The Farahat-Higman Algebra of Centralizers of Symmetric Group Algebras

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Abstract

We define a generalisation of the Farahat-Higman algebra introduced in [FH59] which replaces the role of the center of the group algebra of the symmetric groups with certain centralizer algebras instead. We prove various structural properties of this new algebra and provide an isomorphism which connects it to the degenerate affine Hecke algebra and the algebra of symmetric functions.

1 Introduction

Let $S_n$ denote the symmetric group on $n$ letters. The center $Z(\mathcal{Z}S_n)$ has a basis consisting of conjugacy class sums. In [FH59, Theorem 2.2] the structure constants associated with this basis were shown to be polynomial in $n$. This allowed H. Farahat and G. Higman to construct an algebra, denoted here by $FH_0$, by in some sense replacing the structure constants with their corresponding polynomials. In particular this algebra is given over the subring $R$ of integer-valued polynomials in $\mathbb{Q}[t]$, and via “evaluation at $n$” one has surjective ring homomorphisms $pr_{n,0} : FH_0 \to Z(\mathcal{Z}S_n)$ for each $n \geq 0$. The algebra $FH_0$ is determined by the centers $Z(\mathcal{Z}S_n)$ in the sense that a relation holds in $FH_0$ if and only if its image under each $pr_{n,0}$ holds. The motivation for the construction of such an algebra was to present an alternative proof of Nakayama’s Conjecture regarding the $p$-blocks of symmetric groups. Also, the main result of [FH59] was in establishing a set of generators for $FH_0$, whose non-zero images generate the centers $Z(\mathcal{Z}S_n)$. It was shown in [Jucys74] that these images correspond to the elementary symmetric polynomials in the Jucys-Murphy elements, demonstrating that $Z(\mathcal{Z}S_n)$ is the subalgebra of symmetric polynomials in such elements.

In [Mac95, p.131-132] a certain associated graded algebra $G$ of $FH_0$ is considered. This algebra may be regarded as a $Z$-algebra as the non-constant polynomials appearing in the product of elements of $FH_0$ get filtered out. The algebra $G$ was shown to be isomorphic to the algebra of symmetric functions, and it has been studied in relation to enumeration problems for permutation factorisation, see for example [GL94]. The algebras $FH_0$ and $G$ have seen counterparts in the setting of the spin symmetric group algebras in [TW09], of wreath products of the symmetric groups with any finite group in [W03] and [Ryba21], and of the general linear group over finite fields in [WW19].

In this paper we construct generalisations of $FH_0$ by considering certain centralizer algebras of the symmetric group in place of the centers $Z(\mathcal{Z}S_n)$. For $n \geq m$ let $\text{Stab}_n(m)$ denote the subgroup of $S_n$ of all permutations which fix each element $1, 2, \ldots, m$. Then we consider the centralizer algebra $Z_{n,m}$ consisting of all elements of $\mathcal{Z}S_n$ which commute with $\text{Stab}_n(m)$. When $m = 0$ then $Z_{n,0} = Z(\mathcal{Z}S_n)$. The algebra $Z_{n,m}$ has its own counterparts to class sums which provide a basis, and we prove in Theorem 3.0.3 that the associated structure constants are polynomial in $n$. These polynomials again belong to the subring $R$ of integer-valued polynomials in $\mathbb{Q}[t]$. From that, we can define a new $R$-algebra $FH_m$ which we refer to as the Farahat-Higman algebra associated to $Z_{n,m}$. There exist analogous surjective ring homomorphisms $pr_{n,m} : FH_m \to Z_{n,m}$ for any $n \geq m$, and the algebras $FH_m$ are determined by the centralizer algebras $Z_{n,m}$ in a manner comparable to above. To the best of the author’s knowledge, this is the first time the techniques of Farahat and Higman have been employed on an algebra which is not the center of a group algebra, and the algebra $FH_m$ is in general not commutative.

The remainder of the paper proves various structural properties of $FH_m$. We show that the symmetric group algebra $R\mathcal{S}_m$ and the Farahat-Higman algebra $FH_0$ are subalgebras of $FH_m$. The indexing set for the basis elements of $FH_m$ are called $m$-marked cycle shapes, and have only made brief appearances elsewhere (see for example [K05, p. 8]). We define a monoid structure on the set of $m$-marked cycle shapes, and use...
it to prove a leading term result for the product of basis elements in $\text{FH}_m$. This result, along with other known results of $\text{FH}_m$, allow us to prove the main result Theorem 9.0.3 of the paper we establish an isomorphism between the $R$-algebras $\text{FH}_m$ and $R \otimes \mathbb{Z}(\mathfrak{H}_m \otimes \text{Sym})$ where $\mathfrak{H}_m$ is the degenerate affine Hecke algebra (viewed over the integers) and $\text{Sym}$ is the $\mathbb{Z}$-algebra of symmetric functions. This isomorphism gives us a description of the center of $\text{FH}_m$ and a new basis involving elements which mimic the Jucys-Murphy elements. The $\mathbb{C}$-algebra $\mathbb{C} \otimes \mathbb{Z}(\mathfrak{H}_m \otimes \text{Sym})$ has appeared in the context of the Heisenberg category of [Kho14], where it arises as an endomorphism algebra of certain objects. Also a closely related algebra is studied in [MO01] when investigating a centralizer construction of the degenerate affine Hecke algebra. In this sense the algebra $\text{FH}_m$ gives an organic means of realising the degenerate affine Hecke algebra from a Farahat-Higman perspective. A conscious effort has been made to keep all the results of $\text{FH}_m$ integral (i.e. over $R$ as opposed to its field of fractions $\mathbb{Q}(t)$). The hope is that this will allow $\text{FH}_m$ to remain open as a tool for investigating the modular representations of $\text{Z}_{n,m}$ in future work.

The structure of the paper is as follows: Section 2 proves some technical results regarding the cardinalities of certain orbits. These results are used in Section 3 to prove the polynomial property of the structure constants in the centralizer algebras $\text{Z}_{n,m}$. We also set up some notation including that of an $m$-marked cycle shape and its degree. In Section 4 we define the $R$-algebras $R\mathfrak{S}_m$ and prove that $R\mathfrak{S}_m$ and $\text{FH}_0$ are subalgebras. In Section 5 we describe a monoidal structure on the set of $m$-marked cycles shapes and describe a concrete criterion for a permutation to belong to the orbit associated to a given $m$-marked cycle shape. In Section 6 we use this product to describe a leading term of a product of basis elements in $\text{FH}_m$. Section 7 simply recaps the algebra of symmetric functions and Section 8 recalls the Jucys-Murphy elements of $Z\mathfrak{S}_n$ and their relationship to the Farahat-Higman algebra $\text{FH}_0$. We then end with Section 9 which connects $\text{FH}_m$ with the degenerate affine Hecke algebra and the algebra of symmetric functions.

2 Polynomial Cardinality of $m$-Classes

Given a finite set $A \subset \mathbb{N}$ we write $\mathfrak{S}(A)$ to denote the group of permutations of $A$. For any $n \geq 0$ we write $[n] := \{1, \ldots, n\}$ and $\mathfrak{S}_n := \mathfrak{S}([n])$ for the symmetric group on $n$ letters, with convention that $[0] = \emptyset$ and $\mathfrak{S}_0$ the trivial group. We set $\mathfrak{S}_N := \cup_{n \geq 1} \mathfrak{S}_n$, the group of permutations of $N$ with finite support. We denote the support of any $\pi \in \mathfrak{S}_N$ by $\text{Sup}(\pi) = \{i \in N \mid \pi(i) \neq i\}$ and set $||\pi|| := |\text{Sup}(\pi)|$. For any $r \geq 1$ consider the $r$-fold direct product $\mathfrak{S}_N^{\times r}$. For $\pi = (\pi_i)_{i=1}^r \in \mathfrak{S}_N^{\times r}$ let $\text{Sup}(\pi) := \text{Sup}(\pi_1) \cup \cdots \cup \text{Sup}(\pi_r)$ and set $||\pi|| := |\text{Sup}(\pi)|$. For any $m \geq 0$ let $\text{Stab}(m)$ denote the subgroup of $\mathfrak{S}_N$ consisting of the permutations which fix each element of $[m]$, in particular $\text{Stab}(0) = \mathfrak{S}_N$. The group $\text{Stab}(m)$ acts on the $r$-fold direct product $\mathfrak{S}_N^{\times r}$ by component-wise conjugation. We refer to the respected orbits as the $m$-classes of $\mathfrak{S}_N^{\times r}$. We denote the $m$-class of $\mathfrak{S}_N^{\times r}$ containing $\pi = (\pi_i)_{i=1}^r \in \mathfrak{S}_N^{\times r}$ by

$$\text{CL}_{n,m}(\pi) := \{(\sigma_i)_{i=1}^r \in \mathfrak{S}_N^{\times r} \mid \sigma_i = \tau \pi_i \tau^{-1} \text{ for all } i \in [r] \text{ and some } \tau \in \text{Stab}(m)\}.$$  

Example 2.0.1. Let $m = r = 2$ and consider $\pi = (1,3)(2,4), (3,4,5)) \in \mathfrak{S}_N^{\times 2}$. Then the $m$-class containing $\pi$ is given by $\text{CL}_{2,2}(\pi) = \{(a, b, c) \mid (a, b, c) \in (N \setminus \{2\})^3\}$ where $(N \setminus \{2\})^3$ is the subset of the 3-fold direct product of $N \setminus \{2\}$ consisting of all tuples with pairwise distinct entries.

Given any $m \geq 0$ and $\pi = (\pi_i)_{i=1}^r \in \mathfrak{S}_N^{\times r}$ define

$$\text{Sup}_m(\pi) := \text{Sup}(\pi) \cap [m], \text{ and } \text{Sup}^m(\pi) := \text{Sup}(\pi) \setminus [m],$$

and write $||\pi||_m$ and $||\pi||^m$ for the cardinalities of $\text{Sup}_m(\pi)$ and $\text{Sup}^m(\pi)$ respectively. Let $C$ be an $m$-class of $\mathfrak{S}_N^{\times r}$ and let $\pi = (\pi_i)_{i=1}^r, \sigma = (\sigma_i)_{i=1}^r \in C$. Hence there exists some $\tau \in \text{Stab}(m)$ such that $\sigma_i = \tau \pi_i \tau^{-1}$ for each $i \in [r]$. Conjugating $\pi_i$ by $\tau$ permutes the entries within the cycles of $\pi_i$ according to $\tau$, thus for each $i \in [r]$ the permutations $\pi_i$ and $\sigma_i$ must have the same cycle structure, and the relative positions of the elements of $[m]$ among their cycles must agree. Hence $||\pi|| = ||\sigma||$, $||\pi||_m = ||\sigma||_m$, and $||\pi||^m = ||\sigma||^m$, and so it makes sense to define $||C|| := ||\pi||$, $||C||_m := ||\pi||_m$, and $||C||^m := ||\pi||^m$ for any $\pi \in C$.

Let $n \geq m$, then we set $\text{Stab}_n(m) := \text{Stab}(m) \cap \mathfrak{S}_n$. Similar to the above situation, this group acts on the $r$-fold direct product $\mathfrak{S}_n^{\times r}$ by component-wise conjugation. We refer to the respected orbits as the $m$-classes of $\mathfrak{S}_n^{\times r}$. For any $\pi = (\pi_i)_{i=1}^r \in \mathfrak{S}_n^{\times r}$, we denote the $m$-class of $\mathfrak{S}_n^{\times r}$ containing $\pi$ by

$$\text{CL}_{n,m}(\pi) := \{(\sigma_i)_{i=1}^r \in \mathfrak{S}_n^{\times r} \mid \sigma_i = \tau \pi_i \tau^{-1} \text{ for all } i \in [r] \text{ and some } \tau \in \text{Stab}_n(m)\}.$$
Again we can set \( ||C|| := ||\pi|| \). \( ||C||_X := ||\pi||_X \), and \( ||C||^X := ||\pi||^X \) for any \( m \)-class \( C \) of \( S_N^{x,r} \) and any \( \pi \in C \). Given any \( m \)-class \( C \) of \( S_N^{x,r} \) we write \( C_n := C \cap S_N^{x,r} \).

