-compact elementary extension for an arbitrary algebra $B$. At first, let us remind some more facts from model theory.

**Theorem 7.1** (Corollary from Los’ Theorem \cite{6}). If $B^I/D$ is an ultrapower of an algebra $B$ then the diagonal map $d: B \to B^I/D$, where $d(x) = \bar{x}/D$ and $\bar{x}(i) = x$ for all $i \in I$, is an elementary embedding.

**Proposition 7.2** (\cite{20}). Suppose that $(I, <)$ is a linear order and $(M_i, i \in I)$ is an elementary chain. Then $M = \bigcup_{i \in I} M_i$ is an elementary extension of each $M_i$. 

40
Denote by $T$ the family of all sets of formulas

$$T = S \cup \{ \neg(t_1 = s_1), \ldots, \neg(t_m = s_m) \},$$

where $S \subseteq \text{At}_L(X)$, $(t_1 = s_1), \ldots, (t_m = s_m) \in \text{At}_L(X)$, $|X| < \infty$. For a given L-algebra $B$ let $\mathbb{T}(B)$ be such subfamily of $T$ that $T \in \mathbb{T}(B)$ if and only if $T$ is finitely satisfiable in $B$ but not realized in $B$. So, algebra $B$ is $u_\omega$-compact if and only if $\mathbb{T}(B) = \emptyset$.

For L-algebras $B$ and $C$ we write $B \equiv C$ if $B$ and $C$ are universally equivalent, i.e., $\text{Ucl}(B) = \text{Ucl}(C)$.

**Lemma 7.3.** Let $B$ and $C$ be L-algebras and $B \leq C$. If $B \equiv C$ then $\mathbb{T}(C) \subseteq \mathbb{T}(B)$.

**Proof.** Suppose $T \in \mathbb{T}$ and $T$ is finitely satisfiable in $C$. Then, by Lemma 4.17, $T$ is finitely satisfiable in $B$. If $T \notin \mathbb{T}(B)$, then $T$ is realized in $B$. As $B \leq C$, then $T$ is realized in $C$ and $T \notin \mathbb{T}(C)$. \hfill \Box

**Theorem 7.4.** Let $B$ be an L-algebra. Then there exists an elementary extension $B^*$ of $B$, such that $B^*$ is $u_\omega$-compact (in particularly, $B^*$ is $q_\omega$-compact).

**Proof.** Consider a well-ordering $(I, <)$ on $\mathbb{T}(B)$. Let us construct an elementary chain $(B_i, i \in I)$. At first, take $B_0 = B$. Then $B_1$ is an ultrapower of $B$ where $T_0$ is realized. By Compactness Theorem, such $B_1$ exists and, by Theorem 7.1, $B_1$ is an elementary extension of $B$. Further, $B_2$ is an ultrapower of $B_1$ where $T_1$ is realized, and so on. For an ordinal $\alpha = \beta + 1$ we put $B_\alpha$ as an ultrapower of $B_\beta$ where $T_\beta$ is realized, and $B_\alpha = \bigcup_{\beta < \alpha} B_\beta$ for a limit ordinal $\alpha$. Desired algebra $B^*$ is $\bigcup_{i \in I} B_i$. Indeed, $B^*$ is an elementary extension of $B$, by Theorem 7.1 and Proposition 7.2.

Let us show that $B^*$ is $u_\omega$-compact. By Lemma 7.3, $\mathbb{T}(B^*) \subseteq \mathbb{T}(B)$. Every set of formulas $T$ from $\mathbb{T}(B)$ is realized in $B^*$. So $\mathbb{T}(B^*) = \emptyset$. \hfill \Box

**Corollary 7.5.** For an arbitrary algebra $B$ there exists $u_\omega$-compact algebra $B^*$ which is elementary equivalent to $B$.

In Theorem 7.4 we constructed $u_\omega$-compact extension $B^*$ of $B$ such that $B^*$ is elementary equivalent to $B$. One can modify the idea of Theorem 7.3 and find more constructive $u_\omega$-compact extension $B$ which is universally equivalent to $B$.
Proposition 7.6. Let $B$ be an $L$-algebra. Then there exists an extension $C$ of $B$ such that $C$ is $u_\omega$-compact and $C \equiv \forall B$. Moreover, one can get $C$ by (transfinite) induction in series of extensions \[ B = C_0 < C_1 < C_2 \ldots, \]
where $C_{\beta+1}$ is finitely generated extension of $C_\beta$, and $C_\alpha = \bigcup_{\beta < \alpha} C_\beta$ is the union of the chain for a limit ordinal $\alpha$. Also $C_\alpha \equiv \forall B$ for all $\alpha$.

Proof. Let us construct $C$ by means of transfinite induction on $|T(B)|$. Take $C_0 = B$. Consider an algebra $B_1$ where $T_0$ is realized. Let $b_1, \ldots, b_n \in B_1$ be elements such that $B_1 \models T(b_1, \ldots, b_n)$. Put $C_1$ as the subalgebra of $B_1$ generated by subalgebra $B$ and elements $b_1, \ldots, b_n$. And so on.

If $\alpha = \beta + 1$ then we take $B_\alpha$ as an ultrapower of $C_\beta$ where $T_\beta$ is realized, and $C_\alpha$ is subalgebra of $B_\alpha$ generated by $C_\beta$ and finite set of element in $B_\alpha$ which realize formulas from $T_\beta$. It is easy that $C_\alpha \equiv \forall C_\beta$.

For a limit ordinal $\alpha$ we put $C_\alpha = \bigcup_{\beta < \alpha} C_\beta$ as the union of the chain $(C_\beta, \beta < \alpha)$. In this case $C_\alpha = \lim\rightarrow C_\beta$ is also the direct limit of the direct system $(C_\beta, \beta < \alpha)$, therefore $C_\alpha \in \text{Ucl}(\{C_\beta, \beta < \alpha\})$ [6, Theorem 1.2.9]. Since $C_\beta < C_\alpha$ we have $C_\beta \in \text{Ucl}(C_\alpha)$ for all $\beta < \alpha$. By induction, $C_\beta \equiv \forall C_\gamma$ for any $\beta, \gamma < \alpha$. Therefore, $C_\alpha \equiv \forall C_\beta$ for every $\beta < \alpha$.

At the end of such process we get an extension $C$ of $B$ such that $C \equiv \forall B$ and $T(C) = \emptyset$, i.e., $C$ is $u_\omega$-compact.

The following results are also useful in universal algebraic geometry.

Lemma 7.7. Let $B, C$ be an $L$-algebras. Suppose that $B$ is $q_\omega$-compact, $C \in \text{Qvar}(B)$, and every finitely generated subalgebra $B_0 < B$ is separated by $C$. Then $C$ is $q_\omega$-compact and and $\text{Qvar}(B) = \text{Qvar}(C)$.

Lemma 7.8. Let $B, C$ be an $L$-algebras. Suppose that $B$ is $u_\omega$-compact, $C \in \text{Ucl}(B)$, and every finitely generated subalgebra $B_0 < B$ is discriminated by $C$. Then $C$ is $u_\omega$-compact and $\text{Ucl}(B) = \text{Ucl}(C)$.
Proof. We prove only statement about \( u_\omega \)-compactness. Statement about \( q_\omega \)-compactness may be proven in much the same way. By Theorem 4.2, it is sufficient to show that \( \text{Ucl}(C) \subseteq \text{Dis}(C) \) (inclusion \( \text{Ucl}(C) \supseteq \text{Dis}(C) \) holds anyway). As \( C \in \text{Ucl}(B) \) then \( \text{Ucl}(C) \subseteq \text{Ucl}(B) \) and \( \text{Ucl}(C) \subseteq \text{Ucl}(B) \). Since \( B \) is \( u_\omega \)-compact we have \( \text{Ucl}(B) = \text{Dis}(B) \). If every finitely generated subalgebra \( B_0 < B \) is discriminated by \( C \) then \( \text{Dis}(B) \subseteq \text{Dis}(C) \). Therefore, \( \text{Ucl}(C) \subseteq \text{Dis}(C) \), as desired. Also we got \( \text{Ucl}(B) = \text{Ucl}(C) \) that implies \( \text{Ucl}(B) = \text{Ucl}(C) \).

For \( L \)-algebra \( A \) we denote by \( L_A = L \cup \{ c_a \mid a \in A \} \) the language \( L \) extended by elements from \( A \) as new constant symbols [3, subsection 3.4]. An algebra \( M \) in \( L_A \) is called \( A \)-algebra if the map \( h: A \to M, h(a) = c_a^M, a \in A \), is embedding.

**Proposition 7.9.** Let \( A \) be an \( L \)-algebra. Consider \( A \) as \( A \)-algebra. If \( A \) is \( q_\omega \)-compact (in the language \( L_A \)) then every \( A \)-algebra \( C \) from \( \text{Qvar}_A(A) \) is \( q_\omega \)-compact. If \( A \) is \( u_\omega \)-compact (in the language \( L_A \)) then every \( A \)-algebra \( C \) from \( \text{Ucl}_A(A) \) is \( u_\omega \)-compact.

**Proof.** We use here Lemmas 7.7 and 7.8. Every finitely generated subalgebra \( A_0 \) of \( A \) is an \( L_A \)-algebra, so \( A_0 = A \). Since \( C \) is \( A \)-algebra then \( A \) is separated and discriminated by \( C \) in the obvious way. Thus, we have obtained the looked-for result.

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Algebraic geometry over algebraic structures III: Equationally Noetherian property and compactness

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Abstract

In this paper we discuss some special generalizations of equationally Noetherian property which naturally arise in the universal algebraic geometry. We introduce weakly equationally Noetherian, \( q_\omega \)-compact, \( u_\omega \)-compact, and weakly \( u_\omega \)-compact algebras and then examine properties of such algebras. Also we consider the connections between five classes: the class of equationally Noetherian algebras, the class of weakly equationally Noetherian algebras, the class of \( u_\omega \)-compact algebras, the class of weakly \( u_\omega \)-compact algebras, and the class of \( q_\omega \)-compact algebras.

Keywords: Compactness Theorem, universal closure, quasivariety, algebraic structure, algebraic set, coordinate algebra, (weakly) \( u_\omega \)-compact algebra, \( q_\omega \)-compact algebra, (weakly) equationally Noetherian algebra, logically irreducible set.

Mathematics Subject Classification: 03C05+14A99+08B05

Contents

1 Introduction 2

2 Preliminaries 7
1 Introduction

This paper deals with the universal algebraic geometry. The universal algebraic geometry is a young branch of mathematics. The subject of universal algebraic geometry lies in the solutions of systems of equations over an arbitrary algebraic structure.

Investigations in universal algebraic geometry were started in works by B.I. Plotkin [22, 23, 24] and papers on algebraic geometry over groups by G. Baumslag, O.G. Kharlampovich, A.G. Myasnikov, and V.N. Remeslennikov [14, 15, 16, 17]. After that there were a lot of papers on algebraic geometry over concrete groups, algebras, monoids and so on. Among them there are the famous works by O.G. Kharlampovich,
A. G. Myasnikov [14, 15, 16, 17] and Z. Sela [25, 26, 27] on algebraic geometry over free groups.

In recent years we have achieved more general and systematic point of view on the universal algebraic geometry as on a formalized theory. In this respect we have started a series of works on universal algebraic geometry. This paper is the third one of that series along with [3, 4].

According to [6, 19, 20], in [3] we give a framework of universal algebra and model theory as much as we need it in universal algebraic geometry. At the same time we discuss how notions and ideas from model theory work in universal algebraic geometry. In [4] we introduce the foundation of universal algebraic geometry, basic definitions and constructions of the algebraic geometry over an arbitrary algebraic structure \( B \).

This paper is supposed to be read after the previous ones [3, 4], however for the sake of convenience we present in here some of the most essential notations and definitions (see Section 2).

We consider only first-order functional languages (signatures). Recall that algebraic structures in a functional language are called \( \textit{algebras} \). Typically we denote algebraic structures by capital calligraphic letters (\( A, B, C, \ldots \)) and their universes by the corresponding capital Latin letters (\( A, B, C, \ldots \)).

The main results of papers [3, 4] are so-called the Unification Theorems (Theorem A and Theorem C) which give a description of coordinate algebras by means of several languages.

**Theorem A.** Let \( B \) be an equationally Noetherian algebra in a functional language \( L \). Then for a finitely generated algebra \( C \) of \( L \) the following conditions are equivalent:

1) \( \text{Th}_\forall(B) \subseteq \text{Th}_\forall(C) \), i.e., \( C \in \text{Ucl}(B) \);

2) \( \text{Th}_\exists(B) \supseteq \text{Th}_\exists(C) \);

3) \( C \) embeds into an ultrapower of \( B \);

4) \( C \) is discriminated by \( B \);
5) C is a limit algebra over B;

6) C is an algebra defined by a complete atomic type in the theory \( \text{Th}_\forall(B) \) in \( L \);

7) C is the coordinate algebra of an irreducible algebraic set over \( B \) defined by a system of equations in the language \( L \).

**Theorem C.** Let \( B \) be an equationally Noetherian algebra in a functional language \( L \). Then for a finitely generated algebra \( C \) of \( L \) the following conditions are equivalent:

1) \( C \in \text{Qvar}(B) \), i.e., \( \text{Th}_{qi}(B) \subseteq \text{Th}_{qi}(C) \);

2) \( C \in \text{Pvar}(B) \);

3) \( C \) embeds into a direct power of \( B \);

4) \( C \) is separated by \( B \);

5) \( C \) is a subdirect product of a finitely many limit algebras over \( B \);

6) \( C \) is an algebra defined by a complete atomic type in the theory \( \text{Th}_{qi}(B) \) in \( L \);

7) \( C \) is the coordinate algebra of an algebraic set over \( B \) defined by a system of equations in the language \( L \).

Note that items 5) in both Theorem A and Theorem C give a description of coordinate algebras by means of limit algebras. The limit algebraic structures (groups, as the rule) become the object of intense interest in modern algebra \[2, 5, 7, 8, 9, 10, 11\].

