SQUARE-INTEGRABLE COACTIONS
OF LOCALLY COMPACT QUANTUM GROUPS

ALCIDES BUSS AND RALF MEYER

Abstract. We define and study square-integrable coactions of locally compact quantum groups on Hilbert modules, generalising previous work for group actions. As special cases, we consider square-integrable Hilbert space corepresentations and integrable coactions on $C^*$-algebras. Our main result is an equivariant generalisation of Kasparov’s Stabilisation Theorem.

1. Introduction

This article generalises previous work for group actions by Marc Rieffel and the second author in [13, 15] to coactions of locally compact quantum groups. First we briefly explain the results for group actions we are going to generalise.

Square-integrable group representations have played an important role in representation theory for a long time. Roughly speaking, a unitary representation $\pi$ of a locally compact group $G$ on a Hilbert space $H$ is square-integrable if there are enough vectors $\xi, \eta \in H$ for which the function $c_{\xi\eta}(g) := \langle \pi_g(\xi), \eta \rangle$ belongs to the Hilbert space $L^2(G)$, defined using a left invariant Haar measure on $G$. Irreducible square-integrable representations behave in many respects like irreducible representations of compact groups: they are strongly contained in the regular representation, and the coefficients $c_{\xi\eta}$ satisfy orthogonality relations reminiscent of the relations in the Peter–Weyl Theorem—except that the modular function complicates the formulas for non-unimodular groups.

The definition of $c_{\xi\eta}$ makes perfect sense if $\xi$ and $\eta$ are elements of a Hilbert $B$-module, where $B$ is some $C^*$-algebra with a continuous action of $G$. Square-integrability requires this coefficient to belong to the Hilbert $B$-module $L^2(G, B)$. The main general result about square-integrable group actions on Hilbert modules is that a countably generated $G$-equivariant Hilbert $B$-module $E$ is square-integrable if and only if

$$E \oplus L^2(G, B)^\infty \cong L^2(G, B)^\infty.$$ 

This is an equivariant analogue of Kasparov’s Stabilisation Theorem and generalises the relationship between square-integrable Hilbert space representations and the regular representation.

A group action $\beta$ on a $C^*$-algebra $B$ is integrable if there is a dense subspace of positive elements $x \in B^+$ for which the integral $\int_G \beta_g(x) \, dg$ converges (unconditionally) in the strict topology. This notion is closely related to square-integrability. A group action on a Hilbert module $E$ is square-integrable if and only if the induced...
action on $K(E)$ is integrable. As a consequence, an action on a $C^*$-algebra $B$ is integrable if and only if $B$ is square-integrable as an equivariant Hilbert module over itself.

An action on a commutative $C^*$-algebra $C_0(X)$ is integrable if and only if the corresponding action on $X$ is proper (see [13]). But for non-commutative $C^*$-algebras, integrability is more closely related to stability than to properness. The equivariant stabilisation theorem forces the diagonal action on $E \otimes L^2(G)$ to be square-integrable for any Hilbert module $E$, so that the diagonal action on $B \otimes K(L^2(G)$ is integrable for any $C^*$-algebra $B$. Furthermore, dual actions and actions of compact groups are integrable.

In this article, we are going to extend these results to coactions of locally compact quantum groups. Group actions become coactions of the locally compact quantum group $(C_0(G), \Delta)$ with $\Delta(f)(x, y) := f(x \cdot y)$ for all $x, y \in G$, $f \in C_0(G)$.

The main new difficulty is that the straightforward relationship between $C_0(G)$, $L^2(G)$, and $C^*_r(G)$ becomes less transparent. Let $(G, \Delta)$ be a locally compact quantum group. Then $G$ corresponds to $C_0(G)$. The analogue of $L^2(G)$ is the Hilbert space $L$ associated to a left invariant Haar weight on $G$. It is related to $G$ by a densely defined, closed, unbounded linear map with dense range

$$\Lambda: G \supseteq \text{dom}(\Lambda) \rightarrow L,$$

which is characterised by the condition $\langle \Lambda(x), \Lambda(y) \rangle = \varphi(x^* y)$ for all $x, y \in \text{dom}(\Lambda)$, where $\varphi$ denotes the left invariant Haar weight on $G$. In the group case, $\Lambda$ is the “identical” map that views a function in $\text{dom}(\Lambda) = C_0(G) \cap L^2(G) \subseteq C_0(G)$ as an element of $L^2(G)$. The reduced group $C^*$-algebra $C^*_r(G)$ corresponds to the dual quantum group $\hat{G}$. This has its own left invariant Haar weight $\hat{\varphi}$, and the corresponding Hilbert space may be identified with $L$. This involves another densely defined, closed, unbounded linear map with dense range

$$\hat{\Lambda}: \hat{G} \supseteq \text{dom}(\hat{\Lambda}) \rightarrow L.$$

In the group case, $\text{dom}(\hat{\Lambda})$ contains $L^1(G) \cap L^2(G)$, viewed as a subspace of $C^*_r(G)$, and maps it “identically” to $L^2(G)$.

Using the unbounded linear maps $\Lambda$ and $\hat{\Lambda}$, we carry over the theory of square-integrable Hilbert space representations of groups to corepresentations of $(G, \Delta)$ on Hilbert spaces in Section 4. To prove that these corepresentations have the expected properties, it is convenient to view them as representations of the (universal) dual quantum group of $(G, \Delta)$. The resulting notion of square-integrable representation is a special case of a definition by François Combes [15].

Our proof of the orthogonality relations for irreducible square-integrable corepresentations follows [13]: the crucial ingredient is the equivariant bounded operator $\langle \xi \rangle: H \rightarrow L$ associated to a square-integrable vector $\xi \in H$, where $H$ is a Hilbert space with a coaction of $(G, \Delta)$ and $L$ is equipped with the left regular corepresentation of $(G, \Delta)$. This construction implies immediately that square-integrable irreducible corepresentations embed isometrically and equivariantly into the left regular corepresentation.

For compact quantum groups, square-integrability is no restriction, and our theory specialises to the familiar orthogonality relations for irreducible corepresentations of compact quantum groups that appear in the Peter–Weyl Theorem.

We study integrable coactions on $C^*$-algebras in Section 4. The main ingredient in the definition is the densely defined, unbounded linear map

$$\text{id}_B \otimes \varphi: \mathcal{M}(B \otimes G) \supseteq \text{dom}(\text{id}_B \otimes \varphi) \rightarrow \mathcal{M}(B)$$

described in [10]. We call a positive element $x \in B^+$ integrable with respect to a coaction $\Delta_B$ of $(G, \Delta)$ on $B$ if $\Delta_B(x)$ belongs to the domain of $\text{id}_B \otimes \varphi$. The
coaction $\Delta_B$ is integrable if the set of positive integrable elements is dense in $B^+$. It follows from the left invariance of the Haar weight that the range of $\id_B \otimes \varphi$ consists of $G$-invariant multipliers only. The most important example of an integrable coaction is the coaction $\Delta$ of $G$ on itself. Using some general permanence properties of integrability, this example implies that stable coactions and dual coactions are integrable. Here stable coactions are coactions on $K(B \otimes L)$, where we equip the Hilbert $B$-module $B \otimes L$ with the diagonal coaction, using the left regular coaction on $L$.

Both square-integrable Hilbert space corepresentations and integrable coactions on $C^*$-algebras are special cases of square-integrable coactions on Hilbert modules, which we introduce in Section 1. Let $\mathcal{E}$ be a Hilbert $B$-module with a coaction $\Delta_\mathcal{E}$ of $(G, \Delta)$. We define the coefficient $c_{\xi \eta} \in \mathcal{M}(B \otimes G)$ for $\xi, \eta \in \mathcal{E}$ by

$$c_{\xi \eta} := \Delta_\mathcal{E}(\xi)^*(\eta \otimes 1_G).$$

We call $\xi$ square-integrable if this belongs to the domain of the densely defined unbounded map $\id_B \otimes \Lambda: \mathcal{M}(B \otimes G) \supset \text{dom}(\id_B \otimes \Lambda) \to B \otimes L$ for all $\eta \in \mathcal{E}$; the latter map is also described in [10]. The Hilbert module $\mathcal{E}$ is called square-integrable if square-integrable elements are dense.

We check that $\xi \in \mathcal{E}$ is square-integrable if and only if the compact operator $|\xi\rangle\langle\xi|$ is integrable with respect to the coaction on $\mathcal{K}(\mathcal{E})$ induced by the coaction on $\mathcal{E}$. This allows us to carry over many results on integrable coactions on $C^*$-algebras. In particular, we conclude that $\mathcal{E} \otimes \mathcal{L}$ with the diagonal coaction is square-integrable for any $\mathcal{E}$. We also establish that the operator $\langle\langle \xi \rangle\rangle: \mathcal{E} \to B \otimes \mathcal{L}$ that is defined for square-integrable $\xi$ by $\langle\langle \xi \rangle\rangle(\eta) = (\id \otimes \Lambda)(c_{\xi \eta})$ is adjointable and $G$-equivariant, and we describe its adjoint.

As a consequence, a square-integrable Hilbert module admits many equivariant adjointable maps to the standard Hilbert module $B \otimes \mathcal{L}$. This together with an idea by Mingo and Phillips (see [14]) is the main ingredient in our proof of the equivariant stabilisation theorem, which occupies Section 1. It asserts that a countably generated $G$-equivariant Hilbert $B$-module $\mathcal{E}$ is square-integrable if and only if there is an equivariant isomorphism

$$\mathcal{E} \otimes B \otimes \mathcal{L}^\infty \cong B \otimes \mathcal{L}^\infty.$$ 

Here $B \otimes \mathcal{L}$ carries the diagonal coaction using the left regular corepresentation on $\mathcal{L}$. Thus $B \otimes \mathcal{L}$ plays the role of $L^2(G, B)$ in the group case.

2. Preliminaries on locally compact quantum groups

We use the theory of locally compact quantum groups developed by Johan Kustermans and Stefaan Vaes in [10, 11] throughout this article. In this section, we fix our notation and recall some important facts that we shall need repeatedly.

The theory of square-integrable group representations is based on a close relationship between the spaces $C_0(G)$, $C^*_r(G)$, and $L^2(G)$. Let $f: G \to \mathbb{C}$ be a continuous function with compact support. Such a function can play at least three different roles. First, we may view $f$ as an element of the $C^*$-algebra $C_0(G)$ of continuous functions on $G$ vanishing at infinity. Secondly, we may view $f$ as an element of the reduced group $C^*$-algebra $C^*_r(G)$. And thirdly, we may view $f$ as an element of the Hilbert space $L^2(G) = L^2(G, dx)$ of square-integrable functions on $G$; here $dx$ denotes a left invariant Haar measure on $G$. More formally, the comparison of these three different roles provides us with densely defined, closed, unbounded operators between the Banach spaces $C_0(G)$, $L^2(G)$, and $C^*_r(G)$. We shall need a quantum group generalisation of these unbounded operators.
Let $\mathcal{G}$ be a locally compact quantum group and let $\hat{\mathcal{G}}$ be its dual. Their quantum group structures are encoded by comultiplications

$$\Delta: \mathcal{G} \to \mathcal{M}(\mathcal{G} \otimes \mathcal{G}), \quad \hat{\Delta}: \hat{\mathcal{G}} \to \mathcal{M}(\hat{\mathcal{G}} \otimes \hat{\mathcal{G}}),$$

where $\mathcal{M}$ denotes multiplier algebras and $\otimes$ spatial $C^*$-algebra tensor products. A basic requirement in [10] is the existence of left and right invariant faithful proper weights on $\mathcal{G}$ which satisfy a KMS condition; the existence of such weights for the dual follows. We denote the left invariant Haar weights on $\mathcal{G}$ and $\hat{\mathcal{G}}$ by $\varphi$ and $\hat{\varphi}$.

Any proper weight on a $C^*$-algebra gives rise to a Hilbert space representation by a generalisation of the Gelfand–Naimark–Segal construction (see [4]). When we apply this to the weight $\varphi$, we get a Hilbert space $\mathcal{L}$, a faithful $^*$-representation of $\mathcal{G}$ on $\mathcal{L}$, which we simply denote by left multiplication $\mathcal{G} \times \mathcal{L} \to \mathcal{L}$, $(x, \eta) \mapsto x \cdot \eta$, and a closed unbounded linear map $\Lambda$ between dense subspaces of $\mathcal{G}$ and $\mathcal{L}$.

Actually, we mainly use the extension of $\Lambda$ to the multiplier algebra. The weight $\varphi$ extends uniquely to a strictly lower semi-continuous weight on $\mathcal{M}(\mathcal{G})$, which we still denote by $\varphi$. We let

$$\text{dom } \Lambda := \{ x \in \mathcal{M}(\mathcal{G}) \mid \varphi(x^*x) < \infty \}.$$

There is a linear map $\Lambda: \text{dom } \Lambda \to \mathcal{L}$ that satisfies

$$\langle \Lambda(x), \Lambda(y) \rangle = \varphi(x^*y) \quad \text{for all } x, y \in \text{dom } \Lambda, \text{ and}$$

$$\Lambda(xy) = x \cdot \Lambda(y) \quad \text{for all } x \in \mathcal{M}(\mathcal{G}), y \in \text{dom } \Lambda.$$

The map $\Lambda$ above is closed, densely defined, and has dense range with respect to the strict topology on $\mathcal{M}(\mathcal{G})$ and the norm topology on $\mathcal{L}$. Its restriction to $\mathcal{G}$ itself is closed, densely defined, and has dense range with respect to the norm topologies on both $\mathcal{G}$ and $\mathcal{L}$.

Notice that unlike in [10],[11] our Hilbert space inner products are linear in the second variable—this is standard for Hilbert modules.

In the group case, we have $\mathcal{G} = \mathcal{C}_0(G)$ and $\mathcal{L} = L^2(G)$, where we tacitly use the left invariant Haar measure on $G$. Then $\text{dom } \Lambda = \mathcal{C}_0(G) \cap L^2(G)$, viewed as a subspace of $\mathcal{C}_0(G)$, and $\Lambda$ maps this "identically" to a subspace of $L^2(G)$.

If $G$ is compact, then $\Lambda$ is everywhere defined and bounded, that is, $\mathcal{C}_0(G) = \mathcal{C}(G)$ is contained in $L^2(G)$. This observation generalises to a characterisation for compact quantum groups:

**Remark 2.4.** A quantum group $\mathcal{G}$ is compact if and only if the Haar weight $\varphi$ is bounded, if and only if $\Lambda$ is a bounded linear map $\mathcal{G} \to \mathcal{L}$.

The quantum group structure on $\mathcal{G}$ may also be encoded using a multiplicative unitary. This is the unique unitary operator $W$ on the Hilbert space tensor product $\mathcal{L} \otimes \mathcal{L}$ that satisfies

$$W^*(\Lambda(x) \otimes \Lambda(y)) = (\Lambda \otimes \Lambda)(\Lambda(y) \cdot (x \otimes 1)) \quad \text{for all } x, y \in \text{dom } \Lambda.$$

Then

$$\Delta(x) = W^*(1 \otimes x)W \quad \text{for all } x \in \mathcal{G},$$

where we view both sides as operators on $\mathcal{L}$.

Let $\mathcal{G}^*$ be the dual Banach space of bounded linear functionals on $\mathcal{G}$. Let $\mathbb{B}(\mathcal{L})$ be the algebra of bounded operators on $\mathcal{L}$. We define a map

$$\lambda: \mathcal{G}^* \to \mathbb{B}(\mathcal{L}), \quad \omega \mapsto (\omega \otimes \text{id})(W)$$

as in [11 §1.1]. This becomes an algebra homomorphism if we equip $\mathcal{G}^*$ with the convolution product $\omega * \eta := (\omega \otimes \eta) \circ \Delta$ for $\omega, \eta \in \mathcal{G}$; here we have tacitly extended $\omega \otimes \eta$ to a strictly continuous linear functional on $\mathcal{M}(\mathcal{G} \otimes \mathcal{G})$. 
In the group case, $G^*$ is the Banach space of bounded measures on $G$; the map $\lambda$ lets a measure act on $L^2(G)$ by convolution on the left; and $\ast$ is the usual convolution of measures. To get $C^*_v(G)$, we must restrict attention to the subspace $L^1(G)$ of measures that are absolutely continuous with respect to the Haar measure. Something similar happens in the quantum group case.

If $a, b \in G$, then we define possibly unbounded linear functionals $b \cdot \varphi, \varphi \cdot a$, and $b \cdot \varphi \cdot a$ on $G$ by

$$b \cdot \varphi(x) := \varphi(xb), \quad \varphi \cdot a(x) := \varphi(ax), \quad b \cdot \varphi \cdot a(x) := \varphi(axb).$$

The functional $b \cdot \varphi \cdot a$ is bounded if $a^*, b \in \text{dom } \Lambda$ because

$$(2.7) \quad b \cdot \varphi \cdot a(x) = \langle \Lambda(a^*), x \cdot \Lambda(b) \rangle \quad \text{for all } a^*, b \in \text{dom } \Lambda, x \in G.$$

Let $L^1(G) \subseteq G^*$ be the closed linear span of the set of functionals $b \cdot \varphi \cdot a^*$ for $a, b \in \text{dom } \Lambda$.