**Example 2.0.2.** Continuing from Example 2.0.1 let \( C = CL_{n,2}(\pi) \). For any \( n \geq 2 \) one can deduce that

\[
C_n = \begin{cases} 
\{((1,a),(2,b),(a,b,c)) \mid (a,b,c) \in ([n]\backslash[2])^3\}, & n \geq 5 \\
\emptyset, & 2 \leq n \leq 4 
\end{cases}
\]

When \( n \geq 5 \) the set \( C_n \) is an \( m \)-class of \( S_n^{x^2} \).

As the above example suggests, when \( C \) is an \( m \)-class of \( S_N^{x,r} \) and \( n \geq m \), then the set \( C_n \) is either empty or an \( m \)-class of \( S_n^{x,r} \). The following proposition proves this and gives a criteria for when \( C_n \) is empty.

**Proposition 2.0.3.** Let \( C \) be an \( m \)-class of \( S_N^{x,r} \) and \( n \geq m \). Then \( C_n \) is non-empty if and only if

\[
n \geq ||C||^m + m.
\]

In this case \( C_n \) is an \( m \)-class of \( S_n^{x,r} \) and the \( m \)-classes of \( S_n^{x,r} \) appear uniquely in this manner.

**Proof.** Fix \( \pi = (\pi_i)_{i=1}^r \in C \). All elements of \( C \) are of the form \( (\tau \pi_i \tau^{-1})_{i=1}^r \) for \( \tau \in Stab(m) \). The set \( Sup^m(\tau \pi \tau^{-1}) \) is all elements of \( N \\setminus [m] \) for which at least one \( \tau \pi_i \tau^{-1} \) acts non-trivially. Thus \( \tau \pi \tau^{-1} \in C_n \) if and only if \( Sup^m(\tau \pi \tau^{-1}) \subseteq [n] \setminus [m] \). For this to be the case we must have

\[
|Sup^m(\tau \pi \tau^{-1})| = ||C||^m \leq |[n] \setminus [m]| = n - m.
\]

Rearranging gives Equation (1). So \( C_n = \emptyset \) whenever \( n < ||C||^m + m \). Now assume Equation (1) holds. We have \( Sup^m(\tau \pi \tau^{-1}) = \tau(Sup^m(\pi)) \), the image of \( Sup^m(\pi) \) under \( \tau \). Let \( t : Sup^m(\pi) \to [n] \setminus [m] \) be an injective map, which exists since Equation (1) holds. Then fix a permutation \( \tau_t \in Stab(m) \) such that \( \pi(i) = t(i) \) for all \( i \in Sup^m(\pi) \). Hence \( Sup^m(\tau \pi \tau_t^{-1}) \subseteq [n] \setminus [m] \) and so \( C_n \) is non-empty as it contains \( \sigma := \tau \pi \tau_t^{-1} \).

We now want to prove that \( C_n \) is an \( m \)-class of \( S_n^{x,r} \). Since \( \sigma := \tau \pi \tau_t^{-1} \in C_n \) and \( C_n = C \cap S_n^{x,r} \), any element of \( C_n \) is of the form \( \tau \pi \sigma \tau^{-1} \in Stab(m) \) such that \( Sup^m(\tau \pi \sigma \tau^{-1}) \subseteq [n] \setminus [m] \). If \( \tau \in Stab(m) \) then clearly \( Sup^m(\tau \pi \sigma \tau^{-1}) \subseteq [n] \setminus [m] \), thus we have \( CL_{n,m}(\sigma) \subseteq C_n \). Hence the equality \( CL_{n,m}(\sigma) = C_n \) occurs if one can show that whenever \( \tau \pi \sigma \tau^{-1} \in C_n \) for some \( \tau \in Stab(m) \) there exists a \( \tau' \in Stab(m) \) such that \( \tau \pi \sigma \tau^{-1} = \tau' \sigma \tau'^{-1} \). Given such a \( \tau \) we must have that \( Sup^m(\tau \pi \sigma \tau^{-1}) = \tau(Sup^m(\sigma)) \subseteq [n] \setminus [m] \). Since we also have that \( Sup^m(\sigma) \subseteq [n] \setminus [m] \), let \( \tau' \) be any permutation of \( [n] \setminus [m] \) with the property that \( \tau'(i) = \tau(i) \) for all \( i \in Sup^m(\sigma) \), then it is clear that \( \tau \pi \sigma \tau^{-1} = \tau' \sigma \tau'^{-1} \) and so \( CL_{n,m}(\sigma) = C_n \). Hence for any \( \sigma \in S_n^{x,r} \) we have \( (CL_{n,m}(\sigma))_n = CL_{n,m}(\sigma) \). Since orbits intersect trivially all such \( m \)-classes of \( S_n^{x,r} \) appear as the intersection of a unique \( m \)-class of \( S_N^{x,r} \) with \( S_n^{x,r} \).

We now end this section by presenting a result describing the cardinality of the set \( C_n \) when \( C \) is an \( m \)-class of \( S_N^{x,r} \) and \( n \geq m \). Notably we show that the cardinality of such a set is polynomial in \( n \).

**Proposition 2.0.4.** Let \( C \) be an \( m \)-class of \( S_N^{x,r} \). For \( n \geq m \),

\[
|C_n| = \frac{1}{b(C)}(n - m)(n - m - 1) \cdots (n - m - ||C||^m + 1)
\]

where \( b(C) \in N \) is a constant depending only on the class \( C \) and not on \( n \).

**Proof.** Assume \( n \geq ||C||^m + m \), so \( C_n \) is an \( m \)-class of \( S_n^{x,r} \) by Proposition 2.0.3. Pick \( \pi = (\pi_i)_{i=1}^r \in C_n \), then \( C_n = CL_{n,m}(\pi) \) is the orbit of \( \pi \) in \( S_n^{x,r} \) under the action of \( Stab(m) \) by component-wise conjugation. Thus by the Orbit-Stabilizer Theorem we have

\[
|C_n| = \frac{|Stab(m)|}{|Stab_{Stab(m)}(\pi)|}
\]

where \( Stab_{Stab(m)}(\pi) = \{ \tau \in Stab(m) \mid \tau \pi \tau^{-1} = \pi \} \) is the subgroup of \( Stab(m) \) whose elements fix \( \pi \) under component-wise conjugation. Consider \( Stab(m) \) disjoint union \( Sup^m(\pi) \), the subgroup of \( S_n \) consisting of all
permutations which act trivially on \([m] \cup \text{Sup}^m(\pi)\). Naturally \(\text{Stab}_n([m] \cup \text{Sup}^m(\pi)) \subset \text{Stab}_{\text{Stab}_n(m)}(\pi)\). Now let \(\mathcal{S}(\text{Sup}^m(\pi))\) denote the subgroup of \(\mathcal{S}_n\) consisting of the permutations of \(\text{Sup}^m(\pi)\). Then we have that \(\mathcal{S}(\text{Sup}^m(\pi)) \subset \text{Stab}(m)\), and hence \(\text{Stab}_{\mathcal{S}(\text{Sup}^m(\pi))}(\pi) = \{\tau \in \mathcal{S}(\text{Sup}^m(\pi)) \mid \tau \pi \tau^{-1} = \pi\}\) is also a subgroup of \(\text{Stab}_{\text{Stab}_n(m)}(\pi)\).

**Claim:** We have a group isomorphism

\[
\text{Stab}_{\text{Stab}_n(m)}(\pi) \cong \text{Stab}_n([m] \cup \text{Sup}^m(\pi)) \times \text{Stab}_{\mathcal{S}(\text{Sup}^m(\pi))}(\pi).
\]

Note that the two subgroups \(\text{Stab}_n([m] \cup \text{Sup}^m(\pi))\) and \(\text{Stab}_{\mathcal{S}(\text{Sup}^m(\pi))}(\pi)\) commute and have trivial intersection. So to prove this claim we only need to show that any permutation \(\tau \in \text{Stab}_{\text{Stab}_n(m)}(\pi)\) can be expressed as \(\tau = \tau_1 \tau_2\) for some \(\tau_1 \in \text{Stab}_n([m] \cup \text{Sup}^m(\pi))\) and \(\tau_2 \in \mathcal{S}(\text{Sup}^m(\pi))\). Let \(\tau \in \text{Stab}_{\text{Stab}_n(m)}(\pi)\), then by definition \(\tau \pi_i \tau_i^{-1} = \pi_i\) for each \(i \in [r]\). We seek to show that the sets \([n] \setminus \text{Sup}(\pi)\) and \(\text{Sup}(\pi)\) are invariant under the action of \(\tau\). Suppose for contradiction this is not the case, hence there exists an \(a \in [n] \setminus \text{Sup}(\pi)\) such that \(\tau(a) = b \in \text{Sup}(\pi)\). Then for each \(i \in [r]\),

\[
\pi_i(b) = (\tau \pi_i \tau_i^{-1})(b) = (\tau \pi_i)(a) = \tau(a) = b.
\]

Thus \(\pi_i\) fixes \(b\) for each \(i \in [r]\), but this gives the desired contradiction since \(b \in \text{Sup}(\pi)\). Thus the sets \([n] \setminus \text{Sup}(\pi)\) and \(\text{Sup}(\pi)\) are invariant under the action of \(\tau\), which implies a decomposition \(\tau = \tau_1 \tau_2\) where \(\tau_1\) is a permutation of \([n] \setminus \text{Sup}(\pi)\) and \(\tau_2\) is a permutation of \(\text{Sup}(\pi)\). Naturally \(\tau_1\) and \(\tau_2\) commute, and since \(\tau\) fixes \([m]\), both \(\tau_1\) and \(\tau_2\) must also fix \([m]\). As such \(\tau_1\) is an element of \(\text{Stab}_n([m] \cup \text{Sup}^m(\pi))\) as desired, and \(\tau_2 \in \mathcal{S}(\text{Sup}^m(\pi))\). Lastly note that for each \(i \in [r]\),

\[
\pi_i = \tau \pi_i \tau_i = \tau_2 \pi_i \tau_1 \pi_1^{-1} \tau_2^{-1} = \tau_2 \pi_i \tau_2^{-1}
\]

since \(\tau_1\) commutes with both \(\tau_2\) and \(\pi_i\). Thus \(\tau_2 \pi_i \tau_2^{-1} = \pi_i\) for each \(i \in [r]\), implying that \(\tau_2\) belongs to \(\text{Stab}_{\mathcal{S}(\text{Sup}^m(\pi))}(\pi)\). Hence the claim holds.

Returning to the cardinality of \(\mathcal{C}_n\), recall that \(|\text{Sup}^m(\pi)| = ||C||^m\). Also observe that the size of the group \(\mathcal{S}(\text{Sup}^m(\pi))\) depends only on the class \(\mathcal{C}\), in particular it is independent of \(n\). This implies the same for the cardinality of \(\text{Stab}_{\mathcal{S}(\text{Sup}^m(\pi))}(\pi)\), and so we write \(b(C) := |\text{Stab}_{\mathcal{S}(\text{Sup}^m(\pi))}(\pi)|\). Hence we have that

\[
|\mathcal{C}_n| = \frac{|\text{Stab}_n(m)|}{|\text{Stab}_n([m] \cup \text{Sup}^m(\pi))||\text{Stab}_{\mathcal{S}(\text{Sup}^m(\pi))}(\pi)|} = \frac{(n - m)!}{(n - m - ||C||^m)!b(C)} = \frac{1}{b(C)}(n - m)(n - m - 1) \cdots (n - m - ||C||^m + 1).
\]

We assumed \(n \geq ||C||^m + m\) so that \(\mathcal{C}_n\) is an \(m\)-class of \(\mathcal{S}_n^{\times_r}\). By Proposition 2.7.3 the set \(\mathcal{C}_n\) is empty when \(m \leq n < ||C||^m + m\), and such values of \(n\) are precisely those which give zero in the above formula for \(|\mathcal{C}_n|\). Hence this formula holds for all \(n \geq m\), completing the proof.

\[\square\]

## 3 Centralizer Algebras and Polynomial Structure Constants

Let \(m \geq 0\) and \(\mathcal{C}\) be an \(m\)-class of \(\mathcal{S}_n\) and \(\pi \in \mathcal{C}\). As mentioned before \(\mathcal{C}\) is completely determined by the cycle structure of \(\pi\) and the relative positions of the elements of \([m]\) among the cycles. If one was to take \(\pi\) and replace each of the elements in \(\mathbb{N} \setminus [m]\) among the non-trivial cycles with a formal symbol, say \(*\), then the resulting object would retain the defining characteristics of \(\mathcal{C}\), and so could represent it.

