Spatially Coupled Repeat-Accumulate Codes
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Abstract—In this paper we propose a new class of spatially coupled codes based on repeat-accumulate protographs. We show that spatially coupled repeat-accumulate codes have several advantages over spatially coupled low-density parity-check codes including simpler encoders and slightly higher code rates than spatially coupled low-density parity-check codes with similar thresholds and decoding complexity (as measured by the Tanner graph edge density).

I. INTRODUCTION

Convolutional LDPC codes, otherwise known as spatially coupled LDPC codes (SC-LDPC), were first introduced by Felström and Zigangirov in the late 90’s [1]. Performance results, generated using either density evolution or decoding simulations, have shown that SC-LDPC codes have excellent sum-product decoding thresholds over a range of channels [2]–[4]. Incredibly, and in contrast to standard LDPC codes, these thresholds rapidly improve as a function of the average Tanner graph node degree. This enables the design of iterative error correction codes with both excellent thresholds and very low error floors, something not so far achieved with traditional LDPC or turbo codes.

Recent exciting developments have shown that the iterative decoding threshold of certain SC-LDPC ensembles is actually equal to their MAP threshold on the binary erasure channel (BEC) [5]. I.e., for spatially coupled codes iterative decoding is actually optimal on the BEC. It is conjectured, but not yet proven, that this holds for more general channels as well.

In this paper we consider whether the concept of spatial coupling can apply equally well to another class of iterative error correction codes called repeat-accumulate codes. Repeated-accumulate (RA) codes [6], are error correction codes formed by the serial concatenation of a rate-1/q repetition code and a 1/(1+D) convolutional code, called an accumulator, with an interleaver, II, and (optionally) a rate-a combiner between them. Significantly, RA codes can be encoded using serial concatenation of the constituent encoders, as for serially concatenated turbo codes, and decoded using iterative decoding, as for LDPC codes, thus gaining both the low encoding complexity of turbo codes and the decoding performance of LDPC codes.

In this paper we will consider the spatial coupling of RA codes in such a way to preserve the inherent advantage of RA codes, most importantly their very simple encoding, while obtaining the threshold advantages promised by the idea of spatial coupling. Section II introduces our proposed spatially coupled RA codes, Section III presents threshold results derived using density evolution and Section IV gives simulation results comparing spatially coupled RA and LDPC codes.

II. SPATIALLY COUPLED RA CODES

Spatially coupled RA (SC-RA) codes can be formed in a similar manner to spatially coupled LDPC (SC-LDPC) codes. We consider two ensembles, the first we will use in practice to construct SC-RA codes, and the second is useful to derive density evolution equations.

A. The (q,a,L) Ensemble

The left hand side of fig. 1 shows the protograph of a standard (3,3)-regular RA code. There is one message bit node, shown at the top, a parity bit node, shown at the bottom, and a check node in the middle. A coupled chain of 2L+1 of these protographs, shown on the right hand side of fig. 1, is formed by connecting each message bit to 2l = (q−1)/2 protographs to the left and l protographs to the right [7]. As for coupled LDPC chains we add q − 1 extra check nodes (shown in bold) when forming the coupled chain of protographs. For RA protographs we must also add q − 1 extra parity bit nodes (shown in bold) to avoid creating any degree-1 check nodes.

We could have spatially coupled the parity bit nodes in the same way as the message bit nodes, i.e. by connecting each parity bit node to the check node of the protograph on the right hand side. However, if the parity bit nodes are coupled in this way, the final code will not retain the RA code accumulator structure. Keeping the parity bit nodes uncoupled can be thought of as serially concatenating a spatially coupled low-density generator matrix with a standard accumulator.

A particular code from the (q,a,L) ensemble will be formed using copies of the coupled chain to give a total of M message bits per protograph. Our final code will thus consist of (2L+1)M message bit nodes, (2L+1+2l)aM parity bit nodes and (2L+1+2l)aM check nodes. Hence the code rate,

Fig. 1: Coupled q = 3, a = 3 RA protographs.
assuming every check node results in a linearly independent constraint, is

\[
r_{\text{RA}} = \frac{(2L + 1)M}{(2L + 1)M + (2L + 1 + 2\hat{l})aM} = \frac{(2L + 1)\hat{a}}{(2L + 1)a + (2L + q)q}.
\]  

(1)

When constructing a code from the \((q, a, L)\) ensemble each of the message bit nodes at position \(i \in \{-L, \cdots, L\}\) is connected to exactly one of the check nodes at positions \(j \in \{-i - 1, \cdots, i + 1\}\). The choice of which of the check nodes to connect to at each position can be chosen randomly.