Theorems A and C are formulated for so-called equationally Noetherian algebras (the definition see in Section\[2\]). Equationally Noetherian algebras possess the best opportunity to study the algebraic geometry over them. If a given algebra \( B \) is equationally Noetherian then we have an advantage when investigating the algebraic geometry over \( B \). In this case we may use:

(i) Unification Theorems;
(ii) the decomposition of any algebraic set over $B$ into a finite union of irreducible algebraic sets (Theorem 5.11 below);

(iii) the possibility to study only finite system of equations;

(iv) and some more results [3, 4].

In the case when a given algebra $B$ is not equationally Noetherian we lose some results for equationally Noetherian algebras, while some of them may remain in force. In this paper we introduce four generalizations of the equationally Noetherian property which naturally arise in universal algebraic geometry. These are

(N'): weak equationally Noetherian property that retains (iii);

(Q): $q_\omega$-compactness that retains Unification Theorem C;

(U): $u_\omega$-compactness that retains Unification Theorems A and C;

(U'): and weak $u_\omega$-compactness that retains (iv), namely, some weak form of Unification Theorem A.

We denote by $N$ the class of all equationally Noetherian algebras in a given functional language $L$. By $N'$, $Q$, $U$, $U'$, correspondingly, we denote the classes of algebras with properties above. The picture of connections between classes $Q$, $U$, $U'$, $N'$ and $N$ is presented in Section 6

There exist several equivalent approaches to $q_\omega$- and $u_\omega$-compact algebras. We introduce them in Section 4. One of these approaches rises from some ideas of model theory. It relates to the Compactness Theorem and the notion of compact algebra.

Recall that a set of formulas $T$ in a language $L$ is called satisfiable in a class $K$ of algebraic structures in $L$ (or $T$ is realized in $K$) if one can assign some elements from a particular algebraic structure from $K$ as values to the variables which occur in $T$ in such a way that all formulas from $T$ become true. The set $T$ is called finitely satisfiable in $K$ if every finite subset of $T$ is realized in $K$. 
Compactness Theorem (K. F. Gödel, A. I. Malcev [6]). If a set of first-order formulas $T$ in a language $L$ is finitely satisfiable in a class $K$ of algebraic structures in $L$, then $T$ is satisfiable in an ultraproduct of structures from $K$.

Class $K$ is called compact if every finitely satisfiable in $K$ set of formulas $T$ is satisfiable in $K$. This definition occurs in the book by V. A. Gorbunov [6]. It is natural to name an algebraic structure $B$ compact if the class $\{B\}$ is compact. However, according to W. Hodges [13], algebraic $L$-structure is called compact if its universe is a Hausdorff topological space, in such a way that each function from $L$ is interpreted by a continuous function. The same algebraic structures appear in [6] under the name of topologically compact structures.

Trying to avoid an ambiguity we call an $L$-algebra $B$ logically compact if every finitely satisfiable in $B$ set of formulas $T$ in the language $L$ is satisfiable in $B$. When we modify this definition and consider only special types of sets of formulas $T$ we get definitions of special compactness, such as $q_\omega$- and $u_\omega$-compactness. Short review of the history of “$q_\omega$-compact” notion is represented in Subsection 4.1.

First and foremost in this article we generalize the Unification Theorems to $u_\omega$- and $q_\omega$- compact algebras. In Section 3 we give geometric definitions of $u_\omega$- and $q_\omega$-compactness. In Subsection 3.1 we prove that Theorem A is true for any $u_\omega$-compact algebra $B$ and every algebra $B$ which satisfies Theorem A is $u_\omega$-compact. The similar result that connects $q_\omega$-compact algebras and Theorem C is presented in Subsection 3.2. In Subsection 3.3 for weakly $u_\omega$-compact algebras we formulate and prove a weak analog of Theorem A.

Section 4 is devoted to $q_\omega$- and $u_\omega$-compact algebras. In Subsection 4.1 we put definitions of $q_\omega$- and $u_\omega$-compact algebras in different equivalent forms and prove the equivalence of them in Subsection 4.4. For $u_\omega$-compact algebra $B$ Unification Theorems give a global view to all (irreducible) coordinate algebras over $B$. However, it may happen that one has no $u_\omega$-compact property but some “local $u_\omega$-compact property” which gives result of Theorem A for a certain algebra $C$ (not for all $C$). This idea is developed in Subsection 4.3.

In Section 5 we discuss weak properties: weak equationally Noetherian property (Sub-...
section 5.1) and weak $u_\omega$-compactness (Subsection 5.3). In Subsection 5.2 we introduce logically irreducible algebraic sets. Those sets naturally arise as generalization of irreducible ones. In particularly, we show that the notions of irreducible algebraic set and logically irreducible algebraic set over an algebra $B$ coincide if and only if $B$ is weakly $u_\omega$-compact.

In the last Section 7 we continue discussion about connections between $u_\omega$- and $q_\omega$-compact algebras with the Compactness Theorem and corresponding technique from the model theory. By the way, we construct $u_\omega$-compact elementary extension for an arbitrary algebra $B$.

2 Preliminaries

In this section we remind basic notions and facts from universal algebraic geometry according to [3, 4].

Let $L$ be a first-order functional language, $X = \{x_1, x_2, \ldots, x_n\}$ a finite set of variables, $T_L(X)$ the set of all terms of $L$ with variables in $X$, $T_L(X)$ the absolutely free $L$-algebra with basis $X$ and $A_L(X)$ the set of all atomic formulas of $L$ with variables in $X$.

In universal algebraic geometry atomic formulas from $A_L(X)$ are named equations in $L$ and subsets $S \subseteq A_L(X)$ are named systems of equations in the language $L$.

For a system of equations $S \subseteq A_L(X)$ and an algebra $B$ in the language $L$ we denote by $V_B(S)$ the set of all solutions of $S$ in $B$:

$$V_B(S) = \{(b_1, \ldots, b_n) \in B^n \mid B \models (t(b_1, \ldots, b_n) = s(b_1, \ldots, b_n)) \quad \forall \ (t = s) \in S\}.$$  

It is called the algebraic set over $B$ defined by the system $S$. If $S$ contains of only one equation $(t = s)$ we write $V_B(t = s)$ instead of $V_B(\{(t = s)\})$.

Algebraic set is irreducible if it is not a finite union of proper algebraic subsets; otherwise it is reducible. The empty set is not considered to be irreducible. Hence, according to R. Hartshorne [12], all irreducible algebraic sets are non-empty in our paper.

Two systems $S_1, S_2 \subseteq A_L(X)$ are equivalent over $B$ if $V_B(S_1) = V_B(S_2)$. The radical $\text{Rad}_B(S)$ of a system of equations $S \subseteq A_L(X)$ is the maximal system which is equivalent
to $S$ over $B$. It is also called the radical of algebraic set $Y = V_B(S)$ and denoted by Rad($Y$). By $[S]$ we denote the congruent closure of $S$, i.e., the least congruent subset of $A_{TL}(X)$ that contains $S$.

By $\Phi_{qf,L}(X)$ we denote the set of all quantifier-free formulas in $L$ with variables in $X$. We say that a formula $\phi \in \Phi_{qf,L}(X)$ is a consequence of a system of equations $S \subseteq A_{TL}(X)$ over an $L$-algebra $B$, if $B \models \phi(b_1,\ldots,b_n)$ for all $(b_1,\ldots,b_n) \in V_B(S)$. For example, an atomic formula $(t = s)$, $t,s \in T_L(X)$, is a consequence of $S$ over $B$ if and only if $(t = s) \in \text{Rad}_B(S)$.

For an arbitrary algebraic set $Y \subseteq B^n$ over $B$ the radical $\text{Rad}(Y)$ defines the congruence $\theta_{\text{Rad}(Y)}$ on $T_L(X)$:

$$t_1 \sim_{\theta_{\text{Rad}(Y)}} t_2 \iff (t_1 = t_2) \in \text{Rad}(Y), \quad t_1, t_2 \in T_L(X).$$

The factor-algebra $\Gamma(Y) = T_L(X)/\theta_{\text{Rad}(Y)}$ is called the coordinate algebra of the algebraic set $Y$.

Let $Y \subseteq B^n$ and $Z \subseteq B^m$ be algebraic sets over $B$. One has $\Gamma(Y) \cong \Gamma(Z)$ if and only if algebraic sets $Y$ and $Z$ are isomorphic (we omit here the definition of isomorphism between algebraic sets). Isomorphic algebraic sets are irreducible and reducible simultaneously.

We say that an $L$-algebra $C$ is a coordinate algebra over $B$ if $C \cong \Gamma(Y)$ for some algebraic set $Y$ over $B$, and $C$ is an irreducible coordinate algebra over $B$ if $C \cong \Gamma(Y)$ for some irreducible algebraic set $Y$ over $B$.

One of the principal goals of algebraic geometry over a given algebraic structure $B$ is the problem of classification of algebraic sets over $B$ up to isomorphism. This problem is equivalent to the problem of classification of coordinate algebras of algebraic sets over $B$. Also it is important to classify coordinate algebras of irreducible algebraic sets over $B$. Formulated in Introduction Unification Theorems A and C are very useful for solution of those problems.

In Theorems A and C we claim an algebra $B$ is equationally Noetherian. Thus, let us remind that an $L$-algebra $B$ is called equationally Noetherian, if for every finite set $X$ and every system of equations $S \subseteq A_{TL}(X)$ there exists a finite subsystem $S_0 \subseteq S$ such that $V_B(S_0) = V_B(S)$. Properties of equationally Noetherian algebras are discussed in [3, 4].
An L-algebra \( \mathcal{C} \) is \textit{separated} by L-algebra \( \mathcal{B} \) if for any pair of non-equal elements \( c_1, c_2 \in \mathcal{C} \) there is a homomorphism \( h: \mathcal{C} \to \mathcal{B} \) such that \( h(c_1) \neq h(c_2) \). An algebra \( \mathcal{C} \) is \textit{discriminated} by \( \mathcal{B} \) if for any finite set \( W \) of elements from \( \mathcal{C} \) there is a homomorphism \( h: \mathcal{C} \to \mathcal{B} \) whose restriction onto \( W \) is injective. We are interested in a familiar form of results, so it is useful to put by definition that the trivial algebra \( \mathcal{E} \) is separated by an algebra \( \mathcal{B} \) anyway, and \( \mathcal{E} \) is discriminated by \( \mathcal{B} \) if and only if \( \mathcal{B} \) has a trivial subalgebra.

The definitions of limit algebras and algebras defined by complete atomic types need a large introduction, so we omit them (see \[3\]).

In this paper we use some operators which image a class \( \mathcal{K} \) of L-algebras into another one. For the sake on convenience we collect here the list of all these operators:

- \( S(\mathcal{K}) \) — the class of subalgebras of algebras from \( \mathcal{K} \);
- \( P(\mathcal{K}) \) — the class of direct products of algebras from \( \mathcal{K} \);
- \( P_\omega(\mathcal{K}) \) — the class of finite direct products of algebras from \( \mathcal{K} \);
- \( P_s(\mathcal{K}) \) — the class of subdirect products of algebras from \( \mathcal{K} \);
- \( P_f(\mathcal{K}) \) — the class of filterproducts of algebras from \( \mathcal{K} \);
- \( P_u(\mathcal{K}) \) — the class of ultraproducts of algebras from \( \mathcal{K} \);
- \( L_\rightarrow(\mathcal{K}) \) — the class of direct limits of algebras from \( \mathcal{K} \);
- \( L_\varphi(\mathcal{K}) \) — the class of epimorphic direct limits of algebras from \( \mathcal{K} \);
- \( L_{fg}(\mathcal{K}) \) — the class of algebras in which all finitely generated subalgebras belong to \( \mathcal{K} \);
- \( P\text{var}(\mathcal{K}) \) — the least prevariety including \( \mathcal{K} \);
- \( Q\text{var}(\mathcal{K}) \) — the least quasi-variety including \( \mathcal{K} \), i.e., \( Q\text{var}(\mathcal{K}) = \text{Mod}(\text{Th}_\text{q}(\mathcal{K})) \);
- \( \text{Ucl}(\mathcal{K}) \) — the universal class of algebras generated by \( \mathcal{K} \), i.e., \( \text{Ucl}(\mathcal{K}) = \text{Mod}(\text{Th}_\forall(\mathcal{K})) \);
- \( \text{Res}(\mathcal{K}) \) — the class of algebras which are separated by \( \mathcal{K} \);
- \( \text{Dis}(\mathcal{K}) \) — the class of algebras which are discriminated by \( \mathcal{K} \);
- \( \mathcal{K}_e \) — the addition of the trivial algebra \( \mathcal{E} \) to \( \mathcal{K} \), i.e., \( \mathcal{K}_e = \mathcal{K} \cup \{ \mathcal{E} \} \);
- \( \mathcal{K}_\omega \) — the class of finitely generated algebras from \( \mathcal{K} \).

Here we denote by \( \text{Th}_\text{q}(\mathcal{K}) \) (correspondingly, \( \text{Th}_\forall(\mathcal{K}), \text{Th}_\exists(\mathcal{K}) \)) the set of all quasi-identities (correspondingly, universal sentences, existential sentences) which are true in all structures from \( \mathcal{K} \).
For an arbitrary class $K$ of $L$-algebras one has:

$$\text{Ucl}(K) = SP_u(K), \text{ Dis}(K) \subseteq \text{Ucl}(K),$$

$$\text{Res}(K) = \text{Pvar}(K) = SP(K), \text{ Pvar}(K) \subseteq \text{Qvar}(K).$$

According to Gorbunov [6] and in contrast to [3], we assume that the direct product for the empty set of indexes coincides with the trivial $L$-algebra $\mathcal{E}$. In particular, when we say that an algebra $C$ is a finite direct product of algebras from $K$ (or a subdirect product of a finitely many algebras from $K$) then $C$ may be just the trivial algebra. However, while defining an filterproduct we assume that the set of indexes is non-empty.

3 Generalizations of the Unification Theorems

Unification Theorems A and C are formulated in Introduction above for an equationally Noetherian algebra $B$. Those theorems have been proven in [3, 4].

