Topological Symmetry Breaking
on Einstein Manifolds

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ABSTRACT

It is known that if gauge conditions have Gribov zero modes, then topological symmetry is broken. In this paper we apply it to topological gravity in dimension $n \geq 3$. Our choice of the gauge condition for conformal invariance is $R + \alpha = 0$, where $R$ is the Ricci scalar curvature. We find when $\alpha \neq 0$, topological symmetry is not broken, but when $\alpha = 0$ and solutions of the Einstein equations exist then topological symmetry is broken. This conditions connect to the Yamabe conjecture. Namely negative constant scalar curvature exist on manifolds of any topology, but existence of non-negative constant scalar curvature is restricted by topology. This fact is easily seen in this theory. Topological symmetry breaking means that BRS symmetry breaking in cohomological field theory. But it is found that another BRS symmetry can be defined and physical states are redefined. The divergence due to the Gribov zero modes is regularized, and the theory after topological symmetry breaking become semiclassical Einstein gravitational theory under a special definition of observables.

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1 Introduction

Topological field theories have been studied in these years [1]-[9]. Especially, Seiberg Witten theory is making great development in 4-dim topology, now [3]. But there are few reports in which topological symmetry realize in our world. Hence, the symmetry should be broken in order to have some connection with our world [4]-[5]. In ref.[5], Zaho and Lee add a infinitesimal breaking term to a Lagrangian. They found, that if gauge conditions are well defined and there is no Gribov zero mode then topological symmetry is hold in the limit of zero breaking term, but when there is a zero mode, BRS symmetry, i.e. topological symmetry, is broken. But the theory has such problems that physical meaning is lost after BRS symmetry breaking and divergence appear from Gribov zero modes.

We construct a topological gravitational theory of dimension \( n \geq 3 \) that has breaking phase with the method of Zaho and Lee. Topological gravitational theory we treat is fixed by \( R + \alpha = 0 \) for the conformal symmetry. If we change the gauge condition as \( \alpha \to 0 \), then the theory become ill-defined by Gribov zero modes on some manifolds. Gribov zero modes appear when Fadeev-Popov matrix has zero eigen values. In our theory, this zero eigenvalue equations are Einstein equations. Hence, topological symmetry is broken on only Einstein manifolds. Strictly speaking, solutions of the Einstein equations exist, then topological symmetry is broken and smaller symmetry, diffeomorphism-invariance, are left in this theory. Further, we solve above problems. We recover the physical meaning to define 2nd BRS operator that is constructed by the remaining symmetry, diffeomorphism-invariance. And one method of regularization to avoid divergence from zero modes is introduced in a general case. By using this regularization for gravitational theory, we get semiclassical Einstein gravitational theory in some case.

On the physical point of view, our purposes are to treat Einstein manifolds and to construct quantum gravity of which classical limit become Einstein gravity. When cosmological constant is zero, then scalar curvature \( R \) is zero on Einstein manifolds. If a gauge condition is \( R = 0 \), Gribov zero modes might appear on some manifolds and theory would be ill-defined in former theory [3]-[5]. In our theory, we are able to make the theory well-defined and having broken phase of topological symmetry at \( R = 0 \). And this broken phase is interpreted as semi-classical gravity in a sense. The symmetry breaking conditions are connected to the Yamabe conjecture. We can easily find topological restriction to scalar curvature changing by conformal mode.

This paper is organized as follows. We review Zaho-Lee symmetry breaking theory [5] in a general case, in section 2. The core of this theory is to use singularity of Gribov zero modes. To avoid the zero modes, they added an infinitesimal breaking term to a Lagrangian. Even though the breaking term is infinitesimal, it influence physical amplitude. In section 3, we realize it in topological gravity. As same as some other Witten type topological field theories [4], topological gravity [3]-[6] is constructed by BRS formalism. We fix the conformal symmetry by \( R + \alpha = 0 \), where \( R \) is scalar curvature and \( \alpha \) is cosmological constant. Then if the Einstein equations
and $R = \alpha = 0$ are simultaneously satisfied, topological symmetry is broken. These conditions connect to the Yamabe conjecture [10]. Topological symmetry breaking means BRS symmetry breaking i.e. physical structure lose its meaning, then. In section 4, we define 2nd BRS transformation. So we redefine physical states, then physical states recover with 2nd BRS operator after topological symmetry breaking. In section 5, one method of regularization for the zero modes is given in the general case including our gravitational case. In section 6, we discuss about mathematical meaning. The topological symmetry breaking conditions give some topological information to scalar curvature, which connect with the Yamabe conjecture here. This information is due to the fact that absence of Gribov zero modes become a sufficient condition for functional subspace be a infinite dimensional manifold. In last section, we mention some conclusions, difficulties and prospects.

## 2 General formalism

Zaho, Lee showed that topological symmetry can be broken by singularity of Gribov zero mode [5]. We review it and extend further to a general case in this section. Let $\phi_i(x)$ represent all fields which include unphysical fields like ghost fields. A total lagrangian $\mathcal{L}$ is represented by a classical Lagrangian $\mathcal{L}_{cl}$, BRS operator $S$, and gauge fermions $\Psi_{g.f}$, as

$$\mathcal{L} = \mathcal{L}_{cl} + S\Psi_{g.f}$$

(1)

$\Psi_{g.f}$ is constructed with antighosts $\bar{c}^i$, N-L fields $b_i$, and gauge fixing functions $F_i$. Namely our gauge conditions are $F_i = 0$. In general, it is possible to write $\Psi_{g.f} = i\bar{c}^i (\alpha b_i + F_i)$, where $\alpha$ is a gauge parameter. BRS transformation for $\bar{c}^i$ and $b_i$ is defined by $S\bar{c}^i = b_i, Sb_i = 0$. Then we have

$$\mathcal{L} = \mathcal{L}_{cl} + \imath b^i (\alpha b_i + F_i) + \bar{c}^i (S\phi_j \frac{\delta}{\delta \phi_j} F_i).$$

(2)

We chose Landau gauge, $\alpha = 0$, for simplicity. We demand the total Lagrangian $\mathcal{L}$ has some BRS symmetry. And we assume functional integral major $\mathcal{D}\phi_k$ and a physical observable $O$ is invariant under BRS transformation. If there are Gribov zero modes, naive gauge fixing is not correct, so we need some regularization to avoid it. All gauge conditions $F_i$ do not have to include Gribov zero modes for symmetry breaking, but for simple notation we put regularization terms $\imath e b^i f_i$ for each $F_i$ into the Lagrangian

$$\mathcal{L}_e = \mathcal{L} + \imath e b^i f_i = \mathcal{L}_{cl} + \imath b^i (F_i + \epsilon f_i) + \bar{c}^i (S\phi_j \frac{\delta}{\delta \phi_j} F_i),$$