For each protograph the \(M\) parity-bit nodes are connected to the \(M\) check nodes in a traditional accumulator pattern. We also connect the final bit node in each protograph to the first bit separately terminated the accumulator in each protograph (by the first check node in the following protograph. We could have connected the final bit node in each protograph to the last bit node in the protograph to give \(2L + 1 + 2\hat{l}\) separate size \(M\) accumulators. For large enough \(M\) there should not be a difference in performance, however, a single accumulator avoids the \(2L + 1 + 2\hat{l}\) length \(2M\) cycles.

**Example 1.**

A \((q = 3, a = 3)\) RA protograph is repeated \(2L + 1 + 3\) times to give the coupled chain in fig. 3. Setting \(M = 2\) and randomly choosing an edge permutation for the message bit nodes gives the SC-RA Tanner graph in fig. 3. As SC-RA codes are systematic we form the codeword using the messages bits first, followed by the parity bits. This gives an SC-RA code with parity-check matrix: 

\[
H = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

In practice the edge corresponding to the 1 entry in the top right corner of \(H\) is omitted for ease of encoding.

By slightly re-drawing fig. 1 to push the top row of nodes across to the left immediately shows how to construct SC-RA codes with even values of \(q\). Fig. 3 for example shows a SC-RA code with \(q = 4\).

**B. The \((q, a, L, w)\) Ensemble**

The ensemble \((q, a, L)\) can be modified by adding a “smoothing” parameter \(w\) in a similar method to that for LDPC codes [5]. The \((q, a, L, w)\) ensemble is not used in practice but is useful to simplify the derivation of density evolution equations. Considering this ensemble for SC-RA codes will allow a comparison of the asymptotic performance of SC-RA codes with the SC-LDPC ensembles in [5].

As previously, at each position \([−L, L]\) there are \(M\) message bit nodes. However, the check nodes are considered to be located at all integer positions \(−∞, ∞\) and there are \(\frac{q}{2}M\) check nodes at each position. Only some of these positions actually interact with the message bit nodes. Instead of requiring that each message bit node at position \(i \in \{-L, \cdots, L\}\) is connected to exactly one of the check nodes at positions \(j \in \{-i − 1, \cdots, i + 1\}\) we assume that each of the \(q\) connections of a variable node at position \(i\) is uniformly and independently chosen from the range \([i, \cdots, i + w - 1]\). Similarly, we assume that each of the \(a\) connections of a check node at position \(i\) is independently chosen from the range \([i − w + 1, \cdots, i]\). \(q\) need not be odd. For simplicity we again assume that a parity bit node is associated with every active check node and connected once to that check node and once to the next adjacent active check node on the right.

Using a similar derivation to that for LDPC codes [5], leads to the rate of the \((q, a, L, w)\) RA ensemble as:

\[
t_{\text{RA}, w} = \frac{2L + 1}{2L + 1 + \frac{q}{a} \left[2L - w + 2 \left(w + 1 - \sum_{i=0}^{w-1} \left(\frac{1}{w} \right)^{\hat{a}}\right)\right]}.
\]

(2)

**C. Encoding**

The motivation for considering SC-RA codes is their low encoding complexity. As for traditional repeat-accumulate codes, SC-RA codes can be encoded with complexity linear in the code length by the serial concatenation of a repetition code, interleaver, combiner and \(\frac{1}{\hat{D}}\) convolutional encoder or accumulator.

RA and SC-RA codes are systematic so that the message bits make up the first \(K\) bits in the codeword meaning that codeword bits can be transmitted as soon as message bits are received. The structure of SC-RA codes also has the additional advantage of limiting the number of message bits that must be received before the first parity bit can be encoded. Consider fig. 3 A parity bit in the \(i\)th location is a function only of message bits in the \(i\)th and previous \(q − 1\) locations.
III. Density Evolution

In this section we derive closed form expressions for density evolution for the \((q, a, L, w)\) ensemble on the BEC and show how the multi-edge formulation for LDPC codes can be used to derive thresholds for the \((q, a, L)\) ensemble.

Following a similar approach to that used for the LDPC \(w\)-ensemble [5], gives density evolution equations for the SC-RA \((q, a, L, w)\) ensemble:

\[
x_i^{(t+1)} = \epsilon \left( 1 - \frac{1}{w} \sum_{j=0}^{w-1} \left( 1 - y_{i+j}^{(t)} \right)^2 \right) \cdot \left( 1 - \frac{1}{w} \sum_{k=0}^{w-1} x_{i+k}^{(t)} \right)^{a-1} q^{-1}
\]

\[
y_i^{(t+1)} = \epsilon \left( 1 - \left( 1 - y_i^{(t)} \right) \left( 1 - \frac{1}{w} \sum_{k=0}^{w-1} x_{i-k}^{(t)} \right)^a \right),
\]

where \(x_i^{(t)}\) and \(y_i^{(t)}\) denote the erasure probabilities from message bits and parity bits respectively at position \(i\), at iteration \(t\).