**Question:** Suppose that the algebra $B$ is not equationally Noetherian. When Unification Theorems remain true for $B$?

To answer this question we need to analyze the proofs of Theorems A and C. As it was mentioned in [4], for the reasoning of some implications in Theorems A and C the equationally Noetherian property is not required, namely, one has the following remark.

**Remark 3.1.** Let $B$ be an algebra in a functional language $L$ and $C$ a finitely generated $L$-algebra. Then

- $C$ is the coordinate algebra of an irreducible algebraic set over $B$ defined by a system of equations in the language $L$ **IF AND ONLY IF** $C$ is discriminated by $B$ (Theorem A: $7 \iff 4$);

- **IF** $C$ is discriminated by $B$ **THEN** $C$ is a limit algebra over $B$ (Theorem A: $4 \implies 5$);

- $C$ is the coordinate algebra of an algebraic set over $B$ defined by a system of equations in the language $L$ **IF AND ONLY IF** $C \in \text{Pvar}(B)$ (Theorem C: $7 \iff 2$);
• IF $C$ is a subdirect product of a finitely many limit algebras over $B$ THEN $C \in \text{Qvar}(B)$ (Theorem C: $5 \implies 1$); and so on.

The complete set of implications in Theorems A and C which always remain true is represented as follows:

Theorem A: $\{4 \iff 7\} \implies \{1 \iff 2 \iff 3 \iff 5 \iff 6\}$;

Theorem C: $\{5\} \implies \{1 \iff 6\} \iff \{2 \iff 3 \iff 4 \iff 7\}$.

Further, when proving $1) \implies 4$) in both Theorems A and C, we use not equationally Noetherian property itself, but some weaker properties. What properties exactly? These are $u_\omega$-compactness and $q_\omega$-compactness.

**Definition 3.2.** We say $L$-algebra $B$ is $q_\omega$-compact if for any finite set $X$, any system of equations $S \subseteq \text{At}_L(X)$, and any equation $(t_0 = s_0) \in \text{At}_L(X)$ such that

$$V_B(S) \subseteq V_B(t_0 = s_0)$$

there exists a finite subsystem $S_0 \subseteq S$ such that

$$V_B(S) \subseteq V_B(S_0) \subseteq V_B(t_0 = s_0).$$

Here the finite subsystem $S_0$ may alter depending on equation $(t_0 = s_0)$.

**Definition 3.3.** An $L$-algebra $B$ is termed $u_\omega$-compact if for any finite set $X$, any system of equations $S \subseteq \text{At}_L(X)$, and any equations $(t_1 = s_1), \ldots, (t_m = s_m) \in \text{At}_L(X)$ such that

$$V_B(S) \subseteq V_B(t_1 = s_1) \cup \ldots \cup V_B(t_m = s_m)$$

there exists a finite subsystem $S_0 \subseteq S$ such that

$$V_B(S) \subseteq V_B(S_0) \subseteq V_B(t_1 = s_1) \cup \ldots \cup V_B(t_m = s_m).$$

Here the finite subsystem $S_0$ may alter depending on equations $(t_1 = s_1), \ldots, (t_m = s_m)$.

It is clear that any equationally Noetherian algebra $B$ is $u_\omega$-compact, and any $u_\omega$-compact algebra is $q_\omega$-compact.

The definitions of $u_\omega$-compactness and $q_\omega$-compactness above are given in geometric form. We know some other approaches to these notions that will be discussed in Section 4. In that section will be also represented the etymology of the notion of $u_\omega(q_\omega)$-compactness.
3.1 The generalization of Unification Theorem A

The significance of $u_\omega$-compact algebras in universal algebraic geometry is shown in the following theorem.

**Theorem 3.4** (analog of Theorem A). *Let $\mathcal{B}$ be $u_\omega$-compact algebra in a functional language $L$. Then for a finitely generated algebra $\mathcal{C}$ of $L$ the following conditions are equivalent:*

1) $\text{Th}_\forall(\mathcal{B}) \subseteq \text{Th}_\forall(\mathcal{C})$, i.e., $\mathcal{C} \in \text{Ucl}(\mathcal{B})$;

2) $\text{Th}_3(\mathcal{B}) \supseteq \text{Th}_3(\mathcal{C})$;

3) $\mathcal{C}$ embeds into an ultrapower of $\mathcal{B}$;

4) $\mathcal{C}$ is discriminated by $\mathcal{B}$;

5) $\mathcal{C}$ is a limit algebra over $\mathcal{B}$;

6) $\mathcal{C}$ is an algebra defined by a complete atomic type in the theory $\text{Th}_\forall(\mathcal{B})$ in $L$;

7) $\mathcal{C}$ is the coordinate algebra of an irreducible algebraic set over $\mathcal{B}$ defined by a system of equations in the language $L$.

Moreover, if for an $L$-algebra $\mathcal{B}$ and for all finitely generated $L$-algebras $\mathcal{C}$ the conditions above are equivalent then $\mathcal{B}$ is $u_\omega$-compact.

**Proof.** It follows from Remark 3.1 that conditions 1)–7) are equivalent if and only if one has equivalence 1) $\iff$ 4). The latter means that a finitely generated algebra $\mathcal{C}$ is discriminated by $\mathcal{B}$ if and only if $\mathcal{C} \in \text{Ucl}(\mathcal{B})$, i.e., $\text{Ucl}(\mathcal{B})_\omega = \text{Dis}(\mathcal{B})_\omega$. By Theorem 4.2 below, one has the equality $\text{Ucl}(\mathcal{B})_\omega = \text{Dis}(\mathcal{B})_\omega$ if and only if an algebra $\mathcal{B}$ is $u_\omega$-compact. 

3.2 The generalization of Unification Theorem C

To prove an analog of Theorem C for $q_\omega$-compact algebras we need the following results.
Lemma 3.5 ([3]). Let $\mathcal{C}$ be a limit algebra over an $\mathcal{L}$-algebra $\mathcal{B}$. Then there exists an ultrapower $\mathcal{B}^*$ of $\mathcal{B}$ such that $\mathcal{C}$ embeds into $\mathcal{B}^*$.

Lemma 3.6 ([4]). A finitely generated $\mathcal{L}$-algebra $\mathcal{C}$ is the coordinate algebra of an algebraic set over $\mathcal{L}$-algebra $\mathcal{B}$ if and only if $\mathcal{C}$ is a subdirect product of the coordinate algebras of irreducible algebraic sets over $\mathcal{B}$.

Theorem 3.7 (analog of Theorem C). Let $\mathcal{B}$ be $q_\omega$-compact algebra in a functional language $\mathcal{L}$. Then for a finitely generated algebra $\mathcal{C}$ of $\mathcal{L}$ the following conditions are equivalent:

1) $\mathcal{C} \in Qvar(\mathcal{B})$, i.e., $Th_{qi}(\mathcal{B}) \subseteq Th_{qi}(\mathcal{C})$;

2) $\mathcal{C} \in Pvar(\mathcal{B})$;

3) $\mathcal{C}$ embeds into a direct power of $\mathcal{B}$;

4) $\mathcal{C}$ is separated by $\mathcal{B}$;

5') $\mathcal{C}$ is a subdirect product of limit algebras over $\mathcal{B}$;

6) $\mathcal{C}$ is an algebra defined by a complete atomic type in the theory $Th_{qi}(\mathcal{B})$ in $\mathcal{L}$;

7) $\mathcal{C}$ is the coordinate algebra of an algebraic set over $\mathcal{B}$ defined by a system of equations in the language $\mathcal{L}$.

Moreover, if for an $\mathcal{L}$-algebra $\mathcal{B}$ and for all finitely generated $\mathcal{L}$-algebras $\mathcal{C}$ the conditions above are equivalent then $\mathcal{B}$ is $q_\omega$-compact.

Proof. By Remark 3.1 it is sufficient to prove implications 1) $\implies$ 2), 5') $\implies$ 1), and 7) $\implies$ 5') for $q_\omega$-compact algebra $\mathcal{B}$. By Theorem 4.1 below, we have the identity $Qvar(\mathcal{B})_\omega = Pvar(\mathcal{B})_\omega$ that gives proof of 1) $\implies$ 2). For implication 5') $\implies$ 1) we refer to Lemma 3.5 and the fact that every quasi-variety is closed under ultraproducts, direct products and subalgebras.

For proving 7) $\implies$ 5') suppose that $\mathcal{C}$ is the coordinate algebra of an algebraic set over $\mathcal{B}$. By Lemma 3.6, $\mathcal{C}$ is a subdirect product of coordinate algebras of irreducible algebraic
sets over $B$. By Remark 3.1 (Theorem A: $7 \implies 5$), coordinate algebras of irreducible algebraic sets over $B$ are limit algebras over $B$.

Suppose now that for some L-algebra $B$ we have equivalence $1) \iff 2)$ for all finitely generated L-algebras $C$. It means that $Q\text{var}(B)_{\omega} = P\text{var}(B)_{\omega}$ and, by Theorem 4.1 below, the algebra $B$ is $q_{\omega}$-compact.

**Remark 3.8.** Unfortunately, we are not in a position to formulate Theorem C for $q_{\omega}$-compact algebras in all its fullness, because item 5) essentially needs equationally Noetherian property. We have to weak 5), namely we should erase words “finitely many”.

To establish Remark 3.8 we formulate the following problem.

**Embedding Problem.** Let $B$ be $q_{\omega}$-compact algebra in a functional language $L$. The question: whether or not every coordinate algebra over $B$ subdirectly embeds into a finite direct product of algebras from $U\text{cl}(B)$? If the answer is “not”, then we ask whether or not the same holds for at least $u_{\omega}$-compact algebras.

A. N. Shevlyakov in [28] gives the negative answer to the Embedding Problem both for $q_{\omega}$-compact and $u_{\omega}$-compact algebras.

Let us put an addition to Remark 3.1.

**Remark 3.9.** The following implications and equivalencies from Theorem 3.7 hold for an arbitrary algebra $B$:

$$
\begin{align*}
\{1 \iff 6\} & \iff \{2 \iff 3 \iff 4 \iff 7\} \\
\{5\}' & \iff \{5\}'
\end{align*}
$$

Theorem 3.7 gives a classification of coordinate algebras in terms of quasivarieties. Thereby, any characterizations of quasivariety $Q\text{var}(K)$ of a class $K$ of L-algebras are helpful in universal algebraic geometry. In [6, 19] one can find the identities:

$$
Q\text{var}(K) = SP_{t}(K)_{e} = SPP_{u}(K) = SP_{u}P(K) = SP_{u}P_{\omega}(K) = \\
= SL_{p}P(K) = L_{p}SP(K) = L_{p}P_{n}(K) = L_{p}SP(K).
$$
3.3 Weak generalization of Unification Theorem A

Let $\mathcal{B}$ be an algebra in a functional language $L$. Let us consider the class $\text{Ucl}(\mathcal{B})_\omega$.

By Remark 3.1 for any irreducible algebraic set $Y$ over $\mathcal{B}$ the coordinate algebra $\Gamma(Y)$ belongs to $\text{Ucl}(\mathcal{B})_\omega$. If $\mathcal{B}$ is $u_\omega$-compact algebra then, by Theorem 3.4 every algebra $\mathcal{C}$ from $\text{Ucl}(\mathcal{B})_\omega$ is the coordinate algebra of some irreducible algebraic set $Y$ over $\mathcal{B}$.

Let us apply a weak mode to $u_\omega$-compactness and require that every coordinate algebra $\mathcal{C}$ from $\text{Ucl}(\mathcal{B})_\omega$ is irreducible. Suppose that some algebras from $\text{Ucl}(\mathcal{B})_\omega$ are not coordinate algebras for algebraic sets over $\mathcal{B}$ at all, however, if $\Gamma(Y) \in \text{Ucl}(\mathcal{B})$ then $Y$ is irreducible. Let us introduce a specific name for algebra $\mathcal{B}$ with this type of property.

**Definition 3.10.** We name an $L$-algebra $\mathcal{B}$ *weakly $u_\omega$-compact* if each non-empty algebraic set $Y$ over $\mathcal{B}$ which coordinate algebra $\Gamma(Y)$ belongs to $\text{Ucl}(\mathcal{B})$ is irreducible.

By Theorem 3.4 every $u_\omega$-compact algebra is weakly $u_\omega$-compact. We will discuss weakly $u_\omega$-compact algebras, their properties and equivalent approaches to them in Subsection 5.3.

For weakly $u_\omega$-compact algebras we have just the following weak analog of Theorem A. It allows to describe irreducible coordinate algebras inside the class of all coordinate algebras.

**Theorem 3.11** (weak analog of Theorem A). Let $\mathcal{B}$ be a weakly $u_\omega$-compact algebra in a functional language $L$ and $Y$ a non-empty algebraic set over $\mathcal{B}$. Then the following conditions are equivalent:

1) $\text{Th}_\forall(\mathcal{B}) \subseteq \text{Th}_\forall(\Gamma(Y))$, i.e., $\Gamma(Y) \in \text{Ucl}(\mathcal{B})$;

2) $\text{Th}_\exists(\mathcal{B}) \supseteq \text{Th}_\exists(\Gamma(Y))$;

3) $\Gamma(Y)$ embeds into an ultrapower of $\mathcal{B}$;

4) $\Gamma(Y)$ is discriminated by $\mathcal{B}$;

5) $\Gamma(Y)$ is a limit algebra over $\mathcal{B}$;
6) $\Gamma(Y)$ is an algebra defined by a complete atomic type in the theory $\text{Th}_\nu(B)$ in $L$;

7) $Y$ is irreducible.

Moreover, if for an $L$-algebra $B$ and for every non-empty algebraic set $Y$ the conditions above are equivalent then $B$ is weakly $u_\omega$-compact.

Proof. It follows from Remark 3.1 that conditions 1)–7) are equivalent if and only if one has implication 1) $\implies$ 7). By definition, implication 1) $\implies$ 7) take place if and only if $B$ is weakly $u_\omega$-compact. $\square$

4 $q_\omega$-compact and $u_\omega$-compact algebras

In Section 3 we gave the definitions of $q_\omega$- and $u_\omega$-compact algebras in geometric language. In Subsection 4.1 we gather the numerous another approaches to these notions into two theorems. We will prove these theorems in Subsection 4.4.