(3)

and demand that $\imath e b^i f_i$ is not invariant under BRS transformation. The vacuum expectation value of any observable $O$ is defined by

$$\langle O \rangle_e \equiv \lim_{\epsilon \to 0} \int \mathcal{D}\phi_k O e^{-\int dx b^i \mathcal{L}_e}.$$
If there is no singularity, then $i\epsilon b^i f_i$ never influence, 
\[
\langle O \rangle_\epsilon = \langle O \rangle = \int \mathcal{D}\phi_k \ O e^{-\int dx^D\mathcal{L}}.
\] (5)

In the following, I omit the index $\epsilon$ of $\langle O \rangle_\epsilon$ and $\lim_{\epsilon \to 0}$ for convention. Now we estimate the vacuum expectation value of BRS exact functional $SO$ as,
\[
\langle SO \rangle = \int \mathcal{D}\phi_k \ SO e^{-\int dx^D\mathcal{L}} = \int \mathcal{D}\phi_k \ OS(i\epsilon b^i f_i) e^{-\int dx^D\mathcal{L}}.
\] (6)

Since $SO$ is BRS exact, we usually expect it vanishing in the limit as $\epsilon$ approaches zero. But if there are Gribov zero modes, $\langle SO \rangle \neq 0$ is realized as follows. After $b^i f_i$ integration in(5), we get
\[
\langle SO \rangle = -\epsilon \int \mathcal{D}\phi_k \ O e^{L_{cl}+ie^\epsilon(S\phi_j \frac{\delta}{\delta\phi_j} F_i)}
\]
\[
(S\phi_j \frac{\delta}{\delta\phi_j} f_k)(\frac{\delta}{\delta F_k + \epsilon f_k} \prod_k \delta(F_k + \epsilon f_k))
\]
\[
= -\epsilon \int \mathcal{D}\phi_k \ O e^{-\int dx^D\mathcal{L}_{cl}} \prod_k (S\phi_j \frac{\delta}{\delta\phi_j} f_k)
\]
\[
\left(\frac{\delta}{\delta F_l + \epsilon f_l} \prod_m \delta(F_m + \epsilon f_m)\right)(i)^n (S\phi_j \frac{\delta}{\delta\phi_j} F_l).
\] (7)

Where we assume that the observable $O$ does not contain $b^i$ fields. The second equality in eq.(7) was gotten by $\epsilon^\ell$ integration, and $\phi_i$ represent all fields which were not still integrated. Here, it turns out that when $\frac{\delta F_i}{\delta\phi_j} = 0$ and $F_l = 0$ are simultaneously satisfied then $\langle SO \rangle \neq 0$ from $\frac{\delta}{\delta F_l + \epsilon f_l} \prod_m \delta(F_m + \epsilon f_m)$. It is easily understood by that $\delta(x^2 + \epsilon)$ has strong divergence as the limit $\epsilon \to 0$ and derivative $\frac{\delta}{\delta F_l + \epsilon f_l}$ go up the power of divergence. This is an essence of symmetry breaking. To see this apparently, next we change some of $\phi_i$ to gauge functions $F_i + \epsilon f_i$, and carry out their integral by using $\delta(F_k + \epsilon f_k)$. We find
\[
\langle SO \rangle = i^n \epsilon \int \mathcal{D}\phi_k \prod_k \frac{\delta}{\delta F_k + \epsilon f_k} \left\{ \frac{1}{\mathcal{D}e^\epsilon S(F_i + \epsilon f_i)} \right\}
\]
\[
\prod_l (S\phi_j \frac{\delta}{\delta\phi_j} f_l)(S\phi_j \frac{\delta}{\delta\phi_j} F_m)O e^{-\int dx^D\mathcal{L}_{cl}} \biggl|_{F_i + \epsilon f_i = 0}.
\] (8)

If $b^i f_i$ did not break BRS symmetry i.e.$S\phi_j \frac{\delta}{\delta\phi_j} f_i = 0$ or $\mathcal{D}_{j,\epsilon}^{i}$ defined by $\mathcal{D}_{j,\epsilon}^{i} = \frac{\delta(F_i + \epsilon f_i)}{\delta\phi_j}$ had no zero mode, then $\langle SO \rangle = 0$ in the limit $\epsilon \to 0$ . But now $b_i f_i$ breaks the BRS symmetry , and we assume that there are some $\mathcal{D}_{j,\epsilon}^{i}$ zero modes at $F_i = 0$ . Then $\epsilon$ and $\mathcal{D}_{j,\epsilon}^{i}$ cancel each other. We get some non-zero value $\langle SO \rangle \neq 0$ . This means that BRS symmetry is broken.

For example,
\begin{align}
F_i + \epsilon f_i \big|_{\phi_j = \phi^c_j + \Delta \phi^c_j = 0} & = 0 \quad F_i \big|_{\phi_j = \phi^c_j} = 0 \quad j = 1 \sim n \tag{9} \\
D^i_j \big|_{\phi^c_j = 0} & = 0 \quad j = 1 \sim n \tag{10}
\end{align}

for only one $\phi^c$. Where $n$ is a number of conditions $F_j$ and index “i” is fixed. In this case, as we will see in section 3, $\prod_i \phi_j \delta F_j \sim \Delta \phi^c \sim \epsilon^{\frac{1}{2}}$, and $D^i_{j\epsilon} \sim \Delta \phi^c \sim \epsilon^{\frac{1}{2}}$ when $f_i(\phi^c) \neq 0$. Then the most divergent term of $\frac{\delta}{\delta F_k} (Det D^i_{j\epsilon}^{-1})$ is order $\epsilon^{-1}$. After all we get the order of the $\langle SO \rangle$ as,

$$\langle SO \rangle \sim \frac{\epsilon}{\Delta \phi^c} = 1 \tag{11}$$

We have seen some BRS symmetry is broken by Gribov zero modes in a general case. If we use this way for Witten type topological field theories [2][9], then topological symmetry breaking may be realized.

There are some problems of this method. First, after BRS symmetry was broken, physical states lost their meaning. To solve this, we prepare 2nd-BRS operator for topological gravity in section 4. Second problem is whether partition function $Z$ is finite or not. At first sight it will be divergent. But we will see that it isn’t true.

\begin{align}
Z = & \int D\phi_k e^{-\int dx^D L} \\
= & \int D\phi_k e^{-\int dx^D L_{cl} \left( \frac{\delta (\phi_j - \phi^c_j - \Delta \phi^c_j)}{|Det D^i_{j\epsilon}|} \prod_i S_{\phi_j} \delta_{\phi_j} F_i \right)}
\end{align}

Since $D^i_{j\epsilon} \sim \prod_i S_{\phi_j} \delta_{\phi_j} F_i \sim \epsilon^{\frac{1}{2}}$, they cancel each other. So, the partition function keep finite. Third, as we see in (9), amplitude of some observable is divergent. This fact demand the theory to be regularized. We will give one method of regularization in section 5.