We define an unbounded linear operator $\mu$ from $G$ to $L^1(G) \subseteq G^*$ by letting

$$(2.8) \quad x \in \text{dom } \mu \iff x \cdot \varphi \text{ is bounded and belongs to } L^1(G) \subseteq G^*$$

$$\mu(x) := x \cdot \varphi \quad \text{for all } x \in \text{dom } \mu.$$

It is easy to see that this operator is closed. We will see below that it is densely defined and has dense range.

The $C^*$-algebra $\hat{G}$ is, by definition, the closure of $\lambda(L^1(G))$ in $B(L)$. Its comultiplication is defined by

$$(2.9) \quad \hat{\Delta}(x) := \hat{W}^*(1 \otimes x)\hat{W} \quad \text{with } \hat{W} := \Sigma W^* \Sigma,$$

where $\Sigma$ flips the two tensor factors in $L \otimes L$.

**Remark 2.10.** Since the von Neumann algebra generated by $G$ in $B(L)$ is in standard form (see [16, 10.15]), all normal functionals on it are vector functionals by [17, Theorem V.3.15]). Therefore, any element of $L^1(G)$ is of the form $x \mapsto \langle \eta, x \omega \rangle$ for some $\eta, \omega \in L$. This generalises the fact that any function in $L^1(G)$ is a product of two functions in $L^2(G)$.

Recall that $\varphi$ is a KMS weight with respect to the modular automorphism group $\sigma : \mathbb{R} \times G \to G$, which is determined uniquely by the weight $\varphi$. Extending $\sigma$ analytically from $\mathbb{R}$ to $\mathbb{C}$, the KMS condition implies

$$(2.11) \quad \varphi(yx) = \varphi(x \sigma_{-1}(y)) \quad \text{for all } y \in \text{dom } \sigma_{-1}, x \in \text{dom } \varphi$$

(see [10, Proposition 1.12.3]). Briefly, we have

$$(2.12) \quad y \cdot \varphi = \varphi \cdot \sigma_{-1}(y) \quad \text{for all } y \in \text{dom } \sigma_{-1}.$$

It follows that the functionals $b \cdot \varphi$ and $\varphi \cdot a$, and $b \cdot \varphi \cdot a$ are densely defined for all $a, b \in G$. Moreover, $ab \cdot \varphi(x) = (b \cdot \varphi \cdot \sigma_{-1}(a))(x) = \langle \Lambda(b^*), x \cdot \Lambda(\sigma_{-1}(a)) \rangle$, so that $ab \cdot \varphi$ belongs to $L^1(G) \subseteq \hat{G}$ if $b^* \in \text{dom } \Lambda$ and $a \in \text{dom } (\Lambda \circ \sigma_{-1})$.

The subsets of $ab \cdot \varphi$ and of $ab$ with $a, b$ as above are dense in $L^1(G)$ and in $G$, respectively. This shows that the operator $\mu : G \ni \text{dom } \mu \to L^1(G)$ above is densely defined and has dense range.

We also get a densely defined unbounded linear map $\hat{\Lambda}$ from $\hat{G}$ to $L$ by

$$\hat{\Lambda}(\lambda(x \cdot \varphi)) := \Lambda(x) \quad \text{for } x \in \text{dom } \mu$$

(we have dom $\mu \subseteq \text{dom } \Lambda$). More precisely, we let $\hat{\Lambda}$ be the closure of this linear map, which is a densely defined closed linear operator from $\hat{G}$ to $L$. 

This definition of $\hat{\Lambda}$ is equivalent to the one in [11] page 75. As a consequence, $\hat{\Lambda}$ together with the identical representation of $\hat{G}$ on $L$ is a GNS construction for the Haar weight $\varphi$ on $\hat{G}$, that is,

\[(2.13)\quad \hat{\varphi}(x^*y) = \langle \hat{\Lambda}(x), \hat{\Lambda}(y) \rangle, \quad \hat{\Lambda}(ay) = a(\hat{\Lambda}(y))\]

for all $x,y \in \text{dom} \hat{\Lambda}$, $a \in \hat{G} \subseteq B(L)$. In particular,

\[(2.14)\quad \hat{\varphi}(\lambda(x \cdot \varphi)^* \cdot \lambda(y \cdot \varphi)) = \langle \Lambda(x), \Lambda(y) \rangle = \varphi(x^*y)\]

for all $x,y \in \text{dom} \mu$.

In the group case, $x \mapsto x \cdot \varphi$ is the standard way to map a function on $G$ to a measure on $G$. Both $\mu$ and $\Lambda$ extend “identical” maps on suitable dense subspaces: $\mu$ is the identical map from $C_0(G) \cap L^1(G)$ viewed as a subspace of $C_0(G)$ to the same space viewed as a subspace of $L^1(G)$, and $\Lambda$ extends the identical map from $L^1(G) \cap L^2(G)$ viewed as a subspace of $C^*_r(G)$ to the same space viewed as a subspace of $L^2(G)$.

\[\text{Remark 2.15. Let } \hat{G} \text{ be compact, so that } \Lambda \text{ is bounded. If } x \in \text{dom} \mu, \text{ then}
\]

\[
\|x\varphi\| = \sup\{|x\varphi(y^*)| \mid y \in \hat{G}, \|y\| \leq 1\} = \sup\{|\langle \Lambda(y), \Lambda(x) \rangle| \mid y \in \hat{G}, \|y\| \leq 1\} \\
\quad \leq \|\Lambda\| \sup\{|\langle \Lambda(y), \Lambda(x) \rangle| \mid y \in \hat{G}, \|y\| \leq 1\} = \|\Lambda\| \cdot \|\Lambda(x)\|,
\]

where we used the definition of the norm on $G^*$, the relation $x\varphi(y^*) = \langle \Lambda(y), \Lambda(x) \rangle$, and Riesz’ Theorem for the Hilbert space $L$. Since $\Lambda(\text{dom } \mu)$ is dense in $L$, it follows that there is a bounded map $\Lambda : L \to L^1(\hat{G})$ that maps $\hat{\lambda}(\omega)$ to $\omega$ for all $\omega \in L^1(\hat{G})$.

Since any discrete quantum group is the dual of a compact one, this shows that the map $\Lambda : \hat{G} \supseteq \text{dom} \Lambda \to L$ is the inverse of a bounded map $\Lambda^{-1} : L \to \hat{G}$ for any discrete quantum group $\hat{G}$.

Recall that a (right) corepresentation of $\hat{G}$ on a Hilbert space $\mathcal{H}$ is a unitary operator $U$ on the Hilbert $\hat{G}$-module $\mathcal{H} \otimes \hat{G}$ that satisfies $(\text{id}_\mathcal{H} \otimes \Delta)(U) = U_{12} \cdot U_{13}$, where we use the standard leg numbering notation. This generates a (right) coaction

\[(2.16)\quad \Delta_U : \mathcal{H} \to \mathcal{M}(\mathcal{H} \otimes \hat{G}) := \mathbb{B}(\mathcal{H}, \mathcal{H} \otimes \hat{G}), \quad \eta \mapsto U(\eta \otimes 1_{\hat{G}}).
\]

Conversely, any coaction on $\mathcal{H}$ with the usual properties comes from a unique corepresentation $U$ (see [1] Proposition 2.4). We will mainly use right corepresentations and right coactions in this article.

A right corepresentation of $\hat{G}$ yields a left $G^*$-module structure via

\[(2.17)\quad \omega \ast_U \eta := (\text{id} \otimes \omega)(\Delta_U(\eta)) = (\text{id} \otimes \omega)(U(\eta \otimes 1)) =: (\text{id} \otimes \omega)(U)(\eta).
\]

This module structure is always non-degenerate. The \textit{universal locally compact quantum group} $(\hat{G}_u, \Delta_u)$ associated to the reduced locally compact quantum group $(\hat{G}, \Delta)$ is defined by the universal property that its non-degenerate $^*$-representations correspond exactly to right corepresentations of $\hat{G}$, see [8].

In the group case, a strongly continuous unitary group representation $\pi$ of $G$ on $\mathcal{H}$ is encoded either by the corepresentation

\[U : C_0(G, \mathcal{H}) \to C_0(G, \mathcal{H}), \quad (Uf)(g) := \pi_g(f(g))\]

or by the coaction

\[\Delta_U : \mathcal{H} \to C_0(G, \mathcal{H}), \quad (\Delta_U \eta)(g) := \pi_g(\eta).
\]

The integrated form defined above lets a measure $\omega$ act by $\omega \ast \eta = \int_G \pi_g(\eta) \, d\omega(g)$. The universal dual quantum group $\hat{G}_u$ is the full group $C^*$-algebra, whereas $\hat{G}$ itself is the reduced group $C^*$-algebra of $G$.

The \textit{left regular corepresentation} of $\hat{G}$ on $L$ is a crucial example for us: it is the paradigm of square-integrability. As a right corepresentation, it is the unitary
\[ \Sigma W \Sigma, \text{ where } W \text{ is the multiplicative unitary of } \mathcal{G} \text{ described in (2.3), but viewed as an adjointable operator on the Hilbert } \mathcal{G}\text{-module } \mathcal{G} \otimes \mathcal{L}, \text{ and } \Sigma \text{ is the isomorphism } \mathcal{L} \otimes \mathcal{G} \leftrightarrow \mathcal{G} \otimes \mathcal{L} \text{ that flips the tensor factors. The corresponding right coaction } \Delta_\lambda \text{ is} \]

\[ (2.18) \quad \Delta_\lambda(\eta) = \Sigma W(1 \otimes \eta) \quad \text{for all } \eta \in \mathcal{L}. \]

Let \( \omega \star_\lambda x \) for \( \omega \in \mathcal{G}^* \), \( x \in \mathcal{L} \) denote the resulting non-degenerate left \( \mathcal{G}^* \)-module structure on \( \mathcal{L} \). The module structure \( \star_\lambda \) determines a \( * \)-representation of \( \mathcal{G} \), not just of \( \hat{G}_u \). This is the left regular representation \( \lambda \) we used to define \( \hat{G} \), that is,

\[ (2.19) \quad \omega \star_\lambda \eta = \lambda(\omega)(\eta) \quad \text{for all } \omega \in \mathcal{G}^*, \eta \in \mathcal{L}. \]

This follows from the computation

\[ \omega \star_\lambda \eta = (\text{id} \otimes \omega) \circ \Sigma \circ W(1 \otimes \eta) = (\omega \otimes \text{id})(W(1 \otimes \eta)) \]

\[ = (\omega \otimes \text{id})(W')(\eta) = \lambda(\omega)(\eta). \]

We return to the regular representation. The \textit{left regular representation} \( \lambda \) of the dual quantum group \( \hat{G} \) on \( \mathcal{L} \) is described either by \( \lambda(\omega) = (\omega \otimes \text{id})(W) \) as above or by \( \lambda(\omega)(\hat{A}(x)) = \hat{A}(\omega x) \) for all \( \omega \in \mathcal{G}^, x \in \text{dom } \hat{A} \). Let \( \hat{T} \) be the closure of the unbounded conjugate-linear operator on \( \mathcal{L} \) defined by \( \hat{T}(\hat{A}(x)) := \hat{A}(x^*) \) if \( x, x^* \in \text{dom } \hat{A} \). This operator admits a polar decomposition

\[ (2.20) \quad \hat{T} = \hat{J} \hat{\nabla}^{1/2} = \hat{\nabla}^{-1/2} \hat{J} \]

for some conjugate-unitary \( \hat{J} \), called \textit{modular conjugation}, and a positive unbounded operator \( \hat{\nabla} \) on \( \mathcal{L} \), called the \textit{modular operator} of the KMS weight \( \hat{\varphi} \). This operator is closely related to the \textit{modular automorphism group} \( \hat{\sigma} \) of \( \hat{G} \):

\[ (2.21) \quad \hat{\nabla}^{is} \lambda(\omega) \hat{\nabla}^{-is} = \lambda(\hat{\sigma}_s(\omega)) \]

for all \( s \in \mathbb{C}^, \omega \in \text{dom } \hat{\sigma}_s \).

The \textit{right regular anti-representation} of \( \hat{G} \) is the map

\[ (2.22) \quad \rho : \hat{G} \rightarrow \mathbb{B}(\mathcal{L}), \quad \omega \mapsto \hat{J} \lambda(\omega)^* \hat{J}, \]

which satisfies \( \rho(x^*) = (\rho(x))^* \) and \( \rho(xy) = \rho(y)\rho(x) \) for all \( x, y \in \hat{G} \). It can be turned into a representation using the unitary antipode of \( \hat{G} \). Its range \( \hat{G}^c := \hat{J}\hat{G}\hat{J} \) is the \( C^*-\text{commutant} \) of \( \hat{G} \), which is a locally compact quantum group in its own right (see [11]). We use (2.21) and (2.20) to compute

\[ \rho(\omega)^* \hat{A}(x) = \hat{J} \lambda(\omega) \hat{J} \hat{A}(x) = \hat{J} \hat{\nabla}^{1/2} \lambda(\hat{\sigma}_{1/2}(\omega)) \hat{\nabla}^{-1/2} \hat{J} \hat{A}(x) \]

\[ = \hat{T} \lambda(\hat{\sigma}_{1/2}(\omega)) \hat{T} \hat{A}(x) = \hat{A}(x \cdot \hat{\sigma}_{1/2}(\omega)^*) = \hat{A}(x \cdot \hat{\sigma}_{-1/2}(\omega^*)) \]

providing \( x, x^* \in \text{dom } \hat{A} \) and \( \omega \in \text{dom } \hat{\sigma}_{1/2} \) (then \( x \cdot \hat{\sigma}_{-1/2}(\omega^*) \in \text{dom } \hat{A} \)). As a result,

\[ (2.23) \quad \rho(\hat{\sigma}_{1/2}(\omega)) \hat{A}(x) = \hat{A}(x \cdot \omega) \quad \text{for all } \omega \in \text{dom } \hat{\sigma}_{1/2}, x \in \text{dom } \hat{A}. \]

See also [10], Proposition 1.12.(2).

3. SQUARE-INTEGRABLE HILBERT SPACE REPRESENTATIONS

In the Hilbert space case, it makes essentially no difference whether we study corepresentations of \( \mathcal{G} \) or representations of \( \hat{G}_u \). The latter point of view has the advantage that square-integrable representations in this context are a special case of a general notion due to François Combes [4]. Here we recall this definition, check that the left regular representation is square-integrable, and establish the orthogonality relations for coefficients of square-integrable representations. Our proofs follow those by Marc Rieffel in [15].
The universal dual $\hat{\mathcal{G}}_u$ has its own Haar weight and corresponding GNS construction. Actually, we get these by composing the corresponding maps for $\hat{\mathcal{G}}$ with the canonical quotient mapping $\hat{\mathcal{G}}_u \to \hat{\mathcal{G}}$ (see §1.4). Therefore, we denote the Haar weight on $\hat{\mathcal{G}}_u$ by $\hat{\nu}$ as well, and we still write $\Lambda$ for the unbounded map from $\hat{\mathcal{G}}_u$ to the Hilbert space $\mathcal{L}$. The $C^*$-algebra $\hat{\mathcal{G}}_u$ with the left ideal $\text{dom} \, \Lambda$ and the sesquilinear form $(x, y) \mapsto \hat{\nu}(y^*x)$ is a left Hilbert system in the notation of §1.4.

Let $U \in \mathcal{M}(\mathcal{H} \otimes \mathcal{G})$ be a corepresentation of $\mathcal{G}$ on a Hilbert space $\mathcal{H}$, and let $\rho U : \hat{\mathcal{G}}_u \to \mathcal{B}(\mathcal{H})$ be the corresponding $^*$-representation of $\hat{\mathcal{G}}_u$.

**Definition 3.1** (§1.4). Let $\alpha, \beta \in \mathcal{H}$. The coefficient $c_{\alpha \beta}$ of the representation $\rho U$ is the linear functional

$$\hat{\mathcal{G}}_u \to \mathbb{C}, \quad x \mapsto \langle \beta, \rho U(x)\alpha \rangle.$$  

(Notice that our inner products are linear in the second variable.)

This coefficient is called **square-integrable** if there is $r \in \mathbb{R}_{>0}$ with

$$|c_{\alpha \beta}(x)|^2 \leq r \cdot \hat{\nu}(x^*x) = r \cdot \|\Lambda(x)\|_L^2.$$  

In this case, $\hat{\Lambda}(x) \mapsto c_{\alpha \beta}(x)$ extends uniquely to a bounded linear functional on $\mathcal{L}$. This is of the form $c_{\alpha \beta}(x) = \langle \tilde{c}_{\alpha \beta}, \hat{\Lambda}(x) \rangle$ for some $\tilde{c}_{\alpha \beta} \in \mathcal{L}$ by Riesz’s Theorem.

**Definition 3.2.** We call $\alpha$ square-integrable (with respect to $U$) if $c_{\alpha \beta}$ is square-integrable for each $\beta \in \mathcal{H}$. We call $\alpha$ $U$-bounded if there is $r \in \mathbb{R}_{>0}$ with

$$\|\rho U(\alpha)\|_L^2 \leq r \cdot \|\Lambda(\alpha)\|_L^2 = r \cdot \hat{\nu}(\alpha^*x)$$  

for all $x \in \text{dom} \, \Lambda$.

We let $\mathcal{H}_{si} \subseteq \mathcal{H}$ be the subset of square-integrable vectors.