In Subsection 4.3 we introduce “local $q_\omega(u_\omega)$-compact property” and show its use in universal algebraic geometry. Subsection 4.2 contains some accessory materials.

4.1 Criteria of $q_\omega$- and $u_\omega$-compactness

At first we formulate the theorems and then give the necessary explanations.

Theorem 4.1. For an algebra $B$ in a functional language $L$ the following conditions are equivalent:

1) $B$ is $q_\omega$-compact;

2) for any finite set $X$, any system of equations $S \subseteq \text{At}_L(X)$, and any consequence $c = (t_0 = s_0) \in \text{Rad}_B(S)$ there exists a finite subsystem $S_c \subseteq S$ such that $c \in \text{Rad}_B(S_c)$;

3) for any finite set $X$, any subset $S \subseteq \text{At}_L(X)$, and any atomic formula $(t_0 = s_0) \in \text{At}_L(X)$ if an (infinite) formula

$$\forall x_1 \ldots \forall x_n \left( \bigwedge_{(t=s) \in S} t(\bar{x}) = s(\bar{x}) \implies t_0(\bar{x}) = s_0(\bar{x}) \right)$$
holds in $\mathcal{B}$ then for some finite subsystem $S_c \subseteq S$ the quasi-identity

$$\forall x_1 \ldots \forall x_n \left( \bigwedge_{(t,s) \in S_c} t(\bar{x}) = s(\bar{x}) \rightarrow t_0(\bar{x}) = s_0(\bar{x}) \right)$$

also holds in $\mathcal{B}$;

4) for any finite set $X$, any subset $S \subseteq \text{At}_L(X)$, and any atomic formula $(t_0 = s_0) \in \text{At}_L(X)$ if the set of formulas

$$T = S \cup \{\neg(t_0 = s_0)\}$$

is finitely satisfiable in $\mathcal{B}$ then it is satisfiable in $\mathcal{B}$;

5) every finitely generated algebra from $\text{Qvar}(\mathcal{B})$ is the coordinate algebra of an algebraic set over $\mathcal{B}$;

6) $\text{Qvar}(\mathcal{B})_\omega = \text{Pvar}(\mathcal{B})_\omega$;

7) $\text{Qvar}(\mathcal{B}) = \text{LfgRes}(\mathcal{B})$;

8) $\text{LfgSP}(\mathcal{B}) = \text{LfgSP}(\mathcal{B})$;

9) $\text{LfgSP}(\mathcal{B}) = \text{LfgSP}(\mathcal{B})$;

10) for any finite set $X$ and any system of equations $S \subseteq \text{At}_L(X)$ one has:

$$\text{Rad}_G(S) = \bigcup_{S_0 \subseteq S} \text{Rad}_G(S_0),$$

where $S_0$ runs all finite subsystems of $S$;

11) for any finite set $X$ and any directed system $\{S_i, i \in I\}$ of radical ideals over $\mathcal{B}$ from $\text{At}_L(X)$ the union $S = \bigcup_{i \in I} S_i$ is a radical ideal over $\mathcal{B}$;

12) for any finite set $X$ and any epimorphic direct system $\Lambda = (I, \mathcal{C}_i, h_{ij})$ of coordinate algebras over $\mathcal{B}$ with generating set $X$, and $h_{ij}(x) = x$, $x \in X$, the epimorphic direct limit $\varinjlim \mathcal{C}_i$ is a coordinate algebra over $\mathcal{B}$.  

17
Theorem 4.2. For an algebra $B$ in a functional language $L$ the following conditions are equivalent:

1) $B$ is $u_\omega$-compact;

2) for any finite set $X$, any system of equations $S \subseteq \text{At}_L(X)$, and any consequence $c$ of $S$ over $B$ of the form $c = (t_1 = s_1) \lor \cdots \lor (t_m = s_m)$, $t_i, s_i \in T_L(X)$, there exists a finite subsystem $S_c \subseteq S$ such that $c$ is a consequence of $S_c$ over $B$;

3) for any finite set $X$, any subset $S \subseteq \text{At}_L(X)$, and any atomic formulas $(t_1 = s_1), \ldots, (t_m = s_m) \in \text{At}_L(X)$ if an (infinite) formula

\[ \forall x_1 \ldots \forall x_n \left( \bigwedge_{(t=s) \in S} t(x) = s(x) \rightarrow \bigvee_{i=1}^m t_i(x) = s_i(x) \right) \]

holds in $B$ then for some finite subsystem $S_c \subseteq S$ the universal sentence

\[ \forall x_1 \ldots \forall x_n \left( \bigwedge_{(t=s) \in S_c} t(x) = s(x) \rightarrow \bigvee_{i=1}^m t_i(x) = s_i(x) \right) \]

also holds in $B$;

4) for any finite set $X$, any subset $S \subseteq \text{At}_L(X)$, and any atomic formulas $(t_1 = s_1), \ldots, (t_m = s_m) \in \text{At}_L(X)$ if the set of formulas

\[ T = S \cup \{ \neg(t_1 = s_1), \ldots, \neg(t_m = s_m) \} \]

is finitely satisfiable in $B$ then it is satisfiable in $B$;

5) every finitely generated algebra from $\text{Ucl}(B)$ is the coordinate algebra of an irreducible algebraic set over $B$;

6) $\text{Ucl}(B)_\omega = \text{Dis}(B)_\omega$;

7) $\text{Ucl}(B) = \text{LfgDis}(B)$. 

18
Item 2) in Theorem 4.1 (correspondingly, in Theorem 4.2) gives the definition of $q_\omega$-compact (correspondingly, $u_\omega$-compact) algebra in terms of radicals; item 3) — in terms of infinite formulas; item 5) — in terms of coordinate algebras.

Item 4) shows that the definition of $q_\omega(u_\omega)$-compactness is a compact property relating to special types of sets of formulas $T$, as it is discussed in Introduction. The background of this notion is detailed in [21] for groups. Here we will tell just a few words about it.

The answer for the following question has been attained by V. A. Gorbunov [6].

**Malcev Problem.** When the prevariety $P\text{var}(K)$ generated by class $K$ is a quasivariety?

V. A. Gorbunov has introduced the notion of quasi-compact (q-compact) class $K$ and proved that $P\text{var}(K) = Q\text{var}(K)$ if and only if $K$ is q-compact. Let us compare that result with item 6) in Theorem 4.1.

The definition of q-compact algebra $B$ is much the same as the definition of $q_\omega$-compact algebra given in item 4) of Theorem 4.1. We just bound the set of variables $X$ for defining $q_\omega$-compact algebras: $X$ must be finite. For q-compact algebras $X$ runs sets of all possible cardinalities.

While items 1)–7) in Theorems 4.1 and 4.2 are symmetric, items 10)–12) in Theorem 4.1 are specific for $q_\omega$-compact algebras; 8) and 9) in Theorem 4.1 are just corollaries of 7).

Items 10) and 11) in Theorems 4.1 are close. The family $\{\text{Rad}(S_0)\}$, where $S_0$ runs all finite subsystems of a system $S$, gives an example of a directed system. Let us remind concerned definitions.

A partial ordering $(I, \leq)$ is directed if any two elements from $I$ have an upper bound. A family $\{\theta_i, i \in I\}$ of congruencies on an $L$-algebra $M$ with $i \leq j \iff \theta_i \subseteq \theta_j$ is called directed system of congruencies.

A system $S \subseteq \text{At}_L(X)$ is radical ideal over $B$ if $S = \text{Rad}_B(S)$.

**Definition 4.3.** We say that a family $\{S_i, i \in I\}$ of radical ideals from $\text{At}_L(X)$ is a directed system if the family $\{\theta_{S_i}, i \in I\}$ is a directed system of congruencies on $\mathcal{T}_L(X)$.

Let us prove just a little part of Theorems 4.1 and 4.2.
Lemma 4.4. Let $B$ be an $L$-algebra, $X$ a finite set, $|X| = n$, $S \subseteq \text{At}_L(X)$ a system of equations, and $(t_1 = s_1), \ldots, (t_m = s_m) \in \text{At}_L(X)$ atomic formulas. Then the following conditions are equivalent:

1) $V_B(S) \subseteq V_B(t_1 = s_1) \cup \ldots \cup V_B(t_m = s_m)$;
2) $(t_1 = s_1) \lor \ldots \lor (t_m = s_m)$ is a consequence of $S$ over $B$;
3) the (infinite) formula

$$\forall x_1 \ldots \forall x_n \left( \bigwedge_{(t=s) \in S} t(\bar{x}) = s(\bar{x}) \quad \rightarrow \quad \bigvee_{i=1}^m t_i(\bar{x}) = s_i(\bar{x}) \right)$$

holds in $B$;
4) the set of formulas

$$T = S \cup \{ \neg(t_1 = s_1), \ldots, \neg(t_m = s_m) \}$$

is not satisfiable in $B$;
5) there is no homomorphism $h: \langle \{c_1, \ldots, c_n\} \mid S \rangle \rightarrow B$ such that

$$h(t_i(c_1, \ldots, c_n)) \neq h(s_i(c_1, \ldots, c_n)) \quad \text{for all} \quad i \in \{1, \ldots, m\}.$$ 

Proof. Straightforward.

Corollary 4.5. One has equivalencies 1) $\iff$ 2), 1) $\iff$ 3), 3) $\iff$ 4) in both Theorems 4.1 and 4.2.

Proof. Equivalencies 1) $\iff$ 2), 1) $\iff$ 3) are easy. Note that the statement in item 3) has a form “$A$ implies $B$”. The equivalent statement is “$\neg B$ implies $\neg A$” which gives 4). So we have 3) $\iff$ 4).

From now on, we will use not only geometric definition of $q_\omega$-compact (correspondingly, $u_\omega$-compact) algebra, but also the definitions that items 2), 3), 4) in Theorem 4.1 (correspondingly, in Theorem 4.2) give us.
4.2 $\mathcal{E}$-compact algebras

This subsection is a special excursus. We consider here the following problem.

**Problem.** *When the conditions* $\mathcal{E} \in \text{Ucl}(\mathcal{B})$ *and* $\mathcal{B}$ *has a trivial subalgebra* *are equivalent?*

It is important to note that for a large class of algebras the conditions “$\mathcal{E} \in \text{Ucl}(\mathcal{B})$” and “$\mathcal{B}$ has a trivial subalgebra” are equivalent, but not for all algebras.

**Definition 4.6.** We say an $L$-algebra $\mathcal{B}$ is $\mathcal{E}$-compact if finite satisfiability in $\mathcal{B}$ of the set of all atomic formulas $\text{At}_L(\{x\})$ in one variable $x$ implies its satisfiability in $\mathcal{B}$.

**Lemma 4.7.** An $L$-algebra $\mathcal{B}$ is $\mathcal{E}$-compact if and only if the conditions “$\mathcal{E} \in \text{Ucl}(\mathcal{B})$” and “$\mathcal{B}$ has a trivial subalgebra” are equivalent.

**Proof.** It is sufficient to show that $\text{At}_L(\{x\})$ is satisfiable in $\mathcal{B}$ if and only if $\mathcal{B}$ has a trivial subalgebra, and $\text{At}_L(\{x\})$ is finitely satisfiable if and only if $\mathcal{E} \in \text{Ucl}(\mathcal{B})$.

Suppose that $\text{At}_L(\{x\})$ is satisfiable in $\mathcal{B}$. Then there exists an element $b \in \mathcal{B}$ with $\mathcal{B} \models (t(b) = s(b))$ for all $t, s \in T_L(\{x\})$. Therefore, subalgebra of $\mathcal{B}$ generated by the element $b$ is trivial. Conversely, if $\mathcal{B}$ has a trivial subalgebra $\mathcal{E} = \{e\}$ then the set of all atomic formulas $\text{At}_L(\{x\})$ is realized in $\mathcal{B}$ on the element $e$.

Assume now that $\text{At}_L(\{x\})$ is not finitely satisfiable in $\mathcal{B}$. Then there exists a finite set $S_0$ of atomic formulas such that the universal sentence

$$\forall x \left( \bigvee_{(t=s) \in S_0} \neg (t(x) = s(x)) \right)$$

holds in $\mathcal{B}$. However $\text{false}$ is false in $\mathcal{E}$, so $\mathcal{E} \not\in \text{Ucl}(\mathcal{B})$. Conversely, if the set of all atomic formulas $\text{At}_L(\{x\})$ is finitely satisfiable in $\mathcal{B}$ then by Compactness Theorem it is realized in some ultrapower $\mathcal{B}^\ast$ of $\mathcal{B}$. Hence, $\mathcal{E} \in \text{Ucl}(\mathcal{B})$. \qed

**Corollary 4.8.** The condition “algebra $\mathcal{B}$ is $\mathcal{E}$-compact” means that $\mathcal{B}$ has a trivial subalgebra or $\mathcal{E} \not\in \text{Ucl}(\mathcal{B})$.

Let us note that in “good” signatures all algebras are $\mathcal{E}$-compact.
Lemma 4.9. Suppose a functional language $L$ has at least one constant symbol. Then every algebra in $L$ is $\mathcal{E}$-compact.

Proof. Let $B$ be an $L$-algebra. We need to show that condition $\mathcal{E} \in \text{Ucl}(B)$ implies that $B$ has a trivial subalgebra. Consider the set of formulas

$$T = \{c = c'\} \cup \{F(c, \ldots, c) = c\},$$

where $c, c'$ run all constant symbols from $L$ and $F$ runs all functional symbols from $L$. If $\mathcal{E} \in \text{Ucl}(B)$, then $B \models T$. Therefore, there exists an element $b \in B$ such that $c^B = b$ for all constant symbol $c$ from $L$, and $F(b, \ldots, b) = b$ for all functional symbol $F$ from $L$. Thereby, the element $b$ generates the trivial subalgebra in $B$. □

Lemma 4.10. Suppose $L$ is a finite functional language. Then every algebra in $L$ is $\mathcal{E}$-compact.