### 3 The case of Topological gravity

We show here the theory of the previous section will be realized in topological gravity [1][2]. We use Myers theory [9] that treat spin connection and vierbein as independent fields. Without this property, this theory is almost same as Myers-Periwal theory [8]. In our theory, dimension of the manifold is not essential as far as dimension $n \geq 3$, but 4-dim case is quoted often for a simple example. In these theories, there are BRS operator $S$ and non nilpotents BRS-like operator $\hat{S}$ which is reduced
local orthogonal-transformations and diffeomorphism from $S$. The $S$ is defined as

\begin{align*}
S e^a_{\mu} &= -(w^a_b + P^a_b) e^b_{\mu} + L e^a_{\mu}, \\
S w^a_{\mu} &= L w^a_{\mu} - w^a_c w^c_{\mu} - P^a_c w^c_{\mu} - w^a e^c_{\mu} - (L e^a_{\mu}) e^a_{\mu} - Q^a_{\mu}, \\
S w^a_{\mu b} &= L w^a_{\mu b} - w^a c w^c_{\mu b} - L c e^a_{\mu b} - P^a c e^c_{\mu b} - w^a e^c_{\mu b} - L c e^a_{\mu b}, \\
S \Lambda^a_{\mu b} &= L \Lambda^a_{\mu b} - \nabla^a_\mu P^a_{\mu b} + \nabla^a_\nu R^a_{\nu \mu b}, \\
S \phi^a_{\mu} &= \nabla^a_\mu e^a_{\mu} + \phi^a_{\mu}, \\
S \phi^a_{\mu} &= L \phi^a_{\mu}, \\
SP^a_{\mu b} &= -P^a e P^a_{\mu b} + Q^a_{\mu b} - c^b \Lambda^a_\mu b + \frac{1}{2} c^c e^c_{\mu} R^a_{\mu \nu b}, \\
SQ^a_{\mu b} &= L \phi Q^a_{\mu b} + Q^a e P^a_{\mu b} - e P^a e P^a_{\mu b} + \phi^a \Lambda^a_{\mu b}, \\
x &= y + L e x - \delta P x, \\
y &= L e y - \delta P y - L e x + \delta Q x.
\end{align*}

(13)

Where $L e, L \phi$ denotes the Lie derivative for a fermionic vector field $e^a_{\mu}$ and for a bosonic vector field $\phi^a_{\mu}$. Also, $\delta P, \delta Q$ denote local orthogonal transformations by $P$ and $Q$. $\phi^a_{\mu}$ and $Q^a_{\mu b}$ are second stage ghosts for ghosts $e^a_{\mu}$ and $P^a_{\mu b}$ in the Batalin, Fradkin and Vilkovisky formalism [12]. And $x$ and $y$ stand for all antighosts and N-L fields.

Myers and Periwal induce $\hat{S} = S - L e + \delta P$ and then $\hat{S}$ cohomology represent physical states. After straight forward calculation, we get $\hat{S}^2 = L \phi + \delta Q$. Then, for any scalar functional $h$ up to a total derivative,

$$ Sh = \hat{S} h. $$

(14)

We will show that this $S$ symmetry is broken by the way of section 2. We fix the GL transformation up to diffeomorphism and local orthogonal transformations at the same conditions as Myers and Periwal [6][7], in 4-dim case.

$$ R + \alpha = 0 $$

(15)

$$ W^+_{abcd} = 0 $$

(16)

$$ \nabla^a e^a_{\mu} - \nabla^a e^a_{\mu} = 0 $$

(17)

While we choose the constraints to fix the redundant diffeomorphism and orthogonal transformations

$$ \nabla_a t_{ab} - \frac{1}{2} \nabla_b t = 0, \quad r^a_{\mu b} = 0, $$

(18)

where $t_{ab} = \frac{1}{2} (w_{ab} + w_{ba})$, $t = tr(t_{ab})$, $r_{ab} = \frac{1}{2} (w_{ab} - w_{ba})$. These conditions are all covariant. So diffeomorphism and local orthogonal transformations are still unfixed. In ref. [7], to fix these symmetries, they imposed harmonic condition $\partial_\mu (e e^a_{\mu}) = 0$ and algebraic constraint $\partial_\mu (e e^a_{\mu} e^a_{\mu}) = 0$ where $e^a_{\mu}$ is some fixed background tetrad.

But now, we do not adopt these conditions and the reason will appear in the next
section. So, we assume these symmetries were fixed by some appropriate conditions.

Let us adapt this topological gravity to the way of section 2. On the Landau
gauge, we get delta functions after N-L fields integration. According to the previous
section, some symmetry breaking term should be added. We take it $i\epsilon e\tau f(e^a_\mu, w_{\mu}^{ab})$, where $f$ is some functional of $e^a_\mu$ and $w_{\mu}^{ab}$ that satisfy $Sf \neq 0$, and $\tau$ is an N-L field
used to fix on condition (15). We are able to regard eq.(15) as a fixing condition for
conformal mode. Additional gauge fixing Lagrangian for eq.(15) can be written with
antighost $\rho$ and its N-L fields $\tau = \hat{S}\rho$ as,

$$L_{\alpha} = Se\rho(R + \alpha)$$
$$= \hat{S}e\rho(R + \alpha)$$
$$= e\tau(R + \alpha) - \rho\hat{S}(e(R + \alpha)) \quad (19)$$

The formula (14) was used for second equality of (19), and total divergence
was ignored. Due to the additional symmetry breaking term, the delta function
changes from $\delta(e(R + \alpha))$ to $\delta(e(R + \alpha + \epsilon f))$. So, in this case, symmetry breaking
is only connected to the condition $R + \alpha = 0$. We abbreviate another conditions
for redundant diffeomorphism and local orthogonal symmetry, like eq.(18), and GL
symmetry without conformal mode, like eq.(16) and (17) to “(GL)”. The total
Lagrangian is written as below by using appropriate gauge fermions $x_1(\text{diffeo.}) +
\ x_2(\text{ortho.})$ for fixing the diffeomorphism and local orthogonal transformations,

$$L_{\epsilon} = L_{cl} + L_{\alpha} + \epsilon e\tau f + \hat{S}x_0(\text{GL}) + Sx_1(\text{diffeo.}) + Sx_2(\text{ortho.})$$
$$= L_{cl} + e\tau(R + \alpha + \epsilon f) + \rho\hat{S}(e(R + \alpha))$$
$$+ \hat{S}x_0(\text{GL}) + Sx_1(\text{diffeo.}) + Sx_2(\text{ortho.}) \quad (20)$$

where $x_0, x_1$ and $x_2$ are antighost fields and their tensor property is determined by
gauge functions (GL), (diffeo.) and (ortho.). We get delta functions after N-L
fields $y$ integration

$$\delta(e(R + \alpha + \epsilon f))\prod \delta(\text{GL})\delta(\text{diffeo.})\delta(\text{ortho.}) \quad (21)$$