The corepresentation $U$ is called **square-integrable** if the subspace $\mathcal{H}_{si}$ of square-integrable vectors is dense in $\mathcal{H}$.

**Lemma 3.3** (§1.4). A vector is $U$-bounded if and only if it is square-integrable.

**Proof.** It is clear that bounded vectors are square-integrable.

Conversely, if $\alpha \in \mathcal{H}_{si}$, then there is a linear map

$$\langle \alpha \rangle : \mathcal{H} \to \mathcal{L}, \quad \beta \mapsto \tilde{c}_{\alpha \beta}.$$  

Since its graph is closed, it is bounded. Let $|\alpha\rangle := \langle \alpha \rangle^* : \mathcal{L} \to \mathcal{H}$ be its adjoint. We have

$$\langle \alpha \rangle (\hat{\Lambda}(x)) = \rho U(x)\alpha$$  

for all $x \in \text{dom} \, \Lambda$

because $\langle \beta, \rho U(x)\alpha \rangle = c_{\alpha \beta}(x) = \langle \tilde{c}_{\alpha \beta}, \hat{\Lambda}(x) \rangle = \langle \langle \alpha \rangle^* \beta, \hat{\Lambda}(x) \rangle = \langle \beta, |\alpha\rangle \hat{\Lambda}(x) \rangle$.

The boundedness of $|\alpha\rangle$ means that $\alpha$ is $U$-bounded. \hfill $\square$

The operators $|\alpha\rangle$ and $\langle \alpha \rangle$ introduced in the proof of Lemma 3.3 are the cornerstones of the general theory of square-integrable representations.

Since $\hat{\Lambda}$ factors through the projection $\hat{\mathcal{G}}_u \to \hat{\mathcal{G}}$, Lemma 3.3 implies that any square-integrable representation of $\hat{\mathcal{G}}_u$ factors through $\hat{\mathcal{G}}_u \to \hat{\mathcal{G}}$, so that it is already a representation of $\hat{\mathcal{G}}$. Therefore, we may replace $\hat{\mathcal{G}}_u$ by $\hat{\mathcal{G}}$ in the following.

**Lemma 3.6.** Let $\alpha \in \mathcal{H}_{si}$. The operators $|\alpha\rangle : \mathcal{L} \to \mathcal{H}$ and $\langle \alpha \rangle : \mathcal{H} \to \mathcal{L}$ intertwine the corepresentation $U$ and the left regular corepresentation on $\mathcal{L}$.

**Proof.** The operator $|\alpha\rangle$ is a left $\hat{\mathcal{G}}$-module homomorphism by (3.5). So is its adjoint $\langle \alpha \rangle$ because the representations of $\hat{\mathcal{G}}$ on $\mathcal{H}$ and $\mathcal{L}$ are involutive. But the $\hat{\mathcal{G}}$-module homomorphisms with respect to the integrated forms of two corepresentations are exactly the intertwining operators. \hfill $\square$
Lemma 3.7. The left regular corepresentation of $\mathcal{G}$ on $\mathcal{L}$ is square-integrable.

Proof. Let $\hat{\sigma}$ denote the modular automorphism group of the weight $\hat{\varphi}$ on $\hat{\mathcal{G}}$ and let $\rho$ be the right regular anti-representation of $\hat{\mathcal{G}}$ defined in (2.22). Equation (2.23) yields

$$|\hat{\Lambda}(x)\rangle \Lambda(a) = \hat{\Lambda}(ax) = \rho(\hat{\sigma}_{i/2}(x))\hat{\Lambda}(a)$$

for all $a \in \text{dom } \hat{\Lambda}$, $x \in \text{dom } \hat{\mathcal{L}} \cap \text{dom } \hat{\sigma}_{i/2}$. Since $\rho$ is an isometric *-anti-representation of $\hat{\mathcal{G}}$, this shows that $\|\hat{\Lambda}(x)\| = \|\hat{\sigma}_{i/2}(x)\| < \infty$ for all $x \in \text{dom } \hat{\Lambda} \cap \text{dom } \hat{\sigma}_{i/2}$, so that these vectors are $\lambda$-bounded and hence square-integrable. Since $\hat{\Lambda}$ maps $\text{dom } \hat{\Lambda} \cap \text{dom } \hat{\sigma}_{i/2}$ to a dense subspace of $\mathcal{L}$, the square-integrable vectors are dense for the left regular corepresentation. \hfill $\Box$

Theorem 3.8. A Hilbert space corepresentation is square-integrable if and only if it is a direct sum of sub-corepresentations of the left regular corepresentation, and only if its integrated form $\hat{\mathcal{G}}_a \to \mathbb{B}(\mathcal{H})$ extends to a normal *-representation of the von Neumann algebra completion $\lambda(\hat{\mathcal{G}})^{''}$ of $\hat{\mathcal{G}}$.

Proof. It is easy to see that square-integrability is inherited by sub-corepresentations and direct sums. Hence Lemma 3.7 shows that direct sums of sub-corepresentations of the left regular corepresentation are square-integrable.

Conversely, let $U$ be a square-integrable corepresentation on a Hilbert space $\mathcal{H}$. Any $\xi \in \mathcal{H}_{\text{si}}$ yields a non-zero equivariant linear operator $|\xi\rangle : \mathcal{L} \to \mathcal{H}$. The partial isometry from its polar decomposition is equivariant as well and identifies a non-zero sub-corepresentation of $\mathcal{H}$ with a sub-corepresentation of $\mathcal{L}$. Now we use Zorn’s Lemma to complete the proof: there is a maximal set of orthogonal, non-zero $\mathcal{G}$-invariant subspaces $(\mathcal{H}_i)_{i \in I}$ such that the restriction of $U$ to each $\mathcal{H}_i$ is unitaly equivalent to a sub-corepresentation of the left regular corepresentation. If $\bigoplus \mathcal{H}_i$ is not yet all of $\mathcal{H}$, then its complement contains a square-integrable vector, which yields a sub-corepresentation that we may add to our set of subspaces, contradicting maximality. Hence $\bigoplus \mathcal{H}_i = \mathcal{H}$.

Of course, the left regular representation of $\hat{\mathcal{G}}_a$ on $\mathcal{L}$ extends to a normal *-representation of the von Neumann algebra completion $\lambda(\hat{\mathcal{G}})^{''}$. This remains true for any direct sum of sub-corepresentations of the left regular corepresentation and hence holds for any square-integrable corepresentation. Conversely, it is well-known that any such normal *-representation of $\lambda(\hat{\mathcal{G}})^{''}$ may be decomposed as a direct sum of subrepresentations of the standard representation. \hfill $\Box$

Lemma 3.9. Let $\xi \in \mathcal{H}_{\text{si}}$ and let $\omega \in \mathcal{G}^* \subseteq \mathcal{M}(\hat{\mathcal{G}})$. If $\omega \in \text{dom } \hat{\sigma}_{i/2}$, then

$$\rho_U(\omega)(\xi) = \omega \ast \xi \in \mathcal{H}_{\text{si}} \quad \text{with } |\omega \ast \xi\rangle = |\xi\rangle \circ \rho(\hat{\sigma}_{i/2}(\omega)).$$

Here $\rho$ denotes the right regular anti-representation on $\mathcal{L}$ defined in (2.22). Lemma 3.9 may be proved using the same ingredients as for Lemma 3.7. We omit the details because this is a special case of Lemma 3.30 below, anyway.

Now let $U$ and $U'$ be two irreducible corepresentations of $\hat{\mathcal{G}}$ on Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$. The orthogonality relations for coefficients of square-integrable representations describe the inner products $\langle \hat{c}_{\alpha\beta}, \hat{c}_{\gamma\delta} \rangle$ for $\alpha \in \mathcal{H}_{\text{si}}$, $\beta \in \mathcal{H}$, $\gamma \in \mathcal{H}'_{\text{si}}$, $\delta \in \mathcal{H}'$. Using the operators just introduced, we can rewrite this as

$$\langle \hat{c}_{\alpha\beta}, \hat{c}_{\gamma\delta} \rangle = \langle \langle \alpha | (\beta), (\gamma) | (\delta) \rangle = \langle \beta, | \alpha \rangle \langle \gamma | (\delta) \rangle.$$
Proof. The $\mathcal{G}$-equivariant operator $[\alpha] \langle \gamma | : \mathcal{H} \to \mathcal{H}$ vanishes because $U$ and $U'$ are not equivalent.

The second orthogonality relation deals with the more interesting case $\mathcal{H} = \mathcal{H}'$ and $U = U'$. In that case, Schur's Lemma implies that $[\alpha] \langle \gamma |$ is a scalar multiple of the identity map. Thus there is a sesquilinear form $s$ on $\mathcal{H}_\text{si}$ with $[\alpha] \langle \gamma | = s(\alpha, \gamma)$ for all $\alpha, \gamma \in \mathcal{H}_\text{si}$. Since the representation of $\hat{G}$ on $\mathcal{H}$ is non-degenerate, $[\alpha] \langle \gamma | = 0$ implies $\alpha = 0$ by (3.33). Hence $s(\alpha, \alpha) > 0$ for all $\alpha \in \mathcal{H}_\text{si}$ with $\alpha \neq 0$.

Consequently, there is a positive, self-adjoint, unbounded operator $K$ on $\mathcal{H}$ with $\ker K = \{0\}$ and $s(\alpha, \gamma) = \langle \gamma, K^{-1} \alpha \rangle$ for all $\alpha, \gamma \in \mathcal{H}_\text{si}$. That is,

$$\langle \tilde{e}_{\alpha \beta}, \tilde{e}_{\gamma \delta} \rangle = \langle \beta, (\gamma, K^{-1} \alpha) \cdot \delta \rangle = \langle \beta, \delta \rangle \cdot \langle \gamma, K^{-1} \alpha \rangle = \langle \beta, \delta \rangle \cdot \langle K^{-1/2} \gamma, K^{-1/2} \alpha \rangle$$

for all $\alpha, \gamma \in \mathcal{H}_\text{si}$, $\beta, \delta \in \mathcal{H}$. Taking $\alpha = \gamma$ and $\beta = \delta$, we see that $\mathcal{H}_\text{si} = \text{dom} K^{-1/2}$.

The positive operator $K$ plays the role of the formal dimension for the square-integrable representation $U$, compare [15] §8 for the case of group representations. It remains to describe $K$ or, equivalently, $K^{-1}$ more explicitly. This involves the modular theory of $\hat{G}$.

Since $|\xi\rangle \neq 0$ for square-integrable $\xi \neq 0$ and $|\xi\rangle \langle \xi |$ is a scalar multiple of the identity map, we may choose an auxiliary square-integrable vector $\xi \in \mathcal{H}_\text{si}$ with $|\xi\rangle \langle \xi | = \text{id}_\mathcal{H}$. Equivalently, $\langle \xi | : \mathcal{H} \to \mathcal{L}$ is an isometry.

**Theorem 3.11 (Second Orthogonality Relation).** Let $\mathcal{G}$ be a locally compact quantum group, let $\mathcal{H}$ be a Hilbert space, and let $U$ be an irreducible square-integrable corepresentation of $\hat{G}$ on $\mathcal{H}$. Let $\alpha, \gamma \in \mathcal{H}_\text{si}$, $\beta, \delta \in \mathcal{H}$, and choose $\xi \in \mathcal{H}_\text{si}$ with $|\xi\rangle \langle \xi | = \text{id}_\mathcal{H}$. Let $\tilde{\nabla} : \mathcal{H} \supseteq \text{dom}(\tilde{\nabla}) \to \mathcal{L}$ be the modular operator of the KMS weight $\tilde{\varphi}$ on $\hat{G}$. Then

$$\langle \tilde{e}_{\alpha \beta}, \tilde{e}_{\gamma \delta} \rangle = \langle \beta, \delta \rangle \cdot \langle K^{-1/2} \gamma, K^{-1/2} \alpha \rangle$$

with the positive unbounded operator

$$K^{-1} = ||\xi||^{-2} \langle \xi | \cdot \tilde{\nabla}^{-1} \cdot \langle \xi | : \mathcal{H} \to \mathcal{H}.$$ 

The operator $K$ is independent of the auxiliary vector $\xi$.

**Proof.** Our main task is to relate the matrix coefficients $\tilde{e}_{\alpha \beta}$ and $\tilde{e}_{\beta \alpha}$. It follows easily from the definition that

$$c_{\alpha \beta}(x) = \langle \beta, \rho_U(x) \alpha \rangle = \langle \rho_U(x^*) \beta, \alpha \rangle = \overline{\langle \alpha, \rho_U(x^*) \beta \rangle} = \tilde{e}_{\beta \alpha}(x^*).$$

Equivalently, $\langle \tilde{e}_{\alpha \beta}, \hat{\Lambda}(x) \rangle = \langle \hat{\Lambda}(x^*), \tilde{e}_{\beta \alpha} \rangle$.

Let $\hat{T}$ be the unbounded, conjugate-linear operator on $\mathcal{L}$ that maps $\hat{\Lambda}(x)$ to $\hat{\Lambda}(x^*)$; recall that it has a polar decomposition $\hat{T} = \hat{J} \hat{\nabla}^{1/2} = \hat{\nabla}^{-1/2} \hat{J}$ as in (2.20).

The computation above shows that

$$\tilde{e}_{\alpha \beta} = \hat{T}^*(\tilde{e}_{\beta \alpha})$$

for all $\alpha, \beta \in \mathcal{H}_\text{si}$. Hence

$$\langle \xi, \xi | \cdot \langle K^{-1/2} \beta, K^{-1/2} \alpha \rangle = \langle \tilde{e}_{\xi \alpha}, \tilde{e}_{\xi \beta} \rangle = \langle \hat{T}^* \tilde{e}_{\xi \alpha}, \hat{T}^* \tilde{e}_{\xi \beta} \rangle$$

$$= \langle \hat{J} \hat{\nabla}^{-1/2} \langle \xi | \alpha, J^* \hat{\nabla}^{-1/2} \langle \xi | \beta \rangle = \langle \hat{\nabla}^{-1/2} \langle \xi | \beta, \hat{\nabla}^{-1/2} \langle \xi | \alpha \rangle$$

This implies $||\xi||^2 K^{-1} = \langle \hat{\nabla}^{-1/2} \langle \xi | \cdot \hat{\nabla}^{-1/2} \langle \xi | \cdot = \langle \xi | \hat{\nabla}^{-1} \langle \xi |$. The operator $K^{-1/2}$ is defined independently of $\xi$. Hence $K$ cannot depend on $\xi$. □

**Remark 3.13.** The operator $\hat{\nabla}$ generates the modular automorphism group of $\hat{G}$. Thus $\hat{G}$ is unimodular if and only $\hat{\nabla} = 1$. In this case, we get a scalar formal dimension $K = ||\xi||^2 \cdot \text{id}_\mathcal{H}$, and $\mathcal{H}_\text{si} = \mathcal{H}$ for any irreducible square-integrable representation.
Remark 3.14. If the quantum group $G$ is compact, then $\hat{A}$ is the inverse of a bounded map $L \to L^1(\hat{G})$ by Remark 2.15. Therefore, any bounded linear functional on $\hat{G}_u$ extends to one on $L$, so that any corepresentation of $G$ is square-integrable. Thus Proposition 3.10 and Theorem 3.11 always apply. The two orthogonality relations are part of Woronowicz’s Peter–Weyl Theorem for compact quantum groups in [19, 20].

Now we examine the subspace of $L$ spanned by the coefficients of an irreducible corepresentation $U$ on $H$.

Let $H'$ be the Hilbert space completion of $\text{dom } K^{-1/2} \subseteq H$ with the conjugate linear $\mathbb{C}$-vector space structure and the inner product
\[
(\xi, \eta) := s(\xi, \eta) = (K^{-1/2} \xi, K^{-1/2} \eta).
\]
The Second Orthogonality Relation in Theorem 3.11 yields a linear isometry
\[
\Phi: H \otimes H' \to L, \quad \xi \otimes \eta \mapsto \tilde{e}_{\eta, \xi} = \langle \eta | \xi \rangle.
\]
We claim that $\Phi$ is a bimodule homomorphism with respect to canonical $\hat{G}$-bimodule structures on $H \otimes H'$ and $L$.

The bimodule structure on $L$ is given by the left regular representation and the right regular anti-representation:

\[
(3.15) \quad x \cdot \xi \cdot y := \lambda(x) \circ \rho(y)(\xi) \quad \text{for } x, y \in \hat{G}, \xi \in L,
\]
with $\rho$ defined in (2.22). The bimodule structure on $H \otimes H'$ is defined by
\[
x \cdot (\xi \otimes \eta) \cdot y := \rho_U(x)(\xi) \otimes \rho_U(\tilde{\sigma}_{ij/2}(y)^*)(\eta).
\]
for $x \in \hat{G}, \xi \in H, \eta \in H', y \in \text{dom } \sigma_{ij/2}$.

The equivariance of $\langle \eta | \xi \rangle$ established in Lemmas 3.6 and 3.9 shows that $\Phi$ is compatible with these bimodule structures. As a consequence, the right action of $\text{dom } \sigma_{ij/2}$ extends to a $^*$-anti-representation of $\hat{G}$ because this happens on $L$ and $\Phi$ is isometric.