Proof. After Lemma 4.9 we may assume that $L$ has no constant symbols. Let $B$ an $L$-algebra. If $\mathcal{E} \in \text{Ucl}(B)$ then the existential sentence

$$\exists x \left( \bigwedge_{F \in L} F(x, \ldots, x) = x \right)$$

holds in $B$. Thereby, $B$ has a trivial subalgebra. □

If $L$ is an infinite functional language with no constant symbols, then it is easy to construct an $L$-algebra $B$ that is not $\mathcal{E}$-compact (see Example 5.18 below).

It follows from the definition that all equationally Noetherian algebras are $\mathcal{E}$-compact. Now we state that all $q_\omega$- and $u_\omega$-compact algebras are $\mathcal{E}$-compact. We need the following facts and definitions.

According to V. A. Gorbunov [6], an $L$-algebra $B$ is weakly atomic compact, if for any set $X$ and any subset $S \subseteq \text{At}_L(X)$ finite satisfiability of $S$ in $B$ implies realizability of $S$ in $B$. We say that an $L$-algebra $B$ is weakly atomic $\omega$-compact, if for any finite set $X$ and any subset $S \subseteq \text{At}_L(X)$ finite satisfiability of $S$ in $B$ implies realizability of $S$ in $B$. It is obvious that weak atomic $\omega$-compactness implies $\mathcal{E}$-compactness.
The following result has been proven by M. Kotov \cite{18}.

**Lemma \([18]\).** Every \(q_\omega\)-compact algebra in a functional language \(L\) is weakly atomic \(\omega\)-compact.

**Corollary 4.11.** Let \(B\) be \(q_\omega\)-compact \(L\)-algebra (in particularly, \(B\) may be \(u_\omega\)-compact). Then the universal closure \(\text{Ucl}(B)\) contains the trivial algebra \(E\) if and only if \(B\) has a trivial subalgebra.

Let us note that M. Kotov has proven more general result in his work. We formulate it on geometric language.

**Lemma \([18]\).** Let \(B\) be an \(L\)-algebra and \(S\) a system of equations in \(L\). If \(B\) is \(q_\omega\)-compact and \(V_B(S)\) is a singleton set or the empty set, then there exists a finite subsystem \(S_0 \subseteq S\) which is equivalent to \(S\) over \(B\). If \(B\) is \(u_\omega\)-compact and \(V_B(S)\) is a finite set or the empty set, then there exists a finite subsystem \(S_0 \subseteq S\) which is equivalent to \(S\) over \(B\).

### 4.3 Local compact properties

Let \(X\) be a finite set. Fix a subset \(S \subseteq \text{At}_L(X)\). We will give the definitions of local compact properties with respect to fixed \(S\).

**Definition 4.12.** An \(L\)-algebra \(B\) is called \(q_S\)-compact if for each atomic formula \((t_0 = s_0) \in \text{At}_L(X)\) if the set of formulas

\[
T = S \cup \{\neg(t_0 = s_0)\}
\]

is finitely satisfiable in \(B\) then it is satisfiable in \(B\).

**Definition 4.13.** An \(L\)-algebra \(B\) is called \(u_S\)-compact if for any atomic formulas \((t_1 = s_1), \ldots, (t_m = s_m) \in \text{At}_L(X)\) if the set of formulas

\[
T = S \cup \{\neg(t_1 = s_1), \ldots, \neg(t_m = s_m)\}
\]

is finitely satisfiable in \(B\) then it is satisfiable in \(B\).
It is clear that algebra $B$ is $q_{ω}(u_ω)$-compact if and only if it is $q_{S}(u_S)$-compact for every finite set $X$ and every $S \subseteq \text{At}_L(X)$.

The main results on local compact properties are the following.

**Proposition 4.14.** Let $B$ be an algebra in a functional language $L$, $X$ a finite set, $S \subseteq \text{At}_L(X)$, and $C = \langle X \mid S \rangle$. Then the following conditions are equivalent:

1) $C$ is the coordinate algebra of an algebraic set over $B$ defined by a system of equations in the language $L$;

2) $C$ is separated by $B$;

3) $C \in \text{Qvar}(B)$ and $B$ is $q_{S}$-compact.

**Proposition 4.15.** Let $B$ be an algebra in a functional language $L$, $X$ a finite set, $S \subseteq \text{At}_L(X)$, such that $[S] \neq \text{At}_L(X)$, and $C = \langle X \mid S \rangle$. Then the following conditions are equivalent:

1) $C$ is the coordinate algebra of an irreducible algebraic set over $B$ defined by a system of equations in the language $L$;

2) $C$ is discriminated by $B$;

3) $C \in \text{Ucl}(B)$ and $B$ is $u_{S}$-compact.

Before giving a proof of these propositions we need some remarks. Firstly, equivalence 1) $\iff$ 2) in both Propositions 4.14 and 4.15 have been proven in [4]. Secondly, let us answer the question: when the set of formulas (2) is not finitely satisfiable in $B$? It happens if and only if there exists a finite subset $S_0 \subseteq S$ such that the universal sentence

$$ \forall y_1 \ldots \forall y_n \left( \bigwedge_{(t=s) \in S_0} t(\bar{y}) = s(\bar{y}) \implies \bigvee_{i=1}^{m} t_i(\bar{x}) = s_i(\bar{y}) \right), \quad \text{where } |X| = n, \quad (3) $$

holds in $B$. For example, if $(t_i = s_i) \in [S]$ for some $i \in \{1, \ldots, m\}$, then there exists a finite subset $S_0 \subseteq S$ such that $S_0 \vdash (t_i = s_i)$, in particularly, universal formula (3) holds in $B$. 

24
Thirdly, note that in Propositions 4.15 we claim $[S] \neq \text{At}_L(X)$, but in Propositions 4.14 such restriction is omitted. If $[S] = \text{At}_L(X)$ then $C = \langle X \mid S \rangle$ is the trivial algebra $E$. Moreover, in this case every algebra $B$ is $q_S$- and $u_S$-compact. Since the trivial algebra $E$ is the coordinate algebra of an algebraic set over $B$ anyway and $E$ belongs to each quasi-variety $[4]$, we have no difficulties with $E$ in Propositions 4.14.

Remark 4.16. One can omit restriction $[S] \neq \text{At}_L(X)$ in Proposition 4.15 if and only if $B$ is $E$-compact algebra. Indeed, the trivial algebra $E$ is the coordinate algebra of an irreducible algebraic set over $B$ if and only if $B$ has a trivial subalgebra $[4]$, Lemma 3.22]. By Lemma 4.17 the conditions “$E \in \text{Ucl}(B)$” and “$B$ has a trivial subalgebra” are equivalent if and only if $B$ is $E$-compact.

Now we are going to prove Propositions 4.14 and 4.15. Arguments for them are the similar, so we will prove only Propositions 4.15.

Proof of Propositions 4.15. Let $C \simeq \mathcal{T}_L(X)/\theta_S$, $X = \{c_1, \ldots, c_n\}$, and $[S] \neq \text{At}_L(X)$. By definition $C$ is discriminated by $B$ if for any finite set of atomic formulas $(t_1 = s_1), \ldots, (t_m = s_m) \in \text{At}_L(X) \setminus [S]$ there exists a homomorphism $h: C \to B$, such that $h(t_i(c_1, \ldots, c_n)) \neq h(s_i(c_1, \ldots, c_n))$ for all $i \in \{1, \ldots, m\}$. The existence of such homomorphism $h: C \to B$ means that the set $T$ in (2) is realized in $B$. Note that if we take $(t_i = s_i) \in [S]$ for some $i \in \{1, \ldots, m\}$, then $T$ is not finitely satisfiable in $B$. Anyway, we shown that if $C$ is discriminated by $B$ then $B$ is $u_S$-compact. The occurrence $C \in \text{Ucl}(B)$ follows from the inclusion $\text{Dis}(B) \subseteq \text{Ucl}(B)$.

Suppose now that $C = \langle X \mid S \rangle$ is not discriminated by $B$ and show that $C \notin \text{Ucl}(B)$ or $B$ is not $u_S$-compact. In this case for some atomic formulas $(t_1 = s_1), \ldots, (t_m = s_m) \in \text{At}_L(X) \setminus [S]$ the set $T$ from $[2]$ is not realized in $B$. If at the same time $T$ is finitely satisfiable in $B$ then $B$ is not $u_S$-compact. Assume that $T$ is not finitely satisfiable in $B$. Therefore, there exists a finite subset $S_0 \subseteq S$ such that the universal formula (3) holds in $B$. On the other hand, the formula

$$\bigwedge_{(t = s) \in S_0} t(\bar{y}) = s(\bar{y}) \longrightarrow \bigvee_{i=1}^{m} t_i(\bar{x}) = s_i(\bar{y})$$
is false in $\mathcal{C}$ under the interpretation $y_i \mapsto c_i, i = 1, \ldots, n$, hence $\mathcal{C} \notin \text{Ucl}(\mathcal{B})$.

### 4.4 Proof of the criteria

In this subsection we prove Theorems 4.1 and 4.2 that have been formulated in Subsection 4.1. Remain that equivalencies 1) $\iff$ 2), 1) $\iff$ 3), 3) $\iff$ 4) in both theorems have been proven in Subsection 4.1.

At first we prove the following easy lemma that will be useful below.

**Lemma 4.17.** Let $\mathcal{B}, \mathcal{C}$ be $L$-algebras, $\mathcal{C} \in \text{Ucl}(\mathcal{B})$, and $T$ a set of quantifier-free formulas in $L$. If $T$ is finitely satisfiable in $\mathcal{C}$ then it is finitely satisfiable in $\mathcal{B}$.

**Proof.** Suppose $T$ is finitely satisfiable in $\mathcal{C}$. Then for every finite subset $\{\phi_1, \ldots, \phi_m\} \subseteq T$ the existential sentence

$$\exists x_1 \ldots \exists x_n \ (\phi_1(x_1, \ldots, x_n) \land \ldots \land \phi_m(x_1, \ldots, x_n)) \quad (4)$$

holds in $\mathcal{C}$. Since $\mathcal{C} \in \text{Ucl}(\mathcal{B})$ then (1) holds in $\mathcal{B}$ too. Thereby, $T$ is finitely satisfiable in $\mathcal{B}$.

We start with Theorem 4.2. Consider item 6). It states that $\text{Ucl}(\mathcal{B})_\omega = \text{Dis}(\mathcal{B})_\omega$. As inclusion $\text{Ucl}(\mathcal{B})_\omega \supseteq \text{Dis}(\mathcal{B})_\omega$ holds for an arbitrary algebra $\mathcal{B}$, then item 6) is equivalent to inclusion $\text{Ucl}(\mathcal{B})_\omega \subseteq \text{Dis}(\mathcal{B})_\omega$. On the other hand, $\text{Dis}(\mathcal{B})_\omega$ is the class of all irreducible coordinate algebras over $\mathcal{B}$ [4, Corollary 3.39]. Hence, we have equivalence 5) $\iff$ 6).

Now let us show equivalence 4) $\iff$ 6). Suppose $\mathcal{B}$ is $u_\omega$-compact and $\mathcal{M}$ is a finitely generated algebra from $\text{Ucl}(\mathcal{B})$. If $\mathcal{M}$ is a trivial algebra then, by Corollary 4.11, $\mathcal{B}$ has a trivial subalgebra, therefore, $\mathcal{M}$ is discriminated by $\mathcal{B}$.

For non-trivial algebra $\mathcal{M}$ let us find a presentation $\langle X \mid S \rangle$, where $X$ is a finite set and $S \subseteq \text{At}_L(X)$, $[S] \neq \text{At}_L(X)$. As $\mathcal{B}$ is $u_S$-compact we have $\mathcal{M} \in \text{Dis}(\mathcal{B})$, by Proposition 4.15. Thus we proved inclusion $\text{Ucl}(\mathcal{B})_\omega \subseteq \text{Dis}(\mathcal{B})_\omega$ and implication 4) $\implies$ 6).

We prove the converse implication 6) $\implies$ 4) by contradiction. Suppose that $\mathcal{B}$ is not $u_\omega$-compact. Then there exists a finite set $X$, a subset $S \subseteq \text{At}_L(X)$, and atomic formulas
\[(t_1 = s_1), \ldots, (t_m = s_m) \in \text{At}_L(X), \text{such that the set of formulas} \]
\[T = S \cup \{\neg(t_1 = s_1), \ldots, \neg(t_m = s_m)\},\]

is not realized in \(B\), but every its finite subset is realized in \(B\).

By Compactness Theorem \(T\) is realized in some ultrapower \(B^I/D\) of \(B\). Let \(c_1, \ldots, c_n\) be elements from \(B^I/D\), such that \(B^I/D \models T(c_1, \ldots, c_n)\), and \(C\) subalgebra of \(B^I/D\) generated by the set \(\{c_1, \ldots, c_n\}\). Clearly, \(C\) is finitely generated algebra from \(\text{Ucl}(B)\).

Show that \(C\) is not discriminated by \(B\).

Let \(\langle \{c_1, \ldots, c_n\} \mid R \rangle\) be a presentation of \(C\), i.e., \(C \cong T_L(X) / \theta_R\), \(R \subseteq \text{At}_L(X)\). Since \(C \models T(c_1, \ldots, c_n)\), one has \(S \subseteq R\) and \((t_i = s_i) \notin [R], i = 1, \ldots, m\). Put
\[T' = R \cup \{\neg(t_1 = s_1), \ldots, \neg(t_m = s_m)\}.\]

Since \(T'\) is realized in \(C\) and \(C \in \text{Ucl}(B)\), then, by Lemma 4.17, \(T'\) is finitely satisfiable in \(B\). However, \(T'\) is not satisfiable in \(B\). Thus \(B\) is not \(u_R\)-compact. Hence, by Proposition 4.15, \(C\) is not discriminated by \(B\). We proved \(6) \Rightarrow 4)\).