For simplicity, all delta functions from $\hat{S}x_0(\text{GL}) + Sx_1(\text{diffeo.}) + Sx_2(\text{ortho.})$ are
denoted by $\prod \delta(\text{GL})\delta(\text{diffeo.})\delta(\text{ortho.})$, here. The number of these delta functions
is the same number of components of $e^a_\mu$ and $w_{\mu}^{ab}$, because topological symmetry
permit to transform each components arbitrary. So, if (GL), (diffeo.) and (ortho.)
contain only $e^a_\mu$ and $w_{\mu}^{ab}$, we can rewrite (21) as

$$\delta(e(R + \alpha + \epsilon f))\prod \delta(\text{GL})\delta(\text{diffeo.})\delta(\text{ortho.}) \quad (22)$$
$$= \mathcal{J}^{-1} \prod_{a,b,c,\mu,\nu} \delta(e^a_\mu - e^{a(c)}_\mu - \Delta e^{a(c)}_\mu)\delta(w_{\nu}^{bc} - w_{\nu}^{bc(c)} - \Delta w_{\nu}^{bc(c)}) .$$

6
Where $\mathcal{J}$ is Jacobian,
$$
\mathcal{J} = \begin{vmatrix}
\frac{\delta e(R + \alpha + \epsilon f)}{\delta e^a_{\mu}} & \frac{\delta e^a_{\mu}}{\delta (GL,\text{diff.},\text{ortho.})} \\
\frac{\delta e(R + \alpha + \epsilon f)}{\delta w^a_{\mu}} & \frac{\delta e^a_{\mu}}{\delta (GL,\text{diff.},\text{ortho.})}
\end{vmatrix},
$$

$e^a_{\mu(c)}$, and $w^{bc}_{\nu(c)}$ are solution i.e.

$$
e(R + \alpha) |_{e^a_{\mu(c)} = e^a_{\mu(c)}, w^{bc}_{\nu(c)} = w^{bc}_{\nu(c)} = 0} = 0 \quad \text{(GL, diffeo., ortho.)} \quad \text{for eq.)(23)}
$$

and $\Delta e^a_{\mu(c)} \Delta w^{bc}_{\nu(c)}$ are variation from inducing $\epsilon f$. Note that they are depend on $\epsilon$.

As we saw in section 2, if the Jacobian $\mathcal{J}$ has zero modes, then BRS symmetry, i.e. topological symmetry, is broken. Let us analyze these broken conditions further. We analyze the situation that each component of the first column of the Jacobian matrix vanishes,

$$
\frac{\delta e(R + \alpha + \epsilon f)}{\delta e^a_{\mu}} |_{e^a_{\mu(c)} = e^a_{\mu(c)}, w^{bc}_{\nu(c)} = w^{bc}_{\nu(c)} = 0} = e^a_{\mu(R + \alpha)} + e R^\mu_{\mu(c)} |_{e^a_{\mu(c)}} = 0. \quad \text{(24)}
$$

$$
\frac{\delta e(R + \alpha + \epsilon f)}{\delta w^a_{\mu}} |_{e^a_{\mu(c)} = e^a_{\mu(c)}, w^{bc}_{\nu(c)} = w^{bc}_{\nu(c)} = 0} = -6 e^a_{\mu e^{\nu}_{\lambda}} (D^c_{\nu} e^c_{\lambda} = 0 \quad \text{(25)}
$$

Eq. (24) is Einstein equation with cosmological constant $\alpha$. From eq.(23), eq.(24) become

$$
R^a_{\mu(c)} e^a_{\mu(c)} = 0. \quad \text{(26)}
$$

So we conclude that only if $R = \alpha = 0$ and eq.(25) are satisfied, the topological symmetry is broken. In Calab-Yau manifolds in 6-dim the conditions are satisfied, for example. But in many manifolds, they are not satisfied. This condition connects to Yamabe conjecture, and it will be discussed in section 6.

The equations (25) mean torsion free conditions and these are not contradictory to gauge conditions if we adopt (17). When these conditions are satisfied, Jacobian is of order $\epsilon^4$. Indeed $e(R + \epsilon f)$ can be expanded around $e^a_{\mu(c)}$ as follows,

$$
0 = e(R + \epsilon f) |_{e^a_{\mu(c)} + \Delta e^a_{\mu(c)}, w^{bc}_{\nu(c)} + \Delta w^{bc}_{\nu(c)} = 0}
$$

$$
= \epsilon e f + \epsilon e f \frac{\delta e}{\delta e^a_{\mu}} (\Delta e^a_{\mu(c)}) + \epsilon e f \frac{\delta e}{\delta w^{ab}_{\mu}} (\Delta w^{ab}_{\mu(c)}) + \frac{\delta^2 e(R + \epsilon f)}{\delta e^a_{\mu}} (\Delta e^a_{\mu(c)} \Delta e^b_{\nu(c)}) + \frac{\delta^2 e(R + \epsilon f)}{\delta w^{ab}_{\mu}} (\Delta w^{ab}_{\mu(c)} \Delta w^{cd}_{\nu(c)}) + O(\Delta e^3). \quad \text{(27)}
$$
To leading order in $\epsilon$ we have,

$$
\epsilon = \frac{\delta^2 eR}{\delta e^a_\mu \delta e^b_\nu} (ef)^{-1} \Delta e^{(c)}_{\mu} \Delta e^{(c)}_{\nu} + \frac{\delta^2 eR}{\delta e^a_\mu \delta w^{bc}_\nu} (ef)^{-1} \Delta e^{(c)}_{\mu} \Delta w^{bc}_{\nu}
$$

$$
+ \frac{\delta^2 eR}{\delta w^{ab}_\mu \delta w^{cd}_\nu} (ef)^{-1} \Delta w^{ab}_{\mu} \Delta w^{cd}_{\nu},
$$

(28)

in $f|_{\epsilon^{(c)}} \neq 0$ case. Hence, $\Delta e^{(c)}_{\mu}$ and $\Delta w^{ab}_{\mu}$ should be order $\epsilon^{1/2}$.

We can estimate (24) as,

$$
\delta e(R + ef) \big|_{e^{(c)}, w^{(c)}, \Delta e^{(c)}, \Delta w^{(c)}} = \epsilon \frac{\delta e}{\delta e^a_\mu} (R + ef) \delta^2 e(R + ef) \Delta e^{(c)}_{\mu} + \delta^2 e(R + ef) \Delta w^{bc}_{\nu} + O(\Delta^2 e^2)
$$

(29)

Now, we get $\frac{\delta e(R+ef)}{\delta e^a_\mu} \sim \epsilon^{1/2}$, and similarly $\frac{\delta e(R+ef)}{\delta w^{ab}_\mu} \sim \epsilon^{1/2}$. From the estimation described above, it is concluded that Jacobian $J$ is order $\epsilon^{1/2}$ when $\alpha$ is 0 and a solution of the eq.(26), $R^a_\mu = 0$, exist without contradiction to gauge conditions, then topological symmetry is broken. Note that the singularity from the Gribov zero modes, Jacobian matrix zero eigen values, is contribution from only $\delta (e(R + ef))$. Even if we did not use Jacobian $J$ to estimate the singularity, we could find that the symmetry breaking occur by only this delta function.