We may turn the $\hat{G}$-bimodule structures above into $G$-bicomodule structures, and then $\Phi$ is a homomorphism of $G$-bicomodules. More precisely, the left $\hat{G}$-module structure corresponds to a coaction of $G$, the right $\hat{G}$-module structure corresponds to a left coaction of $G$ or, equivalently, a coaction of the opposite quantum group $G^{\text{op}}$, that is, of $G$ with the opposite comultiplication. The left and right coactions of $G$ commute because the corresponding actions of $\hat{G}$ commute.

Let $P_U: L \to L$ be the projection onto the range of $\Phi$. Since $\Phi$ is a bimodule homomorphism, its range is a sub-bimodule, so that $P_U$ is a bimodule map with respect to the left and right regular representations. Equivalently, $P_U$ belongs to the commutants of both $\lambda(\hat{G})$ and of $\rho(\hat{G})$. Since $\lambda(\hat{G})''$ and $\rho(\hat{G})''$ are commutants of one another, this means that $P_U$ is a central idempotent in the von Neumann algebra completion $\lambda(\hat{G})''$ of $\lambda(\hat{G})$.

It is not hard to see that $P_U$ acts on a square-integrable corepresentation of $G$ as the projection onto the $U$-isotypical subspace.

Since the representations of $\hat{G}$ on $H$ and $H'$ that we use above are both irreducible, the commutant of the left module structure on $H \otimes H'$ is $\mathcal{B}(H')$ acting by $T \mapsto 1 \otimes T$, and the commutant of the right module structure is $\mathcal{B}(H)$ acting by $T \mapsto T \otimes 1$. Using $\Phi$, we see that
\[
(3.16) \quad P_U \lambda(\hat{G})'' \cong \mathcal{B}(H), \quad P_U \rho(\hat{G})'' \cong \mathcal{B}(H').
\]
The First Orthogonality Relation shows that these copies of $\mathcal{B}(H)$ for non-equivalent irreducible square-integrable representations are orthogonal.

Conversely, let $P$ be a central projection in $\lambda(\hat{G})''$ with $P \lambda(\hat{G})'' \cong \mathcal{B}(H)$ for some Hilbert space $H$, then $P \rho(\hat{G})''$ is of type I as well, and the range of a minimal
projection in $P(\hat{G})''$ is an irreducible representation of $\lambda(\hat{G})''$ because its commutant is $C$ by construction. The central projection associated to the corresponding irreducible square-integrable corepresentation of $G$ is exactly $P$. As a result, we get a bijection between the set of equivalence classes of irreducible square-integrable corepresentations of $G$ and the set of direct summands of type I in the direct integral decomposition of the von Neumann algebra $\lambda(\hat{G})''$.

So far, we have worked rather consistently with representations of $\hat{G}_u$ in later sections, we will be equally consistent in using corepresentations of $G$. To see the connection between the two approaches, we reformulate Definition 3.2 in terms of the coaction $\Delta_U: \mathcal{H} \to \mathcal{M}(\mathcal{H} \otimes G): = \mathcal{B}(G, \mathcal{H} \otimes G)$:

**Lemma 3.17.** A vector $\alpha \in \mathcal{H}$ is square-integrable if and only if $(\Delta_U(\alpha), \beta \otimes 1_G)_G$ belongs to $\text{dom } \Lambda$ for all $\beta \in \mathcal{H}$, and in this case $\varepsilon_{\alpha,\beta} = \Lambda((\Delta_U(\alpha), \beta \otimes 1_G)_G)$.

Here we write $(x, y)_G := x^* \circ y \in \mathcal{B}(G, G) = \mathcal{M}(G)$ for $x, y \in \mathcal{M}(\mathcal{H} \otimes G)$.

**Proof.** Recall that the map $\hat{\Lambda}$ is defined so that $\hat{\Lambda}(x\varphi) = \Lambda(x)$ for all $x \in G$ with $x\varphi \in L^1(G)$. Hence

$$\langle \varepsilon_{\alpha,\beta}, \Lambda(x) \rangle = \langle \varepsilon_{\alpha,\beta}, \hat{\Lambda}(x\varphi) \rangle = c_{\alpha,\beta}(x\varphi) = \langle \beta, \rho_U(x\varphi)\alpha \rangle.$$

By definition,

$$\langle \beta, \rho_U(x\varphi)\alpha \rangle = \langle \beta, (\text{id} \otimes x\varphi)\Delta_U(\alpha) \rangle = \varphi((\beta \otimes 1_G, \Delta_U(\alpha))_G \cdot x) = \varphi((\Delta_U(\alpha), \beta \otimes 1_G)_G^* \cdot x).$$

If $(\Delta_U(\alpha), \beta \otimes 1_G)_G \in \text{dom } \Lambda$, then we can rewrite this as

$$\langle \beta, \rho_U(x\varphi)\alpha \rangle = \langle \Lambda((\Delta_U(\alpha), \beta \otimes 1_G)_G), \Lambda(x) \rangle,$$

so that $\alpha$ is square-integrable and $\varepsilon_{\alpha,\beta} = \Lambda((\Delta_U(\alpha), \beta \otimes 1_G)_G)$.

Conversely, suppose that $\alpha$ is square-integrable and let $y := (\Delta_U(\alpha), \beta \otimes 1_G)_G$ for some $\beta$. By assumption, $x \mapsto \langle \beta, \rho_U(x\varphi)\alpha \rangle = \varphi(y^* x)$ is bounded with respect to $\|\Lambda(x)\|$. It remains to check that this implies $y \in \text{dom } \Lambda$.

If $a \in \text{dom } \Lambda \cap \text{dom } \sigma_{-1}$, then $ya \in \text{dom } \Lambda$ as well and

$$\|\Lambda(ya)\|^2 = \varphi(a^* y^* ya) = \varphi(y^* ya \sigma_{-1}(a^*)) \leq C \|\Lambda(ya \sigma_{-1}(a^*))\| \leq C\|a\| \cdot \|\Lambda(ya)\|$$

for some constant $C$, so that $\|\Lambda(ya)\| \leq C\|a\|$ for all $a \in \text{dom } \Lambda$. Since $\text{dom } \Lambda$ is strictly dense in $\mathcal{M}(G)$, we may choose an approximate unit $(u_i)_{i \in I}$ in $\text{dom } \Lambda$. Then $(yu_i)$ converges towards $y$. Our estimate shows that $\Lambda(yu_i)$ converges as well. Since the operator $\Lambda$ is closed, this implies $y \in \text{dom } \Lambda$ as asserted. \hfill $\Box$

Recall that $\Delta_U(\alpha) = U(\alpha \otimes 1_G)$. Hence we can further rewrite

$$\varepsilon_{\alpha,\beta} = \Lambda((\Delta_U(\alpha), \beta \otimes 1_G)_G) = \Lambda((U(\alpha \otimes 1_G), \beta \otimes 1_G)_G).$$

It is shown in [11] that $\hat{\Lambda}^*(\Lambda(x)) = \Lambda(S(x^*))$, where $S$ is the antipode of $G$. Hence the reflection formula [5.12] is equivalent to

$$\langle U(\alpha \otimes 1_G), \beta \otimes 1_G \rangle_G^* = S((\alpha \otimes 1_G, U(\beta \otimes 1_G)_G) \quad \text{for all } \alpha, \beta \in \mathcal{H}_S.$$

**4. Integrable coactions**

In this section, we define and study integrable coactions of locally compact quantum groups on $C^*$-algebras. The main object of interest here is the unbounded map $(\text{id} \otimes \varphi) \circ \Delta_B$, where $\varphi$ is the Haar weight on $G$ and $\Delta_B \in \text{Mor}(B, B \otimes G)$ is a (right) coaction of a locally compact quantum group $(\hat{G}, \Delta)$ on some $C^*$-algebra $B$, that is, a non-degenerate $^*$-homomorphism $\Delta_B: B \to \mathcal{M}(B \otimes G)$ with

$$(\Delta_B \otimes \text{id}) \circ \Delta_B = (\text{id} \otimes \Delta) \circ \Delta_B.$$
For a group action, \((\text{id} \otimes \varphi) \circ \Delta_B\) becomes the map \(B \ni x \mapsto \int_G g \cdot x \, dg\), where the integral is interpreted unconditionally in the strict topology (see [10]). Therefore, we call this map the averaging map. Unless the quantum group \((\mathcal{G}, \Delta)\) is compact, the averaging map is unbounded, and its domain of definition may be small. By definition, integrability means that it is densely defined.

We are going to develop tools to check whether a coaction is integrable. They show, in particular, that dual coactions and stable coactions are integrable. But first, we rigorously define the averaging map, using slices with weights as described in [9, §3]. We only recall this construction briefly here.

The Haar weight is lower semi-continuous by definition. As such, it is a limit of an increasing net of bounded weights. More precisely, we define a partial order for \(\mathcal{M}(\Delta)\) in \([9, \S 3]\). We only recall this construction briefly here.

\[ \text{Sub}(\varphi) := \{ \psi \in \mathcal{G}_+^+ \mid \psi \ll \varphi \} \]

with partial order \(\ll\) is a directed set, and

\[ \varphi(x) = \lim_{\psi \in \text{Sub}(\varphi)} \psi(x) \]

for all \(x \in \mathcal{G}\). Even more, (4.1) is used as a definition to extend \(\varphi\) to the multiplier algebra \(\mathcal{M}(\mathcal{G})\) or to suitable von Neumann algebras.

Given another \(C^*\)-algebra \(B\), we let \(\text{dom}(\text{id} \otimes \varphi)^+\) be the set of all \(x \in \mathcal{M}(B \otimes \mathcal{G})^+\) for which the net \((\text{id} \otimes \psi)(x)\) converges in \(\mathcal{M}(\mathcal{G})\) in the strong topology, and we define

\[ (\text{id} \otimes \varphi)(x) := \lim_{\psi \in \text{Sub}(\varphi)} (\text{id} \otimes \psi)(x) \quad \text{for all } x \in \text{dom}(\text{id} \otimes \varphi)^+. \]

As usual, we let \(\text{dom}(\text{id} \otimes \varphi)\) be the linear span of \(\text{dom}(\text{id} \otimes \varphi)^+\) and extend \(\text{id} \otimes \varphi\) linearly to \(\text{dom}(\text{id} \otimes \varphi)\). Equation (4.2) remains true for \(x \in \text{dom}(\text{id} \otimes \varphi)\); but if \(x\) is not positive, then it is unclear whether convergence in (4.2) suffices for \(x \in \text{dom}(\text{id} \otimes \varphi)\).

**Definition 4.3.** Let \(\Delta_B : B \to \mathcal{M}(B \otimes \mathcal{G})\) be a coaction of \(\mathcal{G}\) on \(B\). We call a positive element \(x \in \mathcal{M}(B)^+\) integrable if \(\Delta_B(x)\) belongs to \(\text{dom}(\text{id} \otimes \varphi)\), and we denote the subset of integrable elements in \(\mathcal{M}(B)^+\) by \(\mathcal{M}(B)^+_i\). Elements in \(\mathcal{M}(B)^+_i := \text{span} \mathcal{M}(B)^+_i\) are also called integrable. We define

\[ \text{Av} : \mathcal{M}(B)^+_i \to \mathcal{M}(B), \quad x \mapsto (\text{id} \otimes \varphi) \circ \Delta_B(x), \]

call this map the averaging map for the coaction \(\Delta_B\) on \(B\).

We also let \(B_1^+ := \mathcal{M}(B)^+_i \cap B \) and \(B_1 := \text{span} B_1^+\).

Although we are mainly interested in the restriction of \(\text{Av}\) to \(B\), working with multipliers all the time creates no additional problems and is sometimes useful. Even for elements of \(B\) we use strict convergence to characterise integrability. The relevant nets are almost never norm convergent.

**Example 4.4.** We consider the standard coaction of a locally compact quantum group \((\mathcal{G}, \Delta)\) on itself. More generally, the same argument works for \(B = D \otimes \mathcal{G}\) and \(\Delta_B = \text{id}_D \otimes \Delta\). The strong form of the left invariance of the Haar weight in [11, Proposition 3.1] asserts that

\[ (\text{id}_D \otimes \varphi) \circ \Delta_B(x) = (\text{id}_D \otimes \varphi)(x) \otimes 1_G \]

holds for all \(x \in \mathcal{M}(D \otimes \mathcal{G})^+\), where both sides belong to the extended positive parts of suitable von Neumann algebras. For our purposes, the important point is that one side is bounded if and only if the other one is. Therefore,

\[ \mathcal{M}(D \otimes \mathcal{G})_i^+ = \text{dom}(\text{id} \otimes \varphi)^+, \quad \mathcal{M}(D \otimes \mathcal{G})_i = \text{dom}(\text{id} \otimes \varphi), \]
and $\text{Av}(x) = (\text{id}_D \otimes \varphi)(x) \otimes 1_G$ for all $x \in (D \otimes G)_i$. In the special case $D = \mathbb{C}$, we get
\begin{equation}
\mathcal{M}(G)_i^+ = \text{dom} \varphi^+, \quad \mathcal{M}(G)_i = \text{dom} \varphi,
\end{equation}
and $\text{Av}(x) = 1_G \cdot \varphi(x)$ for all $x \in \text{dom} \varphi$.

The properties of the slice map $\text{id} \otimes \varphi$ and its domain listed in [9, §3] immediately imply the assertions in the following lemma:

**Lemma 4.7.** The subset of integrable elements $\mathcal{M}(B)_i$ is a hereditary *-subalgebra of $\mathcal{M}(B)_i$, that is, it is a *-subalgebra and $0 \leq x \leq y$ and $y \in \mathcal{M}(B)_i$ imply $x \in \mathcal{M}(B)_i$. If $x \in \mathcal{M}(B)_i$, then the net $(\text{id} \otimes \psi)\Delta_B(x)$ for $\psi \in \text{Sub}(\varphi)$ converges strictly towards $\text{Av}(x)$. Conversely, if $x \in \mathcal{M}(B)_i^+$ is a positive multiplier and the net $(\text{id} \otimes \psi)\Delta_B(x)$ for $\psi \in \text{Sub}(\varphi)$ converges strictly, then $x$ is integrable.

**Proof.** Since $\text{dom}(\text{id} \otimes \varphi)$ is a hereditary *-subalgebra and $\Delta_B$ is a homomorphism, $\mathcal{M}(B)_i$ is a hereditary *-subalgebra. The convergence of $(\text{id} \otimes \psi)\Delta_B(x)$ towards $\text{Av}(x)$ for $x \in \mathcal{M}(B)_i$ follows immediately from the definitions, see also [9, Lemma 3.8]. The converse statement that convergence of this net implies integrability holds because a positive element of $\mathcal{M}(B \otimes G)$ belongs to $\text{dom}(\text{id} \otimes \varphi)$ if and only if it belongs to $\text{dom}(\text{id} \otimes \varphi)^+$; this uses that $\text{dom}(\text{id} \otimes \varphi)^+$ is a hereditary cone. \qed

Recall that a right coaction of $G$ yields a $G^*$-module structure on $\mathcal{M}(B)_i$ via $\psi \star x = (\text{id} \otimes \psi)\Delta_B(x)$. In this notation, $\text{Av}(x)$ becomes the limit of $\psi \star x$ for $\psi \in \text{Sub}(\varphi)$, so that we may also write $\text{Av}(x) = \varphi \star x$.

Let $x \in \mathcal{M}(B)_i^+$. Then $(\psi \star x)_{\psi \in \text{Sub}(\varphi)}$ is an increasing net of positive multipliers. For such nets, there are several equivalent ways to express strict convergence.

First, $(\psi \star x)_{\psi \in \text{Sub}(\varphi)}$ converges strictly towards $y$ if and only if $b^* \cdot (\psi \star x) \cdot b$ converges in norm towards $b^*yb$ for each $b \in B$ (see [9, Result 3.4]).

Secondly, $(\psi \star x)_{\psi \in \text{Sub}(\varphi)}$ converges strictly towards $y \in \mathcal{M}(B)_i$ if and only if it converges weakly, that is, $\lim \theta(\psi \star x) = \theta(y)$ for each $\theta \in B^*$ (see [9, Proposition 3.14]). But mere weak convergence of the net $\psi \star x$ in some ambient von Neumann algebra is not enough: we need the weak limit to be a multiplier of $B$ as well.

**Example 4.8.** Let $G = C_0(G)$ for a locally compact group, with the usual comultiplication. Then continuous $G$-coactions are the same as continuous group actions of $G$. Bounded positive linear functionals on $G$ correspond to bounded measures on $G$. The functionals in $\text{Sub}(\varphi)$ are those that have the form $f \mapsto \int_G f(g) \cdot w(g) \, dg$ for a function $w \in L^1(G)$ with $0 \leq w \leq 1 - \varepsilon$ for some $0 < \varepsilon < 1$; we denote this functional by $w \, dg$. We have $w \, dg \ll w' \, dg$ if and only if $w \leq (1 - \varepsilon)w'$ for some $0 < \varepsilon < 1$.

If $B$ carries a $G$-action and $x \in B$, then $w \, dg \star x = \int_G (g \cdot x)w(g) \, dg$. Integrability means that this net converges strictly; its limit is $\text{Av}(x) = \int_G g \cdot x \, dg$. From this, it is easy to see that our definition of integrability agrees with the usual one in the group case (see [15]).

**Example 4.9.** Let $(G, \Delta)$ be a compact quantum group. Then the Haar weight $\varphi$ is itself bounded. Hence any coaction of $G$ is integrable with $B_1 = B$.