**Definition 3.0.1.** For a finite set \(A\), we call \((a_i)_{i=1}^l \in A^{x l}\), for some \(l \in \mathbb{N}\), a cycle if we only care about the order of the entries up to cyclic shifts, and say it has length \(l\). For \(*\) a formal symbol, we define an \(m\)-marked cycle shape to be a finite collection of cycles with entries in \([m] \cup \{*\}\) with the following properties:

1. The multiset of entries among the cycles equals \([m] \cup \{*^n\}\) for some \(n \in \mathbb{Z}_{\geq 0}\), in particular each element of \([m]\) appears precisely once.
(2) Cycles containing only $*$ must be of length at least two.

We write an $m$-marked cycle shape as a formal product of its cycles by juxtaposition, where the order of the cycles is immaterial. We let $\Lambda(m)$ denote the set of all such $m$-marked cycle shapes.

**Example 3.0.2.** An example of a 6-marked cycle shape is

$$\lambda = (2, 6)(1, *, 4, *)(3, *, *)(*, *, *) \in \Lambda(6).$$

The multiset of entries among the cycles of $\lambda$ is $[6] \cup \{*, 10\}$. Cyclically shifting the entries of any of the cycles or rearranging the cycles in any order will result in an alternative expression of $\lambda$.

The 0-marked cycle shapes only contain the symbol $*$, and we will refer to these as cycle shapes. Clearly one may identify the set $\Lambda(0)$ with the set of proper partitions (i.e., partitions with no parts equal to 1). We consider the empty set $\emptyset$ an element of $\Lambda(0)$ consisting of no cycles. Also the subset of $\Lambda(m)$ consisting of the $m$-marked cycle shapes which contain no symbols $*$ may be identified with $\mathfrak{S}_m$, and so we write $\mathfrak{S}_m \subset \Lambda(m)$.

As discussed above the set of $m$-marked cycle shapes $\Lambda(m)$ provides a natural indexing set for the collection of $m$-classes of $\mathfrak{S}_N$: For $\lambda \in \Lambda(m)$ let $\mathcal{CL}_{\lambda} \in \Lambda(m)$ denote the corresponding $m$-class. The permutations of $\mathcal{CL}_{n,m} \lambda$ are those obtained from $\lambda$ by replacing the symbols $*$ with distinct elements from $\mathbb{N} \setminus [m]$. For example, letting $\lambda$ be the 6-marked cycle shape displayed in Example 3.0.2 we have that

$$\mathcal{CL}_{6,6}(\lambda) = \left\{ (2,6) \{1, a_1, a_2, 4, a_3\} (3, a_4, a_5) (a_6, a_7)(a_8, a_9, a_{10}) \mid (a_i)_{i=1}^{10} \in (\mathbb{N} \setminus [6])^{10} \right\}.$$ 

Recall the quantities $|C|$, $||C||$, and $||C||_m$ for any $m$-class $C$ of $\mathfrak{S}_N$. Then for $\lambda \in \Lambda(m)$ we write

$$||\lambda|| := ||\mathcal{CL}_{n,m}(\lambda)||,$$

$$||\lambda||^m := ||\mathcal{CL}_{n,m}(\lambda)||^m,$$

and

$$||\lambda||_m := ||\mathcal{CL}_{n,m}(\lambda)||_m.$$

The quantity $||\lambda||$ is the number of entries among the cycles of length at least two, $||\lambda||^m$ is the number of $*$ symbols appearing among the cycles, and $||\lambda||_m$ is the number of elements from $[m]$ which appear in cycles of length at least two. We define a map $\deg_m : \Lambda(m) \rightarrow \mathbb{Z}_{\geq m}$ by setting

$$\deg_m(\lambda) := ||\lambda||^m + m.$$ 

We refer to $\deg_m(\lambda)$ as the degree of $\lambda$. We have defined the degree map with the inequality of Proposition 2.0.3 in mind, in particular for any $n \geq m$ we have that the set $\Lambda_{\leq n}(m) := \{ \lambda \in \Lambda(m) \mid \deg_m(\lambda) \leq n \}$ gives us an indexing set for the $m$-classes of $\mathfrak{S}_n$. Given $\lambda \in \Lambda_{\leq n}(m)$ we denote the corresponding $m$-class by $\mathcal{CL}_{n,m}(\lambda) = (\mathcal{CL}_{\lambda})_n$, which consists of all permutations of $\mathfrak{S}_n$ one can obtain from $\lambda$ by replacing the symbols $*$ with distinct elements from $[n] \setminus [m]$.

Consider the $\mathbb{Z}$-algebra $\mathbb{Z}\mathfrak{S}_n$. For any $n \geq m$ we define the $m$-centralizer algebra as

$$Z_{n,m} := \{ z \in \mathbb{Z}\mathfrak{S}_n \mid \tau z = z \tau \text{ for all } \tau \in \mathfrak{S}_n \}.$$ 

When $m = 0$ then $\mathfrak{S}_n = \mathfrak{S}_n$ and $Z_{n,0} = Z(\mathfrak{S}_n)$. For any $\lambda \in \Lambda(m)$ we define the $m$-class sum by

$$K_n(\lambda) := \sum_{\pi \in \mathcal{CL}_{n,m}(\lambda)} \pi.$$ 

Note by Proposition 2.0.3 we have that $K_n(\lambda) \neq 0$ if and only if $\deg_m(\lambda) \leq n$. Also one can deduce that the set of $m$-class sums $\{K_n(\lambda) \mid \lambda \in \Lambda_{\leq n}(m)\}$ provides a $\mathbb{Z}$-basis for the centralizer algebra $Z_{n,m}$. As such the product of any two such elements must decomposed into a linear combination of the same, that is for any $\lambda, \mu \in \Lambda_{\leq n}(m)$ we have that

$$K_n(\lambda)K_n(\mu) = \sum_{\nu \in \Lambda_{\leq n}(m)} a_{\lambda,\mu}(n)K_n(\nu)$$

with structure constants $a_{\lambda,\mu}(n) \in \mathbb{Z}_{\geq 0}$. We now show these structure constants are polynomial in $n$. Let $t$ be a formal variable and $R$ the subring of $\mathbb{Q}[t]$ of all polynomials $f(t)$ such that $f(n) \in \mathbb{Z}$ whenever $n \in \mathbb{Z}$. 

5
Theorem 3.0.3. For each \( \lambda, \mu, \nu \in \Lambda(m) \) there exists a unique polynomial \( f^\nu_{\lambda, \mu}(t) \in R \) such that

\[
K_n(\lambda)K_n(\mu) = \sum_{\nu \in \Lambda \leq n(m)} f^\nu_{\lambda, \mu}(n)K_n(\nu)
\]

in \( Z_{n,m} \) for any \( n \geq m \). We refer to the polynomials \( f^\nu_{\lambda, \mu}(t) \) as the structure polynomials.

Proof. Fix \( \lambda, \mu, \nu \in \Lambda(m) \). Consider the set of pairs

\[
A = \{(\pi_1, \pi_2) \in \mathcal{C}(\lambda) \times \mathcal{C}(\mu) \mid \pi_1 \pi_2 \in \mathcal{C}(\nu) \} \subset \mathcal{S}_n \times \mathcal{S}_n.
\]

When \( A = \emptyset \) we set \( f^\nu_{\lambda, \mu}(t) := 0 \). Assume \( A \neq \emptyset \) and let \((\pi_1, \pi_2) \in A\). For any \( \tau \in \text{Stab}(\nu) \) we have that \((\tau \pi_1 \tau^{-1})(\tau \pi_2 \tau^{-1}) = \tau(\pi_1 \pi_2)\tau^{-1}\) which belongs to \( \mathcal{C}(\nu) \) since \( \pi_1 \pi_2 \) does. So \((\tau \pi_1 \tau^{-1}, \tau \pi_2 \tau^{-1})\) belongs to \( A \) for any \( \tau \in \text{Stab}(\nu) \). Thus for some indexing set \( I \), the set \( A \) is the union of \( m \)-class \( \mathcal{C}(i) \) of \( \mathcal{S}_n \times \mathcal{S}_n \) for each \( i \in I \). For any \((\pi_1^1, \pi_2^i) \in \mathcal{C}(i),
\[
||\mathcal{C}(i)||^m = |\text{Sup}^m(\pi_1^1) \cup \text{Sup}^m(\pi_2^i)| \leq |\text{Sup}^m(\pi_1^1)| + |\text{Sup}^m(\pi_2^i)| = ||\lambda||^m + ||\mu||^m.
\]

Thus by Proposition 2.0.3 we have for any \( n \geq ||\lambda||^m + ||\mu||^m + m \) that \( \mathcal{C}(i) \) is an \( m \)-class of \( \mathcal{S}_n \times \mathcal{S}_n \) for each \( i \in I \). This implies that \( I \) is finite. Also by Proposition 2.0.4 we have for any \( n \geq m \) that

\[
|A \cap (\mathcal{S}_n \times \mathcal{S}_n)| = \sum_{i \in I} \frac{1}{b(\mathcal{C}(i))} (n - m)(n - m - 1)\cdots(n - m - ||\mathcal{C}(i)||^m + 1),
\]

where \( b(\mathcal{C}(i)) \) are constants independent of \( n \). By definition of \( A \), the multiplicity of \( K_n(\nu) \) in the product \( K_n(\lambda)K_n(\mu) \) is \( |A \cap (\mathcal{S}_n \times \mathcal{S}_n)| \) divided by \( |\mathcal{C}(\lambda, \mu, \nu)| \). Again by Proposition 2.0.4 we have that

\[
|\mathcal{C}(\lambda, \mu, \nu)| = \frac{1}{b(\nu)} (n - m)(n - m - 1)\cdots(n - m - ||\nu||^m + 1)
\]

where \( b(\nu) = b(\mathcal{C}(\lambda, \mu, \nu)) \) is a constant independent of \( n \). Now for any \((\pi_1^1, \pi_2^i) \in \mathcal{C}(i),
\[
||\nu||^m = |\text{Sup}^m(\pi_1^1) \cup \text{Sup}^m(\pi_2^i)| \leq |\text{Sup}^m(\pi_1^1)| + |\text{Sup}^m(\pi_2^i)| = ||\mathcal{C}(i)||.
\]

Thus we have that \( |A \cap (\mathcal{S}_n \times \mathcal{S}_n)| \) is given by

\[
b(\nu) \sum_{i \in I} \frac{1}{b(\mathcal{C}(i))} (n - m - ||\nu||^m)(n - m - ||\nu||^m - 1)\cdots(n - m - ||\mathcal{C}(i)||^m + 1).
\]

Hence let \( f^\nu_{\lambda, \mu}(t) \) be the polynomial obtained from the above expression by replacing \( n \) with \( t \). What remains to show is that \( f^\nu_{\lambda, \mu}(t) \) belongs to \( R \). It is known that a polynomial \( f(t) \in \mathbb{Q}[t] \) of degree \( d \) belongs to \( R \) if and only if it is integer valued on \( d + 1 \) consecutive integers. Well we have shown that each \( f^\nu_{\lambda, \mu}(t) \) is integer valued on the infinite set \( \{m, m + 1, \ldots\} \) since the evaluation at such an integer \( n \) gives the coefficient of the term \( K_n(\nu) \) in the product \( K_n(\lambda)K_n(\mu) \) in the \( \mathbb{Z} \)-algebra \( Z_{n,m} \), hence we must have \( f^\nu_{\lambda, \mu}(t) \in R \).