In this section, we have studied only about vacuum condensation in topological gravity by the method of section 2. We found that the topological symmetry breaking was appeared in the process of conformal changing $R \rightarrow 0$ if Ricci flat (26) is realized on the background manifold. Of course, for this theory being well defined as physical theory, some other BRS symmetry should be present. This is a subject of the next section.

### 4 Two-BRS formalism

In section 2, we saw topological symmetry was broken at $R = 0$. However this means that BRS quantization is ill-defined. To clear this problem, we introduce another BRS transformation. $L_c$ operates as diffeomorphism to all fields except anti ghosts. Anti ghost fields were transformed as scalar fields, regardless of those tensor property. As a result of this, in our Lagrangian $\mathcal{L} = \mathcal{L}_c + \tilde{S}\bar{\Psi} + S_{x1}(diff.eo.) + S_{x2}(ortho.)$ where $\tilde{S}\bar{\Psi} = \mathcal{L}_\alpha + \tilde{S}x_0(GL)$, $\mathcal{L}_c$ and $\tilde{S}$ exact gauge fixing term $\tilde{S}\Psi_{g.f.}$ are $L_c$ invariant up to total divergence, because these terms are scalar, ref.[13][3][8]. So, if we can chose $S_{x1}(diff.eo.) + S_{x2}(ortho.)$ to be invariant under $L_c$ and redefine $L_c$ to be nilpotents, then we adopt $L_c$ as new BRS operator and physical states can be redefined with it.
Now one defines appropriate anti ghosts $x$ and Lagrange multipliers $y$, where
\[ Sx_i = y_i \quad Sy_i = 0 \quad i = 1, 2. \]  
(30)
The tensor properties of these $x_i$ and $y_i$ are determined by (diffeo.) and (ortho.). One can require (diffeo.) is scalar under local orthogonal transformations but it has no general covariant property, and (ortho.) is scalar under general coordinate transformations. Under this choice, $Sx_1(\text{diffeo.}) + Sx_2(\text{ortho.})$ is
\[ Sx_1(\text{diffeo.}) + Sx_2(\text{ortho.}) = (L_c + \hat{S})x_1(\text{diffeo.}) + (\delta_p + \hat{S})x_2(\text{ortho.}) \]
\[ = y_1(\text{diffeo.}) + (\delta_p)x_1(L_c + \hat{S})(\text{diffeo.}) + y_2(\text{ortho.}) + (\delta_p)x_2(L_c + \hat{S})(\text{ortho.}) \]
(31)

If we want to regard $L_c$ (or $\delta_p$) as a new BRS operator, it is seen in (31) that gauge fermions (diffeo.) (or (ortho.)) should be $\hat{S}$ cohomology, and $L_c x_i = y_i$ (or $\delta_p x_i = y_i$). But $\hat{S}$ cohomological gauge conditions are not known at least to us. Only we know $\hat{S}\phi_\mu = 0$ where $\phi_\mu$ was induced as ghost for ghost $c_\mu$ in (13), ref.\[1\]. So we adopt some functional of only $\phi_\mu$ for gauge conditions $G_{\mu\nu}(\phi_\mu) = 0$, where tilde means $G_{\mu\nu}$ is not general covariant. Note that $\hat{S}G_{\mu\nu} = \frac{\delta}{\delta\phi_\mu}G_{\mu\nu}\hat{S}\phi_\mu = 0$, and $\delta_p G_{\mu\nu} = 0$ since $G_{\mu\nu}$ has no local coordinate index. To fix the diffeomorphism, we add to Lagrangian with anti ghost $\bar{e}^{\mu\nu}$ and Lagrange multiplier $\mu_{\alpha\beta} = S\epsilon^{\mu\nu}$

\[ S\epsilon^{\mu\nu}G_{\mu\nu} = \epsilon\bar{e}^{\mu\nu}G_{\mu\nu} + (L_c)e^{\mu\nu}G_{\mu\nu} + \epsilon b^{\mu\nu}G_{\mu\nu} + \epsilon\bar{e}^{\mu\nu}L_c G_{\mu\nu} \]  
(32)

Next step, we fix the local orthogonal symmetry. Myers-Periwal fixed it at $\bar{e}^{\mu}_{a\nu}e^{\nu}_{b\mu} = 0$, where $\bar{e}^{\mu}_{a}$ is some back ground tetrad. This condition is not suitable for our purpose. Because, under $L_c$, $\bar{e}^{\mu}_{a}$ do not transform, so $\bar{e}^{\mu}_{a}e^{\nu}_{b\mu}$ is not invariant. i.e. $L_c e^{\mu\nu}e^{\nu}_{b\mu} \neq 0$, where $\bar{P}^{a\mu}$ is anti ghost. For this reason, another condition that include no back ground field, should be induced, here. For example, we fix it at $\bar{e}^{\mu}_{\mu}w^{\mu}_{ab} = 0$ \[4\]. The gauge fixing terms,
\[ S\bar{P}^{a\mu}\bar{\nabla}_{\mu}w^{\mu}_{ab} = \epsilon\bar{P}^{a\mu}\bar{\nabla}_{\mu}w^{\mu}_{ab} + \epsilon q^{a\mu}\bar{\nabla}_{\mu}w^{\mu}_{ab} - \epsilon \bar{P}^{a\mu}(\hat{S} + \bar{\delta}_p)\bar{\nabla}_{\mu}w^{\mu}_{ab} \]  
(33)
where $q^{ab}$ is a Lagrange multiplier, $q^{ab} = SP^{ab}$, is added to Lagrangian.

Everything is ready, for introducing new BRS symmetry. Let us define a new fermionic Lie derivative $L'_c$ for some BRS operator.

definition 1 (new BRS operator $L'_c$) The new BRS operator is defined as follows,
\[ L'_c = L_c \]  
(34)
for all fields $e^{a\mu}_{\mu}, w^{a\mu}_{\mu}, \ldots$ except $\bar{e}^{\mu}_{\mu}, b^{\mu}_{\mu}$
\[ L'_c \bar{e}^{\mu}_{\mu} = t\bar{e}^{\mu}_{\mu} + b^{\mu}_{\mu} \]  
(35)
\[ L'_c b^{\mu}_{\mu} = tb^{\mu}_{\mu} - (L_c t)\bar{e}^{\mu}_{\mu} \]  
(36)
for $\bar{e}^{\mu}_{\mu}, b^{\mu}_{\mu}$.  
9
Here $L'_c$ is nilpotents. From (34), $\mathcal{L}_{cl}$, $\hat{S}\Psi$ and $SeP^{ab}\nabla_{\mu}w^{\mu}_{ab}$ are transformed as scalar by $L'_c$, that is

$$\int d^Dx L'_c(\mathcal{L}_{cl} + \hat{S}\Psi + Se\tilde{P}^{ab}\nabla_{\mu}w^{\mu}_{ab}) = 0.$$  