Equivalence \(6) \iff 7)\) is true in more general case. Let \(K\) and \(K'\) be two classes of \(L\)-algebras (let us have in mind \(K = \text{Ucl}(B)\) and \(K' = \text{Dis}(B)\)), \(K\) is universal axiomatizable and \(K'\) is closed under taking \(L\)-subalgebras. Then \(K = LfgK'\) is equivalent to \(K_\omega = K'_\omega\). Indeed, \(K = LfgK'\) easy implies \(K_\omega = K'_\omega\). Inversely, if \(K_\omega = K'_\omega\) then \(K = LfgK_\omega = LfgK'_\omega = LfgK'\).

Now we begin to prove Theorem 4.1.

Equivalences \(5) \iff 6), 4) \iff 6), 6) \iff 7)\) may be proven by means of the similar reasoning as in Theorem 4.2 (remind that \(\text{Pvar}(B) = \text{Res}(B)\)).

Let us show equivalence \(7) \iff 8) \iff 9\). For an arbitrary algebra \(B\) we have
\[\text{Qvar}(B) = Lfg\text{SP}(B) = L\text{SP}(B)\] (Corollary 2.3.4) and \(\text{Res}(B) = \text{SP}(B)\). So the identity \(\text{Qvar}(B) = Lfg\text{Res}(B)\) is equivalent to \(L\text{SP}(B) = Lfg\text{SP}(B)\) or \(L\text{SP}(B) = Lfg\text{SP}(B)\).

Equivalence \(2) \iff 10\) is easy. Equivalence \(11) \iff 12\) is due to V. A. Gorbunov [6, Proposition 1.4.9]. So, it remains to prove implications \(2) \implies 11)\) and \(11) \implies 10\).
Let $B$ be $\omega$-compact algebra, $\{S_i, i \in I\}$ a directed system of radical ideals from $\text{At}_L(X)$ and $S = \bigcup_{i \in I} S_i$. We show that $S = \text{Rad}(S)$, i.e., $\text{Rad}(S) \subseteq \bigcup_{i \in I} S_i$. Indeed, if $c$ is a consequence of $S$ then there exists a finite subsystem $S_0 \subseteq S$ with $c \in \text{Rad}(S_0)$. Since $I$ is directed there exists an index $i \in I$ such that $S_0 \subseteq S_i$, therefore $c \in S_i$. Thus we have implication $2) \implies 11)$.

To prove implication $11) \implies 10)$ consider an arbitrary system $S \subseteq \text{At}_L(X)$. The family $\{\text{Rad}(S_0)\}$, where $S_0$ runs all finite subsystems of a system $S$, forms a directed system of radical ideals from $\text{At}_L(X)$. Hence $\bigcup_{S_0 \subseteq S} \text{Rad}(S_0)$ is a radical ideal over $B$. Also we have

$$S \subseteq \bigcup_{S_0 \subseteq S} \text{Rad}_B(S_0) \subseteq \text{Rad}_B(S),$$

therefore $\bigcup_{S_0 \subseteq S} \text{Rad}(S_0) = \text{Rad}_B(S)$. So, implication $11) \implies 10)$ has been proven.

## 5 Weakly equationally Noetherian and weakly $\omega$-compact algebras

A weak form of the equationally Noetherian property naturally arises in practice. We discuss algebras with this property in Subsection 5.1.

In Subsection 3.3 we have introduced weakly $\omega$-compact algebras. Now in Subsection 5.3 we present some equivalent approaches to weakly $\omega$-compact algebras.

In Subsection 5.2 we study logically irreducible algebraic sets. It is important to note that logically irreducible algebraic sets inspired the notion of weakly $\omega$-compact algebras.

### 5.1 Weak equationally Noetherian property

**Definition 5.1.** An $L$-algebra $B$ is said to be *weakly equationally Noetherian*, if for any finite set $X$ every system $S \subseteq \text{At}_L(X)$ is equivalent over $B$ to some finite system $S_0 \subseteq \text{At}_L(X)$. Here we do not assume that $S_0$ is a subsystem of $S$.

To make comparison equationally Noetherian and weakly equationally Noetherian properties it is required to reformulate corresponding definitions in the following form.
An L-algebra \( B \) is termed \textit{weakly equationally Noetherian}, if for any finite set \( X \) and any system \( S \subseteq \text{At}_L(X) \) there exists finite system \( S_0 \subseteq \text{Rad}_B(S) \) such that \( V_B(S) = V_B(S_0) \).

An L-algebra \( B \) is termed \textit{equationally Noetherian}, if for any finite set \( X \) and any system \( S \subseteq \text{At}_L(X) \) there exists finite system \( S_0 \subseteq [S] \) such that \( V_B(S) = V_B(S_0) \).

Indeed, for every atomic formula \( c = (t = s) \in [S] \) there exists a finite subsystem \( S_c \subseteq S \) such that \( S_c \vdash (t = s) \). Therefore, if \( V_B(S) = V_B(S_0) \) for a finite system \( S_0 \subseteq [S] \) then one has

\[
V_B(S) = V_B\left( \bigcup_{c \in S_0} S_c \right) \quad (5)
\]

\textbf{Lemma 5.2.} If an L-algebra \( B \) is weakly equationally Noetherian and \( q_\omega \)-compact then it is equationally Noetherian.

\textit{Proof.} As \( B \) is weakly equationally Noetherian, for each system of equations \( S \) there exists a finite system \( S_0 \subseteq \text{Rad}_B(S) \) with \( V_B(S) = V_B(S_0) \). As \( B \) is \( q_\omega \)-compact, for each equation \( c = (t_0 = s_0) \in S_0 \) there exists a finite subsystem \( S_c \subseteq S \) with \( V_B(S_c) \subseteq V_B(t_0 = s_0) \). Thereby, one has \( (5) \). It means that \( B \) is equationally Noetherian algebra. \( \square \)

\textbf{Lemma 5.3.} If an L-algebra \( B \) is weakly equationally Noetherian and \( C \) a subalgebra of some direct power of \( C \) then \( C \) is weakly equationally Noetherian too.

\textit{Proof.} It follows from [4, Lemma 3.7]. \( \square \)

It is clear that every weakly equationally Noetherian algebra is \( E \)-compact.

\textbf{Lemma 5.4.} If an L-algebra \( B \) is weakly equationally Noetherian then

\[
\text{Ucl}(B) \cap \text{Res}(B)_\omega = \text{Dis}(B)_\omega.
\]

\textit{Proof.} Since \( \text{Dis}(B) \subseteq \text{Res}(B) \), \( \text{Dis}(B) \subseteq \text{Ucl}(B) \), and \( \text{Res}(B) = \text{Pvar}(B) \) for any algebra \( B \) [3], we should check that \( \text{Ucl}(B) \cap \text{Pvar}(B)_\omega \subseteq \text{Dis}(B)_\omega \). Let us assume that \( C \) is a finitely generated algebra such that \( C \in \text{Pvar}(B) \setminus \text{Dis}(B) \) and prove \( C \not\in \text{Ucl}(B) \).

If \( C \) is the trivial algebra \( E \) then, by definition, condition \( C \not\in \text{Dis}(B) \) implies that \( B \) has not a trivial subalgebra. Since \( B \) is weakly equationally Noetherian, then \( B \) is \( E \)-compact, and, by Lemma 4.7 \( E \not\in \text{Ucl}(B) \). Thereby, we may assume that \( C \) is non-trivial.
Let $\langle \{c_1, \ldots, c_n \} \mid S \rangle$ be a presentation of $C$, i.e., $C \simeq T_L(X)/\theta_S, \ S \subseteq At_L(X), \ X = \{x_1, \ldots, x_n\}$. Since $C \not\in \text{Dis}(B)$, there exits atomic formulas $(t_1 = s_1), \ldots, (t_m = s_m) \in At_L(X) \setminus [S]$ such that the (infinite) formula
\[
\forall x_1 \ldots \forall x_n \left( \bigwedge_{(t=s) \in S} t(\bar{x}) = s(\bar{x}) \rightarrow \bigvee_{i=1}^m t_i(\bar{x}) = s_i(\bar{x}) \right)
\]
holds in $B$. As one can find a finite system $S_0 \subseteq At_L(X)$ with $V_B(S_0) = V_B(S)$ then the universal sentence
\[
\forall x_1 \ldots \forall x_n \left( \bigwedge_{(t=s) \in S_0} t(\bar{x}) = s(\bar{x}) \rightarrow \bigvee_{i=1}^m t_i(\bar{x}) = s_i(\bar{x}) \right) \tag{6}
\]
holds in $B$.

Since $V_B(S_0) = V_B(S)$ we have $V_C(S_0) = V_C(S) \tag{4}$ Lemma 3.7]. Hence, $(c_1, \ldots, c_n) \in V_C(S_0)$ but $t_i(c_1, \ldots, c_n) \neq s_i(c_1, \ldots, c_n)$ for all $i = 1, \ldots, m$. Therefore, universal formula (6) is not true in $C$, and $C \not\in \text{Ucl}(B)$. 

5.2 Logically irreducible algebraic sets

One of the approaches to $u_\omega$-compact algebras deals with so-called logically irreducible algebraic sets.

**Definition 5.5.** We say that an algebraic set $Y$ over $B$ is **logically irreducible** if its coordinate algebra $\Gamma(Y)$ belongs to $\text{Ucl}(B)$.

In Section 3 we have discussed that every irreducible algebraic set over an arbitrary algebra $B$ is logically irreducible. In Subsection 5.3 we will show that the notions of irreducible and logically irreducible algebraic sets coincide if and only if $B$ is weakly $u_\omega$-compact algebra.

**Lemma 5.6.** Let $B$ be an $L$-algebra. For a finitely generated $L$-algebra $C$ the following conditions are equivalent:

- $C$ is the coordinate algebra of a logically irreducible algebraic set over $B$;
• $C$ belongs to $\text{Ucl}(\mathcal{B}) \cap \text{Pvar}(\mathcal{B})$.

Proof. Indeed, $C$ is the coordinate algebra of an algebraic set over $\mathcal{B}$ if and only if $C \in \text{Pvar}(\mathcal{B})$ \cite[Proposition 3.22]{4}.

Corollary 5.7. The class of all coordinate algebras of logically irreducible algebraic sets over $\mathcal{B}$ coincides with $\text{Ucl}(\mathcal{B}) \cap \text{Pvar}(\mathcal{B})_\omega$.

For irreducible algebraic sets we have the following result.

Lemma 5.8 \cite{3}. Let $\mathcal{B}$ be an $\mathcal{L}$-algebra. Every non-empty algebraic set $Y$ over $\mathcal{B}$ is a union of maximal with respect to inclusion irreducible algebraic sets over $\mathcal{B}$.

Now we try to find a similar decomposition for algebraic sets into a union of maximal logically irreducible algebraic sets. It is clear that Lemma 5.8 gives a decomposition. However, maximal with respect to inclusion irreducible algebraic set may be a proper subset of some logically irreducible algebraic set.

Lemma 5.9. Let $Y_1 \subset Y_2 \subset \ldots$ be an ascending chain of logically irreducible algebraic sets in $B^n$ and $Y$ the least algebraic set containing all these sets. Then $Y$ is logically irreducible algebraic set.

Proof. Note that $Y = V_B(\text{Rad}(\bigcup Y_i))$ and $\text{Rad}(Y) = \bigcap_i \text{Rad}(Y_i)$. Hence, there exists embedding $h : \Gamma(Y) \to \prod_i \Gamma(Y_i)$ \cite[Lemma 3.1]{3}. Index $i$ runs the linearly ordered set $I$. For each $i \in I$ denote by $J_i$ the set $\{j \in I, j \geq i\}$. The family of subsets $\{J_i, i \in I\}$ is centered, hence there exists an ultrafilter $D$ on $I$ containing $J_i$ for all $i \in I$. Let $f : \prod_i \Gamma(Y_i) \to \prod_i \Gamma(Y_i)/D$ be a canonical homomorphism. Let us show that composition $f \circ h : \Gamma(Y) \to \prod_i \Gamma(Y_i)/D$ is embedding.

Indeed, we have $\Gamma(Y) = \mathcal{T}_L(X)/\theta_{\text{Rad}(Y)}$, where $X = \{x_1, \ldots, x_n\}$. If $t_1/\theta_{\text{Rad}(Y)}, t_2/\theta_{\text{Rad}(Y)}$ are distinct elements from $\Gamma(Y)$ then $(t_1 = t_2) \in \text{At}_L(X) \setminus \text{Rad}(Y)$. Since $\text{Rad}(Y_1) \supset \text{Rad}(Y_2) \supset \ldots$, then there exists an index $i_0 \in I$ such that $(t_1 = t_2) \notin \text{Rad}(Y_i)$ for all $i \in J_{i_0}$. It implies that $f \circ h(t_1/\theta_{\text{Rad}(Y)}) \neq f \circ h(t_2/\theta_{\text{Rad}(Y)})$. Thus $f \circ h$ is injective.
Since $\Gamma(Y_i) \in \text{Ucl}(B)$ for each $i \in I$ and $\Gamma(Y) \in \text{SP}_u(\{\Gamma(Y_i), i \in I\})$, then $\Gamma(Y) \in \text{Ucl}(B)$, i.e., $Y$ is logically irreducible algebraic set.

**Lemma 5.10.** Let $B$ be an L-algebra. Every non-empty algebraic set $Y$ over $B$ is a union of maximal with respect to inclusion logically irreducible algebraic sets over $B$.

*Proof.* We will show that for each point $p \in Y$ there exists logically irreducible algebraic set $Z$ such that $p \in Z \subseteq Y$ and $Z$ is maximal with these properties. Denote by $\Omega$ the family of logically irreducible algebraic sets $Z$ with $p \in Z \subseteq Y$ and show that $\Omega$ is not empty and has maximal elements.

Denote by $Z_p$ the closure in the Zariski topology of the set $\{p\}$. One has $p \in Z_p \subseteq Y$. Furthermore, $Z_p$ is irreducible algebraic set [4, Lemma 3.34]. Hence, $Z_p \in \Omega$.