Specialising the above theorem to the case \( m = 0 \) recovers [FH59] Theorem 2.2. It is worth remarking that in [FH59] they instead used reduced partitions as the indexing set for their classes as aposed to cycle shapes, but there is a natural correspondence between them.

4 The Farahat-Higman Algebras \( \mathcal{F} \mathcal{H}_m \)

Knowing that the structure constants of \( Z_{n,m} \) are polynomial in \( n \) allows us to define a new \( R \)-algebra \( \mathcal{F} \mathcal{H}_m \), which we refer to as the Farahat-Higman algebra associated to \( Z_{n,m} \). This is constructed by replacing the structure constants \( a^\nu_{\lambda, \mu}(n) \) with the structure polynomials \( f^\nu_{\lambda, \mu}(t) \). We first define \( \mathcal{F} \mathcal{H}_m \) as a free \( R \)-module equipped with a product, and then shortly prove that such a product realises it as an \( R \)-algebra.
Definition 4.0.1. Let $FH_m$ be the free $R$-module with basis $\{K(\lambda) \mid \lambda \in \Lambda(m)\}$. Equip this module with the product given by the $R$-linear extension of

$$K(\lambda)K(\mu) = \sum_{\nu \in \Lambda(m)} f^\nu_{\lambda,\mu}(t)K(\nu)$$

where $f^\nu_{\lambda,\mu}(t)$ are the structure polynomials given in Theorem 4.0.3.

Since $f^\nu_{\lambda,\mu}(n)$, for any $n \geq m$, gives the multiplicity of $K_n(\nu)$ in the product $K_n(\lambda)K_n(\mu)$, it is clear that $f^\nu_{\lambda,\mu}(t) = 0$ whenever $||\nu|| > ||\lambda|| + ||\mu||$. As such the product described above for $FH_m$ is well-defined since only a finite number of terms will appear in the product of any two elements.

We say a distributive ring is an object satisfying all the axioms of a ring except possibly the associativity of the product and the existence of a multiplicative identity. In particular, a ring is a special case of a distributive ring. From definition we certainly have that $FH_m$ is a distributive ring. We seek to show that the product given in Definition 4.0.1 is associative and that $FH_m$ possess a multiplicative identity. This will follow since $FH_m$ is determined by the algebras $Z_{n,m}$ for all $n \geq m$, as we will now demonstrate.

By definition of the structure polynomials we must have a surjective homomorphism of distributive rings $pr_{n,m} : FH_m \to Z_{n,m}$ given by $pr_{n,m}(K(\lambda)) = K_n(\lambda)$ and $pr_{n,m}(f(t)) = f(n)$ for all $\lambda \in \Lambda(m)$ and $f(t) \in R$.

Lemma 4.0.2. The intersection of the kernels of $pr_{n,m}$ for all $n \geq m$ is trivial, that is

$$\bigcap_{n \geq m} \text{Ker}(pr_{n,m}) = \{0\}$$

Proof. Since $\{K_n(\lambda) \mid \lambda \in \Lambda_{\leq n}(m)\}$ forms a $\mathbb{Z}$-basis of $Z_{n,m}$, it is clear that the kernel of $pr_{n,m}$ is given by the sum $\text{Ker}(pr_{n,m}) = R^{(n)}FH_m + \text{Span}_R\{K(\lambda) \mid \deg_m(\lambda) > n\}$, where $R^{(n)}$ is the subring of $R$ given by all such polynomials with $n$ as a root. Note for any $n_1, n_2 \geq m$ one has that

$$R^{(n_1)}FH_m \cap \text{Span}_R\{K(\lambda) \mid \deg_m(\lambda) > n_2\} \subset R^{(n_1)}FH_m.$$

Therefore we have

$$\bigcap_{n \geq m} \text{Ker}(pr_{n,m}) \subset \left(\bigcap_{n \geq m} R^{(n)}\right)FH_m + \bigcap_{n \geq m} \text{Span}_R\{K(\lambda) \mid \deg_m(\lambda) > n\} = 0,$$

since no element $\lambda$ can have a degree greater than all $n \geq m$, and since the only polynomial with infinitely many roots is the trivial polynomial zero.

Lemma 4.0.3. For any $X, Y \in FH_m$, then $X = Y$ if and only if $pr_{n,m}(X) = pr_{n,m}(Y)$ for all $n \geq m$.

Proof. The forward implication is immediate, while the reverse implication follows since it implies that $X - Y$ belongs to $\cap_{n \geq m} \text{Ker}(pr_{n,m}) = \{0\}$.

Thus we can solve any product in $FH_m$ by computing a corresponding one in $Z_{n,m}$ for arbitrary $n \geq m$.

Example 4.0.4. Consider the 2-marked cycle shapes $\lambda = (1,2)(*,*)$ and $\mu = (1)(2)(*,*)$. We have that $\deg_2(\lambda) = \deg_2(\mu) = 4$, hence the class sums $K_n(\lambda)$ and $K_n(\mu)$ are non-zero if and only if $n \geq 4$. Let $n \geq 2$, then we have that

$$K_n(\lambda)K_n(\mu) = \left(\sum_{(a,b) \subseteq [n]\setminus[2]} (1,2)(a,b)\right)\left(\sum_{(c,d) \subseteq [n]\setminus[2]} (c,d)\right) = \sum_{(a,b) \subseteq [n]\setminus[2]} \sum_{(c,d) \subseteq [n]\setminus[2]} (1,2)(a,b)(c,d).$$

If the two cycles $(a,b)$ and $(c,d)$ are disjoint then the resulting permutation is simply $(a,b)(c,d)$, and there are two such ways to do so. If the two cycles share a single element, then the resulting permutation gives a 3-cycle $(a,b,c)$, and the number of ways to arrive at this 3-cycle from a product of two transpositions is
three. Lastly if the two cycles agree then the result is the identity, and there are as many ways to do this as there are two-element subsets of \([n]\setminus[2]\). Thus altogether we have that

\[ K_n(\lambda)K_n(\mu) = 2K_n(\tau_1) + 3K_n(\tau_2) + \binom{n-2}{2}K_n(\tau_3) \]

where \(\tau_1 = (1,2)(*\,\ast\,\ast)\), \(\tau_2 = (1,2)(\ast\,\ast\,\ast)\), and \(\tau_3 = (1,2)\). Note that

\[ \binom{n-2}{2} = \frac{1}{2}(n-2)(n-3). \]

Thus \(K_n(\lambda)K_n(\mu) = 2K_n(\tau_1) + 3K_n(\tau_2) + \frac{1}{2}(n-2)(n-3)K_n(\tau_3)\) for all \(n \geq 2\), noting that when \(n = 2, 3\) both sides of the equality are zero since \(K_n(\tau) = 0\) for all \(\tau \in \{\lambda, \mu, \tau_2, \tau_3\}\), and the polynomial in \(n\) which is the coefficient of \(K_n(\tau_3)\) has both 2 and 3 as roots. Thus by Lemma 4.0.3 we have that the relation

\[ K(\lambda)K(\mu) = 2K(\tau_1) + 3K(\tau_2) + \frac{1}{2}(z-2)(z-3)K(\tau_3) \]

holds in \(FH_m\), and so \(f^{\tau_1}_{\lambda,\mu}(z) = 2, f^{\tau_2}_{\lambda,\mu}(z) = 3,\) and \(f^{\tau_3}_{\lambda,\mu}(z) = \frac{1}{2}(z-2)(z-3)\).

**Proposition 4.0.5.** The distributive ring \(FH_m\) is an \(R\)-algebra.

**Proof.** We need to prove that the product is associative and a multiplicative identity exists. For the identity, consider \(K(\emptyset_m)\) where \(\emptyset_m\) is the \(m\)-marked cycle shape \((1)(2)\ldots(m)\). The \(m\)-class \(CL_{n,m}(\emptyset_m)\) contains only the identity, so \(K_n(\emptyset_m)\) is the identity of \(Z_{n,m}\) for any \(n \geq m\). Thus \(K(\emptyset_m)\) is the identity of \(FH_m\) by Lemma 4.0.3. For any \((X,Y,W) \in Z_{3,m}\) let \([X,Y,W] := (XY)W = X(YW)\). Then \(pr_{n,m}([X,Y,W]) = 0\) for all \(n \geq m\) as \(Z_{n,m}\) is associative. So \([X,Y,W] = 0\) by Lemma 4.0.3 showing that \(FH_m\) is associative.

We end this section by describing two natural \(R\)-subalgebras of \(FH_m\).

**Lemma 4.0.6.** We have an injective \(R\)-algebra homomorphism \(RS_m \rightarrow FH_m\) defined by the \(R\)-linear extension of \(\pi \mapsto K(\pi)\) for all \(\pi \in S_m(\subseteq \Lambda(m))\).

**Proof.** For \(n \geq m\), \(CL_{n,m}(\pi) = \{\pi\}\) for any \(\pi \in S_m\). Hence \(pr_{n,m}(K(\pi)) = K_n(\pi) = \pi\) showing we have a homomorphism by Lemma 4.0.3. Injectivity follows by construction of \(FH_m\).

Consider the injective map \((-)_m : \Lambda(0) \rightarrow \Lambda(m)\) which sends a cycle shape \(\lambda \in \Lambda(0)\) to the \(m\)-marked cycle shape \(\lambda_m\) which is obtained from \(\lambda\) by adjoining the trivial cycles \((1)(2)\ldots(m)\).

**Lemma 4.0.7.** We have an injective \(R\)-algebra homomorphism \(FH_0 \rightarrow FH_m\) defined by the \(R\)-linear extension of \(K(\lambda) \mapsto K(\lambda_m)\) for all \(\lambda \in \Lambda(0)\).

**Proof.** An application of Lemma 4.0.3 shows that \(f^{\lambda_m}_{\mu,\mu}(t) = 0\) for any \(\lambda, \mu, \nu \in \Lambda(0)\) whenever \(\nu \neq \gamma_m\) for \(\gamma \in \Lambda(0)\). Thus it suffices to show that \(f^{\nu}_{\lambda,\mu}(t) = 0\) for any \(\lambda, \mu, \nu \in \Lambda(0)\). Let \(FH^*_m\) denote the \(R\)-subalgebra of \(FH_m\) generated by \(K(\lambda_m)\) for all \(\lambda \in \Lambda(0)\). Clearly for any \(n \geq 0\) we have that \(pr_{n+m,m}(FH^*_m) = Z(\mathbb{Z}\mathcal{S}([n+m]\setminus[m]))\). Similarly for any \(n \geq 0\) we have that \(pr_{n,0}(FH_0) = Z(\mathbb{Z}\mathcal{S}_n)\). Clearly we have a \(Z\)-algebra isomorphism \(\sigma_n : Z(\mathbb{Z}\mathcal{S}([n+m]\setminus[m])) \cong Z(\mathbb{Z}\mathcal{S}_n)\), which may be realised by the \(R\)-linear extension of conjugation \(\sigma_n : Z(\mathbb{Z}\mathcal{S}([n+m]\setminus[m])) \rightarrow Z(\mathbb{Z}\mathcal{S}_n)\) where \(\sigma_n\) is any permutation of \(\mathcal{S}_{n+m}\) such that \(\sigma([n+m]\setminus[m]) = [n]\). By definition of \((-)_m\) we have that \(K_n(\lambda) = \sigma_n K_n(\lambda_m)\sigma_n^{-1}\) for all \(n \geq 0\) and \(\lambda \in \Lambda(0)\). Applying Lemma 4.0.3 we deduce that \(f^{\lambda_m}_{\lambda,\mu}(t) = f^{\lambda}_{\mu,\mu}(t)\) for any \(\lambda, \mu, \nu \in \Lambda(0)\). Injectivity of \(FH_0 \rightarrow FH_m\) follows since \(\{K(\lambda_m) \mid \lambda \in \Lambda(0)\}\) is \(R\)-linearly independent by construction.