(37)

And by using (36),

$$Se\bar{c}^{\mu\nu}G_{\mu\nu} = L'_c e\bar{c}^{\mu\nu}G_{\mu\nu}.$$  

(38)

we get

$$L'_c Se\bar{c}^{\mu\nu}G_{\mu\nu} = (L'_c)^2 e\bar{c}^{\mu\nu}G_{\mu\nu} = 0,$$  

(39)

from nilpotency of $L'_c$. Our total action is rewritten with $L'_c$, as

$$\int d^Dx (\mathcal{L}_{cl} + \hat{S}\Psi + Se\tilde{P}^{ab}\nabla_{\mu}w^{\mu}_{ab}) + L'_c e\bar{c}^{\mu\nu}G_{\mu\nu}$$  

(40)

and it is invariant under nilpotents operator $L'_c$ as we saw in (37) and (39). Now it is possible to regard $L'_c$ as a new BRS operator, and $\mathcal{L}_{cl} + \hat{S}\Psi + Se\tilde{P}^{ab}\nabla_{\mu}w^{\mu}_{ab}$ is a new classical Lagrangian. In this form, $t^a_b, \phi_{\mu}$ and others except $\bar{c}^{\mu\nu}$ and $b^{\mu\nu}$ become physical fields in addition to $e^a_{\mu}$ and $w^{ab}_{\mu}$, as a result of changing the physical states conditions from $S|_{phys} >= 0$ to $L'_c|_{phys} >= 0$.

Note that using $\phi_{\mu}$ to fix diffeomorphism disturbs rewriting $\delta$ functions with Jacobian like the previous section. Because the number of $\delta$ functions of $e^a_{\mu}, w^{ab}_{\mu}$ is less than one of independent components of fields. But singularity of the $\delta(eR)$ is not changed. It is apparent that topological symmetry is broken.

The symmetry breaking condition is in fact the Einstein equation $R_{ab} = 0$. The solution of a classical Einstein equation exists and it contributes to the path integral, hence the theory of broken topological symmetry is realized in the above. In other words, non-topological phase is defined around classical gravity.

5 Regularization

In the previous section, we got the new BRS operator to define physical states again. It means that the vacuum expectation value of some BRS exact operator is zero. But our theory is still singular due to $\delta(e(R + \epsilon f))$ in (21). So we have to remove this divergence. In this section, we make the prescription to regularize this divergence.

First, we discuss regularization in general formalism. Here, we use same symbols as ones of section 2. There is strong divergence in $\delta(F_i + \epsilon f_i)$ in eq.(7) as $\epsilon \rightarrow 0$ when $\mathcal{D}^{i}_{j}(\phi^c) = 0$. This delta function appeared as a result of $b_i$ integration, where $b_i$ is a Lagrange multiplier of the gauge function $F_i$. So it is evident that more stronger
divergence will appear if an observable contains $b_i$ fields. To get finite vacuum expectation value, we redefine $b_i$ fields as

$$b'_i = \epsilon^{\frac{1}{2}} b_i$$

in $f_i(\phi^c) \neq 0$ case when

$$\frac{\delta F_i}{\delta \phi_j} |_{\phi^c} = \mathcal{D}^i_j(\phi^c) = 0 \quad \frac{\delta F_k}{\delta \phi_j} |_{\phi^c} \neq 0 \quad : k \neq i, j \sim n$$

are satisfied for some $\phi^c$. A fixed index \("i\) means that functional derivative of $F_i$ on some $\phi^c$ vanish like eq.(42) in the following. $\phi^c$, i.e. solutions of $F_j = 0$, do not always satisfy eq.(42), then we put $\phi^z$ and $\bar{\phi}^z$ as

$$\phi^z \in \{ \phi^c \mid \mathcal{D}^i_j(\phi^c) = 0 \} \quad \bar{\phi}^z \in \{ \phi^c \mid \mathcal{D}^i_j(\phi^c) \neq 0 \}$$

In other words, $\phi^z$ is a kernel of $\mathcal{D}^i_j$. By using this $b'_i$, we rewrite the gauge fixing term in the Lagrangian (3) as

$$b^i(F_i + \epsilon f_i) \rightarrow \epsilon^{-\frac{1}{2}} b^i(F_i + \epsilon f_i).$$

Then the delta function is changed as

$$\delta(F_i + \epsilon f_i) \rightarrow \delta(\epsilon^{-\frac{1}{2}}(F_i + \epsilon f_i)) = \epsilon^{\frac{1}{2}} \delta(F_i + \epsilon f_i)$$

Due to eq.(45), delta functions are order 1 for $\phi^z$, but for $\phi^c = \bar{\phi}^z$ they are order $\epsilon^\frac{1}{2}$. In the count of the term $\prod_m S_{\phi_j} \frac{\delta}{\delta \phi_j} F_m$ as similar in eq.(7) all amplitude of observables is order $\epsilon^\frac{1}{2}$ whether there are zero modes or not, as it is. So, also $\bar{c}^i$ fields have to be redefined as

$$\bar{c}'_i = \epsilon^{\frac{1}{2}} \bar{c}_i$$

and the Fadeev Popov terms in the Lagrangian change as

$$\bar{c}' S_{\phi_j} \frac{\delta}{\delta \phi_j} F_i = \bar{c}'(\epsilon^{-\frac{1}{2}} S_{\phi_j} \frac{\delta}{\delta \phi_j} F_i).$$

As a result of these redefinition, order of $\epsilon^{\frac{1}{2}} \delta(F_i + \epsilon f_i)$ and $\prod_m \frac{\delta}{\delta \phi_j} F_m$ are found as

$$\epsilon^{\frac{1}{2}} \delta(F_i + \epsilon f_i) \sim \{ \begin{array}{cl} 1 & : \text{ for } \phi^z \\ \epsilon^{\frac{1}{2}} & : \text{ for } \bar{\phi}^z \end{array}$$

$$\epsilon^{-\frac{1}{2}} S_{\phi_j} \frac{\delta}{\delta \phi_j} F_i \sim \{ \begin{array}{cl} 1 & : \text{ for } \phi^z \\ \epsilon^{-\frac{1}{2}} & : \text{ for } \bar{\phi}^z \end{array}.$$

Before these redefinitions, if an observable contains $b_i$ fields, then singularity is stronger. Indeed from the same reason, in section 2, $\langle SO \rangle$ was non zero, and yet
The partition function was finite. It is easy to understand by the following formulation of a delta function

$$
\lim_{\epsilon \to 0} \int db \ e^{i b (x^2 - \epsilon^2)} = \lim_{\epsilon \to 0} \frac{1}{2\pi} \frac{\partial}{\partial x} \left( \frac{\delta(x - \epsilon) + \delta(x + \epsilon)}{|2x|} \right).
$$