Not surprisingly, the range of the averaging map contains only $G$-equivariant multipliers of $B$:

**Lemma 4.10.** Let $x \in \mathcal{M}(B)_i$. Then $\text{Av}(x) \in \mathcal{M}(B)$ is $G$-invariant, that is, $\Delta_B(\text{Av}(x)) = \text{Av}(x) \otimes 1_G$.\[\]
Proof. We compute
\[
\Delta_B(\Av(x)) = (\Delta_B \otimes \varphi)\Delta_B(x) = (\id_B \otimes \id_G \otimes \varphi)(\Delta_B \otimes \id_G)\Delta_B(x)
\]
\[
= (\id_B \otimes \id_G \otimes \varphi)(\id_B \otimes \Delta)\Delta_B(x) = (\id_B \otimes \varphi)\Delta_B(x) \otimes 1_G = \Av(x) \otimes 1_G,
\]
using the coassociativity of the coaction on \(B\) and the left invariance of the Haar measure in the strong form \((\id_B \otimes \id_G \otimes \varphi) \circ (\id_B \otimes \varphi))(x) = (\id_B \otimes \varphi)(x) \otimes 1_G\) (see \cite[Proposition 3.1]{G}). To justify the first three equalities, we should replace \(\varphi\) by a limit over \(\psi \in \Sub(\varphi)\).

Next we study how integrability behaves with respect to the action of \(G^*\). For group actions, the subspace of integrable elements is \(G\)-invariant; but the \(G\)-action on the space of integrable elements does not integrate to a module structure over \(L^1(G)\) because the modular function intervenes. Therefore, it is to be expected that the modular element of the locally compact quantum group \((G, \Delta)\) appears at this point. Recall that the modular element \(\delta\) is an unbounded multiplier affiliated with \(G\) that is characterised by the property that the unbounded weight
\[
\varphi_{\delta} = \delta^{1/2} \varphi \delta^{1/2}: x \mapsto \varphi(\delta^{1/2} x \delta^{1/2})
\]
is a right invariant Haar weight on \(G\), that is, \(\varphi_{\delta} \ast \omega = \varphi_{\delta} \cdot \omega(1)\) for all \(\omega \in G^*\). The modular element is group-like, that is, \(\Delta(\delta) = \delta \otimes \delta\). This implies \(\Delta(\delta^t) = \delta^t \otimes \delta^t\) for all \(t \in \mathbb{R}\).

Lemma 4.11. Let \(x \in \mathcal{M}(B)\), and let \(\omega \in G^*\) be such that the weight \(\omega_{\delta} = \delta^{1/2} \omega \delta^{1/2}\) is bounded as well. Then \(\omega \ast x \in \mathcal{M}(B)\) and \(\Av(\omega \ast x) = \omega(\delta) \cdot \Av(x)\).

Proof. We may assume without loss of generality that \(x\) and \(\omega\) are positive, the general case follows by polarisation. Then \(\omega \ast x\) is also positive. Thus we must show that the net \(\psi \ast (\omega \ast x)\) for \(\psi \in \Sub(\varphi)\) converges towards \(\omega(\delta)\Av(x) = \omega(\delta) \cdot (\varphi \ast x)\). Since the convolution \(\ast\) is associative, this boils down to the convergence of weights
\[
\lim_{\psi \in \Sub(\varphi)} \psi \ast \omega = \omega(\delta) \cdot \varphi.
\]

Let \(y \in G^+\) satisfy \(\delta^{1/2} y \delta^{1/2} \in \text{dom } \varphi \subseteq G\). Using \(\Delta(\delta^{1/2}) = \delta^{1/2} \otimes \delta^{1/2}\) and the right invariance of \(\varphi_{\delta}\), we get
\[
\lim_{\psi \in \Sub(\varphi)} \psi \ast \omega(\delta^{1/2} y \delta^{1/2}) = \lim_{\psi \in \Sub(\varphi)} (\psi_{\delta} \otimes \omega_{\delta})(\Delta(y))
\]
\[
= \omega_{\delta} = (\varphi_{\delta} \otimes \id) \Delta(y) = \omega_{\delta}(1_G \cdot \varphi_{\delta}(y)) = \omega(\delta) \cdot \varphi(\delta^{1/2} y \delta^{1/2}).
\]

Hence \((\psi \ast \omega)_{\varphi} \in \Sub(\varphi)\) converges towards \(\omega(\delta) \cdot \varphi\) on a dense subset of \(G\). This implies the assertion because this net of weights is increasing.

\]

Lemma 4.12. Let \(\Delta_B\) be a coaction of a locally compact quantum group \(G\) on a \(C^*\)-algebra \(B\). The following statements are equivalent:

1. \(\mathcal{M}(B)\) is norm dense in \(B\);
2. \(\mathcal{M}(B)^+\) is norm dense in \(B^+\);
3. \(\mathcal{M}(B)^+\) is strictly dense in \(\mathcal{M}(B)\);
4. \(\mathcal{M}(B)^+\) is strictly dense in \(\mathcal{M}(B)^+\);

If, in addition, \(B\) is \(\sigma\)-unital, then (1)–(4) are also equivalent to:
5. \(\mathcal{M}(B)^+\) contains a strictly positive element of \(B\);
6. there exists a strictly positive integrable multiplier.

Proof. The equivalences (1) \(\iff\) (2) and (3) \(\iff\) (4) follow because \(\mathcal{M}(B)^+\) is a hereditary cone in \(\mathcal{M}(B)\), and (2)\(\implies\) (4) follows because \(B\) is strictly dense in \(\mathcal{M}(B)\) and the strict topology is weaker than the norm topology.
Now we check that, conversely, (4) $\implies$ (2). Let $x \in \mathcal{M}(B)_1^+$ and $y \in B^+$, then 
$x^{1/2}y^{1/2} \leq x^{1/2}||y||x^{1/2} = ||y|| \cdot x$, so that $x^{1/2}y^{1/2}$ is integrable as well. If $\mathcal{M}(B)_1^+$ is strictly dense in $\mathcal{M}(B)^+$, then the set $\{x^{1/2}y^{1/2} | x \in \mathcal{M}(B)_1^+, y \in B^+\}$ is norm-dense in $B^+$ and consists of integrable elements. Thus (4) implies (2).

A similar argument shows that (6) implies (2) because $x^{1/2}Bx^{1/2}$ is dense in $B$ if $x$ is a strictly positive multiplier of $B$. Since (5)$\implies$(6) is trivial, it remains to check that (2) implies (5) if $B$ is $\sigma$-unital. This ensures that $B$ contains a strictly positive element $h$. By integrability, there is a sequence $(x_n)_{n \in \mathbb{N}}$ of positive integrable elements that converges towards $h$. Then $\sum \lambda_n x_n$ is strictly positive for any sequence of strictly positive numbers $(\lambda_n)$. Choose $(\lambda_n)$ to decay so rapidly that both $\sum \lambda_n ||x_n||$ and $\sum \lambda_n ||\text{Av}(x_n)||$ remain bounded. Then $\sum \lambda_n x_n$ is again integrable, and it is also strictly positive. \hfill $\square$

**Definition 4.13.** The coaction $\Delta_B$ on $B$ is called integrable if the equivalent conditions in Lemma 4.12 are satisfied.

**Proposition 4.14.** Let $B$ and $D$ be $C^*$-algebras with coactions $\Delta_B$ and $\Delta_D$ of $(G, \Delta)$, and let $\pi: \mathcal{M}(B) \to \mathcal{M}(D)$ be a $G$-equivariant, strictly continuous, positive linear map that is non-degenerate in the sense that $\pi(B^+) \cdot D$ is dense in $D$.

Then $\pi(\mathcal{M}(B)_i^+) \subseteq \mathcal{M}(D)_i^+$, and $\pi(\text{Av}(x)) = \text{Av}(\pi(x))$ for all $x \in \mathcal{M}(B)_1$. As a consequence, if $B$ is integrable, so is $D$.

In most applications, $\pi$ is the strictly continuous extension of a non-degenerate equivariant $^*$-homomorphism $B \to \mathcal{M}(D)$. We allow more general maps to prepare for Proposition 4.15 which characterises integrable coactions by the existence of an equivariant, non-degenerate, completely positive map $G \to \mathcal{M}(B)$.

**Proof.** Since $\pi$ is $G$-equivariant, $\pi(\psi \star x) = \psi \star \pi(x)$ for all $x \in \mathcal{M}(B)$, $\psi \in \text{Sub}(\varphi)$. Since $\pi$ is strictly continuous and positive, this shows that $\pi(x)$ is integrable once $x$ is and that $\pi$ intertwines the averaging maps for $B$ and $D$.

Now let $(B, \Delta_B)$ be integrable, that is, $B_1^+$ is dense in $B^+$. Then the set of $\pi(b)d\pi(b)$ with $b \in B_1^+$, $d \in D^+$ is dense in $D^+$ because $\pi$ is non-degenerate. Since $\pi(B_1^+) \subseteq \mathcal{M}(D)_1^+$ and $\mathcal{M}(D)_1^+$ is a hereditary cone, $\pi(b)d\pi(b) \in \mathcal{M}(D)_1^+$ for all $b \in B_1^+$, $d \in D^+$. Hence $D_1^+ = D^+ \cap \mathcal{M}(D)_1^+$ is dense in $D^+$, so that $(D, \Delta_D)$ is integrable. \hfill $\square$

Most of the remaining results in this section are applications of Proposition 4.14.

**Corollary 4.15.** Let $B$ be a $C^*$-algebra with a coaction $\Delta_B$ of $(G, \Delta)$ and let $I \subseteq B$ be a $G$-invariant ideal. Equip $I$ and the quotient $B/I$ with the induced coactions of $G$. If $(B, \Delta_B)$ is integrable, so are $(I, \Delta_I)$ and $(B/I, \Delta_B/I)$.

**Proof.** The $^*$-homomorphisms $B \to \mathcal{M}(I)$ and $B \to B/I$ are non-degenerate and equivariant. Hence the assertion follows from Proposition 4.14. \hfill $\square$

Conversely, suppose that the coactions on $I$ and $B/I$ are integrable. If the extension $I \to B \to B/I$ has an equivariant completely positive section, then Proposition 4.14 applied to the resulting linear isomorphism $I \oplus B/I \to B$ shows that $(B, \Delta_B)$ is integrable.

But without extra assumption, an extension of two integrable coactions need not be integrable. Group actions on commutative $C^*$-algebras provide counterexamples. It is shown by Marc Rieffel in [15] that a group action on a locally compact space $X$ is proper if and only if the induced group action on $C_0(X)$ is integrable. But it is possible to get non-proper actions by gluing together proper actions.
Corollary 4.16. Let \((B, \Delta_B)\) be an integrable coaction of \((G, \Delta)\) and let \(p \in \mathcal{M}(B)\) be a \(G\)-invariant projection, that is, \(\Delta_B(p) = p \otimes 1\). Then the coaction on \(B\) restricts to a coaction on the corner subalgebra \(pBp\), which is again integrable.

Proof. Apply Proposition 4.14 to the map \(x \mapsto pxp\) from \(B\) to \(pBp\), which is completely positive, equivariant, and non-degenerate. \(\square\)

Corner subalgebras are a special case of hereditary subalgebras. But integrability does not appear to be inherited by hereditary subalgebras in general.

We are going to show that dual coactions are integrable. First we briefly recall the definition. The two articles \([2],[8]\) use different conventions at this point: the dual quantum group is defined using the right instead of the left regular representation in \([2]\). Here we follow the conventions in \([10],[13]\), so that we get the \(C^*\)-commutant \(\hat{G}^c\) instead of \(\hat{G}\), which appears in \([2]\).

Let \((G, \Delta)\) be a locally compact quantum group, let \(B\) be a \(C^*\)-algebra, and let \(\Delta_B : B \rightarrow \mathcal{M}(B \otimes G)\) be a coaction of \((G, \Delta)\) on \(B\). So far, the only assumption on \(\Delta_B\) that we needed was that it is non-degenerate and coassociative. To define a crossed product, we also require the coaction to be continuous in the sense that

\[
\text{span}((1_B \otimes G) \cdot \Delta_B(1_B)) = B \otimes \hat{G}.
\]

We represent \(G\) and \(\hat{G}\) on the Hilbert space \(L\) as usual, and view \(B \otimes G\) as a \(C^*\)-algebra of adjointable operators on the Hilbert \(B\)-module \(B \otimes L\). This extends to a \(*\)-representation \(\mathcal{M}(B \otimes G) \rightarrow \mathcal{B}(B \otimes L)\). Let

\[
\hat{G}^c := \hat{J} \hat{G} \hat{J} \subseteq \mathcal{B}(L) \subseteq \mathcal{B}(B \otimes L)
\]

be the \(C^*\)-commutant of \(\hat{G}\). The continuity assumption \((4.17)\) implies that

\[
B \rtimes_r \hat{G}^c := \text{span}(\Delta_B(B) \cdot (1 \otimes \hat{G}^c)) = \text{span}((1 \otimes \hat{G}^c) \cdot \Delta_B(B))
\]

is a \(C^*\)-subalgebra of \(\mathcal{B}(B \otimes L)\). This is the reduced \(C^*\)-crossed product of the coaction \(\Delta_B\) on \(B\). The dual coaction \(\hat{\Delta}_B\) of \(\hat{G}^c\) on \(B \rtimes_r \hat{G}^c\) is defined by

\[
\hat{\Delta}_B(\Delta_B(a) \cdot (1 \otimes x)) := (\Delta_B(a) \otimes 1_G) \cdot (1_B \otimes \hat{\Delta}_c(x)),
\]

where \(\hat{\Delta}_c\) denotes the canonical comultiplication on \(\hat{G}^c\).

Besides the reduced crossed product, there is the full \(C^*\)-crossed product \(B \rtimes \hat{G}_u^c\), which is defined to be universal for covariant representations of \((B, \Delta_B)\) and \((G, \Delta)\). The two crossed products are linked to their constituents by canonical morphisms

\[
j_B \in \text{Mor}(B, B \rtimes \hat{G}_u^c), \quad j_{\hat{G}^c} \in \text{Mor}(\hat{G}_u^c, B \rtimes \hat{G}^c),
\]

\[
j_B^* \in \text{Mor}(B, B \rtimes \hat{G}_u^c), \quad j_{\hat{G}^c}^* \in \text{Mor}(\hat{G}^c, B \rtimes \hat{G}_u^c),
\]

where \(\text{Mor}(A, B)\) denotes the set of non-degenerate \(*\)-homomorphisms \(A \rightarrow \mathcal{M}(B)\) and \((G_u, \Delta_u)\) is the universal locally compact quantum group associated to the reduced quantum group \((G, \Delta)\), see \([8]\).

Proposition 4.20. Let \((G, \Delta)\) be a locally compact quantum group, let \(B\) be a \(C^*\)-algebra, and let \(\Delta_B : B \rightarrow \mathcal{M}(B \otimes G)\) be a continuous coaction of \((\hat{G}^c, \hat{\Delta}_c)\). The associated dual coactions of \((G, \Delta)\) on the reduced and full \(C^*\)-crossed products \(B \rtimes_r G\) and \(B \rtimes G_u\) are both integrable.

Proof. The canonical morphisms \(G \rightarrow \mathcal{M}(B \rtimes_r G)\) and \(G_u \rightarrow \mathcal{M}(B \rtimes G_u)\) are \(G\)-equivariant non-degenerate \(*\)-homomorphisms; here we equip \(G\) and \(G_u\) with the coactions and \((\text{id} \otimes \pi) \circ \Delta_u\), where \(\pi : G_u \rightarrow G\) is the canonical map, and the crossed products carry the dual coactions. By Proposition 4.14 it therefore suffices to check the integrability of the coactions on \(G\) and \(G_u\). The integrability of \((G, \Delta)\) is Example 4.4. Similar assertions hold for \(G_u\) instead of \(G\): it turns out that the following assertions are equivalent for \(x \in G_u\):
• $x$ is integrable;
• $\pi(x) \in \mathcal{G}$ is integrable;
• $\pi(x) \in \text{dom} \varphi$;
• $x \in \text{dom}(\varphi \circ \pi)$.

Hence the canonical coaction on $\mathcal{G}_u$ is integrable as well. \hfill \Box

The next class of examples we consider are \textit{stable coactions}. Let $\Delta_B$ be a right coaction of $(\mathcal{G}, \Delta)$ on a C*-algebra $B$. We equip the Hilbert space $\mathcal{L}$ with the left regular corepresentation of $\mathcal{G}$ defined in (2.18). Then there is a tensor product coaction of $(\mathcal{G}, \Delta)$ on the Hilbert $B$-module $B \otimes \mathcal{L}$, defined by

\[(4.21) \quad \Delta_{B \otimes \mathcal{L}}(x) = (1 \otimes \Sigma W)(\Delta_B \otimes \text{id}_\mathcal{L})(x)\]

for all $x \in B \otimes \mathcal{L}$, where $\Sigma : \mathcal{G} \otimes \mathcal{L} \to \mathcal{L} \otimes \mathcal{G}$ is the coordinate flip and $W$ is the multiplicative unitary for $(\mathcal{G}, \Delta)$ described in (2.5); this tensor product coaction is a special case of [11 Proposition 2.10].