By Zorn Lemma it is sufficiently to show now that family $\Omega$ contains upper boundary for each ascending chain $Y_1 \subset Y_2 \subset \ldots$ of element from $\Omega$. Let $Y_p$ be the least algebraic set that contains union $\bigcup_i Y_i$. By Lemma 5.9, $Y_p$ is logically irreducible. As $Y_p \subseteq Y$ one has $Y_p \in \Omega$.

Thereby, the union $\bigcup_{p \in Y} Y_p$ is desired. $\square$

Let us remind that for equationally Noetherian algebras we have the next result.

**Theorem 5.11.** Let $B$ be an equationally Noetherian algebra. Then any non-empty algebraic set $Y$ over $B$ is a finite union of irreducible algebraic sets (irreducible components): $Y = Y_1 \cup \ldots \cup Y_m$. Moreover, if $Y_i \nsubseteq Y_j$ for $i \neq j$ then this decomposition is unique up to a permutation of components.

It is natural to ask the following question.

**Decomposition Problem.** Let $B$ be a “good” algebra ($u_\omega$-, $q_\omega$-compact, weakly equationally Noetherian, for instance). Is it true that every non-empty algebraic set over $B$ is a finite union of logically irreducible algebraic sets?

In spite of the fact that $u_\omega$-compact and weakly equationally Noetherian algebras are the closest algebras to equationally Noetherian ones we give for them the negative answer to the question above.
Indeed, a decomposition $Y = Y_1 \cup \ldots \cup Y_m$ of algebraic set $Y$ into a union of algebraic sets $Y_1, \ldots, Y_m$ implies the existence of a subdirect embedding $h : \Gamma(Y) \to \Gamma(Y_1) \times \ldots \times \Gamma(Y_m)$ [4]. Suppose that the Decomposition Problem has the positive answer for $u_\omega$-compact algebras. It involves that the Embedding Problem for $u_\omega$-compact algebras has the positive answer too. However, A. N. Shevlyakov has proven the inverse result (see Subsection 3.2). Moreover, he has proven also that the Decomposition Problem for weakly equationally Noetherian algebras has the negative answer [28].

5.3 Weak $u_\omega$-compactness

In the proposition below we gather the different approaches to weakly $u_\omega$-compact algebras.

Proposition 5.12. For an algebra $B$ in a functional language $L$ the following conditions are equivalent:

1) $B$ is weakly $u_\omega$-compact;

2) every non-empty logically irreducible algebraic set over $B$ is irreducible;

3) every non-trivial coordinate algebra over $B$ that belongs to $\text{Ucl}(B)$ is irreducible;

4) $\text{Ucl}(B) \cap \text{Res}(B)_\omega = (\text{Dis}(B)_e)_\omega$.

Proof. Equivalence 1) $\iff$ 2) is evident by definition. Remind that the trivial algebra $E$ is a coordinate algebra over $B$ anyway, moreover, if $Y$ is an algebraic set over $B$ such that $E = \Gamma(Y)$ then $Y$ is irreducible or $Y = \emptyset$ [4] Lemma 3.22]. It implies that we have equivalence 2) $\iff$ 3).

Since $\text{Dis}(B) \subseteq \text{Res}(B)$, $\text{Dis}(B) \subseteq \text{Ucl}(B)$, $\text{Res}(B) = \text{Pvar}(B)$, and $E \in \text{Res}(B)$ for any algebra $B$, then item 4) means that every non-trivial algebra $C$ from $\text{Ucl}(B) \cap \text{Pvar}(B)_\omega$ belongs to $\text{Dis}(A)_\omega$.

As the class of all coordinate algebras of irreducible algebraic sets over $B$ coincides with $\text{Dis}(B)_\omega$ [4] Corollary 3.37], and, by Corollary 5.7 the class of all coordinate algebras
of logically irreducible algebraic sets over \( B \) coincides with \( \text{Ucl}(B) \cap \text{Pvar}(B)_\omega \), we have equivalence 3) \( \iff \) 4).

**Remark 5.13.** Every \( u_\omega \)-compact (as well as \( q_\omega \)-compact, weakly equationally Noetherian) algebra is \( E \)-compact. However, there exist weakly \( u_\omega \)-compact algebras that are not \( E \)-compact (see Example 5.18 below). Suppose an algebra \( B \) is \( E \)-compact. In this case one can omit “non-empty” in item 2), omit “non-trivial” in item 3), and write “\( \text{Ucl}(B) \cap \text{Res}(B)_\omega = \text{Dis}(B)_\omega \)” instead of “\( \text{Ucl}(B) \cap \text{Res}(B)_\omega = (\text{Dis}(B)_e)_\omega \)” in item 4) in the formulation of Proposition 5.12. In this case the empty set is not algebraic over \( B \), or if it is algebraic then its coordinate algebra \( E \) does not belong to \( \text{Ucl}(B) \).

**Lemma 5.14.** If an \( L \)-algebra \( B \) is weakly \( u_\omega \)-compact and \( q_\omega \)-compact then it is \( u_\omega \)-compact.

**Proof.** We need to show that \( \text{Ucl}(B)_\omega \subseteq \text{Dis}(B)_\omega \). Assume that \( C \) is a finitely generated algebra and \( C \not\subseteq \text{Dis}(B) \). Since \( \text{Dis}(B)_\omega = \text{Ucl}(B) \cap \text{Pvar}(B)_\omega \), then \( C \not\subseteq \text{Ucl}(B) \), and we have required, or \( C \not\subseteq \text{Pvar}(B) \). By Theorem 5.11 \( C \not\subseteq \text{Pvar}(B) \) implies that \( C \not\subseteq \text{Qvar}(B) \), hence \( C \not\subseteq \text{Ucl}(B) \). □

The next question is naturally arises. Is there a geometric definition of weak \( u_\omega \)-compactness?

**Definition 5.15.** We name an \( L \)-algebra \( B \) geometrically weakly \( u_\omega \)-compact if for any finite set \( X \), any system of equations \( S \subseteq \text{At}_L(X) \), and any equations \( (t_1 = s_1), \ldots, (t_m = s_m) \in \text{At}_L(X) \) such that

\[
V_B(S) \subseteq V_B(t_1 = s_1) \cup \ldots \cup V_B(t_m = s_m)
\]

and for each \( i \in \{1, \ldots, m\} \)

\[
V_B(S) \not\subseteq V_B(t_i = s_i)
\]

there exists a finite subsystem \( S_0 \subseteq \text{Rad}_B(S) \) such that

\[
V_B(S_0) \subseteq V_B(t_1 = s_1) \cup \ldots \cup V_B(t_m = s_m).
\]
The definition above is evident generalization of both weak equationally Noetherian property and \( u_\omega \)-compactness. It also has analogs in terms of radical, in terms of infinite formulas, and in terms of compactness.

**Lemma 5.16.** For an algebra \( B \) in a functional language \( L \) the following conditions are equivalent:

1) \( B \) is geometrically weakly \( u_\omega \)-compact;

2) for any finite set \( X \), any radical ideal \( S \subseteq \operatorname{At}_L(X) \) over \( B \), and any atomic formulas \((t_1 = s_1), \ldots, (t_m = s_m) \in \operatorname{At}_L(X) \setminus \operatorname{Rad}_B(S)\) if \( c = (t_1 = s_1) \lor \ldots \lor (t_m = s_m) \) is a consequence of \( S \) over \( B \) then there exists a finite subsystem \( S_c \subseteq S \) such that \( c \) is a consequence of \( S_c \) over \( B \);

3) for any finite set \( X \), any radical ideal \( S \subseteq \operatorname{At}_L(X) \) over \( B \), and any atomic formulas \((t_1 = s_1), \ldots, (t_m = s_m) \in \operatorname{At}_L(X) \) if an (infinite) formula

\[
\forall x_1 \ldots \forall x_n \left( \bigwedge_{(t=s) \in S} t(\bar{x}) = s(\bar{x}) \longrightarrow \bigvee_{i=1}^m t_i(\bar{x}) = s_i(\bar{x}) \right)
\]

holds in \( B \), and for each \( i \in \{1, \ldots, m\} \) an (infinite) formula

\[
\forall x_1 \ldots \forall x_n \left( \bigwedge_{(t=s) \in S} t(\bar{x}) = s(\bar{x}) \longrightarrow t_i(\bar{x}) = s_i(\bar{x}) \right)
\]

does not hold in \( B \), then for some finite subsystem \( S_c \subseteq S \) the universal sentence

\[
\forall x_1 \ldots \forall x_n \left( \bigwedge_{(t=s) \in S_c} t(\bar{x}) = s(\bar{x}) \longrightarrow \bigvee_{i=1}^m t_i(\bar{x}) = s_i(\bar{x}) \right)
\]

holds in \( B \);

4) for any finite set \( X \), any radical ideal \( S \subseteq \operatorname{At}_L(X) \) over \( B \), and any atomic formulas \((t_1 = s_1), \ldots, (t_m = s_m) \in \operatorname{At}_L(X) \) if the set of formulas

\[
T = S \cup \{ \neg(t_1 = s_1), \ldots, \neg(t_m = s_m) \}
\]
is finitely satisfiable in $B$ and for each $i \in \{1, \ldots, m\}$ the set of formulas
\[ T_i = S \cup \{\neg(t_i = s_i)\} \]
is realized in $B$ then $T$ is satisfiable in $B$.

Proof. Equivalences 1) $\iff$ 2), 1) $\iff$ 3), 3) $\iff$ 4) follows from Lemma 4.14. Note that the statement in item 3) has a form “$A \& \neg C$ implies $B$”. The equivalent statement is “$\neg B \& \neg C$ implies $\neg A$” which gives 4). So we have 3) $\iff$ 4).

Unfortunately, for weak $u_\omega$-compactness we have no an analog of Theorem 4.2 that holds for $u_\omega$-compact algebras.

Lemma 5.17. If an $L$-algebra $B$ is geometrically weakly $u_\omega$-compact then it is weakly $u_\omega$-compact. The converse statement does not hold.

Proof. Suppose that $B$ is geometrically weakly $u_\omega$-compact and $Y$ a non-empty algebraic set over $B$ such that $\Gamma(Y) \in \text{Ucl}(B)$. We need to show that $\Gamma(Y) \in \text{Dis}(B)$. Let
\[ S = \text{Rad}(Y), \]
then $\Gamma(Y)$ has the presentation $\langle X \mid S \rangle$. If $\Gamma(Y)$ is the trivial algebra, i.e., $S = \text{At}_L(X)$, then $Y$ is irreducible [4, Lemma 3.22] and $\Gamma(Y) \in \text{Dis}(B)$.

Assume now that $\Gamma(Y)$ is non-trivial, i.e., $S \neq \text{At}_L(X)$. As the coordinate algebra $\Gamma(Y)$ is separated by $B$, hence for each atomic formula $(t = s) \in \text{At}_L(X) \setminus S$ the set of formulas $S \cup \{\neg(t = s)\}$ is realized in $B$. Take atomic formulas $(t_1 = s_1), \ldots, (t_m = s_m) \in \text{At}_L(X) \setminus S$. As the set of formulas $T = S \cup \{\neg(t_1 = s_1), \ldots, \neg(t_m = s_m)\}$ is satisfiable in $\langle X \mid S \rangle$, and $\langle X \mid S \rangle \in \text{Ucl}(B)$, then, by Lemma 4.17, $T$ is finitely satisfiable in $B$. It follows from item 4) of Lemma 5.16 that $T$ is satisfiable in $B$. Thereby, algebra $\langle X \mid S \rangle$ is discriminated by $B$.

Example 5.18 below shows that the converse statement does not hold.

The following example is similar to the example by M. V. Kotov [18].
**Example 5.18.** Let $L = \{g_n, n \in \mathbb{N}\}$ be the infinite signature with unary functional symbols and $A$ the $L$-algebra with the universe $\mathbb{N}$ and

$$g_n(x) = \begin{cases} 
nn 
\end{cases}$$

It is clear that $A$ has no trivial subalgebra. At the same time, the set of formulas $\{g_n(x) = x, n \in \mathbb{N}\}$ is finitely satisfiable in $A$, therefore, by Compactness Theorem, it is satisfiable in some ultrapower $A^*$ of $A$. As $A^* \in \text{Ucl}(A)$, then $E \in \text{Ucl}(A)$. Thereby, $A$ is not $E$-compact.

We state that $A$ is weakly $u_\omega$-compact. Indeed, take a non-trivial algebra $C$ from $\text{Ucl}(A) \cap \text{Pvar}(A)$. Since $\text{Pvar}(A) = \text{SP}(A)$ then $C$ is a subalgebra of a direct power of $A$. For any $n, m \in \mathbb{N}, n \neq m$, the universal formula

$$\forall x \ (g_n(x) = x \lor g_m(x) = x)$$

holds in $A$. Therefore, $C$ has a finite universe $\{c_1, c'_1, \ldots, c_d, c'_d\}$ with $c_i = g_{n_i}(c'_i)$ for all $i = 1, d$. The map $h: C \to A$, $h(c_i) = 2n_i$, $h(c'_i) = 2n_i + 1$, $i = 1, d$, is a monomorphism. Thus, $C \in \text{Dis}(A)$, and $A$ is weakly $u_\omega$-compact.

Let us check that $A$ is not geometrically weakly $u_\omega$-compact. Consider the systems of equations $S'(x) = \{g_n(x) = x, n \in \mathbb{N} \setminus \{0\}\}$ and $S(x, y) = S'(x) \cup S'(y)$. We have $V_A(S) = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. Therefore,

$$V_A(S) \subseteq V_A(x = y) \cup V_A(x = g_0(y)),$$

$$V_A(S) \nsubseteq V_A(x = y), \ V_A(S) \nsubseteq V_A(x = g_0(y)).$$

Furthermore, it is not hard to see that

$$\text{Rad}_A(S) = \begin{cases} 
nn 
\end{cases}$$

It is obvious that for any finite subsystem $S_0 \subseteq \text{Rad}_A(S)$ we have

$$V_A(S_0) \nsubseteq V_A(x = y) \cup V_A(x = g_0(y)).$$

37
6 Connections between the classes of algebras Q, U, U', N', and N

Let L be a functional language. We use the following denotations:

- **N** — the class of all equationally Noetherian L-algebras;
- **N'** — the class of all weakly equationally Noetherian L-algebras;
- **Q** — the class of all \(q_{\omega}\)-compact L-algebras;
- **U** — the class of all \(u_{\omega}\)-compact L-algebras;
- **U'** — the class of all weakly \(u_{\omega}\)-compact L-algebras.