(50)

Where existence of $b$ induced a derivative and power of divergence went up. But now, because of redefinition (41) and (46), when an observable contains $b'_i$ fields, i.e. $O = b'_i O'$, then the vacuum expectation value of $O$ is

$$
\langle b'_i O' \rangle = \int \mathcal{D} \phi_k \ O'^i \epsilon^{i \frac{1}{2}} \left( \frac{\partial}{\partial(F^i + \epsilon f^i)} \right) \prod_{k \neq i} \delta(F_k) \prod_m S_{\phi_j} \frac{\delta}{\delta \phi_j} F_m \ e^{\int d^4 \mathcal{L}_{\text{cl}}}.
$$

(51)

Where the power of divergence is unchanged for $\phi^c = \tilde{\phi}^z$ and the power of $\epsilon$ go up for non zero mode $\phi^c \neq \tilde{\phi}^z$. So that, contribution for amplitude is from only Gribov zero modes $\phi^z$, and other contribution from $\tilde{\phi}^z$ vanish. After all, the amplitude (51) is sum over $\phi^z$ and it is order one because

$$
\frac{\partial}{\partial(F^i + \epsilon f^i)} \sim \epsilon^{-1} \text{ less divergence.}
$$

(52)

Hence, we have done the regularization for all observable in general case. Especially there is remarkable property that if an observable contains Lagrange multipliers of $F_i$ then contribution from $\phi_i$ path integration to the amplitude of this observable is only from $\tilde{\phi}^z$. Next we try this regularization in topological gravity case.

We carry out this regularization in the gravitational theory in the same way as the general case. The only things we have to do is to redefine $\rho$ and $\tau$ as follows,

$$
\rho' = \epsilon^{\frac{1}{2}} \rho \quad \tau' = \epsilon^{\frac{1}{2}} \tau
$$

(53)

Then all amplitude is regularized. Note that, if an observable contains $\tau'$, its vacuum expectation value is the sum of solution of $R_{\mu}^a = 0$. This fact is very interesting. In the theory of section 4, we define observables as $L'_{c}$ closed. If we change this definition to $L'_{c}$ closed and containing $\tau'$ fields,

$$
Z = \int \mathcal{D} \phi_k \tau' e^{\int d^4 \mathcal{L}_{\text{cl}}} : \text{partition function}
$$

(54)

$$
L'_{c} O = 0 \text{ and } O = \tau'O' : \text{definition of observable.}
$$

(55)

Then the theory is semiclassical, i.e. path integral contribution for vacuum expectation value is from only solution of the Einstein equation. Note that our theory have many constraints for fixing topological symmetry like eq.(16) and (17). So, after symmetry breaking, these constraints are left as equations of motion. In this meaning, sense of semiclassical gravity is different from usual case.
6 Mathematical interpretation

As we saw in section 3, our theory has broken phase on the condition $R_{ab} = 0$. Let us clarify mathematical meanings of this.

For simplicity, we omit the symmetry breaking term $\epsilon e \tau f$ in this section. As is mentioned in section 3, the Yamabe conjecture is concerned with our theory. We are going to see this fact, as follows. In topological gravity, it is trivial that any physical amplitude is invariant under changing $\alpha$ because gauge condition does not affect physical amplitude. Indeed, a derivative of the partition function with respect $\alpha$ is given by,

$$\frac{\partial}{\partial \alpha} Z = \int \mathcal{D}\phi_k (\hat{S}i\epsilon \rho)e^\int dx^b L = 0$$

(56)

To get the second equality, we use that $\hat{S}$ exact vacuum expectation value vanishes as same as $S$ exact one, ref. [6] [8]. This means that variation of scalar curvature is not restricted by topology. Strictly speaking, our topological gravity may not classify the topology of manifolds perfectly, so we can only say that scalar curvature can be varied without changing class which is classified by our topological theory, as far as the theory is well defined. But our theory is broken at $R_{ab} = R = 0$ as we saw in section 3. From a mathematical viewpoint, gauge conditions restrict back ground manifolds to submanifolds. The inverse mapping theorem demand that Jacobian $J$ is nonzero for the neighborhood of the gauge conditions to be an infinite dimensional manifold. i.e. if the Jacobian is nonzero, then functional $R+\alpha, (GL, diffeo., ortho.)$ and their inverse map are isomorphism, and the inverse map $\{e^a_\mu, w^{ab}_\mu | R + \alpha = 0, (GL, diffeo., ortho.) = 0\}$ become manifold. If $R_{ab} = R = 0$, then the Jacobian is zero and the submanifold is ill-defined. It is known that we can regard a topological field theory as an extended Morse theory [15]. In this point of view if submanifolds are ill-defined then the Morse theory is also ill-defined and topological symmetry is breakdown.

This situation is similar in the Donaldson theory [16] [1]. Homologies of moduli $\mathcal{M} = \{ [A] \mid F^+_A = 0\}$ can be defined, if and only if $\mathcal{M}$ is finite dimensional manifolds, where $[A] = \{A/G\}$. Donaldson invariants are Homology $H_*([A]; \mathbb{Z})$. So, they can be defined in the neighborhood of $F^+_A = 0$ when the Jacobian $\det(d_A^+) \neq 0$, where $d_A^+ = d_A + *d_A^*$. In other words, $d_A^+$ should be surjection.

In our theory, the condition that the Jacobian become zero is $R_{ab} = 0$, and if it’s satisfied topological symmetry is lost.

Yamabe conjectured constant scalar curvature $R$ exist on any compact Rieman manifolds with arbitrary topology of dimension $n \geq 3$ [10]. But it has been corrected by Aubin [17], Schoen [18] and so on. Especially, Kazdan and Warner [19] gave the following theorem that

Theorem 1 (Kazdan Warner theorem) Compact manifolds $M$ of dimension $n \geq 3$ can be divided into three classes,
(A) Any $(C^\infty)$function on $M$ is the scalar curvature of some $(C^\infty)$metric.
A function on $M$ is the scalar curvature of some metric if and only if it is either identically zero or strictly negative somewhere, further more, any metric with vanishing scalar curvature is Ricci-flat.

A function on $M$ is a scalar curvature if and only if it is strictly negative somewhere.

This theorem says that existence of negative scalar curvature do not demand any topological condition. And there is a barrier at $R = 0$. This is consistent with our theory. We are able to classify the type (C) manifolds from the type (A) and (B) manifolds, in our theory. Let us take $R = -\alpha < 0$ first, and makes $\alpha$ to zero. On the type (A)(B) manifolds, topological symmetry is broken , on the other hand, on the type (C) it’s not broken. In other words, our theory may classify manifolds to type (C) and other type, by calculating some vacuum expectation value of $\hat{S}$ exact terms on $R = 0$. If it vanish, the back ground manifold is type(C), and if it’s not zero, then the manifold is type (A)or(B).