The coaction on $B \otimes \mathcal{L}$ induces a coaction on the C*-algebra of compact operators $\mathbb{K}(B \otimes \mathcal{L}) \cong B \otimes \mathbb{K}(\mathcal{L})$. Coactions of this form are called \textit{stable}.

\begin{prop}
Stable coactions are integrable.
\end{prop}

\begin{proof}
We equip $\mathcal{G}$ with the standard coaction $\Delta$ of itself as in Example 4.4. The standard representation $\mathcal{G} \to \mathbb{B}(\mathcal{L})$ is $\mathcal{G}$-equivariant with respect to the right regular corepresentation $V$ of $\mathcal{G}$ on $\mathcal{L}$ because $\Delta(x) = V(x \otimes 1)V^*$ for all $x \in \mathcal{G}$ (see [11 page 86]). The left regular corepresentation is described, as a right corepresentation, by the unitary $\hat{W}^*$. The unitary $U = J\hat{J}$ that combines the modular conjugations of $\mathcal{G}$ and $\hat{\mathcal{G}}$ provides a unitary equivalence between the left and right regular corepresentations, that is, $V = (U^* \otimes \text{id})\hat{W}^*(U \otimes \text{id})$ (see [11 Proposition 2.15]).

Hence we get a $\mathcal{G}$-equivariant, non-degenerate *-homomorphism

\[\mathcal{G} \to \mathbb{B}(B \otimes \mathcal{L}) = \mathcal{M}(\mathbb{K}(B \otimes \mathcal{L})), \quad x \mapsto \text{id}_B \otimes UxU^*\]

The coaction $\Delta$ on $\mathcal{G}$ is integrable by Example 4.4. Finally, Proposition 4.14 shows that the coaction on $\mathbb{K}(B \otimes \mathcal{L})$ is integrable. \hfill \Box

We have chosen to work with coactions of reduced locally compact quantum groups $(\mathcal{G}, \Delta)$. Of course, since its universal companion $(\mathcal{G}_u, \Delta_u)$ also has Haar weights, we could also define and work with integrability in the universal setting. The following result shows that both approaches are equivalent.

\begin{prop}
Let $\Delta_B^u : B \to \mathcal{M}(B \otimes \mathcal{G}_u)$ be a coaction of $(\mathcal{G}_u, \Delta_u)$ on a C*-algebra $B$ and consider its reduced form

\[\Delta_B := (\text{id} \otimes \pi) \circ \Delta_B^u : B \to \mathcal{M}(B \otimes \mathcal{G})\]

where $\pi : \mathcal{G}_u \to \mathcal{G}$ is the canonical map. Then $(B, \Delta_B)$ is integrable if and only if $(B, \Delta_B^u)$ is. Moreover, the spaces of integrable elements in $\mathcal{M}(B)$ with respect to both coactions coincide and $\text{Av}(x) = \text{Av}_u(x)$ for every integrable element $x$, where $\text{Av}_u$ denotes the averaging map for $(B, \Delta_B^u)$.
\end{prop}

\begin{proof}
Recall that the left invariant Haar weight on $\mathcal{G}_u$ is given by $\varphi_u(x) = \varphi(\pi(x))$. It follows that $x \in \text{dom}(\varphi_u)$ if and only if $\pi(x) \in \text{dom}(\varphi)$. This and Proposition 3.14 show that $x \in \text{dom}(\text{id}_B \otimes \varphi_u)$ if and only if $(\text{id}_B \otimes \pi)(x) \in \text{dom}(\text{id}_B \otimes \varphi)$, and in this case

\[(\text{id}_B \otimes \varphi_u)(x) = (\text{id}_B \otimes \varphi)((\text{id}_B \otimes \pi)(x))\]

Now the assertions follow. \hfill \Box
5. Square-integrable coactions on Hilbert modules

Square-integrable coactions on Hilbert modules contain square-integrable corepresentations on Hilbert spaces and integrable coactions on $C^*$-algebras as special cases. Thus they combine the two situations studied in Sections 3 and 4.

First we define coactions on Hilbert modules, following Saad Baaj and Georges Skandalis [1, §2]. Then we define square-integrable vectors in Hilbert modules and associate operators $\langle \xi |$ and $| \xi \rangle$ to them. We provide some examples of square-integrable Hilbert modules and study the general properties of the space of square-integrable vectors and the operators $\langle \xi |$ and $| \xi \rangle$.

5.1. Quantum group coactions on Hilbert modules. Let $(\mathcal{G}, \Delta)$ be a locally compact quantum group and let $B$ be a $C^*$-algebra with a coaction $\Delta_B$ of $\mathcal{G}$. Let $\mathcal{E}$ be a Hilbert $B$-module. We recall the definition and some basic properties of coactions on Hilbert modules. We refer to [1] for further details.

Definition 5.1. A coaction of $(\mathcal{G}, \Delta)$ on $\mathcal{E}$ is a linear map

$$\Delta_\mathcal{E} : \mathcal{E} \to \mathcal{M}(\mathcal{E} \otimes \mathcal{G}) := \mathbb{B}(\mathcal{E} \otimes \mathcal{G})$$

that satisfies the following conditions:

- $\Delta_\mathcal{E}(\xi \cdot b) = \Delta_\mathcal{E}(\xi) \cdot \Delta_B(b)$ for all $\xi \in \mathcal{E}$, $b \in B$;
- $\langle \Delta_\mathcal{E}(\xi), \Delta_\mathcal{E}(\eta) \rangle_{\mathcal{M}(\mathcal{E} \otimes \mathcal{G})} := \Delta_\mathcal{E}(\xi)^* \circ \Delta_\mathcal{E}(\eta) = \Delta_B(\langle \xi, \eta \rangle_B)$ for all $\xi, \eta \in \mathcal{E}$;
- $\operatorname{span} \{ \Delta_\mathcal{E}(\xi) \cdot (B \otimes \mathcal{G}) : \xi \in \mathcal{E} \}$ is a $\mathcal{G}$-equivariant Hilbert $B$-module.
- $\Delta_\mathcal{E} \otimes \operatorname{id}_\mathcal{G} \circ \Delta_\mathcal{E} = (\operatorname{id}_\mathcal{E} \otimes \Delta) \circ \Delta_\mathcal{E}$.

The last condition involves extensions of $\Delta_\mathcal{E} \otimes \operatorname{id}_\mathcal{G}$ and $\operatorname{id}_\mathcal{E} \otimes \Delta$ to the multiplier modules, which exist in the presence of the other conditions.

We also call a Hilbert $B$-module $\mathcal{E}$ with such a coaction of $(\mathcal{G}, \Delta)$ a $\mathcal{G}$-equivariant Hilbert $B$-module.

Example 5.2. The $C^*$-algebra $B$, viewed as a Hilbert module over itself becomes a $\mathcal{G}$-equivariant Hilbert $B$-module for the given coaction $\Delta_B$.

Remark 5.3. We follow [3, 18] and deviate from [1] by not requiring $\Delta_\mathcal{E}(\mathcal{E}) \cdot (1 \otimes \mathcal{G})$ and $(1 \otimes \mathcal{G}) \cdot \Delta_\mathcal{E}(\mathcal{E})$ to be contained in $\mathcal{E} \otimes \mathcal{G}$. Many of the basic assertions on Hilbert module coactions in [1] do not require this conditions.

It is necessary to drop this condition in order to treat non-regular quantum groups because a locally compact quantum group $\mathcal{G}$ is regular if and only if the coaction $\Delta_\lambda$ on $\mathcal{E}$ from the left regular corepresentation satisfies the condition $(1 \otimes \mathcal{G}) \cdot \Delta_\lambda(\mathcal{E}) \subseteq \mathcal{E} \otimes \mathcal{G}$.

A coaction on $\mathcal{E}$ induces a coaction on the $C^*$-algebra of compact operators

$$\Delta_{\mathcal{K}(\mathcal{E})} : \mathbb{K}(\mathcal{E}) \to \mathcal{M}(\mathcal{K}(\mathcal{E}) \otimes \mathcal{G}), \quad |\xi\rangle \langle \eta| \mapsto \Delta_\mathcal{E}(\xi) \circ \Delta_\mathcal{E}(\eta)^*.$$  

Conversely, a coaction of $\mathcal{G}$ on the (linking) $C^*$-algebra $\mathbb{K}(\mathcal{E} \oplus B)$ that restricts to the given coaction on the corner $\mathbb{K}(B) \cong B$ is equivalent to a coaction of $(\mathcal{G}, \Delta)$ on $\mathcal{E}$.

The coaction on $\mathcal{K}(\mathcal{E})$ extends to a map

$$\Delta_{\mathbb{B}(\mathcal{E})} : \mathbb{B}(\mathcal{E}) \to \mathbb{B}(\mathcal{E} \otimes \mathcal{G})$$

that satisfies

$$\Delta_\mathcal{E}(T \xi) = \Delta_{\mathbb{B}(\mathcal{E})}(T) \Delta_\mathcal{E}(\xi) \quad \text{for all } T \in \mathbb{B}(\mathcal{E}), \, \xi \in \mathcal{E}. \quad (5.4)$$

This extension is constructed easily using a third equivalent description of a coaction on $\mathcal{E}$ in terms of a unitary operator

$$V : \mathcal{E} \otimes_{\Delta_B} (B \otimes \mathcal{G}) \to \mathcal{E} \otimes \mathcal{G}. \quad (5.5)$$
We simply put
(5.6) \[ \Delta_{\mathcal{E}}(T) := V(T \otimes_{\Delta_B} \text{id}_{B \otimes \mathcal{G}})V^* \]
Similarly, the coaction \( \Delta_{\mathcal{E}} \) extends to a map
\[ \Delta_{\mathcal{M}(\mathcal{E})} : \mathcal{M}(\mathcal{E}) := \mathbb{B}(B, \mathcal{E}) \to \mathcal{M}(\mathcal{E} \otimes \mathcal{G}) := \mathbb{B}(B \otimes \mathcal{G}, \mathcal{E} \otimes \mathcal{G}) \]
via \( T \mapsto V \circ (T \otimes_{\Delta_B} \text{id}_{B \otimes \mathcal{G}}) \) and the identification \( B \otimes_{\Delta_B} (B \otimes \mathcal{G}) \cong B \otimes \mathcal{G} \).

5.2. Square-integrable elements of equivariant Hilbert modules. The following definition generalises the description of square-integrable vectors for Hilbert space representations in Lemma 3.17. It uses the slice map \( \text{id} \otimes \Lambda : \mathcal{M}(B \otimes \mathcal{G}) \supseteq \text{dom}(\text{id} \otimes \Lambda) \to \mathcal{M}(B \otimes \mathcal{L}) \)
associated to the map \( \Lambda : \mathcal{G} \supseteq \text{dom} \Lambda \to \mathcal{L} \) defined in \([10] \ \S 1.5\). We will describe this map in more detail below.

**Definition 5.7.** Let \( \mathcal{E} \) be a \( \mathcal{G} \)-equivariant Hilbert \( B \)-module with coaction \( \Delta_{\mathcal{E}} \), let \( V : \mathcal{E} \otimes_{\Delta_B} (B \otimes \mathcal{G}) \to \mathcal{E} \otimes \mathcal{G} \) be the unitary operator that describes \( \Delta_{\mathcal{E}} \). We call \( \xi \in \mathcal{E} \) or, more generally, \( \xi \in \mathcal{M}(\mathcal{E}) \) square-integrable if
\[ \Delta_{\mathcal{M}(\mathcal{E})}(\xi)^*(\eta \otimes 1_{\mathcal{G}}) = (\xi \otimes_{\Delta_B} \text{id}_{B \otimes \mathcal{G}})^*V^*(\eta \otimes 1_{\mathcal{G}}) \in \mathcal{M}(B \otimes \mathcal{G}) \]
belongs to the domain of \( \text{id}_B \otimes \Lambda \) for all \( \eta \in \mathcal{E} \), and we define
\[ \langle \xi | \eta \rangle : = \langle (\text{id} \otimes \Lambda)(\Delta_{\mathcal{M}(\mathcal{E})}(\xi)^*(\eta \otimes 1_{\mathcal{G}})) \rangle. \]

We let \( \mathcal{M}(\mathcal{E})_{\text{si}} \) and \( \mathcal{E}_{\text{si}} \) be the subspaces of square-integrable elements. We call \( (\mathcal{E}, \Delta_{\mathcal{E}}) \) square-integrable if \( \mathcal{E}_{\text{si}} \) is norm-dense in \( \mathcal{E} \).

To give meaning to this definition, we must describe the slice map \( \text{id}_B \otimes \Lambda \) and its domain. We do this slightly more directly than in \([9] \ \S 3\].

Recall that the Haar weight \( \varphi \) is the limit of the increasing net of bounded weights \( \langle \psi \rangle_{\varphi \in \text{Sub}(\varphi)} \) and that
\[ \text{dom}(\text{id}_B \otimes \varphi)^+ \} \}
\[ \text{dom}(\text{id}_B \otimes \varphi)^+ := \{ x \in \mathcal{M}(B \otimes \mathcal{G})^+ \mid \text{the net } \left( (\text{id} \otimes \psi)(x) \right)_{\psi \in \text{Sub}(\varphi)} \text{ in } \mathcal{M}(B) \text{ converges strictly} \}. \]
We define
\[ \text{dom}(\text{id}_B \otimes \Lambda) := \{ x \in \mathcal{M}(B \otimes \mathcal{G}) \mid x^*x \in \text{dom}(\text{id}_B \otimes \varphi)^+ \}. \]

For any \( \psi \in \text{Sub}(\varphi) \), there is a vector \( \chi_\psi \in \mathcal{L} \) with
\[ \psi(x^*y) = \langle x\chi_\psi, y\chi_\psi \rangle \quad \text{for all } x, y \in \mathcal{G}, \]
and there is an operator \( T_\psi \) on \( \mathcal{L} \) with \( 0 \leq T_\psi \leq 1 \) that commutes with \( \mathcal{G} \) and satisfies
\[ \langle A(x), T_\psi \Lambda(y) \rangle = \psi(x^*y) \quad \text{for all } x, y \in \text{dom} \Lambda. \]
The net \( (T_\psi)_{\psi \in \text{Sub}(\varphi)} \) is increasing and converges strongly towards \( \text{id}_\mathcal{L} \). The operator \( T_\psi \) and the vector \( \chi_\psi \) are related by
\[ T_\psi^\frac{1}{2} \Lambda(x) = x \cdot \chi_\psi \quad \text{for } x \in \text{dom} \Lambda \]
(see \([10] \ \text{Notation 1.4}\)).

**Lemma 5.11.** We have \( x \in \text{dom}(\text{id}_B \otimes \Lambda) \) if and only if the net \( (x \cdot (1 \otimes \chi_\psi))_{\psi \in \text{Sub}(\varphi)} \) converges strongly in \( \mathbb{B}(B, B \otimes \mathcal{L}) \). Equivalently, \( x \cdot (b \otimes \chi_\psi) \in B \otimes \mathcal{L} \) converges in norm for each \( b \in B \).

This allows us to define \( (\text{id}_B \otimes \Lambda)(x)(b) := \lim_{\psi \in \text{Sub}(\varphi)} x \cdot (b \otimes \chi_\psi) \quad \text{for all } b \in B. \)
Proof. If \((x \cdot (b \otimes \chi_\psi))_{\psi \in \text{Sub}(\varphi)}\) converges in norm for all \(b \in B\), then so does
\[
\langle x(b \otimes \chi_\psi), x(b \otimes \chi_\psi) \rangle_B = b^* (\text{id} \otimes \psi)(x^* x) b,
\]
and this is equivalent to the strict convergence of \((\text{id} \otimes \psi)(x^* x)\) by [9] Result 3.4. Hence \(x \in \text{dom}(\text{id}_B \otimes \Lambda)\).

Conversely, assume that \(x \in \text{dom}(\text{id}_B \otimes \Lambda)\). Let \(\text{id}_B \otimes \Lambda'\) be the map defined in [9]. Then [9] Result 3.21 yields
\[
(\text{id} \otimes T_\psi)(\text{id} \otimes \Lambda'(x)) \cdot b = x \cdot (b \otimes \chi_\psi)
\]
for all \(\psi \in \text{Sub}(\varphi)\), \(b \in B\). Since the net \((T_\psi)\) converges strongly to the identity map, so does \((\text{id}_B \otimes T_\psi)\). Hence the net \(x \cdot (b \otimes \chi_\psi)\) converges towards \(\text{id} \otimes \Lambda'(x) \cdot b\). □

The proof also shows that our definition of \(\text{id} \otimes \Lambda\) agrees with the one in [9].

\[\text{A direct proof that the net } (x \cdot (b \otimes \chi_\psi))_{\psi \in \text{Sub}(\varphi)} \text{ in } B \otimes \mathcal{L} \text{ converges for all } x \in \text{dom}(\text{id}_B \otimes \Lambda) \text{ is similar to the last part of the proof of Lemma 3.17.}\]

The above definition of \(\text{id} \otimes \Lambda\) shows that
\[
(\xi | \eta) := (\text{id} \otimes \Lambda)(\Delta_{\mathcal{M}(\mathcal{E})}(\xi^* (\eta \otimes 1_\varphi))) = \lim_{\psi \in \text{Sub}(\varphi)} \Delta_{\mathcal{M}(\mathcal{E})}(\xi^* (\eta \otimes \chi_\psi))
\]
for all \(\xi \in \mathcal{M}(\mathcal{E})\). It is clear from this that the map \(\eta \mapsto (\xi | \eta)\) is \(B\)-linear.