Density evolution for the \((q, a, L)\) ensembles results in more complicated expressions since the erasure probabilities on edges connected to one protograph cannot be averaged as for the \((q, a, L, w)\) ensemble above. While it is still possible to write the expressions in closed form we instead choose the multi-edge framework to represent the structure of the \((q, a, L)\) ensemble and use multi-edge density evolution to evaluate the decoding thresholds over the erasure channel. For a detailed description of multi-edge density evolution we refer the reader to [7, Sec. 7].

Numerical results are shown in fig. 4 where we compare the decoding thresholds and rates ([5, Eqn. 7], [5, Lemma 3] for the LDPC ensemble and (2) and (3) for the RA ensemble). Each curve corresponds to a value of \(L\) and the markers represent the variable node degree of the message bits. Higher degrees lead to an improved decoding threshold but result in a lower rate due to the increasing number of additional check nodes at the ends of the graph.

The \((q, a, L, w)\) ensembles are shown for the case \(w = q\) and so these ensembles will not have extra check nodes over those in the \((q, a, L)\) ensembles with the same parameters. Consequently the \((q, a, L, w = q)\) ensembles have a slightly higher rate than the \((q, a, L)\) ensembles with the same parameters, due to the likelihood of some check nodes not being active for a given code. When \(w\) is chosen to be larger than \(q\) there is also the likelihood of extra check nodes, outside of those used in the \((q, a, L)\) ensemble, becoming active and thereby slightly reducing the code rate. Thus fig. 4 shows points for LDPC protographs with \(d_i = \{3, 4, 5, 6\}\) and RA protographs with \(q = \{4, 6, 8, 10\}\) (lower degrees correspond to lower thresholds).

We observe that SC-RA codes perform better than SC-LDPC codes giving a higher code rate at the same decoding threshold as the SC-LDPC codes.

IV. Simulation Results

In this section we randomly construct SC-RA codes and compare their decoding performance at finite lengths to SC-LDPC codes. Consider for example the \((q, a, L)\) ensemble with thresholds shown in fig. 4 for \(L = 16\) with \(q = 6\) for the SC-RA code and \(d_i = 4\) for the SC-LDPC code. The SC-RA ensemble has an average variable node degree of 3.86 (compared to 4 for the SC-LDPC code), a higher rate and a similar decoding threshold.
Fig. 5: Erasure correction performance of SC-LDPC and SC-RA codes with $L=16$, for $K=3,300$ and $K=9,900$ using iterative decoding with a maximum of 1000 iterations. Solid lines show the word erasure rate and dashed lines show the bit erasure rate.

Fig. 5 shows the erasure correction performance of (6,6,16) SC-RA codes with $M$ set to 100 and 300 respectively. Also shown is the performance of (4,8,16) SC-LDPC codes with $M$ set to 220 and 660 respectively. (Recall that for SC-LDPC codes $M$ specifies the number of all bit nodes, whereas for SC-RA codes $M$ specifies the number of message bit nodes). For the two shorter codes, each code transmits 3,300 message bits, however the SC-RA code has a slightly higher rate requiring only 7100 codeword bits ($r = 0.4648$) instead of 7260 ($r = 0.4545$). For the two longer codes, each code transmits 9,900 message bits, however the SC-RA code requires only 21300 codeword bits instead of 21780.

Also shown is the threshold for SC-RA codes (from fig. 4) and the iterative decoding threshold for RA codes with the same degree distribution (and rate) as the SC-RA codes but without the spatial coupling.

In fig. 5 we can see that spatial coupling or RA codes does indeed produce codes with excellent iterative decoding performance. We also see that the performance of the SC-RA codes is better than that of the SC-LDPC codes with similar decoding complexity (as measured by the Tanner graph edge density) despite both having the same threshold. We suspect that for finite length codes the structure of the SC-RA codes gives them a further advantage (in addition to the slightly higher rate for the same threshold) over LDPC codes.

V. DISCUSSION

In this paper we have proposed a new class of spatially coupled codes based on repeat-accumulate protographs. We show that spatially coupled repeat-accumulate codes have several advantages over spatially coupled low-density parity-check codes including simpler encoders and slightly better thresholds than spatially coupled low-density parity-check codes with similar rates and decoding complexity. Simulation results for