With this result in mind we will often identify \(FH_0\) as the \(R\)-subalgebra of \(FH_m\) consisting of \(R\)-linear combination of basis elements \(K(\lambda)\) where \(\lambda\) contains the trivial cycles \((1)(2)\ldots(m)\), i.e. indetifying \(FH_0\) with the \(R\)-subalgebra \(FH^*_m\) described in the above proof. Since \(pr_{n,0}(FH_0) = Z(\mathbb{Z}\mathcal{S}_n)\), it is easy to deduce from Lemma 4.0.3 that \(FH_0\) is commutative, and hence a commutative \(R\)-subalgebra of \(FH_m\).
5 Monoid of $m$-Marked Cycle Shapes

To show certain structural properties of the algebras $\text{FH}_m$ it will prove helpful to equip the set of $m$-marked cycle shapes $\Lambda(m)$ with an associative product. To do such we will construct a monoid which we may identify with $\Lambda(m)$. This will also allow us to give a more concrete criteria for a permutation to belong to an $m$-class.

Let $A$ be the free commutative monoid on the set $\{u_1, \ldots, u_m\}$. We have a natural left monoid action $\varphi : \mathfrak{S} \to \text{End}(U_m)$ given by $\varphi(\pi)(u_i) = u_{\pi(i)}$ for all $\pi \in \mathfrak{S}$ and $i \in [m]$, where $\text{End}(U_m)$ is the monoid of all monoid endomorphisms $U_m \to U_m$. Now let $C$ denote the free commutative monoid on the infinite set $\{c_1, c_2, \ldots\}$. Then the monoid we are interested in is $(\mathfrak{S}_m \ltimes \varphi U_m) \times C$, where $\ltimes \varphi$ denotes the semidirect product with respect to the action $\varphi$. The underlying set is $\mathfrak{S}_m \times U_m \times C$ and the product is given by

$$(\pi, p, a)(\sigma, q, b) = (\pi\sigma, \varphi(\sigma)(p)q, ab)$$

for all $(\pi, p, a), (\sigma, q, b) \in \mathfrak{S}_m \times U_m \times C$. We abuse notation and write $\pi = (\pi, 1, 1)$, $p = (1, p, 1)$, and $a = (1, 1, a)$. For any set $A$ we let $\mathbb{Z}_{\geq 0}^A$ denote the set of all functions $f : A \to \mathbb{Z}_{\geq 0}$ with finite support, that is the subset of elements $a \in A$ such that $f(a) \neq 0$ is finite. Then for any $d \in \mathbb{Z}_{\geq 0}^{|m|}$ and $l \in \mathbb{Z}_{\geq 0}^{|n|}$ we let

$$u^d := \prod_{i=1}^m u_i^{|d(i)|} \text{ and } c^l := \prod_{i \in \mathbb{N}} c_{i}^{|l(i)|}$$

which are well-defined by commutativity and since $l$ has finite support. Then as sets we have that

$$(\mathfrak{S}_m \ltimes \varphi U_m) \times C = \left\{ \pi u^d c^l \mid \pi \in \mathfrak{S}_m, d \in \mathbb{Z}_{\geq 0}^{|m|}, l \in \mathbb{Z}_{\geq 0}^{|n|} \right\}.$$

Moreover, the product may be described by

$$\left(\pi u^d c^l\right)(\sigma u^e c^k) = \pi\sigma u^{\pi d + \sigma e} c^{l+k}$$

where $\sigma \circ d \in \mathbb{Z}_{\geq 0}^{|m|}$ is defined by $(\sigma \circ d)(i) = d(\sigma^{-1}(i))$. In particular, the operator $\circ$ realises the set $\mathbb{Z}_{\geq 0}^{|m|}$ as a left action set of $\mathfrak{S}_m$. The set $\Lambda(m)$ of $m$-marked cycle shapes is in a natural one-to-one correspondence with this monoid: Consider the map $(\mathfrak{S}_m \ltimes \varphi U_m) \times C \to \Lambda(m)$ given by sending $\pi u^d c^l$ to the $m$-marked cycle shape consisting, for each $i \in \mathbb{N}$, $u(i)$ many cycles of length $i + 1$ containing only the symbol $\ast$, and where the remaining cycles are constructed from those of $\pi$ by adding $d(i)$ symbols $\ast$ after the entry $i$, for each $i \in [m]$ (see examples below). This map is a bijection since there is a natural inverse to consider. With this correspondence in mind, we identify $\Lambda(m)$ with the monoid $(\mathfrak{S}_m \ltimes \varphi U_m) \times C$.

Examples 5.0.1.

(1) Let $\lambda$ be the 6-marked cycle shape given in Example 3.0.2 then we have the identification

$$(1,4)(2,6)(3)(5)u_1^2u_2u_1c_1^1c_2^1 = (2,6)(5)(1,\ast\ast,\ast,\ast)(\ast)(\ast)(\ast,\ast,\ast),$$

where we have added colours to aid in demonstrating the correspondence.

(2) Consider the 4-marked cycle shapes written as elements of $(\mathfrak{S}_m \ltimes \varphi U_m) \times C$ by $\lambda = (1,2)(3,4)u_1u_2$, and $\mu = (1,4)(2,3)u_3^2c_1$. Their product is given by

$$\lambda \mu = ((1,2)(3,4)u_1u_2)((1,4)(2,3)u_3^2c_1) = (1,3,4,2)u_2^2u_4c_1.$$

Thus when writing $\lambda$ and $\mu$ as $m$-marked cycle shapes we have

$$\lambda \mu = ((1,\ast,\ast,\ast)(3,4))((1,4)(2,3)(\ast)(\ast)(\ast,\ast,\ast)) = (1,3,\ast,\ast,\ast,\ast,\ast,\ast).$$

Given any element $\lambda \in \Lambda(m)$, we will freely move between viewing it as an $m$-marked cycle shape or as an element of $(\mathfrak{S}_m \ltimes \varphi U_m) \times C$, i.e. as an expression of the form $\pi u^d c^l$. As we know, a permutation $\sigma$ belongs to the $m$-class $\text{CL}_n(m, \lambda)$ if and only if $\sigma$ may be obtained from $\lambda$ by replacing the symbols $\ast$ with distinct elements from $\mathbb{N} \setminus [m]$. When expressing $\lambda = \pi u^d c^l$, then from the above discussion it is clear that we have the following equivalent criteria for when $\sigma$ belongs to $\text{CL}_n(m, \lambda)$ given in terms of $\pi$, $d$, and $l$. 


Lemma 5.0.2. We have $\sigma \in CL_{N,m}(\pi u^d c^d)$ if and only if the following hold:

(i) the number of cycles of $\sigma$ of length $i + 1$ containing no elements of $[m]$ is $l(i)$,
(ii) $\sigma^{d(i)+1}(i) = \pi(i)$ for each $i \in [m]$, and $\sigma^n(i) \notin [m]$ for any $1 \leq n \leq d(i)$.

Item (ii) above gives a concrete way of describing the relative positions of the elements of $[m]$ within a permutation $\sigma$, and is completely captured by $\pi$ and $d$. Recall the quantities $||\lambda||$, $||\lambda||_m$, and $||\lambda||^m$ for an $m$-marked cycle shape $\lambda \in \Lambda(m)$. Then when expressing $\lambda = \pi u^d c^d$, one can deduce that

$$||\lambda|| = |\text{Sup}(\pi)| + \sum_{i \in [m]} d(i) + \sum_{i \in \mathbb{N}} (i + 1)l(i), \quad ||\lambda||^m = \sum_{i \in [m]} d(i) + \sum_{i \in \mathbb{N}} (i + 1)l(i), \quad ||\lambda||_m = |\text{Sup}(\pi)|.$$

Recall the degree function $\text{deg}_m : \Lambda(m) \to \mathbb{Z}_{\geq m}$ given by $\text{deg}_m(\lambda) = ||\lambda||^m + m$. The codomain $\mathbb{Z}_{\geq m}$ is a monoid under the addition $a +_m b := a + b - m$ for all $a, b \in \mathbb{Z}_{\geq m}$ with $m$ the identity. Viewing $\Lambda(m)$ as a monoid, then by consulting Equation (2) one can deduce that $\text{deg}_m$ is a monoid homomorphism, and hence provides a grading for $\Lambda(m)$. Lastly, with the identification $\Lambda(m) = (\mathcal{S}_m \times \mathbb{F}_m) \times \mathcal{C}$ we may express the Farahat-Higman algebras by

$$FH_m = \text{Span}_R\{K(\pi u^d c^d) \mid \pi \in \mathcal{S}_m, d \in \mathbb{Z}^m_{\geq 0}, l \in \mathbb{Z}^N_{\geq 0}\}.$$

Note that the $R$-subalgebra generated by the elements $K(d^i)$ for all $l \in \mathbb{Z}^N_{\geq 0}$ is precisely the subspace $FH_m^*$ described in the proof of Lemma 5.0.4, consisting of all basis elements indexed by $m$-marked cycles shapes with the elements within $[m]$ appearing in trivial cycles. Therefore $FH_0 \cong \text{Span}_R\{K(d^i) \mid l \in \mathbb{Z}^1_{\geq 0}\} \subset FH_m$.

6 Leading Terms via Monoid Product

We may extend the degree function $\text{deg}_m : \Lambda(m) \to \mathbb{Z}_{\geq m}$ to one acting on $FH_m$ by setting

$$\text{deg} \left( \sum_{\lambda \in \Lambda(m)} f_\lambda(t)K(\lambda) \right) = \max\{\text{deg}_m(\lambda) \mid f_\lambda(t) \neq 0\}.$$

In particular $\text{deg}_m(K(\lambda)) = \text{deg}_m(\lambda)$. We seek to show that the product of two basis elements of $FH_m$ admits a unique leading term with regards to this degree function. To prove this we will use the following lemma.

Lemma 6.0.1. For $\lambda, \mu \in \Lambda(m)$, let $g \in CL_{N,m}(\lambda)$ and $h \in CL_{N,m}(\mu)$. The equality

$$\text{Sup}^m(g) \cap \text{Sup}^m(h) = \emptyset$$

holds if and only if $gh \in CL_{N,m}(\lambda \mu)$.