**Lemma 5.13.** Let \(\xi \in \mathcal{M}(\mathcal{E})\) and \(\eta \in \mathcal{E}\). Then \((\xi | \eta)\) belongs to \(B \otimes \mathcal{L}\) and not just to \(\mathcal{M}(B \otimes \mathcal{L}) := \mathbb{B}(B, B \otimes \mathcal{L})\).

**Proof.** For any Hilbert \(B\)-module, we have \(\mathcal{E} \cdot B = \mathcal{E}\). Writing \(\eta = \eta' \cdot b\) with \(\eta' \in \mathcal{E}\), \(b \in B\), we get \((\xi | \eta) = (\xi | \eta') \cdot b\) by \(B\)-linearity. □

**Example 14.** The definitions of square-integrable elements and the operator \((\xi | \cdot)\) above generalise the corresponding ones for group actions in [12][13].

To see this, we mainly have to identify the map \(\text{id}_B \otimes \Lambda\) in the case \(\mathcal{G} = C_0(G)\) for a locally compact group \(G\). We describe \(\text{Sub}(\varphi)\) as in Example 4.5. The map \(\text{id}_B \otimes \Lambda\) is defined on a bounded continuous function \(f: G \to B\) if and only if the net \((f \cdot w)_{\psi \in \text{Sub}(\varphi)}\) converges in \(L^2(G, B) := B \otimes L^2(G)\). In [12][13], a net of compactly supported continuous functions \(\chi_i: G \to [0, 1]\) with \(\chi_i \to 1\) uniformly on compact subsets of \(G\) is used instead of the net of functions \(w\) above. But it is not hard to see that both types of cut-off functions yield the same unbounded densely defined map \(\text{id}_B \otimes \Lambda\) from \(C_0(G, B)\) to \(L^2(G, B)\).

Furthermore, \(\Delta_{\mathcal{E}}(\xi^* (\eta \otimes 1))\) for \(\xi, \eta \in \mathcal{E}\) is the function \(g \mapsto \langle g \cdot \xi, \eta \rangle\). Hence our definitions of the subspace \(\mathcal{E}_{\text{si}}\) and the operator \((\xi | \cdot)\) specialise to the corresponding definitions in [12][13] for \(\mathcal{G} = C_0(G)\).

**Example 15.** If the coefficient algebra \(B\) is trivial, then we are dealing with corepresentations of the quantum group \((\mathcal{G}, \Delta)\) on Hilbert spaces. In this case, our definition of a square-integrable vector agrees with the corresponding one in Section 3 by Lemma 3.17. This provides some examples of square-integrable coactions on Hilbert modules. In particular, the left and right regular corepresentations on \(\mathcal{L}\) are square-integrable by Lemma 3.7.

**Example 16.** Let \((\mathcal{G}, \Delta)\) be a compact quantum group, so that \(\mathcal{G}\) is unital and the Haar weight \(\varphi\) is bounded. Then the maps \(\text{id} \otimes \varphi\) and \(\text{id} \otimes \Lambda\) are defined everywhere, so that every element of a Hilbert module is square-integrable. Letting \(\Omega := \Lambda(1_\varphi) \in \mathcal{L}\), we have \(\Lambda(x) = x \cdot \Lambda(1_\varphi) = x \otimes \Omega\) for all \(x \in \mathcal{G}\). Hence
\[
(\xi | \eta) = \Delta_{\mathcal{M}(\mathcal{E})}(\xi^* (\eta \otimes \Omega))
\]
for all \(\xi \in \mathcal{M}(\mathcal{E}), \eta \in \mathcal{E}\).
Our next goal is to show that the operator \( \langle \xi \rangle : \mathcal{E} \to B \otimes \mathcal{L} \) for \( \xi \in \mathcal{M}(\mathcal{E})_{\text{sym}} \) is adjointable and to describe its adjoint. This involves the left \( \mathcal{G} \)-module structure on \( \mathcal{M}(\mathcal{E}) \) defined by \( \omega \cdot \xi := (\text{id} \otimes \omega)\Delta(\mathcal{E})(\xi) \) for all \( \omega \in \mathcal{G}^*, \xi \in \mathcal{M}(\mathcal{E}) \). Here we use the slice map
\[
\text{id} \otimes \omega : \mathbb{B}(B \otimes \mathcal{G}, \mathcal{E} \otimes \mathcal{G}) \to \mathbb{B}(B, \mathcal{E})
\]
for a bounded weight \( \omega \).

**Lemma 5.17.** If \( \xi \in \mathcal{M}(\mathcal{E})_{\text{sym}} \), then the operator \( \langle \xi \rangle : \mathcal{E} \to B \otimes \mathcal{L} \) is adjointable, and the adjoint operator \( \langle \xi \rangle^* \) is determined by
\[
\langle \xi \rangle^*(\hat{b} \otimes \Lambda(x)) = (\text{id}_B \otimes \varphi)(\Delta(\mathcal{E})(\xi) \cdot (\hat{b} \otimes x)) = (x\varphi \ast \xi) \cdot \hat{b}
\]
for all \( b \in B, x \in \text{dom} \mu \), that is, \( x\varphi \) is bounded and belongs to \( L^1(\mathcal{G}) \).

**Proof.** If \( \xi \in \mathcal{M}(\mathcal{E})_{\text{sym}} \), then the graph of the operator \( \langle \xi \rangle : \mathcal{E} \to B \otimes \mathcal{L} \) is closed because the unbounded operator \( \text{id} \otimes \Lambda \) is closed (see [9, Proposition 3.23]). The Closed Graph Theorem shows that \( \langle \xi \rangle \) is bounded.

Let \( b \in B \) and \( x \in \text{dom} \mu \). Then
\[
\langle \xi \rangle^*(\hat{b} \otimes \Lambda(x)) = \langle \xi \rangle^*(\hat{b} \otimes \Lambda(x)) = (\text{id} \otimes \varphi)(\Delta(\mathcal{E})(\xi) \cdot (\hat{b} \otimes x)) = (x\varphi \ast \xi) \cdot \hat{b},
\]
Hence the adjoint of \( \langle \xi \rangle \) is defined on \( b \otimes \Lambda(x) \) and maps it to \( (x\varphi \ast \xi) \cdot \hat{b} \). Since \( B \otimes \Lambda(\text{dom} \mu) \) is dense in \( B \otimes \mathcal{L} \) and \( \langle \xi \rangle \) is bounded, this implies that \( \langle \xi \rangle^* \) is defined on all of \( B \otimes \mathcal{L} \).

For \( \psi \in \text{Sub}(\varphi) \), we may use (5.8) and (5.10) to rewrite
\[
\psi(y) = \langle \chi_\psi, y \cdot \psi \rangle = \langle \chi_\psi, T^{1/2}_\psi \Lambda(y) \rangle = \langle T^{1/2}_\psi \chi_\psi, \Lambda(y) \rangle.
\]
Hence \( \text{id}_B \otimes \psi : B \otimes \mathcal{G} \to B \) factors through the map \( \text{id}_B \otimes \Lambda \) as \( \text{id}_B \otimes \Lambda(y) = \langle 1 \otimes T^{1/2}_\psi \chi_\psi, (\text{id}_B \otimes \Lambda)(y) \rangle \); here we use \( 1 \otimes T^{1/2}_\psi \chi_\psi \in \mathcal{M}(B \otimes \mathcal{L}) := \mathbb{B}(B, B \otimes \mathcal{L}) \), and the inner product actually means that we apply the adjoint of this operator to \( (\text{id}_B \otimes \Lambda)(y) \in B \otimes \mathcal{L} \). Using this notation, we may write
\[
\langle \xi \rangle^*(\hat{x}) = \lim_{\psi \in \text{Sub}(\varphi)} \langle 1 \otimes T^{1/2}_\psi \chi_\psi, \Delta(\mathcal{E})(\xi)(\hat{x}) \rangle \quad \text{for all } x \in B \otimes \mathcal{L}.
\]
This formula works for all elements of \( B \otimes \mathcal{L} \), not just for elementary tensors.

**Proposition 5.20.** Let \( \xi, \eta \in \mathcal{M}(\mathcal{E})_{\text{sym}} \) and consider \( \xi \circ \eta^* \in \mathbb{B}(\mathcal{E}) = \mathcal{M}(K(\mathcal{E})) \). Then \( \xi \circ \eta^* \) is an integrable multiplier of \( K(\mathcal{E}) \) and \( \Lambda^* \xi = \langle \xi \rangle \langle \eta \rangle \). Conversely, if \( \xi \circ \xi^* \in \mathcal{M}(K(\mathcal{E})) \), then \( \xi \) is square-integrable.

In particular, \( \langle \xi \rangle \langle \eta \rangle \in \mathcal{K}(\mathcal{E}) \), for \( \eta, \xi \in \mathcal{E}_{\text{sym}} \), and if \( \xi \in \mathcal{E} \), then \( \xi \in \mathcal{E}_{\text{sym}} \) if and only if \( \langle \xi \rangle \in \mathcal{K}(\mathcal{E}) \).

**Proof.** By polarisation, we may assume without loss of generality that \( \xi = \eta \). We have to show that \( \xi \in \mathcal{M}(\mathcal{E}) \) is square-integrable if and only if \( \xi \circ \xi^* \) is integrable and that \( \Lambda^* \xi = \langle \xi \rangle \langle \xi \rangle \). Recall that \( \xi \) is square-integrable if and only if \( \Delta_{\mathcal{E}}(\xi)(\xi \otimes 1_G) \in \text{dom}(\text{id}_B \otimes \varphi)^+ \). By definition, this is equivalent to \( (\xi \otimes 1_G)^* \Delta_{\mathcal{E}}(\xi) \Delta_{\mathcal{E}}(\xi)(\xi \otimes 1_G) \in \text{dom}(\text{id}_B \otimes \varphi)^+ \). This is equivalent to strict convergence of
\[
(\text{id}_B \otimes \psi)((\xi \otimes 1_G)^* \Delta_{\mathcal{E}}(\xi) \Delta_{\mathcal{E}}(\xi)(\xi \otimes 1_G)) = \hat{\zeta} \ast (\text{id}_B \otimes \psi)(\Delta_{\mathcal{E}}(\xi^*)) \cdot \hat{\zeta}
\]
for \( \psi \in \text{Sub}(\varphi) \), still for all \( \zeta \in \mathcal{E} \). Using [9] Result 3.4, this is equivalent to norm-convergence of \( \hat{b} \ast (\text{id}_B \otimes \psi)(\Delta_{\mathcal{E}}(\xi^*)) \cdot \hat{b} \) for all \( b \in B \), for all \( \zeta \in \mathcal{E} \); using a variant of [9] Result 3.4, whose proof is similar, this is equivalent to convergence of \( (\text{id}_{K(\mathcal{E})} \otimes \psi)(\Delta_{\mathcal{E}}(\xi^*)) \) in the strict topology on \( B(\mathcal{E}) = \mathcal{M}(K(\mathcal{E})) \). By definition,
this means that $\xi \xi^*$ is integrable. Thus $\xi$ is square-integrable if and only if $\xi \xi^*$ is integrable.

Our computation also shows that $\langle \xi, |\xi\rangle_B \langle \xi, |\xi\rangle_B = \langle \xi, A \nu(\xi \xi^*) \rangle_B$ for all $\xi \in E$ if $\xi$ is square-integrable. This implies $|\xi\rangle_B = \nu(\xi \xi^*)$ by polarisation. If $\xi, \eta \in E$, then $\xi \eta^* = |\xi\rangle_B |\eta\rangle_B$, so that we get the assertions in the second paragraph. \hfill \□

5.3. Conditions for square-integrability and examples.

**Proposition 5.21.** A coaction on a Hilbert module $E$ is square-integrable if and only if the induced coaction on $K(E)$ is integrable.

**Proof.** If $E$ is square-integrable, that is, $E_{s}$ is dense in $E$, then the elements $|\xi\rangle_B |\eta\rangle_B$ for $\xi, \eta \in E_s$ span a dense subspace of $K(E)$. Since these elements are integrable by Proposition 5.21, $K(E)$ is dense in $K(E)$, that is, the coaction on $K(E)$ is integrable (see Lemma 4.12).

Conversely, assume that the coaction on $K(E)$ is integrable. Then $K(E)$ contains a strictly positive integrable element $T$ by Lemma 4.12. Since the positive integrable elements form a hereditary cone, $T|\xi\rangle_B |\xi\rangle_B = |T\xi\rangle_B |\xi\rangle_B$ is integrable for all $\xi \in E$, so that $T|\xi\rangle_B$ is square-integrable again by Proposition 5.20. The strict positivity of $T$ means that the range of $T$ is dense in $E$. Since the range of $T$ consists of square-integrable elements, these are dense in $E$. \hfill \□

**Example 5.22.** Let $E = B$ viewed as a Hilbert $B$-module. The given coaction on $B$ turns this into a $G$-equivariant Hilbert $B$-module. Since the natural isomorphism $K(B) \cong B$ is $G$-equivariant, Proposition 5.21 shows that $B$ is square-integrable as a Hilbert $B$-module if and only if the coaction on the $C^*$-algebra $B$ is integrable in the sense of Definition 4.13.

Examples 5.15 and 5.22 show that square-integrable coactions on Hilbert modules contain both square-integrable Hilbert space corepresentations and integrable coactions on $C^*$-algebras as special cases.

**Proposition 5.23.** Let $B$ and $D$ be $C^*$-algebras equipped with coactions of a locally compact quantum group $(G, \Delta)$, let $E$ be a $G$-equivariant Hilbert $B$-module, and let $\pi : D \to \mathbb{B}(E)$ be a $G$-equivariant, non-degenerate $*$-homomorphism. If the coaction on $D$ is integrable, then $E$ is square-integrable.

**Proof.** The coaction on $K(E)$ is integrable by Proposition 4.14. This is equivalent to square-integrability of $E$ by Proposition 5.21. \hfill \□

As in Proposition 4.14, we may replace $\pi$ by a positive, strictly continuous, non-degenerate linear map $\mathcal{M}(D) \to \mathbb{B}(E)$.

**Proposition 5.24.** Let $B$ be a $C^*$-algebra equipped with a coaction of a locally compact quantum group $(G, \Delta)$. Equip $B \otimes \mathcal{L}$ with the diagonal coaction defined in (4.21). Then $B \otimes \mathcal{L}$ is square-integrable. More generally, if $E$ is a $G$-equivariant Hilbert $B$-module, then $E \otimes \mathcal{L}$ with the diagonal coaction is square-integrable.

**Proof.** Tensor product coactions on Hilbert modules are defined in [11]. This specialises to (4.21) for $B \otimes \mathcal{L}$, so that the latter is a $G$-equivariant Hilbert $B$-module. Proposition 4.22 asserts that $K(E \otimes \mathcal{L}) \cong K(K(E) \otimes \mathcal{L})$ is integrable. Hence $E \otimes \mathcal{L}$ is square-integrable by Proposition 5.21. \hfill \□

**Proposition 5.25.** Let $(E_j)_{j \in I}$ be a set of $G$-equivariant Hilbert $B$-modules. Let $E' := \bigoplus_{j \in I} E_j$ be the Hilbert module completion of the algebraic direct sum of the $B$-modules $E_j$, for the obvious inner product $\langle (\xi_j), (\eta_j) \rangle := \sum_{j \in I} \langle \xi_j, \eta_j \rangle$.

The induced coaction of $G$ on $E'$ is square-integrable if and only if $E_j$ is square-integrable for each $j \in I$. 

*SQUARE-INTEGRABLE COACTIONS 23*
Conversely, it is straightforward to see that a vector \( \xi \in L\) is square-integrable if and only if the components \( \xi_j \in E_j \) are square-integrable for all \( j \in I \). Further properties of square-integrable elements.

**Proposition 5.26.** The operators \( \langle \xi \rangle : E \to B \otimes L \) and \( \langle \xi \rangle : B \otimes L \to E \) for square-integrable \( \xi \) are \( \mathcal{G} \)-equivariant.

Here we use the coaction on \( B \otimes L \) defined in (5.21).

**Proof.** A direct proof of this assertion is possible but somewhat unpleasant because the coaction on \( B \otimes L \) is complicated. We avoid this direct proof by a trick. Recall that \( \langle \xi \rangle \langle \eta \rangle = \text{Av}(\xi^* \eta) \) for all \( \xi, \eta \in \mathcal{M}(E) \) by Proposition 5.20. This is \( \mathcal{G} \)-equivariant by Lemma 4.10. Replacing \( E \) by \( E_1 \oplus E_2 \), we see that the same holds for \( \langle \xi \rangle \langle \eta \rangle : E_2 \to E_1 \) for \( \xi \in \mathcal{M}(E_1)_{si} \) and \( \eta \in \mathcal{M}(E_2)_{si} \). Now we consider \( E_2 := B \otimes L \) with the diagonal coaction and use a multiplier of the form \( 1_B \otimes \eta \) with \( \eta \in L_{si} \). It is easy to check that \( \langle 1_B \otimes \eta \rangle = 1_B \otimes \langle \eta \rangle \) in this case, and the equivariance of \( \langle \eta \rangle : L \to L \) is already checked in Lemma 5.6.