Note that in Myers and Periwal [6] observables are topological on $\alpha \neq 0$. On the other hand, they are topological-like but are non topological in a strict sense in our theory. On type (C), they are independent of metric, but on type(A) or (B) they are non topological.

7 Conclusion and discussion

We have constructed a topological-like gravitational theory. Its feature is that the topological symmetry is broken when gauge condition chose $R = 0$ and the Einstein equation $R_{ab} = 0$ has a solution on the back ground manifolds.

Now, the question flowing up naturally is how matter couples. For example, one change gauge condition to $R + \hat{S}_{matter}(\Psi, e^a_\mu) = 0$, where $\hat{S}_{matter}$ is matter action of background fields ($L_c \hat{\Psi} = S\hat{\Psi} = 0$). Then the condition of topological symmetry breaking is the Einstein equation with matter,

$$\frac{\delta e(R + \hat{S}_{matter})}{\delta e^a_\mu} = e e^a_\mu(R + \hat{S}_{matter}) + e (R_{ab} - \bar{T}_{ab}^a) = 0$$

(57)

and the torsion equation ,

$$\frac{\delta e(R + \hat{S}_{matter})}{\delta w^{a \mu}_{ab}} = -6 e [^a_\alpha e^\nu_\beta (D_\nu e^\lambda_\chi) - \bar{\hat{S}}_{ab}^\mu = 0$$

(58)

,where $\bar{T}_{ab}^a = \frac{\delta S_{matter}}{\delta w^{a \mu}_{ab}}$ is a energy momentum tensor and $\bar{\hat{S}}_{ab}^\mu = \frac{\delta S_{matter}}{\delta w^{a \mu}_{ab}}$ is a spin density. To satisfy these (57) and (58), we have to change the gauge conditions of the torsion free condition, eq.(15), and instead of eq.(14) $R_{ab} = 0$ , we need

$$R_{ab} = \bar{T}_{ab}^a = 0$$

(59)
Since we have fixed $R + \bar{S}_{\text{matter}}(\bar{\Psi}, e_\mu^a) = 0$, then the following equation is necessary for symmetry breaking,

$$\bar{S}_{\text{matter}} = \tilde{T}_\mu^a e_\mu^a = tr(\tilde{T}_\mu^a).$$

(60)

For example, Dirac field $\bar{S}_{\text{matter}} = \bar{\Psi} \gamma^\mu \nabla_\mu \bar{\Psi}$ satisfy this condition. Then the topological symmetry breaking occurs depending upon matter fields. We may construct the theory that break topological symmetry by dynamics of matter fields.

This study has constructed topological-like field theory that has broken phase when the solution of the Einstein equations exist. This symmetry breaking is caused by Gribov zero modes, in other words by zero eigen values of Jacobian matrix, that appear as solutions of the Einstein equations. And we found that if one can take gauge fermion cohomological of reduced BRS operator $\hat{S}$, then we can induce new BRS operator by the reduced symmetry. In our case, we got the $L_{c'}$ as a new BRS operator and only symmetry of diffeomorphism is left. To take away the divergence from zero modes, we gave one method of regularization. Hence we could extend the topological gravity which was fixed on $R + \alpha = 0$ ($\alpha > 0$) for conformal symmetry to the topological-like gravity on $R = 0$. It has nontrivial broken phase on some manifolds, and especially when we chose the theory as eq.(54) and (55) then the theory describe semiclassical Einstein gravity. Of course, this theory dose not described the real gravity perfectly, as it is. But the property that breaking topological symmetry depend on back ground manifolds encourages us to apply our theory to other theories. For example, we will have to examine the same methods to Weyl gravity. In our theory, only the conformal symmetry was needed to break the BRS symmetry. So, we might carry out the same methods easily in the Weyl gravity. We might construct quantum gravitational thory that have phase of more real semiclassical Einstein gravity. While the same phenomena will be realized in other theory as well, for example in topological Yang-Mills theory.

In this formalism, the gauge condition directly reflect broken phase physics. There could be some criticisms. First, the way of breaking has many ambiguities of selecting breaking terms. Second, it is unnatural that we have to adopt $\hat{S}$ cohomological gauge fermion for inducing the 2nd BRS operator. But it may be interpreted as follows. If many BRS theory is found in real world, as we had seen in section 3, it’s gauge condition will be restricted as cohomology of reduced BRS. It implies that internal space, i.e.gauge space, have some mechanism or kinematics. It may be that as a result of it, gauge condition is non free from physics. Further, ghost fields become phisical fields after symmetry breaking. We have to adjust and interpretate these new phisical fields to real world.

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References

[1] E.Witten, Commun.Math.Phys.117(1988)353.
[2] E.Witten, Phys.Letts. B206. (1988)601.
[3] N.Seiberg, E.Witten, Nucl.Phys.B436(1994)19;
   N.Seiberg, E.Witten, Nucl.Phys.B431(1994)581;
   E.Witten, Math.Res.Letts. 1(1994)769.
[4] M.Alvarez, J.M.F.Labastida, Phys.Letts. B315(1993)251.
[5] W.Zaho, H.C.Lee, Phys.Rev.Letts. 68(1992)1451.
[6] R.Myers, V.Periwal, Nucl.Phys. B361(1991)290.
[7] R.Myers, Phys.Letts. B252(1990)365.
[8] R.Myers, Int.J.Mod.Phys. A. 5(1990)1369;
   S.Ouvry, R.Stora, and P.van Baal, Phys.Lett. B220(1989)159;
   H.Kanno, Z.Phys. C-Particle and Fields 43(1989)477.
[9] D.Birmingham, M.Blau, M.Rakowski, and
   G.Thompson, Phys.Report. 209(1991)129
[10] A.L.Besse, “Einstein Manifolds,” Springer-Verlag,1987.
[11] J.M.F.Labastida, M.Pernici, Phys.Letts. B213(1988)319.
[12] I.A.Batalin, G.A.Vilkovisky, Phys.Rev.D28(1983)2567;
   I.A.Batalin, G.A.Vilkovisky, Phys.Lett. B102(1981)27; B69(1977)309;
   E.S.Fradkin, G.A.Vilkovisky, Phys.Lett. B55(1975)224;
   E.S.Fradkin, T.A.Fradkin, Phys.Lett. B72(1978)343.
[13] K.Fujikawa, “Quantum Gravity and Cosmology,” World Scientific,1986,p106.
[14] Nakanishi, N., Prog.Theor.Phys. 62, 779(1979)[V].
[15] J.M.F.Labastida, Commun.Math.Phys.123(1989)641.
[16] S.K.Donaldson, P.B.Kronheimer, “The Geometry of Four-Manifolds,”
    Oxford,1990.
[17] T.Aubin, J.Math.Pures Appl.55, 269(1976).
[18] R.Schoen, J.Differential Geometry 20(1984)479.
[19] J.L.Kazdan, F.W.Warner, Inventiones Math.28, 227(1975).