Proof. Suppose that $gh \in CL_{N,m}(\lambda \mu)$, then since $\text{deg}_m$ is a monoid homomorphism

$$|\text{Sup}^m(gh)| = ||\lambda \mu||^m = ||\lambda||^m + ||\mu||^m = |\text{Sup}^m(g)| + |\text{Sup}^m(h)|.$$

Since also $\text{Sup}^m(gh) \subset \text{Sup}^m(g) \cup \text{Sup}^m(h)$, we have $\text{Sup}^m(g) \cap \text{Sup}^m(h) = \emptyset$. This proves the “if” statement. For the “only if” let $\lambda = \pi u^d c^d$ and $\mu = \sigma u^e c^k$ for $\pi, \sigma \in \mathcal{S}_m, d, e \in \mathbb{Z}^m_{\geq 0}$, and $l, k \in \mathbb{Z}^N_{\geq 0}$. By Equation (2),

$$\lambda \mu = \pi \sigma u^{\sigma d + e} c^{l + k}.$$

Hence the result follows if we show that $gh$ satisfies items (i) and (ii) of Lemma 5.0.2 with respect to $\pi \sigma u^{\sigma d + e} c^{l + k}$. For item (i), since Equation (3) is upheld, it is clear that the number of cycles of $gh$ of length $i + 1$ which contain no elements of $[m]$ is $l(i) + k(i)$, since this is the sum of such cycles of $g$ and $h$. For item (ii), pick any $x \in [m]$ and set $y := \sigma(x)$ and $z := \pi(y)$. Since $h \in CL_{N,m}(\pi u^d c^d)$, item (ii) of
Lemma 5.0.2 tells us that \( h : x \mapsto i_1 \mapsto i_2 \mapsto \cdots \mapsto i_{e(x)} \mapsto y \) where \( \{i_1, i_2, \ldots, i_{e(x)}\} \cap [m] = \emptyset \). Similarly since \( g \in \text{CL}_{[n,m]}(\pi u d c^d) \), \( \{j_1, j_2, \ldots, j_{d(y)}\} \cap [m] = \emptyset \). Since Equation (3) is upheld, we must have that 
\[
gh : x \mapsto i_1 \mapsto \cdots \mapsto i_{e(x)} \mapsto j_1 \mapsto \cdots \mapsto j_{d(y)} \mapsto z.
\]

Thus \( (gh)^{d(y)+e(x)+1}(x) = (\pi \sigma)(x) \) and \( (gh)^n(x) \notin [m] \) for any \( 1 \leq n \leq d(y) + e(x) \). One may also note that \( d(y) = d(\sigma^{-1}(x)) = (\sigma \circ d)(x) \), and hence \( gh \) also upholds item (ii) of Lemma 5.0.2.

\[\square\]

Proposition 6.0.2. Let \( \lambda = \pi u d c^d, \mu = \sigma u e c^k \in \Lambda(m) \). In \( \text{FH}_m \) we have that 
\[
K(\lambda)K(\mu) = c_{\lambda,\mu}K(\lambda\mu) + \sum_{\nu \in \Lambda(m)} f_{\lambda,\mu}^\nu(t)K(\nu)
\]
where \( c_{\lambda,\mu} \in \mathbb{N} \) is a constant given by 
\[
c_{\lambda,\mu} = \prod_{i=1}^{n} \left( (l+k)(i) \right) = \prod_{i=1}^{n} \left( l(i) \right).
\]

Proof. Let \( g \in \text{CL}_{[n,m]}(\lambda), h \in \text{CL}_{[n,m]}(\mu) \), then \( \text{Sup}^m(gh) \subseteq \text{Sup}^m(g) \cup \text{Sup}^m(h) \). Thus if \( gh \in \text{CL}_{[n,m]}(\nu) \) for \( \nu \in \Lambda(m) \) then \( ||\nu||^m \leq ||\lambda||^m + ||\mu||^m \). Hence \( \deg_m(\nu) \leq \deg_m(\lambda) + m \deg_m(\mu) = \deg_m(\lambda\mu) \). Thus we have 
\[
K(\lambda)K(\mu) = \sum_{\nu \in \Lambda(m)} f_{\lambda,\mu}^\nu(t)K(\nu).
\]

Suppose \( gh \in \text{CL}_{[n,m]}(\nu) \) with \( \deg_m(\nu) = \deg_m(\lambda\mu) \). This implies that \( ||\nu||^m = ||\lambda\mu||^m \) and hence we must have \( \text{Sup}^m(g) \cap \text{Sup}^m(h) = \emptyset \). Thus by Lemma 6.0.7 we have that \( \nu = \lambda\mu \). Therefore 
\[
K(\lambda)K(\mu) = f_{\lambda,\mu}^{\lambda\mu}(t)K(\lambda\mu) + \sum_{\nu \in \Lambda(m)} f_{\lambda,\mu}^\nu(t)K(\nu).
\]

It remains to show \( f_{\lambda,\mu}^{\lambda\mu}(t) = c_{\lambda,\mu} \). Let \( \omega \in \text{CL}_{[n,m]}(\lambda\mu) \) and \( A_{\lambda,\mu}(\omega) := \{ (g, h) \in \text{CL}_{[n,m]}(\lambda) \times \text{CL}_{[n,m]}(\mu) : gh = \omega \} \). By Equation (2) we have \( \lambda\mu = \pi \sigma u^{\sigma \circ \sigma + e} c^{l+k} \), and so by Lemma 6.0.2 for any \( x \in [m] \), we have that 
\[
\omega : x \mapsto i_1 \mapsto \cdots \mapsto i_{(\sigma \circ \sigma + e)(x)} \mapsto (\pi \sigma)(x), \quad \{i_1, i_2, \ldots, i_{(\sigma \circ \sigma + e)(x)}\} \cap [m] = \emptyset.
\]

Any pair \( (g, h) \in A_{\lambda,\mu}(\omega) \) satisfies Equation (3), and since their product gives \( \omega \) we must have that 
\[
h : x \mapsto i_1 \mapsto \cdots \mapsto i_{e(x)} \mapsto \sigma(x), \quad g : \sigma(x) \mapsto i_{e(x)+1} \mapsto \cdots \mapsto i_{(\sigma \circ \sigma + e)(x)} \mapsto (\pi \sigma)(x).
\]

So if we construct a pair \( (g, h) \in \text{CL}_{[n,m]}(\lambda) \times \text{CL}_{[n,m]}(\mu) \) such that \( gh = \omega \), the cycles containing elements of \([m] \) in \( g \) and \( h \) are predetermined by \( \omega \). Hence we are just concerned with the cycles which contain no elements of \([m] \). In \( \omega \) there are \( (l+k)(i) \) number of such cycles of length \( i+1 \), while \( g \) and \( h \) contain \( l(i) \) and \( k(i) \) such cycles respectively. Thus to construct a pair \( (g, h) \in A_{\lambda,\mu}(\omega) \), it is simply a matter of how one distributes the cycles containing no elements of \([m] \) of \( \omega \) among either \( g \) or \( h \). The binomial coefficient 
\[
\binom{(l+k)(i)}{l(i)}
\]
counts the number of ways to allocate such cycles of length \( i+1 \) of \( \omega \) to the permutation \( g \) (with the remaining cycles allocated to \( h \)). Therefore 
\[
f_{\lambda,\mu}^{\lambda\mu}(t) = |A_{\lambda,\mu}(\omega)| = \prod_{i=1}^{\infty} \binom{(l+k)(i)}{l(i)},
\]
where the product is only formally infinite since the functions \( (l+k), l, \) and \( k \) have finite support.

\[\square\]
Thus the leading term in the product \( K(\lambda)K(\mu) \) is \( c_{\lambda,\mu}K(\lambda\mu) \). So the product of \( \Lambda(m) \) is governing the leading terms. From the formula given above one can deduce that \( c_{\lambda,\mu} = 1 \) if and only if \( \lambda \) and \( \mu \) share no cycles of the same length consisting of only the symbols \(*\), i.e. whenever \( I(i) \neq 0 \) then \( k(i) = 0 \).

If one was to extend \( R \) to a ring which contains the inverses to any natural number, say its field of fractions \( \mathbb{Q}(t) \), then the above leading term result could by applied to easily deduce generating sets for the \( \mathbb{Q}(t)\)-algebra \( \mathbb{Q}(t) \otimes \mathbb{H}_m \). For example, let \( s_i = (i, i + 1) \) for any \( i \in [m] \) be the simple transposition in \( \mathfrak{S}_m \), then by arguing by induction on the degree, Proposition 6.0.3 can be employed to show that the set

\[
\{ K(s_i), K(u_i), K(c_j) \mid i \in [m], j \in \mathbb{N} \}
\]

generates \( \mathbb{Q}(t) \otimes \mathbb{H}_m \) as a \( \mathbb{Q}(t) \)-algebra since \( \{ s_i, u_i, c_j \mid i \in [m], j \in \mathbb{N} \} \) generates \( \Lambda(m) = (\mathfrak{S}_m \ltimes \mathcal{U}_m) \times \mathbb{C} \).

7 Symmetric Functions

Consider the polynomial \( \mathbb{Z} \)-algebra \( \mathbb{Z}[x_1, \ldots, x_n] \) in \( n \) commuting variables. The group \( \mathfrak{S}_n \) acts on such polynomials by permuting the variables. A polynomial is symmetric if it is \( \mathfrak{S}_n \)-invariant. Denote the \( \mathbb{Z} \)-subalgebra of symmetric polynomials by \( \text{Sym}_n \). We are interested in two types of symmetric polynomials, and to describe one type it will be helpful to recall the definition of a partition. We say a tuple \( \alpha = (a_1, a_2, \ldots, a_l) \) of positive integers is a partition whenever \( a_1 \geq a_2 \geq \cdots \geq a_l \). We say \( \alpha \) has length \( l \) and size \( a_1 + \cdots + a_l \). Let \( \mathcal{P}(k, l) \) be the set of partitions of size \( k \) and length \( l \). When \( k = l = 0 \) let \( \mathcal{P}(0, 0) = \{ \emptyset \} \).

**Definition 7.0.1.** For any \( k \geq 0 \), the elementary symmetric polynomials are given by

\[
e_k(x_1, \ldots, x_n) := \sum_{i_1 < i_2 < \ldots < i_k} x_{i_1}x_{i_2}\cdots x_{i_k}
\]

Hence \( e_k \) is the sum of all monomials with \( k \) variables. In particular \( e_0 = 1 \) and \( e_k = 0 \) for \( k > n \).

**Definition 7.0.2.** For \( \alpha = (a_1, a_2, \ldots, a_l) \in \mathcal{P}(k, l) \), the monomial symmetric polynomials are given by

\[
m_\alpha(x_1, \ldots, x_n) := \sum_{\{i_1, i_2, \ldots, n\} \subset [n]} x_1^{a_1}x_2^{a_2}\cdots x_i^{a_i}
\]

Hence \( m_\alpha \) is the sum of all monomials whose exponents match the partition \( \alpha \) up to rearrangement. In particular \( m_\emptyset = 1 \) and \( m_\alpha = 0 \) whenever \( l > n \).

It is well known that \( \text{Sym}_n \) is generated (transcendently) by \( e_1, e_2, \ldots, e_n \). That is we have a \( \mathbb{Z} \)-algebra isomorphism \( \text{Sym}_n \cong \mathbb{Z}[e_1, \ldots, e_n] \) with the latter being a free polynomial algebra in \( n \) commuting generators. One may show that \( \{ m_\alpha \mid \alpha \in \mathcal{P}(k, l), k \geq 0, n \geq l \geq 0 \} \) forms a \( \mathbb{Z} \)-basis for \( \text{Sym}_n \). Assign each variable \( x_i \) a degree of 1, then let \( \text{Sym}_n^k \) denote the degree \( k \) component of \( \text{Sym}_n \). In particular \( e_k, m_\alpha \in \text{Sym}_n^k \) for any \( \alpha \in \mathcal{P}(k, l) \). For any \( N > n \) we have surjective \( \mathbb{Z} \)-module homomorphisms \( \rho_{N,n} : \text{Sym}_N^k \to \text{Sym}_n^k \) given by evaluating the variables \( x_{n+1}, \ldots, x_N \) at zero. One can show that the collection of such morphisms defines an inverse system for the \( \mathbb{Z} \)-modules \( \text{Sym}_n^k \) (with fixed \( k \)), and so we have the inverse limit

\[
\text{Sym}^k := \lim_{\longrightarrow} \text{Sym}_n^k.
\]

Any element of \( \text{Sym}^k \) is of the form \( (f_1, f_2, \ldots) \) with \( f_n \in \text{Sym}_n^k \) and where \( \rho_{N,n}(f_N) = f_n \) for all \( N > n \). One can note that \( \rho_{N,n}(m_\alpha(x_1, \ldots, x_N)) = m_\alpha(x_1, \ldots, x_n) \) for any \( N > n \) and \( \alpha \in \mathcal{P}(k, l) \). Thus we write \( m_\alpha = (m_\alpha(x_1), m_\alpha(x_1, x_2), \ldots) \) and call such elements of \( \text{Sym}^k \) the monomial symmetric functions. The same can be said for \( e_k \) since \( e_k = m_{(1^k)} \) where \( (1^k) \) is the partition of size \( k \) consisting of \( k \) parts equal to 1. The \( \mathbb{Z} \)-algebra of symmetric functions is given by

\[
\text{Sym} := \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \text{Sym}^k.
\]

We have the isomorphism of \( \mathbb{Z} \)-algebras \( \text{Sym} \cong \mathbb{Z}[e_1, e_2, \ldots] \), and a \( \mathbb{Z} \)-basis \( \{ m_\alpha \mid \alpha \in \mathcal{P}(k, l), k, l \geq 0 \} \).
8 Jucys-Murphy Elements and Generators of $\mathrm{FH}_0$

In this section we recall the Jucys-Murphy elements of the symmetric group algebras, and their connections to the Farahat-Higman algebras.