Summing up, if \( \xi \in \mathcal{M}(E)_{si} \), then we know that the operators

\[
\langle \xi \rangle \langle 1_B \otimes \eta \rangle : B \otimes L \to E \quad \text{and} \quad \langle 1_B \otimes \eta \rangle : B \otimes L \to B \otimes L
\]

are \( \mathcal{G} \)-equivariant for all \( \eta \in L_{si} \). Thus

\[
\Delta_E(\langle \xi \rangle \langle 1_B \otimes \eta \rangle | h \rangle) = (| \xi \rangle \langle 1_B \otimes \eta \rangle \otimes 1_G) (| \Delta_B \otimes \text{id}_L \rangle (| h \rangle)) = (| \xi \rangle \otimes 1_G) (\Delta_E(\langle 1_B \otimes \eta \rangle | h \rangle))
\]

for all \( h \in B \otimes L \).

We claim that the sum of the ranges of the operators \( \langle 1_B \otimes \eta \rangle = 1_B \otimes \langle \eta \rangle \) for \( \eta \in L_{si} \) is dense in \( B \otimes L \). Hence

\[
\Delta_E(| \xi \rangle | k \rangle) = (| \xi \rangle \otimes 1_G) \Delta_B \otimes L(k)
\]

holds for a dense set of \( k \in B \otimes L \) and hence for all \( k \in B \otimes L \) by continuity. This shows that \( | \xi \rangle \) is \( \mathcal{G} \)-equivariant. Hence so is its adjoint \( \langle \xi \rangle \).

It remains to check the claim above. It suffices to prove that the ranges of the operators \( \langle \eta \rangle \) for \( \eta \in L_{si} \) span a dense subspace in \( L \). This follows easily from the non-degeneracy of the right regular representation of \( \mathcal{G} \) on \( L \) and the computations in the proof of Lemma 5.7.

**Lemma 5.27.** If \( T \in B(E) \) is \( \mathcal{G} \)-equivariant and \( \xi \in \mathcal{M}(E)_{si} \), then

\[
T \xi \in \mathcal{M}(E)_{si} \quad \text{with} \quad \langle T \xi \rangle = \langle \xi \rangle \circ T^* \quad \text{and} \quad | T \xi \rangle = T \circ | \xi \rangle.
\]

**Proof.** Since \( T \) is equivariant, \( \Delta_E(T \xi) = (T \otimes 1_G) \Delta_E(\xi) \), so that we may rewrite \( \Delta_E(T \xi)^*(\eta \otimes 1_G) = \Delta_E(\xi)^*(T^* \eta \otimes 1_G) \). This implies the assertions. \( \square \)
Lemma 5.28. The subspaces $\mathcal{M}(\mathcal{E})_{si} \subseteq \mathcal{M}(\mathcal{E})$ and $\mathcal{E}_{si} \subseteq \mathcal{E}$ are both complete with respect to the norm $\|\xi\|_{si} := \|\xi\| + \|\xi\|_1$.

Proof. This is the graph norm for the unbounded linear map $\xi \mapsto \|\xi\|$. Hence the assertion boils down to the claim that this operator with domain $\mathcal{M}(\mathcal{E})_{si}$ or $\mathcal{E}_{si} = \mathcal{M}(\mathcal{E})_{si} \cap \mathcal{E}$ is closed. This is true because $\Delta_{\mathcal{M}(\mathcal{E})}$ is bounded and $\text{id}_B \otimes \Lambda$ is closed by [7, Result 2.3].

Lemma 5.29. Let $\xi \in \mathcal{M}(\mathcal{E})_{si}$ and $b \in \mathcal{M}(B)$. Then

$$\langle \xi \cdot b \rangle = \Delta_B(b)^* \circ \langle \xi \rangle, \quad \|\xi \cdot b\| = \|\xi\| \circ \Delta_B(b),$$

where we represent $\Delta_B(b) \in \mathcal{M}(B \otimes \mathcal{G})$ on $B \otimes \mathcal{L}$ using the standard representation of $\mathcal{G}$ on $\mathcal{L}$.

Proof. The subspace $\text{dom}(\text{id}_B \otimes \Lambda)$ is a left ideal in $\mathcal{M}(B \otimes \mathcal{G})$ because the subset $\text{dom}(\text{id}_B \otimes \varphi)^+ = \text{dom}(\text{id}_B \otimes \Lambda)^+$ is a hereditary cone (see also [9]). Furthermore, $\text{id}_B \otimes \Lambda$ is a left module homomorphism. Using $\Delta_{\mathcal{E}}(\xi \cdot b)^* = \Delta_B(b)^* \Delta_{\mathcal{E}}(\xi)^*$, we get $\xi \cdot b \in \mathcal{M}(\mathcal{E})_{si}$ and

$$\langle \xi \cdot b \rangle \eta = (\text{id}_B \otimes \Lambda)(\Delta_{\mathcal{E}}(\xi)^* \cdot (\eta \otimes 1_G)) = (\text{id}_B \otimes \Lambda)(\Delta_B(b)^* \cdot \Delta_{\mathcal{E}}(\xi)^* \cdot (\eta \otimes 1_G)) = \Delta_B(b)^* (\text{id}_B \otimes \Lambda)(\Delta_{\mathcal{E}}(\xi)^* \cdot (\eta \otimes 1_G)) = \Delta_B(b)^* \langle \xi \rangle \eta$$

for all $\eta \in \mathcal{E}$. Hence $\langle \xi \cdot b \rangle = \Delta_B(b)^* \langle \xi \rangle$. The formula for $\|\xi \cdot b\|$ follows by taking adjoints.

Lemma 5.29 can be generalised as follows. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two $\mathcal{G}$-equivariant Hilbert $B$-modules, let $T \in \mathbb{B}(\mathcal{E}_1, \mathcal{E}_2)$ and $\xi \in \mathcal{M}(\mathcal{E}_1) := \mathbb{B}(B, \mathcal{E}_1)$. All relevant Hilbert $B$-modules embed in $\mathcal{E} := \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus B$. We may view $T$ and $\xi$ as multipliers of $\mathbb{E}(\mathcal{E})$. If $T$ is square-integrable as such a multiplier, then Lemma 5.29 yields that $T\xi$ is square-integrable as well and satisfies

$$\langle T\xi \rangle = \Delta_{\mathcal{E}_1}(\xi)^* \langle T \rangle, \quad \|T\xi\| = \|T\| \Delta_{\mathcal{E}_1}(\xi).$$

These formulas originally involve two operators $\mathbb{E}(\mathcal{E}) \otimes \mathcal{L} \to \mathbb{E}(\mathcal{E})$. But it continues to hold if we reinterpret $\|T\|$ as an operator $\mathcal{E}_1 \otimes \mathcal{L} \to \mathcal{E}_2$, $\Delta_{\mathcal{E}_1}(\xi)$ as an operator $B \otimes \mathcal{L} \to \mathcal{E}_1 \otimes \mathcal{L}$, and $\|T\xi\|$ as an operator $B \otimes \mathcal{L} \to \mathcal{E}_2$.

The case $\mathcal{E}_1 = \mathcal{E}_2$ is particularly interesting because we may get square-integrable operators $\xi_1 \to \mathcal{E}_1$ from an equivariant $^*$-representation $\xi_1 \to \mathbb{E}(\mathcal{E}_1)$ and square-integrable multipliers of $D$. As a consequence, $\mathcal{M}(D)_{si} : \mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{E})_{si}$. This yields an alternative proof of Proposition 5.24 that only uses square-integrable elements, not integrable elements.

Let $\xi \in \mathcal{M}(\mathcal{E})_{si}$ and $\omega \in \mathcal{G}^* \subseteq \mathcal{M}(\hat{\mathcal{G}})$. We ask when $\omega \ast \xi$ is square-integrable as well and how to describe $|\omega \ast \xi\rangle$. This requires the modular automorphism group $\hat{\sigma} : \mathbb{R} \times \hat{\mathcal{G}} \to \hat{\mathcal{G}}$ and the right regular anti-representation defined in [22].

Lemma 5.30. Let $\xi \in \mathcal{M}(\mathcal{E})_{si}$ and $\omega \in \mathcal{G}^* \subseteq \mathcal{M}(\hat{\mathcal{G}})$. If $\omega \in \text{dom}(\hat{\sigma})$, then

$$\omega \ast \xi \in \mathcal{M}(\mathcal{E})_{si} \quad \text{with} \quad |\omega \ast \xi\rangle = \|\xi\| \text{id}_B \otimes \rho \circ \hat{\sigma}_{x, \omega}(\omega).$$

Proof. Recall that $\hat{\Lambda}(x) \phi = \Lambda(x) \phi$ for $x \in \text{dom} \mu$. Hence we may rewrite the formula for $|\xi\rangle$ in Lemma 5.17 as $|\xi\rangle (b \otimes \hat{\Lambda}(x)) = (x \ast \xi) \cdot b$ for all $b \in B$, $x \in \mathcal{G}^* \cap \text{dom}(\Lambda)$. For these $b$ and $x$, the formula makes sense regardless whether $\xi$ is square-integrable or not. Hence we may compute $|\omega \ast \xi\rangle (b \otimes \hat{\Lambda}(x))$.

Since the product $\ast$ is associative, we get

$$|\omega \ast \xi\rangle (b \otimes \hat{\Lambda}(x)) = (x \ast (\omega \ast \xi)) \cdot b = |\xi\rangle (b \otimes \hat{\Lambda}(x \omega)).$$
Now we use (2.23) to rewrite the right hand side as 
\[ |\xi\rangle \langle b \otimes \rho \circ \sigma_{ij}(\omega) \hat{A}(x) | = |\xi\rangle \langle b \otimes \rho \circ \sigma_{ij}(\omega)| (b \otimes \hat{A}(x)) \] 
This yields the asserted formula for \(|\omega \ast \xi\rangle\langle \xi|\) and shows that \(|\omega \ast \xi\rangle\langle \xi|\) extends to an adjointable operator \(B \otimes \mathcal{L} \rightarrow \mathcal{E}\). The adjoint operator must be \(|\omega \ast \xi|\), forcing \(\omega \ast \xi\) to be square-integrable. \(\square\)

6. The Stabilisation Theorem

Now we come to our main result, an equivariant generalisation of Kasparov’s Stabilisation Theorem for actions of locally compact quantum groups.

**Theorem 6.1.** Let \((\mathcal{G}, \Delta)\) be a locally compact quantum group, let \(B\) be a \(C^*\)-algebra with a coaction of \((\mathcal{G}, \Delta)\), and let \(\mathcal{E}\) be a countably generated \(\mathcal{G}\)-equivariant Hilbert \(B\)-module. Then the following are equivalent:

1. \(\mathcal{G}\)-equivariant unitary \(\mathcal{E} \oplus B \otimes \mathcal{L}^\infty \cong B \otimes \mathcal{L}^\infty\); here \(B \otimes \mathcal{L}^\infty\) denotes the direct sum of countably many copies of the Hilbert \(B\)-module \(B \otimes \mathcal{L}\), and the latter is equipped with the coaction of \(\mathcal{G}\) described in (4.21).
2. There is an adjointable \(\mathcal{G}\)-equivariant isometry \(\mathcal{E} \rightarrow B \otimes \mathcal{L}^\infty\), that is, \(\mathcal{E}\) is \(\mathcal{G}\)-equivariantly isomorphic to a direct summand of \(B \otimes \mathcal{L}^\infty\).
3. The subspace of square-integrable elements is dense in \(\mathcal{E}\), that is, the coaction on \(\mathcal{E}\) is square-integrable.
4. The subspace of integrable operators is dense in \(\mathbb{K}(\mathcal{E})\), that is, the induced coaction on \(\mathbb{K}(\mathcal{E})\) is integrable.

For a compact quantum group, \(\mathcal{E} \oplus B \otimes \mathcal{L}^\infty \cong B \otimes \mathcal{L}^\infty\) holds for any countably generated equivariant Hilbert \(B\)-module \(\mathcal{E}\) because (3) and (4) are automatically true.

The following technical result is used in the proof of Theorem 6.1.

**Lemma 6.2.** For any \(\mathcal{G}\)-equivariant Hilbert \(B\)-module \(\mathcal{E}\), we have 
\[ \text{span} \{ (\omega \ast \xi) \cdot b \mid \omega, \xi \in \mathcal{E}, b \in B \} = \mathcal{E}. \]

**Proof.** By definition, \((\omega \ast \xi) \cdot b = (\text{id} \otimes \omega)(\Delta \xi)(b \otimes a)\) for \(a \in \mathcal{G}, \omega \in L^1(\mathcal{G}), \xi \in \mathcal{E}, b \in B\). The definition of a Hilbert module coaction contains the requirement that the elements \(\Delta \xi(b \otimes a)\) span a dense subspace of \(\mathcal{E} \otimes \mathcal{G}\). Applying \(\text{id} \otimes \omega\), we get a dense subspace of \(\mathcal{E}\). \(\square\)

**Proof of Theorem 6.1.** The implication (1) \(\Rightarrow\) (2) is trivial, and the equivalence of (3) and (4) is already contained in Proposition 5.21.

We check (2) \(\Rightarrow\) (3). Proposition 5.21 shows that \(B \otimes \mathcal{L}^\infty \cong B^\infty \otimes \mathcal{L}\) is square-integrable. Direct summands of square-integrable equivariant Hilbert module inherit square-integrability by Proposition 5.25. Hence any direct summand of \(B \otimes \mathcal{L}^\infty\) is square-integrable. Thus (2) implies (3).

Finally, we check that (3) implies (1). If \(\xi \in \mathcal{E}\) is square-integrable, then \(|\xi\rangle\langle \xi|\) is an equivariant adjointable map \(B \otimes \mathcal{L} \rightarrow \mathcal{E}\). The formula for \(|\xi\rangle\langle \xi|\) in Lemma 6.2 shows that its range contains \((x \cdot \varphi \ast \xi) \cdot b\) for all \(x \in \text{dom} \mu, b \in B\). Since the set of \(x \cdot \varphi\) with \(x \in \text{dom} \mu\) is dense in \(L^1(\mathcal{G})\) and the set of square-integrable vectors is dense in \(\mathcal{E}\) by assumption, Lemma 6.2 shows that the ranges of the operators \(|\xi\rangle\langle \xi|\) span a dense subspace of \(\mathcal{E}\). This implies (1) using a well-known argument by Mingo and Phillips (see [14]), which is also used in [12] to prove the Equivariant Stabilisation Theorem for Hilbert modules with an action of a locally compact group. The idea of this argument is to construct an equivariant adjointable map \(B \otimes \mathcal{L}^\infty \rightarrow \mathcal{E} \oplus B \otimes \mathcal{L}^\infty\) which is injective and has dense range. Then a polar decomposition provides the desired equivariant unitary. We refer to [12, 14] for further details. \(\square\)
Proposition 6.3. A countably generated $G$-equivariant Hilbert module $E$ is square-integrable if and only if there is a non-degenerate, equivariant, completely positive contractive linear map $G \to B(E)$.

A coaction of $G$ on a $\sigma$-unital $C^*$-algebra $B$ is integrable if and only if there is a non-degenerate, equivariant, completely positive contractive linear map $G \to M(B)$.

Proof. The two assertions are equivalent by Proposition 5.21. Proposition 4.14 shows that the existence of a map $G \to M(B)$ as in the theorem is sufficient for integrability. Conversely, assume that $E$ is a countably generated, square-integrable $G$-equivariant Hilbert $B$-module. The Equivariant Stabilisation Theorem 6.1 provides an equivariant adjointable isometry $T : E \to B \otimes L^\infty$. There are equivariant *-homomorphisms $\pi : G \to B(L) \to B(B \otimes L^\infty)$. Then the map $G \to B(E), \ x \mapsto T^* \pi(x) T$ has the required properties. □

7. Conclusion

We have extended the theory of square-integrable group actions on Hilbert modules to coactions of locally compact quantum groups. This includes basic results on irreducible square-integrable Hilbert space corepresentations. Our main result, the Equivariant Stabilisation Theorem shows that square-integrable coactions on Hilbert modules are closely related to the regular corepresentation and hence to stable coactions: a coaction on a Hilbert $B$-module is square-integrable if and only if it is contained in the diagonal coaction on $B \otimes L^\infty$, where $L$ carries the left regular representation.

It is remarkable that we do not need the quantum group to be regular or the coaction to be continuous (continuity of coactions is only problematic for non-regular quantum groups, see [3]).

In the theory of square-integrable group actions, the next step brings in crossed products. In the quantum group setting, the issue is whether or not the operators $\langle \xi \mid \eta \rangle$ on $B \otimes L$ belong to the canonical representation of the crossed product $C^*$-algebra for sufficiently many square-integrable $\xi, \eta$. If this is the case, we may define a generalised fixed point algebra that is Morita equivalent to an ideal in the crossed product. As in [13], this provides an equivalence between Hilbert modules over the crossed products and $G$-equivariant Hilbert modules over $B$ with a suitable dense subspace. This line of thought is pursued in the first author’s doctoral thesis and will be published elsewhere.

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E-mail address: alcides@mtm.ufsc.br

Departamento de Matemática, Universidade Federal de Santa Catarina, 88.040-900 Florianópolis-SC, Brasil.

E-mail address: rameyer@uni-math.gwdg.de

Mathematisches Institut, Georg-August-Universität Göttingen, Bunsenstrasse 3–5, 37073 Göttingen, Germany