It is clear that

\[ Q \supseteq U \supseteq N \subseteq N'. \]

Moreover, by Lemma 5.2,

\[ N = N' \cap Q = N' \cap U. \]

So, we have exactly the following picture for co-location of classes N, N', Q, U:

- **Q**
- **N**
- **U**
- **N'**

Let us find the place of the class **U'** in the picture above. By Theorem 3.3, Lemma 5.4 and Proposition 5.12 we have

\[ U \subseteq U' \quad \text{and} \quad N' \subseteq U'. \]

It follows from Lemma 5.14 that

\[ Q \cap U' = U. \]

Hence, co-location of classes N, N', Q, U, and **U'** are exactly the following:
In paper [21] A. G. Myasnikov and V. N. Remeslennikov asked the questions for the class of groups:

**Question 1:** $N = Q$ or $N \neq Q$?

**Question 2:** $Q = U$ or $Q \neq U$?

Now we add new questions:

**Question 3:** $N = N'$ or $N \neq N'$?

**Question 4:** $N = U$ or $N \neq U$?

**Question 5:** $U' = U \cup N'$ or $U' \neq U \cup N'$?

The answer to the first question has been given by B. I. Plotkin in [24]. He has constructed $q_\omega$-compact group that is not equationally Noetherian. We will discuss that construction in this section below. Note that B. I. Plotkin uses notation *logically Noetherian* for $q_\omega$-compact algebras and *geometrically Noetherian* for equationally Noetherian algebras.

The second and third questions have been solved by M. V. Kotov [18]. He has constructed examples that show $Q \neq U$ and $N \neq N'$. His examples are original algebraic structures in the language $L = \{g_n, n \in \mathbb{N}\}$ with countable set of unary functional symbols and with universe-sets $\mathbb{R}$ and $\mathbb{N}$.

At these results the fourth question remains open as well as the problem of differentiation of classes $Q, U, N, N'$ for classical varieties: groups, rings, monoids, semigroups. In [28] A. N. Shevlyakov finds the neat examples in the variety of commutative idempotent semigroups in the language with countable set of constants. His examples distinguish classes $N, N', Q, U$.

The algebra $\mathcal{A}$ from Example 5.18 gives an answer to the fifth question. It has been shown that $\mathcal{A} \in U'$, but $\mathcal{A}$ is not $E$-compact. Since all $u_\omega$-compact and weakly equa-
tionally Noetherian algebras are $\mathcal{E}$-compact, then $A \notin U \cup N'$. Another example for $U' \neq U \cup N'$ has been constructed by A. N. Shevlyakov [28] in the class of commutative idempotent semigroups in the language $L$ with countable set of constants. It is important to note that all algebras in the language $L$ are $\mathcal{E}$-compact, by Lemma 4.9.

Let us return to the construction given by B. I. Plotkin. He denotes by $H$ the discrete direct product of all finitely generated groups (in the language of groups $L = \{\cdot, -1, e\}$). Since every finitely generated group $G$ imbeds into $H$, then $G$ is a coordinate group over $H$. By 10) in Theorem 4.1 below, $H$ is $q_\omega$-compact. As there exists a finitely generated group $G$ that is not finitely presented, hence $H$ is not equationally Noetherian.

It is evident that this construction of $H$ may be repeated in other varieties of algebras, where exist finitely generated, not finitely presented algebras. Clearly, the algebraic geometry over objects like $H$ is quite elementary.

7 $q_\omega$-compact and $u_\omega$-compact extensions

In introduction it is given the formulation of the Compactness Theorem and the notion of logical compactness. The Compactness Theorem has a great importance in model theory [13].

For an arbitrary algebra $B$ it is possible with a use of the Compactness Theorem to construct an elementary extension $B^*$ of $B$ such that $B^*$ is logically compact. This algorithm is close to the building of the algebraic closure to a given field $k$.

We use this idea to construct $u_\omega$-compact elementary extension for an arbitrary algebra $B$. At first, let us remind some more facts from model theory.

**Theorem 7.1** (Corollary from Los’ Theorem [6]). If $B^I/D$ is an ultrapower of an algebra $B$ then the diagonal map $d: B \to B^I/D$, where $d(x) = \bar{x}/D$ and $\bar{x}(i) = x$ for all $i \in I$, is an elementary embedding.

**Proposition 7.2** ([20]). Suppose that $(I, \prec)$ is a linear order and $(\mathcal{M}_i, i \in I)$ is an elementary chain. Then $\mathcal{M} = \bigcup_{i \in I} \mathcal{M}_i$ is an elementary extension of each $\mathcal{M}_i$. 40
Denote by $T$ the family of all sets of formulas

$$T = S \cup \{\neg(t_1 = s_1), \ldots, \neg(t_m = s_m)\},$$

where $S \subseteq \text{At}_L(X)$, $(t_1 = s_1), \ldots, (t_m = s_m) \in \text{At}_L(X)$, $|X| < \infty$. For a given $L$-algebra $B$ let $T(B)$ be such subfamily of $T$ that $T \in T(B)$ if and only if $T$ is finitely satisfiable in $B$ but not realized in $B$. So, algebra $B$ is $u_\omega$-compact if and only if $T(B) = \emptyset$.

For $L$-algebras $B$ and $C$ we write $B \equiv C$ if $B$ and $C$ are universally equivalent, i.e., $\text{Ucl}(B) = \text{Ucl}(C)$.

**Lemma 7.3.** Let $B$ and $C$ be $L$-algebras and $B \leq C$. If $B \equiv C$ then $T(C) \subseteq T(B)$.

**Proof.** Suppose $T \in T$ and $T$ is finitely satisfiable in $C$. Then, by Lemma 4.17, $T$ is finitely satisfiable in $B$. If $T \not\in T(B)$, then $T$ is realized in $B$. As $B \leq C$, then $T$ is realized in $C$ and $T \not\in T(C)$.

**Theorem 7.4.** Let $B$ be an $L$-algebra. Then there exists an elementary extension $B^*$ of $B$, such that $B^*$ is $u_\omega$-compact (in particularly, $B^*$ is $q_\omega$-compact).

**Proof.** Consider a well-ordering $(I, \prec)$ on $T(B)$. Let us construct an elementary chain $(B_i, i \in I)$. At first, take $B_0 = B$. Then $B_1$ is an ultrapower of $B$ where $T_0$ is realized. By Compactness Theorem, such $B_1$ exists and, by Theorem 7.1, $B_1$ is an elementary extension of $B$. Further, $B_2$ is an ultrapower of $B_1$ where $T_1$ is realized, and so on. For an ordinal $\alpha = \beta + 1$ we put $B_\alpha$ as an ultrapower of $B_\beta$ where $T_\beta$ is realized, and $B_\alpha = \bigcup_{\beta < \alpha} B_\beta$ for a limit ordinal $\alpha$. Desired algebra $B^*$ is $\bigcup_{i \in I} B_i$. Indeed, $B^*$ is an elementary extension of $B$, by Theorem 7.1 and Proposition 7.2.

Let us show that $B^*$ is $u_\omega$-compact. By Lemma 7.3, $T(B^*) \subseteq T(B)$. Every set of formulas $T$ from $T(B)$ is realized in $B^*$. So $T(B^*) = \emptyset$.

**Corollary 7.5.** For an arbitrary algebra $B$ there exists $u_\omega$-compact algebra $B^*$ which is elementary equivalent to $B$.

In Theorem 7.4 we constructed $u_\omega$-compact extension $B^*$ of $B$ such that $B^*$ is elementary equivalent to $B$. One can modify the idea of Theorem 7.3 and find more constructive $u_\omega$-compact extension $B$ which is universally equivalent to $B$. 

41
Proposition 7.6. Let $\mathcal{B}$ be an $L$-algebra. Then there exists an extension $\mathcal{C}$ of $\mathcal{B}$ such that $\mathcal{C}$ is $u_\omega$-compact and $\mathcal{C} \equiv_\forall \mathcal{B}$. Moreover, one can get $\mathcal{C}$ by (transfinite) induction in series of extensions

$$\mathcal{B} = C_0 < C_1 < C_2 \ldots,$$

where $C_{\beta+1}$ is finitely generated extension of $C_\beta$, and $C_\alpha = \bigcup_{\beta < \alpha} C_\beta$ is the union of the chain for a limit ordinal $\alpha$. Also $C_\alpha \equiv_\forall \mathcal{B}$ for all $\alpha$.

Proof. Let us construct $\mathcal{C}$ by means of transfinite induction on $|T(\mathcal{B})|$. Take $C_0 = \mathcal{B}$. Consider an algebra $\mathcal{B}_1$ where $T_0$ is realized. Let $b_1, \ldots, b_n \in B_1$ be elements such that $B_1 \models T(b_1, \ldots, b_n)$. Put $C_1$ as the subalgebra of $\mathcal{B}_1$ generated by subalgebra $B$ and elements $b_1, \ldots, b_n$. And so on.

If $\alpha = \beta + 1$ then we take $\mathcal{B}_\alpha$ as an ultrapower of $C_\beta$ where $T_\beta$ is realized, and $C_\alpha$ is subalgebra of $\mathcal{B}_\alpha$ generated by $C_\beta$ and finite set of element in $\mathcal{B}_\alpha$ which realize formulas from $T_\beta$. It is easy that $C_\alpha \equiv_\forall C_\beta$.

For a limit ordinal $\alpha$ we put $C_\alpha = \bigcup_{\beta < \alpha} C_\beta$ as the union of the chain $(C_\beta, \beta < \alpha)$. In this case $C_\alpha = \lim \leftarrow C_\beta$ is also the direct limit of the direct system $(C_\beta, \beta < \alpha)$, therefore $C_\alpha \in \text{Ucl}((C_\beta, \beta < \alpha))$ [6, Theorem 1.2.9]. Since $C_\beta < C_\alpha$ we have $C_\beta \in \text{Ucl}(C_\alpha)$ for all $\beta < \alpha$. By induction, $C_\beta \equiv_\forall C_\gamma$ for any $\beta, \gamma < \alpha$. Therefore, $C_\alpha \equiv_\forall C_\beta$ for every $\beta < \alpha$.

At the end of such process we get an extension $\mathcal{C}$ of $\mathcal{B}$ such that $\mathcal{C} \equiv_\forall \mathcal{B}$ and $T(\mathcal{C}) = \emptyset$, i.e., $\mathcal{C}$ is $u_\omega$-compact. \qed

The following results are also useful in universal algebraic geometry.

**Lemma 7.7.** Let $\mathcal{B}, \mathcal{C}$ be an $L$-algebras. Suppose that $\mathcal{B}$ is $q_\omega$-compact, $\mathcal{C} \in \text{Qvar}(\mathcal{B})$, and every finitely generated subalgebra $\mathcal{B}_0 < \mathcal{B}$ is separated by $\mathcal{C}$. Then $\mathcal{C}$ is $q_\omega$-compact and and $\text{Qvar}(\mathcal{B}) = \text{Qvar}(\mathcal{C})$.

**Lemma 7.8.** Let $\mathcal{B}, \mathcal{C}$ be an $L$-algebras. Suppose that $\mathcal{B}$ is $u_\omega$-compact, $\mathcal{C} \in \text{Ucl}(\mathcal{B})$, and every finitely generated subalgebra $\mathcal{B}_0 < \mathcal{B}$ is discriminated by $\mathcal{C}$. Then $\mathcal{C}$ is $u_\omega$-compact and $\text{Ucl}(\mathcal{B}) = \text{Ucl}(\mathcal{C})$. 

42
Proof. We prove only statement about $u_\omega$-compactness. Statement about $q_\omega$-compactness may be proven in much the same way. By Theorem 4.2, it is sufficient to show that $\text{Ucl}(C)_\omega \subseteq \text{Dis}(C)_\omega$ (inclusion $\text{Ucl}(C)_\omega \supseteq \text{Dis}(C)_\omega$ holds anyway). As $C \in \text{Ucl}(B)$ then $\text{Ucl}(C) \subseteq \text{Ucl}(B)$ and $\text{Ucl}(C)_\omega \subseteq \text{Ucl}(B)_\omega$. Since $B$ is $u_\omega$-compact we have $\text{Ucl}(B)_\omega = \text{Dis}(B)_\omega$. If every finitely generated subalgebra $B_0 < B$ is discriminated by $C$ then $\text{Dis}(B)_\omega \subseteq \text{Dis}(C)_\omega$. Therefore, $\text{Ucl}(C)_\omega \subseteq \text{Dis}(C)_\omega$, as desired. Also we got $\text{Ucl}(B)_\omega = \text{Ucl}(C)_\omega$ that implies $\text{Ucl}(B) = \text{Ucl}(C)$. 

For $L$-algebra $A$ we denote by $L_A = L \cup \{ c_a \mid a \in A \}$ the language $L$ extended by elements from $A$ as new constant symbols [3, subsection 3.4]. An algebra $M$ in $L_A$ is called $A$-algebra if the map $h: A \to M$, $h(a) = c_a^M$, $a \in A$, is embedding.

Proposition 7.9. Let $A$ be an $L$-algebra. Consider $A$ as $A$-algebra. If $A$ is $q_\omega$-compact (in the language $L_A$) then every $A$-algebra $C$ from $\text{Qvar}_A(A)$ is $q_\omega$-compact. If $A$ is $u_\omega$-compact (in the language $L_A$) then every $A$-algebra $C$ from $\text{Ucl}_A(A)$ is $u_\omega$-compact.

Proof. We use here Lemmas 7.7 and 7.8. Every finitely generated subalgebra $A_0$ of $A$ is an $L_A$-algebra, so $A_0 = A$. Since $C$ is $A$-algebra then $A$ is separated and discriminated by $C$ in the obvious way. Thus, we have obtained the looked-for result.

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