**Definition 8.0.1.** For each $1 \leq i \leq n$, the $i$-th *Jucys-Murphy element* $L_i$ of $\mathbb{Z}\mathfrak{S}_n$ is defined by

$$L_i := \sum_{j<i} (j,i).$$

Note that $L_1 = 0$. The following relations regarding the Jucys-Murphy elements are well-known.

**Lemma 8.0.2.** The following relations hold within $\mathbb{Z}\mathfrak{S}_n$:

1. $L_i L_j = L_j L_i$ for all $i,j \in [n]$
2. $s_i L_j = L_j s_i$ for all $i \in [n-1]$ and $j \neq i, i+1$
3. $L_i+1 = s_i L_i s_i + s_i$ for all $i \in [n-1]$

where $s_i = (i, i+1)$ is the simple transposition exchanging $i$ and $i+1$ for all $i \in [n-1]$.

It was shown in [Murphy83, Theorem 1.9] that the center is precisely the collection of all symmetric polynomials in the Jucys-Murphy elements. Then from the last section we know that

$$Z(\mathbb{Z}\mathfrak{S}_n) = \langle e_1(L_1, \ldots, L_n), \ldots, e_n(L_1, \ldots, L_n) \rangle.$$  

(5)

We know the center $Z(\mathbb{Z}\mathfrak{S}_n)$ has a $\mathbb{Z}$-basis given by the set $\{K_\lambda(\lambda) \mid \lambda \in \Lambda_n(0)\}$. It is natural to ask how the elementary symmetric polynomials in the Jucys-Murphy elements decompose as a linear combination of class sums. This was answered in [Jucys74, Section 3] as we recall now. Let $\lambda \in \Lambda(0)$ contain $l$ many cycles. We say the *reduced degree* of $\lambda$ is $\text{rd}(\lambda) := \text{deg}_0(\lambda) - l$, i.e. the number of symbols $*$ present in $\lambda$ minus the number of cycles. Then it was shown that

$$e_k(L_1, \ldots, L_n) = \sum_{\lambda \in \Lambda_n(0), \text{rd}(\lambda) = k} K_\lambda(\lambda)$$

for any $k \in [n]$. Recall that $\text{pr}_{n,0}(\mathrm{FH}_0) = Z(\mathbb{Z}\mathfrak{S}_n)$, and from the above formula there is natural elements of $\mathrm{FH}_0$ to consider which project down to the elementary symmetric polynomials in the Jucys-Murphy elements.

**Definition 8.0.3.** For any $k \in \mathbb{Z}_{\geq 0}$ define

$$E_k := \sum_{\lambda \in \Lambda(0), \text{rd}(\lambda) = k} K(\lambda),$$

Note only finitely many elements of $\Lambda(0)$ have reduced degree $k$. Thus $\text{pr}_{n,0}(E_k) = e_k(L_1, \ldots, L_n)$ for any $n \geq 0$ and when $n > k$ both $\text{pr}_{n,0}(E_k)$ and $e_k(L_1, \ldots, L_n)$ are zero. The images of $E_1, E_2, \ldots, E_n$ under $\text{pr}_{n,0}$ thus give a generating set for $Z(\mathbb{Z}\mathfrak{S}_n)$. Relation-wise $\mathrm{FH}_0$ is determined by the centers $Z(\mathbb{Z}\mathfrak{S}_n)$, so one may expect the elements $E_1, E_2, \ldots$ to generate $\mathrm{FH}_0$. This was proved in [FH59], which we record here.

**Theorem 8.0.4.** [Theorem 2.5 of [FH59]] The set of elements $\{E_1, E_2, \ldots\}$ generates $\mathrm{FH}_0$ as an $R$-algebra.

Furthermore, it was proven in [Ryba21, Theorem 3.8] that there is an isomorphism $\mathrm{FH}_0 \cong R \otimes_{\mathbb{Z}} \text{Sym}$ of $R$-algebras which sends $e_k$ to $E_k$. In the next section we will prove an analogous result for $\mathrm{FH}_m$ in Theorem 9.0.3 and to do so it will be helpful to discuss elements of $\mathrm{FH}_0$ which are to the monomial symmetric polynomials $m_\alpha$ where the elements $E_k$ are to the elementary symmetric polynomials $e_k$.

**Lemma 8.0.5.** For any $k,l \geq 0$ and $\alpha \in \mathcal{P}(k,l)$, there exists an element $M_\alpha \in \mathrm{FH}_0$ such that

$$\text{pr}_{n,0}(M_\alpha) = m_\alpha(L_1, \ldots, L_n).$$
Proof. As $\text{Sym} = \mathbb{Z}[e_1, e_2, \ldots]$ then $m_\alpha$ is a finite $\mathbb{Z}$-linear combination of monomials in $e_1, e_2, \ldots$. Letting $M_\alpha$ be obtained from $m_\alpha$ by replacing $e_k$ with $E_k$ gives the element we are looking for.

We wish to say a little more about the elements $M_\alpha$. For any $\alpha = (a_1, \ldots, a_l) \in P(k, l)$ let $\pi \in \Lambda(0)$ denote the cycle shape with $l$ cycles of lengths $a_1 + 1, a_2 + 1, \ldots, a_l + 1$. When $\alpha = \emptyset$ then $\emptyset = \emptyset$ also. We have that $\text{rd}(\pi) = k$, the size of $\alpha$. Then in the proof of Theorem 1.9 of [Murphy83], see also [Ryba21, Proposition 3.11], the following proposition was shown. Let $\mathcal{P} := \cup_{k,l \geq 0} P(k, l)$ and note that the map $(\cdot) : \mathcal{P} \to \Lambda(0)$ sending $\alpha \mapsto \pi$ is a bijection.

**Proposition 8.0.6.** Let $\alpha \in \mathcal{P}$ such that $\deg_0(\alpha) \leq n$, then

$$m_\alpha(L_1, \ldots, L_n) = K_\pi(\pi) + \sum c_\mu(n)K_\mu(\mu),$$

where the sum runs over all $\mu \in \Lambda(0)$ such that $\text{rd}(\mu) < \text{rd}(\pi)$ or that $\text{rd}(\mu) = \text{rd}(\pi)$ and $\mu$ contains less cycles than $\pi$ (noting that only finitely many such $\mu$ exist), and where $c_\mu(n) \in \mathbb{Z}_{\geq 0}$.

Applying Lemma 4.0.3 allows us to deduce the following:

**Lemma 8.0.7.** Let $\alpha \in \mathcal{P}$ then

$$M_\alpha = K(\pi) + \sum c_\mu(t)K(\mu),$$

where the sum runs over all $\mu \in \Lambda(0)$ such that $\text{rd}(\mu) < \text{rd}(\pi)$ or that $\text{rd}(\mu) = \text{rd}(\pi)$ and $\mu$ contains less cycles than $\pi$ (noting that only finitely many such $\mu$ exist), and where $c_\mu(t) \in R$ are such that $c_\mu(n)$ are the coefficients of $K_\mu(\mu)$ in the above proposition.

We may now show that the elements $M_\alpha$ provide an $R$-basis for $\text{FH}_0$.

**Proposition 8.0.8.** The set $\{M_\alpha \mid \alpha \in \mathcal{P}\}$ gives an $R$-basis for $\text{FH}_0$.

Proof. By Equation (6) it is clear that $\{M_\alpha \mid \alpha \in \mathcal{P}\}$ is $R$-linearly independent, hence we only need to show that it spans $\text{FH}_0$. Equip the $R$-basis $\{K(\lambda) \mid \lambda \in \Lambda(0)\}$ of $\text{FH}_0$ with the partial order $\preceq$ defined by letting $K(\mu) \preceq K(\lambda)$ if $\text{rd}(\mu) < \text{rd}(\lambda)$ or if $\text{rd}(\mu) = \text{rd}(\lambda)$ and $\mu$ contains less cycles than $\lambda$. For any $n \in \mathbb{Z}_{\geq 0}$ let $\text{FH}_0^{\leq n}$ denote the $\mathbb{Z}$-module spanned by the elements $K(\lambda)$ such that $\text{rd}(\lambda) \leq n$. We prove that $\{M_\alpha \mid \alpha \in \mathcal{P}\}$ spans $\text{FH}_0^{\leq n}$ by induction on $n$. For the base case we have $\text{FH}_0^{\leq 0} = \text{Span}_R\{K(\emptyset)\}$, and hence this case follows since $M_\emptyset = K(\emptyset) = K(\emptyset)$. Consider $\text{FH}_0^{\leq n}$ for some $n > 1$, and let

$$K = \sum_{\lambda \in \Lambda(0), \text{rd}(\lambda) \leq n} f_\lambda(t)K(\lambda)$$

be an element of $\text{FH}_0^{\leq n}$. Let $K(\lambda')$ be maximal in $\{K(\lambda) \mid f_\lambda(t) \neq 0\}$ with respect to $\preceq$. Then let $\alpha$ be the unique partition of $\mathcal{P}$ such that $\pi = \lambda'$, then $K - f_{\lambda'}(t)M_\alpha$ is an element of $\text{FH}_0^{\leq n}$ whose terms are strictly less that $K(\lambda')$ in the partial order or incomparable. Continuing this procedure of removing the maximal basis element will lead to an element of $\text{FH}_0^{\leq n-1}$, and by induction the resulting element will belong to the span of $\{M_\alpha \mid \alpha \in \mathcal{P}\}$. Since we got their by subtracting some $R$-linear combination of elements $M_\alpha$, the starting element $K$ must also belong to the span of $\{M_\alpha \mid \alpha \in \mathcal{P}\}$ which completes the proof by induction. The proposition follows since $\text{FH}_0$ is the union of $\text{FH}_0^{\leq n}$ for all $n \geq 0$.

**9 An Isomorphism $\text{FH}_m \cong R \otimes_{\mathbb{Z}} (\mathcal{H}_m \otimes \text{Sym})$**

In this section we recall the definition of the degenerate affine Hecke algebra $\mathcal{H}_m$, and some of its structural properties. We define some elements of $\text{FH}_m$ which mimic the variables generators of $\mathcal{H}_m$, and end by proving the isomorphism in the section title.

**Definition 9.0.1.** The degenerate affine Hecke Algebra $\mathcal{H}_m$ is the $\mathbb{Z}$-algebra presented with generating set $\{s_i, y_j \mid 1 \leq i \leq m - 1, j \in [m]\}$ and defining relations
(1i) $s_i^2 = 1$, for $i \in [m - 1]$.
(2i) $y_i y_j = y_j y_i$ for all $i, j \in [m]$.
(1ii) $s_i s_j = s_j s_i$, for $j \neq i, i + 1$.
(2ii) $s_i y_j = y_j s_i$ for all $j \neq i, i + 1$.
(1iii) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, for $i \in [m - 2]$.
(2iii) $y_{i+1} = y_i y_s i + s_i$ for all $i \in [m - 1]$.

The elements $s_i$ are the simple transpositions $(i, i + 1)$ of $S_m$. The algebra $\mathcal{H}_m$ has a basis of the form

$$\{ \pi y_1^{d(1)} \ldots y_m^{d(m)} \mid \pi \in S_m, d \in \mathbb{Z}_{\geq 0} \}.$$ (7)

For a proof see [K05 Theorem 3.2.2]. Hence one can deduce that the subalgebra generated by the elements $s_i$ is a copy of $\mathbb{Z} S_m$, and similarly the subalgebra generated by the variable generators $y_1, \ldots, y_m$ is a copy of the polynomial $\mathbb{Z}$-algebra $\mathbb{Z}[y_1, \ldots, y_m]$. Moreover, as an $\mathbb{Z}$-module we have $\mathcal{H}_m \cong \mathbb{Z} S_m \otimes \mathbb{Z}[y_1, \ldots, y_m]$.

By Lemma 8.0.2 we have a surjective $\mathbb{Z}$-algebra homomorphism $\mathcal{H}_m \rightarrow \mathbb{Z} S_m$ given by $s_i \mapsto s_i$ and $y_j \mapsto L_j$.

We now show that the algebra $\mathcal{H}_m$ contains like-minded elements.

**Definition 9.0.2.** For any $i \in [m]$ define elements $Y_i \in \mathcal{H}_m$ by

$$Y_i := K(u_i) + \sum_{j \in [m] \atop j < i} K((j, i)) = K(u_i) + L_i,$$

where we have identified $R S_m$ with $\{ K(\pi) \mid \pi \in S_m \subset \Lambda(m) \}$.

**Example 9.0.3.** In $\mathcal{H}_3$ we have

$$Y_1 = K((*, 1)), \quad Y_2 = K((*, 2)) + K((1, 2)), \quad Y_3 = K((*, 3)) + K((1, 3)) + K((2, 3)),$$

where we have dropped the trivial cycles (1), (2), and (3) from each of the 3-marked cycle shapes appearing.

Since $\deg_m(u_i) = m + 1$ we have that $\mathcal{H}_m$ has a basis of the form

$$\{ K(\pi) \mid \pi \in S_m \subset \Lambda(m) \}.$$

We now show that the algebra $\mathcal{H}_m$ contains like-minded elements.

**Lemma 8.0.4.** The following relations hold within $\mathcal{H}_m$:

1. $Y_i Y_j = Y_j Y_i$ for all $i, j \in [n]$.
2. $K(s_i) Y_j = Y_j K(s_i)$ for all $i \in [n - 1]$ and $j \neq i, i + 1$.
3. $Y_{i+1} = K(s_i) Y_i K(s_i) + K(s_i)$ for all $i \in [n - 1]$.

where $s_i = (i, i + 1)$ is the simple transposition exchanging $i$ and $i + 1$ for all $i \in [n - 1]$.

**Proof.** (1): For any $n \geq 0$ and $i, j \in [m]$ we have

$$\text{pr}_{n+m,m}(Y_i Y_j) = \sigma_n L_{i+m} L_{j+m} \sigma_n^{-1} = \sigma_n L_{i+m} L_{j+m} \sigma_n^{-1} = \text{pr}_{n+m,m}(Y_j Y_i),$$

since the Jucys-Murphy elements commute. So applying Lemma 8.0.3 shows that $Y_i Y_j = Y_j Y_i$.

(2): It follows as a straight forward application of Lemma 8.0.3 that $K(s_i)$ commutes with $K(u_i)$ whenever $i \in [n - 1]$ and $j \neq i, i + 1$, and we already know that $K(s_i)$ commutes with $L_j$, so (2) follows.

(3): Again by Lemma 8.0.3 one can show that $K(s_i) K(u_i) K(s_i) = K(u_{i+1})$, and hence

$$K(s_i) Y_i K(s_i) = K(u_{i+1}) + s_i L_i s_i = K(u_i) + L_{i+1} - K(s_i) = Y_j - K(s_i),$$

where we used (3) of Lemma 8.0.2. Rearranging gives (3).

**Theorem 9.0.5.** We have an isomorphism of $R$-algebras $\phi : R \otimes (\mathcal{H}_m \otimes \text{Sym}) \rightarrow \mathcal{H}_m$ defined by $s_i \mapsto K(s_i)$, $y_j \mapsto Y_j$, and $e_k \mapsto E_k$ for each $i \in [m - 1]$, $j \in [m]$, and $k \geq 0$. 

$\square$
Proof. To prove \( \phi \) is a homomorphism it suffices to show that the relations of Definition 9.0.1 are respected, and that the elements \( \phi(e_k) \) commute with \( \text{Im}(\phi) \). Relations (1) of Definition 9.0.1 are respected by Lemma 9.0.9 and relations (2) are upheld by Lemma 9.0.9. Also \( \phi(e_k) = \delta_k \in \text{FH}_0 \subseteq \text{FH}_m \), hence the images \( \phi(e_1), \phi(e_2), \ldots \) commute with one another since \( \text{FH}_0 \) is commutative. Moreover, since they belong to \( \text{FH}_0 \), their projections down to \( Z_{n,m} \) by \( \text{pr}_{n,m} \) consist of permutations which act trivially on \([m]\). Therefore \( \phi(e_k) \) commutes with \( \phi(s_i) = K(s_i) \) for any \( k \geq 0 \) and \( i \in [m-1] \). Lastly we have that

\[
[\phi(e_k), \phi(y_j)] = [\phi(e_k), Y_j] = [\phi(e_k), K(u_j)].
\]

We have that \( \text{pr}_{n,m}(K(u_j)) \) is the sum of transpositions \( (a,j) \) for all \( a \in [n]\setminus [m] \), which commutes with any permutation which fixes the elements \([m]\). In particular \( \text{pr}_{n,m}(K(u_j)) \) must commute with \( \text{pr}_{n,m}(\phi(e_k)) \), and so by Lemma 9.0.9 we have that \( K(u_j) \) commutes with \( \phi(e_k) \). Hence \( [\phi(e_k), \phi(y_j)] = 0 \), and so \( \phi(e_k) \) commutes with \( \text{Im}(\phi) \). So \( \phi \) is an \( R \)-algebra homomorphism. For surjectivity we show \( \text{K}(\lambda) \in \text{Im}(\phi) \) for any \( \lambda \in \Lambda(m) \). Write \( \lambda = \pi u^d c^l \) and consider the element \( K(c^l) \) which belongs to the \( R \)-subalgebra \( \text{FH}_0 \) of \( \text{FH}_m \). By Theorem 8.0.4 their exists \( C \in \langle \phi(e_1), \phi(e_2) \ldots \rangle \) such that \( \phi(C) = K(c^l) \). Then by employing the leading term result of Proposition 6.0.2 we have that

\[
\phi \left( \pi y_1^{d(1)} \cdots y_m^{d(m)} C \right) = K(\pi) Y_1^{d(1)} \cdots Y_m^{d(m)} K(c^l) = K(\pi u^d c^l) + T
\]

where \( T \) stands for an \( R \)-linear combination of \( K(\mu) \) such that \( \text{deg}_m(\mu) < \text{deg}_m(\lambda) \). Hence arguing by induction on the degree of \( \lambda \) shows surjectivity, noting that the base case is immediate since the basis elements of degree zero are precisely \( K(\pi) = \phi(\pi) \) for some \( \pi \in \mathfrak{S}_m \subset \Lambda(m) \). For injectivity, by Equation (6) and recalling that the monomial symmetric functions form a \( \Z \)-basis of \( \text{Sym} \), we have that the set

\[
\mathcal{B} := \left\{ \pi y_1^{d(1)} \cdots y_m^{d(m)} \otimes m_\alpha \mid \pi \in \mathfrak{S}_m, d \in \Z_{\geq 0}^m, \alpha \in \mathcal{P} \right\}
\]

forms an \( R \)-basis of \( \text{H}_m \otimes \text{Sym} \). We seek to show that \( \phi(\mathcal{B}) \) is \( R \)-linearly independent. Equip the basis set \( \{ K(\pi u^d c^l) \mid \pi \in \mathfrak{S}_m, d \in \Z_{\geq 0}^m, l \in \N^2 \} \) of \( \text{FH}_m \) with the partial order \( < \) define by \( K(\sigma u^d c^l) < K(\pi u^d c^l) \) if (i) the degree of \( \sigma u^d c^l \) is strictly less than that of \( \pi u^d c^l \), (ii) their degrees agree but \( rd(c^l) < rd(c^l) \), (iii) their degrees agree and \( rd(c^l) = rd(c^l) \) but \( c^l \) contains less cycles than \( c^l \). Note Proposition 6.0.2 tells us that the product of \( K(\lambda) \) and \( K(\mu) \) in \( \text{FH}_m \) results in \( c_{\lambda, \mu} K(\lambda \mu) \) plus addition terms all lower in the order \( \lambda \mu \) does not change the order \( \lambda \mu \). Then by employing the leading term result of Proposition 6.0.2 we have that \( \phi \) acts on a basis element of \( \mathcal{B} \) by

\[
\phi \left( \pi y_1^{d(1)} \cdots y_m^{d(m)} \otimes m_\alpha \right) = K(\pi) Y_1^{d(1)} \cdots Y_m^{d(m)} M_\alpha = K(\pi u^d c^l) + T
\]

where \( T \) is an \( R \)-linear combination of basis elements \( K(\lambda) \) of \( \text{FH}_m \) which are strictly less with respect to \( < \). As such, in the image of any finite \( R \)-linear combination of elements of \( \mathcal{B} \) under \( \phi \), we may pick out a non-zero term which is incomparable or strictly greater than any other term with respect to \( < \), showing that \( \phi(\mathcal{B}) \) is \( R \)-linearly independent in \( \text{FH}_m \), which shows \( \phi \) is injective.

\[\square\]

The \( C \)-algebra \( \mathcal{C} \otimes \Z \left( \text{H}_m \otimes \text{Sym} \right) \) and close variations have made appearances within the literature. In the [MO01] they aimed to give a centralizer construction for the degenerate affine Hecke algebra in a manner comparable to how the Yangians arise from a projective limit of universal enveloping algebras of \( \mathfrak{gl}_n \). No such projective system exists for the group algebras of the symmetric groups, but does for the larger semigroup of partial permutations. Working with these instead algebras \( A_m := A_0 \otimes \text{H}_m \) were constructed, where \( \text{H}_m \) is a degenerate affine counterpart to the semigroup algebra of partial permutations, and \( A_0 \) was shown to be isomorphic to the algebra of shifted symmetric functions. In our setting the lack of a projective system was sidestepped by employing the techniques of Farahat and Higman on the centralizer algebras \( Z_{n,m} \), which allowed us to stay working with the symmetric group itself. The \( C \)-algebra \( \mathcal{C} \otimes \Z \left( \text{H}_m \otimes \text{Sym} \right) \) also appears in the Heisenberg category \( \text{Heis} \) of M. Khovanov defined in [Kho14]. Such a category is a monoidal category generated by two objects \( \uparrow \) and \( \downarrow \), and where the morphism spaces are defined diagrammatically. It was shown in [Kho14] Proposition 4 that the endomorphism algebra \( \text{End}_{\text{Heis}}(\uparrow \otimes \downarrow \otimes m) \) is isomorphic to \( \mathcal{C} \otimes \Z \left( \text{H}_m \otimes \text{Sym} \right) \).

We close this section with some consequences of the above theorem.
**Corollary 9.0.6.** The algebra $FH_m$ has a $R$-basis given by the set

$$\left\{ K(\pi)Y^d_1 \cdots Y^d_m M_\alpha \mid \pi \in S_m, d \in \mathbb{Z}_{\geq 0}^m, \alpha \in P \right\}.$$

**Corollary 9.0.7.** The algebra $FH_m$ is generated by $K(s_i), Y_j,$ and $E_k$ for $i \in [m-1]$, $j \in [m]$, and $k \geq 0$.

**Corollary 9.0.8.** The center of $FH_m$ is $\text{Sym}[Y_1, \ldots, Y_m] \otimes FH_0$, which is generated by the elements $E_k$ and the elementary symmetric polynomials $e_k(Y_1, \ldots, Y_m)$ for all $k \geq 0$.

**Proof.** By Theorem 9.0.5 it is clear that the result follows if $\text{Sym}[y_1, \ldots, y_m]$ is the center of $H_m$, which is shown in [K05] Theorem 3.3.1.

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